Decomposing instantons in two dimensions

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Abstract

We study the Bogomol’nyi–Prasad–Sommerfeld (BPS) vortices in the \((1+1)\)-dimensional \(\mathcal{N} = (2, 2)\) supersymmetric \(U(1)\) gauged \(\mathbb{C}P^1\) nonlinear sigma model. We use the moduli matrix approach to analytically construct the moduli space of solutions and solve numerically the BPS equations. We identify two topologically inequivalent types of magnetic vortices, which we call \(S\) and \(N\) vortices. Moreover, we discuss their relation to instantons (lumps) present in the ungauged case. In particular, we describe how a lump is split into a couple of component \(S–N\) vortices after gauging. We extend this analysis to the case of the extended Abelian Higgs model with two flavors, which is known to admit semi-local vortices. After gauging the relative phase between fields, semi-local vortices are also split into component vortices. We discuss interesting applications of this simple set-up. Firstly, the gauging of nonlinear sigma models reveals a semiclassical ‘partonic’ nature of instantons in \(1+1\) dimensions. Secondly, weak gauging provides for a new interesting regularization of the metric of semi-local vortices.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Instantons play an important role in quantum field theories in various dimensions. In four dimensions, they play a prominent role in defining the properties of the QCD vacuum, and in particular in explaining strong coupling effects like chiral symmetry breaking. In the case of \(\mathcal{N} = 2\) supersymmetric QCD, they give non-perturbative corrections to the exact low-energy effective action [1]. It has been long pointed out that there are similarities between four-dimensional gauge theories and two-dimensional nonlinear sigma models, such as dynamical generation of a mass gap and asymptotic freedom. These similarities extend to the quantitative...
level in the supersymmetric case, where it happens that the exact Bogomol’nyi–Prasad–Sommerfeld (BPS) spectra of super-QCD and nonlinear sigma models are the same [2]. It is not a coincidence that instantons were found in the $O(3)$ sigma model [3] almost at the same time of the discovery of instantons in Yang–Mills theory [4]. The $O(3)$ sigma model is equivalent to the $CP^1 (\approx SU(2)/U(1))$ sigma model and therefore instanton solutions can be extended to the $CP^{N-1} (\approx SU(N)/[SU(N-1) \times U(1)])$ model. In more general terms, when $\pi_2 (M)$ is nontrivial for the target space $M$, the model admits instantons. Instantons in both Yang–Mills theory and sigma models have a scale modulus in addition to orientational moduli in the internal space and position moduli. In particular, the moduli space of instantons in four-dimensional Yang–Mills has real dimension $4N$, while that of two-dimensional sigma model instantons is $2N$. Sigma model instantons are particle-like objects in $2 + 1$ dimensions, and they are called lumps in field theory [5], and skyrmions or coreless vortices in condensed matter physics. The $CP^1$ model can be lifted to a $U(1)$ gauge theory with two complex scalar fields with equal charges, which reduces to the $CP^1$ model in the limit of gauge coupling sent to infinity. The sigma model instanton is lifted to a semi-local vortex [6–10], having the same moduli including a size modulus.

The moduli space dimension of four- and two-dimensional instantons, together with other circumstantial observations, has led to the conjecture that instantons can be more conveniently thought as being formed of $N$ component ‘partons’. The moduli space parameters then describe the positions of these fundamental objects in the Euclidean space. The idea that the functional integral of strongly coupled Yang–Mills is dominated by a liquid of partons which forms the vacuum explains many aspects of confinement and chiral symmetry breaking [11]. The decomposition of sigma model instantons has also been considered more recently [12, 13] and it can occur by deforming the metric of the target space of sigma models. The energy density of the configuration then has subpeaks which can then be interpreted as partons. The proposal of [12] is to identify these component partons as the UV degrees of freedom which may render $2 + 1$ sigma models and $4 + 1$ gauge theories renormalizable. In this paper, we discuss, at the semiclassical level, the decomposition of instantons in a manner different from [12–14]. To this end, we consider a two-dimensional $N = (2, 2)$ supersymmetric $CP^1$ nonlinear sigma model.

Figure 1. $N$ and $S$ vortices. The boundary conditions of the $N$ and $S$ vortices are shown by arrows which represent points on the gauge orbit (on the equator) of $S^2$. $N$ and $S$ denote the north and south poles in the vortex cores.
while $N$ and $S$ vortices wrap upper and lower hemispheres bounded by the vacuum $U(1)$ gauge orbit; see figure 2. Accordingly, each of them has a half (or more generally fractional) instanton charge of $\pi_2(M)$, so that they can be called two-dimensional merons [19], in analogy with merons in four dimensions [20]. A set of $N$ and $S$ vortices can be interpreted as one instanton as can be seen in figure 2, where the distance between them corresponds to the size of the instanton. In this sense, $N$ and $S$ vortices are constituents of an instanton. It was found that the moduli space metric is incomplete at the point where the positions of $N$ and $S$ vortices coincide [17]. In our understanding, this is nothing but a small instanton singularity.

Once the $\mathbb{C}P^1$ model is constructed as the low energy limit of a linear $U(1)$ gauge theory, the $U(1)$ gauged $\mathbb{C}P^1$ model can be formulated as the $U(1) \times U(1)$ gauge theory with two complex scalar fields. In this way, the moduli space of instantons is promoted to that of semi-local vortices, which is regular. Then, the small instanton singularities are resolved. No pathology occurs when $N$ and $S$ vortices coincide, so that the moduli space is regularized.

Another advantage in considering this $U(1) \times U(1)$ linear formulation is that we can obtain a rather interesting regularization of the semi-local vortex metric. An important recent development concerning the aforementioned correspondence between BPS spectra in two and four dimensions regards the role played by non-Abelian semi-local vortices. More precisely, the correspondence holds between four-dimensional $U(N)$ super-QCD with $N_f > N$ flavors and the two-dimensional effective theory on the vortex worldsheet [21, 22] hosted by the theory when put on the Higgs phase. This correspondence has been proved using D-brane constructions of the vortex theory. However, while string theory gives a well-defined effective theory, it is well known that some zero-modes are non-normalizable when the effective theory is derived from field theory [23, 24]. It is then difficult to quantize the vortex theory and check the correspondence in a fully field theoretic framework. A possible approach to the problem has been considered, for example, in [25]. Here we propose weak gauging as an alternative approach to the problem. The weak gauging of a nonlinear sigma model, when considered as the effective theory of a semi-local vortex string, should correspond to a deformation of the four-dimensional bulk theory. This would give new 4D/2D correspondences together with important insights on the physics of non-normalizable modes in the undeformed case.

While we have considered only the supersymmetric case, we can also use our analysis to obtain qualitative answers about non-supersymmetric cases. A more generic potential would introduce, for example, some non-trivial interactions between the component vortices. There exist examples of analogs of these non-supersymmetric vortices in condensed matter systems,
in which also one or both of the $U(1)$ symmetries are global. The case when both $U(1)$s are global is relevant in the case of anti-ferromagnets and two-component Bose–Einstein condensates of ultracold atoms in the anti-ferromagnetic phase [26]. Even if these systems have a global symmetry differing from the supersymmetric gauged $\mathbb{CP}^1$ model, they have similar potential terms. Consequently, topological properties, for instance whether and how instantons are decomposed into constituent vortices, are the same. Another condensed matter example which is similar to our model is given by two-band superconductors. The Landau–Ginzburg Lagrangian proposed to describe them consists of two gaps (complex scalar fields) coupled to the electro-magnetic $U(1)$ gauge field. It has $U(1) \times U(1)$ symmetry, one of which is local electro-magnetic $U(1)$ symmetry while the other $U(1)$ symmetry, a relative phase between the two fields, is a global $U(1)$ which is broken explicitly in the presence of Josephson interactions between two gaps. This model is also known to admit fractional vortices [27] which are mostly the same with ours in the absence of Josephson interactions. The study of our supersymmetric gauged $\mathbb{CP}^1$ model will give insights to these condensed matter systems.

