Deepening the Relationship between SEFE and C-Planarity

Patrizio Angelini and Giordano Da Lozzo

Department of Engineering, Roma Tre University, Italy
{angelini,dalozzo}@dia.uniroma3.it

Abstract. In this paper we deepen the understanding of the connection between two long-standing Graph Drawing open problems, that is, Simultaneous Embedding with Fixed Edges (SEFE) and Clustered Planarity (C-PLANARITY). In his GD’12 paper Marcus Schaefer presented a reduction from C-PLANARITY to SEFE of two planar graphs (SEFE-2). We prove that a reduction exists also in the opposite direction, if we consider instances of SEFE-2 in which the intersection graph is connected. We pose as an open question whether the two problems are polynomial-time equivalent.

1 Introduction

In recent years the problem of displaying together multiple relationships among the same set of entities has turned into a central subject of research in Graph Drawing and Visualization. In this context, the two major paradigms that held the stage are the simultaneous embedding of graphs, in which the relationships are described by means of different sets of edges among the same set of vertices, and the visualization of clustered graphs, in which the relationships are described by means of a set of edges and of a cluster hierarchy grouping together vertices with semantic affinities.

We study the connection between two problems adhering to such paradigms, Simultaneous Embedding with Fixed Edges (SEFE) and Clustered Planarity (C-PLANARITY). Given \( k \) graphs \( G_1(V, E_1), \ldots, G_k(V, E_k) \) the SEFE-\( k \) problem asks whether there exist \( k \) planar drawings \( \Gamma_i \) of \( G_i \), with \( i = 1, \ldots, k \), such that: (i) any vertex \( v \in V \) is mapped to the same point in any \( \Gamma_i \); (ii) any edge \( e \in E_i \cap E_j \) is mapped to the same curve in \( \Gamma_i \) and \( \Gamma_j \) (see [3] for a comprehensive survey on this topic). Given a graph \( G(V, E) \) and a cluster hierarchy over \( V \), the C-PLANARITY problem asks whether a planar drawing of \( G \) exists such that each cluster can be drawn as a simple region enclosing all and only the vertices belonging to it without introducing unnecessary intersections involving clusters and edges (see [54]).

Due to their practical relevance and their theoretical appealing, these problems have attracted a great deal of effort in the research community. However, despite several restricted cases have been successfully settled, the question regarding the computational complexity of the original problems keeps being as elusive as their charm.

In a recent work [8], Marcus Schaefer leveraged the expressive power of SEFE-\( k \) to generalize, in terms of polynomial-time reducibility, several graph drawing problems, including C-PLANARITY. On the other hand, also C-PLANARITY has shown a significant expressive power as it generalizes relevant problems, like STRIP Planarity [2].
most notably, two special cases of $C$-$\text{PLANARITY}$ and $\text{SEFE-2}$ have been proved to be polynomial-time equivalent \cite{Feng1999}, that is, $C$-$\text{PLANARITY}$ with two clusters and $\text{SEFE-2}$ where the common graph is a star. Motivated by such results, we pose the question whether this equivalence extends to the general case.

In this paper we take a first step in this direction, by proving that $\text{C-SEFE-2}$, that is the restriction of $\text{SEFE-2}$ to instances with connected common graph, reduces to $C$-$\text{PLANARITY}$.

The paper is structured as follows. In Section 2 we give basic definitions. In Section 3 we show a polynomial-time reduction from $\text{C-SEFE-2}$ to $C$-$\text{PLANARITY}$. Finally, in Section 4 we give conclusive remarks and present some open problems.

2 Preliminaries

A graph $G = (V,E)$ is a pair, where $V$ is the set of vertices and $E \subseteq V^2$ is the set of edges. A graph without self-loops and multi-edges is called simple. In the following, we will only consider simple graphs. The degree of a vertex is the number of edges incident to it.

A graph is connected if every two vertices are connected by a path. A tree $T$ is a minimally connected graph. Namely, for each two vertices there exists exactly one path connecting them. The degree-1 vertices of $T$ are leaves, while the other vertices are internal vertices. We denote the set of leaves by $\mathcal{L}(T)$.

