Poincaré duality
of the basic intersection cohomology
of a Killing foliation

Martintxo Saralegi-Aranguren‡
Université d’Artois

Robert Wolak§
Uniwersytet Jagiellonski

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Abstract

We prove that the basic intersection cohomology $H^p_p(M/F)$, where $F$ is the singular foliation determined by an isometric action of a Lie group $G$ on a compact manifold $M$, verifies the Poincaré duality property.

Basic cohomology theories are cohomology theories taking into account the particular structure of a foliated manifold and defined using differential forms. The notion was introduced by B. Reinhardt in 1959, cf. [23], in his fundamental study of Riemannian foliations. He proved that the basic cohomology of a compact manifold with a Riemannian foliation is finite dimensional and that the Poincaré duality works in this cohomology.

However, the proofs had some gaps, and the theorems were considered as conjectures. In the paper [8], based his PhD thesis, Carrièrè published an example of a 1-dimensional Riemannian foliation (a Riemannian flow) on a compact 3-manifold for which the top basic cohomology, of degree 2, is trivial. Therefore the basic cohomology of this Riemannian foliation cannot satisfy the Poincaré duality property. The paper also presented a classification of Riemannian flows on compact 3-manifolds.

Using this classification Y. Carrièrè noticed that the basic cohomology of foliations given by isometric flows has the Poincaré duality property, and that the only class of flows whose basic cohomology does not satisfy the Poincaré duality property consists of foliations which cannot be defined by isometric flows.

That remark led him to formulate a conjecture that the non-triviality of the top basic cohomology space, which is necessary for the Poincaré duality to hold, is equivalent to the tautness of the Riemannian foliation, i.e. the existence of a bundle-like metric for which all the leaves of the foliation are minimal sub manifolds. For dimension one foliations it is equivalent to the fact that the foliation is defined by an isometric action, a Killing vector field.

For over a decade, the conjecture was the subject of intensive study by a group of foliators and was finally solved by Masa, cf. [14], and refined Álvarez, cf. [1]. The best account of the development of the theory up to 1995 can be found in Tondeur’s book [33]. For more recent developments see [27].

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§ Laboratoire de Mathématiques de Lens, EA 2462 et Fédération CNRS Nord- Pas-de-Calais FR 2956. Université d’Artois. SP18, rue Jean Souvraz. 62307 Lens Cedex, France. saralegi@euler.univ-artois.fr.
§ Instytut Matematyki. Uniwersytet Jagiellonski. Stanisława Lojasiewicza 6, 30 348 KRAKOW, POLAND. robert.wolak@im.uj.edu.pl.
The case of singular Riemannian foliations (SRF for short) is much more subtle. The suspension of a linear flow on a sphere provides an example of a singular isometric flow for which the top-dimensional basic cohomology group is isomorphic to $\mathbb{R}$, but this cohomology fails to satisfy the Poincaré duality property (see for example [27]). Therefore in the case of SRFs there is no direct connection between an SRF being defined by an isometric action of a Lie group, the non-triviality of the top dimensional basic cohomology group and the Poincaré duality property of the basic cohomology. It seems that the main reason is the fact that the basic cohomology does not take into account the structure of the set of singular orbits. Moreover, V. Miquel Molina and the second author proved that an SRF cannot be taut in the classical sense, cf. [15]. All these facts stress that in the research into the geometrical properties of SRFs and their relation to their topological ones we have to use more subtle instruments which take into account the structure of the set of singularities. The well-developed theory of singularities provide such tools: perverse forms and the intersection cohomology.

The intersection homology was introduced and studied by M. Goresky and R. MacPherson in 80’s in the setting of pseudo manifolds. They established a generalized version of the Poincaré duality property. First versions of the theory used PL machinery [10] and sheaf theory [11]. The first version of the intersection cohomology by means of (perverse) differential forms was proposed by J.-L. Brylinski [7] (see also [28]). This point of view requires some extra data to introduce the notion of perverse form (Thom-Mather system, blow-up, Riemannian metric, . . . ).

