General Relativity Testing in Exoplanetary Systems

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Abstract. In this work discuss the possibility of testing GR with a high degree of accuracy by observing the precession of perihelia in extrasolar planetary systems. Two approaches of measuring perihelion precession of exoplanets are considered- the radial velocity (RV) method and the transit method. In RV, the orbital parameters can be determined by fitting the observed RV curve, which is the reflex motion induced by the orbiting planet. In transit, the time separation of primary and secondary transits is observed to examine the precession. However, the secondary transits are generally weak and difficult to be observed, with amplitudes usually less than 1ppm. Therefore, RV method is more feasible in this study. Focusing on the RV method, we creatively derive an analytic formula $dA_e = K e \sin f(\omega) d\omega$ for evaluating the sensitivity of precession. To demonstrate the detectability, we simulate RV curves for the selected exoplanets and fit the synthetic data. We found that GR effect can be detected in ten years in some promising systems, with an assumption of the host stars being inactive (without any intrinsic noise) and the precision of RV instruments achieving 0.1m/s. Although successful testing of GR in exoplanetary systems depends on ideal situations as stated above, we will be able to verify the GR precession in near future, considering the rapidly increasing population of discovered exoplanets and the improvement in precision of detecting instruments.

1. Introduction

In 1859, French astronomer Urbain Le Verrier detected an anomalous precession rate of the perihelion of Mercury. He analyzed the date of Mercury’s transits from 1697 to 1848 and calculated the rate of precession deviated from that predicted by Newtonian theory by 38 arc-seconds per century. In 1882, this rate was amended to be 43 arc-seconds per century by astronomer Simon Newcomb.

This deviation remained upsetting until Einstein proposed general relativity in 1917. In GR, this precession is explained by gravitation being affected by the curvature of space-time. Einstein showed that the GR prediction agrees closely with the observed precession rate, offering one of the cornerstone tests of GR.

Previously, many astronomers have attempted to test GR using simulated data of exoplanets, notably the work Observability of the General Relativistic Precession of Periastra in Exoplanets, by Andres Jordan and Gaspar Bakos. As the exoplanet database continues to grow, we now have more good candidates for GR testing.

In the universe, a good deal of exoplanets exhibit relatively small semi-major axes and high eccentricities. Due to these features, their perihelion precession rates are expected to be large enough (orders of magnitude larger than the same effect observed in Mercury) to be detectable.
Currently, there are over 4000 exoplanets discovered, but not all of these’s orbital parameters are determined due to observation limitation. We utilize exoplanets with data of orbital period, planetary minimum mass, and eccentricity measured for analysis.

![Figure 1](image1.png)

Figure 1. The scatter diagram shows plot of 1585 sets of data, illustrating how planetary mass relates to the orbital period (log). As the legend shows, the green dots represent parameters of exoplanets systems while the red dots are those in solar system, with Mercury being circled (P=87.7 days). The expression for rate of precession suggests more preferable sample in the left-top corner.

![Figure 2](image2.png)

Figure 2. The plot shows how the orbital eccentricities of exoplanets are distributed with respect to orbital period in days, with the black dots representing the data of 1283 exoplanets and red dots are for those in the solar system. A fair number of exoplanets are with eccentricity larger than 0.4 and orbital period smaller than 10 days.

2. **Spacetime and General Relativity**

2.1. **Space and time**

In classic Physics, space and time are separated. Space spreads infinitely and time elapses evenly, without any interaction with matters in the universe. The classic concept of space, time and inertial frame was implicitly defined by Newton's first law.

Mach's Principle states a relative concept of inertial frames. Its idea was then developed by Albert Einstein (1879-1955) as a systematic theory—General Relativity (GR). GR quantitatively describes the interaction between spacetime and matters. Here, we briefly state the difference of physical properties of space-time between Newton theory, special relativity and general relativity.

Newton's space-time view is based on Euclid distance, which could be formulated as:

$$ds^2 = dx^2 + dy^2 + dz^2$$  \(1\)
In Minkowski’s space-time, i.e. special relativity, space and time are viewed equally. Further, events turn out to be the subjects in this case when talking about distances, which can be seen as the space-time interval:

\[ ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \] (2)

or

\[ ds^2 = c^2 dt^2 - 2dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \] (3)

The space-time in special relativity does not have any interaction with matters neither. However, GR uses the Schwartzchild metric, which measures the space-time outside of a static black hole. In spherical coordinates, Schwartzchild metric, in turn, express the space-time interval as:

\[ ds^2 = (1 - \frac{2GM}{rc^2})c^2 dt^2 - (1 - \frac{2GM}{rc^2})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \] (4)

It is clear that the property of space-time is related to the mass of black hole. Generally speaking, space-time is related to the matters.

