Periodic solutions with prescribed minimal period of vortex type problems in domains

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Abstract
We consider Hamiltonian systems with two degrees of freedom of point vortex type
\[ \kappa_j \dot{z}_j = J \nabla_{\dot{z}_j} H_{\Omega}(z_1, z_2), \quad j = 1, 2, \]
for \( z_1, z_2 \) in a domain \( \Omega \subset \mathbb{R}^2 \). In the classical point vortex context the Hamiltonian \( H_{\Omega} \) is of the form
\[ H_{\Omega}(z_1, z_2) = -\frac{\kappa_1 \kappa_2}{\pi} \log |z_1 - z_2| - 2\kappa_1 \kappa_2 g(z_1, z_2) - \kappa_1^2 h(z_1) - \kappa_2^2 h(z_2), \]
where \( g : \Omega \times \Omega \to \mathbb{R} \) is the regular part of a hydrodynamic Green function in \( \Omega \), \( h : \Omega \to \mathbb{R} \) is the Robin function: \( h(z) = g(z, z) \), and \( \kappa_1, \kappa_2 \) are the vortex strengths. We prove the existence of infinitely many periodic solutions with prescribed minimal period that are superpositions of a slow motion of the center of vorticity close to a star-shaped level line of \( h \) and of a fast rotation of the two vortices around their center of vorticity. The proofs are based on a recent higher dimensional version of the Poincaré–Birkhoff theorem due to Fonda and Ureñá.

Keywords: vortex dynamics, singular first order Hamiltonian systems, periodic solutions, higher dimensional Poincaré–Birkhoff theorem

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(Some figures may appear in colour only in the online journal)
1. Introduction

Given a domain \( \Omega \subset \mathbb{R}^2 \), the dynamics of \( N \) point vortices \( z_1(t), \ldots, z_N(t) \in \Omega \) with vortex strengths \( \kappa_1, \ldots, \kappa_N \in \mathbb{R} \) is described by a Hamiltonian system

\[
\kappa_j \dot{z}_j = J \nabla_z H_\Omega(z_1, \ldots, z_N), \quad j = 1, \ldots, N;
\]

(1.1)

here \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) is the standard symplectic matrix in \( \mathbb{R}^2 \). The Hamiltonian is of the form

\[
H_\Omega(z_1, \ldots, z_N) = -\frac{1}{2\pi} \sum_{j=1}^{N} \sum_{j \neq k \neq j} \kappa_j \kappa_k \log |z_j - z_k| - F(z_1, \ldots, z_N)
\]

where \( F : \Omega^N \to \mathbb{R} \) is a function of class \( C^2 \). The Hamiltonian is defined on the configuration space

\[
\mathcal{F}_N \Omega = \{(z_1, \ldots, z_N) \in \Omega^N : z_j \neq z_k \text{ for } j \neq k \}.
\]

Observe that the system is singular, but of a very different type compared with the singular second order equations from celestial mechanics.

Systems like (1.1) arise as a singular limit problem in fluid mechanics. A model for an incompressible, non-viscous fluid in \( \Omega \) with solid boundary is given by the two dimensional Euler equations

\[
\begin{align*}
\dot{v}_t + (v \cdot \nabla)v &= -\nabla P, & \nabla \cdot v &= 0 & \text{in } \Omega, \\
\nu \cdot \nabla &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

(1.2)

in which \( v(t, x) \in \mathbb{R}^2 \) represents the velocity of the fluid and \( P(t, x) \in \mathbb{R} \) its pressure; \( \nu \) denotes the exterior normal to the domain. Making a point vortex ansatz \( \omega = \sum_{j=1}^{N} \kappa_j \delta_{z_j} \), where \( \delta_z \) is the Dirac delta, for the scalar vorticity \( \omega = \nabla \times v = \partial_1 v_2 - \partial_2 v_1 \), one is led to system (1.1); see [28].

Classically the point vortex equations (1.1) were first derived by Kirchhoff in [23], who considered the case where \( \Omega = \mathbb{R}^2 \) is the whole plane. In this case the function \( F \) in the Hamiltonian is identically zero. On the other hand, when \( \Omega \neq \mathbb{R}^2 \), one has to take account of the boundaries of the domain which leads to

\[
F(z_1, \ldots, z_N) = \sum_{j,k=1}^{N} \kappa_j \kappa_k g(z_j, z_k)
\]

where \( g : \Omega \times \Omega \to \mathbb{R} \) is the regular part of a hydrodynamic Green function in \( \Omega \). An important role plays the Robin function \( h : \Omega \to \mathbb{R} \) defined by \( h(z) = g(z, z) \). In fact, a single vortex \( z(t) \in \Omega \) moves along level lines of \( h \) according to the Hamiltonian system \( \dot{z} = -\kappa_j \nabla h(z) \). This goes back to work of Routh [32] and Lin [25, 26]. The Green function, hence the Hamiltonian \( H_\Omega \) is explicitly known only for a few special domains. Moreover \( h_\Omega \) is not bounded from above nor from below, and its level sets are not compact, except when \( N = 2 \) and the vortex strengths have different signs. Finally the system (1.1) is not integrable in general; see [30, section 3.4] and [37]. We refer the reader to [27, 28, 30, 33] for modern presentations of the point vortex method.

It is worthwhile to mention that systems like (1.1) also arise in other contexts from mathematical physics, e.g. in models from superconductivity (Ginzburg–Landau–Schrödinger equation), or in equations modeling the dynamics of a magnetic vortex system in a thin
ferromagnetic film (Landau–Lifshitz–Gilbert equation); see [7] for references to the literature. The domain can also be a subset of a two-dimensional surface.

Many authors worked on this problem, mostly in the case $\Omega = \mathbb{R}^2$ with $F = 0$. In particular, there is a large amount of work on relative equilibria; see the survey [1] by Aref et al. for examples and references up to 2003, the more recent paper [31] by Roberts on the stability of relative equilibria, or the paper [20] by Hampton, Roberts and Santoprete on various types of relative equilibria in the four-vortex problem, and the references therein. In the presence of boundaries much less is known, except in the case of special domains like the half plane or a radially symmetric domain, i.e. disk or annulus, when the Green function is explicitly known. In the case of two vortices and $k_1k_2 < 0$ the Hamiltonian is bounded below and satisfies $H_\Omega(z_1, z_2) \to \infty$ as $z = (z_1, z_2) \to \partial \Omega$. Consequently all level surfaces of $H_\Omega$ are compact, and standard results about Hamiltonian systems apply. In particular, by a result of Struwe [35] almost every level surface contains periodic solutions. Another simple setting is the case of $\Omega$ being radially symmetric and $N = 2$ whence the system (1.1) is integrable and can be analyzed in detail. For $\Omega$ being a disk this has been done in [16].

Except in the above mentioned special cases even the existence of equilibrium solutions of (1.1) is difficult to prove; see [9, 10]. The problem of finding periodic solutions in a general domain has only recently been addressed in the papers [4–7] where several one parameter families of periodic solutions of the general $N$-vortex problem (1.1) have been found. The solutions found in [4, 5, 7] rotate around their center of vorticity, which is situated near a stable critical point of the Robin function $h$. The periods tend to zero as the solutions approach the critical point of $h$. Recall that $h(z) \to \infty$ as $z \to \partial \Omega$, hence $h$ always has a minimum. It may have arbitrarily many critical points. For a generic domain all critical points are non-degenerate (see [8]), hence in this case the results from [4, 5, 7] produce many one-parameter families of periodic solutions. Moreover, these solutions lie on global continua that are obtained via an equivariant degree theory for gradient maps. A different type of periodic solutions has been found in [6]. There the solutions are choreographies where the vortices move near a compact component of the boundary $\partial \Omega$ almost following a level line $h^{-1}(c)$ with $c \gg 1$.

