Finite-size effects and intermittency in a simple aging system

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We study the intermittent dynamics and the fluctuations of the dynamic correlation function of a simple aging system. Given its size $L$ and its coherence length $\xi$, the system can be divided into $N$ independent subsystems, where $N = \left( \frac{L}{\xi} \right)^d$, and $d$ is the dimension of space. Each of them is considered as an aging subsystem which evolves according to an activated dynamics between energy levels. We compute analytically the distribution of trapping times for the global system, which can take power-law, stretched-exponential or exponential forms according to the values of $N$ and the regime of times considered. An effective number of subsystems at age $t_w$, $N_{\text{eff}}(t_w)$, can be defined, which decreases as $t_w$ increases, as well as an effective coherence length, $\xi(t_w) \sim t_w^{(1-\mu)/d}$, where $\mu < 1$ characterizes the trapping times distribution of a single subsystem. We also compute the probability distribution functions of the time intervals between large decorrelations, which exhibit different power-law behaviours as $t_w$ increases (or $N$ decreases), and which should be accessible experimentally. Finally, we calculate the probability distribution function of the two-time correlator. We show that in a phenomenological approach, where $N$ is replaced by the effective number of subsystems $N_{\text{eff}}(t_w)$, the same qualitative behaviour as in experiments and simulations of several glassy systems can be obtained.

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I. INTRODUCTION

The dynamics of glassy materials such as spin-glasses, structural glasses or amorphous soft materials like gels, pastes or foams has been a subject of considerable study [1, 2, 3]. Considerable effort has been made in order to understand and quantify the out-of-equilibrium character of their temporal relaxation, in particular the absence of time translational invariance (aging), through the study of dynamic correlation functions. On the theoretical side, global dynamic correlations have been described at a mean-field level in disordered systems such as spin-glasses [4], or at a phenomenological level in models such as the Random Energy Model [5, 6]. Both approaches neglect all spatial properties of the system, and are therefore likely to miss any spatial correlations that arise during the dynamics. Alternatively, a phenomenological picture, the droplet model, has been proposed, that focuses on the spatial properties as a key to understand the slow dynamics and the critical properties of spin-glasses [7, 8].

In recent years, interest in the spatial properties of glassy systems has been growing. The size of excitations in finite-dimensional spin-glasses has been studied numerically [9]. In simulations of kinetically constrained glassy systems [10, 11], and of supercooled liquids [12, 13], cooperativity lengths have been identified, and related to the presence of heterogeneities in the dynamics [14, 15].

More generally, a lot of physical questions remain; a crucial one being: in what manner, at a microscopic level, does a glassy system evolve, both spatially and temporally? What are the spatial configurations of the typical rearrangements experienced by a glassy system during its relaxation and how are they affected by aging? Are there any common relaxation mechanisms of glassy systems, though there also exist specificities to given materials? Recently, new results have been obtained in this direction. Focusing on the spatial aspect of glassy relaxation, cooperative rearrangements events have been evidenced, both experimentally [16, 17] and through computer simulations [18], stimulating new research on the challenging question of coherence lengths and cooperativity in glasses. On the temporal side, beautiful experiments have shown evidence of temporal intermittency in colloidal gels and micellar polycrystals [19], and in polycarbonate glasses [20]. It seems now well established that in glasses and gels, relaxation takes place in a discontinuous way, involving sudden rearrangements followed by periods of arrest where almost nothing happens. The precise experimental determination of the distribution of time lags between rearranging events will give insight into the characteristic “trapping times” of the system. In experiments on glasses, this distribution seems to be close to a power-law [21] -which is consistent with a trap model with an exponential distribution of energies [22], whereas it has been found in simulations of supercooled glasses to correspond to a model of traps with either an exponential distribution of energies, or a gaussian distribution of energies [21, 22]. Therefore, quantities of interest are not just average quantities, but also fluctuations, and in particular the full probability distributions of correlations. The study of fluctuations in glassy systems may contain subtle information, as was already realized by Isaacoff and Weissman [23], who analyzed carefully mesoscopic noise in spin-glasses in an attempt to discriminate be-
tween a model of droplets, and a scenario of hierarchical
dynamics.

Recently, probability distributions of two-time corre-
lation functions in gels and glasses have been shown to
exhibit a non-Gaussian behaviour \([19, 21]\). These non-
Gaussian features have been found also in numerical sim-
ulations of disordered systems and kinetically constrained
models \([24, 27, 28]\). They have been tentatively explained
using the analogy between glassy dynamics and critical
dynamics \([24]\), for which universal, non-Gaussian features
can be expected \([27]\).

In this study, we will not advocate any similarity with
critical dynamics, but we will rather try to deduce the
non-Gaussian behaviour merely from finite-size effects in
a simple out-of-equilibrium glassy model.

We will consider a system that can be divided into \(N\)
independent subsystems. Each subsystem is supposed
to represent an independent model of glassy relaxation
between energy traps. Such a model of traps has been
studied extensively \([28]\). We will show that the super-
position of the \(N\) subsystems will have the same aver-
age dynamical correlation as one individual subsystem;
however, its probability distribution function will depend
strongly on \(N\); in the limit \(N \to \infty\), one has to recover
the Gaussian distribution, according to the central limit
theorem. Moreover, the distribution of time intervals be-
tween relaxation events or between decorrelations will
depend crucially on the number of subsystems, i.e on the
value of the internal coherence length.

The paper is organized as follows. In Section II, we
introduce the model and recall the main results of \([6]\). In
Section III, we calculate the distribution of time inter-
vals between successive events in the whole system. In
Section IV, we calculate the distribution of time intervals
between successive decorrelations in the system, the same
way they can be measured in Time Resolved Correlation
experiments on soft glassy materials \([10]\). Section V is
devoted to the probability distribution of the two-time
correlation function. Finally, in Section VI, we summa-
rise and discuss our results.

II. DEFINITION OF THE MODEL

Let us consider a simple model of dynamics of a system
between energy levels. In a generic disordered system, it
is reasonable to assume that low-lying energy levels are
exponentially distributed: this is the case for example
for the lowest energy levels (non-extensive corrections to
the ground state energy) of the Random Energy model
(REM) \([3, 29]\) or in spin-glasses \([30]\). Experimental de-
termination of energy barriers in low-temperature glasses
also seem to support this exponential distribution \([31]\).

More precisely, let us call \(E\) an energy barrier, which
is the difference between a reference energy level which
is taken as the origin for the energies, and a negative
energy level. \(E\) is hence a positive quantity. We choose
the distribution of barriers as \(\rho(E) = \frac{1}{E_0} e^{-E/E_0}\).

Changes in configurations in a disordered or glassy sys-
tem are often attributed to thermally activated events
over energy barriers \([21, 22]\), although other mecanisms
exist, such as kinetically constrained models \([10]\) which
do not require an energy landscape, and are able to re-
produce some of the features of glassy materials. In the
language of activated events, a trapping time \(\tau\) cor-
responding to a barrier \(E\) can be defined as \(\tau = \tau_0 e^{E/k_B T}\),
where \(\tau_0\) is a microscopic time scale, \(k_B\) the Boltzmann
constant, and \(T\) the temperature. For an exponential dis-
bution of barriers, the distribution of trapping times is
equal to \(\psi(\tau) = \mu \frac{\tau^{\mu-1}}{\mu-1}\), where \(\mu = \frac{k_B T}{E_0}\).

The dynamics of a system evolving in such an energy
landscape has been studied by different authors, accord-
ing to which choice of transition rates between energy
levels is made \([6, 29, 32, 33]\). In the following, we will con-
sider only the case where the transition rate from barrier
\(E\) to any other barrier \(E’\) is \(W(E \to E’) = \frac{1}{\tau_0} e^{-E/k_B T}\),
which means that the escape from the initial trap is the
limiting process, whatever the destination. Note that this
family of models does not include any kind of
spatial structure, since energy is not related here to spa-
tial configurations, and that it is mean-field in nature
since transitions to all levels are allowed with the same
probability. Extensions of these models to finite dimen-
sionalities have however been attempted in \([34, 35]\).

We will see in the following that the superposition of several
of such systems can actually introduce (though rather
artificially) a relevant lengthscale.

Dynamical properties of the model have been studied
in detail in \([6]\). In particular, when \(\mu < 1\), the model
exhibits aging (absence of time translation invariance of
the correlations), whereas it is time-translational in-
variant for \(\mu > 1\). In this paper, we will focus on the
case \(\mu < 1\). For a given trajectory of the system, the
two-time correlation function \(C(t_w, t_w + \tau)\) is defined as:
\(C(t_w, t_w + \tau) = 1\) if the system has remained in the same
energy trap between \(t_w\) and \(t_w + \tau\), \(C(t_w, t_w + \tau) = 0\) if
between \(t_w\) and \(t_w + \tau\), the system has left the energy
trap it was in at \(t_w\).

