Forced beams vibrations from vector disturbances

A M Kaziev*, M Kh Dadova, M M Kardanov, I K Mashukov
Kabardino-Balkarian State University named after H.M. Berbekov, 173 Chernyshevskogo str., Nalchik, 360004, Russian Federation

E-mail: kaziev1969@mail.ru

Abstract. The forced transverse vibrations of constant cross section beams taking into account damping are studied. Nonperiodic, periodic, and harmonic beam vibrations from vector disturbances are considered.

Introduction
In connection with the appearance of increasingly complex engineering and technical structures in recent decades, special interest in the structural dynamics issues is shown.

The problems of continuum systems’ vibrations under vector perturbations which components are deterministic or random processes remain poorly studied. In particular, there are only a small number of publications on beam vibrations under vector disturbances containing simultaneously dynamic and kinematic vibrations’ sources. The mathematical problem formulation of the beam vibrations from vector disturbances was considered in [1] and [6]. A vector approach to solving the problem of beam vibrations with friction presence was considered in [2] and [3]. The beams’ fluctuations with longitudinal force presence under vector impacts were solved in [4], [5].

The problem statement and solution
Let us consider the forced steady-state beam disturbances under harmonic perturbations in the general case.

The mathematical model of the problem has the form:

\[ u^{IV} + \gamma \ddot{u} + \varepsilon \dot{u} = f_1(t), \quad f_1(t) = q(t)/EJ, \quad x \in (0, l), \quad t > -\infty, \]

\[ u(0, t) = f_2(t), \quad Eu''(0, t) = f_3(t). \quad u(l, t) = f_4(t), \quad Eu''(l, t) = f_5(t). \]

Here \( \gamma = m/EJ, \varepsilon = \eta m/EJ, m = \rho S \) – is the beam linear mass, \( \eta \) – is a damping coefficient, \( EJ \) – denote bending stiffness, \( f_1(t) \) – corresponds to distributed load (dynamic impact), \( f_2(t), f_3(t) \) – are the functions of vertical movements of the beam’s left and right ends (kinematic effect), \( f_4(t), f_5(t) \) – are the moment loads applied to the beam’s left and right ends (dynamic impact).
Since the system is linear with five input actions, it is advisable to use the superposition principle

\[ u(x,t) = \sum_{j=1}^{5} u_j(x,t), \]

where \( u_j(x,t) \) – is the displacement function with autonomous perturbation \( f_j(t) \). Let the external disturbances be represented by a vector harmonic process

\[ f_k(t) = a_k e^{i(\Omega_k t + \psi_k)} = A_k e^{i\Omega_k t}, \quad k = 1, 2, \ldots, 5. \]  

where

\[ \mathbf{A} = \{A_1, A_2, A_3, A_4, A_5\}, \quad A_k = a_k e^{i\psi_k}, \quad \mathbf{e}(t) = \{e^{i\Omega_1 t}, e^{i\Omega_2 t}, \ldots, e^{i\Omega_5 t}\}, \quad k = 1, 2, \ldots, 5. \]

\( a_k, A_k \) – are the real and complex-valued amplitudes, \( \psi_k \) – denotes initial phases or phase shifts.

Considering (3), the problem (1), (2) becomes a concrete boundary-value problem without initial conditions. The question further will be to find its solution \( u(x,t) \).

The output process \( u(x,t) \) in general case will not be harmonious. At the same time, it will be the sum of five harmonics with different frequencies \( \Omega_k \). Such fluctuations will be periodic only if the relationships \( \Omega_j / \Omega_k \) are the rational numbers.

If all disturbance frequencies are the same, i.e. \( \Omega_1 = \Omega_2 = \ldots = \Omega_5 = \Omega \), then the output process will be harmonious.

For the forced vibrations, three options of steady-state modes are possible:
- non-periodic non-harmonic vibrations;
- periodic non-harmonic vibrations;
- harmonic vibrations.

The superposition principle will be implemented in the following order. We will solve five autonomous problems when only one of the actions with a unit amplitude acts on the system \( e^{i\Omega_k t} \). As a result, we find the harmonics \( v_k(x,t), \quad k = 1, 2, \ldots, 5 \). The problem solution from the vector exposure will be a scalar product of vectors:

\[ u(x,t) = (\mathbf{A}, \mathbf{v}). \]  

Applying the variable separation method, we will search for the solution of \( v_k(x,t) \) in the form:

\[ v_k(x,t) = H_k(x, i\Omega_k) e^{i\Omega_k t}. \]

Let us consider the autonomous tasks.