A drawing of a graph is a mapping of each vertex to a point of the plane and of each edge to a simple curve connecting its endpoints. A drawing is planar if the curves representing its edges do not cross except, possibly, at common endpoints. A graph is planar if it admits a planar drawing. A planar drawing partitions the plane into topologically connected regions called faces. Two planar drawings are said to be equivalent if they determine the same circular order of edges around each vertex. An equivalence class of planar drawings is called an embedding.

Given $k$ planar graphs $G_1(V,E_1), \ldots, G_k(V,E_k)$ such that $E_i \cap E_j = \emptyset$, with $1 \leq i < j \leq k$, a $k$-page book-embedding of graphs $G_i$ consists of a linear ordering $\mathcal{O}$ of $V$ such that for every set $E_i$ there exist no two edges $e_1, e_2 \in E_i$ whose endvertices alternate in $\mathcal{O}$. The Partitioned T-Coherent $k$-Page Book Embedding problem (PTBE-$k$) takes as input a rooted tree $T$ with leaves $\mathcal{L}(T)$ and $k$ sets $E_1, \ldots, E_k$ of edges among leaves such that $E_i \cap E_j = \emptyset$, with $1 \leq i < j \leq k$, and asks whether a $k$-page book-embedding $\mathcal{O}$ of graphs $G_i = (\mathcal{L}(T), E_i)$ exists such that $\mathcal{O}$ is represented by $T$. It is easy to verify that instance $\langle T, E_1, \ldots, E_k \rangle$ admits a partitioned T-coherent $k$-page book-embedding if and only if graphs $G_i = (V(T), E(T) \cup E_i)$ admit a C-SEFE-$k$.

Since $\text{C-SEFE-2}$ and PTBE-2 have been proved to be polynomial-time equivalent \cite{Feng1999}, in order to simplify the description, in the following we will denote an instance $\langle T, E_1, E_2 \rangle$ of PTBE-2 by the corresponding instance $\langle G_1, G_2 \rangle$ of $\text{C-SEFE-2}$, where $G_1 = (V(T), E(T) \cup E_1)$ and $G_2 = (V(T), E(T) \cup E_2)$, and vice versa.

The $C$-$\text{PLANARITY}$ problem, introduced by Feng et al. \cite{Feng2000}, takes as input a clustered graph $C(G,T)$, that is a planar underlying graph $G$ together with a cluster hierarchy $T$, that is a rooted tree whose leaves are the vertices of $G$. Each internal node $\mu$
of $T$ is called cluster and is associated with the leaves of the subtree $T(\mu)$ of $T$ rooted at $\mu$. The problem asks whether a $c$-planar drawing of $C(G, T)$ exists, that is a planar drawing of $G$ together with a drawing of each cluster $\mu$ as a simple region $R(\mu)$ such that: 1. $R(\mu)$ encloses all and only the leaves of $T(\mu)$ and the regions representing the internal nodes of $T(\mu)$; 2. $R(\mu) \cap R(\theta) \neq \emptyset$ if and only if $\theta$ is an internal node of $T(\mu)$; and 3. each edge $(u, v)$ of $G$ intersects $R(\mu)$ at most once. A clustered graph $C(G, T)$ is flat if $T$ is a tree of height 2 (that is, removing all the leaves yields a star graph) and non-flat otherwise.

### 3 Reduction

In this section we prove the main result of the paper. To ease the description, we first prove in Theorem 1 that C-SEFE-2 reduces to C-PLANARITY, where the constructed instance of C-PLANARITY is non-flat. We give an high level view of the reduction. Due to the equivalence between C-SEFE-2 and PTBE-2 [1], the reduction is performed on instances of PTBE-2. The proof exploits two clusters to enforce the placement of the edges of the reduced instance on the pages they are assigned to (similarly to the technique used in [2] to reduce PTBE-2 in which $T$ is a star to C-PLANARITY) and a suitable cluster hierarchy to represent the constraints on the book-embedding imposed by $T$. Then, Theorem 2 states that the reduction of Theorem 1 can be extended to obtain flat instances whose underlying graph is a set of paths.

