Abstract—The control problem of soft robots has been considered a challenging subject because they are of infinite degrees of freedom and highly under-actuated. Existing studies have mainly relied on approximated finite-dimensional models. In this work, we exploit infinite-dimensional feedback control for soft robots. We adopt the Cosserat-rod theory and employ nonlinear partial differential equations (PDEs) to model the kinematics and dynamics of soft manipulators, including their translational motions (for shear and elongation) and rotational motions (for bending and torsion). The objective is to achieve position tracking of the entire manipulator in a planar task space by controlling the moments (generated by actuators). The design is inspired by the energy decay property of damped wave equations and has an inner-outer loop structure. In the outer loop, we design desired rotational motions that rotate the translational component into a direction that asymptotically dissipates the energy associated with position tracking errors. In the inner loop, we design inputs for the rotational components to track their desired motions, again by dissipating the rotational energy. We prove that the closed-loop system is exponentially stable and evaluate its performance through simulations.

I. INTRODUCTION

Soft robots are artificial bodies made of continuously deformable and compliant materials [1]. Compared with conventional rigid-body robots, the compliant structure endows soft robots with unique advantages such as being inherently safe when interacting with humans and being able to adapt to constrained and crowded environments. As a result, soft robots have found many applications including medical surgeries and interventions [2] and underwater maneuvers [3].

Despite the structural advantages, the control problem of soft robots has been considered a challenging subject [4] for two main reasons. First, due to the continuous deformability, soft robots have infinite degrees of freedom and are inherently infinite-dimensional nonlinear systems, yet the inputs are always finite-dimensional because we can only equip finitely many independent actuators on the robots. Second, dimensionality aside, the number of input variables is usually much less than the number of state variables. (Here a “variable” is a continuous function.) In one word, soft robots are highly under-actuated systems.

The existing effort has mainly focused on developing finite dimensional approximations of soft robots’ kinematics and dynamics, such as those that are based on the piecewise constant curvature (PCC) assumption [5] or finite element methods (FEM) [6]. PCC models suggest ignoring the linear strains (like shear) that are sometimes negligible compared with the angular strains (like bending) and further assume that a soft manipulator consists of a finite number of curved segments with constant curvatures. In this way, the configuration space is approximated by a reduced number of finite-dimensional variables. This approach has been widely adopted and produced fruitful results ranging from kinematic control to dynamic control in the last two decades [7]–[9]. However, this over-simplification suffers from low accuracy and might also produce local singularities, especially in the presence of significant body and external loads. FEM is a numerical method for solving partial differential equations, which represents the deformable shape as a very large set of mesh nodes together with the information of their neighbors [10], [11]. While FEM is a powerful tool for simulating deformations of various geometric shapes, it significantly relies on linearization and reduction for control purposes.

Cosserat-rod models, also known as geometrically exact models, are infinite-dimensional models for soft manipulators which are based on continuum mechanics and are considered more accurate [12]–[14]. They describe the kinematics and dynamics of a soft manipulator using a system of two coupled nonlinear partial differential equations (PDEs), one for the translational/linear deformations and the other for the rotational/angular deformations. The PCC and FEM models, to some extent, may be considered as finite-dimensional approximations of the Cosserat-rod models [4]. Moreover, since they are mechanics-based, the role of actuators can be systematically formulated into Cosserat-rod models [15]–[17]. Despite modeling accuracy, these PDEs are nonlinear and highly under-actuated, which are very difficult to control due to the lack of a well-developed control theory for infinite-dimensional nonlinear systems. As a result, the existing effort of the control design based on Cosserat-rod models has mainly relied on discretization [18], [19], or assuming full-actuation [20], [21].

In this work, we design feedback controllers directly based on an under-actuated Cosserat-rod model without approximations. The control objective is to achieve position tracking of the whole manipulator in a planar task space by designing the internal moments (generated by actuators) which are treated as the input variables. We recognize that the complete system has a lower-triangular structure in the sense that the rotational motion can be viewed as a virtual input of the translational component. Therefore, we adopt an inner-outer loop design and exploit the energy decay
property of damped wave equations for each loop. In the outer loop, we design desired rotational motions that rotate the translational motions into a direction that converts the tracking error system into a damped wave equation whose energy is known to decay exponentially. In the inner loop, we design inputs for the rotational component to track their desired motions, again by converting the rotational error into a damped wave equation. We prove that this inner-outer loop feedback controller achieves the exponential stability of position tracking in task space. Simulations are included to validate the performance of the proposed controller.

