Existence and Nonlinear Stability of Steady-States to Outflow Problem for the Full Two-Phase Flow

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Abstract

The outflow problem for the viscous full two-phase flow model in a half line is investigated in the present paper. The existence, uniqueness and nonlinear stability of the steady-state are shown respectively corresponding to the supersonic, sonic or subsonic state at far field. This is different from the outflow problem for the isentropic Navier-Stokes equations, where there is no steady-state for the subsonic state. Furthermore, we obtain either exponential time decay rates for the supersonic state or algebraic time decay rates for supersonic and sonic states in weighted Sobolev spaces.

Key words. Full two-phase flow, outflow problem, stationary solution, nonlinear stability.

1 Introduction

Two-phase flow models play important roles in applied scientific areas, for instance, nuclear, engines, chemical engineering, medicine, oil-gas, fluidization, waste water treatment, biomedical, liquid crystals, lubrication [1, 6, 12, 21, 24], etc. In this paper, we consider the full two-phase flow model which can be formally obtained from a Vlasov-Fokker-Planck equation coupled with the compressible Navier-Stokes equations through the Chapman-Enskog expansion [19].

We consider the initial-boundary value problem (IBVP) for the full two-phase flow model as follows:

$$\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + [\rho u^2 + p_1(\rho)]_x &= \mu u_{xx} + n(v - u), \\
n_t + (nv)_x &= 0, \\
(nv)_t + [nv^2 + p_2(n)]_x &= (nv)_x - n(v - u), \quad x > 0, \; t > 0,
\end{align*}$$

(1.1)

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where \( \rho > 0 \) and \( n > 0 \) stand for the densities, \( u \) and \( v \) are the velocities of two fluids respectively, the constant \( \mu > 0 \) is the viscosity coefficient, and the pressure-density functions take forms

\[
p_1(\rho) = A_1 \rho^\gamma, \quad p_2(n) = A_2 n^\alpha \tag{1.2}
\]

with \( A_1 > 0, \ A_2 > 0, \ \gamma \geq 1 \) and \( \alpha \geq 1 \). The initial data are given by

\[
(\rho, u, n, v)(0, x) = (\rho_0, u_0, n_0, v_0)(x), \quad \inf_{x \in \mathbb{R}_+} \rho_0(x) > 0, \quad \inf_{x \in \mathbb{R}_+} n_0(x) > 0, \tag{1.3}
\]

\[
\lim_{x \to +\infty} (\rho_0, u_0, n_0, v_0)(x) = (\rho_+, u_+, n_+, u_+), \quad \rho_+ > 0, \quad n_+ > 0, \tag{1.4}
\]

and the outflow boundary condition is imposed by

\[
(u, v)(t, 0) = (u_-, u_-), \quad u_- < 0, \tag{1.5}
\]

where \( \rho_+ > 0, \ n_+ > 0, \ u_+ \) and \( u_- < 0 \) are constants.

The condition \( u_- < 0 \) means that the fluids flow out the region \( \mathbb{R}_+ \) through the boundary \( x = 0 \) with the velocity \( u_- \), and therefore the problem (1.1)-(1.5) is called the outflow problem [22]. On the other hand, the similar problem with the case \( u_- > 0 \) is called the inflow problem [22], and the densities on the boundary \( (\rho, n)(t, 0) = (\rho_-, n_-) \) are also imposed for the well-posedness of the inflow problem.

It is an interesting issue to study the outflow/inflow problem. There are many important progress made recently about the existence and nonlinear stability of steady-states and basic waves to the outflow/inflow problem for one-phase flow, such as compressible Navier-Stokes equations [3, 5, 8–11, 15–18, 23, 26–34]. For instance, for the inflow problem of isentropic Navier-Stokes equations, Matsumura-Nishihara [23] showed the existence and stability of steady-states for both subsonic and sonic cases together with the stability of the superposition of steady-states and rarefaction waves under small perturbation, while Fan-Liu-Wang-Zhao [5] investigated that steady-states and rarefaction waves were nonlinear stable under large perturbation, and Huang-Matsumura-Shi [9] obtained the stability of superposition of steady-states and shock waves under small perturbation. For the inflow problem of full compressible Navier-Stokes equations, Nakamura-Nishibata [26] got the existence and stability of steady-states also for both subsonic and sonic cases, and Qin-Wang [31, 32] and Hong-Wang [8] showed the combination of steady-states and rarefaction waves. For the outflow problem of isentropic Navier-Stokes equations, the existence and nonlinear stability of steady-states for both supersonic and sonic cases under small perturbation were proved in [16, 27–29], the convergence rates toward steady-states for supersonic and sonic cases were investigated in [28, 29], the stability of the superposition of steady-states and rarefaction waves were got in [11, 17]. For the outflow problem of full compressible Navier-Stokes equations, Kawashima-Nakamura-Nishibata [15] established the existence and nonlinear stability of steady-states under small perturbation for three cases: supersonic, sonic and subsonic flow while there is no steady-state to the
outflow problem of isentropic Navier-Stokes equations for the subsonic case, in addition they also gained the convergence rates toward the stationary solutions for supersonic and sonic cases [15], and Qin [30], Wan-Wang-Zhao [33] and Wan-Wang-Zou [34] obtained the stability of steady-states, rarefaction waves and their combination under large perturbation.

It is of interest and challenges to investigate the outflow/inflow problem for two-phase flow models due to the coupled motions of two phases. Although it is rather complicated, some important progress has been made about the existence and nonlinear stability of steady-states and basic waves to the outflow/inflow problem for two-phase flow models [4, 7, 20, 35, 37, 38]. For instance, Yin-Zhu [37] obtained the existence of steady-states similar to isentropic Navier-Stokes equations [16], and the nonlinear stability and convergence rates of steady-states for the supersonic case to the outflow problem of the drift-flux model. For the outflow problem of the two-fluid Navier-Stokes-Poisson system, the existence of steady-states which is similar to that of isentropic Navier-Stokes equations [16], and the nonlinear stability of rarefactions waves and steady-states together with the superposition of steady-states and rarefaction waves were proved in [4, 35]. We established existence and nonlinear stability of steady-states to inflow problem of the model (1.1) for supersonic, sonic and subsonic cases in [20], which is a different phenomena compared with the inflow problem for isentropic Navier-Stokes equations [23] and full compressible Navier-Stokes equations [26], where there is no steady-state for the supersonic case.

However, there is no result about the existence and nonlinear stability of steady-states for the outflow problem (1.1)-(1.5). The main purpose of this paper is to prove the existence and nonlinear stability of steady-states for the supersonic, sonic and subsonic case, and obtain either exponential time decay rates for the supersonic flow or algebraic time decay rates for both supersonic and sonic flows. Contrary to the isentropic Navier-Stokes equations, the steady-state to the IBVP (3.3)-(1.5) exists for the subsonic case.

The steady-state \((\bar{\rho}, \bar{u}, \bar{n}, \bar{v})(x)\) to the outflow problem (1.1)-(1.5) satisfies the following system

\[
\begin{align*}
(\bar{\rho}u)_x &= 0, \\
[\bar{\rho}u^2 + p_1(\bar{\rho})]_x &= (\mu \bar{u}u)_x + \bar{n}(\bar{v} - \bar{u}), \\
(\bar{n}v)_x &= 0, \\
[\bar{n}v^2 + p_2(\bar{n})]_x &= (\bar{\n}v)_x - \bar{n}(\bar{v} - \bar{u}), \quad x > 0, 
\end{align*}
\]

(1.6)

with the boundary conditions and spatial far field conditions

\[
(\bar{u}, \bar{v})(0) = (u_-, u_-), \quad \lim_{x \to \infty} (\bar{\rho}, \bar{u}, \bar{n}, \bar{v})(x) = (\rho_+, u_+, n_+, u_+), \quad \inf_{x \in \mathbb{R}^+} \bar{\rho}(x) > 0, \quad \inf_{x \in \mathbb{R}^+} \bar{n}(x) > 0. \quad (1.7)
\]

Integrating (1.6) and (1.6) over \((x, +\infty)\), we have

\[
\bar{u} = \frac{\rho_+}{\bar{\rho}} u_+, \quad \bar{v} = \frac{n_+}{\bar{n}} u_+, \quad (1.8)
\]
which implies that the following relationship
\[ u_+ = \frac{\bar{n}(0)}{n_+} u_- = \frac{\bar{\rho}(0)}{\rho_+} u_- < 0 \]  
(1.9)
is the necessary property of the steady-state to the boundary value problem (BVP) (1.6)-(1.7).

Define the Mach number \( M_+ \) and the sound speed \( c_+ \) as below
\[ M_+ := \frac{|u_+|}{c_+}, \quad c_+ := \left( \frac{A_1 \gamma + A_2 a n_0}{\rho_+ + n_+} \right)^{\frac{1}{2}}. \]  
(1.10)
Then, we have the following results about the existence and uniqueness of the steady-state.

**Theorem 1.1.** Let \( \delta := |u_- - u_+| > 0 \) and \( u_+ < 0 \) hold. Then there exists a set \( \Omega_- \subset \mathbb{R}_- \) such that if \( u_- \in \Omega_- \) and \( \delta \) sufficiently small, there exists a unique strong solution \((\bar{\rho}, \bar{u}, \bar{n}, \bar{v})(x)\) to the BVP (1.6)-(1.7) which satisfies either for the supersonic or subsonic case \( M_+ \neq 1 \) that
\[ |\partial_x^k(\bar{\rho} - \rho_+, \bar{u} - u_+, \bar{n} - n_+, \bar{v} - u_+)| \leq C_1 \delta e^{-c_0 x}, \quad k = 0, 1, 2, 3, \]  
(1.11)
or for the sonic case \( M_+ = 1 \) that
\[ |\partial_x^k(\bar{\rho} - \rho_+, \bar{u} - u_+, \bar{n} - n_+, \bar{v} - u_+)| \leq C_2 \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad k = 0, 1, 2, 3, \]  
(1.12)
and
\[ (\bar{u}_x, \bar{v}_x) = (a \sigma^2(x), a \sigma^2(x)) + O(|\sigma(x)|^3), \]  
(1.13)
where \( \sigma(x) \) is a smooth function satisfying \( \sigma_x = -a \sigma^2 + O(|\sigma|^3) \) and
\[ c_1 \frac{\delta}{1 + \delta x} \leq \sigma(x) \leq C_3 \frac{\delta}{1 + \delta x}, \quad |\partial_x^k \sigma(x)| \leq C_4 \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad k = 0, 1, 2, 3, \]  
(1.14)
and \( C_i > 0, i = 1, 2, 3, c_0 > 0, c_1 > 0, \) and \( a > 0 \) are positive constants.

**Remark 1.1.** Due to the drag force term, the existence of steady-states to the IBVP (1.1)-(1.5) is obtained even for the subsonic case \( M_+ < 1 \), which is different from that of the isentropic Navier-Stokes equations [16]. Moreover, the existence of steady-states to the IBVP (1.1)-(1.5) is similar to that of the full compressible Navier-Stokes equations [15].

Then, we have the nonlinear stability of the steady-state to the IBVP (1.1)-(1.5) for supersonic, sonic and subsonic cases.

**Theorem 1.2.** Let the same conditions in Theorem 1.1 hold and assume that it holds
\[ |p_1'(\rho_+) - p_2'(n_+)| \leq \sqrt{2} |u_+| \min\{(1 + \frac{\rho_+}{n_+})[(\gamma - 1)p_1'(\rho_+) + 1] + \frac{n_+}{\rho_+}[(\alpha - 1)p_2'(n_+)]^{\frac{1}{2}}\} \]  
(1.15)
for the sonic case \( M_+ = 1 \). Then, there exists a small positive constant \( \varepsilon_0 > 0 \) such that if
\[ \| (\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v}) \|_{H^1} + \delta \leq \varepsilon_0, \]  
(1.16)
the IBVP (1.1)-(1.5) has a unique global solution \((\rho, u, n, v)(t, x)\) satisfying

\[
\begin{cases}
(\rho - \bar{\rho}, u - \bar{u}, n - \bar{n}, v - \bar{v}) \in C([0, +\infty); H^1), \\
(\rho - \bar{\rho}, n - \bar{n})_x \in L^2([0, +\infty); L^2), \\
(u - \bar{u}, v - \bar{v})_x \in L^2([0, +\infty); H^1),
\end{cases}
\]

and

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^+} |(\rho - \bar{\rho}, u - \bar{u}, n - \bar{n}, v - \bar{v})(t, x)| = 0. \tag{1.17}
\]

In addition, we have the time convergence rates of the global solution to the IBVP (1.1)-(1.5) for both supersonic and sonic cases.

**Theorem 1.3.** Assume that the same conditions in Theorem 1.1 hold. Then, the following results hold.

(i) For \(M_+ > 1\) and \(\lambda > 0\), if the initial data satisfy

\[
(1 + x)^{\frac{3}{2}}(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}) \in L^2(\mathbb{R}^+)
\]

and

\[
\|(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{H^1} + \delta \leq \varepsilon_0, \tag{1.18}
\]

for a small positive constant \(\varepsilon_0 > 0\), then the solution \((\rho, u, n, v)(t, x)\) to the IBVP (1.1)-(1.5) satisfies

\[
\|(\rho - \bar{\rho}, u - \bar{u}, n - \bar{n}, v - \bar{v})(t)\|_{L^\infty} \leq C_4 \delta_0 (1 + t)^{-\frac{3}{8}}, \tag{1.19}
\]

where \(C_4 > 0\) and \(\delta_0 := \|(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{H^1} + \|(1 + x)^{\frac{3}{2}}(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{L^2}\) are constants independent of time.

(ii) For \(M_+ = 1\), \(1 \leq \lambda < \lambda^*\), \(\lambda^* := 2 + \sqrt{8 + \frac{1}{\mu_+}}\) and \(b := \frac{\rho_+(u_0^2 - p_+(\rho_+))}{|u_+| \sqrt{\mu_+ n_+ n_+}}\), if for arbitrary \(\nu \in (0, \lambda]\), there exists a small positive constant \(\varepsilon_0 > 0\) such that

\[
\|\sigma^{-\frac{3}{8}}(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{H^1} + \delta \|_{L^\infty} \leq \varepsilon_0, \tag{1.20}
\]

then the IBVP (1.1)-(1.5) has a unique global solution \((\rho, u, n, v)(t, x)\) satisfying

\[
\begin{cases}
\sigma^{-\frac{3}{8}}(\rho - \bar{\rho}, u - \bar{u}, n - \bar{n}, v - \bar{v}) \in C([0, +\infty); H^1), \\
\sigma^{-\frac{3}{8}}(\rho - \bar{\rho}, n - \bar{n})_x \in L^2([0, +\infty); L^2), \\
\sigma^{-\frac{3}{8}}(u - \bar{u}, v - \bar{v})_x \in L^2([0, +\infty); H^1),
\end{cases}
\]

and

\[
\|\sigma^{-\frac{3}{8}}(\rho - \bar{\rho}, u - \bar{u}, n - \bar{n}, v - \bar{v})(t)\|_{H^1} \leq C_5 \delta_1 (1 + t)^{-\frac{3}{8b}}, \tag{1.22}
\]

where \(\sigma(x)\) is defined by (1.14) satisfying (2.28), \(C_5 > 0\) is a positive constant independent of time, and \(\delta_1 := \|\sigma^{-\frac{3}{8}}(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{H^1}\) is a constant.
Remark 1.2. For $M_+ > 1$, the exponential time convergence rates of the global solution to the IBVP (1.1)-(1.5) can be established. Indeed, assume that $M_+ > 1, u_+ < 0$ and a certain positive constant $\lambda > 0$ hold. For a certain positive constant $\kappa \in (0, \lambda]$, there exists a small positive constant $\varepsilon_0 > 0$ such that if $e^{\frac{\kappa}{t}}(n_0 - \bar{n}, \rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v}) \in L^2(\mathbb{R}_+)$ and

$$\|(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{H^1} + \delta \leq \varepsilon_0,$$

then the solution $(\rho, u, n, v)(t, x)$ to the IBVP (1.1)-(1.5) satisfies

$$\|(\rho - \bar{\rho}, u - \bar{u}, n - \bar{n}, v - \bar{v})(t)\|_{H^1} \leq C_6 \delta e^{-\frac{\kappa}{t} t},$$

where $C_6 > 0$ and $\kappa_1 \ll \kappa$ are positive constants independent of time, and $\delta_2 := \|(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{H^1} + \|e^{\frac{\kappa}{t}}(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{L^2}$ is a constant.

The proof of (1.23) can be obtained by similar arguments as for (1.19). The details are omitted.

Remark 1.3. In Theorem 1.3, we remove the restriction (1.15), and obtain the nonlinear stability of steady-states and time decay rates of the solution to the IBVP (1.1)-(1.5) for $M_+ = 1$ with the weighted energy method. Moreover, if $p'_1(\rho_+) = p'_2(n_+)$, time decay rates of the solution to the IBVP (1.1)-(1.5) for $M_+ = 1$ are the same as that of isentropic Navier-Stokes equations [28].

We explain main strategies to prove Theorems 1.1-1.3. The system (1.1)-(1.5) can be viewed as two compressible isentropic Navier-Stokes equations coupled with each other through the drag force relaxation mechanisms. Different from the isentropic Navier-Stokes equations [16], we can not reformulate two momentum equations (1.1)\(_2\) and (1.1)\(_4\) into conservation forms due to the influence of drag force, which implies that the steady-state satisfies the system (1.6)-(1.7) consisting of a first-order and a second-order ordinary differential equations instead of only a first-order ordinary differential equation in [16], and it is not straightforward to apply the center manifold theory [2]. To overcome the difficulty, we introduce a new variable $\bar{w} := \bar{u}_x$, get the estimate $|\bar{u}_x(0)| \leq C|u_+ - u_+|$ in Lemma 2.1, and then rewrite the system (1.6) into the $3 \times 3$ system (2.9) of autonomous ordinary differential equations. Since the condition $p'_1(\rho_+) = p'_2(n_+)$ is not necessary, we need subtle analysis to obtain the sign of $\text{Re} \lambda_i, i = 1, 2, 3$, where $\lambda_i, i = 1, 2, 3$ are three eigenvalues of the linearized $3 \times 3$ system of (2.9). It should be noticed that the linearized system of (2.9) has at least one eigenvalue with negative real part due to the effect of drag force, so that we obtain the existence of steady-states for the supersonic, sonic and subsonic case in Theorem 1.1, which is different from the outflow problem of the isentropic Navier-Stokes system [16], where there is no steady-state for the subsonic case.

We establish the uniform estimates of the perturbation $(\phi, \psi, \tilde{\phi}, \tilde{\psi}) := (\rho - \bar{\rho}, u - \bar{u}, n - \bar{n}, v - \bar{v})$ to prove the nonlinear stability of steady-states for the supersonic $M_+ > 1$, sonic case $M_+ = 1$ and subsonic case $M_+ < 1$. For $M_+ = 1$, it is easy to check that $(\bar{\rho}u^2 + p_1(\bar{\rho}))_x$ and $(\bar{n}v^2 + p_2(\bar{n}))_x$ decay slower than...
\(\bar{u}_{xx}\) for \(p'_1(\rho_+) \neq p'_2(n_+)\) owing to (1.6)2, (1.6)4 and (1.12), which implies the term
\[
\int R_2 dx := \int \phi \psi \left( \frac{\rho u^2 + p_1(\rho)}{\hat{\rho}} \right) x + \bar{\phi} \psi \left( \frac{\hat{\rho} v^2 + p_2(\bar{n})}{\bar{n}} \right) x dx
\]
(1.24)
can not be controlled directly as in [16]. With the help of \(\bar{u}_x \geq 0\) and \(\bar{v}_x \geq 0\), we turn to deal with the terms
\[
\int R_2 dx + \int \left( \rho \psi^2 + p_1(\rho) - p_1(\rho) \phi \bar{u}_x \right) dx + \left( n \psi^2 + p_2(n) - p_2(n) \bar{v}_x \right) dx,
\]
the leading terms of which can be rewritten as two positive semidefinite 2 variable quadratic forms (3.21) under the condition (1.15).

By the weighted energy method, we get the exponential or algebraic time decay rates for the supersonic case \(M_+ > 1\) if the initial perturbation belongs to the exponential or algebraic weighted Sobolev space, and obtain algebraic time decay rates for the sonic case \(M_+ = 1\). To get the basic weighted energy estimates, we use (1.11) and the dissipation on the relaxation friction term \(\tilde{\psi} - \psi\), and decompose \(\psi = \bar{\psi} + (\psi - \tilde{\psi})\), as motivated by Li-Wang-Wang [19]. Due to the algebraic decay (1.12) of steady-states, convergence rates of steady-states for the sonic case \(M_+ = 1\) is worse than that of the supersonic case \(M_+ > 1\). It is necessary to use the delicate algebraic decay (1.12)-(1.14) and the dissipation on the drag force \(\tilde{\psi} - \psi\), decompose \(\psi \) as \(\psi = \bar{\psi} + (\psi - \tilde{\psi})\), and obtain more delicate estimates to get the convergence rates for \(M_+ = 1\). It should be noticed that we make full use of the dissipation on drag force term and the viscous terms, and take a linear coordinate transformation

\[
\begin{pmatrix}
\phi \\
\hat{\phi} \\
\tilde{\psi}
\end{pmatrix} = \mathbf{P}
\begin{pmatrix}
\rho \\
\hat{n} \\
\bar{\psi}
\end{pmatrix},
\]
(1.26)
with a invertible matrix \(\mathbf{P}\), to gain the 3 variable quadratic form \(\hat{\lambda}_1 \rho^2 + \hat{\lambda}_2 \bar{n}^2\) with \(\hat{\lambda}_1, \hat{\lambda}_2 > 0\), which plays an important role in basic weighted energy estimates together with some crucial cancellations. In fact, we obtain the algebraic time decay rates of the solution to IBVP (1.1)-(1.5) for \(M_+ = 1\) for the initial perturbation satisfying
\[
\sigma^{-2} \left( \rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v} \right) \in L^2(\mathbb{R}_+), \quad \text{with } \lambda < \lambda^*, \lambda^* := 2 + \sqrt{8} + \frac{1}{1+\mu}, \quad b = \frac{\rho_+ (u^2_+ - p'_1(\rho_+))}{\rho_{+} (\rho_{+} + n_+) n_+}
\]
and the function \(\sigma \geq 0\) satisfying (1.14). This is an interesting phenomena describing the influences of two fluids on each other somehow and it should be emphasized that \(\lambda^* = 5\) is the same as that of the isentropic Navier-Stokes equations [28] for \(u_+^2 = p'_1(\rho_+) = p'_2(n_+)\).

**Notation.** We denote by \(\| \cdot \|_{L^p}\) the norm of the usual Lebesgue space \(L^p = L^p(\mathbb{R}_+), 1 \leq p \leq \infty\). And if \(p = 2\), we write \(\| \cdot \|_{L^2(\mathbb{R}_+)} = \| \cdot \|_2\). \(H^s(\mathbb{R}_+)\) stands for the standard \(s\)-th Sobolev space over \(\mathbb{R}_+\) equipped with its norm
\[
\| f \|_{H^s(\mathbb{R}_+)} = \| f \|_s := \left( \sum_{i=0}^{s} \| \partial^i f \|_2^s \right)^{\frac{1}{s}}.
\]
$C([0, T]; H^1(\mathbb{R}^+))$ represents the space of continuous functions on the interval $[0, T]$ with values in $H^1(\mathbb{R}^+)$. $L^2([0, T]; B)$ denotes the space of $L^2$ functions on the interval $[0, T]$ with values in Banach space $B$. For a scalar function $W(x) > 0$, the weighted $L^2$-spaces $H^1_{W}(\mathbb{R}^+)$ and $H^2_{W}(\mathbb{R}^+)$ are defined as follows:

$$
L^2_{W}(\mathbb{R}^+) := \left\{ f \in L^2(\mathbb{R}^+) \mid \|f\|_{L^2_{W}} := \left( \int_{\mathbb{R}^+} W(x) f^2 dx \right)^{1/2} < +\infty \right\},
$$

$$
H^1_{W}(\mathbb{R}^+) := \left\{ f \in H^1(\mathbb{R}^+) \mid \|f\|_{H^1_{W}} := \left( \int_{0}^{1} \|\partial_x f\|^2_{L^2_{W}} dx \right)^{1/2} < +\infty \right\}.
$$

For a scalar function $W_{a, \nu} := (1 + x)^\nu$ with $\nu \geq 0$, we denote $\|f\|_{a, \nu} := \|(1 + x)^{\frac{\nu}{2}} f\|$.

The rest of this paper will be organized as follows. We prove the existence and uniqueness of steady-states in Section 2, get the nonlinear stability of steady-states in Section 3 for supersonic, sonic and subsonic cases, and obtain convergence rates of steady-states for the supersonic flow in Subsection 4.1 and the sonic flow in Subsection 4.2.

### 2 Existence of Steady-State

We prove Theorem 1.1 on the existence and uniqueness of steady-states to the BVP (1.6)-(1.7) with $u_+ < 0$ and $\delta$ sufficiently small as follows. In order to apply the center manifold theory [2], it is necessary to get the bounds of $\bar{u}_x(0)$ or $\bar{v}_x(0)$.

**Lemma 2.1.** Assume that $u_+ < 0$ and $\delta = |u_- - u_+|$ hold with $\delta$ sufficiently small. Then the steady-state $(\bar{\rho}, \bar{\rho}, \bar{\rho}, \bar{\rho})$ to the BVP (1.6)-(1.7) satisfies

$$
|\bar{u}_x(0)| \leq C|u_- - u_+|, \quad |\bar{v}_x(0)| \leq C|u_- - u_+|,
$$

where $C > 0$ is a positive constant.

**Proof.** Due to $\bar{\rho} = \frac{\rho + u_+}{v}$ and $\bar{\rho} = \frac{n + u_+}{v}$, we have

$$
\begin{cases}
(r_+ u_+ \bar{u} + A_1 \rho_+^2 u_+ \bar{v} - \gamma)_x = (\mu \bar{u}_x)_x + \frac{n + u_+}{v} (\bar{v} - \bar{u}), \\
(n_+ u_+ \bar{v} + A_2 n_+^2 u_+ \bar{v} - \alpha)_x = (n_+ u_+ \frac{\bar{v}}{v})_x - \frac{n + u_+}{v} (\bar{v} - \bar{u}).
\end{cases}
$$

Adding (2.2)1 to (2.2)2 and integrating the resulted equation over $(0, +\infty)$ lead to

$$
\mu \bar{u}_x(0) + n_+ u_+ \frac{\bar{v}_x(0)}{u_-} = \frac{1}{u_+} [(\rho_+ + n_+) u_+^2 - (A_1 \gamma \rho_+ + A_2 \alpha n_+^2)(u_- - u_+) + O(|u_- - u_+|^2)].
$$

With the help of (2.3), we multiply (2.2)1 by $\bar{u}$, (2.2)2 by $\bar{v}$ respectively, then integrate the summation of the resulted equations over $(0, +\infty)$ to gain

$$
\int_0^{+\infty} (\mu \overline{\bar{u}_x^2} + n_+ u_+ \overline{\bar{v}_x^2}) dx + \int_0^{+\infty} \frac{n + u_+}{v} (\bar{v} - \bar{u})^2 dx
$$

$$
= - u_- [\mu \bar{u}_x(0) + n_+ u_+ \frac{\bar{v}_x(0)}{u_-}] + [(\rho_+ + n_+) u_+^2 - (A_1 \gamma \rho_+ + A_2 \alpha n_+^2)(u_- - u_+) + O(|u_- - u_+|^2)]
$$

$$
= O(|u_- - u_+|^2).
$$
Multiplying (2.2) by $\tilde{v}$ and then integrating the resulted equation over $(0, \infty)$ yield
\[
- n_+ u_+ \bar{v}_x^2 (0) + \int_0^{+\infty} A_2 \alpha n_+^\alpha \frac{u_+^{\alpha+2}}{v^{\alpha+2}} \bar{v}_x^2 dx = \int_0^{+\infty} n_+ u_+ \frac{\bar{v}_x^2}{v} dx + \int_0^{+\infty} n_+ u_+ \tilde{v}_x^2 dx
\tag{2.5}
\]
We estimate terms in the right hand side of (2.5). With $\inf_{x \in \mathbb{R}_+} \bar{u} > 0$, $\alpha \geq 1$ and (2.4), we have
\[
\int_0^{+\infty} n_+ u_+ \frac{\bar{v}_x^2}{v} dx + \int_0^{+\infty} n_+ u_+ \frac{\tilde{v}_x^2}{v} dx \\
\leq C \int_0^{+\infty} n_+ u_+ (\bar{v} - \tilde{u})^2 dx + \frac{1}{4} \int_0^{+\infty} A_2 \alpha n_+^\alpha \frac{u_+^{\alpha+2}}{v^{\alpha+2}} \tilde{v}_x^2 dx
\tag{2.6}
\]
Combining (2.5) and (2.6), we get
\[
|\bar{v}_x(0)| \leq C |u_- - u_+|, \quad |\bar{u}_x(0)| \leq C |u_- - u_+|.
\tag{2.7}
\]

Then, we can prove Theorem 1.1 with the above lemma. For (1.6) and (1.6)_4, using $\tilde{\rho} = \frac{\rho u_+}{v}$, $\tilde{n} = \frac{n_+ u_+}{v}$ and integrating the summation of (1.6) and (1.6)_4 over $(x, +\infty)$, we have
\[
\left\{
\begin{array}{l}
\bar{v}_x = \frac{\bar{v}}{n_+ u_+} [\rho_+ u_+(\bar{u} - u_+) + A_1 \rho_+ \bar{u}_x - A_1 \rho_+ \bar{u} + A_2 \alpha n_+^\alpha (\frac{u_+^{\alpha+2}}{v^{\alpha+2}} - 1) - \mu \bar{u}_x], \\
\bar{u}_x = \frac{1}{\mu} [\rho_+ u_+ - A_1 \rho_+ \bar{u}_x + A_1 \rho_+ (\bar{u} - u_+) (1 - \frac{\bar{u}}{\bar{v}})],
\end{array}
\right.
\tag{2.8}
\]
Define $\bar{w} := \bar{u}_x$ and $\bar{U} := (\bar{u}, \bar{w}, \bar{v})^T := (\bar{u} - u_+, \bar{w}, \bar{v} - u_+)^T$. The system (2.8) can be reformulated into the autonomous system as follows
\[
\begin{align*}
\dot{\bar{U}}_x &= J_+ \bar{U} + (0, \bar{g}_2(\bar{U}), \bar{g}_3(\bar{U}))^T, \\
\bar{U}_- := (\bar{u}, \bar{w}, \bar{v})^T (0) &= (u_- - u_+, \bar{u}_x(0), u_- - u_+)^T, \quad \lim_{x \to \infty} \bar{U}_x^T (0, 0, 0),
\end{align*}
\tag{2.9}
\]
where
\[
J_+ = \begin{pmatrix}
0 & \rho_+ u_+^2 - A_1 \gamma \rho_+ \bar{u}_x & 0 \\
\rho_+ u_+^2 - A_1 \gamma \rho_+ \bar{u}_x & \frac{\rho_+ u_+^2}{n_+ u_+} & \frac{n_+}{\mu} \\
\rho_+ u_+^2 & \frac{n_+}{\mu} & n_+ \frac{u_+^{\alpha+2}}{n_+ u_+^{\alpha+2}} - \frac{n_+}{\mu}
\end{pmatrix},
\tag{2.10}
\]
\[
\bar{g}_2(\bar{U}) = \frac{1}{2} (2 \frac{n_+}{\mu} u_+ \bar{v}^2 - 2 \frac{n_+}{\mu} u_+ \bar{w} + 2 A_1 \gamma + 1) \rho_+ \bar{u}_x \bar{v} \bar{u} + O(|\bar{U}|^3),
\tag{2.11}
\]
\[
\bar{g}_3(\bar{U}) = \frac{1}{2} (A_1 \gamma \bar{u}_x (\bar{w} + A_1 \rho_+ \bar{u}_x) + 2 \frac{n_+ u_+^2}{n_+ u_+^{\alpha+2}} \bar{v} \bar{w} + (2 \frac{n_+ u_+^2}{n_+ u_+^{\alpha+2}} - A_2 \alpha n_+^\alpha) \bar{v} \bar{w}) + O(|\bar{U}|^3).
\tag{2.12}
\]
Three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of matrix $J_+$ satisfy

$$\begin{align*}
\lambda_1 \lambda_2 \lambda_3 &= -\frac{(\rho_+ + n_+)u_+^2 - (A_1 \gamma \rho_+^{\gamma-1} + A_2 \alpha n_+^\alpha)}{\mu u_+}, \\
\lambda_1 + \lambda_2 + \lambda_3 &= \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^{\gamma-1} + n_+ u_+^2 - A_2 \alpha n_+^\alpha}{n_+ u_+}, \\
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 &= \frac{\rho_+ (u_+^2 - A_1 \gamma \rho_+^{\gamma-1}(u_+^2 - A_2 \alpha n_+^\alpha - 1))}{\mu u_+^2} - 1 - \frac{n_+}{\mu}.
\end{align*}$$