**Theorem 1.** C-SEFE-2 $\propto$ C-PLANARITY.

**Proof.** Let $\langle T, E_1, E_2 \rangle$ be an instance of PTBE-2 (corresponding to instance $\langle G_1, G_2 \rangle$ of C-SEFE-2) and let $r$ be the root of $T$. We describe how to construct an equivalent instance $C(G, T)$ of C-PLANARITY starting from $\langle T, E_1, E_2 \rangle$. Refer to Fig 1.
Initialize $G$ to a graph $H$ composed of two cycles $C_1 = \langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$ and $C_2 = \langle u_7, u_B, u_8, u^\prime_1, u^\prime_2, u^\prime_3, u_9, u_R, u_{10}, u_\beta, u^\prime_4, u^\prime_5 \rangle$, and of edges $\langle u_1, u_7 \rangle$, $\langle u_2, u_8 \rangle$, $\langle u_3, u^\prime_1 \rangle$, $\langle u_4, u_9 \rangle$, $\langle u_5, u_{10} \rangle$, and $\langle u_6, u^\prime_5 \rangle$. Observe that $H$ is a subdivision of a 3-connected planar graph.

Initialize $T$ to a tree only composed of a root $\lambda$. For $m = 1, \ldots, 10$, add a cluster $\mu_m$ to $T$ as a child of $\lambda$, containing only vertex $u_m$. Also, add clusters $\mu_B$ and $\mu_R$ to $T$ as children of $\lambda$, containing vertices $u_B$ and $u_R$, respectively. Finally, for $\sigma \in \{\alpha, \rho, \beta\}$, add a cluster $\mu_\sigma$ to $T$ as a child of $\lambda$, containing vertices $u^\prime_\sigma$ and $u^\prime_\sigma$.

Then, consider each internal vertex $w_h$ of $T$ according to a top-down traversal of $T$ and add to $T$ a cluster $\theta_h$ either as a child of cluster $\theta_k$, if $w_k \neq r$ is the parent of $w_h$ in $T$, or as a child of cluster $\mu_\rho$, if $r$ is the parent of $w_h$ in $T$. Also, for each leaf vertex $v_i$ of $T$, add to $G$ a path $\langle v^i_\alpha, v^i_\rho, v^i_\beta \rangle$, that we call leaf-path. Add vertices $v^i_\alpha$ and $v^i_\beta$ to clusters $\mu_\alpha$ and $\mu_\beta$, respectively; add $v_i$ to cluster $\theta_h$, if $w_h$ is the parent of $v_i$ in $T$, or to cluster $\mu_\rho$, if $r$ is the parent of $v_i$ in $T$.

Finally, for each edge $\langle v_i, v_j \rangle$ in $E_1$ or in $E_2$, add to $G$ path $\langle v^i_\beta, v^i_{R, j}, v^i_\beta \rangle$ or path $\langle v^i_\alpha, v^i_{B, j}, v^i_\alpha \rangle$, respectively, that we call edge-paths. Add each vertex $v^i_{R, j}$ to $\mu_R$ and each vertex $v^i_{B, j}$ to $\mu_B$.

Suppose that $(T, E_1, E_2)$ admits a SEFE $\langle \Gamma_1, \Gamma_2 \rangle$. We show how to construct a c-planar drawing $\Gamma$ of $C(G, T)$. We will construct the drawing of $G$ contained in $\Gamma$ as a straight-line drawing; hence, we only describe how to place the vertices of $G$. Refer to Fig. 2.

