Abstract

In a recent paper, the partition function (character) of ten-dimensional pure spinor world-sheet variables was calculated explicitly up to the fifth mass-level. In this letter, we propose a novel application of Padé approximants as a tool for computing the character of pure spinors. We get results up to the twelfth mass-level. This work is a first step towards an explicit construction of the complete pure spinor partition function.
1 Introduction

A few years ago, a new formalism was proposed for quantizing the superstring in a manifestly ten-dimensional super-Poincaré covariant manner [1]. This formalism has been used for computing covariant multiloop amplitudes [2], leading to new insights into perturbative finiteness of superstring theory [3].

This pure spinor formalism for superstrings, in the flat background, is based on a set of ten-dimensional superspace variables $x^m$, $\theta^\alpha$ (and its conjugate $p_\alpha$), and a set of bosonic ghosts fields $\lambda^\alpha$ (and its conjugate $w_\alpha$) transforming as an $SO(10)$ spinors and satisfying a pure spinor constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ for $m = 0$ to 9. The need of this constraint comes from nilpotency of a BRST operator constructed from fermionic Green-Schwarz (GS) constraints and the pure spinor $\lambda^\alpha$. It has been shown that the string spectrum found using the BRST cohomology coincides with the light-cone spectrum of the GS string [4, 5, 6]. This fact was also recently proved by using the pure spinor superstring partition function [7] (up to the fifth mass-level).

The information of the spectrum is encoded in the partition function

$$Z(q, t) \equiv Tr_{\mathcal{H}}[(-)^F q^{L_0} t^{J_0}] = \sum_h Z_h(t) q^h,$$

where $J_0 = \oint dz J(z)$ and $L_0 = \oint dz z T(z)$, $J_0$ is the zero mode of the $U(1)$ current $J(z)$, $L_0$ is the zero mode of the Virasoro generator $T(z)$, $F$ the fermion number (which takes account of the statistics), and $\mathcal{H}$ the corresponding Hilbert space. The computation of the pure spinor superstring partition function boils down to compute the partition function of the pure spinor sector since the calculation of the matter sector $(x^m, \theta^\alpha)$ is very simple [7, 8, 9].
There have been many attempts for computing the partition function of pure spinors, for instance, in [8, 9] were computed character formulas \(Z_h(t)\) up to the level \(h = 1\), i.e. the zero mode \(Z_0(t)\) and first massive \(Z_1(t)\) part of the partition function (1.1). In [10, 11], the authors have tried to find a prescription which allows to compute the partition function of curved beta-gamma systems, but the employment of those techniques to the case of pure spinors are still lacking. In [12] it was studied partition functions of certain subset of curved beta-gamma systems (defined by quadratic constraints) as toy models in order to specialize then to the case of pure spinors.

Recently, it has been presented a formal expression for the partition function of pure spinors in terms of an infinite set of free-field ghosts [7]

\[
Z(q, t) = \prod_{k=1}^{\infty} \left[ (1 - t^k)^{-N_k} \prod_{h=1}^{\infty} (1 - q^h t^k)^{-N_k} (1 - q^h t^{-k})^{-N_k} \right], \tag{1.2}
\]

where \(N_k\) are the multiplicities of the ghost fields. The use of these ghosts comes from the resolution of the pure spinor constraint, and the necessity of infinite many of them is because the pure spinor constraint is infinitely reducible [7, 13]. Although it may seem difficult to extract useful information from the formal expression (1.2), by appealing to some regularization procedure in order to guarantee the convergence of the infinite product over \(k\), character formulas \(Z_h(t)\) were calculated up to the fifth mass-level (\(h = 5\)). These first five character formulas were also computed using fixed point techniques [7].

In this paper, we propose a novel application of Padé approximants [14] as a tool for computing higher-level character formulas \(Z_h(t)\) of pure spinors. We show by explicit computations that our first five character formulas are in agreement with the ones found in [7], and in addition, we obtain results up to the twelfth mass-level (\(h = 12\)). It is left as an ambitious challenge to find the complete pure spinor partition function. We hope that our work is a first step towards this explicit construction of the pure spinor partition function.

The paper is organized as follows: In section 2, a short review of \(SO(10)\) pure spinors is given. In section 3, we review some concepts and ideas regarding to the partition function of pure spinors. In section 4, we describe the method of Padé approximants for computing higher level character formulas, and we present our results. A summary and further interesting topics are given in section 5. Finally in appendix A, we present some details involved in the computations of higher level character formulas.

