UNIFIED APPROACH TO THERMODYNAMIC BETHE ANSATZ AND FINITE SIZE CORRECTIONS FOR LATTICE MODELS AND FIELD THEORIES

C. Destri

*Dipartimento di Fisica, Università di Milano and INFN, sezione di Milano*

H.J. de Vega

*Laboratoire de Physique Théorique et Hautes Energies, Paris*

ABSTRACT

We present a unified approach to the Thermodynamic Bethe Ansatz (TBA) for magnetic chains and field theories that includes the finite size (and zero temperature) calculations for lattice BA models. In all cases, the free energy follows by quadratures from the solution of a **single** non-linear integral equation (NLIE). [A system of NLIE appears for nested BA].
We derive the NLIE for: a) the six-vertex model with twisted boundary conditions; b) the XXZ chain in an external magnetic field $h_z$ and c) the sine-Gordon-massive Thirring model (sG-mT) in a periodic box of size $\beta \equiv 1/T$ using the light-cone approach. This NLIE is solved by iteration in one regime (high $T$ in the XXZ chain and low $T$ in the sG-mT model). In the opposite (conformal) regime, the leading behaviors are obtained in closed form. Higher corrections can be derived from the Riemann-Hilbert form of the NLIE that we present.

1. Introduction

The computation of thermodynamical functions for integrable models started with the seminal works of C.N. Yang and C.P. Yang [1], of M. Takahashi [2] and M. Gaudin [3]. In Refs. [2,3] the free energy of the Heisenberg spin chain is written in terms of the solution of an infinite set of coupled nonlinear integral equations, derived on the basis of the so-called “string hypothesis”.

These equation are commonly known as Thermodynamic Bethe Ansatz (TBA) equations. Nowadays, the TBA is mostly used as a nonperturbative tool to investigate Perturbed Conformal Field Theories and has been considerably generalized (see e.g. [4,5]). In all cases some some kind of string hypothesis is used.

In ref.[6], we proposed a simpler way to solve the thermodynamics of the XXZ chain and of the sine–Gordon field theory by means of a single, rigorously derived, nonlinear integral equation. This method can be applied to a wide class of models solvable by Bethe Ansatz, and can provide also the excited–states scaling functions of the sG field theory [7]. Similar methods were used to also determine some correlation lengths at finite temperature in the XXZ chain [8].

The purpose of the present paper is threefold. First we generalize the results of ref.[6] in the presence of external magnetic field for the XXZ Heisenberg model.
Second, we show that this approach provides an unified method to compute both at finite temperature and infinite size and at zero temperature and finite size. Third, we recast our NLIE in two alternative forms: a) as an infinite set of algebraic equations of BA type and b) as a Riemann-Hilbert problem. These forms are appropriate to compute the asymptotic behaviors around the conformal regime (low $T$ for the magnetic chains and high $T$ for field theory). In the recursive regime (the opposite to the conformal one), our NLIE can be easily solved by iteration.

Continuum quantum field theories at finite temperature $T$ are obtained from periodic boundary conditions (PBC) in the imaginary time with period $\beta \equiv T^{-1}$. By Wick rotation, this leads to a periodic spatial box of finite size $\beta$. What we achieve here is the analogous to a Wick rotation on the exact Bethe Ansatz (BA) solution on the lattice.

Our starting point is the BA diagonalization of a row-to-row inhomogeneous and twisted transfer matrix $t(\lambda, \theta_1, \theta_2, \ldots, \theta_N, \omega)$ where there is an inhomogeneity $\theta_n$ attached to each site and a twist $\omega$ in the boundary conditions ($\omega = 0$ corresponds to PBC).

Setting the inhomogeneities $\theta_n = 0$ yields the homogeneous six-vertex transfer matrix.

Choosing the inhomogeneities alternating $\theta_n = (-1)^{n-1} \theta$ yields a diagonal-to-diagonal transfer matrix $T_N(\theta) = t_{2N}(\theta, \theta, 0)$ [20]. For $\theta \to 0$, $T_N(\theta)$ gives at order $\theta$ the XXZ hamiltonian.

$$T_N(\theta) \overset{\theta \to 0}{=} 1 - \frac{\theta}{2J \sin \gamma} H_{XXZ} + O(\theta^2) \quad (1.1)$$

and hence,

$$e^{-\beta H_{XXZ}} = \lim_{N \to \infty} \left[ T_L \left( \frac{2\beta J \sin \gamma}{N} \right) \right]^N$$
This relation permits to write the XXZ partition function

\[ Z \equiv \text{Tr} \left\{ e^{-\beta H_{\text{XXZ}}} \right\} \]

in terms of the diagonal-to-diagonal transfer matrix \( T_L(\theta) \). Moreover, using the crossing symmetry of the statistical weights one can prove that \( Z \) in the infinite volume limit is given just by the ground state eigenvalue of \( T_N(\theta) \) in the \( N \rightarrow \infty \) limit. This alternative way is much simpler than the usual TBA approach, where one has to sum over an infinite number of states with an arbitrary number of holes and an arbitrary number of strings of any length. What we do here is analog to duality in S-matrix theory, where the sum over an infinite number of resonances in the s-channel can be replaced by the exchange of one reggeon in the t-channel.

Setting \( \theta = -i\Theta \) with \( \Theta \) real, we get the lattice field theory evolution operators in the light-cone approach [13,20]

\[
U_R(\Theta) = a(-2i\Theta)^{-N} t_{2N}(-i\Theta, -i\Theta, 0) , \quad U_L(\Theta) = a(2i\Theta)^N t_{2N}(+i\Theta, -i\Theta, 0)^{-1}
\]

where \( a(\pm 2i\Theta)^\pm N \) is a unitarization factor. That is, for imaginary \( \theta \), we interpret the lattice as a discretized Minkowski spacetime [15] and hence the operators \( U_R \) and \( U_L \) describe space and time translations in the light-cone directions of a discretized Minkowski spacetime. If we call \( \delta \) the lattice spacing,

\[
U = e^{-i\delta H} , \quad U_R U_L^{-1} = e^{i\delta P}
\]

(1.2)

define the lattice hamiltonian (\( H \)) and momentum (\( P \)) operators.

Moreover, the operator \( t_{2N}(\lambda, -i\Theta, 0) \) is the generating functional of local conserved charges (including \( H \) and \( P \)) for \( |\text{Im} \lambda| < \gamma/2 \) [21]. In the region \( -\pi - \gamma/2 < \text{Im} \lambda < -\gamma/2 \), \( t_{2N}(\lambda, -i\Theta, 0) \) generates non-local conserved charges [21].
We diagonalize the transfer matrix $t(\lambda, \theta_1, \theta_2, \ldots, \theta_N, \omega)$ by algebraic Bethe Ansatz (BA). The BA expresses eigenvectors and eigenvalues in terms of the roots of $M$ coupled algebraic equations ($1 \leq M \leq N$), the so called Bethe Ansatz equations (BAE).

Using contour integral techniques, we rewrite these BAE as a non-linear integral equation. This equation may take different forms. A compact form is

$$Z(\mu) = \varphi(\mu) + 2 \text{Im} \int_{-\infty}^{+\infty} d\mu' G(\mu - \mu' - i\eta) \log \left[1 + e^{iZ(\mu' + i\eta)}\right]$$ (1.3)

where the kernel $G(\lambda)$ is explicitly given by eq.(2.24) and

$$\varphi(\mu) = \begin{cases} 2N \arctan \left[ \frac{\sinh(\pi \mu/\gamma)}{\cos(\pi \theta/[2\gamma])} \right] - \frac{\pi \omega}{\pi - \gamma} & \text{finite size vertex model with TBC} \\ - \frac{2\pi \tilde{\beta}}{\gamma \sinh(\pi \mu/\gamma)} + \frac{i\pi \beta h}{\pi - \gamma} & \text{XXZ thermodynamics in a magnetic field} \\ m \beta \sinh(\pi \mu/\gamma) & \text{sine-Gordon field theory} \end{cases}$$ (1.4)

$Z(\mu)$ is the unknown in eq.(1.3). The real parameter $\eta$ can be freely chosen in the interval $0 < \eta < \gamma/2$.

We have unified three different problems (finite size vertex model with TBC, the XXZ thermodynamics in a magnetic field and the sine-Gordon field theory in a finite volume) into the single NLIE (1.3)-(1.4).

Once we have the solution $Z(\lambda)$ of eq.(1.3), we can compute the transfer matrix eigenvalues by quadratures. We consider three relevant applications

a) The finite size (homogeneous) six-vertex model with twisted boundary conditions (TBC).

b) The (infinite) XXZ chain at temperature $T = \beta^{-1}$ in an external magnetic field $h$.

c) The massive Thirring-sine Gordon model ($mT$-sG) on a periodic box of size $\beta$. 

5
In the first case, we find for the six vertex partition function on a double periodic square lattice ($T \times 2N$)

$$\lim_{T \to \infty} T^{-1} \log Z_{6V} = \log \tau_{\text{max}} = -2Nf_0(\lambda, \omega) + L_{2N}(\lambda, \omega) \quad (1.5)$$

For large $N$, we obtain that $L_N$ follows the conformal behaviour for TBC with unit central charge [see sec. 7.1]. Namely,

$$L_{2N} \xrightarrow{N \to \infty} \frac{\pi}{6N} \tan \left( \frac{\pi\lambda}{\gamma} \right) \left[ 1 - \frac{6\omega^2}{\pi^2(1 - \gamma/\pi)} \right]$$

In the case b) , we relate the XXZ free energy at temperature $T$ and on an external magnetic field $2h$ in the $z$-direction with the maximal eigenvalue of the alternating transfer matrix $t_{2N}(\pm \theta, \theta, -i\beta h)$ with TBC. Notice that the twist, $-i\beta h$, is purely imaginary for real magnetic fields $h$. This new result allows us to express the XXZ free energy $f(\beta, h)$ in terms of the solution $Z(\lambda)$ of our non-linear integral equation (NLIE) (1.3).

$$f(\beta, h) = h + 2J \cos \gamma + \frac{\sin \gamma}{\beta} \oint_{\Gamma} d\lambda \frac{\log [1 + e^{-iZ(\lambda)}]}{2\pi i \sinh \lambda \sinh(\lambda - i\gamma)} \quad (1.6)$$

Finally, we use the light-cone approach [13, 20, 15, 21] to deal with c). We write the energy in the mT-sG model on a ring of length $\beta$ in terms of the alternating six vertex transfer matrix $[t(\mp i\Theta, -i\Theta, 0)]$ eigenvalues. The basic relation is the formula for the unit evolution operator for lattice field models,

$$U = e^{-i\delta H} = \left[ \frac{a(2i\Theta)}{a(-2i\Theta)} \right]^N t(-i\Theta, -i\Theta, 0) t(i\Theta, -i\Theta, 0)^{-1} \quad (1.7)$$

where $\delta$ is the lattice spacing, $\Theta$ plays the role of UV cutoff and $a(\pm 2i\Theta)$ are unitarizing c-number factors. In the continuum limit, $\delta \to 0$, $N \to \infty$ with
\( \beta \equiv N \delta \) fixed. At the same time \( \Theta \) tends to infinite as

\[
\Theta \simeq \frac{\gamma}{\pi} \log \frac{4N}{m\beta}
\]

We find for the scaling function

\[
E(\beta) \equiv \lim_{N \to \infty} \left[ E_N - E_c \right]
= -\frac{m}{\gamma} \Im \int_{-\infty}^{+\infty} \frac{d\lambda}{\pi} \sinh \left[ \frac{\pi (\lambda + i\eta)}{\gamma} \right] \log \left[ 1 + e^{iZ(\lambda+i\eta)} \right]
\]

where \( Z(\lambda) \) satisfies [cfr. eq.(1.3)]

\[
Z(\lambda) = m\beta \sinh(\pi \lambda / \gamma) + 2 \Im \int_{-\infty}^{+\infty} d\mu G(\lambda - \mu - i\eta) \log \left[ 1 + e^{iZ(\mu+i\eta)} \right]
\]

provided we choose \( 0 < \eta < \frac{1}{2} \min(\gamma, \pi - \gamma) \).

Shifting \( \lambda \) by \( i\gamma/2 \), eq.(1.10) takes the form

\[
\epsilon_f = m\beta \cosh \theta - G_0 \ast L_f + G_1 \ast L_{\bar{f}}
\]

where

\[
\epsilon_f(\theta) = -iZ\left( \frac{\pi}{\gamma} \theta + i\gamma/2 \right), \quad \epsilon_{\bar{f}} = \overline{\epsilon_f}
\]

\[
L_f \equiv \log(1 + e^{\epsilon_f}) , \quad L_{\bar{f}} \equiv \log(1 + e^{\epsilon_{\bar{f}}}) = \overline{L_f}
\]

\[
G_0(\theta) = \frac{1}{\pi} G\left( \frac{\pi}{\gamma} \theta \right) , \quad G_1(\theta) = G_0(\theta + i\pi - i0)
\]

The kernels \( G_0(\theta) \) and \( G_1(\theta) \) are just the \( \theta \) derivative of the fermion-fermion and fermion-antifermion physical phase-shifts. This transparent physical interpretation of eq.(1.11) suggests the form that it should take in other field theory models.

Now we face the problem of solving NLIE like (1.3) or (1.10) in order to get physical results. It must be noticed that we find just one NLIE for finite temperature and finite size situations. This situation contrasts with the standard TBA
which for nonrational values of $\gamma/\pi$ involves an infinite number of nonlinear integral equations for the infinitely many different types of “magnons” describing the energy degeneracies of the physical fermions and antifermions [23]. Therefore, our equation effectively provides a resummation of the magnon degrees of freedom.

There are essentially two regimes in our NLIE. We call them ‘recursive’ and ‘conformal’. In the ‘recursive’ regime the NLIE can be simply solved by iterating the inhomogeneity. This happens for small $\beta$ in the XXZ chain and for large $\beta$ in field theory.

The ‘conformal’ regime is the opposite to the ‘recursive’ regime. In it, the calculations are more subtle. The dominant behaviour (large $\beta$ for the XXZ chain and small $\beta$ in field theory), follows in closed form with the help of the Lemma in sec. 7. This lemma avoids the detailed resolution of the NLIE for the dominant contributions. Higher corrections follow using the Riemann-Hilbert form of our NLIE (sec. 7.4). We find that the energy in the sG-mT model has a series of terms with $(m\beta)^{-1}$ times integer powers of $(m\beta)^{4\gamma/\pi}$ plus the central charge term proportional to $(m\beta)^{-1}$, in agreement with the predictions of Perturbed Conformal Field Theory.

Since the inhomogeneity in the NLIE for field theory is $m\beta \cosh \theta$, it is useful to define for small $m\beta$ kink solutions, $\epsilon_k(\theta)$ of the NLIE with inhomogeneity $e^{\pm \theta}$ [see eq.(7.19)]. We obtain in this way for the energy $E(\beta)$ at high temperatures

$$E_{\beta \to 0} = -\frac{\pi}{6\beta} - \frac{m^2 \beta}{4} \cot \left[ \pi^2 / 2\gamma \right] - \frac{m}{2\pi} \int_{-\infty}^{+\infty} d\theta \, \cosh \theta \left[ \epsilon(\theta) + \epsilon(\theta) \right] \quad (1.13)$$

The first term is the universal conformal Casimir energy, from which we read the correct central charge $c = 1$ of the mT-sG model. The second term, linear in $\beta$, exactly coincides with minus the scaling bulk free energy [see eq.(5.7)]. The last integral in eq.(1.13) represents the resummation of the perturbed conformal field
theory around the massless Thirring model. The fact that it must contain non-integral powers of \( r \), causing non-analyticity at \( r = 0 \), can be established directly from the original equation (7.18), since this cannot be expanded in a Taylor series of \( r \).

In the ‘recursive’ regime, we derive in sec. 6.1, the high temperature expansion for the XXZ chain in the external field \( h \). One only needs the Cauchy theorem to evaluate the high \( T \) expansion coefficients. The first five terms are explicitly computed in sec. 6.1:

\[
f(\beta) = -\beta^{-1} \ln 2 + J \cos \gamma - \beta \left[ J^2 (1 + \frac{1}{2} \cos^2 \gamma) + \frac{1}{2} h^2 \right] + \beta^2 \cos \gamma (J^3 + h^2) - \beta^3 \left[ \left( \frac{1}{4} + \frac{1}{6} \cos^2 \gamma - \frac{1}{12} \cos^4 \gamma \right) J^4 - J^2 h^2 - \frac{1}{12} h^4 \right] + O(\beta^4)
\]

which indeed agrees with the high \( T \) expansion.

