Intrinsic expansions for averaged diffusion processes

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Abstract

We show that the convergence rate of asymptotic expansions for solutions of SDEs is higher in the case of degenerate diffusion compared to the elliptic case, i.e. it is higher when the Brownian motion directly acts only along some directions. In the scalar case, this phenomenon was already observed in \cite{[19]} using Malliavin calculus techniques. Here, we provide a general and detailed analysis by employing the recent study of intrinsic functional spaces related to hypoelliptic Kolmogorov operators in \cite{[36]}. Applications to finance are discussed, in the study of path-dependent derivatives (e.g. Asian options) and in models incorporating dependence on past information.

\textbf{Keywords:} averaged diffusion, hypoelliptic Kolmogorov operators, asymptotic expansion, Asian option

1. Introduction

We study the asymptotic expansion of the conditional expectation

\[ u(t,x) := \mathbb{E}_{t,x}[\varphi(X_T)], \quad (1.1) \]

where \( X = (X_t)_{t \in [0,T]} \) is a continuous \( \mathbb{R}^d \)-valued Feller process and a degenerate diffusion in a sense that will be specified later.

The prototype process we have in mind is \( X = (S,A) \) solution to the SDE

\[
\begin{align*}
\text{d}S_t &= \sigma S_t \text{d}W_t, \\
\text{d}A_t &= S_t \text{d}t,
\end{align*}
\]

where \( W \) is a real Brownian motion. In financial applications, \( S \) and \( A \) represent the price and average processes respectively, in the Black\&Scholes model for arithmetic Asian options. The infinitesimal generator of \( (S,A) \)

\[ A_X := \frac{\sigma^2 s^2}{2} \partial_{ss} + s \partial_a, \quad (s,a) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, \]

is degenerate in two ways: on the one hand, the quadratic form of the second order part is singular (it has rank one) and, on the other hand, it degenerates completely on the half-line \( \{s = 0, a > 0\} \). However, for any \( 0 < a < b \), \( A_X \) is a hypoelliptic operator on the strip \( D := [a,b] \times \mathbb{R}_{>0} \) and coincides on \( D \) with

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an operator that satisfies the Hörmander condition globally, the latter obtained by smoothly perturbing the second order coefficient $\sigma^2 s^2$ outside $D$. By performing a local analysis, we aim at exploiting this fact to prove error estimates, uniform w.r.t. $x = (s,a) \in D$, for the intrinsic asymptotic expansions of the conditional expectation in (1.1).

In general, we assume that the infinitesimal generator of $X$ coincides, on a domain $D$ of $\mathbb{R}^d$, with a differential operator of the form

$$A = \frac{1}{2} \sum_{i,j=1}^{p_0} a_{ij}(t,x) \partial_{x_i x_j} + \sum_{i=1}^{p_0} a_i(t,x) \partial_{x_i} + \langle Bx, \nabla x \rangle, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad (1.3)$$

where $p_0 \leq d$ and $A$ verifies the following

**Assumption 1.1.** $A_0 := (a_{ij}(t,x))_{i,j=1,\ldots,p_0}$ satisfies the non-degeneracy condition

$$\mu M |\xi|^2 \leq \sum_{i,j=1}^{p_0} a_{ij}(t,x) \xi_i \xi_j < M |\xi|^2, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad \xi \in \mathbb{R}^{p_0}, \quad (1.4)$$

for some positive constants $M$ and $\mu$;

**Assumption 1.2.** $B$ is a $(d \times d)$-matrix with constant entries satisfying the following structural condition

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_r & 0 \end{pmatrix} \quad (1.5)$$

where each $B_j$ is a $(p_j \times p_{j-1})$-matrix of rank $p_j$ and

$$p_0 \geq p_1 \geq \cdots \geq p_r \geq 1, \quad \sum_{j=0}^{r} p_j = d.$$

Assumption 1.2 implies that vector fields $\partial_{x_1}, \ldots, \partial_{x_{p_0}}$ and

$$Y := \langle Bx, \nabla x \rangle + \partial_t \quad (1.6)$$

satisfy the Hörmander condition (cf. [26]). Under suitable regularity conditions that will be specified later, the ultra-parabolic operator

$$\mathcal{K} := A + \partial_t \quad (1.7)$$

admits a fundamental solution (see [37] and [8]). In the case $p_0 < d$, which is the focus of this work, this is a remarkable fact as the second order part of $A$ is fully degenerate at any point. Operators $\mathcal{K}$ of this kind are often referred to as Kolmogorov operators.

Our analysis takes advantage of the intrinsic geometry and the related regularity structures induced by the Kolmogorov operator $\mathcal{K}$. These features bring a number of benefits that are explained here below, and distinguish our approach from others in the literature. It is worth to emphasize further that our results are carried out under strictly local assumptions on the generator of $X$, which coincides with a Kolmogorov operator on a domain $D$, not necessarily equal to $\mathbb{R}^d$. This allows to include degenerate models with relevant

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financial applications, such as the well-known CEV model (that is when \( \sigma \) in (1.2) is not a constant but a function of \( S \) of the form \( \sigma(S) = S^\gamma \) for some \( \gamma \in \mathbb{R} \) and the Heston stochastic volatility model as very particular cases. The proof of our main result, Theorem 3.5, will be split in two separate steps: first, in Theorem 3.8, we consider the case \( D = \mathbb{R}^d \) for which we employ some Gaussian upper bounds for the transition density of \( X \); second, we adapt a localization procedure, originally introduced in \( [39] \) and lately extended in \( [3] \), which is based on the Gaussian bounds for a dummy diffusion \( \tilde{X} \) that is generated by \( A \) in (1.3). The latter localization procedure is coherent with what is known in the theory of diffusions as the principle of not feeling the boundary (cf. \( [22, 17] \)).

1.1. Intrinsic Taylor-based asymptotic expansions

Intrinsic Hölder and Sobolev spaces for Kolmogorov operators were studied by several authors, among others \( [9, 2, 30, 29, 33, 31] \). In this paper we use the intrinsic Hölder spaces \( C_B^{n,\alpha} \) in Definition 2.3 below, as defined in \( [36] \) where the authors also proved a Taylor formula with remainder expressed in terms of the homogeneous norm induced by the operator (see Theorem 2.3 below). Deferring precise definitions and statements until Section 2, the \( n \)-th order intrinsic Taylor polynomial, centered at \( \zeta = (s, \xi) \in \mathbb{R} \times \mathbb{R}^d \), of a function \( f \in C_B^{n,\alpha} \) reads as

\[
T_n(f, \zeta)(z) := \sum_{2k+j|\beta|_B \leq n} \frac{1}{k! \beta!} (Y^k \partial_\xi^j f(s, \xi))(t-s)^k (x-e^{(t-s)B} \xi)^\beta, \quad z = (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (1.8)
\]

where \( |\beta|_B \), given in (2.4), is a suitable weight for the multi-index \( \beta \in \mathbb{N}_0^d \). Such Taylor expansion forms the cornerstone of the perturbation technique that we study in this paper. Here below we summarize the intuitive idea behind it and its primary features.

We recall that, under mild assumptions that will be specified in Section 3, the function \( u \) in (1.1) satisfies

\[
\begin{aligned}
\mathcal{K}u &= 0, \quad \text{on } [0, T] \times D, \\
u(T, \cdot) &= \varphi, \quad \text{on } D.
\end{aligned}
\]

(1.9)

Notice that (1.9) is not a standard Cauchy-Dirichlet problem since no lateral boundary conditions are imposed. In a series of papers, two of the authors propose a perturbative method to carry out a closed-from approximation of solutions to (1.9) under the assumption that \( \mathcal{K} \) in (1.3)-(1.7) is locally parabolic, i.e. \( p_0 = d \) and \( B = 0 \) in (1.5) (for a recent and thorough description the reader can refer to \( [28, 34] \)). The basic idea is to approximate the generator by Taylor expanding its coefficients, and take advantage of some symmetry properties of Gaussian kernels. Sharp short-time/small-noise asymptotic estimates for the remainder of the expansion are then proved. In order to generalize the aforementioned technique to the case \( p_0 < d \), we perform an expansion that is compatible with the sub-elliptic geometry induced by Kolmogorov operators. Assuming \( a_{ij}, a_i \in C_B^{N,1} \), we expand the operator \( \mathcal{K} \) through the sequence \( \{\mathcal{K}_n^{(\bar{z})}\}_{0 \leq n \leq N} \) defined as

\[
\mathcal{K}_n^{(\bar{z})} = \frac{1}{2} \sum_{i,j=1}^{p_0} T_n (a_{ij}, \bar{z})(z) \partial_{x_i x_j} + \sum_{i=1}^{p_0} T_{n-1} (a_i, \bar{z})(z) \partial_{x_i} + Y, \quad z = (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (1.10)
\]

where \( T_n (a_{ij}, \bar{z}) \) is the Taylor polynomial of \( a_{ij} \), defined as in (1.8), centered at a fixed point \( \bar{z} \in \mathbb{R} \times \mathbb{R}^d \), and \( T_{-1} (a_i, \bar{z}) \equiv 0 \).

\[\text{We denote by } \mathbb{N} \text{ the set of natural numbers and } \mathbb{N}_0 = \mathbb{N} \cup \{0\}.\]
Remark 1.3. When \( p_0 < d \), the intrinsic space \( C_{B}^{p_0,\alpha} \) is strictly contained into the corresponding Euclidean Hölder space \( C^{p_0,\alpha} \): for this reason, the regularity assumptions on the coefficients are weaker than in the parabolic case.

The leading term of the expansion, \( \mathcal{K}_0^{(\bar{z})} \), is the Kolmogorov operator with constant coefficients

\[
\mathcal{K}_0^{(\bar{z})} = \frac{1}{2} \sum_{i,j=1}^{p_0} a_{ij}(\bar{z}) \partial_{x_i,x_j} + Y,
\]

defined on \( \mathbb{R} \times \mathbb{R}^d \). It is well-known that \( \mathcal{K}_0^{(\bar{z})} \) admits a Gaussian fundamental solution that satisfies some remarkable symmetry properties written in terms of the increments appearing in the intrinsic Taylor polynomials in (1.8). The main result of this paper, Theorem 3.5, provides an explicit approximating expansion for \( u(t,x) \) in (1.1), equipped with sharp short-time error bounds, and can be roughly summarized as:

\[
u(t,x) = u_0(t,x) + \sum_{n=1}^{N} L_n(t,T,x) u_0(t,x) + O\left((T-t)^{\frac{N+1+k}{2}}\right) \quad \text{as} \ t \to T^-,
\]

uniformly with respect to \( x \in D \), where:

- the leading term \( u_0 \) is the solution of the Cauchy problem for \( \mathcal{K}_0^{(\bar{z})} \) with final datum \( \varphi \);  
- \( \{L_n\}_{1 \leq n \leq N} \) is a family of differential operators, acting on \( x \), that can be explicitly computed in terms of the intrinsic Taylor polynomials \( T_n(a_{ij},z) \) and \( T_n(a_{i},z) \) (see Theorem Appendix A.2);  
- the positive exponent \( k \), contributing to the asymptotic rate of convergence, is the intrinsic Hölder exponent of \( \varphi \). Precisely, \( \varphi \in C^k_B \) according to Definition 2.4 below.