In [29] we adapted these notions to the foliated case and we defined basic perverse forms and basic intersection cohomology (BIC for short). We used the fact that on a manifold foliated by an SRF there is a natural stratification defined using the dimension of leaves, the dimension of leaf closures, and holonomy. A remarkable fact is that these perverse forms do not involve any extra data in order to be defined.

In his thesis, [24], J.I. Royo Prieto demonstrated the Poincaré duality property of the BIC for singular Riemannian flows and the singular version of the Molino-Sergiescu theorem, cf. [20]. Inspired by these results, we have started to investigate possible generalizations and to study these cohomologies and their relation to tautness; to this study we dedicated a series of papers, (cf. [25], [26], [27]).

In the paper [29] we studied singular Riemannian foliations with compact leaves on compact manifolds. The leaf space of such a foliation is a pseudomanifold. The basic intersection cohomology is isomorphic to the corresponding intersection cohomology of the leaf space, and the Poincaré duality property in the basic intersection cohomology is equivalent to the Poincaré duality property in the intersection cohomology of the leaf space. The relation between the Poincaré duality property, the non-triviality of the top dimensional BIC and the tautness in the case of SRFs is more complex.

Summing up our main results, let us say that in [30] we demonstrated that the BIC of SRFs defined by an isometric action of an abelian Lie group is finite dimensional and satisfies the Poincaré duality. In the most recent paper [31] we showed that the BIC for SRFs defined by an isometric action of a Lie group is finite dimensional.

In the paper we prove the Poincaré duality property for the BIC of SRF defined by an isometric action of a Lie group $G$ without any restriction imposed on the Lie group. We hope to extend our results to the case of OLF of [19] and general SRFs. However, there is an important technical difference. We have managed to demonstrate our result due to a nice description and properties of the stratification defined by the action of the closure of $G$: tubular neighborhoods of the strata can be assumed to be twisted products. So we can work with trivial tubular neighborhoods, which makes the calculations much easier. In the general case, tubular neighborhoods of the strata of the induced stratification are much more complicated.

Singular Riemannian foliations were introduced and studied by P. Molino. His book [18] contains the best introduction to the subject. The associated stratification was presented and studied in [17]. As a useful reading we also suggest [21, Section 9.3] of the book which contain a lot of information on isometric actions of Lie groups.
The classical reference for a compact group action is [5].

Lie group actions appear in the study of physical models as groups of symmetries of such models. Reduction theory is of great interest and the object of study of numerous papers and books, e.g. [21].

In the present article we use the notation introduced and developed in our previous papers on BIC [29] and isometric actions [30] and [31].

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In the sequel $M$ is a connected, second countable, Haussdorff, without boundary and smooth (of class $C^\infty$) manifold.

1. Killing Foliations.

A smooth action $\Phi: G \times M \to M$ of the Lie group $G$ is an isometric action when there exists a Riemannian metric $\mu$ on $M$ preserved by $G$. Moreover, the isometric action $\Phi$ is tame when the closure of $G$ in $\text{Iso}(M,\mu)$ is compact. This is always the case when the manifold $M$ is compact (cf. [13, Section II, Theorem 1.2]).

The connected components of the orbits of a tame action determine a partition $\mathcal{F}$ on $M$. In fact, this partition is a singular Riemannian foliation that we shall call Killing foliation (cf. [18]).

Notice that $\mathcal{F}$ is also a conical foliation in the sense of [29]. So, its basic intersection cohomology can be used for the study of $\mathcal{F}$. In this work, we prove the Poincaré duality property of this cohomology.

If the action $\Phi$ is tame, then it is a restriction a smooth action $\Phi: K \times M \to M$ where $K$ is a compact Lie group containing $G$. The group $K$ is not unique. We always can choose $K$ in such a way that $G$ is dense in $K$. We shall say that $K$ is a tamer group.