The structure of the paper is as follows. In section 2, we briefly review the construction of two-dimensional instantons as $\mathbb{CP}^1$ lumps. We then employ the moduli matrix formalism to construct vortices in the gauged version of the sigma model. In section 3, we construct a bound state of $N$ and $S$ vortices and show how they emerge from an instanton configuration as we increase the strength of the gauge interactions. In section 4, we lift the $\mathbb{CP}^1$ nonlinear sigma model to a $U(1) \times U(1)$ linear formulation and consider regularization of small instanton singularities. In section 5, we discuss how the gauging of flavor symmetries can provide for a nice regularization of the semi-local vortex metric. Finally, in section 6, we present various possible generalizations where we consider sigma models with higher-dimensional target spaces and gauging of flavor symmetries of higher rank.

2. Solitons in the gauged $\mathbb{CP}^1$ nonlinear sigma model

2.1 Models

Ungauged $\mathbb{CP}^1$. Let us start by considering the standard two-dimensional $\mathbb{CP}^1$ nonlinear sigma model (NL$\sigma$M). The action is as follows [3]:

$$\mathcal{L} = \xi \frac{|\partial_\mu b|^2}{(1 + |b|^2)^2}, \quad \mu = 1, 2. \quad (2.1)$$

In the formula above, $b$ is the holomorphic coordinate parameterizing the target space, and we work with the Euclidean metric. The metric for $b$ is given by the standard round Fubini–Study metric and it is written in a patch which does not include the point at infinity $b = \infty$. We can include this point performing a change of variable $b = 1/b'$, under which the Lagrangian (2.1) is indeed invariant. The standard Fubini–Study metric is of the Kähler type, since it can be derived from a Kähler potential:

$$\mathcal{L} = \partial_\mu \partial_{\mu} K(b, \bar{b}) |\partial_\mu b|^2, \quad K = \xi \log(1 + |b|^2). \quad (2.2)$$

The Kähler property of the metric ensures the existence of a supersymmetric extension with four supercharges of the bosonic NL$\sigma$M [29]. The analyses of this paper can be extended to a study of static string-like solutions with translational symmetry in $3 + 1$ dimensions. Formula (2.1) will then represent the four-dimensional action dimensionally reduced to the plane transverse to the string. From the point of view of constructing solitons, the model we

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3 However, in the presence of Josephson interactions, two kinds of vortices are connected by a domain wall [28].
Consider can be thought of as being either a truncated bosonic sector of an \( N = 2 \) theory in four dimensions or an \( N = (2, 2) \) theory in two dimensions. Since our analysis is limited to the classical level, we never show explicitly the fermionic sector. Supersymmetry however fixes the kinetic terms and the potentials we will consider later.

Since the \( \pi_2 \) of the target space is nontrivial, the \( \mathbb{C}P^1 \ NL\sigma M \) contains stable topological solitons of codimension 2. This means string-like objects in four dimensions or instantons in two Euclidean dimensions. Solitons can be found with a square root completion of the Bogomol’nyi type \[3\]. If we restrict ourselves to consider static solutions, we can rewrite the Lagrangian (2.1) in the following way:

\[
E = 2\xi \frac{|\partial_\xi b|^2 + |\bar{\partial}_\xi b|^2}{(1 + |b|^2)^2} = 2\pi \xi N + 4\xi \frac{|\bar{\partial}_\xi b|^2}{(1 + |b|^2)^2},
\]

\[
z \equiv x_1 + ix_2 \quad \bar{\partial}_\xi \equiv \frac{1}{2} (\partial_1 - i\partial_2).
\] (2.3)

The quantity

\[
N \equiv \frac{1}{\pi} \frac{|\partial_\xi b|^2 - |\bar{\partial}_\xi b|^2}{(1 + |b|^2)^2}
\] (2.4)

gives the degree of the map \( b(z) : \mathbb{C}P^1 \to \mathbb{C}P^1 \) and is the topological integer which characterizes the second homotopy group:

\[
\pi_2(\mathbb{C}P^1) = \mathbb{Z}.
\] (2.5)

Clearly the energy has a lower bound \( E \geq 2\pi \xi N \) which is saturated when the following equation is satisfied:

\[
\bar{\partial}_\xi b = 0.
\] (2.6)

Topological solitons of the type above are BPS saturated \[30, 31\]. They satisfy first order equations of motion obtained from a square root completion and their energy is proportional to a topological integer. In theories with extended supersymmetry, the mass of BPS solitons saturates the bound given by a central extension of the supersymmetry algebra \[32\]. In the language of supersymmetry, lumps are 1/2 BPS, in the sense that they preserve 1/2 of the supersymmetry transformations \[33\]. Fermions and supersymmetry are crucial to preserve BPS saturation once quantum effects are taken into account \[34\].

Since the energy is proportional to the topological integer \( N \), which also counts the number of solitons, there are no static interactions among lumps. The consequence of this fact is the existence of a large set of degenerate solutions (moduli space) parameterized by the positions and orientation of the single-component lumps. This degeneration can be understood easily from a mathematical point of view if we note that the equation above implies that \( b \) is a holomorphic function of the complex variable \( z \). It must contain a finite number of poles and zeros, and must then be given by a holomorphic rational function \[3\]:

\[
b(z) = b_\infty \frac{\sum p_i z^{N-i} + \cdots + p_N}{\sum q_i z^{N-i} + \cdots + q_N}, \quad b_\infty \equiv b(\infty)k
\] (2.7)

where the degree \( N \) of the polynomials is the lump number (2.4). A fundamental lump is given for example by the following choice:

\[
b_0(z) = b_\infty \frac{z - z_1}{z - z_N}.
\] (2.8)

To extract physical quantities such as size and position from the rational map above, we can make use of the \( SU(2) \) isometry enjoyed by the \( \mathbb{C}P^1 \ NL\sigma M \) which acts nonlinearly on the field \( b \):

\[
b \to \frac{v + ub}{u^* - v^* b}, \quad \begin{pmatrix} u^* & -v^* \\ v & u \end{pmatrix} \in SU(2), \quad |u|^2 + |v|^2 = 1.
\] (2.9)
We can then always set $b_\infty \rightarrow 0$. This puts the moduli matrix (2.8) in the form

$$b_0 = \frac{\rho}{z - z_0},$$

which describes a lump of position $z_0$ and the size $\rho$. By explicitly performing this rotation, we obtain

$$z_0 = \frac{z_S + |b_\infty|^2 z_N}{1 + |b_\infty|^2}, \quad \rho = \frac{b_\infty}{1 + |b_\infty|^2} (z_S - z_N).$$

$\textit{Gauged CP}^1$. We now consider the gauged version of the $\text{CP}^1$ NLσM obtained from (2.1) by gauging the following $U(1)$ subgroup of the $\text{SU}(2)$ isometry (2.9):

$$U(1) : \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \quad u = e^{-i\theta}, \quad v = 0,$$

which acts linearly on the field $b$:

$$b \rightarrow e^{-2i\theta} b.$$ (2.12)

The Lagrangian we obtain is then the following [35]:

$$L_{U(1)} = -\frac{1}{4g^2} F^{\mu
u} F_{\mu\nu} + \xi \frac{|\nabla_\mu b|^2}{(1 + |b|^2)^2} - \frac{g^2}{2} \left( \frac{-2|b|^2}{1 + |b|^2} - \zeta \right)^2,$$

where

$$\nabla_\mu = \partial_\mu + 2iA_{\mu\nu}.$$ (2.14)

The complicated potential term is the necessary one for the existence of BPS saturated solitons. It can be more easily derived as a D-term potential which arises as one imposes supersymmetry. In the language of supersymmetry, $\zeta$ is a Fayet–Iliopoulos term [36].

