Flocking in the Cucker-Smale model with self-delay and nonsymmetric interaction weights

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Abstract

We derive a sufficient condition for asymptotic flocking in the Cucker-Smale model with self-delay (also called reaction delay) and with non-symmetric interaction weights. The condition prescribes smallness of the delay length relative to the decay rate of the inter-agent communication weight. The proof is carried out by a bootstrapping argument combining a decay estimate for the group velocity diameter with a variant of the Gronwall-Halanay inequality.

Keywords: Cucker-Smale model, flocking, self-delay, nonsymmetric interaction weights.

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1 Introduction

In this paper we study the asymptotic behavior of the Cucker-Smale model [7, 8] with self-delay, also called reaction-type delay in the previous works [13, 14]. The Cucker-Smale flocking model is a prototypical model of collective behavior [17, 25], describing the dynamics of a group of $N \in \mathbb{N}$ agents, characterized by their positions $x_i \in \mathbb{R}^d$ and velocities $v_i \in \mathbb{R}^d$, $i \in \{1, 2, \ldots, N\}$, with $d \geq 1$. The agents align their velocities to the average velocity of their conspecifics. Motivated by applications in biology and social sciences [1, 3, 22] or engineering problems (for instance, swarm robotics [11, 23, 24]), we introduce a fixed time span $\tau > 0$ for the agents to process the information received from their surroundings and take appropriate action. This leads to the system of delay (functional) differential equations [22],

\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= \sum_{j=1}^{N} \psi_{ij}(t-\tau)(v_j(t-\tau) - v_i(t-\tau)),
\end{align*}

for $i \in [N]$, where here and in the sequel we denote $[N] := \{1, 2, \ldots, N\}$. The system is equipped with the initial datum

\begin{equation}
\begin{aligned}
x_i(t) &= x_i(0), & v_i(t) &= v_i(0), & i \in [N], & t \in [-\tau, 0],
\end{aligned}
\end{equation}

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with prescribed continuous spatial and velocity trajectories \((x_i^0, v_i^0) \in C([−τ, 0]), i = 1, \ldots, N\). We note that we do not require (11) to hold for the initial datum.

The communication weights \(ψ_{ij}\) in (2) measure the intensity of the influence between agents depending on their physical (Euclidean) distance \(|x_i − x_j|\). In the classical setting [7,8] the communication weights are given by

\[
ψ_{ij}(t) := \frac{1}{N} \psi(|x_j(t) − x_i(t))|,\tag{4}
\]

with the nonnegative, bounded and continuous influence function \(ψ : [0, ∞) → [0, ∞)\). We adopt the assumption that \(ψ(s) ≤ 1\) for all \(s ≥ 0\). This, in fact, can always be achieved by an eventual rescaling of time, and, therefore, is without loss of generality.

Another form of the communication weights was introduced in [10], where the scaling by \(1/N\) is replaced by a normalization relative to the influence of all other agents,

\[
ψ_{ij}(t) := \frac{ψ(|x_j(t) − x_i(t))|}{\sum_{ℓ=1}^{N} ψ(|x_ℓ(t) − x_i(t))|}.	ag{5}
\]

Again, the influence function \(ψ\) is assumed to be nonnegative and continuous, and verify \(ψ ≤ 1\) globally. Note that the normalization in (5) leads to nonsymmetric weights, i.e., in general, \(ψ_{ij} \neq ψ_{ji}\).

A generic choice for \(ψ\), introduced in [7,8], is \(ψ(s) = \frac{1}{(1+s^β)^p}\) with \(β > 0\). However, we do not restrict ourselves to this particular form in this paper. Moreover, let us stress that we do not impose any symmetry assumptions on the communication weights \(ψ_{ij}\), i.e., we admit \(ψ_{ij} \neq ψ_{ji}\) for all \(i, j ∈ [N]\).

