ON THE KOLMOGOROV THEOREM FOR SOME INFINITE-DIMENSIONAL
HAMILTONIAN SYSTEMS OF SHORT RANGE

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ABSTRACT. In this paper, it is proved that the infinite KAM torus with prescribed frequency exists in a sufficiently small neighborhood of a given $I^0$ for nearly integrable and analytic Hamiltonian system $H(I, \theta) = H_0(I) + \epsilon H_1(I, \theta)$ of infinite degree of freedom and of short range. That is to say, we will give an extension of the original Kolmogorov theorem to the infinite-dimensional case of short range. The proof is based on the approximation of finite-dimensional Kolmogorov theorem and an improved KAM machinery which works for the normal form depending on initial $I^0$.

1. INTRODUCTION AND MAIN RESULTS

1.1. Motivations. Since Kolmogorov’s work in 1954, remarkable results have been obtained in perturbation theory of integrable Hamiltonian systems. Here we share some of the most important extensions based on the original Kolmogorov theorem for a better understanding of this perturbation theory.

We recall the fact that Kolmogorov announced that most invariant tori for the integrable Hamiltonian systems persist under small perturbations. More concretely, consider the nearly integrable Hamiltonian system of $n$-freedom

\[ H(I, \theta) = H_0(I) + \epsilon H_1(I, \theta), \]

with the symplectic structure $dI \wedge d\theta$ on $\mathbb{R}^n \times \mathbb{T}^n$ and the action-angle variables $(I, \theta)$ belonging to some domain $D \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$. If $H_0(I)$ is analytic and satisfies the non-degenerate condition

\[ \det(\partial^2 H_0(I)) \neq 0, \quad I \in D. \]

Kolmogorov theorem claims that any invariant tori of the unperturbed $H_0$ with prescribed Diophantine frequency $\omega = \omega(I^0) = \frac{\partial H_0(I^0)}{\partial I} \bigg|_{I=I^0}$ for some $I^0 \in D$ persist under a small analytic perturbation $\epsilon H_1(I, \theta)$. Actually, Kolmogorov also gave a precise outline of its proof which is based on a fast convergent Newton scheme. Roughly speaking, one can set up a Newton scheme by replacing $H_1$ with $H_{j+1} = H_j \circ \Phi_j$ after taking symplectic transformation $\Phi_j$ such that the new Hamiltonian $H_{j+1}$ is of super-exponential decay. Taking $j = 1$ as an example, the map $\Phi_1$ is close to the identity and can be regarded as the composition of two “elementary” symplectic transformations: $\Phi_1 = \Phi_1^{(1)} \circ \Phi_1^{(2)}$, where $\Phi_1^{(2)} : (I', \theta') \mapsto (\eta, \xi)$ is the symplectic map generated by $I' \cdot \xi + \epsilon I' \cdot a(\xi)$, while $\Phi_1^{(1)} : (\eta, \xi) \mapsto (I, \theta)$ is the angle-dependent translation generated by $\eta \cdot \theta + \epsilon (b \cdot \theta + s(\theta))$ with real-analytic functions $a(\xi)$ and $s(\theta)$ being of zero average on $\mathbb{T}^n$ and $b \in \mathbb{R}^n$. Obviously, $\Phi_1^{(2)}$ acts in “angle direction” and will be needed to straighten out the flow up to order $O(\epsilon^2)$, while $\Phi_1^{(1)}$ acts in “action direction” and will be needed to keep the frequency $\omega(I^0)$ of the torus fixed. Indeed, the vector $b$ above and the non-degenerate condition (1.2) are sufficient to overcome the frequency drift (which is a key idea introduced by Kolmogorov). See [8] for the details. Later, the rigorous proof was given by Arnold in the analytic category, while Moser also proved it for the finitely differentiable exact symplectic mappings. This theorem now is called the classical KAM theorem. See [6], and the references therein.

Naturally, it is hoped that the KAM theorem of finite-dimensional Hamiltonian systems can be extended to infinite-dimensional ones. Different from the finite-dimensional case, the KAM theorem of infinite-dimensional Hamiltonian systems is generally wrong if it is assumed only that the perturbation

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is small and sufficiently smooth. Hence, two slightly special cases, Hamiltonian partial differential equations (PDEs) and the Hamiltonian systems defined on the infinite lattices, have been taken into account. On the one hand, the infinite-dimensional KAM theory has seen enormous progress with application to Hamiltonian PDEs since Kuksin [22] and Wayne [34]. As an example to which infinite-dimensional KAM theory applies, consider the nonlinear Schrödinger equation (NLS)

$$\sqrt{-1} u_t - \Delta u + V(x, \omega) u + |u|^2 u + h.o.t = 0,$$

subject to Dirichlet condition. Kuksin [24] showed that (1.3) possesses lower dimensional invariant tori around $u = 0$ for “most” parameters $\omega$. See [2, 4, 10, 12, 14, 16, 18, 21, 22, 23, 25, 26, 30, 31, 32, 33, 34, 35, 38, 39] for more related results. In those literatures, the frequency vectors $\omega \in \mathbb{R}^n$ or initial data are regarded as the parameters. In [13], the frequency

$$\omega = \omega_0 t,$$

where $\omega_0 \in \mathbb{R}^n$ is a fixed Diophantine vector and $t \in \mathbb{R}$ is a parameter. Even for the finite-dimensional Hamiltonian, Bourgain [3] showed that, at least, a 1-dimensional parameter is needed to guarantee the existence of the KAM torus with a fixed frequency, which is a progress of the results in [13, 22]. Since the freedom of Hamiltonian PDEs is infinite and the dimension of the obtained KAM tori is finite, the KAM tori are called lower-dimensional tori. Note that the dimension of the invariant tori equals the freedom of Hamiltonian PDEs is infinite and the dimension of the obtained KAM tori is finite, existence of the KAM torus with a fixed frequency, which is a progress of the results in [13, 27]. Since random Fourier multiplier

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On the other hand, let us consider models in mathematical physical which consist of lattices of harmonic oscillators with independent identically distributed random frequencies, subject to un-harmonic coupling force which are of finite range, short range, or hierarchy. Bellissard, Vittot [33], and Fröhlich, Spencer, Wayne [18] showed that there is a set $\Omega \subseteq \mathbb{R}_+^d$ with $\text{Prob}(\Omega) > 0$ (where “Prob” is some probability measure) such that for some $\omega = (\omega_i)_{i \in \mathbb{Z}} \in \Omega$, there exists an infinitely dimensional KAM-torus for the following Hamiltonian

$$H = H(I, \theta) = \sum_{j \in \mathbb{Z}^d} \omega_j I_j + \epsilon P(I, \theta),$$

where $0 < \epsilon \ll 1, d \geq 1$ and $P$ is of short range. Afterwards, different kinds of systems with short range have been deeply investigated by many authors. See [7, 22, 36, 37], for example. It is easily observed that such infinite-dimensional Hamiltonian systems are well approximated by finite-dimensional ones and consequently the classical, finite-dimensional KAM technique works in this case. That is, these infinite invariant tori are obtained by successive small perturbations of finite-dimensional tori. Such results were obtained in [6, 29, 31] by a somewhat similar method. However, the frequencies of those full-dimensional KAM tori are not prescribed, so they are not in the class of Kolmogorov’s.

In the present paper, we will construct a Kolmogorov theorem for the following infinite-dimensional Hamiltonian system

$$H(I, \theta) = H_0(I) + \epsilon H_1(I, \theta), (I, \theta) \in \mathbb{C}^d \times T^d = \mathbb{C}^d \times (\mathbb{C}/2\pi \mathbb{Z})^d$$

provided that $H_0(I), H_1(I, \theta)$ are analytic and of short range. Thus, in order to construct a KAM torus of prescribed frequency, the key difficulty is to eliminate the drift of the frequency. We will overcome the difficulty by advantage of the facts that the Hamilton system (1.5) is of short range and the perturbation is high orders of action variables $(I_j)_{j \in \mathbb{Z}}$ taking an exponential norm with weight $e^{(|j|^{1+\alpha})}$ for any $\alpha > 0$. In other words, the strength of the action variable $I_j$ decays so fast that we can regard an infinite-dimensional Hamiltonian as a finite- dimensional one with the help of short range in each step of KAM iteration. For a finite-dimensional Hamiltonian, the Kolmogorov’s original idea works well in
The \(k\)-th KAM iteration by replacing the symplectic \(\Phi_k\) with a time-1-map \(X_{\Phi_k}\), where \(\Phi_k\) is of the form \(\Phi_k = F_k + (b^k, \theta)\) with a Hamiltonian \(F_k\) to be determined. More exactly, one can choose a length scale \((L^k)\) such that we only need to consider the finite Hamiltonian \(H(\theta, I(k))\) in \(k\)-th KAM iteration, where \(I(k) = (I_j)_{|j| \leq L^k - 1}\). The trouble is estimating vector \(b^k\) whose dimensional number \(2L^k + 1\) tends to \(\infty\) as \(k\) to \(\infty\). Indeed, our work is mainly based on the combination of Kolmogorov’s original idea [21] and that of Fröhlich, Spencer, Wayne [18] and Pöschel [20]. The main aim of the present paper is to prove that there exist full dimensional KAM tori with the prescribed Diophantine frequency \(\omega = \omega(I^0) = \frac{\partial H(I^0)}{\partial I^0}\) for the Hamiltonian \([13]\) in the absence of exterior parameters.

1.2. The main results. To state our results we firstly introduce some notations. In this paper, the positive constant \(\alpha\) is fixed. Note

\[
C^2 := \{I = (I_j)_{j \in \mathbb{Z}} : I_j \in \mathbb{C}\},
\]

where the norms on \(C^2\) are defined by

\[
\|I\| := \sum_{j \in \mathbb{Z}} |I_j| \exp(|j|^{1+\alpha}), \|I\|_{\infty} := \sup_{j \in \mathbb{Z}} |I_j|.
\]

Fix \(I^0 \in C^2\) with \(0 < \|I^0\| < 1\). Given some \(s > 0\) and \(r > 0\), we define domain

\[
\mathcal{D}_{s,r} := \{(I, \theta) \in C^2 \times \mathbb{T}^2 : \|I - I^0\| < s, |Im\theta|_{\infty} < r\},
\]

where \(\mathbb{T}^2 = (\mathbb{C}/2\pi\mathbb{Z})^2\), and a phase space

\[
\mathcal{P} := C^2 \times \mathbb{T}^2,
\]

with

\[
|(I, \theta)|_F := \max(\|I\|, |\theta|_{\infty}).
\]

Given a sequence of length scales \((L^k)\). Let us consider a vector \(I(k) = (I_j)_{|j| \leq L^k - 1}\) and a matrix \(B = (B_{ij})_{|i|, |j| \leq L^k - 1}\). We can expand \(I(k)\) into

\[
\bar{I} = (\bar{I}_j : j \in \mathbb{Z}), \text{ here } \bar{I}_j = \begin{cases}
I_1, |j| \leq L^k - 1 \\
0, |j| \geq L^k,
\end{cases}
\]

and also expand \(B\) into

\[
\bar{B} = (\bar{B}_{ij} : i, j \in \mathbb{Z}), \text{ here } \bar{B}_{ij} = \begin{cases}
B_{ij}, |i|, |j| \leq L^k - 1 \\
0, |i| \text{ or } |j| \geq L^k,
\end{cases}
\]

Define \(\|I(k)\| = \|\bar{I}\|\). Similarly, define \(\|B\| = \|\bar{B}\|\), where \(\|\cdot\|\) is the operator norm reduced by \(\|\cdot\|\) from \(C^2\) to \(C^2\). That is, we can define

\[
\|B\| = \|\bar{B}\| = \sup_{I, \|I\| \neq 0} \frac{\|\bar{B}I\|}{\|I\|}.
\]

For a map \(G : \mathcal{D}_{s,r} \to \mathbb{C}\), we define

\[
|G|_{s,r} := \sup_{(I, \theta) \in \mathcal{D}_{s,r}} |G(I, \theta)|.
\]

Consider an infinite-dimensional Hamiltonian system

\[
H(I, \theta) = H_0(I) + \epsilon H_1(I, \theta), \quad (I, \theta) \in \mathcal{D}_{s,r}
\]

with the standard symplectic structure \(d\theta \wedge dI\) on \(C^2 \times \mathbb{T}^2\). Assume the unperturbed Hamiltonian

\[
H_0(I) = \sum_{j \in \mathbb{Z}} h_{I(j)}(I) = \sum_{j \in \mathbb{Z}} \sum_{|i| \leq L^k - 1} h_{I(i, j)}(I)
\]

satisfies the following conditions:
For any $j \in \mathbb{Z}$, the function $h_{[i,j]} : \mathcal{T}_{s,r} \to \mathbb{C}$ is real for real arguments, analytic in variables $(I, \theta)$.

(A1) Given two sequences of positive numbers $(L^k)$ and $(M^k)$. Denote $\omega = (\omega_j)_{j \in \mathbb{Z}} = (\omega_j(I^0))_{j \in \mathbb{Z}} = \frac{\partial H_0(I^0)}{\partial I^0}$. For all $|j| \leq L^k - 1$, denote $\omega(k) = (\omega_j)_{|j| \leq L^k - 1}$. Frequency $\omega$ satisfies the Diophantine conditions

$$\langle \omega(k), \nu \rangle \geq (1 + \beta)^{-k - 1}$$

for $0 < |\nu| < M^k, k \geq 1$ and $\beta > 0, \gamma > 0$;

(A2) Given a sequence of positive numbers $(L^k)$. Denote $\Omega = \frac{\partial^2 H_0(I^0)}{\partial I^2} = (\Omega_{ij})_{i,j \in \mathbb{Z}}$. For any $i, j \in \mathbb{Z}$, note $\Omega(k) = (\Omega_{ij})_{|i|, |j| \leq L^k - 1}$. The operator $\Omega(k) : \mathbb{C}^2 \to \mathbb{C}^2$ satisfies

$$||\Omega(k)|| \leq \kappa_1,$$

and its inverse operator $\Omega^{-1}(k) : \mathbb{C}^2 \to \mathbb{C}^2$ satisfies

$$||\Omega^{-1}(k)|| \leq \kappa_2,$$

for any $k \geq 1$ and $0 < \kappa_1, \kappa_2 < \infty$.

