COUNTABLE STATE SHIFTS AND UNIQUENESS OF $g$-MEASURES

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Abstract. In this paper we present a new approach to studying $g$-measures which is based upon local absolute continuity. We extend the result in [11] that square summability of variations of $g$-functions ensures uniqueness of $g$-measures. The first extension is to the case of countably many symbols. The second extension is to some cases where $g \geq 0$, relaxing the earlier requirement in [11] that $\inf g > 0$.

1. Introduction

Let $S$ be a countable discrete set, $X = S^{\mathbb{Z}_+}$ the infinite product with the usual Tychonoff product topology, $\mathcal{B}$ the Borel $\sigma$-algebra on $X$ and $T$ the shift on $X$; i.e., $(Tx)_i = x_{i+1}$ for $i \geq 0$. The notion of $g$-measures were introduced into ergodic theory by Keane in [10]. We recall that a $g$-function on $X$ is a measurable function $g$ such that for all $x \in X$, $\sum_{y \in T^{-1}x} g(y) = 1$. For a given $g$-function, a $g$-measure is defined to be a $T$-invariant measure $\mu$ in the space of Borel probability measures $\mathcal{P}(X)$, such that $g(x) = d\mu/d\mu \circ T^{-1}$. In the particular case of $S$ finite and a continuous $g$-function there always exists at least one $g$-measure. The problem of finding sufficient conditions on $0 < g < 1$ for which there is a unique $g$-measure has been extensively studied. In particular, Walters [18] showed that if $\sum_n \text{var}_n(\log g) < +\infty$ then there exists a unique $g$-measure. Recently this result was improved by Johansson and Öberg [11] who showed the same conclusion under the weaker assumption $\sum_n (\text{var}_n(\log g))^2 < +\infty$. Their approach was based upon martingale ideas, rather than the usual transfer operator techniques. Subsequently, Berger, Hoffman and Sidoravicius [3] showed the sharpness of this result by adapting a construction of Bramson and Kalikow [4], to show that

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for any $\epsilon > 0$ there exist $g$-functions satisfying $\sum_n (\text{var}_n(\log g))^2 + \epsilon < +\infty$ and for which there are two distinct $g$-measures.

In this paper we will consider more general settings. In particular, we will allow $S$ to be an infinite set, and we will also allow the possibility that $g$ takes the value 0. In this context, Walters [19], Sarig [13], [14], [15], and Mauldin and Urbanski [12] have proved various existence and uniqueness results with hypotheses similar to that of summable variation. (However, in the infinite state case, existence is no longer automatic due to the lack of compactness of $X^+$.)

In the present paper, one of our main results is a uniqueness result which, in particular, subsumes the uniqueness result in [11]. However, we shall present a new simplified approach based on local absolute continuity. This method has been developed by Shiryaev and co-authors (see for instance [16]), in Probability Theory, but seems novel in the context of Ergodic Theory.

2. Predictable ACS criteria

Consider a discrete filtration $\mathcal{F}_n \nearrow \mathcal{F}$, $n \geq 0$, for a general measure space $(X, \mathcal{F})$. For our purposes we may assume that each $\mathcal{F}_n$ is countable.

A measure $\tilde{\mu} \in \mathcal{M}(X)$ is said to be \emph{locally absolute continuous} with respect to a second measure $\mu \in \mathcal{M}(X)$, written $\tilde{\mu} \ll_{\text{loc}} \mu$, if $\tilde{\mu}|_{\mathcal{F}_n} \ll \mu|_{\mathcal{F}_n}$, for all $n \geq 0$, where $\mu|_{\mathcal{F}_n}$ denote the restriction of $\mu$ to the sub-$\sigma$-algebra $\mathcal{F}_n$. We write conditional expectation relative to a probability measure $\mu$ using the integral notation, \textit{i.e.}, $\int f \, d\mu(x)\mid_{\mathcal{G}}$ denotes the conditional $\mu$-expectation $d(f\mu|_{\mathcal{G}})/d\mu|_{\mathcal{G}}$ of a function $f$ relative to a sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$.

