RELATION BETWEEN IRRATIONALITY AND REGULARITY FOR $C^1$ CONJUGACY OF $C^2$ CIRCLE DIFFEOMORPHISMS TO RIGID ROTATIONS

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ABSTRACT. By introducing the modulus of continuity, we first establish the corresponding cross-ratio distortion estimates under $C^2$ smoothness, and further give a Denjoy-type inequality, which is almost optimal in dealing with circle diffeomorphisms. The latter plays a prominent role in the study of $C^1$ conjugacy to irrational rotations. We also give the explicit integrability correlation between continuity and irrationality for the first time. Further the regularity of the conjugation is also considered and proved to be sharp.

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1. INTRODUCTION

Conjugacy on circle, that is, under what conditions a diffeomorphism on $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z} = [0, 1)$ (regard 1 as equivalent to 0) can be conjugated to a rotation, is one of the most fundamental but difficult topics in dynamical systems. It has a long
research history. It is known since Poincaré that the rotation number $\rho(T) \in T^1$ is always well defined (up to an integer summand) for an orientation-preserving homeomorphism $T$ on $T^1$. More precisely speaking,

$$\rho(T) = \lim_{n \to +\infty} \frac{L_T^n(x_0)}{n},$$

where $L_T(x)$ is the lifting of $T$ from $T^1$ onto $\mathbb{R}$ such that $L_T(x + 1) = L_T(x) + 1$ and $T(x) = \{L_T(x)\}$, where the braces $\{\cdot\}$ denote the fractional part. It is worth mentioning that $\rho(T)$ does not depend on the choice of the initial point $x_0 \in T^1$, and it is irrational if and only if the mapping $T$ has no periodic points. We will focus on this case throughout the present paper. In the study of circle conjugacy, the smooth (even analytic) conjugacy is more important than the topological type. It should be pointed out that not all irrational rotations can be smoothly conjugated, see the counterexamples constructed by Arnol’d [2] and Herman [10]. The latter showed that for a Liouville rotation number $\alpha_1$ (i.e. is not Diophantine and Bruno), there exists an analytical diffeomorphism $\tilde{T}$ of the circle for which the topological conjugacy to the rotation $R_{\alpha_1}(\theta) = \theta + \alpha_1 \mod 1$ is not even absolutely continuous (of course does not admit $C^1$ conjugacy), see also the work in [20, 28]. Denjoy [6] proved that if $T$ is of class $C^2$ then it could topologically conjugate to the rotation $R_{\alpha_2}(\theta) = \theta + \alpha_2 \mod 1$, i.e. there exists a homeomorphism $\phi$ on $T^1$ such that $\phi \circ T = R_{\alpha_2} \circ \phi$. Based on these crucial facts, a natural question to ask is:

**Under what conditions the conjugation is smooth, or even only differentiable, or equivalently what is the relation between regularity of $T$ and irrationality of the rotation number $\rho(T)$ in dealing with at least $C^1$ conjugacy?**

Irrationality can be described in many ways. To consider conjugacy on the circle, Herman [10] established a condition of partial quotients whose corresponding irrational numbers form a set of full Lebesgue measure. Incidentally, it is a breakthrough in global conjugacy, which is different from the KAM approach relying on the localness (i.e., one has to require that $T$ must close to $R_\rho$). Yoccoz [28] further adopted the nonresonant condition of Diophantine’s type for the irrational rotation $\rho \in T^1$ with exponent $\beta > 0$, i.e.,

$$|\rho - \frac{p}{q}| > \frac{C(\rho)}{q^{2+\beta}}$$

for any rational number $p/q$, where $C(\rho)$ is a positive number that depends on $\rho$. In this case the regularity requirement of the mapping $T$ is $C^k$, where $k \geq 3$ and $k > 2\beta + 1$, and the essentially unique diffeomorphism which conjugates $T$ to $\rho$ is of class $C^{k-1-\beta-\varepsilon}$ for every $\varepsilon > 0$. Recently a result in the Hölder’s sense appears in [15] based on Diophantine rotation with exponent $\delta$ (see (1.1)), which weakens $C^k$ smoothness for the mapping $T$ to $C^{2+\alpha}$, through an improved Denjoy’s inequality of Hölder’s type, where $0 \leq \delta < \alpha \leq 1$ and $\alpha - \delta < 1$. The conclusion that the regularity of the conjugation at this point is of class $C^{1+\alpha-\delta}$ is proved to be optimal, which sharpens the work [13] (the corresponding regularity is $C^{1+\alpha-\delta-\varepsilon}$ with any $\varepsilon > 0$), i.e., the exponent $1 + \alpha - \delta$ cannot be higher in
general settings due to the counterexamples in [13]. It should be emphasized that this first result on optimal regularity is based on the use of cross-ratio distortion estimates for $T$ of class $C^{2+\alpha}$ with $\alpha > 0$, which is a conceptually new approach.

However, much recent work require $C^k$ (at least $k \geq 2$) plus certain Hölder continuity for $T$, and one of our motivations is to weaken the regularity to only $C^2$ (and preserves $C^1$ conjugacy simultaneously), which is the weakest case that might be achieved due to certain counterexamples, see [9,11,19]. This is quite necessary because Hölder continuity is not sufficient to deal with all problems of finite smoothness from the perspective of Baire category. Therefore, ones are actually still a long way from the true critical situation. As an illustration, let us take into account some $T$ with continuous $D^2T = -(\log(x(1-x)))^{-1}$ (define $D^2T = 0$ at end points 0 and 1) which cannot be characterized by arbitrary Hölder’s type, since $D^2T \sim (\log x^{-1})^{-1}$ as $x \to 0^+$, that is, Logarithmic Hölder’s type with index 1 in our terminology (see Section 2), and thus this case could not be analyzed through the well known results via only Hölder continuity.

In this paper, we extend cross-ratio distortion estimates to the weakest case and study $C^1$ conjugacy on circle from a different perspective. This corresponds exactly to the opinion proposed by Khanin and Teplynsky in [15], who believed that these powerful tools will prove useful in other problems (different from the Hölder regularity via Diophantine irrationality) involving circle diffeomorphisms. Specifically, by introducing the definition of modulus of continuity in Section 2, we first establish the cross-ratio distortion estimates for any $C^2$ strictly monotone function, and then obtain the corresponding Denjoy-type inequality for any irrational rotation in Section 3. Via these tools, we present the following main result (in Section 4), which contains a sufficient integrability condition under which $C^2$ orientation-preserving circle diffeomorphisms can be $C^1$ conjugated to irrational rotations (characterized by partial quotients rather Diophantine or Bruno condition):

**[Main Theorem]** Let $T$ be a $C^2,\varpi$ orientation-preserving circle diffeomorphism with rotation number $\rho(T) = [k_1,k_2,\ldots]$, where $k_{n+1} = O(\varphi(n))$, and $\varpi$ is a modulus of continuity. Then the circle diffeomorphism in Denjoy’s theory is differentiable if

$$\int_0^1 \left( \int_0^y \varphi(\log_\lambda x) \, dx \right) \frac{\varpi(y)}{y^2} \, dy < +\infty,$$

where $0 < \lambda < 1$ is a definite constant. Detailed definitions and notations are provided in Sections 2 and 4.

Then we further investigate the higher regularity of the conjugation, following the above main theorem, which slightly improves part of the optimal result in [15]. Besides, some explicit examples are provided, including the case of combining partial quotients with probability distributions. Finally, in Subsection 4.6 it is shown certain optimality about our results, from the perspective of preserving and improving $C^1$ conjugacy, etc. To the best of our knowledge, our integrability condition seems to explicitly link irrationality with regularity for the first time and is relatively easy
to verify. Now we obtain the optimal integrability condition about the irrationality and regularity.

Thereby our main theorem develops the classical Denjoy’s and Herman’s theory (see Subsections 4.2 and 4.5 respectively) since the topological conjugacy is elevated to differential’s type and the smoothness assumption of $T$ is reduced to only $C^2$ (in fact, $C^2$ plus some modulus of continuity which is much weaker than the Hölder’s type). Actually, it should be emphasized that the $C^2$ regularity for the mapping $T$ cannot be further weakened, otherwise the conjugation might not be differentiable due to the loss of regularity and irrationality of the rotation and might even become only pure singular, see Subsection 4.6.

2. MODULUS OF CONTINUITY

To state the results in this paper, we first introduce the definition of modulus of continuity, which describes the weaker continuity than Hölder’s type. It has been attracted a lot of attention in dynamical systems, see for instance, Bufetov and Solomyak [4], Duarte and Klein [7], Fan and Jiang [8] and etc. Modulus of continuity characterizes the complexity of dynamical systems, and obviously, not all dynamics could be preserved in dynamical systems of arbitrary complexity. Therefore, finding the inner connection between invariance and complexity and touching criticality is a fundamental but difficult topic. Generally, certain optimal integrability conditions for modulus of continuity always arise in critical cases, and they indeed solve some problems completely and perfectly. Back to our concern, Herman [10] discussed the existence of circle diffeomorphisms via modulus of continuity and provided an stronger answer to Arnol’d’s conjecture. Katznelson and Ornstein [13] also considered problems similar to that in this paper (the terminology $H^{2+\psi}$ with a modulus of continuity $\psi$ in Sections 2, 3 and 4 is equivalent to $C^{2,\psi}(\mathbb{T}^1)$ here). See also Kim and Koberda [16, 17] on circle diffeomorphisms via the tool of modulus of continuity.

Aiming to study the generalized Denjoy-type inequality provided in Subsection 4.3, and further $C^1$ conjugacy for circle diffeomorphisms, even more accurate regularity in this paper (including some optimal cases), we are in a position to formulate the definition of modulus of continuity, see Herman [10]. It should be noted that there are many definitions of modulus of continuity, and there are some subtle differences between them, for example, see [8, 10] and references therein. We do not pursue that.

Throughout this paper, $O(\cdot), o(\cdot)$ and $\sim$ are uniform with respect to $n,m \to +\infty$, or $x \to 0^+$ without causing ambiguity. Note that $a(n,m) \sim b(n,m)$ implies that there exists a universal constant $C_1,C_2 > 0$ independent of $n,m$ (may depend on other parameters), such that $C_1b \leq a \leq C_2b$.

