Abstract. We continue our study of the Springer correspondence in the case of symmetric spaces initiated in [CVX]. In this paper we introduce a certain class of families of Hessenberg varieties and study their monodromy representations in detail in a special case when the Hessenberg varieties can be expressed in terms of complete intersections of quadrics. We obtain decompositions of these monodromy representations into irreducibles and compute the Fourier transforms of the IC complexes associated to these irreducible representations.

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1. Introduction

We continue our study of the Springer correspondence in the case of symmetric spaces initiated in [CVX]. In this paper we introduce a certain class of families of Hessenberg varieties of [GKM] and study their monodromy representations in detail in a special case when the Hessenberg varieties can be expressed in terms of intersections of quadrics.

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Let us recall the main question from [CVX]. For further background we refer to the introduction of [CVX]. Let $G$ be a reductive group and $\theta$ an involution of $G$. We write $K = (G^\theta)^0$ for the connected component of the fixed point set. This gives rise to a symmetric pair $(G, K)$. We also have the corresponding decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ is the fixed point set and $\mathfrak{g}_1$ is the $(-1)$-eigenspace of $\theta$, respectively. We write $N_1 = N \cap \mathfrak{g}_1$ where $N$ is the nilpotent cone in $\mathfrak{g}$. We address the following question which can be regarded as an analogue of the classical Springer correspondence: what are the Fourier transforms of $K$-equivariant IC-sheaves $\mathbb{I}$ on $N_1$? In particular, we would like to understand the case when the Fourier transforms are supported on all of $\mathfrak{g}_1$. We concentrate on the latter questions because we believe, and in fact, conjectured in [CVX], that the other cases can be reduced to this case via induction from smaller groups.

We work in the context of the split symmetric pair $(G, K) = (SL(N), SO(N))$ where $K = G^\theta$ is given by an involution $\theta : G \to G$ and $N$ is odd. Recall that the $K$-orbits in $N_1$ are parametrized by partitions of $N$. In [CVX] we considered the case when the IC-sheaves are supported on nilpotent orbits of order 2, i.e., orbits which correspond to partitions that only involve 2’s and 1’s. In this paper we treat the case of orbits of order 3.

To this end we proposed in [CVX] a general method of analyzing Fourier transforms of IC-sheaves. We replace the Springer resolution and the Grothendieck simultaneous resolution of the classical Springer correspondence by (several) pairs of families of Hessenberg varieties $X$ and $\tilde{X}$ and obtain the following picture:

\[
\begin{array}{ccc}
X & \longrightarrow & \tilde{X} \\
\pi \downarrow & & \downarrow \tilde{\pi} \\
N_1 & \longrightarrow & \mathfrak{g}_1
\end{array}
\]

The image of $\pi$ is a nilpotent orbit closure $\bar{O}$ but neither $\pi$ nor $\tilde{\pi}$ are semi-small in general. In fact, their generic fibers are not just points in general but they form smooth families of varieties. The key to analyzing the Fourier transforms of $K$-equivariant IC-sheaves on $N_1$ in this manner is that the constant sheaf on $\tilde{X}$ is the Fourier transform of the constant sheaf on $X$. Thus, at least as a first approximation, we are reduced to decomposing the push-forwards $\pi_* \mathcal{C}_X$ and $\tilde{\pi}_* \mathcal{C}_{\tilde{X}}$ into direct sums of IC-sheaves; this is possible by the decomposition theorem.

Let us recall the definition of Hessenberg varieties in our setting following [GKM]. Let $x \in \mathfrak{g}_1$, $P$ a parabolic subgroup of $K$, and $\Sigma \subset \mathfrak{g}_1$ a $P$-invariant subspace. The Hessenberg variety associated to the triple $(x, P, \Sigma)$, denoted by $\text{Hess}_x(K/P, \mathfrak{g}_1, \Sigma)$, is by definition the following variety

\[
\text{Hess}_x(K/P, \mathfrak{g}_1, \Sigma) := \{ g \in K/P \mid g^{-1}x \in \Sigma \}.
\]

As $x$ varies over $\mathfrak{g}_1$, we get a family of Hessenberg varieties $\text{Hess}(K/P, \mathfrak{g}_1, \Sigma) \to \mathfrak{g}_1$.

Our results in this paper and the ones in [CVX] provide evidences for the following conjecture:

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1IC=intersection cohomology
**Conjecture 1.1.** The cohomology of smooth Hessenberg varieties can be expressed in terms of Hodge classes coming from the cohomology of partial flag varieties and the cohomology of hyperelliptic curves.

The particular pairs of families of Hessenberg varieties we study here have the following properties. One of the families in the pair, when restricted to the regular semi-simple locus, is isomorphic to a family of complete intersections of quadrics (see Theorem 2.5); this is the family $\tilde{\pi} : \tilde{X} \to \mathfrak{g}_1$ in (1.1). The other family, corresponding to $\pi : X \to N_1$ in (1.1), is supported on the locus of nilpotent elements of order at most three and the fibers of this family admit affine pavings (see §2.4).

The main results in this paper are in sections §4 and §5. When restricted to the locus of regular semi-simple elements $g_{rs}^1$ our particular families of Hessenberg varieties can be interpreted as families of intersections of quadrics in projective spaces. In §4, we study the monodromy representations arising from the primitive cohomology of these families (and their natural double covers). This is accomplished by establishing a relative version of results of T. Terasoma [T] in §3. The resulting monodromy representations of $\pi_1^K(g_{rs}^1)$ can be expressed in terms of monodromy representations of certain families of hyperelliptic curves over $\mathfrak{g}_1^{rs}$. In particular, we see that the cohomology of those Hessenberg varieties can be expressed in terms of cohomology of hyperelliptic curves. This can be viewed as an evidence of the conjecture above. In Theorems 4.1 and 4.2 we describe these monodromy representations completely by decomposing them into irreducible pieces which we call $E_{ij}^N$ and $\tilde{E}_{ij}^N$, respectively.

In §5, we study Fourier transforms of the IC complexes arising from the local systems $E_{ij}^N$ and $\tilde{E}_{ij}^N$. Recall that these were obtained from the primitive cohomology of the particular families of Hessenberg varieties and their double covers. We show that their Fourier transforms are supported on the closed sub-variety $N_3^1 \subset N_1$ consisting of nilpotent elements of order less than or equal to three. Let $\{(\mathcal{O}, \mathcal{E})\}_{\leq 3}$ be the set of all pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O}$ is a $K$-orbit in $N_3^1$ and $\mathcal{E}$ is an irreducible $K$-equivariant local system on $\mathcal{O}$ (up to isomorphism). In this manner we obtain an injective map

\begin{equation}
\{E_{ij}^N\} \cup \{\tilde{E}_{ij}^N\} \hookrightarrow \{(\mathcal{O}, \mathcal{E})\}_{\leq 3}.
\end{equation}

This injection can be interpreted as a generalization of the classical Springer correspondence.

As an interesting corollary (see Example 5.5), we show that the Fourier transform of the IC complex for the unique non-trivial irreducible $K$-equivariant local system on the minimal nilpotent orbit has full support and the corresponding local system is given by the monodromy representation of the universal family of hyperelliptic curves of genus $n$, where $2n + 1 = N$.

The paper is organized as follows. In §2, we introduce certain pairs of families of Hessenberg varieties and prove basic facts about them. In §3 and §4 we establish a relative version of the results of Terasoma [T]. We utilize these results to obtain a decomposition of the monodromy representations into irreducibles. In §5, using the results in previous sections,
we show that Fourier transforms of the IC complexes for the local systems arising from our families of Hessenberg varieties and their double covers are supported on the closed subvariety $N^3_1 \subset N_1$ consisting of nilpotent elements of order at most three. In §6 we give a conjectural (explicit) description of the map in (1.2) (see Conjecture 6.1 and Conjecture 6.3) for $E^N_{ij}$ and we verify the conjectures in various examples.

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2. Hessenberg varieties

In this section we introduce certain families of Hessenberg varieties which naturally arise when computing the Fourier transforms of IC complexes supported on nilpotent orbits of order less than or equal to three, i.e., orbits of the form $O^{3}_{32j1}$. Our main theorem (see Theorem 2.5) says that, generically, these families of Hessenberg varieties are isomorphic to families of complete intersections of quadrics.

2.1. Definition of Hessenberg varieties. Let $G$ be a reductive group and $V$ a representation of $G$. Let $P \subset G$ be a parabolic subgroup and $\Sigma \subset V$ a $P$-invariant subspace. Consider the vector bundle $G \times P \Sigma$ and write

\[(2.1) \quad G \times P \Sigma = \operatorname{Hess}(G/P, V, \Sigma) \to G/P.\]

The natural projection to $V$ gives us a projective morphism

\[(2.2) \quad \operatorname{Hess}(G/P, V, \Sigma) \to V.\]

The fibers of this morphism are called Hessenberg varieties; the fiber over $v$ is given by

\[(2.3) \quad \operatorname{Hess}_v(G/P, V, \Sigma) = \{gP | g^{-1}v \in \Sigma\}.\]

2.2. Hessenberg varieties in our set up. As in [CVX], we consider the following situation. Let $G = SL(N, \mathbb{C})$ and $\theta : G \to G$ the involution such that $K := G^\theta = SO(N, \mathbb{C})$. The pair $(G, K)$ is called a split symmetric pair. As in [CVX], we will assume, starting with §3.2, that $N = 2n + 1$ is odd, mainly for simplicity. The involution $\theta$ induces a grading $g = g_0 \oplus g_1$ on the Lie algebra $g$ of $G$, where $g_i = \{x \in g | d\theta(x) = (-1)^ix\}$. The group $K$ acts on $g_1$ by adjoint action.

The Hessenberg varieties we consider are of the following form. The ambient group is $K$ acting on the vector space $g_1$. Let $T_K$ be a maximal torus of $K$ and consider a co-character $\lambda : \mathbb{G}_m \to T_K$. We write $P = P(\lambda)$ for the parabolic subgroup of $K$ associated to $\lambda$, $\mathfrak{p}$ for the Lie algebra of $P$, and $g_1 = \bigoplus g_{1,j}$ for the grading induced by $\lambda$. For any $i \in \mathbb{Z}$ we define
Let $\Sigma \subset \mathfrak{g}_1$ be a $P$-invariant subspace. The Hessenberg varieties that we are concerned with are of the form $\text{Hess}_v(K/P, \mathfrak{g}_1, \Sigma)$. We have:

**Lemma 2.1 ([GKM]).** Suppose $\Sigma \supset \mathfrak{g}_{1, \geq i}$ for some $i \leq 0$. Then the projective morphism $\text{Hess}(K/P, \mathfrak{g}_1, \Sigma) \to \mathfrak{g}_1$ is smooth over $\mathfrak{g}_1^{rs}$.

**Proof.** This is proved in [GKM, §2.5]. For the reader’s convenience, we recall the argument here. Observe that the Zariski tangent space to $\text{Hess}_v(K/P, \mathfrak{g}_1, \Sigma) \subset K/P$ at a point $x = kP \in \text{Hess}_v(K/P, \mathfrak{g}_1, \Sigma)$ can be identified with the kernel of $[v, -] : T(K/P)|_x \cong K \times^P (\mathfrak{g}_0/\mathfrak{p})|_x \to K \times^P (\mathfrak{g}_1/\Sigma)|_x$, $(k, w) \mapsto (k, [k^{-1}v, w])$.

So it suffices to show that the map above is surjective on the fibers at each point $kP \in \text{Hess}_v(K/P, \mathfrak{g}_1, \Sigma)$ if $v \in \mathfrak{g}_1^{rs}$. For this we show that any $v^* \in \mathfrak{g}_1^*$ that annihilates both $[k^{-1}v, \mathfrak{g}_0]$ and $\Sigma$ is zero. Since $v^*$ annihilates $\Sigma$ and $\Sigma \supset \mathfrak{g}_{1, \geq i}$ for some $i \leq 0$, there exists $\delta > 0$ such that $v^* \in \mathfrak{g}_{1, \geq \delta}^*$, that is, $v^*$ is $K$-unstable. Since $k^{-1}v \in \mathfrak{g}_1^{rs}$ is a “good vector” in $\mathfrak{g}_1$ under the $K$-action, i.e., there is no non-zero $K$-unstable vector $v^* \in \mathfrak{g}_1^*$ that annihilates the subspace $[k^{-1}v, \mathfrak{g}_0] \subset \mathfrak{g}_1$, we have $v^* = 0$. The lemma is proved.$\square$

### 2.3. Families of Hessenberg varieties.

Let us write $(G, K) = (SL(V), SO(V, Q))$, where $Q$ is a non-degenerate quadratic form on $V$. Denote by $\langle , \rangle_Q$ the non-degenerate bilinear form associated to $Q$. For a subspace $U \subset V$, we write $U^\perp = \{v \in V | \langle v, U \rangle_Q = 0\}$.

Let $N$ be the nilpotent cone of $\mathfrak{g}$ and let $N_1 = \mathfrak{g}_1 \cap N$. It is known that the number of $K$-orbits in $N_1$ is finite (see [KR]). Moreover, the $K$-orbits in $N_1$ are parametrized as follows (see [S]). For $N$ odd (resp. even), each partition of $N$ correspond to one $K$-orbit in $N_1$ (resp. except that each partition with only even parts corresponds to two $K$-orbits). In this paper we do not distinguish the two orbits corresponding to the same partition when $N$ is even; thus we write $\mathcal{O}_\lambda$ for an orbit corresponding to $\lambda$.

Let $\{e_i, i = 1, \ldots, N\}$ be a basis of $V$ such that $\langle e_i, e_j \rangle_Q = \delta_{i+j, N+1}$. For any $l \leq \frac{N}{2}$, let $P_l$ be the parabolic subgroup of $K$ that stabilizes the partial flag $0 \subset V_{l-1}^0 \subset V_l^0 \subset V_l^{0, \perp} \subset V_{l-1}^{0, \perp} \subset V = \mathbb{C}^N$, where $V_i^0 = \text{span}\{e_1, \ldots, e_i\}$.

Consider the following two subspaces of $\mathfrak{g}_1$,

$$E_l = \{x \in \mathfrak{g}_1 | xV_l^0 = 0, xV_l^{0, \perp} \subset V_{l-1}^0\} \quad \text{and} \quad O_l = \{x \in \mathfrak{g}_1 | xV_l^0 = 0, xV_{l-1}^{0, \perp} \subset V_{l-1}^0\}.$$ 

Note that both $E_l$ and $O_l$ are $P_l$-invariant. We form the corresponding families of Hessenberg varieties

\begin{align*}
\tau_l^N : \text{Hess}_l^E := \text{Hess}(K/P_l, \mathfrak{g}_1, E_l) \to \mathfrak{g}_1 \\
\sigma_l^N : \text{Hess}_l^O := \text{Hess}(K/P_l, \mathfrak{g}_1, O_l) \to \mathfrak{g}_1.
\end{align*}

\footnote{To see that $v \in \mathfrak{g}_1^{rs}$ is a good vector, we observe that if $v^* \in \mathfrak{g}_1^*$ is a $K$-unstable vector and annihilates $[v, \mathfrak{g}_0]$, then the vector $(v^*, 0) \in \mathfrak{q}^* \cong \mathfrak{g}_1^* \oplus \mathfrak{g}_0^*$ annihilates $[v, \mathfrak{g}] = [v, \mathfrak{g}_0] \oplus [v, \mathfrak{g}_1]$. However, since $v \in \mathfrak{g}_1^{rs} \subset \mathfrak{g}_1^{rs}$ it implies $v^* \in \mathfrak{g}_1^*$ is semisimple. This forces $v^* = 0$ since any $K$-unstable vector is nilpotent.}
A direct calculation shows that

\begin{align*}
(2.6a) \quad \text{Im } \tau^N_i &= \bar{O}_{3^l-12} \in \mathcal{N}_{1} \quad \text{if } 3l \leq N + 1, \quad \text{Im } \sigma^N_i &= \bar{O}_{3^N} \in \mathcal{N}_{1} \quad \text{if } 3l > N + 1 \\
\text{and} \quad \text{Im } \sigma^N_i &= \bar{O}_{3^l-11} \in \mathcal{N}_{1} \quad \text{if } 3l \leq N + 1, \quad \text{Im } \sigma^N_i &= \bar{O}_{3^N} \in \mathcal{N}_{1} \quad \text{if } 3l > N + 1.
\end{align*}

Remark 2.2. When $3l \leq N + 1$, $\tau^N_i$ coincides with Reeder’s resolution of $\bar{O}_{3^l-12} \in \mathcal{N}_{1}$.

Let $E^\perp_i$ and $O^\perp_i$ be the orthogonal complements of $E_i$ and $O_i$ in $g_i$, with respect to the non-degenerate trace form, respectively. Let us now consider the following families of Hessenberg varieties

\begin{align*}
(2.7) \quad \hat{\tau}^N_i : \text{Hess}_{\bar{e}}^{E,\perp} &:= \text{Hess}(K/P_i, g_i, E^\perp_i) \to g_i \\
(2.8) \quad \hat{\sigma}^N_i : \text{Hess}_{\bar{e}}^{O,\perp} &:= \text{Hess}(K/P_i, g_i, O^\perp_i) \to g_i.
\end{align*}

Concretely, we have

$$E^\perp_i = \{ x \in g_i \mid xV^0_i \subset V^0_i, xV^3_i \subset V^0_i \}, \quad O^\perp_i = \{ x \in g_i \mid xV^3_i \subset V^0_i \};$$

and then

$$\text{Hess}_{\bar{e}}^{O,\perp} \simeq \{ (x, 0 \subset V_i - l \subset V_i \subset V^\perp_i \subset V^3_i \subset \mathbb{C}^N) \mid x \in g_i, xV_i - l \subset V_i \},$$

$$\text{Hess}_{\bar{e}}^{E,\perp} \simeq \{ (x, 0 \subset V_i - l \subset V_i \subset V^\perp_i \subset V^3_i \subset \mathbb{C}^N) \mid x \in g_i, xV_i - l \subset V_i, xV_i \subset V^\perp_i \}.$$ Finally, note that, as the notation indicates, the bundle $\text{Hess}_{\bar{e}}^{E,\perp} \to K/P_i$ is the orthogonal complement of the bundle $\text{Hess}_{\bar{e}}^{E} \to K/P_i$ in the trivial bundle $g_i \times K/P_i$ and similarly for $\text{Hess}_{\bar{e}}^{E,\perp}$ and $\text{Hess}_{\bar{e}}^{E}$. Hence, by functoriality of the Fourier transform, we have:

\begin{align*}
(2.9) \quad \hat{\mathcal{F}}((\sigma^N_i)_{\mathbb{C}[-]} \cong (\hat{\sigma}^N_i)_{\mathbb{C}[-]} &\quad \text{and } \hat{\mathcal{F}}((\tau^N_i)_{\mathbb{C}[-]} \cong (\hat{\tau}^N_i)_{\mathbb{C}[-]}.
\end{align*}

2.4. Affine pavings. In this subsection we show that

$$\text{the fibers of } \tau^N_m : \text{Hess}_{\bar{e}}^{E} \to \mathcal{N}_1 \text{ and } \sigma^N_m : \text{Hess}_{\bar{e}}^{O} \to \mathcal{N}_1 \text{ have a paving by affine spaces.}$$

Lemma 2.3. Let $x \in O_{3^l-12} \in \text{Im } \tau^N_m$ (resp. $\text{Im } \sigma^N_m$) and $x_0 \in O_{3^l-12}$. We have

$$\tau^N_m(x) \cong (\tau^N_{m-1})^{-1}(x_0) \quad \text{resp. } \sigma^N_m(x) \cong (\sigma^N_{m-1})^{-1}(x_0).$$

Proof. We prove the lemma for $\tau^N_m$. The argument for $\sigma^N_m$ is entirely similar and omitted. We have

$$\tau^N_m(x) \cong \{ 0 \subset V_{m-1} \subset V_m \subset V^\perp_m \subset V^3_m \subset \mathbb{C}^N \mid xV_m = 0, xV^\perp_m \subset V_{m-1} \}.$$ 

Let $(V_{m-1} \subset V_m) \in (\tau^N_m)^{-1}(x)$. We have that $\text{Im } x \subset (\ker x)^\perp \subset V^\perp_m$. Thus $\text{Im } x^2 \subset V_{m-1}$. 