The existence of BPS solitons requires that the potential term vanishes in the vacuum of the theory. This occurs for

$$|b_\infty|^2 = -\frac{\zeta}{\zeta + 2\xi}.$$ (2.16)

The Fayet–Iliopoulos term must then assume values within the following range:

$$-2\xi \leq \zeta \leq 0.$$ (2.17)

In a generic vacuum, the expectation value of $b$ spontaneously breaks the $U(1)$ gauge symmetry, and the model contains ANO (Abrikosov–Nielsen–Olesen) vortices [15] supported by the following homotopy group:

$$\pi_1(U(1)) = \mathbb{Z}.$$ (2.18)

Note that all the discussions of this section have been carried out considering only one coordinate patch for the $\text{CP}^1$ target space. As already mentioned, to cover the full manifold, we need to take into account another coordinate patch obtained with the change of variable:

$$b' = 1/b.$$ (2.19)

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4 If the expectation value of $b$ is vanishing, we must have a rational map where the degree of the numerator is less than the degree of the denominator.

5 Intuitively, the center of the lump is mapped to $b(z_0) = -1$, the point diametrically opposed to $b_\infty = 1$. The size is given by the displacement from the trivial map (zero size lump) $z_S = z_N = 0$.

6 See the appendix for more details.

7 This is related to the existence of a supersymmetric vacuum in the supersymmetric extension of the model.
In the gauged case, after this change of variables, the Lagrangian (2.14) takes the form
\[
\mathcal{L}_{U(1)} = -\frac{1}{4g^2} F_{\mu \nu} F_{\mu \nu} + \xi \left( \frac{\nabla_\mu b'_i}{(1 + |b'|^2)^2} \right)^2 - \frac{g^2}{2} \left( \frac{\xi - 2|b'|^2}{1 + |b'|^2} - \xi' \right)^2, \tag{2.20}
\]
which has the same form of (2.14) provided that the FI term \( \xi' \) transforms nontrivially while we change the coordinate patch:
\[
\xi' = -\xi - 2\xi. \tag{2.21}
\]

2.2. Vortex equations

The first step in the study of vortex solutions is to employ a Bogomol’nyi completion of the action (2.14)\([30]\):
\[
\mathcal{E}_{U(1)} = \frac{1}{2g^2} \left[ F_{g12} - g^2 \left( \xi - 2|b'|^2 \right) \right]^2 + 4\xi \frac{|\nabla_\mu b'|^2}{(1 + |b'|^2)^2} - \xi F_{g12} + \xi \epsilon_{ij} \partial_i \mathcal{N}_j.
\]
\[\mathcal{N}_j = \frac{i}{2(1 + |b'|^2)} (b \nabla_j \bar{b} - \bar{b} \nabla_j b). \tag{2.22}\]

As usual, vortex equations are given by imposing the vanishing of the squares in the first line of the expression above:
\[
F_{g12} = g^2 \left( \xi - 2|b'|^2 \right), \quad \nabla_g b = 0. \tag{2.23}
\]

The total tension is then given by the two-surface terms in the second line. The first term is proportional to the magnetic flux density. The corresponding second term in linear sigma models is usually discarded as a vanishing boundary term. However, it is non-vanishing for compact NL\(\sigma\)M. To give the correct topological interpretation to the two-surface terms above, we first have to discuss in more detail the topology supporting vortices in the \(\mathbb{C}P^1\) NL\(\sigma\)M. Vortices are undoubtedly characterized by the homotopy group (2.18). However, when we consider compact spaces, equation (2.18) does not give a complete classification of vortices\([17]\). We can in fact unwind the circle \(b = |b| e^{i\theta}\) representing a vortex configuration at the boundary by either crossing the point \(b = 0\) (‘south pole’) or the point \(b = \infty\) (‘north pole’). The topology which correctly describes vortices including core structures in the gauged \(\mathbb{C}P^1\) sigma model is then the following:
\[
\pi_1(U(1)|_{\mathbb{C}P^1}) = \mathbb{Z}_S \times \mathbb{Z}_N. \tag{2.24}\]

We will distinguish vortices with different \(\mathbb{Z}\) charges by labeling them as \(S\) and \(N\) vortices. The corresponding magnetic fluxes (proportional to topological charges), as we will prove soon, may then be written in the following way:
\[
\mathcal{V}_j^S = \mathcal{N}_j; \quad \mathcal{V}_j^N = \mathcal{N}_j + 2A_{gj}, \tag{2.25}\]
and the energy density can be written as
\[
\mathcal{E}_{U(1)} = \frac{\xi + 2\xi}{2} \epsilon_{ij} \partial_i \mathcal{V}_j^S - \xi \frac{\mathcal{V}_j^S}{2} \epsilon_{ij} \partial_i \mathcal{V}_j^N. \tag{2.26}\]
Moduli matrix. We now employ the moduli matrix approach \cite{37, 38} to construct vortices and their moduli space. The moduli matrix construction can be considered as a direct generalization of the rational map construction for lumps to the gauged case \cite{8, 6, 24}.\footnote{Similar connections between the moduli matrix and the rational map construction for monopoles were discussed in \cite{39}.} As usual, we do so by performing the following change of variables:

\[
\begin{align*}
    b(z, \bar{z}) &= s^2(z, \bar{z})b_0(z), \\
    A_\xi &= -i\bar{z}_\xi \log s,
\end{align*}
\]

where \( s \) is a non-vanishing function, while \( b_0 \) is holomorphic. Note that the change of variables above introduces an unphysical ‘V-equivalence’ which scales \( s \) and \( b_0 \) by multiplication with a constant \( V \)

\[
s^2 \to V^{-1}s^2, \quad b_0 \to Vb_0
\]

but leaves all physical quantities unchanged. The second of equations (2.23) is then identically solved, while the first reduces to a second order gauge invariant master equation:

\[
\partial_\xi \partial_{\bar{z}} \log \omega = -\frac{\xi^2}{4} \left( \frac{-2\omega^2|b_0|^2}{1 + \omega^2|b_0|^2} - \zeta \right), \quad \omega = ss^\xi.
\]

The holomorphic function \( b(z) \) is called the moduli matrix, and its complex coefficients parameterize the moduli space of vortices in the system considered. Let us assume that \( b(z) \) has a finite number of zeros and poles. Then, it can be written as a ratio of polynomials

\[
b_0(z) = b_\infty z^{p_0} + p_1^{n_1-1} + \cdots + p_{n_0},
\]

In the expression above, the overall coefficient \( b_\infty \) is not a modulus, since it can be fixed by \( V \)-equivalence. We chose it to be equal to the expectation value of \( b \). The master equation for \( s \) must then be solved (numerically) with the following boundary conditions:

\[
|s| \to |z|^{n_0-n_S} \quad \text{as} \quad |z| \to \infty.
\]

The coefficients \( p_i \) and \( q_j \) represent moduli of the vortex configuration, and the complex dimension of the moduli space is

\[
\text{Dim}_C \mathcal{M} = n_S + n_N.
\]

