GLOBAL FORMALITY AT THE $G_\infty$-LEVEL

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Dedicated to Pierre Deligne’s 65-th birthday

Abstract. In this paper we prove that the sheaf of $\mathcal{L}$-poly-differential operators for a locally free Lie algebroid $\mathcal{L}$ is formal when viewed as a sheaf of $G_\infty$-algebras via Tamarkin’s morphism of DG-operads $G_\infty \to B_\infty$.

In an appendix we prove a strengthening of Halbout’s globalization result for Tamarkin’s local quasi-isomorphism.

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1. INTRODUCTION

Throughout $k$ is a field of characteristic zero. In this paper we extend the global $L_\infty$-formality result [3, Thm 7.4.1] (see also [26]) for Lie algebroids to the $G_\infty$-context.

A similar global $G_\infty$-formality result was obtained in [8] when the Lie algebroid is the tangent bundle of a smooth space (in a suitable context). The methods in loc. cit. are however quite different. In the special case of a $C^\infty$-manifold, the result has also been obtained in [15].

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Throughout \( \mathcal{C} \) is a site equipped with a sheaf of commutative, associative \( k \)-algebras \( \mathcal{O} \). Furthermore \( \mathcal{L} \) is a locally free Lie algebroid over \((\mathcal{C}, \mathcal{O})\) of constant rank \( d \).

Let \( T^\mathcal{L}_{\text{poly}}(\mathcal{O}) \), \( D^\mathcal{L}_{\text{poly}}(\mathcal{O}) \) be respectively the sheaves of \( \mathcal{L} \)-poly-vector fields and \( \mathcal{L} \)-poly-differential operators on \( \mathcal{O} \) (see [2] or §8 below). Then \( T^\mathcal{L}_{\text{poly}}(\mathcal{O}) \) is canonically a sheaf of Gerstenhaber algebras and \( D^\mathcal{L}_{\text{poly}}(\mathcal{O}) \) is canonically a \( B_\infty \)-algebra (see [13] or §3.1 below).

Tamarkin constructs in [24] (see §5) a remarkable morphism of operads \( G_\infty \rightarrow B_\infty \) which is intimately connected to the celebrated (now multiply proved) “Deligne conjecture” [7] which has fundamentally influenced modern deformation theory.

Hence by Tamarkin’s work \( D^\mathcal{L}_{\text{poly}}(\mathcal{O}) \) is then also a strong homotopy Gerstenhaber algebra (\( G_\infty \)-algebra for short, see §4.3 below). Our main result will be the following.

**Theorem 1.1.**

1. There exists a sheaf of \( G_\infty \)-algebras \( t^\mathcal{L} \) on \( \mathcal{C} \) together with \( G_\infty \)-quasi-isomorphisms

\[
T^\mathcal{L}_{\text{poly}}(\mathcal{O}) \longrightarrow t^\mathcal{L} \leftarrow D^\mathcal{L}_{\text{poly}}(\mathcal{O}).
\]

2. The isomorphism

\[
T^\mathcal{L}_{\text{poly}}(\mathcal{O}) \longrightarrow H^*(D^\mathcal{L}_{\text{poly}}(\mathcal{O}))
\]

induced by (1.1) is the usual Hochschild-Kostant-Rosenberg isomorphism [2, 25].

The \( G_\infty \)-structure on \( D^\mathcal{L}_{\text{poly}}(\mathcal{O}) \) depends on the one time choice of a Drinfeld associator. Likewise the quasi-isomorphisms in (1.1) depend on the one time choice of a local formality isomorphism. Once these choices are made the quasi-isomorphisms are canonical.

In Appendix A we prove a strengthening of Halbout’s globalization result [18, Theorem 4.5] for Tamarkin’s local quasi-isomorphism.

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2. **Notations and conventions**

Our grading conventions for Gerstenhaber, \( B_\infty \)- and \( G_\infty \)-algebras are shifted with respect to the usual ones. In our setup the Lie bracket has degree zero and the cupproduct has degree one.

3. **Preliminaries on \( B_\infty \)-algebras**

3.1. \( B_\infty \)-algebras. For a detailed discussion of \( B_\infty \)-algebras we refer to [14, §5.2]. Let \( V \) be a graded vector space and \( T^c(V) \) be the cofree cotensor algebra (with counit). As graded vector spaces we have \( T^c(V) = T(V) \). The comultiplication is given by

\[
\Delta(a_1|\cdots|a_n) = \sum_i (a_1|\cdots|a_i) \otimes (a_{i+1}|\cdots|a_n)
\]

where

\[
(a_1|\cdots|a_n) \overset{\text{def}}{=} a_1 \otimes \cdots \otimes a_n \in T^{c,n}(V)
\]
and $1 = ()$. The counit is given by

$$
\epsilon(a_1|\cdots|a_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}
$$

A $B_\infty$-structure on $V$ consists of a DG-bialgebra structure

$$(T^c(V), \Delta, \epsilon, m, 1, Q)$$

on $T^c(V)$ with unit equal to $1 \in k = T^{c,0}(V)$. We also say that $V$ is $B_\infty$-algebra.

A $B_\infty$-algebra morphism $V \to W$ is a morphism of DG-bialgebras $\psi : T^c(V) \to T^c(W)$. If $\psi$ is obtained by extending a morphism of DG-vector spaces $\psi : V \to W$ then we say that $\psi$ is strict.

If $(T^c(V), \Delta, \epsilon, m, 1, Q)$ is a $B_\infty$-structure on $V$ then $Q$ is determined by the compositions

$$Q^i : T^{c,i}(V) \hookrightarrow T^c(V) \xrightarrow{Q} T^c(V) \xrightarrow{\text{projection}} V$$

and likewise $m$ is determined by

$$m^{p,q} : T^{c,p}(V) \otimes T^{c,q}(V) \hookrightarrow T^c(V) \otimes T^c(V) \xrightarrow{m} T^c(V) \xrightarrow{\text{projection}} V$$

A $B_\infty$-structure leads in a natural way to one unary and two binary operations on $V$ given respectively by a differential $Q^1$, a Lie bracket $[v,w] = m^{1,1}(v,w) - (-1)^{|v||w|}m^{1,1}(w,v)$ of degree zero and a “cup product” $Q^2$ of degree one. The computations in [14, §5.2] show that $(V,Q^1,[\cdot,\cdot],Q^2)$ is a DG-Gerstenhaber algebra up to homotopies expressible in the ternary operations $Q^3, m^{1,2,1}, m^{2,1}$. In particular $H^*(V)$ is a Gerstenhaber algebra.

3.2. Inner $B_\infty$-algebras and brace algebras. The $B_\infty$-algebras which we encounter below are of a special kind. Assume that $(T^c(V), \Delta, \epsilon, m, 1)$ is a bialgebra. Let $\mu \in V_1$ be such that $m(\mu, \mu) = 0$. I.e. $m^{1,1}(\mu, \mu) = 0$. Since $\Delta(\mu) = \mu \otimes 1 + 1 \otimes \mu$ we know that $Q = [\mu, -]$ is a degree one biderivation on $(T^c(V), \Delta, \epsilon, m, 1)$. Hence this gives a $B_\infty$-structure on $V$. We will say that $V$ is an inner $B_\infty$-algebra.

Another major simplification appears when $m^{p,q} = 0$ for $p > 1$. A $B_\infty$-algebra satisfying this condition is called a brace algebra.

It the case of a brace algebra it is easier to express the associativity condition for $m$. We write

$$m^{1,p}(a, (b_1|\cdots|b_p)) = a\{b_1, \ldots, b_p\}$$

and the full multiplication can be expressed by the following identity

$$m((a_1|\cdots|a_p), (b_1|\cdots|b_q)) = \sum_{0 \leq i_1 \leq \cdots \leq i_p \leq q} (-1)^{\epsilon}(b_1|\cdots|b_{i_1}|a_1\{b_{i_1+1}, \ldots\}|\cdots|b_{i_p}|a_p\{b_{i_p+1}, \ldots\}|\cdots|b_q)$$

where $(-1)^{\epsilon}$ is the sign obtained from passing the $a$'s across the $b$'s.

The associativity condition becomes

$$a\{b_1, \ldots, b_q\}\{c_1, \ldots, c_r\} = \sum_{0 \leq i_1 \leq \cdots \leq i_q \leq r} (-1)^{\epsilon} a\{c_1, \ldots, c_{i_1}, b_1\{c_{i_1+1}, \ldots\}, \ldots, c_{i_q}, b_q\{c_{i_q+1}, \ldots\}\}, \ldots, c_r\}$$

where $\epsilon$ is a usual.

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1In this context, a biderivation is a graded linear map that is both a derivation (for the product) and a coderivation (for the coproduct).
4. Preliminaries on $G_\infty$-algebras

4.1. Co-strict homotopy Lie bialgebras. Assume that $(\mathfrak{g}, \delta)$ is a Lie coalgebra. We view the coproduct

$$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$$

as a map of degree $-1$

$$\bar{\delta} : \mathfrak{g}[1] \to \mathfrak{g}[1] \otimes \mathfrak{g}[1]$$

via

$$\bar{\delta}(g) = (-1)^{|g[1]|} g[1] \otimes g[2]$$

where $\delta(g) = g[1] \otimes g[2]$. We can extend $\bar{\delta}$ to a bicoderivation of degree -1

$$\bar{\delta} : S(\mathfrak{g}[1]) \to S(\mathfrak{g}[1]) \otimes S(\mathfrak{g}[1])$$

via the formula

$$\bar{\delta}(g_1 \cdots g_n) = \sum (-1)^{|g_1| + \cdots + |g_{i-1}| + r-1} \Delta(g_1) \cdots \Delta(g_{i-1}) \bar{\delta}(g_i) \Delta(g_{i+1}) \cdots \Delta(g_n)$$

where $\Delta$ is the coshuffle coproduct on $S(\mathfrak{g}[1])$ and where $|g_j|$ refers to the degree of $g_j$ as an element of $\mathfrak{g}$ (and not of $\mathfrak{g}[1]$). In this way $S(\mathfrak{g}[1])$ becomes a Gerstenhaber coalgebra.

A co-strict homotopy Lie bialgebra (CSHLB for short) structure on $\mathfrak{g}$ is a coderivation $Q$ on $S(\mathfrak{g}[1])$ of degree one and square zero such that

$$\bar{\delta} \circ Q + (Q \otimes \text{id} + \text{id} \otimes Q) \circ \bar{\delta} = 0$$

and $Q(1) = 0$. Thus a CSHLB-structure is an $L_\infty$-structure which satisfies a suitable compatibility with respect to the cobracket. A CSHLB-morphism $\mathfrak{g} \to \mathfrak{h}$ is an $L_\infty$-morphism commuting with $\bar{\delta}$.

Example 4.1. If $\mathfrak{g}$ is a DG-Lie bialgebra then its induced $L_\infty$-structure makes it into a CSHLB. A similar statement is true for DG-Lie bialgebra morphisms.

4.2. Cofree Lie coalgebras. Let $V$ be a graded vector space. The shuffle product $m_s$ makes $T^c(V)$ into a bialgebra. The cofree Lie coalgebra cogenerated by $V$ is defined by

$$(4.2) \quad L^c(V) = \ker \epsilon / m_s(\ker \epsilon, \ker \epsilon)$$

with cobracket

$$\delta = \Delta - \Delta^o$$

It is convenient to denote the image of $v_1 \otimes \cdots \otimes v_n$ in $L^c_n(V)$ by $v_1 \otimes \cdots \otimes v_n$.

Remark 4.2.1. The formula (4.2) is dual to the realization of the free Lie algebra $L(V)$ inside the free algebra $T(V)$ as the primitive elements for the coproduct given by $\Delta(v) = v \otimes 1 + 1 \otimes v$ (which is dual to the shuffle product).
4.3. $G_\infty$-algebras. A $G_\infty$-structure on $V$ is a CSHLB-structure on $L^c(V)$. Likewise a $G_\infty$-morphism $V \to W$ is a CSHLB-morphism $L^c(V) \to L^c(W)$. If this morphism is obtained by extending a morphism $V \to W$ of DG-vector spaces then we call it strict. For a detailed study of $G_\infty$-algebras see [16].

A $G_\infty$-structure is determined by maps

$$Q_{p_1,\ldots,p_n} : L^{c,p_1}(V)[1] \otimes \cdots \otimes L^{c,p_n}(V)[1] \to V[1]$$

of degree 1 where $Q^n$ is the sum of all $Q_{p_1,\ldots,p_n}$. We obtain corresponding maps of degree $2-n$

$$l^{p_1,\ldots,p_n} : L^{c,p_1}(V) \otimes \cdots \otimes L^{c,p_n}(V) \to V$$

using standard sign conventions

$$l^{p_1,\ldots,p_n}(\underbrace{1, \ldots, 1}_n) = (-1)^{n+(n-1)|\gamma_1|+(n-2)|\gamma_2|+\cdots+|\gamma_n-1|}Q_{p_1,\ldots,p_n}(\underbrace{1, \ldots, 1}_n).$$

Likewise a $G_\infty$-morphism $\psi : V \to W$ is determined by maps

$$\psi^{p_1,\ldots,p_n} : L^{c,p_1}(V)[1] \otimes \cdots \otimes L^{c,p_n}(V)[1] \to W[1]$$

of degree 0.

Recall that an $L_\infty$-structure on $V$ is given by a coderivation of degree one and square zero on $S(V[1])$. It is determined by maps

$$Q^n : S^n(V[1]) \to V[1]$$

of degree one, or likewise by maps

$$m^n : \bigwedge^l V \to V$$

of degree $2-l$. A $G_\infty$-algebra becomes an $L_\infty$-algebras by putting

$$m^n = l^{1,1,\ldots,1}$$

A $G_\infty$-structure leads in a natural way to one unary and two binary operations on $V$ given respectively by a differential $l^1$, a Lie bracket up to homotopy $l^{1,1}$ of degree zero and a "cupproduct" $l^2$ of degree one.

A computation shows that $(V,l^1,l^{1,1})$ satisfies the Gerstenhaber axioms up to homotopies in terms of the ternary operations $l^3$, $l^{2,1}$, $l^{1,1,1}$. In particular $H^*(V)$ is a Gerstenhaber algebra.

5. Operads and Tamarkin’s morphism

It is easy to see that $B_\infty$-algebras and $G_\infty$-algebras are DG-algebras over certain DG-operads which we denote respectively by $B_\infty$ and $G_\infty$. The operad $B_\infty$ is generated as a graded operad by the operations $m^{p,q} \in B_\infty(p+q)$ (of degree zero) and the operations $Q^n \in B_\infty(n)$, $n \geq 2$ (of degree one).

