Occurrence of distances in vector spaces
over prime fields

Thang Pham ∗ Le Anh Vinh †

Abstract

Let $\mathbb{F}_q$ be an arbitrary finite field, and $E$ be a point set in $\mathbb{F}_q^d$. Let $\Delta(E)$ be the set of distances determined by pairs of points in $E$. By using Kloosterman sums, Iosevich and Rudnev proved that if $|E| \geq 4q^{\frac{d+1}{2}}$ then $\Delta(E) = \mathbb{F}_q$. In general, this result is sharp in odd dimensional spaces over arbitrary finite fields. In this paper, we use the recent point-plane incidence bound due to Rudnev to prove that if $E$ has Cartesian product structure in vector spaces over prime fields, then we can break the exponent $(d+1)/2$ and still cover all distances. We also show that the number of pairs of points in $E$ of any given distance is close to its expected value.

1 Introduction

Let $E$ be a finite subset of $\mathbb{R}^d$ ($d \geq 2$), and $\Delta(E)$ be the distance set determined by $E$. The Erdős distinct distances problem is to find the best lower bound of the size of the distance set $\Delta(E)$ in terms of the size of the point set $E$.

In the plane case, Erdős [7] conjectured that $|\Delta(E)| \gg |E|/\sqrt{\log |E|}$. This conjecture was proved up to log-arithmetic factor by Guth and Katz [10] in 2010. More precisely, they showed that $|\Delta(E)| \gg |E|/\log |E|$. In higher dimension cases, Erdős [7] also conjectured that $|\Delta(E)| \gg |E|^{2/d}$. Interested readers are referred to [26] for results on Erdős distinct distances problem in three and higher dimensions.

∗Department of Mathematics, University of Rochester New York. Email: vpham2@math.rochester.edu
†Vietnam Institute of Educational Sciences. Email: vinhle@vnies.edu.vn
In this paper, we use the following notations: \( X \ll Y \) means that there exists some absolute constant \( C_1 > 0 \) such that \( X \leq C_1 Y \), \( X \gg Y \) means \( X \gg (\log Y)^{-C_2} Y \) for some absolute constant \( C_2 > 0 \), and \( X \sim Y \) means \( Y \ll X \ll Y \).

As a continuous analog of the Erdős distinct distances problem, Falconer \[8\] asked how large the Hausdorff dimension of \( E \subset \mathbb{R}^d \) needs to be to ensure that the Lebesgue measure of \( \Delta(E) \) is positive. He conjectured that for any subset \( E \subset \mathbb{R}^d \) of the Hausdorff dimension greater than \( d/2 \) then \( E \) determines a distance set of a positive Lebesgue measure. This conjecture is still open in all dimensions. We refer readers to \[6, 9\] for recent updates on this conjecture.

Let \( \mathbb{F}_q \) be the finite field of order \( q \), where \( q \) is an odd prime power. Given two points \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) in \( \mathbb{F}_q^d \), we denote the distance between \( x \) and \( y \) by

\[ ||x - y|| := (x_1 - y_1)^2 + \ldots + (x_d - y_d)^2. \]

Note that the distance function defined here is not a metric but it is invariant under translations and actions of the orthogonal group.

For a subset \( E \subset \mathbb{F}_q^d \), we denote the set of all distances determined by \( E \) by

\[ \Delta(E) := \{ ||x - y|| : x, y \in E \}. \]

The finite field analogue of the Erdős distinct distances problem was first studied by Bourgain, Katz, and Tao in 2003 \[2\]. More precisely, they proved that in the prime field \( \mathbb{F}_p \) with \( p \equiv 3 \mod 4 \), for any subset \( E \subset \mathbb{F}_p^2 \) of the cardinality \( |E| = p^\alpha \), \( 0 < \alpha < 2 \), then \( |\Delta(E)| \gg |E|^{\frac{3}{2} + \epsilon} \) for some \( \epsilon = \epsilon(\alpha) > 0 \).

Note that the condition \( p \equiv 3 \mod 4 \) in Bourgain, Katz, and Tao’s result is necessary, since if \( p \equiv 1 \mod 4 \), then there exists \( i \in \mathbb{F}_p \) such that \( i^2 = -1 \). By taking \( E = \{(x, ix) : x \in \mathbb{F}_p\} \), we have \( |E| = p \) and \( \Delta(E) = \{0\} \). For prime \( p \equiv 3 \mod 4 \), this result has been quantified and improved over recent years. The best current result in the range \( |E| \ll p^{1158/1559} \) is

\[ |\Delta(E)| \gtrsim |E|^{\frac{5}{2} + \frac{69}{1559}} \]

due to Iosevich, Koh, and Pham \[18\]. More importantly, they investigated the quantitative
connection between the distance set $\Delta(E)$ and the set of rectangles determined by $E$. In $\mathbb{F}_p^3$, Rudnev [24] proved that for any $E \subset \mathbb{F}_p^3$ that is not supported in a single semi-isotropic plane determines $c \cdot \min\{|E|^{1/2}, p\}$ for some constant $c > 0$.

