WARD-TAKAHASHI IDENTITIES AND NOETHER’S THEOREM IN QUANTUM FIELD THEORY

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Abstract

A gap in the mathematical logic in derivations in quantum field theory arises as consequence of variation before quantization. To close this gap the present paper introduces a mathematically rigorous variational calculus for operator fields. Using quantization before variation it is demonstrated that the so-called naive results are correct; in particular both Noether’s theorem and the Ward-Takahashi identities retain full validity in quantum field theory.

1 Introduction

In classical mechanics Noether’s theorem plays a central role in that, given the Lagrangian, it allows to find the constants of the motion without the need of actually finding solutions to the theory [1]. As emphasized e.g., by Bjorken and Drell, Ref. [2], Chapter 11, one must question whether the classical result, the existence of Noether’s theorem, still applies when going over to quantum fields. The aim of this paper is to demonstrate that Noether’s theorem can be derived in a mathematically rigorous manner directly within the framework of quantum field theory, and that therefore the question of its validity in quantum field theory can be answered in the affirmative.

Elaborating on the above, in classical physics Noether’s theorem is unassailable; it arises by using variational calculus, together with some algebraic manipulations, on the given Lagrangian action. All that involves only well-understood mathematically rigorous operations. Consequently, any results seeming to break that theorem can immediately be declared as hiding a calculational error. In quantum field theory, on the other hand, in the conventional formulation the logical structure of the derivation is not tight in that one performs the variational calculus, and derives the different theorems, using classical fields. Only afterwards one goes over to QFT: the fields are quantized at the end, and the theorems are taken over from the classical level [3]. Thus, the connection with the underlying Lagrangian is not immediate. The chain of mathematical reasoning is broken, and one can argue that the validity of Noether’s theorem in QFT can not

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be taken for granted: the theorems derived within the classical framework may not survive the quantization procedure. It is the aim of the present paper to close this gap in the mathematical logic, and hence to demonstrate the validity of Noether’s theorem in quantum field theory.

Of the mathematical manipulations, algebraic transformations do not pose any difficulties. It is the variational calculus which conventionally is known only for c-number fields. As we will show, it is possible to define rigorously a generalization of the Euler – Lagrange variational calculus to operator fields in such a manner that their commutation relations are maintained; hence the thus defined variational calculus is directly valid for operator fields. It will be demonstrated that in terms of this calculus, together with simple algebraic manipulations, the quantum results turn out to be form-identical with the well-known classical results. This way all the powerful results flowing from the application of the variational calculus valid in classical theory, in particular Noether’s theorem, are strictly valid also in quantum field theory.

The central problem thus is to define a variational calculus for quantum fields, which is the subject of Section 2. For the present purpose one only needs to demonstrate a procedure which is “sufficient”; it is not needed to go to the “necessary” stage. To that end we shall define a variational procedure applicable to quantum, i.e., non-commuting operator fields, which respects the commutation relations. We use this procedure in Section 3 to derive the Euler – Lagrange variational formalism. Using that formalism we re-derive Noether’s theorem in Section 4 in order to exhibit that indeed no mathematically ill-defined steps are required. In the same Section we demonstrate that for the usual quantum fields, i.e., those needed for quantum electrodynamics, the rigorous procedure leads precisely to the results obtained in today’s heuristic, so-called “naive” methods [3]. The case of PCAC is treated in Section 5. This way we show that all steps leading to Noether’s theorem can be carried out in a rigorous manner. Consequently there exists no excuse for considering that Noether’s theorem may be violated in QFT, in particular, that there may exist violations of the so-called “naive” Ward – Takahashi identities. That then has as a consequence that some of the presently disqualified theories can be reconsidered.

2 Variational Derivative for Operator Fields

We will now demonstrate that the variational procedure can be so defined that it respects the commutation relations of the fields. It will turn out that the familiar conjecture is true that the results of variational calculus of c-number fields apply also for operator fields upon taking care of the proper ordering of the factors in products.

