1/(M_R)_{33} expansion of the type-I seesaw mechanism and partial Z_2 symmetry for TM_{1,2} mixing

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We consider an expansion of the type-I seesaw mechanism by the inverse of the 3-3 matrix element 1/(M_R)_{33} of the mass matrix of right-handed neutrinos M_R. Conditions of such a situation are obtained for M_R and the Dirac mass matrix m_D.

In this case, a partial Z_2 symmetry such as Sm_D P = \pm m_D P with a projection matrix P = \text{diag}(1, 1, 0) leads to an approximate Z_2 symmetry by S for the neutrino mass matrix m_\nu. Such a partial Z_2 symmetry is desirable in the context of unified theories because it allows hierarchical m_D and the large mixing of m_\nu simultaneously.

I. INTRODUCTION

The minimal seesaw model has been studied extensively as a toy model of the three-generation seesaw mechanism, because it is a limit where the mass of the heaviest right-handed neutrino M_3 is taken to infinity in the three-generation model. Decoupling effect of contribution from M_3 is discussed in the sequential dominance and in Ref. 36. In this letter, to consider a situation where M_3 is large but its contribution is finite, the type-I seesaw mechanism in a general basis is expanded by the inverse of the largest 3-3 matrix element 1/(M_R)_{33} of the mass matrix of the right-handed neutrinos M_R. If the terms proportional to 1/(M_R)_{33} are sufficiently small, the type-I seesaw mechanism can be described as the sum of the minimal seesaw model and its perturbations. We investigate its phenomenological consequences.

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II. $1/(M_R)_{33}$ EXPANSION

In the type-I seesaw mechanism, the Dirac mass matrix $m_D$ and the symmetric Majorana mass matrix $M_R$ of the right-handed neutrinos $\nu_{R_i}$ are defined as

$$m_D = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \equiv (A, B, C), \quad M_R = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}.$$  \hspace{1cm} (1)

These matrix elements $A_i, B_i, C_i$ and $M_{ij}$ are general complex parameters. By assuming that $M_R$ is hierarchical ($|M_{33}| \gg |M_{ij}|$), the absolute value of $M_{33}$ is close to the heaviest mass value $M_3 \simeq |M_{33}|$. Since this model is reduced to the minimal seesaw model \cite{1, 2, 28} in the limit of $|M_{33}| \to \infty$, we consider isolating its contribution.

First, $M_R$ is divided into the lighter (first and second) and third generations as,

$$M_R \equiv \begin{pmatrix} M_{R0} & u \\ u^T & M_{33} \end{pmatrix} \equiv \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{pmatrix}.$$  \hspace{1cm} (2)

Considering blockwise inversion for $M_R$, we obtain a seesaw-like expression;

$$\begin{pmatrix} M_{R0} & u \\ u^T & M_{33} \end{pmatrix}^{-1} = \begin{pmatrix} M_{R0}^{-1} + M_{R0}^{-1}u(M_{R}^{-1})_{33}u^TM_{R0}^{-1} - M_{R0}^{-1}u(M_{R}^{-1})_{33} \\ -(M_{R}^{-1})_{33}u^TM_{R0}^{-1} & (M_{R}^{-1})_{33} \end{pmatrix},$$  \hspace{1cm} (3)

where

$$(M_{R}^{-1})_{33} = (M_{33} - u^T M_{R0}^{-1} u)^{-1} = \frac{\det M_{R0}}{\det M_R} \equiv \frac{M_{11} M_{22} - M_{12}^2}{(M_{11} M_{22} - M_{12}^2) M_{33} - N},$$  \hspace{1cm} (4)

with $N = M_{22} M_{13}^2 - 2 M_{12} M_{23} M_{13} + M_{11} M_{23}^2$. The inverse of $M_R$ is then

$$M_R^{-1} = \begin{pmatrix} M_{R0}^{-1} & 0 \\ 0 & V \end{pmatrix} + (M_{R}^{-1})_{33} V \otimes V^T,$$  \hspace{1cm} (5)

where

$$V \equiv \begin{pmatrix} -M_{R0}^{-1} u \\ 1 \end{pmatrix} = \begin{pmatrix} (M_{R}^{-1})_{13} \\ (M_{R}^{-1})_{23} \\ (M_{R}^{-1})_{33} \\ 1 \end{pmatrix} = \begin{pmatrix} M_{13} M_{23} - M_{13} M_{22} \\ M_{11} M_{22} - M_{12}^2 \\ M_{12} M_{13} - M_{11} M_{23} \\ M_{11} M_{23} - M_{12} M_{22} \end{pmatrix}.$$  \hspace{1cm} (6)
Since $M_{33}$ only appears in $\det M_R$, it is easy to perform a series expansion by $1/|M_{33}| \ll 1/|M_{ij}|$:

$$M_R^{-1} = \left( \begin{array}{cc} M_{R0}^{-1} & 0 \\ 0 & 0 \end{array} \right) + \frac{1}{M_{33}} \left( 1 + \left( \frac{N}{M_{33} \det M_{R0}} \right) + \left( \frac{N}{M_{33} \det M_{R0}} \right)^2 + \cdots \right) V \otimes V^T. \tag{7}$$