In the setting of arbitrary finite fields $\mathbb{F}_q$, Iosevich and Rudnev [17] showed that Bourgain, Katz, and Tao’s result does not hold. For example, assume that $q = p^2$, one can take $E = \mathbb{F}_p^2$ then $\Delta(E) = \mathbb{F}_p$ or $|\Delta(E)| = |E|^{1/2}$. Thus, Iosevich and Rudnev reformulated the problem in the spirit of the Falconer distance conjecture over the Euclidean spaces. More precisely, they asked for a subset $E \subset \mathbb{F}_q^d$, how large does $|E|$ need to be to ensure that $\Delta(E)$ covers the whole field or at least a positive proportion of all elements of the field?

Using Fourier analytic methods, Iosevich and Rudnev [17] proved that for any point set $E \subset \mathbb{F}_q^d$ with the cardinality $|E| \geq 4q^{(d+1)/2}$ then $\Delta(E) = \mathbb{F}_q$. Hart, Iosevich, Koh, and Rudnev [15] showed that, in general, the exponent $(d + 1)/2$ cannot be improved when $d$ is odd, even if we only want to cover a positive proportion of all the distances. In even dimensional cases, it has been conjectured that the exponent $(d + 1)/2$ can be improved to $d/2$, which is in line with the Falconer distance conjecture in the Euclidean space.

In the plane case, Bennett, Hart, Iosevich, Pakianathan, and Rudnev [3] proved that if $E \subset \mathbb{F}_q^2$ of cardinality $|E| \geq q^{4/3}$, then $\Delta(E)$ covers a positive proportion of all distances.

In a recent note, Murphy and Petridis [20] showed that there are infinite subsets of $\mathbb{F}_q^d$ of size $q^{4/3}$ whose distance sets do not cover the whole field $\mathbb{F}_q$. It is not known whether there exist a small $c > 0$ and a set $E \subset \mathbb{F}_q^d$ with $|E| \geq cq^{3/2}$ such that $\Delta(E) \neq \mathbb{F}_q$. We refer the interested reader to [15, Theorem 2.7] for a construction in odd dimensional spaces.

Chapman et al. [4] broke the exponent $\frac{d+1}{2}$ to $\frac{d^2}{2d-1}$ under the additional assumption that the set $E$ has Cartesian product structure. However, in this case, they can cover only a positive proportion of all distances. In the setting of prime fields, the authors and de Zeeuw [22] proved that for $A \subset \mathbb{F}_p$, we have $|\Delta(A^d)| \geq \frac{1}{c} \cdot \min\{|A|^{2-d^{-2}}, p\}$ with $c = 2 \frac{d^2-1}{d^2-2}$. Therefore, $|\Delta(A^d)| \geq \frac{p}{c}$ under the condition $|A| \geq p^{\frac{d^2-2}{2d-1}}$. However, this result again only gives us a positive proportion of all distances, and does not tell us the number of pairs of any given distance.

In this paper, we will show that if $E \subset \mathbb{F}_p^d$ has Cartesian product structure, we can break the exponent $\frac{d+1}{2}$ due to Iosevich and Rudnev [17] and still cover all possible distances. Our main tool is the recent point-plane incidence bound due to Rudnev [24].
Our first result is for odd dimensional cases.

**Theorem 1.1.** Let $\mathbb{F}_p$ be a prime field, and $A$ be a set in $\mathbb{F}_p$. For an integer $d \geq 3$, suppose the set $A^{2d+1} \subset \mathbb{F}_p^{2d+1}$ satisfies

$$|A^{2d+1}| \gtrsim p \frac{2^{d+1} - 3^{d-2} - d - 1}{2^{d+1} - 1},$$

then we have

- The distance set covers all elements in $\mathbb{F}_p$, namely,

$$\Delta(A^{2d+1}) = (A - A)^2 + \cdots + (A - A)^2 = \mathbb{F}_p.$$ 

- In addition, the number of pairs $(x, y) \in A^{2d+1} \times A^{2d+1}$ satisfying $||x - y|| = \lambda$ is $\sim p^{-1}|A|^{4d+2}$ for any $\lambda \in \mathbb{F}_p$.

**Corollary 1.2.** For $A \subset \mathbb{F}_p$, suppose that $|A| \gtrsim p^{6/11}$, then we have

$$\Delta(A^7) = (A - A)^2 + (A - A)^2 + (A - A)^2 + (A - A)^2 + (A - A)^2 + (A - A)^2 = \mathbb{F}_p.$$ 

Our second result is for even dimensional cases.

**Theorem 1.3.** Let $\mathbb{F}_p$ be a prime field, and $A$ be a set in $\mathbb{F}_p$. For an integer $d \geq 3$, suppose the set $A^{2d} \subset \mathbb{F}_p^{2d}$ satisfies

$$|A^{2d}| \gtrsim p \frac{2^{d+1} - 3^{d-2} - d - 1}{2^{d+1} - 2},$$

then we have

- The distance set covers all elements in $\mathbb{F}_p$, namely,

$$\Delta(A^{2d}) = (A - A)^2 + \cdots + (A - A)^2 = \mathbb{F}_p.$$ 

- In addition, the number of pairs $(x, y) \in A^{2d} \times A^{2d}$ satisfying $||x - y|| = \lambda$ is $\sim p^{-1}|A|^{4d}$ for any $\lambda \in \mathbb{F}_p$. 

