Free Energy Approximations for CSMA networks

B. Van Houdt
Dept. Mathematics and Computer Science
University of Antwerp, Belgium

Abstract—In this paper we study how to estimate the back-off rates in an idealized CSMA network consisting of $n$ links to achieve a given throughput vector using free energy approximations. More specifically, we introduce the class of region-based free energy approximations with clique belief and present a closed form expression for the back-off rates based on the zero gradient points of the free energy approximation (in terms of the conflict graph, target throughput vector and counting numbers).

Next we introduce the size $k_{\text{max}}$ clique free energy approximation as a special case and derive an explicit expression for the counting numbers, as well as a recursion to compute the back-off rates. We subsequently show that the size $k_{\text{max}}$ clique approximation coincides with a Kikuchi free energy approximation and prove that it is exact on chordal conflict graphs when $k_{\text{max}} = n$. As a by-product these results provide us with an explicit expression of a fixed point of the inverse generalized belief propagation algorithm for CSMA networks.

Using numerical experiments we compare the accuracy of the novel approximation method with existing methods.

I. INTRODUCTION

Carrier sense multiple access (CSMA) networks form an attractive random access solution for wireless networks due to their fully distributed nature and low complexity. In order to guarantee a certain set of feasible throughputs for the links part of a CSMA network (defined as the fraction of the time that a link is active), the back-off rate of each link has to be set in the appropriate manner which depends on the network topology, i.e., how the different links in the network interfere with each other. In order to obtain a better understanding of how these back-off rates affect the throughputs, the ideal CSMA network model was introduced (see [2], [5], [6], [7], [19], [20], [27]) and this model was shown to provide good estimates for the throughput achieved in real CSMA like networks [24].

The product form solution of the ideal CSMA model was established long ago [2] (for exponential back-off durations) and the set $\Gamma$ of achievable throughput vectors $\phi = (\phi_1, \ldots, \phi_n)$, where $\phi_i$ is the throughput of link $i$, was characterized in [7]. Further, for each vector $\phi \in \Gamma$ the existence of a unique vector of back-off rates that achieves $\phi$ was proven in [19]. None of the above results indicates how to set the back-off rates to achieve a given vector $\phi \in \Gamma$ (except for very small networks). For line networks with a fixed interference range this problem was solved in [20] for any $\phi = (\alpha, \ldots, \alpha) \in \Gamma$, while [27] presented a closed form expression for the back-off rates to achieve any $\phi \in \Gamma$ in case the conflict graph is a tree. This expression was obtained from the zero gradient points of the Bethe free energy and can be used as an approximation in general conflict graphs, termed the Bethe approximation. The explicit results for line and tree networks were generalized in [21], where a simple formula for the back-off rates was presented for any chordal conflict graph $G$. This formula was subsequently used to develop the local chordal subgraph (LCS) approximation for general conflict graphs.

Another method to approximate unique back-off rates given an achievable target throughput vector exists in using inverse (generalized) belief propagation (I(G)BP) algorithms [8], [25]. These algorithms are message passing algorithms that in general are not guaranteed to converge to a fixed point. In [8] the IBP algorithm for CSMA was argued to converge to the exact vector of back-off rates when the conflict graph is a tree, but convergence of IGBP for loopy graphs to a (unique) fixed point was not established. Belief propagation algorithms are intimately related to free energy approximations as their fixed points can be shown to correspond to the zero gradient points of an associated free energy approximation [26].

The main objective of this paper is to introduce more refined free energy approximations (compared to the Bethe approximation) for the ideal CSMA model that yield closed form approximations for the back-off rates and to compare their accuracy with the Bethe and LCS approximation. Due to space limitations this paper is accompanied by an arXiv preprint [22] that contains some of the proof details. The contributions of the paper are as follows.

First, we introduce a class of region-based free energy approximations with clique belief and a closed form expression for the CSMA back-off rates based on its zero gradient points (Section IV).
Second, we propose the size $k_{\text{max}}$ clique approximation as a special case and present a closed form expression for its counting numbers, as well as a recursive algorithm to compute the back-off rates more efficiently (Section V). Setting $k_{\text{max}} = 2$ reduces the size $k_{\text{max}}$ clique approximation to the Bethe approximation of [27].

Third, we prove that the size $k_{\text{max}}$ clique approximation coincides with a Kikuchi approximation (Section VI). As the Kikuchi approximation used to devise the IGBP algorithm of [8] corresponds to setting $k_{\text{max}} = n$, the size $n$ clique approximation gives a closed form expression for a fixed point of the IGBP algorithm.

Fourth, an exact free energy approximation for chordal conflict graphs is introduced and is proven to coincide with the size $n$ clique approximation (Section VII). This implies that a fixed point of the IGBP algorithm gives exact results on chordal conflict graphs.

Finally, simulation results are presented that compare the accuracy of the size $k_{\text{max}}$ clique approximation with the LCS algorithm presented in [21] (Section VIII). The main observation is that the LCS approximation is less accurate and less robust for denser conflict graphs compared to the size $k_{\text{max}}$ clique approximation.

Before presenting the above results (in Sections IV to VIII), we start with a model description in Section II and a basic introduction on (region-based) free energy approximations in Section III. Conclusions are drawn in Section IX.

We end this section by noting that more advanced free energy approximations have very recently and independently been proposed by other researchers to approximation the back-off rates in CSMA networks. More specifically, in [15] the authors proposed the Kikuchi approximation induced by all the maximal cliques of the conflict graph. In Section VI of this paper we prove that this approximation coincides with the size $n$ clique approximation. Further, the authors also prove the exactness of the maximal clique based Kikuchi approximation on chordal conflict graphs as is done in Section VII in this paper. In [16] the authors generalized their work using the region-based free energy framework of [26] and also consider an approximation that includes the 4-cycles as regions.

II. SYSTEM MODEL

The ideal CSMA model considers a fixed set of $n$ links where a link is said to be active if a packet is being transmitted on the link and inactive otherwise. Whether two links $i$ and $j$ can be active simultaneously is determined by the undirected conflict graph $G = (V, E)$, where $V$ is the set of $n$ links. If $(i, j) \in E$ then link $i$ and $j$ cannot be active at the same time.

