Stone Dualities from Opfibrations

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Abstract. Stone dualities are dual equivalences between certain categories of algebras and those of topological spaces. A Stone duality is often derived from a dual adjunction between such categories by cutting down unnecessary objects. This dual adjunction is called the fundamental adjunction of the duality, but building it often requires concrete topological arguments. The aim of this paper is to construct fundamental adjunctions generically using (co)fibered category theory. This paper defines an abstract notion of formal spaces (including ordinary topological spaces as the leading example), and gives a construction of a fundamental adjunction between the category of algebras and the category of corresponding formal spaces.

1 Introduction

Dual equivalences between categories of spaces and those of algebras are ubiquitous in mathematics - following the famous book by Johnstone \cite{1}, they are collectively called Stone dualities after the Stone Representation Theorem of Boolean algebras. Technically, they often arise as the restriction of dual adjunctions called fundamental adjunctions. For example, the Stone duality between sober topological spaces and spatial frames is obtained by cutting down the fundamental adjunction between the category of topological spaces and that of frames, which is the heart of pointless topology \cite{2}. Categorical settings to capture various fundamental adjunctions of Stone dualities has been studied in \cite{3-8}. The basic idea of these settings is to formulate fundamental adjunctions as dual adjunctions that are representable through functors to \textbf{Set}. The objects representing adjoint functors are called dualizing object \cite{6}.

In this paper, we give a new construction of fundamental adjunctions by \textit{(Grothendieck) opfibrations}\footnote{Grothendieck originally called it cofibred categories, but here we use the word opfibration to avoid confusion with cofibration in homotopy theory.}. Roughly speaking, our construction takes a category of algebras equipped with an abstract notion of subalgebra, then derives...
both the category of spaces and a fundamental adjunction between them. Despite its abstract nature, the constructed fundamental adjunctions reflect several properties seen in concrete ones. One such property is that, in a certain setting, our construction yields fundamental adjunctions that enjoy representability. Another is the full-faithfulness of the algebra-to-space construction. It is characterized in terms of the mono-ness of the unit arrow $X \to \Omega^{A(\cdot, \Omega)}$, where $A$ is a category of algebras and $\Omega$ is the dualizing object. This generalizes the full-faithfulness argument of the constructions of topological spaces from Boolean algebras [1].

![Fig. 1. Sketch of the construction of fundamental adjunction; its input data are on the left and the construction process is on the right](image)

Above we sketch our construction of fundamental adjunctions. It takes two inputs $(c, i, d)$ and $(L, R)$ depicted on the left of Fig. 1. The first is a tuple $(c, i, d)$ called opfibered comprehension (Definition 1). It is an opfibration $c : M \to A$ with further adjunctions $c \dashv i \dashv d$ satisfying certain properties. The category $A$ plays the role of an algebraic category, and is equipped with a notion of subalgebra generalized by an opfibration $c : M \to A$. A typical example of $c$ is the subobject opfibration $\text{cod} : \text{Sub}(A) \to A$. The second input to our construction is an adjunction $L \dashv R : B \to A$. This is often set to the hom-power adjunction $(A(\cdot, \Omega))^{\text{op}} \dashv \Omega^{(\cdot)} : \text{Set}^{\text{op}} \to A$.

Our construction proceeds by taking the pullback of $c$ along $R$ (right of Fig. 1). We call $(R^*M)^{\text{op}}$ the category of formal spaces. The horizontal leg $\hat{R}$ of the pullback has a left adjoint by Hermida’s adjoint lift theorem [9]. The fundamental adjunction then appears as the composite adjunction between $A$ and $R^*M$.

We will illustrate several examples of fundamental adjunctions arising from our construction. The one between the category Frm of frames and that of topological spaces, which is the standard example of Stone duality, is an immediate instance: take $c$ to be the subobject opfibration $c : \text{Sub}(\text{Frm}) \to \text{Frm}$ and $R$ to be the power functor $2^{(-)} : \text{Set}^{\text{op}} \to \text{Frm}$ with the Sierpinski frame $2$.

Related Work. Categorical formulations of fundamental adjunctions via dualizing objects were studied by many authors [3–8]. For the formulation of representable dual adjunctions, see e.g. [5], where Dimov and Tholen also showed
a general condition to obtain fundamental adjunctions using the lift conditions. In [8], Maruyama improved the lift conditions by (1) breaking the symmetry of Dimov and Tholen’s framework, and (2) imposing different conditions on the algebraic and spatial categories.

Categories of Chu spaces [10] are self-dual, and can accommodate various dualities in them. Pratt [11] demonstrates that the self-dual category \( \text{Chu}(\text{Set}, 2) \) can accommodate (1) sets and complete atomic Boolean algebras, (2) Stone spaces and Boolean algebras, and (3) sober spaces and spatial frames. We relate the category of Chu spaces and that of formal spaces in Sect. 4.1. However, our construction does not explain the self-duality.

The theory of natural dualities [12] aims to go roughly the same way as us, to make a “category of spaces” from a given category of algebras. The scope of their theory is narrower than ours, while their theory brings finer results. It is a future work to find connections between their framework and ours. It seems that neither of them can derive the other.

Organization. This paper is organized as follows. In Sect. 2, we define opfibered comprehensions, which is the input of our construction. In Sect. 3, we construct the category of formal spaces from an opfibered comprehension and derive the fundamental adjunction. In Sects. 4 and 5, we list various examples of our framework. In Sect. 6, we show how to relate two fundamental adjunctions. Section 7 summarizes this work and future work.

