Topological spaces compact with respect to a set of filters

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Abstract: If $P$ is a family of filters over some set $I$, a topological space $X$ is sequencewise $P$-compact if for every $I$-indexed sequence of elements of $X$ there is $F \in P$ such that the sequence has an $F$-limit point. Countable compactness, sequential compactness, initial $\kappa$-compactness, $[\lambda, \mu]$-compactness, the Menger and Rothberger properties can all be expressed in terms of sequencewise $P$-compactness for appropriate choices of $P$. We show that sequencewise $P$-compactness is preserved under taking products if and only if there is a filter $F \in P$ such that sequencewise $P$-compactness is equivalent to $F$-compactness. If this is the case, and there exists a sequencewise $P$-compact $T_1$ topological space with more than one point, then $F$ is necessarily an ultrafilter. The particular case of sequential compactness is analyzed in detail.

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1. Introduction

In [13] Kombarov generalized the notion of ultrafilter compactness for topological spaces (see Bernstein [2]) by taking into account a family $P$ of ultrafilters, rather than just a single ultrafilter. We extend Kombarov's notion to families of filters (not necessarily maximal). This provides an essential strengthening: for example, also sequential compactness and the Rothberger property become particular cases.

We assume no separation axiom, if not otherwise specified. In order to avoid trivial exceptions, all topological spaces under consideration are assumed to be nonempty. We now recall the main definitions. If $X$ is a topological space, $I$ is a set, $(x_i)_{i \in I}$ is an $I$-indexed sequence of elements of $X$, and $F$ is a filter over $I$, a point $x \in X$ is an $F$-limit point of the sequence $(x_i)_{i \in I}$ if $\{i \in I : x_i \in U\} \in F$ for every open neighborhood $U$ of $x$. The notion of an $F$-limit...
point belongs to the folklore, and has been used by several mathematicians in distinct contexts; see, e.g., Bernstein [2], Choquet [5], Ginsburg and Saks [11], Katetov [12], just to mention some. If $\mathcal{P}$ is a family of filters over $I$, we say that $X$ is sequencewise $\mathcal{P}$-compact if for every $I$-indexed sequence of elements of $X$ there is $F \in \mathcal{P}$ such that the sequence has an $F$-limit point. Kombarov [13] introduced the above notion under the name $\mathcal{P}$-compactness in the particular case when $\mathcal{P}$ is a family of non principal ultrafilters over $\omega$. As far as we know, for $\mathcal{P}$ a family of ultrafilters over an arbitrary infinite cardinal $\alpha$ the notion has been first considered in García-Ferreira [8, Definition 1.2(1)] (under the name quasi $\mathcal{P}$-compactness). We have chosen the present terminology in hope to avoid any possible ambiguity. Apparently, in the above general form, the case in which $\mathcal{P}$ is a family of filters has never been considered, before we discussed it in [16], under different terminology. Notice that sequencewise $\mathcal{P}$-compactness is trivially closed hereditarily and preserved under surjective continuous images.

We now present some examples. When $\mathcal{P} = \{F\}$ is a singleton, we get the notion of $F$-compactness, particularly studied in the case when $F$ is an ultrafilter [2, 11, 19]. As another example, a topological space is countably compact if and only if, in the present terminology, it is sequencewise $\mathcal{P}$-compact, where $\mathcal{P}$ is the family of all uniform ultrafilters over $I = \omega$ (see Ginsburg and Saks [11, p.404]). More generally, for $\lambda$ regular a topological space satisfies $\text{CAP}_\lambda$ (every subset of cardinality $\lambda$ has a complete accumulation point) if and only if it is sequencewise $\mathcal{P}$-compact for the family $\mathcal{P}$ of all uniform ultrafilters over $I = \lambda$ (see Saks [19, pp.80–81]). The assumption $\lambda$ regular is only for simplicity, similar results hold for $\lambda$ singular, and for pairs of cardinals, as well (see [19]; see also [15, Sections 3, 4]). Caicedo [4] extended some of the above results and simplified many arguments; in particular, it follows easily from [4, Section 3] that a topological space is $[\mu, \lambda]$-compact (see Alexandroff and Urysohn [1], Smirnov [20]) if and only if it is sequencewise $\mathcal{P}$-compact for an appropriate family $\mathcal{P}$ of $[\mu, \lambda]$-regular ultrafilters over $[\lambda]^{<\mu}$ (the set of all subsets of $\lambda$ of cardinality $<\mu$). In particular, the above examples include initial $\lambda$-compactness. Also the Menger, Rothberger and related properties can be given an equivalent formulation in terms of sequencewise $\mathcal{P}$-compactness. See [17].

