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EXTENSIONS OF SUPER LIE ALGEBRAS

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Abstract. We study (non-abelian) extensions of a given super Lie algebra, identify
a cohomological obstruction to the existence, and interpret it in terms of the super
analogon of the Hochschild-Serre spectral sequence. A striking analogy to the setting
of covariant exterior derivatives, curvature, and the Bianchi identity in differential
geometry is spelled out.

1. Introduction. The theory of group extensions and their interpretation in terms
of cohomology is well known, see, e.g., [3], [6], [4], [2]. In a preliminary version of
this paper [1], we spelled out in detail the counterpart for Lie algebras, thereby
trying to establish connections with algebraic analoga of concepts of differential
geometry: covariant exterior derivatives, curvature and the Bianchi identity. We
thought that for the general case of not necessarily abelian extensions no such
explicit formulation has been given in the then existing literature. Kirill Mackenzie
pointed out to us, however, that in fact most of our results are available already
from [5], [15], [19], and that the case of Lie algebroids is treated in [14].

For super Lie algebras the analogy with differential geometry is far less clear,
and the Hochschild-Serre spectral sequence has to be rechecked. We present the
super results in this paper.

2. Super Lie algebras. (See [8], or [16] for an introduction) A super Lie algebra
is a 2-graded vector space \( g = g_0 \oplus g_1 \), together with a graded Lie bracket \([\ , \ ]\) :
\( g \times g \rightarrow g \) of degree 0. That is, \([\ , \ ]\) is a bilinear map with \([g_i, g_j] \subseteq g_{i+j(mod 2)}\), and
such that for homogeneous elements \( X \in g_x, Y \in g_y, \) and \( Z \in g_z \) the identities

\[
[X, Y] = -(-1)^{xy}[Y, X] \quad \text{(graded antisymmetry)}
\]

\[
[X, [Y, Z]] = [[X, Y], Z] + (-1)^{yz}[Y, [X, Z]] \quad \text{(graded Jacobi identity)}
\]

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hold. The graded Jacobi identity, shorter $\sum_{\text{cyclic}}(-1)^{xz}[X,Y,Z] = 0$, says that $\text{ad}_X : g \rightarrow g$, $Y \mapsto [X,Y]$ is a graded derivation of degree $x$, so that $\text{ad}_X[Y,Z] = [\text{ad}_X Y, Z] + (-1)^{xy}[X, \text{ad}_X Z]$. We denote by $\text{der}(g)$ the super Lie algebra of graded derivations of $g$. The notion of homomorphism is as usual, homomorphisms are always of degree 0.

3. Describing extensions, first part. Consider any exact sequence of homomorphisms of super Lie algebras:

$$0 \rightarrow h \rightarrow \epsilon \rightarrow g \rightarrow 0$$

Consider a graded linear mapping $s : g \rightarrow \epsilon$ of degree 0 with $p \circ s = \text{Id}_g$. Then $s$ induces mappings

(3.1) $\alpha : g \rightarrow \text{der}(h)$ (super connection) by $\alpha_X(H) = [s(X), H],$

(3.2) $\rho : \bigwedge^{2}_{\text{graded}} g \rightarrow h$ (curvature) by $\rho(X,Y) = [s(X), s(Y)] - s([X,Y])$

which are easily seen to be of degree 0 and to satisfy:

(3.3) $[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \text{ad}_{\rho(X,Y)}$

(3.4) $\sum_{\text{cyclic}}(-1)^{xz} \left( \alpha_X \rho(Y,Z) - \rho([X,Y], Z) \right) = 0$

Property (3.4) is equivalent to the graded Jacobi identity in $\epsilon$.

