Representations of the Lie algebra of vector fields on a sphere

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Abstract

For an affine algebraic variety \( X \) we study a category of modules that admit compatible actions of both the algebra of functions on \( X \) and the Lie algebra of vector fields on \( X \). In particular, for the case when \( X \) is the sphere \( S^2 \), we construct a set of simple modules that are finitely generated over \( A \). In addition, we prove that the monoidal category that these modules generate is equivalent to the category of finite-dimensional rational \( GL_2 \)-modules.

1 Introduction

In 1986 David Jordan proved simplicity of Lie algebras of polynomial vector fields on smooth irreducible affine algebraic varieties (\cite{8}, see also \cite{9} and \cite{4}). Yet, representation theory for this important class of Lie algebras remains largely undeveloped. The goal of the present paper is to investigate as a test case, representation theory of the Lie algebra of polynomial vector fields on a sphere \( S^2 \). Previously, representation theory was developed only for the classical Lie algebras of Cartan type – polynomial vector fields on an affine space and on a torus. Representations of the Lie algebras of vector fields on an affine space were studied by Rudakov in 1974 \cite{15}.

An important classification result on representations of the Lie algebra of vector fields on a circle was established by Mathieu \cite{12}. The case of an \( N \)-dimensional torus was studied by Larsson \cite{11}, Eswara Rao \cite{6}, \cite{7}, Billig \cite{1}, Mazorchuk-Zhao \cite{13} and Billig-Futorny \cite{2}. The culmination of this work was the proof in \cite{3} of Rao’s conjecture on classification of irreducible weight modules for the Lie algebra of vector fields on a torus with finite-dimensional weight spaces. According to this classification, every such module is either of the highest weight type, or a quotient of a tensor module.

When we move from the torus to general affine algebraic varieties, the first difficulty that arises is the absence of a Cartan subalgebra. It was shown in \cite{1} that even in the case of an affine elliptic curve, the Lie algebra of vector fields does not contain non-zero semisimple or nilpotent elements. This demonstrates that the theory of this class of simple Lie algebras is very different from the classical theory of simple finite-dimensional Lie algebras,
where roots and weights play a fundamental role. When studying representation theory of a simple Lie algebra, one has to impose some reasonable restrictions on the class of modules, since the description of simple modules in full generality is only known for $\mathfrak{sl}_2$.

In case of vector fields on a torus, a natural restriction is the existence of a weight decomposition with finite-dimensional weight spaces. A theorem proved in [3] states that a simple weight module $M$ for the Lie algebra of vector fields on a torus admits a cover $\hat{M} \to M$, where $\hat{M}$ is a module for both the Lie algebra of vector fields and the commutative algebra of functions on a torus. This suggests that a reasonable category of modules for the Lie algebra $\mathcal{V}$ of vector fields on an affine variety $X$ will be those that admit a compatible action of the algebra $A$ of polynomial functions on $X$. We refer to such modules as $AV$-modules. A finiteness condition will be a requirement that the module is finitely generated over $A$.

We begin by discussing the general theory of $AV$-modules, defining dual modules and tensor product in this category. Then the focus of the paper shifts to the study of the case of a sphere $S^2$. A new feature here compared to the torus and circle is that Lie algebra of vector fields on $S^2$, as well as its $AV$-modules, are not free as modules over $A$. We construct a class of tensor modules (geometrically these are modules of tensor fields on a sphere), and prove their simplicity in the category of $AV$-modules. We also show that the monoidal category generated by simple tensor modules on $S^2$ is semisimple and is equivalent to the category of finite-dimensional rational $GL_2$-modules.

The methods that we employ are a combination of Lie theory and commutative algebra. Hilbert’s Nullstellensatz is an essential ingredient in the proof of simplicity of tensor modules.

2 Generalities

Let $X \subset \mathbb{A}^n$ be an algebraic variety over an algebraically closed field $k$ of characteristic zero. Write $A_X$ for the algebra of polynomial functions on $X$, and let $\mathcal{V}_X = \text{Der}_k(A_X)$ be the Lie algebra of polynomial vector fields on $X$. When the variety $X$ is understood by the context we shall drop it as a subscript. Note that $\mathcal{V}$ is an $A$-module and that $A$ is a $\mathcal{V}$-module. For $a \in A$ and $\eta \in \mathcal{V}$ we shall write the latter action as $\eta(a)$.

Consider a vector space $M$ equipped with a module structure for both the associative commutative unital algebra $A$ and for the Lie algebra $\mathcal{V}$, such that the two structures are compatible in the following sense: For every $a \in A$, $\eta \in \mathcal{V}$, and $m \in M$ we have

$$\eta \cdot (a \cdot m) = \eta(a) \cdot m + a \cdot (\eta \cdot m).$$

This is equivalent to saying that $M$ is a module over the smash product algebra $A\#U(\mathcal{V})$, see [14] for details. For simplicity we shall write just $A\mathcal{V}$.
for $A\#U(\mathcal{V})$. A morphism of $A\mathcal{V}$-modules is a map that preserves both the $A$- and the $\mathcal{V}$-structures. The category of $A\mathcal{V}$-modules is clearly abelian since $A$-Mod and $\mathcal{V}$-Mod are.

For two $A\mathcal{V}$-modules $M$ and $N$ we may form the tensor product $M \otimes_A N$ which makes sense since $A$ is commutative. This is equipped with the natural $A$-module structure

$$a \cdot (m \otimes n) = (a \cdot m) \otimes n,$$

where the right side also equals $m \otimes (a \cdot n)$.

The Lie algebra of vector fields acts on the tensor product by

$$\eta(m \otimes n) = (\eta \cdot m) \otimes n + m \otimes (\eta \cdot n),$$

as usual. We verify that these two structures are compatible in the above sense. We have

$$\eta \cdot (a \cdot (m \otimes n)) = \eta \cdot ((a \cdot m) \otimes n) = (\eta \cdot (a \cdot m)) \otimes n + (a \cdot m) \otimes (\eta \cdot n)$$

$$= (\eta(a) \cdot m) \otimes n + (a \cdot (\eta \cdot m)) \otimes n + a \cdot (m \otimes (\eta \cdot n))$$

$$= \eta(a) \cdot (m \otimes n) + a \cdot (\eta \cdot (m \otimes n)).$$

This shows that $M \otimes_A N$ is an $A\mathcal{V}$-module, and that $A\mathcal{V}$-Mod is a monoidal category.