As another example, sequential compactness is equivalent to sequencewise $\mathcal{P}$-compactness for the following choice of $\mathcal{P}$. If $Z$ is an infinite subset of $\omega$, let $F_Z = \{W \subseteq \omega : |Z \setminus W| \text{ is finite}\}$, that is, $F_Z$ is the filter on $\omega$ generated by the Fréchet filter on $Z$. We now get sequential compactness by taking $I = \omega$ and $\mathcal{P} = \{F_Z : Z \text{ an infinite subset of } \omega\}$.

Related subjects have been treated in a great generality in [16], where, under different terminology, we showed that a large class of both covering properties and accumulation properties can be expressed by means of sequencewise $\mathcal{P}$-compactness (see [16, Remark 5.4]). Besides the examples mentioned above, in [16] we also considered various compactness properties defined in terms of ordinal numbers.

A brief summary of the paper follows. In Section 2 we prove some theorems on preservation of sequencewise $\mathcal{P}$-compactness under products. Section 3 discusses many examples, and shows that some classical results can be obtained as a consequence of the theorems in Section 2. The example of sequential compactness is dealt in detail in Section 4; there we show that all products of members of some family $\mathcal{T}$ are sequentially compact if and only if in all members of $\mathcal{T}$ every sequence converges. In Section 5 we characterize the cases in which $\mathcal{P}$ can be taken to consist only of ultrafilters. Finally, the last section contains some additional comments, states some problems, and introduces a pseudocompact-like generalization. Moreover, the connections with Comfort order on ultrafilters are briefly discussed.

## 2. Preservation under products

We state our main result in a form relative to a class $\mathcal{K}$ of topological spaces, since there are significant applications. For a class $\mathcal{K}$ of topological spaces, we say that sequencewise $\mathcal{P}$-compactness and sequencewise $\mathcal{P}'$-compactness are equivalent in $\mathcal{K}$ if for every topological space $X \in \mathcal{K}$, $X$ is sequencewise $\mathcal{P}$-compact if and only if $X$ is sequencewise $\mathcal{P}'$-compact (in the above definition, all members of $\mathcal{P}$ are filters over some set $I$, and all members of $\mathcal{P}'$ are filters over some set $I'$, but we are not necessarily assuming that $I = I'$). Obviously, if $\mathcal{P}$ is a family of filters over $I$, then sequencewise $\mathcal{P}$-compactness is equivalent to sequencewise $\mathcal{P}'$-compactness for an appropriate family $\mathcal{P}'$ of filters over $\kappa = |I|$, henceforth from now on we shall tacitly assume that this is the case. In all product theorems below we allow repetitions, i.e., we allow a space occur multiple times.
Moreover, if (iii) holds, and $X$-compact if and only if $P$. Suppose that Corollary 2.3.

Immediately from Corollary 2.2 with $X = \{x\}$ and the last statement of Theorem 2.1 we have

**Corollary 2.3.**
Suppose that $X$ is a topological space, and $P$ is a nonempty family of filters. Then every power of $X$ is sequencewise $P$-compact if and only if $X^{(\mathbb{P})}$ is sequencewise $P$-compact, if and only if $X$ is $F$-compact for some $F \in P$. If the above conditions hold, and $X$ has two disjoint nonempty closed sets, then any $F$ as above is an ultrafilter.
3. Examples