Also assume the perturbed Hamiltonian

$$H_k = \sum_{[i,j]} f_{[i,j]}(I, I, \theta, \theta) = \sum_{j \in \mathbb{Z}} \sum_{|i-j| \leq L^k} f_{(i,j)}(I, \theta)$$

satisfies the following conditions:

(B0) For any $j \in \mathbb{Z}$, the function $f_{[i,j]} : \mathcal{T}_{s,r} \to \mathbb{C}$ is real for real arguments, analytic in variables $(I, \theta)$;

(B1) For any $j \in \mathbb{Z}$, we have

$$|f_{[i,j]}(I, \theta)| \sim \mathcal{O}(|I_i|^{i+|I_j|+}),$$

for all $[i, j]$ and $i + j + 5$,

where $\mathcal{O}(x)$ means higher order infinitesimal of $x$.

Our main result is as follows

**Theorem 1.1.** Suppose $H = H_0 + \epsilon H_1$ defined in (1.6) satisfies (A0) – (A2) and (B0) – (B1). Then for any $0 < \gamma < \frac{1}{3\beta^2}$ and $I^0 \in \mathbb{C}^2$ with $0 < ||I^0|| < 1$, there exists a constant $c_0 = c_0(s, r, I^0, \gamma) > 0$, such that, for $0 < \epsilon \leq c_0$, there is a set $\mathcal{R}^\infty(I^0)$, a measure $\mu$ with $\mu(\mathcal{R}^\infty) = 1 - \sum_{j=0}^{\infty} \epsilon_j$ ($0 < \kappa < \gamma$) and a real-analytic symplectic transformation $\Psi_* : \mathcal{T}_{s,r} \to \mathcal{T}_{s \frac{1}{2}, r \frac{1}{2}}$, such that for each $\omega(I^0)$ in $\mathcal{R}^\infty$

$$H \circ \Psi_* = e_\ast + \langle \omega(I^0), I - I^0 \rangle + \frac{1}{2} \langle \Omega_*(I - I^0), I - I^0 \rangle + P_\ast(I - I^0, \theta),$$

where $|P_\ast| = \mathcal{O}(|I_i - I_0|^{i+}|I_j - I_0|^{j+}|I_k - I_0|^{k+})$ for $i + j + k + 3$.

Furthermore, one has

$$|\Psi_* - id|_{\mathcal{T}_{s \frac{1}{2}, r \frac{1}{2}}} \ll \epsilon^{3\beta}, |e_\ast - e| \ll 2\epsilon,$$

and the operator $\Omega_* : \mathbb{C}^2 \to \mathbb{C}^2$ satisfies

$$||\Omega_* - \Omega|| \ll \epsilon^{\beta}.$$
Remark 1.3. Theorem 1.1 and corollary 1.2 extend the original Kolmogorov theorem [20] to the infinite-dimensional Hamiltonian systems of short range. It is still open to extend the Kolmogorov theorem to some Hamiltonian PDEs, which is even thought to be a harder problem. See Bourgain [3].

Remark 1.4. The initial conditions in Theorem 1.1 are strongly localized in space. More exactly, we define

\[ |I_j^0| < e^{-j|+\alpha}, \]

for \( \alpha > 0 \), namely the decay is super-exponential. With this fast decay and the fact that the interaction starts with five orders, one can choose a length scale \( (L^0) \) such that we only need to consider the finite Hamiltonian \( H(\theta, I(k)) \) in \( k \)-th KAM iteration. In fact, the methods of Fröhlich-Spencer-Wayne [17] and Pöschel [23] are still valid for the Hamiltonian \( H(I, \theta) \). See Section 4 for more details of the super-exponential decay.

The rest of the paper consists almost entirely of the proofs of the preceding results, which employs the usual Newton type iteration procedure to handle small divisor problems. In section 2 the corresponding homological equation is considered, and in section 3 one step of iterative scheme is described in details. The iteration itself takes place in section 4, and section 5 provides the estimate of measure. In section 6, we prove the main theorem.

2. The Homological Equations

2.1. Derivation of homological equations. The proof of main Theorem employs the rapidly converging iteration scheme of Newton type to deal with small divisor problems introduced by Kolmogorov, involving the infinite sequence of coordinate transformations. Recalling the sequence of length scales, \( L^k \to \infty \), we construct this transformation inductively, attempting, at the \( k \)-th stage of the inductive process, to ”kill” only those part of the interaction, \( f_{[i,j]} \), with the point \( i \) and \( j \) lying inside the box \( B_{j,k} \), which consists of all sites \( j \) with \( |j| < L^k \). In the following, we denote \( | \cdot | \) the sup-norm for any matrices or vectors of finite order. If \( A_1 \) and \( A_2 \) are \( k \times k \) and \( l \times l \) matrices with \( k < l \) respectively, we define

\[ A_1 + A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}_{l \times l} + A_2. \]

In order to provide a formal statement, let us define precisely the analytic function \( f_{[i,j]}(I, \theta) \). Assume \( f_{[i,j]}(I, \theta) = \sum_{\alpha_i, \alpha_j} a_{\alpha_i, \alpha_j}(\theta, \theta_i)(I_i)^{\alpha_i}(I_j)^{\alpha_j} \), for \( \alpha_j \in \mathbb{N} \). We thus have

\[
\begin{align*}
  f_{[i,j]}(I, \theta) &= \sum_{\alpha_i, \alpha_j \geq 5} a_{\alpha_i, \alpha_j}(\theta, \theta_i)(I_i)^{\alpha_i}(I_j)^{\alpha_j} + \sum_{\alpha_j \geq 5} a_{\alpha_j}(\theta_j)(I_j)^{\alpha_j} \\
  &+ \sum_{\alpha_{j+1} \geq 5} a_{\alpha_{j+1}}(\theta_j, \theta_{j+1})(I_j)^{\alpha_{j+1}}(I_{j+1})^{\alpha_{j+1}}.
\end{align*}
\]

Since \( I_j^0 \neq 0 \) for some \( j \in \mathbb{Z} \), we can translate it to the zero point by taking a symplectic transformation \( \Phi \) given by

\[ (I, \theta) \to \begin{cases} 
  \rho = I - I^0 \\
  \theta = \theta.
\end{cases} \]

Hence, (1.6) has the form

\[
H(\rho, \theta) = H_0(I^0 + \rho) + \epsilon H_1(I^0 + \rho, \theta) = H_0(I^0) + \sum_{j \in \mathbb{Z}} \omega_j(I^0)\rho_j + \frac{1}{2} \sum_{j \in \mathbb{Z}, |j| \leq 1} \frac{\partial^2 H_0(I^0)}{\partial I_j \partial I_j} \rho_j \rho_j
\]

\[ + \sum_{\{i,j\} \in \mathbb{Z}} V_{[i,j]}(\rho) + \epsilon \hat{H}_1(\rho, \theta) = \epsilon + \omega(\rho) + \frac{1}{2}(\Omega \rho, \rho) + V(\rho) + \epsilon \hat{H}_1(\rho, \theta), \]

for
For the perturbation $\tilde{H}_1$, we have

\begin{equation}
\tilde{H}_1(\rho, \theta) = \sum_{[i,j]} f_{[i,j]}(\rho_i + \tilde{l}_i^0, \rho_j + \tilde{l}_j^0, \theta_i, \theta_j)
= \sum_{j \in \mathbb{Z}} f_{(j-1,j)}(\rho, \theta) + f_{(j,j)}(\rho, \theta) + f_{(j,j+1)}(\rho, \theta).
\end{equation}

Correspondingly, the new domain turns to be

\[ D_{s,r} := \{ (\rho, \theta) \in \mathbb{C}^2 \times \mathbb{T}^2 : ||\rho|| < s, |\text{Im}\theta| < r \}. \]

For the perturbation $\tilde{H}_1$ with new variables $(\rho, \theta)$, we easily have

\[ f_{(j-1,j)}(\rho, \theta) = \sum_{\alpha_{j-1} + \alpha_j \geq 5} \tilde{a}_{\alpha_{j-1}, \alpha_j}(\theta_{j-1}, \theta_j)(\rho_{j-1} + \tilde{l}_{j-1}^0)^{\alpha_{j-1}}(\rho_j + \tilde{l}_j^0)^{\alpha_j}, \]

\[ f_{(j,j)}(\rho, \theta) = \sum_{\alpha_j \geq 5} \tilde{a}_{\alpha_j}(\theta_j)(\rho_j + \tilde{l}_j^0)^{\alpha_j} = \sum_{\alpha_j \geq 5} \tilde{a}_{\alpha_j}(\theta_j) \left( \sum_{l=0}^{\alpha_j} C_{\alpha_j}^l \rho_j^l (\tilde{l}_j^0)^{\alpha_j-l} \right), \]

and

\[ f_{(j,j+1)}(\rho, \theta) = \sum_{\alpha_j + \alpha_{j+1} \geq 5} \tilde{a}_{\alpha_j, \alpha_{j+1}}(\theta_j, \theta_{j+1})(\rho_j + \tilde{l}_j^0)^{\alpha_j}(\rho_{j+1} + \tilde{l}_{j+1}^0)^{\alpha_{j+1}}, \]

\[ = \sum_{\alpha_j + \alpha_{j+1} \geq 5} \tilde{a}_{\alpha_j, \alpha_{j+1}}(\theta_j, \theta_{j+1}) \left( \sum_{k=0}^{\alpha_j} C_{\alpha_j}^k \rho_j^k (\tilde{l}_j^0)^{\alpha_j-k} \right) \left( \sum_{l=0}^{\alpha_{j+1}} C_{\alpha_{j+1}}^l \rho_{j+1}^l (\tilde{l}_{j+1}^0)^{\alpha_{j+1}-l} \right). \]

Moreover, the analytic function $\tilde{H}_1(\rho, \theta)$ can be expanded into power series

\[ \tilde{H}_1(\rho, \theta) = \tilde{H}_0^r(\theta) + \langle \tilde{H}_0^r(\theta), \rho \rangle + \langle \tilde{H}_0^r(\theta) \rho, \rho \rangle + \sum_{j \in \mathbb{Z}, t \geq 3} W_{j,t}(\rho), \]

where $W(\rho) = \sum_{j \in \mathbb{Z}, t \geq 3} W_{j,t}(\rho)$ with $W_{j,t}(\rho) = \mathcal{O}(|\rho_{j-1}|^{l_{j-1}}|\rho_j|^{l_j}) + \mathcal{O}(|\rho_j|^{l_j}|\rho_{j+1}|^{l_{j+1}})$ of $l_{j-1} + l_j, l_j + l_{j+1} \geq 3$. 

where $V(\rho) = H_0(\rho) - (e + \langle \omega, \rho \rangle + \frac{1}{2}(\Omega \rho, \rho)) = \sum_{[i,j], j \in \mathbb{Z}} V_{[i,j]}(\rho)$ and
From (2.1), one has
\[
\tilde{H}_1^j(\theta) = \sum_{a_{j-1}+a_j \geq 5} a_{a_{j-1}a_j}(\theta_{j-1}, \theta_j)(I^0_{j-1})^{a_j-1}(I^0_j)^{a_j} + \sum_{a_j \geq 5} a_{a_j}(I^0_j)^{a_j} + \sum_{a_j \geq 5} a_{a_j}(\theta_j)(I^0_j)^{a_j};
\]
\[
\tilde{H}_1^\rho(\theta) = \sum_{a_j \geq 5} a_{a_j}(\theta)(I^0_j)^{a_j};
\]
\[
\tilde{H}_1^{\rho_j-1}(\theta) = \frac{1}{2} \sum_{a_j \geq 5} a_{a_j}(\theta_j)(I^0_j)^{a_j-1} + \sum_{a_j \geq 5} a_{a_j}(\theta_j)(I^0_j)^{a_j-1};
\]
\[
\tilde{H}_1^{\rho_j}(\theta) = \sum_{a_j \geq 5} a_{a_j}(\theta_j)(I^0_j)^{a_j-2} + \sum_{a_j \geq 5} a_{a_j}(\theta_j)(I^0_j)^{a_j-2};
\]
\[
\tilde{H}_1^{\rho_j}(\theta) = 0, \text{ for any } |i-j| > 1.
\]

Now consider the Hamiltonian \( H \) of the form
\[
H = N + R,
\]
where
\[
N(\rho) = \epsilon + \langle \omega, \rho \rangle + \frac{1}{2} \langle \Omega \rho, \rho \rangle,
\]
and
\[
R(\rho, \theta) = V(\rho) + \epsilon \tilde{H}_1(\rho, \theta).
\]

More precisely, let
\[
V(\rho, \theta) = V_1(\rho) + V_2(\rho),
\]
with
\[
V_1(\rho) = \sum_{|i,j| \leq L^+ - 1} V_{|i,j|}(\rho),
\]
\[
V_2(\rho) = \sum_{|i,j| \geq L^+} V_{|i,j|}(\rho);
\]
and
\[
\epsilon \tilde{H}_1(\rho, \theta) = R_1(\rho, \theta) + R_2(\rho, \theta),
\]
with
\[
R_1(\rho, \theta) = \epsilon \sum_{|i,j| \leq L^+ - 1} f_{|i,j|}(I^0_i + \rho_i, I^0_j + \rho_j, \theta_i, \theta_j),
\]
\[
R_2(\rho, \theta) = \epsilon \sum_{|i,j| \geq L^+} f_{|i,j|}(I^0_i + \rho_i, I^0_j + \rho_j, \theta_i, \theta_j).
\]

Denote \( R_1(\rho, \theta) = R_{1,\text{low}}(\rho, \theta) + R_{1,\text{high}}(\rho, \theta) \), where
\[
R_{1,\text{low}}(\rho, \theta) = R_1(\theta) + \langle R_1^1(\theta), \rho \rangle + \langle R_1^2(\theta), \rho \rangle,
\]
(2.3)
and
\begin{equation}
R_{1}^{\text{high}}(\rho, \theta) = R_{2}^{\text{low}}(\rho, \theta) + R_{1}^{\text{low}}(\rho, \theta),
\end{equation}
with \( R_{2}^{\text{low}}(\rho, \theta) \) being cubic of \(|\rho_{j}|\) and \( R_{1}^{\text{low}}(\rho, \theta) = O(|\rho_{j}|^{4}) \) for \( l \geq 4 \).

