Boundedness to a logistic chemotaxis system with singular sensitivity

Xiangdong Zhao∗

School of Mathematics, Liaoning Normal University, Dalian 116029, P.R. China

Abstract

In this paper, we study the parabolic-elliptic Keller-Segel system with singular sensitivity and logistic-type source:

\[ \begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v) + ru - \mu u^k, \\
    \tau v_t &= \Delta v - v + u,
\end{align*} \]

under the non-flux boundary conditions in a smooth bounded convex domain \( \Omega \subset \mathbb{R}^n \), \( \chi, r, \mu > 0 \), \( k > 1 \) and \( n \geq 2 \). It is shown that the system possesses a globally bounded classical solution if \( k > \frac{3n-2}{n} \), and \( r > \frac{\chi^2}{4} \) for \( 0 < \chi \leq 2 \), or \( r > \chi - 1 \) for \( \chi > 2 \). In addition, under the same condition for \( r, \chi \), the system admits a global generalized solution when \( k \in (2 - \frac{1}{n}, \frac{3n-2}{n}] \), moreover this global generalized solution should be globally bounded provided \( \frac{r}{\mu} \) and the initial data \( u_0 \) suitably small.

2010MSC: 35K55; 35B45; 35B40; 92C17

Keywords: Keller-Segel system; Singular sensitivity; Logistic source; Boundedness

1 Introduction

Chemotaxis, is a spontaneous cross-diffusion phenomena by which organisms direct their movements in regard to a stimulating chemical. In 1970, Keller and Segel proposed a model to represent the chemotaxis phenomena, i.e., the oriented or partially oriented movement of cells with respect to a chemical signal produced by the cells themselves [1]:

\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega, & t > 0, \\
    \tau v_t &= \Delta v - v + u, & x \in \Omega, & t > 0,
\end{align*}
\]

where \( \tau \in \{0, 1\}, \chi > 0 \). The singular chemotactic sensitive function \( \frac{1}{v^k} \) with \( \chi > 0 \) is derived by the Weber-Fechner law on the response of the cells \( u \) to the stimulating chemical signal \( v \). With the singularity determined by the sensitive function \( \frac{1}{v} \), the cellular movements are governed by the taxis flux \( \frac{\nabla u}{v} \), which may be unbounded when \( v \approx 0 \). Different to the classical Keller-Segel model (i.e., replacing the singular sensitive function \( \frac{1}{v} \) by the constant...
function $\chi$ in (1.1), it is important to obtain a lower bound on $v$ for studying the global dynamical behavior. This can be achieved by a pointwise estimate [2]

$$v(x,t) \geq c_0 \int_{\Omega} u(x,t)dx, \quad x \in \Omega, \ t > 0$$

with $c_0 = c_0(|\Omega|, n) > 0$. Due to the mass conservation of cells $u$ in system (1.1), it is known that the singularity involved in sensitive function $\frac{\chi}{v}$ is in fact absent. Generally, chemotactic sensitive coefficient $\chi > 0$ properly small benefits the global existence-boundedness of solutions to system (1.1), which can be presented in [3–8]. It is pointed that for the parabolic-elliptic case of the system (1.1) ($\tau = 0$) with radial assumption, Nagai and Senba proved that the problem admits a finite time blow-up solution [9] if $\chi > \frac{2n}{n-2}$ with $n \geq 3$, and $\int_{\Omega} u_0|x|^2dx$ sufficiently small.

Consider the chemotaxis system as follows

$$\begin{cases}
    u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + ru - \mu u^k, & x \in \Omega, \ t > 0, \\
    \tau v_t = \Delta v - v + u, & x \in \Omega, \ t > 0,
\end{cases} \quad (1.2)$$

where $\chi, r, \mu > 0$, $k > 1$ and $\tau \in \{0, 1\}$. Such self-limiting growth mechanism involved in the logistic-type source generally benefits the global dynamic of solutions. For parabolic-elliptic case of (1.2) ($\tau = 0$), the system with $k = 2$ possesses a global weak solution if $\mu > 0$ and a global bounded classical solution if $\mu > \frac{2n}{n-2}\chi$ [10]. If $k > 2 - \frac{1}{n}$ with $n \geq 1$, there exists a global very weak solution, which is globally bounded provided $\mu$ sufficiently large and $u_0$ sufficiently small [11]. Replacing $0 = \Delta v - v + u$ in (1.2) by $0 = \Delta v - m(t) + u$ with $m(t) := \frac{1}{|\Omega|} \int_{\Omega} u(x,t)dx$, it is shown with radial assumption that the system admits a finite time blow-up solution if $1 < k < \frac{3}{2} + \frac{1}{2n-2}$ with $n \geq 5$ [12]. For the parabolic-parabolic case of (1.2) ($\tau = 1$), if $k = 2$, $n = 2$ [13], or $n \geq 3$ with $\mu > 0$ sufficiently large [14], the problem possesses globally bounded classical solutions. If $k > 2 - \frac{1}{n}$ with $n \geq 1$, there exists global very weak solutions [15], which are globally bounded provided $\frac{r}{\mu}$ and the initial data all sufficiently small for $n = 3$ [16]. In addition, more properties of solutions to (1.2) can be found in [17–19].

Recall the chemotaxis system with singular sensitivity and logistic source

$$\begin{cases}
    u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v) + ru - \mu u^k, & x \in \Omega, \ t > 0, \\
    \tau v_t = \Delta v - v + u, & x \in \Omega, \ t > 0,
\end{cases} \quad (1.3)$$

where $\chi, r, \mu > 0$, $k > 1$ and $\tau \in \{0, 1\}$. The difficulty in studying global solvability of solutions comes from the hazardous combination of singular sensitive chemotactic mechanism and the self-limiting growth mechanism involved in logistic source. Due to missing the mass conservation for $u$, the singularity contained in chemotactic term may be presence. Similarity to system (1.1), the suitable smallness of chemotactic sensitive coefficient $\chi > 0$ is necessary to establish global existence-boundedness of solutions to system (1.3). If $n, k = 2$, there exists
a unique globally bounded classical solution \([20,21]\), whenever
\[
    r > \begin{cases} 
        \frac{\chi^2}{4}, & 0 < \chi \leq 2, \\
        \chi - 1, & \chi > 2.
    \end{cases} \tag{1.4}
\]

In addition, the author has proved for \(k > 2 - \frac{1}{n}\) that the system (1.3) with \(\tau = 1\) possesses a global very weak solution provided \(\chi\) suitably small related to \(r, k\) [22].

Turn to a chemotaxis-consumption model of the type
\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot \left( \frac{n}{n} \nabla v \right) + ru - \mu u^k, & x \in \Omega, & t > 0, \\
    v_t &= \Delta v - u v, & x \in \Omega, & t > 0,
\end{align*} \tag{1.5}
\]

where \(\chi, r, \mu > 0\) and \(k > 1\). It seems difficult to study the global existence and further global dynamic behavior of solutions to (1.5) than that for the problems (1.2) and (1.3). The troubles lie in the interplay of the consumptive effect with the singular chemotactic mechanism and self-limiting logistic source. Intuitively, the oxygen \(v\) shrinks in (1.5) during evolution, and then enhances the chemotactic strength of the bacteria because of the singular behavior when \(v \approx 0\) in (1.5). This implies the singularity in chemotactic sensitive function \(\frac{\chi}{v}\) should be persistence. Recall some results on the case of \(k = 2\). It is shown for \(n \geq 2\) that there exists a global classical solution if \(0 < \chi < \sqrt{\frac{2}{n}}\) and \(\mu > \frac{n-2}{2n}\), and that for \(n = 1\) the global classical solution is globally bounded if \(\chi, r, \mu > 0\) [23]. Moreover, the problem for \(n \geq 1\) possesses a global generalized solution [24]. If \(k > 1 + \frac{2}{n}\), the author has established the global solvability of classical solutions [25]. We refer to [26–28] for more results on chemotaxis-consumption system without logistic source.

In this paper, we consider the following parabolic-elliptic chemotaxis system with singular sensitivity and logistic-type source
\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot \left( \frac{n}{n} \nabla v \right) + ru - \mu u^k, & x \in \Omega, & t > 0, \\
    0 &= \Delta v - u v + u, & x \in \Omega, & t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, & t > 0, \\
    u(x, 0) &= u_0(x), & x \in \Omega,
\end{align*} \tag{1.6}
\]

where \(\chi, r, \mu > 0, k > 1\). \(\Omega \subset \mathbb{R}^n (n \geq 2)\) is a smooth bounded convex domain, \(\frac{\partial}{\partial \nu}\) denotes the derivation with respect to the outer normal of \(\partial \Omega\), and the initial data
\[
u_0(x) \in C^0(\overline{\Omega}), \ u_0(x) \geq 0 \text{ and } u_0(x) \neq 0, \ x \in \overline{\Omega}. \tag{1.7}
\]

To study the global dynamic behavior of solution to system (1.6) for the more general exponent \(k > 1\) in the logistic-type source \(ru - \mu u^k\), we introduce the generalized solution to (1.6) via the following definitions inspired by [11,24].
Definition 1.1 A pair \((u, v)\) of nonnegative functions

\[ u \in L^1_{\text{loc}}(\Omega \times (0, \infty)), \ v \in L^1_{\text{loc}}((0, \infty); W^{1,1}(\Omega)) \]

will be called a very weak subsolution to (1.6), if

\[ u^k \text{ and } \frac{u}{v} \nabla v \text{ belong to } L^1_{\text{loc}}(\Omega \times (0, \infty)), \]

and moreover

\[
- \int_0^\infty \int_\Omega u \varphi_t - \int_\Omega u \varphi(\cdot, 0) \leq \int_0^\infty \int_\Omega u \Delta \varphi + \chi \int_0^\infty \int_\Omega \frac{u}{v} \nabla u \cdot \nabla \varphi \\
+ r \int_0^\infty \int_\Omega u \varphi - \mu \int_0^\infty \int_\Omega u^k \varphi, \tag{1.8}
\]

\[
- \int_0^\infty \int_\Omega v \psi_t - \int_\Omega v \psi(\cdot, 0) + \int_0^\infty \int_\Omega \nabla v \cdot \nabla \psi + \int_0^\infty \int_\Omega v \psi = \int_0^\infty \int_\Omega u \psi \tag{1.9}
\]

hold for all

\[ \varphi \in C^\infty_0(\bar{\Omega} \times (0, \infty)) \text{ with } \varphi \geq 0 \text{ and } \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, \infty), \tag{1.10} \]

\[ \psi \in L^\infty(\Omega \times (0, \infty)) \cap L^2((0, \infty); W^{1,2}(\Omega)), \text{ and } \psi_t \in L^2(\Omega \times (0, \infty)). \tag{1.11} \]

Definition 1.2 A pair of nonnegative functions \(u \in L^1_{\text{loc}}(\Omega \times (0, \infty)), \ v \in L^1_{\text{loc}}((0, \infty); W^{1,1}(\Omega))\) form a weak logarithmic supersolution to (1.6), if

\[
\frac{u^k}{1 + u}, \frac{\nabla u}{(1 + u)^2} \text{ and } \frac{\nabla v}{v^2} \text{ belong to } L^1_{\text{loc}}(\Omega \times (0, \infty)), \]

and moreover

\[
- \int_0^\infty \int_\Omega \ln(1 + u) \varphi_t - \int_\Omega \ln(1 + u_0) \varphi(\cdot, 0) \geq \int_0^\infty \int_\Omega \frac{\nabla u^2}{(1 + u)^2} \varphi - \chi \int_0^\infty \int_\Omega \frac{u}{(1 + u)^2} \nabla u \cdot \nabla v \varphi \\
- \int_0^\infty \int_\Omega \frac{\nabla u \cdot \nabla \varphi}{1 + u} + \chi \int_0^\infty \int_\Omega \frac{u \nabla u}{(1 + u)v} \cdot \nabla \varphi \\
+ r \int_0^\infty \int_\Omega \frac{u}{1 + u} \varphi - \mu \int_0^\infty \int_\Omega \frac{u^k}{1 + u} \varphi \tag{1.12}
\]

and the equality (1.9) hold for all \(\varphi\) and \(\psi\) satisfying (1.10) and (1.11).