The operad $G_\infty$ is freely generated as a graded operad by the operations $l^{p_1,\ldots,p_n} \in G_\infty(p_1+\cdots+p_n)$, $(p_1,\ldots,p_n) \neq (1)$ (of degree $2-n$) and its homology is the graded operad of Gerstenhaber algebras, which we denote by $G$. Likewise in §3.1 we have indicated that a $B_\infty$-algebra is a Gerstenhaber algebra up to homotopy. This corresponds the existence of a morphism of operads $G \to H^*(B_\infty)$ preserving Lie bracket and cup product.

Tamarkin’s amazing discovery is the following.
**Theorem 5.1** ([24]). There exists a morphism of DG-operads $T : G_\infty \to B_\infty$ such that the following diagram is commutative

$$
\begin{array}{c}
H^*(G_\infty) \\
\downarrow \\
G
\end{array} \xrightarrow{T} \begin{array}{c}
H^*(B_\infty) \\
\downarrow \\
B_\infty
\end{array}
$$

We discuss some further properties of $T$. For more details we refer to [24].

**Lemma 5.2.** One has $T(l_{p_1,\ldots,p_n}) = 0$ for $n > 2$. Furthermore $T(l_{p_1,p_2})$ can be expressed in terms of operations $m_{p_1',p_2'}$.

**Proof.** This follows immediately by degree reasons. Indeed $|l_{p_1,\ldots,p_n}| < 0$ and $B_\infty$ contains no elements of strictly negative degree. Similarly $T(l_{p_1,p_2})$ has degree zero and hence it must be expressible in terms of the generators $m_{p_1',p_2'}$ since the $Q^n$ have strictly positive degree. □

We have mentioned in §4.3 that any $G_\infty$-algebra is an $L_\infty$-algebra. This corresponds to the morphism of operads $L_\infty \to G_\infty$ which sends $m_n$ to $l_{1,\ldots,1}$ with the 1's occurring $n$ times.

Let $L$ be the operad of Lie algebras. Then we also have the usual quasi-isomorphism of operads $L_\infty \to L$ which kills $l^n$ for $n > 2$.

Finally we have a morphism of DG-operads $L \to B_\infty$ which sends $m_2$ to $m_{1,1} - \sigma(m_{1,1})$ where $\sigma = (12)$.

**Lemma 5.3** ([24]). There is a commutative diagram of DG-operads

$$
\begin{array}{ccc}
G_\infty & \xrightarrow{T} & B_\infty \\
\uparrow & & \uparrow \\
L_\infty & \xrightarrow{L} & L
\end{array}
$$

**Proof.** The proof is easy so for the benefit of the reader we will give it here. Since $T(l_{1,1,\ldots,1}) = 0$ the diagram is commutative when evaluated on $l_{1,1,\ldots,1}$.

It remains to prove $T(l_{1,1}) = m_{1,1} - \sigma(m_{1,1})$. Now $T(l_{1,1})$ is an element of $B_\infty(2)$ which has degree zero and is anti-symmetric. Since $B_\infty(2)_0 = km_{1,1} + k\sigma(m_{1,1})$ we deduce $T(l_{1,1}) = \alpha(m_{1,1} - \sigma(m_{1,1}))$ for $\alpha \in k$. Descending to cohomology yields $\alpha = 1$. □

Let $\overline{G}_\infty$ be the quotient of $G_\infty$ by $l_{p_1,\ldots,p_n}$, $n > 2$. By lemma 5.2 we deduce that $T$ factors as follows

$$
\overline{T} : \overline{G}_\infty \to B_\infty
$$

An $\overline{G}_\infty$-structure on $V$ corresponds to a DG-Lie bialgebra structure $(L^c(V), \delta, [-,-], Q)$ on $L^c(V)$ where $\delta$ is the standard cobracket (cf. Example 4.1). So if $V$ is a $B_\infty$-algebra (e.g. a Hochschild complex) then the $G_\infty$-structure on $T_*(V)$ is actually quite special.

**Remark 5.4.** The theory of (de)quantization for Lie bialgebras [9, 10] shows that the operads $\overline{G}_\infty$ and $B_\infty$ are closely related [15]. However they are not isomorphic as

$$
\begin{align*}
\overline{G}_\infty(2)_0 &= kl_{1,1} \\
B_\infty(2)_0 &= km_{1,1} \oplus k\sigma(m_{1,1})
\end{align*}
$$
Thus \( \mathcal{G}_\infty(2)_0 \) is one dimensional and \( B_\infty(2)_0 \) is two dimensional.

We will need the following technical result which is a straightforward generalization of [18, Lemma 4.7].

**Proposition 5.5.** Assume that \( V \) is a \( \mathcal{G}_\infty \)-algebra. In particular \((V, \mathcal{L}^{1,1})\) is a graded Lie algebra. Assume that \( U \) is a perfect Lie subalgebra of \( V \) (i.e. \( \mathcal{L}^{1,1}(\wedge^2 U) = U \)) such that for every \( u \in U, \mathcal{L}^{1,1}(u, -) \) acts as a derivation with respect to the operations \( \mathcal{L}^{1,q} \). Then \( \mathcal{L}^{1,q}(u, -) = 0 \) for all \( u \in U \) and \( q > 1 \).

**Proof.** In the computations below the \( u \)'s are elements of \( U \) and the \( v \)'s are elements of \( V \). Let \((L^c(V), \delta, [-, -])\) be the Lie bialgebra structure on \( L^c(V) \) corresponding to the \( \mathcal{G}_\infty \)-structure.

Let \( \mathcal{L}^{1,1}(u, -) \) act on \( L^c(V) \) by extending the action on \( V \) using the obvious Leibniz rule:

\[
\mathcal{L}^{1,1}(u, v_1 \otimes \cdots \otimes v_q) = \sum_i (-1)^{|v_i|+\cdots+|v_{i-1}|}|u| v_1 \otimes \cdots \otimes \mathcal{L}^{1,1}(u) \otimes \cdots \otimes v_q
\]

Assume that we have shown that \( \mathcal{L}^{1,i}(u, -) = 0 \) for \( 1 < i < q \) for \( q \geq 2 \) (for \( q = 2 \) there is nothing to show). We will prove that \( \mathcal{L}^{1,q}(u, -) = 0 \). A trite computation using the induction hypotheses shows for \( \gamma \in L^{q,c}(V) \)

\[
[u, \gamma] = \mathcal{L}^{1,1}(u, \gamma) + \mathcal{L}^{1,q}(u, \gamma).
\]

Therefore, writing out the Jacobi identity \([u_1, [u_2, \gamma]] - (-1)^{|u_1||u_2|}[[u_2, u_1], \gamma] - [[u_1, u_2], \gamma] = 0\) yields

\[
\begin{align*}
\mathcal{L}^{1,1}(u_1, \mathcal{L}^{1,1}(u_2, \gamma)) + \mathcal{L}^{1,q}(u_1, \mathcal{L}^{1,1}(u_2, \gamma)) &+ \mathcal{L}^{1,1}(u_1, \mathcal{L}^{1,q}(u_2, \gamma)) \\
- (-1)^{|u_1||u_2|} \mathcal{L}^{1,1}(u_2, \mathcal{L}^{1,1}(u_1, \gamma)) - (-1)^{|u_1||u_2|} \mathcal{L}^{1,q}(u_2, \mathcal{L}^{1,1}(u_1, \gamma)) &- (-1)^{|u_1||u_2|} \mathcal{L}^{1,1}(u_2, \mathcal{L}^{1,q}(u_1, \gamma)) \\
- (-1)^{|u_1||u_2|} \mathcal{L}^{1,q}(u_2, \mathcal{L}^{1,1}(u_1, \gamma)) - \mathcal{L}^{1,q}(\mathcal{L}^{1,1}(u_1, u_2), \gamma) &- \mathcal{L}^{1,q}(\mathcal{L}^{1,q}(u_1, u_2), \gamma) = 0.
\end{align*}
\]

After projecting onto \( V \) we obtain

\[
(5.1) \quad \mathcal{L}^{1,q}(u_1, \mathcal{L}^{1,1}(u_2, \gamma)) + \mathcal{L}^{1,1}(u_1, \mathcal{L}^{1,q}(u_2, \gamma)) - (-1)^{|u_1||u_2|} \mathcal{L}^{1,q}(u_2, \mathcal{L}^{1,1}(u_1, \gamma)) \\
- (-1)^{|u_1||u_2|} \mathcal{L}^{1,1}(u_2, \mathcal{L}^{1,q}(u_1, \gamma)) - \mathcal{L}^{1,q}(\mathcal{L}^{1,1}(u_1, u_2), \gamma) = 0.
\]

According to our hypotheses \( \mathcal{L}^{1,1}(u, -) \) is a derivation with respect to the operation \( \mathcal{L}^{1,q} \). Thus

\[
\mathcal{L}^{1,1}(u_1, \mathcal{L}^{1,q}(u_2, \gamma)) = \mathcal{L}^{1,q}(\mathcal{L}^{1,1}(u_1, u_2), \gamma) + (-1)^{|u_1||u_2|} \mathcal{L}^{1,q}(u_2, \mathcal{L}^{1,1}(u_1, \gamma))
\]

and thus we get \( \mathcal{L}^{1,q}(u_1, \mathcal{L}^{1,1}(u_2, \gamma)) = (-1)^{|u_1||u_2|} \mathcal{L}^{1,q}(u_2, \mathcal{L}^{1,1}(u_1, \gamma)) \). Exchanging \( u_1 \) and \( u_2 \) we also have \( \mathcal{L}^{1,q}(u_2, \mathcal{L}^{1,1}(u_1, \gamma)) = (-1)^{|u_1||u_2|} \mathcal{L}^{1,q}(u_1, \mathcal{L}^{1,1}(u_2, \gamma)) \). Then substituting back in (5.1) we get

\[
\mathcal{L}^{1,q}(\mathcal{L}^{1,1}(u_1, u_2), \gamma) = 0.
\]

Finally, we use the fact that \( U \) is perfect to obtain \( \mathcal{L}^{1,q}(U, -) = 0 \). \( \square \)

6. **Twisting**

6.1. **Twisting for \( B_\infty \)-algebras.** Assume that \( V \) is a \( B_\infty \)-algebra. I.e. we have a DG-bialgebra \((T^c(V), \Delta, \epsilon, m, 1, Q)\) with the usual coalgebra structure. Assume that \( \omega \in V_1 \) is a solution of the following Maurer-Cartan equation

\[
Q(\omega) + m(\omega, \omega) = Q^1(\omega) + m^{1,1}(\omega, \omega) = 0
\]
then

\[(T^c(V), \Delta, \varepsilon, m, 1, Q_\omega)\]

with \(Q_\omega(\gamma) = Q + m(\omega, \gamma) - (-1)^{|\gamma|}m(\gamma, \omega)\) defines a new DG-bialgebra. We denote the resulting \(B_\infty\)-structure on \(V\) by \(V_\omega^\prime\). Thus explicitly the new \(B_\infty\)-algebra structure is given by

\[
m_{p,q}^{\omega} = m_{p,q}^{\omega} \delta^p \gamma + m_{1,p}^{\omega} \omega \gamma - (-1)^{|\gamma|}m_{p,1}^{\omega} \gamma \omega
\]

(6.1)

6.2. Twisting for \(G_\infty\)-algebras. Here we will work with infinite series. We assume that our vector spaces are equipped with suitable topologies, that the series we use converge, and furthermore that standard series manipulations are allowed. All these hypotheses hold in our applications. See [3, §4, §6.2] for a more precise description of a setting in which these hypotheses hold.

Assume that \((g, \delta, Q)\) is a CSHLB. Let \(\omega \in g_1\) be a solution of the \(L_\infty\)-Maurer-Cartan equation in \(g\),

\[
\sum \frac{1}{j!}Q^n(\omega^j) = 0
\]

and define \(Q_\omega\) as in [27]. I.e.

\[
Q_\omega^i(\gamma) = \sum_{j \geq 0} \frac{1}{j!}Q^{i+j}(\omega^j \gamma) \quad \text{(for } i > 0)\]

(6.2)

Proposition 6.2.1. If \(\delta(\omega) = 0\), then \(g_\omega := (g, \delta, Q_\omega)\) is a CSHLB.

Proof. It follows from [27] that \((g, Q)\) is an \(L_\infty\)-algebra. So we only have to check that \(\delta \circ Q_\omega + (Q_\omega \otimes \text{id} + \text{id} \otimes Q_\omega) \circ \delta = 0\).

We follow the method of [27]. Let \(S(g[1])^\wedge\) be the completion of \(S(g[1])\) at the ideal generated by \(g[1]\). This is a topological Hopf algebra and the part cogenerated by \(g[1]\) is \(S(g[1])\). One has \(e^\omega \in S(g[1])^\wedge\) and furthermore [27] \(e^\omega\) is group like. I.e.

\[\Delta(e^\omega) = e^\omega \otimes e^\omega\]

(6.3)

In particular multiplication by \(e^\omega\) defines a coalgebra automorphism of \(S(g[1])^\wedge\).

According to [27] one has \(Q_\omega(\gamma) = e^{-\omega}Q(e^\omega \gamma)\). It easily follows that one has to prove

\[\bar{\delta}(e^{\omega} \gamma) = (e^{\omega} \otimes e^{\omega})\bar{\delta}(\gamma)\]

(6.4)

According to the explicit formula (4.1) we have

\[\bar{\delta}(\omega^\gamma) = \Delta(\omega^\gamma)\bar{\delta}(\gamma)\]

(using \(\delta(\omega) = 0\) and hence

\[\bar{\delta}(e^{\omega} \gamma) = \Delta(e^{\omega})\bar{\delta}(\gamma)\]

Invoking (6.3) finishes the proof. \(\Box\)

Assume now that \(\psi : (g, \delta, Q) \to (h, \delta', Q')\) is a CSHLB-morphism and define \(\omega'\) and \(\psi_\omega\) as in [27],

\[
\psi_\omega^i(\gamma) = \sum_{j \geq 0} \frac{1}{j!} \psi^{i+j}(\omega^j \gamma) \quad \text{(for } i > 0)\]

\[
\omega' = \sum_{j \geq 1} \frac{1}{j!} \psi^j(\omega^j)
\]
By [27] \( \omega' \) is a solution to the \( L_\infty \)-Maurer Cartan equation in \( g_\omega \).

**Proposition 6.2.2.** Assume \( \delta(\omega) = 0 \). Then one has \( \delta'(\omega') = 0 \) and \( \psi_\omega \) defines a CSHLB-morphism from \( g_\omega \) to \( b_\omega' \).

