Longitudinal conductivity and Hall coefficient
in two-dimensional metals with spiral magnetic order

Johannes Mitscherling\textsuperscript{1} and Walter Metzner\textsuperscript{1}

\textsuperscript{1}Max Planck Institute for Solid State Research, D-70569 Stuttgart, Germany

(Dated: July 27, 2018)

We compute the longitudinal DC conductivity and the Hall conductivity in a two-dimensional metal with spiral magnetic order. Scattering processes are modeled by a momentum-independent single-particle decay rate $\Gamma$. We derive expressions for the conductivities, which are valid for arbitrary values of $\Gamma$. Both intraband and interband contributions are fully taken into account. For small $\Gamma$, the ratio of interband and intraband contributions is of order $\Gamma^2$. In the limit $\Gamma \to 0$, the conductivity formulae assume a simple quasi-particle form, as first derived by Voruganti et al.\textsuperscript{11} Using the complete expressions, we can show that scattering rates in the regime of recent transport experiments for cuprate superconductors in high magnetic fields are sufficiently small to justify the application of these simplified formulae. The longitudinal conductivity exhibits a pronounced nematicity in the spiral state. The drop of the Hall number as a function of doping observed recently in several cuprate compounds can be described with a suitable phenomenological ansatz for the magnetic order parameter.

PACS numbers:
I. INTRODUCTION

Understanding the “normal” ground state in the absence of superconductivity is the key to understand the fluctuations that govern the anomalous behavior of cuprate superconductors above the critical temperature. Superconductivity can be suppressed by applying a magnetic field, but very high fields are required for a complete elimination in high-temperature superconductors. Recently, magnetic fields up to 88 Tesla were achieved, such that the critical temperature of YBa$_2$Cu$_3$O$_y$ (YBCO) and other cuprate compounds could be substantially suppressed even at optimal doping. Charge transport measurements in such high magnetic fields indicate a drastic reduction of the charge carrier density in a narrow doping range upon entering the pseudogap regime. In particular, Hall measurements at various dopings $p$ yield a drop of the Hall number from $1 + p$ to values near $p$.

The drop in carrier density indicates a phase transition associated with a Fermi surface reconstruction. Storey has pointed out that the observed Hall number drop is consistent with the formation of a Néel antiferromagnet, and also with a Yang-Rice-Zhang (YRZ) state without long-range order. Other possibilities are fluctuating antiferromagnets, fractionalized Fermi liquids, and charge density wave states. As long as no spectroscopic measurements are possible in high magnetic fields, it is hard to confirm or rule out any of these candidates experimentally. For strongly underdoped cuprates, where superconductivity is absent or very weak, neutron scattering probes show that the Néel state is quickly destroyed upon doping, in agreement with theoretical findings. For underdoped YBCO incommensurate antiferromagnetic order has been observed.

On the theoretical side, microscopic calculations yield incommensurate antiferromagnetism as the most robust order parameter in the non-superconducting ground state over a wide doping range away from half-filling. For the two-dimensional Hubbard model, antiferromagnetic order with wave vectors $Q$ away from the Néel point ($\pi, \pi$) was found in numerous mean-field calculations and also by expansions for small hole density, where fluctuations are taken into account. At weak-coupling magnetic order with $Q \neq (\pi, \pi)$ was confirmed by functional renormalization group calculations and at strong coupling by state-of-the-art numerical techniques. Recent dynamical mean-field calculations with vertex corrections suggest that the Fermi surface geometry determines the (generally incommensurate) ordering wave vector not only at weak coupling, but also at strong coupling.
For the two-dimensional $t$-$J$ model, the strong coupling limit of the Hubbard model, expansions for small hole density indicate that the Néel state is stable only at half-filling, and is replaced by a spiral antiferromagnet upon doping.\cite{23,24}

There is a whole zoo of distinct magnetic states. The most favorable, or at least the most popular, are collinear states, combined with charge order to form spin-charge stripes, and planar spiral states. Stripe order has been observed in La-based cuprates.\cite{25} Theoretically, commensurate stripe order was shown to minimize the ground state energy of the strongly interacting Hubbard model with pure nearest-neighbor hopping at doping $1/8$.\cite{21} However, this is a very special choice of parameters, and stripe order is not ubiquitous in cuprates. Recently, it was shown that it is difficult to explain the recent high-field transport experiments in cuprates by collinear magnetic order.\cite{26} Generally, the energy difference between different magnetic states seems to be rather small.

In the present paper we compute the electrical conductivity and the Hall coefficient for planar spiral magnetic states. In a spiral magnet, the electron band is split in only two quasiparticle bands. In this respect, the spiral state is as simple as the Néel state. By contrast, all other magnetically ordered states entail a fractionalization in many subbands, actually infinitely many in case of incommensurate order. Hence, only the spiral magnet forms a metal with a simple Fermi surface, defined by the border of a small number of electron and hole pockets. For a sufficiently large order parameter there are only hole pockets in the hole-doped system. The spectral weight for single-electron excitations is strongly anisotropic, so that the spectral function exhibits Fermi arcs,\cite{27} which are a characteristic feature of the pseudogap phase in high-$T_c$ cuprates.

The electromagnetic response of spiral magnetic states has already been analyzed by Voruganti et al.\cite{1} They derived formulae for the DC conductivity and for the Hall conductivity in the low-field limit, that is, to linear order in the magnetic field. The spiral states were treated in mean-field approximation. The resulting expressions have the same form as for non-interacting electrons, with the bare dispersion relation replaced by quasi-particle bands. Assuming a simple phenomenological form of the spiral order parameter as a function of doping, Eberlein et al.\cite{27} showed that the Hall conductivity computed with the formula derived by Voruganti et al. indeed exhibits a drop of the Hall number consistent with the recent experiments on cuprates in high magnetic fields. Most recently an expression for the thermal conductivity in a spiral state has been derived along the same lines, and a
similar drop in the carrier density has been found.\cite{28}

The expressions for the electrical and Hall conductivities derived by Voruganti et al.\cite{1} have been obtained for small scattering rates. However, the scattering rate in the cuprate samples studied experimentally is sizable. For example, in spite of the high magnetic fields, the product of cyclotron frequency and scattering time $\omega_c \tau$ extracted from the experiments on La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$ (Nd-LSCO) samples is as low as 0.075.\cite{1} Moreover, from the derivation of Voruganti et al., the precise criterion for a “small” decay rate is not clear. Hence, we derive complete expressions for the electrical and Hall conductivities allowing for decay rates of arbitrary size. We assume $\omega_c \tau \ll 1$ such that an expansion to linear order in the magnetic field is indeed sufficient. We find that the relatively simple formulae derived by Voruganti et al. are valid only if the scattering rate is much smaller than the direct gap between the upper and the lower quasi-particle band. Otherwise interband terms yield additional contributions. For a sizable scattering rate the drop in the Hall number caused by the magnetic order is less steep than for a small scattering rate. However, a numerical evaluation of the conductivity formulae shows that for parameters as relevant for cuprate superconductors, the interband contributions play only a minor role. Applying the conductivity formulae to cuprates we show that the longitudinal conductivities exhibit a pronounced nematicity in the spiral state, and the observed Hall number drop can be fitted with realistic parameters.

This article is structured as follows. In Sec. II we derive the formulae for the DC conductivity and the Hall conductivity. We recapitulate the formalism provided already by Voruganti et al.\cite{1} for the sake of a coherent notation and a self-contained presentation. Lengthy algebra is carried out in the appendices. In Sec. III we discuss results with a focus on the role of interband terms in presence of a sizable scattering rate. Here we make contact to the recent experiments in cuprates. A conclusion in Sec. IV closes the presentation.

II. FORMALISM

A. Action

We begin by recapulating the derivation of the action describing electrons with spiral magnetic order coupled to an electromagnetic field.\cite{1} We use natural units such that $\hbar = 1$ and $c = 1$. The kinetic energy of electrons in a tight-binding representation of a single
valence band is given by

\[ H_0 = \sum_{j,j'} \sum_{\sigma} t_{jj'} c_{j,\sigma}^t c_{j',\sigma} = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} a_{\mathbf{p},\sigma}^\dagger a_{\mathbf{p},\sigma} , \]  

(1)

where \( t_{jj'} \) is the hopping amplitude between lattice sites \( j \) and \( j' \). The momentum dependent band energy \( \epsilon_{\mathbf{p}} \) is the Fourier transform of \( t_{jj'} \). The operators \( c \) and \( c^\dagger \) are electron annihilation and creation operators in real space, respectively, while \( a \) and \( a^\dagger \) are the corresponding operators in momentum space. The index \( \sigma \) describes the spin orientation \( \uparrow \) or \( \downarrow \).

The mean-field Hamiltonian for electrons in a spiral magnetic state has the form

\[ H = H_0 - \sum_{\mathbf{p}} \Delta \left( a_{\mathbf{p}+\mathbf{Q}/2,\uparrow}^\dagger a_{\mathbf{p}-\mathbf{Q}/2,\downarrow} + a_{\mathbf{p}-\mathbf{Q}/2,\uparrow}^\dagger a_{\mathbf{p}+\mathbf{Q}/2,\downarrow} \right) . \]  

(2)

The “magnetic gap” \( \Delta \) is a convenient order parameter quantifying the strength of the spiral order. It can be chosen real. In a mean-field solution of the Hubbard model, the magnetic gap is defined by \( \Delta = U \langle a_{\mathbf{p}+\mathbf{Q}/2,\uparrow}^\dagger a_{\mathbf{p}-\mathbf{Q}/2,\downarrow} \rangle \). We have dropped a constant in Eq. (2) which contributes to the total energy, but not to the electromagnetic response.

The action corresponding to the Hamiltonian \( H \) can be written most conveniently by using Grassmann spinors of the form

\[ \Psi_{\mathbf{p}} = \begin{pmatrix} \psi_{i\mathbf{p}_0,\mathbf{p}+\mathbf{Q}/2,\uparrow} \\ \psi_{i\mathbf{p}_0,\mathbf{p}-\mathbf{Q}/2,\downarrow} \end{pmatrix} . \]  

(3)

Here and in the following we use frequency-momentum variables \( p = (i\mathbf{p}_0, \mathbf{p}) \), where \( \mathbf{p}_0 \) is a fermionic Matsubara frequency. The action can then be written as

\[ S[\Psi, \Psi^*] = -\sum_{\mathbf{p}} \Psi_{\mathbf{p}}^* G_{\mathbf{p}}^{-1} \Psi_{\mathbf{p}} , \]  

with the inverse matrix propagator given by

\[ G_{\mathbf{p}}^{-1} = \begin{pmatrix} ip_0 + \mu - \epsilon_{\mathbf{p}+\mathbf{Q}/2} & \Delta \\ \Delta & ip_0 + \mu - \epsilon_{\mathbf{p}-\mathbf{Q}/2} \end{pmatrix} , \]  

(5)

where \( \mu \) is the chemical potential.

We now couple the system to electromagnetic fields. We choose a gauge such that the scalar potential vanishes. The electric and magnetic fields are thus entirely determined by the vector potential \( \mathbf{A}(\mathbf{r},t) \) as \( \mathbf{E}(\mathbf{r},t) = -\partial_t \mathbf{A}(\mathbf{r},t) \) and \( \mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t) \), respectively.
The coupling to the orbital motion of the electrons gives rise to a phase factor multiplying the hopping amplitudes:

\[ t_{jj'}[A] = t_{jj'} \exp \left( i e \int_{r_{j'}}^{r_j} A(r,t) \cdot dr \right), \]

where \( e < 0 \) is the electron charge, and \( r_j \) is the spatial position of the lattice site labeled by \( j \). We discard the Zeemann coupling of the magnetic field to the electron spin. For fields varying slowly between lattice sites connected by \( t_{jj'} \), which is the case we are interested in, we can parameterize \( A(r,t) \) by a link variable \( A_{jj'}(t) = A((r_j + r_{j'})/2, t) \), and approximate the line integral in Eq. (6) by

\[ \int_{r_{j'}}^{r_j} A(r,t) \cdot dr = A_{jj'}(t) \cdot r_{jj'}. \]

with \( r_{jj'} = r_j - r_{j'} \). The action is expressed in terms of the Wick-rotated vector potential, that is, for imaginary times \( \tau \), which we denote by \( A_{jj'}(\tau) \). Expanding Eq. (6) in powers of \( A \), one obtains the complete action

\[ S[\Psi, \Psi^*; A] = - \sum_p \Psi_p^* G_p^{-1} \Psi_p + \sum_{p,p'} \Psi_p^* V_{pp'}[A] \Psi_{p'}, \]

where the coupling to the electromagnetic field has the form

\[ V_{pp'}[A] = \sum_{n=1}^{\infty} \frac{e^n}{n!} \sum_{q_1,\ldots,q_n} \lambda_{pp'}^{q_1,\ldots,q_n} A_{q_1}^{q_1} \cdots A_{q_n}^{q_n} \delta_{p-p', \sum_{i=1}^{n} q_i}. \]

Here and in the following we use Einstein’s summation convention for repeated Greek indices. \( A_q^\alpha \) with \( \alpha = x, y, z \) and \( q = (iq_0, q) \) is the \( \alpha \)-component of the Fourier transform \( A_q \) of \( A_{jj'}(\tau) \), that is,

\[ A_{jj'}(\tau) = \sum_q A_q e^{i[\frac{1}{2} q(r_j + r_{j'}) - q_0 \tau]} . \]

The \( n \)-th order vertices are given by

\[ \lambda_{pp'}^{q_1,\ldots,q_n} = \begin{pmatrix} \epsilon_{p+2q+Q/2} & 0 \\ 0 & \epsilon_{p'+2q'-Q/2} \end{pmatrix}, \]

where \( \epsilon_{p}^{\alpha_1,\alpha_n} = \partial^n \epsilon_p / (\partial p_{\alpha_1} \ldots \partial p_{\alpha_n}) \).

Current-relaxing scattering processes are taken into account in the simplest fashion by adding a fixed decay rate \( \Gamma \) to the (inverse) propagator, such that

\[ G_p^{-1} = \begin{pmatrix} ip_0 + \mu - \epsilon_{p+Q/2} + i\Gamma \text{sgn}(p_0) & \Delta \\ \Delta & ip_0 + \mu - \epsilon_{p-Q/2} + i\Gamma \text{sgn}(p_0) \end{pmatrix}. \]
A decay term of this form is obtained, for example, from scattering at short-ranged impurity potentials in Born approximation.\footnote{\textsuperscript{19}}

B. Current and response functions

The action $S[\Psi, \Psi^*; A]$ in Eq. (8) is quadratic in the fermion fields. The partition function is thus given by a Gaussian integral. Performing the integral, taking the logarithm, and expanding in powers of $V[A]$ yields the grand canonical potential in the form\footnote{\textsuperscript{14}}

$$\Omega[A] = \Omega_0 + T \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}(GV[A])^n,$$

where $T$ is the temperature, and $\Omega_0$ is the grand canonical potential in the absence of $A$. Both $G$ and $V[A]$ are matrices in frequency-momentum and spin space, where $G_{pp'} = \delta_{pp'}G_p$ with $G_p$ given by Eq. (12), and $V_{pp'}[A]$ is given by Eq. (9). The trace is a sum over frequency, momentum, and spin-orientation. Note that $V_{pp'}[A]$ is diagonal in spin space, since the electromagnetic field couples only to the charge.

The charge current is defined by the first derivative of the grand canonical potential with respect to the vector potential,

$$j^\alpha_q = -\frac{1}{L} \frac{\partial \Omega[A]}{\partial A^{-\alpha}_q},$$

where $L$ is the number of lattice sites. We choose units of length such that a single lattice cell has volume one, so that $L$ corresponds to the volume of the system. An expansion of the current in powers of $A$ has the general form

$$j^\alpha_q = -\sum_{n=1}^{\infty} \sum_{q_1, \ldots, q_n} K_{q_1, \ldots, q_n}^{\alpha_1, \ldots, \alpha_n} A_{q_1}^{\alpha_1} \cdots A_{q_n}^{\alpha_n} \delta_{q_1, q_1+\ldots+q_n},$$

In this work we focus on the DC conductivity $\sigma^{\alpha\beta}$ and the DC Hall conductivity $\sigma^{\alpha\beta\gamma}_H$, which describe the current response to homogeneous static electric and magnetic fields,

$$j^\alpha = \left[ \sigma^{\alpha\beta} + \sigma^{\alpha\beta\gamma}_H B^\gamma \right] E^\beta,$$

to leading order in $E$ and $B$. In a microscopic calculation the DC response is obtained as the zero frequency limit of the response to a spatially homogeneous dynamical electric field $E(t)$. A constant magnetic field is associated with a vector potential depending only on space, not on time. Following Voruganti et al.\footnote{\textsuperscript{13}} we therefore split the vector potential
as \( \mathbf{A}(\mathbf{r}, t) = \mathbf{a}^E(t) + \mathbf{a}^B(\mathbf{r}) \), such that \( \mathbf{E}(t) = -\partial_t \mathbf{a}^E(t) \) and \( \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{a}^B(\mathbf{r}) \). The corresponding Fourier transforms are related by \( \mathbf{E}_\omega = i\omega \mathbf{a}^E_\omega \) and \( \mathbf{B}_\mathbf{q} = i\mathbf{q} \times \mathbf{a}^B_\mathbf{q} \).

For imaginary time fields of the form \( \mathbf{A}(\mathbf{r}, \tau) = \mathbf{a}^E(\tau) + \mathbf{a}^B(\mathbf{r}) \), the expansion (15), carried out to the relevant order, can be written as

\[
\begin{align*}
J^\alpha_q &= -\left( \delta_{q,0} K^\alpha_0 + K^\alpha_{q,0} a^B_\mathbf{q} \right) a^E_{iq_0} + \ldots, \\
K^\alpha_0 &= -i\omega \sigma^\alpha_0.
\end{align*}
\]

The Hall conductivity is obtained from the zero frequency and zero momentum limit of the dynamical quantity \( \sigma^\alpha_0 \), related to \( K^\alpha_{q,0} \) by

\[
K^\alpha_{q,0} = -i\omega \sigma^\alpha_0.
\]

C. Diagonalization

To evaluate the trace in Eq. (13), it is convenient to use a basis in which \( G_p \) is diagonal. This can be achieved by the unitary transformation

\[
U_p = \begin{pmatrix} \cos \theta_p & \sin \theta_p \\ -\sin \theta_p & \cos \theta_p \end{pmatrix},
\]

where the rotation angle \( \theta_p \) must satisfy the condition

\[
\tan(2\theta_p) = \frac{2\Delta}{\epsilon_p + \mathbf{Q}/2 - \epsilon_p - \mathbf{Q}/2}.
\]

The transformed propagator has the diagonal form

\[
G_p^{-1} = U_p^\dagger G_p^{-1} U_p = \begin{pmatrix} ip_0 + i\Gamma \text{sgn}(p_0) - E^+_p & 0 \\ 0 & ip_0 + i\Gamma \text{sgn}(p_0) - E^-_p \end{pmatrix},
\]

where \( E^+_p \) and \( E^-_p \) are the two quasi-particle bands in the spiral magnetic state. Their momentum dependence is given by

\[
E^\pm_p = g_p \pm \sqrt{\hbar^2 + \Delta^2 - \mu}.
\]
where \( g_p = \frac{1}{2}(\epsilon_p + \frac{Q}{2} + \epsilon_p - \frac{Q}{2}) \) and \( h_p = \frac{1}{2}(\epsilon_p + \frac{Q}{2} - \epsilon_p - \frac{Q}{2}) \).