2.2. Kepler’s Orbit and GR Precession

Based on direct observations, Kepler derived three laws that describe planetary motions. For the two-body system, all three Kepler's Laws can be derived from Newton's theory of gravitation that the gravitational force follows the inverse-square law.

\[ F_G = \frac{GM_1M_2}{r^2} \] (5)

![Figure 3. Geometry of an elliptical orbit. The orbit's semi-major axis is a, semi-minor axis is b; eccentricity is e, and longitude of pericenter is \( \omega \). After Carl & Stanley [2]](image)

![Figure 4. Motion of a planet in three-dimensional space. I is the inclination; \( \Omega \) is the longitude of the ascending node; and \( \omega \) is the argument of periapsis. After Carl & Stanley [2]](image)

In order to visualize Keplerian motion, a transcendental equation has to be solved. However, it can be solved by using Euler-Cromer method on computer.

As mentioned earlier, the discrepancy between Mercury's observed orbit and the theoretically predicted orbit remained troubling until Einstein proposed the general theory of relativity in 1917. In fact, while the Newtonian central potential is
\[ V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} \]  

(6)

where \( l = r^2 \dot{\phi} \), and \( \dot{\phi} = \frac{dp}{dt} \) is the angular velocity of the rotating mass, the general theory of relativity predicts the effective potential to be

\[ V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} \]  

(7)

which has an additional \(-\frac{Ml^2}{r^3}\) term. Accordingly, the gravitational force under general theory of relativity is

\[ F_G \approx \frac{GM_SM_P}{r^2} \left( 1 + \frac{a}{r^2} \right) \]  

(8)

where \( a \approx 1.1 \times 10^{-8} AU^2 \). From this expression, one can predict that when the distance from the planet to the Sun is large, the additional term \( \frac{GM_SM_P}{r^4} a \) is trivial and can be neglected, so the gravitational force can be approximated as the inverse-square law. However, when the distance is small, this additional term becomes significant. In our solar system, this is especially noticeable for Mercury as it is the planet closest to the Sun.

To calculating the effects of the general relativity on a planet's orbit, we presume the planet is at its apocenter. The planet's subsequent velocity can be derived:

\[ v_x' = v_x - \frac{4\pi^2 x_0}{r_0^3} \left( 1 + \frac{a}{r_0^2} \right) * \Delta t \]  

(9)

\[ v_y' = v_y - \frac{4\pi^2 y_0}{r_0^3} \left( 1 + \frac{a}{r_0^2} \right) * \Delta t \]  

(10)

And the rate of precession of perihelion can be expressed as

\[ \delta \phi_{\text{prec}} = \frac{6\pi G}{c^2} \frac{M}{a(1-e^2)} \]  

(11)

We set \( a = 0.39 \text{AU} \) and \( e = 0.206 \), which corresponds to Mercury's orbit[4]. To exaggerate the effect of general relativity, we set \( a = 0.01 \). The result of the simulation is shown in Figure 5.

![Figure 5. Orbit of a planet under influence general relativity, with physical parameters corresponding to that of Mercury. To show the influence of general relativity more explicitly, we set \( a = 0.01 \text{AU}^2 \) instead of \( 1.1 \times 10^{-8} \text{AU}^2 \) so that the effect of general relativity can be exaggerated. The fact that the planet's orbit is not enclosed demonstrates that general relativity contributes to the planet's precession.](Image)

3. Radial Velocity and Transit Photometry

Among several ways to detect exoplanets, the radial velocity (RV) method and the transit method are more efficient, finding 810 and 3192 exoplanets respectively (approximately 19% and 77% of the total). These two methods will thus be the focus in this study.
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Figure 6. Histogram of discovered exoplanets with respect to method of detection. 810, 3192, 96, 50, 1 exoplanet have been discovered by radial velocity method, transit method, gravitational microlensing method, direct imaging, and astrometry respectively.

