Well-posedness of non-autonomous semilinear input-output systems

Jochen Schmid\textsuperscript{1,2}

\textsuperscript{1} Institut für Mathematik, Universität Würzburg, 97074 Würzburg, Germany
\textsuperscript{2} Fraunhofer Institute for Industrial Mathematics (ITWM), 67663 Kaiserslautern, Germany
jochen.schmid@itwm.fraunhofer.de

We establish well-posedness results for non-autonomous semilinear input-output systems, the central assumption being the scattering-passivity of the considered semilinear system. We consider both systems with distributed control and observation and systems with boundary control and observation. Applications are given to nonlinearly controlled collocated systems and to nonlinearly controlled port-Hamiltonian systems.

Index terms: Well-posedness, non-autonomous systems, nonlinear systems, infinite-dimensional systems, generalized solutions and outputs

1 Introduction

In this paper, we establish well-posedness results for non-autonomous semilinear input-output systems whose input and output operators are linear. We consider semilinear systems with distributed control and observation described by

\[ \dot{x}(t) = A(t)x(t) + f(t, x(t)) + B(t)u(t) \]
\[ y(t) = C(t)x(t), \]

(1.1)

and semilinear systems with boundary control and observation described by

\[ \dot{x}(t) = A(t)x(t) + f(t, x(t)) \]
\[ u(t) = B(t)x(t) \quad \text{and} \quad y(t) = C(t)x(t). \]

(1.2)

In these equations, \( x(t) \in X \) is the state of the system at time \( t \) (\( X \) being a Banach space) and \( u, y \) are the control input and observation output of the system taking values in an input-value and an output-value space \( U, Y \) (Banach spaces) respectively. Also,

\[ A(t) : D(A(t)) \subset X \to X \]
is a linear operator and \( f : \mathbb{R}_0^+ \times X \to X \) is a time-dependent nonlinearity. And finally, in the case (1.1) of distributed control and observation, the input and output operators

\[
B(t) : U \to X \quad \text{and} \quad C(t) : X \to Y \tag{1.3}
\]

are bounded linear operators and, in the case (1.2) of boundary control and observation, the input and output operators

\[
B(t) : D(B(t)) \subset X \to U \quad \text{and} \quad C(t) : D(C(t)) \subset X \to Y \tag{1.4}
\]

are unbounded linear operators. What we are interested in here is the well-posedness of non-autonomous semilinear systems as above. In rough terms, this means that for every initial state \( x_0 \in X \) and every input \( u \in L^2_{\text{loc}}(\mathbb{R}^+_0, U) \) the respective system has a unique generalized solution \( x(\cdot, x_0, u) \in C(\mathbb{R}_0^+, X) \) and a unique generalized output \( y(\cdot, x_0, u) \in L^2_{\text{loc}}(\mathbb{R}_0^+, Y) \) and that these quantities depend continuously on \( (x_0, u) \in X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U) \).

2 Some preliminaries

2.1 Solution concepts and well-posedness

A classical solution to (1.1) or (1.2) for given initial state \( x_0 \in X \) and input \( u \in L^2_{\text{loc}}(\mathbb{R}^+_0, U) \) is a function \( x \in C^1(J, X) \) on some interval \( J \ni 0 \) such that \( x(0) = x_0 \) and such that, for every \( t \in J \),

- \( x(t) \in D(A(t)) \) and (1.1) is satisfied, or
- \( x(t) \in D(A(t)) \cap D(B(t)) \cap D(C(t)) \) and (1.2) is satisfied,

respectively. A generalized solution and a generalized output to (1.1) or (1.2) for given initial state \( x_0 \in X \) and input \( u \in L^2_{\text{loc}}(\mathbb{R}^+_0, U) \) is a function

\[
x \in C(\mathbb{R}_0^+, X) \quad \text{and} \quad y \in L^2_{\text{loc}}(\mathbb{R}_0^+, Y)
\]

such that there exists a sequence \((x_{0n}, u_n)\) of initial states and inputs which converge to \((x_0, u)\):

\[
(x_{0n}, u_n) \xrightarrow{X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U)} (x_0, u),
\]

and for which the system (1.1) or (1.2) has a unique global classical solution \( x(\cdot, x_0, u_n) \) satisfying

\[
x(\cdot, x_0, u_n) \xrightarrow{C(\mathbb{R}_0^+, X)} x \quad \text{and} \quad C(\cdot)x(\cdot, x_0, u_n) \xrightarrow{L^2_{\text{loc}}(\mathbb{R}_0^+, Y)} y.
\]

All the above convergences are w.r.t. the canonical locally convex topologies. Well-posedness of the system (1.1) or (1.2) now means that, for every initial state \( x_0 \in X \) and
every input $u \in L^2_{\text{loc}}(\mathbb{R}^+_0, U)$, the system has a unique generalized solution and generalized output

$$x(\cdot, x_0, u) \in C(\mathbb{R}^+_0, X) \quad \text{and} \quad y(\cdot, x_0, u) \in L^2_{\text{loc}}(\mathbb{R}^+_0, Y)$$

respectively, and that these quantities depend continuously on $(x_0, u)$, that is, the functions

$$(x_0, u) \mapsto x(\cdot, x_0, u) \in C(\mathbb{R}^+_0, X) \quad \text{and} \quad (x_0, u) \mapsto y(\cdot, x_0, u) \in L^2_{\text{loc}}(\mathbb{R}^+_0, Y)$$

are continuous w.r.t. the canonical locally convex topologies.

### 2.2 Semilinear systems without control or observation

We now collect some preliminaries on the solvability of semilinear evolution equations

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)) \quad (2.1)$$

without control inputs or observation outputs, which will be repeatedly used in the sequel.

We recall that a family $A$ of operators $A(t) : D(A(t)) \subset X \to X$ with $t \in \mathbb{R}^+$ is called **locally Kato-stable** [12], [15] iff $A(t)$ is a semigroup generator on $X$ for every $t \in \mathbb{R}^+_0$ and for every $t_0 \in (0, \infty)$ there exist constants $M_{t_0} \in [1, \infty)$ and $\omega_{t_0} \in \mathbb{R}$ such that

$$\left\| e^{A(t_0)s_n} \cdots e^{A(t_1)s_1} \right\| \leq M_{t_0} e^{\omega_{t_0}(s_1 + \cdots + s_n)} \quad (2.2)$$

for all $s_1, \ldots, s_n \in \mathbb{R}^+_0$ and all $t_1, \ldots, t_n \in [0, t_0]$ satisfying $t_1 \leq \cdots \leq t_n$ with arbitrary $n \in \mathbb{N}$.

**Assumption 2.1.** $A(t) = A_0(t)M(t)$ for $t \in \mathbb{R}^+_0$, where $A_0(t) : D(A_0(t)) \subset X \to X$ are linear operators with time-independent domains $D(A_0(t)) = D_0$ and where $M(t) \in L(X)$ are bijective onto $X$ such that

- the family $MA_0$ consisting of the operators $M(t)A_0(t)$ is locally Kato-stable
- $t \mapsto A_0(t)x$ is continuously differentiable for every $x \in D_0$ and $t \mapsto M(t)$ is twice strongly continuously differentiable.

A simple sufficient condition for the above assumption to hold is provided by the following lemma. See Example 2.6 of [9].

**Lemma 2.2.** Suppose that $X$ is a Hilbert space and $A(t) = A_0(t)M(t)$ for $t \in \mathbb{R}^+_0$, where

- $A_0(t) : D(A_0(t)) \subset X \to X$ are contraction semigroup generators on $X$ with time-independent domains $D(A_0(t)) = D_0$ and $t \mapsto A_0(t)x$ is continuously differentiable
- $M(t) \in L(X)$ are symmetric and there exist constants $\underline{m}, \overline{m} \in (0, \infty)$ such that

$$\underline{m} \leq M(t) \leq \overline{m} \quad (t \in \mathbb{R}^+_0), \quad (2.3)$$

and $t \mapsto M(t)$ is twice strongly continuously differentiable.
Assumption 2.1 is then satisfied.

Proof. We have obviously only to prove the local Kato-stability of the family $MA_0$. In order to see that $M(t)A_0(t)$ is a semigroup generator on $X$ for every $t \in \mathbb{R}_0^+$, one can argue as in the proof of Lemma 7.2.3 of [8] and in order to see that the semigroups generated by the operators $M(t)A_0(t)$ satisfy estimates of the form (2.2), one can argue as in the middle part of the proof of Proposition 2.3 from [24].

In the next result, we discuss the classical solvability of the linear problem

$$\dot{x}(t) = A(t)x(t)$$

(2.4)

corresponding to (2.1), that is, of (2.1) with $f(t, x) \equiv 0$. It is based on standard results [11], [13], [12] for non-autonomous linear evolution equations and slightly extends a solvability result from [24] (Proposition 2.8(a)), where the Hilbert space setting from Lemma 2.2 above is assumed. It is most conveniently formulated in terms of (solving) evolution systems. A (solving) evolution system for $A$ on the spaces $D(A(s))$ [4], [20] is, by definition, a family $T$ of bounded operators $T(t, s) \in L(X)$ for $(s, t) \in \Delta := \{(s, t) \in (\mathbb{R}_0^+)^2 : s \leq t\}$ such that

(i) for every $s \in \mathbb{R}_0^+$ and $x_s \in D(A(s))$, the map $[s, \infty) \ni t \mapsto T(t, s)x_s$ is a classical solution of (2.4)

(ii) $T(t, s)T(t, r) = T(t, r)$ for all $(r, s), (s, t) \in \Delta$ and $\Delta \ni (s, t) \mapsto T(t, s)$ is strongly continuous.

