ABSTRACT

We use a recent classification of non-degenerate quasihomogeneous polynomials to construct all Landau-Ginzburg (LG) potentials for $N=2$ superconformal field theories with $c=9$ and calculate the corresponding Hodge numbers. Surprisingly, the resulting spectra are less symmetric than the existing incomplete results. It turns out that models belonging to the large class for which an explicit construction of a mirror model as an orbifold is known show remarkable mirror symmetry. On the other hand, half of the remaining 15% of all models have no mirror partners. This lack of mirror symmetry may point beyond the class of LG-orbifolds.
1 Introduction

One of the most promising strategies for obtaining realistic physical models from the Heterotic string is the consideration of N=2 superconformal field theories (SCFT) with integer left U(1) charges for a description of the internal degrees of freedom, which lead to N=1 space-time supersymmetric models [1]. A large class of N=2 SCFTs with equal rational left and right U(1) charges can be obtained by means of a Landau-Ginzburg (LG) description [2, 3]. N=2 world-sheet superconformal invariance is assumed to imply non-renormalization of the superpotential $W$ in the action

$$\mathcal{L} = \int d^2 z d^4 \theta K(\phi_i, \bar{\phi}_i) + \left( \int d^2 z d^2 \theta W(\phi_i) + c.c. \right)$$

Thus, with a quasi-homogeneous potential $W(\lambda^n \phi_i) = \lambda^d W(\phi_i)$, this action should describe a conformal model at the renormalization group fixed point [4]. Orbifolding these theories by discrete groups containing the canonical $\mathbb{Z}_d$ symmetry of the potential [2, 3] leads to models with the desired property of integral left U(1) charges. Canonically orbifolded LG-theories with 5 variables in the potential are directly related to Calabi-Yau (CY) manifolds described as the zero-locus of the equation $W(z_i) = 0$ in weighted projective space $\mathbb{P}_4$ [7, 8].

An interesting result of the calculation of a large class of Calabi-Yau manifolds in weighted $\mathbb{P}_4$ by Candelas et al. [8] was the observation of an approximate symmetry of the spectra under the exchange of the Hodge numbers $b_{21}$ and $b_{11}$, viz. the 27 and $\bar{27}$ representations of $E_6$. This so-called mirror symmetry (MS), which was predicted from CFT-arguments by Dixon and Gepner, is a powerful computational tool, because only one set of Yukawa couplings is subject to (non-perturbative) quantum corrections. MS may thus be used to calculate non-perturbatively all Yukawa couplings for a Calabi-Yau manifold once the mirror partner is known [9]. Partial results on abelian orbifolds of LG-models [10] showed a further increase of MS (at least in the naive counting of just comparing spectra) to 94%, in accordance with expectations based on recently found techniques for explicit constructions of mirror models [11, 12].

In the present paper we report on a complete computation of all LG-potentials with $c=9$ based on our recent classification of quasi-homogeneous polynomials with non-degenerate critical points [13]. Even if the resulting models are not too promising phenomenologically, our calculations are a valuable step toward more realistic models since they set the stage for a systematic investigation of orbifolds. Our most striking result, however, is an actual decrease of symmetry in the resulting spectra, which we will discuss in some detail below.

In section 2 we recall the essential results on the classification of non-degenerate quasihomogeneous polynomials [13] and describe our methods of calculation. In section 3 we recall how to calculate the Hodge numbers, deriving some useful formulas. We also discuss the conditions for “factorization” of a LG-orbifold, which would lift the zeros of the Hodge diamond. In sections 4 and 5 we present and discuss our results.

2 Calculation of $c=9$ LG-potentials
2.1 Classification of non-degenerate quasihomogeneous polynomials

In the following we summarize recent results [13] on the classification of non-degenerate quasihomogeneous polynomials [14] on which our calculations are based. Some important points are illustrated by examples.

A polynomial \( W(X_i) \) is said to be quasihomogeneous of degree \( d \), if there exist integers \( n_i \) such that \( W(\lambda^{n_i}X_i) = \lambda^d W(X_i) \). \( q_i = n_i/d \) is called the weight of \( X_i \). We call the space of all polynomials with a given weight structure \((n_i, d)\) a configuration. If \( W \) has an isolated critical point at \( X_i = 0 \) it is called non-degenerate. A configuration is called non-degenerate if it contains a non-degenerate member.

