Metropolis Walks on Dynamic Graphs

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Abstract

Recently, random walks on dynamic graphs have been studied because of its adaptivity to dynamical settings including real network analysis. However, previous works showed a tremendous gap between static and dynamic networks for the cover time of a lazy simple random walk: Although $O(n^3)$ cover time was shown for any static graphs of $n$ vertices, there is an edge-changing dynamic graph with an exponential cover time.

We study a lazy Metropolis walk of Nonaka, Ono, Sadakane, and Yamashita (2010), which is a weighted random walk using local degree information. We show that this walk is robust to an edge-changing in dynamic networks: For any connected edge-changing graphs of $n$ vertices, the lazy Metropolis walk has the $O(n^2)$ hitting time, the $O(n^2 \log n)$ cover time, and the $O(n^2)$ coalescing time, while those times can be exponential for lazy simple random walks. All of these bounds are tight up to a constant factor. At the heart of the proof, we give upper bounds of those times for any reversible random walks with a time-homogeneous stationary distribution.

Keywords: Random walk, Markov chain, dynamic graph

1 Introduction

A random walk is a fundamental stochastic process on an undirected graph. A walker starts from a specific vertex of a graph. At each discrete time step, a walker moves to a random neighbor. Because of their locality, simplicity, and low memory overhead, random walks have a wide range of applications including network analysis and distributed algorithms [11, 19]. Since real-world networks change their structure over time, there is a growing interest in the behavior of a random walk on a dynamic network [5, 33, 26, 10, 13, 25].

We study the power of local degree information on exploring dynamic networks. Specifically, we consider the lazy Metropolis walk [30] on a sequence $(G_t)_{t \geq 1}$ of graphs. The transition matrix $P_{LM} = P_{LM}(G)$ of a lazy Metropolis walk on a graph $G$ is given by

$$P_{LM}(u, v) = \begin{cases} \frac{1}{2 \max\{\deg(u), \deg(v)\}} & \text{(if } \{u, v\} \in E(G)\text{)}, \\ \frac{1}{1 - \sum_{w \in N(u)} \frac{1}{2 \max\{\deg(u), \deg(w)\}}} & \text{(if } u = v\text{)}, \\ 0 & \text{(otherwise)}, \end{cases}$$

(1)

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where $N(u)$ is the set of neighbors of $u$ and $\deg(u) = |N(u)|$ is the degree of $u$. Assuming the graphs $(G_t)_{t \geq 1}$ have the same vertex set (i.e., $V(G_t) = V$ for all $t \geq 1$), we obtain tight upper bounds on the mixing, hitting, cover, and coalescing times of the lazy Metropolis walk. Interestingly, these worst-case bounds are the same as known bounds on static graphs up to a constant factor. Moreover, our results exhibit the power of the lazy Metropolis walk since there is a graph sequence such that the hitting and meeting times are exponentially large for the lazy simple random walk.

### 1.1 Model

Let $V$ be a set of $n$ vertices and $P \in [0, 1]^{V \times V}$ be a transition matrix on $V$, i.e., $\sum_{v \in V} P(u, v) = 1$ holds for all $u \in V$. A random walk according to $P$ is the sequence $(X_t)_{t \geq 0}$ of random variables satisfying $\Pr[X_t = v_t | X_0 = v_0, \ldots, X_{t-1} = v_{t-1}] = \Pr[X_t = v_t | X_{t-1} = v_{t-1}] = P(v_{t-1}, v_t)$ for any $t \geq 1$ and $(v_0, \ldots, v_t) \in V^{t+1}$. A stationary distribution of $P$ is a probability distribution $\pi \in (0, 1)^V$ satisfying $\pi P = \pi$. Let $\pi_{\min} := \min_{v \in V} \pi(v)$.

Let $\mathcal{P} = (P_t)_{t \geq 1}$ be a sequence of transition matrices on $V$. In this paper, we consider a random walk according to $\mathcal{P}$: A sequence of random variables $(X_t)_{t \geq 0}$ such that $\Pr[X_t = v_t | X_0 = v_0, \ldots, X_{t-1} = v_{t-1}] = \Pr[X_t = v_t | X_{t-1} = v_{t-1}] = P_t(v_{t-1}, v_t)$ holds for any $t \geq 1$ and $(v_0, \ldots, v_t) \in V^{t+1}$. In other words, we consider the case that the transition matrix changes over time on a static vertex set $V$.

The simple random walk on $G$ is the random walk according to $P_S = P_S(G)$, where $P_S(u, v) := 1_{\{u, v\} \in E} / \deg(u)$ for all $u, v \in V$. $1_Z$ denotes the indicator of $Z$. The lazy simple random walk\(^1\) on $G$ is the random walk according to $P_{LS}(G) := (P_S(G) + I)/2$. The lazy Metropolis walk\(^2\) on $G$ is the random walk according to $P_{LM}(G)$ of (1). Note that the lazy Metropolis walk is equivalent to the lazy simple random walk on a regular graph. The simple random walk on a sequence of graphs $(G_t)_{t \geq 1}$ is the random walk according to\(^3\) $(P_S(G_t))_{t \geq 1}$.

In this paper, we investigate the mixing, hitting, cover, and coalescing times of random walks according to $\mathcal{P} = (P_t)_{t \geq 1}$, denoted by $t^{(\pi)}_{\text{mix}}(\mathcal{P})$, $t_{\text{hit}}(\mathcal{P})$, $t_{\text{cov}}(\mathcal{P})$, and $t_{\text{coal}}(\mathcal{P})$, respectively. The mixing time to $\pi$, denoted by $t^{(\pi)}_{\text{mix}}(\mathcal{P})$, is the minimum time $t \geq 0$ such that the total variation distance between the distribution of the random walk starting from the worst vertex and $\pi$ is at most $1/4$. The hitting time is the maximum expected time for the random walk to visit a specific vertex, where the maximum is taken over the starting and goal vertices. The cover time is the expected time for the random walk to visit all vertices starting from the worst vertex. Note that the hitting time is at most the cover time. We also consider the cover time $t^{(k)}_{\text{cov}}(\mathcal{P})$ of $k$ independent random walks. In the coalescing random walks, $n$ walkers perform independent random walks. Once two or more walkers meet at the same vertex, they merge into one walker. The coalescing time

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\(^1\)The laziness does not change the order of the hitting and cover times, i.e., $t_{\text{hit}}(P_{LS}(G)) = \Theta(t_{\text{hit}}(P_S(G)))$ and $t_{\text{cov}}(P_{LS}(G)) = \Theta(t_{\text{cov}}(P_S(G)))$ for any connected $G$. On the other hand, on connected bipartite graphs, the mixing and coalescing times of the simple random walk is unbounded, while these are bounded for the lazy simple random walk. Hence, we assume the laziness in many cases.

\(^2\)The Metropolis walk on $G$ is the random walk according to $P_M(G) = 2P_{LM}(G) - I$.

\(^3\)Similarly, the Metropolis walk on $(G_t)_{t \geq 1}$ is the random walk according to $(P_M(G_t))_{t \geq 1}$.
Figure 1: The Sisyphus wheel of five vertices. The Sisyphus wheel $G = (G_t)_{t \geq 1}$ is defined as follows: For each $t \geq 1$, let $V = V(G_t) = \{0, \ldots, n - 1\}$, $v(t) = t \mod (n - 1)$, and $E(G_t) = \{\{v(t), i\} : i \in V \setminus \{v(t)\}\}$. Note that the lazy simple random walk starting from the vertex 0 of $G_1$ has to choose the self loop for $\Omega(n)$ consecutive times in order to reach the vertex $n - 1$. Note that the hitting time of the simple random walk on the Sisyphus wheel is unbounded.

$t_{\text{coal}}(P)$ is the expected time for the walkers to merge into one walker. We abbreviate $P$ from these definitions (e.g., $t_{\text{hit}}$ for $t_{\text{hit}}(P)$) if it is clear from the context. See Section 2.2 for formal definitions.

1.2 Previous work

For the simple random walk on any (static) connected graph $G$, it is known that the cover time is $O(n^3)$ [3, 17]. On the other hand, Avin, Kouský and Lotker [6] presented the Sisyphus wheel that is a sequence $(G_t)_{t \geq 1}$ of graphs on which the hitting time is $2^{\Omega(n)}$ for the lazy simple random walk (see Figure 1).

To avoid the issue of an exponential hitting time, they considered the $d_{\text{max}}$-lazy random walk on $G$, where the transition matrix $P = P(G)$ is defined by $P(u, v) = \frac{1}{2d_{\text{max}}}$ if $\{u, v\} \in E(G)$, $P(u, v) = 0$ if $\{u, v\} \notin E(G)$, and $P(u, u) = 1 - \frac{\deg(u)}{2d_{\text{max}}}$. Here, $d_{\text{max}} = d_{\text{max}}(G) = \max_{v \in V(G)} \deg(v)$. They showed that the cover time of this random walk on any sequence of connected graphs is $O(d_*n^3 \log^2 n)$, where $d_* := \max_{t \geq 1} d_{\text{max}}(G_t)$. They also showed that the mixing time to the uniform distribution is $O(d_*n^2 \log n)$ for this walk. Note that the stationary distribution of this walk is the uniform distribution.

For some restricted graph sequences, tight upper bounds were obtained in the work of Sauerwald and Zanetti [33]. For example, consider the lazy simple random walk on a sequence of regular and connected graphs. They showed that both the mixing time to the uniform distribution and the hitting time are $O(n^2)$. Interestingly, these bounds match bounds on static regular graphs [23]. Note that the stationary distribution of $P_{\text{LS}}(G)$ is the uniform distribution for any connected regular graph $G$. They also considered the lazy simple random walk on a sequence $(G_t)_{t \geq 1}$ of connected graphs, where all $P_{\text{LS}}(G_t)$ have the same stationary distribution $\pi$, i.e., the degree distribution does not change over time. They showed that the mixing time to $\pi$ is $O(n/\pi_{\text{min}})$ and the hitting time is $O((n/\pi_{\text{min}}) \log n)$. 
It is known that local degree information provides surprising power with random walks on static connected graphs. Ikeda, Kubo, Okumoto, and Yamashita [21, 22] proposed the $\beta$-random walk on a graph $G$, where the transition matrix $P = P(G)$ is given by $P(u, v) = 1_{\{u, v\} \in E(G)} \cdot \frac{\deg(v)^{-1/2}}{\sum_{w \in N(u)} \deg(w)^{-1/2}}$ for $u, v \in V(G)$. They showed that the hitting time is $O(n^2)$ and the cover times is $O(n^2 \log n)$. Note that there is a graph (the lollipop graph) on which the hitting time is $\Omega(n^3)$ for the simple random walk [8]. Nonaka, Ono, Sadakane, and Yamashita [30] showed proved the $O(n^2)$ hitting time and the $O(n^2 \log n)$ cover time for the Metropolis walk. Abdullah, Cooper, and Draief [1] introduced the minimum-degree random walk (they call it the minimum-degree scheme) on a graph $G$, where the transition matrix $P = P(G)$ is given by $P(u, v) := 1_{\{u, v\} \in E(G)} \cdot \frac{\min\{\deg(u), \deg(v)\}^{-1}}{\sum_{w \in N(u)} \min\{\deg(u), \deg(w)\}^{-1}}$ for any $u, v \in V(G)$. They showed that the hitting time is $O(n^2)$. David and Feige [14] showed that the cover time of this walk is also $O(n^2)$. This $O(n^2)$ cover time is best possible since any random walk on the path has $\Omega(n^2)$ cover time [22].

It is known that the coalescing time is $O(n^3)$ for a lazy simple random walk on any (static) connected graphs [24, 32]. Berenbrink, Giakkoupis, Kermarrec, and Mallmann-Trenn [7] studied the coalescing time of a lazy simple random walk on a connected $(G_t)_{t \geq 1}$ with a time-homogeneous degree distribution in terms of the edge-expansion of $G_t$. For example, on a sequence of regular expander graphs, they showed that walkers coalesce within $O(n)$ steps with probability at least 1/2.

### 1.3 Main Results

Although the $d_{\text{max}}$-lazy random walk has a polynomial cover time for any connected graph sequence $(G_t)_{t \geq 1}$, there remain two issues. First, as mentioned in [6], the $d_{\text{max}}$-lazy random walk requires knowledge of the maximum degree, i.e., a global parameter of $G_t$, at each $t \geq 1$. Second, the known upper bound $O(d_n n^3 \log^2 n)$ of the cover time of $d_{\text{max}}$-lazy random walk is far from the lower bound $\Omega(n^2)$ by [22] (on a static path, any random walk has $\Omega(n^2)$ cover time). We overcome these issues by considering the Metropolis walk.

**Theorem 1.1** (Lazy Metropolis walk on dynamic graphs). Let $(G_t)_{t \geq 1}$ be a sequence of graphs and $P = (P_{\text{LM}}(G_t))_{t \geq 1}$, where $P_{\text{LM}}(G)$ is defined in (1). Suppose that $G_t = (V, E_t)$ is connected at least once in every $C$ steps for some positive constant $C$. Then, we have the following:

(i) $t_{\text{mix}}^{(\pi)}(P) = O(n^2)$ for the uniform distribution $\pi$.

(ii) $t_{\text{hit}}(P) = O(n^2)$.

(iii) $t_{\text{cov}}^{(k)}(P) = O(n^2 + (n^2 \log n)/k)$ for any $k \geq 1$. In particular, $t_{\text{cov}}(P) = O(n^2 \log n)$.

(iv) $t_{\text{coal}}(P) = O(n^2)$.

**Remark 1.2.** Upper bounds of Theorem 1.1(i) to (iv) are tight: On the (static) cycle graph, the lazy Metropolis walk has $t_{\text{mix}}^{(\pi)}(P) = \Omega(n^2)$, $t_{\text{hit}}(P) = \Omega(n^2)$ and $t_{\text{coal}}(P) = \Omega(n^2)$ [2, 24]. In [30], it was shown that there is an (static) example (the glitter star graph) on which the cover time of the Metropolis walk is $\Omega(n^2 \log n)$.
Remark 1.3. Sauerwald and Zanetti [33] obtained upper bounds of the mixing and hitting times of a lazy simple random walk under the assumption of a time-homogeneous stationary distribution. In Theorem 1.1, we further explore $t_{\text{cov}}^{(k)}$ and $t_{\text{coal}}$ using our new technical results Lemmas 4.1 and 4.2, which hold for any reversible random walk with a time-homogeneous stationary distribution. General results are in Section 2.2.

As noted in Section 1.2, the minimum degree random walk of [1, 14] has a faster cover time than the Metropolis walk by an $O(\log n)$ factor on a static graph. However, this is not the case for edge-dynamic graphs: The minimum degree random walk has an exponential cover time on the Sisyphus wheel (Figure 1). Similarly, the Metropolis walk can have an exponentially faster coalescing time than the minimum-degree random walk and other walks. Consider the meeting time $t_{\text{meet}}$, which is the expected time for two independent random walks taken to meet at the same time from worst initial positions. Note that $t_{\text{coal}} \geq t_{\text{meet}}$.

Proposition 1.4. There is a sequence $(G_t)_{t \geq 1}$ on which the lazy minimum degree walk, lazy $\beta$-random walk, and lazy simple random walk have $t_{\text{meet}} = 2^{\Omega(n)}$. Consequently, for this sequence, these random walks satisfy $t_{\text{coal}} \geq t_{\text{meet}} = 2^{\Omega(n)}$.

For graph sequences with good expansions, we give the following bound.