In sec. 6.2 we obtain the ground state energy for the mT-sG model for small \( T \). It is non-analytic at \( T = 0 \). We find up to contributions \( O(e^{-3m\beta}) \)

\[
E(\beta) \xrightarrow{\beta \rightarrow \infty} -\frac{2m}{\pi} K_1(m\beta) + e^{-2m\beta} \sqrt{\frac{m}{\pi \beta}} \left[ \frac{1 + \sqrt{2}}{2} + \sqrt{\frac{\pi}{m\beta}} K_\gamma + O(\frac{1}{\beta}) \right] + O(e^{-3m\beta})
\]

(1.14)

where \( K_1(x) \) stands for a modified Bessell function and the constant \( K_\gamma \) is defined by the integral (6.17). The first term in eq.(1.14) describes free massive bosons, the subsequent terms take into account the interactions.

In secs 4 and 5, we show that our NLIE (1.3) and (1.10) are equivalent to an infinite set of algebraic equations of BA type. These equations differ from the usual BA equations [14, 15] in two essential aspects: i) they are infinite in number, ii) the roots are discretely spaced although they describe models with an infinite number of sites. The root spacing tends to zero only in the conformal limit.

We would like to remark that, contrary to the traditional thermodynamic Bethe Ansatz [2,3], our approach does not rely on the string hypothesis on the structure of the solutions of the BA equations characteristic of the Heisenberg chain. This
makes our approach definitely simpler. Most notably, the whole construction of
the thermodynamics no longer depends on whether $\gamma/\pi$ is a rational or not, unlike
Takahashi approach[2]. This applies equally well to the problem of the ground
state scaling function of the sine–Gordon field theory, since the standard TBA
approach requires the string hypothesis.

Generalizations of the present TBA approach to higher spin chains as well as to
magnetic chains and quantum field theories associated to Lie algebras other than
$A_1$ (nested BA solutions) are relatively straightforward provided crossing symmetry
holds.

Our NLIE has been recently used with success in ref.[24].

2. Nonlinear Integral Equation

Let us consider a horizontal line formed by N sites, and associate to each site
$n = 1, 2, \ldots, N$ a triplet of (vertical) Pauli matrices $(\sigma^x_n, \sigma^y_n, \sigma^z_n)$, with standard
commutation rules

$$\left[\sigma^\alpha_m, \sigma^\beta_n\right] = 2i \delta_{mn} \epsilon^{\alpha\beta\gamma} \sigma^\gamma_n$$

As customary, we also introduce the auxiliary, or horizontal, two-dimensional
space, labeled by the index $n = 0$, on which the matrices $(\sigma^x_0, \sigma^y_0, \sigma^z_0)$ act. For
any complex value $\lambda$ of spectral parameter and an arbitrary set of inhomogeneities
$\theta_1, \theta_2, \ldots, \theta_N$, the local vertices $L_n$ are then written as

$$L_n = R_{0n}(\lambda + \theta_n) P_{0n}$$

in terms of the spin 1/2 $R$–matrices of the six vertex model,

$$R_{kn}(\theta) = \frac{a + c}{2} + \frac{a - c}{2} \sigma^z_k \sigma^z_n + \frac{b}{2} \left( \sigma^x_k \sigma^y_n + \sigma^y_k \sigma^x_n \right)$$

and of the exchange operators,

$$P_{kn} = \frac{1}{2} (1 + \bar{\sigma}_k \cdot \bar{\sigma}_n)$$
The dependence on $\theta$ of $R_{k\, n}(\theta)$ is contained in the the Boltzmann weights $a, b, c$, for which we take the standard trigonometric parametrization

$$a = a(\theta) \equiv \sin(\gamma - \theta) , \quad b = b(\theta) \equiv \sin \theta , \quad c = \sin \gamma$$

(2.3)

where $\gamma$ is the anisotropy parameter. Throughout this paper, unless explicitly stated otherwise, we shall confine our attention to $\gamma$ real and in the interval $(0, \pi/2)$ (by periodicity, the complete relevant interval is $0 \leq \gamma \leq \pi$).

The inhomogeneous and twisted six–vertex transfer matrix associated to this $N$–sites line reads

$$t(\lambda, \theta_1, \theta_2, \ldots, \theta_N, \omega) = e^{i\omega \sigma_0^z} \text{tr}_0 [L_1 L_2 \ldots L_N]$$

(2.4)

The twist angle $\omega$ defines the relation between spin operators after a translation by $N$ sites:

$$\sigma_{n+N}^x \pm i \sigma_{n+N}^y = e^{\pm i \omega} [\sigma_n^x \pm i \sigma_n^y] , \quad \sigma_{n+N}^z = \sigma_n^z$$

(2.5)

Notice also that $R_{k\, n}$ is $U(1)$–invariant,

$$e^{i \phi (\sigma_k^x + \sigma_n^x)} R_{k\, n}(\theta) = R_{k\, n}(\theta) e^{i \phi (\sigma_k^x + \sigma_n^x)}$$

(2.6)

so that the transfer matrix commutes with the $z$–projection of the total spin $S^z = \frac{1}{2} \sum_1^N \sigma_n^z$.

The diagonalization of $t(\lambda, \theta_1, \theta_2, \ldots, \theta_N, \omega)$ is achieved via Quantum Inverse Scattering Method (also called Algebraic Bethe Ansatz) (ref. [14,15]): the eigenstates with $S^z = N/2 - M$, $M = 0, 1, 2, \ldots, \lfloor N/2 \rfloor$, are labelled by the $M$ distinct
roots \(\{\lambda_1, \lambda_2, \ldots, \lambda_M\}\) of the Bethe Ansatz Equations (BAE)

\[
\prod_{n=1}^{N} \frac{\sinh(\lambda_j + i\theta_n + i\gamma/2)}{\sinh(\lambda_j + i\theta_n - i\gamma/2)} = -e^{-2\omega} \prod_{r=1}^{M} \frac{\sinh(\lambda_j - \lambda_r + i\gamma)}{\sinh(\lambda_j - \lambda_r - i\gamma)}
\] (2.7)

while the corresponding eigenvalues \(\tau(\lambda, \theta_1, \theta_2, \ldots, \theta_N, \omega)\) read

\[
\tau(\lambda, \theta_1, \theta_2, \ldots, \theta_N, \omega) = e^{i\omega} \prod_{n=1}^{N} a(\lambda + \theta_n) \prod_{r=1}^{M} \frac{\sinh(i\gamma/2 - \lambda_j + i\lambda)}{\sinh(i\gamma/2 + \lambda_j - i\lambda)} \\
+ e^{-i\omega} \prod_{n=1}^{N} b(\lambda + \theta_n) \prod_{r=1}^{M} \frac{\sinh(3i\gamma/2 + \lambda_j - i\lambda)}{\sinh(-i\gamma/2 - \lambda_j + i\lambda)}
\] (2.8)

We shall primarily be interested in one particular solution with \(S^z = 0\) of the BAE, the so-called antiferromagnetic ground state (a.g.s.) solution, which will be characterized shortly. Since the request \(S^z = 0\) forces \(N\) to be even, we replace \(N\) by \(2N\) throughout. Moreover, for our applications, we shall need to consider only a special type of inhomogeneity structure, the alternating one

\[
\theta_n = (-1)^{n-1} \theta/2
\] (2.9)

where \(\theta\) will either be real, with \(0 \leq \theta < \gamma/2\), or purely imaginary. Correspondingly, the object of our interest is the (twisted) alternating transfer matrix

\[
t_{2N}(\lambda, \theta, \omega) = t(\lambda, \theta_1 = \theta/2, \theta_2 = -\theta/2, \ldots, \theta_{2N} = -\theta/2, \omega)
\] (2.10)

As is well known, this object is a **diagonal to diagonal** transfer matrix with \(N\) links in the horizontal direction [20]. Suppose now that \(N\) distinct roots \(\{\lambda_1, \lambda_2, \ldots, \lambda_N\}\) are given; with them we construct the so-called *counting function*
\[ Z_N(\lambda) = N \left[ \phi(\lambda + i\theta/2, \gamma/2) + \phi(\lambda - i\theta/2, \gamma/2) \right] - \sum_{k=1}^{N} \phi(\lambda - \lambda_k, \gamma) - 2\omega \]  

(2.11)

where

\[ \phi(\lambda, x) \equiv i \log \frac{\sinh(ix + \lambda)}{\sinh(ix - \lambda)} , \quad (\phi(0, x) = 0) \]  

(2.12)

has the cut structure chosen so that it is analytic in the strip \(|\text{Im}\lambda| \leq x\). The a.g.s. solution is now specified by the property

\[ Z_N(\lambda_j) = (-N + 2j - 1)\pi , \quad j = 1, 2, \ldots, N \]  

(2.13)

and enjoys the symmetry (compare with the BAE, eq.(2.7))

\[ \lambda_j(\omega) = \overline{\lambda_j(\overline{\omega})} = -\lambda_{N-j+1}(-\omega) \]  

(2.14)

The existence of this solution is most simply established by numerical calculations, for \(N\) up to the thousands. The uniqueness will become apparent in the sequel. For \(\omega\) real, all \(\lambda_j\) are also real and the a.g.s. can be characterized as the unique solution of the BAE with \(N\) real roots. If \(\text{Im}\omega > 0\) \((< 0)\) then \(\text{Im}\lambda_j > 0\) \((< 0)\) and the a.g.s. goes continuously, by construction, into the real solution as \(\text{Im}\omega \to 0\).

Notice that the symmetry (2.14) implies for the counting function itself

\[ Z_N(\lambda, \omega) = \overline{Z_N(\overline{\lambda}, \overline{\omega})} = -Z_N(-\lambda, -\omega) \]  

(2.15)

which will play an important rôle in the applications. Notice also that all these last statements really hold true for \(\gamma\) in the extended interval \((0, \pi)\).

To gain some understanding on the distribution of roots in this a.g.s. solution one can try to approximate it in the limit of \(N\) large and fixed \((i.e. N-\text{independent})\) \(\theta\) and \(\omega\). The idea is the traditional one that the BA root distribution becomes
the continuous density $\sigma_c(\lambda)$ (normalized to 1) in the limit $N \to \infty$. Then the sum over the roots in the definition of $Z_N(\lambda)$, eq.(2.11), can be approximated by $N$ times the integral over the density $\sigma_c(\lambda)$. At the same time the property (2.13) implies that

$$\lim_{N \to \infty} \frac{1}{N} \frac{d}{d\lambda} Z_N(\lambda) = 2\pi \sigma_c(\lambda)$$

so that one finds the following linear integral equation for $\sigma_c(\lambda)$

$$2\pi \sigma_c(\lambda) = \phi'(\lambda + i\theta/2, \gamma/2) + \phi'(\lambda - i\theta/2, \gamma/2) - \int_{-\infty}^{+\infty} d\mu \phi'(\lambda - \mu, \gamma) \sigma_c(\mu)$$

(2.17)

where $\phi'(\lambda) = d\phi(\lambda)/d\lambda$. Fourier transforming leads to the explicit solution

$$\sigma_c(\lambda) = z_c'(\lambda), \quad z_c(\lambda) = \frac{1}{\pi} \tan^{-1} \left[ \frac{\sinh(\pi \lambda/\gamma)}{\cos(\pi \theta/[2\gamma])} \right]$$

(2.18)

yielding to leading order $Z_N(\lambda) \simeq 2\pi N z_c(\lambda)$. Notice that no dependence on the twist parameter $\omega$ is left over in this continuum approximation. One can attempt to recover $\omega$ by adding to $2\pi N z_c(\lambda)$ the constant function $f_0(\lambda) = -2\omega$, as suggested by the definition itself of $Z_N(\lambda)$, eq.(2.11). However, this continuum approximation need not control the $O(1)$ terms. In fact, to solve (2.17) one has to invert the convolution $(1 + K) * \sigma_c$, where $K$ is defined as

$$(K * f)(\lambda) \equiv \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \phi'(\lambda - x, \gamma) f(x)$$

(2.19)

This implies that the correct constant to add is $(1 + K)^{-1} * f_0$, that is

$$2\pi N z_c(\lambda) - \frac{\pi \omega}{\pi - \gamma} \simeq Z_N(\lambda)$$

(2.20)

Therefore in the continuum approximation the roots are given by

$$\lambda_j \simeq \frac{\gamma}{\pi} \sinh^{-1} \left[ \frac{\cos \frac{\pi \theta}{\gamma} \cot \frac{\pi}{2N}(2j - 1 + \frac{\omega}{\pi - \gamma})}{\cos(\pi \theta/[2\gamma])} \right]$$

(2.21)

Although imprecise, this approximation is useful, for numerical tests show that the actual values of $\lambda_j$ are very close for arbitrary values of $\theta$ and $\omega$. Thus we see
that the roots lay on a smooth curve entirely contained in some horizontal strip $\mathcal{D}$ of the upper (lower) complex plane for $\Im \omega > 0$ ($\Im \omega < 0$). In turn, $\mathcal{D}$ is included inside the strip $\Im \lambda < \gamma/2$. Moreover, from a host of numerical checks as well as from arguments based on the unicity of the a.g.s. configuration, we expect to find no other point, in the strip $|\Im \lambda| < \gamma/2$, besides the roots $\lambda_j$. We shall now look for the modifications to the linear eq.(2.17), which are necessary to go beyond the continuum approximation.

For sake of simplicity, we shall consider the case $N$ even (after all $N \to \infty$ in physical applications) and $\Im \omega > 0$ (recall the property (2.15)). By construction, the function $\exp[iZ_N(\lambda)]$ is analytic in the strip $(-\gamma + \theta)/2 < \Im \lambda < \gamma/2$ and the sum in the last term of eq.(2.11) can be written as a contour integral

$$\sum_{j=1}^{N} \phi(\lambda - \lambda_j, \gamma) = \oint_{\Gamma} \frac{d\mu}{2\pi i} \phi(\lambda - \mu, \gamma) \frac{d}{d\mu} \log[1 + e^{iZ_N(\mu)}]$$

(2.22)

where the closed contour $\Gamma$ encircles counterclockwise all the roots $\lambda_j$ (recall that, by construction, $\exp[iZ_N(\lambda_j)] = -1$). We can take $\Gamma$ to run straight from left to right at $\Im z = -\eta_-, 0 < \eta_- < (\gamma - \Re \theta)/2$ and come back straight at $\Im z = \eta_+$, $\max(\Im \lambda_j) < \eta_+ < \gamma/2$ (of course the exact values of $\eta_-$ and $\eta_+$ do not matter because of the analyticity and the behaviour at real infinity of $e^{iZ_N}$).

Inserting eq.(2.22) in eq.(2.11), using $1 + e^{iZ_N} = e^{iZ_N} (1 + e^{-iZ_N})$ and integrating by parts yields

$$Z_N(\lambda) + \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \phi'(\lambda - x, \gamma) Z_N(x) = N \left[ \phi(\lambda + i\theta, \gamma/2) + \phi(\lambda - i\theta, \gamma/2) \right] - 2\omega$$

$$- i \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \phi'(\lambda - x - i\eta_+, \gamma) \log \left[ 1 + e^{iZ_N(x+i\eta_+)} \right]$$

$$+ i \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \phi'(\lambda - x + i\eta_-, \gamma) \log \left[ 1 + e^{-iZ_N(x-i\eta_-)} \right]$$

(2.23)
Notice that the function $\exp[-iZ_N(\lambda)]$ is analytic in the strip $(\gamma - \text{Re } \theta)/2 > \text{Im } \lambda > -\gamma/2$, so that $\eta_-$ may be chosen to vary up to $\gamma/2$. We next introduce the convolution operation

$$G = K * (1 + K)^{-1}, \quad G(\lambda) = \int_{-\infty}^{+\infty} \frac{dk}{4\pi \sinh(\pi/2 - \gamma)} \frac{\sinh(\pi/2 - \gamma)k}{\sinh(\pi - \gamma)k/2 \cosh(\gamma k/2)} e^{ik\lambda} \quad (2.24)$$

so that eq.(2.23) can be rewritten

$$Z_N(\lambda) = 2\pi N z_c(\lambda) - \frac{\pi \omega}{\pi - \gamma} - i \int_{-\infty}^{+\infty} dx \, G(\lambda - x - i\eta_+) \log \left[ 1 + e^{iZ_N(x+i\eta_+)} \right]$$

$$+ i \int_{-\infty}^{+\infty} dx \, G(\lambda - x + i\eta_-) \log \left[ 1 + e^{-iZ_N(x-i\eta_-)} \right] \quad (2.25)$$

This expression is exact and exhibits (although only implicitly) all the corrections to the continuum approximation (2.20). If $\text{Im } \omega = 0$, so that the roots $\lambda_j$ are real, the property $Z_N(\bar{\lambda}) = Z_N(\lambda)$ allows to interpret the identity (2.25) as a nonlinear integral equation for the counting function itself. In fact, setting $\eta_+ = \eta_- \equiv \eta$, $Z_N(\lambda + i\eta) = i\epsilon(\lambda)$, $G(\lambda + iy) = G_y(\lambda)$ and

$$g(\lambda) = -2\pi i N z_c(\lambda + i\eta) + \frac{i\omega}{1 - \gamma/\pi} \quad (2.26)$$

we obtain, in compact notation

$$\epsilon = g - G * \log(1 + e^{-\epsilon}) + G_{2\eta} * \log(1 + e^{-\epsilon}) \quad (2.27)$$

We recall that $\eta$ can be chosen freely in the interval $0 < \eta < \gamma/2$, since the original assumption $0 < \gamma \leq \pi/2$ implies that $G(\lambda)$ is analytic in the strip $|\text{Im } \lambda| < \gamma$. Eq. (2.27) may nonetheless be extended to the other region $\pi/2 < \gamma < \pi$, where $G(\lambda)$
is analytic for $|\text{Im}\,\lambda| < \pi - \gamma$, simply by requiring that $0 < \eta < \frac{1}{2}\min(\gamma, \pi - \gamma)$. For $\eta \to \gamma/2$, eq.(2.27) agrees with the nonlinear integral equation derived in ref. [18] by different methods.