Preliminaries. The identity functor on a category \( A \) is denoted by \( \text{Id}_A \), and the identity natural transformation on a functor \( f \) is denoted by \( \text{id}_f \). We write \( L \dashv R \colon B \to A \) (or simply \( L \dashv R \)) to mean that the functor \( R \colon B \to A \) has \( L \) as a left adjoint. Its unit and counit are denoted by \( \eta_{L \\dashv R}, \epsilon_{L \\dashv R} \), respectively.

2 Opfibered Comprehension

One of inputs to our construction of fundamental adjunction (Fig. 1) is an opfibered comprehension. 

Definition 1 (opfibered comprehension). An opfibered comprehension is defined to be a tuple \( (c \colon M \to A, i, d) \) of functors such that

1. \( c \colon M \to A \) is an opfibration,
2. \( i \colon A \to M \) is the right adjoint to \( c \) whose counit is the identity, and
3. \( d \colon M \to A \) is the right adjoint to \( i \) whose unit is the identity.

We here recall the definition of (Grothendieck) opfibration; a good reference is [13]. A functor \( c \colon M \to A \) is an opfibration if it satisfies the following cocartesian lifting property: for any \( A \)-arrow \( f : a \to a' \) and \( m \in M \) such that \( cm = a \), there is a cocartesian lifting of \( f \) with \( m \). A cocartesian lifting of \( f \) with \( m \) is the arrow written as \( f(m) : m \to f_*(m) \) satisfying \( c(f_*(m)) = a' \) and the universal property:
for any $A$-arrow $g: a' \to a''$ and $M$-arrow $h: m \to m''$ satisfying $cm'' = a''$ and $ch = g \circ f$, there exists a unique arrow $k: f_*(m) \to m''$ satisfying $ck = g$ and $k \circ f_*(m) = h$.

$$
\begin{aligned}
& M \\
& \downarrow c \\
& A \\
& \downarrow \ \\
& m \\
& \downarrow f(m) \\
& f_*(m) \\
& \downarrow k \\
& m'' \\
& h
\end{aligned}
$$

(1)

The conditions 2 and 3 imply several equalities between functors and natural transformations. The last one is proved in [14].

$$
c \circ i = \text{Id}_A = d \circ i \\
c \eta^{c-i} = \text{Id}_c, \quad \eta^{c-i}_i = \text{Id}_i = \epsilon^{i-d}_i = \text{Id}_d \quad d \eta^{c-i}_i = c \epsilon^{i-d}_i
$$

Perhaps the simplest example is the following:

**Example 1.** We write $A^\to$ for the arrow category of $A$: objects are arrows in $A$, and an arrow from $f$ to $g$ is a pair $(p, q)$ of arrows such that $p \circ f = g \circ q$. The functors $\text{cod}, \text{dom}: A^\to \to A$ respectively map an object arrow $f: a \to a'$ to $a'$ and $a$, and an arrow $(p, q)$ to $p$ and $q$. The functor $\triangle: A \to A^\to$ maps an object $a$ to the identity arrow $\text{id}_a$ on $a$ and an arrow $f: a \to a'$ to $(f, f): \text{id}_a \to \text{id}_{a'}$. It is easy to see that $(\text{cod}: A^\to \to A, \triangle, \text{dom})$ is an opfibered comprehension; the cocartesian lifting of an $A$-arrow $g: a' \to b'$ with an $A^\to$-object $f: a \to a'$ is given by $\bar{g}(f) = (g, \text{id}_a): f \to g \circ f$. We call it the arrow opfibered comprehension.

We can restrict the objects of $A^\to$ to monomorphisms if $A$ has a (strong epi, mono)-factorization system. This is our leading example of an opfibered comprehension.

**Example 2.** We write $\text{Sub}(A)$ for the full subcategory of $A^\to$ whose objects are just monomorphisms in $A$, since an equivalence class of monomorphisms of $A$ is called a *subobject*. If any arrow $f: a \to a'$ of $A$ can be factorized to a strong epimorphism $e(f): a \to \text{Im}(f)$ and a monomorphism $m(f): \text{Im}(f) \to a'$, then $(\text{cod}: \text{Sub}(A) \to A, \triangle, \text{dom})$ is an opfibered comprehension; the cocartesian lifting of an $A$-arrow $f: a \to a'$ with $\text{Sub}(A)$-object $m: x \to a$ is given by $\bar{f}(m) = (f, e(f \circ m)): m \to \text{Im}(f \circ m)$, since for any $\text{Sub}(A)$-object $n: x'' \to a''$ and any arrow $h: x \to x'', g: a' \to a''$ satisfying $g \circ f \circ m = n \circ h$, the property of the strong epimorphism $e(f \circ m)$ implies the existence of the unique arrow $k$ satisfying $k \circ e(f \circ m) = h$ and $n \circ k = g \circ m(f \circ m)$.
We call it the subobject opfibered comprehension.

These two examples above are generalized to the following:

**Example 3.** Let \((E, M)\) be a factorization system on \(A\) in the sense of [15]. We obtain an opfibered comprehension in the same way as Example 2: the triple \((\text{cod}: M \to A, \nabla, \text{dom})\) is an opfibered comprehension, where we regard \(M\) as the full subcategory of \(A^-\) whose objects are arrows in \(M\). We note that the unique diagonal fill-in property of the factorization system guarantees the uniqueness of the mediating morphism \(k\) in (1). In general, \(c\) fails to be an opfibration when \((E, M)\) is merely a weak factorization system in the sense of [16].

On the other hand, in any opfibered comprehension arrows are factored in the following sense.

**Lemma 1.** Let \((c : M \to A, i, d)\) be an opfibered comprehension. Every arrow \(f : a \to a'\) in \(A\) factors as \(f = d(\eta_{f^* (ia)}) \circ d(\overline{(ia)})\).