Lemma 2.3. Suppose that $A(t) = A_0(t)M(t)$ are operators as in Assumption 2.1. Then there exists a unique evolution system $T$ for $A$ on the spaces $D(A(s))$ and for every $t_0 \in (0, \infty)$ there exist constants $M_{t_0} \in [1, \infty)$ and $\omega_{t_0} \in \mathbb{R}$ such that

$$\|T(t, s)\| \leq M_{t_0}e^{\omega_{t_0}(t-s)} \quad ((s, t) \in \Delta_{[0,t_0]} := \Delta \cap [0,t_0]^2).$$

(2.5)

Proof. We have only to observe that the operator

$$A(t) = A_0(t)M(t) = M(t)^{-1}(M(t)A_0(t))M(t)$$

is similar to the operator $M(t)A_0(t)$ and then to combine – in exactly the same way as in the proof of Corollary 2.1.10 of [20] – some standard results for non-autonomous linear evolution equations. We reproduce the arguments from [20] here for the reader’s convenience. Since by Assumption 2.1

$$t \mapsto M(t)A_0(t)x + \tilde{M}(t)M(t)^{-1}x$$

is continuously differentiable for every $x \in D_0$ and $MA_0 + \tilde{M}M^{-1}$ is locally Kato-stable by Proposition 3.5 of [12] with constants $\tilde{M}_{t_0}, \tilde{\omega}_{t_0}$ say, it follows from Theorem 6.1 of [12] that there exists a unique evolution system $\tilde{T}_0$ for $A_0 + MM^{-1}$ on the space $D_0$ and that

$$\|\tilde{T}_0(t, s)\| \leq \tilde{M}_{t_0}e^{\tilde{\omega}_{t_0}(t-s)} \quad ((s, t) \in \Delta_{[0,t_0]}).$$

(2.6)

Set now $T(t, s) := M(t)^{-1}\tilde{T}_0(t, s)M(s)$ for $(s, t) \in \Delta$. As is easily verified, $T$ is an evolution system for $A$ on the spaces $D(A(s))$ and the estimate (2.6) yields the desired estimate (2.5) with appropriate constants $M_{t_0}$ and $\omega_{t_0} := \tilde{\omega}_{t_0}$.
In the next result, we discuss the classical solvability of the full semilinear problem (2.1). It is based on and extends the solvability result from [18], where the linear parts $A(t)$ are assumed to have a time-independent domain.

**Lemma 2.4.** Suppose that $A(t) = A_0(t)M(t) : D(A(t)) \subset X \to X$ are operators as in Assumption 2.1 on a reflexive space $X$ and that $f : \mathbb{R}_0^+ \times X \to X$ is Lipschitz on bounded subsets of $\mathbb{R}_0^+ \times X$. Then for every $s \in \mathbb{R}_0^+$ and $x_s \in D(A(s))$, the system (2.1) has a unique maximal classical solution $x(\cdot, s, x_s) \in C^1([s, T_{s,x_s}), X)$ with initial state $x_s$ at initial time $s$. Additionally, this solution satisfies the integral equation

$$x(t, s, x_s) = T(t, s)x_s + \int_s^t T(t, \tau)f(\tau, x(\tau, s, x_s)) \, d\tau \quad (t \in [s, T_{s,x_s}))$$

(2.7)

with $T$ from Lemma 2.3 and, moreover, this solution exists globally in time, that is, $T_{s,x_s} = \infty$, provided that it is bounded:

$$\sup_{t \in [s, T_{s,x_s})} \|x(t, s, x_s)\| < \infty.$$  

(2.8)

**Proof.** As a first step, we observe that the variable transformation $\xi(t) = M(t)x(t)$ induces a one-to-one correspondence between the maximal classical solutions of (2.1) and the maximal classical solutions of

$$\dot{\xi}(t) = M(t)A_0(t)\xi(t) + M(t)f(t, M(t)^{-1}\xi(t)) + \dot{M}(t)M(t)^{-1}\xi(t).$$

(2.9)

Indeed, it is elementary to verify that for a (maximal) classical solution $x : J \to X$ of (2.1) the function $\xi : J \to X$ defined by $\xi(t) := M(t)x(t)$ is a (maximal) classical solution of (2.9) and that, conversely, for a (maximal) classical solution $\xi : J \to X$ of (2.9) the function $x : J \to X$ is a (maximal) classical solution of (2.1).

As a second step, we show that for every $s \in \mathbb{R}_0^+$ and $x_s \in D(A(s))$, the system (2.1) has a unique maximal classical solution $x(\cdot, s, x_s) \in C^1([s, T_{s,x_s}), X)$ with initial state $x_s$ at initial time $s$. So let $s \in \mathbb{R}_0^+$ and $x_s \in D(A(s))$. We want to apply the solvability result (Theorem 1) from [19] to the transformed equation (2.9). It is clear by Assumption 2.1 that the assumptions of Theorem 1 of [19] are satisfied. It also follows, by the very same arguments as in the autonomous case [17], that (2.9) has a unique maximal mild solution $\xi(\cdot, s, \xi_s) \in C(J_{s,x_s}, X)$ with initial state $\xi_s := M(s)x_s$ at initial time $s$ and that the maximal existence interval $J_{s,x_s}$ is half-open: $J_{s,x_s} = [s, T_{s,x_s})$. Since now

$$\xi_s = M(s)x_s \in M(s)D(A(s)) = D(A_0(s)) = D_0$$

and $X$ is reflexive, Theorem 1 of [19] implies that $\xi(\cdot, s, \xi_s)$ is also a classical solution of (2.9). Since, moreover, classical solutions are well-known to be also mild solutions of (2.9) and since every mild solution of (2.9) with initial state $\xi_s$ is a restriction of the maximal mild solution $\xi(\cdot, s, \xi_s)$, $\xi(\cdot, s, \xi_s)$ even is a unique maximal classical solution. So, by the first step,

$$x(\cdot, s, x_s) := M(\cdot)\xi(\cdot, s, \xi_s) \in C^1([s, T_{s,x_s}), X)$$
is a unique maximal classical solution with initial state $M(s)\xi_s = x_s$, as desired.

As a third step, we prove the integral equation (2.7) for $s \in \mathbb{R}_0^+$ and $x(s) \in D(A(s))$, which says that the classical solutions $x(\cdot, s, x_s)$ is also a mild solution of (2.1). So let $s \in \mathbb{R}_0^+$ and $x_s \in D(A(s))$ and let $t \in [s, T_{s,x_s})$ be fixed. It follows by the right differentiability property (Lemma 2.1.5 of [20]) of the evolution system $T$ for $A$ on the spaces $D(A(s))$ that

$$[s,t] \ni \tau \mapsto T(t, \tau)x(\tau, s, x_s)$$

(2.10)
is continuous and right differentiable with right derivative

$$[s,t] \ni \tau \mapsto T(t, \tau)f(\tau, x(\tau, s, x_s)).$$

(2.11)

Since this right derivative is continuous, it further follows by Corollary 2.1.2 of [17] that (2.10) is continuously differentiable with derivative (2.11). And therefore we obtain (2.7) by the fundamental theorem of calculus.

As a fourth step, we show that if for some $s \in \mathbb{R}_0^+$ and $x_s \in D(A(s))$ the maximal classical solution $x(\cdot, s, x_s)$ does not exist globally in time, that is, if $T_{s,x_s} < \infty$, then it must be unbounded:

$$\sup_{t \in [s,T_{s,x_s})} \|x(t, s, x_s)\| = \infty.$$  (2.12)

So assume that $T_{s,x_s} < \infty$ for some $s \in \mathbb{R}_0^+$ and $x_s \in D(A(s))$. We want to apply the global solvability result (Theorem 6) from [18] to the transformed equation (2.9). It is clear by Assumption 2.1 that the linear and nonlinear part of (2.9) satisfy the assumptions of Theorem 6 of [18]. It also follows, by the first step and the proof of the second step, that $\xi : [s, T_{s,x_s}) \to X$ defined by $\xi(t) := M(t)x(t, s, x_s)$ is a maximal mild solution of (2.9). Since now this maximal mild solution does not exist on $[s, \infty)$ but only on $[s, T_{s,x_s})$, Theorem 6 of [18] implies that

$$\sup_{t \in [s,T_{s,x_s})} \|\xi(t)\| = \infty.$$  

Since, moreover, $t \mapsto M(t)$ is locally bounded and $[0, T_{s,x_s}]$ is compact, we conclude that

$$\sup_{t \in [s,T_{s,x_s})} \|x(t, s, x_s)\| \geq \left( \sup_{t \in [s,T_{s,x_s})} \|M(t)\| \right)^{-1} \sup_{t \in [s,T_{s,x_s})} \|\xi(t)\| = \infty,$$

that is, (2.12) is satisfied, as desired. ■

As a last preliminary, we record two simple facts for later reference. We give the elementary proofs for the sake of completeness.

**Lemma 2.5.** (i) If $f : \mathbb{R}_0^+ \times X \to X$ is Lipschitz on bounded subsets of $\mathbb{R}_0^+ \times X$, then one can choose Lipschitz constants $L_\rho$ of $f|_{[0,\rho] \times \overline{B}_\rho(0)}$ for $\rho \in \mathbb{R}_0^+$ such that $\rho \mapsto L_\rho$ is continuous and monotonically increasing.
(ii) \( C_c(\mathbb{R}_0^+, U) \) is dense in \( L^2_{\text{loc}}(\mathbb{R}_0^+, U) \), where \( U \) is an arbitrary Banach space.

Proof. (i) We have only to apply the elementary and well-known fact that any monotonically increasing function \( l : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) can be majorized by a continuous monotonically increasing function to the particular function

\[
\rho \mapsto L^0_\rho := \min \left\{ L \in \mathbb{R}_0^+ : L \text{ is a Lipschitz constant of } f|_{[0, \rho] \times B_{\rho}(0)} \right\} < \infty.
\]

See Lemma 2.5 of [2], for instance. (ii) We have to show that for a given \( u \in L^2_{\text{loc}}(\mathbb{R}_0^+, U) \) there exists a sequence \( (u_n) \) in \( C_c^2(\mathbb{R}_0^+, U) \) such that

\[
\|u_n - u\|_{[0, t_0], 2} \to 0 \quad (n \to \infty)
\]

for every \( t_0 \in (0, \infty) \). So let \( u \in L^2_{\text{loc}}(\mathbb{R}_0^+, U) \). Since \( u|_{[t,n]} \in L^2([0, n], U) \) and \( C_c^2((0, n), U) \) is dense in \( L^2([0, n], U) \), for every \( n \in \mathbb{N} \) there exists a function \( u_n \in C_c^2(\mathbb{R}_0^+, U) \) with

\[
\text{supp } u_n \subset (0, n) \quad \text{and} \quad \|u_n - u\|_{[0, n], 2} \leq 1/n.
\]

So, for every given \( t_0 \in (0, \infty) \), we have \( \|u_n - u\|_{[0, t_0], 2} \leq \|u_n - u\|_{[0, n], 2} \leq 1/n \) provided that \( n \geq t_0 \) and therefore (2.13) follows.