From Eqs. (3) and (4), the type-I seesaw mechanism can be written as follows:

$$m_\nu = m_D M_R^{-1} m_D^T = m_D \left[ \begin{array}{cc} M_{R0}^{-1} & 0 \\ 0 & 0 \end{array} \right] \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + (M_R^{-1})_{33} V \otimes V^T \right] m_D^T \tag{8}$$

$$= m_D 0 \left( M_{R0}^{-1} m_D^T + (M_R^{-1})_{33} c \otimes c^T \equiv m_{\nu 0} + \delta m_\nu, \tag{9}$$

where

$$m_{\nu 0} \equiv \left( \begin{array}{cc} A & B \end{array} \right), \quad c \equiv m_D V = \left( \frac{M_{R0}^{-1}}{M_R^{-1}} \right)_{13} A + \left( \frac{M_{R0}^{-1}}{M_R^{-1}} \right)_{23} B \equiv c \otimes c^T.$$

This expression of $m_\nu$ is valid in any basis (and even for non-hierarchical $M_R$) and separates minimal seesaw contributions from the rest. Note that factors $(M_{R}^{-1})_{(1,2)3}/(M_{R}^{-1})_{33}$ are finite in the limit of $M_{33} \to \infty$.

Moreover, performing the LDL$^T$ (or generalized Cholesky) decomposition for the minimal seesaw model, we obtain

$$M_{R0}^{-1} = \frac{1}{\det M_{R0}} \left( \begin{array}{cc} M_{22} & -M_{12} \\ -M_{12} & M_{11} \end{array} \right) = \frac{1}{\det M_{R0}} \left( \begin{array}{cc} M_{22} & -M_{12} \\ -M_{12} & M_{11} \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & 1/M_{22} \end{array} \right) \tag{11}$$

$$= \left( \begin{array}{cc} 1 & 0 \\ 0 & 1/M_{22} \end{array} \right) \left( \begin{array}{cc} M_{22} \det M_{R0} & 0 \\ 0 & 1/M_{22} \end{array} \right) \left( \begin{array}{cc} 1 & -M_{12} \\ 0 & 1 \end{array} \right) \equiv LDL^T, \tag{12}$$

and a formula of $m_{\nu 0}$ in a general basis [37, 38]:

$$m_{\nu 0} = \frac{M_{22}}{\det M_{R0}} a \otimes a^T + \frac{1}{M_{22}} B \otimes B^T, \quad a = A - B \frac{M_{12}}{M_{22}}. \tag{13}$$

This formula is used in a partial $Z_2$ symmetry that will be shown later.

**A. Conditions for $\delta m_\nu$ to be considered perturbations**

In Eqs. (8) and (9), let us consider a situation where the absolute values of the second term $\delta m_\nu$ are sufficiently smaller than that of $m_{\nu 0}$. To this end, matrix elements $(\delta m_\nu')_{ij}$ in the
diagonalized basis of $m_{\nu 0}$ must be regarded as perturbations compared to the singular values of $m_{\nu 0}$. Probably such conditions will be solutions to complex equations. Here, to investigate simpler conditions, we focus on the 1-2 block matrix of Eq. (3);

$$
(M^{-1}_R)_{ab} = (M^{-1}_{R0} + M^{-1}_{R0} u(M^{-1}_R)_{33} u^T M^{-1}_{R0})_{ab},
$$

(14)

with $a, b = 1, 2$. These terms yield $m_{\nu 0}$ and a part of $\delta m_{\nu}$, respectively;

$$
m_{\nu 0} = m_{D0} M^{-1}_{R0} M_{D0}^T, \quad \delta m_{\nu 0} \equiv m_{D0} (M^{-1}_{R0} u(M^{-1}_R)_{33} u^T M^{-1}_{R0}) m_{D0}.
$$

(15)

Since $m_{D0}$ is common in $m_{\nu 0}$ and $\delta m_{\nu 0}$, we can consider a condition by comparing the two terms $M^{-1}_{R0}$ and $M^{-1}_{R0} u(M^{-1}_R)_{33} u^T M^{-1}_{R0}$. For each matrix element, it is roughly given by

$$
0.1 |(M^{-1}_{R0})_{ab}| \gtrsim |(M^{-1}_{R0} u(M^{-1}_R)_{33} u^T M^{-1}_{R0})_{ab}|.
$$

(16)