We desire to eliminate the terms \( R_{1}^{\text{low}} \) in (2.3) by the coordinate transformation \( \Psi \), which is obtained as the time-1-map \( X_{\Psi}|_{t=1} \) of a Hamiltonian vector field \( X_{F} \). Write
\begin{equation}
F(\rho, \theta) = \tilde{F}(\rho, \theta) + \langle a, \theta \rangle = F^{0}(\theta) + \langle F^{1}(\theta), \rho \rangle + \langle F^{2}(\theta)\rho, \rho \rangle + \langle a, \theta \rangle,
\end{equation}
where vector \( a \) is chosen to keep the frequency \( \omega \) fixed.

**Remark 2.1.** Since the existing of the term \( \langle a, \theta \rangle \), the function \( F \) is defined on \( \mathbb{C}^{2} \times \mathbb{C}^{2} \) not \( \mathbb{C}^{2} \times T^{2} \), but the flow \( X_{F} \) is perfectly well defined on \( \mathbb{C}^{2} \times \mathbb{C}^{2} \).

Using Taylor’s formula, we have
\begin{equation}
H_{+} = H \circ \Psi = H \circ X_{\Psi}|_{t=1} = H + \{ H, F \} + \int_{0}^{1} (1 - t) \{ \{ H, F \}, F \} \circ X_{F}^t \ dt
\end{equation}
\begin{equation}
= N + \{ N, F \} + \int_{0}^{1} (1 - t) \{ \{ N, F \}, F \} \circ X_{F}^t \ dt
\end{equation}
\begin{equation}
+ R_{1}^{\text{low}} + \int_{0}^{1} \{ R_{1}^{\text{low}}, F \} \circ X_{F}^{t} \ dt
\end{equation}
\begin{equation}
+ R_{1}^{\text{high}} + N_{1} + \{ R_{1}^{\text{high}} + N_{1}, F \} + (N_{2} + R_{2}) \circ X_{F}^{t} \ dt
\end{equation}
\begin{equation}
+ \int_{0}^{1} (1 - t) \{ \{ R_{1}^{\text{high}} + N_{1}, F \}, F \} \circ X_{F}^{t} \ dt.
\end{equation}
Then we obtain the modified homological equation
\begin{equation}
\{ N, F \} + R_{1}^{\text{low}} + \{ R_{1}^{\text{high}} + N_{1}, F \}^{\text{low}} = N_{+} - N,
\end{equation}
where \( N_{+} \) is given below and \( \{ \cdot, \cdot \} \) is the Poisson bracket of functions on \( \mathcal{D}_{s,r} \) computed by the formula
\begin{equation}
\{ F, G \} = \langle F_{s}, G_{r} \rangle - \langle F_{r}, G_{s} \rangle.
\end{equation}
If the homological equation (2.6) is solved, the new perturbation term \( R_{1+} \) can be written as
\begin{equation}
R_{1+} = R_{1}^{\text{high}} + N_{1} + \{ R_{1}^{\text{high}} + N_{1}, F \}^{\text{high}}
\end{equation}
\begin{equation}
+ \int_{0}^{1} (1 - t) \{ \{ N + R_{1}^{\text{high}} + N_{1}, F \}, F \} \circ X_{F}^{t} \ dt
\end{equation}
\begin{equation}
+ \int_{0}^{1} \{ R_{1}^{\text{low}}, F \} \circ X_{F}^{t} \ dt.
\end{equation}
Note that we do not need to eliminate the terms in (2.3) at the next step of KAM procedure, so (2.8) is not necessary to be too small. On the other hand, the remaining terms are either quadratic in \( F \) or bounded by \( \epsilon^{1+\beta} \) (we will prove this in details below). Therefore, we can obtain a non-degenerate normal form of order 2 with a fixed frequency \( \omega \).

Once all the above procedures work well, our new Hamiltonian reads
\begin{equation}
H_{+} = N_{+} + R_{1+} + V_{2} + R_{2}.
\end{equation}
Note \( R_{2} = R_{21} + R_{22} \), where
\begin{equation}
R_{21}(\rho, \theta) = \epsilon \sum_{|i,j|+|L| \leq \text{dist}(i,j), |0| \leq l+1} f_{i,j}(l_{i}^{0} + \rho_{i}, l_{j}^{0} + \rho_{j}, \theta_{i}, \theta_{j}),
\end{equation}
\begin{equation}
R_{22}(\rho, \theta) = \epsilon \sum_{|i,j|+|L| \leq \text{dist}(i,j), |0| \geq l+2} f_{i,j}(l_{i}^{0} + \rho_{i}, l_{j}^{0} + \rho_{j}, \theta_{i}, \theta_{j}),
\end{equation}
with a larger number \( L^{++} \).
It is clear that one has to eliminate the term $R_{1+} + R_{21}$ in the next iterative process and from this term one easily gets that $R_{1+}$ depends only on $\rho_j, \theta_j$ for $|j| \leq L^+$, while $R_{21}$ depends on $\rho_j, \theta_j$ for $|j| \geq L^+$. So in order to know exactly what the new error term depends on and keep the eliminated term unified from 1-th iteration ($R_1$ is of the form like $A + B$), we will rewrite $R_1$ into several terms directly below.

Let

$$R_1 = P(\rho, \theta) + \epsilon Q(\rho, \theta) + \epsilon \sum_{|i| \geq L^+} f_{i,j}(I_i^0 + \rho_i, I_j^0 + \rho_j, \theta_i, \theta_j),$$

(2.11)

with

$$P(\rho, \theta) = \epsilon \left( f_{i,j}(I_i^0 + \rho_i, \theta_i) + \sum_{|j| = L^+} f_{i,j}(I_j^0 + \rho_j, \theta_j) \right),$$

and

$$Q(\rho, \theta) = \sum_{|i| = L^+} f_{i,j}(I_i^0 + \rho_i, I_j^0 + \rho_j, \theta_i, \theta_j).$$

**Remark 2.2.** Since $\|L\| = 1$ and $|f_{i,j}(I, \theta)| \leq K \exp[-l(|j| - 1)^{1+\alpha}]$, we can choose $K = (2^p)^{-1}$ such that $|P|_{L^p} \leq \epsilon$.

More exactly, denote

$$N = N^0 + N^1 + N^2,$$

where

$$N^0 = \sum_{|j| \leq L^+ - 1} \omega_j \rho_j + \sum_{|i|, |j| \leq L^+ - 1} \Omega_{ij} \rho_i \rho_j,$$

$$N^1 = \sum_{|i| = L^+ \text{ or } |j| = L^+} \Omega_{ij} \rho_i \rho_j,$$

$$N^2 = \sum_{|i|, |j| \geq L^+} \omega_{ij} \rho_i \rho_j;$$

and denote

$$V(\rho) = \hat{V}(\rho) + \hat{V}(\rho) + \hat{V}(\rho),$$

where

$$\hat{V}(\rho) = \sum_{|i,j| \leq L^+} V_{i,j}(\rho),$$

$$\hat{V}(\rho) = \sum_{|i| = L^+ \text{ or } |j| = L^+} V_{i,j}(\rho),$$

$$\hat{V}(\rho) = \sum_{|i,j| \geq L^+} V_{i,j}(\rho).$$

For convenience, denote $P(\rho, \theta) = P^{low}(\rho, \theta) + P^{high}(\rho, \theta)$, where

$$P^{low}(\rho, \theta) = \rho^0(\theta) + \langle \rho^1(\theta), \rho \rangle + \langle \rho^2(\theta), \rho \rangle,$$

(2.12)

and

$$P^{high}(\rho, \theta) = \rho^3(\theta) + \rho^4(\theta),$$

(2.13)

with $P^3(\rho, \theta)$ being cubic of $|\rho_j|$ and $P^4(\rho, \theta) = O(|\rho_j|^l)$ for $l \geq 4$.

Similarly, note $Q(\rho, \theta) = Q^{low}(\rho, \theta) + Q^{high}(\rho, \theta)$, where

$$Q^{low}(\rho, \theta) = \rho^0(\theta) + \langle \rho^1(\theta), \rho \rangle + \langle \rho^2(\theta), \rho \rangle,$$

(2.14)
and

\begin{equation}
Q^{\text{high}}(\rho, \theta) = Q^3(\rho, \theta) + Q^4(\rho, \theta),
\end{equation}

with \(Q^3(\rho, \theta)\) being cubic of \(|\rho_j|\) and \(Q^4(\rho, \theta) = \mathcal{O}(|\rho_j|^4)\) for \(l \geq 4\).

Particularly, \(\tilde{V}(\rho)\) has the same form

\[ \tilde{V}(\rho) = \tilde{V}^3(\rho) + \tilde{V}^4(\rho), \]

with \(\tilde{V}^3(\rho)\) is cubic of \(|\rho_j|\) and \(\tilde{V}^4(\rho, \theta) = \mathcal{O}(|\rho_j|^4)\) for \(l \geq 4\);

\[ \tilde{V}(\rho) = \tilde{V}^3(\rho) + \tilde{V}^4(\rho), \]

with \(\tilde{V}^3(\rho)\) being cubic of \(|\rho_j|\) and \(\tilde{V}^4(\rho, \theta) = \mathcal{O}(|\rho_j|^4)\) for \(l \geq 4\); and

\[ \tilde{V}(\rho) = \tilde{V}^3(\rho) + \tilde{V}^4(\rho), \]

with \(\tilde{V}^3(\rho)\) being cubic of \(|\rho_j|\) and \(\tilde{V}^4(\rho, \theta) = \mathcal{O}(|\rho_j|^4)\) for \(l \geq 4\).

Hence, the term we desire to eliminate is \(P^{\text{low}} + \epsilon Q^{\text{low}}\) in (2.6) since (2.11) can be bounded by \(\epsilon^{1+\beta}\) (see below). Furthermore, in order to cure the problem that the measures of \(\omega\) in \(A_1\) will diverge, we reduce the infinite sum of \(v\) to a finite one. In other words, the homological equation of (2.6) turns to be

\begin{equation}
\{N^0, F\} + \Gamma P^{\text{low}} + \epsilon \Gamma Q^{\text{low}} + \{\tilde{V} + \Gamma P^{\text{high}} + \epsilon \Gamma Q^{\text{high}}, F\}^{\text{low}} = N^0 - N^0,
\end{equation}

where \(\Gamma P, \Gamma Q\) mean the truncation of the \(M^+\)-th Fourier coefficient of \(P, Q\) with \(M^+\) given later.

Let \(F^0\) (resp. \(F^1, F^2\)) has the form of \(P^0 + \epsilon Q^0\) (resp. \(P^1 + \epsilon Q^1, P^2 + \epsilon Q^2\)). That is, \(F^0\) (resp. \(F^1, F^2\)) can be expanded to

\begin{equation}
F^0(\theta) = \sum_{0 < |\ell| < M^+} \tilde{F}^0_{\ell} e^{i(\ell, \theta)},
\end{equation}

where \(\tilde{F}^0_{\ell}\) are the \(v\)-th Fourier coefficients of \(F^j (j = 0, 1, 2)\).

Note \((U, \rho, \rho) = \{\hat{V} + P^{\text{high}} + \epsilon Q^{\text{high}}, F\}^{\text{low}} = \{\hat{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F^0 + \langle a, \theta \rangle\}. From the definition of Poisson bracket (2.7), we have the following equations from distinguishing terms of (2.16) by the order of \(\rho\):

\begin{equation}
\begin{cases}
-\partial_\omega F^0 - \langle \omega(+)\rangle, a) + \Gamma P^0 + \epsilon \Gamma Q^0 = 0, \\
-\partial_\omega F^1 - \tilde{\Omega} a + \tilde{\Omega} \partial_\omega F^0 + \Gamma P^1 + \epsilon \Gamma Q^1 = 0, \\
-\partial_\omega F^2 - \tilde{\Omega} \partial_\omega F^1 + U + \Gamma P^2 + \epsilon \Gamma Q^2 = 0,
\end{cases}
\end{equation}

where \(\partial_\omega = \omega \cdot \partial_\theta, \omega(+) = (\omega_j)_{|j| \leq L^+, m} \) and \(\tilde{\Omega} = \Omega^+(+) = (\Omega_j)_{|j| \leq L^+, m}.\)
Therefore, the new Hamiltonian $H_+$ has the form

\begin{equation}
H_+ = H \circ X^1_F |_{t=1}
\end{equation}

\begin{align*}
&= N + \{ N, F \} + \Gamma P^{low} + \epsilon \Gamma Q^{low} + V^3 + \Gamma P^3 + \epsilon \Gamma Q^3 \\
&\quad + \int_0^1 \{ \Gamma P^{low} + \epsilon \Gamma Q^{low}, F \} \circ X^1_F \, dt \\
&\quad + \{ \tilde{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F \} + \{ \tilde{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F \}^3 \\
&\quad + \{ \tilde{V}^4 + \Gamma P^4 + \epsilon \Gamma Q^4, F \} + \{ \tilde{V}^4 + \Gamma P^4 + \epsilon \Gamma Q^4, F \}^4 \\
&\quad + \int_0^1 \{ \Gamma P^{low} + \epsilon \Gamma Q^{low}, F \} \circ X^1_F \, dt \\
&\quad + \int_0^1 (1-t) \{ \{ N + \tilde{V}^3 + V^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F \} \circ X^1_F \, dt \\
&\quad + \left( \sum_{|i| \geq M^+} \tilde{P}_i + \epsilon \tilde{Q}_i \right) \circ X^1_F + \{ N^1, F \} + \{ \tilde{V}^3, F \} \\
&\quad + \left( \epsilon \sum_{|i| \text{ either } |i| \text{ or } |j| \geq \mathbb{L}^+ \text{ but not both}} \frac{f_{i,j}(t^0 + \rho, \theta)}{X^1_F} \right)
\end{align*}

