Gauge-invariant fluctuations of scalar branes

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Abstract

A generalization of the Bardeen formalism to the case of warped geometries is presented. The system determining the gauge-invariant fluctuations of the metric induced by the scalar fluctuations of the brane is reduced to a set of Schrödinger-like equations for the Bardeen potentials and for the canonical normal modes of the scalar-tensor action. Scalar, vector and tensor modes of the geometry are classified according to four-dimensional Lorentz transformations. While the tensor modes of the geometry live on the brane determining the corrections to Newton law, the scalar and and vector fluctuations exhibit non normalizable zero modes and are, consequently, not localized on the brane. The spectrum of the massive modes of the fluctuations is analyzed using supersymmetric quantum mechanics.

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I. INTRODUCTION

Higher dimensional topological defects allow to investigate the properties and implications of warped geometries [1,2]. Fields of various spin can be localized on warped backgrounds. For instance, chiral fermions can be localized on a five-dimensional domain-wall solution [4]. Moreover, also gravitational interactions can be localized on five-dimensional walls [5,6]. For a recent review see [7].

Suppose that a smooth domain-wall solution is used in order to localize fields. Such a geometry may be generated, for instance, by a scalar potential \( V(\phi) \) with spontaneous breaking of \( \phi \to -\phi \) symmetry [4]. Consider first the case where gravity is absent. In this case the scalar field has a normalizable zero mode [4].

If gravity is consistently included the fluctuations of the scalar wall will induce fluctuations in the metric. It is then natural to ask if all the metric fluctuations are localized on the wall. There are two problems in this analysis. The first one is that metric fluctuations are coupled to the fluctuations of the wall. The second one is that the coupled system of equations depends on the coordinate system. In order to discuss the metric fluctuations of scalar-tensor actions, the Bardeen formalism [8] has been very useful in four-dimensional backgrounds. The main idea is to parameterize the metric fluctuations by defining suitable gauge-invariant variables which do not change for infinitesimal coordinate transformations. The Bardeen formalism is also rather effective in order to identify the coordinate systems where the gauge-invariant variables take a simple form. Having generalized the Bardeen formalism to the case of static five-dimensional warped geometries, the localization properties of metric fluctuations can be investigated. The standard lore is that if the zero mode of a given fluctuation is not normalizable, then it will not be localized on the brane and it will not affect the four-dimensional physics.

Explicit physical (thick) brane solutions have been derived in different numbers of transverse dimensions. In five dimensions physical branes can be obtained using a scalar domain-wall where the scalar field depends upon the bulk radius [9–12]. In [13,14] the stability of
scalar domain walls (inspired by gauged supergravity theories) has been analyzed. In more than five dimensions, physical brane solutions including also background gauge fields have been recently analyzed \[16\]. Singular (scalar) brane solutions have been also analyzed from different points of view \[15\]. The present analysis deals with the case of one transverse (bulk) coordinate. The proposed formalism can be however generalized to the case of higher dimensional transverse space.

The generalization of the Bardeen formalism to the case of compact extra-dimensions has been addressed in a cosmological context. The problem was to discuss the evolution of metric fluctuations in cosmological models with compact extra-dimensions with \[17,18\] and without \[19\] bulk scalars. In the present analysis the symmetry of the four-dimensional world is Lorentz invariance which will be used in order to classify the fluctuations of the metric \[20\]. The approach discussed in the present paper is then different from the ones previously discussed, for cosmological applications, in \[17\]–\[19\] and in \[21\].

The plan of this paper is the following. In Section II scalar-tensor models of physical branes will be briefly discussed. Emphasis will be given to the aspects relevant in the analysis of the perturbations. In Section III the Bardeen formalism will be generalized to the case of warped metrics. Variables invariant under infinitesimal coordinate transformations will be discussed. Physical gauge choices will be proposed. In Section IV the evolution for the scalar, tensor and vector modes of the geometry will be derived. The system will be reduced to a set of (decoupled) second order differential equations. In Section V the zero-modes of the gauge-invariant variables will be discussed. General conditions on the normalizability of the gravi-photon and gravi-scalar will be derived. Section VI contains some concluding remarks. In the Appendix useful technical results have been collected.

**II. BACKGROUND MODELS**

In five dimensions only one bulk coordinate is present and smooth domain wall configurations can be found using the following scalar-tensor action:
\[
S = \int d^5x \sqrt{|G|} \left[ -\frac{R}{2\kappa} + \frac{1}{2} G^{AB} \partial_A \varphi \partial_B \varphi - V(\varphi) \right],
\]
whose related equations of motion are
\[
R^B_A - \frac{1}{2} \delta^B_A R = \kappa \left[ \partial_A \varphi \partial^B \varphi - \delta^B_A \left( \frac{1}{2} G^{MN} \partial_M \varphi \partial_N \varphi - V(\varphi) \right) \right],
\]
\[
G^{AB} \nabla_A \nabla_B \varphi + \frac{\partial V}{\partial \varphi} = 0.
\]
Notice that \( \kappa = 8\pi G_5 = 8\pi/M_5^3 \). In order to facilitate the treatment of the fluctuations it is useful to write the five-dimensional metric in conformal coordinates, namely \[^1\]:
\[
ds^2 = G_{AB} dx^A dx^B = a^2(w) [dt^2 - dx^2 - dw^2].
\]
Choosing now natural gravitational units \( 2\kappa = 1 \), the explicit form of Eqs. (2.2)–(2.3) is:
\[
\varphi'' + 3\mathcal{H} \varphi' - \frac{\partial V}{\partial \varphi} a^2 = 0,
\]
\[
\mathcal{H}' + \mathcal{H}^2 = -\frac{1}{6} \left[ \frac{\varphi'^2}{2} + V a^2 \right],
\]
\[
\mathcal{H}^2 = \frac{1}{12} \left[ \frac{\varphi'^2}{2} - V a^2 \right],
\]
where
\[
\mathcal{H} = \frac{a'}{a},
\]
and the prime denotes the derivation with respect to the bulk coordinate \( w \). Eqs. (2.5)–(2.7) are not independent. By combining two of them the third one can be obtained. This property can be used in order to solve the full system.

The equation for \( \varphi \) can be obtained by combining Eqs. (2.6) and (2.7):
\[
\varphi'^2 = 6(\mathcal{H}^2 - \mathcal{H}'),
\]
\[
V a^2 = -3(3\mathcal{H}^2 + \mathcal{H}').
\]
\[^1\]The conventions will be the following. Greek indices run over the four-dimensional space. Latin (uppercase) indices run over all the five dimensions.
Eqs. (2.9) and (2.10) will allow to simplify some of the equations for the fluctuations.

Given a specific form of \( a(w) \), Eq. (2.9) determines \( \varphi(w) \) whereas, from Eq. (2.10), \( V(w) \) is obtained. Inserting \( w(\varphi) \) into \( V(w) \), the specific form of \( V(\varphi) \) is found, once \( a(w) \) is specified.

