Maximal tracial algebras

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Abstract
We introduce and study the notion of maximal tracial algebras. We prove several results in a general setting based on dual pairs and multiplier pairs. In a special case that $X$ is a Banach space we determine the abelian subalgebras of $B(X)$ that are maximal tracial for rank-one tensors. In another special case that $H$ is a Hilbert space we show that a unital weak-operator closed subalgebra $A$ of $B(H)$ is abelian and transitive if and only if it is maximal $e \otimes e$-tracial for every unit vector $e$ in $H$. We also make slight connections between our ideas and the Kadison Similarity Problem and also the Connes’ Embedding Problem.

Keywords Tracial algebra · Dual pair · Multiplier pair · Tracial ultraproduct

Mathematics Subject Classification 46L05 · 47L10 · 46M07 · 46L10

1 Introduction and definitions
Algebras with traces have played an important role in linear algebra and the theory of operator algebras [3,11,15]. We consider a situation where $\mathcal{A}$ is a unital subalgebra of an algebra $\mathcal{B}$ and $\varphi$ is a functional on $\mathcal{B}$ with $\varphi(1) = 1$ that is tracial on $\mathcal{A}$. We study the case when $\mathcal{A}$ is a maximal such algebra for $\varphi$. For the case when $\mathcal{B}$ is the algebra of $2 \times 2$ matrices over an arbitrary field, we completely characterize the pairs $(\mathcal{A}, \varphi)$ so that $\mathcal{A}$ is maximal $\varphi$-tracial. In particular we show some connections between our
ideas in this paper and two very famous problems in Functional Analysis, namely the Kadison Similarity Problem [10] and the Connes’ Embedding Problem [2].

Throughout this paper \( \mathcal{H} \) is a separable Hilbert space over the field of complex numbers \( \mathbb{C} \) and \( B(\mathcal{H}) \) denotes the set of all bounded operators on \( \mathcal{H} \). We show the set of all \( k \times k \) matrices over a field \( \mathbb{F} \) by \( M_k(\mathbb{F}) \) and for \( A \in M_k(\mathbb{F}) \) we use \( Tr (A) \) to denote the usual trace of \( A \). The identity matrix in \( M_k(\mathbb{F}) \) is shown by \( I_k \) and by \( e_{ij} \) \((1 \leq i, j \leq k)\) we mean the standard matrix units in \( M_k(\mathbb{F}) \), i.e. the \( k \times k \) matrix whose all entries are 0’s except the \((i, j)\)-entry which is 1. For \( x, y \in \mathcal{H} \), the rank-one operator \( x \otimes y \) is defined by \( (x \otimes y) h = \langle h, y \rangle x \). Note that if \( A, B \in B(\mathcal{H}) \), then \( A (x \otimes y) B = Ax \otimes B^* y \). More generally, for a Banach space \( X \), a linear functional \( \alpha \) in the normed dual of \( X \), and \( x_0 \in X \) we define the linear map \( x_0 \otimes \alpha : B (X) \to \mathbb{C} \) by \( (x_0 \otimes \alpha) (T) = \alpha (T(x_0)) \). If \( B \) is an algebra over a field \( \mathbb{F} \) and \( S \subseteq B \), then by the commutant \( S' \) of \( S \) in \( B \) we mean \( S' = \{ B \in B : BS = SB, \forall S \in S \} \). If the algebra \( B \) is unital, then we show its unit by 1.

Section one of this paper contains some definitions, notations, terminologies, and a motivating example.

Most of our main results are in section two where we prove several theorems in very general cases for dual pairs. Theorem 2.3 is one of the main results of this section in which we characterize when an abelian algebra is maximal \( \varphi \)-tracial provided that \( \varphi \) is a rank-one functional. This result, for instance, is used in a special case in Corollary 2.6 to determine which abelian subalgebras of \( B(X) \) are maximal tracial for rank-one tensors \( x \otimes \alpha \), where \( X \) is a Banach space, \( x \in X \), and \( \alpha \) is a unital linear functional in the normed dual of \( X \). In this section, we also show that a unital weak-operator closed subalgebra \( A \) of \( B(\mathcal{H}) \) is maximal \( e \otimes e \)-tracial for every unit vector \( e \) in \( \mathcal{H} \) if and only if \( A \) is abelian and transitive. We also provide several interesting examples in this section.

Section three deals with Multiplier Pairs. We prove a general statement in Theorem 3.1 about multiplier pairs that gives a lot of examples of maximal tracial algebras. For instance we use Theorem 3.1 in section four where maximal tracial algebras of von Neumann algebras are studied.

In section four, after proving a few theorems, we raise the following question: If a von Neumann algebra \( \mathcal{M} \) on a Hilbert space \( \mathcal{H} \) is \((e \otimes f)\)-tracial, where \( e, f \in \mathcal{H} \) are cyclic vectors for \( \mathcal{M} \) and \( \langle e, f \rangle = 1 \), then should \( \mathcal{M} \) be maximal \((e \otimes f)\)-tracial? For the case where \( \mathcal{H} \) is separable, we will reduce the problem to the case where \( \mathcal{M} \) is a finite factor von Neumann algebra, and such a factor has a unique norm continuous unital tracial functional. We will also show a slight relationship between some of our ideas in this section and the Kadison Similarity Problem [10].

In the last section of our paper, by introducing a new type of ultraproduct, we make a connection between some of our results in this paper and the Connes’ Embedding Problem [2]. We will show in Theorem 5.1 that the analogue of Connes’ embedding problem has an affirmative answer in the setting of maximal tracial ultraproducts.

**Definition 1.1** Suppose \( \mathcal{A} \) and \( \mathcal{B} \) are unital algebras over a field \( \mathbb{F} \) with \( 1 \in \mathcal{A} \subseteq \mathcal{B} \) and suppose \( \varphi : \mathcal{B} \to \mathbb{F} \) is a linear map such that \( \varphi (1) = 1 \). We say that \( \mathcal{A} \) is a tracial algebra for \( \varphi \) or \( \mathcal{A} \) is \( \varphi \)-tracial (or \( \varphi \) is a tracial functional for \( \mathcal{A} \)) if, for every \( x, y \in \mathcal{A} \),

\[ \langle e_{ij}, e_{kl} \rangle = \delta_{ij} \delta_{kl}, \quad 1 \leq i, j, k, l \leq n \]
\[ \varphi(xy) = \varphi(yx). \]

We say that \( A \) is a \textit{maximal tracial algebra} for \( \varphi \) in \( B \) if \( A \) is tracial for \( \varphi \) and no larger algebra of \( A \) is tracial for \( \varphi \).

The following is our first simple, yet useful, result. If \( B \) is unital algebra over a Hausdorff field \( \mathbb{F} \), then by an \textit{algebra topology} on \( B \) we mean a topology for which addition, scalar multiplication and the multiplication in each variable are all continuous.

**Proposition 1.2** Suppose \( 1 \in B \) is an algebra over a field \( \mathbb{F} \), \( \varphi : B \to \mathbb{F} \) is linear. Then

1. If \( A \subset B \) is a \( \varphi \)-tracial algebra, then \( A + F1 \) is also \( \varphi \)-tracial.
2. If \( \{ A_i : i \in I \} \) is an increasingly directed family of \( \varphi \)-tracial algebras, then also \( \bigcup_{i \in I} A_i \) is \( \varphi \)-tracial.
3. Every \( \varphi \)-tracial algebra is contained in a maximal \( \varphi \)-tracial algebra.
4. Every maximal \( \varphi \)-tracial algebra is unital.
5. If \( \mathbb{F} \) is a Hausdorff field, \( A \) is \( \varphi \)-tracial, and \( \varphi \) is continuous with respect to some Hausdorff algebra topology \( T \) on \( B \), then \( A^{-T} \) is also \( \varphi \)-tracial.
6. If \( \mathbb{F} \) is a Hausdorff field, \( A \) is \( \varphi \)-tracial, and \( \varphi \) is continuous with respect to some Hausdorff algebra topology \( T \) on \( B \), then every maximal \( \varphi \)-tracial algebra in \( B \) is \( T \)-closed.

