Connecting phase transitions between the 3-d O(4) Heisenberg model and 4-d SU(2) lattice gauge theory

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Abstract

SU(2) lattice gauge theory is extended to a larger coupling space where the coupling parameter for horizontal (spacelike) plaquettes, $\beta_H$, differs from that for vertical (Euclidean timelike) plaquettes, $\beta_V$. When $\beta_H \to \infty$ the system, when in Coulomb Gauge, splits into multiple independent 3-d O(4) Heisenberg models on spacelike hyperlayers. Through consideration of the robustness of the Heisenberg model phase transition to small perturbations, and illustrated by Monte Carlo simulations, it is shown that the ferromagnetic phase transition in this model persists for $\beta_H < \infty$. Once it has entered the phase-plane it must continue to another edge due to its symmetry-breaking nature, and therefore must necessarily cross the $\beta_V = \beta_H$ line at a finite value. Indeed, a higher-order SU(2) phase transition is found at $\beta = 3.18 \pm 0.08$, from a finite-size scaling analysis of the Coulomb gauge magnetization from Monte Carlo simulations, which also yields critical exponents. An important technical breakthrough is the use of open boundary conditions, which is shown to reduce systematic and random errors of the overrelaxation gauge-fixing algorithm by a factor of several hundred. The string tension and specific heat are also shown to be consistent with finite-order scaling about this critical point using the same critical exponents.
1 Introduction

It would be very nice to have a local order parameter for confinement. On large lattices the Polyakov loop becomes extremely small in both phases, as do large Wilson loops. In addition, it is difficult to accurately split off the area-law confining term of Wilson loops from the other terms present. For spin models with spontaneously broken symmetries it is much easier to study the ferromagnetic phase transition, due to the local magnetic order parameter which (along with its moments) is easy to measure on any size lattice.

Indeed, there actually is such a local order parameter, the ferromagnetic transition of which signals deconfinement. It requires, however, fixing configurations to minimal Coulomb gauge. Since this is an iterative gauge-fixing procedure, it is fairly costly, but the benefit of a local order parameter may be worth it. Nearly two decades ago, it was envisioned that this order parameter could become a prime means for studying confinement[1]. Difficulties in gauge-fixing have largely prevented these techniques from becoming mainstream.

The minimal Coulomb gauge involves maximizing the trace of all links in three of the four lattice directions (these three are called “horizontal” directions below). The fourth-direction (vertical) links can be reinterpreted as O(4) “spins,” the local order parameter. The minimal Coulomb gauge has a set of remnant SU(2) symmetries, global on the three horizontal-directions, but still local along the fourth. This is because if links are written as $a_0 + i \vec{a} \cdot \vec{\tau}$ the $a_0$ components of the horizontal links are invariant under such a gauge transformation, and this is the quantity being maximized to satisfy the gauge condition. Separate SU(2) transformations on the two ends of the fourth-direction links, which are being treated as spins, generate for them a global $SU(2) \times SU(2) = O(4)$ symmetry, which breaks spontaneously if the spins magnetize. In the limit $\beta \to \infty$, the gauge condition sets the horizontal links to unity, and the plaquette interaction collapses into a spin interaction, a dot-product between the O(4) spins. Thus, in this limit, the SU(2) lattice gauge theory becomes a set of 3-d O(4) Heisenberg models (or non-linear sigma models) at zero temperature (using $T = 1/\beta$ as temperature). As $\beta$ becomes finite, then the primary interaction is still the Heisenberg one, since the horizontal links still are on average close to the identity, but the deviations from the identity introduce new interactions between the spins, which vary spatially. For large $\beta$ these interactions are either small or, if large, rare. There are no direct interactions between vertical links in different hyperlayers; they interact indirectly through shared horizontal links.
For a fixed set of horizontal links, the spin-model now has the appearance of a spin-glass, with the deviations of horizontal links from the identity introducing spatially-varying “disorder interactions.” As $\beta$ is lowered, both the pseudo-temperature, $T = 1/\beta$, and the amount of disorder from the deviation of horizontal links from the identity increases. Thus, one expects that the ferromagnetic phase that must exist at $\beta = \infty$ (since the 3-d O(4) spin model is ferromagnetic at zero temperature), should either have a transition to a paramagnetic phase, or possibly first to a spin-glass ordered non-ferromagnetic phase followed by a second transition to a paramagnetic phase. Because the strong coupling limit is completely random, the system must eventually enter a paramagnetic phase.

Conventional thinking would have the transition to paramagnetic phase happen immediately at $T = 0+$, putting the entire Wilson axis except $\beta = \infty$ in the paramagnetic phase. However, ferromagnetism in the Heisenberg model is quite robust against the introduction of disordering interactions. For instance the $\pm J$ 3-d O(3) spin-glass model remains ferromagnetic at zero temperature until 21% of randomly selected interactions are changed from ferromagnetic to anti-ferromagnetic\[2\]. The O(4) model is expected to behave similarly. Thus it is a little hard to imagine how the infinitesimally small disordering interactions introduced at $T = 0+$ in the gauge theory can so effectively kill the ferromagnetic order. Recently, using spin-glass methods, a spin-glass to paramagnet transition was observed at $\beta = 1.96$, strongly supporting the second hypothesis above\[3\]. Unlike the ferromagnetic phase, which is necessarily unconfined \[4\], the spin-glass phase, which has a hidden pattern of frozen disorder, is still confining. Below, the necessary spin glass to ferromagnet phase transition is searched for. That itself could be at $T = 0+$, again consistent with the whole $\beta$-range being confining, however, the evidence presented below from a variety of angles points to this transition taking place on the infinite lattice around $\beta = 3.2$.