The rest of the paper is organized as follows. We introduce the Cosserat-rod model and the control problem in Section II. The inner-outer loop control design is presented in Section III. In Section IV, simulations are conducted to validate the algorithms. Section V summarizes the contribution.

II. MODELING AND PROBLEM STATEMENT

The special orthogonal group is defined by $SO(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \det R = 1 \}$. The associated Lie algebra is given by $\mathfrak{so}(3) = \{ A \in \mathbb{R}^{3 \times 3} \mid A = -A^T \}$. Define the hat operator $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ by the condition that $u \hat{\otimes} v = u \times v$ for all $u, v \in \mathbb{R}^3$, where $\times$ denotes the cross product. Let $\hat{(\cdot)^\dagger} : \mathfrak{so}(3) \to \mathbb{R}^3$ be its inverse operator, i.e., $(u \hat{\otimes})^\dagger = u$.

Cosserat-rod models are geometrically exact models that describe the dynamic response of deformable rods undergoing external forces and moments [12] and have been widely used to model soft robots [13]–[21]. A Cosserat rod is viewed as a curve and a family of cross-sections (Fig. 1a). Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Let $s \in [0, \ell]$ be the arc length parameter of the undeformed centerline. The position of centerline is specified by $p : [0, \ell] \times [0, T] \to \mathbb{R}^3$. The rotation of each cross-section is specified by $R : [0, \ell] \times [0, T] \to SO(3)$. The columns $\{b_1, b_2, b_3\}$ of $R$ can be seen as a body-attached basis. We thus have two types of coordinate frames: the fixed global frame $\{e_1, e_2, e_3\}$, and a family of body-attached local frames $\{b_1, b_2, b_3\}$. The deformation of the rod consists of linear strains $\gamma$ (shear and extension) and angular strains $\omega$ (bending and torsion), both defined in the local frames (Fig. 1b).

Using the nomenclature in Table I, the kinematics and dynamics of a Cosserat rod are characterized by the following set of PDEs [12], [22]:

$$p_s = Rq$$

$$R_s = Ru_{\hat{\otimes}}$$

$$p_t = Rw_{\hat{\otimes}}$$

$$R_t = Rw_{\hat{\otimes}}$$

$$\mathcal{N}_s + \mathcal{F} = (R \rho_\sigma \gamma)_{\hat{\otimes}}$$

$$\mathcal{M}_s + p_s \times \mathcal{N} + \mathcal{L} = (R \rho_\omega \omega)_{\hat{\otimes}}$$

where $(\cdot)_t := \frac{\partial}{\partial t}(\cdot)$ and $(\cdot)_s := \frac{\partial}{\partial s}(\cdot)$ are partial derivatives. Assuming the rod is initially in a straight configuration on the z-axis, the initial condition is given by

$$p(s, 0) = [0 \ 0 \ 0]^T, \quad p_t(s, 0) = 0,$$

$$R(s, 0) = I, \quad R_t(s, 0) = 0,$$

where $I \in \mathbb{R}^{3 \times 3}$ is the identity matrix. Assume one end $(s = 0)$ is fixed and the other end $(s = \ell)$ is free, the boundary condition is given by

$$p(0, t) = 0, \quad R(0, t) = I,$$

$$\mathcal{N}(\ell, t) = 0, \quad \mathcal{M}(\ell, t) = 0.$$  

We assume the control inputs act on the system as internal forces and moments, denoted by $n_c$ and $m_c$ respectively. This is valid for tendon and fluidic actuators [16]. Following [16], the total internal forces and moments are given by

$$\mathcal{N} = R(n + n_c),$$

$$\mathcal{M} = R(m + m_c).$$

Assuming linear constitutive laws, the internal elastic forces
and moments (caused by deformation) are given by
\[ n = K_1 (q - \bar{q}), \]  
\[ m = K_2 (u - \bar{u}), \]  
where \{\bar{q}(s), \bar{u}(s)\} are the undeformed values of \{q, u\}, \(K_1\) and \(K_2\) are positive-definite and diagonal. In our problem, \(F = \rho \sigma ge_3\) and \(L = 0\) where \(g\) is gravity.

