Error Analysis of Crouzeix–Raviart and Raviart–Thomas Finite Element Methods

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Abstract We discuss the error analysis of the lowest degree Crouzeix–Raviart and Raviart–Thomas finite element methods applied to a two-dimensional Poisson equation. To obtain error estimations, we use the techniques developed by Babuška–Aziz and the authors. We present error estimates in terms of the circumradius and diameter of triangles in which the constants are independent of the geometric properties of the triangulations. Numerical experiments confirm the results obtained.

Keywords Crouzeix–Raviart, Raviart–Thomas, finite element method, error estimation, triangulation, circumradius

Mathematics Subject Classification (2000) 65D05, 65N30

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, and $\mathcal{T}_h$ be a triangulation of $\Omega$ consisting of triangular elements. In this paper, we consider an error analysis of the Raviart–Thomas (RT) and piecewise linear (nonconforming) Crouzeix–Raviart (CR) finite element methods applied to the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,$$  

where $f \in L^2(\Omega)$ is a given function. Let $u$ and $u^{CR}_h$ be the exact and CR finite element solutions, respectively. In standard text books, such as that by Brenner and Scott [4].

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the error of \( u_h^{CR} \), assuming \( u \in H^2(\Omega) \), is estimated as

\[
\| u - u_h^{CR} \|_h := \left( \sum_{K \in T_h} \int_K \| \nabla u - \nabla u_h^{CR} \|^2 \, dx \right)^{1/2} \leq C h |u|_{H^2(\Omega)},
\]

where \( h := \max_{K \in T_h} h_K, h_K := \text{diam}K \), and \( C \) is a constant independent of \( u \) and \( h \) but dependent on the chunkiness parameter of the triangulations \( T_h \), Section 10.3]. The dependence on the chunkiness parameter in (1.2) means that, if a triangulation \( T_h \) contains very “thin” triangles, we cannot apply (1.2). Note that the condition ‘\( u \in H^2(\Omega) \)’ does not hold in general, and we need to assume it explicitly. See Assumption II.

A similar error estimation of the CR finite element method under the maximum angle condition was obtained in [15], in which the constant \( C \) depends on the maximum angle of the triangular elements. Related error estimations were also discussed in [13].

The aim of this paper is to show the estimation

\[
\| u - u_h^{CR} \|_h \leq C(R + h)|u|_{H^2(\Omega)}
\]

holds, where \( R := \max_{K \in T_h} R_K, R_K \) is the circumradius of a triangle \( K \), and the constant \( C \) is independent of \( u \) and \( h \), as well as the geometric properties of \( T_h \). Because \( C \) does not depend on the geometric properties of \( T_h \), we may apply (1.3) even if \( T_h \) contains very “skinny” triangles.

Because the CR finite element method is non-conforming, a lemma similar to Céa’s lemma is not available, and this fact complicates the error analysis of the CR finite element method.

To overcome this difficulty, we first consider the error analysis of the RT finite element method. Because this method is conforming, a Céa’s-lemma-type claim is valid and we shall obtain error estimates of its solutions (Theorem 11). In the proof, we use techniques developed by Babuška and Aziz [2] and the authors [9, 10, 11, 12]. It is well known that the CR and RT FEMs are related [15, 13, 16], and an error estimation of the CR FEM (Theorem 13) is obtained from that of the RT FEM.

Finally, we present results of numerical experiments that are consistent with the theoretical results obtained.

2 Preliminaries

2.1 Notation and function spaces

Let \( \mathbb{R}^2 \) be the two-dimensional Euclidean space with Euclidean norm \( |x| := (x_1^2 + x_2^2)^{1/2} \) for \( x = (x_1, x_2)^\top \in \mathbb{R}^2 \). We always regard \( x \in \mathbb{R}^2 \) as a column vector. For a \( 2 \times 2 \) matrix \( A \) and \( x \in \mathbb{R}^2 \), \( A^\top \) and \( x^\top \) denote their transpositions. For a nonnegative integer \( k \), let \( \mathcal{P}_k \) be the set of two-variable polynomials with degrees of at most \( k \).
Let $\mathbb{N}_0$ be the set of nonnegative integers. For $\delta = (\delta_1, \delta_2) \in \mathbb{N}_0^2$, the multi-index $\partial^\delta$ of partial differentiation (in the sense of the distribution) is defined by

$$
\partial^\delta = \partial^\delta_{x} := \frac{\partial^{\delta_1}}{\partial x_1^{\delta_1}} \frac{\partial^{\delta_2}}{\partial x_2^{\delta_2}}, \quad |\delta| := \delta_1 + \delta_2.
$$

Sometimes $\partial^{(1,0)}v$ and $\partial^{(0,1)}v$ are denoted by $v_x$ and $v_y$, respectively. For a two-variable function $v$, its gradient is denoted by $\nabla v = (v_x, v_y)$. The gradient $\nabla v$ is regarded as a row vector. Also, for a vector $w := (w_1, w_2)^T$, its divergence $w_{1x} + w_{2y}$ is denoted by $\nabla \cdot w$ or $\text{div} w$. Note that $\nabla w$ is a $2 \times 2$ matrix,

$$
\nabla w = \begin{pmatrix} w_{1x} & w_{1y} \\ w_{2x} & w_{2y} \end{pmatrix}.
$$

Let $\Omega \subset \mathbb{R}^2$ be a (bounded) domain. The usual Lebesgue space is denoted by $L^2(\Omega)$. For a positive integer $k$, the Sobolev space $H^k(\Omega)$ is defined by $H^k(\Omega) := \{ v \in L^2(\Omega) \mid \partial^\delta v \in L^2(\Omega), |\delta| \leq k \}$. The norm and semi-norm of $H^k(\Omega)$ are defined by

$$
|v|_{k,\Omega} := \left( \sum_{|\delta| \leq k} \|\partial^\delta v\|^2_{L^2(\Omega)} \right)^{1/2}, \quad \|v\|_{k,\Omega} := \left( \sum_{0 \leq m \leq k} \|v_m\|^2_{L^2(\Omega)} \right)^{1/2}.
$$

For a $2 \times 2$ matrix $A = (a_{ij})_{i,j=1,2}$, its Frobenius norm $\|A\|_F$ is defined by $\|A\|_F^2 = \sum_{i,j=1,2} a_{ij}^2$. Then, for $w \in (H^1(\Omega))^2$,

$$
|w|^2_{1,\Omega} = \int_{\Omega} \|\nabla w\|^2_F \, dx.
$$

The inner products of $L^2(\Omega)$ and $(L^2(\Omega))^2$ are denoted by $(w,v)_{\Omega}$, $w, v \in L^2(\Omega)$, and $(w,q)_{\Omega}$, $w, q \in (L^2(\Omega))^2$. The space $H^1_0(\Omega)$ is the closure of $C_0^\infty(\Omega) \subset H^1(\Omega)$ with respect to the topology of $H^1(\Omega)$. We may use $\|\nabla \phi\|_{0,\Omega} = \|\phi\|_{1,\Omega}$ as the norm of $H^1_0(\Omega)$. Then, $H^{-1}(\Omega)$ is the dual space of $H^1_0(\Omega)$ with norm

$$
\|f\|_{-1,\Omega} := \sup_{\phi \in H^1_0(\Omega)} \frac{\langle f, \phi \rangle}{\|\nabla \phi\|_{0,\Omega}},
$$

where $\langle \cdot, \cdot \rangle$ is the duality pair of $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. We also introduce the function space

$$
H(\text{div}, \Omega) := \left\{ w \in (L^2(\Omega))^2 \mid \text{div} w \in L^2(\Omega) \right\}
$$

with norm

$$
\|w\|_{H(\text{div}, \Omega)} := \left( \|w\|^2_{0,\Omega} + \|\text{div} w\|^2_{0,\Omega} \right)^{1/2}.
$$
2.2 Model equation and its variational formulations

The weak form of (1.1) is

\[ a(u, v) := (\nabla u, \nabla v) = (f, v)_{\Omega}, \quad \forall v \in H^1_0(\Omega). \]  

(2.1)

By the Lax–Milgram lemma, there exists a unique solution \( u \in H^1_0(\Omega) \) for any \( f \in L^2(\Omega) \subset H^{-1}(\Omega) \). From the definitions, the following inequality holds:

\[ \|\nabla u\|_{0, \Omega} = \|f\|_{-1, \Omega} \leq C_{1,1} \|f\|_{0, \Omega}, \quad \forall f \in L^2(\Omega), \]  

(2.2)

where the constant \( C_{1,1} \) comes from Poincaré’s inequality on \( \Omega \).