We will use an approach developed by Shiryaev and his co-authors for extending local absolute continuity to absolute continuity for probability measures. To begin, we recall that for any two probability measures $\mu$ and $\tilde{\mu}$ defined on a $\sigma$-algebra $\mathcal{F}$ of a space $X$, the classical Lebesgue decomposition tells us that we can write $\mu = \lambda_1 + \lambda_2$, where $\lambda_1, \lambda_2$ are probability measures on $\mathcal{F}$ for which $\lambda_1 \ll \tilde{\mu}$ and $\lambda_2 \perp \tilde{\mu}$.

Assuming local absolute continuity allows us to draw a stronger conclusion. More precisely, when $\tilde{\mu} \ll_{\text{loc}} \mu$ we can define the local likelihood ratio process as the $\mathcal{F}_n$-adapted process given by

$$Z_n(x) := \frac{d\tilde{\mu}|_{\mathcal{F}_n}}{d\mu|_{\mathcal{F}_n}}.$$

Since $Z_n$ is a $\mu$-martingale and a $\tilde{\mu}$-submartingale, the limit $\lim_n Z_n$ exists $\mu$-almost surely and $\tilde{\mu}$-almost surely unless $Z_\infty := \limsup Z_n$ equals $\infty$. We can therefore write:

$$\tilde{\mu}(B) = \int_B Z_\infty \, d\mu + \tilde{\mu}(B \cap \{Z_\infty = \infty\}),$$

where $B \in \mathcal{F}$ (Theorem 4 of [5]; or p. 493 of [17]). From this decomposition we immediately have that:

(2.1) \quad $\tilde{\mu} \ll \mu \iff \tilde{\mu}(Z_\infty < \infty) = 1$

(cf. Theorem 5 of [5]; or p. 495 of [17]).

For $x \in X$ and $n \geq 1$, let $\alpha_n(x) = Z_n(x)/Z_{n-1}(x)$ and let

$$d_n(x) := \int (1 - \sqrt{\alpha_k(x)})^2 \, d\mu(x|\mathcal{F}_{k-1}).$$

The predictable increasing process $B_n(x) = \sum_{k=1}^n d_k(x)$, $n \geq 0$, is referred to as the “Hellinger process”. Kabanov, Lipster and Shiryaev in [9] (see also Jacod and Shiryaev [8], p. 253, Theorem 2.36 with $T = \infty$) proved the following consequence of (2.1) that they termed a “predictable ACS-criteria”.

**Lemma 2.1.** If $\tilde{\mu} \ll^{\text{loc}} \mu$, then $\tilde{\mu} \ll \mu$ if and only if $\lim B_n(x) < \infty$ with $\tilde{\mu}$-probability one.

**Proof.** We recall the main steps in the simple proof, adapting pages 496–498 of [17]. By the submartingale property, it suffices to show that the log-likelihood process $\log Z_n = \sum_{k=1}^n \log \alpha_k$ converges $\tilde{\mu}$-a.s. if only if the Hellinger process converges $\tilde{\mu}$-a.s. Furthermore, writing $\log x = \log x$ if $|\log x| < 1$ and $\text{sign}(\log x)$ otherwise, the convergence of $\log Z_n$ occurs precisely when the process

$$Y_n := \sum_{k=1}^n t \log \alpha_k$$

converges.

We now claim that $Y_n$ is a $\tilde{\mu}$-submartingale as well. To see this note first that if $f$ is $\mathcal{F}_n$-measurable then

(2.2) \quad $\int f(x) \, d\tilde{\mu}(x|\mathcal{F}_{n-1}) = \int \alpha_n(x) f(x) \, d\mu(x|\mathcal{F}_{n-1}).$

Hence, Jensen’s inequality gives

$$\int (Y_n - Y_{n-1}) \, d\tilde{\mu}(x|\mathcal{F}_{n-1}) = \int \alpha_n t \log \alpha_n \, d\mu(x|\mathcal{F}_{n-1}) \geq t \log 1 = 0,$$

since $x \log x$ is a convex function for $x \geq 0$. 

