The convergence of a gesture recognizer and the shape of a plane gesture

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Abstract

In this work we develop the mathematical framework of !FTL, a new gesture recognition algorithm, published in [11], and we prove its convergence. Such convergence suggests to adopt a notion of shape for smooth gestures as a complex valued function. However, the idea inspiring that notion came to us from Clifford numbers and not from complex numbers. Moreover, the Clifford vector algebra can be used to extend to higher dimensions the notion of “shape” of a gesture, while complex numbers are useless to that purpose.

1 Introduction

A new gesture recognition algorithm, named !FTL by J.L. Perez-Medina in [11], has a recognition rate aligned to that of the state-of-the-art $P$ recognizer family. Besides, !FTL has proved to be 3 time faster than $P$, thanks to its intrinsic invariance with respect to translation, dilation, and rotation. Indeed, such invariance avoids time consuming rescaling and normalizing pre-processes. In the first part of this article we describe the mathematical framework used to implement !FTL. The notion of shape of a basic gesture is fundamental to !FTL, and recalls that of Lester in [6]. Then, we will show that !FTL is a discretized version of a limit functional which measures the variation between the shapes of two plane gestures; thus, extending Lester’s notion of shape from triangles to gestures. The proof of convergence will be provided using complex numbers. However, while preparing this paper, we were faced to some conflicting items:

- the ideas inspiring our results come from the geometric interpretation\footnote{See [4] and [2], for instance.}
of the Clifford numbers\(^2\), and not from the usual geometry\(^3\) of complex numbers;

- complex numbers are well known, unlike Clifford numbers;
- complex numbers can model plane geometry, while Clifford numbers can model the geometry of a quadratic vector space\(^4\) of any dimension; as a matter of fact, the geometry of Clifford numbers is uniquely determined by the non-degenerate quadratic form defined on the corresponding generating finite-dimensional real vector space; this multidimensional adaptability allows to extend to higher dimensions the notion of shape of a gesture.

As we consider here only gestures in the Euclidean plane, then complex numbers suffice to mimic those Clifford numbers\(^5\) in \(\mathcal{C}\ell(2, 0)\) (the Clifford vector algebra associated to a two-dimensional Euclidean vector space) needed to state and prove our results. That is why we decided to use complex numbers in this work. Nevertheless, our results are deeply rooted in the geometric algebra of Clifford numbers. For this reason, in the end of the article, we will briefly recall the four dimensional Clifford algebra \(\mathcal{C}\ell(2, 0)\), and present our results also in that formalism.

## 2 Preliminary notions

A flat surface can be mathematically modeled by the affine\(^6\) Euclidean plane \(\mathcal{E}\). Thus, the tracing of a single smooth stroke on a flat surface, can be modelled by a function \(G : [0, 1] \rightarrow \mathcal{E}\), such that \(G(t) = O + \vec{g}(t)\), where \(O \in \mathcal{E}\) is an arbitrary reference point in the plane, \([0, 1] = \{t \in \mathbb{R} : 0 \leq t \leq 1\}\), and

\[
\vec{g} : [0, 1] \rightarrow \mathbb{R}^2
\]

is a smooth\(^7\) regular\(^8\) vector valued function. Being the reference point \(O\) arbitrary, we can always consider it as the starting point of the gesture, that is \(\vec{g}(0) = (0, 0)\). In this sense, a gesture is completely determined by the vector-valued function \(\vec{g}\). That is why we give the following definition.

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\(^2\)See \([9]\).

\(^3\)See \([8]\).

\(^4\)See, for instance, \([5]\) or \([7]\).

\(^5\)See, for example, the Lounesto’s article in \([9]\).

\(^6\)See, for instance \([1]\).

\(^7\)That is, twice differentiable on the interval \([0, 1]\), with continuous second derivatives.

\(^8\)That is, whose derivative never vanishes.
Definition 2.1. A plane gesture is a function $\vec{g} : [0, 1] \to \mathbb{R}^2$ which is two times continuously differentiable, and whose derivative $\vec{g}'(t) \in \mathbb{R}^2$ is a vector that never vanishes. Briefly $\vec{g} \in C^2([0, 1]; \mathbb{R}^2)$, and $\vec{g}'(t) \neq \vec{0} = (0, 0)$, for each $t \in [0, 1]$.

When tracing a gesture on a physical device, only a finite number of points are sampled from the input device. A sampled gesture is a finite sequence of points with timestamps; we describe it mathematically as follows.

Definition 2.2. A regular $n$-sample of a plane gesture $\vec{g}$ is a sequence of $n+1$ vectors $\vec{g}_0, \ldots, \vec{g}_k, \ldots, \vec{g}_n$, where

- $\vec{g}_k = \vec{g}(t_k)$,
- $0 = t_0 < \cdots < t_k < t_{k+1} < \cdots < t_n = 1$,
- $\Delta \vec{g}_k = \vec{g}(t_k) - \vec{g}(t_{k-1}) \neq \vec{0} = (0, 0)$, for every $k = 1, \ldots, n$.

The following notion of basic gesture is based on the idea that a shape can arise from at least two consecutive sampled points of a gesture; that is, two vectors in $\mathbb{R}^2$ (see also Remark 2.1).

Definition 2.3. A plane basic gesture is an ordered pair $(\vec{v}_1, \vec{v}_2)$ of non-zero vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$.

