Quantifying Model Uncertainties
in the Space of Probability Measures

Jinqiao Duan\textsuperscript{1}, Ting Gao\textsuperscript{1} and Guowei He\textsuperscript{2}\textsuperscript{*}

\textsuperscript{1. Institute for Pure and Applied Mathematics}
University of California, Los Angeles, CA 90095, USA

\textsuperscript{2. Department of Applied Mathematics, Illinois Institute of Technology}
Chicago, IL 60616, USA

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E-mail: duan@iit.edu  tgao5@iit.edu

\textsuperscript{2. Laboratory for Nonlinear Mechanics}
Institute of Mechanics, Chinese Academy of Sciences
Beijing 100080, China

E-mail: hgw@lnm.imech.ac.cn

Due to lack of scientific understanding, some mechanisms may be missing in mathematical modeling of complex phenomena in science and engineering. These mathematical models thus contain some uncertainties such as uncertain parameters. One method to estimate these parameters is based on pathwise observations, i.e., quantifying model uncertainty in the space of sample paths for system evolution. Another method is devised here to estimate uncertain parameters, or unknown system functions, based on experimental observations of probability distributions for system evolution. This is called the quantification of model uncertainties in the space of probability measures. A few examples are presented to demonstrate this method, analytically or numerically.

\textbf{Keywords}: Model uncertainty; Stochastic differential equations (SDEs); Probability measures; Hellinger distance; Stationary probability density; Parameter estimation; Numerical simulation

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\section{Introduction}

In this chapter we discuss some issues about quantification of model uncertainties in complex dynamical systems.
Mathematical models for scientific and engineering systems often involve with some uncertainties. We may roughly classify such uncertainties into two kinds. The first kind of uncertainties may be called model uncertainty. They are due to physical processes that are not well understood or not well-observed, and thus are not or not well represented in the mathematical models.

The second kind of uncertainties may be called simulation uncertainty. This arises in numerical simulations of multiscale systems that display a wide range of spatial and temporal scales, with no clear scale separation. Due to the limitations of computer power, not all scales of variability can be explicitly simulated or resolved. These uncertainties are sometimes also called unresolved scales, as they are not represented (i.e., not resolved) in modeling or simulation. Although these unresolved scales may be very small or very fast, their long time impact on the resolved simulation may be delicate (i.e., may be negligible or may have significant effects, or in other words, uncertain). Thus, to take the effects of unresolved scales on the resolved scales into account, representations or parameterizations of these effects are desirable.

Model uncertainties have been considered in, for example, Research works relevant for parameterizing unresolved scales include, among others. Stochastically representing unresolved scales in fluid dynamics has considered as well.

In this chapter, we only consider model uncertainties. Specifically, we consider dynamical systems containing uncertain parameters or unknown system functions, and examine how to estimate these parameters, using observed probability distributions of the system evolution.

After briefly comment on estimating uncertain parameters based on observed sample paths for the system evolution in , we then, in propose a method of estimating uncertain parameters based on observed probability distributions (i.e., probability measures) and present a few examples to demonstrate this method, analytically or numerically.

2. Quantifying uncertainty in the space of paths

Since random fluctuations are common in the real world, mathematical models for complex systems are often subject to uncertainties, such as fluctuating forces, uncertain parameters, or random boundary conditions. Stochastic differential equations (SDEs) such as

\[ dX = b(X)dt + \sigma(X)dB_t, \]  

are appropriate models for many of these systems. Here \( B_t \) is a Brownian motion or Wiener process, the drift \( b(X) \) and diffusion \( \sigma(X) \) contain uncertain parameters (or \( b(\cdot) \) and \( \sigma(\cdot) \) are unknown), to be estimated based on observations.

For example, the Langevin type models are stochastic differential equations de-
scribing various phenomena in physics, biology, and other fields. SDEs are used to model various price processes, exchange rates, and interest rates, among others, in finance. Noises in these SDEs may be modeled as a generalized time derivative of some distinguished stochastic processes, such as Brownian motion (BM) or other processes.

