GROUP C*-ALGEBRAS AS DECREASING INTERSECTION OF NUCLEAR C*-ALGEBRAS

YUHEI SUZUKI

Abstract. We prove that for every exact discrete group \( \Gamma \), there is an intermediate C*-algebra between \( C^*_r(\Gamma) \) and \( L(\Gamma) \cap C^u_\nu(\Gamma) \) which is realized as the intersection of a decreasing sequence of isomorphs of the Cuntz algebra \( \mathcal{O}_2 \). In particular, when \( \Gamma \) has the approximation property, the reduced group C*-algebra is realized in this way. We also study extensions of the reduced free group C*-algebras and show that any exact absorbing or unital absorbing extension of it by any stable separable nuclear C*-algebra is realized in this way.

1. Introduction

It is well-known that every exact discrete group admits an amenable action on a compact space [16], and each such action gives rise to an ambient nuclear C*-algebra of the reduced group C*-algebra via the crossed product construction [1]. More generally, it is known that every separable exact C*-algebra is embeddable into the Cuntz algebra \( \mathcal{O}_2 \) [11]. Motivated by these phenomena, we are interested in the following question. How small can we take an ambient nuclear C*-algebra/Cuntz algebra \( \mathcal{O}_2 \) for a given exact C*-algebra? In the present paper, we give an answer to the question for the reduced group C*-algebras of discrete groups with the AP. The next theorem states that an ambient nuclear C*-algebra of the reduced group C*-algebras with the AP can be arbitrarily small in a certain sense.

Theorem A. Let \( \Gamma \) be a countable discrete exact group. Then there is an intermediate C*-algebra \( A \) between the reduced group C*-algebra \( C^*_r(\Gamma) \) and \( L(\Gamma) \cap C^u_\nu(\Gamma) \) satisfying the following properties.

- There is a decreasing sequence of isomorphs of the Cuntz algebra \( \mathcal{O}_2 \) whose intersection is isomorphic to \( A \).
- There is a decreasing sequence \( (A_n)_{n=1}^\infty \) of separable nuclear C*-algebras whose intersection is isomorphic to \( A \) and the sequence admits compatible multiplicative conditional expectations \( (E_n: A_1 \to A_n)_{n=1}^\infty \). Here the compatibility means that the equality \( E_n \circ E_m = E_n \) holds for all \( n \geq m \).

In particular, when the group \( \Gamma \) has the AP, the statements hold for the reduced group C*-algebra \( C^*_r(\Gamma) \).

As a consequence of Theorem A, we obtain the following result.

Corollary B. The decreasing intersection of nuclear C*-algebras need not have the following properties.
The OAP, hence the nuclearity, the CBAP, the WEP, and the SOAP.

The local lifting property.

They can happen simultaneously. The statements are true even when the decreasing sequence admits a compatible family of multiplicative conditional expectations.

Thus the decreasing intersection of nuclear C*-algebras can lost most of good properties. Since the decreasing intersection of injective von Neumann algebras is injective, the analogous results for von Neumann algebras can never be true.

We also give a geometric construction of a decreasing sequence of Kirchberg algebras whose intersection is isomorphic to the hyperbolic group C*-algebra. Although the result follows from Theorem A, this approach has good points. Our decreasing sequence is taken inside the boundary algebra $C(\partial \Gamma) \rtimes \Gamma$. Moreover, the proof does not depend on Kirchberg–Phillips’s $O_2$-absorption theorem and the theory of reduced free products, both of which are used in the proof of Theorem A. Using the sequence constructed by this method, we also study absorbing extensions of the reduced free group C*-algebra by a stable separable nuclear C*-algebras, and prove the following theorem.

**Theorem C.** Let $A$ be a stable separable nuclear C*-algebra and let

$$0 \to A \to B \to C^*_r(\mathbb{F}_d) \to 0$$

be an extension of $C^*_r(\mathbb{F}_d)$ by $A$ ($2 \leq d \leq \infty$). Assume $B$ is exact and the extension is either absorbing or unital absorbing. Then $B$ is realized as a decreasing intersection of isomorphs of the Cuntz algebra $O_2$. In particular, any exact extension of $C^*_r(\mathbb{F}_d)$ by $K$ is realized in this way.

The proof of Theorem C is based on the KK-theory. In the proof, we also obtain the following consequence.

**Theorem D** (Corollary 5.6). For any countable free group $F$, there is a unital embedding of $C^*_r(F)$ into a Kirchberg algebra which implements the KK-equivalence.

**Organization of the paper.** In Section 2, we review some notions and facts used in the paper. In Section 3, we prove Theorem A. We also give few more examples satisfying the conditions in Theorem A. In Section 4, we deal the hyperbolic groups. Based on the study of the boundary action, we construct a decreasing sequence of nuclear C*-algebras inside the boundary algebra $C(\partial \Gamma) \rtimes \Gamma$ whose intersection is the reduced group C*-algebra $C^*_r(\Gamma)$. In Section 5, using the decreasing sequence constructed in Section 4, we prove Theorem C. In Section 6, we study some amenable dynamical systems of the free groups constructed in Section 4.

**Notation.** The symbol ‘⊗’ stands for the minimal tensor product. The symbol ‘∞’ stands for the reduced crossed product of C*-algebras. For a discrete group $\Gamma$ and $g \in \Gamma$, $\lambda_g$ denotes the unitary element of the reduced group C*-algebra $C^*_r(\Gamma)$ corresponding to $g$. For a unital $\Gamma$-C*-algebra $A$ and $g \in \Gamma$, denote by $u_g$ the canonical implementing unitary element of $g$ in the reduced crossed product $A \rtimes \Gamma$. With the same setting, for $x \in A \rtimes \Gamma$ and $g \in \Gamma$, the $g$th coefficient $E(xu_g^*)$ of $x$ is denoted by $E_g(x)$. Here $E: A \rtimes \Gamma \to A$ denotes the canonical conditional
expectation. We denote by $\mathbb{K}$ and $\mathcal{B}$ the C*-algebra of all compact operators and all bounded operators on $\ell^2(\mathbb{N})$ respectively. For a set $X$, denote by $\Delta_X$ the diagonal set $\{(x,x) : x \in X\}$ of $X \times X$. For a subset $Y$ of a topological space $X$, denote by $\text{int}(Y)$ and $\text{cl}(Y)$ the interior and the closure of $Y$ in $X$ respectively.

2. Preliminaries

2.1. Amenability of group actions on compact spaces and C*-algebras. Recall that an action $\Gamma \curvearrowright X$ of a group $\Gamma$ on a compact Hausdorff space is said to be amenable if there is a net $(\zeta_i : X \to \text{Prob}(\Gamma))_{i \in I}$ of continuous maps satisfying

$$\lim_{i \in I} \left( \sup_{x \in X} \| g.\zeta_i(x) - \zeta_i(g.x) \|_1 \right) = 0$$

for all $g \in \Gamma$.

Here $\text{Prob}(\Gamma)$ is equipped with the pointwise convergent topology. More generally, an action of $\Gamma$ on a unital C*-algebra $A$ is said to be amenable if the induced action of $\Gamma$ on the spectrum of the center $Z(A)$ of $A$ is amenable. Here we only review a few properties of amenability of group actions. We refer the reader to [3, Section 4.3] for the details.

Proposition 2.1. Let $\Gamma \curvearrowright A$ be an amenable action of a group $\Gamma$ on a unital C*-algebra $A$. Then the following holds.

- The full and the reduced crossed products coincide.
- The (reduced) crossed product $A \rtimes \Gamma$ is nuclear if and only if $A$ is nuclear.

2.2. Approximation properties for C*-algebras and groups. For C*-algebras $A$, $B$ and a closed subspace $X$ of $B$, we define a subspace $F(A,B,X) \subset A \otimes B$ by

$$F(A,B,X) := \{a \in A \otimes B : \varphi \otimes \text{id}_B(a) \in X \text{ for all } \varphi \in B^*\}.$$ 

A triplet $(A, B, X)$ is said to have the slice map property if the equality $F(A, B, X) = A \otimes X$ holds. We give a definition of the SOAP (strong operator approximation property) and the OAP (operator approximation property) in terms of the slice map property. See [3, Section 12.4] for the details.

Definition 2.2. A C*-algebra $A$ is said to have the SOAP (resp. the OAP) if for any C*-algebra $B$ (resp. for $B = \mathbb{K}$) and for any closed subspace $X$ of $B$, the triplet $(A, B, X)$ satisfies the slice map property.

Obviously, we have the implications.

Nuclearity $\Rightarrow$ CBAP $\Rightarrow$ SOAP $\Rightarrow$ OAP, Exactness

All implications are known to be proper. However, for the reduced group C*-algebras, the SOAP and the OAP are equivalent. The SOAP and the OAP have the strong connection with the property of groups called the AP (approximation property). Here we give the following equivalent condition as a definition of the AP.

Definition 2.3. A discrete group $\Gamma$ is said to have the AP if there exists a net $(\varphi_i)_{i \in I}$ of finitely supported complex valued functions on $\Gamma$ such that $m_{\varphi_i} \otimes \text{id}_\mathbb{K}$ converges to the identity map in the pointwise norm topology. Here $m_{\varphi}(x) := \sum_{g \in \Gamma} \varphi(g) E_g(x) \lambda_g$ is the multiplier of $\varphi$ for a finitely supported function $\varphi$ on $\Gamma$. 
This property is characterized in the following way.