As customary in the context of the Cucker-Smale system [7,8], we define (asymptotic) flocking for solutions of (11)–(2) as the property

\[
\sup_{t ≥ 0} d_x(t) < ∞, \quad \lim_{t → ∞} d_v(t) = 0,\tag{6}
\]

where the position and, resp., velocity diameters of the agent group \(d_x = d_x(t)\) and, resp., \(d_v = d_v(t)\) are defined as

\[
d_x(t) := \max_{i,j ∈ [N]} |x_i(t) − x_j(t)|, \quad d_v(t) := \max_{i,j ∈ [N]} |v_i(t) − v_j(t)|.\tag{7}
\]

Several previous works focused on derivation of sufficient conditions for flocking in the Cucker-Smale system with delay. The papers [2, 4, 5, 6, 15, 18, 19, 20] focus on variants of the model without self-delay (also called propagation- or communication-type delay), where \(v_i\) in (2) is evaluated at time \(t + τ\) instead of \(t + τ\). This leads to qualitatively different dynamics compared to the system (11)–(2). In particular, one has an a-priori bound on the velocity radius \(R_v(t) := \max_{i ∈ [N]} |v_i(t)|\) in terms of the initial datum, independently of the delay length \(τ > 0\). In contrast, the model (11)–(2) with self-delay exhibits, for large enough values of \(τ > 0\), oscillation of the velocities \(v_i = v_i(t)\) with increasing, unbounded amplitude (see Remark 1 below). In other words, the presence of self-delay fundamentally destabilizes the dynamics of the system and flocking can only be expected for small enough values of \(τ\).

The Cucker-Smale model with self-delay was studied in [13, 14], but under the assumption of symmetric communication weights \(ψ_{ij}\). The symmetry leads to conservation of the total momentum \(\sum_{i=1}^{N} v_i\), and the analysis carried out in [13, 14] is utilizes this fact in a
fundamental way. Finally, [9] studies a variant of the model with self-delay and multiplicative noise. However, the authors impose the assumption of a-priori uniform positivity of the communication weights $\psi_{ij}$, which goes against the substance of the Cucker-Smale model (its analysis is interesting precisely for the fact that the communication weight vanishes at infinity).

The main novelty of this paper is that it provides a sufficient condition for flocking in the Cucker-Smale system with self-delay, without the assumption of symmetric communication weights. To our best knowledge, no such result exists in the literature. Our flocking condition is formulated in terms of the delay length $\tau$, the influence function $\psi$ and the position and velocity diameters of the initial datum. It is not explicit, however, can be easily inspected numerically. It will be presented in Section 2 below.

The main difficulties for the analysis stem from the following two properties of system (1)–(2): non-conservation of the total momentum $\sum_{i=1}^{N} v_i$ due to the possible nonsymmetry of the communication weights, so that the asymptotic flocking vector is not known a-priori; and, the instability (possible occurrence of unbounded oscillations) induced by the presence of the self-delay, so that the velocities cannot be bounded a-priori. These difficulties are overcome by a bootstrapping argument combining a decay estimate for the group velocity diameter with a variant of the Gronwall-Halanay inequality, and will be given in Section 3.

2 Sufficient condition for asymptotic flocking

For the case the influence function $\psi$ was not monotone, let us define its nonincreasing rearrangement

$$\Psi(u) := \min_{s \in [0, u]} \psi(s) \quad \text{for } u \geq 0. \quad (8)$$

Moreover, let us denote the initial spatial and velocity diameters

$$d_0^x := \max_{t \in [-\tau, 0]} d_x(t), \quad d_0^v := \max_{t \in [-\tau, 0]} d_v(t), \quad (9)$$

with $d_x$ and $d_v$ defined in (7).

**Theorem 1.** Let the communication weights $\psi_{ij}$ be given by (4) or (5) with a nonnegative, bounded and continuous influence function $\psi \leq 1$. Assume that there exists $C \in (0, 1)$ such that

$$\Psi \left( d_0^x + (1 + 2\tau) \left( \tau + \frac{1}{C} \right) d_0^v \right) - C \geq 4\tau e^{C\tau} e^{C\tau - 1}, \quad (10)$$

with $\Psi$ defined in (8) and $d_0^x$, $d_0^v$ given by (9).