The vertices \( \lambda_{pp'}^{\alpha_1...\alpha_n} \) have to be transformed accordingly as

\[
\tilde{\lambda}_{pp'}^{\alpha_1...\alpha_n} = U_{pp'}^{\dagger} \lambda_{pp'}^{\alpha_1...\alpha_n} U_{pp'}.
\]

In the following we will mainly deal with the first and second order vertices for vanishing momentum transfer, that is, \( p = p' \). The rotated first order vertex has the form

\[
\tilde{\lambda}_p^{\alpha} = \begin{pmatrix}
E^{+,\alpha}_p & F^{\alpha}_p \\
F^{\alpha}_p & E^{-,\alpha}_p
\end{pmatrix},
\]

(25)

where \( E^{\pm,\alpha}_p = \partial E^{\pm}_p / \partial p_\alpha \), and

\[
F^{\alpha}_p = \frac{2\Delta}{E^{+}_p - E^{-}_p} h^{\alpha}_p,
\]

(26)

with \( h^{\alpha}_p = \partial h_p / \partial p_\alpha \). The rotated second order vertex can be written as

\[
\tilde{\lambda}_{p}^{\alpha\beta} = \begin{pmatrix}
E^{+,\alpha\beta}_p - C^{\alpha\beta}_p & H^{\alpha\beta}_p \\
H^{\alpha\beta}_p & E^{-,\alpha\beta}_p + C^{\alpha\beta}_p
\end{pmatrix},
\]

(27)

where \( E^{\pm,\alpha\beta}_p = \partial^2 E^{\pm}_p / (\partial p_\alpha \partial p_\beta) \),

\[
C^{\alpha\beta}_p = \frac{2}{E^{+}_p - E^{-}_p} F^{\alpha}_p F^{\beta}_p = \frac{8\Delta^2}{(E^{+}_p - E^{-}_p)^3} h^{\alpha}_p h^{\beta}_p,
\]

(28)

and

\[
H^{\alpha\beta}_p = \frac{2\Delta}{E^{+}_p - E^{-}_p} h^{\alpha\beta}_p,
\]

(29)

with \( h^{\alpha\beta}_p = \partial^2 h_p / (\partial p_\alpha \partial p_\beta) \).

In subsequent derivations it will be convenient to decompose the vertices in diagonal and off-diagonal parts, such as

\[
\tilde{\lambda}_p^{\alpha} = \mathcal{E}_p^{\alpha} + \mathcal{F}_p^{\alpha}, \quad \text{with}
\]

\[
\mathcal{E}_p^{\alpha} = \begin{pmatrix} E^{+,\alpha}_p & 0 \\ 0 & E^{-,\alpha}_p \end{pmatrix}, \quad \mathcal{F}_p^{\alpha} = \begin{pmatrix} 0 & F^{\alpha}_p \\ F^{\alpha}_p & 0 \end{pmatrix},
\]

(30)

and \( \tilde{\lambda}_{pp}^{\alpha\beta} = \mathcal{E}_p^{\alpha\beta} - C^{\alpha\beta}_p + H^{\alpha\beta}_p \), with \( \mathcal{E}_p^{\alpha\beta}, C^{\alpha\beta}_p \), and \( H^{\alpha\beta}_p \) defined analogously.

D. Ordinary conductivity

Expanding the grand canonical potential \( \Omega[A] \), Eq. (13), to second order in \( A \), and performing the functional derivative with respect to the vector potential yields the current
to linear order in \( A \), from which we can read off the response kernel \( K_{i\omega_0}^{\alpha\beta} \). Using a basis in which the electron propagator is diagonal, one obtains

\[
K_{i\omega_0}^{\alpha\beta} = e^2 \frac{T}{L} \sum_p \text{tr} \left( G_{p,i\omega_0+i\omega_0} \lambda_{pp}^\alpha G_{p,i\omega_0+i\omega_0} \lambda_{pp}^\beta + G_{p,i\omega_0+i\omega_0} \lambda_{pp}^{\alpha\beta} \right).
\] (31)

The first term, known as *paramagnetic* contribution, is due to the second order term from Eq. (13) with first order contributions for the vertices Eq. (9). The second term, known as *diamagnetic* contribution, arises from the first order term in Eq. (13) with the second order contribution for the vertex. Using simple identities (see Appendix C), the response kernel can be rewritten as

\[
K_{i\omega_0}^{\alpha\beta} = e^2 \frac{T}{L} \sum_p \text{tr} \left[ (G_{p,i\omega_0+i\omega_0} - G_{p,i\omega_0}) \mathcal{E}_p^\alpha G_{p,i\omega_0} \mathcal{E}_p^\beta \right]
\] (32)

In this form all terms are quadratic in the propagators, and the property \( K_{i\omega_0}^{\alpha\beta} = 0 \), which is dictated by gauge invariance, is manifestly satisfied.

Performing the analytic continuation to real frequencies and taking the limit \( \omega \to 0 \) (see Appendix C) one obtains the following expression for the DC conductivity

\[
\sigma^{\alpha\beta} = -e^2 \frac{T}{L} \sum_p \int d\epsilon f(\epsilon) \left\{ E_p^{\beta,\alpha} E_p^{\beta,\alpha} [A_p^+(\epsilon)]^2 + E_p^{\beta,-\alpha} E_p^{\beta,-\alpha} [A_p^-(\epsilon)]^2 + 2F_p^\alpha F_p^\beta [A_p^+(\epsilon) A_p^-(\epsilon)] \right\},
\] (33)

where \( f'(\epsilon) \) is the first derivative of the Fermi function \( f(\epsilon) = (e^{\epsilon/T} + 1)^{-1} \), and

\[
A_p^\pm(\epsilon) = \frac{\Gamma/\pi}{(\epsilon - E_p^\pm)^2 + \Gamma^2}
\] (34)

is the spectral function of the quasi-particles. The first two terms in Eq. (33) are *intraband* contributions. They have the same form as for non-interacting electrons with bare electron bands replaced by quasi-particle bands. The last term is an *interband* contribution involving states from both quasi-particle bands. For low temperatures and small \( \Gamma \) only momenta close to the quasi-particle Fermi surface, where \( |E_p^+| \) or \( |E_p^-| \) is small, contribute significantly to the conductivity.

The expression for the conductivity can be further simplified for small \( \Gamma \). The spectral functions \( A^\pm(\epsilon) \) are Lorentzians of width \( \Gamma \). Using \( \pi [A_p^\pm(\epsilon)]^2 \to (2\Gamma)^{-1}\delta(\epsilon - E_p^\pm) \) for \( \Gamma \to 0 \), one can simplify the intraband contribution to

\[
\sigma_{\text{intra}}^{\alpha\beta} \to -e^2 \frac{T}{L} \sum_p \sum_{n=\pm} f'(E_p^n) E_p^{n,\alpha} E_p^{n,\beta},
\] (35)
where $\tau = 1/(2\Gamma)$ is the relaxation time. This simplification holds when $\Gamma$ is so small that the quasi-particle velocities $E_p^{\pm,\alpha}$ are almost constant in a momentum range in which the variation of $E_p^\pm$ is of order $\Gamma$. Under the same assumption, and if in addition $\Gamma \ll E_p^+ - E_p^-$, one has
\begin{equation}
\pi A_p^+(\epsilon)A_p^-(\epsilon) \to \frac{\Gamma}{(E_p^+ - E_p^-)^2} \left[ \delta(\epsilon - E_p^+) + \delta(\epsilon - E_p^-) \right].
\end{equation}

The interband contribution to the conductivity can then be simplified to
\begin{equation}
\sigma_{\text{inter}}^{\alpha\beta} \to -e^2 L \sum_p \sum_{n=\pm} f'(E_p^n) \frac{\epsilon_p^\alpha F_p^{\beta}}{\tau^2(E_p^+ - E_p^-)^2}.
\end{equation}

The interband contribution is thus suppressed by a factor $\tau^{-2}$ compared to the intraband contribution for large $\tau$, and the naive formula for the conductivity, where bare bands are simply replaced by quasi-particle bands, can be applied. $E_p^+ - E_p^- = 2\sqrt{\frac{1}{4}(\epsilon_p+q/2 - \epsilon_p-q/2)^2 + \Delta^2}$ is larger or equal to $2\Delta$, such that $\Gamma \ll E_p^+ - E_p^-$ is satisfied for all $p$ if $\Gamma \ll 2\Delta$. However, even for small $\Gamma$, interband contributions may play a role near the transition between the paramagnetic and the antiferromagnetic phase, where $\Delta$ is also small.

### E. Hall conductivity

Expanding the grand canonical potential $\Omega[A]$ to third order yields the current to quadratic order in $A$. Inserting the decomposition $A = a^F + a^B$ and comparing with Eq. (17), one can read off the response kernel $K_{q,iq_0}^{\alpha\beta\gamma}$. Using the quasi-particle basis in which the electron propagator is diagonal, one obtains
\begin{equation}
K_{q,iq_0}^{\alpha\beta\gamma} = e^3 T \sum_p \text{tr}(G_{p,ip_0} \tilde{\lambda}_{pp}^{\alpha\beta\gamma} + G_{p,ip_0} \tilde{\lambda}_{pp}^{\alpha\beta\gamma}) + G_{p^+,ip_0} \tilde{\lambda}_{p^+p}^{\gamma\beta} - G_{p^-,ip_0} \tilde{\lambda}_{p^-p}^{\alpha\beta} + G_{p,ip_0+iq_0} \tilde{\lambda}_{pp}^{\alpha\beta\gamma}
\end{equation}
Appendix D. For small finite momenta, $K_{q,iq_0}^{\alpha\beta\gamma}$ can be expanded as $K_{q,iq_0}^{\alpha\beta\gamma} = K_{iq_0}^{\alpha\beta\gamma} q_\delta + \ldots$. To determine $\sigma^{\alpha\beta\gamma}_H$ in an uniform magnetic field, it is sufficient to compute the first order coefficient

$$K_{iq_0}^{\alpha\beta\gamma} = \left. \frac{\partial}{\partial q_\delta} K_{q,iq_0}^{\alpha\beta\gamma} \right|_{q=0}. \tag{39}$$

From the six terms in Eq. (38), the first and the third do not contribute to $K_{iq_0}^{\alpha\beta\gamma\delta}$ since they are independent of $q$. The contribution from the second term, which is independent of $q_0$, also vanishes, as can be seen by a short explicit calculation. The evaluation of the remaining three contributions to $K_{iq_0}^{\alpha\beta\gamma\delta}$ is rather involved, since the momentum derivative generates numerous terms. After a lengthy calculation, which is presented in Appendix D, we obtain the comparatively simple result

$$K_{iq_0}^{\alpha\beta\gamma\delta} = -e^3 \frac{T}{4L} \sum_p \text{tr} \left[ (G_p,ip_0+iq_0 - G_p,ip_0-iq_0) \mathcal{C}_p^\alpha G_p,ip_0 \mathcal{C}_p^\delta G_p,ip_0 \mathcal{E}_p^{\beta\gamma} \right],$$

$$-e^3 \frac{T}{4L} \sum_p \text{tr} \left[ (G_p,ip_0+iq_0 - G_p,ip_0-iq_0) \mathcal{F}_p^\alpha G_p,ip_0 \mathcal{C}_p^\delta G_p,ip_0 \mathcal{H}_p^{\beta\gamma} \right],$$

$$-e^3 \frac{T}{2L} \sum_p \text{tr} \left[ (G_p,ip_0+iq_0 - G_p,ip_0-iq_0) \mathcal{F}_p^\alpha G_p,ip_0 \mathcal{F}_p^\delta G_p,ip_0 \mathcal{E}_p^{\beta\gamma} \right],$$

$$-(\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta). \tag{40}$$

$K_{iq_0}^{\alpha\beta\gamma\delta}$ is antisymmetric under exchange of the first two indices ($\alpha$ and $\beta$), as well as under exchange of the last two indices ($\gamma$ and $\delta$). $K_{iq_0}^{\alpha\beta\gamma\delta}$ obviously vanishes for $q_0 = 0$.

Comparing Eq. (40) with Eq. (19), and taking the limit $\omega \to 0$, one obtains the following expression for the DC Hall conductivity,

$$\sigma^{\alpha\beta\nu}_H = \sigma^{\alpha\beta\nu}_H^{\text{intra}} + \sigma^{\alpha\beta\nu}_H^{\text{inter}}, \tag{41}$$

with the intraband contribution

$$\sigma^{\alpha\beta\nu}_H^{\text{intra}} = -e^3 \lim_{\omega \to 0} \frac{1}{\omega} \frac{T}{4L} \sum_{p,p_0} \mathcal{E}^{\nu\gamma\delta} \text{tr} \left[ (G_p,ip_0+iq_0 - G_p,ip_0-iq_0) \right.$$

$$\left. \times \mathcal{C}_p^\alpha G_p,ip_0 \mathcal{C}_p^\delta G_p,ip_0 \mathcal{E}_p^{\beta\gamma} - (\alpha \leftrightarrow \beta) \right] \bigg|_{iq_0 \to \omega+i0^+}, \tag{42}$$
and the interband contributions

\[
\sigma_{H,\text{inter}}^{\alpha\beta\nu} = -e^3 \lim_{\omega \to 0} \frac{1}{4L} \sum_{p, p_0} e^{i\gamma\delta} \text{tr} \left[ \left( G_{p,ip_0 + iq_0} - G_{p,ip_0 - iq_0} \right) \right] \\
\times \mathcal{F}_p^\alpha G_{p,ip_0} \mathcal{E}_p \mathcal{G}_{p,ip_0} \mathcal{H}_{p}^{\beta\gamma} \left( \alpha \leftrightarrow \beta \right) \left|_{iq_0 \to \omega + i0^+} \right.
\]

\[
- e^3 \lim_{\omega \to 0} \frac{1}{2L} \sum_{p, p_0} e^{i\gamma\delta} \text{tr} \left[ \left( G_{p,ip_0 + iq_0} - G_{p,ip_0 - iq_0} \right) \right] \\
\times \mathcal{F}_p^\alpha G_{p,ip_0} \mathcal{E}_p \mathcal{G}_{p,ip_0} \mathcal{E}_{p}^{\beta\gamma} \left( \alpha \leftrightarrow \beta \right) \left|_{iq_0 \to \omega + i0^+} \right.
\]

\[ \tag{43} \]

Already at this point one can see that \( \sigma_{H}^{\alpha\beta\nu} \) vanishes for \( \alpha = \beta \), that is, the magnetic field does not affect the longitudinal conductivity to linear order in \( B \). The Matsubara sum and the analytic continuation to real frequencies, \( iq_0 \to \omega + i0^+ \), can be performed analytically. All contributions to \( \sigma_{H}^{\alpha\beta\nu} \) contain a Matsubara sum of the form

\[
K_{iq_0}^{H} = T \sum_{p_0} \text{tr} \left[ (G_{ip_0 + iq_0} - G_{ip_0 - iq_0}) \mathcal{M}_1 G_{ip_0} \mathcal{M}_2 G_{ip_0} \mathcal{M}_3 \right], \tag{44}
\]

with arbitrary frequency independent matrices \( \mathcal{M}_i \). Momentum dependences are not written here. In Appendix [D] we show that

\[
\lim_{\omega \to 0} \frac{1}{\omega} K_{iq_0 \to \omega + i0^+}^{H} = \frac{-2\pi}{\pi} \int d\epsilon f'(\epsilon) \times \text{tr} \left[ \text{Im} \mathcal{G}_\epsilon^A \mathcal{M}_1 \partial_\epsilon \text{Re} (\mathcal{G}_\epsilon^R \mathcal{M}_2 \mathcal{G}_\epsilon^R) \mathcal{M}_3 \right]. \tag{45}
\]

where \( \mathcal{G}_\epsilon^A \) and \( \mathcal{G}_\epsilon^R \) are the advanced and retarded quasi-particle Green functions, respectively.

Using Eq. (45), and performing a partial integration, the intraband contribution to the Hall conductivity can be written as

\[
\sigma_{H,\text{intra}}^{\alpha\beta\nu} = -e^3 \pi^2 \sum_{p} \sum_{n=\pm} \int d\epsilon f'(\epsilon) \left[ E_{p}^{n,\alpha} E_{p}^{n,\delta} E_{p}^{n,\beta\gamma} - (\alpha \leftrightarrow \beta) \right] \left[ A_{p}^{n}(\epsilon) \right]^3. \tag{46}
\]

For a two-dimensional electron system with a dispersion depending only on \( p_x \) and \( p_y \), and a perpendicular magnetic field (in \( z \)-direction), the relevant component of the Hall conductivity reads

\[
\sigma_{H,\text{intra}}^{xy} = \frac{e^3 \pi^2}{3} \int \frac{d^2p}{(2\pi)^2} \sum_{n=\pm} \int d\epsilon f'(\epsilon) \times \left[ \left( E_{p}^{n,x} \right)^2 E_{p}^{n,yy} - E_{p}^{n,x} E_{p}^{n,y} E_{p}^{n,yy} + (x \leftrightarrow y) \right] \left[ A_{p}^{n}(\epsilon) \right]^3. \tag{47}
\]

Here and in the following \( (x \leftrightarrow y) \) denotes addition of the preceding terms with \( x \) and \( y \) exchanged. Note, however, that \( \sigma_{H}^{xy} \) is antisymmetric in \( x \) and \( y \). For small \( \Gamma \), the product
of spectral functions can be replaced by a delta-function,
\[
[A^\pm(\epsilon)]^3 \to \frac{3}{8\pi^2} \Gamma^{-2} \delta(\epsilon - E_p^n),
\tag{48}
\]
so that the integral over \(\epsilon\) can be performed, yielding
\[
\sigma_{H,\text{intra}}^{xyz} \to e^3 \tau^2 \int \frac{d^2 \mathbf{p}}{(2\pi)^2} \sum_{n=\pm} f'(E_p^n) \left( \left( E_p^{n,x} \right)^2 E_p^{n,yy} - E_p^{n,x} E_p^{n,y} E_p^{n,xy} + (x \leftrightarrow y) \right),
\tag{49}
\]
where \(\tau = 1/(2\Gamma)\). Using \(f'(E_p^n) E_p^{n,\alpha} = \partial_{p\alpha} f(E_p^n)\), and performing a partial integration, this can also be written as
\[
\sigma_{H,\text{intra}}^{xyz} = -e^3 \tau^2 \int \frac{d^2 \mathbf{p}}{(2\pi)^2} \sum_{n=\pm} f(E_p^n) \left[ E_p^{n,xx} E_p^{n,yy} - E_p^{n,xy} E_p^{n,yy} \right].
\tag{50}
\]

Eqs. (49) and (50) agree with the corresponding expressions derived by Voruganti et al.\textsuperscript{10}

Applying Eq. (45) to the interband contribution, one obtains
\[
\sigma_{\text{inter}}^{\alpha\beta\gamma} = -e^3 \pi^2 \frac{\nu \gamma \delta}{L} \sum_{\mathbf{p}} \sum_{n=\pm} \int d\epsilon f'(\epsilon) \left[ A_{\mathbf{p}}^n(\epsilon) \right]^2 A_{\mathbf{p}}^{-n}(\epsilon) \left( F_{\mathbf{p}}^\alpha H_{\mathbf{p}}^{\beta\gamma} - F_{\mathbf{p}}^\beta H_{\mathbf{p}}^{\alpha\gamma} \right) E_p^{n,\delta}
\]
\[
+ 2e^3 \pi^2 \frac{\nu \gamma \delta}{L} \sum_{\mathbf{p}} \sum_{n=\pm} \int d\epsilon f(\epsilon) A_{\mathbf{p}}^n(\epsilon) A_{\mathbf{p}}^{-n}(\epsilon) \frac{A_{\mathbf{p}}^+(\epsilon) - A_{\mathbf{p}}^-(\epsilon)}{E_p^+ - E_p^-}
\]
\[
\times \left[ F_{\mathbf{p}}^\alpha \left( H_{\mathbf{p}}^{\beta\gamma} E_p^{n,\delta} + F_{\mathbf{p}}^\beta E_p^{n,\delta} \right) - (\alpha \leftrightarrow \beta) \right].
\tag{51}
\]

Note that we have chosen a mixed representation with a Fermi function derivative in the first term and the Fermi function in the second. Performing a partial integration on the second term would lead to an integrand with contributions away from the quasi-particle energies, even for small \(\Gamma\). Specifying again to two dimensions with a magnetic field in \(z\)-direction, we get
\[
\sigma_{H,\text{inter}}^{xyz} = -e^3 \pi^2 \int \frac{d^2 \mathbf{p}}{(2\pi)^2} \sum_{n=\pm} \int d\epsilon f'(\epsilon) \left[ A_{\mathbf{p}}^n(\epsilon) \right]^2 A_{\mathbf{p}}^{-n}(\epsilon) \left[ F_{\mathbf{p}}^x \left( H_{\mathbf{p}}^{yz} E_p^{n,y} - H_{\mathbf{p}}^{yy} E_p^{n,x} \right) + (x \leftrightarrow y) \right]
\]
\[
+ 2e^3 \pi^2 \int \frac{d^2 \mathbf{p}}{(2\pi)^2} \sum_{n=\pm} \int d\epsilon f(\epsilon) A_{\mathbf{p}}^n(\epsilon) A_{\mathbf{p}}^{-n}(\epsilon) \frac{A_{\mathbf{p}}^+(\epsilon) - A_{\mathbf{p}}^-(\epsilon)}{E_p^+ - E_p^-}
\]
\[
\times \left[ F_{\mathbf{p}}^x \left( H_{\mathbf{p}}^{yz} E_p^{n,y} - H_{\mathbf{p}}^{yy} E_p^{n,x} \right) + F_{\mathbf{p}}^x E_p^{n,xy} - F_{\mathbf{p}}^y E_p^{n,yx} \right] + (x \leftrightarrow y)].
\tag{52}
\]

For small \(\Gamma\), the products of spectral functions can be replaced by delta functions,
\[
\left[ A_{\mathbf{p}}^n(\epsilon) \right]^2 A_{\mathbf{p}}^{-n}(\epsilon) \to \frac{\delta(\epsilon - E_p^n)}{2\pi^2 (E_p^+ - E_p^-)^2},
\tag{53}
\]
so that the $\epsilon$-integral can be performed, yielding

$$
\sigma_{H,\text{inter}}^{xyz} = -\frac{e^3}{2} \int \frac{d^2p}{(2\pi)^2} \sum_{n=\pm} f'(E^+_p) \left[ F^x_p (H^{yx}_{p} E^{n,y}_p - H^{yy}_{p} E^{n,x}_p) + (x \leftrightarrow y) \right] + \frac{e^3}{2} \int \frac{d^2p}{(2\pi)^2} \sum_{n=\pm} \frac{f(E^+_p) - f(E^-_p)}{(E^+_p - E^-_p)^3} \times \left[ F^x_p H^{yx}_{p} E^{n,y}_p - F^y_p E^{n,x}_p + F^x_p E^{n,y}_p - F^y_p E^{n,x}_p + (x \leftrightarrow y) \right].
$$

(54)

The above simplifications for $\Gamma \to 0$ apply under the same conditions as for the ordinary conductivity, that is, the momentum dependent functions $E^\alpha_p$, $F^\alpha_p$, etc. must be almost constant in the momentum range in which the variation of $E^\pm_p$ is of order $\Gamma$, and $\Gamma$ must be much smaller than $E^+_p - E^-_p$. Here, for the Hall conductivity, the interband contributions are also suppressed by a factor $\tau^{-2}$ compared to the intraband contributions.