Thus $Y_n$ is a submartingale with bounded increments $|Y_n - Y_{n-1}| < 1$. A Doob decomposition $Y_n = A_n + M_n$, where $A_n$ is predictable and increasing and $M_n$ is a martingale, then shows that $\lim Y_n < \infty$ $\tilde{\mu}$-a.s. if and only if

$$\sum_n \int \log \alpha_n \, d\tilde{\mu}(x|\mathcal{F}_{n-1}) + \sum_n \int (\log \alpha_n)^2 \, d\tilde{\mu}(x|\mathcal{F}_{n-1})$$

$$= \sum_n \int [\alpha_n \log \alpha_n + \alpha_n (\log \alpha_n)^2] \, d\mu(x|\mathcal{F}_{n-1}) < \infty,$$

$\tilde{\mu}$-a.s. (The first sum is $A_n$ and the second sum bounds the variance of $M_n$.)

Finally, we have for $x \geq 0$ that

$$(1/C)(1 - \sqrt{x})^2 \leq x \log x + x(\log x)^2 + 1 - x \leq C(1 - \sqrt{x})^2,$$

for some $C > 0$. Since $\int (1 - \alpha_n(x)) \, d\mu(x|\mathcal{F}_{n-1}) = 0$, we see that the sum in (2.3) must converge exactly when the Hellinger process $B_n(x)$ does. □

For $x \in X$, $[x]_{\mathcal{F}_n}$, denotes the minimal element of $\mathcal{F}_n$ containing $x$. We assume that the mapping $[x]_{\mathcal{F}_k} \rightarrow [x]_{\mathcal{F}_{k-1}}$ has countable fibers $S_k(x) = \{[y]_{\mathcal{F}_k} : [y]_{\mathcal{F}_k} \subset [x]_{\mathcal{F}_{k-1}}\}$ and we denote by $p_k$ and $\tilde{p}_k$ the conditional probabilities on $S_k(x)$ induced by $\mu$ and $\tilde{\mu}$, respectively; i.e.

$$p_k(x) := \frac{d\mu|_{\mathcal{F}_k}}{d\mu|_{\mathcal{F}_{k-1}}}(x) = \mu([x]_{\mathcal{F}_k} | [x]_{\mathcal{F}_{k-1}}),$$

and $\tilde{p}_k$ is defined analogously.

We may then write

$$d_n(x) = \rho_H^2(p_k, \tilde{p}_k) := \sum_{[y] \in S_n(x)} (\sqrt{p_k(y)} - \sqrt{\tilde{p}_k(y)})^2,$$

which says that $d_n(x)$ is the squared Kakutani–Hellinger distance between the probabilities $p_k$ and $\tilde{p}_k$ on the countable set $S_n(x)$. To see (2.5), note that $\alpha_k(x) = \tilde{p}_k(x) / p_k(x)$ and, since $\alpha_k$ is $\mathcal{F}_k$-measurable, we obtain

$$\int (1 - \sqrt{\alpha_k})^2 \, d\mu(x|\mathcal{F}_{k-1}) = \sum_{[y] \in S_n(x)} p_k(y)(1 - \sqrt{\tilde{p}_k(y)p_k(y)})^2$$

$$= \sum_{[y] \in S_n(x)} (\sqrt{p_k(y)} - \sqrt{\tilde{p}_k(y)})^2$$
3. Variations of the $g$-function and absolute continuity of $g$-chains

In this section we use the predictable ACS-criteria to derive a criterion for absolute continuity of $g$-measures on the “forward algebra” which we subsequently use to give sufficient conditions for uniqueness. It is convenient to work with the natural extension to the two-sided shift $T$ on the space $X = \mathbb{S}^\mathbb{Z}$ instead of $X^+ = \mathbb{S}^\mathbb{Z}_+$. Thus $(Tx)_i = x_{i+1}$, $i \in \mathbb{Z}$ and $(T^{-1}x)_i = x_{i-1}$. The one-sided shift is recaptured by taking the projection $x \rightarrow x_+ = (x_0, x_1, x_2, \ldots)$ so that $Tx_+ = [Tx]_+$.