For any $A\mathcal{V}$-module $M$ we define

$$M^\circ := \text{Hom}_A(M, A).$$

The algebra $A$ acts naturally on $M^\circ$ by $(a \cdot \varphi)(m) = a \varphi(m)$, and we define an action of $\mathcal{V}$ on $M^\circ$ by

$$(\eta \cdot \varphi)(m) = -\varphi(\eta \cdot m) + \eta(\varphi(m))$$

for all $\eta \in \mathcal{V}$, $\varphi \in M^\circ$, and $m \in M$. These two actions are compatible in the sense defined above, so $M^\circ$ is indeed an $A\mathcal{V}$-module. The contravariant functor $M \mapsto M^\circ$ provides a duality on $A\mathcal{V}$-Mod.

**Chart Parameters**

Let $h \in A$ be a function and consider the corresponding chart for $X$ consisting of points where $h$ does not vanish: $N(h) = \{p \in X|h(p) \neq 0\}$.

**Definition 1.** We shall say that $t_1, \ldots, t_s \in A$ are **chart parameters** in the chart $N(h)$ provided that the following conditions are satisfied:

1. $t_1, \ldots, t_s$ are algebraically independent over $k$, so $k[t_1, \ldots, t_s] \subset A$.
2. Each element of $A$ is algebraic over $k[t_1, \ldots, t_s]$. 


3. For each index $p$, the derivation $\frac{\partial}{\partial p} \in \text{Der}(k[t_1, \ldots, t_s])$ extends to a derivation of the localized algebra $A(h)$.

Note that part 2 also implies that each element of $A(h)$ is algebraic over $k[t_1, \ldots, t_s]$ since algebraic elements are closed under taking inverses. Some further consequences of the definition are given below.

**Lemma 2.** Let $t_1, \ldots, t_s \in A$ be chart parameters in the chart $N(h)$. Then

1. The extension of the derivation $\frac{\partial}{\partial p} \in \text{Der}(k[t_1, \ldots, t_s])$ to the localized algebra $A(h)$ is unique.
2. $\text{Der}(A(h)) = \bigoplus_{p=1}^{s} A(h) \frac{\partial}{\partial p}.

Proof. The uniqueness in the first claim follows from part 2 of Definition [1]. Indeed let $f$ be a non-zero element of $A(h)$ and let $p_nT^n + \cdots + p_1T + p_0$ be its minimal polynomial with $p_i \in k[t_1, \ldots, t_s]$. Let $\mu \in \text{Der}(A(h))$. Then

$$\mu(f) = -\frac{\mu(p_n)f^n + \cdots + \mu(p_1)f + \mu(p_0)}{np_n f^{n-1} + \cdots + p_1},$$

hence every derivation of $A(h)$ is uniquely determined by its values on $k[t_1, \ldots, t_s]$.

The second claim follows from part 3 of Definition [1]. For any $d \in \text{Der}(A(h))$, let $d' = \sum_{i=1}^{s} d(t_i) \frac{\partial}{\partial t_i}$. Then $d$ and $d'$ are both derivations of $A(h)$ which are equal on $k[t_1, \ldots, t_s] \subset A(h)$, hence $d = d'$. Moreover, the expression of a derivation as an $A(h)$-combination of $\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_s} \}$ is unique: this is seen when applying such a combination to $t_1, \ldots, t_s$. \qed

Our prototypical example is the following. Let $X = S^2$ and let $h = z$. Then $x, y$ are chart parameters outside the equator $z = 0$: First $k[x, y] \subset A$. Next we have $q(z) = 0$ for $q(T) = T^2 + (x^2 + y^2 - 1)$ so $z$ is algebraic over $k[x, y]$. Since $x, y, z$ generate the ring $A$, every element of $A$ is algebraic over $k[x, y]$. For the third part we note that $0 = \frac{\partial}{\partial x}(x^2 + y^2 + z^2 - 1) = 2x + 2z \frac{\partial z}{\partial x}$, so for the extension of $\frac{\partial}{\partial x}$ to $A(z)$ we have $\frac{\partial z}{\partial x} = -\frac{z}{x}$ which uniquely determines the derivation $\frac{\partial}{\partial x}$ on $A(z)$.

**An Atlas for X**

Let $X$ be the zero locus of polynomials $g_1, \ldots, g_n \in k[x_1, \ldots, x_m]$ such that $A = k[x_1, \ldots, x_m]/\langle g_1, \ldots, g_n \rangle$, and define

$$J = \left( \frac{\partial g_i}{\partial x_j} \right) \in \text{Mat}_{n \times m}(A).$$

Let $F$ be the field of fractions of $A$ and define $r := \text{rank}_F J$. This means that there exists some $r \times r$-minor which is a nonzero element of $A$. By
the Nullstellensatz we have \( \text{rank}_F J = \max_{P \in X} \text{rank}_k J(P) \), and when \( X \) is smooth, \( \text{rank}_k J(P) \) is independent of the point \( P \) (see [16] Section 2.1.4).

We now consider \( r \times r \)-minors of \( J \). For \( \alpha \subset \{1, \ldots, n\} \) and \( \beta \subset \{1, \ldots, m\} \) we write \( J^{\alpha,\beta} \) for the corresponding \( r \times r \)-minor of \( J \). For \( h \in A \), let \( N(h) = \{ P \in X \mid h(P) \neq 0 \} \).