Since the basis of the variational calculus is the variational derivative we now investigate its
meaning for operator fields. We define it by the following limit procedure: Let \( \varphi(x) \), \( \pi(x) \), be an operator field and its canonical conjugate, respectively, in the Heisenberg picture. We have,

\[
\tilde{\varphi}(x) = \varphi(x) + \delta \varphi(x)
\]

for the varied field and

\[
\tilde{\pi}(x) = \pi(x) + \delta \pi(x)
\]

for the varied conjugate field. We now define the variations as

\[
\delta \varphi(x) = \varepsilon(x) \varphi(x)
\]

and

\[
\delta \pi(x) = \eta(x) \pi(x)
\]

where \( \varepsilon(x) \) and \( \eta(x) \) are arbitrary c-number functions which, as always, are taken to be small, i.e., to approach zero (see below). They can be ordinary functions or generalized functions (for short g-functions). As we will see, these definitions suffice for a consistent variational calculus for operator fields. With our definitions the canonical equal time commutation (or anti-commutation) relations for the varied fields then are preserved in the limit. Indeed,

\[
[\tilde{\varphi}(x), \pi(y)]_{t_x=t_y} = i\delta^{(3)}(x-y) [1 + \varepsilon(x)] ,
\]

or more generally

\[
[\tilde{\varphi}(x), \pi(y)]_{t_x=t_y} = i\delta^{(3)}(x-y) [1 + \varepsilon(x) + \eta(y) + \varepsilon(x) \eta(y)]
\]

and hence are preserved in the limit. The expressions like \( \delta^{(3)}(x-y) \eta(y) \) in these equations which seem to contain products of generalized functions are in fact, as shown below, mathematically well-defined.

There is wide latitude in the choice of the c-number functions. One such possible special choice is

\[
\eta(y) = 0
\]

\[
\varepsilon(x) \Rightarrow \varepsilon_n(x) = \lim_{\varepsilon \to 0} \varepsilon \Delta_n^{(4)}(x-x_0)
\]

where \( \varepsilon = 0 \) is excluded. Further, \( x_0 \) is an arbitrary point in space-time and \( \Delta_n^{(4)}(x) \) is in Lighthill’s terminology [4] a member of a set of “good functions” which for \( n \to \infty \) approach the (four-dimensional) \( \delta \)-function:

\[
\delta^{(4)}(x) = \lim_{n \to \infty} \Delta_n^{(3)}(x) \Delta_n(t) .
\]
The choice (6), (7) will allow the definition of the variational derivative at the point \( x = (x_0, t_0) \) with respect to the operator field \( \varphi(x) \), while maintaining the commutation relations of the fields in the limit \( \varepsilon \to 0 \).

The functional derivative with respect to the conjugate field at the point \( y = (y_0, t_0) \) will arise from the choice

\[
\varepsilon(x) = 0
\]

\[
\eta(y) = \lim_{\eta \to 0} \eta \Delta_n^{(4)}(y - y_0)
\]

Again, the commutation relations are preserved for \( \eta \to 0 \).

We now give the promised demonstration of the mathematical consistency of the terms containing \( \varepsilon(x) \eta(y) \) in (5) when using the choices Eqs. (6), (10). They present no difficulties as they read

\[
\delta^{(3)}(x - y) \varepsilon(x) \varepsilon(y) = \delta^{(3)}(x - y) \lim_{\varepsilon \to 0} \lim_{\eta \to 0} |\varepsilon \eta| \lim_{n \to \infty} \lim_{m \to \infty} \Delta_n(t - t_x) \Delta_m(x - x_0)
\]

\[
\times \Delta_m^n(y - y_0) \Delta^m(x - t_y)
\]

\[
\to \delta^{3}(x - y) \lim_{\varepsilon \to 0} \lim_{\eta \to 0} |\varepsilon \eta| \delta(t_x - t) \delta^{(3)}(x - x_0) \delta(t_y - t)
\]

\[
\to \lim_{\varepsilon \to 0} \lim_{\eta \to 0} \delta(t_x - t) \delta^{(3)}(x - x_0) \delta(t_x - t_y) \delta^{(3)}(x_0 - y_0) |\varepsilon \eta|
\]

Thus it indeed is strictly a second-order term containing ordinary g-functions with non-coincident arguments, and not a product of g-functions, as Eq. (5) seems to imply. Of course, throughout measure \( dx \) and integration over test functions is implied.

