Heat–kernels on the discrete circle and interval

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As is known, the free heat–kernel on the integers (a modified Bessel function) is turned into the periodic free heat–kernel on the discrete circle by factoring, giving a pre–image sum. I generalise existing treatments by making the functions periodic up to a phase, thus introducing an extra parameter into the analysis. Identifying the classical paths form with the conventional eigenfunction expression, I find a combinatorial trace identity which allows various Bessel identities to be extracted, such as a generalisation of the Jacobi–Anger expansion.

The free Dirichlet, Neumann and hybrid Dirichlet–Neumann heat–kernels on a discrete interval are constructed using both modes and images. The Neumann imaging mirror has to be placed at a half–integer.

The corresponding lattice Green functions are expressed in terms of Chebyshev polynomials and the Laplacian matrices extracted. The generating functions for circuits with bumps are evaluated.

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1. Introduction

In the next two sections I summarise and rework some known material, [1], [2], on the free heat–kernels on the integers and on the discrete circle with \( p \) vertices (or \( p \)--cycle). In section 3, I extend the discussion to complex functions on the circle periodic up to a phase. This is mathematically possible and gives extra room for manoeuvre in the analysis and allows for a more general trace identity. Other sections contain some algebraic consequences, in particular Bessel function identities, which might be novel. I also construct the Dirichlet, Neumann and Dirichlet–Neumann kernels on a discrete interval using both modes and images and then, by Laplace transformation, Chebyshev polynomial forms for the lattice Green functions.

I am only concerned here with one dimension but higher dimensions can be reached by cartesian products and, as physical motivation for revisiting this topic, I remark that the spectral problem, \( e.g. \) determinants, on higher dimensional lattices, particularly two dimensional ones, has relevance in statistical mechanics, [3], [4], string theory, [5], [6], network theory and many other areas.

The two–dimensional discrete (periodic) lattice also appears in the description of the generators of SU(N), [7], [8], and can have the interpretation of a phase space. It occurs thus in a finite model of two–dimensional ideal hydrodynamics on the torus, [9], [10]. In [9], the discrete \( \zeta \)--function appears when constructing the stream function of a lattice vortex by analogy with the continuum version.

2. Heat–Kernel on the integers

The free heat–kernel, \( K_{\mathbb{Z}}(j, j'; t) \), on the integers, \( j, j' \in \mathbb{Z} \), has been given by Chung and Yau, [1], for example. The defining equation is the discrete heat, or diffusion, equation\(^2\)

\[
\left( -\nabla \Delta + \frac{\partial}{\partial t} \right) K_{\mathbb{Z}}(j, j'; t) = \delta(t)
\]  

with initial condition,

\[
K_{\mathbb{Z}}(j, j'; 0) = \delta_{jj'}
\]  

and the forward propagation condition, \( K_{\mathbb{Z}}(j, j'; t) = 0 \) for \( t < 0 \).

The discrete Laplacian is

\[
\nabla \Delta y(j) \equiv y(j + 1) - 2y(j) + y(j - 1).
\]  

\(^2\) Some authors have a factor of \( 1/2 \) in the first term.
The explicit expression for \( K \) is derived in [1] and [2]. I give an algebraically slightly different, and more rapid, development.

By translation invariance, it is sufficient to set \( j' = 0 \) and consider \( K_{\mathbb{Z}}(j; t) \equiv K_{\mathbb{Z}}(j, 0; t) \). It is also expedient to remove the middle term in the Laplacian, (3), at the outset by setting,

\[
K_{\mathbb{Z}}(j; t) = e^{-2t} K_{\mathbb{Z}}(j; t),
\]

so that the operator solution of (1) is effectively expressed in,

\[
\mathcal{K}_{\mathbb{Z}}(j; t) = e^{t(E+E^{-1})} \delta(j) = e^{tE} e^{tE^{-1}} \delta(j),
\]

where \( E \) is the usual stepping operator, \( E y(j) = y(j + 1) \), and \( \delta(j) = \delta_{j0} \), in terms of the Kronecker delta.

Expanding the exponentials in (4) it is immediately seen that,

\[
e^{tE^{-1}} \delta(j) = \frac{t^j}{j!}
\]

\[
\mathcal{K}_{\mathbb{Z}}(j; t) = e^{tE} \frac{t^j}{j!} = \sum_{n=0}^{\infty} \frac{t^{2n+j}}{n!(n+j)!} \quad j \geq 0,
\]

which is the standard series expression for the Bessel function, \( I_j(2t) \), and I obtain the published formula for the free heat–kernel on (all) the integers,\(^3\)

\[
K_{\mathbb{Z}}(j; t) = e^{-2t} I_j(2t), \quad t \geq 0.
\]

This result is not surprising, cf [2], in view of the fact that Basset’s modified Bessel function, \( I_\nu \), satisfies the recursion,

\[
(E + E^{-1}) I_\nu(z) = \frac{\partial}{\partial z} I_\nu(z).
\]

Further, the initial condition, (2), is validated using \( I_j(0) = \delta(j) \).

By discrete translation invariance,

\[
K_{\mathbb{Z}}(j, j'; t) = e^{-2t} I_{j-j'}(2t).
\]

Karlsson and Neuhauser [2], note that a probabilistic derivation of this result is given by Feller, [11], and also remark on the relevance of the composition rule,\(^4\)

\[
I_{j-j''}(z_1 + z_2) = \sum_{j'=-\infty}^{\infty} I_{j-j'}(z_1) I_{j'-j''}(z_2).
\]

\(^3\)Equivalently, one requires the term \( E^{-j} \) in \((E + E^{-1})^n \) in order to pull the index \( j \) in \( \delta_{j0} \) back to zero, cf [1].

\(^4\) By an infinite folding, one could obtain an analogue of a path integral.
As is very well known in the continuum case, the heat–kernel on the real line, \( K_{\mathbb{R}} \), can be found in many ways, the standard one being eigenfunction expansion. Equating these various expressions gives identities. I will not do this for the integers because I wish to pass immediately to the factor, \( \mathbb{Z}/p\mathbb{Z} \), which is the discrete circle with \( p \) vertices (a \( p \)-cycle, \( C \)). obtained from the covering, \( \mathbb{Z} \), by identification, exactly as for the continuum.

3. Heat–Kernel on the discrete circle

A standard, geometric, way of obtaining the heat–kernel on the covered space is by a pre–image sum of that on the covering space to give a periodic function.\(^5\) Hence,

\[
K_{\mathbb{Z}/p\mathbb{Z}}(j,j';t) = \sum_{m=-\infty}^{\infty} K_{\mathbb{Z}}(j,j'+mp;t) = e^{-2t} \sum_{m=-\infty}^{\infty} I_{j-j'+mp}(2t),
\]

which can be called the sum over classical paths form, by analogy with the more common continuum situation. I do not distinguish, notationally, between points on \( \mathbb{Z} \) and those on \( \mathbb{Z}/p\mathbb{Z} \). Quantities on the covering space are the thermodynamic limits of those on the covered space and correspond to the direct term, \( m = 0 \), in the sum (6).

A dual representation arises from the eigenfunction expression. The Laplacian eigenproblem on the discrete circle is old, e.g. [12]. I therefore just write down the eigenfunctions, in complex form, and the eigenvalues.