We are interested in estimating parameters contained in the stochastic differential equation (2.1), so that we obtain computational models useful for investigating complex dynamics under uncertainty.

Theoretical results on parameter estimations for SDEs driven by Brownian motion are relatively well developed, and various numerical simulations for these parameter estimations are also implemented. See for a more recent review about estimating and computing uncertain parameters, when dynamical systems are subject to colored or non-Gaussian noises.

These research works on estimating uncertain parameters in dynamical systems are based on observations of sample paths. In the next section, we devise a method to estimate uncertain parameters based on observations of probability distributions of the system evolution.

3. Quantifying uncertainty in the space of probability measures

Consider a dynamical system with model uncertainty, modeled by a scalar SDE

\[ dX = b(X)dt + \sigma(X)dB_t, \quad X(0) = x_0, \]  

(3.1)

where the drift \( b(X) \) and diffusion \( \sigma(X) \) contain uncertain parameters, to be estimated based on observations of probability distributions (i.e., probability measures) of the system paths \( X_t \).

To this end, we need to introduce the Hellinger distance between two probability measures. It is used to quantify the similarity between two probability distributions. This is a metric in the space of probability measures.

For our purpose here, we define the Hellinger distance \( H(f, g) \) between two probability density functions \( p(x) \) and \( q(x) \) as follows

\[ H^2(p, q) \triangleq \frac{1}{2} \int_{\mathbb{R}} (\sqrt{p(x)} - \sqrt{q(x)})^2 dx. \]  

(3.2)

The Hellinger distance \( H(p, q) \) satisfies the property: \( 0 \leq H(p, q) \leq 1 \).

We estimate uncertain parameters by minimizing the Hellinger distance between the true probability density \( p \) for the solution process \( X(t) \) and its observed probability density \( q \). In reality, the probability density \( p \) has to be numerically formulated or discretized. But in order to demonstrate the method, we consider two examples for which the true probability density \( p \) can be analytically formulated. In the first example, we minimize the Hellinger distance between the true stationary probability density for the solution process \( X(t) \) and its observed stationary probability
density, while in the second example, we do this for time-dependent probability densities.

3.1. Observation of stationary probability distributions

Under appropriate conditions on $b$ and $\sigma$ (see, e.g., p.170), such as, $b \leq 0$ and $\sigma \neq 0$ as well as some smoothness requirements, there exists a stationary probability density $p(x)$ for the SDE, as a solution of the steady Fokker-Planck equation

$$p_{xx} + (b \sin(x)p)_x = 0,$$

where the positive normalization constant $C$ is chosen so that $p \geq 0$ and $\int \limits_{\mathbb{R}} p(x)dx = 1$, i.e.,

$$C \equiv 1/\int_{-\infty}^{\infty} \frac{e^{\frac{x^2}{\sigma^2(y)}}}{\sigma^2(x)} dx.$$

Note that $x^*$ here may be an arbitrary point so that the integral $\int x^* \frac{2b(y)}{\sigma^2(y)}dy$ exists (say, take $x^* = 0$ if that is possible).

Example 3.1. (i) A special case: Langevin equation

$$dX = -bX dt + dB_t,$$

with parameter $b > 0$. Given an “observation” of the stationary probability density $q(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$. Find a $b$ so that the Hellinger distance $F(b) = \frac{1}{2} \int \sqrt{p(x) - \sqrt{q(x)}}^2 dx$ is minimized.

(ii) A more general case:

$$dX = b(X) dt + dB_t,$$

with function $b(x) \leq 0$. Given an “observation” of the stationary probability density $q(x) = \frac{1}{\pi (1+x^2)}$ (the Cauchy distribution). Find a function $b(x) \leq 0$ so that the Hellinger distance $F(b(x)) = \frac{1}{2} \int \sqrt{p(x) - \sqrt{q(x)}}^2 dx$ is minimized.

Solution:

(i) The true stationary probability density for the solution process $X_t$ is

$$p(x) = \frac{\sqrt{b}}{\sqrt{\pi}} e^{-bx^2}.$$

Insert $p, q$ into the Hellinger distance $F(b)$, which is now an algebraic function of parameter $b > 0$. Thus we use deterministic calculus to find a minimizer $b$ (possibly by hand, or Matlab if needed). Note: $\int e^{-z^2} dz = \sqrt{\pi}$. 