Definition 2.1. Denote by $\varpi(x)$ a modulus of continuity, which is a strictly monotonic increasing continuous function defined on $\mathbb{R}^+$, such that

1. $\varpi(x+y) \leq \varpi(x) + \varpi(y)$ for $x, y > 0$;
2. $\varpi(px) \leq p\varpi(x)$ for $p \in \mathbb{N}^+$ and $x > 0$;
3. $\varpi(ax) \leq ([a] + 1)\varpi(x)$ for $a \in \mathbb{R}^+$ and $x > 0$. 
The above definition is an extension of the classical Lipschitz and Hölder cases, which correspond to \( \varpi \sim x \) and \( \varpi \sim x^\alpha \) with some \( 0 < \alpha < 1 \), respectively. Further, there are plethora of important examples of that are fail to be characterized by \( \varepsilon \)-Hölder’s type for any \( \varepsilon \in (0, 1) \). For instance, Logarithmic Hölder’s type \( \varpi \sim (\log x^{-1})^{-\alpha} \) with respect to \( 0 < x < 1 \) and index \( \alpha > 0 \), and some examples generated from power series below. Note (1) implies that \( \varpi = O(x) \), that is, could not better than Lipschitz’s type, otherwise all functions with this modulus of continuity must be some constants, see Definition 2.5. In fact, these conditions hold automatically for most monotone functions of the type \( \varpi = O(x) \), we do not pursue this point and focus on the order of \( \varpi \) at \( 0^+ \) instead, which is indeed essential to characterize the complexity of the continuity on a bounded domain throughout this paper, that is, just focus on the case on the interval \((0, \delta]\) with some \( \delta > 0 \) instead of \( \mathbb{R}^+ \).

Although modulus of continuity can be defined in various ways, Dini integrability condition on modulus of continuity is universal and appears frequently in various fields, including dynamical systems and PDEs. Specifically, a modulus of continuity \( \varpi(x) \) on \((0, \delta]\) with \( \delta > 0 \) is said to satisfy the Dini condition (or be of Dini continuous type) if

\[
\int_0^\delta \frac{\varpi(x)}{x} dx < +\infty. \tag{2.1}
\]

As an example, for the Logarithmic Hölder’s type \( \varpi \sim (\log x^{-1})^{-\alpha} \) with \( \alpha > 0 \) above, then \( \varpi \) satisfies the Dini condition (2.1) if and only if \( \alpha > 1 \) at this point. Note that the Dini condition can also be characterized with respect to the summability of evaluated along geometric sequences, see [1]. Namely, (2.1) holds if and only if

\[
\sum_{n=1}^\infty \varpi(\theta^n x) < +\infty
\]

for any \( \theta \in (0, 1) \). Actually we can verify that

\[
\varpi(x) := \sum_{n=1}^\infty \varpi(\theta^n x), \quad x \in (0, \delta] \tag{2.2}
\]

is also a modulus of continuity at this point, see [8]. Such series functions are obviously extremely difficult to analyze in specific problems, so it is necessary to introduce certain integrability conditions, such as (2.1) and etc (in some cases the Dini condition might not be sufficient to preserve certain dynamical properties, such as regularity, and thus stronger integrability conditions must arise as we will see later). Here we introduce a construction method about Dini continuous type, see [1]: Let \( \{a_j\}_{j \in \mathbb{N}^+} \in c_0 \) and \( \{\gamma_j\}_{j \in \mathbb{N}^+} \in \ell_1 \) be sequences of positive numbers, and \( \lim_{j \to +\infty} a_j = 0 \). We further assume that there exist \( \tau, x_\ast > 0 \) such that \( \tau < x_\ast \), and

\[
\sum_{j=1}^\infty a_j x_\ast^{\gamma_j} < +\infty, \quad \sum_{j=1}^\infty a_j^{\frac{\tau}{\gamma_j}} < +\infty.
\]
Then the function
\[ \varpi(x) \sim \sum_{j=1}^{\infty} a_j x^{\gamma_j} \]
(2.3)
is indeed a modulus of continuity on \((0, x^*)\), and satisfies the Dini condition (2.1).

As an explicit illustration, \( \varpi(x) = \sum_{j=1}^{\infty} \sqrt{x} x^{2j} \).

Next we define the “strong” and “weak” properties, as well as the uniform continuity of the function with respect to the modulus of continuity. Denote by \( D \) a connected region in \( \mathbb{R}^d \) with some \( d \in \mathbb{N}^+ \), and \( | \cdot | \) represents the sup-norm.

**Definition 2.2.** Give two modulus of continuity \( \varpi \) and \( \varpi^* \). We say that \( \varpi^* \) is weaker than \( \varpi \) (\( \varpi^* \gtrsim \varpi \) or \( \varpi \lesssim \varpi^* \) for short), if
\[ \lim_{x \to 0^+} \frac{\varpi(x)}{\varpi^*(x)} < +\infty. \]
Further, we say that \( \varpi^* \) is strictly weaker than \( \varpi \) if
\[ \lim_{x \to 0^+} \frac{\varpi(x)}{\varpi^*(x)} = 0. \]

**Remark 2.3.** Obviously any modulus of continuity \( \varpi_0 \) is weaker than \( x \) (\( \varpi_0 \gtrsim x \)) which we forego. Hölder’s type \( \varpi_1 \sim x^\alpha \) with any \( 0 < \alpha < 1 \) is strictly weaker than Lipschitz’s type \( \varpi_2 \sim x \). Logarithmic Hölder’s type \( \varpi_3 \sim (\log x^{-1})^{-\beta} \) with any \( \beta > 0 \) is strictly weaker than \( \varpi_1 \).

**Remark 2.4.** “Strictly” implies that \( \varpi_1 \) cannot be accurately characterized by any modulus of continuity of type \( \varpi_2 \), as we have mentioned in the introduction.

**Definition 2.5.** A function \( f(x) \) is called to be \( C_{k, \varpi} \) with some \( k \in \mathbb{N} \) on \( D \) (\( f \in C_{k, \varpi}(D) \) for short), if \( f(x) \in C^k(D) \) and
\[ \left| f^{(k)}(x) - f^{(k)}(y) \right| \leq \varpi(|x - y|), \quad \forall x, y \in D, \ x \neq y. \]

**Remark 2.6.** The above definition can be easily extended to the higher dimensional (even infinite dimensional) case. Further assume that \( D \) be bounded and closed. If \( f \in C^k(D) \) with some \( k \in \mathbb{N} \), then \( D^k f \) automatically has a modulus of continuity \( \varpi \), i.e., \( f \) is \( C_{k, \varpi} \) on \( D \). However, in the infinite dimensional case, continuity does not imply the existence of modulus of continuity, because the compactness may be absent at this point. But maybe one could choose some appropriate norms to avoid that.

### 3. Cross-ratio estimates via modulus of continuity

Yoccoz [29] introduced the cross-ratio distortion estimates (asymptotics of double ratios) in dynamical systems for the first time, and proved that there are no analytic Denjoy counterexamples. See also [5, 15, 24]. It is worth mentioning that this powerful tool can also be employed to study Denjoy-type inequality under higher smoothness, such as \( C^3 \)'s type based on Schwartz derivatives by Teplinsky [25, 26].
Objectively speaking, the cross-ratio distortion estimates greatly simplify the conjugacy analysis of circle diffeomorphisms and makes the proof fundamental (instead of studying the fundamental segments directly).

We first present some basic notions. Let $f$ be a strictly increasing function and $f'$ does not vanish. The ratio of three pairwise distinct points $x_1, x_2, x_3$ is defined as

$$R(x_1, x_2, x_3) := \frac{x_1 - x_2}{x_2 - x_3},$$

and the ratio distortion with respect to those points and the given $f$ is

$$D(x_1, x_2, x_3; f) := \frac{R(f(x_1), f(x_2), f(x_3))}{R(x_1, x_2, x_3)} \cdot \frac{f(x_1) - f(x_2)}{f(x_2) - f(x_3)}.$$  

The cross-ratio of four pairwise distinct points $x_1, x_2, x_3, x_4$ is denoted by

$$Cr(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_4 - x_1)},$$

and the cross-ratio distortion of those points with respect to $f$ is termed

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = \frac{Cr(f(x_1), f(x_2), f(x_3), f(x_4))}{Cr(x_1, x_2, x_3, x_4)}.$$  

One can easily verify that

$$D(x_1, x_2, x_3; f \circ g) = D(x_1, x_2, x_3; g) \cdot D(g(x_1), g(x_2), g(x_3); f),$$

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = \frac{D(x_1, x_2, x_3; f)}{D(x_1, x_4, x_3; f)},$$  

and

$$\text{Dist}(x_1, x_2, x_3, x_4; f \circ g) = \text{Dist}(x_1, x_2, x_3, x_4; g) \cdot \text{Dist}(g(x_1), g(x_2), g(x_3), g(x_4); f).$$

In fact, the above definitions are also well defined for the case where two (or three) points are identical, as long as $\frac{f(x) - f(x)}{x - x} := f'(x)$ is defined.

**Proposition 3.1** (Cross-ratio estimates). Assume $f$ is $C_{2, \infty}$ with some modulus of continuity $\varpi$, and $f' > 0$ on $[A, B]$. Then for any $x_1, x_2, x_3 \in [A, B]$, the following estimate holds:

$$D(x_1, x_2, x_3; f) = 1 + (x_1 - x_3) \left(\frac{f''}{2f'} + O(\varpi(\Delta))\right),$$

where $\Delta = \max_{1 \leq i \leq 3} \{x_i\} - \min_{1 \leq i \leq 3} \{x_i\}$, and the values of both $f''$ and $f'$ can be taken at any points between $\min_{1 \leq i \leq 3} \{x_i\}$ and $\max_{1 \leq i \leq 3} \{x_i\}$. 

