Existence and uniqueness of limit cycles in a class of second order ODE’s with inseparable mixed terms

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Abstract
We prove a uniqueness result for limit cycles of the second order ODE \( \ddot{x} + \dot{x} \phi(x, \dot{x}) + g(x) = 0 \). Under mild additional conditions, we show that such a limit cycle attracts every non-constant solution. As a special case, we prove limit cycle’s uniqueness for an ODE studied in [5] as a model of pedestrians’ walk. This paper is an extension to equations with a non-linear \( g(x) \) of the results presented in [7].

Keywords: Uniqueness, limit cycle, second order ODE’s, star-shaped function, Conti-Filippov transformation.

1 Introduction
The simplest non-linear continuous dynamical systems originate from the study of planar differential systems,
\[
\begin{align*}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y), \\
P, Q &\in C^1(\mathbb{R}^2, \mathbb{R}^2).
\end{align*}
\]
(1)
Special cases of such systems are Lotka-Volterra ones, and systems equivalent to Liénard equations,
\[
\begin{align*}
\dot{x} &= y - F(x), \\
\dot{y} &= -g(x),
\end{align*}
\]
(2)
or to Rayleigh equations
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -g(x) - f(y).
\end{align*}
\]
(3)
All of them arise as mathematical models of biological, physical, engineering
systems [3]. The study of the dynamics of (1) strongly depends on the existence
and stability properties of special solutions such as equilibrium points and non-
constant isolated periodic solutions. In particular, if an attracting non-constant
periodic solution exists, then it dominates the dynamics of (1) in an open,
connected subset of the plane, its region of attraction. Studying the number
and location of isolated periodic solutions, usually called limit cycles, is by no
means a trivial question, as shown by the resistance of Hilbert XVI problem (see
[9], problem 13). In some cases such a region of attraction can even extend to
cover the whole plane, with the unique exception of an equilibrium point. In such
a case the limit cycle is unique and dominates the system’s dynamics, as in [4].
Uniqueness theorems for limit cycles have been extensively studied (see [2], [13],
[14], for recent results and extensive bibliographies). Limit cycle’s uniqueness
is a relevant feature even in discrete time systems, which are often related to
continuous time systems [10]. Sometimes, suitable symmetry conditions have
been used, in order to simplify the study of such systems. In particular $Z_2$
symmetry, that is orbital symmetry with respect to one axis, has proved to be
useful in approaching similar problems [12].

Most of the results obtained for continuous time dynamical systems in the
plane are concerned with the classical Liénard system (2) and its generalizations,
such as
\[
\dot{x} = \xi(x) \left[ \varphi(y) - F(x) \right], \quad \dot{y} = -\zeta(y)g(x). \quad (4)
\]
Such a class of systems also contain Lotka-Volterra systems and systems equiva-
 lent to Rayleigh equation (3) as special cases.

Even if the systems (4) reach a high level of generality, compared to Van der
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Even if the systems (4) reach a high level of generality, compared to Van der
Pol system,
\[
\dot{x} = y - \epsilon \left( \frac{x^3}{3} - x \right), \quad \dot{y} = -x.
\]
the first one to be investigated in relation to existence and uniqueness of limit
cycles, an evident limitation is given by the fact that the variables $x$ and $y$
appear separately, so that mixed terms are products of single-variable functions.
Since models displaying a different combination of variables do exist, different
methods are desirable, in particular in absence of symmetry conditions.

A recent result [2] is concerned with systems equivalent to
\[
\ddot{x} + \sum_{k=0}^{N} f_{2k+1}(x)x^{2k+1} + x = 0, \quad (5)
\]
with $f_{2k+1}(x)$ increasing for $x > 0$, decreasing for $x < 0$, $k = 0, \ldots, N$. On the
other hand, there exist classes of second order models which are not covered by
previous results. This is the case of a model developed in [5] to describe the
pedestrian’s walk, which leads to the equation
\[\ddot{x} + \epsilon \dot{x}(x^2 + x\dot{x} + \dot{x}^2 - 1) + x = 0, \quad \epsilon > 0. \tag{6}\]