Remark 2.1. A basic gesture can be thought as a particular 2-sample of a plane gesture tracing a triangle. More precisely, we can consider the two vectors of a basic gesture $(\vec{v}_1, \vec{v}_2)$, as a pair of consecutive arrows joining three sampled consecutive points

$$G_0 = O, \ G_1 = O + \vec{v}_1, \ and \ G_2 = O + \vec{v}_2,$$

of a plane gesture $\vec{g}$ tracing the affine plane ordered triangle $G_0G_1G_2$; where $\vec{v}_1 = \Delta \vec{g}_1 = \vec{g}(t_1) - \vec{g}(t_0) \neq \vec{0}$, $\vec{v}_2 = \Delta \vec{g}_2 = \vec{g}(t_2) - \vec{g}(t_1) \neq \vec{0}$, and $0 = t_0 < t_1 < t_2 = 1$. Thus, those three points are the vertexes of a (ordered) triangle, eventually degenerate, whose third oriented side can be traced by the vector $-(\vec{v}_1 + \vec{v}_2)$.

\[\text{\footnotesize{\textsuperscript{9}Besides the unavoidable starting point.}}\]
Figure 1: A basic gesture $(\vec{v}_1, \vec{v}_2)$ tracing an ordered affine triangle $G_0G_1G_2$.

One can note that, following the foregoing procedure, a single generic (unordered) triangle can be traced by six basic gestures, possibly different.

3 The complex number point of view

We recall the well known one-to-one correspondence between vectors in $\mathbb{R}^2$ and complex numbers

$$\mathbb{R}^2 \ni \vec{v} = (x, y) \longleftrightarrow x + iy = \mathbf{v} \in \mathbb{C}, \quad (1)$$

where $x, y \in \mathbb{R}$, and $i = \sqrt{-1}$, that is $i^2 = -1$. The commutative product between two complex numbers $u = r + is$ and $\mathbf{v} = x + iy$ is the complex number

$$uv = (rx - sy) + i(ry + sx).$$

Thus, if $\mathbf{v} \neq 0$, the quotient between $u$ and $\mathbf{v}$ is the complex number

$$\frac{u}{\mathbf{v}} = \frac{rx + sy}{x^2 + y^2} - \frac{ry - sx}{x^2 + y^2}.$$

3.1 The Local Shape Distance

**Definition 3.1.** The **complex shape** of a basic gesture $(\vec{v}_1, \vec{v}_2)$ is the complex number

$$\frac{\mathbf{v}_1}{\mathbf{v}_2},$$

where $\mathbf{v}_1$, $\mathbf{v}_2$ are the two complex numbers corresponding to vectors $\vec{v}_1$, $\vec{v}_2$, respectively, according to the correspondence (1).

**Remark.** The foregoing definition is rooted in Lester’s article [6], where it is shown that two ordered triangles are similar if and only if the basic gestures generating them (in the sense of Remark [2.1]) have the same complex shape.
Definition 3.2. The Local Shape Distance between two basic gestures \((\vec{u}_1, \vec{u}_2)\) and \((\vec{v}_1, \vec{v}_2)\) is the non-negative real number

\[
LSD((\vec{u}_1, \vec{u}_2), (\vec{v}_1, \vec{v}_2)) = \left| \frac{u_1}{u_2} - \frac{v_1}{v_2} \right|_C,
\]

where \(u_i, v_i\) are the complex numbers corresponding to vectors \(\vec{u}_i, \vec{v}_i\), respectively, according to the correspondence \((1)\). We recall that

\[
|u - v|_C = \sqrt{(r - x)^2 + (s - y)^2} = |\vec{u} - \vec{v}|_{\mathbb{R}^2},
\]

where \(u = r + is \in \mathbb{C}, v = x + iy \in \mathbb{C}, \vec{u} = (r, s) \in \mathbb{R}^2, \) and \(\vec{v} = (x, y) \in \mathbb{R}^2,\) according to the correspondence \((1)\). Thus, the Local Shape Distance is simply the distance between the numbers representing the complex shapes of two basic gestures, according to \([6]\).

3.2 The !FTL algorithm

Definition 3.3. Given the \(n\)-samples of two plane gestures \(\vec{f}\) and \(\vec{g}\), the following !FTL algorithm gives a measure of their dissimilarity based on the Local Shape Distance between the complex shapes of basic gestures produced by consecutive pairs of vectors taken from the samples. More precisely, given two isochronous \(10\) \(n\)-samples \(11\)

\[
\vec{f}_0, \ldots, \vec{f}_n, \quad \vec{g}_0, \ldots, \vec{g}_n
\]

of the plane gestures \(\vec{f}(t) = (r(t), s(t)), \vec{g}(t) = (x(t), y(t)) \in \mathbb{R}^2,\) respectively, where \(r, s, x, y \in C^2([0, 1]; \mathbb{R})\), then

\[
!FTL(\vec{f}_0, \ldots, \vec{f}_n, \vec{g}_0, \ldots, \vec{g}_n) = \sum_{k=1}^{n-1} LSD((\Delta \vec{f}_k, \Delta \vec{f}_{k+1}), (\Delta \vec{g}_k, \Delta \vec{g}_{k+1}))
\]

\[
= \sum_{k=1}^{n-1} \left| \frac{\Delta f_k}{\Delta f_{k+1}} - \frac{\Delta g_k}{\Delta g_{k+1}} \right|_C.
\]

where \(\Delta f_k = f(t_k) - f(t_{k-1}), \Delta g_k = g(t_k) - g(t_{k-1}), f(t) = r(t) + is(t), \) and \(g(t) = x(t) + iy(t)\) are complex numbers.