Let $\mathcal{F}^+$ be the $\sigma$-algebra generated by $x_+$, i.e., $\mathcal{F}^+ = \lim_n \mathcal{F}^+_n$, where $\mathcal{F}^+_n = \{[x_0, x_1, \ldots, x_{n-1}]\}$ is the filtration of backward finite cylinders. A $g$-function is a $\mathcal{F}^+$-measurable function $g(x) = g(x_0, x_1, \ldots)$ satisfying $\sum_{y \in \mathcal{F}^{-1}(x_0)} g(y) = 1$ for all $x$. A $g$-measure is a $T$-invariant probability measure on $X$ such that $g = d\mu|_{\mathcal{F}^+}/d\mu|_{T^{-1}\mathcal{F}^+}$. Equivalently,

$$g(x) = \lim_n \frac{\mu[x_0, \ldots, x_n]}{\mu[x_1, \ldots, x_n]},$$

$\mu$-almost everywhere. It is clear that any $g$-measure on $\mathcal{B}(X^+)$ extends uniquely to a $g$-measure on $X$.

If we consider the process

$$x^{(k)} := [T^{-k}x]_+ = (x_{-k}, x_{-k+1}, \ldots, x_0, x_1, \ldots), \quad k \geq 0,$$

we can note that, since $x^{(n-1)} = Tx^{(n)}$, this is a Markov chain on $X^+$, regardless of the underlying probability measure $\mu \in \mathcal{P}(X)$. In this picture, we add the symbol $x_{-n} \in \mathcal{S}$ occuring at position $-n$ at “time” $n$ so that time runs in the reverse with the index of the symbol-sequences. The initial condition is the distribution of $x^{(0)}$, i.e. $\mu|_{\mathcal{F}^+}$. For a $g$-function $g$, we say that a probability $\mu \in \mathcal{P}(X)$ is a $g$-chain if the transition probabilities $\mu(x^{(k)}|x^{(k-1)})$ are given by $g(x^{(k)})$, i.e. if $d\mu|_{T^k\mathcal{F}^+}/d\mu|_{T^{k-1}\mathcal{F}^+} = g \circ T^{-k}$, for $k \geq 1$. A $g$-measure $\mu$ corresponds to a stationary $g$-chain on $X^+$.

We want to apply Theorem 2.1 to the filtration

$$\mathcal{F}^-_n = \{[x_{-n}, x_{-n+1}, \ldots, x_{-1}]\} \nearrow \mathcal{F}^-$$

of finite forward cylinders in $\mathcal{B}(X)$ and thus derive absolute continuity of two $g$-chains with respect to the $\sigma$-algebra $\mathcal{F}^- = \lim_n \mathcal{F}^-_n$ generated by the symbols added during the evolution of the Markov chain $x^{(n)}$. Recall that for $f : X^+ \rightarrow \mathbb{R}$, we denote

$$\operatorname{var}_n f(x) = \sup \{|f(x) - f(y)| : x_i = y_i \text{ for } i \leq n\}.$$
and we also define the \( s \)-variation by

\[
\text{svar}_n f = \sup_x \left( \sum_{\sigma \in S} (\text{var}_{n+1} f(\sigma, x))^2 \right)^{1/2}.
\]

**Lemma 3.1.** Assume that the \( g \)-function \( g \) satisfies

\[
\sum_{n=0}^{\infty} (\text{svar}_n \sqrt{g})^2 < \infty.
\]

Then for any two \( g \)-chains \( \mu \) and \( \tilde{\mu} \), \( \tilde{\mu}|_{\mathcal{F}^+} \ll \mu|_{\mathcal{F}^+} \) provided \( \tilde{\mu}|_{\mathcal{F}^-} \ll \mu|_{\mathcal{F}^-} \), for each \( n \).

Note that the property \( \tilde{\mu}|_{\mathcal{F}^-} \ll \mu|_{\mathcal{F}^-} \) can be given the following interpretation: There is no test that, based on observations of the symbols \( x_{-1}, x_{-2}, \ldots \) added over time, can discern with probability one between the two given initial conditions \( \tilde{\mu}|_{\mathcal{F}^+} \) and \( \mu|_{\mathcal{F}^+} \).

