Non-symmetric and chaotic vibrations of Euler-Bernoulli beams under harmonic and noisy excitations

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Abstract. In this paper we study non-linear dynamics of flexible Euler-Bernoulli beams subjected to harmonic load and white noise. We report that in the case of continuous mechanical systems like the studied Euler-Bernoulli beams the action of white noise yields novel and unexpected phenomena, i.e. symmetric transversal loads and symmetric boundary conditions imply non-symmetric beam vibration. In addition, regions of non-symmetric beam vibrations for two values of white noise intensity are given and the charts of regular and chaotic vibrations of the studied beams are presented.

1. Introduction

Noisy nature of a medium may essentially change the behavior of a dynamical system. This novel type of non-equilibrium transition can be induced by a noise [1]. On the other hand, in deterministic systems a few scenarios of transition from regular to chaotic dynamics are detected and reasonably well studied. Let us mention the Ruelle-Takens scenario [2] where only three frequencies are involved to produce a chaotic attractor. In this scenario noisy (chaotic) system behavior has been associated with an occurrence of a strange attractor arose after three successive period-doubling bifurcations. Many researchers studied simple non-linear physical systems which have been able to generate internal noise (chaos) [3]. Scenarios associated with period doubling (Andronov-Hopf) bifurcations as well as with the intermittency phenomena have been detected and extensively analyzed [4]. However, noisy nature of a medium is capable of inducing much richer and unexpected non-linear dynamical phenomena than the above mentioned. We show that the methods applied to study deterministic classical phase transition can be extended to study noisy systems. Thus, it becomes possible to apply the classical theory of chaos to analyze noise-induced transitions to chaotic vibrations by numerous numerical approaches. In the following paper we are aimed at applying the classical theory of non-linear multi-dimensional deterministic dynamical systems putting emphasis on the noise-induced regular and chaotic dynamics. There exist numerous papers and books devoted to study deterministic chaos exhibited by simple non-linear oscillators. These studies have been directly implemented to analyze also continuous systems by taking their single-mode approximation and by applying the...
Bubnov-Galerkin approach. However, results of this research can be treated as the qualitative ones since the increase in the number of modes may induce their interaction and the arose vibrations may be significantly different from those offered by one degree-of-freedom (1-DoF) approximations.

The problem of non-linear PDEs governing dynamics of the Euler-Bernoulli beams taking into account the geometric non-linearity, which is the subject of our paper, cannot be solved in a closed analytical way. Therefore, numerical methods are applied including FEM (Finite Element Method), FDM (Finite Difference Method), TFE (Time Finite Element), and the Bubnov-Galerkin approaches based on higher order of approximations. In general, the numerical solution requires discretizations of both spatial and time domains. In result, the problem is reduced to a large set of ODEs (Ordinary Differential Equations) with two numerical control parameters, i.e. the spatial grid size and the time step size.

Bar-Yoseph et al. [5] pointed out that nonlinear spatio-temporal dynamical systems governed by non-linear PDEs, may display periodic and aperiodic solutions in time and space including solitons, shock waves, ultrasubharmonic, quasi-periodic and chaotic attractors.

They have employed the space-time spectral element method to solve a simply supported modified Euler-Bernoulli beam undergoing external forced lateral vibrations. It has been shown that the characteristics of the first mode yielded the first period doubling bifurcations, the onset of chaotic strange attractor, and then a stable periodic solution. It has been also reported that the form and size of the first mode strange attractor is sensitive to the time step size, the spatial grid size, and the polynomial order.

Bokhari et al. [6] have studied symmetries and integrability of a fourth-order Euler-Bernoulli equation. They have employed the Lie and Noether symmetry analysis. In our paper we have also addressed this problem, but we have shown that our symmetric system may exhibit a break of symmetry due to action of the white noise.

The literature review does not exhibit research devoted to an interplay of chaotic dynamics of the structural members with white Gaussian noise. However, forced random vibration of beam structures is interest in both theory and application and has been widely documented in the existing literature.