Such an equation can be considered as a special case of a more general class of equations,
\[\ddot{x} + \dot{x}\phi(x, \dot{x}) + g(x) = 0. \tag{7}\]

In this paper we study the class (7), assuming \(\phi(x, \dot{x})\) to have strictly star-shaped level sets and \(xg(x) > 0\) for \(x \neq 0\). We prove a uniqueness result for limit cycles, and, under suitable additional assumptions, we show that a limit cycle exists and attracts every non-constant solution. Since \(\epsilon \dot{x}(x^2 + x\dot{x} + \dot{x}^2 - 1)\) has strictly star-shaped level sets, the model introduced in [5] has a unique limit cycle, attracting every non-constant solution.

The result we present here is as well applicable to several equations of Liéanrd and Rayleigh type, in particular when they have a non-linear \(g(x)\).

This paper is organized as follows. In section 1 we study the equation (7), assuming \(g(x)\) to be linear. We first prove the uniqueness theorem. The main tools applied here is a uniqueness result proved in [6]. Then we introduce some mild additional hypotheses on the sign of \(\phi(x, \dot{x})\), under which the unique limit cycle attracts every non-constant solution. Then, in section 2, we assume \(g(x)\) to be non-linear. We reduce the study of such a case to that of the linear \(g(x)\), by means of Conti-Filippov transformation [8]. The structure of section 2 is very similar to that of section 1, the main difference being the derivation of the condition on \(\phi(x, \dot{x})\) which implies the strict star-shapedness property for the transformed system.

2 Linear \(g(x)\)

Let \(\Omega \subset \mathbb{R}^2\) be a star-shaped set. We denote partial derivatives by subscripts, i.e., \(\phi_x\) is the derivative of \(\phi\) w. r. to \(x\), etc.. We say that a function \(\phi \in C^1(\Omega, \mathbb{R})\) is star-shaped if \((x, y) \cdot \nabla \phi = x\phi_x + y\phi_y\) does not change sign. We say that \(\phi\) is strictly star-shaped if \((x, y) \cdot \nabla \phi \neq 0\), except at the origin \(O = (0, 0)\). We say that \(\gamma(t)\) is positively bounded if the semi-orbit \(\gamma^+ = \{\gamma(t), \ t \geq 0\}\) is contained in a bounded set. Similarly for the negative boundedness. We say that an orbit is an open unbounded orbit if it is both positively and negatively unbounded. We say that a set \(X\) is invariant if every orbit starting at a point of \(X\) is entirely contained in \(X\). For other definitions related to dynamical systems, we refer to [1] We call ray a half-line having origin at the point \((0, 0)\).

In this section we are concerned with the equation
\[\ddot{x} + \dot{x}\phi(x, \dot{x}) + kx = 0, \quad k \in \mathbb{R}, \quad k > 0. \tag{8}\]

Without loss of generality, possibly performing a time rescaling, we may restrict to the case \(k = 1\). Let us consider a system equivalent to the equation (8), for
\( k = 1, \)
\[ \begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - y\phi(x,y).
\end{align*} \]  
(9)

We denote by \( \gamma(t, x^*, y^*) \) the unique solution of the system (9) such that \( \gamma(0, x^*, y^*) = (x^*, y^*) \). We first consider a sufficient condition for limit cycles’ uniqueness. We set
\[ A(x, y) = y\dot{x} - x\dot{y} = y^2 + x^2 + xy\phi(x,y). \]

The sign of \( A(x, y) \) is opposite to that of the angular speed of the solutions of (9). Our uniqueness result comes from theorem 2 of [6], in the form of corollary 6.

**Theorem 1.** Let \( \phi \in (\mathbb{R}^2, \mathbb{R}^2) \) be a strictly star-shaped function. Then (9) has at most one limit cycle.

**Proof.** Without loss of generality, one may assume that, for \((x, y) \neq (0, 0)\), \(x\phi_x + y\phi_y > 0\). The proof can be performed analogously for the opposite inequality.