**Proof.** By [27] one knows that \( \psi_\omega \) is an \( L_\infty \)-morphism. We prove \( \delta'(\omega') = 0 \). From (4.1) we deduce \( \bar{\delta}(e^\omega) = 0 \). Applying \( \psi \) and using \( e^{\omega'} = \psi(e^\omega) \) (see [27]) we deduce \( \bar{\delta}'(e^{\omega'}) = 0 \). Using (4.1) again we find \( \bar{\delta}'(e^{\omega'}) = \Delta(e^{\omega'}) \delta'(\omega') \). Since \( \Delta(e^{\omega'}) = e^{\omega'} \otimes e^{\omega'} \) is invertible we conclude \( \bar{\delta}'(\omega') = 0 \).

To show that \( \bar{\delta}' \circ \psi_\omega = (\psi_\omega \otimes \psi_\omega) \circ \delta \) one uses \( \psi_\omega(\bar{\gamma}) = e^{-\omega} e^{\omega} \gamma \) [27] together with (6.4) (and the corresponding equation for \( \bar{\delta}' \)). \( \square \)

Assume now that \( (S(L^c(V)[1]), Q) \) is a \( G_\infty \)-structure on \( V \) and \( \omega \in V_1 \) is a solution to the \( L_\infty \)-Maurer-Cartan equation

\[
\sum_{i!} \frac{1}{i!} Q^{1, \ldots, 1}_{(i)}(\omega^i) = 0
\]

Since the standard cobracket on \( L^c(V) \) is zero on \( V \) we obtain by Proposition 6.2.1 a new \( G_\infty \)-structure on \( V \) given by \( (S(L^c(V)[1]), Q_\omega) \). We denote this new \( G_\infty \)-structure by \( V_\omega \). Using (6.2) we deduce

\[
Q^p_{\omega^p: \ldots: \omega^p}(\gamma_1, \ldots, \gamma_p) = \sum_{j \geq 0} \frac{1}{j!} Q^{1, \ldots, 1}_{j!(p+1)}(\omega)^j(\gamma_1, \ldots, \gamma_p) \quad \text{for } i > 0
\]

Similarly if \( \psi : V \to W \) is an \( L_\infty \)-morphism and \( \omega \in V_1 \) is a solution to the \( L_\infty \)-Maurer-Cartan equation then we obtain a twisted \( L_\infty \)-morphism \( \psi_\omega : V_\omega \to W_\omega \) where \( \omega' \) and \( \psi_\omega \) are given by the formulas

\[
\psi_{\omega^p: \ldots: \omega^p}^p(\gamma_1, \ldots, \gamma_p) = \sum_{j \geq 0} \frac{1}{j!} Q^{1, \ldots, 1}_{j!(p+1)}(\omega)^j(\gamma_1, \ldots, \gamma_p) \quad \text{for } i > 0
\]

\[
\omega' = \sum_{j \geq 1} \frac{1}{j!} \omega^j
\]

7. **Descent for \( G_\infty \)-algebras and morphisms**

7.1. **Descent for algebras over DG-operads.** Let \( O \) be a DG-operad and let \( \bar{O} \) be its underlying graded operad. If \( V \) is an algebra over \( O \) then an \( S \)-action on \( V \) is a family \( (\iota_s)_{s \in S} \) of \( \bar{O} \)-derivations of degree \(-1\) of \( V \). Put \( L_s = d\iota_s + \iota_s d \) and define

\[
V^S = \{ v \in V \mid \forall s \in S : \iota_s(v) = L_s(v) = 0 \}
\]

A straightforward computation shows that \( V \) is an algebra over \( O \).

7.2. **Descent for \( G_\infty \)-morphisms.** Let \( V, W \) be two \( G_\infty \)-algebras and let \( \psi : V \to W \) be a \( G_\infty \)-morphism. Assume that we are given \( S \)-actions \( (\iota_s)_{s \in S} \) on \( V \) and \( W \). We say that \( \psi \) commutes with these actions if \( \iota_s \) acts as a derivation with respect to \( \psi^{p_1: \ldots: p_n} \) for all \( s \in S \). This is clearly equivalent to

\[
[i_s, \psi] = 0
\]
where the $\bar{i}_s$ are the coderivations commuting with $\bar{\delta}$ on $S(L^c(V)[1])$ and $S(L^c(W)[1])$ obtained by extending $i_s$.

**Lemma 7.2.1.** Assume that $(i_s)_{s \in S}$ commute with $\psi$. Then $\psi$ restricts to a $G_\infty$-morphism $V^S \rightarrow W^S$.

**Proof.** It is sufficient to prove that $\bar{L}_s$ commutes with all $\psi^{p_1,\ldots,p_n}$ or equivalently that

$$[\bar{L}_s, \psi] = 0 \quad (7.2)$$

where $\bar{L}_s$ is the extension of $L_s$ to a coderivation on $S(L^c(V)[1])$ and $S(L^c(W)[1])$ commuting with $\bar{\delta}$. Since $i_s$ commutes with all $Q^{p_1,\ldots,p_n}$ (except perhaps with $Q^1$) we obtain

$$\bar{L}_s = [Q, \bar{i}_s] \quad (7.3)$$

Thus (7.2) follows from (7.3) and (7.1). $\square$

**7.3. Compatibility of descent with twisting of $G_\infty$-algebras.** Here we make the same blanket hypotheses on series manipulations as in §6.2. Let $V$ be a $G_\infty$-algebra, let $\omega \in V$ be a solution of the $L_\infty$-Maurer-Cartan equation (6.5) and define $Q_\omega$ as in (6.6).

**Lemma 7.3.1.** Assume that $(i_s)_{s \in S}$ is an $S$-action on $V$. Assume in addition that for all $(p_1,\ldots,p_n) \neq ()$ we have $Q^{1,\ldots,p_n}(i_s \omega)\gamma_1 \cdots \gamma_n = 0$. Then $(i_s)_{s \in S}$ is an $S$-action on $V_\omega$.

**Proof.** If we compute $i_s(Q^{1,\ldots,p_n}(\gamma_1 \cdots \gamma_n))$ then we see that $i_s$ behaves itself as a derivation with respect to $Q^{1,\ldots,p_n}$ except for extra terms of the form

$$Q^{1,\ldots,1,p_1,\ldots,p_n}(\omega \cdots (i_s \omega) \cdots \omega \gamma_1 \cdots \gamma_n).$$

These are zero by hypotheses. $\square$

**7.4. Compatibility of descent with twisting of $G_\infty$-morphisms.** Let $\psi: V \rightarrow W$ be a morphism of $G_\infty$-algebras. Let $\omega \in V$ be a solution of the $L_\infty$ Maurer-Cartan equation (6.5). Define $\psi_\omega$, $\omega'$ as in (6.7)(6.8).

**Lemma 7.4.1.** Let $(i_s)_{s \in S}$ act on $V$ and $W$ and assume that $\psi$ commutes with it. Assume in addition that for all $(p_1,\ldots,p_n)$ we have $\psi^{1,\ldots,p_n}(i_s \omega)\gamma_1 \cdots \gamma_n = 0$. Then $(i_s)_{s \in S}$ commutes with the twisted $G_\infty$-map $\psi_\omega: V_\omega \rightarrow W_{\omega'}$.

**Proof.** If we compute $i_s(\psi^{1,\ldots,p_n}(\gamma_1 \cdots \gamma_n))$ then we see that $i_s$ behaves itself as a derivation with respect to $\psi^{1,\ldots,p_n}$ except for extra terms of the form

$$\psi^{1,\ldots,1,p_1,\ldots,p_n}(\omega \cdots (i_s \omega) \cdots \omega \gamma_1 \cdots \gamma_n).$$

These are zero by hypotheses. $\square$
8. Poly-vector fields and poly-differential operators

We briefly recall some notations from [3]. For more details the reader is referred to loc. cit.

Let \((\mathcal{C}, \mathcal{O})\) be a ringed site and let \(\mathcal{L}\) be a locally free Lie algebroid of rank \(d\). The enveloping algebra of \(\mathcal{L}\) is denoted by \(U\mathcal{L}\). Its right \(\mathcal{O}\)-module structure is defined to be the same as its left structure. As in [3] \(T^*_\text{poly}(\mathcal{O})\) is the Lie algebra of \(\mathcal{L}\)-poly-vector fields on \((\mathcal{C}, \mathcal{O})\) [2]. I.e. it is the graded vector space \(\wedge \mathcal{L}[1]\) equipped with the graded Lie bracket obtained by extending the Lie bracket on \(\mathcal{L}\). We equip \(T^*_\text{poly}(\mathcal{O})\) with the standard cupproduct (which is of degree one with our shifted grading). In this way \(T^*_\text{poly}(\mathcal{O})\) becomes a sheaf of Gerstenhaber algebras.

Similarly \(D^*_\text{poly}(\mathcal{O})\) is the DG-Lie algebra of \(\mathcal{L}\)-poly-differential operators on \((\mathcal{C}, \mathcal{O})\) [2]. I.e. it is the graded sheaf \(T_\mathcal{O}(U\mathcal{L})[1]\) equipped with a structure of an inner brace algebra

\[
D\{E_1, \ldots, E_n\} = \sum_{i_1 + \cdots + i_n = |P| - n + 1} (-1)^r (\text{id}^\otimes i_1 \otimes \Delta^{|E_1|} \otimes \cdots \otimes \Delta^{|E_n|} \otimes \text{id}^\otimes i_n)(D),
\]

where \(\epsilon = \sum_s (|Q_s| - 1) i_s\). The biderivation \(Q\) on \(T_k(D^*_\text{poly}(\mathcal{O}))\) is given by \([\mu, -]\) where \(\mu = 1 \otimes 1 \in D^*_\text{poly}(\mathcal{O})_1\). One checks that \(Q^n = 0\) for \(n > 2\). So the only non-vanishing \(B_\infty\)-operations are the braces and the cupproduct \(Q^2\). The latter is the ordinary product on \(D^*_\text{poly}(\mathcal{O}) = T_\mathcal{O}(U\mathcal{L})\) (up to a sign).

If \(\mathcal{L}\) is omitted from the notation we assume \(\mathcal{L} = D^r_{\text{tr}}(\mathcal{O}, \mathcal{O})\). In that case \(T^*_\text{poly}(\mathcal{O})\) and \(D^*_\text{poly}(\mathcal{O})\) are the ordinary sheaves of poly-vector fields and poly-differential operators.

We also consider relative variants \(T^\Lambda_{\text{poly}}(\mathcal{B})\), \(D^\Lambda_{\text{poly}}(\mathcal{B})\) of these notations where \(\mathcal{A}, \mathcal{B}\) are sheaves of commutative DG-algebras (and \(\mathcal{B}\) is a DG-\(\mathcal{A}\)-algebra). These will be self explanatory. In this case \(Q\) is given by \(d + [\mu, -]\) where \(d\) is the differential on \(D^\Lambda_{\text{poly}}(\mathcal{B})\) obtained from the DG-structure on \(\mathcal{A}, \mathcal{B}\). It is still true that the braces and the cupproduct are the only non-vanishing \(B_\infty\)-operations on \(D^\Lambda_{\text{poly}}(\mathcal{B})\). Furthermore the differential on \(D^\Lambda_{\text{poly}}(\mathcal{B})\) is of the form \(d_{\text{tot}} = d + d_{\text{Hoch}}\) where \(d_{\text{Hoch}} = [\mu, -]\).

9. Tamarkin’s local formality morphism

Let \(F = k[[t_1, \ldots, t_d]]\). In [24] Tamarkin proved the existence of \(G_\infty\)-quasi-isomorphism

\[
\Psi : T^*_\text{poly}(F) \to T_s(D^*_\text{poly}(F))
\]
such that \(\Psi^1\) is given by the HKR formula.

Moreover, one can construct this quasi-isomorphism in such a way that it has the following properties (see [18, Theorem 4.5] or Theorem A.1.1 below):

\[
\begin{align*}
(P4) \ &\Psi^1_{t_1, \ldots, t_d}(\gamma_1, \ldots, \gamma_n) = 0 \text{ for } \gamma_i \in T^*_\text{poly}(F)_0 \text{ and } n \geq 2, \\
(P5) \ &\Psi^1_{t_1, t_2, \ldots, t_n}(\gamma_1, \ldots, \gamma_n) = 0 \text{ for } n \geq 2, \gamma \in \mathfrak{gl}_d(k) \subset T^*_\text{poly}(F)_0 \text{ and } \omega_d \in L^{\otimes d}T^*_\text{poly}(F).
\end{align*}
\]

\(^2\text{Note that the case } n > 2 \text{ is automatic for degree reasons.}\)
10. Proof of Theorem 1.1

Our proof parallels the proof of the corresponding $L_\infty$-result in [3]. We let $J\mathcal{L}$ be the sheaf of $\mathcal{L}$-jetbundles on $(\mathcal{C}, \mathcal{O})$. Thus

$$J\mathcal{L} = \text{proj lim} \text{Hom}_{\mathcal{O}}((U\mathcal{L})_{\leq n}, \mathcal{O})$$

We show in [3] that $J\mathcal{L}$ is in a natural way a $U\mathcal{L} \otimes_k U\mathcal{L}$ module. Hence we have two commuting actions of $\mathcal{O}$ and $\mathcal{L}$ on $J\mathcal{L}$, depending on whether we embed them in the first or second copy of $U\mathcal{L}$. As in [3] we denote these two actions by $(\mathcal{O}_1, \mathcal{L}_1)$ and $(\mathcal{O}_2, \mathcal{L}_2)$. We may view these as flat connections on $J\mathcal{L}$ and in particular we obtain that $\wedge^0 \otimes_{\mathcal{O}_1} J\mathcal{L}$ is in a natural way a $\wedge^0 \mathcal{L}_1$-DG-algebra (the latter being a natural analogue of the De Rham complex).