To convert eq.(2.25) into a constructive integral equation also when $\text{Im}\,\omega \neq 0$ we can exploit the symmetry (2.15), namely $Z_N(\lambda, \bar{\omega}) = \overline{Z_N(\lambda, \omega)}$ and combine the two cases $\text{Im}\,\omega > 0$ and $\text{Im}\,\omega < 0$. In fact, the change $\omega \to \bar{\omega}$ simply amounts to change the restrictions on $\eta\pm$, which now read $0 < \eta_+ < \gamma/2$ and $-\max(\text{Im}\,\lambda_j) < \eta_- < \gamma/2$. It is again convenient to fix $\eta_+ = \eta_- = \eta$, although the lower bound for $\eta$ actually depends on the roots and introduces an extra nonlinearity in the problem. The choice $\eta = \gamma/2$, however, (recall that we assumed $0 < \gamma < \pi/2$) would eliminate this subtlety. Then we set

\begin{equation}
Z_N(\lambda + i\eta; \text{Re}\,\omega \pm i\text{Im}\,\omega) = i\epsilon_\pm(\lambda)
\end{equation}

\begin{equation}
2\pi N z_c(\lambda + i\eta) - \frac{\pi}{\pi - \gamma}(\text{Re}\,\omega \pm i\text{Im}\,\omega) = ig_\pm(\lambda)
\end{equation}

and arrive at the system of two coupled nonlinear integral equations

\begin{equation}
\epsilon_\pm = g_\pm - G \ast \log(1 + e^{-\epsilon_\pm}) + G_{2\eta} \ast \log(1 + e^{-\bar{\epsilon}_\pm})
\end{equation}

Let us recall that for $\text{Im}\,\omega = 0$ we can choose $\eta \to \frac{1}{2}\min(\gamma, \pi - \gamma)$, while for nonzero $\text{Im}\,\omega$ and $\gamma < \pi/2$ we can fix $\eta \to \gamma/2$. This choices are assumed from now on, unless explicitly stated otherwise.
3. Finite size corrections

The nonlinear integral equations (2.29) are exact for any $N$. In the large $N$ limit they allow to find the finite size corrections to the eigenvalue (2.8) of the transfer matrix for the a.g.s. configuration. For simplicity we shall consider here the homogeneous case (i.e. with $\theta = 0$) which is relevant in the calculation of the partition function $Z_{6V}$ of the standard six–vertex model on a very long torus. Indeed on a doubly periodic square lattice with $T$ sites in the vertical direction and $2N$ sites in the horizontal direction $Z_{6V}$ reads

$$Z_{6V} = \text{Tr} [t_{2N}(\lambda, 0, \omega)]^T$$  \hspace{1cm} (3.1)

Actually, this periodic lattice has a vertical seam across which the vertical statistical variables $\sigma = \pm$ acquire the factor $e^{ \pm i \omega}$, as evident from eq.(2.5). To simplify the discussion, in this section we shall take $\omega$ to be real.

In the cylinder limit $T \to \infty$ at fixed $N$, the largest eigenvalue $\tau_{\text{max}}$ of the transfer matrix dominates and we have

$$\lim_{T \to \infty} T^{-1} \log Z_{6V} = \log \tau_{\text{max}} = -2N f_{0}(\lambda, \omega) + L_{2N}(\lambda, \omega)$$  \hspace{1cm} (3.2)

where $f_{0}$ is the free energy per site in the infinite $N$ limit, and $L_{2N}$ represents the finite size corrections. With our original choice of a real anisotropy $\gamma$, the six vertex model is in the critical regime, so that we expect $L_{2N}$ to be governed for large $N$ by conformal invariance.

It is well known that the $\tau_{\text{max}}$ exactly corresponds to the a.g.s. configuration analyzed in the previous section, so our problem is now to relate $\tau_{\text{max}}$ to the solution of the integral equation (2.27). To this end we first notice that, provided $0 \leq \text{Re} \lambda < \gamma/2$, the second term in the expression (2.8) for the a.g.s. eigenvalue is exponentially small relative to the first one (for an explicit verification see eq.(6.29)
of ref. [21]), and may therefore be dropped. Thus we can write

$$\log \tau_{\text{max}} = i\omega + 2N \log a(\lambda) + E_N$$  \hspace{1cm} (3.3)

where

$$E_N = -i \sum_{j=1}^{N} \phi(\lambda_j + i\lambda, \gamma/2)$$  \hspace{1cm} (3.4)

The sum over the roots can be transformed into integrals by the same procedure used in sect. 2 for the counting function, with the result (we must choose here \(\eta_- < \gamma/2 - \text{Re} \lambda\))

$$E_N = -i \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \phi'(\mu + i\lambda, \gamma/2) Z_N(\lambda) - \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \phi'(\mu + i\lambda + i\eta_+, \gamma/2) \log \left[1 + e^{iZ_N(\mu+i\eta_+)}\right]$$

$$+ \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \phi'(\mu + i\lambda - i\eta_-, \gamma/2) \log \left[1 + e^{-iZ_N(\mu-i\eta_-)}\right]$$

(3.5)

Eq. (2.25) may now be used to eliminate the term linear in \(Z_N(\lambda)\), yielding (we choose \(\eta_+ = \eta_- \equiv \eta\) as usual)

$$E_N = E_c - \int_{-\infty}^{+\infty} d\mu \left\{ \sigma_c(\mu + i\lambda + i\eta) \log \left[1 + e^{iZ_N(\mu+i\eta)}\right] \right. - \sigma_c(\mu + \lambda - i\eta) \log \left[1 + e^{-iZ_N(\mu-i\eta)}\right] \right\}$$

(3.6)

where \(E_c\) is the continuum approximation

$$E_c = -iN \int_{-\infty}^{+\infty} d\mu \phi(\mu + i\lambda, \gamma/2) \sigma_c(\mu)$$

$$= 2N \int_{-\infty}^{+\infty} \frac{dk}{k} \frac{\sinh[(\pi - \gamma)k] \sinh(2\lambda k)}{\sinh(\pi k) \cosh(k\gamma)} , \ |\lambda| < \gamma/2$$

(3.7)
and (recall eq.(2.18) and that $\theta = 0$ here)

$$
\sigma_c(\mu) = \frac{1}{\gamma \cosh(\pi \mu / \gamma)} = \frac{1}{2\pi} [(1 - G) \ast \phi](\mu)
$$

Comparing eqs. (3.2) and (3.6) we obtain

$$
f_0 = -\frac{1}{2N} E_c + \log a(\lambda)
$$

[notice that $E_c/N$ is $N$-independent] and the finite size corrections expressed in terms of the solution of the nonlinear integral equation (2.27):

$$
L_N(\lambda, \omega) = i\omega - \frac{1}{\gamma} \int_{-\infty}^{+\infty} d\mu \left\{ \frac{\log[1 + e^{-\epsilon(\mu)}]}{\cosh[\pi(\mu + i\lambda + i\eta)/\gamma]} - \frac{\log[1 + e^{-\epsilon(\mu)}]}{\cosh[\pi(\mu + i\lambda - i\eta)/\gamma]} \right\}
$$

(3.9)

In section 7.1 we compute the dominant large $N$ behaviour of $L_N(\lambda, \omega)$ which yields the central charge.

4. Thermodynamics of the XXZ spin chain

The Hamiltonian of the periodic XXZ spin chain with $2L$ sites reads

$$
H_{XXZ} = -J \sum_{n=1}^{2L} \left[ \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y - \cos \gamma (\sigma_n^z \sigma_{n+1}^z + 1) \right]
$$

(4.1)

and is very simply related to the transfer matrix $T_L(\theta)$ of the symmetric six–vertex model on a diagonal lattice with $2L$ links in the ”space” direction. In fact $T_L(\theta)$, with a suitable normalization, can be written

$$
T_L(\theta) = e^{-2L\theta} R_{23} R_{45} \ldots R_{2L} R_{12} R_{34} \ldots R_{2L-1} R_{2L}
$$

(4.2)

with the $R$–matrices $R_{kn}$ given in eq.(2.2), and for small $\theta$ one finds

$$
T_L(\theta) \theta \rightarrow 0 \left[ 1 - \frac{\theta}{2J \sin \gamma} H_{XXZ} + O(\theta^2) \right]
$$

(4.3)

Notice that the higher order local operators which follow from the $\theta$ expansion do not commute among themselves nor with $H_{XXZ}$, since diagonal–to–diagonal
transfer matrices do not commute for different values of θ. This is the reason why it is preferable not to call θ spectral parameter in this context.

We now observe that eq.(4.3) allows to write an euclidean path–integral relation between the classical partition function of the six–vertex model and the quantum partition function of the XXZ model [10,11, 12]. Let us consider the XXZ free energy per site, \( f(\beta, h) \), in the presence of an external constant field \( 2h \) coupled to the conserved \( z \)-projection of the total spin \( S_z = \frac{1}{2} \sum_n \sigma_n^z \),

\[
f(\beta, h) = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{2L} \log \left[ \text{Tr} \left\{ e^{-\beta(H_{XXZ} - 2hS_z)} \right\} \right]
\]

(4.4)

Setting \( \tilde{\beta} = \beta J \sin \gamma \), from eq.(4.3) we then read

\[
e^{-\beta H_{XXZ}} = \lim_{N \to \infty} \left[ T_L(2\tilde{\beta}/N) \right]^N
\]

Hence we can write

\[
f(\beta, h) = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{2L} \lim_{N \to \infty} \log Z_{LN}(2\tilde{\beta}/N, h)
\]

(4.5)

where

\[
Z_{LN}(\theta, h) \equiv \text{Tr} \left\{ e^{2\tilde{\beta}S_z} [T_L(\theta)]^N \right\}
\]

(4.6)

is the six–vertex partition function on a square periodic diagonal lattice with \( 2NL \) sites, with the insertion of an horizontal seam along which each link variable \( \sigma = \pm \) is multiplied by \( e^{\pm \beta h} \). Notice that any two neighboring elements of this seam can be freely moved through a common vertex thanks to the \( U(1) \) invariance of the \( R \)-matrices, eq.(2.6). The two limits in eq.(4.6) cannot be interchanged since the degeneracy of \( T_L(0) = 1 \), that is \( 2^{2L} \), is strongly \( L \)-dependent. However, the numerical value of \( Z_{LN}(\theta, h) \) does not change under a rotation by \( \pi/2 \) of the entire lattice, nor under the flipping from \( \pm \) to \( \mp \) of the value of all link variables on, say, the left–oriented diagonals. On the other hand these two operations combined are
equivalent to the substitution $\theta \rightarrow \gamma - \theta$ in each local $R$–matrix $R_{jk}$ (this is called “crossing symmetry”, and $\gamma$ is then identified as crossing point, see e.g. ref.[9]), so that we can write

$$Z_{LN}(\theta, h) = Z_{NL}(\gamma - \theta, h) = \text{Tr} [T_N(\gamma - \theta)^L_h] \quad (4.7)$$

where $T_N(\theta)_h$ is the twisted diagonal–to–diagonal transfer matrix

$$T_N(\theta)_h = c^{-2N}e^{\beta h \sigma^z_1 R_{23} R_{45} \ldots R_{2N-1} e^{2\beta h \sigma^z_1}} R_{12} R_{34} \ldots R_{2N-1} \quad (4.8)$$

Combining eqs. (4.5) and (4.7), we now obtain

$$f(\beta, h) = -\frac{1}{\beta} \lim_{L \to \infty} \frac{1}{2L} \lim_{N \to \infty} \log Z_{NL}(\gamma - 2\tilde{\beta}/N)_h \quad (4.9)$$

It is well known [20] that $T_N(\theta)$ is simply related to the row–to–row (untwisted) alternating transfer matrix $t(\lambda, \theta, \omega = 0)$ introduced in section 2

$$T_N(\theta) = [a(\theta)a(-\theta)/c^2]^N t(\theta/2, \theta/2, 0) t(-\theta/2, \theta/2, 0)^{-1} \quad (4.10)$$

The presence of the external field can be related to twisted boundary conditions [eq.(2.5)] in absence of magnetic field. We consider a purely imaginary twist $\omega = -i\beta h$, and find quite simply

$$T_N(\theta)_h = [a(\theta)a(-\theta)/c^2]^N t(\theta/2, \theta/2, -i\beta h) t(-\theta/2, \theta/2, -i\beta h)^{-1} \quad (4.11)$$

Therefore the eigenvalues of $T_N(\theta)_h$ can be read off the general result (2.8) of the Algebraic BA, with the substitutions

$$\lambda = \theta/2, \quad \theta_n = (-1)^{n-1}\theta/2, \quad \omega = -i\beta h \quad (4.12)$$

Notice that the second term in the r.h.s. of (2.8) then vanishes, so that we have

$$\text{eigenvalue}\{T_N(\theta)_h\} = a(\theta)^N \prod_{r=1}^M \frac{\sinh(\lambda_j + i\theta/2 + i\gamma/2) \sinh(\lambda_j - i\theta/2 - i\gamma/2)}{\sinh(\lambda_j + i\theta/2 - i\gamma/2) \sinh(\lambda_j - i\theta/2 + i\gamma/2)} \quad (4.13)$$

For $0 \leq \theta \leq \gamma$, the largest eigenvalue $\Lambda^{\text{max}}_N(\theta)$ of $T_N(\theta)_h$ is nondegenerate and correspond to the solution of the BAE with $M = N$ roots which we have called
a.g.s. configuration in sect. 2. We see therefore that, for $h > 0$ ($h < 0$) the roots lay in the lower (upper) complex half-plane and are real for $h = 0$. Moreover, by eq.(2.14), $\lambda_j(h) = \bar{\lambda}_j(-h)$ and $\lambda_j = -\bar{\lambda}_{N-j+1}$. Thanks to the latter symmetry the corresponding eigenvalue $\Lambda_N^{\text{max}}(\theta)_h$ remains real, while the former will ensure that $f(\beta, h) = f(\beta, -h)$, as required by the spin inversion symmetry of the XXZ chain. Finally, the nondegeneracy of $\Lambda_N^{\text{max}}(\theta)_h$ implies that the two limits in eq.(4.9) commute [11], so that the limit $L \to \infty$ selects $\Lambda_N^{\text{max}}(\gamma - 2\tilde{\beta}/N)_h$ and one finds [6]

$$f(\beta, h) = -\frac{1}{2\beta} \lim_{N \to \infty} \log \Lambda_N^{\text{max}}(\gamma - 2\tilde{\beta}/N)_h$$

$$= \frac{1}{2\beta} \lim_{N \to \infty} \left[ E_N - 2N \log \frac{\sin(2\tilde{\beta}/N)}{\sin \gamma} \right]$$

(4.14)

where

$$E_N = \sum_{j=1}^{N} \epsilon_0(\lambda_j)$$

(4.15)

$$\epsilon_0(\lambda) = i\phi(\lambda_j + i\gamma/2, \gamma/2 - \tilde{\beta}/N) - i\phi(\lambda_j - i\gamma/2, \gamma/2 - \tilde{\beta}/N)$$

By the same trick used in sec.2 and 3, we transform the sum over the roots into integrals and then use eq.(2.25) to eliminate the term linear in the counting function. We obtain in this way

$$E_N = E_c - i \int_{-\infty}^{+\infty} d\lambda \psi_N(\lambda + i\eta_+) \log \left[ 1 + e^{iZ_N(\lambda+i\eta_+)} \right]$$

(4.16)

$$+ \int_{-\infty}^{+\infty} d\lambda \psi_N(\lambda - i\eta_-) \log \left[ 1 + e^{-iZ_N(\lambda-i\eta_-)} \right]$$

where $E_c$ is the continuum approximation

$$E_c = N \int_{-\infty}^{+\infty} d\lambda \epsilon_0(\lambda)\sigma_c(\lambda) = -2N \int_{-\infty}^{+\infty} \frac{dk}{k} \frac{\sinh(\pi - \gamma)k \sinh(\gamma - 2\tilde{\beta}/N)k}{\sinh \pi k \cosh \gamma k}$$

(4.17)
\[ \psi_N(\lambda) = \frac{1}{2 \pi} [(1 - G) \ast \epsilon'_0](\lambda) = \frac{2}{\gamma} \frac{\sinh(\pi \lambda / \gamma) \cos(\pi \tilde{\beta} / \gamma N)}{\cosh(2 \pi \lambda / \gamma) - \cos(2 \pi \tilde{\beta} / \gamma N)} \]

Let us recall that the counting function \( Z_N(\lambda) \) fulfills now eq.(2.25) where, in the source term \( 2 \pi N z_c(\lambda) \), we must set \( \theta = \gamma / 2 - \tilde{\beta} / N \). We then see that the root density of the continuum approximation

\[ \sigma_c(\lambda) = \frac{2}{\gamma} \frac{\cosh(\pi \lambda / \gamma) \sin(\pi \tilde{\beta} / \gamma N)}{\cosh(2 \pi \lambda / \gamma) - \cos(2 \pi \tilde{\beta} / \gamma N)} \quad (4.18) \]

is strongly peaked at \( \lambda = 0 \) for large \( N \), reflecting the singularity which develops at the origin in the \( N \to \infty \) limit of the source term in eqs.(2.17) when \( \theta = \gamma / 2 - \tilde{\beta} / N \). The density picture correctly describes the BA roots wherever \( \sigma_c(\lambda) \) is of order 1.