**Proposition 2.4.** Let $\Gamma$ be a discrete group. Then the following are equivalent.

1. The group $\Gamma$ has the AP.
2. The $C^*$-algebra $C^*_{\text{r}}(\Gamma)$ has the SOAP.
3. The $C^*$-algebra $C^*_{\text{r}}(\Gamma)$ has the OAP.
4. There is an intermediate $C^*$-algebra between $C^*_{\text{r}}(\Gamma)$ and $L(\Gamma) \cap C^*_{\text{u}}(\Gamma)$ which has the SOAP or the OAP.

See [3, Section 12.4] for the proof. Note that the implication (4) $\Rightarrow$ (1) follows from the proof of (2), (3) $\Rightarrow$ (1).

A group $\Gamma$ is said to have the ITAP (invariant translation approximation property) if we have the equality $L(\Gamma) \cap C^*_{\text{u}}(\Gamma) = C^*_{\text{r}}(\Gamma)$ (in $\mathbb{B}(\ell^2(\Gamma))$).

Here $C^*_u(\Gamma)$ denotes the uniform Roe algebra of $\Gamma$. Note that under the canonical isomorphism $C^*_u(\Gamma) \cong \ell^\infty(\Gamma) \rtimes \Gamma$, the intersection $L(\Gamma) \cap C^*_u(\Gamma)$ is identified with the $C^*$-subalgebra consisting of elements whose coefficients sit in $\mathbb{C}$. It is shown by Zacharias [27] that every group with the AP has the ITAP. See also Proposition 3.4.

We do not know either the ITAP holds or not for groups without the AP.

### 2.3. Reduced free product.

We refer the reader to [3, Section 4.7] for the definition of the reduced free product. First we recall a few terminology related to a theorem we will use. Let $A$ be a $C^*$-algebra and $\varphi$ be a state on $A$. Recall that the centralizer of $\varphi$ is the set of all elements $b \in A$ satisfying the equality $\varphi(ba) = \varphi(ab)$ for all $a \in A$. An abelian $C^*$-subalgebra $D$ of $A$ is said to be diffuse with respect to $\varphi$ if $\varphi|_D$ is a diffuse measure on the spectrum of $D$.

In the proofs of Theorems A and C, we use the reduced free product to make $C^*$-algebras simple. The following two theorems are important in our proof. The first theorem guarantees the nuclearity of the reduced free product under certain conditions. The second one gives a sufficient condition for the simplicity of the reduced free product.

**Theorem 2.5** (Dykema–Smith [3, Exercise 4.8.2]). Let $(A, \varphi)$ be a pair of a unital nuclear $C^*$-algebra and a non-degenerated state on $A$. Let $\psi$ be a pure state on the matrix algebra $M_n$ ($n \geq 2$). Then the reduced free product $(A, \varphi) * (M_n, \psi)$ is nuclear.

**Theorem 2.6** (Dykema [5, Theorem 2]). Let $(A, \varphi)$ and $(B, \psi)$ be pairs of a unital $C^*$-algebra and a non-degenerated state on it. Assume that $B \neq \mathbb{C}$ and the centralizer of $\varphi$ contains a diffuse abelian $C^*$-subalgebra $D$ containing the unit of $A$. Then the reduced free product $(A, \varphi) * (B, \psi)$ is simple.

A good aspect of these theorems is that we only need to force a condition on one of the states. Thus we can apply these theorems at the same time in many situations.

### 2.4. Extensions of $C^*$-algebras.

Here we recall basic facts and terminologies related to the extensions of $C^*$-algebras. We refer the reader to [2, Sections 15, 17] for
the details. Let $A$ be a unital separable C*-algebra, $B$ be a stable (i.e. $B \cong B \otimes \mathbb{K}$) nuclear C*-algebra. Let

$$0 \to B \to C \to A \to 0$$

be an essential extension of $A$ by $B$. Here essential means that the ideal $B$ of $C$ is essential (i.e., $cB = 0$ implies $c = 0$ for $c \in C$).

Let $\sigma: A \to Q(B) := M(B)/B$ be the Busby invariant of the above extension. Here $M(B)$ denotes the multiplier algebra of $B$. As usual, we identify an extension with its Busby invariant. To define the addition of two extensions, we fix an isomorphism $B \cong B \otimes \mathbb{K}$. (Note that up to canonical identifications, the choice of the isomorphism does not affect to the following definitions.)

An extension $\sigma$ is said to be trivial (resp. strongly unital trivial) if it has a *-homomorphism (resp. unital *-homomorphism lifting) $\hat{\sigma}: A \to M(B)$. Two extensions $\sigma_1$ and $\sigma_2$ are said to be strongly equivalent if there is a unitary element $u$ in $M(B)$ satisfying $\text{ad}(\pi(u)) \circ \sigma_1 = \sigma_2$. An extension $\sigma$ is said to be absorbing (resp. unital absorbing) if for any trivial extension (resp. strongly unital trivial extension) $\tau$, $\sigma \oplus \tau$ is strongly equivalent to $\sigma$. On the class of extensions of $A$ by $B$, we define an equivalence relation as follows. Two extensions $\sigma_1$ and $\sigma_2$ are equivalent if there are trivial representations $\tau_1$ and $\tau_2$ such that the direct sums $\sigma_1 \oplus \tau_i$ are strongly equivalent. The quotient $\text{Ext}(A, B)$ of the class of all extensions by this equivalence relation naturally becomes an abelian semigroup.

Kasparov showed that there exists a unital absorbing strongly unital trivial extension $\tau$ of $A$ by $B$ [9, Theorem 6]. Therefore any $[\sigma] \in \text{Ext}(A, B)$ has a unital absorbing representative. Moreover, if $[\sigma]$ contains a unital extension, then $[\sigma]$ has a unital absorbing unital representative. Note that an element $[\sigma] \in \text{Ext}(A, B)$ contains a unital extension if and only if $[\sigma(1)]_0 = 0$ in $K_0(Q(B))$.

A theorem of Kasparov [9, Theorem 2] shows that for a unital absorbing extension $\sigma$, the direct sum $\sigma \oplus 0$ is an absorbing extension. Thus, by the same reason as above, any element of $\text{Ext}(A, B)$ has an absorbing representative. By definition, such representative is unique up to strongly equivalence.

It follows from [9, Theorem 6] that for any unital C*-subalgebra $C \subset A$, the restriction of the absorbing (resp. unital absorbing) extension to $C$ again has the same property.

Let $\text{Ext}(A, B)^{-1}$ be the subsemigroup of $\text{Ext}(A, B)$ consisting invertible elements. Then there is a natural group isomorphism between $\text{Ext}(A, B)^{-1}$ and $KK^1(A, B)$ [2, Corollary 18.5.4].

3. PROOF OF THEOREM A

Let $\Gamma$ be an exact group. Take an amenable action $\Gamma \curvearrowright X$ on a compact metrizable space which contains at least two points. Define $A_n := C(\prod_{k=n}^\infty X) \rtimes \Gamma$ for each $n \in \mathbb{N}$. Here the action $\Gamma \curvearrowright \prod_{k=n}^\infty X$ is given by the diagonal action. We regard $A_{n+1}$ as a C*-subalgebra of $A_n$ in the canonical way. Since the $\Gamma$-space $\prod_{k=n}^\infty X$ is metrizable and amenable, each $A_n$ is separable and nuclear. Put $A := \bigcap_{n=1}^\infty A_n$. We will show that $A$ is isomorphic to an intermediate C*-algebra between $C^*_r(\Gamma)$ and $C_0(\Gamma) \cap L(\Gamma)$, To see this, take an arbitrary point $x \in \prod_{k=1}^\infty X$ and define $\rho: C(\prod_{k=1}^\infty X) \to \ell^\infty(\Gamma)$ by $\rho(f)(s) := f(s, x)$ for $f \in C(\prod_{k=1}^\infty X)$. 
and \( s \in \Gamma \). Then \( \rho \) is a \( \Gamma \)-equivariant \( \ast \)-homomorphism. Hence it induces a \( \ast \)-homomorphism \( \tilde{\rho} : A_1 \to \ell^\infty(\Gamma) \rtimes \Gamma \). Note that for all \( a \in A \) and \( g \in \Gamma \), we have \( E_g(a) \in \bigcap_{n=1}^\infty C(\prod_{k=n}^\infty X) = \mathbb{C} \). Here the last equality follows from the existence of a conditional expectation from \( C(\prod_{k=1}^\infty X) \) onto \( C(\prod_{k=1}^n X) \) for each \( n \). This shows that \( \tilde{\rho} \) is injective on \( A \) and \( \tilde{\rho}(A) \) is contained in \( C_u^*(\Gamma) \cap L(\Gamma) \). Thus \( A \) is isomorphic to the desired \( C^* \)-algebra.

Next we show that there is a compatible family of multiplicative conditional expectations \( (E_n : A_1 \to A_n)_{n=1}^\infty \). Let \( E_n \) be the \( \ast \)-homomorphism induced from the \( \Gamma \)-equivariant \( \ast \)-homomorphism

\[
E_n : C\left(\prod_{k=1}^\infty X\right) \to C\left(\prod_{k=n}^\infty X\right)
\]

defined by

\[
E_n(f)(x_n, x_{n+1}, x_{n+2}, \ldots) := f(x_n, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots),
\]

where, in the right hand side, \( x_n \) is iterated \( n \) times. Then it is not difficult to check that they satisfy the desired conditions.