We finally emphasize that our derivation is valid under the assumption of a momentum-independent magnetic gap $\Delta$ and a momentum-independent decay rate $\Gamma$. A generalization to a momentum-dependent gap and decay rate is not straightforward, since numerous additional terms appear.

### III. RESULTS

We now present and discuss results for the longitudinal conductivity and the Hall conductivity. Motivated by the recent charge transport experiments in cuprates,\textsuperscript{31} we compute these quantities with a phenomenological ansatz for a doping dependent magnetic gap $\Delta(p)$, in close analogy to previous theoretical studies.\textsuperscript{5,27,28} In particular, we will study the size of interband contributions to the conductivities, which were neglected in earlier calculations for Néel and spiral magnetic states. Interband contributions have been taken into account, however, in a calculation of the optical conductivity in a $d$-density wave state,\textsuperscript{32} and in a very recent evaluation of the longitudinal DC conductivity in the spiral state.\textsuperscript{28}

We choose a tight-binding bandstructure

$$
\epsilon_p = -2t(\cos p_x + \cos p_y) - 4t' \cos p_x \cos p_y - 2t''[\cos(2p_x) + \cos(2p_y)],
$$

(55)

where $t$, $t'$, and $t''$ are nearest, second nearest, and third nearest neighbor hopping amplitudes, respectively, on a square lattice with lattice constant $a = 1$. We choose $t$ as our unit of energy. Hopping amplitudes in cuprates have been determined by downfolding ab-initio
band structures on effective single-band Hamiltonians. The ratio $t'/t$ is negative in all cuprate superconductors, ranging from $-0.15$ in LSCO to $-0.35$ in BSCCO.

Theoretical results for spiral states in the two-dimensional $t$-$J$ model suggest a linear doping dependence of the magnetic gap of the form

$$\Delta(p) = D(p^* - p)\Theta(p^* - p),$$

(56)

where $D$ is a prefactor, and $p^*$ is the critical doping at which the magnetic order vanishes. Both $D$ and $p^*$ are material dependent and need to be fitted to experimental data. A linear doping dependence of the gap for $p < p^*$ is also found in resonating valence bond mean-field theory for the $t$-$J$ model. The pseudogap temperature scale $T^*$ seen in experiments also vanishes linearly at a certain critical doping $p^*$. In Ref. [27], a quadratic doping dependence of $\Delta(p)$ was considered, too.

The wave vector of the incommensurate magnetic states obtained in the theoretical literature has the form $Q = (\pi - 2\pi\eta, \pi)$, or symmetry-related, that is, $(-\pi + 2\pi\eta, \pi)$, $(\pi, \pi - 2\pi\eta)$, and $(\pi, -\pi + 2\pi\eta)$. Here $\eta > 0$, the so-called incommensurability, measures the deviation from the Néel wave vector $(\pi, \pi)$. Peaks in the magnetic structure factors seen in neutron scattering experiments are also situated at such wave vectors. The incommensurability $\eta$ is a monotonically increasing function of doping. In Ref. [27], the doping-dependence of $\eta$ was determined by minimizing the mean-field free energy, resulting in $\eta \approx p$, which is roughly consistent with experimental observations in LSCO. In YBCO $\eta$ values below $p$ are observed, and functional renormalization group calculations for the Hubbard model also yield $\eta < p$. Since the precise doping dependence of the incommensurability is not important for our results, we simply choose $\eta = p$.

For small decay rates $\Gamma$, interband contributions are suppressed by a factor $\Gamma^2$ compared to intraband contributions to the conductivities (see Sec. II). To get a feeling for the typical size of $\Gamma$ in the recent high-field experiments, we estimate $\Gamma$ from the experimental result $\omega_c\tau = 0.075$ reported for Nd-LSCO samples at zero temperature by Collignon et al. The cyclotron frequency can be written as $\omega_c = \frac{|e|B}{m_c}$, which defines the cyclotron mass $m_c$. For free electrons $m_c$ is just the bare electron mass $m_e$. Inserting the applied magnetic field of 37.5 Tesla and assuming $m_c = m_e$, one obtains $\Gamma = (2\tau)^{-1} \approx 0.03\, eV$. With the typical value $t \approx 0.3\, eV$ for the nearest neighbor hopping amplitude in cuprates, one thus gets $\Gamma/t \approx 0.1$. The cyclotron mass in cuprates is actually larger than the bare electron mass. Mass ratios
FIG. 1: Longitudinal conductivity $\sigma^{xx}$ at zero temperature as a function of doping $p$ for a doping dependent magnetic order parameter $\Delta(p) = 12t(p^* - p)\Theta(p^* - p)$ with $p^* = 0.19$. The intraband contribution $\sigma_{\text{intra}}^{xx}$ is also shown for comparison. The hopping parameters are $t'/t = -0.3$ and $t''/t = 0.2$. Left: $\Gamma/t = 0.1$. Right: $\Gamma/t = 0.3$. The vertical lines indicate changes of the Fermi surface topology at the three doping values $p^*_e$, $p^*_h$, and $p^*$. $m_c/m_e$ equal to three or even larger have been observed.\textsuperscript{[38]} Hence, $\Gamma/t = 0.1$ is just an upper bound, the actual value can be expected to be even smaller. Indeed, an estimate from the observed residual resistivity in Nd-LSCO yields $\Gamma \approx 0.008\,eV$.\textsuperscript{[39]} The decay rates in cuprate superconductors are actually momentum dependent. However, we do not expect the momentum dependence to affect the order of magnitude of interband contributions.

A. Longitudinal conductivity

In Fig. 1 we show results for $\sigma^{xx}$ as obtained from Eq. \textsuperscript{[33]} at zero temperature for two values of the decay rate $\Gamma$. For the hopping parameters we chose values used for YBCO in the literature. The critical doping $p^* = 0.19$ is the onset doping for the Hall number drop observed in the experiments on YBCO by Badoux et al.\textsuperscript{[8]} The total conductivity $\sigma^{xx}$ is compared to the intraband contribution $\sigma_{\text{intra}}^{xx}$. One can see a pronounced drop of the conductivity for $p < p^*$, as expected from the drop of charge carrier density in the spiral state. For $\Gamma/t = 0.1$ the interband contributions are practically negligible, while for $\Gamma/t = 0.3$ they are already sizable. In particular, the interband contributions shift the drop of $\sigma^{xx}$ induced
FIG. 2: Anisotropy ratio of the longitudinal conductivity $\sigma_{yy}/\sigma_{xx}$ at zero temperature as a function of doping $p$ for three choices of $D$. The band parameters are the same as in Fig. 1 and the decay rate is $\Gamma/t = 0.1$.

by the spiral order toward smaller values of $p$, and they smooth the sharp kink exhibited by $\sigma_{xx}$ at $p^*$ for $\Gamma \rightarrow 0$.

Chatterjee et al. have derived expressions for the electrical and the heat conductivities in the spiral state, for a momentum independent scattering rate, and showed that the two quantities are related by the Wiedemann-Franz law. While their formulae for the conductivities has a different form than ours, we have checked that the numerical results are consistent.

The spiral state exhibits a pronounced nematicity in the longitudinal conductivity. For $Q = (\pi - 2\pi \eta, \pi)$, with an incommensurability in $x$-direction, the conductivity in $y$-direction is larger than in $x$-direction. In Fig. 2 we show the ratio $\sigma_{yy}/\sigma_{xx}$ as a function of doping for the same band parameters as in Fig. 1 and various choices for $D$, with $\Gamma/t = 0.1$. The anisotropy increases smoothly upon lowering the doping from the critical point $p^*$, and it decreases upon approaching half-filling, where $\eta$ vanishes such that the square lattice symmetry is restored. A pronounced temperature and doping dependent in-plane anisotropy of the longitudinal conductivities with conductivity ratios up to 2.5 has been observed in YBCO by Ando et al.

For $p < p^*$, the quasi-particle Fermi surface consists exclusively of hole pockets, while for $p^* < p < p^*$ also electron pockets are present. Note that $p_e^*$ depends (slightly) on the decay rate $\Gamma$, since the relation between the chemical potential $\mu$ and the density depends on $\Gamma$. 

FIG. 3: Quasi-particle Fermi surfaces for $p = 0.09$, $p = 0.115$, and $p = 0.17$ (from top to bottom). Fermi surface sheets surrounding hole (orange) and electron (blue) pockets correspond to zeros of $E_{p+Q/2}^-$ and $E_{p+Q/2}^+$, respectively. The green “nesting” line indicates momenta $p$ satisfying the condition $\epsilon_p = \epsilon_{p+Q}$. The band and gap parameters are the same as in Fig. 1 with $\Gamma/t = 0.1$. 
For \( p < p^*_h \) there are only two hole pockets, while for \( p^*_h < p < p^* \) a second (smaller) pair of hole pockets appear. In Fig. 3 we plot the quasi-particle Fermi surfaces for three choices of the doping \( p \): for \( p < p^*_e \), for \( p^*_e < p < p^*_h \), and for \( p^*_h < p < p^* \). For a more transparent representation of the Fermi surface topology, we shift the momentum by \( Q/2 \) and plot zeros of \( E^\pm_{\mathbf{p}+Q/2} \) instead of zeros of \( E^\pm_{\mathbf{p}} \). The Fermi surfaces look thus similar to those shown in Ref. 27, where the shift by \( Q/2 \) was already included in the definition of the quasi-particle energies. At \( p = p^*_e \) electron and hole pockets merge, and for \( p > p^* \) there is only a single large Fermi surface sheet, which is closed around the unoccupied (hole) states. The doping dependence of the conductivity changes its slope at \( p^*_e \), while there is no pronounced feature at \( p^*_h \). However, choosing a smaller decay rate \( \Gamma \ll 0.1t \), a change of slope of \( \sigma^{xx} \) is visible also at \( p^*_h \), while no pronounced feature of \( \sigma^{yy} \) is visible.

The sequence of Fermi surface topologies as a function of doping depends on the doping dependence of \( \eta \). The above results were obtained for \( \eta = p \). Choosing, for example, a smaller \( \eta(p) \), one may have four (not just two) hole pockets at low doping.

It is instructive to see which quasi-particle states yield the dominant contributions to the conductivity. In two dimensions, the conductivity in Eq. (33) is given by a momentum integral of the form \( \sigma^{\alpha\beta} = \int \frac{d^2p}{(2\pi)^2} \sigma^{\alpha\beta}(\mathbf{p}) \). The Fermi function derivative \( f'(\epsilon) \) restricts the energies \( \epsilon \) up to values of order \( T \). For \( T = 0 \), one has \( f'(\epsilon) = -\delta(\epsilon) \). For small \( \Gamma \) the quasi-particle spectral functions \( A^\pm_{\mathbf{p}}(\epsilon) \) are peaked at the quasi-particle energies. Hence, for low \( T \) and small or moderate \( \Gamma \), the dominant contributions to the conductivity come from momenta where either \( |E^+_{\mathbf{p}}| \) or \( |E^-_{\mathbf{p}}| \) is small, that is, in particular from momenta near the quasi-particle Fermi surfaces.

In Fig. 4 we show color plots of \( \sigma^{xx}_{\text{intra}}(\mathbf{p} + Q/2) \) and \( \sigma^{xx}_{\text{inter}}(\mathbf{p} + Q/2) \) in the Brillouin zone. Although a sizable \( \Gamma/t = 0.3 \) has been chosen, the intraband contributions are clearly restricted to the vicinity of the quasi-particle Fermi surface. Variations of the size of intraband contributions along the Fermi surfaces are due to the momentum dependence of the quasi-particle velocities \( E^{\pm,x}_{\mathbf{p}} = \partial E^{\pm}_{\mathbf{p}} / \partial p_x \). The interband contributions are particularly large near the “nesting line” defined by \( \epsilon_{\mathbf{p}+Q} = \epsilon_{\mathbf{p}} \), where the direct band gap between the quasi-particle energies \( E^+_{\mathbf{p}} \) and \( E^-_{\mathbf{p}} \) assumes the minimal value \( 2\Delta \). For \( p = 0.09 \), the largest interband contributions come from regions on the nesting line remote from the Fermi surfaces. Note, however, that they are much smaller than the intraband contributions, and \( |E^-_{\mathbf{p}}| \) has a local minimum in these regions. For \( p = 0.17 \), the interband contributions are
FIG. 4: Top: Color plot of the momentum resolved intraband contribution to the longitudinal conductivity $\sigma_{\text{intra}}^{xx}(p + Q/2)$ for $p = 0.09$ (left) and $p = 0.17$ (right). Bottom: Interband contribution $\sigma_{\text{inter}}^{xx}(p + Q/2)$ for the same choices of $p$. The band and gap parameters are the same as in Fig. 1, and the decay rate is $\Gamma/t = 0.3$. The Fermi surfaces and the nesting line (cf. Fig. 3) are plotted as thin black lines.

generally larger, and they are concentrated in regions between neighboring electron and hole pockets.

B. Hall conductivity

The Hall conductivity $\sigma_{H}^{yz}$ and the longitudinal conductivities determine the Hall coefficient

$$R_H = \frac{\sigma_{H}^{yz}}{\sigma^{xx}\sigma^{yy}}.$$  \hspace{1cm} (57)
Unlike the longitudinal and Hall conductivities, the Hall coefficient is finite in the limit $\Gamma \to 0$. In the independent electron approximation, there are special cases where the Hall coefficient is determined by the charge density $\rho_c$ via the simple relation $R_H = \rho_c^{-1}$. For free electrons with a parabolic dispersion, this relation holds for any magnetic field, with $\rho_c = e n_e$. For band electrons it still holds in the high-field limit $\omega_c \tau \gg 1$, if the semiclassical electron orbits of all occupied (or all unoccupied) states are closed. For Fermi surfaces enclosing unoccupied states, the relevant charge density is then $\rho_c = |e| n_h$, where $n_h$ is the density of holes. If both electron and hole-like Fermi surfaces are present, one has $\rho_c = e(n_e - n_h)$. Results for the Hall conductivity are thus frequently represented in terms of the so-called Hall number $n_H$, defined via the relation

$$R_H = \frac{1}{|e| n_H}.$$  

(58)

However, $n_H$ is given by the electron and hole densities only in the special cases described above. In particular, in the low-field limit $\omega_c \tau \ll 1$ which applies to the recent “high” magnetic field experiments for cuprates, there is no guarantee that $n_H$ is equal to a charge carrier density.

In Fig. 5 we show results for the Hall number as obtained from the Hall conductivity in Eqs. (47) and Eq. (52). The hopping and gap parameters are the same as in Fig. 1. The Hall number obtained from the total Hall conductivity is compared to the one obtained by neglecting interband contributions, that is, taking only $\sigma_{H,\text{intra}}^{xy}$ and $\sigma_{\text{intra}}^{\alpha\alpha}$ into account. For $p \geq p^*$, where $\Delta = 0$, the Hall number is slightly above the value $1 + p$ corresponding to the density of holes enclosed by the (large) Fermi surface. This is also seen in experiment in YBCO\textsuperscript{43} and Nd-LSCO\textsuperscript{44} Note that $n_H$ is not expected to be equal to $1 + p$ for $\omega_c \tau \ll 1$, since the dispersion $\epsilon_p$ is not parabolic. For $p < p^*$ the Hall number drops drastically. For $\Gamma/t = 0.1$ the interband contributions are again quite small, as already observed for the longitudinal conductivity, and the Hall number gradually approaches the value $p$ upon lowering $p$. Hence, the naive expectation that the Hall number is given by the density of holes in the hole pockets turns out to be correct for sufficiently small $p$. Visible deviations from $n_H = p$ set in for $p > p^*$, where the electron pockets emerge. For $\Gamma/t = 0.3$ interband contributions are sizable. They shift the onset of the drop of $n_H$ to smaller doping.

The Hall conductivity in Eqs. (47) and (52) is given by a momentum integral of the form

$$\sigma_{H}^{xy} = \int \frac{d^2 p}{(2\pi)^2} \sigma_{H}^{xy}(p).$$

To see which momenta, that is, which quasi-particle states, contribute
FIG. 5: Hall number $n_H$ as a function of doping $p$ for a doping dependent magnetic order parameter $\Delta(p) = 12t(p^* - p)\Theta(p^* - p)$ with $p^* = 0.19$. The intraband contribution $n_{H,\text{intra}}$ is also shown for comparison. The straight dashed lines correspond to the naive expectation for large and reconstructed Fermi surfaces, $n_H = 1 + p$ and $n_H = p$, respectively. The vertical lines indicate the three special doping values $p_{e}^*$, $p_h^*$, and $p^*$. The hopping parameters are $t'/t = -0.3$ and $t''/t = 0.2$.

Left: $\Gamma/t = 0.1$. Right: $\Gamma/t = 0.3$. Most significantly to the Hall conductivity, we show color plots of $\sigma_{H,\text{intra}}^{xyz}(p + Q/2)$ and $\sigma_{H,\text{inter}}^{xyz}(p + Q/2)$ for two choices of the hole doping, see Fig. 6. The intraband contributions are concentrated near the quasi-particle Fermi surfaces, due to the peaks in $f'(\epsilon)$ and in the spectral functions, as for the longitudinal conductivity. Contributions from hole pockets count positively, and those from electron pockets negatively, as expected. The contributions are particularly large near crossing points of the Fermi surfaces with the nesting line, where the Fermi surfaces have a large curvature. The interband contributions lie mostly near the nesting line, not necessarily close to Fermi surfaces. For $p = 0.17$ they are concentrated in small regions between electron and hole pockets.

C. Comparison to experiments

So far we have shown results for $p^* = 0.19$, the onset doping for the Hall number drop extracted from the experimental data by Badoux et al. and $D/t = 12$ for the arbitrarily chosen prefactor of the doping dependent magnetic gap. The Hall number drop from values above $1 + p$ for large doping to $p$ for small doping was reproduced by our results, if $\Gamma/t$ is not
FIG. 6: Top: Color plot of the momentum resolved intraband contribution to the Hall conductivity $\sigma_{H,\text{intra}}^{xyz} (p + Q/2)$ for $p = 0.09$ (left) and $p = 0.17$ (right). Bottom: Interband contribution $\sigma_{H,\text{inter}}^{xyz} (p + Q/2)$ for the same choices of $p$. The band and gap parameters are the same as in Fig. 5, and the decay rate is $\Gamma/t = 0.3$. The Fermi surfaces and the nesting line (cf. Fig. 3) are plotted as thin black lines.

too large. To make closer contact to the experiment, we have fitted the parameters $p^*$ and $D$ to obtain quantitative agreement with the observed data points for YBCO. The fit, obtained for a fixed $\Gamma/t = 0.05$, and shown in Fig. 7, is optimal for $p^* = 0.21$ and $D/t = 16.5$.

