Reverse-order law for weighted Moore–Penrose inverse of tensors

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Abstract
The weighted Moore–Penrose inverse of a tensor via the Einstein product is introduced in the literature, very recently. The objective of this paper is to study the reverse-order law for the weighted Moore–Penrose inverse. In this context, a few properties of the weighted Moore–Penrose inverse of an arbitrary order tensor via the Einstein product are discussed. Some new sufficient conditions for the reverse-order law of even-order square tensors are obtained. Several characterizations of the reverse-order law for the weighted Moore–Penrose inverse of tensors via the same product are also presented.

Keywords Tensor · Moore–Penrose inverse · Weighted Moore–Penrose inverse · Einstein product · Reverse-order law

Mathematics Subject Classification 15A69 · 15A09

1 Introduction
There has been active research on tensors for the past four decades. But, a little research contributions on the theory and applications of generalized inverses of tensors are in the literature. In fact, a generalized inverse called the Moore–Penrose inverse of an even-order tensor via the Einstein product was first introduced by Sun et al. [24] in 2016. Then the authors obtained the minimum-norm least-squares solution of some multilinear systems by using the notion of the Moore–Penrose inverse. In the next year, Behera and Mishra [2] continued the same study and proposed different types...
of generalized inverses of tensors. In 2018, Panigrahy and Mishra [19] improved the definition of the Moore–Penrose inverse of an even-order tensor to a tensor of any order via the same product, which also appeared in [17] and [23]. The definition of the Moore–Penrose inverse of an arbitrary order tensor is recalled below.

**Definition 1** (Definition 3.3 [17], Definition 1.1 [19], [23]) Let \( X \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \). Then the unique tensor \( Y \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M} \) satisfying the following four tensor equations:

\[
X *_N Y *_M X = X; \\
Y *_M X *_N Y = Y; \\
(X *_N Y)^H = X *_N Y; \\
(Y *_M X)^H = Y *_M X,
\]

is called as the **Moore–Penrose inverse** of \( X \), and is denoted by \( X^\dagger \).

In the above definition, \((\cdot)^H\) denotes the conjugate transpose of \((\cdot)\) and \(*_N\) denotes the **Einstein product** [7] of tensors, and is defined by

\[
(A *_N B)_{i_1 \cdots i_M k_1 \cdots k_L} = \sum_{j_1 \cdots j_N} a_{i_1 \cdots i_M j_1 \cdots j_N} b_{j_1 \cdots j_N k_1 \cdots k_L}
\]

for tensors \( A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_L} \). In the case of an even-order invertible tensor, Definition 1 coincides with the notion of the inverse which was first introduced by Brazell et al. [5]. Using Lemma 3.1 and Theorem 3.17 of [5], one can compute the inverse of a tensor. The idea of introducing generalized inverses of a tensor originates from the necessity of finding a solution of a given tensor multilinear system (see [2,10,11,24]).

If \( A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) and \( B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) are a pair of invertible tensors, then the equality

\[
(A *_N B)^{-1} = B^{-1} *_N A^{-1}
\]

is called the **reverse-order law** (or two-term reverse-order law) for the ordinary tensor inverse. It is well-known that the reverse-order law does not hold for various classes of generalized inverses of tensors (see [2]). Three articles in the literature ([2,18] and [20]) contain different sufficient or equivalent conditions such that the reverse-order law holds in some sense.

In 2017, Ji and Wei [11] introduced another extension of the Moore–Penrose inverse of an even-order tensor called **weighted Moore–Penrose inverse** (see Definition 3 for its definition) and established the relation between the minimum-norm least-squares solution of a multilinear system and the weighted Moore–Penrose inverse. Very recently, Behera et al. [1] proposed a method of computation of the weighted Moore–Penrose inverse using full rank decomposition of a tensor which was introduced by Liang and Zheng [17]. Among other results, they attempted the problem of three term (or triple) reverse-order law mentioned in the conclusion section of [19] for the weighted Moore–Penrose inverse as well as the two term reverse-order law. The reverse-order law for the
Moore–Penrose inverse plays an important role in the theoretic research and numerical computations of the models involving tensors. In addition, the reverse-order law for the weighted Moore–Penrose inverse is also applied to the generalized least squares problem and the weighted perturbation theory of the singular tensor. For applications and tensor-based methods, we refer the readers to the papers [6,13,14,16,25,27] and the references cited therein. The purpose of this paper is to establish the reverse-order law for the weighted Moore–Penrose inverse of tensors via the Einstein product.

To do this, the rest of the paper is broken down as follows. In Sect. 2, we define mathematical constructs including tensors and a few of its sub classes, and weighted Moore–Penrose inverse of a tensor which are required to state and prove the results in the subsequent sections. Section 3 is of 3-fold. First, a few properties of the weighted Moore–Penrose inverse of any tensor are explained. Second, some sufficient conditions for the reverse-order law of square tensors are provided. Third, the same study is continued but for arbitrary order tensors.

2 Prerequisites

This section contains the basic nomenclature required to prove our main results. A tensor $\mathbf{T}$ is a multidimensional array. An element of $\mathbb{C}^{I_1 \times \cdots \times I_N}$ is an $N$th-order tensor. Here $I_1, I_2, \cdots, I_N$ are dimensions of the first, second, $\cdots,$ Nth way, respectively. The order of a tensor is the number of its dimensions. The scalars, vectors and matrices are respectively zeroth-order, first-order and second-order tensors. The tensors of order three or higher are known as higher-order tensors. These are denoted by calligraphic letters like $\mathcal{A}$. An element of an $N$th order tensor $\mathbf{A}$ at $(i_1, \cdots, i_N)$th position is denoted by $a_{i_1 \cdots i_N}$. For more details, we refer to the recent books [21,22,26] on tensors.

A tensor in $\mathbb{C}^{I_1 \times \cdots \times I_M \times \bar{I}_1 \times \cdots \times \bar{I}_N}$ with entries $(D)_{i_1 \cdots i_N j_1 \cdots j_N} = \prod_{k=1}^{N} \delta_{i_k j_k}$ is called a diagonal tensor if $d_{i_1 \cdots i_N j_1 \cdots j_N} = \begin{cases} k_{i_1 \cdots i_N}, & \text{if } (i_1, \cdots, i_N) = (j_1, \cdots, j_N) \\ 0, & \text{otherwise} \end{cases}$, where $k_{i_1 \cdots i_N} \in \mathbb{C}$. A tensor $\mathcal{T} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times \bar{I}_1 \times \cdots \times \bar{I}_N}$ with entries $(\mathcal{T})_{i_1 \cdots i_N j_1 \cdots j_N} = \prod_{k=1}^{N} \delta_{i_k j_k}$ is called an identity tensor, where $\delta_{i_k j_k} = \begin{cases} 1, & \text{if } i_k = j_k \\ 0, & \text{otherwise} \end{cases}$. The conjugate transpose of a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is defined as $(\mathcal{A}^H)_{j_1 \cdots j_N i_1 \cdots i_M} = \bar{a}_{i_1 \cdots i_M j_1 \cdots j_N}$, where the over-line stands for the conjugate of $a_{i_1 \cdots i_M j_1 \cdots j_N}$. A tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ is Hermitian if $\mathcal{A} = \mathcal{A}^H$ and skew-Hermitian if $\mathcal{A} = -\mathcal{A}^H$. Further, a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ is unitary if $\mathcal{A}^* \mathcal{A} = \mathcal{A} \mathcal{A}^* = I$, and idempotent if $\mathcal{A}^* \mathcal{A} = \mathcal{A} = \mathcal{A}^2$. Ji and Wei [11] defined the class of Hermitian positive definite tensors as below.

**Definition 2** (Definition 1, [11]) Let $\mathcal{P} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$, if there exists a unitary tensor $\mathcal{U}$ such that

$$\mathcal{P} = \mathcal{U}^* \mathcal{D} \mathcal{U}^H,$$

where $\mathcal{D} \in \mathbb{C}^{I_M \times \cdots \times I_M}$ is a diagonal tensor.
where $D$ is a diagonal tensor with positive diagonal entries, then $P$ is said to be Hermitian positive definite.

Let $P$ be a Hermitian positive definite tensor, then define

$$P^{1/2} = U_M D^{1/2} U_M^H,$$

the square root of $P$, where $P = U_M D_M U_M^H$ and $D^{1/2}$ be the diagonal tensor obtained from $D$ by taking the square root of all its diagonal entries. Note that $P^{1/2}$ is always nonsingular and its inverse is denoted by $P^{-1/2}$. Moreover, $P^{1/2}$ is also Hermitian. Ji and Wei [11] introduced the weighted Moore–Penrose inverse for an even-order tensor. Behera et al. [1] then extended it for arbitrary tensor. Hereunder, we restate the definition of the weighted Moore–Penrose inverse for any tensor, which exists for every tensor, by improving its corresponding analog in [11].