1) The disturbance problem from the transverse load (Figure 2) has the form:

\[ v_1^{IV} + \gamma \ddot{v_1} + \alpha \dot{v_1} = e^{i\Omega_1 t}, \quad x \in (0, l), \quad t > -\infty, \]  

Figure 2.
\[ v_1(0, t) = 0, \quad v_1'(0, t) = 0, \quad v_1(l, t) = 0, \quad v_1''(l, t) = 0. \quad (7) \]

Substitution (5) in (6), (7) at \( k = 1 \) gives a boundary value problem with respect to the transfer function \( H_i \) at \( a_i = 1 \)

\[
H_{1V} - b_1^4 H_1 = 1, \quad b_1^4 = -\gamma(i\Omega_1)^2 - \varepsilon(i\Omega_1), \quad x \in (0, l),
\]

\[
H_1(0, i\Omega_1) = 0, \quad H_1'(0, i\Omega_1) = 0, \quad H_1(l, i\Omega_1) = 0, \quad H_1''(l, i\Omega_1) = 0. \quad (8)
\]

The inhomogeneous ordinary differential equation solution consists of the sum of a homogeneous equation solution \( a \) and a particular solution of an inhomogeneous equation

\[ H_1(x, i\Omega_1) = C_1 \sin b_1 x + C_2 \cos b_1 x + C_3 \sh b_1 x + C_4 \ch b_1 x - b_1^{-4}. \quad (9) \]

Here \( C_i \) - defines the unknown integration constants. To use the boundary conditions (8), we differentiate (9) twice

\[ H_1'(x, i\Omega_1) = b_1^2 (-C_1 \sin b_1 x - C_2 \cos b_1 x + C_3 \sh b_1 x + C_4 \ch b_1 x). \quad (10) \]

(6)-(8) give a system of linear equations for \( C_i \) and the solution:

\[
C_2 = C_4 = \frac{1}{2b_1^4}, \quad C_1 = \frac{1}{2b_1^4} \frac{1 - \cos b_1 l}{\sin b_1 l}, \quad C_3 = \frac{1}{2b_1^4} \frac{1 - \ch b_1 l}{\sh b_1 l}.
\]

Therefore, the first desired equation is obtained

\[
H_1(x, i\Omega_1) = \frac{1}{2b_1^4} \left( \frac{1 - \cos b_1 l}{\sin b_1 l} \sin b_1 x + \cos b_1 x + \frac{1 - \ch b_1 l}{\sh b_1 l} \sh b_1 x + \ch b_1 x - 2 \right). \quad (11)
\]

It describes harmonic oscillations, the amplitude of which is defined as the complex function modulus

\[ a_u(x) = |A_1 H_1(x, i\Omega_1)|. \]

2) The right end of the rod performs harmonic disturbances; other disturbances are absent (Fig. 3). The corresponding task functions take the form:

\[ f_1(t) = 0, \quad f_2(t) = 0, \quad f_3(t) = 0, \quad f_4(t) = e^{i\Omega_4 t}, \quad f_5(t) = 0. \]

The function \( v_4(x, t) \) is searched from the task compiled as in previous cases

\[
v_4^{IV} + \gamma v_4' + \varepsilon v_4 = 0 \quad x \in (0, l) \quad (12)
\]

\[ v_4(0, t) = 0, \quad v_4'(0, t) = 0, \quad v_4(l, t) = e^{i\Omega_4 t}, \quad v_4''(l, t) = 0. \quad t > -\infty. \quad (13)
\]

According to the variables’ separation method procedure \( v_4(x, t) \) is presented as a product:

\[ v_4(x, t) = H_4(x, i\Omega_4) e^{i\Omega_4 t}. \quad (14) \]

Its substitution into the main equation (12) and boundary conditions (13) leads to a boundary value problem for the transfer function

\[ H_4^{IV} - b_4^4 H_4 = 0, \quad b_4^4 = -\gamma(i\Omega_4)^2 - \varepsilon(i\Omega_4), \quad x \in (0, l), \]
Its solution has the form:
\[ H_4(x, i\Omega) = C_1 \sin b_4 x + C_2 \cos b_4 x + C_3 \sinh b_4 x + C_4 \cosh b_4 x. \]  \hspace{1cm} \text{(16)}
\[ H'_4(x, i\Omega) = b_4^2 (-C_1 \sin b_4 x - C_2 \cos b_4 x + C_3 \sinh b_4 x + C_4 \cosh b_4 x). \]  \hspace{1cm} \text{(17)}

We substitute (16), (17) into the boundary conditions (15) and obtain
\[ C_2 = C_4 = 0, \quad C_1 = \frac{1}{2 \sin b_4 l}, \quad C_3 = \frac{1}{2 \sinh b_4 l}. \]

That means:
\[ H_4(x, i\Omega) = \frac{1}{2} \left( \frac{\sin b_4 x}{\sin b_4 l} + \frac{\sinh b_4 x}{\sinh b_4 l} \right). \]  \hspace{1cm} \text{(18)}

Figure 4.