Many results can be obtained as particular cases of Theorem 2.1 and Corollary 2.2. For example, by the mentioned characterization of countable compactness, we get that for every class $\mathcal{K}$ every product of members of $\mathcal{K}$ is countably compact if and only if every product of $2^\omega$ members of $\mathcal{K}$ is countably compact, if and only if there is an ultrafilter $F$ uniform over $\omega$ such that every member of $\mathcal{K}$ is $F$-compact. For powers of a single space this result is due to Ginsburg and Saks [11, Theorem 2.6]. In a similar way, by applying the techniques of [19, Sections 2 and 6], for every class $\mathcal{K}$ and every infinite cardinal $\lambda$ we have that every product of members of $\mathcal{K}$ satisfies $\text{CAP}_\lambda$ if and only if every product of $2^{\omega_1}$ members of $\mathcal{K}$ satisfies $\text{CAP}_\lambda$, if and only if there is an ultrafilter $F$ uniform over $\lambda$ such that every member of $\mathcal{K}$ is $F$-compact. A completely analogous characterization of classes $\mathcal{K}$ such that all products of members of $\mathcal{K}$ are $[\mu, \lambda]$-compact is obtained by using $[\mu, \lambda]$-regular ultrafilters over $[\lambda]^{<\mu}$ [4, Theorem 3.4].

In particular, the above results furnish a characterization of classes $\mathcal{K}$ such that all products of members of $\mathcal{K}$ are initially $\lambda$-compact. By [19, Theorem 6.2], there exists some family $\mathcal{P}$ such that a topological space is initially $\lambda$-compact if and only if it is sequencewise $\mathcal{P}$-compact for all $\mathcal{P} \in \mathcal{P}$. García-Ferreira [7, Corollary 2.15] improved this to a single family, and this follows also from [4] in a simpler way (and with an improved bound $2^{\omega_1}$). Moreover, in [7, Theorem 2.17] it is proved that for any given cardinal $\lambda$ if initial $\lambda$-compactness is preserved under products, then there is some ultrafilter $D$ such that $D$-compactness is equivalent to initial $\lambda$-compactness. Theorem 2.1, applied to the case when $\mathcal{K}$ is the class of all topological spaces, together with the characterization of initial $\lambda$-compactness in terms of sequencewise $\mathcal{P}$-compactness, furnishes a simpler proof of [7, Theorem 2.17]; actually, this proof shows that initial $\lambda$-compactness is preserved under products if and only if there is some ultrafilter $D$ such that $D$-compactness is equivalent to initial $\lambda$-compactness (and, if this is the case, then $D$ can be chosen $(\omega, \lambda)$-regular over $[\lambda]^{<\omega} = \lambda$). Moreover, all the above arguments apply to $[\mu, \lambda]$-compactness, too. Results about preservation of $[\mu, \lambda]$-compactness under products are given in [18], where we establish a connection with strongly compact cardinals.

In [17] we provide characterizations of the Menger and Rothberger properties in terms of sequencewise $\mathcal{P}$-compactness. Actually, we consider even more general properties, which depend on three cardinals. Though Theorem 2.1 can be applied in this situation, too, in [17, Corollary 3.2] we are able to obtain stronger bounds in a direct way. Anyway, the above characterizations are good examples of the usefulness of allowing non maximal filters in $\mathcal{P}$; indeed, the characterization of the Rothberger property involves a family $\mathcal{P}$ consisting of filters none of which is maximal [17, Proposition 4.1, and the comments below it]. Moreover, these examples show how significant the difference is between the case in which $\mathcal{P}$ contains some ultrafilter, and the case in which $\mathcal{P}$ contains no ultrafilter (compare the last statement in Theorem 2.1). Indeed, there are $T_1$ spaces all whose powers are Menger (they are exactly the compact spaces); on the contrary, in [17, Corollary 4.2] we prove that if some product of $T_1$ spaces is Rothberger, then all but at most a finite number of the factors are one-element. A somewhat similar situation occurs in the case of sequential compactness, as we are going to discuss soon.

4. Sequential compactness

Recall that a space $X$ is called ultraconnected if no pair of nonempty closed sets of $X$ is disjoint. Equivalently, a space is ultraconnected if and only if $\overline{\{x_0\} \cap \cdots \cap \{x_n\}} \neq \emptyset$ for every $n \in \omega$ and every $(n+1)$-tuple $x_0, \ldots, x_n$ of elements of $X$, where overline denotes closure. We need the following easy lemma, for which we know no reference. Other related results appear in Brandhorst [3].

Lemma 4.1.
For every topological space $X$, the following conditions are equivalent.