3 Semilinear systems with distributed control and observation

3.1 Classical solutions and outputs

Assumption 3.1. \( X \) is a reflexive space and

(i) \( A(t) := \mathcal{A}(t) \) are operators as in Assumption 2.1

(ii) \( t \mapsto \mathcal{B}(t) \) is locally Lipschitz

(iii) \( f \) is Lipschitz on bounded subsets of \( \mathbb{R}_0^+ \times X \).

Lemma 3.2. With the above assumption, for every \( s \in \mathbb{R}_0^+ \) and every classical datum \( (x_s, u) \in \mathcal{D}_s \) defined by

\[
\mathcal{D}_s := D(A(s)) \times C_c^2(\mathbb{R}_0^+, U),
\]

the system (1.1) has a unique maximal classical solution \( x(\cdot, s, x_s, u) \in C^1([s, T_{s,x_s,u}], X) \).

Assumption 3.3. (i) System (1.1) is scattering-passive w.r.t. a continuously differentiable storage function \( V \), that is, \( V \in C^1([0, \infty) \times X, \mathbb{R}_0^+) \) and for some \( \alpha, \beta > 0 \)

\[
\dot{V}(t, x(t, s, x_s, u)) \leq \alpha \|u(t)\|^2_{L^2} - \beta \|y(t, s, x_s, u)\|^2_{L^2} \quad (t \in [s, T_{s,x_s,u}]) \quad (3.1)
\]

for every \( s \in \mathbb{R}_0^+ \) and every \( (x_s, u) \in \mathcal{D}_s \)
(ii) $V(t, \cdot)$ is equivalent to the norm $\| \cdot \|$ of $X$ uniformly w.r.t. $t$, that is, for some $\overline{\psi}, \underline{\psi} \in K_\infty$

\[
\underline{\psi}(\|x\|) \leq V(t, x) \leq \overline{\psi}(\|x\|) \quad ((t, x) \in \mathbb{R}_0^+ \times X).
\] (3.2)

**Assumption 3.4.**

(i) $\partial V : \mathbb{R}_0^+ \times X \rightarrow L(\mathbb{R} \times X)$, the derivative of $V$, is bounded on bounded subsets of $\mathbb{R}_0^+ \times X$.

(ii) $y(\cdot, s, x_s, u) := C(\cdot)x(\cdot, s, x_s, u)$ is measurable for every $s \in \mathbb{R}_0^+$ and $(x_s, u) \in D_s$.

**Lemma 3.5.** With the above assumptions, the maximal classical solution $x(\cdot, s, x_s, u)$ exists globally in time for every $s \in \mathbb{R}_0^+$ and $(x_s, u) \in D_s$, that is, $T_{s, x_s, u} = \infty$. Additionally, there exist $\sigma, \gamma \in K$ such that

\[
\|x(t, s, x_s, u)\| \leq \sigma(\|x_s\|) + \gamma(\|u\|_{[s,t], 2}) \quad (t \in [s, \infty))
\] (3.3)

for every $s \in \mathbb{R}_0^+$ and $(x_s, u) \in D_s$.

### 3.2 Well-posedness: generalized solutions and outputs

We now have to establish existence of generalized solutions and outputs. And to do so, we exploit the following integral equation for classical solutions of (1.2):

\[
x(t, x_0, u) = T(t, 0)x_0 + \int_0^t T(t, s)f(s, x(s, x_0, u)) \, ds + \Phi_t(u)
\] (3.4)

for all $t \in \mathbb{R}_0^+$ and $(x_0, u) \in D_0$, where

\[
\Phi_t(u) := \int_0^t T(t, s)B(s)u(s) \, ds
\] (3.5)

and where $T$ is the evolution system for $A$ on the spaces $D(A(s))$.

**Lemma 3.6.**

(i) $D_0$ is dense in $X \times L^2_{loc}(\mathbb{R}_0^+, U)$

(ii) $\Phi_t : C^2([0, t], U) \rightarrow X$ defined by (3.5) for every $t \in (0, \infty)$ uniquely extends to a bounded linear operator $\overline{\Phi}_t : L^2([0, t], U) \rightarrow X$ and

\[
C_{t_0} := \sup_{t \in [0, t_0]} \| \overline{\Phi}_t \| < \infty
\] (3.6)

for every $t_0 \in (0, \infty)$.

**Proof.** Assertion (i) is a consequence of the density of $C^2_c(\mathbb{R}_0^+, U)$ in $L^2_{loc}(\mathbb{R}_0^+, U)$ (Lemma 2.5) and the density of $D(A(0)) = M(0)^{-1}D(A_0(0))$ in $X$ (Assumption 3.1(i)).

Assertion (ii) immediately follows from the definition of $\Phi_t$. Indeed, let $t_0 \in (0, \infty)$ be fixed and let $t \in [0, t_0]$ and

\[
u_t \in C^2([0, t], U) \quad \text{with} \quad \|u_t\|_{[0,t], 2} \leq 1.
\]
We can then conclude from (3.5) with the help of (2.5) and Assumption 3.1(ii) that

$$
\|\Phi(t)\| \leq M_0 e^{\omega_0 t} b_t \int_0^t \|u(s)\|_U \, ds \leq M_0 e^{\omega_0 t} b_t t^{1/2},
$$

(3.7)

where \(b_t := \sup_{s \in [0,t]} \|B(s)\|_{U,X}\). And from (3.7), in turn, assertion (ii) is clear.

**Theorem 3.7.** With the above assumptions and the additional assumption that \(f(t,0) = 0\) for every \(t \in \mathbb{R}_0^+\), the system (1.1) is well-posed.

**Proof.** (i) We first show that for every \(t_0 \in (0, \infty)\) and every \((x_{01}, u_1), (x_{02}, u_2) \in D_0\) one has the following fundamental estimate:

$$
\|x(\cdot, x_{01}, u_1) - x(\cdot, x_{02}, u_2)\|_{[0,t_0],\infty} \leq \left( M_{t_0} e^{\omega_0 t_0} \|x_{01} - x_{02}\| + C_{t_0} \|u_1 - u_2\|_{[0,t_0],2} \right) \cdot 
\exp \left( M_{t_0} e^{\omega_0 t_0} L_{t_0 + \rho_0(x_{01}, u_1, x_{02}, u_2)} t_0 \right),
$$

(3.8)

where \(M_{t_0}, \omega_0\) are as in Lemma 2.3, \(C_{t_0}\) is as in Lemma 3.6, \(L_{\rho}\) are Lipschitz constants chosen as in Lemma 2.5 and

$$
\rho_0(x_{01}, u_1, x_{02}, u_2) := \sigma(\|x_{01}\|) + \gamma(\|u_1\|_{[0,t_0],2}) + \sigma(\|x_{02}\|) + \gamma(\|u_2\|_{[0,t_0],2})
$$

(3.9)

with \(\sigma, \gamma\) as in Lemma 3.5. So let \(t_0 \in (0, \infty)\) and \((x_{01}, u_1), (x_{02}, u_2) \in D_0\) and write \(x_i := x(\cdot, x_{0i}, u_i)\). It then follows from (3.4) with the help of (2.5), (3.6), and the Lipschitz continuity of \(f\) on bounded subsets (Assumption 3.1(iii)) combined with (3.3) that

$$
\|x_1(t) - x_2(t)\| \leq M_{t_0} e^{\omega_0 t_0} \|x_{01} - x_{02}\| + C_{t_0} \|u_1 - u_2\|_{[0,t_0],2}
$$

$$
+ M_{t_0} e^{\omega_0 t_0} \int_0^t L_{t_0 + \rho_0(x_{01}, u_1, x_{02}, u_2)} \|x_1(s) - x_2(s)\| \, ds
$$

(3.10)

for all \(t \in [0,t_0]\). And therefore the desired estimate (3.8) follows by Grönewall’s lemma. Combining this estimate (3.8) now with the density of \(D_0\) in \(X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U)\) (Lemma 3.6), we immediately see that for every \((x_0, u) \in X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U)\) there exists a unique generalized solution

$$
x(\cdot, x_0, u) \in C(\mathbb{R}_0^+, X).
$$

Since, moreover, the right-hand side of (3.8) depends continuously on \((x_{01}, u_1), (x_{02}, u_2)\), this estimate extends from \(D_0\) to arbitrary \((x_0, u_1), (x_0, u_2) \in X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U)\). And this extended estimate, in turn, yields the continuity of the generalized solution map \((x_0, u) \mapsto x(\cdot, x_0, u) \in C(\mathbb{R}_0^+, X)\).