## 2 Pure spinors: a reminder

The \(SO(10)\) pure spinor \(\lambda^\alpha\) is constrained to satisfy \(\lambda^\alpha \gamma^{m\alpha\beta} \lambda^\beta = 0\), where \(m = 1 \text{ to } 10, \alpha, \beta = 1 \text{ to } 16\). \(\gamma^{m\alpha\beta}\) and \(\gamma^{m\alpha\beta}\) are 16 \times 16 symmetric matrices which are the off-diagonal
blocks of the $32 \times 32$ ten-dimensional $\Gamma$-matrices and satisfy $\gamma^{(m} \gamma^{n)} \gamma^\beta = 2\eta^{mn}\delta^\beta_\alpha$. This implies that $\lambda^\alpha \lambda^\beta$ can be written as

$$\lambda^\alpha \lambda^\beta = \frac{1}{5!} \gamma^{\alpha\beta}_{mnppqr} (\lambda^\gamma \gamma^{mnppqr} \lambda^\delta) \gamma_{\delta}$$

(2.1)

where $\lambda^\gamma \gamma^{mnppqr} \lambda$ defines a 5-dimensional complex hyperplane. This 5-dimensional complex hyperplane is preserved up to a phase by a $U(5)$ subgroup of $SO(10)$ rotations. So projective pure spinors in $d = 10$ Euclidean dimensions parameterize the coset space $SO(10)/U(5)$, which implies that $\lambda^\alpha$ has 11 independent complex degrees of freedom [15].

The pure spinor constraint (2.1) can be solved and we can express $\lambda^\alpha$ in terms of the 11 independent degrees of freedom. This was done in [1] by decomposing the 16 components of $\lambda^\alpha$ into $SU(5) \times U(1)$ representations as

$$\lambda^+ = \gamma, \quad \lambda_{[ab]} = \gamma u_{[ab]}, \quad \lambda_{[abcd]} = -\frac{1}{8} \gamma u_{[ab]udc],$$

(2.2)

where $\gamma$ is an $SU(5)$ scalar, and $u_{[ab]}$ is an $SU(5)$ antisymmetric two-form. Using this decomposition, and by bosonizing the $(\beta, \gamma)$ fields as $(\beta = \partial \xi e^{-\phi}, \gamma = \eta e\phi)$, we can write the formulas for the currents [8]

$$J = -\frac{5}{2} \partial \phi - \frac{3}{2} \eta \xi,$$

$$N^a = v^a,$$

$$N^b_a = -u_{ac}v^{bc} + \frac{3}{4} \eta \xi + \frac{3}{4} \partial \phi,$$

$$N_{ab} = 3\partial u_{ab} + u_{ac}u_{bd}v^{cd} + u_{ab} \left( \frac{5}{2} \eta \xi + \frac{3}{2} \partial \phi \right),$$

$$T = \frac{1}{2} v^{ab} \partial u_{ab} - \frac{1}{2} \partial \phi \partial \phi - \eta \partial \xi + \frac{1}{2} \partial (\eta \xi) - 4 \partial (\partial \phi + \eta \xi),$$

where $T$ is the stress-energy tensor and the worldsheet fields satisfy the OPE’s

$$\eta(\gamma) \xi(z) \sim (y - z)^{-1}, \quad \phi(\gamma) \phi(z) \sim -\log(y - z), \quad v^{ab} u_{cd} \sim \delta_{c}^{[a} \delta_{d]}^{b} (y - z)^{-1}.$$  

(2.4)

Using these parameterizations of a pure spinor, the OPE’s of the currents in (2.3) can be computed to be

$$N_{mn}(y) \lambda^\alpha(z) \sim \frac{1}{2} \frac{1}{y - z} (\gamma_{mn})^\alpha, \quad J(y) \lambda^\alpha(z) \sim \frac{1}{y - z} \lambda^\alpha,$$