Theorem 1.5 (Lazy Metropolis walk on dynamic expanders). Let $(G_t)_{t \geq 1}$ be a sequence of graphs and $P = (P_{LM}(G_t))_{t \geq 1}$, where $P_{LM}(G)$ is defined in (1). Suppose that $G_t = (V, E_t)$ is a connected graph with $\frac{1}{1 - \lambda_2(P_{LM}(G_t))} \leq C'$ at least once in every $C$ steps for some positive constants $C$ and $C'$. Here, $\lambda_2(P)$ denotes the second largest eigenvalue of $P$. Then, we have the following:

(i) $t_{\text{mix}}^{(\pi)}(P) = O(\log n)$ for the uniform distribution $\pi$.

(ii) $t_{\text{hit}}(P) = O(n)$.

(iii) $t_{\text{cov}}^{(k)}(P) = O(\log n + (n \log n)/k)$ for any $k \geq 1$. In particular, $t_{\text{cov}}(P) = O(n \log n)$.

(iv) $t_{\text{coal}}(P) = O(n)$.

Remark 1.6. These bounds are tight: On any (static) constant degree regular graph, the lazy Metropolis walk has $t_{\text{mix}}^{(\pi)}(P) = \Omega(\log n)$. On the (static) complete graph, $t_{\text{hit}}(P) = \Omega(n)$, $t_{\text{cov}}(P) = \Omega(n \log n)$, and $t_{\text{coal}}(P) = \Omega(n)$ [27, 12].

Remark 1.7. Let $G = (V, E)$ be a connected graph. If $\alpha := \frac{d_{\text{max}}}{d_{\text{min}}} \leq 4$ and $\lambda_2(P_S(G)) \leq 2 - \sqrt{\frac{\alpha}{\alpha} - \epsilon}$ for some constant $\epsilon > 0$, then $\frac{1}{1 - \lambda_2(P_{LM}(G))} = O(1)$ (see Lemma B.6 for the proof).

Theorems 1.1 and 1.5 imply the following bounds of lazy simple random walks.

Corollary 1.8 (Lazy simple random walk on dynamic regular graphs). Let $P = (P_{LS}(G_t))_{t \geq 1}$ for a sequence of graphs $(G_t)_{t \geq 1}$. Let $\pi$ be the uniform distribution. Then, we have the following:
(i) If $G_t = (V, E_t)$ is connected and regular for all $t \geq 1$, then $t_{\text{mix}}^{(\pi)}(\mathcal{P}) = O(n^2)$, $t_{\text{hit}}(\mathcal{P}) = O(n^2)$, $t_{\text{cov}}^{(k)}(\mathcal{P}) = O(n^2 + (n^2 \log n/k))$ for any $k \geq 1$, and $t_{\text{coal}}(\mathcal{P}) = O(n^2)$ hold.

(ii) If $G_t = (V, E_t)$ is a connected and regular graph with $\frac{1}{1-\lambda_2(P)} = O(1)$ for all $t \geq 1$, then $t_{\text{mix}}^{(\pi)}(\mathcal{P}) = O(n^2)$, $t_{\text{hit}}(\mathcal{P}) = O(n)$, $t_{\text{cov}}^{(k)}(\mathcal{P}) = O(nk + (n \log n/k))$ for any $k \geq 1$, and $t_{\text{coal}}(\mathcal{P}) = O(n)$ hold. Here, $\lambda_2(\mathcal{P})$ is the second largest eigenvalue of $\mathcal{P}$.

**Remark 1.9.** Except for $t_{\text{cov}}(\mathcal{P}) = O(n^2 \log n)$ in (i), upper bounds in Corollary 1.8 are tight (see Remarks 1.2 and 1.6). It is open whether $t_{\text{cov}}(\mathcal{P}) = O(n^2 \log n)$ is tight or not.

### 1.4 Related work

Cai, Sauerwald, and Zanetti [10] considered the lazy simple random walk on a sequence of edge-Markovian random graphs. They introduced the notion of mixing time on this sequence (note that the stationary distribution changes over time) and obtained several bounds of the mixing time. Lamprou, Martin, and Spirakis [26] study the cover time of the simple random walk on a variant of the edge-Markovian random graphs.

The hitting and cover times on static graphs have been extensively studied for several decades. Consider the simple random walk on connected graph $G$ of $n$ vertices and $m$ edges. Aleliunas, Karp, Lipton, Lovász, and Rackoff [3] showed that the cover time is at most $2m(n-1)$. Later, Kahn, Linial, Nisan, and Saks [23] showed that the cover time is at most by $16mn/d_{\text{min}}$. Hence, the cover time is $O(n^2)$ if $G$ is regular. It is known that the hitting time is at least $(1/2)m/d_{\text{min}}$ (see Corollary 3.3 of Lovász [28]). Brightwell and Winkler [8] presented a graph called the lollipop graph on which the hitting time is approximately $(4/27)n^3$ as $n$ increases. In addition to the trivial relation of $t_{\text{hit}} \leq t_{\text{cov}}$, it is known that $t_{\text{cov}} \leq t_{\text{hit}} \log n$ holds for any $G$ (see Matthews [29]).

The cover time $t_{\text{cov}}^{(k)}$ of $k$ independent simple random walks (on static graphs) has been investigated in [9, 4, 16]. If $k$ walkers start from the stationary distribution, Broder, Karlin, Raghavan, and Upfal [9] showed that the cover time is at most $O\left(\left(\frac{m}{k} \right)^2 \log^3 n \right)$. From any initial positions of $k$ walkers, Alon, Avin, Koucky, Kozma, Lotker, and Tuttle [4] showed that $t_{\text{cov}}^{(k)} = O\left(\frac{t_{\text{hit}} \log n}{k}\right)$ holds if $k = O(\log n)$. For a larger $k$, Elsässer and Sauerwald [16] showed that $t_{\text{cov}}^{(k)} = O\left(\frac{t_{\text{mix}} + t_{\text{hit}} \log n}{k}\right)$ for $k(\leq n)$ lazy simple random walks.

The meeting time and the coalescing time have been investigated well in the context of distributed computation such as leader election and consensus protocols [19]. Consider the simple random walk on a connected and nonbipartite $G$. Tetali and Winkler showed that the meeting time is $(16/27 + o(1))n^3$. Hassin and Peleg [19] showed that $t_{\text{coal}} \leq t_{\text{meet}} \log n$ holds, while $t_{\text{meet}} \leq t_{\text{coal}}$ is trivial. Recent works on the meeting and coalescing times consider the lazy simple random walk on a connected graph $G$ [12, 7, 24, 32]. For example, Kanade, Mallmann-Trenn, and Sauerwald [24] showed $t_{\text{coal}} = O\left(t_{\text{meet}} \left(1 + \sqrt{\frac{t_{\text{mix}}}{t_{\text{meet}}} \log n}\right)\right)$. Recently, Oliveira and Peres [32] proved $t_{\text{coal}} = O(t_{\text{hit}})$. 


Preliminaries and general results

2.1 Notations and definitions

Reversible Markov chain. A matrix $M \in \mathbb{R}^{V \times V}$ is reversible with respect to a vector $\nu \in \mathbb{R}^{V}$ if $\nu(u)M(u,v) = \nu(v)M(v,u)$ holds for any $u,v \in V$. Let $\rho(M)$ be the spectral radius of $M$. If $M$ is reversible with respect to a positive vector, all eigenvalues of $M$ are real numbers (Lemma 12.2 [27]). Let $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$ be the eigenvalues of $M$.

Let $P \in [0,1]^{V \times V}$ be a transition matrix. Note that if $P$ is reversible with respect to a probability distribution $\pi$, $\pi$ is a stationary distribution of $P$ (see, e.g., Proposition 1.20 in [27]). A transition matrix $P$ is irreducible if for any $u,v \in V$ there exists a $t > 0$ such that $P^t(u,v) > 0$ holds. A transition matrix $P$ is aperiodic if for any $v \in V \gcd\{t \geq 0 : P^t(v,v) > 0\} = 1$ holds\footnote{If $P$ is irreducible, there is a unique probability distribution $\pi \in (0,1)^V$ satisfying $\pi P = \pi$. If $P$ is irreducible and aperiodic, $\lim_{t \to \infty} \mu^t P = \pi$ holds for any probability distribution $\mu \in [0,1]^V$ (see, e.g., Corollary 1.17 and Theorem 4.9 in [27]).}. A transition matrix $P$ is lazy if $P(v,v) \geq 1/2$ holds for any $v \in V$. For $P$ which is reversible with respect to $\pi \in (0,1)^V$, let $\lambda_*(P) := \max\{|\lambda_2(P)|,|\lambda_n(P)|\}$ denote the second largest eigenvalue in the absolute value\footnote{If $P$ is a transition matrix, $\rho(P) = 1$ and $\lambda_1(P) = 1$. If $P$ is irreducible, $\lambda_2(P) < 1$. If $P$ is irreducible and aperiodic, $\lambda_*(P) > -1$. If $P$ is lazy, $\lambda_n(P) \geq 0$ (see, e.g., Chapter 12 in [27]).}.

Suppose that $P$ is irreducible and reversible with respect to a probability vector $\pi \in (0,1)^V$. For a pair of vertices $x,y \in V$, we call a sequence $\Gamma = ((v_0,v_1), (v_1,v_2), \ldots, (v_{t-1},v_t))$ a $(P,(x,y))$-path if $v_0 = x, v_t = y$, $P(v_i,v_{i+1}) > 0$ for all $0 \leq i < t$, and $v_i \neq v_j$ for all $0 \leq i < j \leq t$. Let ${{\mathcal I}}_{x,y}$ be the set of all $(P,(x,y))$-paths. Then, for a $(P,(x,y))$-path $\Gamma$, let

$$K_\Gamma(P) := \frac{1}{\pi(u)P(u,v)} \quad \text{and} \quad K(P) := \max_{x,y \in V} \min_{\Gamma \in {{\mathcal I}}_{x,y}} K_\Gamma(P).$$

It is known that $t_{\text{hit}}(P) \leq K(P)$ holds for any irreducible and reversible $P$ (see, e.g., Corollary 3.8 in [2]). Indeed, some well-known upper bounds of the hitting times are given by bounding $K(P)$. For example, $K(P_{LS}(G)) = O(n^2)$ for any connected regular $G$ (see, e.g., Proposition 10.16 in [27]) and $K(P_{LM}(G)) = O(n^2)$ for any connected $G$ (Theorem 4 in [30]).

For a vector $\nu \in \mathbb{R}_+^V$ denote the inner product $\langle \cdot , \cdot \rangle_\nu$ as $\langle f,g \rangle_\nu = \sum_{v \in V} f(v)g(v)\nu(v)$ for $f,g \in \mathbb{R}^V$. For an integer $p \geq 1$ and $f \in \mathbb{R}^V$, let $\|f\|_{p,\nu} = (\sum_{v \in V} \nu(v)|f(v)|^p)^{1/p}$. Let $\mathbb{1}$ denote the $n$-dimensional all-one vector. For $f \in \mathbb{R}^V$ and a positive vector $g \in \mathbb{R}_{\geq 0}^V$, define $\frac{f}{g} \in \mathbb{R}^V$ by $\left(\frac{f}{g}\right)(v) := \frac{f(v)}{g(v)}$ for all $v \in V$.
Transition matrix sequence. Let $\mathcal{P} = (P_t)_{t \geq 1}$ be a sequence of transition matrices. For a property $\mathcal{H}$ of a transition matrix (e.g., being irreducible), a sequence $\mathcal{P}$ satisfies $\mathcal{H}$ if all $P_t$ satisfy $\mathcal{H}$. For example, $\mathcal{P}$ is reversible with respect to $\pi$ if $\pi(u)P_t(u, v) = \pi(v)P_t(v, u)$ holds for any $u, v \in V$ and any $t \geq 1$. Note that if $\mathcal{P}$ is reversible with respect to $\pi$, $\pi$ is a time-homogeneous stationary distribution for $\mathcal{P}$, i.e., $\pi P_t = \pi$ for any $t \geq 1$. A simple but important observation is that, for any graph sequence $(G_t)_{t \geq 1}$, $(P_{LM}(G_t))_{t \geq 1}$ is reversible with respect to the uniform distribution since each $P_{LM}(G_t)$ is a symmetric matrix. For a strictly increasing function $t : \mathbb{N} \to \mathbb{N}$, consider $(P_{t(i)})_{i \geq 1}$ to be a subsequence of $\mathcal{P} = (P_t)_{t \geq 1}$. We say that the subsequence $(P_{t(i)})_{i \geq 1}$ has an interval at most $C$ if $t(i+1) - t(i) \leq C$ holds for any $i \geq 0$ (here, we set $t(0) = 0$). We also use the term “$\mathcal{P}'$ satisfies $\mathcal{H}$” for a subsequence $\mathcal{P}'$. For a pair of integers $a, b \in \mathbb{Z}$, let $[a, b] := \{ z \in \mathbb{Z} : a \leq z \leq b \}$. Write $[b] := [1, b]$ for the abbreviation. For a sequence of matrices $(P_t)_{t \geq 1}$, let $P_{[a,b]} := P_a P_{a+1} \cdots P_b$. Finally, define

$$H_{\text{max}}(\mathcal{P}) := \max_{t \geq 1} t_{\text{hit}}(P_t), \quad \Lambda_{\text{max}}(\mathcal{P}) := \max_{t \geq 1} \lambda_*(P_t), \quad K_{\text{max}}(\mathcal{P}) := \max_{t \geq 1} K(P_t).$$

Here, for a transition matrix $P$, $t_{\text{hit}}(P)$ denotes the hitting time of the random walk according to $P$. We also consider $H_{\text{max}}(\mathcal{P}')$, $\Lambda_{\text{max}}(\mathcal{P}')$, and $K_{\text{max}}(\mathcal{P}')$ for a subsequence $\mathcal{P}' = (P_{t(i)})_{i \geq 1}$ in a similar way, e.g., $H_{\text{max}}(\mathcal{P}') = \max_{i \geq 1} t_{\text{hit}}(P_{t(i)})$.

### 2.2 General results

Now, we introduce the following results on the mixing, hitting, cover and coalescing times for reversible random walks with a time-homogeneous stationary distribution.

**Mixing time.** For an integer $p \geq 1$, a probability vector $\mu \in [0,1]^V$ and a probability vector $\pi \in (0,1]^V$, let $d^{(p, \pi)}(\mu) := \| \mu - 1 \|_{p,\pi} = \left( \sum_{v \in V} \pi(v) \left| \frac{\mu(v)}{\pi(v)} - 1 \right|^p \right)^{1/p}$ be the $\ell^p$-distance between $\mu/\pi$ and $1$. It is known that $d^{(p, \pi)}(\mu) \leq d^{(p+1, \pi)}(\mu)$ holds for any $p \geq 1$ (see, e.g., Section 4.7 in [27]). For example, $\sum_{v \in V} \mu(v) - \pi(v) = d^{(1, \pi)}(\mu) \leq d^{(2, \pi)}(\mu)$. For a sequence of transition matrices $\mathcal{P} = (P_t)_{t \geq 1}$, a probability vector $\pi \in (0,1]^V$, and $\epsilon > 0$, we define the $\ell^2$-mixing time as $t_{\text{mix}}^{(p, \pi)}(\mathcal{P}, \epsilon) := \min \left\{ t \geq 0 : \max_{v \in V} d^{(p, \pi)}(P_{[1,t]}(v, \cdot)) \leq \epsilon \right\}$. Specifically, let $t_{\text{mix}}^{(\pi)}(\mathcal{P}) := t_{\text{mix}}^{(\pi)}(\mathcal{P}, 1/2)$ and $t_{\text{mix}}^{(\pi)}(\mathcal{P}) := t_{\text{mix}}^{(\pi)}(\mathcal{P}, 1/2)$. Henceforth, we use the following parameter for convenience: For any $\mathcal{P}$, $\pi \in (0,1]^V$ and $\epsilon > 0$, let

$$t_{\text{mix}}^{(\pi)}(\mathcal{P}, \epsilon) := \min \left\{ \frac{8K_{\text{max}}(\mathcal{P})}{\lambda_*^2(\mathcal{P})}, \frac{3 \log(\pi_{\text{min}})}{1 - \Lambda_{\text{max}}(\mathcal{P})}, \frac{\log(\epsilon^{-1})}{1 - \Lambda_{\text{max}}(\mathcal{P})} \right\} + \frac{\log(\pi_{\text{min}})}{1 - \Lambda_{\text{max}}(\mathcal{P})}$$

(2)

and $t_{\text{mix}}^{(\pi)}(\mathcal{P}) := t_{\text{mix}}^{(\pi)}(\mathcal{P}, 1/2)$. We show that $t_{\text{mix}}^{(\pi)}(\mathcal{P}, \epsilon)$ is an upper bound of the $\ell^2$-mixing time.