To summarise, we have shown in this Section that it is possible to define a mathematically consistent variational calculus for operator fields which maintains the commutation relations of the field operators as a well-defined mathematical limit. The mathematical steps involved in this procedure are all elementary. As we will see presently, this definition suffices for the applications needed in QFT.

This concludes the definition of the variation of operator fields.
3 The Euler – Lagrange Equation

We now investigate the functional derivative of an operator-field Lagrangian, using to begin with the Klein-Gordon Lagrangian action as an example:

\[
L_0\{\varphi, \partial_\mu \varphi\} = \frac{1}{2} \int d^4x \left[ \left( \frac{\partial \varphi(x)}{\partial x_k} \right)^2 - \left( \frac{\partial \varphi(x)}{\partial t} \right)^2 - m^2 (\varphi(x)) \right].
\] (12)

We have for the variation of the Lagrangian action

\[
\delta L_0 \{\varphi, \partial_\mu \varphi\} = L_0 \{\tilde{\varphi}, \partial_\mu \tilde{\varphi}\} - L_0 \{\varphi, \partial_\mu \varphi\}.
\] (13)

In first order of the variation the mass term yields immediately

\[
\delta m = -\frac{m^2}{2} \int d^4x \left[ (\varphi(x) + \delta \varphi(x)) (\varphi(x) + \delta \varphi(x)) - \varphi(x) \varphi(x) \right]
\] (14)

where in view of (3) both terms are identical. The space derivative terms are, again in first order,

\[
\delta_x = \frac{1}{2} \int d^4x \left[ \frac{\partial (\varphi(x) + \delta \varphi(x))}{\partial x_k} \frac{\partial (\varphi(x) + \delta \varphi(x))}{\partial x_k} - \frac{\partial \varphi(x)}{\partial x_k} \frac{\partial \varphi(x)}{\partial x_k} \right] = \frac{1}{2} \int d^4x \left\{ \frac{\partial \varphi(x)}{\partial x_k} \frac{\partial \delta \varphi(x)}{\partial x_k} + \frac{\partial \delta \varphi(x)}{\partial x_k} \frac{\partial \varphi(x)}{\partial x_k} \right\}
\] (15)

again both terms are identical. The time derivative gives

\[
\delta_t = \frac{1}{2} \int d^4x \left[ \frac{\partial^2 \varphi(x)}{\partial t^2} \delta \varphi(x) + \delta \varphi(x) \left[ \frac{\partial^2 \varphi(x)}{\partial t^2} \right] \right].
\] (16)

Recalling the Fourier expansion of the fields

\[
\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \left[ a_k e^{ikx} + a_k^\dagger e^{-ikx} \right]
\] (17)

we have

\[
\frac{\partial^2}{\partial t^2} \varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \left( -\omega_k^2 \right) \left[ a_k e^{ikx} + a_k^\dagger e^{-ikx} \right]
\] (18)

with

\[
[a_k, a_{k'}^\dagger] = \delta^3(k - k').
\] (19)

The fields and their second time derivatives thus also commute. To recapitulate: in view of the definition (1), (3), and (6) the fields commute with both the space-like and the time-like second derivatives. This way we find

\[
\frac{\delta L}{\delta \varphi} \delta \varphi = \left[ \frac{\partial^2}{\partial x^2_{\mu}} \varphi(x) + m^2 \varphi(x) \right] \delta \varphi(x).
\] (20)
The functional derivative for operator fields thus here is form-identical to that for c-number fields. In other words, the usual results obtained by quantization after variation here are perfectly legitimate, i.e., they agree with those obtained by direct variation of quantum fields.

