Instantons and Wormholes In Minkowski and (A)dS Spaces

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Abstract

Instanton and wormhole solutions are constructed in a $d$-dimensional gravity theory with an axion-dilaton pair of scalar fields. We discuss the cases of vanishing, positive and negative cosmological constant.
1 Introduction

Wormholes may have relevance to many interesting questions in quantum gravity. For instance, in the past it has been argued that they can play role in the renormalisation of the coupling constants in nature, topological fluctuations, quantum decoherence, the question of the vanishing of the cosmological constant and creation of baby universes (see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]).

Instanton solutions in supergravity theories can be responsible for non-perturbative effects in string theories. For example the D-instanton solution of type IIB supergravity found in [13] induces higher derivative non-perturbative corrections of the IIB action [14]. Supergravity solutions corresponding to Euclidean wrapped branes were found in [15, 17, 18, 16]. Axionic wormholes in four dimensions were first considered in [9], where a system consisting of an axion, described by a rank three antisymmetric tensor, coupled to gravity was considered. The case of string theory was considered in [12, 19] where in addition to the axion one also includes a massless dilaton. The solutions are characterized by an integration constant and instanton solutions are extremal in the sense that the integration constant vanishes and some of the (Euclidean) supersymmetry is preserved, whereas wormholes are non-extremal and supersymmetry is broken [20].

In semi-classical quantum gravity, the action of an Euclidean wormhole solution is related to the weight for the insertion of wormhole operators in the path integral (or the rate of baby universe production). For stringy wormholes it was found [12, 19] that the existence of non-singular solutions with finite action depended on the details of the coupling of the dilaton to the axion. In particular for axion-dilaton systems arising from string theory compactification there are no finite action wormholes. However, it was argued that instantons could still contribute when dilaton acquires mass due to supersymmetry breaking. Generalization of axionic instantons to the case of a positive cosmological constant was later obtained in [21, 22].

In this paper we study instantons and wormholes for an axion-dilaton system in $d$ dimensions and obtain a formula for critical values for couplings below which non-singular solutions do exist. We will first consider the Minkowski case and thus obtain generalizations of the known results in four dimensions to an arbitrary $d$-dimensional space. Our
main result is the study of instantons and wormholes in de Sitter and Anti-de Sitter spaces.

2 Axion-Dilaton gravity

We start our analysis by considering the theory of $d$-dimensional gravity coupled to an axion-dilaton system. We will find generalizations of the results of [12, 19] to arbitrary dimensions. For Minkowski signature, the action we consider is given by

$$S_m = \int d^d x \sqrt{-g} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} e^{b \phi} \partial_{\mu} \chi \partial^{\mu} \chi - V(\phi) \right).$$  \hspace{1cm} (2.1)

The potential $V$ will not depend on the axion $\chi$ since we assume an exact shift symmetry $\chi \rightarrow \chi + \epsilon$. The value of $b$ is determined by the particular theory one considers. In $d$ dimensions the axion $\chi$ can be dualized to a $d-1$ form field strength $F_{d-1} = dC_{d-2}$ via $d\chi = e^{-b \phi} \ast F_{d-1}$. The dualized action takes the form

$$S'_m = \int d^d x \sqrt{-g} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{e^{-b \phi}}{2(d-1)!} F_{d-1}^2 - V(\phi) \right).$$ \hspace{1cm} (2.2)

In this form the continuation to Euclidean signature is not problematic [13, 14, 23]. However the dualization and analytic continuation to Euclidean signature do not commute. This leads to the fact that upon continuation to Euclidean space and dualization the kinetic term for the axion $\chi$ changes sign

$$S_{eucl} = \int d^d x \sqrt{g} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} e^{b \phi} \partial_{\mu} \chi \partial^{\mu} \chi - V(\phi) \right).$$ \hspace{1cm} (2.3)

In addition to the bulk term (2.3) there is a boundary term in the action which is important for the proper definition of the variational principle and also for the calculation of the action [13, 14, 12, 19]. The boundary term is given by

$$S_b = \oint \left( e^{b \phi} \chi \partial_n \chi + K \right),$$ \hspace{1cm} (2.4)

where $K$ is the intrinsic curvature on the boundary. The Euclidean equations of motion