From eq.(4.18) we then find as validity interval

\[ |\lambda| \leq O(\sqrt{\beta / N}) \quad (4.19) \]

which shrinks to zero when \( N \to \infty \). Roots \( |\lambda_j| > O(\sqrt{\beta / N}) \) have a spacing of order larger than \( O(1/N) \) and cannot be described by densities. In particular, the roots with largest magnitudes have finite \( N \to \infty \) limits spread by \( O(1) \) intervals (we checked this fact numerically too). Therefore, contrary to the usual situation [15], we must go beyond the density description to obtain a bulk quantity like the free energy per site.

We now consider the limit \( N \to \infty \); \( N \) explicitly enters eqs.(4.16) and (2.25) only through the functions \( \psi_N(\lambda) \) and \( z_c(\lambda) \). Thus, inserting eqs. (4.16) and (4.17) in eq.(4.14) yields

\[ f(\beta, h) = E_{XXZ} + \beta^{-1} L(\beta, h) \quad (4.20) \]

where \( E_{XXZ} \) is the ground–state energy at zero external field \( h \) of (4.1), namely

\[ E_{XXZ} = 2J \left[ \cos \gamma - \sin \gamma \int_{-\infty}^{\infty} dk \frac{\sinh(\pi - \gamma)k}{\sin \pi k \cosh \gamma k} \right] \quad (4.21) \]

while (the spin reflection symmetry allows to take the half sum of the two cases
\( h \geq 0 \) and \( h \leq 0 \), to ensure explicit reality of the final expression

\[
L(\beta, h) = \text{Im} \int_{-\infty}^{+\infty} d\lambda \frac{\log[1 + e^{-\epsilon_+(\lambda)}][1 + e^{-\epsilon_-(\lambda)}]}{2 \gamma \sinh[\pi(\lambda + i\eta)/\gamma]}
\]

(4.22)

The functions \( \epsilon_\pm(\lambda) \) satisfy the limit \( N \to \infty \) of eq.(2.29), which stays formally unchanged

\[
\epsilon_\pm = g_\pm - G \ast \log(1 + e^{-\epsilon_\pm}) + G_{2\eta} \ast \log(1 + e^{-\epsilon_\mp})
\]

(4.23)

but where now

\[
g_\pm(\lambda) = \frac{2\pi i \tilde{\beta}}{\gamma \sinh[\pi \lambda/\gamma]} \pm \frac{\pi \beta h}{\pi - \gamma}
\]

(4.24)

We may also write \( \epsilon_\pm(\lambda) = -iZ(\lambda + i\eta; \mp h) \) (compare with eq.(2.28)), with the analytic function \( Z(\lambda) \) simply related to the original counting function by

\[
Z(\lambda) = \lim_{N \to \infty} [Z_N(\lambda) - \pi N \text{sign} (\lambda)] \quad (\lambda \text{ real})
\]

(4.25)

The problem of calculating the free energy of the (infinite) XXZ spin chain has thus been reduced to the problem of solving the two coupled nonlinear integral equations (4.23) and performing the integral (4.22). Notice that for vanishing as well as for purely imaginary external field, eq.(2.28) entails \( \epsilon_+ = \epsilon_- \), so that eq.(4.23) reduces to a single equation for \( \epsilon(\lambda) = -iZ(\lambda + i\eta) \).

Notice that for \( \lambda \to \pm \infty \), eqs.(2.11) and (4.25) yield

\[
Z(\pm \infty) = -2\omega
\]

One could take the limit \( N \to \infty \) directly in the definition itself of the counting
function, eq.(2.11), which now reads

$$Z_N(\lambda) = S_N(\lambda) - \sum_{k=1}^{N} \phi(\lambda - \lambda_k, \gamma) + 2i\beta h$$  \hspace{1cm} (4.26)$$

with the source term

$$S_N(\lambda) = N \left[ \phi(\lambda + i\gamma/2 - i\tilde{\beta}/N, \gamma/2) + \phi(\lambda - i\gamma/2 + i\tilde{\beta}/N, \gamma/2) \right]$$  \hspace{1cm} (4.27)$$

Since the roots $\lambda_j$ all have finite $N \to \infty$ limits and

$$\lim_{N \to \infty} [S_N(\lambda) - \pi N \text{sign}(\lambda) - N\phi(\lambda, \gamma)] = 2\tilde{\beta}q(\lambda)$$  \hspace{1cm} (4.28)$$

with

$$q(\lambda) = \frac{\sinh 2\lambda}{\cosh 2\lambda - \cos 2\gamma} - \coth \lambda$$  \hspace{1cm} (4.29)$$

we obtain for the limit (4.25) of the counting function

$$Z(\lambda) = 2\tilde{\beta}q(\lambda) - \sum_{j \in \mathbb{Z}} \left[ \phi(\lambda - \xi_j, \gamma) - \phi(\lambda, \gamma) \right] + 2i\beta h$$  \hspace{1cm} (4.30)$$

where the real numbers $\xi_j$, which satisfy $Z(\xi_j) = (2j - 1)\pi$, $j \in \mathbb{Z}$, are the $N \to \infty$ limit of the original BA roots $\lambda_1, \lambda_2, \ldots, \lambda_N$. Eq.(4.30) shows that $Z(\lambda)$ has periodicity $i\pi$ and has, as unique singularity in the strip $|\text{Im}\lambda| < \gamma/2$, a simple pole at the origin with residue $-2\tilde{\beta}$. This fact will play an important rôle in the high temperature expansion of sec. 6.1.

The new BA equations $Z(\xi_j) = (2j - 1)\pi$ embody all the information about the XXZ thermodynamics. They form an infinite set of algebraic equations equivalent to our non-linear integral equations (4.23). We want to stress that these roots are discrete although we have already set $N = \infty$. The root spacing is actually of order $T$ and in the $T \to 0$ limit we recover the continuous distribution of roots associated with the antiferromagnetic ground-state [15]. The roots $\xi_j$ have an accumulation point at $\xi = 0$. That is, $\xi_j = -\tilde{\beta}/[\pi(j - 1/2)]$ for $j \to \infty$. For low temperatures, the largest root, $\xi_1$, is of order $(\gamma/\pi) \ln \tilde{\beta}$. 

26
We can verify that eq.(4.30) is indeed equivalent to eq.(4.23) by means of the usual contour integrations. Taking into account that $Z(\lambda)$ has a simple pole at the origin and using the identity
\[
q(\lambda) = -i\phi'(\lambda - i\gamma/2, \gamma/2) + \frac{1}{2}i\phi'(\lambda, \gamma)
\] (4.31)
one obtains from eq.(4.30), for $\gamma/2 > \text{Im}\lambda > 0$ and $h \geq 0$,
\[
Z(\lambda) = 2i\tilde{\beta} \phi'(\lambda - i\gamma/2, \gamma/2) + 2i\beta h - \oint_{\Gamma} \frac{d\mu}{2\pi i} \phi'(\lambda - \mu, \gamma) \log[1 + e^{-iZ(\mu)}] \] (4.32)
where the contour $\Gamma$ encircles all the roots $\xi_j$ as well as the origin. By deforming $\Gamma$ into straight lines as in sec. 2, applying the convolution $(1 + K)^{-1}$ and observing that
\[
\int_{-\infty}^{+\infty} d\mu (1 + K)^{-1}(\lambda - \mu) \phi'(\mu - i\gamma/2 + i0, \gamma/2) = \frac{i\pi}{\gamma \sinh[\pi(\lambda + i0)/\gamma]}
\]
\[
\int_{-\infty}^{+\infty} d\mu (1 + K)^{-1}(\lambda - \mu) = \frac{\pi}{2(\pi - \gamma)}
\] (4.33)
one recovers the $N \to \infty$ limit of equation (2.25) (with $\omega = -i\beta h$ and $\theta = \gamma/2 - \tilde{\beta}/N$), from which eq.(4.23) has been derived.

As we shall see in sec.6.1, the alternative form (4.32) of the integral equation for $Z(\lambda)$ is most suitable to study the high temperature regime, since it involves a complete contour integral. It is then useful to obtain an analogous formula for the free energy. From eq.(4.22) we obtain (for $h \geq 0$)
\[
L(\beta, h) = \frac{1}{2} \int_{-\infty}^{+\infty} d\lambda \frac{Z(\lambda + i0)}{\gamma \sinh[\pi(\lambda + i0)/\gamma]} - \frac{1}{2i} \oint_{\Gamma} \frac{d\lambda}{\gamma \sinh[\pi\lambda/\gamma]} \log[1 + e^{-iZ(\lambda)}] \] (4.34)
Substituting eq.(4.32) for the term linear in $Z(\lambda)$ and using (4.33), leads to

$$L(\beta, h) = \int_{-\infty}^{+\infty} d\lambda \frac{i\beta \phi'(\lambda - i\gamma/2 + i0, \gamma/2) + i\beta h}{\gamma \sinh[\pi(\lambda + i0)/\gamma]} + \oint_{\Gamma} \frac{d\lambda}{2\pi} \phi'(\lambda - i\gamma, \gamma/2) \log[1 + e^{-iZ(\mu)}]$$

(4.35)

By Fourier transformation, the first integral can be rewritten as

$$\int_{-\infty}^{+\infty} d\lambda \frac{i\beta \phi'(\lambda - i\gamma/2 + i0, \gamma/2) + i\beta h}{\gamma \sinh[\pi(\lambda + i0)/\gamma]} = \beta (2J \cos \gamma - E_{XXZ} + h)$$

(4.36)

so that we finally obtain using eq.(4.20), (4.35) and (4.36)

$$f(\beta, h) = h + 2J \cos \gamma + \frac{\sin \gamma}{\beta} \int_{\Gamma} \frac{d\lambda}{2\pi i} \frac{\log[1 + e^{-iZ(\lambda)}]}{\sinh \lambda \sinh(\lambda - i\gamma)}$$

(4.37)

It is clear that, to go from eq.(4.22) to eq.(4.35), we have just performed in reverse order the same operations that led to (4.22) from the definitions (4.14) and (4.15). However, in the middle, we have taken the limit $N \to \infty$, which is difficult to take directly on eqs.(4.14) and (4.15). The final result, eq.(4.37), can now be transformed, via contour integration, into an infinite sum over the roots $\xi_j$ with the result

$$f(\beta) = h + 2J \cos \gamma + \frac{1}{2\beta} \log \prod_k \left[ 1 + 2 \cos \gamma \left( 1 + 2 \cos \gamma \right) \right]$$

$$\left. \frac{2 \sin \gamma \sin(s_k - 2\gamma)}{\cosh(r_k) - \cos(s_k - 3\gamma)} \right]$$

where $r_k = 2 \text{Re}(\xi_k)$ and $s_k = 2 \text{Im}(\xi_k)$. 

28
5. Ground state scaling function of the massive Thirring (sine–Gordon) model

Let us now reinterpret the diagonal-to-diagonal lattice associated to the transfer matrix $t_{2N}(\lambda, \theta, \omega)$ as the unit–time evolution operator for Minkowski space-time discretized in light-cone coordinates. That is, the axis correspond to $x \pm t$, $(x$ and $t$ being the usual space and time variables). For this purpose we choose $\theta = -2i\Theta$, with $\Theta$ real. Then, the local $R$–matrices (2.2) are rendered unitary upon multiplication by $a(-2i\Theta)^{-1}$.

In this light-cone approach, we start from the discretized Minkowski 2D space-time formed by this regular diagonal lattice of right–oriented and left–oriented straight lines. These represent true world–lines of “bare” objects (pseudo–particles) which are thus naturally divided in left– and right–movers. The right–movers have all the same positive rapidity $\Theta$, while the left–movers have rapidity $-\Theta$. One can regard $\Theta$ as a cut–off rapidity, which will be appropriately taken to infinity in the continuum limit. Furthermore, we shall denote by $V$ the Hilbert space of states of a pseudo–particle (we restrict here to the case in which $V$ is the same for both left– and right–movers and has finite dimension $n = 2$, although more general situations can be considered [20,15] ).

The dynamics of the model is fixed by the microscopic transition amplitudes attached to each intersection of a left– and a right–mover, that is to each vertex of the lattice. This amplitudes can be collected into linear operators $R_{ij}$, the local $R$–matrices, acting non–trivially only on the space $V_i \otimes V_j$ of $i$th and $j$th pseudo–particles. $R_{ij}$ thus represent the relativistic scatterings of left–movers on right–movers and depend on the rapidity difference $\Theta - (-\Theta) = 2\Theta$, which is constant throughout the lattice. Moreover, by space–time translation invariance any other parametric dependence of $R_{ij}$ must be the same for all vertices. We see therefore that attached to each vertex there is a matrix $R(2\Theta)_{ab}^{cd}$, where $a, b, c, d$ are labels for the states of the pseudo–particles on the four links stemming out of the vertex, and take therefore $n$ distinct values. This is the general framework of a vertex
model. These local $R$–matrices (2.2) were rendered unitary upon multiplication by $a(-2i\Theta)^{-1}$.

The difference with the standard statistical interpretation is that the Boltzmann weights are in general complex, since we should require the unitarity of the matrix $R$. In any case, the integrability of the model is guaranteed whenever $R(\lambda)^{cd}_{ab}$ satisfy the Yang–Baxter equations.

For periodic boundary conditions, the one–step light–cone evolution operators $U_L(\Theta)$ and $U_R(\Theta)$, which act on the ”bare” space of states $\mathcal{H}_N = (\otimes V)^{2N}$, ($N$ is the number of sites on a row of the lattice, that is the number of diagonal lines), are built from the local $R$–matrices $R_{ij}$ as [13]

$$U_R(\Theta) = W(\Theta)V, \quad U_L(\Theta) = W(\Theta)V^{-1}, \quad W(\Theta) = R_{12}R_{34} \ldots R_{2N-12N}$$

(5.1)

where $V$ is the one-step space translation to the right. $U_R$ ($U_L$) evolves states by one step in right (left) light–cone direction. $U_R$ and $U_L$ commute and their product $U = U_RU_L$ is the unit time evolution operator.

If $\delta$ stands for the lattice spacing, the lattice hamiltonian $H$ and total momentum $P$ are naturally defined through

$$U = e^{-i\delta H}, \quad U_RU_L^{-1} = e^{i\delta P}$$

(5.2)

These operators directly express in terms of the alternating transfer matrix $t_{2N}(\lambda, -i\Theta, 0)$ as follows

$$U_R(\Theta) = a(-2i\Theta)^{-N} t_{2N}(-i\Theta, -i\Theta, 0), \quad U_L(\Theta) = a(2i\Theta)^N t_{2N}(+i\Theta, -i\Theta, 0)^{-1}$$

Therefore, the unit time evolution operator results

$$U = e^{-i\delta H} = \left[ \frac{a(2i\Theta)}{a(-2i\Theta)} \right]^N t(-i\Theta, -i\Theta, 0) t(i\Theta, -i\Theta, 0)^{-1}$$

(5.3)

Notice that $H$ and $P$ are (non–local) lattice operators. For simplicity we set the twist $\omega = 0$ in this section. Whenever the $R$–matrix, that is the building block of
the whole construction, satisfies the Yang–Baxter equations, this UV–regularized QFT is integrable, and will therefore be so also in the continuum limit. The integrable QFT corresponding to the six–vertex $R$–matrix given above was identified in ref. [13] by Jordan–Wigner transforming the local Pauli matrices into discretized fermionic fields (on the infinite lattice), and then showing that these fields satisfy a lattice version of the field equation characteristic of the massive Thirring model. The bare continuum fields are recovered when $\delta \to 0$ and simultaneously $\Theta \to \infty$ with the bare mass scale $m_0 \sim \delta^{-1} \exp(-2\Theta)$ held fixed. The renormalized, physical continuum limit follows instead by keeping fixed the true mass scale $m \sim \delta^{-1} \exp(-\pi\Theta/\gamma)$ ($\Theta$ plays essentially the rôle of cutoff in rapidity space). All the on–shell calculations can be performed exactly within the Algebraic Bethe Ansatz, leading to the complete spectrum and exact $S$–matrix of the mT model. Let us also recall that the latter is equivalent à la Coleman–Mandelstam to the sine–Gordon model. We shall now show what modifications to the general formulae derived in sect.4 are necessary in order to discuss the thermodynamics of the sG–mT model.