We claim that \( \nabla A(x, y) \) does not vanish on the set \( A_0 = \{(x, y) : A(x, y) = 0\} \setminus \{(0, 0)\} \). In fact, \( \nabla A \) and \( A \) vanish simultaneously at \((x, y)\) if and only if
\[ \begin{align*}
2x + y\phi + xy\phi_x &= 0 \\
2y + x\phi + xy\phi_y &= 0 \\
x^2 + y^2 + xy\phi &= 0
\end{align*} \]  
(10)

Multiplying the first equation by \( y \), the second one by \( x \) and re-ordering terms yields
\[ \begin{align*}
xy\phi &= -x^2(2 + y\phi_x) \\
xy\phi &= -y^2(2 + x\phi_y) \\
xy\phi &= -x^2 - y^2.
\end{align*} \]

Multiplying the third equation by 2 and summing with the first two equations yields
\[ xy(x\phi_x + y\phi_y) = 0. \]

Since, by hypothesis, \( x\phi_x + y\phi_y \neq 0 \) except at \( O \), one has \( xy = 0 \). If \( x = 0 \), then by the third equation in (10) one has \( y = 0 \). Similarly, if \( y = 0 \).

This shows that, at every point, \( A_0 \) is locally a graph. Additionally, every ray \( \{(t \cos \theta, t \sin \theta), t > 0\} \), meets \( A_0 \) at most at a point. In fact, for \( xy \neq 0 \), one has,
\[ A(t \cos \theta, t \sin \theta) = 0 \iff \phi(t \cos \theta, t \sin \theta) = \frac{1}{\cos \theta \sin \theta}, \]

The condition \( x\phi_x + y\phi_y > 0 \) implies that \( \phi \) is an increasing function of \( t \) on every ray. Hence on every ray not contained in an axis there exists at most one \( t \) such that \( \phi(t \cos \theta, t \sin \theta) = \frac{1}{\cos \theta \sin \theta} \). As for \( xy = 0 \), \( A \) vanishes only at \( O \).
Moreover, working as above, one can show that $A_0$ has at a single point in common with the axes, $O$.

The radial derivative $A_r$ of $A$ is given by

$$A_r = \frac{x A_x + y A_y}{r} = \frac{1}{r} \left( 2A + xy(x\phi_x + y\phi_y) \right). \quad (11)$$

Let $(x^*, y^*)$ be a point of the first orthant, i.e. $x^* > 0, y^* > 0$. If $A(x^*, y^*) \geq 0$, then $A_r > 0$ at $(x^*, y^*)$ and at every point $(rx^*, ry^*)$ with $r > 1$, hence $A$ is strictly increasing on the half-line $\{(rx^*, ry^*) : r > 1\}$. Now, let $(x^*, y^*)$ be a point of the second orthant, i.e. $x^* > 0, y^* < 0$. If $A(x^*, y^*) < 0$, then $A_r < 0$ at $(x^*, y^*)$ and at every point $(rx^*, ry^*)$ with $r > 1$, hence $A$ is strictly decreasing on the half-line $\{(rx^*, ry^*) : r > 1\}$. The same argument allows to prove that in the third orthant $A$ behaves as in the first one, and in the fourth orthant $A$ behaves as in the second one.

Assume, by absurd, two distinct limit cycles to exist. The system $\{0\}$ has a unique critical point, hence they are concentric. Let $\mu_1$ be the inner one, $\mu_2$ be the external one. Let $D$ be the annular region bounded by $\mu_1$ and $\mu_2$. We claim that $A(x, y) > 0$ in $D$. We prove it by proving that, for every orbit $\gamma$ contained in $D$, $A(\gamma(t)) > 0$. Let us observe that every orbit in $D$ has to meet every semi-axis, otherwise its positive limit set would contain a critical point different from $O$. On every semi-axis one has $A(x, y) > 0$. Assume first, by absurd, $A(\gamma(t))$ to change sign. Then there exist $t_1 < t_2$ such that $A(\gamma(t_1)) > 0, A(\gamma(t_2)) < 0$, and and $\gamma(t_3), i = 1, 2$ are on the same ray. Assume $\gamma(t_i), i = 1, 2$ to be in the first orthant. Two cases can occur: either $|\gamma(t_1)| < |\gamma(t_2)|$ or $|\gamma(t_1)| > |\gamma(t_2)|$.