The normalised modes on the discrete circle are,

\[
\psi_n(j) = \frac{1}{\sqrt{p}} e^{2\pi inj/p}, \quad n = 0, \ldots, p-1,
\]

with corresponding eigenvalues,

\[
\lambda_n = 4 \sin^2 \frac{\pi n}{p},
\]

so that

\[
K_{\mathbb{Z}/p\mathbb{Z}}(j; t) = \frac{1}{p} \sum_{n=0}^{p-1} e^{-4 \sin^2(\pi n/p)t} e^{2\pi inj/p}.
\]

\(^5\)This procedure has occurred in numerous places at various times.
Equating the two representations of the heat–kernel yields the identity, \[2\],
\[
e^{-2t} \sum_{m=-\infty}^{\infty} I_{j+mp}(2t) = \frac{1}{p} \sum_{n=0}^{p-1} e^{-4\sin^2(\pi n/p) t} e^{2\pi i nj/p}.
\] (10)

Cosmetically re-expressed, this simplifies to,
\[
\sum_{m=-\infty}^{\infty} I_{j+mp}(z) = \frac{1}{p} \sum_{n=0}^{p-1} e^{\cos(2\pi n/p) z} e^{2\pi i nj/p}.
\] (11)

For \(j = 0\), this result is to be found in Al–Jarrah, Dempsey and Glasser, \[13\], together with numerous Bessel series. Their method involves a generalised Poisson formula due to Titchmarsh.

Since the left–hand side of (10) is real (if \(2t\) is) one can conclude,
\[
e^{-2t} \sum_{m=-\infty}^{\infty} I_{j+mp}(2t) = \frac{1}{p} \sum_{n=0}^{p-1} e^{-4\sin^2(\pi n/p) t} \cos 2\pi nj/p
\]
\[
0 = \sum_{n=0}^{p-1} e^{-4\sin^2(\pi n/p) t} \sin 2\pi nj/p.
\] (12)

4. Heat–kernel on the circle for twisted fields.

So far, I have reproduced published results. I wish now to generalise the analysis a little. In the continuum case, according to the results of Schulman, \[14\], Laidlaw and DeWitt, \[15\], and Dowker, \[16\], in the quantum mechanical context the wave functions on the covered space can be be twisted by a unitary representation of the character group of the fundamental group, \(\pi_1\), of the covered manifold.\(^6\) For the circle, \(\pi_1 = \mathbb{Z}\), and the unitary representations are generated by the phase, \(\exp 2\pi i\alpha\), where \(\alpha\) is a real parameter. The wavefunction picks up this phase as its angle argument increases by \(2\pi\), \(i.e.\) as the circle is completed. A possible interpretation of this involves an Aharonov–Bohm flux.

In terms of the propagator, this twisting translates into the statement that each term of the pre–image sum enters with a multiplying factor of a power of this phase.

\(^6\) For line bundles, \(\pi_1\) can be replaced by the more tractable first homology group, \(H_1\) as discussed in \[17\], \[18\].
Here I want to use this flexibility in the discrete case, and for diffusion. That is, I consider a (now complex) function, \( \psi(j) \), defined on the points, \( j \), and satisfying the twisted periodicity condition,

\[
\psi(p) = e^{2\pi i \alpha} \psi(0), \quad \psi(p-1) = e^{2\pi i \alpha} \psi(-1),
\]

which generalises the usual one.\(^7\) At the moment, I present this simply as a mathematical generalisation.\(^8\)

Firstly, the classical paths expression is modified to the twisted form,

\[
K_{\mathbb{Z}/p\mathbb{Z}}(j,j';t) = \sum_{m=-\infty}^{\infty} e^{-2\pi i m\alpha} K_{\mathbb{Z}}(j,j'+mp;t) = e^{-2t} \sum_{m=-\infty}^{\infty} e^{-2\pi i m\alpha} I_{j-j'+mp}(2t).
\]

Then, instead of (7) and (8) the eigenfunctions are,

\[
\psi_{\alpha}^n(j) = \frac{1}{\sqrt{p}} e^{2\pi i (n+\alpha)j/p} , \quad n = 0, \ldots, p-1 , \quad 0 \leq \alpha < 1 ,
\]

with corresponding eigenvalues,

\[
\lambda_n(\alpha) = 4 \sin^2 \frac{\pi (n+\alpha)}{p} . \quad (14)
\]

As before, identifying the paths and eigenfunction expressions yields, instead of (10), the twisted identity, \((z = 2t)\),

\[
e^{-z} \sum_{m=-\infty}^{\infty} e^{-2\pi i m\alpha} I_{j+mp}(z) = \frac{1}{p} \sum_{n=0}^{p-1} e^{-2z \sin^2 \left( \frac{\pi (n+\alpha)}{p} \right)} e^{2\pi i (n+\alpha)j/p} . \quad (15)
\]

I record the special case \( \alpha = 1/2 \), (anti-periodic functions),

\[
e^{-z} \sum_{m=-\infty}^{\infty} (-1)^m I_{j+mp}(z) = \frac{1}{p} \sum_{n=0}^{p-1} e^{-2z \sin^2 \left( \frac{\pi (2n+1)}{2p} \right)} e^{\pi i (2n+1)j/p} = \frac{1}{p} \sum_{n=0}^{p-1} e^{-2z \sin^2 \left( \frac{\pi (2n+1)}{2p} \right)} \cos \left( \pi (2n+1)j/p \right) , \quad (16)
\]

used later.

When \( j = 0 \) see also the computations of Cojocaru, \([20]\).

\(^7\) Fort, \([12]\), discusses the case \( \alpha = 1/2 \) which corresponds to (real) anti-periodic functions.

\(^8\) The use of such twisted boundary conditions is commonplace in statistical physics and string theory, both continuous and discrete. As a typical example I cite \([19]\).
5. Further identities

Before discussing the above results, I outline some known things. The textbook identity,

\[ \prod_{m=0}^{p-1} \left( 4 \sin^2 \left( \frac{\pi (m + \alpha)}{p} \right) + 4 \sinh^2 \gamma \right) = 2 \left( \cosh 2p\gamma - \cos 2\pi \alpha \right), \tag{17} \]

can be proved by elementary trigonometry, e.g. Hobson, [21], Loney, [22], §368, Bromwich, [23], p.211, and so can reasonably be considered as given.

It can be understood as equating two expressions for the determinant of the operator \(-\nabla \Delta + 4 \sinh^2 \gamma\). The left–hand side is the product of the eigenvalues (the definition) and the right–hand side results from a non–eigenvalue evaluation using, say, the Gel’fand–Yaglom technique. This could be considered an alternative derivation to the trigonometric one.

Taking logs and differentiating with respect to \(\gamma\) produces the known,

\[ \frac{1}{4p} \sum_{m=0}^{p-1} \frac{1}{\sin^2 \left( \frac{\pi (m + \alpha)}{p} \right) + \sinh^2 \gamma} \frac{1}{\sinh 2\gamma} = \frac{1}{2} \frac{\sinh 2p\gamma}{\sinh 2\gamma} \frac{1}{\cosh 2p\gamma - \cos 2\pi \alpha} U_{p-1}(\cosh 2\gamma)
\]

\[ = \frac{2(T_{p}(\cosh 2\gamma) - \cos 2\pi \alpha)}{2(T_{p}(\cosh 2\gamma) - \cos 2\pi \alpha)}, \tag{18} \]

where \(T\) and \(U\) are Chebyshev polynomials of the first and second kind.

