Abstract

We consider the problem of condensation of open string tachyon fields which have an $O(D)$ symmetric profile. This problem is described by a boundary conformal field theory with $D$ scalar fields on a disc perturbed by relevant boundary operators with $O(D)$ symmetry. The model is exactly solvable in the large $D$ limit and we analyze its $1/D$ expansion. We find that this expansion is only consistent for tachyon fields which are polynomials. In that case, we show that the theory is renormalized by normal ordering the interaction. The beta-function for the tachyon field is the linear wave operator. We derive an expression for the tachyon potential and compare with other known expressions. In particular, our technique gives the exact potential for the quadratic tachyon profile. It can be used to correct the action which has been derived in that case iteratively in derivatives of the tachyon field.
A classic problem in string theory is to understand how the background space-time on which the string propagates arises in a self-consistent way. For open strings, there are two main approaches to this problem, cubic string field theory [1] and background independent string field theory [2, 3].

The latter approach is defined as a problem in boundary conformal field theory. One begins with the partition function of open-string theory where the world-sheet is a disc. The strings in the bulk are considered to be on-shell and a boundary interaction with arbitrary operators is added. The configuration space of open string field theory is then taken to be the space of all possible boundary operators modulo gauge symmetries and the possibility of field re-definition. Renormalization fixed points, which correspond to conformal field theories, are solutions of classical equations of motion and should be viewed as the solutions of classical string field theory.

Despite many problems which are both technical and matters of principle, background independent string field theory has been useful for finding the classical tachyon potential energy functional and the leading derivative terms in the tachyon effective action [4, 5, 6]. Boundary field theories which can be used to study tachyons in the background independent string field theory framework are the subject of the present paper.

The existence of a tachyon in the bosonic string theory indicates that the 26-dimensional Minkowski space background about which the string is quantized is unstable. An unstable state should decay to something and the nature of both the decay process and the endpoint of the decay are interesting questions. Recently, some understanding of this process has been achieved for the open bosonic string. The key idea is that of Sen [11]. The open bosonic string tachyon reflects the instability of the D-25 brane. This unstable D-brane should decay by condensation of the open string tachyon field. The energy per unit volume released in the decay should be the D-25 brane tension and the end-point of the decay is the closed string vacuum [11, 12, 13]. There are also intermediate unstable states which are the D-branes of all dimensions between zero and 25. To study background independent string field theory, consider the partition function

\[ Z = \int [dX^j(\sigma, \tau)] \exp (-S[X]) \] (1)

where the action is

\[ S[X] = \int d\sigma d\tau \frac{1}{4\pi} \partial_a X(\sigma, \tau) \cdot \partial_a X(\sigma, \tau) + \int_0^{2\pi} \frac{d\tau}{2\pi} T(X(\tau)) \] (2)

Here, the first term in (2) is the bulk action and is integrated over the volume of the unit disc. The second term in (2) is integrated on the circle which is the boundary of the unit disc and describes the interactions. The scalar fields \(X^j\) have \(D\) components with \(j = 1, \ldots, D\) and \(D = 26\) for a critical string. In some of the following computations we will take \(D\) as arbitrary and large, but always for applications to string theory.

\[ ^1\text{For earlier works on tachyon condensation see } [7, 8, 9, 10]. \]
we should eventually put \( D \) to 26. A 1/\( D \)-expansion generally has a finite radius of convergence and we expect it to be quite accurate when \( D = 26 \). We are working in a system of units where \( \alpha' = 1 \).

By classical power-counting the tachyon field has dimension one and is a relevant operator. When it is the only interaction, the field theory is perturbatively super-renormalizable and all ultraviolet divergences can be removed by normal ordering. At this point we must distinguish between the case where \( T(X) \) is a polynomial and a non-polynomial function of \( X \). In the latter case, it can have large anomalous dimensions and generally requires a non-perturbative renormalization beyond normal ordering. It is well known that this renormalization makes the beta-function non-linear in \( T \), so that if vanishing of the beta function is taken as the field equation for \( T \), these nonlinear terms describe tachyon scattering\(^{14, 15}\). This renormalization would also generate higher derivative counter-terms and thereby couple all of the other open string degrees of freedom. In this Paper we will discuss only the case of polynomial potentials with \( O(D) \) symmetry. We find that, in the case of non-polynomial potentials, because of a non-commutativity of the large \( D \) and large cutoff limits, the large anomalous dimensions interfere with the 1/\( D \) expansion. The analysis with polynomial potentials is still sufficient to deduce higher derivative terms in the tachyon effective action.