By the euclidean symmetry of the corresponding functional integral, we know that the free energy density of a 2D QFT at temperature $T$ (on an infinite line) is identical to $T$ times the ground state energy of the same QFT on a ring of circumference $\beta = 1/T$. Of course, such a quantity is UV divergent and must be properly subtracted. Requiring that the free energy $f(\beta)$ vanishes at zero temperature is equivalent to consider only the Casimir energy, that is the difference between the ground state energy on the circle and that of the infinite line. In the context of Perturbed Conformal Field Theory, this Casimir energy is known as ground–state scaling function $E(\beta)$, so that we have $f(\beta) = E(\beta)$. When $T \to \infty$ we expect that UV fixed point characterizing the QFT manifests itself in some universal fashion. In detail, we expect that $E(\beta)$ tends to the Casimir energy of the Conformal Field Theory corresponding to that UV fixed point. From this point of view, the situation of a massive QFT such as the sG–mT model is opposite to that of a gapless spin chain such as the XXZ model at $0 < \gamma < \pi$. In the former the mass
is generated by some IR relevant perturbation of the CFT and becomes irrelevant in the UV limit $T \to \infty$. In the latter the lattice cutoff corresponds to the presence of IR irrelevant perturbations of the CFT those effects become negligible in the $T \to 0$ limit. Thus the basic properties (central charge and scaling dimensions) of the same CFT (the free massless boson) will appear at low temperatures in the gapless XXZ spin chain and at high temperatures in the massive sG–mT model. This dichotomy will be quite clear in the unified description that we are here proposing, since, apart from a different source term, the fundamental integral equation is essentially the same in both cases.

As stated above, the diagonal–to–diagonal transfer matrix $T_N(\theta)$, eq.(4.2), once unitarized into $U$ according to eq.(5.1), describes the real time evolution of a discretized mT model. The continuum relativistic QFT, on the infinite Minkowski plane, is obtained taking first the IR limit $N \to \infty$ at fixed lattice spacing $\delta$ and then the continuum limit $\delta \to 0$ near the critical point $\Theta = \infty$ (with the physical mass scale $m \sim \delta^{-1}\exp(-\pi\Theta/\gamma)$ held fixed). On the other hand, by taking the continuum limit at fixed $N\delta \equiv \beta$, we get instead the same QFT on a ring of length $\beta$. In particular, if we consider the ground state on the lattice, this limit will yield the (UV divergent) bulk ground state energy of order $\beta$ plus the ground state scaling function $E(\beta)$.

Using now eq.(3.4) and (5.3), the lowest value of the energy is then attained in the a.g.s. configuration and take the form

$$E_N = \sum_{j=1}^{N} \left[ \phi(\lambda_j - \Theta, \gamma/2) - \phi(\lambda_j + \Theta, \gamma/2) - 2\pi \right]$$

(5.4)

where the BAE roots $\lambda_j$ are real, since $\omega = 0$. In terms of the counting function $Z_N(\lambda)$, the expression for $E_N$ is identical in structure to eq.(4.16), with only $E_c$
and $\psi_N(\lambda)$ changing to the forms proper of the mT model:

$$E_c = \frac{N^2}{\beta} \left[ -2\pi + \int_{-\infty}^{+\infty} d\lambda \frac{\phi(\lambda + 2\Theta, \gamma/2)}{\gamma \cosh \frac{\pi \lambda}{\gamma}} \right]$$

$$\psi_N(\lambda) = \frac{N}{\gamma\beta} \left[ \text{sech} \frac{\pi}{\gamma}(\lambda - \Theta) - \text{sech} \frac{\pi}{\gamma}(\lambda + \Theta) \right]$$

(5.5)

In the continuum limit $N \to \infty$ the rapidity cutoff $\Theta$ diverges like

$$\Theta \simeq \frac{\gamma}{\pi} \log \frac{4N}{m\beta}$$

(5.6)

where $m$ is the mass of the mT fermion (or sG soliton). Thus

$$E_c = \frac{1}{4}m^2\beta \cot \frac{\pi}{2\gamma} + \text{UV divergent terms}$$

(5.7)

where the first finite term is the scaling bulk energy (the same result was obtained in ref.[19] by completely different means). By 'UV divergent terms' we mean terms that scale as the bare mass ($e^{2\Theta}$) for large $\Theta$. On the other hand

$$E(\beta) \equiv \lim_{N \to \infty} \left[ E_N - E_c \right]$$

$$= -\frac{m}{\gamma} \text{Im} \int_{-\infty}^{+\infty} d\lambda \sinh \left[ \pi(\lambda + i\eta) / \gamma \right] \log \left[ 1 + e^{iZ(\lambda + i\eta)} \right]$$

(5.8)

where $Z(\lambda)$, the $N \to \infty$ limit of the counting function $Z_N(\lambda)$, satisfies

$$Z(\lambda) = m\beta \sinh(\pi\lambda/\gamma) + 2 \text{Im} \int_{-\infty}^{+\infty} d\mu G(\lambda - \mu - i\eta) \log \left[ 1 + e^{iZ(\mu + i\eta)} \right]$$

(5.9)

Let us recall that this result applies to the complete range $0 < \gamma < \pi$, provided we take $0 < \eta < \frac{1}{2}\min(\gamma, \pi - \gamma)$. In the repulsive regime $\gamma < \pi/2$ we can cast
eqs. (5.8) and (5.9) in a very nice form, reminiscent of the standard TBA. Let us choose \( \eta = \gamma/2 \) and set

\[
\begin{align*}
\epsilon_f(\theta) &= -iZ(\frac{2}{\pi} \theta + i \gamma/2), \quad \epsilon_f = \overline{\epsilon_f} \\
L_f &\equiv \log(1 + e^{\epsilon_f}), \quad L_f \equiv \log(1 + e^{\epsilon_f}) = \overline{L_f} \\
G_0(\theta) &= \frac{2}{\pi} G(\frac{2}{\pi} \theta), \quad G_1(\theta) = G_0(\theta + i \pi - i0)
\end{align*}
\]

(5.10)

Then the ground state scaling function reads

\[
E(\beta) = -\frac{m}{2\pi} \int_{-\infty}^{+\infty} d\theta \cosh \theta \left[ L_f(\theta) + L_{\bar{f}}(\theta) \right]
\]

(5.11)

while the nonlinear integral equation becomes

\[
\epsilon_f = g - G_0 \ast L_f + G_1 \ast L_{\bar{f}}
\]

(5.12)

where now \( g(\theta) = m\beta \cosh \theta \) is the source term of the standard TBA, and is directly related to the level density of free relativistic particles. Moreover, from eqs. (2.24) and (5.10),

\[
G_0(\theta) = \int_{-\infty}^{+\infty} \frac{dk}{4\pi} \frac{\sinh \left( \frac{\pi^2}{2\gamma} - 1 \right) k}{\sinh \left( \frac{\pi^2}{2\gamma} - \frac{1}{2} \right) k} \cosh \frac{\pi}{2} k e^{ik\theta}
\]

By construction, the equation for \( \epsilon_f \) is just the complex conjugate of (5.12). Let us observe that \( G_0(\theta) \) is related in the standard way to the fermion–fermion scattering amplitude \( S_{ff}(\theta) \)

\[
G_0(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \log S_{ff}(\theta)
\]

(5.13)

while, up to a sign, \( G_1(\theta) \) coincides with the crossed \( G_0 \), that is \( G_0(i \pi - \theta) \). In other words, we can regard \( \epsilon_f \) and \( \epsilon_{\bar{f}} \) as complex pseudoenergies for the fermions and antifermions, respectively, with eq.(5.12) as TBA equations involving only physical particles. This situation should be compared with the standard TBA
for the mT (or sG) model [23], which for nonrational values of $\gamma/\pi$ involves an infinite number of nonlinear integral equations for the infinitely many different types of “magnons” describing the energy degeneracies of the physical fermions and antifermions. Therefore, our equation effectively provides a resummation of the magnon degrees of freedom.

The results (5.11) and (5.12) may be extended to more general situations in a rather straightforward way. For instance, we could consider a nonzero twist $\omega$ in eqs.(5.3), or introduce a coupling to the conserved $U(1)$ charge $Q$ in the real time evolution defined by $U$. Let us consider first this latter case. The change of the Hamiltonian $H \rightarrow H - hQ$, $(h > 0)$, has a simple consequence: for $h > m$ the ground state contains a definite number of fermions (which have charge $Q = 1$) with the lowest possible kinetic energy. On the light–cone lattice this means that the solution of the BAE must contain holes around $\lambda = 0$, that is roots $\xi_j$ of the counting function $Z_N(\lambda)$, $Z_N(\xi_j) = \cdots$ which do not appear in the sum over BAE roots in the definition of $Z_N(\lambda)$.

As in the XXZ chain, we can find for the sG–mT model an infinite set of algebraic Bethe Ansatz type equations which are equivalent to the non-linear integral equation (5.9). In order to do that, we define

$$\nu(\theta) \equiv Z(\gamma \theta / \pi) - m \beta \sinh(\theta)$$

(5.14)

This function $\nu(\theta)$ vanishes at $\theta = \pm \infty$. We can then write,

$$\text{Im} \log \left[ 1 + e^{iZ(\gamma \theta / \pi + i0)} \right] = \frac{1}{2} \nu(\theta) + \pi D(\theta)$$

(5.15)

where $D(\theta)$ is the (subtracted) discontinuity of $\log \left[ 1 + e^{iZ(\gamma \theta / \pi + i0)} \right]$. That is,

$$D(\theta) = \frac{1}{2\pi i} \log \frac{1 + e^{iZ(\gamma \theta / \pi + i0)}}{1 + e^{iZ(\gamma \theta / \pi - i0)}} + \frac{m \beta}{2\pi} \sinh(\theta)$$

(5.16)
Then,

\[ D'(\theta) = \frac{m\beta}{2\pi} \cosh(\theta) - \sum_k \delta(\theta - \theta_k) \] (5.17)

where the \( \theta_k \) fulfil the equations:

\[ Z(\gamma\theta_k/\pi) = (2k + 1)\pi, \quad k \in \mathbb{Z} \] (5.18)

Since \( Z(-\theta) = -Z(\theta) \), the \( \theta_k \) are symmetrically distributed with respect to the origin:

\[ \theta_k = -\theta_{-k-1} \] (5.19)

Using the asymptotic behaviour \( Z(\gamma\theta/\pi) \to m\beta \sinh(\theta) \) as \( \theta \to \pm\infty \), we find for large roots \( \theta_k \),

\[ \theta_k \stackrel{k \to \pm\infty}{\sim} \pm \log \left[ \frac{2\pi m\beta}{|2k + 1|} \right] \] (5.20)

We see from eq.(5.20) that the term in \( \cosh \theta \) in eq.(5.17) simply ensures the finiteness of \( D'(\theta) \). As eq.(5.20) shows, the roots accumulate at infinity in the mT-sG model, whereas they accumulate at the origin for the XXZ chain at temperature \( T \) (sec.4).

Eq.(5.9) for \( \eta \to 0^+ \) can be now written as

\[ \nu(\theta) = \int_{-\infty}^{+\infty} d\mu \ G_0(\theta - \mu) [\nu(\mu) + 2\pi D(\mu)] \] (5.21)

Upon Fourier transform and pulling \( \nu(\theta) \) to the l.h.s., this yields

\[ \nu(\theta) = (\frac{\gamma}{\pi})^2 \int_{-\infty}^{+\infty} d\mu \ \phi'((\frac{\gamma}{\pi})^2 [\theta - \mu], \gamma) \ D(\mu) \] (5.22)

or

\[ Z(\gamma\theta/\pi) = m\beta \sinh \theta + \int_{-\infty}^{+\infty} d\mu \ \phi((\frac{\gamma}{\pi})^2 [\theta - \mu], \gamma) \ D'(\mu) \] (5.23)

This is the analog of eq.(4.32). Inserting now eq.(5.17) into eq.(5.23) we find for
\( \gamma > \pi/2 \)

\[
Z(\gamma \theta/\pi) = - \sum_k \phi((\frac{\gamma}{\pi})^2 [\theta - \theta_k], \gamma) \tag{5.24}
\]

where the sum is to be understood in principal part. This shows explicitly that \( Z(\lambda) \) has \( \frac{i \pi^2}{\gamma} \) as period. Moreover, setting \( \theta = \theta_l \) in eq.(5.24) and using eq.(5.18), we find

\[
(2l + 1)\pi = - \sum_k \phi((\frac{\gamma}{\pi})^2 [\theta_l - \theta_k], \gamma) \tag{5.25}
\]

always for \( \gamma > \pi/2 \).

6. The Recursive Regime

Depending on the temperature regime chosen, the integral equations (4.32) and (5.12) can be simply solved by iterating the inhomogeneity. This happens for small \( \beta \) in the XXZ chain [eq.(4.32)] and for large \( \beta \) in field theory [eq.(5.12) for the mTm]. We call ‘recursive regimes’ those where such expansions apply.

6.1. High Temperatures in the XXZ chain

We study here the free energy \( f(\beta) \) of the XXZ chain for high temperatures. When \( \beta \) is small it is convenient to use the form (4.32) plus the uniform expansion

\[
Z(\lambda) = \sum_{k=1}^{\infty} \hat{\beta}^k b_k(\lambda) \tag{6.1}
\]

Since the residue of \( Z(\lambda) \) is linear in \( \hat{\beta} \) only, \( b_1(\lambda) \) has a pole at the origin, we have

\[
b_1(\lambda) \xrightarrow{\lambda \to 0} - \frac{2}{\lambda} + O(\lambda) \tag{6.2}
\]

while all the \( b_k(\lambda) \) for \( k \geq 2 \) are analytic there.
From eq.(6.1) we find up to fourth order in $\tilde{\beta}$

$$\log \left[ 1 + e^{-iZ(\lambda)} \right] = \ln 2 - \frac{i\tilde{\beta}}{2}b_1(\lambda) - \frac{\tilde{\beta}^2}{2}\left[ ib_2(\lambda) + \frac{1}{4}b_1(\lambda)^2 \right] - \frac{\tilde{\beta}^3}{2}\left[ ib_3(\lambda) + \frac{1}{2}b_1(\lambda)b_2(\lambda) \right] - \frac{\tilde{\beta}^4}{2}\left[ ib_4(\lambda) + \frac{1}{2}b_1(\lambda)b_3(\lambda) + \frac{1}{4}b_2(\lambda)^2 + \frac{1}{96}b_1(\lambda)^4 \right] + O(\tilde{\beta}^5)$$

(6.3)

We insert this expansion into eq.(4.32) and notice that only the poles at $\lambda = 0$ in eq.(6.3) contribute to the contour integral over $\Gamma$. Since only $b_1(\lambda)$ is singular, the procedure is perfectly recursive and we find

$$b_1(\lambda) = 2q(\lambda) + 2i\tilde{h}$$
$$b_2(\lambda) = -\frac{1}{2}\phi''(\lambda, \gamma) - i\tilde{h}\phi'(\lambda, \gamma)$$
$$b_3(\lambda) = i\tilde{h}\phi'(\lambda, \gamma) \cot \gamma$$
$$b_4(\lambda) = -i\tilde{h}\left( \frac{1}{6} + \frac{1}{2\sin^2 \gamma} - \frac{\tilde{h}^2}{3} \right)\phi'(\lambda, \gamma) + \frac{1}{3}\left( -\frac{1}{3} + \frac{1}{\sin^2 \gamma} + \frac{3}{2}\tilde{h}^2 \right)\phi''(\lambda, \gamma)$$
$$- \frac{i\tilde{h}}{6}\phi'''(\lambda, \gamma) - \frac{1}{72}\phi''''(\lambda, \gamma)$$

(6.4)

where $\tilde{h} = h/(J \sin \gamma)$. Next we insert the expansion (6.3), with the explicit results listed in (6.4), into the formula (4.37) of the free energy. Again all we need to do is to repeatedly apply the residue theorem, with the final result

$$f(\beta) = -\beta^{-1}\ln 2 + J \cos \gamma - \beta \left[ J^2 \left( 1 + \frac{1}{2}\cos^2 \gamma \right) + \frac{1}{2}\tilde{h}^2 \right] + \beta^2 \cos \gamma (J^3 + \tilde{h}^2)$$
$$- \beta^3 \left[ \left( \frac{1}{4} + \frac{1}{6}\cos^2 \gamma - \frac{1}{12}\cos^4 \gamma \right)J^4 - J^2\tilde{h}^2 - \frac{1}{12}\tilde{h}^4 \right] + O(\beta^4)$$