Using the moduli matrix formalism, we can now easily evaluate the flux densities (2.25)

\[
\begin{align*}
    \int d^2z \left( \partial_\xi \mathcal{V}_{2S} - \partial_{\bar{z}} \mathcal{V}_{1S} \right) &= 2 \int d^2z \partial_\xi \partial_{\bar{z}} \log \left( \frac{1 + \omega^2|b_0|^2}{\omega^2|b_0|^2} \right) = 2\pi n_S, \\
    \int d^2z \left( \partial_\xi \mathcal{V}_{2N} - \partial_{\bar{z}} \mathcal{V}_{1N} \right) &= 2 \int d^2z \partial_\xi \partial_{\bar{z}} \log (1 + \omega^2|b_0|^2) = 2\pi n_N,
\end{align*}
\]

which define the integers \( n_S \) and \( n_N \) appearing as the degree of the polynomials in the moduli matrix as the \( S \) and \( N \)-vortex numbers\footnote{Since the function \( s \) is non-vanishing, each zero (pole) of \( b_0(z) \) clearly correspond to the only points where \( b \) equals the value at the south or north pole, roughly corresponding to the cores of \( S \) and \( N \) vortices.}. The total tension is then

\[
T_{U(1)} = \pi (\zeta + 2\xi)n_S - \pi \xi n_N.
\]

Note that the total magnetic flux is

\[
2\pi v_x = -\int \text{d}^2F_{12} = -2 \int d^2z \partial_\xi \partial_{\bar{z}} \log \omega = \pi (n_S - n_N);
\]

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\]

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2\pi v_x = -\int \text{d}^2F_{12} = -2 \int d^2z \partial_\xi \partial_{\bar{z}} \log \omega = \pi (n_S - n_N);
\]
Figure 3. An S-vortex for various values of $\zeta, \xi = 1, \zeta = -1.9, -1.45, -1, 0.55, 0.1, g = 1$

thus, $S$ and $N$-vortices have opposite $U(1)$ charges. We can also define a ‘fractional lump number’

$$v_f = \frac{n_S + n_N}{2}. \quad (2.36)$$

This definition is justified if we note that the integer above is given, in fact, by the integral of following quantity:

$$\frac{(V_{JS} + V_{JN})}{2} = \frac{i}{2(1 + |b|^2)} (b \nabla_j \bar{b} - \bar{b} \nabla_j b) - A_{b}\bar{b} \quad (2.37)$$

which reduces to the lump number given in equation (2.4) for the ungauged case when $A_{b}\bar{b}$ vanishes.

2.3. S(N) isolated vortex

In this section, we construct a fundamental vortex and numerically solve the master equation (2.29). A fundamental $S$-vortex is given by the following choice for the moduli matrix:

$$b_0(z) = b_{\infty}(z - z_S), \quad |\omega|^2 \rightarrow |z|^{-1}. \quad (2.38)$$

In fact, the vortex unwinds around the south pole $b = 0$ at the center $z_S$ of the vortex. Note from equation (2.36) that the $S$-vortex has a fractional lump charge $v_f = 1/2$. Similarly, we have an $N$-vortex with the choice

$$b_0(z) = \frac{1}{z - z_N}, \quad |\omega|^2 \rightarrow |z|, \quad (2.39)$$

where the vortex unwinds around the point $b = \infty$ at the point $z_N$. The $N$-vortex also has a fractional lump charge $v_f = 1/2$.

Let us study a fundamental vortex in terms of $\zeta$. Figure 3 shows the energy density profile of an $S$-vortex as a function of the FI term $\zeta$. The vortex disappears, by dilution, in the limit $\zeta \rightarrow 0$, where the gauge symmetry defined on the $S$-patch is unbroken. In the other limit $\zeta \rightarrow -2\xi$, where the expectation value $b_{\infty}$ goes to infinity, the vortex becomes a spike. There is no need for additional work to study the $N$-vortex. From both the mass formula and the BPS equations, we see that an $N$-vortex is transformed into a physically equivalent $S$-vortex by the formal replacements:

$$b \rightarrow 1/b, \quad \zeta \rightarrow -\zeta - 2\xi. \quad (2.40)$$
A choice of parameters which gives a narrow S-vortex will then give a wide N-vortex and vice versa. This can also be easily understood if we plot the potential of equation (2.14) as in figure 4. If \( \zeta \) is closer to \( 2\xi \), the potential is higher around the S-pole than it is around the N-pole. A narrow S vortex is thus favored to minimize the potential energy with respect to the gradient energy (which tends to be higher for narrower configurations). Vice versa, for an N-vortex a wider configuration is favored. Note that when \( \zeta = -\xi \), the potential is symmetric and both vortices have the same width. We then have two independent effects which determine the size of vortices. A ‘core’ effect is related to the height of the potential which is different at the center of S and N vortices. A ‘tail’ effect, instead, is related to the long distance, exponential fall-off of the vortex and is the same for both S and N vortices. Analogously to the Abelian Higgs model case, we can estimate the typical sizes as follows:

\[
\lambda_S = \frac{1}{g\xi |b_\infty|}, \quad \lambda_N = \frac{1}{g\xi |b'_\infty|} = \frac{|b_\infty|}{g\xi},
\]

(2.41)

if we ignore the effects of the curvature of the target space on the vortex profile. The exponential tail can then be checked analytically expanding the master equation in terms of small fluctuations around the vacuum. We thus proceed expanding \( \omega_\infty \) in the following way:

\[
\omega = \left| \frac{b_\infty}{b_\infty(z)} \right| (1 + \delta \omega + \mathcal{O}(\delta \omega^2)).
\]

(2.42)

The master equation (2.29) then linearizes:

\[
\delta \xi \delta \omega = g^2 \xi \frac{|b_\infty|^2}{(1 + |b_\infty|^2)^2} \delta \omega = -g^2 \xi (\xi + 2\xi) \delta \omega \equiv \frac{\lambda}{4} \delta \omega.
\]

(2.43)

The solution of the equation above is the well-known modified Bessel function of the second kind:

\[
\delta \omega(r) = K_0(\lambda r) \sim e^{-\lambda r}. \quad (2.44)
\]

The exponential decay factor \( \lambda \) corresponds to the mass of the \( b \) field in the vacuum. Note that \( \lambda \) is invariant under the change of coordinate, as it should be since it is a physical quantity.
3. S–N vortex system

If we send the gauge coupling to zero, we have to recover the ungauged sigma model, which does not admit vortices. In fact, the fundamental S or N vortices become wider in this limit, and eventually they vanish.

However, there is something interesting which happens when we take the limit $g \to 0$ in the presence of a composite configuration of S and N vortices: we may recover lump solutions of the ungauged NL$\sigma$M. We will study this phenomenon in this section.

3.1. Well separated S–N vortices

Vortices are well separated when their typical widths are much smaller than their separations:

$$\lambda_{S,N} \ll \Delta z_i.$$  \hfill (3.1)

In this situation, the moduli matrix (2.30) obviously represents a set of well-separated $n_N$ N-vortices located at the poles of $b_0$ and $n_S$ S-vortices located at zeros. We also refer to this situation as ‘strong gauging’, since equation (3.1) can always be satisfied with a sufficiently large value of the gauge coupling (see equation (2.41)).

The simplest configuration is given by the following choice:

$$b_0(z) = b_\infty \frac{z - z_S}{z - z_N}.$$  \hfill (3.2)

A numerical simulation in the case above is shown in figure 5 where an S and an N vortex are clearly identified when the gauge coupling is sufficiently large.

3.2. ‘Gauged’ lumps

Let us now consider the limit $g \to 0$. We want to recover the lump solutions in the original ungauged NL$\sigma$M described by equation (2.7). Looking at the first line of equation (2.27) and the definition of the moduli matrix $b_0$ in equation (2.30) we see that $b$ must reduce to a purely holomorphic function. Assuming that all the holomorphic factors are extracted in the ratio
Moreover, if $b_\infty \neq 0$, the number of $N$ and $S$ vortices must be the same:

$$n_S = n_N = N.$$  

(3.4)

In this situation, the lump number defined in equation (2.4) is a well-defined integer. If the gauge coupling is small enough (weak gauging regime), we expect to find solitonic solutions similar to lumps of the ungauged model at small distances. Nevertheless, at large distances, this soliton will show their true nature of vortices falling off with an exponential factor. This behavior can be confirmed with numerical simulations. Again, figure 5 shows how a lump, in the weak gauging regime, is deformed into a composite state of $S$ and $N$ vortices once the gauge coupling is strong enough.

