How to relate the oscillator and Coulomb systems on spheres and pseudospheres?

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We show that the oscillators on a sphere and pseudosphere are related, by the so-called Bohlin transformation, with the Coulomb systems on the pseudosphere: the even states of an oscillator yields the conventional Coulomb system on pseudosphere, while the odd states yield the Coulomb system on pseudosphere in the presence of magnetic flux tube generating half spin. In the higher dimensions the oscillator and Coulomb(-like) systems are connected in the similar way. In particular, applying the Kustaanheimo-Stiefel transformation to the oscillators on sphere and pseudosphere, we obtained the pseudospherical generalization of MIC-Kepler problem describing three-dimensional charge-dyon system.

I. INTRODUCTION

The $(d-$dimensional) oscillator and Coulomb systems are most known representatives of mechanical systems possessing hidden symmetries which define the $su(d)$ symmetry algebra for the oscillator, and $so(d + 1)$ for the Coulomb system. The hidden symmetry has a very transparent meaning in the case of oscillator, while in the case of the Coulomb system it has a more complicated interpretation in terms of geodesic flows of a $d$-dimensional sphere. On the other hand, the transformation $r = R^2$ converts the $(p + 1)$-dimensional radial Coulomb problem in $2p$-dimensional radial oscillator one, both in classical and quantum cases, where the $r$ and $R$ denote the radial coordinates of Coulomb and oscillator systems, respectively (see, e.g. [1]). In three distinguished cases, $p = 1, 2, 4$, one can establish the complete correspondence between the Coulomb and the oscillator systems, by using the so-called Bohlin (or Levi-Civita) [2], Kustaanheimo-Stiefel [3] and Hurwitz [4] transformations,

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respectively. This transformations assume the reduction of the oscillator system by the action of $Z_2$, $U(1)$, $SU(2)$ groups, respectively, and yield the Coulomb-like systems specified by the presence of monopoles \[5, 6, 7\]. On the other hand, the oscillator and Coulomb systems admit the generalizations to a $d$–dimensional sphere and two-sheet hyperboloid (pseudosphere) with radius $R_0$ given by the potentials \[8, 9\]

$$\begin{align*}
V_{osc} &= \frac{\alpha^2 R_0^2}{2} \frac{x^2}{x_{d+1}^2}, \\
V_C &= -\frac{\gamma}{R_0} \frac{x_{d+1}}{|x|},
\end{align*}$$  

(1)

where $x, x_{d+1}$ are the (pseudo)Euclidean coordinates of ambient space $\mathbb{R}^{d+1}(\mathbb{R}_{d+1})$: $\epsilon x^2 + x_{d+1}^2 = R_0^2, \quad \epsilon = \pm 1$. The $\epsilon = 1$ corresponds to the sphere, and $\epsilon = -1$ to the pseudosphere. These systems possess nonlinear hidden symmetries providing them with the properties similar to those of conventional oscillator and Coulomb systems and have been investigated from many viewpoints (see, e. g. \[10\] and refs therein).

How to relate the oscillator and Coulomb systems on the spheres and pseudospheres?

Recently this problem was considered in Refs.\[11\], where the oscillator and Coulomb systems on spheres were related by some complicated mappings containing the transitions to imaginary coordinates. The geometrical origin of these mapping was not clarified there, as well as the reductions to the Coulomb-like systems with the monopoles, and the relations of the motion constants responsible for hidden symmetries, were not considered there. In our recent paper with G.Pogosyan \[12\] we established the transparent correspondence between oscillator and Coulomb systems on (pseudo)spheres for the simplest, two-dimensional, case ($p = 1$). We have shown that, in the stereographic projection, the conventional Bohlin transformation relates the two-dimensional oscillator on the (pseudo)sphere with the Coulomb systems on pseudosphere, as well as those interacting with specific external magnetic fields. This simple construction allows immediately connect the motion constants defining the hidden symmetry of the systems under consideration, as well as to clarify the mappings suggested in \[11\]. This construction can be straightly used for higher-dimensional cases ($p = 2, 4$), subject to obtain the the pseudospherical analogs of the known Coulomb-like systems, specified by the presence of monopoles: the so-called MIC-Kepler \[13\] and $SU(2)$ Kepler \[7\] problems.