Using this method of integration various exact solutions can be discussed [9–12]. Most of the results reported in this paper will be general but in order to fix the ideas it is useful to have in mind the following regular geometry:

\[
a(w) = \frac{1}{\sqrt{b^2 w^2 + 1}},
\]

(2.11)

where \( b \) is a real number. Using Eq. (2.11) in Eq. (2.9) and integrating once the scalar field corresponding to the geometry (2.11) is

\[
\varphi(w) = \sqrt{6} \arctan bw.
\]

(2.12)

Inserting \( w(\varphi) \) into Eq. (2.10)

\[
V(\varphi) = 3b^2 \left[ 1 - 5 \sin^2 \frac{\varphi}{\sqrt{6}} \right].
\]

(2.13)

Eqs. (2.11)–(2.13) describe a thick domain wall solution. The metric associated with this field configuration is non-singular (i.e. no poles are present in the curvature invariants). In the non-conformal coordinate system the metric is determined recalling that \( a(w)dw = dy \). Integrating once Eq. (2.11)

\[
a(y) = \frac{1}{\cosh by}.
\]

(2.14)

Most of the results reported in the following sections will also hold for singular domain-wall solutions [15] where the scalar field is coupled to the action of a thin wall present for some finite value of the bulk coordinate.

III. FLUCTUATIONS OF THE METRIC AND OF THE BULK FIELD
A. Preliminaries

The fluctuations of the metric (2.4) are coupled to the fluctuations of the bulk scalar \( \varphi \), as it can be appreciated by writing down the equations for the fluctuations in their general form. Contracting once Eq. (2.2) the background equations can be written as

\[
R_{AB} = \frac{1}{2} \partial_A \varphi \partial_B \varphi - \frac{V}{3} G_{AB},
\]

\[
G^{AB} \left( \partial_A \partial_B \varphi - \Gamma^C_{AB} \partial_C \varphi \right) + \frac{\partial V}{\partial \varphi} \chi = 0,
\]

(3.1)

leading to the evolution equations of the fluctuations

\[
\delta R_{AB} = \frac{1}{2} \partial_A \varphi \partial_B \chi + \frac{1}{2} \partial_A \chi \partial_B \varphi - \frac{1}{3} \frac{\partial V}{\partial \varphi} \chi G_{AB} - \frac{V}{3} \delta G_{AB},
\]

\[
\delta G^{AB} \left( \partial_A \partial_B \varphi - \Gamma^C_{AB} \partial_C \varphi \right) + \Gamma^{AB} \left( \partial_A \partial_B \chi - \Gamma^C_{AB} \partial_C \chi - \delta \Gamma^C_{AB} \partial \varphi \right) + \frac{\partial^2 V}{\partial \varphi^2} \chi = 0.
\]

(3.2)

(3.3)

The metric and the scalar field have been separated into their background and perturbation parts:

\[
G_{AB}(x^\mu, w) = \overline{G}_{AB}(w) + \delta G_{AB}(x^\mu, w),
\]

\[
\varphi(x^\mu, w) = \varphi(w) + \chi(x^\mu, w).
\]

(3.4)

In Eqs. (3.2)-(3.3), \( \delta \Gamma^C_{AB} \) and \( \delta R_{AB} \) are, respectively, the fluctuations of the Christoffel connections and of the Ricci tensors, whereas \( \overline{\Gamma}^C_{AB} \) are the background values of the connections

\[
\overline{\Gamma}^C_{Aw} = \mathcal{H} \delta^C_A, \quad \overline{\Gamma}^w_{\mu\nu} = \mathcal{H} \eta_{\mu\nu},
\]

(3.5)

where the metric has been assumed in its conformally flat parametrization as in Eq. (2.4).

The metric fluctuation \( \delta G_{AB} \) contains scalar, vector and tensor modes, namely,

\[
\delta G_{AB}(x^\mu, w) = \delta G_{AB}^{(S)}(x^\mu, w) + \delta G_{AB}^{(V)}(x^\mu, w) + \delta G_{AB}^{(T)}(x^\mu, w).
\]

(3.6)

Since the four-dimensional metric is Lorentz invariant, the scalar, vector and tensor modes of the higher dimensional geometry will be classified according to Lorentz transformations as
previously discussed \[20\] in a different context. The simultaneous use of gauge-invariance and Lorentz invariance is one of the key points of the present analysis.

B. Gauge-invariant variables

The Bardeen formalism for metric fluctuations \[8\] will now be generalized to the case of higher-dimensional metrics containing non compact transverse dimensions of the type of the one given in Eq. (2.4). The fluctuation of the metric can be parametrized as

\[
\delta G_{AB} = a^2(w) \begin{pmatrix}
2h_{\mu\nu} + (\partial_\mu f_\nu + \partial_\nu f_\mu) + 2\eta_{\mu\nu}\psi + 2\partial_\mu \partial_\nu E - D_\mu + \partial_\mu C & 2\xi \\
D_\mu + \partial_\mu C & 2\xi
\end{pmatrix},
\]

(3.7)

where the Lorentz indices run over the four space-time dimensions. The tensor $h_{\mu\nu}$ is traceless and divergence-less

\[
h_{\mu} = 0, \quad \partial_\nu h_{\mu} = 0,
\]

(3.8)
corresponding to five independent components. The vectors $D_\mu$ and $f_\mu$ are both divergence-less

\[
\partial_\mu D^\mu = 0, \quad \partial_\mu f^\mu = 0,
\]

(3.9)

so that they will have, together, six independent components. The scalars $E$, $\psi$, $C$ and $\xi$ lead to four independent components. For infinitesimal coordinate transformations

\[
x_A \rightarrow \tilde{x}^A = x^A + \epsilon^A,
\]

(3.10)

the fifteen independent metric fluctuations change as

\[
\delta \tilde{G}_{AB} = \delta G_{AB} - \nabla_A \epsilon_B - \nabla_B \epsilon_A,
\]

(3.11)

where, for the background metric of Eq. (2.4), $\epsilon_A = a^2(w)(\epsilon_\mu, -\epsilon_\nu)$. In Eq. (3.11) the Lie the covariant derivatives are computed using the background metric.

\[2\] I am grateful to M. Shaposhnikov for valuable comments on this point.
The $\epsilon_\mu$ can be written as the derivative of a scalar plus a vector

$$\epsilon_\mu = \partial_\mu \epsilon + \zeta_\mu, \quad (3.12)$$

where $\partial_\mu \zeta^\mu = 0$.

Using Eqs. (3.11) and (3.12) the explicit transformation properties of the perturbed components of the metric for infinitesimal coordinate transformations can be found. The transverse and traceless tensors are gauge-invariant, i.e. they do not change for infinitesimal gauge transformations

$$\tilde{h}_{\mu\nu} = h_{\mu\nu}, \quad (3.13)$$

whereas the vector transform as

$$\tilde{f}_\mu = f_\mu - \zeta_\mu, \quad (3.14)$$
$$\tilde{D}_\mu = D_\mu - \zeta'_\mu. \quad (3.15)$$

Finally the scalars will transform as

$$\tilde{E} = E - \epsilon, \quad (3.16)$$
$$\tilde{\psi} = \psi - \mathcal{H}\epsilon_w, \quad (3.17)$$
$$\tilde{C}' = C - \epsilon' + \epsilon_w, \quad (3.18)$$
$$\tilde{\xi} = \xi + \mathcal{H}\epsilon_w + \epsilon'_w. \quad (3.19)$$

The prime denotes, as usual, derivation with respect to the bulk coordinate.

Gauge-invariant fluctuations corresponding to scalar and vector modes of a given geometry can be constructed as it was noticed by Bardeen [8] in a four-dimensional context and later generalized to higher (compact) dimensions [17–19]. Since the geometry contains two divergence-less vectors and one gauge function $\zeta_\mu$, only one gauge-invariant variable is allowed. A possible choice of gauge-invariant vector fluctuation is

$$\tilde{V}_\mu = \tilde{D}_\mu - \tilde{f}'_\mu. \quad (3.20)$$
Using Eqs. (3.14) and (3.15), it is easy to see that $\tilde{V}_\mu = V_\mu$.