When \( T(X) \) and the other fields are adjusted so that the sigma model that they define is at an infrared fixed point of the renormalization group, these background fields are a solution of the classical equation of motion of string theory. Witten and Shatashvili \(^2, 3\) have argued that these equations of motion can be derived from an action which is derived from the disc partition function \(^1\) by a prescription which we shall make use of below.

We begin with the observation in ref.\(^{16}\) that the bulk excitations can be integrated out of \(^1\) to get an effective non-local field theory which lives on the boundary. To do this we write the field in the bulk as

\[
X = X_{cl} + X_{qu}
\]

where

\[
-\partial^2 X_{cl} = 0
\]

and \( X_{cl} \) approaches the fixed (for now) boundary value of \( X \),

\[
X_{cl} \to X_{bdry} \text{ and } X_{qu} \to 0
\]

Then, in the bulk, the functional measure is \( dX = dX_{qu} \) and

\[
S = \int \frac{d^2\sigma}{4\pi} \partial X_{qu} \cdot \partial X_{qu} + \int \frac{d\tau}{2\pi} \left\{ \frac{1}{2} X^j |i\partial_\tau| X^j + T(X) \right\}
\]

Then, the integration of \( X_{qu} \) produces a multiplicative constant in the partition function - the partition function of the Dirichlet string, which we shall denote \( Z_{dir} \). The
kinetic term in the boundary action is non-local. The absolute value of the derivative operator is defined by the Fourier transform,

\[ |i\partial|\delta(\tau - \tau') = \sum_n \frac{|n|}{2\pi} e^{in(\tau - \tau')} \]

The partition function of the boundary theory is then

\[ Z = Z_{dir} \int [dX_j] e^{-\frac{1}{2\pi} \sum_0^\infty \frac{1}{|n|^2} \partial^2_i X_j^n |i\partial|X_j^n + T(X) - J \cdot X} \] (4)

where we have added a source \( J^i(\tau) \) so that the path integral can be used as a generating functional for correlators of the fields \( X_j \) restricted to the boundary. In particular, this source will allow us to compute the correlation functions of vertex operators of open string degrees of freedom. The remaining path integral over the boundary \( X^2(\tau) \) defines a one-dimensional field theory with nonlocal kinetic term. If the tachyon field were absent (\( T = 0 \)), the further integration over \( X^2(\tau) \) would give a factor which converts the Dirichlet string partition function to the Neumann string partition function.

**Large \( D \) limit**

It is interesting to consider the boundary conformal field theory in the large \( D \) limit. To do this we must assume that the tachyon is a spherically symmetric function invariant under \( O(D) \) rotations and has the form

\[ T(X(\tau)) = DT \left( \frac{X^2(\tau)}{D} \right) \]

Since a typical configuration of \( X^2(\tau) \) is of order \( D \), a function of this form is itself of order \( D \). In order to study a large \( D \) expansion it is convenient to introduce auxiliary fields. This is done by inserting unity into the path integral

\[ 1 = \int [d\chi] \delta \left( \chi - \frac{X^2}{D} \right) = \int [d\chi d\lambda] \exp \left( i \int_0^{2\pi} \frac{d\tau}{2\pi} (D\lambda \chi - \lambda X^2) \right) \] (5)

Here and later in this paper, we shall make use of zeta-function regularization. The Riemann zeta function is

\[ \zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s} \]

It converges for \( s > 1 \) and has a finite analytic continuation for all real \( s \neq 1 \) where it has a simple pole. It has values

\[ \zeta(0) = -1/2 \ , \ \zeta'(0) = -\frac{1}{2} \ln(2\pi) \ , \ \zeta(2) = \frac{\pi^2}{6} \ , \ \zeta(3) = 1.20206... \]
Using zeta-function regularization, the Jacobian which results from scaling the functional integration variable in (5) by $D$ is

$$\det (D \cdot \delta(\tau - \tau')) = D \sum_{n=-\infty}^{\infty} 1 = D^{1+2\zeta(0)} = 1$$

Inserting (5) into the functional integral results in

$$Z = Z_{\text{dir}} \int [dX d\chi d\lambda] e^{-\int_0^{2\pi} \frac{1}{2} \left( \frac{1}{2} X^j (\partial^2 + 2i\lambda) X^j + DT(\chi) - Di\lambda \chi - J \cdot X \right)}$$

