The Hardy–Littlewood–Pólya inequality of majorization in the context of $\omega$–$m$–star-convex functions

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Abstract. The Hardy–Littlewood–Pólya inequality of majorization is extended for $\omega$–$m$–star-convex functions to the framework of ordered Banach spaces. Several open problems which seem to be of interest for further extensions of the Hardy–Littlewood–Pólya inequality are also included.

Mathematics Subject Classification. Primary 26B25, Secondary 26D10, 46B40, 47B60, 47H07.

Keywords. $\omega$–$m$–star-convex function, Majorization theory, Ordered Banach space, Isotone operator.

1. Introduction

The Hardy–Littlewood–Pólya theorem of majorization is an important result in convex analysis that lies at the core of majorization theory, a subject that has attracted a great deal of attention due to its numerous applications in mathematics, statistics, economics, quantum information etc. See [8, 9, 19, 20, 22–24] to cite just a few books treating this topic.

The relation of majorization was initially formulated as a relation between pairs of vectors with real entries rearranged downward, but nowadays its formulation as a preordering of probability measures.

For the reader’s convenience we briefly recall here the most basic facts concerning the theory of majorization.

Given two discrete probability measures $\mu = \sum_{k=1}^{N} \lambda_k \delta_{x_k}$ and $\nu = \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, supported by a compact interval $[a, b]$, we say that $\mu$ is majorized by $\nu$ (denoted $\mu \prec \nu$) if the following three conditions are fulfilled:
When only conditions (M1) and (M2) occur, we say that $\mu$ is weakly majorized by $\nu$ (denoted $\mu \prec_w \nu$).

Hardy, Littlewood and Pólya [5] used a stronger formulation of (M1), by requiring also that $y_1 \geq y_2 \geq \cdots \geq y_N$. Later, their result was improved by Maligranda, Pečarić and Persson [7] who were able to prove that

$$\mu \prec \nu \text{ implies } \int_{a}^{b} f \, d\mu = \sum_{k=1}^{N} f(x_k) \leq \int_{a}^{b} f \, d\nu = \sum_{k=1}^{N} f(y_k), \quad (\text{HLP})$$

for all continuous convex functions $f : [a, b] \to \mathbb{R}$. Moreover, the same conclusion holds in the case of weak majorization and convex and nondecreasing functions.

Nowadays the inequality HLP is known as the Hardy–Littlewood–Pólya inequality of majorization.

In the early 1950s, the Hardy–Littlewood–Pólya inequality was extended by Sherman [25] to the case of continuous convex functions of a vector variable by using a much broader concept of majorization, based on matrices stochastic on lines. The full details can be found in [19, Theorem 4.7.3, p. 219]. Over the years, many other generalizations in the same vein have been published. See, for example, [3, 14–18, 21].

As was noticed in [12] and [13], the Hardy–Littlewood–Pólya inequality of majorization can be extended to the framework of convex functions defined on ordered Banach spaces alongside the conditions (M1)–(M3). The aim of the present paper is to prove that the same works for the larger class of $\omega$–$m$–star-convex functions.

The main features of these functions are discussed in Sect. 2.

In Sect. 3 we present different types of majorization relations in ordered Banach spaces. The corresponding extensions of the Hardy–Littlewood–Pólya inequality constitute the objective of Sect. 4. The paper ends with mentioning several open problems which seem to be of interest for further extensions of the Hardy–Littlewood–Pólya inequality.

2. Preliminaries on $\omega$–$m$–star-convex functions

Throughout this paper $E$ is a Banach space and $C$ is a convex subset of it.

Definition 1. Let $m$ be a real parameter belonging to the interval $(0, 1]$. A function $\Phi : C \to \mathbb{R}$ is said to be a perturbed $m$–star-convex function with
modulo \( \omega : [0, \infty) \to \mathbb{R} \) (abbreviated as \( \omega-m\)–star-convex function) if it fulfills an estimate of the form
\[
\Phi((1 - \lambda)x + \lambda my) \leq (1 - \lambda)\Phi(x) + m\lambda\Phi(y) - m\lambda(1 - \lambda)\omega(\|x - y\|),
\]
for all \( x, y \in C \) and \( \lambda \in (0, 1) \).