![Fig. 2: Construction of a c-planar drawing of $C(G, T)$ starting from $\langle \Gamma_1, \Gamma_2 \rangle$, where $x = \phi(v_1)$ and $y = \phi(v_j)$.](image-url)

Let $\ell = |\mathcal{L}(T)|$. We first consider cycle $C_1$. Place vertex $u_1$ at point $(-1, \ell + 1)$, $u_2$ at $(\ell + 2, \ell + 1)$, $u_3$ at $(\ell + 2, 0)$, $u_4$ at $(\ell + 2, -\ell - 1)$, $u_5$ at $(-1, -\ell - 1)$, and $u_6$ at $(-1, 0)$. Then, we consider cycle $C_2$. Place vertex $u_7$ at point $(0, \ell)$, $u_B$ at $(\ell, \ell)$, $u_8$ at
Consider the circular order of the leaves of $T$ determined by $\langle \Gamma_1, \Gamma_2 \rangle$ and consider two adjacent leaves $v'$ and $v''$ such that the lowest common ancestor of $v'$ and $v''$ in $T$ is the root $r$ (note that, if $r$ has degree greater than 1, there always exist two such vertices; otherwise, we can obtain an equivalent instance of SEFE by removing $r$ from $T$). Consider the linear order $\mathcal{O}$ of the leaves of $T$ such that $v'$ and $v''$ are the first and the last element of $\mathcal{O}$. Let $\phi : \mathcal{L}(T) \to 1, \ldots, \ell$ be a function such that $\phi(v_i) = k$ if $v_i$ is the $k$-th element in $\mathcal{O}$. For each leaf vertex $v_i$, we draw leaf-path $(v_{i\alpha}', v_{i\beta}', v_{i\gamma}')$ by placing vertex $v_{i\alpha}'$ at point $(x, 0)$, $v_{i\beta}'$ at $(x, 1)$, and $v_{i\gamma}'$ at $(x, -1)$, where $x = \phi(v_i)$.

Then, for each edge $(v_i, v_j) \in E_2$, we draw edge-path $(v_{i\alpha}', v_{i\beta}', v_{i\gamma}')$ by placing vertex $v_{i\beta}'$ at point $(\frac{x+y}{2}, \frac{1}{2} + |x-y|)$, where $x = \phi(v_i)$ and $y = \phi(v_j)$. Symmetrically, for each edge $(v_i, v_j) \in E_1$, we draw edge-path $(v_{i\alpha}', v_{i\beta}', v_{i\gamma}')$ by placing vertex $v_{i\beta}'$ at point $(\frac{x+y}{2}, \frac{1}{2} - |x-y|)$, where $x = \phi(v_i)$ and $y = \phi(v_j)$.

Finally, we draw the region representing each cluster. Consider each cluster $\mu \in \mathcal{T}$ according to a bottom-up traversal and draw $\mu$ as an axis-parallel rectangular region enclosing all and only the vertices and clusters in the subtree of $T$ rooted at $\mu$. Observe that, this is always possible. Namely, for clusters $\mu_m$, with $m = 1, \ldots, 10$, and clusters $\mu_B, \mu_R, \mu_\alpha, \mu_\beta$ this directly follows from the construction. Also, for each cluster $\theta_h$ corresponding to an internal vertex $w_h$ of $T$, this descends from the fact that the ordering of the leaves of $T$ is determined by a SEFE $(\Gamma_1, \Gamma_2)$. Indeed, since the drawing of $T$ is planar in $(\Gamma_1, \Gamma_2)$, for any two vertices $v^i$ and $v^j$ of $G$ belonging to the same cluster, there exists no vertex $v^k$, with $\phi(v_i) < \phi(v_k) < \phi(v_j)$, belonging to a different cluster. Since all leaf-paths are drawn as vertical segments, this implies that no edge-region crossing occurs between a cluster $\theta_h$ and a leaf-path. Once all clusters $\theta_h$ have been drawn, cluster $\mu_\rho$ can be drawn to enclose all and only such clusters.

Further, observe that there exist no two edge-paths $(v_{i\alpha}', v_{i\beta}', v_{i\gamma}')$ and $(v_{j\alpha}', v_{j\beta}', v_{j\gamma}')$, corresponding to edges $(v_i, v_j)$ and $(v_p, v_q)$ of $E_2$, such that pairs $(v_i, v_j)$ and $(v_p, v_q)$ alternate in $\mathcal{O}$. Hence, any two edge-paths are either disjoint or nested. In both cases, by construction, they do not cross (see Fig. 2 for an illustration of the two cases). Similarly, it can be proved that edge-paths corresponding to edges of $E_1$ do not cross. This concludes the proof that $\Gamma$ is a c-planar drawing of $C(G, \mathcal{T})$.