\footnote{For example, in ten dimensional type IIB string theory one has $b = 2$.}
derived from (2.3) are given by

$$\nabla_\mu (e^{b\phi} \partial^\mu \chi) = 0, \quad (2.5)$$

$$\nabla^2 \phi + \frac{b}{2} e^{b\phi} (\partial \chi)^2 - \frac{\partial V(\phi)}{\partial \phi} = 0, \quad (2.6)$$

$$R_{\mu\nu} - \frac{1}{2} \partial_\mu \partial_\nu \phi + \frac{1}{2} e^{b\phi} \partial_\mu \chi \partial_\nu \chi - \frac{1}{d-2} g_{\mu\nu} V(\phi) = 0. \quad (2.7)$$

In the following we will only consider a potential independent of $\phi$ corresponding to a cosmological constant $V = \Lambda$. We are also interested in the most symmetric instanton and wormhole solutions and so we shall consider the following $O(d)$ invariant ansatz for the metric

$$ds^2 = dr^2 + a(r)^2 d\Omega_{d-1}^2, \quad (2.8)$$

where $d\Omega_{d-1}^2$ is the metric of the $(d-1)$ round sphere $S_{d-1}$. In addition we demand that the dilaton $\phi$ and axion $\chi$ depend only on the radial coordinate $r$.

One could imagine an ansatz where the sphere metric $d\Omega_{d-1}^2$ is replaced by the metric on flat space $M_{d-1}$ or the metric on hyperbolic space $H_{d-1}$. However such a slicing would lead to solutions with infinite action coming from (2.4), because the (unit) volume of $M_{d-1}$ and $H_{d-1}$ is infinite and one integrates over the volume in (2.4).

For the ansatz (2.8), the equation of motion for the axion gives

$$\partial_r (e^{b\phi} a^{d-1} \partial_r \chi) = 0, \quad (2.9)$$

which implies a first-order differential equation for the axion field given by

$$\partial_r \chi = \frac{q}{a'^{d-1}} e^{-b\phi}. \quad (2.10)$$

Upon using (2.10), we obtain from the gravitational equations of motion (2.7)

$$-(d-1) \frac{\partial^2 a}{a} - \frac{1}{2} (\partial_r a)^2 + \frac{q^2}{2a^{2d-2}} e^{-b\phi} - \frac{1}{d-2} \Lambda = 0,$$

$$(d-2)(1 - (\partial_r a)^2) - a\partial^2_r a - \frac{1}{d-2} a^2 \Lambda = 0, \quad (2.11)$$

where we have used

$$R_{rr} = -(d-1) \frac{\partial^2 a}{a},$$

$$R_{ij} = [(d-2)(1 - (\partial_r a)^2) - a\partial^2_r a] \delta_{ij}. \quad (2.12)$$
Also, upon using (2.10) and our metric ansatz, the dilaton equation of motion becomes
\[
\partial^2 r \phi + (d - 1) \frac{\partial_r a}{a} \partial_r \phi + \frac{bq^2}{2a^{2d-2}} e^{-b\phi} = 0. \tag{2.13}
\]
It follows from (2.13) that there exists an integral
\[
(\partial_r \phi)^2 - \frac{q^2}{a^{2d-2}} e^{-b\phi} - \frac{c}{a^{2d-2}} = 0, \tag{2.14}
\]
where \(c\) is a constant of integration. Using this integral, the gravitational equations of motion (2.11) give
\[
1 - (\partial_r a)^2 + \frac{c}{2(d - 1)(d - 2)a^{2d-4}} - \frac{\Lambda}{(d - 1)(d - 2)} a^2 = 0. \tag{2.15}
\]
Equation (2.13) can be integrated to find \(a(r)\) and (2.14) can then be solved to find \(\phi(r)\). The detailed properties of these solutions will depend on the choice of \(\Lambda\) and \(c\). For \(c = 0\) one finds an ‘extremal’ instanton whereas for \(c \neq 0\) one finds ‘non-extremal’ wormhole solution. The cases of \(\Lambda = 0\), \(\Lambda < 0\) and \(\Lambda > 0\) correspond to (asymptotically) flat, anti-de Sitter and de Sitter space respectively. All the possible cases will be discussed in the following.