The local algebra of \( W \) is defined as the ring of all polynomials in the \( X_i \) modulo the ideal generated by the gradients \( \partial W/\partial X_i \). It is finite dimensional if and only if \( W \) is non-degenerate. In this case the highest weight occurring in the local algebra is given by the singularity index \( D = \sum_i (1 - 2q_i) \). The central charge of the \( \text{N}=2 \) superconformal field theory whose Landau-Ginzburg potential is \( W \) is given by \( c = 3D \) [3]. The Poincaré polynomial \( P(t) \) is defined as the generating function for the number of basis monomials of the local algebra of a specific degree of quasihomogeneity, i.e. the number of states of a given conformal weight. It can be computed with the formula

\[
P(t) = \prod \frac{1 - t^{d-n_i}}{1 - t^{n_i}}.
\]

Of course, a necessary condition for non-degeneracy is given by the requirement that the r.h.s. of this equation be a polynomial. We will call such a configuration almost non-degenerate. The r.h.s. of eq. (2) will be a polynomial if and only if all zeros of the denominator (counted with their multiplicities) are also zeros of the numerator. This means that the set of all multiples of the numbers \( 1/(d - n_i) \) contains the set of the multiples of the \( 1/n_i \). This is equivalent to the statement that the set of divisors of the \( (d - n_i) \)'s contains the set of divisors of the \( n_i \), (again, multiplicities are to be taken into account).

The main result of ref. [13] uses the following definitions, which suggest a graphical description of the structure of a quasihomogeneous polynomial. A variable \( X \) is called a root if the polynomial \( W \) contains a term \( X^a \). A monomial \( Y^aZ \) is called a pointer at \( Z \). \( a \) is called the exponent of \( X \) or \( Y \), respectively. We recursively define a link between two expressions, which may themselves be variables or links, as a monomial depending only on the variables occurring in these expressions. A link may further be linear in an additional variable \( Z \), which does not count as a variable of the link. In this case we say that the link points at \( Z \), thus extending the previous definition of a pointer. Of course a specific monomial occurring in \( W \) can have more than one interpretation as a link or pointer. Given \( W \), we call any graph (not necessarily the maximal one) whose lines allow the above interpretation in terms of monomials in \( W \) a graphic representation of \( W \). We represent variables by dots, pointers by arrows and links by dashed lines. A graph without links will be called “skeleton graph”.

**Theorem 1:** For a configuration a necessary and sufficient condition for non-degeneracy is that it has a member which can be represented by a graph with:
1. Each variable is either a root or points at another variable.
2. For any pair of variables and/or links pointing at the same variable \( Z \) there is a link joining the two pointers and not pointing at \( Z \) or any of the targets of the sublinks which are joined.

**Example 1:** To illustrate these conditions we consider the configuration \( \{ n_i \} = \{ 5, 6, 7, 8, 9, 11, 12 \} \) and \( d = 29 \). In order that \( X_i \) can point at \( X_j \) \( n_i \) has to divide the \( j^{th} \) entry in the list \( \{ d - n_i \} = \{ 24, 23, 22, 21, 20, 18, 17 \} \). Thus we have, in an obvious notation, \( 1 \to 5 \to 6 \to 3 \to 4 \to 1 \) and \( 7 \to 1 \). For \( X_2 \) there are 2 possibilities: it may point at \( X_1 \) or at \( X_6 \). We choose the first one and arrive at the polynomial

\[
W_{\text{skeleton}} = X_1^4 X_5 + X_2^2 X_6 + X_3^2 X_3 + X_4^3 X_4 + X_5^2 X_1 + X_6^2 X_1 + X_7^2 X_1. \tag{3}
\]

We now have a triple pointer at \( X_1 \) and thus need the 3 links (24)[1], (47)[1] and (27)[1], where numbers in parenthesis represent the variables of a link, whereas square brackets indicate the forbidden targets. \( X_j \) can appear as a target of the first link only if \( d - n_j \) is a multiple of \( \gcd(n_2, n_4) = 2 \), thus the first link can point at \( X_3, X_5 \) or \( X_6 \) and we again choose the first possibility. The targets of the other links are unique (at this stage all links have to be pointers because \( d \) is prime and none of the \( \gcd \)'s of the involved \( n_i \) is 1). We choose to add the following monomials to the potential,

\[
W_{\text{links}} = X_3(X_2X_4^2) + X_5(X_4X_7) + X_6(X_2^2), \tag{4}
\]

which in return generate 3 new double pointers and thus imply the respective links (246)[13], (471)[15] and (275)[16]. The first 2 of these have \( \gcd(n_i, n_j, n_k) = 1 \) and can, e.g., represent the monomials \( X_2^2 X_6 \) and \( X_7^2 X_1 \), which are already present in the potential. The last link has to point at \( X_4 \), which can represent the second monomial in \( W_1 \). Again, the resulting double pointer finally implies the link (2753)[164] with \( \gcd(6, 12, 9, 7) = 1 \). This requires an additional contribution to the potential, which can now be completed to the non-degenerate polynomial

\[
W = W_{\text{skeleton}} + W_{\text{links}} + X_2 X_3^2 X_5. \tag{5}
\]

Our fairly complicated example illustrates some important points: A single monomial can correspond to several different links and neither the choice of a link nor the choice of the corresponding monomial is unique. This does not matter for checking the non-degeneracy criterion. Different choices, however, may of course lead to different discrete symmetries.