Then the system (1)–(3) exhibits flocking in the sense of definition (6). Moreover, the decay of the velocity diameter is exponential with rate $C$,

$$d_v(t) \leq (1 + 2\tau) d_0^v e^{-C(t-\tau)} \quad \text{for all } t \geq \tau. \quad (11)$$

Condition (10) is highly nonlinear and, obviously, not verifiable analytically, apart from trivial cases like $\Psi \equiv 1$. However, observe that the right-hand side in (10) is, for a fixed $C > 0$, an increasing function of $\tau$, and vanishing for $\tau \to 0+$. On the other hand, the left-hand side is, for fixed $d_0^x$ and $d_0^v$, decreasing in $\tau$, and strictly positive for $\tau \to 0+$ if
\( C > 0 \) is small enough and \( \Psi \) is globally positive (but does not need to be uniformly bounded away from zero). Consequently, \((10)\) is to be interpreted, for fixed \( d_0^0 \) and \( d_0^0 \), as a smallness condition for \( \tau \), relative to the decay rate of \( \Psi \). Clearly, since \( \Psi \leq 1 \) by assumption, the necessary condition for \((10)\) to be verified is \( \tau < 1/4 \). In the special case \( \Psi \equiv 1 \), \((10)\) is equivalent to \( \tau < 1/4 \).

A slightly simpler version of \((10)\) is obtained for the case when the initial velocities \( v_0^i \) are all constant on \([-\tau, 0]\). Then, by obvious modifications of the steps carried out in Section 3, one obtains the following simplified version of Theorem 4:

**Theorem 2.** Let the initial velocities \( v_0^i \) be all constant on \([-\tau, 0]\). Let the assumptions of Theorem 1 be verified, with \((10)\) replaced by

\[
\Psi \left( \frac{d_0^0 + d_0^0}{C} \right) - C \geq 4\tau e^{C\tau} e^{C\tau} - \frac{1}{C\tau}.
\]

Then the system \((1) - (3)\) exhibits flocking in the sense of definition \((6)\). Moreover, the decay of the velocity diameter is exponential with rate \( C \),

\[
d_v(t) \leq d_0^0 e^{-C\tau} \quad \text{for all } t \geq 0.
\]

**Remark 1.** Before we proceed with the proof of Theorem 1 let us give a short explanation why we cannot expect flocking to take place in \((1) - (2)\) for arbitrary delay lengths \( \tau > 0 \), even for small initial data. Indeed, considering the simple case of two agents, \( N = 2 \), in one spatial dimension \( d = 1 \), with \( \psi_{12} \equiv \psi_{21} \equiv 1 \), the system \((2)\) reduces to

\[
\dot{w}(t) = -2w(t - \tau)
\]

for \( w := v_1 - v_2 \). Nontrivial solutions of this equation exhibit oscillations whenever \( 2\tau > e^{-1} \) and their amplitude diverges in time if \( 2\tau > \pi/2 \), see, e.g., [22]. In other words, the system never reaches flocking if \( 2\tau > \pi/2 \), apart from the trivial case \( w \equiv 0 \).

### 3 Proof of Theorem 1

Let us start by making two simple observations about the communication weights \( \psi_{ij} \). First, due to the assumption \( \psi \leq 1 \), we have for both the classical \((4)\) and normalized \((5)\) weights the upper bound

\[
\sum_{i=1}^{N} \psi_{ij}(t) \leq 1 \quad \text{for all } i \in [N] \text{ and } t \geq 0. \tag{12}
\]

In fact, for the normalized weights \((5)\) the above holds even with equality, but we do not make use of this property in our proof. Second, due to the assumed continuity of \( \psi \), we have the lower bound

\[
\psi_{ij}(t) \geq \frac{\Psi(d_x(t))}{N} \quad \text{for all } t \geq 0 \text{ and all } i, j \in [N], \tag{13}
\]

with \( \Psi \) given by \((8)\). Indeed, for \((4)\) we have

\[
\psi_{ij}(t) = \frac{1}{N} \psi(|x_i(t) - x_j(t)|) \geq \frac{1}{N} \Psi(|x_i(t) - x_j(t)|) \geq \frac{\Psi(d_x(t))}{N}.
\]
For the same follows due to the assumption ψ ≤ 1,

\[ \psi_{ij}(t) \geq \frac{\psi(|x_j(t) - x_i(t)|)}{N} \geq \frac{\Psi(d_x(t))}{N}. \]

