Microscopic Entropy of Extremal Kerr Black Holes

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Abstract. Kerr black holes are strong candidates for the astrophysical black holes. The understanding of the microscopic structure of this black hole is quite interesting to be investigated. The holographic principle is one way to study the microscopic part of Kerr black holes. There have been a conjecture that there is a duality between near-horizon geometry of extremal Kerr black hole and the conformal field theory (CFT) on its boundary of the horizon, called by Kerr/CFT correspondence. Here, we review this correspondence where in the background geometry, boundary conditions are found for which the algebra of surface charges enhances the $U(1)$ to one copy of the Virasoro algebra with central charge $c_V = 12J$. After finding the conformal temperature, which is $T_V = 1/2\pi$, the entropy is obtained from the Cardy formula. Finally, entropy from the Cardy formula agrees with the Bekenstein-Hawking entropy of the extremal Kerr black hole and also with the corrected one. Kerr/CFT correspondence is also used to find the entropy for another black hole that we show at the end.

1. Introduction

The idea of the Kerr/CFT correspondence was led by the Maldacena's proposal\cite{1} about the AdS/CFT correspondence or AdS/CFT duality that relates the quantum theory of gravity in $D$ dimension to the quantum field theory (QFT) in $(D - 1)$ dimension. The more detail proposal about how QFT is mapped to the quantum theory of gravity in supporting the Maldacena's proposal is provided in\cite{2} and\cite{3}. Some years after the emergence of the idea, AdS/CFT correspondence is extended to the extremal rotating black holes known as Kerr/CFT correspondence\cite{4}. This correspondence can tell the microscopic origin of the of the black hole's entropy by using AdS space-time that is known to be a theoretical space only in our universe, no observational evidence yet. On the other hand, some observational evidence tells that there are some extreme black holes in the universe\cite{5,6}. In the same spirit of AdS/CFT correspondence, based on the Kerr/CFT correspondence, QFT on the boundary of the extremal Kerr black holes or Near Horizon Extremal Kerr (NHEK) black holes can tell the semi-classical dynamics of the Kerr black holes. AdS/CFT correspondence also has been used on the other gravitational objects such as Neutron stars\cite{7} and Boson stars\cite{8} that have a cosmological constant in the action. In\cite{8}, there also a study about the counterpart of the boson stars, i.e. Q-balls with the non-zero cosmological constant. We can also find the other study of the Boson stars and Q-balls in\cite{9,10,11}. Here, we review this Kerr/CFT correspondence based on the work in\cite{4}.

In this correspondence, the main point is the extremal Kerr solution. This metric has an AdS$_2 \times S^2$ structure. This structure leads the study of correspondence between the AdS space-time (part of the quantum gravity theory) and the conformal field theory (CFT) on the black holes. We will study the asymptotic symmetry group of this space-time as that has been proposed by Brown and Hanneux\cite{12}.
for AdS$_3$ space that will produce the central charge. This central charge is a conserved charge that generates the diffeomorphism at infinity. Then by using the Cardy formula, i.e. the entropy from the CFT, we will find the entropy of the extremal Kerr black holes. The temperature of the Cardy formula can be found by using the Frolov-Thorne vacuum for rotating black holes, the generic form of the Hartle-Hawking vacuum [13]. The entropy that emerges from the CFT agrees with the macroscopic Bekenstein-Hawking entropy that is proportional to the quarter of the area of the black holes. Another conformal symmetry also has been studied by Castro et.al. [14]. But the metric is not in the extremal case then it is called by the hidden conformal symmetry. The entropy that is produced also matches the Bekenstein-Hawking entropy.

The study of the microsopic origin of the entropy of black holes seems to be very fascinating. We can calculate the correction of the entropy such as [21] that also agrees with the Bekenstein-Hawking entropy such as that is presented in [20]. We can find some kinds of rotating black holes that are interesting to be studied although there is no observational evidence and beyond the no-hair theorem. These black holes have some parameters like NUT charge, accelerating parameter, cosmological constant, magnetization parameter [15, 16] or merger of some of those parameters [17]. Here we will review some recent studies of the Kerr/CFT correspondence of those kinds of black holes.