**Proof** (1). This follows from the fact that for every \( a, b \in A \) and every \( \alpha, \beta \in \mathbb{F} \) we have

\[(a + \alpha 1)(b + \beta 1) - (b + \beta 1)(a + \alpha 1) = ab - ba.\]

(2) – (4). These are trivial.

(5). Suppose \( a, b \in A^{-T} \). Then there are nets \( \{a_n\}, \{b_m\} \) in \( A \) such that \( a_n \to a \) and \( b_m \to b \) with respect to \( T \). Thus, for each \( m \),

\[a_nb_m \to ab_m \text{ and } b_ma_n \to b_ma.\]

Thus

\[\varphi(b_ma) = \lim_n \varphi(b_ma_n) = \lim_n \varphi(a_nb_m) = \varphi(ab_m).\]

Similarly,

\[\varphi(ba) = \lim_m \varphi(b_ma) = \lim_m \varphi(ab_m) = \varphi(ab)\]

(6). This easily follows from (5). \( \square \)

We now look at a simple matrix example which was the motivating example for this paper.
Example 1.3 Suppose \( \mathbb{F} \) is any field. A linear functional \( \varphi : M_2 (\mathbb{F}) \to \mathbb{F} \) is given by a matrix \( K \in M_2 (\mathbb{F}) \) and is defined by \( \varphi (T) = Tr (TK) \), where \( Tr \) is the usual trace. In this case we write \( \varphi = \varphi_K \). Here

\[
K = \begin{pmatrix}
\varphi (e_{11}) & \varphi (e_{21}) \\
\varphi (e_{12}) & \varphi (e_{22})
\end{pmatrix}
\]

where \( e_{ij} \) (\( 1 \leq i, j \leq 2 \)) are the standard matrix units for \( M_2 (\mathbb{F}) \). The condition \( \varphi_K (1) = 1 \) implies \( Tr (K) = 1 \). Also, if \( A \) is a unital algebra in \( M_2 (\mathbb{F}) \) and \( S \in M_2 (\mathbb{F}) \) invertible, then \( \varphi_K \) is tracial if and only if \( \varphi_S^{-1}KS \) is tracial for \( A \). Hence the problem of finding the tracial unital functionals for \( A \) is the same as for \( SAS^{-1} \).

For a given unital subalgebra \( A \) of \( M_2 (\mathbb{F}) \), we want to find all functionals \( \varphi \) for which \( A \) is maximal \( \varphi \)-tracial.

If \( \text{dim} (A) = 1 \), then \( A \) is abelian, but not maximal abelian, so \( A \) is not maximal tracial for any functional.

If \( \text{dim} (A) = 4 \), then the only tracial functional on \( A \) is \( \lambda Tr (\cdot) \) for some \( \lambda \) in \( \mathbb{F} \). If the characteristic of \( \mathbb{F} \) is 2, then \( \lambda Tr (1) = 0 \neq 1 \), and no unital \( \varphi \) exists, and if the characteristic of \( \mathbb{F} \) is not 2, then \( \varphi = \frac{1}{2} Tr (\cdot) \) is the only tracial unital functional on \( A \).

Next, suppose \( \text{dim} (A) = 2 \). Then there is a matrix \( T \notin \mathbb{F}I_2 \) such that \( A = \mathbb{F}I_2 + \mathbb{F}T \), that is, the algebra of polynomials in \( T \). We want to find the maximal \( \varphi \)-tracial algebras in \( M_2 (\mathbb{F}) \). Let \( p (\lambda) = \lambda^2 - Tr (T) \lambda + \det (T) \) be the characteristic polynomial of \( T \).

If \( p (T) \) splits, then \( A \) is similar to either

\[
D_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{F} \right\} \quad \text{or} \quad T_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{F} \right\}.
\]

If \( \varphi \neq \frac{1}{2} Tr (\cdot) \), then \( D_2 \) or \( T_2 \) is maximal \( \varphi \)-tracial only if there is a 3-dimensional algebra \( D \) that contains \( D_2 \) or \( T_2 \) for which \( \varphi \) is tracial. The only 3-dimensional algebras containing \( D_2 \) are

\[
U_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{F} \right\} \quad \text{and} \quad L_2 = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b \in \mathbb{F} \right\}.
\]

Since \( e_{11}e_{12} = e_{12} \) and \( e_{12}e_{11} = 0 \), then a functional \( \varphi \) is tracial on \( U_2 \) if and only if \( \varphi (e_{12}) = 0 \). Similarly, a functional \( \varphi \) is tracial on \( L_2 \) if and only if \( \varphi (e_{21}) = 0 \). Thus \( D_2 \) is maximal tracial for \( \varphi \) if and only if \( \varphi (e_{12}) \neq 0 \) and \( \varphi (e_{21}) \neq 0 \). Also the only 3-dimensional algebra containing \( T_2 \) is \( U_2 \), so \( T_2 \) is maximal \( \varphi \)-tracial if and only if \( \varphi (e_{12}) \neq 0 \).

If \( p (\lambda) \) doesn’t split, then \( A \) is isomorphic to \( \mathbb{F} [x] / \langle p (x) \rangle \), which is a field. If \( D \) is an algebra and \( A \subsetneq D \), then

\[
\dim_{\mathbb{F}} (D) = 2 \dim_{\mathbb{A}} (D) \geq 4.
\]
Hence $\mathcal{A}$ is a maximal subalgebra of $\mathcal{M}_2(\mathbb{F})$. Thus if $\varphi \neq \frac{1}{2} \text{Tr}()$, then $\mathcal{A}$ is maximal $\varphi$-tracial.

If $\dim(\mathcal{A}) = 3$, then the characteristic polynomial of every $T \in \mathcal{A}$ splits, and $\mathcal{A}$ must be similar to $\mathcal{U}_2$, which is maximal $\varphi$-tracial if and only if $\varphi \neq \frac{1}{2} \text{Tr}()$ and $\varphi(e_{12}) \neq 0$.

## 2 General dual pairs

Dual pairs over a Hausdorff field were studied in [5]. Suppose that $\mathbb{F}$ is a topological field with a Hausdorff topology, e.g., a subfield of the complex numbers with the usual topology, or an arbitrary field with the discrete topology. Next, suppose that $X$ is a vector space over $\mathbb{F}$ and $Y$ is a vector space of linear maps from $X$ to $\mathbb{F}$ such that $\bigcap \{\ker f : f \in Y\} = \{0\}$ (that is, $Y$ separates the points of $X$). Such a pair $(X, Y)$ is called a dual pair over $\mathbb{F}$. In [5] dual pairs were used to unify some of the results on different versions of reflexivity for algebras and linear spaces of $\mathcal{B} (\mathcal{H})$. In this section, for a dual pair $(X, Y)$, we will prove some theorems for continuous linear maps on $X$ and $Y$ that will provide several important results in special cases of dual pairs.