**Proof.** Given two \( g \)-chains \( \mu \) and \( \tilde{\mu} \) we consider the filtration \( \mathcal{F}_n = \mathcal{F}^- \). Our aim is to show that \( d_n(x) \leq (\text{svar}_n \sqrt{g})^2 \) since the lemma then follows from Lemma 2.1.

We see that the probability \( p_n(y) \) (or \( \tilde{p}_n(y) \)), for \( [y]|_{\mathcal{F}_n} \subset [x]|_{\mathcal{F}_{n-1}} \), in \( [x]|_{\mathcal{F}_{n-1}} \) induced from \( \mu \) (or \( \tilde{\mu} \)) is the probability

\[
\pi_n(\sigma|x) = \frac{\mu([\sigma, x_{-n+1}, \ldots, x_{-1}])}{\mu([x_{-n+1}, \ldots, x_{-1}])}
\]

of adding the symbol \( \sigma = y_{-n} \) at place \(-n\) given the symbols \( x_{-n+1}, \ldots, x_{-1} \) at places \(-n+1, \ldots, -1\).

Since \( \pi_n(\sigma|x) = \int g(x^{(n)}) d\mu([x]|_{\mathcal{F}_{n-1}^+}) \) it is clear that \( \pi_n(\cdot|x) \) and \( \tilde{\pi}_n(\cdot|x) \) are weighted averages of probabilities of the form \( g(\cdot, y), y \in X^+ \), and where \( y \) coincide with \( x_{-n+1}, \ldots, x_{-1} \) in the first \( n-1 \) coordinates. Hence, the local Hellinger distance \( d_n(x) \) satisfies

\[
d_n(x) = \rho^2_H(\pi_n, \tilde{\pi}_n) \leq \sup_{(y, \tilde{y})} \rho^2_H(g(\cdot, y), g(\cdot, \tilde{y})),
\]

with the supremum taken over all pairs \((y, \tilde{y})\) in \( X^+ \times X^+ \) that coincide in the first \( n-1 \) coordinates. But this supremum is clearly less than \((\text{svar}_n \sqrt{g})^2\). \( \square \)

4. ** Conditions for uniqueness of \( g \)-measures **

We are now ready to consider the problem of uniqueness of \( g \)-measures. Our main result is the following theorem.
Theorem 4.1. Suppose that \( g \geq 0 \) and that \( \sum_{n=1}^{\infty} (\text{svar}_n \sqrt{g})^2 < \infty \). Then any two distinct ergodic \( g \)-measures must be locally incomparable.

Local incomparability of \( \mu, \tilde{\mu} \in \mathcal{M}(X) \), means that neither \( \mu \ll_{\text{loc}} \tilde{\mu} \) nor \( \tilde{\mu} \ll_{\text{loc}} \mu \). Local incomparability holds for example when the transition operator given by \( g \) is periodic.

Proof. First of all we note that if \( \tilde{\mu}|_{F-} \ll \mu|_{F-} \) then it follows from the Birkhoff Ergodic Theorem that in fact \( \tilde{\mu} \ll \mu \); if we are given a density \( f = d\tilde{\mu}|_{F-}/d\mu|_{F-} \in L^1(\mu) \) then \( h = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n \) must be the density \( d\tilde{\mu}/d\mu \), since it follows that \( \int_C h \, d\mu = \tilde{\mu}(C) \) for any finite cylinder \( C \).

Secondly, it is well known that no two ergodic measures on a compact set can be comparable, so we can not have \( \mu \ll \tilde{\mu} \) or \( \tilde{\mu} \ll \mu \). This result can used in our context as follows: If \( S \) is not finite, then we can denote by \( \bar{S} = S \cup \{\infty\} \) the one point compactification and then \( X \) is contained in the compact space \( \bar{X} = \bar{S}^\mathbb{Z} \). The shift \( \bar{T} : \bar{X} \to \bar{X} \) is continuous and \( \bar{X} - X \) is an invariant set consisting of sequences containing the symbol \( \infty \). The ergodic measures on \( X \) correspond to ergodic measures on \( \bar{X} \) with \( \mu(X) > 0 \) (and by ergodicity \( \mu(X) = 1 \)).