Suppose that $C(G, \mathcal{T})$ admits a c-planar drawing $\Gamma$. We show how to construct a SEFE $(\Gamma_1, \Gamma_2)$ of $(T, E_1, E_2)$. First, observe that all leaf-paths entirely lie inside the face $f$ of $H$ delimited by cycle $C_2$, as $f$ is the only face of $H$ shared by $u_{\alpha}', u_{\beta}', u_{\gamma}'$, and $u_{\sigma}'$. Since all vertices $v_{i\beta}'$ and $v_{i\gamma}'$ are adjacent to vertices of leaf-paths, they also lie inside $f$. Further, since for $\sigma \in \{\alpha, \rho, \beta\}$ cluster $\mu_\sigma$ is represented by a connected region enclosing vertices $u_{\alpha}'$ and $u_{\beta}'$ and not involved in any edge-region and region-region crossing, all the edges connecting vertices of $\mu_\sigma$ to vertices of the same cluster are consecutive in the order of the edges crossing the boundary of $\mu_\sigma$. This implies that the order in which leaf-paths cross the boundary of $\mu_\sigma$ is the reverse of the order in which they cross the boundary of $\mu_\beta$, since no two leaf-paths cross each other in $\Gamma$. To obtain $(\Gamma_1, \Gamma_2)$, we order the leaves $v_i$ of $T$ according to the order in which leaf-paths cross the boundary of $\mu_\sigma$. Let $\mathcal{O}$ be such an order.
First, we show that $O$ can be represented by $T$, which implies that a planar drawing $\Gamma_T$ of $T$ exists respecting $O$. In fact, by construction, for each internal vertex $w_h$ of $T$, the leaves of the subtree $T(w_h)$ of $T$ rooted at $w_h$ belong to the same cluster $\theta_h$. Also, since $T$ is c-planar, all the leaf-paths $(v^i_\alpha, v^1_\beta, v^2_\beta)$ such that $v_i$ is a leaf of $T(w_h)$ are consecutive in the order in which leaf-paths cross the boundary of $\mu_\alpha$, and hence the corresponding leaves $v_i$ are consecutive in $O$. Second, we show how to construct two planar drawings $\Gamma_1$ and $\Gamma_2$ of $G_1$ and $G_2$, respectively, such that the drawing of $T$ contained in $\Gamma_1$ and in $\Gamma_2$ coincides with $\Gamma_T$. We describe the algorithm to construct $\Gamma_2$, the algorithm for $\Gamma_1$ being analogous. Consider two edges $(v_i, v_j)$ and $(v_p, v_q)$ of $E_2$. Since the drawing of $G$ in $\Gamma$ is planar, the corresponding edge-paths $(v^i_\alpha, v^p_\alpha, v^q_\beta, v^j_\beta)$ and $(v^p_\alpha, v^p_\beta, v^q_\alpha)$ do not intersect in $\Gamma$. Also, since the edges belonging to edge-paths are consecutive in the order in which edges incident to vertices of $\mu_\alpha$ cross the boundary of $\mu_\alpha$, the pair of leaves $(v_i, v_j)$ and $(v_p, v_q)$ of $T$ corresponding to vertices $v^i_\alpha, v^p_\alpha$, $v^q_\beta$, and $v^j_\beta$ do not alternate in $O$. Hence, $\Gamma_2$ can be obtained from $\Gamma_T$ by drawing the edges of $E_2$ as curves intersecting neither edges of $T$ nor other edges in $E_2$. Since the drawing of $G_\alpha = T$ is the same in $\Gamma_1$ and in $\Gamma_2$, $(\Gamma_1, \Gamma_2)$ is a SEFE of $(T, E_1, E_2)$. This concludes the proof of the theorem.

In the following we prove that the reduction of Theorem 1 can be modified in such a way that the resulting instance of C-PLANARITY is flat and the underlying graph consists of a set of paths.