Notice that the tamer group is not unique. Consider the two actions $\Phi: \mathbb{T}^2 \times M \to M$ and $\Phi': \mathbb{T}^3 \times M \to M$ on $M = \mathbb{S}^1$ defined by $\Phi((u,v),z) = u \cdot v \cdot z$ and $\Phi'((u,v,w),z) = u \cdot v \cdot w \cdot z$. Take the subgroups $G = \{(e^{2\pi i \alpha}, e^{2\pi i \beta})/t \in \mathbb{R}\}$ and $G' = \{(e^{2\pi i \alpha}, e^{2\pi i \beta}, e^{2\pi i \gamma})/t \in \mathbb{R}\}$ where $\alpha, \beta, \gamma$ are three given reals which are $\mathbb{Q}$-dependents. The two restrictions $\Phi: G \times M \to M$ and $\Phi': G' \times M \to M$ define the one-leaf Killing foliation of $M$ but they have different tamer groups: $\mathbb{T}^2$ and $\mathbb{T}^3$.

For the rest of the work, let $\mathcal{F}$ be a Killing foliation. We fix an effective tame action $\Phi: G \times M \to M$, with $G$ connected, defining $\mathcal{F}$. We also fix a tamer group $K$. In this case, $G$ is normal in $K$ and the quotient group $K/G$ is commutative (see [22, 1.2 and Theorem 1.3]).

There are three key facts for this work: compactness of $K$ (which ensures the existence of invariant metrics, tubular neighborhoods, Molino’s blow-up, isotropy type stratification), density of $G$ in $K$ (facilitates the computation of the intersection cohomology of a twisted product) and the normality of the subgroup $G$ in the group $K$ (which is the key factor in the compatibility between stratifications defined by $G$ and $K$).

We denote by $b = \dim G$ and $m = \dim M$. The induced foliation on the regular stratum $R_\mathcal{F}$ is regular, and its dimension will be denoted by $w = \dim \mathcal{F}$.

2. Stratification.
Classifying the points of $M$ following the dimension of the leaves of $\mathcal{F}$ one gets the stratification $S_F$. It is determined by the equivalence relation $x \sim y \iff \dim G_x = \dim G_y$. The elements of $S_F$ are called strata. The open stratum $R_F$ is the regular stratum, and the other strata are the singular strata.

We fix a base point $p \in R_F$ and we put $(G_p)_0 = L$, where $E_0$ stands for the identity component of the Lie group $E$ (the one containing the identity). When $G$ is abelian then $(G_p)_0 = L$ for each $p \in R_F$. This implies that $L$ acts non-effectively on $R_F$ and therefore on $M$. So, $L = \{e\}$.

This group $L$ is generic in the following sense.

**Proposition 2.1** For each $x \in R_F$ there exists $k \in K$ with $(G_x)_0 = kLk^{-1}$. Moreover, the choice $x \mapsto k$ can be done locally in a smooth way.

**Proof.** We consider a point $x \in R_F$ and we will find an open neighborhood $V \subset R_F$ of $x$ and a smooth map $f : V \to K$ with $(G_x)_0 = f(y)(G_y)_0f(y)^{-1}$, for each $y \in V$.

Since the Lie group $K$ is compact there exists a tube $K \times T$ around the point $x$ ([5, Chapter II, Theorem 5.4]) where $H = K_x$ and $T \subset R_F$ is a transversal to the orbit $K(x)$ containing $x$. In particular, $\dim G_{<z,x>} = \dim G_x$ for each $z \in T$. Here $< k, z >$ is a generic point of the twisted product $K \times T$, and $e$ is the identity of $G$.

The group $G$ being normal in $K$, we get $G_{<k,z>} = G \cap K H_k k^{-1} = k(G \cap H) k^{-1} = k G_{<z,x>} k^{-1}$ and therefore $K \times T \subset R_F$ as $\dim G_{<k,z>} = \dim G_x$.

We consider a neighborhood $W \subset K$ of $e$. This neighborhood can be chosen small enough to ensure the existence of a smooth section $\sigma : \gamma(W) \to W$ of the canonical projection $\gamma : K \to K/H$. Let $\Gamma : K \times T \to K/H$ be the projection. So $V = \Gamma^{-1}(\gamma(W))$ is a neighborhood of $x$ in $K \times T \subset R_F$. Denote by $f : V \to K$ the smooth map defined by $f(y) = \sigma(\Gamma(y))$. Let us consider a point $y = < k, z > \in V$. Since $kH = f(y) H$, then $y = < f(y), z' >$, and therefore $G_y = f(y)(G \cap H)_{<} f(y)$.