The value of $D$ is unreasonably large. For a hopping amplitude $t \approx 0.3 \, \text{eV}$, the magnetic gap $\Delta(p) = D(p^* - p)$ would rise to a value $\Delta \approx 0.5 \, \text{eV}$ at $p = 0.1$. Large values for $D$ were also assumed in previous studies of the Hall effect in Néel and spiral antiferromagnetic states, to obtain a sufficiently steep decrease of the Hall number. The required size of $D$ can be substantially reduced, if the bare hopping $t$ is replaced by a smaller effective
hopping

\[ t_{\text{eff}} = \frac{2p}{1+p} t, \quad (59) \]

where the Gutzwiller factor on the right hand side captures phenomenologically the loss of metallicity in the doped Mott insulator. Such a factor is used in the YRZ-ansatz for the pseudogap phase.\( ^6 \) Replacing \( t \) by \( t_{\text{eff}} \) with \( t = 0.3 \, \text{eV} \), a prefactor \( D = 1.5 \, \text{eV} \) is sufficient to obtain the best fit for \( n_H \), leading to \( \Delta \approx 0.15 \, \text{eV} \) at \( p = 0.1 \). This value is similar to the magnetic energy scale \( J \) in cuprates.

All our results have been computed by evaluating the conductivity formulae with a Fermi function at zero temperature. We have checked that the temperature dependence from the Fermi function is negligible at the temperatures at which the recent transport experiments in cuprates\( ^1 ^4 ^6 \) have been carried out.

IV. CONCLUSION

We have computed electrical DC conductivities in a two-dimensional metal with spiral magnetic order. Scattering processes were modeled by a momentum-independent single-particle decay rate \( \Gamma \). We have derived an expression for the longitudinal conductivity, and, for the first time, a complete formula (including all interband contributions) for the Hall number as a function of doping to the experimental data from Badoux et al.\( ^3 \) For the decay rate chosen as \( \Gamma/t = 0.05 \), best agreement is obtained for \( p^* = 0.21 \) and \( D/t = 16.5 \).
conductivity in the low-field limit $\omega_c \tau \ll 1$.

For small $\Gamma$, interband terms are suppressed by a factor of order $\Gamma^2$ compared to the dominant intraband contributions. In the limit $\Gamma \to 0$, the interband contributions are negligible and the intraband contributions simplify to the formulae derived by Voruganti et al.$^1$ The latter have the same structure as the conductivities for non-interacting electrons in relaxation time approximation, with the bare electron dispersion $\epsilon_p$ replaced by the quasi-particle dispersions $E_p^\pm$ in the spiral state. We expect that this is true for any charge or spin density wave state in mean-field theory.

A numerical evaluation of the conductivities for band parameters as in YBCO and various choices of the decay rate $\Gamma$ shows that interband contributions start playing a significant role only for $\Gamma/t > 0.1$, where $t$ is the nearest neighbor hopping amplitude. Decay rates in the regime of recent high field transport experiments for cuprates are smaller, so that the application of the relatively simple formulae derived by Voruganti et al.$^1$ is justified.

The magnetic order induces a reduction of the longitudinal conductivity and of the Hall number. The longitudinal conductivity in the spiral state exhibits a pronounced doping dependent nematicity in agreement with experimental observations in cuprates.$^{37}$ With a doping dependent magnetic gap of the form $\Delta(p) = D(p^* - p)$ for $p < p^*$, the Hall number drop below the critical doping $p^*$ observed in experiments.$^{31,4}$ can be well described. To fit the experimental data with a realistic (not too large) value of $D$, the reduction of the hopping amplitudes by correlation effects has to be taken into account.

Spiral magnetic order is thus consistent with transport experiments in cuprates, where superconductivity is suppressed by high magnetic fields. We finally note that fluctuating instead of static magnetic order should yield similar transport properties, as long as pronounced magnetic correlations are present.

**Acknowledgments**

We are grateful to A. Eberlein, J. Schmalian, O. Sushkov, L. Taillefer, H. Yamase, and R. Zeyher for valuable discussions.
Appendix A: Momentum expansion of the vertex

We perform a momentum expansion of the vertex $\tilde{\lambda}_{p+aq,p+bq}^{\alpha_1...\alpha_n}$ for small $q$. In this case, the (diagonal) $n$-th order vertex \(^1\) reads

$$\lambda_{p+aq,p+bq}^{\alpha_1...\alpha_n} = \begin{pmatrix} e_{p+aq,p+bq}^{\alpha_1...\alpha_n} & 0 \\ 0 & e_{p+aq,p+bq}^{\alpha_1...\alpha_n} \end{pmatrix}. \tag{A1}$$

Using the basis in which $G_p$ is diagonal, the vertex $\tilde{\lambda}_{p+aq,p+bq}^{\alpha_1...\alpha_n}$ transforms as\(^2\)

$$\tilde{\lambda}_{p+aq,p+bq}^{\alpha_1...\alpha_n} = U_p^\dagger_{p+aq} \lambda_{p+aq,p+bq}^{\alpha_1...\alpha_n} U_{p+bq}, \tag{A2}$$

where

$$U_p = \begin{pmatrix} \cos \theta_p & \sin \theta_p \\ -\sin \theta_p & \cos \theta_p \end{pmatrix}. \tag{A3}$$

The rotation angle $\theta_p$ must satisfy the condition

$$\tan(2\theta_p) = \frac{\Delta}{h_p}, \tag{A4}$$

where $h_p$ was defined in Eq. (23). We perform the three derivatives in Eq. (A2) at $q = 0$. Obviously, we have $\partial_\delta \lambda_{p+aq,p+bq}^{\alpha_1...\alpha_n} \big|_{q=0} = \frac{a+b}{2} \lambda_{p+aq,p+bq}^{\alpha_1...\alpha_n}$. The momentum derivative of the angle $\theta_p$ is $\partial_\delta \theta_p = -\frac{1}{E_p^+ - E_p^-} F_p^\delta$, where we used the definition $F_p^\alpha = 2\Delta h_p^\alpha/(E_p^+ - E_p^-)$ in Eq. (26). Thus, we get

$$\partial_\delta U_{p+aq}^\dagger \big|_{q=0} = \begin{pmatrix} -\sin \theta_p & -\cos \theta_p \\ \cos \theta_p & -\sin \theta_p \end{pmatrix} \partial_\delta \theta_p a = a S_p F_p^\delta U_{p+aq}^\dagger \tag{A5}$$

and

$$\partial_\delta U_{p+bq} \big|_{q=0} = \begin{pmatrix} -\sin \theta_p & \cos \theta_p \\ \cos \theta_p & -\sin \theta_p \end{pmatrix} \partial_\delta \theta_p b = b U_{p+bq} F_p^\delta S_p, \tag{A6}$$

respectively. We introduced the matrix $S_p = 1/(E_p^+ - E_p^-) \sigma^z$. Thus, we end up with

$$\partial_\delta \tilde{\lambda}_{p+aq,p+bq}^{\alpha_1...\alpha_n} = \lambda_{pp}^{\alpha_1...\alpha_n} + q_\delta \partial_\delta \tilde{\lambda}_{p+aq,p+bq}^{\alpha_1...\alpha_n} \big|_{q=0} + \ldots,$$

where

$$\partial_\delta \tilde{\lambda}_{p+aq,p+bq}^{\alpha_1...\alpha_n} \big|_{q=0} = \frac{a+b}{2} \lambda_{pp}^{\alpha_1...\alpha_n} \delta + a S_p F_p^\delta \lambda_{pp}^{\alpha_1...\alpha_n} + b \lambda_{pp}^{\alpha_1...\alpha_n} F_p^\delta S_p. \tag{A7}$$

The components of $\partial_\delta \tilde{\lambda}_{p+aq,p+bq}^{\alpha} \big|_{q=0}$ are explicitly given in Ref. \(^3\)
Appendix B: Analytic continuation

For the analytic continuation of the response functions we use the spectral representation of the imaginary frequency propagator

\[ G_{i\rho_0,\rho} = \int_{-\infty}^{\infty} d\epsilon \frac{A_p(\epsilon)}{i\rho_0 - \epsilon}, \quad (B1) \]

where

\[ A_p(\epsilon) = \begin{pmatrix} A_p^+(\epsilon) & 0 \\ 0 & A_p(\epsilon) \end{pmatrix} \quad (B2) \]

is the matrix of spectral functions of the two quasi-particle bands. For simplicity of notation, we drop the momentum dependence in the following. Real frequency quantities are conveniently formulated in terms of advanced and retarded Green functions,

\[ G_A(\epsilon) = \int_{-\infty}^{\infty} d\epsilon' \frac{A(\epsilon')}{\epsilon - \epsilon' - i0^+}, \quad (B3) \]

\[ G_R(\epsilon) = \int_{-\infty}^{\infty} d\epsilon' \frac{A(\epsilon')}{\epsilon - \epsilon' + i0^+}. \quad (B4) \]

The functions to be continued analytically have the following structure

\[ I_{i\rho_0}^{m,n} = T \sum_{i\rho_1} \text{tr}(G_{i\rho_0+iq_0} M_1 ... G_{i\rho_0+iq_0} M_m G_{i\rho_0} N_1 ... G_{i\rho_0} N_n), \quad (B5) \]

where \( M_1, ... M_m \) and \( N_1, ... N_n \) are frequency-independent \( 2 \times 2 \) matrices, and \( q_0 \) is a bosonic Matsubara frequency. We insert the spectral representation (B1) for each propagator and perform the Matsubara frequency sum over the resulting product of energy denominators. Using Res\(_{i\rho_0}(f(\epsilon)) = -T \), where \( f(\epsilon) = (\epsilon/|T| + 1)^{-1} \) is the Fermi function, we apply the residue theorem to replace the Matsubara frequency sum by a contour integral encircling the fermionic Matsubara frequencies counterclockwise. We then change the contour such that only the poles from the energy denominators are encircled. Applying the residue theorem again yields

\[
T \sum_{i\rho_0} \frac{1}{i\rho_0 + iq_0 - \epsilon_1} ... \frac{1}{i\rho_0 + iq_0 - \epsilon_m} \frac{1}{i\rho_0 - \epsilon_1} ... \frac{1}{i\rho_0 - \epsilon_n} \\
= f(\epsilon_1) \frac{1}{\epsilon_1 - \epsilon_2} ... \frac{1}{\epsilon_1 - \epsilon_m} \frac{1}{-iq_0 + \epsilon_1 - \epsilon'_1} ... \frac{1}{-iq_0 + \epsilon_1 - \epsilon'_n} + ... \\
+ f(\epsilon_m) \frac{1}{\epsilon_m - \epsilon_1} ... \frac{1}{\epsilon_m - \epsilon_{m-1}} \frac{1}{-iq_0 + \epsilon_m - \epsilon'_1} ... \frac{1}{-iq_0 + \epsilon_m - \epsilon'_n} \\
+ f(\epsilon'_1) \frac{1}{iq_0 + \epsilon'_1 - \epsilon_1} ... \frac{1}{iq_0 + \epsilon'_1 - \epsilon_m} \frac{1}{\epsilon'_1 - \epsilon'_2} ... \frac{1}{\epsilon'_1 - \epsilon'_n} + ... \\
+ f(\epsilon'_n) \frac{1}{iq_0 + \epsilon'_n - \epsilon_1} ... \frac{1}{iq_0 + \epsilon'_n - \epsilon_m} \frac{1}{\epsilon'_n - \epsilon'_1} ... \frac{1}{\epsilon'_n - \epsilon'_{n-1}}. \quad (B6)
\]
This expression can be easily continued to real frequencies, replacing \( iq_0 \) by \( \omega + i0^+ \). Performing the integrals over \( \epsilon_1, \ldots, \epsilon_m \) and \( \epsilon'_1, \ldots, \epsilon'_n \) then yields

\[
I_{m,n}^\omega = \int d\epsilon f(\epsilon) \text{tr}\{[A(\epsilon)M_1P(\epsilon)M_2 \ldots P(\epsilon)M_m + \ldots \\
+ P(\epsilon)M_1 \ldots P(\epsilon)M_{m-1}A(\epsilon)M_m]G^{A}_{\epsilon \omega}N_1 \ldots G^{A}_{\epsilon \omega}N_n \}
\]

\[
+ \int d\epsilon f(\epsilon) \{G^{R}_{\epsilon \omega}M_1 \ldots G^{R}_{\epsilon \omega}M_m \{A(\epsilon)N_1P(\epsilon)N_2 \ldots P(\epsilon)N_n + \ldots \\
+ P(\epsilon)N_1 \ldots P(\epsilon)N_{n-1}A(\epsilon)N_n \}\},
\]

where \( P(\epsilon) \) is the principal value integral

\[
P(\epsilon) = \text{P.V.} \int d\epsilon' \frac{A(\epsilon')}{\epsilon - \epsilon'} = \frac{1}{2}(G^A_{\epsilon} + G^R_{\epsilon}).
\]

**Appendix C: Evaluation of ordinary conductivity**

We first present the derivation leading from Eq. (31) to Eq. (32). Splitting the vertices in purely diagonal and off-diagonal contributions, one obtains

\[
K^{\alpha \beta}_{iq_0} = e^2 \text{Tr}_p \left( G_{p,ip_0}^\alpha E^\alpha_p G_{p,ip_0} E^\beta_p + G_{p,ip_0}^\alpha F^\alpha_p G_{p,ip_0} F^\beta_p + G_{p,ip_0} E^\alpha_p - G_{p,ip_0} C^\alpha_p \right),
\]

since only terms with an even number of off-diagonal matrices contribute to the trace. Here and in the following we use the short-hand notation \( \text{Tr}_p = TL^{-1} \sum_p \text{tr}(\ldots) \). Using a partial integration, one obtains the identity

\[
\text{Tr}_p \left( G_{p,ip_0} E^\alpha_p \right) = \text{Tr}_p \left( G_{p,ip_0} \partial_p G_{p,ip_0} E^\beta_p \right) = -\text{Tr}_p \left[ (\partial_p G_{p,ip_0}) E^\beta_p \right] = -\text{Tr}_p \left( G_{p,ip_0} E^\alpha_p G_{p,ip_0} E^\beta_p \right).
\]

A few purely algebraic steps yield the relation

\[
\text{Tr}_p \left( G_{p,ip_0} C^\alpha_p \right) = \text{Tr}_p \left( G_{p,ip_0} F^\alpha_p G_{p,ip_0} F^\beta_p \right).
\]

Hence, the last two (diamagnetic) contributions in Eq. (C1) cancel the first two (paramagnetic) contributions for \( q_0 = 0 \), and we obtain Eq. (32).

We now perform the Matsubara sum and the analytic continuation to real frequencies. The frequency dependence in Eq. (32) has the form

\[
K_{iq_0} = T \sum_{p_0} \text{tr}\left[(G_{ip_0+iq_0} - G_{ip_0})M_1G_{ip_0}M_2\right],
\]

(C4)
where $\mathcal{M}_1$ and $\mathcal{M}_2$ are frequency-independent matrices. We have dropped the momentum dependence, since it does not interfere with the following steps. Applying the general formula Eq. (B7) in Appendix B for the cases $m = n = 1$ and $m = 0$, $n = 2$ yields

$$K_\omega = \int d\epsilon f(\epsilon) \text{tr}[\mathcal{M}_1 A(\epsilon) \mathcal{M}_2 (G_{\epsilon-\omega}^A - G_{\epsilon}^A) + \mathcal{M}_1 (G_{\epsilon+\omega}^R - G_{\epsilon}^R) \mathcal{M}_2 A(\epsilon)] .$$

(C5)

Using the cyclic property of the trace, and the fact that all involved matrices are invariant under the exchange of row and column indices, this can also be written as

$$K_\omega = \int d\epsilon f(\epsilon) \text{tr}[\mathcal{M}_1 A(\epsilon) \mathcal{M}_2 (G_{\epsilon-\omega}^A - G_{\epsilon}^A + G_{\epsilon+\omega}^R - G_{\epsilon}^R)] .$$

(C6)

For the DC conductivity, we need to compute the ratio $K_\omega/i\omega$ in the limit $\omega \to 0$. Using $(G_{\epsilon-\omega}^A - G_{\epsilon}^A)/\omega \to -\partial_\epsilon G_{\epsilon}^A$ and $(G_{\epsilon+\omega}^R - G_{\epsilon}^R)/\omega \to \partial_\epsilon G_{\epsilon}^R$ for $\omega \to 0$, and the relation $G_{\epsilon}^R - G_{\epsilon}^A = -2\pi i A(\epsilon)$, one obtains

$$\lim_{\omega \to 0} \frac{K_\omega}{i\omega} = -2\pi \int d\epsilon f(\epsilon) \text{tr}[\mathcal{M}_1 A(\epsilon) \mathcal{M}_2 \partial_\epsilon A(\epsilon)] .$$

(C7)

Using the relation $\text{tr}[\mathcal{M}_1 A(\epsilon) \mathcal{M}_2 \partial_\epsilon A(\epsilon)] = \text{tr}[\mathcal{M}_1 \partial_\epsilon A(\epsilon) \mathcal{M}_2 A(\epsilon)]$ and a partial integration, one can shift the frequency derivative on the Fermi function,

$$\lim_{\omega \to 0} \frac{K_\omega}{i\omega} = \pi \int d\epsilon f'(\epsilon) \text{tr}[\mathcal{M}_1 A(\epsilon) \mathcal{M}_2 A(\epsilon)] .$$

(C8)

Applying this result to Eq. (32) yields the conductivity in the form

$$\sigma^{\alpha\beta} = -\lim_{\omega \to 0} \frac{K^{\alpha\beta}}{i\omega} = -e^2 \pi \sum_p \int d\epsilon f'(\epsilon) \text{tr}[E^{\alpha}_p A_p(\epsilon) E^{\beta}_p A_p(\epsilon) + F^{\alpha}_p A_p(\epsilon) F^{\beta}_p A_p(\epsilon)] .$$

(C9)

Performing the matrix products and the trace one obtains the formula (33) presented in the main text.

Appendix D: Evaluation of Hall conductivity

1. Vanishing $K^{\alpha\beta\gamma}_{q=0,q_0}$ for $q = 0$

The term $K^{\alpha\beta\gamma}_{q=0,q_0}$ in Eq. (38) should vanish for $q = 0$ as a consequence of gauge invariance. We show that $K^{\alpha\beta\gamma}_{q=0,q_0}$, indeed, vanishes by explicit calculation.
$K_{\alpha\beta\gamma}^{\alpha\beta\gamma}$ in Eq. (38) for $q = 0$ reads

$$K_{q=0,iq_0}^{\alpha\beta\gamma} = e^3 \text{Tr}_p \left[ G_{p,iq_0} \tilde{\lambda}_{\alpha\beta\gamma} + G_{p,iq_0} \tilde{\lambda}_{\gamma\beta\alpha} + G_{p,iq_0} \tilde{\lambda}_{\beta\alpha\gamma} \right] + S_{p,iq_0} \gamma_{\alpha\beta\gamma} (D1)$$

$$+ G_{p,iq_0+iq_0} \tilde{\lambda}_{\alpha\beta\gamma} G_{p,iq_0} \tilde{\lambda}_{\gamma\beta\alpha} G_{p,iq_0} \tilde{\lambda}_{\beta\alpha\gamma} + G_{p,iq_0+iq_0} \tilde{\lambda}_{\gamma\beta\alpha} G_{p,iq_0} \tilde{\lambda}_{\beta\alpha\gamma} G_{p,iq_0} \tilde{\lambda}_{\alpha\beta\gamma} (D2)$$

$$+ G_{p,iq_0+iq_0} \tilde{\lambda}_{\alpha\beta\gamma} G_{p,iq_0} \tilde{\lambda}_{\gamma\beta\alpha} G_{p,iq_0} \tilde{\lambda}_{\beta\alpha\gamma} G_{p,iq_0} \tilde{\lambda}_{\alpha\beta\gamma}$$

$$+ G_{p,iq_0-iq_0} \tilde{\lambda}_{\alpha\beta\gamma} G_{p,iq_0} \tilde{\lambda}_{\gamma\beta\alpha} G_{p,iq_0} \tilde{\lambda}_{\beta\alpha\gamma} G_{p,iq_0} \tilde{\lambda}_{\alpha\beta\gamma} \right]. \quad (D3)$$

Here and in the following we use the short-hand notation $\text{Tr}_p = TL^{-1} \sum_p \text{tr}(...)$. In a first step (i) we show that the first two contributions in Eq. (D1) cancel each other. Thus, $K_{q=0,iq_0}^{\alpha\beta\gamma}$ vanishes.

