Small Field Coleman-Weinberg Inflation driven by Fermion Condensate

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We revisit the small field Coleman-Weinberg (CW) inflation, which has the following two problems. First, the smallness of the slow roll parameter \(\epsilon\) requires the inflation scale to be very low. Second, the spectral index \(n_s\) tends to become smaller compared to the observed value. In this letter, we consider two possible effects on the dynamics of inflation: radiatively generated non-minimal coupling to gravity \(\xi \phi^2 R\) and condensation of fermions coupled to the inflaton as \(\phi \bar{\psi} \psi\). We show that the fermion condensation can solve the above problems.

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Introduction.— The discovery of the standard model (SM) Higgs boson as well as strong constraints on the supersymmetric parameters forces us to reconsider the basic principles of particle physics. In particular the naturalness problem\cite{1} of the electroweak (EW) scale has received renewed interest. The classical conformality principle was advocated by B. Bardeen as an alternative solution to the naturalness problem, and various extensions of the SM based on the Coleman-Weinberg (CW) mechanism\cite{2} are proposed. Since the CW mechanism does not work within the SM due to the large top Yukawa coupling, we anyway need an additional scalar sector in which the symmetry is radiatively broken via the CW mechanism, which triggers the EW symmetry breaking. The theoretical consistency with the naturalness of the EW scale indicates that the breaking scale \(M\) in the additional sector must be much lower than the Planck scale \(M_{Pl}\).

Two types of inflations are possible if the inflaton field \(\phi\) has the CW potential: the large field inflation (LFI) and the small field inflation (SFI). The large field type of the CW inflation is widely studied as a special case of the chaotic inflation models. On the other hand, the small field CW inflation was studied in the early eighties in the non-supersymmetric GUT models\cite{3,4}. Suppose that the inflaton field is trapped at the origin due to thermal corrections to the effective potential generated in the reheating of the LFI. When the fluctuations of the field are dominated by the vacuum energy at \(\phi = 0\), the second inflaton occurs and the radiation generated so far is rapidly diluted. Then the inflaton field \(\phi\) starts to roll down to the true minimum at \(\phi = M\). Since the slow roll parameters in the SFI satisfy \(\epsilon \ll |\eta|\), the amplitude of the scalar perturbations \(\Delta_{s}^{2} = V/24\pi^2 M_{Pl}^2 \epsilon\) becomes very large unless the vacuum energy \(V\) is sufficiently small. The problem can be solved by setting the scale of the inflation much lower than the GUT scale, but then the e-folding number of the inflation is lowered and accordingly the spectral index \(n_s\) of the scalar perturbation becomes smaller than 0.94, which deviates from the current observational bound, \(n_s = 0.942 - 0.976\). The purpose of the present paper is to give a way to reconcile the predicted \(n_s\) in the small field CW inflation with the observation. In particular, we study two effects on the dynamics of SFI: non-minimal coupling to gravity and condensation of fermions coupled to the inflaton field \(\phi\).

Small field CW inflation.— We first give a brief summary of the small field CW inflation and its inherent problem pointed out in \cite{6}. The CW potential is given by

\[ V(\phi) = \frac{A}{4} \phi^4 \left( \ln \frac{\phi^2}{M^2} - \frac{1}{2} \right) + V_0, \quad V_0 = \frac{A M^4}{8}. \]  

The minimum of the potential is located at \(\phi = M\). In this paper, we assume \(M \ll M_{Pl} = 2.4 \times 10^{18}\) GeV. The quartic coupling and its \(\beta\)-function at the scale \(M\) are given by \(6\lambda = V^{(4)}(M) = 22A\) and \(\beta_\lambda = 24A\) respectively. Taking derivatives with respect to \(\phi\), we have

\[ V' = A \phi^3 \ln \frac{\phi^2}{M^2}, \quad V'' = A \phi^2 \left( 2 + 3 \ln \frac{\phi^2}{M^2} \right). \]  