4. Motivation: Lie algebra extensions associated to a principal bundle. In the case of Lie algebras, the extension

$$0 \rightarrow h \rightarrow \epsilon \rightarrow g \rightarrow 0$$

appears in the following geometric situation. Let $\pi : P \rightarrow M = P/K$ be a principal bundle with structure group $K$. Then the Lie algebra of infinitesimal automorphisms $\epsilon = \mathfrak{X}(P)^K$, i.e. the Lie algebra of $K$-invariant vector fields on $P$, is an extension of the Lie algebra $g = \mathfrak{X}(M)$ of all vector fields on $M$ by the Lie algebra $h = \mathfrak{X}_{\text{vert}}(P)^K$ of all vertical $K$-invariant vector fields, i.e., infinitesimal gauge transformations. In this case we have simultaneously an extension of $C^\infty(M)$-modules. A section $s : g \rightarrow \epsilon$ which is simultaneously a homomorphism of $C^\infty(M)$-modules can be considered as a connection, and $\rho$, defined as in 3.2, is the curvature of this connection. This geometric example is a guideline for our approach. It works also for super Lie algebras. See [9], section 11 for more background information. This analogy with differential geometry has also been noticed in [10] and [11] and has been used used extensively in the theory of Lie algebroids, see [14].

5. Algebraic theory of connections, curvature, and cohomology. We want to interpret 3.4 as $\delta_{\alpha,\rho} = 0$ where $\delta_{\alpha}$ is an analogon of the graded version of the Chevalley coboundary operator, but with values in the non-representation $h$; we
shall see that this is exactly the notion of a super exterior covariant derivative. Namely, let $L_{gskew}^{p, y}(\mathfrak{g}; \mathfrak{h})$ be the space of all graded antisymmetric $p$-linear mappings $\Phi : \mathfrak{g}^p \to \mathfrak{h}$ of degree $y$, i.e.

$$\Phi(X_1, \ldots, X_p) \in \mathfrak{h}^{y+x_1+\cdots+x_p},$$

$$\Phi(X_1, \ldots, X_p) = -(-1)^{x_i x_{i+1}} \Phi(X_1, \ldots, X_{i+1}, X_i, \ldots, X_p).$$

In order to treat the graded Chevalley coboundary operator we need the following notation, which is similar to the one used in [12], 3.1: Let $\mathbf{x} = (x_1, \ldots, x_k) \in (\mathbb{Z}_2)^k$ be a multi index of binary degrees $x_i \in \mathbb{Z}_2$ and let $\sigma \in S_k$ be a permutation of $k$ symbols. Then we define the multigraded sign $\text{sign}(\sigma, \mathbf{x})$ as follows: For a transposition $\sigma = (i, i+1)$ we put $\text{sign}(\sigma, \mathbf{x}) = -(-1)^{x_i x_{i+1}}$; it can be checked by combinatorics that this gives a well defined mapping $\text{sign}(\sigma, \mathbf{x}) : S_k \to \{-1, +1\}$. In fact one may define directly $\text{sign}(\sigma, \mathbf{x}) = \text{sign}(\sigma) \text{sign}(\sigma|_{|x_1|, \ldots, |x_k|})$, where $|\cdot| : \mathbb{Z}_2 \to \mathbb{Z}$ is the embedding and where $\sigma|_{|x_1|, \ldots, |x_k|}$ is that permutation of $|x_1| + \cdots + |x_k|$ symbols which moves the $i$-th block of length $|x_i|$ to the position $\sigma i$, and where $\text{sign}(\sigma)$ denotes the ordinary sign of a permutation in $S_k$. Let us write $\sigma \mathbf{x} = (x_{\sigma 1}, \ldots, x_{\sigma k})$, then we have

$$\text{sign}(\sigma \circ \tau, \mathbf{x}) = \text{sign}(\sigma, \mathbf{x}). \text{sign}(\tau, \mathbf{x}),$$

and $\Phi \in L_{gskew}^{p, y}(\mathfrak{g}; \mathfrak{h})$ satisfies

$$\Phi(X_{\sigma 1}, \ldots, X_{\sigma p}) = \text{sign}(\sigma, \mathbf{x}) \Phi(X_1, \ldots, X_p)$$

