On Fano schemes of linear spaces of general complete intersections

FRANCESCO BASTIANELLI, CIRO CILIBERTO, FLAMINIO FLAMINIO, AND PAOLA SUPINO

Abstract. We consider the Fano scheme $F_k(X)$ of $k$-dimensional linear subspaces contained in a complete intersection $X \subset \mathbb{P}^n$ of multi-degree $d = (d_1, \ldots, d_s)$. Our main result is an extension of a result of Riedl and Yang concerning Fano schemes of lines on very general hypersurfaces: we consider the case when $X$ is a very general complete intersection and $\Pi_{i=1}^s d_i > 2$ and we find conditions on $n$, $d$, and $k$ under which $F_k(X)$ does not contain either rational or elliptic curves. At the end of the paper, we study the case $\Pi_{i=1}^s d_i = 2$.

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1. Introduction. In this paper, we are concerned with the Fano scheme $F_k(X)$ $\subset \mathbb{G}(k, n)$, parameterizing $k$-dimensional linear subspaces contained in $X \subset \mathbb{P}^n$, when $X$ is a complete intersection of multi-degree $d = (d_1, \ldots, d_s)$, with $1 \leq s \leq n - 2$. We will avoid the trivial case in which $X$ is a linear subspace, so that $\Pi_{i=1}^s d_i \geq 2$.

Our inspiration has been the following result by Riedl and Yang concerning the case of hypersurfaces:

Theorem 1.1 (cf. [8, Thm. 3.3]). If $X \subset \mathbb{P}^n$ is a very general hypersurface of degree $d$ such that $n \leq \frac{(d+1)(d+2)}{6}$, then $F_1(X)$ contains no rational curves.

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This paper is devoted to generalize Riedl–Yang’s result to complete intersections and to arbitrary $k \geq 1$:

**Theorem 1.2** Let $X \subset \mathbb{P}^n$ be a very general complete intersection of multidegree $d = (d_1, \ldots, d_s)$, with $1 \leq s \leq n - 2$ and $\Pi_{i=1}^s d_i > 2$.

Let $1 \leq k \leq n - s - 1$ be an integer. If

$$n \leq k - 1 + \frac{1}{k+2} \sum_{i=1}^s \left( \frac{d_i + k + 1}{k+1} \right),$$

then $F_k(X)$ contains neither rational nor elliptic curves.

The proof is contained in Section 3. Section 4 concerns the quadric case $\Pi_{i=1}^s d_i = 2$.

We work over the complex field $\mathbb{C}$. As customary, the term “general” is used to denote a point which sits outside a union of finitely many proper closed subsets of an irreducible algebraic variety whereas the term “very general” is used to denote a point sitting outside a countable union of proper closed subsets of an irreducible algebraic variety.

### 2. Preliminaries

Let $n \geq 3$, $1 \leq s \leq n - 2$, and $d = (d_1, \ldots, d_s)$ be an $s$-tuple of positive integers such that $\Pi_{i=1}^s d_i \geq 2$. Let $S_d := \bigoplus_{i=1}^s H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i))$ and consider its Zariski open subset $S_d^* := \bigoplus_{i=1}^s (H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i)) \setminus \{0\})$. For any $u := (g_1, \ldots, g_s) \in S_d^*$, let $X_u := V(g_1, \ldots, g_s) \subset \mathbb{P}^n$ denote the closed subscheme defined by the vanishing of the polynomials $g_1, \ldots, g_s$. When $u \in S_d^*$ is general, $X_u$ is a smooth, irreducible variety of dimension $n - s \geq 2$, so that $S_d^*$ contains an open dense subset parameterizing $s$-tuples $u$ such that $X_u$ is a smooth complete intersection.

For any integer $1 \leq k \leq n - s - 1$, consider the locus

$$W_{d,k} := \{ u \in S_d^* | F_k(X_u) \neq \emptyset \} \subseteq S_d^*$$

and set

$$t(n, k, d) := (k + 1)(n - k) - \sum_{i=1}^s \left( \frac{d_i + k}{k} \right).$$

If no confusion arises, we will denote by $t$ the integer $t(n, k, d)$. In this set-up, we recall the following results:

**Result 2.1** (cf., e.g., [1, 2, 5–7] and [3, Thm. 22.14, p. 294]). Let $n, k, s$, and $d = (d_1, \ldots, d_s)$ be as above.

(a) When $\Pi_{i=1}^s d_i > 2$, one has the following situation.