3.1. Radial Velocity Method

This section intends to investigate the principles behind radial velocity method and analyze how data in radial velocity measurement will correspond to the change of argument of pericenter $\omega$ so as to confirm the presence of general relativity.

A star with a planet moves around the system's center of mass (barycenter) in response to the planet's gravitational force, which leads to a sinusoidal change in the star's radial velocity toward or away from the observer on earth. Due to the Doppler effect, the star's spectrum will have a red shift or blue shift. Depending on the wobbling motion of the star, one can determine the existence of a planet.

Six Keplerian observables can be obtained when observing an exoplanetary orbit, $(a, e, P, \omega, i, \Omega)$. The first two are for the specification of the elliptical orbit; $P$ involves the component masses in the system; the three angles $(\omega, i, \Omega)$ are set based on the reference system in three dimensions (as illustrated in Figure 7). In addition, $t_p$ can also be determined, indicating a specific position of the planet on the elliptical orbit at a given time.

To investigate on the precession of the exoplanetary orbit, we are essentially investigating the change of $\omega$.

Figure 7. The coordinate systems are deliberately set so that the z-axis is lying on the line of sight. $\nu(t)$ in the diagram represents the angle extended from the position of pericenter to the position of the
star. $i$ in the diagram represents the angle of inclination of the orbital plane and the plane set by observation [5].

Resolving the radial distance $r$ onto $z$-axis:

$$r_z = r \sin i \sin (\omega + \nu) \hat{z}$$  \hspace{1cm} (12)

Taking the first derivative of $r_z$ to get the value of radial velocity:

$$v_r = \sin i \left[ r \sin (\omega + \nu) + r \dot{\nu} \cos (\omega + \nu) \right]$$ \hspace{1cm} (13)

We also have

$$r = \frac{a(1-e^2)}{1+\cos \nu}$$ \hspace{1cm} (14)

Take the time derivative of $r$:

$$\dot{r} = \frac{a(1-e^2)}{(1+\cos \nu)} \frac{e \sin \nu}{(1+\cos \nu)} \dot{\nu} = \dot{\nu} r \frac{e \sin \nu}{1+\cos \nu}$$ \hspace{1cm} (15)

According to Kepler's second law, the time derivative of area swept by the radial vector is constant:

$$\frac{\delta A}{\delta t} = \frac{1}{2} r^2 \dot{\nu}.$$ Integrating both sides with an upper limit $P$ would generate:

$$A_P = \frac{1}{2} r^2 \dot{\nu}$$ \hspace{1cm} (16)

where $A$ denotes the area of the ellipse, with a general expression of $A = \pi a^2 \sqrt{1-e^2}$, given that $a$ is the length of the semi-major axis. Replace $A$ with $a$ and $e$ results in

$$r \dot{\nu} = \frac{2\pi a}{P \sqrt{1-e^2}} (1 + \cos \nu)$$ \hspace{1cm} (17)

Simplifying:

$$v_r = \sin i \nu \left[ \cos (\nu + \omega) + \frac{e \sin \nu}{1+\cos \nu} \right]$$

$$= \sin i \frac{2\pi a}{P \sqrt{1-e^2}} \left[ \cos (\nu + \omega) (1 + \cos \nu) + e \cos \nu \sin (\nu + \omega) \right]$$

$$= K \cos (\nu + \omega) + e \cos \omega$$ \hspace{1cm} (18)

where $K$ is the radial velocity semi-amplitude defined as

$$K = \frac{2\pi a \sin i}{P \sqrt{1-e^2}}$$ \hspace{1cm} (19)

Rearrange the above formula with a reasonable approximation for simplification $M_\ast + M_p = M_\ast$, since $M_\ast \gg M_p$:

$$K = \left( \frac{2\pi G}{P} \right)^{1/3} \frac{M_p \sin i}{M_\ast} \frac{1}{(1-e^2)^{1/2}}$$ \hspace{1cm} (20)

With relatively standard parameters (mass of Jupiter, mass of the Sun, period of 1yr) this could be rewritten as:

$$K = 28.4 m s^{-1} \left( \frac{P}{1yr} \right)^{-1/3} \left( \frac{M_p}{M_p} \right) \left( \frac{M_\ast}{M_\odot} \right)^{-2/3}$$ \hspace{1cm} (21)

