ON THE MIURA MAP BETWEEN THE DISPERSIONLESS KP AND DISPERSIONLESS MODIFIED KP HIERARCHIES

Jen-Hsu Chang$^1$ and Ming-Hsien Tu$^2$

$^1$Institute of Mathematics, Academia Sinica, Nankang, Taipei, Taiwan
E-mail: changjen@math.sinica.edu.tw

$^2$Department of Physics, National Chung Cheng University, Minghsiung, Chiayi, Taiwan
E-mail: phymhtu@ccunix.ccu.edu.tw

(April 1, 2022)

Abstract

We investigate the Miura map between the dispersionless KP and dispersionless modified KP hierarchies. We show that the Miura map is canonical with respect to their bi-Hamiltonian structures. Moreover, inspired by the works of Takasaki and Takebe, the twistor construction of solution structure for the dispersionless modified KP hierarchy is given.
I. INTRODUCTION

The dispersionless KP hierarchy (dKP) [1–6] can be thought as the semi-classical limit of the KP hierarchy [7]. There are many mathematical and physical problems associated with the dKP hierarchy and its various reductions, such as Whitham hierarchy, topological field theory and its connections to string theory and 2D gravity [8–12]. Similarly, the dispersionless modified KP (dmKP) hierarchy [13] can be regarded as the semi-classical limit of the modified KP (mKP) hierarchy [14,15]. However, in contrast to dKP, the integrable structures of dmKP are less investigated. This motives us to study the relationships between dKP and dmKP and to gain an insight of dmKP from dKP.

The Miura map [16] has been playing an important role in the development of soliton theory. It’s a transformation between two nonlinear equations, which in general cannot be solved easily. However, knowing the solutions of one of the non-linear systems, one may obtain the solutions of the other one via an appropriate Miura map. A typical example is the Miura map between the KP equation and the mKP equation [17–22]. Motivated by the Miura map between the KP equation and the mKP equation, we will construct the Miura map between dKP and dmKP. (In [23], this Miura map is constructed in different way.) Moreover, since almost all the known integrable systems are Hamiltonian, exploring the Hamiltonian nature of these Miura maps will deepen our understanding of these relations between these integrable systems.

Recently, the canonical property of the Miura map between the mKP and the KP hierarchy has been investigated [22]. It turns out that the Miura map is a canonical map in the sense that the first and second Hamiltonian structures of the mKP hierarchy [24,25] are mapped to the first and second Hamiltonian structures of the KP hierarchy. Since the bi-Hamiltonian structures of mKP and KP have their own correspondences in dmKP and dKP, thus we expect that the bi-Hamiltonian structures of dKP and dmKP are still preserved under the Miura map. We will show, in section 4, that it is indeed the case.

On the other hand, the solution structure of dKP is also an interesting subject. To extend the tau-function theory in KP theory to the semi-classical one in dKP hierarchy and dispersionless analogue of Virasoro constraints [11], Takasaki and Takebe [5] proposed twistor construction of the dKP hierarchy using the Orlov function, which can be regarded as the semi-classical limit of the Orlov operator in KP theory [26,27]. Using the Miura map between dKP and dmKP, we can construct the Orlov function of the dmKP hierarchy and hence establish the twistor construction for dmKP.

Our paper is organized as follows: Section II is background materials for dKP and dmKP; Section III is the Miura map between dKP and dmKP; Section IV proves the canonical property of the Miura map; Section V shows the twistor construction of the dmKP hierarchy; Section VI lists some unsolved problems.

II. BACKGROUND MATERIALS

A. dKP hierarchy

Let’s start with the KP hierarchy. The Lax operator of the KP hierarchy is \( (\partial = \partial_x) \)
\[ L = \partial + \sum_{n=1}^{\infty} u_{n+1} \partial^{-n} \]

and the KP hierarchy is determined by the Lax equations (\( \partial_n = \frac{\partial}{\partial t_n}, t_1 = x \))

\[
\partial_n L = [B_n, L],
\]

(2.1)

where \( B_n = (L^n)_+ \) is the differential part of \( L^n \). The Lax equation (2.1) is equivalent to the existence of the wave function \( \Psi_{KP} \) such that

\[ L \Psi_{KP} = \lambda \Psi_{KP}, \]
\[ \partial_n \Psi_{KP} = B_n \Psi_{KP}. \]

Now for the dKP hierarchy, one can think of fast and slow variables or averaging procedures, by simply taking \( t_n \to \epsilon t_n = T_n(t_1 = x, \epsilon x = X) \) in the KP equation

\[ u_t = \frac{1}{4} u_{xxx} + 3 uu_x + \frac{3}{4} \partial_x^{-1} u_{yy}, \quad (y = t_2, t = t_3) \]

(2.2)

with \( \partial_n \to \epsilon \frac{\partial}{\partial T_n} \) and \( u(t_n) \to U(T_n) \) to obtain

\[ \partial_T U = 3UU_X + \frac{3}{4} \partial_X^{-1} U_{YY} \]

(2.3)

when \( \epsilon \to 0 \) and thus the dispersionless term \( u_{xxx} \) is removed. In terms of hierarchies we write

\[ L_{\epsilon} = \epsilon \partial + \sum_{n=1}^{\infty} u_{n+1}(T/\epsilon)(\epsilon \partial)^{-n} \]

and think of \( u_n(T/\epsilon) = U_n(T) + O(\epsilon) \), etc. One then takes a WKB form for the wave function \( \Psi_{KP} \) with the action \( S_{KP} \)

\[ \Psi_{KP} = \exp \left[ \frac{1}{\epsilon} S_{KP}(T, \lambda) \right]. \]

Now, we replace \( \partial_n \) by \( \epsilon \frac{\partial}{\partial T_n} \) and define \( P = \partial_X S_{KP} \). Then \( \epsilon \partial^n \Psi_{KP} \to P^n \Psi_{KP} \) as \( \epsilon \to 0 \) and the equation \( L \Psi_{KP} = \lambda \Psi_{KP} \) implies

\[ \lambda = P + \sum_{n=1}^{\infty} U_{n+1}(T) P^{-n}. \]

We also note from \( \partial_n \Psi_{KP} = B_n \Psi_{KP} \) that one obtains \( \frac{\partial S_{KP}}{\partial T_n} = B_n(P) = (\lambda^n)_+ \), where the subscript (+) now refers to powers of \( P \). The KP hierarchy goes to

\[ \frac{\partial P}{\partial T_n} = \frac{\partial B_n(P)}{\partial X}. \]

(2.4)

Also, the Lax equation (2.1) goes to

\[ \partial_n \lambda = \{ B_n(P), \lambda \}, \]

(2.5)
where the Poisson bracket \{,\} is defined by
\[
\{f(X,P), g(X,P)\} = \frac{\partial f}{\partial P} \frac{\partial g}{\partial X} - \frac{\partial f}{\partial X} \frac{\partial g}{\partial P}.
\] (2.6)