Given a super connection $\alpha : \mathfrak{g} \to \text{der}(\mathfrak{h})$ as in 3.1, we define the graded version of the covariant exterior derivative by

$$\delta_{\alpha} : L_{gskew}^{p, y}(\mathfrak{g}; \mathfrak{h}) \to L_{gskew}^{p+1, y}(\mathfrak{g}; \mathfrak{h})$$

$$(\delta_{\alpha} \Phi)(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^{x_i x_0 + a_{i}(\mathbf{x})} \alpha_i(\Phi(X_0, \ldots, \widehat{X}_i, \ldots, X_p))$$

$$+ \sum_{i<j} (-1)^{a_{i,j}(\mathbf{x})} \Phi([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_p)$$

$$a_i(\mathbf{x}) = x_i(x_1 + \cdots + x_{i-1}) + i$$

$$a_{i,j}(\mathbf{x}) = a_i(\mathbf{x}) + a_j(\mathbf{x}) + x_i x_j$$

for cochains $\Phi$ with coefficients in the non-representation $\mathfrak{h}$ of $\mathfrak{g}$. In fact, $\delta_{\alpha}$ has the formal property of a super covariant exterior derivative, namely:

$$\delta_{\alpha}(\psi \wedge \Phi) = \delta \psi \wedge \Phi + (-1)^{q} \psi \wedge \delta_{\alpha} \Phi$$

for $\Phi \in L_{gskew}^{p, y}(\mathfrak{g}; \mathfrak{h})$ and $\psi \in L_{gskew}^{q, z}(\mathfrak{g}; \mathbb{R})$ a form of degree $q$ and weight $z$ (we put $\mathbb{R}$ of degree 0), where

$$(\delta \psi)(X_0, \ldots, X_q) = \sum_{i<j} (-1)^{a_{i,j}(\mathbf{x})} \Phi([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_q)$$
is the super analogon of the Chevalley coboundary operator for cochains with values in the trivial $g$-representation $\mathbb{R}$, and where the module structure is given by

$$(\psi \wedge \Phi)(X_1, \ldots, X_{q+p}) =$$

$$= \frac{1}{q!p!} \sum_{\sigma \in S_{q+p}} \text{sign}(\sigma, x)(-1)^b_\sigma(\sigma, x)\psi(X_{\sigma_1}, \ldots, X_{\sigma_q})\Phi(X_{\sigma(q+1)}, \ldots, X_{\sigma(q+p)}),$$

where $b_\sigma(\sigma, x) = |x_{\sigma_1}| + \cdots + |x_{\sigma_q}|.$

Moreover for $\Phi \in L^{p,y}_{gskew}(g; h)$ and $\Psi \in L^{q,z}_{gskew}(g; h)$ we put

$$[\Phi, \Psi]_\lambda(X_1, \ldots, X_{p+q}) =$$

$$= \frac{1}{p!q!} \sum_{\sigma} \text{sign}(\sigma, x)(-1)^b_\sigma(\sigma, x)[\Phi(X_{\sigma_1}, \ldots, X_{\sigma_p}), \Psi(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)})].$$

(5.1) The bracket $[ \ , \ ]_\lambda$ is a $\mathbb{Z} \times \mathbb{Z}_2$-graded Lie algebra structure on

$$L^{p,y}_{gskew}(V, h) = \bigoplus_{p \in \mathbb{Z}_2, y \in \mathbb{Z}_2} L^{p,y}_{gskew}(g; h)$$

which means that the analoga of the properties of section 2 hold for the signs $(-1)^{p_1+q_1} y_1 y_2$. See [12] for more details.

A straightforward computation shows that for $\Phi \in L^{p,y}_{gskew}(g; h)$ we have

(5.2) $\delta_\alpha \delta_\rho(\Phi) = [\rho, \Phi]_\lambda.$

Note that 5.2 justifies the use of the super analogon of the Chevalley cohomology if $\alpha : g \to \text{der}(h)$ is a homomorphism of super Lie algebras or $\alpha : g \to \text{End}(V)$ is a representation in a graded vector space. See [12] for more details.

6. Describing extensions, continued. Continuing the discussion of section 3, we now can describe completely the super Lie algebra structure on $e = h \oplus s(g)$ in terms of $\alpha$ and $\rho$:

(6.1) $[H_1 + s(X_1), H_2 + s(X_2)] =$

$$= ([H_1, H_2] + [H_1, H_2] + \alpha_X(X_1, H_2) - (-1)^{x_1 x_2} \alpha_X(X_1, H_2) + \rho(X_1, X_2)) + s[X_1, X_2].$$

If $\alpha : g \to \text{der}(h)$ and $\rho : \Lambda^2_{\text{graded}} g \to h$ satisfy (3.3) and (3.4) then one checks easily that formula (6.1) gives a super Lie algebra structure on $h \oplus s(g)$.