The case of the Dirac fields is even simpler in that no commuting of fields and varied fields is required. Thus we define in analogy to (1) and (2)

$$\delta \psi(x) = \varepsilon_n(x) \psi(x)$$
$$\delta \bar{\psi}(x) = \eta_n(x) \bar{\psi}(x)$$

where $\varepsilon_n$ and $\eta_n$ as in (6), (7) or (9), (10), to obtain

$$\frac{\delta L}{\delta \psi} \frac{\delta L}{\delta \bar{\psi}} = \bar{\psi}(x) (\gamma \partial_\mu + m) \psi(x)$$

Thus we define in analogy to (1) and (2)

$$\delta \bar{\psi}(x) = \eta_n(x) \bar{\psi}(x)$$

with $\varepsilon_n$ and $\eta_n$ as in (6), (7) or (9), (10), to obtain

$$\frac{\delta L}{\delta \psi} \frac{\delta L}{\delta \bar{\psi}} = \bar{\psi}(x) (-\gamma \partial_\mu + m) \psi(x)$$

Herewith we find that the variation leading to the Euler-Lagrange equations immediately leads to the same results for operator fields as for c-number fields.

The definitions (1) through (5) give an unambiguous meaning to any form one may encounter. Thus, the variation for the most general case, is

$$\delta L \{ \varphi, \partial_\mu \varphi, \pi, \ldots \} = L \{ (\delta \varphi), \partial_\mu (\delta \varphi), \pi, \ldots \} + L \{ \varphi, \partial_\mu (\delta \varphi), \pi, \ldots \}$$

$$+ L \{ \varphi, \partial_\mu \varphi, (\delta \pi), \ldots \} + \ldots .$$

One now can freely perform integrations by parts to free the variations from the derivative operators by the usual rules, i.e., as if the operator fields were c-number fields except that, of course, the order of factors must be maintained. For example,

$$\int d^4 x A (\partial_\mu \delta \varphi) B = - \int d^4 x (\partial_\mu A) \delta \varphi B - \int d^4 x A \delta \varphi ((\partial_\mu B)$$

which, of course, can be interpreted as

$$\int d^4 x \partial_\mu (A \delta \varphi B) = 0 .$$

The resulting expressions are thus precisely those one would obtain for a classical Lagrangian theory. Consequently, for example, Schwinger’s variational treatment of field operators (3) this way can be shown to be rigorously defined. Now the commutation relations, (5), can be used to re-write the expression (26a) if so desired; for example, to implement the normal ordering prescription, or to extricate the variant as in (14) or (15).
As for the interaction terms, they pose no problems as long as they do not contain derivatives, which, e.g., is the case for QED. The cases where they do contain derivatives must be individually investigated, along the lines given in this paper. We emphasize: herewith we have shown that the results derived for c-number fields can be derived in a rigorous manner also directly for operator fields.

4 Noether’s Theorem

Recalling the derivation of Noether’s theorem we will see that nothing beyond the validity of the functional derivative for operator fields is needed. In other words, this will show that all conservation rules derivable from Noether’s theorem are strictly valid both for c-number and for operator fields.

Noether’s theorem concerns the consequences of the symmetries of the Lagrangian. That means that if the Lagrangian is not changed as a consequence of some transformations, be it by a transformation of the coordinates or a transformation of the fields, there exist quantities which are constants of the motion, i.e., conserved quantities; they usually can be formulated as continuity equations. To derive these equations one first notes that the variation with respect to the parameter of the transformation, say \( \alpha \), vanishes:

\[
\frac{\delta L}{\delta \alpha} = 0 \quad (27)
\]

and then uses the Euler-Lagrange equations and some manipulations to cast the conditions (27) in a form of a 4-divergence, i.e., of a differential conservation law, a continuity equation. Here one must pay attention to the boundary conditions, i.e., the possible existence of surface terms, as we will see presently. We shall first derive Noether’s theorem for the case of a translation-invariant Lagrangian, and derive the energy – momentum conservation law. In this case the variation of the fields enters only in deriving the Euler – Lagrange equations of motion.