We organize the remaining parts of the paper as follows. In section 2, we review the part of the derivation of extremal Kerr metric. In section 3, we show the asymptotic symmetry group that will produce the conserved charge. In the next section, the central charge of the extremal Kerr black holes is produced. Then the conformal temperatures are found from the Frolov-Thorne vacuum that is told in the next section. Finally, the Cardy formula and its correction are shown and used for the NHEK metric in section 6. Then we review the recent studies of Kerr/CFT correspondence for the other kind of rotating black holes especially the correction for the Kerr-NUT black holes. Finally, we conclude this paper in the last section.

2. Extremal Kerr Geometry

The study of near horizon geometry of extremal Kerr shows that this geometry has an AdS$_2 \times S^2$ structure. The emergence of the AdS$_2$ structure might be a sign to study the conformal properties of the extremal Kerr metric. In this section, the NHEK metric will be derived from the generic Kerr metric. Because of the event horizon of the Kerr metric is located on

\[
\Delta = M \pm \sqrt{M^2 - a^2}.
\]

The extremality happens when $a = M$ or $J = M^2$. Firstly we will start our derivation from the generic Kerr-Newman black holes in Boyer-Linquist coordinates such as

\[
\begin{align*}
 ds^2 &= \frac{\Delta}{A^2} d\theta^2 + \frac{\Delta}{\Delta} \left( d\phi - \frac{a}{\Delta} d\theta \right)^2 + \Lambda \sin^2 \theta \left( d\phi - \frac{a}{\Lambda} d\theta \right)^2
\end{align*}
\]

Here $\Delta \equiv \hat{r}^2 - 2M\hat{r} + a^2 + \hat{q}^2$, $\Sigma \equiv \hat{r}^2 + a^2 \cos^2 \theta$, $A \equiv (\hat{r}^2 + a^2) - \Delta a^2 \sin^2 \theta$. The Killing vectors of this metric are $\xi_t$ and $\xi_\phi$. Kerr metric is obtained by taking $\hat{q} = 0$ that has the Hawking temperature, surface gravity, and angular momentum [18,19] respectively, such as

\[
T_H = \frac{\kappa}{2\pi}, \kappa = \frac{\hat{r}_+ - M}{2M\hat{r}_+}, \Omega_H = \frac{a}{2M\hat{r}_+}.
\]

This is related to the macroscopic Bekenstein-Hawking entropy

\[
S_{BH} = \frac{A_{BH}}{4} = 2\pi M\hat{r}_+.
\]

But Kaul and Majumdar [20] compute the lowest order corrections to the Bekenstein-Hawking entropy in a particular formulation [21]. They find that the corrected entropy is

\[
S_{BH} = \frac{A_{BH}}{4} - \frac{3}{2} \ln \left( \frac{A_{BH}}{4} \right) + \text{const.}
\]
Furthermore, after taking the extremal case, the event horizon (1) will be on \( r = r_+ = M \). Let
\[
\ell = 2M \frac{\ell}{\lambda} \phi = \phi + \frac{\lambda}{\ell} r, \quad \hat{r} = \frac{\lambda M}{y} + M. \tag{6}
\]
Then we take the limit \( \lambda \to 0 \) so we will have Near-Horizon Extremal Kerr (NHEK) metric in Poincaré-type coordinates
\[
ds^2 = 2J \Omega^2 \left[-dt^2 + \frac{dy^2}{y^2} + d\theta^2 + \Lambda^2 \left(\frac{dt}{\theta} + d\phi\right)^2\right], \tag{7}
\]
where
\[
\Omega = \left(\frac{1 + \cos^2 \theta}{2}\right)^{\frac{1}{2}}, \quad \Lambda = \frac{2 \sin \theta}{1 + \cos^2 \theta},
\]
and now the event horizon is located on \( y = r_+ = 0 \) [22]. This corresponds with the extremal Ernst potential of the NHEK metric shown by Sakti et. al. [23]. For the metric (5), we have
\[
\Delta = 0, \quad \Sigma = M^2(1 + \cos^2 \theta), \quad A = 4M^4, \quad T_H = 0, \quad \kappa = 0, \quad \Omega_H = \frac{1}{2M}.
\tag{8}
\]
Let us scale the time-like coordinate \( dt \to y^2 \, dt \) on the metric (7), so we will have
\[
ds^2 = 2J\Omega^2 \left[-y^2 \, dt^2 + \frac{dy^2}{y^2} + d\theta^2 + \Lambda^2 (d\phi + ydt)^2\right], \tag{9}
\]
which is another form of NHEK metric that is usually used in the Kerr/CFT correspondence such as in [24, 25, 26].
The coordinates (7) and (9) cover only part of NHEK geometry. Global coordinates [28] are given by
\[
y = \frac{1}{r + \sqrt{1 + r^2} \cos \tau}, \quad t = y \sin \tau \sqrt{1 + r^2}, \quad \phi = \varphi + \ln \left(\frac{\cos \tau + r \sin \tau}{1 + \sin \tau \sqrt{1 + r^2}}\right). \tag{10}
\]
Finally, metric (7) is then
\[
ds^2 = 2J\Omega^2 \left[-(1 + r^2) \, d\tau^2 + \frac{dr^2}{1 + r^2} + d\theta^2 + \Lambda^2 (d\phi + r \, d\tau)^2\right]. \tag{11}
\]
In the global coordinates (13), NHEK metric has a periodic time coordinate.