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Corollary 1.4. For $A \subset \mathbb{F}_p$, suppose that $|A| \gtrsim p^{4/7}$, then we have

$$\Delta(A^6) = (A-A)^2 + (A-A)^2 + (A-A)^2 + (A-A)^2 + (A-A)^2 + (A-A)^2 = \mathbb{F}_p.$$ 

Remark 1.1. In the setting of arbitrary finite fields $\mathbb{F}_q$, we can not break the exponent $(d + 1)/2$, and still cover all distances with the method in this paper and the distance energy in [27, Lemma 3.1]. More precisely, for $A \subset \mathbb{F}_q$, one can follow the proofs of Theorems 1.1 and 1.3 to get the conditions $|A^{2d+1}| \gg q^{2d+1/2 + \frac{1}{4d}}$ and $|A^{2d}| \gg q^{2d+1/2 + \frac{1}{4d}}$ for odd and even dimensions, respectively.

Remark 1.2. The Cauchy-Davenport theorem states that for $X, Y \subset \mathbb{F}_p$, we have $|X + Y| \geq \min\{p, |X| + |Y| - 1\}$. It is not hard to check that $\Delta(A^{2d}) = \Delta(A^d) + \Delta(A^d)$. The Chapman et al.'s result [4] tells us that $|\Delta(A^d)| \geq p/2$ whenever $|A| \gg p^{1/4d}$. Therefore, one can apply the Cauchy-Davenport theorem to show that $|\Delta(A^{2d})| \geq p - 1$ under the condition $|A| \geq p^{1/4d}$. However, our set $A^{2d}$ lies on the 2d-dimensional space $\mathbb{F}_p^{2d}$, thus the exponent $\frac{d}{2d-1}$ is worse than $\frac{2d+1}{4d}$. The same happens for odd dimensional spaces. Note that the bound $|\Delta(A^d)| \geq \frac{1}{2} \cdot \min\{|A|^{2 - \frac{1}{2d-1}}, p\}$ with $c = 2^{\frac{d-1}{2d}}$ in [22] is not suitable for this approach since the constant factor $1/c$ is too small.

Let $\mathbb{F}_q$ be an arbitrary finite field, and $E \subset \mathbb{F}_q^d$. The product set of $E$, denoted by $\Pi(E)$, is defined as follows:

$$\Pi(E) := \{x \cdot y : x, y \in E\}.$$ 

Using Fourier analysis, Hart and Iosevich [14] proved that if $|E| \gg q^{\frac{d+1}{2d}}$, then $\Pi(E) \supseteq \mathbb{F}_q \setminus \{0\}$. Moreover, under the same condition on the size of $E$, we have the number of pairs $(x, y) \in E \times E$ satisfying $x \cdot y = \lambda$ is $\sim q^{-1}|E|^2$ for any $\lambda \neq 0$. If $E$ has Cartesian product structure, i.e. $E = A^d$ for some $A \subset \mathbb{F}_q$, then the condition $|E| \gg q^{\frac{d+1}{2d}}$ is equivalent with $|A| \gg q^{\frac{1}{2} + \frac{1}{d}}$.

In the setting of prime fields $\mathbb{F}_p$, if $d = 8$, Glibichuk and Konyagin [12] proved that for $A, B \subset \mathbb{F}_p$, if $|A||B|/2 \geq p$, then we have $8A \cdot B = \mathbb{F}_p$. This result has been extended to arbitrary finite fields by Glibichuk and Rudnev [13].

In this paper, using the techniques in the proofs of Theorems [13], we are able to obtain the following theorem.

Theorem 1.5. For $A \subset \mathbb{F}_p$, suppose that $|A| \gtrsim p^{3/7}$, then we have
• \(6A \cdot A = A \cdot A + A \cdot A + A \cdot A + A \cdot A + A \cdot A + A \cdot A = \mathbb{F}_p\).

• For any \(\lambda \in \mathbb{F}_p\), the number of pairs \((x, y) \in A^6 \times A^6\) such that \(x \cdot y = \lambda\) is \(\sim p^{-1}|A|^{12}\).

Note that our exponent \(4/7\) improves the exponent \(7/12\) of Hart and Iosevich [14] in the case \(d = 6\).

The following is the conjecture due to Iosevich.

**Conjecture 1.6.** Let \(A\) be a set in \(\mathbb{F}_p\), suppose that \(|A| \gg p^{\frac{1}{2}+\epsilon}\) for any \(\epsilon > 0\), then we have

\[
A \cdot A + A \cdot A = \mathbb{F}_p,
\]

\[
(A - A)^2 + (A - A)^2 = \mathbb{F}_p.
\]

In the spirit of sum-product problems, the authors and De Zeeuw [22] proved that for \(A \subset \mathbb{F}_p\), if \(|A| \ll p^{\frac{1}{2}+\frac{1}{5d-1} + \frac{1}{2}}, d \geq 2\), then we have

\[
\max \{|\Delta(A^d)|, |\Pi(A^d)|\} \gg |A|^{2 - \frac{1}{5d-1}}.
\]

Using our energies (Lemmas 2.2 and 2.4 below), and the prime field analogue of Balog-Wooley decomposition energy due to Rudnev, Shkredov, and Stevens [23], we are able to give the energy variant of this result.

