GEOMETRIC AND MEASURE-THEORETICAL STRUCTURES OF MAPS WITH MOSTLY CONTRACTING CENTER

MARCELO VIANA AND JIAGANG YANG

Abstract. We show that every diffeomorphism with mostly contracting center direction exhibits a geometric-combinatorial structure, which we call skeleton, that determines the number, basins and supports of the physical measures. Furthermore, the skeleton allows us to describe how the physical measure bifurcate as the diffeomorphism changes. In particular, we use this to construct examples with any given number of physical measures, with basins densely intermingled, and to analyse how these measures collapse into each other - through explosions of their basins - as the dynamics varies. This theory also allows us to prove that, generically, the basins are continuous functions of the diffeomorphism.

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1. Introduction

The notion of mostly contracting center refers to partially hyperbolic diffeomorphisms and means, roughly, that all Lyapunov exponents along the invariant center bundle are negative. It was introduced by Bonatti, Viana [4] as a more or less technical condition that ensured existence and finiteness of physical measures. Since then, it became clear that maps with mostly contracting center have several distinctive features, that justify their study as a separate class of systems.

For instance, Andersson [2] proved that they form an open set in the space of $C^2$ diffeomorphisms, and that the physical measures vary continuously on an open and dense subset. Castro [9] [10] and Dolgopyat [11] studied the mixing properties of such systems. Moreover, Dolgopyat [12] obtained several limit theorems in a similar context. In addition, Melbourne, Matthew [19] proved an almost sure invariance principle (a strong version of the central limit theorem) for a class of maps that includes some partially hyperbolic diffeomorphisms with mostly contracting center. Burns, Dolgopyat, Pesin [6] studied maps with mostly contracting center in the volume preserving setting, obtaining several interesting results about ergodic components, stable ergodicity, and other aspects of the dynamics. Moreover, Burns, Dolgopyat, Pesin, Pollicott [7] studied stable ergodicity of Gibbs $u$-states, in the general (non-volume preserving) setting.

Date: October 24, 2014.
M.V. and J.Y. were partially supported by CNPq, FAPERJ, and PRONEX.
Before all that, Kan [12] exhibited a whole open set of maps on the cylinder with two physical measures whose basins are both dense in the ambient space. His construction was extended by Ilyashenko, Kleptsyn, Saltykov [15]. See also [9, § 11.1.1]. As it turns out, these maps have mostly contracting center. This construction can also be carried out in manifolds without boundary, but then it is not clear whether coexistence of physical measures can still be a robust phenomenon. This is among the questions we aim to answer in this paper: we find negative answers in some situations.

Systems with mostly contracting center have been found by several other authors. Let us mention, among others: Mañé’s [18] examples of robustly transitive diffeomorphisms that are not hyperbolic (see also [4] and [3, 7]); accessible skew-products $M \times S^1 \to M \times S^1$ over Anosov diffeomorphisms which are not rotation extensions, see [23]. New examples will be given in Section 3.

In the sequel we give the precise statements of our results.

1.1. Partial hyperbolicity, physical measures and skeletons. In this paper, a diffeomorphism $f : M \to M$ is called partially hyperbolic if there is a continuous invariant splitting $TM = E^{cs} \oplus E^u$ of the tangent bundle and there are constants $c > 0$ and $\sigma > 1$ such that

(a) $\|Df^n v^n\| \geq c\sigma^n \|v^n\|$ for every $v^n \in E^u$ and every $n \geq 1$ (we say that $E^u$ is uniformly expanding).

(b) $E^{cs}$ is dominated by $E^u$:

$$\frac{\|Df^n v^n\|}{\|Df^n v^{cs}\|} \geq c\sigma^n \frac{\|v^n\|}{\|v^{cs}\|}$$

for every nonzero $v^n \in E^u$, $v^{cs} \in E^{cs}$, and every $n \geq 1$.

The unstable bundle $E^u$ is automatically uniquely integrable: there exists a unique foliation $\mathcal{F}^u$ of $M$ with $C^1$ leaves tangent to $E^u$ at every point. This unstable foliation $\mathcal{F}^u$ is invariant, meaning that $f(\mathcal{F}^u(x)) = \mathcal{F}^u(f(x))$ for every $x \in M$ and the leaves are, actually, as smooth as the diffeomorphism itself.

We call $u$-disk any embedded disk contained in a leaf of the unstable foliation. A partially hyperbolic map $f : M \to M$ has mostly contracting center (Bonatti, Viana [4]) if, given any $u$-disk $D^u$, one has

$$\limsup_{n \to \infty} \frac{1}{n} \log \|Df^n(x)\| E^{cs}(x) < 0$$

for every $x$ in some positive Lebesgue measure subset $D^u_0 \subset D^u$.

A physical measure for $f : M \to M$ is an invariant probability $\mu$ whose basin

$$B(\mu) = \{x \in M : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \text{ converges to } \mu \text{ in the weak* topology}\}$$

has positive volume. proved that every $C^{1+\varepsilon}$ diffeomorphism with mostly contracting center has a finite number of physical measures, and the union of their basins contains almost every point in the ambient space. The set of Lebesgue density points of $B(\mu)$ will be called essential basin of $\mu$ and will be denoted $B_{es}(\mu)$.

Let $f : M \to M$ be a $C^{1+\varepsilon}$ diffeomorphism with mostly contracting center. We say that a hyperbolic saddle point has maximum index if the dimension of its stable manifold coincides with the dimension of the center-stable bundle $E^{cs}$. A skeleton of $f$ is a collection $\mathcal{S} = \{p_1, \ldots, p_k\}$ of hyperbolic saddle points with maximum index satisfying
(i) For any \( x \in M \) there is \( p_i \in S \) such that the stable manifold \( W^s(\text{Orb}(p_i)) \) has some point of transversal intersection with the unstable leaf \( F^u(x) \) through \( x \).

(ii) \( W^s(\text{Orb}(p_i)) \cap W^u(\text{Orb}(p_j)) = \emptyset \) for every \( i \neq j \), that is, the points in \( S \) have no heteroclinic intersections.

Observe that a skeleton may not exist (for instance if \( f \) has no periodic points). Also, the skeleton needs not be unique, when it exists. On the other hand, existence of a skeleton is a \( C^1 \)-robust property, as we will see in a while.

**Theorem A.** Let \( f \) be a \( C^{1+\varepsilon} \) diffeomorphism with mostly contracting center. Then \( f \) admits some skeleton. Moreover, if \( S = \{p_1, \ldots, p_k\} \) is a skeleton then for each \( p_i \in S \) there exists a distinct physical measure \( \mu_i \), such that

1. The closure of \( W^s(\text{Orb}(p_i)) \) and the homoclinic class of the orbit \( \text{Orb}(p_i) \) both coincide with \( \text{supp} \mu_i \), which is the finite union of disjoint \( u \)-minimal component, i.e., each unstable leaf in every component is dense in this setting.

2. The closure of \( W^s(\text{Orb}(p_i)) \) coincides with the closure of the essential basin of the measure \( \mu_i \).

In particular, the number of physical measures is precisely \( k = \# S \). Moreover, \( \text{supp}(\mu_i) \cap \text{supp}(\mu_j) = \emptyset \) for \( 1 \leq i \neq j \leq k \).

In the proof (Section 2) we just pick, for each physical measure \( \mu_i \) a hyperbolic periodic point \( p_i \in \text{supp} \mu_i \) with maximum index: such points constitute a skeleton. When their stable manifolds are everywhere dense, we get from part (b) of the theorem that there exist several physical measures, whose basins are intermingled. Such examples, that generalize the main observation of Kan [16], are exhibited in Section 3.

1.2. Variation of physical measures. Theorem A provides us with a tool to mirror physical measures into hyperbolic periodic points, and this can be used to describe the way physical measures vary when the dynamics is modified. Starting from a skeleton \( S = \{p_1, \ldots, p_k\} \) for \( f \), we may consider its continuation \( \tilde{S} = \{p_1(g), \ldots, p_k(g)\} \) for any nearby \( g \). Then any maximal subset of \( \tilde{S} \) satisfying condition (ii) is a skeleton for \( g \). That is the main content of the following theorem:

**Theorem B.** There exists a \( C^{1+\varepsilon} \) neighborhood \( U \) of \( f \) such that, for any \( g \in U \), any maximal subset of the continuation \( \{p_1(g), \ldots, p_k(g)\} \) which has no heteroclinic intersections is a skeleton. Consequently, the number of physical measures of \( g \) is not larger than the number of physical measures of \( f \).

In fact, these two numbers coincide if and only if there are no heteroclinic intersections between the continuations \( p_i(g) \). Moreover, in that case, each physical measure of \( g \) is close to some physical measure of \( f \), in the weak* topology.

In addition, restricted to any subset of \( U \) where the number of physical measures is constant, the supports of the physical measures and the closures of their essential basins vary in a lower semi-continuous fashion with the dynamics, both in the sense of the Hausdorff topology.

Of course, this implies that the number of physical measures is an upper semi-continuous function of the dynamics. Consequently, this number is locally constant on an open and dense subset of diffeomorphisms with mostly contracting center. These facts had been proved before by Andersson [2]. One important point in our approach is that we give a definite explanation for possible “collapse” of physical measures: one physical measure is lost for each heteroclinic intersection that is created between the continuations of elements of the skeleton. The precise statements are in Propositions 3.6 and 3.7.
We also want to explain how the basins of the physical measures vary with the dynamics in the following measure theoretical sense. Define the pseudo-distance \( d(A, B) = \text{vol}(A \Delta B) \) in the space of measurable subsets of \( M \).