Notice that both the equations (2.4) and (2.5) are compatible respectively, i.e,
\[\frac{\partial^2 \lambda}{\partial T_n \partial T_m} = \frac{\partial^2 \lambda}{\partial T_m \partial T_n}, \quad \frac{\partial^2 P}{\partial T_n \partial T_m} = \frac{\partial^2 P}{\partial T_m \partial T_n},\] and they both imply the dKP hierarchy
\[
\frac{\partial B_n(P)}{\partial T_m} - \frac{\partial B_m(P)}{\partial T_n} + \{B_n(P), B_m(P)\} = 0.
\] (2.7)

In particular,
\[B_2(P) = P^2 + 2U_2,\]
\[B_3(P) = P^3 + 3U_2P + 3U_3.\]

Then \((T_2 = Y, T_3 = T)\)
\[
\frac{\partial B_2(P)}{\partial T} - \frac{\partial B_3(P)}{\partial Y} + \{B_2(P), B_3(P)\} = 0
\]
becomes
\[U_{3X} = \frac{1}{2} U_{2Y},\]
\[U_{3Y} = \frac{2}{3} U_{2T} - 2U_2U_{2X}\]
and thus
\[\frac{1}{2} U_{2YY} = \frac{2}{3} (U_{2T} - 3U_2U_{2X})_X.\]
This is the dKP equation (2.3) \((U_2 = U)\).

In summary, we define the dKP hierarchy by
\[
\lambda = P + \frac{U_2}{P} + \frac{U_3}{P^2} + \cdots ,
\] (2.8)
\[
\partial_n \lambda = \{B_n(P), \lambda\}.\] (2.9)

Let us define the Hamiltonians \(H_k = 1/k \int \text{res}(\lambda^k)\), where res means the coefficient of \(P^{-1}\),
then the bi-Hamiltonian structure of dKP (2.3) is given by \[28\][13]
\[
\frac{\partial \lambda}{\partial T_k} = \{H_k, \lambda\} = \Theta^{(2)}(dH_k) = \Theta^{(1)}(dH_{k+1}), \quad k = 1, 2, \cdots
\]
where the Hamiltonian one-form \(dH_k\) and the Hamiltonian maps \(\Theta^{(i)}\) are defined by
\[
dH_k = \frac{\delta H_k}{\delta U_2} + \frac{\delta H_k}{\delta U_3} P + \frac{\delta H_k}{\delta U_4} P^2 + \frac{\delta H_k}{\delta U_5} P^3 + \cdots,
\]
\[
\Theta^{(2)}(dH_k) = \lambda \{\lambda, dH_k\} + \{\lambda, \lambda dH_k\} + \lambda \int^X \text{res}(\lambda, dH_k),
\] (2.10)
\[
\Theta^{(1)}(dH_{k+1}) = \lambda \{\lambda, dH_{k+1}\} + \{\lambda, \lambda dH_{k+1}\} + \lambda \int^X \text{res}(\lambda, dH_{k+1}),
\]
the third term of (2.10) being Dirac reduction for \(U_1 = 0\).
B. dmKP

The Lax operator of the mKP hierarchy is defined \[18\] by
\[ K = \partial + v_0 + v_1 \partial^{-1} + v_2 \partial^{-2} + \cdots, \]
which satisfies the Lax equations
\[ \partial_n K = [Q_n, K], \tag{2.11} \]
where \( Q_n = (K^n)_{\geq 1} \) means the part of order \( \geq 1 \) of \( K^n \). Also, the Lax equation (2.11) is equivalent to the existence of wave function \( \Psi_{mKP} \) such that
\[ K \Psi_{mKP} = \mu \Psi_{mKP}, \]
\[ \partial_n \Psi_{mKP} = Q_n \Psi_{mKP}. \]

To obtain the dmKP hierarchy, similarly, one takes \( t_n \to \epsilon t_n = T_n (t_1 = x \to \epsilon t_1 = X) \) in the mKP equation
\[ v_t = \frac{1}{4} v_{xxx} - \frac{3}{2} v_x v_{xx} + \frac{3}{2} v_x \partial_x^{-1} v_y + \frac{3}{4} \partial_x^{-1} v_{yy}, \tag{2.12} \]
with \( \partial_n \to \epsilon \partial / \partial T_n \) and \( v(t_n) \to V(T_n) \) to get
\[ V_T = -\frac{3}{2} V^2 V_X + \frac{3}{2} V_X \partial_X^{-1} V_Y + \frac{3}{4} \partial_X^{-1} V_{YY}, \tag{2.13} \]
when \( \epsilon \to 0 \). Thus, the dispersionless term \( v_{xxx} \) is removed, too. In terms of hierarchies, we write
\[ K_\epsilon = \epsilon \partial + v_1 (T/\epsilon) (\epsilon \partial)^{-1} + v_2 (T/\epsilon) (\epsilon \partial)^{-2} + \cdots \]
and think of \( v_n (T/\epsilon) = V_n (T) + 0(\epsilon) \). One then takes a WKB form for the wave function \( \Psi_{mKP} \) with the action \( S_{mKP} \):
\[ \Psi_{mKP} = \exp \left( \frac{1}{\epsilon} S_{mKP}(T, \mu) \right). \]
Now we replace \( \partial_n \) by \( \epsilon \partial / \partial T_n \) and define \( P = \partial_X S_{mKP} \). Then \( \epsilon^i \partial_X^i \Psi_{mKP} \to P^i \Psi_{mKP} \) as \( \epsilon \to 0 \) and the equation \( K \Psi_{mKP} = \mu \Psi_{mKP} \) yields
\[ \mu = P + \sum_{n=0}^{\infty} V_n (T) P^{-n}. \]

From \( \partial_n \Psi_{mKP} = Q_n \Psi_{mKP} \), one obtains \( \partial S_{mKP} / \partial T_n = Q_n (P) = (\mu^n)_{\geq 1} \), where the subscript \( \geq 1 \) refers to powers \( \geq 1 \) of \( P \). The dmKP hierarchy goes to
\[ \frac{\partial P}{\partial T_n} = \frac{\partial Q_n (P)}{\partial X}. \]

It also can be written as the following zero-curvature form
\[
\frac{\partial Q_n(P)}{\partial T_m} - \frac{\partial Q_m(P)}{\partial T_n} + \{Q_n(P), Q_m(P)\} = 0,
\]
where the Poisson bracket is defined by (2.6). In particular,
\[
Q_2(P) = P^2 + 2PV_0, \\
Q_3(P) = P^3 + 3P^2V_0 + P(V_1 + 3V_0^2).
\]