**Theorem 1.7.** Let \(d \geq 2\) be an integer, \(A\) be a set in \(\mathbb{F}_p\) with \(|A| \leq p^{\frac{1}{2}+\frac{1}{5d-1} + \frac{1}{2}}\). There exist two disjoint subsets \(B\) and \(C\) of \(A\) such that \(A = B \sqcup C\) and

\[
\max \{E_d((B - B)^2), E_d(C \cdot C)\} \lesssim |A|^{4d - 2 + \frac{1}{5d-1}},
\]

where \(E_d((B - B)^2)\) is the number of \(4d\)-tuples \(\{(a_i, b_i, c_i, e_i)\}_{i=1}^d\) with \(a_i, c_i, b_i, e_i \in B\) such that 
\[
(a_1 - b_1)^2 + \cdots + (a_d - b_d)^2 = (c_1 - e_1)^2 + \cdots + (c_d - e_d)^2,
\]

and \(E_d(C \cdot C)\) be the number of \(4d\)-tuples \(\{(a_i, b_i, c_i, e_i)\}_{i=1}^d\) with \(a_i, c_i, b_i, e_i \in C\) such that 
\[
a_1b_1 + \cdots + a_db_d = c_1e_1 + \cdots + c_de_d.
\]

## 2 Preliminaries

Let \(E\) and \(F\) be multi-sets in \(\mathbb{F}_p^2\). We denote by \(\overline{E}\) and \(\overline{F}\) the sets of distinct elements in \(E\) and \(F\), respectively. For any multi-set \(X\), we use the notation \(|X|\) to denote the size of \(X\). For \(\lambda \in \mathbb{F}_p\), let \(N(E, F, \lambda)\) be the number of pairs \(((e_1, e_2), (f_1, f_2)) \in E \times F\) such that
$e_1 f_1 + e_2 + f_2 = \lambda$. In the following lemma, we provide an upper bound and a lower bound of $N(E, F, \lambda)$ for any $\lambda \in \mathbb{F}_p$.

**Lemma 2.1.** Let $E, F$ be multi-sets in $\mathbb{F}_p^2$. For any $\lambda \in \mathbb{F}_p$, we have

$$\left| N(E, F, \lambda) - \frac{|E||F|}{p} \right| \leq p^2 \left( \sum_{(e_1, e_2) \in E} m_E((e_1, e_2))^2 \sum_{(f_1, f_2) \in F} m_F((f_1, f_2))^2 \right)^{1/2},$$

where $m_X((a, b))$ is the multiplicity of $(a, b)$ in $X$ with $X \in \{E, F\}$.

**Proof.** Let $\chi$ be a non-trivial additive character on $\mathbb{F}_p$. We have

$$N(E, F, \lambda) = \sum_{(e_1, e_2) \in E, (f_1, f_2) \in F} \frac{1}{p} m_E((e_1, e_2)) m_F((f_1, f_2)) \sum_{s \in \mathbb{F}_p \backslash \{0\}} \chi(s \cdot (e_1 f_1 + e_2 + f_2 - \lambda)).$$

This gives us

$$N(E, F, \lambda) = \frac{|E||F|}{p} + L,$$

where

$$L = \sum_{(e_1, e_2) \in E, (f_1, f_2) \in F} m_E((e_1, e_2)) m_F((f_1, f_2)) \frac{1}{p} \sum_{s \neq 0} \chi(s \cdot (e_1 f_1 + e_2 + f_2 - \lambda)).$$

If we view $L$ as a sum in $(e_1, e_2) \in E$, then we can apply the Cauchy-Schwarz inequality to derive the following:

$$L^2 \leq \sum_{(e_1, e_2) \in E} m_E((e_1, e_2))^2 \sum_{(e_1, e_2) \in \mathbb{F}_p^2} \frac{1}{p^2} \sum_{s, s' \neq 0} m_F((f_1, f_2)) m_F((f_1', f_2')) \cdot \chi(s \cdot (e_1 f_1 + e_2 + f_2 - \lambda)) \chi(s' \cdot (-e_1 f_1' - e_2 - f_2' + \lambda))$$

$$= \sum_{(e_1, e_2) \in E} m_E((e_1, e_2))^2 \frac{1}{p^2} \sum_{(f_1, f_2) \in \mathbb{F}_p^2} m_F((f_1, f_2)) m_F((f_1', f_2')) \chi(e_1(s f_1 - s' f_1')) \chi(e_2(s - s'))$$

$$\cdot \chi(s(f_2 - \lambda) - s'(f_2' - \lambda))$$
\[ \sum_{(e_1, e_2) \in E} m_E((e_1, e_2))^2 \sum_{s \neq 0}^{} m_F((f_1, f_2))m_F((f'_1, f'_2))\chi(s \cdot (f_2 - f'_2)) = I + II, \]

where \( I \) is the sum over all pairs \( ((f_1, f_2), (f'_1, f'_2)) \) with \( f_2 = f'_2 \), and \( II \) is the sum over all pairs \( ((f_1, f_2), (f'_1, f'_2)) \) with \( f_2 \neq f'_2 \).

It is not hard to check that if \( f_2 \neq f'_2 \), then

\[ \sum_{s \neq 0}^{} \chi(s \cdot (f_2 - f'_2)) = -1, \]

so \( II < 0 \).