(2.5)

$$N^{kl}(y) N^{mn}(z) \sim -\frac{3}{(y - z)^2} (\eta^{[k} \eta^{l]m}) + \frac{1}{y - z} (\eta^{m[l} N^{k]}n - \eta^{n[l} N^{k]}m),$$

$$J(y) J(z) \sim -\frac{4}{(y - z)^2}, \quad J(y) N^{mn}(z) \sim 0,$$

$$N_{mn}(y) T(z) \sim \frac{1}{(y - z)^2} N_{mn}(z), \quad J(y) T(z) \sim -\frac{8}{(y - z)^3} + \frac{1}{(y - z)^2} J(z),$$

$$T(y) T(z) \sim \frac{22}{2} \frac{1}{(y - z)^4} + \frac{2}{(y - z)^2} T(z) + \frac{1}{y - z} \partial T.$$
So the conformal central charge is 22, the ghost-number anomaly is $-8$, the Lorentz central charge is $-3$, and the ghost-number central charge is $-4$. One can verify the consistency of these charges by considering the Sugawara construction of the stress tensor

$$T = \frac{1}{2(k + \delta)} N_{mn} N^{mn} + \frac{1}{8} J J + \partial J , \quad (2.6)$$

where $k$ is the Lorentz central charge, $\delta$ is the dual Coxeter number for $SO(10)$. Setting $k = -3$, one finds that the $SO(10)$ current algebra contributes $-27$ to the conformal central charge and the ghost current contributes 49 to the conformal central charge. So the total conformal central charge is 22 as expected.

### 3 Partition function of pure spinors

In this section, we review some aspects of the pure spinor partition function and indicate the main results we are going to use. The characters of the states (in the pure spinor Hilbert space [7]) we wish to keep track of are their statistics, weights $h$ (Virasoro levels), $t$-charge (measured by $J = w_\alpha \lambda^\alpha$). Introducing formal variables $(q, t)$ for each quantum number, we define the character as

$$Z(q, t) = \text{Tr}(-1)^F q^{L_0} t^{J_0} = \sum_{h \geq 0} Z_h(t) q^h = \sum_{h \geq 0, n} N_{h,n} q^h t^n . \quad (3.1)$$

The trace is taken over the pure spinor Hilbert space. The quantum numbers of the basic operators $\omega$ and $\lambda$ are: $h(\omega, \lambda) = (1, 0)$, $t(\omega, \lambda) = (-1, 1)$. In [7], it was given a prescription for computing the $N_{h,n}$ coefficients [7]. Once these coefficients are known, it is possible to calculate the character of pure spinors at each level $h$

$$Z_h(t) = \sum_n N_{h,n} t^n . \quad (3.2)$$

At the lowest level, the Hilbert space is spanned by non-vanishing polynomials of $\lambda$. Due to the pure spinor constraint, $\lambda$’s can only appear in the “pure spinor representations”

$$\lambda^{(\alpha_1 \lambda^{\alpha_2} \cdots \lambda^{\alpha_n})} = [0000n] t^n , \quad (n \geq 0) . \quad (3.3)$$

Here, we also indicated the $t$-charge of the state, and the symbol $(\alpha_1 \alpha_2 \cdots \alpha_n)$ signifies the “spinorial $\gamma$-traceless condition”, which means that the expression is zero when any

\[^1\]These coefficients represent the number of states at each level $h$ with $n$ t-charge, and its sign tell us whether the states are fermionic or bosonic.
two indices $\alpha_i \alpha_j$ are contracted using $\gamma^\mu_{\alpha_i \alpha_j}$. Since the pure spinor representations have dimension
\[
\dim[0000n] = \frac{(n+7)(n+6)(n+5)^2(n+4)^2(n+3)^2(n+2)(n+1)}{7 \cdot 6 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2}, \tag{3.4}
\]
the level zero character is easily found to be \[ \text{Z}_0 = [8, 9] \]
\[
Z_0(t) = \frac{1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8}{(1 - t)^{16}}. \tag{3.5}
\]

