An Operator model for connective $K$-theory with reality

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Abstract

We construct an explicit model for the connective cover of the spectrum of $K$-theory with reality. This model is a $\mathbb{Z}/2\mathbb{Z}$-equivariant commutative symmetric ring spectrum which identifies with another one due to D. Dugger.

1 Models for $K$-theory with reality

1.1 Periodic $K$-theory with reality

$K$-theory is a generalized cohomology theory that comes in various flavours; the most famous ones are certainly the real $K$-theory $KO$ and the complex $K$-theory $KU$. Many explicit models for a spectrum representing these cohomology theories are known ([AS, HST]). Both $KU$ and $KO$ are contained in a $\mathbb{Z}/2\mathbb{Z}$-equivariant cohomology theory called $K$-theory with reality (denoted by $KR$) discovered by Atiyah [At]. For instance, Bott periodicity for $KR$ (the 1-1 periodicity theorem) implies at once by a clever argument both real and complex Bott periodicity theorems.

The following three papers provide the basis for this exposition:

- [HST, Theorem 3], where a nice operator model for the periodic real $K$-theory spectrum is defined, and models for the spectra of the complex $K$-theory and of $K$-theory with reality are suggested,
- [Mar] where such a model for the real connective real $K$-theory spectrum $ko$ is given,
- [Du] where a model for connective $K$-theory with reality is constructed by an abstract localization method.

Here, a nice operator model for $KO$ refers to a model that forms a commutative symmetric ring spectrum, whose product actually lifts the natural product on $K$-groups. This was first observed by M. Joachim [Jo1, Jo2], while another approach appears in [Mit].

Our aim in this work is to provide an operator model for the spectrum $KR$ as outlined in [HST] and for the corresponding connective version $kr$ following the method in [Mar]. We show that this model for $kr$ coincides with the one constructed in [Du]. Finally, we know from [BR1, BR2] that highly structured ring structures for $KU$, $KO$ and their connective covers exist and are essentially unique. We give here an explicit model for such a cover in the case of $KR$. So our model for $KR$ provides also in particular explicit such coverings for $KO$ and $KU$. In view of these strong uniqueness properties and the universality of Dugger’s construction, it is somehow surprising to find such an explicit model with the required properties, i.e., it is a connective cover as a ring spectrum and it has the ‘right’ equivariant homotopy type.

This altogether suggests that the operator setting emphasized in [HST, Mar] is particularly suited to provide explicit models for $K$-theory ‘in the large’, for instance in the case of $G$-equivariant connective...
K-theory [Cree] We consider the case of kr as a non trivial test case for constructing general connective models for topological K-theory spaces, which will be the subject of future work by the authors.

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1.2 Conventions and notations

We will use the notation (possibly decorated by a subscript) \( H \) to denote a separable, infinite-dimensional \( \mathbb{Z}/2\mathbb{Z} \)-graded real Hilbert space, and use the notation \( L \) for a complex version.

For positive integers \( p, q \), we denote by \( Cl_{p,q} \) the Clifford-algebra on \( \mathbb{R}^{p+q} = \mathbb{R}^p \oplus \mathbb{R}^q \) with the positive definite standard quadratic form on \( \mathbb{R}^p \) and the negative definite standard form on \( \mathbb{R}^q \). That is, \( Cl_{p,q} \) is the quotient of the free associative \( \mathbb{R} \)-algebra on generators \( e_i, f_j, i = 1, \ldots, p, j = 1, \ldots, q \) by the two-sided ideal generated by the relations \( e_i^2 = 1, f_j^2 = 1 \) and the relations given by the fact that any two different generators anti-commute. We shorten notation by using \( Cl_n := Cl_{n,0} \) and \( Cl_{-n} := Cl_{0,n} \) for positive integers \( n \). The complex Clifford algebras \( Cl_{p,q} \) are defined analogously, on \( \mathbb{C}^p \oplus \mathbb{C}^q \). The algebras \( Cl_{p,q} \) and \( Cl_{p,q} \otimes \mathbb{C} \) are canonically isomorphic.

We use the same short notations in the complex case.

The Clifford algebras are \( \mathbb{Z}/2\mathbb{Z} \)-graded with generators in odd degree, the grading involution will be denoted by \( \epsilon \). Note that there is a correspondence between \( \mathbb{Z}/2\mathbb{Z} \)-graded Clifford structures and ungraded ones via an index shift by one (the even part of \( \mathbb{C} \) in the graded setting.

Furthermore, there is a grading-preserving involution \( \epsilon \) on \( Cl_{p,q} \), sending the \((f_j)_{i,j=0,\ldots,q}\) to their negatives. This involution extends uniquely to an antilinear involution \( \theta \) on \( Cl_{p,q} \). This turns the algebras \( Cl_{p,q} \) and \( Cl_{p,q} \) into Real modules over themselves. A Real module \( M \) for \( Cl_{p,q} \) is a complex representation endowed with an antilinear involution \( \epsilon \), such that the Clifford action of \( x \) on \( m \in M \) satisfies \( c(x,m) = \theta(x)c(m) \). One can extend a Real module for \( Cl_{p,q} \) uniquely to a Real module for \( Cl_{p,q} \). All three notions of \( Cl_{p,q} \)-module, Real module for \( Cl_{p,q} \) and Real module for \( Cl_{p,q} \) produce isomorphic module categories. For our \( K \)-theory models, we will use the notion of Real module for \( Cl_{p,q} \).

We refer the reader to [LM] for the general theory of Clifford algebras and their modules.

However one aspect we want to emphasize in advance is the isomorphism of module categories between left \( Cl_n \)-modules and right \( Cl_{-n} \)-modules for all \( n \in \mathbb{Z} \).

We define \( \mathcal{R}_{p,q} := (H \otimes Cl_1) \otimes \mathbb{C} \otimes (H \otimes Cl_{-1}) \otimes \mathbb{C} \cong L_p \otimes \mathbb{C} L_{-q} \cong L \otimes \mathbb{C} L \otimes \mathbb{C} Cl_{p,q} \) where \( \otimes \) denotes the graded tensor product of graded Hilbert spaces (also, suitably completed).

The involution \( \epsilon \) on \( Cl_{p,q} \) extends uniquely to an antilinear involution \( \theta \) on \( \mathcal{R}_{p,q} \). This involution turns \( H_{p,q} \) into a Real module for \( Cl_{p,q} \).

All future mentions about an action of \( \mathbb{Z}/2\mathbb{Z} \) in the following will be correspond to the action induced by the involution \( \epsilon \).

The generators of \( Cl_{p,q} \) act on \( H_{p,q} \) by orthogonal transformations. Moreover \( H_{p,q} \) is a universe for the Clifford algebra \( Cl_{p,q} \) in the sense that it contains each irreducible Real module infinitely many times. Note that any two such universes are non-canonically isomorphic. This does not matter in the homotopy category, because the automorphisms of such a universe form a contractible space, by the natural generalization of a theorem of Kuiper [Kni].

We will need to work with a Real version of these Clifford universes and we will also extend the concept to bi-modules. For instance the Real universe \( \mathcal{R}_{p+k,q+1} \) for \( Cl_{p+k,q+1} \) is also a Real universe for the smaller Clifford algebras \( Cl_{p,q} \). It will be important to keep track of two different Clifford actions by separating them, that is by letting the algebra \( Cl_{p,q} \) act on \( \mathcal{R}_{p+k,q+1} \) from the right as usual, and
thinking of the action of $\mathbb{C}\ell_{k,l}$ as an action of the opposite algebra $\mathbb{C}\ell_{l,k}$ from the left. That is, the Hilbert universe $R_{p+k,q+l}$ becomes a $\mathbb{C}\ell_{l,k} - \mathbb{C}\ell_{p,q}$-bimodule, which is simultaneously a right $\mathbb{C}\ell_{p,q}$-universe and a left $\mathbb{C}\ell_{l,k}$-universe. To express this in the notation we write $R^{k,l}_{p,q}$.

In the following we will consider $R_{p,q} := R_{q,p}$. This is a universe for $\mathbb{C}\ell_{q,p} - \mathbb{C}\ell_{p,q}$ Real bi-modules.

By convention, we set $R_{0,0} := L \otimes_{\mathbb{C}} L$.

### 1.3 Spaces for $K$-theories

Using the notations of [HST] and [Mar], we say that an operator $G$ on a graded Hilbert space is of type $I\hat{nf}$ if it is self-adjoint on a closed domain $\mathcal{D}(G)$ and has finite rank resolvent. Note that self-adjoint in general means densely defined; however the operators in $I\hat{nf}$ are self-adjoint operators on a finite-dimensional closed domain. If one requires the resolvent to be only compact one obtains a larger class $Inf$ of operators which are self-adjoint on a possibly infinite-dimensional closed domain (where they are densely defined). The spectral theorem implies that both types of operators have a spectral decomposition into pairwise orthogonal, finite-dimensional eigenspaces with real eigenvalues which cannot accumulate at infinity (the base point in the one-point compactification $\mathbb{R}$). Even though the domain is part of the information, we will in general suppress notation of the domains. The topology on the spaces of operators of type $Inf$ is the coarsest topology making functional calculus with functions in $C^\infty(\mathbb{R})$ continuous; here $C^\infty(\mathbb{R})$ denotes functions with compact support. This topology allows us to use the spectral decomposition in an intuitive way; we can think of the operators as configurations of discrete points (eigenvalues) on the real line with coefficients (eigenspaces) given by pairwise orthogonal, finite-dimensional subspaces of the background Hilbert space.