(2.13)

If $M_+ > 1$, it is easy to obtain $\lambda_1 \lambda_2 \lambda_3 > 0$ and $u_+^2 > \min\{A_1 \gamma \rho_+^{\gamma-1}, A_2 \alpha n_+^\alpha\}$. Without loss of generality, we assume $A_1 \gamma \rho_+^{\gamma-1} \geq A_2 \alpha n_+^\alpha$. Moreover, we have

$$\lambda_1 + \lambda_2 + \lambda_3 < 0 \text{ for } u_+^2 > A_1 \gamma \rho_+^{\gamma-1}, \quad \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 < 0 \text{ for } A_2 \alpha n_+^\alpha - 1 \leq u_+^2 \leq A_1 \gamma \rho_+^{\gamma-1},$$

(2.14)

which can imply $\text{Re}\lambda_1 < 0$, $\text{Re}\lambda_2 < 0$ and $\lambda_3 > 0$ for $M_+ > 1$. Using similar arguments, we have the following results:

$$\begin{align*}
\text{if } M_+ > 1, \text{ then } \text{Re}\lambda_1 < 0, \text{Re}\lambda_2 < 0, \lambda_3 > 0, \\
\text{if } M_+ < 1, \text{ then } \text{Re}\lambda_1 > 0, \text{Re}\lambda_2 > 0, \lambda_3 < 0, \\
\text{if } M_+ = 1, \text{ then } \lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0.
\end{align*}$$

(2.15)

Then, applying the center manifold theory [2], it is not difficult to show the supersonic or subsonic case $M_+ \neq 1$ in Theorem 1.1 if $\delta$ is small. Finally, we prove the sonic case $M_+ = 1$ in Theorem 1.1 which implies $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0$. The eigenvectors $r_1, r_2, r_3$ of $\lambda_1, \lambda_2, \lambda_3$ are obtained respectively as follows

$$r_1 = \left(\begin{array}{c}
1 \\
-\frac{\mu}{n_+} (\lambda_1^2 - \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^{\gamma-1}}{\mu u_+} \lambda_1) + 1 \\
\end{array}\right), \quad r_2 = \left(\begin{array}{c}
1 \\
-\frac{\mu}{n_+} (\lambda_2^2 - \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^{\gamma-1}}{\mu u_+} \lambda_2) + 1 \\
\end{array}\right), \quad r_3 = \left(\begin{array}{c}
1 \\
0 \\
1 \\
\end{array}\right).$$

(2.16)

Define the matrix $P_1 := [r_1, r_2, r_3]$ and take a linear transformation $Z := (z_1, z_2, z_3)^T = P_1^{-1} \bar{U}$. With (2.11) and (2.12), the system (2.9) can be reformulated as follows

$$\begin{align*}
\frac{d}{dx} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} &= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} g_1(z_1, z_2, z_3) \\ g_2(z_1, z_2, z_3) \\ g_3(z_1, z_2, z_3) \end{pmatrix}, \\
(z_1, z_2, z_3)(0) &= (z_{1-}, z_{2-}, z_{3-}) = (P_1^{-1} \bar{U}_-)^T, \quad \lim_{x \to \infty} (z_1, z_2, z_3) = (0, 0, 0),
\end{align*}$$

(2.17)

where nonlinear functions $g_i(i = 1, 2, 3)$ are denoted by

$$\begin{pmatrix} g_1(z_1, z_2, z_3) \\ g_2(z_1, z_2, z_3) \\ g_3(z_1, z_2, z_3) \end{pmatrix} = P_1^{-1} \begin{pmatrix} 0 \\ \bar{g}_2(\bar{u}, \bar{w}, \bar{v}) \\ \bar{g}_3(\bar{u}, \bar{w}, \bar{v}) \end{pmatrix}.$$

(2.18)
With the help of the manifold theory [2], there exist a local center manifold $W^c(0,0,0)$ and a local stable manifold $W^s_3(0,0,0)$

$$W^c(0,0,0) = \{ (z_1, z_2, z_3) \mid z_1 = f^c_1(z_3), z_2 = f^c_2(z_3), |z_3| \text{ sufficient small} \}, \quad (2.19)$$

$$W^s_3(0,0,0) = \{ (z_1, z_2, z_3) \mid z_1 = f^s_1(z_2), z_3 = f^s_2(z_2), |z_2| \text{ sufficient small} \}, \quad (2.20)$$

where $f^c_i, f^s_i, i = 1, 2$ are smooth functions and $f^c_1(0) = 0, \ Df^c_1(0) = 0, \ f^s_i(0) = 0, \ Df^s_i(0) = 0, \ i = 1, 2.$ Using $U = PZ, (2.21)-(2.12),$ and (2.18), we gain

$$\bar{g}_3(z_3) = az_3^2 + O(|z_1|^2 + |z_2|^2 + |z_3|^3 + |z_1z_3| + |z_2z_3|), \quad (2.21)$$

where

$$a = \frac{A_1(c \gamma + 1) \rho_+^2 + A_2(\alpha + 1)n_+^2}{2u_+^2(1 + b^2)(\mu + n_+)^2}, \quad b := \frac{\rho_+(u_+^2 - p_1'(\rho_+))}{|u_+|\sqrt{(\mu + n_+)^2}}. \quad (2.22)$$

Therefore, the system (2.17) can be reformulated as follows

\[
\begin{align*}
    z_{1x} &= \lambda_1 z_1 + O(|Z|^2), \\
    z_{2x} &= \lambda_2 z_2 + O(|Z|^2), \\
    z_{3x} &= az_3^2 + O(|z_1|^2 + |z_2|^2 + |z_3|^3 + |z_1z_3| + |z_2z_3|), \\
    (z_1, z_2, z_3)(0) := (z_{1-}, z_{2-}, z_{3-}) &= (P_1^{-1}U_\nu)^T, \quad \lim_{x \to \infty} (z_1, z_2, z_3) = (0, 0, 0).
\end{align*}
\]

(2.23)

Let $\sigma_1(x)$ be a solution to (2.23) restricted on the local center manifold satisfying the equation

$$\sigma_{1x} = a\sigma_1^2 + O(\sigma_1^3), \quad \sigma_1(x) \to 0 \text{ as } x \to +\infty. \quad (2.24)$$

which implies that there exists the monotonically increasing solution $\sigma_1(x) < 0$ to (2.24) if $\sigma_1(0) < 0$ holds and $|\sigma_1(0)|$ is sufficiently small. Therefore, if the initial data $(z_{1-}, z_{2-}, z_{3-})$ belongs to the region $\mathcal{M} \subset \mathbb{R}^3$ associated to the local stable manifold and the local center manifold, then we have

\[
\begin{align*}
    z_i &= \sigma_1^2 + O(\delta e^{-cx}), \quad i = 1, 2, \\
    z_3 &= \sigma_1 + O(\delta e^{-cx}),
\end{align*}
\]

(2.25)

with $z_{3-} < 0$, the smallness of $|(z_{1-}, z_{2-}, z_{3-})|$ and

$$C \frac{\delta}{1 + \delta x} \leq |\sigma_1| \leq C \frac{\delta}{1 + \delta x}, \quad |\partial^k \sigma_1| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad C > 0, \ k = 0, 1, 2, 3. \quad (2.26)$$

Due to $\sigma_1(x) \leq 0$, we define

$$\sigma(x) := -\sigma_1, \quad (2.27)$$

which satisfies

$$\sigma_x = -a\sigma^2 + O(|\sigma|^3), \quad \sigma \to 0 \text{ as } x \to +\infty \quad (2.28)$$
It is easy to get
\[ |\partial_x^k(\bar{\rho} - \rho_+, \bar{u} - u_+, \bar{n} - n_+, \bar{v} - u_+)| \leq C \frac{\delta^{k+1}}{(1 + \delta)^{k+1}}, \quad C > 0, \quad k = 0, 1, 2, 3, \tag{2.29} \]
and
\[ (\bar{u} - u_+, \bar{v} - u_+) = (-\sigma(x), -\sigma(x)) + O(|\sigma(x)|^2), \quad (\bar{u}_x, \bar{v}_x) = (a\sigma^2(x), a\sigma^2(x)) + O(|\sigma|^2), \tag{2.30} \]
with the help of \( \bar{U} = PZ \) and (2.16).

3 Nonlinear stability of steady-states

The function space \( \mathcal{Y}(0, T) \) for \( T > 0 \) is denoted by
\[
\mathcal{Y}(0, T) := \{ (\phi, \psi, \bar{\phi}, \bar{\psi}) \mid (\phi, \psi, \bar{\phi}, \bar{\psi}) \in C([0, T]; H^1(\mathbb{R}^+)), \\
(\phi_x, \bar{\phi}_x) \in L^2([0, T]; L^2(\mathbb{R}^+)), \ (\psi_x, \bar{\psi}_x) \in L^2([0, T]; H^1(\mathbb{R}^+)) \}. \tag{3.1} \]

Let
\[ \phi = \rho - \bar{\rho}, \quad \psi = u - \bar{u}, \quad \bar{\phi} = n - \bar{n}, \quad \bar{\psi} = v - \bar{v}. \tag{3.2} \]
Then the perturbation \((\phi, \psi, \bar{\phi}, \bar{\psi})\) satisfies the following system
\[
\begin{align*}
\phi_t + u\phi_x + \rho\psi_x &= -(\psi\bar{\rho}_x + \phi\bar{u}_x), \\
\psi_t + u\psi_x + \frac{p'_1(\rho)}{\rho}\phi_x - \frac{\mu\psi_{xx}}{\rho} - n(\bar{\psi} - \psi) &= F_1, \\
\bar{\phi}_t + v\bar{\phi}_x + n\bar{\psi}_x &= -(\bar{\psi}\bar{n}_x + \bar{\phi}\bar{v}_x), \\
\bar{\psi}_t + v\bar{\psi}_x + \frac{p'_2(n)}{n}\bar{\phi}_x - \frac{(n\bar{\psi}_x)_x}{n} + (\bar{\psi} - \psi) &= F_2,
\end{align*} \tag{3.3} \]
where
\[
\begin{align*}
F_1 &= -[\frac{1}{\rho} - \frac{1}{\bar{\rho}}]n\bar{u}_x + \psi\bar{u}_x + \left(\frac{p'_1(\rho)}{\rho} - \frac{p'_1(\bar{\rho})}{\bar{\rho}}\right)\bar{\rho}_x - \left(\frac{n}{\rho} - \frac{\bar{n}}{\bar{\rho}}\right)(\bar{v} - \bar{u}), \\
F_2 &= -\left[\frac{1}{n} - \frac{1}{\bar{n}}\right](\bar{\psi}\bar{v}_x)_x - \frac{\bar{\psi}\bar{v}_x}{n} + \bar{\psi}\bar{v}_x + \left(\frac{p'_2(n)}{n} - \frac{p'_2(\bar{n})}{\bar{n}}\right)\bar{n}_x.
\end{align*} \tag{3.4} \tag{3.5} \]
The initial and boundary conditions to the system (3.3) satisfy
\[
(\phi, \psi, \bar{\phi}, \bar{\psi})(0, x) := (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) = (\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v}), \tag{3.6} \]
\[
\lim_{x \to \infty} (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) = (0, 0, 0, 0), \quad (\psi, \bar{\psi})(t, 0) = (0, 0). \tag{3.7} \]

**Proposition 3.1.** Assume that the same assumptions in Theorem 1.2 hold. Let \((\phi, \psi, \bar{\phi}, \bar{\psi})\) be the solution to the problem (3.3)-(3.7) satisfying \((\phi, \psi, \bar{\phi}, \bar{\psi}) \in \mathcal{Y}(0, T)\) for any time \( T > 0 \). Then there exist positive constants \( \varepsilon > 0 \) and \( C > 0 \) independent of \( T \) such that if
\[
\sup_{0 \leq t \leq T} \| (\phi, \psi, \bar{\phi}, \bar{\psi})(t) \|_1 + \delta \leq \varepsilon \tag{3.8} \]
is satisfied, then it holds for arbitrary \( t \in [0, T] \) that
\[
\| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + \int_0^t \| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2 dt + \int_0^t \| (\bar{\psi} - \psi, \psi_{xx}, \bar{\psi}_{xx}) \|^2 d\tau \leq C \| (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|_1^2. \tag{3.9}
\]

With the help of (3.8), it is easy to verify the following Sobolev inequality
\[
\| (\phi, \psi, \bar{\phi}, \bar{\psi})(t) \|_{L^\infty} \leq \| (\phi, \bar{\psi}, \bar{\phi}, \psi)(t) \|_{H^1} \leq \sqrt{2}\varepsilon. \tag{3.10}
\]

**Lemma 3.2** ([16]). For any function \( \psi(t, \cdot) \in H^1(\mathbb{R}_+) \), it holds
\[
\int_0^\infty e^{-c_0 x} |\psi|^2 dx \leq C(\| \psi(t, 0) \|^2 + \| \psi_x(t) \|^2), \tag{3.11}
\]
\[
\int_0^\infty \frac{\delta^j}{(1 + \delta x^j)} |\psi|^2 dx \leq C\delta^{-j}(\| \psi(t, 0) \|^2 + \| \psi_x(t) \|^2), \quad j > 2, \tag{3.12}
\]
where \( \delta > 0, c_0 > 0 \) and \( C > 0 \) are positive constants.

With the Lemma 3.2, we can gain the basic \( L^2 \) energy estimates of \( (\phi, \psi, \bar{\phi}, \bar{\psi}) \).

**Lemma 3.3.** Under the same conditions in Proposition 3.1, then the solution \( (\phi, \psi, \bar{\phi}, \bar{\psi}) \) to the problem (3.3)-(3.7) satisfies for \( t \in [0, T] \) that
\[
\| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + \int_0^t \| (\psi_x, \bar{\psi}_x, \bar{\psi} - \psi) \|^2 dt + \int_0^t |\phi(t, 0)|^2 + |\bar{\phi}(t, 0)|^2 d\tau \leq C \| (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|^2 + C(\delta + \varepsilon) \int_0^t \| (\phi_x, \bar{\phi}_x) \|^2 d\tau. \tag{3.13}
\]

**Proof.** Define
\[
\Phi_1(\rho, \bar{\rho}) = \int_0^\rho \frac{p_1(s) - p_1(\bar{\rho})}{s^2} ds, \quad \mathcal{E}_1 = \rho\left(\frac{\psi^2}{2} + \Phi_1\right), \tag{3.14}
\]
\[
\Phi_2(n, \bar{n}) = \int_0^n \frac{p_2(s) - p_2(\bar{n})}{s^2} ds, \quad \mathcal{E}_2 = n\left(\frac{\bar{\psi}^2}{2} + \Phi_2\right). \tag{3.15}
\]

Then, by (1.1) and (1.6), the direct computations lead to
\[
(\mathcal{E}_1 + \mathcal{E}_2)_t + (G_1 + G_2)_x + n(\bar{\psi} - \psi)^2 + \mu\psi_x^2 + n\bar{\psi}_x^2 + R_1 + R_2 = -R_3, \tag{3.16}
\]
where
\[
\begin{align*}
G_1 & := u\mathcal{E}_1 + v\mathcal{E}_2 + (p_1(\rho) - p_1(\bar{\rho}))\psi + (p_2(n) - p_2(\bar{n}))\bar{\psi}, \\
G_2 & := -\left(\mu\psi_{xx} + n\bar{\psi}_x + \bar{\phi}\bar{\psi}_x\right), \\
R_1 & := \left[\rho\psi^2 + p_1(\rho) - p_1(\bar{\rho})\psi\right] u_x + \left[n\bar{\psi}^2 + p_2(n) - p_2(\bar{n}) - \bar{\phi}\bar{\psi}\right] \bar{u}_x, \\
R_2 & := \phi\left[\frac{\rho u_x}{\rho} + \frac{(p_1(\rho))_x}{\rho} + \bar{\phi}\left[\frac{\bar{\psi} x^2}{\rho} + (p_2(\bar{n}))_x\right]\right] + \bar{\phi}\bar{\psi}_{xx}, \\
R_3 & := \bar{\phi}(\bar{\psi} - \psi)(\bar{v} - \bar{n}) + \bar{\phi}\bar{\psi}_x\bar{v}_x.
\end{align*}
\]

Integrating (3.16) in \( x \) over \( \mathbb{R}_+ \) leads to
\[
\frac{d}{dt} \int \mathcal{E}_1 + \mathcal{E}_2 dx - G_1(t, 0) + \int n(\bar{\psi} - \psi)^2 + \mu\psi_x^2 + n\bar{\psi}_x^2 dx + \int R_1 dx + \int R_2 dx = -\int R_3 dx. \tag{3.18}
\]
Under the condition (3.7), we get

$$\begin{align*}
-G_1(t,0) &= -u_0\left[\Phi_1(\rho(0), \rho_0) + \Phi_2(u(0), \bar{u}(0))\right] 
&\geq c(\phi^2(t,0) + \bar{\phi}^2(t,0)).
\end{align*}$$

(3.19)

For the supersonic or subsonic case $M_+ \neq 1$, with the help of (1.11), (3.10) and (3.11), we have

$$\begin{align*}
\int_0^\infty |R_1| + |R_2| + |R_3|dx &\leq C\delta \|(\phi_x, \psi_x, \phi, \psi_x, \psi - \psi_x)\|^2 + C\delta(\phi^2(t,0) + \bar{\phi}^2(t,0)).
\end{align*}$$

(3.20)

For the sonic case $M_+ = 1$ and the restriction $|p'_1(\rho_+)-p'_2(n_+)| \leq \sqrt{2}|u_+| \min\{(1+\frac{\alpha}{\rho_+})\{(\gamma-1)p'_1(\rho_+)^{\frac{\gamma}{\gamma-1}}, (1\rightarrow +\frac{n_+}{\rho_+})\{(\alpha-1)p'_2(n_+)^{\frac{\gamma}{\gamma-1}}\}}\}$, using (1.12)-(1.14), (3.10), and (3.12), we get

$$\begin{align*}
\int R_1 + R_2 + R_3 dx 
&\geq \int (\psi, \phi)M_1(\psi, \phi)^T u_x + (\bar{\psi}, \bar{\phi})M_2(\bar{\psi}, \bar{\phi})^T \bar{u}_x - C \int \frac{\delta^3}{(1+\delta x)\phi^2 + \psi^2 + \bar{\phi}^2 + \bar{\psi}^2} dx
\end{align*}$$

(3.21)

$$\begin{align*}
&\geq -C(\delta + \epsilon)(\phi_x, \psi_x, \phi, \psi_x, \psi - \psi_x)^2 - C\delta(\phi^2(t,0) + \bar{\phi}^2(t,0)),
\end{align*}$$

where $M_1, M_2$ are positive definite or non-negative definite matrices defined by

$$\begin{align*}
M_1 &= \left(\begin{array}{cc}
\frac{\rho_+}{u_+^2 - A_1\gamma\rho_+^{-1}} & \frac{u_+}{2u_+} \frac{A_1\gamma}{A_1\gamma - 1}\rho_+^{-1} \\
\frac{u_+^2 - A_1\gamma\rho_+^{-1}}{2u_+} & \frac{A_1\gamma}{A_1\gamma - 1}\rho_+^{-2}
\end{array}\right),
M_2 &= \left(\begin{array}{cc}
n_+ & \frac{u_+^2 - A_2\alpha n_+^{-1}}{2n_+} \\
\frac{u_+^2 - A_2\alpha n_+^{-1}}{2n_+} & \frac{A_2\alpha}{A_2\alpha - 1}\nu_+^{-2}
\end{array}\right).
\end{align*}$$

(3.22)

Finally, with the help of (3.18)-(3.21), we get (3.13). Hence, the proof of Lemma 3.3 is completed.

In order to complete the proof of Proposition 3.1, we need to establish the high order estimates of $(\phi, \psi, \phi, \psi)$.

**Lemma 3.4.** Under the same conditions in Proposition 3.1, then the solution $(\phi, \psi, \phi, \psi)$ to the problem (3.3)-(3.7) satisfies for $t \in [0, T]$ that

$$\begin{align*}
&\|(\phi_x, \phi^2_x)(t,0)\|^2 + \int_0^t \|(\phi_x, \phi^2_x)(t,0)\|^2 d\tau + \int_0^t \phi^2_2(t,0) + \bar{\phi}_2^2(t,0) d\tau \\
&\leq C\|(\phi_0, \phi_0, \phi_0, \psi_0, \psi_0, \bar{\phi}_0, \bar{\phi}_0, \bar{\psi}_0, \bar{\psi}_0, \bar{\psi}_0)(t,0)\|^2 + C(\epsilon + \delta) \int_0^t \|(\psi_{xx}, \bar{\psi}_{xx})(t,0)\|^2 d\tau.
\end{align*}$$

(3.23)

**Proof.** Differentiating (3.3) in $x$, then multiplying the resulted equation by $\mu \phi_x$, (3.3)2 by $\bar{\rho}^2 \phi_x$ respectively, we gain

$$\begin{align*}
\frac{1}{2}(\mu \phi^2_x)_t + (\mu \phi^2_x)_x + \mu \bar{\rho} \phi_x \psi_{xx} \\
= -\left[\frac{3}{2} \psi_{xx}^2 + \phi \phi_x \psi_{xx} + \left(\frac{1}{2} \phi \tilde{u}_x + \psi \bar{\rho} \phi_x\right) \phi_x + \left(\phi \bar{u}_x + \psi \bar{\rho} \phi_x\right) \phi_x\right],
\end{align*}$$

(3.24)

$$\begin{align*}
(\bar{\rho}^2 \phi_x)_t - (\bar{\rho}^2 \phi_x)_x + \bar{\rho} \phi_x \psi_{xx} + \bar{\rho} \phi_x \psi_{xx} \\
= -(\bar{\rho}^2 \phi_x)_x + \bar{\rho}^2 u \phi_x \psi + 2\bar{\rho} \phi_x \psi_0 + \mu \bar{\rho}^2 (1 - \frac{1}{\bar{\rho}}) \phi_x \psi_{xx} + \bar{\rho}^2 \frac{\bar{\rho}}{\phi} (\psi - \psi_0) \phi_x + F_1 \bar{\rho}^2 \phi_x.
\end{align*}$$

(3.25)
Similarly, differentiating (3.3) in $x$, then multiplying the resulted equation by $\tilde{\phi}_x$, (3.3) by $\tilde{n}\tilde{\phi}_x$ respectively lead to
\begin{equation}
\left(\frac{\partial^2}{2}\right)_x + \left(\frac{v}{2}\right)_x + \tilde{n}\tilde{\phi}_x \tilde{\psi}_x = -\frac{3}{2} \tilde{\psi}_x \phi_x^2 + \tilde{\phi}\tilde{\phi}_x \tilde{\psi}_xx + \left(\frac{1}{2} \tilde{\phi}_x \tilde{v}_x + \tilde{\psi}_x \tilde{n}_x\right)\tilde{\phi}_x - \left(\tilde{\phi}_x v_x + \tilde{\psi}_x \tilde{n}_x\right)\tilde{\phi}_x, \tag{3.26}
\end{equation}
\begin{equation}
(\tilde{n}\tilde{\phi}_x \tilde{\psi}_x)_t - (\tilde{n}\tilde{\phi}_x \tilde{\psi}_x)_x + \frac{p'_{2}(n)}{n} \frac{\partial}{\partial x} - \tilde{n}\tilde{\phi}_x \tilde{\psi}_xx
\end{equation}
\begin{align*}
&= -(\tilde{n}\tilde{\phi}_x \tilde{\psi}_x + \tilde{n}\tilde{v}\tilde{\phi}_x \tilde{\psi}_x + \tilde{n}_x \tilde{\phi}_x \tilde{\psi}_x) + \left[\frac{n}{n}(\tilde{\phi}_x \tilde{\psi}_x)_x + \tilde{n} \left(1 - \frac{1}{n}\right)(\tilde{n}\tilde{\psi}_x)_x - \tilde{n}(\tilde{\psi}_x - \tilde{\psi})\right] \tilde{\phi}_x \\
&+ \tilde{n}_x \tilde{\phi}_x \tilde{\psi}_xx + F_2 \tilde{n}\tilde{\phi}_x.
\end{align*}
Adding (3.24)-(3.27) together, integrating the resulted equation over $\mathbb{R}_+$, we have
\begin{equation}
\frac{d}{dx} \left(\mu \frac{\phi_x^2}{2} + \frac{\phi_x^2}{2} + \rho^2 \phi_x \psi + \tilde{n}\tilde{\phi}_x \tilde{\psi}_x\right) + \int \left(\mu \frac{\phi_x^2}{2} + \frac{\phi_x^2}{2} - \tilde{n}\tilde{\phi}_x \tilde{\psi}_x\right) dx
\end{equation}
\begin{align}
&= - \sum_{i=1}^4 J_i, \tag{3.28}
\end{align}
where
\begin{align*}
J_1 &= - \int \left[p^2(\phi + u\phi_x)\psi_x + \tilde{n}(\tilde{\phi}_x + v\tilde{\phi}_x)\tilde{\psi}_x + 2\tilde{\rho}\tilde{\phi}_x \psi_x + \tilde{n}\tilde{\phi}_x \tilde{\psi}\right] dx, \\
J_2 &= \int -\mu \phi \phi_x \psi_x - \tilde{\phi}_x \tilde{\psi}_x + \mu \rho^2 \frac{1}{\rho^2 - 1}(\phi \psi_x + \tilde{n}\tilde{\psi}_x) + \frac{n}{n}(\tilde{n}\tilde{\psi}_x x \tilde{\phi}_x + \rho^2 \frac{1}{\rho}(\tilde{\psi}_x - \psi)\phi_x - \tilde{n}(\tilde{\psi}_x - \psi)\tilde{\phi}_x dx, \\
J_3 &= - \int \frac{3}{2} \mu \psi_x \phi_x^2 + \frac{3}{2} \psi_x \phi_x^2 dx + \int \frac{\tilde{n}\tilde{\phi}_x x}{n} \tilde{\phi}_x dx, \\
J_4 &= - \int \left[\mu \left(\frac{1}{2} \phi_x \psi_x + \psi_x \tilde{\phi}_x\right) \phi_x + \mu \left(\phi \psi_x + \psi \psi_x\right) \phi_x - F_1 \rho^2 \phi_x \right.
\frac{1}{2} \tilde{\phi}_x \tilde{\phi}_x + \tilde{\psi}_x \tilde{n}_x \tilde{\phi}_x - (\tilde{\phi}_x \tilde{v}_x + \tilde{\psi}_x \tilde{n}_x \tilde{\phi}_x - \tilde{\phi}_x \tilde{v}_x \tilde{n}_x \tilde{\phi}_x - F_2 \tilde{n}\tilde{\phi}_x dx.
\end{align*}
First, we estimate terms in the left side of (3.28). Under the condition (3.7), the terms in the left side is estimated as follows
\begin{equation}
\left|\int \left(\mu u \frac{\phi_x^2}{2} + v \frac{\phi_x^2}{2} - \rho^2 \phi_x \psi - \tilde{n}\tilde{\phi}_x \tilde{\psi}\right) dx\right| = \int \frac{\mu u \frac{\phi_x^2}{2} + v \frac{\phi_x^2}{2} - \rho^2 \phi_x \psi - \tilde{n}\tilde{\phi}_x \tilde{\psi}}{2} 0, \tag{3.29}
\end{equation}
\begin{align}
\int \rho^2 \frac{p'_{2}(\rho)}{\rho} \phi_x^2 + \frac{p'_{2}(n)}{n} \phi_x^2 dx &\geq \rho^2 \frac{p'_{2}(\rho)}{\rho} \phi_x^2 + \frac{p'_{2}(n)}{n} \phi_x^2 dx. \tag{3.30}
\end{align}
We turn Cauchy-Schwartz inequality and Young inequality with $0 < \eta < 1$, we obtain
\begin{align*}
|J_1| &\leq C \left\|\phi_x, \tilde{\phi}_x\right\|^2 + C\delta \left\|\phi_x, \tilde{\phi}_x\right\|^2 + C\delta \left(\phi^2(t, 0) + \phi^2(t, 0)\right), \tag{3.31}
|J_2| &\leq C \left\|\phi, \tilde{\phi}\right\|_{L^\infty} \left\|\left(\phi_x, \tilde{\phi}_x, \phi_x x, \tilde{\phi}_x x\right\|^{2} + C_\eta \left\|\tilde{\psi}_x - \psi\right\|^2 + \eta \left\|\phi_x, \tilde{\phi}_x\right\|^2 + C \delta \left\|\tilde{\phi}_x \tilde{\psi}_x\right\|^2 \right. \right. \\
&\leq C (\varepsilon + \delta + \eta) \left\|\left(\tilde{\phi}_x, \tilde{\psi}_x\right)\right\|^2 + C \varepsilon \left\|\phi_x x, \tilde{\phi}_x x\right\|^2 + C_\eta \left\|\tilde{\psi}_x - \psi\right\|^2 + C \delta \left\|\tilde{\phi}_x \tilde{\psi}_x\right\|^2, \tag{3.32}
|J_3| &\leq C \left\|\phi_x, \tilde{\phi}_x\right\|_{L^\infty} \left\|\left(\phi_x, \tilde{\phi}_x\right\|^{2} + C \left\|\tilde{\phi}_x \tilde{\phi}_x\right\|_{L^\infty} \left\|\left(\phi_x, \tilde{\phi}_x\right\|^{2} \right. \right. \\
&\leq C \left\|\phi_x, \tilde{\phi}_x\right\|_{L^\infty} \left\|\left(\phi_x, \tilde{\phi}_x\right\|^{2} + C \varepsilon \left\|\phi_x x, \tilde{\phi}_x x\right\|^2, \tag{3.33}
\end{align*}
\[ |J_4| \leq C \delta \| (\phi_x, \psi_x, \check{\phi}_x, \check{\psi}_x) \|^2 + C \delta \| \check{\psi}_{xx} \|^2 + C \delta (\phi^2(t, 0) + \check{\phi}^2(t, 0)). \]  

(3.34)

Finally, the substitution of (3.29)-(3.34) into (3.28) for \( \delta, \varepsilon \) and \( \eta \) small enough leads to

\[
\frac{d}{dt} \int (\phi_x^2 + \check{\phi}_x^2 + \rho^2 \phi_x \psi + \bar{n} \phi_x \psi) dx + \| (\phi_x, \check{\phi}_x) \|^2 + \phi_x^2(t, 0) + \check{\phi}_x^2(t, 0) \\
\leq C \| (\psi_x, \check{\psi}_x, \check{\psi}_x - \psi_x) \|^2 + C (\delta + \varepsilon) \| (\psi_{xx}, \check{\psi}_{xx}) \|^2 + C \delta (\phi^2(t, 0) + \check{\phi}^2(t, 0)).
\]

(3.35)

Integrating (3.35) in \( \tau \) over \([0, t] \), and using Lemma 3.3 and Young inequality, we have (3.23).