As above let $F = k[[t_1,\ldots,t_d]]$. Then there exists a natural sheaf of commutative DG-$\wedge^0 \mathcal{L}_1$-algebras $C^{\text{coord}, \mathcal{L}}$ (see [3, §5.2]) such that

$$C^{\text{coord}, \mathcal{L}} \otimes_{\mathcal{O}, \mathcal{L}_1} J\mathcal{L}, d) = (C^{\text{coord}, \mathcal{L}} \otimes_{\mathcal{O}_1} J\mathcal{L}, d) \cong (C^{\text{coord}, \mathcal{L}} \otimes_k F, d \otimes 1 + \omega)$$

where we have denoted the natural differentials by "$d$" and where $\omega$ is a solution of the Maurer-Cartan equation in the DG-Lie algebra

$$C^{\text{coord}, \mathcal{L}} \otimes_k \text{Der}_k(F, F) = C^{\text{coord}, \mathcal{L}} \otimes_k \text{T}_{\text{poly}}(F)_0 \subset C^{\text{coord}, \mathcal{L}} \otimes_k \text{D}_{\text{poly}}(F)_0$$

In [3] we also considered a certain sub-DG $\wedge^0 \mathcal{L}_1$-algebra $C^{\text{aff}, \mathcal{L}}$ of $C^{\text{coord}, \mathcal{L}}$ which can for example be obtained by descent (see §7.1). More precisely for each $v \in \mathfrak{gl}_d(k)$ there exists a $\wedge^0 \mathcal{L}_1$-linear derivation $i_v$ on $C^{\text{coord}}$ (as a graded sheaf of algebras) of degree $-1$ such that

$$C^{\text{aff}} = (C^{\text{coord}}) \otimes_k \mathfrak{gl}_d(k)$$

We now construct some strict $B_\infty$-morphisms (see [3])

$$D^2_{\text{poly}}(\mathcal{O}, \mathcal{L}) \overset{\alpha}{\rightarrow} D_{\text{poly}, \text{aff}, \mathcal{L}}(C^{\text{aff}, \mathcal{L}} \otimes_{\mathcal{O}_1} J\mathcal{L}) \overset{\beta}{\rightarrow} D_{\text{poly}, \text{coord}, \mathcal{L}}(C^{\text{coord}, \mathcal{L}} \otimes_{\mathcal{O}_1} J\mathcal{L}) \overset{\gamma}{\rightarrow} (C^{\text{coord}, \mathcal{L}} \otimes_k \text{D}_{\text{poly}}(F))_\omega$$

The map $\alpha$ is constructed by letting $\mathcal{O}_1, \mathcal{L}_2$ act on $J\mathcal{L}$. This is possible since these actions commute with the $\mathcal{O}_1, \mathcal{L}_1$-actions (which were the only ones we used so far). It has been shown in [3] that $\alpha$ is a quasi-isomorphism.

The map $\beta$ is obtained by extending scalars. It remains to discuss the map $\gamma$. Using (10.1) we obtain an isomorphism

$$D_{\text{poly}, \text{coord}, \mathcal{L}}(C^{\text{coord}, \mathcal{L}} \otimes_{\mathcal{O}_1} J\mathcal{L}) \cong D_{\text{poly}, \text{coord}, \mathcal{L}}(C^{\text{coord}, \mathcal{L}} \otimes F)$$

which commutes with cupproduct and braces and hence it is an isomorphism as sheaves of $B_\infty$ algebras ($B_\infty$ is the underlying graded operad for $B_\infty$, see §7.1). The Hochschild differential on the left is sent to the Hochschild differential on the right. The natural differential $[d, -]$ on the left is sent to $[d + \omega, -]$ on the right.

Thus we get as sheaves of $B_\infty$-algebras,

$$D_{\text{poly}, \text{coord}, \mathcal{L}}(C^{\text{coord}, \mathcal{L}} \otimes_{\mathcal{O}_1} J\mathcal{L}) \cong (D_{\text{poly}, \text{coord}, \mathcal{L}}(C^{\text{coord}, \mathcal{L}} \otimes F), d_{\text{tot}} + [\omega, -])$$

Since $\omega \in C^{\text{coord}, \mathcal{L}} \otimes_k \text{D}_{\text{poly}}(F)_0$ we have for $q > 1$

$$m^{1,q}(\omega, -) = 0$$

$$m^{q,1}(-, \omega) = 0.$$
A simple computation using (6.1) and (10.4) yields as $B_\infty$-algebras
\[ D_{\text{poly},C^{\text{coord},L}}(C^{\text{coord},L} \otimes F)_\omega = (D_{\text{poly},C^{\text{coord},L}}(C^{\text{coord},L} \otimes F), d_{\text{tot}} + [\omega, -]) \]
and combining this with (10.3) we obtain the strict $B_\infty$-isomorphism $\gamma$.

We now apply the functor $T_*$. We get strict maps of $G_\infty$-algebras:
\[ T_* (D_{\text{poly}}^\mathbb{C}(\mathcal{O}_2)) \xrightarrow{\alpha} T_* (D_{\text{poly},C^{\text{aff},L}}(C^{\text{aff},L} \otimes \mathcal{O}_1, J\mathcal{L})) \xrightarrow{\beta} T_* (D_{\text{poly},C^{\text{coord},L}}(C^{\text{coord},L} \otimes \mathcal{O}_1, J\mathcal{L})) \]
\[ \xrightarrow{\gamma} T_* ((C^{\text{coord},L} \otimes D_{\text{poly}}(F))_\omega) = T_* (\text{poly}^{\mathbb{C}}(\text{coord},L) \otimes D_{\text{poly}}(F))_\omega \cong (C^{\text{coord},L} \otimes T_*(D_{\text{poly}}(F))_\omega \]
The equality is an instance of the compatibility of $T_*$ with twisting. It seems this is in general a subtle issue on which we will come back in a future paper. However in this special case we can use an ad hoc argument.

**Lemma 10.1.** We have
\[ T_* ((C^{\text{coord},L} \otimes D_{\text{poly}}(F))_\omega) = T_* (C^{\text{coord},L} \otimes D_{\text{poly}}(F), d_{\text{tot}} + [\omega, -]) = (T_*(C^{\text{coord},L} \otimes D_{\text{poly}}(F), d_{\text{tot}} + [\omega, -]) = (T_*(C^{\text{coord},L} \otimes D_{\text{poly}}(F), d_{\text{tot}})_\omega = T_* (C^{\text{coord},L} \otimes D_{\text{poly}}(F))_\omega \]

**Proof.** The first equality has already been established. The second and fourth equalities are tautologies. Hence it remains to establish the third equality.

From (10.4) we deduce that $[\omega, -] = m^{1,1}(\omega, -) - m^{1,1}(-, \omega)$ behaves as a derivation with respect to the operations $m^{p,q}$. According to Lemma 5.2 the operations $l^{p_1,p_2}$ can be expressed in terms of the $m^{p,q}$. Hence $[\omega, -]$ acts as a derivation with respect to the operations $l^{p_1,p_2}$. It is easy to see that the Lie algebra $D_{\text{poly}}(F)_0$ is perfect and hence the same holds for $C^{\text{coord},L} \otimes_k D_{\text{poly}}(F)_0$ which contains $\omega$. We deduce from Proposition 5.5 that $l^{1,q}(\omega, -) = 0$ for $q > 1$. By Lemma 5.2 we also know that $l^{p_1,\ldots,p_n} = 0$ for $n \geq 3$. Substituting all this information in (6.6) we deduce
\[ Q^{p_1}(\gamma) = Q^{p_1}(\gamma) \quad \text{(if } p_1 > 1) \]
\[ Q^{1}(\gamma) = Q^{1}(\gamma) + Q^{1,1}(\omega, \gamma) \]
\[ Q^{p_1,\ldots,p_n}(\gamma_1 \cdots \gamma_n) = 0 \quad \text{(if } n \geq 2) \]
Translating this back in terms of the $l$’s we see that twisting by $\omega$ does nothing except adding $l^{1,1}(\omega, -) = [\omega, -]$ to the underlying differential. This is precisely the content of the third equality in the statement of the lemma. 

We will now construct similar morphisms of sheaves of DG-Gerstenhaber algebras
\[ T_\mathbb{C}(\mathcal{O}_2) \xrightarrow{\alpha'} T_{\text{poly},C^{\text{aff},L}}(C^{\text{aff},L} \otimes \mathcal{O}_1, J\mathcal{L}) \xrightarrow{\beta'} T_{\text{poly},C^{\text{coord},L}}(C^{\text{coord},L} \otimes \mathcal{O}_1, J\mathcal{L}) \xrightarrow{\gamma'} (C^{\text{coord},L} \otimes kT_{\text{poly}}(F))_\omega \]
Again the map $\alpha'$ is constructed by letting $\mathcal{O}_2, \mathcal{L}_2$ act on $J\mathcal{L}$ and the map $\beta'$ is obtained by extending scalars. It remains to discuss the map $\gamma'$. Using (10.1) we obtain an isomorphism
\[ T_{\text{poly},C^{\text{coord},L}}(C^{\text{coord},L} \otimes \mathcal{O}_1, J\mathcal{L}) \cong T_{\text{poly},C^{\text{coord},L}}(C^{\text{coord},L} \otimes F) \]
which commutes with the cupproduct and the Lie bracket and hence is an isomorphism as sheaves of $G$-algebras. The natural differential $[d, -]$ on the left is sent to $[d + \omega, -]$ on the right. Thus we get as sheaves of $G$-algebras

$$T_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes \Omega_1 J L) \cong (T_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes F), d + [\omega, -])$$

Since $T_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes F)$ is a Gerstenhaber algebra the only operations that are non-zero are $l_i^1, l_i^2$. From the formula (6.6) we deduce that the only effect of twisting by $\omega$ is changing the differential into $d + [\omega, -]$. Thus we obtain

$$T_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes F)_\omega = (T_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes F), d + [\omega, -])$$

Combining this with (10.5) we obtain $\gamma'$.

We will now construct a commutative diagram of $G_\infty$-morphism

$$
\begin{array}{c}
\begin{array}{ccc}
T_{\text{poly}}(\Omega_2) & \xrightarrow{\alpha'} & T_{\text{poly}, C^\text{aff}, L}(C^\text{aff}, L \otimes \Omega_1 J L) & \xrightarrow{\beta'} & T_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes \Omega_1 J L) & \xrightarrow{\gamma'} & (C^\text{coord}, L \otimes T_{\text{poly}}(F))_\omega \\
\phi^\text{aff} & & \phi^\text{coord} & & \phi^\text{coord} & & (id \otimes \psi)_\omega
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
T_*(D_{\text{poly}}(\Omega_2)) & \xrightarrow{\alpha} & T_*(D_{\text{poly}, C^\text{aff}, L}(C^\text{aff}, L \otimes \Omega_1 J L)) & \xrightarrow{\beta} & T_*(D_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes \Omega_1 J L)) & \xrightarrow{\gamma} & (C^\text{coord}, L \otimes T_*(D_{\text{poly}}(F)))_\omega
\end{array}
\end{array}
\end{array}
$$

in which the horizontal arrows are strict. $\phi^\text{coord}$ is defined by

$$\phi^\text{coord} = \gamma^{-1} \circ (id \otimes \psi)_\omega \circ \gamma'$$

and $\phi^\text{aff}$ is derived from $\phi^\text{coord}$ using descent. As graded sheaves we have

$$T_*(D_{\text{poly}, C^\text{aff}, L}(C^\text{aff}, L \otimes \Omega_1 J L)) = C^\text{aff}, L \otimes \Omega_1 D_{\text{poly}, \Omega_1}(J L)$$

$$T_*(D_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes \Omega_1 J L)) = C^\text{coord}, L \otimes \Omega_1 D_{\text{poly}, \Omega_1}(J L)$$

As indicated above there is a $\mathfrak{gl}(k)_v$ action $C^\text{coord}, L$. If $v \in \mathfrak{gl}(k)$ acts by $i_v$ then we obtain a $\mathfrak{gl}(k)$-action on $D_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes \Omega_1 J L)$ as sheaf of $B_\infty$-algebras by letting $\omega$ act as $[i_v, -]$. Under the isomorphism (10.8) this action can also be viewed as the linear extension of the action on $C^\text{coord}, L$. Since the operations of $G_\infty$ can be expressed in those of $B_\infty$ via $T$ it is clear that we obtain a $\mathfrak{gl}(k)$-action on $T_*(D_{\text{poly}, C^\text{aff}, L}(C^\text{aff}, L \otimes \Omega_1 J L))$ as $G_\infty$-algebra. So the formalism of §7 applies.

Combining (10.8) with (10.2) we obtain

$$D_{\text{poly}, C^\text{aff}, L}(C^\text{aff}, L \otimes \Omega_1 J L) = D_{\text{poly}, C^\text{coord}, L}(C^\text{coord}, L \otimes \Omega_1 J L)^{\mathfrak{gl}(k)}$$

For similar reasons the formalism of §7 applies also to $T_{\text{poly}, C^\text{aff}, L}(C^\text{aff}, L \otimes \Omega_1 J L)$. We omit the trivial details.

From (P4) we deduce that $\omega' = \omega$. It remains to show that $\phi^\text{coord}$ commutes with the $\mathfrak{gl}(k)$-action. Then we may as well show that $(id \otimes \psi)_\omega$ commutes with the $\mathfrak{gl}(k)$-action. To this end we can use the criterion given by Lemma 7.4.1. We need to prove that $(id \otimes \psi_1 \otimes \ldots \otimes \psi_n)(i_v \omega_1 \otimes \ldots \otimes \omega_n) = 0$. By [3, Lemma 5.2.1] we have $i_v \omega = 1 \otimes v$. It now suffices to invoke (P8).

As in [3] we put $L = T_*(D_{\text{poly}, C^\text{aff}, L}(C^\text{aff}, L \otimes \Omega_1 J L))$. Now note that if we restrict to the $L_\infty$-setting then the maps we have constructed here are the same as those in [3]. Hence the fact that we obtain quasi-isomorphisms

$$T_{\text{poly}}(\Omega) \xrightarrow{\phi^\text{aff}, \alpha'} T_{\text{aff}}(D_{\text{poly}}(\Omega))$$

as well as part (2) of Theorem 1.1 follows from [3, Thm 7.4.1].
**Appendix A. Proof of the globalization properties**

**A.1. Introduction and statement of the main results.** Since properties (P4)-(P5) (see §9) are crucial for us, and since Halbout’s proof of them is somewhat sketchy (see [18, Theorem 4.5]) we give the proof below. This appendix can be read more or less independently from the main paper.

Our main result is a generalization of Halbout’s result.

**Theorem A.1.1.** Let $W$ be a vector space of dimension $d$. Put $R = SW$ (thus Spec $R = W^*$). Let $A(W)$ be the affine group associated to $W^*$ and let $\mathfrak{a}(W)$ be its Lie algebra. We let $\text{Spec } R$

Theorem A.1.1. Let $A_1$, $A_2$, and $A_3$ be operads over graphs just like Kontsevich’s standard strategy as Halbout in [18]. Put $\mathfrak{g} = D_{\text{poly}}(R)$ and $\mathfrak{h} = T_{\text{poly}}(R)$ and let $Q_0$, $Q_2$ be the codifferentials on $S^\infty(L^\infty(\mathfrak{g})[1])$ and $S^\infty(L^\infty(\mathfrak{h})[1])$ representing the $G_\infty$-structures on $\mathfrak{h}$ and $\mathfrak{g}$.