Now configuration points can be moved continuously (by functional calculus), and coefficients can be changed continuously within the space of projection operators onto modules as above. Whenever two configuration points collide, their coefficients add in algebraic sum, whenever points move to $\infty$, they disappear from the configuration. This is explained in detail in [HST] and [Mar], including precise definitions of the configuration spaces and their topology. Under this topology the spaces $Inf$ and $\hat{Inf}$ are homotopy equivalent via the natural inclusion. We will work throughout with the finite-rank version $\hat{Inf}$. The space of $\hat{Inf}$-operators is naturally pointed, with base point the zero domain operator.

Let $\mathcal{R}\hat{Inf}_{p,q}$ denote the space of odd, $\mathbb{C}\ell_{p,q}$-linear operators on $R_{p,q}$ which are of type $\hat{Inf}$. This is a $\mathbb{Z}/2\mathbb{Z}$-space under the action induced by the involution $\theta$, sending an operator $G$ to its conjugate $\theta G \theta$. This operator has the same eigenvalues as $G$; but where $G$ has eigenspace $G_\lambda$ at $\lambda$, the conjugate has eigenspace $\theta(G_\lambda)$. We set $KR_{p,q}(\mathcal{H}) := \mathcal{R}\hat{Inf}_{p,q}$.

Note that here we only use the right Clifford-action on the universe $R_{p,q} = R^{q,p}_{p,q}$.

The chosen Hilbert universe $\mathcal{H}$ fixed in the beginning is suppressed in the notation, since isomorphisms of Hilbert universes induce homeomorphisms on the operator spaces. The following is implicit in [HST]:

**Proposition 1.1.** For $p,q \geq 0$ and a $\mathbb{Z}/2\mathbb{Z}$-CW-complex $X$, there is a natural bijection of pointed sets

$$KR_{p,q}(X) \cong [X, KR_{p,q}]$$

where homotopy classes are through equivariant homotopies.

Notice that our conventions differ slightly from those of [HST]. Firstly, we use right Clifford actions instead of the corresponding left actions of the opposite algebras. Furthermore, our spaces $KR$ are defined as the finite-rank resolvent version $\hat{Inf}$ of the operator spaces,
whereas in [HST] the authors use the compact resolvent version $Inf$ and show that this is homotopy equivalent to the latter (by an $ad$ hoc adaptation of [HST Proposition 29]).

We now define spaces of operators $\mathcal{R}f_{p,q}^{k,l}$ which will be shown to be connective covers of $\mathcal{R}I\hat{n}f_{p,q}$, in the sense that there are equivariant homotopy equivalences

$$\mathcal{R}I\hat{n}f_{p,q} \xrightarrow{\sim} \Omega^{k,l}\mathcal{R}I\hat{n}f_{p,q}^{k,l}$$

where the spaces $\mathcal{R}I\hat{n}f_{p,q}^{k,l}$ are highly connected (Proposition 2.1).

We set:

$$\mathcal{R}I\hat{n}f_{p,q} := \{G \in \mathcal{R}I\hat{n}f_{p,q}(\mathcal{R}f_{p,q}^{k,l}) | G^2 \text{ is } \mathcal{C}\ell_{l,k} - \mathcal{C}\ell_{p,q} \text{-linear} \}.$$ 

We define our spaces for connective K-theories as

$$kr_{p,q}(\mathcal{H}) := \mathcal{R}I\hat{n}f_{p,q}^{k,l,p,q}.$$ 

As before, we will usually suppress $\mathcal{H}$ from the notation. Observe however that the definition still makes sense if we replace $\mathcal{H}$ by a finite dimensional Hilbert space.

This is again a space of operators on the universe $\mathfrak{R}_{p,q}$ for all $p, q \geq 0$, this time the operators are defined using both Clifford actions.

There is an obvious map

$$\varphi_{p,q} : kr_{p,q} \rightarrow KR_{p,q}$$

obtained by observing that $\mathfrak{R}_{p,q}$ besides being a universe for $\mathcal{C}\ell_{p,q} - \mathcal{C}\ell_{p,q}$-bimodules, is also a universe for right $\mathcal{C}\ell_{p,q}$-modules, hence $kr_{p,q}$ is a subspace of $KR_{p,q}$ for all $p, q \geq 0$.

So the spaces of operators on both source and target of $\varphi$ do not live on the same Hilbert space. We leave it to the reader to keep track of this subtlety, as it does not alter the statements.

### 1.4 Spectra for K-theories

#### 1.4.1 Symmetric structure and multiplication

$\mathcal{R}f_{p,q}^{k,l}$ has an action of the group $\Sigma_p \times \Sigma_q \times \Sigma_k \times \Sigma_l$ by permuting the tensor factors. The symmetric group $\Sigma_p \times \Sigma_q$ acts on $\mathfrak{R}_{p,q} = \mathcal{R}f_{p,q}^{q,p}$ diagonally.

If $\varphi$ is in $\mathcal{R}I\hat{n}f_{p,q}^{k,l}$ and $\psi$ is in $\mathcal{R}I\hat{n}f_{p',q'}^{k',l'}$, then the operator

$$\varphi \ast \psi := \varphi \otimes I + I \otimes \psi$$

is in $\mathcal{R}I\hat{n}f_{p+p',q+q'}^{k+k',l+l'}$. This follows from spectral analysis.

Hence we get for all $p, p', q, q' \geq 0$, a commutative diagram of $\Sigma_p \times \Sigma_q \times \Sigma_p' \times \Sigma_q'$-equivariant maps

$$\begin{array}{ccc}
kr_{p,q} \wedge kr_{p',q'} & \xrightarrow{\ast} & kr_{p+p',q+q'} \\
\varphi_{KN} \downarrow & & \varphi_{KN} \\
KR_{p,q} \wedge KR_{p',q'} & \xrightarrow{\ast} & KR_{p+p',q+q'}
\end{array}$$

The maps $\ast$ are $\mathbb{Z}/2\mathbb{Z}$-equivariant in the sense that the diagrams

$$\begin{array}{ccc}
KR_{p,q} \wedge KR_{p',q'} & \xrightarrow{\theta} & KR_{p+p',q+q'} \\
\varphi_{\wedge \theta} \downarrow & & \varphi_{\wedge \theta} \\
KR_{p,q} \wedge KR_{p',q'} & \xrightarrow{\theta} & KR_{p+p',q+q'}
\end{array}$$

$$\begin{array}{ccc}
KR_{p,q} \wedge KR_{p',q'} & \xrightarrow{\theta} & KR_{p+p',q+q'} \\
\varphi_{\wedge \theta} \downarrow & & \varphi_{\wedge \theta} \\
KR_{p,q} \wedge KR_{p',q'} & \xrightarrow{\theta} & KR_{p+p',q+q'}
\end{array}$$

are commutative diagrams of $\Sigma_p \times \Sigma_q \times \Sigma_p' \times \Sigma_q'$-spaces.
1.4.2 Unit maps

We will use the definition of an equivariant symmetric ring spectrum by [Man].

Let $S^{•,•}$ be the $\mathbb{Z}/2\mathbb{Z}$-equivariant commutative symmetric ring spectrum of spheres, where $S^{p,q}$ is the one-point compactification of the $\mathbb{Z}/2\mathbb{Z}$-representation $\mathbb{R}^{p,q}$ which is $p$ times the trivial representation plus $q$ times the sign representation. We define $\mathbb{Z}/2\mathbb{Z}$-equivariant maps

$$\eta_{kr} : S^{•,•} \rightarrow kr_{•,•}, \quad \eta_{KR} : S^{•,•} \rightarrow KR_{•,•}.$$ 

We proceed as follows (compare [Jo1, HST]). Let $v$ be an element in $\mathbb{R}^{p,q}$. We can see $v$ as an element of the Clifford algebra $\mathbb{C}^{\ell,p,q}$, and let it act via Clifford multiplication from the left by $v$ on the Clifford universe $\mathbb{R}_{p,q}^{q,p}$, $p,q$. We denote this operator by $L_v$. Recall here that $\mathbb{R}_{p,q}^{q,p}$ is a $\mathbb{C}^{\ell,p,q}$-bimodule (a $\mathbb{C}^{\ell,p+q,p+q}$-right module, respectively). We are considering $v$ as elements of the algebra acting from the left (or as the corresponding elements in $\mathbb{C}^{\ell,p+q,p+q}$, respectively).

Let $G_0 \in KO_0$ be an operator of graded index 1 (this is an operator of type $Inf$ as described in the corresponding $KO$ and $ko$ models in [HST, Mar]; compare also with [Jo1, section 3, p. 302], where the Fredholm operator models are used for this construction). Then we take $G_0$ to be the image of $G_0$ via the natural map $ko_0 = KO_0 \rightarrow KR_{0,0} = kr_{0,0}$. We could start directly with any $G_{0,0} \in KR_{0,0}$ of graded index 1 in the fixed points, but by choosing $G_0$ coming from $KO_0$, we ensure that the natural map $KO_0 \rightarrow KR_{0,0}$ is a map of symmetric spectra.