**Proof.** Consider the cocartesian lifting \(f(\overline{(ia)})\) of \(f\) with \(ia\) as above. From the universal property of the cocartesian arrow, we obtain a unique vertical arrow \(k : f^* (ia) \to ia'\) such that \(k \circ \overline{(ia)} = if\). Now we have \(cn_{f^* (ia)} = \text{id}_{a'}\) and \(\eta_{f^* (ia)} \circ f(\overline{(ia)}) = icf (\overline{(ia)}) \circ \eta_{ia} = if\). Therefore \(k = \eta_{f^* (ia)}\). This factorization of \(if\) in \(M\) yields the desired factorization of \(dif = f\) in \(A\).
3 Formal Space and Fundamental Adjunction

Throughout this section, we fix an opfibered comprehension \((c : M \to A, i, d)\) and an adjunction \(L \dashv R : B \to A\). Our first step is to derive the category of spaces from the opfibered comprehension. For this, we take the pullback of \(c\) along \(R : B \to A\), as done in (2). We identify the opposite of the vertex of this pullback as the category of formal spaces and formally continuous maps.

\[
\begin{array}{c}
R^*M \\ \downarrow \pi \\
B \\ \downarrow R \\
A
\end{array}
\quad\quad\quad\quad\quad\quad\quad\quad
\begin{array}{c}
\tilde{R} \\
\downarrow c \\
M \\
\downarrow i \\
A
\end{array}
\]

(2)

**Definition 2.** We define the category \(\text{FS}(R, c)\) of formal spaces to be \((R^*M)^\text{op}\), the opposite of the vertex category of the above pullback in the category \(\text{Cat}\) of locally small categories. We call objects and arrows in \(\text{FS}(R, c)\) formal spaces and formally continuous maps, respectively.

We give the following concrete presentation of \(\text{FS}(R, c)\).

- A formal space is a tuple \((b, m)\) of \(b \in B\) and \(m \in M\) satisfying \(Rb = cm\).
- A formally continuous map from \((b', m')\) to \((b, m)\) is a tuple \((f, g)\) of arrows \(f : b \to b'\) in \(B\) and \(g : m \to m'\) in \(M\) satisfying \(Rf = cg\).

**Example 4.** The leading example of formal spaces is topological spaces. Let \(\text{Frm}\) be the category of frames and frame homomorphisms, and \(2\) be the two-point frame \(\{\bot \leq \top\}\). By applying the above pullback construction to the power functor \(2^{(-)} : \text{Set}^\text{op} \to \text{Frm}\) and the subobject opfibered comprehension \(\text{cod} : \text{Sub}(\text{Frm}) \to \text{Frm}\), we obtain the category \(\text{FS}(2^{(-)}, \text{cod})\) of formal spaces. This is isomorphic to the category \(\text{Top}\) of topological spaces and continuous maps. See details in Sect. 5.

We next construct the fundamental adjunction between \(\text{FS}(R, c)^\text{op}\) and \(A\). The pullback diagram induces two extra adjunctions.

The first is a right adjoint of \(\pi : R^*M \to B\). From \(R \circ \text{Id}_B = c \circ i \circ R\), the universal property of the pullback yields the mediating functor \(\rho : B \to R^*M\) such that \(\pi \circ \rho = \text{Id}_B\) and \(\tilde{R} \circ \rho = i \circ R\).

**Proposition 1.** We have an adjunction \(\pi \dashv \rho\) whose counit is the identity.

**Proof.** We show that \(\pi \dashv \rho\) is the adjunction whose counit is \(\text{id}_b : b = \pi(b, iRb) = \pi pb \to b\). For any \(b' \in B\), \(m \in M\), \(f : b' = \pi(b', m) \to b\) satisfying \(Rb' = cm\), there exists the unique pair of \(h : b' \to b\) in \(B\) and \(g : m \to iRb\) in \(M\) satisfying \(Rh = cg\) : \(cm \to Rb\) and \(\text{id}_b \circ \pi(h, g) = f\), since \(c \dashv i\) and \(\epsilon^{c\circ i} = \text{id}\). \(\Box\)
The second is the left adjoint of \( \tilde{R} : R^* M \to M \). This is the keystone of the fundamental adjunction. The general result of Hermida shows that the horizontal leg of the change-of-base of any opfibration along right adjoint has a left adjoint:

**Theorem 1 (Corollary 3.2.5, [9]).** In (2), \( \tilde{R} : R^* M \to M \) has a left adjoint \( \tilde{L} \) satisfying \( \pi \circ \tilde{L} = L \circ c \).

For reference, we put his construction here. The candidate left adjoint \( \tilde{L} \) maps \( m \in M \) to the pair \( (Lcm, (\eta_{cm})_*(m)) \), where \( (\eta_{cm})_*(m) : m \to (\eta_{cm})_{L*R}(m) = \tilde{R} \tilde{L} m \) satisfies the universal property: for any \( (b, m') \in FS(R, c) \) and \( f : m \to m' = \tilde{R}(b, m') \), there exists the pair of \( g : Lcm \to b \) in \( B \) and \( h : (\eta_{cm})_*(m) \to m' \) in \( M \) satisfying \( Rg = ch \) and \( h \circ (\eta_{cm})_*(m) = f \). The existence and the uniqueness of \( g \) is proven by \( L \dashv R \). The existence and the uniqueness of \( h \) is proven by the cocartesian lifting property of \( (\eta_{cm})_*(m) \).

To summarize, we obtain the following sequence of adjunctions, which factors \( L \dashv R \):

\[
\begin{array}{ccc}
B & \xleftarrow{\rho} & FS(R, c)^{op} & \xleftarrow{\tilde{R}} & M & \xrightarrow{d} & A \\
\pi & \xrightarrow{\pi} & \tilde{L} & \xrightarrow{\tilde{L}} & \pi \end{array}
\]

**Theorem 2 (formal space factorization).** We have the factorization of \( L \dashv R \) as \( R = d \circ \tilde{R} \circ \rho \) and \( L = \pi \circ \tilde{L} \circ i \).