We have also seen that divisibility conditions play an important role for the criticality properties of a configuration. In fact, there is a close relation to the Poincaré polynomial [13]:

**Lemma 1:** The necessary condition for non-degeneracy that the expression (2) for the Poincaré polynomial is a polynomial is equivalent to the criterion of theorem 1 if one omits the requirement that all exponents in the link monomials have to be non-negative.

**Example 2:** To illustrate this connection we consider the configuration \( \{ n_i \} = \{ 1, 1, 6, 14, 21 \} \), \( d = 43 \) for which the expression (2) is a polynomial. As above we choose the “skeleton” polynomial

\[
W_{\text{skeleton}} = X_1^{42} + X_2^{42} + X_3^7 X_1^4 + X_4^3 + X_5 X_2^2 \tag{6}
\]

which has only one double pointer, requiring the link (34)[1]. As \( \gcd(6, 14) = 2 \) this link can only point at \( X_2 \) or at \( X_5 \). In the second case we would need a monomial \( X_3^b X_4^b X_5 \). Consistency
with quasihomogeneity implies $6a + 14b = 22$ which, however, has no solution with both $a$ and $b$ non-negative (by choosing $X_2$ as the target we would be able to find a link, but would be stuck in the same way one step later). Thus we see that this configuration is degenerate, although the divisibility conditions are fulfilled. There are 452 such “almost non-degenerate” configurations with a formal $D$ of 3, most of them have 5 variables.

We now know how to check whether a given configuration is degenerate. In order to be able to calculate all non-degenerate configurations with $c = 9$ we still need to restrict the possible (skeleton) graphs and exponents to a finite set. We call variables trivial if they correspond to terms $X^2$. Trivial variables have weights $q = 1/2$ and therefore do not contribute to $D$, nor to the local algebra, as they can be eliminated by $\partial W/\partial X = 0$. The same is true for Lagrange multipliers (i.e. variables which appear only linearly) in non-degenerate configurations, which may have any weight $q$ and can be eliminated pairwise together with their “target” field with weight $1 - q$. All other fields have weight $q > 1/2$. The following two lemmata complete our review of the results we need from ref. [13].

**Lemma 2:** For every non-degenerate configuration $D$ is greater than or equal to 1/3 times the number of non-trivial variables. For $D = 3$ this means that the number of non-trivial variables is between 4 and 9. In the case of 9 variables the only configuration is $n_i = 1$ and $d = 3$.

**Lemma 3:** Given a positive rational number $D$, there is only a finite number of non-degenerate configurations whose index is $D$. For a non-degenerate quasihomogeneous polynomial with $D = 3$ the number of exponents $a_i > 18$ or $a_i > 84$ is smaller than 3 or 2, respectively.

### 2.2 The calculations

Using these results we have calculated all solutions to $D = 3$ in 3 steps. First we have calculated all inequivalent skeleton graphs which can yield such configurations. As the solution for $N = 9$ variables is unique, we need graphs with between 4 and 8 points, the numbers of which are 19, 47, 130, 343, and 951, respectively. We thus had to investigate 1490 skeleton graphs, of which 175 have no double pointers and thus do not require any links. Remarkably, this comparatively small number of graphs already generates the vast majority of configurations (i.e. 9108 out of 10839).

It would of course be virtually impossible to explicitly construct all the required links for the remaining graphs. Fortunately, however, there is only a finite set of possibilities for the exponents in the skeleton graphs when one requires non-degeneracy, and these exponents already determine the configuration. Lemma 3 restricts all but one of the exponents to be less than 85. One may thus first go through the (inequivalent) choices of a variable for the free exponent $a$ and use the bounds on all other exponents [13] to generate all possible cases and calculate $a$ from the condition $D = 3$. If $a$ turns out integer one can check whether the expression (2) is a polynomial and insert the resulting almost non-degenerate configurations into an ordered list. There is a complication to this procedure by the fact that in some cases $D = 3$ independent of the last exponent. An extreme example of this kind is the polynomial

$$ W = X_1^6X_2 + X_2^6X_3 + X_3^6X_1 + X_1X_4^2 + X_1X_5^2 + X_2X_6^2 + X_2X_7^2 $$

(7)
for which this is even the case for arbitrary $a$ and $b$. A careful examination by hand shows that in these cases the open exponent cannot be greater than 84 without violating the link criterion.