On the other hand, if \( f_2 = f'_2 \), then

\[ \sum_{s \neq 0}^{} \chi(s \cdot (f_2 - f'_2)) = p - 1. \]

In other words,

\[ I \ll p \sum_{(e_1, e_2) \in E}^{} m_E((e_1, e_2))^2 \sum_{(f_1, f_2) \in F}^{} m_F((f_1, f_2))^2, \]

which implies that

\[ |L| \ll p^{1/2} \left( \sum_{(e_1, e_2) \in E}^{} m_E((e_1, e_2))^2 \sum_{(f_1, f_2) \in F}^{} m_F((f_1, f_2))^2 \right)^{1/2}. \]

This completes the proof of the lemma. \( \square \)

For \( A \subset \mathbb{F}_p \), let \( E_d ((A - A)^2) \) be the number of \( 4d \)-tuples \( \{(a_i, b_i, c_i, e_i)\}_{i=1}^d \) with \( a_i, c_i, b_i, e_i \in A \) such that

\[ (a_1 - b_1)^2 + \cdots + (a_d - b_d)^2 = (c_1 - e_1)^2 + \cdots + (c_d - e_d)^2. \]

Similarly, let \( E_d (A \cdot A) \) be the number of \( 4d \)-tuples \( \{(a_i, b_i, c_i, e_i)\}_{i=1}^d \) with \( a_i, c_i, b_i, e_i \in A \).
such that
\[ a_1 b_1 + \cdots + a_d b_d = c_1 e_1 + \cdots + c_d e_d. \]

In our next lemmas, we give recursive formulas for \( E_d((A - A)^2) \) and \( E_d(A \cdot A) \).

**Lemma 2.2.** For \( A \subset \mathbb{F}_p \), we have
\[
E_d((A - A)^2) \lesssim \frac{|A|^{4d}}{p} + |A|^{2d+1} \sqrt{E_{d-1}((A - A)^2)}.
\]

The proof of this lemma will be given in the next section. The following result is a direct consequence, which tells us an upper bound of \( E_d((A - A)^2) \).

**Corollary 2.3.** Let \( A \) be a set in \( \mathbb{F}_p \). For \( d \geq 2 \), suppose that \( |A| \gg p^{1/2} \), then we have
\[
E_d((A - A)^2) \lesssim \frac{|A|^{4d}}{p} + |A|^{4d-2} + \frac{1}{p^{d-1}}.
\]

**Proof.** We proceed by induction on \( d \). The base case \( d = 2 \) follows directly from Lemma 2.2 by using the trivial upper bound \( |A|^3 \) of \( E_1((A - A)^2) \).

Suppose the statement holds for any \( d-1 \geq 2 \), we now prove that it also holds for \( d \). Indeed, by induction hypothesis, we have
\[
E_{d-1}((A - A)^2) \lesssim \frac{|A|^{4(d-1)}}{p} + |A|^{4d-6} + \frac{1}{p^{d-2}}. \tag{1}
\]

On the other hand, it follows from Lemma 2.2 that
\[
E_d((A - A)^2) \lesssim \frac{|A|^{4d}}{p} + |A|^{2d+1} \sqrt{E_{d-1}((A - A)^2)}. \tag{2}
\]

Putting (1) and (2) together, we obtain
\[
E_d((A - A)^2) \lesssim \frac{|A|^{4d}}{p} + |A|^{4d-2} + \frac{1}{p^{d-1}},
\]
whenever \( |A| \gg p^{1/2} \). This completes the proof of the corollary.

Similarly, for the case of product sets, we have
Lemma 2.4. For $A \subset \mathbb{F}_p$, we have

$$E_d(A \cdot A) \lesssim \frac{|A|^{4d}}{p} + |A|^{2d+1} \sqrt{E_{d-1}(A \cdot A)}.$$  

Proof. The proof of this lemma is almost identical with that of Lemma 2.2, so we omit it.

Corollary 2.5. Let $A$ be a set in $\mathbb{F}_p$. For $d \geq 2$, suppose that $|A| \gg p^{1/2}$, then we have

$$E_d(A \cdot A) \lesssim \frac{|A|^{4d}}{p} + |A|^{4d-2} \frac{1}{2d-1}.$$  

Proof. The proof of Corollary 2.5 is identical with that of Corollary 2.3 with Lemma 2.4 in the place of Lemma 2.2, thus we omit it.

2.1 Proof of Lemma 2.2

In the proof of Lemma 2.2, we will use a point-plane incidence bound due to Rudnev [24] and an argument in [25, Theorem 32].

The following is a strengthened version of the Rudnev’s point-plane incidence bound proved by de Zeeuw in [28]. Let us first recall that if $\mathcal{R}$ is a set of points in $\mathbb{F}_p^3$ and $\mathcal{S}$ is a set of planes in $\mathbb{F}_p^3$, then the number of incidences between $\mathcal{R}$ and $\mathcal{S}$, denoted by $I(\mathcal{R}, \mathcal{S})$, is the cardinality of the set $\{(r, s) \in \mathcal{R} \times \mathcal{S} : r \in s\}$.