We now define a map

$$\mathbb{R}^{p,q} \rightarrow kr_{p,q}, \quad v \mapsto L_v \ast G_{0,p+q}.$$ 

and a map

$$\mathbb{R}^{p,q} \rightarrow KR_{p,q}$$

by the commutative diagram of $\Sigma_p \times \Sigma_q$-equivariant maps:

$$\begin{array}{ccc}
\mathbb{R}^{p,q} & \xrightarrow{\eta_{kr}} & kr_{p,q} \\
\downarrow{\eta_{KR}} & & \downarrow{\varphi_{KR}} \\
kr_{p,q} & \rightarrow & KR_{p,q}
\end{array}$$ (3)

These extend to continuous pointed maps

$$\eta_{kr} : \mathbb{R}^{p,q} = S^{p,q} \rightarrow kr_{p,q}, \quad \eta_{KR} : \mathbb{R}^{p,q} = S^{p,q} \rightarrow KR_{p,q}$$

by sending base points to base points. $\mathbb{R}^{p,q} = S^{p,q}$ denotes the one point compactification of the locally compact space $\mathbb{R}^{p,q}$.

In other words, Diagram (3) extends to a commutative diagram of $\Sigma_p \times \Sigma_q$-equivariant maps

$$\begin{array}{ccc}
S^{p,q} & \xrightarrow{\eta_{kr}} & kr_{p,q} \\
\downarrow{\eta_{KR}} & & \downarrow{\varphi_{KR}} \\
kr_{p,q} & \rightarrow & KR_{p,q}
\end{array}$$ (4)

We draw the reader’s attention on a small simplification allowed by considering operators of $Inf$ type instead of for example Fredholm operators as in [Jo1]: we do not need to extend our space of infinitesimal operators to get the unit.

5
1.4.3 Equivariant ring spectrum structure

**Theorem 1.2.** The symmetric sequences of spaces \( kr = (kr_{n,n})_{n \geq 0} \) and \( KR = (KR_{n,n})_{n \geq 0} \) with the above symmetric action, multiplication and units form commutative \( \mathbb{Z}/2\mathbb{Z} \)-equivariant symmetric ring spectra. KR is a representing spectrum for \( K \)-theory with reality. The map \( \varphi_{KR} \) is an equivariant ring map.

We implicitly assert that the spectrum structure of KR obtained from the symmetric structure is exactly the one that corresponds to Bott periodicity (1-1 periodicity, actually), and that the product structure lifts that in \( K \)-groups. This does not exclude the possibility of the existence of different ring structure at the spectrum level although one expects uniqueness as highly structured equivariant ring spectrum in view of [BR1]. In particular, the adjoints of the structure maps for the symmetric \( \mathbb{Z}/2\mathbb{Z} \)-spectrum structure of KR

\[
b_{0,1} : KR_{p,q} \to KR_{p,q+1}, \quad b_{1,0} : KR_{p,q} \to KR_{p+1,q}, \quad b_{1,1} := b_{1,0}b_{0,1} : KR_{p,q} \to KR_{p+1,q+1}
\]

are equivariant weak homotopy equivalences for all \( p,q \geq 0 \) (that is also after taking \( \mathbb{Z}/2\mathbb{Z} \)-fixed points). Moreover, the diagrams

\[
\begin{align*}
\begin{array}{ccc}
kr_{p,q} & \xrightarrow{\varphi} & KR_{p,q} \\
\Omega^0 \varphi_{kr_{p,q+1}} & \xrightarrow{\Omega^0 \varphi} & \Omega^0 \varphi_{KR_{p,q+1}} \\
\end{array}
\end{align*}
\quad \quad \quad
\begin{align*}
\begin{array}{ccc}
kr_{p,q} & \xrightarrow{\varphi} & KR_{p,q} \\
\Omega^1 \varphi_{kr_{p+1,q}} & \xrightarrow{\Omega^1 \varphi} & \Omega^1 \varphi_{KR_{p+1,q}} \\
\end{array}
\end{align*}
\]

commute for \( p,q \geq 0 \).

The fact that our spectrum represents usual KR-theory follows from a sequence of observations. Beginning with the classical fact that the space \( KR_{0,0} \) is a representing space for Atiyah’s KR functor on Real spaces (that is, spaces with \( \mathbb{Z}/2\mathbb{Z} \)-action), we observe that the spaces \( KR_{p,q} \) are representing spaces for certain bundles with Clifford structure on compact spaces.

From [AS, section 5] we see that the multiplication given by the symmetric spectrum structure gives back the tensor product of bundles. Finally \( 1 - 1 \) periodicity corresponds to tensoring with a generator of \( KR_{1,1} \). This means that the spectrum structure and the multiplication are the right one in the homotopy category. We implicitly assume the natural homotopy equivalence between the spaces of operators of \( \hat{I} \) type and the (ungraded version of) operators of Fredholm type. The way to compare our model with the classical models is thoroughly explained in [HST] and [Mar, Section 3.1].

2 Connective \( K \)-theory with reality

In [Du] the author develops a new proof that the natural spectrum map \( KU^{\mathbb{Z}/2\mathbb{Z}} \to KU^{h\mathbb{Z}/2\mathbb{Z}} \) is a weak equivalence. To this end he introduces a spectrum \( kr \), a connected version of KR, which is well behaved under taking fixed points ([Du, section 6, proof of Proposition 6.1]). This produces the following characterization of \( kr = \{kr_{n,n}\}_{n \geq 0} \).
Proposition 2.1. There is up to levelwise \( \mathbb{Z}/2\mathbb{Z} \)-equivariant weak homotopy equivalence a unique \( \mathbb{Z}/2\mathbb{Z} \)-equivariant spectrum \( \text{kr} = \{\text{kr}_{n,n}, b_{1,1}\}_{n \geq 0} \) together with a \( \mathbb{Z}/2\mathbb{Z} \)-equivariant map of spectra \( \{\varphi_{n,n} : \text{kr}_{n,n} \to \text{KR}_{n,n}\} \) such that:

(i) \( \text{kr}_{n,n} \) is \( (2n - 1) \)-connected, \( \text{kr}_{n,n}^{Z/2Z} \) is \( (n - 1) \)-connected,

(ii) \( \pi_i((\varphi)_{n,n}) : \pi_i(\text{kr}_{n,n}) \to \pi_i(\text{KR}_{n,n}) \) is an isomorphism for \( i \geq 2n \) and \( \pi_i(\text{kr}_{n,n}^{Z/2Z}) \to \pi_i(\text{KR}_{n,n}^{Z/2Z}) \) is an isomorphism for \( i \geq n \).

Here the superscript \( [-]^{Z/2Z} \) means fixed points under the \( \mathbb{Z}/2\mathbb{Z} \)-action.

In fact, any spectrum satisfying the properties listed above has a \( \mathbb{Z}/2\mathbb{Z} \)-equivariant indexing of \( \text{kr} \) by \( \mathbb{Z}/2\mathbb{Z} \) such that:

\[ \Omega^{1,1} \text{kr}_{n,n} \to \text{KR}_{n,n} \]

The crucial technical point is the construction of Proposition 2.1. We now give an overview of the proof.

2.1 Overview of the method

The crucial technical point is the construction of \( \mathbb{Z}/2\mathbb{Z} \)-equivariant weak homotopy equivalences (in Section 3.2):

\[
\beta_{1,0} : \mathcal{R} \mathcal{I} \hat{f}_{p,q}^{k,l} \longrightarrow \Omega^{1,0} \mathcal{R} \mathcal{I} \hat{f}_{p,q}^{k,l+1} \\
\beta_{0,1} : \mathcal{R} \mathcal{I} \hat{f}_{p,q}^{k,l} \longrightarrow \Omega^{0,1} \mathcal{R} \mathcal{I} \hat{f}_{p,q}^{k,l+1}
\]

This means that the maps induce weak homotopy equivalences on fixed points as well. The maps will be produced from certain equivariant quasi-fibrations.

Next we observe that the space \( \mathcal{R} \mathcal{I} \hat{f}_{p,q}^{k,l} \) is connected (Section 3.3) for \( p, q, k, l \geq 0 \) and \( k + l \geq 1 \), as well as its fixed point space \( \{\mathcal{R} \mathcal{I} \hat{f}_{p,q}^{k,l}|_{Z/2Z}\} \) for \( k \geq 1 \). This is done by finding explicit paths to the base point (through fixed points if the starting point was a fixed point). Hence by the above equivalences, the space \( \text{kr}_{p,q} \) is at least \( (p + q - 1) \)-connected. On fixed points, we find that \( \text{kr}_{n,n}^{Z/2Z} \) is \( (n - 1) \)-connected. This settles point (2.1)(i) in the characterization of \( \text{kr} \).

