On a non-CP-violating electric dipole moment of elementary particles

J. Giesen, Institut für Theoretische Physik, Bunsenstraße 9, D-37073 Göttingen
(e-mail: giesen@theo-phys.gwdg.de)

Abstract

A description of elementary particles should be based on irreducible representations of the Poincaré group. In the theory of massive representations of the full Poincaré group there are essentially four different cases. One of them corresponds to the ordinary Dirac theory. The extension of Dirac theory to the remaining three cases makes it possible to describe an anomalous electric dipole moment of elementary particles without breaking the reflections.

Introduction

For a long time now there has been great experimental and theoretical interest in an electric dipole moment of elementary particles [1, 2, 3]. We want to characterize an electric dipole moment of an elementary particle by means of two properties. First the dipole moment should show the same behaviour under the reflections (space inversion, time inversion and charge conjugation) as a classical dipole does. This means that an electric dipole moment should not be CP-violating. Second an elementary particle possessing an electric dipole moment should interact with an external electric field.

A description of elementary particles by the Dirac equation easily allows the implementation of external fields. This seems to be the right frame to describe an elementary particle with electric dipole moment. The ordinary Dirac equation only predicts a magnetic dipole moment. Nevertheless one has the possibility to add effective terms to the Lagrangian from which the Dirac equation can be derived. Such effective terms have been used to model the anomalous values of proton and neutron magnetic dipole moments [4]. Landau [5] has shown that it is impossible to find an effective term which allows the description an electric dipole moment. He argued that the spin is the only relativistic observable which is a vector and can be constructed in the particles rest frame. An electric dipole moment which behaves classically should not vanish in the rest frame and cannot behave like the spin under reflections. Thus Landau denied the existence of an electric dipole moment of elementary particles in the sense stated above.

Nowadays [6] one no longer demands the first property. Therefore it becomes possible to describe an anomalous electric dipole moment by means of the spin. The coupling $\sigma E$ then becomes CP-violating.

Wigner [7] has pointed out that the irreducible representations of the Poincaré group should be the basis for a description of elementary particles. Therefore we start with irreducible representations of the Poincaré group and not with the Dirac equation. These representations
are not unique when reflections are taken into account. A careful examination of the representation theory in the massive case shows that essentially one has to distinguish four cases. One of these four cases corresponds to the ordinary Dirac theory. The extension of Dirac theory to the remaining three cases enables us to describe a non-CP-violating elementary particle electric dipole moment.

So far we deal with a one particle description. To get a many particle description one has to quantize the Dirac field. This is possible with the assumption of weak external fields. One can use common techniques to construct a Fock space on which one has a reducible representation of the full Poincaré group.

A more detailed discussion of this work can be found in [8].

Irreducible representations of Poincaré group

In quantum mechanics one is interested in projective representations of symmetry groups. Bargmann [9] was able to show that in the case of the restricted Poincaré group every projective representation is totally characterized by an ordinary representation of the universal covering group. Looking at the full Poincaré group including reflections the situation is a little bit more complicated. Here one has to take into account that one is dealing with projective representations. The irreducible representations for nonvanishing mass have been enumerated by Wigner [7]. He showed that one has to distinguish four cases. In three of them there is a phenomenon called doubling. The three doubled representations may be obtained by the usual one by doubling the number of components. The operators are then written in the form of a direct product $U(\Lambda) = U_0(\Lambda) \otimes I(\Lambda)$ of $2 \times 2$-matrices $I(\Lambda)$ and operators $U_0(\Lambda)$. This subspace consists of normalizable functions of $2s+1$ components. The type of a representation depends on the squares of $U(\Lambda_t)$ and $U(\Lambda_{st})$, where $\Lambda_t$ and $\Lambda_{st}$ denote time inversion and combined space and time inversion, respectively. These squares can be normalized to $\pm 1$. The possible cases are listed in table (1). This means that time inversion and combined space and time inversion are represented by antiunitary operators. In the following we only deal with $s = \frac{1}{2}$ representations.
It is possible to show that the ordinary Dirac equation
\[(i\gamma^\mu \partial_\mu - m)\psi = 0\] (1)
corresponds to the representation of type I. Nevertheless Dirac theory allows one to add further details in a description of elementary particles. First to mention is the prediction of antiparticles, which is expressed in the fact that \(\psi\) is not a two but four component function. In the following we use the representation