Dunne and Ghanbari [7] have compared predicted extreme exceedance possibilities associated with experimental measurements of highly non-linear damped-damped beam vibrations driven by band-limited white noise. It has been shown that by exploiting the Weissman estimator the accurate exceedance probabilities from relatively small amounts of measured data can be obtained.

The stochastic bifurcation exhibited by one flexible beam subject to axial Gauss white noise excitation has been analyzed in reference [8]. The stochastic stability of the trivial system solution has been estimated using the Lyapunov exponents and boundary classifications.

Lan and Qin [9] have employed the method of the energy harvesting from the horizontal coherent resonance of a vertical cantilever beam subjected to the vertical beam excitation. It has been shown that the horizontal coherence resonance can be detected by introducing a vertical white noise.

The existence of a compact random attractor for the floating beam equation with strong damping and white noise has been investigated by Ling Xu and Ma [10].

In the reference [11] global bifurcations and chaotic dynamics of a cantilever beam under axial harmonic load and transversal excitation at its free end have been reported. Rich chaotic dynamics exhibited by plates and shells taking into account their geometric non-linearity has been illustrated and discussed in references [12-14], among other.

In this paper we study vibrations of geometrically non-linear Euler-Bernoulli beams as systems with many DOFs under external excitation in the form of white noise. Here the term “white” refers to white light which has the whole range of frequencies in the visible spectrum. The generalized correlation function governing white noise has the form of: \( B(t) = \sigma^2 \delta(t) \), where \( \sigma^2 \) is a positive constant and \( \delta(t) \) denotes the delta (\( \delta \)) function. Our study may have an impact on non-linear dynamics of plates and shells under action of the sound pressure, which corresponds to the excitations models in the form of Gaussian white noise.
2. Problem formulation
We study a single-layer beam treated as a 2D region of the \( \mathbb{R}^2 \) space with Cartesian coordinate system introduced in the following way: we fix a median line at \( z=0 \), whereas the axis \( OX \) is directed from the left to the right along the median line and the remaining axis \( OZ \) is directed downwards, perpendicularly to \( OX \). In the introduced coordinate system the beam is considered as a two-dimensional area \( \Omega \) defined as follows: \( \Omega = \{ x \in [0, a]; -h \leq z \leq h \} \), \( 0 \leq t \leq \infty \). Here and below we use the following notation: \( 2h \) denotes the height of the beam and \( a \) stands for its length.

![Figure 1. The considered Euler-Bernoulli beam](image)

3. Mathematical model
A mathematical model of the beam has been developed on a basis of the Euler-Bernoulli hypothesis taking into account the non-linear relationship between the beam deformation and displacements in the von Kármán form \([15]\). The system of non-linear PDEs governing beam motion including the energy dissipation has the following form:

\[
2hE \left( \frac{\partial^2 u}{\partial x^2} + L_3(w,w) \right) - 2h^2 \frac{\partial^2 u}{g \partial t^2} = 0,
\]

\[
2hE \left( L_1(u,w) + L_2(w,w) - \frac{(2h)^2}{12} \frac{\partial^4 u}{\partial x^4} \right) + q + \bar{q} - 2h^2 \frac{\partial^2 w}{g \partial t^2} - 2he \frac{\partial w}{g \partial t} = 0,
\]

\[
L_1(u,w) = \frac{\partial^2 u}{\partial x^2} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2}, \quad L_2(w,w) = \frac{3}{2} \frac{\partial^2 w}{\partial x^2} \left( \frac{\partial w}{\partial x} \right)^2, \quad L_3(w,w) = \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x},
\]