Assumption 1 For an arbitrary \( f \in L^2(\Omega) \), the unique solution \( u \) of (1.1) belongs to \( H^2(\Omega) \), and the following inequality holds,

\[ \|u\|_{2, \Omega} \leq C_{1,2} \|f\|_{0, \Omega}, \]  

where \( C_{1,2} \) is a constant independent of \( f \).

It is well known that if \( \Omega \) is convex, then Assumption 1 is valid \([6]\).

The model equation (1.1) has a mixed variational formulation: Find \( (p, u) \in H(div, \Omega) \times L^2(\Omega) \) such that

\[ (p, q)_\Omega + (u, \text{div } q)_\Omega = 0, \quad \forall q \in H(div, \Omega), \]  

\[ (\text{div } p, v)_\Omega + (f, v)_\Omega = 0, \quad \forall v \in L^2(\Omega). \]  

(2.3)

The unique solvability of (2.3) is equivalent to the following inf-sup condition:

\[ \inf_{v \in L^2(\Omega)} \sup_{q \in H(div, \Omega)} \frac{(\text{div } q, v)_\Omega}{\|v\|_{0, \Omega} \|q\|_{H(div, \Omega)}} \geq \beta(\Omega) > 0. \]  

(2.4)

It is easy to verify that \( \beta(\Omega) := (1 + C_{1,1}^2)^{-1/2} \) satisfies the inf-sup condition (2.4). For the mixed variational formulation for the model equation (1.1), readers are referred to textbooks such as \([3]\), \([7]\), and \([8]\).

2.3 Proper triangulation and the finite element methods

Let \( K \subset \mathbb{R}^2 \) be a triangle with vertices \( x_i, \ i = 1, 2, 3; \) let \( e_i \) be the edge of \( K \) opposite to \( x_i \). We always regard \( K \) as a closed set. A proper triangulation \( T_h \) of a bounded polygonal domain \( \Omega \) is a set of triangles that satisfies the conditions,

- \( \overline{\Omega} = \bigcup_{K \in T_h} K. \)

- If \( K_1, K_2 \in T_h \) with \( K_1 \neq K_2 \), we have either \( K_1 \cap K_2 = \emptyset \) or \( K_1 \cap K_2 \) is a common vertex or a common edge.
With this definition, a proper triangulation $\mathcal{T}_h$ is sometimes called a face-to-face triangulation, and there exists no hanging nodes in $\mathcal{T}_h$. The fineness of $\mathcal{T}_h$ is indicated by $h := \max_{K \in \mathcal{T}_h} h_K$, $h_K := \text{diam} K$. We denote the set of edges in $\mathcal{T}_h$ by $\mathcal{E}_h$. We also set

$$\mathcal{E}^b_0 := \{ e \in \mathcal{E}_h \mid e \subset \partial \Omega \}, \quad \mathcal{E}^b_1 := \mathcal{E}_h \setminus \mathcal{E}^b_0.$$  

Let $e \in \mathcal{E}^b_1$ be shared by two triangles $K_1$ and $K_2$ with $e = K_1 \cap K_2$. Suppose that $v_h \in L^2(K_1 \cup K_2)$ satisfies $v_h|_K \in P_1$. Then, the jump of $v_h$ on $e$ is defined and denoted by $[v_h] := \pm (\gamma_{K_1,e}(v) - \gamma_{K_2,e}(v))$, where $\gamma_{K_i,e}(v)$ ($i = 1, 2$) is the trace operator for $v$ on $K_i$ to the edge $e_i$, and the sign is taken arbitrarily and fixed on each $e$. Note that the sign of $[v_h]$ does not affect the following definition of $S^{CR}_h$. The finite element spaces for CR FEM are defined by

$$S^{CR}_h := \left\{ v_h \in L^2(\Omega) \mid v_h|_K \in P_1, \forall K \in \mathcal{T}_h \text{ and } \int_E [v_h] ds = 0, \forall e \in \mathcal{E}^b_1 \right\},$$

$$S^{RT}_{h_0} := \left\{ v_h \in S^{CR}_h \mid \int_E v_h ds = 0, \forall e \in \mathcal{E}^b_1 \right\}.$$  

Note also that, on $e \in \mathcal{E}^b_1$, functions in $S^{CR}_h$ are continuous only at the midpoint of $e$. The CR finite element solution $u^{CR}_h \in S^{CR}_{h_0}$ for the model equation is, for $f \in L^2(\Omega)$,

$$a_h(u^{CR}_h, v_h) := \sum_{K \in \mathcal{T}_h} \int_K \nabla u^{CR}_h \cdot \nabla v_h \, dx = (f, v_h)_\Omega, \quad \forall v_h \in S^{CR}_{h_0}. \quad (2.5)$$

The norm associated with the bilinear form $a_h(\cdot, \cdot)$ is defined by $\|v_h\| := a_h(v_h, v_h)^{1/2}$ for $v_h \in S^{CR}_h$.

Regarding $x \in \mathbb{R}^2$ as variables, let $\mathcal{RT}_0 \subset (P_1)^2$ be defined by

$$\mathcal{RT}_0 := \{a x + b \mid b \in \mathbb{R}^2, a \in \mathbb{R} \} \subset (P_1)^2.$$  

For the RT finite element method, the finite element spaces $S^{RT}_h$ and $S^C_h$ are defined by

$$S^{RT}_h := \left\{ p_h \in (L^2(\Omega))^2 \mid p_h|_K \in \mathcal{RT}_0, \forall K \in \mathcal{T}_h \text{ and } p_h \in \text{div}(\Omega, \Omega) \right\},$$

$$S^C_h := \left\{ v_h \in L^2(\Omega) \mid v_h|_K \in P_0, \forall K \in \mathcal{T}_h \right\}.$$  

For a vector field $\mathbf{q} \in (H^1(\Omega))^2$, its RT interpolation $I^{RT}_K \mathbf{q}$ on each $K \in \mathcal{T}_h$ is defined by

$$\int_{e_i} (\mathbf{q} - I^{RT}_K \mathbf{q}) \cdot \mathbf{n} \, ds = 0, \quad i = 1, 2, 3,$$

where $\mathbf{n}$ is the unit outer normal vector on $\partial K$. As $\text{dim} \mathcal{RT}_0 = 3$, $I^{RT}_K \mathbf{q}$ is determined uniquely. Note also that $(I^{RT}_K \mathbf{q}) \cdot \mathbf{n}$ is a constant on each $e_i$, $i = 1, 2, 3$. Then, the global RT interpolation $I^{RT}_h \mathbf{q} \in S^{RT}_h$ is defined as $I^{RT}_h \mathbf{q}|_K = I^{RT}_K \mathbf{q}$ for each $K \in \mathcal{T}_h$. 