(i) Every sequence in $X$ converges.
(ii) $X$ is ultraconnected and sequentially compact.
(iii) $X$ is ultraconnected and countably compact.
(iv) Every countable open cover of $X$ contains $X$ itself as a member.
Proof. (i) ⇒ (ii) is trivial, since if every sequence in \( X \) converges, then \( X \) is surely sequentially compact. Moreover, if \( X \) is not ultraconnected, say \( C_1, C_2 \subseteq X \) are closed nonempty and disjoint, then it is enough to consider any sequence which takes infinitely many values in \( C_1 \) and infinitely many values in \( C_2 \), in order to get a nonconverging sequence.

(ii) ⇒ (iii) is trivial.

In order to prove (iii) ⇒ (i), suppose that \( X \) is countably compact and ultraconnected, and let \((x_n)_{n \in \omega}\) be a sequence of elements of \( X \). By ultraconnectedness, \( \{x_0\} \cap \cdots \cap \{x_n\} \neq \emptyset \) for every \( n \in \omega \). For each \( n \in \omega \), pick some \( y_n \in \{x_0\} \cap \cdots \cap \{x_n\} \). By countable compactness, \((y_n)_{n \in \omega}\) has some cluster point \( y \in X \). Then it is easy to see that \((x_n)_{n \in \omega}\) converges to \( y \).

The equivalence of (iii) and (iv) is trivial.

Recall that the splitting number \( s \) is the least cardinality of a family \( S \subseteq [\omega]^{\omega} \) such that for every \( A \in [\omega]^{\omega} \) there exists \( S \subseteq S \) with both \( A \cap S \) and \( A \setminus S \) infinite. See, e.g., [6] for further details.

Lemma 4.2.
A product of \( \geq s \) spaces which are not ultraconnected is not sequentially compact.

Proof. Let \( X = \prod_{j \in J} X_j \) be a product of \( \geq s \) spaces which are not ultraconnected. Thus each \( X_j \) has two disjoint closed nonempty subsets \( C_j \) and \( C_j' \). Letting \( Y_j = C_j \cup C_j' \) for \( j \in J \) we have that each \( Y_j \) is a closed subset of \( X_j \), hence \( Y = \prod_{j \in J} Y_j \) is a closed subset of \( X = \prod_{i \in I} X_i \). For each \( j \in J \), we can define a continuous surjective function from \( Y_j \) to the two elements discrete space \( 2 = \{d_1, d_2\} \), by letting \( f_j(y) = d_1 \), \( y \in C_j \), and \( f_j(y) = d_2 \), \( y \in C_j' \), (here we are using the assumption that \( C_j \) and \( C_j' \) are disjoint and nonempty). Naturally, we have a continuous surjective function from \( Y = \prod_{j \in J} Y_j \) to \( 2^J \), \( \lambda = |J| \geq s \); however, \( 2^s \) is well known not to be sequentially compact [6, Theorem 6.1], hence neither \( Y \) nor \( X \) are sequentially compact.

Corollary 4.3.
For every family \( \mathcal{T} \) of topological spaces, the following conditions are equivalent.

(i) All products of members of \( \mathcal{T} \) are sequentially compact.
(ii) For every \( X \in \mathcal{T} \), \( X^s \) is sequentially compact.
(iii) Every \( X \in \mathcal{T} \) has the property that in \( X \) every sequence converges.
(iv) In all products of members of \( \mathcal{T} \) every sequence converges.

Proof. (i) ⇒ (ii) and (iv) ⇒ (i) are trivial.

Had we replaced \( s \) with \( 2^s \) in (ii), the implication (ii) ⇒ (iii) would follow immediately by Corollary 2.3 using the characterization of sequential compactness presented at the beginning, since \( |[\omega]^{\omega}| = 2^\omega \) and, for every \( Z \in [\omega]^{\omega} \), \( F_2 \)-compactness is trivially equivalent to \( F \)-compactness for the Fréchet filter \( F \) over \( \omega \). Then notice that \( F \)-compactness is equivalent to the statement that every sequence converges.

In order to prove (ii) ⇒ (iii) for the improved bound \( s \) in (ii), notice that if (ii) holds, then \( X \) is ultraconnected, by Lemma 4.1. Since \( X \) is trivially sequentially compact, we get that in \( X \) every sequence converges, by Lemma 4.1.