Some natural questions about !FTL arise,

- if gesture \(\vec{g}\) is translated, does the value of !FTL change?

\[10\]That is, they are sampled at the same timestamps \(0 = t_0 < \cdots < t_n = 1.\)

\[11\]See Definition 2.2.
• if gesture $\vec{g}$ is uniformly scaled, does the value of $!\text{FTL}$ change?
• if gesture $\vec{g}$ is rotated, does the value of $!\text{FTL}$ change?
• if one increases the number of the sampled points of the two gestures, does $!\text{FTL}$ has a limit value?
• if such a limit value exists, can we find a closed formula to express it?

In what follows, we will see that $!\text{FTL}$ is invariant with respect to translations, scaling and rotations; moreover, under certain hypothesis, to increase the number of sampled points improves the measure of dissimilarity, which corresponds to a well definite number explicitly expressed as a Riemann integral.

3.2.1 Invariance properties of LSD

In this section, we will show the invariance properties of the complex shape $\frac{\Delta g_k}{\Delta g_{k+1}}$ of a basic gesture $(\Delta \vec{g}_k, \Delta \vec{g}_{k+1})$ coming from the sample of a plane gesture $\vec{g}$. As a matter of fact, let $\vec{g}_0, \ldots, \vec{g}_n$ be an $n$-sample of a plane gesture $\vec{g}(t) = (x(t), y(t))$; then

1. for each vector $\vec{v} \in \mathbb{R}^2$,
   • $\vec{p}(t) = \vec{g}(t) + \vec{v}$ is a plane gesture,
   • $\Delta \vec{p}_k = \Delta \vec{g}_k$, for all $k = 1, \ldots, n$,
   and then $\frac{\Delta p_k}{\Delta p_{k+1}} = \frac{\Delta g_k}{\Delta g_{k+1}}$.

2. for each number $\lambda \in \mathbb{R}$ ($\lambda \neq 0$),
   • $\vec{l}(t) = \lambda \vec{g}(t)$ is a plane gesture,
   • $\Delta \vec{l}_k = \lambda \Delta \vec{g}_k$, for all $k = 1, \ldots, n$,
   and then $\frac{\Delta l_k}{\Delta l_{k+1}} = \frac{\lambda \Delta g_k}{\lambda \Delta g_{k+1}} = \frac{\Delta g_k}{\Delta g_{k+1}}$.

3. for each number $\theta \in \mathbb{R}$,
   • $\vec{q}(t) = (x(t) \cos \theta - y(t) \sin \theta, y(t) \cos \theta + x(t) \sin \theta)$ is a plane gesture,
   • $\vec{q}(t) = (\cos \theta + i \sin \theta)(x(t) + iy(t)) = e^{i\theta} \vec{g}(t)$
\[
\Delta q_k = e^{i\theta} \Delta g_k, \text{ for all } k = 1, \ldots, n,
\]
and then
\[
\frac{\Delta q_k}{\Delta q_{k+1}} = \frac{e^{i\theta} \Delta g_k}{e^{i\theta} \Delta g_{k+1}} = \frac{\Delta g_k}{\Delta g_{k+1}}.
\]
The invariance properties of !FTL follows then from those of the complex shape, thanks to Definition 3.3.

3.2.2 Convergence of !FTL through complex numbers

Theorem 3.1. Given two plane gestures \( \vec{f}(t) = (r(t), s(t)) \in \mathbb{R}^2 \), and \( \vec{g}(t) = (x(t), y(t)) \in \mathbb{R}^2 \), then

\[
\lim_{n \to \infty} !FTL(\vec{f}_0, \ldots, \vec{f}_n, \vec{g}_0, \ldots, \vec{g}_n) = \int_0^1 \left| \frac{f''(t)}{F'(t)} - \frac{g''(t)}{G'(t)} \right| dt
\]

where

- \( \vec{f}_0, \ldots, \vec{f}_n, \vec{g}_0, \ldots, \vec{g}_n \) are isochronous \( n \)-samples of \( \vec{f} \) and \( \vec{g} \), respectively, such that \( t_k = \frac{k}{n} \), for all \( k = 0, 1, \ldots, n \),

- \( f(t) = r(t) + is(t) \), and \( g(t) = x(t) + iy(t) \).

A proof. By hypothesis, the Riemann integral \( \int_0^1 \left| \frac{f''(t)}{F'(t)} - \frac{g''(t)}{G'(t)} \right| dt \) exists; this implies that for every \( \epsilon > 0 \) there exists \( N_\epsilon \in \mathbb{N} \) such that

\[
\left| \sum_{k=1}^n \frac{f''(\xi_k)}{F'(\xi_k)} - \frac{g''(\xi_k)}{G'(\xi_k)} \right| \frac{1}{n} - \left( \int_0^1 \left| \frac{f''(t)}{F'(t)} - \frac{g''(t)}{G'(t)} \right| dt \right) < \epsilon,
\]

provided \( n > N_\epsilon \), and \( \xi_k \in \left[ \frac{k-1}{n}, \frac{k}{n} \right] \), with \( k = 1, \ldots, n \).