Such approximation turns out to be optimal to several extents. In particular, the benefit in exploiting the intrinsic regularity is threefold: first, since the intrinsic Taylor polynomial is typically a projection of the Euclidean one, we avoid taking up terms in the expansion that do not improve the quality of the approximation; secondly, the fact that the increments of the intrinsic Taylor polynomial appear in the symmetries of the fundamental solution of \( \mathcal{K}_0^{(\bar{z})} \) allows to get compact approximation formulas; finally, the asymptotic rate of convergence of the expansion also depends on the intrinsic regularity of the datum \( \varphi \), which is typically higher than the Euclidean regularity. This is particularly relevant in the financial applications (see Remark 1.4 below).

1.2. Applications to finance and comparison with the existing literature

The application of Kolmogorov operators in mathematical finance is particularly relevant in the pricing of Asian-style derivatives. These are financial claims whose payoff is a function not only of the terminal value of an underlying asset, but also of its average over a certain time-period. In most cases of interest, the problem of computing the conditional expectation (1.1), which defines the no-arbitrage price of such financial claims, is not known to have an explicit solution, and thus a considerable amount of literature has been developed in the last decades in order to find accurate and quickly computable approximate solutions. Some of these approaches make use of asymptotic techniques that lead to semi-closed approximation formulas. In this section we aim at firming our results within the existing literature on analytical approximations of Asian-style derivatives. Before to proceed we recall that other financial applications, where averaged-diffusion
processes are employed, include volatility models with path-dependent coefficients, e.g. the Hobson-Rogers model \([31]\).

Let us resume our first example \([1,2]\) and now assume that \(S\) follows the more general dynamics

\[
dS_t = \sigma (t, S_t, A_t) \, dW_t.
\]

In this case, \(a_{11}(t, x_1, x_2) = \sigma^2(t, x_1, x_2)\) and its \(n\)-th order intrinsic Taylor polynomial centered at \(\zeta = (t, x_1, x_2)\) reads as

\[
T_n (a_{11}, \zeta) (t, x_1, x_2) = \sum_{2k+3\beta_0+3\beta_1 \leq n} \frac{(\partial_s + \xi_1 \partial_{\xi_2})^k \partial_{\xi_1}^\beta_0 \partial_{\xi_2} \partial_{a_{11}} a_{11} (s, \xi_1, \xi_2)}{k! \beta_0! \beta_1!} (t-s)^k (x_1-\xi_1)^{\beta_0} (x_2-\xi_2-(t-s)\xi_1)^{\beta_1}.
\]

More explicitly, up to order 3 we have

\[
T_0 (a_{11}, \zeta) (t, x_1, x_2) = a_{11} (\zeta),
\]

\[
T_1 (a_{11}, \zeta) (t, x_1, x_2) = T_0 (a_{11}, \zeta) (t, x_1, x_2) + (x_1 - \xi_1) \partial_{\xi_1} a_{11} (\zeta),
\]

\[
T_2 (a_{11}, \zeta) (t, x_1, x_2) = T_1 (a_{11}, \zeta) (t, x_1, x_2) + \frac{(x_1 - \xi_1)^2}{2} \partial_{\xi_1}^2 a_{11} (\zeta) + (t-s)(\partial_s + \xi_1 \partial_{\xi_2}) a_{11} (\zeta),
\]

\[
T_3 (a_{11}, \zeta) (t, x_1, x_2) = T_2 (a_{11}, \zeta) (t, x_1, x_2) + \frac{(x_1 - \xi_1)^3}{3!} \partial_{\xi_1}^3 a_{11} (\zeta) + (x_2 - \xi_2 - (t-s)\xi_1) \partial_{\xi_2} a_{11} (\zeta) + (t-s)(x_1 - \xi_1)(\partial_s + \xi_1 \partial_{\xi_2}) \partial_{\xi_1} a_{11} (\zeta),
\]

which shows that the increment in the time variable appears only from the 2nd order on, whereas the increment along the average variable appears from the 3rd order on. As it was mentioned above, the operators \(\mathcal{L}_n^{(t)}\) appearing in the asymptotic expansion in \([1,12]\) can be explicitly computed by applying \([A.6]-[A.7]-[A.4]-[5.2]\). In this case they read as

\[
\mathcal{L}_n^{(t)} (t, x_1, x_2) = \frac{1}{2} \int_t^T \left( T_n (a_{11}, \zeta) - T_{n-1} (a_{11}, \zeta) \right) \left( s, \mathcal{M}^{(t)} (s-t, x_1, x_2) \right) \left( \partial_{x_1} - (s-t) \partial_{x_2} \right)^2 ds,
\]

\[
\mathcal{M}^{(t)} (t, x_1, x_2) = \left( x_1 + a_{11} (\zeta) t \partial_{x_1} - a_{11} (\zeta) \frac{t^2}{2} \partial_{x_2}, t x_1 + t x_2 - a_{11} (\zeta) \frac{t^2}{2} \partial_{x_1} + a_{11} (\zeta) \frac{t^3}{6} \partial_{x_2} \right).
\]

In order to show an even more explicit sample, at order 1 we have:

\[
\mathcal{L}_1^{(t)} (t, x_1, x_2) = \frac{\partial_{\xi_1} a_{11} (\zeta)}{2} \int_t^T \left( (x_1 - \xi_1) + a_{11} (\zeta) (s-t) \partial_{x_1} - a_{11} (\zeta) \frac{(s-t)^2}{2} \partial_{x_2} \right) \left( \partial_{x_1} - (s-t) \partial_{x_2} \right)^2 ds.
\]

Two typical arithmetic Asian options are the so-called floating strike and fixed strike Call options, whose payoffs are given respectively by

\[
\varphi_{\text{float}} (x_1, x_2) = \left( x_1 - x_2 / T \right)^+, \quad \varphi_{\text{fixed}} (x_1, x_2) = \left( x_2 / T - K \right)^+,
\]

where \(T\) is the maturity and \(K\) is the strike price.

**Remark 1.4.** The payoff \(\varphi_{\text{fixed}}\) is Lipschitz continuous in the standard Euclidean sense but has higher intrinsic regularity (namely, \(C^2_B\)) according to Definition \([24]\) see also Example \([25]\): this property reflects a higher rate of convergence of the asymptotic expansion \([1.12]\) compared with other expansions based on standard Taylor polynomials. On the other hand, because of its explicit dependence on \(x_1\), the payoff \(\varphi_{\text{float}}\) is only \(C^1_{B,\text{loc}}\).
Even in the simplest case of constant volatility, i.e. in the Black&Scholes model, both the marginal distribution of $A_t$ and the joint distribution of $(S_t, A_t)$ are difficult to characterize analytically. The distribution of $A_t$ was given an integral representation in the pioneering work [43], though that result is of limited practical use in the valuation of Asian options. The approximation formulas that we propose in this paper were applied heuristically in [15], where intensive numerical tests were performed to confirm their accuracy. However, the general hypoelliptic framework that we consider here clearly allows for several generalization, including more general dynamics and more sophisticated Asian style-derivatives including stochastic local volatility models such as the CEV and the Heston models [20]. An interesting example is also given by a generalized type of Asian option, where the average is weighted w.r.t. the volume of traded assets: these options are written on the Volume Weighted Average Price (VWAP), a trading benchmark used especially in pension plans (see, for instance, [32]). The dynamics of the traded volume $V$ are lead by an additional stochastic factor that has to be chosen as to reflect the corresponding volume statistics, and the average process $A$ is then given by

$$A_t = \int_0^t S_\tau V_\tau d\tau \int_0^t V_\tau d\tau.$$

As it was previously argued, our technique makes use of the intrinsic Taylor polynomials in (1.8) in order to be consistent with the subelliptic geometry induced by Kolmogorov operators. This differentiates our approach from others appearing in the literature that are based on classical Euclidean expansions. In the relevant paper [19], Malliavin calculus techniques were employed to derive analytical approximations for the law of a general averaged diffusion. When applied to the pricing of arithmetic Asian options, the approach in [19] returns an expansion whose leading term is the price of a geometric Asian option. Correcting terms are computed by Taylor expanding the coefficients of the diffusion and error estimates depend on standard Euclidean regularity assumptions on the coefficients and on the payoff function. In [41] and [3], the authors followed a different approach and carried out a Taylor based-expansion of the joint distribution $(S_t, A_t)$ to analytically approximate the price of an Asian option (possibly, forward-starting); this technique seems to be limited to the Black&Scholes dynamics. Other approximations, based on Taylor expansions and on Watanabe’s theory, can be found in [25], though no rigorous error bounds are provided.

For sake of completeness, we also give a brief, and by no means exhaustive, overview of the existing literature concerning other approaches to the pricing of Asian options. Within the Black&Scholes framework, [18] derived an analytical expression for the Laplace transform of $A_t$. However, several authors pointed out some stability issues related to the numerical inversion of the Laplace transform, which lacks accuracy and efficiency in regimes of small volatility or short time-to-maturity. This is also a disadvantage of the Laguerre expansion proposed in [11]. [40] used a contour integral approach based on Mellin transforms to improve the accuracy of the results in the case of low volatilities, albeit at a higher computational cost. As opposed to numerical inversion, [27] derived an eigenfunction expansion of the transition density of $A_t$ (see also [10]) by employing spectral theory of singular Sturm-Liouville operators. Although it returns in general very accurate results, Linetsky’s series formula may converge slowly in the case of low volatility and become computationally expensive. Note that, by opposite, the analytical pricing formulas we propose here do not suffer any lack of accuracy or efficiency in these limiting cases. In actual fact, Theorem 3.5 and Remark 3.7 show that the accuracy improves as volatility and/or time to maturity get smaller. Again in the particular case of the Black&Scholes model, and for special homogeneous payoff functions, it is possible to reduce the pricing PDE in [1.9] to a one state variable PDE. PDE reduction techniques were initiated in [29] and
applied to the problem of pricing Asian options by several authors, including [38, 42] and [7]. Eventually, other approaches include the parametrix expansion in [3] and the moment-matching techniques in [12, 6, 16] and [14] among others.