Theorem 2.6 (Rudnev, [24]). Let $\mathcal{R}$ be a set of points in $\mathbb{F}_p^3$ and $\mathcal{S}$ be a set of planes in $\mathbb{F}_p^3$, with $|\mathcal{R}| \leq |\mathcal{S}|$. Suppose that there is no line that contains $k$ points of $\mathcal{R}$ and is contained in $k$ planes of $\mathcal{S}$. Then

$$I(\mathcal{R}, \mathcal{S}) \ll \frac{|\mathcal{R}||\mathcal{S}|}{p} + |\mathcal{R}|^{1/2}|\mathcal{S}| + k|\mathcal{S}|.$$  

Proof of Lemma 2.2. We first have

$$E_d \left( (A - A)^2 \right) = \sum_{t_1, t_2} r_{(d-1)(A-A)^2}(t_1) r_{(d-1)(A-A)^2}(t_2) f(t_1, t_2),$$

where $r_{(d-1)(A-A)^2}(t)$ is the number of $2(d-1)$ tuples $(a_1, \ldots, a_{d-1}, b_1, \ldots, b_{d-1}) \in A^{2d-2}$ such that $(a_1 - b_1)^2 + \cdots + (a_{d-1} - b_{d-1})^2 = t$, and $f(t_1, t_2)$ is the sum $\sum_s r_{(A-A)^2+t_1}(s) r_{(A-A)^2+t_2}(s)$. 

We now split the sum $E_d((A - A)^2)$ into intervals as follows.

\[
E_d((A - A)^2) \ll \sum_{i=1}^{L_1} \sum_{j=1}^{L_2} f(t_1, t_2) r^{(i)}_{(d-1)(A-A)^2}(t_1) r^{(j)}_{(d-1)(A-A)^2}(t_2),
\]

where $L_1 \leq \log(|A|^{2d-2})$, $L_2 \leq \log(|A|^{2d-2})$, $r^{(i)}_{(d-1)(A-A)^2}(t_1)$ is the restriction of the function $r_{(d-1)(A-A)^2}(x)$ on the set $P_i := \{ t : \delta_i \leq r_{(d-1)(A-A)^2}(t) < 2\delta_i \}$ for some $\delta_i > 0$.

Using the pigeon-hole principle two times, there exist sets $P_{i_0}$ and $P_{j_0}$ for some $i_0$ and $j_0$ such that

\[
E_d((A - A)^2) \lesssim \sum_{t_1, t_2} f(t_1, t_2) r^{(i_0)}_{(d-1)(A-A)^2}(t_1) r^{(j_0)}_{(d-1)(A-A)^2}(t_2) \lesssim \delta_{i_0} \delta_{j_0} \sum_{t_1, t_2} f(t_1, t_2) P_{i_0}(t_1) P_{j_0}(t_2).
\]

One can check that the sum $\sum_{t_1, t_2} f(t_1, t_2) P_{i_0}(t_1) P_{j_0}(t_2)$ is equal to the number of incidences between the point set $\mathcal{R}$ of points $(-2a, e, t_1 + a^2 - e^2) \in \mathbb{F}_p^3$ with $a \in A, e \in A, t_1 \in P_{i_0}$, and the plane set $\mathcal{S}$ of planes in $\mathbb{F}_p^3$ defined by

\[
bX + 2cY + Z = t_2 - b^2 + c^2,
\]

where $b \in A, c \in A$ and $t_2 \in P_{j_0}$. Without loss of generality, we can assume that $|P_{i_0}| \leq |P_{j_0}|$.

To apply Theorem 2.6, we need to bound the maximal number of collinear points in $\mathcal{R}$. The projection of $\mathcal{R}$ into the plane of the first two coordinates is the set $-2A \times A$, thus if a line is not vertical, then it contains at most $|A|$ points from $\mathcal{R}$. If a line is vertical, then it contains at most $|P_{i_0}|$ points from $\mathcal{R}$, but that line is not contained in any plane in $\mathcal{S}$. In other words, we can apply Theorem 2.6 with $k = |A|$, and obtain the following

\[
\sum_{t_1, t_2} f(t_1, t_2) P_{i_0}(t_1) P_{j_0}(t_2) \ll \frac{|A|^4|P_{i_0}||P_{j_0}|}{p} + |A|^3|P_{i_0}|^{1/2}|P_{j_0}| + |A|^3|P_{j_0}| \\
\ll \frac{|A|^4|P_{i_0}||P_{j_0}|}{p} + |A|^3|P_{i_0}|^{1/2}|P_{j_0}|.
\]

We now fall into the following cases:

**Case 1:** If the first term dominates, we have

\[
\sum_{t_1, t_2} f(t_1, t_2) P_{i_0}(t_1) P_{j_0}(t_2) \ll \frac{|A|^4|P_{i_0}||P_{j_0}|}{p}.
\]
Case 2: If the second term dominates, we have

\[ \sum_{t_1,t_2} f(t_1,t_2) P_{i_0}(t_1) P_{j_0}(t_2) \ll |A|^3 |P_{i_0}|^{1/2} |P_{j_0}|. \]

Therefore,

\[ E_d ((A - A)^2) \lesssim \delta_{i_0} \delta_{j_0} \left( \frac{|A|^4 |P_{i_0}| |P_{j_0}|}{p} + |A|^3 |P_{i_0}|^{1/2} |P_{j_0}| \right) \]
\[ \lesssim \frac{|A|^{4d}}{p} + |A|^{2d+1} \sqrt{E_{d-1} ((A - A)^2)}. \]

where we have used the facts that

- \( \delta_{i_0} |P_{i_0}|^{1/2} \lesssim \sqrt{E_{d-1} ((A - A)^2)} \),
- \( \delta_{j_0} |P_{j_0}| \lesssim |A|^{2d-2} \),
- \( \delta_{i_0} |P_{i_0}| \lesssim |A|^{2d-2} \).