**Proof.** By the definition of \( \rho \), we have \( d \circ \tilde{R} \circ \rho = d \circ i \circ R = R \). By the definition of \( \tilde{L} \), we have \( \pi \circ \tilde{L} \circ i = \pi \circ \tilde{L} \circ i = L \).

Starting from an opfibered comprehension \( (c : M \to A, i, d) \) and an adjunction \( L \dashv R : B \to A \), this factorization theorem yields an adjunction between \( FS(R, c)^{op} \) and \( A \). This is the main subject of this paper, the fundamental adjunction.
Definition 3 (fundamental adjunction). We call $\bar{L} \circ i \dashv d \circ \bar{R}$ the fundamental adjunction. The left and right fundamental adjoints are denoted by

$$\text{Sp}^{R,c} \triangleq \bar{L} \circ i : A \to \text{FS}(R,c)^{op}, \quad \text{Al}^{R,c} \triangleq d \circ \bar{R} : \text{FS}(R,c)^{op} \to A.$$ 

3.1 Coreflexiveness of Fundamental Adjunction

When the left fundamental adjoint $\text{Sp}^{R,c}$ is full faithful, it yields an equivalence between $A$ and its full image. We here consider when $\text{Sp}^{R,c}$ is full faithful.

As shown in Lemma 1, in any opfibered comprehension $(c : M \to A, i, d)$ an arrow $f : a \to a'$ in $A$ factors as $d(\eta^{c,i}_{f,\cdot}(i)) \circ d(f(ia))$.

Definition 4. Let $(c : M \to A, i, d)$ be an opfibered comprehension. We say that an arrow $f : a \to a'$ in $A$ belongs to $M$ if $d(f(ia)) : a \to d(f(i))$ is invertible.

Example 5. In the arrow opfibered comprehension $\text{cod} : A^\to \to A$, any $A$-arrow belongs to $A^\to$, while for the subobject opfibered comprehension $\text{cod} : \text{Sub}(A) \to A$, an arrow belongs to $\text{Sub}(A)$ if and only if it is mono.

Proposition 2. The left fundamental adjoint $\text{Sp}^{R,c}$ is full faithful if and only if $\eta^{L,R}_{a}$ belongs to $M$ for each $a \in A$.

Proof. The unit of the adjunction $\text{Sp}^{R,c} \dashv \text{Al}^{R,c}$ is $d(\eta^{L,R}_{c}) \circ \eta^{i,d}$. Since $\eta^{i,d} = \text{id}$, it suffices to show that $d(\eta^{L,R}_{c})$ is invertible at each $a \in A$. Now recall that the $m$-th component $\eta^{L,R}_{m}$ of the unit of $\bar{L} \dashv \bar{R}$ is the cartesian lifting $\eta^{L,R}_{m}(m)$ of the unit of $L \dashv R$ with $m$. Therefore $d(\eta^{L,R}_{a}) = d(\eta^{L,R}_{c}(ia)) = d(\eta^{L,R}_{a}(ia))$, and, then the left hand side is invertible, if and only if $\eta^{L,R}_{a}$ belongs to $M$.

Corollary 1. Let $\text{cod} : A^\to \to A$ be the arrow opfibered comprehension and $L \dashv R$ be an adjunction. The left fundamental adjoint $\text{Sp}^{R,c}$ is full faithful.

Corollary 2. Let $\text{cod} : \text{Sub}(A) \to A$ be the subobject opfibered comprehension and $L \dashv R$ be an adjunction. The left fundamental adjoint $\text{Sp}^{R,c}$ is full faithful if and only if each component of the unit of $L \dashv R$ is mono.

3.2 Representability of Fundamental Adjunction

Next, we study the representability of the fundamental adjunction. We say that a dual adjunction $L \dashv R : C^{op} \to D$ is representable through $\gamma : C \to \text{Set}$ and $\delta : D \to \text{Set}$ if there is a pair $\Omega_C \in C, \Omega_D \in D$ of objects such that

$$\gamma \circ L^{op} \cong D(-, \Omega_D), \quad \delta \circ R \cong C(-, \Omega_C).$$

Note that it follows $\gamma \Omega_C \cong \delta \Omega_D$; see e.g. [5, Lemma 2.3].

The fundamental adjunction enjoys this representability when it arises from the following situation. Let $c : M \to A$ be an opfibered comprehension and $\Omega \in A$. We assume that (1) $A$ comes with an adjunction $F \dashv U : A \to \text{Set},$
and (2) the representable functor \((A(\cdot, \Omega))^\text{op} : A \to \text{Set}^\text{op}\) has a right adjoint \(\Omega(\cdot) : \text{Set}^\text{op} \to A\). We call this hom-power adjunction. The second assumption means that \(A\) admits small powers of \(\Omega\); see [17, Section III.4] for detail. Under these assumptions, the fundamental adjunction becomes representable through \(\pi : \text{FS}(\Omega(-), c) \to \text{Set} \) and \(U : A \to \text{Set}\).