In the present paper we consider (1.1) for $N = 2$ vortices in a domain $\Omega \subset \mathbb{R}^2$. We find a new type of solutions that are not (necessarily) located near an equilibrium of $h$ but where the two vortices are close to a level line of $h$. The period of our solutions will be prescribed. More precisely, the solutions that we obtain are essentially superpositions of a slow motion of the center of vorticity along some level line $h^{-1}(c)$ of $h$, and of a fast rotation of the two vortices around their center of vorticity. This will be described in detail; see figure 1 for an illustration when $\Omega$ is a disk. The minimal period of our solution is the period of the center of vorticity. These solutions are of a very different nature from those obtained in [4–7]. The main geometric assumption is that $h^{-1}(c)$ is strictly star-shaped. Our proofs are based on a recent generalization of the Poincaré–Birkhoff theorem due to Fonda–Ureña [19].

A very interesting and challenging problem consists in regularizing these solutions for the Euler equations, that is to obtain regular periodic solutions $v_\epsilon(x, t)$ of (1.2) where the vorticity is concentrated near points $z_{j, \epsilon}(t), \ j = 1, \ldots, N$, and where in the limit $z_{j, \epsilon}(t) \to z_j(t)$ as $\epsilon \to 0$ the vortices move according to one of the periodic solutions of (1.1) found here or in [4–7]. This has been achieved so far only for stationary solutions; see [9, 13, 34]. When $\Omega$ is a disk, then certain uniformly rotating periodic solutions of (1.1) have been regularized to periodic solutions of the Ginzburg–Landau–Schrödinger equation in [36]. This result does not apply to the solutions found in the present paper.

The paper is organized as follows. In section 2 we state and discuss our results about the existence and shape of periodic solutions of (1.1). In section 3 we prove the main theorem 2.2.
about the existence of a periodic solution by an application of [19, theorem 1.2]. This requires the computation of certain rotation numbers which will be done in section 4. The results about the shape of our solutions will be proved in section 5. In the last section 6 we prove various corollaries of theorem 2.2 and its proof.

2. Statement of results

We consider the Hamiltonian system

$$\kappa_j \frac{d}{dt} z_j = J \nabla_z H_\Omega(z_1, z_2), \quad j = 1, 2, \quad \kappa_j \in \mathbb{R} \backslash \{0\}$$

on a domain $\Omega \subset \mathbb{R}^2$ with Hamilton function

$$H_\Omega(z_1, z_2) = -\frac{\kappa_1 \kappa_2}{\pi} \log |z_1 - z_2| - 2\kappa_1 \kappa_2 g(z_1, z_2) - \kappa_1^2 h(z_1) - \kappa_2^2 h(z_2)$$

where $g : \Omega \times \Omega \to \mathbb{R}$ can be any symmetric $C^2$ function, and $h : \Omega \to \mathbb{R}$ is defined by $h(z) = g(z, z)$. The parameters $\kappa_1, \kappa_2 \in \mathbb{R} \backslash \{0\}$ have to satisfy $\kappa_1 + \kappa_2 \neq 0$. We will continue to refer to $z_1, z_2$ as point vortices, even though our results are valid in a more general setting.

Let $C_c \subset h^{-1}(c)$ be a non-constant periodic trajectory of the one degree of freedom Hamiltonian system

$$\dot{z} = -(\kappa_1 + \kappa_2) J \nabla h(z)$$

on the level $c \in \mathbb{R}$. Then $\nabla h(z) \neq 0$ for every $z \in C_c$, hence there exists a neighbourhood $U(C_c) \subset \Omega$ of $C_c$ and $c_0 < c < d_0$ so that

$$C_d := \{z \in U(C_c) : h(z) = d\}, \quad c_0 \leq d \leq d_0.$$
is also the trajectory of a non-constant periodic solution of (2.2). Let \( T(d) > 0 \) be the minimal period of \( \mathcal{C}_d \). Observe that system (2.2) describes the motion of one vortex in \( \Omega \) with strength \( \kappa = \kappa_1 + \kappa_2 \).

We need one geometric assumption on \( h \). A periodic trajectory \( \mathcal{C} \), or any closed \( C^1 \) curve \( \mathcal{C} \subset \mathbb{R}^2 \), is said to be strictly star-shaped if there exists \( z_0 \in \mathbb{R}^2 \) such that for each \( w \in \mathbb{R}^1 \subset \mathbb{R}^2 \) the ray \( z_0 + \mathbb{R}^+ w = \{ z_0 + tw : t \geq 0 \} \) intersects \( \mathcal{C} \) in precisely one point, and the intersection is transversal.

**Assumption 2.1.** The periodic trajectories \( \mathcal{C}_d = h^{-1}(d) \cap \mathcal{U}(\mathcal{C}_c) \), \( c_0 \leq d \leq d_0 \), of (2.2) are strictly star-shaped. The map \( T : [c_0, d_0] \to \mathbb{R}, d \mapsto T(d) \), is strictly monotone.

Clearly, if \( \mathcal{C}_c \) is strictly star-shaped then so is \( \mathcal{C}_d \) for \( d \) close to \( c \). Observe that we do not require that \( \mathcal{C}_c \) is the boundary of a strictly star-shaped set in \( \Omega \). Below we shall provide several examples of domains where assumption 2.1 holds with \( h \) being the Robin function. In order to state our result recall the action integral for a \( T(c) \)-periodic function:

\[
\mathcal{A}(z) = \frac{1}{2} \sum_{j=1}^{2} \int_0^{T(c)} \kappa_j (\dot{z}_j(t), Jz_j(t)) \, dt - \int_0^{T(c)} H_\Omega(z(t)) \, dt.
\]

The main result of the paper is the following.

**Theorem 2.2.** Suppose \( \kappa_1, \kappa_2, \kappa_1 + \kappa_2 \neq 0 \) and that assumption 2.1 holds. Then the system (2.1) has a sequence of periodic solutions \( z^{(n)}(t) \) with minimal period \( T(c) \). These satisfy the following properties.

(a) The center of vorticity \( \mathcal{C}^{(n)}(t) := \frac{\kappa_1}{\kappa_1 + \kappa_2} z_1^{(n)}(t) + \frac{\kappa_2}{\kappa_1 + \kappa_2} z_2^{(n)}(t) \) converges uniformly in \( t \) as \( n \to \infty \) towards a solution \( \mathcal{C}(t) \) of (2.2) with \( \mathcal{C}(t) \in \mathcal{C}_c \).

(b) \( \| z_1^{(n)} - z_2^{(n)} \|_\infty \to 0 \) as \( n \to \infty \), hence \( z_1^{(n)}(t), z_2^{(n)}(t) \to \mathcal{C}(t) \) uniformly in \( t \in [0, T(c)] \).