Averaging over all barrier configurations (which we de-
ote by \(<..>\), one obtains the average two-time corre-
lation function:

\[ \Pi(t_w, t_w + \tau) = < C(t_w, t_w + \tau) > \]

It was shown in \([6]\) that for large \(t_w\), this function is
given by the following formula:

\[ \Pi(t_w, t_w + \tau) \simeq \frac{\sin \pi \mu}{\pi} \int_0^1 dv \frac{1}{v^{1+\mu}} (1-v)^{\mu-1} v^{-\mu} \]

which we will use in sections III and V.
In this regime, \( \tau \) time of the density of events results in the aging of the density is uniform in time and simply equal to the inverse time scale is proportional to the age. This implies that \( \tau \approx c \), with \( c \) a standard quantity defined in the context of renewal theory [36]. For any distribution of trapping times \( \psi(\tau) \) where the trapping times are independent random variables, and given that it does not depend on the age, i.e. on the choice of the time origin, and independently of the type of dynamics used, the following formula holds:

\[
S(t) = \psi(t) + \int_0^t dt_i S(t_i) \psi(t - t_i)
\]

where \( t_i \) stands for the time of the last event to have taken place before \( t \); the first term \( \psi(t) \) corresponds to the special case \( t_i = 0 \).

In the following, we will use the following trapping times distribution: \( \psi(\tau) = \frac{\mu^\tau}{(\tau + \tau_0)^{\mu+1}} \), in order to ensure that \( \tau \) can take values from 0 to \( \infty \). We will also need the large time behaviour of the corresponding sprinkling density \( S(t) \). Following the lines of [37], this can be easily computed using Laplace transforms. Using the notation \( \hat{f}(z) \) to denote the Laplace transform of a function \( f(t) \), equation (1) is equivalent to

\[
\hat{S}(z) = \frac{\hat{\psi}(z)}{1 - \hat{\psi}(z)}
\]

The Laplace transform \( \hat{\psi}(z) \) can be computed as:

\[
\hat{\psi}(z) = \int_0^\infty dt e^{-zt} \psi(t) = \frac{\mu^t}{(1 + \mu z)^{\mu+1}}
\]

Two cases have to be considered before taking the limit \( \tau_0 z \to 0 \).

If \( \mu < 1 \), \( \psi(z) = 1 - \Gamma(1 - \mu)(\tau_0 z) + \frac{1}{1 - \tau_0 z} + o(\tau_0 z) \).

Then \( \hat{S}(z) \approx e^c(1 - \mu z)^{\mu-1} \), and for \( t \gg \tau_0 \), \( S(t) \approx e^c(1 - \mu z)^{\mu-1} \); with \( c(\mu) = \frac{1}{1 - (1 - \mu)(\mu)} = \frac{\sin(\pi \mu)}{\pi} \).

In this regime, \( S(t) \) decreases with time; the decrease in time of the density of events results in the aging of the correlation function \( \Pi(t_w, t_w + \tau) \), which characteristic time scale is proportional to the age \( t_w \).

On the other hand, if \( \mu > 1 \), \( \psi(z) = 1 + \frac{\pi}{\mu - 1} \tau_0 z + o(\tau_0 z) \).

This implies that \( S(z) \approx \frac{\mu}{\mu - 1} \), and that for \( t \gg \tau_0 \), \( S(t) \approx \frac{\mu}{\mu - 1} \). In this non-aging regime, the sprinkling density is uniform in time and simply equal to the inverse trapping time \( \tau > \int_0^\infty d\tau \psi(\tau) = \frac{\tau_0}{\mu - 1} \).

**Definition of the system of study as a superposition of \( N \) subsystems**

Let us now turn to our system of interest. This new system is defined as the superposition of \( N \) subsystems, identical to the one introduced previously, each of which is defined by the same trapping times distribution: \( \psi(\tau) = \frac{\mu^\tau}{(\tau + \tau_0)^{\mu+1}} \). The subsystems are assumed to be independent of each other. One can give an interpretation of such a model in a real space representation: given a system of size \( L \) in \( d \) dimensions, we assume that one can divide this system into \( N \) independent subsystems of length \( \xi \), where \( N = \langle \xi \rangle^d \). \( \xi \) is the typical coherence length of the system, and is considered constant during the time relaxation. However, we will see that this quantity is susceptible to evolve during aging.

During the dynamical evolution, each system relaxes independently, and hence contributes to some extent to the relaxation of the whole system. We will make the following assumptions: (i) all events occurring in a subsystem are also defined as individual events for the whole system, (ii) all events contribute equally to the relaxation of the whole system.

This translates into the following definitions:

- (i) The analog for the whole system of \( \psi(\tau) \) will be denoted as \( P_N(\tau) \): it is the distribution of time intervals between all events (i.e. trapping times). We will see in the next section that this quantity depends in general on the age \( t_w \); we will then call it \( P_N(\tau, t_w) \).

- (ii) The correlation function of the whole system is defined as

\[
C(t_w, t_w + \tau) = \frac{1}{N} \sum_{i=1}^N C_i(t_w, t_w + \tau),
\]

where \( C_i(t_w, t_w + \tau) \) is the correlation function of subsystem \( i \).

Before turning to a detailed calculation of \( P_N(\tau, t_w) \), we can invoke an argument of statistics of extremes. If \( \tau \) is a trapping time of the whole system, then it seems natural to say that \( \tau = \min \{ \tau_i \} \) where \( \tau_i \) is a trapping time of each subsystem \( i \). However, this is true only if all subsystems undergo one event at some time origin, and that one computes the first trapping time of the whole system from this time origin. Hence this argument definitely excludes aging effects, because it neglects any memory effects in the dynamics of the subsystems.

Having made this approximation, one can follow a standard calculation of statistics of extremes, and one can find the distribution of time intervals \( P_N(\tau) \):

\[
P_N(\tau) = N \psi(\tau) \left[ \int_0^\infty dt' \psi(t') \right]^{-N-1} = N \mu \frac{\tau_0^N}{(\tau + \tau_0)^{1+\mu N}}
\]

By expanding around the most probable value \( \tau = 0 \), and setting \( u = \mu N \tau / \tau_0 \), one finds the limiting exponen-
tial distribution \( P(u) = e^{-u} \) for \( u \ll 1 \) (i.e. \( \tau \ll \frac{1}{\rho \mu} \)). This corresponds in fact to the convergence of the probability distribution of extremes towards the Weibull distribution, in the case where the elementary distribution \( \psi(\tau) \) has a finite value for its minimum time \( \tau = 0 \).

In this approximation, the time distribution of events of the whole system simply follows a Poisson process, with a rate proportional to the number of subsystems. In a Poisson process, the conditions of the experiment are supposed to remain constant in time, and all events are independent of each other. However, in this model, although single events in all subsystems are indeed independent of each other, the dynamics is not invariant under time translations (as can be inferred from the sprinkling density \( S(t) \)). As we shall see in the next section, this will give rise to more complicated laws for \( P_N(\tau, t_w) \).

Note finally that we have not been able to find a suitable argument of statistics of extremes for \( C(t_w, t_w + \tau) \) (the statistics of this quantity in the framework of non-equilibrium dynamics have been related to the Gumbel distributions \( 24, 27 \), which are one of the “universal” families of probability distributions of extremes). Instead, we will gain information (see section V) by studying \( C(t_w, t_w + \tau) \) as the sum of \( N \) random variables, reinforcing the idea that it is not an extremal quantity, but rather originates in the contribution of many individual events (as was already pointed out in \( 27 \)).

III. DISTRIBUTION OF TIME INTERVALS BETWEEN ALL EVENTS

Let \( P_N(\tau, t_w) \) be the probability that an event takes place at \( t_w + \tau \) if one took place at \( t_w \), in the system composed of \( N \) independent subsystems.

In this section, it will be more practical for the computations to work with the cumulative probability distribution \( P_N(\tau, t_w) = \int_{-\infty}^{\tau} dt' P_N(\tau', t_w) \). By definition, it is the probability that the time difference between two successive events is larger than \( \tau \). Similarly, we will use \( Q(\tau) = \int_{-\infty}^{\tau} dt' \psi(t' - \tau) = \tau_0^\mu/(\tau_0 + \tau)^\mu \), which is the probability for a trapping time of a subsystem to be larger than \( \tau \), i.e. the probability for a subsystem not to change trap during the period of time \( \tau \).

