Induced Topology on the Hoop Group

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Abstract

A new topology is proposed on the space of holonomy equivalence classes of loops, induced by the topology of the space \( \Sigma \) in which the loops are embedded. The possible role for the new topology in the context of the work by Ashtekar et al. is discussed.

In this short paper we introduce a topology \( \tau \) on the space of holonomy equivalence classes of loops, known in the context of quantum gravity in the Ashtekar variables as the hoop group \( \mathcal{HG} \). The first section is devoted to notation by introducing hoops etc. The main part is the second section in which the topology is defined. In the last section we briefly discuss how this topology might be of interest for the work by Ashtekar et al. regarding the non-linear functional analysis of the quotient space of connections modulo gauge transformations \( \mathcal{A}/\mathcal{G} \).

1 Introduction and Notation

Consider a \( d \)-dimensional real analytical oriented paracompact manifold \( \Sigma \). Let us introduce the notation of continuous, piecewise analytic parametrized

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loos in $\Sigma$

$$\alpha : [0, \lambda_1] \cup [\lambda_1, \lambda_2] \cup \ldots \cup [\lambda_{n-1}, 1] \mapsto \Sigma \quad \alpha(0) = \alpha(1) \quad (1)$$

The mappings are continuous on the whole domain and analytical on the single closed intervals $[\lambda_i, \lambda_{i+1}]$.

We consider only loops which all share the arbitrary but fixed point $x_0 \in \Sigma$. Denote this set of based loops in $\Sigma$ by $L_{x_0}$. The standard composition “$\circ$” of loops makes $L_{x_0}$ a semigroup. The principal fibre bundle $P(\Sigma, G)$, where $G$ is a Lie group, connections and holonomies are defined as usual [3]. The topology on $\Sigma$ is unspecified so we cannot say whether the bundle will be trivial or not. Given a connection $A$ and a based loop $\alpha \in L_{x_0}$ we can compute the holonomy element $H(\alpha, A)$, which takes values in the group $G$. We follow Ashtekar and Lewandowski [1] and introduce holonomy equivalence classes.

**Definition 1.1** A pair of loops $\alpha, \beta \in L_{x_0}$ are holonomy equivalent $\alpha \sim \beta$ if $H(\alpha, A) = H(\beta, A)$ for all $G$-connections $A$. The set of equivalence classes $[\alpha]$ is denoted $\mathcal{H}G = \mathcal{H}G_{x_0} = L_{x_0}/\sim$, the space of hoops.

**Remark:** The hoop space $\mathcal{H}G$ is a group under

$$[\alpha] \cdot [\beta] = [\alpha \circ \beta] \quad , \quad [\alpha_{x_0}] = 1_{\mathcal{H}G_{x_0}} \quad (2)$$

where $\alpha_{x_0} : [0, 1] \mapsto x_0$ is the constant loop.

In the next section we propose a topology on the hoop group, which to our knowledge has not yet been discussed.

2 **The Topology $\tau$**

Let $\tau_{\Sigma}$ denote a fixed topology on $\Sigma$. The open neighbourhood filter of $x_0$ is denoted by $\mathcal{O}(x_0)$ and defined to be the family of open neighbourhoods of $x_0$

$$\mathcal{O}(x_0) = \{U_j\}_{j \in J_{x_0}} \quad (3)$$

where $J_{x_0}$ is an index set depending on $\tau_{\Sigma}$ while $U_j \in \tau_{\Sigma}$.

Now we define the set of “topology generators” $V_j$ in $\mathcal{H}G_{x_0}$. They are induced by $U_j$ in the following way

$$V_j = \{[\alpha] \in \mathcal{H}G_{x_0} \mid \exists \alpha' \in [\alpha], \alpha' \subseteq U_j\} \quad (4)$$
Some additional notational definitions are introduced

\[
V_{j_1 \cdots j_n} = \{ [\alpha] \in \mathcal{HG}_{x_0} \mid \exists \alpha' \in [\alpha], \alpha' \subseteq \bigcap_{i=1,\cdots,n} U_{j_i} \} \quad (5)
\]

\[
V_{(ji)} = \{ [\alpha] \in \mathcal{HG}_{x_0} \mid \exists \alpha' \in [\alpha], \alpha' \subseteq \bigcup_{i \in I_j} U_{j_i} \} \quad (6)
\]

where \( I_j \) is chosen such that for any \( i \in I_j, j_i \in J_{x_0} \). Such an introduction is convenient because generically we have

\[
V_{(j_1,j_2)} \supset \neq V_{\{j_1,j_2\}} \equiv V_{j_1} \cup V_{j_2} \quad (7)
\]

and more generally

\[
V_{(ji)} \supset \neq V_{\{ji\}} = \{ [\alpha] \in \mathcal{HG}_{x_0} \mid \exists \alpha', \exists U_{j_i}, j_i \in I_j, \alpha' \subseteq U_{j_i} \} \quad (8)
\]

