ONE-DIMENSIONAL PACKING:
MAXIMALITY AND RATIONALITY

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Abstract. Every set of natural numbers determines a generating function
convergent for \( q \in (-1, 1) \) whose behavior as \( q \to 1^- \) determines a germ.
These germs admit a natural partial ordering that can be used to compare
sizes of sets of natural numbers in a manner that generalizes both cardinality
of finite sets and density of infinite sets. For any finite set \( D \) of positive integers,
call a set \( S \) “\( D \)-avoiding” if no two elements of \( S \) differ by an element of \( D \). It
is shown that any \( D \)-avoiding set that is maximal in the class of \( D \)-avoiding
sets (with respect to germ-ordering) is eventually periodic. This implies an
analogous result for packings in \( \mathbb{N} \). It is conjectured that for all finite \( D \) there
is a unique maximal \( D \)-avoiding set.

1. Introduction

This article is concerned with two related kinds of optimization problems in \( \mathbb{N} \):
packing problems and distance-avoidance problems. In the former, we are given a
nonempty set \( B \subseteq \mathbb{N} = \{0, 1, 2, \ldots \} \) and we wish to find a collection of disjoint
translates of \( B \) whose union is as big a subset of \( \mathbb{N} \) as possible. In the latter, we
are given a finite set \( D \) of positive integers and we wish to find as big a set \( S \subseteq \mathbb{N} \)
as possible such that no two elements of \( S \) differ by an element of \( D \). In both cases,
the crucial issue is defining what “as big as possible” should mean.

For instance, consider the distance-avoidance problem with \( D = \{3, 5\} \). Three
\( D \)-avoiding sets are

\[ S_0 = \{0, 2, 4, 6, \ldots \}, \quad S_1 = \{1, 3, 5, 7, \ldots \}, \quad \text{and} \quad S_2 = \{0, 1, 2, 8, 9, 10, \ldots \}. \]

(The third of these sets is obtained via the obvious general algorithm for greedily
constructing \( D \)-avoiding sets that considers elements in increasing order, including
each element in the set if it does not introduce a forbidden distance with previous
elements). In terms of subset-inclusion, all three sets are maximal: none of them
can be augmented without violating the \( D \)-avoidance property. We will say \( S_0 \) is
“bigger” than \( S_1 \), which is in turn “bigger” than \( S_2 \), in the sense that

\[ \sum_{n \in S_0} q^n > \sum_{n \in S_1} q^n > \sum_{n \in S_2} q^n \]

for all \( q < 1 \) sufficiently close to 1. That is, we propose to measure of the size of
a set \( S \subseteq \mathbb{N} \) by forming the generating function \( S_q := \sum_{n \in S} q^n \) and examining its
germs “at \( 1^- \).”

For example:

\[ \text{Date: July 18, 2018.} \]
(1) If $S$ is finite, $S_q = |S| + o(1)$, or equivalently, $S_q \rightarrow |S|$ as $q \rightarrow 1^-$. If $S$ is infinite, $S_q$ diverges as $q \rightarrow 1^-$. 
(2) If $S$ is infinite with density $\alpha$, $S_q = \alpha \frac{1}{1-q} + o\left(\frac{1}{1-q}\right)$. 
(3) If $S = \{a, a + d, a + 2d, \ldots\}$ with $a \geq 0$ and $d > 0$, 
$$S_q = \left(\frac{1}{d}\right) \frac{1}{1-q} + \left(\frac{d - 1 - 2a}{2d}\right) + O(1-q).$$

This approach is related to Abel’s method of evaluating divergent series; its application to measuring sets of natural numbers is (apparently) new, but it is likely to hold little novelty for analytic number theorists, who have long used the philosophically similar but technically more recondite notion of Dirichlet density to measure sets of primes. Our definition also has thematic links to work from the earliest days in the study of infinite series. For instance, Grandi’s formula 
$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$$
corresponds to the fact that the germ of $(2N)_q$ exceeds the germ of $(2N+1)_q$ by $1/2 + O(1-q)$ while Callet’s formula 
$$1 + 0 - 1 + 1 + 0 - 1 + \cdots = \frac{2}{3}$$
corresponds to the fact that the germ of $(3N)_q$ exceeds the germ of $(3N+2)_q$ by $2/3 + O(1-q)$.

Our approach resembles the sort of “tame nonstandard analysis” in which $\mathbb{R}$ is replaced by the ordered ring $\mathbb{R}(x)$ where $1/x$ is a formal infinitesimal (also known as “the ring of rational functions ordered at infinity”); our ordering of rational functions corresponds to that of $\mathbb{R}(x)$ if one identifies $1/x$ with $1 - q$.

Theorems 2 and 4 show that for both packing problems and distance-avoidance problems in $\mathbb{N}$, every optimal (that is, germ-maximal) solution is eventually periodic. The proof we give may seem surprisingly complicated, given that the corresponding periodicity property for maximum-density packings and maximum-density distance-avoiding sets is fairly easy. This discrepancy is explained by the fact that the germ-topology does not admit compactness arguments.

We conjecture that for both the packing and distance-avoidance problems, there is a unique optimum subset of $\mathbb{N}$ (guaranteed to be eventually periodic).

The motivation for this work was the study of disk packings. It is our hope that the approach taken here will ultimately lead to results establishing a strong kind of uniqueness for optimal sphere-packings in dimensions 2, 8, and 24. (See [Co] for a survey of the recent breakthroughs in the study of 8- and 24-dimensional sphere-packing.) We also hope that the germ approach will have relevance to the study of densest packings in other dimensions.

For other approaches to measuring efficiency of packings, see [Ku]. The most sophisticated of these approaches is that of Bowen and Radin [Bo]; their ergodic theory approach has attractive features (for instance, it works in spaces with non-amenable symmetry groups), but it does not seem to work so well when the region being packed is not the entire space. Packings in $\mathbb{N}$ could be viewed as special packings of $\mathbb{R}^2$; the lack of symmetry makes it hard to apply the constructions of Bowen and Radin.

See also [Be], [Bl], [Ch], and [Ka] for work on measuring sizes of sets bearing some philosophical similar to ours.

2. Statement of main theorem

Recall that a subset $S$ of $\mathbb{N} = \{0, 1, 2, \ldots\}$ is eventually periodic iff there exist $N \in \mathbb{N}$ and $d \geq 1$ such that for all $n \geq N$, $n \in S$ iff $n + d \in S$. It is easy to show...
that $S$ is eventually periodic if and only if its generating function $S_q := \sum_{n\in S} q^n$ is a rational function of $q$. We call such sets $S$ rational. (Note that this usage coincides with the notion of rationality for subsets of a monoid in automata theory, specialized to the monoid $\mathbb{N}$.) If $S$ is a finite set, then $S$ is rational and $S_q$ is a polynomial. If $S$ is rational and infinite, then $S_q$ has a simple pole at 1, and letting $t = 1 - q$ we can expand $S_q$ as a Laurent series $\sum_{n \geq -1} a_n t^n$ where $a_{-1}$ is the density of $S$. This series converges for all $q$ in $(-1, 1)$, though we will only care about $q$ in $(0, 1)$.