Note that a minimum representation only requires four state variables (or twelve if recalling that each variable has three elements). We choose \{\bar{q}(s), \bar{u}(s)\} as the state variables (or twelve if recalling that each variable has three elements). We choose \{\bar{q}(s), \bar{u}(s)\} as the state variables (or twelve if recalling that each variable has three elements). We choose \{\bar{q}(s), \bar{u}(s)\} as the state variables (or twelve if recalling that each variable has three elements). We choose \{\bar{q}(s), \bar{u}(s)\} as the state variables (or twelve if recalling that each variable has three elements). We choose \{\bar{q}(s), \bar{u}(s)\} as the state variables (or twelve if recalling that each variable has three elements). We choose \{\bar{q}(s), \bar{u}(s)\} as the state variables (or twelve if recalling that each variable has three elements).

In the outer loop, we design desired rotational motions to rotate the translational motions and moments (caused by deformation) are given by
\[ n = K_1 (q - \bar{q}), \]  
\[ m = K_2 (u - \bar{u}), \]  
where \{\bar{q}(s), \bar{u}(s)\} are the undeformed values of \{q, u\}, \(K_1\) and \(K_2\) are positive-definite and diagonal. In our problem, \(F = \rho \sigma ge_3\) and \(L = 0\) where \(g\) is gravity.

The control design relies on recognizing that the complete system (13) but still use the same notations for the reduced variables, e.g., \(p = [p_2 p_3]^T\) and \(w = w_1\) where \((\cdot)_i, i = 1, 2, 3\) is the \(i\)-th element of the original vector. Assuming \(n_c = 0\), the task space representation is given by:
\[ p_t = (RK_3 R^T p_s - RK_3 \bar{q})_s + ge_3, \]
\[ \theta_t = K_4 \theta_{ss} + \frac{1}{\rho J} (m_c)_s + \hat{p}_c RK_5 (R^T p_s - \bar{q}), \]
where \(K_3 = K_1/(\rho \sigma), K_4 = K_2/(\rho J), K_5 = K_1/(\rho J),\) \(p^\wedge := [-p_3 p_2], \theta : [0, \ell] \times [0, T] \rightarrow \mathbb{R}\) is the rotation angle about the \(x\)-axis, and
\[ R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \]

The boundary conditions are correspondingly simplified as:
\[ p(0, t) = 0, \quad \theta(0, t) = 0, \]
\[ (R^T p_s)(\ell, t) - \bar{q} = 0, \quad (K_2 \theta_s + m_c)(\ell, t) = 0. \]

Assume the objective is to track a position trajectory \(p^*(s, t) \in \mathbb{R}^2\) in the task space (alternatively written as \(p_s\) when the superscript position is needed for other notations) which is as smooth as needed with uniformly bounded derivatives and satisfies the following boundary conditions:
\[ p^*(0, t) = 0, \quad p^*_s(\ell, t) = p_s(\ell, t). \]

The condition at \(s = \ell\) is a mild condition to simplify the stability analysis. In practice, for a given desired trajectory \(p_s(s, t)\), it is easy to regulate its value in a small neighborhood of the free end \(s = \ell\) such that the regulated trajectory is almost the same as the original \(p_s\) while satisfying (18). The control problem is stated below.

**Problem 1:** Consider (16). Design \(m_c\) such that \(p(\cdot, t) \rightarrow p^*(\cdot, t)\) as \(t \rightarrow \infty\).
in a direction that asymptotically achieves position tracking. In the inner loop, we design inputs \( m_c \) for the rotational equation to track their desired motions. Both designs are inspired by the energy decay property of damped wave equations. We will assume the states \{q, R\} are available. It is relatively easy to obtain estimates of \( q \) using cameras. One can then estimate \( R \) using the extended Kalman filter for Cosserat-rod models reported in [21].