To minimize the Hellinger distance $F(b)$, we calculate its derivative

$$F'(b) = \frac{1}{2} \int_{\mathbb{R}} e^{-bx^2} \left( \frac{1}{2 \sqrt{\pi b}} - \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(1+2bx^2)} \right) dx - \frac{1}{2 \sqrt{2}} \int_{\mathbb{R}} e \left( \frac{1}{4} b b^2 \sqrt{x^2} - x b^2 \right) dx$$

$$= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{b(x+b)} dx - \frac{1}{2 \sqrt{2}} \int_{\mathbb{R}} e \left( \frac{1}{4} b b^2 \sqrt{x^2} - x b^2 \right) dx = 0.$$

Therefore,

$$b^2 (1+b)^{-\frac{1}{2}} = \frac{b^{-\frac{1}{2}} (1+b)^{-\frac{1}{2}}}{2}.$$  

Thus we obtain the parameter $b = 1$.

(ii) The true stationary probability density for the solution process $X_t$ is

$$p(x) = \frac{e^{2 \int_0^t b(y)dy}}{\int_{-\infty}^\infty e^{2 \int_0^t b(y)dy} dx}.$$  

Insert $p, q$ into the Hellinger distance $F(b(x))$, which is now a functional of $b(x)$ and thus we use calculus of variations (on $F(b(x))$) to find a minimizer $b(x)$. We then derive the Euler-Lagrange equation to be satisfied by $b(x)$, together with appropriate boundary conditions (needed for $p(x) \geq 0$ and $\int_0^1 p(x) dx = 1$).

To this end, we calculate, for an arbitrary “variations” $h(x)$

$$F(b(x) + \varepsilon h(x))$$

$$= \frac{1}{2} \int_{\mathbb{R}} \left[ \frac{e^{2 \int_0^t b(y)dy}}{\int_{-\infty}^\infty e^{2 \int_0^t b(y)dy} dx} - \frac{2e^{\int_0^t b(y)dy} dx}{\int_{-\infty}^\infty e^{2 \int_0^t b(y)dy} dx} + \frac{1}{\pi (1+x^2)} \right] dx.$$  

The Euler-Lagrange equation for $b(x)$ comes from: $\frac{d}{dx} F(b(x) + \varepsilon h(x)) = 0$ for arbitrary “variations” $h(x)$. In fact, the Euler-Lagrange equation for $b(x)$ is

$$\int_{\mathbb{R}} e^{\int_0^t b(w) dw} \left( e^{\int_0^t b(w) dw} - \frac{\sqrt{\int_{\mathbb{R}} e^{2 \int_0^t b(w) dw} dx}}{\sqrt{\pi (1+x^2)}} \right) \cdot e^{\int_0^t b(w) dw} \int_{-\infty}^\infty h(y) dy dz = 0.$$  

After changing the order of integration (first on $y$ and then on $y$ and $x$), we have

$$\left\{ \int_{\mathbb{R}} e^{\int_0^t b(w) dw} \left( e^{\int_0^t b(w) dw} - \frac{\sqrt{\int_{\mathbb{R}} e^{2 \int_0^t b(w) dw} dx}}{\sqrt{\pi (1+x^2)}} \right) dx \cdot \int_{-\infty}^y e^{\int_0^t b(w) dw} dz \right\} h(y) dy = 0$$

holds for all $h(y)$.