Given two sets of natural numbers $S$ and $S'$ (not necessarily rational), write $S \preceq S'$ iff there exists $c > 0$ such that $S_q \leq S'_q$ for all $q$ in the interval $(1 - c, 1)$; we say that $S'$ dominates $S$ in the germ-ordering. The partial ordering $\preceq$ (which we call the germ-ordering at 1−) is a total ordering on the rational subsets of $\mathbb{N}$ that refines the preorder given by comparing density. Also, if two sets have finite symmetric difference they are $\preceq$-comparable. (Both of these assertions are consequences of the fact that the sign of a polynomial can oscillate only finitely many times.) In the case where $S$ and $S'$ are finite, the germ-ordering refines ordering by cardinality; when the finite sets $S$ and $S'$ have the same cardinality $n$, the germ-ordering refines lexicographic ordering of subsets of $\mathbb{N}$ of size $n$. (When $S, S'$ are eventually periodic infinite sets of the same density $c$, there is also a combinatorial criterion for deciding which of $S, S'$ is larger, though it is more complicated.)

The germ-ordering has the “outpacing property” [Ka]; if for all sufficiently large $k$ the $k$th element of $S$ is less than or equal to the $k$th element of $S'$, then $S \preceq S'$.

We mention that, although $\preceq$ is a total ordering for rational subsets of $\mathbb{N}$, the same is not true for unrestricted subsets of $\mathbb{N}$; for instance, if $S$ is the set of natural numbers whose base ten expansion has an even number of digits and $S'$ is its complement, then it can be shown that $S$ and $S'$ are $\preceq$-incomparable.

Given a finite nonempty subset $B$ of $\mathbb{N}$ (a packing body), say that a set $T \subset \mathbb{N}$ is a translation set for $B$ iff the translates $B + n$ ($n \in T$) are disjoint. If $T$ is a translation set, the generating function of $\bigcup_{n \in T}(B + n)$ is just the product of the generating function of $T$ and the generating function of $B$; so if $T$ and $T'$ are translation sets, $T \preceq T'$ iff $\bigcup_{n \in T}(B + n) \preceq \bigcup_{n \in T'}(B + n)$.

**Conjecture 1.** For every packing body $B$, there is a unique germ-maximal translation set for $B$, and it is rational. That is, there is a translation set $T^*$ such that $T^*$ is rational and such that $T \preceq T^*$ for every translation set $T$.

This Conjecture is easy to prove for many specific packing bodies, such as $\{0, 1, \ldots, k-1\}$ for arbitrary $k$ (see Section 3), but we do not have a general proof. Theorem 2 is the best result we currently have that applies to all packing bodies $B$.

**Theorem 2.** For every packing body $B$, every germ-maximal translation set is rational. That is, if $T^*$ is a translation set with the property that there exists no translation set $T \succ T^*$, then $T^*$ is rational.

We hope to (but cannot yet) prove that the collection of translation sets for $B$ contains a maximal element; it is a priori conceivable that there exist translation sets $T_1 \prec T_2 \prec T_3 \prec \ldots$ but no translation set that dominates them all. Thus Theorem 2 does not immediately imply Conjecture 1.

Our proof of Theorem 2 goes by way of a shift of context from the packing problem to the forbidden distance problem (which in $\mathbb{N}$ might with equal aptness
be called the forbidden difference problem). The condition that \( T \) is a translation set for \( B \) is equivalent to the condition that the difference set \( T - T = \{ x - y : x, y \in T \} \) has no element in common with the difference set \( B - B = \{ x - y : x, y \in B \} \) other than 0. Thus the problem of finding the germ-maximal translation set for the packing body \( B \) is a special case of the problem of finding the germ-maximal set \( T \subseteq \mathbb{N} \) that has no differences in the finite set \( D_B \) where \( D_B \) is the set of positive elements of \( B - B \). More generally, for any finite set \( D \) of positive integers, say that \( S \subseteq \mathbb{N} \) is \( D \)-avoiding if there exist no two elements in \( S \) that differ by an element of \( D \). In this setting we can broaden Conjecture 1 and Theorem 2.

**Conjecture 3.** For every finite set \( D \) of positive integers, there is a unique germ-maximal \( D \)-avoiding set \( S^* \) and it is rational.

**Theorem 4.** For every finite set \( D \) of positive integers, every germ-maximal \( D \)-avoiding set is rational.

Of course Conjecture 3 implies Conjecture 1 and Theorem 4 implies Theorem 2. The conclusions of Theorem 2 and Theorem 4 cannot be strengthened to assert that the germ-maximal sets must be periodic, as we show in Section 4 (see Example 10).

3. Proof of main theorem

Our approach to proving Theorem 4 uses a block coding of the kind often employed in dynamical systems theory. Let \( m = \max(D) + 1 \) and replace the indicator sequence of \( S \) (an element of \( \{0, 1\}^\mathbb{N} \)) by a symbolic sequence using a block code of block length \( m \), with an alphabet containing (at most) \( 2^m \) symbols, which we will call letters. More concretely, if the indicator sequence of \( S \) is written as \((b_0, b_1, b_2, \ldots)\) (where \( b_n \) is 1 or 0 according to whether \( n \in S \) or \( n \not\in S \)), then we define the \( m \)-block encoding of \((b_0, b_1, b_2, \ldots)\) to be \((w_0, w_1, w_2, \ldots)\) where the letter \( w_n \) is the \( m \)-tuple \((b_n, b_{n+1}, \ldots, b_{n+m-1})\); we call \( w_n \) a consonant or a vowel according to whether \( b_n = 1 \) or \( b_n = 0 \) (conditions that align with the respective cases \( n \in S \) and \( n \not\in S \)). Say that a letter \( \alpha = (b_1, \ldots, b_m) \) in \( \{0, 1\}^m \) is legal if the set \( \{ i : b_i = 1 \} \) is \( D \)-avoiding; we let \( \mathcal{A} \) be the set of legal letters. Given two letters \( \alpha \) and \( \alpha' \) in \( \mathcal{A} \), say that \( \alpha' = (b'_1, \ldots, b'_m) \) is a successor of \( \alpha = (b_1, \ldots, b_m) \) if \( b'_i = b_{i+1} \) for \( 1 \leq i \leq m - 1 \). For every set \( S \subseteq \mathbb{N} \), the associated block-encoding \( w = (w_0, w_1, w_2, \ldots) \) has the property that for all \( n \geq 1 \), \( w_n \) is a successor of \( w_{n-1} \);

\( S \) is \( D \)-avoiding if and only if \( w \) has the additional property that every letter \( w_n \) is legal. Call such an infinite word \((w_0, w_1, w_2, \ldots)\) \( D \)-legal. Finding a germ-maximal \( D \)-avoiding set is equivalent to finding a \( D \)-legal infinite word for which the set of locations of consonants is germ-maximal. We write \( w \preceq w' \) iff the associated sets \( S, S' \) satisfy \( S \preceq S' \).