A previous attempt at finding this transition showed a possible transition around $\beta = 2.6$\[5\]. However this was not consistent with the apparently obvious confinement in the region $\beta = 2.7$ to 2.85 from the Polyakov loop and interquark potential\[6\] \[7\]. It was later discovered that the gauge-fixing procedure was not working well during Polyakov-loop tunneling events. This depressed the magnetization on smaller lattices resulting in a premature crossing of the Binder cumulant. It is not surprising that the local gauge-fixing algorithm has trouble with the global constraints imposed by the gauge-invariant Polyakov loops. Switching to open boundary conditions seems to completely eliminate this pathology, and as a result not only lessens
the systematic error, but vastly reduces the random error in quantities such as the Binder cumulant (by a factor of several hundred). Evidence is given below that the remaining systematic error is at most of order, and probably much less than, this much-reduced random error, which itself is 100 times smaller than the size of ordinary thermal fluctuations between configurations. Even with the reduction of random error that results from averaging over gauge configurations, the possible systematic error remains less than or of order the net random error. Since the crossings identifying the new phase transition are verified at several $\beta$’s to more than $9\sigma$, systematic error no longer appears to be an issue. Indeed, the use of open boundary conditions seems to have the potential to transform Coulomb-gauge studies to the high-precision realm. The use of open boundary conditions in gauge theories has recently been justified by Lüscher and Shaefer and used successfully by them to reduce barriers to changes in the topological charge. The negative features of open boundary conditions, such as the lack of translational invariance, are not so important for the large lattices in use today, and are rather easily dealt with.

In section 2 the effectiveness of the Coulomb Gauge relaxation using open boundary conditions is detailed. In section 3, the larger coupling space where purely horizontal plaquettes have a different coupling parameter, $\beta_H$, than those with a vertical link, $\beta_V$, is explored. This plane includes both the 3-d O(4) Heisenberg model on the $\beta_H = \infty$ edge, and the ordinary 4-d SU(2) gauge theory along the $\beta_H = \beta_V$ line. A transition very similar to the Heisenberg transition in terms of critical exponents is seen for large but non-infinite $\beta_H$. At this position the transition has not so high critical exponents as seen for the $\beta_V = \beta_H$ case and is very easily studied with standard methods (Binder cumulant crossings etc.). An important point is that once it has been established that the transition enters the $(\beta_H, \beta_V)$ coupling plane from the Heisenberg edge, it must continue until it hits another edge. Due to the symmetry-breaking nature of the transition, the coupling plane must be split into separated symmetry-broken and unbroken regions. As $\beta_H$ is reduced, the critical $\beta_V$ grows, thus a transition at finite $\beta = \beta_V = \beta_H$ has to exist. Because the large $\beta_H$ transition is so similar to the Heisenberg transition, and so easily established with standard Monte Carlo methods, this argument is perhaps the most solid evidence yet that zero-temperature SU(2) lattice gauge theory must have a phase transition at finite $\beta$. This transition breaks the remnant symmetry remaining after the Coulomb gauge fixing. Such a ferromagnetic phase is known to be non-confining. In short, from a statistical mechanics point of view, what is being shown here is that
SU(2) lattice gauge theory, like many other lattice gauge theories, has a lot in common with spin theories with the same symmetry once the gauge is fixed. It is essentially a magnet.

The SU(2) transition itself is studied with Monte-Carlo simulations in section 4. The open boundary conditions allow for much more precise Coulomb-gauge data than previously available. Here the transition, seen at $\beta = 3.18 \pm 0.08$, is somewhat harder to study from a technical standpoint, because the critical exponents and the cumulant crossing values are fairly extreme (in comparison to the Heisenberg model). However, even here the cumulant crossings can be verified to more than 10$\sigma$.

Because the effective O(4) spin models are on 3-d hyperlayers, there are some interesting dimensional issues concerning critical behavior. When these are sorted out, fits give a correlation-length exponent $\nu = 1.7$ for the 4-d gauge theory. This makes the corresponding singularity in the specific heat much too soft to be seen numerically. However, the specific heat (which shows almost no finite lattice size dependence) can still be fit to the infinite-lattice functional form associated with such a transition. In section 5 it is found that the range of $\nu$ to which the sharp rise of specific heat in the crossover region can be fit is consistent with this value, as is the $\nu$ found from a similar fit to string tension measurements and from $\beta$-shift fits to Polyakov-loop transition points for asymmetric lattices. Thus the behavior of these important gauge-invariant quantities is consistent with correlation length scaling associated with a higher order phase transition at $\beta = 3.2$, using the critical exponent found from the Coulomb-gauge magnetization scaling behavior. The final picture that emerges is quite unconventional. The SU(2) lattice gauge theory appears to have three phases: a strong-coupling paramagnetic phase from $\beta = 0$ to 1.96, a spin-glass phase from 1.96 < $\beta$ < 3.2 and a non-confining ferromagnetic phase for $\beta$ > 3.2. The existence of the spin-glass phase explains why there is a gap (the crossover region) between regions that are well-approximated by strong and weak coupling approximations.

2 Coulomb gauge fixing and boundary conditions

The first thing to be mentioned here is that these are not technically “Coulomb gauge simulations.” To actually stay within the Coulomb gauge during Monte Carlo simulation would be difficult and would also require
adding ghost fields to take care of the Fadeev-Popov determinant. Rather, the Coulomb gauge fixing here is part of the post-simulation measurement procedure, applied to a completely conventional gauge-invariant Monte-Carlo simulation. In fact, provided that the gauge fixing algorithm works well, the absolute-value of the resulting fourth-direction link magnetization is actually gauge-invariant in the following sense. If a gauge configuration is subjected to a random gauge transformation and then reprocessed through the Coulomb gauge fixing algorithm, the same result should ensue, provided there is a unique minimum. The direction of the magnetization will be different due to the possible inclusion of a remnant global symmetry transformation, but all important results are dependent only on the absolute value of the magnetization and its even moments, as usual. The minimal Coulomb gauge on a finite lattice is not generally subject to the Gribov ambiguity, because the chances of not having an absolute minimum are quite small. However, it is also numerically difficult to find an absolute minimum and relaxation algorithms will likely find local minima. So, whereas in principle there is a unique solution, in practice many solutions are found and in that practical sense there is still a “Gribov problem” on the lattice.

The method used here to set the Coulomb Gauge is iterative overrelaxation, with a 70% overshoot. There are a number of techniques which have been tried to improve on this basic algorithm. One is to subject the initial configuration to random gauge transformations and then relax each of these separately. This is done, say, 10 times. The one which achieves the best maximization of the traces of horizontal links is then chosen, and the others discarded. Another more-sophisticated approach is simulated annealing. Both are quite demanding of computer time.

The basic overrelaxation algorithm utilizes local gauge transformations at each site, cycling through the lattice multiple times. From 40-4000 sweeps through the lattice are needed for reasonable convergence (the larger number for lower $\beta$). However, with periodic boundary conditions, this does not exhaust all of the symmetry transformations available. There are still the $\mathbb{Z}(2)^4$ Polyakov loop transformations in which a hyperlayer of spins pointing in a particular direction are all multiplied by -1, the non-trivial element of the center. It has been found that adding these transformations in a preconditioning step where the best of the 16 Polyakov-loop sectors is found first, before overrelaxation, improves the final results considerably.