Similarly, we define the projection \( \pi_K^h \) on each \( K \) by

\[
\pi_K^h v := \tilde{v} := \frac{1}{K} \int_K v \, ds \quad \text{or} \quad \int_K (v - \pi_K^h v) \, ds = 0
\]

for \( v \in L^2(K) \). This projection is extended as \( \pi^h : L^2(\Omega) \to S^C_h \) by \( \pi^h v |_K = \pi_K^h v \) on each \( K \in \mathcal{T}_h \) for \( v \in L^2(\Omega) \). Note that \( \pi^h : L^2(\Omega) \to S^C_h \) is an orthogonal projection.

The RT finite element method for the mixed variational equation (2.3) is defined by

\[
\begin{align*}
(p_h, q_h)_\Omega + (u_h^{RT}, \text{div } q_h)_\Omega &= 0, & \forall q_h \in S^RT_h, \\
(\text{div } p_h, v_h)_\Omega + (f, v_h)_\Omega &= 0, & \forall v_h \in S^C_h.
\end{align*}
\] (2.6)

Note that the RT FEM is conforming because \( S_h^{RT} \times S^C_h \subset H(\text{div}, \Omega) \times L^2(\Omega) \). Therefore, we may insert \((q_h, v_h) \in S_h^{RT} \times S^C_h \) into (2.3) and take the difference between (2.3) and (2.6), which implies

\[
\begin{align*}
(p - p_h, q_h)_\Omega + (u - u_h^{RT}, \text{div } q_h)_\Omega &= 0, & \forall q_h \in S^RT_h, \\
(\text{div } (p - p_h), v_h)_\Omega &= 0, & \forall v_h \in S^C_h.
\end{align*}
\] (2.7)

In regard to the convergence of the RT finite element solution, we must consider the discrete inf-sup condition

\[
\inf_{q_h \in S^RT_h} \sup_{v_h \in S^C_h} \frac{(\text{div } q_h, v_h)_\Omega}{\|q_h\|_{L^2(\Omega)} \|v_h\|_{H(\text{div}, \Omega)}} \geq \beta_*,
\]

where \( \beta_* \) is a constant independent of \( h > 0 \). This point will be considered in Section 3.5.

### 2.4 Relationship between the CR and RT finite element methods

It is well-known that the CR and RT finite element methods are closely related. Consider the following finite element equations,

\[
\begin{align*}
\tilde{u}_h^{CR} &\in S^{CR}_h \text{ such that } a_h(\tilde{u}_h^{CR}, v_h) = (u^0_{\Omega}, f), & \forall v_h \in S^{CR}_h, \\
(\tilde{p}_h, \tilde{u}_h^{RT}) &\in S^{RT}_h \times S^C_h \text{ such that } (\tilde{p}_h, q_h)_\Omega + (\tilde{u}_h^{RT}, \text{div } q_h)_\Omega = 0, & \forall q_h \in S^{RT}_h, \\
& \text{(div } \tilde{p}_h, v_h)_\Omega + (\pi^0_{\Omega} f, v_h)_\Omega = 0, & \forall v_h \in S^C_h.
\end{align*}
\] (2.8)

Then, on each \( K \in \mathcal{T}_h \), the equalities

\[
\tilde{u}_h^{RT} = \pi_K^h \tilde{u}_h^{CR} + \frac{x_K^0}{48} \sum_{i=1}^3 |x_i - x_K^i|^2, \quad \tilde{p}_h = \nabla \tilde{u}_h^{CR} - \frac{x_K^0}{2} (x - x_K),
\] (2.10)

hold; here \( x_K := (x_1 + x_2 + x_3)/3 \) is the center of gravity of \( K \). For details, readers are referred to [11, 8, 13, 16].
2.5 Linear transformations of triangles

Let \( \tilde{K} \) be the triangle with vertices \((0, 0)^\top, (1, 0)^\top, (0, 1)^\top \). This \( \tilde{K} \) is called the reference triangle. Let \( \alpha \geq \beta > 0 \) and \( s^2 + t^2 = 1, t > 0 \). An arbitrary triangle on \( \mathbb{R}^2 \) is transformed to the triangle \( K \) with vertices \( \mathbf{x}_1 := (0, 0)^\top, \mathbf{x}_2 := (\alpha, 0)^\top, \mathbf{x}_3 := (\beta s, \beta t)^\top \) by a sequence of parallel translations, rotations, and mirror imaging. Let \( K_{\alpha\beta} \) be the triangle with vertices \((0, 0)^\top, (\alpha, 0)^\top, (0, \beta)^\top \).

We define the \( 2 \times 2 \) matrices as

\[
A := \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix}, \quad B := A^{-1} = \begin{pmatrix} 1 & st^{-1} \\ 0 & t^{-1} \end{pmatrix}, \quad D_{\alpha\beta} := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.
\]

Then, \( \tilde{K} \) is transformed to \( K \) and \( K_{\alpha\beta} \) by the transformations \( y = AD_{\alpha\beta}x \) and \( y = D_{\alpha\beta}x \), respectively. Also, \( K_{\alpha\beta} \) is transformed to \( K \) by \( y = Ax \). Moreover, any function \( w \in H^1(K) \) is pulled-back to a function \( v \in H^1(K_{\alpha\beta}) \) as \( H^1(K) \ni w \mapsto v := w \circ A \in H^1(K_{\alpha\beta}) \). A simple computation shows that \( BB^\top \) has eigenvalues \((1 \pm |s|)/t^2 \). Hence, the chain rule of differentiation implies that \( \nabla_x v = (\nabla_x w)B, |\nabla_x v|^2 = |(\nabla_x w)B|^2, \) and

\[
\frac{1 - |s|}{t^2} |\nabla_x w|^2 \leq |\nabla_x v|^2 \leq \frac{1 + |s|}{t^2} |\nabla_x w|^2.
\]

With \( \det A = t \), we have \( |v|^2_{0,K} = t|w|^2_{0,K_{\alpha\beta}} \) and \( \frac{1 - |s|}{t^2} |v|^2_{1,K_{\alpha\beta}} \leq |w|^2_{1,K_{\alpha\beta}} \). Therefore, we obtain

\[
\frac{|w|^2_{0,K}}{|w|^2_{1,K_{\alpha\beta}}} \leq \frac{t^2|v|^2_{0,K_{\alpha\beta}}}{(1 - |s|)|v|^2_{1,K_{\alpha\beta}}} = (1 + |s|) \frac{|v|^2_{0,K_{\alpha\beta}}}{|v|^2_{1,K_{\alpha\beta}}}.
\]

2.6 Piola transformation

To transform the vector fields, we need to introduce the Piola transformation induced by an affine linear transformation \( y = \varphi(x) := Ax + b \). Suppose that a triangle \( \tilde{K} \) is mapped to \( K \) as \( K := \varphi(\tilde{K}) \). Then, the Piola transformation is the pull-back of the vector field \( p(y) \) (\( y \in K \)) to a vector field \( q(x) \),

\[
q(x) := A^{-1} p(\varphi(x)) = A^{-1} p(y).
\]

By the chain rule, we have

\[
\nabla_x q(x) = A^{-1}(\nabla_y p(\varphi(x)))A.
\]

By a straightforward computation, we confirm the following lemma is valid.