Choose a basis $\{ x^k u_l, k \in [0, 2], l \in [1, l], v_k, x v_k, k \in [1, j], w_l, l \in [1, N - 3i - 2j] \}$ of $V$ as in [CVX] Lemma 2.2. Let $U^0 = \text{span}\{ x^l u_l, l \in [0, 2], k \in [1, i] \}$, $V^0 = \text{span}\{ v_k, x v_k, k \in [1, j], w_l, l \in [1, N - 3i - 2j] \}$.
Then $Q|_{U^0}, Q|_{V^0}$ are non-degenerate and $V^0 = \langle U^0 \rangle$. We have

$$V_m = \text{Im } x^2 \oplus W_{m-i} \text{ and } V_{m-1} = \text{Im } x^2 \oplus W_{m-i-1},$$

where $W_{m-i} = V_m \cap V^0 \supset W_{m-i-1} = V_{m-1} \cap V^0$. We have $\dim W_{m-i} = m - i$ and $\dim W_{m-i-1} = m - i - 1$. Let $x_0 = x|_{V^0}$. Then $x_0 \in \Theta_{21N-3i-2j}$. Note that $xV_m = 0$ if and only if $x_0W_{m-i} = 0$. Now

$$V^\perp_m = \text{span}\{x^t u_k, l = 1, 2, k \in [1, i]\} \oplus W^\perp_{m-i},$$

where $W^\perp_{m-i}$ denotes the orthogonal complement of $W_{m-i}$ in $V^0$ with respect to $Q|_{V^0}$. Thus $xV^\perp_m \subset V_{m-1}$ if and only if $x_0W^\perp_{m-i} \subset W_{m-i-1}$. This gives us the desired isomorphism

$$(\tau^N_m)^{-1}(x) \cong (\tau^N_{m-i})^{-1}(x_0), (V_{m-1}, V_m) \mapsto (\text{pr}_{V^0}(V_{m-1}), \text{pr}_{V^0}(V_m))$$

where $\text{pr}_{V^0}$ is the projection from $V$ to $V^0$ with respect to $V = U^0 \oplus V^0$.

□

Let $O\text{Gr}(k, N)$ denote the orthogonal Grassmannian variety of $k$-dimensional isotropic subspaces in $\mathbb{C}^N$ with respect to a non-degenerate bilinear form on $\mathbb{C}^N$ and $\text{Gr}(k, N)$ denote the Grassmannian variety of $k$-dimensional subspaces in $\mathbb{C}^N$.

By Lemma 2.3 to describe the fibers $(\tau^N_m)^{-1}(x)$ and $(\sigma^N_m)^{-1}(x)$, it suffices to consider the case when $x \in \Theta_{21N-2j}$. We first introduce some notation. Let $x \in \Theta_{21N-2j}$. We write

$$\Sigma := \ker x/\text{Im } x \text{ and } \bar{U} = U/(U \cap \text{Im } x) \text{ for } U \subset \ker x.$$

Define a bilinear from $(,)$ on $\text{Im } x$ by

$$(xv, xw) := (v, xw)_Q.$$  \hfill (2.11)

Using $(\text{Im } x)^\perp = \ker x$, we see that $(,)$ is non-degenerate. For $U \subset \text{Im } x$, we define

$$U^\perp(,) = \{u \in \text{Im } x \mid (u, U) = 0\}.$$

Let us denote

$$\Upsilon^N_{m,j} := (\tau^N_m)^{-1}(x), \quad \Gamma^N_{m,j} := (\sigma^N_m)^{-1}(x), \quad x \in \Theta_{21N-2j}.$$

We partition $\Upsilon^N_{m,j}$ into pieces indexed by the dimension of $V_m \cap \text{Im } x$ by letting

$$\Upsilon^N_{m,k} = \{(0 \subset V_{m-1} \subset V_m \subset V^\perp_m \subset V^\perp_{m-1} \subset \mathbb{C}^N) \in \Upsilon^N_{m,j} \mid \dim(V_m \cap \text{Im } x) = k\}.$$

We now describe the pieces $\Upsilon^N_{m,k}$. To this end, let

$$\Theta^N_{m,k} = \{0 \subset V_m \subset V^\perp_m \subset \mathbb{C}^N \mid \dim(V_m \cap \text{Im } x) = k, xV_m = 0, xV^\perp_m \subset V_m, \dim(xV^\perp_m) \leq m - 1\}.$$

Consider the following map

$$\eta : \Theta^N_{m,k} \rightarrow \text{Gr}(j - k, \text{Im } x) \times \text{Gr}(m - k, \Sigma), \quad (V_m) \mapsto ((V_m \cap \text{Im } x)^\perp, V_m).$$

We claim that

$$\text{Im } \eta \cong O\text{Gr}(j - k, \text{Im } x) \times O\text{Gr}(m - k, \Sigma),$$
where $\text{Im } x$ is equipped with the non-degenerate bilinear form $(,)$. (see (2.11)), and $\Sigma$ is equipped with the non-degenerate bilinear form induced by $\langle \cdot, \cdot \rangle_Q$. It is clear that $V_m \subset \text{OGr}(m - k, \Sigma)$ as $\langle \cdot, \cdot \rangle_Q|_{V_m} = 0$. It is easy to check that $x(V_m^\perp) \subset (V_m \cap \text{Im } x)^{\perp(i)}$ and $\dim x(V_m^\perp) = \dim (V_m \cap \text{Im } x)^{\perp(i)} = j - k$. Thus $x(V_m^\perp) = (V_m \cap \text{Im } x)^{\perp(i)}$. Therefore the condition $xV_m^\perp \subset V_m$ is equivalent to $(V_m \cap \text{Im } x)^{\perp(i)} \subset V_m \cap \text{Im } x$, i.e. $(V_m \cap \text{Im } x)^{\perp(i)} \subset \text{OGr}(j - k, \text{Im } x)$. This proves the claim.

Thus we obtain a surjective map

\[(2.12) \quad \eta : \Theta_{m,j}^{N,k} \rightarrow \text{OGr}(j - k, \text{Im } x) \times \text{OGr}(m - k, \Sigma)\]

and it is easy to see that the fibers of $\eta$ are affine spaces $\mathbb{A}^{k(m - k)}$. Note that the fiber of the natural projection map

\[(2.13) \quad \Upsilon_{m,j}^{N,k} \rightarrow \Theta_{m,j}^{N,k} : (V_{m-1}, V_m) \mapsto V_m\]

at $V_m$ is the projective space $\mathbb{P}(V_m/(xV_m^\perp)) \cong \mathbb{P}^{m-j+k-1}$. It is easy to check using the above maps that each piece $\Upsilon_{m,j}^{N,k}$ has an affine paving. Therefore $\Upsilon_{m,j}^{N,k}$ also has an affine paving.

We can similarly partition $\Gamma_{m,j}^N$ into pieces indexed by the dimension of $V_{m-1} \cap \text{Im } x$. Let

\[
\Gamma_{m,j}^N = \{ (0 \subset V_{m-1} \subset V_m \subset V_m^\perp \subset V_{m-1}^\perp \subset \mathbb{C}^N) \in \Gamma_{m,j}^N | \dim (V_{m-1} \cap \text{Im } x) = k \}\]

\[
\Lambda_{m,j}^N = \{ (0 \subset V_{m-1} \subset V_m^\perp \subset \mathbb{C}^N) | \dim (V_{m-1} \cap \text{Im } x) = k, xV_{m-1} = 0, xV_{m-1}^\perp \subset V_{m-1} \}\.
\]

We have a surjective map

\[\eta' : \Lambda_{m,j}^N \rightarrow \text{OGr}(j - k, \text{Im } x) \times \text{OGr}(m - k - 1, \Sigma), (V_{m-1}) \mapsto ((V_{m-1} \cap \text{Im } x)^{\perp(i)}, \tilde{V}_{m-1}).\]

The fibers of $\eta'$ are affine spaces $\mathbb{A}^{k(m-k-1)}$. The fiber of the natural projection map

\[\Gamma_{m,j}^N \rightarrow \Lambda_{m,j}^N : (V_{m-1}, V_m) \mapsto V_{m-1}\]

at a given $V_{m-1}$ is the variety of isotropic lines in $(V_{m-1}^\perp \cap \ker x)/V_{m-1}$ with respect to the quadratic form induced by $Q$. The same argument as before shows that $\Gamma_{m,j}^N$ is paved by affines.

In particular, we see from the above discussion that

\[(2.14) \quad \Upsilon_{m,j}^{N,k} \neq \emptyset \Leftrightarrow \max\{m + j - N/2, j/2, j + 1 - m\} \leq k \leq \min\{j, m\};\]

\[\Gamma_{m,j}^N \neq \emptyset \Leftrightarrow \max\{m + j - N/2 - 1, j/2, j + 1 - m\} \leq k \leq \min\{j, m - 1\}.\]

Finally, in [CVX] Proof of Proposition 6.5 we have used the following fact

**Lemma 2.4.** For $x_i \in \Theta_{2i,2m-1-2i} \subset \mathbb{C}_{2m-4m+3+i}$ we have

\[\tau_{m+1}^{2n+1}(x_i) \equiv \text{OGr}(m - 1 - i, 2m - 1 - 2i).\]

This can be deduced from the results in this subsection as follows. Using Lemma [2.3 and (2.14) we see that $\tau_{m+1}^{2n+1}(x_i) \equiv \Upsilon_{m-i,2m-1-2i}^{2n+1+3i,m-i}$. The conclusion follows by considering the maps in (2.12) and (2.13).
2.5. Families of complete intersections of quadrics and their identification with Hessenberg varieties. Let \( m \in [1, N - 1] \) be an integer. For any \( s \in g^r_1 \), let

\[
X_{m,s} \subset \mathbb{P}(V) \simeq \mathbb{P}^{N-1}
\]

be the complete intersection of \( m \) quadrics

\[
\langle s^i , -\rangle_Q = 0, \; i = 0, \ldots, m - 1
\]

in \( \mathbb{P}(V) \). As \( s \) varies over \( g^r_1 \), we get a family

\[
\pi_m : X_m \to g^r_1
\]

of complete intersections of \( m \) quadrics in \( \mathbb{P}(V) \).

The families of Hessenberg varieties \( Hess^O_{F} \) and \( Hess^E_{F} \) over \( g^r_1 \) are identified with \( X_m \)'s as follows.

**Theorem 2.5.** Assume that \( k \leq \frac{N-1}{2} \). Then we have

1. There is a \( K \)-equivariant isomorphism \( Hess^O_{k-1} \mid g^r_1 \simeq X_{2k-1} \) of varieties over \( g^r_1 \).
2. There is a \( K \)-equivariant isomorphism \( Hess^E_{k-1} \mid g^r_1 \simeq X_{2k} \) of varieties over \( g^r_1 \).

We begin with the following simple observation.

(2.15) Let \( s \in g^r_1 \). For any isotropic subspace \( 0 \neq U \subset V \), \( \dim(sU \cap U) < \dim(sU) \).

This follows from the fact that \( s \) has no isotropic eigenspaces.

**Proof of Theorem 2.5.** We first define a map from \( X_{2k-1} \) to \( Hess^O_{k-1} \). Let \((s, l) \in X_{2k-1} \), where \( s \in g^r_1 \) and \( l \) is in the complete intersection of \( 2k - 1 \) quadrics \( \langle s^i , -\rangle_Q = 0, \; i = 0, \ldots, 2k - 2 \), in \( \mathbb{P}(V) \). Let \( 0 \neq v \in l \). For \( 1 \leq i \leq k \), consider the subspaces

\[
V_i = \text{span}\{v, sv, \ldots, s^{i-1}v\}.
\]

Note that \( V_i \) is isotropic. We show that \( \dim V_i = i \). We have \( V_i = V_{i-1} \cup sV_{i-1} \). Thus \( \dim V_i = \dim V_{i-1} + \dim sV_{i-1} - \dim (sV_{i-1} \cap V_{i-1}) > \dim V_{i-1} \), where in the last inequality we use (2.15). By induction we see that \( \dim V_i = i \). Hence the assignment \((s, l) \mapsto (s, V_{k-1} \subset V_k)\) defines a map

\[
\iota : X_{2k-1} \to Hess^O_{k-1} \mid g^r_1.
\]

One checks readily that \( \iota \) is \( K \)-equivariant. We prove that \( \iota \) is an isomorphism by constructing an explicit inverse map. Let \((s, V'_{k-1} \subset V'_{k}) \in Hess^O_{k-1} \) with \( s \in g^r_1 \). We define a sequence of subspaces \( 0 \subset V'_1 \subset V'_2 \subset \cdots \subset V'_{k-2} \) recursively. Let us first define \( V'_{k-2} \). Consider the map \( \bar{s} : V'_{k-1} \to V'_{k} \to V'_{k}/V'_{k-1} \). Note that by (2.15), the map \( \bar{s} \) is nonzero, hence surjective as \( \dim V'_{k}/V'_{k-1} = 1 \). Let \( V'_{k-2} = \ker \bar{s} \). We have \( \dim V'_{k-2} = k - 2 \) and \( V'_{k-1} = V'_{k-2} \cup sV'_{k-2} \). By induction we can assume that we have defined \( V'_i \) such that \( \dim V'_i = i \) and \( V'_{i+1} = V'_i \cup sV'_{i} \). Let

\[
V'_{i-1} = \ker (\bar{s} : V'_i \to V'_{i+1} \to V'_{i+1}/V'_i).
\]
The same argument as before shows that \( \dim V'_{i-1} = i - 1 \) and \( V'_i = V'_{i-1} \cup sV'_{i-1} \). Thus in particular we obtain that \( \dim V'_1 = 1 \), and it is easy to see that the map
\[
\text{Hess}_k^E \mid_{\mathfrak{g}^*_1} \rightarrow X_{2k-1}, (s, V'_{k-1} \subset V'_k) \mapsto (s, V'_1).
\]
defines an inverse of \( \iota \). This finishes the proof of (1).

For (2), we observe that, under the isomorphism \( \iota : X_{2k-1} \cong \text{Hess}_k^O \), the equation \( \langle s^{2k-1}v, v \rangle_Q = 0 \) for the divisor \( X_{2k} \subset X_{2k-1} \) becomes \( \langle sV'_k, V'_k \rangle_Q = 0 \), which is the equation for the divisor \( \text{Hess}_k^{E, \perp} \subset \text{Hess}_k^{O, \perp} \). Thus (2) follows.

\[ \square \]

3. Complete intersections of quadrics and their double covers

In §2.3 we have introduced the families \( X_m \rightarrow \mathfrak{g}^{rs}_1 \) of complete intersections of quadrics, which we have identified with families of Hessenberg varieties \( \text{Hess}_n^{E, \perp} \mid_{\mathfrak{g}^*_1}, \text{Hess}_n^{O, \perp} \mid_{\mathfrak{g}^*_1} \). In order to study the monodromy representations of the equivariant fundamental group \( \pi_1^K(\mathfrak{g}^{rs}_1) \) associated with the above families of Hessenberg varieties, we introduce families \( Y_m \) of branched covers of \( \mathbb{P}^{N-m-1} \) and relate them with \( X_m \). We also introduce a family of branched double covers of \( X_m \), denoted by \( \tilde{X}_m \), and relate them with families \( \tilde{Y}_m \) of branched covers of \( \mathbb{P}^{N-m-1} \) which we introduce in §3.3. Our construction can be regarded as a relative version of the construction in [1].

3.1. Some notation. In this section we choose a Cartan subspace \( a \subset \mathfrak{g}_1 \) that consists of diagonal matrices. Let \( a^{rs} = a \cap \mathfrak{g}^{rs} \). We write an element \( a \in a \) with diagonal entries \( a_1, \ldots, a_N \) as \( a = (a_1, \ldots, a_N) \). Thus \( a = (a_1, \ldots, a_N) \in a^{rs} \) if and only if \( a_i \neq a_j \) for \( i \neq j \).

Define
\[
I_N := (\mathbb{Z}/2\mathbb{Z})^N / (\mathbb{Z}/2\mathbb{Z})
\]
where we regard \( \mathbb{Z}/2\mathbb{Z} \) as a subgroup of \( (\mathbb{Z}/2\mathbb{Z})^N \) via the diagonal embedding. For any \( \chi \in I_N \), we define
\[
\text{supp}(\chi) = \{ i \in [1, N] \mid \chi(\xi_i) = -1 \} \text{ and } |\chi| = \# \text{supp}(\chi)
\]
where \( \xi_i \) is the image of \( (0, \ldots, 1, \ldots, 0) \in (\mathbb{Z}/2\mathbb{Z})^N \) in \( I_N \). Note that \( |\chi| \) is even.

If we identify the centralizer \( Z_K(a) \) of \( a \in a^{rs} \) with the kernel of the map \( (\mathbb{Z}/2\mathbb{Z})^N \rightarrow \mathbb{Z}/2\mathbb{Z}, (b_1, \ldots, b_N) \mapsto \sum b_i \), we obtain a natural map
\[
Z_K(a) \rightarrow (\mathbb{Z}/2\mathbb{Z})^N \xrightarrow{pr} I_N.
\]
Note when \( N \) is odd which we will assume from this point on, the map (3.1) is an isomorphism. Therefore, in what follows, we often make the canonical identification \( Z_K(a) \cong I_N \).

To emphasize:

**Assumption.** From now on we assume that \( N \) is odd.
3.2. **Family of curves.** In this subsection we introduce certain families of curves which will be used to construct the families \( Y_m \) and \( Y_m \) of branched covers of projective spaces.