Since there are four scalar fluctuations in the metric and two gauge functions, i.e. $\epsilon$ and $\epsilon_w$, two gauge invariant variables can be defined. The gauge-invariant scalar fluctuations can be parametrized as

$$\tilde{\Psi} = \tilde{\psi} - \mathcal{H}(\tilde{E}' - \tilde{C}), \quad (3.21)$$

$$\tilde{\Xi} = \tilde{\xi} - \frac{1}{a}[a(\tilde{C} - \tilde{E}')]' \quad (3.22)$$

Using Eqs. (3.17) and (3.19) together with Eqs. (3.16) and (3.18) it can be directly obtained that $\tilde{\Xi} = \Xi$ and that $\tilde{\Psi} = \Psi$. The bulk scalar field fluctuation transforms under infinitesimal coordinate shifts:

$$\tilde{\chi} = \chi - \varphi'\epsilon_w \quad (3.23)$$

The gauge-invariant scalar field fluctuation will be

$$\tilde{X} = \tilde{\chi} - \varphi'(\tilde{E}' - \tilde{C}) \quad (3.24)$$

Eqs. (3.20) and (3.22) are reminiscent of the Bardeen potentials [8]. There are, of course, infinite gauge-invariant variables since every combination of gauge-invariant variables is also gauge-invariant. The variables defined in Eqs. (3.20) and (3.22) have the virtue of obeying very simple equations, as it will be shown.

C. Gauge choices

It is useful to fix completely the gauge freedom. Not every gauge fixes completely the coordinate system. In the following, two useful gauge choices will be discussed. If the gauge is completely fixed, five of the fifteen degrees of freedom appearing in the perturbed metric can be eliminated. Once the fluctuations are discussed in a given gauge, the gauge invariant variables can be computed in that specific gauge.
In order to fix completely the coordinate system, $\epsilon$, $\epsilon_w$ and $\zeta_\mu$ should be fixed. Two gauge choices fixing completely the coordinate system will now be discussed. The first one is

$$\tilde{E} = 0, \quad \tilde{f}_\mu = 0, \quad \tilde{\psi} = 0. \quad (3.25)$$

The gauge is called off-diagonal since the scalar contribution of the off-diagonal entry of the perturbed metric, i.e. $\partial_\mu C$ is non vanishing.

Using Eqs. (3.25), respectively into Eqs. (3.14), (3.16) and (3.17) the infinitesimal parameters are fixed to be

$$\epsilon = E, \quad \epsilon_w = \frac{\psi}{H}, \quad f_\mu = \zeta_\mu. \quad (3.26)$$

Another interesting gauge choice is the generalization is the longitudinal gauge where all the off-diagonal (scalar) components of the perturbed metric are vanishing:

$$\tilde{E} = 0, \quad \tilde{C} = 0, \quad \tilde{f}_\mu = 0. \quad (3.27)$$

As before, using Eqs. (3.27) into eqs. (3.16), (3.18) and (3.14) the parameters of the gauge transformation can be totally fixed:

$$\epsilon = E, \quad \epsilon_w = (E' - C), \quad \zeta_\mu = f_\mu. \quad (3.28)$$

Not all the gauge choices have the feature of fixing completely the coordinate system. As an example, consider the gauge

$$\tilde{\psi} = 0, \quad \tilde{C} = 0, \quad \tilde{D}_\mu = 0. \quad (3.29)$$

Using the conditions of Eqs. (3.29) into Eqs. (3.17), (3.18) and (3.15) the parameters of the gauge transformation are fixed but only up to arbitrary integration constants depending only upon $x_\nu$:

$$\epsilon = Q(x^\nu) + \int \left( C + \frac{\psi}{H} \right) dw, \quad \zeta_\mu = P_\mu(x^\nu) + \int D_\mu dw. \quad (3.30)$$
Hence, even after the gauge choice (3.29) has been imposed, $\epsilon$ and $\zeta_\mu$ can still be shifted by an arbitrary function depending upon $x^\nu$. In this sense the gauge-fixing described by Eqs. (3.29) is not complete. Further discussions concerning gauge choices in the specific framework of 3,4 are contained in 22.

IV. EVOLUTION EQUATIONS FOR THE METRIC FLUCTUATIONS

The evolution equations of the fluctuations will now be derived. The coupled system of Eqs. (3.2)–(3.3) will be diagonalized and reduced to a set of (decoupled) second order differential equations. Particular attention will be paid to the evolution of the gauge-invariant variables whose equations of motion should coincide in different gauges. This useful feature of the formalism will be used in order to cross-check the consistency of the resulting equations which will be independently derived in different gauges.

A. Perturbations in the longitudinal gauge

In the longitudinal gauge 3 the Bardeen potentials $\Psi$ and $\Xi$ coincide, respectively, with $\psi$ and $\xi$. This simple fact can be easily appreciated by inserting the gauge fixing of Eqs. (3.27) into Eqs. (3.22). Similarly, the gauge-invariant form of the scalar field fluctuation coincide with $\chi$, i.e. $X \equiv \chi$. Finally, from Eq. (3.20) it can be seen that $V_\mu \equiv D_\mu$ where $V_\mu$ is the gauge-invariant vector fluctuation defined in Eq. (3.20).

The fluctuations of the Christoffel connections and of the Ricci tensors are reported in Appendix A. Using Eqs. (A.2) and (A.3) into Eqs. (3.2)–(3.3) the coupled system of evolution equations of the metric fluctuations can be obtained:

$$h''_{\mu\nu} + 3\dot{H}h'_{\mu\nu} - \partial_\alpha \partial^\alpha h_{\mu\nu} = 0,$$  

(4.1)

$$\psi'' + 7\dot{H}\psi' + \dot{\mathcal{H}}\xi' + 2(\mathcal{H}' + 3\mathcal{H}^2)\xi + \frac{1}{3} \frac{\partial V}{\partial \varphi} a^2 \chi - \partial_\alpha \partial^\alpha \psi = 0,$$  

(4.2)

From now on we will drop the tilde from the variables unless strictly required.
\[ \partial_\alpha \partial^\alpha D_\mu = 0, \quad 4.3 \]

\[ -\partial_\alpha \partial^\alpha \xi - 4[\psi'' + \mathcal{H}\psi'] - 4\mathcal{H}\xi' - \varphi' \chi' - \frac{1}{3} \partial \varphi a^2 \chi + \frac{2}{3} V a^2 \xi = 0, \quad 4.4 \]

\[ \partial_\alpha \partial^\alpha \chi - \chi'' - 3\mathcal{H}\chi' + \frac{\partial^2 V}{\partial \varphi^2} a^2 \chi - \varphi'[4\psi' + \xi'] - 2\xi(\varphi'' + 3\mathcal{H}\varphi') = 0. \quad 4.5 \]

Eq. (4.1) can be immediately reduced to a Schrödinger-like form by defining \( a^{3/2} h_{\mu\nu} = \mu_{\mu\nu} \)

\[ \mu''_{\mu\nu} - \partial_\alpha \partial^\alpha \mu_{\mu\nu} - \frac{(a^{3/2})''}{a^{3/2}} \mu_{\mu\nu} = 0. \quad 4.6 \]

Eqs. (4.1)–(4.5) contain second order derivatives with respect to time. Eqs. (4.1)–(4.4) come from the perturbed form of Einstein equations, i.e. Eq. (3.2). Eq. (4.5) is the explicit form of Eq. (3.3). From the off-diagonal components of the Einstein equations a number of constraints, connecting the first derivatives of different fluctuations, can be obtained:

\[ \partial_\mu \partial_\nu [\xi - 2\psi] = 0, \quad 4.7 \]

\[ D'_\mu + 3\mathcal{H} D_\mu = 0, \quad 4.8 \]

\[ 6\mathcal{H}\xi + 6\psi' + \chi \varphi' = 0. \quad 4.9 \]

Eqs. (4.7)–(4.9) are crucial in order to diagonalize the full system. The constraints (4.7) and (4.9) pertain to scalar modes. This allows to determine the number of propagating degrees of freedom. There are, in this gauge, three scalar functions: \( \psi, \xi \) and \( \chi \). They obey a set of coupled dynamical equations but there are two constraints. Hence, there will be only one physical scalar mode.