The variables $X^j(\tau)$ now appear quadratically and can be integrated out of the partition function. It will be convenient to do this only for the non-zero modes of $X^j$. This will leave behind an explicit integral over the zero mode which is defined by

$$\tilde{X}^j = \int_0^{2\pi} \frac{d\tau}{2\pi} X^j(\tau)$$

The non-zero modes are integrated out to produce the partition function

$$Z = Z_{\text{dir}} \int d\tilde{X}^j [d\chi d\lambda] e^{-S_{\text{eff}}[\tilde{X},\chi,\lambda]}$$

where

$$S_{\text{eff}} = \frac{D}{2} \text{Tr} \ln (\mathcal{P} \left| i\partial \right| + 2i\lambda) \mathcal{P} \mathcal{P} + D \int_0^{2\pi} \frac{d\tau}{2\pi} \left[ T(\chi) - i\lambda \left( \chi - \frac{\tilde{X}^2}{D} \right) - \frac{\tilde{X}}{D} \cdot J \right]$$

$$- \frac{1}{2D} \int_0^{2\pi} \frac{d\tau'}{2\pi} \left( J(\tau) - 2i\lambda(\tau) \tilde{X} \right) (\mathcal{P} \frac{1}{2i\lambda} \mathcal{P} (\partial^2 + 2i\lambda)(J(\tau') - 2i\lambda(\tau') \tilde{X})$$

where $\mathcal{P}$ is the projection operator onto non-zero modes, for example,

$$\mathcal{P} J(\tau) = J(\tau) - \int_0^{2\pi} \frac{d\tau'}{2\pi} J(\tau')$$

The effective action (8) is divergent and has to be renormalized. We will postpone discussion of renormalization momentarily.

In the large $D$ limit, the functional integrals over $\chi$ and $\lambda$ in (8) are evaluated by using the saddle point approximation. The saddle point equations obtained by varying the effective action by $\chi$ and $\lambda$ are

$$T'(\chi) = i\lambda$$

and

$$\chi(\tau) = \left( \frac{\tilde{X} + x(\tau)}{D} \right)^2 + (\tau|\mathcal{P} |i\partial| + 2T'(\chi) \mathcal{P} |\tau)$$

$$+ \left( \frac{1}{2} X^j (\partial^2 + 2i\lambda) X^j + DT(\chi) - Di\lambda \chi - J \cdot X \right)$$
respectively. We have used (9) to eliminate $\lambda$ from eqn. (10). $x(\tau)$ is the induced classical field,

$$x(\tau) = \int_0^{2\pi} \frac{d\tau'}{2\pi} \left( \frac{1}{|i\partial| + 2T'(\chi)} \right) \left( J(\tau') - 2T'(\chi(\tau')) \right) \hat{X}$$

(11)

When eqn. (10) is solved, $\chi$ is a functional of the source $J$ and the tachyon field $T$. There is a stability requirement that the solution is at a minimum of the effective action.

To the leading order in the large $D$ limit, the partition function is

$$Z = Z_{\text{dir}} \int d\hat{X} e^{-S_{\text{eff}}[\chi_0, \lambda_0, \hat{X}]}$$

where $\chi_0$ and $\lambda_0$ are the solutions of equations (10) and (9).

Renormalization

The trace of the logarithm in (8) has ultraviolet divergences that we must renormalize. The divergent parts are

$$\frac{1}{2} \text{Tr} \ln \left( \mathcal{P} |i\partial| + 2T'(\chi) \mathcal{P} \right) = \sum_{n=1}^\infty \ln(n) + \int_0^{2\pi} \frac{d\tau}{2\pi} \left( 2T'(\chi(\tau)) \sum_{n=1}^\infty \frac{1}{n} + F(2T'(\chi(\tau))) \right)$$

(12)

where the finite remainder after subtraction of the divergent terms is

$$F(2T'(\chi)) = -\int_0^{2\pi} \frac{d\tau}{2\pi} \int_0^{2\pi} \frac{d\tau'}{2\pi} G(\tau - \tau'; 0)2T'(\chi(\tau))G(\tau - \tau'; 0)T'(\chi(\tau')) + \ldots$$

(13)

and the Green function

$$G(\tau - \tau'; y(\tau)) = (\tau |\mathcal{P} \frac{1}{|i\partial| + y(\tau)} \mathcal{P}|\tau')$$

(14)

The first of the divergent terms can be handled by zeta-function regularization, where

$$\sum_{n=1}^\infty \ln(n) = -\frac{d}{ds} \zeta(s)|_{s \to 0} = \frac{1}{2} \ln(2\pi)$$

The second one is truly divergent and must be subtracted.