**Definition 3** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, and let $M$, $N$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M}$ and $\mathbb{C}^{J_1 \times \cdots \times J_N}$, respectively. Then the unique tensor $X$ in $\mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$ satisfying

$$A^* N X^* M A = A;$$
$$X^* M A^* N X = X;$$
$$(M^* M A^* N X)^H = M^* M A^* N X;$$
$$(N^* N X^* M A)^H = N^* N X^* M A,$$

is called as the **weighted Moore–Penrose** inverse of tensor $A$, and is denoted by $A_{MN}^\dagger$.

When $M$ and $N$ are both identity tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ and $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$, respectively, then $A_{MN}^\dagger = A^\dagger$. We end this section by recalling a lemma from [11] which describes two properties of the weighted Moore–Penrose inverse of an even-order tensor.

**Lemma 1** (Lemma 2, [11]) Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$, and let $M$ and $N$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$, respectively. Then

(i) $(A_{MN}^\dagger)^\dagger_{MN} = A,

(ii) (A_{MN}^\dagger)^H = (A^H)^\dagger_{N^{-1}M^{-1}}$.

**3 Main results**

This section is divided into three parts. The first part deals with several properties of the weighted conjugate transpose and its related results which will help us in proving a few results in subsequent subsections. The second part concerns with the reverse-order law for the weighted Moore–Penrose inverse of two square tensors via the Einstein
product. The last part gathers various necessary and sufficient conditions for computing the weighted Moore–Penrose inverse of the Einstein product of any two tensors, i.e., the reverse-order law.

### 3.1 Weighted conjugate transpose

Ji and Wei [11] first introduced the notion of the weighted conjugate transpose of an even-order tensor whereas Behera et al. [1] improved the same for any tensor. This is recalled next.

**Definition 4** (Definition 2.8, [1]) The weighted conjugate transpose of the tensor $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, denoted by $A^\#_{MN}$, is defined as

$$A^\#_{MN} = N^{-1} \ast_N A^H \ast_M M,$$

where $M \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ and $N \in \mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ are Hermitian positive definite tensors.

We first state a lemma which establishes two new properties of the weighted conjugate transpose. These are straightforward by using Definition 4.

**Lemma 2** Let $A, B \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, and let $M$ and $N$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ and $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$, respectively. Then

(i) $(A + B)^\#_{MN} = A^\#_{MN} + B^\#_{MN}$,
(ii) $(A^\#_{MN})^H = (A^H)^\#_{N^{-1}M^{-1}}$.

Like the conjugate transpose of a tensor, the weighted conjugate transpose also satisfies the reverse-order law, and is stated below.

**Lemma 3** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, and let $M, N$ and $L$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$, $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ and $\mathbb{C}^{K_1 \times \cdots \times K_L \times K_1 \times \cdots \times K_1}$, respectively. Then

(i) $(A^\#_{MN}^\#)^{\#}_{N^*M} = A$,
(ii) $(A^*_{N}B^\#)^{\#}_{ML} = B^\#_{N^*L} \ast_N A^\#_{MN}$.

**Proof** By using Definition 4, we have $(A^\#_{MN}^\#)^{\#}_{N^*M} = M^{-1} \ast_M (N^{-1} \ast_N A^H \ast_M M)^H \ast_N N$, which on further simplification results the first part of this lemma. Again, applying Definition 4 to $(A^*_{N}B^\#)^{\#}_{ML}$, we get

$$(A^*_{N}B)^{\#}_{ML} = L^{-1} \ast_L B^H \ast_N A^H \ast_M M$$

$$= L^{-1} \ast_L B^H \ast_N N \ast_N N^{-1} \ast_N A^H \ast_M M$$

$$= B^\#_{N} \ast_N A^\#_{MN}.$$
Note that the above lemma (without its proof) also appeared in [1]. During the investigation of the reverse-order law for the weighted Moore–Penrose inverse of tensors, we observe that the weighted conjugate transpose of the Einstein product of a tensor with its weighted Moore–Penrose inverse remains unaltered. This is shown next.

**Theorem 1** Let \( A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \), and let \( M \) and \( N \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \) and \( \mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N} \), respectively. Then

\[
(A^* N A^*)^{\#}_{M M} = A^* N A^* \quad (A^* N A^*)^{\#}_{M M}. \]

**Proof** A simple calculation, in view of the fact \( M \) is Hermitian, leads to

\[
(A^* N A^*)^{\#}_{M M} = A^* N A^* \quad (A^* N A^*)^{\#}_{M M}. \]

The next example illustrates that the above result fails if we take the weighted conjugate transpose with different weights.

**Example 1** Consider a tensor \( A \in \mathbb{C}^{2 \times 3 \times 2 \times 3} \) as follows:

\[
\begin{array}{ccc}
A(:,:,1, 1) & A(:,:,1, 2) & A(:,:,1, 3) \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

and two Hermitian positive definite tensors \( M \) and \( N \) in \( \mathbb{C}^{2 \times 3 \times 2 \times 3} \) as below:

\[
\begin{array}{ccc}
M(:,:,1, 1) & M(:,:,1, 2) & M(:,:,1, 3) \\
13/4 & 0 & 0 & 0 & 7/4 & 0 & 0 & 0 & 13/4 \\
0 & 0 & -3\sqrt{3}/4 & 0 & 3\sqrt{3}/4 & 0 & 3\sqrt{3}/4 & 0 & 0 \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\hat{M}(:,:,2, 1) & \hat{M}(:,:,2, 2) & \hat{M}(:,:,2, 3) \\
0 & 0 & 3\sqrt{3}/4 & 0 & -3\sqrt{3}/4 & 0 & -3\sqrt{3}/4 & 0 & 0 \\
7/4 & 0 & 0 & 0 & 13/4 & 0 & 0 & 0 & 7/4 \\
\end{array}
\]

Then, \( A^*_{MN} \) is given by

\[
\begin{array}{ccc}
\hat{N}(:,:,1, 1) & \hat{N}(:,:,1, 2) & \hat{N}(:,:,1, 3) \\
13/4 & 0 & 0 & 0 & 7/4 & 0 & 0 & 0 & 13/4 \\
0 & 0 & 3\sqrt{3}/4 & 0 & -3\sqrt{3}/4 & 0 & 3\sqrt{3}/4 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\hat{N}(:,:,2, 1) & \hat{N}(:,:,2, 2) & \hat{N}(:,:,2, 3) \\
0 & 0 & 3\sqrt{3}/4 & 0 & -3\sqrt{3}/4 & 0 & -3\sqrt{3}/4 & 0 & 0 \\
7/4 & 0 & 0 & 0 & 13/4 & 0 & 0 & 0 & 7/4 \\
\end{array}
\]
The reverse-order law for weighted Moore–Penrose inverse of tensors is stated as:

\[ A_{\mathcal{M}N}^\dagger(c, ::, 1, 1) \]

\[ A_{\mathcal{M}N}^\dagger(c, ::, 1, 2) \]

\[ A_{\mathcal{M}N}^\dagger(c, ::, 1, 3) \]

The next result shows that the only tensor whose Einstein product with its weighted conjugate transpose results zero tensor is the zero tensor.

\[ A_{\mathcal{M}N}^\dagger \]

\[ A_{\mathcal{M}N}^\dagger(c, ::, 1, 1) \]

\[ A_{\mathcal{M}N}^\dagger(c, ::, 1, 2) \]

\[ A_{\mathcal{M}N}^\dagger(c, ::, 1, 3) \]

The weighted conjugate transpose \((A_{\mathcal{M}N}^\dagger)^{\#}\) is given by

\[ (A_{\mathcal{M}N}^\dagger)^{\#}_{\mathcal{M}N}(c, ::, 1, 1) \]

\[ (A_{\mathcal{M}N}^\dagger)^{\#}_{\mathcal{M}N}(c, ::, 1, 2) \]

\[ (A_{\mathcal{M}N}^\dagger)^{\#}_{\mathcal{M}N}(c, ::, 1, 3) \]

which coincides with \(A_{\mathcal{M}N}^\dagger\). But, \((A_{\mathcal{M}N}^\dagger_{\mathcal{M}N})^{\#}\) is given by

\[ (A_{\mathcal{M}N}^\dagger_{\mathcal{M}N})^{\#}_{\mathcal{N}N}(c, ::, 1, 1) \]

\[ (A_{\mathcal{M}N}^\dagger_{\mathcal{M}N})^{\#}_{\mathcal{N}N}(c, ::, 1, 2) \]

\[ (A_{\mathcal{M}N}^\dagger_{\mathcal{M}N})^{\#}_{\mathcal{N}N}(c, ::, 1, 3) \]

which is not equal with \(A_{\mathcal{M}N}^\dagger_{\mathcal{M}N}\).
Likewise, the left cancellation property is stated below.

