A NEW CLASS OF NON-INJECTIVE POLYNOMIAL LOCAL DIFFEOMORPHISMS ON THE PLANE

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Abstract. In this short note we provide the example with the lowest degree known so far of a non-injective local polynomial diffeomorphism $F = (p, q) : \mathbb{R}^2 \to \mathbb{R}^2$. In our example $p$ has degree 10 and $q$ has degree 15, rather than 10 and 25, respectively, known up to now as the smallest degrees for the coordinates of $F$. Our construction was based on S. Pinchuk celebrated counterexample to the real Jacobian conjecture.

1. Introduction

The real Jacobian conjecture claims that a polynomial local diffeomorphism $F = (p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ must be injective. It was proved false by means of a class of counterexamples, the Pinchuk maps, constructed by Pinchuk [4]. A Pinchuk map $F = (p, q)$ is such that the polynomial $p$ is always the same and the polynomial $q$ varies as the sum of a fixed polynomial with another one that satisfies a suitable differential equation. The degree of $p$ is 10 and it is known that the smallest possible degree of $q$ in a Pinchuk map is 25 [5].

On the other hand it is known that any polynomial local diffeomorphism $F = (p, q)$ with $\deg(p) \leq 4$ satisfies the real Jacobian conjecture [3, 4, 6]. In this note we provide a new class of counterexamples to the real Jacobian conjecture, where the polynomial $p$ is the same as in a Pinchuk map, but now it is possible to take $q$ with degree 15.

2. Construction of the Example

We consider the same three auxiliary polynomials as in Pinchuk’s paper:

$$t = xy - 1, \quad h = xt^2 + t, \quad f = \frac{h^3 + h^2}{h - t} = (xt + 1)^2(t^2 + y).$$

From now on we denote by $J(a, b)$ the Jacobian determinant of a map $(a, b) : \mathbb{R}^2 \to \mathbb{R}^2$, that is,

$$J(a, b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x}.$$

As in [4] Lemma 2.1:

**Proposition 2.1.** The following differential properties are true:

(a) $J(h, t) = h - t$,
(b) $J(f, h) = -f$
Proof. Both items follow from straightforward computations using the chain and the product rules and the linearity of differentiation, as well as the usual determinant properties. For item (a) we have

\[ J(h,t) = J(xt^2 + t,t) = t^2 J(x,t) = xt^2 = h-t. \]

For item (b)

\[ J(f,h) = J \left( \frac{h^3 + h^2}{h-t}, h \right) = (h^3 + h^2) J \left( (h-t)^{-1}, h \right) \]
\[ = - \frac{h^3 + h^2}{(h-t)^2} J(-t,h) \]
\[ = -f, \]

where the last equality follows from item (a).

As in Pinchuk example, we take the first coordinate of our example as

\[ p = f + h \]

which can be shown being, up to scaling, the only linear combination of f and h that has a non-vanishing gradient. Moreover, since the level set \( p = 0 \) is disconnected, as \( xt + 1 \) is a factor of \( p \) and \( xt + 1 = 0 \) is equivalent to \( y = (x-1)/x^2 \) that has two connected components, it follows that for any \( q \) such that \( J(p,q) > 0 \), the map \( (p,q) \) is not a global diffeomorphism, and so it is not injective as we can conclude from the main result of [1].

Proposition 2.2. Let \( m \) be an integer and \( G \) be a real differentiable function, the following holds:

\[ J(p,G(h)f^m) = -f^{m+1}G'(h) + mf^m G(h) \]

Proof. Acting analogously as in the proof of Proposition 2.1,

\[ J(p,G(h)f^m) = J(f,G(h)f^m) + J(h,G(h)f^m) \]
\[ = f^m J(f,G(h)) + G(h) J(h,f^m) \]
\[ = f^m G'(h) J(f,h) + m G(h) f^{m-1} J(h,f), \]

and the claimed identity now follows from item (b) of Proposition 2.1.

The next step in the construction is to take \( q \) as

\[ q = \sum_{i=-2}^{1} f^i M_i(h), \]

where \( M_i(h), i \in \{-2,-1,0,1\} \), are differentiable functions. Choosing \( q \) in this way is convenient because after computing \( J(p,q) \), we can find conditions on \( M_i(h) \) for the Jacobian to be positive and for \( q \) to be a polynomial.

Proposition 2.3. Let \( q \) as in (1), then

\[ J(p,q) = -2 \frac{M_{-2}(h)}{f^2} + \sum_{i=-1}^{1} f^i \left( i M_i(h) - M_{i-1}'(h) \right) - M_1'(h) f^2. \]
Proof. By Proposition 2.2 we have
\[
J(p, q) = \sum_{i=-2}^{1} J(p, f^i M_i(h)) = \sum_{i=-2}^{1} (-f^{i+1}M'_i(h) + if^i M_i(h)),
\]
and the result follows from collecting the powers of \(f\).