(6.5)

which indeed agrees with the high $T$ expansion, as can be derived directly from the definitions (4.4) and (4.1).
6.2. Low Temperatures in field theory

For low temperatures (large $m\beta$) the non-linear integral equation (5.12) valid for $\gamma < \pi/2$,

$$\epsilon_f(\lambda) = m\beta \cosh \lambda + \int_{-\infty}^{+\infty} d\mu \left\{ G_1(\lambda - \mu) \log \left[ 1 + e^{-\epsilon_f(\mu)} \right] - G_0(\lambda - \mu) \log \left[ 1 + e^{-\epsilon_f(\mu)} \right] \right\}$$

(6.6)

can be easily iterated using the inhomogeneity $m\beta \cosh \lambda$ as zeroth order approximation. That is,

$$\epsilon_f(\lambda) = f_0(\lambda) + f_1(\lambda) + f_2(\lambda) + \ldots$$

$$f_0(\lambda) = m\beta \cosh \lambda$$

$$f_1(\lambda) = \int_{-\infty}^{+\infty} d\mu \left[ G_1(\lambda - \mu) - G_0(\lambda - \mu) \right] \log \left[ 1 + e^{-m\beta \cosh \mu} \right]$$

(6.7)

$$f_2(\lambda) = \int_{-\infty}^{+\infty} \frac{d\mu}{e^{f_0(\mu)} + 1} \left[ G_0(\lambda - \mu) f_1(\mu) - G_1(\lambda - \mu) \frac{f_1(\mu)}{f_1(\mu)} \right]$$

The energy $E(\beta)$ is expanded in analogous way as a sum:

$$E(\beta) = E_0(\beta) + E_1(\beta) + E_2(\beta) + \ldots$$

(6.8)

where

$$E_0(\beta) = -\frac{m}{\pi} \int_{-\infty}^{+\infty} d\mu \cosh \mu \log \left[ 1 + e^{-m\beta \cosh \mu} \right]$$

(6.9)

$$E_1(\beta) = \frac{m}{2\pi} \int_{-\infty}^{+\infty} d\mu \cosh \mu e^{-f_0(\mu)} \left[ f_1(\mu) + \frac{f_1(\mu)}{f_1(\mu)} \right]$$

Let us now analyze more explicitly $E_0(\beta)$ and $E_1(\beta)$. From eq.(6.9) we find

$$E_0(\beta) = -\frac{2m}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} K_1(nm\beta)$$

(6.10)

This is the typical behaviour of a massive QFT. Notice that eq.(6.10) for $\gamma = \pi/2$
gives the exact energy $E(\beta)$. (In that free case $G_0(\lambda) = 0$).

For large $m\beta$, eq.(6.10) yields

$$E_0(\beta) \sim -\frac{2m}{\pi} K_1(m\beta) + O(e^{-2m\beta}) = -\sqrt{\frac{2m}{\pi\beta}} e^{-m\beta} \left[ 1 + O\left(\frac{1}{\beta}\right) \right] + O(e^{-2m\beta})$$

(6.11)

It is easy to see that $E_n(\beta)$ is of order $e^{-(n+1)m\beta}$ for large $m\beta$.

We get from eqs.(6.7) and (6.9)

$$E_1(\beta) = -\frac{m}{2\pi} \int_{-\infty}^{+\infty} d\lambda \cosh \lambda e^{-m\beta \cosh \lambda} \int_{-\infty}^{+\infty} d\mu \log \left[ 1 + e^{-m\beta \cosh \mu} \right]$$

$$\left[ 2G_0(\lambda - \mu) - G_1(\lambda - \mu) - \tilde{G}_1(\lambda - \mu) \right]$$

(6.12)

The terms inside braces $[..]$ in eq.(6.12) admit for $\gamma < \pi/2$ an integral representation that follows from eq.(2.24) and eq.(5.10)

$$2G_0(\lambda) - G_1(\lambda) - \tilde{G}_1(\lambda) = -\int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{ix\lambda} \left[ 1 - g\left(\frac{\pi x}{2}\right) \right]$$

(6.13)

where

$$g(y) = \frac{\sinh(4 - \frac{\pi}{2}) y + 3 \sinh\left(\frac{\pi}{2} - 2\right) y}{2 \cosh y \sinh\left(\frac{\pi}{2} - 1\right) y}$$

(6.14)

For large $m\beta$ we can approximate the log in eq.(6.12) by the first term in its power expansion. This yields,

$$E_1(\beta) = -\frac{m}{2\pi} \int_{-\infty}^{+\infty} d\lambda \cosh \lambda e^{-m\beta \cosh \lambda} \int_{-\infty}^{+\infty} d\mu e^{-m\beta \cosh \mu} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{ix(\lambda - \mu)} \left[ 1 - g\left(\frac{\pi x}{2}\right) \right]$$

$$+ O(e^{-3m\beta})$$

(6.15)

This integral can be approximated by the saddle point method for large $m\beta$ with
the following result:

\[ E_1(\beta) \to e^{-2m\beta} \left[ \sqrt{\frac{m}{2\pi \beta}} \frac{K_\gamma}{\beta} + O\left(\frac{1}{\beta^{3/2}}\right) \right] + O(e^{-3m\beta}) \quad (6.16) \]

where

\[ K_\gamma = -\int_{-\infty}^{+\infty} \frac{dy}{\pi^2} g(y) = -\int_{-\infty}^{+\infty} \frac{dy}{\pi^2} \frac{\sinh(4 - \frac{\pi}{\gamma})y + 3 \sinh(\frac{\pi}{\gamma} - 2)y}{2 \cosh y \sinh(\frac{\pi}{\gamma} - 1)y} \quad (6.17) \]

This function of \( \gamma \) can be expressed as an infinite sum of \( \psi \) functions. For rational values of \( \gamma/\pi \) it admits simpler expressions. For example,

\[ K_0 = -\frac{1}{\pi^2}(4 \ln 2 - 1) \quad , \quad K_{\pi/3} = -\frac{2}{\pi^2} \]

In addition, we see from eq.(6.17) that \( K_\gamma < 0 \) for \( \gamma < \pi/2 \) and that \( \lim_{\gamma \to \pi/2} K_\gamma = -\infty \). This divergence indicates a change of regime since \( E_1(\beta) \) is in fact identically zero at \( \gamma = \pi/2 \).

In summary, \( E_n(\beta) \) for large \( m\beta \) is at leading order \( O(e^{-(n+1)m\beta}) \) and contains contributions with arbitrary higher powers of \( e^{-m\beta} \).

From \( E_0(\beta) + E_1(\beta) \) we can write \( E(\beta) \) up to contributions \( O(e^{-3m\beta}) \). We find from eqs. (6.16) and (6.11),

\[ E(\beta) \to e^{-2m\beta} \left[ \sqrt{\frac{m}{2\pi \beta}} \frac{1 + \sqrt{2}}{2} + \sqrt{\frac{\pi}{m\beta}} K_\gamma + O\left(\frac{1}{\beta}\right) \right] + O(e^{-3m\beta}) \]

The full term of order \( e^{-2m\beta} \) in \( E_1(\beta) \) possess the following integral representation:

\[ E_1(\beta) \to \int_{-\infty}^{+\infty} dx \ K_{ix}(m\beta) [K_{1+ix}(m\beta) + K_{1-ix}(m\beta)] \frac{\sinh^2(\pi x/2) \sinh \left( \frac{\pi x}{2\gamma} - 1 \right)}{\cosh(\pi x/2) \sinh \left( \frac{\pi x}{\gamma} - 1 \right)} + O(e^{-3m\beta}) \]

where we used eqs.(6.12), (6.13) and (6.14).
7. The conformal regime

The conformal regime is the opposite to the recursive regime. That is, large $\beta$ for magnetic chains and small $\beta$ in field theory.

In the conformal regime the calculations are more subtle but still straightforward as shown below.

The starting point is again the integral equation (2.25). For simplicity we shall consider here only real twists $\omega$, that is purely imaginary external fields $h$ for the XXZ chain. Then, $Z(\lambda) = Z(\bar{\lambda})$ and we can write for all cases

$$-i \log F(\mu) = \varphi(\mu) + 2 \text{Im} \int_{-\infty}^{+\infty} d\mu' G(\mu - \mu' - i\eta) \log \left[ 1 + F(\mu' + i\eta) \right]$$

(7.1)

where $F(\mu) = e^{iZ(\mu)}$ and

$$\varphi(\mu) = \begin{cases} 
2N \arctan \left[ \frac{\sinh(\pi\mu/\gamma)}{\cos(\pi\theta/[2\gamma])} \right] - \frac{\pi\omega}{\pi - \gamma} & \text{finite size vertex model} \\
- \frac{2\tilde{\beta}}{\gamma \sinh(\pi\mu/\gamma)} + \frac{i\pi\beta h}{\pi - \gamma} & \text{XXZ thermodynamics} \\
m_{\beta} \sinh(\pi\mu/\gamma) & \text{sine-Gordon field theory}
\end{cases}$$

(7.2)

The solution $Z(\lambda)$ must be then inserted in eq.(3.9). [ Recall that $\epsilon(\mu) = -iZ(\mu)$ ]

7.1. Leading finite size corrections for the six-vertex model

Let us derive here the dominant finite size corrections for the six-vertex model free energy [see sec. 3].

We see from eq.(7.2) that $\varphi(\mu) \sim N$ for large $N$ as long as $\mu \leq (\gamma/\pi) \log[N \cos(\frac{\pi\theta}{2\gamma})]$. Hence, $Z(\mu) \sim N$ and we find that this bulk contribution to $L_N$ vanishes to the order $N^0$ and $N^{-1}$.
The contributions to $L_N$ of order $N^{-1}$ (and smaller) come from values of $|\mu|$ larger than $(\gamma/\pi) \log[N \cos(\pi \theta / \gamma)]$, where the previous estimate $Z(\mu) \sim N$ does not hold anymore. In order to compute the contributions from large positive values of $\mu$ one introduces the new function

$$F(x) = e^{-e(\mu)}, \quad x = \mu - \frac{\gamma}{\pi} \log[4N \cos(\pi \theta / \gamma)] \quad (7.3)$$

Then eqs. (7.1) and (3.9) reduce, in the $N \to \infty$ limit, to

$$-i \log F(x) = -e^{-x} - \frac{2\pi \omega}{\pi - \gamma} \int_{-\infty}^{+\infty} dy \, G(x - y - i\eta) \log[1 + F(x + i\eta)] \quad (7.4)$$

$$L_{2N} = i\omega - \frac{i \, e^{-i\omega}}{N \gamma \cos(\pi \lambda / \gamma)} \int_{-\infty}^{+\infty} dx \, e^{-x} \, \text{Im} \left\{ e^{-i\eta} \log[1 + F(x + i\eta)] \right\} \quad (7.5)$$

For simplicity, we shall choose $\eta = 0^+$ and use that $G(x+i0) = G(x)$ and $G(\pm \infty) = 0$ for real $x$. We shall analogously compute below the contributions from large negatives values of $\mu$.

Fortunately, it is not necessary to solve the integral equation (7.4) in order to calculate the integral (7.5). In fact, we can use the following lemma

Lemma. Assume that $F(x)$ satisfies the nonlinear integral equation

$$-i \log F(x) = \varphi(x) + 2 \int_{x_1}^{x_2} dy \, G(x - y) \, \text{Im} \log[1 + F(y + i0)] \quad (7.6)$$

where $\varphi(x)$ is real for real $x$ and $x_1, x_2$ are real numbers. Then the following
relation holds
\[
\text{Im} \int_{x_1}^{x_2} dx \varphi'(x) \log[1 + F(x + i0)] = \\
\frac{1}{2} \text{Im} [\varphi(x_2) \log(1 + F_2) - \varphi(x_1) \log(1 + F_1)] + \frac{1}{2} \text{Re} [\ell(F_1) - \ell(F_2)] \\
+ \int_{x_1}^{x_2} dy [G(x_2 - y) \log(1 + F_2) - G(x_1 - y) \log(1 + F_1)] \text{Im} \log[1 + F(y + i0)]
\]

where \(F_{1,2} = F(x_{1,2})\) and \(\ell(t)\) is a dilogarithm function
\[
\ell(t) \equiv \int_0^t du \left[ \frac{\log(1 + u)}{u} - \frac{\log u}{1 + u} \right]
\]

To prove the lemma we consider the relation
\[
\ell(F_2) - \ell(F_1) = \int_{x_1}^{x_2} dx \left\{ \log[1 + F(x + i0)] \frac{d}{dx} \log F(x) - \log F(x) \frac{d}{dx} \log[1 + F(x + i0)] \right\}
\]

and then use eq.(7.6) and its derivative to substitute \(\log F(x)\) and \(d \log F(x)/dx\). This yields
\[
\ell(F_2) - \ell(F_1) = i \int_{x_1}^{x_2} dx \left\{ \frac{d \varphi(x)}{dx} \log[1 + F(x + i0)] - \varphi(x) \frac{d}{dx} \log[1 + F(x + i0)] \right. \\
\left. + \log[1 + F(x + i0)] \int_{x_1}^{x_2} dy G'(x - y) \text{Im} \log[1 + F(y + i0)] \\
- \frac{d}{dx} \log[1 + F(x + i0)] \int_{x_1}^{x_2} dy G(x - y) \text{Im} \log[1 + F(y + i0)] \right\}
\]

The double integral here cancels upon partial integration and using that \(G'(x - y) = -G'(y - x)\). Finally, taking real part yields eq.(7.7) (related identities were used in ref. [18]).
We can now use the lemma to compute the integral (7.5) setting
\[ \varphi(x) = -e^{-\frac{\pi}{\gamma} x} - \frac{\pi \omega}{\pi - \gamma} \quad \text{and} \quad x_1 = -\infty, \ x_2 = +\infty. \]
We have \( F(x_1) = 0, \ F(x_2) = e^{-2i\omega} \) and
\[ \frac{\pi}{\gamma} \int_{-\infty}^{+\infty} dx \ e^{-\frac{\pi}{\gamma} x} \log[1 + F(x + i0)] = -\frac{1}{4} \left[ \ell(e^{-2i\omega}) + \ell(e^{+2i\omega}) \right] + \frac{\omega^2}{2(1 - \gamma/\pi)} \quad (7.9) \]
In this expression we can apply the formula
\[ \ell(z) + \ell(1/z) = \frac{\pi^2}{3} \quad (7.10) \]
The contribution from large negatives values of \( \mu \) is computed analogously introducing the variable
\[ x' = \mu + \frac{\gamma}{\pi} \log \left[ 4N \cos(\frac{\pi \theta}{\gamma}) \right] \]
We find from eqs. (7.5), (7.9) and (7.10) collecting both contributions:
\[ L_{2N} = \frac{\pi}{6N} \tan \left( \frac{\pi \lambda}{\gamma} \right) \left[ 1 - \frac{6\omega^2}{\pi^2(1 - \gamma/\pi)} \right] \quad (7.11) \]
Hence the central charge turns to be unit. The factor \( \tan \left( \frac{\pi \lambda}{\gamma} \right) \) is a geometric effect due to the kind of diagonal-to-diagonal lattice we are using. (In the row-to-row framework a factor \( \sin \left( \frac{\pi \lambda}{\gamma} \right) \) arises [15]).

Let us prove that this tan factor is just the speed of sound for low-lying excitations. The eigenvalue of \( \log t_{2N}(\lambda, \theta, \omega) \) for a hole excitation normalized to the a.g.s. writes [15]
\[ -2i \arctan \left[ e^{\frac{\pi}{2}(\phi + i\lambda)} \right] \quad (7.12) \]
where \( \phi \) stands for the position of the hole. Since we are here on an euclidean lattice, we can write \( \log t_{2N}(\lambda, \theta, \omega) = h + ip \) to define the time and space generators \( h \).
and $p$, respectively. For large positive or negative $\phi$, the eigenvalues of $h$ and $p$ result from eq.(7.12)

$$
\epsilon = \pm 2 \sin \left( \frac{\pi \lambda}{\gamma} \right) e^{-\frac{\pi |\phi|}{\gamma}}, \quad p = \pm 2 \cos \left( \frac{\pi \lambda}{\gamma} \right) e^{-\frac{\pi |\phi|}{\gamma}}
$$

That is, the speed of sound turns out to be precisely $\tan \left( \frac{\pi \lambda}{\gamma} \right)$.

This result can be interpreted as a proof of the presence of conformal invariance.