**Lemma 3.** Let \( X \) be smooth and let \( r = \text{rank}_F J \). Then the following set of charts forms an atlas for \( X \):

\[ \{ N(\det J^{\alpha,\beta}) \mid |\alpha| = |\beta| = r, \det(J^{\alpha,\beta}) \neq 0 \}. \]

**Proof.** Let \( P \in X \). Since \( X \) is smooth, \( \text{rank}_k J(P) = \text{rank}_F J = r \). Thus there exists a nonzero minor in \( J(P) \): \( \det(J^{\alpha,\beta}(P)) \neq 0 \) for some \( \alpha \) and \( \beta \) of order \( r \), so \( P \in N(\det J^{\alpha,\beta}) \). Thus the above atlas covers \( X \). \( \square \)

From here on we shall fix this atlas for the variety \( X \).

For the sphere \( X = S^2 \), we shall sometimes write \((x, y, z)\) for \((x_1, x_2, x_3)\). We have \( A = k[x, y, z]/(g) \) with \( g = x^2 + y^2 + z^2 - 1 \), so \( J = (2x, 2y, 2z) \). Here \( \text{rank}_F J = 1 \) and there are three nonzero minors: \( 2x, 2y, \) and \( 2z \). So in this case our charts are \( N(x), N(y), \) and \( N(z) \) - each obtained by removing a great circle from the sphere.

As described in [4], the matrix \( J \) also grants an explicit description of the vector fields on \( X \): for \( f_i \in A \), the combination \( \sum_{i=1}^m f_i \frac{\partial}{\partial x_i} \) is a vector field on \( X \) if and only if the vector \((f_1, \ldots, f_m)\) belongs to the kernel of \( J \).

**Lemma 4.** Let \( r = \text{rank}_F J \) and let \( J^{\alpha,\beta} \) be a minor of size \( r \) with \( h = \det J^{\alpha,\beta} \neq 0 \). Then \( \{ x_i \mid i \not\in \beta \} \) are chart parameters in the chart \( N(h) \).

**Proof.** Consider \( \text{Der}_k F \). It is easy to see that for any \( s \in F \) and \( \eta \in \text{Der}(A) \) we have \( s \eta \in \text{Der}_k F \) and conversely, for any \( \mu \in \text{Der}_k F \) there exists \( q \in A \) such that \( q \mu \in \text{Der}(A) \). Then derivations of \( F \) can be written as \( \sum_{i=1}^m f_i \frac{\partial}{\partial x_i} \) where \((f_1, \ldots, f_m)\) are solutions over \( F \) of a system of linear homogeneous equations with matrix \( J \). Since \( \text{rank}_F J = r \), we can keep only the rows \( \alpha \) and choose variables \( \{ f_i \mid i \in \beta \} \) as leading, and \( \{ f_i \mid i \not\in \beta \} \) as free. Writing down the fundamental solutions, we get derivations \( \tau_j = \frac{\partial}{\partial x_j} + \sum_{i \in \beta} f_{i,j} \frac{\partial}{\partial x_i} \in \text{Der}_k F \) for each \( j \not\in \beta \). Note that only \( h \) may appear in the denominators of \( f_{i,j} \), hence \( h \tau_j \in \text{Der}(A) \) for all \( j \not\in \beta \). This implies that \( \{ x_i \mid i \not\in \beta \} \) are algebraically independent. Indeed if \( p \) is a polynomial in \( \{ x_i \mid i \not\in \beta \} \) which vanishes in \( F \), we can apply a sequence of derivations \( \tau_j \) which brings \( p \) to 1 and obtain a contradiction \( 1 = 0 \). If all free variables have zero values, the solution of a homogeneous system is trivial. This implies that every derivation of \( F \) which is zero on \( k[x_i \mid i \not\in \beta] \), is zero on \( F \). By [10] Chapter VIII, Prop. 5.2, \( F \) is algebraic over \( k(x_i \mid i \not\in \beta) \). \( \square \)

**Remark 5.** In the above lemma, we may assume without loss of generality that \( \{ i \mid i \not\in \beta \} = \{1, \ldots, s\} \) and \( \{ i \mid i \in \beta \} = \{s + 1, \ldots, n\} \).
If we treat the chart parameters \( t_i = x_1, \ldots, t_s = x_s \) as independent variables and \( x_{s+1}, \ldots, x_n \) as dependent we can write a derivation

\[
\sum_{i \not\in \beta} f_i \frac{\partial}{\partial x_i} + \sum_{j \in \beta} f_j \frac{\partial}{\partial x_j}
\]
simply as \( \sum_{i=1}^s f_i \frac{\partial}{\partial t_i} \), with understanding that for \( j \in \beta \) we have

\[
\sum_{i=1}^s f_i \frac{\partial x_j}{\partial t_i} = \sum_{i=1}^s f_i \tau_i(x_j).
\]

**Embedding of Vector Fields**

The embedding \( A \subset A(h) \) gives a corresponding embedding of polynomial vector fields:

\[
\text{Vect}(X) \simeq \text{Der}(A) \subset \text{Der}(A(h)) = \bigoplus_{p=1}^n A(h) \frac{\partial}{\partial t_p}.
\]

In other words, a polynomial vector field on \( X \) can be written as \( \sum_{i=1}^n f_i \frac{\partial}{\partial t_i} \in \text{Der}(A(h)) \) for some unique \( f_i \in A(h) \).

**Valuation**

For each point \( P \in N(h) \), define \( \nu_P : A(h) \setminus \{0\} \to \mathbb{N} \) by

\[
\nu_P(f) = \min \left\{ \sum_{i=s}^n \alpha_i \left| \prod_{i=1}^s \left( \frac{\partial}{\partial t_i} \right)^{\alpha_i} f \right| (P) \neq 0 \right\}.
\]

Then \( \nu_P \) is well-defined by the following lemma.