(ii) We first show that for every \(t_0 \in (0, \infty)\) and every \((x_{01}, u_1), (x_{02}, u_2) \in D_0\) one has the following fundamental estimate:

$$
\beta \|y(\cdot, x_{01}, u_1) - y(\cdot, x_{02}, u_2)\|_{[0,t_0],2}^2 \leq \psi(\|x_{01} - x_{02}\|) + \alpha \|u_1 - u_2\|_{[0,t_0],2}^2
$$

$$
+ 2M_{t_0 + \rho_0(x_{01}, u_1, x_{02}, u_2)} \|x(\cdot, x_{01}, u_1) - x(\cdot, x_{02}, u_2)\|_{[0,t_0],\infty} t_0,
$$

(3.11)
where \( \rho_{t_0}(x_{01}, u_1, x_{02}, u_2) \) is defined as in (3.9) and where \( M_{\rho} := K_{\rho}L_{\rho} \) with

\[
K_{\rho} \geq K_{\rho}^0 := \sup \{ \| \partial V(t, x) \| : t + \| x \| \leq \rho \}
\]

chosen such that \( \rho \mapsto K_{\rho} \) is continuous and monotonically increasing (see the proof of Lemma 2.5) and with Lipschitz constants \( L_{\rho} \) chosen as in Lemma 2.5. So let \( t_0 \in (0, \infty) \) and \( (x_{01}, u_1), (x_{02}, u_2) \in D_0 \) and write \( x_i := x(\cdot, x_{0i}, u_i) \) and \( y_i := y(\cdot, x_{0i}, u_i) = C(\cdot)x(\cdot, x_{0i}, u_i) \) as well as \( x_{12} := x_1 - x_2 \) and \( u_{12} := u_1 - u_2 \). It then follows by the differential equation (1.1) that

\[
\frac{d}{ds} V(s, x_{12}(s)) = \partial_s V(s, x_{12}(s)) + \partial_x V(s, x_{12}(s))(A(s)x_{12}(s) + f(s, x_{12}(s)) + B(s)u_{12}(s))
\]

\[
+ \partial_x V(s, x_{12}(s))(f(s, x_1(s)) - f(s, x_2(s)) - f(s, x_{12}(s)))
\]

(3.12)

for all \( s \in \mathbb{R}_0^+ \). Since by the classical solution property of \( x_i \) for (1.1)

\[
x_{12}(s) = x_1(s) - x_2(s) \in D(A(s)) \quad \text{and} \quad u_{12} \in C_c(\mathbb{R}_0^+, U),
\]

we have \((x_{12s}, u_{12}) := (x_{12}(s), u_{12}) \in D_s \) and it thus follows by (3.1) that the first part of the right-hand side of (3.12) can be estimated as follows:

\[
\partial_s V(s, x_{12}(s)) + \partial_x V(s, x_{12}(s))(A(s)x_{12}(s) + f(s, x_{12}(s)) + B(s)u_{12}(s))
\]

\[
= \frac{d}{dt} V(t, x(t, s, x_{12s}, u_{12}))|_{t=s} \leq \alpha \| u_{12}(s) \|_U^2 - \beta \| C(s)x_{12s} \|_Y^2
\]

\[
= \| u_1(s) - u_2(s) \|_U^2 - \beta \| y_1(s) - y_2(s) \|_Y^2 \quad (s \in \mathbb{R}_0^+).
\]

(3.13)

Since, moreover, \( \partial V \) and \( f \) are bounded or Lipschitz, respectively, on bounded subsets of \( \mathbb{R}_0^+ \times X \) (Assumption 3.4(i) and 3.1(iii)!) and \( f(s, 0) = 0 \), it further follows by (3.3) that the second part of the right-hand side of (3.12) can be estimated as follows:

\[
\partial_x V(s, x_{12}(s))(f(s, x_1(s)) - f(s, x_2(s)) - f(s, x_{12}(s)))
\]

\[
\leq 2M_{t_0} + \rho_{t_0}(x_{01}, u_1, x_{02}, u_2) \| x_1(s) - x_2(s) \| \quad (s \in \mathbb{R}_0^+).
\]

(3.14)

Inserting now (3.13) and (3.14) into (3.12) and integrating the resulting estimate (Assumption 3.4(ii)!), we finally obtain the desired estimate (3.11).

Combining this estimate (3.11) now with the density of \( D_0 \) in \( X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U) \) (Lemma 3.6) and the continuity of \((x_0, u) \mapsto x(\cdot, x_0, u) \in C(\mathbb{R}_0^+, X) \) established above, we immediately see that for every \((x_0, u) \in X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U) \) there exists a unique generalized output

\[
y(\cdot, x_0, u) \in L^2_{\text{loc}}(\mathbb{R}_0^+, Y).
\]

Since, moreover, the right-hand side of (3.11) depends continuously on \((x_{01}, u_1), (x_{02}, u_2) \), this estimate extends from \( D_0 \) to arbitrary \((x_{01}, u_1), (x_{02}, u_2) \in X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U) \). And this extended estimate, in turn, yields the continuity of the generalized output map \((x_0, u) \mapsto y(\cdot, x_0, u) \in L^2_{\text{loc}}(\mathbb{R}_0^+, Y) \).
4 Semilinear systems with boundary control and observation

4.1 Classical solutions and outputs

Assumption 4.1. $X$ is a reflexive space and

(i) $\mathcal{D}(A(t)) = \mathcal{D}(\mathcal{B}(t))$ and $A(t) := A(t)|_{\ker \mathcal{B}(t)}$ are operators as in Assumption 2.1

(ii) $\mathcal{B}(t)$ has a bounded linear right-inverse $\mathcal{R}(t) \in \mathcal{L}(U, X)$ for every $t \in \mathbb{R}_0^+$, that is,

$$\mathcal{R}(t)U \subseteq \mathcal{D}(\mathcal{B}(t)) \quad \text{and} \quad \mathcal{B}(t)\mathcal{R}(t)u = u \quad (t \in \mathbb{R}_0^+, u \in U),$$

such that $\mathcal{A}(t)\mathcal{R}(t) \in \mathcal{L}(U, X)$ and such that $t \mapsto \mathcal{R}(t), \mathcal{A}(t)\mathcal{R}(t)$ are locally Lipschitz

(iii) $f$ is Lipschitz on bounded subsets of $\mathbb{R}_0^+ \times X$.

Lemma 4.2. With the above assumption, for every $s \in \mathbb{R}_0^+$ and every classical datum $(x, u) \in \mathcal{D}_s$ defined by

$$\mathcal{D}_s := \{(x, u) \in X \times C^2_c(\mathbb{R}_0^+, U) : x_0 - \mathcal{R}(s)u(s) \in \mathcal{D}(A(s))\},$$

the system (1.2) has a unique maximal classical solution $x(\cdot, s, x, u) \in C^1([s, T_{s,x,u}), X)$.

Assumption 4.3. (i) System (1.2) is scattering-passive w.r.t. a continuously differentiable storage function $V$, that is, $V \in C^1(\mathbb{R}_0^+ \times X, \mathbb{R}_0^+)$ and for some $\alpha, \beta > 0$

$$\dot{V}(t,x(t,s,x,u)) \leq \alpha \|u(t)\|^2_U - \beta \|y(t,s)x(t,u)\|^2_V \quad (t \in [s, T_{s,x,u})) \quad (4.1)$$

$$\tag{4.1}$$

$$y(t,s,x,u) := C(t)x(t,s,u)$$

for every $s \in \mathbb{R}_0^+$ and every $(x, u) \in \mathcal{D}_s$

(ii) $V(t, \cdot)$ is equivalent to the norm $\|\cdot\|$ of $X$ uniformly w.r.t. $t$, that is, for some $\underline{\psi}, \overline{\psi} \in \mathcal{K}_\infty$

$$\underline{\psi}(\|x\|) \leq V(t, x) \leq \overline{\psi}(\|x\|) \quad ((t, x) \in \mathbb{R}_0^+ \times X). \quad (4.2)$$

Assumption 4.4. (i) $\partial V : \mathbb{R}_0^+ \times X \to \mathcal{L}(\mathbb{R} \times X)$, the derivative of $V$, is bounded on bounded subsets of $\mathbb{R}_0^+ \times X$

(ii) $y(\cdot, s, x, u) := C(\cdot)x(\cdot, s, x, u)$ is measurable for every $s \in \mathbb{R}_0^+$ and $(x, u) \in \mathcal{D}_s$.

Lemma 4.5. With the above assumptions, the maximal classical solution $x(\cdot, s, x, u)$ exists globally in time for every $s \in \mathbb{R}_0^+$ and $(x, u) \in \mathcal{D}_s$, that is, $T_{s,x,u} = \infty$. Additionally, there exist $\sigma, \gamma \in \mathcal{K}$ such that

$$\|x(t, s, x, u)\| \leq \sigma(\|x_s\|) + \gamma(\|u\|_{s,t}^2) \quad (t \in [s, \infty)) \quad (4.3)$$

for every $s \in \mathbb{R}_0^+$ and $(x, u) \in \mathcal{D}_s$. 

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4.2 Well-posedness: generalized solutions and outputs

We now have to establish existence of generalized solutions and outputs. And to do so, we exploit the following integral equation for classical solutions of (1.2):

\[
x(t, x_0, u) = T(t, 0)x_0 + \int_0^t T(t, s)f(s, x(s, x_0, u)) \, ds + \Phi_t(u)
\]  

(4.4)

for all \( t \in \mathbb{R}_0^+ \) and \((x_0, u) \in \mathcal{D}_0 \), where

\[
\Phi_t(u) := \int_0^t T(t, s)(A(s)\mathcal{R}(s)u(s) - \mathcal{R}(s)u(s) - \mathcal{R}(s)\dot{u}(s)) \, ds 
- T(t, 0)\mathcal{R}(0)u(0) + \mathcal{R}(t)u(t) 
\]

(4.5)

and where \( T \) is the evolution system for \( A \) on the spaces \( D(A(s)) \).

**Lemma 4.6.**

(i) \( \mathcal{D}_0 \) is dense in \( X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U) \)

(ii) \( \Phi_t : C^2([0, t], U) \to X \) defined by (4.5) for every \( t \in (0, \infty) \) uniquely extends to a bounded linear operator \( \overline{\Phi}_t : L^2([0, t], U) \to X \) and

\[
C_{t_0} := \sup_{t \in [0, t_0]} \| \overline{\Phi}_t \| < \infty
\]

(4.6)

for every \( t_0 \in (0, \infty) \).

**Proof.** Assertion (i) is again a consequence of the density of \( C_c^2(\mathbb{R}_0^+, U) \) in \( L^2_{\text{loc}}(\mathbb{R}_0^+, U) \) (Lemma 2.5) and the density of \( D(A(0)) = M(0)^{-1}D(A_0(0)) \) in \( X \) (Assumption 4.1(i)). Indeed, let \((x_0, u) \in X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U) \). Then there exists a sequence \((u_n) \in C_c(\mathbb{R}_0^+, U) \) with

\[
\begin{align*}
    u_n \overset{L^2_{\text{loc}}(\mathbb{R}_0^+, U)}{\longrightarrow} & u \quad (n \to \infty).
\end{align*}
\]

Since \( D(A(0)) \) is dense in \( X \), so is the subspace \( \mathcal{R}(0)u_n(0) + D(A(0)) \) and therefore for every \( n \in \mathbb{N} \) there exists an \( x_{0n} \in X \) with

\[
    x_{0n} \in \mathcal{R}(0)u_n(0) + D(A(0)) \quad \text{and} \quad x_{0n} \in \overline{\mathcal{B}}_{1/n}(x_0).
\]

Consequently, \((x_{0n}, u_n) \in \mathcal{D}_0 \) and \((x_{0n}, u_n) \overset{X \times L^2_{\text{loc}}(\mathbb{R}_0^+, U)}{\longrightarrow} (x_0, u) \) as \( n \to \infty \), as desired.

