Abstract

I re-examine a recent work by G. Landi and G. E. Landi. [arXiv:1808.06708 [physics.ins-det]], in which the authors claim that the resolution of a tracker can vary linearly with the number of detection layers, \( N \), that is, faster than the commonly known \( \sqrt{N} \) variation, for a tracker of fixed length, in case the precision of the position measurement is allowed to vary from layer to layer, i.e. heteroscedasticity, and an appropriate analysis method, a weighted least squares fit, is used.

keywords:

Tracking, weighted least squares, homoscedasticity, heteroscedasticity, Cramer-Rao Bound

1 Introduction

The momentum of charged particles, including the magnitude and the direction, is one of the very basic observables on which event reconstruction is built in particle physics. In the case of a detector of fixed given length, \( L \), containing a number \( N \) of detection layers, it is common wisdom that the precision on the track angle improves as \( 1/\sqrt{N} \), asymptotically at large \( N \), so that the resolution improves as \( \sqrt{N} \) (e.g. [1], and references therein).

In a recent work, though, G. Landi and G. E. Landi. are claiming that “A very simple Gaussian model is used to illustrate a new fitting result: a linear growth of the resolution with the number \( N \) of detecting layers. This rule is well beyond the well-known rule proportional to \( \sqrt{N} \) for the resolution of the usual fit” [2] (and further developments in [3,4]).

As I didn’t find the graphical pieces of evidence that were presented in [2] to support the allegation quite convincing, I am trying here to re-examine the matter. As in [2], I consider a simple situation of a tracker consisting of equally-spaced parallel layers, without magnetic field, and for which multiple scattering can be neglected.

2 Weighted least squares straight-track fit

Let’s consider a tracker consisting of \( N \) layers, \( i = 1 \cdots N \), equally spaced at position \( x_i \) along the \( x \) axis with spacing \( D \). Each detector \( i \) measures the position in the transverse direction \( y \) of each track traversing it, \( y_i \), with a Gaussian point spread function (PSF) with RMS \( \sigma_i \). We aim at fitting straight tracks

\[
y = ax + b
\]
Figure 1: Homoscedastic trackers: single-Gaussian-distributed measurement precision: variation of the inverse precision, \(1/\sigma_a\), as a function of the number of detectors, for a fixed total detector length \(L\), and for \(\sigma/L = 1\). Left plot: up to \(N - 1 = 12\). Right plot: up to \(N - 1 = 120\).

where \(a\) and \(b\) are the slope and the intercept of the track. Minimization of the \(\chi^2\),

\[
\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - (ax_i + b)}{\sigma_i} \right)^2,
\]

provides the values of \(a\) and \(b\):

\[
a = \frac{s_{xy}s_x - s_x s_y}{s_x^2 s - (s_x)^2} \quad \text{and} \quad b = \frac{s_y s_x^2 - s_x s_{xy}}{s_x^2 s - (s_x)^2}
\]

with precisions

\[
\sigma_a = \sqrt{\frac{s}{s_x^2 s - (s_x)^2}} \quad \text{and} \quad \sigma_b = \sqrt{\frac{s_x^2}{s_x^2 s - (s_x)^2}}
\]

and with

\[
s = \sum_{i=1}^{N} \frac{1}{\sigma_i^2}, \quad s_x = \sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}, \quad s_y = \sum_{i=1}^{N} \frac{y_i}{\sigma_i^2}, \quad s_{xy} = \sum_{i=1}^{N} \frac{x_i y_i}{\sigma_i^2} \quad \text{and} \quad s_x^2 = \sum_{i=1}^{N} \frac{x_i^2}{\sigma_i^2}.
\]

Given that \(x_i = iD\), we have

\[
\sigma_a^2 = \frac{1}{D^2} \left( \frac{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}{\left( \sum_{i=1}^{N} \frac{i^2}{\sigma_i^2} \right)^2} - \left( \sum_{i=1}^{N} \frac{i}{\sigma_i^2} \right)^2 \right)^2
\]

\[\text{(6)}\]