**Lemma 6.** We have \( \nu_P(f) < \infty \) for each nonzero \( f \in A(h) \).

**Proof.** First of all, \( \nu_P \) is finite on \( k[t_1, \ldots, t_s] \subset A(h) \) as it is bounded by the degree function. Now any \( f \in A(h) \) is by definition algebraic over \( k[t_1, \ldots, t_s] \) so we can consider its minimal polynomial \( m(X) = \sum_{i=0}^N a_i X^i \), where \( a_i \in k[t_1, \ldots, t_s] \) and \( m(f) = 0 \) and \( a_0 \neq 0 \). We claim that \( \nu_P(f) \leq \nu_P(a_0) \). To prove this, suppose for contradiction that \( \prod_{i=1}^s \left( \frac{\partial}{\partial t_i} \right)^{\alpha_i} f \) \( (P) = 0 \) for all \( \alpha_1 + \cdots + \alpha_s \leq \nu_P(a_0) \). Pick \( \alpha_i \)'s such that \( \prod_{i=1}^s \left( \frac{\partial}{\partial t_i} \right)^{\alpha_i} a_0 \) \( (P) \neq 0 \). Then for any derivation \( d = \frac{\partial}{\partial t_p} \) we have

\[
d(a_0) = -\left( \sum_{i=1}^N d(a_i) f^i + a_i f^{i-1} d(f) \right).
\]

But by assumption, both \( f(P) = 0 \) and \( d(f)(P) = 0 \) so the right hand side is zero at \( P \), hence also \( d(a_0)(P) = 0 \). Iterating this \( \nu_P(a_0) \) times we get \( \prod_{i=1}^s \left( \frac{\partial}{\partial t_i} \right)^{\alpha_i} a_0 \) \( (P) = 0 \), a contradiction. \( \Box \)
Note that the above proof uses our assumption that \( \text{char } k = 0 \).

## 3 A class of \( \mathcal{A} \mathcal{V} \)-modules

Let \( s := \dim X \). Let \( \{t_1, \ldots, t_s\} \) be chart parameters in a chart \( N(h) \) and let \( A_{(h)} \) be the localization of the algebra \( A \) at \( h \). For any \( \mathfrak{gl}_s \)-module \( U \) we consider the space \( A_{(h)} \otimes_k U \). The algebra \( A \) acts on this space by multiplication on the left factor.

The proof of the following lemma is straightforward and we leave it to the reader.

**Lemma 7.** Let \( U \) be a finite-dimensional \( \mathfrak{gl}_s \)-module. Consider the vector fields \( \mathcal{V} \) as embedded in \( \text{Der}(A_{(h)}) = \bigoplus_{i=1}^s A_{(h)} \frac{\partial}{\partial t_i} \). Define an action of \( \mathcal{V} \) on \( A_{(h)} \otimes U \) by

\[
\left( \sum_{i=1}^s f_i \frac{\partial}{\partial t_i} \right) \cdot (g \otimes u) := \sum_{i=1}^s f_i \frac{\partial g}{\partial t_i} \otimes u + \sum_{p=1}^s \sum_{i=1}^s g \frac{\partial f_i}{\partial t_p} \otimes (E_{p,i} \cdot u). 
\]

Here \( E_{p,i} \) is a standard basis element of \( \mathfrak{gl}_s \), \( g \in A_{(h)} \), and \( u \in U \). This equips \( A_{(h)} \otimes U \) with the structure of an \( \mathcal{A} \mathcal{V} \)-module.

The algebra \( A_{(h)} \) has a natural doubly infinite filtration with respect to powers of \( h \):

\[
\cdots \subset h^{k+1}A \subset h^kA \subset h^{k-1}A \subset \cdots .
\]

For each \( a \in A_{(h)} \) we define its **degree** by

\[
\deg a := \max\{k \in \mathbb{Z} \mid a \in h^kA\}.
\]

We extend our notion of degree to elements of \( A_{(h)} \otimes U \) in the natural way: for nonzero \( m \in A_{(h)} \otimes U \) we define

\[
\deg m := \max\{k \in \mathbb{Z} \mid m \in h^kA \otimes U\}.
\]

If \( M \subset A_{(h)} \otimes U \) is a nonzero submodule, we define

\[
\deg(M) := \inf\{\deg m \mid m \in M, \ m \neq 0\}.
\]

Let \( M \subset A_{(h)} \otimes U \) be a submodule. We shall call \( M \) **bounded** if \( \deg(M) \) is finite. It is easy to see that if \( M \) is finitely generated over \( A \) then it is bounded. Conversely, every bounded submodule in \( A_{(h)} \otimes U \) is finitely generated over \( A \). Indeed, as an \( A \)-module \( h^kA \otimes U \) is isomorphic to \( A \otimes U \) and since \( A \) is noetherian, every submodule in a finitely generated \( A \)-module is finitely generated.

On the other hand, we shall call \( M \) **dense** if \( M \supset h^kA \otimes U \) for some \( k \). Note that \( M \) is both dense and bounded when there exist two integers \( K \geq k \) such that

\[
h^KA \otimes U \subset M \subset h^kA \otimes U.
\]
4 The Sphere

From here on we shall focus on the case when $X$ is the sphere $S^2$. Some results still hold in a more general setting.

Let $X = S^2 \subset \mathbb{A}^3$. With notation as above we have $A = \mathbb{k}[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 - 1)$. However, we shall sometimes write $x, y, z$ for $x_1, x_2, x_3$. Let $\Delta_{ij} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}$. Then it is easy to check that $V$ is generated by $\Delta_{12}, \Delta_{23}, \text{and } \Delta_{31}$ as an $A$-module, and that these generators satisfy

$$[\Delta_{12}, \Delta_{23}] = \Delta_{31}, \quad [\Delta_{23}, \Delta_{31}] = \Delta_{12}, \quad [\Delta_{31}, \Delta_{12}] = \Delta_{23}.$$ 

However, $V$ is not a free $A$-module since we have the relation $x_1 \Delta_{23} + x_2 \Delta_{31} + x_3 \Delta_{12} = 0$.

In the chart $N(z)$ with chart parameters $\{x, y\}$ these generating vector fields are expressed as

$$\Delta_{12} = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad \Delta_{23} = z \frac{\partial}{\partial y}, \quad \Delta_{31} = -z \frac{\partial}{\partial x}.$$ 

5 Explicit construction of modules

The Lie algebra $\mathfrak{sl}_2$ acts naturally on $\mathbb{k}[X, Y]$ by

$$E_{1,2} \cdot f = X \frac{\partial f}{\partial Y}, \quad E_{2,1} \cdot f = Y \frac{\partial f}{\partial X}, \quad (E_{1,1} - E_{2,2}) \cdot f = X \frac{\partial f}{\partial X} - Y \frac{\partial f}{\partial Y}.$$ 

For any $\alpha \in \mathbb{k}$ we may extend this to a $\mathfrak{gl}_2$-module $\mathbb{k}[X, Y]$ by requiring $(E_{1,1} + E_{2,2}) \cdot f = \alpha f$.