Then the equation \( T_2 = Y, T_3 = T \)
\[
\frac{\partial Q_2(P)}{\partial T} - \frac{\partial Q_3(P)}{\partial Y} + \{Q_2(P), Q_3(P)\} = 0,
\]
becomes
\[
V_1X = \frac{3}{2}V_0Y - \frac{3}{2}(V_0^2)X, \\
V_1Y = 2V_0T - 3V_0V_0Y - 2V_1V_0X.
\]
which implies the dmKP (2.13) \((V_0 = V)\).
In summary, we write the dmKP equation as
\[
\mu = P + V_0 + \frac{V_1}{P} + \frac{V_2}{P^2} + \cdots, \\
\partial_n\mu = \{Q_n(P), \mu\}.
\]
If we define the Hamiltonians as \( H_k = \frac{1}{k} \iint \text{res}(\mu^k) \), then the bi-Hamiltonian structure of (2.15) is described by
\[
\frac{\partial \mu}{\partial T_k} = \{H_k, \mu\} = J^{(2)}(dH_k) = J^{(1)}(dH_{k+1})
\]
where
\[
dH_k = \frac{\delta H_k}{\delta V_0}P^{-1} + \frac{\delta H_k}{\delta V_1}P + \frac{\delta H_k}{\delta V_2}P^2 + \cdots, \\
J^{(2)}(dH_k) = \mu\{\mu, dH_k\}_{\geq -1} - \{\mu, (\mu dH_k)_{\geq 1}\}, \\
J^{(1)}(dH_{k+1}) = \{\mu, dH_{k+1}\}_{\geq -1} - \{\mu, (dH_{k+1})_{\geq 1}\}.
\]

### III. DISPERSIONLESS MIURA MAP

It has been shown [17–22] that there exists a gauge transformation (Miura map) between the Lax operator \( L \) of KP and the Lax operator \( K \) of mKP, namely,
\[
K = \Phi^{-1}(t)L\Phi(t),
\]
where \( \Phi(t) \) is an eigenfunction of \( L \), i.e.,
\[
\partial_n\Phi = (L^n)\Phi.
\]
One generalizes this result to dispersionless limit case.

Let

$$\mathcal{L} = P^m + a_{m-1}P^{m-1} + a_{m-2}P^{m-2} + \cdots + a_0 + \frac{a_{-1}}{P} + \frac{a_{-2}}{P^2} + \cdots,$$

where $a_{m-1}, a_{m-2}, \ldots, a_0, a_{-1}, a_{-2}, \cdots$ are functions of $T = (T_1 = X, T_2, T_3, \cdots)$. Also, we suppose $\phi(T)$ (independent of $P$) is any function of $T$. We define

$$\tilde{\mathcal{L}} = e^{-ad\phi(T)} \mathcal{L},$$

$$= \mathcal{L} - \{\phi, \mathcal{L}\} + \frac{1}{2} \{\phi, \{\phi, \mathcal{L}\}\} - \frac{1}{3!} \{\phi, \{\phi, \{\phi, \mathcal{L}\}\}\} + \cdots$$

where the Poisson bracket is defined by (2.6). Since $\phi$ is independent of $P$, a simple calculation gets

$$\tilde{\mathcal{L}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\phi_X)^n \partial_P^n \mathcal{L}. \quad (3.3)$$

**Lemma 1** Let $\tilde{\mathcal{L}}$ be defined as above. Then

$$\tilde{\mathcal{L}}_{\geq 1} = e^{-ad\phi}(\mathcal{L}_{\geq 0}) - \mathcal{L}_{\geq 0}|_{P=\phi_X},$$

where

$$\mathcal{L}_{\geq 0}|_{P=\phi_X} = \phi_X^m + a_{m-1}\phi_X^{m-1} + \cdots + a_1\phi_X + a_0.$$

**Proof.** From (3.3), one knows that $\tilde{\mathcal{L}}_{\geq 0}$ comes from the polynomial part of $\mathcal{L}$. Hence

$$\tilde{\mathcal{L}}_{\geq 1} = \tilde{\mathcal{L}}_{\geq 0} \mathcal{L}_0,$$

$$= e^{-ad\phi}(\mathcal{L}_{\geq 0}) - e^{-ad\phi}(\mathcal{L}_{\geq 0})|_{P=0}.$$

Using (3.3), one knows

$$e^{-ad\phi}(\mathcal{L}_{\geq 0})|_{P=0} = \sum_{n=0}^{\infty} \frac{1}{n!} (\phi_X)^n (\partial_P^n \mathcal{L}_{\geq 0}|_{P=0}),$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (\phi_X)^n (a_n n!),$$

$$= \phi_X^m + a_{m-1}\phi_X^{m-1} + a_{m-2}\phi_X^{m-2} + \cdots + a_1\phi_X + a_0,$$

$$= \mathcal{L}_{\geq 0}|_{P=\phi_X}.$$ 

This completes the lemma. \(\square\)

**Lemma 2** $e^{-ad\phi}\{f(T, P), g(T, P)\} = \{e^{-ad\phi} f(T, P), e^{-ad\phi} g(T, P)\}$. 

7
Proof.

\[
\begin{align*}
\text{r.h.s.} &= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \{ \phi_X^n \partial_P^n f, \phi_P^m \partial_P^m g \}, \\
&= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \phi_X^{m+n} \{ \partial_P^n f, \partial_P^m g \}, \\
&= \sum_{m=0}^{\infty} \phi_X^m \sum_{n=0}^{m} \left( \begin{array}{c} m \\ n \end{array} \right) \{ \partial_P^n f, \partial_P^{m-n} g \}, \\
&= e^{-\text{ad}_\phi \{ f, g \}} = \text{l.h.s.} \quad \Box
\end{align*}
\]

**Theorem 3** Let \( \mathcal{L} \) be defined as above. Then

\[
\tilde{\mathcal{L}}_{T_q} - \{ (\mathcal{L}^q)_{\geq 1}, \mathcal{L} \} = e^{-\text{ad}_\phi} (\mathcal{L}_{T_q} - \{ (\mathcal{L}^q)_+, \mathcal{L} \}) - \{ \phi_{T_q} - (\mathcal{L}^q)|_{P=\phi_X}, \mathcal{L} \},
\]

where the subscript \( T_q \) means \( \partial/\partial T_q \).