Let \( A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( C \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_L \times I_1 \times \cdots \times I_M} \), and let \( M \) and \( N \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \) and \( \mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N} \), respectively. If \( B^{*M}A^{*N}\mathbb{A}^{*M}_{MN} = C^{*M}A^{*N}\mathbb{A}^{*M}_{MN} \), then \( B^{*M}A = C^{*M}A \).

Proof Assume that \( B^{*M}A^{*N}\mathbb{A}^{*M}_{MN} = C^{*M}A^{*N}\mathbb{A}^{*M}_{MN} \). Post-multiplying both sides by \((\mathbb{A}^{*M}_{MN})^{\dagger}\), we obtain \( B^{*M}A^{*N}(\mathbb{A}^{*M}_{MN})^{\dagger} = C^{*M}A^{*N}(\mathbb{A}^{*M}_{MN})^{\dagger} \). Hence, we have \( B^{*M}A = C^{*M}A \) by using Theorem 2 and Definition 3.

Likewise, the left cancellation property is stated below.

Let \( A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \), \( B \in \mathbb{C}^{I_1 \times \cdots \times J_N \times K_L \times I_1 \times \cdots \times I_M} \) and \( C \in \mathbb{C}^{I_1 \times \cdots \times J_N \times K_L \times I_1 \times \cdots \times I_M} \), and let \( M \) and \( N \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathbb{C}^{I_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N} \), respectively. If \( \mathbb{A}^{*M}_{MN} = \mathbb{A}^{*M}_{MN} \), then \( A^{*N}B = A^{*N}C \).

A sufficient condition for the commutativity of \( A^{*M}_{MN} \) and \( B^{*L}B^{*M}_{MN} \) is presented next.

Lemma 6 Let \( A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( B \in \mathbb{C}^{I_1 \times \cdots \times J_N \times K_L \times I_1 \times \cdots \times I_M} \), \( C \in \mathbb{C}^{I_1 \times \cdots \times J_N \times K_L \times I_1 \times \cdots \times I_M} \), and let \( M \) and \( N \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathbb{C}^{I_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N} \), respectively. If \( A^{*M}_{MN} = A^{*M}_{MN} \), then \( A^{*M}_{MN} \) commutes with \( B^{*L}B^{*M}_{MN} \).

Proof Post-multiplying \((\mathbb{A}^{*M}_{MN})^{\dagger}\) to \( A^{*M}_{MN}A^{*N}B^{*L}B^{*M}_{MN} = B^{*L}B^{*M}_{MN}A^{*M}_{MN} \), we obtain

\[
A^{*M}_{MN}A^{*N}B^{*L}B^{*M}_{MN}A^{*M}_{MN} = B^{*L}B^{*M}_{MN}A^{*M}_{MN}. \tag{2}
\]

Taking the weighted conjugate transpose, we then have

\[
A^{*M}_{MN}A^{*N}B^{*L}B^{*M}_{MN}A^{*M}_{MN} = A^{*M}_{MN}A^{*N}B^{*L}B^{*M}_{MN}. \tag{3}
\]

Equations (2) and (3) together yields

\[
A^{*M}_{MN}A^{*N}B^{*L}B^{*M}_{MN} = B^{*L}B^{*M}_{MN}A^{*M}_{MN}. \tag{4}
\]

☐
The next one can be proved analogously.

**Lemma 7** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, and let $M$, $N$ and $L$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$, $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ and $\mathbb{C}^{K_1 \times \cdots \times K_L \times K_1 \times \cdots \times K_L}$, respectively. If $B^{\dagger} \mathcal{N} \mathcal{L} \ast N A^\# \#_{MN} A \ast N B = A^\# \#_{MN} \ast M A \ast N B$, then $B^{\dagger} \mathcal{N} \mathcal{L} \ast N$ commutes with $A^\# \#_{MN} \ast M A$.

Equivalent conditions for the commutativity of $A_{MN}^{\dagger} \ast M A$ and $B^{\dagger} \mathcal{N} \mathcal{L} \ast N$ are shown next.

**Lemma 8** Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, and let $M$, $N$ and $L$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$, $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ and $\mathbb{C}^{K_1 \times \cdots \times K_L \times K_1 \times \cdots \times K_L}$, respectively. Then the commutativity of $A_{MN}^{\dagger} \ast M A$ and $B^{\dagger} \mathcal{N} \mathcal{L} \ast N$ is equivalent to either of the conditions

$$A_{MN}^{\dagger} \ast M A \ast N B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^\# = B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^\#$$

(4)

and

$$B^{\dagger} \mathcal{N} \mathcal{L} \ast N A_{MN}^{\dagger} \ast M A \ast N B = A_{MN}^{\dagger} \ast M A \ast N B.$$  

(5)

**Proof** Suppose that $A_{MN}^{\dagger} \ast M A$ commutes with $B^{\dagger} \mathcal{N} \mathcal{L} \ast N$. Then,

$$A_{MN}^{\dagger} \ast M A \ast N B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^\# = B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^\#,$$

and

$$B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^{\dagger} \ast M A \ast N B = A_{MN}^{\dagger} \ast M A \ast N B.$$  

Conversely, post-multiplying $(A_{MN}^{\dagger})^\#_{N \cdot M}$ to Eq. (4), we obtain

$$A_{MN}^{\dagger} \ast M A \ast N B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^{\dagger} \ast M A = B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^{\dagger} \ast M A.$$  

(6)

Taking the weighted conjugate transpose of both sides of Eq. (6), we get

$$A_{MN}^{\dagger} \ast M A \ast N B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^{\dagger} \ast M A = A_{MN}^{\dagger} \ast M A \ast N B \ast L B_{\mathcal{N} \mathcal{L} \ast N}.$$  

(7)

From Eqs. (6) and (7), we have

$$B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^{\dagger} \ast M A = A_{MN}^{\dagger} \ast M A \ast N B \ast L B_{\mathcal{N} \mathcal{L} \ast N}.$$  

And post-multiplying $B_{\mathcal{N} \mathcal{L} \ast N}$ to Eq. (5), we get

$$B \ast L B_{\mathcal{N} \mathcal{L} \ast N} \ast A_{MN}^{\dagger} \ast M A \ast N B \ast L B_{\mathcal{N} \mathcal{L} \ast N} = A_{MN}^{\dagger} \ast M A \ast N B \ast L B_{\mathcal{N} \mathcal{L} \ast N}.$$  

(8)
Taking the weighted conjugate transpose, we obtain

$$B^*_{L}B^\dagger_{N^L}A^\dagger_{MN^*M}A^*_{N}B^*_{L}B^\dagger_{N^L} = B^*_{L}B^\dagger_{N^L}A^\dagger_{MN^*M}A. \quad (9)$$

From Eqs. (8) and (9), we have

$$A^\dagger_{MN^*M}A^*_{N}B^*_{L}B^\dagger_{N^L} = B^*_{L}B^\dagger_{N^L}A^\dagger_{MN^*M}A. \quad \square$$

The next result provides an absorbing property of a tensor which coincides with its weighted conjugate transpose.

**Lemma 9** Let $M$ and $N$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ and $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$, respectively, and let $P \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ such that $(M^*M)H = M^*M$. If $P^*M = Q$ for $Q \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, then

$$Q^\dagger_{MN^*M}P = Q^\dagger_{MN}.$$

**Proof** Let $X = Q^\dagger_{MN^*M}P$. Using the fact $P^*M = Q$, it is easy to show that $Q^*NX^*M = Q$, $X^*M^*QN = X$ and $(N^*N X^*M)^H = N^*N X^*M$. Again, using $P^*M = Q$ and $(M^*M)^H = M^*M$, we obtain

$$(M^*M Q^*N X)^H = (M^*M Q^*N Q^\dagger_{MN^*M}P)^H$$

$$= (M^*M P^*M Q^*N Q^\dagger_{MN^*M}P)^H$$

$$= [(M^*M P^*M M)^{-1} (M^*M Q^*N Q^\dagger_{MN^*M})^*M^*M (M^*M P)]^H$$

$$= M^*M P^*M M^{-1} (M^*M Q^*N Q^\dagger_{MN^*M})^*M^*M P$$

$$= M^*M Q^*N Q^\dagger_{MN^*M}P$$

$$= M^*M Q^*N X.$$

By Definition 3, we thus have the claim. \quad \square

Likewise, the below one can also be proved.

**Lemma 10** Let $M$ and $N$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ and $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$, respectively, and let $P \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ such that $(N^*N)^H = N^*N$. If $Q^*NP = Q$ for $Q \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, then

$$P^*N Q^\dagger_{MN} = Q^\dagger_{MN}. \quad \copyright Birkhäuser
3.2 Reverse-order law for the weighted Moore–Penrose inverse of square tensors

In this subsection, we provide some sufficient conditions for the reverse-order law of two square tensors via the Einstein product. The first result presents four sufficient conditions.