**Theorem 2.1.** Let $\mathcal{P} = (P_t)_{t \geq 1}$ be a sequence of transition matrices on $V$. Suppose that $\mathcal{P}$ is reversible with respect to a probability vector $\pi \in (0,1]^V$ and contains an irreducible and lazy subsequence $\mathcal{P}' = (P_{t(i)})_{i \geq 1}$ having an interval at most $C$. Then, for any $0 < \epsilon \leq 1$, $t_{\text{mix}}^{(2, \pi)}(\mathcal{P}, \epsilon) \leq C t_{\text{mix}}^{(\pi)}(\mathcal{P}, \epsilon)$.\(^6\)

---

\(^6\)From this definition, $t_{\text{mix}}^{(\pi)}(\mathcal{P}) = \min \{ t \geq 0 : (1/2) \max_{v \in V} \sum_{w \in V} |P_{[1,t]}(v, w) - \pi(w)| \leq 1/4 \}$. 

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For time-homogeneous transition matrices \((P_t = P\) for some \(P\)) the bound of Theorem 2.1 matches known bounds in many cases. For example, \(t_m^*(\mathcal{P}, \epsilon)\) matches a well-known bound of \(\frac{\log(\sigma_{\text{min}}(P)\epsilon^{-1})}{1-\lambda(P)}\) (see, e.g., Theorem 12.4 in [27]).

Kanonische Zeit. Consider the random walk \((X_t)_{t \geq 0}\) according to \(\mathcal{P} = (P_t)_{t \geq 1}\). For a vertex \(w \in V\), let \(\tau_w(\mathcal{P}) = \min\{t \geq 0 : X_t = w\}\) be the first time for the random walk to reach \(w\). Then, the hitting time is defined by \(t_{\text{hit}}(\mathcal{P}) := \max_{v,w \in V} E[\tau_w(\mathcal{P}) | X_0 = v]\).

**Theorem 2.2.** Let \(\mathcal{P} = (P_t)_{t \geq 1}\) be a sequence of transition matrices on \(V\). Suppose that \(\mathcal{P}\) is reversibile with respect to a positive probability vector \(\pi \in (0,1]^V\) and contains an irreducible subsequence \(\mathcal{P}' = (P_t(i))_{i \geq 1}\) having an interval at most \(C\). Then,

\[
t_{\text{hit}}(\mathcal{P}) \leq t_{\text{mix}}^*(\mathcal{P}) + 1 + 2CH_{\text{max}}(\mathcal{P}').
\]

Roughly speaking, Theorem 2.2 gives an upper bound of the hitting time of the random walk according to \(\mathcal{P}\) in terms of \(H_{\text{max}}(\mathcal{P}) = \max_{i \geq 1} t_{\text{hit}}(P_t(i))\). Note that \(t_{\text{hit}}(P_t(i))\) is the hitting time of the random walk \((X_t)_{t \geq 0}\) according to \((\text{a transition matrix}) P_t(i)\).

Koverzeit. Consider \(k\) independent random walks \((X_t(1))_{t \geq 0}, \ldots, (X_t(k))_{t \geq 0}\), where each walk is according to \(\mathcal{P} = (P_t)_{t \geq 1}\). Let \(\tau^{(k)}_{\text{cov}}(\mathcal{P}) = \min\{t \geq 0 : \bigcup_{x=0}^t \bigcup_{i=1}^k \{X_s(i)\} = V\}\) be the first time for \(k\) walkers to visit all vertices. Then, the cover time of \(k\) random walks is defined by \(t^{(k)}_{\text{cov}}(\mathcal{P}) := \max_{x \in V} E \left[\tau^{(k)}_{\text{cov}}(\mathcal{P}) | X_0 = x\right]\). Here, \(X_t = (X_t(1), \ldots, X_t(k)) \in V^k\) is a vector-valued random variable. In particular, let \(t_{\text{cov}}(\mathcal{P}) = t^{(1)}_{\text{cov}}(\mathcal{P})\). Using an upper bound of the mixing time (2), we give the following bound of the cover time.

**Theorem 2.3.** Let \(\mathcal{P} = (P_t)_{t \geq 1}\) be a sequence of transition matrices. Suppose that \(\mathcal{P}\) is reversibile with respect to a positive probability vector \(\pi \in (0,1]^V\) and contains an irreducible and lazy subsequence \(\mathcal{P}' = (P_t(i))_{i \geq 1}\) having an interval at most \(C\). Then, for any \(k\),

\[
t^{(k)}_{\text{cov}}(\mathcal{P}) \leq 100C \left(2t^{(\pi)}_m(\mathcal{P}') + \left[\frac{8H_{\text{max}}(\mathcal{P}') \log(50n)}{k}\right]\right).
\]

If \(P_t = P\) for some \(P\), Theorem 2.3 gives \(O \left(t^{(\pi)}_m(P) + \frac{t_{\text{hit}}(P) \log n}{k}\right)\), which is the same as \(O \left(t^{(\pi)}_{\text{mix}}(P) + \frac{t_{\text{hit}}(P) \log n}{k}\right)\) of [16] in many cases.

Coaleszenzzeit. Let \((C_t(1))_{t \geq 0}, (C_t(2))_{t \geq 0}, \ldots, (C_t(n))_{t \geq 0}\) denote the coaleszenz random walks according to \(\mathcal{P} = (P_t)_{t \geq 1}\). In the coaleszenz random walks, once a walker meets another walker, they start walking together. Formally, from a given initial state \(C_0 = (C_0(1), \ldots, C_0(n)) \in V^n\), we inductively determine \(C_t(a)\) for each \(t \geq 1\) and \(a \in [n]\), as follows. Suppose that \(C_{t-1} = (C_{t-1}(1), \ldots, C_{t-1}(n))\) and \(C_t(1), \ldots, C_t(a-1)\) are determined. If there is some \(b < a\) such that \(C_{t-1}(a) = C_{t-1}(b)\), let \(C_t(a) := C_t(b)\). Otherwise, \(C_t(a)\) is determined by the random walk according to \(P_t\), i.e., \(P_{C_t} = C_t(a) = u = P_{C_t}(u,v)\) for \(u,v \in V\). For \(x = (x_1, x_2, \ldots, x_n) \in V^n\), let \(S(x) := \bigcup_{i=1}^n \{x_i\}\) (e.g., \(S(x) = \{a,b\}\) for \(x = (a,a,b)\)). Then, let \(\tau_{\text{coal}}(\mathcal{P}) = \min\{t \geq 0 : |S(C_t)| = 1\}\) and the coaleszenz time is defined by \(t_{\text{coal}}(\mathcal{P}) := \max_{v \in V^n} E[\tau_{\text{coal}}(\mathcal{P}) | C_0 = v]\).
Theorem 2.4. Let $\mathcal{P} = (P_t)_{t \geq 1}$ be a sequence of transition matrices. Suppose that $\mathcal{P}$ is reversible with respect to a positive probability vector $\pi \in (0, 1]^V$ and contains an irreducible and lazy subsequence $\mathcal{P}' = (P_{t(i)})_{i \geq 1}$ having an interval at most $C$. Then,

$$t_{\text{coal}}(\mathcal{P}) \leq 50 \left( 2C t_m^{(\pi)}(\mathcal{P}') + 48CH_{\max}(\mathcal{P}') + \log_2(n) \right).$$

If $P_t = P$ holds for all $t \geq 1$, the coalescing time is $O(t_m^{(\pi)}(P) + t_{\text{hit}}(P))$ from Theorem 2.4. In many cases, this matches the bound of $O(t_m^{(\pi)}(P) + t_{\text{hit}}(P))$ in [32].

Proof of Theorem 1.1. Let $(G_{t(i)})_{i \geq 1}$ be a subsequence of $(G_t)_{t \geq 1}$, where each $G_{t(i)}$ is connected. Let $\mathcal{P}' = (P_{LM}(G_{t(i)}))_{i \geq 1}$. From the assumption, we can assume that $\mathcal{P}'$ is irreducible and lazy, and has an interval at most $C$. For any $i \geq 1$, combining Lemmas B.1 and B.2 yields

$$\frac{1}{1 - \lambda_2(P_{LM}(G_{t(i)}))} \leq t_{\text{hit}}(P_{LM}(G_{t(i)})) = O(n^2).$$

Note that $\lambda_*(P_{LM}(G_{t(i)})) = \lambda_2(P_{LM}(G_{t(i)}))$ since $P_{LM}(G_{t(i)})$ is lazy. Consequently, we have

$$\frac{1}{1 - \Lambda_{\max}(\mathcal{P}')} = O(n^2), \quad H_{\max}(\mathcal{P}') = O(n^2) \quad \text{and} \quad K_{\max}(\mathcal{P}') = O(n^2).$$

Putting these bounds into Theorems 2.1 to 2.4, we obtain (i) to (iv) of Theorem 1.1, respectively.

Proof of Theorem 1.5. Let $(G_{t(i)})_{i \geq 1}$ be a subsequence of $(G_t)_{t \geq 1}$, where each $G_{t(i)}$ is connected and satisfies $\frac{1}{1 - \lambda_2(P_{LM}(G_{t(i)}))} \leq C'$. Let $\mathcal{P}' = (P_{LM}(G_{t(i)}))_{i \geq 1}$. From the assumption, we suppose that $\mathcal{P}'$ is an irreducible and lazy subsequence having an interval at most $C$. For any $i \geq 1$, we

$$t_{\text{hit}}(P_{LM}(G_{t(i)})) \leq \frac{1}{\pi_{\min}(1 - \lambda_2(P_{LM}(G_{t(i)})))} \leq 2C'n$$

from Lemma B.1. Consequently, we have

$$\frac{1}{1 - \Lambda_{\max}(\mathcal{P}')} = O(1) \quad \text{and} \quad H_{\max}(\mathcal{P}') = O(n).$$

Putting these bounds into Theorems 2.1 to 2.4, we obtain (i) to (iv) of Theorem 1.5, respectively.

3 Mixing time

3.1 Proof of Theorem 2.1

We show Theorem 2.1 in this section. First, we introduce the following simple bound.

Lemma 3.1. Let $(P_t)_{t=1}^T$ be a sequence of transition matrices on $V$. Suppose that $(P_t)_{t=1}^T$ is reversible with respect to a probability vector $\pi \in (0, 1]^V$ and contains an irreducible and aperiodic subsequence $\mathcal{P}' = (P_{t(i)})_{i=1}^L$. Then, for any probability vector $\mu \in [0, 1]^V$,

$$d^{(2, \pi)}(\mu P_{[1,T]}) \leq d^{(2, \pi)}(\mu) \exp\left(-L(1 - \Lambda_{\max}(\mathcal{P}'))\right).$$

Proof. Applying Lemma A.2 repeatedly to $d^{(2, \pi)}(\mu P_{[1,T]}) = \left\| \frac{\mu P_{[1,T]} - \mu}{\pi} - 1 \right\|_{2, \pi}$, we have

$$\frac{\mu P_{[1,T]} - \mu}{\pi} - 1 \leq \frac{\mu P_{[1,T]-1} - \mu}{\pi} \leq \lambda_*(P_T) \leq \cdots \leq \frac{\mu - 1}{\pi} \leq \prod_{i=1}^T \lambda_*(P_i).$$

Since $P_{t(i)}$ irreducible and aperiodic, $\lambda_*(P_{t(i)}) < 1$ holds for any $i \in [L]$. Hence,

$$\prod_{i=1}^L \lambda_*(P_{t(i)}) \leq \prod_{i=1}^L (1 - (1 - \lambda_*(P_{t(i)}))) = \exp\left(-\sum_{i\in[L]} (1 - \lambda_*(P_{t(i)}))\right)$$

holds and we obtain the claim. Note that $\lambda_*(P_t) \leq 1$ holds for any $t \in [T]$.
Next, we introduce the following lemma connecting the $\ell^2$-distance and $K(P)$. For $f \in \mathbb{R}^V$, let $E_\pi(f) := \sum_{v \in V} \pi(v) f(v) = \langle f, 1 \rangle_\pi$ and $\text{Var}_\pi(f) := \sum_{v \in V} \pi(v) (f(v) - E_\pi(f))^2 = \langle f, f \rangle_\pi - \langle f, 1 \rangle_\pi^2$. For $f, g \in \mathbb{R}^V$ and a transition matrix $P$ which is reversible with respect to a probability vector $\pi \in (0, 1)^V$, let $E_{P,\pi}(f, g) := \frac{1}{2} \sum_{u,v \in V} \pi(u) P(u,v) (f(u) - g(v))^2 = \frac{1}{2} (\langle f, P \pi \rangle + \langle g, P \pi \rangle - \langle f, P g \pi \rangle)$. Note that, for any probability vector $\mu \in [0, 1]^V$, we have

$$d^{(2,\pi)}(\mu)^2 = \sum_{v \in V} \pi(v) \left( \frac{\mu(v)}{\pi(v)} - 1 \right)^2 = \sum_{v \in V} \pi(v) \left( \frac{\mu(v)}{\pi(v)} \right)^2 - 1 = \text{Var}_\pi \left( \frac{\mu}{\pi} \right). \quad (3)$$

**Lemma 3.2.** Let $P \in [0, 1]^{V \times V}$ be a transition matrix. Suppose that $P$ is irreducible, lazy and reversible with respect to a positive probability vector $\pi \in (0, 1)^V$. Then, for any probability vector $\mu \in [0, 1]^V$, $\text{Var}_\pi \left( \frac{\mu}{\pi} \right) \leq \frac{\text{Var}_\pi \left( \frac{\mu}{\pi} \right)}{2K(P)} - \text{Var}_\pi \left( \frac{\mu}{\pi} \right)$.

**Proof.** From (20) and Lemma A.3, we have $\text{Var}_\pi \left( \frac{\mu}{\pi} \right) = \text{Var}_\pi \left( \frac{P \mu}{\pi} \right) \leq \text{Var}_\pi \left( \frac{\mu}{\pi} \right) - E_{P,\pi} \left( \frac{\mu}{\pi}, \frac{\mu}{\pi} \right)$. Hence, it suffices to show $E_{P,\pi} \left( \frac{\mu}{\pi}, \frac{\mu}{\pi} \right) \geq \frac{\text{Var}_\pi \left( \frac{\mu}{\pi} \right)}{2K(P)}$. Write $f = \frac{\mu}{\pi}$ for notational convenience. From (3), we have $\text{Var}_\pi(f) = \sum_{v \in V} \pi(v) \left( \frac{\mu(v)}{\pi(v)} \right)^2 - \sum_{v \in V} \pi(v) \left( \frac{\mu(v)}{\pi(v)} \right) \leq f(v_{\max}) - f(v_{\min})$, where $v_{\max}$ and $v_{\min}$ are some vertices that $f(v_{\max}) = \max_{v \in V} f(v)$ and $f(v_{\min}) = \min_{v \in V} f(v)$ hold, respectively. Write $\Gamma_* = ((v_0, v_1), \ldots, (v_{\ell-1}, v_\ell))$ for a $(P, (v_{\max}, v_{\min}))$-path satisfying $K_\Gamma_*(P) = \min_{\ell \in \mathbb{I}_{v_{\max},v_{\min}}} K_\Gamma(P)$. Then, we have

$$E_{P,\pi}(f, f) = \frac{1}{2} \sum_{x, y \in V} \pi(x) P(x, y) (f(x) - f(y))^2 \geq \frac{1}{2} \sum_{i=0}^{\ell-1} \pi(v_i) P(v_i, v_{i+1}) (f(v_i) - f(v_{i+1}))^2 \geq \frac{1}{2} \frac{\left( \sum_{i=0}^{\ell-1} (f(v_i) - f(v_{i+1})) \right)^2}{\sum_{i=0}^{\ell-1} \frac{1}{\pi(v_i) P(v_i, v_{i+1})}} \geq \frac{\text{Var}_\pi(f)^2}{2K(P)}.$$ 

We use the Cauchy-Schwarz inequality in the second inequality.