We now call \( i \) the subsystem in which one event has taken place at \( t_w \). Then let \( \{t_j\}_{j \neq i} \) be the \((N-1)\) times of the last events before \( t_w \) in the other subsystems \( j \). The next event to take place in the whole system will either happen in subsystem \( i \) or in any other subsystem. In order for this next event to occur after a time \( \tau \), one requires the following conditions:

- \( (i) \) subsystem \( i \) has to remain trapped between \( t_w \) and \( t_w + \tau \), with probability \( Q(\tau) \), and

- \( (ii) \) the other subsystems \( j \) have to remain trapped between \( t_j \) and \( t_w + \tau \), with probability \( Q(t_w + \tau) + \int_0^{t_w} dt_j S(t_j) Q(t_w + \tau - t_j) \), \( Q(t_w + \tau) \) being the contribution for the special case \( t_j = 0 \). This last probability is in fact equal to \( \Pi(t_w, t_w + \tau) \), the probability for a system to remain trapped between times \( t_w \) and \( t_w + \tau \) (see section II).

Hence,

\[
P_N^C(\tau, t_w) = Q(\tau) [\Pi(t_w, t_w + \tau)]^{N-1}.
\]

\[
P_N(\tau, t_w) = -\frac{\partial}{\partial \tau} \left[ Q(\tau) [\Pi(t_w, t_w + \tau)]^{N-1} \right].
\]

In the following, we will always consider the case of large \( t_w \): \( t_w \gg \tau_0 \). We now treat separately two different regimes for \( \tau \): \( (i) \) \( \tau \sim \tau_0 \), and \( (ii) \) \( x = \frac{\tau}{t_w} \) finite and smaller than 1.

Case \( \tau \sim \tau_0 \)

In the regime of interest where \( t_w \gg \tau_0 \), \( S(t_w) \simeq 1 + \frac{1}{\nu \tau} \psi(\rho) \) (see section II). So that the leading term in \( t_w \) is:

\[
\Pi(t_w, t_w + \tau) \simeq 1 - \tau S(t_w) \simeq e^{-\tau S(t_w)}
\]

Then, we get the result:

\[
P_N^C(\tau, t_w) \simeq Q(\tau) e^{-(N-1)\frac{\tau}{\nu \tau_0 t_w}}
\]

and

\[
P_N(\tau, t_w) \simeq (\psi(\tau) + (N-1)S(t_w)Q(\tau)) e^{-(N-1)\tau S(t_w)}
\]

We note that for \( N = 1 \) the result \( P_N(\tau, t_w) = \psi(\tau) \) is recovered. As \( N \) increases, the exponential part in \( 24 \) becomes dominant, introducing a rate that is dependent on the waiting time: \( \rho = \frac{e^{\psi(\rho)}(N-1)}{\nu \tau_0 t_w} \).

This can be interpreted as an effective Poisson process, where the number of instances \( N \) is replaced (for large \( N \)) by

\[
n_{eff}(t_w) = N \left( \frac{\tau_0}{t_w} \right)^{1-\mu}.
\]

In other words, the scaling of \( 24 \) suggests that computing at age \( t_w \) the distribution of events of a system composed initially of \( N \) subsystems is the same as computing the distribution of events of a “young” system composed of \( N_{eff}(t_w) \) subsystems. In the case of \( 24 \), one can see that \( N_{eff}(t_w) \) decreases explicitly with the age; in the limiting case where aging disappears \( (\mu \to 1) \),
\( N_{\text{eff}} \) is simply a constant equal to \( N \). The idea of a number of independent subsystems decreasing with the age in non-equilibrium systems is not new. It is intimately related to the concept of a growing lengthscale in an aging system. If one defines a typical length \( \xi \) of a subsystem by \( N = (\xi^d)^d \), one has \( \xi_{\text{eff}} = L N^{-1/d} \), and the dependence of \( N_{\text{eff}} \) on \( t_w \) induces the following power-law for \( \xi_{\text{eff}} \):

\[
\xi_{\text{eff}}(t_w) = LN^{-1/d} \left( \frac{t_w}{\tau_0} \right)^{\frac{1-\mu}{\mu}}
\]

**Case** \( x = \frac{t_w}{\tau} \ll 1 \)

In the regime most accessible experimentally, \( x = \frac{t_w}{\tau} \ll 1 \), and \( \Pi(t_w, t + \tau) \simeq 1 - c(\mu) \left( \frac{t_w}{\tau} \right)^{1-\mu} \), which leads to the result:

\[
P^C_N(\tau, t_w) \simeq Q(\tau) e^{-(N-1)c(\mu)\left( \frac{t_w}{\tau} \right)^{1-\mu}} \quad (3)
\]

and

\[
P_N(\tau, t_w) \simeq [\psi(\tau) + \frac{(N-1)c(\mu)(1-\mu)}{t_w^{-\mu} \tau^\mu} Q(\tau)] e^{-(N-1)c(\mu)\left( \frac{t_w}{\tau} \right)^{1-\mu}}
\]

Again the limiting case \( P^C_N(\tau, t_w) = \psi(\tau) \) for \( N = 1 \) is recovered. In this regime, when \( N \) increases, the distribution \( P^C_N(\tau, t_w) \) evolves towards a stretched exponential in \( \tau \), with a characteristic time proportional to the age \( t_w \). But as in the case of \( \xi_{\text{eff}}(t_w) \) subsystems of size \( \xi_1 \) at time \( t_1 \). This can actually be quantified by a simple argument. The average hopping rate of one subsystem at time \( t_1 \) is \( S(t_1) \). At time \( t_2 \), one defines a coarse-grained subsystem as composed of \( N_1/N_2 \) of the former subsystems. Then the average hopping rate of one coarse-grained subsystem at time \( t_2 \) is \( S(t_2)/N_1/N_2 \). The hopping rates are chosen to be equal, which leads to \( N_2 = N_1 S(t_2)/S(t_1) \). In the special case of \( t_1 = \tau_0 \) and \( t_2 = t_w \), one finds: \( N(t_w) = N(\frac{\tau_w}{\tau})^{1-\mu} \), which is exactly the relation for \( \xi_{\text{eff}}(t_w) \) found from the previous calculations.

To conclude this section, the study of the two cases investigated above show that for a small system, \( P^C_N(\tau, t_w) \) (and \( P_N(\tau, t_w) \)) will still be very close to the power-law characterizing one single subsystem. For a very large system, \( P^C_N(\tau, t_w) \) crosses over from an exponential form to a stretched exponential form at larger \( \tau \).

In general, there will be a crossover in \( P_N(\tau, t_w) \) from an exponential times a power-law, to a stretched exponential times a power-law, as \( \tau \) increases. In all cases, the distributions become fatter with the age, which allows to define an effective number of subsystems \( N_{\text{eff}}(t_w) = N \left( \frac{t_w}{\tau} \right)^{1-\mu} \), or equivalently an effective coherence length

\[
\xi_{\text{eff}}(t_w) = LN^{-1/d} \left( \frac{t_w}{\tau} \right)^{\frac{1-\mu}{\mu}}
\]

For illustration, we plot on Figure 1 the cumulative probability distribution \( P^C_N(\tau, t_w) \) in the two regimes studied for different values of \( N \); the values of the parameters are \( \tau_0 = 1 \), \( \mu = 0.5 \) and \( t_w = 100 \).
IV. DISTRIBUTIONS OF TIME INTERVALS BETWEEN DECORRELATIONS. APPLICATION TO EXPERIMENTALLY ACCESSIBLE DATA

1. Definition of the quantities of interest

When one is not able experimentally to identify individual rearrangement events, it may be easier to turn to the study of the fluctuations of global quantities such as correlation functions. More precisely, in the scattering experiments of [12], non averaged correlation functions called \( c_I(t_w, \tau) \) are computed. \( c_I(t_w, \tau) \) represents the degree of correlation between the speckle field scattered by the sample at time \( t_w \), and the one scattered at time \( t_w + \tau \). The time lag \( \tau \) can be given a fixed value during the analysis of the data, and one computes the time series of \( c_I(t_w, \tau) \) as a function of time, starting from time \( t_w \) (see for example Figure 3(a) in [19]). This allows to compute the probability distribution of \( c_I \), \( P(c_I) \), for a given \( \tau \) and a given age \( t_w \). This is exactly what we compute in section V, if we assume that the function \( C(t_w, t_w + \tau) \) of our model can be identified with \( c_I(t_w, \tau) \) in our model, a threshold value \( \tau^\text{th} \) being accessible experimentally.