In [24] I gave a fragmentary history and some discussion of these expressions in connection with Verlinde’s formula for the dimensions of vector bundles on moduli spaces. In particular it was noted that the values of the twisted \(\zeta\)–function,

\[ \zeta_{\mathbb{Z}/p\mathbb{Z}}(s; \alpha) \equiv \sum_{m=0}^{p-1} \frac{1}{\left( 4 \sin^2 \pi (m + \alpha)/p \right)^s}, \tag{19} \]

on the discrete circle, at positive integers, i.e. ,

\[ 2^{-2n} \sum_{m=0}^{p-1} \csc^{2n} \left( \pi (m + \alpha)/p \right), \quad \alpha > 0, \tag{20} \]

can be obtained by expanding (18) in powers of \(4 \sinh^2 \gamma\). This is just an example of the Euler-Rayleigh technique. See also Dikii, [25].

The quantity \(4 \sinh^2 \gamma\) can be thought of as a mass squared, \(m^2\), a Laplace transform parameter, \(s\), a fugacity, \(z\), or as a spectral parameter, \(\lambda\). One interpretation of the left–hand side of (18) is a resolvent which is the derivative of \(\log \det\) with respect to \(m^2\).
Alternatively, setting $\gamma = 0$ in (17) and taking logs reduces to the ancient, and directly derivable, Kubert identity,
\[ p - \sum_{m=0}^{p-1} \log 2 \sin \pi (m + \alpha)/p = \log 2 \sin \pi \alpha, \]
which can be differentiated, this time with respect to $\alpha$, also to give the cosec sums (20). For general interest, I give two examples, e.g. Fisher, [26],
\begin{align*}
\zeta_{\mathbb{Z}/p\mathbb{Z}}(1; \alpha) &= \frac{1}{4} \left( \csc^2 \pi \alpha/p + p^2 \csc^2 \alpha - \csc^2 \alpha/p \right) \\
\zeta_{\mathbb{Z}/p\mathbb{Z}}(2; \alpha) &= \frac{1}{16} \left( \csc^4 \pi \alpha/p + p^4 \csc^4 \alpha - \frac{2}{3} p^2 (p^2 - 1) \csc^2 \alpha - \csc^4 \alpha/p \right).
\end{align*}
The first term is to be dropped if $\alpha = 0$ as it corresponds to a zero mode and is not included in the $\zeta$–function.

Other relevant formulae are the Fourier series,
\begin{align*}
\frac{\sin 2\pi \alpha}{\cosh 2\gamma - \cos 2\pi \alpha} &= 2 \sum_{n=1}^{\infty} e^{-2\gamma n} \sin 2\pi n \alpha \\
\frac{\sinh 2\gamma}{\cosh 2\gamma - \cos 2\pi \alpha} &= 1 + 2 \sum_{n=1}^{\infty} e^{-2\gamma n} \cos 2\pi n \alpha, \quad \gamma > 0,
\end{align*}
again derivable by school trigonometry, e.g. Loney, [22] §§293–294.

After this recapitulation, I can now return to the results of the earlier sections. Karlsson and Neuhauser, [2], derive the identity, (18), for $\alpha = 0$, by taking the Laplace transform of the heat–kernel equation, (10), evaluated at $j = 0$ (which amounts to taking the trace\footnote{The factor of $p$ is the volume}). The right–hand side gives the resolvent, corresponding to the left–hand side of (18). The transform of the summand in the left–hand side of (10) can be given as a standard closed form and, on summation, effectively gives the right–hand side of (18)\footnote{The relation between variables is that the $s$ of [2] equals $\sqrt{2} \sinh \gamma$ here.}.

It is no harder to show that the more general identity, (18), follows on Laplace transforming the heat–kernel relation, (15), at $j = 0$. As used in [2], in a different notation, the tabulated Laplace transform,
\begin{equation}
\int_0^\infty dz \, e^{-z} I_n(z) e^{-2z \sinh^2 \gamma} = \frac{1}{\sinh 2\gamma} e^{-2\gamma n}, \quad n \geq 0, \tag{22}
\end{equation}
allows the summation over \( m \) to be performed immediately, after this has been written over non-negative integers using \( I_{-n}(z) = I_n(z) \). Simple algebra then quickly delivers (18). Depending on one’s starting point, this can viewed either as a derivation or as a confirmation of (18).

Continuing with this theme, starting with the basic integral form of the modified Bessel,

\[
I_n(z) = \frac{1}{\pi} \int_0^\pi d\phi e^{-z \cos \phi} \cos n \phi,
\]

the Laplace transform, (22), can be derived purely trigonometrically on applying the elementary Fourier series (21). Alternative proofs can be more complicated and may involve contour manipulation, or series expansion. (Consult Gray and Mathews, [27] Chap.VI and Ex.14,p.76.)

I finally note that the heat–kernel relation (15) can be more neatly written as in (11),

\[
\sum_{m=-\infty}^{\infty} e^{-2\pi i m \alpha} I_{j+mp}(z) = \frac{1}{p} \sum_{n=0}^{p-1} e^{z \cos \left(2\pi (n+\alpha)/p\right)} e^{2\pi i (n+\alpha) j/p}.
\]

(23)

Some consequences of this are derived in the next section.

6. Bessel relations

Taking the trace of (23) (i.e. setting \( j = 0 \)) gives the Fourier cosine series,

\[
\frac{1}{2} I_0(z) + \sum_{m=1}^{\infty} \cos 2\pi m \alpha I_{mp}(z) = \frac{1}{2p} \sum_{n=0}^{p-1} e^{z \cos \left(2\pi (n+\alpha)/p\right)}.
\]

(24)

(See also the analysis of Cojocaru [20].)

For example, for \( p = 1 \) and \( p = 2 \), I find,

\[
\frac{1}{2} I_0(z) + \sum_{m=1}^{\infty} \cos 2\pi m \alpha I_m(z) = \frac{1}{2} e^{z \cos 2\pi \alpha}.
\]

(25)

Next, setting \( \alpha = 1/2 \) in these produces the anti-periodic expressions,

\[
\frac{1}{2} I_0(z) + \sum_{m=1}^{\infty} (-1)^m I_m(z) = \frac{1}{2} e^{-z}
\]

(26)

\[
\frac{1}{2} I_0(z) + \sum_{m=1}^{\infty} (-1)^m I_{2m}(z) = \frac{1}{2}.
\]
Adding the first of this last set to the \( p = 1, \alpha = 0 \) equation in (25), gives the \( p = 2, \alpha = 0 \) relation in (25). Subtracting, on the other hand, yields,

\[
2 \sum_{\mu=0}^{\infty} I_{2\mu+1}(z) = \sinh z.
\] (27)

These relations can be obtained from those in [13], which are actually expressed in terms of the \( J_n \) Bessel function. They are listed results, [13] providing the references.