To make terms isomorphic to the Cuntz algebra \( \mathcal{O}_2 \), we first make terms simple. To do this, take a faithful state \( \nu \) on \( A_1 \). Take a compact metric space \( Y \) consisting at least two points and a faithful measure \( \mu \) on \( Y \). On \( (\bigotimes_{k=1}^\infty C(Y)) \otimes A_1 \), define a faithful state \( \varphi \) by \( \varphi := (\bigotimes_{k=1}^\infty \mu) \otimes \nu \). Then define a faithful state \( \varphi_n \) on \( B_n := (\bigotimes_{k=1}^\infty C(Y)) \otimes A_n \) to be the restriction of \( \varphi \). Now take a pure state \( \psi \) on \( M_2 \) and put \( C_n := (B_n, \varphi_n) \ast (\bigotimes_{k=n}^\infty (M_2, \psi)) \). Then by Theorem 2.6, each \( C_n \) is simple. Moreover, since \( C_n \) is the increasing union of finite free products \( ((A_n, \varphi_n) \ast (\bigotimes_{k=n}^\infty (M_2, \psi)))_{m=n}^\infty \), each \( C_n \) is nuclear by Theorem 2.5. For each \( n \in \mathbb{N} \), we have a conditional expectation from \( C_1 \) onto \( (B_1, \varphi) \ast (\bigotimes_{k=1}^\infty (M_2, \psi)) \). This proves the equalities

\[
\bigcap_{n=1}^\infty C_n = \bigcap_{n=1}^\infty B_n = \bigcap_{n=1}^\infty A_n = A.
\]

Finally, to make terms isomorphic to \( \mathcal{O}_2 \), we apply Kirchberg–Phillips’s \( \mathcal{O}_2 \) absorption theorem [11]. We define a new sequence \( (D_n)_{n=1}^\infty \) by \( D_n := C_n \otimes (\bigotimes_{k=n}^\infty \mathcal{O}_2) \). Then each \( D_n \) is isomorphic to \( \mathcal{O}_2 \) and we have

\[
\bigcap_{k=1}^\infty D_n = \bigcap_{k=1}^\infty C_n = A.
\]

\( \square \)

**Remark 3.1.** There is an isomorphism between the decreasing intersection \( A = \bigcap_{n \in \mathbb{N}} (C(\prod_{k=n}^\infty X) \rtimes \Gamma) \) and the \( C^* \)-algebra

\[
B = \{ a \in C(X) \rtimes \Gamma : E_g(a) \in \mathbb{C} \text{ for all } g \in \Gamma \}
\]

that preserves the reduced group \( C^* \)-algebra. To see this, consider the quotient map \( \pi : C(\prod_{k=1}^\infty X) \rtimes \Gamma \to C(X) \rtimes \Gamma \) induced from the diagonal embedding \( X \to \prod_{k \in \mathbb{N}} X \). Then \( \pi \) is injective on \( A \). To see the equality \( \pi(A) = B \), consider the embedding of
C(X) into $C(\prod_{k=1}^{\infty} X)$ induced from the quotient map from $\prod_{k=1}^{\infty} X$ onto the $n$th product component for each $n \in \mathbb{N}$.

Therefore, the question either the equation

$$\bigcap_{n \in \mathbb{N}} \left( C(\prod_{k=n}^{\infty} X) \rtimes \Gamma \right) = C_*^r(\Gamma)$$

holds or not seems difficult when the group $\Gamma$ does not have the AP. Indeed, if the equation holds for every compact metrizable $\Gamma$-space $X$ (when $\Gamma$ is exact, we only need to consider the amenable one), then $\Gamma$ has the ITAP. However, we do not know either a given group has the ITAP or not for groups without the AP.

Now we can prove Corollary [13].

**Proof of Corollary** [13]. We apply Theorem A to $\Gamma := \text{SL}(3, \mathbb{Z})$. (See [3] Section 5.4 for the exactness of $\Gamma$.) This gives an intermediate $C^*$-algebra $A$ between $C_*^r(\Gamma)$ and $L(\Gamma) \cap C_*^u(\Gamma)$ satisfying the conditions in Theorem A. We show that $A$ does not have the OAP and the local lifting property. Since $\Gamma$ does not have the AP, Proposition 3.3 yields that $A$ does not have the OAP.

Next take a subgroup $\Lambda$ of $\Gamma$ isomorphic to the free group $F_2$ [3, Example E.10]. Denote by $p \in B(\ell^2(\Gamma))$ the projection onto the subspace $\ell^2(\Lambda)$. Then the compression by $p$ gives a conditional expectation $E_A^\Lambda : C_u^*(\Gamma) \to C_u^*(\Lambda)$. It is clear from the definition that $E_A^\Lambda$ maps $L(\Gamma) \cap C_u^*(\Gamma)$ into $L(\Lambda) \cap C_u^*(\Lambda)$. Since $\Lambda$ has the AP, we obtain the conditional expectation $\Phi : A \to C_u^*(\Lambda)$.

Since $C_u^*(\Lambda)$ does not have the local lifting property [3, Corollary 3.7.12], $A$ also does not have it.

**Other examples.** We end this section by giving few more examples satisfying the conditions in Theorem A.

**Proposition 3.2.** Let $A$ be a unital separable nuclear $C^*$-algebra, $\Gamma$ be a group with the AP. Then for any action of $\Gamma$ on $A$, the reduced crossed product $A \rtimes \Gamma$ satisfies the conditions mentioned in Theorem A.

Let $A$ be a unital $C^*$-algebra. Let $\Gamma$ be a group and $X$ be a $\Gamma$-set. Consider the reduced crossed product $A^{\otimes X} \rtimes \Gamma$, where $\Gamma$ acts on $A^{\otimes X}$ by the shift of tensor components. We say it the generalized wreath product of $A$ with respect to $X$ and denote it by $A \wr X \Gamma$.

**Proposition 3.3.** The class of unital $C^*$-algebras with the SOAP satisfying the conditions in Theorem A is closed under taking the following operations.

1. Countable (minimal) tensor products.
2. The generalized wreath product with respect to any $\Gamma$-set with $\Gamma$ the AP.

To prove Propositions 3.2 and 3.3 we need the following proposition. The idea of the proof is essentially contained in [27].
**Proposition 3.4.** Let \( \Gamma \) be a group with the AP. Let \( A \) be a \( \Gamma \)-C*-algebra and let \( X \) be a closed subspace of \( A \). Assume that an element \( x \in A \rtimes \Gamma \) satisfies \( E_g(x) \in X \) for all \( g \in \Gamma \). Then \( x \) is contained in the closed subspace 

\[
X \rtimes \Gamma := \overline{\text{span}}\{ xu_g : x \in X, g \in \Gamma \}.
\]

Conversely, if the above implication always holds for any \( \Gamma \)-C*-algebra and its closed subspace, then the group \( \Gamma \) has the AP.

**Proof.** Since \( \Gamma \) has the AP, there is a net \((\varphi_i)_{i \in I}\) of finitely supported functions on \( \Gamma \) satisfying the condition in Definition 2.3. For \( i \in I \), define the linear map \( \Phi_i \colon A \rtimes \Gamma \to A \rtimes \Gamma \) by \( \Phi_i(y) := \sum_{g \in \Gamma} \varphi_i(g) E_g(y) u_g \). We claim that the net \((\Phi_i)_{i \in I}\) converges to the identity map in the pointwise norm topology. To show this, consider the embedding \( \iota : A \rtimes \Gamma \to (A \rtimes \Gamma) \otimes C_b(\Gamma) \) induced from the maps \( a \in A \mapsto a \otimes 1 \) and \( u_g \in \Gamma \mapsto u_g \otimes \lambda_g \). (This indeed defines an embedding by Fell’s absorption principle [3, Prop.4.1.7].) Then the composite \( \iota \circ \Phi_i \) coincides with the composite \((\text{id} \otimes m_n) \circ \iota \). This proves the convergence condition. Now let \( x \) be as stated. Then for any \( i \in I \), we have \( \Phi_i(x) \in X \rtimes \Gamma \). Since the net \((\Phi_i(x))_{i \in I}\) converges in norm to \( x \), we have \( x \in X \rtimes \Gamma \).

To show the converse, apply the above condition to the case \( \Gamma \)-action is trivial. \( \square \)

As a consequence, we obtain a permanence property of the SOAP and the OAP.

**Corollary 3.5.** The SOAP and the OAP are preserved under taking the reduced crossed product of a group with the AP.

**Proof.** We only give a proof for the SOAP. Let \( A \) be a \( \Gamma \)-C*-algebra with the SOAP. Let \( B \) be a C*-algebra and \( X \) be its closed subspace. To show the SOAP of \( A \rtimes \Gamma \), it suffices to prove the inclusion \( F(A \rtimes \Gamma, B, X) \subseteq (A \rtimes \Gamma) \otimes X \). Let \( x \in F(A \rtimes \Gamma, B, X) \). Then \((E_g \otimes \text{id}_B)(x) \in F(A, B, X)\) for all \( g \in \Gamma \). Since \( A \) has the SOAP, we have \( F(A, B, X) \subseteq A \otimes X \). Then from Proposition 3.4 we conclude \( x \in (A \rtimes \Gamma) \otimes X \). Here we use the canonical identification of \((A \rtimes \Gamma) \otimes B\) with \((A \otimes B) \rtimes \Gamma\). \( \square \)

**Remark 3.6.** The similar proofs also show the \( W^* \)-analogues of Proposition 3.4 and Corollary 3.5. We note that the \( W^* \)-analogue of Corollary 3.5 is shown by Haagerup and Kraus for locally compact groups with the AP [8, Theorem 3.2].