In present paper we give a detailed description of this construction for the $p = 1$ case corresponding to the Bohlin transformation (Section 2), for the $p = 2$ case which corresponds to the Kustaanheimo-Stiefel one (Section 3) and discuss the $p = 4$ case corresponding to Hurwitz transformation (Section 4).
II. THE BOHLIN TRANSFORMATION

Let us introduce the complex coordinate $z$ parameterizing the sphere by the complex projective plane $\mathbb{C}P^1$ and the two-sheeted hyperboloid by the Poincaré disks $\mathcal{L}$:

$$\mathbf{x} \equiv x_1 + ix_2 = R_0 \frac{2z}{1 + \epsilon z \bar{z}}, \quad x_3 = R_0 \frac{1 - \epsilon z \bar{z}}{1 + \epsilon z \bar{z}}. \tag{2}$$

In these terms the metric takes the Kähler form

$$ds^2 = R_0^2 \frac{4dzd\bar{z}}{(1 + \epsilon z \bar{z})^2}, \tag{3}$$

while $R_0 x_k$ define the isometries of the Kahler structure ($su(2)$ if $\epsilon = 1$ and $su(1,1)$ if $\epsilon = -1$). The lower hemisphere and the lower sheet of the hyperboloid are parametrized by the unit disk $|z| < 1$, while the upper hemisphere and the upper sheet of hyperboloid, by its outside, and transform into each other by the inversion $z \to 1/z$. Since in the $R_0 \to \infty$ limit the lower hemisphere (the lower sheet of hyperboloid) converts into the whole two-dimensional plane, for the correspondence with conventional oscillator and Coulomb problems, we have to restrict ourselves by those defined on the lower hemisphere and the lower sheet of hyperboloid (pseudosphere).

Let us equip the oscillator’s phase space $T^* \mathbb{C}P^1 (T^* \mathcal{L})$ by the symplectic structure

$$\omega = d\pi \wedge dz + d\bar{\pi} \wedge d\bar{z} \tag{4}$$

and the rotation generators (defining $su(2)$ algebra if $\epsilon = 1$ and $su(1,1)$ if $\epsilon = -1$)

$$J \equiv \frac{iJ_1 - J_2}{2} = \pi + \epsilon \bar{z}\bar{\pi}, \quad J \equiv \frac{\epsilon J_3}{2} = i(z\pi - \bar{z}\bar{\pi}). \tag{5}$$

In these terms, the oscillator’s Hamiltonian is given by the expression

$$H^\epsilon_{osc}(\pi, \bar{\pi}, z, \bar{z}) = \frac{J\bar{J} + \epsilon J^2}{2R_0^2} + \left(\frac{\alpha^2 R_0^2 x^2}{2x^2_3} - \frac{(1 + \epsilon z \bar{z})^2 \pi \bar{\pi}}{2R_0^2} + \frac{2\alpha^2 R_0^2 z \bar{z}}{(1 - \epsilon z \bar{z})^2}\right). \tag{6}$$

The hidden symmetry is given by the complex (or vectorial) constant of motion

$$I = I_1 + iI_2 = \frac{J^2}{2R_0^2} + \frac{\alpha^2 R_0^2 x^2}{2x^2_3}, \tag{7}$$

which defines, together with $J$ and $H_{osc}$, the cubic algebra

$$\{I, J\} = 2iI, \quad \{\bar{I}, I\} = 4i \left(\alpha^2 J + \frac{\epsilon JH_{osc}}{R_0^2} - \frac{J^3}{2R_0^2}\right). \tag{8}$$
The energy surface of the oscillator on the (pseudo)sphere $H^\epsilon_{osc} = E$ reads

\[
\frac{1 - (z\bar{z})^2}{2R_0^4} + 2 \left( \alpha^2 + \frac{E}{E^2} \right) z\bar{z} = \frac{E}{R_0^2} \left( 1 + (z\bar{z})^2 \right). \tag{9}
\]