\( \square \)

**Lemma 3.5.** Under the same conditions in Proposition 3.1, then the solution \((\phi, \psi, \check{\phi}, \check{\psi})\) to the problem (3.3)-(3.7) satisfies for \( t \in [0, T] \) that

\[
\| (\psi_x, \check{\psi}_x) \|^2 + \int_0^t \| (\psi_{xx}, \check{\psi}_{xx}) \|^2 d\tau \leq C \| (\phi_0, \psi_0, \check{\phi}_0, \check{\psi}_0) \|^2.
\]

(3.36)

**Proof.** Multiplying (3.3)\(_2\) by \(-\psi_{xx}\), (3.3)\(_4\) by \(-\check{\psi}_{xx}\), respectively, then adding them together and integrating the resulted equation in \( x \) over \( \mathbb{R}_+ \) imply

\[
\frac{d}{dt} \int \frac{\psi_x^2}{2} + \frac{\check{\psi}_{xx}^2}{2} dx + \int \frac{1}{\rho} \psi_{xx}^2 + \check{\psi}_{xx}^2 dx = \sum_{i=1}^{3} K_i,
\]

(3.37)

where

\[
K_1 = \int [u \psi_x \psi_x - \frac{n}{\rho} (\check{\psi} - \psi) \psi_{xx} + \frac{p'(\rho)}{\rho} \phi_x \psi_{xx} + u \check{\psi}_x \check{\psi}_{xx} + (\check{\psi} - \psi) \check{\psi}_{xx} + \frac{p'_2(n)}{n} \check{\phi}_x \check{\psi}_{xx}] dx,
\]

\[
K_2 = \int [\bar{u} \check{x} \psi_{xx} - \mu \bar{x} \check{x} \left( \frac{1}{\rho} - \frac{1}{\bar{n}} \right) \psi_{xx} + \left( \frac{p'_2(n)}{\rho} - \frac{p'(\bar{n})}{\bar{n}} \right) \psi_{xx} \check{\phi}_x - \left( \frac{n}{\rho} - \frac{\bar{n}}{\bar{n}} \right) (\check{\psi} - \check{\psi}) \psi_{xx} \check{\phi}_x + \bar{\psi}_x \check{\psi}_{xx}
\]

\[
+ (\bar{n} \bar{u} \check{x}) \left( \frac{1}{n} - \frac{1}{\bar{n}} \right) \check{\psi}_{xx} + \left( \frac{p'_2(n)}{n} - \frac{p'(\bar{n})}{\bar{n}} \right) \check{\psi}_{xx} \bar{\check{x}} - \frac{\bar{n} \check{x}}{\bar{n}} \check{\psi}_x \check{\psi}_{xx} - \frac{\check{\psi}_x \check{\psi}_{xx}}{\check{\psi}_x \check{\psi}_{xx}} dx,
\]

\[
K_3 = - \int \left( \frac{\rho}{n} \check{\psi}_{xx} \right) dx + \left( \frac{1}{n} - \frac{1}{\bar{n}} \right) (\bar{n} \check{x}) \check{\psi}_{xx} dx.
\]

We estimate terms in the left side of (3.37). By the decomposition \( \frac{1}{\rho} = \left( \frac{1}{\rho} - \frac{1}{\rho_+} \right) + \frac{1}{\rho_+} \), the second term is estimated as follows:

\[
\frac{d}{dt} \int \frac{\rho}{\rho_+} \psi_{xx}^2 + \check{\psi}_{xx}^2 dx \geq \frac{\mu}{\rho_+} \| \psi_{xx} \|^2 + \| \check{\psi}_{xx} \|^2 - C(\| \phi \|_{L^\infty} + \delta) \| \psi_{xx} \|^2
\]

(3.38)

We turn to estimate terms in the right side of (3.37). With the aid of (1.11), Sobolev inequality and Cauchy-Schwarz inequality, we have

\[
|K_1| \leq \frac{\mu}{16 \rho_+} \| \psi_{xx} \|^2 + \frac{1}{16} \| \check{\psi}_{xx} \|^2 + C \| (\phi_x, \psi_x, \check{\phi}_x, \check{\psi}_x, \psi - \check{\psi}) \|^2,
\]

(3.39)

\[
|K_2| \leq C \delta \| (\phi_x, \psi_x, \check{\phi}_x, \check{\psi}_x) \|^2 + C \delta \| (\psi_{xx}, \check{\psi}_{xx}) \|^2 + C \delta (\phi^2(t, 0) + \check{\phi}^2(t, 0)),
\]

(3.40)

\[
|K_3| \leq C \| \check{\psi} \|_{L^\infty} \| (\check{\psi}_x, \check{\psi}_{xx}) \|^2 + C \| \check{\psi}_x \|_{L^\infty} \| \check{\psi}_{xx} \| \| \check{\phi}_x \| \leq C \| (\check{\psi}_x, \check{\psi}_{xx}) \|^2.
\]

(3.41)
Finally, taking $\delta$ and $\varepsilon$ small enough and substituting of (3.38)-(3.41) into (3.37), we obtain
\[
\frac{d}{dt} \int \psi_t^2 + \tilde{\psi}_t^2 \, dx + \frac{\mu}{2\rho_+} \|\psi_{xx}\|^2 + \frac{1}{2} \|\tilde{\psi}_{xx}\|^2 \leq C\|\phi_x, \psi_x, \phi, \tilde{\phi}, \tilde{\psi} - \phi\|^2 + C\delta(\phi^2(t, 0) + \tilde{\phi}^2(t, 0)). \tag{3.42}
\]
Integrating (3.42) in $\tau$ over $[0, t]$, and using Lemmas 3.3-3.4 and the smallness of $\delta$ and $\varepsilon$, we obtain the desired estimate (3.36). Therefore, we complete the proof of Lemma 3.5.

With the help of Lemmas 3.3-3.5, we get (3.9) and complete the proof of Proposition 3.1.

4 Time convergence rates

4.1 Convergence rate of supersonic steady-state

Proposition 4.1. Assume that the same conditions in Theorem 1.3 for $M_+ > 1$ hold and let $(\phi, \psi, \tilde{\phi}, \tilde{\psi})$ be a solution to the IBVP (3.3)-(3.7) satisfying $(\phi, \psi, \tilde{\phi}, \tilde{\psi}) \in C([0, T]; H^1)$ and $(1 + x)^\varepsilon (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \in C([0, T]; L^2)$ for any time $T > 0$. Then for arbitrary $\nu \in [0, \lambda]$, there exist positive constants $\varepsilon > 0$ and $C > 0$ independent of $T$ such that if
\[
\sup_{0 \leq t \leq T} \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) (t) \|_1 + \delta \leq \varepsilon \tag{4.1}
\]
is satisfied, it holds for arbitrary $t \in [0, T]$ that
\[
(1 + t)^\lambda \nu^{\nu + \theta} \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_1 + \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_{a, \nu}^2 + \nu \int_0^t (1 + \tau)^\lambda \nu^{\nu + \theta} \| (\phi, \tilde{\phi}, \psi, \tilde{\psi}) \|_{a, \nu - 1}^2 \, d\tau \tag{4.2}
\]
\[
\leq C(1 + t)^\theta \| (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) \|_1^2 + \| (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) \|_{a, \lambda}^2,
\]
with $\theta > 0$.

Our first goal is to obtain the basic weighted energy estimates of $(\phi, \psi, \tilde{\phi}, \tilde{\psi})$.

Lemma 4.2. Under the same conditions in Proposition 4.1, then the solution $(\phi, \psi, \tilde{\phi}, \tilde{\psi})$ to the IBVP (3.3)-(3.7) satisfies for $t \in [0, T]$ that
\[
(1 + t)^\xi \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_{a, \nu}^2 + \nu \int_0^t (1 + \tau)^\xi \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_{a, \nu - 1}^2 \, d\tau \tag{4.3}
\]
\[
\leq C\| (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) \|_{a, \lambda}^2 + C\delta \int_0^t (1 + \tau)^\xi \| (\phi_x, \tilde{\phi}x) \|_{a, \lambda}^2 \, d\tau + C\nu \int_0^t (1 + \tau)^\xi \| (\psi_x, \tilde{\psi}_x, \tilde{\phi}, \tilde{\psi} - \phi) \|_{a, \nu - 1}^2 \, d\tau \tag{4.3}
\]
\[
+ C\xi \int_0^t (1 + \tau)^{\xi - 1} \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_{a, \nu}^2 \, d\tau.
\]
with $\xi \geq 0$. 

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\textbf{Proof.} We multiply (3.16) by $W_{a,\nu}$, where $W_{a,\nu} := (1 + x)^\nu$ is a space weight function. We integrate the resulted equality over $\mathbb{R}_+$ to obtain

$$
\frac{d}{dt} \int W_{a,\nu}(E_1 + E_2) - (W_{a,\nu}G_1)(t, 0) - \int W_{a,\nu-1}G_1 dx - \int W_{a,\nu-1}G_2 dx
$$

$$
+ \int W_{a,\nu}\left[\rho_x^2 + (n\bar{\psi}^2 + n(\bar{\psi} - \psi)^2\right]dx
$$

$$
= - \int W_{a,\nu}(R_1 + R_2 + R_3) dx,
$$

where $E_i, i = 1, 2$ are defined by (3.14)-(3.15), and $G_j$ for $j = 1, 2, R_k$ for $k = 1, 2, 3$ are defined by (3.17).

First, we estimate terms on the left hand side of (4.4). Under the condition (3.7), the second term on the left hand side is estimated as

$$
-(W_{a,\nu}G_1)(t, 0) = |u_-[\Phi_1(\rho(t, 0), \bar{\rho}(0)) + \Phi_2(n(t, 0), \bar{n}(0))| \geq c(\phi^2(t, 0) + \bar{\phi}^2(t, 0)),
$$

We decompose $\psi$ as $\psi = \bar{\psi} + (\psi - \bar{\psi})$ and use (1.11) to gain

$$
- \nu \int W_{a,\nu-1}G_1 dx
$$

$$
\geq \nu \int W_{a,\nu-1} - \frac{1}{2}(\phi, \bar{\phi}, \bar{\psi})M_3(\phi, \bar{\phi}, \bar{\psi})^T - \rho_+ u_+ \bar{\psi}(\psi - \bar{\psi}) - A_1\gamma\rho_+^{-1} \phi(\psi - \bar{\psi})|dx
$$

$$
- C(\delta + \varepsilon)(\phi, \psi, \bar{\phi}, \bar{\psi})^2_{a,\nu-1},
$$

where the symmetric matrix $M_3$ is denoted by

$$
M_3 = \begin{pmatrix}
-A_1\gamma\rho_+^{-2}u_+ & 0 & -A_1\gamma\rho_+^{-1} \\
0 & -A_2\alpha n_+^{-2}u_+ & -A_2\alpha n_+^{-1} \\
-A_1\gamma\rho_+^{-1} & -A_2\alpha n_+^{-1} & (\rho_++n_+)u_+
\end{pmatrix}.
$$

It is easy to verify that $M_3$ is a positive definite matrix for $M_+ > 1$. Hence, the estimate of the third term on the left hand side is obtained under the condition $\varepsilon, \delta$ and $\eta$ small enough that

$$
- \nu \int W_{a,\nu-1}G_1 dx
$$

$$
\geq c\nu\|(\phi, \psi, \bar{\phi}, \bar{\psi})^2_{a,\nu-1} - \eta\nu\|(\phi, \bar{\psi})^2_{a,\nu-1} - C\eta\nu\|\bar{\psi} - \psi\|^2_{a,\nu-1} - C(\varepsilon + \delta)(\phi, \psi, \bar{\phi}, \bar{\psi})^2_{a,\nu-1}
$$

$$
\geq c\nu\|(\phi, \psi, \bar{\phi}, \bar{\psi})^2_{a,\nu-1} - C\nu\|\bar{\psi} - \psi\|^2_{a,\nu-1},
$$

By Young inequality with $0 < \eta < 1$, the forth and fifth terms on the left hand side are estimated as

$$
- \nu \int W_{a,\nu-1}G_2 dx \leq \nu\eta\|(\psi, \bar{\psi})^2_{a,\nu-1} + \nu C\eta\|\bar{\psi} - \psi\|^2_{a,\nu-1} + C\delta\|\bar{\phi}, \bar{\psi})^2_{a,\nu-1}.
$$

$$
\int W_{a,\nu}\left[\rho_x^2 + n\bar{\psi}^2 + n(\psi - \bar{\psi})^2\right]dx \geq c\|(\bar{\phi}, \bar{\psi})^2_{a,\nu-1} - C(\varepsilon + \delta)\|(\psi, \bar{\psi}, \bar{\phi}, \bar{\psi})^2_{a,\nu-1}.
$$

By (1.11) and (3.11), it follows from Sobolev inequality and Cauchy-Schwarz inequality that

$$
|\int W_{a,\nu}(R_1 + R_2 + R_3) dx| \leq C\delta \int e^{-\frac{\eta}{\nu}\varepsilon}(\phi^2 + \psi^2 + \bar{\phi}^2 + \bar{\psi}^2 + |\bar{\psi} - \psi|^2)dx
$$

$$
\leq C\delta\|(\phi, \psi, \bar{\phi}, \bar{\psi}, \bar{\psi})^2_{a,\nu-1} + C\delta(\phi^2(t, 0) + \bar{\phi}^2(t, 0)).
$$
Finally, with $\eta$, $\delta$ and $\epsilon$ suitably small, the substitution of (4.5)-(4.11) into (4.4) leads to
\[
\frac{d}{dt} \int W_{a,\nu}(E_1 + E_2) dx + c\nu \| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2_{a,\nu} + c\mid (\psi_x, \bar{\psi}_x, \psi - \bar{\psi}) \mid^2_{a,\nu} + c(\phi^2(t,0) + \bar{\phi}^2(t,0)) \\
\leq C \| (\phi_x, \bar{\phi}_x) \|^2 + C\nu \| (\psi_x, \bar{\psi}_x, \psi - \bar{\psi}) \mid^2_{a,\nu-1}. 
\]
(4.12)

Multiplying (4.12) by $(1 + \tau)^\xi$ and integrating the resulted equation in $\tau$ over $[0,t]$, we gain the desired estimate (4.3).

Similar to Lemmas 3.4-3.5, we get the following high order weighted estimates of $(\phi, \psi, \bar{\phi}, \bar{\psi})$. The details are omitted.

**Lemma 4.3.** Under the same conditions in Proposition 4.1, then the solution $(\phi, \psi, \bar{\phi}, \bar{\psi})$ to the IBVP (3.3)-(3.7) satisfies for $t \in [0,T]$ that
\[
(1 + t)^\xi \| (\phi_x, \bar{\phi}_x) \|^2 + \int_0^t (1 + \tau)^\xi \| (\phi_x, \bar{\phi}_x) \|^2 d\tau \\
\leq C(\| (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|^2_{a,\lambda} + \| (\phi_{0x}, \bar{\phi}_{0x}, \psi_{0x}, \bar{\psi}_{0x}) \|^2) + C\nu \int_0^t (1 + \tau)^\xi \| (\psi_x, \bar{\psi}_x, \psi - \bar{\psi}) \mid^2_{a,\nu-1} d\tau \\
+ C\epsilon \int_0^t (1 + \tau)^\xi \| (\psi_{xx}, \bar{\psi}_{xx}) \|^2_{a,\nu} d\tau + C\epsilon \int_0^t (1 + \tau)^{\xi-1} \| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2_{a,\nu} + \| (\phi_x, \bar{\phi}_x) \|^2_{a,\nu-1} d\tau, 
\]
with $\xi \geq 0$.

**Lemma 4.4.** Under the same conditions in Proposition 4.1 hold, then the solution $(\phi, \psi, \bar{\phi}, \bar{\psi})$ to the IBVP (3.3)-(3.7) satisfies for $t \in [0,T]$ that
\[
(1 + t)^\xi \| (\psi_x, \bar{\psi}_x) \|^2 + \int_0^t (1 + \tau)^\xi \| (\psi_{xx}, \bar{\psi}_{xx}) \|^2 d\tau \\
\leq C(\| (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|^2_{a,\lambda} + \| (\phi_{0x}, \bar{\phi}_{0x}, \psi_{0x}, \bar{\psi}_{0x}) \|^2) + C\nu \int_0^t (1 + \tau)^\xi \| (\psi_x, \bar{\psi}_x, \psi - \bar{\psi}) \mid^2_{a,\nu-1} d\tau \\
+ C\epsilon \int_0^t (1 + \tau)^{\xi-1} \| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2_{a,\nu} + \| (\phi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2_{a,\nu-1} d\tau, 
\]
with $\xi \geq 0$.

**Proof of Proposition 4.1** For $\nu \in [0,\lambda]$ and $\xi \geq 0$, it follows from Lemmas 4.2-4.4 that
\[
(1 + t)^\xi \| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2_{a,\nu} + \| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2_{a,\nu-1} d\tau \\
+ \int_0^t (1 + \tau)^\xi \| (\psi_x, \psi_x, \psi - \bar{\psi}) \|^2_{a,\nu} d\tau + \int_0^t (1 + \tau)^\xi \| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2 d\tau \\
\leq C(\| (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|^2_{a,\lambda} + \| (\phi_{0x}, \psi_{0x}, \bar{\phi}_{0x}, \bar{\psi}_{0x}) \|^2) + C\nu \int_0^t (1 + \tau)^\xi \| (\psi_x, \psi_x, \psi - \bar{\psi}) \mid^2_{a,\nu-1} d\tau \\
+ C\epsilon \int_0^t (1 + \tau)^{\xi-1} \| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2_{a,\nu} + \| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2_{a,\nu-1} d\tau, 
\]
where $C > 0$ is a generic positive constant independent of $T, \nu$, and $\xi$. Hence, applying similar induction arguments as in [3,14,25] to (4.15), we gain the desired estimate (4.2).
Indeed, for any $\lambda > 0$ and $k = 0, 1, 2, \ldots, [\lambda]$, we have

\[
(1 + t)^k \| (\phi, \psi, \chi, \tilde{\psi}) \|^2_{1,\lambda-k} + \| (\phi_x, \psi_x, \chi_x, \tilde{\psi}_x) \|^2 + \nu \int_0^t (1 + t)^k \| (\phi, \psi, \chi, \tilde{\psi}) \|^2_{1,\lambda-k-1} d\tau \\
+ \int_0^t (1 + \tau)^k \| (\psi_x, \chi_x, \psi - \tilde{\psi}) \|^2_{1,\lambda-k} d\tau + \int_0^t (1 + \tau)^k \| (\phi_x, \psi_x, \chi_x, \tilde{\psi}_x) \|^2 d\tau \\
\leq C(\| (\phi_0, \psi_0, \chi_0, \tilde{\psi}_0) \|^2_{1,\lambda} + \| (\phi_{0x}, \psi_{0x}, \chi_{0x}, \tilde{\psi}_{0x}) \|^2),
\]

and

\[
(1 + t)^k \| (\phi, \psi, \chi, \tilde{\psi}) \|^2 + \int_0^t (1 + \tau)^k \| (\phi_x, \psi_x, \chi_x, \tilde{\psi}_x, \psi - \tilde{\psi}) \|^2 d\tau \\
\leq C(\| (\phi_0, \psi_0, \chi_0, \tilde{\psi}_0) \|^2_{1,\lambda} + \| (\phi_{0x}, \psi_{0x}, \chi_{0x}, \tilde{\psi}_{0x}) \|^2).
\]

To prove (4.16) and (4.17), we apply similar induction arguments as in [3, 14, 25] to (4.15).

Step 1. Taking $\xi = 0$, $\nu = \lambda$ in (4.15) and using (3.9), we have (4.16) and (4.17) for $k = 0$. Therefore, (4.16) and (4.17) hold for $0 < \lambda < 1$.

Step 2. Taking $\xi = 1$, $\nu = 0$ in (4.15) and using (4.16) with $k = 0$, we have (4.17) with $k = 1$. Then, taking $\xi = 1$, $\nu = \lambda - 1$ in (4.15) and using (4.17) with $k = 1$ and (4.16) with $k = 0$, we obtain the desired estimate (4.16) with $k = 1$. Therefore, the proof is finished for $1 \leq \lambda < 2$.

Step 3. We repeat the same procedure as in Step 2. The estimate (4.15) (with $\xi = 2$, $\nu = 0$) together with (4.17) (with $k = 1$) lead to (4.17) (with $k = 2$). Also, (4.15) (with $\xi = 2$, $\nu = \lambda - 2$) together with (4.17) (with $k = 2$) and (4.16) (with $k = 1$) lead to (4.16) (with $k = 2$), which proves the estimates (4.16) and (4.17) for $2 \leq \lambda < 3$.

Repeating the same procedure, we get the desired estimates (4.16) and (4.17) for any $\lambda > 0$.

If $\lambda > 0$ is integer, we obtain (4.2) from (4.16) letting $k = \lambda$.

If $\lambda > 0$ is not integer, we obtain (4.2) as follows.

Taking $\nu = 0$ in (4.15), we have

\[
(1 + t)^\xi \| (\phi, \psi, \chi, \tilde{\psi}) \|^2 + \int_0^t (1 + \tau)^\xi \| (\phi_x, \psi_x, \chi_x, \tilde{\psi}_x, \psi - \tilde{\psi}) \|^2 d\tau \\
\leq C(\| (\phi_0, \psi_0, \chi_0, \tilde{\psi}_0) \|^2_{1,\lambda} + \| (\phi_{0x}, \psi_{0x}, \chi_{0x}, \tilde{\psi}_{0x}) \|^2) + C\xi \int_0^t (1 + \tau)^{\xi-1} \| (\phi, \psi, \chi, \tilde{\psi}) \|^2 d\tau.
\]

Using (4.16) with $k = [\lambda]$ and taking $s = 1 - (\lambda - [\lambda])$, we have

\[
\int_0^t (1 + \tau)^{\xi-1} \| (\phi, \psi, \chi, \tilde{\psi}) \|^2 d\tau \\
\leq \int_0^t (1 + \tau)^{\xi-1-[\lambda]} ((1 + t)^{[\lambda]} \| (\phi, \psi, \chi, \tilde{\psi}) \|^2_{1,\lambda-[\lambda]-1} + (1 + t)^{[\lambda]} \| (\phi_x, \psi_x, \chi_x, \tilde{\psi}_x) \|^2) d\tau \\
\leq C(\| (\phi_0, \psi_0, \chi_0, \tilde{\psi}_0) \|^2_{1,\lambda} + \| (\phi_{0x}, \psi_{0x}, \chi_{0x}, \tilde{\psi}_{0x}) \|^2) \int_0^t (1 + \tau)^{\xi-1-[\lambda]} ((1 + t)^{[\lambda]} \| (\phi_x, \psi_x, \chi_x, \tilde{\psi}_x) \|^2 \\
+ (1 + t)^{[\lambda]} \| (\phi, \psi, \chi, \tilde{\psi}) \|^2_{1,\lambda-[\lambda]-1})^{1-s} d\tau.
\]
\[ \leq C(\|\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0\|_{a, \lambda}^2 + \|\phi_{0x}, \psi_{0x}, \bar{\phi}_{0x}, \bar{\psi}_{0x}\|)^2(\int_0^t (1 + \tau)^{\frac{1+|\lambda|}{1+|\lambda|} - 1} d\tau)^{1 + |\lambda| - \lambda} \]
\[ \leq C(\|\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0\|_{a, \lambda}^2 + \|\phi_{0x}, \psi_{0x}, \bar{\phi}_{0x}, \bar{\psi}_{0x}\|)^2(1 + t)^\theta, \]

where we take \( \xi = \lambda + \theta(1 + |\lambda| - \lambda) \) and \( \theta > 0 \).

### 4.2 Convergence rate of sonic steady-state

The function space \( Y_W(0, T) \) for \( T > 0 \) is denoted by

\[ Y_W(0, T) := \{ (\phi, \psi, \bar{\phi}, \bar{\psi}) | (\phi, \psi, \bar{\phi}, \bar{\psi}) \in C([0, T]; H^1_W(\mathbb{R}_+)), (\phi_x, \bar{\phi}_x) \in L^2([0, T]; L^2_W(\mathbb{R}_+)), (\psi_x, \bar{\psi}_x) \in L^2([0, T]; H^1_W(\mathbb{R}_+)) \}. \]

**Proposition 4.5.** Assume that \( 1 \leq \lambda < \lambda^\ast \) with \( \lambda^\ast := 2 + \sqrt{8 + \frac{1}{1+b}}, \)

\( b := \frac{\rho_u(\nu^2 - \rho_v(\nu^*)}{u + \sqrt{\mu_n}} \), and that the same conditions in Theorem 1.3 hold for \( M_+ = 1 \). Let \( (\phi, \psi, \bar{\phi}, \bar{\psi}) \) be a solution to the IBVP (3.3)-(3.7) satisfying \( (\phi, \psi, \bar{\phi}, \bar{\psi}) \in Y^{\sigma - \lambda}(0, T) \) for any time \( T > 0 \). Then for arbitrary \( \nu \in [0, \lambda] \), there exist positive constants \( \varepsilon > 0 \) and \( C > 0 \) independent of \( T \) such that if

\[ \sup_{0 \leq t \leq T} \|\sigma^{\frac{-\lambda}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})(t)\|_1 + \delta^{\frac{1}{2}} \leq \varepsilon \]

(4.21)

is satisfied, it holds for arbitrary \( t \in [0, T] \) that

\[ (1 + \delta t)^{\frac{1}{2} - \beta} \|\sigma^{\frac{-\lambda}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 + \int_0^t (1 + \delta \tau)^{\frac{1}{2} - \beta} \|\sigma^{\frac{-\lambda}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 d\tau \]
\[ + \int_0^t (1 + \delta \tau)^{\frac{1}{2} - \beta} \|\sigma^{\frac{-\lambda}{2}}(\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x)\|^2 d\tau + \int_0^t (1 + \delta \tau)^{\frac{1}{2} - \beta} \|\sigma^{\frac{-\lambda}{2}}(\psi_x, \bar{\psi}_x, \bar{\psi} - \psi)\|^2 d\tau \]
\[ \leq C(1 + \delta t)^{\lambda} \|\sigma^{\frac{-\lambda}{2}}(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0)\|^2, \]

(4.22)

with \( \beta > 0 \).

By the fact \( \lambda \geq 1 \) and (4.21), it is easy to verify the following estimate:

\[ \|\sigma^{\frac{-\lambda}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|_{L^\infty} \leq \|\sigma^{\frac{-\lambda}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|_1 \leq \sqrt{2}\varepsilon. \]

(4.23)

To deal with some nonlinear terms, we use the following inequality as in [15, 27, 36].

**Lemma 4.6.** Let \( \nu \geq 1 \). Then a function \( \sigma^{\frac{-\lambda}{2}}(x)\phi(t, x) \in H^1(\mathbb{R}_+) \) satisfies

\[ \int \sigma^{\frac{-\lambda}{2}}|\phi|^2 dx \leq C\|\sigma^{\frac{-\lambda}{2}}\phi(t, 0)\| (\|\sigma(0)\phi^2(t, 0) + \|\sigma^{\frac{-\lambda}{2}}\phi_x\|^2 + \|\sigma^{\frac{-\lambda}{2}}\phi\|^2), \]

(4.24)

where the function \( \sigma(x) \geq 0 \) is defined by (2.28) with \( \sigma(0) \) small enough.

To gain faster decay rates, it is necessary to use the following Hardy type inequality.
Lemma 4.7 ( [13] ). Let $\zeta \in C^1[0, \infty)$ satisfies $\zeta > 0$, $\zeta_x > 0$ and $\zeta(x) \to \infty$ for $x \to \infty$. Then we have
\[
\int_{\mathbb{R}^+} \psi^2 \zeta_x dx \leq 4 \int_{\mathbb{R}^+} \psi_x^2 \zeta dx
\] (4.25)
for $\psi$ satisfying $\psi(t, 0) = 0$ and $\sqrt{w} \psi \in H^1(\mathbb{R}^+)$, with the function $w := \frac{\zeta^2}{\zeta_x}$.

With the aid of Lemmas 4.6-4.7, we obtain the weighted $L^2$ estimate of $(\phi, \psi, \bar{\phi}, \bar{\psi})$.

Lemma 4.8. Under the same conditions in Proposition 4.5, then the solution $(\phi, \psi, \bar{\phi}, \bar{\psi})$ to the problem (3.3)-(3.7) satisfies for $t \in [0, T]$ that
\[
(1 + \delta t)^{\frac{1}{2}} \left\| \sigma^{-\frac{1}{2}} \left( \phi, \psi, \bar{\phi}, \bar{\psi} \right) \right\|^2 + \int_0^t (1 + \delta t)^{\frac{1}{2}} \left\| \sigma^{-\frac{1}{2}} \left( \phi, \psi, \bar{\phi}, \bar{\psi} \right) \right\|^2 dt \\
+ \int_0^t (1 + \delta t)^{\frac{1}{2}} \left\| \sigma^{-\frac{1}{2}} (\psi_x, \bar{\psi}_x, \bar{\psi} - \psi) \right\|^2 dt + \int_0^t (1 + \delta t)^{\frac{1}{2}} \frac{1}{\delta t} (\phi^2(t, 0) + \bar{\phi}^2(t, 0)) dt \\
\leq C \left\| \sigma^{-\frac{1}{2}} (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \right\|^2 + C \delta \int_0^t (1 + \delta t)^{\frac{1}{2}} \left\| \sigma^{-\frac{1}{2}} (\phi_x, \bar{\phi}_x) \right\|^2 dt \\
+ C \delta \xi \int_0^t (1 + \delta t)^{\frac{1}{2}} \left\| \sigma^{-\frac{1}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|^2 dt,
\] (4.26)
with $\xi \geq 0$.