All of the metrics (7), (9), and (13) are not asymptotically flat at infinity but asymptotically AdS. This fact gives the opportunity to study the AdS/CFT correspondence but here we call it by Kerr/CFT correspondence. In the extremal case, the Killing vectors have transformed and resulted in four new Killing vectors. Those vectors are \( \xi_0 = - \partial_\varphi \) which is the generator of \( U(1) \) and
\[
X_0 = 2 \partial_\tau, \quad X_1 = 2 \sin \tau \frac{r}{\sqrt{1 + r^2}} \partial_\tau - 2 \cos \tau \sqrt{1 + r^2} \partial_r + \frac{2 \sin \tau}{\sqrt{1 + r^2}} \partial_\varphi, \quad X_2 = -2 \cos \tau \frac{r}{\sqrt{1 + r^2}} \partial_\tau - 2 \sin \tau \sqrt{1 + r^2} \partial_r - \frac{2 \cos \tau}{\sqrt{1 + r^2}} \partial_\varphi, \tag{12}
\]
which generate the \( SL(2, R) \) algebra. So, all the vectors build an \( SL(2, R) \times U(1) \) isometry.

3. Asymptotic Symmetry Group

Brown and Hanneux [12] show that by imposing some arbitrary boundary conditions at infinity for AdS_3 space-time where the deviation is the leading term such in BMS/CFT correspondence [29], we can obtain the central term from the Poisson bracket of the conserved charges which are the generators of the diffeomorphism. The central charge of AdS_3 space-time can be obtained by defining the quantum form of the Poisson bracket \( \{ \cdot, \cdot \}_PB \) to \( \{-, \cdot \}_PB \) where \( \hbar = 1 \). One of the important steps in getting this AdS_3 central charge is to set the asymptotic boundary conditions for the metric components of AdS_3 space-time. It is allowed to set some boundary conditions for such space-time due to the fact that this space-time is not flat at infinity.
The non-flat asymptotic condition of NHEK metric is the way to give some appropriate boundary conditions at infinity that will produce the non-trivial central charge after using asymptotic symmetry group (ASG). ASG is defined as the set of allowed symmetry transformations modulo the set of trivial symmetry transformations

\[
\text{ASG} = \frac{\text{Allowed Symmetry Transformations}}{\text{Trivial Symmetry Transformations}}
\]  \hspace{1cm} (13)

Allowed symmetry transformation means that the transformation is consistent with the given boundary conditions while trivial symmetry transformation means that the generator of the transformation vanishes after the boundary conditions are implemented.

