Fracture Roughness and Correlation Length in the Central Force Model

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We measure the roughness exponent and the correlation length exponent of a stress-weighted percolation process in the central force model in 2D. The roughness exponent is found to be \( \zeta = 0.75 \pm 0.03 \) and the correlation length exponent is found to be \( \nu = 1.7 \pm 0.3 \). This result supports a conjecture that the fracture roughness for large scales is controlled by a stress weighted percolation process, and the fracture roughness can be calculated from the correlation length exponent by \( \zeta = 2\nu/(1 + 2\nu) \). We also compare global and local measurements of the fracture roughness and do not find signs of anomalous scaling in the central force model.

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In the early eighties, it was observed that brittle fracture surfaces show self affinity. That is, such a surface will be statistically invariant if the in-plane length scales are changed by a factor \( \lambda \), and the out-of-plane length scale is changed by a factor \( \lambda^2 \), where \( \zeta \) is the roughness or Hurst exponent. In the early nineties, it was proposed that the roughness exponent is universal. This initiated a large effort to further investigate this phenomenon. An early review may be found in Ref. [4].

For large scales the universal value of the roughness exponent is believed to be about 0.80 for brittle fractures. There is also evidence that there may be a different value, 0.5, for small scales, with a well-defined crossover length scale separating these two regimes [5].

Even though the evidence for one or more universal roughness exponents is mounting, the mechanisms that may be responsible are still not known. There have been some proposals for mechanisms, see [6, 7, 8, 9]. It is argued that the breakdown process when the material localizes, and by combining the localization length with the percolation-like description, a relation may be set up between the roughness exponent \( \zeta \) and the correlation length exponent \( \nu \) of the correlated percolation process,

\[
\zeta = \frac{2\nu}{1 + 2\nu}.
\]

This relation has been tested on the two and three-dimensional fuse model [10, 11]. The fuse model consists of a regular network of electrical fuses, each having a burn-out threshold drawn from some spatially uncorrelated statistical distribution [12, 13]. The model has been used quite extensively to study fracture roughness [6, 14, 15, 16, 17]. In this Letter, we test Eq. (1) in the central-force breakdown model, first studied in Ref. [18]. This is a model that is much closer to the fracture problem in that its response is elastic — in the continuum limit it maps onto ordinary Lamé elasticity. It consists of a regular lattice whose bonds are elastic springs free to rotate around the nodes they are connected to. The force on a spring between nodes \( r_i \) and \( r_j \) given as

\[
f_{ij} = \sigma_{ij} (r_j - r_i) \cdot n_{ij} \cdot n_{ij} ,
\]

where \( \sigma_{ij} \) is the spring constant for the spring and \( n \) is the axis vector of the spring. Another elastic model that has been studied is the beam model [20, 21].

Measuring both \( \zeta \) and \( \nu \) independently in the two-dimensional central-force breakdown model, we will be able to test Eq. (1) for this case. For pure central-force rigidity percolation in two dimensions, Moukarzel and Duxbury [22] found \( \nu = 1.16 \) using the pebble game algorithm. As is the case for the fuse model [11, 23], there is no reason whatsoever that the \( \nu \) of the breakdown process should be equal to the percolation correlation exponent. In the fuse model, one finds \( \nu = 1.56 \) in two dimensions to be compared to \( \nu = 4/3 \) in two-dimensional percolation, and \( \nu = 0.83 \pm 0.04 \) in the three-dimensional fuse model, to be compared to \( \nu = 0.88 \) for three-dimensional percolation. As we shall see, we find \( \nu = 1.7 \pm 0.3 \) for the two-dimensional central-force breakdown model.

We simulated fractures in 2D trigonal central force lattice with periodic boundaries in the \( x \)-direction and applied tension in the \( y \)-direction. Each spring in the lattice was assigned a threshold \( t_i \) at random from a power law distribution \( \pi(t) \propto t^{\beta - 1} \) which gives \( t_i = r_i^D \) where \( r_i \) is a random number between 0 and 1 and \( D = \beta - 1 \). Values of \( D \) close to zero gives narrow disorders. The disorder becomes broader when \( D \) increases. We used \( D = 0.7 \) for the narrow disorder and \( D = 20 \) for the broad disorder. For each realisation, a strain of unity is applied in the vertical direction to the boundary lattice points and the central force equations are solved iteratively with the conjugate gradient method [24]. The lattice is periodic.
in the horizontal direction. The spring with \( f_i/t_i \) is identified and the strain is increased (decreased) until \( f_i/t_i = 1 \) and the spring constant for this spring is set to zero. The central force equations are solved again with this new configuration and springs are removed until the elasticity module of the system is zero.