The \( \omega-m\)–star-convex functions associated to an identically zero modulus will be called \( m\)–star-convex. They satisfy the inequality
\[
\Phi((1 - \lambda)x + \lambda my) \leq (1 - \lambda)\Phi(x) + m\lambda\Phi(y),
\]
for all \( x, y \in C \) and \( \lambda \in (0, 1) \).

Notice that the usual convex functions represent the particular case of \( m\)–star-convex functions where \( m = 1 \). On the other hand, every convex function is \( m\)–star-convex (for every \( m \in (0, 1) \)) if \( 0 \in C \) and \( \Phi(0) \leq 0 \). Indeed, we have
\[
\Phi((1 - \lambda)x + \lambda my) = \Phi((1 - \lambda)x + \lambda my + (\lambda - \lambda m)0) \\
\leq (1 - \lambda)\Phi(x) + m\lambda\Phi(y) + (\lambda - \lambda m)\Phi(0) \\
= (1 - \lambda)\Phi(x) + m\lambda\Phi(y).
\]

Every \( \omega-m\)–star-convex function associated to a modulus \( \omega \geq 0 \) is necessarily \( m\)–star-convex. The \( \omega-m\)–star-convex functions whose moduli \( \omega \) are strictly positive except at the origin (where \( \omega(0) = 0 \)) are usually called \emph{uniformly \( m\)–star-convex}. In their case the definitory inequality is strict whenever \( x \neq y \) and \( \lambda \in (0, 1) \).

By reversing the inequalities, one obtains the notions of \( \omega-m\)–star-concave function and \emph{uniformly \( m\)–star-concave function}.

The theory of \( m\)–star-convex functions was initiated by Toader [26], who considered only the case of functions defined on real intervals. For additional results in the same setting see [11] and the references therein.

A simple example of a \( (16/17)\)-star-convex function which is not convex is
\[
\Phi : [0, \infty) \to \mathbb{R}, \quad \Phi(x) = x^4 - 5x^3 + 9x^2 - 5x. \tag{2.1}
\]
See [11, Example 2]. Note that if \( \Phi : C \to \mathbb{R} \) and \( \Psi : C \to \mathbb{R} \) are \( \omega-m\)–star-convex functions and \( \alpha, \beta \in \mathbb{R}_+ \), then
\[
\alpha\Phi + \beta\Psi \quad \text{and} \quad \sup \{ \Phi, \Psi \}
\]
are functions of the same nature. So is
\[
\Phi \times \Psi : C \times C \to \mathbb{R}, \quad (\Phi \times \Psi)(x, y) = \Phi(x) + \Psi(y).
\]
The class of \( \omega-m\)–star-convex functions is also stable under pointwise convergence (when it exists).

Assuming \( C \subset E \) is a convex cone with vertex at the origin, the \emph{perspective} of a function \( f : C \to \mathbb{R} \) is the positively homogeneous function
\[
\tilde{f} : C \times (0, \infty) \to \mathbb{R}, \quad \tilde{f}(x, t) = tf \left( \frac{x}{t} \right).
\]
Lemma 1. The perspective of every \( m \)-star-convex/concave function is a function of the same nature.

Proof. Indeed, assuming (to make a choice) that \( f \) is \( \omega-m \)-star-convex, then for all \((x, s), (y, t) \in C \times (0, \infty) \) and \( \lambda \in [0, 1] \) we have

\[
 f \left( \frac{(1 - \lambda)x + \lambda my}{(1 - \lambda)s + \lambda mt} \right) = f \left( \frac{(1 - \lambda)s x + \lambda mt y}{(1 - \lambda)s + \lambda mt} \cdot \frac{s}{s} + \frac{(1 - \lambda)s + \lambda mt}{(1 - \lambda)s} \cdot \frac{t}{t} \right)
\]

\[
\leq \frac{(1 - \lambda)s}{(1 - \lambda)s + \lambda mt} f \left( \frac{x}{s} \right) + \frac{\lambda mt}{(1 - \lambda)s + \lambda mt} f \left( \frac{y}{t} \right)
\]

that is,

\[
\tilde{f}((1 - \lambda)x + \lambda my, (1 - \lambda)s + \lambda mt) \leq (1 - \lambda) \tilde{f}(x, s) + \lambda m \tilde{f}(y, t).
\]