Now, performing the canonical Bohlin transformation\(^2\)

\[
w = z^2, \quad p = \frac{\pi}{2z}, \tag{10}
\]

we convert the energy surface of the oscillator (9) onto the one of the Coulomb system on the pseudosphere:

\[
\frac{(1 - w\bar{w})^2 p\bar{p}}{2r_0^2} - \frac{\gamma}{r_0} \frac{1 + w\bar{w}}{|w|} = E_C, \tag{11}
\]

where

\[
r_0 = R_0^2, \quad \gamma = \frac{E}{2}, \quad -2E_C = \alpha^2 + \epsilon \frac{E}{r_0}. \tag{12}
\]

The constants of motion of the oscillators, $J$ and $I$ (which are equal on the energy surfaces (9)) converted, respectively into the doubled angular momentum and the doubled Runge-Lenz vector of the Coulomb system

\[
J \to 2J_C, \quad I \to 2A, \quad A = -\frac{i J_C J_C}{r_0} + \gamma \frac{\bar{x}_C}{|x_C|}, \tag{13}
\]

where $J_C, J_C, x_C$ denote the rotation generators and the pseudo-Euclidean coordinates of the Coulomb system.

It is easy to obtain from (13) the symmetry algebra of the reduced system

\[
\{ A, J \} = i A, \quad \{ \bar{A}, A \} = -4i \left( H_C + \frac{J_C^2}{r_0^2} \right) J_C. \tag{14}
\]

\textit{Hence, the Bohlin transformation of the classical isotropic oscillator on the (pseudo)sphere yields the classical Coulomb problem on the pseudosphere.}

The quantum-mechanical counterpart of the energy surface (9) is the Schrödinger equation

\[
H^\epsilon_{osc}(\alpha, R_0|\pi, \bar{\pi}, z, \bar{z})\Psi(z, \bar{z}) = E\Psi(z, \bar{z}), \tag{15}
\]

with the quantum Hamiltonian defined (due to the two-dimensional origin of the system) by the expression (6), where $\pi, \bar{\pi}$ are the momenta operators (hereafter we assume $\hbar = 1$)

\[
\pi = -i \frac{\partial}{\partial z}, \quad \bar{\pi} = -i \frac{\partial}{\partial \bar{z}}. \tag{16}
\]
The energy spectrum of this system is given by the expression (see e.g. [10] and refs therein)

\[ E = \tilde{\alpha}(N+1) + \epsilon \left( \frac{N+1}{2R_0^2} \right)^2, \quad N = 2n_r + |M|, \quad n_r = 0, 1, \ldots \]  \hspace{1cm} (17)

where \( \tilde{\alpha} = \sqrt{\alpha^2 + 1/(4R_0^4)} \), \( M \) is the eigenvalue of \( J \), \( N \) is the principal quantum number, \( n_r \) is the radial quantum number,

\[ |M|, N = 1, \ldots, N_{\text{max}} = \begin{cases} \infty, & \text{if } \epsilon = 1 \\ [2\tilde{\alpha}R_0^2] - 1, & \text{if } \epsilon = -1 \end{cases} \]  \hspace{1cm} (18)

So, the number of levels in the energy spectrum of the oscillator is infinite on the sphere and finite on the pseudosphere.

The quantum-mechanical correspondence between oscillator and Coulomb systems is more complicated, because the Bohlin transformation ([1]) maps the \( z \)-plane into the two-sheeted Riemann surface, since \( \arg w \in [0, 4\pi] \). Thus, we have to supply the quantum-mechanical Bohlin transformation with the reduction by the \( Z_2 \) group action, choosing either even (\( \sigma = 0 \)) or odd (\( \sigma = 1/2 \)) wave functions

\[ \Psi_\sigma(z, \bar{z}) = \psi_\sigma(z^2, \bar{z}^2) \left( \frac{z}{\bar{z}} \right)^{2\sigma} : \psi_\sigma(|w|, \arg w + 2\pi) = \psi_\sigma(|w|, \arg w). \]  \hspace{1cm} (19)