Experimental finding, namely that \( \tau^\text{th} \) is a value that an event has occurred at \( t_w \) if \( C(t_w, t_w + \tau) \) first reaches the value 0 at \( t_w \). We define \( P_0^{(N)}(\tau, t_w, T) \) as the probability per unit time \( \tau \) that such an event takes place at time \( t_w + T \), knowing that such an event happened at \( t_w \). This quantity will be helpful in all the following in order to calculate \( P^{(N)}(\tau, t_w, T) \).

In the case of \( P^{(N)}(\tau, t_w, T) \), the decorrelations in the intervals \([t_w; t_w + \tau]\) and \([t_w + T; t_w + T + \tau]\) are two successive total decorrelations; whereas in the case of \( P_0^{(N)}(\tau, t_w, T) \), they may not be successive in time. Formally speaking, \( P^{(N)}(\tau, t_w, T) \) plays the role of \( \psi(T) \) and \( P_0^{(N)}(\tau, t_w, T) \) the role of \( S(T) \), where \( \psi(T) \) and \( S(T) \) have been introduced in Section II.

Throughout the whole section, we will consider the regime where \( \tau \ll T \). In this case, a good approximation is:

\[
P_0^{(N)}(\tau, t_w, T) \approx \frac{1}{\tau} \left[ \tau S(T) \right]^N.
\]

Therefore, this quantity does not depend on age, and, as in section II, one has the relation:

\[
P_0^{(N)}(\tau, T) = P^{(N)}(\tau, T) + \int_0^T dt' P^{(N)}(\tau, t') P_0^{(N)}(\tau, T-t').
\]

Equivalently, if \( z \) is the Laplace variable conjugated to \( T \), the Laplace transforms are related according to

\[
\hat{P}^{(N)}(\tau, z) = \frac{\hat{P}_0^{(N)}(\tau, z)}{1 + \hat{P}_0^{(N)}(\tau, z)}.
\]

In the case of interest (\( \mu < 1 \), we use the fact that \( S(T) \approx c(\mu)T^{\mu-1}/\tau_0 \) for \( T \gg \tau_0 \), so that


\[ \hat{\mathcal{P}}_0^{(N)}(\tau, z) \simeq \frac{1}{\tau} \left[ \frac{c(\mu) \tau}{\tau_0^\mu} \right]^N \frac{1}{z^2} \int_{\tau_0 z}^\infty du \, e^{-u \tau^2}, \]

where \( x = 1 - N(1 - \mu) \), and we have introduced the lower cut-off \( \tau_0 \) for the case where the integral is divergent at the origin.

According to the value of \( N \), we will see below through the study of the different cases that \( \hat{\mathcal{P}}_0^{(N)}(\tau, T) \) is a power-law with an exponent depending on \( N \).

(a) Case \( 0 < x < 1 : N(1 - \mu) < 1 \)

In this case, the former integral converges when \( \tau_0 z \to 0 \); therefore,

\[ \hat{\mathcal{P}}_0^{(N)}(\tau, z) \simeq \frac{1}{\Gamma(1 - x)} \left( \frac{\tau_0}{\tau(\mu) \tau} \right)^N \]

Hence, if \( \tau_0 z \ll 1 \), \( \hat{\mathcal{P}}_0^{(N)}(\tau, z) \simeq 1 - \Gamma(1 - x)(\tau_0 z)^x \), which leads to

\[ \mathcal{P}^{(N)}(\tau, T) \simeq \frac{\tau_0^x}{(T_0 + T)^{1+x}}. \]

Note that in the special case \( N = 1 \), one has \( x = \mu \) and \( \tau_0 = \tau_0 \), and one recovers the power-law with the initial exponent \( 1 + \mu \).

(b) Case \( 0 < y < 1 : 1 < N(1 - \mu) < 2 \)

The integral is divergent at the origin in the case where \( x < 0 \), and instead of \( x \), we use for convenience \( y = -x = N(1 - \mu) - 1 > 0 \).

For \( 0 < y < 1 \), or \( 1 < N(1 - \mu) < 2 \), we use the following expansion when \( \tau_0 z \to 0 \):

\[ \int_{\tau_0 z}^\infty du \, e^{-u} \approx \frac{1}{y} \frac{1}{(\tau_0 z)^y} - \frac{1}{y} \Gamma(1 - y) \]

We define the time \( T_1 \) such that \( T_1^y = \frac{1}{y} \left[ \frac{\Gamma(\mu)}{\tau_0} \right]^N \)

and we find that \( \hat{\mathcal{P}}_0^{(N)}(\tau, z) \simeq \frac{1}{y} \left( \frac{T_0}{\tau_0} \right)^y - \frac{1}{y} \Gamma(1 - y)(T_1 z)^y \), and

\[ \mathcal{P}^{(N)}(\tau, T) \simeq \frac{1}{y} \left( \frac{T_1}{\tau_0} \right)^y \left[ 1 - \frac{\Gamma(1 - y)}{1 + y} (\tau_0 z)^y \right]. \]

Hence, \( \mathcal{P}^{(N)}(\tau, T) \) is again a power-law at large \( T \):

\[ \mathcal{P}^{(N)}(\tau, T) \simeq \frac{1}{y} \left( \frac{T_1}{\tau_0} \right)^y \left( \frac{T_2}{T_0} \right)^{1+y} (T_2 + T)^{1+y}, \]

where \( T_2 = \frac{\tau_0}{1 + \frac{1}{y} (\tau_0)^y} \).

Note that the constant in front of the power-law is not exact (one would need the expression for all \( T \) of \( S(T) \) to get its exact expression). Moreover, this constant is smaller than 1, which means that the distribution \( \mathcal{P}^{(N)}(\tau, T) \) is not normalized. This comes from the fact that the total number of events (which is equal to \( \hat{\mathcal{P}}_0^{(N)}(\tau, z = 0) \)) is finite; therefore there is a non-zero probability that the time interval between two events is infinite, so that \( \mathcal{P}^{(N)}(\tau, T) \) is not normalized to unity.

(c) Case \( y > 1 : N(1 - \mu) > 2 \)

Now the expansion for \( \tau_0 z \to 0 \) of the integral reads:

\[ \int_{\tau_0 z}^\infty du \, e^{-u} \approx \frac{1}{y} \frac{1}{(\tau_0 z)^y} - \frac{1}{y} \left( \frac{T_1}{\tau_0} \right)^y + \frac{1}{y} \Gamma(2 - y)(T_1 z)^y + o(z^2). \]

in the case where \( 1 < y < 2 \). In general, if \( n = E(y) \) is the integer part of \( y \), the constant part of the expansion is proportional to \( \Gamma(n + 1 - y) \), and is followed by a \( (\tau_0 z)^{n+1-y} \) term. Then, following the case \( 1 < y < 2 \), we find that \( \hat{\mathcal{P}}_0^{(N)}(\tau, z) \simeq \frac{1}{y} \left( \frac{T_0}{\tau_0} \right)^y - \frac{1}{y} T_1^{y-1} z + \frac{1}{y} \Gamma(2 - y)(T_1 z)^y + o(z^2) \). In general, there will always be a singular term in \( z^y \) in between two polynomial terms \( z^n \) and \( z^{n+1} \). Again in the special case where \( 1 < y < 2 \), we find:

\[ \hat{\mathcal{P}}_0^{(N)}(\tau, z) \simeq \frac{1}{y} \left( \frac{T_0}{\tau_0} \right)^y \left[ 1 - \frac{1}{y} z T_2^{1-y} + \frac{1}{y} \Gamma(2 - y)(T_2 z)^y + o(z^2) \right]. \]

As a reminder, let us note that the Laplace transform for small \( z \) of a power-law distribution \( \psi(\tau) \) introduced in Section II is, for \( 1 < \mu < 2 \) : 

\[
\frac{\psi(z)}{\Gamma(2 - \mu)} \simeq \frac{1}{\Gamma(2 - \mu)} \left[ \frac{1}{1 - \frac{1}{\mu - 1}} \right]^N \left[ \frac{\Gamma(\mu)}{\tau_0} \right]^N \left[ \frac{\Gamma(1 - y)}{1 + y} \right]^y (\tau_0 z)^y + o(z^2).
\]

Hence \( \hat{\mathcal{P}}_0^{(N)}(\tau, z) \) cannot be mapped exactly on this type of function, and one can for example add an exponential function to a power-law in order to recover the correct expansion up to order \( z^2 \). This introduces some indeterminacy in the determination of \( \mathcal{P}^{(N)}(\tau, T) \). However, the behaviour at very large \( T \) will be:

\[ \mathcal{P}^{(N)}(\tau, T) \simeq \frac{\alpha}{T^{1+y}}, \]

where \( \alpha \) is an indetermined coefficient. This final power-law behaviour might be in fact rather hard to observe in reality.
3. Special case of the decorrelations with same value

In this section we compute the quantity \( \mathcal{P}^{(k,k)}(\tau, t_w, T) \). Like for the previous case, we define \( \mathcal{P}_0^{(k,k)}(\tau, t_w, T) \) corresponding to \( \mathcal{P}^{(k,k)}(\tau, t_w, T) \), which measures the probability that \( k \) subsystems change traps in the interval \([t_w + T; t_w + T + T]\), knowing that \( k \) subsystems (not necessarily the same) have changed traps in the interval \([t_w; t_w + T]\). More precisely, in this section, we say that an event has occurred at \( t_w \) if \( C(t_w, t_w + T) \) first reaches the value \( 1 - \frac{k}{N} \) at \( t_w \).