**Definition 2.1** Let $X$ be a vector space over a field $\mathbb{F}$, $X'$ be the dual space of $X$, and let $L(X)$ be the set of linear maps on $X$. For a linear map $\alpha : X \to \mathbb{F}$ and $x_0 \in X$ we define the linear maps $x_0 \otimes \alpha : L(X) \to \mathbb{F}$ and $\alpha \otimes x_0 : L(X') \to \mathbb{F}$ by

$$(x_0 \otimes \alpha)(T) = \alpha(T(x_0)) \text{ and } (\alpha \otimes x_0)(S) = S(\alpha)(x_0).$$

For a dual pair $(X, Y)$ over a Hausdorff field $\mathbb{F}$ we let the $\sigma(X, Y)$-topology on $X$ be the smallest topology on $X$ that makes all of the maps in $Y$ continuous on $X$, and we let the $\sigma(Y, X)$-topology on $Y$ be the smallest topology for which the map $f \to f(x)$ is continuous on $Y$ for each $x \in X$. For $A \subseteq X$ and $B \subseteq Y$, we define

$$A^\perp = \{f \in Y : f|_A = 0\} \text{ and } B^\perp = \bigcap \{\ker f : f \in B\}.$$

For a subset $D$ of either $X$ or $Y$, we denote the linear span of $D$ by $sp(D)$.

It was shown in [5, Proposition 1.1] that

$$\left(A^\perp\right)^\perp = sp(\sigma(X,Y)) \text{ and } \left(B^\perp\right)^\perp = sp(\sigma(Y,X))$$

always hold. In particular, if $0 \neq x \in X$, there is an $f \in Y$ such that $f(x) = 1$. It was also shown in [5, Proposition 1.1] that if $f : X \to \mathbb{F}$ is linear and $\sigma(X, Y)$ continuous, then $f \in Y$.

Suppose $(X, Y)$ is a dual pair over a Hausdorff field $\mathbb{F}$. Let $L_{\sigma(X,Y)}(X)$ denote the linear transformations $T : X \to X$ that are $\sigma(X, Y)$-$\sigma(X, Y)$ continuous. We give $L_{\sigma(X,Y)}(X)$ the weak operator topology (WOT) defined as the topology of pointwise $\sigma(X,Y)$ convergence. This means that the weak operator topology is the weak topology on $L_{\sigma(X,Y)}(X)$ induced by the set of linear functionals $\mathcal{E} = \{\alpha\}$. 

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Lemma 2.2 Suppose that \( \{ x \otimes y : x \in X, y \in Y \} \). Consequently, if \( \mathcal{W} = sp(\mathcal{E}) \), then \( L_{\sigma(X,Y)}(X) \), \( \mathcal{W} \) is a dual pair and the weak operator topology is the \( \sigma(L_{\sigma(X,Y)}(X), \mathcal{W}) \) topology. Thus, by [5, Proposition 1.1], a linear map \( \varphi : L_{\sigma(X,Y)}(X) \to F \) is WOT continuous if and only if it is in \( \mathcal{W} \), i.e., of the form \( \varphi = \sum_{k=1}^{n} x_k \otimes \alpha_k \) for \( x_1, \ldots, x_n \in X \) and \( \alpha_1, \ldots, \alpha_n \in Y \).

If we denote the duality of \( X \) and \( Y \) with the notation

\[
\alpha(x) = \langle x, \alpha \rangle
\]

for \( x \in X \) and \( \alpha \in Y \), then, for every \( T \in L_{\sigma(X,Y)}(X) \), there is a \( T^\# \in L_{\sigma(Y,X)}(Y) \) such that, for every \( x \in X \) and every \( \alpha \in Y \),

\[
\langle Tx, \alpha \rangle = \langle x, T^\# \alpha \rangle.
\]

Clearly, \( T^{\#\#} = T \), \( 1^\# = 1 \), \( (ST)^\# = T^\#S^\# \), \( (\lambda S + T)^\# = \lambda S^\# + T^\# \). If \( S \subset L_{\sigma(X,Y)}(X) \), we let \( S^\# \) denote \( \{ S^\# : S \in S \} \). If \( \varphi \) is a WOT continuous linear functional on \( L_{\sigma(X,Y)}(X) \), we define the adjoint functional \( \varphi^\# \) on \( L_{\sigma(Y,X)}(Y) \) by

\[
\varphi^\#(T^\#) = \varphi(T).
\]

Clearly \( (x \otimes \alpha)^\# = \alpha \otimes x \).

The following lemma can be proved easily.

**Lemma 2.2** Suppose that \( (X, Y) \) is a dual pair, \( A \) is a unital subalgebra of \( L_{\sigma(X,Y)}(X) \), and that \( \varphi \) is a WOT continuous linear functional such that \( \varphi(1) = 1 \). Then \( A \) is \( \varphi \)-tracial if and only if \( A^\# \) is \( \varphi^\# \)-tracial and \( A \) is maximal \( \varphi \)-tracial if and only if \( A^\# \) is maximal \( \varphi^\# \)-tracial.

Note that every abelian algebra is \( \varphi \)-tracial for every \( \varphi \). The next theorem characterizes when an abelian algebra is maximal \( \varphi \)-tracial provided \( \varphi \) is a rank-one functional.

**Theorem 2.3** Suppose that \( (X, Y) \) is a dual pair, \( A \) is an abelian unital subalgebra of \( L_{\sigma(X,Y)}(X) \), and that \( x \in X \), \( \alpha \in Y \) with \( (x \otimes \alpha)(1) = \alpha(x) = 1 \). Then \( A \) is maximal tracial with respect to \( x \otimes \alpha \) if and only if the following conditions hold:

1. \( A \) is maximal abelian in \( L_{\sigma(X,Y)}(X) \),
2. \( [Ax]^{-\sigma(X,Y)} = X \), and
3. \( [A^\#x]^{-\sigma(Y,X)} = Y \).

**Proof** Suppose \( A \) is maximal tracial with respect to \( x \otimes \alpha \). Clearly, \( A \) is maximal abelian. Assume, via contradiction, that \( M = [Ax]^{-\sigma(X,Y)} \neq X \). Let

\[
A_1 = \{ T \in L_{\sigma(X,Y)}(X) : T(M) \subset M, T|_M \in A|_M \}.
\]

Then \( A_1 \) is tracial with respect to \( x_0 \otimes \alpha \) and, since \( M \neq X \), \( A_1 \) is not abelian. This contradiction implies \( [Ax]^{-\sigma(X,Y)} = X \).
From Lemma 2.2 we know that if $\mathcal{A}$ is maximal $(x \otimes \alpha)$-tracial, then $\mathcal{A}^\#$ is maximal $(x \otimes \alpha)^\#$-tracial. Since $(x \otimes \alpha)^\# = \alpha \otimes x$, we conclude $[\mathcal{A}^\# \alpha]_{\sigma \left( Y^\# X \right)} = Y$.

Now suppose (1)-(3) hold. Suppose $T \in \ell_{\sigma \left( X \otimes Y \right)} \left( X \right)$ and the algebra generated by $\mathcal{A} \cup \{ T \}$ is tracial for $x \otimes \alpha$. Then, for every $A, B, C \in \mathcal{A}$, we have

$$(x \otimes \alpha) \left( B \left( AT - TA \right) C \right) = (x \otimes \alpha) \left( (AT - TA) CB \right)$$
$$= (x \otimes \alpha) \left( AT \left( CB - T \left( CB \right) A \right) \right)$$
$$= (x \otimes \alpha) \left( \left( CB \right) AT - \left( CB \right) A \right)$$
$$= 0.$$

This means

$$[AT - TA] \left( CX \right) \in \ker \left( B^T \alpha \right)$$

for every $B, C \in \mathcal{A}$. It follows from (2) and (3) that $AT = TA$. Hence, by (1), $T \in \mathcal{A}$.

\[ \square \]

A subalgebra $\mathcal{A}$ of $\ell_{\sigma \left( X \otimes Y \right)} \left( X \right)$ is called transitive if the only $\sigma \left( X \otimes Y \right)$-closed $\mathcal{A}$-invariant linear subspaces are $\{0\}$ and $X$. It is easily shown that $M$ is $\mathcal{A}$-invariant if and only if $M^\perp$ is $\mathcal{A}^\#$-invariant. Thus $\mathcal{A}$ is transitive in $\ell_{\sigma \left( X \otimes Y \right)} \left( X \right)$ if and only if $\mathcal{A}^\#$ is transitive in $\ell_{\sigma \left( Y^\# X \right)} \left( X \right)$.