### 7.2. Low temperatures in the XXZ chain

We shall consider now the low temperature regime. When $\beta \gg 1$ eq. (4.23) indicates that $Z(\lambda) \sim \beta$, so that $\log[1 + e^{iZ(\lambda)}] \simeq iZ(\lambda)$, at least as long as $|\lambda| \leq (\gamma/\pi) \log \beta$. At dominant $\beta \gg 1$ order eq.(4.23) linearizes in the same way as it does for small $\beta$. Inserting then $Z(\lambda) \simeq 2\tilde{\beta}q(\lambda)$ in eq.(4.22), yields $\lim_{\beta \to \infty} L(\beta) = 0$. Therefore eq.(4.20) tells us that

$$
f(\beta) \xrightarrow{\beta \to \infty} E_{XXZ}(\gamma) + O(\beta^{-2}) \quad (7.13)
$$

The contributions of order $\beta^{-2}$ (and smaller) come from values of $\lambda$ larger than $(\gamma/\pi) \log \beta$, where the previous assumption $\log a \sim O(\beta)$ does not hold anymore. It is then convenient to introduce the new function

$$
F(x) = e^{iZ(\lambda)}, \quad x = \lambda - \frac{\gamma}{\pi} \log \frac{4\pi \tilde{\beta}}{\gamma} \quad (7.14)
$$

Then eqs. (4.22) and (4.23) reduce, in the $\beta \to \infty$ limit, to

$$
L(\beta) = \frac{1}{\pi \beta} \int_{-\infty}^{+\infty} dx e^{-\pi x/\gamma} \operatorname{Im} \log[1 + F(x + i0)] \quad (7.15)
$$

and

$$
-i \log F(x) = -e^{-\pi x/\gamma} + 2 \int_{-\infty}^{+\infty} dy G(x - y) \operatorname{Im} \log[1 + F(y + i0)] \quad (7.16)
$$

The integral in eq.(7.15) may now be exactly calculated by using the lemma.
from section 7.1 with $\varphi = -\exp(-\pi x/\gamma)$ and $x_1 = -\infty$, $x_2 = +\infty$. We have $F(x_1) = 0$, $F(x_2) = 1$ and

$$2 \text{Im} \int_{-\infty}^{+\infty} dx \, e^{-\pi x/\gamma} \log[1 + F(x+i0)] = -\frac{\gamma}{\pi} \ell(1) = -\frac{\pi \gamma}{6}$$

Then the free energy for low temperature reads

$$f(\beta) = E_{XXZ}(\gamma) - \frac{\gamma}{12J \sin \gamma} \beta^{-2} + o(\beta^{-2})$$ \hfill (7.17)

in perfect agreement with refs. [2,16,17].

As we have just shown, both the high and the low temperature leading behaviors of the free energy can be derived without much effort from our non–linear integral equation (2.27). The higher order corrections for high $T$ can be obtained in a very systematic way. The situation for the $o(\beta^{-2})$ terms in the low $T$ expansion (7.17) is more involved.

However, the NLIE (4.23) admits a Riemann–Hilbert formulation analogous to the one presented in sec. 7.4 for the sG-mT model. One can establish from such Riemann–Hilbert formulation the analytic structure of the low $T$ expansion.

### 7.3. High temperatures in the mT-sG field theory

Let us recall the integral equation (5.12)

$$\epsilon = r \, \text{ch} - G_0 * L + G_1 * \bar{L}$$ \hfill (7.18)

where $\epsilon \equiv \epsilon_f$, $r \equiv m \beta$ and $\text{ch}$ stands for the hyperbolic cosine, $\text{ch}(\theta) = \cosh \theta$.

For small $r$ (high temperatures), the regions with positive and negative $\theta$ where $r \cosh \theta \sim 1$ are very far apart. It is then useful to separately study eq.(5.12) with
inhomogeneities

\[ \frac{r}{2} e^{\pm \theta} = e^{\pm \theta - \log 2/r}. \]

That is, we define the kink function \( \epsilon_k \) as the solution of the same integral equation with \( r \ ch \) replaced by an exponential

\[ \epsilon_k = e^\theta - G_0 * L_k + G_1 * \bar{L}_k \] (7.19)

where \( L_k = \log(1 + e^{-\epsilon_k}) \). (Analogous kink functions were first considered in [4] and have now become standard in TBA calculations).

Then, we set

\[ \eta(\theta) = \epsilon(\theta) - \epsilon_k(\theta - \log 2/r) - \epsilon_k(-\theta - \log 2/r) \]

\[ \ell(\theta) = L(\theta) - L_k(\theta - \log 2/r) - L_k(-\theta - \log 2/r) + \log 2 \]

From eqs. (7.18) and (7.19), using the translational invariance of eq.(7.19) \( \eta \) and \( \ell \) turn to be related by an homogeneous equation

\[ \eta = -G_0 * \ell + G_1 * \ell \] (7.20)

We then find for the energy

\[ E(\beta) = -\frac{m}{2\pi} \int_{-\infty}^{+\infty} d\theta \cosh \theta \left[ L(\theta) + \bar{L}(\theta) \right] \]

\[ = -\frac{m}{2\pi} \left( f_k + \bar{f}_k \right) - \frac{m}{2\pi} \int_{-\infty}^{+\infty} d\theta \cosh \theta \left[ \ell(\theta) + \bar{\ell}(\theta) \right] \] (7.22)
where the kink contribution reads

\[
f_k = \int_{-\infty}^{+\infty} d\theta e^{\theta} [L_k(\theta - \log 2/r) + L_k(-\theta - \log 2/r) - \log 2]
\]

\[
= \frac{2}{r} \int_{-\infty}^{+\infty} d\theta e^{\theta} L_k(\theta) + \int_{-\infty}^{+\infty} d\theta e^{-\theta} \frac{\partial}{\partial \theta} [L_k(\theta - \log 2/r) - \log 2]
\]

\[
= \frac{2}{r} \int_{-\infty}^{+\infty} d\theta e^{\theta} L_k(\theta) + r \int_{-\infty}^{+\infty} d\theta e^{-\theta} \frac{\partial L_k}{\partial \theta}
\]

(7.23)

The real parts of these two integrals can be calculated without explicit knowledge of the kink function \(\epsilon(\theta)\). To the term proportional to \(r^{-1}\) we can apply the Lemma with the substitutions \(\varphi(\theta) = e^{\theta}\) and \(x_1 = -\infty, \ x_2 = +\infty\). This gives

\[
\int_{-\infty}^{+\infty} d\theta e^{\theta} [L_k(\theta) + L_k^- (\theta)] = \ell(1) = \frac{\pi^2}{6}
\]

(7.24)

To compute the term linear in \(r\) in eq.(7.23) we can adapt to our situation an argument in ref.[4]: to converge for \(\theta \rightarrow -\infty\), the second integral requires that \(\partial L_k/\partial \theta\) decays faster than \(e^{\theta}\), hence also \(\epsilon_k(\theta)\) must tend to 0 faster than \(e^{\theta}\) as \(\theta \rightarrow -\infty\). We then differentiate eq.(7.19) with respect to \(\theta\), let \(\theta \rightarrow -\infty\) and use the asymptotic expansion for large \(|\lambda|\)

\[
G_0(\lambda) \simeq a_1 e^{-|\lambda|} \left[ 1 + O(e^{-2|\lambda|}) \right] + b_1 e^{-2\hat{\gamma}|\lambda|/\pi} \left[ 1 + O(e^{-2\hat{\gamma}|\lambda|/\pi}) \right]
\]

where \(\hat{\gamma} = \gamma (1 - \gamma/\pi)^{-1}\) and

\[
a_1 = \frac{1}{\pi} \tan \left( \frac{\pi^2}{2\gamma} \right), \quad b_1 = \frac{\hat{\gamma}}{\pi^2} \tan \left( \frac{\pi^2}{\pi - \gamma} \right)
\]

Thus we formally obtain for \(\theta \rightarrow -\infty\)

\[
\frac{d\epsilon_k}{d\theta} \simeq \left[ 1 - a_1 \int_{-\infty}^{+\infty} d\lambda e^{-\lambda} \left( \frac{\partial L_k}{\partial \lambda} + \frac{\partial L_k^-}{\partial \lambda} \right) \right] e^\theta + O(e^{2\theta}) + O(e^{2\hat{\gamma}\theta/\pi})
\]

(7.25)

Whether \(e^{2\theta}\) or \(e^{2\hat{\gamma}\theta/\pi}\) is the dominant order depends on who is larger between
\( \gamma \) and \( \pi/3 \), but it does really affect our argument. In any case self–consistency requires that the coefficient of \( (e^{\theta}) \) vanishes, yielding the value of the real part of the second integral in eq.(7.23).

Collecting the results of eqs.(7.24) and (7.25) into eq.(7.22) one obtains in explicit form the kink contribution to the energy

\[
E^{\beta \to 0} = -\frac{\pi}{6\beta} - \frac{1}{4}m^2 \beta \cot \frac{\pi}{2\gamma} - \frac{m}{2\pi} \int_{-\infty}^{+\infty} d\theta \cosh \theta \left[ \ell(\theta) + \ell'(\theta) \right]
\]

(7.26)

The first term is the universal conformal Casimir energy, from which we read the correct central charge \( c = 1 \) of the mT-sG model. The second term, linear in \( \beta \), exactly coincides with minus the scaling bulk free energy (see eq.(5.7)). As is well known, this is not an accident, since the UV–independent part of the complete ground state energy \( E_c + E \simeq \lim_{N \to \infty} E_N \) for small \( r \) can be calculated through Conformal Perturbation Theory (PCT). For the mT model the unperturbed conformal theory is the massless Thirring model, while the perturbation is the mass term. This has noninteger dimension for \( \gamma \neq \pi/2 \), yielding only noninteger powers of \( r \) in the perturbation series for the free energy, apart from the conformal \( \beta^{-1} \) term. Hence no term linear in \( r \) may appear. This argument has been often used to compute the scaling bulk energy of Perturbed Conformal Models from the TBA derived from the corresponding scattering theory. In our case we verify the validity of the argument since the scaling bulk energy can be directly calculated from the microscopic BA solution.

The last integral in eq.(7.26) represents the resummation of the PCT. The fact that it must contain non-integral powers of \( r \), causing non-analyticity at \( r = 0 \), can be established directly from the original equation (7.18), since this cannot be expanded in a Taylor series of \( r \). To establish the absence of any integral power of \( r \), and especially of the first power, without referring to PCT, a more elaborate treatment of eq.(7.18) is necessary. This shall be the subject of next section.
7.4. The NLIE as a Riemann–Hilbert (or Wiener–Hopf) problem.

In this section we recast the non-linear integral equations (5.9) for the mT-sG model as a Riemann-Hilbert (or Wiener-Hopf) problem. The XXZ thermodynamics can be treated analogously. The techniques involved are rather standard, having been used very often to study Bethe Ansatz systems coupled to external fields at zero temperature (see for instance [22]). Here we adapt them to the regime $T > 0$ at zero external field.

Let us define,

$$\rho(\theta) \equiv \frac{1}{2\pi} \frac{d}{d\theta} Z(\gamma \theta/\pi)$$  \hspace{1cm} (7.27)

For large $\beta$, the roots $\theta_k$ defined by eq.(5.18) become a continuous distribution with a density precisely given by $\rho(\theta)$. For arbitrary real values of $\beta$, $\rho(\theta)$ is just a continuous function that we are interested to find. From eqs.(5.14) and (7.27) we find

$$\frac{d\nu}{d\theta} = 2\pi \rho(\theta) - m\beta \cosh \theta$$  \hspace{1cm} (7.28)

This equation combined with the $\theta$ derivative of eq.(5.21) and with eq.(5.17) yields upon partial integration

$$\rho(\theta) - \frac{m\beta}{2\pi} \cosh \theta = \int_{-\infty}^{+\infty} d\mu G_0(\theta - \mu) \left[ \rho(\mu) - \sum_{k \in \mathbb{Z}} \delta(\mu - \theta_k) \right]$$  \hspace{1cm} (7.29)

The asymptotic behaviour given by eq.(5.20) shows that the roots $\theta_k$ accumulate at $\pm \infty$.

The smaller positive root will be called $b \equiv \theta_0$. Then, since there are no roots in the interval $-b \leq \theta \leq b$, it is convenient to write eq.(7.29) as

$$\rho(\theta) - \frac{m\beta}{2\pi} \cosh \theta = \int_{-b}^{+b} d\mu G_0(\theta - \mu) \rho(\mu) = B(\theta)$$  \hspace{1cm} (7.30)
where

\[
B(\theta) \equiv \int_{\{|\mu|>b\}} d\mu G_0(\theta - \mu) \left[ \rho(\mu) - \sum_{k\in\mathbb{Z}} \delta(\mu - \theta_k) \right]
\] (7.31)

For small \(\beta\), eq.(5.20) applies and we find

\[
b^{\beta \to 0} = \log \left( \frac{2\pi}{m\beta} \right) \to +\infty.
\]

Hence, for small \(\beta\) \(B(\theta)\) tends to zero and we can consider it a perturbation.

Eq.(7.30) has the appropriate form to be transformed into a Riemann-Hilbert problem upon Fourier transformation. Notice that the function \(\frac{m\beta}{2\pi} \cosh \theta - \rho(\theta)\) vanishes for large \(\theta\) and can be Fourier transformed as follows:

\[
X_+(\omega) \equiv e^{+i\omega b} \int_{-b}^{+\infty} d\theta \ e^{-i\omega \theta} \left[ \frac{m\beta}{2\pi} \cosh \theta - \rho(\theta) \right]
\]

\[
X_-(\omega) \equiv e^{-i\omega b} \int_{-\infty}^{-b} d\theta \ e^{-i\omega \theta} \left[ \frac{m\beta}{2\pi} \cosh \theta - \rho(\theta) \right]
\] (7.32)

where the functions \(X_\pm(\omega)\) are analytic for \(\pm \text{Im} \omega > 0\). In addition,

\[
X_+(\omega) = X_-(\omega)
\]

Now, Fourier transforming eq.(7.30) yields

\[
[1 - \tilde{G}_0(\omega)] \int_{-b}^{b} e^{-i\omega \theta} \rho(\theta) \ d\theta = \frac{m\beta}{2\pi} \tilde{\chi}_b(\omega) + e^{-i\omega b} X_+(\omega) + e^{+i\omega b} X_-(\omega) + \tilde{B}(\omega)
\] (7.33)

where the function \(\tilde{\chi}_b(\omega)\) is defined by

\[
\tilde{\chi}_b(\omega) \equiv \int_{-b}^{+b} d\theta \ e^{-i\omega \theta} \cosh \theta = \frac{\sinh(1 + i\omega)b}{1 + i\omega} + \frac{\sinh(1 - i\omega)b}{1 - i\omega}
\] (7.34)

52
The kernel $1 - \tilde{G}_0(\omega)$ can be factorized as follows

$$1 - \tilde{G}_0(\omega) = \frac{\sinh\left(\frac{\pi^2 \omega}{2\gamma}\right)}{\sinh\left(\frac{\pi^2 \omega}{2\gamma}\right) + \sinh\left(\frac{\pi}{2\gamma} - 1\right)\pi \omega} = \frac{1}{K_+(\omega)K_-(\omega)}.$$  

The function $K_{\pm}(\omega)$ is analytic and non-zero for $\mp \text{Im}(\omega) > 0$, $K_+(\omega) = K_-(\omega)$, and explicitly one finds

$$K_{\pm}(\omega) = \sqrt{2\pi} \left(\frac{\gamma}{\pi}\right)^{-i\omega/2} \left(1 - \frac{\gamma}{\pi}\right)^{1/2 - i(\pi/\gamma - 1)\omega/2} \frac{\Gamma\left(1 - \frac{i\pi \omega}{2\gamma}\right)}{\Gamma\left(\frac{1}{2} - \frac{i\omega}{2}\right) \Gamma\left(1 - \frac{1}{2} + \frac{1}{\gamma} - \frac{i\omega}{2}\right)}.$$  

Let us define

$$f_{\pm}(\omega) \equiv e^{\mp i\omega b} \int_{-b}^{+b} d\theta e^{-i\omega \theta} \rho(\theta).$$

The function $f_{\pm}(\omega)$ is analytic in $\mp \text{Im}\omega > 0$. Eq.(7.33) yields a pair of equations

$$\frac{f_+}{K_+} = K_- \left[ e^{-i\omega b} \frac{m\beta}{2\pi} \tilde{c}_b + X_+ e^{-2i\omega b} + X_- + e^{-i\omega b} \tilde{B} \right]$$

$$\frac{f_-}{K_-} = K_+ \left[ e^{+i\omega b} \frac{m\beta}{2\pi} \tilde{c}_b + X_- e^{+2i\omega b} + X_+ + e^{+i\omega b} \tilde{B} \right].$$  

(7.35)

Projecting eqs.(7.35) into the half-plane $\text{Im}\omega > 0$ yields

$$K_- X_- + \left[ K_- \left( \frac{m\beta}{2\pi} \tilde{c}_b + \tilde{B} \right) e^{-i\omega b} \right]_+ + \left[ K_- X_+ e^{-2i\omega b} \right]_- = 0$$

$$\frac{f_-}{K_-} - \left[ K_+ \left( \frac{m\beta}{2\pi} \tilde{c}_b + \tilde{B} \right) e^{i\omega b} \right]_- - \left[ K_+ X_- e^{+2i\omega b} \right]_+ = 0.$$  