The former, $|\gamma(t_1)| > |\gamma(t_2)|$, contradicts the fact that $A$ is radially increasing in the first orthant, hence one has $|\gamma(t_1)| > |\gamma(t_2)|$. The orbit $\gamma(t_1)$ crosses the segment $\Sigma = \{r\gamma(t_1), 0 < r < 1\}$, going towards the positive $y$-semi-axis. Let $G$ be the region bounded by the positive $y$-semi-axis, the ray $\{r\gamma(t_1), r > 0\}$ and the portions of $\mu_1, \mu_2$ meeting the $y$-axis and such a ray. The orbit $\gamma$ cannot remain in $G$, since in that case $G$ would contain a critical point different from $O$. Also, $\gamma$ cannot leave $G$ crossing the positive $y$-semi-axis, because $A(x, y) > 0$ on such an axis. Hence $\gamma$ leaves $G$ passing again through the segment $\Sigma$. That implies the existence of $t_3 > t_2$, such that $\gamma(t_3)$ lies on the ray $\{r\gamma(t_1), 0 < r\}$. Again, one cannot have $|\gamma(t_3)| < |\gamma(t_2)|$, since $A(\gamma(t_1)) > 0$ implies $A$ increasing on the half-line $r\gamma(t_3))$, $r > 1$, hence one has $|\gamma(t_3)| > |\gamma(t_2)|$. Also, one cannot have $|\gamma(t_3)| > |\gamma(t_1)|$, otherwise $\gamma(t_3)$ would enter a positively invariant region, bounded by the curve $\gamma(t)$, for $t_1 < t < t_3$, and by the segment with extrema $\gamma(t_1), \gamma(t_3)$, hence there would exist a critical point different from $O$. As a consequence, one has $|\gamma(t_3)| > |\gamma(t_1)|$. Since $A(x, y) > 0$ on the segment joining $\gamma(t_1)$ and $\gamma(t_3)$, such a segment, with the portion of orbit joining $\gamma(t_1)$ and $\gamma(t_3)$ bounds a region which is negatively invariant for $\{0\}$, hence contains a critical point different from $O$, contradiction.

This argument may be adapted to treat also the case of a ray in the second orthant, replacing the positive $y$-semi-axis with the positive $x$-semi-axis, and reversing the relative positions of the points $\gamma(t_1), \gamma(t_2), \gamma(t_3)$. In the other orthants one repeats the arguments of the first and second orthants, respectively.
Finally, assume that $A(\gamma(t)) = 0$ at some point $(x^*, y^*)$. If $(x^*, y^*)$ is interior to $D$, then on the ray $\{(rx^*, ry^*), r > 0\}$, there exist points interior to $D$ with $A(rx^*, ry^*) < 0$. Then we can apply the above argument to the orbits starting at such points. If $(x^*, y^*)$ belongs to the boundary of $D$, then it is on $\mu_1$ or on $\mu_2$. Assume $(x^*, y^*) = \mu_1(t^*)$, for some $t^*$ (the argument works similarly on $\mu_2$). Since $A(\mu_1(t^*)) = 0$, $\mu_1$ is tangent to the ray $\{(rx^*, ry^*), r > 0\}$. On the other hand, $A_r = \frac{r y (x \phi_x + y \phi_y)}{r} \neq 0$ at $(x^*, y^*)$, hence $A_0$ is neither tangent to the ray $\{(rx^*, ry^*), r > 0\}$, nor to $\mu_1$ at $\mu_1(t^*)$. This implies that $A_0$ and $\mu_1$ are transversal at $(x^*, y^*)$, so that a portion of $A_0$ enters $D$. Since $A_0$ separates points where $A > 0$ from points where $A < 0$, also in this case there exist points interior to $D$ with $A < 0$.