**Theorem 4** Let \( A, B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \), and let \( M \) and \( N \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_N} \). If

\[
\begin{align*}
A^*_{\mathcal{N}}(B^*_{\mathcal{N}}B^\dagger_{\mathcal{MN}}) &= (B^*_{\mathcal{N}}B^\dagger_{\mathcal{MN}})A^*_{\mathcal{N}}, \quad (10) \\
A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}}(B^*_{\mathcal{N}}B^\dagger_{\mathcal{MN}}) &= (B^*_{\mathcal{N}}B^\dagger_{\mathcal{MN}})A^\dagger_{\mathcal{MN}}, \quad (11) \\
B^*_{\mathcal{N}}(A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}}) &= (A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}})B^\dagger_{\mathcal{MN}}, \quad (12) \\
B^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}}(A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}}) &= (A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}})B^\dagger_{\mathcal{MN}}, \quad (13)
\end{align*}
\]

then \((A^*_{\mathcal{N}}B)\dagger_{\mathcal{MN}} = B^\dagger_{\mathcal{MN}}A^\dagger_{\mathcal{MN}}\).

**Proof** Suppose that Eqs. (10)–(13) hold. Let \( \mathcal{X} = A^*_{\mathcal{N}}B \) and \( \mathcal{Y} = B^\dagger_{\mathcal{MN}}A^\dagger_{\mathcal{MN}} \). It can be easily seen that \( \mathcal{X}^*_{\mathcal{N}}\mathcal{Y}^*_{\mathcal{N}}\mathcal{X} = \mathcal{X} \) and \( \mathcal{Y}^*_{\mathcal{N}}\mathcal{X}^*_{\mathcal{N}}\mathcal{Y} = \mathcal{Y} \), by using Eqs. (12) and (13), and Eqs. (10) and (11), respectively. We then get \((M^*\mathcal{X}^*\mathcal{Y})^H = M^*\mathcal{X}^*\mathcal{Y}^*\mathcal{X}^*\mathcal{Y}^*\mathcal{X} \) by using Eq. (10), which again reduces to \((M^*\mathcal{X}^*\mathcal{Y})^H = M^*\mathcal{X}^*\mathcal{Y}\) by Eq. (11). Likewise, we obtain \((N^*\mathcal{Y}^*\mathcal{X})^H = N^*\mathcal{Y}^*\mathcal{X}\) by using Eqs. (12) and (13). Hence the result follows by Definition 3.

The next result shows that the assumptions (10) and (11) in the above result can be replaced by a single one.

**Theorem 5** Let \( A, B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \), and let \( M \) and \( N \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_N} \). If

(i) \((M^*\mathcal{X}^*\mathcal{Y})^H = M^*\mathcal{X}^*\mathcal{Y}\),
(ii) \(B^*_{\mathcal{N}}(A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}}) = (A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}})B^\dagger_{\mathcal{MN}},\)
(iii) \(B^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}}(A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}}) = (A^\dagger_{\mathcal{MN}}A^*_{\mathcal{N}})B^\dagger_{\mathcal{MN}},\)

then \((A^*_{\mathcal{N}}B)^\dagger_{\mathcal{MN}} = B^\dagger_{\mathcal{MN}}A^\dagger_{\mathcal{MN}}\).

**Proof** Let \( \mathcal{X} = A^*_{\mathcal{N}}B \) and \( \mathcal{Y} = B^\dagger_{\mathcal{MN}}A^\dagger_{\mathcal{MN}} \). Clearly, \( \mathcal{X}^*_{\mathcal{N}}\mathcal{Y}^*_{\mathcal{N}}\mathcal{X} = \mathcal{X} \), \( \mathcal{Y}^*_{\mathcal{N}}\mathcal{X}^*_{\mathcal{N}}\mathcal{Y} = \mathcal{Y} \) and \((N^*\mathcal{Y}^*\mathcal{X})^H = N^*\mathcal{Y}^*\mathcal{X}\), by using (ii) and (iii). By Definition 3 and the assumption (i), we thus have \((A^*_{\mathcal{N}}B)^\dagger_{\mathcal{MN}} = B^\dagger_{\mathcal{MN}}A^\dagger_{\mathcal{MN}}\).

The last two conditions of Theorem 4 can also be replaced by a single one. This is stated next.

**Theorem 6** Let \( A, B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \), and let \( M \) and \( N \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_N} \). If
Theorem 7 Let $A$, $B \in \mathbb{C}^{I_1 \times \cdots \times I_N}$, and let $\mathcal{M}$ and $\mathcal{N}$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_N}$ such that $(\mathcal{M} \ast \mathcal{N}) = (\mathcal{N} \ast \mathcal{M})^H$.

(i) $A_N (B_N B_M^\dagger) = (B_N B_M^\dagger) A_N$.

(ii) $A_M^\dagger \ast \mathcal{M} \ast \mathcal{N} = (B_N B_M^\dagger) A_M^\dagger$.

(iii) $(N_B M_N \ast N_N A_B M_N \ast N_N B_M^\dagger) = (N_B M_N \ast N_N A_B M_N \ast N_N A_M^\dagger)$.

then $(A_N B_M^\dagger) = B_M^\dagger A_M^\dagger$.

The next result illustrates that if a tensor $A$ further satisfies $\mathcal{M} \ast \mathcal{N} A_M^\dagger = (\mathcal{M} \ast \mathcal{N} A)^H$, then (12) and (13) are sufficient to hold the reverse-order law.

Theorem 8 Let $A$, $B \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$, and let $\mathcal{M}$ and $\mathcal{N}$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ such that $(\mathcal{M} \ast \mathcal{N}) = (\mathcal{N} \ast \mathcal{M})^H$.

(i) $A_N (B_N B_M^\dagger) = (B_N B_M^\dagger) A_N$.

(ii) $A_M^\dagger \ast \mathcal{M} \ast \mathcal{N} = (B_N B_M^\dagger) A_M^\dagger$.

(iii) $(N_B M_N \ast N_N A_B M_N \ast N_N B_M^\dagger) = (N_B M_N \ast N_N A_B M_N \ast N_N A_M^\dagger)$.

then $(A_N B_M^\dagger) = B_M^\dagger A_M^\dagger$.

The Proof of Theorems 7 and 8 will be given in the next section.
Reverse-order law for weighted Moore–Penrose inverse of tensors

Therefore, \( X_{\mathcal{MN}}^\dagger = \mathcal{Y} \) by Definition 3. \( \square \)

One of any assumptions (10)–(13) is sufficient to hold the reverse-order law for tensors \( \mathcal{A} \) and \( \mathcal{B} \) satisfying \( \mathcal{M}_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger = (\mathcal{M}_{\ast \mathcal{N}} A)^H \) and \( \mathcal{N}_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger = (\mathcal{N}_{\ast \mathcal{N}} B)^H \). This is demonstrated below.

**Theorem 9** Let \( \mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \), and let \( \mathcal{M} \) and \( \mathcal{N} \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) such that \( \mathcal{M}_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger = (\mathcal{M}_{\ast \mathcal{N}} A)^H \) and \( \mathcal{N}_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger = (\mathcal{N}_{\ast \mathcal{N}} B)^H \). If at least one of the following holds

\[
\mathcal{A}_{\ast \mathcal{N}} (\mathcal{B}_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger) = (\mathcal{B}_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger)_{\ast \mathcal{N}} \mathcal{A},
\]

(18)

\[
\mathcal{A}_{\mathcal{MN}}^\dagger (\mathcal{B}_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger) = (\mathcal{B}_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger)_{\ast \mathcal{N}} \mathcal{A}_{\mathcal{MN}}^\dagger,
\]

(19)

\[
\mathcal{B}_{\ast \mathcal{N}} (\mathcal{A}_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}}) \mathcal{A} = (\mathcal{A}_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}})_{\ast \mathcal{N}} \mathcal{B},
\]

(20)

\[
\mathcal{B}_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}} (\mathcal{A}_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}}) \mathcal{A} = (\mathcal{A}_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}})_{\ast \mathcal{N}} \mathcal{B}_{\mathcal{MN}}^\dagger,
\]

(21)

then \( (\mathcal{A}_{\ast \mathcal{N}} \mathcal{B})^\dagger_{\mathcal{MN}} = B_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger \).