To find the allowed diffeomorphism, we need to specify the boundary conditions by defining the deviations of NHEK metric components \( h_{\mu \nu} = O(r^p) \) where \( p \in \mathbb{Z} \). Different boundary conditions will also result in different physics but sometimes will produce the same central charge [30]. Here we choose the boundary conditions [4]

\[
h_{\mu \nu} = \begin{pmatrix}
    0 & O(r^2) & O(r^{-1}) & O(1) \\
    O(r) & 0 & O(r^{-2}) & O(r^{-1}) \\
    O(r) & O(r^{-3}) & O(r^{-2}) & O(r^{-1}) \\
    O(r) & O(r) & O(r^{-1}) & O(1)
\end{pmatrix}
\]  \hspace{1cm} (14)

Then the deviations (14) are used in Lie derivative \( \mathcal{L}_\xi \tilde{g}_{\mu \nu} \sim h_{\mu \nu} \) (conformal Killing equation) where \( g_{\mu \nu} = \tilde{g}_{\mu \nu} + h_{\mu \nu} \) and the Killing vector ansatz is expanded in radial component

\[
\zeta^\mu (r, \tau, \theta, \varphi) = \sum_{n=-1}^{\infty} \zeta^\mu (\tau, \theta, \varphi) \frac{1}{r^n}.
\]  \hspace{1cm} (15)

This fashion is similar with the fashion to find the asymptotic symmetry for spherically symmetric space-time [31]. As an example, we show one component of the deviation to find the diffeomorphism, i.e.

\[
\mathcal{L}_\xi \tilde{g}_{rr} = \zeta^\theta \partial_\theta \tilde{g}_{rr} + \zeta^r \partial_r \tilde{g}_{rr} + 2 \tilde{g}_{rr} \partial_r \zeta^r,
\]

\[
O(r^{-3}) = \left( \zeta^\theta_1 r + \cdots \right) \left( \frac{4 f \Omega^2}{1 + r^2} \right) + \left( \zeta^r_1 r + \cdots \right) \left( \frac{-4 f \Omega^2 r}{(1 + r^2)^2} \right)
\]

\[
+ \left( \frac{4 f \Omega^2}{1 + r^2} \right) \partial_r (\zeta^r_1 r + \cdots),
\]

\[
0 = \zeta^\theta_{-1} r + \zeta^\theta_0.
\]  \hspace{1cm} (16)

Collect each of the equation based on the degree of \( r \) and using the asymptotic property \( r \to \infty \) in equation (16) will result in \( \zeta^\theta_{-1} = \zeta^\theta_0 = 0 \).

For all deviations (14), the components of asymptotic Killing vectors are

\[
\zeta^r_{-1} = 0, \zeta^\theta_0 = 0, \zeta^r_1 = C_1, \zeta^r_2 = 0, \zeta^r_3 = O(r^{-3}),
\]

\[
\zeta^\theta_{-1} = C_1(\varphi), \zeta^\theta_0 = O(1),
\]

\[
\zeta^r_{-1} = 0, \zeta^\theta_0 = 0, \zeta^\theta_1 = O(r^{-1})
\]

\[
\zeta^\theta_{-1} = 0, \zeta^\theta_0 = C_2(\varphi), \zeta^\theta_1 = 0, \zeta^\theta_2 = O(r^{-2}).
\]  \hspace{1cm} (17)
Finally, the general diffeomorphisms which preserve the boundary conditions (14) are of the form

$$\xi^\mu \partial_\mu = [C + O\left(\frac{1}{r}\right)] \partial_\tau + [-r \epsilon'(\phi) + O(1)] \partial_\phi + O\left(\frac{1}{r}\right) \partial_\phi + \left[\epsilon(\phi) + O\left(\frac{1}{r}\right)\right] \partial_\phi,$$

(18)

where $C_1(\phi) = -\epsilon'(\phi)$, $C_2(\phi) = \epsilon(\phi)$ and $C$ is constant that we take to 0. The subleading terms on (18) are considered as trivial diffeomorphisms and the leading terms are

$$\xi_\epsilon = \epsilon(\phi) \partial_\phi - r \epsilon'(\phi) \partial_\phi,$$

(19)

that will be the part of ASG of NHEK metric. The Virasoro algebra has an $U(1)$ isometry but not $SL(2, \mathbb{R})$. Because of the (19), NHEK metric transforms as

$$\delta_s ds^2 = 4f \Omega^2 \left[ r^2 (1 - \Lambda^2) \partial_\phi \epsilon d\tau^2 - \frac{r \partial^2_\phi \epsilon}{1 + r^2} d\phi dr + \Lambda^2 \partial_\phi \epsilon d\phi^2 - \frac{\partial_\phi \epsilon}{(1 + r^2)^2} \right].$$