For any \( a = (a_1, ..., a_N) \in a^r s \), there are natural isomorphisms
\[
\pi_1^{ab}(\mathbb{P}^1 - \{a_1, ..., a_N\}) \otimes \mathbb{Z}/2\mathbb{Z} \simeq I_N \quad \pi_1^{ab}(\mathbb{P}^1 - \{a_1, ..., a_N, a_{N+1} = \infty\}) \otimes \mathbb{Z}/2\mathbb{Z} \simeq I_{N+1}.
\]
The isomorphisms are given by assigning to a small loop around each \( a_i \) the element in \( I_N \) (resp. \( I_{N+1} \)) with only non-trivial coordinate in position \( i \). Let
\[
C_a \to \mathbb{P}^1 \quad \text{(resp. } \tilde{C}_a \to \mathbb{P}^1)\]
be the abelian covering of \( \mathbb{P}^1 \) ramified at \( \{a_1, ..., a_N\} \) (resp. \( \{a_1, ..., a_{N+1} = \infty\} \)) with Galois groups given by \( I_N \) (resp. \( I_{N+1} \)). Concretely, \( C_a \) (resp. \( \tilde{C}_a \)) is the smooth projective curve corresponding to the function field
\[
\mathbb{C}(t)((\frac{t-a_i}{t-a_1})^{1/2}_{i=2,...,N}) \quad \text{(resp. } \mathbb{C}(t)((\frac{t-a_i}{t-a_1})^{1/2}_{i=1,...,N})).
\]
The group \( I_N \) (resp. \( I_{N+1} \)) acts on \( C_a \) (resp. \( \tilde{C}_a \)). For any \( \chi \in I_N \) (resp. \( \chi \in I_{N+1} \)) we define
\[
C_{a,\chi} = C_a / \ker \chi \quad \text{(resp. } \tilde{C}_{a,\chi} = \tilde{C}_a / \ker \chi),
\]
which is a branched double cover of \( \mathbb{P}^1 \) with branch locus \( \{a_i \mid i \in \text{supp}(\chi)\} \). Concretely, \( C_{a,\chi} \) (resp. \( \tilde{C}_{a,\chi} \)) is isomorphic to the smooth projective hyperelliptic curve with affine equation
\[
y^2 = \prod_{i \in \text{supp} \chi} (x - a_i) \quad \text{(resp. } y^2 = \prod_{i \in \text{supp} \chi, i \neq N+1} (x - a_i)).
\]
We have \( \dim H^1(C_{\chi}, \mathbb{C}) = |\chi| - 2 \) (resp. \( \dim H^1(\tilde{C}_{\chi}, \mathbb{C}) = |\chi| - 2 \)).

As \( a \) varies over \( a^{rs} \), we obtain a family of curves
\[
C \to a^{rs} \quad \text{(resp. } \tilde{C} \to a^{rs})
\]
and we similarly obtain families of hyperelliptic curves
\[
C_{\chi} \to a^{rs} \quad \text{(resp. } \tilde{C}_{\chi} \to a^{rs}) \quad \text{for any } \chi.
\]
We also note that the Weyl group \( W = S_N \) acts naturally on \( C \) (resp. \( \tilde{C} \)) making the projection \( C \to a^{rs} \) (resp. \( \tilde{C} \to a^{rs} \)) a \( W \)-equivariant map.

We will now associate monodromy representations to these families. Let us fix \( a \in a^{rs} \) and a character \( \chi \in I_N \) (resp. \( \chi \in I_{N+1} \)) and we recall that \( \pi_1(a^{rs}, a) \) is the pure braid group \( P_N \). The monodromy representation of the family \( C_{\chi} \to a^{rs} \) factors through the symplectic group and we denote it by \( \rho_{C_{\chi}} : P_N \to Sp(H^1(C_{a,\chi}, \mathbb{C})) \simeq Sp(2m - 2) \) where \( m = \frac{|\chi|}{2} \). Similarly we obtain a monodromy representation \( \rho_{\tilde{C}_{\chi}} : P_N \to Sp(H^1(\tilde{C}_{a,\chi}, \mathbb{C})) \simeq Sp(2m - 2) \).

We claim:

The images of the representations \( \rho_{C_{\chi}} \) and \( \rho_{\tilde{C}_{\chi}} \) are Zariski dense
\[
(3.2) \quad \text{in } Sp(H^1(C_{a,\chi}, \mathbb{C})) \text{ and } Sp(H^1(\tilde{C}_{a,\chi}, \mathbb{C})), \text{ respectively}.
\]
In particular, the representations \( \rho_{C_{\chi}} \) and \( \rho_{\tilde{C}_{\chi}} \) are irreducible.
We see this as follows. Consider the following subvariety $a^r s(a, \chi) \subset a^r s$

(3.3) $a^r s(a, \chi) = \{a' \in a^r s | a'_i = a_i \text{ if } i \notin \text{supp}(\chi)\}$.

It suffices to show that the monodromy representation of the restriction of $C^r_s \to a^r s$ to $a^r s(a, \chi)$ has Zariski dense image. Now, $a^r s(a, \chi)$ is an open subset of the space $M_{2m}$ of $2m$ distinct marked points in $\mathbb{C}$ and the family $C^r_s \times a^r s(a, \chi)$ is the restriction of the universal family of hyperelliptic curves parametrized by $M_{2m}$. Note further, that $M_{2m}$ itself is an open subset of the space $\tilde{M}_{2m}$ of $2m$ distinct marked points in $\mathbb{P}^1$ carrying its own family of hyperelliptic curves. Now, by [A] (see also [KS, Theorem 10.1.18.3]), the monodromy representation on $\tilde{M}_{2m}$ is irreducible and has Zariski dense image. Therefore $\rho_{C^r_s}$, as a restriction to an open subset, has the same property. The argument in the case $C^r_s \to a^r s$ is completely analogous except one has to take into account that $a_{N+1} = \infty$.

Finally, there is a unique character $\chi_0 \in I^\vee_{N+1}$ with $|\chi_0| = N + 1$ (here we use the assumption that $N$ is odd). The character $\chi_0$ is invariant under the Weyl group action. Thus we can pass to a quotient of $\widetilde{C}^r_s \to a^r s$ under the $W$ action and in this way obtain a family $\widetilde{C}_{X_0} \to c^r s = a^r s/W$. The family $\widetilde{C}_{X_0} \to c^r s = a^r s/W$ is the universal family of hyperelliptic curves $y^2 = \prod_{i=1}^{N} (x - a_i)$ and $\widetilde{C}_{X_0} \to a^r s$ is a similar universal family with marked ramification points.

3.3. Branched cover $y_m$ of projective spaces and $X_m$. Define

$$I_{N-m-1}^N = \ker(\text{sum} : I_N^{N-m-1} \to I_N),$$

where $\text{sum}$ is the summation map. Fix $a = (a_1, \ldots, a_N) \in a^r s$. Let $C_a \to \mathbb{P}^1$ be the curve introduced in §3.2. The semi-direct product $I_{N-m-1}^N \rtimes S_{N-m-1}$ acts naturally on $C_a^{N-m-1}$ and we define

$$y_{m,a} = C_a^{N-m-1} / I_{N-m-1}^N \rtimes S_{N-m-1}.$$  

We have a natural map

$$\iota_a : y_{m,a} \to C_a^{N-m-1} / I_{N-m-1}^N \rtimes S_{N-m-1} \simeq \mathbb{P}^{N-m-1}.$$

According to [T] Proposition 2.4.4], for a suitable choice of homogeneous coordinates $[x_1, \ldots, x_{N-m}]$ of $\mathbb{P}^{N-m-1}$, each ramification point $a_i$ defines a hyperplane

(3.4) $H_{a,i} = x_1 + a_i x_2 + \cdots + a_i^{N-m-1} x_{N-m} = 0$

in $\mathbb{P}^{N-m-1}$ and the map $\iota_a$ is an $I_N$-branched cover of $\mathbb{P}^{N-m-1}$ with branch locus $\{H_{a,i} = 0\}_{i=1,\ldots,N}$. As $a$ varies over $a^r s$, we get a $a^r s$-family of $I_N$-branched covers of $\mathbb{P}^{N-m-1}$

$$\begin{array}{ccc}
\mathbb{P}^{N-m-1} & \xrightarrow{\iota} & \mathbb{P}^{N-m-1} \\
\downarrow \quad \iota_a \quad \downarrow & & \downarrow \quad \iota_a \quad \downarrow \\
\mathbb{P}^{N-m-1} & \xrightarrow{\iota_a} & \mathbb{P}^{N-m-1} \\
\end{array}$$

where

$$y_m = C^{N-m-1} / I_{N-m-1}^N \rtimes S_{N-m-1}.$$
and the base change of $\iota$ to $a$ is equal to $\iota_a$. Observe that the $W$-action on $C$ induces a $W$-action on $\mathcal{Y}_m$ making the projection $\mathcal{Y}_m \to \mathfrak{a}^{rs}$ a $W$-equivariant map.

Let $X_m \to \mathfrak{g}_1^{rs}$ be the family of complete intersections of quadrics introduced in (3.4). For $a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}$ the equation of $X_{m,a}$ is given by

$$a_i v_1^2 + \cdots + a_N v_N^2 = 0, \quad i = 0, \ldots, m - 1.$$ 

Consider the map

$$s : \mathbb{P}(V) \to \mathbb{P}(V), \quad [v_1, \ldots, v_N] \mapsto [v_1^2, \ldots, v_N^2].$$

The image $s(X_{m,a})$ is equal to $\mathbb{P}(V_{m,a})$, where

$$V_{m,a} = \{ v \in V \mid a_i v_1 + \cdots + a_i v_N = 0, \quad i = 0, \ldots, m - 1 \} \subset V.$$ 

The resulting map

$$s_a : X_{m,a} \to \mathbb{P}(V_{m,a})$$

is an $I_N$-branched cover with branch locus $\{ v_i = 0 \}_{i=1,\ldots,N}$. As $a$ varies over $\mathfrak{a}^{rs}$, we obtain

$$X_m \mid_{\mathfrak{a}^{rs}} \rightarrow \mathbb{P}(V_m) \quad \downarrow \quad s_a \quad \rightarrow \quad \mathfrak{a}^{rs}$$

Here $V_m \to \mathfrak{a}^{rs}$ is the vector bundle over $\mathfrak{a}^{rs}$ whose fiber over $a$ is $V_{m,a}$ and $\mathbb{P}(V_m)$ is the associated projective bundle.

The two families $X_m \mid_{\mathfrak{a}^{rs}}$ and $\mathcal{Y}_m$ are related as follows. Let

$$(\tilde{a}^{rs})' = \{(a, c) \mid a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}, c = (c_1, \ldots, c_N) \in \mathbb{C}^N, \quad c_i^2 = d_i := \prod_{j \neq i} (a_j - a_i)\}.$$ 

The projection $(\tilde{a}^{rs})' \to \mathfrak{a}^{rs}, \quad (a_1, \ldots, a_N, c_1, \ldots, c_N) \mapsto (a_1, \ldots, a_N)$ realizes $(\tilde{a}^{rs})'$ as a $(\mathbb{Z}/2\mathbb{Z})^N$-torsor over $\mathfrak{a}^{rs}$. Consider the following $I_N$-torsor over $\mathfrak{a}^{rs}$

$$\tilde{a}^{rs} := (\tilde{a}^{rs})' / (\mathbb{Z}/2\mathbb{Z}) \to \mathfrak{a}^{rs};$$

here we view $\mathbb{Z}/2\mathbb{Z}$ as a subgroup of $(\mathbb{Z}/2\mathbb{Z})^N$ via the diagonal embedding. The Weyl group $W$ acts naturally on $\tilde{a}^{rs}$ making the projection to $\mathfrak{a}^{rs}$ a $W$-equivariant map. We observe that the $I_N$-torsor $\tilde{a}^{rs}$ of (3.6) gives rise to the following canonical map

$$\rho : \pi_1(\mathfrak{a}^{rs}, a) \cong P_N \to I_N.$$ 

**Proposition 3.1.** We have an $I_N \times W$-equivariant isomorphism

$$X_m \mid_{\mathfrak{a}^{rs}} \simeq (\mathcal{Y}_m \times_{\mathfrak{a}^{rs}} \tilde{a}^{rs}) / I_N := \mathcal{Y}_m' / I_N,$$

where $W$ (resp. $I_N$) acts on $\mathcal{Y}_m'$ by the diagonal action (resp. on the first factor).

**Proof.** Following [1, §5], we consider the family

$$X_m' \mid_{\mathfrak{a}^{rs}} \to \mathfrak{a}^{rs}.$$
whose fiber over \( a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs} \) is the complete intersection of \( m \) quadrics in \( \mathbb{P}(V) \) given by
\[
\frac{a_i^j}{d_i}v_1^2 + \cdots + \frac{a_N^j}{d_N}v_N^2 = 0, \quad i = 0, \ldots, m - 1,
\]
where \( d_i := \prod_{j \neq i} (a_j - a_i) \). One can think of \( X_m|_{\mathfrak{a}^{rs}} \) as a twist of \( X_m'|_{\mathfrak{a}^{rs}} \). More precisely, we have a natural map
\[
\tilde{\pi}^{rs} \times_{\mathfrak{a}^{rs}} X_m'|_{\mathfrak{a}^{rs}} \to X_m|_{\mathfrak{a}^{rs}}, \quad (a, c, [v_1, \ldots, v_N]) \mapsto (a, [v_1/c_1, \ldots, v_N/c_N])
\]
and it is not hard to see that it descends to a canonical \( I_N \)-equivariant isomorphism
\[
(3.9) \quad X_m|_{\mathfrak{a}^{rs}} \simeq (\tilde{\pi}^{rs} \times_{\mathfrak{a}^{rs}} X_m'|_{\mathfrak{a}^{rs}})/I_N.
\]
Here \( I_N \) acts on the product via the diagonal action. The Weyl group \( W = S_N \) acts naturally on \( X_m|_{\mathfrak{a}^{rs}} \), \( X_m'|_{\mathfrak{a}^{rs}} \), \( \tilde{\pi}^{rs} \), and \( (\tilde{\pi}^{rs} \times_{\mathfrak{a}^{rs}} X_m'|_{\mathfrak{a}^{rs}})/I_N \), making the projections to \( \mathfrak{a}^{rs} \) equivariant maps under the \( W \)-actions. Moreover, the isomorphism in (3.9) is also \( W \)-equivariant. In §3.6 (see Proposition 3.3), we show that there is an \( I_N \rtimes W \)-equivariant isomorphism
\[
(3.10) \quad X_m'|_{\mathfrak{a}^{rs}} \simeq Y_m.
\]
Combining (3.9) with (3.10) we obtain (3.8). \( \square \)

3.4. Branched double covers \( \tilde{X}_m \) of complete intersections of quadrics. We introduce a branched double cover of \( X_m \) as follows. Let \( \tilde{V} = V \oplus \mathbb{C} \). For any \( s \in \mathfrak{g}_1^{rs} \), consider the following quadrics in \( \mathbb{P}(V) \),
\[
\tilde{Q}_i(v, \epsilon) = \langle s^iv, v \rangle_Q = 0, \quad i = 0, \ldots, m - 1 \quad \tilde{Q}_m(v, \epsilon) = \langle s^mv, v \rangle_Q - \epsilon^2 = 0.
\]
We define \( \tilde{X}_{m,s} \) to be the complete intersection of \( m + 1 \) quadrics \( \tilde{Q}_i = 0, \ i = 0, \ldots, m \). As \( s \) varies over \( \mathfrak{g}_1^{rs} \), we get a family
\[
\tilde{\pi}_m : \tilde{X}_m \to \mathfrak{g}_1^{rs}
\]
of complete intersections of \( m + 1 \) quadrics in \( \mathbb{P}(V) \). The projection \( \tilde{V} = V \oplus \mathbb{C} \to V, \ (v, \epsilon) \mapsto v \), induces a map \( p_m : \tilde{X}_m \to X_m \) which is a branched double cover with branch locus \( X_{m+1} \subset X_m \).

The map \( \tilde{V} = V \oplus \mathbb{C} \to \tilde{V} \) given by \( (v, \epsilon) \mapsto (v, -\epsilon) \) defines an involution on \( \tilde{X}_m \). We denote this involution by \( \sigma \).

The group \( K = SO(V, Q) \) acts naturally on both \( X_m \) and \( \tilde{X}_m \). The maps \( \pi_m : X_m \to \mathfrak{g}_1^{rs} \) and \( \tilde{\pi}_m : \tilde{X}_m \to \mathfrak{g}_1^{rs} \) are \( K \)-equivariant. In particular, the centralizer \( Z_K(s) \) acts on the fibers \( X_{m,s} \) and \( \tilde{X}_{m,s} \).
3.5. **Branched cover \(\tilde{Y}_m\) of projective spaces and \(\tilde{X}_m\).** In this subsection we generalize Proposition 3.1 to the branched double cover \(\tilde{X}_m\) introduced in §3.4.

For \(a = (a_1, \ldots, a_N) \in \mathfrak{a}^{rs}\) the equations of \(\tilde{X}_{m,a} \subset \mathbb{P}^{N-m-1}(\tilde{V})\) (recall that \(\tilde{V} = V \oplus \mathbb{C}\)) are given by

\[
a_i^1 v_1^2 + \cdots + a_N v_N^2 = 0, \quad i = 0, \ldots, m - 1, \quad a_i^m v_1^2 + \cdots + a_N v_N^2 - \epsilon^2 = 0.
\]

Consider the map

\[\tilde{s} : \mathbb{P}(\tilde{V}) \to \mathbb{P}(\tilde{V}), \quad [v_1, \ldots, v_N, \epsilon = v_{N+1}] \mapsto [v_1^2, \ldots, v_N^2, v_{N+1}^2].\]

We have \(\tilde{s}(\tilde{X}_{m,a}) \simeq \mathbb{P}(\tilde{V}_{m,a})\), where \(\tilde{V}_{m,a} \subset \tilde{V}\) is the subspace defined by the equations

\[
a_i^1 v_1 + \cdots + a_N v_N = 0, \quad i = 0, \ldots, m - 1, \quad a_i^m v_1 + \cdots + a_N v_N - v_{N+1} = 0.
\]

The map

\[
\tilde{s}_a : \tilde{X}_{m,a} \to \mathbb{P}(\tilde{V}_{m,a})
\]

is a \(I_{N+1}\)-branched cover with branch locus \(\{v_i = 0\}_{i=1,\ldots,N+1}\). As \(a\) varies over \(\mathfrak{a}^{rs}\), we obtain

\[
\tilde{X}_m|_{\mathfrak{a}^{rs}} \xrightarrow{\tilde{s}} \mathbb{P}(\tilde{V}_m) \xleftarrow{a^{rs}} \mathfrak{a}^{rs}
\]

here \(\tilde{V}_m\) is the vector subbundle of the trivial bundle \(\tilde{V} \times \mathfrak{a}^{rs}\) whose fiber over \(a\) is the subspace \(\tilde{V}_{m,a}\), and \(\mathbb{P}(\tilde{V}_m)\) is the associated projective bundle.