The theory of minimal models shows that there is a $G_\infty$-structure $Q_1$ on $\mathfrak{h} \cong H^*(\mathfrak{g})$ and a $G_\infty$-quasi-isomorphism $\Psi' : (\mathfrak{h}, Q_1) \to (\mathfrak{g}, Q_2)$ such that $\Psi'^1 = \text{HKR}$.

The construction of such a minimal model can be made very explicit using trees (see §A.2 and also [4, 21, 22]). From this construction it follows easily that $\Psi'$ satisfies (P1-5) and $Q_1$ is given by affine invariant $R$-poly-differential operators.

Now $\mathfrak{h}$ equipped with the usual zero differential is a (shifted) Schouten algebra and Tamarkin shows that it is in fact rigid. This means that there is a $G_\infty$-isomorphism $\Psi'' : (\mathfrak{h}, Q_0) \to (\mathfrak{h}, Q_1)$ such that $\Psi''^1 = \text{id}_\mathfrak{h}$. For the proof of Tamarkin’s theorem it now suffices to take $\Psi = \Psi' \Psi''$. 

The crucial rigidity result is proved by showing that the corresponding deformation complex
\[
\text{coder}(S^c(L^c(b)[1]), S^c(L^c(b)[1]))^\delta, [Q_0, -]) = (\text{Hom}(S^c(L^c(b)[1]), b[1]), [Q_0-])
\]
is essentially acyclic (by the superscript $\delta$ we mean that we only consider coderivations compatible with $\delta$, see §4.1). In fact its sole cohomology is a copy of $k$ located in degree $-2$.

Since we want $\Psi$ to satisfy stronger conditions we have to choose $\Psi''$ more carefully. This amounts to working with smaller deformation complexes. It turns out (P4) follows from (P1)(P3)(P5) so we will not worry about it in this introduction.

To take care of (P1)(P3)(P5) we have to use the complex
\[
\text{Diff}(S^c(L^c(b)/a(W)[1]), b[1]), [Q_0, -])^\mathcal{A}(W)
\]
where "Diff" means we only consider cochains which yield $R$-poly-differential operators $L^{c,p_1}(b) \otimes \cdots \otimes L^{c,p_n}(b) \to b$ for all $(p_i)_i$. We prove the following result.

**Theorem A.1.2.** We have
\[
H^\ast(\text{Diff}(S^c(L^c(b)/a(W)[1]), b[1]), [Q_0, -])^\mathcal{A}(W) \cong S gl(W)[-2]^{GL(W)[2]}
\]
Thus the complex (A.1) is not acyclic. Luckily we find that its cohomology lives only in even degree and as the obstruction against rigidity lives in odd degree we are saved.

Below we keep the notations introduced in this introduction. If $V$ is a $GL(W)$-representation then if confusion is possible we denote the corresponding $\mathcal{A}(W)$-representation by $V$ (thanks to the projection $\mathcal{A}(W) \to GL(W) \cong \mathcal{A}(W)/\text{translation}$). E.g. as an $\mathcal{A}(W)$-module $T_{poly}(R) = R \otimes k \wedge W^*$.

**A.2. Minimal models and trees.** Minimal models for strong homotopy algebras over operads can be constructed using trees (see [4, 21, 22]). This approach was popularized by Kontsevich and Soibelman for $A_\infty$-algebras in [21]. Below we give a straightforward account of this theory, which shows for example that the side conditions sometimes imposed on the homotopy are unnecessary.

Let $O$ be a graded operad. Thus we have maps
\[
O(m) \otimes O(n_1) \otimes \cdots \otimes O(n_m) \to O(n_1 + \cdots + n_m)
\]
For simplicity we assume $O(0) = 0$ (no constants), $O(1) = k$ (no non-trivial unary operations). We also assume dim $O(n) < \infty$ for all $n$.

A coalgebra $C$ over $O$ is a collection of $S_n$-invariant “operations”
\[
\Delta_m : O(m) \otimes C \to C^{\otimes m}
\]
satisfying the standard axioms.

The cofree $O$-coalgebra over the graded vector space $V$ is given by the formula\footnote{For cofree coalgebras it would seem more natural to use a definition in terms of $S_n$-invariants. However in characteristic zero there is a canonical identification between invariants and coinvariants.}
\[
F^c(V) = \bigoplus_n O(n)^* \otimes_{S_n} V^{\otimes n}
\]
We remind the reader of the formula for the $O$-coaction.
For \( \phi \in \mathcal{O}(n_1 + \cdots + n_m) \) write
\[
\sum \phi_{(0)} \otimes \phi_{(1)} \otimes \cdots \otimes \phi_{(m)}
\]
for the image of \( \phi \) under the map
\[
\mathcal{O}(n_1 + \cdots + n_m)^* \to \mathcal{O}(m)^* \otimes \mathcal{O}(n_1)^* \otimes \cdots \otimes \mathcal{O}(n_m)^*
\]
dual to (A.2). Then, for any \( \gamma \in \mathcal{O}(m) \) and \( \phi \in \mathcal{O}(n)^* \), we have
\[
\Delta_m(\gamma \otimes \phi \otimes v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in S_n, \sum_{i=1}^n n_i = n} m! n_1! \cdots n_m! ((\sigma \phi)_{(0)}(\gamma) \otimes v_{\sigma^{-1}(1)}(1) \otimes \cdots \otimes v_{\sigma^{-1}(n_1)}(n_1)) \otimes \\
\cdots \otimes ((\sigma \phi)_{(m)}(v_{\sigma^{-1}(n_1+\cdots+n_{m-1}+1)}) \otimes \cdots \otimes v_{\sigma^{-1}(n)})
\]
where here and below the sign is controlled by the Koszul convention.\(^5\)

For use below we give the formula for a coalgebra morphism \( \psi : F^c(V) \to F^c(W) \) determined by maps
\[
\psi^n : F^{c,n}(V) \to W
\]
(A.4)
\[
\psi(\phi \otimes v_1 \otimes \cdots \otimes v_n) = \sum_{m \geq 2, \sigma \in S_n, \sum_{i=1}^m n_i = n} m! n_1! \cdots n_m! \psi^n \circ (\sigma \phi)_{(0)}(\gamma) \otimes v_{\sigma^{-1}(1)}(1) \otimes \cdots \otimes v_{\sigma^{-1}(n_1)}(n_1) \otimes \\
\cdots \otimes \psi^n((\sigma \phi)_{(m)}(v_{\sigma^{-1}(n_1+\cdots+n_{m-1}+1)}) \otimes \cdots \otimes v_{\sigma^{-1}(n)})
\]

Now we discuss minimal models. Let \( Q_2 : F^c(V) \to V[1] \) be a codifferential and let \( H^*(V) \) be the cohomology of the complex \((V, Q_2^1)\). Then \( Q_2^1 Q_2^2 + Q_2^2 Q_2^1 = 0 \) and hence \( H^*(Q_2^3) \) is well defined and yields a map \( F^{c,2}(H^*(V)) \to H^*(V) \).

Furthermore we have \( Q_2^2 Q_2^1 + (Q_2^2)^2 + Q_2^1 Q_2^3 = 0 \). Thus \((Q_2^3)^2\) is homotopic to the zero map. Hence \( H^*(Q_2^3)^2 = 0 \).

For a series of maps \( \psi^i : F^{c,i}(H^*(V)) \to V, i = 1, \ldots, n \) we let \( \psi^{\leq i} \) be the corresponding coalgebra morphism \( F^c(H^*(V)) \to F^c(V) \) (the higher Taylor coefficients are assumed to be zero). We use similar conventions for coderivations.

We now choose a decomposition \((V, Q_2^1) = (H^*(V), 0) \oplus W\) as complexes and we let \( i : H^*(V) \to V, p : V \to H^*(V) \) be the corresponding injection and projection map. Furthermore we choose a homotopy \( H : V \to V[-1] \) such that
\[
ip - \text{id} = Q_2^1 H + HQ_2^1
\]
Let \( P : F^c(V) \to V \) be the projection and let \( I_n : F^{n,c}(V) \to F^c(V) \) be the injection. We use the same notation with \( V \) replaced by \( H^*(V) \).

**Proposition A.2.1.** Define a coderivation \( Q_1 \) of degree one on \( F^{c,n}(H^*(V)) \) and a graded coalgebra map \( \psi : F^c(H^*(V)) \to F^c(V) \) recursively as follows: \( \psi^1 = i \), \( Q_1^1 = 0 \) and for \( n \geq 2 \)
\[
\psi^n = HPQ_2^2 \psi^{\leq n-1} I_n
\]
(A.5)
\[
Q_1^n = pPQ_2^2 \psi^{\leq n-1} I_n
\]
Then
\[
(1) \ Q_2 \psi = \psi Q_1.
\]
\(^5\)In such formulas it is possible to get rid of the inverse factorials by restricting the summation to shuffles.
(2) \((Q_1)^2 = 0\).
(3) The functor \(H^*(-)\) applied to \(\psi^1 : H^*(V) \to V\) yields the identity on \(H^*(V)\).
(4) \(Q_1^2 : F_{c,S}(H^*(V)) \to H^*(V)\) coincides with \(H^*(Q_2^1)\).

**Proof.** (3)(4) are trivial so we concentrate on (1) and (2).

(1) is equivalent to

\[ PQ_2\psi I_n = P\psi Q_1 I_n \]

for all \(n \geq 1\). We prove this by induction on \(n\), the case \(n = 1\) being clear.

We compute (with obvious notations)

\[
PQ_2\psi I_n - P\psi Q_1 I_n = Q_2^1\psi I_n + PQ_2\psi Q_1 I_n - PQ_2\psi Q_1 I_n - P\psi^1 I_n - P\psi Q_2\psi Q_1 I_n
\]

\[
= (\text{id} + Q_2^1H)PQ_2\psi I_n - P\psi Q_2\psi Q_1 I_n - P\psi Q_2\psi Q_1 I_n
\]

\[
= -Q_2^1PQ_2\psi I_n - P\psi Q_2\psi Q_1 I_n
\]

\[
= -Q_2^1PQ_2\psi Q_1 I_n - P\psi Q_2\psi Q_1 I_n
\]

\[
= -Q_2^1PQ_2\psi Q_1 I_n - P\psi Q_2\psi Q_1 I_n
\]

\[
= 0
\]

The argument for (2) is as in [15]. We include it for completeness. It is sufficient to prove

\[
P(Q_1)^2 I_{n+1} = 0
\]

for all \(n \geq 1\). Again \(n = 1\) is clear and we use induction for \(n \geq 2\). Since \(Q_1^1 = 0\) and \(\psi^1\) is injective it suffices to prove \(\psi^1(Q_1^{<\text{in}})^2 I_{n+1} = \psi^n(Q_1^{<\text{in}})^2 I_{n+1} = 0\). Since \(Q_1^{<\text{in}}\) maps \(F_{c,S}(H^*(V))\) to \(F_{c,S}(H^*(V))\) we have

\[
\psi^n Q_1^{<\text{in}} I_{n+1} = Q_2^1 Q_1^{<\text{in}} I_{n+1} \quad \text{ (using (1))}
\]

\[
= Q_2(\psi^n Q_1^{<\text{in}} - Q_2\psi^n I_{n+1})
\]

\[
= Q_2(\psi^n Q_1^{<\text{in}} - Q_2\psi^n I_{n+1})
\]

where the last equality follows from the fact that \((\psi^n Q_1^{<\text{in}} - Q_2\psi^n) I_{n+1}\) takes values in \(V\) (again using (1)).

We conclude

\[
\text{im } \psi^1(Q_1^{<\text{in}})^2 I_{n+1} \subseteq \text{im } \psi^1 \cap \text{im } Q_2^1 = 0
\]

which finishes the proof. \(\square\)

Combining (A.4) with Proposition A.2.1 the recursion relations can be written as

\[
\psi^n(\phi \otimes v_1 \otimes \cdots \otimes v_n) =
\]

\[
\sum_{m \geq 2, \sigma \in S_n} \frac{1}{m! n_1! \cdots n_m!} HQ_2^m (\{\sigma\phi\}(m) \otimes \psi^n (\{\sigma\phi\}(1) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n_1)}) \otimes \\
\cdots \otimes \psi^n (\{\sigma\phi\}(m) \otimes v_{\sigma^{-1}(n_1+\cdots+n_{m-1}+1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}))
\]

\[
(\text{A.6})
\]

\[
\psi^n(\phi \otimes v_1 \otimes \cdots \otimes v_n) =
\]

\[
\sum_{m \geq 2, \sigma \in S_n} \frac{1}{m! n_1! \cdots n_m!} HQ_2^m (\{\sigma\phi\}(m) \otimes \psi^n (\{\sigma\phi\}(1) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n_1)}) \otimes \\
\cdots \otimes \psi^n (\{\sigma\phi\}(m) \otimes v_{\sigma^{-1}(n_1+\cdots+n_{m-1}+1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}))
\]

\[
\text{A.6)
\]
If $Q_T$ given by \((1(23))4\). Then we have

\[
\sum_{m \geq 2, \sigma \in S_m} \pm \frac{1}{m! n_1 \cdots n_m} \cdot Q_2^m((\sigma \phi)(0) \otimes \psi^{n_1}((\sigma \phi)(1) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n_1)}) \otimes \\
\cdots \otimes \psi^{n_m}((\sigma \phi)(m) \otimes v_{\sigma^{-1}(n_1+ \cdots + n_{m-1} + 1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}) \}
\]

Let $T_n$ be the set of planar rooted trees whose leaves are indexed from 1 to $n$ and whose internal vertices have at least two branches. Iterating (A.6)(A.7) it is clear that we will get $\psi^n(\phi \otimes v_1 \otimes \cdots \otimes v_n)$, $Q_1^n(\phi \otimes v_1 \otimes \cdots \otimes v_n)$ as sums indexed by elements of $S_n \times T_n$.