Therefore,

$$\int_{\mathbb{R}} e^{\int_0^t b(w) dw} \left( e^{\int_0^t b(w) dw} - \frac{\sqrt{\int_{\mathbb{R}} e^{2 \int_0^t b(w) dw} dx}}{\sqrt{\pi (1+x^2)}} \right) dx \cdot \int_{-\infty}^y e^{\int_0^t b(w) dw} dz = 0.$$
Since $\int_{-\infty}^{\infty} e^{2\int_{0}^{y} b(w)dw} dz > 0$, we further obtain

$$\int_{0}^{\infty} e^{\int_{0}^{y} b(w)dw} \left( e^{\int_{0}^{\infty} b(w)dw} - \frac{\sqrt{\int_{\mathbb{R}} e^{2\int_{0}^{y} b(w)dw} dx}}{\sqrt{\pi(1 + x^2)}} \right) dx = 0$$

for $y \in \mathbb{R}$. Then taking the derivative of the above equation with respect to $y$, we arrive at

$$e^{\int_{0}^{y} b(w)dw} = \frac{\sqrt{\int_{\mathbb{R}} e^{2\int_{0}^{y} b(w)dw} dx}}{\sqrt{\pi(1 + y^2)}}, \ \forall y \in \mathbb{R}^1.$$

Thus, after taking the ‘square’ and ‘ln’ on both sides of the above equation, we get

$$2 \int_{0}^{y} b(w)dw = \ln(\int_{\mathbb{R}} e^{2\int_{0}^{y} b(w)dw} dx) - \ln(\pi(1 + y^2)).$$

Finally taking the derivative with respect to $y$, we have

$$b(y) = -\frac{\frac{y}{\pi(1 + y^2)}}{y \in \mathbb{R}^1}.$$

Also note that we only need $\int_{0}^{y} b(w)dw < 0$ for all $y \in \mathbb{R}^1$ for the stationary probability density to make sense.

### 3.2. Observation of time-dependent probability distributions

Consider a scalar SDE

$$dX = b(X)dt + \sigma(X)dB_t, \ X(0) = x_0. \quad (3.3)$$

The Fokker-Planck equation for the probability density $p(x, t) \triangleq p(x, t; x_0, 0)$ for the solution $X(t, x_0)$ is

$$p_t = \frac{1}{2}(\sigma^2(x)p(x, t))_{xx} - (b(x)p(x, t))_x, \ p(x, 0) = \delta(x_0). \quad (3.4)$$

With an observation of $p(x, t)$, we can estimate parameters, or $b(\cdot)$, or $\sigma(\cdot)$, by examining the inverse problem of the Fokker-Planck equation (3.4). For more information about inverse problems of partial differential equations, see.\(^{29}\)

Let us look at a specific example.

**Example 3.2.** Consider a scalar SDE

$$dX = -b \sin(X)dt + \sqrt{2} dB_t, \ X(0) = 0. \quad (3.5)$$
(i) Assume that an observation obtained for $p$ to be

$$q_1(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}. \quad (3.6)$$

Find the parameter $b$ by minimizing the Hellinger distance $H(p, q_1)$.

(ii) Assume that another observation obtained for $p$ to be

$$q_2(x,t) = \frac{\sqrt{t}}{\pi(t+x^2)}. \quad (3.7)$$

Find the parameter $b$ by minimizing the Hellinger distance $H(p, q_2)$.

**Solution:**

The Fokker-Planck equation for (3.5) is

$$p_t = p_{xx} + (b\sin(x)p)_x, \quad p(x,0) = \delta(0). \quad (3.8)$$

In this case, we define the Hellinger distance:

$$H_i^2(b) = \max_{t\in[0,T]} \int_{-\infty}^{\infty} (\sqrt{q_i(x,t)} - \sqrt{p(x,t,b)})^2 dx$$

where $i = 1, 2$, and $T$ is the time period when $q_i(x,t)$ are observed. We numerically find $b$ by minimizing $H_i(b)$.

The observation $q_1(x,t)$ is plotted in Figure 0.1.
By the definition $H_1^2(b) = \max_{t \in [0,T]} \int_{-\infty}^{\infty} (\sqrt{q_1(x,t)} - \sqrt{p(x,t,b)})^2 dx$, we have the plot of $H_1(b)$ in Figure 0.2. And whatever $T$ is, $H_1(b)$ is always minimized when $b = 0$. This gives us the parameter value $b = 0$.