**Proof of Proposition 3.2** Replace \( C(\prod_{k=1}^\infty X) \) by \( C(\prod_{k=1}^\infty X) \otimes A \) with the diagonal \( \Gamma \)-action in the proof of Theorem 3.1. \( \square \)

**Proof of Proposition 3.3** We only prove the second claim.

First take a decreasing sequence \((A_n)_{n=1}^\infty\) of separable nuclear C*-algebras whose intersection is isomorphic to \( A \) and that admits a compatible family of multiplicative conditional expectations. We will use \( C(\prod_{k=1}^\infty X) \otimes A_n^{\otimes X} \) instead of \( C(\prod_{k=1}^\infty X) \) in the proof of Theorem A. To do this, we remark that the equality

\[
\bigcap_{n=1}^\infty \left(C(\prod_{k=1}^\infty X) \otimes (A_n^{\otimes X})\right) = A^{\otimes X}
\]

holds since C*-algebras \( A \) and \( A_n \) have the SOAP. \( \square \)
4. Hyperbolic group case

In this section, we give a geometric construction of a decreasing sequence of Kirchberg algebras whose decreasing intersection is isomorphic to the hyperbolic group C*-algebra. We construct such sequence inside the boundary algebra $C(\partial \Gamma) \rtimes \Gamma$. To find such sequence, we construct amenable quotients of the boundary space. The proof does not depend on both the reduced free product theory and Kirchberg–Phillips’s $O_2$-absorption theorem. We will use the sequence constructed in this section for the free group case in the next two sections.

For the definition and basic properties of hyperbolic groups, we refer the reader to [3, Section 5.3] and [7]. (For the reader who is only interested in the free group case, we recommend to concentrate on that case. In this case, some arguments related to geodesic paths becomes much simpler.) Here we only recall a few facts. (See 8.16, 8.21, 8.28, and 8.29 in [7].) For a torsion free element $t$ of a hyperbolic group $\Gamma$, the sequence $(t^n)_{n=1}^{\infty}$ is quasi-geodesic. The boundary action of $t$ has exactly two fixed points. They are the points represented by the quasi-geodesic paths $(t^n)_{n=1}^{\infty}$ and $(t^{-n})_{n=1}^{\infty}$. We denote them by $t^+_{\infty}$ and $t^-_{\infty}$ respectively. For any neighborhoods $U_{\pm}$ of $t_{\pm}^{\infty}$, there is $n \in \mathbb{N}$ such that for any $m \geq n$, $t^m(\partial \Gamma \setminus U_-) \subset U_+$ holds.

For a metric space $(X, d)$ and its points $x, y, z \in X$, we denote by $\langle y, z \rangle_x$ the Gromov product $(d(y, x) + d(z, x) - d(y, z)) / 2$ of $y, z$ with respect to $x$.

We recall the following criteria for the Housdorffness of a quotient space. We left the proof to the reader.

**Proposition 4.1.** Let $X$ be a compact Hausdorff space. Let $\mathcal{R}$ be an equivalence relation on $X$. Assume that each equivalence class of $\mathcal{R}$ is compact and the quotient map $\pi: X \to X/\mathcal{R}$ is closed. Then the quotient space $X/\mathcal{R}$ is Hausdorff.

The next lemma guarantees the amenability of certain quotients of amenable dynamical systems. We are grateful to Narutaka Ozawa for letting us known Lusin’s theorem [23, Theorem 5.8.1].

**Lemma 4.2.** Let $\Gamma$ be a group, $X$ be an amenable compact metrizable $\Gamma$-space. Let $\mathcal{R}$ be a $\Gamma$-invariant equivalence relation on $X$ such that the quotient space $X/\mathcal{R}$ is Hausdorff. Assume that each equivalence class of $\mathcal{R}$ is finite. Then $X/\mathcal{R}$ is again an amenable compact $\Gamma$-space.

To prove Lemma 4.2, we need the following characterization of amenability due to Anantharaman-Delaroche [11, Theorem 4.5]. See also [3, Prop. 5.2.1] for a generalized version.

**Proposition 4.3.** Let $\alpha: \Gamma \curvearrowright X$ be an action of $\Gamma$ on a compact metrizable space $X$. Then $\alpha$ is amenable if and only if there exists a net $(\zeta_i: X \to \text{Prob}(\Gamma))_{i \in I}$ of Borel maps which satisfies the following condition.

$$\lim_{i \in I} \int_X \|g.\zeta_i(x) - \zeta_i(g.x)\|_1 \; d\mu = 0$$

for all $\mu \in \text{Prob}(X)$ and $g \in \Gamma$.

Here $\text{Prob}(X)$ denotes the set of all Borel probability measures on $X$.

**Proof of Lemma 4.2.** Since $\mathcal{R}$ is closed in $X \times X$ and each equivalence class is finite, Lusin’s theorem [23, Theorem 5.8.1] tells us that $\mathcal{R}$ is presented as a countable
disjoint union of graphs of Borel maps between Borel subsets of \( X \). Then it is not hard to check that for each \( f \in C(X) \), the function \( \tilde{f} \) on \( X/R \) defined by

\[
\tilde{f}([x]) := \frac{1}{\sharp[x]} \sum_{y \in [x]} f(y)
\]

is Borel. By the same reason, the similar formula also defines the map \( \Phi \) from \( C(X, \text{Prob}(\Gamma)) \) to \( B(X/R, \text{Prob}(\Gamma)) \). Here \( C(X, \text{Prob}(\Gamma)) \) denotes the set of all continuous maps from \( X \) into \( \text{Prob}(\Gamma) \) and \( B(X/R, \text{Prob}(\Gamma)) \) denotes the set of all Borel maps from \( X/R \) into \( \text{Prob}(\Gamma) \).

Let \((\zeta_i : X \to \text{Prob}(\Gamma))_{i \in I}\) be a net of continuous maps that satisfies the condition in the definition of the amenability for \( \Gamma \curvearrowright X \). Consider the net \((\Phi(\zeta_i))_{i \in I}\). Then for any \( g \in \Gamma \), \( x \in X \), and \( i \in I \), we have

\[
\| (g.\Phi(\zeta_i))(\{x\}) - \Phi(\zeta_i)(g.\{x\}) \|_1 \leq \frac{1}{\sharp[x]} \sum_{y \in [x]} \| g.\zeta_i(y) - \zeta_i(g.y) \|_1.
\]

Thus, for each \( g \in \Gamma \), the norms \( \| (g.\Phi(\zeta_i))(\{x\}) - \Phi(\zeta_i)(g.\{x\}) \|_1 \) converge to 0 uniformly on \( X/R \) as \( i \) tends to \( \infty \). In particular, the net \((\Phi(\zeta_i))_{i \in I}\) satisfies the condition in Proposition 3.3.

**Lemma 4.4.** Let \( \Gamma \) be a hyperbolic group. Let \( T \) be a finite set of torsion free elements of \( \Gamma \). Then the set

\[
R_T := \Delta \partial \Gamma \cup \{(g.t^{+\infty}, g.t^{-\infty}) : g \in \Gamma, t \in T \cup T^{-1}\}
\]

is a \( \Gamma \)-invariant equivalence relation on \( \partial \Gamma \). Moreover, the quotient space \( \partial \Gamma/R_T \) is a Hausdorff space.

**Proof.** Clearly \( R_T \) is \( \Gamma \)-invariant. Let \( s, t \) be torsion free elements of \( \Gamma \). Then the two sets \( \{s^{+\infty}\} \) and \( \{t^{+\infty}\} \) are either disjoint or the same [7, 8.30]. Therefore the set \( R_T \) is an equivalence relation. Note that this shows that each equivalence class of \( R_T \) contains at most two points.

For Hausdorffness of the quotient space, since each equivalence class only consists finitely many points, it suffices to show that the quotient map \( \pi : \partial \Gamma \to \partial \Gamma/R_T \) is closed. Let \( A \) be a closed subset of \( \partial \Gamma \). Then \( \pi^{-1}(\pi(A)) = A \cup B \), where

\[
B := \{g.t^{-\infty} \in \partial \Gamma : g \in \Gamma, t \in T \cup T^{-1}, g.t^{+\infty} \in A\}.
\]