## 3 Instantons

For \(c = 0\), (2.13) can be solved and one finds the three solutions corresponding to flat, de Sitter and anti-de Sitter space respectively.

\[
\Lambda = 0 : \quad a^2(r) = r^2, \quad (3.1)
\]
\[
\Lambda > 0 : \quad a^2(r) = \frac{(d - 1)(d - 2)}{\Lambda} \sin^2 \left( \sqrt{\frac{\Lambda}{(d - 1)(d - 2)}} r \right), \quad (3.2)
\]
\[
\Lambda < 0 : \quad a^2(r) = \frac{(d - 1)(d - 2)}{|\Lambda|} \sinh^2 \left( \sqrt{\frac{|\Lambda|}{(d - 1)(d - 2)}} r \right). \quad (3.3)
\]

If \(c = 0\) in (2.14), the solution of the dilaton is of the form
\[
e^{\frac{b}{2}\phi} = \text{const} - \frac{|bq|}{2} \int \frac{dr}{a(r)^{d-1}}. \quad (3.4)
\]
3.1 Flat space

For $\Lambda = 0$, the metric (in the Einstein frame) is flat space. The dilaton (3.4) is then given by

$$e^{\frac{b}{2}\phi} = e^{\frac{b}{2}\phi_\infty} + \frac{|bq|}{2(d - 2)} \frac{1}{r^{d-2}},$$

(3.5)

where $\phi_\infty$ is the value of the dilaton at infinity. The fact that the equations of motion are equivalent to a first order equation is a hint that this solution is related to a Killing spinor equation. This implies that in a supergravity theory these solutions are BPS and preserve some supersymmetry. We also note that the dilaton diverges as $r \to 0$. However the action (2.4) of the instanton is finite and solely given by a contribution from infinity

$$S_{\text{inst}} = \frac{\text{Vol}(S_{d-1})|q|}{|b|} e^{-\frac{b}{2}\phi_\infty}.$$  

(3.6)

Such instantons are responsible for non-perturbative effects. The broken supersymmetries are related to fermionic zero modes and integration over fermionic zero modes induces higher dimensional terms in the effective action. Such terms are weighted by the instanton action (3.6). Often such effects can be attributed to Euclidean wrapped branes or D-instantons [18, 17, 15, 16].

3.2 Anti-de Sitter space

In this case the dilaton solution for the instanton in AdS space is given by

$$e^{\frac{b}{2}\phi} = \text{const} - \frac{|bq|}{2} \int \frac{dr}{a(r)^{d-1}}$$

$$= \text{const} - \frac{|bq|}{2} \left( \frac{|\Lambda|}{(d - 1)(d - 2)} \right)^{\frac{d-1}{2}} \int \frac{dr}{\sinh^{d-1} \left( \sqrt{\frac{|\Lambda|}{(d-1)(d-2)}} r \right)}.$$  

(3.7)

The behavior of the instanton is very similar to the one of flat space, in particular the action of the instanton is again given by a boundary term

$$S_{\text{inst}} = \frac{\text{Vol}(S_{d-1})|q|}{|b|} e^{-\frac{b}{2}\phi_\infty}.$$ 

(3.8)

In the calculation of the action of the above Euclidean gravitational instantons, one has to include the contribution of the gravitational action which diverges because of the infinite
volume of AdS. To render the action finite one needs to employ the counterterms subtraction method which was proposed for computing the boundary stress tensor associated with a gravitating system \cite{24,25,26,27}. The appearance of the logarithmically divergent contributions has a physical meaning. They are related to the conformal anomaly of the dual field theory living on the boundary. In some sense the AdS/CFT correspondence relates the infinite volume singularity in the AdS to the UV divergence in the CFT. This regularization is independent of the presence of an instanton and the instanton ($q$ dependent) part of the action is given by \eqref{eq:3.3}.

### 3.3 de Sitter space

In this case the dilaton solution for the instanton in dS space is given by

$$e^{\frac{b}{2}\phi} = \text{const} - \frac{|bq|}{2} \int \frac{dr}{a(r)^{d-1}}$$

$$= \text{const} - \frac{|bq|}{2} \left( \frac{\Lambda}{(d-1)(d-2)} \right)^{d-1} \int \frac{dr}{\sin^{d-1} \left( \sqrt{\frac{\Lambda}{(d-1)(d-2)}} r \right)}.$$  \quad (3.9)

Since Euclidean $d$-dimensional de Sitter space is simply the $d$-dimensional sphere the range of $r$ is $r \in [0, \pi \sqrt{\frac{(d-1)(d-2)}{\Lambda}}]$. The integral on the right hand side ranges between $-\infty$ and $+\infty$. Since $e^{\frac{b}{2}\phi}$ ranges between 0 and $+\infty$, the dilaton must become singular and therefore instantons do not exist in de Sitter space.