In a third step we have checked non-degeneracy using theorem 1 and calculated the Hodge numbers (see below). As the calculation of skeleton graphs and the criterion for non-degeneracy are recursive it was straightforward to implement the sketched procedure. All of our programs are written in the language C. Of course one has to be careful to avoid integer overflow. With the variable type “long int” we were, however, safe by a factor of 100, which we derived both theoretically (from limits on the exponents) and “experimentally”. A complete run of our programs takes more than a week on an HP 9000/720 workstation.

In order to check our results we have also implemented an independent algorithm for finding all configurations. Once the limits on $d$ are known a simple program can generate all partitions of $d(N - 3)/2$. For these one can check whether the $n_i$ have no common divisor and whether (2) is a polynomial. If an upper bound for $d$ is known one can thus generate the list of all almost non-degenerate configurations. Unfortunately the number of possibilities grows roughly like $d^N/N!$. For small $d$, however, this is a very simple and efficient procedure. One can work on the efficiency for large $d$ by requiring from the start that the partition admits a pointer structure. With such an improved algorithm we have checked our results up to $d$ equal to 5000, 2000, 700, 250 and 200 for $N = 4, \ldots, 8$. This has to be compared to the maximal values of $d$ in the direct computation which are 3486, 1743, 1806, 600 and 384 for the respective numbers of variables.

3 Calculation of Hodge numbers

In order to use a LG-model for constructing a consistent Heterotic string we need to project onto integral (left) charges \[Q, F\]. The resulting chiral ring determines the gauge multiplets: The chiral primary fields with $(q_L, q_R) = (1, 1)$ yield 27’s of $E_6$, whereas the states with charges (1,2), which are related to the (1,-1) states by spectral flow, end up as anti-generations. In the following we will only consider the simplest case of the canonical $\mathbb{Z}_d$ orbifold. In case of 5 variables its chiral ring is related to the cohomology ring of the corresponding Calabi-Yau manifold in the weighted projective space $\mathbb{P}_4$ \[4\]. We will call the respective dimensions Hodge numbers also in the present LG context.

The formulas for calculating these numbers were given by Vafa \[4\]. The numbers $p_{ij}$ of states with $(q_L, q_R) = (i, j)$ are the coefficients of $t^i \bar{t}^j$ in the generalized Poincaré polynomial \[1\]

$P(t, \bar{t}) = \text{tr} t^0 \bar{t}^0 = Q(t^{1/2}, \bar{t}^{1/2})|_{int}$,

(8)

with

$Q(t^{1/2}, \bar{t}^{1/2}) = \sum_{0 \leq i < d, \delta_i \in \mathbb{Z}} \prod_{0 \leq i < d, \delta_i \in \mathbb{Z}} \frac{1 - (t \bar{t})^{1-q_i}}{1-(t \bar{t})} \prod_{\delta_i \in \mathbb{Z}} (t \bar{t})^{\delta_i - q_i} \left( \frac{t}{\bar{t}} \right)^{\delta_i - \frac{1}{2}}$,

(9)

\[1\]When comparing with the corresponding Calabi-Yau one should note that $p_{11}$ corresponds to $b_{12}$ and vice versa.
where \( \theta_i = lq_i \) and \( \tilde{\theta}_i = \theta_i - [\theta_i] \) is the non-integer part of \( lq_i \). The subscript \( \text{int} \) means that only integral powers of \( t \) and \( \bar{t} \) are kept in the expression for \( P \).

It is well known that for Calabi-Yau manifolds \( b_{01} = b_{02} = 0 \) at least for \( \chi \neq 0 \), where \( \chi \) is the Euler number \( \chi = 2(b_{11} - b_{12}) \) \([17]\). In the context of LG models formula (9) implies that the exponent of \( t \) can only vanish if \( \sum_{\bar{\theta}_i \in \mathbb{Z}} (\bar{\theta}_i - q_i) = 0 \). For \( \theta_i = q_i \) this gives \( p_{01} = 1 \). For the other \( p_{0j} \) we have

**Theorem 2**: Contributions to \( p_{01} \) and \( p_{02} \) in (8) can only arise if there is a subset of the \( q_i \) with \( \sum_{\text{subset}} (1 - 2q_i) \in \mathbb{Z} \) and an element of the \( \mathbb{Z}_d \) which acts on this subset like the generator of the \( \mathbb{Z}_d \) and does not act on the other fields. In this case the \( \mathbb{Z}_d \) orbifold factorizes into a product of canonical \( \mathbb{Z}_{d_i} \) orbifolds of LG-models with integer indices \( D_i \) with \( \gcd(d_i, d_j) = 1 \) and \( \sum D_i = D \).

For \( D = 3 \) this implies vanishing Euler number because one of the factors must have \( D = 1 \).