2. Kolmogorov operators and intrinsic Hölder spaces

In this section we collect some known facts about the intrinsic geometry of Kolmogorov operators. We also recall the definition of intrinsic Hölder spaces and the Taylor formula recently proved in [36]. We consider the prototype Kolmogorov operator obtained by (1.3)-(1.7) with $A_0$ equal to a scalar $(p_0 \times p_0)$-matrix and $a_i \equiv 0$, $i = 1, \ldots, p_0$, i.e.

$$\mathcal{K}^\Lambda := \frac{\Lambda}{2} \sum_{i=1}^{p_0} \partial^2_{x_i} + \langle Bx, \nabla_x \rangle + \partial_t, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad \Lambda > 0. \quad (2.1)$$

In this case we say that $\mathcal{K}^\Lambda$ is a constant coefficients Kolmogorov operator. By Assumption 1.2, the vector fields $\partial_{x_1}, \ldots, \partial_{x_{p_0}}$ and $Y$ in (1.6) satisfy the Hörmander’s condition and therefore $\mathcal{K}^\Lambda$ is hypoelliptic. As it was first observed in [26], $\mathcal{K}^\Lambda$ has remarkable invariance properties with respect to the homogeneous Lie group $G_B = (\mathbb{R} \times \mathbb{R}^d, \circ, (D(\lambda))_{\lambda > 0})$ where “$\circ$” is the group law defined as

$$(t, x) \circ (s, \xi) = (t + s, e^{sB}x + \xi), \quad (t, x), (s, \xi) \in \mathbb{R} \times \mathbb{R}^d,$$

and $(D(\lambda))_{\lambda > 0}$ are the dilations given by

$$D(\lambda) = \text{diag}(\lambda^2, \lambda^3 I_{p_1}, \ldots, \lambda^{2p_1+1} I_{p_r}),$$

where $I_{p_j}$ denote the $(p_j \times p_j)$-identity matrices. Precisely, it was proved in [26] that $\mathcal{K}^\Lambda$ is invariant with respect to the left translations and homogeneous of degree two with respect to the dilations $(D(\lambda))_{\lambda > 0}$. Notice that $G_B$ is completely determined by the matrix $B$; moreover, the identity element in $G_B$ is $\text{Id} = (0, 0)$ and the inverse is $(t, x)^{-1} = (-t, -e^{-tB}x)$. For convenience, we also denote by

$$D_0(\lambda) = \text{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \ldots, \lambda^{2p_1+1} I_{p_r}),$$

the “spatial part” of $D(\lambda)$. A homogeneous norm on $G_B$ is defined as follows:

$$\| (t, x) \|_B = |t|^{1/2} + |x|_B, \quad |x|_B := \sum_{j=1}^d |x_j|^{1/\sigma_j}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad (2.2)$$

where $(\sigma_j)_{1 \leq j \leq d}$ are the integers such that

$$D_0(\lambda) = \text{diag}(\lambda^{\sigma_1}, \ldots, \lambda^{\sigma_d}), \quad (2.3)$$

that is $\sigma_1 = \cdots = \sigma_{p_0} = 1$, $\sigma_{p_0+1} = \cdots = \sigma_{p_0+p_1} = 3$ and so forth.

In the general setting of homogeneous Lie groups, Hölder spaces and intrinsic Taylor polynomials can be defined as in [13] and [1]. For the particular case of homogeneous Lie groups induced by Kolmogorov operators, [36] provides a deeper analysis of the intrinsic Taylor polynomials under optimal regularity assumptions.
For any Lipschitz vector field $Z$ on $\mathbb{R} \times \mathbb{R}^d$, we denote by $\delta \mapsto e^{\delta Z}(z)$ the integral curve of $Z$ starting from $z$: in particular, we have
\[
e^{\delta \partial_{e_i}}(t, x) = (t, x + \delta e_i), \quad i = 1, \cdots, p_0, \quad e^{\delta Y}(t, x) = (t + \delta, e^{\delta B}x),
\]
where $e_i$ denotes the $i$-th element of the natural Euclidean basis of $\mathbb{R}^d$. We say that a function $u$ is $Z$-differentiable at $z$ if $\delta \mapsto u(e^{\delta Z}(z))$ is differentiable at 0 and in that case $\frac{d}{d\delta} u(e^{\delta Z}(z)) \big|_{\delta=0}$ is referred to as the Lie derivative of $u$ at $z$ along $Z$. Since the vector fields $\partial_{x_1}, \cdots, \partial_{x_{p_0}}$ and $Y$ are $D(\lambda)$-homogeneous of degree one and two respectively, we associate to $u$ the norms:

\[
\|u\|_{C^0_B} := \sup_{\mathbb{R}^1 \times \mathbb{R}^d} |u| + \sup_{\mathbb{R}^1 \times \mathbb{R}^d} \frac{|u(e^{\delta Z}(z)) - u(z)|}{|\delta|^m},
\]

if $Y u \in C^n_B$ we define the intrinsically Hölder spaces on the homogeneous group $G_B$.

**Definition 2.1.** Let $\alpha \in [0,1]$ and $n \in \mathbb{N}$ with $n \geq 2$, then:

i) $u \in C^{0,\alpha}_B$ if $u \in C^0_Y$ and $u \in C^{\alpha}_{\partial_{x_i}}$ for any $i = 1, \cdots, p_0$;

ii) $u \in C^{1,\alpha}_B$ if $u \in C^1_Y$ and $\partial_{x_i} u \in C^{0,\alpha}_B$ for any $i = 1, \cdots, p_0$;

iii) $u \in C^{n,\alpha}_B$ if $Y u \in C^{n-2,\alpha}_B$ and $\partial_{x_i} u \in C^{n-1,\alpha}_B$ for any $i = 1, \cdots, p_0$.

We also introduce the norms:

\[
\|u\|_{C^{0,\alpha}_B} := \|u\|_{C^0_Y} + \sum_{i=1}^{p_0} \|u|_{C^{\alpha}_{\partial_{x_i}}},
\]

\[
\|u\|_{C^{1,\alpha}_B} := \|u\|_{C^1_Y} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C^{0,\alpha}_B},
\]

\[
\|u\|_{C^{n,\alpha}_B} := \|Y u\|_{C^{n-2,\alpha}_B} + \sum_{i=1}^{p_0} \|\partial_{x_i} u\|_{C^{n-1,\alpha}_B}.
\]

**Remark 2.2.** Notice that $C^{n+1,\alpha}_B \subseteq C^{n,\alpha}_B$ for any $n \in \mathbb{N}_0$.

For any multi-index $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}^d$, we define the $B$-length of $\beta$ as

\[
|\beta|_B := \sum_{j=1}^{d} \sigma_j \beta_j,
\]

with $\sigma_j$ as in (2.3). We are now in position to state the intrinsic Taylor theorem that was proved in (3.6).

**Theorem 2.3.** Let $\alpha \in [0,1]$ and $n \in \mathbb{N}_0$. If $u \in C^{n,\alpha}_B$ then the derivatives

\[
Y^k \partial^j u \in C^{n-2k-|\beta|,\alpha}_B \quad \text{for} \quad 0 \leq 2k + |\beta|_B \leq n,
\]

exist and therefore, for any point $\zeta = (s, \xi)$, the $n$-th order $B$-Taylor polynomial $T_n(u, \zeta)(\cdot)$ in (1.8) is well defined. Moreover, we have

\[
|u(z) - T_n(u, \zeta)(z)| \leq c_B \|u\|_{C^{n,\alpha}_B} \|\xi^{-1} \circ z\|^{n+\alpha}_B, \quad z, \zeta \in \mathbb{R} \times \mathbb{R}^d,
\]

where $c_B$ is a positive constant that only depends on $B$. 

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Definition 2.1 and Theorem 2.3 will be used in the next section, respectively, to specify suitable regularity conditions on the coefficients of $K$ in (1.3)-(1.7), and to expand them as in (1.10). However, as anticipated in Section 1.1, the intrinsic regularity of the terminal datum $\varphi$ plays as well a key role in the error analysis of the expansion (1.12). This motivates the following

**Definition 2.4.** Let $k \in ]0, 2r + 1]$. We denote by $C^k_B(\mathbb{R}^d)$ the space of functions $\varphi$ on $\mathbb{R}^d$ such that

$$|\varphi(x) - \varphi(y)| \leq C|x - y|^k_B, \quad x, y \in \mathbb{R}^d,$$

for some positive constant $C$, where $\cdot_B$ is the norm on $\mathbb{R}^d$ defined in (2.2). We also set

$$\|\varphi\|_{C^k_B(\mathbb{R}^d)} = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^k_B}.$$

Moreover, by convention, $C^0_B(\mathbb{R}^d)$ is the set of bounded and continuous functions on $\mathbb{R}^d$ and $\|\varphi\|_{C^0_B(\mathbb{R}^d)} = \|\varphi\|_{L^\infty(\mathbb{R}^d)}$.

**Example 2.5.** Consider the case of arithmetic Asian options with fixed strike discussed in Section 1.2, i.e.

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi_{\text{fixed}}(x_1, x_2) = (x_2/T - K)^+.$$

According to Definition 2.4, $\varphi_{\text{fixed}} \in C^3_B(\mathbb{R}^2)$ even if it is only Lipschitz continuous in the Euclidean sense.

3. Approximate solutions and error bounds

Let $X$ be a Feller process as defined in the introduction: in particular, we assume that the infinitesimal generator of $X$ coincides with operator $A$ in (1.3) on a fixed domain $D$ of $\mathbb{R}^d$. Moreover, $A$ satisfies Assumptions 1.1 and 1.2. Throughout this section $N \in \mathbb{N}_0$ and $T > 0$ are fixed and we also require the following assumptions to be in force:

**Assumption 3.1.** The coefficients $a_{ij}, a_i$ of $A$ belong to $C^{N, 1}_B$ and

$$\|a_{ij}\|_{C^{N, 1}_B}, \|a_i\|_{C^{N, 1}_B} \leq M,$$

with $M$ as in (1.4).

**Assumption 3.2.** The final datum $\varphi$ is a continuous function with sub-exponential growth such that $u = u(t, x)$ in (1.1) is well defined and belongs to $L^\infty([0, T] \times D)$. Moreover, there exists $\psi \in C^k_B(\mathbb{R}^d)$, with $k \in [0, 2r + 1]$, such that $\varphi = \psi$ on $D$. The following preliminary result can be proved as in [24] or [35], using the Schauder estimates and the results on Green functions proved in [9].