**Theorem 2.4** A subalgebra $\mathcal{A}$ of $\ell_{\sigma \left( X \otimes Y \right)} \left( X \right)$ is maximal abelian and transitive if and only if $\mathcal{A}$ is maximal $x \otimes \alpha$-tracial for every $x \in X$ and $\alpha \in Y$ with $\alpha \left( x \right) = (x \otimes \alpha) \left( 1 \right) = 1$.

**Proof** Suppose $\mathcal{A}$ is maximal $x \otimes \alpha$-tracial for every $x \in X$ and $\alpha \in Y$ with $\alpha \left( x \right) = (x \otimes \alpha) \left( 1 \right) = 1$. Let $S, T \in \mathcal{A}$. Assume, via contradiction, that $ST \neq TS$. Then there is an $x \in X$ such that $(ST - TS) x \neq 0$. We know that there are $\alpha_1, \alpha_2 \in Y$ such that $\alpha_1 \left( (ST - TS) x \right) \neq 0$ and $\alpha_2 \left( x \right) \neq 0$. Thus there is an $\alpha \in \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \}$ such that $\alpha \left( x \right) \neq 0$ and $\alpha \left( (ST - TS) x \right) \neq 0$. By dividing by $\alpha \left( x \right)$ we can assume that $\alpha \left( x \right) = 1$ and thus

$$(x \otimes \alpha) \left( ST - TS \right) = \alpha \left( (ST - TS) x \right) \neq 0,$$

a contradiction. Thus $\mathcal{A}$ is abelian. Since $\mathcal{A}$ is maximal tracial for every $x \otimes \alpha$ with $\alpha \left( x \right) = 1$, it follows that $\mathcal{A}$ is maximal abelian.

Next suppose $x \in X$ and $x \neq 0$. Then there is an $\alpha \in Y$ such that $\alpha \left( x \right) = 1$. Since $\mathcal{A}$ is abelian and maximal $(x \otimes \alpha)$-tracial, we know from Theorem 2.3 that $[\mathcal{A} x]_{\sigma \left( X \otimes Y \right)} = X$. Thus $\mathcal{A}$ is transitive.

The proof of the other direction is straightforward by using Theorem 2.3. \[ \square \]

**Corollary 2.5** Suppose $\mathcal{H}$ is a Hilbert space and $A \subset B \left( \mathcal{H} \right)$ is a unital weak-operator closed algebra. Then $A$ is maximal $e \otimes e$-tracial for every unit vector $e$ in $\mathcal{H}$ if and only if $A$ is abelian and transitive.
Proof Following the proof of Theorem 2.4, we just need to show $\mathcal{A}$ is abelian. If $S, T \in \mathcal{A}$ and $ST - TS \neq 0$, then the numerical range of $ST - TS$ is not $\{0\}$, which means that there is a unit vector $e \in \mathcal{H}$ such that

$$(e \otimes e) (ST - TS) = \langle (ST - TS) e, e \rangle \neq 0,$$

which contradicts the assumption that $\mathcal{A}$ is $(e \otimes e)$-tracial. Hence $\mathcal{A}$ is abelian and the rest follows as in the proof of Theorem 2.4.

We need to interpret the preceding results for Banach spaces. Suppose $X$ is a Banach space and $Y$ is the dual space of $X$. Then the $\sigma (X, Y)$-topology on $X$ is the weak topology and the $\sigma (Y, X)$-topology on $Y$ is the weak*-topology. Hence $L_{\sigma (X,Y)} (X)$ is the set of all linear transformations $T$ on $X$ that are weak-weak continuous, which, by the closed graph theorem, simply means that $L_{\sigma (X,Y)} (X)$ is the set of all linear transformations $T$ on $X$ that are bounded on $X$. However, $L_{\sigma (Y,X)} (Y)$ is the set of all linear transformations on $Y$ that are weak*-weak* continuous, which are precisely the adjoints of transformations in $B (X)$. Thus $L_{\sigma (Y,X)} (Y) \neq B (Y)$, except when $X$ is a reflexive Banach space. If $T \in L_{\sigma (X,Y)} (X)$, then $T^\#$ is the usual Banach-space adjoint of $T$. However, if $S \in L_{\sigma (Y,X)} (Y)$, then $S = T^\#$ for some $T \in B (X)$, and our definition of $S^\#$ is $S^\# = T$. Nonetheless, the Banach-space adjoint of $S$ acts on normed dual of $Y$, which is the second dual of $X$. For example, C. Read [14] has constructed a transitive operator $T$ on $\ell^1$. But no operator $A$ is transitive on the dual $\ell^\infty$ of $\ell^1$, since, for any nonzero $x \in \ell^\infty$ the closed linear span $M$ of $\{x, Ax, \ldots\}$ is nonzero, and not $\ell^\infty$ ($M$ is separable) and $A$-invariant. However, $T^\#$ has no nontrivial weak*-closed invariant subspaces of $\ell^\infty$.

Corollary 2.6 Suppose $X$ is a Banach space with normed dual $Y$, $\mathcal{A} \subset B (X)$ is an abelian algebra, and $x \in X$, $\alpha \in Y$ with $\alpha (x) = 1$. Then a maximal abelian algebra $\mathcal{A} \subset B (X)$ is a maximal $x \otimes \alpha$-tracial algebra if and only if $[Ax]^{-\|\cdot\|} = X$ and $[A^\# \alpha]^{-\text{weak}^*} = Y$.

Example 2.7 Suppose $(\Omega, \Sigma, \mu)$ is a probability space and $\alpha$ is a normalized gauge norm on $L^\infty (\mu)$, i.e., $\alpha$ is a norm such that $\alpha (1) = 1$, and for every $f \in L^\infty (\mu)$, $\alpha (f) = \alpha (|f|)$. We can extend the domain of $\alpha$ to every measurable function $f$ by

$$\alpha (f) = \sup \{\alpha (h) : h \in L^\infty (\mu), |h| \leq |f| \ \text{a.e.} \ (\mu)\}.$$ 

Then $L^\alpha (\mu) = \{f : \alpha (f) < \infty\}$ is a Banach space and $L^\alpha (\mu)$ is defined as the $\alpha$-closure of $L^\infty (\mu)$ in $L^\alpha (\mu)$. We say that the gauge norm $\alpha$ is continuous if

$$\lim_{\mu (E) \to 0} \alpha (\chi_E) = 0.$$

It is well-known that if $f \in L^\alpha (\mu)$ and $h \in L^\infty (\mu)$, then

$$\alpha (hf) \leq \alpha (f) \|h\|_\infty.$$
Thus $L^\infty (\mu)$ acts, via multiplication, as operators on $L^\alpha (\mu)$. It is shown in [8] that $L^\infty (\mu)$ is actually a maximal abelian algebra of operators on $L^\alpha (\mu)$. The norm $\alpha$ has a dual norm $\alpha'$ on $L^\infty (\mu)$ defined by

$$\alpha' (f) = \sup \{ \| fh \|_1 : h \in L^\infty (\mu), \alpha (h) \leq 1 \}.$$ 