**Proof**: If \( X_i \) occurs in the potential as \( X_i^\alpha, \tilde{\theta}_i \) will be of the form \( \lambda/\alpha \) and therefore larger than or equal to \( q_i = 1/\alpha \). If \( X_i \) occurs as \( X_i^\alpha \), we consider \( \delta_i = \tilde{\theta}_i - q_i \). Obviously if \( \delta_i < 0 \) then \( \delta_j = 1 - \alpha \tilde{\theta}_i - (1 - \alpha q_i) = -\alpha \delta_i \), yielding \( \delta_i + \delta_j = -(\alpha - 1) \delta_i > 0 \). There is a slight subtlety concerning the possibility that several different \( X_i \) with \( \delta_i < 0 \) might belong to the same \( X_j \). If we consider all \( X_i \) with \( \delta_i \) smaller than some given negative value \( \delta \), calculate \( dW \) and then set all other \( X_k \) to zero, non-degeneracy implies that we must have at least as many equations left as there are \( X_i \). These equations must come from \( dW/dX_j \) with \( \delta_j \geq -2\delta \). Therefore, if we introduce a ranking of the \( X_i \) with respect to the absolute value of \( \delta_i \) and a corresponding ranking of the \( X_j \), we see that \( \delta_i + \delta_j \) will always be positive for corresponding \( i \)’s and \( j \)’s, thereby proving the first statement of the theorem. Now consider an integer \( l \) in (9) and an appropriate ordering of the variables such that \( \tilde{\theta}_i = q_i \) for \( 0 < i \leq I \) and \( \tilde{\theta}_i = 0 \) for \( I < i \leq N \). We define \( D_1 = \sum_{i < I} (1 - 2q_i) \) and represent \( q_i \) by simplified fractions \( q_i = r_i/s_i \). This implies \( \gcd(s_i, s_j) = 1 \) for \( i \leq I < j \), as \( s_i \) divides \( l - 1 \) and \( s_j \) divides \( l \). With \( d_1 \) and \( d_2 \) being the least common multiples of the \( s_i \) with \( i \leq I \) and \( i > I \), respectively, \( \mathbb{Z}_d = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \) and the complete expression (8) factorizes into a product of canonical LG-orbifolds with integer indices. //.

In the following we assume \( p_{01} = 0 \) (we will reconsider factorization at the end of section 4), thus

\[
P(t, \bar{t}) = (1 + t^3)(1 + \bar{t}^3) + n_g(t\bar{t} + t^2\bar{t}^2) + \bar{n}_g(t\bar{t}^2 + t^2\bar{t}).
\]  

(10)

Vafa has also given a formula for the Euler number which is much simpler than the ones above for the Hodge numbers. This formula and an analogous one for the sum of all Hodge numbers, which in our context is equal to \( 4 + 2n_g + 2\bar{n}_g \), can be derived in the following way: The projection of \( Q \), which is a polynomial in \( t^\frac{1}{d} \) and \( \bar{t}^\frac{1}{d} \), onto \( P \), can be achieved through

\[
P(t, \bar{t}) = d^{-2} \sum_{j=1}^{d} \sum_{k=1}^{d} Q(e^{2\pi i j/\bar{d}}, e^{2\pi i k/d}).
\]  

(11)

With \( P(1, 1) = 4 + 2n_g + 2\bar{n}_g \) and \( P(-1, -1) = 2(n_g - \bar{n}_g) \) we get

\[
4 + 2n_g + 2\bar{n}_g = d^{-2} \sum_{j=1}^{d} \sum_{k=1}^{d} Q(e^{2\pi i j/\bar{d}}, e^{2\pi i k/d})
\]  

(12)

and

\[
2n_g - 2\bar{n}_g = d^{-2} \sum_{j=1}^{d} \sum_{k=1}^{d} Q(e^{2\pi i (2j+1)/\bar{d}}, e^{2\pi i (2k+1)/d}).
\]  

(13)
The calculation of these expressions is straightforward, using ∑(1 − 2qi) = 3 and

\[
\frac{1 - e^{2\pi i j(1-q_i)}}{1 - e^{2\pi i j q_i}} = \frac{1 - q_i}{q_i} \quad \text{or} \quad e^{-2\pi i j q_i}
\]

for \( j q_i \in \mathbb{Z} \) or \( j q_i \not\in \mathbb{Z} \), respectively. We find (with the symbol “∩” for the greatest common divisor of two integers)

\[
4 + 2n_g + 2\bar{n}_g = d^{-1} \sum_{k=1}^{d} \sum_{l=1}^{d} (-1)^{(N-1)k(l-1)+N_l-N_{l\cap k}} \prod_{(l\cap k)q_i \in \mathbb{Z}} \frac{1 - q_i}{q_i}
\]