Take the Lagrangian \( \mathcal{L}(\varphi, \partial_\mu \varphi) \), and assume that for the replacement

\[
x_\mu \rightarrow x'_\mu = x_\mu + \delta x_\mu \quad (28)
\]

the Lagrangian is not changed. Then we have

\[
\int_{\Omega'} \mathcal{L}'(x') \, d^4x' = \int_{\Omega} \mathcal{L}(x) \, d^4x \quad . \quad (29)
\]

Hence, renaming the integration variable we obtain

\[
0 = \delta L = \int_{\Omega'} \mathcal{L}'(x') \, d^4x' - \int_{\Omega} \mathcal{L}(x) \, d^4x
\]
\[ \int_{\Omega'} L'(x) \, d^4x - \int_{\Omega} L(x) \, d^4x = 0. \quad (30) \]

We now add and subtract \( \int_{\Omega} L'(x) \, d^4x \):

\[ \delta L = \left( \int_{\Omega'} L'(x) \, d^4x + \int_{\Omega} [L'(x) - L(x)] \, d^4x \right). \quad (31) \]

We recognize that the first term of (31) together with (28) is simply a surface term. Herewith, up to first order in the variation (\( L' \rightarrow L \) in the surface term)

\[ \delta L = \int_{\Gamma} \delta L(x) \, d\sigma + \int_{\Omega} \delta L(x) \, d^4x. \quad (32) \]

Consider now the second term of (32). The variation \( \delta L \) here results only from the variation of the fields

\[ \delta \varphi(x) = \varphi'(x) - \varphi(x), \quad (33) \]

which is treated as in Section 2. Hence

\[ \delta L = \frac{\partial L}{\partial \varphi} \delta \varphi + \frac{\partial L}{\partial (\partial_\mu \varphi)} \partial_\mu \delta \varphi. \quad (34) \]

Recalling the Euler – Lagrange equation

\[ \frac{\partial L}{\partial \varphi} = \partial_\mu \frac{\partial L}{\partial (\partial_\mu \varphi)} \]

we obtain

\[ \delta L = \partial_\mu \left[ \frac{\partial L}{\partial (\partial_\mu \varphi)} \delta \varphi \right]. \quad (35) \]

As the last step we use the Gauss theorem

\[ \int_{\Sigma} f_\mu \, d\sigma_\mu = \int_{\Omega} \partial_\mu f_\mu \, d^4x = 0, \quad (36) \]

to convert the surface integral of (32) into a volume integral

\[ \delta L = 0 = \int_{\Omega} \partial_\mu \left( L(x) \, \delta x_\mu + \partial_\mu \frac{\partial L}{\partial (\partial_\mu \varphi)} \delta \varphi \right) \, d^4x. \quad (37) \]

which, owing to the arbitrariness of the variations yields

\[ \partial_\mu \left( L(x) \, \delta x_\mu + \partial_\mu \frac{\partial L}{\partial (\partial_\mu \varphi)} \delta \varphi \right) = - \partial_t P_0 + \nabla \tilde{P} = 0. \quad (38) \]

This is the promised differential form of the continuity equation. Indeed, no operations beyond functional derivation described in Section 2, and algebraic manipulations, are needed.
The integral form which gives directly the constants of the motion is achieved by integrating (38) over “all” 3-space, i.e., over that volume which contains the fields, and over the time coordinate between $t_1$ and $t_2$:

$$0 = \int_{t_1}^{t_2} dt \int d^3 x \, \partial_\mu P_\mu$$

$$= - \int dt \int d^3 x \left[ \partial_t \left( L(x) \delta t + \frac{\partial L}{\partial (\partial_\nu \phi)} \delta \phi \right) \right]$$

$$= - \left\{ \int d^3 x \left[ L(x) \delta t + \frac{\partial L}{\partial \phi} \delta \phi \right]_{t_2} + \int d^3 x \left[ L(x) \delta t + \frac{\partial L}{\partial \phi} \delta \phi \right]_{t_1} \right\} . \quad (39)$$