Notice that, to evaluate each shape \( \frac{\Delta g_k}{\Delta g_{k+1}} \), the extremities of two adjacent intervals are needed. In particular, we can write

\[
\sum_{k=1}^{2m-1} \frac{\Delta g_k}{\Delta g_{k+1}} = \sum_{h=1}^{m} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} + \sum_{h=1}^{m-1} \frac{\Delta g_{2h}}{\Delta g_{2h+1}},
\]

when \( n = 2m \) is even \(^{12}\). Thus, to estimate the difference between local shape distances and terms of a Riemann sum, we have to consider the latter on couples of adjacent intervals. In order to simplify notations, we will consider in the following only the case \( n = 2m \) (\( n \) even). However, our arguments can be applied similarly to the case: \( n \) odd. If \( n > 2N_\epsilon \), then the integral can be estimated both by

\(^{12}\) A similar expression holds when \( n \) is odd.
• \[
\left| \sum_{h=1}^{m} \left\{ \frac{f''(\xi_h^0)}{f'(\xi_h^0)} - \frac{g''(\xi_h^0)}{g'(\xi_h^0)} \right\} \frac{1}{n} - \frac{1}{2} \int_0^1 \frac{f''(t)}{f'(t)} - \frac{g''(t)}{g'(t)} \, dt \right| < \frac{\epsilon}{2},
\]
where \( \xi_h^0 \in \left[ \frac{2(h-1)}{n}, \frac{2h}{n} \right] \), with \( h = 1, \ldots, m \), and

• \[
\left| \sum_{h=1}^{m-1} \left\{ \frac{f''(\xi_h^0)}{f'(\xi_h^0)} - \frac{g''(\xi_h^0)}{g'(\xi_h^0)} \right\} \frac{1}{n} - \frac{1}{2} \int_0^1 \frac{f''(t)}{f'(t)} - \frac{g''(t)}{g'(t)} \, dt \right| < \frac{\epsilon}{2},
\]
where \( \xi_h^0 \in \left[ \frac{2h-1}{n}, \frac{2h+1}{n} \right] \), with \( h = 1, \ldots, m \).

Then, to obtain the thesis, it suffices to see how to estimate the following quantity,

\[
\left| \frac{\Delta f_{2h}}{\Delta f_{2h+1}} - \frac{\Delta g_{2h}}{\Delta g_{2h+1}} \left\{ \frac{g''(\xi_h^0)}{g'(\xi_h^0)} - \frac{f''(\xi_h^0)}{f'(\xi_h^0)} \right\} \frac{1}{n} \right| =
\]

\[
\left| \frac{\Delta g_{2h}}{\Delta g_{2h+1}} \left( \frac{f''(\xi_h^0)}{f'(\xi_h^0)} - \frac{1}{n} \right) + \left( 1 - \frac{\Delta g_{2h}}{\Delta g_{2h+1}} \frac{g''(\xi_h^0)}{g'(\xi_h^0)} \right) \frac{1}{n} \right|,
\]
for each \( h = 1, \ldots, m \). In particular, we can observe that, assuming \( \delta = \frac{1}{n} \), then

\[
1 - \frac{\Delta g_{2h}}{\Delta g_{2h+1}} = 1 - \frac{g(t_{2h}) - g(t_{2h} - \delta)}{g(t_{2h} + \delta) - g(t_{2h})} = \frac{g(t_{2h} + \delta) - 2g(t_{2h}) + g(t_{2h} - \delta)}{g(t_{2h} + \delta) - g(t_{2h})} =
\]

\[
= \frac{g(t_{2h} + \delta) - 2g(t_{2h}) + g(t_{2h} - \delta)}{\delta} \frac{\delta}{g(t_{2h} + \delta) - g(t_{2h})},
\]

By hypothesis, the function \( g \) is twice differentiable and \( g' \neq 0 \), thus we have that, for every \( t \in [0, 1] \)

\[
\lim_{\delta \to 0} \frac{\frac{g(t+\delta)-2g(t)+g(t-\delta)}{\delta^2}}{\frac{g(t+\delta)-g(t)}{\delta}} = \frac{g''(t)}{g'(t)},
\]
as the limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero. So, we have that, for every \( \epsilon > 0 \) there exists \( \delta_\epsilon \), such that if \( \delta < \delta_\epsilon \), then

\[
\left| 1 - \frac{\Delta g_{2h}}{\Delta g_{2h+1}} \right| \frac{g''(\xi_h^0)}{g'(\xi_h^0)} < \frac{\epsilon}{2} \delta,
\]
and this prove the thesis, provided \( \delta < \min\{\delta_\epsilon, \frac{1}{2N} \} \). □

The foregoing proof can also be used to prove other results, such as the following one.

\(^{13}\) A similar argument can be applied for the function \( g \).
Corollary 3.1. Given a plane gesture $\vec{g}$, then\footnote{We adopt the same notations of Theorem 3.1}

$$
\lim_{n \to \infty} \sum_{k=1}^{n-1} \frac{\Delta g_k}{\Delta g_{k+1}} = 2 - \int_0^1 \frac{g''(t)}{g'(t)} \, dt \in \mathbb{C}.
$$

A more general proof of the foregoing result will be given with Theorem 3.2. Corollary 3.1 makes then reasonable to give the following definition.

Definition 3.4. The Complex Shape of a plane gesture $\vec{g}(t) = (x(t), y(t)) \in \mathbb{R}^2$, $x, y \in C^2([0,1]; \mathbb{R})$, is the following complex valued function

$$
1 - \frac{g''(t)}{2g'(t)},
$$

where $g(t) = x(t) + iy(t)$ and $t \in [0,1]$.