### 4 Wormholes

If $c \neq 0$ the equations become more complicated. As we shall see solutions in this case correspond to wormholes. This means that the metric factor $a(r)$ reaches a minimal value $a_0 > 0$ where $\partial_r a(r) = 0$, defining the neck of the wormhole. Here

$$\partial_r a = \pm \sqrt{1 + \frac{c}{2(d-1)(d-2)a^{2d-4}} - \frac{\Lambda}{(d-1)(d-2)} a^2}.$$ \quad (4.1)

The counterterms subtraction method of \cite{24} was employed in \cite{25} and explicit calculations have been performed for the gravitational action for our metric in the dimensions $d = 3, 4, 5, 6, 7$. Logarithmically divergent contributions have been obtained for $d = 3, 5, 7$. 2

\[ \]
Hence there is a sphere of non-zero minimal size. The dilaton can be determined from \( (2.14) \) and we get

\[
\int \frac{d\phi}{\sqrt{q^2 e^{-b\phi} + c}} = \pm \int \frac{dr}{a^{d-1}}. \tag{4.2}
\]

In the following we will solve these equations for all possible values of \( \Lambda \) and \( c \).

An Euclidean wormhole corresponds to a tunneling event between two asymptotic spaces. Alternatively, one can cut the wormhole in half along the minimal sphere. Such an Euclidean configuration provides the bounce for the creation of a baby universe. The continuation to Minkowski signature is consistent since the momentum components of the fields normal to the minimal sphere vanish.\(^3\) The Minkowskian evolution of the baby universe will be of FRW type.

### 4.1 Flat space

For \( \Lambda = 0 \), the space will be asymptotically flat. Recall from \((2.13)\) that

\[
1 - (a')^2 + \frac{c}{2(d - 1)(d - 2)} a^{2d-4} = 0. \tag{4.3}
\]

If \( c < 0 \), one has a minimal size sphere at \( a_0 = \left( \frac{2(d - 1)(d - 2)}{|c|} \right)^{\frac{1}{2d-4}} \), where \( \partial_r a = 0 \). This is the neck of the wormhole, connecting two asymptotically flat regions, located at \( r \to \pm \infty \).

Integrating the \( \phi \) equation \((2.14)\) gives

\[
\frac{1}{|c|^{1/2}|b|} \int \frac{d\tilde{\phi}}{e^{-\tilde{\phi}} - 1} = \pm \int \frac{da}{a^{d-1} \sqrt{1 - \frac{|c|}{2(d - 1)(d - 2)} \frac{1}{a^{2d-4}}}}, \tag{4.4}
\]

where we have defined

\[
e^{b\phi} = \frac{q^2}{|c|} e^{\tilde{\phi}}. \tag{4.5}
\]

Equation \((4.4)\) can be easily integrated. Using the fact that \( a \to \infty \) corresponds to \( \phi \to \phi_\infty \) one finds the following solution

\[
\arcsin \left( \sqrt{\frac{|c|}{q^2}} e^{\frac{b\phi(r)}{2}} \right) - \arcsin \left( \sqrt{\frac{|c|}{q^2}} e^{\frac{b\phi_\infty}{2}} \right) = \mp |b| \sqrt{\frac{d - 1}{2(d - 2)}} \arcsin \left( \sqrt{\frac{|c|}{2(d - 1)(d - 2)}} \frac{1}{a(r)^{d-2}} \right). \tag{4.6}
\]

\(^3\)For the axion it is convenient to dualize the to a \( d - 1 \) form whose indices are tangent to the minimal sphere.
As one approaches the neck of the wormhole \((a \to a_0)\), the argument of the arcsin on right hand side of (4.6) becomes one. The dilaton should be regular on the neck. This implies that there is a 'critical value' \(b_c\) of the dilaton coupling and for non-singular solutions we must have\(^4\)

\[ |b| < b_c = \sqrt{\frac{2(d-2)}{d-1}}. \tag{4.7} \]

Note that the integration constant \(c\) does not appear in (4.7). The Euclidean action of the wormhole is still given by a boundary term only, however in contrast to the instanton case the region of the neck of the wormhole does contribute.

\[ S_{\text{eucl}} = \frac{2}{|b|} \text{Vol}(S_{d-1}) \left( \sqrt{q^2 e^{-b \phi_\infty} + c} - \sqrt{q^2 e^{-b \phi_0} + c} \right). \tag{4.8} \]

In order to have a positive wormhole action (4.8) and a smooth limit to the instanton solution as \(|c| \to 0\) we have to choose the plus sign in (4.6).