The equations of motion for the scalar modes represented by \( \xi, \psi \) and \( \chi \) can be decoupled using the following procedure. Eq. (4.7) implies that \( \xi = 2\psi \). By summing up Eq. (4.2) and Eq. (4.4) the following equation can be obtained:

\[ \psi'' + \mathcal{H}\psi' + \partial_\alpha \partial^\alpha \psi + \frac{\chi' \varphi'}{3} = 0. \quad 4.10 \]

In order to get to Eq. (4.10), the background relations (2.9)–(2.10) have been used.

Using Eq. (4.7), the constraint of Eq. (4.9) can be expressed as

\[ \chi = -\frac{6}{\varphi'}(\psi' + 2\mathcal{H}\psi), \quad 4.11 \]
which implies, if inserted into Eq. (4.10), that

$$\psi'' + [3\mathcal{H} - 2\frac{\phi''}{\phi'}]\psi' + [4\mathcal{H}' - 4\mathcal{H}\frac{\phi''}{\phi'}]\psi - \partial_\alpha \partial^\alpha \psi = 0. \quad (4.12)$$

Since, in the longitudinal gauge, $\Psi = \psi$ Eq. (4.12) holds also for $\Psi$.

Defining the rescaled variable

$$\Phi = a^{3/2} \frac{\phi'}{\psi}, \quad (4.13)$$

from Eq. (4.12) the following equation can be obtained (see Appendix B):

$$\Phi'' - \partial_\alpha \partial^\alpha \Phi - z \left( \frac{1}{z} \right)'' \Phi = 0, \quad (4.14)$$

where

$$z = a^{3/2} \frac{\phi'}{\mathcal{H}}, \quad (4.15)$$

is an interesting function which controls the localization properties of the scalar zero modes as it will be discussed in Section V. An equation analogous to (4.12) holds also for the other Bardeen potential, namely $\Xi$, thanks to the constraint of Eq. (4.7).

Inserting Eq. (4.7) and recalling Eqs. (2.9)–(2.10) also the equation for $\chi$ can be expressed in a more tractable form, namely:

$$\partial_\alpha \partial^\alpha \chi - \chi'' - 3\mathcal{H}\chi + \frac{\partial^2 V}{\partial \phi^2} a^2 \chi - 6\phi'\psi' - 4\psi \frac{\partial V}{\partial \phi} a^2 = 0. \quad (4.16)$$

Eq. (4.16) can be reduced to a Schrödinger-like form in terms of the following variable:

$$\mathcal{G} = a^{3/2} \chi - z\psi, \quad z = a^{3/2} \frac{\phi'}{\mathcal{H}} \quad (4.17)$$

The equation obeyed by $\mathcal{G}$ is simply:

$$\mathcal{G}'' - \partial_\alpha \partial^\alpha \mathcal{G} - \frac{z''}{z} \mathcal{G} = 0. \quad (4.18)$$

Eq. (4.18) can be obtained by expressing $\psi$ as

$$\psi = \frac{a^{3/2}}{z} \chi - \frac{\mathcal{G}}{z}. \quad (4.19)$$
From this last equation, the derivatives of $\psi$ with respect to $w$

$$
\psi'(\chi, \chi', G, G'), \quad \psi''(\chi, \chi'', G, G', G''),
$$

(4.20)
can be inserted back into Eq. (4.10). Using now simultaneously the constraints (4.9) and Eq. (4.10), Eq. (4.18) is obtained after some algebra involving the repeated use of Eqs. (2.9)–(2.10).

The canonical variable $G$ is gauge-invariant. Recall in fact that under infinitesimal gauge transformations

$$
\tilde{\chi} = \chi - \epsilon_w \varphi',
$$

$$
\tilde{\psi} = \psi - \epsilon_w \mathcal{H},
$$

(4.21)
so that $\tilde{G} = G$. Recall now that the gauge-invariant fluctuations connected with $\psi$ and $\chi$ (i.e. $\Psi$ and $X$) are given, respectively, by Eqs. (3.21) and (3.24). Since it has been shown that $G$ is gauge-invariant, it is possible to express it as a direct combination of gauge-invariant quantities

$$
G = \frac{a^{3/2} X}{3} - 2\Psi.
$$

(4.22)

**B. Perturbations in the off-diagonal gauge**

The evolution equations for the coupled system given in Eqs. (3.2)–(3.3) will now be studied in the off-diagonal gauge. The interest of this exercise is not only academic since it represents an important cross-check of the validity of the formalism. In fact, by definition of gauge-invariant variable, the evolution of the Bardeen potentials should have the same form in any gauge.

The fluctuations of the Christoffel connections and of the Ricci tensors are reported in the Appendix A. Using Eqs. (A.6) and (A.7) into Eqs. (3.2)–(3.3) the explicit form of the perturbed equations of motion in the off-diagonal gauge reads:
\[ h''_{\mu \nu} + 3H h'_{\mu \nu} - \partial_\beta \partial^\beta h_{\mu \nu} = 0, \tag{4.23} \]
\[ \mathcal{H} \xi' + 2(\mathcal{H}' + 3\mathcal{H}^2)\xi - \mathcal{H} \partial_\beta \partial^\beta C + \frac{1}{3} \frac{\partial V}{\partial \varphi} a^2 \chi = 0, \tag{4.24} \]
\[ \partial_\beta \partial^\beta D_\mu = 0, \tag{4.25} \]
\[ -\partial_\alpha \partial^\alpha \xi - 4\mathcal{H} \xi' + \mathcal{H} \partial_\alpha \partial^\alpha C + (\partial_\alpha \partial^\alpha C)' - \varphi' \chi' - \frac{1}{3} \frac{\partial V}{\partial \varphi} a^2 \chi + \frac{2}{3} Va^2 \xi = 0, \tag{4.26} \]
\[ \partial_\alpha \partial^\alpha \chi - \chi'' - 3H \chi' + \frac{\partial^2 V}{\partial \varphi^2} a^2 \chi + \varphi'(\partial_\alpha \partial^\alpha C - \xi') - 2\xi(\varphi'' + 3\mathcal{H} \varphi') = 0. \tag{4.27} \]

Eqs. (4.23)–(4.26) are derived from Eq. (3.2), whereas, Eq. (4.27) is derived from Eq. (3.3) and it is the perturbed form of the scalar field equation. The constraints connecting the first derivatives of the fluctuations will be:
\[ \xi - 3\mathcal{H} C - C' = 0, \tag{4.28} \]
\[ D'_\mu + 3\mathcal{H} D_\mu = 0, \tag{4.29} \]
\[ 6\mathcal{H} \xi + \chi \varphi' = 0. \tag{4.30} \]