(c) Consider the difference \( \mathcal{D}^{(n)}(t) := z_1^{(n)}(t) - z_2^{(n)}(t) = \rho^{(n)}(t) (\cos \theta^{(n)}(t), \sin \theta^{(n)}(t)) \) in polar coordinates and set \( d_n = \| z_1^{(n)}(0) - z_2^{(n)}(0) \| \). Then the angular velocity \( \dot{\theta}^{(n)} \) satisfies

\[
d_n^2 \dot{\theta}^{(n)}(t) = \frac{\kappa_1 \kappa_2}{\pi} + o(1) \quad \text{as} \quad n \to \infty \quad \text{uniformly in} \ t.
\]

(d) The action of the solution satisfies \( \mathcal{A}(z^{(n)}) \to -\sigma \infty \) as \( n \to \infty \), where \( \sigma = \text{sgn}(\kappa_1 \kappa_2) \) is the sign of \( \kappa_1 \kappa_2 \).

**Remark 2.3.**

(a) This result can be interpreted as follows, using the notation of theorem 2.2. The solutions

\[
z_1^{(n)}(t) = C^{(n)}(t) + \frac{\kappa_2}{\kappa_1 + \kappa_2} \mathcal{D}^{(n)}(t) \quad \text{and} \quad z_2^{(n)}(t) = C^{(n)}(t) - \frac{\kappa_1}{\kappa_1 + \kappa_2} \mathcal{D}^{(n)}(t)
\]

are superpositions of a slow motion of the center of vorticity with minimal period \( T(c) \), and of a fast rotation of the two vortices around their center of vorticity. The trajectory of the center of vorticity converges (as \( n \to \infty \)) towards the level line \( \mathcal{C}_c \) of \( h \). The angular velocity of the two vortices around their center of vorticity is asymptotic to \( \frac{\kappa_1 \kappa_2}{\pi} \) as \( d_n \to 0 \) where \( d_n \) is the distance of the initial positions of the two vortices. The rotation number of \( z_1^{(n)}(t) - z_2^{(n)}(t) \) in \([0, T(c)]\) is asymptotic to \( \frac{\kappa_1 \kappa_2 T}{2\pi d_n^2} \) and tends to infinity as \( d_n \to 0 \).
(b) In the case $\kappa_1\kappa_2 < 0$ the center of vorticity does not lie between the two vortices. If $\kappa_1 + \kappa_2$ is close to 0 then the two vortices are relatively far away from their center of vorticity, compared with their distance from each other.

(c) Clearly the theorem holds for any $\epsilon \in (\epsilon_0, \epsilon_d)$ instead of $\epsilon$.

(d) If $\kappa_1\kappa_2 < 0$ then $H_\Omega(z) \to \infty$ as $z \to \partial F(\Omega)$, hence the level surfaces $H_\Omega^{-1}(c)$ are compact. Therefore a result of Struwe [35, theorem 1.1] can be applied and yields that for almost every $c > \inf H_\Omega$ there exists a periodic solution of (2.1) on $H_\Omega^{-1}(c)$. Even in that case theorem 2.2 is new in that we localize the solutions and describe their shape.

(e) It is an interesting problem whether it is possible to weaken or to drop the condition that $C_\epsilon$ is strictly star-shaped. We refer the reader to [17, 24, 29] for results and discussions of this delicate issue in the setting of the Poincaré–Birkhoff fixed point theorem for non-autonomous one degree of freedom Hamiltonian systems. Although star-shapedness is essential for the multidimensional Poincaré–Birkhoff fixed point theorem [19, theorem 1.2] we believe that it is not essential in our special case; see also [18].

(f) It is also an interesting problem to consider more than two vortices. One might conjecture that, given a periodic solution $Z_j(t) = e^{-\omega_j t} z_j$, $j = 1, \ldots, N$, $\omega \in \mathbb{R}$, $z_1, \ldots, z_N \in \mathbb{R}^2$, of the Hamiltonian system

$$J \dot{z}_j = -\frac{1}{2\pi} \sum_{k=1, k \neq j}^N \kappa_k \frac{J(z_j - z_k)}{|z_j - z_k|^2}, \quad j = 1, \ldots, N,$$

in the plane, there exist solutions $z_j(t) \in \Omega$, $j = 1, \ldots, N$, of the shape

$$z_j(t) = C(t) + rZ_j(t/r^2) + o(r) \quad \text{as} \quad r \to 0, \quad j = 1, \ldots, N.$$

Here $C(t)$ is a periodic solution of the Hamiltonian system $\dot{C} = -\kappa J\nabla h(C)$, where $\kappa = \sum_{j=1}^N \kappa_j$ is the total vorticity. Such a result has been proved in [4, 5, 7] in the case when $Z(t) \equiv a_0 \in \Omega$ is an equilibrium, i.e. when $a_0 \in \Omega$ is a critical point of the Robin function $h$. The methods from these papers do not seem to be applicable, however, when $C(t)$ has minimal period $T > 0$. Since the minimal period of $Z(t)$ is $2\pi/\omega$, the superposition $C(t) + rZ_j(t/r^2)$ is periodic or quasiperiodic depending on whether or not $2\pi r^2/\omega T$ is rational.

(g) As mentioned in the introduction it would be extremely interesting to regularize our solutions to obtain regular solutions of the Euler equations. One approach could be via vortex patch solutions; see [14, 15, 21, 22] and the references cited therein. Certain uniformly rotating periodic solutions of (1.1) in the disk have been regularized in [36] to periodic solutions $u_\epsilon(x, t) \in \mathcal{C}$ of the Ginzburg–Landau–Schrödinger equation

$$i u_t(x, t) = \Delta u(x, t) + \frac{1}{\epsilon^2} u(x, t) \left(1 - |u(x, t)|^2\right). \quad (2.3)$$

However this result does not apply to the solutions from theorem 2.2 because these are not uniformly rotating. The question of regularizing our solutions to periodic solutions of (1.2) or (2.3) is open so far, even in the case of the disk.

It is easy to construct functions $g$ on an arbitrary domain $\Omega \subset \mathbb{R}^2$ so that the assumptions of theorem 2.2 hold. We shall now present several examples where these assumptions can be verified for $g$ being the regular part of the Dirichlet Green function for the Laplace operator in $\Omega$ and $h$ being the associated Robin function.
Let us begin with the case of a bounded convex domain $\Omega$. It is well known that the Robin function $h : \Omega \to \mathbb{R}$ is strictly convex and that it has a unique non-degenerate minimum $z_0$, the harmonic center of $\Omega$ (see [12]). Moreover $h(z) \to \infty$ as $z \to \partial \Omega$. We set $m(\Omega) := h(z_0) = \min h$. The level sets $h^{-1}(c)$ with $c > m(\Omega)$ are connected and strictly star-shaped with respect to $z_0$. For $c > m(\Omega)$ we may therefore define $T(c)$ to be the minimal period of the solution of (2.2) with trajectory $h^{-1}(c)$. The following lemma shows that the assumptions of theorem 2.2 are satisfied.

**Lemma 2.4.** For a bounded convex domain $\Omega$ the function $(m(\Omega), \infty) \to \mathbb{R}$, $c \mapsto T(c)$, defined above is strictly decreasing with $T(m(\Omega)) := \lim_{c \to m(\Omega)} T(c) = \frac{2\pi}{\sqrt{|k_1 + k_2|} \det h''(z_0)}$ and $T(c) \to 0$ as $c \to \infty$. Here $z_0$ is the harmonic center of $\Omega$.

The lemma will be proved in section 6 below. As a consequence of this lemma we can apply theorem 2.2 in an arbitrary bounded convex domain.

**Corollary 2.5.** Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. Then for every $0 < T < T(m(\Omega))$ system (2.1) has infinitely many periodic solutions $z^{(n)}$ with minimal period $T$ and having the properties stated in theorem 2.2, where $C = h^{-1}(c)$ and $c > m(\Omega)$ is uniquely determined by the equation $T(c) = T$. If $T \to T(m(\Omega))$ then $c \to m(\Omega) = \min h$, and $z^{(n)}$ converges towards the harmonic center of $\Omega$.

