On the critical behavior of a lattice prey-predator model

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The critical properties of a simple prey-predator model are revisited. For some values of the control parameters, the model exhibits a line of directed percolation like transitions to a single absorbing state. For other values of the control parameters one finds a second line of continuous transitions toward infinite number of absorbing states, and the corresponding steady-state exponents are mean-field like. The critical behavior of the special point \( T \) (bicritical point), where the two transition lines meet, belongs to a different universality class. The use of dynamical Monte-Carlo method shows that a particular strategy for preparing the initial state should be devised to correctly describe the physics of the system near the second transition line. Relationships with a forest fire model with immunization are also discussed.

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I. INTRODUCTION

The study of prey-predator systems has attracted a lot of attention since the pioneering works of Lotka [1] and Volterra [2]. Working at a mean-field level (homogeneous populations) they showed that, depending on the initial state, the system can evolve toward a simple steady-state or a limit cycle, in which the populations oscillate periodically in time.

An important question is the understanding of the role played by the local environment on the dynamics (spatial effects) [3] and, accordingly, many extended prey-predator models have been studied during the past years [4–9]. Recently, a simple prey-predator model was introduced by some of us [10]. Although governed by only two control parameters, this model exhibits a rich phase diagram. As a function of the two control parameters \( \lambda_a \) and \( \lambda_b \), which are the growth rates of prey and predator respectively, two different phases are observed: a pure prey phase (P), and a coexistence phase of prey and predator in which an oscillatory (O) region and a non-oscillatory (NO) region can be distinguished. For a system size \( L \to \infty \), these three different domains meet at a particular point, called \( T = (\lambda_a^T, \lambda_b^T) \) (precise definitions are given below). It was shown [10] that \( \lambda_a^T = 0 \) and \( \lambda_b^T \approx 5.0 \pm 0.3 \). For \( \lambda_a > 0 \), a phase transition line between the pure prey phase and the coexistence phase is present, and the critical exponents along this line are the ones of directed percolation (DP) [3]. However, it was also observed that when the growth rate of prey \( \lambda_a \to 0 \) and \( \lambda_b > \lambda_b^T \), the model undergoes a non-DP continuous phase transition. Since the directed percolation is a generic universality class for models with absorbing states (unless some special conditions are satisfied [12]), existence of such a transition is certainly surprising. These two lines of different continuous non-equilibrium phase transition meet at the bicritical point \( T' \) [13], and one forecast that the critical behavior at this particular point may also belong to a new universality class.

The goal of this paper is to study in more details the properties of these non-DP phase transitions. First, we perform extensive steady-state simulations, which confirm the non-DP character of the transition in the limit \( \lambda_a \to 0 \) and \( \lambda_b \geq \lambda_b^T \).

As it was already shown [10], in this limit the model exhibits oscillatory behavior. But, in addition to that, for \( \lambda_a = 0 \) the model has infinitely many absorbing states. These two properties are responsible for a rather peculiar behavior of the model, which becomes particularly transparent when the model is examined using the dynamical Monte Carlo method. When applied to models with infinitely many absorbing states, this dynamical method uses the so-called natural absorbing states, which are the most likely states to be reached by the dynamical evolution of the system. We show, however, that this common but somehow heuristic procedure fails here. Indeed, for the present model, natural absorbing states contain only short-ranged islands of prey on which spreading is not critical. To restore criticality of spreading we generated short-ranged islands of prey on which spreading is not critical. A thorough investigation of the critical behavior has been

\[\lambda \in \text{conditions are given below). It was shown [10] that absorbing states (unless some special conditions are sat-\]
made using two different complementary approaches. In Sec. II, the critical behavior is investigated using steady state properties while in Sec. III one uses dynamical Monte Carlo method. It is shown that for $\lambda_a \to 0$ and $\lambda_b \geq \lambda_T^b$ the steady-state exponents are indeed mean-field-like, while the dynamical exponents are non-universal depending continuously upon $\lambda_b$. Nevertheless, trace of the mean-field character of the transition shows up in scaling relation among dynamical exponents. The critical behavior at point $T$ is also investigated, and it turns out that the corresponding exponents belong to a new universality class. Finally, physical arguments explaining the above findings are given in Sec. IV.