As it was analyzed in [7, 13], the pure spinor constraint can be resolved by introducing an infinite chain of free-field ghosts. The multiplicities $N_k$ of the ghosts can be obtained by writing the level zero character (3.5) as [7, 8]
\[
Z_0(t) = \prod_{k=1}^{\infty} (1 - t^k)^{-N_k}. \tag{3.6}
\]
The fields at ghost number $k$ will have $|N_k|$ components, and will be bosons for $N_k > 0$ and fermions for $N_k < 0$. The multiplicities $N_k$ contain the information about the Virasoro central charge, as well as the ghost current algebra:
\[
\frac{1}{2} c_{\text{vir}} = \sum_{k=1}^{\infty} N_k, \quad a_{\text{ghost}} = -\sum_{k=1}^{\infty} kN_k, \quad c_{\text{ghost}} = -\sum_{k=1}^{\infty} k^2 N_k. \tag{3.7}
\]
We can easily deduce from (3.5) and (3.6):
\[
N_1 = 16, \quad N_2 = -10, \quad N_3 = 16, \quad N_4 = -45, \quad N_5 = 144, \quad N_6 = -456, \quad N_7 = 1440, \ldots \tag{3.8}
\]
For the computations to be done in the appendix A, we will need to know the value of the moments of the $N_k$'s, i.e. we want: $\sum_{k=1}^{\infty} \ k^{s+1} N_k$. This was analyzed in [8]. The moments of (3.8) are given by
\[
\sum_{k=1}^{\infty} k^{s+1} N_k = 12 - 2^{s+1} - \frac{1}{\zeta(-s)} \sum_{k=1}^{\infty} k^s \left( (-2 - \sqrt{3})^k + (-2 + \sqrt{3})^k \right) \tag{3.9}
\]
\[
= 12 - 2^{s+1} - \frac{Li_{-s}(-2 - \sqrt{3}) + Li_{-s}(-2 + \sqrt{3})}{\zeta(-s)},
\]
where $Li_s(z)$ is the so-called polylogarithm (also known as de Jonquier’s function), it is a special function defined by the sum
\[
Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.
\]
Another way to get $\sum_{k=1}^{\infty} k^{s+1} N_k$ is by considering the general expression of the form we analyzed in (3.5) and (3.6):

$$\prod_{k=1}^{\infty} (1 - t^k)^{-N_k} = \frac{P(t)}{Q(t)},$$

where $P$ and $Q$ are some polynomials. We have

$$\sum_{k=1}^{\infty} N_k \log(1 - e^{kx}) = - \log \frac{P(e^x)}{Q(e^x)}.$$

Since

$$\log(1 - e^x) = \log(-x) + \frac{x}{2} + \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g)!} x^{2g},$$

where $B_k$ are Bernoulli numbers, we have:

$$\log(x) \sum_{k=1}^{\infty} N_k + \sum_{k=1}^{\infty} \log(-k) N_k + \frac{x}{2} \sum_{k=1}^{\infty} k N_k + \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g)!} x^{2g} \sum_{k=1}^{\infty} k^{2g} N_k = - \log \frac{P(e^x)}{Q(e^x)}.$$  \hspace{1cm} (3.10)

Using (3.5) and expanding the right hand side RHS of (3.10), we obtain the following value for the moments

$$\sum_{k=1}^{\infty} N_k = 11, \quad \sum_{k=1}^{\infty} k N_k = 8, \quad \sum_{k=1}^{\infty} k^2 N_k = 4, \quad \sum_{k=1}^{\infty} k^4 N_k = -4,$$

$$\sum_{k=1}^{\infty} k^6 N_k = 4, \quad \sum_{k=1}^{\infty} k^8 N_k = \frac{68}{3} \quad \text{and} \quad \sum_{k=1}^{\infty} k^{10} N_k = -396.$$ \hspace{1cm} (3.11)

The first three moments of $N_k$’s given in (3.11) contain the information about the conformal central charge $c_{\text{vir}}$, the ghost-number anomaly $a_{\text{ghost}}$, and the ghost-number central charge $c_{\text{ghost}}$ (3.7). As we can check, the value of these current central charges are in agreement with the non-covariant calculation (2.5).

In a recent paper [7], using the ghosts, it was constructed a BRST operator and since this operator was given such that it carries zero $t$-charge, the partition function of its cohomology is equal to that of the total Hilbert space of (now unconstrained) pure spinors and the ghosts. Therefore, the partition function of pure spinors was formally written as

$$Z(q, t) = \prod_{k=1}^{\infty} \prod_{h=0}^{\infty} (1 - q^h t^k)^{-N_k} (1 - q^h t^{k-1})^{-N_k} (1 - q^h t^{k-2})^{-N_k} \cdots$$

$$Z(q, t) = \prod_{k=1}^{\infty} (1 - q^h t^{k-1})^{-N_k} \prod_{h=1}^{\infty} (1 - q^h t^{k-2})^{-N_k} = \sum_{h=0}^{\infty} Z_h(t) q^h.$$
Using (3.11) and (3.12), an elementary calculation shows

\[
Z(q, t) = -t^{-8}Z(q, 1/t), \quad (3.13)
\]

\[
Z(q, t) = -t^{-4}q^{2}Z(q, q/t), \quad (3.14)
\]