**Lemma 2** Let \( A \) be a \( 2 \times 2 \) regular matrix and \( b \in \mathbb{R}^2 \). Suppose that a triangle \( \tilde{K} \subset \mathbb{R}^2 \) is transformed to \( K \) by the affine linear transformation \( \varphi(x) = Ax + b \). Let \( \tilde{e}_i \) be the edges of \( \tilde{K} \), and \( e_i := \varphi(\tilde{e}_i), (i = 1, 2, 3) \). Suppose that a vector field \( p \)
and a function $v$ on $K$ are pulled-back to $q(x) := A^{-1}p(\varphi(x))$ and $\tilde{v}(x) := v(\varphi(x))$, respectively. Then, the following equalities holds:

$$
\int_K v \text{ div } p \, d\gamma = (\text{det } A)^{-1} \int_{\tilde{K}} \tilde{v} \text{ div } q \, dx,
$$

$$
\int_K \nabla v \cdot p \, d\gamma = (\text{det } A)^{-1} \int_{\tilde{K}} \nabla \tilde{v} \cdot q \, dx,
$$

$$
\int_{\partial K} p \cdot \mathbf{n} \, ds = (\text{det } A)^{-1} \int_{\partial \tilde{K}} q \cdot \tilde{\mathbf{n}} \, ds, \quad i = 1, 2, 3,
$$

where $\mathbf{n}$ and $\tilde{\mathbf{n}}$ are the unit outer normal vectors of $K$ and $\tilde{K}$, respectively.

### 3 Error analysis of the CR and RT finite element methods

#### 3.1 Babuška–Aziz’s technique

In this section, we use the technique introduced by Babuška and Aziz to claim that squeezing the reference triangle perpendicularly does not reduce properties of the approximation of the interpolations.

Let $\Xi^{(1,0)}_2 \subset H^1(\tilde{K})$ be defined by, for \( \gamma = (1, 0) \) or \( (0, 1) \),

$$
\Xi^{(1,0)}_2 := \left\{ v \in H^1(\tilde{K}) \left| \int_0^1 v(s, 0) \, ds = \int_{\partial \tilde{K}} v \, d\gamma = 0 \right. \right\},
$$

$$
\Xi^{(0,1)}_2 := \left\{ v \in H^1(\tilde{K}) \left| \int_0^1 v(0, s) \, ds = \int_{\partial \tilde{K}} v \, d\gamma = 0 \right. \right\}.
$$

Then, the constant $A_2$ is defined by

$$
A_2 := \sup_{v \in \Xi^{(1,0)}_2} \frac{|v|^0_{0, \tilde{K}}}{|v|^1_{1, \tilde{K}}} = \sup_{v \in \Xi^{(0,1)}_2} \frac{|v|^0_{0, \tilde{K}}}{|v|^1_{1, \tilde{K}}},
$$

and called the Babuška–Aziz constant. According to Liu–Kikuchi [14], $A_2$ is the maximum positive solution of the equation $1/x + \tan(1/x) = 0$, and $A_2 \approx 0.49291$.

For the Babuška-Aziz constant, the following lemma is known.

**Lemma 3 (Babuška–Aziz)** $A_2 < \infty$.

Similarly, for $\tilde{K}$ and $K_{ab}$, we define the following sets:

$$
\Xi^{(1)}_2 := \left\{ v \in (H^1(\tilde{K}))^2 \left| \int_{\partial \tilde{K}} v \cdot \mathbf{n} \, ds = 0, \quad i = 2, 3 \right. \right\},
$$

$$
\Xi_{ab}^{(0)} := \left\{ v \in (H^1(K_{ab}))^2 \left| \int_{\partial \tilde{K}} v \cdot \mathbf{n} \, ds = 0, \quad i = 2, 3 \right. \right\}.
$$
Note that $v = (v_1, v_2)^T \in \mathcal{H}_2$ if and only if $v_1 \in \mathcal{H}_2^{(0,1)}$ and $v_2 \in \mathcal{H}_2^{(1,0)}$. We thus realize that
\[
\sup_{q \in \mathcal{H}_2} |q|_{0, K} = A_2 < \infty, \quad (3.1)
\]  
For $K \subset \mathbb{R}^2$, moreover, we define the following sets:
\[
\begin{align*}
\mathcal{X}_2^{(1)}(K) &:= \left\{ v \in H^1(K) \middle| \int_K v \, dx = 0 \right\}, \\
\mathcal{X}_2^{(2)}(K) &:= \left\{ q \in (H^1(K))^2 \middle| \int_{\partial K} q \cdot n \, ds = 0, \; i = 1, 2, 3 \right\}.
\end{align*}
\]  
From the definitions, we obviously have $\mathcal{X}_2^{(2)}(\hat{K}) \subset \mathcal{H}_2$, and $\mathcal{X}_2^{(2)}(K_{\text{opt}}) \subset \mathcal{H}_2^{(0)}$. Hence, by (12) Lemma 4.2 and (3.1), the following lemma holds.

**Lemma 4** The constants
\[
\begin{align*}
B^{(1)}_2(\hat{K}) &:= \sup_{v \in \mathcal{X}_2^{(1)}(\hat{K})} \frac{|v|_{0, \hat{K}}}{|v|_{1, \hat{K}}} < \infty, \\
B^{(2)}_2(\hat{K}) &:= \sup_{q \in \mathcal{X}_2^{(2)}(\hat{K})} \frac{|q|_{0, \hat{K}}}{|q|_{1, \hat{K}}} \leq \sup_{q \in \mathcal{H}_2} \frac{|q|_{0, \hat{K}}}{|q|_{1, \hat{K}}} = A_2 < \infty,
\end{align*}
\]  
that indicate the approximation efficiency of several interpolations on $\hat{K}$ are bounded.

Let an arbitrary $v \in \mathcal{X}_2^{(1)}(K_{\text{opt}})$ be pulled-back to $\hat{v} := v \circ D_{\text{opt}} \in \mathcal{X}_2^{(1)}(\hat{K})$. We immediately note that
\[
|v|_{0, K_{\text{opt}}}^2 = \alpha \beta |v|_{2, K_{\text{opt}}}^2, \quad |v|_{1, K_{\text{opt}}}^2 = \beta^2 |v|_{0, K_{\text{opt}}}^2, \quad |v|_{1, K_{\text{opt}}}^2 = \alpha^2 |v|_{0, K_{\text{opt}}}^2,
\]  
which yields
\[
\frac{|v|_{0, K_{\text{opt}}}^2}{|v|_{1, K_{\text{opt}}}^2} = \frac{\beta}{\alpha} |v|_{0, K_{\text{opt}}}^2 + \frac{1}{\beta^2} |v|_{0, K_{\text{opt}}}^2 \leq (\max(\alpha, \beta))^2 \frac{|v|_{0, K_{\text{opt}}}^2}{|v|_{1, K_{\text{opt}}}^2} \leq (\max(\alpha, \beta))^2 B^{(1)}_2(\hat{K})^2.
\]  
Therefore, we obtain
\[
B^{(1)}_2(K_{\text{opt}}) = \sup_{v \in \mathcal{X}_2^{(1)}(K_{\text{opt}})} \frac{|v|_{0, K_{\text{opt}}}^2}{|v|_{1, K_{\text{opt}}}^2} \leq \max(\alpha, \beta) B^{(1)}_2(\hat{K}).
\]  
Noting $\mathcal{X}_2^{(2)}(K_{\text{opt}}) \subset \mathcal{H}_2^{(0)}$, we similarly obtain
\[
B^{(2)}_2(K_{\text{opt}}) := \sup_{q \in \mathcal{X}_2^{(2)}(K_{\text{opt}})} \frac{|q|_{0, K_{\text{opt}}}^2}{|q|_{1, K_{\text{opt}}}^2} \leq \max(\alpha, \beta) \sup_{q \in \mathcal{H}_2^{(0)}} \frac{|q|_{0, \hat{K}}^2}{|q|_{1, \hat{K}}^2} = \max(\alpha, \beta) A_2.
\]  
Gathering the above inequalities, we obtain the following lemma.