Furthermore, the first equation in (25) is recognised as a real form of the Jacobi–Anger expansion which, as is well known, yields a generating function for \( I_m \) if one puts \( t = e^{2\pi i\alpha} \) when it turns into the basic,

\[
\sum_{m=-\infty}^{\infty} I_m(z) t^m = e^{z(t+t^{-1})/2}.
\]

The other interesting value is \( \alpha = 1/4 \) for which (25) becomes,

\[
1 = I_0(z) + 2 \sum_{m=1}^{\infty} (-1)^m I_{2m}(z)
\]

\[
cosh(z/\sqrt{2}) = I_0(z) + 2 \sum_{m=1}^{\infty} (-1)^m I_{4m}(z).
\] (28)

The first of these agrees with the second of (26).

For comparison, I present an elementary derivation of the first equation in (28). First I note by simple cancellation and recursion that, (all arguments are \( z \)),

\[
I_1 = \sum_{m=0}^{\infty} (-1)^{m+1} I_{2m+1} + \sum_{m=1}^{\infty} (-1)^m I_{2m+1} = \sum_{m=1}^{\infty} (-1)^m (I_{2m-1} + I_{2m+1}) = \sum_{m=1}^{\infty} (-1)^m I'_{2m}.
\]

Then, using \( I_1 = I'_0/2 \), integration yields the required relation after setting \( z = 0 \) to fix the unknown constant.

\[ \text{[11]} \] The value \( \alpha = 1/4 \) is a turning point between the ‘boson’ value, \( \alpha = 0 \), and the ‘fermion’, \( \alpha = 1/2 \).
Finally in this section, I give two further examples of (24), this time for $p = 3$,
\[
\frac{1}{2} I_0(z) + \sum_{m=1}^{\infty} (-1)^m I_{3m}(z) = \frac{1}{6} \left( e^{z/2} + 2e^{-z/4} \cosh 3z/4 \right)
\]
\[
\frac{1}{2} I_0(z) + \sum_{m=1}^{\infty} (-1)^m I_{6m}(z) = \frac{1}{6} \left( 1 + 2 \cosh \sqrt{3}z/2 \right).
\]

For other uses of similar Bessel relations see [28].

7. Heat–Kernel on the discrete Dirichlet interval

It is straightforward to find the Dirichlet (D) heat–kernel on the discrete interval, or path, $P$, defined to be a set of $p + 1$ vertices, labelled by $j \in \mathbb{Z}$. The end points, $j = 0$ and $j = p$, are boundary vertices at which the function, and hence the heat–kernel, vanishes. The remaining, internal vertices, $j = 1, \ldots, p - 1$, are the dynamical ones. Then $P$ is a $(p - 1)$-path.

Translation invariance is lost and one has to consider $K^{D}_P(j, j'; t)$ which, just as in the continuum case, can be constructed from $K_{\mathbb{Z}}(j, j'; t)$ by images. The positioning of the continuum images is classic and given e.g. in Thomson, [29], and has been used frequently in heat conduction and hydrodynamical problems. (See Hicks, [30], Carslaw and Jaeger, [31], Basset, [32] §57.) In the discrete situation, the mirrors are also placed at $j = 0$ and $j = p$, enclosing the $p - 1$ dynamical, free vertices and it is easy to see that the discrete Dirichlet conditions hold at the end points.

This construction results in the image summation,
\[
K^{D}_P(j, j'; t) = \sum_{m=-\infty}^{\infty} \left( K_{\mathbb{Z}}(j, j' - 2mp; t) - K_{\mathbb{Z}}(j, -j' - 2mp; t) \right)
\]
\[
= e^{-2t} \sum_{m=-\infty}^{\infty} \left( I_{j-j'+2mp}(2t) - I_{j+j'+2mp}(2t) \right).
\]

Dual to this image form is the eigenfunction expression. The eigenvalues and modes here are genuinely historic, being,
\[
\lambda_n = 4 \sin^2 \frac{\pi n}{2p}, \quad n = 1, \ldots, p - 1,
\]
and
\[
y_n(j) = \sqrt{\frac{2}{p}} \sin \frac{j\pi n}{p},
\]
respectively, e.g. Fort, [12], so that the heat–kernel is,

\[ K^D_p(j, j'; t) = \frac{2}{p} \sum_{n=1}^{p-1} e^{-4t \sin^2(\pi n/2p)} \frac{n\pi j}{p} \frac{n\pi j'}{p} \sin \frac{n\pi j}{p} \sin \frac{n\pi j'}{p} \]

\[ = \frac{1}{p} \sum_{n=1}^{p-1} e^{-4t \sin^2(\pi n/2p)} \left( \cos \frac{n\pi (j - j')}{p} - \cos \frac{n\pi (j + j')}{p} \right), \tag{31} \]

which can be seen to be equivalent to the image form, (29), by virtue of the identity, (12), given earlier, after setting \( p \to 2p \). I give a few details, as they are not quite obvious. I first note that the \( n = 0 \) term in (12) cancels by the subtraction in (31). Then I remark on the identity,

\[ \sum_{n=1}^{p-1} \cos(\pi n j/p) (e^{-4t \sin^2(\pi n/2p)} - (-1)^j e^{-4t \cos^2(\pi n/2p)}) \]

\[ = e^{-2t} \sum_{n=1}^{p-1} \cos(\pi n j/p) \left( e^{2t \cos \pi n/p} - (-1)^j e^{-2t \cos \pi n/p} \right) = 0. \]

which comes into play when the sum \( \sum_{n=1}^{2p-1} \) in (12) is split into one from \( n = 1 \) to \( p - 1 \) plus one from \( p \) to \( 2p - 1 \) and shows that the two sums are equal so that (12) becomes,

\[ e^{-2t} \sum_{m=-\infty}^{\infty} I_{j+2mp}(2t) = \frac{1}{2p} + \frac{1}{p} \sum_{n=1}^{p-1} e^{-4 \sin^2(\pi n/p) t} \cos 2\pi n j/p. \tag{32} \]

The required equivalence of (31) and (29) then follows easily.

In the continuum case, this equivalence is effectively Poisson summation, as has been known for about 150 years, in various guises, and the identity (10) is a discrete analogue, [1], [2].

Taking the trace, i.e. setting \( j = j' \) and summing over \( j \) from 1 to \( p - 1 \), yields the image form,

\[ \sum_{j=1}^{p-1} K^D_p(j, j; t) = e^{-2t}(p - 1) \sum_{m=-\infty}^{\infty} I_{2mp}(2t) - e^{-2t} \sum_{m=-\infty}^{\infty} \sum_{j=1}^{p-1} I_{2j+2mp}(2t) \]

\[ = e^{-2t} \sum_{m=-\infty}^{\infty} I_{2mp}(2t) - e^{-2t} \sum_{m=-\infty}^{\infty} \sum_{j=0}^{p-1} I_{2j+2mp}(2t) \]

\[ = e^{-2t} \sum_{m=-\infty}^{\infty} I_{2mp}(2t) - e^{-2t} \sum_{m=-\infty}^{\infty} I_{2m}(2t) \]

\[ = e^{-2t} \sum_{m=-\infty}^{\infty} I_{2mp}(2t) - e^{-2t} \cosh 2t \tag{33} \]
where I have used the listed (25), for $\alpha = 0$ and have split all the integers into residue classes mod $p$, $N = j + mp$, $m \in \mathbb{Z}$ and $0 \leq j < p$.