**Outer loop.** Define the following translational error term:

\[
\tilde{p} = p - p^*.
\]

By the first equation of (16), we obtain that \( \tilde{p} \) satisfies:

\[
\dot{\tilde{p}}_{tt} = (RK_3 R^T p_s - RK_3 \bar{q})_s + ge_3 - \tilde{p}_{tt},
\]

which is known to converge exponentially under suitable assumptions on the coefficients \{K_q, K_v, K_p\} and the boundary condition. This inspires us to design the desired rotational motion \( R^*(s, t) \) (alternatively \( R_s \)), in the form of

\[
R^* = \begin{bmatrix}
\cos \theta^* & -\sin \theta^* \\
\sin \theta^* & \cos \theta^*
\end{bmatrix},
\]

such that at every \( t \), the following ODE holds

\[
\dot{R}^* = (RK_3 R^T p_s - R^* K_3 \bar{q})_s + ge_3 - \theta^*_{tt}.
\]  

We view \( R \) as a virtual input to this system and use it to reshape the translational dynamics such that \( \tilde{p} \) satisfies the following damped wave equation:

\[
\ddot{\tilde{p}}_{tt} = (K_q \tilde{p}_s)_s - K_v \ddot{p}_t - K_p \tilde{p},
\]

for some positive-definite matrix functions \( K_q(s), K_v(s, t), K_p(s) \in \mathbb{R}^{2 \times 2} \), with boundary conditions \( \theta^*(0, t) = 0 \) and \( \theta^*_x(\ell, t) = \dot{\theta}_s(\ell, t) \) to ensure that the desired rotational trajectory is trackable. It is important that \( K_v \) can be a function of \( t \) which ensures that (20) has a solution. We should point out that if \( K_v \) is prescribed, (20) may not admit a solution for \( R^* \). The correct procedure is to treat both \( R^*(s, t) \) and \( K_v(s, t) \) as independent variables and solve for them simultaneously at every \( t \) with the constraints that \( R^* \) is a rotation matrix and \( K_v > 0 \).

**Inner loop.** Define the following rotational error term:

\[
\tilde{\theta} = \theta - \theta^*.
\]

By the second equation of (16), \( \tilde{\theta} \) satisfies:

\[
\dot{\tilde{\theta}}_{tt} = K_q \tilde{\theta}_s + \frac{1}{\rho J} (m_c)_s + p_c^* RK_3 (R^T p_s - \bar{q}) - \theta^*_{tt}.
\]  

We can use the same idea to reshape the rotational dynamics into a damped wave equation. This motivates us to design the input \( m_c \) such that

\[
K_q \tilde{\theta}_s + \frac{1}{\rho J} (m_c)_s + p_c^* RK_3 (R^T p_s - \bar{q}) - \theta^*_{tt} = (k_u \tilde{\theta}_s)_s - k_w \dot{\theta}_t - k_\theta \tilde{\theta},
\]

for some functions \( k_u(s), k_w(s), k_\theta(s) > 0 \) with the boundary condition \( m_c(\ell, t) + K_2 \dot{\theta}_s(\ell, t) = 0 \) according to (17).

Equivalently, at every \( t \), after substituting the boundary condition, \( m_c \) has a closed-form solution given by:

\[
m_c(s, t) = -\rho J \int_s^t \left[ (k_u \tilde{\theta}_s)_s - k_w \dot{\theta}_t - k_\theta \tilde{\theta} \\
- p_c^* RK_3 (R^T p_s - \bar{q}) + \theta^*_{tt} \right] (\tau, t) d\tau
- K_2 \dot{\theta}_s(s, t).
\]  

We can prove that the closed-loop system under such an inner-outer loop control is exponentially stable.

**Theorem 1:** Consider (16). Let \( R^* \) be computed by (20) and \( m_c \) be given by (23). Let the smallest eigenvalue of \( K_q \) be sufficiently large such that \( (K_q + RK_3 R^T - R^* K_3 R^T) \) is positive-definite for all \( s = t = 0 \). Then as \( t \to \infty \), the following convergence holds exponentially,

\[
\left( \|\tilde{\theta}(\cdot, t)\|_{L^\infty}, \|\tilde{\theta}_t(\cdot, t)\|_{L^\infty}, \|\tilde{\theta}_tt(\cdot, t)\|_{L^2} \right) \to 0,
\]

\[
\left( \|\tilde{p}(\cdot, t)\|_{L^\infty}, \|\tilde{p}_t(\cdot, t)\|_{L^\infty}, \|\tilde{p}_tt(\cdot, t)\|_{L^2} \right) \to 0.
\]

**Proof:** The proof mainly consists of two arguments. In the inner loop, \{\( \tilde{\theta}, \tilde{\theta}_t, \tilde{\theta}_s \)\} converge exponentially under (23). In the outer loop, \{\( \tilde{p}, \tilde{p}_t, \tilde{p}_s \)\} become input-to-state stable [25] after at most a finite time (once \( \{\tilde{\theta}, \tilde{\theta}_t\} \) become small) and eventually converges exponentially.