Remark. We decided to scale (3) in half so that the complex shape of a rectilinear gesture would be 1, regardless of whether it is considered as “basic” or not. Indeed, (3) is the double of the complex shape simply because of a kind of double counting of intervals in relation (2).

Example. The complex shape of the circled plane gesture

$$
\vec{g}(t) = \left( x_0 + r \cos (2\pi (t - \phi)), y_0 + r \sin (2\pi (t - \phi)) \right),
$$

is the constant value $1 - \pi i$. Notice that it is independent from the radius $r \in \mathbb{R}^+$, the center $(x_0, y_0) \in \mathbb{R}^2$, and the phase $\phi \in \mathbb{R}$, thanks to the invariant properties of the complex shape of a basic gesture.

3.2.3 The case of non-uniformly spaced timestamps

The !FTL algorithm is suited for uniform n-samplings, that is, when $t_k - t_{k-1}$ is independent of index $k$. However, most of sampling devices are multitasking; this implies that the Central Processing Unit is not always sampling points; so, $t_k - t_{k-1}$ may depend on $k$. In this situation, it is reasonable to explore the weighted complex shape

$$
\frac{t_{k+1} - t_k}{t_k - t_{k-1}} \frac{\Delta g_k}{\Delta g_{k+1}} \in \mathbb{C},
$$

of a basic gesture

$$
(\Delta \vec{g}_k, \Delta \vec{g}_{k+1}) = (\vec{g}(t_k) - \vec{g}(t_{k-1}), \vec{g}(t_{k+1}) - \vec{g}(t_k)),
$$
taken from the $n$-sample of a plane gesture $\vec{g}$ (without assuming that the timestamps are uniformly spaced). Of course, such weighted complex shape coincides with the complex shape when $t_{k+1} - t_k = t_k - t_{k-1}$. Moreover, we will show that, such weighted complex shape is still convergent to the same value of $\vec{g}$. 

**Lemma 3.1.** Given a plane gesture $\vec{g}(t)$ then, for each $t \in (0,1)$, we have that

$$
\lim_{{\tau_0 \to t \ , \ \tau_1 \to t \ , \ \tau_2 \to t \atop \tau_0 \neq \tau_1 \ , \ \tau_1 \neq \tau_2 \ , \ \tau_2 \neq \tau_0}} \left(1 - \frac{\tau_2 - \tau_1}{\tau_1 - \tau_0} \frac{g(\tau_1) - g(\tau_0)}{g(\tau_2) - g(\tau_1)} \right) \frac{1}{\tau_2 - \tau_0} = \frac{1}{2} \frac{g''(t)}{g'(t)}
$$

A proof. After rewriting

$$
\left(1 - \frac{\tau_2 - \tau_1}{\tau_1 - \tau_0} \frac{g(\tau_1) - g(\tau_0)}{g(\tau_2) - g(\tau_1)} \right) \frac{1}{\tau_2 - \tau_0} = \frac{\frac{g(\tau_2) - g(\tau_1)}{\tau_2 - \tau_1} - \frac{g(\tau_1) - g(\tau_0)}{\tau_1 - \tau_0}}{\tau_2 - \tau_0} = \frac{1}{\tau_2 - \tau_0} \left(\frac{g(\tau_2) - g(\tau_1)}{\tau_2 - \tau_1} - \frac{g(\tau_1) - g(\tau_0)}{\tau_1 - \tau_0}\right) \tag{4}
$$

we notice that

$$
\frac{g(\tau_2) - g(\tau_1)}{\tau_2 - \tau_1} - \frac{g(\tau_1) - g(\tau_0)}{\tau_1 - \tau_0} \tag{5}
$$

is the second divided difference\textsuperscript{15} of the complex valued function $g$ at points $\tau_0$, $\tau_1$, and $\tau_2$. Being the function twice continuously differentiable, it suffices to apply the Mean Value Theorem for divided differences\textsuperscript{16} to real and imaginary parts of $g$, to obtain that

$$
\lim_{{\tau_0 \to t \ , \ \tau_1 \to t \ , \ \tau_2 \to t \atop \tau_0 < \tau_1 < \tau_2}} \frac{\frac{g(\tau_2) - g(\tau_1)}{\tau_2 - \tau_1} - \frac{g(\tau_1) - g(\tau_0)}{\tau_1 - \tau_0}}{\tau_2 - \tau_0} = \frac{g''(t)}{2}.
$$

Notice that we can always assume condition $\tau_0 < \tau_1 < \tau_2$; as a matter of fact, the second divided difference \textsuperscript{[5]} is symmetric with respect points $\tau_0$, $\tau_1$, and $\tau_2$. As the limit of quotient \textsuperscript{[4]} is the quotient of the limits, provided the limit of the denominator is not zero, one obtains the thesis. \hfill \Box

**Theorem 3.2.** Given a plane gesture $\vec{g}(t) = (x(t), y(t)) \in \mathbb{R}^2$, then

$$
\lim_{{\delta \to 0^+}} \sum_{k=1}^{n-1} \frac{t_{k+1} - t_k}{t_k - t_{k-1}} \frac{\Delta g_k}{\Delta g_{k+1}} = 2 - \int_0^1 \frac{g''(t)}{g'(t)} \, dt \in \mathbb{C},
$$

where $0 = t_0 < \cdots < t_{k-1} < t_k < \cdots < t_n = 1$, and $\delta = \max_{{1 \leq k \leq n}} \{t_k - t_{k-1}\}$.