Now we get back to a general domain $\Omega$. Here we obtain solutions near a non-degenerate local minimum.

**Corollary 2.6.** Let $z_0$ be a non-degenerate local minimum of $h$ and set $m := h(z_0)$.

$T(m) := \frac{2\pi}{\sqrt{|k_1 + k_2|} \det h''(z_0)}$. There exists $\varepsilon > 0$ and a neighborhood $U(z_0)$ of $z_0$ such that for $c \in (m, m + \varepsilon)$ system (2.1) has infinitely many periodic solutions $z^{(n)}$ with trajectories in $U(z_0)$ and minimal period $T(c) < T(m)$. The solutions have the properties stated in theorem 2.2 with $C_c = h^{-1}(c) \cap U(z_0)$.

**Remark 2.7.**

(a) Since the Robin function satisfies $h(z) \to \infty$ as $z \to \partial \Omega$ in a bounded domain there always exists a minimum. It is not difficult to produce examples of domains so that the associated Robin function has many local minima. Moreover, for a generic domain all critical points are non-degenerate; see [8]. Therefore corollary 2.6 applies to generic domains.

(b) Corollary 2.6 in particular yields solutions $z^{(n)}(t)$ approaching the local minimum $z_0$ of $h$, i.e. $z^{(n)}(t) \to z_0$ as $n \to \infty$, $k = 1, 2$. The minimal periods of these solutions converge towards $T_{z_0} = \frac{2\pi}{\sqrt{|k_1 + k_2|} \det h''(z_0)}$. In [4, 5, 7] the authors also obtained periodic solutions converging towards $z_0$. More precisely, they produced a family of $T_r$-periodic solutions $z^{(n)}(t)$, parameterized over $r \in (0, r_0)$ with $|z^{(n)}(t) - z_0| = r + o(r)$ and $T_r \to 0$ as $r \to 0$. Therefore these solutions are different from those obtained in the present paper. Also the method of proof is very different. In [4, 5, 7] variational methods or degree methods were used whereas we apply a multidimensional version of the Poincaré–Birkhoff theorem. Consequently, here we do not obtain continua of periodic solutions. Instead we obtain infinitely many periodic solutions with prescribed period.

In our last corollary we consider the case when $\partial \Omega$ has a component that is strictly star-shaped.
Corollary 2.8. Suppose \( \partial \Omega \) has a compact component \( \Gamma \) that is of class \( C^2 \) and is strictly star-shaped. Then there exist \( M > 0 \) and a neighbourhood \( U(\Gamma) \) of \( \Gamma \) such that for \( c > M \) system (2.1) has infinitely many periodic solutions \( z^{(c)} \) with trajectories in \( U(\Gamma) \) and minimal period \( T(c) \). The solutions have the properties stated in theorem 2.2 with \( C_c = h^{-1}(c) \cap U(\Gamma) \). Moreover \( T(c) \to 0 \) as \( c \to \infty \).

Remark 2.9.

(a) Corollary 2.8 applies to the typical multiply connected circular domains \( \Omega = \Omega_0 \setminus \bigcup_{i=1}^{m} \Omega_i \) where all \( \Omega_i \) are strictly star-shaped, \( \Omega_1, \ldots, \Omega_m \subset \Omega_0 \) are compactly contained in \( \Omega_0 \), and \( \overline{\Omega}_1, \ldots, \overline{\Omega}_m \) are disjoint. One can take \( \Gamma = \partial \Omega_0 \), for every \( i = 1, \ldots, m \). If \( \Omega_0 \) is bounded one can also take \( \Gamma = \partial \Omega_0 \).

(b) In [6] the authors also obtain periodic solutions near a compact component \( \Gamma \) of the boundary. It is not required that \( \Omega \) is star-shaped, and the authors could deal with \( N \geq 2 \) vortices. On the other hand, in [6] the vorticities had to be identical. For \( r > 0 \) small they obtain \( T_r \)-periodic solutions where the vortices \( z_1, \ldots, z_N \) all follow the same trajectory \( \Gamma_r = \{ z(t) : t \in \mathbb{R} \} \) with a time shift \( z_r(t) = z_1(t + (j-1)T_r/N) \). At first order (in \( r \)) the trajectory \( \Gamma_r \), consists of the points \( z \in \Omega \) with distance \( r \) from \( \Gamma \). These solutions are very different from those obtained in corollary 2.8, however. In particular, for \( j \neq k \) the distance \( |z_r(t) - z_r(t)| \) is of order \( r/N + o(1) \) as \( r \to 0 \) where \( L \) is the length of \( \Gamma \).

3. Proof of theorem 2.2

For the proof of theorem 2.2 we may assume that the trajectories \( \mathcal{C}_d, c_0 \leq d \leq d_0 \), are strictly star-shaped with respect to \( \mathcal{C}_d = 0 \). We may also assume that \( \kappa_1 + \kappa_2 = 1 \). If \( \kappa_1 + \kappa_2 \neq 1 \) then apply theorem 2.2 to the system with \( \mathcal{C}_j = \frac{\kappa_j}{\kappa_1 + \kappa_2} \) instead of \( \kappa_j, j = 1, 2 \). A solution \( \tilde{z}(t) \) of this system yields a solution \( z(t) = \tilde{z}(\kappa_1 + \kappa_2) \) of the original system; recall that we assume \( \kappa_1 + \kappa_2 \neq 0 \). Finally we set \( \sigma = \text{sgn}(\kappa_1 \kappa_2) \).

Let \( E_2 \) be the \( 2 \times 2 \) identity matrix, and set \( E_2^*: = \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \). The transformation

\[
A = \begin{pmatrix} \sqrt{\kappa_1 \kappa_2} E_2^* & -\sqrt{\kappa_1 \kappa_2} E_2^* \\ \frac{1}{\kappa_1} E_2 & -\frac{1}{\kappa_2} E_2 \end{pmatrix} \in \mathbb{R}^{4 \times 4}
\]

transforms the system (2.1) to a Hamiltonian system

\[
\dot{w}_j = J \nabla_{w_j} H_j(w_1, w_2) \quad \text{for} \quad j = 1, 2,
\]

with Hamiltonian

\[
H_j(w_1, w_2) = -\frac{\kappa_1 \kappa_2}{\pi} \log |w_1| - 2\kappa_1 \kappa_2 g (A^{-1} w) - \kappa_2^2 h \left( \Pi_1 (A^{-1} w) \right) - \kappa_1^2 h \left( \Pi_2 (A^{-1} w) \right).
\]

where \( \Pi_j : \mathbb{R}^4 \to \mathbb{R}^2, \Pi_j(z_1, z_2) = z_j, \) for \( j = 1, 2 \). The transformation \( A \) is defined on \( \mathcal{F}_2 \Omega := A(\mathcal{F}_2 \Omega) \). Note that \( w_2 = \kappa_1 z_1 + \kappa_2 z_2 \in \Omega \) provided \( |z_1 - z_2| < \frac{1}{\kappa_1 \kappa_2} \text{dist}(z_2, \partial \Omega) \), because \( \kappa_1 + \kappa_2 = 1 \). Moreover, given a compact subset \( K \subset \Omega \) there exists \( \delta > 0 \) so that \( (B_\delta(0) \setminus \{0\}) \times K \subset A(\mathcal{F}_2 \Omega) \). Here \( B_\delta(0) \) denotes the closed disk around 0 with radius \( \delta \).