Counterterms must be introduced to cancel this divergence. This is achieved by the renormalization transform

$$T(\chi) \rightarrow T \left( \chi - 2 \sum_{n=1}^\infty \frac{1}{n} - 2c_1 \right)$$
where \( c_1 \) is an arbitrary finite constant. When we substitute this into the effective action and translate the variable \( \chi \), up to an irrelevant constant it has the form

\[
S_{\text{eff}} = D \left\{ \int_0^{2\pi} \frac{d\tau}{2\pi} \left( (\chi - \frac{\dot{X}^2}{D} + 2c_1) : T'(\chi) : + F(2 : T'(\chi) :) 
\right.
\right.
\]
\[
\left. - \frac{1}{2D} (J - 2 : T'(\chi) : \dot{X}) \frac{1}{|\partial|} \right. 
\]
\[
\left. + 2 : T'(\chi) : (J - 2 : T'(\chi) : \dot{X} - \frac{\dot{X}}{D} \cdot J) \right\} 
\]

(15)

The appearance of the arbitrary constant \( c_1 \) in the action reflects the arbitrariness involved in subtracting the divergent terms. This arbitrariness was discussed in ref. [6]. The constant \( c_1 \) should eventually be fixed by some renormalization prescription.

The replacement of the tachyon field \( T \) by \( : T : \) amounts to the large \( D \) limit of normal ordering. We can introduce an ultraviolet cutoff \( \Lambda \) and renormalization scale \( \mu \) by the notation

\[
\sum_1^{\infty} \frac{1}{n} = \ln \frac{\Lambda}{\mu} 
\]

(16)

Then, taking a logarithmic derivative of \( : T : \) by \( \mu \) leads to a simple linear beta function for the tachyon field at this order [14, 15]

\[
\beta(T) = - : T : - 2 : T : 
\]

which, we shall show in the following is just the large \( D \) limit of the tachyon wave operator. In the following, we will assume that this renormalization procedure has been done and will drop the normal ordering dots from \( T \). The net effect has been to replace the divergent term linear in \( T' \) in the trace-log term in the action by \( T' \) times a finite arbitrary constant. Now, the effective action and the eqn.(10) which determines its minimum are free of infrared divergences.

**Small derivative expansion**

A transparent way to understand the content of the classical partition function is to consider the limit where \( T(X) \) is a smooth function and to expand in derivatives of \( T \). To do this, we set the source \( J \) to zero. Then, we expect that the condensate \( \chi \) is a constant, independent of \( \tau \). Then, the Green function can easily be evaluated. It is most useful to consider an expansion of (10) (after renormalization)

\[
\chi = \frac{\dot{X}^2}{D} - 2c_1 + 2 \sum_{p=1}^{\infty} \zeta(p + 1) (-2T'(\chi))^p 
\]

\[
= \frac{\dot{X}^2}{D} - 2 (c_1 + \gamma + \psi(2T' + 1)) 
\]

(17)

where \( \gamma \) is the Euler-Mascheroni constant, \( \psi(x) = d \ln \Gamma(x)/dx \) is the Psi (Digamma) function. The terms on the right-hand-side of this equation have increasing numbers
of derivatives of $T$. We can easily solve it iteratively to obtain $\chi$ to any order in an
expansion in the number of derivatives of $T$ that is desired. For example, to order three we have:

$$\chi = \frac{\hat{X}^2}{D} - 2c_1 - 4\zeta(2)T' \left( \frac{\hat{X}^2}{D} - 2c_1 \right) + 8\zeta(3) \left( T' \left( \frac{\hat{X}^2}{D} - 2c_1 \right) \right)^2 + \ldots$$  

(18)

This can then be plugged into equation (8) to get

$$Z = Z_{\text{dir}} \int d\hat{X} e^{-\frac{\hat{X}^2}{D} \left( 1 - 2c_1 DT' \left( \frac{\hat{X}^2}{D} \right) + 2D\zeta(2) \left[ T' \left( \frac{\hat{X}^2}{D} \right) \right]^2 + \ldots \right)}$$  

(19)

where the omitted terms denoted by dots are at least for orders in derivatives of $T$ by
its argument.