From Eq. (12) and $(M^{-1}_R)_{33} \simeq M^{-1}_{33}$, the condition becomes

$$
\left| \frac{0.1}{M_{33} \det M_{R0}} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{12} & M_{11} \end{pmatrix} \right| \gtrsim \left| \begin{pmatrix} (M^{-1}_R)_{13} \\ (M^{-1}_R)_{23} \end{pmatrix} \otimes \begin{pmatrix} (M^{-1}_R)_{13} \\ (M^{-1}_R)_{23} \end{pmatrix} \right|.
$$

(17)

After all, this inequality is equivalent to comparing the matrix elements of $M^{-1}_{R0}$ with the coefficients of $A, B$ in Eq. (10). Given this inequality, $0.1 |m_{\nu 0}| \gtrsim |\delta m_{\nu 0}|$ holds for any $A$ and $B$. However, this condition does not make sense if there is a zero texture such as $M_{11} = 0$.

A more concise condition is obtained by comparing the matrix elements in the diagonalized basis of $M^{-1}_{R0}$. At first we define $D = \text{diag}(M^0_1, M^0_2)$ where $M^0_{1,2}$ is the mass singular values restricted to the minimal seesaw mechanism. By substituting $M^{-1}_{R0} = U^T D^{-1} U$ in Eq. (14) and multiplying $D U^*$ and $U^T D$ from the left and right, the 1-2 block matrix becomes

$$
U^T D^{-1} U + U^T D^{-1} U u(M^{-1}_R)_{33} u^T U^T D^{-1} U,
$$

(18)

$$
D + U u(M^{-1}_R)_{33} u^T U.
$$

(19)

By comparing the diagonal elements, another condition for $\delta m_{\nu 0}$ is obtained as

$$
0.1 \begin{pmatrix} M^0_1 \\ M^0_2 \end{pmatrix} \gtrsim \frac{1}{M_{33}} \begin{pmatrix} |(U u)_1|^2 \\ |(U u)_2|^2 \end{pmatrix}.
$$

(20)

This can be regarded as a perturbative condition in a basis where $M^{-1}_{R0}$ is diagonal. Given the hierarchy $|M_{22}| \gg |M_{12}|, |M_{11}|$, the mass values are roughly evaluated as $M^0_1 \simeq$
\[
|\det M_{R0}/M_{22}|, M_2^0 \simeq |M_{22}| \text{ and } U \sim 1 \text{ holds. Thus the perturbative condition is approximately}
\[
0.1|M_{33}| \left( \left| \frac{\det M_{R0}}{M_{22}} \right| \right) \gtrsim \left( \frac{|M_{13}|^2}{|M_{23}|^2} \right).
\]

Eqs. (17) and (21) are inequalities for absolute values of the 2-dimensional vectors \( M_{R0}^{-1}u = U^T D^{-1} U u \) and \( U u \). Thus, if \( U \) is small mixing \( U \sim 1 \), the two inequalities are approximately equivalent through the scale transformation by \( D \).

In the limit where the off-diagonal elements \( M_{ij} \) are small, the upper bound of Eq. (21) corresponds to the singular values \( M_{1,2,3} \) of \( M_R \). Thus, these bounds approximately lead to the following form of \( M_R \):
\[
M_R \sim \begin{pmatrix}
\epsilon * |M_{13}| & \lesssim 0.3 \sqrt{\epsilon M_{33}} \\
\delta |M_{23}| & \lesssim 0.3 \sqrt{\delta M_{33}} \\
M_{33}
\end{pmatrix}.
\]

Here, \( O(1) \) coefficients are omitted and there is not much restriction on \( M_{12} \) as long as \( |M_{12}/M_{22}| \lesssim 1 \) holds.

Similarly, focusing on the 3-3 element of Eq. (3) or the term of \( C \otimes C^T \) in Eq. (10), we obtain a new condition for \( C \);
\[
\left| \frac{C_i^2}{M_{33}} \right| \lesssim 0.1 \left\{ \frac{A_i^2 M_{22}}{\det M_{R0}}, \frac{B_i^2 M_{11}}{\det M_{R0}} \right\}.
\]

In other words, the condition is simply written as \( |C_i|^2 \lesssim 0.1 |M_{33}| \{ \sqrt{\Delta m_{21}^2}, \sqrt{\Delta m_{31}^2} \} \) by the mass squared differences \( \Delta m_{ij}^2 \). If the magnitude of \( C_3 = (m_D)_{33} \) is close to the mass of the top quark \( m_t \), we obtain a lower bound for \( M_3 \):
\[
M_3 \gtrsim \frac{10 m_t^2}{\sqrt{\Delta m_{31}^2}} \simeq 5.81 \times 10^{15} \text{ GeV}.
\]