Lemma 1, allows us to easily produce nontrivial examples of \( m \)-star-convex functions of several variables with some nice properties. For example, starting from (2.1), we conclude that

\[
\Phi(x, t) = x^4 - 5x^3t + 9x^2t^2 - 5xt^3
\]
is a \((16/17)\)-star-convex function on \([0, \infty) \times (0, \infty)\).

Under the presence of Gâteaux differentiability, \( \omega-m \)-star-convex functions generate specific gradient inequalities that play a prominent role in our generalization of the Hardy–Littlewood–Pólya inequality of majorization.

Lemma 2. Suppose also that \( C \) is an open convex subset of the Banach space \( E \) and \( \Phi : C \rightarrow \mathbb{R} \) is a function both Gâteaux differentiable and \( \omega-m \)-star-convex. Then

\[
m \Phi(y) \geq \Phi(x) + d \Phi(x)(my - x) + m \omega \left( \|x - y\| \right), \quad (2.2)
\]

for all points \( x, y \in C \).

Proof. Indeed, we have

\[
\frac{\Phi((1 - \lambda)x + m \lambda y) - \Phi(x)}{\lambda} \leq - \Phi(x) + m \Phi(y) - m(1 - \lambda) \omega \left( \|x - y\| \right)
\]
and the proof ends with passing to the limit as \( \lambda \rightarrow 0^+ \).

Remark 1. Lemma 2 shows that the critical points \( x \) of the differentiable \( \omega-m \)-star-convex functions are those for which \( \omega \geq 0 \) fulfill the property

\[
m \inf_{y \in C} \Phi(y) \geq \Phi(x).
\]
Unlike the case of convex functions of one real variable, when the isotonicity of the differential is automatic, for several variables, this is not necessarily true in the case of a differentiable convex function of a vector variable. See [12, Remark 4].

In this paper we deal with functions defined on ordered Banach spaces, that is, on real Banach spaces endowed with order relations \( \leq \) that make them ordered vector spaces such that positive cones are closed and

\[ 0 \leq x \leq y \implies \|x\| \leq \|y\|. \]

The Euclidean \( N \)-dimensional space \( \mathbb{R}^N \) has a natural structure of an ordered Banach space associated to coordinatewise ordering. The usual sequence spaces \( c_0, c, \ell^p \) (for \( p \in [1, \infty] \)) and the function spaces \( C(K) \) (for \( K \) a compact Hausdorff space) and \( L^p(\mu) \) (for \( 1 \leq p \leq \infty \) and \( \mu \) a \( \sigma \)-additive positive measure) are also examples of ordered Banach spaces (with respect to coordinatewise/pointwise ordering and natural norms).

A map \( T : E \to F \) between two ordered vector spaces is called isotone (or order preserving) if

\[ x \leq y \text{ in } E \implies T(x) \leq T(y) \text{ in } F \]

and antitone (or order reversing) if \( -T \) is isotone. When \( T \) is a linear operator, \( T \) is isotone if and only if \( T \) maps positive elements into positive elements (abbreviated, \( T \geq 0 \)).

For basic information on ordered Banach spaces see [13]. The interested reader may also consult the classical books of Aliprantis and Tourky [1] and Meyer–Nieberg [10].

As was noticed by Amann [2, Proposition 3.2, p. 184], the Gâteaux differentiability offers a convenient way to recognize the property of isotonicity of functions acting on ordered Banach spaces: the positivity of the differential. We state here his result (following the version given in [12, Lemma 4]):

**Lemma 3.** Suppose that \( E \) and \( F \) are two ordered Banach spaces, \( C \) is a convex subset of \( E \) with nonempty interior \( \text{int} \, C \) and \( \Phi : C \to F \) is a convex function, continuous on \( C \) and Gâteaux differentiable on \( \text{int} \, C \). Then \( \Phi \) is isotone on \( C \) if and only if \( \Phi'(a) \geq 0 \) for all \( a \in \text{int} \, C \).