These symmetries (3.13) and (3.14) of the partition function are referred as field-antifield and ∗-conjugation symmetry respectively [7]. In the forthcoming sections, starting from the formal expression of the partition function (3.12), we are going to describe a method for computing higher level character formulas \( Z_{h}(t) \).

4 \hspace{1em} \textbf{Padé approximants}

The Padé approximation seeks to approximate the behavior of a function by a ratio of two polynomials. This ratio is referred to as the Padé approximant. This approximation works nicely even for functions containing poles, because the use of rational functions allows them to be well-represented. Recently, the Padé approximation has been applied to string field theory to analyze the tachyon condensation [16, 17, 18].

Let us now consider the general equations of the Padé approximation. Given some function \( f(t) \), its \([M/N]\) Padé approximant denoted by \( f^{[M/N]}(t) \) is a rational function of the form [14]

\[
f^{[M/N]}(t) = \frac{1 + \sum_{j=1}^{M} p_{j}t^{j}}{\sum_{j=0}^{N} q_{j}t^{j}}, \quad (4.1)
\]

where the coefficients \( p_1, p_2, \ldots, p_{M}, q_0, q_1, \ldots, q_{N} \), are obtained by solving a system of \( M + N + 1 \) algebraic equations

\[
\frac{d^{n}f^{[M/N]}}{dt^{n}}(a) = f^{(n)}(a), \quad n = 0, 1, 2, \ldots, M + N. \quad (4.2)
\]

The equations (4.2) came from equating the coefficients of \((t - a)^{n}\) (up to the order \( n = M + N \)) in the Taylor expansion of the functions \( f(t) \) and \( f^{[M/N]}(t) \) around some point \( t = a \) (which usually is taken at \( t = 0 \)).

Having sketched briefly the method to approximate functions by means of rational functions. Next, we are going to use this method for computing higher level character formulas of pure spinors. Let us start by written the formal expression (3.12) for the partition function of pure spinors like the following

\[
Z(q, t) = Z_{0}(t) \left[ 1 + \sum_{h=1}^{\infty} f_{h}(t)q^{h} \right], \quad (4.3)
\]
where the level $h$ function $f_h(t)$ is defined by

$$f_h(t) = \frac{1}{h!} \frac{\partial^h}{\partial q^h} \tilde{Z}(q, t)|_{q=0}$$

where $\tilde{Z}(q, t) = \prod_{k=1}^{\infty} \prod_{h=1}^{\infty} (1 - q^h t^k)^{-N_k} (1 - q^h t^{-k})^{-N_k}.$ (4.4)

As we know by a previous work [7], up to the level $h = 5$, these level $h$ functions are given by rational functions. Therefore, this result is an indication that these level $h$ functions can be computed by means of Padé approximants. In fact this is the case as it is shown in the appendix A, the functions $f_1(t), f_2(t), f_3(t), \cdots$ can be calculated using Padé approximants. As our main result, we have noted that these functions can be written like the following

$$f_h(t) = \frac{\sum_{i=0}^{2h+6} C_{i,h} t^i}{t^{h+2}(1 + 4t + t^2)},$$

(4.5)

where the value of the coefficients $C_{i,h}$ up to the level $h = 12$ are given in the following tables