To be more precise write for $v_1, \ldots, v_n \in V$

\[
\phi(v_1, \ldots, v_n) = Q_2^n(\phi \otimes v_1 \otimes \cdots \otimes v_n)
\]

Then we get

\[\psi^n(\phi \otimes v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in S_n, T \in T_n} \pm \frac{1}{w_T} \psi_T(\sigma \phi \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})\]

\[Q_1^n(\phi \otimes v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in S_n, T \in T_n} \pm \frac{1}{w_T} Q_{1,T}(\sigma \phi \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})\]

where $w_T$ is the product of the factorials of the internal vertices and where $\psi_T$, $Q_{1,T}$ are inductively defined as follows: $\psi_1 = i$ (here $\bullet$ is the single vertex tree) and

\[
\psi_T(\phi \otimes v_1 \otimes \cdots \otimes v_n) := H_{(\phi)(0)}(\psi_T_1(\phi(1) \otimes v_1 \otimes \cdots \otimes v_{|T_1|}), \ldots, \psi_T_m(\phi(m) \otimes v_1 + \sum_{|T_i| = 1}^{n} v_i \otimes \cdots \otimes v_n))
\]

\[
Q_{1,T}(\phi \otimes v_1 \otimes \cdots \otimes v_n) := p_{\phi}(0)(\psi_T_1(\phi(1) \otimes v_1 \otimes \cdots \otimes v_{|T_1|}), \ldots, \psi_T_m(\phi(m) \otimes v_1 + \sum_{|T_i| = 1}^{n} v_i \otimes \cdots \otimes v_n))
\]

if $T = B_+(T_1 \cdots T_m)$ is obtained from $m$ trees by grafting them on a common new root. Here we have denoted by $|T|$ the number of leaves of a given tree $T$.

We illustrate this with an example.

**Example A.2.2.** Let $T$ be the tree with corresponding parenthesized expression given by \(((1(23))4\). Then we have

\[
\psi_T(\phi \otimes v_1 \otimes v_2 \otimes v_3 \otimes v_4) = H_{\phi(0)}(H_{\phi(1)(0)}(i(v_1), H_{\phi(1)(2)(0)}(i(v_2), i(v_3)), i(v_4)))
\]

where we have used $\mathcal{O}(1) = k$ and $\psi^1 = i$ and where $\phi(0) \otimes \phi(1)(0) \otimes 1 \otimes \phi(1)(2)(0) \otimes 1 \otimes 1 \otimes 1$ is the image of $\phi$ under the compositions

\[
\mathcal{O}(4)^* \rightarrow \mathcal{O}(2)^* \otimes \mathcal{O}(3)^* \otimes \mathcal{O}(1)^*
\]

\[
\rightarrow \mathcal{O}(2)^* \otimes (\mathcal{O}(2)^* \otimes \mathcal{O}(1)^* \otimes \mathcal{O}(2)^*) \otimes \mathcal{O}(1)^*
\]

\[
\rightarrow \mathcal{O}(2)^* \otimes \mathcal{O}(2)^* \otimes \mathcal{O}(1)^* \otimes (\mathcal{O}(2)^* \otimes \mathcal{O}(1)^* \otimes \mathcal{O}(1)^*) \otimes \mathcal{O}(1)^*
\]
Pictorially $\psi_T$ is given by

In this case $w_T = 2! \times 2! \times 2!$. The pictorial representation for $Q_{1,T}$ is similar. The only difference is that the root edge is decorated by $p$ instead of $H$.

Although it is not relevant for the discussion below we note that (A.8)(A.9) can be more elegantly written in terms of arbitrary trees instead of planar trees. To be more precise let $\tilde{T}_n$ be the set isomorphism classes of trees with leaves given by $\{1, \ldots, n\}$. Then there is a map

$$\gamma : S_n \times T_n \to \tilde{T}_n$$

which associates to $(\sigma, T) \in S_n \times T_n$ the tree obtained from $T$ by replacing the leaf vertices by $1, \ldots, n$ by $\sigma^{-1}(1), \ldots, \sigma^{-1}(n)$.

It is clear that $\gamma$ is surjective but not injective. The cardinalities of the fibers are precisely given by the numbers $w_T$ introduced above.

If $T = \gamma(\sigma, T)$ then we put

$$\psi_T(\phi \otimes v_1 \otimes \cdots \otimes v_n) = \pm \psi_T(\sigma(\phi) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})$$

$$Q_{1,T}(\phi \otimes v_1 \otimes \cdots \otimes v_n) = \pm Q_{1,T}(\sigma(\phi) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})$$

One may check that this is well defined using the operad axioms. (A.8)(A.9) then become

$$\psi^n(\phi \otimes v_1 \otimes \cdots \otimes v_n) = \sum_{T \in \tilde{T}_n} \psi_T(\phi \otimes v_1 \otimes \cdots \otimes v_1)$$

$$Q^n_{1}(\phi \otimes v_1 \otimes \cdots \otimes v_n) = \sum_{T \in \tilde{T}_n} Q_{1,T}(\phi \otimes v_1 \otimes \cdots \otimes v_n)$$

A.3. The construction of $\Psi'$. We revert to the notations from the introduction. We prove the following result

**Proposition A.3.1.** There exists a $G_\infty$-structure $Q_1$ on $\mathfrak{h}$ and $G_\infty$-quasi-isomorphism

$$\Psi' : (\mathfrak{h}, Q_1) \to (\mathfrak{g}, Q_2)$$
such that \( \Psi' \) satisfies the obvious analogues of (P1-3); \( H^*(Q_1^{1,1}) = \text{HKR} \circ Q_1^{1,1} \circ \text{HKR}^{-1} \) and \( H^*(Q_2^2) = \text{HKR} \circ Q_1^2 \circ \text{HKR}^{-1} \); the components of \( Q_1 \) are \( \mathcal{A}(W) \)-invariant poly-differential operators and in addition:

- (R1) \( Q_1^{p_1,p_2,\ldots,p_n}(\alpha_1,\ldots,\alpha_n) = 0 \) for \( \gamma \in h_0 \) and \( \alpha_i \in L^{c,p_i}(\mathfrak{h}) \), except when \( n = 2 \) and \( p_2 = 1 \).
- (R2) \( \Psi^{1,2,\ldots,p_n}(\alpha_1,\ldots,\alpha_n) = 0 \) for \( \gamma \in h_0 \) and \( \alpha_i \in L^{c,p_i}(\mathfrak{h}) \) and \( n \geq 2 \).

Note that in contrast to the statement of (P5) in this case \( \gamma \) is an arbitrary, not necessarily linear, vector field.

**Proof.** Define \( \mathcal{F}^{m,n}(\mathfrak{g}) \subset \mathfrak{g} \) as the vector space of poly-differential operators of degree \( \leq n \) with \( m \) arguments. Clearly \( \mathcal{F}^{m,n}(\mathfrak{g}) \) is a finite free \( R \)-module. A linear map \( \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathfrak{g} \) is called a local \( R \)-poly-differential operator if its restriction to any \( \mathcal{F}^{m_1,n_1}(\mathfrak{g}) \times \mathcal{F}^{m_2,n_2}(\mathfrak{g}) \times \cdots \) is an \( R \)-poly-differential operator. Local poly-differential operators are compatible with all operations we use below.

It is easy to verify that structure maps in the \( B_\infty \)-structure on \( \mathfrak{g} \) are \( \mathcal{A}(W) \)-invariant local \( R \)-poly-differential operators. Applying the morphism of DG-operads \( G_\infty \to B_\infty \) we obtain the same result for the \( G_\infty \)-structure on \( \mathfrak{g} \).

If \( V \) is a graded vector spaces then the cofree coalgebra cogenerated by \( V[1] \) for the Gerstenhaber algebra operad is equal to \( S^c(L^c(V)[1]) \). Hence if \( V \) is a \( G_\infty \)-algebra represented by a codifferential of degree one on \( S(L(V)[1]) \) then we may use the (A.8)(A.9) to construct a minimal model for \( V \).

We apply this with \( V = \mathfrak{g} \) and we put \( i = \text{HKR} \). We claim that we may choose \( H \) and \( p \) to be \( \mathcal{A}(W) \)-equivariant and \( R \)-linear such that in addition we have \( Hi = 0 \).

It is easy to write down a formula for \( p \) but \( H \) is another matter. There exists explicit, but quite non-trivial, formulas for \( H \) which have the required properties [17, 6]. As the explicit form of \( H \) is not required for us we may also use the following non-explicit argument.

As \( \mathcal{A}(W) \)-representations

\[
T_{\text{poly}}(R) = R \otimes_k \Lambda \mathfrak{W}
\]

and

\[
D_{\text{poly}}(R) = R \otimes_k T_k(S\mathfrak{W}^*)
\]

and \( i = \text{HKR} \) is clearly obtained by base extension from its restriction

\[
i' : \Lambda \mathfrak{W} \to T_k(S\mathfrak{W}^*)
\]

In particular \( i' \) is still a quasi-isomorphism. We choose a \( \text{GL}(W) \) invariant quasi-inverse

\[
p' : T_k(S\mathfrak{W}^*) \to \Lambda \mathfrak{W}
\]

such that \( p'i' = \text{id} \). This is possible since \( \text{GL}(W) \) is reductive. Then we choose a \( \text{GL}(W) \)-equivariant homotopy

\[
H' : T_k(S\mathfrak{W}^*) \to T_k(S\mathfrak{W}^*)[-1]
\]

between the identity and \( i'p' \) satisfying \( H'i' = 0 \). Finally we let \( p, H \) be the base extensions of \( p' \), \( H' \).

Now we construct \( Q_1 \) and \( \Psi' \) using (A.8) (A.9). We obtain (R1) and (R2) (which are a priori not part of the conclusions of Proposition A.3.1) using the following two facts:
• $Q_2^{p_2,\ldots,p_n}(i(\gamma)\alpha_2\cdots\alpha_n) = 0$ for $\gamma \in \mathfrak{h}_0$ and $\alpha_i \in L^c p_i(\mathfrak{g})$, except when $n = 2$ and $p_2 = 1$. This follows from Proposition 5.5 and Lemma 5.2.

• $HQ_2^{1,1}(i(\gamma)i(\gamma')) = 0$ for $\gamma \in \mathfrak{h}_0$ and $\gamma' \in \mathfrak{h}$. This follows from the fact that $HQ_2^{1,1}(i(\gamma)i(\gamma')) = H[i(\gamma), i(\gamma')] = H[i(\gamma, \gamma')] = 0$ where the second equality follows from the fact that the HKR map is compatible with $[\delta, -]$ when $\delta$ is a vector field.

A.4. Sketch of Tamarkin’s argument. We will first remind the reader of Tamarkin’s argument (without supplying all details) which yields a $G_\infty$-isomorphism $\Psi'' : (\mathfrak{h}, Q_0) \to (\mathfrak{h}, Q_1)$ such that $\Psi'' = \text{id}_\mathfrak{h}$, but satisfying a priori no additional conditions. Then we will modify Tamarkin’s argument to make $\Psi''$ satisfy stronger conditions.

Put $F = S^c(L^c(\mathfrak{h})[1])$. We first consider the “deformation complex” of $\mathfrak{h}$

(A.10) \[ (\text{coder}(F,F)^d, [Q_0, -]) = (\text{Hom}(F, \mathfrak{h}[1]), [Q_0, -]) \]

Proposition A.4.1 (Tamarkin [24]). The deformation complex of $\mathfrak{h}$ is exact, except for a copy of $k$ located in cohomological degree $-2$ which has as generating vector the composition $F \to k \to \mathfrak{h}[1][-2] = S(W \oplus W^*[1])$.

We sketch the proof below but for further reference we first state and prove the motivating corollary.

Corollary A.4.2. There exists a $G_\infty$-isomorphism $\Psi'' : (\mathfrak{h}, Q_0) \to (\mathfrak{h}, Q_1)$ such that $\Psi'' = \text{id}_\mathfrak{h}$.

Proof. We consider $F$ to be graded by $\mathfrak{h}$-homogeneity. Below superscripts refer to this grading. Clearly $Q_0$ raises the grading by one$^6$ on $F$. Hence $[Q_0, -]$ maps $\text{Hom}(F^n, \mathfrak{h}[1])$ to $\text{Hom}(F^{n+1}, \mathfrak{h}[1])$.

In other words, we deduce that there is a complex of complexes with zero differential

(A.11) \[ 0 \to k[2] \to \text{Hom}(F^0, \mathfrak{h}[1]) \xrightarrow{[Q_0, -]} \text{Hom}(F^1, \mathfrak{h}[1]) \xrightarrow{[Q_0, -]} \text{Hom}(F^2, \mathfrak{h}[1]) \xrightarrow{[Q_0, -]} \text{Hom}(F^3, \mathfrak{h}[1]) \to \cdots \]

Since the total complex is exact by Tamarkin’s result, this is in fact an exact sequence of complexes with zero differential.

Now we perform a standard computation. Choose $n \geq 3$ minimal such that $(Q_1 - Q_0)|F^n \neq 0$. Then we have on $F^{\leq n}$: $0 = [Q_1, Q_1] = [Q_0, Q_0] + 2[Q_0, Q_1 - Q_0] = 2[Q_0, Q_1 - Q_0]$. Hence by (A.11) we find $Z \in \text{Hom}(F^{n-1}, \mathfrak{h}[1])$ such that $[Q_0, H] = Q_1 - Q_0$. Put $\theta = eZ$. Then we find on $F^n$: $\theta Q_1 \theta^{-1} = Q_0 - [Q_0, Z] + (Q_1 - Q_0) = Q_0$.

Replacing $Q_1$ by $\theta Q_1 \theta^{-1}$ and repeating this procedure we eventually find the requested $G_\infty$-isomorphism.

Proof of Proposition A.4.1 (sketch). The crucial point is that $Q_0 = Q_0^\text{cup} + Q_0^\text{Lie}$ where $Q_0^\text{cup}$ and $Q_0^\text{Lie}$ are respectively obtained from the product and Lie bracket on $\mathfrak{h}$ and where $[Q_0^\text{Lie}, Q_0^\text{cup}] = 0$.

$^6$This depends crucially on the fact that $Q_0$ is built only of binary operations. I.e. $\mathfrak{h}$ it is not itself a strong homotopy algebra.
The commutative algebra structure on $\mathfrak{h}[-1]$ makes $\mathfrak{h}[-1]$ into a $C_{\infty}$-algebra. This yields a codifferential $Q_{0}^{\text{cup}}$ on $L^c(\mathfrak{h})$. Using Leibniz rule the codifferential may be extended to a codifferential on $F$, also denoted by $Q_{0}^{\text{cup}}$.

The Lie algebra structure on $\mathfrak{h}$ extends to a Lie algebra structure on $L^c(\mathfrak{h})$ such that $L^c(\mathfrak{h}) \to \mathfrak{h}$ is a Lie algebra morphism. For the explicit formula of the Lie bracket (which is not entirely obvious) we refer to [11, §3.1].

In particular $L^c(\mathfrak{h})$ is an $L_\infty$-algebra and hence there is a corresponding codifferential $Q_{0}^{\text{lie}}$ on $F$.