**Theorem C.** Let \( I \) be a closed subset of \( C^{1+\varepsilon} \) diffeomorphisms with mostly contracting center. Then there is a residual subset \( R \subset I \), such that for any \( f \in R \):

(i) every nearby diffeomorphism \( g \in I \) has the same number of physical measures as \( f \);

(ii) if \( g \in I \) is close to \( f \) then the basins of its physical measures are close to the basins of the physical measures of \( f \), relative to the pseudo-distance \( d \).

In Subsection 3.3 we will show how this theory can be applied to various examples, including those of Kan \cite{16}. In particular, Theorem C shows that the basins are quite stable from a measure-theoretical point of view, even though their topology may change drastically.

### 2. Geometric structure of physical measures

Let \( f \) be a \( C^{1+\varepsilon} \) partially hyperbolic diffeomorphism with mostly contracting center. As before, \( E^{cs} \oplus E^u \) denotes the corresponding invariant splitting and \( i_{cs} = \dim E^{cs} \). We call Gibbs \( u \)-state of \( f \) any invariant probability absolutely continuous along strong unstable leaves. It follows that the support is \( u \)-saturated, that is, it consists of entire unstable leaves.

The notion of Gibbs \( u \)-state goes back to Pesin, Sinai \cite{20} and was used by Bonatti, Viana \cite{4} to construct the physical measures of diffeomorphisms with mostly contracting center. Indeed, they showed that such diffeomorphisms have finitely many ergodic Gibbs \( u \)-states, and these are, precisely, the physical measures. Gibbs \( u \)-states also provide an alternative definition of mostly contracting center: \( f \) has mostly contracting center if and only if all Lyapunov exponents along the bundle \( E^{cs} \) are negative for every ergodic Gibbs \( u \)-state. This is related to the fact that, given any disk \( D \) inside an unstable leaf, any Cesaro accumulation point of the iterates of (normalized) Lebesgue measure on \( D \) is a Gibbs \( u \)-state. In fact, more is true: every accumulation point of

\[
\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}
\]

is a Gibbs \( u \)-state, for almost every \( x \in D \). Another useful property is that the space \( G(f) \) of all Gibbs \( u \)-states is convex and weak* compact. The extremal elements are the ergodic Gibbs \( u \)-states. Moreover, \( G(f) \) is an upper semi-continuous function of \( f \), in the sense that the set \( \{(f, \mu) : \mu \in G(f)\} \) is closed. Proofs of these facts can be found in Chapter 11 of \cite{3}.

The following fact will be used several times in what follows:

**Proposition 2.1** (Viana, Yang \cite{23}). If \( f \) is a \( C^{1+\varepsilon} \) diffeomorphism with mostly contracting center then the supports of its physical measures, \( \mu_1, \ldots, \mu_l \) are pairwise disjoint. Moreover, the support of every \( \mu_i \) has finitely many connected components and each connected component is minimal for the unstable foliation (every unstable leaf is dense).

#### 2.1. Proof of Theorem A

The first step is to construct a skeleton:

**Proposition 2.2.** Every \( C^{1+\varepsilon} \) partially hyperbolic diffeomorphism with mostly contracting center admits some skeleton.

**Proof.** Since the center Lyapunov exponents are all negative, every physical measure \( \mu_i \), \( 1 \leq i \leq l \) is a hyperbolic measure (meaning that all the Lyapunov exponents...
are different from zero). So, by Katok [17], there exist periodic points \( q_i \) with maximum index and whose stable manifold intersects transversely the unstable leaf of some point in the support of \( \mu_i \). Since the support is \( u \)-saturated, invariant, and closed, it follows that \( q_i \in \text{supp} \mu_i \). For each \( i \) we choose one such periodic point \( q_i \); we are going to show that \( \{q_1, \ldots, q_l\} \) is a skeleton for \( f \).

Consider any \( x \in M \) and let \( D \) be a disk around \( x \) inside the corresponding unstable leaf. Let \( \mu \) be any Cesaro accumulation point of the iterates of the volume measure \( \text{vol}_D \) on \( D \). As observed before, \( \mu \) is a Gibbs \( u \)-state and, hence, may be written as \( \mu = \sum_{i=1}^l a_i \mu_i \). Choose \( i \) such that \( a_i \) is non-zero. Let \( B \) be a neighborhood of \( q_i \) small enough that the unstable leaf through any point in \( B \) intersects the stable manifold \( W^s(q_i) \) transversely. Then \( \mu_i(B) > 0 \), because \( q_i \in \text{supp} \mu_i \), and so \( \mu(B) > 0 \). Consequently, there is \( n \) arbitrarily large such that \( f^n \text{vol}_D(B) > 0 \). This implies that the unstable manifold of \( f^n(x) \) intersects \( W^s(q_i) \) transversely. By invariance, it follows that \( F^n(x) \) intersects transversely the stable manifold of some iterate of \( q_i \). This proves condition (i) in the definition of skeleton.

Condition (ii) is easy to prove. Indeed, on the one hand, \( W^u(\text{Orb}(q_j)) \) is contained in \( \text{supp} \mu_j \). On the other hand, this support can not intersect \( W^s(\text{Orb}(q_j)) \) for any \( j \neq i \): otherwise, \( q_j \) would be in \( \text{supp} \mu_i \), which would contradict the fact that the supports are pairwise disjoint. Thus, there can indeed be no heteroclinic connections. \( \square \)

Now, we use the skeleton to analyse the physical measures:

**Proposition 2.3.** Let \( f \) be a \( C^{1+\varepsilon} \) diffeomorphism with mostly contracting center. Suppose that \( \mathcal{S} = \{p_1, \ldots, p_k\} \) is a skeleton of \( f \). Then

(a) \( \#\mathcal{S} \) coincides with the number of physical measure of \( f \);
(b) the closure of \( W^u(\text{Orb}(p_i)) \) coincides with \( \text{supp}(\mu_i) = H(p_i, f) \);
(c) the closure of \( W^s(\text{Orb}(p_i)) \) coincides with the closure of \( \text{Bass}(\mu_i) \).

**Proof.** To prove claim (a) it suffices to show that all skeletons have the same number of elements (because the claim holds for the skeleton constructed in Proposition 2.2).

Let \( \mathcal{S}' = \{q_1, \ldots, q_l\} \) be any other skeleton. By condition (i) in the definition, for each \( q_j \in \mathcal{S}' \) there is some \( p_i \in \mathcal{S} \) such that \( W^u(q_j) \) intersects \( W^s(\text{Orb}(p_i)) \) transversely. Choose any such \( p_i \) (we will see in a while that the choice is unique). For the same reason, for this \( p_i \) there exists some \( q_j \in \mathcal{S}' \) such that \( W^u(p_i) \) intersects \( W^s(\text{Orb}(q_j)) \) transversely. It follows that \( W^u(q_j) \) accumulates on \( \text{Orb}(q_j) \), which, by condition (ii) in the definition, can only happen if \( q_j = q_i \). Thus, \( p_i \) and \( q_j \) are heteroclinically related to one another. Since different elements of either skeleton do not have heteroclinic intersections, this implies that \( p_i \) is unique and the map \( q_j \mapsto p_i \) is injective. Reversing the roles of the two skeletons, we also get an injective map \( p_i \mapsto q_j \) which, by construction, is the inverse of the previous one. Thus, these maps are bijections and, in particular, \( \#\mathcal{S} = \#\mathcal{S}' \).

Now take \( \mathcal{S}' \) to be the skeleton obtained in Proposition 2.2. Up to renumbering, we may assume that the \( i = j \) in the previous construction. Also by construction, each \( p_i \) is contained in the closure of \( W^u(\text{Orb}(q_i)) \), which coincides with the support of \( \mu_i \). Since the unstable foliation is minimal in each connected component of the support, this implies that the closure of \( W^u(\text{Orb}(p_i)) \) coincides with \( \text{supp}(\mu_i) \). To finish the proof of claim (b) it remains to show that this coincides with the homoclinic class of \( p_i \). We only have to prove that \( H(p_i) \) contains the closure of \( W^u(\text{Orb}(p_i)) \), since the converse is an immediate consequence of the definition of homoclinic class.

To this end, let \( D \) be any disk contained in the unstable manifold of \( \text{Orb}(p_i) \). Let \( \mu \) be any Cesaro accumulation point of the iterates \( f^n \text{vol}_D \). This is a Gibbs \( u \)-state and it gives full measure to \( \text{supp} \mu_i \) (because \( W^u(\text{Orb}(p_i)) \subset \text{supp} \mu_i \) and
the definition of skeleton. Let us call

2.2. Proof of Theorem B.

Since $D$ unstable leaf $F$ of saddles with maximum index satisfying condition (i), that is, such that

some $i$

Lemma 2.4.

intersections between any of its points.

Then $f^{-n}(W_{loc}^{s}(x_0))$ accumulates on $W^{s}(\text{Orb}(p_i))$ and the essential basin is $f$-invariant, it follows that $W^{s}(\text{Orb}(p_i))$ is contained in the closure of $B_{\text{ess}}(\mu_i)$.