When one uses multiple relaxations, most give similar results, however now and then a really bad solution is found (local minimum far from the global one). So it is likely that using only a single relaxation would result
in too many outliers; using more relaxations will reduce these but at an obvious computational cost. The technique chosen for the current study is somewhat different. The idea is to couple the gauge-fixing more closely with the original simulation. Gauge fixing is performed after each sweep of the simulation, which uses a Metropolis algorithm with about 50% acceptance. In most cases, new gauge elements are not chosen from the full group, but from a limited neighborhood of the original link. Throughout most of this study, this neighborhood was chosen to be the nearest half of the group space, but the adjustability of this hit-size also gives one a handle to test the efficacy of the gauge-fixing algorithm. The previous gauge-fixed configuration is used as the starting point for the Monte Carlo sweep. Thus, since only half of the links are changed, and those that do are mostly changed by small amounts in each sweep, the gauge is still “partially fixed” at the beginning of the next overrelaxation. This better starting point seems to avoid the “really bad minima” which are sometimes found when starting from a random gauge transformation. Since the overrelaxation is done even during the initial equilibration portion of the simulation, it has a long time to find a good “groove” and maintain it while the gauge configuration slowly evolves. Although the preconditioning step mentioned above is included as well, it turns out not to work with this close-coupled algorithm, because, since after only one sweep the memory of the previous relaxation is still there, the algorithm always prefers not to jump to another Polyakov loop sector. One could possibly spawn 16 new configurations and relax each one, but that is simply too costly.

This algorithm seems to perform well most of the time, but it was noticed that at high-beta ($\beta > 2.9$), where the magnetization was quite large and histograms peaked very far from zero, occasional “storms” of low values occurred. This produced a small secondary peak about zero magnetization (Fig. 1-lower). At first this was possibly thought to be a real phenomenon, and evidence for a first-order phase transition, but the persistence of this second peak over a large range of $\beta$ was not consistent with that interpretation. It was later discovered that the observed “storms” of low magnetization appeared to occur during Polyakov loop tunneling events. The known sensitivity of the gauge-fixing algorithm to Polyakov loop sector changes pointed to it as the cause of the storm - it was simply having trouble finding a good minimum during the tunneling. Since this represents a global shift in the vacuum, akin to a quantum phase transition, it is not surprising that the local gauge-fixing algorithm has trouble initially following it. After a few tens of sweeps it finds a new groove, but even a few percent of bad sweeps
with very different values will affect moments of the magnetization unacceptably. The memory effect of the close-coupled algorithm is a negative feature when it comes to following sudden vacuum tunnelings. These problems only appear to occur at values of $\beta$ above the point at which the Polyakov loop is broken on a given lattice (e.g. $\beta > 2.7$ on the $16^4$ lattice).

In order to test this further, the algorithm was tried with open boundary conditions (OBC). Results were beyond all expectations. The magnetization distribution completely loses the suspicious second peak (Fig. 1 - upper graph), and the gauge condition function no longer has large excursions. These reduced fluctuations also resulted in much lower statistical errors on quantities such as susceptibility and Binder cumulant. This suggests that much of the observed error in the periodic boundary condition (PBC) simulations did indeed come from fluctuations in the efficacy of the gauge setting algorithm.

The magnetization order parameter is defined first by defining an $O(4)$ vector $\vec{a} = (a_0, a_1, a_2, a_3)$ from each 4th-direction pointing SU(2) link $a_0 \mathbf{1} + i \sum_{j=1}^{3} a_j \tau_j$ and averaging these over each 3-d perpendicular hyperlayer.

$$\vec{m} = \frac{1}{L^3} \sum_{\text{hyperlayer}} \vec{a}.$$  

Figure 1: Coulomb gauge magnetization histograms at $\beta = 3.5$ on $16^4$ lattices for periodic boundary conditions (lower) and open boundary conditions (upper). The lower peak on the lower graph contains very few actual configurations, bars being magnified by the $1/|\vec{m}|^3$ geometry factor. Left scale is for lower graph and right for upper.
Figure 2: Results of multiple gauge relaxations on the same gauge configuration after random gauge transformations for (a) periodic boundary conditions and (b) open boundary conditions. The single square datapoint on each graph is the result from the closely coupled algorithm. Trend lines indicate the remaining degree of correlation between magnetization and gauge function.

Expectation values are then taken as ensemble averages over both hyperlayers and gauge configurations. In Fig. 2, scatter plots are shown for multiple relaxations following random gauge transformations on the same gauge configuration, one taken from a $16^4$ lattice at $\beta = 3.5$. (Note it is the high $\beta$ region that is problematic in the PBC simulations and data in this region is also the most important for this paper). The gauge function being maximized is shown on the horizontal axis, and the magnetization $|\vec{m}|$ on the vertical axis with (a) showing the PBC case and (b) the OBC case. Note the large difference in scales. The scatter for the OBC case is seen to be smaller by a factor of about 200. Also for OBC there is very little correlation between the small remaining fluctuations in gauge function with the observable of interest, the magnetization. This would seem to indicate that the residual error is mostly random, rather than systematic, and thus easily observed. This contrasts with the rather strong correlation seen with PBC, which suggests a systematic error at least as large as the random. In addition, the OBC fluctuations are around 100 times smaller than the ordinary thermal fluctuations in magnetization between different gauge configurations, adding less than one part in $10^4$ to the susceptibility, which is completely negligible.
in comparison to the ordinary statistical errors in this study. Thus, switching from PBC to OBC would seem to transform use of Coulomb gauge with Monte Carlo data from a relatively rough and problematic method into a precise one, where the additional spurious fluctuations in quantities due to imprecision in gauge fixing are negligible compared to the ordinary thermal (quantum) fluctuations.