where

\begin{equation}
N_+ = \tilde{c} + N + [P^2] + \epsilon [Q^2] + [U],
\end{equation}

with $\tilde{c} = [P^{0}] + \epsilon [Q^0] + \langle \omega (+), a \rangle$, the $[P^j] + \epsilon [Q^j]$, $[U]$ being respectively the 0-th Fourier coefficients of $P^j + \epsilon Q^j$ ($j = 0, 1, 2$), $U$, and

\begin{equation}
P_+ : = V^3 + \tilde{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3 + \tilde{V}^4 + \Gamma P^4 + \epsilon \Gamma Q^4 \\
+ \{ \tilde{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F \} + \{ \tilde{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F \}^3 \\
+ \{ \tilde{V}^4 + \Gamma P^4 + \epsilon \Gamma Q^4, F \} + \{ \tilde{V}^4 + \Gamma P^4 + \epsilon \Gamma Q^4, F \}^4 \\
+ \int_0^1 \{ \Gamma P^{low} + \epsilon \Gamma Q^{low}, F \} \circ X^1_F \, dt \\
+ \int_0^1 (1-t) \{ \{ N + \tilde{V}^3 + V^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F \} \circ X^1_F \, dt \\
+ \left( \sum_{|i| \geq M^+} \tilde{P}_i + \epsilon \tilde{Q}_i \right) \circ X^1_F + \{ N^1, F \} + \{ \tilde{V}^3, F \} \\
+ \left( \epsilon \sum_{|i| \text{ either } |i| \text{ or } |j| \geq \mathbb{L}^+ \text{ but not both}} \frac{f_{i,j}(t^0 + \rho, \theta)}{X^1_F} \right)
\end{equation}

where $P_+$ depends only on $\rho_j, \theta_j$ for $|j| \leq \mathbb{L}^+$.

### 2.2 The solvability of the homological equations

In this subsection, we will estimate the solutions of the homological equations. To avoid a flood of constants we will write $a \ll b$, if there exists a constant $C \geq 1$ depending only on $\alpha, \gamma$ such that $a \leq Cb$. Moreover, one has the following estimates.

Recalling that

\begin{align*}
P^{low}(\rho, \theta) &= P^0(\theta) + (P^1(\theta), \rho) + (P^2(\theta), \rho) \\
&= \sum_{0 \leq |j| \leq 1} P^{0|j}(\theta) + \sum_{0 \leq |j| \leq 1} P^{1|j}(\theta) \rho_j + \sum_{0 \leq |j| \leq 1} P^{2|j}(\theta) \rho_0 \rho_j,
\end{align*}
where $P^{0\theta_j}(\theta) = P^{0\theta_j}(\theta_{j-1}, \theta_j, \theta_{j+1})$, one has the estimates
\begin{align*}
|P^{0\theta_j}(\theta)| &\leq \epsilon, \\
|\rho P^{1\rho_j}(\theta)| &\leq \epsilon, \\
|P^{2\rho_j\rho_j}(\theta)| &\leq \epsilon, \\
|P^{2\rho_j\rho_j}(\theta)| &\leq \epsilon
\end{align*}
for $0 \leq |j| \leq 1$.

Similarly, since
\begin{align*}
\Omega_j &\leq 1.
\end{align*}

\begin{align*}
Q^{\rho} = Q^{0}(\theta) &+ (Q^{1}(\theta), \rho) + (Q^{2}(\theta)\psi, \rho) \\
&+ \sum_{1 \leq |j| \leq L_j^+ - 1} Q^{1\rho_j}(\theta) + \sum_{1 \leq |j| \leq L_j^+ - 1} Q^{2\rho_j}(\theta) + \sum_{1 \leq |j| \leq L_j^+ - 1} Q^{2\rho_j\rho_j}(\theta),
\end{align*}

where $Q^{0\theta_j}(\theta) = Q^{0\theta_j}(\theta_{j-1}, \theta_j, \theta_{j+1})$, one obtains
\begin{align*}
|Q^{0\theta_j}(\theta)| &\leq |I_{j-1}^0|, \\
|Q^{1\rho_j}(\theta)| &\leq |I_{j-1}^1|, \\
|Q^{2\rho_j}(\theta)| &\leq |I_{j-1}^2|, \\
|Q^{2\rho_j\rho_j}(\theta)| &\leq |I_{j-1}^2|, \text{for } |i - j| \leq 1,
\end{align*}

for $1 \leq |i|, |j| \leq L_j^+ - 1$.

**Lemma 2.3.** For the definition of the operator $\widetilde{\Omega}(k) : \mathbb{C}^n \to \mathbb{C}^n$ with the norm $\|\cdot\|$ satisfying
\begin{align}
\|\widetilde{\Omega}(k)\| &\leq \kappa_1, \|\widetilde{\Omega}^{-1}(k)\| \leq \kappa_2, \forall k, \\
\|\Omega(k)\| &\leq 2\kappa_1, \|\Omega^{-1}(k)\| \leq 2\kappa_2, \forall k,
\end{align}
we have
\begin{align}
\|\Omega(k)\| \leq 2\kappa_1, \|\Omega^{-1}(k)\| \leq 2\kappa_2, \forall k,
\end{align}
where $\|\cdot\|$ denotes the sup-norm for any finite matrices.

**Proof.** For any given $k$, the matrix $\Omega(k)$ is symmetric by the definition of $\Omega(k)$. For any $j$, take
\begin{align*}
I = (0, \ldots, e^{-j})^{[i, n]} \text{ with } \|I\| = 0.
\end{align*}
Then one has
\begin{align*}
\|\Omega(k)\| &= \Omega_{(j-1)j} [e_j^{\beta_j-1}]^{1+n} + \Omega_{jj} [e_j^{\beta_j-1}]^{1+n} + \Omega_{(j+1)j} [e_j^{\beta_j-1}]^{1+n} \leq \kappa_1,
\end{align*}
which implies
\begin{align*}
|\Omega_{jj}| + |\Omega_{(j+1)j}| &\leq \kappa_1, \; j \geq 1,
\end{align*}
or
\begin{align*}
|\Omega_{(j-1)j}| + |\Omega_{jj}| &\leq \kappa_1, \; j \leq -1.
\end{align*}
Particularly, for $j = 0$, we have
\begin{align*}
|\Omega_{(-1)0}| + |\Omega_{00}| + |\Omega_{01}| &\leq \kappa_1.
\end{align*}
That is, one also has
\begin{align*}
|\Omega_{(j-1)j}| + |\Omega_{jj}| &\leq \kappa_1, \; j \geq 1,
\end{align*}
and
\begin{align*}
|\Omega_{jj}| + |\Omega_{(j+1)j}| &\leq \kappa_1, \; j \leq -1.
\end{align*}
Since $\Omega_{(j-1)} = \Omega_{(j-1)j}$ and $\Omega_{jj} = \Omega_{(j+1)j}$, one finally obtains
\begin{align*}
|\Omega_{(j-1)j}| + |\Omega_{jj}| + |\Omega_{(j+1)j}| &\leq 2\kappa_1, \; \forall j.
\end{align*}
Therefore, one has

\[ |\Omega(k)| \leq 2k_1. \]

The remaining proof is similar. \(\square\)

**Lemma 2.4.** Given some positive parameters \(0 < \beta < \frac{1}{10}, \epsilon > 0\) and \(0 < \tilde{\sigma} < s, 0 < \sigma < r\), one has

\[ |\{F, G\}|_{s-\tilde{\sigma}, r-\sigma} \leq 2e^{-\frac{1+\beta}{\tilde{\sigma}} \sigma^{-1}} |F|_{s, r} |G|_{s, r}, \]

where \(F\) and \(G\) are functions which are of the form \(G^0 + (G^\rho, \rho^*) + (G^{pp}_\rho, \rho^*)\) with \(\rho^* = (\rho_j)_{|j| \leq L^+ - 1}\) of \(L^+ = (\frac{1+\beta}{\tilde{\sigma}} \ln \epsilon)^{\frac{1}{1-\beta}}\).

**Proof.** Let

\[ F = F^0 + \langle F^\rho, \rho^* \rangle + \langle F^{pp}_\rho, \rho^* \rangle \]

and

\[ G = G^0 + \langle G^\rho, \rho^* \rangle + \langle G^{pp}_\rho, \rho^* \rangle. \]

Recall that

\[ \{F, G\} = \langle F_\rho, G_\rho \rangle - \langle F_\rho, G_\rho \rangle. \]

Considering the term \(\{F_\rho, G_\rho\}\), we have

\[
|\{F_\rho, G_\rho\}|_{s-\tilde{\sigma}, r-\sigma} = \left| \sum_{j \in \mathbb{Z}} \frac{\partial F}{\partial \theta_j} \frac{\partial G}{\partial \theta_j} \right|_{s-\tilde{\sigma}, r-\sigma} \\
\leq \left| \sum_{|j| \leq L^+ - 1} \tilde{\sigma}^{-1} e^{j|\tilde{\sigma}|} |G|_{s, r-\sigma} |F_\rho|_{s-\tilde{\sigma}, r-\sigma} \right| \\
\leq \tilde{\sigma}^{-1} e^{\tilde{\sigma} - 1 \sigma} \sigma^{-1} |G|_{s, r-\sigma} |F|_{s-\tilde{\sigma}, r} \\
\leq e^{-\frac{1+\beta}{\tilde{\sigma}} \sigma^{-1}} |G|_{s, r-\sigma} |F|_{s-\tilde{\sigma}, r}.
\]

Similarly, one has

\[
|\langle F_\rho, G_\rho \rangle|_{s-\tilde{\sigma}, r-\sigma} = \left| \sum_{j \in \mathbb{Z}} \frac{\partial F}{\partial \theta_j} \frac{\partial G}{\partial \theta_j} \right|_{s-\tilde{\sigma}, r-\sigma} \\
\leq \left| \sum_{|j| \leq L^+ - 1} \tilde{\sigma}^{-1} e^{j|\tilde{\sigma}|} |F_\rho|_{s, r-\sigma} |G_\rho|_{s-\tilde{\sigma}, r-\sigma} \right| \\
\leq \tilde{\sigma}^{-1} e^{\tilde{\sigma} - 1 \sigma} \sigma^{-1} |F|_{s, r-\sigma} |G|_{s-\tilde{\sigma}, r} \\
\leq e^{-\frac{1+\beta}{\tilde{\sigma}} \sigma^{-1}} |F|_{s, r-\sigma} |G|_{s-\tilde{\sigma}, r}.
\]

It follows that

\[ |\{F, G\}|_{s-\tilde{\sigma}, r-\sigma} \leq 2e^{-\frac{1+\beta}{\tilde{\sigma}} \sigma^{-1}} |F|_{s, r} |G|_{s, r}. \]

\(\square\)

**Lemma 2.5.** Let \(\omega\) be Diophantine with \(\gamma > 0\) (see (A$_1$) for \(k = 1\)), and choose \(L^+ = (\frac{1+\beta}{\tilde{\sigma}} \ln \epsilon)^{\frac{1}{1-\beta}}\). Then for any \(0 < s < 1, r > 0\) and \(0 < \sigma < \frac{1}{2r}\), the solutions of the homological equations which are given by (2.17), satisfy

\[
|F^0|_{r-\sigma} \leq \frac{1}{\sigma^{2r+\gamma-1}} e^{1-\gamma},
\]

\[
|F^1|_{r-3\sigma} \leq \frac{1}{\sigma^{4r+\gamma-1}} e^{1-2\gamma},
\]

\[
|F^2|_{r-5\sigma} \leq \frac{1}{\sigma^{6r+\gamma-1}} e^{1-3\gamma}.
\]
Moreover, one has
\begin{equation}
\label{eq:2.27}
|\hat{F}|_{r, r-5\sigma} \leq \frac{1}{\sigma^{2L+1}} e^{1-3\gamma},
\end{equation}
where \(\hat{F} = F^0 + (F^1, \rho) + (F^2, \rho, \rho)\).