**Proof.** Using (3.3), we have

\[
\begin{align*}
\frac{\partial \tilde{\mathcal{L}}}{\partial T_q} &= e^{-\text{ad}_\phi} \frac{\partial \mathcal{L}}{\partial T_q} + \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial T_q} (\phi_X^n) \partial_P^n \mathcal{L}, \\
&= e^{-\text{ad}_\phi} \frac{\partial \mathcal{L}}{\partial T_q} + \left( \frac{\partial^2 \phi}{\partial T_q \partial X} \right) \sum_{n=0}^{\infty} \frac{1}{n!} \phi_X^n \partial_P^{n+1} \mathcal{L}, \\
&= e^{-\text{ad}_\phi} \frac{\partial \mathcal{L}}{\partial T_q} - \{ \phi_{T_q}, e^{-\text{ad}_\phi} \mathcal{L} \}.
\end{align*}
\]

Then, by Lemmas 1 and 2, we have

\[
\begin{align*}
\tilde{\mathcal{L}}_{T_q} - \{ (\mathcal{L}^q)_{\geq 1}, \mathcal{L} \} &= e^{-\text{ad}_\phi} (\mathcal{L}_{T_q} - \{ \phi_{T_q}, \mathcal{L} \}) - \{ e^{-\text{ad}_\phi} (\mathcal{L}^q)_+ - (\mathcal{L}^q)_+|_{P=\phi_X}, e^{-\text{ad}_\phi} \mathcal{L} \}, \\
&= e^{-\text{ad}_\phi} (\mathcal{L}_{T_q} - \{ (\mathcal{L}^q)_+, \mathcal{L} \}) + \{ (\mathcal{L}^q)_+|_{P=\phi_X} - \phi_{T_q}, \tilde{\mathcal{L}} \}.
\end{align*}
\]

This completes the theorem. \( \Box \)

**Corollary 4** Let

\[
\mathcal{L} = P + \frac{U_2}{P} + \frac{U_3}{P^2} + \frac{U_4}{P^3} + \cdots
\]

and suppose that \( U_i(T) \) satisfy the dKP hierarchy (2.3) (\( \lambda = \mathcal{L} \)) and \( \phi(T) \) satisfies the equation

\[
\frac{\partial \phi}{\partial T_n} = (\mathcal{L}^n)_+|_{P=\phi_X}. \quad \text{(3.4)}
\]

Then \( \tilde{\mathcal{L}} = e^{-\text{ad}_\phi} \mathcal{L} \) will satisfy the dmKP hierarchy (2.13) (\( \mu = \tilde{\mathcal{L}} \)).
Proof. Obvious. □

From the corollary, one calls the map
\[ \mathcal{L} \to e^{-ad\phi} \mathcal{L} \] (3.5)

the dispersionless Miura map between dKP and dmKP. It’s because one can think the map (3.5) as the dispersionless limit of equation (3.1) and, moreover, the equation (3.4) can be regarded as the dispersionless limit of equation (3.2). As in the case of KP and mKP, the dispersionless Miura map gives rise to a transformation between dKP and dmKP in terms of "dispersionless" eigenfunction \( \phi(T) \). If one assumes that
\[
\tilde{L} = P + V_0 + \frac{V_1}{P} + \frac{V_2}{P^2} + \frac{V_3}{P^3} + \cdots,
\]

then, after some calculations, one gets
\[
V_0 = \phi_X,
V_1 = U_2, \quad (3.6)
V_2 = U_3 + \phi_X U_2,
V_3 = U_4 + 2\phi_X U_3 + \phi_X^2 U_2,
V_4 = U_5 + 3\phi_X U_4 + 3\phi_X^2 U_3 + \phi_X^3 U_2,
\]
\[
\vdots
V_n = \sum_{i=0}^{n-1} \binom{n-1}{i} \phi_X^i U_{n+1-i}, \quad n \geq 1.
\]

Finally, it is well known that Miura-type transformations between (2.2) and (2.12) are
\[
u_1 = \frac{3}{2}(-v^2 - v_x + \partial_x^{-1}v_y),
\]
\[
u_2 = \frac{3}{2}(-v^2 + v_x + \partial_x^{-1}v_y).
\]

In the dispersionless limit, the term \( v_x \) is removed and we obtain the only transformation
\[ U = \frac{3}{2}(-V^2 + \partial_y^{-1}V_x). \] (3.7)

Notice that we can also obtain the equation (3.7) from (2.14) and (3.6). Furthermore, since the term corresponding to \( v_x \) is removed, this would explain why we cannot find the auto-Bäcklund transformation of the dKP hierarchy as one did in the ordinary case [21].

IV. CANONICAL PROPERTY OF THE MIURA MAP

Having constructed the dispersionless Miura map between the dKP hierarchy and the dmKP hierarchy in the Lax formulation, which provides a connection of solutions associated with dKP and dmKP, we next would like to investigate the canonical property of
the Miura map. As we have seen that both dKP and dmKP hierarchies equip a compatible bi-Hamiltonian structure, thus it is quite natural to ask whether their bi-Hamiltonian structures are still preserved under the Miura map.

To proceed the discussion, it is convenient to rewrite the dispersionless Miura map as

$$G : \mu(T, P) \rightarrow \lambda(T, P) = e^{ad\phi(T)} \mu(T, P) \quad (4.1)$$

where \( \lambda \) and \( \mu \) are Lax operators of the dKP and dmKP hierarchies respectively and the function \( \phi(T) = \int X V_0 \) is independent of \( P \). In the following, the symbols \( A, B \) and \( C \) will stand for arbitrary Laurent series without further mention.

Lemma 5 \( e^{-ad\phi(T)} e^{ad\phi(T)} A = A \).

Proof. By definition,

$$e^{-ad\phi(T)} e^{ad\phi(T)} A = e^{-ad\phi(T)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \phi_X^n \partial_P^n A,$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{m! n!} \phi_X^{m+n} \partial_P^{m+n} A,$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \phi_X^m \partial_P^m A \sum_{n=0}^{m} (-1)^n \binom{m}{n} = A. \quad \square$$

Lemma 6 \( e^{-ad\phi}(AB) = (e^{-ad\phi} A)(e^{-ad\phi} B) \).

Proof.

$$e^{-ad\phi}(AB) = \sum_{n=0}^{\infty} \frac{\phi_X^n}{n!} \partial_P^n (AB),$$

$$= \sum_{n=0}^{\infty} \frac{\phi_X^n}{n!} \left[ \sum_{m=0}^{n} \binom{n}{m} \left( \partial_P^m A \right) \left( \partial_P^{n-m} B \right) \right],$$

$$= \sum_{n=0}^{\infty} \left[ \sum_{m=0}^{n} \frac{\phi_X^{n-m} \phi_X^m}{m! (n-m)!} \left( \partial_P^m A \right) \left( \partial_P^{n-m} B \right) \right],$$

$$= (\sum_{m=0}^{\infty} \frac{\phi_X^m}{m!} \partial_P^m A) (\sum_{n=0}^{\infty} \frac{\phi_X^n}{n!} \partial_P^n B),$$

$$= (e^{-ad\phi} A)(e^{-ad\phi} B). \quad \square$$

Lemma 7 \( \int \text{res}(A\{B, C\}) = \int \text{res}(\{A, B\}C) \)

Proof.

$$l.h.s. = \int \text{res} \left[ A \left( \frac{\partial B}{\partial P} \frac{\partial C}{\partial X} - \frac{\partial B}{\partial X} \frac{\partial C}{\partial P} \right) \right],$$

$$= \int \text{res} \left[ -\frac{\partial}{\partial X} \left( A \frac{\partial B}{\partial P} \right) C + \frac{\partial}{\partial P} \left( A \frac{\partial B}{\partial X} \right) C \right],$$

$$= r.h.s. \quad \square$$
To investigate the canonical property of the Miura map (4.1) we shall first construct the tangential map between the tangent spaces (to which $\delta \lambda$ and $\delta \mu$ belong) of the corresponding phase space manifolds.