Definition 1.3 A couple of function \((u, v)\) will be called a generalized solution to (1.6) if it

is both a very weak subsolution and a weak logarithmic supersolution of (1.6).

To obtain the global dynamic behavior of solution to system (1.6) for general \(k > 1\), we
will at first establish a positive uniform-in-time lower bound for chemical signal \(v\). With the
aid of a crucial ODE inequality \[29\], this will be realized by a uniform-in-time upper estimate
on the integral \(\int_\Omega u^{-m}dx\) with some \(m > 0\). Furthermore, via a standard process on energy
estimate for \( \int_{\Omega} u^p dx \) with \( p > 1 \), we conclude the global boundedness of the classical solutions if \( k > \frac{3n-2}{n} \) and \( \chi > 0 \) suitably small relative to \( r > 0 \).

In order to deal with the global generalized solution of classical parabolic-elliptic system (1.2) (i.e., without singular sensitive function \( \frac{1}{p} \) in (1.6)) [15], the crucial step is to conclude the relative compactness of the solution \( \{v_\epsilon\}_{\epsilon \in (0,1)} \) to the corresponding regularization problem in \( L^k_{\text{loc}}((0,\infty);W^{1,k}_{\text{loc}}(\Omega)) \) with respect to the strong topology for \( k \in (2 - \frac{1}{n}, 2) \). Differently, for the system (1.6), we will firstly show that \( \{v_\epsilon\}_{\epsilon \in (0,1)} \) has a uniform-in-time lower bound (independent of \( \epsilon \in (0,1) \)). Secondly, it will be derived that for some \( p > \frac{nk}{n-1} \) with \( k > 2 - \frac{1}{n} \) that \( \{\nabla v_\epsilon\}_{\epsilon \in (0,1)} \) is relatively compact in \( L^p_{\text{loc}}(\Omega \times (0,\infty)) \) with respect to the weak topology.

Finally, upon selecting a suitable subsequence, we will obtain a global generalized solution for \( k > 2 - \frac{1}{n} \) with \( n \geq 2 \) by a standard compactness argument. Furthermore, if \( \frac{r}{\mu} \) and the initial data \( u_0 \) suitably small, this global generalized solution is in fact globally bounded.

Now, we state the main results of this paper.

**Theorem 1** Let \( n \geq 2 \) and \( k > \frac{3n-2}{n} \). If \( r, \chi > 0 \) satisfying

\[
 r > \begin{cases} 
 \chi^2, & 0 < \chi \leq 2, \\
 \chi - 1, & \chi > 2,
\end{cases}
\]  

(1.13)

the problem (1.6) possesses a unique globally bounded classical solution.

**Theorem 2** Let \( n \geq 2, k > 2 - \frac{1}{n} \) and \( r, \chi > 0 \) satisfy (1.13). Then the system (1.6) admits at least a global generalized solution.

**Theorem 3** Let \((u, v)\) be the global generalized solution for \( k \in (2 - \frac{1}{n}, \frac{3n-2}{n}] \) established in Theorem 2. Then for \( p > \frac{n(n+2)}{2(n+1)} \) there exist \( \eta, \lambda > 0 \) small such that this solution is globally bounded provided \( \frac{r}{\mu} < \eta \) and \( \int_{\Omega} u_0^p dx < \lambda \).

**Remark 1** Since \( \frac{3n-2}{n} = 2 \) for \( n = 2 \), we conclude by Theorem 1 with [20] that the classical solution to (1.6) for the case \( n = 2 \) must be globally bounded if \( k \geq 2 \). In addition, Theorems 2 and 3 show that \( k < 2 \) is permitted for the global existence-boundedness of solution to (1.6). This extends the global boundedness results for (1.6) with \( k = 2 \) obtained in [20].

**Remark 2** Recall from [10, 11, 30] that the classical parabolic-elliptic chemotaxis system (1.2) possesses a global or further globally bounded classical (or generalized) solution if \( k > 1 \) and \( \mu > 0 \) suitably large in logistic source \( ru - \mu u^k \). While Theorems above say that the global solvability of solution to the system (1.6) requires not only the restriction on logistic kinetics but also the chemotactic sensitive coefficient \( \chi > 0 \) properly small relative to \( r > 0 \), and the same were observed for the parabolic-parabolic case of system (1.3) [22]. Here the difficulty due to the singular sensitivity is substantial.
The rest part of the paper is arranged as follows. In Section 2, we will establish a uniform-in-time lower bound estimate on \( v \) and demonstrate the global boundedness of the classical solution. Section 3 deals with the global existence of classical solution to the corresponding regularization problems. Then we prove the global existence and boundedness to the generalized solution to system (1.6) in Section 4.

2 Global boundedness of classical solutions

We at first give a lemma on the local existence of classical solutions to system (1.6) without proof, which can be obtained by the contraction argument as that in [20, Lemma 2.1].

**Lemma 2.1** Assume that \( u_0 \) satisfies (1.7). If \( k > 1, r, \chi, \mu > 0 \), then there exist \( T_{\text{max}} \in (0, +\infty) \) and a unique pair \((u, v)\) of functions

\[
\begin{align*}
u \in C^0(\Omega \times [0, T_{\text{max}}]) & \cap C^{2,1}(\Omega \times (0, T_{\text{max}})), \\
v \in C^{2,0}(\Omega \times (0, T_{\text{max}})),
\end{align*}
\]

fulfilling (1.6) in the classical sense with \( u, v > 0 \) in \( \Omega \times (0, T_{\text{max}}) \). Moreover, either \( T_{\text{max}} = \infty \), or \( \limsup_{t \to T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \), or \( \liminf_{t \to T_{\text{max}}} \inf_{x \in \Omega} v(x, t) = 0 \). □

Let \((u, v)\) be the local classical solution in this section. Without loss of generality, assume that \( T_{\text{max}} > 1 \). Then we have the following a priori estimates.

**Lemma 2.2** For \( k > 1 \) and \( r, \chi, \mu > 0 \), it holds that

\[
\begin{align*}
\int_\Omega u \, dx & \leq m^*, \quad t \in (0, T_{\text{max}}), \\
\int_t^{t+1} \int_\Omega u^k \, dx \, ds & \leq M_1 \quad \text{for all } t \in (0, T_{\text{max}} - 1)
\end{align*}
\]

with \( m^* = \max \left\{ \int_\Omega u_0 \, dx, |\Omega| \left( \frac{r}{\mu} \right)^{\frac{1}{k-1}} \right\} \) and \( M_1 = \frac{(1+r)m^*}{\mu} \).

**Proof.** Integrate (1.6) over \( \Omega \) to know

\[
\frac{d}{dt} \int_\Omega u \, dx = r \int_\Omega u \, dx - \mu \int_\Omega u^k \, dx
\]

\[
\leq r \int_\Omega u \, dx - \frac{\mu}{|\Omega|^{k-1}} \left( \int_\Omega u \, dx \right)^k, \quad t \in (0, T_{\text{max}})
\]

by the H"older inequality. We get (2.1) by the Bernoulli inequality with (2.4). The estimate (2.2) comes directly from by integrating (2.3) from \( t \) to \( t + 1 \). □

To study the dynamic behavior of solutions to (1.6) for \( k > 1 \), we should pay attention to establish a positive uniform-in-time lower bound for \( v \). With the aid of the following crucial ODE inequality [29], this will be realized by a uniform-in-time upper estimate on the integral \( \int_\Omega u^{-m} \, dx \) with some \( m > 0 \) [20].
Lemma 2.3  [29, Lemma 3.4] Let $T > 0$, and suppose that $y$ is a nonnegative absolutely continuous function on $[0, T)$ satisfying

$$y'(t) + ay(t) \leq f(t)$$

for almost every $t \in (0, T)$

with some $a > 0$ and a nonnegative function $f \in L^1_{\text{loc}}([0, T))$ for which there exists $b > 0$ such that

$$\int_t^{t+1} f(s)dx \leq b$$

for all $t \in [0, T - 1)$.

Then

$$y(t) \leq \max\{y(0) + b, \frac{b}{a} + 2b\}$$

for all $t \in (0, T)$.

Now, we have:

Lemma 2.4  Let $k > 1$, $\mu > 0$ and $r, \chi > 0$ satisfy

$$r > \begin{cases} \frac{\chi^2}{1}, & 0 < \chi \leq 2, \\ \chi - 1, & \chi > 2. \end{cases}$$

(2.5)

Then there exists some $\delta_0 > 0$ such that

$$v(x, t) \geq \delta_0$$

for all $(x, t) \in \Omega \times (0, T_{\text{max}})$.  

(2.6)

Proof. Since $u \in C^0(\bar{\Omega} \times [0, T_{\text{max}}])$ by Lemma 2.1, we know that there exists $t_0 \in (0, T_{\text{max}})$ such that

$$\int_\Omega u(x, t)dx \geq \frac{1}{2} \int_\Omega u_0dx, \quad t \in (0, t_0]$$

(2.7)

Invoke the pointwise lower bound estimate in [2] to know

$$v(x, t) \geq c_1 \int_\Omega u(x, t)dx$$

for all $x \in \Omega$ and $t \in (0, T_{\text{max}})$

(2.8)

with some $c_1 > 0$. Hence, (2.8) with (2.7) yields

$$v(x, t) \geq c_1 \int_\Omega u(x, t)dx \geq \frac{c_1}{2} \int_\Omega u_0dx := \beta_0$$

for all $x \in \Omega$ and $t \in (0, t_0]$.  