The argument of pericenter $\omega$, together with $K$, would result in a horizontal (phase) and vertical shift in the radial velocity curve. A more direct effect of change of $\omega$ is shown in Figure 8. It can be seen that a vertical shift of the periodic curve can be relatively more significant than that on the horizontal axis.
The parameters chosen so that the semi-amplitude $K = 50 m s^{-1}$, $P = 1 \text{ day}$, and $e = 0.5$. Since $\omega$ essentially represents the phase of the perturbation, it can be deduced that the black curve ($\omega = 120 \text{ degree}$) the turned upside down from the green curve ($\omega = 60 \text{ degree}$). Notice that the shift of curve around $\omega = 90 \text{ degree}$ can be more significant than others, showing how radial velocity change could be related to value of $\omega$. Produced with RadVel, a Python software for the Radial Velocity Modeling Toolkit in Fulton et al. 2018.

It can be noticed from Fig. 8 that when measurements are taken at different times, the difference in radial velocity would differ significantly. For instance, a rather obvious shift of the curves would occur at the peak, so that the data taken at the very period of time could give a more accurate curve-fitting, which will be discusses in detail in section 3.

To quantitatively analyse how the varying longitude of pericenter would affect the vertical translation of the curve, we can take the derivative of $v_r = K[\cos(\nu + \omega) + e\cos\omega]$, which would give:

$$dA_r = Ke \sin \omega d\omega$$

where $A_r$ denotes the vertical shift of the graph. It is worth-noting that the horizontal translation term is deliberately ignored, which can be comprehensible since the term $K\cos(\nu + \omega)$ is zero when taking integration over a complete orbit.

A relationship between the detectability and seven observables can then be deduced, formulated as:

$$d\omega = \frac{6\pi G M_*}{c^2 a(1-e^2)} dt$$

Since the relativistic effect on planetary motion is minute, detection highly depends on how accurate instrumentations can reach. Among the most advanced radial velocity instruments, both currently used and still under construction, accuracy for detection can reach as high as 0.1 meters per second. This limitation on accuracy of detection sets the basis on which how we would select our potential candidates.

3.2. Transit Photometry

When a planet transits directly between its host star and the observer, it blocks a portion of the star's light, which leads to a slight decrease in the observed brightness of the star. This is defined as the primary eclipse. Then the planet gradually moves to the back of the star. As the star starts blocking the planet, secondary eclipse occurs. This leads to a small decrease in total flux as the light reflected from the planet is blocked. If a star is observed to dim periodically, the star may have a planet.
Figure 9. Schematic of a transit. Starting at position 1 on the upper graph when the planet's projection first contacts with the star's rim, the planet blocks certain fraction of light from the star, which reduces the total flux; the second and third contact refer to the position 2 and 3 on the upper graph respectively, denoting the times when the planet's projection lies entirely inside of the star on ingress and egress; and the fourth contact corresponding to the position 4 refers to the point of exit of the planet's projection. After the egress, the total flux increases. After Winn (2009, Figure 1).

Figure 10. Transit curve for the case in which the period is 2 days, impact parameter is 0.1, planet stellar radius ratio is 0.05, and eccentricity is 0. The troughs in the Figure correspond to the eclipses, with the deeper troughs being primary eclipses and shallower troughs being secondary eclipses. Produced with k transit, a Python exoplanet transit modelling tool, implemented [6].

The time interval between primary and secondary eclipses in elliptical orbits can be approximated in terms $e$ and $\omega$ [7]:

$$ (t_2 - t_1) - \frac{P}{2} = \frac{P}{\pi} e \cos \omega (1 + \csc^2 i) $$

(24)

From the preceding sections, we've discussed how general relativity disturbs planets' planets' orbits by perturbing its longitude of percenter; specifically, we have

$$ d\omega = \frac{6\pi G}{c^2} \frac{M}{a(1-e^2)} dt $$

(25)

Moreover, as planet's the longitude of pericenter $\omega$ is directly correlated with the time interval between primary and secondary eclipses, given that general relativity mainly affects $\omega$ and only has negligible effects on eccentricity $e$ and orbital period $P$, the effect of general relativity can be visualized on the transit light curve by inspecting changes in the time interval between primary and secondary eclipses explicitly.
Figure 11. The elliptical orbit of the planet HR 6469 with \( e = 0.672 \). The time interval between primary and secondary eclipse is not constant in this case, because the perturbation of \( \omega \) leads to varying phase differences between two eclipses at different revolutions. After Scarfe et al. (1994, Figure 4).