Assertion (ii) now does not immediately follow from the definition of \( \Phi_t \) anymore, but from (4.4) instead. Indeed, let \( t_0 \in (0, \infty) \) be fixed and let \( t \in [0, t_0] \) and

\[
    u_t \in C^2([0, t], U) \quad \text{with} \quad \| u_t \|_{[0, t], 2} \leq 1.
\]

Then, of course, there exists a \( u \in C^2(\mathbb{R}_0^+, U) \) with \( u_{[0, t]} = u_t \) and, by the density of \( \mathcal{R}(0)u(0) + D(A(0)) \) in \( X \), there also exists an \( x_0 \in X \) with

\[
    x_0 \in \mathcal{R}(0)u(0) + D(A(0)) \quad \text{and} \quad x_0 \in \overline{\mathcal{B}}_1^X(0).
\]
Consequently, \((x_0, u) \in D_0\). We can thus conclude from (4.4) with the help of (4.3), (2.5), and Assumption 4.1(iii) that

\[
\|\Phi_t(u_1)\| = \|\Phi_t(u)\| \leq \rho_0(x_0, u) + M_0 e^{\omega_1 t_0} \|x_0\| + M_0 e^{\omega_1 t_0} \cdot \int_0^t L(t_0 + \rho_0(x_0, u)) (t_0 + \rho_0(x_0, u)) + \|f(0, 0)\| \text{ ds}
\]

\[
\leq \rho_0 + M_0 e^{\omega_1 t_0} \left(1 + L(t_0 + \rho_0(t_0 + \rho_0) + \|f(0, 0)\|\right)
\]

(4.7)

where we used that

\[
\rho_0(x_0, u) := \sigma (\|x_0\|) + \gamma (\|u\|)_{[0, t_0]} \leq \sigma(1) + \gamma(1) =: \rho_0
\]

and that the Lipschitz constants \(L_\rho\), chosen according to Lemma 2.5, are monotonically increasing in \(\rho\). And from (4.7), in turn, assertion (ii) is clear.

\[\blacksquare\]

**Theorem 4.7.** With the above assumptions and the additional assumption that \(f(t, 0) = 0\) for every \(t \in \mathbb{R}_0^+\), the system (1.2) is well-posed.

**Proof.** (i) We first show that for every \(t_0 \in (0, \infty)\) and every \((x_0, u_1), (x_0, u_2) \in D_0\) one has the following fundamental estimate:

\[
\|x(\cdot, x_0, u_1) - x(\cdot, x_0, u_2)\|_{[0, t_0], \infty} \leq \left(M_0 e^{\omega_0 t_0} \|x_0 - x_0\| + C_0 \|u_1 - u_2\|_{[0, t_0], 2}\right) \cdot \\
\cdot \exp \left(M_0 e^{\omega_0 t_0} L(t_0 + \rho_0(x_0, u_1, x_0, u_2), t_0), 0\right)
\]

(4.8)

where \(M_0, \omega_0\) are as in Lemma 2.3, \(C_0\) is as in Lemma 4.6, \(L_\rho\) are Lipschitz constants chosen as in Lemma 2.5 and

\[
\rho_0(x_0, u_1, x_0, u_2) := \sigma (\|x_0\|) + \gamma (\|u_1\|)_{[0, t_0]} + \sigma (\|x_0\|) + \gamma (\|u_2\|)_{[0, t_0]}\]  

(4.9)

with \(\sigma, \gamma\) as in Lemma 4.5. So let \(t_0 \in (0, \infty)\) and \((x_0, u_1), (x_0, u_2) \in D_0\) and write \(x_i := x(\cdot, x_0, u_i)\). It then follows from (4.4) with the help of (2.5), (4.6), and the Lipschitz continuity of \(f\) on bounded subsets (Assumption 4.1(iii)) combined with (4.3) that

\[
\|x_1(t) - x_2(t)\| \leq M_0 e^{\omega_0} \|x_0 - x_0\| + C_0 \|u_1 - u_2\|_{[0, t_0], 2} + M_0 e^{\omega_0} \left(1 + L(t_0 + \rho_0(x_0, u_1, x_0, u_2), t_0)\right) \|x_1(s) - x_2(s)\| \text{ ds}
\]

(4.10)

for all \(t \in [0, t_0]\). And therefore the desired estimate (4.8) follows by Grönwall’s lemma. Combining this estimate (4.8) now with the density of \(D_0\) in \(X \times L_{\text{loc}}^2(\mathbb{R}_0^+, U)\) (Lemma 4.6), we immediately see that for every \((x_0, u) \in X \times L_{\text{loc}}^2(\mathbb{R}_0^+, U)\) there exists a unique generalized solution

\[
x(\cdot, x_0, u) \in C(\mathbb{R}_0^+, X).
\]

Since, moreover, the right-hand side of (4.8) depends continuously on \((x_0, u_1), (x_0, u_2)\), this estimate extends from \(D_0\) to arbitrary \((x_0, u_1), (x_0, u_2) \in X \times L_{\text{loc}}^2(\mathbb{R}_0^+, U)\). And
this extended estimate, in turn, yields the continuity of the generalized solution map \((x_0, u) \mapsto x(\cdot, x_0, u) \in C(\mathbb{R}_0^+, X)\).

(ii) We first show that for every \(t_0 \in (0, \infty)\) and every \((x_{01}, u_1), (x_{02}, u_2) \in D_0\) one has the following fundamental estimate:

\[
\beta \|y(\cdot, x_{01}, u_1) - y(\cdot, x_{02}, u_2)\|^2_{[0, t_0], 2} \leq \bar{v}(\|x_{01} - x_{02}\|) + \alpha \|u_1 - u_2\|^2_{[0, t_0], 2} + 2M_{t_0 + \rho_{01}(x_{01}, u_1, x_{02}, u_2)} \|x(\cdot, x_{01}, u_1) - x(\cdot, x_{02}, u_2)\|_{[0, t_0], \infty} t_0,
\]

(4.11)

where \(\rho_{t_0}(x_{01}, u_1, x_{02}, u_2)\) is defined as in (4.9) and where \(M_\rho := K_\rho L_\rho\) with

\[
K_\rho \geq K_\rho^0 := \sup\{|\partial V(t, x)| : t + \|x\| \leq \rho\}
\]

chosen such that \(\rho \mapsto K_\rho\) is continuous and monotonically increasing (see the proof of Lemma 2.5) and with Lipschitz constants \(L_\rho\) chosen as in Lemma 2.5. So let \(t_0 \in (0, \infty)\) and \((x_{01}, u_1), (x_{02}, u_2) \in D_0\) and write \(x_i := x(\cdot, x_{0i}, u_i)\) and \(y_i := y(\cdot, x_{0i}, u_i) = C(\cdot)x(\cdot, x_{0i}, u_i)\) as well as \(x_{12} := x_1 - x_2\) and \(u_{12} := u_1 - u_2\). It then follows by the differential equation (1.2) that

\[
\frac{d}{ds} V(s, x_{12}(s)) = \partial_s V(s, x_{12}(s)) + \partial_x V(s, x_{12}(s)) (A(s)x_{12}(s) + f(s, x_{12}(s)))
+ \partial_x V(s, x_{12}(s)) (f(s, x_1(s)) - f(s, x_2(s)) - f(s, x_{12}(s)))
\]

(4.12)

for all \(s \in \mathbb{R}_0^+\). Since by the classical solution property of \(x_i = x_i - R u_i\) for (??) \(x_{12}(s) - R(s)u_{12}(s) = x_1(s) - R(s)u_1(s) - x_2(s) - R(s)u_2(s) \in D(A(s))\) and \(u_{12} \in C_c^2(\mathbb{R}_0^+, U)\), we have \((x_{12}, u_{12}) := (x_{12}(s), u_{12}) \in D_s\) and thus it follows by (4.1) that the first part of the right-hand side of (4.12) can be estimated as follows:

\[
\partial_s V(s, x_{12}(s)) + \partial_x V(s, x_{12}(s)) (A(s)x_{12}(s) + f(s, x_{12}(s)))
= \frac{d}{dt} V(t, x(t, s, x_{12}, u_{12}))|_{t=s} \leq \alpha \|u_{12}(s)\|_U^2 - \beta \|C(s)x_{12s}\|_Y^2
= \|u_1(s) - u_2(s)\|_U^2 - \beta \|y_1(s) - y_2(s)\|_Y^2
\]

(4.13)

Since, moreover, \(\partial V\) and \(f\) are bounded or Lipschitz, respectively, on bounded subsets of \(\mathbb{R}_0^+ \times X\) (Assumption 4.4(i) and 4.1(iii)) and \(f(s, 0) = 0\), it further follows by (4.3) that the second part of the right-hand side of (4.12) can be estimated as follows:

\[
\partial_x V(s, x_{12}(s)) (f(s, x_1(s)) - f(s, x_2(s)) - f(s, x_{12}(s))
\leq 2M_{t_0 + \rho_{01}(x_{01}, u_1, x_{02}, u_2)} \|x_1(s) - x_2(s)\| (s \in \mathbb{R}_0^+).
\]

(4.14)

Inserting now (4.13) and (4.14) into (4.12) and integrating the resulting estimate (Assumption 4.4(ii)), we finally obtain the desired estimate (4.11).