Another test used for finding systematic error was to vary the Metropolis hit size for the run, using the closely-coupled algorithm where the gauge relaxation is performed after each sweep. This is the maximum amount by which new gauge elements are allowed to differ from old ones in an update. Hit sizes vary from -1 to 1, with -1 meaning no change allowed at all and 1 meaning the full group (it is the negative of the minimum $a_0$ component of the update matrix). One would expect that the gauge setting algorithm would perform better when smaller changes were being made, since it would be less likely to find a local minimum far from the global one, and basically have more tries on a more restricted set of configurations. These runs were compared with those using a single relaxation after a random gauge transformation, and also the best of five and best of ten relaxations after random gauge transformations. The latter were only run at a hit size of zero because they would not be expected to show a dependence on hit size. The results are shown in Fig. 3 for Binder cumulant and magnetization. It is clear that there is very little difference between any of these algorithms - even the single relaxation after random gauge transformation gives results consistent with the others. For the magnetization, a slightly larger value is obtained with best of ten method than for a single relaxation. The best of ten value agrees with that from the closely-coupled algorithm. This suggests that the closely-coupled algorithm with zero hit size, as used in this paper is competitive with the best of ten algorithm. However the differences are so small it appears that even the single relaxation after random gauge transformation could be used in the OBC case. These runs were long enough (50,000 sweeps) that they should have encountered exceptional configurations if such still exist, one or two bad values from which would have had a large effect on the higher moments. So the data seem to indicate that the previous difficulties with Coulomb gauge relaxations no longer occur when using open boundary conditions. Indeed using open boundary conditions opens the path to highly precise measurements in the Coulomb gauge in which the efficacy of the gauge fixing algorithm is no longer a factor.
Figure 3: Binder cumulant, $U$, and magnetization vs. hit size for the closely-coupled algorithm on $16^4$ lattices at $\beta = 3.5$ (diamonds). Also shown is relaxation after a single random gauge transformation (open circle), best of five ($\times$) and best of 10 (filled circle). Points are horizontally offset slightly at hit size 0 for clarity. Also shown are $24^4$ values for $U$ (triangles), demonstrating that algorithmic differences, if they exist, are much smaller than the gap between $16^4$ and $24^4$ values.
3 Expanding the parameter space to connect the spin model to the gauge theory

The possible existence of a zero-temperature phase transition in the usual 4-d SU(2) gauge theory can be put into clearer focus by considering a larger coupling space in which horizontal and vertical plaquettes have different coupling parameters, $\beta_H$, $\beta_V$. Vertical plaquettes are those that include the fourth (Euclidean time) direction. As discussed above, for $\beta_H = \infty$ horizontal links are locked to unity in the Coulomb gauge, which transforms the 4-link gauge interaction into a 2-link nearest-neighbor spin interaction (scalar product) among the vertical links. Thus the theory is exactly a set of multiple non-interacting copies of the O(4) 3-dimensional Heisenberg model (the quantum version of which is the non-linear sigma model). A similar connection of the SU(2) gauge theory to the Heisenberg spin model in one fewer dimensions was explored analytically in [11]. The 3-d O(4) classical Heisenberg model has a well-known higher-order ferromagnetic phase transition at $\beta_V = 0.936(1)$ with critical exponents $\nu = 0.7479(90)$, $\gamma/\nu = 1.9746(38)$, $\beta/\nu = 0.5129(11)$, and $\alpha = 2 - d\nu = -0.244(27)$ [12]. Since $\alpha < 0$ there is no infinite singularity in the specific heat, but rather a cusp. As mentioned before, the long-range ferromagnetic order of the Heisenberg model is quite robust to the addition of a significant number of randomly chosen antiferromagnetic links, deleted links, or added next neighbor interactions[2, 13]. (Although the O(3) model is usually studied, the O(4) model appears to differ only in detail.) Thus, it would be quite surprising if the transition did not enter the $(\beta_V, \beta_H)$ coupling plane as $\beta_H$ is backed off from $\infty$. In other words one would expect the transition to still exist for large but finite $\beta_H$. In this region, the horizontal gauge fields will still be very close to unity, so the Hamiltonian would still be primarily the Heisenberg nearest-neighbor interaction. Small additional terms of order $1/\beta_H$ would be present which would add disorder and likely add next-neighbor interactions if the horizontal gauge fields were to be integrated out. A renormalization of the pseudotemperature $(1/\beta_V)$ would also be expected. Because the integrated horizontal gauge fields do not break the remnant O(4) symmetry which remains after Coulomb gauge fixing, these effective spin interactions cannot favor domain formation as a random external magnetic field would[14]. If the phase transition does enter the coupling plane for $\beta_H < \infty$, then due to its symmetry-breaking nature it cannot end except at another edge of the phase-plane (it must divide the phase-plane into two disconnected regions of different symmetry). As $\beta_H$ is lowered from infinity, the critical value of $\beta_V$ must increase to compensate
for the additional disorder, so clearly these must then meet at a finite value \( \beta_c = \beta_{Vc} = \beta_{Hc} \), which would be the critical point of the SU(2) gauge theory.

Since a ferromagnetic phase in Coulomb gauge is necessarily deconfined,[4] this calls into question the customary assumption of confinement in the continuum limit (\( \beta \to \infty \)). If the conventional interpretation of no phase transition in SU(2) is correct, then the O(4) critical point would have to exist only for \( \beta_H = \infty \) and cease to exist for any \( \beta_H < \infty \). In other words the slightest \( 1/\beta_H > 0 \) coupling must destroy the long-range order. Since the large \( \beta_H \) case turns out to be simpler and more easily studied than the \( \beta_V = \beta_H \) symmetric case, it seems worth focusing on it first to investigate this important question. It might, for instance, be possible to extend the proof of long range order in the Heisenberg model[15] to the small \( 1/\beta_H \) region, which by the above argument would prove the existence of an SU(2) transition.

A Monte Carlo simulation at \( \beta_H = 20 \) was performed to look for a Heisenberg-like transition in the large \( \beta_H \) region. Such a transition is easily found using the Coulomb gauge method with open boundary conditions. As previously mentioned, the order parameter is simply the fourth-direction pointing links which act as O(4) spins. The “gold standard” method of finding a Binder cumulant crossing from simulations on different size lattices is used to find the position of the phase transition on the infinite lattice, just as is done for the case of the spin model itself. The Binder cumulant[16], \( U = 1 - < |\vec{m}|^4 > / (3 < |\vec{m}|^2 >^2) \), is plotted in Fig. 4, showing a definite crossing near \( \beta_V = 1.01 \).

For the O(4) order parameter, Binder cumulant values range from 0.5 in the deep paramagnetic region to 2/3 in the deep ferromagnetic region[17]. Fig. 5 shows scaling plots for Binder cumulant, \( U \), subtracted susceptibility \( \chi = V( < |\vec{m}|^2 > - < |\vec{m}| >^2 ) \), and magnetization \( < |\vec{m}| > \), demonstrating good collapse to finite size scaling ansätze[18] (see axis labels for detailed scaling functions, with \( T = 1/\beta_V \) and \( T_c = 1/\beta_{Vc} \)).