Homogeneous components are preserved by this action, and $\mathbb{k}_\alpha[X, Y] = \bigoplus_{m \geq 0} U^\alpha_m$ where $U^\alpha_m$ is the homogeneous component of degree $m$. Explicitly, for $m \in \mathbb{N}$ and for $0 \leq i \leq m$, we define $v^m_i := \binom{m}{i} X^i Y^{m-i}$ (we shall drop the upper $m$ when it is understood by the context). Then $U^\alpha_m = \text{span}\{v^m_0, \ldots, v^m_m\}$ is the homogeneous component of degree $m$, and the action on these basis elements is given by

$$E_{1,1} \cdot v_i = \frac{1}{2}(\alpha + m - 2i)v_i, \quad E_{1,2} \cdot v_i = (m - i + 1)v_{i-1},$$

$$E_{2,1} \cdot v_i = (i + 1)v_{i+1}, \quad E_{2,2} \cdot v_i = \frac{1}{2}(\alpha - m + 2i)v_i.$$ 

In particular this implies that

$$(E_{1,1} - E_{2,2}) \cdot v_i = (m - 2i)v_i, \quad (E_{1,1} + E_{2,2}) \cdot v_i = \alpha v_i.$$ 

Here $v_{i-1}$ and $v_{m+1}$ are 0 by definition. When restricted to $\mathfrak{sl}_2$, $U^\alpha_m$ is the unique simple module of dimension $m + 1$. In particular, any finite-dimensional simple $\mathfrak{gl}_2$-module is isomorphic to $U^\alpha_m$ for a unique $\alpha$ and $m$. 

8
Modules of rank 1

We first consider the case \( m = 0 \). Here \( U_0^\alpha \) is one dimensional and the identity acts by \( \alpha \). We consider the chart \( N(z) \) on \( S^2 \) with chart parameters \( \{x, y\} \).

**Proposition 8.** The module \( A(z) \otimes U_0^\alpha \) contains a bounded AV-submodule if and only if \( \alpha \in 2\mathbb{Z} \).

**Proof.** Let \( M \) be a bounded submodule of \( A(z) \otimes U_0^\alpha \). Let \( w \) be a non-zero element of \( M \) of lowest possible degree \( k \). Then \( w \) can be expressed as

\[
w = \sum_{i \geq k} z^i a_i \otimes v_0 \text{ where } a_k \neq 0.
\]

We compute

\[
\Delta_{2,3}(z^k a_k \otimes v_0) = z \frac{\partial}{\partial y}(z^k a_k) \otimes v_0 + z^k a_k \frac{\partial z}{\partial y} \otimes E_{2,2} v_0
\]

\[
= z \left( -z^k \frac{\partial a_k}{\partial y} - ka_k \frac{z^{k-1} y}{z} \right) \otimes v_0 - z^k a_k \frac{y}{z} \otimes \frac{\alpha}{2} v_0,
\]

which modulo the space \( z^k A \otimes U_0^\alpha \) equals

\[
-(k + \frac{\alpha}{2}) a_k y z^{k-1} \otimes v_0.
\]

So by the minimality of \( k \) we must have \( \alpha = -2k \). On the other hand it is easy to check that for \( \alpha = 2k \) the space \( A^\alpha = z^k \otimes U_0^\alpha \) is an AV-submodule in \( A(z) \otimes U_0^\alpha \). \qed

Higher rank

**Module of 1-forms on \( X \)**

In this section we consider for a moment an arbitrary \( s \)-dimensional variety \( X \). The space of 1-forms \( \Omega \) is an AV-module where \( A \) acts by left multiplication and vector fields act as follows:

\[
\left( \sum_{i=1}^s f_i \frac{\partial}{\partial t_i} \right) \cdot \left( \sum_{j=1}^s g_j dt_j \right) = \sum_{i,j=1}^s f_i \frac{\partial g_j}{\partial t_i} dt_j + g_j d \left( f_i \frac{\partial}{\partial t_i} t_j \right)
\]

\[
= \sum_{i,j=1}^s f_i \frac{\partial g_j}{\partial t_i} dt_j + \delta_{ij} g_j d(f_i) = f_i \frac{\partial g_j}{\partial t_i} dt_j + \delta_{ij} g_j \sum_{p=1}^n \frac{\partial f_i}{\partial t_p} dt_p.
\]

By identifying \( e_i \leftrightarrow dt_i \) we see that \( \Omega \subset A_{(h)} \otimes V \), where the action on \( V \) now is \( E_{p,i} e_j = \delta_{i,j} e_p \) which shows that \( V \) is the natural \( gl_s \)-module.

For \( X = S^2 \) this means that \( \Omega \subset A(z) \otimes U_1^1 \). In this case, the submodule \( \Omega \) is generated by \( dz = -z^{-1} x dx - z^{-1} y dy \).
Module of vector fields on $X$

Similarly, the Lie algebra $\mathcal{V}$ of vector fields themselves forms an $A\mathcal{V}$-module in a natural way: $A$ acts by left multiplication, and $\mathcal{V}$ acts adjointly. We may rewrite this $\mathcal{V}$-action in the following way:

\[
\left( \sum_{i=1}^{s} f_i \frac{\partial}{\partial t_i} \right) \cdot \left( \sum_{j=1}^{s} g_j \frac{\partial}{\partial t_j} \right) = \sum_{i,j=1}^{s} f_i g_j \left( \frac{\partial}{\partial t_i} \right) \cdot \left( \frac{\partial}{\partial t_j} \right) - g_j \frac{\partial f_i}{\partial t_i} \cdot \frac{\partial}{\partial t_j}.
\]

Comparing with the definition (1), we see that $\mathcal{V}$ is isomorphic to a submodule of $A(h) \otimes U$ with $v_i \leftrightarrow \frac{\partial}{\partial t_i}$.