Now we can restrict to the annular region $D$ and divide the vector field of (9) by $A(x, y)$, as in corollary 6 in [6]. In order to apply such a corollary, one has to compute the expression

$$\nu = P(xQ_x + yQ_y) - Q(xP_x + yP_y),$$

where $P$ and $Q$ are the components of the considered vector field. For system (9), one has

$$\nu = y(\begin{matrix} -x - xy\phi_x - y\phi - y^2\phi_y \\ -y^2 (x\phi_x + y\phi_y) \end{matrix}) \leq 0.$$

The function $\nu$ vanishes only for $y = 0$. For both cycles one has:

$$\int_0^{T_i} \nu(\mu_i(t))dt < 0, \quad i = 1, 2,$$

where $T_i$ is the period of $\mu_i$, $i = 1, 2$. Hence both cycles, by theorem 1 in [6], are attractive. Let $A_1$ be the region of attraction of $\mu_1$. $A_1$ is bounded, because it is enclosed by $\mu_2$, which is not attracted to $\mu_1$. The external component of $A_1$’s boundary is itself a cycle $\mu_3$, because [9] has just one critical point at $O$. Again,

$$\int_0^{T_3} \nu(\mu_3(t))dt < 0,$$

hence $\mu_3$ is attractive, too. This contradicts the fact that the solutions of (9) starting from its inner side are attracted to $\mu_1$. Hence the system (9) can have at most a single limit cycle.

The angular velocity of the solutions need not be negative at every point of the plane. In fact, even a simple system as that one studied in [6] has negative angular velocity only in a proper subset of the plane. In Figure 1 we have plotted some orbits of the equation (9) tending at the limit cycle, together with the two components of the curve $A_0$. The orbits cross $A_0$ at the points where their angular velocity changes sign.
In the example of Figure 1 we have chosen $\epsilon = 1$. In general, the system studied in [5] has just one limit cycle, for $\epsilon > 0$. In fact, in this case one has

$$\nu = x\phi_x + y\phi_y = 2\epsilon y^2(x^2 + xy + y^2) > 0 \quad \text{for} \quad y \neq 0.$$  

It should be noted that even if the proof is essentially based on a stability argument, the divergence cannot be used in order to replace the function $\nu$. In fact, the divergence of system (9) is

$$\text{div} \left( y, -x - y\phi(x, y) \right) = -\phi - y\phi_y,$$

which does not have constant sign, under our assumptions. Moreover, the divergence cannot have constant sign in presence of a repelling critical point and an attracting cycle.

Now we care about the existence of limit cycles. Let us denote by $D_r$ the disk $\{(x, y) : x^2 + y^2 \leq r^2\}$, and by $\partial D_r$ its boundary $\{(x, y) : x^2 + y^2 = r^2\}$. Let us consider the function $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Its derivative along the solutions of (9) is

$$\dot{V}(x, y) = -y^2\phi(x, y).$$

**Lemma 1.** Let $U$ be a bounded set, with $\sigma := \sup \{\sqrt{x^2 + y^2}, (x, y) \in U\}$. If $\phi(x, y) \geq 0$ out of $U$, and $\phi(x, y)$ does not vanish identically on any $\partial D_r$, for $r > \sigma$, then every $\gamma(t)$ definitely enters the disk $D_\sigma$ and does not leave it.

**Proof.** The level curves of $V(x, y)$ are circumferences. For every $r \geq \sigma$, the disk $D_r$ contains $U$. Since $\dot{V}(x, y) = -y^2\phi(x, y) \leq 0$ on its boundary, such a disk is positively invariant. Let $\gamma$ be an orbit with a point $\gamma(t^*)$ such that $d^* = \text{dist}(\gamma(t^*), O) > \sigma$. Then $\gamma$ does not leave the disk $D_{d^*}$, hence it is positively bounded. Moreover $\gamma(t)$ cannot be definitely contained in $\partial D_r$, for
any $r > \sigma$, since $\dot{V}(x,y)$ does not vanish identically on any $\partial D_r$, for $r > \sigma$. Now, assume by absurd that $\gamma(t)$ does not intersect $B_\sigma$. Then its positive limit set is a cycle $\gamma(t)$, having no points in $D_\sigma$. The cycle $\gamma(t)$ cannot cross outwards any $\partial D_r$, for some $r > \sigma$, contradicting the fact that $V(x,y)$ does not vanish identically on any $\partial D_r$, for $r > \sigma$. Hence there exists $t^+ > t^*$ such that $\gamma(t^+) \in D_\sigma$. Then, for every $t > t^+$, one has $\gamma(t) \in D_\sigma$, because $\dot{V}(x,y) \leq 0$ on $B_\sigma$.