and

\[
2n_g - 2\bar{n}_g = d^{-1} \sum_{k=1}^{d} \sum_{l=1}^{d} (-1)^{(N-1)(kl-k-l)+N-N_{l\cap k}} \prod_{(l\cap k)q_i \in \mathbb{Z}} \frac{1 - q_i}{q_i},
\]

where \( N \) denotes the total number of fields and \( N_l \) denotes the number of fields for which \( lq_i \in \mathbb{Z} \). By adding a trivial term \( X^2 \) to our potential, we can always make \( N \) odd\(^2\), thereby getting rid of the first expression in the exponent of \((-1)\). We find that for odd \( N \)

\[
\chi = d^{-1} \sum_{k=1}^{d} \sum_{l=1}^{d} \prod_{(l\cap k)q_i \in \mathbb{Z}} \frac{q_i - 1}{q_i},
\]

\[
4 + 2n_g + 2\bar{n}_g = d^{-1} \sum_{k=1}^{d} (-1)^{N_k} \sum_{l=1}^{d} \prod_{(l\cap k)q_i \in \mathbb{Z}} \frac{q_i - 1}{q_i}.
\]

One should note that these formulas allow calculations of \( n_g \) and \( \bar{n}_g \) in a much easier way. It is quite remarkable that a generalisation of these formulas, which make no reference to the actual structure of the local algebra / chiral ring, seems to work for arbitrary orbifolds.

A useful estimate for the number of antigenerations is obtained with the following argument: If \( g \) is the generator of the canonical group and \( a \cap d = 1 \), the action of \( g^a \) will leave no \( X_i \) invariant. Therefore the vacuum in the sector twisted by such an element will be invariant under the action of any group element. Each \( g^a \) will add a term with coefficient 1 to the Poincaré polynomial. The elements corresponding to \( a = 1 \) and \( a = d - 1 \) give \( \bar{t}^3 \) and \( t^3 \), respectively, while all others contribute to \( \bar{n}_g \). The number \( \bar{n}_g \) of antigenerations therefore obeys \( \bar{n}_g \geq \phi(d)/2 - 1 \), where \( \phi \) is Euler’s function, i.e. \( \phi(d) \) is the number of integers \( a \) with \( 0 < a < d \) and \( a \cap d = 1 \).

## 4 Results

Table I lists the numbers of different types of potentials that we have found. The numbers on top of the columns indicate the numbers of variables. The columns 1g, 2g and 3g show the numbers of 1-, 2- and 3-generation models, respectively. The column c3 shows the number of models with

\(^2\)For the \( \mathbb{Z}_d \)-orbifolds this cannot lead to a doubling of the ground state because \( d = (2 \sum n_i)/(N - 3) \) has to be even for even \( N \).
Euler numbers that are odd multiples of 6 (these are probably the best candidates for 3-generation orbifold models).

|                | 4    | 5    | 6    | 7    | 8    | 9    | 1g   | 2g   | 3g   | c3   | Total |
|----------------|------|------|------|------|------|------|------|------|------|------|-------|
| Non-degenerate | 2390 | 5165 | 2567 | 669  | 47   | 1    | 1    | 26   | 40   | 496  | 10839 |
| Invertible     | 2069 | 4191 | 2239 | 568  | 40   | 1    | 0    | 5    | 20   | 286  | 9108  |
| Not Invertible | 321  | 974  | 328  | 101  | 7    | 0    | 1    | 21   | 20   | 210  | 1731  |
| Degenerate     | 14   | 418  | 3    | 17   | 0    | 0    | 0    | 1    | 4    | 125  | 452   |
| Almost Non-deg.| 2404 | 5583 | 2570 | 686  | 47   | 1    | 1    | 27   | 44   | 621  | 11291 |

Table I: Numbers of various types of models.

By invertible we mean those configurations which contain polynomials which do not require links for non-degeneracy (i.e., they have as many monomials as variables). These are precisely the models where each point in their graphical representation is hit by not more than one pointer, i.e. where an inversion of the direction of the arrows in the graph yields again a sensible graph. According to Berglund and H"ubsch [12], these models have mirrors that can be represented as orbifolds of the model that one obtains by inverting the directions of the arrows. By degenerate we mean the almost non-degenerate configurations generated by our program which violate the link criterion. It is remarkable that these models yield integer Hodge numbers when one naively applies formulas [13] and [14].

In figs. 1 – 4 we have plotted $d$ versus $\bar{n}_g$ for the non-degenerate models with 4, 5, 6 or 7 variables, respectively. There is an apparent quantization of directions in the $d - \bar{n}_g$ plane, with different ranges covered for $N$ even and for $N$ odd. In fig. 5 we have plotted $\phi(d)/2 - 1$ over $\bar{n}_g$ for all non-degenerate models. The plot shows excellent agreement with the theoretical estimate of section 2. A plot of $n_g + \bar{n}_g$ versus $\chi$ has, of course, the symmetric form familiar from refs. [8] and [14] and is not given here.

