PARTIAL MONOIDS AND DOLD-THOM FUNCTORS

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Abstract. Dold-Thom functors are generalizations of infinite symmetric products, where integer multiplicities of points are replaced by composable elements of a partial abelian monoid. It is well-known that for any connective homology theory, the machinery of Γ-spaces produces the corresponding linear Dold-Thom functor. In this note we show how to obtain such functors directly from Ω-spectra.

1. Introduction

Let \( SP^\infty X \) be the infinite symmetric product of a pointed connected topological space \( X \). Then, according to the Dold-Thom Theorem [2], the homotopy groups of \( SP^\infty X \) coincide, as a functor, with the reduced singular homology of \( X \).

The construction of the infinite symmetric product can be generalized so as to produce an arbitrary connective homology theory. Such generalized symmetric products were defined by G. Segal in [3]: essentially, these are labelled configuration spaces, with labels in a Γ-space. If the Γ-space of labels is injective (see [8]) it gives rise to a partial abelian monoid; it has been proved that in [8] that for each connective homology theory there exists an injective Γ-space. In this case Segal’s generalized symmetric product can be thought of as a space of configurations of points labelled by composable elements of a partial monoid. An explicit example discussed in [7] is the space of configurations of points labelled by orthogonal vector spaces: it produces connective \( K \)-theory.

We shall call the generalized symmetric product functor with points having labels, or “multiplicities”, in a partial monoid \( M \) the **Dold-Thom functor with coefficients in** \( M \). Certainly, the construction of such functors via Γ-spaces is most appealing. However, if we start with a spectrum, constructing the corresponding Dold-Thom functor using Γ-spaces is not an entirely straightforward procedure since the Γ-space naturally associated to a spectrum is not injective. The purpose of the present note is to show how a connective spectrum \( M \) gives rise to an explicit partial monoid \( M \) such that the homotopy of the Dold-Thom functor with coefficients in \( M \) coincides, as a functor, with the homology with coefficients in \( M \). The construction is based on a trivial observation: if \( Y \) is a space and \( X \) is a pointed space, the space of maps from \( Y \) to \( X \) has a commutative partial multiplication with a unit.

2. Definitions and statement of results

2.1. Partial monoids. Most of the following definitions appear in [5]. A **partial monoid** \( M \) is a space equipped with a subspace \( M_{(2)} \subseteq M \times M \) and an addition map \( M_{(2)} \to M \), written as \( (m_1, m_2) \to m_1 + m_2 \), and satisfying the following two conditions:

- there exists \( 0 \in M \) such that \( 0 + m \) and \( m + 0 \) are defined for all \( m \in M \) and such that \( 0 + m = m + 0 = m \);
- for all \( m_1, m_2 \) and \( m_3 \) such that \( m_1 + (m_2 + m_3) \) is defined, \( (m_1 + m_2) + m_3 \) is also defined, and both are equal.

We shall say that a partial monoid is **abelian** if for all \( m_1 \) and \( m_2 \) such that \( m_1 + m_2 \) is defined, \( m_2 + m_1 \) is also defined, and both expressions are equal.
The classifying space $BM$ of a partial monoid $M$ is defined as follows. Let $M_{(k)}$ be the subspace of $M^k$ consisting of composable $k$-tuples. The $M_{(k)}$ form a simplicial space, with the face operators $\partial_i : M_{(k)} \to M_{(k-1)}$ and the degeneracy operators $s_i : M_{(k)} \to M_{(k+1)}$ defined as

$$\partial_i(m_1, \ldots, m_k) = (m_1, \ldots, m_i, m_{i+1}, \ldots, m_k) \quad \text{if } i = 0$$
$$= (m_1, \ldots, m_i + m_{i+1}, \ldots, m_k) \quad \text{if } 0 < i < k$$
$$= (m_1, \ldots, m_{k-1}) \quad \text{if } i = k$$

and

$$s_i(m_1, \ldots, m_k) = (m_1, \ldots, m_i, 0, m_{i+1}, \ldots, m_k) \quad \text{if } 0 \leq i \leq k.$$ 

The classifying space $BM$ is the realization of this simplicial space. In the case when $M$ is a monoid, $BM$ is its usual classifying space. If $M$ is a partial monoid with a trivial multiplication (that is, the only composable pairs of elements in $M$ are those containing 0), the space $BM$ coincides with the reduced suspension $\Sigma M$.

A map $f : M \to N$ between two partial monoids is a homomorphism if, whenever $m_1 + m_2$ is defined, $f(m_1) + f(m_2)$ is also defined and equal to $f(m_1 + m_2)$. If $f$, considered as a map of sets, is an inclusion, we say that $M$ is a partial submonoid of $N$. A partial monoid $M$ is filtered if $M = \cup M(i)$ where $M(i)$ is a sequence of partial monoids such that $M(i)$ is a partial submonoid of $M(i+1)$ for all $i$, and such that any two elements of $M$ are composable if and only if they are composable in $M(i)$ for some $i$. A filtered partial monoid $M = \cup M(i)$ has constant filtration if the inclusion of $M(i)$ into $M(i+1)$ is an isomorphism for all $i$. Every partial monoid can be considered as a filtered monoid, with the constant filtration.