where \( w(x,t) \) is the beam deflection, \( u(x,t) \) stands for the beam displacement along the axis \( OX \), \( \varepsilon \) is the damping coefficient, \( q = q(x,t) \) denotes beam transversal load, \( \bar{q}(t) \) is the applied white noise in the form \( \bar{q} = w \cdot (2.0 \text{ rand}())/(\text{RAND\_MAX}+1.0)-1.0 \), \( E \) is Young’s modulus, \( \nu \) denotes the Poisson coefficient, whereas \( \rho, \gamma \) stand for the density and the volumetric weight of the beam material, \( g \) – the gravity of Earth, \( w_n \) – white noise intensity. The following dimensionless quantities are introduced:

\[
\bar{w} = \frac{w}{2h}, \quad \bar{u} = \frac{ua}{(2h)^2}, \quad \bar{x} = \frac{x}{a}, \quad \lambda = \frac{a}{2h}, \quad \bar{q} = \frac{q}{(2h)^4 E}, \quad \bar{t} = \frac{t}{\tau}, \quad \tau = \frac{a}{c}, \quad c = \sqrt{\frac{Eg}{\gamma}}, \quad \bar{\varepsilon} = \varepsilon \frac{a}{c}.
\]

Using (2) in (1) we get

\[
\frac{\partial^2 u}{\partial x^2} + L_3(w,w) - \frac{\partial^2 u}{\partial t^2} = 0,
\]

\[
\frac{1}{\lambda^2} \left( L_1(u,w) + L_2(w,w) - \frac{1}{12} \frac{\partial^4 w}{\partial x^4} \right) - \frac{\partial^2 w}{\partial t^2} - \bar{\varepsilon} \frac{\partial w}{\partial t} + q + \bar{q} = 0,
\]

where bars above dimensionless quantities are omitted for simplification (we have fixed \( \lambda = 50, \bar{\varepsilon} = 1.0 \)).
In (3) we consider the following boundary conditions applied on the beam ends

\[ w(0,t) = w(1,t) = u(0,t) = u(1,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(1,t)}{\partial x^2} = 0, \]  

and the following initial conditions

\[ w(x,t)|_{t=0} = \frac{\partial w(x,t)}{\partial t} |_{t=0} = u(x,t)|_{t=0} = \frac{\partial u(x,t)}{\partial t} |_{t=0} = 0. \]  

4. Methods of solution

In what follows we substitute the differential operators with respect to the variable \( x \) as well as their derivatives by the difference operators with approximation \( O(h^2) \). Therefore, system (3) is reduced to the following set of ordinary differential equations (ODEs):

\[ \ddot{w}_i + \varepsilon \dot{w}_i = q + \tilde{q} + \frac{1}{\lambda} \left\{ L_{1,h}(u_i, w_i) + L_{2,h}(w_i, \dot{w}_i) - \frac{1}{12} L_1(w_i) \right\}, \quad \ddot{u}_i = L_{3,h}(w_i, \dot{w}_i) + l_2(u_i), \quad i = 0, \ldots, n, \]  

where \( n \) denotes a number of nodes regarding the space coordinate, \( L_{1,h}, L_{2,h} \) and \( L_{3,h} \) stand now for non-linear difference operators, \( l_i(w), l_i(\dot{w}) \) and \( l_i(u) \) are linear difference operators given in a standard form. ODEs (6) together with boundary (4) and initial (5) conditions are derived through application of the FDM with approximation \( O(h^2) \) and they are solved using the standard 4th order Runge-Kutta method. The carried out numerical experiments have shown that the FDM convergence regarding nodes number for \( x \in [0;1] \) achieved for \( n = 40 \) by the Runge-Kutta methods of 4th and 6th orders is the same. This is why further the 4th order Runge-Kutta method has been applied, since it requires less computational time.

5. Analysis of the simulation results

In order to study dynamics of non-linear beams subjected to harmonic load \( q = q_0 \sin(\omega_p t) \) and white noise we apply the so-called charts of beam vibration types versus the control parameters \( \{q_0, \omega_p, w_n\} \). To get a point of this chart we have to analyze the frequency power spectrum as well as the Lyapunov exponents (LEs) for each set of \( \{q_0, \omega_p, w_n\} \).