**Proposition 3.3.** Let Assumptions 1.1, 1.2, 3.1 and 3.2 be in force. Then, $u \in C([0, T] \times D) \cap C^{N+2, 1}_{B, \text{loc}}$ and satisfies (1.9).

As was mentioned in the introduction, the idea behind our approximation of $u = u(t, x)$ in (1.1) is to expand the generator of $X$ by approximating the coefficients $a_{ij}$ and $a_i$ in (1.3) by means of their intrinsic
Taylor polynomials in (1.8). Thus we fix \( \bar{z} = (\bar{t}, \bar{x}) \in \mathbb{R} \times \mathbb{R}^d \) and consider the sequence \( (K_n(\bar{z}))_{0 \leq n \leq N} \) in (1.10). We recall that, by Assumptions 1.1 and 1.2, \( K_n(\bar{z}) \) in (1.11) has a fundamental solution \( \Gamma_0(\bar{z}) \) that is the \( d \)-dimensional Gaussian density

\[
\Gamma_0^{(\bar{z})}(t, x; T, y) = \frac{1}{\sqrt{(2\pi)^d|C_{\bar{z}}(T-t)|}} \exp\left(-\frac{1}{2}(C_{\bar{z}}^{-1}(T-t)(y - e^{(T-t)B}x), (y - e^{(T-t)B}x))\right)
\]

(3.1)

with covariance matrix \( C_{\bar{z}}(t) \) given by

\[
C_{\bar{z}}(t) = \int_0^t e^{sB}A(\bar{z})e^{sB^*}ds,
\]

(3.2)

Next we formally expand the expected value \( u \) in (1.1) as

\[
u \approx U_N^{(\bar{z})} := \sum_{n=0}^N u_n^{(\bar{z})}.
\]

(3.3)

Inserting (1.10), (3.3) into (1.9) and formally collecting terms of the same order, we find that the functions \( u_n^{(\bar{z})} \) satisfy the following sequence of nested Cauchy problems

\[
\begin{align*}
K_0^{(\bar{z})}u_0^{(\bar{z})} &= 0, & \text{on} \ [0, T] \times \mathbb{R}^d, \\
u_0^{(\bar{z})}(T, \cdot) &= \varphi, & \text{on} \ \mathbb{R}^d,
\end{align*}
\]

(3.4)

and

\[
\begin{align*}
K_0^{(\bar{z})}u_n^{(\bar{z})} &= -\sum_{h=1}^n (K_h^{(\bar{z})} - K_{n-h}^{(\bar{z})})u_{n-h}^{(\bar{z})}, & \text{on} \ [0, T] \times \mathbb{R}^d, \\
u_n^{(\bar{z})}(T, \cdot) &= 0, & \text{on} \ \mathbb{R}^d.
\end{align*}
\]

(3.5)

The explicit representation of the terms \( u_n^{(\bar{z})} \) of the expansion is given in Theorem \ref{appendix:2}.

**Remark 3.4.** In the above construction, the approximation in (3.3) is defined in terms of a sequence of Cauchy problems that admit a unique non-rapidly increasing solution. Conversely, equations (1.9) do not have a unique solution unless additional lateral boundary conditions are posed. Nevertheless, Theorem 3.5 below states that the above expansion is asymptotically convergent in the limit of short-time, uniformly on compact subsets of \( D \). This is in line with the so-called principle of not feeling the boundary (cf. \cite{22, 17}).

Basically, the same asymptotic result would hold for any bounded solution of equations (1.9), with error bounds depending on the \( L^\infty \)-norm of the solution. Of course, knowing the boundary conditions would allow to construct an approximate sequence that is also accurate near the boundary; this is the case of barrier options in the financial applications.

The choice of the basis point \( \bar{z} \) is somewhat arbitrary, but only some particular choices allow for performing a rigorous error analysis. For instance, here below we consider the case \( \bar{z} = z = (t, x) \). However, although we omit to write separate proofs, the same results hold by setting \( \bar{z} = (T, x) \). In the following statement, we put

\[
U_N(z) := U_N^{(\bar{z})}(z), \quad z \in [0, T] \times D,
\]

(3.6)

with \( U_N^{(\bar{z})} \) defined by (3.3), (3.4), (3.5).
Theorem 3.5. Let Assumptions 1.1, 1.2, 3.1 and 3.2 be in force. Then for any compact subset $K$ of $D$, we have
\[
|u(t,x) - U_N(t,x)| \leq C(T-t)^{\frac{N+1}{2}}, \quad (t,x) \in [0,T] \times K,
\]
where $C$ is a positive constant that depends only on $M, \mu, B, T, N, K, \|\varphi\|_{C^2_B(\mathbb{R}^d)}$ and $\|u\|_{L^\infty([0,T] \times D)}$.

Theorem 3.5 will be proved in Section 3.2.

Remark 3.6. As shown in Example 2.5, for a fixed-strike Asian option we have $\varphi \in C^3_B(\mathbb{R}^2)$ and therefore we get $(T-t)^{\frac{N+1}{2}}$ in the error estimate (3.7). This is coherent with the previous results proved in [19] in the scalar case for $N \leq 2$, and sheds some light on why the order of convergence of Asian call options is improved w.r.t. their European counterparts, for which the error is of order $(T-t)^{\frac{N+2}{2}}$. When placed within our framework, this improvement of convergence can be seen as part of a wider phenomenon related to the intrinsic geometry of Kolmogorov operators.

Remark 3.7. If the coefficients $a_{ij}, a_i$ only depend on the first $p_0$ variables, then it is possible to prove the error bounds in (3.7) to be also asymptotic in the limit of small $M$. Precisely,
\[
|u(t,x) - U_N(t,x)| \leq C(M(T-t))^{\frac{N+1}{2}}, \quad (t,x) \in [0,T] \times K,
\]
with $C$ independent of $M$ as $M \to 0^+$. This is the case, for instance, of classical volatility models for Asian options where the volatility coefficient depends at most on the underlying asset $S_t$ (local volatility) and on some exogenous factors (stochastic volatility), but not on the average process $A_t$.

In the global case, when $D = \mathbb{R}^d$, we have some stronger results. Aside from the error bounds in (3.7) becoming global in space, we are also able to obtain analogous asymptotic error bounds for the transition density of $X$. We start by observing that when $D = \mathbb{R}^d$ our assumptions imply that $X$ has a transition density $\Gamma$ that coincides with the fundamental solution of $\mathcal{K}$ as in (1.3)–(1.7) (see, for instance, [17]). We denote by $\Gamma_N$ the $N$-th order approximation of $\Gamma$ defined as
\[
\Gamma_N(t,x;y) = \sum_{n=0}^{N} u_n(t,x;T,y) \quad 0 \leq t < T, \ x, y \in \mathbb{R}^d,
\]
where $u_0(t,x;T,y) = \Gamma^{0}(t,x;T,y)$ in (3.1), and the correcting terms $u_n(t,x;T,y)$ are defined recursively by (3.5) with $\bar{z} = (t, x)$. We have the following

Theorem 3.8. Let Assumptions 1.1, 1.2, 3.1 and 3.2 be in force with $D = \mathbb{R}^d$. Then, we have
\[
|u(t,x) - U_N(t,x)| \leq C(T-t)^{\frac{N+1}{2}}, \quad (t,x) \in [0,T] \times \mathbb{R}^d,
\]
where $C$ depends only on $M, \mu, B, T, N$ and $\|\varphi\|_{C^2_B(\mathbb{R}^d)}$. Moreover, for any $c > 1$, we have
\[
|\Gamma(t,x;y) - \Gamma_N(t,x;y)| \leq C(T-t)^{\frac{N+1}{2}}\Gamma^c(t,x;y), \quad (t,x) \in [0,T] \times \mathbb{R}^d,
\]
where, for any $\Lambda > 0$, $\Gamma^\Lambda$ denotes the fundamental solution of the constant-coefficient Kolmogorov operator $\mathcal{K}^\Lambda$ as defined in (2.1), and $C$ is a positive constant that depends only on $M, \mu, B, T, N$ and $c$. 

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Proposition 3.9. For any $k \in \mathbb{R}_{\geq 0}$, $c > 1$ and $\beta \in \mathbb{N}_0^d$, with $|\beta|_B \leq N + 2$, we have

$$\left[ y - e^{(T-t)B \cdot x}\right]_B^k |D_x^\beta \Gamma(t, x; T, y)| \leq C(T-t)^{\frac{|\beta|_B - 1}{2}} \Gamma^{cM}(t, x; T, y), \quad 0 \leq t < T, \quad x, y \in \mathbb{R}^d,$$

where $\Gamma^{cM}$ is the fundamental solution of the operator in (2.1) and $C$ is a positive constant, only dependent on $M, \mu, B, T, N, k$ and $c$.

The following result is proved in Appendix B.

Proposition 3.10. Let $\varphi \in C^k_b(\mathbb{R}^d)$ with $k \in [0, 2r + 1]$ and $n \in \mathbb{N}$ with $n \leq N$. Then we have

$$|D_x^\beta u^{(\varphi)}(t, x)| \leq C(T-t)^{\frac{|\beta|_B - 1}{2}} \left[ (T-t)^{\frac{n}{2}} + |x - e^{(T-t)B \cdot x}|_B^n \right], \quad 0 \leq t < T, \quad x \in \mathbb{R}^d,$$

where $C$ is a constant that depends only on $M, \mu, B, T, N, |\beta|_B$ and $\|\varphi\|_{C^k_b(\mathbb{R}^d)}$.

Proof of Theorem 3.8. To keep formulas at a reasonable size we suppose that the functions $a_i$, $i = 1, \ldots, p$, in (1.3) are identically zero. We first remark that a straightforward computation (see Lemma 6.3 in [28]) shows that

$$u(t, x) - U_N(t, x) = \sum_{n=0}^{N} E_n^{(\varphi)}(t, x),$$

where

$$E_n^{(\varphi)}(t, x) := \int_t^T \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi) \left( \mathcal{X} - \mathcal{X}_n^{(\varphi)} \right) u^{(\varphi)}_{N-n}(s, \xi) d\xi ds$$

and

$$E_n^{(\varphi)}(t, x) = \frac{1}{2} \sum_{i,j=1}^{p_0} \int_t^T \int_{\mathbb{R}^d} \Gamma(t, x; s, \xi) \left( a_{ij}(s, \xi) - \mathcal{T}_n(a_{ij}(\bar{z})(s, \xi)) \right) \partial_{\xi_i} \xi_j u^{(\varphi)}_{N-n}(s, \xi) d\xi ds.$$
For instance, in the case of a fixed-strike Asian option (see Example 2.5), we have

\[ |D_x^2 u(t, x) - D_x^2 U_N(z) (t, x)|_{z=(t,x)}| \leq C (T - t)^{N+1-|\alpha|B}, \quad |\alpha|B \leq N. \]  

(3.12)

The proof of this formula is analogous to the proof of Theorem 3.8 once \( D_x^2 \) is applied to the representation formulas (3.10) and (3.11). When \( u(t, x) \) represents the price of an arithmetic Asian option, formula (3.12) provides error bounds on the approximate sensitivities or, as they are usually called in finance, the Greeks.