Thus, \( v_4(x, t) \) by (14) became known.

3) The moment load is applied to the right end (Figure 4), the remaining perturbations are zero. Therefore:
\[ f_i(x, t) = f_2(t) = f_3(t) = f_4(t) = 0, \quad f_5(t) = e^{i\Omega_5 t}, \]
\[ v_3(0, t) = 0, \quad v'_3(0, t) = 0, \quad v_3(l, t) = 0, \quad EJv'_3(l, t) = e^{i\Omega_5 t}. \]

First, as in previous cases, we obtain the boundary value problem with respect to the transfer function
\[ H^V_5 - b_5^4 H_5 = 0, \quad b_5^4 = -(i\Omega_5)^2 - e(i\Omega_5), \quad x \in (0, l), \]
\[ H_5(0, i\Omega_5) = 0, \quad H'_5(0, i\Omega_5) = 0, \quad H_5(l, i\Omega_5) = 0, \quad EJH'_5(l, i\Omega_5) = 1. \]  \hspace{1cm} \text{(19)}

It is easy to show that its solution has the form
\[ H_5(x, i\Omega_5) = C_1 \sin b_5 x + C_2 \cos b_5 x + C_3 \sinh b_5 x + C_4 \cosh b_5 x. \]  \hspace{1cm} \text{(20)}
\[ H'_5(x, i\Omega_5) = b_5^2 (-C_1 \sin b_5 x - C_2 \cos b_5 x + C_3 \sinh b_5 x + C_4 \cosh b_5 x). \]  \hspace{1cm} \text{(21)}

Substituting (20), (21) into the boundary conditions (19), gives:
\[ C_2 = C_4 = 0, \quad C_1 = -\frac{1}{2EJb_5^2 \sin b_5 l}, \quad C_3 = \frac{1}{2EJb_5^2 \sinh b_5 l}. \]

It means, that:
\[ H_5(x, i\Omega_5) = \frac{1}{2EJb_5^2} \left( \frac{\sinh b_5 x}{\sinh b_5 l} - \frac{\sin b_5 x}{\sin b_5 l} \right). \]  \hspace{1cm} \text{(22)}

The function \( v_5(x, t) \) and the vibration amplitude became known.

The missing Transfer Functions \( H_2(x, i\Omega_2), H_3(x, i\Omega_3) \), can be written out by substituting in (18) and (22) \( x \) by \( l-x \), namely:
\[ H_2(x, i\Omega_2) = \frac{1}{2} \left( \frac{\sin b_2(l-x)}{\sin b_2 l} + \frac{\sin b_2(l-x)}{\sin b_2 l} \right), \quad b_2^4 = -\gamma(i\Omega_2)^2 - \varepsilon(i\Omega_2). \]  
\[ H_3(x, i\Omega_3) = \frac{1}{2EJb_3^2} \left( \frac{\sin b_3(l-x)}{\sin b_3 l} - \frac{\sin b_3(l-x)}{\sin b_3 l} \right), \quad b_3^4 = -\gamma(i\Omega_3)^2 - \varepsilon(i\Omega_3). \]  

With equal frequencies, i.e. \( \Omega_1 = \Omega_2 = \ldots = \Omega_5 = \Omega \), the total oscillations will be harmonic, and the formula (2) will take the form:

\[ u(x, t) = [A, H(x, i\Omega)] e^{i\Omega t}, \]

moreover, the vector \( H(x, i\Omega) \) components are determined by (11), (18), (22), (23), (24). The corresponding amplitude is written out easily.

\[ a_u(x) = |[A, H(x, i\Omega)]|. \]  

Example. Take a steel beam \((\rho = 7800 \text{ kg/m}^3, E = 200 \text{ GPa}, \eta = 1 \text{ c}^{-1})\) from the I-beam No. 14, span \( l = 6 \text{ m} \).

The first elements of the eigenfrequency spectrum have the following values:

\( \{79, 59, 318, 39, 716, 37, \ldots\} \text{ c}^{-1} \).

Let the disturbances’ characteristics be as follows”

\( a_1 = 700 \text{ N/m}, \quad a_2 = 5 \text{ mm}, \quad a_3 = 4900 \text{ N/m}, \quad a_4 = 10 \text{ mm}, \quad a_5 = 980 \text{ N/m}. \)

The impact values are selected so that the displacements caused by them are commensurate. First, we consider the non-harmonic non-periodic disturbances when the frequencies and initial phases have the values:

\( \Omega = \{5, \ 5\sqrt{2}, \ 5\pi, \ 23, 4, \ 33, 32\} \text{ c}^{-1}, \quad \psi = \{0, \ 0, \ \pi, \ 0, \ \pi\}. \)

Figure 5.