The numbers of spectra for the different types of models are: 2997 different spectra from 10839 non-degenerate models, 2339 different spectra from 9108 invertible models and 3371 different spectra from 11291 almost non-degenerate models. 77% of all spectra coming from non-degenerate, 92% of all spectra coming from invertible and only 69% of all spectra coming from our almost non-degenerate models have mirrors in their respective lists of spectra. If we do not count spectra but configurations, this asymmetry is even more striking: In our set of non-degenerate configurations, only 216 out of 9107 invertible models (that’s approximately 2%) have no mirror partners, whereas 856 out of 1731 non-invertible models, i.e. quite exactly half of them, are singles.

The non-degenerate models gave rise to 342, the invertible models to 268 and all almost non-degenerate models to 411 different Euler numbers. In our list of non-degenerate configurations we did not find one Euler number (namely 22) contained in the list in [8]. The 75 Euler numbers which are not contained there are shown in table II.
Apart from one exception, all of our 3-generation models come from potentials with 5 points. They are given in tables III and IV. The remaining 3 generation model comes from a configuration with 7 variables requiring links: \( \{n_i\} = \{3, 4, 6, 6, 7, 7, 9\} \) and \( d = 21 \) with \( \bar{n}_g = 19, n_g = 16 \) and \( \chi = 6 \).

All candidates for 3 generation models belong to polynomials in 5 or 7 variables, whereas the 2 generation models come from configurations with 4 – 6 non-trivial variables. Surprisingly, there is exactly one 1 generation model, namely the model we analysed in example 1: \( \{n_i\} = \{5, 6, 7, 8, 9, 11, 12\} \) and \( d = 29 \) with \( \bar{n}_g = 13, n_g = 12 \) and \( \chi = 2 \).

Table V shows the models with the lowest total numbers of particles. It is remarkable that they all belong to configurations with more than 5 variables. The 5 variable models with the lowest value of \( n_g + \bar{n}_g \) are the mirror pair of three generation models from table II with spectra (16, 13; 6) and (13, 16; -6).
In Table VI we list the spectra which appear 20 or more times in our list of (irreducible) non-degenerate models. All Euler numbers among these spectra are multiples of 24. We have also given the sum of all Hodge numbers \( \bar{\chi} = P(1, 1) = 2(n_g + \bar{n}_g + 2) \). It turns out that this number also tends to being very “non-prime”. In addition there are 88 configurations with \( n_g = \bar{n}_g = 21 \) and \( p_{01} = 1 \).

| \( n_1 \) | \( n_2 \) | \( n_3 \) | \( n_4 \) | \( n_5 \) | \( n_6 \) | \( n_7 \) | \( n_8 \) | \( d \) | \( n_g \) | \( \bar{n}_g \) | \( \chi \) | \( n_g + \bar{n}_g \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 11 | 12 | 14 | 18 | 21 | 54 | 10 | 10 | 0 | 20 |
| 7 | 11 | 12 | 16 | 20 | 24 | 60 | 14 | 8 | 12 | 22 |
| 6 | 7 | 9 | 13 | 16 | 21 | 48 | 11 | 11 | 0 | 22 |
| 7 | 18 | 22 | 24 | 30 | 34 | 90 | 11 | 11 | 0 | 22 |
| 7 | 9 | 11 | 12 | 15 | 17 | 19 | 45 | 11 | 11 | 0 | 22 |
| 4 | 4 | 5 | 5 | 6 | 7 | 7 | 19 | 8 | 17 | -18 | 25 |
| 5 | 6 | 7 | 8 | 9 | 11 | 12 | 29 | 13 | 12 | 2 | 25 |
| 4 | 6 | 7 | 9 | 10 | 15 | 34 | 7 | 19 | -24 | 26 |
| 3 | 7 | 8 | 10 | 14 | 15 | 38 | 8 | 18 | -20 | 26 |
| 6 | 10 | 11 | 13 | 15 | 20 | 50 | 17 | 9 | 16 | 26 |
| 7 | 7 | 8 | 8 | 12 | 12 | 36 | 16 | 10 | 12 | 26 |
| 8 | 9 | 12 | 17 | 20 | 24 | 60 | 19 | 7 | 24 | 26 |
| 7 | 10 | 12 | 16 | 20 | 25 | 60 | 7 | 19 | -24 | 26 |
| 5 | 5 | 6 | 6 | 8 | 9 | 9 | 24 | 13 | 13 | 0 | 26 |
| 6 | 9 | 13 | 14 | 16 | 17 | 21 | 48 | 13 | 13 | 0 | 26 |
| 8 | 8 | 11 | 11 | 12 | 12 | 14 | 14 | 36 | 16 | 10 | 12 | 26 |