We end this section by noting that more advanced free energy approximations have very recently and independently been proposed by other researchers to approximation the back-off rates in CSMA networks. More specifically, in [15] the authors proposed the Kikuchi approximation induced by all the maximal cliques of the conflict graph. In Section VI of this paper we prove that this approximation coincides with the size $n$ clique approximation. Further, the authors also prove the exactness of the maximal clique based Kikuchi approximation on chordal conflict graphs as is done in Section VII in this paper. In [16] the authors generalized their work using the region-based free energy framework of [26] and also consider an approximation that includes the 4-cycles as regions.

The focus in this paper is not on computing $p_1(1)$ given the back-off rates $(\nu_1, \ldots, \nu_n)$, but on the inverse problem: how to set/estimate $(\nu_1, \ldots, \nu_n)$ such that a given target throughput vector $(\phi_1, \ldots, \phi_n) \in \Gamma$ is achieved, that is, such that $p_1(1) = \phi_i$ for all $i$. In [7]

1It is worth noting that a considerable body of work exists that considers unsaturated CSMA networks where each link maintains its own buffer to store packets that arrive according to a Poisson process (e.g., [14], [11], [3], [10], [4]).
the set of achievable throughput vectors was shown to equal
\[
\Gamma = \left\{ \sum_{\vec{x} \in \Omega} \xi(\vec{x}) \bigg| \sum_{\vec{x} \in \Omega} \xi(\vec{x}) = 1, \xi(\vec{x}) > 0 \text{ for } \vec{x} \in \Omega \right\},
\]
(2)
where \( \Omega = \{(x_1, \ldots, x_n) \in \{0, 1\}^n | x_i x_j = 0 \text{ if } (i, j) \in E\} \). In other words, a throughput vector \( \vec{\theta} \) is achievable if and only if it belongs to the interior of the convex hull of the set \( \Omega \).

III. FREE ENERGY AND REGION-BASED APPROXIMATIONS

We start with a brief introduction on (region-based) free energy approximations and describe these in the context of factor graphs, we refer the reader to [26] for a more detailed exposition.

A factor graph [9] is a bipartite graph that contains a set of variable and factor nodes that represents the factorization of a function. The factor graph associated with (1) contains a variable node for each variable \( x_i \), for \( i = 1, \ldots, n \), and \( n + |E| \) factor nodes \( f_a \): one for each factor \( f_i \) and \( f_{(i,j)} \). Further, a variable node \( x_i \) is connected to a factor node \( f_a \) if and only if \( x_i \) is an argument of \( f_a \). As illustrated in Figure 1 the factor graph of (1) can be obtained from the conflict graph \( G \) by labeling node \( i \) as \( x_i \), replacing each edge \( (i, j) \) by a factor node \( f_{(i,j)} \), by connecting \( f_{(i,j)} \) to \( x_i \) and \( x_j \) and by adding the factor nodes \( f_i \), where \( f_i \) is connected to \( x_i \).

Given a distribution \( p \) on \( \{0,1\}^n \) (as in (1)) with an associated factor graph with variable nodes \( \{x_1, \ldots, x_n\} \) and factor nodes \( \{f_1, \ldots, f_M\} \), the Gibbs free energy \( F(b) \), where \( b \) is a distribution on \( \{0,1\}^n \), is defined as \( F(b) = U(b) - H(b) \), where

\[
U(b) = -\sum_{x \in \{0,1\}^n} b(x) \sum_{a=1}^M \ln(f_a(x_a)),
\]
is the Gibbs average energy and the subset of the elements of \( \{x_1, \ldots, x_n\} \) that are an argument of the function \( f_a \) is denoted as \( x_a \) and

\[
H(b) = -\sum_{x \in \{0,1\}^n} b(x) \ln(b(x)),
\]
is the Gibbs entropy. For the factor graph of (1) we have
\[
\sum_{a=1}^M \ln(f_a(x_a)) = \sum_{i=1}^n x_i \ln(\nu_i) + \sum_{(i,j) \in E} \ln(1 - x_i x_j).
\]
It is well-known that the Gibbs free energy associated with a factor graph is minimized when the distribution \( b \) matches \( p \). Although the minimizer \( p \) of the Gibbs free energy \( F(b) \) may be known explicitly as in the CSMA setting, computing marginal distributions of the form
\[
p_S(x) = \sum_{x_i: i \notin S} p(x_1, \ldots, x_n),
\]
where \( S \) is a subset of \( \{1, \ldots, n\} \), is often computationally prohibitive. As such, approximations for the Gibbs free energy have been developed that allow approximating marginal distributions of the form \( p_S(x) \) at a (much) lower computational cost. Such approximations have also been used to attack the inverse problem which attempts to estimate the model parameters (e.g., the back-off rates \( \nu_i \) in CSMA or the couplings in the Ising model) given some values for some of the marginal distributions (e.g., the target throughput vector in CSMA or the magnetizations and correlations in the Ising model). For the Bethe approximation this has led to explicit formulas for the approximate solution of the inverse problem for both the ideal CSMA model [27] and the Ising model [13].

The class of free energy approximations that is used in this paper for the inverse problem is the class of region-based free energy approximations [26]. A region-based free energy approximation is characterized by a set \( \mathcal{R} \) of regions and a counting number \( c_R \) for each \( R \in \mathcal{R} \). Each region \( R \) has an associated set of variables \( V_R \), which is a subset of the variable nodes in the factor graph, and a set of factors denoted as \( \mathcal{F}_R \), which is a subset of the factor nodes in the factor graph. The following three conditions must be met for the sets \( V_R \) and \( \mathcal{F}_R \). First, if \( f_a \in \mathcal{F}_R \), then the arguments of \( f_a \) must belong to \( V_R \). Second, the set \( \cup_{R \in \mathcal{R}} V_R = \{x_1, \ldots, x_n\} \) and \( \cup_{R \in \mathcal{R}} \mathcal{F}_R = \{f_1, \ldots, f_M\} \), in other words each variable node and factor node must belong to at least one region. Third, the counting numbers \( c_R \) are integers such that for each factor node \( f_a \) and variable node \( x_i \) we have
\[
\sum_{R \in \mathcal{R}} c_R 1_{\{f_a \in \mathcal{F}_R \}} = \sum_{R \in \mathcal{R}} c_R 1_{\{x_i \in V_R \}} = 1.
\]
For example for the Bethe approximation of [27] one associates a single region \( R \) with every node in the bipartite factor graph. For the region \( R \) associated with a factor node \( f_a \) one sets \( \mathcal{F}_R = \{f_a\} \), \( V_R = \{x_i | x_i \text{ is an argument of } f_a \} \) and \( c_R = 1 \). For the region \( R \) corresponding to a variable node \( x_i \) one sets
\( F_R = \emptyset, \mathcal{V}_R = \{ x_i \} \) and \( c_R = -d_i \), where \( d_i \) is the number of neighbors of node \( i \) in the conflict graph \( G \) such that (4) holds.