(i) **Cancelation of Eq. (D1)** – Using the derivative of the $n$-order vertex in Eq. (A7) with $a = b = 1$ we have $\tilde{\lambda}_{\alpha\beta\gamma} = \partial_{\alpha} \tilde{\lambda}_{\beta\gamma} - S_{\alpha\beta\gamma} \gamma_{\beta\gamma} - \tilde{\lambda}_{\gamma\beta\alpha} \gamma_{\beta\gamma} S_{p}$. The first expression in Eq. (D1) then reads

$$\text{Tr}_p \left[ G_{p,iq_0} \tilde{\lambda}_{\alpha\beta\gamma} \right] = \text{Tr}_p \left[ G_{p,iq_0} \partial_{\alpha} \tilde{\lambda}_{\beta\gamma} - G_{p,iq_0} S_{p} \gamma_{\beta\gamma} - G_{p,iq_0} \tilde{\lambda}_{\gamma\beta\alpha} \gamma_{\beta\gamma} S_{p} \right] \quad (D4)$$

$$= -\text{Tr}_p \left[ \partial_{\alpha} \left( G_{p,iq_0} \tilde{\lambda}_{\beta\gamma} \right) + \left( S_{p} G_{p,iq_0} \gamma_{\beta\gamma} + F_{p} G_{p,iq_0} S_{p} \right) \tilde{\lambda}_{\alpha\beta\gamma} \right] \quad (D5)$$

We performed a partial integration in $p_{\gamma}$ in the first part. The boundary contribution vanishes. In the second part we used that $S_{p}$ and $G_{p,iq_0}$ commutes. In the last part we reversed the matrix order under the trace and commute $S_{p}$ and $G_{p,iq_0}$. The derivative of a Green function is $\partial_{\alpha} G_{p,iq_0} = G_{p,iq_0} \epsilon_{p}^{\alpha\gamma} G_{p,iq_0}$, which can be shown easily using definition Eq. (22). The last two expression can be combined using the purely algebraic relation

$$S_{p} G_{p,iq_0} F_{p} + F_{p} G_{p,iq_0} S_{p} = G_{p,iq_0} F_{p} G_{p,iq_0}. \quad (D6)$$

Thus,

$$\text{Tr}_p \left[ G_{p,iq_0} \tilde{\lambda}_{\alpha\beta\gamma} \right] = -\text{Tr}_p \left[ G_{p,iq_0} \epsilon_{p}^{\alpha\gamma} G_{p,iq_0} \tilde{\lambda}_{\gamma\beta\alpha} + G_{p,iq_0} F_{p} G_{p,iq_0} \tilde{\lambda}_{\alpha\beta\gamma} \right]$$

$$= -\text{Tr}_p \left[ G_{p,iq_0} \epsilon_{p}^{\alpha\gamma} G_{p,iq_0} \tilde{\lambda}_{\gamma\beta\alpha} \right], \quad (D6)$$

where we finally used the definition $\tilde{\lambda}_{\alpha\beta\gamma} = \epsilon_{p}^{\alpha\gamma} + F_{p}^{\gamma}$. We see that the second part of Eq. (D1) is exactly canceled.

(ii) **Cancelation of Eq. (D2) and Eq. (D3)** – We start with the expressions in Eq. (D2). Using the derivative of the $n$-order vertex in Eq. (A7) with $a = b = 1$ we have $\tilde{\lambda}_{\alpha\beta\gamma} = \partial_{\alpha} \tilde{\lambda}_{\beta\gamma} - S_{p} F_{p} \tilde{\lambda}_{\beta\gamma} - \tilde{\lambda}_{\gamma\beta\alpha} F_{p} S_{p}$. We use this relation for both parts and perform a partial
integration in $p_q$ in the first part. The boundary contribution vanishes. We get

$$
\text{Tr}_p \left[ G_{p,ip_0+iq_0} \tilde{\lambda}^\alpha_{pp} G_{p,ip_0} \tilde{\lambda}^\gamma_{pp} \right] + \text{Tr}_p \left[ G_{p,ip_0+iq_0} \tilde{\lambda}^\alpha_{pp} G_{p,ip_0} \tilde{\lambda}^\gamma_{pp} \right] \quad (D7)
$$

$$
= - \text{Tr}_p \left[ \partial_\gamma (G_{p,ip_0+iq_0} \tilde{\lambda}^\alpha_{pp} G_{p,ip_0}) \tilde{\lambda}^\beta_{pp} + G_{p,ip_0+iq_0} \tilde{\lambda}^\beta_{pp} \partial_\gamma (S_p F^\alpha_{p} F^\beta_{p} + \tilde{\lambda}^\alpha_{pp} F^\beta_{p} S_p) \right] \quad (D8)
$$

$$
- \text{Tr}_p \left[ -G_{p,ip_0+iq_0} \tilde{\lambda}^\alpha_{pp} G_{p,ip_0} (\partial_\gamma \tilde{\lambda}^\beta_{pp}) + G_{p,ip_0+iq_0} \tilde{\lambda}^\gamma_{pp} G_{p,ip_0} (S_p F^\alpha_{p} \tilde{\lambda}^\beta_{pp} + \tilde{\lambda}^\beta_{pp} F^\alpha_{p} S_p) \right] \quad (D9)
$$

Performing the derivative in Eq. (D8) we see that $\partial_\gamma \tilde{\lambda}^\beta_{pp}$ is canceled by the first part in Eq. (D9) when we reverse the matrix order under the trace. We have two derivatives of Green functions where we can use $\partial_\gamma G_{p,ip_0} = G_{p,ip_0} \Sigma^\alpha_{p} G_{p,ip_0}$ and $\partial_\gamma G_{p,ip_0+iq_0} = G_{p,ip_0+iq_0} \Sigma^\alpha_{p} G_{p,ip_0+iq_0}$, respectively. The remaining parts in both Eq. (D8) and (D9) can be recombined by commuting matrices and reversing their order such that we can use the purely algebraic relation

$S_p G_{p,ip_0} F^\gamma_{p} G_{p,ip_0} S_p = G_{p,ip_0} F^\gamma_{p} G_{p,ip_0}$ and the corresponding one replacing $ip_0 \to ip_0 + iq_0$.

We end up with

$$
\text{Tr}_p \left[ G_{p,ip_0+iq_0} \tilde{\lambda}^\alpha_{pp} G_{p,ip_0} \tilde{\lambda}^\gamma_{pp} \right] + \text{Tr}_p \left[ G_{p,ip_0+iq_0} \tilde{\lambda}^\alpha_{pp} G_{p,ip_0} \tilde{\lambda}^\gamma_{pp} \right] \quad (D10)
$$

$$
= - \text{Tr}_p \left[ G_{p,ip_0+iq_0} \Sigma^\alpha_{p} G_{p,ip_0+iq_0} \tilde{\lambda}^\beta_{pp} G_{p,ip_0} \tilde{\lambda}^\alpha_{pp} - G_{p,ip_0+iq_0} \tilde{\lambda}^\beta_{pp} G_{p,ip_0} \Sigma^\alpha_{p} G_{p,ip_0} \tilde{\lambda}^\alpha_{pp} \right] \quad (D11)
$$

$$
- \text{Tr}_p \left[ G_{p,ip_0+iq_0} F^\gamma_{p} G_{p,ip_0+iq_0} \tilde{\lambda}^\alpha_{pp} G_{p,ip_0} \tilde{\lambda}^\gamma_{pp} - G_{p,ip_0+iq_0} \tilde{\lambda}^\beta_{pp} G_{p,ip_0} F^\gamma_{p} G_{p,ip_0} \tilde{\lambda}^\alpha_{pp} \right] \quad (D12)
$$

$$
= - \text{Tr}_p \left[ G_{p,ip_0+iq_0} \tilde{\lambda}^\alpha_{pp} G_{p,ip_0+iq_0} \tilde{\lambda}^\beta_{pp} G_{p,ip_0} \tilde{\lambda}^\gamma_{pp} - G_{p,ip_0+iq_0} \tilde{\lambda}^\beta_{pp} G_{p,ip_0} \tilde{\lambda}^\gamma_{pp} G_{p,ip_0} \tilde{\lambda}^\alpha_{pp} \right] \quad (D13)
$$

where we combined the two lines Eq. (D11) and (D12) using the definition $\tilde{\lambda}^\alpha_{pp} = \Sigma^\alpha_{p} + F^\gamma_{p}$.

After shifting the Matsubara summation by $-iq_0$ in the first part and reversing the order in the second part we see that Eq. (D13) is equal to Eq. (D3) up to the sign and, thus, cancels.

### 2. Derivation leading from Eq. (38) to Eq. (40)

We present the derivation leading from Eq. (38) to Eq. (40). As shown in the main text, $K^{\alpha\beta\gamma}_{q,iq_0}$ in Eq. (38) reduces to three contributions in an uniform magnetic field $q \to 0$. We combine these in a convenient form for the following calculation defining

$$
\frac{\partial}{\partial q_0} K^{\alpha\beta\gamma}_{q,iq_0} \bigg|_{q=0} = e^3 \left[ (K^{\alpha\beta\gamma}_{iq_0})^{(1)} + (K^{\alpha\beta\gamma}_{iq_0})^{(2)} \right] \quad (D14)
$$

with

$$
(K^{\alpha\beta\gamma}_{iq_0})^{(1)} = \frac{\partial}{\partial q_0} \text{Tr}_p \left[ +\frac{1}{2} G_{p^+,ip_0+iq_0} \tilde{\lambda}^\alpha_{p^+} G_{p^-,ip_0} \tilde{\lambda}^\beta_{p^-} + \tilde{\lambda}^\gamma_{p^+} G_{p^+,ip_0} \tilde{\lambda}^\gamma_{p^+} \right] \bigg|_{q=0} \quad (D15)
$$
and

$$(K_{\alpha\beta\gamma\delta}^{(2)}) = \frac{\partial}{\partial q_\delta} \left[ \text{Tr}_p \left[ \cdots \right] \right]_{q=0}.$$  

(D16)

Here and in the following we use the short-hand notation $\text{Tr}_p = TL^{-1} \sum_p \text{tr}(\ldots)$. We used the short notation $p^\pm = p \pm \frac{1}{2} q$. In order to obtain Eq. (D15) we splitted the corresponding term in Eq. (38) into two equal parts, shifted the Matsubara summation by $iq_0$ and reversed the order of the matrices under the trace.$^{[12]}$

The following evaluation is rather involved, since the momentum derivative and the $2 \times 2$ matrix structure generates numerous terms. In order to tackle the later problem, we work in the matrix form as long as possible. The evaluation is mainly guided by a careful and structured decomposition of the momentum derivatives and the expected antisymmetry in the indices $\gamma \leftrightarrow \delta$ and $\alpha \leftrightarrow \beta$.

The calculation is structured as follows: We first (a) decompose $(K_{\alpha\beta\gamma\delta}^{(2)})$ into five different contributions. In the lengthy part (b) we reduce the decomposition of $(K_{\alpha\beta\gamma\delta}^{(2)})$ to these contributions in order to be able to combine them in the last step (c).

**a. Decomposition of $(K_{\alpha\beta\gamma\delta}^{(1)})$**

Let us first focus on Eq. (D15). We use that the second term of Eq. (D15) has the same derivative as the first term up to an overall minus sign when exchanging $iq_0 \rightarrow -iq_0$. In the following, we denote this as $(iq_0 \rightarrow -iq_0)$

$$(K_{\alpha\beta\gamma\delta}^{(1)}) = \frac{1}{2} \left[ \text{Tr}_p \left[ \cdots \right] \right]_{q=0}.$$  

(D17)

We have to perform four derivatives. For practical reasons we will split them into three parts: (i) the derivative of the Green functions and (ii) the derivative of the first order vertex and (iii) the derivative of the second order vertex. In step (iv) we combine the results of step (i) to (iii).

(i) **Derivative of the Green functions** – Using that $\partial_\delta G_{p^\pm, i p_0} |_{q=0} = \pm \frac{1}{2} G_{p, i p_0} E_\delta G_{p, i p_0}$, which...
follows directly by the definitions Eq. (22) and (30), we get
\[
\frac{1}{4} \text{Tr}_p \left[ \mathcal{G}_{p,p_0+iq_0} \mathcal{E}_p^\delta \mathcal{G}_{p,p_0+iq_0} \tilde{\lambda}_{pp}^\alpha \mathcal{G}_{p,p_0} \tilde{\lambda}_{pp}^{\beta\gamma} - \mathcal{G}_{p,p_0+iq_0} \tilde{\lambda}_{pp}^\alpha \mathcal{G}_{p,p_0} \mathcal{G}_{p,p_0} \tilde{\lambda}_{pp}^{\beta\gamma} - (iq_0 \to -iq_0) \right]. \tag{D18}
\]

The two terms are equal under the trace, which can be seen by shifting the Matsubara summation by \( iq_0 \), exchanging the sign via \( (iq_0 \to -iq_0) \) and reversing the matrix order under the trace. Thus,
\[
-\frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,p_0+iq_0} \tilde{\lambda}_{pp}^\alpha \mathcal{G}_{p,p_0} \mathcal{E}_p^\delta \mathcal{G}_{p,p_0} \mathcal{E}_p^\delta G_{p,p_0} \tilde{\lambda}_{pp}^{\beta\gamma} - (iq_0 \to -iq_0) \right]. \tag{D19}
\]

When splitting the vertices in their diagonal and off-diagonal parts, only an even number of diagonal matrices leads to a non-zero trace. One obtains three contributions:
\[
-\frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,p_0+iq_0} \mathcal{E}_p^\alpha \mathcal{G}_{p,p_0} \mathcal{E}_p^\delta \mathcal{G}_{p,p_0} \mathcal{E}_p^{\beta\gamma} - (iq_0 \to -iq_0) \right], \tag{D20}
\]
\[
+\frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,p_0+iq_0} \mathcal{E}_p^\alpha \mathcal{G}_{p,p_0} \mathcal{E}_p^\delta \mathcal{G}_{p,p_0} \mathcal{C}_p^{\beta\gamma} - (iq_0 \to -iq_0) \right], \tag{D21}
\]
\[
-\frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,p_0+iq_0} \mathcal{F}_p^\delta \mathcal{G}_{p,p_0} \mathcal{E}_p^\delta \mathcal{G}_{p,p_0} \mathcal{H}_p^{\beta\gamma} - (iq_0 \to -iq_0) \right]. \tag{D22}
\]

(ii) Derivative of the single vertex – Using the general expansion Eq. (A7), the derivative of the single vertex \( \tilde{\lambda}_{pp}^\alpha \) is
\[
\frac{\partial}{\partial q_\delta} \tilde{\lambda}_{pp}^\alpha \bigg|_{q=0} = \frac{1}{2} \left( \mathcal{S}_p \mathcal{F}_p^\delta \tilde{\lambda}_{pp}^\alpha - \tilde{\lambda}_{pp}^\alpha \mathcal{F}_p^\delta \mathcal{S}_p \right) = \frac{1}{2} \left( \mathcal{S}_p \mathcal{F}_p^\delta \mathcal{E}_p^\alpha - \mathcal{E}_p^\alpha \mathcal{F}_p^\delta \mathcal{S}_p \right) \tag{D23}
\]
where the off-diagonal part of \( \tilde{\lambda}_{pp}^\alpha \) cancels. We obtain
\[
\frac{1}{4} \text{Tr}_p \left[ \mathcal{G}_{p,p_0+iq_0} \mathcal{S}_p \mathcal{F}_p^\delta \mathcal{G}_{p,p_0} \mathcal{E}_p^\alpha \mathcal{G}_{p,p_0} \tilde{\lambda}_{pp}^{\beta\gamma} - \mathcal{G}_{p,p_0+iq_0} \mathcal{E}_p^\alpha \mathcal{F}_p^\delta \mathcal{S}_p \mathcal{G}_{p,p_0} \tilde{\lambda}_{pp}^{\beta\gamma} - (iq_0 \to -iq_0) \right]. \tag{D24}
\]

The two terms are equal under the trace, which can be seen by shifting the Matsubara summation by \( iq_0 \), exchanging the sign via \( (iq_0 \to -iq_0) \) and reversing the matrix order under the trace. The vertex \( \tilde{\lambda}_{pp}^{\beta\gamma} \) only contributes with its off-diagonal part, since only an even number of off-diagonal matrices leads to a non-zero trace. We commute \( \mathcal{S}_p \) and \( \mathcal{G}_{p,p_0+iq_0} \) and get
\[
\frac{1}{2} \text{Tr}_p \left[ \mathcal{S}_p \mathcal{G}_{p,p_0+iq_0} \mathcal{F}_p^\delta \mathcal{E}_p^\alpha \mathcal{G}_{p,p_0} \mathcal{H}_p^{\beta\gamma} - (iq_0 \to -iq_0) \right]. \tag{D25}
\]
A few purely algebraic steps yield the relation \( \mathcal{S}_p \mathcal{G}_{p,p_0+iq_0} \mathcal{F}_p^\delta = \mathcal{G}_{p,p_0+iq_0} \mathcal{F}_p^\delta \mathcal{G}_{p,p_0} \mathcal{G}_{p,p_0+iq_0} - \mathcal{F}_p^\delta \mathcal{G}_{p,p_0+iq_0} \mathcal{S}_p \). Using this, the second term leads to a contribution, which cancels under
the trace with the corresponding term in \((iq_0 \leftrightarrow -iq_0)\) when shifting the Matsubara summation. We end up with
\[
-\frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_p \mathcal{G}_p \mathcal{F}_p \mathcal{G}_p \mathcal{H}_p^\beta - (iq_0 \rightarrow -iq_0) \right],
\]
where we have again shifted the Matsubara summation and reversed the matrix order under the trace.