This implies that the range of definition of \( w \) can be restricted, without loss of generality, to \( \arg w \in [0, 2\pi] \). In that case, the resulting system is the Coulomb problem on the hyperboloid given by the Schrödinger equation

\[ H_C(\gamma, r|p_\sigma, \bar{p}_\sigma, w, \bar{w})\psi_\sigma = \mathcal{E}_C\psi_\sigma \]  \hspace{1cm} (20)

where \( \gamma, \mathcal{E}_C, r \) are given by ([12]), and the momenta operators are of the form

\[ p_\sigma = -i \frac{\partial}{\partial w} - \frac{\sigma}{iw}, \quad \bar{p}_\sigma = -i \frac{\partial}{\partial \bar{w}} + \frac{\sigma}{iw}. \]  \hspace{1cm} (21)

Hence, the resulting Coulomb system includes the interaction with the magnetic vortex (an infinitely thin solenoid) with the magnetic flux \( \pi \sigma \) and zero strength \( \text{rot} \sigma/w = 0 \). Such a composites are typical representatives of the anyonic systems with the spin \( \sigma \). So, we get a conventional 2d Coulomb problem on the hyperboloid at \( \sigma = 0 \) and those with half spin generated by the magnetic flux, at \( \sigma = 1/2 \). Taking into account the relations ([12]), one can rewrite the oscillator’s energy spectrum ([17]) as follows

\[ \sqrt{\frac{1}{4r_0^2} - \epsilon \frac{2\gamma}{r_0} - 2\mathcal{E}_C} = \frac{2\gamma}{N + 1} - \epsilon \frac{N + 1}{2r_0}. \]  \hspace{1cm} (22)
From this expression one can easily obtain the energy spectrum of the reduced system on the pseudosphere

\[ \mathcal{E}_C = -\frac{N_\sigma(N_\sigma + 1)}{2r_0^2} - \frac{\gamma^2}{2(N_\sigma + 1/2)^2}, \]  

(23)

where

\[ N_\sigma = n_r + m_\sigma, \quad m_\sigma = M/2, \quad n_r, m_\sigma - \sigma, N_\sigma - \sigma = 0, 1, \ldots, N_{\sigma}^{\text{max}} - \sigma. \]  

(24)

Here \( m_\sigma \) denotes the eigenvalue of the angular momentum of the reduced system, and \( n_r \) is the radial quantum number of the initial (and reduced) system. Notice, that the magnetic vortex shifts the energy levels of the two-dimensional Coulomb system which is nothing else than the reflection of Aharonov-Bohm effect.

It is seen, that the whole spectrum of the oscillator on pseudosphere (\( \epsilon = -1 \)) transforms in the spectra of the constructed Coulomb systems on the pseudosphere, while for the oscillator on the sphere (\( \epsilon = 1 \)) the positivity of l. h. s. of (22) restrict the admissible values of \( N_\sigma \). So, only the part of the spectrum of the oscillator on the sphere transforms into the spectrum of Coulomb system. Hence, in both cases we get the same result

\[ N_{\sigma}^{\text{max}} = \left[ \sqrt{r_0^2 \gamma} - (1/2 + \sigma) \right]. \]  

(25)

### III. KUSTAANHEIMO-STIEFEL TRANSFORMATION

It is easy to see that the \( 2p \)-dimensional oscillator on (pseudo)sphere can be connected with the \( (p + 1) \)-dimensional Coulomb-like systems on pseudosphere likewise in the higher dimensions \( p = 2, 4 \). Indeed, in stereographic coordinates, the oscillator on \( 2p \)-dimensional (pseudo)sphere is described by the Hamiltonian system given by [4], [5], where the following replacement is performed \( (z, \pi) \rightarrow (z^a, \pi_a), a = 1, \ldots, p \) with the summation over these indices. Consequently, the oscillator’s energy surfaces are of the form [4]. Further reduction to the \( (p + 1) \)-dimensional Coulomb-like system on pseudosphere must be similarly followed in the corresponding reduction in the flat case [6], [7]. Since \( |\mathbf{u}| = z\bar{z} \) in all three cases, we can interpret \( \mathbf{u} \) as the stereographic coordinates of the reduced system, consequently interpreting the last one as the Coulomb-like system on \( (p + 1) \)-dimensional pseudosphere.