| $i$  | $C_{i,0}$ | $C_{i,1}$ | $C_{i,2}$ | $C_{i,3}$ | $C_{i,4}$ | $C_{i,5}$ | $C_{i,6}$ | $C_{i,7}$ |
|------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0    | 0         | 0         | -1        | -16       | -126      | -672      | -2772     | -9504     |
| 1    | 0         | 0         | 12        | 146       | 920       | 3996      | 13440     | 37224     |
| 2    | 1         | 0         | -67       | -536      | -2411     | -7616     | -18358    | -35184    |
| 3    | 4         | 46        | 248       | 822       | 1852      | 3270      | 7752      | 33356     |
| 4    | 1         | 40        | 319       | 1200      | 1745      | -5944     | -48147    | -179648   |
| 5    | 0         | 46        | 628       | 4114      | 17000     | 48206     | 91948     | 87730     |
| 6    | 0         | 0         | 319       | 3720      | 21767     | 82112     | 210717    | 326760    |
| 7    | 0         | 0         | 248       | 4114      | 32356     | 162662    | 585464    | 1575690   |
| 8    | 0         | 0         | -67       | 1200      | 21767     | 162552    | 778424    | 2706944   |
| 9    | 0         | 0         | 12        | 822       | 17000     | 162662    | 977032    | 4215020   |
| 10   | 0         | 0         | -1        | -536      | 1745      | 82112     | 778424    | 4454624   |
| 11   | 0         | 0         | 0         | 146       | 1852      | 48206     | 585464    | 4215020   |
| 12   | 0         | 0         | 0         | -16       | -2411     | -5944     | 210717    | 2706944   |
| 13   | 0         | 0         | 0         | 0         | 920       | 3270      | 91948     | 1575690   |
| 14   | 0         | 0         | 0         | 0         | -126      | -7616     | -48147    | 326760    |
| 15   | 0         | 0         | 0         | 0         | 0         | 3996      | 7752      | 87730     |
| 16   | 0         | 0         | 0         | 0         | 0         | -672      | -18358    | -179648   |
| 17   | 0         | 0         | 0         | 0         | 0         | 13440     | 33356     | 37224     |
| 18   | 0         | 0         | 0         | 0         | 0         | -2772     | -35184    |          |
| 19   | 0         | 0         | 0         | 0         | 0         | 0         | 37224     |          |
| 20   | 0         | 0         | 0         | 0         | 0         | 0         | -9504     |          |
We have defined the values of the $C_{i,0}$ coefficients such that the level zero function is defined as $f_0(t) = 1$. It is interesting to note that the coefficients $C_{i,h}$ satisfy the following identities

$$C_{i,h} = C_{2h+6-i,h}, \quad (4.8)$$

$$\sum_{i=0}^{h'} \phi_{h'-i} C_{i,h} = -\sum_{i=0}^{h} \phi_{h-i} C_{i,h'}, \quad (4.9)$$

which can be derived by using the two symmetries of the partition function (3.13), (3.14).
and verified by using the coefficients shown in the above tables. The coefficient $\phi_m$ is generated by

$$
\frac{Z_0(t)}{1 + 4t + t^2} = \sum_{n=0}^{\infty} \phi_n t^n,
$$

(4.10)

and it is given explicitly by the formula

$$
\phi_n = \frac{(1 + n)(2 + n)(3 + n)(4 + n)(5 + n)^2(6 + n)(7 + n)(8 + n)(9 + n)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7}.
$$

(4.11)

The importance of the identity (4.9) is as follows. If we know the coefficients $C_{i,0}$, $C_{i,1}$, $\cdots$, $C_{i,h'}$, it is possible to compute explicitly the coefficients $C_{0,h}$, $C_{1,h}$, $\cdots$, $C_{h',h}$. For instance, setting $h' = 0$ in equation (4.9), we get

$$
C_{0,h} = -\sum_{i=0}^{h} \phi_{h-i} C_{i,0},
$$

(4.12)

using (4.11) and the value of the coefficients $C_{i,0}$ given in the table (4.6) into the equation (4.12), we obtain

$$
C_{0,h} = \frac{(1 - h)h(1 + h)^2(2 + h)^2(3 + h)^2(4 + h)(5 + h)}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7}.
$$

(4.13)

By employing the same steps given above, setting $h' = 1$ in equation (4.9), we arrive to the following expression for the $C_{1,h}$ coefficient

$$
C_{1,h} = \frac{(h - 1)h(1 + h)^2(2 + h)(3 + h)(4 + h)(108 + 10h + 12h^2 - h^3)}{2^5 \cdot 3^3 \cdot 5 \cdot 7}.
$$

(4.14)

Finally, it would be important to find an explicit expression for a general coefficient $C_{i,h}$ (for all $h \geq 0$ and $i \geq 0$). It is clear that if we know explicitly $C_{i,h}$, it should be possible to write a compact expression for the complete pure spinor partition function

$$
Z(q, t) = \frac{1 + t}{(1 - t)^{11}} \sum_{h=0}^{\infty} \sum_{i=0}^{2h+6} C_{i,h} t^{i-h-2} q^h,
$$

(4.15)

where the factor $(1 + t)/(1 - t)^{11}$, in front of our formula (4.15), comes from substitution of equations (3.5) and (4.5) into the equation (4.3).