Applying Lemma 3.2 repeatedly, we obtain the following lemma. The idea of the proof is essentially same as that of Theorem 3.3 in [33].

**Lemma 3.3.** Let $(P_t)_{t=1}^T$ be a sequence of transition matrices on $V$. Suppose that $(P_t)_{t=1}^T$ is reversible with respect to a probability vector $\pi \in (0, 1)^V$ and contains an irreducible and lazy subsequence $\mathcal{P}' = (P_{(l)})_{l=1}^L$. Then, for any probability vector $\mu \in [0, 1]^V$, $d^{(2,\pi)}(\mu P_{[1:T]}) \leq 1$ if $L \geq \left[ 8K_{\max}(\mathcal{P}') \right] + \left[ 3 \log \left( \pi_{\min}^{-1} \right) \right]$.

**Proof.** Write $\epsilon_t = d^{(2,\pi)}(\mu P_{[1:t]})^2 = \text{Var}_\pi(\mu P_{[1:t]}/\pi)$ and $\epsilon_0 = d^{(2,\pi)}(\mu)^2$ for notational convenience. From Lemma A.2, $\epsilon_t$ is non-increasing, i.e., $\epsilon_{t+1} \leq \epsilon_t$ holds for any $t \geq 0$. Let $L_j := \min\{i \geq 0 :
\( \epsilon_t \leq \epsilon_0/2^j \) for \( j \geq 0 \). Then, for any \( t \leq t(L_j - 1) < t(L_j) \), \( \epsilon_t \geq \epsilon_t(L_j - 1) > \epsilon_0/2^j \) holds. Applying Lemma 3.2 yields

\[
\frac{\epsilon_0}{2^j} < \epsilon_t(L_j - 1) \leq \epsilon_t(L_j - 1) - \frac{\epsilon_t(L_j - 1) - \epsilon_t(L_j - 2)}{2K(P_t(L_j - 1))} < \epsilon_t(L_j - 2) - \frac{\epsilon_0}{2^{K_{\max}(P')}2^j}
\]

\[
< \cdots < \epsilon_t(L_j - 1) - (L_j - L_j - 1)\frac{\epsilon_0}{2^{K_{\max}(P')}2^j}
\]

\[
\leq \frac{\epsilon_0}{2^{j-1}} - (L_j - L_j - 1)\frac{\epsilon_0}{2^{K_{\max}(P')}2^j}.
\]

Hence, \( L_j - L_j - 1 \leq 1 + \frac{2K_{\max}(P')\epsilon_0}{\epsilon_0}2^j \) holds for any \( j \geq 1 \). This implies that

\[
L_{[\log_2(\epsilon_0)]} = \sum_{j=1}^{[\log_2(\epsilon_0)]} (L_j - L_j - 1) \leq [\log_2(\epsilon_0)] + \frac{2K_{\max}(P')}{\epsilon_0}2^{[\log_2(\epsilon_0)]} + 1
\]

\[
\leq [\log_2(\epsilon_0)] + 8K_{\max}(P').
\]

Consequently, for any \( T \geq t(L_{[\log_2(\epsilon_0)]}) \), \( d^{(2,\pi)}(\mu P_{[1,T]})^2 = \epsilon_T \leq \epsilon_t(L_{[\log_2(\epsilon_0)]}) \leq 1 \) holds and we obtain the claim. Note that \( \log_2(\epsilon_0) \leq \frac{\log((1/\pi_{\min})^2)}{\log 2} \leq 3 \log(1/\pi_{\min}) \). □

Combining Lemmas 3.1 and 3.3, we obtain the following lemma, which immediately gives Theorem 2.1.

**Lemma 3.4.** Let \( (P_i)_{i=1}^T \) be a sequence of a transition matrices on \( V \). Suppose that \( (P_i)_{i=1}^T \) is reversible with respect to a probability vector \( \pi \in (0,1)^V \) and contains an irreducible and lazy subsequence \( P' = (P_{i(t(i))})_{i=1}^L \). Then, for any probability vector \( \mu \in [0,1]^V \) and \( 0 < \epsilon \leq 1 \), \( d^{(2,\pi)}(\mu P_{[1,T]}) \leq \epsilon \) if \( L \geq t_{m}(P', \epsilon) \), where \( t_{m}(\pi) \) is given by (2).

Proof. Suppose \( L \geq L_1 + L_2 \) for \( L_1 = \min \left\{ \left[ \frac{\log(\pi_{\min}^{-1})}{1-\Lambda_{\max}(P')} \right], \left[ 8K_{\max}(P') \right] + \left[ 3 \log(\pi_{\min}^{-1}) \right] \right\} \) and \( L_2 = \left[ \frac{\log(\epsilon^{-1})}{1-\Lambda_{\max}(P')} \right] \). First, consider the case that \( L_1 = \left[ \frac{\log(\pi_{\min}^{-1})}{1-\Lambda_{\max}(P')} \right] \) holds. Then, we obtain the claim immediately from Lemma 3.1 since \( L \geq L_1 + L_2 \geq \left[ \frac{\log(\pi_{\min}^{-1}\epsilon^{-1})}{1-\Lambda_{\max}(P')} \right] \).

Consider the other case that \( L_1 = \left[ 8K_{\max}(P') \right] + \left[ 3 \log(\pi_{\min}^{-1}) \right] \). From Lemma 3.1,

\[
d^{(2,\pi)}(\mu P_{[1,T]}) = d^{(2,\pi)}(\mu P_{[1,t(L_1)]}P_{[t(L_1)+1,T]})
\]

\[
\leq d^{(2,\pi)}(\mu P_{[1,t(L_1)]}) \exp \left( -\left( L_2(1-\Lambda_{\max}(P')) \right) \right) \leq d^{(2,\pi)}(\mu P_{[1,t(L_1)]}) \epsilon.
\]

Note that \( (P_i)_{i=t(L_1)+1}^T \) contains \( (P_{i(t(i))})_{i=t(L_1)+1}^{L_1+L_2} \) since \( t(L_1) + 1 \leq t(L_1 + 1) \) and \( L_1 + L_2 \leq L \leq T \). Furthermore, since \( (P_i)_{i=1}^{t(L_1)} \) contains \( (P_{i(t(i))})_{i=1}^{L_1}, d^{(2,\pi)}(\mu P_{[1,t(L_1)]}) \leq 1 \) from Lemma 3.3. Hence, we obtain the claim. □

**Proof of Theorem 2.1.** For any \( T \), \( (P_i)_{i=1}^T \) contains \( (P_{i(t(i))})_{i=1}^T \) with \( L \geq \left[ T/C \right] \). Hence, taking \( T = Ct_{m}(P', \epsilon) \), we have \( d^{(2,\pi)}(P_{[1,T]}(v, \cdot)) \leq \epsilon \) for any \( v \in V \) from Lemma 3.4. □
3.2 Separation distance

We introduce the following proposition that we will use in Sections 5 and 6.

**Proposition 3.5.** Let \( (P_t)_{t \geq 1} \) be a sequence of transition matrices on \( V \). Suppose that \( (P_t)_{t \geq 1} \) is reversible with respect to a positive probability vector \( \pi \in (0,1]^V \) and contains an irreducible and lazy subsequence \( \mathcal{P}' = (P_{t(i)})_{i \geq 1} \) having an interval at most \( C \). Then, for any \( 0 < \epsilon \leq 1 \),

\[
T \geq Ct_{\mu}(\mathcal{P}',\epsilon),
\]

and \( u,v \in V \), \( P_{[1,2T]}(u,v) \geq (1-\epsilon)^2\pi(v) \) holds.

**Proof.** From the Cauchy-Schwarz inequality,

\[
\left( \frac{(P_{[1,2T]}(u,v)}\pi(v) \right) = \sum_{w \in V} (P_{1} \cdots P_{2T})(u,w)(P_{T+1} \cdots P_{2T})(w,v) \pi(v)
\]

\[
= \sum_{w \in V} (P_{1} \cdots P_{T})(u,w)(P_{2T} \cdots P_{T+1})(v,w) \pi(w)
\]

\[
\geq \left( \sum_{w \in V} \sqrt{(P_{1} \cdots P_{T})(u,w)(P_{2T} \cdots P_{T+1})(v,w)} \right)^2
\]

holds. Note that \( \pi(v)(P_{T+1} \cdots P_{2T})(v,w) = \pi(w)(P_{2T} \cdots P_{T+1})(w,v) \) holds from the reversibility.

Write \( \mu_v(w) = (P_{1} \cdots P_{T})(u,w) \) and \( \nu_v(w) = (P_{2T} \cdots P_{T+1})(v,w) \). From Lemma 3.4, we have \( d^{(2,\pi)}(\mu_v) = \epsilon \) and \( d^{(2,\pi)}(\nu_v) = \epsilon \). Hence,

\[
\sum_{w \in V} \sqrt{\mu_v(w)\nu_v(w)} \geq \sum_{w \in V} \min\{\mu_v(w), \nu_v(w)\} = 1 - \frac{1}{2} \sum_{w \in V} |\mu_v(w) - \nu_v(w)|
\]

\[
\geq 1 - \frac{1}{2} \sum_{w \in V} |\mu_v(w) - \pi(w)| + \frac{1}{2} \sum_{w \in V} |\pi(w) - \nu_v(w)|
\]

\[
= 1 - \frac{d^{(1,\pi)}(\mu_u) + d^{(1,\pi)}(\nu_v)}{2} \geq 1 - \frac{d^{(2,\pi)}(\mu_u) + d^{(2,\pi)}(\nu_v)}{2}
\]

\[
\geq 1 - \epsilon \geq 0.
\]

The first equality follows since \( \mu_u, \nu_v \) are probability distributions (see, e.g., (4.13) in [27]). \( \square \)

4 Hitting time and meeting time

We introduce Lemmas 4.1 and 4.2, which enable us to estimate the hitting, cover and coalescing times for the random walk \( (X_t)_{t \geq 0} \) according to a sequence of transition matrices \( \mathcal{P} \). For \( w \in V \), define \( D_w \in \{0,1\}^{V \times V} \) by \( D_w(u,v) := 1_{u=v}1_{u \neq w} \) for each \( u,v \in V \). For \( \mathcal{P} = (P_t)_{t \geq 1} \) and a sequence of vertices \( W = (w_t)_{t \geq 0} \), let \( Q_{W,t} := D_{w_{t-1}}P_tD_{w_t} \) and \( Q_{W,[a,b]} := Q_{W,a}Q_{W,a+1} \cdots Q_{W,b} \) for \( 1 \leq a \leq b \). Then,

\[
\Pr \left[ \bigwedge_{t=a-1}^b \{ X_t \neq w_t \} | X_{a-1} = \cdot \right] = D_{w_{a-1}}P_aD_{w_a}P_{a+1}D_{w_{a+1}} \cdots P_bD_{w_b}1 = Q_{W,[a,b]}1 \quad (4)
\]

holds for any \( 1 \leq a \leq b \). If \( W = (w_t)_{t \geq 0} \) satisfies \( w_t = w \in V \) for all \( t \), write \( Q_{w,t} = D_wPD_w \) and \( Q_{w,[a,b]} = Q_{w,a}Q_{w,a+1} \cdots Q_{w,b} \) for abbreviation.
4.1 Hitting time

We introduce the following lemma, which plays key role to estimate the hitting time (Theorem 2.2) and the cover time (Section 5). Note that we do not assume the laziness.

**Lemma 4.1.** Let \( (P_t)_{t=a}^b \) be a sequence of transition matrices on \( V \). Suppose that \( (P_t)_{t=a}^b \) is reversible with respect to a probability vector \( \pi \in (0,1)^V \) and contains an irreducible subsequence \( \mathcal{P}' = (P_{t(i)})_{i=1}^b \). Then, for any \( w \in V \),

\[
\sum_{v \in V} \pi(v) (Q_{w,[a,b]} \mathbb{1})(v) \leq \|Q_{w,[a,b]} \mathbb{1}\|_{2,\pi} \leq \exp \left( -\frac{L}{H_{\text{max}}(\mathcal{P}')} \right).
\]

**Proof.** The first inequality is trivial from the Cauchy-Schwarz inequality. For the second inequality, applying Lemma A.1 repeatedly yields

\[
\|Q_{w,[a,b]} \mathbb{1}\|_{2,\pi} = \|Q_{w,a}Q_{w,a+1,b} \mathbb{1}\|_{2,\pi} \leq \rho(Q_{w,a}) \|Q_{w,a+1,b} \mathbb{1}\|_{2,\pi} \leq \cdots \leq \prod_{t=a}^b \rho(Q_{w,t}).
\]

From Lemma A.5, \( \rho(D_{w,t(i)}) \leq 1 - \frac{1}{t_{\text{hit}}(P_{t(i)})} \) holds for any \( i \in [L] \). Hence,

\[
\prod_{t=a}^b \rho(Q_{w,t}) \leq \prod_{i\in[L]} \rho(D_{w,t(i)}) \leq \prod_{i\in[L]} \left( 1 - \frac{1}{t_{\text{hit}}(P_{t(i)})} \right) \leq \exp \left( -\sum_{i \in [L]} \frac{1}{t_{\text{hit}}(P_{t(i)})} \right)
\]

holds and we obtain the claim. Note that \( \rho(Q_{w,t}) \leq 1 \) holds for any \( t \).

**Proof of Theorem 2.2.** From (4), it holds for any \( x, w \in V \) that

\[
\Pr[\tau_w(\mathcal{P}) > T | X_0 = x] = \Pr\left[ \bigwedge_{t=0}^T \{ X_t \neq w \} \bigg| X_0 = x \right] = (Q_{w,[1,T]} \mathbb{1})(x).
\]

Write \( t_m = t_{\text{mix}}(\mathcal{P}) \). For any \( T \geq t_m \),

\[
(Q_{w,[1,T]} \mathbb{1})(x) = \sum_{v \in V} Q_{w,[1,t_m]}(x,v) (Q_{w,[t_m+1,T]} \mathbb{1})(v)
\]

\[
\leq \sum_{v \in V} P_{[1,t_m]}(x,v) (Q_{w,[t_m+1,T]} \mathbb{1})(v)
\]

\[
\leq \sqrt{\sum_{v \in V} \frac{P_{[1,t_m]}(x,v)^2}{\pi(v)} \sum_{v' \in V} \pi(v') (Q_{w,[t_m+1,T]} \mathbb{1})(v')^2}
\]

\[
= \sqrt{1 + d^{(2,\pi)}(P_{[1,t_m]}(x,\cdot))^2} \|Q_{w,[t_m+1,T]} \mathbb{1}\|_{2,\pi}
\]

\[
\leq \sqrt{1 + 1/4 \exp \left( -\frac{|(T-t_m)/C|}{H_{\text{max}}(\mathcal{P}')} \right)} \leq 2 \exp \left( -\frac{T-t_m}{CH_{\text{max}}(\mathcal{P}')} \right)
\]

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holds from Lemma 4.1. We use (3) in the second equality. Note that \((P_t^{T})_{t=m+1}^{T}\) contains an irreducible subsequence with length at least \(\lfloor (T - t_m)/C \rfloor\). Hence, for any \(x \in V\),

\[
\mathbb{E}[\tau_w(P)|X_0 = x] = \sum_{T=0}^{\infty} \Pr[\tau_w(P) > T|X_0 = x] \leq t_m + 1 + \sum_{T=t_m+1}^{\infty} 2 \exp \left(-\frac{T - t_m}{CH_{\max}(P')}\right) \leq t_m + 1 + 2CH_{\max}(P').
\]

The last inequality follows since \(\sum_{t=1}^{\infty} \exp(-tc) \leq \frac{\exp(-c)}{1-\exp(-c)} = \frac{1}{c} \leq \frac{1}{c}\) holds.