For example, if \( p = 2 \), we should reduce the four-dimensional oscillator by the Hamiltonian action of \( U(1) \) group given by the generator

\[ J = i(z\pi - \bar{z}\bar{\pi}). \]
For this purpose, we have to fix the level surface

\[ J = 2s \]  

(26)

and factorize it by the $U(1)$-Hamiltonian flow, choosing six $U(1)$-invariant stereographic coordinates in the form of conventional Kustaanheimo-Stiefel transformation \[3, 8\]

\[ \mathbf{u} = z \vec{\mathbf{\sigma}} \tilde{z}, \quad \mathbf{p} = \frac{z \vec{\mathbf{\sigma}} \pi + \pi \vec{\mathbf{\sigma}} \tilde{z}}{2(z \tilde{z})}, \]  

(27)

where $\mathbf{\sigma}$ are Pauli matrices.

As a result, the reduced symplectic structure reads

\[ d\mathbf{p} \wedge du + s \frac{(\mathbf{u} \times du) \wedge du}{|\mathbf{u}|^3}, \]  

(28)

the oscillator’s energy surface takes the form

\[ \frac{(1 - \mathbf{u}^2)^2}{8r_0^2}(\mathbf{p}^2 + \frac{s^2}{u_2}) - \frac{\gamma}{r_0} \frac{1 + \mathbf{u}^2}{2|\mathbf{u}|} = E_C, \]  

(29)

where $\mathbf{u}$ denote the stereographic coordinates of three-dimensional pseudosphere, while $r_0$, $\gamma$, $E_C$ are defined by the expressions (12).

So, we get the energy surface of the pseudospherical analog of a Coulomb-like system describing the interaction of two non-relativistic dyons, which was proposed in 13 and is known as the MIC-Kepler system.

In the coordinates of ambient space the potential of pseudospherical MIC-Kepler system looks as follows

\[ V_{MIC} = \frac{s^2}{r_0^2} \left( \frac{x^2}{2|x|^2} - 2 \right) - \frac{\gamma}{r_0} \frac{x^2}{|x|} \]  

(30)

To quantize the system, we should replace the equations (9),(26) by the following spectral problem

\[ \hat{H}_{osc}(\pi, \bar{\pi}, z, \bar{z})\Psi(z, \bar{z}) = E_{osc}\Psi(z, \bar{z}), \quad \hat{J}_0(\pi, \bar{\pi}, z, \bar{z})\Psi(z, \bar{z}) = 2s\Psi(z, \bar{z}) \]  

(31)

where the momenta $\pi_\alpha, \bar{\pi}_\alpha$ are replaced by the operators

\[ \pi_\alpha = -i \frac{\partial}{\partial z^\alpha}, \quad \bar{\pi}_\alpha = -i \frac{\partial}{\partial \bar{z}^\alpha}, \]  

(32)

and the appropriate ordering in the Hamiltonian is assumed.
The second equation in the \((31)\) can be resolved by the substitution of the anzats
\[
\Psi_s(z, \bar{z}) = \psi_s(u)e^{is\lambda} \quad [\hat{J}_0 \lambda] = i, \quad \lambda = is \log \frac{z^1}{\bar{z}^1},
\]
which reduces the first equation in \((31)\) (i.e., the oscillator’s Schrödinger equation) to those corresponding to the generalized MIC-Kepler system \((29)\), where
\[
\hat{p}_s = e^{-is\lambda} \hat{p} e^{is\lambda} = -i \frac{\partial}{\partial u} - sA(u),
\]
with \(A(u)\) being the vector potential of Dirac’s monopole with singularity directed along axes \(u_3\), and \(\hat{p}\) be defined by the second expression in \((27)\), where \(\pi, \bar{\pi}\) are given by operators \((32)\) placed at right.

The requirement that \(\Psi\) to be single-valued wave function, leads \(s\) to be integer or half-integer, i.e. the Dirac’s quantization condition.