One disadvantage of using open boundary conditions is that lack of translational invariance makes the Fourier transform impossible, and thus the second moment correlation length[19], which also yields scaling information, cannot be defined. However, the unsubtracted susceptibility, \( \chi_b = V < |\vec{m}|^2 > \) when raised to the appropriate power can be used as a surrogate for the correlation length,

\[
\xi_L = \chi_b^{\nu/\gamma}.
\]

The subscript \( L \) refers to the linear lattice size. The quantity \( \xi_L/L \) can be then used in the collapse fits to further constrain the determination of
exponents $\gamma$ and $\nu$ (see Fig. 5d). Once $\gamma/\nu$ has been determined, $\chi_L/L$ may be used to see crossings at a fixed point corresponding to the infinite lattice critical point. Fig. 6 gives a very clear crossing at a coupling consistent with that of the Binder cumulant. The overall fit giving the best collapse of the four quantities, shown in Fig. 5, gives $\beta_{Vc} = 1.01(2)$, $\nu = 0.97(17)$, $\gamma/\nu = 2.00(25)$, $\beta/\nu = 0.51(14)$, with an overall $\chi^2/d.f. = 2.4$. Only the $20^4$ and $24^4$ data are used in the scaling fits as the $16^4$ appears a bit too small to follow the universal scaling functions within the statistical errors here (higher order corrections are sometimes needed for smaller lattices). These runs were all for 50,000 sweeps following 10,000 equilibration sweeps. The $\gamma/\nu$ and $\beta/\nu$ values are consistent with those for the $O(4)$ Heisenberg model, whereas the value of $\nu$ appears somewhat larger. This makes sense considering that the value of $\nu$ seen below for the $\beta_H = \beta_V$ SU(2) case is much higher, around 3.42. Thus the $\beta_H = 20$ transition appears to have already started evolving in this direction but is otherwise very similar to the Heisenberg transition itself. Of course the main result here is that the $\beta_H = 20$ transition simply exists at a finite $\beta_{Vc}$. As argued above, due to the symmetry breaking nature of the phase transition, this is a sufficient condition for the existence of an SU(2) transition at finite $\beta$. Binder cumulant crossings and scaling collapse plots are
Figure 5: Finite size scaling collapse plots for $U$, $\chi$, $\langle |\vec{m}| \rangle$, and $\xi_L/L$ for the $\beta_H = 20$ theory.
tried and true methods for finding phase transitions. Calling into question the $\beta_H = 20$ result would also seem to call into question the Heisenberg results themselves, since the methods and data are so similar. The reliability of these methods has been demonstrated in a large variety of models. If something drastically different were to happen on very large lattices, it is unlikely that such clean crossings would have been observed here. In order for the transition to disappear, the behavior of $U$ and $\xi_L/L$ in the weak coupling region would have to switch from the observed increasing function with lattice size to a decreasing function. Such a change in behavior would be highly unusual, if not unprecedented.

The suggested phase diagram in the extended plane is given in Fig. 7. The Heisenberg transition is seen on the upper boundary with the $\beta_H = 20$ transition, just described, below it. Further down, along the $\beta_V = \beta_H$ symmetry line, is the SU(2) transition at $\beta = 3.2$ to be described in the next section. Also shown is the spin glass transition for SU(2) at $\beta = 1.96$ described in [3]. The point where the spin-glass phase first splits off has not yet been determined, nor has the behavior below the symmetry line. The bottom boundary is a set of one-dimensional O(4) spin models, which are in the unbroken phase except at $\beta_V = \infty$. So a likely place for the transitions to end up is this point, but further investigation is needed.

Figure 6: Crossing graph for $\xi_L/L$ for $\beta_H = 20$. The $24^4$ value exceeds the $20^4$ by $21\sigma$ and the $16^4$ by $43\sigma$ at $\beta = 1.06$. 
Figure 7: Suggested phase diagram from the phase transitions seen in this paper at $\beta_H = 20$ (square) and $\beta_H = \beta_V$ SU(2) gauge theory (triangle). Also shown are the 3-d O(4) Heisenberg model transition on the upper axis (star) and the paramagnet to spin-glass transition seen in ref.[3] (diamond).

4 Monte Carlo simulations

A similar study was performed for the regular SU(2) theory, again using open boundary conditions. Some additional details, which also apply to the simulations of the last section are given here. Although opening the boundary conditions might seem a rather drastic step, when larger lattices are used it is certainly practical and causes fewer difficulties than one might expect. The outer three layers were discarded in order to lessen the effects of the open boundary. The vast majority of the boundary-effect occurs here. In other words, for a $16^4$ lattice quantities are only measured on the inner $10^4$ and for finite-size scaling purposes the size is $10^4$. This can be thought of as a soft open boundary. All remaining boundary effects can be absorbed into the finite size scaling applied to different lattice sizes. In principle one does not need to exclude boundary layers, but that would require larger lattices for sensible results. One also loses the translational invariance, which, as mentioned above, prevents the calculation of the second moment correlation length. Otherwise analysis is relatively unchanged.

Lattices of $16^4$, $20^4$ and $24^4$ were measured for 50,000 sweeps, after an initial equilibration of 10,000 sweeps, for $\beta$ between 2.3 and 3.6. The gauge overrelaxation was very slow at $\beta = 2.3$, taking roughly 4000 sweeps per Monte Carlo sweep. For $\beta > 3.0$, 100 or fewer overrelaxation sweeps were
needed. Overrelaxation was terminated when the per-link value of the gauge condition changed by less than $2 \times 10^{-8}$. Test runs with a criterion ten times smaller showed no difference. Of course for many of these large $\beta$ the correlation length is larger than the largest lattice size. Simulations designed to explore continuum physics must be run in a region where the correlation length is smaller than the lattice to avoid finite lattice and finite temperature effects. However, what is being sought here is a possible critical point, where the correlation length is infinite. So, just as in the study of magnetic theories, one must march right through the suspected critical region with finite lattices and interpret the data using finite-size scaling. If the true critical point is at $\beta = \infty$ then finite-size scaling should be able to see that too. In that case, the scaling in the entire range studied should look like the unbroken region with no apparent critical point. In particular, the Binder cumulant should everywhere be a decreasing function of lattice size.