If $\| . \|_1 \leq \alpha$, then $\alpha'$ is a normalized gauge norm, and, for $f \in L^\alpha (\mu)$ and $h \in L^{\alpha'} (\mu)$,

$$\| fh \|_1 \leq \alpha (f) \alpha' (h).$$

If, in addition, $\alpha$ is continuous, then the normed dual of $L^\alpha (\mu)$ is $L^{\alpha'} (\mu)$, given for $h \in L^{\alpha'} (\mu)$, the functional $\psi_h$ on $L^\alpha (\mu)$ defined by

$$\psi_h (f) = \int_{\Omega} fh d\mu.$$ 

Suppose $f \in L^\alpha (\mu)$ and $h \in L^{\alpha'} (\mu)$, then $L^\infty (\mu)$ is maximal $f \otimes h$-tracial if and only if both $L^\infty (\mu)$ $f$ is dense in $L^\alpha (\mu)$ and $L^\infty (\mu)$ $h$ is weak*-dense in $L^{\alpha'} (\mu)$. This is equivalent to $f (\omega) \neq 0$ a.e. $(\mu)$ and $h (\omega) \neq 0$ a.e. $(\mu)$. To see this, suppose $f \neq 0$ a.e. $(\mu)$. Suppose $\psi \in L^\alpha (\mu)$ and $\psi |_{L^\infty (\mu)} f = 0$. Then there is an $h \in L^{\alpha'} (\mu)$ such that, $\psi = \psi_h$. Hence, for every $u \in L^\infty (\mu)$,

$$\int_{\Omega} ufh d\mu = 0.$$ 

Since $fh \in L^1 (\mu)$, this means $fh = 0$ a.e. $(\mu)$, which implies $h = 0$ a.e. $(\mu)$. Thus $\psi = 0$. A similar proof shows that if $h \in L^{\alpha'} (\mu)$, then $L^\infty (\mu)$ $h$ is weak*-dense in $L^\alpha (\mu)$.

**Example 2.8** Let $H^2$ be the classical Hardy space on the unit disk. It is known that $H^\infty$ (acting as multiplications) is a maximal abelian algebra in $B (H^2)$. The cyclic vectors for $H^\infty$ are the outer functions (see [9]). There are many cyclic vectors in $H^2$ for $(H^\infty)^\#$. We see that $H^\infty$ is $f \otimes h$-maximal tracial if and only if $f$ is outer and $h$ is a cyclic vector for $(H^\infty)^\#$. Since 1 is not cyclic for $(H^\infty)^\#$, we see that $H^\infty$ is not $1 \otimes 1$-maximal tracial.

**Example 2.9** In the preceding example, $H^\infty$ is the unital weakly closed algebra generated by the unilateral shift operator on $\ell^2$. Suppose $X \in \{c_0\} \cup \{\ell^p : 1 \leq p < \infty\}$, and let

$$e_0 = (1, 0, \ldots), e_1 = (0, 1, 0, \ldots), \ldots.$$ 

Suppose $\{\beta_n\}_{n \geq 0}$ is a bounded sequence of positive numbers, and define $Te_n = \beta_n e_{n+1}$, i.e.

$$T (a_0, a_1, a_2, \ldots) = (0, \beta_0 a_0, \beta_1 a_1, \ldots).$$
Then $T$ is a bounded linear operator and $\mathcal{A} = \{ T \}' = \{ p( T ) : p \in \mathbb{C}[ z] \}^{WOT}$ is a maximal abelian algebra in $B( X )$. Clearly, $e_0$ is a cyclic vector for $T$. There are also many weak*-cyclic vectors for $T^h$ of the form $h = (1, d_1, d_2, \ldots )$. Thus $\mathcal{A}$ is maximal $e_0 \otimes h$-tracial for many choices of $h$.

**Example 2.10** Suppose $I$ is an uncountable set. Then $\ell^\infty( I )$ is a maximal abelian algebra of operators on $\ell^2( I )$, but, since any vector in $\ell^2( I )$ has countable support, $\ell^\infty( I )$ has no cyclic vector. Hence $\ell^\infty( I )$ is not maximal tracial for $x \otimes y$ for any $x, y \in \ell^2( I )$, or for any weak*-continuous linear $\varphi$ functional on $B( \ell^2( I ) )$ since such a $\varphi$ involves only countably many vectors.

**Definition 2.11** Suppose $( X, Y )$ is a dual pair over a Hausdorff field $\mathbb{F}$, and $\mathcal{A}$ is a WOT-closed unital subalgebra of $L_{\sigma( X, Y )}( X )$. We say that $\mathcal{A}$ has a complemented invariant subspace lattice if and only if, for every $\mathcal{A}$-invariant $( X, Y )$-closed linear subspace $M$ of $X$, there is an idempotent $P \in L_{\sigma( X, Y )}( X )$ such that $P( X ) = M$ and $P \in \mathcal{A}'$, where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$.

**Theorem 2.12** Suppose $( X, Y )$ is a dual pair over a Hausdorff field $\mathbb{F}$, and $\mathcal{A}$ is a WOT-closed unital subalgebra of $L_{\sigma( X, Y )}( X )$ with a complemented invariant subspace lattice. If $\mathcal{A}$ is maximal tracial for $e \otimes f$, with $e \in X$, $f \in Y$ and $f( e ) = 1$, then $[ \mathcal{A} e ]^{-\sigma( X, Y )} = X$ and $[ \mathcal{A}^h f ]^{-\sigma( Y, X )} = Y$.

**Proof** Let $M = [ \mathcal{A} e ]^{-\sigma( X, Y )}$. Then $M$ is $\mathcal{A}$-invariant. Thus there is an idempotent $P \in L_{\sigma( X, Y )}( X )$ such that $P( X ) = M$ and $P \in \mathcal{A}'$. With respect to the decomposition $X = M \oplus \ker P$, we can write every $T \in B( X )$ as an operator matrix

$$T = \begin{pmatrix} B & C \\ D & E \end{pmatrix}.$$ 

If $A \in \mathcal{A}$ we have

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}.$$ 

Let $\mathcal{S}$ be the set of all the operators of the form $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ and let $\mathcal{D} = \mathcal{A} + \mathcal{S}$. Since $e \in M$, then it follows that $\mathcal{S}$ is a two-sided ideal in $\mathcal{D}$ and that $\mathcal{S} \subset \ker ( e \otimes f )$. Thus for every $A, B \in \mathcal{A}$ and every $S_1, S_2 \in \mathcal{S}$ we have

$$(A + S_1)(B + S_2) - (B + S_2)(A + S_1) - (AB - BA) \in \mathcal{S}.$$ 

Hence $\mathcal{D}$ is $e \otimes f$-tracial. Since $\mathcal{A}$ is maximal $e \otimes f$-tracial, then $P = 1$ and $M = X$.

Similarly, suppose $N = [ \mathcal{A}^h f ]^{-\sigma( Y, X )}$. Then $N_\perp$ is $\mathcal{A}$-invariant. Thus, there is an idempotent $P \in L_{\sigma( X, Y )}( X ) \cap \mathcal{A}'$ such that $P( X ) = N_\perp \subset \ker f$. Consequently $(1 - P^h)(Y) = N$. Writing the matrix representations of the elements of $L_{\sigma( X, Y )}( X )$ and $\mathcal{A}$ as above, we can let $T$ be the set of all operators of the form $\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}$. Since the
range of any operator in $T$ is contained in $N_{⊥} \subset \ker f$, we see that $T \subset \ker (e \otimes f)$. Therefore $\mathcal{E} = \mathcal{A} + T$ is $e \otimes f$-tracial, and, since $\mathcal{A}$ is maximal $(e \otimes f)$-tracial, we see that $N_{⊥} = \{0\}$, and $N = (N_{⊥})^\perp = Y$. 