The Witten-Shatashvili action is given by

$$S = \left( 1 + \int \beta(T) \frac{\delta}{\delta T} \right) Z$$

$$= Z_{\text{dir}} \int d\hat{X} e^{-\frac{\hat{X}^2}{D} \left\{ 1 + DT' \left( \frac{\hat{X}^2}{D} \right) + 2DT' \left( \frac{\hat{X}^2}{D} \right) \left[ 1 - c_1 DT' \left( \frac{\hat{X}^2}{D} \right) \right] \right\}}$$

It is not difficult to show that this action exactly coincides with the one found in [6],
where the ambiguity $c_1$ was first discussed. In fact, rescaling $T$ as $DT(\frac{\hat{X}^2}{D}) = T(\hat{X})$
and rewriting the derivative with respect to the argument in terms of derivatives with
respect to $\hat{X}^{\mu}$, $S$ becomes

$$S = \int d\hat{X} e^{-T \left( 1 + T + (1 + c_1)\partial_{\mu}T \partial^{\mu}T - c_1 T\partial_{\mu}T \partial^{\mu}T + O((\partial T)^2) \right)}$$  

(20)

**Higher orders in $1/D$**

To compute higher orders in $1/D$, we must consider fluctuations of the consensate
$\chi$. For this computation, in this section we will set the source $J$ equal to zero, so that
the value of $\chi$ in the leading order is a constant which satisfies the equation (11). Thus,
we write

$$\chi(\tau) = \chi_0 + \delta\chi$$

$$i\lambda = T'(\chi_0) + i\delta\lambda$$

$\delta\chi$ and $\delta\lambda$ are the quantum variable that have to be integrated. $\chi_0$ satisfies

$$\chi_0 = \frac{\hat{X}^2}{D} + (\tau) \frac{1}{|i\partial| + 2T'(\chi_0)} |\tau|'$$  

(21)
and, being a constant, can be written explicitly as

\[ \chi_0 = \frac{\hat{X}^2}{D} + 2 \ln \frac{\Lambda}{\mu} + 2H[2T'(\chi_0)] \]  

where we used (16),

\[ H(x) = \frac{\partial F}{\partial x} = \sum_{k=1}^{\infty} (-1)^k x^k \zeta(k + 1) = -\gamma - \psi(x + 1) \]  

and \( F(2T'(\chi_0)) \) is (13) the finite part appearing in (12) for a constant \( \chi \)

\[ F(x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^{k+1} \zeta(k + 1)}{k + 1} = -\gamma x - \ln \Gamma(x + 1) \]  

Then in the partition function (7) the integral over \( \chi \) is computed in the large \( D \) limit by the saddle point approximation. It is given by the leading order

\[ \frac{Z}{Z_{\text{dir}}} = \int d\hat{X}_j \exp \left\{ -D \left[ T(\chi_0) - H(2T'(\chi_0))2T'(\chi_0) + F(2T'(\chi_0)) \right] \right\} \]

multiplied by the determinant of the operator

\[ \begin{pmatrix} 2P(\tau - \tau') + 4\hat{X}^2 G(\tau - \tau') & -i \\ -i & T''(\chi_0) \end{pmatrix} \]

where

\[ P(\tau - \tau'; 2T'(\chi_0)) \equiv (G(\tau - \tau'; 2T'(\chi_0)))^2 = \left( (\tau|P|\tau') + \frac{1}{2}T''(\chi_0) \right)^2 \]

is the bubble diagram which contributes to the \( \chi-\chi \) correlator. Eqs. (16) and (26) can be used to estimate the divergent part of the \( 1/D \) terms

\[ \frac{1}{2} \text{Tr} \ln \left[ 1 + 2T'' \left( P + 2\frac{\hat{X}^2}{D} G \right) \right] = \frac{1}{2} \text{Tr} \ln \left[ 1 + 2T''(\chi_0) \left( P + 2\frac{\hat{X}^2}{D} G \right) \right] \]

\[ + 4T''(\chi_0) \left[ \ln \frac{\Lambda}{\mu} + H(2T'(\chi_0)) \right] \left[ \ln \frac{\Lambda}{\mu} + H(2T'(\chi_0)) + \frac{\hat{X}^2}{D} \right] \]

where the first term is finite and contains higher powers of \( T''(\chi_0) \). These terms do not participate to the renormalization program. To the order \( 1/D \) the partition function then reads

\[ \frac{Z}{Z_{\text{dir}}} = \int d\hat{X}_j \exp \left\{ -D \left[ T(\chi_0) - H(2T'(\chi_0))2T'(\chi_0) + F(2T'(\chi_0)) \right] \right\} \]
\[ + \frac{4}{D} T''(\chi_0) \left[ \ln \frac{\Lambda}{\mu} + H(2T'(\chi_0)) \right] \left[ \ln \frac{\Lambda}{\mu} + H(2T'(\chi_0)) + \frac{\hat{X}^2}{D} \right] \]
\[ + \frac{1}{2D} \hat{\text{Tr}} \ln \left[ 1 + 2T''(\chi_0) \left( P + 2 \frac{\hat{X}^2}{D} G \right) \right] \right\} \] (28)