For instance, in the case of a fixed-strike Asian option (see Example 2.5), we have \( k = 3 \) and thus

\[ |\Delta - \partial_x U_N(z) (t, x)|_{z=(t,x)}| \leq C (T - t)^{N+3}, \quad |\Gamma - \partial_{x_1, x_2} U_N(z) (t, x)|_{z=(t,x)}| \leq C (T - t)^{N+2}, \]

where \( \Delta := \partial_x u \) and \( \Gamma := \partial_{x_1, x_2} u \).

### 3.2. Proof of Theorem 3.5

Throughout this section we suppose the assumptions of Theorem 3.5 to be in force. The proof of Theorem 3.5 is based on some estimates on short cylinders initially introduced in [39] for uniformly parabolic operators and later generalized to Kolmogorov operators in [4].

First, we introduce the “cylinder” of radius \( R \) and height \( h \) centered in \((s, \xi) \in \mathbb{R} \times \mathbb{R}^d\) and its lateral and parabolic boundaries, respectively:

\[
H_{h,R}(s, \xi) := \{(t, x) \in \mathbb{R} \times \mathbb{R}^d | s-h < t < s, [x - e^{(t-s)B}]_B < R\}, \\
\Sigma_{h,R}(s, \xi) := \{(t, x) \in \mathbb{R} \times \mathbb{R}^d | s-h < t < s, [x - e^{(t-s)B}]_B = R\}, \\
\partial_p H_{h,R}(s, \xi) := \Sigma_{h,R}(s, \xi) \cup \{(s, x) \in \mathbb{R} \times \mathbb{R}^d | [x - \xi]_B < R\}.
\]

We explicitly observe that these cylinders are invariant with respect to the left translations in \( G_B \), meaning that \( z \circ H_{h,R}(\zeta) = H_{h,R}(z \circ \zeta) \) for any \( z, \zeta \in \mathbb{R} \times \mathbb{R}^d \). We also recall the following inequality (see Proposition 2.1 in [39]):

\[ \|z \circ \zeta\|_B \leq c_B (\|z\|_B + \|\zeta\|_B), \quad z, \zeta \in \mathbb{R} \times \mathbb{R}^d, \]  

(3.13)

where \( c_B \geq 1 \) is a constant that depends only on the matrix \( B \). In particular, taking \( z = (0, x) \) and \( \zeta = (t, 0) \), (3.13) implies that

\[ [e^{tB} x]_B \leq \|(t, e^{tB} x)\|_B = \|z \circ \zeta\|_B \leq c_B (\|t\|_B + \|x\|_B), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d. \]

(3.14)

**Lemma 3.12.** There exist \( C > 0, \varepsilon \in [0, 1] \), only dependent on \( M, \mu, B \), and a nonnegative function \( v \in C([0, T] \times \mathbb{R}^d) \cap C_{B;\text{loc}}^{2,1}\) such that, for every \( R > 0 \) we have

\[
X u(t, x) = 0 \quad (t, x) \in H_{\varepsilon R^2, R}(T, 0), \]

(3.15)

\[
v(t, x) \geq 1, \quad (t, x) \in \Sigma_{\varepsilon R^2, R}(T, 0), \]

(3.16)

\[
v(t, x) \leq C \exp \left( -\frac{R^2}{C(T-t)} \right), \quad (t, x) \in H_{\varepsilon R^2 - \frac{R}{c_B}, B}(T, 0), \]

(3.17)

where \( c_B \) is the constant in (3.13).
Proof. Let $\Gamma$ denote the fundamental solution of $\mathcal{K}$ in (1.7): $\Gamma$ can be thought as the transition density of a dummy process $\tilde{X}$ whose infinitesimal generator is $\mathcal{A}$ and can be used to approximate the original process $X$ locally on $D$. The proof of the lemma is based on a Gaussian upper bound for $\Gamma$. More precisely, since $\mathcal{K}$ is a global Kolmogorov operator, by Proposition 3.9 we have: there exists a positive constant $c^+$, only depending on $M, \mu$ and $B$, such that

$$\Gamma(t, x; s, \xi) \leq c^+ \Gamma^A(t, x; s, \xi), \quad 0 \leq t < s \leq T, \ x, \xi \in \mathbb{R}^d,$$

(3.18)

where $\Gamma^A$ is the fundamental solution of the constant coefficients Kolmogorov operator in (2.1) and $A$ is strictly greater than $M$, say $A = 2M$.

Next, we set

$$v(t, x) = 2 \int_{\mathbb{R}^d} \Gamma(t, x; T, y) \chi_R(y) dy, \quad t < T, \ x \in \mathbb{R}^d,$$

where $\chi_R \in C^\infty(\mathbb{R}^d, [0, 1])$ is a cut-off function such that $\chi_R(y) = 0$ if $|y|_B < \frac{R}{2}$ and $\chi_R(y) = 1$ if $|y|_B > \frac{3}{4}R$. By definition, it is clear that $v$ satisfies (3.15). Moreover, we have

$$\lim_{t \to T^-} v(t, x) = 2 \chi_R(x) = 2,$$

(3.19)

uniformly w.r.t. $x \in \mathbb{R}^d$ such that $|x|_B = R$: this follows by noting that

$$|v(t, x) - 2 \chi_R(x)| \leq 2 \int_{\mathbb{R}^d} \Gamma(t, x; T, y) |\chi_R(y) - \chi_R(x)| dy \leq 2 c^+ \int_{\mathbb{R}^d} \Gamma^A(t, x; T, y) |\chi_R(y) - \chi_R(x)| dy.$$

(by (3.18))

Now, by (3.19) there exists $\varepsilon > 0$, which we can safely assume to be less than $\frac{1}{16 c_B^2}$ and $\frac{1}{64 c_B^2}$, such that (3.16) holds.

The proof of (3.17) depends on the reverse triangle inequality for the norm $|\cdot|_B$:

$$|y - e^{tB}x|_B \geq \frac{1}{c_B} |y|_B - c_B (|t|^\frac{3}{2} + |x|_B), \quad t \in \mathbb{R}, \ x,y \in \mathbb{R}^d,$$

whose proof is an easy consequence of (3.14). In particular, if $|y|_B \geq \frac{R}{2}$ and $(t, x) \in H_{xR^2, 2R (T, 0)}$, then in light of the first bound for $\varepsilon$ we get

$$|y - e^{(T-t)B}x|_B \geq \frac{R}{8c_B}. \quad (3.20)$$

Hence, for such $(t, x)$ we get

$$v(t, x) \leq 2 c^+ \int_{\mathbb{R}^d} \Gamma^A(t, x; T, y) \chi_R(y) dy \leq 2 c^+ \int_{\mathbb{R}^d} \Gamma^A(t, x; T, y) dy$$

$$= \frac{2 c^+ (2\pi)^{-d/2}}{\sqrt{|C(T-t)|}} \int_{|y|_B \geq 4} \exp \left( -\frac{1}{2} (C^{-1}(T-t)(y - e^{(T-t)B}x), (y - e^{(T-t)B}x)) \right) dy$$

(by (3.20) and denoting by $C$ the matrix in (3.2) with $A_0 = \Lambda I_{p_0}$ and $I_{p_0}$ being the $(p_0 \times p_0)$ identity matrix)

$$\leq \frac{2 c^+ (2\pi)^{-d/2}}{\sqrt{|C(T-t)|}} \int_{|y - e^{(T-t)B}x|_B \geq \frac{R}{8c_B}} \exp \left( -\frac{1}{2} (C^{-1}(T-t)(y - e^{(T-t)B}x), (y - e^{(T-t)B}x)) \right) dy$$

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(by the change of variables \( \eta = D_0(1)\eta \)) and the homogeneity relation (B.11)

\[
\frac{2e^{-(2\pi)^{-\frac{1}{2}}}}{\sqrt{|C(1)|}} \int_{|\eta| B \geq \frac{1}{R}} \exp \left(-\frac{1}{2}(C^{-1}(1) \eta, \eta)\right) \, d\eta.
\]

Since we are assuming \( T - t \leq \varepsilon R^2 \), thanks to the second bound on \( \varepsilon \) we have \( |\eta| B \geq \frac{R}{8 \varepsilon B} \varepsilon R - t \), such that

\[
(C^{-1}(1) \eta, \eta) \geq C_0 |\eta|^2 = C_0 \sum_{j=1}^{d} \frac{|\eta_j|^2}{|\eta| B} |\eta|^{2\sigma_j} = C_0 \sum_{j=1}^{d} \left( \frac{|\eta_j| \sigma_j}{|\eta| B} \right)^{2\sigma_j} |\eta|^{2\sigma_j}.
\]

\[
\geq C_0 |\eta|^{2\sigma} = C_0 d^{2\sigma} \left( \sum_{j=1}^{d} \frac{|\eta_j| \sigma_j}{|\eta| B} \right)^{2(2\sigma+1)} = \frac{C_0}{d^{2\sigma+1}} |\eta|^{2\sigma}.
\]

Setting \( C_1 := \frac{C_0}{d^{2\sigma+1}} \), we get

\[
\int_{|\eta| B \geq \frac{1}{R}} \exp \left(-\frac{1}{2}(C^{-1}(1) \eta, \eta)\right) \, d\eta \leq \int_{|\eta| B \geq \frac{1}{R}} \exp \left(-\frac{1}{2} C_1 |\eta|^2 \right) \, d\eta
\]

\[
\leq \max_{|\eta| B \geq \frac{1}{R}} \exp \left(-\frac{1}{2} C_1 |\eta|^2 \right) \int_{|\eta| B \geq \frac{1}{R}} \exp \left(-\frac{1}{2} C_1 |\eta|^2 \right) \, d\eta
\]

\[
\leq \exp \left(-\frac{C_1 R^2}{24 \varepsilon R^2 (T-t)} \right) \int_{\mathbb{R}^d} \exp \left(-\frac{1}{4} C_1 |\eta|^2 \right) \, d\eta,
\]

which, combined with (3.21), proves (3.24).