(iii) Derivative of the second order vertex – Using the general expansion Eq. \((A7)\) the derivative of the double vertex \(\tilde{\lambda}_{p^-p^+}^{\beta\gamma}\) is
\[
\left. \frac{\partial}{\partial q_\delta} \tilde{\lambda}_{p^-p^+}^{\beta\gamma} \right|_{q=0} = -\frac{1}{2} \left( \mathcal{S}_p F_p^\delta \tilde{\lambda}_{p^-p^+}^{\beta\gamma} - \tilde{\lambda}_{p^-p^+}^{\beta\gamma} F_p^\delta S_p \right) = -\frac{1}{2} \left( \mathcal{S}_p F_p^\delta \mathcal{E}_p^\beta - \mathcal{E}_p^\beta F_p^\delta S_p \right)
\]
In the second step we used that the parts \(\mathcal{E}_p^\beta\) and \(\mathcal{H}_p^\beta\) of \(\tilde{\lambda}_{p^-p^+}^{\beta\gamma}\) drop out. We have a very similar structure as in the single vertex case (ii). After the same type of manipulations we obtain
\[
-\frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_p \mathcal{F}_p \mathcal{G}_p \mathcal{F}_p \mathcal{E}_p^\beta - (iq_0 \rightarrow -iq_0) \right].
\]

(iv) Combining the results of steps (i)–(iii) – The contribution \((K_{iq_0}^{\alpha\beta\delta})^{(1)}\) decomposes into five contributions
\[
(K_{iq_0}^{\alpha\beta\delta})^{(1)} = (K_{iq_0}^{\alpha\beta\gamma})^{(1,I)} + (K_{iq_0}^{\alpha\beta\gamma})^{(1,II)} + (K_{iq_0}^{\alpha\beta\gamma})^{(1,III)} + (K_{iq_0}^{\alpha\beta\gamma})^{(1,IV)} + (K_{iq_0}^{\alpha\beta\gamma})^{(1,V)},
\]
where we define
\[
(K_{iq_0}^{\alpha\beta\gamma})^{(1,I)} = -\frac{1}{2} \text{Tr}_p \left[ (\mathcal{G}_p \mathcal{G}_p \mathcal{F}_p \mathcal{G}_p \mathcal{E}_p^\alpha \mathcal{G}_p \mathcal{E}_p^\beta \mathcal{E}_p^\gamma) \right],
\]
\[
(K_{iq_0}^{\alpha\beta\gamma})^{(1,II)} = +\frac{1}{2} \text{Tr}_p \left[ (\mathcal{G}_p \mathcal{G}_p \mathcal{F}_p \mathcal{G}_p \mathcal{E}_p^\alpha \mathcal{G}_p \mathcal{E}_p^\beta \mathcal{C}_p^\gamma) \right],
\]
\[
(K_{iq_0}^{\alpha\beta\gamma})^{(1,III)} = -\frac{1}{2} \text{Tr}_p \left[ (\mathcal{G}_p \mathcal{G}_p \mathcal{F}_p \mathcal{F}_p \mathcal{G}_p \mathcal{H}_p^\gamma) \right],
\]
\[
(K_{iq_0}^{\alpha\beta\gamma})^{(1,IV)} = -\frac{1}{2} \text{Tr}_p \left[ (\mathcal{G}_p \mathcal{G}_p \mathcal{F}_p \mathcal{F}_p \mathcal{G}_p \mathcal{E}_p^\beta \mathcal{H}_p^\gamma) \right],
\]
\[
(K_{iq_0}^{\alpha\beta\gamma})^{(1,V)} = -\frac{1}{2} \text{Tr}_p \left[ (\mathcal{G}_p \mathcal{G}_p \mathcal{F}_p \mathcal{F}_p \mathcal{G}_p \mathcal{E}_p^\beta \mathcal{E}_p^\gamma) \right].
\]
b. Decomposition of \( (K_{\alpha\beta\gamma\delta})^{(2)} \)

We continue with the decomposition of Eq. (D16). We use that the second term of Eq. (D16) has the same derivative as the first term up to an overall minus sign when exchanging \( iq_0 \leftrightarrow -iq_0 \). In the following, we denote the second term as \( (iq_0 \leftrightarrow -iq_0) \):

\[
(K_{\alpha\beta\gamma\delta})^{(2)} = \text{Tr}_p \left[ \frac{\partial}{\partial q_0} \left( \mathcal{G}_{p^+,ip_0+iq_0} \tilde{\lambda}_{p^+,p^-} \mathcal{G}_{p^-,ip_0} \tilde{\lambda}_{p^+,p^+} \mathcal{G}_{p^+,ip_0} \tilde{\lambda}_{p^+,p^+} \right) - (iq_0 \rightarrow -iq_0) \right]_{q_0=0}.
\]

(D35)

We have to perform six derivatives. For practical reasons we will perform them in the following order: (i) Derivative of the vertex \( \tilde{\lambda}_{p^+,p^+} \), (ii) derivative of the vertex \( \tilde{\lambda}_{p^-} \), (iii) derivative of the vertex \( \tilde{\lambda}_{p^+,p^+} \), (iv) derivative of the Green functions \( \mathcal{G}_{p^+,ip_0} \), (v) derivative of the Green function \( \mathcal{G}_{p^+,ip_0+iq_0} \). In a sixth and seventh step we will simplify the contributions: (vi) Reducing the number of Green functions in the individual expressions and (vii) recombining the contributions.

(i) Derivative of the vertex \( \tilde{\lambda}_{p^+,p^+} \) – Using the general expansion Eq. (A7) the derivative of the vertex \( \tilde{\lambda}_{p^+,p^+} \) is

\[
\frac{\partial}{\partial q_0} \tilde{\lambda}_{p^+,p^+} \bigg|_{q_0=0} = \frac{1}{2} \left( \tilde{\lambda}_{p^+,p^+} \mathcal{F}_{p^+} \tilde{\lambda}_{p^+,p^+} + \tilde{\lambda}_{p^+,p^+} \mathcal{S}_{p} \right)
\]

(D36)

\[
= \frac{1}{2} \left( \mathcal{E}_{p}^{\beta} + \mathcal{H}_{p}^{\beta} + \mathcal{S}_{p} \mathcal{F}_{p}^{\beta} \mathcal{E}_{p}^{\beta} + \mathcal{E}_{p}^{\beta} \mathcal{F}_{p}^{\beta} \mathcal{S}_{p} \right).
\]

(D37)

In the second step we used that the \( \mathcal{F}_{p}^{\beta} \) contributions of \( \tilde{\lambda}_{pp} \) exactly cancels the \( \mathcal{C}_{p}^{\beta\delta} \) in the second order vertex \( \tilde{\lambda}_{pp}^{\beta\delta} \), which can be directly seen by using the definition Eq. (28) and \( \mathcal{S}_{p} = 1/(E_{p}^{+} - E_{p}^{-}) \sigma^{z} \).

In a first step, let us focus on the first two contributions containing \( \mathcal{E}_{p}^{\beta} \) and \( \mathcal{H}_{p}^{\beta} \). The former one is a diagonal matrix. Thus, in order to get a non-zero trace, the two remaining vertices must both contribute with their diagonal or off-diagonal part. The latter one is an off-diagonal matrix so that the two remaining vertices must contribute with their diagonal
and off-diagonal part or vice versa. We obtain four contributions

\[ + \frac{1}{2} \text{Tr}_p \left[ (G_{p,ip_0+iq_0} - G_{p,ip_0-iq_0}) E_p^\alpha G_{p,ip_0} E_p^\gamma G_{p,ip_0} E_p^\beta \right] = -(K_{iq_0}^{\alpha\beta\gamma})^{(1,I)}, \tag{D38} \]

\[ + \frac{1}{2} \text{Tr}_p \left[ (G_{p,ip_0+iq_0} - G_{p,ip_0-iq_0}) F_p^\alpha G_{p,ip_0} F_p^\gamma G_{p,ip_0} E_p^\delta \right] = -(K_{iq_0}^{\alpha\beta\delta})^{(1,V)}, \tag{D39} \]

\[ + \frac{1}{2} \text{Tr}_p \left[ (G_{p,ip_0+iq_0} - G_{p,ip_0-iq_0}) E_p^\alpha G_{p,ip_0} F_p^\gamma G_{p,ip_0} H_p^\delta \right] = -(K_{iq_0}^{\alpha\beta\gamma})^{(1,IV)}, \tag{D40} \]

\[ + \frac{1}{2} \text{Tr}_p \left[ (G_{p,ip_0+iq_0} - G_{p,ip_0-iq_0}) F_p^\alpha G_{p,ip_0} E_p^\gamma G_{p,ip_0} H_p^\delta \right] = -(K_{iq_0}^{\alpha\beta\delta})^{(1,III)}, \tag{D41} \]

which can immediately be identified as the antisymmetric counterparts under exchange of \( \gamma \leftrightarrow \delta \) of the expressions we obtained for \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{(1)} \) given in Eq. \( (D30)-(D34) \). Note that a contribution for \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{(1,II)} \) is missing.

We will continue with the remaining two contributions of the vertex derivative containing \( S_p F_p^\alpha E_p^\beta \) and \( E_p^\beta F_p^\delta S_p \). As these already include an off-diagonal matrix \( F_p^\delta \) one of the two remaining vertices \( \tilde{\lambda}_p^\alpha \) or \( \tilde{\lambda}_p^\gamma \) has to contribute with its off-diagonal part \( F_p \) whereas the other one contributes with its diagonal part \( E_p \) in order to get a non-zero trace. We get

\[ (K_{iq_0}^{\alpha\beta\gamma\delta})^{(v,1)} = + \frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F_p^\alpha G_{p,ip_0} E_p^\gamma G_{p,ip_0} S_p F_p^\delta E_p^\beta - (iq_0 \rightarrow -iq_0) \right], \tag{D42} \]

\[ (K_{iq_0}^{\alpha\beta\gamma\delta})^{(v,2)} = + \frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F_p^\alpha G_{p,ip_0} E_p^\gamma G_{p,ip_0} E_p^\beta F_p^\delta S_p - (iq_0 \rightarrow -iq_0) \right], \tag{D43} \]

\[ (K_{iq_0}^{\alpha\beta\gamma\delta})^{(v,*)} = + \frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} E_p^\alpha G_{p,ip_0} F_p^\gamma G_{p,ip_0} S_p F_p^\delta E_p^\beta - (iq_0 \rightarrow -iq_0) \right], \tag{D44} \]

\[ (K_{iq_0}^{\alpha\beta\gamma\delta})^{(v,**)} = + \frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} E_p^\alpha G_{p,ip_0} F_p^\gamma G_{p,ip_0} E_p^\beta F_p^\delta S_p - (iq_0 \rightarrow -iq_0) \right]. \tag{D45} \]

The first two we will simplify via combining them with the contributions produced by the derivative of the Green functions, which we will perform in step (iv) and (v). The later two will be reconsidered in step (ii). The upper label \( v \) shall indicate that they were obtained by a vertex derivative.

(ii) Derivative of the vertex \( \tilde{\lambda}_p^\alpha \) \( \rightarrow \) \( \tilde{\lambda}_p^\alpha \) \( \rightarrow \) – Using the general expansion Eq. \( (A7) \) the derivative of the vertex \( \tilde{\lambda}_p^\alpha \) is

\[ \frac{\partial}{\partial q_{i\delta}} \tilde{\lambda}_p^\alpha \bigg|_{q=0} = + \frac{1}{2} \left( S_p F_p^\delta \tilde{\lambda}_p^\alpha - \tilde{\lambda}_p^\alpha F_p^\delta S_p \right) = + \frac{1}{2} \left( S_p F_p^\delta E_p^\alpha - E_p^\alpha F_p^\delta S_p \right). \tag{D46} \]

In the second step we used that the off-diagonal part of \( \tilde{\lambda}_p^\alpha \) cancels. Using this derivative and the fact that only an even number of off-diagonal matrices leads to a non-zero trace we
obtain the four contributions

\[ (K^\alpha_{\lambda_0}^\gamma_{\lambda_0})^{(v,**)} = +\frac{1}{2} \text{Tr}_p \left[ G_{p,p_0+iq_0} S_p F_p^\delta \epsilon_0^p G_{p,p_0} F_p^\gamma \epsilon_0^p - (iq_0 \to -iq_0) \right], \]

\[ (K^\alpha_{\lambda_0}^\gamma_{\lambda_0})^{(v,***)} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,p_0+iq_0} \epsilon_0^p F_p^\delta S_p G_{p,p_0} F_p^\gamma \epsilon_0^p - (iq_0 \to -iq_0) \right], \]

\[ (K^\alpha_{\lambda_0}^\gamma_{\lambda_0})^{(v,****)} = +\frac{1}{2} \text{Tr}_p \left[ G_{p,p_0+iq_0} S_p F_p^\delta \epsilon_0^p G_{p,p_0} F_p^\gamma \epsilon_0^p - (iq_0 \to -iq_0) \right], \]

\[ (K^\alpha_{\lambda_0}^\gamma_{\lambda_0})^{(v,*)} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,p_0+iq_0} \epsilon_0^p F_p^\delta S_p G_{p,p_0} F_p^\gamma \epsilon_0^p - (iq_0 \to -iq_0) \right]. \]

The second contribution \((K^\alpha_{\lambda_0}^\gamma_{\lambda_0})^{(v,****)}\) cancels with Eq. \((\ref{D44})\) by using that \(\epsilon_0^p\) respectively \(\epsilon_0^\gamma\) commute with \(G_{p,p_0+iq_0}\) and reversing the matrix order under the trace. In order to see that \((K^\alpha_{\lambda_0}^\gamma_{\lambda_0})^{(v,**)}\) is canceled by Eq. \((\ref{D45})\) we have to perform the matrix trace explicitly.

This leads to

\[ (K^\alpha_{\lambda_0}^\gamma_{\lambda_0})^{(v,**)} + (K^\alpha_{\lambda_0}^\gamma_{\lambda_0})^{(v,**)} = \frac{1}{2} TL^{-1} \sum_p F_p^\gamma F_p^\delta \left[ (G_{p,p_0+iq_0}^+ - G_{p,p_0-iq_0}^-) G_{p,p_0}^+ G_{p,p_0}^- \left( E_p^+ E_p^- + E_p^- E_p^+ \right) 
- (G_{p,p_0+iq_0}^- - G_{p,p_0-iq_0}^+) G_{p,p_0}^+ G_{p,p_0}^- \left( E_p^+ E_p^- + E_p^- E_p^+ \right) \right]. \]

From the purely algebraic relation \(G_{p,p_0}^+ G_{p,p_0}^- = (G_{p,p_0}^- - G_{p,p_0}^+)/ (E_p^+ - E_p^-)\) it immediately follows that the two type of Green function products are equal under the Matsubara summation,

\[ T \sum_{p_0} G_{p,p_0}^+ G_{p,p_0}^- (G_{p,p_0+iq_0}^+ - G_{p,p_0-iq_0}^-) = T \sum_{p_0} G_{p,p_0}^+ G_{p,p_0}^- (G_{p,p_0+iq_0}^- - G_{p,p_0-iq_0}^+). \]

Thus, Eq. \((\ref{D52})\) is zero. We will come back to the other two contributions \((\ref{D49})\) and \((\ref{D50})\) after performing the derivative of the Green functions in step (iv) and (v).

(iii) Derivative of the vertex \(\lambda^\gamma_{p-p^+}\) — Using the general expansion Eq. \((\ref{A7})\) the derivative of the vertex \(\lambda^\gamma_{p-p^+}\) is

\[ \frac{\partial}{\partial q_{\delta}} \lambda^\gamma_{p-p^+} \bigg|_{q=0} = -\frac{1}{2} \left( S_p F_p^\delta \lambda_{p-p}^\gamma - \lambda_{p-p}^\gamma F_p^\delta S_p \right) = -\frac{1}{2} \left( S_p F_p^\delta \epsilon_0^p - \epsilon_0^p F_p^\delta S_p \right). \]

In the second step we used that the off-diagonal part of \(\lambda_{p-p}^\gamma\) cancels. Using the vertex derivative and the fact that only an even number of off-diagonal matrices leads to a non-
zero trace we obtain the four contributions

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(v,5)} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \epsilon_{\gamma}^\gamma G_{p,ip_0} \epsilon_{\beta}^\beta - (iq_0 \rightarrow -iq_0) \right], \quad (D55)
\]

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(v,6)} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \epsilon_{\gamma}^\gamma G_{p,ip_0} \epsilon_{\beta}^\beta - (iq_0 \rightarrow -iq_0) \right], \quad (D56)
\]

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(v,7)} = +\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \epsilon_{\gamma}^\gamma G_{p,ip_0} \epsilon_{\beta}^\beta - (iq_0 \rightarrow -iq_0) \right], \quad (D57)
\]

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(v,8)} = +\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \epsilon_{\gamma}^\gamma G_{p,ip_0} \epsilon_{\beta}^\beta - (iq_0 \rightarrow -iq_0) \right]. \quad (D58)
\]

We will come back to these four contributions after performing the derivative of the Green functions in step (iv) and (v).

(iv) Derivative of the \( G_{p^\pm,ip_0} \) – We have to perform the derivative of three Green functions in \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{(2)} \) given in Eq. (D35). We start with the derivative of the two Green functions independent of \( iq_0 \). Using the derivative of a Green function \( \partial_0 G_{p^\pm,ip_0} \big|_{q=0} = \pm \frac{1}{2} G_{p,ip_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \), we get

\[
-\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \tilde{\lambda}_{pp}^\alpha G_{p,ip_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \tilde{\lambda}_{pp}^\gamma G_{p,ip_0} \tilde{\lambda}_{pp}^\beta - (iq_0 \rightarrow -iq_0) \right] \\
+ \frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \tilde{\lambda}_{pp}^\alpha G_{p,ip_0} \tilde{\lambda}_{pp}^\gamma G_{p,ip_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \tilde{\lambda}_{pp}^\beta - (iq_0 \rightarrow -iq_0) \right]. \quad (D59)
\]

Expanding the vertex \( \tilde{\lambda}_{pp}^\gamma \) into its diagonal \( \epsilon_{\delta}^\delta \) and off-diagonal part \( F_{p}^\delta \) we see that the diagonal part cancels as \( \epsilon_{\delta}^\delta G_{p,ip_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} = \epsilon_{\alpha}^\alpha G_{p,ip_0} \epsilon_{\delta}^\delta G_{p,ip_0} \). Expanding the remaining two vertices into their diagonal and off-diagonal parts we get four contributions with non-zero trace:

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(4,1)} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \epsilon_{\delta}^\delta G_{p,ip_0} \epsilon_{\gamma}^\gamma G_{p,ip_0} \epsilon_{\beta}^\beta - (iq_0 \rightarrow -iq_0) \right], \quad (D60)
\]

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(4,2)} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} \epsilon_{\delta}^\delta G_{p,ip_0} \epsilon_{\gamma}^\gamma G_{p,ip_0} \epsilon_{\beta}^\beta - (iq_0 \rightarrow -iq_0) \right], \quad (D61)
\]

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(4,3)} = +\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} F_{p}^\gamma G_{p,ip_0} \epsilon_{\gamma}^\gamma G_{p,ip_0} \epsilon_{\beta}^\beta - (iq_0 \rightarrow -iq_0) \right], \quad (D62)
\]

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(4,4)} = +\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0} F_{p}^\gamma G_{p,ip_0} \epsilon_{\gamma}^\gamma G_{p,ip_0} \epsilon_{\beta}^\beta - (iq_0 \rightarrow -iq_0) \right]. \quad (D63)
\]

The upper label g shall indicate that they were obtained within the Green function derivative. We will come back to these contributions in step (vi).

(v) Derivative of the \( G_{p^+,ip_0+iq_0} \) – Using \( \partial_0 G_{p^+,ip_0+iq_0} \big|_{q=0} = \frac{1}{2} G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0+iq_0} \) we get

\[
+\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \epsilon_{\alpha}^\alpha G_{p,ip_0+iq_0} \tilde{\lambda}_{pp}^\gamma G_{p,ip_0} \tilde{\lambda}_{pp}^\gamma G_{p,ip_0} \tilde{\lambda}_{pp}^\beta - (iq_0 \rightarrow -iq_0) \right]. \quad (D64)
\]
We decompose the remaining three vertices in their diagonal and off-diagonal parts. The contribution of fully-diagonal matrices (all vertices with its diagonal part $\mathcal{E}_p$) cancels with the corresponding term in $(iq_0 \leftrightarrow -iq_0)$ after shifting the Matsubara summation. Thus, two of the three vertices $\tilde{\lambda}^\alpha_{pp}$, $\tilde{\lambda}^\gamma_{pp}$ and $\tilde{\lambda}^\beta_{pp}$ have to contribute with their off-diagonal part to get a non-zero trace.

When the vertex $\tilde{\lambda}^\gamma_{pp}$ contribute with its off-diagonal part, the two other vertices $\tilde{\lambda}^\alpha_{pp}$ and $\tilde{\lambda}^\beta_{pp}$ contribute with their diagonal and off-diagonal parts or vice versa. We label these as

\begin{equation}
(K^{\alpha\beta\gamma\delta}_{iq_0}(g,5)) = \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,ip_0+iq_0} \mathcal{E}^\delta_p \mathcal{G}_{p,ip_0+iq_0} \mathcal{F}^\alpha_p \mathcal{G}_{p,ip_0} \mathcal{E}^\gamma_p \mathcal{G}_{p,ip_0} \mathcal{E}^\beta_p - (iq_0 \rightarrow -iq_0) \right], \tag{D65}
\end{equation}

\begin{equation}
(K^{\alpha\beta\gamma\delta}_{iq_0}(g,6)) = \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,ip_0+iq_0} \mathcal{E}^\delta_p \mathcal{G}_{p,ip_0+iq_0} \mathcal{F}^\alpha_p \mathcal{G}_{p,ip_0} \mathcal{E}^\gamma_p \mathcal{G}_{p,ip_0} \mathcal{F}^\beta_p - (iq_0 \rightarrow -iq_0) \right]. \tag{D66}
\end{equation}

We will come back to them in step (vi).