Suppose \( S \) is some \( D \)-avoiding subset of \( \mathbb{N} \) that is germ-maximal in the collection of \( D \)-avoiding subsets of \( \mathbb{N} \). Let \( w = (w_0, w_1, \ldots) \) be the associated infinite word in \( \mathcal{A}^\mathbb{N} \). Assume for simplicity that the letter \( w_0 = \alpha \) occurs infinitely often in \( w \). (The last paragraph of the proof addresses what happens if this assumption fails.)

Let \( K = \{ k \in \mathbb{N} : w_k = \alpha \} = \{ k_0, k_1, k_2, \ldots \} \), where \( k_0 = 0 \) and \( k_0 < k_1 < k_2 < \cdots \). We divide the infinite word \( w \) into infinitely many subwords \((w_{k_0}, w_{k_0+1}, \ldots, w_{k_1-1}), (w_{k_1}, w_{k_1+1}, \ldots, w_{k_2-1}), (w_{k_2}, w_{k_2+1}, \ldots, w_{k_3-1}), \ldots \). Each of these finite words is associated with the word \( c_k := (w_{k_{i-1}}, w_{k_{i-1}+1}, \ldots, w_{k_i-1}, w_{k_i}) \) (for \( k \geq 1 \)) that both begins and ends with the letter \( \alpha \); define a circular word as
a word whose first and last letters are the same. (Note that we are not modding out by cyclic shift of such words.) Let $C$ be the set of all circular words beginning and ending with $\alpha$. We define the length of a circular word to be the number of letters it contains, counting its first and last letter as a single letter. (Thus, if $\alpha, \beta, \gamma$ are letters, the circular word $(\alpha, \beta, \gamma, \alpha)$, which for brevity we may also write as $\alpha \beta \gamma \alpha$, has length 3.) If $c \in C$ has length $a$ and $c' \in C$ has length $a'$, let $c : c'$ denote the circular word of length $a + a'$ in $C$ obtained by concatenating $c$ and $c'$ (where the final $\alpha$ in $c$ gets identified with the initial $\alpha$ in $c'$). The operation $\cdot$ is associative, and indeed, the word $w$ itself can be written as $c_1 : c_2 : c_3 : \ldots$, where the circular words $c_i$ are primitive (i.e., each $c_i$ contains $\alpha$ only at the beginning and at the end). We also use ""," to denote concatenation of noncircular words.

Every circular word $c \in C$ is associated with a polynomial $P_\alpha = P_\alpha(q)$ (sometimes we will omit the subscript or will write $P_i$ to mean $P_{c_i}$) whose degree is at most the length $a$ of the circular word $c$ and whose coefficients are 0’s and 1’s according to whether the respective letters in the circular word are vowels or consonants; we call $P_\alpha$ the generating function of $c$. If $w = c_1 : c_2 : c_3 : \ldots$ is the $D$-legal infinite word representing the $D$-avoiding set $S$, $S_\alpha$ can be written as $P_1 + q^{a_1} P_2 + q^{a_1 + a_2} P_3 + \cdots = P_1 + A_1 P_2 + A_1 A_2 P_3 + \cdots$ where $a_i$ is the length of $c_i$ and $A_i$ is $q^{a_i}$.

For any circular word $c$ with length $a$, we define $|c| := P_\alpha(q)/(1-q^a)$; it is equal to the generating function of the infinite periodic word $c : c : c : \ldots$. Given two periodic words $c, c' \in C$ (possibly of different lengths), write $c \preceq c'$ iff $|c| \leq |c'|$; call this the germ-ordering on circular words. It is easy to check that $|c| = |c| : |c:c : c : \ldots$. We have $|c| = |c'|$ iff $c : c : c : \ldots = c' : c' : c' : \ldots$.

The following two Lemmas are the linchpins of the proof of Theorem 4.

**Lemma 5.** If $c \preceq c'$, then $c \preceq c : c' \preceq c' : c \preceq c'$.

**Proof:** Write $|c| = P/(1 - A)$ and $|c'| = P'/(1 - A')$: we also have $|c : c'| = (P + AP')/(1 - AA')$ and $|c' : c| = (P' + A'P)/(1 - AA')$. The stipulated relation $c \preceq c'$ is equivalent to $P/(1 - A) \preceq P'/(1 - A')$, or

\[ P(1 - A') \preceq P'(1 - A); \]

the desired relations $c \preceq c' : c, c : c' \preceq c' : c$, and $c' : c \preceq c'$ are respectively equivalent to

\[ P/(1 - A) \preceq (P + AP')/(1 - AA'), \]

\[ (P + AP')/(1 - AA') \preceq (P' + A'P)/(1 - AA'), \]

\[ (P' + A'P)/(1 - AA') \preceq P'/(1 - A'). \]

To prove (2), note that (by cross-multiplying, expanding, and cancelling terms) we can write it equivalently as $-AA'P \preceq AP' - AP - AAP'$, which is just (1) multiplied by $A$. The two denominators in (3) are identical, so (3) is equivalent to $P + AP' \preceq P' + A'P$, which in turn is equivalent to (1). The proof of (4) is similar to the proof of (2).

Note that the proof also tells us that if $c \prec c'$, then $c \prec c : c' \prec c' : c \prec c'$.