To show closedness of \( \pi(A) \), which is equivalent to that of \( \pi^{-1}(\pi(A)) \), it suffices to show that \( \text{cl}(B) \subset A \cup B \). Fix a finite generating set \( S \) of \( \Gamma \) and denote by \( | \cdot | \) and \( d(\cdot, \cdot) \) the length function and the left invariant metric on \( \Gamma \) determined by \( S \) respectively. Take \( \delta > 0 \) with the property that every geodesic triangle in \( (\Gamma, d) \) is \( \delta \)-thin [3, Proposition 5.3.4]. Let \( x \in \text{cl}(B) \) and take a sequence \( (g_n.t_n^{-\infty})_{n=1}^{\infty} \) in \( B \) which converges to \( x \). By passing to a subsequence, we may assume that there is \( t \in T \cup T^{-1} \) with \( t_n = t \) for all \( n \in \mathbb{N} \). Replace \( g_n \) by \( g_n^{l(n)} \) for some \( l(n) \in \mathbb{Z} \) for each \( n \in \mathbb{N} \), we may further assume \( |g_n| \leq |g_n|^{t_k} \) for all \( k \in \mathbb{Z} \) and \( n \in \mathbb{N} \). If the sequence \( (g_n)_{n=1}^{\infty} \) has a bounded subsequence, then it has a constant subsequence. Hence we have \( x \in B \). Assume \( |g_n| \to \infty \). For each \( k \in \mathbb{Z} \), take a geodesic path \([e, t^k]\) from \( e \) to \( t^k \). Since \( t \) is torsion free, the sequences \((t^n)_{n=1}^{\infty} \)
forces that the subgroup $G$ be an infinite torsion subgroup, a contradiction. Thus we can take $g$ satisfies $\text{int}(F) \subseteq \partial G \setminus \{0\}$. The order of $g$ must be finite. Assume $F_{g_1} = L_\Lambda$. This means that the kernel of $\varphi_\Lambda$ is not nontrivial. Since it cannot contain a torsion free element, it is a nontrivial torsion subgroup. Therefore it must be finite. This contradicts $\langle G, t \rangle$ as above. Hence $\varphi_\Lambda$ is not topologically free. Take an element $g_1 \in \Lambda \setminus \{e\}$ such that the set $F_{g_1} := \{x \in L_\Lambda : g_1 x = x\}$ has a nontrivial interior. Since $L_\Lambda$ does not have an isolated point, the order of $g_1$ must be finite. Assume $F_{g_1} = L_\Lambda$. This means that the kernel of $\varphi_\Lambda$ is nontrivial. Since it cannot contain a torsion free element, it is a nontrivial torsion subgroup. Therefore it must be finite. This contradicts the ICC condition. For a subgroup $G$ of $\Lambda$, we set $F_G := \bigcap_{g \in G} F_g$. Note that for a subgroup $G$ of $\Lambda$ and $g \in \Lambda$, we have $F_{gGg^{-1}} = gF_G$. Set $G_1 := \langle g_1 \rangle$. Then $\text{int}(F_{G_1}) = \text{int}(F_{g_1}) \neq \emptyset$. We will show that there is $g_2 \in \Lambda$ satisfying

\[ \emptyset \neq g_2(\text{int}(F_{G_1})) \cap \text{int}(F_{G_1}) \subseteq \text{int}(F_{G_1}). \]

Indeed, if such $g_2$ does not exist, then the family $\{g(\text{int}(F_{G_1})) : g \in \Lambda\}$ makes an open covering of $L_\Lambda$ whose members are mutually disjoint. (Note that if $g \in \Lambda$ satisfies $\text{int}(F_{G_1}) \not\subseteq g(\text{int}(F_{G_1}))$, then $g^{-1}$ satisfies the required condition.) This forces that the subgroup

$A_0 := \{g \in \Lambda : g(\text{int}(F_{G_1})) = \text{int}(F_{G_1})\}$

has finite index in $\Lambda$. Since $\Lambda$ is ICC, the subgroup $G := \langle gG_1g^{-1} : g \in A_0 \rangle$ must be infinite. Moreover, by definition, we have $\text{int}(F_G) = \text{int}(F_{G_1}) \neq 0$. Hence $G$ must be an infinite torsion subgroup, a contradiction. Thus we can take $g_2 \in \Lambda$ as above. Set $G_2 = \langle G_1, g_2G_1g_2^{-1} \rangle$. Then we have $\emptyset \neq \text{int}(F_{G_2}) \subseteq \text{int}(F_{G_1})$. This shows that
$G_2$ is still finite and is larger than $G_1$. Continuing this argument inductively, we obtain a strictly increasing sequence $(G_n)_{n=1}^\infty$ of finite subgroups of $\Lambda$. Then the union $\bigcup_{n=1}^\infty G_n$ is an infinite torsion subgroup of $\Lambda$, again a contradiction. □

**Remark 4.6.** Conversely, if $\Lambda$ is not ICC, then the action on the limit set $L_\Lambda$ is not faithful. In this case, $\Lambda$ contains a finite index subgroup $\Lambda_0$ with the nontrivial center $Z_0$. Since $L_{\Lambda_0} = L_\Lambda$, the group $Z_0$ acts on $L_\Lambda$ trivially.

**Lemma 4.7.** For $\Lambda$ as in Lemma 4.5, the equivalence relation

$$\mathcal{R} := \left( \bigcup_{t \in \Lambda, \text{torsion free}} \mathcal{R}_t \right) \cap (L_\Lambda \times L_\Lambda)$$

on $L_\Lambda$ is dense in $L_\Lambda \times L_\Lambda$.

**Proof.** Let $s$ and $t$ be two torsion free elements in $\Lambda$ which do not have a common fixed point. For any neighborhoods $U_\pm$ of $s^{\pm\infty}$ and neighborhoods $V_\pm$ of $t^{\pm\infty}$ with the properties $U_+ \cap V_- = \emptyset$ and $U_- \cap V_+ = \emptyset$, take a natural number $N$ satisfying $s^N(\partial \Lambda \setminus U_-) \subseteq U_+$ and $t^N(\partial \Lambda \setminus V_+) \subseteq V_-$. Then, for any $m \in \mathbb{N}$, we have $(s^N t^N)^m(\partial \Lambda \setminus V_-) \subseteq U_+$ and $(s^N t^N)^{-m}(\partial \Lambda \setminus U_+) \subseteq V_-$. This shows that the element $s^N t^N$ is torsion free, $(s^N t^N)^{+\infty} \in \text{cl}(U_+)$, and $(s^N t^N)^{-\infty} \in \text{cl}(V_-)$. Thus the product $\text{cl}(U_+) \times \text{cl}(V_-)$ intersects with $\mathcal{R}$. This proves density of $\mathcal{R}$. □

Recall that an action $\Gamma \curvearrowright X$ of a group on a compact Hausdorff space is called locally boundary if for any nonempty open set $U \subset X$, there is an open set $V \subset U$ and an element $t \in \Gamma$ such that $\text{cl}(t.V) \subsetneq V$ holds [12, Definition 6].

**Lemma 4.8.** Let $\Lambda$ and $\Gamma$ be as in Lemma 4.5. Let $T$ be a finite set of torsion free elements of $\Lambda$. Then the action $\Gamma \curvearrowright (L_\Lambda/(\mathcal{R}_T \cap L_\Lambda \times L_\Lambda))$ is locally boundary.

**Proof.** Let $s$ be a torsion free element of $\Lambda$ whose fixed points are not equal to $gt^{\pm\infty}$ for any $g \in \Lambda$ and $t \in T$. Then $\pi(s^{+\infty}) \neq \pi(s^{-\infty})$. Hence, on the set $\pi(L_\Lambda \setminus \{s^{+\infty}\})$, the sequence $(s^n x)_{n=1}^\infty$ converges to $\pi(s^{+\infty})$ uniformly on compact subsets. Thus for any neighborhood $U$ of $\pi(s^{+\infty})$ whose closure does not contain $\pi(s^{-\infty})$, there is $n \in \mathbb{N}$ such that $s^n(\text{cl}(U)) \subsetneq U$. From the minimality of $\Gamma \curvearrowright L_\Lambda$, now it is easy to conclude that the action is locally boundary. □

**Theorem 4.9.** Let $\Lambda$ be a subgroup of a hyperbolic group $\Gamma$. Then there is a decreasing sequence of nuclear $C^*$-subalgebras of $C(L_\Lambda) \rtimes \Lambda$ whose intersection is equal to $C^*_r(\Lambda)$. Moreover, if $\Lambda$ is ICC, then we can find such sequence with the terms Kirchberg algebras in the UCT class.

**Proof.** Let $(\overline{\mathcal{R}}_n)_{n=1}^\infty$ be an increasing sequence of finite subsets of torsion free elements whose union contains all torsion free elements. Define $\mathcal{R}_n := \mathcal{R}_n \cap (L_\Lambda \times L_\Lambda)$ for each $n$. Put $A_n := C(L_\Lambda/\mathcal{R}_n) \rtimes \Lambda$. Then by Lemmas 1.2 and 1.3, each $A_n$ is nuclear. Moreover, by Lemma 4.7, we have $\bigcap_{n=1}^\infty C(L_\Lambda/\mathcal{R}_n) = \mathbb{C}$. Since every hyperbolic group is weakly amenable [17], we have the equality

$$\bigcap_{n=1}^\infty A_n = C^*_r(\Lambda).$$
When $\Lambda$ is ICC, a similar proof to that of Lemma 4.5 shows the topological freeness of $\Lambda \rtimes L_\Lambda$. Since each $L_\Lambda / \mathcal{R}_n$ is local boundary, Theorem 9 of [12] yields that each $A_n$ is a Kirchberg algebra. Since each $A_n$ is the groupoid $C^*$-algebra of an amenable étale groupoid, they are in the UCT class by Tu’s theorem [25]. □

5. Extensions of $C^*_\ell (\mathbb{F}_d)$ by nuclear $C^*$-algebras

In this section, we prove Theorem C.

We first consider the case $d$ is finite. We deal the case $d = \infty$ in the end of this section. Denote by $S$ the set of all canonical generators of $\mathbb{F}_d$. We denote by $| \cdot |$ the length function on $\mathbb{F}_d$ determined by $S$. To prove Theorem C first we compute the $K$-theory of the crossed product $C(\partial \mathbb{F}_d / \mathcal{R}_S) \rtimes \mathbb{F}_d$.