(20)

We know that the azimuthal coordinate is periodic $\phi \sim \phi + 2\pi$ hence we can define $\epsilon_0(\phi) = -e^{-i n \phi}$ and $\xi_\epsilon = \xi_\epsilon(\epsilon_n)$. It is clear that there are so many boundary conditions that can be applied to the metric but it will produce the different physics. By Lie bracket, the asymptotic symmetry generators (19) satisfy the following Virasoro algebra

$$\{\xi_m, \xi_n\}_L = (m - n) \xi_{m+n},$$

(21)

without the central term. Note that $\xi_0$ generates the $U(1)$ rotational isometry. Actually in the general diffeomorphisms (18), the translation of time $\tau$ is included that is generated by $\partial_\tau$. The time translation is generated by the anti-holomorphic conserved quantity (right-mover), denoted by $E_R$, that measures the deviation $M^2 - J$ of the NHEK metric. Because we want to study the extremal Kerr black holes, $E_R$ must vanish. So the supplementary boundary condition must be added besides (14), that is

$$Q_{\partial_\tau} = 0.$$

(22)

4. Central Charge

Diffeomorphism $\xi$ is generated by a conserved charge $Q_\xi$. The Poisson bracket between two conserved charges, say $Q_\xi[g]$ and $Q_\xi[g]$, is found to be the conserved charge of commutation between two isomorphisms plus a central term [32],

$$\{Q_\xi, Q_\xi\}_{PB} = Q_{\{\xi, \xi\}} + K[\xi, \xi],$$

(23)

where the central term is given by

$$K[\xi, \xi] = -\frac{1}{8\pi} \oint_{\partial^\nu} k_\xi (\bar{\partial}_{\mu} \xi, \bar{\partial}_{\nu} \xi) .$$

(24)

$k_\xi$ is a 2-form that is given by

$$k_\xi = \frac{\sqrt{-g}}{8} \epsilon_{\mu \nu \alpha \beta} k_\xi^{\mu \nu} dx^\alpha \wedge dx^\beta,$$

(25)

where the sign $\wedge$ is the wedge product that has property $a \wedge b = -b \wedge a$ and the superpotential

$$k_\xi^{\mu \nu} = \xi_\nu \partial_{\mu} h - \xi_\nu \partial_{\mu} h^{\rho \phi} h^{\mu \rho} + \frac{h}{2} \partial_{\nu} \xi_\mu - \partial_{\nu} \xi_\mu h^{\rho \phi} + \xi_\nu D_\rho \xi_\mu + \partial_{\nu} D_\rho \xi_\mu + \partial_{\nu} D_\rho h^{\mu \rho} - \left(\xi_\nu D_\mu h - \xi_\nu D_\mu h^{\rho \phi} + \frac{h}{2} \partial_{\nu} \xi_\mu - \partial_{\nu} \xi_\mu h^{\rho \phi} + \partial_{\nu} D_\rho \xi_\mu + \partial_{\nu} D_\rho h^{\mu \rho}\right).$$

(26)

Note that $h_{\mu \nu} = h_{\mu \nu}(\xi_\xi)$, $\xi_\epsilon = \xi_\epsilon(\epsilon_n)$, and $\epsilon_m(\phi) = -e^{-im \phi}$. The superpotential (26) can be obtained from the perturbation of the Einstein field equation. Here we use $h^{\rho \sigma} = \bar{\partial}^\rho \bar{\partial}^\sigma h_{\mu \nu}$ and $h = h^{\mu \nu} \bar{\partial}_{\mu \nu}$.
In computing the central term, Mei [33] shows some derivations of formula (for general rotating extremal black holes) that we need. Then Lie derivative $\mathcal{L}_\xi \hat{g}_{\mu\nu} = h_{\mu\nu}$ can be obtained as

$$h_{rr} = \xi^r_{,r} \partial_r \hat{g}_{rr} + 2 \hat{g}_{rr} \partial_r \xi^r_{,r} = -\frac{4i e^{-i\phi} \Omega^2}{(1 + r^2)^2},$$

$$h_{r\phi} = \hat{g}_{rr} \partial_{\phi} \xi^r_{,r} = -\frac{2n^2 r e^{-i\phi} \Omega^2}{1 + r^2},$$

$$h_{r\tau} = \xi^r_{,r} \partial_{\tau} \hat{g}_{rr} = 4im^2 e^{-i\phi} \Omega^2 (1 - \Lambda^2),$$

$$h_{\phi\phi} = 2 \hat{g}_{\phi\phi} \partial_{\phi} \xi^\phi_{,\phi} = -4ie^{-i\phi} \Omega^2 \Lambda^2.$$  