Now we prove the converse inequality. Let $x_0$ be any Lebesgue density point of the basin of $\mu_i$ in ambient space. Using the fact that the unstable foliation is absolutely continuous (see [5]), we can find a small disk $D$ around $x_0$ inside the corresponding unstable leaf such that $\text{Leb}_D(D \cap B(\mu_i)) > 0$. Let $B$ be a neighborhood of $p_i$ small enough that $F^{n}(y)$ intersects $W^{s}(p_i)$ transversely, for every $y \in B$. Take $x \in D \cap B(\mu_i)$. While proving part (b) we have shown that for such a point there exists arbitrarily large values of $n \geq 1$ such that $f^{n}(x) \in B$. Then $f^{n}(D)$ intersects $W^{s}(p_i)$ transversely and, hence, $W^{s}(\text{Orb}(p_i))$ intersects $D$. Since $D$ is arbitrary, it follows that $x_0$ is in the closure of $W^{s}(\text{Orb}(p_i))$. \hfill \Box

Combining Propositions 2.2 and 2.3 yields Theorem A.

2.2. Proof of Theorem B

It will be convenient to separate the two conditions in the definition of skeleton. Let us call pre-skeleton any finite collection $\{p_1, \ldots, p_k\}$ of saddles with maximum index satisfying condition (i), that is, such that every unstable leaf $F^{n}(x)$ has some point of transverse intersection with $W^{s}(\text{Orb}(p_i))$ for some $i$. Thus a pre-skeleton is a skeleton if and only if there are no heteroclinic intersections between any of its points.

One reason why this notion is useful is that the continuation of a pre-skeleton is always a pre-skeleton:

Lemma 2.4. Let $f$ be a partially hyperbolic diffeomorphism which has a pre-skeleton $\mathcal{S} = \{p_1, \ldots, p_k\}$. Let $p_i(g)$, $i = 1, \ldots, k$ be the continuation of the saddles $p_i$ for nearby diffeomorphism $g$. Then $\mathcal{S}(g) = \{p_1(g), \ldots, p_k(g)\}$ is a pre-skeleton for every $g$ in a neighborhood of $f$.

Proof. This is a really a simple consequence of the fact that the unstable foliation depends continuously on the point and the dynamics. Let us detail the argument. Given any $x \in M$, take $i$ such that the unstable leaf $F^{n}(x)$ has some transverse intersection $a_x$ with the stable manifold of some point in the orbit of $p_i \in \mathcal{S}$. Fix $R_x > 0$ large enough so that $a_x$ is in the interior of the $R_x$-neighborhood $F^{-n}_{R_x}(x)$
of $x$ inside $F^u(x)$ and in the interior of the $R_x$-neighborhood $W^u_{R_x}(\text{Orb}(p_i))$ of the orbit of $p_i$ inside its stable manifold. Then, since unstable leaves vary continuously with the point, for any $y$ in a small neighborhood $U_x$ of $x$, there exists $a_y$ close to $a_x$ such that $F^q_{R_x}(y)$ and $W^u_{R_x}(\text{Orb}(p_i))$ intersect transversely at $a_y$. Let $\{U(x_1), \ldots, U(x_m)\}$ be a finite covering of $M$ and let $R = \max\{R_{x_1}, \ldots, R_{x_m}\}$. Thus, $F^q_R(x)$ has some transverse intersection with $\bigcup_{i=1}^k W^u_R(\text{Orb}(p_i))$ for every $x \in M$. Moreover, up to replacing $f$ measure $\mu$ proves the first part of the theorem. \hfill \Box

Another reason why the notion of pre-skeleton is useful to us is that every pre-skeleton contains some skeleton. To prove this it is convenient to introduce the following partial order relation, which will also be useful later on. For any two elements of a pre-skeleton $S = \{p_1, \ldots, p_k\}$ define: $p_i \prec p_j$ if and only if $W^u(\text{Orb}(p_i))$ has some transverse intersection with $W^s(\text{Orb}(p_j))$.

We say that $p_i \in S$ is a maximal element if $p_j \prec p_i$ for every $p_j \in S$ such that $p_i \prec p_j$. Two maximal elements $p_i$ and $p_j$ are equivalent if $p_i \prec p_j$ and $p_j \prec p_i$. We call slice of $S$ any subset that contains exactly one element in each equivalence class of maximal elements.

**Lemma 2.5.** Let $f$ be a partially hyperbolic diffeomorphism which has a pre-skeleton $S = \{p_1, \ldots, p_k\}$. Any slice of $S$ is a skeleton.

**Proof.** Let $S'$ be a subset as in the statement. Begin by noting that $S'$ is also a pre-skeleton. Indeed, since $S$ is assumed to be a pre-skeleton, for any $x \in M$ there exists $p_i \in S$ such that $F^u(x)$ has some transverse intersection with $W^s(\text{Orb}(p_i))$. Moreover, there exists some maximal element $p_j$ of $S$ such that $p_i \prec p_j$. Using the $\lambda$-lemma, it follows that $F^u(x)$ has some transverse intersection with $W^s(\text{Orb}(p_j))$. Moreover, up to replacing $p_j$ by some other maximal element equivalent to it, we may suppose that $p_j \in S'$. This proves our claim. Finally, by definition, there is no heteroclinic intersection between the elements of $S'$. So, $S'$ is indeed a skeleton. \hfill \Box

Now we are ready to give the proof of Theorem B. The set $S = \{p_1, \ldots, p_k\}$ is a pre-skeleton of $f$, of course. So, by Lemma 2.4, there is a $C^1$ neighborhood $V$ of $f$ such that $S(g) = \{p_1(g), \ldots, p_k(g)\}$ is a pre-skeleton for every $g \in V$. Since diffeomorphisms with mostly contracting center form a $C^{1+\varepsilon}$ open set (by Andersson), we may find a $C^{1+\varepsilon}$ neighborhood $U \subset V$ such that every $g \in U$ has mostly contracting center. By Lemma 2.5, every slice $S'(g)$ of $S(g)$ is a skeleton for $g$. Since $\#S'(g) \leq \#S(g) = \#S(f)$, it follows from Theorem A that the number of physical measures of $g \in U$ is not larger than the number of physical measures of $f$. Indeed, these two numbers coincide if and only if $S(g)$ is a skeleton for $g$, that is, if there are no heteroclinic intersections between the continuations $p_i(g)$. This proves the first part of the theorem.

Now let $(f_n)_n$ be a sequence of diffeomorphisms converging to $f$ in the $C^{1+\varepsilon}$ topology and suppose that $S(f_n) = \{p_1(f_n), \ldots, p_k(f_n)\}$ is a skeleton of $f_n$ for any large $n$. Let $\mu_1(f_n), \ldots, \mu_k(f_n)$ be the physical measures (ergodic Gibbs $u$-states). By Theorem A, we may number these measures in such a way that each $\mu_i$ is supported on the closure of $W^u(\text{Orb}(p_i(f_n)))$. Up to restricting to a subsequence, we may assume that $\mu_i(f_n)$ converges, in the weak* topology, to some $f$-invariant measure $\mu_i$. By semicontinuity of the space of Gibbs $u$-states, every $\mu_i$ is a Gibbs $u$-state for $f$. Write $\mu$ as a convex combination $\mu_i = \sum_{j=1}^k a_j \mu_j$ of the physical measures of $f$. We claim that $a_i = 1$. Indeed, suppose that there is $j \neq i$ such that
\[ a_j > 0. \text{ Then } \limsup_n \supp(\mu_j(f_n)) \supset \supp(\mu_j(f)). \]

By Theorem A, we have that \( \supp(\mu_i(f_n)) = \text{closure of } W^u(\text{Orb}(p_i), f_n) \). For \( n \) large, this implies that \( W^u(\text{Orb}(p_i(f_n)), f_n) \) has some transverse intersection with \( W^{\text{loc}}_u(\text{Orb}(p_j(f)), f) \), because the unstable manifolds of hyperbolic periodic points vary continuously with the dynamics. Using the corresponding fact for stable manifolds, we conclude that \( W^u(\text{Orb}(p_i(f_n)), f_n) \) has some transverse intersection with \( W^{\text{loc}}_s(\text{Orb}(p_j(f)), f_n) \). This contradicts the assumption that \( S(f_n) = \{p_1(f_n), \ldots, p_k(f_n)\} \) is a skeleton of \( f_n \). This proves our claim, which yields the second part of the theorem.

The third part is an easy consequence of parts (b) and (c) of Proposition 2.3, together with the general fact that the closures of the stable and the unstable manifolds of hyperbolic saddles vary lower semicontinuously with the diffeomorphism (because the invariant manifolds themselves vary continuously on compact parts). The proof of Theorem B is complete.

2.3. Local description of the continuation of physical measures. Our next goal will be to analyse how physical measures and their basins vary with the dynamics. Here we find a couple of conditions that ensure continuous dependence. This is a prelude to the next section, where we will analyse how physical measures may collapse as their basins explode.

Take \( f \) to be a diffeomorphism with mostly contracting center with a skeleton \( S = \{p_1, \ldots, p_k\} \). Let \( S(g) = \{p_1(g), \ldots, p_k(g)\} \) be its continuation for nearby diffeomorphisms \( g \).

**Corollary 2.6.** There is a \( C^{1+\varepsilon} \) neighborhood \( V \) of \( f \), such that for \( g \in V \), if \( \{p_i(g)\} \) is an equivalence class of maximal elements consisting of a single point, then \( g \) has a physical measure \( \mu_i(g) \) on the closure of \( W^u(\text{Orb}(p_i)) \), which is close to \( \mu_i(f) \) in the weak* topology.