Proof: One trivially verifies that the conclusion holds automatically as long as there exist \( x_i = x_j \) with \( i \neq j \), we thus assume that \( x_i \neq x_j \) if \( i \neq j \). Three different cases need to be discussed. We first prove a basic case, and the rest of the cases can be directly obtained through it. Let \( x^* \) arbitrarily lie between \( \min_{1 \leq i \leq 3} \{ x_i \} \) and \( \max_{1 \leq i \leq 3} \{ x_i \} \), and denote \( \Delta = \max_{1 \leq i \leq 3} \{ x_i \} - \min_{1 \leq i \leq 3} \{ x_i \} \).

**Case1:** \( x_2 \) lies between \( x_1 \) and \( x_3 \). By applying the Mean Value Theorem there exist (i) \( \zeta_1 \) between \( x_1 \) and \( x_2 \), (ii) \( \zeta_2, \zeta_3 \) between \( x_2 \) and \( x_3 \), (iii) \( \zeta_4 \) between \( \zeta_3 \) and \( x^* \), such that

\[
\frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{f(x_2) - f(x_3)}{x_2 - x_3} = \left( f'(x_2) + \frac{1}{2} f''(\zeta_1) (x_1 - x_2) \right) - \left( f'(x_2) + \frac{1}{2} f''(\zeta_2) (x_3 - x_2) \right)
\]

\[
= \frac{1}{2} \left( x_1 - x_3 \right) \left( f''(x^*) + f''(\zeta_1) - f''(x^*) \right) + \left( f''(\zeta_1) - f''(\zeta_2) \right) \frac{x_3 - x_2}{x_1 - x_3}
\]

\[
= (x_1 - x_3) \left( \frac{1}{2} f''(x^*) + O(\varpi(\Delta)) \right), \tag{3.2}
\]

and

\[
\left( \frac{f(x_2) - f(x_3)}{x_2 - x_3} \right)^{-1} = \frac{1}{f'(\zeta_3)} = \frac{1}{f'(x^*)} \frac{f'(\zeta_3)}{f'(x^*)} = \frac{1}{f'(x^*)} \frac{f'(\zeta_3) + f''(\zeta_4) (x^* - \zeta_3)}{f'(\zeta_3)} = \frac{1}{f'(x^*)} (1 + O(\Delta)). \tag{3.3}
\]

At this point, it follows that

\[
D(x_1, x_2, x_3; f) = 1 + \left( \frac{f(x_1) - f(x_2)}{x_1 - x_2} - \frac{f(x_2) - f(x_3)}{x_2 - x_3} \right) \cdot \frac{f(x_2) - f(x_3)}{x_2 - x_3}
\]

\[
= 1 + (x_1 - x_3) \left( \frac{1}{2} f''(x^*) + O(\varpi(\Delta)) \right) \cdot \frac{1}{f'(x^*)} (1 + O(\Delta))
\]

\[
= 1 + (x_1 - x_3) \left( f''(x^*) + O(\varpi(\Delta)) \right). \tag{3.4}
\]

**Case2:** \( x_1 \) lies between \( x_2 \) and \( x_3 \). In view of (3.2) and (3.3) in Case1 we get

\[
D(x_1, x_2, x_3; f) = 1 + \left[ \frac{x_1 - x_3}{x_2 - x_3} \left( \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_3)}{x_1 - x_3} \right) \right] \cdot \frac{f(x_2) - f(x_3)}{x_2 - x_3}
\]

\[
= 1 + \left[ \frac{x_1 - x_3}{x_2 - x_3} \left( \frac{1}{2} f''(x^*) + O(\varpi(\Delta)) \right) \right] \cdot \frac{1 + O(\Delta)}{f'(x^*)}.
\]
\[= 1 + (x_1 - x_3) \left( \frac{f''(x^*)}{2f'(x^*)} + \mathcal{O}(\varpi(\Delta)) \right).\]

**Case 3:** \(x_3\) lies between \(x_1\) and \(x_2\). The conclusion holds by the same token in view of Case 2 because

\[
D(x_1, x_2, x_3; f) = 1 + \left[ \frac{x_1 - x_3}{x_1 - x_2} \left( \frac{f(x_1) - f(x_3)}{x_1 - x_3} - \frac{f(x_2) - f(x_3)}{x_2 - x_3} \right) \right] \cdot \frac{f(x_2) - f(x_3)}{x_2 - x_3}.
\]

This completes the proof. \(\square\)

**Proposition 3.2** (Cross-ratio distortion estimates). Assume \(f\) is \(C_{2,\infty}\) with some modulus of continuity \(\varpi\), and \(f' > 0\) on \([A, B]\). Then for any \(x_1, x_2, x_3, x_4 \in [A, B]\), the following estimate holds:

\[
\text{Dist}(x_1, x_2, x_3, x_4; f) = 1 + (x_1 - x_3) \mathcal{O}(\varpi(\Delta)),
\]

where \(\Delta = \max\{x_i\} - \min\{x_i\}\).

**Proof.** Recall (3.1). Then direct calculation gives that

\[
\text{Dist}(x_1, x_2, x_3, x_4; f) = \frac{D(x_1, x_2, x_3; f)}{D(x_1, x_4, x_3; f)}
\]

\[
= \left(1 + (x_1 - x_3) \frac{f''(x^*)}{2f'(x^*)} + (x_1 - x_3) \mathcal{O}(\varpi(\Delta))\right)
\]

\[
\cdot \left(1 - (x_1 - x_3) \frac{f''(x^*)}{2f'(x^*)} + (x_1 - x_3) \mathcal{O}(\varpi(\Delta))\right)
\]

\[
= 1 + (x_1 - x_3) \left(\mathcal{O}(\varpi(\Delta)) + \mathcal{O}(\Delta) + \mathcal{O}(\Delta^2)\right)
\]

\[
= 1 + (x_1 - x_3) \mathcal{O}(\varpi(\Delta)).
\]

\(\square\)

### 4. Circle Diffeomorphisms

#### 4.1. Notations and lemmas. For a given \(\rho \in \mathbb{T}^1 \setminus \{0\}\), the following continued fraction of \(\rho\) is uniquely determined:

\[
\rho = \frac{1}{k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \ddots}}} := [k_1, k_2, \ldots, k_n, \ldots],
\]

where \(k_n\) with \(n \in \mathbb{N}^+\) are called the partial quotients of \(\rho\). Denote by \(p_n/q_n = [k_1, k_2, \ldots, k_n]\) the \(n\)th approximant of \(\rho\) (rational approximations), then \(p_n\) and \(q_n\) satisfy the following recurrence relations for all \(n \in \mathbb{N}\)

\[
p_{n+1} = k_{n+1} p_n + p_{n-1}, \quad q_{n+1} = k_{n+1} q_n + q_{n-1}
\]

(4.1)

with \(p_1 = 1, p_0 = 0\) and \(q_1 = k_1, q_0 = 1\) (here we define \(p_{-1} = 1, q_{-1} = 0\) for convenience).
Note that the rotation number $\rho(T)$ of an orientation-preserving homeomorphism $T$ on $\mathbb{T}^1$ is always well-defined, we therefore could study the conjugacy directly via the partial quotients as well as the approximant of the rotation. Consider a marked point $\xi_0 \in \mathbb{T}^1$ and its trajectory $\xi_i = T^i \xi_0$ with $i \in \mathbb{N}^+$, and pick out of it the sequence of the dynamical convergents $\xi_{q_n}$ for $n \in \mathbb{N}$, indexed by the denominators of the $n$th approximant $p_n/q_n$ of the rotation $\rho(T)$. Here we define $\xi_{q_n} = \xi_0 - 1$ for convenience. The arithmetic properties of rational approximations together with the combinatorial equivalence between $T$ and the rigid rotation $R_\rho(T) : \theta \to \theta + \rho(T) \mod 1$ (i.e., the order of points on the circle for any trajectory coincides with the order of points for $R_\rho(T)$) show that the dynamical convergents approach the initial point $\xi_0$ from both sides:

$$\xi_{q_0} < \xi_{q_1} < \cdots < \xi_{q_{2^n-1}} < \cdots < \xi_0 < \cdots < \xi_{q_2} < \xi_{q_1} < \xi_{q_0}. \quad (4.2)$$

In view of (4.2), we denote by $\Delta^n(\xi)$ the $n$th fundamental segment, where $\Delta^n(\xi) = [\xi, T^n \xi]$ if $n$ is even and $\Delta^n(\xi) = [\xi, T^n \xi] \mod 1$ if $n$ is odd. For a given marked point $\xi_0 \in \mathbb{T}^1$, we denote that $\Delta_0^{(n)} = \Delta_n(\xi_0)$ and $\Delta_i^{(n)} = \Delta_n(\xi_i) = T_i \Delta_0^{(n)}$ for all $i \in \mathbb{N}^+$. Further, define

$$l_n := l_n(T) = \max_{\xi \in \mathbb{T}^1} |\Delta_n(\xi)| = \|T^q - \text{id}\|_{C^0}$$

and

$$\Delta_n := l_n(R_\rho) = |q_n \rho - p_n| = (-1)^n (q_n \rho - p_n).$$

At this point we have $l_{-1} = \Delta_{-1} = 1$, and $l_n, \Delta_n \in (0, 1)$ for all $n \in \mathbb{N}^+$. Recall the recurrence relations (4.1) of $p_n$ and $q_n$, we arrive at $\Delta_n = k_{n+2} \Delta_{n+1} + \Delta_{n+2}$ for $-1 \leq n \in \mathbb{Z}$. Since $\Delta_n$ is strictly monotonically decreasing, we could extend it as a continuous function $\Delta(\cdot) : [1, +\infty) \to (0, 1]$ and denote by $\Delta^{-1}$ the inverse of $\Delta$.

Next we provide several basic but well-known lemmas about the combinatorics of trajectories as well as estimates for fundamental segments. Their proofs are classical and can be found in [15] (see Lemmas 1, 2, 3, 4, 5, 7 respectively), which we omit here for the sake of brevity since they do not involve irrationality or modulus of continuity.