Using these explicit description we may compute the actions of $[Q_{0}^{\text{cup}}, -]$ and $[Q_{0}^{\text{lie}}, -]$ on $\text{Hom}(F, \mathfrak{h}[1])$. This is somewhat tricky to get right and we only list the results. We refer to Tamarkin’s paper for details.

1. $[Q_{0}^{\text{lie}}, -]$ is simply the Cartan-Eilenberg differential for the $L^c(\mathfrak{h})$ representation $\mathfrak{h}$.

2. To describe $[Q_{0}^{\text{cup}}, -]$ we put for clarity $A = \mathfrak{h}[-1] = S(W \oplus W^*[-1])$ and consider $A$ as an associative algebra. Then

$$\text{Hom}(F, \mathfrak{h}[1]) = \text{Hom}_A(A \otimes_k F, \mathfrak{h}[1])$$
$$= \text{Hom}_A(S_A^e(A \otimes_k L^c(\mathfrak{h})[1]), \mathfrak{h}[1])$$

On $A \otimes_k L^c(\mathfrak{h}) = A \otimes_k L^c(A[1])$ we have the A-linear Harrison differential $d_{\text{Harr}}$ which is obtained from the Hochschild differential on $A \otimes T^c(A[1])$ (using the surjection $A \otimes_k T^c(A[1]) \to A \otimes_k L^c(A[1])$).

We extend $d_{\text{Harr}}$ to a differential on $S_A^e(A \otimes_k L^c(\mathfrak{h})[1])$ which we also denote by $d_{\text{Harr}}$. Then $[Q_{0}^{\text{cup}}, -]$ is the dual of $d_{\text{Harr}}$.

We now relate the complex $(\text{Hom}_A(S_A^e(A \otimes_k L^c(\mathfrak{h})[1]), \mathfrak{h}[1]), [Q_{0}, -])$ to the complex computing the Lichnerowisz-Poisson cohomology of $A$. This is explained [11, §1.4.9] but we present a slightly different point of view in terms of Lie algebroid cohomology.

If $\mathfrak{l}$ is a Lie algebra acting on a commutative ring $S$ then $S \otimes_k \mathfrak{l}$ carries a natural structure of a Lie algebroid. In our case the Lie algebra $\mathfrak{h}$ acts on $A$ (via the Lie bracket on $\mathfrak{h} = A[-1]$). Hence $L^c(\mathfrak{h})$ acts on $A$ via the projection map $L^c(\mathfrak{h}) \to \mathfrak{h}$ which is a Lie algebra homomorphism. It follows $A \otimes L^c(\mathfrak{h})$ is a Lie algebroid. Using the explicit formula for the Lie bracket on $L^c(\mathfrak{h})$ in [11, §3.1] one checks that $(A \otimes_k L^c(\mathfrak{h}), d_{\text{Harr}})$ is in fact a DG-Lie algebroid. The complex $(\text{Hom}_A(S_A^e(A \otimes_k L^c(\mathfrak{h})[1]), \mathfrak{h}[1]), [Q_{0}, -])$ is the complex computing the cohomology of this Lie algebroid.

As $\mathfrak{h}$ is a (shifted) Gerstenhaber algebra we find that $\Omega_A[1]$ is a Lie algebroid with Lie bracket and anchor map determined by $[df, dg] = d[f, g]$, $\rho(df)(g) = [f, g]$. The Lie algebroid cohomology of $\Omega_A[1]$ is the (shifted) Lichnerowisz-Poisson cohomology of $A$. Furthermore one has morphisms of DG-Lie algebroids

\begin{equation}
(A \otimes_k L^c(A[1]), d_{\text{Harr}}) \to (A \otimes A[1], 0) \xrightarrow{a \otimes b - adb} (\Omega_A[1], 0)
\end{equation}

Dualizing we obtain a morphism of complexes

\begin{equation}
\text{Hom}_A(S_A^e(\Omega_A[2]), \mathfrak{h}[1]), d_{\text{Poisson}}) \to (\text{Hom}_A(S_A^e(A \otimes_k L^c(\mathfrak{h})[1]), \mathfrak{h}[1]), [Q_{0}, -])
\end{equation}

where $d_{\text{Poisson}}$ is the differential computing the (shifted) Lichnerowisz-Poisson cohomology of $A$. 

We claim that (A.13) is in fact a quasi-isomorphism. Using a spectral sequence argument it is sufficient to prove that
\[ \text{Hom}_A(S_c(\Omega_A[2]), h[1]), 0) \to (\text{Hom}_A(S_c(A \otimes_k L^c(h)[1]), h[1]), \{Q_0^{\text{sup}}, -\}) \]
is a quasi-isomorphism. This follows from the fact that (A.12) is a homotopy equivalence. To prove this last statement we note that it is well-known that (A.12) is a quasi-isomorphism (e.g. [5, Lemma 2.5.10]). Then it suffices to observe that \( A \otimes L^c(A[1]) \) is a colimit of finite extensions of shifts of \( A \) and hence is homotopically projective. Thus a quasi-isomorphism with source \( A \otimes L^c(A[1]) \) is automatically a homotopy equivalence.

Using all this we are reduced to computing the (shifted) Lichnerowisz-Poisson cohomology of \( A \). Now the Gerstenhaber bracket is in fact symplectic and it is well-known that Lichnerowisz-Poisson cohomology for a symplectic form is the same as De Rham cohomology (see [1]). The fact that the De Rham cohomology of \( A \) is trivial then finishes the proof of Proposition A.4.1. We refer to [24] for more details.

□

A.5. The differential deformation complex. We define \( \text{Diff}(S^c(L^c(h)[1]), h[1]) \) as the graded subspace of \( \text{Hom}(S^c(L^c(h)[1]), h[1]) \) such that all the occurring maps \( L^c(m_1)(h) \otimes \cdots \otimes L^c(m_1)(h) \to h \) are \( R \)-poly-differential operators.

**Lemma A.5.1.** \( \text{Diff}(S^c(L^c(h)[1]), h[1]) \) is closed under \( [Q_0^{\text{sup}}, -] \) and \( [Q_0^{\text{Lie}}, -] \).

**Proof.** Ultimately \( [Q_0^{\text{sup}}, -] \) and \( [Q_0^{\text{Lie}}, -] \) are derived from the cupproduct and Lie bracket on \( h \). Since the latter are \( R \)-poly-differential operators, one obtains that the same holds for \( [Q_0^{\text{sup}}, -] \) and \( [Q_0^{\text{Lie}}, -] \). □

We have two aims in this section. The first one is to give a more convenient expression for the differential deformation complex. See (A.15) and Lemma A.5.2. The second aim is to show that the analogue of Proposition A.4.1 holds for the differential deformation complex. See (A.16). These two results will be used in the next section.

We have
\[
\text{Diff}(L^c(A[1]), A) \hookrightarrow \text{Diff}(T^c(A[1]), A)
\]
\[= T_A(\text{Diff}(A, A)[-1])
\]
\[= D_{\text{poly}}(A)[-1]
\]
Since \( A = S(W \oplus W^*[1]) \) we have\(^7\) \( \text{Diff}(A, A) = A \otimes S(W^* \oplus W[1]) \) and thus
\[(A.14) \quad T_A(\text{Diff}(A, A)[-1]) = A \otimes T(S(W^* \oplus W[1])[-1])
\]

**Lemma A.5.2.** Denote the free Lie algebra generated by a graded vector space \( V \) by \( L(V) \). Then with the identification (A.14) we have
\[
\text{Diff}(L^c(A[1]), A) = A \otimes L(S(W^* \oplus W[1])[-1])
\]

**Proof.** Since \( L^c(A) \) is equal to \( T^c(A) \) modulo shuffles, one quickly establishes that \( \text{Diff}(L^c(A[1]), A) \) is the set of primitive elements for the coshuffle coproduct on \( T_A(\text{Diff}(A, A)[-1]) \). This yields the desired result. □

\(^7\)It is easy to see that \( A \)-differential operators are the same as \( R \)-differential operators.
Put $L(A) = \text{Diff}(L^c(A[1]), A)$. Then we get
\[
(A.15)\quad \text{Diff}(S^c(L^c(h)[1]), h[1]) = S_A(L(A)[-1])[2]
\]
The differential $[Q^\text{cup}_0, -]$ on $S_A(L(A)[-1])[2]$ is obtained by extending the Harrison differential on $L(A)$ (obtained from the Hochschild differential on $D_{\text{poly}}(A)$).

The HKR quasi-isomorphism
\[
S_A(\text{Hom}_A(\Omega_A, A)[-1]) \rightarrow D_{\text{poly}}(A)[-1]
\]
restricts to a quasi-isomorphism
\[
\text{Hom}_A(\Omega_A, A)[-1] \rightarrow L(A)
\]
so that we get a quasi-isomorphism
\[
(S_A(\text{Hom}_A(\Omega_A, A)[-2]), 0) \rightarrow (S_A(L(A)[-1]), [Q^\text{cup}_0, -])
\]
and one checks that this is compatible with $Q^\text{Lie}_0$ using the fact that it is the corestriction of (a shifted version of) (A.13).

So ultimately using (A.13) we obtain a quasi-isomorphism
\[
(S_A(\text{Hom}_A(\Omega_A[2], A))[2], d_{\text{Poisson}}) \rightarrow (S_A(L(A)[-1])[2], [Q_0, -]) = (\text{Diff}(S^c(L^c(h)[1]), h[1]), [Q_0, -])
\]
and since the cohomology is of the left hand side is $k[2]$ (see \S A.4) one obtains that the analogue of Proposition A.4.1 holds for the differential deformation complex.
\[
(A.16)\quad (\text{Diff}(S^c(L^c(h)[1]), h[1]), [Q_0, -]) \cong k[2]
\]

Remark A.5.3. Being a symmetric algebra the ring $R$ has a natural ascending filtration. So $\text{Diff}(F, h[1])$ has a corresponding descending filtration (we view $h[1]$ as not filtered) and the completion for this filtration is $\text{Hom}(F, h[1])$. It seems not unlikely that (with some more work) this observation could be used to deduce (A.16) from Proposition A.4.1.

A.6. The construction of $\Psi''$. Consider
\[
\mathfrak{d} = \{Q \in \text{coder}(S^c(L^c(h)[1]), S^c(L^c(h)[1]))^\delta \mid \forall \gamma \in a(W) : Q^{1,p_2,\ldots,p_n}(\gamma, \ldots) = 0\}
\]
Then $\mathfrak{d}$ is clearly a Lie subalgebra of $\text{coder}(S^c(L^c(h)[1]), S^c(L^c(h)[1]))^\delta$ and hence so is $\mathfrak{d} = \mathfrak{d}_A(W)$.

We have $Q^\text{cup}_0 \in \mathfrak{d}$ and hence $\mathfrak{d}$ and $\mathfrak{d}$ are closed under $[Q^\text{cup}_0, -]$. On the other hand $Q^\text{Lie}_0 \notin \mathfrak{d}$. Nonetheless one checks that $\mathfrak{d}$ is closed under $[Q^\text{Lie}_0, -]$ (but not $\mathfrak{d}$).

In this way we obtain the “affine equivariant deformation complex”
\[
(\mathfrak{d}, [Q_0, -]) = \text{Hom}(S^c(L^c(h)/a(W)[1]), h[1])^{A(W), [Q_0, -]}
\]
This complex can be defined in yet another way. For simplicity write $d = [Q_0, -]$. For $\gamma \in a(W)$ define the endomorphism $i_\gamma$ of degree $-1$ of $\text{Hom}(S^c(L^c(h)[1]), h[1])$ which sends $Q$ to $Q(\gamma, \ldots)$. The one checks that $\gamma \in a(W)$ acts by $L_\gamma \overset{\text{def}}{=} di_\gamma + i_\gamma d$ and hence
\[
\mathfrak{d} = \{Q \in \text{Hom}(S^c(L^c(h)[1]), h[1]) : \forall \gamma \in a(W) \mid i_\gamma Q = L_\gamma Q = 0\}
\]
In this way we see the connection with equivariant cohomology.

In order to construct a $\Psi''$ satisfying (P3)(P5) we would have to analyze the cohomology of $\mathfrak{d}$. However we want to satisfy also (P1). Therefore we consider the subcomplex
\[
(A.17)\quad D = (\text{Diff}(S^c(L^c(h)/a(W)[1]), h[1])^{A(W), [Q_0, -]})
defined as usual by requiring that all occurring maps are \( R \)-poly-differential operators. The cohomology of \( D \) is given by Theorem A.1.2. We will prove that theorem after we have proved the next corollary.

**Corollary A.6.1.** (to Theorem A.1.2) Assume that \( Q_1 \) was chosen as in Proposition A.3.1. Then there exists a \( G_\infty \)-isomorphism \( \Psi'' : (\mathfrak{g}, Q_0) \to (\mathfrak{g}, Q_1) \) such that \( \Psi'' \) is an \( \mathfrak{g} \)-homogeneity isomorphism and satisfies (P1)(P3)(P4)(P5) hold for \( \Psi'' \).

**Proof.** Of course we proceed as in Corollary A.4.2. The grading by \( \mathfrak{g} \)-homogeneity we used on \( \text{Hom}(F, \mathfrak{g}[1]) \) induces a grading on \( D \). Then we have a complex of complexes with zero differential

\[
A.18 \quad 0 \to D^0 \xrightarrow{[Q_0,-]} D^1 \xrightarrow{[Q_0,-]} D^2 \xrightarrow{[Q_0,-]},
\]

Then by (R1) we have that \( Q_1|F^n \in D^n \) for \( n \geq 3 \). Since \( Q_0|F^n = 0 \) for \( n \geq 3 \) and \( Q_0|F^n = Q_1|F^n \) for \( n \leq 2 \) we find \( Q_1 - Q_0 \in D \).

Choose \( n \geq 3 \) minimal such that \( Q_1 - Q_0|F^n \neq 0 \). As in the proof of Corollary A.4.2 we find \( [Q_0, Q_1 - Q_0] = 0 \) on \( F^{n+1} \). Now the point is that \( Q_1 - Q_0 \) has cohomological degree one (not Hochschild degree one). Since by Theorem A.1.2 the total complex of (A.18) has no cohomology in odd degree we find \( Z \in D^{n-1} \) such that \( [Q_0, Z] = Q_1 - Q_0|F^n \).

We put \( Q'_1 = e^Z Q_1 e^{-Z} = Q_1 + \sum [Z, Q_1] \). Now \( [Z, Q_1] = -[Q_0, Z] + [Z, Q_1 - Q_0] \in D \). Hence \( Q'_1 \in D \). Replacing \( Q'_1 \) and iterating we find a \( \Psi'' \) satisfying (P1)(P3)(P5).