**Theorem 3.** We define the object \(\Omega_{\text{FS}} \in \text{FS}(\Omega(-), c)\) to be \(\text{Sp}^{\Omega(-), c}(F1)\), where \(1\) is the terminal object of \(\text{Set}\). Then

\[
\pi \circ (\text{Sp}^{\Omega(-), c})^\text{op} = A(-, \Omega), \quad U \circ \text{A}l^{\Omega(-), c} \cong \text{FS}(\Omega(-), c)(- , \Omega_{\text{FS}}).
\]

Proof. The first is by the factorization of \(A(-, \Omega)\) (Theorem 2). We show the second.

\[
U \circ \text{A}l^{\Omega(-), c} \cong \text{Set}(1, U \circ \text{A}l^{\Omega(-), c}(-)) \cong \text{FS}(\Omega(-), c)^\text{op}(\text{Sp}^{\Omega(-), c}(F1), -) = \text{FS}(\Omega(-), c)(- , \Omega_{\text{FS}}).
\]

4 Formal Spaces from Arrow Opfibered Comprehension

When the construction of the category of formal spaces is applied to some concrete arrow opfibered comprehension \(\text{cod} : A^{-} \to A\), derived concepts of formal space coincide with existing structures. We illustrate two such examples: one is Chu spaces [10] and the other is topological systems introduced by Vickers [18]. These suggests that, for the arrow opfibered comprehension, our construction can be regarded as a “non-symmetric” generalization of Chu construction.

4.1 Chu Spaces

Let \((A, I, \otimes, [-, -])\) be a symmetric monoidal closed category, and \(\Sigma \in A\) be an object. It plays the role of a dualizing object. In [19], Pavlovic showed that a category of Chu spaces can be obtained as the comma category of \(\text{Id}_A : A \to A\) and the internal hom functor \([-, \Sigma] : A^\text{op} \to A\). In general, the comma category of the form \(\text{Id}_A \downarrow F\) is isomorphic to the vertex of the pullback of the codomain functor \(\text{cod} : A^{-} \to A\) along \(F\). Therefore the category \(\text{Chu}(A, \Sigma)\) of Chu spaces in [19] is isomorphic to the category \(\text{FS}([- , \Sigma], \text{cod})\) of formal spaces.

4.2 Topological Systems

In [18], Vickers introduces the category of topological system to model state spaces paired with notions of observations. This category is defined by the following, and has a similar flavor to the category of Chu spaces.

- A topological system is a tuple \((x, a, s : x \times a \to 2)\) of a set, a frame, and a function such that, for each \(p \in x\), the function \(s(p, -) : a \to 2\) preserves finite meets and arbitrary unions.
A map from a topological system \((x', a', s')\) to another one \((x, a, s)\) is a tuple \((f, g)\) such that \(f : x' \to x\) is a function, \(g : a \to a'\) is a frame homomorphism and they satisfy \(s \circ (f \times a) = s' \circ (x' \times g)\).

We can easily see that it is isomorphic to the category \(\text{FS}(2(-), \text{cod})\) of formal spaces, where \(\text{cod} : \text{Frm} \to \text{Frm}\) is the arrow opfibered comprehension and \(2(-) : \text{Set}^{\text{op}} \to \text{Frm}\) is the power functor to \(2\).

The left fundamental adjoint \(\text{Sp}^{2(-), \text{cod}} : \text{Frm} \to \text{FS}(2(-), \text{cod})^{\text{op}}\) sends each frame to the corresponding locale as defined in [18]. By Proposition 2 we can see that the functor is fully faithful, which implies that the definition of the category of locales there is indeed equivalent to the more common definition: \(\text{Loc} = \text{Frm}^{\text{op}}\).

5 Formal Spaces from Subobject Opfibered Comprehension

Let \(U : A \to \text{Set}\) be a monadic functor and \(\Omega \in A\). Then (1) \(U\) has a left adjoint, (2) \(A\) admits powers of \(\Omega\) since \(A\) has small limits, and (3) \(A\) admits a strong epi-mono factorization. Therefore from such \(U\) and \(\Omega\), we obtain two ingredients needed for constructing the category of formal spaces and the fundamental adjunction: (1) the subobject opfibered comprehension \(\text{cod} : \text{Sub}(A) \to A\), and (2) the hom-power adjunction \((A(-), \Omega)\)^{\text{op}} \vdash \Omega(-).

The derived category \(\text{FS}(\Omega(-), \text{cod})\) of formal spaces and the fundamental adjunction have the following concrete description.

- A formal space is a pair \((I, X)\) of a set \(I\) and a subobject \(X\) of \(\Omega^I\).
- A formally continuous map \(f : (I, X) \to (J, Y)\) is a function \(f : I \to J\) such that \(\Omega^J : \Omega^J \to \Omega^I\) restricts to an \(A\)-arrow of type \(Y \to X\).
- The right fundamental adjoint satisfies \(\text{Al}^{\Omega(-), \text{cod}}(I, X) = X\), simply extracting the subobject part.
- The left fundamental adjoint takes \(X \in A\) and computes the following push-forward in the opfibration \(c\) (recall the notation \(e(f), m(f), \text{Im}(f)\) about the factorization of \(f\) in Example 2):

\[
\begin{array}{c}
\text{Sub}(A) \\
\downarrow \text{cod} \\
A
\end{array}
\begin{array}{c}
\Delta X \\
\eta_X^{A(-, \Omega) + \Omega(-)} \text{Im}(\eta_X^{A(-, \Omega) + \Omega(-)}) \\
\eta_X^{A(-, \Omega) + \Omega(-)} \text{Im}(\eta_X^{A(-, \Omega) + \Omega(-)})
\end{array}
\begin{array}{c}
\text{m}(\eta_X^{A(-, \Omega) + \Omega(-)}) \\
\Omega^A(X, \Omega)
\end{array}
\begin{array}{c}
\eta_X^{A(-, \Omega) + \Omega(-)} \\
\eta_X^{A(-, \Omega) + \Omega(-)}
\end{array}
\begin{array}{c}
\text{Im}(\eta_X^{A(-, \Omega) + \Omega(-)})
\end{array}
\end{array}
\]

Then it returns the formal space \((A(X, \Omega), \text{Im}(\eta_X^{A(-, \Omega) + \Omega(-)})\).
From Theorem 3, the fundamental adjunction is representable, and from Corol-
larly 2, the left fundamental adjoint $\text{Sp}^{\Omega(-), \text{cod}}$ is fully faithful if and only if the
unit $\eta : X \to \Omega^{A(X, \Omega)}$ of the hom-power adjunction is mono.