If we change the linear section $s$ to $s' = s + b$ for linear $b : g \to h$ of degree zero, then we get

(6.2) $\alpha'_X = \alpha_X + \text{ad}_b(X)$

(6.3) $\rho'(X, Y) = \rho(X, Y) + \alpha_X(b(Y)) - (-1)^{x_1} \alpha_Y b(X) - b([X, Y]) + [bX, bY]$

$$= \rho(X, Y) + (\delta_\alpha b)(X, Y) + [bX, bY].$$

$\rho' = \rho + \delta_\alpha b + \frac{1}{2}[b, b]_\lambda.$
7. Proposition. Let \( \mathfrak{h} \) and \( \mathfrak{g} \) be super Lie algebras. Then isomorphism classes of extensions of \( \mathfrak{g} \) over \( \mathfrak{h} \), i.e. short exact sequences of Lie algebras 
\[
0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{e} \longrightarrow \mathfrak{g} \longrightarrow 0
\]
modulo the equivalence described by commutative diagrams of super Lie algebra homomorphisms

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathfrak{h} \\
\downarrow & & \downarrow \phi \\
0 & \longrightarrow & \mathfrak{e} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathfrak{g} \\
\end{array}
\]

correspond bijectively to equivalence classes of data of the following form:

(7.1) A linear mapping \( \alpha : \mathfrak{g} \rightarrow \text{der}(\mathfrak{h}) \) of degree 0,

(7.2) A graded skew-symmetric bilinear mapping \( \rho : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{h} \) of degree 0, such that

\[
[\alpha X, \alpha Y] - \alpha([X,Y]) = \text{ad}_\rho(X,Y)
\]

(7.3) \[
\sum_{\text{cyclic}} (-1)^{xy} \left( \alpha_X \rho(Y,Z) - \rho([X,Y],Z) \right) = 0.
\]

On the vector space \( \mathfrak{e} := \mathfrak{h} \oplus \mathfrak{g} \) a Lie algebra structure is given by

\[
[H_1 + X_1, H_2 + X_2]_\mathfrak{e} = \\
= ([H_1, H_2] + \alpha X_1 H_2) + (-1)^{x_2} \alpha X_2 H_1 + \rho(X_1, X_2) + [X_1, X_2]_\mathfrak{g},
\]

and the associated exact sequence is

\[
0 \\rightarrow \mathfrak{h} \\rightarrow \mathfrak{e} \\rightarrow \mathfrak{g} \\rightarrow 0.
\]

Two data \((\alpha, \rho)\) and \((\alpha', \rho')\) are equivalent if there exists a linear mapping \( b : \mathfrak{g} \rightarrow \mathfrak{h} \) of degree 0 such that

\[
\alpha'_X = \alpha X + \text{ad}^b_{b(X)},
\]

\[
\rho'(X,Y) = \rho(X,Y) + \alpha_X b(Y) - (-1)^{xy} \alpha_Y b(X) - \rho([X,Y], Z) + [b(X), b(Y)]
\]

(7.7) \( \rho' = \rho + \delta_b b + \frac{1}{2}[b, b]_\lambda \),

the corresponding isomorphism being

\[
\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{g} \rightarrow \mathfrak{h} \oplus \mathfrak{g} = \mathfrak{e'}, \quad H + X \mapsto H - b(X) + X.
\]

Moreover, a datum \((\alpha, \rho)\) corresponds to a split extension (a semidirect product) if and only if \((\alpha, \rho)\) is equivalent to a datum of the form \((\alpha', 0)\) (then \( \alpha' \) is a homomorphism). This is the case if and only if there exists a mapping \( b : \mathfrak{g} \rightarrow \mathfrak{h} \) such that

\[
\rho = \delta_\alpha b - \frac{1}{2}[b, b]_\lambda.
\]

Proof. Direct computations. \( \square \)
8. Corollary. Let \( g \) and \( h \) be super Lie algebras such that \( h \) has no (graded) center. Then isomorphism classes of extensions of \( g \) over \( h \) correspond bijectively to homomorphisms of super Lie algebras

\[
\bar{\alpha} : g \to \text{out}(h) = \text{der}(h)/\text{ad}(h).
\]