(iii) ⇒ (iv) By (iii), every \( X \in \mathcal{T} \) is \( F \)-compact for the Fréchet filter \( F \) over \( \omega \), and this implies (iv), since \( F \)-compactness is preserved under taking products.

Notice that the example \( \mathcal{T} = \{2\} \) shows that the value \( s \) in (ii) of Corollary 4.3 cannot be improved, since \( 2^s \) is sequentially compact for every \( \lambda < s \) [6, Theorem 6.1]. The equivalence of conditions (i), (iii) and (iv) in Corollary 4.3 can be also obtained as a consequence of results from Brandhorst [3].
Problem 4.4.
(i) Is sequential compactness equivalent to sequencewise $\mathcal{P}$-compactness for some $\mathcal{P}$ with $|\mathcal{P}| < 2^\omega$? In case of affirmative answer, which is the smallest possible cardinality of such $\mathcal{P}$?
(ii) Can we have $\mathcal{P}$ of smallest cardinality as above with $\mathcal{P} \subseteq \{F_Z : Z \subseteq [\omega]^\omega\}$? (Recall from the introduction that for $Z \subseteq [\omega]^\omega$ we have put $F_Z = \{W \subseteq \omega : |Z \setminus W| \text{ is finite}\}$.)

Of course, if we could have $|\mathcal{P}| = s$ in (i) above, then the implication (ii) $\Rightarrow$ (iii) in Corollary 4.3 would be a direct consequence of Corollary 2.3. On the other hand, the equivalence of (i) and (ii) in Corollary 2.2 shows that sequential compactness is not equivalent to sequencewise $\mathcal{P}$-compactness for some $\mathcal{P}$ with $|\mathcal{P}| < s$, since, again $\mathcal{K} = \{2\}$ would give a counterexample to [6, Theorem 6.1].

5. Maximal vs. non maximal filters

We now show that if sequencewise $\mathcal{P}$-compactness is equivalent to sequential compactness for a certain $\mathcal{P}$, then $\mathcal{P}$ contains no ultrafilter. This follows from the more general result below. Thus allowing $\mathcal{P}$ to contain non maximal filters is an effective generalization. Notice that the next proposition also shows that if sequencewise $\mathcal{P}$-compactness and sequencewise $\mathcal{P}'$-compactness are equivalent, then $\mathcal{P}$ contains some ultrafilter if and only if so does $\mathcal{P}'$.

Proposition 5.1.
For every nonempty family $\mathcal{P}$ of filters, the following conditions are equivalent.
(i) $\mathcal{P}$ contains at least one ultrafilter.
(ii) Sequencewise $\mathcal{P}$-compactness is equivalent to sequencewise $\mathcal{P}'$-compactness for some $\mathcal{P}'$ which contains at least one ultrafilter.
(iii) Every compact topological space is sequencewise $\mathcal{P}$-compact.
(iv) $2^{|\mathcal{P}|}$ is sequencewise $\mathcal{P}$-compact.

Proof. (i) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (iii) follows immediately from the well-known fact that every compact topological space is $D$-compact for every ultrafilter $D$. (iii) $\Rightarrow$ (iv) is an immediate consequence of Tychonoff compactness theorem. (iv) $\Rightarrow$ (i) is immediate from Corollary 2.3.