**Proof of Theorem 3.3** Since the statement is a short-time estimate on a compact subset, it is enough to prove (3.24) for \( (t, x) \in H_{\varepsilon R^2, R}(T, \xi) \subseteq [0, T] \times D \) for suitably small \( \varepsilon, R > 0 \). Secondly, we can suppose \( \xi = 0 \). In fact, if \( u \) is a solution to \( \mathcal{K} u = 0 \) in \( H_{\varepsilon R^2, R}(T, \xi) \) then \( w(t, x) = u(t, x - e^{-TB} \xi) \) solves on \( H_{\varepsilon R^2, R}(T, 0) \) the operator obtained through \( \mathcal{K} \) by translating its coefficients.

Let us denote by \( u^0 \) the unique solution (with polynomial growth) to the Cauchy problem

\[
\begin{cases}
\mathcal{K} f = 0, & \text{on } [0, T] \times \mathbb{R}^d, \\
f(T, \cdot) = \psi, & \text{on } \mathbb{R}^d,
\end{cases}
\]

with \( \psi \) as in Assumption 3.2 and by \( U_N^0 \) its \( N \)-th order approximation as defined in Section 3. By triangular inequality we have

\[
|u - U_N| \leq |u - u^0| + |u^0 - U_N^0| + |U_N^0 - U_N|.
\]

We now aim at estimating each of the terms in the sum above.

We start with \( |u - u^0| \). Let \( v \) be the function appearing in Lemma 3.12. By Proposition 3.3 and (3.15), \( u - u^0 \) and \( v \) solve \( \mathcal{K} w = 0 \) in \( H_{\varepsilon R^2, R}(T, 0) \) and are continuous on \( \overline{H_{\varepsilon R^2, R}(T, 0)} \). Moreover, \( (u - u^0)(T, x) = 0 \) if \( [x] R < R \), and thus, by setting

\[
C_1 := \max_{\Sigma_{\varepsilon R^2, R}(T, 0)} |u - u^0|,
\]

we get \( |u - u^0| \leq C_1 v \) on \( \partial_p H_{\varepsilon R^2, R}(T, 0) \). Therefore, by the Feynman-Kac theorem we have

\[
|u - u^0|(t, x) = |E_{t,x} [u - u^0](\tau, X_\tau)| \leq C_1 E_{t,x} [v(\tau, X_\tau)] = C_1 v(t, x),
\]

which is the desired estimate.
where \( \tau \) denotes the exit time from \( H_{\mathbb{R}^2,R}(T,0) \) of the process \((s,X_s)\) starting from \((t,x)\) \(\in H_{\mathbb{R}^2,R}(T,0)\).

By estimate (3.17) of Lemma 3.12 we obtain

\[
|u(t,x) - \hat{u}(t,x)| \leq C_1 C_2 \exp \left( -\frac{R^2}{C_2 (T-t)} \right), \quad (t,x) \in H_{\mathbb{R}^2,R}(T,0),
\]

(3.23)

with \( C_2 > 0 \) depending only on \( M, \mu, B. \)

We continue by estimating \( |u^\psi - U_N^\psi| \). By Theorem 3.8 there exists \( C_3 > 0 \), only dependent on \( M, \mu, B, T, N \) and \( \|\psi\|_{C^3_{\mathbb{R}^d}} \), such that

\[
|u^\psi(t,x) - U_N^\psi(t,x)| \leq C_3 (T-t)^{3+\frac{3}{2}}, \quad (t,x) \in [0,T] \times \mathbb{R}^d.
\]

(3.24)

We conclude by estimating \( |U_N^\psi - U_N| \). First observe that, by (A.1), for any multi-index \( \alpha \in \mathbb{N}_0^d \) we have

\[
D^\alpha \left( u_0^{(\psi)} - u_0^{(\hat{\psi})} \right)(t,x) = D^\alpha \int_{\mathbb{R}^d} \Gamma_0^{(\hat{\psi})}(t,x;T,y) \left( \varphi(y) - \psi(y) \right) dy = \int_{\mathbb{R}^d} D^\alpha \Gamma_0^{(\hat{\psi})}(t,x;T,y) \left( \varphi(y) - \psi(y) \right) dy,
\]

with \( \Gamma_0^{(\hat{\psi})} \) as in (3.1). Now, \( \Gamma_0^{(\hat{\psi})} \) is the fundamental solution of the constant-coefficients Kolmogorov operator \( \mathcal{K}_0^{(\hat{\psi})} \) in (1.12), for which Assumptions (1.12) and (3.1) are trivially satisfied. Therefore, the bounds in Lemma 3.24 also apply to \( \Gamma_0^{(\hat{\psi})} \) and yield

\[
|D^\alpha \left( u_0^{(\psi)} - u_0^{(\hat{\psi})} \right)(t,x)| \leq C_4 (T-t)^{3+\frac{3}{2}} w(t,x), \quad z \in \mathbb{R} \times \mathbb{R}^d, \quad (t,x) \in [0,T] \times \mathbb{R}^d,
\]

(3.25)

with

\[
w(t,x) := \int_{\mathbb{R}^d} \Gamma^{2M}(t,x;T,y) \left| \left( \varphi(y) - \psi(y) \right) \right| dy,
\]

where \( \Gamma^{2M} \) is the fundamental solution of the Kolmogorov operator \( \mathcal{K}^{2M} \) as in (2.1), and \( C_4 > 0 \) only depends on \( M, \mu, B, T, |\alpha|_B \). Now note that, by (3.6) and (A.3), we have

\[
(U_N^\psi - U_N)(t,x) = \left( u_0^{(\psi)} \right. - \left. u_0^{(\hat{\psi})} \right)(t,x) + \sum_{n=1}^{N} C_n^{(\hat{\psi})(\psi)} \left( u_n^{(\psi)} - u_n^{(\hat{\psi})} \right)(t,x)
\]

|\( z = (t,x) \).

Thus by Lemma [Appendix B.7] with (3.26) we get

\[
\left| (U_N^\psi - U_N)(t,x) \right| \leq C_5 w(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^d,
\]

where \( C_5 > 0 \) only depends on \( M, \mu, B, T \) and \( N \). By repeating step by step the same proof of (3.23) it is straightforward to obtain an estimate for \( |w(t,x)| \) analogous to (3.23), which finally yields

\[
\left| (U_N^\psi - U_N)(t,x) \right| \leq C_5 C_6 C_7 \exp \left( -\frac{R^2}{C_7 (T-t)} \right), \quad (t,x) \in H_{\mathbb{R}^2,R}(T,0),
\]

(3.26)

with \( C_7 > 0 \) depending only on \( M, \mu, B, T, N \), and

\[
C_6 := \max_{\Sigma_{\mathbb{R}^2,R}(T,0)} |w|.
\]

Plugging (3.23)-(3.24)-(3.25) into (3.26) yields (3.7) for \((t,x) \in H_{\mathbb{R}^2,R}(T,0)\) and concludes the proof.\(\square\)
Appendix A. Analytical approximation formulas

We show that the functions \( u_n^0(z) \) in (3.3) can be explicitly computed at any order. It is clear that the leading term \( u_0^0(z) \) is given by

\[
u_0^0(t, x) = \int_{\mathbb{R}^d} \Gamma_0^0(t, x; T, y) \varphi(y) dy, \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\]

where \( \Gamma_0^0(z) \) is the Gaussian density in (A.1). For \( n \in \mathbb{N} \) with \( n \leq N \), the explicit representation for the correcting terms \( u_n^0(z) \) can be derived using the following notable symmetry properties of \( \Gamma_0^0(z) \).

**Lemma Appendix A.1.** For any \( x, y \in \mathbb{R}^d \), \( t < s \) and \( \bar{z} = (\bar{t}, \bar{x}) \in \mathbb{R} \times \mathbb{R}^d \), we have

\[
\begin{align*}
\nabla_x \Gamma_0^0(t, x; s, y) &= -e^{(s-t)B^*} \nabla_y \Gamma_0^0(t, x; s, y), \\
y \Gamma_0^0(t, x; s, y) &= M^0(s-t, x) \Gamma_0^0(t, x; s, y),
\end{align*}
\]

where \( M^0(t, x) \) is the operator defined as

\[
M^0(t, x) = e^{tB}(x + M(t) \nabla_x), \quad M(t) = e^{-tB}C_z(t)e^{-tB^*}.
\]

**Proof.** Using the explicit expression of \( \Gamma_0^0(z) \), the proof is a direct computation. \( \square \)

The following result provides an explicit representation of \( u_n^0(z) \) in (3.3): remarkably, it can be written as a finite sum of spatial derivatives acting on \( u_0^0(z) \).

**Theorem Appendix A.2.** Let Assumptions [A.1, A.2, and A.3] be in force. Then, for any \( n \in \mathbb{N} \) with \( n \leq N \), and for any \( \bar{z} \in \mathbb{R} \times \mathbb{R}^d \), we have

\[
u_n^0(t, x) = \mathcal{L}_n^0(t, T, x)u_0^0(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d.
\]

In (A.5), \( \mathcal{L}_n^0(t, T, x) \) denotes the differential operator

\[
\mathcal{L}_n^0(t, T, x) = \sum_{h=1}^{n} \int_{t_s}^{T} ds_1 \int_{s_1}^{T} ds_2 \cdots \int_{s_{h-1}}^{T} ds_h \sum_{i \in I_{n, h}} G_i^0(t, s_1, x) \cdots G_i^0(t, s_h, x),
\]

where

\[I_{n, h} = \{i = (i_1, \ldots, i_h) \in \mathbb{N}^h \mid i_1 + \cdots + i_h = n\}, \quad 1 \leq h \leq n,
\]

and

\[
G_i^0(t, s, x) = \frac{1}{2} \sum_{j, k=1}^{p_0} (T_n(a_{ij}, \bar{z}) - T_{n-1}(a_{ij}, \bar{z}))(s, M^0(s-t, x))(e^{(s-t)B^*} \nabla_x)_{ij}(e^{-(s-t)B^*} \nabla_x)_j + \sum_{i=1}^{p_0} (T_{n-1}(a_i, \bar{z}) - T_{n-2}(a_i, \bar{z}))(s, M^0(s-t, x))(e^{-(s-t)B^*} \nabla_x)_i,
\]

with \( M^0(t, x) \) as in (A.4) and, by convention, \( T_{-1}f \equiv 0 \).