We first prove a result on the decay of the velocity diameter \( d_v = d_v(t) \).

\[ \textbf{Lemma 1.} \text{ Let the communication weights } \psi_{ij} \text{ satisfy (14) and for a fixed } T > \tau \text{ denote } \psi := N \min_{t \in [\tau, T]} \min_{i \neq j \in [N]} \psi_{ij}(t - \tau). \]

Then, along the solutions of (1–2),

\[ \frac{d}{dt} d_v(t) \leq 4 \int_{t-\tau}^{t} d_v(s - \tau) ds - \bar{\psi} d_v(t) \quad (15) \]

for almost all \( t \in (\tau, T) \).

\[ \textbf{Proof.} \text{ For the sake of legibility, in this section we shall use the shorthand notation } \tilde{v}_j := v_j(t - \tau), \text{ while } v_j \text{ stands for } v_j(t). \]

Due to the continuity of the velocity trajectories \( v_i = v_i(t) \), there is an at most countable system of open, mutually disjoint intervals \( \{ \mathcal{I}_\sigma \}_{\sigma \in \mathbb{N}} \) such that

\[ \bigcup_{\sigma \in \mathbb{N}} \mathcal{I}_\sigma = [\tau, \infty) \]

and for each \( \sigma \in \mathbb{N} \) there exist indices \( i(\sigma), k(\sigma) \) such that

\[ d_v(t) = |v_{i(\sigma)}(t) - v_{k(\sigma)}(t)| \quad \text{for } t \in \mathcal{I}_\sigma. \]

Then, using the abbreviated notation \( i := i(\sigma), k := k(\sigma) \), we have for every \( t \in \mathcal{I}_\sigma \),

\[ \frac{1}{2} \frac{d}{dt} d_v(t)^2 = (\dot{v}_i - \dot{v}_k) \cdot (v_i - v_k) \]

\[ = \left( \sum_{j=1}^{N} \psi_{ij}(t)(\tilde{v}_j - \tilde{v}_i) - \sum_{j=1}^{N} \psi_{kj}(t)(\tilde{v}_j - \tilde{v}_k) \right) \cdot (v_i - v_k). \quad (16) \]

We process the first term of the right-hand side as follows,

\[ \sum_{j=1}^{N} \psi_{ij}(t)(\tilde{v}_j - \tilde{v}_i) \cdot (v_i - v_k) = \sum_{j=1}^{N} \psi_{ij}(t)(\tilde{v}_j - v_j + v_i - \tilde{v}_i) \cdot (v_i - v_k) \]

\[ + \sum_{j=1}^{N} \psi_{ij}(t)(v_j - v_i) \cdot (v_i - v_k). \quad (17) \]
Noting that \( t \geq \tau \), we estimate the difference \( |\tilde{v}_j - v_j| \) by
\[
|\tilde{v}_j - v_j| \leq \int_{t-\tau}^{t} |\dot{\tilde{v}}_j(s)| \, ds
\leq \int_{t-\tau}^{t} \sum_{\ell=1}^{N} \psi_{j\ell}(s) |v_{\ell}(s-\tau) - v_j(s-\tau)| \, ds
\leq \int_{t-\tau}^{t} \sum_{\ell=1}^{N} \psi_{j\ell}(s) \, dv(s-\tau) \, ds
\leq \int_{t-\tau}^{t} dv(s-\tau) \, ds,
\]
where for the last equality we used the property (12).

Performing an analogous estimate for the term \( |\tilde{v}_i - v_i| \) and using the Cauchy-Schwarz inequality and, again, (12), we arrive at
\[
N \sum_{j=1}^{N} \psi_{ij}(t) (|\tilde{v}_j - v_j| + |\tilde{v}_i - v_i|) |v_i - v_k| \leq 2 d_v(t) \int_{t-\tau}^{t} dv(s-\tau) \, ds.
\]