2.2. Partial monoids and spectra. For a pointed space $X$ let $\Omega \varepsilon X$ be the space of all maps $\mathbb{R} \to X$ supported (that is, attaining a value distinct from the base point of $X$) inside the interval $[-\varepsilon, \varepsilon]$. We denote by $\Omega X'$ the union of the $\Omega \varepsilon X$ for all $\varepsilon > 0$. The usual loop space $\Omega X$ can be identified with $\Omega \varepsilon X$ for any $\varepsilon > 0$; we shall think of $\Omega X$ as of $\Omega 1 X$. The inclusion $\Omega X \hookrightarrow \Omega' X$ is a homotopy equivalence.

If $X$ is an abelian partial monoid and $Y$ is a space, the space of all continuous maps $Y \to X$ is also an abelian partial monoid. Two maps $f, g$ are composable (in the sense of the monoid product, not as maps) if at each point of $Y$ their values are composable. In particular, if $X$ is a pointed topological space, then, considering $X$ as a monoid with trivial multiplication, we see that $\Omega' X$ is an abelian partial monoid; two maps in $\Omega' X$ are composable if their supports are disjoint.

The space $\Omega' X$ is much better behaved with respect to this partial multiplication than the usual loop space $\Omega X$: while a generic element of $\Omega X$ is only composable with the base point, each element of $\Omega' X$ is composable with a big (in a sense that we need not make precise here) subset of $\Omega' X$.

Let $\mathcal{M}$ be a connective perfect $\Omega'$-spectrum, that is, a sequence of pointed spaces $\mathcal{M}_i$ with $i \geq 0$, together with homeomorphisms $\mathcal{M}_i = \Omega' \mathcal{M}_{i+1}$ for all $i$. We have a sequence of partial monoid structures on the infinite loop space $\mathcal{M}_0$: the partial monoid $M(m)$ is obtained by considering the trivial multiplication on $\mathcal{M}_m$ and then taking the induced partial multiplication on $\mathcal{M}_0 = \Omega^m \mathcal{M}_m$. These partial monoid structures are compatible in the sense that the map $M(m) \to M(m+1)$ given by the identity map on $\mathcal{M}_0$, is a homomorphism. Let $M$ be the partial abelian monoid structure on $\mathcal{M}_0$ obtained “as a limit” of the $M(m)$: the sum of two points of $\mathcal{M}_0$ exists if and only if for some $m$ these points are composable in $M(k)$ with $k \geq m$; if defined, it coincides with their sum in $M(k)$.

Let us say that $\mathcal{M}$ has a good base point if the base point is a neighbourhood deformation retract in each $\mathcal{M}_i$ in such a way that the retractions commute with the structure maps of $\mathcal{M}$. Standard arguments show that each connective homology theory can be represented by a perfect $\Omega'$-spectrum with a good base point.
2.3. **Dold-Thom functors.** Let $M$ be an abelian partial monoid and $X$ a connected topological space. We define the configuration space $M_n[X]$ of at most $n$ points in $X$ with labels in $M$, as follows. For $k > 0$ let $W_k$ be the subspace of the symmetric product $SP^k(X \wedge M)$ consisting of points $\sum_{i=1}^k (x, m_i)$ such that the $m_i$ are composable; $W_0$ is defined to be a point. The space $M_n[X]$ is the quotient of $W_k$ by the relation

$$(x, m_1) + (x, m_2) + \ldots = (x, m_1 + m_2) + \ldots,$$

where the omitted terms on both sides are understood to coincide. This quotient map commutes with the inclusions of $W_n$ into $W_{n+1}$ coming, in turn, from the inclusions $SP^n(X \wedge M) \to SP^{n+1}(X \wedge M)$, and, hence, $M_n[X]$ is a subspace of $M_{n+1}[X]$.

The **Dold-Thom functor of $X$ with coefficients in $M$** is the space

$$M[X] = \bigcup_{n>0} M_n[X].$$

The Dold-Thom functor with coefficients in the monoid of non-negative integers is the infinite symmetric product. If $M$ has trivial multiplication, we have $M[X] = M_1[X] = M \wedge X$.

The composability of labels in a configuration is essential for the functoriality of $M[X]$. A based map $f : X \to Y$ induces a map $M[f] : M[X] \to M[Y]$ as follows: a point $\sum (x_j, m_j)$ is sent to the point $\sum (y_j, n_j)$ where the label $n_j$ is equal to the sum of all the $m_i$ such that $f(x_i) = y_j$.

Apart from the infinite symmetric products, Dold-Thom functors generalize classifying spaces: for any partial monoid $M$ its classifying space $BM$ is homeomorphic to $M[S^1]$. To construct the homeomorphism, take the lengths of the intervals between the particles to be the barycentric coordinates in the simplex in $BM$ defined by the labels of the particles. Similarly, the classifying space of an arbitrary $\Gamma$-space can be constructed in this way (modulo some technical details), see Section 3 of [6]. The identification of $BM$ with $M[S^1]$ also makes sense for non-abelian monoids. The construction of a classifying space for a monoid as a space of particles on $S^1$ was first described in [4].

If $M = \cup M(i)$ is a filtered partial abelian monoid, we define $M_n[X]$ as the union of the spaces $M(i)_n[X]$. Then, as above, the Dold-Thom functor with coefficients in $M$ associates to a space $X$ the space $M[X] = \cup M_n[X]$. As a set, $M[X]$ does not depend on the filtration $M(i)$; however, different filtrations on $M$ may give rise to different topologies on $M[X]$. In particular, the space $M[X]$ for the same partial monoid $M$ can have different topologies according to whether $M$ is considered as a filtered or a non-filtered partial monoid.