Simulations of transit light curves shown in Figure 12 directly show the effects of changing \( \omega \) on \( (t_2 - t_1) \). The red line showing the curve of the planet when \( \omega = 8.13 \) degrees, the green line showing the curve of the planet when \( \omega = 80 \) degrees. The planet has an eccentricity of 0.71, orbital period of 1.5 days, impact parameter of 0.1, and planet stellar radius ratio of 0.05. The differing phases of the interval between the primary eclipses and secondary eclipses indicate the effect of general relativity on longitude of pericenter can be extracted from the transit light curve.

Figure 12. The simulated transit light curves show the effects of changing \( \omega \) on \( (t_2 - t_1) \).

4. Simulation and Data Analysis

4.1. Selecting Process

Recall the equation for choosing exoplanet systems \( dA_{\rho} = Ke \sin \omega \, d\omega \). Also, we have

\[
K = 28.4 \, m \, s^{-1} \left( \frac{P}{1 \, yr} \right)^{1/3} \left( \frac{M_P \sin i}{M_J} \right) \left( \frac{M_*}{M_\odot} \right)^{-2/3}
\]

and

\[
\frac{d\omega_{GR}}{dt} = \frac{7.78}{(1-e^2)} \left( \frac{M_*}{M_\odot} \right) \left( \frac{a}{0.05 \, AU} \right)^{-1} \left( \frac{P}{p_{day}} \right)^{-1}
\]

We imported the exoplanet archive that has been updated on May 7th, 2020 from NASA, and filtrated out the systems with only one exoplanet being detected and is detected by radial velocity. 438
samples are left for selection. We took out the observables, including orbital period, mass of planet, mass of star, and eccentricity. We omitted the candidates that are lack of the values of these parameters and are left with 287 samples in total. Applying the analysis equation to select the best candidates.

We deliberately omit the candidates with $e < 0.1 \text{ or } P > 150 \text{ so as to magnify the change of radial velocity detected, and further shorten the time span needed for detection. Calculate the semi-major amplitude } dA_r \text{ under general relativistic effect of each candidate and sort them descendingly would generate. Overall, only HD108147b has radial velocity semi-amplitude exceeding 0.1 meters per second change every century, which could be achieved with cutting-edge observational instruments. The observables of the first ten exoplanets are further investigated with fiducial simulation to investigate the observability of } d \omega. \text{ We omitted the candidates that are lack of the values of } s_i n \omega \text{ when coming to real detection.}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
pl\_name & pl\_ordper & pl\_orbsmax & pl\_orbeccen & pl\_orblper & pl\_bmass & st\_mass & pl\_rvamp & omega\_pr & amp\_pr \\
\hline
13 & HD 108147 b & 10.898500 & 0.102000 & 0.53000 & 308.00 & 0.26100 & 1.27 & 25.10 & 0.618012 & 0.113072 \\
1 & HD 72892 b & 39.475000 & 0.228000 & 0.42300 & 344.35 & 5.45000 & 1.02 & 318.40 & 0.053692 & 0.034047 \\
28 & HD 39855 b & 3.249800 & 0.041000 & 0.14000 & 102.97 & 0.02674 & 0.87 & 4.08 & 2.590747 & 0.025169 \\
7 & HD 185269 b & 6.837760 & 0.077000 & 0.22900 & 176.20 & 1.01000 & 1.30 & 93.30 & 1.013636 & 0.025051 \\
24 & HD 24085 b & 2.045500 & 0.034000 & 0.22000 & 8.92 & 0.03713 & 1.22 & 5.40 & 7.170948 & 0.023655 \\
0 & TYC 4282-006051-b & 101.540000 & 0.422000 & 0.28000 & 288.94 & 10.78000 & 0.97 & 495.20 & 0.009555 & 0.021871 \\
4 & HD 35759 b & 82.467000 & 0.389000 & 0.38900 & -104.00 & 3.76000 & 1.15 & 173.90 & 0.016431 & 0.018824 \\
17 & GJ 436 b & 2.643883 & 0.029100 & 0.13827 & 351.00 & 0.07000 & 0.47 & 17.09 & 2.422678 & 0.015631 \\
14 & HD 33283 b & 18.199100 & 0.150800 & 0.39900 & 155.50 & 0.32900 & 1.38 & 22.40 & 0.232640 & 0.015049 \\
22 & GJ 674 b & 4.693800 & 0.039000 & 0.20000 & 143.00 & 0.03500 & 0.35 & 8.70 & 0.774742 & 0.014159 \\
\hline
\end{tabular}
\caption{Best 10 exoplanets, sorted based on the magnitude of } d A_r \text{. The value of } \omega \text{ for planet HD185269b approaches 180 degrees, therefore the value of sine approaches zero, resulting in a much smaller } A_r \text{, same for HD24085b. For HD 39855b, the relatively small rate of precession (2.590747 degrees per century) is strengthened by a large value of } s i n \omega \text{ when coming to real detection.}