(2.9)

For $m > 0$, it is known from (1.6)1 that

$$\frac{1}{m} \frac{d}{dt} \int_\Omega u^{-m}dx = -\int_\Omega u^{-m-1}[\Delta u - \chi \nabla \cdot \frac{u}{v} \nabla v] + ru - \mu u^k]dx$$

$$= -(m + 1) \int_\Omega u^{-m-2} |\nabla u|^2 dx + \chi(m + 1) \int_\Omega u^{-m-1} \frac{\nabla u \cdot \nabla v}{v} dx$$

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\[-r \int_{\Omega} u^{-m} \, dx + \mu \int_{\Omega} u^{-m-1+k} \, dx, \quad t \in (t_0, T_{\max}). \tag{2.10}\]

If \( a \in (0, \chi) \), we get from (1.6) that
\[
(m + 1)a \int_{\Omega} u^{-m-1} \nabla u \cdot \nabla v \, dx = -\frac{(m + 1)a}{m} \int_{\Omega} \nabla u^{-m} \cdot \nabla v \, dx \\
\leq -\frac{(m + 1)a}{m} \int_{\Omega} u^{-m} |\nabla v|^2 \, dx + \frac{(m + 1)a}{m} \int_{\Omega} u^{-m} \, dx, \tag{2.11}\]
and by Young’s inequality that
\[
(m + 1)(\chi - a) \int_{\Omega} u^{-m-1} \nabla u \cdot \nabla v \, dx \leq (m + 1) \int_{\Omega} u^{-m-2} |\nabla u|^2 \, dx \\
+ \frac{(m + 1)(\chi - a)^2}{4} \int_{\Omega} u^{-m} |\nabla v|^2 \, dx. \tag{2.12}\]

Let \( m := \frac{4a}{(\chi - a)^2} \) for \( a \in (0, \chi) \). Then \( \frac{(m+1)(\chi-a)^2}{4} = \frac{(m+1)a}{m} \). Combining (2.10)–(2.12), we have
\[
\frac{1}{m} \frac{d}{dt} \int_{\Omega} u^{-m} \, dx \leq -(r - \frac{(m+1)a}{m}) \int_{\Omega} u^{-m} \, dx + \mu \int_{\Omega} u^{-m-1+k} \, dx, \quad t \in (t_0, T_{\max}). \tag{2.13}\]

Denote
\[
f(a) := -4(r - \frac{(m+1)a}{m}) = a^2 - (2\chi - 4)a + \chi^2 - 4r.
\]

A direct calculation shows that \( \Delta = 16(r + 1 - \chi) > 0 \) for \( r > \max\{\chi - 1, 0\} \), and hence \( f(a) < 0 \) for \( a \in (a_-, a_+) \), here \( a_\pm = \chi - 2 \pm 2\sqrt{r + 1 - \chi} \). By the Viète formula, we know \( a_- < 0 < a_+ \) if \( r > \frac{\chi^2}{4} \) with \( \chi > 0 \), and \( 0 < a_- < \chi < a_+ \) if \( \chi - 1 < r \leq \frac{\chi^2}{4} \) with \( \chi > 2 \). Therefore, if \( r, \chi > 0 \) satisfying (2.5), there exists some \( c_0 > 0 \) such that
\[-(r - \frac{(m+1)a}{m}) = \frac{f(\rho)}{4} \leq -c_0 < 0 \quad \text{for} \quad a \in (0, \chi) \cap (a_-, a_+).
\]

If \( k - 1 - m = 0 \), it is known from (2.13) that
\[
\frac{1}{m} \frac{d}{dt} \int_{\Omega} u^{-m} \, dx \leq -c_0 \int_{\Omega} u^{-m} \, dx + \mu |\Omega|, \quad t \in (t_0, T_{\max}). \tag{2.14}\]

If \( k - 1 - m < 0 \), we obtain by Young’s inequality with (2.13) that
\[
\frac{1}{m} \frac{d}{dt} \int_{\Omega} u^{-m} \, dx \leq -c_0 \int_{\Omega} u^{-m} \, dx + \mu |\Omega| \left( \frac{2\mu}{c_0} \right)^{\frac{m+1-k}{k+1}}, \quad t \in (t_0, T_{\max}). \tag{2.15}\]

Similar process for the case of \( k - 1 - m > 0 \), we get
\[
\frac{1}{m} \frac{d}{dt} \int_{\Omega} u^{-m} \, dx \leq -c_0 \int_{\Omega} u^{-m} \, dx + \int_{\Omega} u^k \, dx + \mu \frac{k}{m+1} |\Omega|, \quad t \in (t_0, T_{\max}). \tag{2.16}\]

The estimates (2.14)–(2.16) show for \( k > 1 \) that
\[
\frac{1}{m} \frac{d}{dt} \int_{\Omega} u^{-m} \, dx \leq -\frac{c_0}{2} \int_{\Omega} u^{-m} \, dx + \int_{\Omega} u^k \, dx + C_1, \quad t \in (t_0, T_{\max}). \tag{2.17}\]
with $C_1 = \mu|\Omega|(\frac{2\mu}{c_0})^{\frac{m+1-k}{k-1}} + \mu^{\frac{k}{m+1}}|\Omega|$. Based on Lemma 2.3 with (2.2) and (2.17), then
\[
\int_{\Omega} u^{-m}dx \leq C_2, \quad t \in (t_0, T_{\max}) \tag{2.18}
\]
with $C_2 = \{(C_1 + M_1)m + \mu \int_{\Omega} u(x, t_0)^{-m}dx, \quad (C_1 + M_1)m + 2^{m^2(C_1 + M_1)}\}$. Let $\alpha := \frac{m}{m+1} \in (0, 1)$. Then for $k > 1$ we obtain from (2.18) and (2.8) that
\[
v(x, t) \geq c_1 \int_{\Omega} ud\bar{x} \geq c_1|\Omega|^{\frac{m+1}{m}} \left( \int_{\Omega} u^{-m}dx \right)^{-\frac{1}{m}} \geq c_1 C_2^{-\frac{1}{m}}|\Omega|^{\frac{m+1}{m}} =: \eta_0
\]
by the Hölder inequality for all $(x, t) \in \Omega \times (t_0, T_{\max})$. This together with (2.9) concludes (2.6) with $\delta_0 = \min\{\beta_0, \eta_0\}$. □

Based on the uniform-in-time lower bound estimate for $v$ in Lemma 2.4, we establish the following $L^p$-estimate for $u$.

**Lemma 2.5** If $k > \frac{3n-2}{n}$, $\mu > 0$ and $\alpha, \chi > 0$ satisfying (2.5), then for $p > 1$ there exists some $M_2 > 0$ such that
\[
\int_{\Omega} u^pdx \leq M_2, \quad t \in (0, T_{\max}). \tag{2.19}
\]

**Proof.** A simple calculation with (1.6)1 and (2.6) shows
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^pdx = -(p-1) \int_{\Omega} u^{p-2}|\nabla u|^2dx + \chi(p-1) \int_{\Omega} \frac{u^{p-1}}{v} \nabla u \cdot \nabla vdx + r \int_{\Omega} u^pdx - \mu \int_{\Omega} u^{p+k-1}dx
\]
\[
\leq -\frac{1}{p} \int_{\Omega} u^pdx + \frac{\chi^2(p-1)}{4\delta_0^2} \int_{\Omega} u^p|\nabla v|^2dx - \frac{\mu}{2} \int_{\Omega} u^{p+k-1}dx + C_3
\]
\[
\leq -\frac{1}{p} \int_{\Omega} u^pdx + C_4 \int_{\Omega} |\nabla v|^{\frac{2(p+k-1)}{k-1}}dx - \frac{\mu}{4} \int_{\Omega} u^{p+k-1}dx + C_3 \tag{2.20}
\]
with $C_3 = (r+1)(\frac{2(r+1)}{\mu})^{\frac{p}{k-1}}$ and $C_4 = \frac{\chi^2(p-1)}{4\delta_0^2}(\frac{\chi^2(p-1)}{\delta_0^2})^{\frac{2p}{k-1}}$ for $t \in (0, T_{\max})$. Since $\int_{\Omega} v(x, t)dx = \int_{\Omega} u(x, t)dx \leq m^*, \quad t \in (0, T_{\max})$, invoking the classical result by Brézis and Strauss [31] and the Minkowski inequality, we get for $r \in (1, \frac{n}{n-1})$ that
\[
\|v\|_{W^{1, r}(\Omega)} \leq C_{BS}\|\Delta v\|_{L^1(\Omega)} \leq C_{BS}\|v - u\|_{L^1(\Omega)} \leq 2C_{BS}m^* \tag{2.21}
\]
with some $C_{BS} > 0$. According to the standard elliptic $L^p$-theory, we know from (1.6)2 for $m \geq 1$ that
\[
\|v\|_{W^{2, m}(\Omega)} \leq C_5\|u\|_{L^m(\Omega)} \tag{2.22}
\]
with some $C_5 > 0$, and thus by the Gagliardo-Nirenberg inequality with (2.21) that
\[
\|\nabla v\|_{L^{\frac{2(p+k-1)}{p+k}}(\Omega)} \leq C_{GN}\|v\|_{W^{2, p+k-1}(\Omega)}^{a}\|\nabla v\|_{L^{r}(\Omega)}^{1-a} \leq C_6\|u\|_{L^{p+k-1}(\Omega)}^{a} \leq C_0\|u\|_{L^{p+k-1}(\Omega)} \tag{2.23}
\]
with some $C_6 > 0$, where
\[ a = \frac{n}{\tau} - \frac{(k-1)n}{2(p+k-1)} \cdot \frac{2}{1 - \frac{n}{p+k-1} + \frac{n}{\tau}}. \]
If $k > \frac{3n-2}{n}$, then $a \in (0, 1)$ and $\frac{2a}{k-1} < 1$. Consequently, we obtain from (2.20) and (2.23) that
\[ \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx \leq \frac{1}{p} \int_{\Omega} u^p dx + C_7, \quad t \in (0, T_{\text{max}}), \]
by Young’s inequality with $C_7 = C_3 + C_4 |\Omega| \left( \frac{4C_4}{\mu} \right)^\frac{2a}{k-1-2a} C_6^{\frac{2(p+k-1)}{k-1-2a}}$. This concludes (2.19) by the Bernoulli inequality with some $M_2 > 0$. \( \square \)