Collecting the results of the above statements, we may state a theorem of existence and uniqueness for limit cycles of a class of second order equations. We say that an equilibrium point $O$ is negatively asymptotically stable if it is asymptotically stable for the system obtained by reversing the time direction.

**Theorem 2.** If the hypotheses of theorem 1 and lemma 1 hold, and $\phi(0,0) < 0$, then the system (9) has exactly one limit cycle, which attracts every non-constant solution.

**Proof.** By the above lemma, all the solutions are definitely contained in $D_\sigma$. The condition $\phi(0,0) < 0$ implies by continuity $\phi(x,y) < 0$ in a neighbourhood $N_O$ of $O$. This gives the negative asymptotic stability of $O$ by Lasalle’s invariance principle [11], since $\dot{V}(x,y) \geq 0$ in $N_O$, and the set $\{\dot{V}(x,y) = 0\} \cap N_O = \{y = 0\} \cap N_O$ does not contain any positive semi-orbit. The system has just one critical point at $O$, hence by Poincaré-Bendixson theorem there exist a limit cycle. By theorem 1, such a limit cycle is unique. ♣

This proves that every non-constant solution to the equation (6) studied in [5] is attracted to the unique limit cycle. We can produce more complex systems with such a property. Let us set

$$\phi(x,y) = -M + \sum_{k=1}^{n} H_{2k}(x,y),$$

with $H_{2k}(x,y)$ is a homogeneous function of degree $2k$, positive except at $O$, $M$ is a positive constant. Then, by Euler’s identity, one has

$$\nu = \sum_{k=1}^{n} \left( x \frac{\partial H_{2k}}{\partial x} + y \frac{\partial H_{2k}}{\partial y} \right) = \sum_{k=0}^{n} 2k H_{2k}(x,y) > 0 \quad \text{for} \quad (x,y) \neq (0,0).$$

If $\phi(x,y)$ does not vanish identically on any $\partial D_{\sigma}$, for instance if $H_{2k}(x,y) = (x^2 + xy + y^2)^k$, then the corresponding system [9] has a unique limit cycle. In general, it is not necessary to assume the positiveness of all of the homogeneous functions $H_{2k}(x,y)$, as the following example shows. Let us set $Q(x,y) = x^2 + xy + y^2$. Then take

$$\phi(x,y) = -1 + Q - Q^2 + Q^3.$$ 

One has

$$\nu = x \phi_x + y \phi_y = 2Q - 4Q^2 + 6Q^3 = Q(2 - 4Q + 6Q^2).$$
The discriminant of the quadratic polynomial $2 - 4Q + 6Q^2$ is $-32 < 0$ hence $\nu > 0$ everywhere but at $O$. Moreover, $\phi(x, y)$ does not vanish identically on any circumference, hence the corresponding system (9) has a unique limit cycle.

3 Non-linear $g(x)$

Even if the equation we consider in this section are of a more general type, we actually derive our result from that of the previous section, so that we can consider what follows a corollary of the previous result. Let us consider the equation

$$\ddot{x} + \dot{x}\Phi(x, \dot{x}) + g(x) = 0. \quad (12)$$

We assume that $xg(x) > 0$ for $x \neq 0$, $g \in C^1(\mathbb{R}, \mathbb{R})$, $g'(0) \neq 0$. We could consider equations defined in smaller subset of the plane, without essential changes.