As in [26] we denote sums of the form

\[
\sum_{x \in \{0,1\}^{\mathcal{V}_R}} g(x),
\]

where \( g \) is a function from \( \{0,1\}^{\mathcal{V}_R} \) to \( \mathbb{R} \) and \( R \) is a region, as \( \sum_{x_R} g(x_R) \). Using this notation, the region-based free energy is a function of the set of beliefs \( \{ b_R(x_R) | R \in \mathcal{R} \} \), where \( b_R \) is a distribution on \( \{0,1\}^{\mathcal{V}_R} \), and is defined as

\[
F_R(\{ b_R \}) = U_R(\{ b_R \}) - H_R(\{ b_R \}),
\]

where \( U_R(\{ b_R \}) \) is the region-based average energy defined as

\[
U_R(\{ b_R \}) = - \sum_{R \in \mathcal{R}} c_R \sum_{x_R} b_R(x_R) \sum_{f_a \in \mathcal{F}_R} \ln(f_a(x_a)),
\]

and \( H_R(\{ b_R \}) \) is the region-based entropy given by

\[
H_R(\{ b_R \}) = - \sum_{R \in \mathcal{R}} c_R \sum_{x_R} \ln(b_R(x_R)).
\]

Note that the requirement that \( f_a \in \mathcal{F}_R \) implies that the arguments of \( f_a \) must belong to \( \mathcal{V}_R \) is necessary for (6) to be well defined.

The beliefs \( b_R(x_R) \) are used as approximations for the marginal probabilities \( p_R(x_R) = \sum_{x_i \in \mathcal{V}_R} p(x_1, \ldots, x_n) \). The approximation exists in finding the beliefs \( b_R(x_R) \) such that the region-based free energy is minimized over the set \( \Delta_R \) of consistent beliefs \( b_R(x_R) \) defined as

\[
\Delta_R = \left\{ \{ b_R, R \in \mathcal{R} \} \mid b_R(x_R) \geq 0, \sum_{x_R} b_R(x_R) = 1, \sum_{x_i \in \mathcal{V}_R \setminus \mathcal{V}_R'} b_R(x_R') = \sum_{x_j \in \mathcal{V}_R \setminus \mathcal{V}_R'} b_R(x_R) \right\}.
\]

We note that having a consistent set of beliefs does not imply that they are the marginals of a single distribution \( b(x) \) on \( \{0,1\}^n \) [26, Section V.A]. Further, the average energy given by (6) is known to be exact, that is, equal to the Gibbs free energy \( U(p) \), if \( b_R(x_R) = p_R(x_R) \) for all \( R \) and \( x_R \). This condition is however not sufficient for the region-based entropy to be exact (that is, equal to the Gibbs entropy \( H(p) \)) [26].

IV. CLIQUE BELIEF

In this section we introduce the notion of clique belief and indicate how to select the back-off rates to obtain a zero gradient point of the region-based free energy under clique belief. These back-off rates, presented in Theorem 1, are used as an approximation for the vector of back-off rates that achieves a given throughput vector \( (\phi_1, \ldots, \phi_n) \).

Clique belief is defined as the belief that all the nodes \( i \in V \) with \( x_i \in \mathcal{V}_R \) form a clique in the conflict graph \( G \) for any \( R \in \mathcal{R} \), meaning that any two nodes within a region \( R \) are active at the same time is zero. More specifically, we define the set of clique beliefs \( \Delta^C_R \) as the set of beliefs \( \{ b_R \} \) for which \( b_R(x_R) \) has the form

\[
b_R(x_R) = \begin{cases} 1 - \sum_{i:x_i \in \mathcal{V}_R} \phi_i & \text{for } \sum_{x_i \in \mathcal{V}_R} x_i = 0, \\ \phi_i & \text{for } x_i = 1 \text{ and } \sum_{x_j \in \mathcal{V}_R} x_j = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

for some set \( \{ \phi_1, \ldots, \phi_n \} \) with \( \phi_i \geq 0 \) and \( \sum_{i:x_i \in \mathcal{V}_R} \phi_i < 1 \) for all \( R \in \mathcal{R} \). Clique beliefs are clearly consistent, that is, \( \Delta_R^C \subseteq \Delta_R \).

The next condition limits the set of region-based free energy approximations considered somewhat by putting some minor conditions on the manner in which the regions are selected.

Condition 1. The set of regions \( \mathcal{R} \) is such that \( \mathcal{R} = \{ R_{f_1}, \ldots, R_{f_n} \} \cup \{ R_{x_1}, \ldots, R_{x_n} \} \) with

1) \( \mathcal{V}_{R_{f_i}} = \mathcal{V}_{R_{x_i}} = \{ x_i \} \), \( \mathcal{F}_{R_{f_i}} = \{ f_i \} \) and \( \mathcal{F}_{R_{x_i}} = \emptyset \), for \( i = 1, \ldots, n \),
2) for \( R \in \mathcal{R}' \) we have \( \emptyset \neq \mathcal{F}_R \subseteq \{ f_{(i,j)}(i, j) \in E \} \) and \( \mathcal{V}_R = \{ x_i \} \),
3) \( c_{R_{x_j}} = 1 - (1 + \sum_{R \in \mathcal{R}'} c_R 1_{\{ x_j \in \mathcal{V}_R \}}) \) and \( c_{R_{f_i}} = 1 \), for \( i = 1, \ldots, n \).

Note that this condition states that there are \( 2n \) special regions for which the set of variable nodes, factor nodes as well as the counting numbers are fixed. For each of the remaining regions (that is, for each \( R \in \mathcal{R}' \)) the set of variable and factor nodes is determined by some nonempty set of edges, while its counting number can be chosen arbitrarily as long as (4) holds.