**Proof of the Theorem 1** By the variation-of-constants formula for $u$ and the order preserving of the Neumann heat semigroup \( \{e^{t\Delta}\}_{t \geq 0} \) with the positivity of $u$, we know
\[ u(x, t) = e^{t\Delta} u_0 - \chi \int_0^t e^{(t-s)\Delta} \cdot \left( \frac{\mu}{v} \nabla v \right) ds + \int_0^t e^{(t-s)\Delta} (ru - \mu u^k) ds, \]
\[ \leq e^{t\Delta} u_0 - \chi \int_0^t e^{(t-s)\Delta} \cdot \left( \frac{\mu}{v} \nabla v \right) ds + r \int_0^t e^{(t-s)\Delta} u ds, \quad (x,t) \in \Omega \times (0, T_{\text{max}}). \quad (2.24) \]
Let $p > n$ in Lemma 2.5. Then we have by the classical elliptic equation theory to (1.6)$_2$ with (2.22) and (2.19) that
\[ \|\nabla v\|_{L^\infty(\Omega)} \leq C_9 \|v\|_{W^{2,p}(\Omega)} \leq C_9 C_5 \|u\|_{L^p(\Omega)} \leq C_{10}, \quad t \in (0, T_{\text{max}}) \quad (2.25) \]
with some $C_9, C_{10} > 0$. Consequently, invoking the homogeneous Neumann semigroup estimates in [32, Lemma 1.3] with (2.1), it is known from (2.24) with (2.6), (2.19) and (2.25) that
\[
\|u\|_{L^\infty(\Omega)} \leq \|e^{t\Delta} u_0\|_{L^\infty(\Omega)} + \chi \int_0^t \|e^{(t-s)\Delta} \cdot \left( \frac{\mu}{v} \nabla v \right)\|_{L^\infty(\Omega)} ds + r \int_0^t \|e^{(t-s)\Delta} u\|_{L^\infty(\Omega)} ds \\
\leq \|u_0\|_{L^\infty(\Omega)} + \chi K \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{1}{p}}) e^{-\lambda_1(t-s)} \|u\|_{L^2(\Omega)} ds + \frac{rm^*}{|\Omega|} \\
+ r K \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{1}{p}}) e^{-\lambda_1(t-s)} \|u - u^*\|_{L^2(\Omega)} ds + \frac{rm^*}{|\Omega|} \\
\leq \bar{C}, \quad t \in (0, T_{\text{max}}) \quad (2.26) \]
with some $\bar{C} > 0$. This concludes $T_{\text{max}} = \infty$ by Lemma 2.1, i.e., the classical solution $(u,v)$ is globally bounded. \( \square \)
3 Regularization problem

To deal with the global existence-boundedness of generalized solution to (1.6) for \( k > 1 \), we introduce an appropriate regularization problem related to (1.6)

\[
\begin{cases}
  u_{\epsilon t} = \Delta u_{\epsilon} - \chi \nabla \cdot \left( \frac{u_{\epsilon}}{v_{\epsilon}} \nabla v_{\epsilon} \right) + ru_{\epsilon} - \mu u_{\epsilon}^{k} - \epsilon u_{\epsilon}^{k+1}, & x \in \Omega, \ t > 0, \\
  0 = \Delta v_{\epsilon} - v_{\epsilon} + u_{\epsilon}, & x \in \Omega, \ t > 0, \\
  \frac{\partial u_{\epsilon}}{\partial \nu} = \frac{\partial v_{\epsilon}}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
  u_{\epsilon}(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\]

with \( \epsilon \in (0, 1) \). The local classical solution of the regularization problem (3.1) with general \( k > 1 \) can be obtained in the similar arguments [20]. That is:

**Lemma 3.1** Assume that \( u_0 \) satisfies (1.7). Let \( k > 1, r, \chi, \mu > 0 \). Then for each \( \epsilon \in (0, 1) \) there exist \( T_{\max, \epsilon} \in (0, +\infty) \) and a unique pair \((u_{\epsilon}, v_{\epsilon})\) of functions satisfying (1.6) in the classical sense with \( u_{\epsilon}, v_{\epsilon} > 0 \) in \( \overline{\Omega} \times (0, T_{\max, \epsilon}) \). Moreover, either \( T_{\max, \epsilon} = \infty \), or \( \limsup_{t \to T_{\max, \epsilon}} \| u_{\epsilon}(\cdot, t) \|_{L^\infty(\Omega)} = \infty \), or \( \liminf_{t \to T_{\max, \epsilon}} \inf_{x \in \Omega} v_{\epsilon}(x, t) = 0 \). □

Let \((u_{\epsilon}, v_{\epsilon})\) is the local classical solution to system (3.1) for each \( \epsilon \in (0, 1) \) and \( k > 1 \). Without loss of generality, assume that \( T_{\max, \epsilon} > 1 \) for each \( \epsilon \in (0, 1) \). Then we have the following a priori estimates.

**Lemma 3.2** With \( k > 1 \) and \( \mu, r, \chi > 0 \), it holds for each \( \epsilon > 0 \) that

\[
\begin{align*}
  \int_{\Omega} u_{\epsilon} dx & \leq m^*, \ t \in (0, T_{\max, \epsilon}), \\
  \int_{0}^{T} \int_{\Omega} u_{\epsilon}^{k} dx ds & \leq M_1(1 + T), \ T \in (0, T_{\max, \epsilon} - 1) \\
  \int_{t}^{t+1} \int_{\Omega} u_{\epsilon}^{k} dx ds & \leq M_1, \ t \in (0, T_{\max, \epsilon})
\end{align*}
\]

with \( m^*, M_1 > 0 \) defined in Lemma 2.2.

**Proof.** Integrate (3.1) over \( \Omega \) with the Hölder inequality to know that

\[
\begin{align*}
  \frac{d}{dt} \int_{\Omega} u_{\epsilon} dx &= r \int_{\Omega} u_{\epsilon} dx - \mu \int_{\Omega} u_{\epsilon}^{k} dx - \epsilon \int_{\Omega} u_{\epsilon}^{k+1} dx \\
  &\leq r \int_{\Omega} u_{\epsilon} dx - \frac{\mu}{|\Omega|^{k-1}} \left( \int_{\Omega} u_{\epsilon} dx \right)^k, \ t > 0.
\end{align*}
\]
We get (3.2) by the Bernoulli inequality with (3.6). The estimates (3.3) and (3.4) come from by integrating (3.5) with (3.2). □

In order to deal with the global existence of classical solution to (3.1) for each \( \epsilon \in (0,1) \), we give a uniform-in-time lower bound of \( v_\epsilon \) for all \((x,t) \in \Omega \times (0,T_{max,\epsilon}).\)

**Lemma 3.3** Let \( k > 1, \mu > 0 \) and \( r, \chi > 0 \) satisfy (2.5). Then for each \( \epsilon \in (0,1) \) there exists some \( \delta_1 > 0 \) such that

\[
v_\epsilon(x,t) \geq \delta_1 \quad \text{for all} \quad (x,t) \in \Omega \times (0,T_{max,\epsilon}). \tag{3.7}
\]

**Proof.** Since \( u_\epsilon \in C^0(\bar{\Omega} \times [0,T_{max,\epsilon}) \) by Lemma 3.1 for each \( \epsilon \in (0,1) \), we know by a continuous argument that there exists \( t_0 \in (0,1) \) such that

\[
\int_\Omega u_\epsilon(x,t)dx \geq \frac{1}{2} \int_\Omega u_0dx, \quad t \in (0,t_0]. \tag{3.8}
\]

If \( m > 0 \), a direct computation with (3.1) shows

\[
\frac{1}{m} \frac{d}{dt} \int_\Omega u_\epsilon^{-m}dx = -\int_\Omega u_\epsilon^{-m-1}[\Delta u_\epsilon - \chi \nabla \cdot (\frac{u_\epsilon}{v_\epsilon} \nabla v_\epsilon)]dx
\]

\[
- r \int_\Omega u_\epsilon^{-m}dx + \mu \int_\Omega u_\epsilon^{-m-1+k}dx + \epsilon \int_\Omega u_\epsilon^{-m+k}dx
\]

\[
= -(m+1) \int_\Omega u_\epsilon^{-m-2}|\nabla u_\epsilon|^2dx + \chi(m+1) \int_\Omega \frac{u_\epsilon^{-m-1}}{v_\epsilon} \nabla u_\epsilon \cdot \nabla v_\epsilon dx
\]

\[
- r \int_\Omega u_\epsilon^{-m}dx + \mu \int_\Omega u_\epsilon^{-m-1+k}dx + \epsilon \int_\Omega u_\epsilon^{-m+k}dx, \quad t \in (t_0,T_{max,\epsilon}) \tag{3.9}
\]

for all \( \epsilon \in (0,1) \). If \( m = \frac{4k}{(\chi-a)r} \) with \( a \in (0,\chi) \) and \( r, \chi > 0 \) satisfying (1.7), there exists some \( c_0 > 0 \) such that \(- (r - \frac{(m+1)a}{m}) \leq -c_0 < 0 \) by a similar argument as that in Lemma 2.4, and hence it holds from (3.9) that

\[
\frac{1}{m} \frac{d}{dt} \int_\Omega u_\epsilon^{-m}dx \leq -c_0 \int_\Omega u_\epsilon^{-m}dx + \mu \int_\Omega u_\epsilon^{-m-1+k}dx + \int_\Omega u_\epsilon^{-m+k}dx, \quad t \in (t_0,T_{max,\epsilon}) \tag{3.10}
\]

for all \( \epsilon \in (0,1) \).

If \( k - m \leq 0 \), we have from (3.10) by the Young’s inequality that

\[
\frac{1}{m} \frac{d}{dt} \int_\Omega u_\epsilon^{-m}dx \leq -c_0 \int_\Omega u_\epsilon^{-m}dx + \left(1 + \left(\frac{4}{c_0}\right)^{\frac{m-k}{k}} + \mu \left(\frac{4\mu}{c_0}\right)^{\frac{m+1-k}{k-1}}\right)|\Omega|. \tag{3.11}
\]

Similarly, if \( k - 1 - m \leq 0 < k - m \), we get

\[
\frac{1}{m} \frac{d}{dt} \int_\Omega u_\epsilon^{-m}dx \leq -c_0 \int_\Omega u_\epsilon^{-m}dx + \int_\Omega u_\epsilon^{k}dx + \left(2 + \mu \left(\frac{2\mu}{c_0}\right)^{\frac{m+1-k}{k-1}}\right)|\Omega|, \tag{3.12}
\]

and if \( k - 1 - m > 0 \),

\[
\frac{1}{m} \frac{d}{dt} \int_\Omega u_\epsilon^{-m}dx \leq -c_0 \int_\Omega u_\epsilon^{-m}dx + \int_\Omega u_\epsilon^{k}dx + \left(\mu \left(\frac{k}{m+1}\right)^{\frac{k-1-m}{m+1}} + 2 \left(\frac{k-m}{m}\right)^{k-m}\right)|\Omega|. \tag{3.13}
\]

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The estimates (3.11)–(3.13) yield for \( k > 1, \mu > 0 \) and \( r, \chi > 0 \) satisfying (2.5) that
\[
\frac{1}{m} \frac{d}{dt} \int_{\Omega} u_\epsilon^{-m} dx \leq -\frac{c_0}{2} \int_{\Omega} u_\epsilon^{-m} dx + \int_{\Omega} u_\epsilon^{\frac{\mu}{2}} dx + C_{11}, \ t \in (t_0, T_{\text{max,} \epsilon}) \tag{3.14}
\]
with some \( C_{11} > 0 \) for all \( \epsilon \in (0, 1) \). By a similar discussion as that in Lemma 2.4 with (3.4), we conclude the uniformly lower bound estimate (3.7) with some \( \delta_1 > 0 \). \( \square \)

If \( k > 2 - \frac{2}{n} \), the local classical solution \((u_\epsilon, v_\epsilon)\) for each \( \epsilon \in (0, 1) \) is in fact global.