Proposition 5.2.
For every family $\mathcal{P}$ of filters containing at least one ultrafilter, the following conditions are equivalent.
(i) Sequencewise $\mathcal{P}$-compactness is equivalent to sequencewise $\mathcal{P}'$-compactness, where $\mathcal{P}'$ is the set of members of $\mathcal{P}$ which are ultrafilters.
(ii) Sequencewise $\mathcal{P}$-compactness is equivalent to sequencewise $\mathcal{P}'$-compactness for some $\mathcal{P}'$ which contains only ultrafilters.
(iii) The product of a compact topological space with a sequencewise $\mathcal{P}$-compact space is still sequencewise $\mathcal{P}$-compact.
(iv) The product of $2^\lambda$ with a sequencewise $\mathcal{P}$-compact space is still sequencewise $\mathcal{P}$-compact, where $\lambda = |\{F \in \mathcal{P} : F \text{ is not an ultrafilter}\}|$.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial.
(ii) $\Rightarrow$ (iii) Let $X$ be compact, $Y$ be sequencewise $\mathcal{P}$-compact, and let $(x_z)_{z \in X}$ be a sequence of elements of $X \times Y$. Since $Y$ is sequencewise $\mathcal{P}'$-compact by (ii), then there is some ultrafilter $D \in \mathcal{P}'$ such that the second projection of $(x_z)_{z \in X}$ has some $D$-limit point in $Y$. Since, as we mentioned, every compact topological space is $D$-compact for every ultrafilter $D$, then also the first projection of $(x_z)_{z \in X}$ has some $D$-limit point in $X$, and this implies that $(x_z)_{z \in X}$ has some $D$-limit point in $X \times Y$. Thus $X \times Y$ is sequencewise $\mathcal{P}'$-compact, hence sequencewise $\mathcal{P}$-compact, by (ii).
In order to finish the proof, suppose that (i) fails; we will show that (iv) fails, too. If (i) fails, then, with \( \mathcal{P} \) defined as in (i), there is some sequencewise \( \mathcal{P} \)-compact space \( X \) which is not sequencewise \( \mathcal{P}^\prime \)-compact (since sequencewise \( \mathcal{P}^\prime \)-compactness trivially implies sequencewise \( \mathcal{P} \)-compactness). In particular, \( X \) has a sequence \( (x'_a)_{a \in \kappa} \) such that for every ultrafilter \( D \in \mathcal{P} \), \( (x'_a)_{a \in \kappa} \) has no \( D \)-limit point in \( X \). Moreover, for every \( F \in \mathcal{P} \) which is not an ultrafilter, there is a sequence \( (x_F,a)_{a \in \kappa} \) in \( 2 = \{d_1, d_2\} \) which has no \( F \)-limit point in \( 2 \): just choose \( Z \subseteq \kappa \) in such a way that neither \( Z \) nor \( \kappa \setminus Z \) belong to \( F \); this is possible since \( F \) is not ultra. Then let \( x_F,a = d_1 \) if \( a \in Z \) and \( x_F,a = d_2 \) if \( a \in \kappa \setminus Z \). If in \( 2^2 \times X \) we consider a sequence \( (x_a)_{a \in \kappa} \), whose projections are \( (x_F,a)_{a \in \kappa} \) on the copies of \( 2 \), and \( (x'_a)_{a \in \kappa} \) on \( X \), then \( (x_a)_{a \in \kappa} \) cannot have an \( F \)-limit point in \( 2^2 \times X \) for \( F \in \mathcal{P} \), thus (iv) fails.

The above propositions solve a problem raised by an anonymous referee, asking whether, given a family of filters \( \mathcal{P} \), there is a family of ultrafilters \( \mathcal{P}^\prime \) such that sequencewise \( \mathcal{P} \)-compactness and sequencewise \( \mathcal{P}^\prime \)-compactness are equivalent. Proposition 5.1 applied, say, to the case of sequential compactness shows that \( \mathcal{P}^\prime \) as above does not always necessarily exists. Proposition 5.2 characterizes exactly the cases in which such \( \mathcal{P}^\prime \) exists.

Propositions 5.1 and 5.2 can be used to get another proof of some results from [17]. There we considered the following relation approximating the Menger property. A topological space \( X \) is said to satisfy \( R(\lambda, \mu; < \kappa) \) if for every sequence \( (\mathcal{U}_a)_{a \in \lambda} \) of open covers of \( X \) such that \( |\mathcal{U}_a| \leq \mu \) and for every \( a \in \lambda \) there are subsets \( \mathcal{V}_a \subseteq \mathcal{U}_a \), \( a \in \lambda \), such that \( \bigcup_{a \in \lambda} \mathcal{V}_a \) is a cover of \( X \), and \( |\mathcal{V}_a| < \kappa \) for every \( a \in \lambda \). Thus the Menger property is \( R(\omega, \mu; \omega) \) for every \( \mu \). Notice also that \( R(1, \mu; < \kappa) = [\kappa, \mu] \)-compactness.

**Corollary 5.3.**

For every infinite cardinals \( \mu \) and \( \kappa \), and nonzero cardinal \( \lambda \), there is a family \( \mathcal{P} \) of ultrafilters such that sequencewise \( \mathcal{P} \)-compactness is equivalent to \( R(\lambda, \mu; < \kappa) \).

