BLACK HOLES IN THE EINSTEIN-GAUSS-BONNET
THEORY AND THE GEOMETRY OF THEIR
THERMODYNAMICS-II

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In the present work we study (i) charged black hole in Einstein-Gauss-Bonnet (EGB) theory, known as
einstein-Maxwell-Gauss-Bonnet (EMGB) black hole and (ii) black hole in EGB gravity
with Yang-Mills field. The thermodynamic geometry of these two black hole solutions has been
investigated, using the modified entropy in Gauss-Bonnet theory.

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I. INTRODUCTION

The interest in the black hole thermodynamics results due to the nice similarities between a black hole
and a thermodynamical system. Also the thermodynamical quantities, namely the temperature (known
as Hawking temperature) and entropy of a black hole are related to the geometry of the event horizon:
the temperature is proportional to the surface gravity of the event horizon while the entropy is related
to the area of the event horizon [1, 2] and they satisfy the first law of thermodynamics [3]. However, it is
still a challenging problem to find any statistical origin of the black hole thermodynamics.

In this context, Ruppeiner [4] introduced a metric as the Hessian matrix of the thermodynamic entropy
(known as Ruppeiner metric) and showed that the corresponding geometry has physical relevance in the
fluctuation theory of equilibrium thermodynamics. More precisely, the Ruppeiner geometry is related
to the phase structure of thermodynamic system : the scalar curvature ($R$) of the Ruppeiner metric
diverges (i.e. $R \to -\infty$) at the phase transition and critical point while $R \to 0$ (i.e. flat Ruppeiner
metric) indicates no statistical interaction of the thermodynamical system. Also the inverse Ruppeiner
metric gives the second moments of fluctuations. It should be mentioned in this context that Weinhold
[5] first introduced the geometric concept into ordinary thermodynamics by introducing a Riemannian
metric (known as Weinhold metric) as the Hessian of the internal energy(mass parameter here), having
no physical meaning in equilibrium thermodynamics. However, the Ruppeiner metric is conformally
related to the Weinhold metric. In this paper we deal with five dimensional black hole solutions in (i)
Einstein-Maxwell-Gauss-Bonnet (EMGB) theory with a cosmological constant and (ii) Einstein-Yang-
Mills-Gauss-Bonnet (EYMGB) theory and the corresponding thermodynamics with modified form of
entropy [6]. The entropy is modified by the entropy of the analogous Schwarzschild black hole solution
in EGB theory. The black hole solution in EMGB theory and the geometry of its thermodynamic has
been studied in section II while section III contains the black hole solution in EYMGB theory and the
corresponding thermodynamic geometry.

II. GEOMETRIC IDEA OF 5-D EMGB BLACK HOLE THERMODYNAMICS

We present the black hole solution and its properties in five dimensional Einstein-Maxwell-Gauss-
Bonnet theory and subsequently the thermodynamics of this black hole is studied.

A five dimensional spherically symmetric solution in Einstein-Maxwell theory with Gauss-Bonnet term
was obtained by Wiltshire [7] with metric ansatz

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2_3$$ (1)
where,
\[ B(r) = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha} \sqrt{1 + \frac{16m\alpha}{\pi r^4} - \frac{8q^2\alpha}{3r^6} + \frac{4\Lambda}{3}} \]  
(2)

and \( d\Omega^2_3 \) is the metric of unit three sphere.

The electric field is chosen along the radial direction having non vanishing components of the electromagnetic tensor \( F_{\mu\nu} \) (in an orthonormal frame) are
\[ F_{\hat{t}\hat{r}} = -F_{\hat{r}\hat{t}} = \frac{q}{4\pi r^3} \]

The two parameters \( m(> 0) \) and \( q \) are identified as the mass and charge of the system. In the limit \( \alpha \to 0 \) the above solution reduces to the five dimensional Reissner-Nordström solution with a cosmological constant. Note that the solution (2) is well defined if the expression within the square root is positive definite. As a result the solution is valid for \( r > r_0 \) where \( r_0 \) is the largest positive root of the equation
\[ (3 + 4\Lambda) r^6 + \left(\frac{48m\alpha}{\pi}\right) r^2 - 8\alpha q^2 = 0 \]  
(3)

This hyper surface \( r = r_0 \) corresponds to curvature singularity which may be covered by the event horizon (for black hole solution) or may represent a naked singularity.