Next, we sketch the proof of Theorem Appendix A.2 that is based on the symmetry properties of the Gaussian density \( \Gamma_0(z) \) in (A.1), combined with an extensive use of other very general relations such as the Duhamel’s principle and the Chapman-Kolmogorov equation. Since the choice of \( \bar{z} \) is unimportant through this section, we drop the explicit dependence on \( \bar{z} \) in the following formulas. First, we generalize formula (A.4) to polynomial functions \( p \) with time-dependent coefficients, that is \( p = p(t, \cdot) \) is a polynomial for every fixed \( t \in \mathbb{R} \); this will be used to deal with the operators \( \mathcal{K}_n \) in (1.10) that have coefficients of this form.
Proposition Appendix A.3. For any \( t, s, s_1 \in [0, T] \), with \( t < s, x, y \in \mathbb{R}^d \), we have

\[
p(s_1, y)\Gamma_0(t; x; s, y) = p(s_1, M(s - t, x))\Gamma_0(t; x; s, y).
\] (A.8)

Proof. Let us recall that operator \( M(t, x) \) acts only on the variable \( x \). First, we prove that the components \( M_i(t, x), i = 1, \ldots, d \), commute when applied to \( \Gamma_0 = \Gamma_0(t; x; s, y) \) and its derivatives (notice however that this is not true in general when they are applied to a generic function). Notice also that formula (A.2) expresses an \( x \)-derivative as a linear combination of \( y \)-derivatives with coefficients that depend only on \( t \) and \( s \). This is obviously true also for higher orders and we express it through the differential operator \( S^\beta_y(s - t) \), acting on \( y \), defined by

\[
D^2_y \Gamma_0(t; x; s, y) = S^\beta_y(s - t) \Gamma_0(t; x; s, y).
\]

Now we have

\[
M_i(s - t, x) M_j(s - t, x) D^2_y \Gamma_0 = M_i(s - t, x) M_j(s - t, x) S^\beta_y(s - t) \Gamma_0 = S^\beta_y(s - t) M_i(s - t, x) M_j(s - t, x) \Gamma_0 = S^\beta_y(s - t) (y_j M_i(s - t, x) \Gamma_0) = S^\beta_y(s - t) (y_j \Gamma_0) = M_j(s - t, x) M_i(s - t, x) D^2_y \Gamma_0.
\]

Since \( p(s_1, \cdot) \) is a polynomial by definition, we therefore have that the operators \( p(s_1, M(s - t, x)) \) are defined unambiguously when applied to \( \Gamma_0(t; x; s, y) \) and to its derivatives. Moreover, clearly (A.8) is now a straightforward consequence of (A.3). \( \square \)

Remark Appendix A.4. By Proposition Appendix A.3, the operators \( \mathcal{G}_n(t; s, x) \) are defined unambiguously when applied to \( \Gamma_0 = \Gamma_0(t; x; s, y) \), to its derivatives and, more generally, by the representation formula (A.1), to solutions of the Cauchy problem (3.4).

The next proposition, essentially based on the symmetries of Lemma Appendix A.1 is the key of the proof of Theorem Appendix A.2.

Proposition Appendix A.5. For any \( x, y \in \mathbb{R}^d \), \( t < s \) and \( n \in \mathbb{N} \) with \( n \leq N \), we have

\[
\int_{\mathbb{R}^d} \Gamma_0(t; x; s, \xi)((\mathcal{K}_n - \mathcal{K}_{n-1}) f)(s, \xi) d\xi = \mathcal{G}_n(t; s, x) \int_{\mathbb{R}^d} \Gamma_0(t; x; s, \xi) f(\xi) d\xi,
\] (A.9)

for any \( f \in C^2_0(\mathbb{R}^d) \).

Proof. To keep formulas at a reasonable size we suppose that the functions \( a_i, i = 1, \ldots, p_0 \), in (1.3) are identically zero. By the definition (1.10) we have

\[
\int_{\mathbb{R}^d} \Gamma_0(t; x; s, \xi)((\mathcal{K}_n - \mathcal{K}_{n-1}) f)(s, \xi) d\xi = \frac{1}{2} \sum_{i,j=1}^{p_0} \int_{\mathbb{R}^d} (T_n(a_{ij}, \xi) - T_{n-1}(a_{ij}, \xi)) (s, \xi) \Gamma_0(t; x; s, \xi) \partial_{\xi_i} \partial_{\xi_j} f(\xi) d\xi
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{p_0} (T_n(a_{ij}, \xi) - T_{n-1}(a_{ij}, \xi)) (s, M(s - t, x)) \int_{\mathbb{R}^d} \Gamma_0(t; x, y; s, \xi, \omega) \partial_{\xi_i} \partial_{\xi_j} f(\xi) d\xi
\] (by (A.8))
\[
\frac{1}{2} \sum_{i,j=1}^{p_n} \left( T_n(a_{ij}, \bar{z}) - T_{n-1}(a_{ij}, \bar{z}) \right)(s, \mathcal{M}(s-t, x)) \int_{\mathbb{R}^d} \partial_{\xi_1, \xi_j} \Gamma_0(t, x; s, \xi) f(\xi) d\xi \quad \text{(by parts)}
\]

\[
= G_n(t, s, x) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) f(\xi) d\xi.
\]

(by (A.2) and (A.7))

The proof of Theorem Appendix A.2 consists of mostly formal and tedious computations that are totally analogous to those given for the parabolic case in Section 5 in [28]. This may not be surprising since our framework contains the parabolic one as a special case. Therefore, we only give a proof for \(n = 1\), which still sheds light on the origin of the operators \(L_n\).

By definition, \(u_1\) is the solution of the Cauchy problem (3.3) with \(n = 1\). By Duhamel’s principle we have

\[
\begin{align*}
\quad u_1(t, x) &= \int_t^T \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) (\mathcal{K}_1 - \mathcal{K}_0) u_0(s, \xi) d\xi ds \\
&= \int_t^T \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) u_0(s, \xi) d\xi ds \quad \text{(by (A.9) with } n = 1) \\
&= \int_t^T \int_{\mathbb{R}^d} \theta_1(t, s, x) \int_{\mathbb{R}^d} \Gamma_0(s, \xi; T, y) \varphi(y) dy d\xi ds \quad \text{(by (A.1))} \\
&= \int_t^T \int_{\mathbb{R}^d} \varphi(y) \int_{\mathbb{R}^d} \Gamma_0(t, x; s, \xi) \Gamma_0(s, \xi; T, y) d\xi dy ds \quad \text{(Fubini’s theorem)} \\
&= \int_t^T \theta_1(t, s, x) ds u_0(t, x) \quad \text{(Chapman-Kolmogorov and (A.1))} \\
&= \mathcal{L}_1(t, T) u_0(t, x). \quad \text{(by (A.9))}
\end{align*}
\]

\[\square\]

Appendix B. Proof of Proposition 3.10

In this section we prove some preliminary estimates on the spatial derivatives of solutions of constant coefficient-Kolmogorov operators: in particular, we prove estimates for the derivatives of \(u_n^{(z)}\) defined by (3.4)-(3.5). Throughout this section \(z \in \mathbb{R} \times \mathbb{R}^d\) is fixed.

Proposition Appendix B.1. Let \(k \in [0, 2r + 1]\), \(\beta \in \mathbb{N}_0^d\) with \(|\beta|_{B} > 0\). If \(\psi \in C^k_B(\mathbb{R}^d)\) then the solution \(u_0^{(z)}\) of the Cauchy problem (3.4) satisfies

\[
\left| D^\beta_x u_0^{(z)}(t, x) \right| \leq C(T - t)^{k - \frac{|\beta|_{B}}{2}}, \quad 0 \leq t < T, \; x \in \mathbb{R}^d,
\]

where \(C\) is a positive constant that depends only on \(M, \mu, B, T, \beta\) and \(\|\psi\|_{C^k_B(\mathbb{R}^d)}\).

Proof. We prove the case \(k \in [0, 2r + 1]\) since the case \(k = 0\) is straightforward. We first note that, since \(\Gamma_0^{(z)}\) is a density and \(|\beta|_{B} > 0\), we have

\[
D^\beta_x \int_{\mathbb{R}^d} \Gamma_0^{(z)}(t, x; T, y) dy = 0.
\]
and therefore
\[
D_x^2 u_0^{(\varepsilon)}(t, x) = \int_{\mathbb{R}^d} \psi(y) D_x^2 \Gamma_0^{(\varepsilon)}(t, x; T, y) dy \\
= \int_{\mathbb{R}^d} \left( \psi(y) - \psi(e^{(T-t)B} x) \right) D_x^2 \Gamma_0^{(\varepsilon)}(t, x; T, y) dy.
\]

Since \( \psi \in C^k_B(\mathbb{R}^d) \), we obtain
\[
\left| D_x^2 u_0^{(\varepsilon)}(t, x) \right| \leq \| \psi \|_{C^k_B(\mathbb{R}^d)} \int_{\mathbb{R}^d} \left[ y - e^{(T-t)B} x \right]_B^k \left| D_x^2 \Gamma_0^{(\varepsilon)}(t, x; T, y) \right| dy \\
\leq C \| \psi \|_{C^k_B(\mathbb{R}^d)} (T-t)^{\frac{k-|\beta|}{2}} \int_{\mathbb{R}^d} \Gamma^{2M}(t, x; T, y) dy,
\]
where the second inequality follows from a direct estimate on the derivatives of \( \Gamma_0^{(\varepsilon)} \) (see, for example, Section 2 in [37]) and \( \Gamma^{2M} \) is the fundamental solution of the Kolmogorov operator \( \mathcal{K}^{2M} \) as defined in (2.1).

In the next lemmas we will use the following result proved in [26].

Lemma Appendix B.2. The following homogeneity relations hold
\[
C_{\bar{z}}(t) = D_0(\sqrt{t}) C_{\bar{z}}(1) D_0(\sqrt{t}), \quad (B.1)
\]
\[
M_{\bar{z}}(t) = D_0(\sqrt{t}) M_{\bar{z}}(1) D_0(\sqrt{t}), \quad (B.2)
\]
\[
e^{tB} = D_0(\sqrt{t}) e^{B} D_0 \left( \frac{1}{\sqrt{t}} \right), \quad (B.3)
\]
for any \( t > 0 \).

Notation Appendix B.3. From now to the end of this section, we use the Greek letters \( \alpha, \beta, \gamma, \delta, \nu \) to denote multi-indexes in \( \mathbb{N}_0^d \), and \( |\alpha| = \sum_{i=1}^d \alpha_i \) is the standard Euclidean height of \( \alpha \). To simplify notations, if \( I \) is any family of indexes, we use the unconventional notation
\[
\sum_{\ell \in I} \pi_\ell = \sum_{\ell \in I} c_{\ell} \pi_\ell
\]
for a sum where the constants \( c_{\ell} \) depend only on \( \bar{z}, B, N, T, a_{ij}, a_i \) and are uniformly bounded by a constant that depends only on \( M, \mu, B, T, N \) and \( B \).