The point (ii) in Proposition 2.1 is slightly more involved, and relies on further properties of the quasi-fibrations. We let

\[
\beta_{1,1} := \beta_{1,0}\beta_{0,1} : \text{kr}_{p,q} = \mathcal{R} \mathcal{I} \hat{f}_{p,q}^{k,l} \longrightarrow \Omega^{1,1} \mathcal{R} \mathcal{I} \hat{f}_{p,q+1}^{k,l+1} = \Omega^{1,1} \text{kr}_{p+1,q+1}
\]

In Section 3.4, we show that the following diagram of \( \mathbb{Z}/2\mathbb{Z} \)-equivariant maps commutes

\[
\begin{array}{ccc}
\text{kr}_{n,n} & \xrightarrow{\beta_{1,1}} & \text{kr}_{n,n} \\
\downarrow{\beta_{1,1}} & & \downarrow{\varphi_{n,n}} \\
\Omega^{1,1}\text{kr}_{n+1,n+1} & \xrightarrow{\beta_{1,1}} & \Omega^{1,1}\text{KR}_{n+1,n+1}
\end{array}
\]
Under forgetting the $\mathbb{Z}/2\mathbb{Z}$-action, we get inductively, beginning with $kr_{0,0} = KR_{0,0}$, that $\pi_k(\varphi_{n,n}) : \pi_k(kr_{n,n}) \to \pi_k(KR_{n,n})$ is an isomorphism for $k \geq 2n$. Indeed, the commutation of Diagram (6) together with the fact that $\beta_{1,1}$ and $b_{1,1}$ are $\mathbb{Z}/2\mathbb{Z}$-equivariant weak equivalences, yields the following. If $\pi_k(\varphi_{n,n}) : \pi_k(kr_{n,n}) \to \pi_k(KR_{n,n})$ for $k \geq 2n$, as $\pi_k(\Omega^{1,1} \varphi_{n+1,n+1})$ and $\pi_k(\varphi_{n,n})$ are isomorphisms in the same dimensions $k$, and as $\Omega^2 = \Omega^{1,1}$ by forgetting the action, we get that

$$\pi_k(\varphi) : \pi_k(kr_{n+1,n+1}) \to \pi_k(KR_{n+1,n+1})$$

is an isomorphism for $k \geq 2n + 2$.

Taking fixed points in Diagram (6) yields by a similar but slightly more complicated argument that $\pi_k([\varphi]^{\mathbb{Z}/2\mathbb{Z}}) : \pi_k([kr_{n,n}]^{\mathbb{Z}/2\mathbb{Z}}) \to \pi_k([KR_{n,n}]^{\mathbb{Z}/2\mathbb{Z}})$ is an equivalence in degrees $k \geq n$, which is the last statement to be proved in the characterization. The details of this argument are deferred to Section 3.4.

### 3 Proofs

#### 3.1 CW-structures

It is an important if somewhat technical point to check that our spaces have the homotopy type of CW-complexes with a nice filtration by subcomplexes.

This property is needed later on to allow us to use the Dold-Thom criterion for quasi-fibrations. It was shown in [Mar] in the case of the spaces for connective real K-theory $ko$ building on a similar proof in [HST section 7]. The approach works under minor modifications. We give a short outline here, in order to enable the reader to trace the differences in the $\mathbb{Z}/2\mathbb{Z}$-equivariant context. The filtration is constructed as follows: let

$$\ldots \subset \mathcal{H}(i) \subset \mathcal{H}(i+1) \subset \ldots \subset \mathcal{H}$$

for $i \geq 0$ be a filtration of $\mathcal{H}$ by finite dimensional subspaces of increasing dimension. This induces a filtration

$$\ldots \subset \mathcal{R}^{k,l}_{p,q}(i) \subset \mathcal{R}^{k,l}_{p,q}(i+1) \subset \ldots \subset \mathcal{R}^{k,l}_{p,q}$$

by graded free $\mathcal{R}_{q,p} - \mathcal{R}_{k,l}$-submodules, for any $p,q,k,l \geq 0$. In the following, this indexing does not change and so we omit the indices of the Clifford-structures. We obtain a filtration

$$\ldots \subset R \hat{I} \hat{n} f(i) \subset R \hat{I} \hat{n} f(i+1) \subset \ldots \subset R \hat{I} \hat{n} f$$

where $R \hat{I} \hat{n} f(i) := R \hat{I} \hat{n} f(\mathcal{R}(i))$. Recall that the definition of the $R \hat{I} \hat{n} f$ spaces also makes sense in the context of a finite dimensional Hilbert universe. Observe first that the natural map

$$\text{colim}_i ([R \hat{I} \hat{n} f(i)]^{\mathbb{Z}/2\mathbb{Z}}) \to [\text{colim}_i (R \hat{I} \hat{n} f(i))]^{\mathbb{Z}/2\mathbb{Z}}$$

is a homeomorphism, because the colimit is taken over monomorphisms.

**Proposition 3.1.** The spaces $R \hat{I} \hat{n} f(i)$ as well as their fixed point spaces $[R \hat{I} \hat{n} f(i)]^{\mathbb{Z}/2\mathbb{Z}}$ have the homotopy type of CW complexes. The natural inclusions

$$\text{colim}_i (R \hat{I} \hat{n} f(i)) \to R \hat{I} \hat{n} f, \quad \text{colim}_i ([R \hat{I} \hat{n} f(i)]^{\mathbb{Z}/2\mathbb{Z}}) \to [R \hat{I} \hat{n} f]^{\mathbb{Z}/2\mathbb{Z}}$$

are homotopy equivalences.
In particular, \( \text{colim}_i \mathcal{R} \hat{I} n f(i) \simeq \mathcal{R} \hat{I} n f \) and \( \text{colim}_i (\mathcal{R} \hat{I} n f(i))^{\mathbb{Z}/2\mathbb{Z}} \simeq \mathcal{R} \hat{I} n f^{\mathbb{Z}/2\mathbb{Z}} \) have both the homotopy type of CW complexes.

**Proof of Proposition** [HST] The fact that the inclusions of the colimits are homotopy equivalences is an ad hoc adaptation of the arguments in [HST] Proposition 29 and Remark 30.

The fact that the spaces are CW-complexes up to homotopy is more involved. The spaces \( \mathcal{R} \hat{I} n f(i) \) are homeomorphic to realizations of nerves of certain topological categories \( \mathcal{C}(i) \) (of course these categories should be indexed by the appropriate Clifford structure, i.e. \( \mathcal{C}(k,l)(i) \), we suppress this again).

\[
\begin{array}{cccccc}
\cdots & \mathcal{R} \hat{I} n f(i) & \mathcal{R} \hat{I} n f(i+1) & \cdots & \text{colim}_i \mathcal{R} \hat{I} n f(i) & \sim & \mathcal{R} \hat{I} n f \\
\approx & \approx & & \text{colim}_i [\mathcal{NC}(i)] & \text{colim}_i [\mathcal{NC}(i+1)] & \cdots & \text{colim}_i [\mathcal{NC}(i)]
\end{array}
\]

The nerve of these categories are simplicial topological spaces \( m \mapsto \mathcal{NC}(i)[m] \) which are good in Segal’s sense, and are levelwise CW-complexes. Goodness in this sense implies that the realization of \( \mathcal{NC}(\cdot) \) is homotopy equivalent to the fat realization (see [Seg]), which itself preserves \( \mathbb{Z}/2\mathbb{Z} \)-complex structure. Since we can show this structure levelwise, we obtain CW-complexes in the lower line. The maps in the bottom colimit system come from the realization of levelwise inclusions of CW-complexes.

We define the small topological categories \( \mathcal{C}(i) \) analogous to the internal space category of [HST]. The objects of \( \mathcal{C}(i) \) are Clifford-linear (w.r.t. the full \( \mathbb{C}(\ell_{p,q}) \) structure) subspaces of \( \mathcal{R}^{k,l}(i) \), topologized as space of finite-rank, Clifford-linear orthogonal projection operators. The set of morphisms between \( W_0 \) and \( W_1 \) is empty except if \( W_0 \subseteq W_1 \), in which case its elements are odd, orthogonal involutions on \( W_1 - W_0 := W_0^\perp \) which preserve the right Clifford action (\( \mathbb{C}(\ell_{p,q}) \)-linear), that is

\[
\mathcal{C}(i)(W_0, W_1) = \{ R \in \mathcal{O}^{\text{odd}}(W_1 - W_0) \mid R^2 = 1d, R e_i = e_i R, i = 1, \ldots, p, R f_j = f_j R, j = 1, \ldots, q \}.
\]

The morphism sets are again topologized as subspaces of bounded operators on \( \mathcal{R}(i) \). The action of \( \theta \) on the \( \mathcal{R} \hat{I} n f(i) \) induces an action on object and morphism spaces of \( \mathcal{C}(i) \).

For the fixed-point case the categories consist of \( \mathbb{Z}/2\mathbb{Z} \)-invariant subspaces together with involutions which commute with \( \theta \).

The nerve of \( \mathcal{C}(i) \) is the usual simplicial topological space made of chains of composable morphisms, with the product topology.

In particular, a point in \( \mathcal{NC}(i)[m] \) is a length-\( m \) flag \( (W_0 \subset \ldots \subset W_m) \) of Clifford-modules together with Clifford-right-linear involutions on the steps \( W_i - W_{i-1} \).

The realization of this simplicial space is the quotient

\[
\prod_{m \geq 0} \mathcal{NC}(i)[m] \times \Delta^m / \sim
\]

by the standard relation \( \sim \) [HST]. We think of \( \Delta^m \) as \( m \)-tuple \( (t_1 \leq \ldots \leq t_m) \) of points in \( \hat{\mathbb{R}} = [0, +\infty] \).