Theorem 1. Let \( \mathcal{R} \) be a set of regions that meets Condition 1. For the zero gradient points of the region-based free energy \( F(\{ b_R \}) \) defined by (5) over the set \( \Delta^C_R \) of clique beliefs we have

\[
\nu_i = \frac{\phi_i}{(1 - \phi_i)^{1+c_{R_{x_i}}}} \prod_{R \in \mathcal{R}'} \left( 1 - \sum_{j:x_j \in \mathcal{V}_R} \phi_j \right)^{-c_R}.
\]
Proof. Under clique belief the entropy given by (7) equals

\[ H_\mathcal{R}(\{b_R\}) = - \sum_{R \in \mathcal{R}} c_R \sum_{i : x_i \in V_R} \phi_i \ln(\phi_i) \]
\[ - \sum_{R \in \mathcal{R}} c_R \left( 1 - \sum_{i : x_i \in V_R} \phi_i \right) \ln(1 - \sum_{i : x_i \in V_R} \phi_i) \]

due to (8) and (4) yields

\[ H_\mathcal{R}(\{b_R\}) = - \sum_{i=1}^{n} \phi_i \ln(\phi_i) \]
\[ - \sum_{R \in \mathcal{R}} c_R \left( 1 - \sum_{i : x_i \in V_R} \phi_i \right) \ln(1 - \sum_{i : x_i \in V_R} \phi_i). \]

To determine \( U_\mathcal{R}(\{b_R\}) \) first note that \( \ln(f_{(i,j)}(x_i, x_j)) \) is zero unless \( x_i = x_j = 1 \). When \( x_i = x_j = 1 \), we have \( b(x_R) = 0 \) if \( f_{(i,j)} \in F_R \) for \( R \in \mathcal{R} \). Using the common convention that \( \ln(0) = \lim_{x \rightarrow 0} x \ln(x) = 0 \) yields

\[ - \sum_{R \in \mathcal{R}} c_R \sum_{x_R} b(x_R) \sum_{f_{a} \in F_R} \ln(f_a(x_a)) = 0 \]

As \( \ln(f_i(0)) = 0 \), \( \ln(f_i(1)) = \ln(\nu_i) \) and \( b_{R_{i}}(1) = \phi_i \), one therefore finds

\[ U_\mathcal{R}(\{b_R\}) = - \sum_{i \in V} \phi_i \ln(\nu_i). \]  

(10)

Demanding that the partial derivatives \( d F_\mathcal{R}(\{b_R\})/d\phi_i \) are equal to zero is equivalent to the requirement that

\[ \ln(\nu_i) = - \sum_{R \in \mathcal{R}, x_i \in V_R} c_R \left( 1 + \ln(1 - \sum_{j : x_j \in V_R} \phi_j) \right) \]
\[ + 1 + \ln(\phi_i). \]

The expression in (9) therefore follows from (4). \( \square \)

In general the region-based free energy \( F(\{b_R\}) \) defined by (5) is not a convex function and therefore there is no guarantee that the zero gradient point corresponds to a (global) extremal point. For some region based free energy approximations it is possible to give sufficient conditions (in terms of the counting numbers) for the free energy to be convex on the set of consistent beliefs [12, Theorem 3]. These can be used to show that the free energy of the Bethe approximation presented in Section V-A is convex if the conflict graph contains at most one loop.

Formula (9) proposes an approximation for the back-off rates \( \nu_i \) if the target throughputs of the links are given by the vector \( (\phi_1, \ldots, \phi_n) \) provided that \( \sum_{i : x_i \in V_R} \phi_i < 1 \) for all \( R \in \mathcal{R} \). Depending on the choice of the regions in \( \mathcal{R}' \), this condition may be more restrictive than demanding that \( (\phi_1, \ldots, \phi_n) \in \Gamma \). However for the size \( k_{\max} \) clique approximation introduced in the next section, the requirement \( \sum_{i : x_i \in V_R} \phi_i < 1 \) holds for any \( (\phi_1, \ldots, \phi_n) \in \Gamma \) as for the size \( k_{\max} \) clique approximation each region \( R \) corresponds to a clique and the sum of the throughputs of all the nodes belonging to a clique is clearly bounded by one in \( \Gamma \).

It is important to stress that formula (9) often leads to a distributed computation of \( \nu_i \) as node \( i \) only needs to know its own target throughput, the target throughput of any node \( j \) sharing a region with \( i \) (that is, any \( j \) for which there exists an \( R \in \mathcal{R}' \) such that \( x_i, x_j \in V_R \)) as well as the counting numbers \( c_R \) for the regions \( R \) to which it belongs. The size \( k_{\max} \) clique approximation presented in the next sections is such that two nodes only belong to the same region if they are neighbors in the conflict graph \( G \) and the required counting numbers can be computed from the subgraph induced by a node and its one hop neighborhood.

Thus, for the size \( k_{\max} \) clique approximation a node can compute its approximate back-off rate using information from its one-hop neighbors only for any feasible throughput vector.

V. SIZE \( k_{\max} \) CLIQUE APPROXIMATION

In this section we introduce a region-based free energy approximation for general conflict graphs \( G \), called the size \( k_{\max} \) clique approximation and present explicit expressions for the counting numbers. We start by considering two special cases.

A. Bethe approximation

A first special case is to define \( \mathcal{R} \) such that Condition 1 is met and setting \( \mathcal{R}' = \{ R_{(i,j)} : (i, j) \in E \} \) such that \( V_{R_{(i,j)}} = \{ x_i, x_j \} \) and \( F_{R_{(i,j)}} = \{ f_{(i,j)} \} \). The associated counting numbers are \( c_{R_{(i,j)}} = 1 \), which implies that \( c_{R_{ij}} = -d_i \), where \( d_i \) denotes the number of neighbors of \( i \) in \( G \).

The entropy given in (10) therefore becomes

\[ H_\mathcal{R}(\{b_R\}) = - \sum_{(i,j) \in E} (1 - \phi_i - \phi_j) \ln(1 - \phi_i - \phi_j) \]
\[ + \sum_{i=1}^{n} [(d_i - 1)(1 - \phi_i) \ln(1 - \phi_i) - \phi_i \ln(\phi_i)] \]

and (9) implies that the back-off rate, denoted as \( \nu^{(2)}_i \), should be set as

\[ \nu^{(2)}_i = \frac{\phi_i (1 - \phi_i)^{d_i-1}}{\prod_{(i,j) \in E} (1 - \phi_i - \phi_j)}. \]  

(12)

125
The above expression corresponds to the Bethe approximation for CSMA networks proposed in [27]. It is worth noting that (12) is a fixed point of the inverse belief propagation (IBP) algorithm presented in [8]. More specifically, the update rule in [8, Section IV.C] can be written as

\[ \frac{m_{ij}(0)}{m_{ij}(1)} = 1 + \frac{m_{ij}(0)}{m_{ij}(1)} \frac{\phi_j}{1 - \phi_j}, \]

where \( \phi_j \) is the target throughput of node \( j \). It is easy to check that this update rule has \( m_{ij}(0)/m_{ij}(1) = (1 - \phi_j)/(1 - \phi_i - \phi_j) \) as a fixed point and if we plug this into Equation (8) of [8, Section IV.C] we obtain (12).