To estimate the second term of the right-hand side of (17), observe that, using the Cauchy-Schwarz inequality, we have
\[
(v_j - v_i) \cdot (v_i - v_k) = (v_j - v_k) \cdot (v_i - v_k) - |v_i - v_k|^2
\leq |v_i - v_k| \left( |v_j - v_k| - |v_i - v_k| \right) \leq 0,
\]
since, by definition, \( |v_j - v_k| \leq d_v = |v_i - v_k| \). Then, with (14), we have
\[
N \sum_{j=1}^{N} \psi_{ij}(t) (v_j - v_i) \cdot (v_i - v_k) \leq \frac{\psi}{N} \sum_{j=1}^{N} (v_j - v_i) \cdot (v_i - v_k).
\]

Repeating the same steps for the second term of the right-hand side of (16), we finally arrive at
\[
\frac{1}{2} \frac{d}{dt} d_v(t)^2 \leq 4 d_v(t) \int_{t-\tau}^{t} dv(s-\tau) \, ds + \frac{\psi}{N} \left( \sum_{j=1}^{N} (v_j - v_i) \cdot (v_i - v_k) - \sum_{j=1}^{N} (v_j - v_k) \cdot (v_i - v_k) \right)
= 4 d_v(t) \int_{t-\tau}^{t} dv(s-\tau) \, ds - |v_i - v_k|^2
\leq 4 d_v(t) \int_{t-\tau}^{t} dv(s-\tau) \, ds - \psi d_v(t)^2,
\]
from which (15) directly follows, for almost all \( t \in (\tau, T) \).

The proof of Theorem 1 shall be based on the decay estimate of Lemma 11 combined with the following variant of the Gronwall-Halanay lemma 10.
Lemma 2. Fix \( \tau > 0 \) and let \( u \in C([-\tau, \infty)) \) be a nonnegative continuous function with piecewise continuous derivative on \((\tau, \infty)\), such that for almost all \( t > \tau \) the integro-differential inequality is satisfied,

\[
\frac{d}{dt}u(t) \leq \frac{\alpha}{\tau} \int_{t-\tau}^{t} u(s-\tau) \, ds - \beta u(t),
\]

with constants \( 0 < \alpha < \beta \). Then there exists a unique \( \gamma \in (0, \beta - \alpha) \) such that

\[
\beta - \gamma = \alpha e^{\gamma \tau} \frac{e^{\gamma \tau} - 1}{\gamma \tau},
\]

and the estimate holds

\[
u(t) \leq \left( \max_{s \in [-\tau, \tau]} u(s) \right) e^{-\gamma(t-\tau)} \quad \text{for all } t \geq \tau.
\]

Proof. The proof is obtained as a slight generalization of [4, Lemma 2.5] and [12, Lemma 3.3].

Lemma 3. Along the solutions of (11)–(12), we have

\[
\max_{s \in [-\tau, \tau]} d_\nu(s) \leq (1 + 2\tau)d_\nu^0,
\]

with \( d_\nu^0 \) defined in (9).

Proof. From (24) we have for all \( t \in (0, \tau) \) and \( i \in [N] \),

\[
|v_i(t)| \leq \sum_{j=1}^{N} \psi_{ij}(t-\tau) |v_j(t-\tau) - v_i(t-\tau)| \leq \sum_{j=1}^{N} \psi_{ij}(t-\tau) d_\nu(t-\tau) \leq d_\nu^0,
\]

where the last inequality follows from (12). Therefore, still for \( t \in (0, \tau) \),

\[
|v_i(t) - v_j(t)| \leq |v_i(0) - v_j(0)| + \int_{0}^{t} (|v_i(s)| + |v_j(s)|) \, ds
\]

\[
\leq d_\nu(0) + 2\tau d_\nu^0 \leq (1 + 2\tau)d_\nu^0,
\]

and taking a maximum over \( i, j \in [N] \) yields (21).
By (11) we readily have
\[ d_x(t - \tau) \leq d_x^0 + \int_0^{t-\tau} d_v(s)ds. \] (24)

The bounds (21) and (22) imply for all \( t \in (\tau, T) \),
\[ \int_0^{t-\tau} d_v(s)ds = \int_0^{\min\{\tau, t-\tau\}} d_v(s)ds + \int_{\min\{\tau, t-\tau\}}^{t-\tau} d_v(s)ds \leq (1 + 2\tau) \min\{\tau, t-\tau\} d_v^0 + \frac{(1 + 2\tau)d_v^0}{C} \leq (1 + 2\tau) \left( \tau + \frac{1}{C} \right) d_v^0. \]