Now it is clear how to obtain an operator in \( \mathcal{R} \hat{I} n f(i) \) from a point in the realization: we send \((x; t) \in \mathcal{NC}(i)[m] \times \Delta^m \) to the operator \( G(x; t) \) with spectral configuration given by:

- the eigenvalues \( 0, \pm t_i \) for \( i = 1, \ldots, m \) (notice that these can be listed with multiplicities, \( 0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \infty \); an eigenspace at infinity for \( t_i = \infty \) is formally included in the configuration with eigenspace denoting the complement of the domain).
• the eigenspace at 0 is \( W_0 \), the eigenspaces for \( \pm t \), are the \( \pm 1 \)-eigenspaces of the involution \( R_t \) on the module \( W_t - W_{t-1} \) (to the formal eigenspace at infinity we add the complement of \( W_m \)). Notice that the single eigenspaces are linear w.r.t. the right Clifford action, however the eigenspaces \( W_t - W_{t-1} \) of the square \( G(x,t)^2 \) are linear w.r.t. both actions).

This produces a continuous map on the quotient \( [\mathfrak{NC}(i)] \) which is a homeomorphism (compare \cite[HST Chap. 7, lemma 98]{HST}). Again it is obvious that this construction, when applied to the smaller categories in the fixed-point case, produces operators in \( [\mathcal{RI}f(i)]^{\mathbb{Z}/2\mathbb{Z}} \) since all eigenspaces will be invariant under the action.

Now one shows:

- for each \( m \geq 0 \), the space \( \mathfrak{NC}(i)[m] \) is a CW-complex,
- the simplicial space \( m \mapsto \mathfrak{NC}(i)[m] \) is good in Segal’s sense.

The idea for the first part is that each space \( \mathfrak{NC}(i)[m] \) is the total space of a fiber bundle over a flag space, where both base and fiber have the homotopy type of CW-complexes. Namely, there is an obvious map \( p_m(i) : \mathfrak{NC}(i)[m] \rightarrow \mathcal{Flag}_m(i) \) taking an element in \( \mathfrak{NC}(i)[m] \) to its flag \( (W_0 \subset \ldots \subset W_m) \).

The flag space \( \mathcal{Flag}_m(i) = \bigsqcup_{d=\left(d_0 \leq d_1 \leq \ldots \leq d_m\right) \in \mathbb{N}^{m+1}} \mathcal{Flag}_m^d(i) \) is the space of Clifford-linear flags of length \( m \) in \( \mathcal{RI}f(i) \). It consists of components determined by the Clifford dimension of the successive subspaces of a flag.

Now the map \( p_m(i) \) restricts to a fiber bundle over each component \( \mathcal{Flag}_m^d(i) \), where the fiber is the space of odd, orthogonal, \( Cl_{p,q} \)-linear involutions on the flags (i.e. involutions on \( W_m - W_0 \) which keep the subspaces \( W_t - W_{t-1} \) of fixed dimensions invariant).

The flag spaces \( \mathcal{Flag}_m^d(i) \) are CW since they are themselves total spaces of bundles over the \( d_m \)-dimensional Clifford-linear Grassmannian of \( \mathcal{RI}f(i) \), which is CW, with CW fibers. The subspace of involutions in \( \mathcal{O}^{p,q}_{Cl}(W_m - W_0) \) preserving a specific flag in \( \mathcal{Flag}_m(i) \) also has a CW-complex decomposition.

Furthermore, the simplicial space \( \mathfrak{NC}(i) \) is good, which means that the inclusions of the degenerated subspaces

\[
\mathfrak{NC}(i)[m,k] := s_k(\mathfrak{NC}(i)[m-1]) \hookrightarrow \mathfrak{NC}(i)[m]
\]

are strict neighbourhood retracts in \( \mathfrak{NC}(i)[m] \). This can be easily seen by observing that the spaces \( \mathfrak{NC}(i)[m,k] \) map surjectively onto connected components of \( \mathfrak{NC}(i)[m] \), namely those which lie over connected components \( \mathcal{Flag}_m^d(i) \) of \( \mathcal{Flag}_m(i) \) with dimension vectors \( d \) of the form

\[
d = (d_0, \ldots, d_k, d_{k+1}, \ldots, d_m - 1).
\]

Using \cite{PS}, we observe that all spaces which we need to be CW-complex are defined by algebraic equations and are actually \( \mathbb{Z}/2\mathbb{Z} \)-semialgebraic sets, hence possess a \( \mathbb{Z}/2\mathbb{Z} \)-CW complex structure. In particular, under taking \( \mathbb{Z}/2\mathbb{Z} \)-fixed points, the whole argument works again, and this yields the result stated in Proposition 3.1.

### 3.2 Constructing the structure maps \( \beta_{1,0} \) and \( \beta_{0,1} \).

Here we define two maps

\[
\phi : \mathbb{R}^{k,l}_{p,q} \rightarrow \mathcal{RI}f^{k+1,l}_{p-1,q}, \quad \psi : \mathbb{R}^{k,l}_{p,q} \rightarrow \mathcal{RI}f^{k,l+1}_{p,q-1}
\]
which we prove to be quasi-fibrations with the same fibers $\mathcal{R}\hat{I}n f_{p,q}^{k,l}$ and with contractible total spaces. This implies weak homotopy equivalences (see explicit Formulas (11) and (12))

$$\beta_{1,0} : \mathcal{R}\hat{I}n f_{p,q}^{k,l} \to \Omega^{1,0}\mathcal{R}\hat{I}n f_{p,q+1}^{k+1,l} \quad \beta_{0,1} : \mathcal{R}\hat{I}n f_{p,q}^{k,l} \to \Omega^{0,1}\mathcal{R}\hat{I}n f_{p,q-1}^{k,l+1}.$$  

In a non-equivariant context, one can compare a quasifibration with contractible total space with the path-loop fibration of the base space; this yields two long exact sequences in homotopy and one obtains weak homotopy equivalences as above by the five-lemma, since both total spaces are contractible. In the $\mathbb{Z}/2\mathbb{Z}$-equivariant case, we have to take care that the actions are consistent. This leads to the two different versions of the loop space.

We can reformulate the maps $\beta_{1,0}$ and $\beta_{0,1}$ to adjust the indexing.

This is done by using $(1,1)$-periodicity first, and then applying the maps $\phi$ or $\psi$, respectively. This produces the weak equivalences

$$\beta_{1,0} : \mathcal{R}\hat{I}n f_{p,q}^{k,l} \to \Omega^{1,0}\mathcal{R}\hat{I}n f_{p,q+1}^{k+1,l} \quad \beta_{0,1} : \mathcal{R}\hat{I}n f_{p,q}^{k,l} \to \Omega^{0,1}\mathcal{R}\hat{I}n f_{p,q-1}^{k,l+1}.$$  

For $kr_{p,q}$, this yields the deloopings we need.

The index shift using $(1,1)$-periodicity can be written out, analogously to [Jo1], section 5], as the equivariant homotopy equivalence

$$\mathcal{R}\hat{I}n f_{p,q}^{k,l} \simeq \mathcal{R}\hat{I}n f_{p+1,q+1}^{k,l}, \quad G \mapsto (Id_{\mathcal{C}^{1,1}} \hat{\otimes} G_{0,0}) * G \quad (7)$$  

where $G_{0,0}$ is the generator of $\text{KR}_{0,0}$ as before. Note that $\hat{G}_{0,0} := Id_{\mathcal{C}^{1,1}} \hat{\otimes} G_{0,0}$ is defined on $\mathcal{R}_{1,1}$, therefore the image above is an operator on $\mathcal{R}_{p,q} \hat{\otimes} \mathcal{R}_{1,1} \simeq \mathcal{R}_{p+1,q+1}^{k,l}$ as wished. The fact that the above map is a homotopy equivalence is fully explained in [Jo1] and relies in turn heavily on the seminal paper [AS].

To prove that the maps a quasifibration, we use the following version of a well-known theorem of Dold and Thom [DT]:

**Theorem 3.2.** Let $q : X \to Y$ be surjective. Then $q$ is a quasifibration, if there is an increasing filtration $\{F_iY\}$ of $Y$, where $\text{colim}_i F_iY \sim Y$ is a homotopy equivalence, such that the following holds:

(i) For every open subset $U$ of $B_i := F_i - F_{i-1}$, the restriction $p^{-1}(U) \to U$ is a fibration.

(ii) For every $i$, there exist neighbourhoods $N_i \subset F_{i+1}$ of $F_i$ and a contracting homotopy $h : N_i \times I \to N_i$ with $h_0 = id_{N_i}$ and $h_1(N_i) \subset F_i$.

(iii) This deformation $h$ is covered by a homotopy $H : q^{-1}(N_i) \times I \to q^{-1}(N_i)$, $H_0 = id_{q^{-1}(N_i)}$, such that for each point $x \in N_i$

$$H_1 : q^{-1}(x) \to q^{-1}(h_1(x))$$

is a weak homotopy equivalence.

In the following, given an increasing filtration $\{X_i\}_{i \geq 0}$ of a space $X$, the spaces $X_{i+1} - X_i$ are called the filtration strata and the space $X_i$ will be called the $i$th filtration step.

It is not difficult to show that the maps $\phi$ and $\psi$ are surjective and have the correct fiber over the base point. As we will see below, there is also an obvious way to filter the base space, namely by filtering the operators by the Clifford dimension of their domains. If we think of the operators as given by their spectral configuration, this means that we filter such configurations by the Clifford dimension of the algebraic sum of eigenspaces. Neighborhoods of one such filtration level within the next larger one are given by operators
whose spectrum has at least one pair of eigenvalues within a fixed contractible neighborhood of the base point \( \infty \) of \( \mathbb{R} \).