The observation $q_2(x,t)$ is plotted in Figure 0.3. By the definition $H_2^2(b) = \max_{t \in [0,T]} \int_{-\infty}^{\infty} (\sqrt{q_2(x,t)} - \sqrt{p(x,t,b)})^2 dx$, we have the plot of $H_2(b)$ in Figure 0.4. So we see that if $T = 5$, $H_2(b)$ is minimized when $b = 0.7$ and if $T = 30$, $H_2(b)$ is minimized when $b = 0.6$.

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Fig. 0.2. Squared Hellinger distance between \( p(x,t) \) and the observation \( q_1(x,t) \): \( T = 5 \) (top) and \( T = 30 \) (bottom).

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Fig. 0.3. Observation $q_3(x, t)$: $t = 5$ (top) and $t = 30$ (bottom).

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Fig. 0.4. Squared Hellinger distance between $p(x, t)$ and the observation $q_2(x, t)$: $T = 5$ (top) and $T = 30$ (top).

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Quantifying Model Uncertainties in the Space of Probability Measures

Jinqiao Duan¹, Ting Gao¹ and Guowei He²

1. Institute for Pure and Applied Mathematics
University of California, Los Angeles, CA 90095, USA

Department of Applied Mathematics, Illinois Institute of Technology
Chicago, IL 60616, USA
E-mail: duan@iit.edu  tgao5@iit.edu

2. Laboratory for Nonlinear Mechanics
Institute of Mechanics, Chinese Academy of Sciences
Beijing 100080, China
E-mail: hgw@lnm.imech.ac.cn

Due to lack of scientific understanding, some mechanisms may be missing in mathematical modeling of complex phenomena in science and engineering. These mathematical models thus contain some uncertainties such as uncertain parameters. One method to estimate these parameters is based on pathwise observations, i.e., quantifying model uncertainty in the space of sample paths for system evolution. Another method is devised here to estimate uncertain parameters based on experimental observations of probability distributions for system evolution. This is called the quantification of model uncertainties in the space of probability measures. A few examples are presented to demonstrate this method, analytically or numerically.

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1. Introduction

In this chapter we discuss some issues about quantification of model uncertainties in complex dynamical systems.

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Mathematical models for scientific and engineering systems often involve with some uncertainties. We may roughly classify such uncertainties into two kinds. The first kind of uncertainties may be called *model uncertainty*. They are due to physical processes that are not well understood or not well-observed, and thus are not or not well represented in the mathematical models.

The second kind of uncertainties may be called *simulation uncertainty*. This arises in numerical simulations of multiscale systems that display a wide range of spatial and temporal scales, with no clear scale separation. Due to the limitations of computer power, not all scales of variability can be explicitly simulated or resolved. These uncertainties are sometimes also called *unresolved scales*, as they are not represented (i.e., not resolved) in modeling or simulation. Although these unresolved scales may be very small or very fast, their long time impact on the resolved simulation may be delicate (i.e., may be negligible or may have significant effects, or in other words, uncertain). Thus, to take the effects of unresolved scales on the resolved scales into account, representations or parameterizations of these effects are desirable.

Model uncertainties have been considered in, for example, \[18,25,40,43-45\]. Research works relevant for parameterizing unresolved scales include, \[6-8,17,24,26,30,38,49,56,57\] among others. Stochastically representing unresolved scales in fluid dynamics has also been considered as well. \[34,39,51\]

In this chapter, we only consider model uncertainties. Specifically, we consider dynamical systems containing uncertain parameters and examine how to estimate these parameters, using observed probability distributions of the system evolution. After briefly comment on estimating uncertain parameters based on observed sample paths for the system evolution in \(\S 2\), we then, in \(\S 3\), propose a method of estimating uncertain parameters based on observed probability distributions (i.e., probability measures) and present a few examples to demonstrate this method, analytically or numerically.

### 2. Quantifying uncertainty in the space of paths

Since random fluctuations are common in the real world, mathematical models for complex systems are often subject to uncertainties, such as fluctuating forces, uncertain parameters, or random boundary conditions. \[18,25,52,55\] Stochastic differential equations (SDEs) such as

\[dX = b(X)dt + \sigma(X)dB_t,\]

are appropriate models for many of these systems. \[54,55\] Here \(B_t\) is a Brownian motion or Wiener process, the drift \(b(X)\) and diffusion \(\sigma(X)\) contain uncertain parameters, to be estimated based on observations.