**Lemma 5** The following inequalities hold:
\[
B^{(1)}_2(K_{\text{opt}}) \leq \max(\alpha, \beta) B^{(1)}_2(\hat{K}), \quad B^{(2)}_2(K_{\text{opt}}) \leq \max(\alpha, \beta) A_2.
\]  

Lemma 5 means that squeezing the reference triangle $\hat{K}$ perpendicularly does not diminish the effectiveness of the approximation through the interpolations on $K_{\text{opt}}$. 

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**Error Analysis of Crouzeix–Raviart and Raviart–Thomas Finite Element Methods**

9

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\[\begin{align}
\sup_{q \in \mathcal{H}_2} |q|_{0, K} = A_2 < \infty, \\
\sup_{v \in \mathcal{X}_2^{(1)}(\hat{K})} \frac{|v|_{0, \hat{K}}}{|v|_{1, \hat{K}}} < \infty, \\
\sup_{q \in \mathcal{X}_2^{(2)}(\hat{K})} \frac{|q|_{0, \hat{K}}}{|q|_{1, \hat{K}}} \leq \sup_{q \in \mathcal{H}_2} \frac{|q|_{0, \hat{K}}}{|q|_{1, \hat{K}}} = A_2 < \infty, \\
\frac{|v|_{0, K_{\text{opt}}}^2}{|v|_{1, K_{\text{opt}}}^2} = \frac{\beta}{\alpha} |v|_{0, K_{\text{opt}}}^2 + \frac{1}{\beta^2} |v|_{0, K_{\text{opt}}}^2 \leq (\max(\alpha, \beta))^2 \frac{|v|_{0, K_{\text{opt}}}^2}{|v|_{1, K_{\text{opt}}}^2} \leq (\max(\alpha, \beta))^2 B^{(1)}_2(\hat{K})^2.
\end{align}\]
3.2 Estimations on the general triangle.

As stated in Section 2.2, an arbitrary triangle on $\mathbb{R}^2$ is transformed to the triangle $K$ with vertices $x_1 := (0,0)^T$, $x_2 := (\alpha,0)^T$, $x_3 := (\beta s,\beta)^T$ by a sequence of parallel translations, rotations, and mirror imaging, where $s^2 + t^2 = 1$, $0 < \beta \leq \alpha$, and $t > 0$. Then, $K$ is obtained from $K_{\alpha\beta}$ by the linear transformation $y = Ax$, where $A$ is the matrix defined in (2.11). Let $w \in H^1(K)$ be pulled-back to $v \in H^1(K_{\alpha\beta})$ as $v(x) = w(Ax)$. Combining (2.12) and Lemma 5, we have

$$B_{\alpha\beta}^{(1)}(K) := \sup_{v \in X^{(2)}(K)} \frac{|v_{0,1}|}{|v_{1,1}|} \leq \sqrt{2} \sup_{v \in X^{(2)}(K_{\alpha\beta})} \frac{|v_{0,1,K_{\alpha\beta}}|}{|v_{1,1,K_{\alpha\beta}}|} \leq \sqrt{2} h_K B_{\alpha\beta}^{(1)}(\hat{K}).$$

Here, we used the assumption $0 < \beta \leq \alpha \leq h_K := \text{diam}K$.

To consider $B_{\alpha\beta}^{(2)}(K)$, we need to introduce the Piola transformation $q(x) := A^{-1}p(Ax)$ for $p \in X^{(2)}(K_{\alpha\beta})$. By Lemma 2, we have $q \in X^{(2)}(K_{\alpha\beta})$ and

$$|p|_{0,K}^2 = \int_K |p(y)|^2 dy = (\det A) \int_{K_{\alpha\beta}} |p(Ax)|^2 dx \quad = (\det A) \int_{K_{\alpha\beta}} |Aq(x)|^2 dx \leq (1 + |s|) t \int_{K_{\alpha\beta}} |q(x)|^2 dx \leq (1 + |s|) \sup_{K_{\alpha\beta}} |q|^2_{0,K_{\alpha\beta}}.$$

In the above inequalities, we used the fact that the singular values of $A$ are $(1 \pm |s|)^{1/2}$ and $(1 - |s|)|q|^2 \leq |Aq|^2 \leq (1 + |s|)|q|^2$.

Let $Y$ and $Z$ be $2 \times 2$ matrices with $Y := AXA^{-1}$. Given the singular values of $A$ and $A^{-1}$, we have

$$\|Y\|_{F}^2 = \|AXA^{-1}\|_{F}^2 \geq (1 - |s|)|XA^{-1}|_{F}^2 \geq \frac{(1 - |s|)^2}{t^2} \|X\|_{F}^2.$$

Hence, it follows from $\nabla_Y q(x) = A^{-1} \nabla_Y p(Ax) A$ that

$$|p|_{1,K}^2 = \int_K |\nabla_Y p(y)|^2 dy = (\det A) \int_{K_{\alpha\beta}} |\nabla_Y p(Ax)|^2 dx \quad = t \int_{K_{\alpha\beta}} |A(\nabla_Y q(x))A^{-1}|_{F}^2 dx \geq \frac{(1 - |s|)^2}{t} \|q^2_{0,K_{\alpha\beta}}.

We thus obtain

$$|p|_{2,K}^2 \leq \frac{(1 + |s|)^2}{2} \|q|_{0,K_{\alpha\beta}}^2 \quad \frac{(1 + |s|)^2}{2} \|q|_{1,K_{\alpha\beta}}^2 \leq \frac{(1 + |s|)^2}{1 - |s|} \|q|_{1,K_{\alpha\beta}}^2.$$

Assuming that the edge connecting $x_2$ and $x_3$ is the longest edge of $K$, the following inequality holds [10, Lemma 3.2]:

$$\frac{1 + |s|}{\sqrt{1 - |s|}} \leq 4 \frac{\sqrt{2} R_K}{\alpha},$$

(3.2)
To show this inequality, we first confirm that the following inequality is valid:

\[ \sqrt{1 + |s|} \leq \sqrt{2} \sqrt{1 + \gamma^2 - 2\gamma s}, \quad 0 < \forall y \leq 1, \quad -1 < \forall s \leq \frac{\gamma}{2}. \]

We then insert \( \gamma := \beta/\alpha \) and use the laws of sines and cosines.