We now introduce another family \(\tilde{Y}_m\) of branched covers of \(\mathbb{P}^{N-m-1}\). Let \(\text{sum} : I_{N+1}^{N-m-1} \to I_{N+1}\) be the summation map and define \(\tilde{I}_{N+1}^{N-m-1} = \ker(\text{sum})\). For any \(a \in \mathfrak{a}^{rs}\) let \(\tilde{C}_a \to \mathbb{P}^1\) be the \(I_{N+1}\)-branched cover of \(\mathbb{P}^1\) introduced in §3.2. The semi-direct product \(I_{N+1}^{N-m-1} \rtimes S_{N-m-1}\) acts naturally on \((\tilde{C}_a)^{N-m-1}\). We define

\[
\tilde{Y}_{m,a} = (\tilde{C}_a)^{N-m-1}/\tilde{I}_{N+1}^{N-m-1} \rtimes S_{N-m-1}.
\]

Similar to the case of \(Y_{m,a}\), the natural map

\[
\tilde{i}_a : \tilde{Y}_{m,a} \to (\tilde{C}_a)^{N-m-1}/\tilde{I}_{N+1}^{N-m-1} \rtimes S_{N-m-1} \simeq \mathbb{P}^{N-m-1}
\]

is an \(I_{N+1}\)-branched cover of \(\mathbb{P}^{N-m-1}\) with branch locus \(\{H_{a,i} = 0\}_{i=1,\ldots,N+1}\), here \(H_{a,i} = 0\) for \(i = 1, \ldots, N\) are the hyperplanes as before (see §3.4) and \(H_{a,N+1} := x_{N-m} = 0\) is the hyperplane corresponding to the ramification point \(a_{N+1} = \infty\).

As \(a\) varies over \(\mathfrak{a}^{rs}\), we get an \(\mathfrak{a}^{rs}\)-family of \(I_{N+1}\)-branched cover of \(\mathbb{P}^{N-m-1}\)
We will again make use of (3.6) to relate the two families \( \tilde{X}_m|_{a^rs} \) and \( \tilde{Y}_m \). The Weyl group \( W \) acts naturally on \( \tilde{X}_m|_{a^rs} \) and \( \tilde{Y}_m \), making the projections to \( a^{rs} \) equivariant with respect to these \( W \)-actions. We let \( I_N \) act on \( \tilde{Y}_m \) via the map

\[
\kappa : I_N \cong \mathbb{Z}_K(a) \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^{N+1} \xrightarrow{pr} I_{N+1},
\]

where the first arrow is given by \( (\zeta_1, \ldots, \zeta_N) \mapsto (\zeta_1, \ldots, \zeta_N, 0) \). We also let \( W \) act on \( I_{N+1} \) by permuting the first \( N \) coordinates and we use this convention to form the semi-direct product \( I_{N+1} \rtimes W \).

**Proposition 3.2.** There is an \( I_{N+1} \rtimes W \)-equivariant isomorphism

\[
\tilde{X}_m|_{a^rs} \simeq \tilde{Y}_m := (\tilde{Y}_m \times_{a^rs} a^{rs})/I_N,
\]

where \( W \) (resp. \( I_{N+1} \)) acts on \( \tilde{Y}_m \) by the diagonal action (resp. on the first factor).

**Proof.** Let us consider the following twist of \( \tilde{X}_m|_{a^rs} \):

\[
\tilde{X}_m'|_{a^rs} \to a^{rs}
\]

whose fiber over \( a = (a_1, \ldots, a_N) \) is the complete intersection of quadrics given by

\[
\frac{a_1^i}{d_1} v_1^2 + \cdots + \frac{a_N^i}{d_N} v_N^2 = 0, \quad i = 0, \ldots, m - 1, \quad \frac{a_1^m}{d_1} v_1^2 + \cdots + \frac{a_N^m}{d_N} v_N^2 - \epsilon^2 = 0
\]

where \( d_i \) is defined as before, i.e., \( d_i = \prod_{j \neq i} (a_j - a_i) \). Similar to the case of \( X_m \), we have a canonical isomorphism

\[
\tilde{X}_m|_{a^rs} \simeq (\tilde{a}^{rs} \times_{a^rs} \tilde{X}_m'|_{a^rs})/I_N.
\]

Here the \( I_N \)-action on \( \tilde{X}_m'|_{a^rs} \) is defined as the composition of \( I_N \to I_{N+1} \) in (3.12) with the natural action of \( I_{N+1} \) on \( \tilde{X}_m \). The Weyl group \( W \) acts naturally on \( \tilde{X}_m \), \( \tilde{X}_m' \) and \( \tilde{Y}_m \), making the projections to \( a^{rs} \) equivariant maps under the \( W \)-actions. Thus we obtain \( I_{N+1} \rtimes W \)-actions on \( \tilde{X}_m \), \( \tilde{X}_m' \) and \( \tilde{Y}_m \) and the projections to \( a^{rs} \) are \( \tilde{I}_{N+1} \rtimes W \)-equivariant. Moreover, the isomorphism in (3.14) is \( I_{N+1} \rtimes W \)-equivariant. In [3.6](see Proposition 3.3), we show that there is an \( I_{N+1} \rtimes W \)-equivariant isomorphism \( \tilde{X}_m'|_{a^rs} \simeq \tilde{Y}_m \). Combining this with (3.14) we obtain (3.13).

\[
\square
\]

### 3.6. The families \( \tilde{X}_m' \) and \( \tilde{Y}_m \)

In this subsection we state and prove the following proposition which was used in the previous subsections.

**Proposition 3.3.** We have an \( I_{N+1} \rtimes W \)-equivariant isomorphism \( \tilde{X}_m'|_{a^rs} \simeq \tilde{Y}_m \). In particular, it induces an \( I_N \rtimes W \)-equivariant isomorphism on the quotient

\[
X_m'|_{a^rs} \simeq \tilde{X}_m'|_{a^rs}/(\mathbb{Z}/2\mathbb{Z}) \simeq \tilde{Y}_m/(\mathbb{Z}/2\mathbb{Z}) \simeq \tilde{Y}_m.
\]

Here \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \tilde{X}_m' \) and \( \tilde{Y}_m \) via the map \( \mathbb{Z}/2\mathbb{Z} \to I_{N+1} \) given by \( 1 \mapsto (0, \ldots, 0, 1) \).
We follow closely the argument in [1] \S 2. We begin by introducing some auxiliary spaces and maps. Let \( \tilde{t}_m \subset \tilde{V} \times a^{rs} \) be the vector sub-bundle whose fiber over \( a \in a^{rs} \) is the subspace \( \tilde{v}_{a} \subset \tilde{V} \) defined by the equations

\[
\frac{a_1 v_1}{d_1} + \cdots + \frac{a_N v_N}{d_N} = 0, \quad i = 0, \ldots, m-1, \quad \frac{a_m v_1}{d_1} + \cdots + \frac{a_N v_N}{d_N} - v_{N+1} = 0.
\]

The map \( \tilde{s} : \mathbb{P}(\tilde{V}) \times a^{rs} \to \mathbb{P}(\tilde{V}) \times a^{rs} \) (see (3.11)) maps \( \tilde{X}_{m,a} \) to \( \mathbb{P}(\tilde{V}_m) \) and the resulting map

\( \tilde{s}' : \tilde{X}_{m,a} \to \mathbb{P}(\tilde{V}_m) \)

is an \( I_{N+1} \)-branched cover with branch locus \( \{ v_i = 0, \quad i = 1, \ldots, N+1 \} \).

Let \( C(\mathbb{P}(\tilde{V}_m)) \) be the function field of \( \mathbb{P}(\tilde{V}_m) \). Then the function field of \( \tilde{X}_{m,a} \) is given by the following field extension

\[
F := C(\mathbb{P}(\tilde{V}_m))((\frac{v_i}{v_{N+1}})^{1/2}_{i=1,\ldots,N}) \supseteq C(\mathbb{P}(\tilde{V}_m)).
\]

Since \( \tilde{X}_{m,a} \) is smooth and \( \tilde{s}' \) is finite, it follows that

\[ \tilde{X}_{m,a} \text{ is the normalization of } \mathbb{P}(\tilde{V}_m) \text{ in } F. \]

The group \( I_{N+1} \) acts on \( F \) by \( \zeta : \frac{v_i^{1/2}}{v_{N+1}} \mapsto (-1)^{\zeta_i} \frac{v_i^{1/2}}{v_{N+1}}, \zeta = (\zeta_1, \ldots, \zeta_{N+1}) \in I_{N+1} \) and \( W \) acts on \( F \) by \( w : (\frac{v_i}{v_{N+1}})^{1/2} \mapsto (\frac{v_i}{v_{N+1}}^{1/2}). \)

Similarly, let \( k_\eta \) be the function field of \( a^{rs} \) and let \( \eta = (a_1, \ldots, a_N) \in a^{rs}(k_\eta) \) be the corresponding generic point. Then the function field of \( \tilde{y}_m \) is given by the following field extension

\[
F' = C(\mathbb{P}^{N-m-1})(\frac{H_{\eta,i}}{H_{\eta,N+1}})^{1/2}_{i=1,\ldots,N} \supseteq C(\mathbb{P}^{N-m-1}).
\]

Here, \( H_{\eta,i}, i = 1, \ldots, N \) are the hyperplanes associated to \( \eta \in a^{rs} \) in (3.4), \( H_{\eta,N+1} = x_{N-m} \), and \( \frac{H_{\eta,i}}{H_{\eta,N+1}} \) are rational functions on \( \mathbb{P}^{N-m-1}_\eta \), regraded as elements in \( C(\mathbb{P}^{N-m-1}) = C(\mathbb{P}^{N-m-1}_\eta) \). Since \( \tilde{y}_m \) is smooth and \( \tilde{r} : \tilde{y}_m \to \mathbb{P}^{N-m-1} \) is finite, it follows that

\( \tilde{y}_m \) is the normalization of \( \mathbb{P}^{N-m-1} \) in \( F' \).

The group \( I_{N+1} \) acts on \( F' \) by \( \zeta : H_{\eta,i}^{1/2} \mapsto (-1)^{\zeta_i} H_{\eta,i}^{1/2}, \zeta = (\zeta_1, \ldots, \zeta_{N+1}) \in I_{N+1} \) and \( W \) acts on \( F \) by \( w : (\frac{H_{\eta,i}}{H_{\eta,N+1}})^{1/2} \mapsto (\frac{H_{\eta,i}}{H_{\eta,N+1}}^{1/2}). \)

By the discussion above, to prove Proposition 3.3 it is enough to prove the following statement.

\[
(3.16) \quad (\mathbb{P}(\tilde{V}_{m,a}), \{ v_i \}_{i=1,\ldots,N+1}) \quad \text{and} \quad (\mathbb{P}^{N-m-1}, \{ H_{a,i} \}_{i=1,\ldots,N+1}) \quad \text{are equivalent.}
\]

\[ \text{Recall for any irreducible variety } X \text{ and } K \text{ a finite extension of the function field } C(X), \text{ there exists a unique normal variety } Y \text{ and a finite morphism } f : Y \to X \text{ such that the induced map } C(X) \to C(Y) = K \text{ is the given field extension. We call } Y \text{ the normalizati on of } X \text{ in } K. \]
That is, there is an isomorphism (or trivialization) of vector bundles $\phi: \mathbb{C}^{N-m} \times \mathfrak{a}^r \simeq \tilde{V}_m'$ over $\mathfrak{a}^r$ such that for any $k$-point $a \in \mathfrak{a}^r(k)$, $k$ a field, the induced map on the dual fibers $\phi_a^*: (\tilde{V}_{m,a}')^* \simeq k^{N-m}$ satisfies $\phi_a^*(v_i) = H_{a,i}$ for $i = 1, ..., N + 1$.

To prove (3.16), we need to construct, for each $S$-point $a \in \mathfrak{a}^r(S)$, a functorial isomorphism $\phi_a: \mathbb{C}^{N-m} \times S \simeq \tilde{V}_{m,a}'$ satisfying the desired property. For notational simplicity we construct such isomorphisms on the level of $k$-points. The argument for general $S$-points is the same.

Consider the following map
$$\psi_a: \tilde{V} \otimes k \xrightarrow{pr} V \otimes k \simeq V \otimes k$$
where $pr: \tilde{V} \otimes k = (V \otimes k) \oplus k \rightarrow V \otimes k$ is the projection map and the second isomorphism is given by multiplying the diagonal matrix $d = \text{diag}(d_1^{-1}, ..., d_N^{-1}) \in GL(V \otimes k)$ (recall for $a = (a_1, ..., a_N) \in \mathfrak{a}^r$, $d_i = \prod_{j \neq i} (a_j - a_i)$). One can check that $\psi_a$ maps $\tilde{V}_{m,a}'$ isomorphically onto $V_{m,a}$ (see (3.15) and (3.5) for the definitions of $\tilde{V}_{m,a}'$ and $V_{m,a}$, respectively) and the resulting isomorphism $\psi_a: \tilde{V}_{m,a}' \simeq V_{m,a}$ satisfies
$$\psi_a^*(d_i \cdot v_i) = v_i$$
for $i = 1, ..., N$ and $\psi_a(H_\infty) = v_{N+1}$, where $H_\infty := a_1^m v_1 + \cdots + a_N^m v_N$. Thus we are reduced to show that
$$(\mathbb{P}(V_{m,a}), \{d_i \cdot v_i\}_{i=1,\ldots,N} \cup H_\infty)$$
and
$$(\mathbb{P}(k^{N-m}), \{H_{a,i}\}_{i=1,\ldots,N+1})$$
are equivalent, that is, there is an isomorphism
$$\gamma_a: k^{N-m} \simeq V_{m,a}$$
such that $\gamma_a^*(d_i \cdot v_i) = H_{a,i}$, $i = 1, ..., N$ and $\gamma_a^*(H_\infty) = H_{a,N+1}$.

Consider the basis $u_i = (a_1^i, ..., a_N^i)$, $i = 0, ..., N - 1$ of $V \otimes k$. Then the isomorphism $V \otimes k \simeq (V \otimes k)^*$ given by the paring $\langle (v_i), (w_i) \rangle = \sum v_i w_i$, induces the following isomorphism
$$f_1: V_{m,a}^* \simeq V \otimes k/\langle u_0, ..., u_{m-1} \rangle \simeq k\langle u_m, ..., u_{N-1} \rangle.$$ 
Let $s_i$ be the elementary symmetric polynomial in $a_1, ..., a_N$ of degree $i$ and let $A = (a_{ij}) \in GL_{N-m}(k)$ be the matrix with entries $a_{ij} = (-1)^{i-1} s_{j-i}$ if $j \geq i$ and $a_{ij} = 0$ otherwise. Consider the following isomorphism
$$f: V_{m,a}^* \xrightarrow{f_1} k\langle u_m, ..., u_{N-1} \rangle \xrightarrow{f_2} k^{N-m} \xrightarrow{f_3} k^{N-m},$$
here $f_2: k\langle u_m, ..., u_{N-1} \rangle \simeq k^{N-m}$ is the isomorphism given by $u_{N-i} \mapsto (-1)^{i-1} x^i$ and $f_3$ is the isomorphism given by right multiplication by $A^{-1}$. We claim that the dual
$$\gamma_a := f^*: k^{N-m} \simeq V_{m,a}$$
is the desired isomorphism, i.e., we have $f(d_i \cdot v_i) = H_{a,i}$ for $i = 1, ..., N$, and $f(H_\infty) = H_{a,N+1}$. Note that the configuration $(\mathbb{P}(V_{m,a}), \{d_i \cdot v_i\}_{i=1,\ldots,N})$ (resp. $(\mathbb{P}(k^{N-m}), \{H_{a,i}\}_{i=1,\ldots,N})$) is equal to the configuration $(P, H_1, ..., H_N)$ (resp. $(P', H'_1, ..., H'_N)$) in [2.1] §2.1. Moreover, the map

\[ \text{Here we regard } x_i \text{ as the } i\text{-th coordinate vector of } k^{N-m}. \]

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$f$ is the one used in [11 Proof of Theorem 2.1.1] to show that the above configurations are equivalent. Thus according to [11 Proof of Theorem 2.1.1] we have

$$f(d_i \cdot v_i) = H_{a, i} \quad \text{for} \quad i = 1, ..., N.$$ 

So it remains to show that $f(H_{\infty}) = H_{a, N+1}$. For this we observe that $f_1(H_{\infty}) = u_m$. Hence

$$f(H_{\infty}) = f_3 \circ f_2 \circ f_1(H_{\infty}) = f_3 \circ f_2(u_m) = (-1)^{N-m-1} \cdot f_3(x_{N-m}) = (-1)^{N-m-1} \cdot x_{N-m} \cdot A = x_{N-m} = H_{a, N+1}.$$ 

This proves (3.16).

The proof of Proposition [3.3] is complete.

4. Monodromy of families of Hessenberg varieties

In this section we study the monodromy representation of $\pi_1^K(\mathfrak{g}^{rs}_1, a) = Z_K(a) \ltimes B_N$ on the primitive cohomology of complete intersections of quadrics $X_m$ (and on the primitive cohomology of their branched double covers $\tilde{X}_m$). By Theorem [2.5] this gives us a complete description of the monodromy representations of $\pi_1^K(\mathfrak{g}^{rs}_1)$ associated with the families of Hessenberg varieties $\text{Hes}_n^{O,\perp}$ and $\text{Hes}_n^{E,\perp}$.

To state the result, let us recall, from (3.2), the monodromy representations $\rho_{C, \chi} : P_N \to Sp(H^1(C_{a, \chi}, \mathbb{C})) \simeq Sp(2i - 2)$ and $\rho_{\tilde{C}, \chi} : P_N \to Sp(H^1(\tilde{C}_{a, \chi}, \mathbb{C})) \simeq Sp(2i - 2)$ where $i = \frac{|a|}{2}$. Recall further that, by (3.2), these representations are irreducible with Zariski dense image. Let us consider the irreducible representation of $Sp(2i - 2)$ associated to the fundamental weight $\omega_j$. Composing $\rho_{C, \chi}$ and $\rho_{\tilde{C}, \chi}$ with this fundamental representation we obtain irreducible representations $P^j_\chi$ and $\tilde{P}^j_\chi$ of the pure braid group $P_N$.

For a character $\chi$ of an abelian group we write $V_\chi$ for the corresponding one dimensional representation. Recall that the group $Z_K(a)$ can be naturally identified with $I_N$ as explained in (3.11). We also relate the characters of $I_N$ and $I_{N+1}$ using the map $\kappa$ defined in (3.12). From these considerations we conclude that

$$(4.1) \quad Z_K(a)^\vee = I_N^\vee \quad \text{and we have a map} \quad \tilde{\kappa} : I_{N+1}^\vee \to I_N^\vee.$$ 

In particular, characters of $I_N$ and $I_{N+1}$ can be regarded as characters of $Z_K(a)$. To state the main theorems of this section we define two $Z_K(a) \ltimes P_N$-representations as follows:

$$E_{ij}^N \simeq \bigoplus_{\chi \in I_N, |\chi| = 2i} P^j_\chi \otimes V_\chi \quad \text{and} \quad \tilde{E}_{ij}^N = \bigoplus_{\chi \in I_{N+1}, |\chi| = 2i, N+1 \in \text{supp}_N} \tilde{P}^j_\chi \otimes V_\chi,$$

where the $I_N$ acts on $\tilde{E}_{ij}^N$ via the map $\tilde{\kappa} : I_{N+1}^\vee \to I_N^\vee$ of (4.1), and $P_N$ acts on $V_\chi$ via the map $\rho : P_N \to I_N$ of (3.7). Lemmas [4.3] and [4.6] show that the $Z_K(a) \ltimes P_N$ actions on $E_{ij}^N$ and $\tilde{E}_{ij}^N$ extend naturally to $Z_K(a) \ltimes B_N$-actions.