**Proof** The proof is based on the assumption Eq. (18). Let \( \mathcal{X} = \mathcal{A}_{\ast \mathcal{N}} \mathcal{B} \) and \( \mathcal{Y} = B_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger \). Using Eq. (18), we obtain

\[
\mathcal{X}_{\ast \mathcal{N}} \mathcal{Y}_{\ast \mathcal{N}} \mathcal{X} = \mathcal{X} \text{ and } \mathcal{Y}_{\ast \mathcal{N}} \mathcal{X}_{\ast \mathcal{N}} \mathcal{Y} = (B_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger)_{\ast \mathcal{N}} (B_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger)_{\ast \mathcal{N}} \mathcal{A}_{\mathcal{MN}}^\dagger.
\]

Then, we get \( \mathcal{Y}_{\ast \mathcal{N}} \mathcal{X}_{\ast \mathcal{N}} \mathcal{Y} = \mathcal{Y} \) since \( \mathcal{M}_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger = (\mathcal{M}_{\ast \mathcal{N}} A)^H \). Again, using the hypothesis \( \mathcal{M}_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger = (\mathcal{M}_{\ast \mathcal{N}} A)^H \), we obtain \( (\mathcal{M}_{\ast \mathcal{N}} \mathcal{X}_{\ast \mathcal{N}} \mathcal{Y})^H = \mathcal{M}_{\ast \mathcal{N}} \mathcal{X}_{\ast \mathcal{N}} \mathcal{Y} \). Likewise, the assumption \( \mathcal{N}_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger = (\mathcal{N}_{\ast \mathcal{N}} B)^H \) yields \( (\mathcal{N}_{\ast \mathcal{N}} \mathcal{Y}_{\ast \mathcal{N}} \mathcal{X})^H = \mathcal{N}_{\ast \mathcal{N}} \mathcal{Y}_{\ast \mathcal{N}} \mathcal{X} \). Therefore, \( X_{\mathcal{MN}}^\dagger = \mathcal{Y} \), by Definition 3.

We conclude this subsection with the note that one can also show that if any one of the Eqs. (19), (20) and (21) holds together with \( \mathcal{M}_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger = (\mathcal{M}_{\ast \mathcal{N}} A)^H \) and \( \mathcal{N}_{\ast \mathcal{N}} B_{\mathcal{MN}}^\dagger = (\mathcal{N}_{\ast \mathcal{N}} B)^H \), then \( (\mathcal{A}_{\ast \mathcal{N}} \mathcal{B})_{\mathcal{MN}}^\dagger = B_{\mathcal{MN}}^\dagger_{\ast \mathcal{N}} A_{\mathcal{MN}}^\dagger \). \( \square \)

### 3.3 Reverse-order law for the weighted Moore–Penrose inverse of arbitrary tensors

In this subsection, we obtain some necessary and sufficient conditions for the reverse-order law of the Einstein product of two tensors. The very first result was proved in [1] using the range and null space of a tensor, however we present a new proof without using the notion of the range and null space of a tensor.
Theorem 10 Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, and let $M$, $N$ and $L$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$, $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ and $\mathbb{C}^{K_1 \times \cdots \times K_L \times K_1 \times \cdots \times K_L}$, respectively. Then

$$(A \ast_N B)_{\mathcal{ML}}^\dagger = B_{\mathcal{NL}^* N}^\dagger A_{\mathcal{MN}}^\dagger$$

if and only if

$$(A_{\mathcal{MN}}^\ast_M A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\#) = B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\#.$$

Proof For any tensor $G \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, we have

$$G_{\mathcal{MN}}^\dagger A_{\mathcal{MN}}^\ast_M G_{\mathcal{MN}}^\# = (G_{\mathcal{MN}}^\dagger A_{\mathcal{MN}}^\ast_M G_{\mathcal{MN}}^\#)^{\mathcal{N}^* \mathcal{N}^* N}_{\mathcal{MN}^*}.$$

Similarly, one can also show that $G_{\mathcal{MN}}^\# G_{\mathcal{MN}}^\dagger = G_{\mathcal{MN}}^\#$. Suppose that $(A \ast_N B)_{\mathcal{ML}}^\dagger = B_{\mathcal{NL}^* N}^\dagger A_{\mathcal{MN}}^\dagger$. Setting $G = A \ast_N B$, we now obtain

$$(A \ast_N B)_{\mathcal{ML}}^\dagger = (A \ast_N B)_{\mathcal{ML}^* M}^\dagger (A \ast_N B)_{\mathcal{ML}}^\dagger,$$

so we have

$$B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\# = B_{\mathcal{NL}^* N}^\dagger A_{\mathcal{MN}}^\dagger M A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\#.$$

Pre-multiplying both sides by $A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# B$, we get

$$A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\dagger M A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\# = A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\dagger M A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\#,$$

which yields

$$A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# (I - A_{\mathcal{MN}}^\dagger M A) \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\# = 0.$$

Using the fact $I - A_{\mathcal{MN}}^\dagger M A$ is idempotent and $(I - A_{\mathcal{MN}}^\dagger M A)^{\mathcal{N}^* \mathcal{N}^* N}_{\mathcal{MN}^*} = I - A_{\mathcal{MN}}^\dagger M A$, the last equation can be written as

$$[(I - A_{\mathcal{MN}}^\dagger M A) \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\dagger M A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\# = 0.$$

Thus,

$$A_{\mathcal{MN}^* M}^\dagger A \ast_N B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\# = B \ast_L B_{\mathcal{NL}^* N}^\# A_{\mathcal{MN}}^\#.$$
Again,
\[ A \ast N B = (A^\dagger_{MN})_N \ast M \ast (B^\dagger_{NL})_L \ast N (B^\dagger_{NL})_L \ast N A^\dagger_{MN} \ast M A \ast N B \]
\[ = (A^\dagger_{MN})_N \ast M \ast B^\dagger_{NL} \ast N A^\dagger_{MN} \ast M A \ast N B. \]

Now, pre-multiplying \( B^\#_{NL} \ast N A^\#_{MN} \ast M A \ast N A^\dagger_{MN} \) to
\[ A \ast N B = (A^\dagger_{MN})_N \ast M \ast B^\dagger_{NL} \ast N A^\dagger_{MN} \ast M A \ast N B, \]
we get
\[ B^\#_{NL} \ast N A^\#_{MN} \ast M A \ast N A^\dagger_{MN} \ast M A \ast N B = B^\#_{NL} \ast N A^\#_{MN} \ast M A \ast N B \ast L B^\dagger_{NL} \ast N A^\dagger_{MN} \ast M A \ast N B, \]
which implies
\[ B^\#_{NL} \ast N A^\#_{MN} \ast M A \ast N (I - B^\ast L B^\dagger_{NL}) \ast N A^\dagger_{MN} \ast M A \ast N B = O. \]

Since \( I - B^\ast L B^\dagger_{NL} \) is idempotent and \( (I - B^\ast L B^\dagger_{NL})^\#_{NN} = I - B^\ast L B^\dagger_{NL} \), so
\[ [(I - B^\ast L B^\dagger_{NL}) \ast N A^\#_{MN} \ast M A \ast N B]^\#_{NL} \ast N (I - B^\ast L B^\dagger_{NL}) \ast N A^\dagger_{MN} \ast M A \ast N B = O. \]

Thus,
\[ B^\ast L B^\dagger_{NL} \ast N A^\dagger_{MN} \ast M A \ast N B = A^\dagger_{MN} \ast M A \ast N B. \]

Conversely, pre-multiplying and post-multiplying Eq. (22) by \( B^\dagger_{NL} \) and \((A \ast N B)^\#_{ML})^\dagger_{LM}, \) respectively, we get
\[ B^\dagger_{NL} \ast N A^\dagger_{MN} \ast M A \ast N B = (A \ast N B)^\dagger_{ML} \ast M (A \ast N B). \] \hspace{1cm} (24)

Taking the weighted conjugate transpose of Eq. (23), we have
\[ B^\#_{NL} \ast N A^\#_{MN} \ast M A \ast N B \ast L B^\dagger_{NL} = B^\#_{NL} \ast N A^\#_{MN} \ast M A. \] \hspace{1cm} (25)

Pre-multiplying and post-multiplying Eq. (25) by \((A \ast N B)^\#_{ML} \ast N A^\dagger_{MN} \ast N A \ast N B^\dagger_{NL} \), respectively, we obtain
\[ A \ast N B \ast L B^\dagger_{NL} \ast N A^\dagger_{MN} = A \ast N B \ast L (A \ast N B)^\dagger_{ML}. \] \hspace{1cm} (26)

In order to show \( (A \ast N B)^\dagger_{ML} = B^\dagger_{NL} \ast N A^\dagger_{MN} \), we have to show that \( B^\dagger_{NL} \ast N A^\dagger_{MN} \) satisfies Definition 1. This is done now. Using Eq. (24), we have
\[ A \ast N B \ast L B^\dagger_{NL} \ast N A^\dagger_{MN} \ast M A \ast N B = A \ast N B. \]