The main tools is the so-called Conti-Filippov transformation, which acts on the equivalent system

$$\dot{x} = y, \quad \dot{y} = -g(x) - y\Phi(x, y), \quad (13)$$

in such a way to take the conservative part of the vector field into a linear one. Let us set $G(x) = \int_0^x g(s)ds$, and denote by $\sigma(x)$ the sign function, whose value is $-1$ for $x < 0$, 0 at 0, 1 for $x > 0$. Let us define the function $\alpha : \mathbb{R} \to \mathbb{R}$ as follows:

$$\alpha(x) = \sigma(x)\sqrt{2G(x)}.$$ 

Then Conti-Filippov transformation is the following one,

$$(u, v) = \Lambda(x, y) = (\alpha(x), y). \quad (14)$$

Since we assume that $g \in C^1(\mathbb{R}, \mathbb{R})$, one has $\alpha \in C^1(\mathbb{R}, \mathbb{R})$. The function $u = \alpha(x)$ is invertible, due to the condition $xg(x) > 0$. Let us call $x = \beta(u)$ its inverse. The condition $g'(0) > 0$ guarantees the differentiability of $\beta(u)$ at $O$. For $x \neq 0$, that is for $u \neq 0$, one has,

$$\alpha'(x) = \sigma(x)g(x)\sqrt{2G(x)}, \quad \beta'(u) = \frac{1}{\alpha'(\beta(u))} = \frac{\sigma(x)\sqrt{2G(x)}}{g(x)} = \frac{u}{g(\beta(u))}. \quad (15)$$

For $x = u = 0$ one has,

$$\alpha'(0) = \sqrt{g'(0)}, \quad \beta'(0) = \sqrt{\frac{1}{g'(0)}}.$$

Finally,

$$\lim_{u \to 0} \frac{g(\beta(u))}{u} = \sqrt{g'(0)} > 0.$$
Theorem 3. Assume $g \in C^1(\mathbb{R}, \mathbb{R})$, with $g'(x) > 0$ and $xg(x) > 0$ for $x \neq 0$. Let $\Phi \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ satisfy
\[
\frac{\sigma(x)\sqrt{2G(x)}}{g(x)} \left[ 2G(x) \frac{\Phi_x(x,y)g(x) - \Phi(x,y)g'(x)}{g(x)^2} + \Phi(x,y) \right] + y\Phi_y(x,y) \neq 0.
\]
Then (13) has at most one limit cycle.

Proof. For $u \neq 0$, the transformed system has the form
\[
\dot{u} = v \frac{g(\beta(u))}{u}, \quad \dot{v} = -g(\beta(u)) - v\Phi(\beta(u), v). \tag{16}
\]
For $u = 0$, the above form is extended by continuity. In the following we consider only the case $u \neq 0$, that is $x \neq 0$, since the case $u = 0 = x$ is obtained by continuity. We may multiply the system (16) by $\frac{u}{g(\beta(u))}$, obtaining a new system having the same orbits as (16),
\[
\dot{u} = v, \quad \dot{v} = -u - v \frac{u\Phi(\beta(u), v)}{g(\beta(u))}. \tag{17}
\]
Such a system is of the type (9), so that we may apply theorem 1 to get uniqueness of solutions. This reduces to require the strict star-shapedness of the function
\[
\frac{u\Phi(\beta,u)}{g(\beta(u))},
\]
that is,
\[
u \left[ \frac{u\Phi(\beta(u), v)}{g(\beta(u))} \right] + v\Phi_v(\beta(u), v) > 0.
\]
Since $\beta(u) = x$, $v = y$, the second term in the above sum is just $y\Phi_y(x,y)$. As for the the first one, one has
\[
\left[ \frac{u\Phi(\beta(u), v)}{g(\beta(u))} \right]_u = \frac{u\Phi_u(\beta(u), v)\beta'(u) + \Phi(\beta(u), v)}{g(\beta(u))^2} g'(\beta(u)) - u\Phi(\beta(u), v)g'(\beta(u))\beta'(u).\]
Replacing $\beta(u)$ with $x$ and applying the formulae (15) one has
\[
\frac{\sigma(x)\sqrt{2G(x)}}{g(x)} \left[ \frac{\sigma(x)\sqrt{2G(x)}\Phi_x(x,y)}{g(x)^2} + \Phi(x,y) \right] + \frac{\Phi(x,y)}{g(x)}.
\]
Since $\sigma(x)^2 = 1$ everywhere but at 0, the above formula reduces to

$$\begin{bmatrix} u\Phi(\beta(u, v)) & g(\beta(u)) \end{bmatrix} = \frac{2G(x)}{g(x)} \Phi_x(x, y)g(x) - \Phi(x, y)g'(x) + \Phi(x, y).$$