Although this fact has been mentioned in Ref. [36], interestingly, the lower bound is quite close to the scale of grand unified theories (GUTs). When these two perturbative conditions (Eq. (17) or (21) and Eq. (23)) is satisfied, the contribution from the off-diagonal block in Eq. (3) (or cross terms such as \( A \otimes C^T \) and \( B \otimes C^T \) in Eq. (10)) is automatically small. Therefore two conditions are sufficient.
B. Partial $Z_2$ symmetry

In such a “quasi-minimal” seesaw model, a partial $Z_2$ symmetry of $m_D$ leads to an approximate $Z_2$ symmetry of $m_\nu$. The following TM$_{1,2}$ mixing \cite{33-41} is well discussed in many seesaw models \cite{42-47};

$$U_{T1} = U_{TBM}U_{23}, \quad U_{T2} = U_{TBM}U_{13}, \quad (25)$$

where

$$U_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & 0 \\ 0 & -s_{23}e^{i\phi} & c_{23} \end{pmatrix}, \quad U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\phi} \\ 0 & 1 & 0 \\ -s_{13}e^{i\phi} & 0 & c_{13} \end{pmatrix}, \quad (26)$$

with $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$. The mass matrix $m_\nu$ that predicts TM$_{1,2}$ mixing has a $Z_2$ symmetry $S_{1,2}m_\nu S_{1,2} = m_\nu$ with \cite{48-51}

$$S_1 = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \quad S_2 = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}. \quad (27)$$

In the model where the contribution of $1/M_{33}$ can be regarded as perturbations, the conditions for $Z_2$ symmetry will be relaxed. If $\tilde{m}_\nu = (a, B, C)$ has the following partial $Z_2$ symmetry, $m_\nu$ has the $Z_2$ symmetry in a good approximation;

$$S_{1,2}\tilde{m}_\nu P = \pm\tilde{m}_\nu \quad \text{or} \quad S_{1,2}\tilde{m}_\nu P' = \pm\tilde{m}_\nu, \quad (28)$$

where $P = \text{diag}(1, 1, 0)$ and $P' = \text{diag}(1, -1, 0)$ are projections to the lighter generations. This is because, from Eq. (13),

$$S_{1,2}\delta m_\nu S_{1,2} = S_{1,2}\tilde{m}_\nu P \begin{pmatrix} M_{22} & 0 & 0 \\ 0 & \text{det} M_{00} & 0 \\ 0 & 0 & M_{12} \end{pmatrix} P\tilde{m}_\nu S_{1,2} = m_\nu, \quad (29)$$

and $\delta m_\nu$ can be regarded as a perturbation. Although the first condition $S_{1,2}\tilde{m}_\nu P = \pm\tilde{m}_\nu P$ leads to partial $Z_2$-symmetric $m_D$ (without tilde), the case $S_{2}\delta m_D P = \pm m_D P$ is unsuitable because the limit of $M_{33} \rightarrow \infty$ predicts $m_2 = 0$ or $m_{1,3} = 0$ \cite{52}. Such a situation is similar
to the constrained sequential dominance [53–57]. This partial $Z_2$ symmetry can be regarded as its extension because it predicts $T_{M_{1,2}}$ for general parameter regions and general basis where $M_R$ is not diagonal. Such a partial $Z_2$ symmetry is desirable in the context of GUTs because it allows hierarchical Dirac neutrino mass $m_D$ and the large mixing of $m_\nu$ simultaneously.

III. SUMMARY

To summarize, in this letter, we consider an expansion of the type-I seesaw mechanism by the inverse of the 3-3 matrix element $1/(M_R)_{33}$ of the right-handed neutrinos. If the terms proportional to $1/(M_R)_{33}$ are sufficiently small, the type-I seesaw mechanism can be described as the sum of the minimal seesaw model and its perturbations. We obtain perturbative conditions of such a situation for the mass matrix of the right-handed neutrinos $M_R$ and the Dirac mass matrix $m_D$. The conditions are approximately written by $|(M_R)_{13}|^2 \lesssim 0.1 M_1 M_3$, $|(M_R)_{23}|^2 \lesssim 0.1 M_2 M_3$ and $(m_D)^2 \lesssim 0.1 M_3 \{\sqrt{\Delta m_{21}^2}, \sqrt{\Delta m_{31}^2}\}$ with the mass singular values $M_i$ of $M_R$.

Moreover, conditions that the neutrino mass matrix $m_\nu$ has $Z_2$ symmetry is relaxed. It is found that $m_\nu$ has an approximate $Z_2$ symmetry when the deformed Dirac mass matrix $\tilde{m}_D$ has a partial $Z_2$ symmetry such as $S\tilde{m}_DP = \pm \tilde{m}_DP$, restricted by a projection matrix $P = \text{diag}(1, 1, 0)$ to the two lighter generations.

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