**Remark 2.** If the ordered Banach space \( E \) has finite dimension, then the statement of Lemma 3 remains valid when the interior of \( C \) is replaced by the relative interior of \( C \). See [19, Exercise 6, p. 81].

As was noticed in [11, Example 7], the function

\[ \gamma : (-\infty, 1] \to \mathbb{R}, \quad \gamma(x) = -2x^3 + 5x^2 + 6x \]

is convex on \( (-\infty, 5/6] \), concave on \( [5/6, 1] \), and \( m \)-star-convex on \( (-\infty, 1] \), with \( m = 27/28 \). The last assertion follows from a formula due to Mocanu,

\[ m = \inf \left\{ \frac{x\gamma'(x) - \gamma(x)}{y\gamma'(x) - \gamma(y)} : y\gamma'(x) - \gamma(y), x, y \in I \right\}, \]

where
mentioned at the bottom of page 72 in [11].

Proceeding like in Lemma 1, one can prove that the function associated to \( \gamma \),

\[ \Upsilon : (-\infty, 1] \times [1, \infty) \rightarrow \mathbb{R}, \quad \Upsilon(x, y) = -\frac{2x^3}{y^2} + \frac{5x^2}{y} + 6x, \]

is 27/28-star-convex. The function \( \Upsilon \) is also Gateaux differentiable, with

\[ (x, y) = \left( \frac{1}{y^2} \left( -6x^2 + 10xy + 6y^2 \right), \frac{x^2}{y^3} \left( 4x - 5y \right) \right). \]

According to Lemma 3, the map 

\[ d\Upsilon : (-\infty, 1] \times [1, \infty) \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

is isotone on the domain where \( d^2\Upsilon = d(d\Upsilon) \) is positive, that is, where the Hessian of \( \Upsilon \),

\[ \begin{pmatrix} -\frac{2}{y^2} (6x - 5y) & 2\frac{x}{y^2} (6x - 5y) \\ 2\frac{x}{y^2} (6x - 5y) & -2\frac{x^2}{y^3} (6x - 5y) \end{pmatrix}, \]

has nonnegative entries only. Therefore \( d\Upsilon \) is isotone on \( (-\infty, 1] \times [1, \infty) \).

3. The majorization relation on ordered Banach spaces

In this section we discuss the concept of majorization in the framework of ordered Banach spaces. Since in an ordered Banach space not every string of elements admits a decreasing rearrangement, in this paper we will concentrate on the case of pairs of discrete probability measures at least one of which is supported by a monotone string of points. The case where the support of the left measure consists of a decreasing string is defined as follows.

**Definition 2.** Suppose that \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \) and \( \sum_{k=1}^{N} \lambda_k \delta_{y_k} \) are two discrete Borel probability measures that act on the ordered Banach space \( E \) and \( m \in (0, 1] \) is a parameter. We say that \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \) is weakly \( mL \)-majorized by \( \sum_{k=1}^{N} \lambda_k \delta_{y_k} \) (denoted \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{w mL} \sum_{k=1}^{N} \lambda_k \delta_{y_k} \)) if the left hand measure is supported by a decreasing string of points

\[ x_1 \geq \cdots \geq x_N \quad (3.1) \]

and

\[ \sum_{k=1}^{n} \lambda_k x_k \leq \sum_{k=1}^{n} \lambda_k m y_k \quad \text{for all } n \in \{1, \ldots, N\}. \quad (3.2) \]
We say that \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \) is \( mL \)-majorized by \( \sum_{k=1}^{N} \lambda_k \delta_{y_k} \) (denoted \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{mL} \sum_{k=1}^{N} \lambda_k \delta_{y_k} \)) if in addition
\[
\sum_{k=1}^{N} \lambda_k x_k = \sum_{k=1}^{N} \lambda_k m y_k. \tag{3.3}
\]

Notice that the context of Definition 2 makes it necessary that all the weights \( \lambda_1, \ldots, \lambda_N \) belong to \( (0, 1] \) and \( \sum_{k=1}^{N} \lambda_k = 1 \).