We can also perform, to some extent, an analytical analysis of the weak coupling limit, by expanding the non-holomorphic term $w$ and the master equation (2.29) as follows:

$$w = 1 + \delta w + O(g^2),$$

$$\bar{\partial}_z \delta \omega = -\frac{g^2}{4} \left( \frac{2|b_0|^2}{1 + |b_0|^2} - \zeta \right) + O(g^2).$$

(3.5)

Let us study the equation above in the proximity of a zero (pole) around $z = z_0$. The second term can be approximated by a constant in the vicinity of $z_0$:

$$\bar{\partial}_z \delta \omega = \frac{g^2}{4} (\zeta + \xi \pm \xi), \quad |z - z_0| \ll \Delta z_i,$$

(3.6)

where the $\Delta z_i$ are the typical separations between zeros and poles. The plus and minus signs respectively apply to the case of zeros and poles. The equation above can then be easily solved for $\delta \omega$:

$$\delta \omega \sim \frac{g^2}{4} (\zeta + \xi \pm \xi)|z - z_0|^2 + \text{harmonics},$$

(3.7)

where with harmonics we mean the solution of the homogenous equation $\bar{\partial}_z \delta \omega = 0^{10}$. Since the non-homogeneous term grows with the distance from $z_0$, the lump approximation is valid when

$$|z - z_0|^2 \ll |\Delta z_i|^2 \ll \left( \frac{g^2}{4} (\zeta + \xi \pm \xi) \right)^{-1}.$$  

(3.8)

The condition above also tells us that a generic configuration of $S$ and $N$ vortices is approximated by a lump solution whenever the typical distances $|\Delta z_i|$ are small enough. Note also that the approximation above is valid at short distances only. At sufficiently large distances, in fact, we have to spot the true ‘local’ nature of the configuration for finite $g$. Indeed, the second term of equation (3.5) vanishes at large distances, and we have to take into account terms linear in $\delta \omega$ in the expansion. We then recover equation (2.42) which gives the correct exponential tail.

The metric on the moduli space of composite $S$ and $N$ vortices was also previously studied in [17, 18], where it was found that a singularity develops when a couple of $S$–$N$ vortices coincide. In this section, we have shown that a configuration of two very close vortices (a pole and a zero) is indistinguishable from a lump (see equation (2.11)). The singularity discovered in [17, 18] is then exactly identified as a small lump singularity. As is well known, small lump singularities can be ‘resolved’ by introducing an appropriate UV completion for the NL$\sigma$M.

In the next section, we will consider this possibility explicitly.

---

10 These are fluctuating solutions which are not important in this argument.
4. Linear formulation

4.1. Model

It is simple to guess the correct UV completion of the gauged $\mathbb{C}P^1$ NL$\sigma$M. First let us consider the ungauged NL$\sigma$M. It can easily be written as the strong gauge limit $e \to \infty$ of the so-called extended Abelian Higgs model with two flavors:

$$
L = -\frac{1}{4e^2} F_{\mu\nu}^\phi F_{\mu\nu}^\phi - \frac{1}{4g^2} F_{\mu\nu}^g F_{\mu\nu}^g + |\nabla_\mu \phi_1|^2 + |\nabla_\mu \phi_2|^2 - \frac{e^2}{2} (|\phi_1|^2 + |\phi_2|^2 - \xi^2).
$$

(4.1)

As is well known, the model above admits semi-local vortices which are a particular type of strings which admit size moduli [6–10, 24]. They are indeed very similar to lump solutions of the underlying $\mathbb{C}P^1$ NL$\sigma$M (to which they reduce in the $e \to \infty$ limit). However, while a lump becomes a singular spike in the zero size limit, a semi-local vortex remains regular and reduces to a traditional (sometimes called ‘local’) Abrikosov–Niellesen–Olsen vortex, which has a typical size of order $1/e\sqrt{\xi}$. In this sense, the moduli space of semi-local vortices is considered to be a regularization of the singularities (small lump singularities) of the lump moduli space.

Semi-local vortices ultimately exist because the vacuum manifold, in this case $S^3$, is simply connected:

$$
\pi_1(S^3) = 0,
$$

(4.2)

while the second homotopy group of the moduli space of vacua $S^3/U(1) = \mathbb{C}P^1$ is nontrivial [10]

$$
\pi_2(\mathbb{C}P^1) = \mathbb{Z}.
$$

(4.3)

Note that the $SU(2)$ isometry of the $\mathbb{C}P^1$ NL$\sigma$M is now seen as a flavor symmetry which rotates the fields $\phi_i$. Moreover, we can identify the holomorphic coordinate $b$ appearing in equation (2.1) as

$$
b = \frac{\phi_2}{\phi_1}.
$$

(4.4)

It is then clear that to obtain the model (2.14) in the $e \to \infty$, we have to gauge a $U(1)$ flavor symmetry with the charge assignments given in table 1.

The resulting model is then the following linear gauged model:

$$
L = -\frac{1}{4e^2} F_{\mu\nu}^{\phi_1} F_{\mu\nu}^{\phi_1} - \frac{1}{4g^2} F_{\mu\nu}^{\phi_2} F_{\mu\nu}^{\phi_2} + |\nabla_\mu \phi_1|^2 + |\nabla_\mu \phi_2|^2

- \frac{e^2}{2} (|\phi_1|^2 + |\phi_2|^2 - \xi) - \frac{g^2}{2} (|\phi_1|^2 - |\phi_2|^2 - \chi)^2,
$$

(4.5)

where $\chi$ is a second FI term for the new gauge group $U(1)_2$ and

$$
\nabla_\mu \phi_1 = (\partial_\mu - iA_{\mu}^{ext} - iA_{\mu}^{gs})\phi_1, \quad \nabla_\mu \phi_2 = (\partial_\mu - iA_{\mu}^{ext} + iA_{\mu}^{gs})\phi_2.
$$

(4.6)

By comparing the second potential term in the equation above with the potential term arising in the gauged NL$\sigma$M Lagrangian (2.14), we also obtain the following relationship between the FI terms $\chi$ and $\zeta$:

$$
\chi = \zeta + \xi.
$$

(4.7)
The vacuum of the theory is then given by
\[ |\phi_{1\infty}|^2 = \frac{\xi + \chi}{2}, \quad |\phi_{2\infty}|^2 = \frac{\xi - \chi}{2}. \] (4.8)
Both the \( U(1) \) gauge groups are then spontaneously broken; we thus have for the first homotopy group
\[ \pi_1 \left( \frac{U(1) e \times U(1) g}{\mathbb{Z}_2} \right) = \mathbb{Z}_S \times \mathbb{Z}_N. \] (4.9)
Note that, due to the presence of a \( \mathbb{Z}_2 \) common factor, the smallest closed paths are obtained with a \( \pi \) rotation into the group \( U(1)_e \) and either a \( \pi \) or a \( -\pi \) rotation into the group \( U(1)_g \). These two paths represent the smallest elements of \( \mathbb{Z}_S \) and \( \mathbb{Z}_N \), respectively. The choice of notation is not a coincidence either: the two factors characterize two type of vortices which correspond to the \( S \) and \( N \) vortices we already studied in the previous sections in the \( e \to \infty \) limit. The \( S \) and \( N \) vortices are a particular type of what is called, in the literature, the fractional vortex. They were first discovered in two-component superconductors within a Landau–Ginzburg model [27], which can be considered as a non-relativistic version of the model (4.5) where, in the case of superconductors, the group \( U(1)_e \) is global\(^{11} \). This definition is due to the fact that the fundamental vortices carry only one-half of the magnetic flux of both gauge groups (in particular of the group \( U(1)_s \)). For generic values of the gauge couplings, the \( S \) and \( N \) vortices do not feel static interactions, even if they form a coupled system. Non-static interactions are, on the other hand, non-trivial and can be studied in the moduli space approximation [40]. However, when the gauge couplings coincide \( e = g \), the \( S \) and \( N \) vortices completely decouple and do not interact at all, at least at the classical level.