If \( r_h \) is the radius of the event horizon then it is related to the mass \( (m) \) and charge \( (q) \) of the black hole by the relation
\[ \frac{\Lambda}{3} r_h^6 - 2r_h^4 + \left(\frac{4m}{\pi} - 4\alpha\right) r_h^2 - \frac{2q^2}{3} = 0 \]  
(4)

(Note that event horizon exits if the above equation has at least one positive root)

So we can write the mass parameter as
\[ m = \pi\alpha + \frac{\pi q^2}{6} r_h^{-2} + \frac{\pi r_h^{-2}}{2} - \frac{\pi\Lambda}{12} r_h^4 \]  
(5)

Now using the entropy of the horizon [6] of the of the spherically symmetric black hole solution (Schwarzschild solution) in Einstein-Gauss-Bonnet theory, the entropy of the present black hole takes the form (choosing the Boltzmann constant appropriately)
\[ S = r_h^3 + 6\alpha r_h \]  
(6)

where, \( \tilde{\alpha} = (n - 2)(n - 3)\alpha \) and \( \alpha \) is the Gauss-Bonnet coupling parameter having dimension \((\text{length})^2\).

From equation (5) and (6) we can say that the mass parameter \( (m) \) is in principle a function of entropy \( (S) \) and charge \( (q) \). Using the energy conservation law of the black hole (i.e. \( dm = TdS + \phi dq \)) one obtains the temperature and electric potential of the black hole on the event horizon as
\[ T = \left(\frac{\partial m}{\partial S}\right)_q = \frac{\pi}{9r_h^3} \left(\frac{3r_h^2 - \Lambda r_h^6 - q^2}{r_h^2 + 2\tilde{\alpha}}\right) \]  
(7)

and
\[ \phi = \left(\frac{\partial m}{\partial q}\right)_S = \frac{\pi q}{3} r_h^{-2} \]  
(8)

The Weinhold metric [5, 10], the Hessian of the mass parameter, i.e.,
\[ g^{(W)}_{ij} = \partial_i\partial_j m(S, q) \]  
(9)

has the explicit form
\[ ds^2_W = \frac{\pi}{9r_h^3} \left[ \frac{U(r_h)}{3r_h^2 (r_h^2 + 2\tilde{\alpha})^2} dS^2 - 4qSdq + 3r_h (r_h^2 + 2\tilde{\alpha}) dq^2 \right] \]  
(10)
with,
\[ U(r_h, q) = -\Lambda r_h^8 - 3r_h^6 (1 + 2\alpha\Lambda) + 6\alpha r_h^4 + 5q^2 r_h^2 + 6\alpha^2q^2 \]

The transformation,
\[ S = r_h^3 + 6\alpha r_h, \quad x = r_h, \quad y = qr_h^{-2} \] (11)
reduces the above metric into the diagonal form as,
\[ ds_W^2 = \frac{\pi}{2} \left[ -\left\{ 4y^2 - \frac{U_0(x, y)}{x^4 (x^2 + 2\alpha)} \right\} dx^2 + x^2 dy^2 \right] \] (12)
with
\[ U_0(x, y) = -\Lambda x^8 - 3x^6 (1 + 2\alpha\Lambda) + 6\alpha x^4 + 5y^2 x^6 + 6\alpha^2 y^2 x^4 \]

As the Ruppeiner metric, the Hessian of the entropy function, i.e.,
\[ g^{(R)}_{ij} = \partial_i \partial_j S(m, q) \]
is conformally related to the Weinhold metric by the conformal factor \( T^{-1} \) so we have,
\[ ds_R^2 = \frac{1}{T} ds_W^2 \]

Thus the explicit form of Ruppeiner metric is
\[ ds_R^2 = \frac{3 (x^2 + 2\alpha) (3 - \Lambda x^2 - y^2)}{x^3} \left[ -\left\{ 4y^2 - \frac{U_0(x, y)}{x^4 (x^2 + 2\alpha)} \right\} dx^2 + x^2 dy^2 \right] \] (13)

The non-flat nature of the Ruppeiner metric suggests that the statistical mechanics description is possible for the thermodynamics of the present black hole. As the expression for scalar curvature \( (R) \) is very complicated so we have not presented it here but one thing to note that \( R \) diverges at \( y^2 = \pm \sqrt{3 - \Lambda x^2} \).

Further for a given charge, the expression for the heat capacity is
\[ c_q = \frac{3x^5 (x^2 + 2\alpha)^2 (3 - \Lambda x^2 - y^2)}{U_0(x, y)} \] (14)

Thus at \( y^2 = 3 - \Lambda x^2 \), the metric coefficients as well as the curvature scalar becomes singular while \( c_q \) changes sign at that point. So there is a phase transition [11] corresponding to \( r_h \) given by
\[ q^2 + \Lambda r_h^6 - 3r_h^4 = 0 \]

Further, note that for positive \( \Lambda \), \( U_0 \) has one zero at some positive \( x \), so in some part of state space \( c_q \) will be positive, i.e., black hole will be a stable one while other part corresponds to unstable black hole.