The conserved charges that generate vectors $\partial_{\tau}$ and $\zeta_{\epsilon}$, respectively, are

$$Q_{\partial_{\tau}} = -\frac{1}{8\pi} \oint_{\partial r} k_{\partial_{\tau}}, Q_{\zeta_{\epsilon}} = -\frac{1}{8\pi} \oint_{\partial r} k_{\zeta_{\epsilon}}.$$  

The component of 2-form in the central term (28) for $\zeta_{\epsilon}$ that will survive after performing integration with respect to the $\theta$ coordinate and taking asymptotic $r$ is

$$k_{\zeta} = 2 J^2 \Lambda^4 k^{rr} d\phi \wedge d\theta,$$

where

$$k^{rr} = -\frac{1}{8\beta^2 \Lambda^4} \left[ 2 \Lambda^2 e'(\phi) r h_{\phi\phi} - \epsilon(\phi) \Lambda^2 \left\{ \frac{\Lambda^2 h_{\tau\tau}}{r^2} + (\Lambda^2 + 1) h_{\phi\phi} + 2r \partial_{\phi} h_{\tau\phi} \right\} \right].$$

Then the 2-form (29) in spatial infinity becomes

$$k_{\zeta} = ie^{-i(m+n)\phi} \sin\theta \left\{ n^2 (n - m) + \frac{8n^2 \sin^2\theta}{(1 + \cos^2\theta)^2} \right\}.$$  

The central term now becomes

$$K[\zeta, \xi] = \int_0^{2\pi} \int_0^\pi i e^{-i(m+n)\phi} \sin\theta \left\{ n^2 (n - m) + \frac{8n^2 \sin^2\theta}{(1 + \cos^2\theta)^2} \right\} d\theta d\phi$$

and we have used the identity

$$\int_0^{2\pi} i e^{-i(m+n)\phi} = 2\pi \delta_{m+n,0}.$$  

By defining

$$Q_{\zeta_n} = L_{\zeta_n} - \frac{3J}{2} \delta_{n,0},$$

and using (21), the algebra of the conserved charges is then

$$[L_m, L_n] = (m - n)L_{m+n} + jm (m^2 - 1) \delta_{m+n,0}.$$  

Easily we can read off the central charge for NHEK from (35), that is

$$c_L = 12 J.$$  

Some of the astrophysical black holes in our universe are observed that approach the extremal case, for example, XTE J1650-500 [35] and GRS 1915+105 [6], so the estimation of the central charge can be obtained. For GRS 1915 +105, the central charge is $c_L = (2 \pm 1) \times 10^7$, with the uncertainty coming from the uncertainty in the measured mass of the black hole.

5. Conformal Temperature

For exerting Cardy formula for the entropy of CFT, the conformal temperature must be determined. For rotating black holes, we use the generalization of Hartle-Hawking vacuum for Schwarzschild black hole [13], i.e. Frolov-Thorne vacuum. Constructing the Frolov-Thorne vacuum for generic Kerr starts by expanding the quantum fields in eigenmodes of the asymptotic energy $\omega$ and angular momentum $m$. As an example, we could write an expansion for scalar field $\Phi$ as
\[ \Phi = \sum_{\omega,m,t} \phi_{\omega,m,t} e^{-i\omega t + im\phi} f_1(r, \theta). \] (37)