4.2. Curve-fitting

In the process of simulation in following slides, we made the following assumptions. First, the host stars are inactive so that jitter is low- 0.1 m/s. Second, the observations are perfectly precise so that the error of measurement can be approximated to zero. Third, RV instruments have a precision of 0.1 m/s. The entire simulation process is based on module RadVel, an open-source package on python, for modelling Keplerian orbits in radial velocity timeseries [10]. Here we use the exoplanet HD108147b as an example.

Step one: Plot theoretical radial velocity

The relationship between the true anomaly $\nu(t)$ and radial velocity $v_r$ is formulated as:

$$v_r = K [\cos(\omega + \nu) + e \cos \omega] \quad (28)$$

As is illustrated in Fig. 13, the semi-amplitude $K$ can be represented by half the length of vertical distance from the peak to trough, which is 25 ms$^{-1}$ in the case of HD108147---the candidate who has the largest $dA_r$, due to its relatively large eccentricity and short orbiting period.
Step Two: Generate synthetic data

We introduce error bars on Figure 13 to simulate the white noise occurred in real observations. A typical jitter $\sigma_l$ of $1m/s$ (M. Oshagh, 2017) is chosen, and the observational error $\sigma_{obs}$ is assumed to be zero (Oliva et al. (2015a), Marconi et al. (2016)). 100 observation points are chosen to be synthetic data randomly with uniform possibility. The observations are represented in Figure 14 as yellow spots.

Step Three: Perform curve-fitting

We imported the submodule radvel.fitting to find the set of orbital parameters using maximum likelihood fit. We initiated an appropriate guess of $K = 26$, $\sqrt{e}\cos\omega = 0.45$, $\sqrt{e}\sin\omega = -0.57$ and $t_p = 0.1$ to perturb the initial guess. If the initial guess deviates from the real model too much, it will be impossible to have the parameters converge to the optimal set of parameters. The fitted-curve is shown in Figure 15:
Figure 15. The green spots are the synthetic data. It is clear how the stellar jitter can affect the evolution and how the fitted curve deviates from the initial guess model.

Fitting with the synthetic observation data, the difference between the output value of \( \omega \) and input value \( \omega_0 \) is 2.5 degrees. We repeated the entire simulation and curve-fitting process for 1000 times for top ten exoplanets in table 1, took the \( \Delta \omega \) of each loop, and plotted the histogram of the set of data. The probability density appears to be obeying normal distribution. More details on \( \sigma_\omega \) for each exoplanet would be presented in section 3.3.

4.3. Detectability of Precession in Exoplanets
This section aims to examine whether general relativistic precession of \( \omega \) can be significant enough to be detectable in tens of years or less. If so, the exoplanetary systems can be promising objects to efficiently examine the general relativity by monitoring their perihelion precession in acceptable lengths of time.

Obtaining 1000 values of \( \Delta \omega \) for each exoplanet in Table 2, we can accordingly plot the histogram. Figure 16 shows the histogram plotted using simulation parameters of exoplanet HD 24085 b. The blue columns show the probability density of each interval of \( \Delta \omega \), and the dotted orange line is the best-fit line for the planet's probability distribution. As indicated by the best-fit curve, the probability distribution of \( \Delta \omega \) tends to exhibit characteristics of normal distribution, with mean \( \mu = 0.00672 \) and standard deviation \( \sigma = 0.130 \). Moreover, as shown in the Appendix A, the probability distributions of the other eight exoplanets we simulated show similar patterns. As the \( \omega \) distributions are consistent with normality in Shapiro-Wilk normality tests, using \( \sigma_\omega \), the expected uncertainty in the longitude of pericenter, to obtain confidence levels appears to be justified (Andrés, 2008).