**Theorem 2.** C-SEFE-2 $\cong$ C-PLANARITY with flat cluster hierarchy and underlying graph that is a set of paths.

**Proof.** Let $(T, E_1, E_2)$ be an instance of C-SEFE-2. We describe how to construct an equivalent instance $C(G, T)$ of C-PLANARITY with flat cluster hierarchy and underlying graph that is a set of paths starting from $(T, E_1, E_2)$.

First, we construct an instance $C^*(G^*, T^*)$ of C-PLANARITY with non-flat cluster hierarchy by applying the reduction shown in Theorem 1. We describe how to transform $C^*$ into an equivalent instance $C(G, T)$ of C-PLANARITY with the required properties.

For vertices $u^\prime_\mu, u^\prime_\rho$, and for all vertices $v^i_\alpha$ and $v^j_\beta$ having degree at least 2, consider the parent cluster $\theta$ of any such vertex $v$ in $T$. Add a cluster $\mu_v$ to $T$ as a child of $\theta$ and containing only vertex $v$. The obtained instance $C'(G' = G, T')$ is obviously equivalent to $C^*$.

Let $\Delta$ be the set of all clusters $\tau \in T'$ such that $T' = \tau$ has only one leaf $t$. Note that, $\Delta$ consists of all clusters $\mu_m$, with $m = 1, \ldots, 10$, and all clusters added at the previous step to obtain $C'$. For each cluster $\tau \in \Delta$, we perform the following procedure. For each edge $(t, \tau)$ of $G'$ such that $t \in \tau$, add a vertex $t_\tau$ to $\tau$ and add edge $(t_\tau, \tau)$ to $G'$. Finally, remove vertex $t$ and its incident edges from $C'$. This can be seen as replacing $t$ with $\deg(t)$ copies of it. For simplicity, in the following we keep the same notation $(v^i_\alpha, v^1_\beta, v^2_\beta)$ for leaf-paths, and $(v^p_\alpha, v^p_\beta, v^q_\alpha)$ and $(v^p_\alpha, v^p_\beta, v^q_\alpha)$ for edge-paths, where their endvertices have been naturally replaced by the appropriate copy. See Fig. 3(a) for an illustration of this step.

Observe that, the constructed instance $C''(G'', T'')$ is such that $G''$ consists of a set of paths. In fact, after performing the two steps described above, each vertex of $G''$
has either degree 1 or degree 2. Also, every vertex of degree 2 is the middle vertex of either a leaf-path or an edge-path. Hence, no cycle is created. Further, $C''(G'', T'')$ is equivalent to $C'$, as in any c-planar drawing of $C''$ a vertex $t$ that has been removed from a cluster $\tau$ can be reinserted inside $R(\tau)$ and connected to all the vertices of $\tau$ while maintaining c-planarity (the other direction being trivial).

We now show how to construct instance $C$ starting from $C''$. For each vertex $v^i_\alpha$ whose parent $\tau^i_\alpha$ in $T''$ is different from $\mu_\alpha$, we subdivide edge $(v^i_\alpha, v^i)$ with a vertex $z^i_\alpha$; we add $z^i_\alpha$ to cluster $\mu_\alpha$; and we remove $\tau^i_\alpha$ from the children of $\mu_\alpha$ and add $\tau^i_\alpha$ as a child of the root $\lambda$. For each vertex $v^i_\beta$ whose parent $\tau^i_\beta$ in $T''$ is different from $\mu_\beta$, we subdivide edge $(v^i_\beta, v^i)$ with a vertex $z^i_\beta$; we add $z^i_\beta$ to cluster $\mu_\beta$; and we remove $\tau^i_\beta$ from the children of $\mu_\beta$ and add $\tau^i_\beta$ as a child of the root $\lambda$.

Let $\mu'$ and $\mu''$ be the parent clusters of the 3 copies of $u'_\rho$ and $u''_\rho$, respectively, in $T''$. Subdivide the edge connecting a vertex in $\mu'$ to $u'_\beta$ with a new vertex and the edge connecting a vertex in $\mu''$ to $u''_\beta$ with a new vertex, and add both such vertices to $\mu_\rho$. Also, remove $\mu'$ and $\mu''$ from the children of $\mu_\rho$ and add them as children of the root $\lambda$.