Then \( \mathcal{P}_0^{(k,k)}(\tau, t_w, T) \) is the probability per unit time \( \tau \) that such an event takes place at time \( t_w + T \), knowing that an event happened at \( t_w \). In this case \((k \neq N)\), all quantities will depend on \( t_w \) and a relation of the type of (1) is a priori not correct. However, in the limit that we consider, where both \( T \) and \( \tau \) are small compared to \( t_w \), and if \( k \) is not too far from \( N \) (experimentally, the threshold will be taken low enough in order to get rid of experimental noise), we will assume that such a relation can still be valid. We will write:

\[
\mathcal{P}_0^{(k,k)}(\tau, t_w, T) = \mathcal{P}^{(k,k)}(\tau, t_w, T) + \int_0^T dt' \mathcal{P}^{(k,k)}(\tau, t_w, t') \mathcal{P}_0^{(k,k)}(\tau, t_w, T - t') \tag{4}
\]

Like in the previous section, we will first compute \( \mathcal{P}_0^{(k,k)}(\tau, t_w, T) \) and deduce \( \mathcal{P}^{(k,k)}(\tau, t_w, T) \) from its Laplace transform thanks to the relation:

\[
\tilde{\mathcal{P}}^{(k,k)}(\tau, t_w, z) = \frac{\tilde{\mathcal{P}}_0^{(k,k)}(\tau, t_w, z)}{1 + \tilde{\mathcal{P}}_0^{(k,k)}(\tau, t_w, z)} \tag{5}
\]

In order to compute \( \mathcal{P}_0^{(k,k)}(\tau, t_w, T) \), we call \( n \) the number of subsystems that changed traps in the interval \([t_w; t_w + T]\), but stayed trapped in the interval \([t_w + T; t_w + T + T]\). Hence \( k-n \) subsystems change traps both in the intervals \([t_w; t_w + T]\) and \([t_w + T, t_w + T + T]\); \( n \) subsystems change traps in the interval \([t_w + T; t_w + T + T]\) but where trapped in the interval \([t_w + T; t_w + T]\); finally, \( N - k - n \) subsystems were trapped in both the intervals \([t_w; t_w + T]\) and \([t_w + T, t_w + T + T]\).

Therefore, if \( k_{\text{sup}} = \min(k, N - k) \),

\[
\mathcal{P}_0^{(k,k)}(\tau, t_w, T) = \frac{C_{k_{\text{sup}}}^{k}}{\tau} \sum_{n=0}^{k_{\text{sup}}} C_k^n C_{N-k}^n [\tau S(T)]^{k-n} \times [1 - \tau S(T)]^n [\tau S(t_w + T)]^n [1 - \tau S(t_w + T + T)]^{N-k-n}.
\]

Such a finite sum being analytically untractable, we will take its continuous limit, by defining \( x = k/N \) and \( y = n/N \), and by replacing \( \sum_{n=0}^{k_{\text{sup}}} \) by \( N \int_0^{x_{\text{sup}}} \) in the limit where \( N \) is large.

First, if we consider the lowest order in \( \tau \) only, we reduce the expression to:

\[
\mathcal{P}_0^{(k,k)}(\tau, t_w, T) = \frac{C_{N}^{k}}{\tau} \sum_{n=0}^{k_{\text{sup}}} C_k^n C_{N-k}^n [\tau S(T)]^{k-n} [\tau S(t_w + T)]^n.
\]

Then by using the Stirling formula \( N! \sim N^N e^{-N/2} \sqrt{2\pi N} \), and taking the continuous limit, we get:

\[
\mathcal{P}_0^{(k,k)}(\tau, t_w, T) \simeq \frac{\tau^{k-1}}{(2\pi)^{3/2} N^{1/2}} \int_0^{x_{\text{sup}}} dy \exp[-N \{ (x - y) \ln(x - y) + 2y \ln y + (1 - x - y) \ln(1 - x - y) - (x - y) \ln S(T) - y \ln S(T + t_w) \}]
\]

By using a saddle point method, we find that the former integral is maximized for \( y^* \) such that

\[
(x - y^*)(1 - x - y^*) = y^{*2} \gamma(t_w, T)
\]

where \( \gamma(t_w, T) = \frac{S(T)}{S(T + T)} \).

We keep the positive solution, having checked that we have both \( y^* \lesssim x \) and \( y^* \lesssim 1 - x \):

\[
y^* = \frac{1}{2(\gamma - 1)} [-1 + \sqrt{1 + 4x(1-x)(\gamma - 1)}].
\]

Replacing \( y^* \) in the last expression for \( \mathcal{P}_0^{(k,k)}(\tau, t_w, T) \) leads to a complicated expression that can be simplified in our regime of interest, \( t_0 \ll T \ll t_w \).

First, by using the expression for \( S(T) \) at large times and \( \mu < 1 \), we find that, for \( \tau_0 \ll T \ll t_w \), \( \gamma(t_w, T) \simeq (t_w/T)^{1-\mu} \), and \( y^*(T, t_w) \simeq \left( \frac{2(1-x)}{\gamma(t_w, T)} \right)^{1/2} \). Then, if \( \frac{t_w}{T} \) is large enough, we can always assume to have \( y^* \lesssim \min(x, 1 - x) \), so that equation (8) can be expanded in \( y^* \). Finally, we find the result:

\[
\mathcal{P}_0^{(k,k)}(\tau, t_w, T) \simeq \frac{C_{k}^{k}}{2\pi \tau} S(T)^{k} \exp \left[ K(k) \left( \frac{T}{t_w} \right)^{1/\mu} \right].
\]

The Laplace transform with respect to \( T \) reads:

\[
\tilde{\mathcal{P}}_0^{(k,k)}(\tau, t_w, z) \simeq \frac{C_{k}^{k}}{2\pi \tau} \left[ \frac{c(\mu)_{\gamma}}{\tau_0} \right]^{k} I(t_w, z),
\]

where

\[
I(t_w, z) = \int_0^{\infty} dT \frac{e^{-z T}}{T^{(1-\mu)k}} \exp \left[ K(k) \left( \frac{T}{t_w} \right)^{1/\mu} \right],
\]
and $K(k) = 2\sqrt{k(N - k)}$. 
$I(t_w, z)$ is a convergent integral for $k(1 - \mu) < 1$, which is the first case we consider.

(a) Case $k(1 - \mu) < 1$

Since we are interested in the case $T \ll t_w$, we will make an expansion for $z t_w \gg 1$.

Then we need to consider two separate cases for the evaluation of $P^{(k,k)}(\tau, t_w, z)$ from relation (6). We report here the main results, and refer to Appendix A for the calculations.

\[(a1) \text{Case } 0 < k < \frac{1+\mu}{2(1-\mu)}
\]

$P^{(k,k)}(\tau, t_w, T)$ is a power-law:

$$P^{(k,k)}(\tau, t_w, T) \approx C_{k}^{\beta} \left[ \frac{c(\mu)\tau}{\tau_0^\mu} \right]^k \frac{A t_w^\alpha}{\Gamma(-\beta)} T^{1+\beta}$$  \hspace{1cm} (8)

with $\alpha = \frac{k(1-\mu)^2}{1+\mu}$ and $\beta = \frac{2k(1-\mu)}{1+\mu} - 1$.

\[(a2) \text{Case } \frac{1+\mu}{2(1-\mu)} < k < \frac{1}{1-\mu}
\]

$P^{(k,k)}(\tau, t_w, T) \approx e^{\alpha(\tau/T_w) - \beta K(t_w, \tau) + \beta T/T_w^\mu}$  \hspace{1cm} (9)

which is a stretched exponential since $\cos(\frac{\tau}{T_w^\mu}) < 0$.