**Lemma 4.1.** For any $\xi \in \mathbb{T}^1$ and $0 < i < q_{n+1}$, the segments $\Delta_i^{(n)}(\xi)$ and $\Delta_i^{(n)}(T^i \xi)$ are disjoint (except at the endpoints). In particular, for any fixed $\xi_0$, all the segments $\Delta_i^{(n)}$, $0 \leq i < q_{n+1}$, are disjoint.

**Lemma 4.2.** If a point $\xi_i$ with some $i > 0$ belongs to the fundamental segment $\Delta_0^{(n)}$, then $i$ can be expanded in the form $i = q_n + \sum_{s=n+1}^{n+m} \hat{k}_s q_s$ with some integer $0 \leq \hat{k}_s < k_{s+1}$, $n + 1 \leq s \leq n + m$, $m \geq 1$.

**Lemma 4.3.** $l_n \geq \Delta_n$.

**Lemma 4.4.** For any $0 \leq j - i < q_{n+1}$, there holds

$$\left|\Delta_i^{(n+m)}\right| \sim \frac{\left|\Delta_j^{(n+m)}\right|}{\left|\Delta_j^{(n)}\right|}.$$
Lemma 4.5. \[ \frac{\Delta_0^{(n+m)}}{\Delta_0^{(n)}} = O \left( \frac{t_{n+m}}{t_n} \right). \]

Lemma 4.6. \[ \frac{t_{n+m}}{t_n} = O (\lambda^m) \text{, where the constant } \lambda \in (\frac{1}{2}, 1) \text{ is defined in the classical Denjoy theory below, see Statement (B).} \]

4.2. Classical Denjoy’s theory. Here we go back to the classical Denjoy theory [6] which will be used later, see details from Lecture 10 (Homeomorphisms and Diffeomorphisms of the Circle) in the book [22].

For any orientation-preserving circle diffeomorphism \( T \in C^{1+BV} (\mathbb{T}^1) \) (BV stands for bounded variation) with any irrational rotation number \( \rho \) (i.e., does not have to satisfy any nonresonant conditions such as the Diophantine’s type), the following estimates hold:

(A) \( \log(T^{q_n})' (\xi_0) = O (1) \).

(B) There exists \( \lambda \in (\frac{1}{2}, 1) \) such that \[ \frac{\Delta_0^{(n+m)}}{\Delta_0^{(n)}} = O (\lambda^m) \]. More precisely, we can write \( \lambda \) explicitly:

\[ \lambda = \frac{1}{\sqrt{1 + e^{-C}}} \quad \text{where} \quad C = \int_0^1 \left| \frac{d}{dx} \log T' (x) \right| dx. \]

(C) There exists a homeomorphism \( \phi \) such that conjugates \( T \) to \( R_\rho \), i.e., \( \phi \circ T \circ \phi^{-1} = R_\rho \). This is called the topological (or continuous) equivalence of \( T \) and \( R_\rho \).

4.3. Denjoy-type inequality via modulus of continuity. To establish the generalized Denjoy-type inequality in the case of modulus of continuity beyond H"older’s type, we need some basic lemmas based on the cross-ratio distortion estimates we introduced forego.

Lemma 4.7. For any fixed \( \xi_0 \), define functions:

\[
\begin{aligned}
M_n (\xi) &= D (\xi_0, \xi, \xi_{q_n-1}; T^{q_n}) , \quad \xi \in \Delta_0^{(n-1)}, \\
K_n (\xi) &= D (\xi_0, \xi, \xi_{q_n-1}; T^{q_n-1}) , \quad \xi \in \Delta_0^{(n-2)}.
\end{aligned}
\] (4.3)

Then we have the following equations:

\[ M_n (\xi_0) \cdot M_n (\xi_{q_n-1}) = K_n (\xi_0) \cdot M_n (\xi_{q_n}) , \quad (4.4) \]

\[ K_{n+1} (\xi_{q_n-1}) - 1 = \frac{\Delta_0^{(n+1)}}{\Delta_0^{(n-1)}} (M_n (\xi_{q_n+1}) - 1) , \quad (4.5) \]

\[ \frac{(T^{q_n})' (\xi_0)}{M_n (\xi_0)} - 1 = \frac{\Delta_0^{(n)}}{\Delta_0^{(n-1)}} \left( 1 - \frac{(T^{q_n-1})' (\xi_0)}{K_n (\xi_0)} \right). \quad (4.6) \]
**Proof.** Through the definitions in (4.3) we arrive at the following expressions:

\[
M_n (\xi_0) = D (\xi_0, \xi_0, \xi_{qn-1}; T^{qn}) = (T^{qn})' (\xi_0) : \frac{\Delta_{qn}^{(n-1)}}{\Delta_0^{(n-1)}},
\]

\[
M_n (\xi_{qn-1}) = D (\xi_0, \xi_{qn-1}, \xi_{qn-1}; T^{qn}) = \frac{\Delta_{qn}^{(n-1)}}{\Delta_0^{(n-1)}} : (T^{qn})' (\xi_{qn-1}),
\]

\[
M_n (\xi_{qn+1}) = D (\xi_0, \xi_{qn+1}, \xi_{qn-1}; T^{qn}) = \frac{\Delta_{qn}^{(n+1)}}{\Delta_0^{(n+1)}} : (T^{qn})' (\xi_{qn-1}),
\]

\[
K_n (\xi_0) = D (\xi_0, \xi_0, \xi_{qn}; T^{qn-1}) = (T^{qn-1})' (\xi_0) : \frac{\Delta_{qn-1}^{(n)}}{\Delta_0^{(n)}},
\]

\[
K_n (\xi_{qn}) = D (\xi_0, \xi_{qn}, \xi_{qn}; T^{qn-1}) = \frac{\Delta_{qn-1}^{(n-1)}}{\Delta_0^{(n-1)}} : (T^{qn-1})' (\xi_{qn}),
\]

\[
K_{n+1} (\xi_{qn-1}) = D (\xi_0, \xi_{qn-1}, \xi_{qn+1}; T^{qn}) = \frac{\Delta_{qn-1}^{(n-1)}}{\Delta_0^{(n-1)}} : \frac{\Delta_{qn}^{(n-1)}}{\Delta_0^{(n-1)}} - \frac{\Delta_{qn+1}^{(n)}}{\Delta_0^{(n)}}.
\]

Further, one notices that

\[(T^{qn-1})' (\xi_{qn}) \cdot (T^{qn})' (\xi_0) = (T^{qn})' (\xi_{qn-1}) : (T^{qn-1})' (\xi_0),\]

and

\[\left| \frac{\Delta_{qn}^{(n-1)}}{\Delta_0^{(n-1)}} \right| + \left| \frac{\Delta_{qn}^{(n)}}{\Delta_0^{(n)}} \right| = \left| \frac{\Delta_{qn-1}^{(n-1)}}{\Delta_0^{(n-1)}} \right| + \left| \frac{\Delta_{qn-1}^{(n)}}{\Delta_0^{(n)}} \right|,
\]

then the conclusions (4.4), (4.5) and (4.6) can be directly derived. \(\square\)

**Lemma 4.8.**

\[
\begin{align*}
\{ \log \text{Dist} (\xi_0, \xi, \xi_{qn-1}, \eta : T^{qn}) = O (\varpi (l_{n-1})), \quad &\xi, \eta \in \Delta_0^{(n-1)}, \\
\log \text{Dist} (\xi_0, \xi, \xi_{qn}, \eta : T^{qn-1}) = O (\varpi (l_n)), \quad &\xi, \eta \in \Delta_0^{(n-2)}. 
\end{align*}
\]

**Proof.** Follows from Proposition 3.2, 4.1 and 4.2. The proof is exactly the same as Lemma 6 in [15]. \(\square\)

Now we are in a position to establish the crucial Denjoy-type inequality via modulus of continuity, i.e., a stronger version of Statement (A). It can be seen later that it plays a significant role in dealing with the \(C^1\) smoothness of the homeomorphism \(\phi\). Actually, considering the work of Teplinski [26], the result below could be extended to the case of \(C_{3,\infty}\) in studying \(C^2\) conjugacy, we do not pursue that.

**Theorem 4.9** (Denjoy-type inequality via modulus of continuity). Assume \(T\) to be a \(C_{2,\infty}\) orientation-preserving circle diffeomorphism with an irrational rotation.
number (does not have to satisfy any nonresonant conditions) and a modulus of continuity $\varpi$. Then

$$ (T^{q_n})' (\xi) = 1 + O(\tau_n), \quad \tau_n := \sum_{k=0}^{n} \frac{l_n}{l_{n-k}} \varpi (l_{n-k-1}). $$

In particular,

$$ \tau_n = O \left( \chi_n \int_{\chi_n}^1 y^{-2} \varpi (y) \, dy \right) = o(1). \quad (4.7) $$

**Remark 4.10.** In fact, the regularity of $T$ is only needs to be $C^2$, because there automatically exists a modulus of continuity $\varpi$ such that $T \in C_{2,\varpi}(\mathbb{T}^1)$ due to Remark 2.6. Conclusion (4.7) implies that we do not have any additional restrictions for the above $\varpi$, which is different from Theorem 4.13, etc.

**Remark 4.11.** Direct calculation gives that $\tau_n = O(\lambda^n)$ if $\varpi_1 (x) \sim x^\alpha$ with some $0 < \alpha < 1$ (Hölder’s type), and $\tau_n = O(n\lambda^n)$ if $\varpi_2 (x) \sim x$ (Lipschitz’s type). In both cases $\tau_n$ decreases exponentially, and they are estimated to be optimal, as commented in [15]. As to the Logarithmic Hölder’s type $\varpi_3 \sim (\log x^{-1})^{-1-\varepsilon}$ with $\varepsilon > 0$ that satisfies the Dini condition (2.1), $\tau_n = O(n^{-1-\varepsilon})$ (the same as that in [13]) due to the asymptotic behavior $\int_{2}^{X} (\log z)^{-1-\varepsilon} \, dz \sim X (\log X)^{-1-\varepsilon}$ for $X$ sufficiently large.