Further, as long as there exists a cluster $\mu \neq \mu_\rho \in T''(\mu_\rho)$ such that all the children of $\mu$ are leaves, we perform the following procedure. We add a new cluster $\mu'$ to $T''$ as a child of the root $\lambda$. Consider the parent $\theta$ of $\mu$ in $T''$. For each vertex $v^i \in \mu$, we remove $v^i$ from the children of $\mu$ and add it as a child of $\theta$; also, we subdivide the unique edge $(v^i, x)$ incident to $v^i_\beta$ with a new vertex that we add to cluster $\mu'$. Finally, we remove $\mu$ from $T''$. The instance $C(G, T)$ obtained by applying the reduction to C-SEFE-2 instance of Fig. 1(a) can be seen in Fig. 3(b). In order to prove that $C$ is equivalent to $C''$, observe that paths $(v^i_\alpha, z^i_\alpha, v^i, z^i_\beta, \ldots, v^i_\beta)$ obtained from leaf-paths

Fig. 3: Construction of an equivalent instance $C(G, T)$ with the desired properties. (a) Obtaining an instance whose underlying graph is a set of paths. (b) Obtaining a flat instance.
(\(v^i_{\alpha}, v^j_{\alpha}, v^{j'}_{\beta}\)) are bounded to cross the boundary of \(R(\mu_{\alpha})\) in \(C\) in the same order in which the corresponding leaf-paths are bounded to cross the boundary of \(R(\mu_{\alpha})\) in \(C''\). Namely, for each cluster \(\mu \in T''\) there exists a cluster \(\mu' \in T\) imposing the same consecutivity constraint on the ordering in which paths cross the boundary of \(R(\mu_{\alpha})\). This concludes the proof of the theorem. □

Corollary 1. PTBE-2 \(\propto\) C-PLANARITY with flat cluster hierarchy and underlying graph that is a set of paths.

4 Conclusions and Open Problems

In this paper we show that C-SEFE-2 is polynomial-time reducible to C-PLANARITY even in the case in which the cluster hierarchy is flat and the underlying graph is a set of paths.

We regard as an intriguing open question whether a polynomial-time reduction exists from general instances of SEFE-2 to instances of C-PLANARITY, which would prove, together with the reduction by Schaefer [8], these two problems to be ultimately the same. Moreover, as our reduction produces instances of C-PLANARITY with a number of clusters depending linearly in the size of the reduced C-SEFE-2 instance, it is worth of interest asking whether a sublinear or constant number of clusters would suffice.

References

1. Angelini, P., Di Battista, G., Frati, F., Patrignani, M., Rutter, I.: Testing the simultaneous embeddability of two graphs whose intersection is a biconnected or a connected graph. J. of Discrete Algorithms 14, 150–172 (2012)
2. Angelini, P., Da Lozzo, G., Di Battista, G., Frati, F.: Strip planarity testing. In: Wismath, S., Wolff, A. (eds.) Graph Drawing. LNCS, vol. 8242, pp. 37–48. Springer (2013)
3. Blasius, T., Kobourov, S.G., Rutter, I.: Simultaneous embedding of planar graphs. In: Tamassia, R. (ed.) Handbook of Graph Drawing and Visualization. CRC Press (2013)
4. Di Battista, G., Frati, F.: Efficient c-planarity testing for embedded flat clustered graphs with small faces. JGAA 13(3), 349–378 (2009)
5. Feng, Q., Cohen, R.F., Eades, P.: Planarity for clustered graphs. In: ESA’95. LNCS, vol. 979, pp. 213–226 (1995)
6. Feng, Q.W., Cohen, R.F., Eades, P.: How to draw a planar clustered graph. In: COCOON’95. LNCS, vol. 959, pp. 21–30 (1995)
7. Hong, S., Nagamochi, H.: Two-page book embedding and clustered graph planarity. TR [2009-004], Dept. of Applied Mathematics and Physics, University of Kyoto, Japan (2009)
8. Schaefer, M.: Toward a theory of planarity: Hanani-tutte and planarity variants. JGAA 17(4), 367–440 (2013)