Consequently,
\[ d_x(t - \tau) \leq d_x^0 + (1 + 2\tau) \left( \tau + \frac{1}{C} \right) d_v^0. \] (25)

Then, using (25) in (13) and recalling that, by definition, \( \Psi \) is a nonincreasing function, gives
\[ \psi_{ij}(t - \tau) \geq \frac{1}{N} \Psi \left( d_x^0 + (1 + 2\tau) \left( \tau + \frac{1}{C} \right) d_v^0 \right) \] (26)
for all \( i, j \in [N] \) and \( t \in (\tau, T) \). Therefore, denoting \( \Psi_C := \Psi \left( d_x^0 + (1 + 2\tau) \left( \tau + \frac{1}{C} \right) d_v^0 \right) \), we have
\[ N \min_{t \in [\tau, T]} \min_{i \neq j \in [N]} \psi_{ij}(t - \tau) \geq \Psi_C \]
and Lemma 1 gives
\[ \frac{d}{dt} d_v(t) \leq 4 \int_{t-\tau}^t d_v(s - \tau)ds - \Psi_C d_v(t) \quad \text{for} \quad t \in (\tau, T). \]

Noting that assumption (10) with \( C \in (0, 1) \) implies \( \Psi_C > 4\tau \), we apply Lemma 2 with \( \alpha := 4\tau \) and \( \beta := \Psi_C \). This leads to
\[ d_v(t) \leq \left( \max_{s \in [-\tau, \tau]} d_v(s) \right) e^{-\gamma(t-\tau)} \leq (1 + 2\tau)d_v^0 e^{-\gamma(t-\tau)} \quad \text{for} \quad t \in [\tau, T], \] (27)
with \( \gamma \in (0, \Psi_C - 4\tau) \) the unique solution of
\[ \Psi_C - \gamma = 4\tau e^{\gamma \tau} \frac{e^{\gamma \tau} - 1}{\gamma \tau}. \]

Comparing with (10), we have
\[ 4\tau e^{C\tau} \frac{e^{C\tau} - 1}{C\tau} + C \leq 4\tau e^{\gamma \tau} \frac{e^{\gamma \tau} - 1}{\gamma \tau} + \gamma, \]
and the monotonicity of the above expression implies \( C \leq \gamma \). But then (27) gives
\[
\int_T^\tau d_v(t) dt \leq (1 + 2\tau) d_v^0 \int_T^\tau e^{-C(t-\tau)} dt = \frac{(1 + 2\tau)d_v^0}{C} \left( 1 - e^{-C(T-\tau)} \right) < \frac{(1 + 2\tau)d_v^0}{C},
\]
which is a contradiction to (23). Thus, we conclude that \( T = \infty \), i.e., that (22) holds for all \( t > 0 \). Consequently, by (24) we have
\[
\sup_{t \geq 0} d_x(t) \leq d_x^0 + \int_0^\infty d_v(s) ds < d_x^0 + (1 + 2\tau) \left( \tau + \frac{1}{C} \right) d_v^0,
\]
and so the first condition in the definition 6 of flocking is verified. Moreover, since (26) holds for all \( t \geq 0 \), Lemma 4 with \( \psi := \psi_C \) can be applied globally, and a subsequent application of Lemma 2 gives the exponential decay of \( d_v = d_v(t) \) as claimed by (11).

Remark 2. In fact, for our analysis we do not need to restrict the form of the interaction weights \( \psi_{ij} \) to either (4) or (5). Instead, the proof of Theorem 1 is only based on the lower (12) and upper (13) bounds on \( \psi_{ij} \). Consequently, its statement remains valid for any form of \( \psi_{ij} \), as long as they verify (12) and (13).

References

[1] S. Camazine, J. L. Deneubourg, N.R. Franks, J. Sneyd, G. Theraulaz and E. Bonabeau: Self-Organization in Biological Systems. Princeton University Press, Princeton, NJ, 2001.

[2] M. R. Cartabia: The Cucker-Smale model with time delay. arxiv.org/abs/2008.09530 (2020).