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3
\]

and

\[
\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3
\]

of the Dirac matrices. Then the notation of the reflections in the Dirac theory of type I leads to:

- **space inversion:** \(\psi(x, t) \mapsto \gamma^0\psi(-x, t) =: \psi_\text{s}(x, t)\)
- **time inversion:** \(\psi(x, t) \mapsto \gamma^1\gamma^3J\psi(x, -t) =: \psi_\text{t}(x, t)\)
- **charge conjugation:** \(\psi(x, t) \mapsto i\gamma^2J\psi(x, t) =: \psi_\text{c}(x, t)\)

Important for our aim to describe an electric dipole moment of an elementary particle is to implement an interaction with an external electromagnetic field. This is done by minimal coupling:

\[(\gamma^\mu (i\partial_\mu + A_\mu) - m)\psi = 0\] (2)

\(^1C\) is a representation of \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) in \(\text{SU}(2)\) in a space of dimension \(2s+1\).

\(^2J\) is the operator of complex conjugation.
Here $A = (A_{\mu})$ is a four-potential, which should transform under the reflections as

- space inversion: $A(x, t) \mapsto \mathbb{1}_{1,3}A(-x, t)$
- time inversion: $A(x, t) \mapsto \mathbb{1}_{1,3}A(x, -t)$
- charge conjugation: $A(x, t) \mapsto -A(x, t)$

To every four-potential there is an electromagnetic field tensor $F_{\mu\nu}(x, t) = \partial_\mu A_\nu(x, t) - \partial_\nu A_\mu(x, t)$. Its behaviour under reflections follows from the behaviour of the corresponding four-potential.

- space inversion: $F_{\mu\nu}(x, t) \mapsto \begin{cases} F_{\mu\nu}(-x, t) : & \mu, \nu = 1, \ldots, 3 \\ -F_{\mu\nu}(-x, t) : & \text{else} \end{cases}$
- time inversion: $F_{\mu\nu}(x, t) \mapsto \begin{cases} -F_{\mu\nu}(x, -t) : & \mu, \nu = 1, \ldots, 3 \\ F_{\mu\nu}(x, -t) : & \text{else} \end{cases}$
- charge conjugation: $F_{\mu\nu}(x, t) \mapsto -F_{\mu\nu}(x, t)$

Further it is important that the Dirac equation can be derived from a Lagrangian. The Lagrangian

$$\mathcal{L} = \bar{\psi}^+(\gamma^\mu(i\partial_\mu + eA_\mu) - m)\psi, \quad \bar{\psi}^+ = \gamma^0\psi^*$$

leads to equation (2), for which the following statement holds:

**Note 1** Let $\psi$ be a solution of equation (2) then $\psi_\text{s}$ and $\psi_\text{t}$ are solutions of the reflected equations. These are the equations one gets by replacing $(x, t)$ by $(-x, t)$ or $(x, -t)$ and the external fields by the reflected ones.

**Proof:** Follows immediately from the behaviour of the four-potential under the reflections and the commutation relations of the gamma matrices.

The concept of a Lagrangian allows one to add effective terms which are covariant and gauge invariant. So the addition of

$$\mathcal{L}_{\text{eff}} = -d\bar{\psi}^+\gamma^\mu\gamma^\nu\gamma_5F_{\mu\nu}\psi$$

to the Lagrangian (3) leads to the equation

$$(\gamma^\mu(i\partial_\mu + eA_\mu) - d\gamma^\mu\gamma^\nu\gamma_5F_{\mu\nu} - m)\psi = 0 \ .$$

An elementary particle with an electric dipol moment is described by this equation. In the non-relativistic limit the coupling with an external electric field is of the form $\hat{\sigma}E$. Here is

$$\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3), \quad \hat{\sigma}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

with ordinary Pauli matrices $\sigma_i$. As a consequence of the added term one finds:
**Note 2** Let $\psi$ be a solution of equation (5) then $\psi_s$ and $\psi_t$ are no longer solutions of the reflected equations.