The three conditions (3.1), (3.2) and (3.3) imply \( m y_N \leq x_N \leq x_1 \leq m y_1 \) but not the ordering \( y_1 \geq \cdots \geq y_N \). For example, when \( N = 3 \), one may consider the case where
\[
m = 1, \quad \lambda_1 = \lambda_2 = \lambda_3 = 1/3, \quad x_1 = x_2 = x_3 = x
\]
and
\[
y_1 = x, \quad y_2 = x + z, \quad y_3 = x - z,
\]
z being any positive element.

Under these circumstances it is natural to introduce the following companion to Definition 2, involving the ascending strings of elements as support for the right hand measure.

**Definition 3.** The relation of weak \( mR \)-majorization,
\[
\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{wmR} \sum_{k=1}^{N} \lambda_k \delta_{y_k},
\]
between two discrete Borel probability measures means the fulfillment of the condition (3.2) under the presence of the ordering \( y_1 \leq \cdots \leq y_N \); (3.4)
assuming in addition the condition (3.3), we say that \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \) is \( mR \)-majorized by \( \sum_{k=1}^{N} \lambda_k \delta_{y_k} \) (denoted \( \sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{mR} \sum_{k=1}^{N} \lambda_k \delta_{y_k} \)).

When every element of \( E \) is the difference of two positive elements, the weak majorization relations \( \prec_{wmL} \) and \( \prec_{mR} \) can be augmented so as to obtain majorization relations.

### 4. The extension of the Hardy–Littlewood–Polya inequality of majorization

The objective of this section is to consider the corresponding extensions of the Hardy–Littlewood–Pólya inequality of majorization for \( \prec_{wmL}, \prec_{mL}, \prec_{wmR} \) and \( \prec_{mR} \). Moreover, we also present also a Sherman type inequality.

The proof of the following theorem is inspired by the techniques succesfully used in [7,12].
Theorem 1. Suppose that $\sum_{k=1}^{N} \lambda_k \delta_{x_k}$ and $\sum_{k=1}^{N} \lambda_k \delta_{y_k}$ are two discrete probability measures whose supports are included in an open convex subset $C$ of the ordered Banach space $E$. If $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{mL} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, then

$$m \sum_{k=1}^{N} \lambda_k \Phi(y_k) \geq \sum_{k=1}^{N} \lambda_k \Phi(x_k) + \sum_{k=1}^{N} \lambda_k \omega(\|x_k - y_k\|), \quad (4.1)$$

for every Gâteaux differentiable $\omega$–$m$–star-convex function $\Phi : C \to F$ whose differential is isotone and satisfies the hypotheses of Lemma 2.

The conclusion (4.1) still works under the weaker hypothesis $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{wmL} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, provided that $\Phi$ is also an isotone function.

Proof. According to the gradient inequality (2.2), we have

$$m \sum_{k=1}^{N} \lambda_k \Phi(y_k) - \sum_{k=1}^{N} \lambda_k \Phi(x_k) = \sum_{k=1}^{N} \lambda_k (m \Phi(y_k) - \Phi(x_k))$$

$$\geq \sum_{k=1}^{N} \Phi'(x_k)(\lambda_k m y_k - \lambda_k x_k) + \sum_{k=1}^{N} \lambda_k \omega(\|x_k - y_k\|),$$

whence, by using Abel’s trick of interchanging the order of summation [19, Theorem 1.9.5, p. 57], one obtains

$$\sum_{k=1}^{N} \lambda_k m \Phi(y_k) - \sum_{k=1}^{N} \lambda_k \Phi(x_k) - \sum_{k=1}^{N} \lambda_k \omega(\|x_k - y_k\|) \geq \Phi'(x_1)(\lambda_1 m y_1 - \lambda_1 x_1)$$

$$+ \sum_{m=2}^{N} \Phi'(x_m) \left[ \sum_{k=1}^{m} (\lambda_k y_k - \lambda_k x_k) - \sum_{k=1}^{m-1} (\lambda_k y_k - \lambda_k x_k) \right]$$

$$= \sum_{m=1}^{N-1} \left[ (\Phi'(x_m) - \Phi'(x_{m+1})) \sum_{k=1}^{m} (\lambda_k m y_k - \lambda_k x_k) \right]$$

$$+ \Phi'(x_N) \left( \sum_{k=1}^{N} (\lambda_k m y_k - \lambda_k x_k) \right).$$