Here \(\phi_0\) is the value of the dilaton at the neck and is given by

\[ e^{b \phi(r_0)} = \left| \frac{q}{c} \right|^\frac{d}{2} \sin \left( \arcsin \left( \sqrt{\frac{|c|}{q^2} e^{b \phi_\infty}} \right) + |b| \frac{\pi}{2} \sqrt{\frac{d-1}{2(d-2)}} \right). \tag{4.9} \]

The finiteness of the wormhole action depends on the value of \(b\). If \(|b| > b_c\) the Euclidean action is infinite and the wormholes are completely suppressed in the semi-classical approximation. The above analysis constitutes the generalization of the results known in the literature for four dimensions to \(d\) dimensions.

### 4.2 Anti de Sitter space

For \(c < 0\), the existence of the integral (2.14) implies the relation

\[ \frac{1}{\sqrt{|c||b|}} \int \frac{d\tilde{\phi}}{\sqrt{e^{-\tilde{\phi}} - 1}} = \pm \int da\frac{d^4-1}{\sqrt{1 - \frac{|c|}{2(d-1)(d-2)} a^{d-4}}} + \frac{|A|}{(d-1)(d-2)a^2}. \tag{4.10} \]

The above equation can be easily integrated in terms of elementary functions for the case of \(d = 3\). In this case we obtain

\(^4\)Note that in [12], the derivative of the dilaton was assumed to vanish at the throat of the wormhole. Moreover, our dilaton solution for the case of \(d = 4\), differs from that given in [19] though we agree on the critical value \(b_c\).
\[
\text{arcsin} \left( \sqrt{\frac{|c|}{q^2}} e^{\frac{b}{2} \phi(r)} \right) - \text{arcsin} \left( \sqrt{\frac{|c|}{q^2}} e^{\frac{b}{2} \phi_{\infty}} \right)
= \pm \frac{|b|}{2} \left( \text{arcsin} \left( \frac{-\frac{|c|}{2} + a^2(r)}{a^2(r) \sqrt{1 + \frac{|c| \Lambda}{2}}} \right) - \text{arcsin} \left( \frac{1}{\sqrt{1 + \frac{|c| \Lambda}{2}}} \right) \right). \quad (4.11)
\]

This wormhole solutions for certain dilaton couplings would not be finite if one turns on \( \Lambda \). The neck is at
\[
a_0^2 = \sqrt{1 + \frac{|\Lambda|}{2}} - 1. \quad (4.12)
\]

At the neck we obtain
\[
\text{arcsin} \left( \sqrt{\frac{|c|}{q^2}} e^{\frac{b}{2} \phi_0} \right) - \text{arcsin} \left( \sqrt{\frac{|c|}{q^2}} e^{\frac{b}{2} \phi_{\infty}} \right)
= \pm \frac{|b|}{2} \left( \frac{\pi}{2} + \text{arcsin} \left( \frac{1}{\sqrt{1 + \frac{|c| \Lambda}{2}}} \right) \right). \quad (4.13)
\]

In this case the critical value of \(|b| < b_c\) is given by
\[
b_c = 2 \left[ 1 + \frac{2}{\pi} \arcsin \left( \frac{1}{\sqrt{1 + \frac{|c| \Lambda}{2}}} \right) \right]^{-1}. \quad (4.14)
\]

Hence setting \( \Lambda \to 0 \) one gets \( b_c = 1 \) which is the correct flat space value (See (4.7) for \( d = 3 \)).

The choice of the negative sign in (4.11) leads to a positive action and smooth limit to the anti de Sitter instanton solution\(^5\). The range of \( a(r) \) is \([a_0, \infty]\) and as \( r \to \infty \) the metric approaches the euclidean AdS metric, hence the wormhole connects two asymptotically AdS spaces.

Note however that for non-zero \( \Lambda \) the value of \( b_c \) decreases. These wormhole solutions for certain dilaton couplings would not be finite if one turns on a negative cosmological

\(^5\)For the instanton solution in anti-de Sitter three dimensional space we have
\[
e^{\frac{b}{2} \phi(r)} - e^{\frac{b}{2} \phi_{\infty}} = \frac{|b|}{2} \sqrt{\frac{|\Lambda|}{2}} \left( \cosh \sqrt{\frac{|\Lambda|}{2}} r - 1 \right) \right)
\]
constant $\Lambda$. For $d = 4, 5$ the integral on the right hand side of (4.10) can still be solved in terms of elliptic integrals, however the results are not very illuminating, the general feature that there is a critical value of $b_c$ which is smaller than the flat space value persist. For $d > 5$ one can only solve the integrals numerically.