We always use the following standard picture of the Gromov boundary $\partial \mathbb{F}_d$.

$$\partial \mathbb{F}_d := \left\{ (x_n)_{n=1}^\infty : x_n \neq x_{n+1}^{-1} \text{ for all } n \in \mathbb{N} \right\}$$

equipped with the relative product topology. For $w \in \mathbb{F}_d$, we denote by $p[w]$ the characteristic function of the clopen set

$$\{(x_n)_{n=1}^\infty \in \partial \mathbb{F}_d : x_1 \cdots x_{|w|} = w\}$$

equipped with the relative product topology. For $w \in \mathbb{F}_d$, we denote by $p[w]$ the characteristic function of the clopen set

$$\left\{ (x_n)_{n=1}^\infty \in \partial \mathbb{F}_d : x_1 \cdots x_{|w|} = w \right\}$$

and set $q[w] := p[w] + p[w^{-1}]$. Throughout this section, we identify $C(\partial \mathbb{F}_d / \mathcal{R}_S)$ with the $\mathbb{F}_d$-$C^*$-subalgebra of $C(\partial \mathbb{F}_d)$ in the canonical way. Under this identification, it is not difficult to check that for $s \in S$, $q[s]$ is contained in $C(\partial \mathbb{F}_d / \mathcal{R}_S)$. We denote the action $\mathbb{F}_d \rtimes C(\partial \mathbb{F}_d)$ by $\mathcal{P}$.

**Lemma 5.1.** The $C^*$-algebra $C(\partial \mathbb{F}_d / \mathcal{R}_S)$ is generated by the set

$$\mathcal{P} := \{wq[s] : w \in \mathbb{F}_d, s \in S\}.$$

In particular, the space $\partial \mathbb{F}_d / \mathcal{R}_S$ is homeomorphic to the Cantor set.

**Proof.** By the Stone–Weierstrass theorem, it suffices to show that the set $\mathcal{P}$ separates the points of $\partial \mathbb{F}_d / \mathcal{R}_S$. Let $x = (x_n)_{n=1}^\infty$ and $y = (y_n)_{n=1}^\infty$ be two elements in $\partial \mathbb{F}_d$ satisfying $(x, y) \notin \mathcal{R}_S$. If $x \notin \{ws^{\pm \infty} : w \in \mathbb{F}_d, s \in \mathbb{S} \}$, then take $n \in \mathbb{N}$ with $x_n \neq y_n$. Let $m$ be the smallest integer greater than $n$ satisfying $x_m \neq y_m$ (which exists by assumption). Then the projection $(x_1 \cdots x_{m-1})q[x_m]$ separates $x$ and $y$. Next consider the case $x = zs^{\pm \infty}$, $y = wt^{\pm \infty}$, where $s, t \in \mathbb{S}$ and $z, w$ are elements of $\mathbb{F}_d$ whose last alphabets are not equal to $s^{\pm 1}, t^{\pm 1}$, respectively. Assume $|z| \geq |w|$. Note that the equality $z = w$ implies $s \neq t^{\pm 1}$ by assumption. Hence the projection $zq[s]$ separates $x$ and $y$. Thus $\mathcal{P}$ satisfies the required condition.

The last assertion now follows from the following fact. A topological space is homeomorphic to the Cantor set if and only if it is compact, metrizable, totally disconnected, and does not have any isolated point. □

**Lemma 5.2.** The $K_0$-group of $C(\partial \mathbb{F}_d / \mathcal{R}_S) \rtimes \mathbb{F}_d$ is generated by $\{[q[s]]_0 : s \in S\}$.

**Proof.** By Lemma 5.1 and the Pimsner–Voiculescu exact sequence [19], the $K_0$-group is generated by the elements represented by a projection in $C(\partial \mathbb{F}_d / \mathcal{R}_S)$. Let $r$ be a projection in $C(\partial \mathbb{F}_d / \mathcal{R}_S)$. Then $r$ can be presented as a sum $\sum_{w \in F} p[w]$, where $F$ is
a subset of $\mathbb{F}_d \setminus \{e\}$ whose elements have the same lengths. Let $w$ be an element of $\mathbb{F}_d$
whose reduced form is $s_1^{n(1)} \ldots s_k^{n(k)}$, where $s_i \in S \cup S^{-1}$, $n(i) \in \mathbb{N}$, and $s_i \neq s_{i+1}$ for all $i$. We define $\hat{w} \in \mathbb{F}_d$ by $s_1^{n(1)} \ldots s_k^{n(k)}$. We will show that $w \in F$ implies $\hat{w} \in F$. Indeed, if $w \in F$, then $r(ws_k^{-\infty}) = 1$. Hence we must have $r(ws_k^{-\infty}) = 1$. This implies $\hat{w} \in F$ as desired. Since $w \neq \hat{w}$ and $[p[w] + p[\hat{w}]_0] = [q[s_k^{n(k)}]]_0$, it suffices to show that for $s \in S$ and $n \in \mathbb{N}$ the element $[q[s^n]]_0$ is contained in the subgroup generated by $[q[s]]_0, s \in S$. This follows from the equations
$$q[s^2] = sq[s] + s^{-1}q[s] + q[s] - 2$$
and
$$q[s^k] = sq[s^{k-1}] + s^{-1}q[s^{k-1}] - q[s^{k-2}]$$
for $s \in S$ and $k > 2$. \hfill $\Box$

We denote the triplet $(K_0, [1]_0, K_1)$ by $K_s$.

**Theorem 5.3.** The $K_s(C(\partial \mathbb{F}_d/R_S) \rtimes \mathbb{F}_d)$ is isomorphic to $(\mathbb{Z}^d, (1, 1, \ldots, 1), \mathbb{Z}^d)$.

**Proof.** We first compute the pair $(K_0, [1]_0)$. By Lemma 5.2 it suffices to show the linear independence of the family $([q[s]]_0)_{s \in S}$. Let
$$\eta: C(\partial \mathbb{F}_d, \mathbb{Z})^{\mathbb{N}S} \rightarrow C(\partial \mathbb{F}_d, \mathbb{Z})$$
be the additive map defined by $(f_s)_{s \in S} \mapsto \sum_{s \in S}(f_s - s(f_s))$ and denote by $\tau$ the restriction of $\eta$ to $C(\partial \mathbb{F}_d/R_S, \mathbb{Z})^{\mathbb{N}S}$. Then the Pimsner–Voiculescu exact sequence [19] shows that the canonical map
$$C(\partial \mathbb{F}_d/R_S, \mathbb{Z}) \rightarrow K_0(C(\partial \mathbb{F}_d/R_S) \rtimes \mathbb{F}_d)$$
is surjective and its kernel is equal to $\operatorname{im}(\tau)$. Hence it suffices to show that $\operatorname{im}(\tau)$ does not contain a nontrivial linear combination of the projections $q[s], s \in S$. The isomorphisms $\ker(\eta) \cong K_1(C(\partial \mathbb{F}_d) \rtimes \mathbb{F}_d) \cong \mathbb{Z}^d$ (see [4] [19], [22]) show that $\ker(\eta) = \{(f_s)_{s \in S} : \text{each } f_s \text{ is constant}\}$. Now let $r = \sum_{s \in S} n(s)q[s]$ be a nontrivial linear combination of $q[s]$. If $\sum_{s \in S} n(s) \neq 0 \mod (d-1)$, then $r \not\in \ker(\eta)$ by [4], [22]. If $\sum_{s \in S} n(s) = (d-1)m$ for some $m \in \mathbb{Z}$, then $\sum_{s \in S} n(s)q[s] = g((g_s)_{s \in S})$, where
$$g_s := (n(s) - m)p[s^{-1}]$$
for $s \in S$. Hence $\eta^{-1}\{r\} = (g_s)_{s \in S} + \ker(\eta)$, which does not intersect with $C(\partial \mathbb{F}_d/R_S, \mathbb{Z})^{\mathbb{N}S}$. Thus we have $r \not\in \ker(\tau)$ in either case.

The isomorphism of the $K_1$-group follows from the Pimsner–Voiculescu exact sequence [19] and the equality $\ker(\tau) = \ker(\eta)$.

**Proof of Theorem C:** the case $d$ is finite. Let $A$ be a stable separable nuclear C*-algebra. Let
$$\iota: C^*_r(\mathbb{F}_d) \rightarrow C(\partial \mathbb{F}_d/R_S) \rtimes \mathbb{F}_d$$
be the inclusion map. Then the above computation yields that the homomorphism $\iota_{*,0}$ has a left inverse and the homomorphism $\iota_{*,1}$ is an isomorphism. Consequently, the homomorphism
$$\operatorname{Hom}(K_i(C(\partial \mathbb{F}_d/R_S) \rtimes \mathbb{F}_d), K_{1-i}(A)) \rightarrow \operatorname{Hom}(K_i(C^*_r(\mathbb{F}_d)), K_{1-i}(A))$$
induced from $\iota$ is surjective for $i = 0, 1$. Recall that both $C^*_r(\mathbb{F}_d)$ and $C(\partial \mathbb{F}_d/R_S) \rtimes \mathbb{F}_d$ satisfies the universal coefficient theorem [21], Corollary 7.2. Since $K_i(C^*_r(\mathbb{F}_d))$ is a
free $\mathbb{Z}$-module for $i = 0, 1$, the universal coefficient theorem \[21\] yields that the canonical homomorphism

$$\text{Ext}(C^*_r(F_d), A)^{-1} \to \bigoplus_{i=0, 1} \text{Hom}(K_i(C^*_r(F_d)), K_{1-i}(A))$$

is an isomorphism. Combining these facts, we see that the homomorphism

$$\iota^*: \text{Ext}(C(\partial F_d/R_S) \times F_d, A) \to \text{Ext}(C^*_r(F_d), A)^{-1}$$

induced from $\iota$ is surjective.