Using Lemma 6, we get \( B^\ast L B^\dagger_{NL} \ast N A^\dagger_{MN} \ast M A \ast N B \ast L B^\dagger_{NL} \ast N A^\#_{MN} \ast N A^\dagger_{MN} \ast M A \ast N B^\dagger_{NL} \ast N A^\#_{MN} \), which on pre-multiplication of \( B^\dagger_{NL} \ast N A^\dagger_{MN} \ast M A \ast N B \ast L B^\dagger_{NL} \ast N A^\#_{MN} \).
Thus have \( A_{MN}^\# = B_{NL}^\# * N A_{MN}^\# \). Again, pre-multiplying \((B_{NL}^\#)_{MN}^\# \) and post-multiplying \((A_{MN}^\#)_{MN}^\# \), we obtain

\[
B_{NL}^\# * N A_{MN}^\# * M A_{N}^* B * L B_{NL}^\# * N A_{MN}^\# = B_{NL}^\# * N A_{MN}^\#.
\]

Now, with the help of Eq. (26), we get

\[
(M * M A_{N}^* B * L B_{NL}^\#)_{MN}^\# = M * M A_{N}^* B * L B_{NL}^\# A_{MN}^\#,
\]

and using Eq. (24), we have

\[
(L * B_{NL}^\# * N A_{MN}^\# * M A_{N}^* B)^{H} = L * B_{NL}^\# * N A_{MN}^\# * M A_{N}^* B.
\]

\[\Box\]

We next present a simpler characterization of Theorem 10.

**Theorem 11** Let \( A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L} \), and let \( M \), \( N \) and \( L \) be Hermitian positive definite tensors in \( \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathbb{C}^{K_1 \times \cdots \times K_L \times K_1 \times \cdots \times K_L} \), respectively. Then

\[
(A_{N}^* B)_{MC} = B_{NL}^\# * N A_{MN}^\#
\]

if and only if

\[
(A_{MN}^\# M A_{N}^* B * L B_{NL}^\#)_{MN}^\# = A_{MN}^\# M A_{N}^* B * N B_{NL}^\#
\]

and

\[
(A_{MN}^\# M A_{N}^* B * L B_{NL}^\#)_{MN}^\# = A_{MN}^\# M A_{N}^* B * L B_{NL}^\#.
\]

**Proof** Since \((A_{MN}^\# M A_{N}^* B * L B_{NL}^\#)_{MN}^\# = A_{MN}^\# M A_{N}^* B * L B_{NL}^\# \), we so have

\[
A_{MN}^\# M A_{N}^* B * L B_{NL}^\# = B_{NL}^\# * N A_{MN}^\# M A,
\]

which after post-multiplication of \( A_{MN}^\# M A_{N}^* B * L B_{NL}^\# \) yields Eq. (22). Again, \((A_{MN}^\# M A_{N}^* B * L B_{NL}^\#)_{MN}^\# = A_{MN}^\# M A_{N}^* B * L B_{NL}^\# \) implies \( B_{NL}^\# * N A_{MN}^\# M A \) which after post-multiplication of \( B \) yields Eq. (23). We thus have \((A_{N}^* B)_{MC} = B_{NL}^\# * N A_{MN}^\# \) by Theorem 10.

Conversely, we have Eqs. (22) and (23) by Theorem 10. Post-multiplying \((A_{MN}^\# M A_{N}^* B * L B_{NL}^\#)_{MN}^\# \) to Eq. (22) yields \( (A_{MN}^\# M A_{N}^* B * L B_{NL}^\#)_{MN}^\# = A_{MN}^\# M A_{N}^* B * L B_{NL}^\# \), while post-multiplying \( B_{NL}^\# \) to Eq. (23) results

\[
(A_{MN}^\# M A_{N}^* B * L B_{NL}^\#)_{MN}^\# = A_{MN}^\# M A_{N}^* B * L B_{NL}^\#.
\]

\[\Box\]

In Theorem 10 and Theorem 11, two conditions are required to hold the reverse-order law for the weighted Moore–Penrose inverse. The next result says that these two conditions can be replaced by a single one.
Theorem 12 Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, and let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ be Hermitian positive definite tensors in $\mathbb{C}^{J_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$, $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ and $\mathbb{C}^{K_1 \times \cdots \times K_L \times K_1 \times \cdots \times K_L}$, respectively. Then $(A_{*N}B)_{\mathcal{M}\mathcal{L}} = B^\dagger_{\mathcal{N}\mathcal{L}} A_{\mathcal{M}\mathcal{N}}^\dagger$ if and only if

$$A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\# * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger = B_{*L} B_{\mathcal{N}\mathcal{L}}^\# * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger.$$

(27)

Proof Pre-multiplying and post-multiplying Eq. (27) by $A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A$ and $A_{\mathcal{M}\mathcal{N}}^\dagger$, respectively, we get Eq. (22). Again, pre-multiplying and post-multiplying Eq. (27) by $B_{\mathcal{N}\mathcal{L}}^\dagger$ and $B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger$, respectively, we obtain

$$B_{\mathcal{N}\mathcal{L}}^\dagger * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger = B_{\mathcal{N}\mathcal{L}}^\dagger * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger,$$

which yields Eq. (23) after taking the weighted conjugate transpose. Thus $(A_{*N}B)_{\mathcal{M}\mathcal{L}} = B_{\mathcal{N}\mathcal{L}}^\dagger * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger$ by Theorem 10.

Conversely, we have Eq. (22) which follows from Theorem 10. Post-multiplying this by $A$, we get

$$A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} (A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B)^\#_{\mathcal{N}\mathcal{L}} = B_{*L} B_{\mathcal{N}\mathcal{L}}^\# * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger.$$

We also have Eq. (23). Using this, we now obtain

$$A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\# * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger = B_{*L} B_{\mathcal{N}\mathcal{L}}^\# * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger.$$

(27)

We next present another characterization of the reverse-order law.

Theorem 13 Let $A \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, and let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ be Hermitian positive definite tensors in $\mathbb{C}^{J_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$, $\mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ and $\mathbb{C}^{K_1 \times \cdots \times K_L \times K_1 \times \cdots \times K_L}$, respectively. Then $(A_{*N}B)_{\mathcal{M}\mathcal{L}} = B_{\mathcal{N}\mathcal{L}}^\dagger * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger$ if and only if

$$A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} = B_{*L} (A_{*N} B)^\dagger_{\mathcal{M}\mathcal{L}} * \mathcal{M} A_{*N} B_{*L}$$

(28)

and

$$B_{*L} B_{\mathcal{N}\mathcal{L}}^\dagger * \mathcal{N} A_{\mathcal{M}\mathcal{N}}^\dagger = A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} (A_{*N} B)^\dagger_{\mathcal{M}\mathcal{L}}.$$

(29)

Proof By Theorem 10, Eqs. (22) and (23) hold. Now, post-multiplying $(A_{*N} B)^\dagger_{\mathcal{M}\mathcal{L}}$ to Eq. (22), we obtain

$$A_{\mathcal{M}\mathcal{N}}^\dagger * \mathcal{M} A_{*N} B_{*L} (A_{*N} B)^\#_{\mathcal{M}\mathcal{L}} * \mathcal{M} (A_{*N} B)^\dagger_{\mathcal{M}\mathcal{L}} = B_{*L} (A_{*N} B)^\dagger_{\mathcal{M}\mathcal{L}} * \mathcal{M} (A_{*N} B)^\dagger_{\mathcal{M}\mathcal{L}} * \mathcal{M} (A_{*N} B)^\dagger_{\mathcal{M}\mathcal{L}}.$$
which in turn implies \( A_{MN}^\dagger A_{NM} A_{NB} = B_{LM}(A_{NB})^\dagger_{ML} A_{NB} \). Post-multiplying \((A_{NB})^\dagger_{ML}\) to Eq. (23), we obtain

\[
A_{MN}^\dagger A_{NB} B_{LM}(A_{NB})^\dagger_{ML} = (B_{LM}(A_{NB})^\dagger_{ML})^\dagger_{ML} A_{MN} A_{NB} B_{LM}(A_{NB})^\dagger_{ML} = (B_{LM}(A_{NB})^\dagger_{ML})^\dagger_{ML} A_{MN} A_{NB}
\]

Conversely, post-multiplying \( B_{NL}^\dagger A_{MN}^\# \) to Eq. (28), we have

\[
A_{MN}^\dagger A_{NB} B_{LM}(A_{NB})^\dagger_{ML} A_{MN}^\# = B_{LM}(A_{NB})^\dagger_{ML} A_{MN}^\# A_{NB} B_{LM}(A_{NB})^\dagger_{ML} = B_{LM}(A_{NB})^\dagger_{ML} A_{MN}^\# A_{NB} = B_{LM} A_{MN}^\# A_{NB}.
\]

Again, post-multiplying \( A_{NB} \) to Eq. (29), we obtain

\[
B_{LM} A_{MN}^\# A_{NB} = A_{MN}^\# A_{NB} B_{LM}(A_{NB})^\dagger_{ML} A_{NB} = A_{MN}^\# A_{NB}.
\]

We thus have \((A_{NB})^\dagger_{ML} = B_{NL}^\dagger A_{MN}^\#\) by Theorem 10. \( \square \)

Next theorem replaces the first condition of Theorem 10 by a new condition.