The renormalization of this partition function can be achieved by normal ordering. Up to order \(1/D\)
\[ : T(\chi) := \exp \left\{ \left( \ln \frac{\Lambda}{\mu} + c_1 \right) \nabla^2 \right\} T(\chi) = T(\chi + 2 \ln \frac{\Lambda}{\mu} + 2c_1) + \frac{4}{D} \left( \ln \frac{\Lambda}{\mu} + c_1 \right) \left( \chi + \ln \frac{\Lambda}{\mu} + c_1 \right) T''(\chi + 2 \ln \frac{\Lambda}{\mu} + 2c_1) \] (29)

It is simple to see that in terms of the normal ordered : \(T(\chi)\) : the effective action is finite and reads
\[ S_{\text{eff}} = D \left\{ T(\chi_0) - H(2T'(\chi_0))2T'(\chi_0) + F(2T'(\chi_0)) \right\} \]
\[ - \frac{4}{D} T''(\chi_0) \left( c_1^2 + c_1 \chi_0 - \frac{\hat{X}^2}{D} H(2T'(\chi_0)) + \left( H(2T'(\chi_0)) \right)^2 \right) \]
\[ + \frac{1}{2D} \hat{\text{Tr}} \ln \left[ 1 + 2T''(\chi_0) \left( P + 2 \frac{\hat{X}^2}{D} G \right) \right] \} \] (30)

where now
\[ \chi_0 = \frac{\hat{X}^2}{D} - 2c_1 + 2H[2T'(\chi_0)] \] (31)

Moreover any power of \(T''\) in the expansion of \(\hat{\text{Tr}} \ln[1 + 2T''(P + 2G\hat{X}^2/D)]\) in (28) can be computed exactly.

**The limit of quadratic tachyon profile**

As a check of our calculation we take the spherically symmetric tachyon profile considered in \[2\]
\[ T(\chi) = \frac{a}{D} + u\chi \]
For this potential only the leading term in the \(1/D\) expansion survives (higher order terms contain derivatives with respect to \(\chi\) of order 2 and higher and vanish) and from (23,24) and (28) one gets
\[ \frac{Z}{Z_{\text{dir}}} = e^{-a+2Du(c_1+\gamma)}(2u)^{D/2} [\Gamma(2u)]^D \] (32)

namely Witten’s result with the ambiguity due to renormalization kept into account.

**Why the beta function is linear**

We shall now perform the calculation of the effective potential in powers of the tachyon field \(T\) and in the large \(D\) expansion. We shall show that these two expansions
do not commute and that the $1/D$ expansion fails in reproducing the large anomalous dimension of non-polynomial tachyon profiles.

To compare our approach to that of Klebanov and Susskind \[14\] let us consider the partition function of the boundary theory in the presence of a constant source term of the form $J(k) = -ik$.

$$
\frac{Z(k)}{Z_{\text{dir}}} = \int [dX_j] e^{-\int_0^{2\pi} \frac{d\tau}{2\pi} \left[ \frac{1}{2} X^j_i \partial \partial X^j_i + T(X) \right] - ik \hat{X}}
$$

and expand the exponential in powers of $T(X)$. The first non-trivial term is

$$
\frac{Z^{(1)}(k)}{Z_{\text{dir}}} = -\int [dX_j] \int \frac{dk_1}{(2\pi)^D} \int_0^{2\pi} \frac{d\tau_1}{2\pi} T(k_1)e^{-\int_0^{2\pi} \frac{d\tau}{2\pi} \left[ \frac{1}{2} X^j_i \partial \partial X^j_i \right] - ik \hat{X} + ik_1 X(\tau_1)}
$$

The functional integral over the non-zero modes of $X(\tau)$ gives

$$
\frac{Z^{(1)}(k)}{Z_{\text{dir}}} = -\int d\hat{X}_j \int \frac{dk_1}{(2\pi)^D} T(k_1)e^{-\frac{k^2}{2}G(0) + ik_1 \hat{X}}
$$

the integrals over the zero-modes give a $D$-dimensional $\delta$-function so that

$$
-\frac{Z^{(1)}(k)}{Z_{\text{dir}}} \equiv T_R(k) = T(k)e^{-\frac{k^2}{2}G(0)}
$$

This equation provides the renormalized coupling $T_R$ in terms of the bare coupling $T$, to the lowest order in perturbation theory. It corresponds to normal ordering\(^2\).