Combining (3.2) with Lemma 5 we obtain

\[ B_{2}^{(2)}(K) := \sup_{p \in X_{h}(K)} \frac{|p|_{0,K}}{|p|_{1,K}} \leq \frac{1 + |s|}{\sqrt{1 - |s|^2}} \sup_{q \in X_{h}(K)} \frac{|q|_{0,K}}{|q|_{1,K}} \leq 4 \sqrt[4]{\frac{R_{K}}{\alpha}} B_{2}^{(2)}(K_{\alpha\beta}) \leq 4 \sqrt{2} R_{K} A_{2}. \]

**Theorem 6** Let \( K \) be the triangle with vertices \( x_{1} := (0,0)^{T}, x_{2} := (\alpha,0)^{T}, x_{3} := (\beta\alpha,\beta\beta)^{T} \) such that \( 0 < \beta \leq \alpha, \quad s^{2} + t^{2} = 1, \quad t > 0 \). Suppose that the edge connecting \( x_{2} \) and \( x_{3} \) is the longest edge of \( K \). Then, there exist positive constants \( C_{2}^{(i)}, \quad i = 1, 2 \) that are independent of \( K \) such that the following estimates hold:

\[ B_{2}^{(1)}(K) := \sup_{v \in X_{h}^{(1)}(K)} \frac{|v|_{0,K}}{|v|_{1,K}} \leq C_{2}^{(1)} h_{K}, \quad B_{2}^{(2)}(K) := \sup_{q \in X_{h}^{(2)}(K)} \frac{|q|_{0,K}}{|q|_{1,K}} \leq C_{2}^{(2)} R_{K}, \]

where \( h_{K} := \text{diam}K \) and \( R_{K} \) is the circumradius of \( K \).

**Remark:** Because \( A_{2} = 0.49291 \), we have \( C_{2}^{(2)} \approx 2.7883 \).

**Remark:** We present an example for the error estimates of \( B_{2}^{(2)}(K) \). Let \( K \) be the triangle with vertices \( (1,0)^{T}, (-1,0)^{T}, \) and \( (0,h)^{T} \). Let

\[ u := xy - \frac{1 + h^2}{2h} x, \quad q := \nabla u = \left( y - \frac{1 + h^2}{2h} \right). \]

Then, it is straightforward to verify

\[ \int_{\partial K} q \cdot nds = 0, \quad i = 1, 2, 3, \]

that is, \( \int_{\partial K} q \cdot nds = 0, \) and

\[ |q - I_{K}^{RT} q|_{0,K}^{2} = \frac{(3 + h^2)(1 + h^2)}{12h}, \quad |q - I_{K}^{RT} q|_{1,K}^{2} = 2h, \]

\[ \frac{|q - I_{K}^{RT} q|_{0,K}}{|q - I_{K}^{RT} q|_{1,K}} \geq \frac{1 + h^2}{2 \sqrt{6} h} = \frac{R_{K}}{2 \sqrt{6}}. \]

Because \( q - I_{K}^{RT} q \in X_{h}^{(2)} \), this inequality means that the estimate \( B_{2}^{(2)}(K) \leq C_{2}^{(2)} R_{K} \) in Theorem 6 cannot be further improved; that is, the parameter \( R_{K} \) is the best possible parameter to measure the convergence of solutions.
3.3 Poincaré–Wirtinger’s inequality on triangles

We consider the error analysis of the projection \( \pi_K^0 : L^2(K) \to P_0 \) on \( K \) and its extension to \( \pi_\Omega^0 : L^p(\Omega) \to S^2_h \) on \( \Omega \).

From the definition, we have \( \int_K (f(x) - \bar{f}) \, dx = 0 \), and therefore \( f - \bar{f} \in X^{(1)}_2(K) \). Hence, we obtain the following theorem from Theorem 6. See also [12, Corollary 4.4].

**Theorem 7**

\[
\| f - \pi_K^0 f \|_{0,K} \leq C_2^{(1)} h_K |f|_{1,K}, \quad \forall f \in H^1(K), \quad \forall K \in \mathcal{T}_h,
\]

\[
\| f - \pi_\Omega^0 f \|_{0,\Omega} \leq C_2^{(1)} h |f|_{1,\Omega}, \quad \forall f \in H^1(\Omega),
\]

where \( h := \max_{K \in \mathcal{T}_h} h_K \), and \( C_2^{(1)} \) is the constant appearing in Theorem 6.

**Corollary 8** For an arbitrary proper triangulation \( \mathcal{T}_h \) of a bounded polygonal domain \( \Omega \), the following estimation holds:

\[
\| f - \pi_\Omega^0 f \|_{-1,\Omega} \leq C_2^{(1)} h |f|_{0,\Omega}, \quad \forall f \in L^2(\Omega),
\]

where \( C_2^{(1)} \) is the constant appearing in Theorem 6.

**Proof** Because the projection \( \pi_\Omega^0 : L^2(K) \to S^2_h \) is orthogonal, we have, for an arbitrary \( \phi \in H^1_0(\Omega) \),

\[
\left( f - \pi_\Omega^0 f, \phi \right)_\Omega = \left( f - \pi_\Omega^0 f, \phi - \pi_\Omega^0 \phi \right)_\Omega \\
\quad \leq \| f - \pi_\Omega^0 f \|_{0,\Omega} \| \phi - \pi_\Omega^0 \phi \|_{0,\Omega} \leq C_2^{(1)} h |f|_{0,\Omega} \| \nabla \phi \|_{0,\Omega},
\]

and

\[
\| f - \pi_\Omega^0 f \|_{-1,\Omega} := \sup_{\phi \in H^1_0(\Omega)} \frac{\left( f - \pi_\Omega^0 f, \phi \right)_\Omega}{\| \nabla \phi \|_{0,\Omega}} \leq C_2^{(1)} h |f|_{0,\Omega}.
\]

In the above inequality, we used the fact \( |f - \pi_\Omega^0 f|_{0,\Omega} \leq |f|_{0,\Omega} \). □

The important feature in the above Theorem 7 and Corollary 8 is that constant \( C_2^{(1)} \) does not depend on the geometry of the triangulation \( \mathcal{T}_h \) at all.

3.4 Error analysis of the RT interpolation

By definition, the RT interpolation \( I^{RT}_K \) satisfies

\[
\mathbf{q} - I^{RT}_K \mathbf{q} \in X^{(1)}_2(K), \quad \mathbf{q} \in (H^1(\Omega))^2
\]
on each triangle $K \in \mathcal{T}_h$. Therefore, it follows from Theorem 6 that, for each $K \in \mathcal{T}_h$,

$$|q - I_{RT}^K q|_{0,K} \leq C^{(2)}_2 R_K |q - I_{RT}^K q|_{1,K}. \quad (3.3)$$

Moreover, the definition of $X^{(2)}_2(K)$ and the divergence theorem yield

$$\int_K \text{div}(q - I_{RT}^K q) \, dx = \int_{\partial K} (q - I_{RT}^K q) \cdot \mathbf{n} \, ds = 0. \quad (3.4)$$

Note that $\text{div}(I_{RT}^K q) \in P_0$ because $I_{RT}^K q \in RT_0$. Hence, we realize that

$$\text{div}(I_{RT}^K q) = \frac{1}{|K|} \int_K \text{div} q \, dx = \pi_0^K(\text{div} q). \quad (3.5)$$

Setting constant $a_q$ to $a_q := \pi_0^K(\text{div} q)/2$, we then have

$$(I_{RT}^K q)(x) = a_q x + b_q, \quad b_q \in \mathbb{R}^2.$$ 

Therefore, for $q(x) = (q_1(x), q_2(x))^T$ and $x = (x, y)^T$, we have

$$|q - I_{RT}^K q|_{1,K}^2 = |q|^2_{1,K} + 2 \int_K a_q^2 dx - 2a_q \int_K \text{div} q \, dx
= |q|^2_{1,K} + 2 \int_K a_q^2 dx - 2a_q \int_K \text{div} (I_{RT}^K q) \, dx
= |q|^2_{1,K} - 2 \int_K a_q^2 dx \leq |q|^2_{1,K}.$$

In the above inequalities, we used the equality (3.4). Gathering (3.3), (3.5), Theorem 7, and the above inequality, we obtain the following theorem.