**Theorem 14** Let \( A \in C^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( B \in C^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L} \), and let \( M, N \) and \( L \) be Hermitian positive definite tensors in \( C^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( C^{K_1 \times \cdots \times K_L} \), respectively. Then \((A_{NB})^\dagger_{ML} = B_{NL}^\dagger A_{MN}^\#\) if and only if

(i) \((L_{LM} B_{NL}^\dagger A_{MN}^\# A_{NB})^H = L_{LM} B_{NL}^\dagger A_{MN}^\# A_{NB}\),

(ii) \(B_{LM} B_{NL}^\dagger A_{MN}^\# A_{NB} = A_{MN}^\# A_{NB}\).

**Proof** Suppose that \((A_{NB})^\dagger_{ML} = B_{NL}^\dagger A_{MN}^\#\). By Definition 3, it is clear that

\[
(L_{LM} B_{NL}^\dagger A_{MN}^\# A_{NB})^H = L_{LM} B_{NL}^\dagger A_{MN}^\# A_{NB}.
\]

Theorem 10 also yields \(B_{LM} B_{NL}^\dagger A_{MN}^\# A_{NB} = A_{MN}^\# A_{NB}\). Conversely, (ii) is equivalent to

\[
B_{NL}^\dagger A_{MN}^\# A_{NB} B_{LM} B_{NL}^\dagger = B_{NL}^\dagger A_{MN}^\# A_{NB}.
\]
Pre-multiplying and post-multiplying this by \((A_{*N}B)_{\mathcal{ML}}^\dagger\) and \(A_{\mathcal{MN}}^\dagger\), respectively, we get

\[
A_{*N}B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}}^\dagger = A_{*N}B_{*L}(A_{*N}B)_{\mathcal{ML}}^\dagger.
\]  

(31)

Let \(X = A_{*N}B\) and \(Y = B_{\mathcal{NL}*N}A_{\mathcal{MN}}^\dagger\). In view of Eq. (31), we have \(X_{*L}Y_{*M}X = X\) and \((M_{*M}X_{*L}Y)^H = M_{*M}X_{*L}Y\). From Eq. (30), we also obtain

\[
B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}*M}^\dagger A_{*N}B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}*M}^\dagger A = B_{\mathcal{NL}*N}A_{\mathcal{MN}*M}^\dagger A
\]

on considering Lemma 7. Pre-multiplying and post-multiplying the last equation by \((B_{\mathcal{NL}})^{\#}_{\mathcal{LN}}\) and \(A_{\mathcal{MN}}^\dagger\), respectively, we have

\[
B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}*M}^\dagger A_{*N}B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}*M}^\dagger A = B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}*M}^\dagger A
\]

Further, pre-multiplying and post-multiplying this equation by \(B_{\mathcal{NL}}^\dagger\) and \((A_{\mathcal{MN}}^\dagger)^{\#}_{\mathcal{MN}}\), respectively, we get \(Y_{*M}X_{*L}Y = Y\). At last \((L_{*L}Y_{*M}X)^H = L_{*L}Y_{*M}X\) by (i). Thus, \(X_{\mathcal{ML}} = Y\).

\[\Box\]

Similarly, we replace the second condition of Theorem 10 by a new condition.

**Theorem 15** Let \(A \in \mathbb{C}^{J_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}\) and \(B \in \mathbb{C}^{J_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M, \mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}\) and \(C \in \mathbb{C}^{K_1 \times \cdots \times K_L \times K_1 \times \cdots \times K_L}\), respectively. Then \((A_{*N}B)_{\mathcal{ML}}^\dagger = B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}}^\dagger\) if and only if

(i) \(A_{\mathcal{MN}*M}A_{*N}B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}}^\dagger = B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}}^\dagger\),

(ii) \((M_{*M}A_{*N}B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}}^\dagger)^H = M_{*M}A_{*N}B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}}^\dagger\).

It is interesting to note that the fact “\(M_{*M}A_{*N}B_{*L}B_{\mathcal{NL}*N}^\dagger A_{\mathcal{MN}}^\dagger\) is Hermitian” is a weaker condition than Eq. (23) and “\((L_{*L}Y_{*M}X)^H = L_{*L}Y_{*M}X\)” is weaker than Eq. (22). The next example verifies the above fact.

**Example 2** Consider the tensors \(A\) and \(B\) in \(\mathbb{C}^{2 \times 3 \times 2 \times 3}\) as follows:

| \(A(:, :, 1, 1)\) | \(A(:, :, 1, 2)\) | \(A(:, :, 1, 3)\) |
|-----------------|-----------------|-----------------|
| 1 0 1           | 0 1 0           | 5 0 0           |
| 1 1 0           | 0 0 0           | 0 0 0           |
| \(A(:, :, 2, 1)\) | \(A(:, :, 2, 2)\) | \(A(:, :, 2, 3)\) |
| 0 1 0           | 0 0 0           | 1 0 0           |
| 0 0 0           | 1 0 0           | 0 1 1           |

and
Also, consider three Hermitian positive definite tensors \( \mathcal{M} \), \( \mathcal{N} \) and \( \mathcal{L} \) in \( \mathbb{C}^{2 \times 3 \times 2 \times 3} \), such that

\[
\begin{bmatrix}
\mathcal{M}(\cdot, \cdot, 1, 1) & \mathcal{M}(\cdot, \cdot, 1, 2) & \mathcal{M}(\cdot, \cdot, 1, 3) \\
1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4 \\
0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{M}(\cdot, \cdot, 2, 1) & \mathcal{M}(\cdot, \cdot, 2, 2) & \mathcal{M}(\cdot, \cdot, 2, 3) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{N}(\cdot, \cdot, 1, 1) & \mathcal{N}(\cdot, \cdot, 1, 2) & \mathcal{N}(\cdot, \cdot, 1, 3) \\
1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4 \\
0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{N}(\cdot, \cdot, 2, 1) & \mathcal{N}(\cdot, \cdot, 2, 2) & \mathcal{N}(\cdot, \cdot, 2, 3) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\mathcal{L}(\cdot, \cdot, 1, 1) & \mathcal{L}(\cdot, \cdot, 1, 2) & \mathcal{L}(\cdot, \cdot, 1, 3) \\
4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{L}(\cdot, \cdot, 2, 1) & \mathcal{L}(\cdot, \cdot, 2, 2) & \mathcal{L}(\cdot, \cdot, 2, 3) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Then, \( A_{\mathcal{M} \mathcal{N}} \) is given by

\[
\begin{bmatrix}
A_{\mathcal{M} \mathcal{N}}(\cdot, \cdot, 1, 1) & A_{\mathcal{M} \mathcal{N}}(\cdot, \cdot, 1, 2) & A_{\mathcal{M} \mathcal{N}}(\cdot, \cdot, 1, 3) \\
1 & 0 & 5/4 & 0 & 4 & 0 & 4 & 0 & 0 \\
0 & 0 & 1/4 & 0 & 4 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_{\mathcal{M} \mathcal{N}}(\cdot, \cdot, 2, 1) & A_{\mathcal{M} \mathcal{N}}(\cdot, \cdot, 2, 2) & A_{\mathcal{M} \mathcal{N}}(\cdot, \cdot, 2, 3) \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Now, \( A_{\mathcal{M} \mathcal{N}}^\dagger \) and \( B_{\mathcal{N} \mathcal{L}}^\dagger \) are as follows:
\[ A_{MN}^\dagger(:, :; 1, 1) \quad A_{MN}^\dagger(:, ; 1, 2) \quad A_{MN}^\dagger(:, ; 1, 3) \]

\[
\begin{array}{ccc}
0 & 0 & 1/5 \\
0 & 0 & 0 \\
0 & 0 & 1/2 \\
0 & 0 & 0 \\
5/6 & 0 & -2/5 \\
0 & -5/6 & -1/6
\end{array}
\]

\[
A_{MN}^\dagger(:, ; 2, 1) \quad A_{MN}^\dagger(:, ; 2, 2) \quad A_{MN}^\dagger(:, ; 2, 3) \]