(b) Case $k(1 - \mu) > 1$

$P^{(k,k)}(\tau, t_w, T)$ is simply an exponential, with a weak dependence on $t_w$ (which vanishes for large $t_w$):

$$P^{(k,k)}(\tau, t_w, T) \approx C_{k}^{\alpha} \left[ \frac{c(\mu)\tau}{\tau_0^\mu} \right]^k \frac{A t_w^\alpha}{\Gamma(-\beta)} T^{1+\beta}$$  \hspace{1cm} (10)

where $C(t_w, \tau) = \frac{C_{k}^{\beta}}{C_{k}^{\alpha}} \left[ \frac{c(\mu)\tau}{\tau_0^\mu} \right]^k \frac{K(k)\left(\frac{c(\mu)}{\tau_0^\mu} \right)}{\tau_0^\mu}$.

We note here that it is not possible to make a clear continuity between the cases $k = N$ and the case $k$ different than $N$ treated in this section. This is due to the fact that we use a saddle-point approximation of $I(t_w, z)$, which is defined only if $K(k)$ is different from 0, hence $k$ is different from $N$ and 0. In this section, the approximations are more numerous and are susceptible to be valid only for some intermediate values of $k$. Moreover, we are interested in the aging regime and now the correct expansion is to be made for $z t_w \gg 1$, and not anymore for $z t_0 \ll 1$.

4. Generalization

The most general case, as introduced in the beginning of Section IV, consists in computing $P^{(k)}(\tau, t_w, T)$ the distribution of time intervals between successive decorrelations larger than $\frac{\tau}{N}$.

As before, we will use the $P_0$ quantities relative to the $P$ quantities we are looking for, and we will assume that relations between them similar to equation (4) still hold.

Therefore, we will start by computing

$$P^{(0)}_0(\tau, t_w, T) = \sum_{k'\geq k} \sum_{k''\geq k'} P^{(k',k'')}_0(\tau, t_w, T)$$  \hspace{1cm} (11)

where $P^{(k',k'')}_0(\tau, t_w, T)$ is the probability that $k'$ subsystems have changed traps in the interval $[t_w, t_w + \tau]$ and $k''$ subsystems have changed traps in the interval $[t_w + T, t_w + T + \tau]$. We refer the reader to Appendix B for technical details of the computation, which is similar to the one explained in section IV.3.

From the calculation, we get

$$P^{(k)}_0(\tau, t_w, T) = P^{(k^*)}_0(\tau, t_w, T),$$

$$P^{(k)}(\tau, t_w, T) = P^{(k^*)}(\tau, t_w, T),$$

where $k^*$ is a growing function of $k$, and is equal to $k$ if $\tau S(T) < 1$ and $k > N/2$.

Therefore the results of the previous section can be used. We conclude that, according to the value of $k^*$ (which will be equal to $k$ in most cases of interest), $P^{(k)}(\tau, t_w, T)$ can either take a power-law form (equation (5)), a stretched exponential form (equation (8)), or an exponential form (equation (10)), where $k$ has to be replaced by $k^*$.

V. PROBABILITY DISTRIBUTION OF THE TOTAL CORRELATOR

In section III, we used the notion of “events”, and calculated their time distribution, because it was the most natural quantity to compute in the framework of the model presented here. However, it may be difficult in an experiment to identify such events, or even to find a reasonable definition of individual events.

A well-defined quantity that is more readily accessible is the distribution probability of a correlator $C(t_w, t_w + \tau)$, as has been recently investigated in numerical simulations 24.

In our model, each subsystem $i$ is a two-level system, where the correlation $C_i(t_w, t_w + \tau)$ can only take the
values 0 or 1. More precisely, the probability distribution of the correlation of a subsystem $i$ is:

$$P(C_i) = f\delta(C_i - 1) + (1-f)\delta(C_i),$$

where the parameter $f$ coincides with the average value of the correlation for the values of $t_w$ and $\tau$ considered: $f = \Pi(t_w, t_w + \tau) = < C_i(t_w, t_w + \tau) >$. In the following we will omit temporarily the dependence of $f$ on $t_w$ and $\tau$, for simplicity in the notations.

For the whole system (the superposition of the $N$ independent subsystems), the definition of $C(t_w, t_w + \tau)$ has been given in Section II. One can actually rewrite this quantity as:

$$C(t_w, t_w + \tau) = \frac{m(t_w, t_w + \tau)}{N} = 1 - \frac{n(t_w, t_w + \tau)}{N},$$

where $m(t_w, t_w + \tau)$ is the number of subsystems that remained in the same energy trap between $t_w$ and $t_w + \tau$, and $n(t_w, t_w + \tau)$ the number of subsystems that changed trap between $t_w$ and $t_w + \tau$.

Then, in order to find the value $C$ for the correlation between times $t_w$ and $t_w + \tau$, one has to draw $m$ subsystems among $N$ instances that are trapped during this time interval, with probability $f$, and $N - m$ subsystems that changed trap during this time interval, with probability $1 - f$. Hence, the probability distribution of $C$, $P(C)$, is simply the binomial distribution: $P(C) = NP(m)$, with

$$P(m) = C_N^m f^m (1-f)^{N-m},$$

and $C_N^m$ is the binomial coefficient.

Note that in the limit of a large number of subsystems $N$, the central limit theorem holds and the limiting distribution is a Gaussian:

$$P(C) \rightarrow \sqrt{\frac{N}{2\pi f(1-f)}} \exp \left[ -\frac{N(C - f)^2}{2f(1-f)} \right].$$

We show on Figure 2 the distributions $P(C)$ for different values of the average correlation: $< C > = 0.1, 0.5, 0.9$. The full lines correspond to $N = 10$ and the dotted lines to $N = 100$, showing a closer resemblance to a Gaussian distribution for $N = 100$.

For comparison with experiments or numerical simulations, we computed the variance $\sigma^2(C)$, the skewness $s(C)$ and the kurtosis $\kappa(C)$ of the distribution $P(C)$. They can be easily computed by noticing that the cumulants of independent random variables are additive quantities:

$$\sigma^2(C) = \frac{1}{N} f(1-f)$$
$$s(C) = \frac{1}{\sqrt{N}} \frac{1-2f}{\sqrt{f(1-f)}}$$
$$\kappa(C) = \frac{1}{N} \frac{6f^2 - 6f + 1}{f(1-f)}$$

At this stage, one would like to plot the previous quantities as a function of $\tau$ for different values of the age $t_w$, in order to compare with existing results in the literature 19, 24, 39. For given $\tau$ and $t_w$, $f = \Pi(t_w, t_w + \tau)$ is explicitly defined using the relation of section II. Moreover, we use the results of Section III that suggest that at age $t_w$, the system can actually be considered as a superposition of $N_{eff}(t_w) = N \left( \frac{t_w}{t_{w_0}} \right)^{1-\mu}$ independent subsystems. Hence in the preceding formulae for $\sigma^2(C)$, $s(C)$, $\kappa(C)$, we replace $N$ by $N_{eff}(t_w)$. We want to stress that it is not proved here that this mapping can be applied to the case of the full probability of the correlation functions, it is only a phenomenological attempt to try to extend this coarse-graining to the calculation of other quantities than the distribution of times. This argument actually leads to results compatible to what is found in experiments.

In Figures 3, 4 and 5, $\sigma^2(C)$, $s(C)$, and $\kappa(C)$ are shown as a function of $\tau$ for different values of $t_w$, for $\mu = 0.8$ and $N = 10$. As observed in recent experiments on foams 19, 33, one can see that the variance is maximum at a time $\tau$ of the order of the relaxation time (which in our model is proportional to the age $t_w$), and goes to 0 at large $\tau$; this maximum increases with the age. The skewness is negative at small $\tau$, crosses the origin at intermediate times when $P(C)$ becomes symmetric and then becomes positive; the negative skewness is
more and more pronounced when \( t_w \) increases. The kurtosis is positive at small times, then negative, and positive again at large \( \tau \), while the negative part becomes more and more pronounced with the age. All these results are compatible with recent numerical simulations \[24\], and seem to be compatible with preliminary experiments on colloidal gels \[19\]. Note again that the use of \( N_{\text{eff}}(t_w) \) contains crucial information: if \( N \) was kept constant, the curves would simply be superimposed by a \( \tau/t_w \) rescaling, in particular, the maxima and minima of the variance, skewness and kurtosis would not depend on the age.