II. MODEL

The model used in Ref. [10] is defined as follows. Each cell of a two-dimensional square lattice (of size $L \times L$, with periodic boundary condition), labeled by the index $i$, can be at time $t$, in one of the three following states: $\sigma_i = 0$, 1, and 2. A cell in state 0, 1, or 2 corresponds, respectively, to a cell which is empty, occupied by prey, or simultaneously occupied by prey and predators. The transition rates for site $i$ are (i) $0 \to 1$ at rate $\lambda_a(n_{i,1} + n_{i,2})/4$, (ii) $1 \to 2$ at rate $\lambda_b(n_{i,1}/6)$, and (iii) $2 \to 0$ at rate 1, where $n_{i,\sigma}$ denotes the number of nearest neighbor sites of $i$ which are in the state $\sigma$. The first two processes model the spreading of prey and predators. The third process represents the local depopulation of a cell due to overly greedy predators. The rate of the third process is chosen to be 1, which sets the time scale, hence $t$, as well as $\lambda_a$ and $\lambda_b$ are dimensionless quantities.

The properties of this model have been investigated both by mean-field and Monte-Carlo methods [11]. The Monte-Carlo result extrapolated to the case $L \to \infty$ is summarized in Fig. 1. The transition line between the prey phase and the coexistence phase, $\lambda_T^b(\lambda_a)$, belongs to the directed percolation universality class [13], as expected, and terminates at the point $T \equiv (\lambda_T^d = 0, \lambda_T^b)$, where the P, O, and NO domains meet. For $\lambda_b > \lambda_T^b$, the transition between the oscillatory domain of the coexistence phase and the prey phase takes place at $\lambda_a = 0$. Along this transition line the predator density approaches zero as a power law $b \sim \lambda_T^b \beta_2$, with $\beta_2 \approx 1$, and so, does not belong to the DP class, which is somehow unexpected. The value $\beta_2 = 1$ lead to the conjecture [11] that this second transition could be mean-field-like. There is a crossover between the O and NO parts of the coexistence phase. The purpose of the present study is to give a complete description of the nature of the transitions near the line $\lambda_a = 0$ for $\lambda_b \geq \lambda_T^b$.

It is worth to mention, that our model is closely related to a model introduced by Drossel et al. [14] to investigate the effect of immunization in an extension of the simple forest-fire model [15]. This three-state model (0: empty site, 1: tree, and 2: burning tree) differs from our model in some details: the growth rate of a tree ($\sigma : 0 \to 1$) is $p$, independently of the environment, and a tree is ignited ($\sigma : 1 \to 2$) with rate $(1 - g)\Theta(n_2)$, ($\Theta$ is the usual Heaviside-function). This second process models the immunization of trees against fire. The third process ($\sigma : 2 \to 0$) happens at rate 1. For non zero immunity and $p > 0$ Albano [13] showed that the transition toward a single absorbing state is DP like, while for $p = 0$ (at the end point of the DP transition line), the transition belongs to the dynamical percolation universality class, and the absorbing state is not unique.

III. STEADY STATE STUDY OF THE CRITICAL BEHAVIOR

Extensive Monte-Carlo simulations for system sizes up to $4000 \times 4000$ have been performed to investigate the behavior of the predator density $b$, for $\lambda_a \to 0$ and three different values of $\lambda_b$, namely, $\lambda_b = 4.67$, 5.0, and 6.0, following trajectories of type 2 and 4 in Fig. 1. The value $\lambda_a = 0, \lambda_b = 4.67$ corresponds to the best determination of the end point $T$, obtained by the dynamical approach described below. Owing to the oscillatory behavior near the critical line ($\lambda_a = 0, \lambda_b > \lambda_T^b$), the system very easily evolves into an absorbing state, where the predators are extinct, therefore careful initialization is needed in the simulations. Usually $10^4$ MCS "thermalization" were applied following $10^4$ MCS "approaching" time where $\lambda_a$ was decreased continuously. The densities and the fluctuations of prey and predators were averaged over $\sim 2 \times 10^5$ MCS for each $\lambda_a$, $\lambda_b$ points. It is found...
that \( b \sim \lambda_a^{2\beta} \) for \( \lambda_a \to 0 \). In order to see corrections to scaling, we compute the effective exponent

\[
\beta_{eff}(\lambda_a(i)) = \frac{\ln b(\lambda_a(i)) - \ln b(\lambda_a(i-1))}{\ln \lambda_a(i) - \ln \lambda_a(i-1)},
\]

where \( \lambda_a(i) \) and \( \lambda_a(i-1) \) are two consecutive values of the control parameter \( \lambda_a \).