In order to study the large $D$ limit, when $T(X)$ is a spherically symmetric function, $T(X) = DT(X^2/D)$, it is useful to introduce the Fourier transform of $T(k)$ in the form

$$
T(k) = \int d\hat{X} e^{-ik \hat{X}} \int d\rho DT(\rho)e^{-i\rho \cdot X^2} = \int d\rho \left( \frac{\pi D}{i\rho} \right)^{D/2} DT(\rho)e^{\frac{k^2}{4\rho}}
$$

so that the renormalized coupling as a function of $\rho$ reads

$$
T_R(\rho) = \left( 1 + \frac{i2\rho G(0)}{D} \right) T(\rho)
$$

The second order term in $T$ is given by

$$
\frac{Z^{(2)}(k)}{Z_{\text{dir}}} = \int [dX] \int_0^{2\pi} \frac{d\tau_1}{4\pi} \frac{d\tau_2}{2\pi} \int \frac{dk_1 dk_2}{(2\pi)^D} T(k_1)T(k_2) < e^{ik_1 X(\tau_1)} e^{ik_2 X(\tau_2)} > e^{-ik \hat{X}}
$$

The integral over the non-zero modes can be performed to give

$$
\frac{Z^{(2)}(k)}{Z_{\text{dir}}} = \int_0^{2\pi} \frac{d\tau_1}{4\pi} \frac{d\tau_2}{2\pi} \int \frac{dk_1 dk_2}{(2\pi)^D} \int d\hat{X} \int d\rho_1 d\rho_2 DT(\rho_1)T(\rho_2) \left( -\frac{\pi^2 D^2}{\rho_1 \rho_2} \right)^{D/2}
$$

\(^2\)To make the comparison with the results of ref.\[14\] more straightforward, in this section we choose the renormalization constant ambiguity $c_1 = 0$.  

\[11\]
\[
\exp \left[ i(k_1 + k_2 - k)\hat{X} + i\frac{k_1^2 D}{4\rho_1} + i\frac{k_2^2 D}{4\rho_2} - \frac{1}{2} \left( k_1^2 + k_2^2 \right)G(0) - k_1k_2G(\tau_1 - \tau_2) \right] \tag{40}
\]

Integrating over \(k_1\) and \(k_2\) and expanding for large \(D\) one gets

\[
\frac{Z^{(2)}(k)}{Z_{\text{dir}}} = \int_0^{2\pi} \frac{d\tau_1}{4\pi} \int d\hat{X} \int d\rho_1 d\rho_2 D^2 T_R(\rho_1) T_R(\rho_2) e^{-\frac{2m^2}{D}G^2(\tau_1 - \tau_2) - ik\hat{X} - i(\rho_1 + \rho_2)\hat{X}^2} \tag{41}
\]

Performing at this step the integral over \(\tau_1\) and \(\tau_2\), for \(\rho_1\rho_2 < 0\) one finds a logarithmic divergence which would reproduce the large anomalous dimension. This divergence would provide the non-linear part of the beta function for non-polynomial tachyon fields. If, instead, in eq. (41) one first expands the exponential in powers of \(1/D\) and then performs the integral over \(\tau_1\) and \(\tau_2\), the result is finite and reads

\[
\frac{Z^{(2)}(k)}{Z_{\text{dir}}} = \int d\hat{X} \int d\rho_1 d\rho_2 D T_R(\rho_1) T_R(\rho_2) \left[ \frac{D}{2} - 2\zeta(2)\rho_1\rho_2 \right] e^{-ik\hat{X} - i(\rho_1 + \rho_2)\hat{X}^2/D} \tag{42}
\]

In terms of \(T(\hat{X}^2/D)\), this can be rewritten as

\[
\frac{Z^{(2)}(k)}{Z_{\text{dir}}} = \int d\hat{X} \left[ \frac{D^2}{2} \left( T_R(\hat{X}^2/D) \right)^2 + 2\zeta(2)D \left( T_R(\hat{X}^2/D) \right) \right] e^{-ik\hat{X}} \tag{43}
\]

Eq. (43) coincides with eq. (19) when \(c_1 = 0\), the exponential is expanded up to the second order in \(T\) and a constant source is introduced.

**Conclusions**

We have studied the problem of tachyon condensation in bosonic open string theory in the case where the tachyon condensate has spherical symmetry. Our analysis is limited to the case where the condensate field is a polynomial. In this case, the beta function is linear. Furthermore, the only fields which are polynomial and are fixed points are \(T = 0\) and \(T = \infty\). These are the usual perturbative and stable vacua which are found in the special case where the tachyon is a quadratic function of \(X\). In the present work, we have an elaboration of that case to higher order polynomial potentials.