**B. Triangle approximation**

A second special case, called the triangle approximation, is obtained by extending the set of regions \( \mathcal{R}' \) as defined in the previous subsection with the regions \( \{R_{i,j,k}(i,j,k) \in E \} \) such that \( V_{R_{i,j,k}} = \{x_i,x_j,x_k\} \) and \( F_{R_{i,j,k}} = \{f_{i,j},f_{i,k},f_{j,k}\} \). The counting numbers are now set as \( c_{R_{i,j,k}} = 1 \) and \( c_{R_{i,j}} = 1 - t_{i,j} \) such that \( c_{R_{i,j}} = t_i - d_j \), where \( t_i \) and \( t_{i,j} \) are the number of triangles in \( G \) that contain node \( i \) and edge \( (i,j) \), respectively. Note that these counting numbers obey the requirement given in (4) as \( \sum_{j \in N_i} t_{i,j} = 2t_i \).

For the triangle approximation the back-off rates, denoted as \( v_t^{(3)} \), given by (9) correspond to

\[ v_t^{(3)} = \left( \frac{1 - \phi_i}{1 - \phi_i - \phi_j} \right)^{n-1}, \]

where \( \Delta_E \) denotes the set of triangles in \( E \).

**C. General case**

The idea behind the Bethe and triangle approximation can be generalized to cliques of larger sizes, at the expense of an increased complexity to compute the back-off rates. In this section the set \( \mathcal{R}' \) corresponds to the set of all the cliques \( K \) in the conflict graph \( G \) with a size in \( \{2, \ldots, k_{\text{max}}\} \), where \( k_{\text{max}} \) is a predefined maximum allowed clique size. Note that setting \( k_{\text{max}} = 2 \) and 3 corresponds to the previous two approximations. If \( K \) is a clique of size \( k \in \{2,3, \ldots, k_{\text{max}}\} \) and \( R(K) \) its associated region, then \( V_{R(K)} = \{x_i | i \in K\} \) and \( F_{R(K)} = \{f_{i,j} | i,j \in K\} \).

The counting number \( c_{R(K)} \) associated with a clique \( K \) of size \( k_{\text{max}} \). For a region \( R(K) \) corresponding to a clique \( K \) of size \( k \) with \( 1 \leq k < k_{\text{max}} \), we set \( c_{R(K)} \)

\[ c_{R(K)} = 1_{\{k>1\}} - \sum_{R' \in \mathcal{R}'} c_{R'1_{V_{R(K)} \subseteq V_{R'}}}. \]

Note, for any maximal clique \( K \) of size \( 2 \leq k \leq k_{\text{max}} \), we have \( c_{R(K)} = 1 \) irrespective of its size. The next proposition provides an explicit expression for \( c_{R(K)} \).

**Theorem 2.** If \( K \) is a clique of size \( k \in \{1, \ldots, k_{\text{max}}\} \) then

\[ c_{R(K)} = 1_{\{k>1\}} + \sum_{s=k+1}^{k_{\text{max}}} (-1)^{s-k} n_{K,s}, \]

where \( n_{K,s} \) denotes the number of cliques \( K' \) in \( G \) with \( |K'| = s \) and \( K \subseteq K' \).

**Proof.** The result clearly holds for \( k = k_{\text{max}} \). We use backward induction on \( |K| = k \) to find

\[ -c_{R(K)} + 1_{\{k>1\}} = \sum_{u=k+1}^{k_{\text{max}}} \sum_{\text{cliques } K'} (1 + \sum_{s=u+1}^{k_{\text{max}}} (-1)^{s-u} n_{K',s}) = \]

\[ \sum_{u=k+1}^{k_{\text{max}}} n_{K,u} + \sum_{u=k+1}^{k_{\text{max}}} \sum_{s=u+1}^{k_{\text{max}}} (-1)^{s-u} \sum_{\text{cliques } K'} n_{K',s} \]

(15)

The latter sum equals \( (s-k) n_{K,s} \) as we can pick \( u-k \) elements from the \( s-k \) elements not belonging to \( K \) of a size \( s \) clique that contains \( K \) to obtain a \( K' \). Hence, switching sums in (15) yields

\[ -c_{R(K)} + 1_{\{k>1\}} = \sum_{s=k+1}^{k_{\text{max}}} n_{K,s} + \sum_{s=k+1}^{k_{\text{max}}} \sum_{u=k+1}^{s-1} (-1)^{s-u} (s-k) n_{K,s} = \]

\[ \sum_{s=k+1}^{k_{\text{max}}} \left[ (1 + (-1)^{s-k} \sum_{z=1}^{s-k-1} (-1)^{z} \frac{s-k}{z}) n_{K,s} \right] \]

(16)

By noting that \( \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \), (16) becomes

\[ c_{R(K)} = 1_{\{k>1\}} + \sum_{s=k+1}^{k_{\text{max}}} (1 + (-1)^{s-k} (1 - (-1)^{s-k})) n_{K,s} \]

\[ = 1_{\{k>1\}} + \sum_{s=k+1}^{k_{\text{max}}} (-1)^{s-k} n_{K,s}, \]

as required. \( \square \)

Note that increasing \( k_{\text{max}} \) by one simply adds one additional term to \( c_{R(K)} \) in (14), which allows us to compute the back-off rates of the size \( k_{\text{max}} \) clique approximation in a recursive manner as follows.
Corollary 1. Let $\nu_i^{(k_{\text{max}})}$ be the back-off rate for node $i$ corresponding with the size $k_{\text{max}}$ clique approximation, then
\[
\nu_i^{(k_{\text{max}})} = \nu_i^{(k_{\text{max}}-1)} \prod_{\text{cliques } K \text{ in } G: i \in K, 1 \leq |K| \leq k_{\text{max}} \atop K \subseteq K'} \left(1 - \sum_{s \in K} \phi_s \right)^{n_{K,k_{\text{max}}},(1),k_{\text{max}}-|K|}_{K'} \prod_{K \subseteq K' \atop i \in K} \left(1 - \sum_{s \in K} \phi_s \right)^{(-1)^{k_{\text{max}}-|K|+1}_{K'}},
\]
where $\nu_i^{(1)} = \phi_i/(1 - \phi_i)$ and $n_{K,k_{\text{max}}}$ denotes the number of size $k_{\text{max}}$ cliques $K'$ in $G$ with $K \subset K'$. Proof. The result follows from (9) when combined with (14).