\[
\beta_{eff}(\lambda_a(i)) \sim \lambda_a^{2\beta}.
\]

(1)

As the Fig. 2 shows, for \( \lambda_b = 5 \) and 6, the linear extrapolation of \( \beta_{eff} \) converges to \( \approx 1 \) within statistical errors

\[
\beta(\lambda_b = 5.0) = 1.01(1), \quad \beta(\lambda_b = 6.0) = 0.96(2).
\]

(2)

(3)

For \( \lambda_b = 4.67 \) (trajectory of type 2 in Fig. 1) it goes to a somewhat higher value, and one finds

\[
\beta(\lambda_b = 4.67) = 1.33(4).
\]

(4)

The measurement of the fluctuations

\[
\chi_b = L^2 ((b - \langle b \rangle)^2) \sim \lambda_a^{-\gamma}
\]

are less precise and we estimate \( \gamma = -0.6(16) \) at \( \lambda_b = 4.67 \), and \( \gamma \sim 0 \) for \( \lambda_b = 5, 6 \).

For \( \lambda_b > \lambda_b^c \) these values are consistent with the previous prediction \( \beta_2 \approx 1 \). However, at the bicritical point \( T \) the value of \( \beta_2 \) is completely different and thus belongs to a new universality class.

IV. DYNAMICAL STUDY OF THE CRITICAL BEHAVIOR

A very useful technique to study the critical properties of a system with absorbing states is the so-called dynamical Monte Carlo method \[17\]. In this approach, the system is prepared in an initial state, which is one of the absorbing states up to one site, which is set to be in the active state. One considers an ensemble of trials starting from the same initial state. Certain dynamical quantities exhibit a power law behavior when the system is critical. For example, the survival probability behaves as

\[
P(t) \sim t^{-\delta}.
\]

(6)

The deviation from this power law behavior, when the system is off-critical, provides a very precise way to locate the critical point.

The number of active sites \( N(t) \) behaves as

\[
N(t) \sim t^\eta
\]

(7)

while, for the mean square spreading from the origin \( R^2(t) \)

\[
R^2(t) \sim t^z
\]

(8)

where the dynamical exponent \( z = 2\nu_\perp/\nu_\parallel \) is the ratio of the critical exponents of spatial (\( \nu_\perp \)) and temporal (\( \nu_\parallel \)) correlation lengths. Some scaling relations between these exponents can be also derived \[18\].

We made various simulations using the dynamical Monte Carlo method and our results are summarized below.

A. Case of \( \lambda_a = 0 \)

\[
\begin{array}{c}
\log_{10}(t) \\
\log_{10}(P(t))
\end{array}
\]

Fig. 3. The survival probability \( P(t) \) as a function of \( t \) obtained for \( \lambda_a = 0 \) and (from the top) \( \lambda_b = 4.6, 4.65, 4.67, 4.7, \) and 4.75 (trajectory of type 3 in Fig. 1). We used system size \( L = 3000 \), and up to \( 10^5 \) independent runs were made for each value of \( \lambda_b \). The dotted line has the slope corresponding to \( \delta = 0.092 \).
First, we simulated the model on the $\lambda_d = 0$ line, taking, as an absorbing state, a lattice filled with prey (trajectory of type 3 in Fig. 1). Measuring the survival probability $P(t)$, we found that $\lambda_0 = \lambda_{b}^T \approx 4.67$ is the critical point, which separates the absorbing phase ($\lambda_b < \lambda_{b}^T$) and the phase with annular growth ($\lambda_b > \lambda_{b}^T$). Measuring the slope at $\lambda_b = \lambda_{b}^T$ (see Fig. 3), we estimate $\delta \approx 0.095(5)$, which is very close to the value obtained for dynamical percolation, for which, in two dimensions, $\delta = 0.092$ [18]. Moreover, using (7) and (8) we obtained $\eta = 0.60(5)$ and $z = 1.72(4)$, which are also very close to the dynamical percolation values.