Combining this estimate (4.11) now with the density of \(D_0\) in \(X \times L^2_{loc}(\mathbb{R}_0^+, U)\) (Lemma 4.6) and the continuity of \((x_0, u) \mapsto x(\cdot, x_0, u) \in C(\mathbb{R}_0^+, X)\) established above, we
immediately see that for every \((x_0, u) \in X \times L^2_{\loc}(\mathbb{R}_0^+, U)\) there exists a unique generalized output 
\[ y(\cdot, x_0, u) \in L^2_{\loc}(\mathbb{R}_0^+, Y). \]
Since, moreover, the right-hand side of (4.11) depends continuously on \((x_{01}, u_1), (x_{02}, u_2)\), this estimate extends from \(D_0\) to arbitrary \((x_{01}, u_1), (x_{02}, u_2) \in X \times L^2_{\loc}(\mathbb{R}_0^+, U)\). And this extended estimate, in turn, yields the continuity of the generalized output map 
\((x_0, u) \mapsto y(\cdot, x_0, u) \in L^2_{\loc}(\mathbb{R}_0^+, Y)\).

\[\Box\]

5 Some applications

We now present two (classes of) applications of our abstract well-posedness results: one for the case of distributed control and observation and one for the case of boundary control and observation. In both cases, the considered systems arise as closed-loop systems by coupling a linear system \(\mathcal{S}\) to a nonlinear controller \(\mathcal{S}_c\) with a standard feedback interconnection, that is, the output \(y\) of the linear system is the input \(u_c\) of the controller, the input \(u\) of the linear system is minus the output \(-y_c\) of the controller plus the (external) input \(u\) of the closed-loop system, and \(y\) is also the (external) output of the closed-loop system. In short,
\[ y(t) = u_c(t) \quad \text{and} \quad -y_c(t) + u(t) = u(t) \quad \text{and} \quad y(t) = y(t) \quad (5.1) \]
and in pictures such a closed-loop system can be represented as in the figure below. Also, in both cases, the considered systems will be even strictly impedance-passive (instead of only scattering-passive) w.r.t. a continuously differentiable storage function, that is, \(V \in C^1(\mathbb{R}_0^+ \times X, \mathbb{R}_0^+)\) and \(U = Y\) is a Hilbert space such that for some \(\varsigma > 0\)
\[ \dot{V}(t, x(t, s, x_s, u)) \leq \text{Re} \langle u(t), y(t) \rangle_U - \varsigma \| y(t, s, x_s, u) \|_U^2 \quad (s \in [s, T_{x, x_s, u}]) \quad (5.2) \]
for every \(s \in \mathbb{R}_0^+\) and \((x_s, u) \in D_s\). Clearly, this implies the scattering-passivity estimates (3.1) and (4.1) with \(\alpha := \frac{1}{2\varsigma}\) and \(\beta := \frac{\varsigma}{2}\).

5.1 Case with distributed control and observation

5.1.1 Setting: open-loop system and controller

As our open-loop system, we consider a non-autonomous linear collocated system with distributed control and observation. Such a system evolves according to the differential
equation
\[
\dot{x}(t) = A(t)x(t) + B(t)u(t)
\] (5.3)
in the state space \(X\) with the additional observation condition
\[
y(t) = C(t)x(t).
\] (5.4)
In these equations, \(A(t) := A(t)\) are operators as in Lemma 2.2 (in particular, \(X\) is a Hilbert space) and \(B(t) \in L(U, X), C(t) \in L(X, Y)\) with a Hilbert space \(U = Y\) such that
\[
C(t) = B(t)^*
\] (5.5)
(which, in concrete examples, typically means \([16]\) that the observation takes place at the same location as the control or, in other words, that control and observation are collocated). Additionally, \(t \mapsto M(t)\) from Lemma 2.2 is assumed to be monotonically decreasing, that is, for every \(x \in X\)
\[
\langle M(t)x, x \rangle_X \leq \langle M(s)x, x \rangle_X \quad (s \leq t)
\] (5.6)
and \(t \mapsto B(t)\) is assumed to be locally Lipschitz. We now couple our open-loop system (5.3)-(5.4) to a nonlinear static controller described by the input-output relation
\[
y_c(t) = g(u_c(t)),
\] (5.7)
where \(g : U \to U\) is Lipschitz on bounded subsets of \(U\) and strictly damping in the sense that for some \(\varsigma > 0\)
\[
\langle y, g(y) \rangle_U \geq \varsigma \|y\|_U^2 \quad (y \in U).
\] (5.8)

5.1.2 Closed-loop system
Choosing the coupling of the controller (5.7) to the open-loop system (5.3)-(5.4) to be a standard feedback interconnection (5.1), we see that the arising closed-loop system is described by a differential equation of the form
\[
\dot{x}(t) = A(t) + f(t, x(t)) + B(t)u(t)
\] (5.9)
in the state space \(X\) with the additional observation condition
\[
y(t) = C(t)x(t),
\] (5.10)
where \(A(t), B(t), C(t)\) are as above and \(f : \mathbb{R}_0^+ \times X \to X\) is defined by
\[
f(t, x) := -B(t) g(B(t)^*x).
\]

**Corollary 5.1.** With the above assumptions, the closed-loop system (5.9)-(5.10) is well-posed.
Proof. We verify the assumptions of Theorem 3.7. As a first step, we observe that
Assumption 3.1 is satisfied. Indeed, this immediately follows from our assumptions above
and Lemma 2.2.

As a second step, we show that Assumption 3.3 is satisfied with
\[ V(t, x) := \frac{1}{2} \langle M(t)x, x \rangle_X \quad ((t, x) \in \mathbb{R}_0^+ \times X). \]  
Indeed, \( V \in C^1(\mathbb{R}_0^+ \times X, \mathbb{R}_0^+) \) and for every \( s \in \mathbb{R}_0^+ \) and \((x, u) \in D_s\) we see with
\( x(t) := x(t, s, x_s, u) \) and \( y(t) := y(t, s, x_s, u) \) that
\[
\frac{d}{dt} V(t, x(t)) = \frac{1}{2} \langle \dot{M}(t)x(t), x(t) \rangle_X + \text{Re} \langle M(t)x(t), A(t)x(t) \rangle_X
\]
\[
- \text{Re} \langle x(t), B(t) g(B(t)^* x(t)) \rangle_X + \text{Re} \langle x(t), B(t)u(t) \rangle_X
\]
\[
\leq - \text{Re} \langle C(t)x(t), g(C(t)x(t)) \rangle_U + \text{Re} \langle C(t)x(t), u(t) \rangle_U
\]
\[
\leq \text{Re} \langle u(t), y(t) \rangle_U - \varepsilon \|y(t)\|_U^2 \tag{5.12}
\]
for all \( t \in [s, T_{s,x,u}) \). (In the first inequality, we used the monotonicity (5.6) and the
contraction semigroup generation assumption from Lemma 2.2 and in the second inequality
we used the strict damping assumption (5.8).) Consequently, Assumption 3.3(i) is
satisfied with \( \alpha := \frac{1}{2} \) and \( \beta := \frac{1}{2} \). And in view of (2.3), Assumption 4.3(ii) is satisfied
as well.

As a third step, we observe that Assumption 3.4 is satisfied. Assumption 3.4(i) is
obviously satisfied and in order to verify the Assumption 4.4(ii) we have only to use that
\( t \mapsto C(t) = B(t)^* \) is locally Lipschitz continuous.

5.2 Case with boundary control and observation

5.2.1 Setting: open-loop system and controller

As our open-loop system, we consider a non-autonomous linear port-Hamiltonian system
of order \( N \in \mathbb{N} \) on a bounded interval \((a, b)\) with control and observation at the boundary [9]. Such a system evolves according to the differential equation
\[ \dot{x}(t) = A(t)x(t) = P_N \partial^N_C (H(t)x(t)) + \cdots + P_1 \partial_C (H(t)x(t)) + P_0 H(t)x(t) \tag{5.13} \]
in the state space \( X := L^2((a, b), \mathbb{R}^m) \) with the additional control and observation conditions
\[ u(t) = B(t)x(t) \quad \text{and} \quad y(t) = C(t)x(t). \tag{5.14} \]
In these equations, \( A(t) : D(A(t)) \subset X \to X \) is the linear operator defined by
\[ A(t)x := A_0 H(t)x := P_N \partial^N_C (H(t)x) + \cdots + P_1 \partial_C (H(t)x) + P_0 H(t)x \]
\[ D(A(t)) := \{ x \in X : H(t)x \in H^N((a, b), \mathbb{R}^m) \text{ and } W_{B,1}(H(t)x)|_{\partial} = 0 \} \tag{5.15} \]
and $B(t), C(t) : D(A(t)) \subset X \rightarrow \mathbb{R}^k$ are the linear boundary control and observation operators defined by

$$B(t)x := B_0H(t)x := W_{B,2}(H(t)x)|_\partial \quad \text{and} \quad C(t)x := C_0H(t)x := W_C(H(t)x)|_\partial$$

where $W_{B,1} \in \mathbb{R}^{(mN-k)\times 2mN}$ and $W_{B,2}, W_C \in \mathbb{R}^{k \times 2mN}$ and where, for a function $f \in H^N((a,b), \mathbb{R}^m)$, the symbol $f|_\partial$ denotes the (column) vector consisting of the boundary values of the first $N - 1$ derivatives of $f$, more precisely:

$$f|_\partial := (f(b)^\top, f'(b)^\top, \ldots, f^{(N-1)}(b)^\top, f(a)^\top, f'(a)^\top, \ldots, f^{(N-1)}(a)^\top)^\top \in \mathbb{R}^{2mN}.$$

As usual, $P_0, P_1, \ldots, P_N \in \mathbb{R}^{m \times m}$ are matrices such that $P_N$ is invertible and $P_1, \ldots, P_N$ are alternately symmetric and skew-symmetric while $P_0$ is dissipative:

$$P_l^\top = (-1)^{l+1}P_l \quad (l \in \{1, \ldots, N\}) \quad \text{and} \quad P_0^\top + P_1 \leq 0. \quad (5.16)$$

Also, $H(t)(\zeta) \in \mathbb{R}^{m \times m}$ are symmetric matrices for $(t, \zeta) \in \mathbb{R}_0^+ \times (a,b)$ satisfying the following assumptions:

- there exist positive finite constants $\underline{m}, \overline{m} \in (0, \infty)$ such that $\underline{m} \leq H(t)(\zeta) \leq \overline{m}$ \quad $(t, \zeta) \in \mathbb{R}_0^+ \times (a,b)) \quad (5.17)$