4.2 Meeting time lemma for lazy chains

Consider two independent random walks \((X_t(1))_{t \geq 0}\) and \((X_t(2))_{t \geq 0}\), where each walk is according to \(P = (P_t)_{t \geq 1}\). The meeting time is defined by \(t_{\text{meet}}(P) := \max_{x,y \in V} \mathbb{E}[\tau_{\text{meet}}(P)|X_0(1) = x, X_0(2) = y]\), where \(\tau_{\text{meet}}(P) := \min\{t \geq 0 : X_t(1) = X_t(2)\}\). Now, we have the following: For any \(0 \leq s < a \leq b\) and \(x, y \in V\), there is a sequence of vertices \(W^* = \{w_t^\star\}_{t \geq 0}\) such that

\[
\Pr \left[ \bigwedge_{t=a-1}^{b} \{X_t(1) \neq X_t(2)\} \left| X_{a-1}(1) = x, X_{s}(2) = y \right. \right] \leq (Q_{W^*,[a,b]}1)(x)
\]

holds. It is because

\[
\Pr \left[ \bigwedge_{t=a-1}^{b} \{X_t(1) \neq X_t(2)\} \left| X_{a-1}(1) = x, X_{s}(2) = y \right. \right] = \sum_{(w_t^\star)_{t=a-1}^{b} \in V_{b-a+2}} \Pr \left[ \bigwedge_{t=a-1}^{b} \{X_t(1) \neq w_t\} \left| X_{a-1}(1) = x \right. \right] \Pr \left[ \bigwedge_{t=a-1}^{b} \{X_t(2) = w_t\} \left| X_{s}(2) = y \right. \right]
\]

\[
\leq \max_{(w_t^\star)_{t=a-1}^{b} \in V_{b-a+2}} \Pr \left[ \bigwedge_{t=a-1}^{b} \{X_t(1) \neq w_t\} \left| X_{a-1}(1) = x \right. \right] = (Q_{W^*,[a,b]}1)(x)
\]

holds for some \(W^* = \{w_t^\star\}_{t \geq 0}\). The following lemma plays key role to estimate (5) and the coalescing time (Section 6).

**Lemma 4.2.** Let \((P_t^b)_{t=a}^{b}\) be a sequence of transition matrices on \(V\). Suppose that \((P_t^b)_{t=a}^{b}\) is reversible with respect to a positive probability vector \(\pi \in (0,1)^V\) and contains an irreducible and lazy subsequence \(P' = (P_t^{L})_{t=1}^{L}\). Then, for any sequence of vertices \(W = (w_t^b)_{t=a-1}^{b}\),

\[
\sum_{v \in V} \pi(v) \left( Q_{W,[a,b]}1 \right)(v) \leq \|Q_{W,[a,b]}1\|_{2,\pi} \leq \exp \left(-\frac{L}{H_{\max}(P')}\right).
\]
Proof. The first inequality is trivial from the Cauchy-Schwarz inequality. In the following, we show the second inequality. For any \( i \in [L] \), applying Lemmas A.4 and A.5 yields

\[
\|Q_{V,[t(i),t(i)+1,\beta]}\|_{2,\pi}^2 = \|Q_{V,[t(i),t(i)+1,\beta]}\|_{2,\pi}^2
\]

\[
= \|D_{w_{t(i)-1}}P_{t(i)}D_{w_{t(i)}}Q_{V,[t(i)+1,\beta]}\|_{2,\pi}^2
\]

\[
\leq \rho \left( D_{w_{t(i)-1}}P_{t(i)}D_{w_{t(i)}} \right) \|Q_{V,[t(i)+1,\beta]}\|_{2,\pi}^2
\]

\[
\leq \left( 1 - \frac{1}{t_{hit}(P_{t(i)})} \right)^2 \|Q_{V,[t(i)+1,\beta]}\|_{2,\pi}.
\]

Since

\[
\|Q_{V,[t,\beta]}\|_{2,\pi} = \|Q_{V,t}Q_{V,[t+1,\beta]}\|_{2,\pi}
\]

\[
\leq \rho \left( Q_{V,t} \right) \|Q_{V,[t+1,\beta]}\|_{2,\pi}
\]

\[
\leq \|Q_{V,[t+1,\beta]}\|_{2,\pi}
\]

(6)

holds for any \( t \in [a, b] \), applying (6) and (7) repeatedly yields

\[
\|Q_{V,[a,b]}\|_{2,\pi}^2 \leq \prod_{i \in [L]} \left( 1 - \frac{1}{t_{hit}(P_{t(i)})} \right)^2 \leq \exp \left( -\sum_{i \in [L]} \frac{2}{t_{hit}(P_{t(i)})} \right).
\]

and we obtain the claim. Note that \( t_{hit}(P_{t(i)}) \leq H_{max}(P') \) for all \( i \in [L] \) from definition. \(\square\)

4.3 Lower bound of meeting time

In this section, we prove Proposition 1.4.

Proof of Proposition 1.4. For a graph \( H \) and a permutation \( \sigma \) on \( V(H) \), let \( \sigma(H) \) be the graph given by \( V(\sigma(H)) = V(H) \) and \( E(\sigma(H)) = \{ \{ \sigma(u), \sigma(v) \} : \{ u, v \} \in E(H) \} \).

For an integer \( m \in \mathbb{N} \), define a graph \( G \) by \( V(G) := \{ u_0, \ldots, u_m \} \cup \{ v_0, \ldots, v_m \} \) and

\[
E(G) := \{ u_0, v_0 \} \cup \bigcup_{i=1}^{m} \{ u_i, u_0 \} \cup \bigcup_{j=1}^{m} \{ v_j, v_0 \}.
\]

(8)

Let \( \sigma \) be the permutation on \( V \) defined as \( \sigma(u_i) = u_{(i+1) \mod (m+1)} \) and \( \sigma(v_i) = v_{(i+1) \mod (m+1)} \). We claim that the lazy simple random walk on the sequence \( (G_t)_{t \geq 1} \) given by \( G_1 = G \) and \( G_{t+1} = \sigma(G_t) \) \((t \geq 1)\) has the desired property (see Figure 2). Consider two independent lazy simple random walks \( (X_t(1))_{t \geq 0} \) and \( (X_t(2))_{t \geq 0} \) with initial points \( (X_0(1), X_0(2)) = (u_m, v_m) \). Suppose \( \tau_{\text{meet}} \geq t \).

Then, there is \( t' \leq t \) such that either \( X(1) \) or \( X(2) \) moves along the edge \( \{ u_j, v_j \} \) for \( j = t' \mod (m+1) \) (i.e., either \( X_{t'-1}(1), X_{t'}(1) = (u_j, v_j) \) or \( X_{t'-1}(2), X_{t'}(2) = (v_j, u_j) \) holds). Focus on the walk \( X(1) \). To reach \( u_j \), the walker \( X(1) \) must choose the self loop for \( m \) consecutive times, which occurs with probability \( 2^{-m} \). Therefore, \( \text{Pr} [ \tau_{\text{meet}} \leq t ] \leq 2t \cdot 2^{-m} \) and we have \( t_{\text{meet}} = 2^{\Theta(m)} \). The same argument holds for the lazy minimum degree walk and lazy \( \beta \)-random walk. \(\square\)
5 Cover time

We prove Theorem 2.3. Consider \( k \) independent random walks \((X_t(1))_{t \geq 0}, \ldots, (X_t(k))_{t \geq 0}\) according to \( \mathcal{P} \). Let \((X_t)_{t \geq 0}\) be a random variable defined as \( X_t = (X_t(1), \ldots, X_t(k)) \in V^k \).

Lemma 5.1. Let \( \mathcal{P} = (P_t)_{t \geq 1} \) be a sequence of transition matrices. Suppose that \( \mathcal{P} \) is reversible with respect to a probability vector \( \pi \in (0, 1)^V \) and contains an irreducible and lazy subsequence \( \mathcal{P}' = (P_t(i))_{i \geq 1} \) having an interval at most \( C \). Then, for any \( k \), \( T \geq 2C t_m^{(\pi)}(\mathcal{P}') + C \left[ \sum_{i=1}^{8n(\mathcal{P})} \log(50n) \right] \), and \( x \in V^k \),

\[
\Pr \left[ \bigvee_{w \in V} \bigwedge_{i \in [k]} \bigwedge_{t=0}^T \{X_t(i) \neq w\} \bigg| X_0 = x \right] \leq \frac{99}{100} \text{ holds.}
\]

Proof. Write \( t_m = 2C t_m^{(\pi)}(\mathcal{P}') \) for convenience. From Proposition 3.5, there is a transition matrix \( q \in [0, 1]^{V \times V} \) such that

\[
(P_1P_2 \cdots P_{t_m})(x, u) = \frac{1}{4}\pi(u) + \frac{3}{4}q(x, u)
\]

holds for any \( x, u \in V \). Write \( E \) denote the event that \( \bigvee_{w \in V} \bigwedge_{i \in [k]} \bigwedge_{t=0}^T \{X_t(i) \neq w\} \) for convenience. Now, we apply Lemma B.5 to \( E \) with the following settings: For \( S \subseteq [k] \) and \( i \in [k] \), define the probability distribution \( \mu_{S,i} \in [0, 1]^V \) by \( \mu_{S,i} := 1\mathbb{I}_{i \in S}\pi + 1\mathbb{I}_{i \notin S}g(x_i, \cdot) \). Let \( I_1, I_2, \ldots, I_k \) be \( k \) independent binary random variables such that \( \Pr[I_i = 1] = 1/4 \) holds for any \( i \in [k] \) and let \( I = \{ i \in [k]: I_i = 1 \} \). Let \( \mathcal{U} = \{ S \subseteq [k] : |S| \geq k/8 \} \). From Lemma B.5,

\[
\Pr[ E | X_0 = x ] \\
\leq \Pr[I \notin \mathcal{U}] + \max_{S \in \mathcal{U}} \sum_{u=(u_1, \ldots, u_k) \in V^k} \Pr[ E | X_{t_m} = u, X_0 = x ] \prod_{i \in [k]} \mu_{S,i}(u_i) \tag{9}
\]

holds for any \( x \in V^k \). From the Chernoff inequality (Lemma B.3), it is easy to see that

\[
\Pr[I \notin \mathcal{U}] = \Pr\left[ \sum_{i=1}^k I_i \leq k/8 \right] \leq \exp\left( -\frac{k}{32} \right). \tag{10}
\]

Note that \( E \left[ \sum_{i=1}^k I_i \right] = k/4 \). For the second term of (9), from the union bound and the Markov
property, we have
\[
\Pr[\mathcal{E}|X_{t_m} = u, X_0 = x] \leq \sum_{u \in V} \Pr[\bigwedge_{i \in [k]} \bigwedge_{t=t_m}^T \{X_t(i) \neq w\} | X_{t_m} = u] \\
\leq \sum_{u \in V} \prod_{i \in [k]} \sum_{u_i \in V} \prod_{i \in [k]} \mu_{S,i}(u_i) \prod_{i \in [k]} \left(Q_{w,[t_m+1,T]}(u_i)\right) \\
= \sum_{u \in V} \prod_{i \in [k]} \sum_{u_i \in V} \pi(u_i) \left(Q_{w,[t_m+1,T]}(u_i)\right)
\]

Note that the equality follows from (4). Hence,
\[
\sum_{u=(u_1,\ldots,u_k) \in V^k} \Pr[\mathcal{E}|X_{t_m} = u, X_0 = x] \prod_{i \in [k]} \mu_{S,i}(u_i) \\
\leq \sum_{u \in V} \sum_{u_1 \in V} \cdots \sum_{u_k \in V} \prod_{i \in [k]} \mu_{S,i}(u_i) \prod_{i \in [k]} \left(Q_{w,[t_m+1,T]}(u_i)\right) \\
= \sum_{u \in V} \prod_{i \in [k]} \sum_{u_i \in V} \mu_{S,i}(u_i) \left(Q_{w,[t_m+1,T]}(u_i)\right) \\
\leq \sum_{u \in V} \prod_{i \in S} \sum_{u_i \in V} \pi(u_i) \left(Q_{w,[t_m+1,T]}(u_i)\right)
\]

holds for any \(S \in \mathcal{U}\). From assumption, \((P_t)_{t=t_m+1}^T\) contains an irreducible subsequence with length of \([(T - t_m)/C]\) = \(\left\lceil \frac{8H_{\max}(P') \log(50n)}{k} \right\rceil \geq \frac{8H_{\max}(P') \log(50n)}{k}\). Applying Lemma 4.1,
\[
\sum_{u \in V} \prod_{i \in S} \sum_{u_i \in V} \pi(u_i) \left(Q_{w,[t_m+1,T]}(u_i)\right) \\
\leq \sum_{u \in V} \prod_{i \in S} \exp \left(-\frac{8 \log(50n)}{k}\right) = \sum_{w \in V} \exp \left(-|S| \frac{8 \log(50n)}{k}\right) \leq \frac{1}{50}.
\]

holds for any \(S \in \mathcal{U}\). Combining (9) to (12), \(\Pr[\mathcal{E}|X_0 = x] \leq \exp \left(-\frac{1}{\delta_0}\right) + \frac{1}{50} \leq \frac{99}{100}\). \(\square\)

**Proof of Theorem 2.3.** For \(\mathcal{X} = (X_t)_{t \geq 0} = (X_t(1),\ldots,X_t(k))_{t \geq 1}\), let \(\mathcal{E}(\mathcal{X}, [a, b])\) be the event of \(\bigwedge_{w \in V} \bigwedge_{i \in [k]} \bigwedge_{t=0}^T \{X_t(i) \neq w\}\). Then, for any \(x \in V^k\), \(s \geq 0\), we have \(\Pr[\mathcal{E}(\mathcal{X}, [s, s + T]) | X_s = x] \leq 99/100\). Here, \(T := 2CT_m(\pi') + C \left\lceil \frac{8H_{\max}(P') \log(50n)}{k} \right\rceil \). To see this, consider the following arguments. Let \(P^* = (P_t^*)_{t \geq 1}\) be a sequence of transition matrices, where \(P_t^* := P_{s+t}\) for all \(t \geq 1\). Let \((X_t^1(1))_{t \geq 0}, \ldots, (X_t^k(k))_{t \geq 0}\) be \(k\) independent random walks, where each walk is according to \(P^*\). Write \(X_t^i = (X_t^i(1),\ldots,X_t^i(k))\) for each \(t \geq 0\). Then, since \(\Pr[X_{s+t}(i) = v | X_s(i) = u] = \Pr[X_t^i(i) = v | X_t^i(i) = u]\) holds for any \(u, v \in V\) and \(i \in [k]\), we obtain \(\Pr[\mathcal{E}(\mathcal{X}, [s, s + T]) | X_s = x] = \Pr[\mathcal{E}(\mathcal{X}^*, [0, T]) | X_0^* = x] \leq 99/100\). Here, we use Lemma 5.1.