**Lemma 3.4** Let \( k > 2 - \frac{2}{n}, \mu > 0 \) and \( r, \chi > 0 \) satisfy (2.5). Then for each \( \epsilon \in (0, 1) \) the system (3.1) possesses a global classical solution \((u_\epsilon, v_\epsilon)\).

**Proof.** For each \( \epsilon \in (0, 1) \), let \((u_\epsilon, v_\epsilon)\) be the local classical solution to the regularization problem (3.1) with general \( k > 1 \). Since \( k + 1 > \frac{3m - 2}{n} \) for \( k > 2 - \frac{2}{n} \), replacing \( \delta_0 \) by \( \delta_1 \) in (2.20) and (2.26), we can prove for each \( \epsilon \in (0, 1) \) that the solution \((u_\epsilon, v_\epsilon)\) is global by a similar arguments in Lemma 2.5 and (2.24)–(2.26). \( \square \)

**Corollary 3.1** Let \( k > 2 - \frac{2}{n}, \mu > 0 \) and \( r, \chi > 0 \) satisfy (2.5). Then the estimates in Lemmas 3.2 and 3.3 are valid with \( T_{\text{max,} \epsilon} = \infty \) and for all \( \epsilon \in (0, 1) \). For convenience, we omit the new marks on these estimates.

**Proof.** For each \( \epsilon \in (0, 1) \), it is shown from Lemma 3.4 that the regularization problem (3.1) possesses a global classical \((u_\epsilon, v_\epsilon)\). This yields \( T_{\text{max,} \epsilon} = \infty \) for all \( \epsilon \in (0, 1) \). Since the constants \( m^*, M_1 > 0 \) are not dependent on \( \epsilon \), the estimates (3.2)–(3.4) in Lemma 3.2 are valid for all \( \epsilon \in (0, 1) \). In addition, we know \( u_\epsilon \in C^0(\bar{\Omega} \times [0, 1]) \) for all \( \epsilon \in (0, 1) \), which concludes the estimate (3.8) with some \( t_0 > 0 \) independent of \( \epsilon \). A similar arguments (the constants there are all independent of \( \epsilon \)) from (3.9) to (3.14), and in Lemma 2.4 with (3.4) indicate the estimate (3.7) for all \( \epsilon \in (0, 1) \). \( \square \)

Now, we deal with a spatio-temporal integral estimate on \( \nabla \ln v_\epsilon \) for all \( \epsilon \in (0, 1) \).

**Lemma 3.5** Let \( k > 2 - \frac{2}{n}, \mu > 0 \) and \( r, \chi > 0 \) satisfy (2.5). Then
\[
\int_0^T \int_{\Omega} \frac{|\nabla v_\epsilon|}{v_\epsilon^2} dx ds \leq |\Omega| T, \ T > 0 \tag{3.15}
\]
for all \( \epsilon \in (0, 1) \).

**Proof.** Multiply (3.1) by \( \frac{1}{v_\epsilon} \) and integrate by part to get
\[
0 = \int_{\Omega} \frac{1}{v_\epsilon} [\Delta v_\epsilon - v_\epsilon + u_\epsilon] dx = \int_{\Omega} \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2} dx - |\Omega| + \int_{\Omega} u_\epsilon dx, \ t > 0. \tag{3.16}
\]
This yields conclusion (3.15) by integrating (3.16) from 0 to \( T \) for all \( \epsilon \in (0, 1) \). \( \square \)

We proceed to derive another spatio-temporal integral estimate on \( u_\epsilon \) for all \( \epsilon \in (0, 1) \).
Lemma 3.6 For $k > 2 - \frac{2}{n}$, $\mu > 0$ and $r, \chi > 0$ satisfying (2.5), there exists some $M_3 > 0$ such that

$$
\int_0^T \int_\Omega \frac{\|\nabla u_\epsilon\|^2}{(1 + u_\epsilon)^2} dx dt \leq M_3(1 + T), \quad T > 0
$$

(3.17)

for all $\epsilon \in (0, 1)$.

Proof. A direct calculation with (3.1)_1 shows

$$
\frac{d}{dt} \int_\Omega \ln(1 + u_\epsilon) dx = \int_\Omega \frac{1}{1 + u_\epsilon} [\Delta u_\epsilon - \chi \nabla \cdot (\frac{u_\epsilon}{v_\epsilon} \nabla v_\epsilon) + ru_\epsilon - \mu u_\epsilon^k + \epsilon u_\epsilon^{k+1}] dx
$$

$$
= \int_\Omega \frac{\|\nabla u_\epsilon\|^2}{(1 + u_\epsilon)^2} dx - \chi \int_\Omega \frac{u_\epsilon}{v_\epsilon} \nabla u_\epsilon \cdot v_\epsilon dx
$$

$$
+ r \int_\Omega \frac{u_\epsilon}{1 + u_\epsilon} dx - \mu \int_\Omega \frac{u_\epsilon^k}{1 + u_\epsilon} dx - \epsilon \int_\Omega \frac{u_\epsilon^{k+1}}{1 + u_\epsilon} dx, \quad t > 0.
$$

(3.18)

By Young’s inequality, we have

$$
\chi \int_\Omega \frac{u_\epsilon}{(1 + u_\epsilon)^2 v_\epsilon} \nabla u_\epsilon \cdot v_\epsilon dx \leq \frac{1}{2} \int_\Omega \frac{\|\nabla u_\epsilon\|^2}{(1 + u_\epsilon)^2} dx + \frac{\chi^2}{2} \int_\Omega \frac{\|\nabla v_\epsilon\|^2}{v_\epsilon^2} dx.
$$

(3.19)

It is known from (3.18) and (3.19) that

$$
\int_\Omega \frac{\|\nabla u_\epsilon\|^2}{(1 + u_\epsilon)^2} dx \leq 2 \int_\Omega \ln(1 + u_\epsilon) dx + \chi^2 \int_\Omega \frac{\|\nabla v_\epsilon\|^2}{v_\epsilon^2} dx
$$

$$
- 2r \int_\Omega \frac{u_\epsilon}{1 + u_\epsilon} dx + 2 \mu \int_\Omega \frac{u_\epsilon^k}{1 + u_\epsilon} dx + 2 \epsilon \int_\Omega \frac{u_\epsilon^{k+1}}{1 + u_\epsilon} dx, \quad t > 0.
$$

(3.20)

Combining (3.20) with (3.2), (3.3) and (3.15), then we get with the fact $0 < \ln(1 + a) \leq a$ for $a > 0$ that

$$
\int_0^T \int_\Omega \frac{\|\nabla u_\epsilon\|^2}{(1 + u_\epsilon)^2} dx ds \leq 2 \int_\Omega \ln(1 + u_\epsilon(\cdot, t)) dx - 2 \int_\Omega \ln(1 + u_0) dx + \chi^2 \int_0^T \int_\Omega \frac{\|\nabla v_\epsilon\|^2}{v_\epsilon^2} dx ds
$$

$$
- 2r \int_0^T \int_\Omega \frac{u_\epsilon}{1 + u_\epsilon} ds + \mu \int_0^T \int_\Omega \frac{u_\epsilon^k}{1 + u_\epsilon} ds + \epsilon \int_0^T \int_\Omega \frac{u_\epsilon^{k+1}}{1 + u_\epsilon} ds
$$

$$
\leq 2 \int_\Omega u_\epsilon(\cdot, t) dx + (1 + \mu) \int_\Omega u_\epsilon^k dx + \chi^2 \int_\Omega u_\epsilon^{k+1} dx, \quad T > 0
$$

(3.21)

with $M_3 > 0$. □

Next, we deal with the estimate on the time derivative of $\ln(1 + u_\epsilon)$ for all $\epsilon \in (0, 1)$.

Lemma 3.7 For $k > 2 - \frac{2}{n}$, $\mu > 0$ and $r, \chi > 0$ satisfying (2.5), there exists $M_4 > 0$ such that

$$
\int_0^T \left\| \frac{d}{dt} \ln(1 + u_\epsilon) \right\|_{(W_0^{n+1.2}(\Omega))^*} ds \leq M_4(1 + T), \quad T > 0
$$