The space-like components $\vec{P}$ do not survive since the fields supposedly vanish at infinity (or, alternatively, the contribution from the boundaries cancel when using periodic boundary conditions). This way we have obtained the result that

$$Q = \int d^3 x \left[ L(x) \delta t + \frac{\partial L}{\partial \phi} \delta \phi \right] \quad (40)$$

is time-independent, i.e., is a conserved quantity, a constant of the motion. Again, all operations are fully defined.

We now specify to a “global” translation:

$$x'_\mu = x_\mu + \varepsilon_\mu ; \quad \delta x_\mu = \varepsilon_\mu \quad (41)$$

and

$$\phi'(x') = \phi(x) \quad (42)$$

which, for example, for a plane wave would read with (33), in first order

$$\phi'(x') = (1 - ik_\mu \varepsilon_\mu) \phi(x') \simeq e^{-ik_\mu \varepsilon_\mu} e^{ik_\mu x'_\mu} = e^{ikx} = \phi(x) .$$

Now we manipulate (42) as

$$0 = \phi'(x') - \phi(x) = \phi'(x') - \phi(x') + \phi(x') - \phi(x)$$

$$= \delta \phi(x') + \varepsilon_\nu \partial_\nu \phi(x) \quad (43)$$

which, inserted in (40) leads to

$$\vec{P}_0 = \int d^3 x \left[ -i L(x) \varepsilon_4 - \frac{\partial L}{\partial \phi} \varepsilon_\nu \partial_\nu \phi \right]$$

$$= \int d^3 x \varepsilon_\nu \left[ -i \delta_{\nu 4} L(x) - \frac{\partial L}{\partial \phi} \partial_\nu \phi \right] \quad (44)$$
as the conserved quantity. Returning to (38) we re-write it for our case using (43) as

\[ 0 = \partial_{\mu} \left( L(x) \varepsilon_{\mu} - \frac{\partial L}{\partial (\partial_{\mu} \varphi)} \varepsilon_{\nu} \partial_{\nu} \varphi \right) = \partial_{\mu} \left( L \delta_{\mu\nu} - \frac{\partial L}{\partial (\partial_{\mu} \varphi)} \partial_{\nu} \varphi \right) \varepsilon_{\nu} \]

\[ \equiv \partial_{\mu} T_{\mu\nu} \varepsilon_{\nu} . \]  

(45)

Owing to the arbitrariness of \( \varepsilon_{\nu} \) there must hold

\[ \partial_{\mu} T_{\mu\nu} = 0 \]  

(46)

with

\[ T_{\mu\nu} = L \delta_{\mu\nu} - \frac{\partial L}{\partial (\partial_{\mu} \varphi)} \partial_{\nu} \varphi \]  

(47)

and, comparing with (44)

\[ P_{\mu} = -i \int T_{4\mu} \, d^{3}x . \]  

(48)

We recapitulate: using the validity of (41) we have derived the conservation law (47), and, more specifically (44), needing no mathematical operations beyond the functional derivative and algebraic manipulations. If the symmetry of (41) is the only symmetry of the Lagrangian then the above conservation laws are the only ones guaranteed by the Lagrangian. Since, in order to be useful in the description of Nature, a theory must guarantee energy-momentum conservation, it suggests itself to identify \( P_{\mu} \), (48), with the energy – momentum four-vector, and \( T_{\mu\nu} \), (45), with the stress tensor. Once the identification of \( T_{\mu\nu} \) as the stress tensor has been accepted, it must be demanded to be valid in any and every theory, now in the precise form: the conservation law arising from the translation invariance (if it exists in the considered Lagrangian) concerns and yields the energy – momentum conservation law of the theory. And fully generally: if the Lagrangian has some symmetry leading to a conservation law as in (40), then, if a solution seems to violate that law, the calculation must contain an error.