**Lemma 6.** If the concatenation $w = c_1 : c_2 : c_3 : \ldots$ is germ-maximal in the set of $D$-legal words, then we must have $c_1 \succeq c_2 \succeq c_3 \succeq \ldots$ in the germ-ordering.
Proof: We will show that $c_1 \geq c_2$ since that contains the idea of the general argument. If $c_1 = c_2$ there is nothing to prove, so assume $c_1 \neq c_2$, and let $w' = c_2 : c_1 : c_3 : \ldots$, which must be $D$-legal if $w$ is (indeed, the whole reason for the block coding was to make this claim true). The sets $S$ and $S'$ respectively associated with $w$ and $w'$ have finite symmetric difference, so $w$ and $w'$ must be comparable. Since we are assuming $w$ is germ-maximal, we must have $w \geq w'$ in the germ ordering. That is, we must have $P_1 + A_1P_2 \geq P_2 + A_2P_1$ (all the later terms match up and cancel). But this is equivalent to $P_1/(1 - A_1) \geq P_2/(1 - A_2)$, so $c_1 \geq c_2$ as claimed. $lacksquare$

Proof of Theorem 4: By an easy pigeonhole argument, for all $N$ there must exist $i, j \geq N$ with $i < j$ such that the sum of the lengths of the words $c_i, c_{i+1}, \ldots, c_j$ is a multiple of the length of $c_1$, say $r$ times the length of $c_1$. Let $w'$ be the word obtained from $w$ by replacing the $j - i + 1$ letters $c_i, c_{i+1}, \ldots, c_j$ by $r$ occurrences of the letter $c_1$. Let $S$ and $S'$ be the sets associated with $w$ and $w'$, respectively. Lemma 6 tells us that $c_1 \geq c_i \geq c_{i+1} \geq \cdots \geq c_j$, so repeated application of Lemma 5 gives $|c_1 : c_i : c_{i+1} : \cdots : c_j| \geq |c_1 : c_{i+1} : \cdots : c_j|$. If strict inequality holds, then $w' \succ w$, contradicting maximality of $w$. (Here we use the fact that the difference $S_q - S_q$ can be expressed as $1 - q^n$ times $|c_1 : c_2 : \ldots : c_1| - |c_1 : c_{i+1} : \cdots : c_j|$, where $n$ is the common value of $r a_1$ and $a_i + a_{i+1} + \cdots + a_j$.) So we must have $|c_1 : c_1 : \cdots : c_1| = |c_1 : c_{i+1} : \cdots : c_j|$, implying that $c_1, c_{i+1}, \ldots, c_j$ are all the circular word $c_1$. Since the circular words $c_i$ are in germ-decreasing order, this means that $c_1, c_2, \ldots, c_N$ are all equal. Since this is true for all $N$, we must have $w = c_1 : c_1 : \cdots : c_1$; that is, $w$ is periodic.

The above argument was predicated on the assumption that $\alpha$ occurs infinitely often. If this assumption fails, then a version of the argument still goes through, but it is slightly more complicated: one finds the smallest $i$ for which the letter $w_i$ occurs infinitely often in $w$ (guaranteed to exist), and then one applies the preceding argument to the letters $w_i, w_{i+1}, w_{i+2}, \ldots$, ignoring the letters $w_0, \ldots, w_{i-1}$. Instead of concluding that $w$ is periodic, we obtain the weaker conclusion that $w$ is eventually periodic. $lacksquare$

4. Existence and uniqueness in special cases

In this section we discuss some progress in the direction of Conjecture 3. We give examples of $D$ for which there is a unique germ-maximal $D$-avoiding set $S$. In some cases $S$ is periodic and in other cases $S$ is eventually periodic but not periodic.

For $D$ a fixed finite set, we write $\|D\|$ for the largest element of $D$. In this section we refer to sets $S \subseteq \mathbb{N}$ in terms of their indicator functions rather than the block encoding used in the proof of Theorem 4 thus we view $S$ as an element of $\{0, 1\}^\mathbb{N}$ and we write $S$ and subsets of $S$ of the form $S \cap [a, b]$ as bit strings (of length $b - a + 1$ in the latter case).

4.1. Some periodic winners, including the symmetric case. A $D$-avoiding bit string $s$ that is finite and has length greater than $\|D\|$ is called repeatable if the concatenation $s : s$ is $D$-avoiding. Note that in this case the infinite concatenation $s : s : s : \cdots$ is also $D$-avoiding.

Theorem 7. Fix $D$. If there is an integer $m > \|D\|$ such that the (germ-)best $D$-avoiding string of length $m$ is repeatable, then the infinite string obtained by
repeating this string is the unique maximal $D$-avoiding element in the germ order. In particular the unique maximal element exists and is periodic.

An example will demonstrate the idea of the proof. With $D = \{3, 5\}$ and window size $m = 8$, the best $D$-avoiding string is 10101010, which is repeatable. Therefore the infinite periodic string $S$ with repetend 10 is the unique maximal $D$-avoiding string, meaning that the set of even integers is the unique maximal $D$-avoiding set.

To see why this is true, let $S$ be the offset of symmetry. We claim that every string $s$ satisfying the germ-maximal $D$-avoiding string of length $m$ is either identically zero or else positive on the interval $(1 − \epsilon, 1)$. The claim follows.

Theorem 7 applies to many, but definitely not all, sets $D$. For instance we call $D$ symmetric if there is an integer $k > \|D\|$ such that $i \in D$ iff $k − i \in D$. (In other words $D$ is symmetric if $−D$ is a translate of $D$.) We call the number $k$ the offset of symmetry for $D$. Theorem 7 applies to all symmetric sets.

Corollary 8. If $D$ is symmetric then there is a unique germ-maximal $D$-avoiding set, and this set is periodic with period dividing the offset of symmetry.

Proof. Fix symmetric $D$ and let $k$ be the offset of symmetry. We claim that every legal string of length $k$ is repeatable, so in particular the best legal string of length $k$ is repeatable. Thus Theorem 7 applies with $m = k$.

To see why this is true, let $s$ be a $D$-legal string of length $k$ and consider the string $s : s$, whose two halves correspond to the sets $S$ and $S + k$. Suppose this is not $D$-legal. Then $s : s$ must have elements $x$ in the first half and $y$ in the second half with $y − x \in D$. Note that $y − x \in D$ implies $0 < y − x < k$. Since the length of $s$ is $k$, we must have $y − k$ in $S$, so the distance $x − (y − k) = k − (y − x)$ mustn’t be in $D$, even though $y − x$ is in $D$. This contradicts the symmetry of $D$. □

For example, if $D = \{1, 2, \ldots, k − 1\}$, then by Corollary 8 there is a unique germ-maximal $D$-avoiding set, which is easily seen to consist of the multiples of $k$.

Example 9. For non-symmetric $D$, it is still sometimes possible to apply Theorem 7. For instance suppose $D = \{1, 2, n\}$. If $n \not\equiv 0 \mod 3$, then the winner is clearly periodic with repetend 100. For $D = \{1, 2, 3n\}$ we take $m = 3n + 1$ and find (exercise) that the best $D$-avoiding string of length $m$ is $(100)^n0$. This is repeatable, so Theorem 7 tells us we have the winner.