\textbf{Proof.} First of all, we consider the first equation of \eqref{eq:2.18}. From \eqref{eq:2.17}, we can solve the equation
\[\partial_{\nu} F^0 = \Gamma F^0 + \epsilon \Gamma Q^0 - ([P^0] + [Q^0] + \langle \omega^\alpha + a \rangle).\]
That is, the solution of the homological equation is given by
\begin{equation}
\label{eq:2.28}
F^0(\theta) = \sum_{0 < |v| < M^+} \frac{P^0_v + \epsilon Q^0_v}{1(\omega(\nu), v)} e^{i \langle v, \theta \rangle}.
\end{equation}
From the Diophantine condition \((A_1)\), one has
\[|F^0(\theta)|_{r-\sigma} \leq \epsilon^{-\gamma} \sum_{0 < |v| < M^+} \left( \epsilon + \epsilon \sum_{|j| \leq L^+ - 1} |P^0_{[j]}|^2 \right) e^{-r|v|} |e^{i(r-\sigma)v}| \leq \frac{1}{\sigma^{2L+1}} e^{1-\gamma}, 0 < \sigma < r,
\]
which finishes the proof of \eqref{eq:2.28}.
Moreover, combining with Cauchy estimate
\[|\partial_{\nu} F^0(\theta)|_{r-2\sigma} \leq \frac{1}{\sigma} |F^0(\theta)|_{r-\sigma},\]
where \(|\partial_{\nu} F^0(\theta)|_{r} = \max_{|j| \leq L^+ - 1} \sup_{\theta \in \mathbb{C}^*} |\partial_{\theta_j} F^0(\theta)|\), one finally obtains
\begin{equation}
\label{eq:2.29}
|\partial_{\nu} F^0(\theta)|_{r-2\sigma} \leq \frac{e^{1-\gamma}}{\sigma^{2L+1}}.
\end{equation}
Next we consider the second equation of \eqref{eq:2.18}. Since \(\hat{F}_0^1 = 0\), we can choose a vector \(a\) such that \(\hat{\Omega} a - [P^1] - \epsilon [Q^1] = 0\). Then we have
\begin{equation}
\label{eq:2.30}
a = (\hat{\Omega})^{-1} ([P^1] + \epsilon [Q^1]),
\end{equation}
with the estimate
\[|a| = |(\hat{\Omega})^{-1}||[P^1] + \epsilon [Q^1]| \leq 2\kappa_2 \langle \langle [P^1] + \epsilon [Q^1] \rangle \rangle \leq \epsilon.
\]
Let \(\hat{F}^0 = -\hat{\Omega} \partial_{\nu} F^0\). We get
\[|\hat{F}^0(\theta)|_{r-2\sigma} \leq |\hat{\Omega}| |\partial_{\nu} F^0(\theta)|_{r-2\sigma} \leq \frac{1}{\sigma^{2L+1}} e^{1-\gamma},\]
where the last equality is based on \eqref{eq:2.29}.
Similarly to \eqref{eq:2.28}, we can solve the equation
\begin{equation}
\label{eq:2.31}
F^1(\theta) = \sum_{0 < |v| < M^+} \frac{\hat{P}^1_v + \epsilon \hat{Q}^1_v + \hat{F}^0_v e^{i \langle v, \theta \rangle}}{1(\omega(\nu), v)} e^{i \langle v, \theta \rangle},
\end{equation}
and easily obtain the estimate
\[|F^1(\theta)|_{r-3\sigma} \leq \epsilon^{-\gamma} \sum_{0 < |v| < M^+} \left( \epsilon + \frac{1}{\sigma^{2L+1}} e^{1-\gamma} \right) e^{-r(2\sigma)|v|} |e^{i(r-3\sigma)v}| \leq \frac{1}{\sigma^{4L+1}} e^{1-2\gamma},\]
which finishes the proof of (2.25).
Furthermore, we have

$$|\partial_\theta F^1(\theta)|_{r-4\sigma} \leq \frac{1}{\sigma^{4L+\gamma}} e^{1-2\gamma},$$

where $|\partial_\theta F^1(\theta)|_{r} = \max_{|j| \leq L^+ - 1} \sum \sup_{\theta \in \mathbb{T}} |\partial_\theta j F^1(\theta)|$.

Now we turn to the third equation of (2.18). Let $\tilde{F}^1 = -\hat{\Omega} \partial_\theta F^1$. On $D_{s-\hat{\sigma}, r-4\sigma}$, we have

$$|\tilde{F}^1(\theta)|_{r-4\sigma} \leq |\hat{\Omega}| |\partial_\theta F^1(\theta)|_{r-4\sigma} \leq \frac{1}{\sigma^{4L+\gamma}} e^{1-2\gamma}.$$

Since

$$\langle \tilde{\rho}, \rho \rangle = \{\tilde{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F^0 + \langle a, \theta \rangle\},$$

and

$$\{\tilde{V}^3, F^0 + \langle a, \theta \rangle\} = \sum_{i,j,l} \partial_3 \tilde{V}^3 \partial_\rho_i \partial_\rho_j \partial_\rho_l (\partial_\theta l F^0 + a_l) \rho_i \rho_j,$$

we easily estimate that

$$|\{\tilde{V}^3, F^0 + \langle a, \theta \rangle\}|_{\rho \rho} \leq \sup_{j} \left(\left| \frac{\partial^3 \tilde{V}^3}{\partial \rho_i \partial \rho_j \partial \rho_l} \right| + \left| \frac{\partial^3 \tilde{V}^3}{\partial \rho_j \partial \rho_l} \right| \right) \left( |\partial_\theta F^0| + |a| \right) \leq |\partial_\theta F^0| + |a|$$

and

$$|\{\Gamma P^3 + \epsilon \Gamma Q^3, F^0 + \langle a, \theta \rangle\}|_{\rho \rho} = \max_{i} \sum_{|j| \leq L^+ - 1} \sum_{l} \left| \partial^3 (\Gamma P^3 + \epsilon \Gamma Q^3) \partial_\rho_i \partial_\rho_j \partial_\rho_l \right| (\partial_\theta l F^0 + a_l) \leq \sum_{|j| \leq L^+ - 1} \epsilon |l| \left( |\partial_\theta F^0| + |a| \right) \leq |\partial_\theta F^0| + |a|.$$

From (2.33) and (2.34), one has

$$|U|_{r-2\sigma} \leq \frac{1}{\sigma^{2L+\gamma}} e^{1-\gamma}.$$

Similarly to (2.28), the solution of the third equation of (2.18) is given by

$$F^2(\theta) = \sum_{0 < |v| < M^+} \frac{\hat{F}^2 + \epsilon \hat{Q}^2 + \hat{F}^1_v + \hat{U}_v}{\imath(\omega(+), v)} e^{\imath (v, \theta)},$$

and the corresponding estimate is

$$|F^2(\theta)|_{r-5\sigma} \leq \frac{1}{\sigma^{6L+\gamma}} e^{1-3\gamma}.$$
which finishes the proof of (2.26).
Consequently, from (2.24), (2.25) and (2.26), one obtains
\[ |\tilde{F}|_{r-5\sigma} \leq |F_0|_{r-5\sigma} + |F_1|_{r-5\sigma} \left( \sum_{|j| \leq L^+ - 1} |\rho_j| \right) + |F_2|_{r-5\sigma} \left( \sum_{|i|, |j| \leq L^+ - 1} |\rho_i||\rho_j| \right) \]
\[ < \frac{1}{\sigma^{2L^+ - 1}} e^{1-\gamma} + \frac{1}{\sigma^{4L^+ - 1}} e^{1-2\gamma} \left( \sum_{|j| \leq L^+ - 1} e^{-|j|^{1+\alpha}} \right) \]
\[ + \frac{1}{\sigma^{6L^+ - 1}} e^{1-3\gamma} \left( \sum_{|j| \leq L^+ - 1} e^{-|j|^{1+\alpha}} \right)^2 \]
\[ \leq \frac{1}{\sigma^{6L^+ - 1}} e^{1-3\gamma}. \]

\[ \square \]

2.3. The derivatives of \( F \). On \( D_{r-5\sigma} \), from Lemma 2.5, we obtain the estimate
\[ (2.37) \quad |\tilde{F}|_{r-5\sigma} \leq \frac{1}{\sigma^{6L^+ - 1}} e^{1-3\gamma}. \]

From the equation of motion
\[ (2.38) \quad \dot{\theta} = F_\rho(\rho, \theta), \dot{\rho} = F_\beta(\beta, \theta) = \tilde{F}_\beta(\beta, \theta) + \alpha, \]
vector \( \alpha \) has to belong to \( C^2 \). That is, we have the estimate
\[ (2.39) \quad \|\alpha\| = \sum_{|j| \leq L^+ - 1} \sum_{|i| \leq L^+ - 1} (\tilde{\Omega}_{ij})^{-1} (|P^1|^{\rho_i} + \epsilon|Q^1|^{\sigma_{ij}}) e^{1+\alpha} \]
\[ \leq \kappa_2 \sum_{|j| \leq L^+ - 1} (|P^1|) e^{1+\alpha} \]
\[ \leq \kappa_2 (\epsilon + e^{1+\alpha} \sum_{0 \leq |j| \leq 1} e^{1+\alpha}) \]
\[ \leq \kappa_2 (\epsilon + \frac{4\theta}{5} e^{1+\alpha}) \]
\[ \leq \epsilon^{\frac{4}{5} - \gamma - \frac{1}{5} \delta}. \]

Hence, on \( D_{r-6\sigma} \), one has
\[ \|F_\beta\| = \sum_{|j| \leq L^+ - 1} |\tilde{F}_{\beta j} + \alpha_j e^{1+\alpha} | \]
\[ \leq \sum_{|j| \leq L^+ - 1} |\tilde{F}_{\beta j}| e^{1+\alpha} + \|\alpha\| \]
\[ \leq \sum_{|j| \leq L^+ - 1} |\tilde{F}_{\beta j}| e^{(L^+ - 1)^{1+\alpha}} + \|\alpha\| \]
\[ \leq e^{-\frac{4\theta}{5}} (\sigma^{-1} |\tilde{F}|_{r-5\sigma} + \epsilon^{\frac{4}{5} - \gamma - \frac{1}{5} \delta}) \]
\[ \leq \frac{1}{\sigma^{6L^+ - 1}} e^{1-3\gamma - \frac{1}{5} \delta}. \]
Similarly, on $D_{s-\delta,r-5\sigma}$, we obtain the estimate
\[
|F_p| \leq \sup_j |F_{p_j}|
\]
\[
\leq \sup_j \sigma^{-1} e^{\|I_j\|^{1+\alpha} F_{|s,r-5\sigma}}
\]
\[
\leq \epsilon \frac{1+\sigma}{\sigma \sigma^{6+1}} e^{1-3\gamma}
\]
\[
\leq \frac{1}{\sigma \sigma^{6+1}} \epsilon^{1-3\gamma-\frac{3}{2}}.
\]
Recalling the estimates $F_p, F_\theta$, we thus have
\[
\sigma^{-1} |F_p|, \sigma^{-1} |F_\theta| \leq \frac{1}{\sigma \sigma^{6+1}} \epsilon^{1-3\gamma-\frac{3}{2}}
\]
uniformly on $D_{s-\delta,r-6\sigma}$.

Since $(I, \theta)|_p = \max(||I||, ||\theta||_\infty)$, note $W = \text{diag}(\sigma^{-1} I_\lambda, \sigma^{-1} I_\lambda)$, and then the above estimates are equivalent to
\[
|WX_F|_p \leq \frac{1}{\sigma \sigma^{6+1}} \epsilon^{1-3\gamma-\frac{3}{2}}
\]
on $D_{s-\delta,r-6\sigma}$.

Considering the Hamiltonian vector-field $X_F$ associated with $F$, the time-1 map can be written as
\[
\Psi : D_b = D_{s-2\delta,r-7\sigma} \to D_a = D_{s-\delta,r-6\sigma},
\]
and the estimate
\[
|W(\Psi - \text{id})|_{p,D_b} \leq \frac{1}{\sigma \sigma^{6+1}} \epsilon^{1-3\gamma-\frac{3}{2}}
\]
holds, where $D_b$ and $D_a$ are the domain of $|W| \cdot |p|$-distance.

3. The new Hamiltonian

In views of (2.12) and (2.19), we obtain the new Hamiltonian
\[
H_+ = N_+ + P_+ + \tilde{V} + \epsilon \sum_{i,j} f_{i,j} (I_i^0 + \rho_i, I_j^0 + \rho_j, \theta_i, \theta_j),
\]
where $N_+$ and $P_+$ are given in (2.20) and (2.21) respectively.

3.1. The new normal form $N_+$. From (2.20), $N_+$ is given by
\[
N_+ = \hat{e} + N + \langle [P^2] \rho, \rho \rangle + \epsilon \langle [Q^2] \rho, \rho \rangle + \langle U \rangle \rho, \rho \rangle
\]
\[
= e_+ + \sum_{j \in \mathbb{Z}} \omega_j \rho_j + \frac{1}{2} \sum_{j \in \mathbb{Z}, |i-j| \leq 1} \Omega_{ij} \rho_i \rho_j + \sum_{|i| \leq 1} \hat{\Omega}_{ij} \rho_i \rho_j,
\]
\[
= e_+ + \langle \omega(+), \rho(+), \rho(+) \rangle + \frac{1}{2} \langle \Omega^+ \rho(+), \rho(+) \rangle + \sum_{|j| \geq L^+} \omega_j \rho_j + \frac{1}{2} \sum_{|i| \leq 1} \Omega_{ij} \rho_i \rho_j,
\]
where $\hat{\Omega}_{ij} = [P^2]^{\rho_i \rho_j} + [Q^2]^{\rho_i \rho_j} + [U]^{\rho_i \rho_j}$ and $\Omega^+_{ij} = \Omega_{ij}(+) + \hat{\Omega}_{ij}$ for $|i|, |j| \leq L^+ - 1$.