The developed algorithm allows to detect regions of periodicity, period doubling bifurcation regions, zones of quasi-periodicity as well as zones of chaos. Below we report time histories \( w(t) \) in the middle of the beam for the beam length \( x \in [0;1] \) divided into 40 parts. The time history refers to the time interval \( t \in [0;1024] \) with the boundary conditions (4). To obtain the charts of vibration character depending on the control parameters \( q_0 \in [100; 60000] \) with the division into 600 parts and \( \omega_p \in [0.0003; 12] \) with the division into 350 parts one needs to study \( 2.1 \times 10^5 \) times the governing differential equations. In other words, to obtain the above mentioned chart each of the points from a set of \( 2.1 \times 10^5 \) points has been analyzed with the help of the frequency power spectrum, the Morlet wavelet, modal phase portraits, autocorrelation functions, the Lyapunov exponents and the Poincaré maps \( w_{i,T}(t_i) \), where \( T \) denotes a period of regular excitation.
In Fig. 2 the charts of vibration types in the case of white noise absence $w_n=0$ (Fig. 2a) are shown accompanied by the charts for two values of white noise amplitude $w_n=0.01$ and $w_n=1$ (Fig. 2b,d) as well as by zones of non-symmetric forms of vibrations for the white noise amplitude $w_n=0.1$ (Fig. 2c) and $w_n=1$ (Fig. 2e). We have chosen two points with the following coordinates: point 1 $(x_1=0.503, x_2=43300)$, point 2 $(x_1=5303.5, x_2=56700)$. Tables 1, 2 report the wavelets, time histories, power spectra, Poincaré maps, phase portraits and snapshots of the beam deflections $w(x,t), x \in [0;1]$ at the given time instants for the point 1 and 2, respectively. The overall charts analysis shows that the presence of white noise significantly reduces zones of periodic vibrations but variations of the intensity of white noise have little effect on the change of zones of periodicity. It should be noted that an increase in the noise amplitude up to $w_n=1$ in the resonance zone yields an increase in periodic waves regions. It means that we may control periodic vibrations of the beam by adding white noise properly. It is particularly important to note that adding white noise leads to the occurrence of large zones of non-symmetric beams vibrations under symmetric loads and symmetric boundary conditions applied. Zones of non-symmetric vibrations are shown in Fig. 2c,e.

The carried out analysis of the charts $q_o(\omega_o)$ for $w_n = [0.0;0.01;0.1;1]$ have shown that for a lack of the white noise a zone of non-symmetric vibrations is very small and it cannot be easily detected through the applied numerical approaches. However, an occurrence of the white noise of small intensity yields a birth of large zones of non-symmetric vibrations. Further increase of the white noise intensity causes a slight further grow of the non-symmetric zones (see Fig. 3). In the carried out further numerical experiments for two vibrational regimes, i.e. for periodic vibrations (point 1) and chaotic vibrations (point 2) shown in Fig. 3 we have investigated influence on the noise action on the
vibrations of the beam for two sets of the beams excitations represented by points 1, 2. In both cases, i.e. for periodic (point 1) and chaotic (point 2) vibrations the beam is subjected to the white noise of intensity $w_n = 0.01$. The snapshots of the bending forms of the beam $w(0 \leq x \leq 1)$ are reported in Figures 4, 6 for two time instance given in those figures, where solid (dashed) curves correspond to $w_n = 0$ ($w_n = 0.01$). In addition, in Figures 5, 7 the beam center time histories $w(x = 0.5, t)$ associated with the points 1,2 shown in Fig. 3, i.e. for the regular and chaotic beams vibrations, are reported. Dark points correspond to time instants, where the birth of the non-symmetric vibrations have been detected. Remarkably, their number grows in time yielding a collapse of the previous symmetric beam vibrations.