(i) For $t < 0$, $W_{d,k} \not\subseteq S_d^*$, so, for $u \in S_d^*$ general, $F_k(X_u) = \emptyset$.

(ii) For $t \geq 0$, $W_{d,k} = S_d^*$ and, for $u \in S_d^*$ general, $F_k(X_u)$ is smooth with $\dim(F_k(X_u)) = t$ and it is irreducible when $t \geq 1$.

(b) When $\Pi_{i=1}^s d_i = 2$, one has the following situation.

(i) For $\left\lfloor \frac{n-s}{2} \right\rfloor < k \leq n - s - 1$, $W_{d,k} \not\subseteq S_d^*$; more precisely, for $u \in S_d^*$ such that $X_u$ is smooth, one has $F_k(X_u) = \emptyset$. 
(ii) For $1 \leq k \leq \left\lceil \frac{n-s}{2} \right\rceil$, $W_{d,k} = S^*_d$ and, for $u \in S^*_d$ such that $X_u$ is smooth, $F_k(X_u)$ is smooth, (equivariantly) with $\dim(F_k(X_u)) = t = (k+1)(n-s - \frac{3k}{2})$. Moreover $F_k(X_u)$ is irreducible unless $\dim(X_u) = n-s$ is even and $k = \frac{n-s}{2}$, in which case $F_k(X_u)$ consists of two disjoint irreducible components. In this case, each irreducible component of $F_k(X_u)$ is isomorphic to $F_{k-1}(X'_u)$, where $X'_u$ is a general hyperplane section of $X_u$.

**Result 2.2** (cf., e.g., [2, Rem. 3.2, (2)]). Let $n, k, s$, and $d = (d_1, \ldots, d_s)$ be as above with $\Pi^r_{i=1} d_i \geq 2$. When $u \in S^*_d$ is such that $X_u \subset \mathbb{P}^n$ is a smooth, irreducible complete intersection of dimension $n-s$ and $F_k(X_u)$ is not empty, smooth, and of (expected) dimension $t$, the canonical bundle of $F_k(X_u)$ is given by

$$\omega_{F_k(X_u)} = \mathcal{O}_{F_k(X_u)} \left(-n - 1 + \sum_{i=1}^s \frac{d_i + k}{k+1}\right)$$

(2.1)

where $\mathcal{O}_{F_k(X_u)}(1)$ is the hyperplane line bundle of $F_k(X_u)$ in $\mathbb{P}^{(n+1)}$ via the Plücker embedding of $\mathbb{G}(k,n)$.

We will also need the following:

**Result 2.3** (cf. [8, Prop. 3.5]). Let $n, m$ be positive integers with $m \leq n$. Let $B \subset \mathbb{G}(m,n)$ be an irreducible subvariety of codimension at least $\epsilon \geq 1$. Let $C \subset \mathbb{G}(m-1,n)$ be a non-empty subvariety satisfying the following condition: for every $(m-1)$-plane $c \subset C$, if $b \in \mathbb{G}(m,n)$ is such that $c \subset b$, then $b \in B$. Then the codimension of $C$ in $\mathbb{G}(m-1,n)$ is at least $\epsilon + 1$.

### 3. The proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2, which uses a strategy similar to the one in [8, Thm. 3.3].

**Proof of Theorem 1.2.** Let $\mathbb{G} := \mathbb{G}(k,n)$ be the Grassmannian of $k$-planes in $\mathbb{P}^n$ and $\mathbb{H} := \mathbb{H}_{d,n}$ be the irreducible component of the Hilbert scheme whose general point parameterizes a smooth, irreducible complete intersection $X \subset \mathbb{P}^n$ of dimension $n-s \geq 2$ and multi-degree $d$ ($\mathbb{H}$ is the image of an open dense subset of $S^*_d$ via the classifying morphism). Let us consider the incidence correspondence

$$U_{k,n,d} := \{([\Lambda], [X]) \in \mathbb{G} \times \mathbb{H} | \Lambda \subset X \} \subset \mathbb{G} \times \mathbb{H}$$

with the two projections $\mathbb{G} \leftarrow U_{k,n,d} \leftarrow \mathbb{H}$.

Note that $U_{k,n,d}$ is smooth and irreducible, because the map $\pi_1$ is surjective and has smooth and irreducible fibres which are all isomorphic via the action of the group of projective transformations.