We have the following generalization of the Dold-Thom theorem:

**Theorem.** Let $M$ be the filtered partial abelian monoid corresponding to a perfect connective $\Omega'$-spectrum $\mathcal{M}$ with a good base point. Then the spaces $M(S^n)$ form a connective spectrum $M[S]$, weakly homotopy equivalent to $\mathcal{M}$. The functor $X \to \pi_* M[X]$ coincides on connected spaces with the reduced homology with coefficients in $M$.

### 2.4. Uniform partial monoids.

The nice properties of a partial monoid coming from a perfect connective $\Omega'$-spectrum with a good base point can be summarized in the notion of a **uniform partial abelian monoid**. Essentially, uniform partial abelian monoids are those whose pairs of non-composable elements form a subspace of infinite codimension in the space of all pairs. The precise definition of this infinite codimension property is somewhat technical.

Given a partial abelian monoid $M$, a subset $Z \subset M$, and a homotopy $s_t : M \to M$, with $t \in [0, 1]$, we say that $s_t$ is a **deformation of $M$, admissible with respect to $Z$** if

- $s_0 = 1d$ and $s_t(0) = 0$ for all $t$;
- for any set of composable elements $m_i \in M$, the set $s_t(m_i)$ is also composable for all $t$;
- if a set of composable elements $m_i \in M$ is composable with $m' \in Z$, then the set $s_t(m_i)$ is composable with $m'$ for all $t$. 

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A filtered partial abelian monoid \( M = \bigcup M(i) \) is uniform if it has the following properties:

1. as sets, \( M(i) = M(i+1) \) for all \( i \);
2. for each \( i \) and for any compact subset \( Z \subset M \) there exists a deformation \( d_i \) of \( M(i) \), admissible with respect to \( Z \), such that any element of \( d_i(M) \) is composable in \( M(i+1) \) with any element of \( Z \);
3. there exists a neighbourhood \( U \) of 0 in \( M \) together with a deformation \( r \) of \( M(i) \) (the same for all \( i \)), admissible with respect to the whole \( M(i) \), that retract \( U \) to 0.

**Theorem 1.** If a filtered partial abelian monoid \( M \) is uniform, the spaces \( M[S^n] \) form a connective spectrum \( M[S] \) and the functor \( X \to \pi_*M[X] \) coincides on connected spaces with the reduced homology with coefficients in this spectrum.

Our main theorem is a consequence of this fact and of the following statement:

**Theorem 2.** Let \( M \) be the filtered partial abelian monoid corresponding to a perfect connective \( \Omega' \)-spectrum \( M \) with a good base point. Then \( M \) is uniform and the spectrum \( M[S] \) is weakly equivalent to the spectrum \( M \).

### 3. Quasifibrations of Dold-Thom functors

The proof of the fact that a uniform partial monoid gives rise to a homology theory follows the original argument of Dold and Thom \[^2\], see \[^1\] for the translation of the original proof into English. It is based on the following fact:

**Proposition 3.** If \( M \) is uniform, and \( A \) is a connected neighbourhood deformation retract in \( X \), the map \( M[X] \to M[X/A] \) is a quasifibration with the fibre \( M[A] \).

Then the homotopy long exact sequence of a quasifibration becomes the homology long exact sequence of a cofibration etc, cf. \[^2\]. The rest of this section is dedicated to the proof of Proposition \[^3\]

We denote by \( p \) the projection map of \( X \) to \( X/A \) and by \( \pi \) the induced map \( \pi = M[p] : M[X] \to M[X/A] \). We shall prove by induction on \( n \) that \( \pi \) is a quasifibration over \( M_n[X/A] \). This, by Satz 2.15 of \[^2\] (or Theorem A.1.17 of \[^1\]) will imply that \( \pi \) is a quasifibration over the whole \( M[X/A] \).

Assume that \( \pi \) is a quasifibration over \( M_{n-1}[X/A] \). According to Satz 2.2 of \[^2\] (or Theorem A.1.2 of \[^1\]) it is sufficient to prove that \( \pi \) is a quasifibration over \( M_n[X/A] - M_{n-1}[X/A] \), over a neighbourhood of \( M_{n-1}[X/A] \) in \( M_n[X/A] \) and over the intersection of this neighbourhood with \( M_n[X/A] - M_{n-1}[X/A] \).

It will be convenient to speak of delayed homotopies. A delayed homotopy is a map \( f_t : A \times [0,1] \to B \) such that for some \( \varepsilon > 0 \) we have \( f_t = f_0 \) when \( t \leq \varepsilon \). A map \( p : E \to B \) is said to have the delayed homotopy lifting property if it has the homotopy lifting property with respect to all delayed homotopies of finite CW-complexes into \( B \). It is clear that a map that has the delayed homotopy lifting property is a quasifibration.

**Lemma 4.** Let \( B \) be an arbitrary subset of \( M_n[X/A] - M_{n-1}[X/A] \). The map \( \pi \), when restricted to \( \pi^{-1}(B) \), has the delayed homotopy lifting property.

**Proof.** Let \( f_t : Z \times [0,1] \to B \) be a delayed homotopy of a finite CW-complex \( Z \) into \( B \), such that \( f_t = f_0 \) for \( t \leq \varepsilon \), and let \( f_0 : Z \to \pi^{-1}(B) \) be its lifting at \( t = 0 \).

Notice that \( M_n[X/A] - M_{n-1}[X/A] \) can be thought of as the subspace of \( M_n[X] \) consisting of configurations of \( n \) distinct points, all outside \( A \) and with non-trivial labels. Therefore, we can think of \( B \) as of a subspace of \( M_n[X] \).