(3.22)

for all $\epsilon \in (0, 1)$.
Proof. Let \( \phi \in W^{n+1,2}_0(\Omega) \). Then we have from (3.1) that

\[
\int_\Omega \frac{d}{dt} \ln(1 + u_\epsilon) \phi \, dx = \int_\Omega \frac{1}{1 + u_\epsilon} \phi [\nabla u_\epsilon - \chi \nabla \cdot (\frac{u_\epsilon}{v_\epsilon} \nabla v_\epsilon) + ru_\epsilon - \mu u_\epsilon^k - \epsilon u_\epsilon^{k+1}] \, dx
\]

\[
= -\int_\Omega \nabla \left( \frac{\nabla u_\epsilon}{1 + u_\epsilon} \right) \cdot (\nabla u_\epsilon - \chi \frac{u_\epsilon}{v_\epsilon} \nabla v_\epsilon) \, dx + r \int_\Omega \frac{u_\epsilon}{1 + u_\epsilon} \phi \, dx - \mu \int_\Omega \frac{u_\epsilon^k}{1 + u_\epsilon} \phi \, dx - \epsilon \int_\Omega \frac{u_\epsilon^{k+1}}{1 + u_\epsilon} \phi \, dx
\]

\[
= \int_\Omega \frac{\nabla u_\epsilon^2}{(1 + u_\epsilon)^2} \phi \, dx - \int_\Omega \frac{\nabla u_\epsilon \cdot \nabla \phi}{1 + u_\epsilon} \, dx - \chi \int_\Omega \frac{u_\epsilon}{(1 + u_\epsilon)^2 v_\epsilon} \nabla u_\epsilon \cdot \nabla v_\epsilon \phi \, dx
\]

\[
+ \chi \int_\Omega \frac{u_\epsilon}{(1 + u_\epsilon)v_\epsilon} \nabla v_\epsilon \cdot \nabla \phi \, dx + r \int_\Omega \frac{u_\epsilon}{1 + u_\epsilon} \phi \, dx - \mu \int_\Omega \frac{u_\epsilon^k}{1 + u_\epsilon} \phi \, dx - \epsilon \int_\Omega \frac{u_\epsilon^{k+1}}{1 + u_\epsilon} \phi \, dx
\]

\[
\leq \left( \int_\Omega \frac{|\nabla u_\epsilon|^2}{(1 + u_\epsilon)^2} \, dx \right) \|\phi\|_{L^\infty(\Omega)} + \left( \int_\Omega \frac{|\nabla u_\epsilon|^2}{(1 + u_\epsilon)^2} \, dx \right)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\Omega)}
\]

\[
+ \chi \left( \int_\Omega \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2} \, dx \right)^{\frac{1}{2}} \|\nabla \phi\|_{L^2(\Omega)} + \left( \int_\Omega \frac{|\nabla u_\epsilon|^2}{(1 + u_\epsilon)^2} \, dx \right)^{\frac{1}{2}} \left( \int_\Omega \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2} \, dx \right)^{\frac{1}{2}} \|\phi\|_{L^\infty(\Omega)}
\]

\[
+ (r + (1 + \mu) \int_\Omega u_\epsilon^k \, dx) \|\phi\|_{L^\infty(\Omega)}, \quad t > 0
\]

by the Hölder inequality. Since \( W^{n+1,2}_0(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \), it is known by Young’s inequality with (3.23) that

\[
\left| \int_\Omega \frac{d}{dt} \ln(1 + u_\epsilon) \phi \, dx \right| \leq C_{12} \left( 1 + \int_\Omega u_\epsilon^k \, dx + \int_\Omega \frac{|\nabla u_\epsilon|^2}{(1 + u_\epsilon)^2} \, dx + \int_\Omega \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2} \, dx \right) \|\phi\|_{W^{n+1,2}_0(\Omega)}
\]

with \( C_{12} > 0 \) for \( t > 0 \). Integrating (3.24) from 0 to \( T \), we obtain from (3.3), (3.15), (3.17) and (3.24) that

\[
\int_0^T \left\| \frac{d}{dt} \ln(1 + u_\epsilon) \right\|_{(W^{n+1,2}_0(\Omega))'} \, ds \leq \sup_{\phi \in W^{n+1,2}_0(\Omega), \|\phi\|_{W^{n+1,2}_0(\Omega)} \leq 1} \int_0^T \left| \int_\Omega \frac{d}{dt} \ln(1 + u_\epsilon) \phi \, dx \right| \, ds
\]

\[
\leq C_{12} \left( T + \int_0^T \int_\Omega u_\epsilon^k \, dx \, ds + \int_0^T \int_\Omega \frac{|\nabla u_\epsilon|^2}{(1 + u_\epsilon)^2} \, dx \, ds + \int_0^T \int_\Omega \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2} \, dx \, ds \right)
\]

\[
\leq M_4(1 + T), \quad T > 0
\]

with some \( M_4 > 0 \). The proof is complete. \( \square \)

Based on Lemma 3.3, we further prove the following estimates on \( v_\epsilon \) for all \( \epsilon \in (0, 1) \).

**Lemma 3.8** Let \( k > 2 - \frac{2}{n} \), \( \mu > 0 \) and \( r, \chi > 0 \) satisfy (2.5). Then for \( q \in \left( 2, \frac{nk}{n-1} \right) \) there exists \( M_5 > 0 \) such that

\[
\int_0^T \int_\Omega \left| \frac{\nabla v_\epsilon}{v_\epsilon} \right|^q \, dx \, ds \leq M_5(1 + T), \quad T > 0
\]

for all \( \epsilon \in (0, 1) \).
Proof. Similar argument for (2.21) and (2.22) in Lemma 2.5, we know for \( r \in (1, \frac{n}{n-1}) \) that

\[
\|v_\epsilon(\cdot,t)\|_{W^{1,r}(_{\Omega})} \leq C_{BS}\|\Delta v_\epsilon\|_{L^1(_{\Omega})} \leq C_{BS}\|v_\epsilon - u_\epsilon\|_{L^1(_{\Omega})} \leq 2C_{BS}m^*, \ t > 0
\]  (3.26)

and for \( m \geq 1 \) that

\[
\|v_\epsilon(\cdot,t)\|_{W^{2,m}(_{\Omega})} \leq C_5\|u_\epsilon(\cdot,t)\|_{L^m(_{\Omega})}, \ t > 0
\]  (3.27)

for all \( \epsilon \in (0,1) \). Let \( q \in (\frac{(n+1)k}{n}, \frac{nk}{n-1}) \). Then \( r := \frac{(n+1)k}{n} \in (1, \frac{n}{n-1}) \), and hence by the Gagliardo-Nirenberg inequality with (3.27), we know

\[
\|\nabla v_\epsilon(\cdot,t)\|_{L^q(_{\Omega})} \leq C_{GN}\|v_\epsilon(\cdot,t)\|_{W^{2,k}(_{\Omega})}^{\frac{a}{2}}\|\nabla v_\epsilon(\cdot,t)\|_{L^r(_{\Omega})}^{1-\frac{a}{2}}
\]

\[
\leq C_5C_{GN}\|u_\epsilon(\cdot,t)\|_{L^q(_{\Omega})}^{\frac{a}{2}}\|\nabla v_\epsilon(\cdot,t)\|_{L^r(_{\Omega})}^{1-\frac{a}{2}}, \ t > 0
\]

with some \( C_{GN} > 0 \) for all \( \epsilon \in (0,1) \), where \( a = \frac{n-2}{1+\frac{q-1}{2}} \equiv \frac{k}{q} \in (0,1) \). This together with (3.26) and (3.3) indicates

\[
\int_0^T \int_\Omega |\nabla v_\epsilon|^q dxdt \leq C_{13}(1+T), \ T > 0
\]  (3.28)

with some \( C_{13} > 0 \). The proof is complete by (3.28) and (3.7) with \( M_5 = \frac{C_{10a}}{\delta_{l}} > 0 \). \( \square \)

We now perform a subsequence extraction procedure to obtain a limit object \((u,v)\), i.e., a generalized solution to the problem (1.6).

Lemma 3.9 Let \( k > 2 - \frac{2}{\mu} \), \( \mu > 0 \) and \( r, \chi > 0 \) satisfy (2.5). Then for \( p \in (1,k) \) with \( q \in (2, \frac{nk}{k-1}) \) there exist \( u \in L^1_{loc}(_{\Omega \times (0,\infty)}) \) and \( v \in L^1_{loc}(_{0,\infty},W^{1,1}(_{\Omega})) \) such that

\[
\ln(1+u_\epsilon) \rightharpoonup \ln(1+u), \quad \text{in } L^2_{loc}([0,\infty);W^{1,2}_v(_{\Omega})),
\]

\[
u_\epsilon \rightharpoonup u, \quad \text{in } L^k_{loc}(_{\Omega \times [0,\infty)}),\]

\[
u_\epsilon \rightarrow u, \quad \text{a.e. in } \Omega \times (0,\infty) \text{ and } L^p_{loc}(_{\Omega \times [0,\infty)}),\]

\[
\nabla v_\epsilon \rightharpoonup v, \quad \text{a.e. in } \Omega \times (0,\infty) \text{ and in } L^1_{loc}([0,\infty);W^{1,1}_v(_{\Omega})),
\]

\[
\frac{u^2_\epsilon|\nabla v_\epsilon|^2}{(1+u_\epsilon)^2v^2_\epsilon} \rightharpoonup \frac{u^2|\nabla v|^2}{(1+u)^2v^2}, \quad \text{in } L^1_{loc}(_{\Omega \times [0,\infty)}))
\]  (3.34)

for \( \epsilon = \epsilon_j \downarrow 0 \).

Proof. Let \( T > 0 \). The conclusions (3.29), (3.30) and (3.33) are the direct results from (3.17), (3.3) and (3.28). Since \( W^{1,1}(_{\Omega}) \hookrightarrow L^2(_{\Omega}) \), we have by the Aubin-Lions lemma with (3.17) and (3.22) that \( \ln(1+u_\epsilon) \rightharpoonup \ln(1+u) \) in \( L^2(_{\Omega \times (0,T)}) \), and moreover \( u_\epsilon \rightarrow u \) a.e. in \( \Omega \times (0,T) \), as \( \epsilon = \epsilon_j \downarrow 0 \). For \( p \in (1,k) \), it is known from (3.3) that

\[
\int_0^T \int_\Omega u^p_\epsilon dxdt \leq |\Omega|T + \int_0^T \int_\Omega u^k_\epsilon dxdt \leq C(1+T), \ T > 0
\]
by Young’s inequality for all $\epsilon \in (0, 1)$, i.e., $\{u^\epsilon\}_{\epsilon \in (0, 1)} \subset L^k_{loc}(\Omega \times [0, \infty))$. This together with $u^\epsilon \to u$ a.e. in $\Omega \times (0,T)$ indicates (3.31) by the Vitali convergence theorem. The estimates (3.27) and (3.31) imply that there exists some nonnegative $v$ defined on $\Omega \times (0,T)$ such that (3.32) holds. Consequently, we note from (3.25) for $q \in (2, \frac{nk}{k-1})$ that $\frac{\nabla u^\epsilon}{v^\epsilon} \to \frac{\nabla v}{v}$ in $L^\frac{2}{r}_{loc}(\Omega \times [0, \infty))$, which concludes (3.34) due to $\frac{u}{1+u^\epsilon} \to \frac{u}{1+u}$ in $L^m_{loc}(\Omega \times [0, \infty))$ for every $m > 1$ by (3.31) as $\epsilon = \epsilon_j \searrow 0$. □

4 Global existence and boundedness to generalized solution

In this section we begin with proving that the function $(u,v)$ determined in Lemma 3.9 just is the global generalized solution of (1.6).