(i) Inner loop. We prove \( (\|\tilde{\theta}\|_{L^\infty}, \|\tilde{\theta}_t\|_{L^\infty}, \|\tilde{\theta}_s\|_{L^\infty}) \to 0 \). Substituting (22) into (21), we obtain

\[
\dot{\tilde{\theta}}_{tt} = (k_u \tilde{\theta}_s)_s - k_w \dot{\theta}_t - k_\theta \tilde{\theta},
\]

with boundary conditions \( \tilde{\theta}(0, t) = 0 \) and \( \tilde{\theta}_s(\ell, t) = 0 \). Consider a Lyapunov functional

\[
V_1(t) = \frac{1}{2} \int_0^\ell k_u \tilde{\theta}_s^2 + \theta^*_{tt} + 2c \tilde{\theta}_t + k_\theta \tilde{\theta}_t^2 ds
\]

\[
= \frac{1}{2} \int_0^\ell k_u \tilde{\theta}_s^2 + \theta^*_{tt} + \left[ \tilde{\theta}_t^T \begin{bmatrix} 1 & c \\
0 & k_\theta \end{bmatrix} \tilde{\theta}_t \right] ds,
\]

where \( c > 0 \) is a constant to be determined later. We have

\[
\frac{d}{dt}V_1 = \int_0^\ell k_u \dot{\tilde{\theta}}_s \tilde{\theta}_st + \dot{\theta}_{tt} + c \theta_{tt} + c \tilde{\theta}_t^2 + k_\theta \tilde{\theta}_t ds
\]

\[
= \int_0^\ell k_u \dot{\tilde{\theta}}_s \tilde{\theta}_st + (\tilde{\theta}_t + c \tilde{\theta})(k_u \tilde{\theta}_s)_s - k_w \dot{\theta}_t - k_\theta \tilde{\theta} + c \tilde{\theta}_t^2 + k_\theta \tilde{\theta}_t ds.
\]

Using integration by parts and the boundary condition,

\[
\int_0^\ell (\tilde{\theta}_t + c \tilde{\theta})(k_u \tilde{\theta}_s)_s ds = \int_0^\ell -k_u(\tilde{\theta}_{tt} + c \tilde{\theta}_{tt}) \tilde{\theta}_s ds.
\]

Then,

\[
\frac{d}{dt}V_1 = \int_0^\ell -ck_u \tilde{\theta}_s^2 - (k_w - c) \tilde{\theta}_t^2 - c k_u \tilde{\theta}_t \tilde{\theta}_t - ck_\theta \tilde{\theta}_t^2 ds
\]

\[
= \int_0^\ell ck_u \tilde{\theta}_s^2 + \left[ \tilde{\theta}_t^T \begin{bmatrix} k_w - c \\
ck_u & c_\theta \end{bmatrix} \tilde{\theta}_t \right] ds.
\]

We can choose \( c \) such that

\[
0 < c < \inf \left\{ \sqrt{k_\theta(s), \frac{k_w k_\theta}{2k_w}}(s) \right\}.
\]  