Hence the inflaton mass is given by \(m_\phi^2 = V''(M) = \beta_\lambda M^2\). The slow roll parameters are calculated to be

\[ \epsilon = \frac{M_{Pl}^2}{2} \left( \frac{V'}{V} \right)^2 \approx 32 \left( \frac{M_{Pl}}{M} \right)^2 \left( \frac{\phi}{M} \right)^6 \left( \ln \frac{\phi^2}{M^2} \right)^2, \]

\[ \eta = M_{Pl}^2 \left( \frac{V''}{V} \right) \approx 24 \left( \frac{M_{Pl}}{M} \right)^2 \left( \frac{\phi}{M} \right)^2 \ln \frac{\phi^2}{M^2}. \]

Here we used \(V \approx V_0\) in the region \(\phi \ll M\). The slow roll conditions \(\epsilon, |\eta| < 1\) require that the field value \(\phi\) during inflation must be much smaller than \(M\); thus the relation \(\epsilon \ll |\eta|\) follows. Inflation stops at \(|\eta| = 1\) where the slow roll condition is violated.\cite{4} can be approximately solved as

\[ \frac{\phi^2}{M^2} \approx \frac{\ln(24M_{Pl}^2/|\eta|M^2)}{(M/M_{Pl})^2} \approx 10^{-3} |\eta| (M/M_{Pl})^2 \ll 1. \]
In the last equality, we put $M = 10^{19}$ GeV and $|\eta| = 0.02$, but the coefficient $10^{-3}$ is insensitive to these values. The slow roll parameter $\epsilon$ is given by

$$\epsilon = \frac{|\eta|^3}{432 \ln(24M^2_{Pl}/|\eta|M^2)} \left(\frac{M}{M_{Pl}}\right)^4 \ll 1. \quad (6)$$

In order to make the amplitude of the scalar perturbation

$$\Delta^2_R \approx \frac{V_0}{24\pi^2 M^4_{Pl}} \epsilon = \frac{94A \ln(24M^2_{Pl}/|\eta|M^2)}{4\pi^2|\eta|^3} \quad (7)$$

consistent with the Planck data \footnote{See also Ref. \cite{R} for a solution by using supersymmetry.}, $\Delta^2_R = 2.215 \times 10^{-9}$ at the pivot scale $k_0 = k_{CMB} = 0.05$ Mpc$^{-1}$, the coefficient $A$ must be extremely small $A \sim 10^{-15}$. Hence the potential height is given by $V_0^{1/4} \sim 10^{-4} M$. Hereafter, the subscript “CMB” means the value evaluated at the pivot scale $k = k_{CMB}$.

The e-folding number $N$ is given by

$$N = \frac{1}{M^4_{Pl}} \int_{\phi_{end}}^{\phi} \frac{V}{\sqrt{V}} d\phi \approx \frac{3}{2} \left(\frac{1}{|\eta|} - \frac{1}{\eta_{end}}\right). \quad (8)$$

By putting $|\eta_{end}| = 1$, we have $\eta = -1/(2N/3 + 1)$. Since $\epsilon \ll |\eta|$, the spectral index of the scalar perturbation is given by $n_s = 1 + 2\eta$. Hence $n_s \sim 0.96$ \footnote{The authors \cite{R} considered a brane world scenario to reconcile the prediction with the measurement \cite{R}.} requires a large e-folding number $N = 3/(1 - n_s) - 3/2 = 73.5$ in the small field CW inflation.

On the other hand, the e-folding number at the pivot scale of CMB measurement is given by

$$N_{CMB} = 61 + \frac{2}{3} \ln \left(\frac{V_0^{1/4}}{10^{16}\text{GeV}}\right) + \frac{1}{3} \ln \left(\frac{T_R}{10^{16}\text{GeV}}\right). \quad (9)$$

where we assumed that there was an epoch of the inflaton field’s oscillation induced by its mass term after the inflation and before the reheating. After the reheating, we also assumed that the radiation dominated epoch continues until the matter-radiation equality epoch. The smallness of the vacuum energy $V_0^{1/4} \sim 10^{-4} M \ll M_{Pl}$ suggests a small e-folding number, which is inconsistent with the above large e-folding number $N = 73.5$. The authors \footnote{The authors \cite{R} considered a brane world scenario to reconcile the prediction with the measurement \cite{R}.} considered a brane world scenario to reconcile the prediction with the measurement \cite{R}. In this letter, we study the following two effects on the dynamics of inflation: a negative non-minimal coupling to gravity and condensations of fermions. The first gives a negative quadratic term $-6|\xi|\phi^2$ in $V(\phi)$ while the second induces a linear term $-C\phi$.