Table V: The models with lowest total numbers of particles

As a by-product of our classification of \( D = 3 \) models we have also obtained all \( D = 2 \) configurations. It turns out that among these 124 models (which have 3 to 6 variables) there is exactly one model, \( \{ n_i \} = \{ 3, 3, 4, 4, 4 \} , d = 12 \), with \( b_{01} = 2 \) and \( b_{11} = 4 \), whereas all other models have \( b_{01} = 0 \) and \( b_{11} = 20 \). The first model factorizes and thus corresponds to a torus, whereas
all others have the same Hodge-diamond as the K3-surface \[15\]. This result is in fact known for
the 17 models in our list which correspond to minimal models [19] and can be understood from
the fact that, topologically, the only 2-dimensional complex manifolds of SU(2) holonomy are the
torus and the K3 surface. It also implies that the only case in which formula (18) need not apply
is when \(\chi = 0\) and \(P(1,1) = 96\): For \(D = 3\) complete factorization, i.e. a 3-dimensional torus,
cannot occur without further orbifolding, as for \(D = 1\) the only possible values of \(d\) are 3, 4 and
6.

5 Discussion

The most striking result of our calculations is definitely the strong violation of mirror symmetry
by non-invertible models. Whereas mirror symmetry turned out to be nearly exact for invert-
ible models, which were already known to have mirror partners at least as orbifolds due to the
Berghlund-H"ubsch-(BH-)construction [12], half of the non-degenerate models requiring links turned
out not to have mirror partners. This fact makes it seem doubtful that all non-degenerate models
have mirrors that can be described as Landau-Ginzburg orbifolds.

Exactly half of our 3 generation models, namely 20 out of 40, require links, whereas among all
non-degenerate models only 1 out of 6 does so. Note that in [8] only about 1000 configurations
of the 7555 Calabi-Yau’s in weighted \(\mathbb{P}_4\) were missing, whereas these authors only found 25 of
the 39 models with \(|\chi| = 6\) in this class. With the help of the BH-construction we can easily
construct at least 9 further 3 generation models as orbifolds of the invertible models of table III
(3 of them are already given in [10]).

Our work might be useful for the construction of even more 3 generation models due to the
following considerations: As already observed by Candelas et al. [8], the Euler number is a
multiple of 12 (or even 24) in most cases. On the other hand, all known 3 generation orbifolds
come from potentials where the Euler number is an odd multiple of 6. These configurations, of
which we have 496 in our list, may be the most promising for further investigation. A mirror
pair of such orbifolds is given in [10]. The fact that all 3 generation models and candidates for
3 generation models come from 5- or 7-point polynomials is a consequence of the fact that only
polynomials with an odd number of variables can have odd \(d\) and only models with odd \(d\) allow
Euler numbers which are not divisible by 4.

Although we have restricted attention to the canonical \(\mathbb{Z}_d\) orbifold, our results yield an implicit
classification of all abelian orbifolds, because any abelian group action can be diagonalized. Thus
we can assume that such a group acts as a phase symmetry and a calculation of all possible
systems of links generates all points of maximal abelian symmetry. This procedure may take us
to different points in moduli space, as the following example illustrates: Consider \(W = X_1^3 + X_2^3\)
and the symmetry \(X_1 \leftrightarrow X_2\), which acts diagonally on \(\tilde{X}_1 = X_1 + X_2\) and \(\tilde{X}_2 = X_1 - X_2\). In
these new variables \(W = \tilde{X}_1^3/4 + 3\tilde{X}_1\tilde{X}_2^2/4\).

Several questions are raised by our work: Are there interpretations of all concepts of singularity
theory, specifically of our link criterion, in terms of N=2 SCFT? What is the role of the almost
non-degenerate configurations: Could we conclude from the applicability of formulas 15 and 16 that almost non-degenerate configurations correspond to genuine SCFTs? (Including their spectra would not improve MS: Instead of providing partners we would rather generate new singles.) Most pressing, however, is the question where we should look for the mirror partners of the non-invertible models.

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Fig. 1: A plot of $d$ vs. the number of $27$ representations for 4 variables.
Fig. 2: A plot of $d$ vs. the number of $27$ representations for 5 variables.

Fig. 3: A plot of $d$ vs. the number of $27$ representations for 6 variables.
Fig. 4: A plot of $d$ vs. the number of $\bar{27}$ representations for 7 variables.

Fig. 5: A plot of $p = \phi(d)/2 - 1$ vs. the number of $\bar{27}$ representations for all non-degenerate configurations.