Since $\Pi^r_{i=1} d_i > 2$, from Result 2.1(a), if $t \leq 0$, then $F_k(X)$ is either empty or a zero-dimensional scheme, so in particular the statement holds true. We can therefore assume $t \geq 1$. In this case, from Result 2.1(a.ii), the map $\pi_2$ is dominant and the fibre over the general point $[X] \in \mathbb{H}$ is isomorphic to $F_k(X)$, hence it is smooth, irreducible of dimension $t$. Thus $U_{k,n,d}$ dominates $\mathbb{H}$ via $\pi_2$ and

$$\dim(U_{k,n,d}) = \dim(\mathbb{H}) + t.$$  

(3.1)
Consider now
\[ \mathcal{R}_{k,n,d} := \left\{ ([\Lambda], [X]) \in U_{k,n,d} \mid \text{there exists a rational curve in } F_k(X) \text{ containing } [\Lambda] \right\} \subseteq U_{k,n,d} \]
and similarly
\[ \mathcal{E}_{k,n,d} := \left\{ ([\Lambda], [X]) \in U_{k,n,d} \mid \text{there exists an elliptic curve in } F_k(X) \text{ containing } [\Lambda] \right\} \subseteq U_{k,n,d}. \]

Notice that both \( \mathcal{R}_{k,n,d} \) and \( \mathcal{E}_{k,n,d} \) are (at most) countable unions of irreducible locally closed subvarieties. Let us see this for \( \mathcal{R}_{k,n,d} \), the case of \( \mathcal{E}_{k,n,d} \) being analogous. Look at the morphism \( \pi_2 : U_{k,n,d} \to \mathbb{H} \), whose fibre over a point \([X] \in \mathbb{H}\) is isomorphic to the Fano scheme \( F_k(X) \). Consider the locally closed subset \( \mathcal{H} \) in the Hilbert scheme consisting of the set of points parameterizing irreducible rational curves in \( U_{k,n,d} \) contained in fibres of \( \pi_2 \). Then \( \mathcal{H} \), as well as the Hilbert scheme, is a countable union of irreducible varieties. Consider the incidence correspondence
\[ \mathcal{I} = \{ (([\Lambda], [X]), [C]) \in U_{k,n,d} \times \mathcal{H} \mid ([\Lambda], [X]) \in C \}. \]

Since \( \mathcal{H} \) is a countable union of irreducible varieties, also \( \mathcal{I} \) is a countable union of irreducible varieties. Finally \( \mathcal{R}_{k,n,d} \) is the image of \( \mathcal{I} \) via the projection on \( U_{k,n,d} \), and therefore it is (at most) a countable union of irreducible varieties.

We point out that proving the assertion for very general complete intersections \( X \subset \mathbb{P}^n \) of multi-degree \( d = (d_1, \ldots, d_s) \) is equivalent to proving it for very general complete intersections \( Y \subset \mathbb{P}^M \) of multi-degree
\[ d^n = d^n_M := (1, \ldots, 1, d_1, \ldots, d_s) \]
for some \( M \geq n \). In particular, condition (1.1) gives equivalent inequalities and \( t := t(n, k, d) = t(M, k, d^n_M) \). Moreover, in the light of this fact, we can assume \( d_1, \ldots, d_s \geq 2 \).

We claim that, under our assumptions, there exists \( M \geq n \) such that both \( \mathcal{R}_{k,M,d^n_M} \) and \( \mathcal{E}_{k,M,d^n_M} \) have codimension at least \( t+1 \) in \( U_{k,M,d^n_M} \), which proves the statement. Indeed, consider the case of \( \mathcal{R}_{k,M,d^n_M} \) (the same reasoning applies to the case of \( \mathcal{E}_{k,M,d^n_M} \)); if \( \dim \mathcal{U}_{k,M,d^n_M}(\mathcal{R}_{k,M,d^n_M}) \geq t+1 \), then, by formula (3.1), we have that \( \dim(\mathcal{R}_{k,M,d^n_M}) \leq \dim(\mathbb{H}_{d^n_M,M}) - 1 \). Thus \( \dim(\pi_2(\mathcal{R}_{k,M,d^n_M})) \leq \dim(\mathbb{H}_{d^n_M,M}) - 1 \), proving the assertion for very general complete intersections \( Y \subset \mathbb{P}^M \) of multi-degree \( d^n_M \), and hence for very general complete intersections \( X \subset \mathbb{P}^n \) of multi-degree \( d^n_M \).