Write \(\mathcal{E}_{a,b} = \mathcal{E}(\mathcal{X}, [a, b])\) and \(\mathcal{E}[\ell] = \bigwedge_{i=1}^\ell \mathcal{E}(i-1)_{T,iT}\) for convenience. Then, \(\Pr[\tau_{cov}(\mathcal{P}) >
\[ \ell T | X_0 = x \leq \Pr[\mathcal{E}[\ell] | X_0 = x] \]

\[
\Pr[\mathcal{E}[\ell] | X_0 = x] \\
= \sum_{x' \in V^k} \Pr[\mathcal{E}(\ell-1)_{T,T}, \mathcal{E}[\ell-1], X_{(\ell-1)T} = x' | X_0 = x] \\
= \sum_{x' \in V^k} \Pr[\mathcal{E}(\ell-1)_{T,T} | \mathcal{E}[\ell-1], X_{(\ell-1)T} = x', X_0 = x] \Pr[\mathcal{E}[\ell-1], X_{(\ell-1)T} = x' | X_0 = x] \\
= \sum_{x' \in V^k} \Pr[\mathcal{E}(\ell-1)_{T,T} | X_{(\ell-1)T} = x'] \Pr[\mathcal{E}[\ell-1], X_{(\ell-1)T} = x' | X_0 = x] \\
\leq 99/100 \Pr[\mathcal{E}[\ell-1] | X_0 = x] \leq \cdots \leq (99/100)^\ell.
\]

Hence, we have
\[
\mathbb{E}[\tau_{\text{cov}}(P) | X_0 = x] = \sum_{\ell=0}^{\infty} \sum_{t=0}^{T-1} \Pr[\tau_{\text{cov}}(P) > \ell T + t | X_0 = x] \leq T \sum_{\ell=0}^{\infty} (99/100)^\ell \leq 100T.
\]

\[ \square \]

6 Coalescing time

We show Theorem 2.4 in this section. Recall that \((C_t(1))_{t \geq 0}, (C_t(2))_{t \geq 0}, \ldots, (C_t(n))_{t \geq 0}\) denote the coalescing random walks according to \(\mathcal{P} = (P_t)_{t \geq 1}\), which is defined in Section 2.2.

**Lemma 6.1.** Let \(\mathcal{P} = (P_t)_{t \geq 1}\) be a sequence of transition matrices. Suppose that \(\mathcal{P}\) is reversible with respect to a positive probability vector \(\pi \in (0, 1)^V\) and contains an irreducible and lazy subsequence \(\mathcal{P}' = (P_{t(i)})_{t \geq 1}\) having an interval at most \(C\). Let \(T \geq 2Ct_m(\pi)(\mathcal{P}', 1/2) + 48CH_{\max}(\mathcal{P}') + \log_2 n\). Then, for any \(x \in V^n\), \(\Pr[|S(C_T)| \geq 2 | C_0 = x] \leq \frac{49}{50}\) holds.

**Proof.** Let \((X_t)_{t \geq 0} = ((X_t(1), \ldots, X_t(n)))_{t \geq 0}\) be \(n\) independent random walks according to \(\mathcal{P}\). To estimate \(|S(C_T)|\), we introduce two random processes called random walks with killings and random walks with a list of allowed killings, introduced in [31].

In the random walks with killings \((Y_t)_{t \geq 0} = ((Y_t(1), \ldots, Y_t(n)))_{t \geq 0}\), a walker is killed when it meets another walker with a smaller index. Formally, let \(Y_t(1) = X_t(1)\) for all \(t \geq 0\). For \(t \geq 0\) and \(a \geq 1\), \(Y_t(a)\) is inductively defined as follows. Suppose that \((Y_s)_{s=0}^{t-1}\) and \(Y_t(1), \ldots, Y_t(a-1)\) are determined. Let \(\partial \notin V\) be a coffin state. Then, let
\[
\tau(a) := \min\{t \geq 0 : X_t(a) = Y_t(b) \text{ for some } b < a\}, \quad \text{and}
\]
\[
Y_t(a) := \begin{cases} X_t(a) & \text{if } t < \tau(a) \\ \partial & \text{if } t \geq \tau(a) \end{cases}.
\]

We define random walks \((Z_t)_{t \geq 0} = ((Z_t(1), \ldots, Z_t(n)))_{t \geq 0}\) with a list of allowed killings. Let \(k := \lfloor \log_2(n) \rfloor\). We split the set of walkers \(\{1, 2, \ldots, n\}\) into \(W_k, W_{k-1}, \ldots, W_0\), where \(W_k := \{2^k, 2^k + 1, \ldots n\}\) and \(W_i := \{j : 2^i \leq j < 2^{i+1}\}\) for \(0 \leq i \leq k - 1\). Define the rounds
Let $R_k, R_{k-1}, \ldots, R_0$ recursively as $R_k := \{0, 1, \ldots, 2t_m - 1\}$ and, for $0 \leq i \leq k - 1$, let $R_i := \{ \sum_{j=i+1}^{k} |R_j| + t : 0 \leq t < C \left[ \frac{KH_{\max}(P')}{2^i} \right] + 1 \}$. Here, $t_m := 2Ct_m(P')$ and $K$ is some constant which we will determine later. $(Z_t)_{t \geq 0}$ is controlled by a list of allowed killings $(A_t)_{t \geq 0}$, defined as follows. For $t \in R_k$, let $A_t = \emptyset$. Let $A_t := \{(b, a) : b \in W_i, a \in \bigcup_{j=i+1}^{k} W_j\}$ for $t \in R_j$ with $0 \leq j \leq k - 1$. In the random walks with a list of allowed killings, a walker $a \in W_j$ with $j = i + 1, \ldots, k$ is killed when it meets another walker $b \in W_i$ during the round $i$. Formally, let $Z_t(1) = X_t(1)$ for all $t \geq 0$. For $t \geq 0$ and $a \geq 1$, $Z_t(a)$ is inductively defined as follows. Suppose that $(Z_a)_{a=0}^{t-1}$ and $Z_t(1), \ldots, Z_t(a-1)$ are determined. Then, let

$$
\tau^A(a) := \min\{t \geq 0 : X_t(a) = Z_t(b) \text{ for some } (b, a) \in A_t\}, \quad \text{and}
$$

$$
Z_t(a) := \begin{cases} X_t(a) & \text{if } t < \tau^A(a) \\ \partial & \text{if } t \geq \tau^A(a) \end{cases}.
$$

For a vector $x \in (V \cup \{\partial\})^n$, let $\mathcal{S}(x) = \{i \in [n] : x(i) \neq \partial\}$. Obviously, $\Pr[|\mathcal{S}(C_t)| \geq k|C_0 = x] = \Pr[|\mathcal{S}(Y_t)| \geq k|X_0 = x]$, holds for any $t \geq 0$, $k \geq 0$ and $x \in V^n$. Furthermore, $\Pr[|\mathcal{S}(Y_t)| \geq k|X_0 = x] \leq \Pr[|\mathcal{S}(Z_t)| \geq k|X_0 = x]$, holds for any $t \geq 0$, $k \geq 0$ and $x \in V^n$. To see this, consider using $(C_t)_{t \geq 0}$ instead of $(X_t)_{t \geq 0}$ in definitions of both $(Y_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$. Then, $|\mathcal{S}(Y_t)| \leq |\mathcal{S}(Z_t)|$ holds for any $t \geq 0$. Note that $\Pr[C_t(i) = v|C_0(i) = u] = \Pr[X_t(i) = v|X_0(i) = u]$ holds for any $u, v \in V$ and $i \in [n]$.

Let $T$ be the time when the round 0 is ended, i.e., $T = t_m + \sum_{j=0}^{k-1} \left( C \left[ \frac{KH_{\max}(P')}{2^j} \right] + 1 \right) \leq t_m + 48CH_{\max}(P') + \log_2 n$. We proceed to show that $\Pr[|\mathcal{S}(Z_T)| \geq 2|X_0 = x] \leq 49/50$ holds. For $b \in W_i$, let $s(b) := \min R_i$ denote the initial time of the round $i$. Since $b \in W_i$ is not killed during the rounds $k, k-1, \ldots, i$, $Z_t(b) = X_t(b)$ holds for any $b \in W_i$ and $t \in \bigcup_{j=i}^{k} R_j$. From the same reason, $Z_s(b) = X_s(b)$ holds for any $b \in [n]$. Furthermore, $s(b) \geq t_m$ holds for any $b \in W_i$ with $i = k-1, k-2, \ldots, 0$ from $|R_k| = t_m$. Hence, from Proposition 3.5, there is a transition matrix $q_b \in [0, 1]^{V \times V}$ such that

$$
\Pr[X_s(b) = u_b|X_0(b) = x_b] = \frac{1}{4}\pi(u_b) + \frac{3}{4}q_b(x_b, u_b)
$$

holds for each $b \in \{2^k - 1\}$ ($b \in W_i$ with $i = k-1, \ldots, 0$). Now, we apply Lemma B.5 to the event of $|\mathcal{S}(Z_T)| \geq 2$ with the following setting. For $S \subseteq \{2^k - 1\}$ and $b \in \{2^k - 1\}$, define the probability distribution $\mu_{S, b} \in [0, 1]^V$ by $\mu_{S, b} = \mathbf{1}_{b \in S}\pi_b + \mathbf{1}_{b \notin S}q_b(x_b, \cdot)$. Let $I_1, I_2, \ldots, I_{2k-1}$ be be $2^k - 1$ be independent binary random variables such that $\Pr[I_i = 1] = 1/4$ holds for all $i \in \{2^k - 1\}$ and $I = \{i \in \{2^k - 1\} : I_i = 1\}$. Let $U := \{S \subseteq \{2^k - 1\} : |W_i \cap S| \geq |W_i|/8 \text{ holds for all } 0 \leq i \leq k - 1\}$. For notational convenience, write $X_{s(\cdot)} := (X_{s(1)}(1), X_{s(2)}(2), \ldots, X_{s(2^k-1)}(2^k-1))$. Then, from Lemma B.5,

$$
\Pr[|\mathcal{S}(Z_T)| \geq 2|X_0 = x] \leq \Pr[I \notin U] + \max_S \sum_{u \in V^{2^k-1}} \Pr[|\mathcal{S}(Z_T)| \geq 2|X_{s(\cdot)} = u, X_0 = x] \prod_{b \in \{2^k-1\}} \mu_{S, b}(u_b)
$$

(13)

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holds. Applying the Chernoff inequality (Lemma B.3), we have

\[
\Pr[I \in \mathcal{U}] = \Pr \left[ \bigwedge_{i=0}^{k-1} \{|I \cap W_i| \geq |W_i|/8\} \right] = \prod_{i=0}^{k-1}\Pr \left[ \sum_{j \in W_i} I_i \geq 2^i/8 \right]
\geq \prod_{i=0}^{k-1} \left( 1 - \exp \left( -\frac{2^i}{32} \right) \right) = \prod_{i=1}^{k-6} (1 - \exp \left( -\frac{2^i}{2^2} \right))
\geq \prod_{i=1}^{k-6} (1 - 2^{-i}) \geq \exp \left( -\sum_{i=1}^{k-6} \frac{2}{2^i} \right) \geq \exp(-2).
\]

(14)

Note that \( E \left[ \sum_{j \in W_i} I_j \right] = |W_j|/4 = 2^i/4. \)

Now, we estimate the second term of (13). For \( b \in W_i \), write \( R(b) := R_i \). From definition, \( Z_t(b) = X_t(b) \) holds for any \( t \in R(b) \). Then, from the union bound and the Markov property,

\[
\Pr \left[ |S(Z_T)| \geq 2|X_{s(\cdot)} = u, X_0 = x| \right]
= \Pr \left[ \bigvee_{i=1}^{k} \bigvee_{a \in W_i} \bigwedge_{b \in [2^i-1]} \bigwedge_{t \in R(b)} \{Z_t(b) \neq Z_t(a)\} \left| X_{s(\cdot)} = u, X_0 = x \right| \right]
\leq \sum_{i=1}^{k} \sum_{a \in W_i} \Pr \left[ \bigwedge_{b \in [2^i-1]} \bigwedge_{t \in R(b)} \{X_t(b) \neq X_t(a)\} \left| X_{s(\cdot)} = u \right| \right]
= \sum_{i=1}^{k} \sum_{a \in W_i} \prod_{b \in [2^i-1]} \Pr \left[ X_{s(b)}(b) = u_b, X_{s(a)}(a) = a_a \right]
\leq \sum_{i=1}^{k} \sum_{a \in W_i} \prod_{b \in [2^i-1]} \left( Q_{W_{s(b)}^a[\{s(b)+1,s(b)+|R(b)|-1\}]1 \right) (u_b).
\]

We use (5) in the last inequality: For each \( b \), \( W_b^* \) denotes a sequence of vertices maximizing the corresponding probability. Hence, we have

\[
\sum_{u \in V^{2k-1}} \Pr \left[ |S(Z_T)| \geq 2|X_{s(\cdot)} = u| \prod_{b \in [2^k-1]} \mu_{S,b}(u_b) \right]
\leq \sum_{i=1}^{k} \sum_{a \in W_i} \sum_{u \in V^{2k-1}} \prod_{b \in [2^i-1]} \mu_{S,b}(u_b) \prod_{b \in [2^i-1]} \left( Q_{W_{s(b)}^a[\{s(b)+1,s(b)+|R(b)|-1\}]1 \right) (u_b)
= \sum_{i=1}^{k} \sum_{a \in W_i} \prod_{j=0}^{i-1} \prod_{b \in W_j} \sum_{u_b \in V} \mu_{S,b}(u_b) \left( Q_{W_{s(b)}^a[\{s(b)+1,s(b)+|R(b)|-1\}]1 \right) (u_b)
\leq \sum_{i=1}^{k} \sum_{a \in W_i} \prod_{j=0}^{i-1} \prod_{b \in W_j \cap S} \sum_{u_b \in V} \pi(u_b) \left( Q_{W_{s(b)}^a[\{s(b)+1,s(b)+|R(b)|-1\}]1 \right) (u_b).
\]

(15)

Since \( (P_t)_{t=s(b)+1}^{|R(b)|-1} \) contains an irreducible and lazy subsequence with length \( |(|R(b)| - 1)/C| \) and \(|R(b)| - 1 = C[KH_{\text{max}}(P')/2^l]| \) for any \( b \in W_j \) with \( j = k - 1, \ldots, 0 \), applying Lemma 4.2
Thus, we have $49$ random walks according to $P$ random walks including the lazy simple random walk and the lazy minimum degree random walk. Presented an example of a sequence of graphs with an exponential coalescing time for well-known time. All of those bounds match known upper bounds on static graphs and are tight. We also showed that the lazy Metropolis walk on a sequence of connected edge-changing graphs has the following:

$$\sum_{\pi(u_b) \leq f(u_b)} \prod_{i=1}^{k} \prod_{a \in W_i} \prod_{j=0}^{i-1} \prod_{b \in W_i \cap S} \pi(u_b) = \sum_{\pi(u_b) \leq f(u_b)} \prod_{i=1}^{k} \prod_{a \in W_i} \prod_{j=0}^{i-1} \exp\left(-\frac{K}{2j}\right) = \sum_{\pi(u_b) \leq f(u_b)} \prod_{i=1}^{k} \prod_{a \in W_i} \prod_{j=0}^{i-1} \exp\left(-\frac{K}{8}\right)$$

Combining (13) to (16) with $K = 24$, $\Pr[|S(Z_T)| \geq 2|X_0 = x] \leq 1 - \frac{1}{e^2} + \frac{2}{e^2-2} \leq \frac{49}{50}$ holds. \qed

**Proof of Theorem 2.4.** Let $P^*_t = (P^*_t)^{t\geq1}$, where $P^*_t = P_{t+s}$ for all $t$. Let $(C^*_t)^{t\geq0}$ be the coalescing random walks according to $P^*$. Then, $\Pr[|S(C_{s+T})| \geq 2|C_s = x] = \Pr[|S(C^*_T)| \geq 2|C^*_0 = x] \leq 49/50$ from Lemma 6.1. Hence,

$$\Pr[|S(C_{kT})| \geq 2|C_0 = x] = \sum_{y \in V^n} \Pr[|S(C_{kT})| \geq 2, |S(C_{(k-1)T})| \geq 2, C_{(k-1)T} = y|C_0 = x]$$

$$= \sum_{y \in V^n: |S(y)| \geq 2} \Pr[|S(C_{kT})| \geq 2, C_{(k-1)T} = y|C_0 = x] \cdot \Pr[|S(C_{(k-1)T})| \geq 2, C_{(k-1)T} = y|C_0 = x]$$

$$\leq (49/50) \Pr[|S(C_{(k-1)T})| \geq 2|C_0 = y] \leq \cdots \leq (49/50)^k.$$

Thus, we have

$$\mathbb{E}[t_{\text{coal}}(P)|C_0 = x] = \sum_{k=0}^{\infty} \sum_{t=0}^{T-1} t_{\text{coal}}(P) \geq kT + t|C_0 = x] \leq T \sum_{k=0}^{\infty} (49/50)^k \leq 50T.$$ \qed

**7 Conclusion**

We showed that the lazy Metropolis walk on a sequence of connected edge-changing graphs has the $O(n^2)$ mixing time, the $O(n^2)$ hitting time, the $O(n^2 \log n)$ cover time and the $O(n^2)$ coalescing time. All of those bounds match known upper bounds on static graphs and are tight. We also presented an example of a sequence of graphs with an exponential coalescing time for well-known random walks including the lazy simple random walk and the lazy minimum degree random walk.