Proposition 9. Let $M$ be an $A\mathcal{V}_{S^2}$-submodule of $A(z) \otimes U$, where $U$ is a finite-dimensional $\mathfrak{gl}_2$-module. Then for $\sum_k g_k \otimes u_k \in M$ we also have $\sum_k (z g_k \otimes E_{i,j} \cdot u_k) \in M$ for all $1 \leq i, j \leq 2$. In other words, $M$ is closed under the operators $z \otimes E_{i,j}$.

Proof. It suffices to prove the statement for a single term $g \otimes u$. For each vector field $\mu \in \mathcal{V}$ and for each function $f \in A$ we have

\[
(f \mu) \cdot (g \otimes u) - f(\mu \cdot (g \otimes u)) \in M.
\]

Taking $\mu = \Delta_{2,3}$ we obtain the following element in $M$:

\[
f_z \frac{\partial g}{\partial y} \otimes u + f \frac{\partial z}{\partial x} \otimes E_{1,2}u + zg \frac{\partial f}{\partial x} \otimes E_{1,2}u + fg \frac{\partial z}{\partial y} \otimes E_{2,2}u + zg \frac{\partial f}{\partial y} \otimes E_{2,2}u - zf \frac{\partial g}{\partial y} \otimes u - fg \frac{\partial z}{\partial x} \otimes E_{1,2}u = zg \frac{\partial f}{\partial x} \otimes E_{1,2}u + zg \frac{\partial f}{\partial y} \otimes E_{2,2}u.
\]

Taking $f = x$ we obtain $zg \otimes E_{1,2}u \in M$. If we instead take $f = y$ we obtain $zg \otimes E_{2,2}u$. Analogously, by taking $\mu = \Delta_{3,1}$ and $f = x, y$ we obtain $zg \otimes E_{1,1}u \in M$ and $zg \otimes E_{2,1}u \in M$. 

In what follows we shall use the following version of Hilbert’s Nullstellensatz, see [15, Section 1.2.2].
Lemma 10. Let $I \triangleleft A$ be an ideal. Suppose that $g \in A$ satisfies $g(P) = 0$ at all points $P \in X$ for which $f(P) = 0$ for all $f \in I$. Then $g^k \in I$.

Proposition 11. Let $U$ be a finite-dimensional irreducible $\mathfrak{gl}_2$-module. Then every nonzero $A\mathcal{V}_{S^2}$-submodule of $A(z) \otimes U$ is dense.

Proof. Since $U$ is simple and finite-dimensional it has form $U_n^m$ as above. Let $M \subset A(z) \otimes U$ be a submodule and define

$$I = \{ f \in A | f(A \otimes U) \subset M \}.$$

Then $I$ is an ideal of $A$. To show that $M$ is dense we need to show that $z^N \in I$ for some $N$.

Let $v \in M$ and express this element in the form $v = \sum_{i=0}^m f_i \otimes v_i$, with $f_i \in A(z)$. In fact we may assume that $f_i \in A$ (otherwise just multiply by a power of $z$). By Lemma 9, $M$ is closed under the operator $z \otimes E_{1,2}$, so we obtain $z^k f_0 \otimes v_0 \in M$ for some $k$ and for some nonzero $f_0$. Acting by $z \otimes E_{2,1}$ repeatedly on this element we get $z^{k+i} f_0 \otimes v_i$, so in particular we have $z^N f_0 \otimes U \subset M$, which shows that $z^N f_0 \in I$ so $I$ is non-zero.

We now aim to apply Hilbert’s Nullstellensatz to the function $g = z$. Fix $P \in S^2$ with nonzero $z$-coordinate. We need to show that there exists $f \in I$ with $f(P) \neq 0$. We had already found $z^N f_0 \in I$ so if $f_0(P) \neq 0$ we are done. Otherwise, consider the element $z \frac{\partial}{\partial z}(z^{N+1} f_0 \otimes v_0) \in M$. This expands as

$$z^{N+2} \frac{\partial f_0}{\partial x} \otimes v_0 - z^N f_0 ((N+1)x \otimes v_0 + x \otimes E_{1,1} v_0 - y \otimes E_{2,1} v_0),$$

and since the second term lies in $z^N f_0(A \otimes U) \subset M$, we also get $z^{N+2} \frac{\partial f_0}{\partial x} \otimes v_0 \in M$. This shows that $z^{N'} \frac{\partial f_0}{x} \in I$ for some $N'$, and by symmetry we also get $z^{N'} \frac{\partial f_0}{y} \in I$. Since $\nu_P(f_0)$ is finite there is some product $d$ of derivations with $d(f_0)(P) \neq 0$. So acting repeatedly with elements as above we eventually obtain $z^K d(f_0) \in M$ and $z^K d(f_0)$ is nonzero at $P$. By the Nullstellensatetz this means that $z^K \in I$ for some $K$, which in turn means that $M$ is dense. \hfill \Box

Corollary 12. When $U$ is an irreducible $\mathfrak{gl}_s$-module there exists at most one simple $A\mathcal{V}_{S^2}$-submodule of $A(z) \otimes U$.

Proof. Let $M$ and $M'$ be simple submodules in $A(z) \otimes U$. By Proposition 11 both modules are dense, so they both contain $z^N A \otimes U$ for large enough $N$. Thus $M \cap M'$ is a nonzero submodule of both $M$ and $M'$ so by simplicity we must have $M = M'$. \hfill \Box

6 Tensor modules

Consider two charts $N(h)$ and $N(\tilde{h})$ in our atlas for $X$. Let $t_1, \ldots, t_s$ be chart parameters for $N(h)$ and let $\tilde{t}_1, \ldots, \tilde{t}_s$ be chart parameters for $N(\tilde{h})$. **11**
Let $V$ be the natural $\mathfrak{gl}_s$-module with basis $\{e_1, \ldots, e_s\}$. Define an $A_{(\hat{h})}$-linear map $C : A_{(h, \hat{h})} \otimes V \to A_{(h, \hat{h})} \otimes V$ by

$$Ce_i = \sum_{j=1}^{s} \frac{\partial t_i}{\partial \hat{t}_j} e_j.$$ 

Note that $C$ is invertible.