Equating this to the trace obtained from the eigenfunction form yields an identity that contains nothing new over the periodic case, as the equivalence, proved just above, shows.

8. Heat–Kernel on the discrete Neumann interval

Again I approach the construction from the image side and now offset the mirrors by $1/2$ to the half–integer points $1/2$ and $p+1/2$ so as, this time, to enclose a $p$–path, $\mathcal{P}$ ($j = 1$ to $j = p$) of $p$ free vertices. The end points, $j = 0$ and $j = p + 1$, which are sometimes referred to as fictional, or ghost, points, are now to be regarded as images and tied to the free vertices at $j = 1$ and $j = p$, respectively, thus reproducing the standard discrete Neumann conditions. In the continuum limit the $1/2$ offset becomes irrelevant.

It is easily concluded that the even images of the point $j$ are at $j - 2mp$ and the odd ones at $-j - 2mp + 1$ ($m = -\infty \rightarrow \infty$), so as to yield the image form,

$$K^N_P(j, j'; t) = \sum_{m=-\infty}^{\infty} \left( K_{\mathbb{Z}}(j, j' - 2mp; t) + K_{\mathbb{Z}}(j, -j' + 1 - 2mp; t) \right)$$

$$= \sum_{m=-\infty}^{\infty} \left( K_{\mathbb{Z}}(j - j' + 2mp; t) + K_{\mathbb{Z}}(j + j' - 1 + 2mp; t) \right),$$

where I now look at the eigenfunction expression. The eigenvalues are as in (30) except that the mode label extends to $n = 0$, corresponding to a zero mode.

The $N$–eigenfunctions on the $p$–path are again standard,

$$y^N_n(j) = \sqrt{\frac{2}{p}} \cos \frac{n\pi(2j - 1)}{2p}, \quad n = 1, \ldots, p - 1$$

$$= \frac{1}{\sqrt{p}}, \quad n = 0,$$

and the free, off–diagonal heat–kernel can be written,

$$K^N_P(j, j'; t) = \frac{1}{p} + 2 \frac{1}{p} \sum_{n=1}^{p-1} e^{-4t \sin^2(\frac{\pi n}{2p})} \cos \frac{n\pi(2j - 1)}{2p} \cos \frac{n\pi(2j' - 1)}{2p}$$

$$= \frac{1}{p} + \frac{1}{p} \sum_{n=1}^{p-1} e^{-4t \sin^2(\frac{\pi n}{2p})} \left( \cos \frac{n\pi(j + j' - 1)}{p} + \cos \frac{n\pi(j - j')}{p} \right).$$

(36)
To transform this into an image form one proceeds as in the last section by using (32) to obtain,

$$K^N_P(j, j'; t) = e^{-2t} \sum_{m=-\infty}^{\infty} \left( I_{j-j'+2mp}(2t) + I_{j+j'-1+2mp}(2t) \right)$$

$$= \sum_{m=-\infty}^{\infty} \left( K_Z(j - j' + 2mp; t) + K_Z(j + j' - 1 + 2mp; t) \right),$$

in agreement with (34).

The trace can be constructed, but, again, yields nothing new and so I will not write it out.

9. The D–N interval

Having treated the D–D and the N–N intervals, I now complete the set by tackling the hybrid, D–N case, although nothing especially new is expected to emerge.

I choose $j = 0$ as the D end point, i.e. $y(0) = 0$, and set the mirrors at $j = 0$ and at $j = p + 1/2$ enclosing $p$ free vertices, as in the N–N case. The images are located at $j - 2m(2p+1)$ for even reflections and at the opposite values, $-j - 2m(2p+1)$, for odd, $(-\infty \leq m \leq \infty)$. The even terms enter with the sign, $(-1)^m$, while the odd ones have $(-1)^{m+1}$. Hence the heat–kernel is,

$$K^ND_P(j, j'; t) = \sum_{m=-\infty}^{\infty} (-1)^m \left( K_Z(j, j' - m(2p+1); t) - K_Z(j, -j' - m(2p+1); t) \right)$$

$$= \sum_{m=-\infty}^{\infty} (-1)^m \left( K_Z(j - j' + m(2p+1); t) - K_Z(j + j' + m(2p+1); t) \right).$$

$$e^{-2t} \sum_{m=-\infty}^{\infty} (-1)^m \left( I_{j-j'+m(2p+1)}(2t) - I_{j+j'+m(2p+1)}(2t) \right).$$

(38)

The transformation of this into the eigenfunction form provides a use for the particular twisted identity, (16), corresponding to anti–periodic functions. I repeat it here, with $p \rightarrow 2p + 1$,

$$e^{-z} \sum_{m=-\infty}^{\infty} (-1)^m I_{j+m(2p+1)}(z) = \frac{1}{2p + 1} \sum_{n=0}^{2p} e^{-2z \sin^2 \left( \frac{\pi (2n+1)}{2(2p+1)} \right)}$$

$$\times \cos \frac{\pi (2n+1)j}{2p + 1}. \quad (39)$$
Substitution into (38) produces, at first,

\[ K^{ND}_P(j, j'; t) = \frac{1}{2p + 1} \left[ \sum_{n=0}^{p-1} + \sum_{n=p+1}^{2p} \right] e^{-4t \sin^2 \left( \frac{\pi(2n+1)}{2(2p+1)} \right)} \times \left( \cos \frac{\pi(2n+1)(j - j')}{2p + 1} - \cos \frac{\pi(2n+1)(j + j')}{2p + 1} \right), \]

after noting that the \( n = p \) summand is zero. As earlier, the two sums are equal and so,

\[ K^{ND}_P(j, j'; t) = \frac{4}{2p + 1} \sum_{n=0}^{p-1} e^{-4t \sin^2 \left( \frac{\pi(2n+1)}{2(2p+1)} \right)} \times \sin \frac{\pi(2n+1)j}{2p + 1} \sin \frac{\pi(2n+1)j'}{2p + 1}, \]

from which the eigenvalues and eigenfunctions can be read off, in a standard way, as,

\[ \lambda_n^{DN} = 4 \sin^2 \frac{\pi(2n+1)}{2(2p+1)}, \quad n = 0, \ldots, p - 1 \]

and

\[ y_n^{DN}(j) = \frac{2}{\sqrt{2p + 1}} \sin \frac{\pi(2n+1)j}{2p + 1}. \]

These agree with available expressions e.g. [4], [6].

10. Lattice Green functions

In the continuum, the Laplace transform of the heat–kernel is the Green function, and this holds in the discrete case too.