Proof. We multiply (3.16) by the space weight function $\sigma^{-\nu}$, where the space weight function $\sigma \geq 0$ satisfies (1.14) and (2.28). Then, we integrate the resulted equation over $\mathbb{R}^+$ to get
\[
\frac{d}{dt} \left( \int \sigma^{-\nu} (\mathcal{E}_1 + \mathcal{E}_2) dx - (\sigma^{-\nu} G_1)(t, 0) - av \int \sigma^{-\nu-1} G_1 dx - av \int \sigma^{-\nu-1} G_2 dx \\
+ \int \sigma^{-\nu} n(\psi - \psi)^2 dx + \int \sigma^{-\nu} (\mu \nu^2_x + \nu^2_x) dx + \int \sigma^{-\nu} R_1 dx \\
- \int \sigma^{-\nu} R_2 dx - \int \sigma^{-\nu} R_3 dx, \right)
\] (4.27)
where $\mathcal{E}_i$, $i = 1, 2$ are defined by (3.14)-(3.15), and $G_j$ for $j = 1, 2$, $R_k$ for $k = 1, 2, 3$ are defined by (3.17). First, we estimate terms on the left hand side of (4.27). Under the condition (3.7), the second term on the left hand side is estimated as
\[
-(\sigma^{-\nu} G_1)(t, 0) \geq C \frac{\sigma}{\delta t}(\phi^2(t, 0) + \bar{\phi}^2(t, 0)) \geq 0.
\] (4.28)

For the third term on the left hand side, using (4.24) and $\psi = \bar{\psi} + (\psi - \bar{\psi})$ yields
\[
-av \int \sigma^{-\nu-1} G_1 dx \\
\geq av \int \sigma^{-\nu-1} \left\{ \frac{1}{2} (\phi, \bar{\phi}, \psi, \bar{\psi}) M_4(\phi, \bar{\phi}, \psi, \bar{\psi})^T - \rho_+ u_+ \psi(\psi - \bar{\psi}) - A_1 \gamma \rho_+^{-1} \phi(\psi - \bar{\psi}) \right\} dx \\
+ av \int \sigma^{-\nu-1} \left\{ -(A_1 \gamma \rho_+^{-2} u_1 - A_1 \gamma \rho_+^{-2} u_+ \frac{\phi^2}{2} - (A_2 \alpha \alpha + A_2 \alpha \alpha + A_2 \alpha \alpha + A_2 \alpha \alpha ) \frac{\phi^2}{2} \\
- (A_1 \gamma \rho_+^{-1} - A_1 \gamma \rho_+^{-1}) \phi \psi - (A_2 \alpha \alpha - A_2 \alpha \alpha - A_2 \alpha \alpha - A_2 \alpha \alpha ) \phi \psi \right\} dx - C \varepsilon \frac{\sigma}{\delta t}(\phi^2(t, 0) + \bar{\phi}^2(t, 0)) \\
- C \varepsilon (\sigma^{-\frac{1}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|^2 + \| \sigma^{-\frac{1}{2}} (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2),
\] (4.29)
where the symmetric matrix $M_4$ is defined as

$$M_4 = \begin{pmatrix} -A_1 \gamma \rho_+^{-2} u_+ & 0 & -A_1 \gamma \rho_+^{-1} \\ 0 & -A_2 \alpha n_+^{\alpha - 2} u_+ & -A_2 \alpha n_+^{\alpha - 1} \\ -A_1 \gamma \rho_+^{-1} & -A_2 \alpha n_+^{\alpha - 1} & -(\rho_+ + n_+) u_+ \end{pmatrix}. \tag{4.30}$$

Owing to $M_+ = 1$, it is easy to check that three eigenvalues of the matrix $M_4$ satisfy: $\hat{\lambda}_1 > 0$, $\hat{\lambda}_2 > 0$, $\hat{\lambda}_3 = 0$. Take the coordinate transformation

$$\begin{pmatrix} \phi \\ \bar{\phi} \\ \bar{\psi} \end{pmatrix} = P \begin{pmatrix} \hat{\rho} \\ \hat{n} \\ \hat{v} \end{pmatrix}, \tag{4.31}$$

where the matrix $P$ is denoted by

$$P = \begin{pmatrix} r_{11} & r_{21} & -\frac{\rho_+}{u_+} \\ r_{12} & r_{22} & -\frac{n_+}{u_+} \\ r_{13} & r_{23} & 1 \end{pmatrix} \text{ with constants } r_{ij} \text{ for } 1 \leq i \leq 2, \ 1 \leq j \leq 3, \tag{4.32}$$

such that

$$(\phi, \bar{\phi}, \bar{\psi}) M_4 (\phi, \bar{\phi}, \bar{\psi})^T = (\hat{\rho}, \hat{n}, \hat{v}) \begin{pmatrix} \hat{\lambda}_1 & 0 & 0 \\ 0 & \hat{\lambda}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{\lambda}_1 \hat{\rho}^2 + \hat{\lambda}_2 \hat{n}^2. \tag{4.33}$$

By (1.13), (4.23), (4.29), and (4.31)-(4.33), the third term is estimated as

$$-av \int \sigma^{-(\nu-1)}G_1 dx \geq av \int \sigma^{-(\nu-1)} \left( \frac{\hat{\lambda}_1 - \hat{\rho}^2}{\rho_+^2} + \hat{\lambda}_2 \hat{n}^2 \right) dx + av \int \sigma^{-(\nu-2)} \frac{A_1 \gamma (\gamma + 1) \rho_+^{\gamma - 1} + A_2 \alpha (\alpha + 1) n_+^{\alpha - 1}}{2|u_+|^2} \hat{\rho}^2 dx$$

$$+ av \int \sigma^{-(\nu-1)} \frac{\rho_+ (u_+^2 - A_1 \gamma \rho_+^{\gamma - 1})}{|u_+|^2} \hat{v} dx - C\delta^2 \|\sigma^{-\frac{\nu-1}{2}}(\hat{\rho}, \hat{n})\|^2 - C\delta^2 \|\sigma^{-\frac{\nu-2}{2}} \hat{v}\|^2$$

$$- C\varepsilon \frac{1}{\delta^2} (\hat{\rho}^2(t, 0) + \hat{v}^2(t, 0)), \tag{4.34}$$

where we have used the following facts

$$- (A_1 \gamma \rho_+^{\gamma - 1} - A_1 \gamma \rho_+^{\gamma - 1}) \geq \frac{A_1 \gamma (\gamma - 1) \rho_+^{\gamma - 1}}{|u_+|} \sigma - C\sigma^2,$$

$$- (A_2 \alpha n_+^{\alpha - 1} - A_2 \alpha n_+^{\alpha - 1}) \geq \frac{A_2 \alpha (\alpha - 1) n_+^{\alpha - 1}}{|u_+|} \sigma - C\sigma^2,$$

$$- (A_1 \gamma \rho_+^{\gamma - 2} u_+ - A_1 \gamma \rho_+^{\gamma - 2} u_+) \geq A_1 \gamma (3 - \gamma) \rho_+^{\gamma - 2} \sigma - C\sigma^2,$$

$$- (A_2 \alpha n_+^{\alpha - 2} u_+ - A_2 \alpha n_+^{\alpha - 2} u_+) \geq A_2 \alpha (3 - \alpha) n_+^{\alpha - 2} \sigma - C\sigma^2.$$
With the help of (1.12)-(1.14), (3.7), (4.21), (4.23)-(4.24), (4.31)-(4.33), and \( \psi = \bar{\psi} + (\psi - \bar{\psi}) \), it holds that

\[
- a v \int \sigma^{-(\nu-1)} G_2(t,x) dx
\]

\[
\geq - a^2 \frac{\mu + n + \nu(\nu - 1)}{2} \| \sigma^{-\frac{\nu}{2}} \bar{\psi} \|^2 - C(\delta + \varepsilon) \| \sigma^{-\frac{\nu}{2}} \psi_x \|^2 - C \varepsilon \| \sigma^{-\frac{\nu}{2}} \bar{\psi}_x \|^2 - C \delta \| \sigma^{-\frac{\nu}{2}} (\bar{\psi} - \psi) \|^2
\]

\[
\geq - a^2 \frac{\mu + n + \nu(\nu - 1)}{2} \| \sigma^{-\frac{\nu}{2}} \bar{\psi} \|^2 - C \delta \| \sigma^{-\frac{\nu}{2}} (\bar{\psi} - \psi) \|^2,
\]

(4.35)

\[
\int \sigma^{-\nu} n(\bar{\psi} - \psi)^2 dx \geq [n_+ - C(\delta + \varepsilon)] \| \sigma^{-\frac{\nu}{2}} (\bar{\psi} - \psi) \|^2,
\]

(4.36)

\[
\int \sigma^{-\nu} R_1 dx
\]

\[
\geq a \int \sigma^{-(\nu-2)} \frac{A_1 \gamma(\gamma + 1) \rho^2 + \rho_+ \psi^2 + A_2 \alpha(\alpha - 1) n^2}{2} d\bar{\psi}^2 + n_+ \bar{\psi}^2 dx
\]

\[
- C(\delta + \varepsilon) \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2
\]

(4.37)

For \( \nu \in (0, 3) \), with the help of (1.13) and (2.22), we add (4.34)-(4.37) together to have

\[
- a v \int \sigma^{-(\nu-1)} G_1 dx - a v \int \sigma^{-(\nu-1)} G_2 dx + \int \sigma^{-\nu} n(\bar{\psi} - \psi)^2 dx + \int \sigma^{-\nu} R_1 dx
\]

\[
\geq c \| \sigma^{-\frac{\nu}{2}} (\bar{\psi}, \bar{\psi}) \|^2 + \frac{1}{4} \left\{ a \left( \frac{A_1 \gamma(\gamma + 1) \rho^2 + A_2 \alpha(\alpha - 1) n^2}{2 |u_+|^2} [1 + \nu - \frac{\nu(\nu - 1)}{2(1 + b^2)}] \right) \| \sigma^{-\frac{\nu}{2}} \bar{\psi} \|^2
\]

\[
+ n_+ \| \sigma^{-\frac{\nu}{2}} (\bar{\psi} - \psi) \|^2 \right\} + \int \sigma^{-\nu} (\psi - \bar{\psi}) M_5 (\psi - \bar{\psi})^T dx - C \delta \| \sigma^{-\frac{\nu}{2}} (\bar{\psi}, \bar{\psi}) \|^2
\]

\[
- C \delta \| \sigma^{-\frac{\nu}{2}} \bar{\psi} \|^2 - C(\varepsilon + \delta) \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\psi}, \bar{\psi}) \|^2
\]

\[
- C \varepsilon \frac{\delta}{\sigma^2} (\sigma^2(t,0) + \bar{\sigma}^2(t,0)) - C(\varepsilon + \delta) \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2
\]

\[
\geq c \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\bar{\psi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\bar{\psi} - \psi) \|^2
\]

\[
- C(\delta + \varepsilon) \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 - C\varepsilon \frac{\sigma^2(t,0) + \bar{\sigma}^2(t,0)}{\delta \sigma^2},
\]

(4.38)

where the positive definite matrix \( M_5 \) is defined by

\[
M_5 = \left( \frac{\frac{1}{2} n_+}{\sqrt{\mu + \nu n} a b v \sigma} \frac{\sqrt{\mu + \nu n} a b v \sigma}{3} \frac{A_1 \gamma(\gamma + 1) \rho^2 + A_2 \alpha(\alpha - 1) n^2}{2 |u_+|^2} [(1 + \nu - \frac{\nu(\nu - 1)}{2(1 + b^2)}] \right).
\]

(4.39)

Then, we consider the case \( \nu \in [3, 2 + \sqrt{8 + \frac{1}{\nu + 3}}) \) using the Lemma 4.7 with \( \xi = \sigma^{-(\nu-1)} \). Therefore,
with the aid of (2.22), the sixth term is estimated as below:

\[
\int \sigma^{-\nu}(\mu \psi_x^2 + n \psi_x^2)dx \\
\geq a^2(\mu + n_+) \left( \frac{\nu - 1}{4} \right)^2 \| \sigma^{-\nu} \tilde{\psi} \|^2 - C\delta \frac{\| \sigma^{-\nu} \tilde{\psi} \|^2 + \| \sigma^{-\nu} (\hat{\rho}, \hat{\nu}) \|^2}{\| \sigma^{-\nu} (\hat{\rho}, \hat{\nu}) \|^2} \\
- C(\delta + \varepsilon) \| \sigma^{-\varepsilon} (\tilde{\psi} - \psi, \tilde{\psi}_x) \|^2. 
\] (4.40)

For \( \nu \in (3, \lambda] \), adding (4.38) to (4.40), taking \( k = \nu B[4(1 + b^2)\nu + 4b^2 + 5 - \nu^2]^{-1} \in (0, 1) \), and using (2.22) and \( c(\phi, \tilde{\phi}, \tilde{\psi}) \leq \| (\hat{\rho}, \hat{\nu}, \hat{\psi}) \| \leq C(\| \phi, \tilde{\phi}, \tilde{\psi} \|) \), we have

\[
- a\nu \int \int \sigma^{-\nu} \sigma G_1 dx - a\nu \int \int \sigma^{-\nu} \sigma G_2 dx + \int \sigma^{-\nu} \sigma n(\tilde{\psi} - \psi)^2 dx + \int \sigma^{-\nu} R_1 dx \\
+ \int \sigma^{-\nu} (\mu \psi_x^2 + n \psi_x^2) dx \\
\geq c\| \sigma^{-\nu} (\hat{\rho}, \hat{\nu}) \|^2 + \int \sigma^{-\nu} (\psi - \hat{\psi})^2 (\psi - \hat{\psi})^2 dx + (1 - k) \{ n_+ \| \sigma^{-\varepsilon} (\psi - \tilde{\psi}) \| ^2 \\
+ a \frac{A_1 \gamma (\gamma + 1) \rho_x^2 + A_2 \alpha (\alpha + 1) n_+^2}{2n_+^2} \left[ 1 + \nu \frac{\nu - 1}{2(1 + b^2)} + \frac{(\nu - 1)^2}{4(1 + b^2)} \| \sigma^{-\nu} \tilde{\psi} \|^2 \right] \} \\
- C\delta \frac{\| \sigma^{-\varepsilon} (\hat{\rho}, \hat{\nu}) \|^2}{\| \sigma^{-\nu} \tilde{\psi} \|^2} - C(\varepsilon + \delta) \| \sigma^{-\varepsilon} (\phi_x, \tilde{\psi}_x, \tilde{\phi}_x, \tilde{\psi}_x) \|^2 \\
- C(\varepsilon + \delta) \| \sigma^{-\varepsilon} (\psi - \hat{\psi}) \|^2 - C\frac{\sigma}{\delta} \| \sigma^{-\varepsilon} (\phi_x, \tilde{\phi}_x, \tilde{\psi}_x) \|^2 \\
\geq c\| \sigma^{-\varepsilon} (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|^2 + c\| \sigma^{-\varepsilon} (\hat{\rho}, \hat{\nu}) \|^2 + c\| \sigma^{-\varepsilon} (\psi - \hat{\psi}, \tilde{\psi}_x, \tilde{\phi}_x) \|^2 \\
- C(\varepsilon + \delta) \| \sigma^{-\varepsilon} (\phi_x, \tilde{\phi}_x, \tilde{\psi}_x) \|^2 - C\frac{\sigma}{\delta} \| \sigma^{-\varepsilon} (\phi_x, \tilde{\phi}_x, \tilde{\psi}_x) \|^2,
\] (4.41)

where \( \delta \) and \( \varepsilon \) are small enough, and the positive definite matrix \( M_6 \) is defined as

\[
M_6 = \left( \begin{array}{cc}
\frac{kn_+}{\sqrt{\rho \rho_x^2}} & \frac{\sqrt{\rho \rho_x^2}}{abu} \\
\sqrt{\rho \rho_x^2} & \frac{ka \gamma (\gamma + 1) \rho_x + A_2 \alpha (\alpha + 1) n_+^2}{2n_+^2} \left[ 1 + \nu \frac{\nu - 1}{2(1 + b^2)} + \frac{(\nu - 1)^2}{4(1 + b^2)} \right] \sigma^{-\nu} \tilde{\psi} \end{array} \right).
\] (4.42)

By (1.12)-(1.13), (4.31)-(4.33), Cauchy-Schwarz inequality and \( M_+ = 1 \), we estimate terms on the right hand side as

\[
| \int \sigma^{-\nu} R_2 dx + \int \sigma^{-\nu} R_3 dx | \\
\leq C\delta \| \sigma^{-\varepsilon} (\hat{\rho}, \hat{\nu}) \|^2 + C\delta \| \sigma^{-\varepsilon} (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|^2 + C\delta \| \sigma^{-\varepsilon} (\psi - \hat{\psi}) \|^2.
\] (4.43)

Finally, taking \( \delta \varepsilon \) and \( \varepsilon \) small enough, and combining (4.28)-(4.43), we obtain

\[
\frac{d}{dt} \int \sigma^{-\nu}(E_1 + E_2) dx + c\| \sigma^{-\varepsilon} (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|^2 + c\| \sigma^{-\varepsilon} (\psi_x, \tilde{\psi}_x, \psi - \tilde{\psi}) \|^2 + c\| \sigma^{-\varepsilon} (\phi_x, \tilde{\phi}_x, \phi - \tilde{\phi}) \|^2 + c\| \sigma^{-\varepsilon} (\psi_x, \tilde{\psi}_x, \psi - \tilde{\psi}) \|^2 \\
\leq C(\delta + \varepsilon) \| \sigma^{-\varepsilon} (\phi_x, \tilde{\phi}_x) \|^2.
\] (4.44)

Multiplying (4.44) by \( (1 + \delta \tau)^{\frac{3}{2}} \) and integrating the resulted equation in \( \tau \) over \([0, t] \), we obtain (4.26). The proof of Lemma 4.8 is completed. \( \square \)
In order to show Proposition 4.5, we need to obtain the high order weighted estimates of $(\phi, \psi, \bar{\phi}, \bar{\psi})$.

**Lemma 4.9.** Under the same conditions in Proposition 4.5, then the solution $(\phi, \psi, \bar{\phi}, \bar{\psi})$ to the IBVP (3.3)-(3.7) satisfies for $t \in [0,T]$ that

\[
(1 + \delta t)^{\xi} \| \sigma^{-\frac{2}{3}} (\phi_x, \bar{\phi}_x) \|^2 + \int_0^t (1 + \delta \tau)^{\xi} \| \sigma^{-\frac{2}{3}} (\phi_x, \bar{\phi}_x) \|^2 d\tau \\
\leq C \| \sigma^{-\frac{2}{3}} (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|^2 + \| \sigma^{-\frac{2}{3}} (\phi_0, \psi_0) \|^2 + C(\varepsilon + \delta) \int (1 + \delta \tau)^{\xi} \| \sigma^{-\frac{2}{3}} (\psi_{xx}, \bar{\psi}_{xx}) \|^2 d\tau \quad (4.45)
\]

\[
+ \delta \xi \int_0^t (1 + \delta \tau)^{\xi-1} \| \sigma^{-\frac{2}{3}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 d\tau,
\]

with $\xi \geq 0$.

**Proof.** Adding (3.24)-(3.27) together, and multiplying the resulted equation by $\sigma^{-\nu}$ with the weight function $\sigma \geq 0$ satisfying (1.14) and (2.28), we integrate the resulted equation in $x$ over $\mathbb{R}_+$ to obtain

\[
\frac{d}{dt} \int \sigma^{-\nu} \left( \frac{\mu \phi_x^2}{2} + \frac{\phi_x^2}{2} + \rho^2 \phi_x \psi + \bar{n} \bar{\phi}_x \bar{\psi} \right) dx - \left[ \sigma^{-\nu} \left( \frac{\mu \phi_x^2}{2} + \frac{\phi_x^2}{2} - \rho^2 \phi_t \psi - \bar{n} \bar{\phi}_t \bar{\psi} \right) \right] (t,0) \\
- a \nu \int \sigma^{-\nu-1} \left[ \frac{\mu \phi_x^2}{2} + \frac{\phi_x^2}{2} - \rho^2 \phi_t \psi - \bar{n} \bar{\phi}_t \bar{\psi} \right] dx + \int \sigma^{-\nu} \left( \frac{\rho \phi_x^2}{\rho} \frac{\mu \phi_x^2}{2} + \frac{\phi_x^2}{2} \right) dx
\]

\[= \sum_{i=1}^6 J_i,
\]

where

\[
J_1 = - \int \sigma^{-\nu} \left( \frac{\mu \phi_x^2}{2} + \frac{\phi_x^2}{2} + \rho^2 \phi_x \psi + \bar{n} \bar{\phi}_x \bar{\psi} \right) \psi_x + 2 \rho \rho \phi_t \psi_x + \bar{n} \bar{n} \bar{\phi}_t \bar{\psi}_x dx,
\]

\[
J_2 = \int \sigma^{-\nu} \left[ - \mu \phi \phi_x \psi_{xx} - \bar{n} \bar{n} \bar{\phi}_x \bar{\psi}_{xx} + \mu \rho^2 \left( \frac{1}{\rho} - \frac{1}{\rho^2} \right) \phi_x \psi_{xx} + \bar{n} \left( \frac{1}{n} - \frac{1}{n} \right) \bar{n} \bar{n} \bar{\phi}_x \bar{\psi}_{xx} \right] dx,
\]

\[
J_3 = \int \sigma^{-\nu} \left[ \frac{n}{\rho} \left( \psi - \psi \right) \phi_x - \bar{n} \left( \bar{\psi} - \bar{\psi} \right) \bar{\phi}_x \right] dx,
\]

\[
J_4 = - \int \sigma^{-\nu} \left( 3 \mu \psi \phi_x^2 + \frac{3}{2} \phi_x \psi_x^2 \right) dx + \int \sigma^{-\nu} \left( \frac{\phi_x \psi_{xx}}{n} \phi_x \right) dx, \quad J_5 = \int \sigma^{-\nu} \left( F_1 \rho^2 \phi_x + F_2 \bar{n} \bar{\phi}_x \right) dx,
\]

\[
J_6 = - \int \sigma^{-\nu} \left[ \frac{1}{2} \phi_x \psi_x + \psi_x \psi_{xx} \right] \phi_x + \mu \left( \psi_{xx} + \psi_{xx} \right) \phi_x \bar{\phi}_x \bar{\psi}_x \\
- \bar{n} \bar{n} \bar{n} \bar{\phi}_x \bar{\psi}_{xx} + \left( \frac{1}{2} \phi_x \bar{\psi}_x + \psi_x \bar{\psi}_{xx} \right) \bar{\phi}_x - \left( \tilde{\phi}_x \bar{\psi}_x + \psi_x \bar{n} \bar{\phi}_x \right) \bar{\phi}_x \bar{\psi}_x dx.
\]

Owing to (1.12), (3.7), and (4.23), we obtain the estimates for terms on the left hand side as below

\[
- \left[ \sigma^{-\nu} \left( \frac{\mu \phi_x^2}{2} + \frac{\phi_x^2}{2} - \rho \rho \phi_x \psi - \bar{n} \bar{n} \bar{\phi}_x \bar{\psi} \right) \right] (t,0) \geq \frac{C}{\rho} \left( \phi_x^2(t,0) + \phi_x^2(t,0) \right) \geq 0, \quad (4.47)
\]

\[
- a \nu \int \sigma^{-\nu-1} \left( \frac{\mu \phi_x^2}{2} + \frac{\phi_x^2}{2} \right) dx
\]

\[
\geq \frac{a \nu}{2} \int \sigma^{-\nu-1} \left( \mu \phi_x^2 + \phi_x^2 \right) dx - C(\varepsilon + \delta) \| \sigma^{-\frac{2}{3}} (\phi, \bar{\phi}_x) \|^2, \quad (4.48)
\]

\[
- a \nu \int \sigma^{-\nu-1} \left( - \rho \rho \phi_t \psi - \bar{n} \bar{n} \bar{\phi}_t \bar{\psi} \right) dx
\]

\[
\geq - C \varepsilon \sigma^{-\frac{2}{3}} (\phi, \bar{\phi}_x) \|^2 - C \| \sigma^{-\frac{2}{3}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 - C \| \sigma^{-\frac{2}{3}} (\psi, \bar{\psi}) \|^2, \quad (4.49)
\]
Finally, the substitution of \((4.47)-(4.56)\) into \((4.46)\) for \(\delta\) and \(\varepsilon\) small enough leads to that
\[
d\frac{d}{dt} \int \sigma^{-\frac{\nu}{2}} \left[ \frac{\mu_2}{2} \sigma_{\phi_x}^2 + \frac{\mu_2}{2} \sigma_{\phi_x}^2 \right] + \frac{A_1 \gamma_1^2}{2} \lVert \sigma^{-\frac{\nu}{2}} \phi_x \rVert^2 + \frac{A_2 \gamma_2^2}{2} \lVert \sigma^{-\frac{\nu}{2}} \phi_x \rVert^2 \leq C(\varepsilon + \delta) \lVert \sigma^{-\frac{\nu}{2}} (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \rVert^2 + C \lVert \sigma^{-\frac{\nu}{2}} (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \rVert^2.
\]
Multiplying \((4.57)\) by \((1 + \delta \tau)^{\xi}\) and integrating the resulted equation in \(\tau\) over \([0, t]\), and using Cauchy-Schwarz inequality and Lemma 4.8, we obtain the desired estimate \((4.45)\). The proof of Lemma 4.9 is completed. \qed

**Lemma 4.10.** Under the same conditions in Proposition 4.5, then the solution \((\phi, \psi, \tilde{\phi}, \tilde{\psi})\) to the IBVP \((3.3)-(3.7)\) satisfies for \(t \in [0, T]\) that
\[
(1 + \delta t)^{\xi} \lVert \sigma^{-\frac{\nu}{2}} (\psi_x, \tilde{\psi}_x) \rVert^2 + \int_0^t (1 + \delta \tau)^{\xi} \lVert \sigma^{-\frac{\nu}{2}} (\psi_{xx}, \tilde{\psi}_{xx}) \rVert^2 d\tau \leq C \lVert \sigma^{-\frac{\nu}{2}} (\phi_0, \psi_0, \phi_{0x}, \psi_{0x}, \tilde{\phi}_0, \tilde{\psi}_0) \rVert^2
\]
\[
+ C \delta \xi \int_0^t (1 + \delta \tau)^{\xi - 1} \lVert \sigma^{-\frac{\nu}{2}} (\phi, \psi, \phi_x, \psi_x, \tilde{\phi}, \tilde{\psi}, \phi_{xx}, \psi_{xx}) \rVert^2 d\tau,
\]
with \(\xi \geq 0\).

**Proof.** Multiplying \((3.3)\) by \(-\sigma^{-\nu} \psi_{xx}\), \((3.3)\) by \(-\sigma^{-\nu} \tilde{\psi}_{xx}\) respectively with the function \(\sigma\) satisfying \((1.14)\) and \((2.28)\), then adding them together and integrating the resulted equation in \(x\) over \(\mathbb{R}_+\) lead to
\[
\frac{d}{dt} \int \sigma^{-\nu}(\frac{\psi^2}{2} + \tilde{\psi}^2) dx - \nu \int \sigma^{-(\nu - 1)}(\psi \psi_x + \psi \tilde{\psi} \psi) dx + \int \sigma^{-\nu}(\frac{\mu}{\rho} \psi^2_{xx} + \tilde{\psi}^2_{xx}) dx = \sum_{i=1}^3 K_i,
\]

\[
\int \sigma^{-\nu}(\frac{\psi^2}{2} + \tilde{\psi}^2) dx - \nu \int \sigma^{-(\nu - 1)}(\psi \psi_x + \psi \tilde{\psi} \psi) dx + \int \sigma^{-\nu}(\frac{\mu}{\rho} \psi^2_{xx} + \tilde{\psi}^2_{xx}) dx = \sum_{i=1}^3 K_i,
\]

\[
(4.59)
\]
where

\[ K_1 = \int \sigma^{-\nu} \left[ u \psi_x \psi_{xx} - \frac{n}{\rho} (\bar{\psi} - \psi) \psi_{xx} + \frac{p'_1(\rho)}{\rho} \phi_x \psi_{xx} + \nu \phi_x \bar{\psi}_{xx} + (\bar{\psi} - \psi) \bar{\psi}_{xx} + \frac{p'_2(n)}{n} \bar{\phi}_x \bar{\psi}_{xx} \right] dx, \]

\[ K_2 = \int \sigma^{-\nu} \left[ \bar{u} \psi_x \psi_{xx} - \frac{1}{\rho} \psi_{xx} + \left( \frac{p'_1(\rho)}{\rho} - \frac{p'_2(n)}{n} \right) \psi_{xx} \bar{\phi}_x - \frac{n}{\rho} (\bar{\psi} - \bar{u}) \psi_{xx} + \bar{u} \bar{\psi}_{xx} \right] dx, \]

\[ + \left( \bar{n} \bar{v}_x \right) \left[ \frac{1}{n} \bar{\psi}_{xx} + \left( \frac{p'_2(n)}{n} - \frac{p'_2(n)}{n} \right) \bar{\psi}_{xx} \bar{n}_x - \frac{\bar{n}}{n} \bar{\psi}_{xx} \bar{n}_x - \frac{\bar{\phi} \bar{v}_x}{n} \bar{\psi}_{xx} \right] dx, \]

\[ K_3 = - \int \sigma^{-\nu} \left[ \frac{\bar{\phi} \bar{v}_x}{n} \bar{\psi}_{xx} + \left( \frac{1}{n} - \frac{1}{n} \bar{\psi}_x \bar{\psi}_{xx} \right) dx. \]

First, we estimate terms in the left side of (4.59). The second term is estimated as follows:

\[ a \nu \int \sigma^{-\nu-1} (\psi \psi_x + \bar{\psi} \bar{\psi}_x) dx \]

\[ \leq C \delta \left\| \sigma^{-\frac{\nu}{2}} (\psi_{xx}, \bar{\psi}_{xx}) \right\|^2 + C \delta \left\| \sigma^{-\frac{\nu}{2}} (\psi_{xx}, \bar{\phi}_x, \bar{\psi}_x, \bar{\psi} - \psi) \right\|^2 + C \delta \left\| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|^2, \quad (4.60) \]

where we have used (1.12), (3.3)2, (3.3)4, (4.23) and Cauchy-Schwarz inequality.

With the help of \( \delta \) and \( \varepsilon \) small enough, the third term is estimated as follows:

\[ \int \sigma^{-\nu} \left( \frac{\mu}{\rho} \psi^2_{xx} + \bar{\psi}^2_{xx} \right) dx \geq \frac{\mu}{2 \rho_+} \left( \| \sigma^{-\frac{\nu}{2}} (\psi_{xx}) \|^2 + \| \sigma^{-\frac{\nu}{2}} (\bar{\psi}_{xx}) \|^2 \right). \quad (4.61) \]

We turn to estimate terms on the right hand side of (4.59). With the help of (1.12), (4.23) and Cauchy-Schwarz inequality, we obtain

\[ |K_1| \leq \frac{\mu}{8 \rho_+} \left\| \sigma^{-\frac{\nu}{2}} (\psi_{xx}) \right\|^2 + \frac{1}{8} \left\| \sigma^{-\frac{\nu}{2}} (\psi_{xx}) \right\|^2 + C \left\| \sigma^{-\frac{\nu}{2}} (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \bar{\psi} - \psi) \right\|^2, \quad (4.62) \]

\[ |K_2| \leq C \delta \left\| \sigma^{-\frac{\nu}{2}} (\psi_{xx}, \bar{\psi}_{xx}) \right\|^2 + C \delta \left\| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|^2, \quad (4.63) \]

\[ |K_3| \leq C (\varepsilon + \delta) \left\| \sigma^{-\frac{\nu}{2}} (\bar{\psi}_{xx}) \right\|^2. \quad (4.64) \]

Finally, we substitute (4.60)-(4.64) into (4.59) to gain under the condition \( \delta \) and \( \varepsilon \) small enough that

\[ \frac{d}{dt} \int \sigma^{-\nu} \left( \frac{\psi^2_{xx}}{2} + \bar{\psi}^2_{xx} \right) dx + \frac{\mu}{4 \rho_+} \left\| \sigma^{-\frac{\nu}{2}} (\psi_{xx}) \right\|^2 + \frac{1}{4} \left\| \sigma^{-\frac{\nu}{2}} (\bar{\psi}_{xx}) \right\|^2 \]

\[ \leq C \left\| \sigma^{-\frac{\nu}{2}} (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \bar{\psi} - \psi) \right\|^2 + C \delta \left\| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|^2. \quad (4.65) \]

Multiplying (4.65) by \((1 + \delta \tau)^{\xi}\) and integrating the resulted inequality in \( \tau \) over \([0, t] \), and using Lemmas 4.8-4.9 and the smallness of \( \delta \) and \( \varepsilon \), we obtain (4.58). The proof of Lemma 4.10 is completed. \( \square \)

**Proof of Proposition 4.5** With the help of Lemmas 4.8-4.10, it holds for \( \delta \) and \( \varepsilon \) suitably small that

\[ (1 + \delta \tau)^{\xi} \left\| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|^2_1 + \int_0^t (1 + \delta \tau)^{\xi} \left\| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|^2_2 d\tau + \int_0^t (1 + \delta \tau)^{\xi} \left\| \sigma^{-\frac{\nu}{2}} (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \right\|^2_2 d\tau + C \int_0^t (1 + \delta \tau)^{\xi} \left\| \sigma^{-\frac{\nu}{2}} (\psi_{xx}, \bar{\psi}_{xx}, \bar{\psi} - \psi) \right\|^2_2 d\tau + C \left( \frac{1}{\delta^2} \right) \int_0^t (1 + \delta \tau)^{\xi} (\phi^2(t, 0) + \bar{\phi}^2(t, 0)) d\tau \]

\[ \leq C \left\| \sigma^{-\frac{\nu}{2}} (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \right\|^2_1 + C \delta \xi \int_0^t (1 + \delta \tau)^{\xi-1} \left\| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|^2_2 d\tau, \quad (4.66) \]
where $C > 0$ is a positive constant independent of $T$, $\nu$ and $\xi$. Applying similar induction arguments as in [3,14,25] to (4.66), we have

\begin{equation}
(1 + \delta t)\frac{\lambda}{2} + \beta \|\sigma^{-\frac{1}{2}}(\phi, \psi, \phi_x, \psi_x, \tilde{\phi}, \tilde{\psi}, \tilde{\phi}_x, \tilde{\psi}_x)\|^2 + \int_{0}^{t}(1 + \delta \tau)\frac{\lambda}{2} + \beta \|\sigma^{-\frac{1}{2}}(\phi, \tilde{\phi}, \psi, \tilde{\psi})\|^2 \\
+ \int_{0}^{t}(1 + \delta \tau)\frac{\lambda}{2} + \beta \|\sigma^{-\frac{1}{2}}(\phi_x, \psi_x, \tilde{\phi}_x, \tilde{\psi}_x, \bar{\psi}, \bar{\psi}_x)\|^2 \\
\leq C(1 + \delta t)^{\beta} \|\sigma^{-\frac{1}{2}}(\phi_0, \psi_0, \phi_{0x}, \bar{\phi}_0, \bar{\psi}_0, \phi_{0x}, \bar{\psi}_{0x})\|^2.
\end{equation}

(4.67)

for $\beta > 0$, which implies

\begin{equation}
\|\sigma^{-\frac{1}{2}}(\phi, \psi, \phi_x, \psi_x, \tilde{\phi}, \tilde{\psi}, \tilde{\phi}_x, \tilde{\psi}_x)(t)\|^2 \leq C(1 + \delta t)^{-\frac{\lambda}{1-\nu}} \|\sigma^{-\frac{1}{2}}(\phi_0, \psi_0, \phi_{0x}, \bar{\phi}_0, \bar{\psi}_0, \phi_{0x}, \bar{\psi}_{0x})\|^2.
\end{equation}

(4.68)

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Existence and Nonlinear Stability of Steady-States to Outflow Problem for the Full Two-Phase Flow

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Abstract

The outflow problem for the viscous full two-phase flow model in a half line is investigated in the present paper. The existence, uniqueness and the nonlinear time stability of the steady-state are shown corresponding to supersonic, sonic or subsonic flow at far field. Furthermore, we can obtain either the exponential time decay rate for supersonic state or the algebraic time decay rate for supersonic state and sonic state in the weighted Sobolev space.