For the middle of the beam \((x = \frac{l}{2} = 3 \text{ m})\) the calculations by the formula (4) for the time interval \( t \in [0; 2] \text{ c} \), are performed and the results are presented by the curve in Figure 5. It is seen that the vibrations are non-harmonic, non-periodic in nature and represent the sum of harmonics with different frequencies and amplitudes.

With the same data, but at the disturbance frequencies - 

\( \Omega = \{85, 68, 51, 34, 17\} \text{ s}^{-1} \)
the fluctuations will be periodic, since they are multiple 17 s\(^{-1}\). The calculations performed at disturbance amplitudes -
\[
\begin{aligned}
    a_3 &= 700 \text{ N/m}, & a_2 &= 5 \text{ mm}, \\
    a_4 &= 2900 \text{ N/m}, & a_4 &= 10 \text{ mm}, \\
    a_5 &= 980 \text{ N/m},
\end{aligned}
\]
gave the results presented by the graphs in Figure 6.

![Figure 6](image)

It is easy to notice that the curve is again the sum of harmonics, but the disturbances are already periodic. The disturbance period is determined by \(\Omega_5\), namely
\[
T = \frac{2\pi}{\Omega_5} = \frac{2 \cdot 3.141}{17} = 0.3695 \text{ c}^{-1}.
\]

Now we turn to harmonic disturbances when all the disturbance frequencies are equal to each other
\[\Omega_1 = \Omega_2 = \ldots = \Omega_5 = \Omega.\]
Let the initial phases be the same as before.

![Figure 7](image)

According to the formulas (25), the calculations were made for the disturbances’ amplitudes, the results of which are presented by graphs in Fig. 7. The curves obtained at increasing frequencies are shown as:
\[
\Omega = 1 \text{ s}^{-1} \text{ (curve 1), } 36 \text{ s}^{-1} \text{ (2), } 45 \text{ s}^{-1} \text{ (3), } 307 \text{ s}^{-1} \text{ (4), } 625 \text{ s}^{-1} \text{ (5),}
\]
and perturbation amplitudes

\[ a_1 = 700 \text{ N/m}, \quad a_2 = 5 \text{ mm}, \quad a_3 = 2900 \text{ N/m}, \quad a_4 = 10 \text{ mm}, \quad a_5 = 980 \text{ N/m} \]

while saving other parameters.

At a relatively low perturbation frequency (curve 1), the deviations are close to static (for example, in the middle of the span) \( u_{\text{stat}} \approx 25.6 \text{ mm} \). With an increase in the perturbation frequency, it approaches the first natural frequency, and therefore, the amplitudes gradually increase (curves 2, 3). At disturbance frequencies exceeding the first eigenfrequency but lower than the second eigenfrequency, the first eigenform is basically preserved, then the second eigenform appears and plays the main role (curve 4). A further increase in perturbation frequencies leads to the higher vibrational modes’ emergency (curve 5).

**Summary**

The new models for the problems in their mathematical formulation and mechanical content that adequately describe the disturbance processes of homogeneous beams with combined dynamic and kinematic perturbations of a deterministic nature have been developed. The examples on which the results from such effects are shown, are considered. The found transfer functions from autonomous dynamic and kinematic actions will subsequently make it possible to solve the beam vibrations’ problems in random vector processes [5].

**Acknowledgements**

The forced transverse beams vibrations with constant cross section to damping were investigated. Non-periodic, periodic and harmonic beams oscillations from vector perturbations were considered.

**References**

[1] Biderman V L 1979 *Applied theory of mechanical vibrations* (Higher school, Moscow) 416.

[2] Kulteberaev H P, Kaziev A M 2003 On harmonic beams vibrations excited by vector perturbations *Light building constructions* (Rostov-on-Don, Rostov State University of civil engineering) 146-154.

[3] Kulteberaev H P 1993 Kinematically excited bending vibrations of rods *News of universities. North Caucasus region. Natural Sciences* 3 83-84.

[4] Kaziev A M 2003 Free vibrations of a longitudinally loaded beam in the presence of friction *Bulletin of KBSU, A series of technical sciences* (Nalchik, Kabardino-Balkarian State University) 5 103 - 105.

[5] Kaziev A M 2002 On the analogies between harmonic and random vibrations of beams *Materials of the All-Russian Scientific Conference of Students, Graduate Students and Young Scientists “Perspective - 2002”* (Nalchik, Kabardino-Balkarian State University) II 24-27.

[6] Timoshenko S P 1967 *Fluctuations in engineering* (Nauka, Moscow) 444.