From now on we shall understand all $\mathrm{GL}_s$-modules (resp. $\mathfrak{gl}_s$-modules) as $\mathrm{GL}(V)$-modules (resp. $\mathfrak{gl}(V)$-modules).

For a finite-dimensional rational $\mathrm{GL}_s$-module $U$, write $\rho : \mathrm{GL}_s \to \mathrm{GL}(U, k)$ for the corresponding representation. Then we may consider $\rho(C)$ as an element of $\mathrm{GL}(U, A_{(h)})$. We can also consider it as a map $A_{(h, \hat{h})} \otimes U \to A_{(h, \hat{h})} \otimes U$.

Denote by $\mathcal{T}$ the full subcategory of $A V \text{-Mod}$ consisting of those objects $M$ that satisfy

- $M$ is finitely generated as an $A$-module,
- For each chart $N(h)$ in our fixed atlas there exists an injective $A V$-module homomorphisms $\varphi_h : M \to A_{(h)} \otimes U$,
- The following diagram commutes for each pair of charts $(N(h), N(\hat{h}))$:

$$\begin{array}{ccc}
M & \xrightarrow{\varphi_h} & A_{(h)} \otimes U \\
\downarrow \varphi_h & & \downarrow \rho(C) \\
A_{(h)} \otimes U & \xrightarrow{\varphi_h} & A_{(h, \hat{h})} \otimes U
\end{array}$$

The objects of the category $\mathcal{T}$ will be called tensor modules.

We point out that the modules $\Omega$ and $V$ are in fact tensor modules. For $\Omega$, the natural $\mathfrak{gl}_s$-module appearing in its construction is identified with $V$ via $e_i \leftrightarrow dt_i$. Then the transformation $C$ corresponds to the usual change of variables formula for the differentials. Clearly all compatibility conditions are satisfied and $\Omega$ is a tensor module.

The $\mathfrak{gl}_s$-module corresponding to $V$ is the dual of the natural module $(V^*, \rho^*)$. It can easily be checked that $\rho^*(C)$ corresponds to the change of variables formula for partial derivatives.

The category of tensor modules is also closed under taking tensor products and duals, in particular we have $\Omega \simeq V^\otimes$.

Finally, for the sphere the rank 1 modules $A^\alpha$ (for $\alpha \in 2\mathbb{Z}$) are also tensor modules with $\rho(C) = \det(C)^\frac{\alpha}{2}$. 

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**Theorem 13.** Let $M \in \mathcal{S}$ be a tensor module on $S^2$ corresponding to a simple rational $GL_2$-module $U$. Then $M$ is a simple $AV$-module.

**Proof.** Let $M'$ be a nonzero submodule of $M$ and define

$$I = \{ f \in A | fM \subset M' \}.$$  

Then $I$ is an ideal and it does not depend on the chart we use. Since $M$ is bounded, Proposition 11 implies that $z^N \in I$. By symmetry we also have $x^N \in I$ and $y^N \in I$ for some large enough $N$. But then the set of common zeros $V(I) \subset X$ is empty, and Hilbert’s weak Nullstellensatz gives $1 \in I$. In view of the definition of $I$ this says that $M = M'$.

**Theorem 14.** When $\frac{m-\alpha}{2} \in \mathbb{Z}$ the vector

$$w_m := \sum_{i=0}^{m} z^{-\frac{\alpha+m}{2}} x^{m-i} y^i \otimes v_i$$

generates a bounded $AV$-submodule inside $A(z) \otimes U_m^\alpha$. On the other hand, when $\frac{m-\alpha}{2} \notin \mathbb{Z}$, the module $A(z) \otimes U_m^\alpha$ contains no bounded submodules.

**Proof.** We first prove the second part. Let $M$ be a nonzero bounded submodule of $A(z) \otimes U_m^\alpha$. Since $M$ is dense, it contains a vector of form $z^N \otimes v_0$ for large enough $N$. We may pick such a vector with minimal $N$.

We now compute

$$\Delta_{2,3}(z^N \otimes v_0) = -z^{N-1} y \left( N + \frac{1}{2}(\alpha - m) \right) \otimes v_0.$$

By the minimality of $N$ we must have $N + \frac{1}{2}(\alpha - m) = 0$ which since $N$ is an integer means that $\frac{m-\alpha}{2} \notin \mathbb{Z}$.

To prove the first statement, assume that $\frac{m-\alpha}{2} \in \mathbb{Z}$. First note that $w_m$ gives a correct generator for the cases discussed above: vector fields $V$, 1-forms $\Omega$, and all rank 1-modules $A^\alpha$. We now proceed by induction on $m$.

Dropping the tensor signs we may write $w_m$ as

$$w_m = z^{-\frac{\alpha+m}{2}} \sum_{i=0}^{m} \binom{m}{i} x^{m-i} y^i X^{-i} = z^{-\frac{\alpha+m}{2}} (xY + yX)^m.$$  

Consider three $\mathfrak{sl}_2$-submodules $U_m$, $U_n$, and $U_{m+n}$ of $k[X,Y]$. The map $\varphi : U_m \otimes U_n \to U_{m+n}$ given by multiplication in $k[X,Y]$, $(f \otimes g \mapsto fg)$, is a surjective homomorphism of $\mathfrak{sl}_2$-modules. Introducing an action of the identity gives a corresponding surjective homomorphism of $\mathfrak{gl}_2$-modules: $U_m^\alpha \otimes U_n^\beta \to U_{m+n}^{\alpha+\beta}$. This morphism can now be further extended to a surjective homomorphism of $AV$-modules:

$$\varphi : (A(z) \otimes U_m^\alpha) \otimes (A(z) \otimes U_n^\beta) \to A(z) \otimes U_{m+n}^{\alpha+\beta}. $$
By the inductive assumption, \( w_m \) and \( w_n \) generate bounded submodules in \( A(z) \otimes U^\alpha_m \) and \( A(z) \otimes U^\beta_n \) respectively, so \( w_m \otimes w_n \) generates a bounded submodule in \( (A(z) \otimes U^\alpha_m) \otimes (A(z) \otimes U^\beta_n) \).