\textsuperscript{15}See \textsuperscript{[3]} at page 123.

\textsuperscript{16}See Theorem 2.10 in \textsuperscript{[10]}, at page 60.
A proof. By hypothesis, the complex valued Riemann integral \( \int_0^1 \frac{g''(t)}{g'(t)} \, dt \) exists; this implies that for every \( \epsilon > 0 \) there exists \( \delta_\epsilon > 0 \) such that
\[
\left| \int_0^1 \frac{g''(t)}{g'(t)} \, dt - \sum_{k=1}^n \frac{g''(\xi_k)}{g'(\xi_k)} (t_k - t_{k-1}) \right|_C < \epsilon ,
\]
provided the partition
\[
0 = t_0 < \cdots < t_{k-1} < t_k < \cdots < t_n = 1
\]
is such that \( t_k - t_{k-1} < \delta_\epsilon \), and \( \xi_k \in [t_{k-1}, t_k] \) for each \( k = 1, \ldots, n \).
Notice that, to evaluate each shape \( \frac{\Delta g_k}{\Delta g_{k+1}} \), the extremities of two adjacent intervals are needed. This implies that
\[
\sum_{k=1}^{2m-1} \frac{\Delta g_k}{\Delta g_{k+1}} = \sum_{h=1}^{m} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} + \sum_{h=1}^{m-1} \frac{\Delta g_{2h+1}}{\Delta g_{2h+2}} ,
\]
when \( n \) is even\(^{17} \). Thus, to estimate the difference between complex shapes and Riemann sums, we need to consider the latter on couples of adjacent intervals; one with even-indexed extremities, the other with odd-indexed extremities. In order to simplify notations, we will consider in the following only partitions of \([0, 1]\) having an even number of points \( (n = 2m) \), that is
\[
0 = t_0 < \cdots < t_{k-1} < t_k < \cdots < t_{2m} = 1 .
\]
However, our arguments can be applied similarly to partitions of \([0, 1]\) having an odd number of points. If partition (7) is such that
\[
\max \left\{ \max_{1 \leq h \leq m} (t_{2h} - t_{2(h-1)}) , \ max_{1 \leq h \leq m} (t_{2h+1} - t_{2h-1}) \right\} < \delta_\epsilon ,
\]
then we can estimate the Riemann sum both
\begin{itemize}
  \item on “even indexed” intervals
  \[
  \left| \int_0^1 \frac{g''(t)}{g'(t)} \, dt - \sum_{h=1}^m \frac{\Delta g''(\xi_h)}{\Delta g'_{(2h-1)}} (t_{2h} - t_{2(h-1)}) \right|_C < \epsilon ,
  \]
  whatever are \( \xi_{h} \in [t_{2(h-1)}, t_{2h}] \) when \( h = 1, \ldots, m \), and
\end{itemize}
\(^{17}\)A similar expression old when \( n \) is odd.
Now, let us focus on the first term of the right expression in (6). In order to get the thesis, we need to estimate each term

\[
\int_0^1 \frac{g''(t)}{g'(t)} dt - \frac{g''(\xi^0_0)}{g'(\xi^0_0)} (t_1 - t_0) - \frac{g''(\xi^0_m)}{g'(\xi^0_m)} (t_2m - t_{2m-1}) - \sum_{h=1}^{m-1} \frac{g''(\xi^h_h)}{g'(\xi^h_h)} (t_{2h+1} - t_{2h-1}) < \epsilon ,
\]

whatever are \(\xi^0_h \in [t_{2h-1}, t_{2h+1}]\), with \(h = 1, \ldots, m - 1\), \(\xi^0_0 \in [t_0, t_1]\), and \(\xi^0_m \in [t_{2m-1}, t_{2m}]\).

Now, let us focus on the first term of the right expression in (6). In order to get the thesis, we need to estimate each term

\[
\frac{t_{2h} - t_{2h-1}}{t_{2h-1} - t_{2(h-1)}} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} - (t_{2h} - t_{2(h-1)}) \left(1 + \frac{1}{2} \frac{g''(\xi^h_h)}{g'(\xi^h_h)} (t_{2h} - t_{2(h-1)}) \right) = \left(1 + \frac{1}{2} \frac{g''(\xi^h_h)}{g'(\xi^h_h)} \right) (t_{2h} - t_{2(h-1)}) .
\]

If one considers Lemma 3.3 with \(\tau_0 = t_{2(h-1)}\), \(\tau_1 = t_{2h-1}\), and \(\tau_2 = t_{2h}\), we have the estimate

\[
\left| \frac{t_{2h} - t_{2h-1}}{t_{2h-1} - t_{2(h-1)}} \frac{\Delta g_{2h-1}}{\Delta g_{2h}} - 1 - \frac{1}{2} \frac{g''(\xi^h_h)}{g'(\xi^h_h)} \right| < \epsilon ,
\]

which is independent from index \(h\), thanks to the uniform continuity of \(\frac{g''}{g'}\).