Now let $B$ be the exact $C^*$-algebra obtained by an extension $\sigma$ of $C^*_r(F_d)$ by $A$ which is either absorbing or unital absorbing. Since $A$ is nuclear and $C^*_r(F_d)$ is exact, the Effros–Haagerup lifting theorem \[6, Theorem B and Prop. 5.5\] shows that $[\sigma] \in \text{Ext}(C^*_r(F_d), A)$ is invertible in the semigroup $\text{Ext}(C^*_r(F_d), A)$. Note that in either case, the direct sum $\sigma \oplus 0$ is absorbing. Thus, by surjectivity of $\iota^*$, the direct sum $\sigma \oplus 0$ extends to a $*$-homomorphism $\varphi: C(\partial F_d/R_S) \times F_d \to M_2(Q(A))$. Then, since $\varphi(1) = \sigma(1) \oplus 0 \leq 1 \oplus 0$, the map

$$\tilde{\sigma}: C(\partial F_d/R_S) \times F_d \ni x \mapsto \varphi(x)_{1,1} \in Q(A)$$

defines a $*$-homomorphism which extends $\sigma$.

We next show that $B$ is realized as a decreasing intersection of separable nuclear $C^*$-algebras. Take a decreasing sequence $(A_n)_{n=1}^{\infty}$ of nuclear $C^*$-subalgebras of $C(\partial F_d/R_S) \times F_d$ whose decreasing intersection is equal to $C^*_r(F_d)$. Put $B_n := \tilde{\sigma}^{-1}(\tilde{\sigma}(A_n))$ for each $n$. Then, since nuclearity is preserved under taking the extension, each $B_n$ is nuclear. Moreover, we have the equality

$$\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \tilde{\sigma}^{-1}(\tilde{\sigma}(A_n)) = B.$$

For unital case, the rest of the proof is similarly done to the proof of Theorem \[X\]. For non-unital case, let $(B_n)_{n=1}^{\infty}$ be a decreasing sequence of separable nuclear $C^*$-algebras whose intersection is $B$. Denote by 1 the unit of the unitization $\widetilde{B}_1$ of $B_1$. Define $C^*$-subalgebras $C_n$ of $\widetilde{B}_1 \oplus \ell^\infty(\mathbb{N})$ by

$$C_n := C^*(B_n, \{1 \oplus p_k : k \in \mathbb{N}\}),$$

where $p_k$ is the characteristic function of the set $\{l \in \mathbb{N} : l \geq k\}$. Set $D_n := C_n \otimes \bigotimes_{k=n}^{\infty} C(X)$ for each $n$, where $X$ is a compact metrizable space consisting at least two points. Take a faithful state $\phi$ on $C_1$ and a faithful measure $\mu$ on $X$. Then define a state $\varphi$ on $D_1$ by $\varphi := \phi \otimes \bigotimes_{k=1}^{\infty} \mu$. Put $\varphi_n := \varphi|_{D_n}$. Now take a pure state $\psi$ on $\mathbb{M}_2$ and define

$$E_n := q_n((D_n, \varphi|_{D_n}) * (\bigotimes_{k=n}^{\infty} (\mathbb{M}_2, \psi))) q_n,$$

where $q_n := (1 \oplus p_n) \in D_n$. Then, being as a corner of a simple unital separable nuclear $C^*$-algebra, each $E_n$ also has these properties. Now put $F_n := E_n \otimes \bigotimes_{k=n}^{\infty} \mathcal{O}_2$. Then each $F_n$ is isomorphic to $\mathcal{O}_2$ \[11\]. Now it is easy to see that the intersection of the decreasing sequence $(F_n)_{n=1}^{\infty}$ is isomorphic to $B$.

Finally, when $A = \mathbb{K}$, by Voiculescu’s theorem \[26\], any essential unital extension is absorbing and any essential non-unital extension is absorbing. Moreover, since
$C^*_r(F_d)$ is simple [20], the only non-essential extension is the zero extension $C^*_r(F_d) \oplus K$. In this case, the claim follows from the above argument. □

We remark that in the proofs of Theorems A and C, the following is implicitly proved. Here we record it as a proposition.

**Proposition 5.4.** Let $A$ be a (possibly non-unital) C*-algebra which is realized as a decreasing intersection of separable nuclear C*-algebras. Then it is realized as a decreasing intersection of isomorphs of the Cuntz algebra $O_2$.

Now consider the case $d = \infty$.

**Proof of Theorem C: the case $d = \infty$.** Let $\Lambda$ be the commutator subgroup of $F_2$. Then $\Lambda$ is isomorphic to $F_\infty$. Therefore we only need to show the claim for $\Lambda$. Let $S$ be the canonical generator of $F_2$ and consider the restriction $\alpha$ of the action $F_2 \curvearrowright \partial F_2/RS$ to $\Lambda$. Let $\iota: C^*_r(\Lambda) \to C(\partial F_2/RS) \rtimes \Lambda$ denote the inclusion. We will show that the induced homomorphism $\iota_*$ on the K-theory is left invertible. To show the claim for the $K_0$-group, consider the following inclusion map

$$\tilde{\iota}: C^*_r(\Lambda) \to C(\partial F_2/RS) \rtimes F_2.$$  

Then by Theorem 5.3, the homomorphism $\tilde{\iota}_{*,0}$ is left invertible. This proves the left invertibility of $\iota_{*,0}$.

To show the claim for the $K_1$-group, first take a free basis $A$ of $\Lambda \cong F_\infty$. Define the homomorphism

$$\eta: C(\partial F_2/RS, \mathbb{Z})^\oplus A \to C(\partial F_2/RS, \mathbb{Z})$$

by $\eta((f_a)_{a \in A}) := \sum_{a \in A} (f_a - a(f_a))$. Then by the Pimsner–Voiculescu six term exact sequence, we obtain an isomorphism

$$K_1(C(\partial F_2/RS) \rtimes \Lambda) \cong \ker(\eta)$$

which maps $[u_a]_1$ to $(\delta_{a,b}1)_{b \in A}$ for each $a \in A$. Since the subgroup generated by 1 is a direct summand of the group $C(\partial F_2/RS, \mathbb{Z})$, the homomorphism $\mathbb{Z}^\oplus A \to \ker(\eta)$ given by $\delta_a \mapsto (\delta_{a,b}1)_{b \in A}$ is left invertible. Consequently, the homomorphism $\iota_{*,1}$ is left invertible. Now the rest of the proof is similarly done to the case $d$ is finite. □

**Remark 5.5.** Let $\sigma$ be the Busby invariant as before. Denote by

$$\varphi_i: K_i(C^*_r(F_d)) \to K_{1-i}(A)$$

the homomorphisms corresponding to $\sigma$. Then the six-term exact sequence gives the formula of the K-theory of the extension $B$ as follows.

$$K_i(B) \cong \coker(\varphi_{1-i}) \oplus \mathbb{Z}^\oplus d_i (i = 0, 1)$$

where $d_i := \text{rank}(\ker(\varphi_i))$. Furthermore, when $B$ is unital, the unit element $[1]_0$ corresponds to the element

$$u = \begin{cases} 
0 & \text{if } d_0 = 0, \\
0 \oplus (1, 0, \ldots, 0) & \text{otherwise}.
\end{cases}$$
Next consider two triplets \((G^{(j)}_0, u^{(j)}, G^{(j)}_1), j = 1, 2\), where \(G^{(j)}_i\) are countable abelian groups and \(u^{(j)} \in G^{(j)}_0\). Then, by the theorem of Kirchberg [10] and Phillips [18] Theorem 4.1], every homomorphism between them is implemented by a unital \(*\)-homomorphism between Kirchberg algebras in the UCT class. Combining this fact with our results in this section, we obtain the following consequence.

**Corollary 5.6.** For any countable free group \(F\), there is a unital embedding of \(C^*_v(F)\) into a Kirchberg algebra which implements the \(KK\)-equivalence.

6. Consequences to amenable minimal Cantor systems of free groups

Let \(d\) be a natural number greater than 1. Again let \(S\) denote the set of canonical generators of \(F_d\). Let \(\mathfrak{F} \subset S\) be a nonempty proper subset. Then a similar proof to Theorem 5.3 shows that the space \(\partial F_d / R_{\mathfrak{F}}\) is the Cantor set and \(K_1(C(\partial F_d / R_{\mathfrak{F}}) \rtimes F_d)\) is isomorphic to \((\mathbb{Z}^d, \{1, 0, \ldots, 0\}, \mathbb{Z}^d)\). (The set \(\{[1]_0, [p[s]]_0, [q[t]]_0 : s \in S \setminus \mathfrak{F}, t \in \mathfrak{F}'\}\) is a basis of the \(K_0\)-group for any subset \(\mathfrak{F}'\) of \(\mathfrak{F}\) with the cardinality \(#\mathfrak{F}' - 1\). We remark that the equality \((d - \infty^d - 1)[1]_0 = -\sum_{s \in \mathfrak{F}}[q[s]]_0\) holds in the \(K_0\)-group.)