Applying our multiplication map we conclude that \( \varphi(w_m \otimes w_n) \) generates a bounded submodule in \( A(z) \otimes U^{\alpha+\beta}_{m+n} \). But

\[
\varphi(w_m \otimes w_n) = \varphi \left( z^{-\frac{\alpha+m}{2}} (xY + yX)^m \otimes z^{-\frac{\beta+n}{2}} (xY + yX)^n \right) = z^{-\frac{(\alpha+\beta)+m+n}{2}} (xY + yX)^{m+n} = w_{m+n}.
\]

This concludes the proof. \( \square \)

7 Tensor product decomposition

Note in \( A(h) \otimes_A A(h) \) we have

\[
h^{-1} \otimes 1 = h^{-1} \otimes hh^{-1} = h^{-1} h \otimes h^{-1} = 1 \otimes h^{-1}.
\]

This shows that \( A(h) \otimes_A A(h) \simeq A(h) \). Therefore, from the classical decomposition of \( \mathfrak{gl}_2 \)-modules it follows that

\[
(A(h) \otimes U^\alpha_m) \otimes_A (A(h) \otimes U^\beta_n) = A(h) \otimes (U^\alpha_m \otimes U^\beta_n) = \bigoplus_{i=0}^n A(h) \otimes U^{\alpha+\beta}_{m+n-2i}.
\]

We shall show that our AV-modules which appear as submodules in \( A(h) \otimes U \) respect this decomposition.

**Theorem 15.** *In the category of tensor modules on the sphere, tensor products of simple modules decompose as a direct sums of simple tensor modules.*

**Proof.** Let \( M \) and \( N \) be simple tensor modules embedded as \( M \subset A(z) \otimes U^\alpha_m \) and \( N \subset A(z) \otimes U^\beta_n \) with \( m \geq n \). Then

\[
M \otimes_A N \subset (A(z) \otimes U^\alpha_m) \otimes_A (A(z) \otimes U^\beta_n) = A(z) \otimes (U^\alpha_m \otimes U^\beta_n) = \bigoplus_{i=0}^n A(z) \otimes U^{\alpha+\beta}_{m+n-2i}.
\]

Write \( \pi_k \) for the projection onto the \( k \)-th direct summand:

\[
\pi_k : A(z) \otimes (U^\alpha_m \otimes U^\beta_n) \to A(z) \otimes U^{\alpha+\beta}_{m+n-2k}.
\]

Then \( \text{id}_{A(z) \otimes (U^\alpha_m \otimes U^\beta_n)} = \bigoplus_{k=0}^n \pi_k \). We shall show that this decomposition still holds when restricted to the subspace \( M \otimes_A N \). For this it suffices to check that \( \pi_k(M \otimes N) \subset M \otimes N \). Let \( v \in M \otimes_A N \). By the density of \( M \) and \( N \), we see that \( M \otimes N \) is dense in \( \bigoplus_{i=0}^n A(z) \otimes U^{\alpha+\beta}_{m+n-2i} \) as well. This implies that \( \pi_k(M \otimes N) \) is nonzero for each \( k \), so \( \pi_k(M \otimes N) \) is a nonzero
bounded submodule of $A(z) \otimes U_{m+n-2k}^\alpha$, and by previous results $\pi_k(M \otimes N)$ is dense and simple.

Now for arbitrary $j \geq 0$ we have $z^j v = \sum_k \pi_k(z^j v)$, and for $j$ large enough we get $\pi_k(z^j v) \in M \otimes N$ by the density of $M \otimes N$.

Next, by the simplicity of $\pi_k(M \otimes N)$, we have

$$\pi_k(v) \in A^\# U(V) \cdot \pi_k(z^j v) \subset M \otimes N.$$ 

Thus we have shown that $id_{M \otimes N} = \bigoplus \pi_k|_{M \otimes N}$ and hence

$$M \otimes N = \bigoplus \pi_k(M \otimes N).$$

\[ \square \]

**Corollary 16.** The category of finite-dimensional rational $GL_2$-modules is equivalent to a full subcategory of the category $\mathfrak{T}$ of tensor modules on the sphere. Moreover, this subcategory is generated by $A^{-2}$ and $\Omega$ as a monoidal abelian category.

**Proof.** The equivalence is provided by the functor $F : U \mapsto \text{soc}(A(z) \otimes U)$. This is a bijection between simple finite-dimensional rational $GL_2$-modules and simple tensor modules.

For the second statement we note that the tensor products of $A\mathcal{V}$-modules of rank 1 is given by

$$A^\alpha \otimes A^\beta \simeq A^{\alpha+\beta}.$$ 

In particular, we get $A^{2k} = (A^2)^\otimes k$ and $A^{-2k} = (A^{-2})^\otimes k$ for any $k \in \mathbb{N}$. We also have $\mathcal{V} \otimes_A A^2 \simeq \Omega$ and $\Omega \otimes_A A^{-2} \simeq \mathcal{V}$.

Let $M$ be a bounded submodule of $A(z) \otimes U_m^\alpha$. Then $\frac{m-a}{2}$ is an integer by Lemma [1]. Now let $V = U^1$ be the natural $\mathfrak{gl}_2$-module. It is well known that $U_m \subset V^\otimes m$ as $\mathfrak{sl}_2$-modules, so we get $U_m^\alpha \subset A^{a-m} \otimes V^\otimes n$. This in turn implies that $M \subset A^{a-m} \otimes_A \Omega^\otimes m$.

Thus the category $\mathfrak{T}$ is generated by $A^2$, $A^{-2}$, and $\Omega$ as a monoidal abelian category. However, $A^2$ is a direct summand of $\Omega \otimes \Omega$ so it may be dropped as a generator. \[ \square \]

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