Because of its importance we derive one more conservation law, which will provide an example of the analysis concerning “internal” symmetries.

We consider the Lagrangian

\[ L = -\bar{\psi} \left[ \gamma_{\mu} \left( \partial_{\mu} - i e A_{\mu} \right) + m \right] \psi + L(F_{\mu\nu}) \]

\[ = -\bar{\psi} \gamma_{\mu} \partial_{\mu} \psi + i e \bar{\psi} \gamma_{\mu} A_{\mu} \psi - \bar{\psi} \psi m + L(F_{\mu\nu}) . \]  

(49)

Here \( \psi(x) \) is a Dirac spinor field, and \( A(x) \) stands for the electromagnetic vector potential. Hence this Lagrangian is said to describe “spinor electrodynamics.”
Since $\psi$, in contrast to $\varphi$ above, is complex, and since in observables a change of the phase is irrelevant, (49) should be invariant under the transformation

$$\psi \rightarrow \psi' = e^{i \alpha} \psi \quad .$$  

(50a)

$$\bar{\psi} \rightarrow \bar{\psi}' = e^{-i \alpha} \bar{\psi} \quad .$$  

(50b)

We thus require

$$\delta L = 0 = -\int \left[ \delta \bar{\psi} \frac{\partial L}{\partial \bar{\psi}} + \frac{\partial L}{\partial \psi} \delta \psi + \partial_\mu (\delta \bar{\psi}) \frac{\partial L}{\partial (\partial_\mu \psi)} + \frac{\partial L}{\partial (\partial_\mu \psi)} \partial_\mu (\delta \psi) \right] + \frac{\partial L}{\partial A} \delta A + \frac{\partial L}{\partial (\partial_\mu A)} \partial_\mu (\delta A) \right] d^4 x \quad .$$  

(51)

Taking $\alpha$ to be infinitesimal we have from (50)

$$\frac{\delta \bar{\psi}}{\delta \alpha} = -i \bar{\psi}$$

$$\frac{\delta \psi}{\delta \alpha} = i \psi$$

$$\partial_\mu \delta \psi = (i \partial_\mu \alpha) \psi + i \alpha \partial_\mu \psi$$

$$\partial_\mu \frac{\delta \psi}{\delta \alpha} = i \frac{\delta \psi}{\delta \alpha}$$

and thus

$$\delta L = 0 = -\int \left\{ -i \alpha \bar{\psi} \left[ \gamma_\mu (\partial_\mu - ie A_\mu) + m \right] \psi + \bar{\psi} \gamma_\mu [(i \partial_\mu \alpha) \psi + i \alpha \partial_\mu \psi] \right\}$$

$$-i e \bar{\psi} \gamma_\mu \psi \delta A_\mu \right\} \quad .$$  

(53)

Since $\alpha$ is an arbitrary function of $x$, (53) imposes the condition

$$e \delta A_\mu = \partial_\mu \alpha$$  

(54)

for the Lagrangian (49) to be invariant under the transformation (50). Owing to the antisymmetry of $F_{\mu \nu}$ the last term of (49) does not yield a contribution.