- $\zeta \mapsto H(t)(\zeta) \in \mathbb{R}^{m \times m}$ is measurable for every fixed $t \in \mathbb{R}_0^+$

- $t \mapsto H(t) \in L(X)$ is twice strongly continuously differentiable and monotonically decreasing, that is, for every $x \in X$

$$\langle H(t)x, x \rangle_2 \leq \langle H(s)x, x \rangle_2 \quad (s \leq t). \quad (5.18)$$

(In these assumptions, we used the symbol $H(t)$ not only to denote the measurable function $H(t) : (a,b) \rightarrow \mathbb{R}^{m \times m}$ but, as usual, also to denote the corresponding multiplication operator $X \ni x \mapsto H(t)x \in X$, which belongs to $L(X)$ by virtue of (5.17.b.).) We further assume that the boundary matrix

$$W := \begin{pmatrix} W_B \\ W_C \end{pmatrix} \in \mathbb{R}^{(mN+k)\times 2mN} \quad \text{with} \quad W_B := \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix} \quad (5.19)$$

is a matrix of full row rank $mN + k$. And finally, we assume that our open-loop system $(5.13)$-$(5.14)$ with $H(t)(\zeta) \equiv I$ is impedance-passive, that is,

$$\langle x, A(t)x \rangle_2 \leq \langle B(t)x, x \rangle_2 \quad (x \in D(A_0)). \quad (5.20)$$

Concrete examples of open-loop systems that satisfy all the above assumptions will be given below (Example 5.3 and 5.4). We now couple our open-loop system $(5.13)$-$(5.14)$ to a nonlinear dynamic controller described by the ordinary differential equation

$$\dot{v}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} K_cv_2(t) \\ -\nabla P_c(v_1(t)) - R_c(t, K_cv_2(t)) + B_cu_c(t) \end{pmatrix} \quad (5.21)$$
In the state space \( X_c := \mathbb{R}^{mc} \times \mathbb{R}^{mc} \) with the additional input-output relation
\[
y_c(t) = B_c^T K_c v_2(t) + S_c u_c(t). \tag{5.22}
\]
In these equations, \( K_c \in \mathbb{R}^{mc \times mc} \), \( B_c \in \mathbb{R}^{mc \times k} \), \( S_c \in \mathbb{R}^{k \times k} \) represent a generalized mass matrix, an input matrix, and a direct feedthrough matrix respectively satisfying \( K_c > 0 \) and \( S_c > 0 \) and, in particular,
\[
y^T S_c y \geq \varsigma |y|^2 \quad (y \in \mathbb{R}^k), \tag{5.23}
\]
where \( \varsigma > 0 \) is the smallest eigenvalue of \( S_c \). Also, the potential energy \( \mathcal{P}_c : \mathbb{R}^{mc} \to \mathbb{R}_0^+ \) is differentiable such that \( \nabla \mathcal{P}_c \) is locally Lipschitz continuous and \( \mathcal{P}_c(0) = 0 \) and the damping function \( \mathcal{R}_c : \mathbb{R}_0^+ \times \mathbb{R}^{mc} \to \mathbb{R}^{mc} \) is locally Lipschitz continuous such that \( \mathcal{R}_c(t,0) = 0 \) for all \( t \). And finally, we assume that

- \( \mathcal{P}_c \) is positive definite and radially unbounded, that is, \( \mathcal{P}_c(v_1) > 0 \) for all \( v_1 \in \mathbb{R}^{mc} \setminus \{0\} \) and \( \mathcal{P}_c(v_1) \to \infty \) as \( |v_1| \to \infty \)
- \( \mathcal{R}_c(t, \cdot) \) is damping, that is, \( v_2^T \mathcal{R}_c(t, v_2) \geq 0 \) for all \( (t,v_2) \in \mathbb{R}_0^+ \times \mathbb{R}^{mc} \).

### 5.2.2 Closed-loop system

Choosing the coupling of the controller (5.21)-(5.22) to the open-loop system (5.13)-(5.14) to be a standard feedback interconnection (5.1), we see that the arising closed-loop system is described by a differential equation of the form
\[
\dot{x}(t) = \mathcal{A}(t)x(t) + f(t,x(t)) \tag{5.24}
\]
in the state space \( X := \mathbb{R} \times X_c \) with the following additional conditions for the in- and output \( u, y \) of the closed-loop system:
\[
u(t) = B(t)x(t) \quad \text{and} \quad y(t) = C(t)x(t). \tag{5.25}
\]
In these equations, \( \mathcal{A}(t) : D(\mathcal{A}(t)) \subset X \to X \) and \( f : \mathbb{R}_0^+ \times X \to X \) are the linear and nonlinear operator defined respectively by
\[
\mathcal{A}(t)x := \begin{pmatrix} \mathcal{A}(t)x \\ K_c v_2 \\ -v_1 + B_c C(t)x \end{pmatrix} \quad \text{and} \quad f(t,x) := \begin{pmatrix} 0 \\ 0 \\ v_1 - \nabla \mathcal{P}_c(v_1) - \mathcal{R}_c(t,K_c v_2) \end{pmatrix},
\]
with \( D(\mathcal{A}(t)) := D(\mathcal{A}(t)) \times X_c \) and \( B(t), C(t) : D(\mathcal{A}(t)) \subset X \to \mathbb{R}^k \) are the linear input and output operators defined by
\[
B(t)x := B(t)x + B_c^T K_c v_2 + S_c C(t)x \quad \text{and} \quad C(t)x := C(t)x,
\]
where \( x \) and \( v \) denote the components of \( x \), that is, \((x,v_1,v_2) = (x,v) = x \).

**Corollary 5.2.** With the above assumptions, the closed-loop system (5.24)-(5.25) is well-posed.
Proof. We verify the assumptions of Theorem 4.7. As a first step, we show that Assumption 4.1 is satisfied. We first observe that the linear part \( \mathcal{A}(t) \) and the in- and output operators \( \mathcal{B}(t), \mathcal{C}(t) \) of our closed-loop system (5.24)-(5.25) factorize in the form

\[
\mathcal{A}(t) = \mathcal{A}_0 M(t), \quad \mathcal{B}(t) = \mathcal{B}_0 M(t), \quad \mathcal{C}(t) = \mathcal{C}_0 M(t)
\]

where \( I \) is the identity operator on \( X_c \). We also observe that \( \mathcal{A}_0 \) and \( \mathcal{B}_0, \mathcal{C}_0 \) are the linear part and the in- and output operators of the closed-loop system (2.16)-(2.17) from [21] with \( \mathcal{H}(\zeta) \equiv I \), respectively, and that by virtue of our assumptions above the assumptions from [21] – and in particular Condition 2.1 and 3.1 from [21] – are satisfied with \( \mathcal{H}(\zeta) \equiv I \). So, it follows that

- \( \mathcal{A}_0 := \mathcal{A}_0 \vert_{\ker \mathcal{B}_0} \) is a contraction semigroup generator on \( X \) w.r.t. the scalar product \( \langle \cdot, \cdot \rangle_X \) defined by \( \langle x, y \rangle_X := \langle x, y \rangle_2 + v_1^T w_1 + v_2^T K_c w_2 \) (Lemma 2.3 of [21]!)

- \( \mathcal{B}_0 \) has a linear bounded right-inverse \( \mathcal{R}_0 \), that is, \( \mathcal{R}_0 \in L(\mathbb{R}^k, X) \) with \( \mathcal{R}_0 \mathbb{R}^k \subset \mathcal{D}(\mathcal{B}_0) = \mathcal{D}(\mathcal{A}_0) \) and \( \mathcal{B}_0 \mathcal{R}_0 u = u \) for all \( u \in \mathbb{R}^k \) and, of course, \( \mathcal{A}_0 \mathcal{R}_0 \in L(\mathbb{R}^k, X) \) (remark after Condition 3.1 of [21]!)

With these observations at hand, we now see first that the operators

\[
\mathcal{A}(t) := \mathcal{A}(t) \vert_{\ker \mathcal{B}(t)} = \mathcal{A}_0 \vert_{\ker \mathcal{B}_0} M(t) = \mathcal{A}_0 M(t)
\]

satisfy the assumptions from Lemma 2.2 and thus Assumption 4.1(i) and second that the operators

\[
\mathcal{R}(t) := M(t)^{-1} \mathcal{R}_0
\]

satisfy Assumption 4.1(ii). And finally, Assumption 4.1(iii) is obviously satisfied by virtue of our regularity assumptions on \( \mathcal{P}_c \) and \( \mathcal{R}_c \).

As a second step, we show that Assumption 4.3 is satisfied with

\[
V(t, x) := \frac{1}{2} \left( \mathcal{H}(t) x, x \right)_2 + \mathcal{P}_c(v_1) + \frac{1}{2} v_2^T K_c v_2 \quad ((t, x) \in \mathbb{R}_0^+ \times X).
\]

Indeed, \( V \in C^1(\mathbb{R}_0^+ \times X, \mathbb{R}_0^+) \) and for every \( s \in \mathbb{R}_0^+ \) and \( (x, u) \in \mathcal{D}_s \) we see with

\[
\begin{align*}
\frac{d}{dt} V(t, x(t)) &= \frac{1}{2} \langle \mathcal{H}(t) \dot{x}(t), \dot{x}(t) \rangle_2 + \langle \mathcal{H}(t) x(t), \mathcal{A}(t) x(t) \rangle_2 \\
& \quad + \nabla \mathcal{P}_c(v_1(t))^\top \dot{v}_1(t) + (K_c v_2(t))^\top \dot{v}_2(t) \\
& \leq \langle \mathcal{B}(t) x(t), \mathcal{C}(t) x(t) \rangle + \langle \mathcal{B}_0^\top K_c v_2(t) \rangle^\top \mathcal{C}_0(t) x(t) - (K_c v_2(t))^\top \mathcal{R}_c(t, K_c v_2(t)) \\
& \leq \langle \mathcal{B}(t) x(t) - S \mathcal{C}(t) x(t) \rangle^\top \mathcal{C}(t) x(t) \leq u(t)^\top y(t) - \varsigma |y(t)|^2
\end{align*}
\]
for all \( t \in [s, T_{s,x,u}) \). (In the first inequality, we used the monotonicity and impedance-passivity assumption (5.18) and (5.20), in the second inequality we used the damping assumption on \( R_c \), and in the last inequality we used (5.23).) Consequently, Assumption 4.3(i) is satisfied with \( \alpha := \frac{1}{C} \) and \( \beta := \frac{1}{C} \). And in view of (5.17) and our assumptions on \( P_c \), Assumption 4.3(ii) is satisfied as well (invoke Lemma 2.5 of [2], for instance, to see that \( P_c \) is equivalent to the norm \( \| \cdot \| \) of \( R^m \)).