Summing up Eqs. (4.24) and (4.26) the following equation is obtained:
\[ \mathcal{H} \xi' - \partial_\alpha \partial^\alpha \xi - 4\mathcal{H} \xi' + (\partial_\alpha \partial^\alpha C)' - \varphi' \chi' = 0. \tag{4.31} \]

Using Eq. (4.28), the constraint given in Eq. (4.30) can be expressed as
\[ \chi = -\frac{6\mathcal{H}}{\varphi'} (3\mathcal{H} C + C'). \tag{4.32} \]

Inserting Eq. (4.32) into Eq. (4.31)
\[ C'' + 3\mathcal{H} C' + 3\mathcal{H}' C + \partial_\alpha \partial^\alpha C + \frac{\varphi' \chi'}{3\mathcal{H}} = 0. \tag{4.33} \]

Finally, using the constraint provided by Eq. (4.32) in order to eliminate \( \chi \) the following decoupled equation can be obtained:
\[ C'' + \left[ 3\mathcal{H} + \frac{2\mathcal{H}'}{\mathcal{H}} - 2\frac{\varphi''}{\varphi'} \right] C' + 3\left[ 3\mathcal{H}' - 2\mathcal{H} \frac{\varphi''}{\varphi'} \right] C - \partial_\alpha \partial^\alpha C = 0. \tag{4.34} \]

In order to check for the consistency of the formalism the evolution equation for the gauge-invariant variables can be obtained. In the off-diagonal gauge \( \Psi = \mathcal{H} C \). From Eq.
the equation for $\Psi$ is derived after the repeated use of the background relations given in Eqs. (2.9) and (2.10):

$$
\Psi'' + [3\mathcal{H} - 2\frac{\varphi''}{\varphi'}] \Psi' + [4\mathcal{H}' - 4\mathcal{H} \frac{\varphi''}{\varphi'}] \Psi - \partial_\alpha \partial^\alpha \Psi = 0.
$$

Notice, as it should, that this equation is exactly Eq. (4.12) obtained in the longitudinal gauge. This shows that the equations of the gauge-invariant variables are the same in any gauge provided the gauge freedom is completely fixed in each of the selected gauges.

Finally, in the off-diagonal gauge, the gauge-invariant normal mode $\mathcal{G}$ defined in Eq. (4.22) becomes

$$
\mathcal{G} = a^{3/2} \chi.
$$

Hence, gauge-invariance suggests that also in the off-diagonal gauge the equation for $\mathcal{G}$ will be given by Eq. (4.18). In fact, the constraints (4.28) and (4.30) can be used into Eq. (4.27) in order to eliminate the dependence on $\xi$ and $C$. The resulting equation for $a^{3/2} \chi$ is exactly Eq. (4.18) once we recall that, from Eq. (2.5), the double derivative of the potential with respect to $\varphi$ can be expressed as

$$
a^2 \frac{\partial^2 V}{\partial \varphi^2} = \left[ \mathcal{H} \varphi'' - 6\mathcal{H}^2 + \frac{\varphi'''}{\varphi'} + 3\mathcal{H}' \right].
$$

C. Canonical normal modes of the action

The investigation of the canonical normal modes of the action (2.1) is rather long and, here, only the results will be reported. The same relations obtained from the equations of motion could be obtained by perturbing the action (2.1) to second order in the amplitude of the fluctuations. In discussing the fluctuations of (2.1) it is better to perturb the Einstein-Hilbert term in the form

$$
G^{AB} \left( \Gamma^D_{AC} \Gamma^C_{BD} - \Gamma^C_{AB} \Gamma^D_{CD} \right)
$$

(4.38)
to second order in the amplitude of the fluctuations. This form of the gravity action automatically eliminates the total derivatives.

The normal modes for tensors turn out to be for each of the polarizations

$$\delta^{(2)} S_T = \int d^4 x dw \frac{1}{2} \left[ \eta^\alpha\beta \partial_\alpha \mu \partial_\beta \mu - \mu^4 - \frac{(a^{3/2})''}{a^{3/2}} \mu^2 \right]$$  \hspace{1cm} (4.39)

where $\mu = a^{3/2} h$ and $\eta_{\alpha\beta}$ is the Minkowski metric.

Fixing the longitudinal gauge the normal modes for the vector action will be

$$\delta^{(2)} S_V = \int d^4 x dw \frac{1}{2} \left[ \eta^\alpha\beta \partial_\alpha D_\mu \partial_\beta D_\mu \right]$$  \hspace{1cm} (4.40)

where $D_\mu = a^{3/2} D_\mu$. The normal modes for the scalar action will be

$$\delta^{(2)} S_T = \int d^4 x dw \frac{1}{2} \left[ \eta^\alpha\beta \partial_\alpha G \partial_\beta G - G_2 \right]$$  \hspace{1cm} (4.41)

where $G$ is the variable previously defined and $z = a^{3/2} \phi'/\mathcal{H}$. In order to get these normal modes the longitudinal gauge has been selected. The calculation can be also done without assuming any specific gauge (as discussed in [17], in a different context). The advantage of working with the full (i.e. non gauge-fixed) action is that the constraints will naturally appear.

Suppose, for instance, that the action of Eq. (2.1) is perturbed to second order in the amplitude of the vector fluctuations without fixing the specific gauge. Hence, in the perturbed action, after integration by parts and up to total derivative terms, a term like

$$\int a^3 d^4 x \; dw \; \partial_{(\mu} f_{\nu)} \left[ \partial^{(\mu} D^{\nu)}' + 3 \mathcal{H} \partial^{(\mu} D^{\nu)} \right]$$  \hspace{1cm} (4.42)

will appear on top of the kinetic term for $D_\mu$. By taking the derivative of the perturbed action with respect to $\partial_\mu f_\nu$ the constraint of Eq. (4.8) can be obtained.

V. ZERO MODES OF THE FLUCTUATIONS, LOCALIZATION AND NORMALIZABILITY
A. Localization of the tensor modes

The evolution equation of each tensor polarization can be written as

\[ \mu'' - \partial_\alpha \partial^\alpha \mu - \frac{s''}{s} \mu = 0. \quad (5.1) \]

where \( sh = \mu \) and with \( s = a^{3/2} \). The zero mode of this equation corresponds to

\[ \mu_0(w) = Ka^{3/2}, \quad (5.2) \]

where \( K \) is a constant which should be fixed by normalization. The normalization condition reads

\[ \int |\mu_0(w)|^2 dw = K^2 \int_{-\infty}^{+\infty} a^3(w) dw = 1. \quad (5.3) \]

The zero mode is then normalized provided \( a^3(w) \) goes to zero to infinity faster than \( 1/w \)

\[ \lim_{|w| \to \infty} a^3(w) = \mathcal{O}\left(\frac{1}{w^\lambda}\right), \quad \lambda > 1, \quad (5.4) \]

and provided \( a(w) \) is regular in all the domain of definition of \( w \). Suppose, for instance, that \( a(w) = (b^2 w^2 + 1)^{-1/2} \) as obtained, in a particular case, in Eq. (2.11). In this case \( K = \sqrt{2b} \).