The followings are questions left on this topic:

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1. Does the lazy simple random walk on any sequence of connected edge-changing regular graphs have the $O(n^2)$ cover time? Corollary 1.8 implies an $O(n^2 \log n)$ cover time.

2. Is there a random walk (using local degree information) having the $O(n^2)$ cover time on any sequence of connected edge-changing graphs?

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References

[1] M. Abdullah, C. Cooper, and M. Draief. Speeding up cover time of sparse graphs using local knowledge. *In Proceedings of the International Workshop on Combinatorial Algorithms (IWOCA)*, 1:1–12, 2015.

[2] D. J. Aldous and J. A. Fill. Reversible Markov chains and random walks on graphs. https://www.stat.berkeley.edu/users/aldous/RWG/book.html.

[3] R. Aleliunas, R. M. Karp, R. J. Lipton, L. Lovász, and C. Rackoff. Random walks, universal traversal sequences, and the complexity of maze problems. *In Proceedings of 20th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 218–223, 1979.

[4] N. Alon, C. Avin, M. Koucky, G. Kozma, Z. Lotker, and M. Tuttle. Many random walks are faster than one. *Combinatorics, Probability and Computing*, 20(4):2623–2641, 2011.

[5] C. Avin, M. Kouský, and Z. Lotler. How to explore a fast-changing world (cover time of a simple random walk on evolving graphs). *In Proceedings of the 35th International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 121–132, 2008.

[6] C. Avin, M. Kouský, and Z. Lotler. Cover time and mixing time of random walks on dynamic graphs. *Random Structures & Algorithms*, 52(4):576–596, 2018.

[7] P. Berenbrink, G. Giakkoupis, A.-M. Kermarrec, and F. Mallmann-Trenn. Bounds on the voter model in dynamic networks. *In Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming (ICALP)*, 2016.

[8] G. Brightwell and P. Winkler. Maximum hitting time for random walks on graphs. *Random Structures & Algorithms*, 1(3):263–276, 1990.

[9] A. Broder, A. Karlin, P. Raghavan, and E. Upfal. Trading space for time in undirected s-t connectivity. *SIAM Journal on Computing*, 23(2):324–334, 1994.
[10] L. Cai, T. Sauerwald, and L. Zanetti. Random walks on randomly evolving graphs. In Proceedings of the 27th International Colloquium on Structural Information and Communication Complexity (SIROCCO), 2020.

[11] C. Cooper. Random walks, interacting particles, dynamic networks: Randomness can be helpful. In Proceedings of the 18th International Colloquium on Structural Information and Communication Complexity (SIROCCO), pages 1–14, 2011.

[12] C. Cooper, R. Elsässer, H. Ono, and T. Radzik. Coalescing random walks and voting on connected graphs. SIAM Journal on Discrete Mathematics, 27(4):1748–1758, 2013.

[13] C. Cooper and A. Frieze. Crawling on simple models of web graphs. Internet Mathematics, 1(1):57–90, 2003.

[14] R. David and U. Feige. Random walks with the minimum degree local rule have $O(n^2)$ cover time. SIAM Journal on Computing, 47(3):755–768, 2018.

[15] B. Doerr and F. Neumann. Theory of evolutionary computation: Recent developments in discrete optimization. Springer International Publishing, 2020.

[16] R. Elsässer and T. Sauerwald. Tight bounds for the cover time of multiple random walks. Theoretical Computer Science, 412(24):2623–2641, 2011.

[17] U. Feige. A tight upper bound on the cover time for random walks on graphs. Random Structures & Algorithms, 6(1):51–54, 1995.

[18] J. A. Fill. Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process. The Annals of Applied Probability, pages 62–87, 1991.

[19] Y. Hassin and D. Peleg. Distributed probabilistic polling and applications to proportionate agreement. Information and Computation, 171(2):248–268, 2001.

[20] R. Horn and C. Johnson. Matrix Analysis: Second Edition. Campridge University Press, 2012.

[21] S. Ikeda, I. Kubo, N. Okumoto, and M. Yamashita. Impact of local topological information on random walks on finite graphs. In Proceedings of the 30th International Colloquium on Automata, Languages and Programming (ICALP), pages 1054–1067, 2003.

[22] S. Ikeda, I. Kubo, and M. Yamashita. The hitting and cover times of random walks on finite graphs using local degree information. Theoretical Computer Science, 410(1):94–100, 2009.

[23] J. Kahn, N. Linial, N. Nisan, and M. Saks. On the cover time of random walks on graphs. Journal of Theoretical Probability, 2:121–128, 1989.
[24] V. Kanade, F. Mallmann-Trenn, and T. Sauerwald. On coalescence time in graphs: When is coalescing as fast as meeting? *In Proceedings of the 30th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 956–965, 2019.

[25] S. Kijima, N. Shimizu, and T. Shiraga. How many vertices does a random walk miss in a network with moderately increasing the number of vertices? *In Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 106–122, 2021.

[26] I. Lamprou, R. Martin, and P. Spirakis. Cover time in edge-uniform stochastically-evolving graphs. *Algorithms*, 11(10):149, 2018.

[27] D. A. Levin and Y. Peres. *Markov Chain and Mixing Times: Second Edition*. The American Mathematical Society, 2017.

[28] L. Lovász. Random walks on graphs: A survey. *Combinatorics, Paul Erdős is Eighty*, 2:1–46, 1993.

[29] P. Matthews. Covering problems for Markov chains. *The Annals of Probability*, 16(3):1215–1228, 1988.

[30] Y. Nonaka, H. Ono, K. Sadakane, and M. Yamashita. The hitting and cover times of Metropolis walks. *Theoretical Computer Science*, 411(16–18):1889–1894, 2010.

[31] R. I. Oliveira. On the coalescing time of reversible random walks. *Transactions of the American Mathematical Society*, 364:2109–2128, 2012.

[32] R. I. Oliveira and Y. Peres. Random walks on graphs: New bounds on hitting, meeting, coalescing and returning. *In Proceedings of the 16th Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*, pages 119–126, 2019.

[33] T. Sauerwald and L. Zanetti. Random walks on dynamic graphs: Mixing times, hitting times, and return probabilities. *In Proceedings of the 46th International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 93:1–93:15, 2019.

A Tools for Lemmas 3.1, 3.2, 4.1 and 4.2

The following lemmas, which we use in the proofs of Lemmas Lemmas 3.1, 3.2, 4.1 and 4.2, are rephrased versions of previous works. Note that we do not assume some assumptions of those works, e.g., the irreducibility of $P$ in Lemmas A.2 to A.4. Thus, we put the proofs of these lemmas for completeness.
Lemma A.1. Let $M \in \mathbb{R}^{V \times V}$ be a matrix and $\nu \in \mathbb{R}_{>0}^{V}$ be a positive vector. Suppose that $M$ is reversible with respect to $\nu$. Then,

$$
\langle Mf, f \rangle_\nu \leq \rho(M) \langle f, f \rangle_\nu \quad \text{and} \quad \|Mf\|_{2,\nu} \leq \rho(M) \|f\|_{2,\nu}
$$

hold for any $f \in \mathbb{R}^{V}$. Furthermore, if $M$ is a transition matrix,

$$
\langle Mf, f \rangle_\nu \leq \lambda_{\ast}(M) \langle f, f \rangle_\nu \quad \text{and} \quad \|Mf\|_{2,\nu} \leq \lambda_{\ast}(M) \|f\|_{2,\nu}
$$

hold for any $f \in \mathbb{R}^{V}$ satisfying $\langle f, 1 \rangle_\nu = 0$.

Proof. Since $M$ is reversible with respect to $\nu$, $\langle Mf, g \rangle_\nu = \langle f, Mg \rangle_\nu$ holds for any $f, g \in \mathbb{R}^{V}$. From the spectral theorem, the inner product space $(\mathbb{R}^{V}, \langle \cdot, \cdot \rangle_\nu)$ has an orthonormal basis of real-valued eigenvectors $\{\psi_{i}\}_{i=1}^{V}$ corresponding to real eigenvalues $\{\lambda_{i}(M)\}_{i=1}^{V}$ (see, e.g., Lemma 12.1 in [27]). In other words, for any $i, j$ and $f \in \mathbb{R}^{V}$, we have $M\psi_{i} = \lambda_{i}(M)\psi_{i}$, $\langle \psi_{i}, \psi_{j} \rangle_{\nu} = 1_{i=j}$, and $f = \sum_{i=1}^{V} \langle f, \psi_{i} \rangle_{\nu} \psi_{i}$. For any $f \in \mathbb{R}^{V}$, we have

$$
\|f\|_{2,\nu}^{2} = \langle f, f \rangle_{\nu} = \left(\sum_{i=1}^{V} \langle f, \psi_{i} \rangle_{\nu} \psi_{i}, f \right)_{\nu} = \sum_{i=1}^{V} \langle f, \psi_{i} \rangle_{\nu}^{2},
$$

(17)

$$
\|Mf\|_{2,\nu}^{2} = \sum_{i=1}^{V} \langle Mf, \psi_{i} \rangle_{\nu}^{2} = \sum_{i=1}^{V} \langle Mf, M\psi_{i} \rangle_{\nu}^{2} = \sum_{i=1}^{V} \lambda_{i}(M)^{2} \langle f, \psi_{i} \rangle_{\nu}^{2},
$$

(18)

$$
\langle Mf, f \rangle_{\nu} = \sum_{i=1}^{V} \langle Mf, \psi_{i} \rangle_{\nu} \psi_{i}, f \rangle_{\nu} = \sum_{i=1}^{V} \langle f, M\psi_{i} \rangle_{\nu} \langle \psi_{i}, f \rangle_{\nu} = \sum_{i=1}^{V} \lambda_{i}(M) \langle f, \psi_{i} \rangle_{\nu}^{2}.
$$

(19)

Combining (17) and (19), $\langle Mf, f \rangle_{\nu} \leq \rho(M) \langle f, f \rangle_{\nu}$ holds. Combining (17) and (18), $\|Mf\|_{2,\nu}^{2} \leq \rho(M)^{2} \|f\|_{2,\nu}^{2}$ holds. If $M$ is a transition matrix, we have $\lambda_{1}(M) = 1$ and $\psi_{1} = 1$. Furthermore, $|\lambda_{i}(M)| \leq 1$ holds for all $1 \leq i \leq |V|$. Combining (17) and (19), $\langle Mf, f \rangle_{\nu} \leq \lambda_{1}(M) \langle f, \psi_{1} \rangle_{\nu} + \lambda_{\ast}(M) \langle f, f \rangle_{\nu}$ holds. Combining (17) and (18), $\|Mf\|_{2,\nu}^{2} \leq \lambda_{1}(M)^{2} \langle f, \psi_{1} \rangle_{\nu}^{2} + \lambda_{\ast}(M)^{2} \langle f, f \rangle_{\nu}$ holds.

Lemma A.2 (See, e.g., (12.8) of [27]). Let $P \in [0, 1]^{V \times V}$ be a transition matrix. Suppose that $P$ is reversible with respect to a positive probability vector $\pi \in (0, 1)^{V}$. Then, for any probability vector $\mu \in [0, 1]^{V}$,

$$
\left\| \frac{\mu P}{\pi} - 1 \right\|_{2,\pi}^{2} \leq \lambda_{\ast}(P)^{2} \left\| \frac{\mu}{\pi} \right\|_{2,\pi}^{2}.
$$

Proof. Since $P$ is reversible with respect to $\pi$,

$$
\left( \frac{\mu P}{\pi} \right)(v) = \sum_{u \in V} \frac{\mu(u) P(u, v)}{\pi(v)} = \sum_{u \in V} P(v, u) \frac{\mu(u)}{\pi(u)} = \left( P \left( \frac{\mu}{\pi} \right) \right)(v)
$$

(20)

holds for any $v \in V$, i.e., $\frac{\mu P}{\pi} = P \left( \frac{\mu}{\pi} \right)$ holds. Combining (20) and Lemma A.1, we have

$$
\left\| \frac{\mu P}{\pi} - 1 \right\|_{2,\pi}^{2} = \left\| P \left( \frac{\mu}{\pi} \right) - P1 \right\|_{2,\pi}^{2} = \left\| P \left( \frac{\mu}{\pi} - 1 \right) \right\|_{2,\pi}^{2} \leq \lambda_{\ast}(P)^{2} \left\| \frac{\mu}{\pi} - 1 \right\|_{2,\pi}^{2}.
$$

Note that the third equality follows since $\langle \frac{\mu}{\pi} - 1, 1 \rangle_{\pi} = \sum_{v \in V} \pi(v) \left( \frac{\mu(v)}{\pi(v)} - 1 \right) = 0$. \qed
Lemma A.3 (See, e.g., Proposition 2.5 in [18]). Let $P \in [0,1]^{V \times V}$ be a transition matrix. Suppose that $P$ is lazy and reversible with respect to $\pi \in (0,1)^V$. Then for any $f \in \mathbb{R}^V$,

$$\text{Var}_\pi(Pf) \leq \text{Var}_\pi(f) - \mathcal{E}_{P,\pi}(f,f).$$

Proof. It is straightforward to see that

$$\text{Var}_\pi(Pf) = \langle Pf, Pf \rangle_\pi - \langle Pf, 1 \rangle_\pi^2 = \langle P^2f, f \rangle_\pi - \langle f, Pf \rangle_\pi^2$$

$$= (\langle f, f \rangle_\pi - \langle f, 1 \rangle_\pi^2) - (\langle f, f \rangle_\pi - \langle Pf, f \rangle_\pi) = \text{Var}_\pi(f) - \mathcal{E}_{P^2,\pi}(f,f)$$

holds. From (17) and (19), $\mathcal{E}_{P,\pi}(f,f) = \langle f, f \rangle_\pi - \langle Pf, f \rangle_\pi = \sum_{i=2}^{|V|} (1 - \lambda_i(P)) \langle f, \psi_i \rangle_\pi^2$. Hence, we have $\mathcal{E}_{P^2,\pi}(f,f) = \sum_{i=2}^{|V|} (1 - \lambda_i(P))^2 \langle f, \psi_i \rangle_\pi^2 \geq \sum_{i=2}^{|V|} (1 - \lambda_i(P)) \langle f, \psi_i \rangle_\pi^2 = \mathcal{E}_{P,\pi}(f,f)$. Thus, we obtain the claim.