Key words. Full two-phase flow, outflow problem, stationary solution, nonlinear stability.

1 Introduction

Two-phase flow models play important roles in applied scientific areas, for instance, nuclear, engines, chemical engineering, medicine, oil-gas, fluidization, waste water treatment, liquid crystals, lubrication, biomedical flows \([8,13,22]\), etc. In this paper, we consider the full two-phase flow model which can be formally obtained from a Vlasov-Fokker-Planck equation coupled with the compressible Navier-Stokes equations through the Chapman-Enskog expansion \([20]\).

We are concerned with the initial-boundary value problem (IBVP) for the full two-phase flow model as follows:

\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + [\rho u^2 + p_1(\rho)]_x &= (\mu u)_x + n(v - u), \\
n_t + (nv)_x &= 0, \\
(nv)_t + [nv^2 + p_2(n)]_x &= (nv)_x - n(v - u),
\end{aligned}
\]

where \( x > 0, \ t > 0, \rho, u, n, p_1, p_2, \mu, \) and \( v \) are positive bounded functions, \( \mu \) is a positive constant.

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where \( \rho > 0 \) and \( n > 0 \) stand for the densities of two fluids, and \( u, v \) are the velocities of two fluids. The constant \( \mu > 0 \) is the viscosity coefficient and the pressure-density functions take the form

\[
p_1(\rho) = A_1 \rho^\gamma, \quad p_2(n) = A_2 n^\alpha
\]

with \( A_1 > 0, A_2 > 0, \gamma \geq 1 \) and \( \alpha \geq 1 \). The initial data are given by

\[
(\rho, u, n, v)(0, x) = (\rho_0, u_0, n_0, v_0)(x), \quad \inf_{x \in \mathbb{R}^+} \rho_0(x) > 0, \quad \inf_{x \in \mathbb{R}^+} n_0(x) > 0,
\]

and the outflow boundary condition is imposed

\[
(u, v)(t, 0) = (u_-, u_-), \quad u_- < 0,
\]

where \( \rho_+ > 0, n_+ > 0, u_+ \) and \( u_- < 0 \) are constants.

There are important progress made recently about the global existence and time decay rates of solutions for the two-phase flow models \([3, 5–8, 14, 29, 31, 32, 34–36, 39]\). For instance, the global existence of three-dimensional strong solutions to Cauchy problem for Euler-NS model was proved in \([3]\). Later, time decay rates of the three-dimensional strong solution to Cauchy problem for Euler-NS model were obtained in \([31]\). The global existence and time decay rates of three-dimensional strong solutions to the initial boundary problem for Euler-NS model was proved in \([14]\). The global well-posedness of strong solutions to Cauchy problem in Besov space or Sobolev space was obtained in \([8, 32, 39]\) for drift-flux models. The existence of multi-dimensional global weak solutions for drift-flux models to both Cauchy problem \([6, 34]\) and the initial boundary value problem \([29]\) was proved. The global weak solution to the free boundary value problem in 1D was shown in \([5, 7, 35, 36]\) for drift-flux models.

There are important progress made recently about the existence and nonlinear stability of steady-states for the inflow/outflow problem of the compressible Navier-Stokes system \([2, 4, 9–12, 15, 18, 19, 21, 23, 25–28, 30, 33, 38, 40]\). Among them, the existence and nonlinear stability of steady-states to the inflow problem of Navier-Stokes equations in a half line was obtained in \([10, 11, 21, 25, 30]\) for both subsonic and sonic cases. Concerning with the outflow problem of Navier-Stokes equations, the nonlinear stability of the steady-states for both supersonic and sonic cases in a half line was proved in \([12, 19, 26]\), and the stability of the multidimensional planar stationary solution for both supersonic and sonic cases was obtained in \([15]\).

The existence and time decay rates of the solution to the outflow problem for Navier-Stokes equations in a half line were shown in \([27, 28]\) for supersonic and sonic cases. The existence and nonlinear stability of steady-states to the outflow problem for full compressible Navier-Stokes equations \([2, 18]\) were shown for supersonic, sonic and subsonic cases, and the time decay rates for the supersonic case or sonic case were obtained.
However, there are also important results about the outflow/inflow problem for two-phase flow models except for [4, 9, 33, 38, 40]. Among them, the existence of steady-states for supersonic and sonic cases, and the nonlinear stability of steady-states for the supersonic case to the outflow problem were gained in [38] concerning with the drift-flux model. For the two-fluid Navier-Stokes-Poisson system, the existence and nonlinear stability of steady-states to the outflow problem for supersonic and sonic cases in [4, 33, 40] were proved. The existence and nonlinear stability of steady-states to inflow problem of the two-fluid Navier-Stokes-Poisson system for both subsonic and sonic cases were shown in [9].

However, there is no result about the nonlinear time stability and time decay rates of steady-states for the outflow problem (1.1)-(1.5). The main purpose of this paper is to prove the nonlinear time stability of steady-states for supersonic, sonic and subsonic cases, and obtain either the exponential time decay rate for the supersonic flow or the algebraic time decay rate for both supersonic and sonic flows.

The steady-state $(\tilde{\rho}, \tilde{\mu}, \tilde{n}, \tilde{v})(x)$ to the problem (1.1)-(1.5) satisfies the following system

\[
\begin{aligned}
(\tilde{\rho}u)_x &= 0, \\
[p\tilde{u}^2 + p_1(\tilde{\rho})]_x &= (\mu\tilde{u}_x)_x + \tilde{n}(\tilde{v} - \tilde{u}), \\
(n\tilde{v})_x &= 0, \\
[p\tilde{v}^2 + p_2(\tilde{n})]_x &= (\tilde{n}\tilde{v})_x - \tilde{n}(\tilde{v} - \tilde{u}), & x > 0,
\end{aligned}
\] (1.6)

for the boundary conditions and spatial far field conditions

\[
(\tilde{u}, \tilde{v})(0) = (u_-, u_-), \quad \lim_{x \to +\infty} (\tilde{\rho}, \tilde{\mu}, \tilde{n}, \tilde{v})(x) = (\rho_+, u_+, n_+, u_+). \quad \inf_{x \in \mathbb{R}_+} \tilde{n}(x) > 0, \quad \inf_{x \in \mathbb{R}_+} \tilde{n}(x) > 0, \quad (1.7)
\]

Integrating (1.6)₁ and (1.6)₃ over $(x, +\infty)$ leads to

\[
\tilde{u} = \frac{\rho_+}{\tilde{\rho}} u_+, \quad \tilde{v} = \frac{n_+}{\tilde{n}} u_+, \quad (1.8)
\]

which implies the following relationship

\[
u_+ = \frac{\tilde{n}(0)}{n_+} u_- = \frac{\tilde{\rho}(0)}{\rho_+} u_- < 0 \quad (1.9)
\]
is necessary for the existence of the steady-state to BVP (1.6)-(1.7).

Define the generalized Mach number $M_+$ and the generalized sound speed $c_+$ as below

\[
M_+ := \frac{|u_+|}{c_+}, \quad c_+ := \left(\frac{A_1 \gamma \rho_+^\gamma + A_2 \alpha n_+^\alpha}{\rho_+ + n_+}\right)^{1/2}, \quad (1.10)
\]

Then, we have the following results about the existence and uniqueness of the steady-state.

**Theorem 1.1.** Let $\delta := |u_- - u_+| > 0$ and $u_+ < 0$ hold. Then there exists a set $\Omega_- \subset \mathbb{R}_-$ such that if $u_- \in \Omega_-$ and $\delta$ sufficiently small, there exists a unique strong solution $(\tilde{\rho}, \tilde{\mu}, \tilde{n}, \tilde{v})(x)$ to the boundary value problem (1.6)-(1.7) which satisfies either for $M_+ \neq 1$ that

\[
|\partial_x^k (\tilde{\rho} - \rho_+, \tilde{u} - u_+, \tilde{n} - n_+, \tilde{v} - u_+)| \leq C_1 \delta e^{-c_0 x}, \quad k = 0, 1, 2, 3, \quad (1.11)
\]
or for $M_+ = 1$ that
\[ |\partial_x^k (\tilde{\rho} - \rho_+, \tilde{u} - u_+, \tilde{n} - n_+, \tilde{v} - u_+)| \leq C_2 \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad k = 0, 1, 2, 3, \quad (1.12) \]

and
\[ (\tilde{u}_x, \tilde{v}_x) = (a\sigma^2(x), a\sigma^2(x)) + O(|\sigma(x)|^3), \quad (1.13) \]

where $\sigma(x)$ is a smooth function satisfying $\sigma_x = -a\sigma^2 + O(|\sigma|^3)$ and
\[ c_1 \frac{\delta}{1 + \delta x} \leq \sigma(x) \leq C_3 \frac{\delta}{1 + \delta x}, \quad |\partial_x^k \sigma(x)| \leq C_3 \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad k = 0, 1, 2, 3,..., \quad (1.14) \]

and $C_i > 0$, $i = 1, 2, 3$, $c_0 > 0$ $c_1 > 0$ and $a > 0$ are positive constants.

Then, we have the nonlinear stability of the steady-state to the IBVP (1.1)-(1.5) for supersonic, sonic and subsonic cases.

**Theorem 1.2.** Let the same conditions in Theorem 1.1 hold. In addition, assume that $|p'_1(\rho_+)-p'_2(n_+)| \leq \sqrt{2}|u_+| \min \{ (1 + \frac{\rho}{n_+})[(\gamma - 1)p_1'(\rho_+)]^\frac{1}{2}, (1 + \frac{\rho}{\rho_+})[(\alpha - 1)p_2'(n_+)]^\frac{1}{2} \}$ hold if $M_+ = 1$. Then, there exists a small positive constant $\varepsilon_0 > 0$ such that if
\[ \|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v})\|_{H^1 + \delta} \leq \varepsilon_0, \quad (1.15) \]

the initial boundary value problem (1.1)-(1.5) has a unique global solution $(\rho, u, n, v)(t, x)$ satisfying
\[
\begin{align*}
(\rho - \tilde{\rho}, u - \tilde{u}, n - \tilde{n}, v - \tilde{v}) & \in C([0, +\infty); H^1), \\
(\rho - \tilde{\rho}, n - \tilde{n}) & \in L^2([0, +\infty); L^2), \\
u - \tilde{u}, v - \tilde{v}) & \in L^2([0, +\infty); H^1),
\end{align*}
\]

and
\[ \lim_{t \to +\infty} \sup_{x \in \mathbb{R}_+} |(\rho - \tilde{\rho}, u - \tilde{u}, n - \tilde{n}, v - \tilde{v})(t, x)| = 0. \quad (1.16) \]

In addition, we have the time convergence rates of the steady-state to the IBVP (1.1)-(1.5) for both supersonic and sonic cases.

**Theorem 1.3.** Assume that the same conditions in Theorem 1.1 hold. Then, the following results hold.
\[(i) \text{ For } M_+ > 1 \text{ and } \lambda > 0, \text{ if the initial data satisfy } (1 + x)^\frac{\gamma}{2}(\rho_0 - \tilde{\rho}), (1 + x)^\frac{\gamma}{2}(u_0 - \tilde{u}), (1 + x)^\frac{\gamma}{2}(n_0 - \tilde{n}), (1 + x)^\frac{\gamma}{2}(v_0 - \tilde{v})) \in L^2(\mathbb{R}_+) \text{ and} \]
\[ \|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v})\|_{H^1 + \delta} \leq \varepsilon_0, \quad (1.17) \]

for a small positive constant $\varepsilon_0 > 0$, then the solution $(\rho, u, n, v)(t, x)$ to the IBVP (1.1)-(1.5) satisfies
\[ \|(\rho - \tilde{\rho}, u - \tilde{u}, n - \tilde{n}, v - \tilde{v})(t)\|_{L^\infty} \leq C_4 \delta_0 (1 + t)^{-\frac{\lambda}{2}}, \quad (1.18) \]
where $C_4 > 0$ is a positive constant independent of time and $\delta_0 := \| (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}) \|_{H^1} + \| (1 + x)^\frac{3}{4} (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}) \|_{L^2}$ is a non-negative constant.

(ii) For $M_+ = 1$, $1 \leq \lambda < 2 + \sqrt{8 + \frac{1}{1 + b^2}}$ and $b := \frac{\rho_+ (\kappa_+^2 - p_+^{(\rho_+)}))}{|w_+| \sqrt{(\mu + \nu_+^2)} n_+}$, if for $\nu \in (0, \lambda]$, there exists a small positive constant $\varepsilon_0 > 0$ such that if

$$\| \sigma^\frac{2}{\lambda} (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}) \|_{H^1} + \delta^\frac{2}{\lambda} \leq \varepsilon_0,$$

(1.19) then the IBVP (1.1)-(1.5) has a unique global solution $(\rho, u, n, v)(t, x)$ satisfying

$$\left\{ \begin{array}{l}
(\sigma^\frac{2}{\lambda} (\rho - \tilde{\rho}), \sigma^\frac{2}{\lambda} (u - \tilde{u}), \sigma^\frac{2}{\lambda} (n - \tilde{n}), \sigma^\frac{2}{\lambda} (v - \tilde{v})) \in C([0, +\infty); H^1), \\
(\sigma^\frac{2}{\lambda} (\rho - \tilde{\rho}), \sigma^\frac{2}{\lambda} (n - \tilde{n})) \in L^2([0, +\infty); L^2), \\
(\sigma^\frac{2}{\lambda} (u - \tilde{u}), \sigma^\frac{2}{\lambda} (v - \tilde{v})) \in L^2([0, +\infty); H^1)
\end{array} \right.$$

(1.20)

and

$$\| \sigma^\frac{2}{\lambda} (\rho - \tilde{\rho}, u - \tilde{u}, n - \tilde{n}, v - \tilde{v})(t) \|_{H^1} \leq C_5 \delta (1 + t)^{-\frac{\lambda}{2\lambda}},$$

(1.21)

where $\sigma(x)$ is defined by (1.14) satisfying (2.28), $C_5 > 0$ is a positive constant independent of time, and $\delta_1 := \| \sigma^\frac{2}{\lambda} (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}) \|_{H^1}$ is a non-negative constant.

Remark 1.1. For $M_+ > 1$, the nonlinear stability of the steady-state $(\tilde{\rho}, \tilde{u}, \tilde{n}, \tilde{v})$ for the outflow problem (1.1)-(1.5) with an exponential time decay rate can be gained. Indeed, assume that $M_+ > 1$, $u_+ < 0$ and a certain positive constant $\lambda > 0$ hold. For a certain positive constant $\kappa \in (0, \lambda]$, there exists a small positive constant $\varepsilon_0 > 0$ such that if $(e^{\frac{2}{\lambda} x} (n_0 - \tilde{n}, \rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v})) \in L^2(\mathbb{R}_+)$ and

$$\| (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}) \|_{H^1} + \delta \leq \varepsilon_0,$$

then the solution $(\rho, u, n, v)(t, x)$ to the IBVP (1.1)-(1.5) satisfies

$$\| (\rho - \tilde{\rho}, u - \tilde{u}, n - \tilde{n}, v - \tilde{v})(t) \|_{H^1} \leq C_5 \delta e^{-\frac{\lambda}{2\lambda} t},$$

(1.22)

where $C_5 > 0$ and $\kappa_1 \ll \kappa$ are positive constants independent of time, and $\delta_2 := \| (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}) \|_{H^1} + \| e^{\frac{2}{\lambda} x} (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}) \|_{L^2}$ is a non-negative constant.

The proof of (1.22) can be obtained by similar arguments as (1.18). The details are omitted.

We explain the strategies to prove above main Theorems. The system (1.1)-(1.5) can be viewed as the two Navier-Stokes systems coupled by the drag force. On the one hand, compared with the single Navier-Stokes system, it is more difficult to prove the existence of the steady-states to IBVP (1.1)-(1.5). Indeed, it is easy to check that there is no trivial steady-states which are constructed by steady-states to the Navier-Stokes system compared with [4.9,33,40]. In order to use the center manifold theory [1], we need to introduce the new variable $\tilde{w} := \tilde{u}_x$, get the estimates $|\tilde{u}_x(0)| \leq C |u_+ - u_+|$ in Lemma 2.1, and reformulate the system (1.6) into the first-order autonomous system (2.9). On the other hand, the
linearized reformulated system has at least one eigenvalue with negative real part due to the effect of the drag force. Thus, we obtain the existence of the steady-states for supersonic, sonic and subsonic cases in Theorem 1.1. The steady-states exist for the subsonic case, which is different from the single Navier-Stokes system [19].

To prove the nonlinear stability of the steady-states in Theorem 1.2, it is necessary to obtain the uniform estimates on the perturbations around steady-states \((\phi, \psi, \tilde{\phi}, \tilde{\psi})\). In fact, compared with the single Navier-Stokes system, it is more difficult to get the uniform estimates for sonic flow due to the drag force. Indeed, the bad term can be controlled by the good term

\[
\int R_2 dx = \int \phi \psi \left( \frac{\rho \bar{u}^2 + \rho_1(\bar{\rho})}{\bar{\rho}} + \phi \bar{v} \frac{\bar{u}^2 + p_2(\bar{n})}{\bar{n}} \right) dx = \int \phi \psi \frac{\mu \bar{u}_x}{\bar{\rho}} + \left( \bar{\rho} \bar{u} \bar{v} - \bar{n} (\bar{u} - \bar{\rho}) \right) - \phi \psi \frac{\bar{v} \bar{u}_x}{\bar{n}} \right) dx
\]

under the assumption \(|p_1'(\rho_+) - p_2'(n_+)| \leq \sqrt{2} \min\{1 + \frac{\nu}{\nu + \mu}, ((\gamma - 1)p_1'(\rho_+))^{\frac{\gamma}{\gamma - 1}}, (1 + (\frac{n}{\rho_+})^{\frac{\gamma}{\gamma - 1}})((\gamma - 1)p_2'(n_+))^{\frac{\gamma}{2}}\}.

With the weighted energy method, we get the exponential or algebraic time decay rates for the supersonic case. For sonic case, we obtain the existence and the algebraic time decay rates of the solution to IBVP (1.1)-(1.5) without the restriction \(|p_1'(\rho_+) - p_2'(n_+)| \leq \sqrt{2} \min\{1 + \frac{\nu}{\nu + \mu}, ((\gamma - 1)p_1'(\rho_+))^{\frac{\gamma}{\gamma - 1}}, (1 + (\frac{n}{\rho_+})^{\frac{\gamma}{\gamma - 1}})((\gamma - 1)p_2'(n_+))^{\frac{\gamma}{2}}\} by the weighted energy method. It is difficult to obtain the convergence rate for the supersonic case by the ideas used in compressible Navier-Stokes equations [28] except for \(p_1'(\rho_+) = p_2'(n_+)\).

And for the sonic case, the ideas in compressible Navier-Stokes equations [28] can't work except for the special case \(n_+ = \rho_+, \mu = n_+, p_1'(\rho_+) = p_2'(n_+)\) and \(\gamma = \alpha\). To get decay rates for both supersonic and sonic case, we make full use of the drag force term. Indeed, one important observation is decomposing \(\psi\) as

\[
\psi = \tilde{\psi} + (\psi - \tilde{\psi})
\]
in basic energy estimates for both supersonic and sonic cases. For the sonic case, inspired by [27, 28], we take the linear coordinate transformation

\[
\begin{pmatrix}
\frac{\phi}{\tilde{\phi}} \\
\frac{\psi}{\tilde{\psi}}
\end{pmatrix} = \begin{pmatrix}
\rho \\
\tilde{n}
\end{pmatrix}
\]

with an invertible matrix \(P\). With the help of \(\psi = \tilde{\psi} + (\psi - \tilde{\psi})\), the coordinate transformation and (1.13), we can get the good term for \(\nu \in (0, 3)\) that

\[
\frac{A_1 \gamma (\gamma + 1) \rho_+^\gamma + A_2 \alpha (\alpha + 1) \nu_+^{\alpha + 1}}{2|\nu_+|^2} \left[ 1 + \nu - \frac{\nu (\nu - 1)}{2(1 + b^2)} \right] \int \sigma^{-(\nu - 2)} \tilde{\psi}^2 dx + c \int \sigma^{-(\nu - 1)} (\tilde{\rho}^2 + \tilde{n}^2) dx,
\]

to control the bad term

\[
\int \sigma^{-\nu} R_2 = \int \sigma^{-\nu} \left[ \phi \psi \frac{(\rho \bar{u}^2 + \rho_1(\bar{\rho})}{\bar{\rho}} + \phi \bar{v} \frac{\bar{u}^2 + p_2(\bar{n})}{\bar{n}} \right] dx,
\]

(1.23)
where the constants $a, b$ are defined by (2.22) and the space weight function $\sigma \geq 0$ satisfies (1.14) and (2.28). However, it is more difficult to get decay rates for $p_1'(\rho_+) \neq p_2'(\rho_+)$. due to the term

$$a \nu \int \sigma^{-(\nu-1)}[-\rho_+ u_+ \tilde{\psi}(\psi - \bar{\psi}) - A_1 \gamma \rho_+^{\gamma-1} \phi(\psi - \bar{\psi})]dx$$

$$\geq a \nu \int \sigma^{-(\nu-1)} \frac{\rho_+ (u_+^2 - A_1 \gamma \rho_+^{\gamma-1})}{|u_+|} \bar{\phi}(\psi - \bar{\psi})dx - C \delta^2 \sigma^{-(\nu-2)}(\bar{\rho}^2 + \bar{n}^2)dx + \int \sigma^{-(\nu-2)} \bar{\phi}^2 dx,$$

which is caused by the decomposition $\psi = \bar{\psi} + (\psi - \bar{\psi})$. To overcome the difficulty, it is necessary to control the bad term by using good terms for $\nu \in (0, 3]$ that

$$\frac{a_1 \gamma (\gamma + 1) \rho_+^{\gamma+1} + A_2 \sigma (\alpha + 1) n_+^{\alpha+1}}{2|u_+|^2} \left[ 1 + \nu - \frac{\nu(\nu-1)}{2(1+b^2)} \right] \int \sigma^{-(\nu-2)} \bar{\phi}^2 dx + n_+ \int \sigma^{-\nu}(\psi - \bar{\psi}) dx$$

$$+ c \int \sigma^{-(\nu-1)}(\bar{\rho}^2 + \bar{n}^2)dx,$$

where $a > 0, b \geq 0$ are constants defined by (2.22), and the space weight function $\sigma \geq 0$ satisfies (1.14) and (2.28).

**Notation.** We denote by $\| \cdot \|_{L^p}$ the norm of the usual Lebesgue space $L^p = L^p(\mathbb{R}_+), 1 \leq p \leq \infty$. And if $p = 2$, we write $\| \cdot \|_{L^2(\mathbb{R}_+)} = \| \cdot \|$. $H^s(\mathbb{R}_+)$ stands for the standard $s$-th Sobolev space over $\mathbb{R}_+$ equipped with its norm

$$\|f\|_{H^s(\mathbb{R}_+)} = \|f\|_s = \left( \sum_{i=0}^s \| \partial^i f \|^2 \right)^{\frac{1}{2}}.$$

$C([0, T]; H^1(\mathbb{R}_+))$ represents the space of continuous functions on the interval $[0, T]$ with values in $H^s(\mathbb{R}_+)$. $L^2([0, T]; B)$ denotes the space of $L^2$ functions on the interval $[0, T]$ with values in the Banach space $B$. For a scalar function $W(x) > 0$, the weighted $L^2(\mathbb{R}_+)$ and $H^1(\mathbb{R}_+)$ spaces are defined as follows:

$$L_W^2(\mathbb{R}_+) := \left\{ f \in L^2(\mathbb{R}_+) \mid \| f \|_{L^2_W} := \left( \int_{\mathbb{R}_+} W(x) f^2 dx \right)^{\frac{1}{2}} < +\infty \right\},$$

$$H_W^1(\mathbb{R}_+) := \left\{ f \in H^1(\mathbb{R}_+) \mid \| f \|_{H_W^1} := \left( \sum_{i=0}^1 \| \partial_i f \|_{L^2_W}^2 \right)^{\frac{1}{2}} < +\infty \right\}.$$  

For a scalar function $W_{a, \nu} := (1 + x)^\nu$ with $\nu \geq 0$, we denote $\| f \|_{a, \nu} := \|(1 + x)^{\frac{\nu}{2}} f\|$.

The rest of this paper will be organized as follows. We prove the existence and uniqueness of the steady-states in Section 2, get the nonlinear stability of the steady-states in Section 3 for supersonic, sonic and subsonic cases, and obtain the convergence rates of the steady-states for supersonic flow in Subsection 4.1 and sonic flow in Subsection 4.2.

## 2 Existence of Stationary Solution

We prove Theorem 1.1 on the existence and uniqueness of the stationary solution to (1.6)-(1.7) with $u_+ < 0$ and $\delta$ sufficiently small as follows. First, to apply the center manifold theory [1], it is necessary to get the bounds of $\bar{u}_x(0)$ or $\bar{v}_x(0)$.  

7
Lemma 2.1. Assume that $u_+ < 0$ and $\delta = |u_- - u_+|$ hold with $\delta$ sufficiently small. Let $(\tilde{\rho}, \tilde{u}, \tilde{n}, \tilde{v})$ be the solution to the BVP (1.6)-(1.7). Then we have

$$|\tilde{u}_x(0)| \leq C|u_- - u_+|, \quad |\tilde{v}_x(0)| \leq C|u_- - u_+|,$$

where $C > 0$ is a positive constant.