The Binder cumulant, $U$, shows an apparent crossing at quite a high cumulant value somewhere between $\beta = 3.1$ and 3.3 (fig. 8). The crossing around $\beta = 3.2$ is a bit messy, but for $\beta = 3.4$ to 3.6 $U$ is definitely an increasing function of lattice size indicating this is in the spontaneously broken region. Errors at these $\beta$’s are very small. At $\beta = 3.4$ $U_{24}$ exceeds $U_{20}$ by $4\sigma$ and $U_{16}$ by $11\sigma$. At $\beta = 3.5$ these are $5\sigma$ and $14\sigma$ respectively. Magnetization histograms are given in Fig. 9, showing a pattern typical of a higher order transition. For this O(4) order parameter one must take into account the geometrical factor (from solid angle) that biases the distribution toward larger magnitudes. In the unbroken phase, the distribution of magnetization mod-


To more easily see the Gaussian behavior, the probability distribution $P(|\vec{m}|)$ is obtained by histogramming, and the quantity $P(|\vec{m}|)/|\vec{m}|^3$ is plotted. The value of $|\vec{m}|$ for each bin is not taken at the center, but at a value that would produce a flat histogram in an $|\vec{m}|^3$ distribution, regardless of bin-size choice. This is

$$|\vec{m}|^3_{\text{bin}} = \frac{1}{4} \frac{(m_2 - m_1^4)}{(m_2^2 - m_1^2)}, \quad (3)$$

where $m_2$ and $m_1$ are the bin edges. This detail affects only the first couple of bins in the histograms. Magnetization and subtracted susceptibility curves are shown in Fig. 10 with the susceptibility showing a peak growing with lattice size and shifting toward weaker couplings. Scaling collapse fits were then performed for $\chi$ and $<|\vec{m}|>$ using $U$ itself as the scaling variable. This can be done since $U$ has no anomalous dimension, and has the advantage of being a single parameter fit (no fitting of $\beta_c$ is needed). From these fits, values of $\gamma/\nu = 2.951(3)$ and $\beta/\nu = 0.0250(8)$ were found. Since these are consistent with the hyperscaling relationship $\gamma/\nu + 2\beta/\nu = d = 3$, it was decided to redo the fits with only a single degree of freedom by enforcing hyperscaling (Fig. 11), which yielded $\gamma/\nu = 2.950(2)$ with $\chi^2$/d.f. = 1.7.

Once $\gamma/\nu$ has been determined, a correlation length can be defined from the bare susceptibility as above (Eqn. 2). Plots of $\xi/L$ should cross at an infinite lattice critical point. Such a plot is given in Fig. 12, which shows...

Figure 9: Magnetization histograms at $\beta = 2.3, 2.5, 2.9$ and 3.5 on $24^4$ lattices. The latter three use the right-side vertical scale.
Figure 10: Magnetization and susceptibility for 4-d SU(2).

Figure 11: Magnetization and susceptibility scaling collapse fits using $U$ as the scaling variable.
a rather clear crossing, also around $\beta = 3.2$. These crossings are verified by even larger confidence intervals than for $U$, with the $24^4$ data exceeding the $20^4$ by $18\sigma$ and the $16^4$ by $32\sigma$ at $\beta = 3.5$. Finally a full four-quantity collapse fit for $\beta_c$ and $\nu$ was performed utilizing $U$, $\chi$, $<|\vec{m}|>$, and $\xi_L/L$ (Fig. 13). This yields $\beta_c = 3.18(8)$ and $\nu = 3.42(+0.32 - 0.23)$, with a $\chi^2/d.f. = 3.2$. Again, the $16^4$ data were excluded from the fit. The slightly high $\chi^2/d.f.$ probably indicates the presence of some higher order corrections to scaling still operating at these lattice sizes. Errors in critical exponents and $\beta_c$ were determined by forcing the given quantity higher or lower until the $\chi^2/d.f.$ doubled, with the other parameters free to change. This somewhat more conservative method than “adding unity” to $\chi^2/d.f.$ guards against any possible underestimate of statistical errors, which were from plateaus of binned fluctuations.

5 Fits to gauge invariant quantities

When one thinks of confinement, the quantity that immediately comes to mind is the string tension. In particular, the inverse square root of the string tension defines a correlation length. If the phase transition seen above is to be consistent, the correlation length in the confining phase should match that determined from the string tension. If the string tension scales with the same exponent given above, that would help corroborate the existence of the phase transition, since the string tension is determined from gauge
Figure 13: SU(2) scaling collapse plots for $U$, $\chi$, $\langle |\tilde{m}| \rangle$, and $\xi_L/L$. 
invariant Wilson loops and does not involve any sort of gauge fixing. One can also attempt fit the “singular part” of the specific heat in this region to the expected scaling behavior. Finally, the positions of the deconfinement transition observed on lattices with one short dimension, usually interpreted as a finite temperature transition can be analyzed as finite-lattice $\beta$-shifts of the infinite lattice transition seen above, which also depend on the exponent $\nu$. It will be shown that all three of these gauge invariant quantities scale consistently with the above transition seen at $\beta_c = 3.18$ with $\nu = 3.4$.

However, first the three-dimensional scaling of the hyperlayers needs to be reconciled with the four-dimensionality of the original system. For instance what value of $d$ should be used in the hyperscaling relationship for the specific heat exponent $\alpha = 2 - d\nu$? The key is to consider the relationship between the correlation length being measured from the spin correlations in the 3-d hyperlayers and that of the four-dimensional theory (being measured by the string tension for instance). The magnetization correlation function can be considered similar to the 1xN Wilson loop or, more accurately, a unit length partial Polyakov loop (PPL) or “gaugeon[1].” In the Coulomb gauge a segment of links of any length, $aT$ in the fourth direction is an observable. Here $T$ is the time extent in units of the lattice spacing. The correlation function between two PPL’s a distance $R$ apart is expected to behave as

$$
\exp(-\sigma R a^2)
$$

where $\sigma$ becomes the usual asymptotic string tension for large $T[1]$. For $T = 1$, $\sigma$ will differ somewhat but the dimensionality will still be the same. The $\sigma$ for $T = 1$, sometimes referred to as the Coulomb string tension, has been observed to approximately scale with the asymptotic string tension $[20]$. It is therefore an inverse length-squared object. However, this same object was treated as the inverse correlation length itself in the 3-d analysis, because there the 4-d links were treated as “spins” with no inherent length associated with them, whereas in the 4-d theory they are associated with the lattice spacing. Therefore the correlation length determined from the spin analysis is really an area in the 4-d theory. Thus the correct 4-dimensional exponent $\nu_4$ for a true length, $1/\sqrt{\sigma}$, should be half of the 3-d measured value of 3.42, i.e. $\nu_4 = 1.71$.