It would be straightforward to obtain boundary states for D-branes with a tachyon condensate in the large \(D\) limit which we have considered here. When \(D = 26\) these would not be exact boundary states, but would be systematically correctable to any order in \(1/D\). It would be interesting to examine whether, off-shell, they require extra constraints to define them as was discussed for the quadratic tachyon profile in refs. [17].

**References**

[1] E. Witten, “Noncommutative Geometry And String Field Theory,” Nucl. Phys. B 268, 253 (1986).
[2] E. Witten, “On background independent open string field theory,” Phys. Rev. D 46, 5467 (1992) [arXiv:hep-th/9208027]; E. Witten, “Some computations in background independent off-shell string theory,” Phys. Rev. D 47, 3405 (1993) [arXiv:hep-th/9210063].

[3] S. L. Shatashvili, “Comment on the background independent open string theory,” Phys. Lett. B 311, 83 (1993) [arXiv:hep-th/9303143]; S. L. Shatashvili, “On the problems with background independence in string theory,” [arXiv:hep-th/9311177].

[4] A. A. Gerasimov and S. L. Shatashvili, “On exact tachyon potential in open string field theory,” JHEP 0010, 034 (2000) [arXiv:hep-th/0009103]; A. A. Gerasimov and S. L. Shatashvili, “Stringy Higgs mechanism and the fate of open strings,” JHEP 0101, 019 (2001) [arXiv:hep-th/0011009]; A. A. Gerasimov and S. L. Shatashvili, “On non-abelian structures in field theory of open strings,” JHEP 0106, 066 (2001) [arXiv:hep-th/0105245]. E. T. Akhmedov, A. A. Gerasimov and S. L. Shatashvili, “On unification of RR couplings,” JHEP 0107, 040 (2001) [arXiv:hep-th/0105228].

[5] D. Kutasov, M. Marino and G. W. Moore, “Some exact results on tachyon condensation in string field theory,” JHEP 0010, 045 (2000) [arXiv:hep-th/0009148].

[6] A. A. Tseytlin, “Sigma model approach to string theory effective actions with tachyons,” J. Math. Phys. 42, 2854 (2001) [arXiv:hep-th/0011033].

[7] K. Bardakci, “Dual Models And Spontaneous Symmetry Breaking I,” Nucl. Phys. B 70, 397 (1974).

[8] K. Bardakci and M. B. Halpern, “Explicit Spontaneous Breakdown In A Dual Model,” Phys. Rev. D 10, 4230 (1974).

[9] K. Bardakci and M. B. Halpern, “Explicit Spontaneous Breakdown In A Dual Model. 2. N Point Functions,” Nucl. Phys. B 96, 285 (1975).

[10] K. Bardakci, “Spontaneous Symmetry Breakdown In The Standard Dual String Model,” Nucl. Phys. B 133, 297 (1978).

[11] A. Sen, “Descent relations among bosonic D-branes,” Int. J. Mod. Phys. A 14, 4061 (1999) [arXiv:hep-th/9902107].

[12] S. Elitzur, E. Rabinovici and G. Sarkisian, “On least action D-branes,” Nucl. Phys. B 541, 246 (1999) [arXiv:hep-th/9807161].

[13] J. A. Harvey, S. Kachru, G. W. Moore and E. Silverstein, “Tension is dimension,” JHEP 0003, 001 (2000) [arXiv:hep-th/9909072].

[14] I.R. Klebanov and L. Susskind, “Renormalization Group And String Amplitudes,” Phys. Lett. B 200, 446 (1988).

[15] V. A. Kostelecky, M. Perry and R. Potting, “Off-shell structure of the string sigma model,” Phys. Rev. Lett. 84, 4541 (2000) [arXiv:hep-th/9912243].
[16] C. G. Callan and L. Thorlacius, “Sigma Models And String Theory,” Print-89-0232 (PRINCETON) In *Providence 1988, Proceedings, Particles, strings and supernovae, vol. 2* 795-878. (see Conference Index); C. G. Callan and L. Thorlacius, “Open String Theory As Dissipative Quantum Mechanics,” Nucl. Phys. B 329, 117 (1990); C. G. Callan and L. Thorlacius, “Using Reparametrization Invariance To Define Vacuum Infinities In String Path Integrals,” Nucl. Phys. B 319, 133 (1989).

[17] M. Laidlaw and G. W. Semenoff, “The boundary state formalism and conformal invariance in off-shell string theory,” arXiv:hep-th/0112203. G. W. Semenoff, “On a modification of the boundary state formalism in off-shell string theory,” arXiv:hep-th/0106033.