Proof. Due to $\tilde{\rho} = \frac{\rho_+ u_+}{u_+ u_-}$ and $\tilde{n} = \frac{n_+ u_+}{u_+}$, we can obtain

$$
\begin{align*}
& (\rho_+ u_+ \tilde{u} + A_1 \rho_+^3 u_+^3 \tilde{u} - \gamma) = x = (\mu \tilde{u}_x)_x + \frac{n_+ u_+}{\bar{v}} (\bar{v} - \tilde{u}), \\
& (n_+ u_+ \tilde{v} + A_2 n_+^a u_+^a \tilde{v} - \alpha) = x = (n_+ u_+ \tilde{v}_x)_x - \frac{n_+ u_+}{\bar{v}} (\bar{v} - \tilde{u}).
\end{align*}
$$

(2.2)

Adding (2.2) to (2.2) and integrating the resulted equation over $(0, +\infty)$ lead to

$$
\mu \tilde{u}_x(0) + \frac{n_+ u_+}{u_+} \tilde{v}_x(0) = \frac{1}{\mu} [(\rho_+ + n_+) u_+^2 - (A_1 \gamma \rho_+^3 + A_2 n_+^a)] (u_- - u_+) + O(|u_- - u_+|^2)
$$

(2.3)

Multiplying (2.2) by $\tilde{u}$, (2.2) by $\tilde{v}$, respectively, then adding them together and integrating the resulted equation over $(0, +\infty)$, we can gain

$$
\int_0^{+\infty} (\mu \tilde{u}_x^2 + n_+ u_+ \tilde{v}_x^2) dx + \int_0^{+\infty} \frac{n_+ u_+}{\bar{v}} (\bar{v} - \tilde{u})^2 dx

= -[\mu \tilde{u}_x(0) + n_+ u_+ \tilde{v}_x(0)] + [(\rho_+ + n_+) u_+^2 - (A_1 \gamma \rho_+^3 + A_2 n_+^a)] (u_- - u_+) + O(|u_- - u_+|^2)

= O(|u_- - u_+|^2),
$$

(2.4)

where we have used (2.3). Multiplying (2.2) by $\frac{\tilde{u}}{\bar{v}}$ and then integrating the resulted equation over $(0, \infty)$ yield

$$
\int_0^{+\infty} \frac{n_+ u_+}{\bar{v}} (\bar{v} - \tilde{u}) \tilde{v}_x^2 dx = \int_0^{+\infty} n_+ u_+ \tilde{v}_x^2 dx + \int_0^{+\infty} n_+ u_+ \tilde{v}_x^2 dx

$$

(2.5)

We estimate terms in the right hand side of (2.5). Using $\inf_{x \in \mathbb{R}} \tilde{n} > 0$, $\alpha \geq 1$ and (2.3), we have

$$
\int_0^{+\infty} \frac{n_+ u_+}{\bar{v}} (\bar{v} - \tilde{u}) \tilde{v}_x^2 dx + \int_0^{+\infty} n_+ u_+ \tilde{v}_x^2 dx

\leq C \int_0^{+\infty} \frac{n_+ u_+}{\bar{v}} (\bar{v} - \tilde{u})^2 dx + \frac{1}{4} \int_0^{+\infty} A_2 n_+^a \frac{u_+^a}{\bar{v}^a+2} \tilde{v}_x^2 dx

\leq C|u_- - u_+|^2 + \frac{1}{4} \int_0^{+\infty} A_2 n_+^a \frac{u_+^a}{\bar{v}^a+2} \tilde{v}_x^2 dx.
$$

(2.6)

By using (2.5) and (2.6), we can get

$$
|\tilde{v}_x(0)| \leq C|u_- - u_+|, \quad |\tilde{u}_x(0)| \leq C|u_- - u_+|.
$$

(2.7)

□
Then, with the above lemma, we can prove Theorem 1.1. For (1.6)$_2$ and (1.6)$_4$, using $\bar{\rho} = \frac{\bar{\rho}_+ u_+}{\mu}$, \nolinebreak $\bar{n} = \frac{n_+ u_+}{\bar{\rho}}$, and integrating the summation of (1.6)$_2$, (1.6)$_4$ over $(x, +\infty)$, we have

$$\begin{align*}
\bar{v}_x &= \frac{\bar{v}}{n_+ u_+}[\rho_+ u_+ (\bar{u} - u_+) + A_1 \rho_+^{\gamma - 1} - 1] + n_+ u_+ (\bar{v} - u_+) + A_2 n_+^{\alpha} (\frac{u_+^{\alpha}}{\bar{\rho}_+^{\alpha}} - 1 - \mu \bar{u}_x], \\
\bar{u}_{xx} &= \frac{1}{\mu} [\rho_+ u_+ - A_1 \rho_+^{\gamma - 1} \bar{u}^{\gamma - 1}] u_x - n_+ u_+ (1 - \frac{\bar{u}}{\bar{v}}). 
\end{align*}$$

(2.8)

Define $\tilde{w} := \bar{u}_x$ and $\tilde{U} := (\bar{u}, \bar{w}, \bar{v})^T := (\bar{u} - u_+, \bar{w}, \bar{v} - u_+)^T$. The system (2.8) can be reformulated into the autonomous first-order system as follows

$$\begin{align*}
\begin{cases}
\tilde{U}_x = J_+ \tilde{U} + (0, \tilde{g}_2(\tilde{U}), \tilde{g}_3(\tilde{U}))^T, \\
\lim_{x \to \infty} \tilde{U}^T = (0, 0, 0),
\end{cases}
\end{align*}$$

(2.9)

where

$$J_+ = \begin{pmatrix}
0 & 1 & 0 \\
\frac{n_+}{\mu} u_+ & \frac{\rho_+ u_+^{\gamma - 1} - A_1 \rho_+^{\gamma - 1}}{n_+ u_+} & \frac{n_+}{\mu} \\
\frac{n_+}{\mu} & \frac{\mu_+}{n_+} & \frac{n_+ u_+^{\gamma - 1} - A_2 \alpha n_+^{\alpha}}{n_+ u_+}
\end{pmatrix},$$

(2.10)

$$\tilde{g}_2(\tilde{U}) = \frac{1}{2} \left(2 \frac{n_+}{\mu} \bar{u}^2 - \frac{n_+}{\mu} \bar{u} \bar{v} + 2 \frac{A_1(\gamma + 1) \rho_+^{\gamma - 1}}{\mu u_+^2}(\bar{w} \bar{u}) + O(|\bar{U}|^3),
\right.$$

(2.11)

$$\tilde{g}_3(\tilde{U}) = \frac{1}{2} \left(2 A_1(\gamma + 1) \rho_+^{\gamma - 1} \bar{u}^2 - 2 \frac{\rho_+ u_+^{\gamma - 1} - A_1 \gamma \rho_+^{\gamma - 1} \bar{v}}{n_+ u_+^2} \bar{v} \bar{u} + (2 \frac{n_+ u_+^{\gamma - 1} - A_2 \alpha n_+^{\alpha}}{n_+ u_+^2} \\
\right. + \frac{A_2 \alpha (\alpha + 1) n_+^{\alpha}}{n_+ u_+^2} \bar{v} \bar{u}^2 - 2 \frac{\mu}{n_+ u_+} \tilde{w} \tilde{v} + O(|\bar{U}|^3).
\right)$$

(2.12)

Three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of matrix $J_+$ satisfy

\nolinebreak

$$\begin{align*}
\lambda_1 \lambda_2 \lambda_3 &= -\frac{(n_+ + \rho_+) u_+^{\gamma - 1} - (A_1 \gamma \rho_+^{\gamma - 1} + A_2 \alpha n_+^{\alpha})}{\mu u_+}, \\
\lambda_1 + \lambda_2 + \lambda_3 &= \frac{\rho_+ u_+^{\gamma - 1} - A_1 \gamma \rho_+^{\gamma - 1}}{\mu u_+} + \frac{n_+ u_+^{\gamma - 1} - A_2 \alpha n_+^{\alpha}}{n_+ u_+}, \\
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 &= \frac{\rho_+ (u_+^{\gamma - 1} - A_1 \gamma \rho_+^{\gamma - 1})(u_+^{\gamma - 1} - A_2 \alpha n_+^{\alpha - 1})}{\mu u_+} - 1 - \frac{n_+}{\mu}.
\end{align*}$$

(2.13)

If $M_+ > 1$, it is easy to obtain $\lambda_1 \lambda_2 \lambda_3 > 0$ and $u_+^{\gamma - 1} > \min \{A_1 \gamma \rho_+^{\gamma - 1}, A_2 \alpha n_+^{\alpha - 1}\}$. Without loss of generality, we assume $A_1 \gamma \rho_+^{\gamma - 1} > A_2 \alpha n_+^{\alpha - 1}$. Moreover, we have

$$\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 > 0 \text{ for } u_+^{\gamma - 1} > A_1 \gamma \rho_+^{\gamma - 1}, \\
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 < 0 \text{ for } A_2 \alpha n_+^{\alpha - 1} - u_+^{\gamma - 1} \\
\end{align*}$$

(2.14)

which can imply Re$\lambda_1 < 0$, Re$\lambda_2 < 0$ and $\lambda_3 > 0$ for $M_+ > 1$. Using similar arguments, we have the following cases:

$$\begin{align*}
\begin{cases}
\text{if } M_+ > 1, \text{ then } \text{Re} \lambda_1 < 0, \text{Re} \lambda_2 < 0, \lambda_3 > 0, \\
\text{if } M_+ < 1, \text{ then } \text{Re} \lambda_1 > 0, \text{Re} \lambda_2 > 0, \lambda_3 < 0, \\
\text{if } M_+ = 1, \text{ then } \lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0.
\end{cases}
\end{align*}$$

(2.15)
Then, applying the center manifold theory [1], it is not difficult to show the case \( M_+ \neq 1 \) in Theorem 1.1 if \( \delta \) is small. Finally, we prove case \( M_+ = 1 \) in Theorem 1.1 which implies \( \lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0 \). The eigenvectors \( r_1, r_2, r_3 \) of \( \lambda_1, \lambda_2, \lambda_3 \) are obtained respectively as follows

\[
\begin{align*}
    r_1 &= \begin{pmatrix} 1 \\ \lambda_1 \\ -\frac{\mu u_2}{n_+} (\lambda_1^2 - \frac{\rho^+ u_3^2 - A_1 \gamma^2}{\mu n_+} \lambda_1) + 1 \end{pmatrix}, \\
    r_2 &= \begin{pmatrix} 1 \\ \lambda_2 \\ -\frac{\mu u_2}{n_+} (\lambda_2^2 - \frac{\rho^+ u_3^2 - A_1 \gamma^2}{\mu n_+} \lambda_2) + 1 \end{pmatrix}, \\
    r_3 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{2.16}
\end{align*}
\]

Define the matrix \( P_1 := [r_1, r_2, r_3] \) and take a linear change of coordinate \( Z := (z_1, z_2, z_3)^T = P_1^{-1} \bar{U} \).

With (2.11) and (2.12), the system (2.9) can be reformulated as follows

\[
\begin{align*}
\frac{d}{dx} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} &= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} g_1(z_1, z_2, z_3) \\ g_2(z_1, z_2, z_3) \\ g_3(z_1, z_2, z_3) \end{pmatrix}, \\
( z_1, z_2, z_3 ) (0) &= ( z_{1-}, z_{2-}, z_{3-} ) = ( P_1^{-1} \bar{U} )^T, \lim_{x \to \infty} ( z_1, z_2, z_3 ) = (0, 0, 0),
\end{align*}
\]

where nonlinear functions \( g_i (i = 1, 2, 3) \) are denoted by

\[
\begin{align*}
    g_1(z_1, z_2, z_3) &= P_1^{-1} \begin{pmatrix} 0 \\ \bar{g}_2(\bar{u}, \bar{v}, \bar{w}) \\ \bar{g}_3(\bar{u}, \bar{v}, \bar{w}) \end{pmatrix}. \tag{2.18}
\end{align*}
\]

With the help of the manifold theory [1], there exist a local center manifold \( W^c(0, 0, 0) \) and a local stable manifold \( W^s_\delta(0, 0, 0) \)

\[
\begin{align*}
    W^c(0, 0, 0) &= \{ (z_1, z_2, z_3) \mid z_1 = f_1^c(z_3), z_2 = f_2^c(z_3), |z_3| \text{ sufficient small} \}, \tag{2.19}
    W^s_\delta(0, 0, 0) &= \{ (z_1, z_2, z_3) \mid z_1 = f_1^s(z_2), z_3 = f_2^s(z_2), |z_2| \text{ sufficient small} \}, \tag{2.20}
\end{align*}
\]

where \( f_i^c, f_i^s, i = 1, 2 \) are smooth functions and \( f_1^c(0) = 0, Df_1^c(0) = 0, f_1^s(0) = 0, Df_1^s(0) = 0, i = 1, 2 \).

Using \( \bar{U} = PZ, (2.12), (2.18) \), we can gain

\[
\bar{g}_3(z_3) = az_3^2 + O(|z_1|^2 + |z_2|^2 + |z_3|^3 + |z_1z_3| + |z_2z_3|), \tag{2.21}
\]

where

\[
a = \frac{A_1\gamma(\gamma + 1) \rho^+_1 + A_2\alpha(a + 1) \mu_+}{2u_+^2(1 + b^2)(\mu + n_+)}, \quad b := \frac{\rho^+ u_3^2 - \rho^+_1 (\rho^+)}{|u_+| \sqrt{(\mu + n_+)} n_+}. \tag{2.22}
\]

Therefore, the system (2.17) can be reformulated as follows

\[
\begin{align*}
    z_{1x} &= \lambda_1 z_1 + O(|Z|^2), \\
    z_{2x} &= \lambda_2 z_2 + O(|Z|^2), \\
    z_{3x} &= az_3^2 + O(|z_1|^2 + |z_2|^2 + |z_3|^3 + |z_1z_3| + |z_2z_3|), \\
( z_1, z_2, z_3 ) (0) &= ( z_{1-}, z_{2-}, z_{3-} ) = ( P_1^{-1} \bar{U} )^T, \lim_{x \to \infty} ( z_1, z_2, z_3 ) = (0, 0, 0). \tag{2.23}
\end{align*}
\]
Let $\sigma_1(x)$ be a solution to $(2.23)_1$ restricted on the local center manifold satisfying the equation

$$\sigma_{1x} = a\sigma_1^2 + O(\sigma_1^3), \quad \sigma_1(x) \to 0 \text{ as } x \to +\infty.$$  \hspace{1cm} (2.24)

which implies that there exists the monotonically increasing solution $\sigma_1(x) < 0$ to (2.24) for $\sigma_1(0) < 0$ and $|\sigma_1(0)|$ sufficiently small. Therefore, if the initial data $(z_{1-}, z_{2-}, z_{3-})$ belongs to the region $\mathcal{M} \subset \mathbb{R}^3$ associated to the local stable manifold and the local center manifold, then we have

$$z_i = O(\sigma_1^2 + O(\delta e^{-cx}), \quad i = 1, 2,$$

$$z_3 = \sigma_1 + O(\delta e^{-cx}),$$

with $z_{3-} < 0$, the smallness of $|(z_{1-}, z_{2-}, z_{3-})|$ and

$$c\frac{\delta}{1 + \delta x} \leq |\sigma_1| \leq C\frac{\delta}{1 + \delta x}, \quad |\partial^k \sigma_1| \leq C\frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad C > 0, \quad k = 0, 1, 2, 3.$$  \hspace{1cm} (2.25)

Due to $\sigma_1(x) \leq 0$, we define

$$\sigma(x) := -\sigma_1,$$  \hspace{1cm} (2.27)

which satisfies

$$\sigma_x = -a\sigma^2 + O(|\sigma|^3), \quad \sigma \to 0 \text{ as } x \to +\infty$$  \hspace{1cm} (2.28)

It is easy to get

$$|\partial^k_x (\rho - \rho+, \bar{u} - u+ - n+ - v+ - u+)| \leq C\frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad C > 0, \quad k = 0, 1, 2, 3,$$  \hspace{1cm} (2.29)

and

$$(\bar{u} - u+, \bar{v} - u+) = (-\sigma(x), -\sigma(x)) + O(|\sigma(x)|^2), \quad (\bar{u}_x, \bar{v}_x) = (a\sigma^2(x), a\sigma^2(x)) + O(|\sigma|^3),$$  \hspace{1cm} (2.30)

with the help of $\bar{U} = PZ$ and (2.16).

### 3 Nonlinear stability of steady-states

The function space $Y(0, T)$ for $T > 0$ is denoted by

$$Y(0, T) := \{ (\phi, \psi, \phi_0, \psi_0) \mid (\phi, \psi, \phi_0, \psi_0) \in C([0, T]; H^1),$$

$$\phi_x, \phi_0 \in L^2([0, T]; L^2), \quad \psi_x, \psi_0 \in L^2([0, T]; H^1) \}.$$  \hspace{1cm} (3.1)

Let

$$\phi = \rho - \bar{\rho}, \quad \psi = u - \bar{u}, \quad \phi_0 = n - \bar{n}, \quad \psi_0 = v - \bar{v}.$$  \hspace{1cm} (3.2)
Then the perturbation \((\phi, \psi, \tilde{\phi}, \tilde{\psi})\) satisfies the following system

\[
\begin{aligned}
\phi_t + u\phi_x + \rho \psi_x &= -(\psi \tilde{\rho}_x + \phi \tilde{u}_x), \\
\psi_t + u\psi_x + \frac{p'_1(\rho)}{\rho} \phi_x &= -\frac{\mu \psi_{xx}}{\rho} - n(\tilde{\psi} - \psi) = F_1, \\
\tilde{\phi}_t + v \tilde{\phi}_x + n \tilde{\psi}_x &= -(\tilde{\psi} \tilde{n}_x + \tilde{\phi} \tilde{v}_x), \\
\tilde{\psi}_t + v \tilde{\psi}_x + \frac{p'_2(n)}{n} \tilde{\phi}_x &= -\frac{(n \tilde{\psi}_x)_x}{n} + (\tilde{\psi} - \psi) = F_2,
\end{aligned}
\]  

(3.3)

where

\[
\begin{aligned}
F_1 &= -[-\mu(\frac{1}{\rho} - \frac{1}{\rho}) \tilde{u}_{xx} + \psi \tilde{u}_x + (\frac{p'_1(\rho)}{\rho} - \frac{p'_1(\rho)}{\rho}) \tilde{\rho}_x - (\frac{n}{\rho} - \frac{\tilde{n}}{\rho})(\tilde{\psi} - \tilde{u})], \\
F_2 &= -[-(\frac{1}{n} - \frac{1}{n})(\tilde{n} \tilde{v}_x)_x - \psi \tilde{\psi}_x + (\frac{p'_2(n)}{n} - \frac{p'_2(n)}{n}) \tilde{n}_x].
\end{aligned}
\]  

(3.4)

(3.5)

The initial and boundary conditions to the system (3.3) satisfy

\[
(\phi, \psi, \tilde{\phi}, \tilde{\psi})(0, x) := (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) = (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}),
\]  

(3.6)

\[
\lim_{x \to \infty} (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) = (0, 0, 0, 0), \quad (\psi, \tilde{\psi})(t, 0) = (0, 0).
\]  

(3.7)

**Proposition 3.1.** Assume that the same assumptions in Theorem 1.2 hold. Let \((\phi, \psi, \tilde{\phi}, \tilde{\psi})\) be the solution to the problem (3.3)-(3.7) satisfying \((\phi, \psi, \tilde{\phi}, \tilde{\psi}) \in Y(0, T)\) for any time \(T > 0\). Then there exist positive constants \(\varepsilon > 0\) and \(C > 0\) independent of \(T\) such that if

\[
\sup_{0 \leq t \leq T} \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_1 + \delta \leq \varepsilon
\]

is satisfied, it holds for arbitrary \(t \in [0, T]\) that

\[
\| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_1 + \int_0^t \| (\phi_x, \psi_x, \tilde{\phi}_x, \tilde{\psi}_x) \|^2 \, d\tau + \int_0^t \| (\psi - \tilde{\psi}, \psi_{xx}, \tilde{\psi}_{xx}) \|^2 \, d\tau \leq C \| (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) \|_1^2.
\]  

(3.8)

(3.9)

Under the condition (3.8), it is easy to verify the following Sobolev inequality

\[
\| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_{L^\infty} \leq \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_{H^1} \leq \sqrt{2}\varepsilon.
\]  

(3.10)

**Lemma 3.2 ( [19] ).** The function \(\psi(\cdot, t) \in H^1(\mathbb{R}_+)\) satisfies for the constants \(\delta > 0\) and \(c_0 > 0\) that

\[
\int_0^\infty \delta e^{-c_0 x} |\psi|^2 \, dx \leq C \delta (|\psi(0, t)|^2 + ||\psi_x(t)||^2),
\]  

(3.11)

\[
\int_0^\infty \frac{\delta^j}{(1 + \delta x)^j} |\psi|^2 \, dx \leq C \delta^{-2} (|\psi(0, t)|^2 + ||\psi_x(t)||^2), \quad j > 2,
\]  

(3.12)

where \(C > 0\) is a positive constant independent of time.

With the above Lemma, we can gain the basic \(L^2\) energy estimates of \((\phi, \psi, \tilde{\phi}, \tilde{\psi})\).
Lemma 3.3. Under the same conditions in Proposition 3.1, then the solution \((\phi, \psi, \bar{\phi}, \bar{\psi})\) to the problem (3.3)-(3.7) satisfies for \(t \in [0, T]\) that

\[
\| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + \int_0^t \| (\psi_x, \bar{\psi}_x, \bar{\psi} - \psi) \|^2 \, dt + \int_0^t | \phi(t, 0) |^2 + | \bar{\phi}(t, 0) |^2 \, dt
\leq C \| (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|^2 + C(\delta + \varepsilon) \int_0^t \| (\phi_x, \bar{\phi}_x) \|^2 \, dt .
\]

(3.13)

Proof. Define

\[
h_1(\rho) = \rho \int_\beta^\rho \frac{p_1(s) - p_1(\bar{\rho})}{s^2} \, ds , \quad \Phi_1(\rho, \bar{\rho}) = \int_\beta^\rho \frac{p_1(s) - p_1(\bar{\rho})}{s^2} \, ds , \quad \mathcal{E}_1 = \rho \left( \frac{\psi^2}{2} + \Phi_1 \right),
\]

(3.14)

\[
h_2(n) = n \int_\bar{n}^n \frac{p_2(s) - p_2(\bar{n})}{s^2} \, ds , \quad \Phi_2(n, \bar{n}) = \int_\bar{n}^n \frac{p_2(s) - p_2(\bar{n})}{s^2} \, ds , \quad \mathcal{E}_2 = n \left( \frac{\bar{\psi}^2}{2} + \Phi_2 \right).
\]

(3.15)

Then, by (1.1) and (1.6), the direct computations lead to that

\[
(\mathcal{E}_1 + \mathcal{E}_2)_t + (G_1 + G_2)_x + n(\bar{\psi} - \psi)^2 + \mu \bar{\psi}_x^2 + n\bar{\psi}_x^2 + R_1 + R_2 = -R_3 ,
\]

(3.16)

where

\[
\begin{align*}
G_1 := & \ u_1 + \rho u_2 + (p_1(\rho) - p_1(\bar{\rho})) \psi + (p_2(n) - p_2(\bar{n})) \bar{\psi} , \\
G_2 := & \ -(\mu \psi_x + n \bar{\psi}_x + \bar{\phi} \bar{\psi}_x) , \\
R_1 := & \ \frac{1}{\rho} \left( p_1(\rho) - p_1(\bar{\rho}) \right) \psi \bar{u}_x + \left( n \bar{\psi}^2 + p_2(n) - p_2(\bar{n}) \right) \bar{\psi} \bar{u}_x + \left( n \bar{\psi}^2 + p_2(n) - p_2(\bar{n}) \right) \bar{\phi} \bar{v}_x , \\
R_2 := & \ \phi \bar{\psi} \bar{u}_x + \frac{(p_1(\bar{\rho}) \bar{\psi} \bar{u}_x + \bar{\phi} \bar{\psi} \bar{\alpha} + \bar{\phi} \bar{\psi} \bar{v}_x)(\psi_x + \bar{\psi}_x)}{n} , \\
R_3 := & \ \phi \bar{v}_x - \bar{\psi} \bar{v}_x ,
\end{align*}
\]

(3.17)

Integrating (3.16) in \(x\) over \(\mathbb{R}_+\) leads to

\[
\frac{d}{dt} \int \mathcal{E}_1 + \mathcal{E}_2 \, dx - G_1(t, 0) - \int n(\bar{\psi} - \psi)^2 + \mu \bar{\psi}_x^2 + n\bar{\psi}_x^2 \, dx + \int R_1 \, dx + \int R_2 \, dx = - \int R_3 \, dx .
\]

(3.18)

Under the condition (3.7), we get

\[
- G_1(t, 0) = - u_1 [ \Phi_1(\rho(t, 0), \bar{\rho}(0)) + \Phi_2(n(t, 0), \bar{n}(0))] \geq c(\phi^2(t, 0) + \bar{\phi}^2(t, 0)).
\]

(3.19)

For the case \(M_+ \neq 1\), with the help of (1.11), (3.10) and (3.11), we can obtain

\[
\int_0^\infty |R_1| + |R_2| + |R_3| \, dx \leq C \delta \| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \bar{\psi} - \psi) \|^2 + C \delta (\phi^2(t, 0) + \bar{\phi}^2(t, 0)).
\]

(3.20)

For the case \(M_+ = 1\) and the restriction \(|p_1'(\rho_+) - p_2'(n_+)| \leq \sqrt{2} |u_+| \min\left\{ (1 + \frac{\rho_+}{\rho_+})[(\gamma - 1)p_1'(\rho_+)]^{\frac{1}{2}}, (1 + \frac{n_+}{n_+})[(\alpha - 1)p_2'(n_+)]^{\frac{1}{2}} \right\} \), using (1.12)-(1.14), (3.10), (3.11), we can get

\[
\begin{align*}
& \int R_1 + R_2 + R_3 \, dx , \\
\geq & \ \int (\psi, \phi) M_1(\psi, \phi)^T \bar{u}_x + (\bar{\psi}, \bar{\phi}) M_2(\bar{\psi}, \bar{\phi})^T \bar{v}_x \, dx - C \int \frac{\delta^3}{(1 + \delta x)^3} (\phi^2 + \psi^2 + \bar{\phi}^2 + \bar{\psi}^2) \, dx \\
& \ - C \delta \int |\bar{\psi} - \psi|^2 + \psi^2 \, dx - C \delta^{\frac{3}{2}} \| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 (\phi^2(t, 0) + \bar{\phi}^2(t, 0)) + \| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2.
\end{align*}
\]
\[
\geq \int (\psi, \phi)M_1(\psi, \phi)^T \ddot{u}_x + (\bar{\psi}, \bar{\phi})M_2(\bar{\psi}, \bar{\phi})^T \ddot{v}_x \, dx - C(\delta + \varepsilon) \| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \bar{\psi} - \psi) \|^2 \\
- C\delta(\phi^2(t, 0) + \bar{\phi}^2(t, 0))
\]  
(3.21)

where $M_1$, $M_2$ are non-negative definite matrices defined by

\[
M_1 = \left( \begin{array}{cc} \rho_- & \frac{u_-^2 - A_1 \gamma \rho_-^{-1}}{2u_+} \\ \frac{u_-^2 - A_1 \gamma \rho_-^{-1}}{2u_+} & \frac{u_-^2 - A_2 \alpha \gamma \rho_-^{-1}}{2u_+} \end{array} \right), \quad M_2 = \left( \begin{array}{cc} \frac{u_-^2 - A_1 \gamma \rho_-^{-1}}{2u_+} & \frac{u_-^2 - A_2 \alpha \gamma \rho_-^{-1}}{2u_+} \\ \frac{u_-^2 - A_1 \gamma \rho_-^{-1}}{2u_+} & \frac{u_-^2 - A_2 \alpha \gamma \rho_-^{-1}}{2u_+} \end{array} \right).
\]

Finally, with the help of (3.18)-(3.21), we get

\[
\| (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + \int_0^t \| (\psi_x, \bar{\phi}_x, \bar{\psi} - \psi) \|^2 \, d\tau + \int_0^t \phi^2(t, 0) + \bar{\phi}^2(t, 0) \, d\tau
\]
\[
\leq C\| (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|^2 + C\delta \int_0^t \| (\phi_x, \bar{\phi}_x) \|^2 \, d\tau.
\]

Hence, we complete the proof of Lemma 3.3.

In order to complete the proof of Proposition 3.1, we need to obtain the estimates of $(\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x)$.

**Lemma 3.4.** Under the same conditions in Proposition 3.1, then the solution $(\phi, \psi, \bar{\phi}, \bar{\psi})$ to the problem (3.3)-(3.7) satisfies for $t \in [0, T]$ that

\[
\| (\phi_x, \bar{\phi}_x) \|^2 + \int_0^t \| (\phi_x, \bar{\phi}_x) \|^2 \, d\tau + \int_0^t \phi^2_x(t, 0) + \bar{\phi}^2_x(t, 0) \, d\tau
\]
\[
\leq C\| (\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0) \|^2 + C(\varepsilon + \delta) \int_0^t \| (\psi_x, \bar{\psi}_x) \|^2.
\]

**Proof.** Differentiating (3.3)$_1$ in $x$, then multiplying the resulted equation by $\mu \phi_x$, (3.3)$_2$ by $\bar{\rho}^2 \phi_x$ respectively, we gain

\[
(\mu^2 \phi^2_x) + (\mu u \phi^2_x) + \mu \bar{\rho} \phi_x \psi_x = -\mu^3 \psi_x \phi^2_x + (\phi \psi_{xx} + \frac{1}{2} \phi \bar{u}_x + \psi_x \bar{\rho}_x) \phi_x + (\phi \bar{u}_x + \psi_x \bar{\rho}_x) \phi_x,
\]

(3.25)

Similarly, differentiating (3.3)$_3$ in $x$, then multiplying the resulted equation by $\bar{\phi}_x$, (3.3)$_4$ by $\bar{n} \bar{\phi}_x$ respectively lead to

\[
(\bar{n} \bar{\phi}_x \bar{\psi})_t + (\bar{n} \bar{\phi}_x \bar{\psi}_x) + \bar{n} \bar{\phi}_x \bar{\psi}_x = \left[ \frac{3}{2} \bar{\psi}_x \bar{\phi}^2_x + (\bar{\phi}_x \bar{v}_x + \frac{1}{2} \bar{\phi}_x \bar{v}_x + \psi_x \bar{n}_x) \bar{\phi}_x - (\bar{\phi}_x \bar{v}_x + \psi_x \bar{n}_x) \bar{\phi}_x \right],
\]

(3.27)

\[
(\bar{\phi}_x \bar{\psi})_t + (\bar{\phi}_x \bar{\psi}_x) + \bar{n} \bar{\phi}_x \bar{\psi}_x = \left[ \frac{3}{2} \bar{\psi}_x \bar{\phi}^2_x + (\bar{\phi}_x \bar{v}_x + \frac{1}{2} \bar{\phi}_x \bar{v}_x + \psi_x \bar{n}_x) \bar{\phi}_x - (\bar{\phi}_x \bar{v}_x + \psi_x \bar{n}_x) \bar{\phi}_x \right],
\]

(3.28)
Adding (3.25)-(3.28) together and integrating over $\mathbb{R}_+$ yield

$$
\frac{d}{dt}\int \left(\frac{\phi_x^2}{2} + \frac{\bar{\phi}_x^2}{n} \right) d\tau + \int \left(\mu \phi_x^2 + v \bar{\phi}_x^2 - \rho^2 \phi_x \psi - \phi \bar{\psi}_x \psi \right) dx + \int \left(\mu \bar{\phi}_x^2 + v \phi_x^2 - \rho^2 \bar{\phi}_x \psi - \bar{\phi} \phi_x \psi \right) dx
$$

$$
+ \int \left(\rho^2 \frac{\phi_1'(\rho)}{n} \phi_x^2 + n \frac{\phi_2(n)}{n} \bar{\phi}_x^2 \right) dx = \sum_{i=1}^5 J_i,
$$

(3.29)

where

$$
J_1 = - \int \left(\rho^2 (\psi_t + u \phi_x) \psi_x + n (\bar{\phi}_t + v \bar{\phi}_x) \bar{\psi}_x + 2 \bar{\rho} \bar{\phi}_t \phi_x + n \bar{\phi}_x \bar{\psi}_x \right) dx,
$$

$$
J_2 = \int -\mu \phi_x \psi_{xx} - \bar{\phi}_x \bar{\psi}_{xx} + \rho^2 \left(\frac{1}{\rho} - \frac{1}{\rho} \right) \phi_x \psi_{xx} + \bar{\Delta} + \frac{n}{n} \bar{\psi}_{xx} \psi_x + \rho^2 \left(\frac{1}{\rho} \right) \phi_x \psi_{xx} - \bar{\psi} \psi_x \bar{\psi}_x dx,
$$

$$
J_3 = - \int \frac{3}{2} \mu \phi_x \phi_x^2 + \frac{3}{2} \bar{\psi}_x \bar{\psi}_x^2 dx + \int \frac{n (\bar{\psi}_x \psi_x)}{n} \phi_x dx, \quad J_4 = \int F_1 \rho^2 \phi_x + F_2 \bar{\phi}_x dx,
$$

$$
J_5 = - \int \left[\mu (\frac{1}{2} \phi_x \bar{\psi}_x + \psi_x \bar{\phi}_x) \phi_x + \mu (\bar{\phi}_x \bar{\psi}_x + \bar{\psi}_x \psi_x) \bar{\phi}_x
$$

$$
+ \left(\frac{1}{2} \bar{\phi}_x \bar{\psi}_x + \psi_x \bar{\phi}_x \bar{\psi}_x \right) \bar{\phi}_x - \left(\bar{\phi}_x \psi_x + \bar{\psi}_x \psi_x \right) \psi_x - \bar{\phi}_x \bar{\psi}_x \psi_x \right] dx.
$$

First, we estimate terms in the left side of (3.29). Under the condition (3.7), the second term in the left side is estimated as follows

$$
\int \left(\mu \phi_x^2 + v \bar{\phi}_x^2 - \rho^2 \phi_x \psi - \phi \bar{\psi}_x \psi \right) dx = -u \int \left(\frac{\phi_2^2(0, t)}{2} + \bar{\phi}_2^2(0, t) \right) \geq 0,
$$

(3.30)

The third term is estimated as follows

$$
\int \rho^2 \frac{\phi_1'(\rho)}{n} \phi_x^2 + n \frac{\phi_2(n)}{n} \bar{\phi}_x^2 dx \geq c \|\phi_x, \bar{\phi}_x\|^2 - C(\varepsilon + \delta) \|\phi_x, \bar{\phi}_x\|^2.
$$

(3.31)

We turn to estimate terms in the right hand side of (3.29). By (1.11)-(1.12), (3.3), (3.3), (3.11)-(3.12), Cauchy-Schwartz inequality and Young inequality, we can obtain

$$
|J_1| \leq C \|\psi_x, \bar{\psi}_x\|^2 + C \delta \|\phi_x, \bar{\phi}_x\|^2 + C \delta \|\phi(t, 0) + \bar{\phi}(t, 0)\|^2,
$$

(3.32)

$$
|J_2| \leq C \|\phi, \bar{\phi}\|_{L^\infty} \|\phi_x, \bar{\phi}_x, \psi_x, \bar{\psi}_x\|^2 + C \|\phi_x, \bar{\phi}_x\|_{L^\infty} \|\psi - \bar{\psi}\|^2 + \eta \|\phi_x, \bar{\phi}_x\|^2 + C \delta \|\phi_x, \bar{\phi}_x\|^2
$$

$$
\leq C(\varepsilon + \delta + \eta) \|\phi_x, \bar{\phi}_x\|^2 + C \varepsilon \|\psi_x, \bar{\psi}_x\|^2 + C \|\phi_x, \bar{\phi}_x\|^2 + C \delta \|\phi_x, \bar{\phi}_x\|^2
$$

(3.33)

$$
|J_3| \leq C \|\psi_x, \bar{\psi}_x\|_{L^\infty} \|\phi_x, \bar{\phi}_x\|^2 + C \|\phi_x, \bar{\phi}_x\|_{L^\infty} \|\psi_x, \bar{\psi}_x\|^2
$$

$$
\leq C \|\psi_x, \bar{\psi}_x\|^2 + C \varepsilon \|\psi_x, \bar{\psi}_x\|^2 + C \delta \|\phi(t, 0) + \bar{\phi}(t, 0)\|^2.
$$