If the vertex $\tilde{\lambda}^\gamma_{pp}$ contributes with its diagonal part, the contribution with non-zero trace reads

\begin{equation}
+ \frac{1}{4} \text{Tr}_p \left[ \mathcal{G}_{p,ip_0+iq_0} \mathcal{E}^\delta_p \mathcal{G}_{p,ip_0+iq_0} \mathcal{F}^\alpha_p \mathcal{G}_{p,ip_0} \mathcal{E}^\gamma_p \mathcal{G}_{p,ip_0} \mathcal{F}^\beta_p - (iq_0 \rightarrow -iq_0) \right]. \tag{D67}
\end{equation}

We continue simplifying this. We split the contribution into two equal parts, rename the sum over Matsubara frequencies $iq_0 \rightarrow -iq_0$ of the second part and reverse the matrix order under the trace. We end up with an antisymmetric form in $\delta \leftrightarrow \gamma$, which we denote as

\begin{equation}
+ \frac{1}{4} \text{Tr}_p \left[ \mathcal{G}_{p,ip_0+iq_0} \mathcal{E}^\delta_p \mathcal{G}_{p,ip_0+iq_0} \mathcal{F}^\alpha_p \mathcal{G}_{p,ip_0} \mathcal{E}^\gamma_p \mathcal{G}_{p,ip_0} \mathcal{F}^\beta_p - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right]. \tag{D68}
\end{equation}

We can introduce two derivatives of the Green functions with respect to $\delta$ and $\gamma$ and get

\begin{equation}
+ \frac{1}{4} \text{Tr}_p \left[ (\partial_\delta \mathcal{G}_{p,ip_0+iq_0}) \mathcal{F}^\alpha_p (\partial_\gamma \mathcal{G}_{p,ip_0}) \mathcal{F}^\beta_p - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right]. \tag{D69}
\end{equation}

We perform a partial integration in $p_5$. The boundary terms vanish under the momentum summation. Thus,

\begin{equation}
- \frac{1}{4} \text{Tr}_p \left[ \mathcal{G}_{p,ip_0+iq_0} \partial_\delta \left[ \mathcal{F}^\alpha_p (\partial_\gamma \mathcal{G}_{p,ip_0}) \mathcal{F}^\beta_p \right] - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right]. \tag{D70}
\end{equation}

After performing the product rule of the derivative the term including $\partial_\delta \partial_\gamma \mathcal{G}_{p,ip_0}$ vanishes due to corresponding term in $(\delta \leftrightarrow \gamma)$. The two other contributions containing $\partial_\delta \mathcal{F}^\alpha_p$ and $\partial_\delta \mathcal{F}^\beta_p$ are equal up to an exchange of $\alpha \leftrightarrow \beta$ when reversing the matrix order under the trace. We denote the second contribution as $(\alpha \leftrightarrow \beta)$ and get

\begin{equation}
- \frac{1}{4} \text{Tr}_p \left[ \mathcal{G}_{p,ip_0+iq_0} (\partial_\delta \mathcal{F}^\alpha_p) \mathcal{G}_{p,ip_0} \mathcal{E}^\gamma_p \mathcal{G}_{p,ip_0} \mathcal{F}^\beta_p + (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right]. \tag{D71}
\end{equation}
Using the definition \( F_\alpha = 2\Delta h_p^2/(E_p^+ - E_p^-) \) one immediately finds the identity \( \partial_\gamma F_\alpha = H_\alpha + S_p F_\alpha^\delta \mathcal{E}_\delta + \mathcal{E}_\alpha F_\alpha^\delta S_p \), where \( S_p = 1/(E_p^+ - E_p^-) \sigma^z \). The first part containing \( H_\alpha \) can be identified with \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{(1,III)} \) in Eq. (D32) by reversing the matrix order under the trace. Thus,

\[
- \frac{1}{4} \text{Tr}_p \left[ G_{p,ip_0+iq_0} H_\alpha^\gamma G_{p,ip_0} G_{p,ip_0} F_\alpha^\gamma + (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) - (iq_0 \leftrightarrow iq_0) \right] \\
= -\frac{1}{2} \left( (K_{iq_0}^{\alpha\beta\gamma\delta})^{(1,III)} + (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) \right). \tag{D73}
\]

The second part containing \( S_p F_\alpha^\gamma \mathcal{E}_\delta \) is canceled by the corresponding term in \( (\gamma \leftrightarrow \delta) \) since \( \mathcal{E}_\delta G_{p,ip_0} G_{p,ip_0} \mathcal{E}_\gamma^n = G_{p,ip_0} G_{p,ip_0} \mathcal{E}_\gamma^n \). The third part containing \( \mathcal{E}_\delta F_\alpha^\gamma S_p \) gives

\[
- \frac{1}{4} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \mathcal{E}_\delta^\gamma F_\alpha^\gamma S_p G_{p,ip_0} G_{p,ip_0} F_\alpha^\gamma + (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) - (iq_0 \to -iq_0) \right]. \tag{D74}
\]

Reversing the matrix order under the trace in the \( (\alpha \leftrightarrow \beta) \) part and using that all matrices except of \( F_\alpha^\gamma \) and \( F_\alpha^\delta \) commute we get

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{g,7} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \mathcal{E}_\delta^\gamma F_\alpha^\gamma S_p G_{p,ip_0} G_{p,ip_0} F_\alpha^\gamma - (\gamma \leftrightarrow \delta) - (iq_0 \to -iq_0) \right]. \tag{D75}
\]

This contribution will be reconsidered in step (vii).

(vi) Reducing the number of Green functions – The six contributions \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{g,1} \) to \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{g,6} \) in Eq. (D60)–(D63) and Eq. (D65)+(D66) involve four Green functions each. In order to be able to recombine these with the expressions \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{v,1} \) to \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{v,8} \) in step (vii) we have to reduce the Green function number by one. As all six contributions have a similar structure, we can do this in the same fashion for all of them using the relation

\[
G_{p,ip_0} F_\alpha^\gamma G_{p,ip_0} = F_\alpha^\gamma G_{p,ip_0} S_p + S_p G_{p,ip_0} F_\alpha^\gamma, \tag{D76}
\]

which can be verified by purely algebraic steps.

Let us start with the two contributions \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{g,5} \) and \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{g,3} \), given in Eq. (D65) and Eq. (D62). The further one was defined as

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(g,5)} = +\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \mathcal{E}_\delta G_{p,ip_0} G_{p,ip_0} F_\alpha^\gamma G_{p,ip_0} - (iq_0 \to -iq_0) \right]. \tag{D77}
\]

Using the relation (D76) the contribution \( (K_{iq_0}^{\alpha\beta\gamma\delta})^{(g,5)} \) splits up into the two parts

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})^{(g,5)} = +\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \mathcal{E}_\delta G_{p,ip_0} G_{p,ip_0} F_\alpha^\gamma G_{p,ip_0} S_p F_\alpha^\delta - (iq_0 \to -iq_0) \right] \\
+ \frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} \mathcal{E}_\delta G_{p,ip_0} G_{p,ip_0} F_\alpha^\gamma S_p G_{p,ip_0} F_\alpha^\delta - (iq_0 \to -iq_0) \right]. \tag{D78}
\]
We perform the following manipulations: We first shift the Matsubara frequency summation by \( iq_0 \) and change sign using the antisymmetric part \( (iq_0 \leftrightarrow -iq_0) \). In the second part we commute the diagonal matrices \( S_p \) and \( G_{p,ip_0+iq_0} \). Finally, we reverse the order of the matrices in both parts and end up with

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})(g,5) = -\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F^\alpha_p G_{p,ip_0} E^\delta_p G_{p,ip_0} E^\beta_p S_p - (iq_0 \to -iq_0) \right]
- \frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F^\alpha_p G_{p,ip_0} E^\delta_p G_{p,ip_0} E^\beta_p S_p - (iq_0 \to -iq_0) \right].
\] (D79)

Note that we can identify the second part as the antisymmetric counterpart of \((K_{iq_0}^{\alpha\beta\gamma\delta})(v,2)\) in Eq. (D43) under the exchange of \( \gamma \leftrightarrow \delta \),

\[
-\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F^\alpha_p G_{p,ip_0} E^\delta_p G_{p,ip_0} E^\beta_p S_p - (iq_0 \to -iq_0) \right] = -(K_{iq_0}^{\alpha\beta\gamma\delta})(v,2).
\] (D80)

We continue applying the relation (D76) to \((K_{iq_0}^{\alpha\beta\gamma\delta})(g,3)\) in Eq. (D62). As before, we can identify the second part as an antisymmetric counterpart, now for \((K_{iq_0}^{\alpha\beta\gamma\delta})(v,6)\) in Eq. (D56) by simplify commuting \( S_p \) and \( G_{p,ip_0} \). Thus, we end up with

\[
(K_{iq_0}^{\alpha\beta\gamma\delta})(g,3) = +\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F^\alpha_p G_{p,ip_0} S_p E^\delta_p G_{p,ip_0} E^\beta_p - (iq_0 \to -iq_0) \right]
- (K_{iq_0}^{\alpha\beta\gamma\delta})(v,6).
\] (D81)

Note, that the first part of (D79) and (D81) exactly cancel each other since \( F^\alpha_p F^\alpha_p = F^\alpha_p F^\alpha_p \) and the other matrices are diagonal and, thus, trivially commute.

We apply relation (D76) to the remaining four contributions using the same type of manipulations. The contribution \((K_{iq_0}^{\alpha\beta\gamma\delta})(g,2)\) in Eq. (D61) gives the antisymmetric counterparts to \((K_{iq_0}^{\alpha\beta\gamma\delta})(v,8)\) in Eq. (D58) and \((K_{iq_0}^{\alpha\beta\gamma\delta})(v,1)\) in Eq. (D42). The contribution \((K_{iq_0}^{\alpha\beta\gamma\delta})(g,4)\) in Eq. (D63) gives the antisymmetric counterparts to \((K_{iq_0}^{\alpha\beta\gamma\delta})(v,4)\) in Eq. (D50) and \((K_{iq_0}^{\alpha\beta\gamma\delta})(v,5)\) in Eq. (D55). The two contributions \((K_{iq_0}^{\alpha\beta\gamma\delta})(g,6)\) in Eq. (D66) and \((K_{iq_0}^{\alpha\beta\gamma\delta})(g,1)\) in Eq. (D60) give the antisymmetric counterparts to \((K_{iq_0}^{\alpha\beta\gamma\delta})(v,3)\) in Eq. (D49) and \((K_{iq_0}^{\alpha\beta\gamma\delta})(v,7)\) in Eq. (D57) and

\[
-\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F^\beta_p F^\alpha_p S_p - (iq_0 \to -iq_0) \right]
- \frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F^\beta_p F^\alpha_p S_p - (iq_0 \to -iq_0) \right].
\] (D82)

These can be combined by using the relation \( F^\beta_p F^\alpha_p S_p + S_p F^\beta_p F^\alpha_p = C_{p\gamma}^{\beta\gamma} \), which follows from the definition Eq. (28). We identify

\[
-\frac{1}{2} \text{Tr}_p \left[ G_{p,ip_0+iq_0} F^\beta_p F^\alpha_p S_p - (iq_0 \to -iq_0) \right] = -(K_{iq_0}^{\alpha\beta\gamma\delta})(1,11).
\] (D84)
which exactly cancels the \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(1,11)}\) in Eq. (D31).

(vii) **Recombining the contributions** – In step (i) and (ii) we found eight contributions \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,1)}\) to \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,8)}\), for which we could find antisymmetric counterparts in \(\gamma \leftrightarrow \delta\) in step (vi). For simplicity, we will label these antisymmetric contributions with the same symbols in the following. Furthermore, we have the remaining contribution \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(g,7)}\) of step (v).

We start with \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,1)}\) and \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,2)}\) in Eq. (D42) and Eq. (D43), which sum up to

\[
(K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,1)} + (K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,2)} = \frac{1}{2} \text{Tr} \left[ G_{p_0} G_{p_0} \mathcal{F}^{\alpha_0}_{p} \mathcal{G}^{\beta_0}_{p} \mathcal{E}^{\gamma_0}_{p} S_{p_0} \left( \mathcal{S}^{\delta_0}_{p} \mathcal{F}^{\alpha_0}_{p} \mathcal{E}^{\beta_0}_{p} + \mathcal{E}^{\delta_0}_{p} \mathcal{F}^{\beta_0}_{p} \mathcal{S}^{\alpha_0}_{p} \right) - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right].
\]

(D85)

In order to further simplify this expression we need to exchange the indices \(\delta \leftrightarrow \beta\) via the identity \(S^{\delta}_{p} F^{\beta}_{p} \mathcal{E}^{\delta}_{p} = S^{\beta}_{p} F^{\beta}_{p} \mathcal{E}^{\delta}_{p} + \mathcal{E}^{\delta}_{p} \mathcal{F}^{\beta}_{p} S^{\alpha}_{p}\). The identity can be varified by the following steps: By definition Eq. (23) one obtains the derivative of the dispersion \(E^{\pm}_{p,\sigma} = \frac{\hbar^{\sigma}_{p} \pm 2h^{\sigma}_{p} h^{\delta}_{p}}{\Delta} (E^{+}_{p} - E^{-}_{p}).\) Thus, with definition Eq. (26) we have

\[
E^{+\sigma}_{p} - E^{-\sigma}_{p} = \frac{2h^{\sigma}_{p}}{\Delta} F^{\sigma}_{p}.
\]

(D87)

This immediately leads to

\[
F^{\delta}_{p} (E^{+\beta}_{p} - E^{-\beta}_{p}) = F^{\beta}_{p} (E^{+\delta}_{p} - E^{-\delta}_{p}),
\]

(D88)

which is equivalent to the identity, which we had to show.

After exchanging the indices \(\delta \leftrightarrow \beta\) the first part of Eq. (D85) containing \(S^{\delta}_{p} F^{\beta}_{p} \mathcal{E}^{\delta}_{p}\) then cancels the contribution \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(g,7)}\) in Eq. (D75) after reversing the matrix order under the trace. The second part of Eq. (D85) containing \(\mathcal{E}^{\delta}_{p} F^{\beta}_{p} S^{\alpha}_{p}\) is zero due to the corresponding one in \((\gamma \leftrightarrow \delta)\). We sum up:

\[
(K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,1)} + (K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,2)} + (K^{\alpha_0\beta_0\gamma_0\delta_0})^{(g,7)} = 0.
\]

(D89)

Let us continue with the four antisymmetrized contributions \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,5)}\) to \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,8)}\) in Eq. (D55) to Eq. (D58). Note that \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,6)}\) and \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,7)}\) as well as \((K^{\alpha_0\beta_0\gamma_0\delta_0})^{(v,5)}\) and
where \((K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,8)}\) are the antisymmetric counterparts of each other under the exchange of \(\alpha \leftrightarrow \beta\),

\[
(K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,6)} + (K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,7)} = -\frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,q_0+iq_0}^{\alpha} \mathcal{F}_p^{\delta} \mathcal{G}_{p,q_0}^{\gamma} \mathcal{S}_p \mathcal{F}_p^{\beta} \mathcal{E}_p^{\gamma} \mathcal{G}_{p,q_0}^{\delta} \mathcal{E}_p^{\beta} \right] - (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) ,
\]

\[
(K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,5)} + (K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,8)} = \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,q_0+iq_0}^{\alpha} \mathcal{F}_p^{\delta} \mathcal{G}_{p,q_0}^{\gamma} \mathcal{S}_p \mathcal{F}_p^{\beta} \mathcal{E}_p^{\gamma} \mathcal{G}_{p,q_0}^{\delta} \mathcal{E}_p^{\beta} \right] - (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) ,
\]

which can be seen by reversing the matrix order under the trace. We perform the trace over bands explicitly. Using that the obtained Green functions are equal under the Matsubara summation as shown in Eq. \(D53\), we can combine the four parts and get

\[
(K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,6)} + (K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,7)} + (K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,5)} + (K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,8)} = 0 .
\]

We can exchange the indices \(\delta \leftrightarrow \beta\) using Eq. \(D88\). Then, the expression cancels by its antisymmetric counterpart in \(\alpha \leftrightarrow \beta\). We sum up:

\[
(K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,6)} + (K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,7)} + (K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,5)} + (K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,8)} = 0 .
\]

We continue with the last two remaining contributions \((K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,3)}\) and \((K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,4)}\) in Eq. \(D49\) and Eq. \(D50\). We shift the Matsubara sum by \(-iq_0\) and change sign by using \((iq_0 \rightarrow -iq_0)\) of \((K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,4)}\) and get

\[
(K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,3)} = + \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,q_0+iq_0}^{\alpha} \mathcal{F}_p^{\delta} \mathcal{G}_{p,q_0}^{\gamma} \mathcal{S}_p \mathcal{F}_p^{\beta} - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right] ,
\]

\[
(K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,4)} = + \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,q_0}^{\alpha} \mathcal{F}_p^{\delta} \mathcal{S}_p \mathcal{G}_{p,q_0+iq_0}^{\gamma} \mathcal{F}_p^{\beta} - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right] .
\]

We can reintroduce a derivative with respect to \(p_\gamma\) using that \(\mathcal{G}_{p,q_0}^{\alpha} \mathcal{S}_p \mathcal{G}_{p,q_0+iq_0}^{\gamma} \mathcal{F}_p^{\beta} = \partial_\gamma \mathcal{G}_{p,q_0}^{\alpha} \mathcal{F}_p^{\beta}\). Thus,

\[
(K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,3)} = + \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,q_0+iq_0}^{\alpha} \mathcal{F}_p^{\delta} \mathcal{G}_{p,q_0}^{\gamma} \mathcal{S}_p \mathcal{F}_p^{\beta} (\partial_\gamma \mathcal{G}_{p,q_0}) \mathcal{F}_p^{\beta} - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right] ,
\]

\[
(K_{i;q_0}^{\alpha\beta\gamma\delta})^{(v,4)} = + \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,q_0}^{\alpha} \mathcal{F}_p^{\delta} \mathcal{S}_p (\partial_\gamma \mathcal{G}_{p,q_0+iq_0}) \mathcal{F}_p^{\beta} - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right] .
\]
We perform an integration by part in \((K_{i\gamma_0}^{\alpha\beta\gamma\delta})_{(v,3)}\) with respect to \(p_\gamma\). The derivative of the Green’s function \(\partial_\gamma G_{p,i\gamma_0+iq_0}\) is canceled by \((K_{i\gamma_0}^{\alpha\beta\gamma\delta})_{(v,4)}\) when reversing the matrix order under the trace. We get four parts

\[
(K_{i\gamma_0}^{\alpha\beta\gamma\delta})_{(v,3)} + (K_{i\gamma_0}^{\alpha\beta\gamma\delta})_{(v,4)} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,i\gamma_0+iq_0} \left( \partial_\gamma S_p \right) \mathcal{F}_p^\delta \mathcal{E}_p^\alpha G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\beta - (\gamma \leftrightarrow \delta) - (i\gamma_0 \to -i\gamma_0) \right]
\]

\[
-\frac{1}{2} \text{Tr}_p \left[ G_{p,i\gamma_0+iq_0} S_p \left( \partial_\gamma \mathcal{F}_p^\delta \right) \mathcal{E}_p^\alpha G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\beta - (\gamma \leftrightarrow \delta) - (i\gamma_0 \to -i\gamma_0) \right]
\]

\[
-\frac{1}{2} \text{Tr}_p \left[ G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\delta \left( \partial_\gamma \mathcal{E}_p^\alpha \right) G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\beta - (\gamma \leftrightarrow \delta) - (i\gamma_0 \to -i\gamma_0) \right]
\]

\[
-\frac{1}{2} \text{Tr}_p \left[ G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\delta \mathcal{E}_p^\alpha G_{p,i\gamma_0+iq_0} \left( \partial_\gamma \mathcal{F}_p^\beta \right) - (\gamma \leftrightarrow \delta) - (i\gamma_0 \to -i\gamma_0) \right] .
\]