The massive modes of Eq. (5.1) obey a Schrödinger-like equation which can be written as

\[ -\frac{d^2 \mu_m}{dw^2} + V(w) \mu_m = m^2 \mu_m, \quad (5.5) \]

where the effective “potential” term is

\[ V(w) = \frac{s''}{s} \equiv \mathcal{J}^2 - \mathcal{J}', \quad \mathcal{J} = -\frac{s'}{s} \quad (5.6) \]

In Eq. (5.6) \( \mathcal{J} \) is the superpotential usually defined in the context of supersymmetric quantum mechanics [24]. Using supersymmetric quantum mechanics Eq. (5.3) can be written as
\[ A^\dagger A \mu_m = m^2 \mu_m, \]  
\begin{equation}
\tag{5.7}
\end{equation}

where

\[ A^\dagger = \left( -\frac{d}{dw} + J \right), \quad A = \left( \frac{d}{dw} + J \right). \]  
\begin{equation}
\tag{5.8}
\end{equation}

The fact that the Schrödinger equation can be written in this form excludes the presence of tachyonic modes with \( m^2 < 0 \).

**B. Localization of the vector zero mode**

The equation describing the vector modes are of the type

\[ \partial_\alpha \partial^\alpha D = 0, \quad D' + 3\mathcal{H}D = 0, \]  
\begin{equation}
\tag{5.9}
\end{equation}

for each polarization of the gauge-invariant vectors. Since the equations are first order, the normal modes can be read-off from the perturbed action of Eq. (4.40). They are \( D = a^{3/2} D \).

Since, from Eq. (5.9) \( D_0 = K/a^3 \) the normalization relation for the zero mode will be

\[ \int dw |D_0(w)|^2 = K^2 \int_{-\infty}^{+\infty} \frac{dw}{a^3(w)} = 1. \]  
\begin{equation}
\tag{5.10}
\end{equation}

This relation would imply that

\[ \lim_{|w| \to \infty} \frac{1}{a^3(w)} = \mathcal{O}\left(\frac{1}{w^\lambda}\right), \quad \lambda > 1. \]  
\begin{equation}
\tag{5.11}
\end{equation}

Eq. (5.11) cannot be true together with Eq. (5.4). Eq. (5.4) has to be correct for independent reasons: it guarantees that the four-dimensional Planck mass is finite since, in our parametrization\(^4\)

\[ M_P^2 \sim M_5^3 \int a^3(w) dw. \]  
\begin{equation}
\tag{5.12}
\end{equation}

Hence, the zero mode of the vector fluctuations is not localized on the brane if the four-dimensional Planck mass is finite.

---

\(^4\)Notice that we always worked in natural five-dimensional units where \( 2\kappa = 16\pi/M_5^3 = 1 \).
C. Localization of the scalar zero modes

The evolution equation for the Bardeen potentials can be written as

$$\Phi'' - \partial_\alpha \partial^\alpha \Phi - z \left(\frac{1}{z}\right)'' \Phi = 0.$$  \hspace{1cm} (5.13)

The same equation holds also for the rescaled $\Xi$. From Eq. (5.13) the zero mode is given by $\Phi_0 = K/z(w)$ where $K$ is a constant. The normalization condition is given by,

$$\int |\Phi_0(w)^2| dw = K^2 \int \frac{dw}{z^2} \equiv K^2 \int_{-\infty}^{+\infty} \frac{\mathcal{H}^2}{a^3 \varphi'^2} = 1, \hspace{1cm} (5.14)$$

which implies that the scalar zero mode is normalizable provided $1/z^2(w)$ goes to zero for $|w| \to \infty$ faster than $1/w$, i.e.

$$\lim_{|w|\to\infty} \frac{\mathcal{H}^2}{a^3 \varphi'^2} = \mathcal{O}\left(\frac{1}{w^\lambda}\right), \hspace{0.5cm} \lambda > 1, \hspace{1cm} (5.15)$$

and provided $z(w)$ is everywhere regular. In the background given by Eqs. (2.11)–(2.12) the scalar zero mode is not normalizable. In fact the normalizability condition requires that

$$\frac{K^2}{6} \int_{-\infty}^{+\infty} dw b^2 w^2 (1 + b^2 w^2)^{3/2}, \hspace{1cm} (5.16)$$

is finite. This does not happen since the integrand is not convergent for $|w| \to \infty$. Since Eq. (5.13) is second order there will be also a second (linearly independent) solution to Eq. (5.13) for the zero mode, namely $\Phi_0(w) \sim z^{-1}(w) \int^w z^2(w')dw'$. If the first solution (going as $z^{-1}(w)$) is not normalizable, also the second one will not be normalizable [23]. For instance, in the case of the solution (2.11)–(2.13) the second zero mode blows up, for large $|w|$ as $|w|^{5/2}$.

The massive modes of $\Phi$ will obey a Schrödinger-like equation

$$-\frac{d^2\Phi_m}{dw^2} + V_\Phi(w)\Phi_m = m^2\Phi_m, \hspace{1cm} (5.17)$$

whose associated effective potentials and superpotentials are

$$V_\Phi(w) = \mathcal{J}^2 - \mathcal{J}', \hspace{0.5cm} J_\Phi(w) = \frac{z'}{z}. \hspace{1cm} (5.18)$$
The equation for the canonical normal mode $\mathcal{G}$ reads

$$
\mathcal{G}'' - \partial_\alpha \partial^\alpha \mathcal{G} - \frac{z''}{z} \mathcal{G} = 0.
$$

(5.19)

which implies that the zero mode should go as $\mathcal{G}(w) = K z(w)$. This implies that the normalization condition is

$$
K^2 \int dw |z(w)|^2 = K^2 \int_{-\infty}^{+\infty} \frac{a^3 \phi'^2}{\mathcal{H}^2} = 1
$$

(5.20)

This normalization integral implies that $z(w)$ has to be always finite for any $w$ and that it should go to zero, for large $|w|$, faster than $1/w$. Since the integrand appearing in Eq. (5.20) is exactly the inverse of the one appearing in Eq. (5.14), it is clear that they cannot be simultaneously finite for a given physical model. However, they can be simultaneously divergent. An example is, again the model discussed in Eqs. (2.11)–(2.12). In this case the integrand appearing in Eq. (5.20) is, up to numerical factors,

$$
\frac{1}{b^2 w^2 (b^2 w^2 + 1)^2},
$$

(5.21)

which is not convergent for $|w| \to 0$.

The massive modes related to $\mathcal{G}$ can be, again, described by a Schrödinger-like equation of the type of the one reported in Eq. (5.17). Notice that there is an interesting “duality” relation among the effective potentials. The effective potential for the equation for $\mathcal{G}$ is simply

$$
V_\mathcal{G} = \frac{z''}{z}.
$$

(5.22)

For $z(w) \to 1/z(w)$,

$$
V_\mathcal{G}(w) \to V_\Phi(w) = z(z^{-1})''.
$$

(5.23)

In terms of the superpotential this symmetry implies that for $z \to 1/z$

$$
V_\Phi(w) = \mathcal{J} - \mathcal{J}' \to V_\mathcal{G}(w) = \mathcal{J}^2 + \mathcal{J}'.
$$

(5.24)
The two Schrödinger-like equations obeyed by the massive modes of $G$ and $\Phi$ can then be written, in terms of the operators defined in Eq. (5.8) as

$$\mathcal{A}^\dagger \mathcal{A} \Phi_m = m^2 \Phi_m,$$

$$\mathcal{A} \mathcal{A}^\dagger G_m = m^2 G_m,$$  \hfill (5.25)

with $J = z'/z$. In the context of supersymmetric quantum mechanics these two potentials $V_\Phi$ and $V_G$ are supersymmetric partners potentials [24]. This property implies that the spectra are related.