Solving the Schrödinger equation, one gets the oscillator’s energy spectrum
\[
E = \tilde{\alpha}(N + 2) + \epsilon \frac{(N + 2)^2 - 2}{2R_0^2}, \quad N = 2n_r + |L|,
\]
where \(\tilde{\alpha} = \sqrt{\alpha^2 + 1/(4R_0^2)}\), \(L\) is the eigenvalue of complete angular momentum, \(N\) is the principal quantum number, \(n_r\) is the radial quantum number,
\[
N_{\text{max}} = \left\{ \begin{array}{ll}
\infty, & \text{if } \epsilon = 1 \\
\tilde{\alpha} R_0^2 \left( 1 + \sqrt{1 + 2/(\tilde{\alpha} R_0^2)^2} \right) - 2, & \text{if } \epsilon = -1
\end{array} \right.
\]

Completely similar to the previous case, we can get from this expression the energy spectrum of the MIC-Kepler system on pseudosphere:
\[
E_C = -\frac{(n_r + |L_s|)(n_r + |L_s| + 2)}{2r_0^2} - \frac{\gamma^2}{2(n_r + |L_s| + 1)^2},
\]
where
\[
l_s = L/2, \quad |l_s|, n_r + |l_s| = |s|, |s| + 1, \ldots, N_{s,\text{max}}.
\]
It is convenient to introduce the new quantum number

\[ k \equiv n_r + |L_s| - |s|, \quad k = 0, 1, \ldots, N_s^{\text{max}} - |s|, \]

and re-write the expression (37) as follows

\[ \mathcal{E}_C = -\frac{(k + |s|)(k + |s| + 2)}{2r_0^2} - \frac{\gamma^2}{2(k + |s| + 1)^2}. \] (39)

It is seen, that the degeneracy of the reduced system the same, as in the usual MIC-Kepler problem \[6\], viz \( k(k + |s| - 1) \).

It is pleasure to notice, that the spherical generalization of MIC-Kepler system has also been presented on the Colloquium, which was constructed by V. Gritsev, Yu. Kurochkin and V. Otchik \[14\].

IV. DISCUSSION: THE HURWITZ TRANSFORMATION

We have shown, that applying the standard Bohlin/Kustaanheimo-Stiefel transformations to the stereographic (conformal-flat) coordinates of the two-/four-dimensional oscillators on sphere and pseudosphere yield the pseudospherical two-dimensional Coulomb and (three-dimensional) MIC-Kepler systems, respectively. It is obvious, from above-presented consideration, that the relation of eight-dimensional oscillator on (pseudo)sphere and of the pseudospherical analog of the so-called \( SU(2) \) Kepler (or Yang-Coulomb) system \[7\] would be completely similar to the mentioned cases.

For establishing such a connection (and constructing the pseudospherical \( SU(2) \) Kepler system) we should perform the Hamiltonian reduction of the eight-dimensional oscillator by the \( SU(2) \) group action action

\[ z^a \rightarrow z^a g, \quad g\bar{g} = 1, \quad g \in \mathbb{H}, \quad z^a \in \mathbb{H}^2 \] (40)

where \( z^1, z^2 \) are quaternions, parameterizing stereographic coordinates of eight dimensional (pseudo)sphere. The spatial stereographic coordinates of the reduced system should be chosen in the form of standard Hurwitz transformation \[4,7\]

\[ u = 2z_1 \bar{z}_2, \quad u_5 = z_1 \bar{z}_1 - z_2 \bar{z}_2, \quad u \in \mathbb{H}, \quad u_5 \in \mathbb{R} \] (41)
and completed with the conjugated momenta and isospinning coordinates as well. The potential of the pseudospherical SU(2) Kepler system would be of the form similar to the MIC-Kepler one,

$$V_{SU(2)-Kepler} = \frac{j(j+1)}{r_0^2} \left( \frac{x_6^2}{2|x|^2} - 2 \right) - \frac{\gamma}{r_0} \frac{x_6}{2|x|} \quad (42)$$

where \((x, x_6)\) denote the coordinates the ambient space of five-dimensional pseudosphere, \(j(j+1)\) is the eigenvalue of operator \(J_i^2\) defining the SU(2) group action \([10]\), while \(r_0, \gamma\), are given by the expressions \([12]\). The kinetic term of the Hamiltonian would include the interaction with the vector potential of five-dimensional SU(2) monopole \([15]\).

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