**Proof.** Suppose there exists a sequence \( \{f_n\} \) converging to \( f \) such that \( (\mu_i(f_n))_n \) does not converge to \( \mu_i \). We may assume that the sequence converges to some measure \( \mu \). Then \( \mu \) is a Gibbs u-state of \( f \) and so we may write it as

\[ \mu = a_1 \mu_1 + \cdots + a_k \mu_k. \]

Since \( \mu \neq \mu_i \), there exists \( j \neq i \) such that \( a_j \neq 0 \). By the same argument as in the proof of Theorem B, we have that \( W^u(\text{Orb}(p_j(f_n)), f_n) \) intersects \( W^s(\text{Orb}(p_i(f)), f) \) transversely at some point, for every large \( n \). Consequently, if \( n \) is large enough then \( W^u(\text{Orb}(p_j(f_n)), f_n) \) has some transverse intersection with \( W^s(\text{Orb}(p_i(f_n)), f_n) \). This implies that \( p_i(f_n) \prec p_j(f_n) \), which contradicts the assumption that \( p_i(f_n) \) is maximal and its equivalence class is formed by a single point. \( \square \)

Given \( r \geq 1 \) and two saddle points \( p(f) \) and \( q(f) \) of diffeomorphism \( f \), we say that \( q \) is not \( C^r \) attainable from \( p \) if there is a \( C^r \) neighborhood \( V \) of \( f \) such that \( W^u(p(g), g) \cap W^s(q(g), g) = \emptyset \) for all \( g \in V \), where \( p(g) \) and \( q(g) \) are the analytic continuations of \( p(f) \) and \( q(f) \), respectively.

**Corollary 2.7.** Assume that \( p_i(f) \in S \) is not \( C^{1+\varepsilon} \) attainable from any \( p_j(f) \in S \) with \( j \neq i \). Then the physical measure \( \mu_i(f) \) is stable, in the sense that for every \( g \) in a \( C^{1+\varepsilon} \) neighborhood of \( f \) there exists a physical measure \( \mu_i(g) \) which is close to \( \mu_i(f) \) in the weak* topology.

**Proof.** Let \( V \) be a neighborhood of \( f \) as in the definition of non-attainability. Let \( S'(g) \) be any slice of \( S(g) \). By Lemma 2.5, \( S'(g) \) is a skeleton for \( g \). The assumption implies that \( p_i(g) \) is a maximal element of \( S'(g) \) and its equivalence class consists of a single point. So, the conclusion follows from Corollary 2.6. \( \square \)
3. Exploding basins

We start by giving a geometric and measure-theoretical criterion for a partially hyperbolic diffeomorphism to have mostly contracting center, using the notion of skeleton and a local version of the mostly contracting center property. Then we use this criterion to give new examples of diffeomorphisms with any finite number of physical measures, whose basins are all dense in the ambient space.

Such examples are not stable: the number of physical measures may decrease under perturbation. Indeed, for any proper subset of physical measures one can find a small perturbation of the original diffeomorphism for which those physical measures disappear (their basins are engulfed by the basins of the physical measures that do remain).

Using different perturbations, one can approximate the original diffeomorphism $f$ by other diffeomorphisms $f_n$ having a unique physical measure $\mu_n$, in such a way that $(\mu_n)_n$ converges to any given Gibbs $u$-state of $f$. In particular, such examples are statistically unstable: the simplex generated by all the physical measures does not vary continuously. They are also stochastically unstable, as we will also comment upon later.

3.1. Criterion. Take $f$ to be a partially hyperbolic diffeomorphism with invariant splitting $E^s \oplus E^c$. As before, denote $i_{cs} = \dim E^c$. We start with a semi-local version of the notion of mostly contracting center.

Let $\Lambda$ be a compact $u$-saturated $f$-invariant subset of $M$. We say that $f$ has mostly contracting center at $\Lambda$ if the center Lyapunov exponents are negative for every ergodic Gibbs $u$-state supported on $\Lambda$. Then, we say that $\Lambda$ is an elementary set if there exists exactly one ergodic Gibbs $u$-state $\mu$ supported in $\Lambda$ and it satisfies $\operatorname{supp} \mu = \Lambda$.

The same arguments as in Theorem 3.1 also yield a corresponding semi-local statement: If $\Lambda$ is an elementary set and $\mu$ is the corresponding Gibbs $u$-state, then

- $\mu$ is a physical measure;
- $\Lambda$ has finitely many connected components and the unstable foliation is minimal in each connected component;
- if $p \in \Lambda$ is any hyperbolic saddle with maximum index, then the closure of $W^s(\operatorname{Orb}(p))$ coincides with the closure of the essential basin of $\mu$.

A contains some hyperbolic saddle with maximum index, by arguments in the proof of Proposition 2.2.

**Proposition 3.1.** Let $\Lambda_1, \ldots, \Lambda_k$ be elementary sets, $\mu_1, \ldots, \mu_k$ be the corresponding Gibbs $u$-states, and $p_i \in \Lambda_i$, $i = 1, \ldots, k$ be hyperbolic saddles with maximum index. If $\{p_1, \ldots, p_k\}$ is a pre-skeleton, then it is a skeleton, and $f$ has mostly contracting center. Moreover, $\{\mu_1, \ldots, \mu_k\}$ are the physical measures of $f$, and their basins cover a full Lebesgue measure subset.

**Proof.** It is part of the definition of elementary set that the center Lyapunov exponents of $\mu_j$ are all negative, for every $j = 1, \ldots, k$. So, to prove that $f$ has mostly contracting center it suffices to show that $f$ has no any other ergodic Gibbs-$u$ states. Suppose there exists some ergodic Gibbs-$u$ state $\mu \notin \{\mu_1, \ldots, \mu_k\}$. It follows from the definition that there exists a $u$-disk $D$ contained in some unstable leaf that intersects the basin of $\mu$ on a full Lebesgue measure set $D_0 \subset D$. We claim that there exist $n_0 \geq 1$ and $1 \leq i \leq k$ such that $f^{n_0}(D_0)$ intersects the basin of $\mu_i$. Of course, this contradicts the fact that $\mu \neq \mu_i$. Thus, we are left to justify our claim.

Since $\{p_1, \ldots, p_k\}$ is a pre-skeleton, there exist $n \geq 1$ and $1 \leq i \leq k$ such that $f^n(D)$ intersects $W^s(p_i)$ transversely at some point (otherwise, the Hausdorff limit of $f^n(D)$ would contain some unstable leaf disjoint from $\bigcup_{i=1}^k W^s(O(p_i))$, which
would contradict the definition of pre-skeleton). Again by the definition of Gibbs u-state, there exists a u-disk $D' \subseteq \text{supp }\mu_i$ and a full Lebesgue measure subset $D_0' \subseteq D'$ formed by regular points of $\mu_i$. Since the center Lyapunov exponents are negative, it follows from Pesin theory that there exists a lamination whose laminae are local stable manifolds $W^s_{\text{loc}}(x)$ of almost every point $x \in D_0'$. Moreover, this stable lamination is absolutely continuous.

By Proposition 2.4, $W^s(\text{Orb}(p_i))$ is dense in $\text{supp }\mu_i$. Assuming that $n_0$ is large enough, $f^{n_0}(D)$ is close to $W^u(\text{Orb}(p_i))$ and, in particular, it cuts $\bigcup_{x \in D_0'} W^s_{\text{loc}}(x)$. The intersection is contained in the basin of $\mu_i$, since $W^s_{\text{loc}}(x) \subset B(\mu_i)$ for every $x \in D_0'$. Moreover, by absolute continuity of the lamination, the intersection has positive Lebesgue measure. This implies that $f^n(D_0)$ intersects the basin of $\mu_i$. $\square$

3.2. New Kan-type examples. In this subsection, we use Proposition 3.1 to construct new examples of diffeomorphisms with mostly contracting center and several physical measures, such that every basin intersects every open set on a positive measure subset

**Proposition 3.2.** For any $k \geq 1$, there is a diffeomorphism $f \in \text{Diff}^2(T^2 \times S^2)$ such that $f$ has mostly contracting center and $k$ physical measures $\mu_1, \ldots, \mu_k$ such that $\text{supp }\mu_i = T^2 \times A_i$ for some $A_i \subset S^2$ and the basin $B(\mu_i)$ is dense in $T^2 \times S^2$, for every $i$. Moreover, the same remains true for any diffeomorphism in a $C^{\omega}$-neighborhood which preserves the set $T^2 \times A_i$ for all $i = 1, \ldots, k$.

**Proof.** Let $k$ be fixed and $g \in \text{Diff}^1(T^2)$ be a $C^\omega$ Anosov diffeomorphism with $2k$ fixed points, denoted as $p_1, p_1', \ldots, p_k, p_k'$. Our example will be a partially hyperbolic skew product map

$$f : T^2 \times S^2 \to T^2 \times S^2, \quad f(x, y) = (g(x), h_x(y)),$$

whose center foliation is the vertical foliation by spheres, $W^c(x) = \{x\} \times S^2$. It is easy to see that, for any $x \in T^2$,

$$W^s(W^c(x), f) = W^s(x, g) \times S^2 \quad \text{and} \quad W^u(W^c(x), f) = W^u(x, g) \times S^2.$$

For $x$ and $\tilde{x}$ in the same stable manifold of $g$, let $H^s_{x, \tilde{x}} : W^c(x) \to W^c(\tilde{x})$ be the stable holonomy, defined as the projection along strong stable leaves of $f$. Let the unstable holonomy $H^u_{x, \tilde{x}} : W^c(x) \to W^c(\tilde{x})$ be defined analogously, for $x$ and $\tilde{x}$ in the same unstable leaf of $g$.