Define \( g : Z \to M[X] \) as the difference \( g(z) = f_0(z) - f_0(z) \). The map \( g \) is well-defined, continuous and its image belongs to \( M[A] \). Since \( Z \) is compact, the image of \( f_0 \) belongs to \( M(k)[X] \) for some
and \( \leq \) for 0.

The property (2) of uniform partial monoids guarantees the existence of a deformation \( d_k^t : M \to M \) such that each point in \( d_k^t(M) \) is composable with \( M(k + 1) \) with each point in the image of \( Z \times [0, 1] \) under \( f_t \). There is an induced deformation \( M[A] \to M[A] \) which we also denote by \( d_k^t \).

Define the homotopy \( g_t : Z \to M[A] \) as \( d_k^t \circ g \) for \( 0 \leq t < \varepsilon \) and as \( d_k^t \circ g \) for \( \varepsilon \leq t \leq 1 \). The property (2) of uniform partial monoids implies that \( g_t \) is well-defined and is composable with \( f_t \) for all \( t \). Consider the map \( \tilde{f}_t = f_t + g_t : Z \to M[X] \). By construction, it lifts \( f_t \).

It remains to see that \( \pi \) is a quasifibration over some neighbourhood of \( M_{n-1}[X/A] \).

Let \( V \) be a neighbourhood of \( A \) in \( X \) that retracts onto \( A \) by means of the deformation \( q_t : X \to X \) with \( q_0 = Id, q_1(V) \subset A \) and such that \( q_t \) is identity on \( A \). Let \( U \) be a neighbourhood of the base point in \( M \) as in the property (3) of uniform monoids, and let \( r_t : M \to M \) be its retraction onto 0 with \( r_0 = Id, r_1(U) = 0 \). Denote by \( Q_n \subset M_n[X/A] \) the subspace of points \( \sum (\bar{x}_i, m_i) \) with at least one of the \( \bar{x}_i \) in \( pV \) or one of the \( m_i \) in \( U \).

**Lemma 5.** The map \( \pi \) is a quasifibration over \( Q_n \).

**Proof.** According to Hilfssatz 2.2 of [2] (or Lemma A.1.11 of [1]) it is sufficient to prove that there is a deformation of \( \pi^{-1}(Q_n) \) into \( \pi^{-1}(M_{n-1}[X/A]) \) which respects \( \pi \) and, at the final value of the parameter, induces weak homotopy equivalences on the fibres of \( \pi \).

Consider the deformations \( \tilde{q}_t, \tilde{r}_t \), given by

\[
\tilde{q}_t \left( \sum (x_i, m_i) \right) = \sum (q_t(x_i), m_i),
\]

and

\[
\tilde{r}_t \left( \sum (x_i, m_i) \right) = \sum (x_i, r_t(m_i)),
\]

for \( 0 \leq t \leq 1 \). Here \( x_i \in X, m_i \in M \). It follows from the property (3) of uniform monoids that applying first \( \tilde{q}_t \) and then \( \tilde{r}_t \) we obtain a deformation of \( \pi^{-1}(Q_n) \) into \( M_{n-1}[X] \). Clearly, this deformation sends fibres of \( \pi \) into fibres. We now need to show that these maps of fibres are weak homotopy equivalence.

This is a consequence of the following observation. Let \( M_{<a>}[A] \subset M[A] \) be the subset of configurations whose coefficients are composable with \( a \in M \). Then the inclusion of \( M_{<a>}[A] \) into \( M[A] \) induces isomorphisms on all homotopy groups.

We need to show that the image of any map \( f \) of a finite CW-complex \( X \) into \( M[A] \) can be deformed into \( M_{<a>}[A] \). Assume that the image of \( f \) is contained in \( M(i)[A] \). Then, applying the deformation \( d_k^t \) from the second property of the uniform monoids, with \( Z = \{ a \} \), to all the labels of the \( f(x) \), where \( x \in X \), we get a homotopy of \( f \) to a map of \( X \) into \( M_{<a>}[A] \).

Now, we can claim that \( \tilde{q}_t \) is a weak homotopy equivalence on the fibres of \( \pi \). Indeed, the fibre of \( \pi \) over a point \( \sum m_i b_i \), where \( m_i \in M \) and \( b_i \in X/A \) are distinct form the base point, can be identified with \( M_{<\sum m_i}>[A] \). The deformation \( q_t \) (or, more precisely, the deformation that \( q_t \) induces on \( X/A \)) moves some of the points \( b_i \) into the base point at \( t = 1 \). Therefore, applying \( \tilde{q}_1 \) to the fibre over the base point \( \sum m_i b_i \) where the \( m_i \) are a subset of the \( m_i \) that corresponds to the points not moved to the base point. The fibre over \( \sum m_i b_i \) is naturally identified with \( M_{<\sum m_i}>[A] \), and \( \tilde{q}_1 \) induces the natural inclusion \( M_{<\sum m_i}>[A] \hookrightarrow M_{<\sum m_i}>[A] \). This inclusion commutes with the inclusions of both spaces into \( M[A] \), and, therefore, induces isomorphisms on all homotopy groups.

Quite similarly, the map \( \tilde{r}_t \) is a weak homotopy equivalence on the fibres of \( \pi \), since for any finite set of composable elements \( m_i \in M \) the deformation \( r_t \) induces for any \( t \in [0, 1] \) a weak homotopy equivalence \( M_{<\sum m_i}>[A] \to M_{<\sum r_t(m_i)>}[A] \).
4. Perfect Ω'-spectra

Proposition 6. Let \( \mathcal{B} \) be a connective spectrum. There exists a perfect Ω'-spectrum with a good base point, weakly equivalent to \( \mathcal{B} \).