### 4.3 de Sitter space

In this section we will consider wormhole solutions for de Sitter spaces, i.e., for the case of positive cosmological constant. In this case the equation of the metric takes the form (for $\Lambda > 0$)

$$1 - (\partial_r a)^2 + \frac{c}{2(d-1)(d-2)} \frac{1}{a^{2d-4}} - \frac{\Lambda}{(d-1)(d-2)} a^2 = 0. \quad (4.15)$$

The instanton for which $c = 0$ was discussed in section 3.3. For $c \neq 0$, the dilaton is determined by the integral

$$\int \frac{d\phi}{\sqrt{q^2 e^{-b \phi} + c}} = \pm \int \frac{da}{a^{d-1} \sqrt{1 + \frac{c}{2(d-1)(d-2)} a^{2d-4} - \frac{\Lambda}{(d-1)(d-2)} a^2}}. \quad (4.16)$$

In what follows we will consider the three dimensional case for simplicity. It is clear that the two cases $c > 0$ and $c < 0$ are qualitatively different and will discuss both of them in turn. For $c < 0$, and for the case of $d = 3$, (4.13) can have either two or no values of $a$ where $\partial_r a = 0$, if

$$0 < |c\Lambda| < 2, \quad (4.17)$$

then we have two real zeros given by

$$a^2_{\pm} = 1 \pm \frac{1 - |c\Lambda|}{\Lambda}. \quad (4.18)$$

The solution for the dilaton is given by

$$\arcsin \sqrt{\frac{|c|}{q^2} e^{\frac{b}{2}(r)}} - \arcsin \sqrt{\frac{|c|}{q^2} e^{\frac{b}{2}r}} = \pm \frac{|b|}{2} \left( \arcsin \left( \frac{2a^2(r) - |c|}{2a^2(r) \sqrt{1 - \frac{|c\Lambda|}{2}}} \right) + \frac{\pi}{2} \right), \quad (4.19)$$

where $\phi \to \phi_\pm$ as $a \to a_\pm$. Note that a non-singular dilaton configuration is only possible if the dilaton coupling $b$ satisfies

$$|b| < b_c = 1. \quad (4.20)$$
Here we have only displayed the results for $d = 3$. For $d = 4$ and $d = 5$, the relevant integrals can be solved in terms of elliptic integrals, however the results are not very illuminating. For $d > 5$, the integrals can only be solved numerically.

The range of $a(r)$ is $a \in [a_-, a_+]$ and the topology of the wormhole is the one of a cylinder. A wormhole with opposite axion charge could be glued onto the first one. As pointed out in [12], the dilaton makes this impossible since it would have to have vanishing derivative at both necks in order to be continuous. This is only possible if the dilaton becomes singular.

Now for $c > 0$, there is only one value of $a$ where $\partial_r a = 0$,

$$a_0^2 = \frac{1 + \sqrt{1 + \frac{c\Lambda}{2}}}{\Lambda}, \quad (4.21)$$

hence the metric function has the range $a \in [0, a_0]$. The solution for the dilaton equation in this case reads

$$\arcsinh \left( \sqrt{\frac{c}{q^2}} e^{\frac{q}{2} \phi(r)} \right) - \arcsinh \left( \sqrt{\frac{c}{q^2}} e^{\frac{q}{2} \phi_0} \right) = \mp \frac{|b|}{2} \text{arcosh} \left( \frac{2a^2(r) + c}{2a^2(r) \sqrt{1 + \frac{c\Lambda}{2}}} \right). \quad (4.22)$$

where $\phi_0$ is the value of the dilaton at $a = a_0$. Note however that this solution contains a curvature singularity at $a = 0$. At this point $a(r) \sim r^{1/(d-1)}$ as $r \to 0$ and it is easy to show the Ricci scalar diverges. Since only non-singular solutions are admissible as semiclassical saddle points, the solution with $c > 0$ cannot be used as an Euclidean wormhole.