\section{3 Multiplier pairs}

Multiplier pairs were studied in [7]. A multiplier pair is a pair $(X, Y)$ where $X$ is a Banach space, $Y$ is a Hausdorff topological vector space, $X \subset Y$, and there is a multiplication on $X$ with values in $Y$, i.e., a bilinear map

$$\cdot : X \times X \rightarrow Y$$

such that

1. $\cdot$ is continuous in each variable
2. There is an identity element $e \in X$ such that, for every $a \in X$,
   $$a \cdot e = e \cdot a = a$$
3. The sets $L_0 = \{a \in X : a \cdot X \subset X\}$ and $R_0 = \{b \in X : X \cdot b \subset X\}$ are dense in $X$,
4. There are dense subsets $E \subset L_0$, $F \subset X$, $G \subset R_0$, such that, for all $a \in E$, $b \in F$, $c \in G$, we have
   $$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

If $a \in L_0$ and $b \in R_0$, the operators $L_a$ and $R_b$ defined on $X$ by

$$L_a x = a \cdot x$$ and $$R_b x = x \cdot b$$

are bounded on $X$ (see [7, Theorem 1]). Moreover, it was shown in [7, Theorem 1] that if $\mathcal{L} = \{L_a : a \in L_0\}$ and $\mathcal{R} = \{R_b : b \in R_0\}$, then $\mathcal{L}$, $\mathcal{R} \subset B(X)$ and

$$\mathcal{L}' = \mathcal{R}$$ and $$\mathcal{R}' = \mathcal{L}.$$ 

In particular, if the multiplication is commutative, then $\mathcal{L} = \mathcal{R}$ is maximal abelian in $B(X)$. In this case the next theorem reduces to Theorem 2.3.

We now prove a result about multiplier pairs that gives a lot of examples of maximal tracial algebras. In many of the examples in [7], condition (1) in the following theorem holds.

**Theorem 3.1** Suppose $(X, Y)$ is a multiplier pair with identity element $e$, and suppose $\alpha \in X^\#$ and $\alpha(e) = 1$. Also suppose

1. $\{R_B : B \in L_0 \cap R_0\}$ is WOT-dense in $\mathcal{R}$
2. $[L^\#_\alpha]^\text{weak}^* = X^\#$, and

\begin{itemize}
    \item \text{Birkhäuser}
(3) $\mathcal{L}$ is $(e \otimes \alpha)$-tracial.

Then $\mathcal{L}$ is maximal $(e \otimes \alpha)$-tracial.

**Proof** Suppose $B \in \mathcal{L}_0 \cap \mathcal{R}_0$.

**Claim** $L_B^* \alpha = R_B^* \alpha$.

**Proof of Claim** Suppose $A \in \mathcal{L}_0$. Then

$$\left( L_B^* \alpha \right)(A) = \alpha(BA) = (e \otimes \alpha)(L_B L_A) = (e \otimes \alpha)(L_A L_B) = \alpha(AB) \tag{1}$$

$$= \left( R_B^* \alpha \right)(A).$$

Since $\mathcal{L}_0$ is dense in $X$, the claim is proved.

Suppose $D \subset B(X)$ is a $(e \otimes \alpha)$-tracial algebra containing $\mathcal{L}$, and suppose $T \in D$. Choose $B \in \mathcal{L}_0 \cap \mathcal{R}_0$ and $C, D \in \mathcal{L}_0$. We then have

$$\left( T R_B(C), L_D^* \alpha \right) = \left( T (CB), L_D^* \alpha \right) = \left( T L_C L_B e, L_D^* \alpha \right) = \left( L_D T L_C e, L_B^* \alpha \right) = \left( L_D T L_C e, R_B^* \alpha \right) = \left( (R_B T)(C), L_D^* \alpha \right).$$

Since $\mathcal{L}^* \alpha$ is weak* dense in $X^*$ and $\mathcal{L}_0$ is dense in $X$, we see that

$$T R_B = R_B T$$

for every $B \in \mathcal{L}_0 \cap \mathcal{R}_0$. It follows from (1) that $T \in \mathcal{R}' = \mathcal{L}$. Thus, $\mathcal{L}$ is maximal $(e \otimes \alpha)$-tracial. \(\square\)

### 4 von Neumann algebras

Suppose $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space $H$ with a faithful normal tracial state $\tau$. There is a notion of *convergence in measure* on $\mathcal{M}$ (see [13]), defined by saying a net $\{T_\lambda\}$ in $\mathcal{M}$ converges to 0 in measure, denoted by $T_\lambda \rightharpoonup 0$ (\(\tau\)), if and only if there is a net $\{P_\lambda\}$ of projections in $\mathcal{M}$ such that $\tau(P_\lambda) \to 0$ and $\|T_\lambda(1 - P_\lambda)\| \to 0$. The completion of $\mathcal{M}$, denoted by $\hat{\mathcal{M}}$, with respect to this topology, is a *-algebra, in which all of the algebraic operations on $\mathcal{M}$ have continuous extensions (see [13]). Moreover, every $A \in \hat{\mathcal{M}}$ has a polar decomposition $A = U|A|$ where $U$ is a unitary in $\mathcal{M}$ and $|A| = (A^* A)^{1/2}$. A norm $\beta$ on $\mathcal{M}$ is called

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a normalized unitarily invariant norm on $\mathcal{M}$ if, for every $T \in \mathcal{M}$ and all unitary operators $U, V \in \mathcal{M}$,

$$\beta (UTV) = \beta (T).$$

Examples of such norms are the $p$-norms, $1 \leq p < \infty$, defined on $\mathcal{M}$ by

$$\|T\|_p = \tau \left( |T|^p \right)^{1/p},$$

where $|T| = (T^* T)^{1/2}$. We define $L^\beta (\mathcal{M}, \tau)$ to be the completion of $\mathcal{M}$ with respect to $\beta$. When $\beta \geq \|T\|_1$, we have that $L^\beta (\mathcal{M}, \tau)$ is a multiplier pair and $L^0_\beta = \mathcal{R}_0 = \mathcal{M}$ and $e = 1$. Suppose $T \in \mathcal{M}$. We define $\varphi_T \in L^\beta (\mathcal{M}, \tau)^\#$, by

$$\varphi_T (A) = \tau (TA).$$

If we let $\alpha = \varphi_1$, we see that $1 \otimes \alpha = \tau$ is tracial on $\mathcal{M}$. Also, if $A \in L^p (\mathcal{M}, \tau) \subset L^1 (\mathcal{M}, \tau)$ with polar decomposition $U |A|$, and if

$$\left( L^\#_D \alpha \right) (A) = 0$$

for every $D \in \mathcal{M}$, then

$$0 = L^\#_{U^*} (A) = \tau (U^* A) = \tau (|A|).$$

This implies $A = 0$. Thus $\{ L^\#_D \alpha : D \in \mathcal{L}_0 \}$ is weak* dense in $L^\beta (\mathcal{M}, \tau)$. If we apply Theorem 3.1, we obtain the following result.

**Theorem 4.1** Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state. If $\beta \geq \|T\|_1$ is a normalized unitarily invariant norm, then $\mathcal{M} = \{ L_A : A \in \mathcal{M} \}$ is a maximal $1 \otimes \varphi_1$-tracial algebra in $B \left( L^\beta (\mathcal{M}, \tau) \right)$.

An example of a von Neumann algebra with a faithful normal tracial state is $L^\infty (\mu)$ where $\Omega, \mu$ is a probability space and $\tau : L^\infty (\mu) \to \mathbb{C}$ is defined by

$$\tau (f) = \int_\Omega f \, d\mu.$$
Corollary 4.2 Suppose \((\Omega, \mu)\) is a probability space, \(\beta \geq \|\|_1\) is a normalized gauge norm on \(L^\infty(\mu)\), \(f \in L^\beta(\mu)\) and \(\alpha \in L^\beta(\mu)\) satisfy

\[ f \neq 0 \text{ a.e. (}\mu\text{)} \text{ and } L^\infty(\mu)^\# \alpha \text{ is weak* dense in } L^\beta(\mu)^\#. \]

Then \(L^\infty(\mu)\) is maximal \((f \otimes \alpha)\)-tracial.

We now consider an important problem. Suppose \(\mathcal{M}\) is a von Neumann algebra on a Hilbert space \(\mathcal{H}\) and \(e, f \in \mathcal{H}\) and \(\langle e, f \rangle = 1\). If \(e \otimes f\) is tracial on \(\mathcal{M}\), when is \(\mathcal{M}\) maximal \((e \otimes f)\)-tracial?