The main results of this section are the following.
Theorem 4.1. For $1 \leq m \leq N - 1$, the monodromy representation of $\pi_1^K(\mathfrak{g}_1^{rs}, a)$ on $P(X_m) := H_{prim}^{N-m-1}(X_{m,a}, \mathbb{C})$ decomposes into irreducible representations in the following manner:

$$P(X_m) \cong \bigoplus_{i} \bigoplus_{j \equiv N-m-1 \mod 2} E^N_{ij},$$

with $N - m + 1 \leq 2i \leq N$, $l = \min\{N - m - 1, -N + m + 2i - 1\}$.

To state the second main result, we set $P(\tilde{X}_m) := H_{prim}^{N-m-1}(\tilde{X}_{m,a}, \mathbb{C})$.

Recall that there is an involution action $\sigma$ on $\tilde{X}_m$ and the projection map $p_m : \tilde{X}_m \to X_m$ is a branched double cover with Galois group $\langle \sigma \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ (see [3.4]). Then $P(\tilde{X}_m) = P(\tilde{X}_m)^{\sigma = id} \oplus P(\tilde{X}_m)^{\sigma = -id}$ and we have $P(\tilde{X}_m)^{\sigma = id} = P(X_m)$. The next theorem describes $P(\tilde{X}_m)^{\sigma = -id}$.

Theorem 4.2. For $1 \leq m \leq N - 1$, the monodromy representation of $\pi_1^K(\mathfrak{g}_1^{rs}, a)$ on $P(\tilde{X}_m)^{\sigma = -id}$ decomposes into irreducible representations in the following manner:

$$P(\tilde{X}_m)^{\sigma = -id} \cong \bigoplus_{i} \bigoplus_{j \equiv N-m-1 \mod 2} \tilde{E}^N_{ij},$$

with $N - m + 1 \leq 2i \leq N + 1$, $l = \min\{N - m - 1, -N + m + 2i - 1\}$.

4.1. Proof of Theorem 4.1. Let us start with the following proposition which is a consequence of Proposition [3.1]

Proposition 4.3. There is an isomorphism of representations of $\pi_1^K(\mathfrak{g}_1^{rs}, a) \simeq I_N \rtimes B_N$

$$H^i(X_{m,a}, \mathbb{C}) \cong \bigoplus_{\chi \in I'_N} H^i(Y_{m,a}, \mathbb{C})_\chi \otimes V_\chi.$$  

The group $I_N$ acts on the summand $H^i(Y_{m,a}, \mathbb{C})_\chi \otimes V_\chi$ via the character $\chi \in I'_N$.

Proof. Observe that the families $X_m|_{a^{rs}} \to a^{rs}$, $Y_m \to a^{rs}$, and $\tilde{a}^{rs} \to a^{rs}$ are all $W$-equivariant. Hence their cohomology groups $H^i(X_{m,a}, \mathbb{C})$, $H^i(Y_{m,a}, \mathbb{C})$, $H^0((\tilde{a}^{rs})_a, \mathbb{C})$ carry an action of the braid group $B_N \simeq \pi_1^W(a^{rs}, a)$. Let

$$H^i(X_{m,a}, \mathbb{C}) = \bigoplus_{\chi \in I'_N} H^i(X_{m,a}, \mathbb{C})_\chi \quad \quad H^i(Y_{m,a}, \mathbb{C}) = \bigoplus_{\chi \in I'_N} H^i(Y_{m,a}, \mathbb{C})_\chi$$

$$H^0((\tilde{a}^{rs})_a, \mathbb{C}) = \bigoplus_{\chi \in I'_N} V_\chi$$
be the decompositions with respect to the action of $I_N$; for the last identity we recall that $\tilde{a}^r s \to a^r s$ is an $I_N$-torsor. For $\chi \in I_N^\times$ and $b \in B_N$, we write $b \cdot \chi$ for the action of $b$ on $\chi$. Then the braid group action on $H^i(Y_{m,a}, \mathbb{C})$ is described as follows

$$b \in B_N : H^i(Y_{m,a}, \mathbb{C})_\chi \mapsto H^i(Y_{m,a}, \mathbb{C})_{b \cdot \chi}.$$ 

The $B_N$-actions on $H^i(X_{m,a}, \mathbb{C})$ and $H^0((\tilde{a}^r s)_a, \mathbb{C})$ are described in the same manner.

By the Künneth formula, the cohomology of the fiber of $Y_m = (Y_m \times_{a^r s} \tilde{a}^r s)/I_N$ over $a \in a^r s$ is canonically isomorphic to

$$H^i(Y_{m,a}, \mathbb{C}) \cong \bigoplus_{\chi \in I_N^\times} H^i(Y_{m,a}, \mathbb{C}) \otimes V_\chi.$$ 

Thus by (3.8) we obtain the desired $\pi_1^K(g_1^{rs}, a) \cong I_N \acts B_N$-equivariant isomorphism.

The isomorphism in Proposition 4.3 implies the following isomorphism of monodromy representations

$$P(X_m) \cong \bigoplus_{\chi \in I_N^\times} P(Y_m) \otimes V_\chi,$$

where

$$P(Y_m) := H^{N-m-1}_{prim}(Y_{m,a}, \mathbb{C}), \quad P(Y_m)_{\chi} := H^{N-m-1}(Y_m, \mathbb{C})_{\chi} \cap P(Y_m).$$

Our goal is to decompose the representation above into irreducible representations. Observe that each summand $P(Y_m)_{\chi}$ is invariant under the action of the pure braid group $P_N$. According to [11, Theorem 2.5.1], there is an isomorphism of representations of $P_N$

$$P(Y_m)_{\chi} \cong \wedge^{N-m-1} H^1(C_a, \mathbb{C})_{\chi} \cong \wedge^{N-m-1} H^1(C_{a,\chi}, \mathbb{C}),$$

where $P_N$ acts on $H^1(C_{a,\chi}, \mathbb{C})$ via the map $\rho_{C_{\chi}} : P_N \to Sp(2i-2), i = |\chi|/2$.

As an $Sp(2i-2)$-representation $\wedge^{N-m-1} H^1(C_{a,\chi}, \mathbb{C})$ decomposes into a direct sum of fundamental representations in a well-known manner. This implies the following decomposition of $P(Y_m)_{\chi}$ into irreducible representations of $P_N$:

$$P(Y_m)_{\chi} \cong \wedge^{N-m-1} H^1(C_{a,\chi}, \mathbb{C}) = \bigoplus_{j \equiv N-m-1 \mod 2, \ j \in [0, l]} P^j_{\chi},$$

where $l = \min\{N - m - 1, -N + m + |\chi| - 1\}$.

Combining (4.2) with (4.3), we obtain the following decomposition

$$P(X_m) \cong \bigoplus_{\chi \in I_N^\times} P(Y_m)_{\chi} \otimes V_\chi \cong \bigoplus_{\chi \in I_N^\times} \bigoplus_{j \in [0, l]} P^j_{\chi} \otimes V_\chi.$$
Using the notation from the beginning of this section the decomposition \((4.4)\) can be rewritten as

\[
P(X_m) \simeq \bigoplus_{i,j} E_{ij}^N,
\]

where \(N - m + 1 \leq 2i \leq N\) and \(l = \min\{N - m - 1, -N + m + 2i - 1\}\).

We have the following

**Lemma 4.4.**

1. Each \(E_{ij}^N\) is an irreducible representation of \(\pi_1^K(g_i^r, a)\). We denote by \(\rho_{ij}^N : \pi_1^K(g_i^r, a) \to GL(E_{ij}^N)\) the corresponding map.

2. Suppose \(j > 0\). Let \(H := \overline{\rho_{ij}^N(P_N)} \subseteq GL(E_{ij}^N)\) be the Zariski closure of \(\rho_{ij}^N(P_N)\) in \(GL(E_{ij}^N)\) (recall \(P_N \subseteq \pi_1^K(g_i^r, a)\) is the pure braid group). Then we have \(\text{Lie} H \simeq \mathfrak{sp}(2i - 2)\). In particular, the image \(\rho_{ij}^N(\pi_1^K(g_i^r, a))\) is infinite.

**Proof.** We begin with the proof of (1). We first show that \(E_{ij}^N\) is a \(\pi_1^K(g_i^r, a)\)-invariant subspace of \(P(X_m)\). For this, we observe that the decomposition in \((4.4)\) is compatible with the action of \(B_N\), that is, for \(b \in B_N\),

\[
b : P_{\chi}^j \otimes V_\chi \mapsto P_{b\chi}^j \otimes V_{b\chi}.
\]

Since the braid group \(B_N\) acts transitively on the set \(\{\chi \in I_N^\times | \chi| = 2i\}\), it follows that the subspace

\[
E_{ij}^N = \bigoplus_{\chi \in I_N^\times, |\chi| = 2i} P_{\chi}^j \otimes V_\chi
\]

is stable under the action of \(\pi_1^K(g_i^r, a)\). Now since each summand \(P_{\chi}^j \otimes V_\chi\) is irreducible as a representation of \(P_N\), it follows that each \(E_{ij}^N\) is an irreducible representation of \(\pi_1^K(g_i^r, a)\).

We prove (2). For each \(\chi \in I_N^\times\), we define \(\rho_\chi : P_N \xrightarrow{\rho} I_N \xrightarrow{\chi} \mu_2\). Here \(\rho\) is the map in \((3.7)\). Define

\[
\psi_1 := (\rho_{C_\chi}, \bigoplus_{|\chi| = 2i} \rho_\chi) : P_N \to Sp(2i - 2) \times \mu_2^{(N)}.
\]

Let \(V_{ij}\) denote the irreducible representation of \(Sp(2i - 2)\) associated to the fundamental weight \(\omega_j\). Then the restriction of \(\rho_{ij}^N : \pi_1^K(g_i^r, a) \to GL(E_{ij}^N)\) to \(P_N\) can be identified with

\[
\psi : P_N \xrightarrow{\psi_1} Sp(2i - 2) \times \mu_2^{(N)} \xrightarrow{\psi_2} GL(V_{ij})^{\times N}
\]

where \(\psi_2\) maps \(Sp(2i - 2)\) diagonally into \(GL(V_{ij})^{\times N}\) and \(\psi_2\) maps \(\mu_2 = \{\pm 1\}\) to \(\pm id \in GL(V_{ij})\). Since \(\rho_{C_\chi}(P_N) = Sp(2i - 2)\), it implies that the connected component \(\psi_1(P_N) = Sp(2i - 2)\). So to prove (2), it suffices to show that \(\text{Lie}(\text{Im}(\psi_2)) \simeq \mathfrak{sp}(2i - 2)\) for \(j > 0\). This follows from the fact that the induced map \(d\psi_2 : \mathfrak{sp}(2i - 2) \to \bigoplus \mathfrak{gl}(V_{ij})\) on the Lie algebras is injective.
It follows from the lemma above that (4.5) is the decomposition of the monodromy representation $P(X_m)$ into irreducible subrepresentations. This completes the proof of Theorem 4.1.

4.2. Proof of Theorem 4.2. The proof is similar to the case of $X_m$. First using the isomorphism (3.13) and the same argument as in the case of $X_m$, we obtain the following proposition.

**Proposition 4.5.** There is an isomorphism of $I_{N+1} \rtimes B_N$-representations

$$H^i(\tilde{X}_{m,a}, \mathbb{C}) \simeq \bigoplus_{\chi \in I_{N+1}^\lor} H^i(\tilde{Y}_{m,a}, \mathbb{C}) \chi \otimes V_\chi,$$

where for $V_\chi$, we regard $\chi$ as an element in $I_N^\lor$ via the map $\kappa: I_{N+1}^\lor \to I_N \subset (4.1)$, and the group $I_{N+1}$ acts on the summand $H^i(\tilde{Y}_{m,a}, \mathbb{C}) \chi \otimes V_\chi$ via the character $\chi \in I_{N+1}^\lor$.

Set

$$P(\tilde{Y}_m) := H_{prim}^{N-m-1}(\tilde{Y}_{m,a}, \mathbb{C}).$$

By Proposition 4.5 there is an isomorphism of $I_{N+1} \rtimes B_N$-representations

$$P(\tilde{X}_m) \simeq \bigoplus_{\chi \in I_{N+1}^\lor} P(\tilde{Y}_m) \chi \otimes V_\chi. \tag{4.6}$$

For any $\chi \in I_{N+1}^\lor$ with $|\chi| = 2i$, let $\tilde{C}_{a,\chi}$ be the hyperelliptic curve defined in 3.2 and let $\rho_{\tilde{C}_\chi}: P_N \to Sp(H^1(\tilde{C}_{a,\chi}, \mathbb{C})) \simeq Sp(2i - 2)$ denote the monodromy representation for the family $\tilde{C}_\chi \to a^x$. Again by (4.1), we have an isomorphism of $P_N$-representations

$$P(\tilde{Y}_m) \chi \simeq \wedge^{N-m-1} H^1(\tilde{C}_{a,\chi}, \mathbb{C}) \simeq \wedge^{N-m-1} H^1(\tilde{C}_{a,\chi}, \mathbb{C}). \tag{4.7}$$

Combining (4.6) with (4.7) we obtain the following decomposition

$$P(\tilde{X}_m) \simeq \bigoplus_{\chi \in I_{N+1}^\lor} \wedge^{N-m-1} H^1(\tilde{C}_{a,\chi}, \mathbb{C}) \otimes V_\chi. \tag{4.8}$$

We describe the monodromy representation $P(\tilde{X}_m)^{\sigma = -id}$ (recall that $\sigma$ is the involution on $\tilde{X}_m$). For this, we first observe that the involution action of $\langle \sigma \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ on $\tilde{X}_m$ is equal to the composition of

$$i_\infty: \mathbb{Z}/2\mathbb{Z} \to I_{N+1}, \ 1 \mapsto (0, \ldots, 0, 1),$$

with the action of $I_{N+1}$ on $\tilde{X}_m$. Hence by (4.8) we have

$$P(\tilde{X}_m)^{\sigma = -id} = \bigoplus_{\chi \in I_{N+1}^\lor} P(\tilde{X}_m) \chi \simeq \bigoplus_{\chi \in I_{N+1}^\lor} \wedge^{N-m-1} H^1(\tilde{C}_{a,\chi}, \mathbb{C}) \otimes V_\chi. \tag{4.9}$$
Again, since $\rho_{\tilde{C}_x}(P_N)$ is Zariski dense in $Sp(H^1(\tilde{C}_x, \mathbb{C}))$, we have the following decomposition
\[
\wedge^{N-m-1}H^1(\tilde{C}_{a,x}, \mathbb{C}) = \bigoplus_{j \equiv N-m-1 \mod 2, \; j \in [0, l]} \tilde{P}_j,
\]
where $l = \min\{N - m - 1, -N + m + |\chi| - 1\}$.

Using the notation from the beginning of this section the decomposition (4.9) can be rewritten as
\[
P(\tilde{X}_m)^{\sigma = \text{id}} \simeq \bigoplus_{i \equiv N-m-1 \mod 2, \; j \in [0, l]} \tilde{E}_{ij}^N,
\]
where $N - m + 1 \leq 2i \leq N + 1$, $l = \min\{N - m - 1, -N + m + 2i - 1\}$.

The same argument as in the proof of Lemma 4.4 shows the following

Lemma 4.6.

1. $\tilde{E}_{ij}^N$ is an irreducible representation of $\pi_1^K(\mathfrak{g}_1^s, a)$. We denote by $\tilde{\rho}_{ij}^N : \pi_1^K(\mathfrak{g}_1^s, a) \to GL(E_{ij}^N)$ the corresponding map.

2. Suppose $j > 0$. Let $H := \overline{\tilde{\rho}_{ij}^N(P_N)} \subset GL(\tilde{E}_{ij}^N)$ be the Zariski closure of $\tilde{\rho}_{ij}^N(P_N)$ in $GL(\tilde{E}_{ij}^N)$. Then we have $Lie H \simeq \mathfrak{sp}(2i - 2)$. In particular, the image $\tilde{\rho}_{ij}^N(\pi_1^K(\mathfrak{g}_1^s, a))$ is infinite.

This completes the proof of Theorem 4.2.

4.3. The local systems $E_{ij}^N$ and $\tilde{E}_{ij}^N$. In this subsection, we show that from the constructions in previous sections, we have obtained the following set consisting of pairwise non-isomorphic irreducible $K$-equivariant local systems on $\mathfrak{g}_1^s$

(4.10) \[\left\{E_{ij}^{2n+1}, \; i \in [1, n], \; j \in [0, i - 1]; \; \tilde{E}_{ij}^{2n+1}, \; i \in [1, n + 1], \; j \in [1, i - 1], \; \tilde{E}_{n+1,0}^{2n+1} \cong \mathbb{C}\right\}.
\]
For this, we first observe that
\[E_{ij}^N \simeq E_{i'j'}^N \text{ and } \tilde{E}_{ij}^N \simeq \tilde{E}_{i'j'}^N \text{ if and only if } i = i', j = j'.\]

In fact, assume that $E_{ij}^N \simeq E_{i'j'}^N$. Then we must have $i = i'$, otherwise, the centralizer $Z_K(a) \cong I_N$ would act differently on $E_{ij}^N$ and $E_{i'j'}^N$. Now regarding $E_{ij}^N$ and $E_{i'j'}^N$ as $P_N$-representations we see that $j = j'$. Similar argument applies to $\tilde{E}_{ij}^N$.

It remains to prove the following.

Lemma 4.7. We have $E_{i,j}^N \simeq E_{i',j'}^N$ if and only if $i + i' = (N + 1)/2$ and $j = j' = 0$. 

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Proof. Recall

\[ E_{ij}^N = \bigoplus_{\chi \in I_N^\vee, |\chi| = 2i} P_{\chi}^j \otimes V_{\chi}, \quad \tilde{E}_{ij}^N = \bigoplus_{\chi \in I_{N+1}^\vee, |\chi| = 2i, N+1 \in \text{supp } \chi} \tilde{P}_{\chi}^j \otimes V_{\chi}. \]

For \( V_{\chi'} \) we regard \( \chi' \) as an element in \( I_N^\vee \) via the map \( \kappa : I_{N+1}^\vee \to I_N^\vee \) in (4.11). Observe that for \( \chi' \in I_{N+1}^\vee \) and \( N+1 \in \text{supp } \chi' \), we have

\[ \kappa(\chi') = \chi \text{ if and only if } \text{supp } \chi = \{1, \ldots, N+1\} \backslash \text{supp } \chi'. \]

Thus the map \( \kappa \) maps the subset \( \{ \chi' \in I_{N+1}^\vee, |\chi'| = 2i', N+1 \in \text{supp } \chi' \} \) bijectively to the subset \( \{ \chi \in I_N^\vee, |\chi| = 2i \} \) where \( 2i + 2i' = N+1 \). Hence we have

\[ E_{i,0}^N \cong \tilde{E}_{i',0}^N \text{ for } i + i' = (N + 1)/2. \]

Conversely, we observe that \( E_{ij}^N \cong \tilde{E}_{ij'}^N \) implies \( P_{\chi}^j \otimes V_{\chi} \cong \tilde{P}_{\chi'}^{j'} \otimes V_{\chi'} \) (as representations of \( I_N \times P_N \)) for some \( \chi \in I_N^\vee \) and \( \chi' \in I_{N+1}^\vee \) with \( N+1 \in \text{supp } \chi' \). This implies that \( \kappa(\chi') = \chi \) and it follows from (4.11) that \( \text{supp } \chi \cap \text{supp } \chi' = \emptyset \). Therefore the monodromy representation of the restriction of \( C_{\chi} \to a^{rs} \) (resp. \( \tilde{C}_{\chi'} \to a^{rs} \)) to the subvariety \( a^{rs}(a, \chi') \) (resp. \( a^{rs}(a, \chi) \)) in (3.3) is trivial. On the other hand, the monodromy representation of the restriction of \( \tilde{C}_{\chi'} \to a^{rs} \) (resp. \( C_{\chi} \to a^{rs} \)) to \( a^{rs}(a, \chi') \) (resp. \( a^{rs}(a, \chi) \)) has Zariski dense image (see (3.2)). This forces \( j = j' = 0 \) and the desired claim follows again from (4.12).

\[ \square \]

4.4. The local systems \( E_{i,0}^{2n+1}, \tilde{E}_{i,0}^{2n+1} \) and the \( \mathcal{L}_i \)'s, \( \mathcal{F}_i \)'s in [CVX]. Recall that in [CVX §5], we have defined the local systems \( \mathcal{L}_i \) and \( \mathcal{F}_i \) on \( g_1^{rs} \). We have the following.