Note that the usual $\beta$ exponent for dynamical percolation, which takes the value $\beta \sim 0.14$, is defined through $b \sim (\lambda_{b}^T - \lambda_b)^3$, which differs from our definition of $\beta_2$ in Sec. III. Thus it is not surprising that these two exponents differ. Note also that the dynamical estimation of the critical endpoint $\lambda_b = \lambda_{b}^T$ has been used in the static approach of Sec. III.

B. Case of $\lambda_a > 0$

The same scheme was used for $\lambda_a > 0$. The critical point was located for $\lambda_a = 0.5$ and 1 (trajectory of type 1 in Fig. 1), and they agree with the steady state results of Ref. [10]. Measuring the slope at criticality (see Fig. 4) we estimate $\delta \approx 0.45$, i.e., the value compatible with DP [18]. The fact, that for $\lambda_a > 0$ the phase transition belongs to the DP universality class, was already confirmed using the static calculations [10].

C. Inhomogeneous absorbing states

Static simulations suggest that the model becomes critical on the line $\lambda_b > \lambda_{b}^T$ and $\lambda_a = 0$. Moreover, let us notice that for $\lambda_d = 0$ there are infinitely many absorbing states: indeed, any configuration without predators is an absorbing state. It is well known that dynamical Monte Carlo method can be applied also to models with infinitely many absorbing states. However, as we will see below, applicability of this method to the criticality on this line requires serious reconsiderations.

First, let us recall that the dynamical Monte Carlo method for models with infinitely many absorbing states usually uses the so-called natural absorbing states, i.e. states which are reached by the model’s dynamics. Numerical evidence suggests that for such states the dynamical critical point coincides with the static one. Moreover, dynamical exponents $\delta$ and $\eta$, measured on such states, take universal values.

Following this prescription, we generated natural absorbing states for $\lambda_a = 0$ and $\lambda_b > \lambda_{b}^T$, and then used such states to perform dynamical simulations. An initial configuration was chosen randomly, with equal probabilities of a site being empty, occupied by prey, or simultaneously by prey and predator. Fixing $\lambda_a(= 0)$ and $\lambda_b$, we then allowed the system to evolve until an absorbing configuration was reached (i.e. all predators die out). Our results, presented in Fig. 5, show, however, that the spreading of activity is not critical (i.e. power-law), but rather exponential. But we can argue that this is not surprising. Indeed, a random initial configuration (with probability of prey equal 1/3) is below the percolation threshold with respect to the clusters of prey, and contains only finite clusters of them [13]. On such clusters activity has certainly finite duration (for $\lambda_a = 0$ prey do not grow) and the exponential decay of $P(t)$, as it can be seen in Fig. 3, is an expected feature.

Clearly, the lack of criticality in $P(t)$ is due to the finiteness of prey clusters in the natural absorbing states. In principle, we can cure this effect starting from random initial configurations but containing larger fraction of prey. For sufficiently large concentrations the system will be above the percolation threshold and activity will be able to spread infinitely. Probably, for a certain concentration of prey we can tune the system to have power-law decay for $P(t)$. Such a procedure, however, is somehow artificial, and criticality of spreading will not be related with the static criticality of the system.
FIG. 5. The survival probability $P(t)$ as a function of $t$ obtained for $\lambda_a = 0$ and for $\lambda_b = 6$ (solid line) and 8 (dotted line). Absorbing states were obtained using continuous cooling with the cooling rates (from the top) $r = 0.0001$, 0.0001, 0.001, 0.01, and $\infty$. For the slowest cooling we used the system size $L = 1000$. For each cooling rate we generated $10^3$ absorbing states and for each absorbing state we generated from $10^2$ to $10^5$ independent runs.