On the other hand, $G_x = G \cap K_x = G \cap H$ and $\dim G_x = \dim G_y$ give $\dim(G \cap H) = \dim(G \cap H)_{<}$ and therefore:

$$(G_y)_0 = f(y)((G \cap H)_{<})_0 f(y)^{-1} = f(y)(G \cap H)_0 f(y)^{-1} = f(y)(G_x)_0 f(y)^{-1}.$$  

This ends the proof.

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### 3. Presentation of the Poincaré duality property.

The basic intersection cohomology $H^*_\pi(M/\mathcal{F})$, relatively to the perversity $\pi$, was introduced in [29, Section 3] for the study of conical foliations\(^1\). It coincides with the usual intersection cohomology when the leaves are compact [29, Theorem 1].

We define the support of a perverse form $\omega \in \Sigma_\pi(M)$ as $\text{supp } \omega = \overline{\{x \in M \setminus \Sigma_\pi / \omega(x) \neq 0\}}$, where the closure is taken in $M$. We denote by $\Omega^*_\pi(M/\mathcal{F}) = \{\omega \in \Omega_\pi(M/\mathcal{F}) / \text{ supp } \omega \text{ is compact}\}$ the complex of intersection basic differential forms with compact support relatively to the perversity $\pi$. The cohomology $H^*_\pi(M/\mathcal{F})$ of this complex is the basic intersection cohomology with compact support of $(M, \mathcal{F})$, relatively to the perversity $\pi$\(^2\).

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\(^1\)We refer the reader to [31] for notation and main properties of this notion.

\(^2\)We refer the reader to [30] for notation and main properties of this notion.
The goal of this work is to construct a non-degenerate pairing \( P_M : \mathbb{H}^*_p(M/F) \times \mathbb{H}^*_{\mathbb{Q},c}(M/F) \to \mathbb{R} \), inducing the Poincaré duality
\[
P_M : \mathbb{H}^*_p(M/F) \to \text{Hom}\left(\mathbb{H}^*_{\mathbb{Q},c}(M/F), \mathbb{R}\right).
\]
Here, \( \bar{p} \) and \( \bar{q} \) are complementary perversities, that is, \( \bar{p} + \bar{q} = \bar{I} \), with
\[
\bar{I}(S) = \text{codim}_M F - \text{codim}_S F_S - 2,
\]
where \( S \) is a singular stratum and \( F_S \) the restriction of \( F \) to \( S \).

We have seen in [31, Theorem 4.4] that the discussed intersection cohomologies are finite dimensional when \( M \) is compact. So the above Poincaré duality becomes the isomorphism
\[
\mathbb{H}^*_p(M/F) \cong \mathbb{H}^*_{\mathbb{Q},c}^*(M/F).
\]

4. Twisted product.

Twisted products are local building blocks of SRFs. Their basic intersection cohomology have been calculated in [31, Proposition 5]. Below we present the compact support version of this result.

Let \( K \) be a compact Lie group, \( G \) a normal subgroup of \( K \) and \( H \) a closed subgroup of \( K \). We consider a twisted product \( K \times_H N \), where \( N \) is a manifold endowed with an effective smooth action \( \Theta : H \times N \to N \). The restriction \( \Theta_0 : (G \cap H) \times N \to N \) is a tame action, where the tamer group \( H' \) is the closure of \( (G \cap H) \) in \( H \). The associated Killing foliation is denoted by \( N \). We denote by \( \Phi : G \times (K \times_H N) \to (K \times_H N) \) the associated tame action, and the induced foliation by \( \mathcal{W} \). The foliation defined by the tame (left or right!) action of \( G \) on \( K \) is denoted by \( \mathcal{K} \). The foliation defined by the tame right action of \( GH^3 \) on \( K \) is denoted by \( \mathcal{E} \).

**Proposition 4.1** \( \mathbb{H}^*_{\mathbb{Q},c}(K \times_H N/\mathcal{W}) = \left( H^*(K/\mathcal{E}) \otimes \mathbb{H}^*_{\mathbb{Q},c}(N/N) \right)^{H/H_0} \).