The classification theorem of Kirchberg and Phillips [10, 18] shows that the crossed products of the dynamical systems \(\varphi_{\mathfrak{F}}: F_d \curvearrowright \partial F_d / R_{\mathfrak{F}}\) are mutually isomorphic for nonempty subsets \(\mathfrak{F}\) of \(S\). Moreover, these crossed products are isomorphic to a Cuntz–Krieger algebra [14 Lemma 3.7].

Recall that two minimal topologically free actions on the Cantor set is continuously orbit equivalent if and only if their transformation groupoids are isomorphic as \(\alpha\)-étale groupoids [24, Def.5.4]. See [15] and [24] for relevant topics. We will show that they are not continuously orbit equivalent. Hence we obtain examples of amenable minimal Cantor \(F_d\)-systems which are not continuously orbit equivalent but have isomorphic crossed products.

Let \(\alpha: \Gamma \actson X\) be an action of a group on the Cantor set. Recall that the topological full group \([[\alpha]]\) of \(\alpha\) is the group consisting homeomorphisms \(h \in X\) satisfying the following condition. For any \(x \in X\), there is a neighborhood \(U\) of \(x\) and \(g \in \Gamma\) such that \(h(y) = g.y\) for all \(y \in U\).

**Theorem 6.1.** Let \(\mathfrak{F}\) and \(\mathfrak{F}'\) be nonempty subsets of \(S\). Then \(\varphi_{\mathfrak{F}}\) and \(\varphi_{\mathfrak{F}'}\) are continuously orbit equivalent if and only if \(#\mathfrak{F} = #\mathfrak{F}'\).

**Proof.** Let \(\mathfrak{F}\) be a nonempty subset of \(S\) and denote by \([y]\) the element of \(\partial F_d / R_{\mathfrak{F}}\) represented by \(y \in \partial F_d\). Set
\[
X := \{x \in \partial F_d / R_{\mathfrak{F}} : \text{there is } g \in F_d \setminus \{e\} \text{ with } g.x = x\}.
\]
Notice that the definition of \(X\) only depends on the structure of the transformation groupoid \(\partial F_d / R_{\mathfrak{F}} \rtimes F_d\). It is clear from the definition that
\[
X = \{[w^{+\infty}] : w \in F_d \setminus \{e\}\}.
\]
For each \(x \in X\), consider the following condition.
\[(*)\] There is \(F \in \{[\varphi_{\mathfrak{F}}]\}\) which fixes \(x\) and acts nontrivially on any neighborhood of \(x\), and both \((F^m)_{m=1}^{\infty}\) and \((F^{-n})_{n=1}^{\infty}\) do not uniformly converge to the constant map \(y \mapsto x\) on any neighborhood of \(x\).
Then it is easy to check that
\[ Y := \{ x \in X : x \text{ satisfies the condition } (\ast) \} = \{ gw^+ \infty : g \in F_d, w \in \mathcal{F} \}. \]
Now the cardinality of $\mathcal{F}$ is recovered as the number of the $F_d$-orbits in $Y$. □

**Remark 6.2.** It follows from Matui’s theorem [15, Theorem 3.10] and Theorem 6.1 that the topological full groups of $\varphi_\mathcal{F}$ and $\varphi_{\mathcal{F}'}$ are not isomorphic when $\# \mathcal{F} \neq \# \mathcal{F}'$.

Now consider a one-sided irreducible finite shift $\sigma_A$ with $K_*(O_A) \cong (\mathbb{Z}^d, (1, 1, \ldots, 1), \mathbb{Z}^d)$. (Such one exists [14, Lemma 3.7] and is unique up to continuously orbit equivalence [14, Theorem 3.6].) Then, thanks to the classification theorem of Kirchberg and Phillips [10], for each nonempty subset $\mathcal{F}$ of $S$, the crossed product $C(\partial F_d / R_\mathcal{F}) \rtimes F_d$ is isomorphic to the Cuntz–Krieger algebra $O_A$. Thus the Cartesian subalgebra $C(\partial F_d / R_\mathcal{F}) \subset C(\partial F_d / R_\mathcal{F}) \rtimes F_d$ provides a Cartan subalgebra of $O_A$ whose spectrum is the Cantor set. On the other hand, the transformation groupoid $G_A$ of $\sigma_A$ (see [14, Section 2.2] for instance) does not admit a point satisfying the condition $(\ast)$ stated in the proof of Theorem 6.1. Thus our Cartan subalgebras are not conjugacy to the canonical one $C(X_A) \subset O_A$.

**Acknowledgement.** The author would like to thank to Hiroki Matui, who turns the author’s interest to amenable quotients of the boundary actions. He also thanks to Narutaka Ozawa for helpful discussions on hyperbolic groups and approximation theory. He also thanks to Caleb Eckhardt who raised a question about properties of the decreasing intersection of nuclear C*-algebras with conditional expectations. He is supported by Research Fellow of the JSPS (No.25-7810) and the Program of Leading Graduate Schools, MEXT, Japan.

**References**

[1] C. Anantharaman-Delaroche. *Systèmes dynamiques non commutatifs et moyennabilité.* Math. Ann. 279 (1987), 297–315.
[2] B. Blackadar, *K-theory for operator algebras.* Cambridge University Press, Cambridge (1998).
[3] N. P. Brown, N. Ozawa, *C*-algebras and finite-dimensional approximations. Graduate Studies in Mathematics 88. American Mathematical Society, Providence, RI, 2008. xvi+509 pp.
[4] J. Cuntz, *A class of C*-algebras and topological Markov chains II: reducible chains and the Ext-functor for C*-algebras.* Invent. Math. 63 (1981), 25–40.
[5] K. Dykema, *Simplicity and the stable rank of some free product C*-algebras.* Trans. Amer. Math. Soc. 351 (1999), 1–128.
[6] E. G. Effros, U. Haagerup, *Lifting problems and local reflexivity for C*-algebras.* Duke Math. J. 52 (1985), 103–128.
[7] E. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d’après Mikhael Gromov.* Progress in Math. 83, (1990), Birkhäuser.
[8] U. Haagerup, J. Kraus, *Approximation properties for group C*-algebras and group von Neumann algebras.* Trans. Amer. Math. Soc. 344 (1994), 667–699.
[9] G. Kasparov, *Hilbert C*-modules: Theorems of Stinespring and Voiculescu.* J. Operator Theory 4 (1980), 133–150.
[10] E. Kirchberg, *The classification of purely infinite C*-algebras using Kasparov’s theory.* Preprint.
[11] E. Kirchberg, N. C. Phillips, *Embedding of exact C*-algebras in the Cuntz algebra O_2.* J. Reine Angew. Math. 525 (2000), 17-53.
[12] M. Laca, J. Spielberg, *Purely infinite $C^*$-algebras from boundary actions of discrete groups.* J. Reine Angew. Math. **480** (1996), 125–139.

[13] V. Lafforgue, M. de la Salle, *Noncommutative $L^p$-spaces without the completely bounded approximation property.* Duke Math. J. **160** (2011), 71–116.

[14] K. Matsumoto, H. Matui, *Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras.* To appear in Kyoto J. Math. [arXiv:1307.1299]

[15] H. Matui, *Topological full groups of one-sided shifts of finite type.* To appear in J. Reine Angew. Math.

[16] N. Ozawa, *Amenable actions and exactness for discrete groups.* C. R. Acad. Sci. Paris Ser. I Math. **330** (2000), 691–695.

[17] N. Ozawa, *Weak amenability of hyperbolic groups.* Groups Geom. Dyn. **2** (2008), 271–280.

[18] N. C. Phillips, *A classification theorem for nuclear purely infinite simple $C^*$-algebras.* Doc. Math. **5** (2000), 49–114.

[19] M. Pimsner, D. Voiculescu, *$K$-groups of reduced crossed products by free groups.* J. Operator Theory **8** (1982), 131–156

[20] R. T. Powers, *Simplicity of the $C^*$-algebra associated with the free group on two generators.* Duke Math. J. **42** (1975), 151–156.

[21] J. Rosenberg, C. Schochet, *The Kunneth theorem and the universal coefficient theorem for Kasparov’s generalized $K$-functor.* Duke Math. J. **55** (1987), no. 2, 431–474.

[22] J. Spielberg, *Free product groups, Cuntz–Krieger algebras, and covariant maps.* Internat. J. Math. **2** (1991), 457–476.

[23] S. M. Srivastava, *A course on Borel Sets.* Grad. Texts in Math. 180 (1998), Springer.

[24] Y. Suzuki, *Amenable minimal Cantor systems of free groups arising from diagonal actions.* To appear in J. Reine Angew. Math. [arXiv:1312.7098]

[25] J.-L. Tu, *La conjecture de Baum–Connes pour les feuilletages moyennables.* $K$-theory **17** (1999), 215–264.

[26] D. Voiculescu, *A non-commutative Weyl–von Neumann theorem.* Rev. Roumaine Math. Pures Appl. **21** (1976), no. 1, 97–113.

[27] J. Zacharias, *On the invariant translation approximation property for discrete groups.* Proc. Amer. Math. Soc. **134** (2006), 1909–1916.

Department of Mathematical Sciences, University of Tokyo, Komaba, Tokyo, 153-8914, Japan

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan

*E-mail address: suzukiyu@ms.u-tokyo.ac.jp*