**CW inflation with non-minimal coupling to gravity.—**

So far we have implicitly assumed that the scalar field is minimally coupled to the gravity. But the assumption of $\xi = 0$, where $\xi$ is a non-minimal coupling to gravity $L_\xi = -\xi \phi^2 R/2$, cannot be maintained in quantum field theories since the parameter $\xi$ receives radiative corrections \footnote{See also Ref. \cite{R} for a solution by using supersymmetry.}. The $\beta$ function of $\xi$ is given by $\beta_\xi = (\xi - 1/6)\beta_{m_2}$ where $\beta_{m_2}$ is the $\beta$-function of the mass term. Hence $\xi$ gets renormalized unless $\xi = 1/6$. For example, in the minimal $B - L$ model \footnote{The authors \cite{R} considered a brane world scenario to reconcile the prediction with the measurement \cite{R}.}, if we start from $\xi = 0$ at the UV scale, a negative $\xi$ of order $O(10^{-3})$ can be generated at the scale of inflation. Then an effective mass term $m^2\phi^2/2$ with $m^2 = 12\xi H^2$ is induced in the CW potential \footnote{The authors \cite{R} considered a brane world scenario to reconcile the prediction with the measurement \cite{R}.} during inflation with the Hubble constant $H$.

In the following, we study the small field CW inflation with a negative mass term $m^2 = 12\xi H^2 < 0$. Since $\phi \ll M_{Pl}$, the mass term is negligible compared to the original vacuum energy, $m^2\phi^2 \ll V_0$. Hence $V$ in the definitions of the slow roll parameters can be safely replaced by $V_0$. The first and the second derivatives of $V(\phi)$ are modified due to the mass term. Since the inequality $\epsilon \ll |\eta|$ still holds, the condition $|\eta_{end}| = 1$ determines the end of the inflation. The slow roll parameter $\eta$ becomes

$$\eta = 24 \left(\frac{M_{Pl}}{M}\right)^2 \left(\frac{\phi}{M}\right)^2 \ln \left(\frac{\phi}{M^2} + \frac{m^2M^2_{Pl}}{V_0}\right). \quad (10)$$

The second term is constant; $m^2M^2_{Pl}/V_0 = m^2/3H^2$. It must be smaller than 1 because otherwise the slow roll condition $|\eta| < 1$ is always violated. The inflation ends when the first term in $\eta$ grows over 1. Thus $\phi_{end}$ is the same as in the CW inflation with no mass term \footnote{The authors \cite{R} considered a brane world scenario to reconcile the prediction with the measurement \cite{R}.}.

The e-folding number is given by

$$N = \frac{1}{M^4_{Pl}} \int_{\phi_{end}}^{\phi} \frac{V_0 d\phi}{A\phi^3 \ln(M^2/\phi^2) - m^2\phi} \approx \frac{1}{8|\xi|} \ln \left(\frac{\tilde{A}\phi^2}{A\phi^2 - m^2}\right)_{\phi_{end}} \phi_{end}. \quad (12)$$

where $\tilde{A} = A\ln(M^2/\phi^2)$. In the second equality we performed the integration by using an approximation that $\ln(M^2/\phi^2)$ is almost constant during the inflation. The approximation is shown to be very good by comparing it with numerical calculations.