We are therefore left to show that there exists \( M \geq n \) such that both \( \mathcal{R}_{k,M,d^n_M} \) and \( \mathcal{E}_{k,M,d^n_M} \) have codimension at least \( t+1 \) in \( U_{k,M,d^n_M} \). In what follows, we will focus on \( \mathcal{R}_{k,M,d^n_M} \) (the same arguments work for the case of \( \mathcal{E}_{k,M,d^n_M} \)).

If \( \mathcal{R}_{k,n,d} = \emptyset \), we are done. So assume \( \mathcal{R}_{k,n,d} \neq \emptyset \) and take \(([\Lambda_0], [X_0]) \in \mathcal{R}_{k,n,d} \) a very general point in an irreducible component of \( \mathcal{R}_{k,n,d} \). We note that for any \( M \geq n \), we can embed \( X_0 \subset \mathbb{P}^n \) into an \( n \)-plane of \( \mathbb{P}^M \), and hence we can identify \( X_0 \) with a complete intersection in \( \mathbb{P}^M \) of multi-degree \( d^n_M \), so that \(([\Lambda_0], [X_0]) \in \mathcal{R}_{k,M,d^n_M} \). Therefore it suffices to find some \( M \geq n \) and an irreducible subvariety \( \mathcal{F} \subset U_{k,M,d^n_M} \) such that
\(([A_0], [X_0]) \in \mathcal{F}\) and \(\text{codim}_\mathcal{F}(\mathcal{R}_{k,M,d^n} \cap \mathcal{F}) \geq t + 1\).

To do so, we start with the following remark: if \(X \subset \mathbb{P}^n\) is a very general complete intersection of multi-degree \(d\), it follows from Results 2.1(a.ii) and 2.2 and formula (2.1) that \(\omega_{F_k(X)}\) is ample if and only if

\[
n \leq \sum_{i=1}^s \left( \frac{d_i + k}{k + 1} \right) - 2.
\]

We set

\[
m = m(d,k) := \sum_{i=1}^s \left( \frac{d_i + k}{k + 1} \right) - 2
\]

and, in view of the obvious equality \((\frac{d_i + k + 1}{k + 1}) = (\frac{d_i + k}{k + 1}) + (\frac{d_i + k}{k})\), we notice that the hypothesis (1.1) reads \(m - n \geq t\). Hence we have \(m > n\).

Let \(Y' \subset \mathbb{P}^m\) be a very general complete intersection of multi-degree \(d'\) and let \(\Lambda' \subset Y'\) be a very general \(k\)-plane of \(Y'\), i.e., \([\Lambda']\) corresponds to a very general point in \(F_k(Y')\). By Result 2.1 and the fact that \(m > n\), we have that \(F_k(\Lambda')\) is smooth, irreducible with \(\dim(F_k(\Lambda')) = (k + 1)(m - k) - \sum_{i=1}^s (\frac{d_i + k}{k + 1}) > t \geq 1\). Notice that there are no rational curves in \(F_k(\Lambda')\) through \(\Lambda'\) since \(F_k(\Lambda')\) is smooth and of general type, because \(\omega_{F_k(\Lambda')} = \mathcal{O}_{F_k(\Lambda')}(1)\) by the choice of \(m\), and \([\Lambda']\) is very general on \(F_k(\Lambda')\). In other words, \(([\Lambda'], [Y']) \in \mathcal{U}_{k,m,d} \setminus \mathcal{R}_{k,m,d}\).

Take now an integer \(M \gg m > n\) and let \(Y'' \subset \mathbb{P}^M\) be a smooth, complete intersection of multi-degree \(d''\), containing a \(k\)-plane \(\Lambda''\), such that \(X_0\) is an \(n\)-plane section of \(Y''\). \(Y''\) is an \(m\)-plane section of \(Y''\), and \(\Lambda'' = \Lambda_0 = \Lambda'\).

For any integer \(r \geq n\), let \(Z_r \subset \mathbb{G}(r,M)\) be the variety of \(r\)-planes in \(\mathbb{P}^M\) containing \(\Lambda_0\) and let \(Z_r' \subseteq Z_r\) be the subset of those \(r\)-planes \(\Lambda \in Z_r\) such that the Fano scheme \(F_k(\Lambda'') \cap \Lambda\) of the complete intersection \(Y'' \cap \Lambda \subset \mathbb{P}^M\) contains a rational curve through the point \([A_0] \in F_k(\Lambda'') \cap \Lambda\). Note that \(Z_r\) is isomorphic to \(\mathbb{G}(r - k - 1, M - k - 1)\).