4.2. Vortices: BPS equations and moduli matrices

As usual we can perform a Bogomol’nyi completion of the action
\[ E = \frac{1}{2e^2} [F_{e12} - e^2 (|\phi_1|^2 + |\phi_2|^2 - \xi)]^2 + \frac{1}{2g^2} [F_{g12} - g^2 (|\phi_1|^2 - |\phi_2|^2 - \chi)]^2 + 4|\nabla \phi_1|^2 + 4|\nabla \phi_2|^2 - \xi F_{e12} - \chi F_{g12} + 2 \partial_\xi \partial_\chi (|\phi_1|^2 + |\phi_2|^2), \] (4.10)
which leads to BPS equations [16]
\[
\begin{align*}
F_{e12} &= e^2 (|\phi_1|^2 + |\phi_2|^2 - \xi) \\
F_{g12} &= g^2 (|\phi_1|^2 - |\phi_2|^2 - \chi)
\end{align*}
\] (4.11)
\[ \nabla \phi_1 = \nabla \phi_2 = 0. \]
The tension is then given by the following topological term:
\[ T = 2\pi \xi v_e + 2\pi \chi v_g, \] (4.12)
where \( v_e \) and \( v_g \) are respectively the winding numbers for the gauge groups \( U(1)_e \) and \( U(1)_g \). Note that the last term in equation (4.10) is a total derivative with vanishing contribution.

The moduli matrix formalism is implemented as usual with the substitutions [41]
\[
\begin{align*}
\phi_1 (z, \bar{z}) &= s_e^{-1}(z, \bar{z}) s_g^{-1}(z, \bar{z}) \phi_{10}(z), \\
\phi_2 (z, \bar{z}) &= s_e^{-1}(z, \bar{z}) s_g(z, \bar{z}) \phi_{20}(z), \\
A_e &= -i \partial_\xi \log s_e, \\
A_g &= -i \partial_\chi \log s_g, \\
\omega_e &= |s_e|^2, \omega_g &= |s_g|^2.
\end{align*}
\] (4.13)

\(^{11}\)Different types of fractional vortices were also introduced in [12, 13].
\( \phi_{01}(z) \) and \( \phi_{02}(z) \) are the moduli matrices for \( \phi_1 \) and \( \phi_2 \), respectively. The general prescription to find the right boundary conditions on the moduli matrix was described in [41] and gives in the present case

\[
\phi_{01}(z) = \sqrt{\frac{\xi + \chi}{2}} (z^{\eta_s} + p_1 z^{\eta_s-1} + \ldots + p_{\eta_s})
\]

\[
\phi_{02}(z) = \sqrt{\frac{\xi - \chi}{2}} (z^{\eta_s} + q_1 z^{\eta_s-1} + \ldots + q_{\eta_s})
\]

\[
\omega_\xi \to |z|^{\eta_s+\eta_s}, \quad \omega_\chi \to |z|^{\eta_s-\eta_s}, \quad (4.14)
\]

where \( n_S \) and \( n_N \) are the vortex numbers

\[
v_\xi = -\frac{1}{2\pi} \int \partial \bar{\partial} \log \omega_\xi = -\frac{1}{\pi} \int \partial \bar{\partial} \log \omega_\xi = \frac{n_S + n_N}{2},
\]

\[
v_\chi = -\frac{1}{2\pi} \int \partial \bar{\partial} \log \omega_\chi = -\frac{1}{\pi} \int \partial \bar{\partial} \log \omega_\chi = \frac{n_S - n_N}{2}. \quad (4.15)
\]

In the language of the NL\( \sigma \) M, the integers above correspond to the (fractional) vortex and lump numbers defined in the previous section:

\[
v_\xi = v_\nu, \quad v_\chi = v_\nu. \quad (4.16)
\]

Finally, let us write the BPS equations in terms of the moduli matrices:

\[
\partial \bar{\partial} \log \omega_\xi = \frac{e^2}{4} \left( \omega_\xi^{-1} \omega_\xi^{-1} |\phi_{01}|^2 + \omega_\xi^{-1} \omega_\xi |\phi_{02}|^2 - \xi \right),
\]

\[
\partial \bar{\partial} \log \omega_\chi = \frac{g^2}{4} \left( \omega_\chi^{-1} \omega_\chi^{-1} |\phi_{01}|^2 - \omega_\xi^{-1} \omega_\xi |\phi_{02}|^2 - \chi \right). \quad (4.17)
\]

### 4.3. Resolving the small lump singularity

One can easily check that all the analysis of this section correctly reduces to that done for the gauged \( \mathbb{C}^1 \) NL\( \sigma \) M in the limit \( e \to \infty \). Moreover, at finite \( e \), but in the regime \( e \gg g \), the results of the sections about the gauged NL\( \sigma \) M qualitatively apply to the linear case. We have just to substitute lumps with the very similar semi-local vortices in the discussions of the previous sections. A semi-local vortex is thus split into a couple of \( S \) and \( N \) vortices upon gauging of a \( U(1) \) flavor symmetry.

However, the crucial difference is that small lump singularities are now removed. As we have seen in the previous section, these singularities arise when a couple of \( S \) and \( N \) vortices have coincident positions: there, a singular spiky lump develops. At finite \( e \), however, this singularity is substituted by the insertion of a regular local vortex. To see this, let us consider the moduli matrices in the case of two coincident \( S \) and \( N \) vortices. For simplicity, we can set \( \chi = 0 \). Then, we have

\[
\phi_{01} = \phi_{02} = \frac{\xi}{2} (z - z_0) \quad (4.18)
\]

and the BPS equations (4.17) can be reduced to the following:

\[
\omega_\xi \equiv 1
\]

\[
\partial \bar{\partial} \log \omega_\xi = -\frac{e^2}{4} \left( 2 \omega_\xi^{-1} |\phi_{01}|^2 - \xi \right) = -\frac{e^2}{2} \left( \omega_\xi^{-1} |\phi_{01}|^2 - \xi/2 \right), \quad (4.19)
\]

which represent the master equation for a standard, regular vortex for the \( U(1)_e \) group.
5. Regularization of the semi-local vortex metric

As is well known, the effective theory on the worldsheet of semi-local vortices (and similarly of lumps) has an infrared logarithmic divergence due to the slow polynomial decay of the fields \([8, 6, 24, 23]\). As usual for effective theories with at least four supercharges, the effective theory can be conveniently written in terms of a Kähler potential. In the case of a semi-local vortex, the Kähler potential can be computed exactly at each order in a power expansion in terms of \(1/(e^{|\rho|\sqrt{\xi}})\), where \(|\rho|\) is the size modulus of the semi-local vortex \([42]\). At the leading order in this expansion, the Kähler potential for a semi-local vortex in the Abelian extended Higgs model (4.1), is as follows\(^{12}\):

\[
K(|z-z_0|^2, |\rho|^2) = 2\pi \xi |z-z_0|^2 + \pi \xi |\rho|^2 \ln \left( \frac{L^2}{|\rho|^2} \right) + \pi \xi |\rho|^2, \tag{5.1}
\]

where we included the first term which describes the position \(z_0\) of the vortex. In the expression above, the divergent logarithm has been regularized with the introduction of a large infrared cut-off \(L\).