Lemma Appendix B.4. Let
\[
\mathcal{W}(t) = e^{-tB^*} \nabla_x, \quad t \in \mathbb{R},
\]
denote the differential operators appearing in (A.7) and by \( \mathcal{W}^\alpha(t) \) the composition\(^3\)
\[
\mathcal{W}^\alpha(t) = \mathcal{W}_1^{\alpha_1}(t) \cdots \mathcal{W}_d^{\alpha_d}(t). \quad (B.4)
\]
The following representation holds true:
\[
\mathcal{W}^\beta(t) = \sum_{|\alpha| = |\beta|} t^{\frac{|\alpha| B - |\beta| B}{|\alpha| B \geq |\beta| B}} D_x^\alpha.
\]

\(^3\)Operator \( \mathcal{W}^\alpha(t) \) in (B.4) is well defined since the components of \( \mathcal{W}(t) \) commute.
Proof. It suffices to prove the statement for a single $W_i(t)$. Using the relations in Lemma Appendix B.2, we have

$$ W_i(t) = \sum_{j=1}^{d} D_0 \left( \frac{1}{\sqrt{t}} \right) e^{-B^{\ast} D_0} \partial_{x_j} $$

with $\sigma_i$ as in (2.3). The result follows noting that the intrinsic order of $\partial_{x_j}$ is exactly $\sigma_j$. Moreover, as the matrix $e^{-B^{\ast}}$ is upper triangular the sum actually ranges over $j = i, \ldots, d$ and thus $\sigma_j - \sigma_i$ is always a nonnegative integer.

Next step is the study of the operator $\mathcal{M}^{(j)}(t, x)$: we recall that, by Proposition Appendix A.3 the components of $\mathcal{M}^{(j)}(t, x)$ commute when applied to $\Gamma^{(j)}_0$ and more generally to $u_n^{(j)}$ and its derivatives.

Lemma Appendix B.5. For any $\beta \in \mathbb{N}_0^d$, we have

$$ \left( \mathcal{M}^{(j)}(s-t, x) - e^{(s-t)B} x \right) = \sum_{|\delta| + |\alpha| \leq |\beta|} (s-t)^{\frac{|\delta| + |\alpha| - |\beta|}{2}} \left( x - e^{(t-s)B} x \right)^{\delta} D^\alpha_x. \quad (B.5) $$

Proof. First of all, let us note that

$$ \mathcal{M}^{(j)}(s-t, x) - e^{(s-t)B} x = e^{(s-t)B} \left( x - e^{(t-s)B} x + M_x(s-t) \nabla_x \right), $$

and it is not restrictive to take $x = 0$ and $t = 0$. We proceed now by induction on $|\beta|$. If $|\beta| = 1$ then $\beta = e_i$ where $e_i$ is the $i$-th element of the canonical basis of $\mathbb{R}^d$. A direct computation shows

$$ \left( \mathcal{M}^{(j)}(s, x) \right)^{e_i} = \sum_{|\delta| + |\alpha| \leq |\beta|} s^{\frac{|\delta| + |\alpha| - |\beta|}{2}} D^\alpha_x \left( x + M_x(s) \nabla_x \right)^{\delta} \quad (\text{by (B.3)}) $$

$$ = \sum_{|\delta| + |\alpha| \leq |\beta|} s^{\frac{|\delta| + |\alpha| - |\beta|}{2}} \left( x + s^{\frac{|\delta|}{2}} \sum_{|\nu| = 1}^\infty s^{\frac{|\nu|}{2}} D^\nu_x \right), \quad (\text{by (B.2)}) $$

which proves (B.5) with $\beta = e_i$. We now assume the statement to hold for $|\beta| \leq n$, and prove it true for $\beta + e_i$. By inductive hypothesis applied to both $\beta$ and $e_i$ we get

$$ \left( \mathcal{M}^{(j)}(s, x) \right)^{\beta + e_i} = \sum_{|\delta| + |\alpha| \leq |\beta|} \sum_{|\delta| + |\alpha| \leq |\beta|} s^{\frac{|\delta| + |\alpha| - |\beta|}{2}} s^{\frac{|\delta| + |\alpha| - |\beta|}{2}} x^{\delta} D^\alpha_x \left( x^2 D^\alpha_x \right) $$

$$ = \sum_{|\delta| + |\alpha| \leq |\beta + e_i|} s^{\frac{|\delta| + |\alpha| - |\beta| + |e_i|}{2}} D^\alpha_x, \quad (\text{setting} \; \delta = \delta^1 + \delta^2 \; \text{and} \; \alpha = \alpha^1 + \alpha^2). $$

Lemma Appendix B.6. For any $n \in \mathbb{N}$, with $n \leq N$, we have the following representation

$$ G_n(t, s, x) = \sum_{(\alpha, \delta) \in I_n} (s-t)^{\frac{|\alpha| + |\delta| - |\beta| - n}{2}} \left( x - e^{(t-s)B} x \right)^{\delta} D^\alpha_x, \quad (B.6) $$

where

$$ I_n = \{ (\alpha, \delta) \in \mathbb{N}_0^d \times \mathbb{N}_0^d \mid 1 \leq |\alpha| \leq n + 2, |\delta| + |\alpha| \leq n, |\alpha| - |\delta| + n - 2 \geq 0 \}. $$
Appendix B.4 and Appendix B.5.

For greater convenience we recall the expression of the result will then readily follow. We only consider the case $\bar{x} = 0$. Plugging equation (B.6) into the definition of $L_{h,i}$ we obtain

$$L_{h,i}(t, T, x) = \sum_{(\alpha, \delta) \in J_n}^{\bullet} (T-t)^{\alpha B + \delta + n} \left( x - e^{(t-i)B} \bar{x} \right)^{\delta} D_x^{\alpha}.$$  \hfill (B.7)

where

$$J_n = \{ (\alpha, \delta) \in \mathbb{N}^d \times \mathbb{N}^d | 1 \leq |\alpha| \leq 3n, |\delta|_B \leq n, |\alpha|_B - |\delta|_B + n \geq 0 \}. \quad (B.8)$$

Proof. For greater convenience we recall the expression of $\mathcal{L}_n(t, T, x)$ as given in (A.6):

$$\mathcal{L}_n^{(2)}(t, T, x) = \sum_{h=1}^{n} \sum_{i \in I_{n, h}} L_{h,i}(t, T, x),$$

where

$$L_{h,i}(t, T, x) := \int_t^T ds_1 \int_{s_1}^T ds_2 \cdots \int_{s_{h-1}}^T ds_h \mathcal{L}_n^{(2)}(t, s_1, x) \cdots \mathcal{L}_n^{(2)}(t, s_h, x),$$

and $I_{n, h} = \{ i = (i_1, \ldots, i_h) \in \mathbb{N}^h | i_1 + \cdots + i_h = n \}$, for $1 \leq h \leq n$. We prove that, for fixed $h \in \{1, \ldots, n\}$ and $i \in I_{n, h}$ it holds

$$L_{h,i}(t, T, x) = \sum_{(\alpha, \delta) \in J_n}^{\bullet} (T-t)^{\alpha B + \delta + n} \left( x - e^{(t-i)B} \bar{x} \right)^{\delta} D_x^{\alpha}.$$  \hfill (B.9)

the result will then readily follow. We only consider the case $\bar{x} = 0$. Plugging equation (B.6) into the definition of $L_{h,i}$ we obtain

$$L_{h,i}(t, T, x) = \sum_{(\alpha, \delta) \in I_{i_1}}^{\bullet} \cdots \sum_{(\alpha^h, \delta^h) \in I_{i_h}}^{\bullet} x^{\delta} D_x^{\alpha^1} \left( x^{\delta^2} D_x^{\alpha^2} \left( \cdots \left( x^{\delta^h} D_x^{\alpha^h} \right) \right) \right) \times$$

$$\times \int_t^T \cdots \int_{s_{h-1}}^{s_h} \prod_{j=1}^{h} (s_j - t)^{\alpha_j - |\delta_j| B + n_j} ds_1 \cdots ds_h.$$  \hfill (B.10)

Now, setting $\alpha = \alpha^1 + \cdots + \alpha^h$, $\delta = \delta^1 + \cdots + \delta^h$ and recalling that $i_1 + \cdots + i_h = n$, the integral above can be easily computed to be equal to

$$(T-t)^{\alpha B + \delta + n},$$

times a constant. The statement follows applying Leibniz rule and noticing that $(\alpha, \delta) \in J_n$ if $(\alpha^j, \delta^j) \in I_{i_j}$ for $j = 1, \ldots, h$.  \hfill \square

Proof of Proposition 3.10 By (A.3)-(B.7), we get

$$D_x^\delta u_n^{(2)}(t, x) = D_x^\delta \sum_{(\alpha, \delta) \in J_n}^{\bullet} (T-t)^{\alpha B + \delta + n} \left( x - e^{(t-i)B} \bar{x} \right)^{\delta} D_x^{\alpha} u_0^{(2)}(t, x)$$

(by applying Leibniz rule and reordering the indexes of $J_n$ in (B.8))

$$= \sum_{(\alpha, \delta) \in J_n}^{\bullet} (T-t)^{\alpha B + \delta + n} \left( x - e^{(t-i)B} \bar{x} \right)^{\delta - \nu} D_x^{\alpha + \delta - \nu} u_0^{(2)}(t, x),$$

$$\nu \leq \min \{ \beta, \delta \}.$$
where $\nu \leq \min\{\beta, \delta\}$ means that $\nu_i \leq \min\{\beta_i, \delta_i\}$ for any $i = 1, \ldots, d$. Now, by applying Proposition Appendix B.1 and the property

$$|y^\delta| = \prod_{i=1}^{d} |y_i|^\delta_i \leq \prod_{i=1}^{d} |(y^\beta_i)|_B = |y|_B^\beta,$$

we obtain

$$|D_x^\delta u_n^{(2)}(t, x)| \leq \sum_{(\alpha, \delta) \in J_n} (T - t)^{-|\beta| B + n + k - |\delta| B + |\nu| B} [x - e^{(t - \bar{t})B\bar{x}}]_B^{\delta B - |\nu| B}$$

and the statement follows by the elementary inequality

$$a^m b^{n-m} \leq a^n + b^n, \quad a, b \in \mathbb{R}_{>0}, \quad 0 \leq m \leq n.$$
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