On the other hand the following equation is valid

\[
V_{j_1 \cdots j_n} = \bigcap_{i=1,\cdots,n} V_{j_i} \quad (9)
\]

The concept "topology generator" introduced above is accounted for by our main claim

**Theorem 2.1** \( \tau = \tau(\{V_j\}_{j \in J_{x_0}}) \) is a topology on \( \mathcal{HG}_{x_0} \) where \( \{V_j\}_{j \in J_{x_0}} \) is a base of \( \tau \) in the usual (topological) sense.

**Proof:** We only have to check that \( \{V_j\}_{j \in J_{x_0}} \) is stable under finite intersections. By construction (4) and (9) this is obvious

\[
\bigcap_{i=1,\cdots,n} V_{j_i} = \{ [\alpha] \in \mathcal{HG}_{x_0} \mid \exists \alpha' \in [\alpha], \alpha' \subseteq \bigcap_{i=1,\cdots,n} U_{j_i} \}
\]

\[
\equiv \{ [\alpha] \in \mathcal{HG}_{x_0} \mid \exists \alpha' \in [\alpha], \alpha' \subseteq U_k \}
\]

\[
= V_k \quad \Box \quad (10)
\]

**Remark:** By this definition the empty set \( \emptyset \) is not an open set, but by the enlargement \( \tau' = \tau \cup \emptyset \) we obtain a perhaps more conventional type of topology.

Because \([\alpha_{x_0}] \in V_j \) for any \( j \in J_{x_0}, (\mathcal{HG}_{x_0}, \tau) \) is not a Hausdorff topological space.

\footnotesize{\(^1\)According to and others it is strictly speaking not necessary to include \( \emptyset \) to have a topology.}
3 Outlook

The question of topology on the hoop group is interesting in its own right, but our motivation has come from the use of the hoop group in the context of quantum gravity in the Ashtekar formalism. In this context $\Sigma$ is chosen as a 3-manifold and usually one chooses $G = SU(2)$ whereby $P(\Sigma, G)$ is trivial. $\Sigma$ plays the role as a space-slice in the canonical 3+1 dimensional formulation of quantum gravity. The topology of classical space-time is chosen to be $\Sigma \times \mathbb{R}$. The role of the hoop group is related to the problem of constructing measures on the configuration space, which is the essence of a quantum field theory.

In the recent work by Ashtekar et.al. [1] [2] a non-linear duality between loops and connections has been explored, in order to generalize the notation from ordinary QFT [3] to the non-linear configuration space of Ashtekar gravity and full Yang-Mills theory in 4 dimensions. The holonomy defines this duality and by using $\mathcal{H}\mathcal{G}$ as the dual space to $A/\mathcal{G}$, Ashtekar and Lewandowski have introduced cylindrical measures on $A/\mathcal{G}$ [1]. The holonomy can be viewed as a functional $T_\alpha(A) = tr H([\alpha], [A]_G)$ on $A/\mathcal{G}$.

The characteristic function is a functional on $\mathcal{H}\mathcal{G}$ [2]

$$\chi([\alpha]) = \int_{A/\mathcal{G}} T_{\alpha}([A]_G)d\mu_{AL}([A]_G)$$

where the bars denote that the integral and the $T_{\alpha}$ functional are extended to a larger space, here the algebraic dual of $A/\mathcal{G}$. $d\mu_{AL}$ is the Ashtekar-Lewandowski measure, induced by the Haar measure on $SU(2)^n/Ad$ in the projective limit. As stated in [2] the function is only known to be continuous in the discrete topology on $\mathcal{H}\mathcal{G}$. We see that hoops play the role as test-functions in this scheme and the topology on the hoop group plays a role similar to the topology on the Schwarz space in ordinary QFT. The discrete topology is known to be too simple for the formulation of an interesting quantum theory. A more promising topology has been introduced by Barret [6] and discussed by Lewandowski [7], though one has not yet been able to use it in the study of continuity of $\chi([\alpha])$.

It will be interesting to know whether $\chi([\alpha])$ is continuous or not in the topology $\tau_\Sigma$ (depending on $\tau_\Sigma$) proposed in this paper. This and other questions are under current investigation [8].
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