Proof the Theorem 2. For $k > 2 - \frac{1}{n}$, we will firstly demonstrate that the function $(u,v)$ obtained in Lemma 3.9 is a very weak subsolution of (1.6) in $\Omega \times (0,T)$ for $T > 0$. Let $\varphi$ satisfy (1.10). Multiplying (3.1) by $\varphi$ and integrating by parts, then we have for all $\epsilon \in (0, 1)$ that

$$-\int_0^T \int_\Omega u^\epsilon \varphi_t - \int_\Omega u^\epsilon \varphi(\cdot, 0) = \int_0^T \int_\Omega u^\epsilon \Delta \varphi + \chi \int_0^T \int_\Omega \frac{u^\epsilon}{v^\epsilon} \nabla v^\epsilon \cdot \nabla \varphi$$
$$+ r \int_0^T \int_\Omega u^\epsilon \varphi - \mu \int_0^T \int_\Omega u^k \varphi - \epsilon \int_0^T \int_\Omega u^{k+1} \varphi. \quad (4.1)$$

By (3.30), we know

$$-\int_0^T \int_\Omega u^\epsilon \varphi_t \to -\int_0^T \int_\Omega u \varphi_t, \quad (4.2)$$
$$\int_0^T \int_\Omega u^\epsilon \Delta \varphi \to \int_0^T \int_\Omega u \Delta \varphi, \quad (4.3)$$
$$r \int_0^T \int_\Omega u^\epsilon \varphi \to r \int_0^T \int_\Omega u \varphi \quad (4.4)$$

as $\epsilon = \epsilon_j \searrow 0$. Since $\frac{nk}{n-1} > \frac{k}{k-1}$ for $k > 2 - \frac{1}{n}$, we know by (3.31) with (3.33) that

$$\chi \int_0^T \int_\Omega \frac{\nabla v^\epsilon}{v^\epsilon} \cdot \nabla \varphi \to \chi \int_0^T \int_\Omega \frac{\nabla v}{v} \cdot \nabla \varphi \quad \text{as} \quad \epsilon = \epsilon_j \searrow 0. \quad (4.5)$$

Consequently, in view of (4.2)–(4.5) with the Fatou lemma and the positivity of $\epsilon \int_0^T \int_\Omega u^{k+1} \varphi$ for $\epsilon \in (0, 1)$, we obtain

$$\mu \int_0^T \int_\Omega u^k \varphi \leq \mu \lim inf_{\epsilon = \epsilon_j \searrow 0} \int_0^T \int_\Omega u^k \varphi$$
$$= \int_0^T \int_\Omega u \varphi_t + \int_\Omega u_0 \varphi(\cdot, 0) + \int_0^T \int_\Omega u \Delta \varphi$$
\[ + \chi \int_0^T \int_\Omega u \frac{\nabla v}{v} \cdot \nabla \varphi + r \int_0^T \int_\Omega u \varphi. \]  

(4.6)

Take \( \psi \) satisfying (1.11). Multiply (3.1)_2 by \( \psi \) and integrate by parts, then we

\[- \int_0^T \int_\Omega v \psi_t - \int_\Omega v_0 \psi(\cdot, 0) + \int_0^T \int_\Omega \nabla v \cdot \nabla \psi + \int_0^T \int_\Omega v \psi = \int_0^T \int_\Omega u \psi. \]

(4.7)

According to (3.32) and (3.30), we get (1.9) by taking \( \epsilon = \epsilon_j \searrow 0 \). This together with (4.6) indicates that \((u, v)\) is a very weak subsolution of (1.6).

Taking \( \varphi \) in (1.10) and multiplying (1.6)_1 by \( \frac{\varphi}{1 + u_\epsilon} \), we have

\[- \int_0^T \int_\Omega \ln(1 + u_\epsilon) \varphi_t - \int_\Omega \ln(1 + u_0) \varphi(\cdot, 0) = \int_0^T \int_\Omega \frac{|\nabla u_\epsilon|^2}{(1 + u_\epsilon)^2} \varphi - \chi \int_0^T \int_\Omega \frac{u_\epsilon}{(1 + u_\epsilon)^2} \nabla u_\epsilon \cdot \nabla v \varphi \]

\[- \int_0^T \int_\Omega \frac{\nabla u_\epsilon \cdot \nabla \varphi}{1 + u_\epsilon} + \chi \int_0^T \int_\Omega \frac{u_\epsilon}{(1 + u_\epsilon)v_\epsilon} \nabla v_\epsilon \cdot \nabla \varphi \]

\[+ r \int_0^T \int_\Omega \frac{u_\epsilon}{1 + u_\epsilon} \varphi - \mu \int_0^T \int_\Omega \frac{u_\epsilon^k}{1 + u_\epsilon} \varphi - \epsilon \int_0^T \int_\Omega \frac{u_\epsilon^{k+1}}{1 + u_\epsilon} \varphi \]

\[= \int_0^T \int_\Omega \left( \frac{\nabla u_\epsilon \cdot \nabla \varphi}{1 + u_\epsilon} - \frac{\chi u_\epsilon \nabla v_\epsilon}{2(1 + u_\epsilon)v_\epsilon} \right) \varphi - \frac{\chi^2}{4} \int_0^T \int_\Omega \frac{u_\epsilon^2 \nabla v_\epsilon^2}{(1 + u_\epsilon)^2} \varphi \]

\[- \int_0^T \int_\Omega \frac{\nabla u_\epsilon \cdot \nabla \varphi}{1 + u_\epsilon} + \chi \int_0^T \int_\Omega \frac{u_\epsilon}{(1 + u_\epsilon)v_\epsilon} \nabla v_\epsilon \cdot \nabla \varphi \]

\[+ r \int_0^T \int_\Omega \frac{u_\epsilon}{1 + u_\epsilon} \varphi - \mu \int_0^T \int_\Omega \frac{u_\epsilon^k}{1 + u_\epsilon} \varphi - \epsilon \int_0^T \int_\Omega \frac{u_\epsilon^{k+1}}{1 + u_\epsilon} \varphi. \]

(4.8)

By (3.30), we know

\[- \int_0^T \int_\Omega \ln(1 + u_\epsilon) \varphi_t \to - \int_0^T \int_\Omega \ln(1 + u) \varphi_t, \]

(4.9)

\[r \int_0^T \int_\Omega \frac{u_\epsilon}{1 + u_\epsilon} \varphi \to r \int_0^T \int_\Omega \frac{u}{1 + u} \varphi, \]

(4.10)

\[- \mu \int_0^T \int_\Omega \frac{u_\epsilon^k}{1 + u_\epsilon} \varphi \to - \mu \int_0^T \int_\Omega \frac{u^k}{1 + u} \varphi. \]

(4.11)

as \( \epsilon = \epsilon_j \searrow 0 \), whereas (3.29) implies that

\[- \int_0^T \int_\Omega \frac{\nabla u_\epsilon \cdot \nabla \varphi}{1 + u_\epsilon} \to - \int_0^T \int_\Omega \frac{\nabla u \cdot \nabla \varphi}{1 + u} \]

(4.12)

as \( \epsilon = \epsilon_j \searrow 0 \). It follows from (3.34) that

\[- \frac{\chi^2}{4} \int_0^T \int_\Omega \frac{u_\epsilon^2 \nabla v_\epsilon^2}{(1 + u_\epsilon)^2} \varphi \to - \frac{\chi^2}{4} \int_0^T \int_\Omega \frac{u^2 \nabla v^2}{(1 + u)^2} \varphi; \]

\[\chi \int_0^T \int_\Omega \frac{u_\epsilon}{(1 + u_\epsilon)v_\epsilon} \nabla v_\epsilon \cdot \nabla \varphi \to \chi \int_0^T \int_\Omega \frac{u}{(1 + u)v} \nabla v \cdot \nabla \varphi \]

(4.13) 

(4.14)
as $\epsilon = \epsilon_j \searrow 0$. In addition, a simple calculation with (3.3) shows that

$$-\epsilon \int_0^T \int_\Omega \frac{u_{\epsilon}^{k+1} \varphi}{1 + u_{\epsilon}} \leq \epsilon \|\varphi\|_{L^\infty(\Omega \times (0,T))} \int_0^T \int_\Omega \frac{u_{\epsilon}^k}{1 + u_{\epsilon}}$$

$$\leq \epsilon M_1 (1 + T) \|\varphi\|_{L^\infty(\Omega \times (0,T))} \to 0 \quad (4.15)$$

as $\epsilon = \epsilon_j \searrow 0$. Consequently, we obtain from (4.8) with (4.9)--(4.15) and the Fatou lemma that

$$\int_0^T \int_\Omega \left( \frac{\nabla u}{1 + u} - \frac{\chi u \nabla v}{2(1 + u)v} \right)^2 \varphi \leq \liminf_{\epsilon = \epsilon_j \searrow 0} \int_0^T \int_\Omega \left( \frac{\nabla u_{\epsilon}}{1 + u_{\epsilon}} - \frac{\chi u_{\epsilon} \nabla v_{\epsilon}}{2(1 + u_{\epsilon})v_{\epsilon}} \right)^2 \varphi$$

$$= - \int_0^T \int_\Omega \ln(1 + u) \varphi_t - \int_\Omega \ln(1 + u_0) \varphi(\cdot,0) + \frac{\chi^2}{4} \int_0^T \int_\Omega \frac{u^2}{(1 + u)^2} \nabla v^2 \varphi$$

$$+ \int_0^T \int_\Omega \frac{\nabla u \cdot \nabla \varphi}{1 + u} - \chi \int_\Omega \frac{u}{1 + u} \nabla v \cdot \nabla \varphi$$

$$- r \int_0^T \int_\Omega \frac{u}{1 + u} \varphi + \mu \int_0^T \int_\Omega \frac{u_{\epsilon}^k}{1 + u} \varphi \quad (4.16)$$

as $\epsilon = \epsilon_j \searrow 0$. This together with (4.7) yields that $(u,v)$ is a weak logarithmic supersolution of (1.6) as well.

The proof is complete. □

Next, we will prove that the global generalized solution to (1.6) is globally bounded. At first, we give a crucial estimate on $T \int_0^T \int_\Omega u_{\epsilon}^p dx$ for all $\epsilon \in (0,1)$.

**Lemma 4.1** Let $(u_{\epsilon}, v_{\epsilon})$ be the global very weak solution of the problem (1.6) established in Theorem 2. Then for $p > \frac{n(n+2)}{2(n+1)}$, we have

$$\frac{d}{dt} \int_\Omega u_{\epsilon}^p dx \leq - \int_\Omega u_{\epsilon}^p dx + M_6 \left( \int_\Omega u_{\epsilon}^p dx \right)^{\frac{2n+4}{2n+2}}$$

$$+ M_6 \left( \int_\Omega u_{\epsilon}^p dx \right)^{\frac{4}{p}} + M_6 \left( \int_\Omega u_{\epsilon}^p dx \right)^{\frac{2q}{p^q}} + M_6 \int_\Omega u_{\epsilon} dx, \ t > 0 \quad (4.17)$$

for all $\epsilon \in (0,1)$ with some $M_6 > 0$.