**Table 1. Various algebraic categories**

| Category | Object          | Arrow                          |
|----------|----------------|--------------------------------|
| BA       | Boolean algebras | Boolean homomorphisms          |
| SLat     | Join semilattices | Join-preserving functions      |
| Lat      | Bounded lattices | Bounded lattice homomorphisms  |
| DLat     | Distributive bounded lattices | Bounded lattice homomorphisms |
| CSLat    | Complete lattices | Join-preserving functions      |
| Frm      | Frames          | Frame homomorphisms            |

We demonstrate that some known fundamental adjunctions between cate-
gories of algebras and those of spaces are instances of the above fundamen-
tal adjunction. To save space, we write $\text{FS}(\Omega.A), \text{Sp}^{\Omega.A}$ and $\text{Al}^{\Omega.A}$ to mean $\text{FS}(\Omega(-), \text{cod}), \text{Sp}^{\Omega(-), \text{cod}}$ and $\text{Al}^{\Omega(-), \text{cod}}$, respectively. Let us introduce various
categories by Table 1. These categories have the special object $2 = (\{\perp, \top\}, \perp \leq \top)$. In particular, $\text{FS}(2, BA)$ is isomorphic to the category $\text{Fld}$ of fields of sets and Boolean homomorphisms [20], and $\text{FS}(2, Frm) \cong \text{Top}$ as in Exam-
ple 4. Since the lattice $2^X$ and its sublattices are always distributive, we have $\text{FS}(2, DLat) \cong \text{FS}(2, Lat)$.

**Example 6.** When $A = BA, SLat, DLat$, or $CSLat$, $L = A(-, 2)$ is faithful and the left fundamental adjoint $\text{Sp}^{2.A}$ is fully faithful by Proposition 2. On the other hand, when $A = Lat$, $L$ is not faithful, since components for non distributive bounded lattices of the unit $\eta^{\text{Lat}(-, 2)-2(-)}$ are monomorphisms [21]. When $A = Frm$, $L$ is not faithful and $\text{Sp}^{2.Frm}$ is not fully faithful, since components for non spatial frames of the unit $\eta^{\text{Sp}^{2.Frm}-\text{Al}^{2.Frm}}$ are not isomorphisms [1].

**Example 7.** We write $\text{Poset}$ for the category of partially ordered sets and mono-
tone maps. The representable functor $L = \text{Frm}(-, 2) : \text{Frm} \to \text{Poset}^{\text{op}}$ for $\text{Poset}$-enriched $\text{Frm}$ has the right adjoint $R = \text{Up}$, whose $\text{Up}(X, \leq)$ is the frame of all up-closed subsets of $(X, \leq)$. Then, $\text{FS}(\text{Up}, \text{cod})$ is the following category $\text{PoTop}$:

- its objects are $(X, \leq, \alpha)$ such that $(X, \leq) \in \text{Poset}$, $(X, \alpha) \in \text{Top}$, and $\alpha \subseteq \text{Up}(X, \leq)$.
- its arrows $f : (X, \leq, \alpha) \to (Y, \leq, \beta)$ satisfy $f : (X, \leq) \to (Y, \leq) \in \text{Poset}$ $f : (X, \alpha) \to (Y, \beta) \in \text{Top}$.

**Example 8.** A fundamental adjunction $\text{Sp}^{\Omega.A} \dashv \text{Al}^{\Omega.A} : \text{FS}(\Omega.A)^{\text{op}} \to A$ is extendible by composing another adjunction $F \dashv U : A \to A'$, for example, ideal
completion. A subset of a join semilattice \( X = (X, \lor, \bot) \) is called an ideal [21] in \( X \), if it is down-closed and finite join closed. The set of all ideals in \( X \) forms a complete join semilattice, where for a set \( \alpha \) of ideals, its join \( \lor \alpha \) is given not by its union \( \bigcup \alpha \), but by \( \{ x \mid \exists \beta \subseteq \bigcup \alpha, \beta \text{ is finite}, x \leq \lor \beta \} \). This construction gives adjunctions ideals \( \dashv \text{forget} : \mathrm{CSLat} \to \mathrm{SLat} \) and ideals \( \dashv \text{forget} : \mathrm{Frm} \to \mathrm{DLat} \). Therefore, we have the following extended fundamental adjunctions.

\[
\begin{align*}
\mathrm{Sp}^{2,\mathrm{CSLat}} \circ \text{ideals} & \dashv \text{forget} \circ \mathrm{Al}^{2,\mathrm{CSLat}} : \mathrm{FS}(2, \mathrm{CSLat})^{\text{op}} \to \mathrm{SLat} \\
\mathrm{Sp}^{2,\mathrm{Frm}} \circ \text{ideals} & \dashv \text{forget} \circ \mathrm{Al}^{2,\mathrm{Frm}} : \mathrm{Top}^{\text{op}} \to \mathrm{DLat} \\
\mathrm{Sp}^{\mathrm{Up},\text{cod}} \circ \text{ideals} & \dashv \text{forget} \circ \mathrm{Al}^{\mathrm{Up},\text{cod}} : \mathrm{PoTop}^{\text{op}} \to \mathrm{DLat}
\end{align*}
\]

6 Change of Bases

In this section, we give some construction of adjunctions among categories of different formal spaces. Below we show that an adjunction between opfibered comprehensions induces an adjunction between categories of formal spaces.