**Proof.** It suffices to follow [31, Proposition 5] taking into the account the fact that, given a differential form \( \omega \) on \( K \times_H N \), we have:
\[
\bigcap^* \omega \in \Omega^*_{\mathbb{Q},c}(K \times N/\mathcal{E} \times N) \iff \omega \in \Omega^*_{\mathbb{Q},c}(K \times_H N/\mathcal{W}).
\]
It is so as the canonical projection \( \prod : K \times N \to K \times_H N \) is an onto map and that the Lie groups \( K \) and \( H \) are compact. \qed

5. Tangent volume form.

In order to construct the pairing giving the Poincaré duality we need to introduce a particular tangent volume form of the orbits of \( \Phi \).

We fix a bi-invariant metric \( \nu \) on \( \mathfrak{f} \), the Lie algebra of \( K \) which exists since \( K \) is compact (see for example [3, pag. 247]). Consider \( \{u_1, \ldots, u_f\} \) an orthonormal basis of \( \mathfrak{f} \) where \( \{u_1, \cdots, u_b\} \) is a basis of \( \mathfrak{g} \), \( \{u_1, \cdots, u_{w}\} \) is a basis of \( \mathfrak{l}^+ \), the orthogonal complement of \( \mathfrak{l} \) in \( \mathfrak{g} \), and \( \{u_{w+1}, \cdots, u_b\} \) is a basis of \( \mathfrak{l} \). Here, \( \mathfrak{g} \) (resp. \( \mathfrak{l} \)) denotes the Lie algebra of \( G \) (resp. \( L \)). We take \( \tau = u_1^* \wedge \cdots \wedge u_w^* \), the associated volume form of \( \mathfrak{l}^+ \).

\(^3\)This is the minimal subgroup of \( K \) containing \( G \) and \( H \). Normality of \( G \) implies that \( GH = \{g \cdot h : g \in G, h \in H\} \). The Lie algebra of \( GH \) is the sum of the Lie algebras of \( G \) and \( H \).
We denote by $V_u$ the fundamental vector field on $M$ associated to $u \in g$. A tangent volume form of $\Phi$ is a $G$-invariant differential form $\eta \in \Pi^\omega_\Phi(M)$ verifying:

$$\eta(V_{v_1}(x), \ldots, V_{v_n}(x)) = \tau(\text{Ad}(\ell^{-1}) \cdot v_1, \ldots, \text{Ad}(\ell^{-1}) \cdot v_n),$$

where $\{v_1, \ldots, v_n\} \subset g$, $x \in R_F$ and $(G_x)_0 = \ell L \ell^{-1}$. The existence of a tangent volume form implies that the foliation $\mathcal{F}$ is tangentially orientable on the regular stratum.

In the next Proposition we prove the existence of a tangent volume form under a suitable orientation conditions on the manifold and on the foliation. The tame action $\Phi$ is said to be orientable if

(i) the manifold $M$ is orientable, and

(ii) the adjoint action $\text{Ad} : N_K(L) \times 1 \to 1$ is orientation preserving.

We say that a Killing foliation is orientable if it is induced by an orientable action.

5.1 Remarks.

(a) Condition (ii) does not depend on the choice of the point $p$ defining $L$. Let us verify that fact. Choose another point $p' \in R_F$, defining $L' = (G_{p'})_0$. Without loss of generality we can suppose that $p'$ is near enough to $p$ (the connectedness of $R_F$) in order to apply [5, Chapter II, Corollary 5.4] and find $k \in K$ with $k^{-1}G_{p'}k \subset K_p$. Since $G$ is normal in $K$, we get $k^{-1}G_{p'}k \subset G_p$, and therefore $k^{-1}(G_{p'})_0k \subset (G_p)_0$. Since $p, p' \in R_F$, we conclude that $k^{-1}(G_{p'})_0k = (G_p)_0$, that is $k^{-1}L'k = L$. This gives the claim since $\text{Ad} : N_K(L') \times 1' \to 1'$ becomes $\text{Ad} : kN_K(L)k^{-1} \times \text{Ad}(k)(1) \to \text{Ad}(k)(1)$.