Note however that because $D$ is not symmetric, $D$-avoiding strings $s$ of this (or any) length are not guaranteed to be repeatable. So, some (admittedly minimal) analysis is required to determine (a) exactly which one is best and (b) that it happens to be repeatable.
Some other $D$'s for which we can manually determine that the optimal sequence (of some length $w$) is repeatable:

- $\{1, 3, 4\}$ ($m = 7, s = 1010000$)
- $\{2, 3, 5\}$ ($m = 7, s = 1100000$)
- $\{2, 3, 6\}$ ($m = 9, s = 110001000$)
- $\{2, 3, 7\}$ ($m = 10, s = 1100011000$)
- $\{3, 4, 7\}$ ($m = 10, s = 1110000000$)

Note: in each of these cases the optimum is obtained by the greedy algorithm.

In the next subsection we show that there exist 3-element $D$'s for which the optimal $D$-avoiding set is not periodic (but rather only eventually periodic). Thus Theorem 4 cannot be used, and in particular, for such a $D$, there can be no (finite) window size $w$ such that the optimal $D$-avoiding string of length $w$ is repeatable.

### 4.2. An optimal non-periodic sequence

We now describe the only other technique we currently have for proving strings are optimal. It is not clear how broadly this approach can be applied.

**Example 10.** Let $D = \{4, 7, 11\}$. Then the unique germ-optimal $D$-avoiding string is the eventually periodic string

\[ S^* = 110 \ 100 \ 100 \ 100 \ 100 \ 000 \ 100 \ 100 \ 100 \ldots, \]

with repetend 100. The string $S^*$ beats the (best) periodic string $S^{**} = 100 \ 100 \ 100 \ldots$ by an infinitesimal amount (that is, the difference $|S^*|_q - |S^{**}|_q$ goes to 0 as $q \to 1^-$ and is positive for all $q < 1$ sufficiently close to 1).

**Proof.** (Computer assisted.) Consider the two strings of length 15:

(5) \[ A = 110 \ 100 \ 100 \ 100 \ 000 \]

(6) \[ B = 100 \ 100 \ 100 \ 100 \ 100 \]

We will show that the optimal string is

\[ S^* = ABBBB\ldots \]

One verifies the following computational facts:

**Fact 1.** $ABB$, $ABB$, and $ABB$ are optimal for their respective lengths.

**Fact 2.** Let $P, Q, R$ be any three blocks of length 15 such that $PQR$ is $D$-avoiding. Then at least one of these statements holds:

- $R \preceq B$
- $QR \preceq BB$
- $PQR \preceq BBB$

Fact 1 is verified using Algorithm 11 (next subsection). To verify Fact 2, first create a list of $R$ such that $R \succ B$. For each of these, create a list of $QR$ such that $QR \succ BB$. For each of these $QR$, create a list of $PQR$ such that $PQR \succ BBB$. One sees at this point that the third list is empty.

Let $S$ be any $D$-avoiding string. We wish to show that $S^* \succeq S$. Write

\[ S = S_1S_2S_3\ldots \]
where each $S_i$ is a string of length 15, which we refer to as a **block**. In this proof we will use the word **span** to mean a collection of consecutive blocks. Thus Fact 1 says that $S^*$ beats or ties $S$ in the spans consisting of blocks 1–3, blocks 1–4 and blocks 1–5.

For any $k \geq 3$, we inductively construct a partition $P_k$ of the set $\{1, 2, ..., k\}$ into spans such that $S^*$ beats or ties $S$ on each of these spans. If $k = 3, 4,$ or $5$, then we let $P_k$ consist of a single span $1, \ldots, k$ (see Fact 1). Now let $k \geq 6$, and assume we have partitions $P_n$ for all $n < k$. Let

$$PQR = S_{k-2}S_{k-1}S_k.$$  

Note that $S^*$ has $BBB$ in the corresponding locations. Now we apply Fact 2, and define $P_k$ as follows:

- If $R \preceq B$, we let $P_k$ be $P_{k-1}$ together with singleton $\{k\}$.
- Otherwise, if $QR \preceq BB$, we let $P_k$ be $P_{k-2}$ together with doubleton $\{k-1, k\}$.
- Otherwise, we must have $PQR \preceq BBB$, and we let $P_k$ be $P_{k-3}$ together with tripleton $\{k-2, k-1, k\}$.

To show $S^* \succeq S$, it is sufficient to construct to a partition of the positive integers into spans of length $\leq 3$ such that $S^*$ beats $S$ on each span. For this purpose, we employ König’s Lemma. The construction above of the partitions $P_k$ is represented by an acyclic graph with vertex set $\{3, 4, 5, \ldots\}$ and such that each $k \geq 6$ is connected to exactly one of $k-1, k-2, \text{ or } k-3$. Note that if $j$ is connected to $k$ with $j < k$, then $P_j$ is a subpartition of $P_k$. An infinite path $\{j_1, j_2, \ldots\}$ in this graph exists by König’s Lemma. By construction, the partitions $P_{j_1}, P_{j_2}, \ldots$ are nested. Taking the union of this infinite set of nested partitions, we obtain the desired partition of the positive integers. □

The Fibonacci case $D = \{5, 8, 13\}$ is similar. In this case, we find that with blocks of length 18, Computational Fact 2 is exactly the same, while Fact 1 requires going a little further out to $k = 5, 6, 7$.

We do not know which or how many (non-symmetric) $D$’s, even of size 3, will yield to this approach.

### 4.3. A finite set of candidates for an optimal sequence

We next present a fast algorithm for finding the germ-optimal $D$-avoiding bit string of any fixed length.

We note two properties of our setup:

- The condition for $D$-avoiding is a local condition, i.e. to check whether the string $S$ is $D$-avoiding can be done locally by checking the condition on contiguous substrings of $S$ of length no more than $\|D\| + 1$.
- The germ order has the property that given two strings of the same length $A$ and $B$, if $A$ is bigger than $B$ in germ order then $AX$ is bigger than $BX$ for all strings $X$.

These two properties are enough to allow for a standard dynamic programming algorithm on a line to compute, in linear time in the length $l$, a finite list of sequences of length $l$ each of which is optimal conditioned on the values of its final $\|D\|$ bits. The list has one sequence ending with each legal string of length $\|D\|$, so the size of the list is constant in $l$. The optimal sequence of length $l$ can then be found by comparing the sequences on the list.
Algorithm 11. Given a finite set \( D \subset \mathbb{Z}_{\geq 0} \), fix any integer \( m > \| D \| \). Let \( \{ \sigma_1, \ldots, \sigma_r \} \) denote the set of all \( D \)-avoiding strings of length \( m \). Then there is an efficient algorithm to compute, for any \( k \geq 1 \), the set \( S_k = \{ s_{k,1}, \ldots, s_{k,r} \} \), where \( s_{k,i} \) is the optimal \( D \)-avoiding string of length \( km \) that ends in the substring \( \sigma_i \). The optimal \( D \)-avoiding string of length \( km \) is then the best element of the finite set \( S_k \).