This phenomenon is called violation of invariance under space respectively time inversion. The situation is different when looking at representations of type III. First one has to expand the Dirac theory to representations with doubling. This is done for the type III case by

$$(i\Gamma^\mu \partial_\mu - m)\psi = 0, \quad \Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}. \quad (7)$$

Here the notation of the reflections leads to:

- **space inversion**: $\psi(x,t) \mapsto \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix} \psi(x,t) =: \psi_s(x,t)$
- **time inversion**: $\psi(x,t) \mapsto \begin{pmatrix} 0 & \gamma^1 \gamma^3 \\ -\gamma^1 \gamma^3 & 0 \end{pmatrix} J\psi(x,-t) =: \psi_t(x,t)$
- **charge conjug.**: $\psi(x,t) \mapsto i \begin{pmatrix} \gamma^2 \\ 0 \end{pmatrix} J\psi(x,t) =: \psi_c(x,t)(x,t)$

Now $\psi$ is an eight component function. An elementary particle described by equation (7) has the same properties as a particle described by equation (1), i.e. the same magnetic dipole moment with the same behaviour under the reflections. Again one can get the Dirac equation from a Lagrangian. Adding the covariant and gauge invariant term

$$L_{\text{eff}} = -d\psi^+ \begin{pmatrix} 0 & \gamma^\mu \gamma^\nu \gamma_5 \\ \gamma^\mu \gamma^\nu \gamma_5 & 0 \end{pmatrix} F_{\mu\nu} \psi$$

(8)

to the Lagrangian

$$L = \psi^+(\Gamma^\mu (i\partial_\mu + eA_\mu) - m)\psi$$

(9)

leads to the field equation

$$\left(\Gamma^\mu (i\partial_\mu + eA_\mu) - d \begin{pmatrix} 0 & \gamma^\mu \gamma^\nu \gamma_5 \\ \gamma^\mu \gamma^\nu \gamma_5 & 0 \end{pmatrix} F_{\mu\nu} - m\right) \psi = 0. \quad (10)$$

This equation again describes an elementary particle with an electric dipole moment. In the non-relativistic limit the coupling with an external electric field is of the form

$$\begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix} E,$$

with $\hat{\sigma}$ from (8). In contrast now to the non-doubled case one has:

**Note 3** Let $\psi$ be a solution of equation (10) then $\psi_s$ and $\psi_t$ are solutions of the reflected equations.

This means that equation (10) is invariant both under space and time inversion. Since equations (8) and (10) are invariant under charge conjugation, which is seen by using the commutation relations of the gamma matrices, the second equation describes contrary to the first one a non-CP-violating electric dipole moment of elementary particles.
Quantization of the Dirac Field

We write the field equation (10) in terms of a Hamiltonian which can be checked to be hermitian. That means:

\[ i \partial_t \psi = H \psi \]  

(11)

with

\[
H = \Gamma^0 \Gamma^j (i \partial_j - eA_j) + eA_0 + \Gamma^0 m - d \Gamma^0 \begin{pmatrix} 0 & \gamma^\mu \gamma^\nu \gamma_5 \\ \gamma^\mu \gamma^\nu \gamma_5 & 0 \end{pmatrix} F_{\mu \nu}
\]  

(12)

The spectrum of the Hamiltonian \( H \) depends on the external fields. We want to assume that the Hilbert space \( \mathcal{H} \) of the Dirac equation can be split into two orthogonal spectral subspaces of the Hamiltonian,

\[
\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- ,
\]

such that \( \mathcal{H}_+ \) can be interpreted as a Hilbert space for a particle. This is the assumption of weak external fields. For a more rigorous treatment see [10].

Now one is able to build the Fock space \( F \) over \( \mathcal{H} \). The unitary or antiunitary operators belonging to Poincaré transformations including the reflections can be implemented by unitary and antiunitary operators, respectively, because these operators do not mix the Hilbert spaces \( \mathcal{H}_+ \) and \( \mathcal{H}_- \). See [10] again for further details.