Finally, we computed the probability distribution \( P(C) \) for a given \( \tau/t_w \) ratio - i.e a given value of the average correlation \( < C >_\tau \), at different ages, therefore for different \( N_{\text{eff}}(t_w) \); the system is made initially of \( N = 1000 \) independent subsystems. Figure 6 shows such probability distributions, for \( \mu = 0.8 \), \( f = < C > = 0.8 \), \( N = 1000 \), plotting \( P(C)/\sigma(C) \) versus \( (C - < C >)/\sigma(C) \). One can see that even for \( N = 1000 \), systematic deviations exist, though they may be hard to distinguish when an experimental noise is present. Deviations are systematic as \( t_w \) increases, which is not surprising since \( N_{\text{eff}}(t_w) \) decreases with the age; the central limit theorem is less and less valid as \( t_w \) increases. Note that we tried to fit these curves with Gumbel distributions like in \[24\], but we didn’t find a very precise agreement in our case.

Finally, let us note that our results fail to reproduce the expected zero limit of the skewness and kurtosis of \( P(C) \) when \( t_w \to \infty \). These two quantities actually diverge in our case for any finite \( N \), because \( P(C) \) tends to a delta function \( \delta(C) \) and \( \sigma(C) \to 0 \). However, the limit \( t_w \to \infty \) is formally equivalent to \( N_{\text{eff}}(t_w) \to \infty \), which restores the correct limit of the central limit theorem. This problem would be absent if \( P(C_i) \) for one individual system had a non-zero variance when \( C_i \to 0 \). In practice, this will be always the case, because of the presence of white thermal noise.

VI. DISCUSSION AND CONCLUSION

We have studied a system made of the superposition of \( N \) subsystems characterized individually by a slow dynamics and a power-law distribution of times between jumps. This system can in turn be characterized by its distribution of time intervals between events \( P_N(\tau, t_w) \). For small \( N \), the power-law behaviour of a single subsystem is the dominant behaviour. For large \( N \), at very small time intervals, this distribution is exponential and crosses over at larger time intervals to a stretched exponential. In the two regimes, a natural scaling between \( N \) and \( t_w \) appears, which allows to define an age dependent effective number of subsystems \( N_{\text{eff}}(t_w) \) which decreases with \( t_w \), or equivalently, an effective coherence length.
which grows slowly with the age: $\xi(t_w) \sim t_w^{(1-\mu)/d}$. It would be interesting to see whether experimental results also display the different regimes calculated here. In particular, one could observe an exponential or a stretched exponential distribution for a young system, and a power-law distribution for an aged system. As explained in section III, the interpretation of $\xi_{eff}(t_w)$ comes from the aging of all individual subsystems: the systems that hop are more rare when $t_w$ increases, and hence further apart from each other; this introduces a relevant lengthscale $\xi(t_w)$ that grows with $t_w$.

We also computed $P^{(k)}(\tau, t_w, T)$ the distribution of time intervals between successive decorrelations over an interval $\tau$, larger than $\frac{1}{t_w}$. This quantity, like $P(C)$, is accessible through scattering experiments. Although the derivation of this quantity is a bit technical, the results can be summarized in the following way. In the extreme case where one only counts the largest decorrelations ($k = N$), the distribution of time intervals between successive decorrelations crosses over from a power law with a small exponent at small $N$ (or large $t_w$) to a power-law with a large exponent at large $N$ (or small $t_w$). It is hoped that in experiments, these regimes will be observed, when one varies the age or other parameters. For example, if the temperature is decreased in a glass, or the density is increased in a gel, or if one increases the quench rate, one expects that the coherence length increases, i.e the number of independent subsystems decreases. For $k \neq N$, the results are more difficult to obtain, due to technical complications, and are likely to be reliable only for intermediate values of $k$. If $k$ is small (i.e one counts only the decorrelations corresponding to a few subsystems), the distribution $P^{(k)}(\tau, t_w, T)$ is a power-law; if $k$ is larger, it is a stretched exponential and for $k$ large it is an exponential. Equivalently, since one expects the number of subsystems to decrease with the age, $P^{(k)}(\tau, t_w, T)$ is typically an exponential at small $t_w$, then a stretched exponential and finally a power-law at large $t_w$, for a given value of the correlation threshold $C_{th} = 1 - k/N$.

Finally, we also computed analytically the probability distribution of the correlator $P(C)$, which reproduces many features of the experimental findings of [19] and of the numerical results of [24]. We used the fact (not shown rigorously) that at $t_w$ the system can be described as a collection of $N_{eff}(t_w)$ independent subsystems. No rescaling to some time-evolving Gumbell distribution seems to be relevant here. Since the full probability is calculated here, it would be interesting to compare it more quantitatively to experimental results. In particular, we find that the variance grows with the age like $\sigma^2 \sim N_{eff}^{-1} \sim t_w^{-\mu}$.

Although these results are derived from a simplified theoretical model, we think that such a description captures the essential intermittent character of the relaxation of a realistic glassy system, together with the existence of a coherence lengthscale growing with the age, due to the decrease of the rate of events in time. We hope that these findings will be helpful in analyzing more quantitatively some experimental results. In particular, the results of Section IV and Section V might be of interest for the Time Resolved Correlation experiments of [19]. However, since the individual trap model is probably a very crude model for the relaxation of realistic materials (< $C(t_w, t_w + \tau >)$ is a priori unlikely to coincide with the experimental average relaxation function of colloidal gels), it may be interesting in a further study to replace it with a more realistic dynamical model for jammed colloidal gels [40, 20]. Concerning the intermittency dynamics studied in [20], the quantities calculated in Section III are maybe the most relevant ones to compare with the experimental results.

The model studied here has the major disadvantage that the relevant coherence length of the system is introduced by hand through the superposition of several subsystems. However, we were able to extract useful information from this picture. It would be more satisfactory to solve a model where an aging coherence length builds up during the dynamics, resulting from microscopic dynamics. Candidates could be special cases of kinetically constrained models, though analytical calculations will be quite hard [41]. However, numerical simulations provide now a useful alternative tool for the precise investigation, in microscopic models of glasses, of the quantities calculated in this paper [24].

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APPENDIX A

(a) Case \( k(1-\mu) < 1 \)

In this case, we can perform a saddle-point calculation in order to evaluate \( I(t_w, z) \), which leads to:

\[
I(t_w, z) \simeq A t_w^\alpha z^\beta e^{-\frac{B}{zt_w^{1+\gamma}}},
\]

with \( A = [(1-\mu)K(k)/2]^{\frac{1}{1+\gamma}}, B = [(1-\mu)K(k)/2]^{\frac{1}{1+\gamma}}, \alpha = \frac{k(1-\mu)^2}{1+\mu}, \beta = \frac{2k(1-\mu)}{1+\mu} - 1, \) and \( \gamma = \frac{1}{1+\mu} \).

Since we are interested in the case \( T \ll t_w \), we expand \( I(t_w, z) \) for \( zt_w \gg 1 \): \( I(t_w, z) \simeq A t_w^\alpha z^\beta \left[ 1 - \frac{B}{zt_w^{1+\gamma}} \right] \).

Then we need to consider two separate cases for the evaluation of \( \tilde{p}(k,k)(\tau, t_w, z) \) from relation (1):

(a1) Case \( 0 < k < \frac{1+\mu}{2(1-\mu)} \)

For \( 0 < k < \frac{1+\mu}{2(1-\mu)} \) (i.e. \( \beta < 0 \)), one gets for \( zt_w \gg 1 \):

\[
\tilde{p}(k,k)(\tau, t_w, z) \simeq A \frac{c_k}{2\pi^2} \left[ \frac{(\mu)\tau}{\pi^\frac{\mu}{\sqrt{\mu}}} \right] t_w^{\frac{\mu}{\sqrt{\mu}}} e^{-\frac{B}{zt_w^{1+\gamma}}}.
\]

Therefore, \( p(k,k)(\tau, t_w, T) \) is a power-law:

\[
p(k,k)(\tau, t_w, T) \simeq C_k \left[ \frac{(\mu)\tau}{\pi^\frac{\mu}{\sqrt{\mu}}} \right] t_w^{\frac{\mu}{\sqrt{\mu}}} e^{-\frac{B}{zt_w^{1+\gamma}}}.
\]

(a2) Case \( \frac{1+\mu}{2(1-\mu)} < k < \frac{1}{1+\mu} \)

In this case, we find that \( \tilde{p}(k,k)(\tau, t_w, z) \simeq \exp \left[ -\frac{K(t_w, z)}{zt_w} \right] \), with \( K(t_w, \tau) = \frac{2\pi^\frac{\mu}{\sqrt{\mu}}}{c_k \tau^\frac{\mu}{\sqrt{\mu}}} \).