Given \( 0 < a_1 < b_1 \) we define the annulus

\[
A_1(a_1, b_1) := \{ w_1 \in \mathbb{R}^2 : a_1 \leq |w_1| \leq b_1 \},
\]

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and for \(c_0 \leq c_1 < c < d_1 \leq d_0\) we define the annular region
\[
\mathcal{A}_2(c_1, d_1) := \{w_2 \in \mathcal{U} : c_1 \leq h(w_2) \leq d_1\}.
\]
From now on we fix some \(c_1 \in (c_0, c)\) and some \(d_1 \in (c, d_0)\) arbitrarily. Suitable values \(b_1 > a_1 > 0\) will be carefully chosen later.

**Lemma 3.1.** The gradient of \(H_1\) with respect to \(w_2\) satisfies
\[
\nabla_{w_2} H_1(w) = -\nabla h(w_2) + Q(w),
\]
with \(Q(w) = o(1)\) as \(w_1 \to 0\) uniformly for \(w_2\) in compact subsets of \(\Omega\).

**Proof.** Recall that \(\kappa_1 + \kappa_2 = 1\). A direct computation shows
\[
\nabla_{w_2} H_1(w) = -2\kappa_1\kappa_2 \nabla z_1 g(A^{-1}w) - 2\kappa_1\kappa_2 \nabla z_2 g(A^{-1}w) - \kappa_1^2 \nabla h(\Pi_1(A^{-1}w))
\]
\[
- \kappa_2^2 \nabla h(\Pi_2(A^{-1}w)).
\]
The Taylor expansion for \(h\) near \(w_2\) yields
\[
\nabla h(\Pi_1(A^{-1}w)) = \nabla h(w_2) + o(1) \quad \text{as} \quad w_1 \to 0,
\]
and
\[
\nabla h(\Pi_2(A^{-1}w)) = \nabla h(w_2) + o(1) \quad \text{as} \quad w_1 \to 0.
\]
This implies
\[
\kappa_1^2 \nabla h(\Pi_1(A^{-1}w)) + \kappa_2^2 \nabla h(\Pi_2(A^{-1}w)) = (\kappa_1^2 + \kappa_2^2) \nabla h(w_2) + o(1) \quad \text{as} \quad w_1 \to 0.
\]
Using the symmetry of \(g(z_1, z_2)\) and \(h(z) = g(z, z)\) we obtain analogously
\[
\nabla z_1 g(A^{-1}w) + \nabla z_2 g(A^{-1}w) = \nabla h(w_2) + o(1) \quad \text{as} \quad w_1 \to 0.
\]
This yields \(Q(w) = o(1)\) as \(w_1 \to 0\). Since all functions are of class \(C^2\) the convergence is uniform for \(w_2\) in a compact subset of \(\Omega\). \(\square\)

Now let \(W(t; w) \in \mathcal{A}_2 \mathcal{F}_2 \Omega\) be the solution of the initial value problem for (3.1) with initial condition \(W(0; w) = w\). We write \(J_u\) for its maximal existence interval.

**Lemma 3.2.**

(a) For all \(\varepsilon > 0\) there exists \(0 < \delta < \varepsilon\) such that \(\overline{B_\delta(0) \setminus \{0\}} \times \mathcal{A}_2(c_1, d_1) \subset \mathcal{A}_2 \mathcal{F}_2 \Omega\). Moreover, if \(0 < |w_1| \leq \delta\) and \(w_2 \in \mathcal{A}_2(c_1, d_1)\) then
\[
|W(t; w)| < \varepsilon \quad \text{for every} \quad t \in J_u \quad \text{with} \quad W_2(t; w) \in \mathcal{A}_2(c_0, d_0).
\]

(b) If \(\inf J_u < \infty\) for some \(w \in \overline{B_\delta(0) \setminus \{0\}} \times \mathcal{A}_2(c_1, d_1) \subset \mathcal{A}_2 \mathcal{F}_2 \Omega\) then there exists \(T(w) < \inf J_u\) such that \(W(t; w) \notin \overline{B_\delta(0) \setminus \{0\}} \times \mathcal{A}_2(c_1, d_1)\) for \(T(w) < t < \inf J_u\).

**Proof.**

(a) By contradiction, suppose that for some \(\varepsilon > 0\) there exist sequences \(w_n = (w_{1,n}, w_{2,n})\), \(t_n \in J_u\), with \(|w_{1,n}| \to 0\) as \(n \to \infty\), \(w_{2,n} \in \mathcal{A}_2(c_1, d_1)\) and
\[
|W_1(t_n, w_n)| \geq \varepsilon, \quad W_2(t_n, w_n) \in \mathcal{A}_2(c_0, d_0).
\]
Then
\[ H_1(W_1(t_n, w_n), W_2(t_n, w_n)) = H_1(w_{1,n}, w_{2,n}), \]

because the Hamiltonian is constant along a solution. But in this last equality the left hand side is bounded for all \( n \) as a consequence of (3.2) whereas the right hand side tends to \( \sigma\infty \) as \( n \to \infty \).

(b) This follows from a similar energy argument.

For \( w_2 \in \Omega \) let \( Z(t; w_2) \) be the solution of the initial value problem
\[ \dot{Z}(t; w_2) = -J\nabla h(Z(t; w_2)) \quad \text{in} \quad Z(0; w_2) = w_2. \]

(3.3)

If \( w_2 \in \mathcal{A}_2(c_0, d_0) \) this is defined for all \( t \in \mathbb{R} \). The following lemma concerns the existence of \( W(t; w) \) for \( t \) in the prescribed time interval \([0, T(c)]\) and the behaviour \( W_2(t; w) \) as \( w_1 \to 0 \).

**Lemma 3.3.**

(a) There exists \( \delta > 0 \) with \( (B_\delta(0) \setminus \{0\}) \times \mathcal{A}_2(c_1, d_1) \subset AF_2 \Omega \) and such that the solution \( W(t; w) \) exists for \( t \in [0, T(c)] \) provided \( 0 < |w_1| \leq \delta \) and \( w_2 \in \mathcal{A}_2(c_1, d_1) \). Moreover, \( W_2(t; w) \in \mathcal{A}_2(c_0, d_0) \) for all \( t \in [0, T(c)] \).

(b) For \( w_2 \in \mathcal{A}_2(c_1, d_1) \) there holds \( W_2(t; w) \to Z(t; w_2) \) as \( w_1 \to 0 \) uniformly on \([0, T(c)]\), and uniformly for \( w_2 \in \mathcal{A}_2(c_1, d_1) \).

**Proof.**

(a) Set \( \varepsilon := \frac{1}{4} \text{dist}(\mathcal{A}_2(c_1, d_1), \mathcal{A}_2(c_0, d_0)) > 0 \) and let
\[ U_\varepsilon(\mathcal{A}_2(c_1, d_1)) = \{ w \in \Omega : \text{dist}(w, \mathcal{A}_2(c_1, d_1)) \leq \varepsilon \} \subset \mathcal{A}_2(c_0, d_0) \]
be the closed \( \varepsilon \)-neighbourhood of \( \mathcal{A}_2(c_1, d_1) \). We proceed in three steps.

**Step 1:** there exists \( \delta_0 > 0 \) and \( t_0 > 0 \) so that \( W(t; w) \) exists for \( t \in [0, t_0] \) provided \( 0 < |w_1| \leq \delta_0 \) and \( w_2 \in U_\varepsilon(\mathcal{A}_2(c_1, d_1)) \).