## 5 Axionic Solutions

The case of $b = 0$ is special since then there is no coupling of the axion to the dilaton and the dilaton can be consistently set to zero. The axion is still given by the equation

$$\partial_r \chi = \frac{q}{a^{d-1}}. \quad (5.1)$$

The role of the integration constant $c$ is now simply played by $-q^2$. The gravitational equations are equivalent to

$$(\partial_r a)^2 = 1 - \frac{1}{2(d-1)(d-2)} \frac{q^2}{a^{2d-4}} - \frac{\Lambda a^2}{(d-1)(d-2)}. \quad (5.2)$$
Here it is useful to perform a change of variable $r’ = a(r)$, and we obtain for the metric

$$ds^2 = \frac{1}{1 - \frac{1}{2} \frac{q^2}{(d-1)(d-2)} r^{2d-4}} - \frac{\Lambda r^2}{(d-1)(d-2)} dr^2 + r^2 d\Omega_{d-1}^2. \quad (5.3)$$

In these coordinates (5.1) can be solved

$$\chi = \text{const} \pm \int dr \frac{q}{r^{d-1}} \frac{1}{\sqrt{1 - \frac{1}{2} \frac{q^2}{(d-1)(d-2)} r^{2d-4} - \frac{\Lambda r^2}{(d-1)(d-2)}}}. \quad (5.4)$$

The integral (5.4) is very similar to the ones discussed in the previous section. For $\Lambda = 0$ the solution is

$$\chi - \chi_\infty = \mp \sqrt{\frac{2(d-1)}{(d-2)}} \arcsin \sqrt{\frac{1}{2} \frac{q^2}{(d-1)(d-2)} r^{2d-4}}. \quad (5.5)$$

For $\Lambda \neq 0$, explicit formulae can be obtained for $d = 3, 4, 5$ as discussed in section 4.2 and 4.3. Again only the case of $d = 3$ has a simple realization in terms of elementary functions. For example for $\Lambda < 0$ and $d = 3$ one finds

$$\chi - \chi_\infty = \pm \left( \arcsin \left( \frac{-\frac{q^2}{2} + r^2}{r^2 \sqrt{1 + \frac{|q^2\Lambda|}{2}}} \right) - \arcsin \left( \frac{1}{\sqrt{1 + \frac{|q^2\Lambda|}{2}}} \right) \right). \quad (5.6)$$

### 6 Discussion

In this paper we have discussed instanton and wormhole solutions in axion-dilaton gravity theories. Such theories are abundant in string theories and supergravities in various dimensions. Instanton solutions exist for flat and AdS spaces in arbitrary dimensions, however in dS space there are no non-singular instanton solutions.

The existence of wormholes depends on the dimensionality of spacetime and the dilaton coupling constant $b$. There is a critical value $b_c$ above which the instanton solution becomes singular and has an infinite action. Wormholes induce non-local interactions in the theory and there may be a connection between wormholes and non-local string theories found in [28, 29]. It would be interesting to see whether one can find an example in string theory where $b$ is in the range where non-singular wormholes exist.

In certain theories (e.g. in gauged supergravities), the dilaton can acquire a non-trivial potential,

$$S_m = \int d^d x \sqrt{-g} \left( R - \frac{1}{2} \partial_{\mu} \phi \partial^\mu \phi - \frac{1}{2} \epsilon^{\lambda\phi} \partial_{\mu} \chi \partial^\mu \chi - V(\phi) \right). \quad (6.1)$$
Using the ansatz for the metric given by (2.8), the dilaton equation of motion is given by

\[ \partial_r^2 \phi + (d - 1) \frac{\partial_r a}{a} \partial_r \phi + \frac{b q^2}{2 a^{2d-2}} e^{-b \phi} \frac{\partial V(\phi)}{\partial \phi} = 0, \]  

(6.2)

and it is not possible to find a simple integral like (2.14) which allowed for the decoupling of the metric and dilaton equations. Hence the equations in these cases are considerably more complicated. However, it may be possible to find first order differential equations as in the case for domain walls \[30,31\]. Also, it might be interesting to study these equations further and in particular investigate whether it is possible to stabilize the dilaton and get wormhole solutions which are more like axionic ones. We hope to report on these issues in a future publication.

Acknowledgments:

The work of M.G. was supported in part by DOE grant DE-FG02-91ER40655. M.G. gratefully acknowledges the hospitality of the Stanford Theory Group during the final stages of this work. We are grateful to U. Theis, S. Vandoren and K. Behrndt for pointing out an error in a previous version.

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