So far, various different approaches have been considered to consistently deal with these divergences. Two possibilities were discussed in \([23]\). One is to consider strings of finite length \(L\). This would naturally cut-off the divergences as in (5.1). However, this approach obviously spoils the BPS nature of the string. A more convenient way to proceed is to introduce a twisted cut-off for the size modulus \(|\rho|\). The logarithm is then cut off at the scale \(L \sim 1/m\). The clear advantage of this approach is that the massive deformation preserves the BPS nature of the vortex. However, as a drawback, this regularization is ultimately due to the fact that we really lift the moduli space once we give a mass to the size parameter \(\rho\). Another recent proposal takes advantage of the presence of the divergencies and of the possibility of eliminating them with an appropriate change of variables \([42]\). The effective action is then exactly given by just these rescaled divergent terms.

In this work, we propose a new rather interesting regularization of the metric on the moduli space of semi-local vortices which preserves the BPS nature of the string and does not lift its moduli space. As a consequence, \(\rho\) remains a genuinely massless zero mode. This regularization is obtained by weakly gauging a flavor symmetry. While the regularization through a massive deformation corresponds to the inclusion \(F\)-terms, or potential terms, into the effective action, the proposed regularization through weak gauging corresponds to a deformation of the Kähler potential. The example we studied in the previous section is the explicit realization of this idea for the semi-local vortex in the extended Abelian Higgs model with two flavors, when we can gauge the relative phase between the two fields.

Recalling the discussion of the previous sections, it is straightforward to determine how the divergent Kähler potential in (5.1) gets deformed, at least in two special limits. The first one is when the size of the vortex is small \(|\rho|^2 \ll 1/g^2 \xi\):

\[
K(|z-z_0|^2, |\rho|^2) = 2\pi \xi |z-z_0|^2 + \pi \xi |\rho|^2 \ln \left( \frac{1}{|\rho|^2 g^2 \xi} \right) + \pi \xi |\rho|^2. \tag{5.2}
\]

The expression above can be easily guessed if we note that after gauging the power law fall-off of the fields is cut off with an exponential behavior, at distances of the order of the mass of the lightest particles in the bulk. In the limit of large size \(|\rho|^2 g^2 \xi \gg 1\), on the other hand, the semi-local vortex is split into two local vortices, and the Kähler potential reduces to that of two isolated Abelian vortices\(^{13}\):

\[
K(|z-z_N|^2, |z-z_S|^2) = \pi \xi |z-z_N|^2 + \pi \xi |z-z_S|^2 = 2\pi \xi |z-z_0|^2 + 2\pi \xi |\rho|^2, \tag{5.3}
\]

\(^{12}\) High order corrections have been calculated in \([42]\).

\(^{13}\) We have set \(\chi = 0\) for simplicity, throughout this section.
Table 2. Field content and charges.

| $U(1)^d$ | $b_1$ | $b_d$ | $b_{N-1}$ |
|----------|-------|-------|------------|
| $U(1)_1$ | -2    | 0     | 0          |
| $U(1)_d$ | 0     | 0     | -2         |

The Kähler potential at generic values of the coupling $g$ must then be calculated numerically using the expression

$$K(|z - z_0|^2, |\rho|^2) = \int d^2x \left\{ \xi \ln \omega_x + \chi \ln \omega_y + \frac{1}{e^2} \partial_i \ln \omega_x \partial_i \ln \omega_y + \frac{1}{g^2} \partial_i \ln \omega_y \partial_i \ln \omega_y + \omega_x^{-1} \omega_y^{-1} |\phi_0|^2 + \omega_x^{-1} \omega_y |\phi_0|^2 \right\},$$

(5.4)

whose numerical evaluation must interpolate between expressions (5.2) and (5.3).

6. Generalizations

In the case of the $\mathbb{C}P^1$ NL$\sigma$M, the $U(1)$ sub-group of the $SU(2)$ isometry is the maximal symmetry which can be gauged without breaking supersymmetry [35]. This fact can be intuitively understood by recalling that gauging in supersymmetric theories generically reduces the complex dimension $D$ of a target manifold to $D - d$, where $d$ is the dimension of the group being gauged$^{14}$. At most, we can then gauge $D$ isometries.

The discussion of the previous sections can be generalized in the case of NL$\sigma$Ms with higher-dimensional target spaces $D > 1$ in a variety of ways. Gauging in a different way the isometries of the target space will reveal different interesting connections between various type of vortices and lumps. In this section, we will have a brief look at some interesting possibilities.

6.1. $\mathbb{C}P^{N-1}$ NL$\sigma$M

The $\mathbb{C}P^{N-1}$ NL$\sigma$M has an $SU(N)$ isometry. As already explained, gauging of the full isometry will break supersymmetry [35]. We can however safely consider the gauging of a subgroup of dimension at most $N - 1$. We will consider, as particular examples, gauging of Abelian and non-Abelian subgroups.

**Gauging of an Abelian isometry.** A simple generalization of the model discussed in section 2 is the one given by weakly gauging a $U(1)^d$ subgroup with charge assignments given by table 2.

After gauging, the target space of $\mathbb{C}P^{N-1}$ is effectively reduced to $\mathbb{C}P^{N-1-d}$. When gauge couplings are sent to infinity, then the model supports $\mathbb{C}P^{N-1-d}$ lumps. Moreover, at finite values of gauge couplings, the Abelian gauge symmetries are generically spontaneously broken, and the model will also support Abelian vortices:

$$\pi_1(U(1)^d) = \mathbb{Z}^d.$$  

(6.1)

$^{14}$ Gauging in a supersymmetric compatible way requires the introduction of a D-term potential, whose minimization gives generically $d$ real constraints, in addition to the $d$ degrees of freedom eaten by gauge invariance.
Table 3. Field content and charges.

| $b_1, \ldots, b_d$ | $b_{d+1}, \ldots, b_{N-1}$ |
|---------------------|-----------------------------|
| $U(d)$              | $\square$                   |
|                     | 0                           |

Much in the same way as the $\mathbb{C}P^1$ case, a lump of the ungauged $\mathbb{C}P^{N-1}$ sigma model will thus be split into a composite state of a semi-local vortex (which reduces to a $\mathbb{C}P^{N-1-d}$ lump in the infinite gauge coupling limit) plus $d$ Abelian vortices. In other words, $d$ size moduli of the original lump are transformed into an equivalent number of position moduli of Abelian vortices.

**Gauging of a non-Abelian isometry.** Another interesting possibility is to gauge a non-Abelian $U(d)$ isometry, with $d^2 \leq N - 1$. As shown in table 3, we can arrange $d^2$ fields as $d$ fundamentals of $U(d)$, while the rest are singlets. Again after gauging the target space effectively reduces to $\mathbb{C}P^{N-1-d^2}$, which still supports lumps. Moreover, non-Abelian vortices are supported by the non-trivial homotopy:

$$\pi_1(U(d)) = \mathbb{Z}.$$ (6.2)

After gauging of a $U(d)$ isometry, lumps of $\mathbb{C}P^{N-1}$ sigma models reduce to a composite state of a semi-local vortex (similar to a $\mathbb{C}P^{N-1-d^2}$ lump) plus $d$ $U(d)$ non-Abelian vortices. The presence of $d$ vortices is due to the fact that non-Abelian vortices have $1/d$ charge with respect to an Abelian vortex. It can also be guessed by matching the dimensions of the moduli spaces. $d^2$ size moduli of the original lump are translated into $d$ orientations of the $d$ non-Abelian vortices.