The question arises here, whether it is possible to generate absorbing states on which spreading would be critical, and where this criticality would be generated “more naturally”? Hopefully, criticality of spreading on absorbing states obtained during such a procedure should be related with the steady-state criticality of the model. In the following we suggest a procedure which imitate the quasi-static approach to the critical point on the line $\lambda_a = 0$. In our approach we gradually reduce $\lambda_a$ according to the formula:

$$\lambda_a = \lambda_a^0 \exp(-rt), \quad (9)$$

where $\lambda_a^0 = 1$ and $r$ is the ‘cooling’ rate. [We expect that the detailed time dependence in Eq. (9) is not relevant as long as it is a slow process]. We terminate the cooling when an absorbing state is reached. When the cooling is slow, the system have enough time to build large clusters of prey. Our simulations for $\lambda_b = 6$ suggest (see Fig. 3) that in the limit $r \to 0$, such absorbing states are critical, with $\delta = 0.59(10)$ (along trajectories of type 5 in Fig. 1). Measuring the number of active sites $N(t)$, and using Eq. (9) we estimate $\eta = 0.34(10)$ (see Fig. 4). The departure of the curves from a straight line observed for large values of time is related to the finiteness of the cooling rate. Moreover, we measured the averaged squared distance $R^2(t)$, and using Eq. (8) we obtain $z = 2.0(1)$ (see Fig. 5). Actually, we expect that the correct value of this exponent is $z = 2$. Indeed, in Eq. (8) one makes the average only over surviving runs, thus the long-time contributions to $R^2(t)$ come from the rare events, when the activity happened to be placed on a large island of prey. Since $\lambda_b > \lambda_b^c$, the activity on such islands spreads in a deterministic way (annular growth), which leads to $z = 2$.

The same procedure for $\lambda_b = 8$ yields: $\delta = 0.35(10)$, $\eta = 0.67(10)$ and $z = 2.0(1)$. (Relatively large errors of estimations of critical exponents are due to several, and difficult to estimate factors, as finite time, finite cooling rate $r$, and statistical fluctuations.) Such results confirm that $z = 2$, and suggest that exponents $\delta$ and $\eta$ might change continuously with $\lambda_b$. Non-universality of these exponents is a well-known property for some other models belonging to the directed percolation universality class [20]. Note, however, that the situation is different in our case, because the non-universality is related to the value of $\lambda_b$, rather than to the choice of the initial state. Such control parameter dependence has already been observed in other models [21]. Note that non-universal behavior
is not present along the DP line since the corresponding absorbing state is unique. Let us finally note that δ + η seems to be close to unity, which is an exact mean-field result (δ_{MF} = 1, η_{MF} = 0). This is the only dynamical trace of the mean-field nature of the transition observed in the steady state. Let us emphasize, however, that the criticality of spreading appears only if we prepare the absorbing states using the method which mimics the quasi-steady-state evolution of the model.

V. CONCLUSIONS

The detailed investigation of the critical properties of a simple prey-predator model introduced in Ref. [10] showed the presence of three different types of nonequilibrium phase transitions between active and absorbing states. First, the existence of a usual DP-like transition line was confirmed at λ^a (κ^a) for λ^a > 0. Second, a mean-field-like transition was observed for λ^a → 0, λ^b > λ^0. The mean-field character of this transition can be explained in terms of oscillations present in the model. Indeed, as described in Ref. [10], when λ^a → 0, the system is subject to large density oscillations. These oscillations are generating an important local mixing of the possible states, leading to a mean-field-like behavior.

The criticality along the λ^a = 0 line was confirmed with the dynamical approach using specially prepared inhomogeneous initial states. Some dynamical trace of the mean-field nature of this transition was also observed. Third, at the bicritical point η^c, where the two different critical lines meet, we found a dynamical percolation type transition moving along the λ^a = 0 line, while approaching the T point from finite λ^a’s we observed a new type of critical behavior.

The measured exponents corresponding to the λ^a = 0 line are summarized in Table I. The best numerical estimates for the critical exponents of the two-dimensional dynamical percolation are given for comparison: δ = 0.092, η = 0.586, and z = 1.771.

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| Exponent | λ^a | 4.67 | 5.0 | 6.0 | 8.0 |
|----------|-----|------|-----|-----|-----|
| β_2      | 1.33(4) | 1.01(1) | 0.96(4) | –   |
| γ        | -0.65(10) | -0.11(1) | -0.05(5) | –   |
| δ        | 0.093(5) | –   | 0.39(10) | 0.35(10) |
| η        | 0.60(5) | –   | 0.34(10) | 0.67(10) |
| z        | 1.72(4) | –   | 2.0(1) | 2.0(1) |

TABLE I. Critical exponents around λ^a = 0

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