As a third step, we show that Assumption 4.4 is satisfied. Assumption 4.4(i) is obviously satisfied and in order to verify the Assumption 4.4(ii) we use that the graph norm \( \| \cdot \|_{A_0} := \| \cdot \|_X + \| A_0 \|_X \) of the port-Hamiltonian operator \( A_0 \) is equivalent to the norm of \( H^N((a,b),R^m) \), that is, for some \( \xi, \tau \in (0,\infty) \)

\[
\xi \| f \|_{H^N((a,b),R^m)} \leq \| f \|_{A_0} \leq \tau \| f \|_{H^N((a,b),R^m)} \quad (f \in D(A_0))
\]

(Lemma 3.2.3 of [1]). With this equivalence and the embedding \( H^N((a,b),R^m) \hookrightarrow C^{N-1}([a,b],R^m) \), we get for every \( s \in \mathbb{R}_+^\times \) and \((x_s,u) \in D_s \) with \((x(t),v_1(t),v_2(t)) := x(t) := x(t,s,x_s,u) \) that

\[
|((H(t)x(t))|_a - (H(t_0)x(t_0))|_a) \leq \| H(t)x(t) - H(t_0)x(t_0) \|_{C^{N-1}([a,b],R^m)}
\]

\[
\leq C \| H(t)x(t) - H(t_0)x(t_0) \|_{A_0}
\]

\[
= C \| H(t)x(t) - H(t_0)x(t_0) \|_X + C \| A(t)x(t) - A(t_0)x(t_0) \|_X
\]

(5.28)

for all \( t, t_0 \in [s, T_{s,x,u}) \). Since \( t \mapsto x(t) \in X \) and \( t \mapsto A(t)x(t) = \dot{x}(t) \in X \) are continuous and \( t \mapsto H(t) \) is strongly continuous, it follows from (5.28) that the classical output

\[
t \mapsto y(t,s,x_s,u) = C(t)x(t) = W_C(H(t)x(t))|_a
\]

is continuous and hence measurable, as desired.

\[\blackBox\]

**Example 5.3.** Consider a vibrating string with possibly time-dependent material coefficients \([9]\), that is, the transverse displacement \( w(t,\zeta) \) of the string at position \( \zeta \in (a,b) \) evolves according to the partial differential equation

\[
\partial_t (\rho(t,\zeta)\partial_\zeta w(t,\zeta)) = \partial_\zeta (T(t,\zeta)\partial_\zeta w(t,\zeta)) \quad (t \in [0,\infty), \zeta \in (a,b))
\]

(5.29)

(vibrating string equation). In these equations, the material coefficients \( \rho, T \) are the mass density and the Young modulus of the string, respectively. We assume that for some \( m_0, \bar{m} \in (0,\infty) \)

\[
m_0 \leq \rho(t,\zeta), T(t,\zeta) \leq \bar{m} \quad ((t,\zeta) \in \mathbb{R}_0^+ \times (a,b)),
\]

that for \( l = 0,1,2 \) the partial derivatives \( \partial^l_t \rho, \partial^l T \) exist and are continuous on \( \mathbb{R}_0^+ \times (a,b) \) and that \( t \mapsto \rho(t,\zeta) \) is monotonically increasing while \( t \mapsto T(t,\zeta) \) is monotonically decreasing for every \( \zeta \in (a,b) \). Also, assume that the string is clamped at its left end, that is,

\[
\partial_\zeta w(t,a) = 0 \quad (t \in \mathbb{R}_0^+)
\]

(5.30)
and that the control input $u(t)$ and observation output $y(t)$ are given respectively by the force and by the velocity at the right end of the string, that is,

$$u(t) = T(t, b) \partial_t w(t, b) \quad \text{and} \quad y(t) = \partial_t w(t, b)$$  \hspace{1cm} (5.31)

for all $t \in \mathbb{R}_0^+$. With the choices

$$\mathcal{E}(t)(\zeta) := \begin{pmatrix} \rho(t, \zeta) \partial_t w(t, \zeta) \\ \partial_t w(t, \zeta) \end{pmatrix}, \quad \mathcal{H}(t, \zeta) := \begin{pmatrix} 1/\rho(t, \zeta) & 0 \\ 0 & T(t, \zeta) \end{pmatrix}, \quad P_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $P_0 := 0 \in \mathbb{R}^{2 \times 2}$, the pde (5.29) takes the form (5.13) of a port-Hamiltonian system of order $N = 1$ and, moreover, the boundary condition (5.30) and the in- and output conditions (5.31) take the desired form (5.15) and (5.14), with matrices $W_{B,1}, W_{B,2}, W_C \in \mathbb{R}^{1 \times 4}$. It is straightforward to verify that the impedance-passivity condition (5.20) is satisfied, that the matrix $W \in \mathbb{R}^{3 \times 4}$ from (5.19) has full rank, and that all the assumptions on $\mathcal{H}$, especially the bounds (5.17) and the monotonicity (5.18), are satisfied. So, as soon as the controller is chosen as in Section 5.2.1 above, the resulting closed-loop system will be well-posed by Corollary 5.2. 

**Example 5.4.** Consider a Timoshenko beam with possibly time-dependent material coefficients [9], that is, the transverse displacement $w(t, \zeta)$ and the rotation angle $\varphi(t, \zeta)$ of the beam at position $\zeta \in (a, b)$ evolve according to the partial differential equations

$$\begin{align*}
\partial_t \left( \rho(t, \zeta) \partial_t w(t, \zeta) \right) &= \partial_t \left( K(t, \zeta) \left( \partial_t w(t, \zeta) - \varphi(t, \zeta) \right) \right) \quad \hspace{1cm} (5.32) \\
\partial_t \left( I_r(t, \zeta) \partial_t \varphi(t, \zeta) \right) &= \partial_t \left( EI(t, \zeta) \partial_t \varphi(t, \zeta) + K(t, \zeta) \left( \partial_t w(t, \zeta) - \varphi(t, \zeta) \right) \right) \quad (5.33)
\end{align*}$$

for $t \in [0, \infty), \zeta \in (a, b)$ (Timoshenko beam equations). In these equations, $\rho$, $E$, $I$, $I_r$, $K$ are the mass density, the Young modulus, the moment of inertia, the rotatory moment of inertia, and the shear modulus of the beam, respectively. We assume that for some $\underline{m}, \overline{m} \in (0, \infty)$

$$\underline{m} \leq \rho(t, \zeta), EI(t, \zeta), I_r(t, \zeta), K(t, \zeta) \leq \overline{m} \quad ((t, \zeta) \in \mathbb{R}_0^+ \times (a, b)),$$

that for $l = 0, 1, 2$ the partial derivatives $\partial_t^l \rho, \partial_t^l EI, \partial_t^l I_r, \partial_t^l K$ exist and are continuous on $\mathbb{R}_0^+ \times (a, b)$ and that $t \mapsto \rho(t, \zeta), I_r(t, \zeta)$ are monotonically increasing while $t \mapsto EI(t, \zeta), K(t, \zeta)$ are monotonically decreasing for every $\zeta \in (a, b)$. Also, assume that the beam is clamped at its left end, that is,

$$\partial_t w(t, a) = 0 \quad \text{and} \quad \partial_t \varphi(t, a) = 0 \quad (t \in [0, \infty)) \hspace{1cm} (5.34)$$

(velocity and angular velocity at the left endpoint $a$ are zero), and that the control input $u(t)$ is given by the force and the torsional moment at the right end of the beam and the observation output $y(t)$ is given by the velocity and angular velocity at the right end of the beam, that is,

$$u(t) = \begin{pmatrix} K(t, b) \left( \partial_t w(t, b) - \varphi(t, b) \right) \\ EI(t, b) \partial_t \varphi(t, b) \end{pmatrix}, \quad y(t) = \begin{pmatrix} \partial_t w(t, b) \\ \partial_t \varphi(t, b) \end{pmatrix}$$  \hspace{1cm} (5.35)
for all $t \in (0, \infty)$. With the choices

$$x(t)(\zeta) := \begin{pmatrix} \partial_\zeta w(t, \zeta) - \varphi(t, \zeta) \\ \rho(t, \zeta) \partial_t w(t, \zeta) \\ \partial_\zeta \varphi(t, \zeta) \\ L(t, \zeta) \partial_t \varphi(t, \zeta) \end{pmatrix}, \quad H(\zeta) := \begin{pmatrix} K(t, \zeta) & 0 & 0 & 0 \\ 0 & 1/\rho(t, \zeta) & 0 & 0 \\ 0 & 0 & EI(t, \zeta) & 0 \\ 0 & 0 & 0 & 1/I_r(t, \zeta) \end{pmatrix},$$

and the same choice of $P_1, P_0 \in \mathbb{R}^{4 \times 4}$ as in [8], the pde (5.32)-(5.33) take the form (5.13) of a port-Hamiltonian system of order $N = 1$ and, moreover, the boundary condition (5.34) and the in- and output conditions (5.35) take the desired form (5.15) and (5.14) with matrices $W_{B,1}, W_{B,2}, W_C \in \mathbb{R}^{2 \times 8}$. It is straightforward to verify that impedance-passivity condition (5.20) is satisfied, that the matrix $W \in \mathbb{R}^{6 \times 8}$ from (5.19) has full rank, and that all the assumptions on $H$, especially the bounds (5.17) and the monotonicity (5.18), are satisfied. So, as soon as the controller is chosen as in Section 5.2.1 above, the resulting closed-loop system will be well-posed by Corollary 5.2. ▽

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