Proof. Our arguments are standard; we follow [3].

First, let us replace inductively the spaces \( \mathcal{B}_i \) by the mapping cylinders of the structure maps \( \Sigma \mathcal{B}_{i-1} \to \mathcal{B}_i \). This allows us to assume that all the structure maps \( \Sigma \mathcal{B}_{i-1} \to \mathcal{B}_i \) are inclusions. Moreover, it can be assumed that in each \( \mathcal{B}_i \) there is a neighbourhood \( V_i \) of the base point, together with a deformation retraction \( s_i : \mathcal{B}_i \times [0,1] \to \mathcal{B}_i \) collapsing \( V_i \) onto the base point, such that the structure map sends \( \Sigma V_{i-1} \) into \( V_i \) and commutes with the retractions \( s_i \).

Since \( \Sigma \mathcal{B}_{i-1} \to \mathcal{B}_i \) is an inclusion, the composition \( \mathcal{B}_{i-1} \to \Omega \mathcal{B}_i \to \Omega' \mathcal{B}_i \) is an inclusion too. Therefore, we have inclusions \( \Omega'^j \mathcal{B}_i \to \Omega'^{j+1} \mathcal{B}_{i+1} \) for all \( i,j \geq 0 \). Define the spectrum \( \mathcal{M} \) by setting \( \mathcal{M}_i = \cup_j \Omega'^j \mathcal{B}_{i+j} \), with the structure maps induced by those of \( \mathcal{B} \). By construction, the natural map \( \mathcal{B} \to \mathcal{M} \) is induces isomorphisms in homotopy groups, so it is a weak equivalence.

Since the deformations \( s_i \) commute with the structure maps in \( \mathcal{B} \), they give rise to deformations \( r_i : \mathcal{M}_i \times [0,1] \to \mathcal{M}_i \). In each \( \mathcal{M}_i \) there is a neighbourhood of the base point \( U_i = \cup_j \Omega'^j V_{i+j} \) which is collapsed by \( r_i \) onto the base point. We have \( \Omega' U_i = U_{i-1} \) by construction and the \( r_i \) commute with the structure maps of \( \mathcal{M} \) since the \( s_i \) commute with those of \( \mathcal{B} \). This means that \( \mathcal{M} \) has a good base point.

Proposition 7. A filtered partial abelian monoid coming from a perfect Ω'-spectrum with a good base point is uniform.

Proof. Property (1) is satisfied by definition.

To establish property (2), choose a positive integer \( i \) and take an arbitrary compact subset \( Z \subset \mathcal{M}_0 \). Consider points of \( Z \) (and, more generally, of \( \mathcal{M}_0 \)) as maps of \( \mathbb{R}^{i+1} \) to \( \mathcal{M}_{i+1} \); let \( x_1, \ldots, x_{i+1} \) be the coordinates in \( \mathbb{R}^{i+1} \). Since \( Z \) is compact there exists \( a \in \mathbb{R} \) such that \( f \in Z \) implies that the support of \( f \) is contained in the half-space \( x_{i+1} < a \).

Now, define \( d^i_t \) for \( 0 \leq t \leq 1 \) by
\[
d^i_t(f) = f(x_1, \ldots, x_i, (1 - 2t)x_{i+1} + 2t(e^{x_{i+1}} + a)).
\]
The support of each element of \( d^i_1(\mathcal{M}_0) \) is contained in the half-space \( x_{i+1} > a \), therefore each element of \( d^i_1(\mathcal{M}_0) \) is composable with each element of \( Z \) in \( \mathcal{M}(i+1) \). Also, the deformation \( d^i_t \), as a deformation of \( \mathcal{M}(i) \), is admissible with respect to \( Z \). Indeed, if we consider \( f \in \mathcal{M}_0 \) as a map \( \mathbb{R}^i \to \mathcal{M}_i \), then \( d^i_t \) does not change the support of \( f \), and the composability of two elements of \( \mathcal{M}(i) \) only depends on whether their supports are disjoint or not.

Finally, the property (3) follows from the fact that \( \mathcal{M} \) has a good base point. Indeed, we have a deformation \( r_t : \mathcal{M}_0 \to \mathcal{M}_0 \) that collapses a neighbourhood of a base point and has the property that if \( \mathcal{M}_0 \) is considered as \( \Omega^0 \mathcal{M}_i \), then \( r_t \) is induced by a deformation of \( \mathcal{M}_i \). Hence, for any map \( f : \mathbb{R}^i \to \mathcal{M}_i \) the support of \( r_t(f) \) is contained in the support of \( f \); this means that \( r_t \) preserves composability in \( \mathcal{M}(i) \) for all \( i \). In particular, \( r_t \) can be taken as the deformation in the property (3).

□

5. On the spectrum \( M[S] \)

5.1. The spectrum \( M[S] \) and the proof of Theorem [1] Proposition [3] with \( X = D^n \) and \( A = \partial D^n \) gives the quasifibration
\[
M[S^{n-1}] \to M[D^n] \to M[S^n].
\]
The space \( M[D^n] \) is contractible, and therefore, we have weak homotopy equivalences \( M[S^{n-1}] \simeq \Omega M[S^n] \) and the spaces \( M[S^i] \) for \( i > 0 \) form an \( \Omega \)-spectrum which we denote by \( M[S] \).
More generally, given any connected $X$ and any inclusion map $i : A \hookrightarrow X$ with $A$ connected, the cofibration $X \to CX \to \Sigma X$ gives rise to a weak homotopy equivalence $M[X] \simeq \Omega M[\Sigma X]$, and the cofibration $A \to Cyl(i) \to X \cup_i CA$ gives rise to an exact sequence $\pi_* M[A] \to \pi_* M[X] \to \pi_* M[X \cup_i CA]$. Here $CX$ is the cone on $X$ and $Cyl(i)$ is the cylinder of the map $i$. Since $\pi_* M[X]$ is, clearly, a homotopy functor, this means that the groups $\pi_* M[X]$ form a reduced homotopy theory for connected spaces.