VI. CONCLUDING REMARKS

If physical walls in five dimensions are described using Eqs. (2.5)–(2.7), then in order to have localized zero modes of the metric the following integrals should be simultaneously convergent:

$$I_{\text{tens}} = \int_{-\infty}^{+\infty} a^3(w)dw,$$  \hfill (6.1)

$$I_{\text{vec}} = \int_{-\infty}^{+\infty} \frac{dw}{a^3(w)},$$  \hfill (6.2)

$$I_\Phi = \int_{-\infty}^{+\infty} \frac{dw}{z^2(w)},$$  \hfill (6.3)

$$I_G = \int_{-\infty}^{+\infty} z^2(w)dw,$$  \hfill (6.4)

where

$$z(w) = \frac{a^{3/2} \varphi'}{\mathcal{H}},$$  \hfill (6.5)

These equations do not assume any specific background solution but only the form (2.4) of the metric together with Eqs. (2.5)–(2.7).

The integrals of Eqs. (6.1)–(6.4) cannot be all simultaneously convergent. Since the four dimensional Planck mass should be finite, the integral in Eq. (6.1), i.e. $I_{\text{tens}}$, should converge. This implies that tensor modes are localized on the brane and that they lead to
ordinary four-dimensional gravity. The convergence of the integral in Eq. (6.2), i.e. \( I_{\text{vec}} \), would insure the normalizability of the vector zero mode. This is not possible as long as the four dimensional Planck mass is finite.

Using the explicit form of \( z(w) \), and recalling that, from Eq. (2.10), \( \varphi'^2 = 6(\mathcal{H}^2 - \mathcal{H}') \), Eq. (6.3) can be written as

\[
I_{\Phi} = \frac{1}{6} \int_{-\infty}^{+\infty} \frac{\mathcal{H}^2}{a^3(\mathcal{H}^2 - \mathcal{H}')} dw. \tag{6.6}
\]

From Eq. (6.1) \( a(w) \), should decay at infinity. If \( a(w) \sim w^{-\alpha} \), for \( |w| \to \infty \), \( \mathcal{H}^2/(\mathcal{H}^2 - \mathcal{H}') \) goes to a constant and the integrand blows up as \( w^{3\alpha} \). If \( a(w) \) decays exponentially the integrand appearing in (6.6) blows up exponentially.

Eq. (6.4) can be written as

\[
I_G = \int_{-\infty}^{+\infty} \frac{a^3 \varphi'^2}{\mathcal{H}^2} dw \equiv 6 \int_{-\infty}^{+\infty} \left\{ a^3 \frac{\mathcal{H}^2 - \mathcal{H}'}{\mathcal{H}^2} \right\}. \tag{6.7}
\]

If \( I_{\text{tens}} \) is convergent at infinity, then \( I_G \) will be convergent in the same limit but this does not imply the localization as discussed in Section V.

The integral relations obtained in the present investigation are of general relevance for warped scalar-tensor backgrounds and they can be exploited in order to check for the normalizability of the scalar and vector zero modes of the geometry. Depending upon the specific model, the range of variation of \( w \) can be changed. However, the localization of the zero modes will always be controlled by the functions derived in the present analysis.

In this paper, the Bardeen formalism has been extended to the case of warped metrics and the localization of scalar and vector modes of the geometry has been investigated. All the results have been expressed in terms of gauge-invariant, i.e. coordinate independent, variables. The coupled system of gauge-invariant perturbations has been reduced to a set of Schrödinger-like equations whose effective potential determines the localization properties of the corresponding zero modes. Supersymmetric quantum mechanics has been used in order to show the absence of tachyonic states in the spectrum and duality relations among the effective potentials have been derived. Possible extensions of the present results to related problems are in progress.
The results of the present investigation can be generalized to higher dimensions. Of particular interest, in this framework, are six-dimensional models which are relevant both in the context of compact \cite{25,27} and non compact \cite{28,33} extra-dimensions. Six-dimensional (non-singular) brane solutions have been recently derived in the context of the Abelian-Higgs model \cite{16} and their fluctuations can be investigated with suitable extensions of the techniques discussed in the present analysis.

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APPENDIX A: FLUCTUATIONS IN THE LONGITUDINAL AND OFF-DIAGONAL GAUGES

In the longitudinal gauge the fluctuation of the metric is:

\[
\delta G_{AB} = a^2(w) \begin{pmatrix} 2h_{\mu\nu} + 2\eta_{\mu\nu}\psi & D_{\mu} \\ D_{\mu} & 2\xi \end{pmatrix}. \tag{A.1}
\]

The Christoffel connections perturbed to first order in \( \delta G_{AB} \) are

\[
\begin{align*}
\delta^{(1)} \Gamma^{w}_{ww} & = -\xi', \\
\delta^{(1)} \Gamma^{w}_{\mu w} & = -\partial_{\mu} \xi + \mathcal{H} D_{\mu}, \\
\delta^{(1)} \Gamma^{w}_{\mu v} & = h'_{\mu v} + 2\mathcal{H} h_{\mu v} + \eta_{\mu v} [\psi' + 2\mathcal{H} \psi + 2\mathcal{H} \xi] - \frac{1}{2} (\partial_{\mu} D_{\nu} + \partial_{\nu} D_{\mu}), \\
\delta^{(1)} \Gamma^{w}_{\alpha \beta} & = -\partial^{\mu} h_{\alpha \beta} + \partial_{\alpha} h^{\mu}_{\beta} + \partial_{\beta} h^{\mu}_{\alpha} + \delta^{\mu}_{\alpha} \partial_{\alpha} \psi + \delta^{\mu}_{\beta} \partial_{\beta} \psi + \eta_{\alpha \beta} [-\mathcal{H} D^{\mu} - \partial^{\mu} \psi], \\
\delta^{(1)} \Gamma^{w}_{ww} & = -\partial^{\mu} \xi + D^{\mu'} + \mathcal{H} D^{\mu}, \\
\delta^{(1)} \Gamma^{w}_{w\alpha} & = h^{\mu'}_{\alpha} + \delta^{\mu'}_{\alpha} \psi' + \frac{1}{2} (\partial_{\alpha} D^{\mu} - \partial_{\mu} D_{\alpha}). \tag{A.2}
\end{align*}
\]

Using Eqs. (A.2) the Ricci tensors perturbed to first order in the metric fluctuations can be computed with some trivial algebra involving the use of the Palatini identities:

\[
\begin{align*}
\delta^{(1)} R_{\mu \nu} & = h^{\nu}_{\mu} + 3\mathcal{H} h'_{\mu \nu} + h_{\mu \nu} (2\mathcal{H}' + 6\mathcal{H}^2) - \partial_{\nu} \partial^{\alpha} h_{\mu \nu} + \partial_{\mu} \partial_{\nu} [\xi - 2\psi] \\
& \quad + \eta_{\mu \nu} [\psi'' + 7\mathcal{H} \psi' + (2\mathcal{H}' + 6\mathcal{H}^2)(\xi + \psi) - \partial_{\alpha} \partial^{\alpha} \psi + \mathcal{H} \xi'] \\
& \quad - \frac{1}{2} [(\partial_{\mu} D_{\nu} + \partial_{\nu} D_{\mu})' + 3\mathcal{H} (\partial_{\mu} D_{\nu} + \partial_{\nu} D_{\mu})], \\
\delta^{(1)} R_{\mu w} & = \partial_{\mu} [-3\mathcal{H} \xi - 3\psi'] - \frac{1}{2} \partial_{\alpha} \partial^{\alpha} D_{\mu} + D_{\mu} (\mathcal{H}' + 3\mathcal{H}^2), \\
\delta^{(1)} R_{ww} & = -\partial_{\alpha} \partial^{\alpha} \xi - 4(\psi'' + \mathcal{H} \psi') - 4\mathcal{H} \xi', \tag{A.3}
\end{align*}
\]

where the background value of the Christoffel connections

\[
\Gamma^{A}_{Bw} = \mathcal{H} \delta^{A}_{B}, \quad \Gamma^{w}_{\alpha \beta} = \mathcal{H} \eta_{\alpha \beta}. \tag{A.4}
\]

has been used.