Concluding, one has

$$u \left[ \frac{u\Phi(\beta(u, v))}{g(\beta(u))} \right] = \frac{\sigma(x)\sqrt{2G(x)}}{g(x)} \left[ 2G(x) \frac{\Phi_x(x, y)g(x) - \Phi(x, y)g'(x)}{g(x)^2} + \Phi(x, y) \right].$$

Hence, the star-shapedness conditions reduces to

$$\frac{\sigma(x)\sqrt{2G(x)}}{g(x)} \left[ 2G(x) \frac{\Phi_x(x, y)g(x) - \Phi(x, y)g'(x)}{g(x)^2} + \Phi(x, y) \right] + y\Phi_y(x, y) \neq 0.$$

\[\blacklozenge\]

If $g(x) = x$, then $G(x) = \frac{x^2}{2}$ and $\frac{\sigma(x)\sqrt{2G(x)}}{g(x)} = 1$, for $x \neq 0$. In this case the star-shapedness condition just reduces to what considered in the previous section, since

$$2G(x) \frac{\Phi_x(x, y)g(x) - \Phi(x, y)g'(x)}{g(x)^2} + \Phi(x, y) = x\Phi_x(x, y).$$

Now we prove the non-linear analogous of lemma 1. Let us set

$$E(x, y) = G(x) + \frac{y^2}{2}$$

For $r > 0$, we set $\Delta_r = \{(x, y) : 2E(x, y) < r^2\}$ and $\partial\Delta_r = \{(x, y) : 2E(x, y) = r^2\}$

**Lemma 2.** Let $U$ be a bounded set, with $\sigma := \sup\{\sqrt{2E(x, y)}, (x, y) \in U\}$. If $\Phi(x, y) \geq 0$ out of $U$, and $\Phi(x, y)$ does not vanish identically on any $\partial\Delta_r$, for $r > \sigma$, then every $\gamma(t)$ definitely enters the set $\Delta_\sigma$ and does not leave it.

**Proof.** Performing Conti-Filippov transformation, the sets $\Delta_r$ are taken into the sets $D_r$, as well as the boundaries $\partial\Delta_r$ are taken into the boundaries $\partial D_r$. The function $\Phi(x, y)$ does not vanish identically on any $\partial\Delta_r$ if and only if the function $\phi(u, v) = \Phi(\beta(u, v))$ does not vanish identically on any $\partial D_r$. Then one can apply lemma 1 to the system \[17\]. In fact, the derivative of the Liapunov function $V(u, v) = \frac{1}{2}(u^2 + v^2)$ along the solutions of \[17\] is just

$$\dot{V}(u, v) = -v^2 \frac{u\Phi(\beta(u, v))}{g(\beta(u))}.$$

The function $\frac{u\Phi(\beta(u, v))}{g(\beta(u))}$ is positive for $u \neq 0$, hence the hypotheses of lemma 1 are satisfied by the system \[17\]. As a consequence, the conclusions of lemma 1 hold
for the system \( (17) \), and applying the inverse transformation \( \Lambda^{-1} \) one obtains the thesis.

Now we can conclude proving the analogue of theorem 2 for the equation with a non-linear \( g(x) \).

**Theorem 4.** If the hypotheses of theorem 3 and lemma 2 hold, and \( \phi(0,0) < 0 \), then the system \( (13) \) has exactly one limit cycle, which attracts every non-constant solution.

**Proof.** As the proof of theorem 2 replacing the Liapunov function \( V(x,y) \) with the Liapunov function \( E(x,y) \).

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