We first note that if \(E\) is an invariant subspace for \(\mathcal{M}\), then \(E^\perp\) is also invariant, so Theorem 2.12 implies that a necessary condition for \(\mathcal{M}\) to be maximal \((e \otimes f)\)-tracial is that \(e\) and \(f\) be cyclic vectors for \(\mathcal{M}\). The real question is whether the converse is true.

Problem 4.3 Suppose that \(\mathcal{M}\) is a von Neumann algebra on a separable Hilbert space \(\mathcal{H}\) and \(e, f\) are cyclic vectors for \(\mathcal{M}\) with \(\langle e, f \rangle = 1\). If \(\mathcal{M}\) is \((e \otimes f)\)-tracial, then is \(\mathcal{M}\) maximal \((e \otimes f)\)-tracial?

In the case where \(\mathcal{H}\) is separable, we will reduce the problem to the case where \(\mathcal{M}\) is a finite factor von Neumann algebra, and such a factor has a unique norm continuous unitial tracial functional.

Suppose \(\mathcal{M}\) is a von Neumann algebra on a separable Hilbert space \(\mathcal{H}\) and \(e, f\) are cyclic vectors for \(\mathcal{M}\) with \(\langle e, f \rangle = 1\). Let

\[
\mathcal{J} = \{ A \in \mathcal{M} : (e \otimes f)(AB) = 0 \text{ for all } B \in \mathcal{M} \}
= \{ A \in \mathcal{M} : Ae \perp \mathcal{M}f \}
= \{ A \in \mathcal{M} : Ae = 0 \}.
\]

Clearly, \(\mathcal{J}\) is a WOT-closed linear space. Also, if \(e \otimes f\) is tracial on \(\mathcal{M}\), then \(\mathcal{J}\) is a two-sided ideal in \(\mathcal{M}\). Hence there is a projection \(P\) in the center \(Z(\mathcal{M})\) such that \(\mathcal{J} = PM\). Thus \(PMe = MPe = 0\). Since \(Me\) is dense in \(\mathcal{H}\), it follows that \(P = 0\) and \(\mathcal{J} = 0\). Similarly, if \(A \in \mathcal{M}\) and \(Af = 0\), then \(A = 0\). Thus \(e\) and \(f\) are cyclic separating vectors for \(\mathcal{M}\).

If we write the central decomposition for \(\mathcal{M}\), we get

\[
\mathcal{H} = \int_\Omega^{\oplus} \mathcal{H}_\omega d\mu(\omega), \quad \mathcal{M} = \int_\Omega^{\oplus} \mathcal{M}_\omega d\mu(\omega),
\]

\[
e = \int_\Omega^{\oplus} e_\omega d\mu(\omega), \quad f = \int_\Omega^{\oplus} f_\omega d\mu(\omega), \quad \text{and}
\]

and

\[
Z(\mathcal{M}) = \int_\Omega^{\oplus} C_\omega d\mu(\omega),
\]

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where each $M_{\omega}$ is a factor von Neumann algebra. Since $e$ and $f$ are separating vectors for $M$, we have $e_\omega \neq 0$ and $f_\omega \neq 0$ almost everywhere. We also get $e_\omega$ and $f_\omega$ are cyclic and separating for $M_{\omega}$ a.e. ($\mu$). We also have $e_\omega \otimes f_\omega$ is tracial for almost every $\omega$. Since a factor von Neumann algebra $M_{\omega}$ can have at most one nonzero continuous unital tracial state $\tau_\omega$, we know that almost every $M_{\omega}$ is a finite factor and has a unique tracial functional $\tau_\omega$ of norm 1. We see that $\langle e_\omega, f_\omega \rangle \neq 0$ a.e. ($\mu$). Thus $\tau_\omega = (\frac{1}{\langle e_\omega, f_\omega \rangle}) e_\omega \otimes f_\omega$.

Suppose now that $M \subset D$ and $D$ is $(e \otimes f)$-tracial. Suppose $P$ is a central projection in $M$. Then, for all $A, B \in M$, we have

$$(e \otimes f) (A (PT - TP) B) = (e \otimes f) (APT B) - (e \otimes f) (ATPB)$$

$$= (e \otimes f) ((BAPT)) - (e \otimes f) ((B PAT))$$

$$= (e \otimes f) ((PBAT)) - (e \otimes f) ((PBAT))$$

$$= 0.$$ 

Thus

$$\langle (PT - TP) (Be) , (A^* f) \rangle = 0$$

for every $A, B \in M$. Since $e$ and $f$ are cyclic, we see that $PT - TP = 0$ for every projection $P \in Z (M)$. Thus $T \in Z (M)'$. Hence we can write $T = \int_{\Omega} T_\omega d\mu (\omega)$. Since the algebra generated by $T$ and $M$ is $(e \otimes f)$-tracial, it follows that, for almost all $\omega \in \Omega$, the algebra generated by $T_\omega$ and $M_{\omega}$ is $(e_\omega \otimes f_\omega)$-tracial. It now easily follows that $M$ is maximal $(e \otimes f)$-tracial if and only if $M_{\omega}$ is maximal $\left(\frac{1}{\langle e_\omega, f_\omega \rangle} e_\omega \otimes f_\omega\right)$-tracial a.e. ($\mu$). Hence when $H$ is separable, this reduces the problem to the case where $M$ is a finite factor von Neumann algebra.

Suppose now that $M$ is a finite factor and $\tau$ is the unique continuous unital tracial functional on $M$. Since $e$ and $f$ are separating cyclic vectors, we can assume, via unitary equivalence [12], that $H = L^2 (M, \tau)$ and $e, f \in L^2 (M, \tau) \subset L^1 (M, \tau) \subset \hat{M}$ as above. We now have $e \otimes f = \tau$ on $M$. This means that, for every $A \in M$,

$$\tau (f^* Ae) = \tau (Ae f^*) = \tau (A).$$

Thus

$$\tau (A (1 - ef^*)) = 0.$$ 

It follows $ef^* = 1$ (where the multiplication is in $\hat{M}$). Since $M$ is finite, $f^* e = 1$. Thus $e, e^{-1} \in L^2 (M, \tau)$ and $f^* = e^{-1}$, so $f = (e^{-1})^*$. Thus $e \otimes f = e \otimes (e^{-1})^*$. Now the question asks whether $M$ is maximal $\left( e \otimes (e^{-1})^* \right)$-tracial.

If our question has an affirmative answer, it would imply that if $M \subset B (H)$ is a von Neumann algebra and $S \in B (H)$ is invertible and $A = S^{-1} MS$ and $(e \otimes f)$ is a unital tracial functional on $A$, then $A$ is maximal $(e \otimes f)$-tracial.
There is a slight relationship between these ideas and a famous problem in $C^*$-algebra theory called the Kadison Similarity Problem [10], which asks if every bounded unital homomorphism $\rho : A \to B (\mathcal{H})$ from a $C^*$-algebra $A$ is similar to a $*$-homomorphism. This is equivalent to the question when $A$ is a von Neumann algebra acting on a separable Hilbert space, and $\rho$ is weak*-weak* continuous [10]. A beautiful theorem of U. Haagerup [4] shows that if $\rho (A)$ has a cyclic vector, then $\rho$ is similar to a $*$-homomorphism, and, thus, for some invertible operator $S \in B (\mathcal{H})$ the map $A \mapsto S^{-1} \rho (A) S$ is a weak*-weak* continuous $*$-homomorphism, which means that $\mathcal{M} = S^{-1} \rho (A) S$ is a von Neumann algebra. A functional $u \otimes v$ on $\rho (A)$ corresponds to the functional $Su \otimes (S^*)^{-1} v$ on $\mathcal{M}$.