**Theorem 9** For an arbitrary $q \in (H^1(\Omega))^2$ with $\text{div} q \in H^1(\Omega)$, the following estimates hold:

$$\begin{align*}
|q - I_{RT}^K q|_{0,K} \leq C^{(2)}_2 R_K |q|_{1,K}, & \quad \text{div} q - \text{div} I_{RT}^K q|_{0,K} \leq C^{(2)}_2 h_K |\text{div} q|_{1,K}, \\
|q - I_{RT}^h q|_{0,\Omega} \leq C^{(2)}_2 R |q|_{1,\Omega}, & \quad \text{div} q - \text{div} I_{RT}^h q|_{0,\Omega} \leq C^{(2)}_2 h |\text{div} q|_{1,\Omega},
\end{align*}$$

where $R := \max_{K \in \mathcal{T}_h} R_K$, $h := \max_{K \in \mathcal{T}_h} h_K$, and $C^{(i)}_2$, $i = 1, 2$ are the constants appearing in Theorem 6 that are independent of $\mathcal{T}_h$ and $q$.

### 3.5 Discrete inf-sup condition for the RT finite elements

Following Mao–Shi [15], we now discuss the discrete inf-sup condition for the RT finite elements. Take $v_h \in S_h^C$ as arbitrary, and consider the following Poisson problem:

$$-\Delta w = v_h \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega.$$
Let \( q := -\nabla w \). Suppose that Assumption 1 holds and \( C^2 R \leq 1 \). Then, recalling that \( \text{div}(I_h^{RT}q) = \pi_w^0(\text{div} q) \), it follows from Theorems 7 and 9 that

\[
\|I_h^{RT}q\|_{H(\text{div},\Omega)} = \left( \|I_h^{RT}q\|_{0,\Omega}^2 + |\text{div}(I_h^{RT}q)|^2_{0,\Omega} \right)^{1/2} \\
\leq (2)|q|_{0,\Omega}^2 + 2|q - I_h^{RT}q|_{0,\Omega}^2 + |\text{div}(I_h^{RT}q)|_{0,\Omega}^2 \\
\leq (2)|q|_{0,\Omega}^2 + 2C^2 R^2|q|^2_{1,\Omega} + |\text{div}q|_{0,\Omega}^2 \\
\leq (2)|w|_{0,\Omega}^2 + |v_h|_{0,\Omega}^2)^{1/2} \leq (2 + C^2)\|v_h\|_{0,\Omega}.
\]

Moreover, we realize that

\[
\left( \text{div}(I_h^{RT}q), v_h \right)_\Omega = \left( \pi_w^0(\text{div} q), v_h \right)_\Omega = (\text{div} q, v_h)_\Omega = (v_h, v_h)_\Omega = |v_h|_{0,\Omega}^2,
\]

and therefore,

\[
\frac{\left( \text{div}(I_h^{RT}q), v_h \right)_\Omega}{\|I_h^{RT}q\|_{H(\text{div},\Omega)}} \geq \frac{|v_h|_{0,\Omega}}{(2 + C^2)^{1/2}}.
\]

Hence, we finally conclude that

\[
\sup_{q_h \in S_h^{RT}} \frac{(\text{div} q_h, v_h)_\Omega}{\|q_h\|_{H(\text{div},\Omega)}} \geq \frac{|v_h|_{0,\Omega}}{(2 + C^2)^{1/2}},
\]

and obtain the following theorem.

**Theorem 10.** On the regularity of the solutions of the model problem, we impose Assumption 1. Then, for a triangulation \( T_h \) of \( \Omega \) that satisfies \( C^2 R \leq 1 \) with \( R := \max_{K \in T_h} R_K \), the following discrete inf-sup condition holds:

\[
\inf_{v_h \in S_h^C} \sup_{q_h \in S_h^{RT}} \frac{(\text{div} q_h, v_h)_\Omega}{|v_h|_{0,\Omega}\|q_h\|_{H(\text{div},\Omega)}} \geq \frac{1}{(2 + C^2)^{1/2}} \beta_*.
\]

Here, \( C^2 \) is the constant appearing in Theorem 6 and \( C_{1,2} \) is the constant appearing in Assumption 1.

### 3.6 Error analysis of the RT finite element method

Because of the inclusion \( S_h^{RT} \times S_h^C \subset H(\text{div},\Omega) \times L^2(\Omega) \), the RT finite element method (2.6) for the mixed variational formulation (2.3) is conforming, and the following Cea’s-lemma-type estimation is known [5, Lemma 2.44].
Theorem 11 Let \((p, u) \in H(\text{div}, \Omega) \times L^2(\Omega)\) be the exact solution of (2.3), and \((p_h, u_h^{RT}) \in S_h^{RT} \times S_h^C\) be the solution of the RT finite element method (2.6). Suppose that Assumption 1 holds and \(C_{2}^{12}R \leq 1\). Then, we have the following error estimations:

\[
\|p - p_h\|_{0, \Omega} \leq 2(1 + \beta_h^{-1}) \inf_{q_h \in S_h^C} \|p - q_h\|_{H(\text{div}, \Omega)},
\]

\[
|u - u_h^{RT}|_{0, \Omega} \leq (1 + \beta_h^{-1}) \inf_{w_h \in S_h^C} |u - w_h|_{0, \Omega} + \beta_h^{-1} \inf_{q_h \in S_h^C} |p - q_h|_{0, \Omega},
\]

where \(\beta_h\) is the constant of the discrete inf-sup condition appearing in (3.6).

Gathering Theorems 7, 9, 10, and 11, we immediately obtain the following corollary.

Corollary 12 Under the assumptions of Theorem 7, we have the following error estimations:

\[
\|p - p_h\|_{0, \Omega} \leq CR|f|_{0, \Omega}, \quad |u - u_h^{RT}|_{0, \Omega} \leq C(R + h)|f|_{0, \Omega},
\]

where constant \(C\) depends on \(C_{2}^{(i)}\), \(i = 1, 2\) and \(C_{12}\), but is independent of \(h, R, f, \) and the geometric properties of \(T_h\).

3.7 Error analysis of the CR finite element method

In this section, we estimate the error \(\|u - u_h^{CR}\|_h\), where \(u \in H^2(\Omega) \cap H_0^0(\Omega)\) is the exact solution of (2.1), and \(u_h^{CR} \in S_h^{CR}\) is the CR finite element solution defined by (2.5). We impose Assumption 1.