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/6 \\
0 & 1 & 0 \\
1/6 & 0 & -1/15 \\
1/6 & 0 & -2/15
\end{array}
\]

and

\[
B_{NL}^\dagger(:, ; 1, 1) \quad B_{NL}^\dagger(:, ; 1, 2) \quad B_{NL}^\dagger(:, ; 1, 3) \]

\[
\begin{array}{ccc}
0 & -5/21 & -1/7 \\
0 & 0 & 5/21 \\
0 & 0 & 0 \\
21/4 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}
\]

\[
B_{NL}^\dagger(:, ; 2, 1) \quad B_{NL}^\dagger(:, ; 2, 2) \quad B_{NL}^\dagger(:, ; 2, 3) \]

\[
\begin{array}{ccc}
0 & 8/21 & 3/7 \\
0 & 0 & -8/21 \\
0 & 0 & 8/21 \\
0 & 0 & 0 \\
4/7 & 0 & 0 \\
0 & 0 & 1
\end{array}
\]

Then \( M_M A_N B_L B_{NL}^\dagger * N A_{MN}^\dagger = \mathcal{U} \) is

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\]

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\]

and its conjugate transpose, \( \mathcal{U}^H \), is

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\]

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\]

Thus, \( \mathcal{U}^H = \mathcal{U} \), i.e., \((M_M A_N B_L B_{NL}^\dagger * N A_{MN}^\dagger)^H = M_M A_N B_L B_{NL}^\dagger * N A_{MN}^\dagger \). But \( B_L B_{NL}^\dagger * N A_{MN}^\dagger M_M A_N B = \mathcal{V} \) is

\[
\begin{array}{ccc}
-22/21 & 4 & 5/4 \\
52/21 & 8/21 & 3/2 \\
25/21 & 0 & 25/4 \\
-40/21 & 10/21 & 5/4 \\
-25/21 & 4 & 0 \\
61/21 & 11/21 & 0
\end{array}
\]

\[
\begin{array}{ccc}
-22/21 & 4 & 5/4 \\
52/21 & 8/21 & 3/2 \\
25/21 & 0 & 25/4 \\
-40/21 & 10/21 & 5/4 \\
-25/21 & 4 & 0 \\
61/21 & 11/21 & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
10/21 & 0 & 5/4 \\
-16/21 & 4/21 & 3/2 \\
101/21 & -4 & 15/2 \\
-170/21 & 32/21 & 7/4
\end{array}
\]

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
10/21 & 0 & 5/4 \\
-16/21 & 4/21 & 3/2 \\
101/21 & -4 & 15/2 \\
-170/21 & 32/21 & 7/4
\end{array}
\]
and $A_{MN}^* M A_N^* B$ is

| $A_{MN}^* M A_N^* B(:\ldots, 1, 1)$ | $A_{MN}^* M A_N^* B(:\ldots, 1, 2)$ | $A_{MN}^* M A_N^* B(:\ldots, 1, 3)$ |
|--------------------------|--------------------------|--------------------------|
| 2 4 5/4                  | 5 0 25/4                 | 1 4 0                    |
| 4 0 3/2                  | 0 0 5/4                  | 4 1/4 0                 |

| $A_{MN}^* M A_N^* B(:\ldots, 2, 1)$ | $A_{MN}^* M A_N^* B(:\ldots, 2, 2)$ | $A_{MN}^* M A_N^* B(:\ldots, 2, 3)$ |
|--------------------------|--------------------------|--------------------------|
| 0 0 0                    | 0 0 2                    | 13 4 15/2                |
| 0 0 0                    | 2 0 5/4                  | -4 1/2 7/4              |

Hence, $V \neq A_{MN}^* M A_N^* B$, i.e., $B_L^* B_{MN}^* A_{MN}^* M A_N^* B \neq A_{MN}^* M A_N^* B$.

The last result of this paper shows that the reverse-order law is a sufficient condition for the commutativity of $A_{MN}^* M A$ and $B_L^* B_{MN}^*$.

**Theorem 16** Let $A \in \mathbb{C}^{J_1 \times \cdots \times J_N}$ and $B \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, and let $M$, $N$, and $L$ be Hermitian positive definite tensors in $\mathbb{C}^{I_1 \times \cdots \times I_M}$, $\mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, respectively. If $(A_N^* B)^{\dagger}_{MN} = B_{MN}^* N A_{MN}^*$, then $A_{MN}^* M A_N^* B$ and $B_L^* B_{MN}^*$ commute.

**Proof** By Theorem 13, we have

$$A_{MN}^* M A_N^* B = B_L^* B_{MN}^* N A_{MN}^* M A_N^* B.$$ 

Post-multiplying $B_{MN}^* N A_{MN}^* M A_N^* B$ and taking the weighted conjugate transpose in both the sides of the above equation, we get

$$B_L^* B_{MN}^* N A_{MN}^* M A_N^* B = B_L^* B_{MN}^* N A_{MN}^* M A_N^* B * L B_{MN}^*$$

$$= A_{MN}^* M A_N^* B * L B_{MN}^*.$$ 

Note that, in general the converse of the above theorem is not true. It is illustrated by the following example.

**Example 3** Consider the tensors $A$ and $B$ in $\mathbb{C}^{2 \times 3 \times 2 \times 3}$ as follow:

| $A(:\ldots, 1, 1)$ | $A(:\ldots, 1, 2)$ | $A(:\ldots, 1, 3)$ |
|-----------------|-----------------|-----------------|
| 1 0 0           | 0 1 0           | 0 0 2           |
| 0 0 0           | 0 0 0           | 0 0 0           |

| $A(:\ldots, 2, 1)$ | $A(:\ldots, 2, 2)$ | $A(:\ldots, 2, 3)$ |
|-----------------|-----------------|-----------------|
| 0 0 0           | 0 0 0           | 0 0 0           |
| 1 0 0           | 0 1 0           | 0 0 2           |

and

| $B(:\ldots, 1, 1)$ | $B(:\ldots, 1, 2)$ | $B(:\ldots, 1, 3)$ |
|-----------------|-----------------|-----------------|
| 0 0 0           | 0 1 0           | 1 -1 0          |
| 1 1 0           | 0 0 0           | 0 0 0           |

| $B(:\ldots, 2, 1)$ | $B(:\ldots, 2, 2)$ | $B(:\ldots, 2, 3)$ |
|-----------------|-----------------|-----------------|
| 1 0 -1          | 0 0 0           | 0 0 0           |
| 0 0 0           | 0 0 0           | 0 0 1           |
Also, consider three Hermitian positive definite tensors \( \mathcal{M}, \mathcal{N} \) and \( \mathcal{L} \) in \( \mathbb{C}^{2 \times 3 \times 2 \times 3} \), such that

\[
\begin{array}{ccc}
\mathcal{M}(\vdots, 1, 1) & \mathcal{M}(\vdots, 1, 2) & \mathcal{M}(\vdots, 1, 3) \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{M}(\vdots, 2, 1) & \mathcal{M}(\vdots, 2, 2) & \mathcal{M}(\vdots, 2, 3) \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{N}(\vdots, 1, 1) & \mathcal{N}(\vdots, 1, 2) & \mathcal{N}(\vdots, 1, 3) \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{N}(\vdots, 2, 1) & \mathcal{N}(\vdots, 2, 2) & \mathcal{N}(\vdots, 2, 3) \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{L}(\vdots, 1, 1) & \mathcal{L}(\vdots, 1, 2) & \mathcal{L}(\vdots, 1, 3) \\
4 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{L}(\vdots, 2, 1) & \mathcal{L}(\vdots, 2, 2) & \mathcal{L}(\vdots, 2, 3) \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

Then, \( \mathcal{A}_2 \mathcal{B} \) is given by

\[
\begin{array}{ccc}
\mathcal{A}_2 \mathcal{B}(\vdots, 1, 1) & \mathcal{A}_2 \mathcal{B}(\vdots, 1, 2) & \mathcal{A}_2 \mathcal{B}(\vdots, 1, 3) \\
0 & 0 & 0 \\
1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{A}_2 \mathcal{B}(\vdots, 2, 1) & \mathcal{A}_2 \mathcal{B}(\vdots, 2, 2) & \mathcal{A}_2 \mathcal{B}(\vdots, 2, 3) \\
1 & 0 & -2 \\
1 & 0 & 0 \\
\end{array}
\]

Now, \( \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger \) and \( \mathcal{B}_{\mathcal{N}\mathcal{L}}^\dagger \) are as follows:

\[
\begin{array}{ccc}
\mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 1, 1) & \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 1, 2) & \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 1, 3) \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 2, 1) & \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 2, 2) & \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 2, 3) \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 1, 1) & \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 1, 2) & \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 1, 3) \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 2, 1) & \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 2, 2) & \mathcal{A}_{\mathcal{M}\mathcal{N}}^\dagger(\vdots, 2, 3) \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

and
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