To complete our discussion we want to give the transformation formulas of the Dirac operator \( \psi(x) \) under the reflections in the type III representation. Therefore we write the Dirac operator as

\[
\psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 \vec{p}}{2\omega_\vec{p}} \left( \exp(i(\omega_\vec{p}x^0 - \vec{p} \vec{x})) \sum_{r,s=1}^2 v^{rs}(\vec{p}) b^+_{rs}(\vec{p}) \\
+ \exp(-i(\omega_\vec{p}x^0 - \vec{p} \vec{x})) \sum_{r,s=1}^2 u^{rs}(\vec{p}) a_{rs}(\vec{p}) \right)
\]  

(13)

with

\[
v^{1s}(\vec{p}) = \begin{pmatrix} \hat{v}^s(\vec{p}) \\ 0 \end{pmatrix}, \quad v^{2s}(\vec{p}) = \begin{pmatrix} 0 \\ \hat{v}^s(\vec{p}) \end{pmatrix}
\]

\[
 u^{1s}(\vec{p}) = \begin{pmatrix} \hat{u}^s(\vec{p}) \\ 0 \end{pmatrix}, \quad u^{2s}(\vec{p}) = \begin{pmatrix} 0 \\ \hat{u}^s(\vec{p}) \end{pmatrix}
\]

where \( \hat{v}^s(\vec{p}) \) and \( \hat{u}^s(\vec{p}) \) are solutions of the Fourier-transformed field equation (1). The unitary operators \( U_p, U_c \) and the antiunitary operator \( U_t \) are specified by

\[
U_p \psi(x) U_p^{-1} = \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix} \psi(-x,t)
\]

\[
U_t \psi(x) U_t^{-1} = \begin{pmatrix} 0 & \gamma^1 \gamma^3 \\ -\gamma^1 \gamma^3 & 0 \end{pmatrix} \psi(x,-t)
\]

\[
U_c \psi(x) U_c^{-1} = i \begin{pmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{pmatrix} J \psi(x,t).
\]  

(14)
The field equations for the Dirac operator $\psi(x)$ remain invariant under the reflections. A little calculation which is similar to the case of the type I representation leads to

\[
\begin{align*}
U_p a_{rs}^+ U_p^{-1} &= a_{rs}^+ (-\vec{p}) \\
U_p b_{rs}^+ U_p^{-1} &= -b_{rs}^+ (-\vec{p}) \\
U_t a_{rs}^+ U_t^{-1} &= a_{rs}^+ (-\vec{s}) (-\vec{p}) \\
U_t b_{rs}^+ U_t^{-1} &= b_{rs}^+ (-\vec{s}) (-\vec{p}) \\
U_c a_{rs}^+ U_c^{-1} &= b_{rs}^+ (\vec{p}) \\
U_c b_{rs}^+ U_c^{-1} &= a_{rs}^+ (\vec{p})
\end{align*}
\tag{15}
\]

If we require that these equations leave the vacuum $\Omega \in F$ invariant,

\[
U_p \Omega = U_t \Omega = U_c \Omega = \Omega,
\tag{16}
\]

then $U_p, U_t$ and $U_c$ are again unitary or antiunitary operators in the Fock space.

**Acknowledgement**

I thank H. Reeh, who proposed to look at the representations of the Poincaré group.

**References**

[1] W. Bernreuther, M. Suzuki: Rev. Mod. Phys. 13 (1991) 313
[2] N. F. Ramsey: Rep. Prog. Phys. 45 (1982) 95
[3] E. E. Salpeter: Phys. Rev. 112 (1950) 1642
[4] W. Pauli: Rev. Mod. Phys. 13 (1941) 203
[5] L. Landau: Nucl. Phys. 3 (1957) 127
[6] J. M. Frère: preprint CERN-TH6815/93 (1993)
[7] E. P. Wigner: Group Theoretical Concepts and Methods in Elementary Particle Physics (Editor F. Gürsey) Gordon and Breach 1962
[8] J. Giesen: diploma thesis, unpublished (1994)
[9] V. Bargmann: Gruppentheoretische Analyse der Lorentzianvarianz. ETH-Zürich: Vorlesungsausarbeitung 1963
[10] B. Thaller: The Dirac equation. Berlin, Heidelberg, New York: Springer 1992