**Proof.** The existence of such a family \( \mathcal{P} \), possibly containing non maximal filters, follows from [16, Theorem 5.8 and Remark 5.4]. \( \mathcal{P} \) must contain an ultrafilter by (iii) of Proposition 5.1; \( \mathcal{P} \) can then be chosen to consist only of ultrafilters by the equivalence of (ii) and (iii) in Proposition 5.2.

An explicit description of \( \mathcal{P} \) as given by Corollary 5.3 can be found in [17, Theorem 2.3]. On the other hand, by the remarks after [17, Proposition 4.1] and by Proposition 5.1, if \( \mu \geq 2 \) and \( R(\lambda, \mu; < 2) \) is equivalent to sequencewise \( \mathcal{P} \)-compactness, then \( \mathcal{P} \) contains no ultrafilter.

6. Concluding remarks

The problem of deciding whether, in some class \( \mathcal{K} \), sequencewise \( \mathcal{P} \)-compactness and sequencewise \( \mathcal{P}^\prime \)-compactness are equivalent for certain families \( \mathcal{P} \) and \( \mathcal{P}^\prime \), might involve very difficult problems, sometimes of purely set-theoretical nature (even just when \( \mathcal{K} \) is the class of all topological spaces). Other particularly interesting cases are given by the class of topological groups, and the class of normal spaces.

For a class \( \mathcal{K} \) of topological spaces, and for \( F, G \) filters (not necessarily over the same set), define the following (pre-)order: \( F \leq_{C, \mathcal{K}} G \) if and only if every \( G \)-compact topological space in \( \mathcal{K} \) is \( F \)-compact. Strictly speaking, \( \leq_{C, \mathcal{K}} \) is not an order relation, but it induces an order on the equivalence classes modulo the relation \( \equiv_{C, \mathcal{K}} \) defined by \( F \equiv_{C, \mathcal{K}} G \) if and only if both \( F \leq_{C, \mathcal{K}} G \) and \( G \leq_{C, \mathcal{K}} F \). When \( \mathcal{K} \) is the class of all Tychonoff spaces, and \( F \) and \( G \) are ultrafilters, \( \leq_{C, \mathcal{K}} \) is the Comfort (pre-)order. See [10] for a survey, in particular Section 3 therein.

Trivially, if sequencewise \( \mathcal{P} \)-compactness is equivalent to \( F \)-compactness in \( \mathcal{K} \), then \( F \leq_{C, \mathcal{K}} G \) for every \( G \in \mathcal{P} \). Hence, by (i) \( \Leftrightarrow \) (iii) of Theorem 2.1, if \( \mathcal{P} \) is a class of filters with no minimum with respect to \( \leq_{C, \mathcal{K}} \) (here "minimum" is intended modulo equivalence), then sequencewise \( \mathcal{P} \)-compactness is not preserved under taking products of spaces in \( \mathcal{K} \).

However, the existence of such a minimum in \( \mathcal{P} \) is not a sufficient condition for preservation under taking products, as already the example of sequential compactness shows (indeed, \( F_\infty \equiv_{C, \mathcal{K}} F_{\infty'} \), for every \( Z, Z' \in [\omega]^{\omega} \) and every \( \mathcal{K} \)). Using a result by García-Ferreira [9], we can give an example in which all members of \( \mathcal{P} \) are ultrafilters. For every non principal ultrafilter \( D \) over \( \omega \), [9, Example 2.3] constructs a space which is not \( D \)-compact, but which is sequencewise \( \mathcal{P} \)-compact,
where $\mathcal{P}$ is the family of all ultrafilters over $\omega$ which are Rudin–Keisler equivalent to $D$ ($\mathcal{P}$ is called $T_{\mathcal{RK}}(D)$ in [9]). A fortiori, all elements of $\mathcal{P}$ are Comfort equivalent, hence each of them is a minimum in $\mathcal{P}$, under equivalence. However, sequencewise $\mathcal{P}$-compactness is not preserved under products, since otherwise, by Theorem 2.1, it would be equivalent to $D'$-compactness for some $D' \in \mathcal{P}$. But $D'$-compactness is equivalent to $D$-compactness, hence $\mathcal{P}$-compactness would be equivalent to $D$-compactness, and this is false as shown by the space constructed by García-Ferreira.