**Theorem 8** For the Miura map $G$, the linearized map $G'$ and its transposed map $G'^\dagger$ are given by

$$G' : B \rightarrow e^{ad(T)} B + \{ \int^X b_0, \lambda \}, \quad (4.2)$$

$$G'^\dagger : A \rightarrow e^{-ad(T)} A + P^{-1} \int^X \text{res}\{A, \lambda\} \quad (4.3)$$

where $b_0 \equiv (B)_0$ and $\dagger$ is the transposed operation defined by $\int \text{res}(AG'B) = \int \text{res}((G'^\dagger A)B)$.

**Proof.** Let $B = \delta \mu$ be an infinitesimal deformation of the Lax operator $\mu$, then under the Miura map $G$ we have

$$\mu + B \rightarrow e^{ad(\phi + \int^X b_0)}(\mu + B),$$

$$= e^{ad\phi} \mu + e^{ad\phi} B + \{ \int^X b_0, \lambda \} + O(B^2).$$

which implies the linearized map (4.2). On the other hand, using Lemmas 5-7 and the fact $\text{res}(e^{ad\phi} A) = \text{res}(A)$ we have

$$\int \text{res}(AG'B) = \int \text{res}(A(e^{ad\phi} B)) + \int \text{res}(A\{\int^X b_0, \lambda\}),$$

$$= \int \text{res}((e^{-ad\phi} A)B) + \int b_0 \int^X \text{res}\{A, \lambda\},$$

$$= \int \text{res}((e^{-ad\phi} A)B) + \int \text{res}((P^{-1} \int^X \text{res}\{A, \lambda\})B)$$

where we have used integration by part and $b_0 = \text{res}(BP^{-1})$ to reach the last line. Comparing the last line with $\int \text{res}((G'^\dagger A)B)$ we obtain (L3). □

Now we are in a position to investigate the canonical property of the Miura map.

**Theorem 9** The Miura map $G$ maps the bi-Hamiltonian structure of the dmKP hierarchy given by $J^{(1)}$ and $J^{(2)}$ to the bi-Hamiltonian structure of the dKP hierarchy given by $\Theta^{(1)}$ and $\Theta^{(2)}$ respectively, i.e., they are related by

$$\Theta^{(1)} = G' J^{(1)} G'^\dagger, \quad (4.4)$$

$$\Theta^{(2)} = G' J^{(2)} G'^\dagger \quad (4.5)$$

where $G'$ and $G'^\dagger$ are transformations defined in Theorem 8.

**Proof.** To prove the first structure, let us act the right hand side of (L4) on an arbitrary Laurent series $A$, then $G' J^{(1)} (G'^\dagger A) = G'B$ where

$$B \equiv J^{(1)} (G'^\dagger A),$$

$$= \{\mu, G'^\dagger A\}_{\geq -1} - \{\mu, (G'^\dagger A)_{\geq 1}\},$$

$$= e^{-ad\phi} \left(\{\lambda, A\}_+ - \{\lambda, A_+\} + \{\lambda, (e^{-ad\phi} A)_0\} \right).$$

11
and thus
\[ \int X b_0 = \int X (B)_0 = (e^{-a\phi} A)_0. \]  
(4.7)

Substituting (4.6) and (4.7) into (4.2) we have
\[ G'J^{(1)}(G^{\dagger A}_1) = e^{a\phi} B + \{(e^{-a\phi} A)_0, \lambda\}, \]
\[ = \{\lambda, A\}_+ - \{\lambda, A_+\} = \Theta^{(1)}(A). \]

This completes the first part of the proof. For the second Hamiltonian structure, using (2.16) and (4.3) we have
\[ B \equiv J^{(2)}(G^{\dagger A}_1), \]
\[ = \{\mu, G^{\dagger A}_1\}_{+\mu} - \{\mu, (\mu G^{\dagger A}_1)_{+}\} + \{\mu, (\mu G^{\dagger A}_1)_0\} + \mu P^{-1}\text{res}\{\mu, G^{\dagger A}_1\} \]
(4.8)

where each term in (4.8) can be calculated as follows:
\[ (1) = e^{-a\phi} \{\lambda, A\}_+ + \lambda, \]
\[ (2) = -e^{-a\phi} \{\lambda, (A\lambda)_+\} + \lambda, \int X \text{res}\{A, \lambda\}, \]
\[ (3) = e^{-a\phi} \{\lambda, (e^{-a\phi} (A\lambda)_0) + \{\lambda, \int X \text{res}\{A, \lambda\}\}, \]
\[ (4) = 0. \]

Then
\[ B = (1) + (2) + (3) + (4), \]
\[ = e^{-a\phi} \{\lambda, A\}_+ + \lambda - \{\lambda, (A\lambda)_+\} + \{\lambda, (e^{-a\phi} (A\lambda)\}_0\}, \]
(4.9)

and
\[ \int X b_0 = \left(e^{-a\phi} (\lambda A)\right)_0 + \int X \text{res}\{A, \lambda\}. \]
(4.10)

Substituting (4.9) and (4.10) into (4.2) we get
\[ G'J^{(2)}(G^{\dagger A}_1) = \lambda \{\lambda, A\}_+ - \{\lambda, (\lambda A)_+\} + \{\lambda, \int X \text{res}\{\lambda, A\}\} = \Theta^{(2)}(A). \]

This completes the theorem. □

V. SOLUTION STRUCTURE OF DISPERSIONLESS MKP

In [5,6], it is shown that the twistor construction exists for the solution structure of dKP hierarchy. Based on the dispersionless Miura map described in Section III, we can also find a similar twistor construction for solution structure of dmKP. This is the purpose of this section.
First of all, let’s recall the twistor construction of dKP in \([5,6]\). Here we change slightly the symbols used in those papers. Let’s consider the dKP (2.9). It can be shown that there exists a Laurent series \(\psi(T, P)\) (dressing function) such that

\[
\lambda = e^{ad\psi}(P),
\]

where \(\psi(T, P)\) has the form

\[
\psi(T, P) = \sum_{n=1}^{\infty} \psi_n(T) P^{-n}.
\]

Such Laurent series \(\psi(T, P)\) is not unique up to a constant Laurent series \(\sum_{i=0}^{\infty} c_i P^{-i}\). The Orlov function of dKP is by definition a formal Laurent series \(\sum_{n=1}^{\infty} nT_n P^{n-1}\).

\[
M = e^{ad\psi}(\sum_{n=1}^{\infty} nT_n P^{n-1}).
\]

It’s convenient to expand \(M\) into a Laurent series of \(\lambda\) as

\[
M = \sum_{n=1}^{\infty} nT_n \lambda^{n-1} + \sum_{i=1}^{\infty} h_i(T) \lambda^{-i}. \tag{5.1}
\]

It can be also shown that the series \(M\) satisfies the Lax equation

\[
\frac{\partial M}{\partial T_n} = \{B_n, M\} \tag{5.2}
\]

and the canonical Possion relation

\[
\{\lambda, M\} = 1. \tag{5.3}
\]

To get the solution structure of dKP hierarchy, let’s consider a pair of two functions \((f(P, X), g(P, X))\) such that they are arbitrary holomorphic functions defined in a neighborhood of \(P = \infty\) except at \(P = \infty\) itself. Then we have the following fact (twistor construction of dKP hierarchy).