For example, the Langevin type models are stochastic differential equations describing various phenomena in physics, biology, and other fields. SDEs are used to
model various price processes, exchange rates, and interest rates, among others, in finance. Noises in these SDEs may be modeled as a generalized time derivative of some distinguished stochastic processes, such as Brownian motion (BM) or other processes.

We are interested in estimating parameters contained in the stochastic differential equation (2.1), so that we obtain computational models useful for investigating complex dynamics under uncertainty.

Theoretical results on parameter estimations for SDEs driven by Brownian motion are relatively well developed, and various numerical simulations for these parameter estimations are also implemented. See for a more recent review about estimating and computing uncertain parameters, when dynamical systems are subject to colored or non-Gaussian noises.

These research works on estimating uncertain parameters in dynamical systems are based on observations of sample paths. In the next section, we devise a method to estimate uncertain parameters based on observations of probability distributions of the system evolution.

3. Quantifying uncertainty in the space of probability measures

Consider a dynamical system with model uncertainty, modeled by a scalar SDE

\[ dX = b(X)dt + \sigma(X)dB_t, \quad X(0) = x_0, \]  

where the drift \( b(X) \) and diffusion \( \sigma(X) \) contain uncertain parameters, to be estimated based on observations of probability distributions (i.e., probability measures) of the system paths \( X_t \).

To this end, we need to introduce the Hellinger distance between two probability measures. It is used to quantify the similarity between two probability distributions.

This is a metric in the space of probability measures.

3.1. Stationary case

Under appropriate conditions on \( b \) and \( \sigma \) (see p.170.), such as, \( b \leq 0 \) and \( \sigma \neq 0 \) as well as some smoothness requirements, there exists a stationary probability density \( p(x) \) for the SDE (3.1), as a solution of the steady Fokker-Planck equation,

\[ p(x) = \frac{C}{\sigma^2(x)} e^{\int_{x_0}^{x} \frac{2b(y)}{\sigma^2(y)} dy}, \]

where the positive normalization constant \( C \) is chosen so that \( p \geq 0 \) and \( \int_R p(x)dx = 1 \), i.e.,

\[ C \equiv 1/ \int_{-\infty}^{\infty} e^{\int_{x_0}^{x} \frac{2b(y)}{\sigma^2(y)} dy} dx. \]
Note that \( x_0 \) here may be an arbitrary point so that the integral \( \int_{x_0}^{\cdot} \frac{2b(y)}{\sigma^2(y)} dy \) exists (say, take \( x_0 = 0 \) if that is convenient).

**Example 3.1.** (a) A special case: Langevin equation

\[
dX = -bX dt + dB_t,
\]

with parameter \( b > 0 \). Given an “observation” of the stationary probability density \( q(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \). Find a \( b \) so that the Hellinger distance \( F(b) = \frac{1}{2} \int_{R} (\sqrt{p(x)} - \sqrt{q(x)})^2 dx \) is minimized.

*Hint:* Insert \( p, q \) into the Hellinger distance. \( F(b) \) is an algebraic function of parameter \( b > 0 \) and thus we use deterministic calculus to find a minimizer \( b \) (possibly by hand, or Matlab if needed). Note: \( \int_{R} e^{-z^2} dz = \sqrt{\pi} \).

(b) A more general case:

\[
dX = b(X) dt + dB_t,
\]

with function \( b(x) \leq 0 \). Given an “observation” of the stationary probability density \( q(x) = \frac{1}{\pi(1+x^2)} \) (the Cauchy distribution). Find a function \( b(x) \leq 0 \) so that the Hellinger distance \( F(b(x)) = \frac{1}{2} \int_{R} (\sqrt{p(x)} - \sqrt{q(x)})^2 dx \) is minimized.