**Proof.** It follows from Lemma 4.8 that

$$ \frac{M_n (\xi)}{M_n (\eta)} = \text{Dist} (\xi_0, \xi, \xi_{q_{n-1}}, \eta; T^{q_n}) = 1 + O (\varpi (l_{n-1})), $$

$$ \frac{K_n (\xi)}{K_n (\eta)} = \text{Dist} (\xi_0, \xi, \xi_{q_n}, \eta; T^{q_{n-1}}) = 1 + O (\varpi (l_n)). $$

Recall (A), i.e., $\log (T^{q_n})' (\xi_0) = O(1)$, then there exist $c_1, c_2 \in \mathbb{R}$ independent of $n$ such that $c_1 \leq M_n, K_n \leq c_2$, which gives that

$$ M_n (\xi) = m_n + O (\varpi (l_{n-1})), \quad (4.8) $$

$$ K_n (\xi) = m_n + O (\varpi (l_n)), \quad (4.9) $$

where $m_n^2$ is the products in (4.4), i.e.,

$$ m_n = \sqrt{M_n (\xi_0) \cdot M_n (\xi_{q_{n-1}})} = \sqrt{K_n (\xi_0) \cdot K_n (\xi_{q_n})}. $$

Through (4.5), (4.8), (4.9) and Lemma 4.5, one can verify that

$$ m_{n+1} - 1 = \left| \frac{\Delta_0^{(n+1)}}{\Delta_0^{(n-1)}} (m_n - 1) + O (\varpi (l_{n+1})) \right|, $$

which leads to

$$ m_n - 1 = O \left( \sum_{k=0}^{n} \varpi (l_{n-k}) \left| \frac{\Delta_0^{(n)}}{\Delta_0^{(n-1)}} \right| \left| \frac{\Delta_0^{(n-1)}}{\Delta_0^{(n-k)}} \right| \right). $$
\[ O \left( \sum_{k=0}^{n} \varpi \left( \frac{l_{n-k}}{l_{n-k-1}} \cdot \frac{l_{n-1}}{l_{n-k-1}} \right) \right) \]  
\[ = O \left( \varpi \left( \sum_{k=0}^{n} \frac{\zeta \left( l_{n-k} \right)}{\zeta \left( l_{n-k-1} \right)} \lambda^k \right) \right) \]
\[ = O \left( \varpi \left( \sum_{k=0}^{n} \lambda^k \right) \right) \]
\[ = O \left( \varpi \left( l_n \right) \right), \]  
where \( \zeta \left( x \right) := x / \varpi \left( x \right) \), and we assume that \( \zeta \) is monotonically increasing without loss of generality since \( \varpi \) is a modulus of continuity, hence \( \zeta \left( l_n \right) \leq \zeta \left( l_{n-k} \right) \) for sufficiently large \( n, k \) thanks to \( \ln = O \left( \lambda^k l_{n-k} \right) \leq \ln \).

Substituting (4.11) into (4.8) and (4.9) we arrive at
\[ M_n \left( \xi \right) = 1 + O \left( \varpi \left( l_n \right) \right) + O \left( \varpi \left( l_{n-1} \right) \right) \]
\[ = 1 + O \left( \varpi \left( l_{n-1} \right) \right), \]  
and
\[ K_n \left( \xi \right) = 1 + O \left( \varpi \left( l_n \right) \right) + O \left( \varpi \left( l_n \right) \right) \]
\[ = 1 + O \left( \varpi \left( l_n \right) \right). \]  

Substituting (4.12) and (4.13) into (4.6) we derive
\[ \left( T^{q_{n+1}} \right)' \left( \xi_0 \right) = 1 + O \left( \varpi \left( l_n \right) \right) \]
\[ = 1 + O \left( \varpi \left( l_n \right) \right), \]  
and
\[ \left( T^{q_n} \right)' \left( \xi_0 \right) = 1 + O \left( \varpi \left( l_n \right) \right), \]  
which leads to
\[ \left( T^{q_{n+1}} \right)' \left( \xi_0 \right) - 1 = \frac{\left| \Delta_0^{(n+1)} \right|}{\left| \Delta_0^{(n)} \right|} \left( 1 - \left( T^{q_n} \right)' \left( \xi_0 \right) \right) + O \left( \varpi \left( l_n \right) \right) \]  
because of (4.14). By iterating (4.15), we get
\[ \left( T^{q_n} \right)' \left( \xi_0 \right) = 1 + O \left( \sum_{k=0}^{n} \frac{\left| \Delta_0^{(n-k)} \right|}{\left| \Delta_0^{(n-k)} \right|} \varpi \left( l_{n-k-1} \right) \right) \]
\[ = 1 + O \left( \sum_{k=0}^{n} \frac{l_n}{l_{n-k}} \varpi \left( l_{n-k-1} \right) \right) \]  
(4.16)
\[ = 1 + O(\tau_n), \]
where Lemma 4.5 is used in (4.16), and
\[ \tau_n := \sum_{k=0}^{n} \frac{l_n}{l_{n-k}} \varpi (l_{n-k-1}). \]

As to (4.7), we first assume that \( x^{-2} \varpi (x) \) is decreasing on \((0, 1)\) without loss of generality due to Definition 2.1, therefore by applying Lemma 4.6 we obtain that
\[ \tau_n = \sum_{k=0}^{n} \frac{l_n}{l_{n-k}} \varpi (l_{n-k-1}) = O \left( \sum_{k=0}^{n} \lambda^k \varpi (\lambda^{n-k}) \right) \]
\[ = O \left( \lambda^n \sum_{k=0}^{n} (\lambda^k - \lambda^{k+1}) \cdot \lambda^{-k} \varpi (\lambda^k) \right) \]
\[ = O \left( \lambda^n \int_{\lambda^n}^{1} y^{-2} \varpi (y) dy \right). \]

Additionally, it can be proved according to L’Hospital’s rule and \( \varpi (0+)=0 \) that
\[ \lim_{n \to +\infty} \lambda^n \int_{\lambda^n}^{1} y^{-2} \varpi (y) dy = \lim_{z \to +\infty} \frac{\int_{z-1}^{1} y^{-2} \varpi (y) dy}{z} = \lim_{z \to +\infty} \varpi (z^{-1}) = 0, \]
which gives (4.7).

This completes the proof of Theorem 4.9. \[ \square \]

4.4. \( C^1 \) conjugacy through the convergence of \( \sum_{n=0}^{\infty} k_{n+1} \tau_n \). To obtain the \( C^1 \) regularity of \( \phi \) in Statement (C), we employ the method of constructing the continuous density \( h : T^1 \to \mathbb{R}^+ \) of the invariant probability measure for \( T \) and the corresponding homological equation, see for instance, [2, 15, 18, 23]. Specifically, the convergence of \( \sum_{n=0}^{\infty} k_{n+1} \tau_n \) implies \( C^1 \) conjugacy.

**Theorem 4.12.** The circle diffeomorphism \( \phi \) in Statement (C) is \( C^1 \) if
\[ \sum_{n=0}^{\infty} k_{n+1} \tau_n < \infty. \] (4.17)

**Proof:** Let \( \xi_0 \) be arbitrary fixed and consider the corresponding trajectory \( \Theta := \{ T^i \xi_0 : i \in \mathbb{N} \} \). If \( \Theta \) fills up the circle in a dense way by ergodicity, then the homeomorphism \( \phi \) in the classical Denjoy’s theory can be defined uniquely up to a rotation \( \rho \), and the conjugation \( \phi \) can be written as
\[ \phi (\xi) = \int_{\xi_0}^{\xi} \mu (dx), \]
where \( \mu \, (dx) \) is a normalized measure invariant with respect to \( T \).

Firstly, define a discrete mapping \( \gamma \) on \( \Theta \) that depends on \( \xi_0 \):

\[
\gamma (\xi_0) = 0, \quad \gamma (\xi_{i+1}) - \gamma (\xi_i) = -\log T'(\xi_i), \quad i \in \mathbb{N}^+.
\] (4.18)

Then we obtain the following by Lemma 4.2, Theorem 4.9 and Cauchy theorem through the convergence in (4.17), as long as \( \xi_j \in \Delta^{(n)} \) with \( j > i \):

\[
|\gamma (\xi_j) - \gamma (\xi_i)| \leq \sum_{s=n}^{\infty} k_{s+1} |\log T'(\xi_s)|
= \mathcal{O} \left( \sum_{s=n}^{\infty} k_{s+1}\tau_s \right) = o(1), \quad n \to +\infty.
\]

This implies that \( \gamma \in C(\Theta) \). Noting the arbitrary choice of \( \xi_0 \) and the density of \( \Theta \), we can continuously extend the mapping \( \gamma \) onto \( T^1 \). Now define the density \( h \) as:

\[
h (\xi) = \frac{e^{\gamma (\xi)}}{\int_{T^1} e^{\gamma (x)} \, dx}.
\]

Obviously \( h > 0 \) is continuous on \( T^1 \), and

\[
\int_{T^1} h (\xi) \, d\xi = 1,
\]

which implies that \( h \) is indeed a normalized invariant measure, and

\[
\gamma (T\xi) - \gamma (\xi) = -\log T'(\xi)
\] (4.19)

by (4.18). Note that by (4.19) we obtain the desired homological equation

\[
h (T\xi) = \frac{e^{\gamma (T\xi)}}{\int_{T^1} e^{\gamma (x)} \, dx} = \frac{e^{\gamma (\xi) - \log T'(\xi)}}{\int_{T^1} e^{\gamma (x)} \, dx} = \frac{h (\xi)}{T'(\xi)}.
\] (4.20)

Define a mapping

\[
\phi (\xi) := \int_{\xi_0}^{\xi} h (x) \, dx \equiv \int_{\xi_0}^{\xi} \mu (dx)
\]

as we forego. One can verify that \( \phi \) is indeed a \( C^1 \) diffeomorphism such that the following conjugate holds

\[
\phi \circ T \circ \phi^{-1} = R_\rho,
\] (4.21)

because through the homological equation (4.20) we have

\[
(\phi \circ T)' = \phi' (T\xi) \cdot T'(\xi) = h (T\xi) \cdot T'(\xi) = h (\xi) = (R_\rho \circ \phi)'.
\]