[3] C. Castellano, S. Fortunato and V. Loreto: Statistical physics of social dynamics. Rev. Mod. Phys., 81, (2009), 591–646.

[4] Y.-P. Choi, J. Haskovec: Cucker-Smale model with normalized communication weights and time delay. Kinetic and Related Models 10 (2017), 1011-1033.

[5] Y.-P. Choi, J. Haskovec: Hydrodynamic Cucker-Smale model with normalized communication weights and time delay. SIAM J. Math. Anal., Vol. 51, No. 3 (2019), 2660–2685.

[6] Y.-P. Choi and C. Pignotti: Emergent behavior of Cucker-Smale model with normalized weights and distributed time delays. Networks and Heterogeneous Media, Vol. 14 (2019), pp. 789–804.

[7] F. Cucker and S. Smale, Emergent behaviour in flocks, IEEE T. on Automat. Contr., 52 (2007), 852–862.

[8] F. Cucker and S. Smale, On the mathematics of emergence, Jap. J. Math., 2 (2007), 197–227.

[9] R. Erban, J. Haskovec and Y. Sun, A Cucker-Smale model with noise and delay, SIAM J. Appl. Math., 76 (2016), 1535–1557.
[10] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*. Academic Press, New York London, 1966.

[11] H. Hanman: *Swarm Robotics: A Formal Approach*. Springer, 2018.

[12] J. Haskovec, A simple proof of asymptotic consensus in the Hegselmann-Krause and Cucker-Smale models with normalization and delay. SIAM J. on Applied Dynamical Systems, 20:1 (2021), pp. 130–148.

[13] J. Haskovec and I. Markou, Asymptotic flocking in the Cucker-Smale model with reaction-type delays in the non-oscillatory regime. Kinetic and Related Models 13 (2020), 795–813.

[14] J. Haskovec and I. Markou, Exponential asymptotic flocking in the Cucker-Smale model with distributed reaction delays. Mathematical Biosciences and Engineering 17:5 (2020), pp. 5651–5671.

[15] Y. Liu, and J. Wu, Flocking and asymptotic velocity of the Cucker-Smale model with processing delay, *J. Math. Anal. Appl.*, 415 (2014), 53–61.

[16] S. Motsch and E. Tadmor: A New Model for Self-organized Dynamics and Its Flocking Behavior. J. Stat. Phys. 144 (2011).

[17] G. Naldi, L. Pareschi and G. Toscani (eds.): *Mathematical Modeling of Collective behavour in Socio-Economic and Life Sciences*, Series: Modelling and Simulation in Science and Technology, Birkhäuser, 2010.

[18] C. Pignotti and I. Reche Vallejo: Asymptotic analysis of a Cucker-Smale system with leadership and distributed delay. In: Trends in Control Theory and Partial Differential Equations, Springer Indam Series, Vol. 32 (2019), pp. 233–253.

[19] C. Pignotti and I. Reche Vallejo: Flocking estimates for the Cucker-Smale model with time lag and hierarchical leadership. Journal of Mathematical Analysis and Applications, Vol. 464 (2018), pp. 1313–1332.

[20] C. Pignotti and E. Trelat: Convergence to consensus of the general finite-dimensional Cucker-Smale model with time-varying delays. Comm. Math. Sci. 16 (2018), 2053–2076.

[21] A. Seuret, V. Dimos, V. Dimarogonas and K.H. Johansson: Consensus under Communication Delays. Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, Dec. 9-11, 2008.

[22] H. Smith: *An Introduction to Delay Differential Equations with Applications to the Life Sciences*. Springer New York Dordrecht Heidelberg London, 2011.

[23] C. Somarakis and J. Baras: Delay-independent convergence for linear consensus networks with applications to non-linear flocking systems. In Proceedings of the 12th IFAC Workshop on Time Delay Systems, pp. 159–164, Ann Arbor (2015).

[24] K. Szwaykowska, I.B. Schwartz, L.M. Romero, C.R. Heckman, D. Mox and M. Ani Hsieh: Collective motion patterns of swarms with delay coupling: theory and experiment. Phys. Rev. E 93, 032307.

[25] T. Vicsek and A. Zafeiris: *Collective motion*. Phys. Rep., 517 (2012), 71–140.