Proof. Choose a linear lift \( \alpha : g \to \text{der}(h) \) of \( \bar{\alpha} \). Since \( \alpha : g \to \text{der}(h)/\text{ad}(h) \) is a homomorphism, there is a uniquely defined skew symmetric linear mapping \( \rho : g \times g \to h \) such that \( [\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \text{ad}_{\rho(X,Y)} \). Condition (7.4) is then automatically satisfied. For later use we record the simple proof:

\[
\begin{align*}
\sum_{\text{cyclic }X,Y,Z} (-1)^{xz} & \left[ \alpha_X \rho(Y,Z) - \rho([X,Y],Z), H \right] \\
= \sum_{\text{cyclic }X,Y,Z} (-1)^{xz} & \left( \alpha_X[\rho(Y,Z), H] - (-1)^{(x+y+z)}[\rho(Y,Z), \alpha_X H] - \right. \\
& \left. - [\rho([X,Y], Z), H] \right) \\
= \sum_{\text{cyclic }X,Y,Z} (-1)^{xz} & \left( \alpha_X[\alpha_Y, \alpha_Z] - \alpha_X \alpha_{[Y,Z]} - (-1)^{(x+y+z)}[\alpha_Y, \alpha_Z]\alpha_X + \\
& + (-1)^{(x+y+z)}\alpha_{[Y,Z]} \alpha_X - [\alpha_{[X,Y], \alpha_Z} + \alpha_{[[X,Y],Z]} \right)H \\
= \sum_{\text{cyclic }X,Y,Z} (-1)^{xz} & \left( [\alpha_X, [\alpha_Y, \alpha_Z]] - [\alpha_X, \alpha_{[Y,Z]}] - [\alpha_{[X,Y], \alpha_Z} + \alpha_{[[X,Y],Z]} \right)H = 0.
\end{align*}
\]

Thus \( (\alpha, \rho) \) describes an extension, by 7. The rest is clear. \( \square \)

9. Remark. If the super Lie algebra \( h \) has no center and a homomorphism \( \bar{\alpha} : g \to \text{out}(h) = \text{der}(h)/\text{ad}(h) \) is given, the extension corresponding to \( \bar{\alpha} \) is given by the pullback diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & h & \longrightarrow & \text{der}(h) \times_{\text{out}(h)} g & \longrightarrow & 0 \\
\text{pr}_1 \downarrow & & \downarrow \text{pr}_2 & & \alpha \downarrow & & \\
0 & \longrightarrow & h & \longrightarrow & \text{der}(h) & \longrightarrow & \text{out}(h) \longrightarrow 0
\end{array}
\]

where \( \text{der}(h) \times_{\text{out}(h)} g \) is the Lie subalgebra

\[
\text{der}(h) \times_{\text{out}(h)} g := \{(D,X) \in \text{der}(h) \times g : \pi(D) = \tilde{\alpha}(X)\} \subset \text{der}(h) \times g.
\]

We owe this remark to E. Vinberg.

If the super Lie algebra \( h \) has no center and satisfies \( \text{der}(h) = h \), and if \( h \) is an ideal in a super Lie algebra \( e \), then \( e \cong h \oplus e/h \), since \( \text{Out}(h) = 0 \).

10. Theorem. Let \( g \) and \( h \) be super Lie algebras and let

\[
\bar{\alpha} : g \to \text{out}(h) = \text{der}(h)/\text{ad}(h)
\]
be a homomorphism of super Lie algebras. Then the following are equivalent:

(10.1) For one (equivalently: any) linear lift $\alpha : \mathfrak{g} \to \text{der}(\mathfrak{h})$ of degree 0 of $\overline{\alpha}$ choose $\rho : \bigwedge^2_{\text{graded}} \mathfrak{g} \to \mathfrak{h}$ of degree 0 satisfying $([\alpha_X,\alpha_Y] - \alpha_{[X,Y]} = \text{ad}_{\rho(X,Y)}$. Then the $\delta_\alpha$-cohomology class of $\lambda = \lambda(\alpha,\rho) := \delta_\alpha \rho : \bigwedge^3 \mathfrak{g} \to Z(\mathfrak{h})$ in $H^3(\mathfrak{g};Z(\mathfrak{h}))$ vanishes.