*Hint:* Insert \( p, q \) into the Hellinger distance. \( F(b(x)) \) is a functional of \( b(x) \) and thus we use calculus of variations (on \( F(b(x)) \)) to find a minimizer \( b \). Derive the Euler-Lagrange equation to be satisfied by \( b(x) \), together with appropriate boundary conditions (needed for \( p \geq 0 \) and \( \int_{R} p dx = 1 \)). Can you devise an algorithm to simulate for \( b(x) \)?

**Solution:**

(a) From the Fokker-Planck equation of

\[
 dX_t = -bX_t dt + dB_t, \quad X_0 = 0,
\]

we have the stationary probability density

\[
 P(x) = \frac{1}{\sqrt{\pi}} e^{-bx^2}.
\]

To minimize the Hellinger distance \( F(b) \), we need

\[
 F'(b) = \frac{1}{2} \int_{R} e^{-bx^2} \left( \frac{1}{2\sqrt{\pi}b} - \frac{\sqrt{b}}{\sqrt{\pi}x^2} \right) dx - \frac{1}{2} \int_{R} e^{-\frac{(1+b)x^2}{2}} \left( \frac{1}{4} b^{-\frac{3}{2}} - \frac{3}{2} b^2 \right) dx
\]

\[
 = \frac{1}{\sqrt{2}} b^\frac{1}{2} (1+b)^{-\frac{3}{2}} - \frac{1}{2\sqrt{2}} b^{-\frac{3}{2}} (1+b)^{-\frac{5}{2}} = 0.
\]

Therefore,

\[
 b^\frac{1}{2} (1+b)^{-\frac{3}{2}} = \frac{b^{-\frac{3}{2}} (1+b)^{-\frac{5}{2}}}{2} \Rightarrow b = 1.
\]
(b) From the Fokker-Planck equation of
\[ dX_t = b(X_t) dt + dB_t, \quad X_0 = 0, \]
we could have the stationary probability density
\[ P(x) = \frac{e^{2 \int_0^x b(y) dy}}{\int_{-\infty}^\infty e^{2 \int_0^x b(y) dy} dx}. \]

Using calculus of variation, we need
\[
F(b(x) + \epsilon h(x))
= \frac{1}{2} \int_{-\infty}^\infty \left[ \frac{e^{2 \int_0^x b(y) + \epsilon h(y) dy}}{\int_{-\infty}^\infty e^{2 \int_0^x b(y) + \epsilon h(y) dy} dx} - \frac{2 e^{2 \int_0^x b(y) + \epsilon h(y) dy} dx}{\sqrt{\pi} \sqrt{1 + x^2} \sqrt{\int_{-\infty}^\infty e^{2 \int_0^x b(y) + \epsilon h(y) dy} dy} dx} + \frac{1}{\pi (1 + x^2)} \right] dx
= 0.
\]

This is equivalent to
\[
\int_{-\infty}^\infty e^{\int_0^x b(w) dw} \left( e^{\int_0^x b(w) dw} - \frac{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dx}{\sqrt{\pi} \sqrt{1 + x^2} \sqrt{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dy} dx} \right) \int z h(y) dy dz = 0.
\]

When changing the order of integration (first on \( y \) and \( z \) then on \( y \) and \( x \)), we have
\[
\left\{ \int_{-\infty}^\infty e^{\int_0^x b(w) dw} \left( e^{\int_0^x b(w) dw} - \frac{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dx}{\sqrt{\pi} \sqrt{1 + x^2} \sqrt{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dy} dx} \right) dx \cdot \int y e^{\int_0^x b(w) dw} dz \right\} h(y) dy = 0
\]
holds for all \( h(y) \).