Choose \( \delta_1 > 0 \) such that \( (B_{\delta_1}(0) \setminus \{0\}) \times \mathcal{A}_2(c_0, d_0) \subset AF_2 \Omega \) and set
\[ C := \sup_{\varepsilon < |w_1| < \delta_1, w_2 \in U_\varepsilon(\mathcal{A}_2(c_1, d_1))} |\nabla_{w_2}H_1(w_1, w_2)|. \]

(3.4)

Note that \( C < \infty \) because \( \nabla_{w_2}H_1 \) is defined and continuous also for \( |w_1| = 0 \). By lemma 3.2 (a) we can find \( \delta_0 > 0 \) such that if \( 0 < |w_1| \leq \delta_0 \) and \( w_2 \in U_\varepsilon(\mathcal{A}_2(c_1, d_1)) \), \( W_2(t; w) \in \mathcal{A}_2(c_0, d_0) \), then \( |W_2(t; w)| < \delta_1 \). Now lemma 3.2 (b) implies that \( W(t; w) \) exists for \( t \in [0, \varepsilon/C] \). Setting \( t_0 = \varepsilon/C \) we proved **Step 1**.

**Step 2:** if \( w^{(n)}_0 \to 0 \) and \( w^{(n)}_2 \in U_\varepsilon(\mathcal{A}_2(c_1, d_1)) \) with \( w^{(n)}_2 \to w_2 \), \( w_2 \in U_\varepsilon(\mathcal{A}_2(c_1, d_1)) \), then \( W_2(t; w^{(n)}_2) \to Z(t; w_2) \), uniformly for \( t \in [0, t_0] \), and uniformly for \( w_2 \in \mathcal{A}_2(c_1, d_1) \). In fact, using the equation for \( w_2 \) in integral form we have for \( t \in [0, t_0] \):
\[
|W_2(t; w^{(n)}_2) - W_2(t; w^{(m)}_2)|
\leq |w^{(n)}_2 - w^{(m)}_2| + \int_0^t |\nabla_{w_2}H_1(W(s; w^{(n)}_2)) - \nabla_{w_2}H_1(W(s; w^{(m)}_2))|ds.
\]

Note that \( \{W(t; w) : t \in [0, t_0], w \in (B_{\delta_1}(0) \setminus \{0\}) \times U_\varepsilon(\mathcal{A}_2(c_1, d_1))\} \subset AF_2 \Omega \) is a relatively compact subset in \( \Omega \times \Omega \). Since \( \nabla_{w_2}H_1 \) is defined on \( \Omega \times \Omega \) and is Lipschitz continuous on compact sets there exists \( \varepsilon > 0 \) such that
\[ |W_2(t; w^{(n)}) - W_2(t; w^{(m)})| \leq |w_2^{(n)} - w_2^{(m)}| + k \int_0^t |W_1(s; w^{(n)}) - W_1(s; w^{(m)})| + |W_2(s; w^{(n)}) - W_2(s; w^{(m)})| \, ds. \]

Now Gronwall’s lemma yields for \( t \in [0, t_0] \):

\[ |W_2(t; w^{(n)}) - W_2(t; w^{(m)})| \leq \left( |w_2^{(n)} - w_2^{(m)}| + k \int_0^t |W_1(s; w^{(n)}) - W_1(s; w^{(m)})| \right) e^{kt_0}. \]

This implies that \( W_2(t; w^{(n)}) \) converges as \( n \to \infty \) uniformly for \( t \in [0, t_0] \). The limit \( Z(t; w_2) \) satisfies the equation (3.3) because

\[ \nabla_{w_2} H_1(W(t; w^{(n)})) \to -\nabla h(Z(t; w_2)) \text{ as } n \to \infty; \]

see lemma 3.1. This proves Step 2.

**Step 3:** there exists \( \delta > 0 \) such that if \( 0 < |w_1| \leq \delta \) and \( w_2 \in A_2(c_1, d_1) \) then \( W_2(t; w) \in A_2(c_0, d_0) \) for all \( t \in [0, T(c)] \).

Arguing by contradiction, suppose there exist \( w_1^{(n)} \to 0, w_2^{(n)} \to w_2 \in A_2(c_1, d_1) \) and \( t_n \to t_0 \) such that \( W_2(t_n; w^{(n)}) \in \partial A_2(c_0, d_0) \). Step 2 implies \( W_2(t; w^{(n)}) \to Z(t; w_2) \) as \( n \to \infty \) uniformly on \([0, t_0]\). Then there exists \( n_1 \) such that for all \( n \geq n_1 \) we have \( W_2(t_0; w^{(n)}) \in \mathcal{U}_c(A_2(c_1, d_1)) \). This implies that \( t_n \geq 2t_0 \) for all \( n \geq n_1 \). So we can apply again Step 2 and obtain that \( W_2(t; w^{(n)}) \to Z(t; w_2) \) uniformly on \([0, 2t_0]\). By induction the procedure continues until we obtain in a finite number of steps that \( W_2(t; w^{(n)}) \to Z(t; w_2) \) uniformly on \([0, T(c)]\), which gives the contradiction and proves Step 3.

(b) This follows from Gronwall’s lemma as in Step 2.

Since \( W_i(t; w) \neq 0 \) for any \( t, w \) there exists a continuous choice of the argument of \( W_i(t; w) \) and we may define the rotation number

\[ \text{Rot}(W_i(t; w); [0, T(c)]) := \frac{1}{2\pi} \left( \arg(W_i(T(c); w)) - \arg(w_1) \right) \in \mathbb{R}. \]

And since \( W_2(t; w) \in A_2(c_0, d_0) \) for \( 0 < |w_1| \leq \delta, w_2 \in A_2(c_1, d_1), t \in [0, T(c)] \) we may also define the rotation number

\[ \text{Rot}(W_2(t; w); [0, T(c)]) := \frac{1}{2\pi} \left( \arg(W_2(T(c); w)) - \arg(w_2) \right) \in \mathbb{R}. \]

In the next section we shall prove the following result; here \( \delta > 0 \) is from lemma 3.3 (a).

**Proposition 3.4.** For every \( a_0 > 0 \) there exist \( 0 < a_1 < b_1 < \min\{a_0, \delta\} \) arbitrarily small and there exists \( \nu \in \mathbb{Z} \) such that the following holds for \( w \in A_1(a_1, b_1) \times A_2(c_1, d_1) \).

(a) If \( \sigma > 0 \) then

\[ \text{Rot}(W_i(t; w); [0, T(c)]) \begin{cases} > \nu, & \text{if } |w_1| = a_1 \\ < \nu, & \text{if } |w_1| = b_1. \end{cases} \]

The inequalities are reversed if \( \sigma < 0 \).

(b) If \( T(\delta) \) is strictly increasing for \( \delta \in (c_0, d_0) \) then
\[ \text{Rot}(W_2(t; w); [0, T(c)]) \begin{cases} > 1, & \text{if } w_2 \in C_{c_1} \\ < 1, & \text{if } w_2 \in C_{d_1}. \end{cases} \]

The inequalities are reversed if \( T(d) \) is strictly decreasing for \( d \in (c_0, d_0) \).