(10.2) There exists an extension $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$ inducing the homomorphism $\overline{\alpha}$. If this is the case then all extensions $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$ inducing the homomorphism $\overline{\alpha}$ are parameterized by $H^2(\mathfrak{g},(Z(\mathfrak{h}),\overline{\alpha})), the second graded Chevalley cohomology space of the super Lie algebra $\mathfrak{g}$ with values in the graded $\mathfrak{g}$-module $(Z(\mathfrak{h}),\overline{\alpha})$.

Proof. From the computation in the proof of corollary 8 it follows that $\text{ad}(\lambda(X,Y,Z)) = \text{ad}(\delta_\alpha \rho(X,Y,Z)) = 0$ so that $\lambda(X,Y,Z) \in Z(\mathfrak{h})$. The super Lie algebra $\text{out}(\mathfrak{h}) = \text{der}(\mathfrak{h})/\text{ad}(\mathfrak{h})$ acts on the center $Z(\mathfrak{h})$, thus $Z(\mathfrak{h})$ is a graded $\mathfrak{g}$-module via $\overline{\alpha}$, and $\delta_\alpha$ is the differential of the Chevalley cohomology. Using 5.2, then 5.1 we see $\delta_\alpha = \delta_\alpha \delta_\alpha \rho = [\rho,\rho]_\lambda = -(-1)^{2+0}0[\rho,\rho]_\lambda = 0$, so that $[\lambda] \in H^3(\mathfrak{g};Z(\mathfrak{h}))$.

Let us check next that the cohomology class $[\lambda]$ does not depend on the choices we made. If we are given a pair $(\alpha,\rho)$ as above and we take another linear lift $\alpha' : \mathfrak{g} \to \text{der}(\mathfrak{h})$ then $\alpha'_X = \alpha_X + \text{ad}_{b(X)}$ for some linear $b : \mathfrak{g} \to \mathfrak{h}$. We consider $\rho' : \bigwedge^2_{\text{graded}} \mathfrak{g} \to \mathfrak{h}$, $\rho'(X,Y) = \rho(X,Y) + (\delta_\alpha b)(X,Y) + [b(X),b(Y)]$.

Easy computations show that $[\alpha'_X,\alpha'_Y] - \alpha'_{[X,Y]} = \text{ad}_{\rho'(X,Y)}$ $\lambda(\alpha,\rho) = \delta_\alpha \rho = \delta_\alpha \rho' = \lambda(\alpha',\rho')$ so that even the cochain did not change. So let us consider for fixed $\alpha$ two linear mappings $\rho,\rho' : \bigwedge^2_{\text{graded}} \mathfrak{g} \to \mathfrak{h}$, $[\alpha_X,\alpha_Y] - \alpha_{[X,Y]} = \text{ad}_{\rho(X,Y)} = \text{ad}_{\rho'(X,Y)}$. Then $\rho - \rho' = : \mu : \bigwedge^2_{\text{graded}} \mathfrak{g} \to Z(\mathfrak{h})$ and clearly $\lambda(\alpha,\rho) - \lambda(\alpha,\rho') = \delta_\alpha \rho - \delta_\alpha \rho' = \delta_\alpha \mu$. If there exists an extension inducing $\overline{\alpha}$ then for any lift $\alpha$ we may find $\rho$ as in proposition 7 such that $\lambda(\alpha,\rho) = 0$. On the other hand, given a pair $(\alpha,\rho)$ as in...
(1) such that \([\lambda(\alpha, \rho)] = 0 \in H^3(\mathfrak{g}, (Z(\mathfrak{h}), \tilde{\alpha}))\), there exists \(\mu : \bigwedge^2 \mathfrak{g} \to Z(\mathfrak{h})\) such that \(\delta_\alpha \mu = \lambda\). But then
\[
\text{ad}_{(\rho - \mu)(X,Y)} = \text{ad}_{\rho(X,Y)}, \quad \delta_\alpha (\rho - \mu) = 0,
\]
so that \((\alpha, \rho - \mu)\) satisfy the conditions of 7 and thus define an extension which induces \(\tilde{\alpha}\).