It follows from (2.35) that
\[
|\hat{\Omega}| \leq \langle [P^2] \rangle + \epsilon \langle [Q^2] \rangle + \langle U \rangle \leq \epsilon + \frac{1}{\sigma \sigma^{6+1}} \epsilon^{1-\gamma} \leq \frac{3}{2}.
\]
Consequently, since $\Omega^+ = \Omega(+) + \hat{\Omega}$, the matrix $\Omega^+$ satisfies
\[
|\Omega^+| = |\Omega(+) + \hat{\Omega}|
\]
\[
= |\Omega(+) + |E + \Omega^{-1}(+)|\hat{\Omega}|
\]
\[
\leq \kappa_1 (1 + 2\kappa_2 \epsilon^{\frac{3}{2}}),
\]
and its inverse \((\Omega^+)^{-1}\) satisfy

\[
|(\Omega^+)^{-1}| = |(\Omega(+) + \hat{\Omega})^{-1}|
\]

\[
= |(E + \Omega^{-1}(+)\hat{\Omega})^{-1}\Omega^{-1}(+)|
\]

\[
\leq |(E + \Omega^{-1}(+)\hat{\Omega})^{-1}| |\Omega^{-1}(+)|
\]

\[
\leq \frac{\kappa_2}{1 - 2\kappa_2\epsilon^2}.
\]

Moreover, let

\[
\bar{\Omega}^+ = \left( \begin{array}{cc} \Omega^+ & 0 \\ 0 & 0 \end{array} \right)_{\infty \times \infty},
\]

and then the operator \(\bar{\Omega}^+ : \mathbb{C}^Z \rightarrow \mathbb{C}^Z\) satisfies

\[
\left\| \bar{\Omega}^+ \right\| = \sup_{\|I\| \neq 0} \frac{\|\bar{\Omega}^+ I\|}{\|I\|} \leq \sup_{\|I\| \neq 0} \frac{\|\Omega^+ I\| + \|\hat{\Omega} I\|}{\|I\|}
\]

\[
\leq \kappa_1 + \sup_{\|I\| \neq 0} \sum_{|i|, |j| \leq L^+ - 1} |\Omega_{ij}| |e_i| |e_j|^{1+\alpha}
\]

\[
\leq \kappa_1 + \sup_{\|I\| \neq 0} \left( \sum_{|j| \leq L^+ - 1} \frac{q - \frac{L^+}{d}}{\|I\|} \right) \left( \sum_{|i| \leq L^+ - 1} |I_i| |e_i|^{1+\alpha} \right)
\]

\[
\leq \kappa_1 + \epsilon^+.
\]

Correspondingly, let

\[
(\bar{\Omega}^+)^{-1} = \left( \begin{array}{cc} (\Omega^+)^{-1} & 0 \\ 0 & 0 \end{array} \right)_{\infty \times \infty},
\]

and the related operator \((\bar{\Omega}^+)^{-1} : \mathbb{C}^Z \rightarrow \mathbb{C}^Z\) satisfies

\[
\left\| (\bar{\Omega}^+)^{-1} \right\| = \sup_{\|I\| \neq 0} \frac{\left\| (\bar{\Omega}^+)^{-1} I \right\|}{\|I\|} = \sup_{\|I\| \neq 0} \frac{\|\Omega^+ I(+)\|}{\|I\|}
\]

\[
= \sup_{\|I\| \neq 0} \frac{\|E + \Omega^{-1}(+)\hat{\Omega})^{-1}\Omega^{-1}(+) I(+))\|}{\|I\|}
\]

\[
\leq \sup_{\|I\| \neq 0} \frac{\|E + \Omega^{-1}(+)\hat{\Omega})^{-1}\Omega^{-1}(+) \hat{\Omega})^{-1} b(+)\|}{\|I\|}
\]

(by letting \(b(+) = \Omega^{-1}(+) I(+))\))

\[
\leq \sup_{\|I\| \neq 0} \frac{\|\sum_{k=0}^{\infty} (\Omega^{-1}(+)\hat{\Omega})^k b(+)\|}{\|I\|}
\]

\[
\leq \frac{1}{1 - 2\kappa_2\epsilon^2} \left( \sup_{\|I\| \neq 0} \frac{\|b(+)\|}{\|I\|} \right)
\]

\[
\leq \frac{\kappa_2}{1 - 2\kappa_2\epsilon^2}.
\]
3.2. The new perturbation $P_+$. Recall that the new term $P_+$ is given by (2.21), i.e.,

\[ P_+ = \hat{V}^3 + \tilde{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3 + \hat{V}^4 + \tilde{V}^4 + \Gamma P^4 + \epsilon \Gamma Q^4 \]

\[ + \{ \hat{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F \}^3 + \{ \hat{V}^3 + \Gamma P^3 + \epsilon \Gamma Q^3, F \}^4 \]

\[ + \{ \hat{V}^4 + \Gamma P^4 + \epsilon \Gamma Q^4, F \}^3 + \{ \hat{V}^4 + \Gamma P^4 + \epsilon \Gamma Q^4, F \}^4 \]

(3.2) \[ + \int_0^1 \{ \Gamma P^{low} + \epsilon \Gamma Q^{low}, F \} \circ X^i_F \ dt \]

(3.3) \[ + \int_0^1 (1 - t) \{ \{ N^0 + N^1 + \hat{V} + \tilde{V} + \Gamma P^{high} + \epsilon \Gamma Q^{high}, F \}, F \} \circ X^i_F \ dt \]

(3.4) \[ + ( \sum_{|m| \geq M^+} \hat{P}_v + \epsilon \hat{Q}_v) \circ X^i_F \]

(3.5) \[ + \{ N^1, F \} + \{ \tilde{V}, F \} + (\epsilon \sum_{|i| or |j| \geq 1^+} \text{either but not both} \]

By estimates (2.27)-(2.40), one has

\[ |H \circ X^i_F|_{s_2 - 2\delta, r - 7\sigma} \leq |H|_{s_2 - \delta, r - 6\sigma}. \]

(3.6)

Hence, with this assumption and Lemma 2.3 one can estimate new error term $P_+$ by

\[ |H \circ X^i_F|_{s_2 - 2\delta, r - 7\sigma} \leq |H|_{s_2 - \delta, r - 6\sigma}. \]

(3.7) \[ \left| \int_0^1 \{ \Gamma P^{low} + \epsilon \Gamma Q^{low}, F \} \circ X^i_F \ dt \right|_{s_2 - \delta, r - 7\sigma} \]

\[ \leq \int_0^1 \left| \{ \Gamma P^{low} + \epsilon \Gamma Q^{low}, F \} \right|_{s_2 - \delta, r - 6\sigma} dt \]

\[ \leq \epsilon - \frac{1 + \gamma_3}{\sigma} - 1 (|F|_{s_2 - \delta, r - 5\sigma} + |a|) \left| \Gamma P^{low} + \epsilon \Gamma Q^{low} \right|_{s_2, r} \]

\[ \leq \frac{1}{\sigma^{6L^+}} \left( \frac{2}{\sigma} + 3\gamma_3 + \frac{1}{\hat{\beta}} \right) \left| \Gamma P^{low} + \epsilon \Gamma Q^{low} \right|_{s_2, r} \]

\[ \leq \frac{1}{\sigma^{6L^+}} \left( \frac{2}{\sigma} + 3\gamma_3 + \frac{1}{\hat{\beta}} \right), \]

and

\[ \left| \sum_{|m| \geq M^+} (\hat{P}_v + \epsilon \hat{Q}_v) e^{t \langle v, \theta \rangle} \right|_{s_2 - \delta, r - 2\sigma} \]

\[ \leq \left| \sum_{|m| \geq M^+} (\hat{P}_v + \epsilon \hat{Q}_v) e^{t \langle v, \theta \rangle} \right|_{s_2 - \delta, r - \sigma} \]

\[ \leq \epsilon \left| \sum_{M \geq M^+} M^{2L^+} e^{-\sigma M} \right| \]

\[ \leq \epsilon^2, \]

by choosing $M^+ = \frac{2}{\sigma} |\ln \epsilon|$. 

Similarly, by estimates (3.7), (3.9), (3.8), and (3.10), we obtain (3.12)

\[
\left| \sum_{\gamma \geq 1} P_j \right|_{s-3\sigma, r - 8\sigma} \lesssim 1 + \frac{1}{\sigma \sigma 6 \gamma} \epsilon^{\frac{4}{3} - 3\gamma - \frac{1}{2}} + \left( \frac{1}{\sigma \sigma 6 \gamma} \epsilon^{\frac{4}{3} - 3\gamma - \frac{1}{2}} \right)^2.
\]

Thus, if we choose \( \epsilon_1 = \epsilon^{1 + \beta} \), (3.11) will be bounded by \( \epsilon_1 \).

4. Iteration and Convergence

Let \( 0 < \beta < \frac{1}{10} \) and \( 0 < \gamma < \frac{1}{901} \) be constants. In the following, we display the various inductive constants in the list:

1. \( \epsilon_k = \epsilon^{(1 + \beta)^k} \): \( \epsilon_k \) bounds the size of the interaction after \( k \) iterations;

2. Given \( 0 < s_0 = s \leq 1 \), \( s_{k+1} = s_k - 3 \sigma_k = s_k - \frac{(k+1)^{-2}}{2 \sum_{j=1}^{k} f_j} s_0 \) for \( k = 0, 1, 2, ... \) \( s_k \) measures the size of the analyticity domain in the action variables after \( k \) iterations, and \( 3 \sigma_k \) is the amount by which the domain shrinks in the \( k \)-th step;
Lemma 4.1. (Iterative Lemma) There exists \( \varepsilon_0 > 0 \) such that if \( \varepsilon < \varepsilon_0 \), then, for every \( k \geq 0 \), there exists a canonical transformation, \( \Psi^k \), which is analytic and invertible on \( \mathcal{D}_k \) and maps this set into \( \mathcal{D}_0 \). The Hamiltonian \( H_k = H_0 \circ \Psi^k \) has the form

\[
H_k = N_k + R_k
\]

where

\[
N_k = \varepsilon_k + \langle \omega(k), \rho(k) \rangle + \frac{1}{2} \langle \Omega^k \rho(k), \rho(k) \rangle + \sum_{|j| \geq \mathcal{L}_k} \omega_j \rho_j + \frac{1}{2} \sum_{|j| \geq \mathcal{L}_k} \Omega_j \rho^2_j,
\]

with \( \Omega^k = \Omega(k) + \sum_{s=0}^{k-1} \hat{\Omega}^s = \Omega(k) + \sum_{s=0}^{k-1} [P_s^2] + \sum_{s=0}^{k-1} [Q_s^2] + [U_s] \), and

\[
R_k = P_k(\rho, \theta) + \tilde{V}_k(\rho) + \varepsilon \sum_{|i,j| \geq \mathcal{L}^k} f_{|i,j|}(I_i^0 + \rho_i, I_j^0 + \rho_j, \theta_i, \theta_j),
\]

with \( P_k \) depending only on \( (\rho_j, \theta_j) \) for \( |j| \leq \mathcal{L}_k \) and \( \tilde{V}_k(\rho) = \sum_{|i,j| \geq \mathcal{L}_k} V_{|i,j|}(\rho) \). Rewrite \( R_k \) into the following form

\[
R_k = P_k + \tilde{V}_k(\rho) + \tilde{V}_k(\rho) + \tilde{V}_k(\rho) + R_{k,1} + R_{k,2},
\]

where

\[
\tilde{V}_k(\rho) = \sum_{|i,j| \geq \mathcal{L}_k, |i|, |j| \leq \mathcal{L}^k + 1 - 1} V_{|i,j|}(\rho),
\]

and

\[
R_{k,1} = \varepsilon \sum_{|i,j| \geq \mathcal{L}_k, |i|, |j| \leq \mathcal{L}^k + 1 - 1} f_{|i,j|}(I_i^0 + \rho_i, \theta_i), \quad R_{k,2} = \varepsilon \sum_{|i,j| \geq \mathcal{L}_k, |i|, |j| \geq \mathcal{L}^k + 1} f_{|i,j|}(I_i^0 + \rho_i, \theta_i).
\]

Note

\[
R_{k,1} = \varepsilon Q_k(\rho, \theta) + \varepsilon \sum_{|i| \geq L_k + 1} f_{|i,j|}(\rho, \theta),
\]

with

\[
Q_k = \sum_{|i,j| \geq \mathcal{L}_k, |i|, |j| \leq \mathcal{L}^k + 1 - 1} f_{|i,j|}(I_i^0 + \rho_i, I_j^0 + \rho_j, \theta_i, \theta_j).
\]
Suppose $P^\text{low}_k(\rho, \theta)$ satisfies the smallness assumption

$$|P^\text{low}_k|_{s_k, r_k} < \epsilon_k,$$

and $P^\text{high}_k(\rho, \theta)$ satisfies

$$|P^\text{high}_k|_{s_k, r_k} < 1 + \sum_{i=0}^{k-1} \epsilon_i^{1/2}.$$

Then for $\omega = \omega(k + 1) \in (A_1)$, the $k$-th homological equation

$$\{N_k, F_k\} + \Gamma P^\text{low}_k + \epsilon \Gamma Q^\text{low}_k + \{V_k + \Gamma P^\text{high}_k + \epsilon \Gamma Q^\text{high}_k, F_k\}^\text{low}$$

$$= [P^0_k] + \epsilon [Q^0_k] + [P^0_k] + \epsilon [Q^0_k] + [P^0_k] + [Q^0_k] + [U_k],$$

with $(U_k, \rho) = \{V_k + \Gamma P^\text{high}_k + \epsilon \Gamma Q^\text{high}_k, F_k\}^\text{low}$, has a solution $F_k = \tilde{F}_k + \langle a^k, \theta \rangle = F^0_k + F^1_k + F^2_k + \langle a^k, \theta \rangle$ with the estimates

$$\||a^k|| \leq \frac{3}{k} - \frac{1}{2} - \gamma,$$

and

$$|	ilde{F}_k|_{s_{k+1}, r_{k+1}} \leq \epsilon_{k+1}.$$

Moreover,

$$H_{k+1} = N_{k+1} + R_{k+1},$$

where

$$N_{k+1} = \epsilon_{k+1} + \langle \omega(k + 1), \rho(k + 1) \rangle + \frac{1}{2} \Omega(k + 1, \rho(k + 1))$$

$$+ \sum_{|j| \geq L^k} \omega_j \rho_j + \frac{1}{2} \sum_{|i| \neq |j| \geq L^k} \Omega_{i,j} \rho_i \rho_j,$$

with $\Omega(k + 1) = \Omega(k + 1) + \sum_{s=0}^{k} \Omega_s = \Omega(k + 1) + \sum_{s=0}^{k} [P^2_s] + \epsilon [Q^2_s] + [U_s]$, and

$$R_{k+1} = V_{k+1}(\rho) + P_{k+1}(\rho, \theta)$$

$$+ \epsilon \sum_{|i,j| = |(i,j)| \geq L^k} f_{i,j} (P^0_i + \rho_i, P^0_j + \rho_j, \theta_i, \theta_j),$$

in which $V_{k+1}(\rho) = \sum_{|i,j| = |(i,j)| \geq L^k} V_{i,j}(\rho)$, with the following estimates hold:

1. The symplectic map $\Psi^{k+1} = \Psi^k \circ \Psi_k$ satisfies

$$|W_0(\Psi^{k+1} - \Psi^k)| \leq \epsilon_k^{17}.$$

2. The $(2L^{k+1} - 1) \times (2L^{k+1} - 1)$ matrix $\Omega^{k+1}$, the relative operator $\bar{\Omega}^{k+1} : \mathbb{C}^Z \to \mathbb{C}^Z$ respectively satisfy

$$|\Omega^{k+1}| \leq \kappa_1 (1 + 2\kappa_2 \sum_{s=0}^{k} \epsilon_s^k), |||\Omega^{k+1}||| \leq \kappa_1 + \sum_{s=0}^{k} \epsilon_s^k,$$

and its inverse matrix $(\Omega^{k+1})^{-1}$, inverse operator $(\bar{\Omega}^{k+1})^{-1} : \mathbb{C}^Z \to \mathbb{C}^Z$ respectively satisfy

$$|((\Omega^{k+1})^{-1})| \leq \frac{\kappa_2}{1 - 2\kappa_2 \sum_{i=0}^{k} \epsilon_i^k}, |||(\Omega^{k+1})^{-1}||| \leq \frac{\kappa_2}{1 - 2\kappa_2 \sum_{i=0}^{k} \epsilon_i^k}.$$

3. The perturbation $P^\text{low}_{k+1}$ satisfies

$$|P^\text{low}_{k+1}|_{s_{k+1}, r_{k+1}} \leq \epsilon_{k+1},$$

and $P^\text{high}_{k+1}(\rho, \theta)$ satisfies

$$|P^\text{high}_{k+1}|_{s_{k+1}, r_{k+1}} \leq 1 + \sum_{i=0}^{k} \epsilon_i^{1/2}.$$
Proof. First of all, distinguishing terms of (4.4) by the order of ρ, we have
\[
\begin{align*}
\left\{
\begin{array}{l}
-\partial_{\rho} F_0^k - \langle \omega(k+1), a_k^0 \rangle + \Gamma P_0^k + \epsilon \Gamma Q_0^k = 0, \\
-\partial_{\rho} F_1^k - \hat{\Omega}^k a_k^1 - \hat{\Omega}^k \partial_{\theta} F_0^k + \Gamma P_1^k + \epsilon \Gamma Q_1^k = 0, \\
-\partial_{\rho} F_2^k - \hat{\Omega}^k \partial_{\theta} F_1^k + U_k + \Gamma P_2^k + \epsilon \Gamma Q_2^k = 0,
\end{array}
\right.
\end{align*}
\] (4.13)
where \(\hat{\Omega}^k = \Omega(k+1) + \sum_{j=0}^{k-1} \hat{\Omega}^j\).

Now we will prove the Lemma by the following steps.

**Step 1. Proof of (4.14).** We consider the first equation of (4.13). Since \(\omega\) satisfies the Diophantine condition (A1), we can solve the first equation of (4.13),
\[
(4.14) \quad \partial_{\rho} F_0^k = \Gamma P_0^k + \epsilon \Gamma Q_0^k - ([P_0^k] + \epsilon [Q_0^k] + \langle \omega, a_k^0 \rangle),
\]
then the solution of the homological equation is given by
\[
(4.15) \quad F_0^k(\theta) = \sum_{0<|v|<M_k+1} P_{0,v}^k + \epsilon Q_{0,v}^k e^{i(\omega, \theta)}. 
\]
From Diophantine condition (A1), we have
\[
|F_0^k(\theta)|_{r_k-\sigma_k} \leq \varepsilon_k^\gamma \sum_{0<|v|<M_k+1} \left( \varepsilon_k + \epsilon \sum_{L_k \leq |j| \leq L_k+1-1} |F_j^0|_{-1} \right) e^{-r_k|v|} e^{r_k-\sigma_k|v|}
\]
\[
\leq \sigma_k^{-1} (2M_k+1)^{2k+1} \varepsilon_k^{-1-\gamma}
\]
(4.16)
From Cauchy estimate, one has
\[
|\partial_{\theta} F_0^k(\theta)|_{r_k-2\sigma_k} \leq \frac{1}{\sigma_k} |F_0^k(\theta)|_{r_k-\sigma_k},
\]
which together with (4.16) implies
\[
|\partial_{\theta} F_0^k(\theta)|_{r_k-2\sigma_k} \leq \frac{\varepsilon_k^{1-2\gamma}}{\sigma_k^2}.
\]
(4.17)
We next consider the second equation of (4.13).
Since \(\hat{F}_k^{l,0} = 0\), we can choose a vector \(a_k^0\) such that \((\hat{\Omega}^k)a_k^0 - [P_k^0] - \epsilon [Q_k^0] = 0\). Then we have
\[
(4.18) \quad a_k^0 = (\hat{\Omega}^k)^{-1}([P_k^0] + \epsilon [Q_k^0]),
\]
with the estimate
\[
|a_k^0| = |(\hat{\Omega}^k)^{-1}([P_k^0] + \epsilon [Q_k^0])|
\]
\[
\leq |\Omega^{-1}(k+1)| |(E + \sum_{s=0}^{s-1} \hat{\Omega}^s)^{-1}([P_k^0] + \epsilon [Q_k^0])|
\]
\[
\leq \varepsilon_k.
\]
Let \(\hat{F}_k^0 = -\hat{\Omega} \cdot \partial_{\theta} F_k^0\). We get
\[
|\hat{F}_k^0(\theta)|_{r_k-2\sigma_k} \leq |\hat{\Omega}| |\partial_{\theta} F_k^0(\theta)|_{r_k-2\sigma_k} \leq \frac{1}{\sigma_k} \varepsilon_k^{1-2\gamma}.
\]
Similarly, we can solve the second equation (4.13)
\[
(4.19) \quad F_1^k(\theta) = \sum_{0<|v|<M_k+1} \hat{P}_{0,v}^k + \epsilon \hat{Q}_{0,v}^k + \hat{F}_k^{l,0} e^{i(\omega, \theta)}. 
\]
From (4.17), we then obtain the estimate

\[
|F_k^3(\theta)|_{r_k - 3\sigma_k} \leq \epsilon_k^{-\gamma} \sum_{0 < |v| \leq M^{k+1}} \left( \epsilon_k^2 + e^{-4(L_k - 1) + \alpha} \epsilon_k^2 \right) e^{-(r_k - 3\sigma_k)|v|} e^{(r_k - 3\sigma_k)|v|} \leq \frac{1}{\sigma_k} \epsilon_k^{-4\gamma},
\]

and

(4.20) \quad |\partial_\theta F_k^3(\theta)|_{r_k - 4\sigma_k} \leq \frac{1}{\sigma_k} \epsilon_k^{-4\gamma}.

Now we consider the third equation of (4.13). Let \( \tilde{F}_k^3 = -\tilde{Q}_k \partial_\theta F_k^3 \). We have

\[
|\tilde{F}_k^3(\theta)|_{r_k - 4\sigma_k} \leq |\tilde{Q}_k| |\partial_\theta F_k^3(\theta)|_{r_k - 4\sigma_k} \leq \frac{1}{\sigma_k} \epsilon_k^{-4\gamma}.
\]

On \( D_{x_k - \theta_k, r_k - 4\sigma_k} \),

Since \( \langle U_k \rho, \rho \rangle = \{ \hat{V}_k^3 + \Gamma P_k^3 + \epsilon \Gamma Q_k^3, F_k^0 + \langle a^k, \theta \rangle \} \),

and

\[
\{ \hat{V}_k^3, F_k^0 + \langle a^k, \theta \rangle \} = \sum_{i,j,l} \frac{\partial^3 \hat{V}_k^3}{\partial \rho_i \partial \rho_j \partial \rho_l} (\partial_{\theta_j} F_k^0 + a_j^k) \rho_i \rho_j,
\]

we easily have

\[
|\{ \hat{V}_k^3, F_k^0 + \langle a^k, \theta \rangle \}^{\rho \rho}| \leq \sup_j \left( \left| \frac{\partial^3 \hat{V}_k^3}{\partial \rho_i \partial \rho_j \partial \rho_l} \right| + \left| \frac{\partial^3 \hat{V}_k^3}{\partial \rho_{j+1} \partial \rho_{j+1} \partial \rho_{j+1}} \right| \right) (|\partial_\theta F_k^0| + |a^k|),
\]

and

\[
|\{ \Gamma P_k^3 + \epsilon \Gamma Q_k^3, F_k^0 + \langle a^k, \theta \rangle \}^{\rho \rho}| = \max_i \sum_{|j| \leq L_k} \left| \frac{\partial^3 (\Gamma P_k^3 + \epsilon \Gamma Q_k^3)}{\partial \rho_i \partial \rho_{j+1} \partial \rho_{j+1}} (\partial_{\theta_j} F_k^0 + a_j^k) \right| (|\partial_\theta F_k^0| + |a^k|)
\]

\[
\leq \left( \sum_{|j| \leq L_k} \epsilon_k^4 \sum_{L_k \leq |j| \leq L_k} \epsilon_k^4 \right) (|\partial_\theta F_k^0| + |a^k|)
\]

\[
|\partial_\theta F_k^0| + |a^k|.
\]

One then obtains

(4.21) \quad |U_k|_{r_k - 2\sigma_k} \leq \frac{1}{\sigma_k} \epsilon_k^{-2\gamma}.

Therefore, we can solve the third equation of (4.13)

\[
F_k^2(\theta) = \sum_{0 < |v| \leq M^{k+1}} \tilde{F}_{k,v}^2 + \epsilon \tilde{Q}_{k,v}^2 + \tilde{F}_{k,v}^1 + \tilde{G}_{k,v} + \tilde{F}_{k,v}^1 e^{(v, \theta)},
\]
and we also obtain
\[
|F_k^2(\theta)|_{r_k-5\sigma_k} \leq \epsilon_k^{-\gamma} \sum_{0<|\nu|<M^{k+1}} \left( \sum_{|j| \leq L^k} \epsilon_k^4 \epsilon_k^{-3(L^k - 1)^{\alpha}} + \frac{K_2}{\sigma_k^4} \epsilon_k^{-4\gamma} \right) e^{-(r_k-5\sigma_k)|\nu|} + e^{-(r_k-4\sigma_k)|\nu|} e^{(r_k-5\sigma_k)|\nu|} \leq \frac{1}{\sigma_k^4} \epsilon_k^{-\gamma}.
\]
Consequently, one obtains
\[
|F_k^2|_{r_k-5\sigma_k} |F_k^1|_{r_k-5\sigma_k} \leq \sigma_k^{-1} \epsilon_k^{-2\gamma} + \frac{1}{\sigma_k^4} \epsilon_k^{-4\gamma} \left( \sum_{|j| \leq L^k+1} \epsilon_k^{-|j|^{1+\alpha}} \right) + \frac{1}{\sigma_k^4} \epsilon_k^{-6\gamma} \left( \sum_{|j| \leq L^k+1} \epsilon_k^{-|j|^{1+\alpha}} \right)^2 \leq \frac{1}{\sigma_k^4} \epsilon_k^{-\gamma} \leq \frac{1}{\sigma_k^4} \epsilon_k^{-\gamma}.
\]

**Step 2. Proof of (1.5).** From the equation of motion
\[
(\hat{\theta} = F_{k,\rho}(\rho, \theta), \hat{\rho} = F_{k,\rho}(\rho, \theta) = \hat{F}_{k,\rho}(\rho, \theta) + a^k),
\]
vector \(a^k\) has to belong to \(C^2\).

Recall that
\[
(\hat{\Omega}^k)^{-1} = (\Omega(k + 1) + \sum_{s=0}^{k-1} \hat{\Omega}^s)^{-1} = (E + (\Omega^k)^{-1} \sum_{s=0}^{k-1} \hat{\Omega}^s)^{-1} (\Omega^k)^{-1} (k + 1),
\]
Thus, one has
\[
\|(\hat{\Omega}^k)^{-1}\| \leq 4K_2
\]
and
\[
\|a^k\| = \sum_{|i| \leq L^{k+1}+1} \left| \sum_{|j| \leq L^k} (\Omega^k)^{-1} ([P_k^1]_{ij} + \epsilon [Q_k^1]_{ij}) e^{ij^{1+\alpha}} \right| \leq \sum_{|i| \leq L^k} \left| ([\hat{\Omega}^k_{ij}]^{-1} [P_k^1]_{ij} + \epsilon [Q_k^1]_{ij}) e^{ij^{1+\alpha}} \right| + \epsilon \sum_{L^k \leq |i|, |j| \leq L^{k+1}+1} \left| ([\hat{\Omega}^k_{ij}]^{-1} [Q_k^1]_{ij}) e^{ij^{1+\alpha}} \right| \leq A + B,
\]
where
\[
A = \sum_{|i|, |j| \leq L^k} \left| ([\hat{\Omega}^k_{ij}]^{-1} [P_k^1]_{ij}) e^{ij^{1+\alpha}} \right|,
\]
\[
B = \epsilon \sum_{L^k \leq |i|, |j| \leq L^{k+1}+1} \left| ([\hat{\Omega}^k_{ij}]^{-1} [Q_k^1]_{ij}) e^{ij^{1+\alpha}} \right|.
\]
By some simple calculations, one gets
\begin{align*}
A & \leq 4\kappa_2 \sum_{|j| \leq L^k} \left( \epsilon_{k-1}^\alpha \epsilon_k \right) |j|^{1+\alpha} \\
& \leq 4\kappa_2 \epsilon_{k-1} \left( \sum_{|j| \leq L_k} e^{-|j|^{1+\alpha}} \right) \\
& \leq 4\kappa_2 \left( 2L_k \epsilon_{k-1} \right) \\
& \lesssim \epsilon_{k-1}^{3-\gamma},
\end{align*}
and
\begin{align*}
B & \leq 4\kappa_2 \epsilon \sum_{L^k \leq |j| \leq (L^k)^{1+1}} |\tilde{F}_{k,\theta}|^2 |j|^{1+\alpha} \\
& \leq 4\kappa_2 \epsilon e^{-4L^k (1+\alpha)} \sum_{L^k \leq |j| \leq (L^k)^{1+1}} e^{-|j|^{1+\alpha}} \\
& \leq 4\kappa_2 \epsilon \left( 2L_k \right)^{1+1} \\
& \lesssim \epsilon_{k-1}^{3-\gamma}.
\end{align*}