III. THERMODYNAMICS OF 5-D EYMBG BLACK HOLE FROM GEOMETRIC ASPECT

In EYMBG gravity theory, the 5-D spherically symmetric solution obtained recently by Mazhamousavi and Halisoy [12] has the metric ansatz
\[ ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2 d\Omega_3^2 \] (15)
where
\[ U(r) = 1 + \frac{r^2}{4\alpha} \pm \sqrt{\left( \frac{r^2}{4\alpha} \right)^2 + \left( 1 + \frac{m}{2\alpha} \right) + \frac{q^2 \ln r}{\alpha}} \] (16)
m is a constant of integration and q is the only non-zero gauge charge such that
\[ F_{\alpha \beta}^i F^{i \alpha \beta} = \frac{6q^2}{r^4} \]

with \( F_{\alpha \beta}^i F^{i \alpha \beta} \), the Yang-Mills field 2-form.

In the limit \( \alpha \to 0 \), we obtain Einstein-Yang-Mills solution with \( m \) as a mass of the system provided negative branch of the above solution is chosen. Note that if the Gauss-Bonnet coupling parameter \( \alpha \) is positive definite then the solution is well defined for all \( r \) but for \( \alpha < 0 \), the geometry has a curvature singularity at the hyper surface \( r = r_s \) where, \( r_s \) is the largest value of the radial coordinate such that the expression within the square root is positive. If \( r_{h} \) is the radius of the event horizon (\( r_h \) is the positive root of the equation \( U(r) = 0 \), the least one if there are more than one positive roots) then black hole will exit if \( r_s < r_h \), otherwise the singularity will be naked.

Now putting \( U(r) = 0 \), the event horizon satisfies
\[ r_h^2 - m - 2q^2 \ln r_h = 0 \]  \hspace{1cm} (17)

which is independent of the coupling parameter \( \alpha \). Now using the entropy for the Schwarzschild solution in EGB gravity, we have (choosing the unit properly) as before
\[ S = r_h^3 + 6\alpha r_h \]  \hspace{1cm} (18)

Thus the expressions for the thermodynamic quantities namely temperature, electric potential, heat capacity are given by
\[ T = \frac{2 \left( r_h^2 - q^2 \right)}{3r_h \left( r_h^2 + 2\alpha \right)} \]
\[ \phi = -4q \ln r_h \]
\[ c_q = \frac{3r_h \left( r_h^2 - q^2 \right) \left( r_h^2 + 2\alpha \right)^2}{P(r_h, q)} \]  \hspace{1cm} (19)

with
\[ P(r_h, q) = -r_h^4 + r_h^2 (2\alpha + 3q^2) + 2q^2 \alpha \]

Now the explicit form of the Weinhold metric is
\[ ds^2_W = \left[ \frac{2P(r_h)}{9r_h^2 \left( r_h^2 + 2\alpha \right)^2} dS^2 - \frac{8q}{3r_h \left( r_h^2 + 2\alpha \right)} dS dq - 4 \ln r h dq \right] \]  \hspace{1cm} (20)

The transformation
\[ S = r_h^3 + 6\alpha r_h \ , \ x = \ln r_h \ , \ y = q \cdot x \]  \hspace{1cm} (21)

makes the Weinhold metric and hence the Ruppeiner metric (conformally related to the Weinhold metric) into diagonal form as
\[ ds^2_R = \frac{3e^x \left( e^{2x} + 2\alpha \right)}{\left( e^{2x} - \frac{x^2}{e^{2x}} \right)} \left[ \left\{ \frac{p_0(x, y)}{e^{2x} \left( e^{2x} + 2\alpha \right)} + \frac{q^2 y^2}{x^2} \right\} dx^2 - 2 \frac{dy^2}{x} \right] \]  \hspace{1cm} (22)

with
\[ p_0(x, y) = -e^{4x} + e^{2x} \left( 2\alpha + \frac{3y^2}{x^2} \right) + 2\alpha \frac{y^2}{x^2} \]. As the curvature scalar (that we have not presented here due to its complicated and lengthy form) as well as the metric coefficients have singularity at \( y^2 = x^2 e^{2x} \) and the heat capacity changes sign in
crossing the above singularity so there is a phase transition at the singularity. Further, the expression for \( c_\eta \) shows that it has singularities at

\[
\frac{r_h^4 - (2\tilde{\alpha} + 3q^2)}{r_h^2 - 2q^2\tilde{\alpha}} = 0
\]

i.e., at

\[
\frac{r_h^2}{\left(\frac{3q^2}{2}\right)^2} + \left(\frac{\tilde{\alpha} + \frac{3q^2}{2}}{2q^2\tilde{\alpha}}\right)^2 + 2q^2\tilde{\alpha} = 0
\]

Thus \( c_\eta \) is not positive definite so the black hole is not a stable one in the whole admissible state space.

For both the black holes the variation of the thermodynamical parameters (namely mass, Temperature and heat capacity) are different from those of our earlier work [13]. The basic difference in the two papers is the choice of the entropy function in the previous paper; Bekenstein-Hawking entropy relation has been used while in the present paper we have chosen the entropy of a Schwarzschild like black hole in EGB gravity. In fact, heat capacity shows a significant variation and as a result we may conclude that choice of entropy has an important role for the stability as well as phase transition of a black hole.

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