(3.34)

$$
|J_4 + J_5| \leq C \|\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x\|^2 + C \delta \|\psi_x, \bar{\psi}_x\|^2 + C \delta \|\phi(t, 0) + \bar{\phi}(t, 0)\|^2.
$$

(3.35)

Finally, the substitution of (3.30)-(3.35) into (3.29) for $\delta, \varepsilon$ and $\eta$ small enough leads to

$$
\frac{d}{dt} \int \left(\phi_x^2 + \bar{\phi}_x^2 + \rho^2 \phi_x \psi + \bar{n} \bar{\phi}_x \bar{\psi} \psi \right) dx + \|\phi_x, \bar{\phi}_x\|^2 + \phi_x^2(0, t) + \bar{\phi}_x^2(t, 0)
$$

$$
\leq C \|\psi_x, \bar{\psi}_x, \psi - \bar{\psi}\|^2 + C(\varepsilon + \delta) \|\psi_x, \bar{\psi}_x\|^2 + C \delta \|\phi(t, 0) + \bar{\phi}(t, 0)\|^2.
$$

(3.36)
Integrating (3.36) over \([0,t]\), we can obtain

\[
\|
\phi_x, \bar{\phi}_x \n\|^2 + \int_0^t \|
\phi_x, \bar{\phi}_x \n\|^2 \, d\tau \leq C\|
\phi_0, \bar{\phi}_0, \psi_0, \bar{\psi}_0, \phi_0, \bar{\phi}_0 \n\|^2 + C(\delta + \epsilon) \int_0^t \|
\psi_x, \bar{\psi}_x \n\|^2 \, d\tau, \tag{3.37}
\]

where we have used (3.13) and Young inequality. Hence, we complete the proof of Lemma 3.4. \(\square\)

**Lemma 3.5.** Under the same conditions in Proposition 3.1, then the solution \((\phi, \psi, \bar{\phi}, \bar{\psi})\) to the problem (3.3)-(3.7) satisfies for \(t \in [0,T]\) that

\[
\|
\psi_x, \bar{\psi}_x \n\|^2 + \int_0^t \|
\psi_x, \bar{\psi}_x \n\|^2 \, d\tau \leq C\|
\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0 \n\|^2. \tag{3.38}
\]

**Proof.** Multiplying (3.3)\(_2\) by \(-\psi_{xx}\), (3.3)\(_4\) by \(-\bar{\psi}_{xx}\), respectively, then adding them together and integrating the resulted equation in \(x\) over \(\mathbb{R}_+\) imply

\[
\frac{d}{dt} \int \frac{\psi_x^2}{2} + \frac{\bar{\psi}_{xx}^2}{2} \, dx + \int \mu \frac{1}{\rho} \psi_{xx}^2 + \bar{\psi}_{xx}^2 \, dx = \sum_{i=1}^3 K_i, \tag{3.39}
\]

where

\[
K_1 = \int [u \psi_x \psi_{xx} - \frac{n}{\rho} (\psi - \psi) \psi_{xx} + \frac{p'_1(\rho)}{\rho} \phi_x \psi_{xx} + \psi \bar{\psi}_x \psi_{xx} + (\bar{\psi} - \psi) \bar{\psi}_{xx}] \, dx,
\]

\[
K_2 = \int [\bar{u} \psi_x \psi_{xx} - \mu \bar{\psi}_{xx} (\frac{1}{\rho} - \frac{1}{\rho}) \psi_{xx} + \frac{p'_1(\rho)}{\rho} \phi_x \psi_{xx} - \frac{p'_2(\rho)}{\rho} \bar{\psi}_x \psi_{xx} - \frac{n}{\rho} (\bar{\psi} - \psi) \bar{\psi}_{xx} + \bar{\psi} \psi \bar{\psi}_{xx}]
\]

\[
\quad + (\bar{n} \bar{\psi}_x) (\frac{1}{n} - \frac{1}{\rho} - \frac{1}{\rho}) \psi_{xx} + \frac{p'_1(n)}{n} \bar{\psi}_{xx} \bar{n}_x - \frac{n}{n} \bar{\psi}_{xx} \bar{n}_x - \frac{\bar{n}}{n} \bar{\psi}_{xx} \bar{n}_x - \frac{\bar{n}}{n} \bar{\psi}_{xx} \bar{n}_x \, dx,
\]

\[
K_3 = -\int [\frac{\psi_x}{n} \psi_{xx} + \frac{1}{n} \frac{1}{\rho} (\bar{n} \bar{\psi}_x) \bar{n}_x \psi_{xx}] \, dx.
\]

We estimate terms in the left side of (3.39). By the decomposition \(\frac{1}{\rho} = \frac{(1 - \frac{1}{\rho})}{\rho} + \frac{1}{\rho} - \frac{1}{\rho} + \frac{1}{\rho} - \frac{1}{\rho} + \frac{1}{\rho}\), the second term is estimated as follows:

\[
\int \mu \frac{\psi_x^2}{\rho} + \frac{\bar{\psi}_{xx}^2}{\rho} \, dx \geq \mu \frac{\rho_+}{\rho_+} \| \psi_{xx} \|^2 + \| \bar{\psi}_{xx} \|^2 - C(\| \phi \|_{L_\infty} + \delta) \| \psi_{xx} \|^2 \tag{3.40}
\]

\[
\geq \mu \frac{\rho_+}{\rho_+} \| \psi_{xx} \|^2 + \| \bar{\psi}_{xx} \|^2 - C(\| \epsilon + \delta \| \psi_{xx} \|^2.
\]

We turn to estimate terms in the right side of (3.39). With the aid of (1.11), Sobolev inequality and Cauchy-Schwarz inequality, we have

\[
|K_1| \leq \frac{\mu}{16\rho_+} \| \psi_{xx} \|^2 + \frac{1}{16} \| \psi_{xx} \|^2 + C(\| \phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \psi - \bar{\psi} \|)^2, \tag{3.41}
\]

\[
|K_2| \leq C\| \phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x \|^2 + C\| \psi_{xx}, \bar{\psi}_{xx} \|^2 + C\| \phi^2(t,0) + \bar{\phi}^2(t,0) \|, \tag{3.42}
\]

\[
|K_3| \leq C\| \psi_{xx} \|^2 + C\| \psi_{xx} \|^2 \| \bar{\psi}_{xx} \|^2 \leq C\| \bar{\psi}_{xx} \|^2. \tag{3.43}
\]

Finally, taking \(\delta\) and \(\epsilon\) small enough and substituting of (3.40)-(3.43) into (3.39), we obtain

\[
\frac{d}{dt} \int \psi_x^2 + \bar{\psi}_{xx}^2 \, dx + \frac{\mu}{2\rho_+} \| \psi_{xx} \|^2 + \| \psi_{xx} \|^2 \leq C(\| \phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \psi - \bar{\psi} \|^2 + C\| \phi^2(t,0) + \bar{\phi}^2(t,0) \|). \tag{3.44}
\]
Integrating (3.44) in $\tau$ over $[0, t]$ leads to
\[
\|(\psi_x, \bar{\psi}_x)^2 + \int_0^t \|(\psi_{xx}, \bar{\psi}_{xx})\|^2 d\tau \leq C\|(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0)\|^2_1,
\]
where we have used (3.13), (3.24) and the smallness of $\delta$ and $\varepsilon$. Thus, we complete the proof of Lemma 3.5.

With the help of Lemmas 3.3-3.5, we get (3.9) and complete the proof of Proposition 3.1.

### 4 Time convergence rates

#### 4.1 Convergence rate of supersonic steady-state

**Proposition 4.1.** Assume that the same conditions in Theorem 1.3 for $M_+ > 1$ hold and let $(\phi, \psi, \bar{\phi}, \bar{\psi})$ be a solution to the IBVP (3.3)-(3.7) satisfying $(1 + x)^\frac{1}{2}(\phi, \psi, \bar{\phi}, \bar{\psi}) \in C([0, T]; L^2)$ for any time $T > 0$. Then for $\nu \in [0, \lambda]$, there exist positive constants $\varepsilon > 0$ and $C > 0$ independent of $T$ such that if
\[
\sup_{0 \leq t \leq T} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|_1 + \delta \leq \varepsilon \tag{4.1}
\]
is satisfied, it holds for arbitrary $t \in [0, T]$ that
\[
\begin{align*}
(1 + t)^{\lambda - \nu + \theta} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|_1 + \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 d\tau + \nu \int_0^t (1 + \tau)^{\lambda - \nu + \theta} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a, \nu-1} d\tau \\
+ \int_0^t (1 + \tau)^{\lambda - \nu + \theta} \|(\psi_x, \bar{\psi}_x, \bar{\psi} - \psi)\|^2_{a, \nu} d\tau + \int_0^t (1 + \tau)^{\lambda - \nu + \theta} \|(\phi_x, \psi_{xx}, \bar{\phi}_x, \bar{\psi}_{xx})\|^2_{a, \nu} d\tau \\
\leq C(1 + t)^{\delta} \|(\phi_0, \psi_0, \bar{\phi}_0, \psi_0)\|^2 + \|(\phi_0, \psi_0, \bar{\phi}_0, \psi_0)\|^2_{a, \lambda},
\end{align*}
\]
with $\theta > 0$.

Our first goal is to obtain the basic weighted energy estimates of $(\phi, \psi, \bar{\phi}, \bar{\psi})$.

**Lemma 4.2.** Under the same conditions in Proposition 4.1, then the solution $(\phi, \psi, \bar{\phi}, \bar{\psi})$ to the IBVP (3.3)-(3.7) satisfies for $t \in [0, T]$ that
\[
\begin{align*}
(1 + t)^{\xi} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a, \nu} + \nu \int_0^t (1 + \tau)^{\xi} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a, \nu-1} d\tau \\
+ \int_0^t (1 + \tau)^{\xi} \|(\psi_x, \bar{\psi}_x, \bar{\psi} - \psi)\|^2_{a, \nu} d\tau + \int_0^t (1 + \tau)^{\xi} (\phi_x^2(t, 0) + \phi_{xx}^2(t, 0)) d\tau \\
\leq C\|(\phi_0, \psi_0, \bar{\phi}_0, \psi_0)\|^2_{a, \lambda} + C\delta \int_0^t (1 + \tau)^{\xi} \|(\phi_x, \bar{\phi}_x)\|^2 d\tau \\
+ C\nu \int_0^t (1 + \tau)^{\xi} \|(\psi_x, \bar{\psi}_x, \bar{\psi} - \psi)\|^2_{a, \nu-1} d\tau + C\xi \int_0^t (1 + \tau)^{\xi - 1} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a, \nu} d\tau,
\end{align*}
\]
with $\xi \geq 0$. 

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Proof. We multiply (3.16) by $W_{a, \nu}$, where $W_{a, \nu} := (1 + x)^\nu$ is a space weight function. We integrate the resulted equality over $\mathbb{R}_+$ to obtain

\[
\frac{d}{dt} \int W_{a, \nu}(E_1 + E_2) - (W_{a, \nu}G_1)(t, 0) - \int W_{a, \nu-1}G_1 dx - \int W_{a, \nu-1}G_2 dx \\
= - \frac{d}{dt} \int W_{a, \nu}(R_1 + R_2 + R_3) dx,
\]

where $G_i$ for $i = 1, 2$, $R_j$ for $j = 1, 2, 3$ and $E_k$, $k = 1, 2$ are defined by (3.17).

First, we estimate terms on the left hand side of (4.4). Under the condition (3.7), the second term on the left hand side is estimated as

\[
-W_{a, \nu}G_1(t, 0) = |u_-|\left|\Phi_1(\rho(t, 0), \tilde{\rho}(0)) + \Phi_2(n(t, 0), \tilde{n}(0))\right| \geq c(\phi^2(t, 0) + \tilde{\phi}^2(t, 0)),
\]

We decompose $\psi$ as $\psi = \tilde{\psi} + (\psi - \tilde{\psi})$ and use (1.11) to gain

\[
-\nu \int W_{a, \nu-1}G_1 dx \\
\geq \nu \int W_{a, \nu-1} \left(-\frac{u_+}{2}n_+\tilde{\phi}^2 - \frac{u_+}{2}A_1\gamma \rho_+^{-2}\phi^2 - A_1\gamma \rho_+^{-1}\phi\tilde{\phi} - \frac{u_+}{2}\rho_+\tilde{\phi}^2 - A_2\alpha_+^{-1}\phi\tilde{\phi}\
\right.
\\
\left. - \frac{u_+}{2}A_2\alpha_+^{-2}\phi^2 - \frac{u_+}{2}\rho_+|\psi - \tilde{\psi}|^2 - \rho_+u_+\tilde{\psi}(\psi - \tilde{\psi}) - A_1\gamma \rho_+^{-1}\phi(\psi - \tilde{\psi})\right) dx
\]

\[
\geq \nu \int W_{a, \nu-1} \left(\frac{1}{2}(\phi, \tilde{\phi}, \psi)M_3(\phi, \tilde{\phi}, \psi)^T - \rho_+u_+\tilde{\psi}(\psi - \tilde{\psi}) - A_1\gamma \rho_+^{-1}\phi(\psi - \tilde{\psi})\right) dx
\]

\[
-C(\delta + \varepsilon)||\phi, \tilde{\phi}, \psi||^2_{a, \nu-1},
\]

where the symmetric matrix $M_3$ is denoted by

\[
M_3 = \begin{pmatrix}
-A_1\gamma \rho_+^{-2}u_+ & 0 & -A_1\gamma \rho_+^{-1} \\
0 & -A_2\alpha_+^{-2}u_+ & -A_2\alpha_+^{-1} \\
-A_1\gamma \rho_+^{-1} & -A_2\alpha_+^{-1} & -(\rho_++n_u)u_+
\end{pmatrix}.
\]

It is easy to verify that $M_3$ is a positive definite matrix under the condition $M_+ > 1$. Hence, the estimate of the third term on the left hand side is obtained under the condition $\varepsilon$, $\delta$ and $\eta$ small enough that

\[
-\nu \int W_{a, \nu-1}G_1 dx \\
\geq c\nu||\phi, \tilde{\phi}, \psi||^2_{a, \nu-1} - \eta\nu||\phi, \tilde{\phi}, \psi||^2_{a, \nu-1} - C\eta\nu||\psi - \tilde{\psi}||^2_{a, \nu-1} - C(\varepsilon + \delta)||\phi, \tilde{\phi}, \psi||^2_{a, \nu-1}
\]

\[
\geq c\nu||\phi, \tilde{\phi}, \psi||^2_{a, \nu-1} - C\nu||\psi - \tilde{\psi}||^2_{a, \nu-1},
\]

The forth and the fifth terms on the left hand side are estimated as below

\[
-\nu \int W_{a, \nu-1}G_2 dx \leq \nu\eta||\phi, \tilde{\phi}, \psi||^2_{a, \nu-1} + \nu C\eta||\psi, \tilde{\psi}, \tilde{\phi}||^2_{a, \nu-1} + C\varepsilon||\phi, \tilde{\phi}, \psi||^2_{a, \nu-1}.
\]
\[
\int W_{a,\nu} [\psi_x^2 + n \psi_x^2 + n(\psi - \bar{\psi})^2] dx \geq c \|(\psi_x, \bar{\psi}_x, \psi - \bar{\psi})\|_{a,\nu}^2 - C(\varepsilon + \delta) \|(\psi_x, \bar{\psi}_x, \psi - \bar{\psi})\|_{a,\nu}^2. \tag{4.10}
\]

With the help of (1.11), (3.11), Sobolev inequality and Cauchy-Schwarz inequality, we have the estimate of the term on the right hand side of (4.4) as

\[
\left| \int W_{a,\nu}(R_1 + R_2 + R_3) dx \right| \leq C\delta \int e^{-a_0' t} (\phi^2 + \phi_x^2 + \bar{\phi}_x^2 + |\psi - \bar{\psi}|^2) dx
\]

\[
\leq C\delta \|(\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \psi - \bar{\psi})\|^2 + C\delta (\phi^2(0) + \bar{\phi}^2(0)). \tag{4.11}
\]

Finally, with \(\eta, \delta\) and \(\varepsilon\) suitably small, the substitution of (4.5)-(4.11) into (4.4) leads to

\[
\frac{d}{dt} \int W_{a,\nu}(E_1 + E_2) dx + C\nu \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|_{a,\nu-1}^2 + c \|(\psi_x, \bar{\psi}_x, \psi - \bar{\psi})\|_{a,\nu}^2 + c(\phi^2(t,0) + \bar{\phi}^2(t,0))
\]

\[
\leq C\delta \|(\phi_x, \bar{\phi}_x)\|^2 + C\nu \|(\psi_x, \bar{\psi}_x, \psi - \bar{\psi})\|^2_{a,\nu-1}. \tag{4.12}
\]

Multiplying (4.12) by \((1 + \tau)^{\xi}\) and integrating the resulted equation in \(\tau\) over \([0,t]\) yield for \(\xi \geq 0\) that

\[
(1 + t)^{\xi} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a,\nu} + \nu \int_0^t (1 + \tau)^{\xi} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a,\nu-1} d\tau
\]

\[
+ \int_0^t (1 + \tau)^{\xi} \|(\psi_x, \psi, \psi - \bar{\psi})\|_{a,\nu}^2 d\tau + \int_0^t (1 + \tau)^{\xi} (\phi^2(t,0) + \bar{\phi}^2(t,0)) d\tau
\]

\[
\leq C\|(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0)\|^2_{a,\nu} + C\delta \int_0^t (1 + \tau)^{\xi} \|(\phi_x, \bar{\phi}_x)\|^2 d\tau
\]

\[
+ C\nu \int_0^t (1 + \tau)^{\xi} \|(\psi_x, \psi, \psi - \bar{\psi})\|^2_{a,\nu-1} d\tau + C\xi \int_0^t (1 + \tau)^{\xi-1} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a,\nu} d\tau,
\]

Hence we obtain (4.3) and complete the proof of Lemma 4.2.

To prove Proposition 4.1, we need to obtain the following weighted energy estimates of \((\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x)\).

By similar arguments as showing Lemmas 3.4-3.5, we get the following Lemmas. The details are omitted.

**Lemma 4.3.** Under the same conditions in Proposition 4.1, then the solution \((\phi, \psi, \bar{\phi}, \bar{\psi})\) to the IBVP (3.3)-(3.7) satisfies for \(t \in [0, T]\)

\[
(1 + t)^{\xi} \|(\phi_x, \bar{\phi}_x)\|^2 + \int_0^t (1 + \tau)^{\xi} \|(\phi_x, \bar{\phi}_x)\|^2 d\tau
\]

\[
\leq C \|(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0)\|^2_{a,\lambda} + \|(\phi_0x, \bar{\phi}_0x)\|^2 + C\xi \int_0^t (1 + \tau)^{\xi-1} \|(\psi_x, \bar{\psi}_x)\|^2_{a,\nu} d\tau
\]

\[
+ C\xi \int_0^t (1 + \tau)^{\xi-1} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a,\nu} + \|(\phi_x, \bar{\phi}_x)\|^2 d\tau,
\]

with \(\xi \geq 0\).

**Lemma 4.4.** Under the same conditions in Proposition 4.1 hold, then the solution \((\phi, \psi, \bar{\phi}, \bar{\psi})\) to the IBVP (3.3)-(3.7) satisfies for \(t \in [0, T]\)

\[
(1 + t)^{\xi} \|(\psi_x, \bar{\psi}_x)\|^2 + \int_0^t (1 + \tau)^{\xi} \|(\psi_x, \bar{\psi}_x)\|^2 d\tau
\]

\[
\leq C \|(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0)\|^2_{a,\lambda} + \|(\phi_0x, \bar{\phi}_0x, \psi_0x, \bar{\psi}_0x)\|^2
\]

\[
+ C\xi \int_0^t (1 + \tau)^{\xi-1} \|(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2_{a,\nu} + \|(\phi_x, \bar{\phi}_x, \psi_x, \bar{\psi}_x)\|^2 d\tau
\]

\[
\tag{4.15}
\]
with $\xi \geq 0$.

**Proof of Proposition 4.1** For $\nu \in [0, \lambda]$ and $\xi \geq 0$, with the three Lemmas 4.2-4.4 above leads to

\[
(1 + t)^\xi \left( \left\| (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|_{a,\nu}^2 + \left\| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \right\|^2 \right) + \nu \int_0^t (1 + \tau)^\xi \left( \left\| (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|_{a,\nu-1}^2 \right) d\tau \\
+ \int_0^t (1 + \tau)^\xi \left( \left\| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \right\|^2 \right) d\tau \\
\leq C \left( \left\| \phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0 \right\|_{a,\lambda}^2 + \left\| (\phi_{0x}, \psi_{0x}, \bar{\phi}_{0x}, \bar{\psi}_{0x}) \right\|^2 \right) + C\nu \int_0^t (1 + \tau)^\xi \left( \left\| (\psi_x, \bar{\psi}_x, \bar{\psi} - \psi) \right\|^2 \right) d\tau \\
+ C\xi \int_0^t (1 + \tau)^{\xi-1} \left( \left\| (\phi, \psi, \bar{\phi}, \bar{\psi}) \right\|_{a,\nu} + \left\| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \right\|^2 \right)
\]

(4.16)

where $C > 0$ is a generic positive constant independent of $T, \nu$, and $\xi$. Hence, applying induction arguments similar to those used in [2,17,24] to (4.16) lead to the desired estimate (4.2).

### 4.2 Convergence rate of sonic steady-state

The function space $Y_W(0, T)$ for $T > 0$ is denoted by

\[
Y_W(0, T) := \{ (\phi, \psi, \bar{\phi}, \bar{\psi}) \mid (\phi, \psi, \bar{\phi}, \bar{\psi}) \in C([0, T]; H^1_W), \\\n(\phi_x, \bar{\phi}_x) \in L^2([0, T]; L^2_W), (\psi_x, \bar{\psi}_x) \in L^2([0, T]; H^1_W) \}.
\]

(4.17)

**Proposition 4.5.** Assume that $1 \leq \lambda < 2 + \sqrt{8 + \frac{2b}{1+a}}$ with $b := \frac{\mu(a^2 - \rho_0^2)}{|\mu + a|}$ and that the same conditions in Theorem 1.3 hold for $M_+ = 1$. Let $(\phi, \psi, \bar{\phi}, \bar{\psi})$ be a solution to the IBVP (3.3)-(3.7) satisfying $(\phi, \psi, \bar{\phi}, \bar{\psi}) \in Y_{\sigma, \frac{3}{2}}(0, T)$ for any time $T > 0$. Then for arbitrary $\nu \in (0, \lambda]$, there exist positive constants $\varepsilon > 0$ and $C > 0$ independent of $T$ such that if

\[
\sup_{0 \leq t \leq T} \| \sigma^{-\frac{1}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})(t) \|_1 + \frac{\delta}{2} \leq \varepsilon
\]

(4.18)

is satisfied, it holds for arbitrary $t \in [0, T]$ that

\[
(1 + \delta t)^{\frac{\lambda-\rho}{\mu} + \beta} \| \sigma^{-\frac{1}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + \int_0^t (1 + \delta \tau)^{\frac{\lambda-\rho}{\mu} + \beta} \| \sigma^{-\frac{1}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 d\tau \\
+ \int_0^t (1 + \delta \tau)^{\frac{\lambda-\rho}{\mu} + \beta} \| \sigma^{-\frac{1}{2}}(\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2 d\tau \\
\leq C(1 + \delta t)^{\beta} \| \sigma^{-\frac{1}{2}}(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0, \psi_{0x}, \bar{\psi}_{0x}, \bar{\psi}_{0x}) \|^2,
\]

(4.19)

with $\beta > 0$.

Under the condition $\lambda \geq 1$ and (4.18), it is easy to gain the following estimate:

\[
\| \sigma^{-\frac{1}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi}) \|_{L^\infty} \leq \| \sigma^{-\frac{1}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi}) \|_1 \leq \sqrt{2} \varepsilon.
\]

(4.20)

To deal with some nonlinear terms, we use the following inequality as in [18,26,37].
Lemma 4.6 ([37]). Let \( \nu \geq 1 \). Then a function \( \phi(t, x) \in H^{1}_{\sigma^{-\nu}}(\mathbb{R}_{+}) \) satisfies
\[
\int \sigma^{-\frac{\nu-1}{2}}|\phi|^2 dx \leq C \|\sigma^{-\frac{\nu}{2}}\phi\| (\sigma(0)\phi^2(t, 0) + \|\sigma^{-\frac{\nu}{2}}\phi_x\|^2 + \|\sigma^{-\frac{\nu-2}{2}}\phi_x\|^2),
\]
where the function \( \sigma(x) \geq 0 \) is defined by (2.28) with \( \sigma(0) \) small enough.

To gain faster decay rates, it is necessary to use the following Hardy type inequality which is proved in [16].

Lemma 4.7 ([16]). Let \( \zeta \in C^{1}[0, \infty) \) satisfies \( \zeta > 0, \zeta_x > 0 \) and \( \zeta(x) \to \infty \) for \( x \to \infty \). Then we have
\[
\int_{\mathbb{R}_{+}} \psi^2 \zeta_x dx \leq 4 \int_{\mathbb{R}_{+}} \psi_x^2 \zeta_x dx
\]
for \( \psi \) satisfying \( \psi(t, 0) = 0 \) and \( w\psi \in H^{1}(\mathbb{R}_{+}), \) with the function \( w := \frac{\zeta^2}{\zeta_x} \).

With the aid of Lemmas 4.6-4.7, we obtain the weighted \( L^2 \) estimate of \( (\phi, \psi, \bar{\phi}, \bar{\psi}) \).

Lemma 4.8. Under the same conditions in Proposition 4.5, then the solution \( (\phi, \psi, \bar{\phi}, \bar{\psi}) \) to the problem (3.3)-(3.7) satisfies for \( t \in [0, T] \) that
\[
(1 + \delta \tau)^{\xi}||\sigma^{-\frac{\nu}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})||^2 + \int_{0}^{t} (1 + \delta \tau)^{\xi}||\sigma^{-\frac{\nu}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})||^2 d\tau
\]
\[
+ \int_{0}^{t} (1 + \delta \tau)^{\xi}||\sigma^{-\frac{\nu}{2}}(\psi_x, \psi_x, \bar{\psi}_x - \psi)\|^2 d\tau + \int_{0}^{t} (1 + \delta \tau)^{\xi} \frac{1}{\delta^\nu}(\phi^2(t, 0) + \bar{\phi}^2(t, 0))d\tau
\]
\[
\leq C||\sigma^{-\frac{\nu}{2}}(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0)||^2 + C\delta \int_{0}^{t} (1 + \delta \tau)^{\xi}||\sigma^{-\frac{\nu}{2}}(\phi_x, \phi_x)\|^2 d\tau
\]
\[
+ C\delta \xi \int_{0}^{t} (1 + \delta \tau)^{\xi-1}||\sigma^{-\frac{\nu}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})||^2 d\tau.
\]

with \( \xi \geq 0 \).

Proof. We multiply (3.16) by the space weight function \( \sigma^{-\nu} \), where the space weight function \( \sigma \geq 0 \) satisfies (1.14) and (2.28). Then, we integrate the resulted equation over \( \mathbb{R}_{+} \) to get
\[
\frac{d}{dt} \int \sigma^{-\nu}(E_1 + E_2) dx - (\sigma^{-\nu}G_1)(t, 0) - \delta^\nu \int \sigma^{-\nu-1}G_1 dx - \delta^\nu \int \sigma^{-\nu-1}G_2 dx
\]
\[
+ \int \sigma^{-\nu}n(\bar{\psi} - \psi)^2 dx + \int \sigma^{-\nu}(\mu \psi_x^2 + n\bar{\psi}_x^2) dx + \int \sigma^{-\nu}R_1 dx
\]
\[
= - \int \sigma^{-\nu}R_2 dx - \int \sigma^{-\nu}R_3 dx,
\]
where \( G_i \) for \( i = 1, 2, R_j \) for \( j = 1, 2, 3 \) and \( E_k, k = 1, 2 \) are defined by (3.17).