The homotopies \( h \) contracting these neighbourhoods are induced by contracting the neighborhood of \( \infty \); we apply functional calculus with such a homotopy to the operator space, this is continuous and produces a homotopy as wished.

It is also easy to see that this can be covered by a corresponding homotopy \( H \) in the total spaces. Thus the constructive parts of this criterion in Theorem 3.2 are not difficult (including the point ii) of this criterion); it remains to show that the maps are fibration on the filtration strata (i.e. on the subspaces of operators with fixed Clifford dimension of domain), and that the homotopies contracting neighborhoods of filtration levels induce weak homotopy equivalences in the fibers.

Indeed the fibers over different filtration strata are not homeomorphic and we need an elaborate technical argument to show this last point. We show the isomorphism on homotopy groups by studying maps from spheres into the fiber spaces in question.

Details of this proof are explained in [Mar], where it is done for the ordinary real connective theory \( ko \), and we will omit them.

In the next section we will provide the definitions of the total spaces and maps and the essential parts of the proofs with particular respect to the adjustments in the \( \mathbb{Z}/2\mathbb{Z} \)-equivariant context.

### 3.2.1 Definition of the quasi-fibration maps.

We define an operator class \( \mathcal{R} \hat{I} f_{p,q}^{\mathbb{R}^0} \) of \( \mathbb{R} \) \( \mathcal{C}_p \)-linear operators of type \( \hat{I} f \), with eigenvalues of finite multiplicity in \( (-\infty, +\infty) \). This means that we can write the operator as a formal sum \( E = \sum \lambda \pi E \), where \( \lambda \in [-\infty, +\infty) \).

The complex equivalent of this is a class \( \mathcal{R} \hat{I} f_{p,q}^{\mathbb{R}^0} \mathcal{C}_p \) of even, skew-adjoint Clifford-linear operators with finite-resolvent and spectrum in \( [-i\infty, +i\infty) \).

As a set, the total space \( \hat{E}_{p,q} \) consists of triples \( (D; G, E) \) such that \( \mathcal{D}(G) = \mathcal{D}(E) = \mathcal{D} \) and \( [G, E] = 0 \) on \( \mathcal{D} \). The set \( \hat{E}_{p,q} \) is defined analogously as triples \( (D; G, F) \) of commuting operators \( G \in \mathcal{R} \hat{I} f_{p,q}^{\mathbb{R}^0} \), \( F \in \mathcal{R} \hat{I} f_{p,q}^{\mathbb{R}^0} \), defined on a common domain \( \mathcal{D} \).

Just as we can think of an operator \( G \in \mathcal{R} \hat{I} f_{p,q}^{\mathbb{R}^0} \) as a configuration of points in \( \mathbb{R} \) (eigenvalues) with coefficients (projections onto eigenspaces), we can think of a triple \( (D; G, E) \in \hat{E}_{p,q} \) as a configuration of points on the cone \( [-\infty, +\infty] \subset \mathbb{R} \) with coefficients as before (the base point of \( [-\infty, +\infty] \) is at \( +\infty \)).

The points and coefficients now correspond to the eigenvalues and eigenspaces in a simultaneous spectral decomposition of the two commuting operators \( G \) and \( E \). In the second case the same holds true if we replace the cone by a complex cone \( [-i\infty, i\infty] \subset \mathbb{R} \). This small difference will be crucial for the commutativity with the non-trivial \( \mathbb{Z}/2\mathbb{Z} \)-action on the loop space \( \Omega^{1,1} \) (compare Sections 2.2).

The topology of the total spaces is defined in terms of these configuration spaces (compare remarks following the definition of the spaces \( \mathcal{R} \hat{I} f_{p,q} \)). To put a precise expression in the notation\(^1\) of [Mar], we want to have

\[
\hat{E}^{k,l}_{p,q} := C \hat{f}^{\mathbb{R}^0}_{p+q+1} \left( [-\infty, +\infty] \subset \mathbb{R}; \mathcal{R}^{k,l}_{p,q} \right) \quad \hat{E}_{p,q}^{k,l} := C \hat{f}^{\mathbb{R}^0}_{p+q} \left( [-i\infty, i\infty] \subset \mathbb{R}; \mathcal{R}^{k,l}_{p,q} \right).
\]

The sets of triples and configurations as above are homeomorphic. For our calculations we will use the operator description. Both total spaces are \( \mathbb{Z}/2\mathbb{Z} \)-spaces by the involution \( \theta : (\cdot, -\cdot) \mapsto (\theta - \theta, \theta - \theta) \).

Now the maps are defined as

\[
\phi : \hat{E}^{k,l}_{p,q} \hookrightarrow \mathcal{R} \hat{I} f_{p-1,q+1}^{\mathbb{R}^0}; (D; G, E) \mapsto (G + E e_{p+q})
\]

\(^1\)The superscript \( \mathbb{Z}/2\mathbb{Z} \) from [Mar] in the notation for the configuration spaces refers to a \( \mathbb{Z}/2\mathbb{Z} \)-grading. To make things clear, we use the notation \( [-1/2] \) to express fixed point spaces in this work.
ψ : \hat{\mathbb{F}}_{p,q}^{k,l} \to \mathcal{R}I\hat{\mathcal{I}}_p^{k,l+1} : (D; G, F) \mapsto (G + F\hat{f}_q)

where the image is defined on \( D \cap \text{Esp}(E; -\infty)^\perp \) (or \( D \cap \text{Esp}(F; -i\infty)^\perp \) respectively). Here \( e_p \) is the \( p \)th generator of \( \mathbb{C} \ell_{p,q} \) and we think of \( e_p \) as the \((k+1)\)st generator of \( \mathbb{C}\ell_{l,k+1} \) acting from the left, while \( f_q \) is the \( q \)th generator of \( \mathbb{C} \ell_{p,q} \) and we think of \( f_q \) as the \((l+1)\)st generator of \( \mathbb{C}\ell_{l+1,k} \) acting from the left.

Note that an image operator \( \phi(G, E) \) satisfies \((\phi(G, E))^2 = G^2 + E^2\), while for an element \( \psi(G, F) \) in the image of \( \psi \) we have \((\psi(G, F))^2 = G^2 - F^2\). This means that the operator in the image of either of the maps has positive real eigenvalues of the shape \( \pm \sqrt{\lambda^2 + \mu^2} \) for eigenvalues \( \lambda \) of \( G \) and \( \mu \) of \( E \) (\( \lambda \) of \( G \) and \( i\mu \) of \( F \), respectively, in the case of \( \psi \)) and the corresponding eigenspaces are the intersections of eigenspaces of \( G \) and \( E \) (resp. \( F \)) at these eigenvalues.

One can determine the fibers over a point \( P \) in the base space of either fibration

\[
\phi^{-1}(P) \approx \mathcal{R}I\hat{\mathcal{I}}_p^{k,l}(\mathcal{D}(P)^\perp) \quad \psi^{-1}(P) \approx \mathcal{R}I\hat{\mathcal{I}}_p^{k,l}(\mathcal{D}(P)^\perp)
\]

by calculating the graded commutator of \( P \) with \( e_p \) (or \( f_q \) respectively), which recaptures the information of the pre-image operators on the part \( \mathcal{D}(P) \) of their domain.

Furthermore, as configuration spaces on cones, the spaces \( \hat{\mathbb{F}}_{p,q}^{k,l} \) and \( \hat{\mathbb{F}}_{p,q}^{k,l} \) are contractible by applying functional calculus with a suitable contraction of the cones to the operators. This can be done \( \mathbb{Z}/2\mathbb{Z} \)-equivariantly, since functional calculus affects only eigenvalues, not eigenspaces, and therefore commutes with \( \theta \).

We now check the part i) and iii) of the Dold-Thom criterion (Theorem 3.2) for our maps.

### 3.2.2 Proof of part (i): fibration on filtration strata

We use filtration levels

\[
F_i = \left\{ P \in \mathcal{R}I\hat{\mathcal{I}}_p^{k,l+1} : \dim_{\mathbb{C}\ell_{p+k,q+l}} \mathcal{D}(P) \leq i \right\}
\]

and filtration strata

\[
B_i = \left\{ P \in \mathcal{R}I\hat{\mathcal{I}}_p^{k,l+1} : \dim_{\mathbb{C}\ell_{p+k,q+l}} \mathcal{D}(P) = i \right\}
\]

in the first case and analogously in the second (with different indexing of the algebras). To satisfy 3.2(i) we have to show that the maps are fibrations on the filtration strata.

Within a filtration stratum \( B_i \) the domains of the operators form a subspace of a finite Grassmannian \( \mathcal{G}r_1(\mathcal{R}_{p,q}^{k,l}) \) of \( \mathbb{C}\ell_{p+k,q+l} \)-linear subspaces of \( \mathcal{R}_{p,q}^{k,l} \). The pullback of the complement of the tautological bundle over this Grassmannian determines the subspaces \( \mathcal{D}(\cdot)^\perp \) which distinguish the fibers.

The local trivializations of this bundle induce local trivializations of both \( \phi \) and \( \psi \) on the filtration stratum \( B_i \) by conjugating the operators.