Combining (10) and (12), we can express $\eta$ in terms of $N$ as

$$\eta(N) = \frac{-12\xi}{1 - e^{-8CN \xi + \xi} + 4\xi}. \quad (13)$$

Since $\epsilon \ll |\eta|$, the spectral index is given by $n_s = 1 + 2\eta_{CMB}$. By using Eq. (13), we can rewrite the e-folding number $N$ in terms of the symmetry breaking scale $M$. Here we assume $T_R = V_0^{1/4}$ for simplicity. In Figure 1 we plot $n_s$ as a function of $M$. The dashed blue line is the analytical result (13) based on the approximation that the logarithmic term is constant. The red solid line is the numerical one without using the above approximation. We also plot the CW result without the non-minimal coupling $\xi = 0$ (green
valley along the direction of mixing term mixed with the Higgs states of quarks. When the SM singlet scalar field $\phi$, fermions.

where the inflaton is coupled with strongly interacting initial term. A similar mechanism will work in a general model. Then the inflaton potential

\[ V = \phi^6 \ln(\phi^2/M^2) - C. \]

Since $\phi \ll M$, the inequality $\epsilon \ll \eta$ still holds and the inflaton ends at the same value of the field $\phi_{\text{end}}$ as in the original CW inflation. Since $V''$ is unchanged, the field value at a fixed $\eta$ is independent of the value of $C$. But the relation between the e-folding number $N$ and the field value $\phi$ is modified as

\[ N = \frac{1}{M_{\text{Pl}}^2} \int_{\phi_{\text{end}}}^{\phi_{\text{end}}} \frac{V_0 d\phi}{A\phi^3 \ln(M^2/\phi^2) + C}. \]  

The second term in the denominator reduces $N$ for fixed $\phi$. It corresponds to the fact that, in presence of the linear term, $\phi_{\text{CMB}}$ becomes smaller than in the original CW inflation so that $|\eta|$ becomes smaller. We parameterize $C$ as $C \equiv AC\phi^6(M/M_{\text{Pl}})^3$ for convenience. Two terms in the denominator of $N$ balance for $C \sim 10^{-6}$ and $\phi_{\text{end}}$, or for $C \sim 10^{-3}$ and $\phi_{\text{end}}$.

In Figure 2 we plot the spectral index $n_s$ and its running $\alpha_s \approx -2\xi(2)$ with $\xi(2) \equiv V''V''''M_{\text{Pl}}^4/V^2$ by changing $C$ for various values of $M$. The scale of $M$ is varied between $10^7$ and $10^{13}$ GeV. The predicted spectral index $n_s$ becomes consistent with the observation when $C \sim 10^{-5}$.

To summarize the linear term induced by the fermion condensate can solve the small $n_s$ problem in the small field CW inflation. The slow roll parameter $\epsilon$ is made bigger about 10 times, but the inequality $\epsilon \ll |\eta|$ still holds. Thus the predicted tensor-to-scalar ratio is negligibly small. The magnitude of the scalar perturbation again requires a very small quartic coupling $A \sim 1.3 \times 10^{-14}(C/10^{-5})^2$. In addition, as shown in Figure 2 this model can be tested by the future 21cm and CMB B-mode observations on the running of the spectral index $n_s$.

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**Fermion condensates.**— Another possibility to increase $n_s$ is a generation of a linear term in the inflaton potential $V(\phi)$ due to condensation of fermions coupled to the inflaton field. We will give two examples that may realize such a possibility.

First, in the $B-L$ model, the RH neutrinos $N_i$ are coupled to $\phi$ by $\phi\overline{\nu}_Y N_i N_i$. By integrating out $\phi$, four-Fermi interaction $G(\overline{N}_e N)(\overline{N}_e N)$ is induced with $G \sim Y_{i\nu}^2/\mu_f^2$. If the Majorana Yukawa coupling is large enough $Y_{N_i} \sim O(10)$, the RH neutrinos may condense $\phi$. Then the inflaton potential $V(\phi)$ acquires a linear term $-C\phi$ where $C = Y_{N_i} N_i N_i$. The minus sign is a convention to determine the direction of the linear potential. A similar mechanism will work in a general model where the inflaton is coupled with strongly interacting fermions.