(7.36)

The projection $[\ldots]_{\pm}$ can be explicitly performed as

$$[F(\omega)]_{\pm} = \pm \int \frac{d\omega'}{2\pi i} \frac{F(\omega')}{\omega - \omega'}, \quad \mp \text{Im}\omega > 0.$$  

Equations analogous to (7.36) follow upon the exchange $\omega \leftrightarrow -\omega$, $\pm \leftrightarrow \mp$. 53
More explicitly, the first of eqs. (7.36) reads

$$K_-(\omega) X_-(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi i} \frac{e^{-i\omega'b}}{\omega - \omega'} K_-(\omega') \left[ \frac{m\beta}{2\pi} \tilde{c}_{b'}(\omega') + \tilde{B}(\omega') \right] + \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi i} \frac{e^{-2i\omega'b}}{\omega - \omega'} K_-(\omega') X_+(\omega') , \quad \text{Im} \, (\omega) > 0 . \tag{7.37}$$

Then, inserting eq. (7.34) in eq. (7.37) and evaluating by residues part of the terms, yields

$$K_-(\omega) X_-(\omega) = \frac{im\beta}{4\pi} \left[ e^b \frac{K_- (\omega) - K_- (i)}{\omega - i} + e^{-b} \frac{K_- (\omega)}{\omega + i} \right] + \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi i} \frac{e^{-2i\omega'b}}{\omega - \omega'} K_-(\omega') \left[ X_-(\omega') - \frac{im\beta}{4\pi} \left( \frac{e^b}{-\omega' - i} + \frac{e^{-b}}{-\omega' + i} \right) \right] + \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi i} \frac{e^{-i\omega'b}}{\omega - \omega'} K_-(\omega') \tilde{B}(\omega') . \tag{7.38}$$

Setting

$$v(\omega) \equiv K_-(\omega) \left[ X_- (\omega) - \frac{im\beta}{4\pi} \left( \frac{e^b}{\omega - i} + \frac{e^{-b}}{\omega + i} \right) \right], \tag{7.39}$$

we see that $v(\omega)$ is analytic in $\text{Im} \, \omega > 0$ except for a simple pole at $\omega = i$. We then obtain from eqs. (7.38) and (7.39)

$$v(\omega) = -\frac{im\beta e^b}{4\pi} \frac{K_-(i)}{\omega - i} + \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi i} \frac{e^{2i\omega'b}}{\omega + \omega'} \alpha(\omega') v(\omega') + \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi i} \frac{e^{i\omega'b}}{\omega + \omega'} K_+(\omega') \tilde{B}(\omega') \tag{7.40}$$

where

$$\alpha(\omega) \equiv \frac{K_+(\omega)}{K_-(\omega)}$$

The ground state energy $E$ can be conveniently expressed in terms of the residue of $X_-(\omega)$ at $\omega = -i$. In fact, using eqs. (5.8) for $\eta \rightarrow 0^+$, (5.15), (7.28) and (5.17),
\[ E = m \int_{-\infty}^{+\infty} d\theta \cosh \theta \left[ \rho(\theta) - \sum_{k \in \mathbb{Z}} \delta(\theta - \theta_k) \right] \]

Comparing this with eqs. (7.29) and (7.32), we obtain

\[ X_-(\omega) \xrightarrow{\omega \to -i} -iE \frac{\pi^2}{m\pi} \tan \frac{b}{2\gamma \omega + i} \]

that is

\[ E = \frac{im\beta}{\pi} e^b \cot \frac{\pi^2}{2\gamma} \lim_{\omega \to -i} (\omega + i)X_-(\omega) \]

We can also express \( E \) in terms of \( v(-i) \)

\[ E = m \cot \frac{\pi^2}{2\gamma} \left[ -\frac{m\beta}{4} + \frac{i\pi e^b}{\xi} v(-i) \right] \quad (7.41) \]

where

\[ \xi \equiv \lim_{\omega \to -i} \frac{K_-(\omega)}{\omega + i} = \frac{\pi}{\sqrt{2\gamma}} \left( 1 - \frac{\gamma}{\pi} \right) \Gamma \left( \frac{1 - \frac{\pi}{2\gamma}}{2} \right) \frac{\Gamma \left( \frac{3}{2} - \frac{\pi}{2\gamma} \right)}{\Gamma \left( \frac{1 - \frac{\pi}{2\gamma}}{2} \right)} \]

Eqs.(7.40) and (7.41) are exact and provide the starting point to systematically compute the high temperature (large \( b \)) behaviour. Here we limit ourselves to a qualitative analysis: to start we notice that the first term in eq.(7.41) exactly provides minus the bulk free energy, which supposedly is the only term regular at \( m\beta = 0 \). Next we observe that the integration path in the first integral of eq.(7.40) can be deformed into the path \( C_+ \) which runs around the positive imaginary axis. Thus we see that this integral gives poles to \( v(\omega) \) in the half-plane \( \text{Im} \omega < 0 \) in correspondence with the poles of \( K_+(\omega) \) for \( \text{Im} \omega > 0 \) (at \( \omega = i, K_+(\omega) \) has a simple zero). These poles are located at \( \text{Im} \omega = \xi_n \equiv (2\gamma/\pi) n, n = 1, 2, \ldots \), so that the integral contributes a series of terms proportional to \( v(i\xi_n) \exp(-2b\xi_n) \)
to \( v(-i) \) and hence to the energy. Neglecting for the moment the second integral in eq.(7.40), we evaluate \( v(\omega) \) iteratively for large \( b \), and conclude that \( v(-i) \) has an asymptotic expansion in powers of \( \exp(-2b\xi_1) \simeq (m\beta/2\pi)^{4\gamma/\pi} \), including the power zero. Inserting this in eq.(7.41) we find that the energy has a term proportional to \( (m\beta)^{-1} \) plus a series of terms with \( (m\beta)^{-1} \) times integer powers of \( (m\beta)^{4\gamma/\pi} \). The effects of the second integral in eq. (7.40) can be taken into account by applying the Euler-Maclaurin formula to the definition of \( B(\theta) \), eq.(7.31). To second order in the Euler-Maclaurin expansion we then find

\[
\hat{B}(\omega) \simeq \left[ -\cos \omega + \frac{\omega \sin \omega b}{6\rho(b)} + \ldots \right]
\]

where \( \rho(b) \) can be evaluated to this order by self–consistency. This leads to changes in the coefficients of the expansion of \( v(-i) \) and hence of the energy, but leaves unchanged the analytic structure. The same would happen by pushing further the Euler-Maclaurin expansion. The structure of this expansion and the powers \( (m\beta)^{4\gamma n/\pi} \) agree with Perturbed Conformal Field Theory, since the perturbing operator (the mass term in the mT model or the cosine term in the sG model) has exactly the scale dimension \( 2(1 - \gamma/\pi) \).

8. Comparison against standard TBA in the IR limit \( m\beta \to \infty \).

We can write the non-linear integral equation (5.12) for the massive Thirring model as

\[
\nu_f \equiv \epsilon_f - r \cosh \theta = -G_0 * L_f + G_1 * L_{\bar{f}}
\]

(8.1)

where \( r \equiv m\beta \). In Fourier space the two kernels are related by \( \tilde{G}_1(k) = \tilde{G}_0(k)e^{\pi k} \), so that eq.(8.1) reads there

\[
\tilde{\nu}_f = \tilde{G}_0 \left( -\tilde{L}_f + e^{\pi k} \tilde{L}_{\bar{f}} \right)
\]

(8.2)

We have to lowest order
\[ L_f(\theta) = \log[1 + e^{-\epsilon_f(\theta)}] \simeq e^{-r \cosh \theta} \equiv g(\theta) \]

and thus, to the same first order in the uniformly small \( g \)

\[ \tilde{\nu}_f \simeq \tilde{G}_0(-1 + e^{\pi k}) \tilde{g} \]

Hence, to second order in \( g \)

\[ L_f + L_{\bar{f}} \simeq e^{\epsilon_f} - \frac{1}{2} e^{-2\epsilon_f} + e^{\epsilon_f} - \frac{1}{2} e^{-2\epsilon_f} \]
\[ \simeq g(2 - \nu_f - \nu_f b) - g^2 \] \hspace{1cm} (8.3)
\[ = 2g(1-g) + gW * g \]

where the kernel of the convolution operator \( W \) has Fourier transform

\[ \tilde{W}(k) = 1 - 4\tilde{G}_0(k) \sinh^2 \frac{\pi k}{2} = 1 - \frac{2 \sinh \left( \frac{\pi^2 k}{2\gamma} - 1 \right) k \sinh^2 \frac{\pi}{2} k}{\sinh \left( \frac{\pi^2 k}{2\gamma} - \frac{1}{2} \right) k \cosh \frac{\pi}{2} k} \]

The traditional TBA is based on the string hypothesis. According to the work of Takahashi and Suzuki [2], one needs to write \( \gamma/\pi \) as continued fraction whose structure will fix the type of string configurations allowed as roots of the BAE at finite temperature \( T = 1/\beta \). Standard entropy arguments then lead to a generically infinite set of coupled nonlinear integral equations [2]. When \( \gamma/\pi \) is a rational number, this set of equations can be reduced to a finite system. The simplest case is obtained when \( \gamma = \pi/n \), with \( n \) an integer larger than 2. One then deals with the following system of coupled equations for the real pseudoenergies \( \epsilon_j(\theta) \), \( j = 1, 2, \ldots, n \)

\[ \epsilon_1(\theta) = r \cosh \theta - D * L_2(\theta) \]
\[ \epsilon_j = -D * (L_{j-1} + L_{j+1}) , \quad j = 2, 3, \ldots, n-3 \]
\[ \epsilon_{n-2} = D * (L_{n-3} + 2L_n) \]
\[ \epsilon_{n-1} = \epsilon_n = -D * L_{n-2} \] \hspace{1cm} (8.4)

where \( L_j = \log(1 + e^{-\epsilon_j}) \) and \( D(\theta) = [2\pi \cosh(\theta)]^{-1} \). The structure of this system
is that of the $D_n$ Dynkin diagram. In the limit $r \to \infty$ the functions $L_j$ all tend to constants, with $L_1 \to 0$ in particular. Since $\tilde{D}(0) = 1/2$ we then get the system of numerical equations

$$
\epsilon_j = -\frac{1}{2} \log \left( 1 + e^{-\epsilon_{j-1}} \right) \left( 1 + e^{-\epsilon_{j+1}} \right), \quad j = 2, 3, \ldots, n - 3
$$

$$
\epsilon_{n-2} = -\frac{1}{2} \log \left( 1 + e^{-\epsilon_{n-3}} \right) \left( 1 + e^{-\epsilon_n} \right)^2
$$

$$
\epsilon_{n-1} = \epsilon_n = -\frac{1}{2} \log \left( 1 + e^{-\epsilon_{n-2}} \right)
$$

with solution

$$
\epsilon_j = -\log \left( j^2 - 1 \right), \quad j = 1, 2, \ldots, n - 2
$$

$$
\epsilon_{n-1} = \epsilon_n = -\log(n - 2)
$$

We now linearize eqs. (8.4) around this $r = \infty$ solution, by setting

$$
L_1 = \ell_1 = 2g
$$

$$
L_j = \log j^2 + \ell_j, \quad \epsilon_j = -\log(j^2 - 1) - \frac{j^2}{j^2 - 1} \ell_j, \quad j = 2, 3, \ldots, n - 2
$$

$$
L_{n-1} = L_n = \log(n - 1) + \ell_{n-1}, \quad \epsilon_{n-1} = -\log(j^2 - 1) - \frac{n - 2}{n - 1} \ell_{n-1}
$$

so that the functions $\ell_j$ fulfill the linear system

$$
\frac{j^2}{j^2 - 1} \ell_j = D \ast (\ell_{j-1} + \ell_{j+1}), \quad j = 2, 3, \ldots, n - 3
$$

$$
\frac{(n - 2)^2}{(n - 2)^2 - 1} \ell_{n-2} = D \ast (\ell_{n-3} + 2\ell_{n-1})
$$

$$
\frac{n - 2}{n - 1} \ell_{n-1} = D \ast \ell_{n-2}
$$

Clearly all $\ell_j$ are of order $g$, so that we obtain the order $g^2$ for the function $L_1$ from

$$
L_1 \simeq e^{-\epsilon_1} - \frac{1}{2} e^{-2\epsilon_1} \simeq 2g(1 - g) + gD \ast \ell_2
$$

since $\epsilon_1 \simeq r \cosh \theta - \log 2 - D \ast \ell_2$. Agreement with the result (8.3) of our TBA,
requires

\[ D \ast \ell_2 = W \ast g \]  

(8.9)
as soon as \( \gamma = \pi/n \). This requirement is indeed satisfied: by assuming \( \ell_2 \) to be given by eq.(8.9) we can easily calculate in Fourier space all other \( \ell_j \)'s from the first \( n - 3 \) equations of the system (8.7). The last equation of that system must therefore reduce to an identity, and it does.

REFERENCES

1. C.N. Yang and C.P. Yang, *J. Math. Phys.* 10 (1969) 1115.
2. M. Takahashi, *Prog. Theor. Phys.* 46 (1971) 401.
   M. Takahashi and M. Suzuki, *Prog. Theor. Phys.* 48 (1972) 2187.
3. M. Gaudin, *Phys. Rev. Lett.* 26 (1971) 1302.
4. Al.B. Zamolodchikov, *Nucl. Phys.* B342 (1990) 695.
5. Al.B. Zamolodchikov, *Nucl. Phys.* B358 (1991) 524.
   P. Christe and M. Martins, *Mod. Phys. Lett.* A5 (1990) 2189.
   M. Martins, *Phys. Rev. Lett.* 65 (1990) 2091.
   T. Klassen and E. Melzer, *Nucl. Phys.* B338 (1990) 485;
   *Nucl. Phys.* B350 (1991) 635.
   F. Ravanini, *Phys. Lett.* 282B (1992) 73.
   V. A. Fateev and Al. B. Zamolodchikov, *Phys. Lett.* 271B (1991) 91.
   P. Fendley, K. Intriligator *Nucl. Phys.* B372 (1992) 533.
   F. Ravanini, R. Tateo and A. Valleriani, *Int. J. Mod. Phys.* A8 (1993) 1707.
6. C. Destri and H. J. de Vega, *Phys. Rev. Lett.* 69 (1992) 2313.
7. C. Destri and H. J. de Vega, work in progress.
8. A. Klumper, *Z. Phys.* B 91 (1993) 507.
9. R.J. Baxter, “Exactly solved models in Statistical Mechanics”, Academic Press, 1982.
10. M. Suzuki, *Prog. Theor. Phys.* **56** (1976) 1454.

11. J. Suzuki, Y. Akutsu and M. Wadati, *J. Phys. Soc. Jpn.* **59** (1990) 2667.

12. T. Koma, *Prog. Theor. Phys.* **81** (1988) 783.

   M. Takahashi, *Phys. Rev.* **B 43**, 5788 (1990).

13. C. Destri and H.J. de Vega, *Nucl. Phys.* **B290** (1987) 363.

14. L. D. Faddeev and L. A. Takhtadzhyan, *Russian Math. Surveys* **34** (1979) 11.

15. H.J. de Vega, *Int. J. Mod. Phys.* **A** (1989) 2371; *Int. J. Mod. Phys.* **B** (1990) 735.

16. H.J. de Vega and F. Woynarovich, *Nucl. Phys.* **B251** (1985) 439.

17. H.J. de Vega and M. Karowski, *Nucl. Phys.* **B280** (1987) 225.

18. A. Klumper, M.T. Batchelor and P.A. Pearce, *J. Phys.* **A24** (1991) 3111.

19. C. Destri and H.J. de Vega, *Nucl. Phys.* **B258** (1991) 251.

20. C. Destri and H.J. de Vega, *J. Phys.* **A22** (1989) 1329.

21. C. Destri and H. J. de Vega, *Nucl. Phys.* **B406** (1993) 566.

22. A. Polyakov and P. Wiegmann, *Phys. Lett.* **141B** (1984) 223.

   N. Reshetikhin and P. Wiegmann, *Phys. Lett.* **189B** (1987) 125.

   G. Japaridze, A. Nersesyan and P. Wiegmann, *Nucl. Phys.* **B230** (1984) 511.

   P. Hasenfratz, M. Maggiore and F. Niedermayer, *Phys. Lett.* **245B** (1990) 522.

   P. Hasenfratz and F. Niedermayer *Phys. Lett.* **245B** (1990) 529.

23. M. Fowler and X. Zotos, *Phys. Rev.* **B24** (1981) 2634 and **B25** (1982) 5806.

24. Al. B. Zamolodchikov, *Phys. Lett.* **335B** (1994) 436 and Montpellier preprints 1994.