If the zero set \( \{x \in X : g(x) = 0\} \) has empty interior, then every \( g \)-measure \( \mu \) must assign a positive probability to each cylinder \([x_0, \ldots, x_n] \). Hence all \( g \)-measures are locally comparable and the above theorem applies. In particular, if \( g \) is strictly positive then we can deduce the following corollary.

Corollary 4.2. If, in addition to the assumptions of the theorem, \( g > 0 \) then there is at most one \( g \)-measure.

Remark 1. If we consider a more general subshift of finite type, then we would need to impose suitable recurrence conditions on the associated transition matrix \([12], [14]\).

In the case \( S \) is finite and \( g \) continuous the last corollary reduces to the condition of square summability of variations in \([14]\):

Corollary 4.3. If \( S \) is finite and \( g > 0 \) satisfies \( \sum_n (\text{var}_n (\log g))^2 < +\infty \) then there is precisely one \( g \)-measure.

To see this it is enough to note that the compactness of \( X \) implies that \( g(x) \) is bounded away from 0 and 1 and that in this case \( \text{var}_n \log g \) and \( (\text{svar}_{n-1} \sqrt{g})^2 \) are of the same order.
Example 1. Let $X^+ = \mathbb{Z}_+^\mathbb{Z}$. Fix a sequence $p_i > 0$, $i \geq 1$, such that $\sum_{i=1}^{\infty} p_i = 1$ and a number $\alpha > 0$. Define

$$g(i, x) = \begin{cases} p_i b(x) & i \geq 1 \\ 1 - b(x) & i = 0 \end{cases}$$

where

$$b(x) = \frac{1}{\zeta(3 + \alpha)} \sum_{k=1}^{\infty} \frac{1}{k^{3+\alpha}} \frac{1}{1 + x_{k-1}}.$$ 

Then $(\text{svar}_n \sqrt{g})^2$

$$= \sup_{x_k = x_{k+1}} \left\{ \left( \sqrt{b(x)} - \sqrt{b(x)} \right)^2 + \left( \sqrt{1 - b(x)} - \sqrt{1 - b(x)} \right)^2 \right\}$$

$$\leq \sqrt{\text{var}_n b(x)} = O(n^{1+\alpha/2})$$

and, since this is summable, there is at most one $g$-measure by Corollary 4.2.

However, since

$$\lim_{N \to \infty} \left( \limsup_{x_k = x_{k+1}} b(x) \right) = 0$$

we see that $\text{var}_n \log g = \text{var}_n \log b = \infty$, for all $n \geq 1$, so any condition on the variations of $\log g$ does not apply in this case.

As we will see in the next two examples, conditions on the variations of $g$ are not always necessary to ensure uniqueness of a $g$-measure.

Example 2. Let $a_n \to a$, with $0 < a_n, a < 1$. Let $X^+ = \{0, 1\}^\mathbb{N}$ then we can define $g : X^+ \to (0, 1)$ by

$$g(x) = a_n \quad \text{if} \quad x_0 = \cdots = x_{n-1} = 0 \text{ and } x_n = 1$$

$$= 1 - a_n \quad \text{if} \quad x_1 = \cdots = x_{n-1} = 0 \text{ and } x_0 = x_n = 1$$

Hulse [7] observed that there is a unique $g$-measure without any hypotheses on $a_n$. In particular, no conditions on the variations are required.