When a sample is fractured, elasticity module equal zero, the fracture surface is created in the dual lattice for the triangular lattice. Before the fracture width is zero, the fracture surface is created in the dual lattice. The central force equations are solved again with this new configuration and springs are removed until the elasticity module of the system is zero. Assume that the fracture process has produced a fracture of length \( \eta < L \). The roughness measured over a window of size \( l < \eta \) will be

\[
W(l) = (\langle (y(x))^2 \rangle_l - \langle y(x) \rangle^2_l)^{1/2} \propto L^\zeta, \quad (3)
\]

and the local fracture roughness exponent with the scaling of the local width with window size

\[
w(l) = (\langle (y(x))^2 \rangle_l - \langle y(x) \rangle^2_l)^{1/2} \propto L^{\zeta_{loc}}, \quad (4)
\]

and the average wavelet coefficients (AWC) method \[27\]

\[
W[y](a) \propto a^{\zeta_{loc}+1/2}. \quad (5)
\]

The simulations for measuring the fracture roughness was done with a narrow disorder of \( D = 0.70 \). For the global exponent we measured \( \zeta = 0.75 \pm 0.03 \).

We apply the AWC method and the local window method to fracture profiles of lattice size \( L = 256 \). We measured a local fracture roughness of \( \zeta_{loc} = 0.72 \pm 0.02 \) for the AWC method and \( \zeta_{loc} = 0.70 \pm 0.02 \) for the local window method, see Figs. 2 and 3. Both of these methods underestimate the roughness exponent, with around 0.03 for the AWC method and around 0.05 for the local window method. This was checked by creating artificial surfaces with known roughnesses using a wavelet method for generating the surfaces. \[23\] For the local window method the deviations we found for the measured roughness exponent is consistent with the one found by Schmittbuhl et al. \[24\].

We therefore find the difference between \( \zeta \) and \( \zeta_{loc} \) to be close to zero, which is a sign that anomalous scaling, as found by Zapperi et al. \[30\] in the 2D fuse model, is not found in the 2D central force model, and that there is only one roughness exponent, \( \zeta = 0.75 \pm 0.03 \), for the central force model.

Anomalous scaling may be interpreted in the following way. Assume that the fracture process has produced a fracture of length \( \eta < L \). The roughness measured over a window of size \( l < \eta \) will be

\[
w(l) = A(\eta) L_{k(loc)}, \quad (6)
\]

where the prefactor \( A(\eta) \) depends on \( \eta \) as a power law,

\[
A(\eta) \sim \eta^{\zeta - \zeta_{loc}}, \quad (7)
\]

so that \( w(l = \eta) \sim \eta^\delta \). Imagine we now fix a window size \( l < \eta \) and follow the fracture roughness \( w(l) \) as \( \eta \) grows.

The roughness within the window changes because the prefactor \[6\] changes. This is not possible unless the cause of the anomalous scaling has its origin in statistical non-stationarity \[31\], that is, the larger \( \eta \) is, the larger the sample over which the window \( l \) is applied to, and this will change the average on which the prefactor is built. Using the wavelet-based method proposed in Ref.
breaks down in the limit $L \to \infty$, and $p_{\text{eff}}$ is effective critical density at which the network breaks down for a finite $L$. This implies that the fluctuations in the density of broken springs at fracture will scale as

$$
\sigma(p_{\text{eff}}) = \langle (p_{\text{eff}}^2) \rangle - \langle p_{\text{eff}} \rangle^2 \propto L^{-1/\nu}.
$$

Using a disorder of $D = 20$ which gives flat damage density profiles, indicating that there is no localization in the fracture process, we obtain $1/\nu = 0.57 \pm 0.10$, see Fig. 4. This corresponds to a value for $\nu$ equal to $1.7 \pm 0.3$, which by Eq. (11) gives $\zeta \in (0.73, 0.80)$, which is consistent with the direct measurement of $\zeta$. The value we found for $\nu$ is different from the rigidity percolation value of 1.16 found by Moukarzel and Duxbury [22].

To conclude: We have studied the central force breakdown model to test Eq. (11) and found support for it measuring the roughness exponent of $\zeta = 0.75 \pm 0.03$. The difference in global and local roughness exponent is small and a check for anomalous scaling of the wavelet coefficients show no sign of such. The correlation length exponent for breakdown process was found to be $\nu = 1.7 \pm 0.3$ giving consistent $\zeta$ values using Eq. (11). The value for $\zeta$ and $\nu$ are close to those of the fuse model, suggesting that these exponents might be similar for these models.

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