Using proposition 3.4 we can now prove theorem 2.2. For any \( w_2 \in A_2(c_1, d_1) \) the rotation number of \( W_1(t; w) \) in the interval \([0, T(c)]\) passes 1 as \( w_1 \) goes from the inner boundary of \( A_1(a_1, b_1) \) to the outer boundary of \( A_1(a_1, b_1) \). Similarly, for any \( w_1 \in A_1(a_1, b_1) \) the rotation number of \( W_2(t; w) \) in the interval \([0, T(c)]\) passes \( \nu \in \mathbb{Z} \) as \( w_2 \) goes from one boundary curve of \( A_2(c_1, d_1) \) to the other one. This is precisely the setting of the generalized Poincaré–Birkhoff theorem [19, theorem 1.2]. As a consequence we deduce that the Hamiltonian system (3.1) has a \( T(c) \)-periodic solution with initial condition \( w \in A_1(a_1, b_1) \times A_2(c_1, d_1) \). Lemma 3.3 implies that \( W_2(t; w) \in A_2(c_0, d_0) \) for all \( t \in \mathbb{R} \), provided \( b_1 \) is small.

Now recall that \( c_1 \in (c_0, c) \) and \( d_1 \in (c, d_0) \) were chosen arbitrarily, whereas \( 0 < a_1 < b_1 \) could be chosen arbitrarily small. Therefore we can consider sequences \( c_n \nearrow c \) and \( d_n \searrow c \) and can construct sequences \( 0 < a_n < b_n < a_{n-1} \to 0 \) such that (3.1) has a \( T(c) \)-periodic solution \( w^{(n)}(t) \) with \( w^{(n)}(0) = A_1(a_n, b_n) \times A_2(c_n, d_n) \) and \( W_2^{(n)}(t) \in A_2(c_{n-1}, d_{n-1}) \) for all \( t \in \mathbb{R} \). Let \( z^{(n)}(t) = A^{-1}w^{(n)}(t) \) be the corresponding solution of (2.1). Parts (a) and (b) of theorem 2.2 follow immediately. Parts (c) and (d) will be proved in section 5.

4. Proof of proposition 3.4

It will be useful to introduce polar coordinates for \( W_1, W_2 \). We set \( e(\theta) = (\cos \theta, \sin \theta) \) and fix initial conditions \( w_1 = \rho_1 e(\theta_1), w_2 = \rho_2 e(\theta_2) \). Then setting \( \rho = (\rho_1, \rho_2) \in (\mathbb{R}^+)^2 \) and \( \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 \) we define \( R_j(t; \rho, \theta) = (W_j(t; \rho_1 e(\theta_1), \rho_2 e(\theta_2))) \) and let \( \Theta_j(t; \rho, \theta) \) be a continuous choice of the argument of the \( W_j(t; \rho_1 e(\theta_1), \rho_2 e(\theta_2)) \). Thus we can write
\[
W_j(t; w) = R_j(t; \rho, \theta)e(\Theta_j(t; \rho, \theta)) \quad \text{for } j = 1, 2.
\]

We will also write \( R(t; \rho, \theta) = (R_1, R_2)(t; \rho, \theta) \) and \( \Theta(t; \rho, \theta) = (\Theta_1, \Theta_2)(t; \rho, \theta) \).

Next we describe the radial component of the boundary curves of \( A_2(c_1, d_1) \) as a function of the angle, obtaining functions \( r_j : \mathbb{R} \to (0, \infty) \) defined by \( r_1(\theta)e(\theta) \in C_{c_1} \) and \( r_2(\theta)e(\theta) \in C_{d_1} \). Since both boundary curves are strictly star-shaped with respect to the origin, \( r_j \) is well defined. Clearly \( r_j \) is \( 2\pi \)-periodic and there holds
\[
C_{c_1} = \{ r_1(\theta)e(\theta) : \theta \in \mathbb{R} \}, \quad C_{d_1} = \{ r_2(\theta)e(\theta) : \theta \in \mathbb{R} \}.
\]

We also set
\[
A_2^{\text{pol}}(c_1, d_1) := \{ (\rho_2, \theta_2) \in \mathbb{R}^+ \times \mathbb{R} : \rho_2 e(\theta_2) \in A_2(c_1, d_1) \}.
\]

Proposition 3.4 is now equivalent to the following result.

**Proposition 4.1.** For every \( a_0 > 0 \) there exist \( 0 < a_1 < b_1 < a_0 \) arbitrarily small and there exists \( \nu \in \mathbb{Z} \) such that the following holds for \( w \in A_1(a_1, b_1) \times A_2(c_1, d_1) \).

(a) If \( \sigma > 0 \) then
\[
\Theta_1(T(c); \rho_1, \rho_2, \theta_1, \theta_2) - \theta_1 \begin{cases} > 2\pi \nu, & \text{if } \rho_1 = a_1, \ (\rho_2, \theta_2) \in A_2^{\text{pol}}(c_1, d_1), \\ < 2\pi \nu, & \text{if } \rho_1 = b_1, \ (\rho_2, \theta_2) \in A_2^{\text{pol}}(c_1, d_1). \end{cases}
\]

The inequalities are reversed if \( \sigma < 0 \).
(b) If \( T(d) \) is strictly increasing for \( d \in (c_0, d_0) \) then

\[
\Theta_2(T(c); \rho_1, \rho_2, \theta_1, \theta_2) - \theta_2 \begin{cases} > 2\pi, & \text{if } \rho_1 \in [a_1, b_1], \rho_2 = r_1(\theta_2), \\ < 2\pi, & \text{if } \rho_1 \in [a_1, b_1], \rho_2 = r_2(\theta_2). \end{cases}
\]

The inequalities are reversed if \( T(d) \) is strictly decreasing for \( d \in (c_0, d_0) \).

**Proof.** We begin with the proof of part (b) because this determines the choice of \( b_1 \) which will then be used in the proof of part (a) where we choose \( a_1 \). Suppose \( T(d) \) is strictly increasing for \( d \in (c_0, d_0) \). For \( \rho_2 = r_1(\theta_2) \), that is

\[
w_2 = \rho_2 e(\theta_2) \in \mathcal{C}_{c_1} \subset \partial A_2(c_1, d_1),
\]

the solution \( Z(t; w_2) \) of the initial value problem (3.3) has the period \( T(c_1) \). Now lemma 3.3 implies that \( W_2(T; w) \to Z(T; w_2) \) as \( w_1 \to 0 \). Since \( T(c_1) < T(c) \) the argument \( \Theta_2 \) of \( W_2 \) satisfies

\[
\Theta_2(T(c); \rho_1, \rho_2, \theta_1, \theta_2) - \theta_2 > 2\pi \tag{4.1}
\]

for \( \rho_1 = |w_1| \) small. Similarly, for \( \rho_2 = r_2(\theta_2) \), that is

\[
w_2 = \rho_2 e(\theta_2) \in \mathcal{C}_{d_1} \subset \partial A_2(c_1, d_1),
\]

the solution \( Z(t; w_2) \) of the initial value problem (3.3) has the period \( T(d_1) > T(c) \), so \( W_2(T(c), w) \to Z(T(c), w_2) \) as \( w_1 \to 0 \) implies

\[
\Theta_2(T(c); \rho_1, \rho_2, \theta_1, \theta_2) - \theta_2 < 2\pi \tag{4.2}
\]

for \( \rho_1 = |w_1| \) small. Part (b) follows provided we choose \( b_1 \) so small that (4.1) and (4.2) hold for \( \rho_1 = |w_1| < b_1 \). The case that \( T(d) \) is strictly decreasing for \( d \in (c_0, d_0) \) can be proved analogously.