### 6.2. **Grassmannian NL$\sigma M$**

The Grassmannian ($Gr_{N,\tilde{N}}$) can be considered as a generalization of $\mathbb{C}P^{N-1}$ manifolds. $Gr_{N,\tilde{N}}$ is the set of $N$-dimensional complex planes in an $(N + \tilde{N})$-dimensional space. They can be algebraically defined as the set of $N \times (\tilde{N} + N)$ matrices modulo an $SU(N)$ equivalence:

$$Gr_{N,\tilde{N}} = \{M_{N,\tilde{N}+N} | M \sim GM, G \in GL(C, N)\}. \quad (6.3)$$

The complex dimension of the manifold is then $N\tilde{N}$. Grassmannian manifolds enjoy an $SU(N + \tilde{N})$ isometry. If we write the matrix $M$ as follows:

$$M = (A|B), \quad (6.4)$$

the isometry action is then

$$M \rightarrow M_{SU(N+\tilde{N})}. \quad (6.5)$$

The matrices $A$ and $B$ have the role of homogeneous coordinates. Using the $GL(N, \mathbb{C})$ action, and assuming that $A$ is invertible, we can always reduce $M$ to the following form:

$$M \sim (1|B'), \quad (6.6)$$

where now the $B'$ fields are the independent $N\tilde{N}$ holomorphic coordinates of the Grassmannian\textsuperscript{15}. Grassmannian lumps were studied in connection to non-Abelian semi-local vortices in [24, 44]. The dimension of the moduli space of a Grassmannian lump is $N + \tilde{N}$.

\textsuperscript{15} Grassmannian manifolds are interesting in quantum field theory since they give the Higgs branch of supersymmetric $U(N) \mathcal{N} = 2$ QCD with $N + \tilde{N}$ fundamental flavors [43].
Assuming $\tilde{N} \leq N$, we can choose to gauge a $U(\tilde{N})$ subgroup of the $SU(N \times \tilde{N})$ isometry acting on the fields like

$$A \rightarrow Ae^{-i\alpha}, \quad B \rightarrow BG_{SU(\tilde{N})}e^{i\alpha}.$$  

(6.7)

The original $Gr_{N,\tilde{N}}$ manifold is then reduced to a gauged $Gr_{N-N,\tilde{N}}$ manifold. The gauged sigma model will then support non-Abelian $U(\tilde{N})$ semi-local vortices, whose moduli space dimension is $\tilde{N} + (N - \tilde{N}) = N$. However, the original Grassmannian lump will split into a composite state of a $U(\tilde{N})$ semi-local vortex (which reduces to a $Gr_{N-N,\tilde{N}}$ lump in the infinite gauge coupling limit) plus an additional $U(\tilde{N})$ local vortex.

7. Summary and conclusions

In this paper, we have explicitly constructed vortices in the $U(1)$ gauged $\mathbb{C}P^1$ nonlinear sigma model and studied them numerically. We identified the configuration given by the superposition of $N$ and $S$ vortices as a two-dimensional instanton (lump) of the original ungauged sigma model. In this way, we identified a general mechanism to decompose instantons into smaller constituents by gauging an isometry of the target space of a nonlinear sigma model.

We can perform a similar analysis in the linear case of the Abelian extended Higgs model with two flavors where a $U(1)$ flavor symmetry is also gauged. This model can be seen as an UV completion of the gauged $\mathbb{C}P^1$ sigma model. The extended Higgs model is well known to contain semi-local vortices [6, 7]. In this context, we identified the gauging of flavor symmetries as a general mechanism to regularize the metric of semi-local vortices. Contrarily to other known methods, the one we propose does not lift any moduli space parameter.

We also briefly discussed how the above ideas can be generalized to the case of nonlinear sigma models with higher-dimensional target spaces. In general, lumps are split into component objects which can be identified as being vortices, non-Abelian vortices or semi-local vortices, depending on the chosen isometry one decides to gauge.

It is tantalizing to try to extend these ideas to the four-dimensional case. An interesting direct connection between two-dimensional lumps and four-dimensional instantons has been found in [45] through non-Abelian vortices [38, 46, 22, 21, 47]. Precisely, instantons living in the unbroken phase of four-dimensional non-Abelian $SU(N)$ gauge theories are confined on the two-dimensional string worldsheet of non-Abelian vortices once one enters a broken Higgs phase. The effective theory of non-Abelian vortices is described by a two-dimensional $\mathbb{C}P^N$ nonlinear sigma model and instantons are then confined as lumps of the effective theory. Due to the results of this paper, it is natural to expect that gauging of $U(1)$ flavor symmetries in the four-dimensional theory may lead to instantons splitting into meron-like objects. A situation where this set-up can be realized in nature is for example the color–flavor-locked superconducting phase in high density QCD [48], where non-Abelian vortices [49] admit $\mathbb{C}P^3$ moduli in the worldsheets [50] and the role of the additional $U(1)$ gauge symmetry is played by the electromagnetic interactions. Fractional instantons, or merons, on the worldsheet of non-Abelian strings have already been considered in [51] in the different context of $\mathcal{N} = 1^*$ $SU(N)$ gauge theories, where non-Abelian vortices are described by a non-supersymmetric, ungauged $O(3)$ sigma model [52]. In this case, merons are identified as global vortices instead of our local vortices. Moreover, the setup is completely non-supersymmetric. We strongly believe that future research along this line may lead to new crucial understandings of the vacuum structure of strongly coupled gauge theories.
Appendix. Superfield formalism

Let us explicitly derive the Bogomol’nyi equations for a general $U(1)$ gauged $\mathcal{N} = 2$ nonlinear sigma-model. Using an $\mathcal{N} = 1$ superfield formalism, and restricting to the relevant bosonic content, the action of the model is given in terms of a Kähler potential and a gauge kinetic term [35]:

$$
\mathcal{L}_K = \int d^4 \theta \left[ K(e^{-V/\Phi^I}) + \zeta V \right] + \frac{1}{4g^2} \left( \int d^2 \theta W^a \bar{W}_a + \text{c.c.} \right)
$$

$$
W^a = \frac{1}{4} \bar{D}D_a V,
$$

(A.1)

where the $\Phi_I$ are charged chiral superfields and $V$ is the vector superfield. The bosonic part is then the following:

$$
\mathcal{L}_K = -\frac{1}{4g^2} F_{g12}^\mu \bar{F}_{g12}^\nu + g_{IJ} \nabla_\mu \phi^I \bar{\nabla}^\mu \phi^J + \frac{g^2}{2} \left( \partial_{|\phi^I|^2} K(|\phi^I|^2) - \zeta \right)^2,
$$

$$
g_{IJ} = \partial_{\phi^I} K(|\phi^I|^2).
$$

(A.2)

One can then employ a Bogomol’nyi completion:

$$
\mathcal{E}_K = \frac{1}{2g^2} \left[ \mathcal{F}_{g12}^\mu - g^2 \left( \partial_{|\phi^I|^2} K(|\phi^I|^2) - \zeta \right) \right]^2 + 4g_{IJ} \bar{\nabla}^\mu \phi^I \bar{\nabla}_\mu \phi^J - \zeta \mathcal{F}_{g12} + \epsilon_{ij} \partial_j \mathcal{N}_i.
$$

(A.3)

As usual, vortex equations are given by imposing the vanishing of the squares in the first line of the expression above:

$$
\mathcal{F}_{g12} = g^2 \left( \partial_{|\phi^I|^2} K(|\phi^I|^2) - \zeta \right) \nabla_{\mu} \phi^I = 0,
$$

(A.4)

where the second follows since the metric $g_{IJ}$ is assumed to be positive definite.

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