□

Lemma A.4 (See, e.g., Theorem 4.1 in [32]). Let $V$ be a vertex set and let $P \in [0,1]^{V \times V}$ be a transition matrix over $V$. Suppose that $P$ is lazy and reversible with respect to a positive probability vector $\pi \in (0,1)^V$. Then for any $x, y \in V$ and any $f \in \mathbb{R}^V$,

$$\|D_x P D_y f\|_{2,\pi}^2 \leq \rho(D_x P D_x) \rho(D_y P D_y) \|f\|_2^2.$$ 

Proof. Since $P$ is reversible with respect to $\pi$, the inner product space $(\mathbb{R}^V, \langle \cdot, \cdot \rangle_\pi)$ has an orthonormal basis of real-valued eigenvectors $\{\psi_i\}_{i=1}^{|V|}$ corresponding to real eigenvalues $\{\lambda_i(P)\}_{i=1}^{|V|}$. Hence $P(v,u) = \pi(u) \sum_{i=1}^{|V|} \lambda_i(P) \psi_i(v) \psi_i(u)$. Let $\sqrt{P} \in [0,1]^{V \times V}$ be the positive semidefinite square root of $P$, i.e., $\sqrt{P}(v,u) = \pi(u) \sum_{i=1}^{|V|} \sqrt{\lambda_i(P)} \psi_i(v) \psi_i(u)$. Note that all eigenvalues are nonnegative since $P$ is lazy. It is easy to see that $\langle \sqrt{P} \rangle^2 = P$ and $\pi(v) \sqrt{P}(v,u) = \pi(u) \sqrt{P}(u,v)$ holds for any $u, v \in V$. Hence

$$\pi(v) \langle D_w \sqrt{P} \rangle (v,u) = \pi(v) D_w(v,v) \sqrt{P}(v,u) = \pi(u) D_w(v,v) \sqrt{P}(u,v) = \pi(u) (\sqrt{P} D_w)(u,v)$$

holds for any $u, v \in V$, i.e., $\langle D_w \sqrt{P} f, g \rangle_\pi = \langle f, \sqrt{P} D_w g \rangle_\pi$ holds for any $f, g \in \mathbb{R}^V$. Thus for any $g \in \mathbb{R}^V$, applying Lemma A.1 yields

$$\|D_w \sqrt{P} g\|_{2,\pi}^2 = \langle D_w \sqrt{P} g, D_w \sqrt{P} g \rangle_\pi = \langle D_w P D_w g, g \rangle_\pi \leq \rho(D_w P D_w) \|g\|_{2,\pi}^2,$$

$$\|\sqrt{P} D_w g\|_{2,\pi}^2 = \langle \sqrt{P} D_w g, \sqrt{P} D_w g \rangle_\pi = \langle g, D_w P D_w g \rangle_\pi \leq \rho(D_w P D_w) \|g\|_{2,\pi}^2.$$ 

Consequently,

$$\|D_x P D_y f\|_{2,\pi}^2 = \|D_x \sqrt{P} \sqrt{P} D_y f\|_{2,\pi}^2 \leq \rho(D_x P D_x) \|\sqrt{P} D_y f\|_{2,\pi}^2 \leq \rho(D_x P D_x) \rho(D_y P D_y) \|f\|_{2,\pi}^2$$

holds and we obtain the claim. □

Lemma A.5 (See, e.g., Section 3.6.5 in [2]). Let $V$ be a vertex set and let $P \in [0,1]^{V \times V}$ be a transition matrix over $V$. Suppose that $P$ is irreducible. Then, for any $w \in V$,

$$\rho(D_w P D_w) \leq 1 - \frac{1}{t_{\text{hit}}(P)}.$$
Proof. Define \( P_w \in [0, 1]^{V \setminus \{w\} \times V \setminus \{w\}} \) by \( P_w(u, v) = P(u, v) \) for any \( u, v \in V \setminus \{w\} \). Since \( P_w \) is nonnegative, \( \lambda = \rho(P_w) \) is an eigenvalue of \( P_w \) and there is a nonnegative nonzero vector \( g \in \mathbb{R}^{V \setminus \{w\}} \) such that \( P_w g = \lambda g \) holds (see, e.g., Theorem 8.3.1 in [20]). Write \( Q_w = D_w P D_w \) for convenience. Define \( h \in \mathbb{R}^V \) by \( h(w) = 0 \) and \( h(v) = \frac{g(v)\pi(v)}{Z} \) for any \( v \in V \setminus \{w\} \), where \( Z = \sum_{v \in V \setminus \{w\}} g(v)\pi(v) \). Then, \( h \) is a probability vector. Furthermore,

\[
(hQ_w)(v) = \sum_{u \in V} h(u)Q_w(u, v) = \sum_{u \in V \setminus \{w\}} \frac{g(u)\pi(u)}{Z} P_w(u, v)
= \sum_{u \in V \setminus \{w\}} \frac{g(u)\pi(v)}{Z} P_w(v, u) = \frac{\pi(v)}{Z} (P_w g)(v) = \frac{\pi(v)}{Z} \lambda g(v) = \lambda h(v)
\]

holds for any \( v \in V \setminus \{w\} \). Since \( (hQ_w)(w) = 0 = \lambda h(w) \), we have \( hQ_w = \lambda h \). Hence, \( hQ_w \) holds for any \( t \geq 1 \). This implies that

\[
\Pr_h[\tau_w > t] = \Pr_h[\bigwedge_{i=0}^t \{X_i \neq w\}] = \sum_{v \in V} h(v) \sum_{u \in W} Q_w^t(v, u) = \lambda^t \sum_{u \in V} h(u) = \lambda^t
\]

holds for any \( t \geq 1 \). Since \( P \) is irreducible, there is a \( t^* \geq 1 \) such that \( \Pr_h[\tau_w > t^*] < 1 \). Hence, \( \lambda < 1 \) and we have

\[
E_h[\tau_w] = \sum_{t=0}^{\infty} \Pr_h[\tau_w > t] = \frac{1}{1 - \lambda}.
\]

Note that \( \Pr_h[\tau_w > 0] = 1 \) holds from \( h(w) = 0 \). Thus, \( \rho(D_w P D_w) = \lambda = 1 - \frac{1}{E_h[\tau_w]} \leq 1 - \frac{1}{t_{hit}(P)} \) holds and we obtain the claim. Note that we have \( E_h[\tau_w] = \sum_{v \in V} h(v) E_v[\tau_w] \leq \sum_{v \in V} h(v) t_{hit}(P) = t_{hit}(P) \). \( \square \)

B Other tools

Lemma B.1 (Lemmas 4.24 and 4.25 in [2]). Suppose that \( P \) is irreducible and reversible with respect to \( \pi \). Then, \( \frac{1}{1 - \lambda_2(P)} \leq t_{hit}(P) \leq \frac{2}{\pi_{\min}(1 - \lambda_2(P))} \) holds.

Lemma B.2 (See, e.g., Theorem 4 in [30]). For any connected graph \( G \) of \( n \) vertices, \( t_{hit}(P_{LM}(G)) \leq K(P_{LM}(G)) \leq 12n^2 \) holds.

Lemma B.3 (The Chernoff inequality (see, e.g., Theorem 1.10.5 in [15])). Let \( X_1, X_2, \ldots, X_n \) be independent random variables taking values in \([0, 1]\). Let \( X = \sum_{i=1}^n X_i \). Let \( \delta \geq 0 \). Then, \( \Pr[X \leq (1 - \epsilon) E[X]] \leq \exp \left(-\frac{\epsilon^2 E[X]}{2} \right) \) holds.

Lemma B.4 (Corollary 4.3.15 in [20]). For any symmetric matrices \( A, B \in \mathbb{R}^{V \times V} \) and \( 1 \leq i \leq |V| \), \( \lambda_i(A) + \lambda_1(B) \leq \lambda_i(A + B) \leq \lambda_i(A) + \lambda_{|V|}(B) \) holds.
Lemma B.5. Let \((p_i)_{i \in [n]}\) and \((q_i)_{i \in [n]}\) be sequences of probability distributions over \(V\). Let \(X_1, X_2, \ldots, X_n\) be \(n\) independent random variables such that \(\Pr[X_i = v] = \frac{1}{2}p_i(v) + \frac{3}{4}q_i(v)\) holds for all \(i \in [n]\) and \(v \in V\). Write \(X = (X_1, X_2, \ldots, X_n)\). Let \(I_1, I_2, \ldots, I_n\) be \(n\) independent binary random variables such that \(\Pr[I_i = 1] = 1/4\) holds for all \(i \in [n]\) and let \(I = \{i \in [n] : I_i = 1\}\). For \(S \subseteq [n]\) and \(i \in [n]\), define the probability distribution \(\mu_{S,i} \in [0,1]^V\) by \(\mu_{S,i} = 1_{i \in S}p_i + 1_{i \notin S}q_i\). Then, for any event \(E\) and \(U \in 2^{[n]}\),

\[
\Pr[E] \leq \Pr[I \notin U] + \max_{S \in U} \sum_{(u_1, \ldots, u_n) \in V^n} \Pr[E|X = (u_1, \ldots, u_n)] \prod_{i \in [n]} \mu_{S,i}(u_i).
\]

Proof. Then, for any \((u_1, \ldots, u_n) \in V^n\), we have

\[
\Pr[X = (u_1, \ldots, u_n)] = \prod_{i \in [n]} \Pr[X_i = u_i] = \prod_{i \in [n]} \left(\frac{1}{4}p_i(u_i) + \frac{3}{4}q_i(u_i)\right) = \sum_{S \subseteq [n]} \left(\prod_{i \in S} \frac{1}{4}p_i(u_i)\right) \left(\prod_{i \notin S} \frac{3}{4}q_i(u_i)\right)
\]

\[
= \sum_{S \subseteq [n]} \left(\frac{1}{4}\right)^{|S|} \left(\frac{3}{4}\right)^{|S|} \left(\prod_{i \in S} p_i(u_i)\right) \left(\prod_{i \notin S} q_i(u_i)\right) = \sum_{S \subseteq [n]} \Pr[I = S] \prod_{i \in [n]} \mu_{S,i}(u_i).
\]

Hence, from the definition of the conditional probability,

\[
\Pr[E] = \sum_{u \in V^n} \Pr[X = u] \Pr[E|X = u]
\]

\[
= \sum_{S \subseteq [n]} \Pr[I = S] \sum_{u \in V^n} \Pr[E|X = u] \prod_{i \in [n]} \mu_{S,i}(u_i)
\]

\[
\leq \sum_{S \notin U} \Pr[I = S] + \sum_{S \in U} \Pr[I = S] \sum_{u \in V^n} \Pr[E|X = u] \prod_{i \in [n]} \mu_{S,i}(u_i)
\]

\[
\leq \Pr[I \notin U] + \max_{S \in U} \sum_{u \in V^n} \Pr[E|X = u] \prod_{i \in [n]} \mu_{S,i}(u_i)
\]

holds and we obtain the claim. Here, we use \(u = (u_1, \ldots, u_n) \in V^n\) for convenience. \(\square\)

Lemma B.6. Let \(G = (V, E)\) be a connected graph. Suppose that \(\alpha := \frac{d_{\max}}{d_{\text{min}}} < 4\). Then, for any \(1 \leq i \leq n\), \(|\lambda_i(P_S) - \lambda_i(P_M)| \leq 1 - \frac{\sqrt{\alpha}}{\alpha}\).

Proof. Let \(\pi_S\) denote the stationary distribution of \(P_S\), i.e., \(\pi_S(v) = \deg(v)/(2|E|)\) for all \(v \in V\). Define diagonal matrices \(D_{\pi_S}^{-1/2}, D_{\pi_S}^{1/2} \in \mathbb{R}^{V \times V}\) by \(D_{\pi_S}^{-1/2}(v,v) := \pi_S(v)^{-1/2}\) and \(D_{\pi_S}^{1/2}(v,v) := \pi_S(v)^{1/2}\), respectively. Since \(P_S\) is reversible with respect to \(\pi_S\), \(D_{\pi_S}^{1/2}P_SD_{\pi_S}^{-1/2}\) is symmetric. Indeed, \(D_{\pi_S}^{1/2}P_SD_{\pi_S}^{-1/2} = \sqrt{\frac{\pi_S(v)}{\pi_S(y)}} P_S(x, y) = \frac{1}{\sqrt{\deg(x) \deg(y)}}\) holds for any \(\{x, y\} \in E\). Furthermore, \(D_{\pi_S}^{1/2}P_SD_{\pi_S}^{-1/2}\) and \(P_S\) have the same eigenvalues (see, e.g., Lemma 12.2 in [27]). From Lemma B.4, we have

\[
|\lambda_i(P_S) - \lambda_i(P_M)| = \left|\lambda_i \left(D_{\pi_S}^{1/2}P_SD_{\pi_S}^{-1/2}\right) - \lambda_i(P_M)\right| \leq \rho \left(D_{\pi_S}^{1/2}P_SD_{\pi_S}^{-1/2} - P_M\right)
\]

\[
\leq \max_{x \in V} \sum_{y \in V} \left|D_{\pi_S}^{1/2}P_SD_{\pi_S}^{-1/2} - P_M\right|(x, y).
\]

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For the last inequality, see, e.g., Corollary 6.1.5 in [20]. Since $P_M(x, x) = 1 - \sum_{y \in N(x)} P_M(x, y) = \sum_{y \in N(x)} (P_S(x, y) - P_M(x, y))$, we obtain
\[
\sum_{y \in V} \left| (D_{\pi S}^{1/2} P_S D_{\pi S}^{-1/2} - P_M)(x, y) \right| \\
= \sum_{y \in N(x)} \left( (D_{\pi S}^{1/2} P_S D_{\pi S}^{-1/2})(x, y) - P_M(x, y) \right) + P_M(x, x) \\
= \sum_{y \in N(x)} \left( P_S(x, y) + (D_{\pi S}^{1/2} P_S D_{\pi S}^{-1/2})(x, y) - 2P_M(x, y) \right)
\]
holds for any $x$. Write $d_v = \deg(v)$ for convenience. Then, we have
\[
\sum_{y \in N(x)} \left( \frac{1}{d_x} + \frac{1}{\sqrt{d_x d_y}} - \frac{2}{\max\{d_x, d_y\}} \right) \\
= \sum_{y \in N(x) : d_x > d_y} \frac{1}{d_x} \left( \sqrt{\frac{d_x}{d_y}} - 1 \right) + \sum_{y \in N(x) : d_x < d_y} \frac{1}{d_x} \left( 1 + \frac{1}{\sqrt{d_x d_y}} - \frac{2}{d_y} \right) \\
\leq (\sqrt{\alpha} - 1) \left\{ y \in N(x) : d_x < d_y \right\} + \left( 1 + \frac{1}{\sqrt{\alpha}} - \frac{2}{\alpha} \right) \left\{ y \in N(x) : d_x > d_y \right\} \leq 1 + \frac{1}{\sqrt{\alpha}} - \frac{2}{\alpha}.
\]
The first inequality follows since $\sqrt{\frac{d_x}{d_y}} \geq \frac{1}{\sqrt{\alpha}} > 1/2$. Note that $f(x) = 1 + x - 2x^2$ is nonincreasing for $x \geq 1/4$. \qed