When $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{mL} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$, the last term vanishes and the fact that $D \geq 0$ is a consequence of the isotonicity of $\Phi$. When $\sum_{k=1}^{N} \lambda_k \delta_{x_k} \prec_{wmL} \sum_{k=1}^{N} \lambda_k \delta_{y_k}$ and $\Phi$ is isotone, one applies Lemma 3 (a) to infer that
\[ \Phi'(x_N) \left( \sum_{k=1}^{N} (\lambda_k m y_k - \lambda_k x_k) \right) \geq 0. \]

The other cases can be treated in a similar way. \(\square\)

**Remark 3.** Even in the context of usual convex functions, the isotonicity of the differential is not only sufficient but also necessary for the validity of Theorem 1. See [12, Remark 5].

We leave it to the reader as an exercise to formulate the variant of Theorem 1 in the case of relations \(\preccurlyeq_{wmR}^\uparrow\) and \(\preccurlyeq_{mR}^\uparrow\).

### 5. Further results and open problems

The aim of this section is to mention some open problems which might be of interest for further research.

Notice first that any perturbation of an \(\omega-m\)-star-convex function \(\Phi\) satisfying the hypotheses of Theorem 1 by a bounded function \(\Pi\) fulfil an inequality of majorization very close to (4.1). Precisely, if \(|\Pi| \leq \delta\) and \(\sum_{k=1}^{N} \lambda_k \delta_{x_k} \preccurlyeq_{mL} \sum_{k=1}^{N} \lambda_k \delta_{y_k}\), then \(\Psi = \Phi + \Pi\) will satisfy the relation
\[
\sum_{k=1}^{N} \lambda_k \Psi(x_k) \geq \sum_{k=1}^{N} \lambda_k \Phi(x_k) + \sum_{k=1}^{N} \lambda_k \omega(\|x_k - y_k\|) - (1 + m)\delta.
\]

This directs attention to the following class of *approximately* \(\omega-m\)-star-convex functions:

**Definition 4.** A function \(\Phi : C \to \mathbb{R}\) is said to be a \(\delta-\omega-m\)-star-convex function if it verifies an estimate of the form
\[
\Phi((1 - \lambda)x + \lambda my) \leq (1 - \lambda)\Phi(x) + m\lambda\Phi(y) - m\lambda(1 - \lambda)\omega(\|x - y\|) + \delta,
\]
for some \(\delta \geq 0\) and all \(x, y \in C\) and \(\lambda \in (0, 1)\).

The above definition extends (for \(\omega = 0\) and \(m = 1\)) the concept of \(\delta\)-convex function, first considered by Hyers and Ulam [6] in a paper dedicated to the stability of convex functions. It is natural to raise the question whether their result extends to the framework of \(\delta-\omega-m\)-star-convex functions:

**Problem 1.** Suppose that \(C\) is a convex subset of \(\mathbb{R}^N\). Is that true that every \(\delta-\omega-m\)-star-convex function \(\Phi : C \to \mathbb{R}\) can be written as \(\Phi = \Psi + \Pi\), where \(\Psi\) is an \(\omega-m\)-star-convex function and \(\Pi\) is a bounded function whose supremum norm is not larger than \(k_N\delta\), where the positive constant \(k_N\) depends only on the dimension \(N\) of the underlying space?
Of some interest seems to be the concept of local approximate $m$–star-convexity suggested by [4, Definition 1], which clearly yields new extensions of the majorization inequality:

**Definition 5.** A function $\Phi : C \to \mathbb{R}$ is called locally approximately $m$–star-convex if for every $x_0 \in C$, and every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y$ in the open ball with center $x_0$ and radius $\delta$ and all $\lambda \in (0, 1)$,

$$
\Phi((1 - \lambda)x + m\lambda y) \leq (1 - \lambda)\Phi(x) + m\lambda\Phi(y) + t(1 - t)\|x - y\|.
$$

The whole discussion above can be placed in the more general context of $M_p$-convexity.