Neither the string tension nor the specific heat show much finite-size variation in the $\beta$ region in which they have been measured (the region where the correlation length is less than $L$). Therefore, they can be expected to rather closely follow infinite-lattice scaling laws. Fig. 14 shows a plot of $\sqrt{\sigma a^2}$ vs. $\beta$ with a one-parameter fit to the form $c(1/\beta - 1/\beta_c)^{\nu_4}$. String tensions were
Figure 14: String tension fit to finite-order scaling law.

taken from refs [7, 21]. A somewhat reasonable, though not entirely satisfactory, fit is obtained for the values $\beta_c = 3.18$, $\nu_4 = 1.71$ obtained above from the 3-d magnetization. The fit particularly differs with the $\beta = 2.85$ point (6 standard deviations off). Without this point the $\chi^2$/$d.f.$ = 3.5. If $\nu$ is allowed to vary, only a slightly better fit is obtained for $\nu_4 = 1.67$ ($\chi^2$/$d.f.$ = 4.08 - fit improvement does not compensate for increasing d.f. by 1). Therefore the string tension appears to be selecting essentially the same value for the critical exponent as obtained from the Coulomb gauge magnetization. Since string tensions are measured with a variety of methods and on different lattice sizes, some systematic differences between the datapoints may be present, which could explain some of the difficulties with the fit. Similar fits can be made for a fairly broad range of $\beta_c$, so the string tension is not good at pinning that down further.

The specific heat exponent is expected from hyperscaling to be $\alpha = 2 - 4\nu_4 = -4.84$. The high negative value indicates a very soft singularity. Five derivatives of the specific heat would have to be taken before an infinite singularity would be seen. Such a weak singularity would be virtually impossible to detect directly using numerical methods and certainly explains how this transition could have been missed before. Such high values of $\nu$ are unusual in spin models, but more common in spin glasses, because the lower critical dimension (lcd) of many spin glasses is thought to lie around 2.5 [22], and $\nu \to \infty$ at the lcd. Since it is 3-d hyperlayers that are being studied here, and a spin-glass to ferromagnet transition is being hypothesized, the
closeness of the dimensionality to the lcd could explain the unusually high value of $\nu$. For a finite singularity, scaling can still be checked. Since the finite size dependence of the specific heat is known to be very small, one can fit to the expected infinite lattice scaling function. The following fitting function was used:

$$C(\beta) = C_0 (\frac{1}{\beta} - \frac{1}{\beta_c})^{(4\nu_4 - 2)} + C_1 + C_2/\beta + C_3/\beta^2 + C_4/\beta^3. \quad (5)$$

The last four terms are needed to fit the non-singular part. $C_1$ is set to the perturbative value of 0.75, but $C_2$ through $C_4$ are allowed to differ from the perturbative values to account for higher-order neglected terms. Fig. 15 shows that data in the range $2.44 \leq \beta \leq 3.2$ fits well to this form with the previously determined $\nu_4 = 1.71$ ($\chi^2/d.f. = 0.58$). Data were from plaquette fluctuations on $24^4$ lattice for runs of 500,000 sweeps or more. Again if $\nu_4$ is allowed to vary, the best fit is found at a nearby value of 1.63 ($\chi^2/d.f. = 0.57$). Without the $1/\beta^3$ term a value of 1.53 is preferred ($\chi^2/d.f. = 0.62$) The specific heat is not very sensitive to the exact value of $\beta_c$.

Finally one can look at the scaling of $\beta_c$ on lattices with one short dimension - what has usually been interpreted as a finite temperature deconfinement transition. If there is an infinite-lattice critical point at $\beta = 3.2$, then the phase transition on the distorted lattice would not be a finite-temperature transition, since the continuum limit is already deconfined at zero temperature. Rather it is simply the same transition $\beta$-shifted by the finite size shift effect. The shift in the critical point for a finite lattice is expected to be $L^{-1/\nu_4}$

$$|\beta_c - \beta_{\tau}| \propto L^{-1/\nu_4} \quad (6)$$

Figure 15: Specific heat finite-order scaling fit. Lower curve is non-singular part of fit.
Figure 16: Scaling fit for finite-temperature transition points according to the finite-order zero-temperature phase transition hypothesis (straight lines). Diamonds are for the Wilson action and squares are a Z2 monopole suppressed action. Curves are the two-loop renormalization group prediction of the no phase transition hypothesis, made to pass through the highest $L_\tau$ points.

for $L_\tau << L$, where $L_\tau$ is the temporal lattice size. For large $\nu$ this can be substantial and dies only slowly with lattice size. If one plots $\beta_{c\tau}$ vs. $L_\tau^{-1/\nu_4}$, with $\nu_4 = 1.71$ (Fig. 16), a reasonable linear fit is seen to Wilson-action data from Ref. [6], with an intercept giving $\beta_c = 3.11(4)$, close to the previously identified value of 3.18(8). Also shown is the two-loop weak-coupling renormalization group prediction for the no phase transition hypothesis, which does not fit the data as well. In addition the same quantity but for an alternative action which suppresses $Z_2$ monopoles, as studied by Gavai[23], is plotted. These data also fit well to a linear fit with an intercept $\beta_c = 3.13(4)$, in agreement with the Wilson value, and do not fit at all well with the perturbative renormalization group. The monopole suppressed action has the same weak coupling perturbation series as the Wilson action. For weak couplings, results should merge with the Wilson action ones, so the $\beta_c$ values would be expected to be close, though not necessarily identical. The system with $L \to \infty$ with $L_\tau$ finite is, of course, a true 3-d system and thus will have different critical exponents from the 4-d theory. As $L_\tau \to L$, a dimensional crossover is to be expected with critical exponents morphing to their 4-d values[24].
Therefore, the gauge invariant quantities of string tension, specific heat, and finite-lattice shift all scale reasonably in accordance with the zero temperature phase transition seen by the Coulomb gauge magnetization order parameter. This lends further credence to the existence of this phase transition - in particular that gauge fixing is not necessary to see it. It is merely a very convenient and probably necessary step to take in order to define a local order parameter.