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Then $V_1$ is positive-definite and $\frac{d}{dt} V_1$ is negative-definite. We obtain that $\langle \hat{\theta}(\cdot, t), \hat{\theta}(\cdot, t) \rangle_{L_2}, \langle \hat{\theta}(\cdot, t), \hat{\theta}(\cdot, t) \rangle_{L_2} \to 0$ exponentially. Note that $\langle \hat{\theta}(\cdot, t), \hat{\theta}(\cdot, t) \rangle_{L_2}, \langle \hat{\theta}(\cdot, t), \hat{\theta}(\cdot, t) \rangle_{L_2} \to 0$ implies $\|\tilde{\theta}(\cdot, t)\|_{L_\infty} \to 0$. Next, define $\eta = \hat{\theta}_t$. Since all the coefficients in (24) are independent of $t$, one can take the time derivative on both sides and find that $\eta$ satisfies the same equation as $\hat{\theta}$. By the same argument, we can prove that $\langle \tilde{\eta}(\cdot, t), \tilde{\eta}(\cdot, t) \rangle_{L_2}, \langle \tilde{\eta}(\cdot, t), \tilde{\eta}(\cdot, t) \rangle_{L_2} \to 0$ exponentially and hence $\|\tilde{\theta}_t(\cdot, t)\|_{L_\infty} = \|\tilde{\eta}(\cdot, t)\|_{L_\infty} \to 0$ exponentially.

(ii) Outer loop. We prove $\langle \bar{\theta}(\cdot, t), \bar{\theta}(\cdot, t) \rangle_{L_2}, \langle \bar{\theta}(\cdot, t), \bar{\theta}(\cdot, t) \rangle_{L_2} \to 0$. Substituting (20) into (19), we obtain

$$\bar{\theta}(\cdot, t) = \left( C\eta + \Phi \right) \bar{\theta} + K_v \bar{\theta}_t + K_p \bar{\theta} + \Psi,$$

where

$$\Phi = RK_3 R^T - R^T K_3 R_\ast,$$

$$\Psi = \left( \Phi \bar{\theta}_s \right) + \left( R - R^T \right) K_3 \bar{\theta}.$$

According to (18), the boundary condition is given by $\bar{\theta}(0, t) = 0$ and $\bar{\theta}(\cdot, 0) = 0$. Consider a Lyapunov functional

$$V_2(t) = \frac{1}{2} \int_0^t \bar{\theta}_s^T (K_q + \Phi) \bar{\theta}_s + \bar{\theta}_t^T (C + \Phi \bar{\theta}_s) + \bar{\theta}_s^T \bar{\theta}_s + 2 \bar{\theta}^T C \bar{\theta}_t + \bar{\theta}^T K_p \bar{\theta} ds,$$

where $C > 0$ is a positive-definite constant matrix such that

$$K_p^2 - C \text{ and } K_v - C - \frac{K_v C K_p^{-1} K_v}{4}$$

are both positive-definite for all $s$, which is essentially the matrix version of condition (25). (The existence of such an $C$ can be easily verified.) By assumption, $V_2$ is positive-definite at $t = 0$. Note that $\|\bar{\theta}(\cdot, t)\|_{L_\infty} \to 0$ exponentially implies that $\|\Phi(\cdot, t)\|_{L_\infty} \to 0$ exponentially. Hence, $V_2$ is positive-definite for all $t \geq 0$. Next, by taking the time derivative and using integration by parts in a similar way as for $\frac{d}{dt} V_1$, we obtain

$$\frac{d}{dt} V_2 = \int_0^t \bar{\theta}_s^T (K_q + \Phi) \bar{\theta}_s + \bar{\theta}_t^T (C + \Phi \bar{\theta}_s) + \bar{\theta}_s^T \bar{\theta}_s + 2 \bar{\theta}^T C \bar{\theta}_t + \bar{\theta}^T K_p \bar{\theta} ds$$

where the last term is negative-definite by our selection of $C$. Note that $(C K_q + \Phi - \frac{1}{2} \Phi \bar{\theta}_s)$ is not necessarily positive-definite. However, $\|\Phi(\cdot, t)\|_{L_\infty} \to 0$ exponentially implies that $\|\Phi(\cdot, t)\|_{L_\infty} \to 0$ exponentially. This means that there exists a finite time $T > 0$ such that when $t > T$, $(C K_q + \Phi - \frac{1}{2} \Phi \bar{\theta}_s)$ becomes positive-definite for all $s$ and $V_2$ becomes input-state stable [25] with respect to $\|\Phi(\cdot, t)\|_{L_2} \to 0$. Since $\|\tilde{\eta}(\cdot, t)\|_{L_\infty}, \|\tilde{\eta}(\cdot, t)\|_{L_2} \to 0$ exponentially implies that $\|\tilde{\theta}_t(\cdot, t)\|_{L_\infty} \to 0$ exponentially, we conclude that $\|\bar{\theta}(\cdot, t)\|_{L_2}, \|\bar{\theta}_t(\cdot, t)\|_{L_2}, \|\tilde{\theta}_s(\cdot, t)\|_{L_2} \to 0$ exponentially.