It should perhaps also be noted that under the condition $\sum_n (\text{svar}_n \sqrt{g})^2 < \infty$ we obtain absolute continuity on the forward algebra $\mathcal{F}^-$ also for any non-stationary $g$-chain. It could also be noted that there are $g$-functions that have a unique $g$-measure, for which some initial conditions gives chains which are not a. c. on $\mathcal{F}^-$. 
Example 3. Let $S = \{+1, -1\}$, $X = S^\mathbb{Z}$ and
\[
g(\pm 1, x_1, x_2, \ldots) := \phi(\pm \sum_{i=1}^\infty a_i x_i)
\]
where $\phi(x) = e^x/(e^x + e^{-x})$ and where we require that $a_n > 0$, $\sum_n a_n < \infty$ but
\[(4.1) \quad \sum_n (\sum_{i=n}^\infty a_i)^2 = \infty.
\]
Then it is not too hard to see that if $\mu$ is the distribution of the Markov chain $x^{(k)}$, $k \geq 0$, given that $x^{(0)} = (+1, +1, +1, \ldots)$ and $\tilde{\mu}$ the distribution starting in $x^{(0)} = (-1, -1, -1, \ldots)$, then, by (4.1), $d_n(x) \geq c \cdot (\sum_{i=n}^\infty a_i)^2$. Hence, by Lemma 2.1 $\mu$ and $\tilde{\mu}$ must indeed be mutually singular on $\mathcal{F}^-$. However the condition of Dobrushin (see, for instance, [6]) can be used to deduce a unique ergodic measure if we in addition assume that $\sum_i a_i < 1$.

5. Existence of $g$-measures

In the case of countable shift state spaces, we cannot rely on the Schauder–Tychonoff fixed point theorem to produce a $g$-measure, due to the lack of compactness of $X$.

Example 4. Consider a subshift of finite type $\Sigma \subset X^+$ of sequences $(x_n)_{n=0}^\infty \in \mathbb{Z}^\mathbb{Z}_+$ where $x_n$ can be followed by $x_{n+1}$ iff $|x_n - x_{n+1}| \leq 1$. The function $g(x) = 1/3$ is a $g$-function. However, the properties of the simple random walk on $\mathbb{Z}$ implies that there is no finite $g$-measure.

In this paper we do not want to rely upon summability of variations (or local Hölder continuity), as in Sarig [13], Sarig [15] and Mauldin and Urbanski [12], to derive the existence of $g$-measures. We can do without this assumption if we assume instead that the $g$-function $g$ can be continuously extended to a compactification of $X$.

We give below a sufficient condition for existence, which is weak enough to demonstrate that our uniqueness conditions are not vacuous in the case of countable state shifts. Let $\tilde{S}$ be a one-point compactification of $S$, i.e., $\tilde{S} = S \cup \{\infty\}$ and let $\tilde{X} = S^\mathbb{Z}_+$. It is clear that a continuous function $f : X \to \mathbb{R}$ can be continuously extended to $\tilde{X}$ if and only if the following hold: For every $\epsilon > 0$ and $n \in \mathbb{Z}_+$ there is some finite set $B \subset S$ such that $|f(x) - f(y)| < \epsilon$ whenever $x_n, y_n \notin B$ and $y_i = x_i$ for $i \neq n$. 

Theorem 5.1. Suppose that $g$ is continuous $g$-function on $X$ such that $g$ can be extended to a continuous function on $\bar{X}$. Suppose further that for every $x \in X$ we have

$$g(\sigma x) \leq K \pi(\sigma),$$

where $K \geq 1$ and where $\pi$ is a fixed probability measure on the symbol set $S$, i.e., $\sum_\sigma \pi(\sigma) = 1$. Then there exists at least one $g$-measure on $X = S^{\mathbb{Z}^+}$.

Proof. By the Dominated Convergence Theorem in the context of functions in $\ell_1(S)$ it follows from (5.1) that the continuous extension of $g$ must be a $g$-function on $\bar{X}$, i.e. the extended $g$ has, in addition to continuity, the property that $\sum_{\sigma \in \mathcal{S}} g(\sigma, x) = 1$ for any $x \in \bar{X}$. We can therefore assume a $g$-measure $\mu \in \mathcal{P}(\bar{X})$ implied by the Schauder-Tychonoff fixed point theorem.