Assuming that $h_x$ is uniformly close to the identity in the $C^2$ topology, the partially hyperbolic map $f$ is center bunched (see [22] or [8]), so that these holonomy maps are all $C^1$ diffeomorphisms; moreover, they are close to the identity in the $C^3$ topology. In what follows we consider $k \geq 3$: the cases $k = 1, 2$ are easier.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two smooth circles in $S^2$ intersecting transversely on exactly $k$ points, $A_1, \ldots, A_k$. Take these points to be listed in cyclic order. Then consider points $B_1, \ldots, B_k \in \mathcal{C}$, such that each $B_i$ lies in the circle segment between $A_i$ and $A_{i+1}$ (with $A_{k+1} = A_1$). For each $i = 1, \ldots, k$, let $X_i$ be a Morse-Smale vector field on the sphere such that:

(i) $\Omega(X_i) = \{A_1, B_1, \ldots, A_k, B_k\}$;

(ii) $A_i$ is a sink, $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_k$ are saddles and $B_1, \ldots, B_k$ are sources;

(iii) the basin of the attractor $A_i$ is the complement of segment $S_i \subset \mathcal{C}$ connecting all the saddles and sources.

Figure 1 illustrates the case $k = 3$ and $i = 1$: then $S_i$ is just the segment of $\mathcal{C}$ from $B_1$ to $B_3$ that does contain $A_1$.

Analogously, consider points $B_1', \ldots, B_k' \in \mathcal{C}'$, such that each $B_i'$ lies in the segment of $\mathcal{C}'$ between $A_i$ and $A_{i+1}$. Then let $X_i'$, $i = 1, \ldots, k$, be a Morse-Smale
vector field on the sphere satisfying (i), (ii) and (iii), with $B_i$ replaced by $B'_i$ and $S_i$ replaced by a segment $S'_i \subset C'$. Using these two vector fields, we are going to construct the partially hyperbolic skew-product $f : T^2 \times S^2 \to T^2 \times S^2$ satisfying

1. $A_1, \ldots, A_k$ are fixed points of $h_x(\cdot)$ for any $x \in T^2$. Thus, $T_i = T^2 \times \{A_i\}$ is an $F$-invariant torus, restricted to which $F$ is an Anosov map.

2. $\int_{T_i} \log \|Dh_x(A_i)\| \, d\mu(x) < 0$ for $i = 1, \ldots, k$, where $\mu_i$ denotes the (unique) Gibbs $u$-state of $F|_{T_i}$.

3. $h_{p_i} = \text{time-}\varepsilon$ map of $X_i$ and $h_{p'_i} = \text{time-}\varepsilon$ of $X'_i$, for some small $\varepsilon > 0$.

4. $f$ is $C^2$ close to $(g(x), \text{id})$ (this implies that $\varepsilon$ in (3) should be small).

Condition (2) implies that the center Lyapunov exponents of every $\mu_i$ are negative, and so $T_i$ is an elementary set.

Lemma 3.3. The set $\{p_i \times A_i\}_{i=1}^k$ is a skeleton.

Proof. As a first step, we prove that every strong unstable leaf $F^u(z)$ has a point of transverse intersection with the stable manifold of some $(p_i, A_i)$. Observe that $W^s(W^u(p_1), f) = W^s(p_1, g) \times S^2$. Also, $W^u(p_1, g)$ intersects the $g$-unstable manifold of any point in $T^2$ transversely (recall that $g : T^2 \to T^2$ is Anosov). It follows that, for any $z \in M$, there exists some point $a \in M$ where $F^u(z)$ and $W^u(W^c(p_1), f)$ intersect transversely. There are three possibilities:

(a) $a \in W^s((p_1, A_1), f)$;
(b) $a \in W^s((p_1, A_i), f)$ for some $i \neq 1$;
(c) $a \in F^s(B_j)$ for some $1 \leq j \leq k$.

In case (a) we are done. As for case (b), we claim that it implies that $F^u(z)$ has some transverse intersection $W^s((p_1, A_i), f)$. Indeed, the hypothesis implies that the iterates $f^n(F^u(z))$ accumulate on the unstable leaf $F^u(p_1, A_i)$ of the fixed point $(p_1, A_i)$. The latter is contained in the Anosov torus $T_i$, which also contains $(p_i, A_i)$ and its strong stable leaf $F^s(p_i, A_i)$. In fact, $F^u(p_1, A_i)$ and $F^s(p_i, A_i)$ are transverse inside $T_i$. Thus, it follows that the iterates $f^n(F^u(z))$ accumulate on $(p_i, A_i)$. Since $(p_i, A_i)$ has stable index 3, we get that $f^n(F^u(z))$ has some
transverse intersection with $W^s((A - i, p_i), f)$ for every large $n$. Taking pre-images, we get our claim. Thus, in case (b) we are done as well.

Now, we consider case (c). For $n$ large, $f^n(F^u(z)) = F^u(F^n(a)) = F^u(f^n(a))$ is close to $F^u(p_1, B_1)$. Let $q \in \mathbb{T}^2$ be a point of transverse intersection between $W^u(p_1, g)$ and $W^c(\tilde{p}_1, g)$. Then the consider the map

$$H = H^s_{q, \tilde{p}_1} \circ H^u_{p_1, q} : W^c(p_1) \to W^c(\tilde{p}_1).$$

As observed above, under our assumptions the map $H$ is $C^1$ close to the identity map in the second coordinate. So, in view of our conditions on $C$ and $C'$ (more specifically, the assumption that they meet at $A_1, \ldots, A_N$ only, and so transversely), we have that $H(B_1) \notin C'$. Consequently, $H(B_1) \in W^s((\tilde{p}_1, A_1))$. This means that the strong unstable leaf $F^n(p_1, B_1)$ has some transverse intersection with $W^s((\tilde{p}_1, A_1), f)$. Then the same is true for $f^n(F^u(z))$ if $n$ is large enough. Now observe that $(p_1, A_1)$ and $(\tilde{p}_1, A_1)$ are homoclinically related, meaning that the unstable manifold of any point has some transverse intersection with the stable manifold of the other. So, the previous conclusion implies that $f^n(F^u(z))$ has some transverse intersection with $W^s((p_1, A_1), f)$. This reduces the present situation to case (a).

Thus, we have shown that $\{p_i \times A_i\}_{i=1}^k$ is a pre-skeleton. Next, notice that $W^u((p_i, A_i), f) = F^n(p_i \times A_i)$ is contained in $T_i$ for every $i$. Since these tori are pairwise disjoint, and each one of them is fixed under $f$, we have that $T_i$ is in the complement of $W^s((p_i, A_i), f)$ for every $j \neq i$. So, the points $(p_i, A_i)$ can have no heteroclinic intersections. This finishes the proof that $\{p_i \times A_i\}_{i=1}^k$ is a skeleton. □

Let us proceed with the proof of Proposition 3.2. Applying Proposition 3.1 to the elementary sets $T_i$ and the skeleton $\{p_i \times A_i\}_{i=1}^k$ provided by Lemma 3.3, we find that $f$ has mostly contracting center with $k$ physical measures $\mu_1, \ldots, \mu_k$ such that $\text{supp } \mu_i = \mathbb{T}_i$ for every $i$.

**Lemma 3.4.** $W^s((p_i, A_i), f)$ is dense in $\mathbb{T}^2 \times S^2$ for every $i = 1, \ldots, k$.

**Proof.** By construction, the stable manifold of $A_i$ for the flow $X$ is dense in the sphere $W^c(p_i)$; recall Figure 11. It follows that the stable manifold $W^s((p_i, A_i), f)$ is dense in $W^s(W^c(p_i), f)$. Moreover, the latter is dense in $\mathbb{T}^2 \times S^2$ because it coincides with $W^s(p_i, g) \times S^2$ and the stable manifold $W^s(p_i, g)$ is dense in $\mathbb{T}^2$. This proves the lemma. □

Then, by Theorem A the basin of each physical measure $\mu_i$ is dense in $\mathbb{T}^2 \times S^2$. This completes the proof of Proposition 3.2 in what concerns the map $f$. We are left to show that the conclusions extend to any $C^{1+\alpha}$ diffeomorphism $\tilde{f}$ in a neighborhood which leaves every $\mathbb{T}_i$ fixed.

Begin by observing that $\tilde{f} \mid \mathbb{T}_i$ is close to $f \mid \mathbb{T}_i$ and, in particular, it is Anosov. It follows that $\tilde{f}$ admits a unique Gibbs $u$-state supported on $\mathbb{T}_i$ (the physical measure of that Anosov diffeomorphism) and that Gibbs $u$-state is close to $\mu_i$. The latter ensures that the center Lyapunov exponents remain negative, and so $\mathbb{T}_i$ remains an elementary set for $\tilde{f}$. Each fixed point $(p_i, A_i)$ admits a continuation $(p_i(\tilde{f}), A_i)$ for $\tilde{f}$. By Lemma 2.4, these points form a pre-skeleton for $\tilde{f}$. So, we are still in a position to use Proposition 3.1 to conclude that $\tilde{f}$ has mostly contracting center and exactly $k$ physical measures, $\tilde{\mu}_1, \ldots, \tilde{\mu}_k$, with $\tilde{\mu}_i$ supported on $\mathbb{T}_i$ for every $i$. The proposition also states that $\{p_i(\tilde{f}) \times A_i\}_{i=1}^k$ is actually a skeleton for $\tilde{f}$.