**Theorem 4 (base change theorem for opfibered comprehensions).** Let

- \( (c : M \to A, i, d) \) and \( (c' : M' \to A', i', d') \) be opfibered comprehensions,
- \( F_A \dashv U_A : A \to A' \) and \( F_M \dashv U_M : M \to M' \) be adjunctions and
- \( R : B \to A \) be a functor

such that \( (c, c') \) is a map of adjunction (see [17, Section IV.7]) from \( F_M \dashv U_M \) to \( F_A \dashv U_A \), that is, the following equalities hold:

\[
\begin{align*}
&c' \circ U_M = U_A \circ c, \quad c \circ F_M = F_A \circ c', \quad c \epsilon_{F_M \dashv U_M} = \epsilon_{F_A \dashv U_A}.
\end{align*}
\]

Then there is an adjunction \( F^* \dashv U^* : R^* M \to R'^* M' \) satisfying \( \pi' \circ U^* = \pi \) and \( R' \circ U^* = U_M \circ R \), where \( R' = U_A \circ R \).
Proof. $U_M$ satisfies $c' \circ U_M \circ \tilde{R} = U_A \circ \pi \circ \tilde{R} = U_A \circ R \circ \pi = R' \circ \pi$. Since $(R^*M', \pi', \tilde{R}')$ is the pullback of $R'$ and $c'$ in $\mathbf{Cat}$, there exists the unique $U^*$ satisfying $\pi' \circ U^* = \pi$ and $\tilde{R}' \circ U^* = U_M \circ \tilde{R}$, where $U^*$ maps an object $(b, m)$ to $(b, U_M m)$ and maps arrows similarly.

To save notational burden, let $\eta^A$ and $\epsilon^A$ be the unit and the counit of $F_A \dashv U_A$, and $\eta^M$ and $\epsilon^M$ be the unit and the counit of $F_M \dashv U_M$. That $(c, c')$ is a map of adjunction implies that $\eta^A_{c'}: c' \Rightarrow U_A F_A c'$ is the same as $\epsilon^M_{c'}: c' \Rightarrow c' U_M F_M$.

For $(b, m') \in R^*M'$, $(b, (\epsilon^A_{Rb})_* F_M m')$ is an object of $R^*M$. Note that $m'$ is above $U_A Rb$, $F_M m'$ is above $F_A U_A Rb$, and $(\epsilon^A_{Rb})_* F_M m'$ is above $Rb$. Let $e_{m'}: F_M m' \to (\epsilon^A_{Rb})_* F_M m'$ be the cartesian arrow above $\epsilon^A_{Rb}$ defined canonically. Here,

$$c'(U_M e_{m'} \circ \eta^M_{m'}) = c' U_M e_{m'} \circ \eta^M_{m'} = U_A c e_{m'} \circ \eta^A_{m'} = U_A \epsilon^A_{Rb} \circ \eta^A_{U_A Rb} = \text{id}_{U_A Rb}$$

holds. Thus, we have an arrow

$$(\text{id}_b, U_M e_{m'} \circ \eta^M_{m'}): (b, m') \to (b, U_M ((\epsilon^A_{Rb})_* F_M m')) = U^*(b, (\epsilon^A_{Rb})_* F_M m')$$

in the category $R^*M'$.

We will show that this is a universal arrow from $(b, m')$ to $U^*$. Assume that we have an arrow $(f, g): (b, m') \to (b_2, U_M m) = U^*(b_2, m)$. Using the adjunction $F_M \dashv U_M$, we can decompose $g: m' \to U_M m$ to

$$m' \xrightarrow{\eta^M_{m'}} U_M F_M m' \xrightarrow{U_M F_M g} U_M F_M U_M m \xrightarrow{U_M \epsilon^M_{m'}} U_M m.$$

The composite

$$F_M m' \xrightarrow{F_M g} F_M U_M m \xrightarrow{\epsilon^M_{m'}} m$$

in $M$ is sent by $c$ to

$$F_A U_A Rb \xrightarrow{F_A U_A Rf} F_A U_A Rb_2 \xrightarrow{\epsilon^A_{Rb_2}} Rb_2$$

in $A$. By naturality it is equal to

$$F_A U_A Rb \xrightarrow{\epsilon^A_{Rb}} Rb \xrightarrow{Rf} Rb_2.$$

Thus, by the universality of cocartesian lifting, $\epsilon^M_{m'} \circ (F_M g)$ decomposes through $\epsilon_{m'}$. Combining this with the first decomposition yields the decomposition we want. Uniqueness can be shown by a similar means.

Therefore, $U^*$ has a left adjoint $F^*$ satisfying $F^*(b, m') = (b, (\epsilon^A_{Rb})_* F_M m')$.

$\square$

By specializing Theorem 4, we have some more usable results.
Corollary 3 (base change theorem for arrow opfibered comprehension). Let $F_A \dashv U_A : A \to A'$ be an adjunction and $R : B \to A$ be a functor. There exist adjunctions $F_{A \dashv U_A} : A \to (A')^\sim$ and $F^* \dashv U^* : R^*A \to R'^*(A')^\sim$ satisfying the conditions in Theorem 4. Moreover, $F^*(b, m') = (b, \epsilon_{Rb} \circ F_{Am'})$ for each $(b, m') \in R^*(A')^\sim$.

Proof. We can canonically obtain $F_{A \dashv U_A} : A \to (A')^\sim$ by defining their object parts by the arrow parts of $F_A \dashv U_A : A \to A'$. These satisfy the conditions in Theorem 4, so we can apply it to obtain $F^* \dashv U^* : R^*A \to R'^*(A')^\sim$.