Recall that the weighted $M_p$-mean is defined for every pair of positive numbers $a, b$ by the formula

$$
M_p(a, b; 1 - \lambda, \lambda) = \begin{cases} 
((1 - \lambda)a^p + \lambda b^p)^{1/p}, & \text{if } p \in \mathbb{R} \setminus \{0\} \\
\lambda^p, & \text{if } p = 0 \\
\max\{a, b\}, & \text{if } p = \infty,
\end{cases}
$$

where $\lambda \in [0, 1]$. If $p > 0$, then it is usual to extend $M_p$ to all pairs of nonnegative numbers.

**Definition 6.** A function $\Phi : C \to \mathbb{R}$ is called $\omega$–$m$–$M_p$–star-convex if there exist a number $p \in \mathbb{R}$ and a modulus $\omega : [0, \infty) \to \mathbb{R}$ such that

$$
\Phi((1 - \lambda)x + \lambda y) \leq ((1 - \lambda)\Phi(x)^p + m\lambda\Phi(y)^p)^{1/p} - m\lambda(1 - \lambda)\omega(\|x - y\|)
$$

for all $x, y \in C$ and $\lambda \in (0, 1)$.

Reversing the inequality one obtains the concept of $\omega$–$m$–$M_p$–star-concave functions.

The usual $M_p$-convex/$M_p$-concave functions represent the particular case where $m = 1$ and $\omega = 0$.

It is worth noticing that the $M_p$-convex ($M_p$-concave) functions for $p \neq 0$ are precisely the functions $\Phi$ such that $\Phi^p$ is convex (concave), while the $M_0$-convex ($M_0$-concave) functions are nothing but the log-convex (log-concave) functions. Notice also that the $M_\infty$-convex ($M_{-\infty}$-concave) functions are precisely the quasi-convex (quasi-concave) functions.

The next result represents the extension of Lemma 2 to the case of $\omega$–$m$–$M_p$–star-convex functions.

**Lemma 4.** Suppose that $C$ is an open convex subset of the Banach space $E$ and $\Phi : C \to \mathbb{R}_+$ is a function both Gâteaux differentiable and $\omega$–$m$–$M_p$–star-convex. If $p \neq 0$, then $\Phi$ satisfies the inequality

$$
\Phi^p(y) \geq \Phi^p(x) + p\Phi(x)^{p-1}d\Phi(x)(y - x) + m\omega(\|x - y\|),
$$

for all $x, y \in C$. 

The analogue of this result for $p = 0$ and $\omega = 0$ requires the strict positivity of the function $\Phi$ and can be stated as
\[
\log \Phi(y) - \log \Phi(x) \geq \frac{d\Phi(x)(y - x)}{\Phi(x)},
\]
for all $x, y \in C$. The last two inequalities work in the reverse direction in the case of $\omega - m - M_p$-star-concave functions.

While it is clear that Lemma 4 allows us to prove Hardy–Littlewood–Pólya type inequalities more general than those provided by Theorem 1, the exploration of the world of $\omega - m - M_p$-star-convex/concave functions for $\omega \neq 0$ and $m \in (0, 1)$ is just at the beginning.

Acknowledgements

The work of G. M. Lachescu and I. Roventa have been supported by a grant of the Romanian Ministry of Research, Innovation and Digitalization (MCID), project number 22 - Nonlinear Differential Systems in Applied Sciences, within PNRR-III-C9-2022-18.

Author contributions Yongge Tian wrote the main manuscript text.

Data availability The authors declare that data supporting the findings of this study are available within the article and its supplementary information files.

Declarations

Conflict of interest The authors declare no competing interests.