After we take the trace over the region inside the horizon, the vacuum is a diagonal density matrix in the energy-angular momentum eigenbasis with a Boltzmann weighting factor \( \exp\left(\frac{m\Omega_H - \omega}{T_H}\right) \) where \( T_H \) is the Hawking temperature. For Schwarzschild black hole, we can set \( \Omega_H \). A procedure to take the limit of the near horizon region and near-extremal black hole (6) allows us to have
\[ e^{-i\omega t + im\phi} = e^{i\left(2M \omega - m\right)t + im\phi} = e^{-in_R t + in_L \phi}, \] (38)

where \( n_L = m \) and \( n_R = \frac{1}{2}(2M \omega - m) \) are the left and right charges associated to \( \partial \phi \) and \( \partial \chi \) in the near-horizon region, respectively. In terms of these variables, the Boltzmann factor is given by
\[ \exp\left(\frac{m\Omega_H - \omega}{T_H}\right) = \exp\left(-\frac{n_R}{T_R} = \frac{n_L}{T_L}\right), \] (39)

where the dimensionless left and right temperatures are then
\[ T_L = \frac{r_+ - M}{2\pi(r_+ - \alpha)}, \quad T_R = \frac{r_+ - M}{2\pi \lambda r_+}. \] (40)

In extremal limit \( a \to M \), so \( r_+ = M \), the temperatures (40) reduce to
\[ T_L = \frac{1}{2\pi}, \quad T_R = 0. \] (41)

6. Entropy from Cardy Formula

In the previous sections, only the left temperature remains non-zero and also the central charge. Via the famous Cardy formula, we can use both of them to find the corresponding entropy of NHEK metric. The general Cardy formula for entropy via CFT description is
\[ S = 2\pi \left( \frac{c_L \delta}{6} + \frac{c_R \delta}{6} \right) = \frac{\pi^2}{3} (c_L T_L + c_R T_R), \] (42)

where \( \delta, \delta \) are the energy. In order to obtain the entropy in terms of \( T \), we use the thermodynamic relation \( d\delta = T \ dS \). For extremal Kerr, we exert the central charge (36) and the temperature (41), the microscopic entropy (42) is then
\[ S = 2\pi J. \] (43)

Entropy (43) is in agreement with the macroscopic entropy Bekenstein-Hawking (4). But substantively, as we have told in section two, there are corrections in the macroscopic entropy formula for black holes. Accordingly, there must be the same correction in the derivation of the Cardy formula. In the Cardy formula, the corrections come from the addition of the lowest eigenvalue \( \delta_0 \) of the conformal operator \( L_0 \), that often but not always has a zero value [21]. This correction will give the partition function to be
\[ Z(t', \bar{t}') = \sum_{\Delta, \bar{\Delta}} \rho(\Delta, \bar{\Delta}) e^{i\Delta t' (\Delta - \Delta_0)} e^{-i\Delta \bar{t}' (\bar{\Delta} - \bar{\Delta}_0)}, \] (44)

By assuming \( \Delta_0 \ll c_L, \bar{\Delta}_0 \ll c_R \) in the derivation, this results in
\[ S = S_L + S_R - \frac{3}{2} \ln S_L S_R + \ln c_L c_R + \text{const.}, \] (45)

where \( S_L = 2\pi \sqrt{c_L \bar{\Delta}/6}, S_R = \sqrt{c_R \bar{\Delta}/6}. \) In the Kerr/CFT Correspondence, we can see that \( c_R = 0 \), so we will have \( S_L = A_{BH}/4 \). The constant \( c_L \) is actually universal in the sense of being independent of the area of the black holes. The third term above can be considered as a constant too, so this will be equal to (5) as the corrected Bekenstein-Hawking entropy. If we add this correction to NHEK metric, we will have
\[ S = 2\pi J - \frac{3}{2} \ln 2\pi J + \text{const.} \] (46)
7. Kerr/CFT Correspondence on Other Black Holes

In the previous section, it has been presented that the Kerr/CFT correspondence tool has been successfully used to derive entropy of the black hole through a CFT description. Until now, the black hole solution that best fits the experimental evidence and realistic is the Kerr black hole or perhaps its counterpart in the presence of an external magnetic field, the Magnetized Kerr. In fact, mathematically there are a variety of black hole solutions that have various parameters that indicate a particular physical phenomenon. Stephani et al. [34] classify a variety of black holes with certain symmetries. On the Kerr/CFT correspondence, we focus on black holes that have the stationary and axially symmetric solutions.