**Question** Suppose $\mathcal{M} \subseteq B (\mathcal{H})$ is a von Neumann algebra, and $e, f \in \mathcal{H}$ satisfy $\langle e, f \rangle = 1$, $\mathcal{M}$ is $(e \otimes f)$-tracial, and $e$ and $f$ are cyclic for $\mathcal{M}$, must $\mathcal{M}$ be maximal $(e \otimes f)$-tracial?

5 Maximal tracial ultraproducts

The theory of ultraproducts has played a fundamental role in many areas of mathematics, logic, Banach spaces, von Neumann algebras and $C^*$-algebras. For von Neumann algebras, the tracial ultraproducts have been extremely important. We define a new type of ultraproduct. Suppose $\{ A_i : i \in I \}$ is an infinite collection of unital $C^*$-algebras. Also suppose $\alpha$ is a nontrivial ultrafilter on $I$. We let $A = \prod_{i \in I} A_i$ be the $C^*$-direct product, and we define

$$J = \{ \{ a_i \} \in A : \lim_{i \to \alpha} \| a_i \| = 0 \}.$$  

Then $J$ is a closed two-sided ideal in $A$, and $A/J$ is called the ultraproduct of the $C^*$-algebras $A_i$, and is denoted by

$$\prod_{i \in I} A_i.$$  

Next, suppose for each $i \in I$ we have $\tau_i$ is a tracial state on $A_i$. If we define

$$J_\alpha = \{ \{ a_i \} \in A : \lim_{i \to \alpha} \tau_i (a_i^* a_i) = 0 \},$$  

then $J_\alpha$ is a closed two-sided ideal in $A$, and $A/J_\alpha$ is called the tracial ultraproduct of the $C^*$-algebras $A_i$, and is denoted by

$$\prod_{i \in I} (A_i, \tau_i).$$  

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Sakai [16] proved that if each $A_i$ is a factor von Neumann algebra, then $\prod_{i \in I} (A_i, \tau_i)$ is a factor von Neumann algebra. More recently, the first author and Li [6] proved that every tracial ultraproduct of $C^*$-algebras is a von Neumann algebra.

Another way to view a tracial ultraproduct, is by defining a tracial state $\tau_\alpha$ on $\prod_{i \in I} A_i$ by

$$\tau_\alpha (\{a_i\}) = \lim_{i \to \alpha} \tau_i (a_i).$$

Then $J_\alpha = \{a \in \prod_{i \in I} A_i : \tau_\alpha (a^*a) = 0\}$, which is the first part of the GNS construction for $\tau_\alpha$.

We now consider the case where, for each $i \in I$, $\varphi_i$ is a state on $A_i$. We define $\varphi_\alpha$ on $\prod_{i \in I} A_i$ by

$$\varphi_\alpha (\{a_i\}) = \lim_{i \to \alpha} \varphi_i (a_i).$$

Let $\mathcal{M} \subset \mathcal{A}$ be a maximal tracial $C^*$-algebra with respect to $\varphi_\alpha$. We define the maximal tracial ultraproduct with respect to $\mathcal{M}$, denoted by

$$\prod_\mathcal{M} (A_i, \varphi_i),$$

to be $\mathcal{M}/ [\mathcal{M} \cap J_\alpha]$. Since $\varphi_\alpha|_\mathcal{M}$ is tracial, $\varphi_\alpha$ induces a tracial state $\hat{\varphi}_\alpha$ on the quotient $\mathcal{M}/ [\mathcal{M} \cap J_\alpha]$.

If each $\varphi_i$ is tracial, then $\mathcal{A}$ is $\varphi_\alpha$-tracial and $\prod_\mathcal{A} (A_i, \varphi_i)$ is the usual tracial ultraproduct. If $\varphi$ is tracial on $\mathcal{A}$, then $\prod_\mathcal{A} (A_i, \varphi_i)$ is the tracial ultraproduct defined by H. Ando and E. Kirchberg [1].

One of the biggest open problems in the theory of von Neumann algebras is Connes’ Embedding Problem [2], which asks if every von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space with a faithful normal tracial state $\tau$ can be tracially embedded into a tracial ultraproduct $\prod_{i \in I} (A_i, \tau_i)$ with each $A_i$ finite-dimensional. This means there is a unital $\ast$-homomorphism $\pi : \mathcal{R} \to \prod_{i \in I} (A_i, \tau_i)$ such that $\tau = \hat{\tau}_\alpha \circ \pi$.

The analogue of Connes’ embedding problem easily has an affirmative answer in the setting of maximal tracial ultraproducts. However, this sheds no new light on the original problem.

**Theorem 5.1** Every von Neumann algebra $\mathcal{R}$ acting on a separable Hilbert space with a faithful normal tracial state $\tau$ can be tracially embedded into a maximal tracial ultraproduct of finite-dimensional algebras.

**Proof** We can assume that $\mathcal{R} \subset B(\mathcal{H})$ for some separable infinite-dimensional Hilbert space $\mathcal{H}$ and that there is a unit vector $f \in \mathcal{H}$ such that, for every $T \in \mathcal{R}$,

$$\tau (T) = \langle Tf, f \rangle.$$

We can also assume that $f$ is a cyclic vector for $\mathcal{R}$. Let $E = \{e_1, e_2, \ldots\}$ be an orthonormal basis for $\mathcal{H}$ such that $e_1 = f$. For each positive integer $n$, let $P_n$ be the
orthogonal projection onto the linear span of \{e_1, \ldots, e_n\}. Let \( A_n = P_n B (H) P_n \) and define a state \( \varphi_n : A_n \to \mathbb{C} \) by
\[
\varphi_n (A) = \langle Af, f \rangle = \langle Ae_1, e_1 \rangle.
\]
Let \( \alpha \) be any free ultrafilter on \( \mathbb{N} \). Let
\[
\mathcal{M}_0 = \left\{ \{A_n\} \in \prod_{n \in \mathbb{N}} A_n : \{A_n\} \text{ is } ^*\text{SOT convergent, } \lim_{n \to \infty} A_n \in \mathcal{R} \right\}.
\]
Since \( \tau \) is a trace on \( \mathcal{R} \), for \( S = \{A_n\}, T = \{B_n\} \in \mathcal{M}_0 \) with \( A_n \to A \in \mathcal{R} \) and \( B_n \to B \in \mathcal{R} \) \((^*\text{SOT})\), we have
\[
A_n B_n \to AB \text{ and } B_n A_n \to BA \ (^*\text{SOT}).
\]
Thus
\[
\varphi_\alpha (ST) = \tau (AB) = \tau (BA) = \varphi_\alpha (TS).
\]
Hence \( \varphi_\alpha \) is tracial for the \( C^* \)-algebra \( \mathcal{M}_0 \). Then there is a maximal \( \varphi_\alpha \)-tracial \( C^* \)-algebra \( \mathcal{M} \subset \prod_{n \in \mathbb{N}} A_n \).

Let \( \eta : \mathcal{M} \to \mathcal{M}/(\mathcal{M} \cap \mathcal{J}_\alpha) = \prod_{n \in \mathbb{N}} (A_n, \varphi_n) \) be the quotient map and define \( \pi : \mathcal{R} \to \prod_{n \in \mathbb{N}} (A_n, \varphi_n) \) by
\[
\pi (A) = \eta (\{P_n AP_n\}).
\]
We easily see that, for every \( A \in \mathcal{R} \),
\[
\hat{\varphi}_\alpha (\pi (A)) = \lim_{\alpha} \langle P_n AP_n f, f \rangle = \langle Af, f \rangle = \tau (A).
\]
Thus \( \hat{\varphi}_\alpha \circ \pi = \tau \). Since \( \tau \) is faithful on \( \mathcal{R} \), \( \pi \) is an embedding.

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