The transform of the right–hand side of (15) (times \( 1/2 \)) is the Green function for the \( p \)-cycle, \( C \),

\[ G^c_P(j, j', \alpha, \gamma) = \frac{1}{4p} \sum_{n=0}^{p-1} \frac{e^{2\pi i(n+\alpha)(j-j')/p}}{\sin^2 \left( \frac{\pi(n + \alpha)/p}{\pi(n + \alpha)/p} + \sinh^2 \gamma \right)} \]

\[ = G_p(j - j', 0, \alpha, \gamma) \equiv G_p(j - j', \alpha, \gamma). \]

In order to apply the transform to the left–hand side of (15) the sum has to be arranged so that the order of the Bessel function is always non–negative i.e. \( m \) is
such that \( j + mp \geq 0 \). Those terms for which \( j + mp \) is negative can be converted to positive \( j + mp \) using \( I_{-j-mp} = I_{j+mp} \).

It is best to use residues mod \( p \) and set \( j = Jp + r \) with \( J \in \mathbb{Z} \) and \( 0 \leq r < p \). The positive terms then correspond to \( m \geq -J + 1 \) and the negative ones to \( m \leq -J - 1 \). The zero term, \( m = -J \), gives an \( I_r \) and the sum therefore can be rearranged to,

\[
\sum_{m=-J+1}^{\infty} \left( e^{-2\pi i \alpha} I_{Jp+r+mp} + e^{2\pi i \alpha + 4\pi i J \alpha} I_{Jp-r+mp} \right) + e^{2\pi i J \alpha} I_r(z),
\]

allowing the Laplace transform to be taken giving,

\[
\frac{1}{2 \sinh 2\gamma} \sum_{m=-J+1}^{\infty} e^{-2\gamma mp} \left( e^{-2\gamma Jp} \left( e^{-2\gamma r} e^{-2\pi i \alpha} + e^{2\gamma r} e^{2\pi i \alpha + 4\pi i J \alpha} \right) + e^{2\pi i J \alpha} e^{-2\gamma r} \right),
\]

(43)

The \( m \)-summation can be done as before and, taking the real part, yields the identity,

\[
\frac{1}{4p} \sum_{n=0}^{p-1} \cos \frac{2\pi(n + \alpha)j}{p} \frac{1}{\sin^2 \left( \frac{\pi(n + \alpha)}{p} \right) + \sinh^2 \gamma} = \cos \frac{2\pi \alpha J}{2 \sinh 2\gamma} \left( \frac{\cosh 2\gamma r \sinh 2p\gamma}{\cosh 2p\gamma - \cos 2\pi \alpha} - \sinh 2\gamma r \right)
\]

\[
- \frac{\sin \frac{2\pi \alpha J}{2 \sinh 2\gamma} \sin \frac{2\pi \alpha \sinh 2\gamma}{\cosh 2\gamma \cosh 2p\gamma - \cos 2\pi \alpha}}{2 \left( T_p(\cosh 2\gamma) - \cos 2\pi \alpha \right)}
\]

(44)

again in terms of the Chebyshev polynomials, which are the basic building blocks for the Green functions of the free equation.

As a check, when \( j = 0 \) \((J = 0 = r)\) equating this to the real part of (42) gives the identity, (18). Also, when \( \alpha = 0 \), it is periodic in \( J \) of period 1 \((i.e. \ \text{in} \ j \ \text{of period} \ p)\) as it should be. Actually it is then independent of \( J \) the expression being\(^{12}\)

\[
\text{Re} G_C^p(Jp + r, 0, \gamma) = \frac{\cosh(2r - p)\gamma}{2 \sinh 2\gamma \sinh p\gamma} = \frac{U_{p-r-1}(\cosh 2\gamma) + U_{r-1}(\cosh 2\gamma)}{2(T_p(\cosh 2\gamma) - 1)}. \tag{45}
\]

\(^{12}\) This is the standard result found in Feller, [11]. See Montroll and Weiss, [33], equ'n II.19.
I also write out the anti–periodic Green function, \((\alpha = 1/2)\),

\[
\text{Re} \, G^C_p(Jp + r, 1/2, \gamma) = (-1)^J \frac{\sinh(p - 2r)\gamma}{2\sinh2\gamma \cosh p\gamma} \times \frac{U_{p-r-1}(\cosh 2\gamma) - U_{r-1}(\cosh 2\gamma)}{2(T_p(\cosh 2\gamma) + 1)} .
\]

(46)

When \(\alpha \neq 0\), the twisted periodicity appears in its SO(2) aspect and, as a by–product, yields the imaginary part of (42) which can be confirmed by performing the summation, (43). I will not expose the easily found expressions.

This lattice Green function has also been evaluated using Laplace transforms by Cojocaru, [20], equn.(20).

The formula (44) generalises that of Bendito, Encinas and Carmona, [34], who give the untwisted (45).

The interval Green functions, which are real, can be obtained by combining the cycle expressions, (45) and (46), according to the image forms, (29), (37) and (38).

The Dirichlet Green function for the \(p\)–path, \(P\), is found to be, after some simple trigonometric rearrangement,

\[
G^D_p(r, r', \gamma) = \frac{U_{p-r}(\cosh 2\gamma) U_{r'-1}(\cosh 2\gamma)}{U_p(\cosh 2\gamma)}
\]

(47)

where \(r \geq r'\). If \(r < r'\), the symmetry of the Green function in \(r\) and \(r'\) can be employed. This result agrees with that of Chung and Yau, [1], derived in a more involved way, and that of Bendito, Encinas and Carmona, [34], obtained via a Sturm–Liouville approach which illuminates the structure of (47) and avoids mention of the eigenproblem.

The Neumann Green function is found to be, again after simple trigonometric manipulation,

\[
G^N_p(r, r', \gamma) = \frac{V_{p-r}(\cosh 2\gamma) V_{r'-1}(\cosh 2\gamma)}{2(\cosh 2\gamma - 1) U_{p-1}(\cosh 2\gamma)} , \quad r \geq r' ,
\]

(48)

which agrees with [34]. The third kind Chebyshev polynomial, \(V\), arises through the Neumann offset \(r \rightarrow r - 1/2, r' \rightarrow r' - 1/2\).

---

13 On the interval \(j = r, j' = r'\). As a tactical point, I find it easier to use trigonometric formulae than relations between Chebyshev polynomials.
Finally I find the N–D Green function to be,

\[ G^{ND}_p(r, r', \gamma) = \frac{V_{p-r}(\cosh 2\gamma) U_{r'-1}(\cosh 2\gamma)}{V_p(\cosh 2\gamma)}, \quad r \geq r'. \]  

(49)

To attain this form, one sees from (38) that it is the anti–periodic cycle Green function, (46), that is required here.

Bass, [35], gives the Green functions for all three interval types, but has a more complicated structure. He obtains the form for the Dirichlet and Neumann cases in terms of the periodic one by algebra.

11. Graph considerations

The stepping operator \( E \) is the coordinate (or vertex) space representation of the abstract stepping operator \( \mathbf{E} \), and \( \mathbf{E} + \mathbf{E}^{-1} \) is the adjacency operator, \( \mathbf{A} \), having the usual adjacency matrix representation on the integer lattice, \( \mathbb{Z} \), coordinates.