This completes the proof of the lemma. \( \square \)

3 Proof of Theorem 1.1

Proof of Theorem 1.1: Let \( \lambda \) be an arbitrary element in \( \mathbb{F}_p \). Let \( E \) be the multi-set of points \((2x, x^2 + (y_1 - z_1)^2 + \cdots + (y_d - z_d)^2) \in \mathbb{F}_p^2 \) with \( x, y_i, z_i \in A \), and \( F \) be the multi-set of points \((-t, t^2 + (u_1 - v_1)^2 + \cdots + (u_d - v_d)^2) \in \mathbb{F}_p^2 \) with \( t, u_i, v_i \in A \). We have \( |E| = |F| = |A|^{2d+1} \).

It follows from Lemma 2.1 that

\[ \left| N(E, F, \lambda) - \frac{|E||F|}{p} \right| \leq p^\frac{1}{2} \left( \sum_{(e_1,e_2) \in \mathcal{E}} m_E((e_1,e_2))^2 \sum_{(f_1,f_2) \in \mathcal{F}} m_F((f_1,f_2))^2 \right)^{1/2}. \quad (3) \]

We observe that if \( N(E, F, \lambda) \) is equal to the number of pairs \((x,y) \in A^{2d+1} \times A^{2d+1} \) such that \( ||x - y|| = \lambda \).

From the setting of \( E \) and \( F \), it is not hard to see that
\[
\sum_{(e_1, e_2) \in E} m_E((e_1, e_2))^2 = |A| E_d((A - A)^2), \quad \sum_{(f_1, f_2) \in F} m_F((f_1, f_2))^2 = |A| E_d((A - A)^2). \tag{4}
\]

Putting (3) and (4) together, we have
\[
\left| N(E, F, \lambda) - \frac{|A|^{4d+2}}{p} \right| \leq p^{\frac{1}{2}} |A| E_d((A - A)^2). \tag{5}
\]

On the other hand, Corollary 2.3 gives us
\[
E_d((A - A)^2) \lesssim \frac{|A|^{4d}}{p} + |A|^{4d-2+\frac{1}{d}-1}. \tag{6}
\]

Substituting (6) into (5), we obtain
\[
N(E, F, \lambda) \sim |A|^{4d+2} p^{-1} \text{ whenever } |A|^{2d+1} \geq p^{2d+2-d-1}. \tag{7}
\]

Since \(\lambda\) is arbitrary in \(\mathbb{F}_p\), the theorem follows. \(\square\)

## 4 Proofs of Theorems 1.3 and 1.5

The proof of Theorem 1.3 is similar to that of Theorem 1.1, but we need a higher dimensional version of Lemma 2.1.

Let \(E\) and \(F\) be multi-sets in \(\mathbb{F}_p^3\). For \(\lambda \in \mathbb{F}_p\), let \(N(E, F, \lambda)\) be the number of pairs \(((e_1, e_2, e_3), (f_1, f_2, f_3)) \in E \times F\) such that \(e_1 f_1 + e_2 f_2 + e_3 + f_3 = \lambda\). One can follow step by step the proof of Lemma 2.1 to obtain the following.

**Lemma 4.1.** Let \(E, F\) be multi-sets in \(\mathbb{F}_p^3\). For any \(\lambda \in \mathbb{F}_p\), we have
\[
\left| N(E, F, \lambda) - \frac{|E||F|}{p} \right| \leq p \left( \sum_{(e_1, e_2, e_3) \in E} m_E((e_1, e_2, e_3))^2 \sum_{(f_1, f_2, f_3) \in F} m_F((f_1, f_2, f_3))^2 \right)^{1/2}.
\]

We are now ready to prove Theorem 1.3.
Proof of Theorem 1.3: Let $\lambda$ be an arbitrary element in $\mathbb{F}_p$. Let $E$ be the multi-set of points $(2x_1, 2x_2, x_1^2 + x_2^2 + (y_1 - z_1)^2 + \cdots + (y_{d-1} - z_{d-1})^2) \in \mathbb{F}_p^3$ with $x_i, y_i, z_i \in A$, and $F$ be the multi-set of points $(-t_1 - t_2, t_1^2 + t_2^2 + (u_1 - v_1)^2 + \cdots + (u_{d-1} - v_{d-1})^2) \in \mathbb{F}_p^3$ with $t_i, u_i, v_i \in A$. We have $|E| = |A|^{2d}$ and $|F| = |A|^{2d}$.

It follows from Lemma 4.1 that

$$\left| N(E, F, \lambda) - \frac{|E||F|}{p} \right| \leq p \left( \sum_{(e_1, e_2, e_3) \in E} m_E((e_1, e_2, e_3))^2 \sum_{(f_1, f_2, f_3) \in F} m_F((f_1, f_2, f_3))^2 \right)^{1/2}. \quad (7)$$

We observe that if $N(E, F, \lambda)$ is equal to the number of pairs $(x, y) \in A^{2d} \times A^{2d}$ such that $||x - y|| = \lambda$.