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References

[1] Aliprantis, C.D., Tourky, R.: Cones and Duality, Graduate Studies in Mathematics 84. American Mathematical Society, Providence (2007)
[2] Amann, H.: Multiple positive fixed points of asymptotically linear maps. J. Funct. Anal. 17(2), 174–213 (1974)
[3] Bradanović, I., Latif, S., Pečarić, N., Pečarić, J.: Sherman’s and related inequalities with applications in information theory. J. Inequal Appl. 2018(1), article 98 (2018)
[4] Danilidis, A., Georgiev, P.: Approximate convexity and submonotonicity. J. Math. Anal. Appl. 291(1), 292–301 (2004)
[5] Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities, 2nd ed. Cambridge University Press (1952). Reprinted 1988
[6] Hyers, D.H., Ulam, S.M.: Approximately convex functions. Proc. Am. Math. Soc. 3, 821–828 (1952)
[7] Maligranda, L., Pečarić, J., Persson, L.-E.: Weighted Favard’s and Berwald’s inequalities. J. Math. Anal. Appl. 190(1), 248–262 (1995)
[8] Marshall, A.W., Olkin, I.: Inequalities: Theory of Majorization and Its Applications, vol. 143. Academic Press, New York (1979)
[9] Marshall, A.W., Olkin, I., Arnold, B.: Inequalities: Theory of Majorization and Its Applications, 2nd ed. Springer Series in Statistics. Springer, New York (2011)
[10] Meyer-Nieberg, P.: Banach Lattices. Springer, Berlin (1991)
[11] Mocanu, P.T., Serb, I., Toader, G.: Real star-convex functions. Stud. Univ. Babes-Bolyai Math. 43, 65–80 (1997)
[12] Niculescu, C.P.: A new look at the Hardy–Littlewood–Pólya inequality of majorization. J. Math. Anal. Appl. 501(2), article 125211 (2021)
[13] Niculescu, C.P., Olteanu, O.: From the Hahn–Banach extension theorem to the isotonicity of convex functions and the majorization theory. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 114(4), 1–19 (2020)
[14] Niculescu, C.P., Popovici, F.: The extension of majorization inequalities within the framework of relative convexity. J. Inequal. Pure Appl. Math. 7, Article No. 27 (Electronic only) (2006)
[15] Niculescu, C.P., Roventa, I.: An approach of majorization in spaces with a curved geometry. J. Math. Anal. Appl. 411, 119–128 (2014)
[16] Niculescu, C.P., Roventa, I.: Relative convexity and its applications. Aequationes Math. 89, 1389–1400 (2015)
[17] Niculescu, C.P., Roventa, I.: Relative Schur convexity on global NPC spaces. Math. Inequal. Appl. 18, 1111–1119 (2015)
[18] Niculescu, C.P., Roventa, I.: Hardy–Littlewood–Pólya theorem of majorization in the framework of generalized convexity. Carpathian J. Math. 33, 87–95 (2017)
[19] Niculescu, C.P., Persson, L.-E.: Convex Functions and Their Applications. A Contemporary Approach, 2nd ed. CMS Books in Mathematics vol. 23. Springer, New York (2018)
[20] Nielsen, F., Bhatia, R. (eds.): Matrix Information Geometry. Springer, Heidelberg (2013)
[21] Niezgoda, M.: Nonlinear Sherman-type inequalities. Adv. Nonlinear Anal. 9, 168–175 (2020)
[22] Pečarić, J.E., Proschan, F., Tong, Y.C.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, San Diego (1992)
[23] Petz, D.: Quantum Information Theory and Quantum Statistics. Springer (2007)
[24] Rüschendorf, L.: Mathematical Risk Analysis. Springer Series in Operations Research and Financial Engineering. Springer, Heidelberg (2013)
[25] Sherman, S.: On a theorem of Hardy, Littlewood, Pólya, and Blackwell. Proc. Natl. Acad. Sci. USA 37, 826–831 (1951)
[26] Toader, G.: Some generalizations of the convexity. In: Proceedings of the Colloquium on Approximation and Optimization, pp. 329–338. Univ. Cluj-Napoca, Cluj-Napoca (1985)
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Received: June 20, 2022
Revised: January 22, 2023
Accepted: January 25, 2023