We now apply the Euler-Lagrange equation

$$\frac{\delta L}{\delta \psi} = 0 \Rightarrow \frac{\partial L}{\partial \psi} = 0 \quad (55a)$$

$$\frac{\delta L}{\delta \psi} = 0 \Rightarrow \frac{\partial L}{\partial \psi} - \frac{\partial}{\partial \mu} \frac{\partial L}{\partial (\partial_\mu \psi)} = 0 \quad (55b)$$
to obtain
\[
\left( \frac{\partial L}{\partial \psi} \right) \delta \psi = \left( \frac{\partial}{\partial \mu} \frac{\partial L}{\partial (\partial_\mu \psi)} \right) \delta \psi .
\] (55c)

Inserting this in (51) we find
\[\delta L = 0 \Rightarrow \left[ \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi)} \right) \delta \psi + \frac{\partial L}{\partial (\partial_\mu \psi)} \partial_\mu \delta \psi \right] = 0\]

\[\Rightarrow i \frac{\partial}{\partial \mu} \left[ \frac{\partial L}{\partial (\partial_\mu \psi)} \psi \right] = 0\]

(56)

the last step again owing to the arbitrariness of \(\alpha\). This yields, in view of (49)
\[j_\mu = i \frac{\partial L}{\partial (\partial_\mu \psi)} \psi = i \bar{\psi} \gamma_\mu \psi\]

(57)
as expected. And, of course, the Lagrangian, being translation invariant, leads to a stress tensor analogous to (17).

Again, only the functional derivative and some algebraic manipulations are needed as mathematical operations in the above derivations.

5 Axial Currents

As the last example we investigate the question of axial current anomalies \([6],[7]\). As is well-known, they violate the Ward – Takahashi identities which arise directly as consequences of the Euler – Lagrange equations. Indeed, writing out the Euler – Lagrange equations (55a), (55b), which arise from the Lagrangian Eq. (49), we have:
\[0 = \frac{\partial}{\partial \psi} \left( -\bar{\psi} \left[ \gamma_\mu \left( \partial_\mu - i e A_\mu \right) + m \right] \psi + \mathcal{L}(F_{\mu\nu}) \right) \]
\[= \gamma_\mu \partial_\mu \psi + i e \gamma_\mu A_\mu \psi - \bar{\psi} m \]

(58a)

and
\[0 = \left( \frac{\partial}{\partial \bar{\psi}} - \frac{\partial}{\partial \mu} \frac{\partial}{\partial (\partial_\mu \psi)} \right) \left( -\bar{\psi} \left[ \gamma_\mu \left( \partial_\mu - i e A_\mu \right) + m \right] \psi + \mathcal{L}(F_{\mu\nu}) \right) \]
\[= -\partial_\mu \bar{\psi} \gamma_\mu + i e \bar{\psi} \gamma_\mu A_\mu - \bar{\psi} m \]

(58b)

Multiplying (58a) on the left by \(\bar{\psi} \gamma_5\) and (58b) on the right by \(\gamma_5 \psi\) and adding these equations we obtain
\[\partial_\mu \bar{\psi} \gamma_\mu \gamma_5 \psi = 2 i m \bar{\psi} \gamma_5 \psi\]

(59)
which is the basis of the usual, so-called “naive” axial-vector Ward–Takahashi identity. Again, after the variational derivative only strictly rigorous mathematical operations are needed in the derivation. For the discussion of the mathematical inaccuracy responsible for the anomalous breaking of the Ward–Takahashi identity Eq. (59) see ref [8].

This way, in all the above examples, no mathematically ill-defined, questionable operations are required in the derivations.

6 Conclusions

All the results obtained in the examples of this paper, from the definition of the variational calculus for quantum fields, up to the derivation of Noether’s theorem, were obtained without the use of any ill-defined mathematical steps or concepts. Thus there is no need to check whether the conservation laws obtained on the c-number level from Noether’s theorem “are consistent with the commutation relations” [2]. The previous gap in the mathematical logic has been closed since the variational calculus has been constructed precisely so as to be applicable directly to operator fields, i.e., to ensure that the commutation relations are maintained in the variational procedure. And, as we have shown, the results are those one would have obtained for c-number fields.

This way, all results obtained in the so-called “naive” manner, i.e., performing “quantization after variation”, remain valid; in particular, all “naive” Ward–Takahashi identities retain validity in QFT. Thus, Noether’s theorem is fully valid in quantum field theory.
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