**Proof.** It follows from (3.1) and (3.7) for $1 < p < q$ that

$$\frac{1}{p} \frac{d}{dt} \int_\Omega u_{\epsilon}^p dx = \int_\Omega u_{\epsilon}^{p-1} [\Delta u_{\epsilon} - \chi \nabla \cdot (\frac{u_{\epsilon}}{v_{\epsilon}}) \nabla v_{\epsilon}] + r u_{\epsilon} - \mu u_{\epsilon}^{k-1} + e u_{\epsilon}^{k+1}] dx$$

$$\leq - \frac{1}{p} \int_\Omega u_{\epsilon}^p dx - (p-1) \int_\Omega u_{\epsilon}^{p-2} |\nabla u_{\epsilon}|^2 dx + \chi (p-1) \int_\Omega \frac{u_{\epsilon}^{p-1}}{v_{\epsilon}^2} \nabla u_{\epsilon} \cdot \nabla v_{\epsilon} dx$$

$$+ (r+1) \int_\Omega u_{\epsilon}^p dx - \mu \int_\Omega u_{\epsilon}^{p+k-1} dx$$

$$\leq - \frac{1}{p} \int_\Omega u_{\epsilon}^p dx - \frac{p-1}{2} \int_\Omega u_{\epsilon}^{p-2} |\nabla u_{\epsilon}|^2 dx + \frac{\chi^2 (p-1)}{2\delta_1} \int_\Omega u_{\epsilon}^p |\nabla v_{\epsilon}|^2 dx$$
\[ + (r + 1) \int_{\Omega} u_\epsilon^r dx - \frac{\mu}{2} \int_{\Omega} u_\epsilon^{p+k-1} dx \]
\[ \leq - \frac{1}{p} \int_{\Omega} u_\epsilon^p dx - \frac{p-1}{2} \int_{\Omega} u_\epsilon^{p-2} |\nabla u_\epsilon|^2 dx + \frac{\chi^2(p-1)}{2\delta^2} \int_{\Omega} u_\epsilon^q dx \]
\[ + \frac{\chi^2(p-1)}{2\delta^2} \int_{\Omega} |\nabla u_\epsilon|^{\frac{2q}{q-p}} dx + C_{14} \int_{\Omega} u_\epsilon dx, \quad t > 0 \]  

(4.18)

by Young’s inequality with \( C_{14} = (r + 2) \frac{p+k-1}{k+1} \left( \frac{2}{p} \right)^{\frac{n}{k+1}} \). Invoking the Gagliardo-Nirenberg inequality, we get

\[ \|u_\epsilon\|_{L^q(\Omega)} = \|u_\epsilon^\frac{p-2}{2}\|_{L^\frac{2q}{q-p}(\Omega)} \leq C_{GN} \|u_\epsilon^\frac{p-2}{2}\|_{L^\frac{2q}{q-p}(\Omega)} \leq 2^{\frac{na}{2}} C_{GN} \left( \|u_\epsilon^\frac{p-2}{2}\|_{L^\frac{2q}{q-p}(\Omega)} \right). \]

(4.19)

If \( 1 < p < q < \frac{2n+2}{n}p \), we know \( a = \frac{n}{2} - \frac{2n}{q} \in (0, 1) \) and \( \frac{2na}{p} < 1 \). This fact together with (4.19) yields

\[ \frac{\chi^2(p-1)}{2\delta^2} \int_{\Omega} u_\epsilon^q dx \leq \frac{p-1}{2} \int_{\Omega} u_\epsilon^{p-2} |\nabla u_\epsilon|^2 dx + C_{15} \left( \int_{\Omega} u_\epsilon^p dx \right)^{\frac{q(1-a)}{p-qa}} + C_{16} \left( \int_{\Omega} u_\epsilon^p dx \right)^{\frac{2}{p}} \]

(4.20)

by Young’s inequality with \( C_{15} = (p-1) \frac{p}{2} \left( \frac{2q+2\mu}{2} \right) \frac{q}{\delta} \frac{C_{GN}^{\frac{q}{p}}} {\|u_\epsilon\|_{L^\frac{2q}{q-p}(\Omega)}} \) and \( C_{16} = \frac{q}{p} C_{GN} \frac{q}{\delta} (p-1) \). Now, let \( p \in (\frac{n(n+2)}{2(n+1)}, n] \) with \( p < q < \frac{2n+2}{n}p \). Then \( \frac{2na}{q-p} < \frac{np}{n-p} \). By the classical imbedding Theorem with (3.27), we obtain

\[ \frac{\chi^2(p-1)}{2\delta^2} \int_{\Omega} |\nabla u_\epsilon|^{\frac{2q}{q-p}} dx = \frac{\chi^2(p-1)}{2\delta^2} \|\nabla u_\epsilon\|_{L^\frac{2q}{q-p}(\Omega)}^{\frac{2q}{q-p}} \]
\[ \leq C_{17} \frac{\chi^2(p-1)}{2\delta^2} \|\nabla u_\epsilon\|_{W^{1,p}(\Omega)}^{\frac{2q}{q-p}} \]
\[ \leq C_{18} \left( \int_{\Omega} u_\epsilon^p dx \right)^{\frac{2q}{p(q-p)}}, \quad t > 0 \]

(4.21)

with some \( C_{17}, C_{18} > 0 \). Combing (4.18) with (4.20) and (4.21), we have

\[ \frac{1}{p} \frac{d}{dt} \int_{\Omega} u_\epsilon^p dx \leq - \frac{1}{p} \int_{\Omega} u_\epsilon^p dx + C_{15} \left( \int_{\Omega} u_\epsilon^p dx \right)^{\frac{q-na}{p-qa}} \]
\[ + C_{16} \left( \int_{\Omega} u_\epsilon^p dx \right)^{\frac{2}{p}} + C_{18} \left( \int_{\Omega} u_\epsilon^p dx \right)^{\frac{2q}{p(q-p)}} + C_{14} \int_{\Omega} u_\epsilon dx, \quad t > 0. \]

(4.22)

This completes the conclusion (4.17) with \( M_6 = p \max\{C_{14}, C_{15}, C_{16}, C_{18}\} \). □

Now, we establish the following uniform-in-time estimate on \( \int_{\Omega} u_\epsilon^p dx \) for all \( \epsilon \in (0, 1) \), with the initial data \( u_0 \) and \( \frac{n}{p} \) suitably small.
Lemma 4.2 Let \((u_\epsilon, v_\epsilon)\) be the global very weak solution of the problem (1.6) established in Theorem 2. Then for \(p \in \left(\frac{n(n+2)}{2(n+1)}, n\right]\) there exist \(\eta, \lambda > 0\) such that

\[
\int_\Omega u_\epsilon^p dx \leq M_7, \quad t > 0,
\]

provided \(\frac{p}{p} < \eta\) and \(\int_\Omega u_0^p dx < \lambda\), for all \(\epsilon \in (0, 1)\) with \(M_7 > 0\).

Proof. Let \(F_\epsilon(t) := \int_\Omega u_\epsilon(x, t)^p dx\), \(t > 0\). Then we have from (4.22) and (3.2) that

\[
\begin{aligned}
F_\epsilon'(t) &\leq -F_\epsilon(t) + M_6 F_\epsilon(t)^{\frac{q-a}{p-a}} + M_6 F_\epsilon(t)^{\frac{q}{p}} + M_6 F_\epsilon(t)^{\frac{2q}{p(q-a)}} + M_6 m^*, \quad t > 0, \\
F_\epsilon(0) &= \int_\Omega u_0^p dx.
\end{aligned}
\] (4.24)

Since \(p \in \left(\frac{n(n+2)}{2(n+1)}, n\right]\) and \(p < q < \frac{n+2}{n} p\), we know \(\frac{q-a}{p-a}, p, \frac{2q}{p(q-a)} > 1\). Denote

\[
\begin{align*}
h(s, m^*) &:= -s + M_6 s^{\frac{q-a}{p-a}} + M_6 s^{\frac{q}{p}} + M_6 s^{\frac{2q}{p(q-a)}} + M_6 m^*, \quad s > 0.
\end{align*}
\]

Then there exists \(m_0^* > 0\) such that \(h(s, m_0^*)\) has the unique positive root \(s_0\). Furthermore, \(M(t) \equiv s_0\) verifies the ODE problem

\[
\begin{aligned}
M'(t) &= h(M(t), m_0^*), \quad t > 0, \\
M(0) &= s_0.
\end{aligned}
\] (4.25)

If \(m^* < m_0^*\), it follows by a continuous dependence argument that the function \(h(s, m^*)\), with \(h(s, m^*) < h(s, m_0^*)\), has exactly two positive roots \(0 < s_1 < s_0 < s_2\). Now let

\[
\eta := \left(\frac{m_0^*}{|\Omega|}\right)^{k-1} \quad \text{and} \quad \lambda := \min\left\{s_0, \frac{m_0^*}{|\Omega|^{p-1}}\right\}
\]

with \(\frac{p}{p} < \eta\) and \(\int_\Omega u_0^p dx < \lambda\). Then

\[
\begin{aligned}
\int_\Omega u_0^p dx &< |\Omega|^{\frac{p+1}{p}} \left(\int_\Omega u_0^p dx\right)^{\frac{1}{p}} < m_0^* \quad \text{and} \quad \int_\Omega u_\epsilon dx \leq \max\left\{\int_\Omega u_0 dx, \left(\frac{\mu}{\mu + 4}\right)^{\frac{1}{p-1}} |\Omega|\right\} < m_0^*
\end{aligned}
\]

for all \(\epsilon \in (0, 1)\). Consequently, we obtain from these estimates with problems (4.24) and (4.25) that

\[
F_\epsilon(t) = \int_\Omega u_\epsilon^p dx \leq s_1, \quad t > 0
\]

for all \(\epsilon \in (0, 1)\) by an ODE comparison principle. The proof is complete. \(\square\)

Proof of Theorem 3 Based on the estimate \(\int_\Omega u_\epsilon^p dx \leq s_1\) for \(p > \frac{n(n+2)}{2(n+1)}\) in Lemma 4.1 and uniformly in time lower-bound estimate of \(v_\epsilon\), we obtain the global boundedness of solutions to the regularization problem (3.1) via a similar argument as that in [11, Lemma 2.3], i.e., \(\|u_\epsilon\|_{L^\infty(\Omega)} \leq \tilde{C}\) with some \(\tilde{C} > 0\) for all \(t > 0\) and \(\epsilon \in (0, 1)\). Consequently, we conclude that the generalized solution \((u, v)\) is globally bounded as well by taking \(\epsilon = \epsilon_j \searrow 0\). \(\square\)

Acknowledgements

This work was supposed by the Doctoral Scientific Research Foundation of Liaoning Normal University (Grant No. 203070091907).
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