Of course, in any single particular application of Theorem 2.1 the needed results can be proved directly. Nevertheless, we believe that the theory presented here has some intrinsic interest. At the very least, it has the advantage of presenting many distinct results in a unified way providing a conceptual clarification.

Apart from this, the notion of sequencewise $\mathcal{P}$-compactness is of some use in at least two respects. First, we have showed that, when studying the satisfiability of a topological property in products, it is convenient to translate this property (if possible) in terms of sequencewise $\mathcal{P}$-compactness. This provides a standard method for dealing with the problem and is what Bernstein [2], Ginsburg and Saks [11], Saks [19] and Caicedo [4], among others, have essentially done in $P$-compactness. This provides a standard method for dealing with the problem and is what Bernstein [2], Ginsburg and Saks [11], Saks [19] and Caicedo [4], among others, have essentially done in particular cases, as we mentioned in the introduction. We have continued this line of research in [17] for the Menger and Rothberger properties, here for sequential compactness, and in [18] for $[\mu, \lambda]$-compactness.

Second, the theory presented here naturally leads to new problems. For example, as mentioned above, it stresses the importance of studying when sequencewise $\mathcal{P}$-compactness and sequencewise $\mathcal{P}'$-compactness are equivalent for different $\mathcal{P}$ and $\mathcal{P}'$. This should be particularly interesting when restricted to special classes $\mathcal{X}$ of spaces. Just to present a simply stated but intriguing case, really little is known about Comfort order restricted to the class $\mathcal{X}$ of normal spaces, that is, $\leq_{C, X}$. Notice that this is just a particular case in which both $\mathcal{P}$ and $\mathcal{P}'$ are singletons.

Another kind of problems arise as follows. So far, we have considered certain given topological properties and showed that there exists an appropriate family $\mathcal{P}$ which characterizes them in terms of sequencewise $\mathcal{P}$-compactness. One can also try to follow the other direction, that is, take some interesting classes of ultrafilters and consider the associated topological properties.

We can also introduce a "pseudocompact-like" version of sequencewise $\mathcal{P}$-compactness. If $X$ is a topological space, $I$ is a set, $(Y_i)_{i \in I}$ is an $I$-indexed sequence of subsets of $X$, and $F$ is a filter over $I$, a point $x \in X$ is an $F$-limit point (see Choquet [5]) of the sequence $(Y_i)_{i \in I}$ if $\{i \in I : Y_i \cap U \neq \emptyset\} \in F$ for every open neighborhood $U$ of $x$. If $\mathcal{P}$ is a family of filters over $I$, we say that $X$ is sequencewise $\mathcal{P}$-pseudocompact if for every $I$-indexed sequence of nonempty open subsets of $X$ there is $F \in \mathcal{P}$ such that the sequence has an $F$-limit point. Examples include pseudocompactness and $D$-pseudocompactness [11]; see [14, 15] and [16, Sections 4, 5] for further examples. Sometimes the above examples are presented in equivalent formulations; however, they can be recast in terms of sequencewise $\mathcal{P}$-pseudocompactness by a remark analogous to [16, Remark 5.4] and, in case, using [16, Theorem 5.9]. Though the study of the behavior of sequencewise $\mathcal{P}$-pseudocompactness with respect to products goes beyond the scope of the present note, let us notice that, in general, results about sequencewise $\mathcal{P}$-compactness do not necessarily generalize, as they stand. As a classical example, products of countably compact spaces and of pseudocompact spaces behave in a different way with respect to ultrafilter convergence [11, Theorem 2.6 and Example 4.4]. More elaborate examples (and a possible explanation for the asymmetry) can be found in [15, Sections 1.2, 5] and [14, Section 4].

As a final remark, let us mention that all results of the present note can be easily generalized to the case of $\kappa$-box products, provided that we consider only $\kappa$-complete filters and ultrafilters. The $\kappa$-box product $\prod_{i \in I} X_i$ is defined on the set $\prod_{i \in I} X_i$, and its topology has $\{\prod_{i \in I} O_i : O_i$ is open in $X_i$ and $|\{i \in I : O_i \neq X_i\}| < \kappa\}$ as a base.

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