First, we estimate terms on the left hand side of (4.24). Under the condition (3.7), the second term on the left hand side is estimated as
\[
- (\sigma^{-\nu}G_1)(t, 0) \geq \frac{\sigma}{\delta^\nu}(\phi^2(t, 0) + \bar{\phi}^2(t, 0)) \geq 0.
\]
For the third term on the left hand side, using (4.21) and \( \psi = \vec{\psi} + (\psi - \vec{\psi}) \) yields

\[
- \alpha \nu \int \sigma^{-(\nu-1)} G_1 dx \\
\geq \alpha \nu \int \sigma^{-(\nu-1)} \frac{1}{2} (\phi, \vec{\phi}, \psi) M_4 (\phi, \vec{\phi}, \psi)^T - \rho_+ u_+ \vec{\psi} (\psi - \vec{\psi}) - A_1 \gamma \rho_+^{\gamma-1} \phi (\psi - \vec{\psi}) \] \] dx
\]

\[+ \alpha \nu \int \sigma^{-(\nu-1)} \left[- (A_1 \gamma \rho_+^{\gamma-2} \vec{\psi} - A_1 \gamma \rho_+^{\gamma-2} u_+) \phi \right] - \frac{(2 \alpha \alpha \alpha - 2 \vec{\psi} - A_2 \alpha \alpha \alpha - 2 u_+) \vec{\phi}^2}{2} \]
\]

\[- (A_1 \gamma \rho_+^{\gamma-1} - A_1 \gamma \rho_+^{\gamma-1}) \phi \psi - (A_2 \alpha \alpha \alpha - A_2 \alpha \alpha \alpha - 1) \vec{\phi} \vec{\psi} \] \] dx - \frac{C \varepsilon (\sigma^{-(\nu-2)} (\phi, \psi, \vec{\phi}, \vec{\psi})^2)}{2} + \| \sigma^{-(\nu-2)} (\phi, \psi, \vec{\phi}, \vec{\psi}) \|^2 \]
\]

where the symmetric matrix \( M_4 \) is defined as

\[
M_4 = \begin{pmatrix}
-A_2 \alpha \gamma \rho_+^{\gamma-2} u_+ & 0 & -A_1 \gamma \rho_+^{\gamma-1} \\
0 & -A_2 \alpha \alpha \alpha - 2 u_+ & -A_2 \alpha \alpha \alpha - 1 \\
-A_1 \gamma \rho_+^{\gamma-1} & -A_2 \alpha \alpha \alpha - 1 & - (\rho_+ + n_+) u_+ \\
\end{pmatrix} \]

(4.27)

Under the condition \( M_+ = 1 \), it is easy to check that three eigenvalues of the matrix \( M_4 \) satisfy: \( \lambda_1 > 0 \), \( \lambda_2 > 0 \), \( \lambda_3 = 0 \). Take the coordinate transformation

\[
\begin{pmatrix}
\phi \\
\vec{\phi} \\
\vec{\psi}
\end{pmatrix} = \begin{pmatrix}
\hat{\rho} \\
\hat{n} \\
\hat{\hat{\psi}}
\end{pmatrix}
\]

(4.28)

where the matrix \( P \) is denoted by

\[
P = \begin{pmatrix}
\rho_{11} & \rho_{12} & -\frac{\rho_{13}}{u_+} \\
\rho_{21} & \rho_{22} & -\frac{\rho_{23}}{u_+} \\
\rho_{31} & \rho_{32} & 1
\end{pmatrix}
\]

(4.29)

such that

\[
(\phi, \vec{\phi}, \vec{\psi}) M_4 (\phi, \vec{\phi}, \vec{\psi})^T = (\hat{\rho}, \hat{n}, \hat{\hat{\psi}}) \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix} (\hat{\rho}, \hat{n}, \hat{\hat{\psi}})^T = \lambda_1 \rho^2 + \lambda_2 \hat{n}^2 \]

(4.30)

By (1.13), (4.20), (4.26), (4.28) and (4.29), the third term is estimated as

\[
- \alpha \nu \int \sigma^{-(\nu-1)} G_1 dx \\
\geq \alpha \nu \int \sigma^{-(\nu-1)} \left( \frac{1}{2} \rho^2 + \frac{1}{2} \hat{n}^2 \right) dx + \alpha \nu \int \sigma^{-(\nu-1)} \left( \frac{A_1 \gamma (\gamma + 1) \rho_+^{\gamma} + A_2 \alpha (\alpha + 1) u_+^\alpha}{2 |u_+|^2} \right) \hat{\psi}^2 dx
\]

\[+ \alpha \nu \int \sigma^{-(\nu-1)} \left( \frac{2 \alpha \alpha \alpha - 2 \vec{\psi} - A_2 \alpha \alpha \alpha - 2 u_+ \vec{\phi}^2}{2} \right) \hat{\psi} dx - C \delta \| \sigma^{-(\nu-2)} (\vec{\phi}, \vec{\psi}) \|^2 - C \delta \| \sigma^{-(\nu-2)} \hat{\hat{\psi}} \|^2
\]

\[- C \varepsilon \| \sigma^{-(\nu-2)} (\phi, \psi, \vec{\phi}, \vec{\psi}) \|^2 - C \varepsilon \| \sigma^{-(\nu-2)} f_\theta (\phi_x, \psi_x, \vec{\phi}_x, \vec{\psi}_x) \|^2
\]

\[- C \varepsilon \frac{1}{\delta \phi^2 (t, 0) + \delta \hat{\hat{\psi}}^2 (t, 0)},
\]

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where we have used the following facts:

\[- (A_1 \gamma \rho_+^{-1} - A_1 \gamma \rho_0^{-1}) \geq \frac{A_1 \gamma (\gamma - 1) \rho_0^{-1}}{|u_+|} \sigma - C \sigma^2,\]

\[- (A_2 \alpha \omega \rho_+^{-1} - A_2 \alpha \omega \rho_0^{-1}) \geq \frac{A_2 \alpha (\alpha - 1) \omega \rho_0^{-1}}{|u_+|} \sigma - C \sigma^2,\]

\[- (A_1 \gamma \rho_+^{-2} \bar{u} - A_1 \gamma \rho_0^{-2} u_+) \geq A_1 \gamma (3 - \gamma) \rho_0^{-2} \sigma - C \sigma^2,\]

\[- (A_2 \alpha \omega \rho_+^{-2} \bar{v} - A_2 \alpha \omega \rho_0^{-2} u_+) \geq A_2 \alpha (3 - \alpha) \omega \rho_0^{-2} - C \sigma^2.\]

(4.32)

With the help of (1.21), (3.7), (4.20), (4.21), (4.28), (4.29) and \(\psi = \bar{\psi} + (\psi - \bar{\psi})\), we get estimates of the forth, the fifth and seventh terms on the left hand side as

\[- av \int \sigma^{-(\nu - 1)} G_2(t, x) dx\]

\[\geq - a^2 \nu (\nu - 1)(\frac{\mu}{2}) \| \sigma^{-\frac{\nu}{2}} \psi \|^2 + \frac{n_+}{2} \| \sigma^{-\frac{\nu}{2}} \bar{\psi} \|^2 - C(\delta + \varepsilon) \| \sigma^{-\frac{\nu}{2}} \bar{\psi} \|^2 - C \varepsilon \| \sigma^{-\frac{\nu}{2}} \bar{\psi}_x \|^2\]

\[\geq - a^2 \sigma^\nu R_3 dx\]

\[\geq a \int \sigma^{-(\nu - 2)} \left[ \frac{A_1 \gamma (\gamma - 1) \rho_0^{-1}}{2} \delta^2 + \rho_+ \psi^2 + \frac{A_2 \alpha (\alpha - 1) \omega \rho_0^{-1}}{2} \delta^2 + n_+ \psi^2 \right] dx\]

\[- C(\delta + \varepsilon) \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 - C \delta \| \sigma^{-\frac{\nu}{2}} (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2 - C \varepsilon \| \sigma^{-\frac{\nu}{2}} \bar{\psi}_x \|^2\]

\[\geq a \int \sigma^{-(\nu - 2)} \left[ \frac{A_1 \gamma (\gamma - 1) \rho_0^{-1}}{2} \delta^2 + \rho_+ \psi^2 + \frac{A_2 \alpha (\alpha - 1) \omega \rho_0^{-1}}{2} \delta^2 + n_+ \psi^2 \right] dx\]

\[- C(\delta + \varepsilon) \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 - C \delta \| \sigma^{-\frac{\nu}{2}} (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2 - C \varepsilon \| \sigma^{-\frac{\nu}{2}} \bar{\psi}_x \|^2\]

(4.34)

(4.35)

For \(\nu \in (0, 3]\), adding (4.33) and (4.35) to the third term in the left side of (4.24) leads to

\[- av \int \sigma^{-(\nu - 1)} G_1 dx - av \int \sigma^{-(\nu - 1)} G_2 dx + \int \sigma^{-(\nu - 2)} (\psi - \bar{\psi})^2 dx + \int \sigma^{-(\nu - 1)} R_1 dx\]

\[\geq c \| \sigma^{-\frac{\nu}{2}} (\bar{\phi}, \bar{\psi}) \|^2 + \frac{1}{4 \alpha} \left( \frac{A_1 \gamma (\gamma + 1) \rho_0^2 + A_2 (\alpha + 1) n_+ \omega}{2 |u_+|^2} \right) \left( 1 + \nu - \frac{\nu (\nu - 1)}{2(1 + b^2)} \right) \| \sigma^{-\frac{\nu}{2}} \bar{\psi} \|^2\]

\[\int \sigma^{-(\nu - 1)} G_1 dx - av \int \sigma^{-(\nu - 1)} G_2 dx + \int \sigma^{-(\nu - 2)} (\psi - \bar{\psi})^2 dx + \int \sigma^{-(\nu - 1)} R_1 dx\]

\[\geq c \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\bar{\phi}, \bar{\psi}) \|^2 + \frac{1}{4 \alpha} \left( \frac{A_1 \gamma (\gamma + 1) \rho_0^2 + A_2 (\alpha + 1) n_+ \omega}{2 |u_+|^2} \right) \left( 1 + \nu - \frac{\nu (\nu - 1)}{2(1 + b^2)} \right) \| \sigma^{-\frac{\nu}{2}} \bar{\psi} \|^2\]

\[\geq c \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\psi - \bar{\psi}) \|^2\]

\[\geq c \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\psi - \bar{\psi}) \|^2\]

\[\geq c \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\psi - \bar{\psi}) \|^2\]

\[\geq c \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\psi - \bar{\psi}) \|^2\]

\[\geq c \| \sigma^{-\frac{\nu}{2}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\bar{\phi}, \bar{\psi}) \|^2 + c \| \sigma^{-\frac{\nu}{2}} (\psi - \bar{\psi}) \|^2\]

(4.36)
where the positive definite matrix $M_5$ is defined by
\[
M_5 = \left( \begin{array}{cc}
\frac{3}{4}n_+ & \frac{a\sqrt{(\mu+n_+)n_+b}}{2}\sigma \\
\frac{a\sqrt{(\mu+n_+)n_+b}}{2}\sigma & \frac{3}{4}n_+ + A_1\gamma(\gamma+1)n_+^2 + A_2\alpha(\alpha+1)n_+^2
\end{array} \right).
\]  
(4.37)

We will consider the case $\nu \in [3, 2+\sqrt{8+\frac{1}{1+b^2}}]$ using the Lemma 4.7 with $\zeta = \sigma^{-(\nu-1)}$. Therefore, with the aid of (2.22), the sixth term is estimated as below:
\[
\int \sigma^{-\nu}(\mu \psi_x^2 + \nu \psi_x^2)dx \\
\geq a^2(\mu + n_+)(\nu - 1)^2 \sigma^{-\nu}(\phi - 2\Phi) + C\delta \sigma^{-\nu}((\phi - 2\Phi) - C(\delta + \varepsilon)\sigma^{-\nu}(\phi - 2\Phi))
\]
\[
\geq a^2(\mu + n_+)(\nu - 1)^2 \sigma^{-\nu}(\phi - 2\Phi) + C\delta \sigma^{-\nu}((\phi - 2\Phi) - C(\delta + \varepsilon)\sigma^{-\nu}(\phi - 2\Phi))
\]  
(4.38)

Under the condition $\nu \in (3, \lambda]$, adding (4.36) to (4.38) and using $c[(\phi, \phi, \phi)] \leq |(\hat{\rho}, \hat{n}, \hat{v})| \leq C|(\phi, \phi, \phi)|$, we have for $k = \nu B[4(1+b^2)\nu + 4b^2 + 5 - \nu^2]^{\frac{1}{2}} < 1$ that
\[
-\nu \int \sigma^{-\nu}G_1dx + \nu \int \sigma^{-\nu}G_1dx + \int \sigma^{-\nu}n(\psi - \psi_x)^2dx + \int \sigma^{-\nu}R_1dx
\]  
(4.39)

where $\delta$ and $\varepsilon$ are small enough, and the positive definite matrix $M_6$ is defined as
\[
M_6 = \left( \begin{array}{cc}
\frac{k_{n_+}}{2} & \frac{aA_1\gamma(\gamma+1)n_+^2 + A_2\alpha(\alpha+1)n_+^2}{2(n_++b)}
\end{array} \right).
\]  
(4.40)

By (1.12), (1.13), (4.29), Cauchy-Schwarz inequality and $u_+^2 = \frac{A_1\gamma\rho_+ + A_2\alpha n_+^2}{\rho_+ + n_+}$, we estimate terms on the right hand side as
\[
|\int \sigma^{-\nu}R_2dx + \int \sigma^{-\nu}R_3dx|
\]  
(4.41)
Finally, with $\delta > 0$ and $\varepsilon > 0$ small enough, and combining (4.25)-(4.41), we obtain

\begin{equation}
\frac{d}{dt} \int \sigma^{-\nu}(E_1 + E_2)dx + c\|\sigma^{-\nu}2(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|^2 + c\|\sigma^{-\nu}2(\psi_x, \tilde{\psi}_x, \bar{\psi} - \psi)\|^2
\end{equation}

\begin{equation}
+ \frac{c}{\delta}\sum_{i}^7 (\phi^2(0,t) + \tilde{\phi}^2(t,0))
\end{equation}

\begin{equation}
\leq C(\delta + \varepsilon)\|\sigma^{-\nu}(\phi_x, \tilde{\phi}_x)\|^2.
\end{equation}

Multiplying (4.42) by $(1 + \delta\tau)^{\xi}$ and integrating the resulted equation in $\tau$ over $[0, t]$, we obtain

\begin{equation}
(1 + \delta\tau)^{\xi}\|\sigma^{-\nu}(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|^2 + \int_0^t (1 + \delta\tau)^{\xi}\|\sigma^{-\nu}2(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|^2d\tau
\end{equation}

\begin{equation}
+ \int_0^t (1 + \delta\tau)^{\xi}\|\sigma^{-\nu}(\psi_x, \tilde{\psi}_x, \bar{\psi} - \psi)\|^2d\tau + \frac{1}{\delta\nu}\int_0^t (1 + \delta\tau)^{\xi}(\phi^2(t,0) + \tilde{\phi}^2(t,0))d\tau
\end{equation}

\begin{equation}
\leq C\|\sigma^{-\nu}(\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0)\|^2 + C(\delta + \varepsilon)\int_0^t (1 + \delta\tau)^{\xi}\|\sigma^{-\nu}(\phi_x, \tilde{\phi}_x)\|^2d\tau
\end{equation}

\begin{equation}
+ C\delta\varepsilon\int_0^t (1 + \delta\tau)^{\xi-1}\|\sigma^{-\nu}(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|^2d\tau,
\end{equation}

which leads to the completeness for the proof of Lemma 4.8.

In order to show Proposition 4.5, we need to obtain the weighted energy estimates of $(\phi_x, \psi_x, \tilde{\phi}_x, \tilde{\psi}_x)$.

**Lemma 4.9.** Under the same conditions in Proposition 4.5, then the solution $(\phi, \psi, \tilde{\phi}, \tilde{\psi})$ to the IBVP (3.3)-(3.7) satisfies for $t \in [0, T]$ that

\begin{equation}
(1 + \delta\tau)^{\xi}\|\sigma^{-\nu}(\phi_x, \tilde{\phi}_x)\|^2 + \int_0^t (1 + \delta\tau)^{\xi}\|\sigma^{-\nu}(\phi_x, \tilde{\phi}_x)\|^2d\tau
\end{equation}

\begin{equation}
\leq C\|\sigma^{-\nu}(\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0)\|^2 + C\|\sigma^{-\nu}(\phi_{xx}, \tilde{\phi}_{xx})\|^2 + C(\delta + \varepsilon)\int_0^t (1 + \delta\tau)^{\xi}\|\sigma^{-\nu}(\psi_{xx}, \tilde{\psi}_{xx})\|^2d\tau
\end{equation}

\begin{equation}
+ \delta\varepsilon\int_0^t (1 + \delta\tau)^{\xi-1}\|\sigma^{-\nu}(\phi, \psi, \tilde{\phi}, \tilde{\psi})\|^2d\tau,
\end{equation}

with $\xi \geq 0$.

**Proof.** Adding (3.25)-(3.28) together, then multiplying the resulted equation by $\sigma^{-\nu}$ and integrating over $\mathbb{R}_+$ with the space weight function $\sigma \geq 0$ satisfying (1.14) and (2.28), we obtain

\begin{equation}
\frac{d}{dt} \int \sigma^{-\nu}(\phi_x^2/2 + \tilde{\phi}_x^2/2 + \rho^2\phi_x\psi + \tilde{n}\tilde{\phi}_x\tilde{\psi})dx - \frac{\sigma^{-\nu}}{\delta\nu} \int \rho^2\phi_t\psi - \tilde{n}\tilde{\phi}_t\tilde{\psi} |dx + \int \sigma^{-\nu}(\rho^2\phi_x \bar{\phi}_x + \tilde{n}\tilde{\phi}_x \bar{\tilde{\phi}})dx
\end{equation}

\begin{equation}
= \sum_{i=1}^6 J_i,
\end{equation}

where

\begin{align*}
J_1 &= - \int \sigma^{-\nu}|\rho^2(\phi + u\phi_x)\psi_x + \tilde{n}(\tilde{\phi}_t + \tilde{\phi}_x\tilde{\psi})\tilde{\psi}_x + 2\rho\tilde{\phi}_x\phi_t\psi + \tilde{n}\tilde{\phi}_t\tilde{\psi} |dx, \\
J_2 &= \int \sigma^{-\nu}|\rho^2(\phi_x\psi_{xx} - \tilde{n}\tilde{\phi}_x\psi_{xx} + \rho\rho^2(\phi_t \psi - \tilde{n}\tilde{\phi}_t\tilde{\psi})dx + \int \sigma^{-\nu}(\rho^2\phi_x \bar{\phi}_x + \tilde{n}\tilde{\phi}_x \bar{\tilde{\phi}})dx
\end{align*}

\begin{equation}
= \sum_{i=1}^6 J_i,
\end{equation}

\begin{align*}
J_3 &= \int \sigma^{-\nu}|\rho^2(\psi - \tilde{n})(\psi - \tilde{\psi})\phi_x |dx,
\end{align*}

\begin{equation}
= \sum_{i=1}^6 J_i.
\end{equation}
\[ J_4 = - \int \sigma^{-\nu} \left( \frac{3}{2} \mu \psi_x \frac{\partial^2}{2} + \frac{3}{2} \psi_x \partial^2 \right) dx + \int \sigma^{-\nu} \left( \frac{\partial \psi x}{n} \right) \partial x dx, \quad J_5 = \int \sigma^{-\nu} (F_1 \rho^2 \phi_x + F_2 \bar{n} \bar{\phi}_x) dx, \]

\[ J_6 = - \int \sigma^{-\nu} \left( \frac{1}{2} \phi_x \tilde{\nu}_x + \psi_x \bar{\nu}_x \right) \phi_x + \mu (\phi \bar{\nu}_x + \psi \bar{\nu}_x) \phi_x \]

\[ - \bar{n} \bar{\phi}_x \bar{\psi}_x + \left( \frac{1}{2} \bar{\partial}_x \bar{\nu}_x + \psi_x \bar{\nu}_x \right) \bar{\phi}_x - \left( \bar{\phi} \bar{\nu}_x + \bar{\psi} \bar{\nu}_x \right) x \bar{\phi}_x dx. \]

Using (1.12), (3.7), and (4.20), we obtain the estimates for terms on the left hand side as below

\[ - [\sigma^{-\nu} (\mu \partial_x^2 + v \partial_x^2 - \bar{\rho} \partial_x \psi - \bar{n} \partial_x \bar{\psi})] (t, 0) \geq \frac{C}{\sigma^2} (\bar{\sigma}^2 (t, 0) + \bar{\sigma}_x^2 (t, 0)), \quad (4.46) \]

\[ - \nu \int \sigma^{-(\nu-1)} (\mu \partial_x^2 + v \partial_x^2) dx \]

\[ \geq \frac{\alpha \nu |u|}{2} \int \sigma^{-(\nu-1)} (\mu \partial_x^2 + v \partial_x^2) dx - C(\varepsilon + \delta) \| \sigma^{-\frac{\nu}{2}} (\phi_x, \bar{\phi}_x) \|^2, \quad (4.47) \]

\[ - \nu \int \sigma^{-(\nu-1)} (-\partial^2 \phi \psi - \bar{n} \bar{\phi} \bar{\psi}) dx \]

\[ \geq -C \varepsilon \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2 - C \| \sigma^{-\overline{\Theta}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 - C \| \sigma^{-\overline{\Theta}} (\psi_x, \bar{\psi}_x) \|^2, \quad (4.48) \]

\[ \int \sigma^{-\nu} \left( \rho^2 \frac{\partial}{\rho} \phi_x + \bar{n} \frac{\partial (n-\bar{\phi} \bar{\psi})}{n} \partial_x^2 \right) dx \geq \frac{A_1 \gamma \rho^4_1}{2} \| \sigma^{-\overline{\Theta}} \phi_x \|^2 + \frac{A_2 \alpha n^{-1}}{2} \| \sigma^{-\overline{\Theta}} \bar{\phi}_x \|^2, \quad (4.49) \]

under the condition \( \delta \) and \( \varepsilon \) small enough.

We turn to estimate terms on the right hand side of (4.45). With the help of (1.12), (4.20), Young inequality and Cauchy-Schwarz inequality, we gain

\[ |J_1| \leq C \delta \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2 + C \| \sigma^{-\overline{\Theta}} (\psi_x, \bar{\psi}_x) \|^2 + C \| \sigma^{-\overline{\Theta}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2, \quad (4.50) \]

\[ |J_2| \leq C(\| \phi, \bar{\phi} \| \| \psi, \bar{\psi} \| \| \overline{\Theta} \phi_x, \bar{\phi}_x, \psi_x, \bar{\psi}_x \| \|^2 + C \delta \| \sigma^{-\overline{\Theta}} \bar{\psi}_x \|^2 \]

\[ \leq C(\delta + \varepsilon) \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2 + C(\delta + \varepsilon) \| \sigma^{-\overline{\Theta}} (\psi_x, \bar{\psi}_x) \|^2 + C \delta \| \sigma^{-\overline{\Theta}} \bar{\psi}_x \|^2, \quad (4.51) \]

\[ |J_3| \leq \frac{A_1 \gamma \rho^4_1}{8} \| \sigma^{-\overline{\Theta}} \phi_x \|^2 + \frac{A_2 \alpha n^{-1}}{8} \| \sigma^{-\overline{\Theta}} \bar{\phi}_x \|^2 + C \| \sigma^{-\overline{\Theta}} \bar{\psi}_x \|^2, \quad (4.52) \]

\[ |J_4| \leq C(\| \psi_x, \bar{\psi}_x \| \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2 + C \| \phi \| \| \psi \| \| \sigma^{-\overline{\Theta}} (\bar{\phi}_x, \bar{\psi}_x) \|^2 \]

\[ \leq C \varepsilon \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2 + C \varepsilon \| \sigma^{-\overline{\Theta}} (\psi_x, \bar{\psi}_x) \|^2 + C \delta \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2, \quad (4.53) \]

\[ |J_5| \leq C \delta \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2 + C \delta \| \sigma^{-\overline{\Theta}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2, \quad (4.54) \]

\[ |J_6| \leq C \delta \| \sigma^{-\overline{\Theta}} (\psi_x, \psi_x) \|^2 + C \delta \| \sigma^{-\overline{\Theta}} \psi_x \|^2 + C \delta \| \sigma^{-\overline{\Theta}} (\psi_x, \bar{\psi}_x) \|^2 + C \delta \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2. \quad (4.55) \]

Finally, the substitution of (4.46)-(4.55) into (4.45) for \( \delta \) and \( \varepsilon \) small enough leads to that

\[ \frac{d}{dt} \int \sigma^{-\nu} [\mu \partial_x^2 + \bar{\rho}^2 \phi_x + \bar{n} \bar{\phi}_x \bar{\psi}] dx + c \| \sigma^{-\overline{\Theta}} (\phi_x, \bar{\phi}_x) \|^2 \]

\[ + \frac{A_1 \gamma \rho^4_1}{4} \| \sigma^{-\overline{\Theta}} \phi_x \|^2 + \frac{A_2 \alpha n^{-1}}{4} \| \sigma^{-\overline{\Theta}} \bar{\phi}_x \|^2 \]

\[ \leq C(\varepsilon + \delta) \| \sigma^{-\overline{\Theta}} (\psi_x, \bar{\psi}_x) \|^2 + C \| \sigma^{-\overline{\Theta}} (\phi, \psi, \bar{\phi}, \bar{\psi}) \|^2 + C \| \sigma^{-\overline{\Theta}} (\psi_x, \bar{\psi}_x, \bar{\psi} - \psi) \|^2. \]
Multiplying (4.56) by \((1 + \delta r)^\xi\) and integrating the resulted equation in \(\tau\) over \([0, t]\) imply
\[
(1 + dt)^\xi \|\sigma^{-\xi}(\phi_x, \phi_x)\|^2 + \int_0^t (1 + \delta \tau)^\xi \|\sigma^{-\xi}(\phi_x, \phi_x)\|^2 d\tau + \int_0^t (1 + \delta \tau)^\xi \|\sigma^{-\xi}(\phi_x, \phi_x)\|^2 d\tau
\]
\[
\leq C \|\sigma^{-\xi}(\phi_0, \psi_0, \psi_{0z}, \phi_0, \phi_0, \phi_{0z}, \phi_{0z}, \psi_{0z})\|^2 + C(\varepsilon + \delta) \int_0^t (1 + \delta \tau)^\xi \|\sigma^{-\xi}(\psi_{xx}, \psi_{xx})\|^2 d\tau
\]
\[
+ \delta \xi \int_0^t (1 + \delta \tau)^{\xi - 1} \|\sigma^{-\xi}(\phi, \psi, \phi_x, \phi, \phi, \psi, \phi, x_x)\|^2 d\tau,
\]
with the help of Cauchy-Schwarz inequality and (4.23).

\[\square\]

**Lemma 4.10.** Under the same conditions in Proposition 4.5, then the solution \((\phi, \psi, \phi, \psi)\) to the IBVP (3.3)-(3.7) satisfies for \(t \in [0, T]\) that
\[
(1 + dt)^\xi \|\sigma^{-\xi}(\phi_x, \phi_x)\|^2 + \int_0^t (1 + \delta \tau)^\xi \|\sigma^{-\xi}(\phi_x, \phi_x)\|^2 d\tau
\]
\[
\leq C \|\sigma^{-\xi}(\phi_0, \psi_0, \phi_{0x}, \phi_0, \psi_0, \phi_{0x}, \phi_{0x}, \psi_{0x})\|^2 + C\xi \int_0^t (1 + \delta \tau)^{\xi - 1} \|\sigma^{-\xi}(\phi, \phi, \phi_x, \phi, \phi, \psi, \phi, \psi)\|^2 d\tau.
\]

with \(\xi \geq 0\).

**Proof.** Multiplying \((3.3)_2\) by \(-\sigma^{-\nu} \psi_{xx}\), \((3.3)_4\) by \(-\sigma^{-\nu} \psi_{xx}\), respectively for the function \(\sigma\) satisfying (1.14) and (2.28), then adding them together and integrating the resulted equation in \(x\) over \(\mathbb{R}_+\) lead to
\[
\frac{d}{dt} \int \sigma^{-\nu} \left(\frac{\psi_x^2}{2} + \frac{\psi_{xx}^2}{2}\right) dx - \nu \int \sigma^{-\nu-1}(\psi_{xx} + \psi_{xx}) dx + \int \sigma^{-\nu} \left(\frac{\mu}{\rho} \psi_{xx}^2 + \psi_{xx}^2\right) dx = \sum_{i=1}^3 K_i, \tag{4.59}
\]
where
\[
K_1 = \int \sigma^{-\nu}[u \psi_x \psi_{xx} - \frac{n}{\rho} (\psi - \psi) \psi_{xx} + \frac{p_1(\rho)}{\rho} \phi_x \psi_{xx} + \nu \bar{\psi} \psi_{xx} + (\bar{\psi} - \psi) \psi_{xx}] dx,
\]
\[
K_2 = \int \sigma^{-\nu} \left[ (\bar{u} \psi_x) \psi_{xx} - \mu \bar{u} \psi_{xx} - \frac{1}{\rho} (\frac{p_1(\rho)}{\rho} - \frac{p_1(\rho)}{\rho}) \psi_{xx} \psi_{xx} - \frac{p_2(n)}{\rho} \frac{n}{\rho} \bar{\psi} \psi_{xx} \right. \]
\[
\left. + \frac{1}{\rho^2} \sigma^{-\nu} \left[ \frac{(\bar{p}_2(n))}{n} \bar{\psi}_x \bar{\psi}_x + \frac{1}{\rho} \bar{\psi}_{xx} \bar{\psi}_{xx} - \frac{1}{n} \bar{p}_3(n) \bar{\psi}_x \bar{\psi}_x - \frac{1}{n} \bar{p}_3(n) \bar{\psi}_x \bar{\psi}_x \right] dx, \]
\[
K_3 = - \int \sigma^{-\nu} \left[ \frac{\partial \psi_{xx}}{n} \frac{\bar{\psi}_x}{n} + \frac{1}{n} \sigma^{-\nu} (\bar{\psi}_x \bar{x}_{xx}) dx. \right.
\]

First, we estimate terms in the left side of (4.59). The second term is estimated as follows:
\[
av \int \sigma^{-\nu-1}(\psi_{xx} + \psi_{xx}) dx \tag{4.60}
\]
\[
\leq C\xi \|\sigma^{-\xi}(\phi_x, \phi_x, \phi_x, \phi_x, \phi_x, \phi_x, \phi_x, \psi - \psi)\|^2 + C\xi \|\sigma^{-\xi}(\phi, \psi, \phi, \psi, \phi, \phi, \psi, \psi)\|^2,
\]
where we have used (1.12), (3.3)_2, (3.3)_4, (4.20) and Cauchy-Schwarz inequality. The third term is estimate as follows:
\[
\int \sigma^{-\nu} \left(\frac{\mu}{\rho} \psi_{xx}^2 + \psi_{xx}^2\right) dx \geq \left[ \frac{\mu}{\rho_+} - C(\varepsilon + \delta) \right] \|\sigma^{-\xi} \psi_{xx}\|^2 + \|\sigma^{-\xi} \psi_{xx}\|^2 \tag{4.61}
\]
\[
\geq \frac{\mu}{2\rho_+} \|\sigma^{-\xi} \psi_{xx}\|^2 + \|\sigma^{-\xi} \psi_{xx}\|^2.
\]
if $\delta$ and $\varepsilon$ are small enough.

We turn to estimate terms on the right hand side of (4.59). With the help of (1.12), (4.20) and Cauchy-Schwarz inequality, we obtain

\[
|K_1| \leq \frac{\mu}{8\rho_+} \|\sigma^{-\frac{3}{2}} \psi_{xx}\|^2 + \frac{1}{8} \|\sigma^{-\frac{3}{2}} \psi_{xx}\|^2 + C \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi}, \psi - \bar{\psi})\|^2,
\]

(4.62)

\[
|K_2| \leq C\delta \|\sigma^{-\frac{3}{2}}(\psi_{xx}, \bar{\psi}_{xx})\|^2 + C\delta \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 + C\delta \|\sigma^{-\frac{3}{2}}(\bar{\phi}, \bar{\psi})\|^2,
\]

(4.63)

\[
|K_3| \leq C(\varepsilon + \delta) \|\sigma^{-\frac{3}{2}}(\bar{\phi}, \bar{\psi})\|^2.
\]

(4.64)

Finally, we substitute (4.60)-(4.64) into (4.59) to gain under the condition $\delta$ and $\varepsilon$ small enough that

\[
\frac{d}{dt} \int \sigma^{-\nu} \left( \frac{\psi_x^2}{2} + \frac{\bar{\psi}_x^2}{2} \right) dx + \frac{\mu}{4\rho_+} \|\sigma^{-\frac{3}{2}} \psi_{xx}\|^2 + \frac{1}{4} \|\sigma^{-\frac{3}{2}} \psi_{xx}\|^2 \leq C \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 + C\delta \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2
\]

(4.65)

Multiplying (4.65) by $(1 + \delta\tau)^{\xi}$ and integrating the resulted inequality in $\tau$ over $\tau \in [0, t]$, we can obtain

\[
(1 + \delta\tau)^{\xi} \|\sigma^{-\frac{3}{2}}(\psi, \bar{\psi})\|^2 + \int_0^t (1 + \delta\tau)^{\xi} \|\sigma^{-\frac{3}{2}}(\psi_{xx}, \bar{\psi}_{xx})\|^2 d\tau \leq C \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 t_1 + C\delta \xi \int_0^t (1 + \delta\tau)^{\xi-1} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 d\tau,
\]

(4.66)

where we have used (4.23), (4.44) and the smallness of $\delta$ and $\varepsilon$. Hence we finish the proof.

**Proof of Proposition 4.5** We sum up the estimates (4.23), (4.44) and (4.58) and take $\delta$ and $\varepsilon$ suitably small to obtain

\[
(1 + \delta\tau)^{\xi} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 t_1 + \int_0^t (1 + \delta\tau)^{\xi} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 d\tau
\]

\[
+ \int_0^t (1 + \delta\tau)^{\xi} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 + C \int_0^t (1 + \delta\tau)^{\xi} \|\sigma^{-\frac{3}{2}}(\psi_{xx}, \bar{\psi}_{xx}, \psi - \bar{\psi})\|^2
\]

\[
+ C \frac{1}{\delta\nu} \int_0^t (1 + \delta\tau)^{\xi} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 d\tau
\]

\[
\leq C \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 t_1 + C\delta \xi \int_0^t (1 + \delta\tau)^{\xi-1} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 d\tau,
\]

(4.67)

for a positive constant $C > 0$ independent of $T$, $\nu$ and $\xi$. Applying induction arguments in [2,17,24] to (4.67), we have

\[
(1 + \delta\tau)\frac{\Delta u + \beta}{\nu} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 + \int_0^t (1 + \delta\tau)\frac{\Delta u + \beta}{\nu} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2
\]

\[
+ \int_0^t (1 + \delta\tau)\frac{\Delta u + \beta}{\nu} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2 \leq C(1 + \delta\tau)^{\beta} \|\sigma^{-\frac{3}{2}}(\phi, \psi, \bar{\phi}, \bar{\psi})\|^2
\]

(4.68)
for $\beta > 0$, which implies

$$\|\sigma^{-\frac{2}{\beta}}(\phi, \psi, \phi_x, \psi_x, \phi_x, \psi_x)\|^2 \leq C(1 + \delta t)^{-\frac{4}{\beta}} \|\sigma^{-\frac{2}{\beta}}(\phi_0, \psi_0, \phi_{0x}, \psi_{0x}, \phi_{0x}, \psi_{0x})\|^2.$$  \hspace{1cm} (4.69)

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