**Fact:** (K. Takasaki and T. Takebe, [5]) Suppose

(i) \(\lambda\) and \(M\) has the form (2.8) and (5.1).

(ii) \(f(P, X)\) and \(g(P, X)\) described as above satisfy the canonical relation

\[
\{f(P, X), g(P, X)\} = 1. \tag{5.4}
\]

Then the following functional equations (in \(P\))

\[
f(\lambda, M)_{\leq -1} = 0 \quad g(\lambda, M)_{\leq -1} = 0 \tag{5.5}
\]

will imply equations (2.9), (2.2) and (5.3), i.e, the pair \((\lambda, M)\) gives a solution of dKP hierarchy. We call \((f(P, X), g(P, X))\) the twistor data of this solution.}

Conversely, each solution of dKP hierarchy possesses a twistor data corresponding to the solution, i.e, if \((\lambda, M)\) is a solution of (2.9), (2.2) and (5.3), then there exists a pair
\((f(P, X), g(P, X))\) which satisfies (5.4) and (5.3). In fact, if we let \(e^{a\psi(T,P)}\) be the dressing operator corresponding to \((\lambda, M)\), then the twistor data \((f, g)\) of this solution will be

\[
\begin{align*}
    f(P, X) &= e^{-a\psi_0(X,P)} P, \\
    g(P, X) &= e^{-a\psi_0(X,P)} X,
\end{align*}
\]  

(5.6)

where \(\psi_0(X, P) = \psi(T_1 = X, T_2 = T_3 = T_4 = \cdots = 0, P)\).

Next, we consider the dispersionless Miura map (3.5) from dKP to dmKP. Let us define

\[
\begin{align*}
    \mu &= e^{-a\phi(T)} \lambda, \\
    \tilde{M} &= e^{-a\phi(T)} M.
\end{align*}
\]  

(5.7)

Then \(\mu\) satisfies dmKP hierarchy (Theorem 3) and from Lemma 2 we have

\[
\{\mu, \tilde{M}\} = 1.
\]  

(5.8)

Moreover, a similar argument of Theorem 3 can also show that

\[
\frac{\partial \tilde{M}}{\partial T_n} = \{Q_n(P), \tilde{M}\}.
\]  

(5.9)

Now, we want to construct a pair of twistor data \((\tilde{f}(P, X), \tilde{g}(P, X))\) corresponding to \(\mu\) and \(\tilde{M}\) defined in (5.7).

**Theorem 10** Let \((\lambda, M)\) be a solution of (2.9), (5.2), and (5.3) and \(\mu, \tilde{M}\) is defined by the Miura map (5.7). If we define

\[
\begin{align*}
    \tilde{f}(P, X) &= e^{-a\psi_0(X,P)} e^{a\phi_0(X)} P, \\
    \tilde{g}(P, X) &= e^{-a\psi_0(X,P)} e^{a\phi_0(X)} X = g(P, X),
\end{align*}
\]

where \(\psi_0(X, P)\) is defined in (5.4) and \(\phi_0(x) = \phi(T_1 = X, T_2 = T_3 = \cdots = 0, \) (obviously, we have \(\{\tilde{f}, \tilde{g}\} = 1\).) then

\[
\begin{align*}
    \tilde{f}(\mu, \tilde{M})_{\leq 0} &= 0, \\
    \tilde{g}(\mu, \tilde{M})_{\leq -1} &= 0.
\end{align*}
\]

**Proof.** For convenience, we let \(T = 0\) mean \(T_2 = T_3 = T_4 = \cdots = 0\). Since

\[
\begin{align*}
    \lambda(T = 0) &= e^{a\psi_0} P, \\
    M(T = 0) &= e^{a\psi_0} X,
\end{align*}
\]

then we have

\[
\begin{align*}
    \mu(T = 0) &= e^{-a\phi_0} \lambda(T = 0) = e^{-a\phi_0} e^{a\psi_0} P, \\
    \tilde{M}(T = 0) &= e^{-a\phi_0} \tilde{M}(T = 0) = e^{-a\phi_0} e^{a\psi_0} X.
\end{align*}
\]

Therefore, by the Lemma 5 and assumptions, we have
\[ \tilde{f}(\mu(T = 0), \tilde{M}(T = 0)) = e^{-ad\phi_0}e^{ad\psi_0} \tilde{f}(P, X), \]
\[ = e^{-ad\phi_0}e^{ad\psi_0}(e^{-ad\psi_0}e^{ad\phi_0} P) = P, \]
\[ \tilde{g}(\mu(T = 0), \tilde{M}(T = 0)) = e^{-ad\phi_0}e^{ad\psi_0} \tilde{g}(P, X), \]
\[ = e^{-ad\phi_0}e^{ad\psi_0}(e^{-ad\psi_0}e^{ad\phi_0} X) = X. \] (5.10)

Now, we prove that \( \tilde{f}(\mu, \tilde{M}) \leq 0 \). Since \( \mu \) and \( \tilde{M} \) satisfy equations (2.15) and (5.9) respectively, we have
\[ \frac{\partial \tilde{f}(\mu, \tilde{M})}{\partial T_n} = \{Q_n(P), \tilde{f}(\mu, \tilde{M})\}. \]

Using (5.10), we see that \( \frac{\partial \tilde{f}(\mu, \tilde{M})}{\partial T_n}|_{T=0} \) will only contain powers \( \geq 1 \) of \( P \). In this way, we can prove, by induction, that \( (\partial/\partial T)^\alpha \tilde{f}(\mu, \tilde{M})|_{T=0} \), i.e., coefficients of Taylor expansion at \( T = 0 \), will only contain powers \( \geq 1 \) of \( P \) for any multi-index \( \alpha \). Thus, we have proved that \( \tilde{f}(\mu, \tilde{M}) \leq 0 \). As for \( \tilde{g}(\mu, \tilde{M}) \leq -1 \), we notice that the powers of \( P \) of \( \{Q_n(P), X\} \) are \( \geq 0 \). Then it can be proved in the same way. \( \square \)

This theorem shows the possibility of twistor construction for the solution structure of dmKP without using dispersionless Miura map. Indeed, we have the following main theorem of this section.