For any integer \(r \geq n\), we define the morphism \(\phi_r : Z_r \to \mathcal{U}_{k,m,d''}\) sending the \(r\)-plane \(\Lambda\) to \(([A_0], [\Lambda'']) \in \mathcal{U}_{k,m,d''}\). It is clear that \(\phi_r\) maps \(Z_r\) isomorphically onto its image \(\mathcal{F}_r := \phi_r(Z_r)\), where the inverse map sends a pair \(([A_0], [\Lambda]) \in \mathcal{F}_r\) to the point \([\Lambda] \in Z_r\) corresponding to the linear span \(\Lambda = \langle Y \rangle\) (recall that we assumed \(d_1, \ldots, d_s \geq 2\)). Then the image of \(Z_r'\) in \(\mathcal{U}_{k,m,d''}\) is \(\mathcal{F}_r \cap \mathcal{R}_{k,m,d''}\). We will set \(\mathcal{F} := \mathcal{F}_n\).

By construction of the pair \(([A_0], [Y']) \in \mathcal{U}_{k,m,d''} \setminus \mathcal{R}_{k,m,d''}\), we have that \(\text{codim}_{\mathcal{F}_m}(\mathcal{F}_m \cap \mathcal{R}_{k,m,d''}) = \epsilon \geq 1\). Now we apply Result 2.3 to \(\mathcal{F}_m \cong \mathbb{G}(r - k - 1, M - k - 1)\), \(\tilde{B} = \mathcal{F}_m \cap \mathcal{R}_{k,m,d''}\), and \(C = \mathcal{F}_{m-1} \cap \mathcal{R}_{k,m,d''-1}\), because clearly \(B\) and \(C\) verify the condition stated there. We deduce that \(\text{codim}_{\mathcal{F}_{m-1}}(\mathcal{F}_{m-1} \cap \mathcal{R}_{k,m,d''-1}) \geq \epsilon + 1\). Iterating this argument, we see that \(\text{codim}_{\mathcal{F} \cap \mathcal{R}_{k,m,d''}}(\mathcal{F} \cap \mathcal{R}_{k,m,d''}) \geq m - n + 1\). Now, as we already noticed, \(m - n + 1 \geq t + 1\) is equivalent to the condition (1.1), hence we have \(\text{codim}_{\mathcal{F} \cap \mathcal{R}_{k,m,d''}}(\mathcal{F} \cap \mathcal{R}_{k,m,d''}) \geq t + 1\), as wanted. This completes the proof. \(\square\)

**Remark 3.1.** Let \(X \subset \mathbb{P}^n\) be a general complete intersection of multi-degree \(d = (d_1, \ldots, d_s)\). If \(\sum_{i=1}^s (\frac{d_i + k}{k + 1}) \leq n\), then, by (2.1), \(F_k(X)\) is a smooth Fano
variety and therefore, by [4], it is rationally connected, i.e., there is a rational curve passing through two general points of it.

4. The quadric case. Here we prove the following result:

**Theorem 4.1.** Let \( X \subset \mathbb{P}^n \) be a smooth complete intersection of multi-degree \( d = (d_1, \ldots, d_s) \), with \( 1 \leq s \leq n - 2 \) and \( \prod_{i=1}^{s} d_i = 2 \). Let \( k \) be an integer such that \( 1 \leq k \leq n - s - 1 \). Then:

(i) for \( \left\lfloor \frac{n-s}{2} \right\rfloor < k \leq n - s - 1 \), \( F_k(X) \) is empty, whereas

(ii) for \( 1 \leq k \leq \left\lfloor \frac{n-s}{2} \right\rfloor \), the single component or both components of \( F_k(X) \) (see Result 2.1(b.ii)) are rationally connected.

**Proof.** Since \( \prod_{i=1}^{s} d_i = 2 \), we have that \( X \subset \mathbb{P}^{n-s+1} \) is a smooth quadric hypersurface. From Result 2.1(b), if \( \left\lfloor \frac{n-s}{2} \right\rfloor < k \leq n - s - 1 \), \( F_k(X) \) is empty.

Next we assume \( 1 \leq k \leq \left\lfloor \frac{n-s}{2} \right\rfloor \). From Result 2.2 and formula (2.1), we have

\[
\omega_{F_k(X)} = \mathcal{O}_{F_k(X)}(-n + s + k).
\]

Since \( k \leq \frac{n-s}{2} \), \( -n + s + k \leq -1 \), therefore the single component or both components of \( F_k(X) \) (see Result 2.1(b.ii)) are smooth Fano varieties, hence they are rationally connected by [4]. \( \square \)

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