There is a natural transformation of the homology with coefficients in $M[S^n]$ to $\pi_* M[X]$, induced by the obvious map $M[S^n] \wedge X \to M[S^n \wedge X]$ that sends $(\sum m_i z_i, x)$ to $\sum m_i(z_i, x)$:

$$\lim_{i \to \infty} \pi_{k+i}(M[S^n] \wedge X) \to \lim_{i \to \infty} \pi_{k+i}M[S^n \wedge X] = \pi_k M[X].$$

If $X$ is a sphere, the Freudenthal Theorem implies that this is an isomorphism. Hence, $\pi_* M[X]$ coincides as a functor, on connected CW-complexes, with the homology with coefficients in $M[S]$.

### 5.2. The weak equivalence of $M[S]$ and $\mathcal{M}$. We shall first construct a weak homotopy equivalence between the infinite loop spaces of the spectra $\mathcal{M}$ and $M[S]$ and then show that there exists an inverse to this equivalence, which is induced by a map of spectra. This will establish Theorem 2.

Let $I$ be the interval $[-1/2, 1/2]$.

#### Proposition 8. The map $I \to S^1$ which identifies the endpoints of $I$ induces a quasifibration $M[I] \to M[S^1]$ with the fibre $\mathcal{M}_0$.

The proof of Proposition 8 is the same as that of Proposition 3. Here $\mathcal{M}_0$ should be thought of as $M[S^0]$ (which is not defined as $S^0$ is not connected).

Since $M[I]$ is contractible, it follows that $\mathcal{M}_0$ is weakly homotopy equivalent to $\Omega M[S^1]$; the weak equivalence is realized by the map $\psi$ that sends $m \in \mathcal{M}_0$ to the loop (parameterized by $I$) whose value at the time $t \in I$ is the configuration consisting of one point with coordinate $-t$ and label $m$.

Let us now define the inverse to the above weak equivalence $\psi$. It will be a map $\Phi : M^* [S] \to \mathcal{M}$ where $M^*[S]$ is a certain subspectrum of $M[S]$, weakly equivalent to $M[S]$.

Let us think of $S^n$ as the $n$-dimensional cube $I^n = [-1/2, 1/2]^n \subset \mathbb{R}^n$, modulo its boundary. Fix a homeomorphism of the interior of $I^n$ with $\mathbb{R}^n$, say, by sending each coordinate $u_k$ to tan $\pi u_k$.

Then, for any $i \geq 0$ the labels of the configurations in $M[S^n]$ (that is, elements of $M$) can be thought of as maps $I^n \times \mathbb{R}^i \to \mathcal{M}_{n+i}$. We define $M^{(i)}[S^n]$ to be the subspace of $M[S^n]$ consisting of the configurations $\sum m_\alpha x_\alpha$, with $x_\alpha \in I^n$, such that all points $m_\alpha(q_\alpha) \in \Omega^k \mathcal{M}_{n+i}$ are composable (that is, have disjoint supports in $\mathbb{R}^i$) for any choice of the $q_\alpha \in I^n$.

Let $M^*[S^n]$ be the union of all $M^{(i)}[S^n]$.

The spaces $M^*[S^n]$ form a sub-spectrum $M^*[S]$ of $M[S]$. Indeed, the structure map of $M[S]$ sends $\sum m_\alpha x_\alpha \in M[S^n]$ to the loop $t \mapsto \sum m_\alpha(x_\alpha, t)$, where $t \in I$. If $\sum m_\alpha x_\alpha \in M^{(i)}[S^n]$, its image is readily seen to be a loop in $M^{(i-1)}[S^{n+1}]$.

#### Proposition 9. The inclusion map $M^*[S^n] \to M[S^n]$ is a weak homotopy equivalence for all $n > 0$.

The proof of this fact is rather technical. Let us postpone it until the next section and first define the map of spectra $\Phi : M^*[S] \to \mathcal{M}$.

Take $n > 0$ and think, as above, of a point in $M^*[S^n]$ as of a sum $\sum m_\alpha x_\alpha$, where $x_\alpha \in I^n$ and $m_\alpha$ are maps of $I^n$ to $\mathcal{M}_n$. There exists $i$ such that the elements $m_\alpha(-x_\alpha)$ are composable as maps from $\mathbb{R}^i$ to $\mathcal{M}_{n+i}$. Their sum is a well-defined point of $\mathcal{M}_n$ which does not depend on $i$. We set $\Phi_{n}(\sum m_\alpha x_\alpha)$ to be equal to this sum.