**Theorem 11** Let
\[ \mu = P + V_0 + \frac{V_1}{P} + \frac{V_2}{P^2} + \cdots, \]
\[ \mathcal{M}_{dmkp} = \sum_{n=1}^{\infty} nT_n \mu^{n-1} + \sum_{i=1}^{\infty} S_i(T) \mu^{-i} \]

(\( \mathcal{M}_{dmkp} \) can be defined as the Orlov function of dmKP). Suppose that
\[ \{f(P, X), g(P, X)\} = 1. \] (5.11)

Then the functional equations
\[ f(\mu, \mathcal{M}_{dmkp}) \leq 0, \]
\[ g(\mu, \mathcal{M}_{dmkp}) \leq -1 \] (5.12)
can get a solution of
\[ \partial_{T_n} \mu = \{Q^0_{\geq 1}(P), \mu\}, \]
\[ \partial_{T_n} \mathcal{M}_{dmkp} = \{Q^0_{\geq 1}(P), \mathcal{M}_{dmkp}\}, \]
\[ \{\mu, \mathcal{M}_{dmkp}\} = 1. \]

**Proof.** For convenience, we let
\[ \tilde{\mu} = f(\mu, \mathcal{M}_{dmkp}), \]
\[ \tilde{\mathcal{M}}_{dmkp} = g(\mu, \mathcal{M}_{dmkp}). \] (5.13)
We first derive the canonical Poisson relation. By differentiating the last equations with respect to $P$ and $X$, we have

\[
\begin{pmatrix}
\frac{\partial f(\mu, M_{dnkp})}{\partial \mu} & \frac{\partial f(\mu, M_{dnkp})}{\partial M_{dnkp}} \\
\frac{\partial g(\mu, M_{dnkp})}{\partial \mu} & \frac{\partial g(\mu, M_{dnkp})}{\partial M_{dnkp}}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \mu}{\partial P} & \frac{\partial \mu}{\partial M_{dnkp}} \\
\frac{\partial M_{dnkp}}{\partial P} & \frac{\partial M_{dnkp}}{\partial X}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \tilde{\mu}}{\partial P} & \frac{\partial \tilde{\mu}}{\partial M_{dnkp}} \\
\frac{\partial M_{dnkp}}{\partial P} & \frac{\partial M_{dnkp}}{\partial X}
\end{pmatrix}.
\]

(5.14)

Since the determinant of the first matrix on the left hand side is 1 because of (5.11), the determinants of both hand sides give

\[
\{\mu, M_{dnkp}\} = \{\tilde{\mu}, \tilde{M}_{dnkp}\}.
\]

(5.15)

One can calculate the left hand side as

\[
\{\mu, M_{dnkp}\} = \frac{\partial \mu}{\partial P} \frac{\partial M_{dnkp}}{\partial X} - \frac{\partial M_{dnkp}}{\partial P} \frac{\partial \mu}{\partial X},
\]

where we have used the fact that the terms containing \(\frac{\partial M_{dnkp}}{\partial \mu}\) in the last line cancel. Moreover, the Laurent expansions of \(\tilde{\mu}\) and \(\tilde{M}_{dnkp}\) contain only non-negative powers of \(P\) because of the functional equations (5.12). Therefore strictly negative powers of \(P\) in the last line should be absent, thus

\[
\{\mu, M_{dnkp}\} = \{\tilde{\mu}, \tilde{M}_{dnkp}\} = 1.
\]

(5.15)

This gives the desired canonical Poisson relation. We now show that the Lax equation for \(\mu\) and \(M_{dnkp}\) are indeed satisfied. Differentiating equations (5.13) with respect to \(T_n\) gives

\[
\begin{pmatrix}
\frac{\partial f(\mu, M_{dnkp})}{\partial \mu} & \frac{\partial f(\mu, M_{dnkp})}{\partial M_{dnkp}} \\
\frac{\partial g(\mu, M_{dnkp})}{\partial \mu} & \frac{\partial g(\mu, M_{dnkp})}{\partial M_{dnkp}}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \mu}{\partial T_n} & \frac{\partial \mu}{\partial M_{dnkp}} \\
\frac{\partial M_{dnkp}}{\partial T_n} & \frac{\partial M_{dnkp}}{\partial X}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial \tilde{\mu}}{\partial T_n} & \frac{\partial \tilde{\mu}}{\partial M_{dnkp}} \\
\frac{\partial M_{dnkp}}{\partial T_n} & \frac{\partial M_{dnkp}}{\partial X}
\end{pmatrix}.
\]

(5.16)

Combining equations (5.14) and (5.16), one can eliminate the derivative matrix of \((f, g)\) by \((\mu, M_{dnkp})\) and obtain the matrix relation

\[
\begin{pmatrix}
\frac{\partial \mu}{\partial P} & \frac{\partial \mu}{\partial M_{dnkp}} \\
\frac{\partial M_{dnkp}}{\partial P} & \frac{\partial M_{dnkp}}{\partial X}
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{\partial \mu}{\partial T_n} & \frac{\partial \mu}{\partial M_{dnkp}} \\
\frac{\partial M_{dnkp}}{\partial T_n} & \frac{\partial M_{dnkp}}{\partial X}
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{\partial \tilde{\mu}}{\partial P} & \frac{\partial \tilde{\mu}}{\partial M_{dnkp}} \\
\frac{\partial M_{dnkp}}{\partial P} & \frac{\partial M_{dnkp}}{\partial X}
\end{pmatrix}.
\]

Since the the determinants of the \(2 \times 2\) matrices on both sides are 1 because of (5.13), the inverse can also be written explicitly. In components, thus, the above matrix relation gives
The left hand side of equation (5.17) can be calculated just as we have done above for derivatives in \((P, X)\). For the first equation of (5.17),

\[
\frac{\partial M_{dmpk}}{\partial X} \frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial X} \frac{\partial M_{dmpk}}{\partial T_n} = - \frac{\partial (\mu^n)_{\geq 1}}{\partial X} + \text{powers of } P \leq 0.
\]

By the functional equations (5.12), we know that the right hand side of the first equation of (5.17) has Laurent expansion with only powers of \(\geq 1\). Therefore only powers of \(P \geq 1\) should survive. Hence

\[
\frac{\partial M_{dmpk}}{\partial X} \frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial X} \frac{\partial M_{dmpk}}{\partial T_n} = - \frac{\partial (\mu^n)_{\geq 1}}{\partial X} = - \frac{\partial Q_n}{\partial X}.
\]  

(5.18)