From the setting of $E$ and $F$, it is not hard to see that

$$\sum_{(e_1, e_2, e_3) \in E} m_E((e_1, e_2, e_3))^2 = |A|^{2d}E_{d-1}((A-A)^2), \quad \sum_{(f_1, f_2, f_3) \in F} m_F((f_1, f_2, f_3))^2 = |A|^{2d}E_{d-1}((A-A)^2). \quad (8)$$

Putting (7) and (8) together, we have

$$\left| N(E, F, \lambda) - \frac{|A|^{4d}}{p} \right| \leq p|A|^{2d}E_{d-1}((A-A)^2). \quad (9)$$

On the other hand, Corollary 2.3 gives us

$$E_{d-1}((A-A)^2) \lesssim \frac{|A|^{4d-4}}{p} + |A|^{4d-6+\frac{1}{2d-2}}. \quad (10)$$

Substituting (10) into (9), we obtain $N(E, F, \lambda) \sim |A|^{4d}p^{-1}$ whenever

$$|A|^{2d} \gtrsim p \frac{2d+1}{2d+1} \frac{2^{d-\frac{d-1}{2}}}{2^{d+1/2}}.$$

Since $\lambda$ is arbitrary in $\mathbb{F}_p$, the theorem follows. \qed

Proof of Theorem 1.5: The proof of Theorem 1.3 is similar to that of Theorem 1.3 with Corollary 2.5 in the place of Corollary 2.3 \qed
5 Proof of Theorem 1.7

Let us first recall the prime field analogue of Balog-Wooley decomposition energy due to Rudnev, Shkredov, Stevens [23].

Theorem 5.1 ([23]). Let \( A \) be a set in \( \mathbb{F}_p \) with \( |A| \leq p^{5/8} \). There exist two disjoint subsets \( B \) and \( C \) of \( A \) such that \( A = B \sqcup C \) and

\[
\max \{ E^+(B), E^x(C) \} \lesssim |A|^{14/5},
\]

where \( E^+(B) = |\{(a, b, c, d) \in B^4 : a + b = c + d\}| \), and \( E^x(C) = |\{(a, b, c, d) \in C^4 : ab = cd\}| \).

We refer the interested reader to [1] for the result over \( \mathbb{R} \) due to Balog and Wooley.

The following is another corollary of Lemma 2.2.

Corollary 5.2. Let \( A \) be a set in \( \mathbb{F}_p \), and \( B \) be a subset of \( A \). For an integer \( d \geq 2 \), suppose that \( |A| \ll p^{\frac{1}{2} + \frac{1}{2d-1}} \) and \( E^+(B) \lesssim |A|^{14/5} \), then we have

\[
E_d((B - B)^2) \lesssim |A|^{4d - 2 + \frac{1}{5 \cdot 2^{d-1}}}.
\]

Proof. We proceed by induction on \( d \). The base case \( d = 2 \) follows directly from Lemma 2.2 and the facts that \( E_1((B - B)^2) \ll E^+(B) \) and \( |B| \leq |A| \).

Suppose the corollary holds for \( d - 1 \geq 2 \), we now show that it also holds for the case \( d \).

Indeed, it follows from Lemma 2.2 that

\[
E_d((B - B)^2) \lesssim \frac{|B|^{4d}}{p} + |B|^{2d+1} \sqrt{E_{d-1}((B - B)^2)}.
\]

On the other hand, by induction hypothesis, we have

\[
E_{d-1}((B - B)^2) \lesssim |A|^{4d - 6 + \frac{1}{5 \cdot 2^{d-1}}},
\]

Thus, using the fact that \( |B| \leq |A| \), we obtain

\[
E_d((B - B)^2) \lesssim \frac{|A|^{4d}}{p} + |A|^{4d - 2 + \frac{1}{5 \cdot 2^{d-1}}} \lesssim |A|^{4d - 2 + \frac{1}{5 \cdot 2^{d-1}}},
\]
whenever $|A| \ll p^{\frac{3}{2} \frac{1}{5} - \frac{1}{2}}$.

Using the same argument, we also have another corollary of Lemma 2.4.

**Corollary 5.3.** Let $A$ be a set in $\mathbb{F}_p$, and $C$ be a subset of $A$. For an integer $d \geq 2$, suppose that $|A| \ll p^{\frac{3}{2} \frac{1}{5} - \frac{1}{2}}$ and $E^\times(C) \lesssim |A|^{14/5}$, then we have

$$E_d(C \cdot C) \lesssim |A|^{4d - 2 + \frac{1}{5} \frac{1}{2} - \frac{3}{2}}.$$

We are now ready to prove Theorem 1.7.

**Proof of Theorem 1.7.** It follows from Theorem 5.1 that there exist two disjoint subsets $B$ and $C$ of $A$ such that $A = B \sqcup C$ and $\max\{E^+(B), E^\times(C)\} \lesssim |A|^{14/5}$. One now can apply Corollaries 5.2 and 5.3 to derive

$$\max\{E_d((B - B)^2), E_d(C \cdot C)\} \lesssim |A|^{4d - 2 + \frac{1}{5} \frac{1}{2} - \frac{3}{2}}.$$

This completes the proof of the theorem.

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