We now briefly discuss the complexity to compute the back-off rate of node $i$ when using the $k_{\text{max}}$ clique approximation. Node $i$ can be part of at most $\min(2d_i, d_i^{k_{\text{max}}-1})$ cliques $K$ with $|K| \leq k_{\text{max}}$, where $d_i$ is the number of neighbors of $i$ in $G$. This set can be computed by first listing the $d_i$ size 2 cliques containing $i$. Having obtained the set of size $k$ cliques that contain $i$, the set of size $k+1$ cliques is found by considering all its one element extensions. By using an ordered list \(\{i_1, \ldots, i_{k-1}\}\) of the $k-1$ other nodes belonging to a size $k$ clique containing $i$, only one element extensions with a node $j > i_{k-1}$ need to be considered and the creation of identical cliques of size $k+1$ is avoided. Having obtained the list of cliques $K$ that contain $i$ with $|K| \leq k_{\text{max}}$, the back-off rate given by (17) can be readily computed by noting that
\[
\nu_i^{(k_{\text{max}})} = \nu_i^{(k_{\text{max}}-1)} \prod_{\text{cliques } K' \text{ in } G: i \in K', |K'| = k_{\text{max}} \atop K \subseteq K'} \left(1 - \sum_{s \in K} \phi_s \right)^{n_{K,k_{\text{max}}},(1),k_{\text{max}}-|K'|}_{K'} \prod_{K \subseteq K' \atop i \in K} \left(1 - \sum_{s \in K} \phi_s \right)^{(-1)^{k_{\text{max}}-|K'|+1}_{K'}.}
\]

VI. KIKUCHI APPROXIMATIONS

The IGBP algorithm of [8] is a message passing algorithm to estimate the back-off rates to achieve a given throughput vector $(\phi_1, \ldots, \phi_n) \in \Gamma$. This algorithm is based on a so-called Kikuchi free energy approximation. In this section we show that the size $k_{\text{max}}$ clique approximation also coincides with a Kikuchi approximation. In fact for $k_{\text{max}} = n$ this Kikuchi approximation corresponds to the one associated to the IGBP algorithm. As such the expression for the back-off rates of the size $n$ clique approximation gives us an explicit expression for a fixed point of the IGBP algorithm in [8, Section VI.B] due to [26, Section VII] (as the Bethe approximation did for the IBP algorithm).

In a Kikuchi approximation (see [26, Appendix B] for more details) the set of regions $\mathcal{R}$ can be written as $\mathcal{R} = \bigcup_{i=0}^{n} \mathcal{R}_i$, for some $s$. We state that a region $R$ is a subset of a region $R'$ if $V_R \subseteq V_{R'}$ and $\mathcal{F}_R \subseteq \mathcal{F}_{R'}$. The regions in $\mathcal{R}_0$ fully characterize a Kikuchi approximation as follows. The regions in $\mathcal{R}_{i+1}$ for $i = 0, \ldots, s$, are constructed from the sets $\mathcal{R}_0, \ldots, \mathcal{R}_i$ by taking all the different intersections $\mathcal{R}_i \cap \mathcal{R}_j \neq \emptyset$, with $\mathcal{R}_i \subseteq \mathcal{R}_j$ and $\mathcal{R}_j \subseteq \mathcal{R}_i$, of the regions $\mathcal{R}_i \in \mathcal{R}_i$ with the regions $\mathcal{R}_j \in \bigcup_{k=0}^{s} \mathcal{R}_k$ and subsequently removing the sets $\mathcal{R} \in \mathcal{R}_{i+1}$ for which there exists an $R' \in \mathcal{R}_{i+1}$ with $R \subseteq R'$. Note, $s$ is the smallest integer such that $\mathcal{R}_{s+1}$ is empty. The counting number $\tilde{c}_R$ of region $R \in \mathcal{R}_i$ in a Kikuchi approximation is given by
\[
\tilde{c}_R = 1 - \sum_{R' \in R: R \subseteq R'} \tilde{c}_{R'},
\]
as a region $R \in \mathcal{R}_i$ cannot be a subset of a region $R' \in \mathcal{R}_j$ with $j > i$ (since this would imply the existence of a superset of $R$ in $\mathcal{R}_i$).

Theorem 3. The size $k_{\text{max}}$ clique approximation coincides with a Kikuchi approximation with $\mathcal{R}_0 = \{R_1, \ldots, R_n\} \cup \{R(K) | K \in K_G(k_{\text{max}})\}$, where $K_G(k_{\text{max}})$ is the union of the set of all the cliques of size $k_{\text{max}}$ and the set of the maximal cliques of size $k \in \{2, \ldots, k_{\text{max}} - 1\}$ in $G$.

Proof. See Appendix A of [22].

The above theorem shows that the maximal clique based Kikuchi approximation considered in [15], [16] coincides with the size $n$ clique approximation. We note that no explicit expression for the counting numbers or a recursive scheme similar to (17) is presented in [15], [16].

VII. CHORDAL CONFLICT GRAPHS

In this section we establish two results: (a) we show that the exact explicit expressions for the back-off rates for chordal conflict graphs, presented in [21], corresponds to a zero gradient point of a region-based free energy approximation defined for chordal conflict graphs only and (b) we prove that the size $n$ clique approximation coincides with this chordal free energy approximation. This implies that the size $n$ clique approximation (and therefore also a fixed point of the IGBP algorithm of [8]) provide exact results for chordal conflict graphs $G$.

A graph $G$ is chordal if and only if all cycles consisting of more than 3 nodes have a chord. A chord of a cycle is an edge joining two nonconsecutive nodes of the cycle. Let $K_G = \{K_1, \ldots, K_m\}$ be the set of maximal cliques of $G$. A clique tree $T = (K_G, \mathcal{E})$ is a tree in which the nodes correspond to the maximal
cliques and the edges are such that the subgraph of \( T \) induced by the maximal cliques that contain the node \( v \) is a subtree of \( T \) for any \( v \in V \). A graph \( G \) is chordal if and only if it has at least one clique tree (see Theorem 3.1 in [1]).