We are left to prove that the basin of every $\tilde{\mu}_i$ is dense. By Theorem A it suffices to show that the stable manifold of every $(p_i, A_i)$ is dense. The center foliation of $f$ coincides with the trivial fibration $\{x\} \times S^2$, which is normally hyperbolic and smooth. Thus, by the stability theorem of Hirsch, Pugh, Shub 14, the perturbation
\( \hat{f} \) admits an invariant center foliation of \( T^2 \times S^2 \) whose leaves are \( C^{1+\alpha} \) spheres uniformly close to the trivial fibers. In particular, the center leaf through each point \((p_t(\hat{f}), A_t)\) is close to \( (p_t) \times S^2 \). That implies that the restriction of \( f \) to that center leaf is Morse-Smale and the stable manifold of \((p_t(\hat{f}), A_t)\) is dense in it. So, the stable manifold of \((p_t(\hat{f}), A_t)\) is dense in the stable manifold of \( W^s(p_t(\hat{f}, A_t), f) \). The stability theorem also says that there exists a homeomorphism \( h \) of \( T^2 \times S^2 \) that maps the center leaves of \( f \) to the center leaves of \( \hat{f} \) and which is a leaf conjugacy:

\[
h(W^c(z, f)) = W^c(h(z), \hat{f}) \quad \text{for every } z \in T^2 \times S^2.
\]

Then the stable manifold of \( W^c(p_t(\hat{f}, A_t), \hat{f}) \) is just the image under \( h \) of the stable manifold of \( W^c(p_t, f) \). That guarantees that the stable manifold of \( W^c(p_t(\hat{f}, A_t), \hat{f}) \) is dense in \( T^2 \times S^2 \). In this way we have recovered all the ingredients we used for \( f \) and so at this point our arguments extend to \( \hat{f} \), as claimed.

\[\square\]

### 3.3. Collapse of measures and explosion of basins

In this subsection, we prove that the examples we have just constructed are statistically unstable: the simplex generated by all the physical measures does not vary continuously with the dynamics, as physical measures may collapse, with their basins of attraction exploding, after small perturbations of the diffeomorphism. In fact, we obtain two different instability results:

- For any proper subset of physical measures, one can find a small perturbation of the original diffeomorphism for which those physical measures vanish: their basins are engulfed by the ones of the remaining physical measures.
- For any Gibbs state \( \mu \) of the original diffeomorphism (not necessarily ergodic), one can find diffeomorphisms \( f_n \) converging to \( f \), such that every \( f_n \) has a unique physical measure \( \mu_n \) and the sequence \( (\mu_n) \) converges to \( \mu \) in the weak-* topology.

In all that follows \( f : M \to M \) is a partially hyperbolic diffeomorphism with \( k = 3 \) physical measures, as constructed in the previous section (the constructions extend to arbitrary \( k \) in a straightforward way). Let us first describe our perturbation technique. It is designed to create new heteroclinic intersections, thus reducing the number of saddle points in the skeleton.

For distinct \( i,j \in \{1,2,3\} \), let \( q_{i,j} \in T^2 \) be a point of transverse intersection of \( W^u(p_i, g) \) and \( W^s(p_j, g) \). Consider a smooth flow \( Y_{t}^{i,j} \) on \( T^2 \times S^2 \) such that:

1. \( Y_{t}^{i,j} \) is supported on a small neighborhood of \((q_{i,j}, A_i)\);
2. \( Y_{t}^{i,j} \) preserves the center foliation of \( f \);
3. For any \( t > 0 \), the map \( Y_{t}^{i,j} \) sends \((q_{i,j}, A_i)\) to some \((q_{i,j}, C_i)\) with \( C_i \notin C \).

We will always consider perturbations \( f_{t_1,t_2,t_3} \) of the original \( f \) of the form

\[
f_{t_1,t_2,t_3} = Y_{t_1}^{1,2} \circ Y_{t_2}^{2,3} \circ Y_{t_3}^{1,3} \circ f \quad t_1, t_2, t_3 \text{ close to zero}.
\]

Observe that \( f_{t_1,t_2,t_3}|p_i \times S^2 = f|p_i \times S^2 \), since \( p_i \times S^2 \), \( i = 1,2,3 \) are away from the regions of perturbation. By Lemma \([24]\) \( \{p_i \times A_i\} \) is a pre-skeleton of \( f_{t_1,t_2,t_3} \). Denote \( p_4 = p_1 \) and \( A_4 = A_1 \) and \( q_{3,4} = q_{3,1} \).

**Lemma 3.5.** The strong unstable leaf \( F^u((p_i, A_i), f_{t_1,t_2,t_3}) \) has some transverse intersection with \( W^s((p_i, A_i), f_{t_1,t_2,t_3}) \), for every \( t_i > 0 \).

**Proof.** Let \( j = i + 1 \). By construction, the strong unstable leaf of \((p_i, A_i)\) for \( t_{i+1} \) contains the point \( Y_{t_i}^{i,j}(q_{i,j}, A_i) \), which is the strong stable leaf of some point in \( \{p_j\} \times (S^2 \setminus C) \). The latter is in the stable manifold of \((p_j, A_j)\). Clearly, the two manifolds intersect transversely at this point.

\[\square\]

We are ready to state and prove our first instability result:
Proposition 3.6. Given any proper subset $\Gamma$ of the set $\{\mu_1, \mu_2, \mu_3\}$ of physical measures of $f$, one can find $\hat{f}$ arbitrarily close to $f$ such that the set of physical measures of $\hat{f}$ is $\{\mu_1, \mu_2, \mu_3\} \setminus \Gamma$.

Proof. First, suppose that $\#\Gamma = 1$, say, $\Gamma = \{\mu_1\}$. Consider $\hat{f} = f_{t_1, 0, 0}$ with $t_1 > 0$. The measures $\mu_2$ and $\mu_3$ are still ergodic Gibbs-$u$ states and physical measures for $\hat{f}$, since $\hat{f}$ coincides with $f$ on the neighborhood of their supports, $T_2$ and $T_3$. Moreover, the unstable manifolds of $(p_2, A_2)$ and $(p_3, A_3)$ are still contained in $T_2$ and $T_3$, respectively, and so these points have no heteroclinic intersections. On the other hand, by Lemma 3.5, $(p_1, A_1)$ has a unique physical measure, $(p_2, A_2), (p_3, A_3)$ is a skeleton of $\hat{f}$, by Lemma 2.5. So, by Theorem A, the diffeomorphism $\hat{f}$ has exactly two physical measures, $\mu_1$ and $\mu_2$.

Now suppose that $\#\Gamma = 2$, say, $\Gamma = \{\mu_1, \mu_2\}$. Consider $\hat{f} = f_{t_1, t_2, 0}$ with $t_1 > 0$ and $t_2 > 0$. Then, just as before, $(p_1, A_1) ∼ (p_2, A_2)$ and $(p_2, A_2) ∼ (p_3, A_3)$ for $\hat{f}$. Then, by Lemma 2.5, $\{(p_3, A_3)\}$ is a skeleton of $\hat{f}$ and, by Theorem A, the map $\hat{f}$ has a unique physical measure, $\mu_3$.

The same arguments show that if $t_1, t_2, t_3$ are all positive then $f_{t_1, t_2, t_3}$ has a unique physical measure (the points $(p_1, A_1)$ are all heteroclinically related), which need not be close to any of the physical measures of the original map $f$.

Proposition 3.7. Let $\nu$ be any Gibbs-$u$ state of $f$, there is a small perturbation $\hat{f}$ of $f$, such that $\hat{f}$ has a unique physical measure $\mu$ which is close to $\nu$ in the weak-* topology.

Proof. Notice that $\nu$ is an element of the simplex

$$\Delta = \{(s_1 \mu_1 + s_2 \mu_2 + s_3 \mu_3) : s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, s_1 + s_2 + s_3 = 1\},$$

since every Gibbs $u$-state is a linear combination of the ergodic Gibbs $u$-states and, for $f$, these are precisely the physical measures. Clearly, it is no restriction to suppose that $\nu$ belongs to the interior of $\Delta$. Fix $\rho > 0$ such that $\nu$ is in

$$\Delta_\rho = \{(s_1 \mu_1 + s_2 \mu_2 + s_3 \mu_3) : s_1 \geq \rho, s_2 \geq \rho, s_3 \geq \rho, s_1 + s_2 + s_3 = 1\}.$$

Consider the hexagon

$$H = \{(t_1, t_2, t_3) : t_1 \geq 0, t_2 \geq 0, t_3 \geq 0, t_1 + t_2 + t_3 = \varepsilon, t_1 + t_2 \geq \delta, t_1 + t_3 \geq \delta, t_2 + t_3 \geq \delta\}.$$ 

with $0 < \delta \ll \varepsilon \ll \rho$. Every triple $(t_1, t_2, t_3) \in H$ has at least two positive coordinates. Hence, by the same arguments as in the proof of Proposition 3.6, the corresponding $f_{t_1, t_2, t_3}$ has exactly one Gibbs $u$-state $\mu_{t_1, t_2, t_3}$, which is also the unique physical measure. This defines a map $\Phi(t_1, t_2, t_3) = \mu_{t_1, t_2, t_3}$ from $H$ to the space of probability measures on $T^2 \times S^2$. By upper semi-continuity of the space of Gibbs $u$-states, $\Phi$ is continuous and its image is contained in a neighborhood of $\Delta$.