### 3.2.3 Proof of part (iii): weak equivalence of fibers under change of filtration level

As discussed above we now have to prove that a deformation \( h \) which contracts a neighbourhood \( N_i \) of \( F_i \) into \( F_{i-1} \) is covered by a homotopy \( H \) such that for each operator \( P \in N_i \), the change in the fibers of \( \phi \) (\( \psi \) respectively)

\[
H_1 : \phi^{-1} P \to \phi^{-1} h_1(P)
\]

is a weak homotopy equivalence.

We do this for \( \phi \), the proof works analogously for the map \( \psi \). The crucial difference in the fibers \( \phi^{-1} P \) and \( \phi^{-1} h_1(P) \) is that the domains of \( P \) and \( h_1(P) \) differ by some finite-dimensional \( \mathbb{C}\ell_{p+k,q+l} \)-linear subspace \( V \) (the homotopy \( h \) removes those summands in the spectral decomposition of \( P \) whose eigenvalues are within a given neighbourhood of \( \infty \)).

Now we can show in general that the following holds:
Proposition 3.3. An inclusion \( i : R \hookrightarrow R \oplus V \) of infinite-dimensional graded Real \( \mathcal{C}l_{i,k} - \mathcal{C}l_{p,q} \)-bimodules with finite codimension induces a weak homotopy equivalence

\[
i_* : \mathcal{R}I\widehat{n}_{p,q}(R) \xrightarrow{\sim} \mathcal{R}I\widehat{n}_{p,q}(R \oplus V).
\]

This implies in particular that there is a weak homotopy equivalence of the fibers

\[
\phi^{-1}h_1(P) \xrightarrow{\sim} \phi^{-1}P
\]

induced by the inclusion \( \mathcal{D}(h_1(P))^\perp \hookrightarrow \mathcal{D}(P)^\perp \).

The proof of proposition 3.3 is rather technical and uses the fact that the involved spaces have the homotopy type of \( CW \)-complexes as shown in proposition 3.1.

As infinite-dimensional Clifford-modules the two label spaces \( R \) and \( R \oplus V \) are isomorphic. Choosing such an isomorphism \( \tau \) gives a homeomorphism and a diagram:

\[
\begin{array}{ccc}
\mathcal{R}I\widehat{n}_{p,q}(R) & \xrightarrow{i_*} & \mathcal{R}I\widehat{n}_{p,q}(R \oplus V) \\
\tau_* & & \\
\end{array}
\]

We claim that these maps induce inverse to each other isomorphisms on homotopy groups. For any map \( c \) that represents an element in \( \pi_*(\mathcal{R}I\widehat{n}_{p,q}(R)) \) we check that it is homotopic to \( \tau_* \circ i_* \circ c \). To see this we first transform the problem: \( c \) factors up to homotopy as shown

for some \( j \), by Proposition 3.1. We observe that this is the point where we need our spaces to have a nice \( CW \)-filtration.

This means that we can instead show that the factorization of \( c \) is homotopic to the factorization of \( \tau_* \circ i_* \circ c \). This can be solved on the level of Real Hilbert spaces

\[
\begin{array}{ccc}
\mathcal{R}k_{p,q}(\mathcal{R}k_{p,q}(i)) & \xrightarrow{j_*} & \mathcal{R}k_{p,q}(R) & \xrightarrow{i_*} & \mathcal{R}k_{p,q}(R \oplus V) \\
\tau_* & & & & \\
\end{array}
\]

where the maps \( j \) and \( \tau \circ i \circ j \) are \( \mathbb{Z}/2\mathbb{Z} \)-equivariantly homotopic as \( \mathbb{Z}/2\mathbb{Z} \)-equivariant embeddings of \( \mathcal{C}l_{p+k,q+l} \)-modules of infinite codimension (the space of such embeddings is equivariantly contractible).

This induces homotopies on the maps of operators (compare general properties of the operator spaces, [Mar]).

\[\square\]

Thus we can complete the proof that both maps are indeed (non-equivariantly) quasi-fibrations in the same fashion as in [Mar].
3.2.4 $\mathbb{Z}/2\mathbb{Z}$-equivariance

The quasifibrations given by the Formulas (8) and (9) induce weak homotopy equivalences

$$\beta_{1,0} : \mathcal{R}I\hat{n}_{p,q}^{k,l} \longrightarrow \Omega^{1,0} \mathcal{R}I\hat{n}_{p,q}^{k,l+1} : G \mapsto (t \mapsto (\beta_{1,0} G)_t := G + E_t e_p, \ t \in [-\infty, +\infty]),$$

$$\beta_{0,1} : \mathcal{R}I\hat{n}_{p,q}^{k,l} \longrightarrow \Omega^{0,1} \mathcal{R}I\hat{n}_{p,q}^{k,l+1} : G \mapsto (t \mapsto (\beta_{0,1} G)_t := G + F_t f_q, \ t \in [-\infty, +\infty]),$$

where $E_t := tI \text{d}(G)$ and $F_t := tI d(G)$.

We check the commutation with the $\mathbb{Z}/2\mathbb{Z}$-actions. The map $\beta_{1,0}$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant: we have

$$(\Omega^{1,0} \theta(\beta_{1,0} G))_t = \theta(G + E_t e_p) \theta$$

$$= \theta G \theta + \theta E_t \theta e_p$$

$$= \theta G \theta + E_t e_p$$

$$= (\beta_{1,0} \theta(G))_t,$$

since $E_t$ commutes with $\theta$

$$\theta E_t \theta(x \otimes v) = \theta E_t (\pi \otimes \theta(v))$$

$$= \theta (t \pi \otimes \theta(v))$$

$$= tx \otimes v$$

$$= E_t (x \otimes v).$$

In the case of $\beta_{0,1}$ we want to see that the diagram commutes

$$\mathcal{R}I\hat{n}_{p,q}^{k,l} \xrightarrow{\beta_{0,1}} \Omega^{0,1} \mathcal{R}I\hat{n}_{p,q}^{k,l+1}$$

$$\mathcal{R}I\hat{n}_{p,q}^{k,l} \xrightarrow{\beta_{0,1}} \Omega^{0,1} \mathcal{R}I\hat{n}_{p,q}^{k,l+1}$$

where now $\Omega^{0,1} \theta(\gamma)(t) = \theta(\gamma(-t))$ for $\gamma$ a loop in $\Omega^{0,1} \mathcal{R}I\hat{n}_{p,q}^{k,l+1}, \ t \in [-\infty, +\infty]$. We have

$$(\Omega^{0,1} \theta(\beta_{0,1} G))_t = \theta((\beta_{0,1} G)_{-t})$$

$$= \theta(G + F_{-t} f_q) \theta$$

$$= \theta G \theta - \theta F_{-t} \theta f_q$$

$$= \theta G \theta + F_t f_q$$

$$= (\beta_{0,1} \theta(G))_t.$$

The first equation follows since $\theta$ commutes with $e$ but anti-commutes with the multiplication by $f_q$ (where $f_q$ is in the $(-1)$-eigenspace of $\theta$). The second identity follows from

$$(\theta E_t \theta)(x \otimes v) = \theta E_t (\pi \otimes \theta(v))$$

$$= \theta (-t \pi \otimes \theta(v))$$

$$= -t \pi \otimes v$$

$$= F_t (x \otimes v).$$
Thus $\psi$ indeed induces a map into the $\Omega^{0,1}$-loop space.

Finally, combining the maps from Formula (11) and (12) with the $(1,1)$-periodicity given in formula (7), we obtain the (weak) homotopy equivalences which constitute the structure maps

$$\beta_{1,0} : \mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q}^{k,l} \to \Omega^{1,0} \mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q+1}^{k,l+1} : G \mapsto \left( t \mapsto (\beta_{1,0}) G^t := \widetilde{G}^t \star G + E_t \epsilon p+1, \ t \in [-\infty, +\infty] \right),$$

$$\beta_{0,1} : \mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q}^{k,l} \to \Omega^{0,1} \mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p-1,q}^{k+1,l} : G \mapsto \left( t \mapsto (\beta_{0,1}) G^t := \widetilde{G}^t \star G + E_t \epsilon f_{p+1}, \ t \in [-\infty, +\infty] \right),$$

where $E_t := tId_{\mathcal{D}(\hat{\mathcal{G}}_{\mathcal{A}_0} \ast G)}$ and $\hat{F}_t := tId_{\mathcal{D}(\hat{\mathcal{G}}_{\mathcal{A}_0} \ast G)}$.

3.2.5 Quasi-fibration on fixed points

Finally we want to see that by restricting to fixed-points of the $\mathbb{Z}/2\mathbb{Z}$-action, the map $\phi$ is also a quasi-fibration. Recall that the action on $\mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f$ was given by conjugating with $\theta$. Fixed points therefore are operators $G$ with $\theta \mathcal{G} \theta = \theta G \theta$: thus they have the same eigenvalues, but their eigenspaces are closed under the involution $\theta$. Taking fixed points produces a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}/2\mathbb{Z} & \xrightarrow{\lambda} & \mathbb{Z}/2\mathbb{Z} \\
\downarrow \phi & & \downarrow \phi \\
\mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q}^{k,l} & \xrightarrow{\lambda} & \mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q}^{k,l} \\
\end{array}$$

We have already seen that the map $\phi$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant, thus it restricts to a continuous map on fixed-points. One can check with little effort that all parts of the proof of the quasi-fibration properties still work upon taking fixed points; that is all homotopies and constructions used are $\mathbb{Z}/2\mathbb{Z}$-equivariant.