For chordal conflict graphs \( G \) we can define a region-based free energy approximation, called the **chordal region-based free energy approximation**, by making use of any clique tree \( T = (K_G, E) \) of \( G \) in the following manner. We define a set \( R \) containing \( 2n + 2|K_G| - 1 \) regions: one region \( R_K \) for each maximal clique \( K \in K_G \), one region \( R_{(K, K')} \) for each edge \( (K, K') \in E \), one region \( R_f \) for each factor node \( f \), and one region \( R_s \) for each variable node \( x \). Let \( V_R \) and \( F_R \) denote the set of variable and factor nodes associated with region \( R \in \mathcal{R} \), then \( V_{R_K} = \{ x_i | i \in K \} \) and

\[
F_{R_K} = \{ f_{(i,j)} | i, j \in K \},
\]

for \( K \in K_G \), \( V_{R_{(K, K')}} = \{ x_i | i \in K \cap K' \} \) and

\[
F_{R_{(K, K')}} = \{ f_{(i,j)} | i, j \in K \cap K' \},
\]

for \( (K, K') \in E \) and \( V_{R_f} = V_{R_s} = \{ x_i \} \), \( F_{R_f} = \emptyset \) and \( F_{R_s} = \{ f \} \). The counting numbers \( c_R \) are defined as follows: \( c_{R_K} = c_{R_f} = 1 \) and \( c_{R_{(K, K')}} = c_{R_s} = -1 \).

Note the set of regions \( \mathcal{R} \) fulfills Condition 1, therefore under clique belief we have \( U_{\mathcal{R}}(\{ b_R \}) = -\sum_{i \in V} \phi_i \ln(\nu_i) \). As the nodes in \( K \) and \( K' \) form a clique, the clique belief matches the exact marginal probabilities \( p_R(x_K) \) for each \( R \in \mathcal{R} \) when

\[
\phi_i = \sum_{x \in \{0,1\}^n} p(x) 1_{\{ x_i = 1 \}} \quad \text{for } i \in V.
\]

As the beliefs \( b_{R_{(K, K')}}(x_I) \) and \( b_{R_s}(x_I) \) are the same and \( c_{R_f} = -c_{R_s} \), these regions cancel each other in the expression for the entropy. Thus for the entropy we have

\[
H_{\mathcal{R}}(\{ b_R \}) = -\sum_{i \in V} \phi_i \ln(\nu_i) + \sum_{(K, K') \in E} \left( 1 - \sum_{s \in K \cap K'} \phi_s \right) \ln(1 - \sum_{s \in K \cap K'} \phi_s) - \sum_{K, K' \in K_G} \left( 1 - \sum_{s \in K} \phi_s \right) \ln(1 - \sum_{s \in K} \phi_s).
\]

(18)

As noted before, even when the beliefs are equal to the exact marginal probabilities, the region-based entropy is in general not exact. Below we prove that the entropy (and therefore also the energy) is exact in this particular case by leveraging existing results on the junction graph approximation method [26, Appendix A].

**Theorem 4.** The expression for the region-based entropy \( H_{\mathcal{R}}(\{ b_R \}) \) given by (18) is equal to the Gibbs entropy \( H(p) \) defined by (3). Further, \( H(p) = -\ln(\frac{1}{2} \prod_{i \in V} \nu_i^{p_i(1)}) \) and if \( x \in \{0,1\}^n \) such that \( x_i x_j = 0 \) if \( (i, j) \in E \) \( (p(x) = 0 \) otherwise), we have

\[
p(x) = \frac{\prod_{K \in K_G} (\vartheta_1(K) + \vartheta_0(K))}{\prod_{(K, K') \in E} (\vartheta_1(K \cap K') + \vartheta_0(K \cap K'))},
\]

(19)

where \( \vartheta_1(S) = \sum_{i \in S} p_i(1) 1_{\{ x_i = 1, \sum_{s \in S} x_s = 1 \}} \) and \( \vartheta_0(S) = (1 - \sum_{s \in S} p_s(1)) 1_{\{ \sum_{i \in S} x_i = 0 \}} \).

**Proof.** See Appendix B of [22].

**Corollary 2.** For a chordal conflict graph \( G \) with clique tree \( (K_G, E) \), the normalizing constant \( Z \) is given by

\[
Z = \frac{\prod_{(K, K') \in E} (1 - \sum_{s \in K \cap K'} p_s(1))}{\prod_{K \in K_G} (1 - \sum_{s \in K} p_s(1))},
\]

(20)

and the back-off rate \( \nu_i \) obeys

\[
\nu_i = p_i(1) \frac{\prod_{(K, K') \in E \cap K \cap K'} (1 - \sum_{s \in K \cap K'} p_s(1))}{\prod_{K \in K_G; i \in K} (1 - \sum_{s \in K} p_s(1))},
\]

(21)

where the marginal probability \( p_i(1) \) is the throughput of link \( i \).

**Proof.** The expression for \( Z \) follows from equating (1) and (19) with \( x = (0, \ldots, 0) \). The back-off rate \( \nu_i \) is found in the same way using \( x \) with \( x_i = 1 \) and \( x_j = 0 \) for \( i \neq j \).

Note the above formula for the back-off rate \( \nu_i \) for chordal conflict graphs \( G \) was derived earlier in [21] and corresponds to (9).

**Theorem 5.** When the conflict graph \( G \) is chordal the Kikuchi approximation with \( \mathcal{R}_0 = \{ R_1, \ldots, R_f \} \cup \{ R(K)|K \in K_G \} \), where \( K_G \) is the set of the maximal cliques in \( G \), coincides with the chordal region-based free energy approximation (defined for chordal conflict graphs only).

**Proof.** See Appendix C of [22].

**Corollary 3.** When \( G \) is chordal, the back-off rates given by (9) for the size \( k_{max} = n \) clique approximation or the Kikuchi approximation with \( \mathcal{R}_0 = \{ R_1, \ldots, R_f \} \cup \{ R(K)|K \in K_G \} \), where \( K_G \) is the set of the maximal cliques in \( G \), are both equal to (21) and are therefore exact.