For the second equation of (5.17), we have similarly

\[
\frac{\partial M_{dmpk}}{\partial P} \frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial P} \frac{\partial M_{dmpk}}{\partial T_n} = - \frac{\partial (\mu^n)_{\geq 1}}{\partial P} + \text{powers of } P,
\]

\[
= - \frac{\partial (\mu^n)_{\geq 1}}{\partial P} + \text{powers of } P.
\]

By the functional equations (5.12), noticing the partial derivative \(\partial/\partial P\), we see that the right hand side of the second equation of (5.17) have Laurent expansion with only nonnegative powers of \(P\). Hence only nonnegative powers of \(P\) should survive. Thus

\[
\frac{\partial M_{dmpk}}{\partial P} \frac{\partial \mu}{\partial T_n} - \frac{\partial \mu}{\partial P} \frac{\partial M_{dmpk}}{\partial T_n} = - \frac{\partial (\mu^n)_{\geq 1}}{\partial P} = - \frac{\partial Q_n}{\partial P}.
\]  

(5.19)

Using (5.15), equations (5.18) and (5.19) can be readily solved:

\[
\frac{\partial \mu}{\partial T_n} = - \frac{\partial \mu}{\partial P} \frac{\partial Q_n}{\partial X} + \frac{\partial \mu}{\partial X} \frac{\partial Q_n}{\partial P} = \{Q_n, \mu\},
\]

\[
\frac{\partial M_{dmpk}}{\partial T_n} = - \frac{\partial M_{dmpk}}{\partial P} \frac{\partial Q_n}{\partial X} + \frac{\partial M_{dmpk}}{\partial X} \frac{\partial Q_n}{\partial P} = \{Q_n, M_{dmpk}\}.
\]

This completes the theorem. \(\Box\)
VI. CONCLUDING REMARKS

We have studied the Miura map between the dKP and dmKP hierarchies. We show that the Miura map not only preserves the Lax formulation of these two hierarchies but also is a canonical map in the sense that the bi-Hamiltonian structure of the dmKP hierarchy is mapped to the bi-Hamiltonian structure of the dKP hierarchy. We further use the twistor construction developed by Takasaki and Takebe to investigate the solution structure of the dmKP hierarchy.

In spite of the results obtained in the paper, there are some related problems deserve further investigations. We list some of them in the following.

(1) In [29], it is shown that the second Hamiltonian structure $\Theta^{(2)}$ of dKP has free field realizations. Since the Miura map is canonical, this suggests the possibility of free field realizations of second Hamiltonian structure $J^{(2)}$ of dmKP [30].

(2) In [31], we know that bi-Hamiltonian structure of Dubrovin-Novikov (DN) type [32] has geometric structure of Frobenius manifold [10]. A natural question is: what’s the geometric meaning of the Miura map between bi-Hamiltonian structures of DN type?

(3) The dmKP theory should be investigated without using Miura map. The quasi-classical $\tau$-function for dKP has been established in [5,11,12]. The basic question for dmKP theory is: Does the quasi-classical $\tau$-function theory exist? We notice that the Hirota bilinear equations for KP and mKP are essentially different [14]. Also, in [8], the dispersionless Hirota equation for dKP is obtained. Is there an analogue for dmKP?

Acknowledgements

We would like to thank Prof. J.C Shaw for useful discussions. JHC thanks for the support of the Academia Sinica and MHT thanks for the support of the National Science Council of Taiwan under Grant No. NSC 89-2112-M194-018.
REFERENCES

[1] R. Carroll, Jour. Nonlin. Sci. 4, 519 (1994).
[2] R. Carroll and Y. Kodama, J. Phys. A 28, 6373 (1995).
[3] Y. Kodama and J. Gibbons, Integrability of dispersionless KP hierarchy, Proceedinds Fourth Workshop on Nonlinear and Turbulent Process in Physics, 166 (World Scientific, 1990).
[4] D. Lebedev and Yu. Manin, Phys. Lett. A 74, 154 (1979).
[5] T. Takasaki and T. Takebe, SDiff(2) KP Hierarchy, Proceedings of RIMS Research Project, *Infinite Analysis*, (World Scientific, 1991).
[6] K. Takasaki and T. Takebe, Rev. Math. Phys. 7, 743 (1995).
[7] L.A. Dickey, *Soliton Equations and Hamiltonian Systems*, (World Scientific, 1991).
[8] S. Aoyama and Y. Kodama, Commun. Math. Phys. 182, 185 (1996).
[9] B. Dubrovin, Nucl. Phys. B 379, 627 (1992).
[10] B. Dubrovin, Geometry of 2d topological field theories, *Integrable systems and Quantum Groups*, Lecture Notes in Math. 1620, 120 (Springer, 1996).
[11] I. Krichever, Commun. Math. Phys. 143, 415 (1992).
[12] I. Krichever, Commun. Pure Appl. Math. 47, 437 (1994).
[13] Luen-Chau Li, Commun. Math. Phys. 203, 573 (1999).
[14] M. Jimbo and T. Miwa, Publ. RIMS, Kyoto University, 19, 943 (1983).
[15] B.A. Kupershmidt, Commun. Math. Phys. 99, 51 (1985).
[16] R.M. Miura, J. Math. Phys. 9, 1202 (1968).
[17] K. Kiso, Prog. Theor. Phys. 83, 1108 (1990).
[18] B. Konopelchenko and W. Oevel, Publ. RIMS, Kyoto Univ. 29, 581 (1993).
[19] B.A. Kupershmidt, Commun. Math. Phys. 167, 351 (1995).
[20] W. Oevel and C. Rogers, Rev. Math. Phys. 157, 299 (1993).
[21] Jiin-Chang Shaw and Ming-Hsien Tu, J. Math. Phys. 38, 5756 (1997).
[22] Jiin-Chang Shaw and Ming-Hsien Tu, J. Phys. A 30, 4825 (1997).
[23] B.A. Kupershmidt, J. Phys. A 23, 871 (1990).
[24] W. Oevel, Phys. Lett. A 186, 79 (1994).
[25] W. Oevel and W. Strampp, Comm. Math. Phys. 157, 51 (1993).
[26] A.Y. Orlov and E.I. Schulman, Lett. Math. Phys. 12, 171 (1986).
[27] P. Van Moerbeke, Integrable fundations of string theory, *Lectures on Integrable Systems*, Eds. O. Babelon et. el, (World Scientific, 1994).
[28] J.M. Figueroa-O’Farrill and E. Ramos, Phys. Lett. B 282, 357 (1992).
[29] Y. Cheng and Z.F. Li, Lett. Math. Phys. 42, 73 (1997).
[30] Jen-Hsu Chang and Ming-Hsien Tu, in preparation.
[31] B. Dubrovin, Flat pencils of metrics and Frobenius manifolds, *Integrable systems and algebraic geometry*, Kobe/Kyoto, 1997, 47-72, (World Scientific, 1998).
[32] B. Dubrovin and S.P. Novikov, Russ. Math. Surv. 44, 35 (1989).