This completes the proof. \( \square \)
4.5. \(C^1\) conjugacy and further regularity through irrational rotation \(\rho\) of \(\varphi\)-type. Let \(\rho = [k_1, k_2, \ldots] \in \mathbb{T}^1\) be an irrational number. The simplest case is that, \(\rho\) is termed of constant type, or equivalently, is termed Diophantine of exponent 2, if \(k_n = O(1)\), as pointed out by Petersen in [21]. This situation appears in many fields, including dynamical systems, number theory, asymptotic analysis, and even stochastic fields, because its good properties can lead to some fantastic results, such as the Hardy-Littlewood series \(\sum_{n=1}^{\infty} (n \sin (\pi n \rho))^{-1}\) and extensions [3]. A more complicated case could be considered as \(k_n = O(n^\nu)\) with some \(\nu > 0\), i.e., the polynomial type. These two situations have attracted a lot of research in conjugacy of circle diffeomorphisms, see for instance, Herman’s Theorem [10], i.e., a \(C^{2+\gamma}\) mapping \(T\) in Theorem 4.9 with a polynomial type irrational rotation \(\rho\) can lead to \(C^1\) conjugacy in Denjoy’s theory. See also Sinai and Khanin [14, 23] for irrationality of polynomial type. This is somewhat different in characterizing irrationality from Diophantine, Bruno and even Liouville conditions, but they are obviously more convenient and straightforward for studying rotational \(C^1\) conjugacy than the latter, we thus apply the Denjoy-type inequality from this point of view. In addition, for some special irrational numbers, their partial quotients are very characteristic, such as the constant type

\[
\phi = \frac{\sqrt{5} - 1}{2} = [1, 1, 1, \ldots] \quad \text{(Golden Number)}
\]

and

\[
\sqrt{2} - 1 = [2, 2, 2, \ldots].
\]

More generally, for any \(k \in \mathbb{N}^+\), one can trivially verify that

\[
\frac{\sqrt{k^2 + 4} - k}{2} = [k, k, k, \ldots] \in \{ x : x^2 + kx - 1 = 0 \}.
\]

As to the polynomial type, see for instance,

\[
\tanh 1 = \frac{e - 1}{e + 1} = [1, 3, 5, 7, \ldots]
\]

by Gauss in 1812, \(k_n = O(n)\) at this point. To highlight the conjugacy of most (almost) irrational rotations, one has to show whether irrational numbers are many in the sense of Lebesgue measure (or full measure) in the perspective of direct study of partial quotients. Fortunately for almost irrational numbers, their partial quotients satisfy \(k_n = O(n^\nu)\) with any \(\nu > 0\), see Lemma 5.1 in Appendix (or Corollary A2.2 in [13] for more precise explicit examples). Although this is sufficient to represent almost irrational numbers, it is still of interest to study the more general cases of irrationality, where our regularity requirements for the mapping \(T\) are different in dealing with \(C^1\) conjugacy. Namely, for a nondecreasing continuous function \(\varphi\) on \(\mathbb{R}^+\), we say that an irrational number \(\rho \in \mathbb{T}^1\) is of \(\varphi\)-type if its partial quotients satisfy \(k_n = O(\varphi(n))\). Through the Denjoy inequality via modulus of continuity established forego, we could give a criterion on the required regularity for \(C^1\) conjugacy which is different from known results, thanks to the cross-ratio distortion estimates. They not only simplify the proof and derive the
optimal results of Hölder’s type, but also bring the new perspective in the sense of weaker continuity below.

Next, we present an adequacy integrability condition on regularity of $T$ for $C^1$ conjugacy based on the general $\varphi$-type irrationality, and provide several explicit examples. In other words, there exist different regularity requirements for $T$ when $k_{n+1}$ has different $\varphi$-forms. Due to the optimality of Denjoy-type inequality claimed in [15], our integrability criterion might also be accurate since we cover the case of Hölder’s type.

**Theorem 4.13** (Main Theorem). Let $T$ be a $C_{2,\infty}$ orientation-preserving circle diffeomorphism with rotation number $\rho = [k_1, k_2, \ldots], k_{n+1} = O(\varphi(n))$. Then the circle diffeomorphism $\phi$ in (4.21) is $C^1$ if the following integrability condition holds:

$$
\int_0^1 \left( \int_0^y \varphi \left( \log_\lambda x \right) dx \right) \frac{\varphi(y)}{y^2} dy < +\infty,
$$

where $0 < \lambda < 1$ is given in Statement (B).

**Remark 4.14.** Note that $C^1$ conjugacy can be directly obtained from Theorems 4.9 and 4.12, as long as

$$
\sum_{n=0}^{\infty} k_{n+1} \lambda^n \int_{\lambda^n}^{\lambda^n} y^{-2} \varphi(y) dy = O(1). \tag{4.23}
$$

We still establish the $\varphi$-type theorem because the convergence condition is completely explicit (we suspect it is almost sharp in general settings), although it may not be accurate for some given irrational numbers, since $k_n$ may be large only at a very small number of points (although they are infinite many). We do not pursue that.

**Proof.** In view of (4.22), we derive that

$$
\sum_{n=0}^{\infty} k_{n+1} \tau_n = O \left( \sum_{n=0}^{\infty} \varphi(n) \tau_n \right)
$$

$$
= O \left( \sum_{n=0}^{\infty} \varphi(n) \lambda^n \int_{\lambda^n}^{1} \frac{\varphi(y)}{y^2} dy \right)
$$

$$
= O \left( \sum_{n=0}^{\infty} \left( \lambda^n - \lambda^{n+1} \right) \varphi \left( \log_\lambda \lambda^n \right) \int_{\lambda^n}^{1} \frac{\varphi(y)}{y^2} dy \right)
$$

$$
= O \left( \int_0^1 \varphi \left( \log_\lambda x \right) dx \int_x^1 \frac{\varphi(y)}{y^2} dy \right)
$$

$$
= O \left( \int_0^1 \left( \int_0^y \varphi \left( \log_\lambda x \right) dx \right) \frac{\varphi(y)}{y^2} dy \right)
$$

$$
= O(1).
$$

Then the conclusion follows from Theorem 4.12 directly. \qed
Integrability condition (4.22) on modulus of continuity seems to be complicated, but for many common cases one could simplify (4.22) explicitly through asymptotic analysis, see the Corollary 4.15 given below. As it can be seen that, to obtain \(C^1\) conjugacy on \(T^1\), the larger \(k_n\) is, the stronger the required regularity has to be.

**Corollary 4.15.** The \(C_{2,\infty}\) mapping \(T\) in Theorem 4.13 admits \(C^1\) conjugacy, if the corresponding irrational rotation number \(\rho \in T^1\) is of the following type and the modulus of continuity satisfies the integrability condition:

\((C1)\) Constant type, i.e., \(k_{n+1} = O(1)\):

\[
\int_0^1 \frac{\varpi(y)}{y} dy < +\infty.
\]

For instance, Lipschitz’s type \(\varpi \sim x\); Hölder’s type \(\varpi \sim x^{\alpha}\) with any \(0 < \alpha < 1\); Logarithmic Hölder’s type \(\varpi \sim (\log x^{-1})^{-\alpha}\) with any \(\alpha > 1\); and even

\[
\varpi \sim \frac{1}{(\log x^{-1})(\log \log x^{-1}) \cdots (\log \cdots \log x^{-1})^{1+\sigma}}
\]

for any \(\ell \in \mathbb{N}^+\) and \(\sigma > 0\).

\((C2)\) Polynomial type, i.e., \(k_{n+1} = O(n^\nu)\) with some \(\nu > 0\):

\[
\int_0^1 \frac{(-\log y)^\nu \varpi(y)}{y} dy < +\infty.
\]

For instance, Lipschitz’s type \(\varpi \sim x\); Hölder’s type \(\varpi \sim x^\alpha\) with any \(0 < \alpha < 1\); Logarithmic Hölder’s type \(\varpi \sim (\log x^{-1})^{-\alpha}\) with any \(\alpha > \nu + 1\); and even

\[
\varpi \sim \frac{1}{(\log x^{-1})^\nu+1(\log \log x^{-1}) \cdots (\log \cdots \log x^{-1})^{1+\sigma}}
\]

for any \(\ell \in \mathbb{N}^+\) and \(\sigma > 0\).

\((C3)\) Exponential type, i.e., \(k_{n+1} = O(a^n)\) with some \(1 < a < \lambda^{-1}\). Then for any fixed \(b > 0\) arbitrarily small, one has to require that

\[
\int_0^1 \frac{\varpi(y)}{y^{1+b}} dy < +\infty.
\]

For instance, Lipschitz’s type \(\varpi \sim x\); Hölder’s type \(\varpi \sim x^\alpha\) with any \(0 < \alpha < 1\).

**Remark 4.16.** \((C1)\) is the same as the Dini condition (2.1), see (2.2) and (2.3) for more examples. \((C2)\) shows that arbitrary Hölder continuity (or even Logarithmic Hölder continuity \((\log(x^{-1}))^{-1-\varepsilon}\) with any \(\varepsilon > 0\)) is sufficient for \(C^1\) conjugacy of almost irrational rotations which we forego.