We go through the four parts of \((D100)\): (1) The first line containing \(\partial_\gamma S_p\) cancels by the corresponding part in \((\gamma \leftrightarrow \delta)\) since \((\partial_\gamma S_p) \mathcal{F}_p^\delta = \partial_\gamma \left( \frac{1}{E_p^+ - E_p^-} \right) F_p^\delta \sigma^x\sigma^x = -\frac{1}{(E_p^+ - E_p^-)^2} \frac{2\gamma_p}{E_p^-} F_p^\delta F_p^\beta \sigma^x\sigma^x\), where \(\sigma^x\) and \(\sigma^z\) are the Pauli-matrices, which follows from Eq. \((D87)\). (2) In order to see the cancelation of the second line containing \(\partial_\gamma \mathcal{F}_p^\alpha\) we use \(\mathcal{F}_p^\delta = F_p^\delta \sigma^x\sigma^x = \frac{2\Delta_{\mathcal{F}_p}}{E_p^+ - E_p^-} h_p^\delta \sigma^x\sigma^x\) defined in Eq. \((26)\). The derivative of \(1/(E_p^+ - E_p^-)\) as well as of \(h_p^\delta\) cancels again by the corresponding part in \((\gamma \leftrightarrow \delta)\). (3) The derivative in the third line gives by definition \(\partial_\gamma \mathcal{E}_p^\alpha = \mathcal{E}_p^{\alpha\gamma}\). (4) For the fourth line we again use the definition of \(\mathcal{F}_p^\delta\) like for the second part. Whereas the derivative of \(1/(E_p^+ - E_p^-)\) cancels due to the corresponding part in \((\gamma \leftrightarrow \delta)\), the derivative of \(h_p^\delta\) now produces the off-diagonal matrix of the second order vertex \(\mathcal{H}_p^{\beta\gamma} = \frac{2\Delta_{\mathcal{F}_p}}{E_p^+ - E_p^-} h_p^\beta \sigma^x\sigma^x\) defined in Eq. \((29)\). Thus, the contributions finally reduce to

\[
(K_{i\gamma_0}^{\alpha\beta\gamma\delta})_{(v,3)} + (K_{i\gamma_0}^{\alpha\beta\gamma\delta})_{(v,4)} = -\frac{1}{2} \text{Tr}_p \left[ G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\delta \mathcal{E}_p^\alpha G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\beta - (\gamma \leftrightarrow \delta) - (i\gamma_0 \to -i\gamma_0) \right]
\]

\[
-\frac{1}{2} \text{Tr}_p \left[ G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\beta \mathcal{E}_p^\alpha G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\beta - (\gamma \leftrightarrow \delta) - (i\gamma_0 \to -i\gamma_0) \right] .
\]

We reinstall three Green functions by using the purely algebraic relation \(S_p G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\delta = G_{p,i\gamma_0+iq_0} \mathcal{F}_p^\delta G_{p,i\gamma_0+iq_0} - \mathcal{F}_p^\delta G_{p,i\gamma_0+iq_0} S_p\). The former part of this decomposition containing \(S_p\) cancels by the corresponding term in \((i\gamma_0 \leftrightarrow -i\gamma_0)\) when shifting the Matsubara summation and commuting the matrices. We also shift the Matsubara summation in the latter parts.
and change their sign by the \((iq_0 \leftrightarrow iq_0)\). Thus, we get

\[
+ \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,ip_0+iq_0} \mathcal{F}_p^{\beta} \mathcal{G}_{p,ip_0} \mathcal{F}_p^{\delta} \mathcal{G}_{p,ip_0} \mathcal{E}_p^{\alpha \gamma} - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right]
= - \left( (K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,V)} - (\gamma \leftrightarrow \delta) \right),
\]

(D103)

\[
+ \frac{1}{2} \text{Tr}_p \left[ \mathcal{G}_{p,ip_0+iq_0} \mathcal{E}_p^{\alpha} \mathcal{G}_{p,ip_0} \mathcal{F}_p^{\delta} \mathcal{G}_{p,ip_0} \mathcal{H}_p^{\beta \gamma} - (\gamma \leftrightarrow \delta) - (iq_0 \rightarrow -iq_0) \right]
= - \left( (K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,V)} - (\gamma \leftrightarrow \delta) \right),
\]

(D104)

where we indentified that the first one antisymmetrize the contribution \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,V)}\) in Eq. (D34) with respect to \(\alpha \leftrightarrow \beta\). The second one exactly cancels \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,V)}\) in Eq. (D33).

**Result** – We summarize all remaining contributions of \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(2)}\) in Eq. (D35) expressed in terms of the five contributions of \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(1)}\) in Eq. (D17):

\[
(K_{iq_0}^{\alpha \beta \gamma \delta})^{(2)} = -(K_{iq_0}^{\alpha \beta \delta \gamma})^{(1,I)} - (K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,III)} - (K_{iq_0}^{\alpha \beta \delta \gamma})^{(1,IV)} - (K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,V)}
\]

(D105)

\[
- (K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,II)}
\]

(D106)

\[
- \left[ (K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,IV)} - (K_{iq_0}^{\alpha \beta \delta \gamma})^{(1,IV)} \right]
\]

(D107)

\[
- \frac{1}{2} \left[ (K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,III)} - (K_{iq_0}^{\alpha \beta \delta \gamma})^{(1,III)} \right] - \frac{1}{2} \left[ (K_{iq_0}^{\beta \alpha \gamma \delta})^{(1,III)} - (K_{iq_0}^{\beta \alpha \delta \gamma})^{(1,III)} \right]
\]

(D108)

\[
- \left[ (K_{iq_0}^{\alpha \beta \gamma \delta})^{(1,V)} - (K_{iq_0}^{\alpha \beta \delta \gamma})^{(1,V)} \right].
\]

(D109)

The contributions of \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(2)}\) (i) antisymmetrizes \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(1)}\) with respect to \(\gamma \leftrightarrow \delta\) [D105], (ii) cancels the contribution II [D106] and IV [D107] and (iii) antisymmetrizes contribution III [D108] and V [D109] with respect to \(\alpha \leftrightarrow \beta\).

c. **Combining \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(1)}\) and \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(2)}\)**

We started with Eq. (38). We had split \(K_{iq_0}^{\alpha \beta \gamma \delta} = \frac{\partial}{\partial q_5} K^{\alpha \beta \gamma \delta}_{q,iq_0} \bigg|_{q=0}\) into the two parts \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(1)}\) and \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(2)}\). In a first step, we have identified five contributions [D30]–[D34] in decomposing \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(1)}\) and were able to express the result of decomposing \((K_{iq_0}^{\alpha \beta \gamma \delta})^{(2)}\)
with these contributions \( \text{(D105)} - \text{(D109)} \). Thus, we finish with the following final result
\[
K_{i|q_0}^{\alpha \beta \gamma \delta} = (K_{i|q_0}^{\alpha \beta \gamma \delta})^{(1)} + (K_{i|q_0}^{\alpha \beta \gamma \delta})^{(2)}
\]
\[
= - \frac{1}{4} \text{Tr}_p \left[ (G_{p, i|p_0 + i|q_0} - G_{p, i|p_0 - i|q_0}) E_p^\alpha G_{p, i|p_0} E_p^\beta G_{p, i|p_0} E_p^\gamma G_{p, i|p_0} E_p^\delta \right]
\]
\[
= - \frac{1}{4} \text{Tr}_p \left[ (G_{p, i|p_0 + i|q_0} - G_{p, i|p_0 - i|q_0}) F_p^\alpha G_{p, i|p_0} F_p^\delta G_{p, i|p_0} F_p^\gamma G_{p, i|p_0} F_p^\delta \right]
\]
\[
= - \frac{1}{2} \text{Tr}_p \left[ (G_{p, i|p_0 + i|q_0} - G_{p, i|p_0 - i|q_0}) F_p^\alpha G_{p, i|p_0} F_p^\delta G_{p, i|p_0} F_p^\gamma G_{p, i|p_0} F_p^\delta \right]
\]
\[
- \left( \alpha \leftrightarrow \beta \right) - \left( \gamma \leftrightarrow \delta \right),
\]
where we antisymmetrized the \( (K_{i|q_0}^{\alpha \beta \gamma \delta})^{(1,1)} \) in Eq. \( \text{(D30)} \) with respect to \( \alpha \leftrightarrow \beta \) using that all matrices are diagonal and that the other derivatives cancel by the corresponding part in \( (\gamma \leftrightarrow \delta) \) after partial integration in \( p_\gamma \). This final result is given in Eq. \( \text{(40)} \).

3. Matsubara sum and analytic continuation

All contributions in Eq. \( \text{(40)} \) contain a Matsubara sum of the form
\[
K_{i|q_0}^H = T \sum_{p_0} \text{tr} \left[ (G_{i|p_0 + i|q_0} - G_{i|p_0 - i|q_0}) M_1 G_{i|p_0} M_2 G_{i|p_0} M_3 \right],
\]
with arbitrary frequency independent matrices \( M_i \). Momentum dependences are not written here. We apply the general formula Eq. \( \text{(B7)} \) in Appendix \( \text{B} \) for the case \( m = 1, n = 2 \) and \( m = 2, n = 1 \), after shifting the Matsubara summation by \( q_0 \). Using the relation \( \mathcal{A}(\epsilon) \mathcal{M}_2 \mathcal{P}(\epsilon) + \mathcal{P}(\epsilon) \mathcal{M}_2 \mathcal{A}(\epsilon) = - \frac{1}{2\pi i} (G_{\epsilon}^R M_2 G_{\epsilon}^R - G_{\epsilon}^A M_2 G_{\epsilon}^A) \), we get
\[
K_{i|q_0 \rightarrow \omega + i|0+}^H = \frac{1}{2\pi i} \int d\epsilon f(\epsilon) \text{tr} \left[ - (G_{\epsilon}^R - G_{\epsilon}^A) \mathcal{M}_1 (G_{\epsilon - \omega}^A M_2 G_{\epsilon - \omega}^A - G_{\epsilon - \omega}^A M_2 G_{\epsilon - \omega}^A) \mathcal{M}_3 \right.
\]
\[
+ (G_{\epsilon}^R - G_{\epsilon}^A) \mathcal{M}_1 (G_{\epsilon + \omega}^R M_2 G_{\epsilon + \omega}^R - G_{\epsilon + \omega}^R M_2 G_{\epsilon + \omega}^R) \mathcal{M}_3
\]
\[
- (G_{\epsilon}^R - G_{\epsilon}^A) \mathcal{M}_1 (G_{\epsilon}^R M_2 G_{\epsilon}^R - G_{\epsilon}^A M_2 G_{\epsilon}^A) \mathcal{M}_3
\]
\[
+ (G_{\epsilon}^A - G_{\epsilon}^A) \mathcal{M}_1 (G_{\epsilon}^R M_2 G_{\epsilon}^A - G_{\epsilon}^A M_2 G_{\epsilon}^A) \mathcal{M}_3] .
\]
For the DC Hall conductivity, we need to compute the ratio \( K_{i|q_0 \rightarrow \omega + i|0+}^H / \omega \) in the limit \( \omega \rightarrow 0 \). We use \( (G_{\epsilon - \omega}^A - G_{\epsilon}^A) / \omega \rightarrow - \partial_{\epsilon} G_{\epsilon}^A, (G_{\epsilon + \omega}^R - G_{\epsilon}^R) / \omega \rightarrow \partial_{\epsilon} G_{\epsilon}^R, (G_{\epsilon - \omega}^A M_2 G_{\epsilon - \omega}^A - G_{\epsilon}^A M_2 G_{\epsilon}^A) / \omega \rightarrow - \partial_{\epsilon} (G_{\epsilon}^A M_2 G_{\epsilon}^A) \) and \( (G_{\epsilon + \omega}^R M_2 G_{\epsilon + \omega}^R - G_{\epsilon}^R M_2 G_{\epsilon}^R) / \omega \rightarrow \partial_{\epsilon} (G_{\epsilon}^R M_2 G_{\epsilon}^R) \) for \( \omega \rightarrow 0 \). Writing real
and imaginary parts explicitly, one obtains Eq. (45).

1. P. Voruganti, A. Golubentsev, and S. John, Conductivity and Hall effect in the two-dimensional Hubbard model, Phys. Rev. B 45, 13945 (1992).
2. D. M. Broun, What lies beneath the dome?, Nat. Phys. 4, 170 (2008).
3. S. Badoux, W. Tabis, F. Laliberté, G. Grissonnanche, B. Vignolle, D. Vignolles, J. Béard, D. A. Bonn, W. N. Hardy, R. Liang, N. Doiron-Leyraud, L. Taillefer, and C. Proust, Change of carrier density at the pseudogap critical point of a cuprate superconductor, Nature (London) 531, 210 (2016).
4. C. Collignon, S. Badoux, S. A. A. Afshar, B. Michon, F. Laliberté, O. Cyr-Choinière, J.-S. Zhou, S. Licciardello, S. Wiedmann, N. Doiron-Leyraud, and L. Taillefer, Fermi-surface transformation across the pseudogap critical point of the cuprate superconductor La_{1.6-x}Nd_{0.4}Sr_xCuO_4, Phys. Rev. B 95, 224517 (2017).
5. J. G. Storey, Hall effect and Fermi surface reconstruction via electron pockets in the high \( T_c \) cuprates, Europhys. Lett. 113, 27003 (2016).
6. K.-Y. Yang, T. M. Rice, and F.-C. Zhang, Phenomenological theory of the pseudogap state, Phys. Rev. B 73, 174501 (2006).
7. Y. Qi and S. Sachdev, Effective theory of Fermi pockets in fluctuating antiferromagnets, Phys. Rev. B 81, 115129 (2010).
8. S. Chatterjee and S. Sachdev, Fractionalized Fermi liquid with bosonic chargons as a candidate for the pseudogap metal, Phys. Rev. B 94, 205117 (2016).
9. S. Caprara, C. Di Castro, G. Seibold, and M. Grilli, Dynamical charge density waves rule the phase diagram of cuprates, Phys. Rev. B 95, 224511 (2017).
10. D. Haug, V. Hinkov, A. Suchaneck, D. S. Inosov, N. B. Christensen, C. Niedermayer, P. Bourges, Y. Sidis, J. T. Park, A. Ivanov, C. T. Lin, J. Mesot, and B. Keimer, Magnetic-Field-Enhanced Incommensurate Magnetic Order in the Underdoped High-Temperature Superconductor YBa_2Cu_3O_{6.45}, Phys. Rev. Lett. 103, 017001 (2009).
11. D. Haug, V. Hinkov, Y. Sidis, N. B. Christensen, A. Ivanov, T. Keller, C. T. Lin, and B. Keimer, Neutron scattering study of the magnetic phase diagram of underdoped YBa_2Cu_3O_{6+x}, New J. Phys. 12, 105006 (2010).
12 H. J. Schulz, Incommensurate Antiferromagnetism in the Two-Dimensional Hubbard-Model, Phys. Rev. Lett. **64**, 1445 (1990).

13 M. Kato, K. Machida, H. Nakanishi, and M. Fujita, Soliton lattice modulation of incommensurate spin density wave in two-dimensional Hubbard model – a mean-field study, J. Phys. Soc. Jpn. **59**, 1047 (1990).

14 R. Fresard, M. Dzierzawa, and P. Wölfle, Slave-Boson Approach to Spiral Magnetic Order in the Hubbard Model, Europhys. Lett. **15**, 325 (1991).

15 M. Raczkowski, R. Frésard, and A. M. Oleś, Interplay between incommensurate phases in the cuprates, Europhys. Lett. **76**, 128 (2006).

16 P. A. Igoshev, M. A. Timirgazin, A. A. Katanin, A. K. Arzhnikov, and V. Yu. Irkhin, Incommensurate magnetic order and phase separation in the two-dimensional Hubbard model with nearest- and next-nearest-neighbor hopping, Phys. Rev. B **81**, 094407 (2010).

17 A. V. Chubukov and D. M. Frenkel, Renormalized perturbation theory of magnetic instabilities in the two-dimensional Hubbard model at small doping, Phys. Rev. B **46**, 11884 (1992).

18 A. V. Chubukov and K. A. Musaelian, Magnetic phases of the two-dimensional Hubbard model at low doping, Phys. Rev. B **51**, 12605 (1995).

19 W. Metzner, M. Salmhofer, C. Honerkamp, V. Meden, and K. Schönhammer, Functional renormalization group approach to correlated fermion systems, Rev. Mod. Phys. **84**, 299 (2012).

20 H. Yamase, A. Eberlein, and W. Metzner, Coexistence of Incommensurate Magnetism and Superconductivity in the Two-Dimensional Hubbard Model, Phys. Rev. Lett. **116**, 096402 (2016).

21 B.-X. Zheng, C.-M. Chung, P. Corboz, G. Ehlers, M.-P. Qin, R. M. Noack, H. Shi, S. R. White, S. Zhang, and G. K.-L. Chan, Stripe order in the underdoped region of the two-dimensional Hubbard model, Science **358**, 1155 (2017).

22 D. Vilardi, C. Taranto, and W. Metzner, Dynamically enhanced magnetic incommensurability: Effects of local dynamics on non-local spin-correlations in a strongly correlated metal, Phys. Rev. B **97**, 235110 (2018).

23 B. I. Shraiman and E. D. Siggia, Spiral Phase of a Doped Quantum Antiferromagnet, Phys. Rev. Lett. **62**, 1564 (1989).

24 V. N. Kotov and O. P. Sushkov, Stability of the spiral phase in the two-dimensional extended $t$-$J$ model, Phys. Rev. B **70**, 195105 (2004).

25 J. Tranquada, B. Sternlieb, J. Axe, Y. Nakamura, and S. Uchida, Evidence for stripe correlations
of spins and holes in copper oxide superconductors, Nature \textbf{375}, 561 (1995).

26 M. Charlebois, S. Verret, A. Foley, O. Simard, D. Sénéchal, and A.-M. S. Tremblay, Hall effect in cuprates wit an incommensurate collinear spin-density wave, Phys. Rev. B \textbf{96}, 205132 (2017).

27 A. Eberlein, W. Metzner, S. Sachdev, and H. Yamase, Fermi Surface Reconstruction and Drop in the Hall number due to Spiral Antiferromagnetism in High-$T_c$ Cuprates, Phys. Rev. Lett. \textbf{117}, 187001 (2016).

28 S. Chatterjee, S. Sachdev, and A. Eberlein, Thermal and electrical transport in metals and superconductors across antiferromagnetic and topological quantum transitions, Phys. Rev. B \textbf{96}, 075103 (2017).

29 K. G. Wilson, Confinement of quarks, Phys. Rev. D \textbf{10}, 2445 (1974).

30 G. Rickayzen, \textit{Green's Functions and Condensed Matter} (Academic Press, London, 1980).

31 The quasi-particle spectral function must not be confused with the spectral function of the electrons appearing in the spectral representation of the Green function defined with bare (unrotated) electron operators.

32 D. N. Aristov and R. Zeyher, Optical conductivity of unconventional charge-density-wave systems: Role of vertex corrections, Phys. Rev. B \textbf{72}, 115118 (2005).

33 O. K. Andersen, A. I. Liechtenstein, O. Jepsen, and F. Paulsen, LDA energy bands, low-energy Hamiltonians, $t', t'', t_{\perp}(k)$, and $J_{\perp}$, J. Phys. Chem. Solids, \textbf{56}, 1573 (1995).

34 E. Pavarini, I. Dasgupta, T. Saha-Dasgupta, O. Jepsen, and O. K. Andersen, Band-Structure Trend in Hole-Doped Cuprates and Correlation with $T_{c,\max}$, Phys. Rev. Lett. \textbf{87}, 047003 (2001).

35 O. P. Sushkov and V. N. Kotov, Superconducting spiral phase in the two-dimensional $t$-$J$ model, Phys. Rev. B \textbf{70}, 024503 (2004).

36 K. Yamada, C. H. Lee, K. Kurahashi, J. Wada, S. Wakimoto, S. Ueki, H. Kimura, Y. Endoh, S. Hosoya, G. Shirane, R. J. Birgeneau, M. Greven, M. A. Kastner, and Y. J. Kim, Doping dependence of the spatially modulated dynamical spin correlations and the superconducting transition temperature in La$_{2-x}$Sr$_x$CuO$_4$, Phys. Rev. B \textbf{57}, 6165 (1998).

37 Y. Ando, K. Segawa, S. Komiya, and A. N. Lavrov, Electric Resistivity Anisotropy from Self-Organized One Dimensionality in High-Temperature Superconductors, Phys. Rev. Lett. \textbf{88}, 137005 (2002).

38 See, for example, T. D. Stanescu, V. Galitski, and H. D. Drew, Effective Masses in a Strongly Anisotropic Fermi Liquid, Phys. Rev. Lett. \textbf{101}, 066405 (2008), and references therein.
39 L. Taillefer, private communication.

40 N. Ashcroft and N Mermin, *Solid State Physics* (Saunders College, Philadelphia, 1976).

41 S. Verret, O. Simard, M. Charlebois, D. Sénéchal, and A.-M. S. Tremblay, Phenomenological theories of the low-temperature pseudogap: Hall number, specific heat, and Seebeck coefficient, Phys. Rev. B 96, 125139 (2017).

42 For hermitian matrices $M_1$ to $M_m$ one can easily prove the following equality under the trace:
$$\text{tr}(M_1 \ldots M_m) = \text{tr}(M_m \ldots M_1).$$
All 2x2-matrices we consider (e.g. $\varepsilon_p$, $\mathbf{F}_p$, ...) in this paper are hermitian. In the text, we refer to this kind of manipulation as “reverse the matrix order under the trace”.