The exactness of the above Kikuchi approximation for chordal conflict graphs was established independently in [15], [16].
we also note that node $k$ it uses the size to compute its back-off rate irrespective of whether $R$ the ones used in [21]: the throughputs observed during simulation. relative error between the given target throughputs model with the back-off rates set as estimated by chordal subgraph (LCS) approximation presented in VIII. EXPERIMENTAL EVALUATION

To study the accuracy of the size $k_{max}$ clique approximation we perform simulation experiments similar to one ones presented in [8, Section IV.E] for IGBP, except that we consider a different set of conflict graphs and also compare with the local chordal subgraph (LCS) approximation presented in [21]. More specifically, we simulate the ideal CSMA model with the back-off rates set as estimated by each approximation method and compute the mean relative error between the given target throughputs (used as input by the approximation method) and the throughputs observed during simulation.

The set of conflict graphs considered is similar to the ones used in [21]: the $n$ nodes of the conflict graph are placed randomly in a square of size 1 and there exists an edge between two nodes if and only if the Euclidean distance between them is less than some threshold $R$. In the experiments we set $n = 100$ nodes with $R$ values equal to 0.15, 0.2 and 0.25. At this point we also note that node $i \in V$ requires the same input to compute its back-off rate irrespective of whether it uses the size $k_{max}$ clique approximation or the LCS approximation: it needs to construct the subgraph $G_i = (V_i, E_i)$ induced by node $i$ and its $d_i$ neighbors. Hence both approximations can be implemented in a fully distributed manner.

In a first set of experiments the target throughput of each node was set equal to $\phi$ divided by the size of the largest clique in the graph, where $\phi$ equals 0.55, 0.7 and 0.85. Note that for $\phi > 1$ this vector does not belong to $\Gamma$, the set of achievable throughput vectors. Figure 3 presents the mean relative error of the LCS and size $k_{max}$ clique approximation for various combinations of $R$ and $\phi$ (the corresponding conflict graph for $R = 0.20$ is shown in Figure 2). For each value of $R$, the largest value for $k_{max}$ presented in this figure corresponds to the size of the largest clique in the graph, that is, the same result is obtained when setting $k_{max} = n$. Recall that we showed earlier that the size $n$ clique approximation coincides with a fixed point of the IGBP algorithm of [8]. Thus, the rightmost bar in Figure 3 corresponds to a fixed point of the IGBP algorithm. Figure 3 shows that the size $k_{max}$ clique approximation is more accurate that the LCS approximation in this setting, except for small $k_{max}$, at the expense of being more complex.

We further note that the relative errors grow as the graph becomes more dense (increasing $R$) and this growth seems more pronounced for the LCS approximation. We also note that the approximation becomes worse as the target throughput of the links increases (increasing $\phi$). Nevertheless the mean relative error of the size $n$ clique approximation remains below 2% in all cases. This is somewhat higher than the values reported in [8] for IGBP, but this is mostly due to the fact that more dense conflict graphs are considered here (for $R = 0.15$ we have 301 edges, while for $R = 0.25$ we have as many as 788 edges).

While the results in Figure 3 are based on three conflicts graphs only, Figure 4 compares the accuracy of the LCS and size $k_{max}$ approximations with $k_{max} = 2,5, n$ on a set of 75 conflict graphs: 25 for each of the three $R$ values. The target throughput of node $i$ is set equal to $0.85/(1 + d_i)$, meaning not all nodes have the same target throughput (as opposed to Figure 3). For each $R$ value and approximation method considered, Figure 4 depicts the mean relative throughput error (obtained by simulation) for the 2 conflict graphs that resulted in the smallest and largest mean relative throughput error, as well as the average taken over the 25 conflict graphs.

The results in Figure 4 are in agreement with Figure 3: the LCS approximation outperforms the Bethe approximation, increasing $k_{max}$ reduces the
relative throughput errors and the LCS error increases more significantly when the graph becomes denser compared to the size $n$ approximation. We further note that the size $k_{\text{max}} = 5$ approximation produces errors close to the size $n$ approximation, which is a useful observation in case we wish to limit the time needed to compute the required back-off rates.

To get an idea on the computation times of the back-off vector for the different approximations, we generated 1000 conflict graphs with $n = 100$ and $R \in (0, 0.25]$. Figure 5 depicts the average time needed to compute the vector of back-off rates for all the conflict graphs with a maximum clique size between 4 and 15 (only 9 of the 1000 conflict graphs contained a 15+ clique). The results show that the computation times of the LCS and Bethe approximation have a similar shape. They also highlight that for denser graphs limiting $k_{\text{max}}$ may offer an attractive trade-off between the computation times and accuracy of the approximation.

We end by noting that the time complexity to compute the back-off rate for node $i$ only depends on the structure of the subgraph $G_i$ induced by node $i$ and its neighbors. Thus the overall network size $n$ has no impact on the computation times and as such the approximation method is also suitable for very large graphs as long as the size of the one hop neighborhood does not scale with the overall network size. Further the complexity to compute $\nu_i$ for the size $n$ approximation is similar to performing a single iteration of the IGBP algorithm, for which convergence to a (unique) fixed point is not guaranteed and the number of iterations grows with the density of the conflict graph (see [8, Tables IV and V]).

IX. CONCLUSIONS

In this paper we presented the class of region-based free energy approximations for the ideal CSMA model, which contains the Bethe approximation of [27] as a special case. We obtained a closed form expression for the vector of back-off rates that corresponds to a zero gradient point of the free energy within the set of clique beliefs (in terms of its counting numbers).

We subsequently focused on the size $k_{\text{max}}$ clique approximation (which can be implemented in a fully distributed manner) and derived explicit expressions for its counting numbers as well as a recursive method to compute the back-off rates more efficiently. We further showed that this approximation is exact on chordal conflict graphs and coincides with a Kikuchi approximation. The latter result implies that the size $k_{\text{max}}$ clique approximation with $k_{\text{max}} = n$ gives an explicit expression for a fixed point of the IGBP algorithm of [8]. The paper also contains an alternate proof for the back-off rates needed to achieve any achievable throughput vector in a chordal graph $G$ presented in [21].

There are a number of possible extensions to the work presented in this paper. First, while this paper has focused on achieving a given target throughput vector, it should be possible to consider utility maximization problems as in [27]. Second, one could try to relax the conflict graph based interference model considered in this paper to a more realistic SINR (signal-to-interference-plus-noise ratio) model. In fact, such a relaxation of the Bethe approximation presented in [27] was recently developed in [17]. Finally, other free energy approximation techniques such as the tree-based reparameterization framework of [23] could be considered as well.