**Proof.** Explicit examples are easy to verify and we thus omit the proof here.
(C1) Note $\varphi(x) = 1$. Therefore by (4.22) we get
\[
\int_0^1 \left( \int_0^y \varphi(\log_\lambda x) \, dx \right) \frac{\varpi(y)}{y^2} \, dy = \int_0^1 \left( \int_0^y 1 \, dx \right) \frac{\varpi(y)}{y^2} \, dy = \int_0^1 \frac{\varpi(y)}{y} \, dy < +\infty.
\]

(C2) Note $\varphi(x) = x^\nu$ with some $\nu > 0$, and
\[
\int_0^y \varphi(\log_\lambda x) \, dx = \int_0^y (\log_\lambda x)^\nu \, dx
= \mathcal{O}\left( \int_0^y (-\log x)^\nu \, dx \right)
= \mathcal{O}\left( \int_{-\log y}^{+\infty} y^\nu e^{-z} \, dz \right)
= \mathcal{O}\left( (-\log y)^\nu y \right).
\]

Then by (4.22) we have
\[
\int_0^1 \left( \int_0^y \varphi(\log_\lambda x) \, dx \right) \frac{\varpi(y)}{y^2} \, dy = \mathcal{O}\left( \int_0^1 \frac{(-\log y)^\nu \varpi(y)}{y} \, dy \right)
= \mathcal{O}(1).
\]

(C3) For any fixed $b > 0$ arbitrarily small, let $a_* = e^{-b \ln \lambda} \in \left( 1, \lambda^{-1} \right)$. Note that $\varphi(x) = a_*^x$, and
\[
\int_0^y \varphi(\log_\lambda x) \, dx = \int_0^y x^{-b} \, dx = \mathcal{O}\left( y^{1-b} \right).
\]

Then it follows that
\[
\int_0^1 \left( \int_0^y \varphi(\log_\lambda x) \, dx \right) \frac{\varpi(y)}{y^2} \, dy = \int_0^1 \frac{\varpi(y)}{y^{1+b}} \, dy
= \mathcal{O}(1).
\]

Additionally, we could further discuss the regularity of the conjugation $\phi$ based on the above Theorem 4.13 as a byproduct. We emphasize that the integrability condition (4.22) is indeed crucial. As pointed out in Remark 2.6, $\phi$ has a modulus of continuity $\varpi_h$, but it’s hard to give an explicit form in general. The following Theorem 4.17 provides a way to construct an explicit one $\tilde{\varpi}$ such that $\tilde{\varpi} \gtrsim \varpi_h$. As shown by Corollary 4.19, optimal estimates in special cases could be derived, but it seems difficult to obtain optimality in general settings. We also hold the opinion that the cross-ratio distortion estimates can extend the optimal results in [15] based on Diophantine rotations, but we will not discuss that here.

**Theorem 4.17** (Higher regularity). Define
\[
\mathcal{R}(n) := \int_0^{\lambda^n} \left( \varphi(\log_\lambda x) \int_x^1 \frac{\varpi(y)}{y^2} \, dy \right) \, dx.
\]
If $\hat{\varphi} := R \circ \Delta^{-1} (\cdot)$ is indeed a modulus of continuity, then $\phi$ in Theorem 4.13 belongs to $C_{1, \hat{\varphi}} (\mathbb{T}^1)$.

**Remark 4.18.** Obviously $R$ is well-defined through the integrability condition (4.22) and satisfies $R(0+) = 0$ by applying Fubini’s Theorem and Cauchy’s Theorem. Easy to verify that $\hat{\varphi} (0+) = 0$. Notice that $\Delta$ can be replaced by a function $\tilde{\Delta}$ that satisfies $\tilde{\Delta} \leq \Delta$, e.g., $\tilde{\Delta} (n) := \prod_{j=n-1}^{n+1} (k_{j+2} + 1)^{-1}$.

**Proof.** In this case only the regularity of the density $h$ needs to be estimated. Let $\eta_1, \eta_2 \in \mathbb{T}^1$ be fixed such that they are sufficiently close to each other. Then there exist a unique $n \in \mathbb{N}$ such that $\Delta_n \leq |\phi (\eta_1) - \phi (\eta_2)| < \Delta_{n+1}$, and let $k \in \mathbb{N}^+$ be the largest number such that $k \Delta_n \leq |\phi (\eta_1) - \phi (\eta_2)|$. Obviously $1 \leq k \leq k_{n+1}$.

In view of Lemma 4.2, Theorem 4.9, the continuity of $h$ (see Theorem 4.12) and the integrability condition (4.22), we derive the following similar to that in Theorem 4.13:

$$|\log h (\eta_2) - \log h (\eta_1)| = \mathcal{O} \left( k \tau_n + \sum_{s=n+1}^{\infty} k_{s+1} \tau_s \right)$$

$$= \mathcal{O} \left( \sum_{s=n}^{\infty} k_{s+1} \tau_s \right)$$

$$= \mathcal{O} \left( \sum_{s=n}^{\infty} \varphi (s) \lambda s \int_{\lambda s}^{1} y^{-2} \varpi (y) dy \right)$$

$$= \mathcal{O} \left( \int_0^{\lambda n} (\varphi (\log_{\lambda} x) \int_{x}^{1} y^{-2} \varpi (y) dy) dx \right)$$

$$= \mathcal{O} (R (n)) .$$

The same estimate is true for $|h (\eta_2) - h (\eta_1)|$ because $\log h (\eta_1) = \mathcal{O} (1)$. Since $\phi$ has already been proved to be a diffeomorphism in Theorem 4.13, we therefore get

$$|h (\eta_2) - h (\eta_1)| = \mathcal{O} (R (n))$$

$$= \mathcal{O} (\hat{\varphi} \circ \Delta (n))$$

$$= \mathcal{O} (\hat{\varphi} (k \Delta_n))$$

$$= \mathcal{O} (\hat{\varphi} (|\phi (\eta_1) - \phi (\eta_2)|))$$

$$= \mathcal{O} (\hat{\varphi} (|\eta_1 - \eta_2|)) ,$$

which gives that $h \in C_{0, \hat{\varphi}} (\mathbb{T}^1)$, and thus one arrives at $\phi \in C_{1, \hat{\varphi}} (\mathbb{T}^1)$.

The proof is completed. $\square$

It should be emphasized that we always study circle diffeomorphisms in terms of partial quotients $k_n$ throughout this paper. Since $\Delta (n)$ is uniquely determined for the known $k_n$, some precise estimates might be obtained by applying Theorem 4.17 at this point. As an illustration, let us consider the simplest case $k_n = \mathcal{O} (1)$ (1-type irrationality), i.e., there exist $K_1, \ldots, K_{\nu} \in \mathbb{N}^+$ with $\nu \in \mathbb{N}^+$ such that $k_n \in \mathcal{S} =$
\{ K_1, \ldots, K_\nu \}, \text{ and we may could accurately estimate the asymptotic behavior of } \Delta \text{ based on the probability of the numbers appearing in } S \text{ (or only derive the rough estimates of } k_n \text{ without considering the probability). One therefore obtains a } C_{1,\varpi} \text{ conjugation } \phi \text{ for the mapping } T, \text{ and the modulus of continuity } \tilde{\varpi} \text{ might be more accurate than any known result, or even optimal. We emphasize that the asymptotic analysis is crucial in dealing with the regularity of } \phi. \text{ Based on the above, Corollary 4.19 below discusses regularity of both Hölder’s type and Logarithmic Hölder’s type, and only the simplest form of probability is studied (nevertheless, we still further improve the known optimal regularity results so far, see Remark 4.20). It can be seen later that the probability at this point does not affect the latter, but the former depends on it explicitly. This phenomenon also shows from the side that it is necessary to study the classification. The more general irrationality such as } \varphi \text{-type cases could also be considered, but we will not discuss here.}

**Corollary 4.19 (1-type irrationality).** Assume that \( k_n \in S = \{ K_1, \ldots, K_\nu \} (\nu \geq 2, \text{ and } K_i \neq K_j \text{ for } i \neq j) \) satisfy the nonuniform distribution:

\[
\lim_{N \to +\infty} \frac{|\{ n = K_j : 1 \leq n \leq N \}|}{N} = p_j \in (0, 1), \quad 1 \leq j \leq \nu, \quad (4.24)
\]

where \( \sum_{j=1}^{\nu} p_j = 1 \). Then there hold:

(C4) If \( T \in C_{2,\varpi_1}(\mathbb{T}^1) \) with \( \varpi_1 \sim x^{\alpha} \) and \( 0 < \alpha < 1 \), then \( \phi \in C_{1,\varpi_1}(\mathbb{T}^1) \) with \( \varpi_1 \sim x^\beta \), where

\[
\alpha \leq \beta := \min \left\{ 1, \frac{\alpha \log \lambda}{\log \vartheta} \right\}, \quad \vartheta := \prod_{j=1}^{\nu} K_j^{-p_j}.
\]

(C5) If \( T \in C_{2,\varpi_2}(\mathbb{T}^1) \) with \( \varpi_2 \sim (-\log x)^{-1-\sigma} \) and \( \sigma > 0 \), then \( \phi \in C_{1,\varpi_2}(\mathbb{T}^1) \) with \( \varpi_2 \sim (\log x^{-1})^{-\sigma} \).

**Remark 4.20.** Distribution (4.24) can actually be extended further to the strongly nonuniform case similar to that in Remark 4.14, where the probability depends on the position of the points, but we do not pursue that. Once (4.24) is removed, the diffeomorphism \( \phi \) in (C4) can be of \( C_{1,\varpi^*}(\mathbb{T}^1) \) with \( \gamma = \min \left\{ 1, \frac{-\alpha \log \lambda}{\log M} \right\} \), where \( M = \max_{1 \leq j \leq \nu} K_\nu \) (since positive integers are uniformly distributed). Similar to the following proof we have \( \inf_{M \in \mathbb{N}^+} \gamma \geq \alpha \). This implies that Diophantine’s type of exponent 2 (\( k_n = O(1) \)) admits \( C^{1,\alpha} \) conjugacy, which has been shown to be optimal in [13], see also [15], and it shows that our theorem is a little bit more accurate than the known optimal results. Additionally, (C5) is independent of the probability.