We briefly spell out the dynamic program. Initially define \( s_{1,i} = \sigma_i \), so that \( S_1 = \{ \sigma_1, \ldots, \sigma_r \} \). Assuming we have \( S_k = \{ s_{k,1}, \ldots, s_{k,r} \} \) where \( s_{k,i} \) is the optimal \( D \)-avoiding string of length \( km \) that ends in the substring \( \sigma_i \), we efficiently generate \( S_{k+1} \) as follows. For each \( i \), define \( s_{k+1,i} \) to be the \( D \)-avoiding string of biggest germ order from the set \( S_k \sigma_i \) (that has \( r \) elements) consisting of the elements of \( S_k \) concatenated with \( \sigma_i \). Then define \( S_{k+1} = \{ s_{k+1,1}, \ldots, s_{k+1,r} \} \). The optimality of \( s_{k+1,i} = \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_k} \sigma_i \) follows from the guarantee that if \( s_{k+1,i} \) is the optimal string of length \( (k + 1)m \) ending in \( \sigma_i \) then the substring \( \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_k} \) must be optimal among strings that end in \( \sigma_{j_k} \) and hence was one of the elements considered in \( S_k \).

In practice, these lists tend to stabilize fairly quickly.

4.4. Convergence and maximality of local optimization. Consider a finite difference set \( D \) and two “boundary” strings \( \alpha, \beta \in \{0,1\}^{\| D \|} \). Observe that for any \( \ell \geq \| D \| \) there is a unique maximum string \( \gamma \in \{0,1\}^\ell \) for which

\[
\alpha \gamma \beta \geq \alpha x \beta
\]

for all strings \( x \) of length \( \ell \). We introduce the notation \( \Gamma_\ell(\alpha, \beta) \) for this maximum string. It follows that if \( w \in \{0,1\}^* \) (or \( \{0,1\}^\omega \)) is a maximal \( D \)-avoiding string then any appearance of the strings \( \alpha \) and \( \beta \) in \( w \) separated by exactly \( \ell \) symbols must enclose the string \( \Gamma_\ell(\alpha, \beta) \). (Note that it makes sense to define this notion for \( \ell < \| D \| \), though in this case one must focus on consistent pairs \( (\alpha, \beta) \) for which there exists at least one such \( x \).

In general, for two strings \( w, w' \in \{0,1\}^\omega \), we write

\[
w \vdash^\ell_\alpha \beta w'
\]

if \( w \) can be written \( x\alpha y\beta z \) for a string \( y \in \{0,1\}^\ell \) so that \( w' = x\alpha \gamma \beta z \), where \( \gamma = \Gamma_\ell(\alpha, \beta) \). We likewise define

\[
w \vdash_\ell w'
\]

if \( w \vdash^\alpha_\ell w' \) for some pair \( \alpha, \beta \in \{0,1\}^{\| D \|} \). Observe that

\[
w \vdash_\ell w' \Rightarrow w \preceq w'
\]

(in the germ order).

Theorem 12. Let \( D \) be a finite subset of \( \mathbb{N} \) and \( \ell \geq \| D \| \). Let \( w = w^{(0)} \) be a \( D \)-avoiding string and let \( w^{(1)}, w^{(2)}, \ldots \) be a sequence of elements of \( \{0,1\}^* \) for which

\[
w^{(0)} \vdash_\ell w^{(1)} \vdash_\ell w^{(2)} \vdash_\ell \ldots
\]

Then this sequence converges in the sense that there is a string \( w^* \) so that for any position \( t \), \( w^*_t = w^{(k)}_t \) for all sufficiently large \( k \).
Proof. Define $g^{(i)}$ to be the power series associated with $w^{(i)}$. Then for each $i$ we may write

$$g^{(i+1)} = g^{(i)} + X^t p(X)$$

where $p(X)$ is a polynomial of degree no more than $\ell - 1$ with coefficients in $\{-1,0,1\}$. When $w^{(i)} \neq w^{(i+1)}$, the value of $t$ is determined by the length of the common prefix of the two strings. As $w^{(i+1)} \geq w^{(i)}$, $X^t p(X) \geq 0$ and hence $p(X) \geq 0$ in the germ order.

Let $P_\ell = \{a_0 X^t + \cdots + a_\ell \mid a_i \in \{-1,0,1\}\}$ denote the set of all polynomials of degree at most $\ell$ with coefficients in $\{-1,0,1\}$, let

$$R_\ell = \left\{ x \in \mathbb{R} \mid p(x) = 0 \text{ for some } p(X) \in P_\ell \setminus \{0\} \right\},$$

and define

$$\epsilon_\ell = \max(\{x \in R_\ell \mid x < 1\}).$$

Observe that if $p(X) \in P_\ell$ exceeds 0 in the germ order, then $p(x) > 0$ for all $x \in (\epsilon_\ell, 1)$. The same can be said for any polynomial of the form $X^t p(X)$, and we conclude that for any point $x_0 \in (\epsilon_\ell, 1)$, the values $g^{(i)}(x_0)$ are monotonically increasing. As $g^{(i)}(x_0) \leq 1 + x_0 + x_0^2 + \cdots = 1/(1-x_0)$, the monotone sequence $g^{(i)}(x_0)$ is bounded and hence converges to a particular value $g^*(x_0)$.

Finally, for a fixed point $x_0$, define

$$\epsilon_0 = \min \left( \left\{ |q(x_0)| \mid q(X) \in P_{k-1} \setminus \{0\} \right\} \right).$$

Now consider two strings $w$ and $\tilde{w}$ for which $w \sqsubseteq \tilde{w}$ corresponding to a substring replacement starting at position $t$; then the power series $g$ and $\tilde{g}$ associated with these strings satisfy $\tilde{g}(x_0) = g(x_0) + x_0^t g(x_0)$ for a nonzero polynomial $q(X) \in P_{k-1}$ and hence $\tilde{g}(x_0) \geq g(x) + x_0^t \epsilon_0$. Then observe that if

$$\left| g^{(i)}(x_0) - g^*(x_0) \right| < x_0^t \epsilon_0$$

for all $i \geq k$ then then $t$th bit of all strings $w^{(i)}$ must agree for $i \geq k$. It follows that the sequence $w^{(i)}$ converges pointwise to a particular string $w^*$.