Therefore, we obtain
\[ \|a^k\| \leq \epsilon_{k-1}^{3-\gamma}. \]

On \( D_{\delta_k r_k - 6\sigma_k} \), one also has
\[ \|F_{k,\theta}\| = \sum_{|j| \leq (L^k)^{1+1}} |\tilde{F}_{k,\theta} + a_k| |j|^{1+\alpha} \]
\[ \lesssim \sum_{|j| \leq (L^k)^{1+1}} |\tilde{F}_{k,\theta}| e^{-|j|^{1+\alpha}} + \|a^k\| \]
\[ \lesssim \epsilon_{k-1}^{1+\gamma} \|\tilde{F}_{k,\theta}\|_{\delta_k r_k - 5\sigma_k} + \epsilon_{k-1}^{3-\gamma} \]
\[ \lesssim \sigma_{k-1}^{5} \epsilon_{k-1}^{3-\gamma} + \epsilon_{k-1}^{3-\gamma}. \]

On \( D_{\delta_k - \delta_k} \), we obtain the estimate
\[ |F_{k,\rho}|_\infty = \sup_j |\tilde{F}_{k,\rho}|_j \]
\[ \lesssim \sup_j \sigma_{k-1}^{-1} \epsilon_{k-1} |j|^{1+\alpha} \|\tilde{F}_{k,\rho}\|_j \]
\[ \lesssim \epsilon_{k-1}^{1+\gamma} \sigma_{k-1}^{-1} \|	ilde{F}_{k,\rho}\|_{\delta_k r_k - 5\sigma_k} \]
\[ \lesssim \epsilon_{k-1}^{3-\gamma} \sigma_{k-1}^{-5} \epsilon_{k-1}^{3-\gamma}. \]

Recalling the estimates for \( \tilde{F}_{k,\rho}, F_{k,\theta} \), we thus have
\[ \sigma_{k-1}^{-1} |F_{k,\rho}|_\infty, \sigma_{k-1}^{-1} \|F_{k,\theta}\| \lesssim \sigma_{k-1}^{-6} \epsilon_{k-1}^{3-\gamma} - \frac{3}{2} \beta \]
uniformly on \( D_{\delta_k - \delta_k} \).

Noting \( W_k = diag(\sigma_{k}^{-1} I_\Lambda, \sigma_{k}^{-1} I_\Lambda) \), the above estimates are equivalent to
\[ |W_k X F_k|_p < \sigma_{k-1}^{-1} \sigma_{k}^{-5} \epsilon_{k-1}^{3-\gamma} - \frac{3}{2} \beta \]
on \( D_{\delta_k - \delta_k} \).

Considering the Hamiltonian vector-field \( X F_k = \Psi_k \) associated with \( F_k \), the time-1-map can be written as
\[ \Psi_k : D_{\delta_k r_k - 7\sigma_k} \rightarrow D_{\delta_k - \delta_k} \cup D_{\delta_k - \delta_k}. \]
for which the estimate
\[
|W_k(\Psi_k - id)|_{\mathcal{D}_{k+1}} \leq \sigma_k^{-1} \sigma_k^{-\gamma} \epsilon_k^{\gamma - \beta}
\]
holds.

Since \(\Psi^{k+1} = \Psi^k \circ \Psi_k\), write
\[
|W_0(\Psi^{k+1} - \Psi^k)|_{\mathcal{D}_{k+1}} = |W_0(\Psi^k \circ \Psi_k - \Psi^k)|_{\mathcal{D}_{k+1}} \leq |W_0 R^k W_k^{-1}|_{\mathcal{D}_k} |W_k(\Psi_k - id)|_{\mathcal{D}_{k+1}}.
\]

By the inductive construction, \(\Psi^k = \Psi_0 \circ \cdots \circ \Psi_{k-1}\), and
\[
|W_0 D\Psi_k W_k^{-1}|_{\mathcal{D}_{v+1}} \leq 1 + \epsilon_k^{\frac{1}{25}}, \text{ (in view of (4.24))}
\]
thus, we obtain
\[
|W_0(\Psi^{k+1} - \Psi^k)|_{\mathcal{D}_{k+1}} \leq \prod_{v=0}^{k-1} |W_0 D\Psi_v W_v^{-1}|_{\mathcal{D}_{v+1}} |W_v(\Psi_v - id)|_{\mathcal{D}_{v+1}} \leq (1 + \epsilon_k^{\frac{1}{25}})^k \sigma_k^{-\gamma} \epsilon_k^{\gamma - \frac{\beta}{2}} \lesssim \epsilon_k^{\frac{1}{25}}.
\]

**Step 3. Proofs of (4.9) and (4.10).** On \(\mathcal{D}_{k-\theta_k, r_k-2\theta_k}\), it follows from (4.24) that
\[
|\hat{\Omega}^k| \leq \epsilon_k^{\frac{1}{25}} \sigma_k^{-\gamma} \epsilon_k^{\frac{1}{25}} \lesssim \epsilon_k^{\frac{1}{25}}.
\]

Therefore, since \(\hat{\Omega}^{k+1} = \hat{\Omega}(k+1) + \sum_{s=0}^{k} \hat{\Omega}^s\) for \(\hat{\Omega}^s = (\hat{\Omega}_{ij})\) of \(|i|, |j| \leq L^s + 1\), the matrix \(\hat{\Omega}^{k+1}\) satisfies
\[
|\hat{\Omega}^{k+1}| = |\hat{\Omega}(k+1) + \sum_{s=0}^{k} \hat{\Omega}^s| = |\hat{\Omega}(k+1)| \left| E + \hat{\Omega}^{-1}(k+1) \sum_{s=0}^{k} \hat{\Omega}^s \right| \lesssim \kappa_1 (1 + 2\kappa_2 \sum_{s=0}^{k} \hat{\epsilon}_k^{\frac{1}{25}}),
\]
and its inverse \((\hat{\Omega}^{k+1})^{-1}\) satisfies
\[
|((\hat{\Omega}^{k+1})^{-1})| = \left| \left( \hat{\Omega}(k+1) + \sum_{s=0}^{k} \hat{\Omega}^s \right)^{-1} \right| \leq \left| \left( E + \hat{\Omega}^{-1}(k+1) \sum_{s=0}^{k} \hat{\Omega}^s \right)^{-1} \Omega^{-1}(k+1) \right| \lesssim \frac{\kappa_2}{1 - 2\kappa_2 \sum_{s=0}^{k} \hat{\epsilon}_k^{\frac{1}{25}}}.
\]

Moreover, since
\[
\bar{\Omega}^{k+1} = \begin{pmatrix} \Omega^{k+1} & 0 \\ 0 & 0 \end{pmatrix}_{\infty \times \infty},
\]
the operator $\hat{\Omega}^{k+1}: \mathbb{C}^\mathbb{Z} \to \mathbb{C}^\mathbb{Z}$ satisfies
\[
|||\hat{\Omega}^{k+1}||| = \sup_{I, ||I|| \neq 0} \frac{||\hat{\Omega}^{k+1} I||}{||I||} \leq \sup_{I, ||I|| \neq 0} \frac{||\hat{\Omega}(k+1) I|| + ||\sum_{s=0}^{k} \hat{\Omega}^s I||}{||I||} \\
\leq \kappa_1 + \sup_{I, ||I|| \neq 0} \sum_{s=0}^{k} \left( \sum_{|i|, |j| \leq L^{*+1} - 1} |\hat{\Omega}^s_i| |e^{i+j}|^{1+\alpha} \frac{||I||}{||I||} \right) \\
\leq \kappa_1 + \frac{\sum_{s=0}^{k} \epsilon^s_2}{15},
\]

Similarly, as
\[
(\hat{\Omega}^{k+1})^{-1} = \begin{pmatrix}
(\hat{\Omega}^{k+1})^{-1} - 1 & 0 \\
0 & 0
\end{pmatrix}_{\infty \times \infty},
\]

the operator $(\hat{\Omega}^{k+1})^{-1}: \mathbb{C}^\mathbb{Z} \to \mathbb{C}^\mathbb{Z}$ satisfies
\[
|||(\hat{\Omega}^{k+1})^{-1}||| = \sup_{I, ||I|| \neq 0} \frac{||(\hat{\Omega}^{k+1})^{-1} I||}{||I||} = \sup_{I, ||I|| \neq 0} \frac{||(\hat{\Omega}^{k+1})^{-1} I(k+1) + \sum_{s=0}^{k} \hat{\Omega}^s I(k+1)||}{||I||} \\
\leq \sup_{I, ||I|| \neq 0} \frac{||(E + \Omega^{-1}(k+1) \sum_{s=0}^{k} \hat{\Omega}^s)^{-1} b(k+1)||}{||I||} \\
(\text{by letting } b(k+1) = \Omega^{-1}(k+1) I(k+1)) \\
\leq \frac{1}{1 - 2\kappa_2 \sum_{s=0}^{k} \epsilon^s_2} \left( \sup_{I, ||I|| \neq 0} \frac{||b(k+1)||}{||I||} \right) \\
\leq \frac{1}{1 - 2\kappa_2 \sum_{s=0}^{k} \epsilon^s_2}.
\]

Step 4. Proofs of (4.11) and (4.12). From (4.6), (4.8) and accordingly to the same process of (3.11), (5.12), we can obtain
\[
(4.28) \quad |P_{k+1}^{d, w}|_{s_k - 3s_k r_k - 8s_k} \ll \epsilon^{1+\beta}_k \ll \epsilon^{1+\beta}_{k+1},
\]

and
\[
(4.29) \quad \sum_{j \geq 3}^{4} |P_{k+1}^j|_{s_k - 3s_k r_k - 8s_k} \ll 1 + \sum_{s=0}^{k} \epsilon^{1+\beta}_s,
\]

while we omit the details. \(\square\)

5. THE MEASURE ESTIMATE

We construct a measure with support at the origin. Let
\[
d\sigma(x) = \frac{1}{\sqrt{2\pi}} \exp^{-x^2/2} \, dx
\]
be the standard gaussian measure on the real line with mean zero and variance one, and set
\[ d\mu(\omega) = \prod_{i \in \mathbb{Z}} \frac{1}{\sqrt{2\pi}} \exp^{-\omega_i^2/2} d\omega_i. \]

Note that
\[ \tilde{N}_v^k = \{ \omega : |\omega(k) \cdot v| \leq (\epsilon(1+\beta)^{-1})^\gamma \}, \]
and
\[ N^k = \bigcup_{0 < |v| \leq M^k} \tilde{N}_v^k \]

As in the finite-dimensional case, (5.1) yields
\[ \mu(\tilde{N}_v^k) \leq (\epsilon(1+\beta)^{-1})^\gamma |v| \leq C^{2L_k} \epsilon_k^{-1} \gamma, \]
where \(||v||_e\) denotes the Euclidean length.

Since the number of \(v\) in (5.2) is bounded by \(2M^k \), we obtain
\[ \mu(N^k) \leq C^{2L_k} \epsilon_k^{-1} \cdot (2M^k)^{2L_k} \leq (\epsilon_k^{-1})^k, \]
for some \(0 < \kappa < \gamma\).

Define
\[ \mathcal{R}^k = \mathcal{R}^0 \setminus \bigcup_{1 \leq j \leq k} \tilde{N}_j. \]
Thus, we have
\[ \mu(\mathcal{R}^k) \leq 1 - \sum_{j=0}^{k-1} (\epsilon_j)^k, \]
for some \(0 < \kappa < \gamma\).

6. Proof of Theorem 1.1

In this section, we prove the Theorem 1.1 by applying Iterative lemma to the Hamiltonian system defined in (1.6).

Proof. Let \( \mathcal{D}_* = \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \subset \cap_{s \geq 0} \mathcal{D}_{s_k, r_k}, \mathcal{R}_* = \cap \mathcal{R}^j, \Omega_* = \Omega + \sum_{j=1}^{\infty} \tilde{\Omega}^j \) and \( \Psi_* = \prod_{s=0}^{\infty} \Psi^s \). By the standard argument, we conclude that \( \Psi_* , D\Psi_* , \Omega_* , H_k \) converge uniformly on the domain \( \mathcal{D}_{\frac{1}{2}, \frac{1}{2}} \). Let
\[ H_* = N_* + R_* , \]
where
\[ N_* = \epsilon_* + \omega, \rho + \frac{1}{2} (\Omega_* \rho, \rho), \]
and
\[ R_* = P_* (\rho, \theta), \]
with \( P_* = O(|\rho_1|^{t_1} |\rho_2|^{t_2} |\rho_3|^{t_3}) \) for \( t_i + t_j + t_k \geq 3 \).
Moreover, by the standard KAM proof, we obtain the following estimates:
(1) The symplectic map \( \Psi_* \) satisfies
\[ |W_0(\Psi_* - id)|_{\rho, \rho_*} \leq \epsilon^{\frac{1}{2}}. \]
(2) The operator \( \Omega_* : C^2 \rightarrow C^2 \) satisfies
\[ ||\Omega_* - \Omega|| \leq \sum_{s=0}^{\infty} \epsilon_s^{\frac{1}{2}} \leq \epsilon^{\frac{1}{4}}. \]
The measure of $\mathcal{R}^\infty$ satisfies

$$(6.4) \quad \mu(\mathcal{R}^\infty) \geq 1 - \sum_{j=0}^{\infty} (\epsilon_j)^\kappa,$$

for some $0 < \kappa < \gamma$. \hfill $\square$

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