Finally, suppose that (10.1) is satisfied, and let us determine how many extensions there exist which induce \(\tilde{\alpha}\). By proposition 7 we have to determine all equivalence classes of data \((\alpha, \rho)\) as described there. We may fix the linear lift \(\alpha\) and one mapping \(\rho : \bigwedge^2_{\text{graded}} \mathfrak{g} \to \mathfrak{h}\) which satisfies (7.3) and (7.4), and we have to find all \(\rho'\) with this property. But then \(\rho - \rho' = \mu : \bigwedge^2_{\text{graded}} \mathfrak{g} \to Z(\mathfrak{h})\) and
\[
\delta_\alpha \mu = \delta_\alpha \rho - \delta_\alpha \rho' = 0 - 0 = 0
\]
so that \(\mu\) is a 2-cocycle. Moreover we may still pass to equivalent data in the sense of proposition 7 using some \(b : \mathfrak{g} \to Z(\mathfrak{h})\). The corresponding \(\rho'\) is, by (7.7), \(\rho' = \rho + \delta_\alpha b + \frac{1}{2}[b, b] = \rho + \delta_\alpha b\). Thus only the cohomology class of \(\mu\) matters.

11. **Corollary.** Let \(\mathfrak{g}\) and \(\mathfrak{h}\) be super Lie algebras such that \(\mathfrak{h}\) is abelian. Then isomorphism classes of extensions of \(\mathfrak{g}\) over \(\mathfrak{h}\) correspond bijectively to the set of all pairs \((\alpha, [\rho])\), where \(\alpha : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h}) = \text{der}(\mathfrak{h})\) is a homomorphism of super Lie algebras and \([\rho] \in H^2(\mathfrak{g}, \mathfrak{h})\) is a graded Chevalley cohomology class with coefficients in the \(\mathfrak{g}\)-module \(\mathfrak{h}\).

**Proof.** This is obvious from theorem 10.

12. **An interpretation of the class \(\lambda\).** Let \(\mathfrak{h}\) and \(\mathfrak{g}\) be super Lie algebras and let a homomorphism of super Lie algebras \(\tilde{\alpha} : \mathfrak{g} \to \text{der}(\mathfrak{h})/\text{ad}(\mathfrak{h})\) be given. We consider the extension
\[
0 \to \text{ad}(\mathfrak{h}) \to \text{der}(\mathfrak{h}) \to \text{der}(\mathfrak{h})/\text{ad}(\mathfrak{h}) \to 0
\]
and the following diagram, where the bottom right hand square is a pullback (compare with remark 9):

```
0 \arrow{r} \arrow{d} & 0 \arrow{d} \\
\text{Z(}\mathfrak{h}\text{)} \arrow{r} \arrow{d} & \text{Z(}\mathfrak{h}\text{)} \arrow{d} \\
0 \arrow{r} \arrow{u} & \mathfrak{h} \arrow{r} \arrow{u} & \mathfrak{g} \arrow{r} \arrow{u} & 0 \\
0 \arrow{r} \arrow{d} & \text{ad}(\mathfrak{h}) \arrow{r} \arrow{d} \arrow{u} & \mathfrak{g} \arrow{r} \arrow{d} \arrow{u} \arrow{d} & 0 \\
0 \arrow{r} \arrow{u} \arrow{d} & \text{der}(\mathfrak{h}) \arrow{r} \arrow{u} \arrow{d} \arrow{d} & \text{der}(\mathfrak{h})/\text{ad}(\mathfrak{h}) \arrow{r} \arrow{u} \arrow{d} \arrow{d} & 0
```