**Lemma 4.8.** We have

\[ (4.13) \quad E_{i,0}^{2n+1} \cong \mathcal{L}_i \text{ if } 1 \leq 2i \leq n, \quad E_{i,0}^{2n+1} \cong \mathcal{L}_{2n-2i+1} \text{ if } n+1 \leq 2i \leq 2n, \]

\[ (4.14) \quad \tilde{E}_{i,0}^{2n+1} \cong \mathcal{F}_j \text{ for } 1 \leq j \leq n. \]

**Proof.** We begin with the proof of (4.13). Recall from loc. cit. that we have

\[ (\pi_{2n})_* \mathbb{C}|_{g_1^{rs}} \cong \bigoplus_{i=0}^n \mathcal{L}_i \text{ and } \dim \mathcal{L}_i = \binom{2n+1}{i} \]

where

\[ \pi_{2n+1} : K \times_{P_K} [n_P, n_P]^{\perp} := \{(x, 0 \subset V_n \subset V_n^{\perp} \subset \mathbb{C}^{2n+1}) \mid x \in g_1, xV_n \subset V_n^{\perp} \} \to g_1. \]

On the other hand, recall the \( I_N \)-torsor over \( a^{rs} \) in (3.3)

\[ \tilde{\pi} : \tilde{a}^{rs} = \{(a, c) \mid a = (a_1, \ldots, a_N) \in a^{rs}, c = (c_1, \ldots, c_N), c_i^2 = \prod_{j \neq i} (a_j - a_i) / (\mathbb{Z}/2\mathbb{Z}) \to a^{rs}. \]
We have
\[(4.16) \quad \tilde{\pi}_* \mathbb{C}|_{\mathfrak{g}^{rs}_1} \simeq \mathbb{C} \oplus \bigoplus_{i=1}^{n} E^{2n+1}_{i,0} \quad \text{and} \quad \dim E^{2n+1}_{i,0} = \binom{2n+1}{2i}.
\]

We show that there is a $I_N \times W$-equivariant isomorphism
\[(4.17) \quad \tilde{\alpha}^{rs} \simeq K \times P_K [n_P, n_P]^{\perp}_{a^{rs}}.
\]
Then \[(4.13)\] follows from \[(4.15), \ (4.16)\], and dimension considerations of the representations.

Using the identities $\sum_{i=1}^{2n+1} a_i^k c_i^{-2} = 0$, $k = 0, ..., 2n - 1$, it is easy to check that the map
\[\tilde{\alpha}^{rs} \to X_{2n|a^{rs}}, \ (a, c) \in \tilde{\alpha}^{rs} \mapsto [c^{-1}_1, ..., c^{-1}_{2n+1}]
\]
defines a $I_N \times W$-equivariant isomorphism
\[\tilde{\alpha}^{rs} \simeq X_{2n|a^{rs}}.
\]

On the other hand, by the description of the Hessenberg varieties $\text{Hess}_{n}^{E, \perp}$ in \[(2.3)\] we have a natural map
\[\text{Hess}_{n}^{E, \perp} \to K \times P_K [n_P, n_P]^{\perp}_{a^{rs}}, \ (x, V_{n-1} \subset V_n) \mapsto (x, V_n).
\]
It is easy to check that the map above is a $K$-equivariant isomorphism over $\mathfrak{g}^{rs}_1$. The desired isomorphism \[(4.17)\] follows from the following compositions of isomorphisms
\[\tilde{\alpha}^{rs} \simeq X_{2n|a^{rs}} \cong \text{Hess}_{n}^{E, \perp} \simeq K \times P_K [n_P, n_P]^{\perp}_{a^{rs}}.
\]
This completes the proof of \[(4.13)\].

To prove \[(4.14)\], we observe that
\[\tilde{E}_{n+1,i}^{2n+1} \simeq \tilde{\mathcal{D}}^i_{X_0} \otimes V_{X_0} \cong (\wedge^j H^1(\tilde{C}_{a, \chi_0}, \mathbb{C}))_{prim} \otimes V_{X_0},
\]
where $\chi_0 \in I_N^+$ is the unique character such that $|\chi_0| = 2n + 2$, and $\tilde{C}_{a, \chi_0}$ is the hyperelliptic curve of genus $n$ with affine equation $y^2 = \prod_{i=1}^{2n+1} (x - a_i)$. By \[(4.11)\], $\chi_0$, when regarded as an element in $I_N^+$ (see \[(4.11)\]), is trivial. Hence $I_N$ acts trivially on $\tilde{E}_{n+1,i}^{2n+1}$ and $V_{X_0}$, i.e.
\[(4.18) \quad \tilde{E}_{n+1,i}^{2n+1} \simeq (\wedge^j H^1(\tilde{C}_{a, \chi_0}, \mathbb{C}))_{prim} \simeq \mathcal{F}_j
\]
where the last isomorphism follows from the discussion above and the definition of $\mathcal{F}_j$'s in \textit{loc.cit.}

\[\square\]

\textbf{Remark 4.9.} In \cite{CVX} Proof of Proposition 6.5] we used the fact that among the IC($\mathfrak{g}_1$, $\mathcal{L}_1$)'s ($i \geq 1$), only IC($\mathfrak{g}_1$, $\mathcal{L}_{2j-1}$), $1 \leq j \leq m$, appear in the decomposition of $(\tilde{\tau}_m)_* \mathbb{C}[-]$, where $\tilde{\tau}_m = \tilde{\tau}_{m+1}$ and $2m \leq n + 1$. To prove this fact, it suffices to show that in the decomposition of the monodromy representation $P(X_{2m})$, only the above mentioned local systems appear. Applying Theorem \[(4.11)\] to $P(X_{2m})$ with $N = 2n + 1$ we see that among the $E_{i}$'s only those with $n - m + 1 \leq i \leq n$ appear. The desired conclusion follows from \[(4.13)\] and the fact that $2m \leq n + 1$. 

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5. Computation of the Fourier transforms

Let \( \mathcal{F} : D_K(\mathfrak{g}_1) \to D_K(\mathfrak{g}_1) \) denote the Fourier transform, where we identify \( \mathfrak{g}_1 \) and \( \mathfrak{g}_1^* \) via a \( K \)-invariant non-degenerate bilinear form on \( \mathfrak{g}_1 \). The Fourier transform \( \mathcal{F} \) induces an equivalence of categories \( \mathcal{F} : \text{Perv}_K(\mathfrak{g}_1) \to \text{Perv}_K(\mathfrak{g}_1) \).

In this section we study the Fourier transforms of \( \text{IC}(\mathfrak{g}_1, E_{ij}^N) \) and \( \text{IC}(\mathfrak{g}_1, \tilde{E}_{ij}^N) \). We show that they are supported on \( N_3 \subset N_1 \), more precisely, on \( N_3 \subset N_1 \), the closed subvariety consisting of nilpotent elements of order less than or equal to 3. Thus we obtain many more examples of IC complexes supported on nilpotent orbits whose Fourier transforms have both full support and infinite monodromy (see also \([CVX]\)). As an interesting corollary (see Example 5.5), we show that the Fourier transform of the IC extension of the unique non-trivial irreducible \( K \)-equivariant local system on the minimal nilpotent orbit has full support and its monodromy is given by a universal family of hyperelliptic curves.

The main result of this section is the following theorem.

**Theorem 5.1.** Let \( N_3^3 \subset N_1 \) be the closed subvariety consisting of nilpotent elements of order less than or equal to 3. Then \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, E_{ij}^N)) \) and \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, \tilde{E}_{ij}^N)) \) are supported on \( N_3^3 \).

We first argue the case \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, E_{ij}^N)) \) for \( m \leq \frac{N-1}{2} \), consider the families of Hessenberg varieties

\[
\sigma_m^N : \text{Hess}_m^O \to \mathfrak{g}_1, \quad \tau_m^N : \text{Hess}_m^E \to \mathfrak{g}_1
\]

and

\[
\hat{\sigma}_m^N : \text{Hess}_m^{O,\perp} \to \mathfrak{g}_1, \quad \hat{\tau}_m^N : \text{Hess}_m^{E,\perp} \to \mathfrak{g}_1
\]

defined in (2.3). We have (see (2.9))

\[
\mathcal{F}((\hat{\sigma}_m^N)_\ast \mathbb{C}[-]) = (\sigma_m^N)_\ast \mathbb{C}[-], \quad \mathcal{F}((\hat{\tau}_m^N)_\ast \mathbb{C}[-]) = (\tau_m^N)_\ast \mathbb{C}[-].
\]

By Theorem 2.5 over \( \mathfrak{g}_1^\ast \), we have \( \text{Hess}_m^{O,\perp} \simeq X_{2m-1}, \text{Hess}_m^E \simeq X_{2m} \). Hence the decomposition theorem implies that

\[
\text{IC}(\mathfrak{g}_1, P(X_{2m-1})) \text{ is a direct summand of } (\hat{\sigma}_m^N)_\ast \mathbb{C}[-]
\]

\[
\text{IC}(\mathfrak{g}_1, P(X_{2m})) \text{ is a direct summand of } (\hat{\tau}_m^N)_\ast \mathbb{C}[-].
\]

Therefore the Fourier transforms \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, P(X_{2m-1}))) \) and \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, P(X_{2m}))) \) appear as direct summands of \( (\sigma_m^N)_\ast \mathbb{C}[-] \) and \( (\tau_m^N)_\ast \mathbb{C}[-] \). Now in view of (2.6a) and (2.6b), we see that \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, P(X_{2m-1}))) \) and \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, P(X_{2m}))) \) are supported on \( N_3^3 \). Since each local system \( E_{ij}^N \) appears in \( P(X_m) \) for some \( m \) (see Theorem 4.1), we conclude that \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, E_{ij}^N)) \) is supported on \( N_3^3 \).

It remains to consider the case \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, \tilde{E}_{ij}^N)) \). Since each local system \( \tilde{E}_{ij}^N \) appears in \( P(\tilde{X}_m)^{\sigma=-id} \) for some \( m \), we are reduced to proving the following proposition:

**Proposition 5.2.** \( \mathcal{F}(\text{IC}(\mathfrak{g}_1, P(\tilde{X}_m)^{\sigma=-id})) \) is supported on \( N_3^3 \).

The proof of this proposition occupies the remainder of this section.
5.1. **Proof of Proposition 5.2 when $m$ is odd.** Recall that in \([2, 3]\) we have introduced the families of Hessenberg varieties

\[
\text{Hess}_{k}^{O, \perp} := \{(x, 0 \subset V_{k-1} \subset V) \mid x \in \mathfrak{g}_1, xV_{k-1} \subset V\}
\]

\[
\text{Hess}_{k}^{E, \perp} := \{(x, 0 \subset V_{k-1} \subset V) \mid x \in \mathfrak{g}_1, xV_{k-1} \subset V\}
\]

and the natural projection maps $\sigma^N_k, \tilde{\sigma}^N_k : \text{Hess}_{k}^{O, \perp} \to \mathfrak{g}_1$, $\tau^N_k, \tilde{\tau}^N_k : \text{Hess}_{k}^{E, \perp} \to \mathfrak{g}_1$.

Our first goal is to show that $\text{IC}(\mathfrak{g}_1, P(\tilde{X}_{2k-1}))$ appears as a direct summand in the push forward of certain intersection cohomology complex on $\text{Hess}_{k}^{O, \perp}$ along $\tilde{\sigma}^N_k$.

Let $[\mathbb{G}_a/\mathbb{G}_m^{[2]}]$ be the stack quotient, where $\mathbb{G}_m^{[2]} \cong \mathbb{G}_m$ acts on $\mathbb{G}_a$ via the square map, i.e., for $t \in \mathbb{G}_m$ and $x \in \mathbb{G}_a$, $t : x \mapsto t^2x$. We first introduce a map

\[\alpha : \text{Hess}_{k}^{O, \perp} \to [\mathbb{G}_a/\mathbb{G}_m^{[2]}].\]

Recall that such a map is equivalent to a pair $(\text{Hess}_{k}^{O, \perp}, \phi)$, where $\text{Hess}_{k}^{O, \perp}$ is a $\mathbb{G}_m$-torsor over $\text{Hess}_{k}^{O, \perp}$ and $\phi : \text{Hess}_{k}^{O, \perp} \to \mathbb{G}_a$ is a map such that

\[\phi(t \cdot v) = t^2\phi(v) \quad \text{for} \quad v \in \text{Hess}_{k}^{O, \perp} \quad \text{and} \quad t \in \mathbb{G}_m.\]

To construct such a pair, we set

\[\text{Hess}_{k}^{O, \perp} := \{(x, V_{k-1} \subset V_k, l) \mid (x, V_{k-1} \subset V_k) \in \text{Hess}_{k}^{O, \perp}, \ 0 \neq l \in V_k/V_{k-1} \cong \mathbb{C}\},\]

where the action of $\mathbb{G}_m$ on $\text{Hess}_{k}^{O, \perp}$ is given by $t \cdot (x, V_{k-1} \subset V_k, l) = (x, V_{k-1} \subset V_k, tl)$ for $t \in \mathbb{G}_m$. Define

\[\phi : \text{Hess}_{k}^{O, \perp} \to \mathbb{G}_a, \ (x, V_{k-1} \subset V_k, l) \mapsto \langle xl, l \rangle_{Q}.\]

Note that the above pairing is well-defined since $xV_{k-1} \subset V_k$ and $xV_k \subset V_{k-1}$. One checks easily that $\phi$ satisfies (5.1). This finishes the construction of $(\text{Hess}_{k}^{O, \perp}, \phi)$, hence that of the map $\alpha : \text{Hess}_{k}^{O, \perp} \to [\mathbb{G}_a/\mathbb{G}_m^{[2]}]$. By construction, the map $\alpha$ is $K$-equivariant (where $K$ acts trivially on $[\mathbb{G}_a/\mathbb{G}_m^{[2]}]$), moreover it factors through $\text{Hess}_{k}^{E, \perp}$, i.e.,

\[\alpha : \text{Hess}_{k}^{O, \perp} \to \text{Hess}_{k}^{O, \perp} / \text{Hess}_{k}^{E, \perp} \xrightarrow{\tilde{\alpha}} [\mathbb{G}_a/\mathbb{G}_m^{[2]}].\]

There is a unique non-trivial irreducible local system $\mathcal{L}$ on $[\mathbb{G}_m/\mathbb{G}_m^{[2]}] \subset [\mathbb{G}_a/\mathbb{G}_m^{[2]}]$. We denote by $\text{IC}([\mathbb{G}_a/\mathbb{G}_m^{[2]}], \mathcal{L})$ the corresponding intersection cohomology complex on $[\mathbb{G}_a/\mathbb{G}_m^{[2]}]$. Let

\[\mathcal{K} := (\tilde{\sigma}^N_k)^* \alpha^* \text{IC}([\mathbb{G}_m/\mathbb{G}_m^{[2]}], \mathcal{L}) \in D_K(\mathfrak{g}_1).\]

The factorization in (5.2) and the functorial properties of Fourier transform (see [KaS, Proposition 3.7.14]) imply the following

\[\mathfrak{Y}(\mathcal{K}) \text{ is supported on } \text{Im}(\tilde{\tau}^N_k) \subset N_1^3.\]

Thus to show that $\mathfrak{Y}(\text{IC}(\mathfrak{g}_1, P(\tilde{X}_{2k-1})))$ is supported on $N_1^3$, it suffices to show that

\[\text{the complex } \mathcal{K} \text{ contains } \text{IC}(\mathfrak{g}_1, P(\tilde{X}_{2k-1})^{\sigma=-id}) \text{ as a direct summand.}\]

Let

\[\tilde{\pi}_{2k-1} : \tilde{X}_{2k-1} \xrightarrow{\pi_{2k-1}} X_{2k-1} \xrightarrow{\pi_{2k-1}} \mathfrak{g}_1^{rs}.\]
be the branched double cover of $X_{2k-1}$ and $\sigma$ the involution on $\widetilde{X}_{2k-1}$ defined in \(3.4\). We have that \(((\pi_{2k-1})_* C)^{\sigma = -id}\) contains $P(\widetilde{X}_{2k-1})^{\sigma = -id}$ as a direct summand. The statement \((5.4)\) follows from the following claim

\[
\mathcal{K}|_{\mathfrak{g}_1^o} \simeq (\pi_{2k-1})_* C
\]

To prove the claim, let $s : [G_a/G_m] \to [G_a/G_m^2]$ be the descent of the map $G_a \to G_a$, $t \mapsto t^2$. Then from the definitions of $\widetilde{X}_{2k-1}$ and the map $\alpha$, one can check that, under the isomorphism $X_{2k-1} \simeq \text{Hess}_{k-1}^O |_{\mathfrak{g}_1^o}$ in Theorem \(2.3\), the branched double cover $\widetilde{X}_{2k-1}$ can be identified with the following fiber product

\[
\begin{array}{ccc}
\widetilde{X}_{2k-1} & \xrightarrow{p_{2k-1}} & [G_a/G_m^2] \\
\downarrow & & \downarrow s \\
X_{2k-1} & \xrightarrow{\alpha|_{\mathfrak{g}_1^o}} & [G_a/G_m^2].
\end{array}
\]

Since

\[
s_* C = (s_* C)^{\sigma = id} \oplus (s_* C)^{\sigma = -id} = C \oplus \text{IC}([G_a/G_m^2], \mathcal{L}),
\]

by proper base change we have

\[
(\alpha|_{\mathfrak{g}_1^o})^* \text{IC}([G_a/G_m^2], \mathcal{L}) \simeq ((p_{2k-1})_* C)^{\sigma = -id}.
\]

This implies that

\[
\mathcal{K}|_{\mathfrak{g}_1^o} \simeq (\pi_{2k-1})_* (\alpha|_{\mathfrak{g}_1^o})^* \text{IC}([G_a/G_m^2], \mathcal{L}) \simeq ((\pi_{2k-1})_* C)^{\sigma = -id}.
\]

This proves \((5.4)\).