One sees from the definition (4) that \( \mathbf{K}(t) \) is a generating function for the powers of the adjacency operator, \( \mathbf{K}(t) = e^{\mathbf{A}_z t} \),

with

\[ \mathbf{K}_\mathbb{Z}(j, j'; t) = \langle j | \mathbf{K}_\mathbb{Z}(t) | j' \rangle. \]

A similar relation holds on the \( p \)-cycle projection, \( \mathbb{C}_p = \mathbb{Z}/p\mathbb{Z} \). According to (6),

\[ \mathbf{K}_\mathbb{C}(t) = e^{\mathbf{A}_c t}, \]  

(50)

which are operators in the \( p \)-dimensional vector space describing \( \mathbb{C}_p \). \( \mathbf{A}_c \) is represented by the usual periodic adjacency matrix.

The trace of the expansion of (50) yields,

\[ \text{tr} \mathbf{K}_\mathbb{C}(t) = \sum_{k=0}^{\infty} t^k \frac{k!}{k!} \text{tr} \mathbf{A}_c^k, \]  

(51)

and, from the definition of adjacency matrix, \( \text{tr} \mathbf{A}_c^k \) is the number of closed paths (circuits) of length \( k \) on the \( p \)-cycle, which equals, by homogeneity (or transitivity) \( p \) times the number of circuits starting and finishing on any one vertex. Denoting this latter number by \( g_k \), one can derive identities from the various forms of the
left–hand side of (51) which allow the computation of $g_k$, or, equivalently, of the series generating function defined by,

$$g(\sigma) = \sum_{k=0}^{\infty} g_k \sigma^k.$$  

The Laplace transform of (51) produces,

$$g(\sigma) = \frac{1}{\sigma} \int_{0}^{\infty} dt e^{-t/\sigma} \text{tr} \mathbf{K}_c(t),$$

which is the trace of the cycle lattice Green function, already computed, in (18) or (45), the relation between variables being $2 \cosh 2\gamma = 1/\sigma \geq 2$. Hence one has the explicit rational formula for the generating function,

$$g(\sigma) = \frac{1}{2\sigma} \sinh 2p\gamma \frac{1}{\sinh 2\gamma \cosh 2p\gamma - 1} = \frac{1}{2\sigma} \frac{U_{p-1}(1/2\sigma)}{T_p(1/2\sigma) - 1}. \quad (52)$$

A more general quantity in graph theory is the number of circuits of length $k$ with $l$ bumps (or immediate backtrackings), attached to a vertex. Denoting this number by $f_{kl}$, the corresponding generating function is,

$$f(u, \sigma) = \sum_{k,l=0}^{\infty} f_{lk} u^l \sigma^k,$$

so that, as a special case, $g(\sigma) = f(1, \sigma)$ and for the number of circuits with no backtrackings, $f(\sigma) \equiv f(0, \sigma)$.

If the graph is $d$–regular (for the cycle $d = 2$) there is the relation, [36],

$$\frac{f(1-u, \sigma)}{1-u^2\sigma^2} = g\left(\frac{\sigma}{1+u(d-u)\sigma^2}\right), \quad (53)$$

which allows the more interesting $f(u, \sigma)$ to be obtained from the simpler $g(\sigma)$. Trigonometric expansion (e.g. Loney, [22] §§293–294, can be used to evaluate this but it is simpler to employ Chebyshev polynomials as rapid computation is available through,

$$U_n(x) = \text{tr} \left( C^n (x) Q \right), \quad T_n(x) = \frac{1}{2} \text{tr} C^n(x), \quad V_n(x) = \text{tr} \left( C^n (x) R \right),$$

where $C$, $Q$, $R$ and $S$ are given by,

$$C(x) = \begin{pmatrix} 0 & 1 \\ -1 & 2x \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$ 

14 I introduce $S$ because the denominator of the Neumann Green function, (48), can be written $U_p + U_{p-2} - 2U_{p-1} = \text{tr} (C^p S)$. 

18
Hand evaluation quickly produces the first two examples,

\[
f_1(u, \sigma) = \frac{1 + (1-u)\sigma}{1 - (1+u)\sigma}, \quad f_2(u, \sigma) = \frac{1 + (1-u^2)\sigma^2}{1 - (1+u^2)\sigma^2},
\]

where the subscript refers to the number of vertices, i.e. \( p \).

For larger \( p \), it is sufficient to give the \( g_p \) from which the \( f_p(u, \sigma) \) can readily be found on applying (53). I list the results for \( p = 1 \) to \( p = 12 \),

\[
\begin{align*}
&\frac{1}{2\sigma - 1}, \quad \frac{1}{4\sigma^2 - 1}, \quad \frac{\sigma - 1}{2\sigma^2 + \sigma - 1}, \quad \frac{2\sigma^2 - 1}{4\sigma^2 - 1}, \quad \frac{\sigma^2 + \sigma - 1}{2\sigma^3 - 3\sigma^2 - \sigma + 1}, \\
&\frac{3\sigma^2 - 1}{8\sigma^4 - 6\sigma^2 + 1}, \quad \frac{4\sigma^4 - 5\sigma^2 + 1}{2\sigma^4 + 3\sigma^3 - 4\sigma^2 - \sigma + 1}, \quad \frac{\sigma^4 + 2\sigma^3 - 3\sigma^2 - \sigma + 1}{8\sigma^4 - 6\sigma^2 + 1}, \\
&\frac{2\sigma^5 - 5\sigma^4 - 4\sigma^3 + 5\sigma^2 + \sigma - 1}{5\sigma^4 - 5\sigma^2 + 1}, \quad \frac{4\sigma^6 - 13\sigma^4 + 7\sigma^2 - 1}{2\sigma^5 - 5\sigma^4 - 4\sigma^3 + 5\sigma^2 + \sigma - 1}.
\end{align*}
\]

The ensuing \( f_p(u, \sigma) \) agree with Bartholdi, [36], (who uses more involved algebra) except that his even expressions appear to have misprints.

Taylor expansion allows the coefficients \( f_{lk} \) to be read off and the correctness of \( f_4(u, \sigma) \), say, can be checked. For interest, I write it out,

\[
\begin{align*}
f_4(u, \sigma) &= 1 + 2u\sigma^2 + 2\left(u^3 + u^2 + u + 1\right)\sigma^4 \\
&\quad + 2\left(u^5 + 2u^4 + 4u^3 + 6u^2 + 6u\right)\sigma^6 + \ldots.
\end{align*}
\]

It is possible to confirm numerically the determinantal identity (17) by computing the Laplacian determinant using the explicit forms of the generating functions (54). This is no more than a check of algebra of course, but it makes a comforting exercise.