We introduce the following auxiliary equations: for \(f \in L^2(\Omega)\),

\[-\Delta \bar{u} = \pi_0^0 f \quad \text{in} \ \Omega, \quad \bar{u} = 0 \quad \text{on} \ \partial\Omega.\]

The CR FEM for this equation is defined by (2.8). Note that \(u - \bar{u}\) satisfies

\[a(u - \bar{u}, v) = (f - \pi_0^0 f, v)_\Omega, \quad \forall v \in H_0^1(\Omega).\]

Therefore, from (2.2) and the Poincaré–Wirtinger inequality (Corollary 8), we have

\[|u - \bar{u}|_{1, \Omega} \leq C_{1,1} \left\|f - \pi_0^0 f\right\|_{-1, \Omega} \leq C_{1,1}C_{2}^{(1)}|f|_{0, \Omega}. \quad (3.7)\]

Because \(\pi_0^0 : L^2(\Omega) \rightarrow S_h^C\) is an orthogonal projection, we have, for an arbitrary \(v_h \in S_h^{CR}\),

\[
(f - \pi_0^0 f, v_h)_\Omega = (f - \pi_0^0 f, v_h - \pi_0^0 v_h)_\Omega, \quad \left|f - \pi_0^0 f\right|_{0, \Omega} \leq |f|_{0, \Omega}.
\]

Hence, Poincaré–Wirtinger’s inequality (Theorem 7) yields

\[
\left|v_h - \pi_0^0 v_h\right|_{0, \Omega}^2 = \sum_{K \in T_h} \int_K \left|v_h - \pi_0^0 v_h\right|^2 \, dx \leq C_{2}^{(1)} \sum_{K \in T_h} h_K^2 \int_K |\nabla v_h|^2 \, dx \leq C_{2}^{(1)}h^2\|v_h\|_h^2.
\]
Therefore, we finally conclude
\[ \| \tilde{u}_h - \tilde{u}_h^{CR} \|_h \leq C(R + h) \| f \|_{L^2(\Omega)}. \tag{3.9} \]

Gathering the estimations (3.7), (3.8), (3.9) with the triangle inequality, we obtain the following theorem.
Theorem 13  Let \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) be the exact solution of (1.1), and \( u_h^{CR} \in S_h^{CR} \) be the CR finite element solution (2.5). Suppose that Assumption 1 holds and \( C_2^{(2)} R \leq 1 \). Then, we have the following error estimations:
\[
\| u - u_h^{CR} \|_h \leq C(R + h) \| f \|_{H^1(\Omega)},
\]
where constant \( C \) depends on \( C^{(i)}_2, i = 1, 2, C_{1,1}, \) and \( C_{1,2} \), but is independent of \( h, R, f, \) and the geometric properties of \( \mathcal{T}_h \).

4 Numerical experiments

In this section, we present the results of numerical experiments that confirm the obtained error estimations. Let \( \Omega := (0, 1) \times (0, 1) \). We compute the \( P_1 \) Lagrange and the CR finite element solutions, \( u_h^L \) and \( u_h^{CR} \), respectively, for the model problem
\[
-\Delta u = 2x(1 - x) + 2y(1 - y) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]
which has the exact solution \( u = x(1 - x)y(1 - y) \).

To this end, we triangulate \( \Omega \) with triangles of height \( 1/N \) and baseline length \( 1/M \) (see Figure 1), with a positive integer \( M \) and a positive and even integer \( N \). The triangulation has \((2M + 1)N\) elements, and the numbers of freedom are \( MN + M + 3N/2 + 1 \) and \( 3MN + M + 5N/2 \) for the \( P_1 \) Lagrange and the CR elements, respectively. In Figure 2, we give the finite element solutions obtained.

![Fig. 1 Triangulation used for \( \Omega \) with \( M = 4 \) and \( N = 20 \). The dots on the left indicate the degrees of freedom of the \( P_1 \) Lagrange elements, and those on the right indicate the degrees of freedom of the CR elements.](image)

Setting \( N \) to be the closest even integer to \( M^\alpha \) with \( \alpha = 1.5 \), we compute the error of \( |u - u_h^L|_{1,\Omega} \) and \( |u - u_h^{CR}|_h \) for various \( M \). The results are given in Tables 1 and 2. Note that because \( C_2^{(2)} \approx 2.7883 \) as stated before, the condition \( C_2^{(2)} R \leq 1 \) is satisfied if \( R \leq 0.3586 \). We clearly see that the behavior of the errors is consistent with [9].
Fig. 2 Numerical solutions: \( P_1 \) Lagrange elements (left), CR elements (right). Note that the values of the finite element solutions are somewhat exaggerated.

Theorem 3] and [10, Theorem 1.1] for \( P_1 \) Lagrange elements, and Theorem 13 for the CR finite elements. In particular, we emphasize that \(|u - u_h|_{1,\Omega}/R\) and \(\|u - u_h^{CR}\|_h/R\) look stable, whereas \(|u - u_h|_{1,\Omega}/h\) and \(\|u - u_h^{CR}\|_h/h\) seem to diverge. This means that the error estimations in terms of the circumradius \(R\) are essential and the best possible.

Table 1 Error \(|u - u_h^{L,2}|\) with various \(M\) and \(N \approx M^{1.5}\).

| \(M\) | \(N\) | \(h\) | \(R\) | \(|u - u_h^{L,2}|_{1,\Omega}\) | \(|u - u_h^{L,2}|_{1,\Omega}/h\) | \(|u - u_h^{L,2}|_{1,\Omega}/R\) |
|---|---|---|---|---|---|---|
| 10 | 32 | 0.1000 | 0.1756 | 0.0167277 | 0.1672776 | 0.0952370 |
| 20 | 90 | 0.0500 | 0.1180 | 0.0108233 | 0.2164462 | 0.0916713 |
| 30 | 164 | 0.0333 | 0.0941 | 0.0085646 | 0.2569403 | 0.0905888 |
| 40 | 252 | 0.0250 | 0.0807 | 0.0073229 | 0.2929178 | 0.0907044 |
| 50 | 354 | 0.0200 | 0.0722 | 0.0065410 | 0.3270520 | 0.0905805 |
| 60 | 464 | 0.0167 | 0.0655 | 0.0059329 | 0.3559770 | 0.0905489 |
| 70 | 586 | 0.0142 | 0.0606 | 0.0054905 | 0.3843359 | 0.0905290 |
| 80 | 716 | 0.0125 | 0.0566 | 0.0051271 | 0.4101745 | 0.0905289 |
| 90 | 854 | 0.0111 | 0.0533 | 0.0048257 | 0.4343167 | 0.0905366 |
| 100 | 1000 | 0.0100 | 0.0505 | 0.0045726 | 0.4572635 | 0.0905472 |

Table 2 Error \(\|u - u_h^{CR}\|_h\) with various \(M\) and \(N \approx M^{1.5}\).

| \(M\) | \(N\) | \(h\) | \(R\) | \(\|u - u_h^{CR}|_{1,\Omega}\) | \(\|u - u_h^{CR}|_{1,\Omega}/h\) | \(\|u - u_h^{CR}|_{1,\Omega}/R\) |
|---|---|---|---|---|---|---|
| 10 | 32 | 0.1000 | 0.1756 | 0.0167791 | 0.1677918 | 0.0953598 |
| 20 | 90 | 0.0500 | 0.1180 | 0.0104671 | 0.2093425 | 0.0908627 |
| 30 | 164 | 0.0333 | 0.0941 | 0.0081263 | 0.2440346 | 0.0863037 |
| 40 | 252 | 0.0250 | 0.0807 | 0.0068669 | 0.2746769 | 0.0850560 |
| 50 | 354 | 0.0200 | 0.0722 | 0.0059329 | 0.3041381 | 0.0842343 |
| 60 | 464 | 0.0167 | 0.0655 | 0.0054905 | 0.3343359 | 0.0838012 |
| 70 | 586 | 0.0142 | 0.0606 | 0.0051271 | 0.3543016 | 0.0834546 |
| 80 | 716 | 0.0125 | 0.0566 | 0.0048257 | 0.3770886 | 0.0832266 |
| 90 | 854 | 0.0111 | 0.0533 | 0.0045726 | 0.3984651 | 0.0830631 |
| 100 | 1000 | 0.0100 | 0.0505 | 0.0043183 | 0.4188380 | 0.0829382 |
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