The idea that the pure gauge SU(2) lattice gauge theory might have a lattice-artifact driven phase transition in analogy to U(1) has been considered before\cite{25, 26}. Perhaps the best previous evidence to date that such a transition must exist involves another expansion of the coupling space, the fundamental-adjoint plane. For a sufficiently large adjoint coupling a first order bulk transition appears in the SU(2) theory. The scaling of latent heat with the size of the hysteresis region was shown in Ref. \cite{27} to be inconsistent with the endpoint of the first order line being an ordinary critical point. Rather it is only consistent with being a tri-critical point associated with an exactly broken symmetry, for which the transition cannot just end, but must continue as a higher-order one, again until it hits an edge of the phase diagram. Slopes of lines of constant physics predict a crossing of the Wilson axis. Consistent with this hypothesis, when “finite temperature” transition points are followed into the fundamental-adjoint plane they appear to head toward this tricritical point\cite{28}. Exact joining is difficult to establish, however, and has been disputed\cite{29}, but if the endpoint of the first-order line is truly a tricritical point then joining must take place.

6 Conclusion

It has often proven useful to connect theories by using an extended coupling space, especially when the behavior of one theory is better known than the other. This paper extends an old suggestion by Darhuus and Fröhlich relating the 4-d SU(2) lattice gauge theory to the 3-d O(4) Heisenberg model\cite{11}, by allowing the couplings of horizontal and vertical plaquettes to differ. The enlarged model becomes the 3-d O(4) Heisenberg model in the $\beta_H \to \infty$ limit and the 4-d SU(2) lattice gauge theory in the $\beta_H = \beta_V$ limit. A Monte Carlo simulation at $\beta_H = 20$ shows a phase transition at $\beta_V c = 1.01$. In order to follow the same ferromagnetic order parameter as in the spin model, the lattice gauge theory must be transformed into Coulomb gauge, in which the 4th direction pointing links act as O(4) spins exhibiting a 3-d global symmetry, the remnant symmetry of the gauge fixing. Although Coulomb gauge fixing
has been plagued with large systematic and random errors introduced by
the gauge fixing algorithm, it is found that using open boundary conditions
solves these problems - with errors hundreds of times less than when using
periodic boundary conditions, nearly eliminating the lattice Gribov problem.
The local order parameter allows standard methods for finding phase tran-
sitions such as Binder cumulant crossings and finite-size scaling collapse fits.
Not surprisingly, because of the high value of $\beta_H$, the $\beta_H = 20$ transition
is very similar to the Heisenberg transition at $\beta_H = \infty, \beta_V = 0.936, \beta_{Vc}$
with similar critical exponents. However, the fact that the transition still exists
for non-infinite $\beta_H$ has far-reaching implications. The ferromagnetic transi-
tion breaks an exact symmetry, the remnant gauge symmetry left over after
Coulomb gauge fixing. Therefore, it must divide the coupling plane into two
distinct regions - symmetry broken and unbroken. The line of phase trans-
itions must persist until it hits another edge. The Heisenberg model itself
has only one critical point, so the line of phase transitions must cross the
line $\beta_V = \beta_H$ at a finite value (somewhere between 1.01 and 20 - as $\beta_H$ de-
creases, $\beta_{Vc}$ increases). Indeed a high-order transition is seen around $\beta = 3.2$,
showing Binder cumulant and $\xi_L/L$ crossings, and good collapse of various
graphs to scaling functions about this point. The transition also predicts
scaling laws consistent with the behavior of the specific heat, string-tension,
and even finite-temperature transition points (reinterpreted as finite-lattice
shifts) in the crossover region.

Much of the lattice gauge theory program has been based on the assump-
tion that the non-abelian gauge theories SU(2) and SU(3) have no phase
transition and the continuum limit is confining. Of course the SU(3) case
needs to be checked in detail, but if SU(2) has a zero-temperature phase
transition it is likely SU(3) has one as well. A non-confining and symmetry-
broken continuum limit would require a new mechanism of quark confine-
ment, since it would no longer be a property of the gluon sector acting alone.
One possibility is that the light-quark chiral condensate is itself the source
of confinement, a hypothesis which has been advanced by Gribov[30] and
others[31]. A strong enough force, whether confining or not can cause quark-
antiquark pairs to condense in the vacuum, breaking chiral symmetry. If this
chiral vacuum expels strong color fields, of which there is some evidence[32],
then the chiral vacuum itself could cause a bag-like pressure around strong
color sources, with the lowest energy configuration of vacuum + hadron being
one in which the color fields are confined to a small volume. In most cases
unbreaking of chiral symmetry from thermal effects and deconfinement are
coincident, which supports such a linkage. Thus it is possible that simulations
with light quarks are exploring correct continuum physics (whereas the pure-glue simulations in the confining phase are exploring a strong-coupling phase not connected to the continuum limit). To be sure, alternative gluon actions need to be explored which eliminate, if possible whatever lattice artifacts are causing the pure-glue phase transition, in analogy with the monopoles of the U(1) theory. Preliminary data suggest that eliminating both negative plaquettes (Z2 vortices and monopoles) and a different kind of monopole defined from the non-abelian Bianchi identity, which I have called SO(3)-Z2 monopoles, may force the SU(2) theory to remain in the ferromagnetic phase for all couplings.

The symmetry breaking which occurs in the weak coupling phase is not unexpected. The U(1) theory in the Coulomb gauge almost certainly is also magnetized in the same way, becoming paramagnetic as the coupling becomes stronger at the monopole-induced phase transition to a confining theory, but being magnetized in the weak-coupling region down to the weak-coupling continuum limit. In this gauge, at least, the photon is a spin-wave like Goldstone boson associated with the breaking of the remnant symmetry left over after Coulomb gauge fixing, as opposed to the original bare gauge field the theory started out with. This picture has been previously explored in the continuum. The Gupta-Bleuer quantization method may be an alternative way to see how the elementary fields build into a Goldstone boson, and how a linked gauge and Lorentz symmetry is “reborn.” A similar picture may hold for the pure-gauge SU(2) and SU(3) gauge theories, with the gauge fields remaining massless and the running coupling having an infrared fixed point. However, when light quarks are added, this vacuum could become unstable and confinement may emerge as a byproduct of chiral symmetry breaking, following Gribov’s or a similar mechanism.

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