By applying the same lemma for the odd terms involving \(\xi^h_h\), the thesis follows. □

**Remark.** The foregoing theorem provide a new algorithm (let us call it \(\text{WFTL}^{18}\)), to measure the dissimilarity between the not-necessarily uniformly sampled isochronous \(n\)-samples, of two plane gestures \(\vec{f}\) and \(\vec{g}\)

\[
\text{WFTL}(\vec{f}_0, \ldots, \vec{f}_n, \vec{g}_0, \ldots, \vec{g}_n) = \sum_{k=1}^{n-1} \frac{t_{k+1} - t_k}{t_{k+1} - t_k} \text{LSD}((\Delta \vec{f}_k, \Delta \vec{f}_{k+1}), (\Delta \vec{g}_k, \Delta \vec{g}_{k+1}))
\]

\[
= \sum_{k=1}^{n-1} \frac{t_{k+1} - t_k}{t_{k+1} - t_k} \left| \frac{\Delta \vec{f}_k}{\Delta \vec{f}_{k+1}} - \frac{\Delta \vec{g}_k}{\Delta \vec{g}_{k+1}} \right| .
\]

We can then claim the following result.

**Corollary.** Given two plane gestures \(\vec{f}(t) = (r(t), s(t)) \in \mathbb{R}^2\), and \(\vec{g}(t) = (x(t), y(t)) \in \mathbb{R}^2\), then

\[
\lim_{\delta \to 0^+} \text{WFTL}(\vec{f}_0, \ldots, \vec{f}_n, \vec{g}_0, \ldots, \vec{g}_n) = \int_0^1 \left| \frac{f''(t)}{f'(t)} - \frac{g''(t)}{g'(t)} \right| dt
\]

where

\[\text{LSD}((\Delta f_k, \Delta f_{k+1}), (\Delta g_k, \Delta g_{k+1}))\]

\[\text{The letter “W” stands for “weighted”.}\]
\[ \vec{f}_0, \ldots, \vec{f}_n, \vec{g}_0, \ldots, \vec{g}_n \text{ are isochronous } n\text{-samples of } \vec{f} \text{ and } \vec{g}, \text{ respectively,} \]

\[ 0 = t_0 < \cdots < t_{k-1} < t_k < \cdots < t_n = 1, \quad \delta = \max_{1 \leq k \leq n} \{t_k - t_{k-1}\}, \]

\[ f(t) = r(t) + is(t), \text{ and } g(t) = x(t) + iy(t). \]

4 The Clifford number point of view

As we have seen, in order to define the complex shape of a basic gesture, we had to consider the components of a vector \( \vec{v} = (x, y) \in \mathbb{R}^2 \) as the real and imaginary parts of the complex number \( v = x + iy \). This twisted construction allows us to use the quotient between complex numbers to encode the concept of “shape” as a complex number. However, it is possible to reach the notion of shape of a basic gesture directly from the Euclidean vector space. As a matter of fact a Euclidean vector space is a particular Quadratic space; that is a vector space with a non-degenerate symmetric bilinear form\(^\text{19} \). To each Quadratic space it is associated a unique Clifford vector algebra that we denote by the symbol \( \mathcal{C}_\ell(p, q) \), where \( (p, q) \in \mathbb{N}^2 \) is the signature of the non-degenerate quadratic form associated to the symmetric bilinear form. Here, we are just interested in the Euclidean plane case: \( p = 2 \) and \( q = 0 \). The positive definite symmetric bilinear form, between vectors \( \vec{u} \) and \( \vec{v} \) in the two dimensional Euclidean vector space \( \mathbb{E}_2 \), is denoted by a dot: \( \vec{u} \cdot \vec{v} \). Let \( \{\vec{e}_1, \vec{e}_2\} \) be a fixed orthonormal basis for \( \mathbb{E}_2 \). Then, every element \( X \in \mathcal{C}_\ell(2, 0) \) can be uniquely expressed as

\[ X = \alpha + x_1 \vec{e}_1 + y \vec{e}_2 + \beta I, \]

where \( \alpha, x_1, x_2, \beta \) are real numbers, and \( I = \vec{e}_1 \vec{e}_2 \). With such representation, the Euclidean space \( \mathbb{E}_2 \) is a vector subspace of \( \mathcal{C}_\ell(2, 0) \), and the field of scalars \( \mathbb{R} \) is a subalgebra of \( \mathcal{C}_\ell(2, 0) \). The associative and distributive (but not necessarily commutative) product in \( \mathcal{C}_\ell(2, 0) \) is uniquely generated by the following simple rule:

\[ \vec{v} \vec{u} = \vec{u} \cdot \vec{v} \]

for every \( \vec{v} \in \mathbb{E}_2 \subset \mathcal{C}_\ell(2, 0) \).

The foregoing rule has several important consequences. In particular,

\[ \frac{1}{2}(\vec{u} \vec{v} + \vec{v} \vec{u}) = \vec{u} \cdot \vec{v}, \text{ for each } \vec{u}, \vec{v} \in \mathbb{E}_2; \]

\(^{19}\text{Which, moreover, is positive definite.}\)
\[ (\vec{e}_1)^2 = 1 = (\vec{e}_2)^2, \text{ and } \vec{e}_1\vec{e}_2 = -\vec{e}_2\vec{e}_1, \quad I^2 = -1; \]

\[ \vec{e}_1I = \vec{e}_1\vec{e}_2 = \vec{e}_2 \in \mathbb{E}_2 \text{ and, similarly, } \vec{e}_2I = -\vec{e}_1 \in \mathbb{E}_2; \]

- if we define \( \vec{u} \wedge \vec{v} = \frac{1}{2}(\vec{u}\vec{v} - \vec{v}\vec{u}) \), then we have that
\[
- \vec{u} \wedge \vec{v} = \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} I, \text{ where } \vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2, \quad \vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2;
- \vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}. \]