Our goal now is to find hypothesis on \(q\) in order to have \(J(p, q) > 0\). In his construction, Pinchuk uses the fact that \(J\) assumes coefficients of \(M\) then \(q\) then \(f\) to construct \(P, Q, N\) then \(t\) is a suitable polynomial. Instead, by using Groebner basis we looked for other algebraic combinations of \(t\) and \(h\) in a way that these combinations did not vanish simultaneously with \(f\). We found out that \(f\) and
\[
c(h^2 + h) - t = \frac{h^3 + h^2}{f} - h + c(h + h^2)
\]
have this property, where \(c \in \mathbb{R}\):

**Proposition 2.4.** Let \(c \in \mathbb{R}\), then \(f\) and \(c(h^2 + h) - t\) do not vanish simultaneously.

**Proof.** By definition, if \(f = 0\), we have that \(xt + 1 = 0\) or \(t^2 + y = 0\). In both cases \(t \neq 0\). However, if \(xt + 1 = 0\) then \(h = 0\), and if \(t^2 + y = 0\) then \(h + 1 = 0\). In both cases \(c(h^2 + h) = 0\) concluding that \(f\) and \(c(h^2 + h) - t\) do not vanish simultaneously.

Next theorem provides conditions on \(q\), defined by (1), in a way that
\[
J(p, q) = \left(\frac{h^3 + h^2}{f} - h + c(h^2 + h)\right)^2 + \left(\frac{h^3 + h^2}{f} + N_0(h) + N_1(h)f\right)^2 + f^2,
\]
where \(N_0\) and \(N_1\) are suitable differentiable functions, and therefore \(J(p, q) > 0\). It will also show conditions for \(q\) to be a polynomial.

**Theorem 2.5.** Let \(q\) be as in (1), and \(N_0\) a differentiable function, defining
\[
M_{-2}(h) = -(h^3 + h^2)^2,
M_{-1}(h) = 2(h^3 + h^2)((3 - c)(h^2 + h) - N_0(h)),
N_1(h) = \frac{M'_{-1}(h) + h^2(ch + c - 1)^2 + N_0(h)^2}{2(h^3 + h^2)},
M_1(h) = -\int_0^h N_1(s)^2 ds - h,
M_0(h) = \int_0^h (M_1(s) - 2N_0(s)N_1(s))ds,
\]
then \(J(p, q)\) satisfies (3). Moreover, given any \(K(h) \in \mathbb{R}[h]\), if we additionally assume
\[
N_0(h) = -h + (h^2 + h)K(h),
\]
then \(q\) is a polynomial in \((x, y)\).

**Proof.** By acting as if \(f\) and \(h\) are independent variables, it suffices to compare the coefficients of \(f^m\) in (2) and (3) to verify the first part of the result. For the second part we first observe that \(M_{-2}(h)/f^2\) and \(M_{-1}(h)/f\) are polynomials in \((x, y)\) since
(h^3 + h^2)/f is a polynomial in (x, y). Now by unwinding M′_1(h) and completing
squares of N_1(h) on the N_1 expression it follows that
\[ N_1(h) = -\frac{3h^2 + 2h - N_0(h)^2}{2(h^3 + h^2)} - \frac{1}{2} (c - 3) (ch + c - 7h - 5) + N'_0(h). \]
Under the hypothesis it now follows that N_1 is a polynomial in \( h \), so M_0 and M_1
are polynomials, and therefore \( q \) is a polynomial as we wanted. □

**Corollary 2.6.** If we take \( p = f + h \) and
\[ q = -\frac{(h^3 + h^2)^2}{f^2} - \frac{2h^2 (h^3 + h^2)}{f} + 4h^3 + \frac{3h^2}{2} - 5hf, \]
the map \((p, q)\) is a non-injective polynomial local diffeomorphism. The degree of \( p \)
is 10 and the degree of \( q \) is 15.

**Proof.** It suffices to take \( c = 1 \) and \( K(h) = 3 \), hence \( N_0(h) = 3h^2 + 2h \), on Theorem
[2.5]. The polynomial \( q \) has degree 15, since the degrees of \( h \) and \( f \) are 5 and 10,
respectively. □

**Remark 1.** If we take \((p, q)\) as in Corollary 2.6 we have
\[ J(p, q) = \left( \frac{h^3 + h^2}{f} + h^2 \right)^2 + \left( \frac{h^3 + h^2}{f} + 3h^2 + 2h - 2f \right)^2 + f^2. \]

**Remark 2.** We point out that by taking \( c = 0 \) and \( K(h) = 0 \) on Theorem 2.5 \( q \)
is the Pinchuk map of degree 25 that can be found on 2-5, so our construction
generalizes Pinchuk’s construction up to triangular automorphisms (see 5, Section 2).

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