5 Summary and discussions

We have given in detail a prescription for computing the partition function of pure spinors. This prescription is mainly based in the knowledge of the zero mode part $Z_0(t)$ of the
partition function. From this level zero character formula, we have extracted the ghosts multiplicities $N_k$, and using the moments $(\sum_k k^{s+1}N_k)$ of those multiplicities, by employing a novel application of Padé approximants, we were able to compute higher level character formulas $Z_h(t)$ of pure spinors (up to the twelfth mass-level $h = 12$). We have found that our results are in agreement with the results found in [7] (up to the fifth mass-level $h = 5$) where the fixed point technique was used.

We mention a subtle computational issue related to the fixed point formula [7, 8]. In general for $SO(2d)$ pure spinors the number of fixed points is $2^{d-1}$ and the complexity of summing over these fixed points (as it was also noted in [8]) grows exponentially with $N = 2^{d-1}$. On the other hand the computations shown in this letter are less complicated, so our technique can be used as an alternative (to the fixed point technique) easier (computationally) way to get the character formulas for higher-dimensional pure spinors.

So far, in this work, we have computed the partition function without the spin dependence on the states. Spin dependence is crucial if we want to prove that the full partition function (including the contribution of the worldsheet matter sector) correctly reproduces the light cone superstring spectrum [7]. Therefore, it would be interesting to know the character formula with the spin dependence in the ghosts-for-ghosts scheme. We left this issue as a future work.

It would be nice to see whether our technique will also apply to other constrained systems like strings moving on algebraic surfaces. Another possible application would be the computation of the partition function of eleven-dimensional pure spinors, this can be an interesting issue because an attempt at quantization of the supermembrane was given by using eleven-dimensional pure spinors [19].

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### A Computation of higher level character formulas

Higher level character formulas $Z_h(t)$ can be obtained from the formal expression (3.12) as follows. Performing a Taylor expansion of the expression (3.12) around $q = 0$, we have

$$Z(q, t) = Z_0(t) + \sum_{h=1}^{\infty} \frac{q^h}{h!} \frac{\partial^h}{\partial q^h} Z(q, t) \bigg|_{q=0}$$

$$= Z_0(t) \left[ 1 + \sum_{h=1}^{\infty} f_h(t) q^h \right],$$

where the level $h$ function $f_h(t)$ has been defined as the expression (4.4). To obtain these level $h$ functions, we are going to use a method based on Padé approximants. Let us explain our method by computing in detail the level one function $f_1(t)$.

From the expression (4.4), we derive the following expression for the level one function

$$f_1(t) = \sum_{k=1}^{\infty} N_k (t^k + t^{-k}),$$

(A.2)

expanding the RHS of (A.2) around $t = 1$ and keeping terms up to some order (relevant for the computations to be done next), we get

$$f_1(t) = 2 \sum_{k=1}^{\infty} N_k + (t - 1)^2 \sum_{k=1}^{\infty} k^2 N_k - (t - 1)^3 \sum_{k=1}^{\infty} k^2 N_k + \cdots$$

(A.3)

Applying the formula (3.9) to find the even moments $\sum_k N_k, \sum_k k^2 N_k$ and replacing them into the equation (A.3), we obtain

$$f_1(t) = 22 + 4(t - 1)^2 - 4(t - 1)^3 + \cdots$$

(A.4)

Using Padé approximants, we express the function $f_1(t)$ as a rational function

$$f_1(t) \approx f_1^{[M/N]}(t) = \frac{1 + \sum_{j=1}^{M} p_j t^j}{\sum_{j=0}^{N} q_j t^j},$$