The pushout in the construction in Theorem 4 turns out to coincide with postcomposition. This yields the equality for $F^*$.

\[ \square \]

Corollary 4 (base change theorem for subobject opfibered comprehension). Let $F_A \dashv U_A : A \to A'$ be an adjunction and $R : B \to A$ a functor. Assume that

- $A$ and $A'$ have (strong epi, mono)-factorization systems,
- $F_A$ preserves monomorphisms, and
- the counit of $F_A \dashv U_A$ is componentwise monic.

Then there exist adjunctions $F_{\text{Sub}(A) \dashv U_{\text{Sub}(A)}} : \text{Sub}(A) \to \text{Sub}(A')$ and $F^* \dashv U^* : R^*\text{Sub}(A) \to R'^*\text{Sub}(A')$ satisfying the conditions in Theorem 4. Moreover, $F^*(b, m') = (b, \epsilon_{Rb} \circ F_{Am'})$ for each $(b, m') \in R'^*\text{Sub}(A')$.

Proof. Since both $F_A$ and $U_A$ preserve monomorphisms, we can let $F_{\text{Sub}(A)}(m') = m'$ and $U_{\text{Sub}(A)}(m) = m$. These satisfy the conditions in Theorem 4, so we can apply it to obtain $F^* \dashv U^* : R^*\text{Sub}(A) \to R'^*\text{Sub}(A')$.

By the assumption on the counit, the pushout in the construction in Theorem 4 turns out to coincide with postcomposition. This yields the equality for $F^*$.

\[ \square \]

Example 9. The leading example of Corollary 4 is the adjunction between $\text{Fld} \cong \text{FS(2, BA)}$ and $\text{FS(2, DLat)}$.

The forgetful functor $\text{forget} : \text{BA} \to \text{DLat}$ has a right adjoint $\text{comp}$, which maps a distributive lattice $(X, \lor, \land, \bot, \top)$ to the Boolean algebra of its complemented elements [20], that is to say, $\{x \in X \mid \exists x' \in X, x \lor x' = \top, x \land x' = \bot\}$. Since this adjunction satisfies the condition of Corollary 4, we have the adjunction $\text{forget}^* \dashv \text{comp}^* : \text{FS(2, DLat)}^{\text{op}} \to \text{FS(2, BA)}^{\text{op}}$ satisfying $\text{forget}^* \circ \text{Sp}_2^{\text{BA}} = \text{Sp}_2^{\text{DLat}} \circ \text{forget}$.

\[ \begin{array}{ccc}
\text{FS(2, DLat)}^{\text{op}} & \xrightarrow{2^\sim} & \text{Sub(DLat)} \\
\text{forget}^* \downarrow & & \downarrow \text{dom} \\
\text{FS(2, BA)}^{\text{op}} & \xrightarrow{2^\sim} & \text{Sub(BA)} \\
\text{BA} & \xleftarrow{\text{DLat}^\sim} & \text{Sub(DLat)} \\
\end{array} \text{op} \xrightarrow{\top} \text{Sub(DLat)} \xleftarrow{\top} \text{Sub(BA)} \xrightarrow{\top} \text{BA} \]

\[ \begin{array}{ccc}
\text{dom} & \xrightarrow{\top} & \text{DLat} \\
\Delta & \downarrow & \Delta' \\
\text{dom}' & \xleftarrow{\top} & \text{BA} \\
\end{array} \text{op} \xrightarrow{\top} \text{Sub(DLat)} \xleftarrow{\top} \text{Sub(BA)} \xrightarrow{\top} \text{BA} \]

\[ \begin{array}{ccc}
\text{BA}(\sim, 2) & \xrightarrow{2^\sim} & \text{DLat}(\sim, 2) \\
\text{forget} \downarrow & & \downarrow \text{comp} \\
\text{FS(2, BA)}^{\text{op}} & \xrightarrow{2^\sim} & \text{Sub(BA)} \\
\end{array} \text{op} \xrightarrow{\top} \text{Sub(DLat)} \xleftarrow{\top} \text{Sub(BA)} \xrightarrow{\top} \text{BA} \]
By composing $\text{forget} \dashv \text{comp}$ with the adjunction $\text{Sp}^2,\text{Frm} \circ \text{ideals} \dashv \text{forget} \circ \text{Al}^2,\text{Frm}$: $\text{Top}^{op} \rightarrow \text{DLat}$ in Example 8, we also have the following adjunction.

$$\text{Sp}^2,\text{Frm} \circ \text{ideals} \circ \text{forget} \dashv \text{comp} \circ \text{forget} \circ \text{Al}^2,\text{Frm}; \text{Top}^{op} \rightarrow \text{BA}$$

7 Conclusion and Future Work

This paper has defined the notion of opfibered comprehension $(c: M \rightarrow A, i, d)$ including the example of arrows $M = A^{-}\rightarrow$ or the example of subobjects $M = \text{Sub}(A)$. For any functor $R: B \rightarrow A$, we have constructed its formal spaces $\text{FS}(R, c)$ and a fundamental adjunction $\text{Sp}^{R, c} \dashv \text{Al}^{R, c}: \text{FS}(R, c)^{op} \rightarrow A$. The leading example is the adjunction between $\text{Top}^{op}$ and $\text{Frm}$.

We also have given the sufficient condition to construct different formal spaces $\text{FS}(R, c)$ and $\text{FS}(R', c')$. Its leading example is the adjunction between $\text{FS}(2, \text{DLat})$ and $\text{FS}(2, \text{BA})$.

It is future work to compare with other dualities, for example, Priestley duality [21], algebra/coalgebra duality [22], natural dualities [12], and so on.

Acknowledgments. This work was the supported by JSPS KAKENHI Grant Number JP24700017. The second and third authors carried out this research under the support of JST ERATO HASUO Metamathematics for Systems Design Project (No. JPMJER1603).

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