Table 2. Parameters of exoplanets investigated. The rightmost column is \( \dot{\omega} \), the rate of precession due to general relativity.

| Planet name          | Period (days) | \( \alpha \) (AU) | \( M_* \) (Solar mass) | \( e \) | \( K \) (m/s) | \( \omega \) (deg) | \( \dot{\omega} \) (deg) |
|----------------------|---------------|------------------|------------------------|--------|--------------|-------------------|------------------------|
| HD 108147 b          | 10.899        | 0.102            | 1.27                   | 0.530  | 25.1         | 308.0             | 0.618                  |
| HD 72892 b           | 39.475        | 0.228            | 1.02                   | 0.423  | 318.4        | 344.4             | 0.0537                 |
| HD 39855 b           | 3.250         | 0.041            | 0.87                   | 0.140  | 4.1          | 103.0             | 2.591                  |
| HD 24085 b           | 2.046         | 0.034            | 1.22                   | 0.220  | 5.4          | 8.9               | 7.171                  |
| TYC 4282-00605-1 b   | 101.540       | 0.422            | 0.97                   | 0.280  | 495.2        | 288.9             | 0.010                  |
| HD 35759 b           | 82.467        | 0.389            | 1.15                   | 0.389  | 173.9        | -104.0            | 0.016                  |
| GJ 436 b             | 2.644         | 0.029            | 0.47                   | 0.138  | 17.1         | 351.0             | 2.423                  |
| HD 33283 b           | 18.199        | 0.151            | 1.38                   | 0.399  | 22.4         | 155.5             | 0.233                  |
| GJ 674 b             | 4.694         | 0.039            | 0.35                   | 0.200  | 8.7          | 143.0             | 0.775                  |
Figure 16. The probability distribution of HD 24085 b.

The second column in Table 3 summarizes the simulation results of $\sigma_\omega$ for the nine exoplanets we've chosen to investigate. In order for the influence of general relativity to be detectable, it is essential for it to reach at least $3\sigma_\omega$ detection. The last two columns in Table 3 shows the time needed for general relativity to reach $3\sigma_\omega$ and $5\sigma_\omega$ respectively, which are calculated using expressions $\frac{3\sigma_\omega}{\omega}$ or $\frac{5\sigma_\omega}{\omega}$.

| Planet name   | $\sigma_\omega$ | Time to achieve $3\sigma_\omega$ | Time to achieve $5\sigma_\omega$ |
|---------------|-----------------|----------------------------------|----------------------------------|
| HD 108147 b   | 0.080           | 38.835                           | 64.725                           |
| HD 72892 b    | 0.007           | 39.106                           | 65.177                           |
| HD 39855 b    | 1.520           | 175.994                          | 293.323                          |
| HD 24085 b    | 0.130           | 5.439                            | 9.064                            |
| TYC 4282-00605-1 b | 0.006 | 180.000                          | 300.000                          |
| HD 35759 b    | 0.148           | 2775.000                         | 4625.000                         |
| GJ 436 b      | 0.358           | 44.325                           | 73.875                           |
| HD 33283 b    | 19.079          | 24565.236                        | 40942.060                        |
| GJ 674 b      | 0.497           | 192.387                          | 320.645                          |

As shown in Table 3, for some exoplanets, such as HD 108147 b, the general relativistic precession can be detected within decades. The most promising exoplanet, HD 24085 b, can exhibit the effects of general relativity that reach $5\sigma_\omega$ level in approximately 9 years. These results can be explained by the exoplanets' low orbital periods, small semi-major axis, eccentric orbits, their own great masses, or their massive host stars. Some exoplanets, such as TYC 4282-00605-1 b and HD 35759 b, however, despite their high semi-amplitudes, can hardly exhibit notable effects of general relativity within decades of timeframe; these cases occur primarily due to their fewer desirable parameters, such as their long orbital periods.