The pull back \(\beta\) of \(\alpha\) is determined by
\[
\beta : \text{ad}(\mathfrak{h}) \to \text{der}(\mathfrak{h})/\text{ad}(\mathfrak{h})
\]
The left hand vertical column describes \( \mathfrak{h} \) as a central extension of \( \text{ad}(\mathfrak{h}) \) with abelian kernel \( Z(\mathfrak{h}) \) which is moreover killed under the action of \( \mathfrak{g} \) via \( \tilde{\alpha} \); it is given by a cohomology class \([\nu] \in H^2(\text{ad}(\mathfrak{h}); Z(\mathfrak{h}))^\mathfrak{g}\). In order to get an extension \( \mathfrak{e} \) of \( \mathfrak{g} \) with kernel \( \mathfrak{h} \) as in the third row we have to check that the cohomology class \([\nu] \) is in the image of \( i^* : H^2(\tilde{\mathfrak{e}}; Z(\mathfrak{h})) \rightarrow H^2(\text{ad}(\mathfrak{h}); Z(\mathfrak{h}))^\mathfrak{g} \), which is the case if and only if \([\nu] \) is in the kernel of the transgression homomorphism \( \lambda : H^2(\text{ad}(\mathfrak{h}); Z(\mathfrak{h}))^\mathfrak{g} \rightarrow H^3(\mathfrak{g}; Z(\mathfrak{h}))^\mathfrak{g} \) in the following exact sequence, which is a special case of 13 below:

\[
0 \rightarrow H^1(\mathfrak{g}; Z(\mathfrak{h})) \xrightarrow{p^*} H^1(\tilde{\mathfrak{e}}; Z(\mathfrak{h})) \xrightarrow{i^*} H^1(\text{ad}(\mathfrak{h}); Z(\mathfrak{h}))^\mathfrak{g} \xrightarrow{\delta_1=0} \delta_1=0 \rightarrow H^2(\mathfrak{g}; Z(\mathfrak{h})) \xrightarrow{p^*} H^2(\tilde{\mathfrak{e}}; Z(\mathfrak{h})) \xrightarrow{i^*} H^2(\text{ad}(\mathfrak{h}); Z(\mathfrak{h}))^\mathfrak{g} \xrightarrow{\delta_2=\lambda} \delta_2=\lambda \rightarrow H^3(\mathfrak{g}; Z(\mathfrak{h})) \xrightarrow{p^*} H^3(\tilde{\mathfrak{e}}; Z(\mathfrak{h})) \xrightarrow{i^*} H^3(\text{ad}(\mathfrak{h}); Z(\mathfrak{h}))^\mathfrak{g}
\]

Note that, if \([\nu] \) is in the kernel of \( \lambda \), then \((i^*)^{-1}(\nu)\) is a coset in \( H^2(\tilde{\mathfrak{e}}; Z(\mathfrak{h})) \) which is isomorphic to \( H^2(\mathfrak{g}; Z(\mathfrak{h})) \).

### 13. Theorem

Let \( 0 \rightarrow \mathfrak{h} \xrightarrow{i} \mathfrak{e} \xrightarrow{p} \mathfrak{g} \rightarrow 0 \) be an exact sequence of super Lie algebras, and let \( V \) be a graded \( \text{der}(\mathfrak{h})/\text{ad}(\mathfrak{h}) \)-module, and let \( \tilde{\alpha} : \mathfrak{g} \rightarrow \text{der}(\mathfrak{h})/\text{ad}(\mathfrak{h}) \) be the homomorphism induced by the extension. Then the following sequence is exact:

\[
0 \rightarrow H^1(\mathfrak{g}; V) \xrightarrow{p^*} H^1(\tilde{\mathfrak{e}}; V) \xrightarrow{i^*} H^1(\mathfrak{h}; V)^\mathfrak{g} \xrightarrow{\delta_1=0} H^2(\mathfrak{g}; V) \xrightarrow{p^*} \delta_1=0 \rightarrow H^2(\tilde{\mathfrak{e}}; V) \xrightarrow{i^*} H^2(\mathfrak{h}; V)^\mathfrak{g} \xrightarrow{\delta_2=\lambda} \delta_2=\lambda \rightarrow H^3(\mathfrak{g}; V) \xrightarrow{p^*} H^3(\tilde{\mathfrak{e}}; V) \xrightarrow{i^*} H^3(\mathfrak{h}; V)^\mathfrak{g}.
\]

This is a prolonged version of a special case of the super version of the Hochschild-Serre exact sequence, see [7]. From this source one can splice together the result above for Lie algebras. The whole paper [7] is valid for super Lie algebras.

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