Define the map $\Phi_0$ simply as $\Omega \Phi_1$. Then it is clear from the construction that $\Phi_0 \circ \psi$ is the identity map on $\mathcal{M}_0$. Therefore it only remains to prove Proposition 9.
5.3. Proof of Proposition 9. Let us first prove an auxiliary result. Let \( f \) be a map of a \( k \)-dimensional disk \( D^k \) to \( M[S^n] \), and \( c \) be an integer such that the image of \( f \) lies in \( M(c)[S^n] \). Let us say that a triangulation of the disk \( D^k \) is compatible with \( f \) if on each closed simplex of triangulation the map \( f \) is represented as a sum of several continuous functions with values in \( S^n \cap M(c) \), and, if for each \( i \)-simplex \( \delta \) (\( i < k \)) and each \( i + 1 \)-simplex \( \tau \) with \( \delta \subset \partial \tau \) this representation of \( f \) on \( \delta \) as such is just the restriction of the representation of \( f \) on \( \tau \), possibly with some summands grouped into one.

**Lemma 10.** Given a triangulation of \( D^k \) and a map \( f : D^k \to M[S^n] \) compatible with it on \( \partial D^k \), there exists a refinement of this triangulation and a map \( f' \) homotopic to \( f \) such that \( f' \) is compatible with the refined triangulation. If the image of \( f \) belongs to \( M^*[S^n] \), the images of \( f' \) and of the homotopy between \( f \) and \( f' \) can be chosen so as belong to \( M^*[S^n] \).

*Proof.* We shall prove Lemma 10 not only for \( D^k \), but for any subset \( A \subset \mathbb{R}^k \) which is a union of \( k \)-dimensional simplices. Assume that we have proved the Lemma for all functions \( A \to M[S^n] \) with images in \( M_r[S^n] \), and consider now a function \( f : A \to M[S^n] \) whose image lies in \( M_{r+1}[S^n] \).

Denote by \( C \) the closed set \( f^{-1}(M_r[S^n]) \subset A \). There exists an open neighbourhood \( V \subset A \) of \( C \) and a deformation \( f_t \) of \( f \) over \( V \) to a function \( f_1 : V \to M_r[S^n] \), such that if the image of \( f \) belongs to \( M^*[S^n] \), the image of \( f_t \) for all \( t \) also does. This neighbourhood, together with the deformation, can be constructed as follows.

Roughly speaking, there are two ways for a sequence of configurations in \( M_{r+1}[S^n] - M_r[S^n] \) to converge to a configuration in \( M_r[S^n] \). It may happen that at least one of the labels of points in a configuration tends to the unit in \( M \). Alternatively, it might happen that none of the labels tend to the unit but two points tend to each other, or a point tends to the base point in \( S^n \). Let us write \( \partial C = (C \cap \partial D^k) \cup C_0 \cup C_1 \) where \( C_0 \) consists of those points of \( C \) that are not on the boundary of \( D^k \) and that are limits of sequences with at least one label tending to the unit. (Then \( C_1 \) consists of limits of configurations with points colliding or disappearing at the base point.)

Consider a deformation \( R_t \) of \( M[S^n] \) that is induced by contracting the neighbourhood \( U \) of the unit in \( M \). It sends \( M_r[S^n] \) to itself and takes the image (under \( f \)) of an open neighbourhood \( V_0 \) of \( C_0 \) into \( M_r[S^n] \).

Let \( C_1' \) be a compact subset of \( C_1 \) such that \( C_1 - C_1' \subset V_0 \). There exists an open neighbourhood \( W \) of \( C_1' \) in \( D^k \) such that for each \( x \in W \) the labels of the configuration \( f(x) \) lie outside some neighbourhood of the unit in \( M \). There is a continuous map of \( f(W) \) to the space of finite subsets of \( S^n \) of cardinality at most \( r + 1 \) (denoted by \( \exp_{r+1}[S^n] \)) which replaces a configuration of labelled points \( \sum m_\alpha \cdot x_\alpha \) by the set \( \{ x_\alpha \} \). It is known that \( \exp_{S^n} \) is a simplicial subcomplex of the simplicial complex \( \exp_{r+1}[S^n] \) and therefore a neighbourhood deformation retract. If this neighbourhood retraction is parameterized by \( t \in [0,1] \), we can assume that it is one-to-one for \( 0 \leq t < 1 \). It follows that it can be lifted to a deformation \( D_t \) of \( f(W) \) into \( M_{r+1}[S^n] \), provided that \( W \) is chosen to be sufficiently small. Choose open neighbourhoods \( V_1 \subset V_1 \subset W \) of \( C_1' \) such that

\[
\hat{V}_1 - V_1 = \partial V_1 \times [0,1].
\]

For \( x \in \hat{V}_1 - V_1 \) denote by \( \tau(x) \) the second coordinate in this decomposition; we assume that \( \tau(x) = 1 \) for \( x \in \partial V_1 \). We know that \( f(V_1) \) can be deformed into \( M_r[S^n] \) by means of \( D_t \). This deformation can be extended to a deformation \( D_t^A \) of the function \( f \) over the whole set \( A \): we set it to be the identity outside \( \hat{V}_1 \) and on \( \hat{V}_1 - V_1 \) we set \( D_t^A(x) = D_t^{\tau(x)}(x) \).

Finally, take \( V = V_0 \cup V_1 \) and define \( f_t \) as \( D_t^A \) followed by the deformation induced by \( R_t \). Then \( f_{1} \) sends \( V \) to \( M_r[S^n] \). Notice that, by construction, if the image of \( f \) is contained in \( M^*[S^n] \), the image of \( f_t \) also is.