The most extreme case in the table is that of HD 33283 b, whose general relativistic effects can only be detected in tens of thousands of years; this is because this exoplanet has an $\omega_0$ that is close to 180 degrees, rendering the value of $\sin\omega$ to approach zero, exhibiting a much smaller $dA_r$. As illustrated by this example, we can see that the observability of general relativistic precession on an exoplanet does not only depend on semi-major axis, eccentricity, stellar mass, and orbital period, but also depends on its longitude of pericenter. In our simulations that utilize radial velocity methods, although the general pattern of the observability follows the rules stated in section 3.1 (the shorter the semi-major axis, bigger the eccentricity, greater the stellar mass, and shorter the period, the more
pronounced and hence more observable is the general relativistic precession), the longitude of pericenter also has a great influence: the closer $\omega_0$ to 180 degrees, the more difficult it is to readily observe the general relativistic precession. HD 33283 b is an ideal target for observation of general relativistic precession if we neglect its longitude of pericenter, but, unluckily, because of the fact that its $\omega_0$ is close to 180 degrees, it becomes an undesirable subject.

Thus, given above discussions, with the effects of general relativistic precession can be detected in a number of exoplanets, represented by HD 108147 b, HD 72892 b, HD 24085 b, and GJ 436 b, it can be concluded that using radial velocity method on exoplanets is a promising way of verifying the correctness of general relativity in short periods of time.

5. Conclusion
In this work, we discuss the possibility of testing general relativity by observing the precession of perihelion in extrasolar planetary systems. The precession of Mercury's perihelion is a classic test of GR.

Up to now, thousands of planets outside the solar system have been discovered. The exoplanets present diversities in terms of their orbits and other physical properties. There are many exoplanets with mass as large as Jupiter yet residing in smaller orbits than Mercury, termed “Hot Jupiter”. Also, many exoplanets with extremely high eccentricities have been observed. Some exoplanetary systems will be exceptional testbeds for GR, allowing us to even improve on the classic observations on precession of perihelion due to GR effect.

We consider two approaches towards measuring the precession of perihelion in exoplanetary systems- the radial velocity (RV) technique and the transit technique. These are currently the most efficient methods to detect exoplanets. In the RV technique, the orbital elements can be determined by fitting the observed RV curve, which is the Doppler shift of the host star as its reflex motion induced by the orbiting planets. The principle is measuring the longitude of periastron at different time points to seek for its variation predicted by GR. In the Transit technique, the timing separation of primary and secondary transits are monitored to examine the precession of perihelion, as our line of sight intercepts the orbits with different angles. However, the secondary transits are normally extremely weak, with an amplitude less than 1ppm, and are very difficult to observe.

Therefore, we focus on the RV technique in the rest of our study. Our achievements can be summarized as follows:
- We creatively derived analytic formula $dAr$ evaluating the sensitivity of perihelion's precession in RV measurements;
- Using the formula, we selected the best exoplanets in the current exoplanetary database for GR testing;
- We simulated the radial velocity curves for the selected exoplanets and fitted the synthetic data to explore the precession of perihelion in the simulated observations;
- By assuming the host stars are inactive and the precision of RV instruments achieves 0.1m/s, we find that the GR effect can be detected within a time span of ten years in some exoplanetary systems.

Although successful testing of GR in exoplanetary systems depends on some ideal situations as stated above, we eventually are able to identify ideal exoplanetary systems and verify the GR precession of perihelion with high-precision RV instruments in near future, considering the fast-increasing population of discovered exoplanets and the improvement of technique.

6. Appendices
The simulations and curve-fittings for nine exoplanets are shown below. Their standard deviations and means are summarized in Table 4:
Probability Distribution of HD 35759 b

- best-fit line, \( \mu=0.0011, \sigma=0.148 \)
- probability density

Probability Distribution of HD39855b

- best-fit line, \( \mu=-0.0163, \sigma=1.52 \)
- simulated data

\( \omega-\omega_0 \) [deg]
Table 4. Summary of standard deviation $\sigma_\omega$ and mean $\mu$ for the nine exoplanets discussed.

| Planet name          | $\sigma_\omega$ (deg) | $\mu$ (deg) |
|----------------------|-----------------------|--------------|
| HD 108147 b          | 0.080                 | 0.00262      |
| HD 72892 b           | 0.007                 | 0.000062     |
| HD 39855 b           | 1.520                 | -0.0163      |
| HD 24085 b           | 0.130                 | 0.00672      |
| TYC 4282-00605-1 b   | 0.006                 | 0.0003       |
| HD 35759 b           | 0.148                 | 0.0011       |
| GJ 436 b             | 0.358                 | -0.00102     |
| HD 33283 b           | 19.079                | 4.4036       |
| GJ 674 b             | 0.497                 | 0.000608     |

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