**Remark 5.3.** The construction of the map $\alpha$ was inspired by discussions with Zhiwei Yun. In particular, the idea of making use of the local system $\mathcal{L}$ on $[G_a/G_m^2]$ was explained to one of us by him.

### 5.2. Proof of Proposition \(5.2\) when $m$ is even.

Let us consider the following family of Hessenberg varieties

\[
H = \{ (x, 0 \subset V_{k-1} \subset V_k \subset V_{k+1} \subset V_{k+1}^+ \subset V_k^+ \subset V_{k-1}^+ \subset V = \mathbb{C}^N) | x \in \mathfrak{g}_1, xV_{k-1} \subset V_k, xV_k \subset V_k^+ \}.
\]

Note that the natural map

\[
p : H \to \text{Hess}_{k-1}^E, (x, V_{k-1} \subset V_k \subset V_{k+1}) \mapsto (x, V_{k-1}^+ \subset V_k)
\]

realizes $H$ as a quadric bundle over $\text{Hess}_{k}^E$.

We first construct a map $\beta : H \to [G_a/G_m^2]$. The construction is very similar to that of the map $\alpha$ in \(5.1\) and we use the notations there. Set

\[
\tilde{H} := \{ (x, V_{k-1} \subset V_k \subset V_{k+1}, l) | (x, V_{k-1} \subset V_k \subset V_{k+1}) \in H, 0 \neq l \in V_k/V_{k-1} \simeq \mathbb{C} \},
\]

and the following diagram

\[
\begin{array}{ccc}
\tilde{H} & \xrightarrow{\tilde{p}} & [G_a/G_m^2] \\
\downarrow & & \downarrow s \\
H & \xrightarrow{p} & [G_a/G_m^2].
\end{array}
\]
where the action of $\mathbb{G}_m$ on $\widetilde{H}$ is given by $t \cdot (x, V_{k-1} \subset V_k \subset V_{k+1}, l) = (x, V_{k-1} \subset V_k \subset V_{k+1}, tl)$. Define

$$
\phi : \widetilde{H} \to \mathbb{G}_a, \ (x, V_{k-1} \subset V_k \subset V_{k+1}, l) \mapsto \langle xl, xl \rangle_Q.
$$

Note that the above pairing is well-defined since $x V_{k-1} \subset V_k$ and $x V_k \subset V_k^\perp$. One checks that $\phi$ satisfies (5.1). This finishes the construction of $(\widetilde{H}, \phi)$. Hence we obtain a map $\beta : H \to [\mathbb{G}_a/\mathbb{G}_m^2]$.

Let $f : H \to g_1$ be the natural projection map. Define

$$
\mathcal{F} := f_* \beta^* IC([\mathbb{G}_a/\mathbb{G}_m^2], L) \in D_K(g_1).
$$

We show that (5.5) $\mathcal{F}(\mathcal{F})$ is supported on $N^3_1$, and

(5.6) the complex $\mathcal{F}$ contains $IC(g_1, P(\tilde{X}_{2k})^{\sigma=\text{id}})$ as a direct summand.

The proposition then follows from (5.5) and (5.6).

To prove (5.5), let

$$
H' = \left\{ (x, 0 \subset V_{k-1} \subset V_k \subset V_{k+1} \subset V_{k+1}^\perp \subset V \subset V_{k-1} \subset V = \mathbb{C}^N) \mid x \in g_1, x V_{k-1} \subset V_k, x V_k \subset V_{k+1} \right\}.
$$

Note that $H' \subset H$ is a sub-bundle. By construction, the map $\beta$ factors through $H'$, i.e.,

$$
\beta : H \to H/H' \xrightarrow{\beta'} [\mathbb{G}_a/\mathbb{G}_m^2].
$$

Let $\tilde{f}'$ be the natural projection map

$$
\tilde{f}' : (H')^\perp := \left\{ (x, 0 \subset V_{k-1} \subset V_k \subset V_{k+1} \subset V_{k+1}^\perp \subset V_{k-1} \subset V = \mathbb{C}^N) \mid x \in g_1, x V_k = 0, x V_{k+1} \subset V_{k-1}, x V_{k+1}^\perp \subset V_k \right\} \to N_1.
$$

A direct calculation shows that

(5.7) $\text{Im } \tilde{f}' = \widetilde{O}_{3k+1-N} \text{ if } 3k \leq N \text{ and } \text{Im } \tilde{f}' = \widetilde{O}_{3N-2k,2k-N} \text{ if } 3k \geq N + 1$.

The standard properties of Fourier transform imply that

$$
\mathfrak{F}(\mathcal{F}) \text{ is supported on } \text{Im}(\tilde{f}') \subset N^3_1.
$$

This proves (5.5).

It remains to prove (5.6). Notice that the map $\beta$ factors as $\beta : H \xrightarrow{p} \text{Hess}_{k}^{E,\perp} \xrightarrow{\bar{\beta}} [\mathbb{G}_a/\mathbb{G}_m^2]$. Consider the following diagram

$$
\begin{array}{ccc}
\beta : H & \xrightarrow{p} & \text{Hess}_{k}^{E,\perp} & \xrightarrow{\bar{\beta}} & [\mathbb{G}_a/\mathbb{G}_m^2] \\
\downarrow & & & \downarrow & \\
f \downarrow & & & \downarrow & \\
\mathfrak{g}_1 & \xrightarrow{\mathfrak{f}_N} & [\mathbb{G}_a/\mathbb{G}_m^2].
\end{array}
$$

We have

$$
\mathcal{F} := f_* \beta^* IC([\mathbb{G}_a/\mathbb{G}_m^2]) \simeq (\mathfrak{f}_N)_* p_* \beta^* (IC([\mathbb{G}_a/\mathbb{G}_m^2], L))
$$
which is isomorphic to \((\tau_k^N)_*(\beta^*(\text{IC}([G_a/G_m^2], \mathcal{L})) \otimes p_\ast \mathcal{C}).\) Since \(\mathcal{C}\) is a direct summand of \(p_\ast \mathcal{C},\) it implies that \((\tau_k^N)_*(\beta^*(\text{IC}([G_a/G_m^2], \mathcal{L})))\) is a direct summand of \(\mathcal{F}.\) So it is enough to show that

\[
\text{IC}(\mathcal{g}_1, P(\widetilde{X}_{2k})^{\sigma=-id}) \text{ is a direct summand of } (\tau_k^N)_*(\beta^*(\text{IC}([G_a/G_m^2], \mathcal{L}))).
\]

This follows from the same argument as in the proof of (5.3), replacing \(X_{2k-1}\) (resp. \(\widetilde{X}_{2k-1}\)) there by \(X_{2k}\) (resp. \(\widetilde{X}_{2k}\)). Thus the proof of the proposition is complete.

5.3. Matching for \(\text{IC}(\bar{O}_{2|12n+1-2i}, \mathcal{E}_i), i \text{ odd.}\) Here we complete the proof of [CVX] Theorem 6.2 by treating the case of odd \(i.\) In [CVX] we treated the even case of the proposition below and showed that there exists a permutation \(s\) of the set \(\{2j+1 | 1 \leq 2j + 1 \leq n\},\) such that \(\mathfrak{F}(\text{IC}(\bar{O}_{2|12n+1-2i}, \mathcal{E}_i)) = \text{IC}(\mathcal{g}_1, \mathcal{F}_s(i))\) (see Theorem 5.2 and Theorem 6.2 in loc.cit.).

**Proposition 5.4.** We have that

\[
\mathfrak{F}(\text{IC}(\bar{O}_{2|12n+1-2i}, \mathcal{E}_i)) = \text{IC}(\mathcal{g}_1, \mathcal{F}_i),
\]

where \(\mathcal{E}_i\) denotes the unique non-trivial irreducible \(K\)-equivariant local system on \(O_{2|12n+1-2i}.\)

**Proof.** It remains to prove the proposition for odd \(i.\) Assume that \(2m \leq n + 1.\) By (5.3) and (5.4), we see that the Fourier transform of \(\text{IC}(\mathcal{g}_1, P(\widetilde{X}_{2m-1})^{\sigma=-id})\) is supported on \(\text{Im} \tau_m = \bar{O}_{3m-12|12n+2-3m}\) (see (2.6a)). Using Theorem 4.2 and (4.14) we obtain that

\[
\text{IC}(\mathcal{g}_1, \mathcal{F}_i) \text{ is a direct summand of } \text{IC}(\mathcal{g}_1, P(\widetilde{X}_{2m-1})^{\sigma=-id}) \text{ if and only if } i \text{ is odd and } 1 \leq i \leq 2m - 1.
\]

This implies that the Fourier transform of \(\text{IC}(\mathcal{g}_1, \mathcal{F}_{2j-1}), 1 \leq j \leq m,\) is supported on \(\bar{O}_{3m-12|12n+2-3m}.\) Now it is easy to check that \(O_{2|12n+1-2i} \subset \bar{O}_{3m-12|12n+2-3m}\) if and only if \(i \leq 2m - 1.\) In view of [CVX] Theorems 5.2 and 6.2, the proposition follows by induction on \(m.\)

\[\square\]

**Example 5.5.** Let \(O_{min} = O_{2|12n-1}.\) By the above proposition, we have

\[
\mathfrak{F}(\text{IC}(\mathcal{g}_1, \mathcal{F}_1)) \simeq \text{IC}(O_{min}, \mathcal{E}_1),
\]

where

\[
\mathcal{F}_1 \simeq \widehat{E}_{n+1}^2 \simeq H^1(\tilde{C}_{a,\chi_0}, \mathbb{C}) \text{ (see (4.18))}
\]

is isomorphic to the monodromy representation associated with \(\tilde{C}_{\chi_0} \to \mathfrak{c}^{rs},\) the universal family of hyperelliptic curves in (3.2).
6. Conjectures and examples

Let $N = 2n + 1$ and let $E_{ij}^{2n+1}$ (resp. $\tilde{E}_{ij}^{2n+1}$) be the monodromy representations of $\pi_1^K(g_i^s)$ constructed from the families of complete intersections of quadrics in $\mathbb{P}^{2n}$ (resp. their double covers), see \textsuperscript{41}. Let $\{(\mathcal{O}, \mathcal{E})\}_{\leq 3}$ denote the set of pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O}$ is a $K$-orbit in $N_1^3$ and $\mathcal{E}$ is an irreducible $K$-equivariant local system on $\mathcal{O}$ (up to isomorphism). Using Theorem \ref{inj} we establish an injective map

\begin{equation}
(6.1) \quad \mathcal{S} : \begin{cases} E_{ij}^{2n+1}, \ i \in [1, n], \ j \in [0, i - 1]; \\ \tilde{E}_{ij}^{2n+1}, \ i \in [1, n + 1], \ j \in [i, i - 1], \ E_{n+1,0}^{2n+1} \cong \mathbb{C} \end{cases} \hookrightarrow \{(\mathcal{O}, \mathcal{E})\}_{\leq 3},
\end{equation}

where $\mathcal{S}(E_{ij}^{2n+1}) = (\mathcal{O}, \mathcal{E})$ if and only if $\mathcal{S}(g_1, E_{ij}^{2n+1}) = \text{IC}(\mathcal{O}, \mathcal{E})$, similarly for $\tilde{E}_{ij}^{2n+1}$. Here the $K$-equivariant local systems on $g_i^s$ in the left hand side of (6.1) are pairwise non-isomorphic, see (4.10).

In this section we state two conjectures (Conjecture \ref{6.1} and Conjecture \ref{6.3}) that describe the map $\mathcal{S}$ in (6.1) in the case of $\{E_{ij}^{2n+1}\}$ explicitly. We verify our conjectures in several examples by studying various families of Hessenberg varieties.

In what follows we make use of the following observation:

\begin{equation}
(6.2) \quad \text{an orbit } O_{3^k 2^l 1^{2n+1-3k-2l}} \subset N_1^3 \text{ is odd dimensional } \iff k \text{ is odd and } l \text{ is even.}
\end{equation}

6.1. Complete intersections of even number of quadrics and conjectural matching. Recall that the local systems $E_{i,2j}^{2n+1}$, where $i \in [1, n]$ and $2j \in [0, i - 1]$, are constructed from families of complete intersections $X_{2m}$ of even number of quadrics in $\mathbb{P}^{2n}$ for $m \in [1, n]$.

We first show that

\begin{equation}
(6.3) \quad \text{IC}(g_1, E_{i,2j}^{2n+1}) \text{ is supported on an even dimensional } K\text{-orbit in } N_1^3.
\end{equation}

To this end we first note that each $\text{IC}(g_1, E_{i,2j}^{2n+1})$ is a direct summand of $\text{IC}(g_1, P(X_{2m}))$ for some $m$, which in turn is a direct summand of $(\tau_m^N)^*\mathbb{C}[-]$. One readily checks that

$$\dim \text{Hess}^E_{m, n} = m (4n - 3m + 5) - 2n - 2,$$

which is even.

Note also that $\dim X_{2m,a}$ is even. Now (6.3) follows from the decomposition theorem and the fact that the fibers of $\tau_m^N$ have non-vanishing cohomology only in even degrees (see \textsuperscript{2.1}).

Thus (6.2) puts a restriction on nilpotent orbits which can support $\text{IC}(g_1, E_{i,2j}^{2n+1})$. Our first conjecture is:

**Conjecture 6.1.** We have that

\begin{align*}
\text{IC}(g_1, E_{i,2j}^{2n+1}) &\cong \text{IC}(\tilde{E}_{i,2j}^{2n+1}, \mathbb{C}) \text{ if } i + j \geq n + 1 \\
\text{IC}(g_1, E_{i,2j}^{2n+1}) &\cong \text{IC}(E_{i,2j}^{2n+1}, \mathbb{C}) \text{ if } i + j \leq n \text{ and } 2i - j \geq n + 1 \\
\text{IC}(g_1, E_{i,2j}^{2n+1}) &\cong \text{IC}(\tilde{E}_{i,2j}^{2n+1}, \mathbb{C}) \text{ if } i + j \leq n \text{ and } 2i - j \leq n.
\end{align*}
Remark 6.2. The nilpotent orbits appearing in the conjecture above exhaust all the non-zero even dimensional orbits of the form $O_{3^i2^j1^k}$, where the partition $3^i2^j1^k$ has no gaps.

Note that the conjecture above holds for $E_{i,0}^{2n+1}$. This follows from (4.13) and Theorem 6.1, i.e., we have

$$\mathfrak{F}(IC(g_1, E_{i,0}^{2n+1})) = IC(\bar{O}_{2i+2n-4i+1}, \mathbb{C}) \text{ if } 2i \leq n$$

$$\mathfrak{F}(IC(g_1, E_{i,0}^{2n+1})) = IC(\bar{O}_{2n-2i+1+14i-2n-1}, \mathbb{C}) \text{ if } 2i \geq n + 1.$$  

Below we verify the conjecture in a simple case that involves nilpotent orbits of order 3.

6.2. Complete intersection of 4 quadrics, $n \geq 3$. In this subsection we show that

$$\mathfrak{F}(IC(g_1, E_{n,2}^{2n+1})) = IC(\bar{O}_{3^i2^j1^k}, \mathbb{C}).$$  

Let us write

$$\tau = \tau_2^{2n+1} : Hess^E_2 \rightarrow \bar{O}_{3^i2^j1^k}, \quad \tilde{\tau} = \tilde{\tau}_2^{2n+1} : Hess_2 \rightarrow g_1.$$  

We have $\mathfrak{F}(\tau_2^\tau C[-]) \cong \tilde{\tau}_2^\tau C[-]$ and

$$\tilde{\tau}_2 C[-] = IC(g_1, E_{n,2}^{2n+1} \oplus E_{n,0}^{2n+1} \oplus E_{n-1,0}^{2n+1} \oplus \bigoplus_{a=0}^{2n-4} IC(g_1, \mathbb{C})[2n - 4 - 2a] \oplus \cdots$$

where $\cdots$ is a direct sum of IC complexes with smaller support. We have

$$\bar{O}_{3^i2^j1^k} = \bar{O}_{3^i2^j1^k} \cup \bar{O}_{3^i2^j1^k} \cup \bar{O}_{2i+2n-4i+1}.$$  

In view of Proposition 5.4, Lemma 4.7 (4.14) and (6.4), we conclude that $\mathfrak{F}(IC(g_1, E_{n,2}^{2n+1}))$ is not supported on $\bar{O}_{2i+2n-4i+1}$'s. Now it follows from (6.2) and (6.3) that

$$\mathfrak{F}(IC(g_1, E_{n,2}^{2n+1}))$$

is supported on $\bar{O}_{3^i2^j1^k}$. Thus (6.5) follows the fact that the only IC complex supported on $\bar{O}_{3^i2^j1^k}$ appearing in $\tau_2 C[-]$ is $IC(\bar{O}_{3^i2^j1^k}, \mathbb{C})$ as $\tau$ is a resolution of $\bar{O}_{3^i2^j1^k}$.

6.3. Complete intersections of odd number of quadrics and a conjectural matching. Recall that the local systems $E_{i,j}^{2n+1}$, where $i \in [1,n]$ and $2j \in [2, i]$, are constructed from complete intersections $X_{2m-1}$ of odd number of quadrics in $\mathbb{P}^{2n}$, $m \in [1,n]$.

Using that $\dim Hess^O_m = m(2n - 3m + 5) - 2n - 3$, which is odd, and arguing as in (6.3), we obtain that

$$\mathfrak{F}(IC(g_1, E_{i,j}^{2n+1}))$$

is supported on an odd dimensional $K$-orbit in $X_1^3$.

Let $\mathcal{O} \subset N_1^3$ be an odd dimensional $K$-orbit. To describe our second conjecture, let us first label the non-trivial irreducible $K$-equivariant local systems on $\mathcal{O}$ as follows. By (6.2), we can assume that $\mathcal{O} = \mathcal{O}_{2k-1+2j+2n+4-6k-4i}$. 


Let $x \in \mathbb{O}_{3k-12l+2n+4-6k-4l}$, $k \geq 1$. We first define representatives for the component group $A_K(x) = Z_K(x)/Z_K(x)^0$. Take a basis

$$x^i u_j, i \in [0, 2], j \in [1, 2k - 1], x^i v_j, i \in [0, 1], j \in [1, 2l] \text{ and } w_i, i \in [1, 2n + 4 - 6k - 4l]$$

of $V$ as in [CVX, Lemma 2.2]. Define $\gamma_i \in Z_K(x), i = 1, 2$ as follows

$$\gamma_1(w_1) = w_2, \gamma_1(w_2) = w_1, \gamma_1(x^i u_j) = -x^i u_j, i \in [0, 2], j \in [1, 2k - 1],$$

$$\gamma_2(x^j v_1) = x^j v_2, \gamma_2(x^j v_2) = x^j v_1, j \in [0, 1],$$

and $\gamma_1$ (resp. $\gamma_2$) acts as identity on all other basis vectors.