Another example is to use conventional chiral condensates of quarks. When the SM singlet scalar field $\phi$ is mixed with the Higgs $h$ with a very small (and negative) mixing term $\lambda_{\text{mix}}\phi^2 h^2$, the potential $V(\phi, h)$ has a valley along the direction of $h^2 = (|\lambda_{\text{mix}}|/2\lambda_h)\phi^2$ (see e.g. [12]). If the chiral condensate $\langle \bar{qq} \rangle \neq 0$ occurs near the origin of the potential, it generates a linear term $-C_0 h$ in the Higgs potential with $C_0 \sim y\langle \bar{qq} \rangle$ where $y$ is the Yukawa coupling. Then the scalar mixing induces a linear term in the direction of the valley $\phi$ with a coefficient $C = \sqrt{|\lambda_{\text{mix}}|/2\lambda_h} C_0 = (246/M(\text{GeV})C_0$.

**CW inflation with fermion condensate.**— In the following, we suppose that an appropriate magnitude of a linear term exists in the inflaton potential (see [14] for a role played by the linear term in a different context). Then a constant term is added to the first derivative of $V$: $V' = A\phi^3 \ln(\phi^2/M^2) - C$. Since $\phi \ll M$, the inequality $\epsilon \ll \eta$ still holds and the inflaton ends at the same value of the field $\phi_{\text{end}}$ as in the original CW inflation. Since $V''$ is unchanged, the field value at a fixed $\eta$ is independent of the value of $C$. But the relation between the e-folding number $N$ and the field value $\phi$ is modified as

\[ N = \frac{1}{M_{\text{Pl}}^2} \int_{\phi_{\text{end}}}^{\phi_{\text{end}}} \frac{V_0 d\phi}{A\phi^3 \ln(M^2/\phi^2) + C}. \]  

2 The Higgs field acquires a small VEV; $\langle h \rangle = (C_0/\lambda_h)^{1/3} < 246$ GeV. It breaks the EW symmetry near the origin of the potential and modifies the orbit of the classical motion on $h, \phi$ plane from the path along the valley. If $C_0^{-1/3} \gg \phi_{\text{end}}$, it invalidates the generation of a linear term in the inflaton potential. Detailed studies in specific models are left for future investigations.
FIG. 2. The relations between the spectral index $n_s$ and its running $\alpha_s$ in the SFI model with a linear potential $C \phi$ are plotted. Each red curve corresponds to a different symmetry breaking scale $M = 10^{7−11}$ GeV. The relation of $N_{\text{CMB}}$ and $M$ is given by Eq. (9) with $T_R \sim V_0^{1/4}$. A different $C$ corresponds to a different point on the red curve; $C \equiv A \tilde{C} M^3 (M/M_{\text{Pl}})^3$. By the future 21cm and CMB B-mode observations, the sensitivity on $\alpha_s$ will become $\delta \alpha_s = 5.3 \times 10^{-4}$ [15] which is denoted by the horizontal solid line.

Summary.— In this paper, we studied effects of a non-minimal coupling to gravity and fermion condensation on the small field CW inflation. The original small field CW inflation predicts rather small spectral index $n_s < 0.94$, compared to the Planck measurement $n_s = 0.942 \pm 0.008$ [7] in case with the running of $n_s$. The effect of a non-minimal coupling to gravity with a negative value $|\xi| \sim O(10^{-3})$ is shown to increase the spectral index $n_s$ by 0.005 but not sufficient to reconcile the prediction with the data for $M < 10^{13}$ GeV. We then studied the effect of condensation of fermions coupled to the inflaton field $y_\psi \phi \bar{\psi}$. If $\langle \bar{\psi} \psi \rangle \neq 0$, a linear term is generated in the inflaton potential $V(\phi)$. In particular, when the inflaton and the Higgs are mixed, the chiral condensate of quarks induce a linear term in the inflaton potential. We showed that an appropriate magnitude of condensation can make the theoretical prediction of $n_s$ consistent with the observational data. The tensor-to-scalar ratio $r = 16 \epsilon$ is negligibly small. However, the running of the spectral index will be tested by the future 21cm and CMB B-mode observations [15].

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