The Laplacian graph eigenvalues, denoted \( \lambda_n \), \( (n = 1 \rightarrow p) \), are related to the adjacency eigenvalues, \( \mu_n \) by \( \lambda_n = 2 - \mu_n \), in this cycle case. The first \( \lambda_n \), i.e. \( \lambda_1 \), is zero and so \( \mu_1 = 2 \). The log Laplacian determinant, defined omitting the zero eigenvalue, is, therefore,

\[
\log \det_L'(p) = \sum_{n=2}^{p} \log \lambda_n = \sum_{n=2}^{p} \log(2 - \mu_n)
\]

\[
= (p - 1) \log 2 - \sum_{k=1}^{\infty} \sum_{n=2}^{p} \frac{1}{k2^k} \mu_n^k
\]

\[
= (p - 1) \log 2 - \sum_{k=1}^{\infty} \frac{1}{k2^k} \left(\text{tr} A_k^k - 2^k\right)
\]

\[
= (p - 1) \log 2 - \sum_{k=1}^{\infty} \frac{1}{k2^k} \left(pg_k - 2^k\right).
\]
Now define the subtracted generating function, $g^*$, by
\[
p g_p^*(\sigma) = p g_p(\sigma) - \frac{1}{1 - 2\sigma}
\]
which is finite at $\sigma = 1/2$, then (55) becomes,
\[
det'_L(p) = 2^{p-1} \exp \left( - \int_0^{1/2} \frac{d\sigma}{\sigma} (p g_p^*(\sigma) - p + 1) \right)
\]
which can be evaluated and gives the right answer\textsuperscript{15}, $\det'_L(p) = p^2$, case by case, using (54). The integration is the reverse of the step taking (17) into (18) for arbitrary $p$.

A further check of the expressions, if one were needed, is provided by computing the graph adjacency matrix from the expansion of the Green function in powers of $\rho \equiv \sigma/(1 - 2\sigma)$. The coefficient of $-\rho^2$ is the combinatorial Laplacian matrix, $L$, from which the adjacency matrix can be found. The definition of Laplacian I use is the operator relation,
\[
G = \rho/(I + \rho L).
\]
I do not give the details since the answers are as expected. However, for the $p$–path, I do write out the Laplacian matrices for the D–D, N–N and D–N boundary conditions computed from (47), (48) and (49), using $p = 3$ as an illustration,
\[
L_D = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, L_N = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, L_{DN} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.
\]
The adjacency matrix is obtained by replacing the diagonal elements by zeros.

The Neumann diagonal elements correspond directly to the degrees of the free vertices whereas the first and last Dirichlet elements include the effects of the two fixed end points, which are not part of the free graph and serve only to make $L_D$ non–singular.\textsuperscript{16} This is confirmed by the D–N matrix.

A combinatorial expression for the number of closed circuits, $g_k$, on the $p$–cycle

\textsuperscript{15} According to one version of Kirchhoff’s matrix-tree theorem, it gives the number of spanning trees, $\det'_L(p)/p$, correctly as $p$ for the cycle.

\textsuperscript{16} In the terminology of Chung and Yau, [37], the graph is an induced subgraph of a host graph, which includes the end, or boundary, points. This can be seen from the form of the matrices in (57). $L_D$ is obtained from $L_N$, in two higher dimensions, by striking out the rows and columns that refer to the end points.
also follows directly from the operator definition,

\[
g_k = \frac{1}{p} \text{tr} \ A^k_c = \frac{1}{p} \text{tr} \ (E + E^{-1})^k, \quad E^p = 1
\]

\[
= \sum_{m=\text{ even}}^{k/p} \left( \frac{k}{k - mp} \right) \left( \frac{k}{k - mp} \right)
\]

12. The thermodynamical limit

As mentioned before, the covering space (an infinite lattice) corresponds to the thermodynamic limit of the covered (a finite lattice) (e.g. [4]). For example, very easily, the Laplace transform of the heat–kernel on \( \mathbb{Z} \), (5), gives the thermodynamic limit of the cycle free Green function on \( \mathbb{Z}/p\mathbb{Z} \) directly as,

\[
G(j, j', \gamma) = e^{-2\gamma(j-j')} \frac{1}{2 \sinh 2\gamma}, \quad j \geq j'.
\]

The parameter, \( \gamma \), can be extended into the complex plane so long as it retains a positive real part (assuming \( j \geq j' \)) and then one can compare with the formulae in Katsura and Inawashiro, [38], Appendix, (although I have problems with equation (A.4)).

13. Remarks

Most of the preceding expressions and relations are rather elementary examples of more general structures that have been investigated for many years, in many different areas, as mentioned in the introduction. For example, (bulk) lattice Green functions, in all dimensions, have been rewritten in terms of Bessel functions both for numerical and formal reasons, an early reference being van der Pol and Bremmer, [39], p.368, in connection with electrical networks. Later works include [40] and [38] where Mellin–Barnes type integrals for Bessel products are utilised to give more explicit representations of the lattice Green functions, a topic of some considerable activity and manipulative complexity. See the reviews, Guttmann, [41], and Zucker, [42].

Incidentally, van der Pol and Bremmer also take the infinite square lattice expression to an elliptic integral, see [39], equn.(91). A number of forms for this
quantity have been obtained over the years, the earliest being, perhaps, by McCrea and Whipple, [43], for a random walk problem. More mathematical are the computations by Stöhr, [44], who touches on the resistor network realisation which itself has also seen a goodly amount of development. Flanders, [45], lists some square expressions.

14. Potentials, higher dimensions and continuum limit

I wish to mention possible extensions to the preceding calculations which I will not be carrying out here.

The analysis in this work has all been concerned with free propagation. An obvious development would be to consider the case with a potential described by the recurrence,

\[
\left( -\nabla \Delta + V(j) + \frac{\partial}{\partial t}\right)K_{\mathbb{Z}}(j, j'; t) = \delta(t),
\]

instead of (1). In order to project down to \(\mathbb{Z}/p\mathbb{Z}\), for example, \(V(j)\) would have to be periodic, \(V(j + p) = V(j)\). Working on the interval, \(P\), would also be possible directly and related to a Sturm–Liouville problem.

Unless exact solutions were available, one might be interested in the discrete analogue of the short–time behaviour of the heat–kernel, and the resulting expansion coefficients. One possible structure has been analysed by Iliev, [46].

I have also restricted myself to one dimension. Higher (free) dimensions can be assembled in the usual way by taking the cartesian products of one–dimensional structures, for example a torus from the circle, a rectangle from the interval and and cubic lattices from the integers. Some basic notions are outlined in [1] and a more extensive discussion of tori in Chinta, Jorgenson and Karlsson, [47], [48] (see also Chaumard, [49]), who are particularly concerned with the continuum limit, another important topic I have not addressed.

15. Conclusion

I have made a slight extension to the computation of the free heat–kernel on a \(p\)–cycle and have used the resulting flexibility to derive a bigger variety of Bessel relations, for example.

An organisational point is that, if one is interested just in the discrete resolvent (the left–hand side of (18)) then an appeal can be made to the identity (17) and the Bessel form can be bypassed.
I have also calculated the heat–kernel on a discrete interval for various boundary conditions using both modes and images. As usual, the Neumann images are all positive while the Dirichlet ones alternate in sign (positive for even reflections and negative for odd). In the continuum, when adding Neumann to Dirichlet the odd D images cancel corresponding N ones leaving just the images for a periodic circle of circumference twice the size of the interval, as is well known. This cancellation does not happen in the discrete case due to the offset of 1/2 to the Neumann mirrors.

Expressions for the free Green functions in terms of Chebyshev polynomials are derived. That on the D–N path is perhaps novel. As a check, the combinatorial Laplacian matrices are extracted.

Relatedly, rational generating functions for circuits of specific length and number of bumps on the cycle graph are readily computed and compared with those of Bartholdi, [36].

Paths and cycles are graphs and these calculations illustrate very general graph–theoretic procedures. I intend to expand on this aspect elsewhere.
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