Remark 2: Our control design takes place in the task space so we completely avoid the inverse kinematics problem encountered if using PCC models [5]. The control algorithm in either loop may be replaced by alternative algorithms as long as they have suitable stabilizing properties. For example, the inner loop is exponentially stable and the outer loop is input-to-state stable (at least after a finite time).

The control input given by (23) is infinite-dimensional. In the implementation, it needs to be approximated by a finite number of actuators. Take fluidic actuators as an example. Then one needs to find values of the air pressure $\{P_i\}_{i=1}^\infty$ such that the generated actuation $m_c$ (finite-dimensional) according to (15) approximates the designed $m_c$ (infinite-dimensional) in (23). Intuitively, we can achieve better approximations with more actuators and suitable placements. On the other hand, if a position trajectory is not achievable based on the current actuator allocation, then it is impossible to accurately approximate the designed $m_c$. This allocation problem highly depends on the actuators and is under study.

IV. SIMULATION STUDY

A simulation study is performed on MATLAB to verify the proposed control algorithms.

Setup. The system parameters for (16) are chosen as $\ell = 0.5$, $K_3 = \text{diag}[1, 1.5]$, $K_4 = 1$, and $K_5 = 1.5$. The system (16) is simulated using finite differences where we set $ds = 0.05$ and $dt = 0.005$. In other words, we use $N = 11$ points to represent a rod. (Note that these system parameters are set to be small values to avoid numerical instability and may not reflect a real robot.) The manipulator is initially undeformed and lies on the $z$-axis.

Algorithm implementation. In the outer loop, we set the control gains to be $K_q = 1$ and $K_p = 4$. To solve for the desired rotation matrix $R^\ast$ using (20), we notice that at every $t$, (20) can be written as an ODE of $s$ in the following form:

$$\frac{d}{ds} (R^s T K_3 R^T a - R^s b) + K_v c + d = 0,$$

where $a, b, c, d : [0, \ell] \to \mathbb{R}^3$ are given vector fields. We let $K_v = \text{diag}[K_v^1, K_v^2]$ be a diagonal matrix. After spatial discretization, each component of a vector field becomes an $N$-dimensional vector, and the operator $\frac{d}{ds}$ is represented by a matrix $A \in \mathbb{R}^{N \times N}$. We thus obtain the following $N$-dimensional algebraic equation for each component $i$:

$$A (R^s T K_3 R^T a - R^s b)^i + K_v^i c^i + d^i = 0, \quad i = 1, 2,$$

subject to the constraints that $\theta^j_i \in [-\pi, \pi]$, $K_v^{1, j}$, $K_v^{2, j} > 0$, $j = 1, \ldots, N$, which can then be solved using nonlinear optimization algorithms such as “lsqnonlin” in MATLAB. Note that since the deformation is continuous, $\{\theta^j_i, K_v^{1, j}, K_v^{2, j}\}$ are supposed to change continuously in time. Thus, using the computed solution from the last time step to start the search of the current time step significantly improves the efficiency of the optimization algorithm. In the inner loop, the control gains are chosen to be $k_u = 0.5$, $k_\theta = 4$ and $k_w = 2$.

Result. The position tracking task is illustrated in Fig. 2. The desired position is initially bent and the manipulator is able to converge to the desired position. The $L^2$ norms of the position and velocity errors are given in Fig. 3.
V. CONCLUSION

In this work, we explored infinite-dimensional control theory for soft manipulators that are modeled by nonlinear Cosserat-rod PDEs. We presented an inner-outer loop control design for position tracking in planar task space. We first designed desired rotational motions that asymptotically achieve position tracking and then designed inputs for the rotational component to track their desired motions. Both designs were inspired by the energy decay property of damped wave equations. Exponential stability was proved. These results suggested the promising role of infinite-dimensional control theory for soft robots. Our future work is to extend the inner-outer loop design to the 3D case and test the control algorithms on a real platform.

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