Then, the associative and distributive product \( U \, V \) between the Clifford numbers \( U = u_0 + \vec{u} + u_3I \) and \( V = v_0 + \vec{v} + v_3I \) in \( Cl(2, 0) \), can be written as follows

\[
u_0v_0 + \vec{u} \cdot \vec{v} - u_3v_3 + \\
u_0\vec{v} + v_0\vec{u} + v_3\vec{u}I + u_3I\vec{v} + \\
u_0\vec{v} + v_0\vec{u} + u_3v_3I + u_3v_3I, \quad (8)
\]

\[
u_0\vec{v} + v_0\vec{u} + v_3\vec{u}I + u_3I\vec{v}, \quad (9)
\]

\[
u_0\vec{v} + v_0\vec{u} + u_3v_3I + u_3v_3I, \quad (10)
\]

where \((8) \in \mathbb{E}, (9) \in \mathbb{E}_2, \text{ and } (10) \text{ is scalar multiple of } I. \quad \text{Moreover,}
\]

\[ U \cdot V = u_0v_0 + \vec{u} \cdot \vec{v} + u_3v_3 \]

defines a positive definite bilinear form on \( Cl(2, 0) \), whose Euclidean norm \(^{20}\)

\[ |U|_{Cl(2, 0)} = \sqrt{(u_0)^2 + (|\vec{u}|_{\mathbb{E}_2})^2 + (u_3)^2}. \]

This apparently messy situation hide an algebraic structure that is richer than that of complex numbers, and can encode many different geometric notion of the Euclidean plane within a single algebraic frame. Here, we want to point out just few properties:

- every non-zero vector \( \vec{v} \in \mathbb{E}_2 \) is invertible in \( Cl(2, 0) \) and \((\vec{v})^{-1} = \frac{1}{|\vec{v}|_{\mathbb{E}_2}^2} \vec{v}\)

- if \( \vec{u}, \vec{v} \in \mathbb{E}_2 \), and \( \vec{v} \neq 0 \), then \( \vec{u}(\vec{v})^{-1} = \vec{u} / \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|_{\mathbb{E}_2}^2} + \frac{1}{|\vec{v}|_{\mathbb{E}_2}^2} \vec{u} \wedge \vec{v}. \)

Since \( i \in \mathbb{C} \) has the same algebraic properties of \( -I \in Cl(2, 0) \), we can consider the shape of a basic gesture \((\vec{u}, \vec{v})\) “directly” as the quotient, in the Clifford algebra \( Cl(2, 0) \), of the two vectors; as a matter of fact

\[
\mathbb{C} \ni \frac{u}{v} = \frac{rx + sy}{x^2 + y^2} - i \frac{ry - sx}{x^2 + y^2} \leftrightarrow \frac{rx + sy}{x^2 + y^2} + i \frac{ry - sx}{x^2 + y^2} = \vec{u} / \vec{v} \in Cl(2, 0),
\]

where \( u = r + is, v = x + iy, \vec{u} = re_1 + se_2, \) and \( \vec{v} = xe_1 + ye_2 \).

\(^{20}\) Note that, if \( U \in \mathbb{R}, \) then \( |U|_{Cl(2, 0)} = |U|_{\mathbb{R}}, \) and if \( U \in \mathbb{E}_2, \) then \( |U|_{Cl(2, 0)} = |U|_{\mathbb{E}_2}. \)
Definition 4.1. The shape of a basic gesture $(\vec{v}_1, \vec{v}_2)$ is the Clifford number

$$\vec{v}_1(\vec{v}_2)^{-1} = \vec{v}_1/\vec{v}_2 \in \mathbb{C}_\ell(2,0).$$

Definition 4.2. The Shape of a plane gesture $\vec{g}(t) \in \mathbb{E}_2$, is the following multivector-valued function

$$1 - \frac{1}{2} \left( \frac{\vec{g}''(t)}{\vec{g}'(t)} \right) \in \mathbb{C}_\ell(2,0).$$

In order to express our previous convergence theorems in terms of Clifford numbers, it suffices to rewrite the Local Shape Distance in terms of scalar products.\footnote{As was done in [11] for the Javascript implementation of the algorithm (see appendix B.1 to [11]).}

Proposition. Given two basic gestures $(\vec{u}_1, \vec{u}_2)$ and $(\vec{v}_1, \vec{v}_2)$, then

$$LSD((\vec{u}_1, \vec{u}_2), (\vec{v}_1, \vec{v}_2)) = \left| \vec{u}_1/\vec{u}_2 - \vec{v}_1/\vec{v}_2 \right|_{\mathbb{C}_\ell(2,0)} = \sqrt{\frac{|\vec{u}_1|^2|\vec{v}_2|^2 + |\vec{u}_2|^2|\vec{v}_1|^2 - 2 \left( \vec{u}_1 \cdot \vec{u}_2 \right) \left( \vec{v}_1 \cdot \vec{v}_2 \right) - \left( \vec{u}_1 \cdot \vec{v}_2 \right) \left( \vec{u}_2 \cdot \vec{v}_1 \right) + \left( \vec{u}_1 \cdot \vec{v}_1 \right) \left( \vec{u}_2 \cdot \vec{v}_2 \right)}}{\left| \vec{u}_2 \right|^2|\vec{v}_2|^2},$$

where $|\cdot| = |\cdot|_{\mathbb{E}_2}$.

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