**Proof.** We first establish an explicit expression about \( \mathcal{R} \), based on a universal modulus of continuity \( \varpi \) which is strictly weaker than the Lipschitz’s type (because it is trivial and can be discussed separately, we therefore assume that \( x/\varpi(x) \) is monotonically decreasing to 0 as \( x \to 0^+ \) without loss of generality), and \( k_n = O(1) \) (without considering the probability).
Note that \( \inf_{0 < z < 2^{-1}} \int_{z}^{1} d \left( y^{-1} \varpi(y) \right) < 0 \) by the assumption we make forego, and
\[
\int_{z}^{1} y^{-2} \varpi(y) \, dy = z^{-1} \varpi(z) - \int_{z}^{1} d \left( y^{-1} \varpi(y) \right).
\]
Then it follows from L’Hospital’s rule that
\[
\lim_{z \to 0^+} \frac{\int_{0}^{z} y^{-1} \varpi(y) \, dy}{z \int_{z}^{1} y^{-2} \varpi(y) \, dy} = \lim_{z \to 0^+} \frac{z^{-1} \varpi(z)}{\int_{z}^{1} y^{-2} \varpi(y) \, dy - z^{-1} \varpi(z)} = +\infty.
\]
This gives
\[
\mathcal{R}(n) = \int_{0}^{\lambda^{n}} \left( \varphi \left( \log_{\lambda} x \right) \int_{x}^{1} y^{-2} \varpi(y) \, dy \right) \, dx
\]
\[
= \int_{0}^{\lambda^{n}} \left( \int_{x}^{1} y^{-2} \varpi(y) \, dy \right) \, dx
\]
\[
= \int_{0}^{\lambda^{n}} \left( \int_{0}^{y} y^{-2} \varpi(y) \, dy \right) \, dy + \int_{0}^{\lambda^{n}} \left( \int_{0}^{y} y^{-2} \varpi(y) \, dx \right) \, dy
\]
\[
= \int_{0}^{\lambda^{n}} y^{-1} \varpi(y) \, dy + \lambda^{n} \int_{0}^{\lambda^{n}} y^{-2} \varpi(y) \, dy
\]
\[
= O \left( \int_{0}^{\lambda^{n}} y^{-1} \varpi(y) \, dy \right).
\]

Next we provide an estimate of \( \Delta \) based on (4.24) and Remark 4.18. Let \( c > 0 \) be a generic constant that does not affect the estimates. Since \( \Delta_{n} = k_{n+2} \Delta_{n+1} + \Delta_{n+2} < (k_{n+2} + 1) \Delta_{n+1} \), we therefore obtain that \( \Delta_{n} \geq c \vartheta^{n} \) (obviously \( 0 < \vartheta < 1 \)), which gives
\[
\Delta^{-1}(x) := \frac{\log c^{-1} x}{\log \vartheta} \sim \frac{\log x^{-1}}{\log \vartheta^{-1}}.
\]
Note that \( \Delta_{n} = O(\lambda^{n}) \) since \( \frac{\Delta_{n}}{\Delta_{n-m}} = O(\lambda^{m}) \). This implies that \( \lambda \geq \vartheta \) (in fact \( \lambda > \frac{1}{2} \geq \vartheta \) by Statement (B)), and therefore \( \beta := \min \left\{ 1, \frac{\alpha \log \lambda}{\log \vartheta} \right\} \geq \alpha \).

Finally, denote \( \mathcal{R}_{1} \) and \( \mathcal{R}_{2} \) with respect to \( \varpi_{1} \) and \( \varpi_{2} \), respectively. For (C4), direct calculation gives that
\[
\mathcal{R}_{1}(n) = O \left( \int_{0}^{\lambda^{n}} y^{-1-\alpha} \, dy \right) = O(\lambda^{\alpha n}),
\]
therefore,
\[
\varpi_{1} \sim \mathcal{R}_{1} \circ \Delta^{-1} = O \left( \lambda^{\alpha \frac{\log x^{-1}}{\log \vartheta}} \right) = O \left( x^{\beta} \right).
\]
As to (C5), one notices that
\[
\mathcal{R}_{2}(n) = O \left( \int_{0}^{\lambda^{n}} y^{-1}(- \log y)^{-1-\sigma} \, dy \right) = O \left( n^{-\sigma} \right)
\]
by asymptotic analysis. This leads to
\[ \tilde{\omega}_2 \sim R_2 \circ \Delta^{-1} = O \left( (\log x^{-1})^{-\sigma} \right). \]
One easily to verify that both \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \) are modulus of continuity, then the conclusions directly follows from Theorem 4.17. \( \square \)

4.6. Optimality about our integrability condition. In this subsection, we will show certain optimality for our integrability condition (4.22), namely
\[ \int_0^1 \left( \int_0^y \varphi(\log_\lambda x) \, dx \right) \frac{\varphi(y)}{y^2} \, dy < +\infty. \]
Let us start by reviewing some classic and important results.

(D1) Theorem 1.1 in Yoccoz [27] (page 126) concludes that, if the mapping \( T \) considered throughout this paper is only \( C^2 \), and \( \rho \in \mathbb{T}^1 \) is a number of constant type (i.e., 1-type in our terminology, or equivalently, \( k_n = O(1) \)), then the conjugation \( \phi \) to the rotation with \( \rho \) is indeed absolutely continuous (therefore, must be differentiable a.e.). This is somewhat different from the classical Denjoy theory in Subsection 4.2 since the regularity for both \( T \) and \( \phi \) are higher. However, \( C^1 \) conjugacy has not yet been achieved at this point.

(D2) Meanwhile, Katznelson and Ornstein [12] emphasized that, the condition of boundedness of the continued-fraction coefficients of \( \rho \) is essential to obtain absolutely continuous conjugacy (stronger than topological conjugacy but weaker than \( C^1 \) conjugacy). Otherwise, for a given \( \rho \in \mathbb{T}^1 \) with unbounded coefficients, that is, there exists a sequence \( \{k_{n_j}\}_j \) satisfying \( \lim_{j \to +\infty} k_{n_j} = +\infty \), one can construct a \( C^2 \) mapping \( T \) on \( \mathbb{T}^1 \) such that the conjugation \( \phi \) to the rotation with \( \rho \) is purely singular, i.e., maps a map of zero measure onto a map of full measure, not to mention a admitting of continuity. Additionally, the mapping \( T \) can be chosen arbitrarily close to the rotation with \( \rho \). See related work, Hawkins and Schmidt [9], Katznelson [11] and Lazutkin [19]. Even for the rotation number of constant type (Diophantine index is 0), in order to preserve \( C^1 \) conjugacy, the regularity requirement of mapping \( T \) cannot be lower than that of \( C^2 \). More precisely, the counterexample in [13] (Appendix 3) shows that the regularity of \( C^{2-\varepsilon} \) with any \( 0 < \varepsilon < 1 \) for \( T \) cannot admit \( C^1 \) conjugacy at this point.

(D3) Katznelson and Ornstein [13] (Section 3) also discussed circle diffeomorphisms of \( C^2 \)'s type, provided certain modulus of continuity. There the Denjoy-type inequality is not optimal. Besides, they were more concerned with the case based on Diophantine irrationality, and the higher Hölder regularity of conjugations. Khanin and Teplinsky [15] established the optimal Denjoy-type inequality under \( C^{2+\alpha} \) smoothness, where \( 0 < \alpha < 1 \). However, as we mentioned in the introduction and Section 2, Hölder regularity is not sufficient to characterize continuity in general.

Here we touch the optimality, which can be summarized as:
Note that $\mathcal{R}$ in Theorem 4.17 (higher regularity than $C^1$) is strongly related to the integrability condition (4.22), and we derive more accurate estimates than the known optimal regularity results (at least for constant type, i.e., 1-type irrationality), see Corollary 4.19 and Remark 4.20. This shows that our integrability condition has certain optimality in the sense of obtaining higher regularity of $C^1$ conjugacy.

Assuming only $C^2$ (without any extra Hölder continuity) and for constant type irrational numbers, we could indeed improve the absolute continuous conjugacy in (D1) to $C^1$ conjugacy, as long as the modulus of continuity naturally possessed by $D^2T$ (Remark 2.6) satisfies the Dini condition (2.1), see conclusion (C1) in Corollary 4.15 and Remark 4.16.

The counterexamples in (D2) also illustrate the optimality of our integrability condition, that is, to preserve the differentiable conjugacy, $C^2$ regularity for the mapping $T$ cannot be further weakened. And in particular, for $\rho$ with unbounded $k_n$, we can still obtain $C^1$ conjugacy due to Theorem 4.13, as long as the irrationality and regularity satisfy the equilibrium in the integrability condition (4.22) (and thus this is not inconsistent with the counterexample constructed by Hawkins and Schmidt [9] in (D2)), while the mapping $T$ we consider is still only $C^2$ (without any extra Hölder continuity). Besides, the counterexample constructed by Katznelson and Ornstein [13] in (D2) also reflects our optimality (note that the constant type rotation corresponds to $\varphi(x) = 1$ in our integrability condition (4.22)). The above arguments show that our integrability condition is somewhat strong, in the sense of preserving $C^1$ conjugacy (note the huge gap between pure singularity and differentiability).

Recall (D3). Fortunately, via the cross-ratio distortion estimates introduced by Khanin and Teplinsky in [15], we extend Denjoy-type inequality to the weakest case (only $C^2$ regularity) and obtain the optimal result (see [15]). Then we further apply this powerful tool to investigate $C^1$ conjugacy, and propose the optimal integrability condition (4.22), which reveals the explicit relation between irrationality and regularity in preserving $C^1$ conjugacy for the first time, and it is also obviously easier to verify, see Corollary 4.15 for explicit examples. Additionally, due to the optimality of Denjoy-type inequality, the weaker but implicit condition (4.23) in Remark 4.14 is indeed better than the results in [13], in the sense of only $C^2$ without any extra Hölder continuity for the mapping $T$. It should be emphasized that our analysis process is much simpler than traditional approaches.

5. Appendix

Lemma 5.1. Let $\rho = [k_1(\rho), k_2(\rho), \ldots] \in T^1 \setminus \{0\}$ be arbitrarily given. Assume that $K_n \geq 0$. Then

$$\sum_{n=1}^{\infty} k_n(\rho) K_n < +\infty, \rho - a.e. \ in \ T^1 \quad (5.1)$$
holds if
\[ \sum_{n=1}^{\infty} K_n \log n < +\infty. \] (5.2)

Further, if \( \{K_n\} \) is monotonically increasing, then (5.1) is in fact equivalent to (5.2).

**Proof**: See details from Lemma A.2.1 and its Remark in [13]. \( \square \)

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