Now we can prove part (a). The proof of this part is similar to the proof of the main result in [11]. Suppose first that \( \sigma > 0 \). With \( b_1 \) determined above we choose \( \nu \in \mathbb{Z} \) satisfying

\[
2\pi \nu > \max \left\{ \Theta_1(T(c); b_1, \rho_2, \theta_1, \theta_2) - \theta_1 : \theta_1 \in [0, 2\pi], (\rho_2, \theta_2) \in A_{c_1}^{\nu_0}(c_1, d_1) \right\}. \tag{4.3}
\]

Setting

\[
z_1(R, \Theta) = \frac{\kappa_2}{\sqrt{\kappa_2 \kappa_1}} R_1 e(\Theta_1) + R_2 e(\Theta_2),
\]

\[
z_2(R, \Theta) = -\frac{\kappa_1}{\sqrt{\kappa_2 \kappa_1}} R_1 e(\Theta_1) + R_2 e(\Theta_2),
\]

and

\[
k(R, \Theta) = 2 \left( \kappa_2 \sqrt{\kappa_1 \kappa_2} |\nabla z_1| - \kappa_1 \sqrt{\kappa_1 \kappa_2} |\nabla z_2| \right) g(z_1(R, \Theta), z_2(R, \Theta))
\]

\[
+ \kappa_1 \sqrt{\kappa_1 \kappa_2} \nabla h(z_1(R, \Theta)) - \kappa_2 \sqrt{\kappa_1 \kappa_2} \nabla h(z_2(R, \Theta)),
\]

the equations for \( R_1, \Theta_1 \) are given by
The following proposition implies part (c) of theorem 2.2.

**Proposition 5.1.** Let \( z^{(n)}(t) \) be a sequence of T-periodic solutions of (2.1) with the property that \( z_1^{(n)}(0), z_2^{(n)}(0) \to C_0 \in \Omega \) and such that the solution \( C(t) \) of (2.2) with initial condition \( C(0) = C_0 \) is non-stationary periodic. Then setting \( d_n = |z_1^{(n)}(0) - z_2^{(n)}(0)| \) the angular velocity of the difference \( D^{(n)}(t) := z_1^{(n)}(t) - z_2^{(n)}(t) = \rho^{(n)}(t)(\cos \theta^{(n)}(t), \sin \theta^{(n)}(t)) \) satisfies

\[
\dot{\theta}^{(n)}(t) = \frac{k_1 k_2}{\pi} + o(1) \quad \text{as} \quad n \to \infty \quad \text{uniformly in} \ t.
\]
Proof. Define

\[ u_n(s) := \frac{1}{d_n} D^{(n)}(d_n^2 s). \]

Then \( u_n \) satisfies

\[ \dot{u}_n = -\frac{\kappa_1 \kappa_2}{\pi} J \frac{u_n}{|u_n|^2} - o(1) \quad \text{as } n \to \infty, \text{ uniformly in } [0, T]. \]

Note that \( |u_n(0)| = 1 \) for all \( n \), so up to a subsequence \( u_n(0) \to \bar{u} \) with \( |ar{u}| = 1 \). By a straightforward calculation we obtain that \( \frac{d}{ds} |u_n(s)|^2 = o(1) \) as \( n \to \infty \), uniformly in \([0, T]\). Thus there exists \( \varepsilon > 0 \) such that for \( n \) sufficiently large we have \( |u_n(s)| \geq \varepsilon \) uniformly for \( s \in [0, T] \). Next let \( u_\infty \) be the solution of the initial value problem

\[ \begin{aligned}
\dot{u}_\infty &= -\frac{\kappa_1 \kappa_2}{\pi} J \frac{u_\infty}{|u_\infty|^2} \\
u_\infty(0) &= \bar{u}.
\end{aligned} \]

We now deduce easily that \( u_n \to u_\infty \) uniformly on \([0, T]\). Note that \( \frac{d}{ds} \arg(u_\infty(s)) = \frac{\kappa_1 \kappa_2}{\pi} \), which implies \( d_n^2 \dot{\theta}^{(n)}(s) \to \frac{\kappa_1 \kappa_2}{\pi} \).

\[ \square \]

Proof of theorem 2.2 (d). This is a straightforward computation using

\[ z_1^{(n)}(t) = C^{(n)}(t) + \frac{\kappa_2}{\kappa_1 + \kappa_2} D^{(n)}(t) \quad \text{and} \quad z_2^{(n)}(t) = C^{(n)}(t) - \frac{\kappa_1}{\kappa_1 + \kappa_2} D^{(n)}(t), \]

and parts (a) and (b) of theorem 2.2.

\[ \square \]

6. Proof of the remaining results

Proof of lemma 2.4. First we transform the equation (2.2) using the canonical coordinate change \((\rho, \theta) \mapsto \sqrt{2} \rho e^{\theta} \). Setting \( h_1(\rho, \theta) = (\kappa_1 + \kappa_2) h(\sqrt{2} \rho e^{\theta}) \) this leads to the system

\[ \begin{aligned}
\dot{\rho} &= -\frac{\partial}{\partial \theta} h_1(\rho, \theta) \\
\dot{\theta} &= \frac{\partial}{\partial \rho} h_1(\rho, \theta).
\end{aligned} \]

In convex domains the Robin function \( h \) is strictly convex by [12], hence \( \frac{\partial}{\partial \rho} h_1(\rho, \theta) \) is strictly increasing in \( \rho \). This implies that the minimal period \( T(c) \) is decreasing with respect to \( c \).

Moreover, since the origin is a nondegenerate minimum of \( h \), we can apply the Hartman–Grobman theorem, which tells us that the flow of the system near the hyperbolic critical point is topologically equivalent to the flow of the linearized system

\[ \dot{\zeta} = -(\kappa_1 + \kappa_2) J h''(0) \zeta. \]

The solution of this harmonic oscillator is periodic with period \( T_m = \frac{2\pi}{|\kappa_1 + \kappa_2| \sqrt{\det h''(0)}} \). The lemma follows.
Proof of corollary 2.6. Since $h''(z_0)$ is positive definite the Robin function is strictly convex in a neighbourhood $U$ of $z_0$. Therefore the level lines $h^{-1}(c) \cap U$ for $c > c_0 = h(z_0)$ close to $c_0$ are convex. As in the proof of lemma 2.4 the period $T(c)$ of the solution of (2.2) with trajectory $h^{-1}(c) \cap U$ is strictly decreasing in $c$ for $c > c_0$ close to $c_0$. The corollary follows now from theorem 2.2.

Proof of corollary 2.8. Let $U(\Gamma) \subset \mathbb{R}^2$ be a tubular neighbourhood of $\Gamma$ and $p : U(\Gamma) \rightarrow \Gamma$ be the orthogonal projection. Moreover let $\nu : \Gamma \rightarrow \mathbb{R}^2$ be the exterior normal. It is well known

\[ \nabla h(z) = \frac{\nu(p(z))}{2\pi d(z, \Gamma)} + O(1) \quad \text{as} \quad d(z, \Gamma) = \text{dist}(z, \Gamma) \rightarrow 0; \]

see [3]. Therefore the level lines $h^{-1}(c) \cap U(\Gamma)$ for $c > c_0$ are also strictly star-shaped with respect to $z_0$, if $c_0$ is large enough. Moreover the period $T(c)$ of the solution of (2.2) with trajectory $h^{-1}(c) \cap U(\Gamma)$ is strictly decreasing in $c$ due to (6.1). Consequently the corollary follows from theorem 2.2.

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