Let $D$ be a finite subset of $\mathbb{N}$ and $\ell \geq \|D\|$. For a $D$-avoiding string $w \in \{0,1\}^*$ and a position $t > \ell$, let $r_t(w)$ be the string obtained by replacing bits $t, t+1, \ldots, t + \ell - 1$ with the best possible legal alternative, i.e. $r_t(w)$ is defined by $w \leftarrow_{\alpha, \beta}^t r_t(w)$ with $\alpha = w_{t-\|D\|} \cdots w_{t-1}$ and $\beta = w_{t+\ell} \cdots w_{t+\ell+\|D\|-1}$.

**Corollary 13.** Let $D$ be a finite subset of $\mathbb{N}$ and $\ell \geq \|D\|$. Let $w \in \{0,1\}^*$ be a $D$-avoiding string. Let $t_1, t_2, \ldots$ be a sequence of integers so that $t_i > \ell$ for each $i$ and each integer in the set $\{\ell + 1, \ldots\}$ appears infinitely often in the sequence. Then the sequence

$$w^{(0)} = w,$$

$$w^{(i)} = r_{t_i}(w^{(i-1)}),$$

converges to an $\ell$-maximal element $w^*$, which is to say that $r_t(w^*) = w^*$ for all $t > \ell$. 

4.5. A topological aside. We mentioned in the introduction that germs do not come with a nice topology. As an illustration of this (related to the famous Ross-Littlewood Paradox), consider the sequence of sets \( S_n = \{n, n + 1, \ldots, 10n\} \): we have \( S_1 \prec S_2 \prec S_3 \prec \ldots \), but it is unclear what the limit of the \( S_n \)'s should be. Surely it is not the pointwise limit of the sets, since that is the null set! One way to understand what is going on here is to note that, even though for each \( n \) there exists \( \epsilon_n > 0 \) such that \((S_n)_q < (S_{n+1})_q\) for all \( q \) in \((1 - \epsilon_n, 1)\), we have \( \inf \epsilon_n = 0 \), so that the intersection of the intervals \((1 - \epsilon_n, 1)\) is empty.

This sort of situation comes into play when one tries to prove Conjecture 3 by showing that \( c \geq c_1, c_2, c_3, \ldots \) implies \( c : c : c : \cdots \geq c_1 : c_2 : c_3 : \cdots \). If we take \( \epsilon_n \) satisfying \(|c| \geq |\epsilon_n|\) for all \( q \) in \((1 - \epsilon_n, 1)\), and the infimum of the \( \epsilon_n \) not known to be positive, then the obvious approach to proving the implication fails.

5. Truncating the germs

In our approach, a rational set \( S \subseteq \mathbb{N} \) is replaced by the power series \( \sum_{n \in S} q^n \), which is rewritten as the Laurent series \( \sum_{n \geq -1} a_n (1 - q)^n \), and the coefficients \( a_{-1}, a_0, a_1, a_2, \ldots \) are used to put a total ordering on the rational sets. The coefficients \( a_n \) carry finer and finer information as \( n \) increases, so it is natural to discard this information after some point. The classical theory of packings retains only \( a_{-1} \) (the density of \( S \)); we suggest that it is natural to retain both \( a_{-1} \) and \( a_0 \). That is, we define a non-Archimedean valuation \( \nu \) from the set of rational subsets of \( \mathbb{N} \) to \( \mathbb{Q} \times \mathbb{Q} \), where we view \( \mathbb{Q} \times \mathbb{Q} \) as the lexicographic product of the ordered ring \( \mathbb{Q} \) with itself. It can be shown that the pairs \((a_{-1}, a_0)\) that occur are those of the form \((0, k)\) or \((1, -k)\) where \( k \) is a nonnegative integer, along with pairs of the form \((p, q)\) where \( p \) is a rational number strictly between 0 and 1 and where \( q \) is an arbitrary rational number. This valuation is not translation-invariant; if \( \nu(S) = (p, q) \), then \( \nu(S + 1) = (p, q - p) \). Note that under this valuation, the sets \( \{3, 6, 9, 12, 15, 18\} \) and \( \{1, 3, 6, 9, 15, 18\} \) discussed at the end of section 2 have the same size. The valuation is emphatically not countably additive, as can for instance be seen by viewing \( \mathbb{N} \) as a union of singleton sets.

One can try to extend this valuation to various classes of sets that include but are not limited to the rational subsets of \( \mathbb{N} \). One way to do this without directly invoking the expansion of \( \sum_{n \in S} q^n \) as a Laurent series in \( 1 - q \) is to define a partial preorder on the power set of \( \mathbb{N} \) (the \( \lim \inf \) preorder) such that \( S \) dominates \( S' \) in the \( \lim \inf \) preorder iff \( \lim \inf_{q \to 1^-} (\sum_{n \in S} q^n - \sum_{n \in S'} q^n) \geq 0 \). This partial preordering, restricted to the rational sets, coincides with the total preordering obtained by factoring the germ-ordering through the valuation \( \nu \).

An important rationale for truncating the germs comes from considering the role played by the choice of regularization scheme. If one wanted to extend our theory from packings in \( \mathbb{N} \) to packings in \( \mathbb{Z} \) (with a view toward eventually looking at packings in \( \mathbb{R}^d \)), a different regularization scheme would be required (since for \( S \subseteq \mathbb{Z} \), \( \sum_{n \in S} q^n \) diverges for all \( q \) in \((0, 1)\) unless \( S \) is bounded below). Two natural choices are the germ of \( \sum_{n \in S} q^n \) as \( q \to 1^- \) ("\( L^1 \)-regularization") and the germ of \( \sum_{n \in S} q^n \) as \( q \to 0 \) ("\( L^2 \)-regularization"). It can be shown that, for rational sets \( S \subseteq \mathbb{Z} \) (defined in the natural way from the monoid structure of \( \mathbb{Z} \)) the pair \((a_{-1}, a_0)\) is the same for \( L^2 \)-regularization and \( L^1 \)-regularization, while later coefficients \( a_n \) are different in the two theories. Indeed, the valuation \( \nu \) we constructed earlier, mapping the set of rational subsets of \( \mathbb{N} \) to \( \mathbb{Q} \times \mathbb{Q} \), is quite robust; most sensible
regularization schemes give rise to \( \nu \). This is just a restatement of the fact that the Grandi series and its variants have the same value under most sensible ways of summing divergent series.