Assume that $l \geq 1$ and $2n + 4 - 6k - 4l \neq 0$. Then $A_K(x) \cong \{1, \gamma_1, \gamma_2, \gamma_1 \gamma_2\} \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let

$$E_{k,l}^1 \text{ (resp. } E_{k,l}^2, E_{k,l}^3)$$

denote the irreducible $K$-equivariant local system on $\mathbb{O}_{3k-12l+2n+4-6k-4l}$ corresponding to the irreducible character of $A_K(x)$

$$\chi_1 \text{ (resp. } \chi_2, \chi_3) \text{ with } \chi(\gamma_1) = -1 \text{ (resp. } -1, 1) \text{ and } \chi(\gamma_2) = 1 \text{ (resp. } -1, -1).$$

Assume that $l = 0$ and $2n + 4 - 6k \neq 0$. Then $A_K(x) \cong \{1, \gamma_1\} \cong \mathbb{Z}/2\mathbb{Z}$. We denote by $E_{k,0}^1$ the irreducible $K$-equivariant local system on $\mathbb{O}_{3k-12n+4-6k}$ corresponding to the irreducible character $\chi$ of $A_K(x)$ with $\chi(\gamma_1) = -1$.

Assume that $l \geq 1$ and $2n + 4 - 6k - 4l \neq 0$. Then $A_K(x) \cong \{1, \gamma_2\} \cong \mathbb{Z}/2\mathbb{Z}$. We denote by $E_{k,l}^3$ the irreducible $K$-equivariant local system on $\mathbb{O}_{3k-12n+2-3k}$ corresponding to the irreducible character $\chi$ of $A_K(x)$ with $\chi(\gamma_2) = -1$.

We will simply write $E_i$, $i = 1, 2, 3$, when the supports of these local systems are clear.

Our second conjecture is the following.

**Conjecture 6.3.** We have that

$$\mathfrak{F}(\text{IC}(g_1, E_{i,2j-1}^{2n+1})) \cong \text{IC}(\mathbb{O}_{32(n-i)+12(n+j-n-1)12i-4j+2}, E_1^1) \text{ if } i + j \geq n + 1$$

$$\mathfrak{F}(\text{IC}(g_1, E_{i,2j-1}^{2n+1})) \cong \text{IC}(\mathbb{O}_{32(n-i)+12(n+j-n-1)14i-2j-2n}, E_2^2) \text{ if } i + j \leq n \text{ and } 2i - j \geq n + 1$$

$$\mathfrak{F}(\text{IC}(g_1, E_{i,2j-1}^{2n+1})) \cong \text{IC}(\mathbb{O}_{32j-12(2n-4i+2)}, E_3^3) \text{ if } i + j \leq n \text{ and } 2i - j \leq n.$$ 

**Remark 6.4.** In particular, the conjecture above implies that the set of all Fourier transforms $\mathfrak{F}(\text{IC}(g_1, E_{i,2j-1}^{2n+1}))$ coincides with the set of all IC complexes supported on odd dimensional orbits in $\mathbb{N}_1^3$, with non-trivial local systems.

In the following subsections we verify the conjecture above in two simple examples, see (6.7) and (6.8). We also prove a lemma (Lemma 6.7) that is compatible with our conjecture.
6.4. Complete intersection of 3 quadrics, \( n \geq 2 \). In this subsection we show that
\[
\mathfrak{F} (\text{IC}(g_1, E_{n,1}^{2n+1})) = \text{IC}(\bar{\mathcal{O}}_{3^12^{n-2}}, E^1).
\]
Let us write
\[
\sigma = \sigma_2^{2n+1} : \text{Hess}_2^O \rightarrow \bar{\mathcal{O}}_{3^12^{n-2}}, \quad \tilde{\sigma} = \sigma_2^{2n+1} : \text{Hess}_2^{O,1} \rightarrow g_1.
\]
The fiber \( \sigma^{-1}(x) \) at \( x \in \mathcal{O}_{3^12^{n-2}} \) is a non-singular quadric in \( \mathbb{P}^{2n-3} \). Thus in the decomposition of \( \sigma_* \mathbb{C}[-] \), we have the following direct summands
\[
\bigoplus_{a=0}^{2n-4} \text{IC}(\bar{\mathcal{O}}_{3^12^{n-2}}, \mathbb{C})[2n - 4 - 2a] \oplus \text{IC}(\bar{\mathcal{O}}_{3^12^{n-2}}, E^1).
\]
We have \( \mathfrak{F}(\sigma_* \mathbb{C}[-]) \cong \tilde{\sigma}_* \mathbb{C}[-] \) and
\[
\tilde{\sigma}_* \mathbb{C}[-] \cong \text{IC}(g_1, E_{n,1}^{2n+1}) \oplus \cdots
\]
Note that \( \mathcal{O}_{3^12^{n-2}} \) is the only odd-dimensional orbit contained in \( \bar{\mathcal{O}}_{3^12^{n-2}} \) and there is a unique non-trivial irreducible \( K \)-equivariant local system on \( \mathcal{O}_{3^12^{n-2}} \), denoted by \( E^1 \). In view of (6.6), the equation (6.7) follows from the fact that the support of \( \mathfrak{F}(\text{IC}(\bar{\mathcal{O}}_{3^12^{n-2}}, \mathbb{C})) \) is a proper subset of \( g_1 \) (see [CVX, Theorem 4.9]).

Remark 6.5. Here we see that Fourier transform of IC complexes supported on nilpotent orbits \( \lambda \), where \( \lambda \) has gaps, with nontrivial local systems can have full support (compare with [CVX, Theorem 4.9]).

6.5. Complete intersection of 5 quadrics, \( n \geq 4 \). In this subsection, we show that
\[
\mathfrak{F} (\text{IC}(\bar{\mathcal{O}}_{3^12^21^{2}2^{n-6}}, E^1)) = \text{IC}(g_1, E_{n,3}^{2n+1}), \quad \mathfrak{F} (\text{IC}(\bar{\mathcal{O}}_{3^12^21^{2}2^{n-6}}, E^2)) = \text{IC}(g_1, E_{n,1}^{2n+1})
\]
\[
\mathfrak{F} (\text{IC}(\bar{\mathcal{O}}_{3^12^21^{2}2^{n-6}}, E^3)) = \text{IC}(g_1, E_{7,1}^{2n+1}).
\]
Let us write
\[
\sigma = \sigma_3^{2n+1} : \text{Hess}_3^O \rightarrow \bar{\mathcal{O}}_{3^12^21^{2}2^{n-5}}, \quad \tilde{\sigma} = \sigma_3^{2n+1} : \text{Hess}_3^{O,1} \rightarrow g_1.
\]
We have \( \mathfrak{F}(\sigma_* \mathbb{C}[-]) \cong \tilde{\sigma}_* \mathbb{C}[-] \) and
\[
\tilde{\sigma}_* \mathbb{C}[-] \cong \text{IC}(g_1, E_{n,1}^{2n+1} \oplus E_{n,3}^{2n+1} \oplus E_{n-1,1}^{2n+1}) \oplus \cdots
\]
The odd dimensional orbits contained in \( \text{Im} \sigma = \bar{\mathcal{O}}_{3^12^21^{2}2^{n-5}} \) are \( \mathcal{O}_{3^12^21^{2}2^{n-6}} \) and \( \mathcal{O}_{3^12^12^{n-2}} \). In view of (6.7), the equation (6.8) follows from Lemma 6.7 (see (6.6) and the following statement.

The IC complexes supported on \( \mathcal{O}_{3^12^21^{2}2^{n-6}} \), that appear in the decomposition (6.9)
\[
\text{IC}(\mathcal{O}_{3^12^21^{2}2^{n-6}}, E^1 \oplus E^2)
\]
It remains to prove (6.9). Note that there is no orbit \( \Theta \) such that \( \Theta_{3^12^12^{n-6}} < \Theta < \mathcal{O}_{3^12^12^{n-5}} \).

The fiber \( \sigma^{-1}(x_2) \) at \( x_2 \in \mathcal{O}_{3^12^21^{2}2^{n-5}} \) is a nonsingular quadric in \( \mathbb{P}^{2n-6} \). Thus in the decomposition of \( \sigma_* \mathbb{C}[-] \),
\[
\text{IC}(\mathcal{O}_{3^12^21^{2}2^{n-5}}) = \bigoplus_{a=0}^{2n-7} \text{IC}(\bar{\mathcal{O}}_{3^12^12^{n-5}}, \mathbb{C})[2n - 7 - 2a].
\]
The fiber at \( x_1 \in \mathcal{O}_{\mathbb{P}^{2n-6}} \) is a quadric bundle over \( \pi_0^{-1}(x_1) \) with fibers being a quadric \( Q \) of rank \( 2n - 8 \) in \( \mathbb{P}^{2n-6} \). Here \( \pi_0 \) is Reeder’s resolution of \( \mathcal{O}_{\mathbb{P}^{2n-6}} \), i.e.

\[
\pi_0 : \{(x, 0 \subset V_2 \subset V_2^\perp \subset V) \mid x \in \mathfrak{g}_1, xV_2 = 0, xV_2^\perp \subset V_2\} \to \mathcal{O}_{\mathbb{P}^{2n-6}}.
\]

It is easy to check that the map \( \pi_0 \) is small. Thus we have

\[
H^k_{\pi_0}(\mathcal{O}_{\mathbb{P}^{2n-6}}, \mathbb{C}) = H^k(\pi_0^{-1}(x_1), \mathbb{C}).
\]

Note that \( H^\text{odd}(\pi_0^{-1}(x_1), \mathbb{C}) = 0 \), \( H^\text{odd}(\sigma^{-1}(x_1), \mathbb{C}) = 0 \), and

\[
H^{2k}(\sigma^{-1}(x_1), \mathbb{C}) = \bigoplus_{a=0}^{2n-7} H^{2a}(\sigma^{-1}(x_1), \mathbb{C}) \otimes H^{2k-2a}(\pi_0^{-1}(x_1), \mathbb{C})
\]

\[
\cong \bigoplus_{a=0}^{2n-7} H^{2k-2a}(\pi_0^{-1}(x_1), \mathbb{C}) \otimes (H_{\text{prim}}^{2n-6}(\pi_0^{-1}(x_1), \mathbb{C}) \otimes H^{2k-2n+6}(\pi_0^{-1}(x_1), \mathbb{C})).
\]

We have \( \text{codim}_{H^s} \mathcal{O}_{\mathbb{P}^{2n-6}} = 2n - 6 \) and \( \pi_0^{-1}(x_1) \) consists of two points. Moreover, \( A_K(x_1) \) acts on \( H^{2n-6}_{\text{prim}}(\pi_0^{-1}(x_1), \mathbb{C}) \) as \( \chi_1(1 + \chi_3) = \chi_1 \oplus \chi_2 \). The equation (6.9) follows. This finishes the proof of (6.8).

**Remark 6.6.** Note that (6.8) shows that all three IC complexes supported on \( \mathcal{O}_{\mathbb{P}^{2n-6}} \) with nontrivial local systems correspond to the monodromy representations constructed from complete intersections of odd number of quadrics.

### 6.6. The case of a curve.

In this subsection we prove the following lemma by considering the family \( X_{2n-1} \) of complete intersections of quadrics in \( \mathbb{P}^n \).

**Lemma 6.7.** For each \( i \in [1, n-1] \), there exists some \( 1 \leq j \leq \left[ \frac{n-1}{2} \right] \) and a nontrivial local system \( \mathcal{E}_j^s \) (2 or 3) on \( \mathcal{O}_{\mathbb{P}^{2n-6}} \) such that

\[
\mathfrak{F}(\mathcal{O}(\mathbf{g}_1, \mathcal{E}_{i,1}^{2n+1})) \cong \mathcal{O}(\mathcal{O}_{\mathbb{P}^{2n-6}}^{2n+1}, \mathcal{E}_j^s).
\]

Let us write

\[
\sigma = \sigma_n^{2n+1} : \text{Hess}_n^O \to \mathcal{O}_{\mathbb{P}^{2n-1}} \quad \text{and} \quad \tilde{\sigma} = \tilde{\sigma}_n^{2n+1} : \text{Hess}_n^{O,\perp} \to \mathfrak{g}_1.
\]

Assume that \( n \geq 3 \). We have \( \dim \text{Hess}_n^O = n^2 + 3n - 3 \). We show that

\[
\sigma_\ast \mathcal{C}[-] \cong \bigoplus_{a=0}^{n-3} \mathcal{O}(\mathcal{O}_{\mathbb{P}^{2n-6}}^{-n-1}, \mathcal{C})[n - 3 - 2a] \bigoplus \mathcal{O}(\mathcal{O}_{2j=1}, \mathcal{E}^3)
\]

\[(6.10) \quad \bigoplus_{2 \leq j \leq n-2} \mathcal{O}(\mathcal{O}_{2j-n+1}, \mathcal{C})[j] \oplus \mathcal{O}(\mathcal{O}_{1n+1}, \mathcal{C})[-1].
\]

The lemma follows from the decomposition above, the equations \( \mathfrak{F}(\sigma_\ast \mathcal{C}[-]) \cong \sigma_\ast \mathcal{C}[-] \), \( \tilde{\sigma}_\ast \mathcal{C}[-] \cong \bigoplus \text{IC}(\mathbf{g}_1, \mathcal{E}_{i,1}^{2n+1}) \oplus \cdots \), and (6.7).

In the remainder of this subsection we prove (6.10). Consider first Reeder’s resolution of \( \mathcal{O}_{\mathbb{P}^{2n-1}} \) given by

\[
\rho : \{(x, 0 \subset V_1 \subset V_2 \subset V_2^\perp \subset \mathbb{C}^{2n+1}) \mid x \in \mathfrak{g}_1, xV_1 = 0, xV_2^\perp \subset V_2 \} \to \mathcal{O}_{\mathbb{P}^{2n-1}}.
\]
It is easy to check that \( \rho \) is a small map. Thus for \( x_j \in \mathcal{O}_{3^j2^i} \), we have

\[
\mathcal{H}^{k-n^2-2n} \mathcal{O}(\mathcal{O}_{3^j2^i}, \mathbb{C}) \cong H^k(\rho^{-1}(x_j), \mathbb{C}).
\]

Now we study the map \( \sigma \) and the decomposition of \( \sigma_\ast \mathcal{C}[-] \). The fiber \( \sigma^{-1}(x_{n-1}) \) at \( x_{n-1} \in \mathcal{O}_{3^j2^i} \) is a nonsingular quadric in \( \mathbb{P}^{n-2} \) and \( \text{codim}_{\text{Hess}} \mathcal{O}_{3^j2^i} = n - 3 \). Thus in the decomposition of \( \sigma_\ast \mathcal{C}[-] \), the following IC complexes supported on \( \mathcal{O}_{3^j2^i} \) appear,

\[
\bigoplus_{a=0}^{n-3} \text{IC}(\mathcal{O}_{3^j2^i}, \mathbb{C})[n-3-2a] \text{ for all } n \text{ and } \text{IC}(\mathcal{O}_{3^j2^i}, \mathcal{E})[-] \text{ if } n \text{ is odd},
\]

where \( \mathcal{E} \) is the unique nontrivial irreducible \( K \)-equivariant local system on \( \mathcal{O}_{3^j2^i} \).

We have that \( \sigma^{-1}(x_j) \) (\( x_j \in \mathcal{O}_{3^j2^i} \)) is a \( Q_j \)-bundle over \( \rho^{-1}(x_j) \) for \( j \geq 1 \), where \( Q_j \) is a quadric \( \sum_{k=1}^j a_k^2 = 0 \) in \( \mathbb{P}^{n-2} := \{ [a_1, \ldots, a_{n-1}] \} \).

For \( j \) odd, or \( j \geq 2 \) even and \( 2k > \text{codim}_{\text{Hess}} \mathcal{O}_{x_j} \), we have

\[
H^{2k}(\sigma^{-1}(x_j), \mathbb{C}) \cong \bigoplus_{a=0}^{n-3} H^{2k-2a}(\rho^{-1}(x_j), \mathbb{C}) \cong \bigoplus_{a=0}^{n-3} \text{IC}(\mathcal{O}_{3^j2^i}, \mathbb{C})[n^2-2n-2a]
\]

where in the second isomorphism we use (6.11). Thus in view of (6.12), IC complexes supported on \( \mathcal{O}_{3^j2^i} \), for odd \( j < n-1 \), do not appear in the decomposition of \( \sigma_\ast \mathcal{C} \).

For \( j \geq 2 \) even, and \( 2k = \text{codim}_{\text{Hess}} \mathcal{O}_{x_j} \), we have

\[
H^{2k}(\sigma^{-1}(x_j), \mathbb{C}) \cong \bigoplus_{a=0}^{n-3} \mathcal{H}^{2k-n^2-2n} \text{IC}(\mathcal{O}_{3^j2^i}, \mathbb{C})[2a]
\]

\[
\oplus (H^{2n+j-4}(Q_j, \mathbb{C}) \otimes H^{2 \text{dim} \rho^{-1}(x_j)}(\rho^{-1}(x_j), \mathbb{C})).
\]

Note that \( \rho^{-1}(x_j) \) has two irreducible components. Moreover \( A_j \) acts on \( H^{2n+j-4}(Q_j, \mathbb{C}) \) via the character \( \chi_3 \), and acts on \( H^{2 \text{dim} \rho^{-1}(x_j)}(\rho^{-1}(x_j), \mathbb{C}) \) via \( 1 \otimes \chi_1 \). In view of (6.12), we conclude that IC complexes \( \text{IC}(\mathcal{O}_{3^j2^i} \mathcal{E}_j, \mathcal{E}^2) \) and \( \text{IC}(\mathcal{O}_{3^j2^i} \mathcal{E}_j, \mathcal{E}^3) \), for \( j \) even, appear in the decomposition of \( \sigma_\ast \mathcal{C}[-] \).

For \( j = 0 \) and \( 2k = \text{codim}_{\text{Hess}} \mathcal{O}_{x_0} = n^2 - n - 2 \), since \( 2k - 2a > 2 \text{dim} \rho^{-1}(x_0) \) for all \( 0 \leq a \leq n-3 \), we have

\[
\bigoplus_{a=0}^{n-3} \mathcal{H}^{2k-2a-n^2-2n} \text{IC}(\mathcal{O}_{3^j2^i}, \mathbb{C}) = 0.
\]

We have \( 2 \text{dim} \sigma^{-1}(x_0) = \text{codim}_{\text{Hess}} \mathcal{O}_{x_0} \) and \( \sigma^{-1}(x_0) \cong \{ 0 \subset W_{n-2} \subset W_{n-1} \subset W_{n-1}^\perp \subset W_{n-2}^\perp \subset \mathbb{C}^{2n-2} \} \). Note that \( \sigma^{-1}(x_0) \) has two connected components and \( A_k(x_0) \) permutes them. We conclude that the IC complexes supported on \( \mathcal{O}_{x_0} \) appearing in \( \sigma_\ast \mathcal{C}[-] \) are \( \text{IC}(\mathcal{O}_{3^j2^i}, \mathbb{C}) \oplus \text{IC}(\mathcal{O}_{3^j2^i}, \mathcal{E}_1) \).
The decomposition (6.10) follows from the above discussion and the fact that none of the IC complexes supported on $\mathcal{O}_{2^{n+1}i-2}$, $i \geq 1$ can appear in the decomposition of $\sigma_*\mathcal{C}[-]$. The proof of Lemma 6.7 is complete.

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