Let $I_1, I_2, I_3$ and $J_1, J_2, J_3$ be the boundary segments of $H$, with $I_l$ contained in $\{t_l = 0\}$ and $J_l$ contained in $\{t_l + t_{l+1} = \delta\}$. Observe that $\mu_l$ is the unique physical measure of $f_{t_1, t_2, t_3}$ for every $(t_1, t_2, t_3) \in I_l$ (because $t_l$ is the unique vanishing parameter). Thus, $P(\Phi)(I_l) = \{\mu_l\}$ for every $l$. The set of Gibbs $u$-states of $f_{0,0,0}$ is the line segment $[\mu_1, \mu_2]$ joining $\mu_1$ to $\mu_2$. Then, by upper semi-continuity of the set of Gibbs-$u$-states, $P(\Phi)(I_l)$ is close to $[\mu_1, \mu_2]$ for every $(t_1, t_2, t_3) \in J_l$. Thus, the restriction of $P \cdot \Phi$ to $J_1$ is a continuous curve joining $\mu_1$ to $\mu_2$ inside a small neighborhood of $[\mu_1, \mu_2]$. Similar observations hold for the images of $J_2$ and $J_3$, of course. By a topological degree argument, it follows that $P \cdot \Phi$ contains $\Delta_\rho$ (otherwise, $P(\Phi)(H)$ would be retractable to the boundary of $\Delta$, which is nonsense) In particular, $P(\mu_{t_1, t_2, t_3}) = \nu$ for some $(t_1, t_2, t_3) \in H$. This proves our claim. \qed
4. Generic continuity of basins

In this section, we show that the basins of physical measures vary continuously with the dynamics, restricted to residual subsets of any complete subspace of diffeomorphisms with mostly contracting center. Continuity is with respect to the pseudo-distance \(d(A, B) = \text{vol}(A \Delta B)\) in the family of measurable subsets of \(M\).

**Proposition 4.1.** Let \(\mathcal{I}\) be a complete subspace of the space of \(C^{1+\alpha}\) diffeomorphisms of \(M\) with mostly contracting center. There is a residual subset of diffeomorphisms \(\mathcal{R} \subset \mathcal{I}\), such that the following holds.

Let \(f \in \mathcal{R}\) have exactly \(k\) physical measures \(\mu_1, \ldots, \mu_k\), and let \(\{f_n\}_{n=1}^\infty \subset \mathcal{I}\) be any sequence converging to \(f\) in the \(C^{1+\alpha}\) topology. Then there exists \(N \geq 1\) such that every \(f_n, n \geq N\) has exactly \(k\) physical measures, \(\mu_{n, 1}, \ldots, \mu_{n, k}\). Moreover, up to suitable numbering,

\[d(B(f_n, \mu_{n, i}), B(f, \mu_i)) \to 0 \quad \text{for every} \; 1 \leq i \leq k.\]

The conclusion holds, in particular, within the family of examples constructed in Subsection 3.2.

**Proof.** By Theorem 5.2, the number of physical measures is an upper semi-continuous function of the dynamics. Thus, the number of physical measures is locally constant on some open and dense subset \(\mathcal{O} \subset \mathcal{I}\). We will construct \(\mathcal{R} \subset \mathcal{O}\).

Given any diffeomorphism \(f_0 \in \mathcal{O}\), let \(\mu_1(f_0), \ldots, \mu_k(f_0)\) be its physical measures and \(\{p_i(f_0), \ldots, p_k(f_0)\}\) be a skeleton of \(f_0\) with \(p_i(f_0) \in \text{supp} \mu_i(f_0)\) for each \(i\). As we have seen before, the continuations \(p_i(g), 1 \leq i \leq k\) of the saddle points \(p_i(f_0)\) constitute a skeleton for every \(g\) in a small neighborhood \(U \subset \mathcal{O}\) (because the number of physical measures remains the same). Let \(\mu_1(g), \ldots, \mu_k(g)\) denote the physical measures of \(g\).

For each \(1 \leq i \leq k\), choose a small neighborhood \(V_i\) of \(p_i(f_0)\). Fix \(\rho > 0\) small, such that the \(\rho\)-neighborhood \(W^u_{\rho}(p_i(g), g)\) of \(p_i(g)\) inside its unstable manifold \(W^u(p_i(g), g)\) is contained in \(V_i\) for every \(g \in U\) and every \(i\). For each \(m, N \geq 1\) and \(g \in U\), define

\[\Lambda(m, N, g) = \{x \in W^u_{\rho}(p_i(g), g) : \prod_{i=0}^{n-1} \|Dg^n|E^c\text{sg}(g^{nN}(x))\| \leq e^{-n/m} \text{ for any } n \geq 0\}.

**Lemma 4.2.** For every \(m, N \geq 1\) and \(g \in U\), the set \(\Lambda_1(m, N, g)\) is closed and \(g \mapsto \Lambda_1(m, N, g)\) is upper semi-continuous (relative to the Hausdorff topology), for every fixed \(m\) and \(N\).

**Proof.** The claims that \(\Lambda_1(m, N, g)\) is closed and varies semi-continuously follow immediately from the definition and the observation that \(W^u_{\rho}(p_i(g), g)\) varies continuously with \(g\). \(\square\)

**Corollary 4.3.** Let \(\text{vol}_g^u\) denote the Lebesgue measure on strong unstable leaves of \(g\). For every \(m\) and \(N\), the function \(L_{m, N, i} : g \mapsto \text{vol}_g^u(\Lambda_1(m, N, g))\) is upper semi-continuous.

**Proof.** By semi-continuity in the Hausdorff topology (Lemma 4.2), if \(g'\) is close to \(g\) then \(\Lambda_1(m, N, g')\) is contained in a small neighborhood of \(\Lambda_1(m, N, g)\). Observing that the unstable manifold \(W^u_{\rho}(p_i(g'), g')\) is also close to \(W^u_{\rho}(p_i(g), g)\) in the \(C^1\) topology, it follows that the Lebesgue measure of \(\Lambda_1(m, N, g')\) can not be much larger than the Lebesgue measure of the closure of \(\Lambda_1(m, N, g)\), which is \(\Lambda_1(m, N, g)\) itself. \(\square\)

**Lemma 4.4.** For each \(i\), the set \(\bigcup_{m, N \geq 1} \Lambda_i(m, N, g)\) has full Lebesgue measure in \(W^u_{\rho}(p_i(g), g)\) for every \(m, N \geq 1\) and \(g \in U\).
Proof. Since \( \{p_1(g), \ldots, p_k(g)\} \) is a skeleton for \( f \), each point \( p_i(g) \) belongs to the support of a physical measure \( \mu_i(g) \). Moreover, by [3] Theorem 11.16, Lebesgue almost every point in \( W^u(p_i(g), g) \) is in the basin of some Gibbs \( u \)-state which, given the previous observation, must be \( \mu_i(g) \). Since \( g \) has mostly contracting center, 

\[
\lim_n \int \frac{1}{n} \log \|Dg^n|E^{cs}\| \, d\mu_i = \lambda^c_i < 0,
\]

where \( \lambda^c_i \) denotes the largest center Lyapunov for \( \mu_i(g) \). Fix \( n \geq 1 \) such that the integral on the left hand side is negative. The measure \( \mu_i(g) \) need not be ergodic for \( g^n \) but there always is some \( g^n \)-ergodic probability measure \( \hat{\mu}_i \) such that

\[
\mu_i(g) = \sum_{j=0}^{n-1} g^n_j \hat{\mu}_i(g).
\]

and, up to replacing \( \hat{\mu}_i \) by some iterate,

\[
\int \log \|Dg^n|E^{cs}\| \, d\hat{\mu}_i < 0.
\]

Then, for Lebesgue almost every \( x \in W^u(p_i(g), g) \) there is \( j \in \{0, \ldots, n-1\} \) such that \( g^j(x) \) belongs to the basin of \( \hat{\mu}_i \) relative to \( g^n \) (hence \( x \) belongs to \( B(g, \mu_i(g)) \)) and, in particular,

\[
\lim_k \frac{1}{k} \sum_{s=0}^{k-1} \log \|Dg^n|E^{cs}(g^{s+n}(x))\| = \int \log \|Dg^n|E^{cs}\| \, d\hat{\mu}_i < 0.
\]

Thus, using the fact that \( \log \| \cdot \| \) is subadditive, we can find some (large) multiple \( N \) of \( n \) such that

\[
\frac{1}{k} \sum_{s=0}^{k-1} \|Dg^n|E^{cs}(g^{sN}(x))\| < 0
\]

for every \( k \). Hence, \( x \in \Lambda_i(m, N, g) \) for some \( m \geq 1 \). \( \square \)

By semi-continuity (Lemma 4.2 and Corollary 4.3), there exists a residual subset \( \mathcal{R}_{m,N,i} \) of \( \mathcal{U} \) such that each \( g \in \mathcal{R}_{m,N,i} \) is a continuity point of \( L_{m,N,i} \) and \( \Lambda_i(m, N, \cdot) \). Consider the residual subset \( \mathcal{R} = \cap_{m,N,i} \mathcal{R}_{m,N,i} \). We will show that, any diffeomorphism \( g \in \mathcal{R} \) satisfies the requirements in the conclusion of Proposition 4.1.