Table 1. The Morlet wavelets, time histories, frequency power spectrum, Poincaré map, 3D phase plot, and beam vibration snapshots in time instant $t = 804.145$ (point 1)

| $w_n = 0, \omega_p = 1.0503, q_0 = 43300$ | $w_n = 0.01, \omega_p = 1.0503, q_0 = 43300$ |
|---|---|
| ![Wavelet](image1.png) | ![Wavelet](image2.png) |
| ![Time History](image3.png) | ![Time History](image4.png) |
| ![Frequency Spectrum](image5.png) | ![Frequency Spectrum](image6.png) |
| ![Poincaré Map](image7.png) | ![Poincaré Map](image8.png) |
| ![Phase Plot](image9.png) | ![Phase Plot](image10.png) |
| ![Beam Vibration Snapshot](image11.png) | ![Beam Vibration Snapshot](image12.png) |
Table 2. The Morlet wavelets, time histories, frequency power spectrum, Poincaré map, 3D phase plot, and beam vibration snapshots in time instant $t=804.145$ (point 2)

| $w_n = 0, \omega_p = 5.5303, q_0 = 56700$ |
|------------------------------------------|
| ![Wavelet](image1) ![Time History](image2) ![Frequency Spectrum](image3) ![Poincaré Map](image4) ![Phase Plot](image5) ![Vibration Snapshots](image6) |

| $w_n = 0.01, \omega_p = 5.5303, q_0 = 56700$ |
|------------------------------------------|
| ![Wavelet](image1) ![Time History](image2) ![Frequency Spectrum](image3) ![Poincaré Map](image4) ![Phase Plot](image5) ![Vibration Snapshots](image6) |
Figure 3. Charts of zones, where non-symmetric beam vibrations appear at least in one instant of time.

Figure 4. Snapshots of beam center vibrations (point 1) in different time instants.
Figure 5. Time history of the beam center vibrations with the marked (in blue) points of symmetry braking (point 1).

Figure 6. Snapshots of beam center vibrations (point 2) for two different time instants.

Figure 7. Time history of the beam center vibrations with the marked (in blue) points of symmetry braking (point 2).
Observe that in both points 1, 2 the detected time instants of the non-symmetric vibrations occurrence are approximately the same. In Figure 4 two snapshots of the beam bending forms are reported corresponding to either lack of white noise (solid curve) or action of the white noise intensity $w_n = 0.01$. Figure 6 presents the analogous results for the point 2 (chaotic vibrations), whereas in Figures 5, 7 time histories of the symmetric and non-symmetric beam center vibrations are shown.

Table 3. FFT and Morlet spectra for points 1, 2 with the noise intensity $w_n = 0$, $w_n = 0.01$.

| Point 1  | $\omega_p = 1.0853, q_0 = 50000$ |
|----------|---------------------------------|
|          | $w_n = 0$                       |
|          | $w_n = 0.01$                    |
| Point 2  | $\omega_p = 9.5903, q_0 = 56400$|
|          | $w_n = 0$                       |
|          | $w_n = 0.01$                    |

Finally, in Table 3 the Fourier spectrum (FFT) as well as the Morlet wavelet spectrum $\omega(t)$ for the points 1, 2 are given. It should be emphasized that the vibrational regime of both chosen points do not exhibit qualitatively new phenomena. Though in the case of regular beam vibrations (point 1) the noise action does not change the vibration qualitatively, but in the case of point 2 (chaos) the beam chaotic dynamics increases in time, and the chaotic vibrations are transited into hyper-chaotic vibrations.

6. Concluding remarks
We have detected, illustrated and discussed novel non-linear phenomena exhibited by geometrically non-linear Euler-Bernoulli beams under combined harmonic and white noise excitations. In particular, we have reported non-symmetric beam vibrations under symmetrically distributed loads and symmetric boundary conditions applied. Furthermore, the vibration characteristics shown in Tables 1, 2 do not exhibit noisy effects in the case of regular vibrations (point 1), which contradicts the observations of the beam characteristics in the case of point 2 (chaos).

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