Now, by refining the triangulation of \( A \), find triangulated neighbourhoods \( A' \subset A'' \subset V \) of \( C \) such that \( A'' - A' \) is homeomorphic to \( \partial A' \times [0,1] \). The function \( f \) on \( A \) can be modified to coincide
with \( f_1 \) on \( A' \), with \( f \) on \( A - A'' \) and to interpolate between \( f \) and \( f_1 \) on \( A'' - A' \). Outside \( A' \) this modified function is represented locally as a sum of \( r + 1 \) functions into \( S^n \wedge M(c) \) since \( f \) sends \( A - A' \) to \( M_{r+1}[S^n] - M_r[S^n] \). On \( A' \) the modified function takes values in \( M_r[S^n] \) and can be represented locally as a sum by the induction assumption. \( \square \)

In fact, it is easy to see that the homotopy in the Lemma 10 can be take to be constant on \( \partial D^k \).

**Proof of Proposition 7.** We need to prove that any map of a \( k \)-dimensional disk \( f : D^k \to M[S^n] \) whose restriction to \( \partial D^k \) has its image in \( M^*[S^n] \), can be deformed into \( M^*[S^n] \), and the deformation can be chosen to be constant on \( \partial D^k \). Let \( c \) be an integer such that the image of \( f \) lies in \( M(c)[S^n] \).

It follows from Lemma 10 that we can assume that there exists a triangulation compatible with \( f \). Let \( B_i \) be the union of \( \partial D^k \) with all simplices of triangulation of dimensions \( \leq i \). The map \( f \) can be deformed so that it maps some neighbourhood of \( B_{i+1} = \partial D^k \) to \( M^*[S^n] \) and in such a way that the deformed map is compatible with some triangulation of \( D^k \). Indeed, if \( d_t \) with \( t \in [0, 1] \) contracts a collar of \( \partial D^k \) to \( \partial D^k \), then \( f \circ d_t \) is the required deformation of \( f \). Now, assume that \( f \) maps \( B_{i+1} \), together with some neighbourhood \( V_i \) of this set into \( M^*[S^n] \). In order to “improve” \( f \) on some neighbourhood of \( B_i \) we perform the following two-step construction.

Let \( \delta \) be an \( i \)-dimensional face of a \( k \)-simplex \( \sigma \). Denote by \( U_{\delta, \sigma} \) the subspace of \( \sigma \) consisting of the points that are closer, or are as close, to \( \delta \) as they are to any other \( i \)-dimensional face of \( \sigma \). Let \( l_t \) be the uniform linear contraction towards \( \delta \) along the subspace perpendicular to \( \delta \), with \( l_0 = \text{Id} \) and \( l_1 \) being the projection on \( \delta \). Denote by \( U_{\delta, \sigma} \subset U_{\delta, \sigma} \) the subspace \( l_{i/2}(U_{\delta, \sigma}) \) and let \( s_t(\delta, \sigma) \) be the deformation \( \sigma \to \sigma \), which is the identity outside \( U_{\delta, \sigma} \), coincides with \( l_t \) on \( U_{\delta, \sigma} \), and on \( U_{\delta, \sigma} - U_{\delta, \sigma} \) it interpolates the identity and \( l_t \) by a uniform contraction towards \( \delta \), linear in the subspaces perpendicular to \( \delta \). Denote by \( U \) the union of all \( U_{\delta, \sigma} \) for all pairs of incident simplices of the triangulation \( \delta, \sigma \) of dimensions \( i \) and \( k \) respectively. Similarly, let \( U' \) be the union of all \( U_{\delta, \sigma} \) and \( s_t : D^k \times [0, 1] \to D^k \) be the deformation that restricts to the \( s_t(\delta, \sigma) \) on \( U \) and is the identity outside \( U \) (note that the \( s_t(\delta, \sigma) \) agree on the boundaries of the \( k \)-simplices.) The first step consists of replacing \( f \) by \( f \circ s_1 \).

Now, for each \( i \)-simplex \( \delta \) choose a function \( h_\delta : D^k \to \mathbb{R} \) such that \( h_\delta(x) = 1 \) for all \( x \in \delta \) such that \( x \notin V_{i-1} \), and \( h_\delta(x) = 0 \) unless \( x \in U_{\delta, \sigma} \) for some \( k \)-simplex \( \sigma \) incident with \( \delta \). The effect of the first step of our construction is that over the union of all \( U_{\delta, \sigma} \) where \( \sigma \) is incident with \( \delta \), the function \( f \) can now be written as a sum \( \sum_{q=0}^{Q} f_q \) where \( f_q \) are functions to \( S^n \wedge M(c) \) for some \( c \). Consider the elements of the monoid \( M(c) \) as maps of \( \mathbb{R}^{c+1} \) to \( M_{c+1} \). Since \( \delta \) is compact, the \( M(c) \)-components of the \( f_q \) are supported in the region \( |u_{c+1}| < A/2 \) for some \( A \) (recall that by \( u_i \) we denote the coordinates in the euclidian space). Let \( T_v \) be the continuous map \( S^n \wedge M(c) \to S^n \wedge M(c) \) induced by composing an element of \( M(c) \) with the translation by a real number \( v \) along the \( c + 1 \)st coordinate. Consider the deformation \( d_t(f) = \sum T_{h(x)}f_q(f_q(x)) \), which carries the \( M(c) \)-components of the \( f_q \) away from each other along the \( c + 1 \)st coordinates. The second step of our construction consists of replacing \( f = \sum f_q \) on \( U' \) by \( d_1(f) = \sum T_{h(x)}f_q(f_q(x)) \). Repeat this step for all \( i \)-simplices \( \delta \). The new map to \( M[S^n] \) is compatible with the same triangulation, it is homotopic to the original map and the image of \( B_i \), together with some neighbourhood \( V_i \), is contained in \( M^*[S^n] \). \( \square \)

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