For every $\epsilon > 0$ we have a finite set $B_\epsilon \subset S$ such that $\pi(S \setminus B_\epsilon) \leq \frac{\epsilon}{K}$ and let $B^X_\epsilon = \{x_0 \in B_\epsilon\}$ be the sequences in $X = S^{\mathbb{Z}^+}$ which have their last symbols in $B_\epsilon$. Then we have that

$$\mu(X \setminus B^X_\epsilon) \leq \sum_{\sigma \in X \setminus B_\epsilon} \int_X g(\sigma x) \, d\mu(x) \leq \epsilon.$$ 

which immediately implies $\mu(\{x_0 = \infty\}) = 0$ and hence, by translation invariance, $\mu(\{\exists n : x_n = \infty\}) = 0$. In other words, $\mu(X) = 1$ and $\mu$ corresponds to a $g$-measure in $\mathcal{P}(X)$. \hfill \Box

Remark 2. It is easy to check that the $g$-function in Example satisfies the conditions above. We can hence deduce that this $g$-function admits precisely one $g$-measure.

Remark 3. In particular, if $\text{var} \log g < +\infty$ then we can fix $x_0 \in X$ and write $g(\sigma x) \leq e^{\text{var} \log g} g(\sigma x_0)$. Thus, the hypotheses hold with $K = e^{\text{var} \log g}$ and $\pi(\sigma) = g(\sigma x_0)$.

References

[1] H. Berbee, Chains with Infinite Connections: Uniqueness and Markov Representation, Probab. Theory Related Fields 76 (1987), 243–253.

[2] H. Berbee, Uniqueness of Gibbs measures and absorption probabilities, Ann. Probab. 17 (1989), no. 4, 1416–1431.

[3] N. Berger, C. Hoffman and Sidoravicius, Nonuniqueness for specifications in $l^2 + \epsilon$. Preprint available on www.arxiv.org (PR/0312344).

[4] Bramson and S. Kalikow, Nonuniqueness in $g$-functions, Israel J. Math. 84 (1993), 153–160.
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[5] H.J. Engelbert and A.N. Shiryaev, On absolute continuity and singularity of probability measures, *Banach Cent. Publ.* 6 (1980), 121–132.

[6] R. Fernandez and G. Maillard, Chains with Complete Connections: General Theory, Uniqueness, Loss of Memory and Mixing Properties, *J. Statist. Phys.* 118 (2005), no. 3-4, 555–588.

[7] P. Hulse *Ph.D. Thesis*, Warwick University (1980).

[8] J. Jacod and A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, 2nd ed., Grundlehren der mathematischen Wissenschaften 288, Springer-Verlag 2003.

[9] Yu. Kabanov, R.S. Lipster and A.N. Siryaev, On the question of the absolute continuity and singularity of probability measures (Russian), *Mat. Sb. (N.S.)* 104(146) (1977), no. 2(10), 227–247.

[10] M. Keane, Strongly Mixing $g$-Measures, *Invent. Math.* 16 (1972), 309–324.

[11] A. Johansson and A. Öberg, Square summability of variations of $g$-functions and uniqueness of $g$-measures, *Math. Res. Lett.* 10 (2003), no. 5-6, 587–601.

[12] D. Mauldin and M. Urbanski, Gibbs states on the symbolic space over an infinite alphabet, *Israel J. Math.* 125 (2001), 93–130.

[13] O. Sarig, Thermodynamic formalism for countable Markov shifts, *Ergodic Theory Dynam. Systems* 19 (1999), no. 6, 1565–1593.

[14] O. Sarig, Thermodynamic formalism for null recurrent potentials, *Israel J. Math.* 121 (2001), 285–311.

[15] O. Sarig, Existence of Gibbs measures for countable Markov shifts, *Proc. Amer. Math. Soc.* 131 (2003), no. 6, 1751–1758.

[16] A.N. Shiryaev, Absolute Continuity and Singularity of Probability Measures in Functional Spaces, *Proceedings of the International Congress of Mathematicians*, Helsinki 1978, pp. 209–225.

[17] A.N. Shiryaev, *Probability*, Second edition. Graduate Texts in Mathematics 95, Springer-Verlag, New York, 1996.

[18] P. Walters, Ruelle’s operator theorem and $g$-measures, *Trans. Amer. Math. Soc.* 214 (1975), 375–387.

[19] P. Walters, Invariant measures and equilibrium states for some mappings which expand distances, *Trans. Amer. Math. Soc.* 236 (1978), 121–153.

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