Besides via the Einstein field equations, the stationary black holes solution can be derived through the Ernst equation [35]. The black hole solution that has sufficiently complete parameters is shown by Plebanski and Demianski [17]. In this solution there are mass $M$, NUT charge $n$, spin $a$, electric charge $e$, magnetic charge $g$, and the cosmological constant $\Lambda$. When there is an external magnetic field, then there is a constant $B$ for magnetic field parameter that appears in the solution. In addition, there is a black hole solution that appears in the string theory wherein the Lagrangian, there are the dilaton field, electromagnetic field, and anti-symmetric electromagnetic tensor field known as Kerr-Sen black holes [36]. The entropy of black holes with these parameters is also derived by the Kerr/CFT correspondence. In addition, we show a solution that has some of the parameters for which the entropy has been obtained through this correspondence.

One of the rotating black hole solutions is Kerr-NUT space-time which has spin $a$ and NUT charge $n$ [37]. The NUT parameter corresponds with a twisting property of the surrounding space-time or the gravitomagnetic charge of the central mass. The Kerr-NUT metric is defined by

$$ds^2 = -\frac{\Delta}{\rho^2} [dt + \{2n (\cos \theta - 1) - a \sin^2 \theta \}d\phi]^2 + \frac{\rho^2 dr^2}{\Delta} + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [adt - \{r^2 + (n + a)^2 \} d\phi]^2,$$

where

$$\rho^2 = r^2 + (n + a \cos \theta)^2, \Delta = r^2 - 2Mr + a^2 - n^2.$$

We obtain (47) from the general accelerating and rotating black hole [17] by setting $b, e, g, \Lambda = 0$. This space-time (47) has the event horizon on $r_+ = M \pm \sqrt{M^2 + n^2 - a^2}$. When extreme value occurs, we can adopt Kerr/CFT correspondence to obtain the entropy where the structure is $AdS_2 \times S^2$ and has an $SL(2, R) \times U(1)$ isometry generated from the algebra of the same Killing vector as (12). The entropy is then

$$S_L = \frac{A_{BH}}{4} = 2\pi (M^2 + n^2 + an) = 2\pi a (a + n).$$

This entropy has a difference from [37] because we do not neglect the product of spin and NUT charge in the term $(n + a)^2$. By taking into account the correction, it will be obtained

$$S = 2\pi a (a + n) - \frac{3}{2} \ln 2\pi a (a + n) + \text{const.}$$

When the NUT charge vanishes, it will reduce to the entropy of the extremal Kerr (46).

This correspondence has also been successfully used on Kerr-Newman-Dyonic-(A) dS [38] and Kerr-Newman-NUT black holes. The existence of the gauge field causes us to require the asymptotic boundary conditions of this field. But from the calculations will be found that the contribution of the central charge of the gauge field will disappear. Furthermore, in the Kerr-Sen black holes [36], the resulting entropy is equal to the Kerr/CFT correspondence because the contribution of the dilaton field, the gauge field, and the anti-symmetric electromagnetic tensor field will disappear on its central charge. The acceleration and external magnetization parameters appear on the Magnetized Accelerating Kerr-Newman-Dyonic black holes [26] for which the entropy agrees with the Bekenstein-Hawking entropy too.
8. Concluding Remarks

In this paper, we review the conjectured Kerr/CFT correspondence. We find explicitly the near-horizon metric of extremal space-time by taking the near-horizon transformation. The near-horizon geometry has $\text{AdS}_2 \times S^2$ structure. By imposing the appropriate boundary conditions, we can obtain the diffeomorphisms that generate a Virasoro algebra without any central term. The generator of diffeomorphism which is a conserved charge can be used to construct an algebra under Dirac brackets. This algebra is the same as diffeomorphism algebra but with an additional central term. The conserved charge for the right-mover is assumed to be zero in order to match the extremality condition of extremal Kerr metric. The conformal temperatures are found from Frolov–Thorne vacuum. Then the central charge together with the temperature allows us to obtain the microscopic origin of the entropy of the extremal Kerr. The entropy from CFT and the corrected one agree with the Bekenstein-Hawking entropy. We also briefly show the tool of the Kerr/CFT correspondence on the Kerr-NUT space-time for which the entropy also agrees with the Bekenstein-Hawking entropy.

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