The evaluation of the inverse Laplace transform leads to:

\[
p(k,k)(\tau, t_w, T) \simeq e^{\cos(\frac{\pi\tau}{\sqrt{\mu}})(\beta K(t_w, \tau))^{\frac{1}{1+\gamma}} \left( \frac{\pi}{\sqrt{\mu}} \right)^\frac{1}{1+\gamma}}
\]

which is a stretched exponential since \( \cos(\frac{\pi\tau}{\sqrt{\mu}}) < 0 \).

(b) Case \( k(1-\mu) > 1 \)

Finally, we consider the case \( k(1-\mu) > 1 \): then we have to introduce the lower-cut-off \( \tau_0 \) in \( I(t_w, z) \), which leads to:

\[
I(t_w, z) \simeq z^{(1-\mu)k-1} \int_{\tau_0 z}^{\infty} du \frac{e^{-u}}{u(1-\mu)k} e^{-uK(k)(\pi u)\frac{1}{1+\gamma}}.
\]

The saddle point solution is \( u^* = \frac{[(1-\mu)K(k)/2]^{\frac{1}{1+\gamma}}}{z^{\tau_0}} \). In the limit of very large \( zt_w \), one will at some point reach the situation where \( u^* < \tau_0 z \). Hence, for \( zt_w \gg 1 \), the integral will be best evaluated by its lower cut-off value:

\[
I(t_w, z) \simeq \frac{1}{z^{\tau_0}} \frac{1}{\tau_0 z^{1+\gamma}} e^{-\tau_0 z K(k)(\pi)\frac{1}{1+\gamma}}.
\]

Since \( T \gg \tau_0 \), we finally get:

\[
\tilde{p}(k,k)(\tau, t_w, z) \simeq \frac{C_{k,k}^{\tau_0} \left[ \frac{(\mu)\tau}{\pi^\frac{\mu}{\sqrt{\mu}}} \right]}{\tau_0 \gamma},
\]

where

\[
C(\tau, T) = \frac{\gamma}{2\pi^2} \left[ \frac{(\mu)\tau}{\pi^\frac{\mu}{\sqrt{\mu}}} \right] K(k)(\pi)\frac{1}{1+\gamma}.
\]

Finally, \( p(k,k)(\tau, t_w, T) \) is simply an exponential, with a weak dependence on \( t_w \) (which vanishes for large \( t_w \)):

\[
p(k,k)(\tau, t_w, T) \simeq C(\tau, T)e^{-C(\tau, T)T}.
\]

APPENDIX B

For the computation of \( p_0^{(k',k'')}(\tau, t_w, T) \), we will proceed in the same way as in the last section for \( p_{0,k}(\tau, t_w, T) \). Using again combinatorial arguments, we can write that:

\[
p_0^{(k',k'')}(\tau, t_w, T) = \frac{C_k}{\tau} \sum_{n_{inf}=0}^{n_{sup}} C_{k'} C_{N-k'-n} \left[ \tau S(T) \right]^{k'-n} \left[ 1 - \tau S(T) \right]^{N-k'-n} \left[ \tau S(t_w + T) \right]^{k'-n}.
\]

where \( n_{inf} = \text{sup}(0, k'-k'') \) and \( n_{sup} = \text{inf}(k', N-k') \).

As before, we keep only the leading order in \( \tau \) and go to the large \( N \) limit by introducing \( x' = \frac{k'}{N}, x'' = \frac{k''}{N} \) and \( y = \frac{\mu}{\sqrt{\mu}} \). This leads to:

\[
p_0^{(k',k'')}(\tau, t_w, T) \simeq \frac{\tau^{x''-1}}{(2\pi)^{3/2}N^{3/2}} \int_{x_{inf}}^{x_{sup}} dy e^{-N S(x', x'', y)},
\]

where

\[
S(x', x'', y) = y \ln y + (x'' - x' + y) \ln(1 - x' + y) + (x' - y) \ln(x' - y) + (1 - x'' - y) \ln(1 - x'' - y) - (x' - y) \ln S(T) - (x'' - x' + y) \ln S(t_w + T).
\]
We use a saddle-point approximation to minimize \( S(x’, x’’, y) : \frac{\partial S}{\partial y} |_{y^*} = 0 \) leads to:
\[
(x’ - y^*)(1 - x’ - y^*) = y^*(x’’ - x’ + y)\gamma(t_w, T)
\]
which is the generalization of equation (7).

The positive solution is therefore:
\[
y^* = \frac{1}{2(\gamma - 1)}[-1 - (\gamma - 1)(x’’ - x’)]
+ \sqrt{(1 + (\gamma - 1)(x’’ - x’)^2 + 4x’(1 - x’’)(\gamma - 1)]}
\]
and it is easy to check that \( x_{inf} \leq y^* \leq x_{sup} \). Note that in this equation and in the following, we drop the \((t_w, T)\) dependence of \( \gamma(t_w, T) \) for convenience.

We now come back to the calculation of \( \mathcal{P}_0^{(k)}(\tau, t_w, T) \) through equation (11). For large \( N \), we have:
\[
\mathcal{P}_0^{(k)}(\tau, t_w, T) \simeq N^2 \int_x^1 dx’ \int_x^1 dx’’ \mathcal{P}_0^{(k, k’)}(\tau, t_w, T).
\]

Direct integration is not possible here, so we use again a saddle point treatment of the double integral. The saddle points \( x^{*o} \) and \( x^{*o}’o \) are solutions of the set of equations formed by: \( \frac{\partial S}{\partial x}(x^{*o}, x^{*o}’o) = 0 \) and \( \frac{\partial S}{\partial x’}(x^{*o}, x^{*o}’o) = 0 \), or:
\[
\frac{\partial y^*}{\partial x’} \ln \frac{y^*(x’’ - x’ + y^*)\gamma}{(1 - x’ - y^*)(x’ - y^*)} = \ln \frac{\gamma(x’’ - x’ + y^*)}{(x’ - y^*)}
\]
\[
\frac{\partial y^*}{\partial x’’} \ln \frac{y^*(x’’ - x’ + y^*)\gamma}{(1 - x’ - y^*)(x’ - y^*)} = \ln \frac{\tau S(T + t_w)(1 - x’’ - y^*)}{x’’ - x’ + y^*}
\]
\[\text{(12)}\]
\[\text{(13)}\]

Here, \( \frac{\partial y^*}{\partial x} |_{(x^{*o}, x^{*o}’o)} \) and \( \frac{\partial y^*}{\partial x’} |_{(x^{*o}, x^{*o}’o)} \) can be calculated using the expression for \( y^* \).

At this point, taking \( \frac{T_s}{T_c} \ll 1 \) limit enables us to make simplifications in equations (12) and (13). A rather lengthy calculation leads to the conclusion that there is no couple of solutions \( (x^{*o}, x^{*o}’o) \) compatible with equations (12) and (13), and such that \( x^{*o} \neq x^{*o}’o \).

Consequently, we keep only the terms such that \( k’ = k’’ \) in the sum of (14), and try to find the value for \( k’ \) that maximizes this sum, knowing the expression of \( \mathcal{P}_0^{(k, k’)}(\tau, t_w, T) \) from section IV.3:
\[
\mathcal{P}_0^{(k)}(\tau, t_w, T) \propto \int_x^1 dx’ \exp \{N[-x’ \ln x’ - (1 - x’)] \ln S(T) + 2\left(\frac{x’(1 - x’)}{\gamma(t_w, T)}\right)^{1/2}\}
\]
The saddle-point equation for \( x^{*o} \) is now:
\[
\ln \left(\frac{x^{*o}}{1 - x^{*o}}\right) = \frac{1 - 2x^{*o}}{\sqrt{x^{*o}(1 - x^{*o})\gamma(t_w, T)}} + \ln[\tau S(T)].
\]

If \( \tau S(T) = 1 \), this equation has the only solution \( x^{*o} = \frac{1}{2} \). For \( \tau S(T) > 1 \), there is a solution \( x^{*o} > \frac{1}{2} \), and for \( \tau S(T) < 1 \), there is a solution \( x^{*o} < \frac{1}{2} \). In general, \( x^{*} = \sup(x, x^{*o}) \) will maximize the expression for \( \mathcal{P}_0^{(k)}(\tau, t_w, T) \), and can be estimated numerically. In particular, if \( \tau S(T) < 1 \) and \( k > N/2 \), we have \( x^{*} = x \).

Finally,
\[
\mathcal{P}_0^{(k)}(\tau, t_w, T) = \mathcal{P}_0^{(k’)}(\tau, t_w, T),
\]
\[
\mathcal{P}^{(k)}(\tau, t_w, T) = \mathcal{P}^{(k’)}(\tau, t_w, T),
\]
where \( k^{*} = \sup(k, k^{*o}) \).

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