We claim that (P4) is satisfied. We first observe that for \( n > 2 \), (P4) is automatic for degree reasons. For \( n = 2 \) we observe that \( \Psi'' \) is an affine invariant differential operator

\[
A.19 \quad \bigwedge^2 T_{\text{poly}}(R) \to R
\]

Any such affine invariant differential operator \( P \) is of the form

\[
P = f^{i,j}_I(x) \partial_{x_i} \partial^I_\underline{x} \wedge \partial_{x_j} \partial^J_\underline{x},
\]

where \( x = (x_1, \ldots, x_n) \) is a coordinate system on \( W^* \) and the \( \xi_i \)'s are the corresponding odd coordinates on \( W[1] \); and \( \partial^I_\underline{x} = \prod_{i=1}^k \partial_{x_{i_i}} \) for \( I = (i_1, \ldots, i_k) \).

Translation invariance implies that \( f^{i,j}_I \) is actually a constant polynomial, and \( GL(W) \)-invariance imposes that \( P \) can only be (up to a scalar factor) \( \partial_{\xi_i} \wedge (\partial_{\xi_j} \partial_{x_i} \partial_{x_j}) \), which does not happen to satisfy (P5). The corollary is therefore proved. \( \square \)

**Proof of Theorem A.1.2.** Put

\[
\text{Diff}_0(A, A) = \{ D \in \text{Diff}(A, A) \mid D|\mathfrak{a}(W) = 0 \}
\]

**Sublemma.** There is a a split exact sequence of \( (A, A(W)) \)-modules

\[
0 \to \text{Diff}_0(A, A) \to \text{Diff}(A, A) \to A \otimes \mathfrak{a}(W)^* \to 0
\]

where the rightmost non-trivial map is given by \( D \mapsto D|\mathfrak{a}(W) \).

**Proof.** This sequence is obviously left exact. To prove the claims it is sufficient to construct the splitting of the right most non-trivial map.

Choose a basis \( (t_i)_{i=1}^d \) for \( W \). Then \( A \) is the graded commutative algebra generated by \( t_i \) and \( \partial_i = \partial/\partial t_i \). Denote the partial derivatives on \( A \) with respect to \( t_i \) and \( \partial_i \) by \( \partial_{t_i} \) and \( \partial_{\partial_i} \).
A basis of $a(W) \subset A$ is given by $(\partial_i)$ and $(t_i\partial_j)_{ij}$. Let $E = \sum t_i\partial_i \in \text{Diff}(A, A)$. We send the element of $A \otimes a(W)^*$, given by $\partial_i \mapsto a_i$, $t_i\partial_j \mapsto a_{ij}$ to $\sum a_i(1-E)\partial_i + \sum_{ij} a_{ij}\partial_i \partial_j \in \text{Diff}(A, A)$. This is an $A$-linear splitting and one checks that it is indeed $\mathcal{A}(W)$ equivariant.

If we put

$$\mathcal{L}_0(A) = \text{Diff}(L^c(A[1])/a(W), A)$$

we obtain an exact sequence

(A.20) \hspace{1cm} 0 \rightarrow \mathcal{L}_0(A) \rightarrow \mathcal{L}(A) \rightarrow A \otimes a(W)^* \rightarrow 0

still split as $(A, \mathcal{A}(W))$-modules. Similarly as in (A.15) we get

$$\text{Diff}(S^c(L^c(h)/a(W)[1]), h[1])^{\mathcal{A}(W)} = (S_A(\mathcal{L}_0(A)[-1]))^{\mathcal{A}(W)[2]}$$

**Sublemma.** The inclusion $S_A(\mathcal{L}_0(A)[-1]) \hookrightarrow S_A(\mathcal{L}(A)[-1])$ extends to a quasi-isomorphism

(A.21) \hspace{1cm} (S_A(\mathcal{L}_0(A)[-1]))^{\mathcal{A}(W)} \rightarrow ((S_A(\mathcal{L}(A)[-1])) \otimes_k S(a(W)^*[-2]))^{\mathcal{A}(W)}

where the differential on the righthand side is given by

(A.22) \hspace{1cm} d \otimes 1 + \sum_i i e_i \otimes e_j^*

with $(e_j)_{ij}$ an arbitrary basis for $a(W)$.

**Proof.** This is a standard observation in equivariant cohomology. One first check that the square of (A.22) is indeed zero and that (A.21) is indeed a morphism of complexes.

To prove that it is a quasi-isomorphism we use an appropriate spectral sequence argument to reduce to proving that

$$((S_A(\mathcal{L}_0(A)[-1]))^{\mathcal{A}(W)}, 0) \rightarrow ((S_A(\mathcal{L}(A)[-1]) \otimes_k S(a(W)^*[-2]))^{\mathcal{A}(W)}, \sum_j i e_j \otimes e_j^*)$$

is a quasi-isomorphism. Before taking $\mathcal{A}(W)$-invariants this is obviously a quasi-isomorphism since it is simply a kind of Koszul resolution. But since (A.20) is split is actually an $\mathcal{A}(W)$-equivariant homotopy equivalence. Hence it remains a homotopy equivalence after taking invariants.

Let $T(W) \subset \mathcal{A}(W)$ be the translation group and let $t(W)$ be its corresponding Lie algebra.

**Sublemma.** Assume that $V$ is a rational $t(W)$-representation. Then $V \otimes_k R$ is injective in the category of all $t(W)$-representations (not just rational ones). In particular

$$\text{Ext}^i_{t(W)}(k, V \otimes R) = \begin{cases} (V \otimes R)^{T(W)} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** Let $U(t(W))$ be the enveloping algebra of $t(W)$. The category of $t(W)$-representations is nothing but the category of $U(t(W))$-modules. We have $R = SW$ and $U(t(W)) = SW^*$ and the action of $U(t(W))$ on $SW$ is given by contraction. It is then well-known that $SW$ is an injective $U(t(W))$-module.
Since $U(t(W))$ is noetherian, a direct limit of injectives is injective, so we may assume that $V$ is finite dimensional. Then we have

$$\text{Hom}_{U(t(W))}(-, V \otimes R) = \text{Hom}_{U(t(W))}(V^* \otimes -, R)$$

which is an exact functor. So we are done. □

We have a two-step $\mathcal{A}(W)$-invariant filtration on $a(W)^*$ given by

$$0 \to \mathfrak{g}(W)^* \to a(W)^* \to t(W)^* \to 0$$

which induces a filtration by ideals on $S(a(W)^*[-2])$ and hence a filtration on $S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(a(W)^*[-2])$ with associated graded given by

$$S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2]) \otimes_k S(\mathfrak{g}(W)^*[-2])$$

Using the second sublemma we find that this is compatible with taking $\mathcal{A}(W)$-invariants. Thus we obtain

$$\bigl(\text{gr}(S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(a(W)^*[-2]))\bigr)^{\mathcal{A}(W)}$$

We now claim that the only cohomology of $((S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2])))^{T(W)}$ is a copy of $k$ in degree zero. To this end we use the following sublemma.

**Sublemma. The inclusion**

$$((S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2])))^{T(W)} \hookrightarrow S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2])$$

**extends to a quasi-isomorphism**

$$\bigl((S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2]))^{T(W)} \to S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2]) \otimes S(t(W)^*[-1])\bigr)$$

**where the differential on the righthand side is given**

$$d \otimes 1 \otimes 1 + \sum_j i_{e_j} \otimes e_j^* \otimes 1 + \sum_k L_{e_k} \otimes 1 \otimes e_k^* - \sum_l 1 \otimes e_l^* \otimes \partial_{e_l}$$

for an arbitrary basis $(e_i)_i$ of $t(W)$.

Note that the complex (A.25) is the (unrestricted) BRST model for equivariant $T(W)$-cohomology [19, §3]. Normally we would need to take a certain subcomplex to get the correct result but in this case this is not necessary because of the $t(W)$ injectivity of $R$ which is essentially a manifestation of the contractibility of $T(W)$.

**Proof.** One first checks that (A.25) has square zero. Then one filters (A.24) according to $t(W)^*$ homogeneity in $S(t(W)^*[-1])$. Then we have to show that

$$((S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2]))^{T(W)}, 0)$$

$$\to (S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2]) \otimes S(t(W)^*[-1]), \sum_k L_{e_k} \otimes 1 \otimes e_k^*)$$

is a quasi-isomorphism.

Now the righthand side of (A.26) computes

$$\text{Ext}_1^*(k, (S_{\mathcal{A}}(\mathcal{L}(A)[-1]) \otimes_k S(t(W)^*[-2])))$$
To see this replace t(W) by U(t(W)) = SW* and then replace k by its Koszul resolution over SW*.

Since \( S^*_A(\mathcal{L}(A)) \cong V_n \otimes R \) for suitable rational t(W) representations \( V_n \) we may conclude by the second sublemma above.

To finish the computation of the cohomology of \( S_A(\mathcal{L}(A)[-1]) \otimes_k S((t(W))^*[-2])^{T(W)} \) we observe that the inclusion \( k \subset S_A(\mathcal{L}(A)[-1]) \) extends to a morphism of complexes

\[
 k \otimes_k S((t(W))^*[-2]) \otimes S((t(W))^*[-1]) \rightarrow S_A(\mathcal{L}(A)[-1]) \otimes_k S((t(W))^*[-2]) \otimes S((t(W))^*[-1])
\]

where the righthand side is as above and the lefthand side has differential \( \sum_i 1 \otimes e_i^* \otimes \partial v_i \). We claim that this is again a quasi-isomorphism. Considering an appropriate filtration this follows from the fact that \( k \rightarrow \mathcal{L}(A)[-1] \) is a quasi-isomorphism by (A.16).

So it now remains to compute the cohomology of \( (S((t(W))^*[-2]) \otimes S((t(W))^*[-1]), \sum_i e_i^* \otimes \partial v_i) \), but this is just an ordinary Koszul complex, so its cohomology is \( k \).

We now return to the spectral sequence derived from (A.23). Its first page becomes by the above discussion

\[
 k \otimes_k S((t(W))^*[-2]))^{GL(W)}
\]

and after that it degenerates. This finishes the proof (taking into account that \( \mathfrak{gl}(W)^* \cong \mathfrak{gl}(W) \) as GL(W)-representations).

The proof of Theorem now follows easily by putting \( \Psi = \Psi'\Psi'' \) and combining Proposition A.3.1 with Corollary A.6.1.

### A.7. Kontsevich graphs

In this section we describe in more detail the R-poly-differential operators between the R-modules \( T_{\text{poly}}(R) \) and \( D_{\text{poly}}(R) \) which are equivariant under the affine group. This material is well-known to experts. See for example [12, 23].

An “admissible” graph (or “Kontsevich graph”) is an oriented graph with the following properties.

1. There are \( t \) vertices of the “first type” labeled by \( 1, \ldots, t \). Vertex \( v \) has \( n_v \) outgoing edges.
2. There are \( n \) vertices of the “second type”, labeled by \( 1, \ldots, n \), with no outgoing edges.

We write \( \Gamma_i \) for the vertices of type \( i \) in an admissible graph \( \Gamma \).

Fix a basis \( (t_i)_i \) for \( W \) and write \( \partial_i = \partial/\partial t_i \). For \( a_v = \sum_{s_1, \ldots, s_{n_v}} a_v^{s_1 \cdots s_{n_v}} \partial_{s_1} \cdots \partial_{s_{n_v}} \in T_{\text{poly}}^n(R) \) and \( f_1, \ldots, f_n \in R \) we define

\[
 U_{\Gamma}(a_1, \ldots, a_t)(f_1, \ldots, f_n) = \prod_{\text{In}(v) = r_1, \ldots, r_d} \partial_{r_1} \cdots \partial_{r_d} a_v^{s_1 \cdots s_{n_v}} \prod_{\text{Out}(v) = s_1, \ldots, s_u} \partial_{s_1} \cdots \partial_{s_u} f_v
\]

Then clearly \( U_{\Gamma}(a_1, \ldots, a_t) \) is an element of \( D_{\text{poly}}^n(R) \) and hence \( U_{\Gamma} \) defines a map

\[
 T_{\text{poly}}^{n_1}(R) \times \cdots \times T_{\text{poly}}^{n_t}(R) \rightarrow D_{\text{poly}}^n(R)
\]

which is an \( A(W) \) invariant poly-differential operator.
**Proposition A.7.1.** An affine invariant poly-differential operators $T_{\text{poly}}(R) \rightarrow D_{\text{poly}}(R)$ is a linear combination of operators $\mathcal{U}_\Gamma$ where $\Gamma$ runs through the admissible graphs.

**Proof.** We have as $\mathcal{A}(W)$ representations

$$T_{\text{poly}}^n(R) = R \otimes_k \Lambda^n W^*$$

and

$$D_{\text{poly}}^n(R) = R \otimes_k (S W^*) \otimes^n$$

Thus for an arbitrary $R$-module the $R$-poly-differential operators from $T_{\text{poly}}^n(R) \times \cdots \times T_{\text{poly}}^n(R)$ to $N$ are given by

$$\Lambda^n W \otimes_k \cdots \otimes_k \Lambda^n W \otimes_k (S W^*) \otimes^n \otimes N$$

Hence the $\mathcal{A}(W)$-invariant $R$-poly-differential operators from $T_{\text{poly}}^n(R) \times \cdots \times T_{\text{poly}}^n(R)$ to $D_{\text{poly}}^n(R)$ are given by

$$M = \left( \Lambda^n W \otimes_k \cdots \otimes_k \Lambda^n W \otimes_k (S W^*) \otimes^n \otimes_k R \right)^{\mathcal{A}(W)}$$

If follows from Schur-Weyl duality that an $M$ is spanned by elements $m_\Gamma$ associated to “admissible” graphs $\Gamma$ where $m_\Gamma$ is defined as follows. Write $dt_i$ for the element $t_i$ considered as an element of $W^*$. Then

$$m_\Gamma = \prod_{v \in \Gamma_1} dt_{s_v} \cdots \otimes \prod_{v \in \Gamma_1} \partial_{r_v} \cdots \otimes \prod_{v \in \Gamma_2} \partial_{u_v} \cdots \partial_{u_v}$$

A moment inspection reveal that $m_\Gamma$ corresponds to $\mathcal{U}_\Gamma$. \hfill \Box

**Remark A.7.2.** Since the Euler operator is invariant under $\text{GL}(W)$ but not under $\mathcal{A}(W)$, the conclusion of Proposition (A.7.1) does not hold if we only demand that the poly-differential operators are invariant under $\text{GL}(W)$.

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