Given \( g \in \mathcal{R} \), take \( N \) and \( m \) such that \( \operatorname{vol}_g^m(\Lambda_i(m, N, g)) > 0 \). There exist a neighborhood \( \mathcal{V} \subset \mathcal{U} \) of \( g \) and \( \delta = \delta(m, N, \mathcal{V}) \) such that for any \( h \in \mathcal{V} \), the Pesin stable manifold of every \( y \in \Lambda_i(m, N, h) \) has uniform size \( \delta \) (meaning that it contains a \( \dim E^{cs} \)-disk of radius \( \delta \) around \( y \)). In fact, the uniform bound on the size of the stable manifold follows from the same arguments as [4] Lemma 3.7, applied to the inverse of \( h \).

For any sequence \((g_n)_n \rightarrow g \),

(a) \( \Lambda_i(m, N, g_n) \) converges to \( \Lambda_i(m, N, g) \) in the Hausdorff topology;
(b) \( \operatorname{vol}_g^m(\Lambda_i(m, N, g_n)) \) converges to \( \operatorname{vol}_g^m(\Lambda_i(m, N, g)) \).

Let \( \mathcal{F} : [-1,1]^{\dim E^{cs}} \times [-1,1]^{\dim E^u} \rightarrow V_i \) be a foliation chart, that is, a \( C^1 \) local chart of \( M \) such that:

(i) \( \mathcal{F}(0, 0) = p_i(g) \) and \( \mathcal{F}(0 \times [-1,1]^{\dim E^u}) = W^u_{2p_i}(p_i, g); \)
(ii) the tangent space to the image \( \mathcal{F}((x) \times [-1,1]^{\dim E^u}) \) of every vertical is uniformly close to \( E^u \);
(iii) the tangent space to the image \( \mathcal{F}([-1,1]^{\dim E^{cs}} \times \{y\}) \) of every horizontal is uniformly close to \( E^s \).
For $h \in \mathcal{V}$, consider the lamination $W^u_{\Lambda}(m, N, h)$ whose laminae are the local stable manifolds $W^s_i(y, h)$ with $y \in \Lambda_i(m, N, h)$. Denote by $W^u_{\Lambda}(m, N, h)$ the support of this lamination, that is, the union of all its laminae.

Fix $\varepsilon > 0$ much smaller than $\delta$ and consider any $h \in \mathcal{V}$ and $a \in [-\varepsilon, \varepsilon]^{\dim E^{cs}}$. Denote $\mathcal{F}^a_i = \mathcal{F}^\ast([a] \times [-1, 1])^{\dim E^{cs}}$ and also $\mathcal{F}^a_i = \mathcal{F}^\ast([-\varepsilon, \varepsilon]^{\dim E^{cs}} \times [-1, 1])^{\dim E^{cs}}$. The lamination $W^u_{\Lambda}(m, N, h)$ induces a holonomy map $\Phi_{h, a}$ from $\Lambda_i(m, N, h) \subset W^u_{2\delta^i}(p_i, h)$ to $\mathcal{F}^a_i$. By absolute continuity of the Pesin stable lamination (see [21]), this holonomy map $\Phi_{h, a}$ is absolutely continuous, with Jacobian

\[
\mathcal{J}\Phi_{h, a}(y) = \lim_k \frac{\det(Dh^k | T_y \mathcal{F}^a_i(p_i, h))}{\det(Dh^k | T_{\Phi_{h, a}(y)} \mathcal{F}^a_i))},
\]

uniformly bounded from zero and infinity and depending continuously on $(h, a, y)$.

**Lemma 4.5.** For any $a \in [-\varepsilon, \varepsilon]^{\dim E^{cs}}$, the volume of the symmetric difference

\[
\text{vol}\left(\Phi_{g,a}(\Lambda_i(m, N, g)) \Delta \Phi_{h,a}(\Lambda_i(m, N, h))\right) \to 0.
\]

**Proof.** From (a) above, together with the fact that the holonomy varies continuously with the dynamics, we get that

(2) \[\Phi_{g,a}(\Lambda_i(m, N, g)) \to \Phi_{h,a}(\Lambda_i(m, N, h))\]

in the Hausdorff topology. From (b) above, together with the fact that the Jacobian also varies continuously with the dynamics we get that

(3) \[\text{vol}\left(\Phi_{g,a}(\Lambda_i(m, N, g))\right) \to \text{vol}\left(\Phi_{h,a}(\Lambda_i(m, N, h))\right).
\]

We leave it to the reader to check that (2) and (3) imply the claim of the lemma. □

By Fubini, it follows from Lemma 4.5 that

\[
\text{vol}\left(\mathcal{F}^a_i \cap (W^u_{\Lambda}(m, N, g) \Delta W^u_{\Lambda}(m, N, h))\right) \to 0.
\]

**Lemma 4.6.** For any $\varepsilon > 0$ and $h \in \mathcal{V}$,

\[
\bigcup_{k=0}^{\infty} h^{-k} (\mathcal{F}^a_i \cap W^u_{\Lambda}(m, N, h)) = B(h, \mu_i(h)) \text{ up to zero Lebesgue measure}.
\]

**Proof.** As observed in the course of proving Lemma 4.3, Lebesgue almost every $x \in W^u_{2\delta}(p_i(g), g)$ belongs to the basin $B(h, \mu_i(h))$. Since the basin is saturated by stable sets, and the lamination $W^u_{\Lambda}(m, N, h)$ is absolutely continuous, it follows that

\[
W^u_{\Lambda}(m, N, h) \subset B(h, \mu_i(h)) \text{ up to zero Lebesgue measure}.
\]

Since the basin is an invariant set, this implies the inclusion $\subset$ in the statement.

The converse is a corollary of Proposition 6.9 of [23]. Indeed, this proposition states that $\cup_j \cup_{n \in \mathbb{N}} h^{-n}W^u_{\Lambda_j}(m, N, h)$ contains a full Lebesgue measure subset of every strong-unstable disk. By the absolute continuity of the strong unstable foliation, this implies that $\cup_j \cup_{n \in \mathbb{N}} h^{-n}W^u_{\Lambda_j}(m, N, h)$ contains a full volume subset of the ambient manifold. Since we already know that each $\cup_{n \in \mathbb{N}} h^{-n}W^u_{\Lambda_j}(m, N, h)$ is contained in the corresponding basin $B(h, \mu_j(h))$, and the basins are pairwise disjoint, it follows that $\cup_{n \in \mathbb{N}} h^{-n}W^u_{\Lambda_j}(m, N, h) = B(h, \mu_j(h))$ up to measure zero. The proof is complete. □

This already allows us to prove that the volume of basins is a lower semi-continuous function of the dynamics:

**Lemma 4.7.** $\text{vol}(B(g, \mu_i(g)) \setminus B(g_n, \mu_i(g_n)) \to 0$ as $n \to \infty$. 

Proof. By Lemma 4.6 given any $\varepsilon > 0$ we may take $N_\varepsilon$ such that
\[
\operatorname{vol}(B(g, \mu_i(g)) \setminus \bigcup_{k=0}^{N_\varepsilon} g^{-k}(F^s_\varepsilon \cap W^s_{\Lambda_i}(m, N, g))) \leq \varepsilon.
\]
By Lemma 4.5 we also have
\[
\operatorname{vol}(F^s_\varepsilon \cap (W^s_{\Lambda_i}(m, N, g_n) \Delta W^s_{\Lambda_i}(m, N, g))) \to 0.
\]
This implies that
\[
\operatorname{vol}(\bigcup_{k=0}^{N_\varepsilon} g^{-k}(F^s_\varepsilon \cap W^s_{\Lambda_i}(m, N, g_n))) \to 0.
\]
Hence, for $n$ sufficiently large, we have
\[
\operatorname{vol}(B(g, \mu_i(g)) \setminus \bigcup_{k=0}^{N_\varepsilon} g^{-k}(F^s_\varepsilon \cap W^s_{\Lambda_i}(m, N, g_n))) \leq 2\varepsilon.
\]
Moreover, the definition of $W^s_{\Lambda_i}(m, N, g_n)$ ensures that
\[
\bigcup_{k=0}^{N_\varepsilon} g^{-k}(F^s_\varepsilon \cap W^s_{\Lambda_i}(m, N, g_n)) \subset B(g_n, \mu_i(g_n)).
\]
Since $\varepsilon$ can be taken arbitrarily small, this proves the claim. \qed

The rest of the proof is quite straightforward. Since, for both $g_n$ and $g$, the basins are pairwise disjoint and their union has total measure,
\[
B(g_n, \mu_i(g_n)) \setminus B(g, \mu_i(g)) \subset \bigcup_{j \neq i} B(g, \mu_j(g)) \setminus B(g_n, \mu_i(g_n))
\]
up to measure zero, for every $i$. Using Lemma 4.7 it follows that $\operatorname{vol}(B(g_n, \mu_i(g_n)) \setminus B(g, \mu_i(g)))$ converges to zero. This proves upper semi-continuity of the volume of basins, and so the argument is complete. \qed

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**IMPA, Est. D. Castorina 110, 22460-320 Rio de Janeiro, Brazil**  
*E-mail address: viana@impa.br*  

**Departamento de Geometria, Instituto de Matemática e Estatística, Universidade Federal Fluminense, Niterói, Brazil**  
*E-mail address: yangjg@impa.br*