In the off-diagonal gauge the perturbed metric takes the form

25
\[ \delta G_{AB} = a^2(w) \begin{pmatrix} 2h_{\mu\nu} & D_{\mu} + \partial_{\mu}C \\ D_{\mu} + \partial_{\mu}C & 2\xi \end{pmatrix}. \] (A.5)

Following the same steps outlined in the discussion of the longitudinal gauge, the fluctuation of the Christoffel connection can be easily obtained

\[
\begin{align*}
\delta^{(1)} \Gamma^{w}_{ww} &= -\xi', \\
\delta^{(1)} \Gamma^{w}_{\mu\nu} &= h'_{\mu\nu} + 2\mathcal{H}h_{\mu\nu} - \partial_{\mu}\partial_{\nu}C - \frac{1}{2}\left( \partial_{\mu}D_{\nu} + \partial_{\nu}D_{\mu} \right) + 2\eta_{\mu\nu}\mathcal{H}\xi, \\
\delta^{(1)} \Gamma^{w}_{\mu w} &= -\partial_{\mu}\xi + \mathcal{H}(\partial_{\mu}C + D_{\mu}), \\
\delta^{(1)} \Gamma^{\mu}_{\alpha\beta} &= \left( -\partial^{\mu}h_{\alpha\beta} + \partial_{\beta}h^{\mu}_{\alpha} + \partial_{\alpha}h^{\mu}_{\beta} \right) - \mathcal{H}\eta_{\alpha\beta}(\partial^{\mu}C + D^{\mu}), \\
\delta^{(1)} \Gamma^{\mu}_{ww} &= -\partial^{\mu}\xi + \mathcal{H}(\partial^{\mu}C + D^{\mu}) + (\partial^{\mu}C + D^{\mu})', \\
\delta^{(1)} \Gamma^{\mu}_{w\alpha} &= h^{\mu}_{\alpha} + \frac{1}{2}(\partial_{\alpha}D^{\mu} - \partial^{\mu}D_{\alpha}). \quad (A.6)
\end{align*}
\]

From Eq. (A.6) the fluctuations of the Ricci tensors are, after some algebra,

\[
\begin{align*}
\delta^{(1)} R^{\mu\nu} &= [h''_{\mu\nu} + 3\mathcal{H}h'_{\mu\nu} + h_{\mu\nu}(2\mathcal{H}' + 6\mathcal{H}^2)\partial_{\beta}\partial^{\beta}h_{\mu\nu}] + \eta_{\mu\nu}[\mathcal{H}\xi' + 2(\mathcal{H}' + 3\mathcal{H}^2)\xi - \mathcal{H}\partial_{\beta}\partial^{\beta}C] \\
&\quad + \partial_{\beta}\partial_{\nu}[\xi - 3\mathcal{H}C - C'] - \frac{1}{2}\left[ (\partial_{\mu}D_{\nu} + \partial_{\nu}D_{\mu})' + 3\mathcal{H}(\partial_{\nu}D_{\mu} + \partial_{\mu}D_{\nu}) \right], \\
\delta^{(1)} R^{ww} &= -\partial_{\alpha}\partial^{\alpha}\xi - 4\mathcal{H}\xi' + \mathcal{H}\partial_{\alpha}\partial^{\alpha}C + (\partial_{\alpha}\partial^{\alpha}C)', \\
\delta^{(1)} R^{\mu w} &= -\frac{1}{2}\partial_{\alpha}\partial^{\alpha}D_{\mu} + (\mathcal{H}' + 3\mathcal{H}^2)D_{\mu} + \partial_{\mu}[-3\mathcal{H}\xi + (\mathcal{H}' + 3\mathcal{H}^2)C]. \quad (A.7)
\end{align*}
\]

APPENDIX B: DECOUPLED EQUATION FOR THE BARDEEN POTENTIAL

Often in the course of this investigation the background equations have been used in order to reduce complicated expressions to simple ratios of background functions. In order to give an example consider the equation for the Bardeen potential obtained in Eq. (4.10):

\[
\Psi'' + \left( 3\mathcal{H} - 2\frac{\mathcal{H}'}{\mathcal{\varphi}'} \right)\Psi' + \left( 4\mathcal{H}' - 4\mathcal{H}\frac{\mathcal{H}'}{\mathcal{\varphi}'} \right)\psi - \partial_{\alpha}\partial^{\alpha}\Psi = 0. \quad (B.1)
\]

To eliminate the first derivative term is trivial since it suffice to redefine \( \Psi \) as

\[
\Phi = \frac{a^{3/2}}{\mathcal{\varphi}'}\Psi. \quad (B.2)
\]
Then, the equation for $\Phi$ will be

$$\Phi'' - \partial_{\alpha} \partial^{\alpha} \Phi + \Phi \left[ \frac{5}{2} \mathcal{H}' + \frac{\phi''}{\phi'} - \frac{9}{4} \mathcal{H}^2 - \mathcal{H} \frac{\phi''}{\phi'} - 2 \left( \frac{\phi''}{\phi'} \right)^2 \right] = 0. \quad (B.3)$$

It will now be shown that the quantity in squared bracket appearing in Eq. (B.3) equals exactly $-z(z^{-1})''$ where $z = a^{3/2} \phi'/\mathcal{H}$. Notice, in fact, that

$$-z(z^{-1})'' = -\frac{9}{4} \mathcal{H}^2 + \frac{9}{2} \mathcal{H}' + \frac{\phi''''}{\phi'} - 2 \left( \frac{\phi''}{\phi'} \right)^2 - 2 \frac{\mathcal{H}'}{\mathcal{H}} \frac{\phi''}{\phi'} - 3 \frac{\mathcal{H} \phi''}{\phi'} - \frac{\mathcal{H}''}{\mathcal{H}}. \quad (B.4)$$

Recalling now, from Eqs. (2.9)–(2.10), that $\phi'^2 = 6(\mathcal{H}^2 - \mathcal{H}')$ we can obtain (taking the derivative with respect to $w$):

$$\frac{\mathcal{H}''}{\mathcal{H}} = 2 \mathcal{H}' - \frac{\phi'' \phi'}{3 \mathcal{H}}. \quad (B.5)$$

Inserting Eq. (B.5) into Eq. (B.4) we get

$$-z(z^{-1})'' = -\frac{9}{4} \mathcal{H}^2 + \frac{\phi''''}{\phi'} + \frac{5}{2} \mathcal{H}' - 2 \left( \frac{\phi''}{\phi'} \right)^2 + \frac{\phi''}{\phi'} \left( 2 \frac{\mathcal{H}'}{\mathcal{H}} - 3 \mathcal{H} + \frac{\mathcal{H}^2}{3 \mathcal{H}} \right) \quad (B.6)$$

Using $\phi'^2 = 6(\mathcal{H}^2 - \mathcal{H}')$ in the last bracket of Eq. (B.6), we find that $-z(z^{-1})''$ is exactly what appears in the squared bracket of Eq. (B.3). Similar algebraic manipulations are involved in the derivation of all the other decoupled equations discussed in the bulk of the paper.
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