The proof therefore applies to the restriction $[\phi]^\mathbb{Z}/2\mathbb{Z}$.

3.3 Connectivity properties of the spaces

**Proposition 3.4.** The spaces $\mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q}^{k,l}$ are connected for $p, q, l \geq 0$ and $k + l \geq 1$. The space $[\mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q}^{k,l}]^\mathbb{Z}/2\mathbb{Z}$ is connected for all $k \geq 1$, $l \geq 0$.

Let $G = \sum \lambda \pi_{G, \lambda}$ be the spectral decomposition of an operator $G \in \mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q}^{k,l}$.

Then

$$H_t(G) := \left( \frac{1}{1-t} \right) G + \left( \frac{t}{1-t} \right) \pi_{\ker(G)} f = \sum_{\lambda \neq 0} \frac{\lambda}{1-t} \pi_{G, \lambda} \pm \left( \frac{t}{1-t} \right) \pi_{\ker(G)^\pm}, \ t \in [0, 1]$$

defines a path from $G$ to the base point using left multiplication by one generator $f$ of $\mathcal{C}\ell_{l,k}$ (we have at least one such generator by assumption). This splits the $\mathcal{C}\ell_{l,k} \ast \mathcal{C}\ell_{p,q}$-module $\ker(G) = \ker(G^2)$ into two $\mathcal{C}\ell_{p,q}$-modules $\ker(G)^\dagger \oplus \ker(G)^\dagger = \ker(G)$ (the eigenspaces of the action of $f$).

Along the path, all eigenvalues move to infinity, symmetrically about 0: the eigenvalue 0 itself splits in two which move out as well. This move is continuous and depends continuously $t$ (but not in $G$, see [Mar]).

Note that all properties of $\mathcal{R}\mathcal{I}\mathcal{N}_{p,q} f_{p,q}^{k,l}$ are preserved throughout the path, even though $\ker(G)^\dagger$ are not $\mathcal{C}\ell_{l,k} \ast \mathcal{C}\ell_{p,q}$-modules (their sum is).
If \( G \) is a fixed point of the \( \mathbb{Z}/2\mathbb{Z} \)-action, i.e. if all eigenspaces are \( \mathbb{Z}/2\mathbb{Z} \)-invariant subspaces, we need to use a generator of the left \( C_{0,k} \)-action which commutes with the action \( \theta \). This is the case for the generators of \( C_{0,k} \) acting from the left. Our assumptions guarantee that there is at least one. Then the split parts will again be \( \mathbb{Z}/2\mathbb{Z} \)-invariant and all operators in the path will also be fixed points.

**Corollary 3.5.** The space \( R\hat{\Gamma}f_{p,q}^{k,l} \) is at least \((k + l - 1)\)-connected. The space \([R\hat{\Gamma}f_{p,q}^{k,l}]\mathbb{Z}/2\mathbb{Z}\) is at least \((k - 1)\)-connected.

This follows by induction from the deloopings in the previous section. On fixed-points, the induction stops after \( k - 1 \) steps.

### 3.4 Commutation with Bott maps

The diagram

\[
\begin{array}{ccc}
kr_n,n & \xrightarrow{\beta_{1,1}} & \Omega^{1,1}kr_{n+1,n+1} \\
\downarrow & & \downarrow \\
KR_n,n & \xrightarrow{b_{1,1}} & \Omega^{1,1}KR_{n+1,n+1}
\end{array}
\]  

(18)

commutes. This is done by a direct computation.

Notice that the vertical maps in the diagram are essentially inclusions, forgetting the extra linearity properties of the operators with respect to the left Clifford actions. The map \( b_{1,1} \) comes from the structure of KR as a symmetric spectrum, as explained in Section 1.4.2. This map associates to an operator \( F \in KR_{n,n} \) the map

\[
b_{1,1}(F) : S^{1,1} \to KR_{p+1,q+1}; \quad v \mapsto L_v \star G_{0,0}^* \star F.
\]

On the other hand, the map \( \beta_{1,1} \) is given by combining the formulas (16) and (17), i.e. we apply twice the index shift (formula (7)) and the two maps induced by the quasi-fibrations. This associates to an operator \( F \in kr_{n,n} \) the map \( \beta_{1,1}(F) \):

\[
\begin{align*}
\mathbb{S}^{1,1} & \to kr_{n+1,n+1} \\
v = (v_1, v_2) & \mapsto \tilde{G}_{0,0} \star \tilde{G}_{0,0} \star F + Id_{\Delta(G_{0,0}^*G_{0,0}^*F)} v_1 \epsilon e_{n+1} \\
& \quad + Id_{\Delta(G_{0,0}^*G_{0,0}^*F)} iv_2 \epsilon f_{n+1}
\end{align*}
\]

which can be rewritten as:

\[
\beta_{1,1}(F)(v) = Id_{C_{0,2}} \hat{\otimes} G_{0,0}^* \star F + L_v \hat{\otimes} Id_{\Delta(G_{0,0}^*F)} \\
= L_v \star G_{0,0}^* \star F
\]

since multiplication with \( v_1 \epsilon e \) and \( iv_2 \epsilon f_{n+1} \) as a right action corresponds to multiplication by \( v = v_1 + iv_2 \) as a left action (on the particular copy of \( C_{0,2} \) specified by the generators \( e \) and \( f \)).

This implies that \( kr_{n,n} \to KR_{n,n} \) models the \((2n - 1)\)-connective cover of \( KU_{2n} = \mathbb{Z} \times BU \). In fact, we actually prove that \( kr_{p,q} \to KR_{p,q} \) models the \((p + q - 1)\)-connective cover of \( \mathbb{Z} \times BU \) (if \( p + q \) is even) or \( U \) (if \( p + q \) is odd).
As announced in Section 2.1, we now check that \( \pi_k(\mathbb{F}_{n,n}) \) is an isomorphism for \( k \geq n \). We begin with the observation \([At]\), that there is an equivariant cofibration sequence for all \( s, t \geq 0 \)

\[
S^{s,t} \to S^{s,t+1} \to S^{s+1,t+1} \times \mathbb{Z}/22Z
\]

Applying mapping spaces into a \( \mathbb{Z}/22\mathbb{Z} \)-space \( X \), we get after taking fixed points the natural fibration sequences of spaces

\[
\Omega^{s+t+1}X \to \Omega^{s+t+1}X/Z2Z \to \Omega^{s,t}X/Z2Z
\]

where \( \Omega X \) refers to the non-equivariant loop space of the space \( X \) (i.e., forget all actions).

We note firstly that \( kr_{p,q} \) is a connective cover of \( KR_{p,q} \), and that \( KR_{p,q} \) is \( \mathbb{Z} \times BU \) or \( U \) depending whether \( p+q \) is even or not. Hence equivariantly, the map \( \varphi_{p,q} : kr_{p,q} \to KR_{p,q} \) is equivariantly an isomorphism on homotopy groups \( \pi_i \) for \( i \geq p+q \). In fact, because our spaces \( U \) and \( BU \) have homotopy vanishing every second degree, and more precisely in odd degrees for \( BU \). We therefore even get \( \varphi_{n,n} : kr_{n,n} \to KR_{n,n} \) is equivariantly an isomorphism on homotopy groups \( \pi_i \) for \( i \geq 2n-1 \).

Secondly we know that \([kr_{n,0}]Z2Z \to [KR_{n,0}]Z2Z\) models the cover map \( ko \to KO \) so that \([\mathbb{F}_{n,0}]Z2Z\) is an isomorphism on homotopy groups \( \pi_i \) for \( i \geq n \). Consider the commutative diagram

\[
\begin{array}{ccc}
\Omega^n\hat{kr}_{n,n} & \to & [\Omega^n\mathbb{F}_{n,n}]Z2Z \\
\downarrow & & \downarrow \\
\Omega^nKR_{n,n} & \to & [\Omega^nKR_{n,n}]Z2Z
\end{array}
\]

The left vertical map is an isomorphism on homotopy groups \( \pi_i \) for \( i \geq n-1 \). The middle one identifies with \( ko \to KO \) by the natural equivalences \( \Omega^0\mathbb{F}_{n,n} \simeq kr_{n,0} \) and \( \Omega^0KR_{n,n} \simeq KR_{n,0} \) (plus the observation that \( KR_{n,0} \) is none but \( KU_0 \) acted on by complex conjugation, and the same for \( kr_{n,0} \), and in this case fixed point give back the case of \( ko \to KO \), as in \([Mar]\), hence is an isomorphism on homotopy groups \( \pi_i \) for \( i \geq n \). So we get by the five lemma applied to the long exact sequence in homotopy of theses fibrations that the rightmost vertical map is also an isomorphism on \( \pi_i \) for \( i \geq n \). Beginning \([\Omega^n\mathbb{F}_{n,n}]Z2Z\), we induct decreasingly on \( k \) to get that \( \pi_i([\Omega^k\mathbb{F}_{n,n}]Z2Z) \) is an isomorphism for \( i \geq n \). This gives the desired result for \( k = 0 \).

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