Therefore,
\[
\int_{-\infty}^\infty e^{\int_0^x b(w) dw} \left( e^{\int_0^x b(w) dw} - \frac{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dx}{\sqrt{\pi} \sqrt{1 + x^2} \sqrt{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dy} dx} \right) dx \cdot \int y e^{\int_0^x b(w) dw} dz = 0.
\]

And since \( \int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dz > 0 \), we have
\[
\int_{-\infty}^\infty e^{\int_0^x b(w) dw} \left( e^{\int_0^x b(w) dw} - \frac{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dx}{\sqrt{\pi} \sqrt{1 + x^2} \sqrt{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dy} dx} \right) dx = 0
\]
for \( y \in \mathbb{R} \). When taking the derivative of the above equation with respect to \( y \), we have
\[
e^{\int_0^x b(w) dw} = \frac{\sqrt{\int_{-\infty}^\infty e^{2 \int_0^x b(w) dw} dx}}{\sqrt{\pi} (1 + y^2)}, \quad \forall y \in \mathbb{R}.
\]
Therefore, after taking ‘square’ and ‘ln’ on both sides of the above equation, we have

\[ 2 \int_0^y b(w) dw = \ln(\int e^{\int_0^y b(w) dw} dx) - \ln(\pi(1 + y^2)). \]

Taking derivative with respect to \( y \), we have

\[ b(y) = -\frac{y}{\pi(1 + y^2)} \quad y \in \mathbb{R}. \]

only need \( \int_0^y b(w) dw < 0 \) for all \( y \in \mathbb{R} \).

3.2. Time-dependent case

Consider a scalar SDE

\[ dX = b(X) dt + \sigma(X) dB_t, \quad X(0) = x. \quad (3.2) \]

The Fokker-Planck equation

\[ p_t = \frac{1}{2}(\sigma^2(x)p(x,t))_{xx} - (b(x)p(x,t))_x, \quad (3.3) \]

Example 3.2. Consider a scalar SDE

\[ dX = -b \sin(X) dt + \sqrt{2} dB_t, \quad X(0) = 0. \quad (3.4) \]

(1) Assume that an observation obtained for \( p \) to be

\[ q_1(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}. \quad (3.5) \]

(2) Assume that another observation obtained for \( p \) to be

\[ q_2(x,t) = \frac{\sqrt{t}}{\pi(t + x^2)}. \quad (3.6) \]

The Fokker-Planck equation for (3.4) is

\[ p_t = p_{xx} + (b \sin(x)p)_x. \quad (3.7) \]

Find \( b \) based on the above two observation.

Solution: In this case, we define the Hellinger distance:

\[ H_i(b) = \max_{t \in [0,T]} \int_{-\infty}^{\infty} (\sqrt{q_i(x,t)} - \sqrt{p(x,t,b)})^2 dx \]
where $i = 1, 2$. We need to calculate $b$ by minimizing $H_i(b)$.

Now it’s not easy to get the analytical solution of $p(x, t)$, so we do some computer simulations instead. Assuming that the initial distribution $p(x, 0)$ is a delta function $\delta_0$ on $(-\infty, \infty)$, we can solve the above Fokker-Planck equation numerically, the solution is in the following figures.

![Fig. 0.1. Solutions of Fokker-Planck equation when $b = 1$. In the plot, $t = 5$ (left) and $t = 30$ (right).](image)

For the observation $q_1(x, t)$, it’s the solution of $p_t = p_{xx}$ with delta function $\delta_0$ as the initial condition. Hence, the corresponding SDE of $q_1(x, t)$ is $dX = \sqrt{2} \, dB, \; X(0) = 0$. We have its distribution shown below:
Fig. 0.2. Observation distribution $q_1(x, t)$. In the plot, $t = 5$ (left) and $t = 30$ (right).

By the definition $H_1(b) = \max_{t \in [0, T]} \int_{-\infty}^{\infty} (\sqrt{q_1(x, t)} - \sqrt{p(x, t, b)})^2 dx$, we have the plot of $H_1(b)$ in the following figures. And whatever $T$ is, $H_1(b)$ is always minimized when $b = 0$.

For the observation $q_2(x, t)$, we have its distribution like

By the definition $H_2(b) = \max_{t \in [0, T]} \int_{-\infty}^{\infty} (\sqrt{q_2(x, t)} - \sqrt{p(x, t, b)})^2 dx$, we have the plot of $H_2(b)$ in the following figures. So we can see that if $T = 5$, $H_2(b)$ is minimized when $b = 0.7$ and if $T = 30$, $H_2(b)$ is minimized when $b = 0.6$.

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