### 3 Homoscedastic trackers

In case the precisions \(\sigma_i\) are the same for all layers (homoscedasticity) and equal to a common value \(\sigma\), eq. \[6\] simplifies and the precision of the measurement of the track angles boils down to
\[
\sigma_a = \frac{2\sigma}{L} \sqrt{\frac{3(N-1)}{N(N+1)}} = \frac{2\sigma}{D} \sqrt{\frac{3}{(N-1)N(N+1)}},
\]  

(7)

where \( L = (N-1)D \) is the total length of the detector. For \( N = 1 \), \( \sigma_a \) is undefined as was expected for an angle measurement. For trackers with a large number of detection layers, and for a total length \( L \) being kept constant, the precision of the measurement of the angle varies asymptotically as \( \sqrt{12/N} \sigma/L \), and the resolution as \( \sqrt{N/12} L/\sigma \).

It seems clear from the variation of the inverse precision, \( 1/\sigma_a \), as a function of the number of detectors, Fig. 1 that the linear variation with \( N \) alluded in [2] is an “impression” when focusing attention on the very smallest numbers of detectors (left plot), while the asymptotic \( \sqrt{N} \) variation is clearly visible for larger numbers (right plot).

4 Heteroscedastic trackers

I now turn to the two-Gaussian toy model that G. Landi and G. E. Landi [2] have used as being a good approximation of tracking with silicon strip detectors, and with which they say they observe a linear growth. The point spread function consists of two Gaussians with different standard deviations, the first one with \( \sigma_1 = 0.18 \) with a probability of 80% and the second one with \( \sigma_2 = 0.018 \) with a probability of 20% [2].

With this model, fitting each track with a weighted least squares provides values of a with a Gaussian probability density function with standard deviation \( \sigma_a \) given by eq. (6), (as demonstrated by the pull distribution, that is found to follow a perfect \( \mathcal{N}(0,1) \) distribution) but the value of \( \sigma_a \) varies from track to track, depending on the distributions of the precisions of the measurements (\( \sigma_1 \) or \( \sigma_2 \)) along the track. As the \( \alpha \) distribution of the whole event sample is not Gaussian-distributed, I use the same method as in [2] to obtain a samplewise estimate of \( 1/\sigma_a \), that is, the maximum of the \( \alpha \) distribution.

![Figure 2: Heteroscedastic trackers: double-Gaussian-distributed measurement precision: variation of the height of the \( \alpha \) peak at maximum, as a function of the number of detectors, for a fixed total detector length \( L \). Left plot: up to \( N - 1 = 12 \). Right plot: up to \( N - 1 = 120 \).](image)

For a small number of measurements, I do seem to observe a linear growth (Fig. 2 left), as claimed...
in [2], but also in the same way as for the homoscedastic single-Gaussian measurement examined in the previous section. At large values of $N$, I obtain a $\sqrt{N}$-like variation, something which is more easily observed on the variation with $\sqrt{N}$ (Fig. 3).

![Graph showing variation](image)

Figure 3: Variation with $\sqrt{N}$ of the estimators shown in Figs. 1 and 2 up to $N - 1 = 120$. Left: Homoscedastic trackers: single-Gaussian-distributed measurement precision: variation of the inverse precision, $1/\sigma_a$. Right: Heteroscedastic trackers: double-Gaussian-distributed measurement precision: variation of the height of the $a$ peak at maximum.

5 Conclusion

The present work does confirm that for tracking detectors consisting of a small number of layers, the angle resolution seems to vary as $N$ as was shown in Fig. 2 of [2]. Examination of detectors consisting of a large number of layers, though, shows a variation of the resolution as $\sqrt{N}$, compatible with common wisdom. Neither homoscedasticity nor heteroscedasticity are found to play any role in the matter, in contrast with what alleged in [2].

References

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