(A.5)

for instance, as a pedagogical illustration let us compute explicitly the $[2/1]$ Padé approximant of $f_1(t)$

$$f_1^{[2/1]}(t) = \frac{1 + p_1 t + p_2 t^2}{q_0 + q_1 t},$$

(A.6)

expanding the RHS of (A.6) around $t = 1$, we get

$$f_1^{[2/1]}(t) = \frac{1 + p_1 + p_2}{q_0 + q_1} + \frac{p_1 q_0 + 2 p_2 q_0 - q_1 + p_2 q_1}{(q_0 + q_1)^2} (t - 1)$$

$$+ \frac{p_2 q_0^2 - p_1 q_0 q_1 + q_1^2}{(q_0 + q_1)^3} (t - 1)^2 - \frac{p_2 q_0^3 - p_1 q_0^2 q_1^2 + q_1^3}{(q_0 + q_1)^4} (t - 1)^3 + \cdots$$

(A.7)
Equating the coefficients of \((t-1)^0, (t-1)^1, (t-1)^2, (t-1)^3\) in equations (A.4) and (A.7), we get 4 equations for the unknown coefficients \(p_1, p_2, q_0, q_1\)

\[
\frac{1 + p_1 + p_2}{q_0 + q_1} = 22, \\
\frac{p_1 q_0 + 2p_2 q_0 - q_1 + p_2 q_1}{(q_0 + q_1)^2} = 0, \\
\frac{p_2 q_0^2 - p_1 q_0 q_1 + q_1^2}{(q_0 + q_1)^3} = 4, \\
\frac{p_2 q_1 q_0^2 - p_1 q_0 q_1^2 + q_1^3}{(q_0 + q_1)^4} = 4,
\]

solving these system of equations (A.8) we obtain

\[
p_1 = \frac{7}{2}, \quad p_2 = 1, \quad q_0 = 0, \quad q_1 = \frac{1}{4}.
\]

Computations of higher Padé approximants follows in the same way as it was shown above. The results of these computations are given in the following table

| \([M/N]\) | \(p_1, p_2, \cdots, p_M\) | \(q_0, q_1, \cdots, q_N\) |
|----------|----------------|----------------|
| 2/1      | 7/2, 1         | 0, 1/4         |
| 2/2      | 20/23, 1       | 1/46, 2/23, 1/46 |
| 3/1      | 1, 18/25, -2/25 | 1/50, 1/10   |
| 1/3      | 43/23          | 75/5566, 875/5566, -135/2783, 1/121 |
| 3/2      | 20/23, 1, 0    | 1/46, 2/23, 1/46 |
| 2/3      | 20/23, 1       | 1/46, 2/23, 1/46, 0 |
| 3/3      | 20/23, 1, 0    | 1/46, 2/23, 1/46, 0 |
| 4/4      | 20/23, 1, 0, 0 | 1/46, 2/23, 1/46, 0, 0 |

As we can see by explicit computations, the Padé approximants are approaching to the rational function \((46 + 40t + 46t^2)/(1 + 4t + t^2)\), and therefore we take this function as being the level one function \(f_1(t)\)

\[
f_1(t) = \frac{46 + 40t + 46t^2}{1 + 4t + t^2}.
\]

By multiplying this function (A.11) with the level zero character \(Z_0(t)\), we get

\[
Z_1(t) = \frac{46 - 144t + 116t^2 + 16t^3 - 16t^5 - 116t^6 + 144t^7 - 46t^8}{(t - 1)^{16}},
\]

and therefore, we correctly reproduce the level one character formula given in \([7, 9]\).
For the next level $h = 2$, by using the same strategy shown above, we have found that the Padé approximant computation gives the following result for the level two function

$$f_2(t) = \frac{-1 + 12t - 67t^2 + 248t^3 + 319t^4 + 628t^5 + 319t^6 + 248t^7 - 67t^8 + 12t^9 - t^{10}}{t^4(1 + 4t + t^2)}.$$  

By multiplying this level two function $f_2(t)$ with the level zero character $Z_0(t)$, we correctly reproduce the level two character formula found in [7].

Computation of higher level functions $f_h(t)$ by means of Padé approximants, suggest us that these functions can be written like

$$f_h(t) = \sum_{i=0}^{2h+6} C_{i,h} t^i \left(1 + 4t + t^2\right). \quad (A.13)$$

We have computed the $C_{i,h}$ coefficients up to the level $h = 12$, the results are given in the tables (4.6) and (4.7) of section 4. Multiplying the functions $f_h(t)$ with the level zero character formula $Z_0(t)$, we obtain the characters $Z_h(t)$. We have compared our first five character formulas with the formulas given in [7] and we have found agreement.

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