It is easy to show that the germ-based partial ordering on subsets of \( \mathbb{N} \) gives rise to a total ordering when restricted to rational subsets of \( \mathbb{N} \). That is because the difference between the generating functions of two rational subsets of \( \mathbb{N} \) is a rational function, and hence can undergo only finitely many changes of sign as \( q \) approaches 1 from the left. To decide which of two sets \( S, S' \) is larger, express \( |S|_q - |S'|_q \) in the form \( (1 - q)^r p(q)/(1 - q)^n \) where the polynomial \( p(q) \) is not divisible by \( 1 - q \); then \( p(q) \) is non-zero, and \( S > S' \) or \( S < S' \) according to whether \( p(1) \) is positive or negative.

It is a diverting exercise to show that the range of \( \nu(S) \) as \( S \) varies over all rational subsets of \( \mathbb{N} \) is the union of three sets: the set of pairs \( (0, n) \) where \( n \) ranges over the nonnegative integers; the set of pairs \( (r, s) \) where \( r \) ranges over the rational numbers strictly between 0 and 1 and \( s \) ranges over all rational numbers; and the set of pairs \( (1, -n) \) where \( n \) ranges over the nonnegative integers.

Recall that the first term of the ordered pair \( \nu(S) \) is the coefficient of \( (1 - q)^{-1} \) in the asymptotic expansion of the generating function \( |S|_q \), and that this coefficient is precisely the density of \( S \). The second term of \( \nu(S) \) is the coefficient of \( (1 - q)^0 \), that is, the constant term in the asymptotic expansion. To compare two rational subsets \( S, S' \), first see which has bigger density. If they have the same density, compare the constant terms. If those are the same as well, then the comparison is more subtle. As an illustration of what’s involved, one may consider the periodic approximation to the Thue-Morse sequence. Each of the periodic strings 1010101010101010, 1001011001101001, . . . is slightly larger than the next.

6. Connection to sphere-packing

In the case of packing \( \mathbb{N} \) with translates of \( B = \{0, 1, 2, \ldots, k - 1\} \), there is an appreciable efficiency gap between the best packing and all other packings (where an element \( x \) of a non-Archimedean ordered ring extending \( \mathbb{R} \) is said to be appreciable when there exist positive \( r, s \) in \( \mathbb{R} \) with \( r < x < s \):

**Theorem 14.** For \( k \geq 1 \) and \( D = \{1, 2, \ldots, k - 1\} \), if \( S^* \) is the \( D \)-avoiding set \( \{0, k, 2k, 3k, \ldots\} \) and \( S \) is any other \( D \)-avoiding set, \( S_q \gtrless (S^*)_q - \frac{1}{k} + O(1 - q) \).

**Proof:** We focus on the case \( k = 2 \) for clarity. Let \( S^* = \{0, 2, 4, \ldots\} \) and let \( S \) be some \( \{1\}\)-avoiding set other than \( S^* \). We can write \( S \) as the disjoint union of two sets, one of the form \( \{0, 2, \ldots, 2(m - 1)\} \) (empty if \( m = 0 \)) and one of the form \( \{t_1, t_2, t_3, \ldots\} \) (with \( t_1 < t_2 < t_3 < \ldots \)) satisfying \( t_1 \geq 2m + 1, t_2 \geq 2m + 3, t_3 \geq 2m + 5, \ldots \). The germ of \( S \) is dominated by the germ of \( \{0, 2, \ldots, 2(m - 1)\} \cup \{2m + 1, 2m + 3, 2m + 5, \ldots\} \); but this germ is the same (up to \( O(1 - q) \)) as the germ of \( \{1, 3, 5, \ldots\} \), which falls short of the germ of \( \{0, 2, 4, \ldots\} \) by \( \frac{1}{2} + O(1 - q) \). The case \( k > 2 \) is similar. ■

Packing problems and distance-avoidance problems in \( \mathbb{N} \) were chosen as a testbed for ideas about packing problems and distance-avoidance problems in \( \mathbb{R}^n \), and more specifically, sphere-packing problems. Note that the problem of packing spheres of radius 1 in \( \mathbb{R}^n \) is equivalent to the problem of packing points in \( \mathbb{R}^n \) so that no two are at distance less than 2 (the points are the centers of the spheres). We will not
pursue the topic of sphere-packing here, but we will mention the conjectures that motivated this work.

**Conjecture 15.** Let $S$ be a subset of $\mathbb{R}^2$, no two of whose points are at distance less than 2, and let $S^*$ be the set of center-points in a hexagonal close-packing of disks of radius 1 in $\mathbb{R}^2$. Let

$$\delta(S) = \liminf_{s \to \infty} \left( \sum_{(x,y) \in S^*} e^{- (x^2 + y^2) / s^2} - \sum_{(x,y) \in S} e^{- (x^2 + y^2) / s^2} \right).$$

Then either $S$ is related to $S^*$ by an isometry of $\mathbb{R}^2$, in which case $\delta(S) = 0$, or else $S$ is not related to $S^*$ by an isometry of $\mathbb{R}^2$, in which case $\delta(S) > 0$.

**Remark:** In private communication, Henry Cohn has shown that when $S$ is related to $S^*$ by an isometry of $\mathbb{R}^2$, $\delta(S)$ is indeed 0.

**Conjecture 16.** In the previous Conjecture, “$\delta(S) > 0$” can be replaced by “$\delta(S) \geq 1$” in the conclusion.

That is, there is an appreciable efficiency-gap for 2-dimensional sphere-packing.

The dichotomy between $\delta(S) = 0$ and $\delta(S) \geq 1$ in Conjecture 16 might at first seem to contradict the continuity of the summands as a function of the positions of the points; if all the points move continuously, won’t the lim inf also change continuously? The catch is that the lim inf can (and often does) diverge. For instance, if one obtains $S$ from $S^*$ by translating a half-plane’s worth of points by $\epsilon > 0$, or dilating the configuration $S^*$ by a factor of $c > 1$, then the lim inf diverges, no matter how close $\epsilon$ is to 0, or how close $c$ is to 1.

Clearly the bound in Conjecture 16 cannot be improved, since removing a single point from $S^*$ gives a set $S$ for which the lim inf is exactly 1.

We close by mentioning that the germ of $\sum_{n \in S} q^n$ as $q \to 1^-$ can also be thought of as the germ of $\sum_{n \in S} e^{-n/s}$ as $s \to +\infty$. The function $e^{-t}$ is a natural regularizer to use on $[0, \infty)$; the function $e^{-t^2}$ is a natural regularizer to use on $(-\infty, \infty)$; and higher-dimensional Gaussian kernels are natural regularizers to use in higher-dimensions. Some aspects of the theory are insensitive to the choice of regularizer.

**Acknowledgments:** This work has benefited from conversations with Tibor Beke, Ilya Chernykh, Henry Cohn, David Feldman, Boris Hasselblatt, Alex Iosevich, Sinai Robins, and Omer Tamuz.

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