(b) By connectedness, the group $K$ preserves the orientation of $g$. So the condition (ii) is equivalent to

(ii') the adjoint action $\text{Ad} : N_K(L) \times 1^+ \to 1^+$ is orientation preserving.

(c) Condition (ii) is verified in the case where $N_K(L)$ is connected. In particular, when $G$ is abelian or when $1 = 0$, that is, if $\dim \mathcal{F} = \dim G$.

(d) There are non-orientable actions on orientable manifolds. For example, consider the action of $G = S^3$, on the twisted product $M = S^3 \times_{S^1} S^3 = S^2 \times_{S^1} S^2$ (cf. [5, pag. 80]). Here, $L = S^1$ and the element $j \in N_{S^3}(S^1)$ acts on $1 = \mathbb{R}$ by multiplying by $-1$, which does not preserve the orientation. Since $M/\mathcal{F} = \mathbb{R}P^2$ then the Poincaré duality property does not hold.

Proposition 5.2 For any orientable action there exists a $K$-invariant tangent volume form.

Proof. Let $\Phi : G \times M \to M$ be an orientable action.

In order to decrease the depth of the stratification, we are going to use the Molino’s blow up $\tilde{M}$ and the stratification $S_{k,M}$ defined by the action of $K$ (see Appendix). We prove the following statement by induction on depth $S_{k,M}$:

“There exists a $K$-invariant differential form $\tilde{\eta} \in \Pi^\omega_{\Phi \times 1}(M \times [0, 1[)$ verifying

$$\tilde{\eta}((V_{v_1}(x), 0), \ldots, (V_{v_n}(x), 0)) = \tau(\text{Ad}(\ell^{-1}) \cdot v_1, \ldots, \text{Ad}(\ell^{-1}) \cdot v_n),$$

where $\{v_1, \ldots, v_n\} \subset g$, $x \in R_F$ and $(G_x)_0 = \ell L \ell^{-1}$ with $\ell \in K$.”
The existence of $\eta$ is proven by taking $p = 0$.

**First case:** depth $S_{k,m} = 0$.

Since $M = R_F$ then we have $\Pi_*^1(M \times [0, 1]^p) = \Omega^1((M \times [0, 1]^p)/(\mathcal{F} \times I)).$ Take $T(\mathcal{F} \times I)$ the sub-bundle of the tangent bundle $T(M \times [0, 1]^p)$ formed by vectors tangents to the leaves of the foliation $\mathcal{F} \times I$. Here $I$ denotes the point wise foliation of $[0, 1]^p$. Since the foliation $\mathcal{F}$ is $K$-invariant, then it suffices to define $\overline{\eta}$ on $T(\mathcal{F} \times I)$ and to extend it by 0 to a $K$-invariant complement of $T(\mathcal{F} \times I)$ in $T(M \times [0, 1]^p)$. In fact, this restriction is given by (3). It remains to prove that $\overline{\eta}$ is well-defined, smooth on the tangent bundle $T(\mathcal{F} \times I)$ and $K$-invariant. Let us check that.

- **The definition (3) does not depend on $\ell$.** Let us consider $\ell' \in K$ with $(G_{\chi})_0 = \ell' L \ell'^{-1}$. Then $\ell'^{-1} \ell \in N_K(L)$. This gives

$$\tau(\text{Ad}(\ell'^{-1}) \cdot v_1, \ldots, \text{Ad}(\ell'^{-1}) \cdot v_w) = \tau(\text{Ad}(\ell'^{-1}) \ell \text{Ad}(\ell^{-1}) \cdot v_1, \ldots, \text{Ad}(\ell'^{-1}) \ell \text{Ad}(\ell^{-1}) \cdot v_w).$$

Since the metric $v$ has been chosen to be bi-invariant, then the element $\text{Ad}(\ell'^{-1}) \ell$ preserves the metric $v$. It also preserves the orientation of $\mathfrak{l}^\perp$ (see (ii')). So, we get

$$\tau(\text{Ad}(\ell'^{-1}) \cdot v_1, \ldots, \text{Ad}(\ell'^{-1}) \cdot v_w) = \tau(\text{Ad}(\ell^{-1}) \cdot v_1, \ldots, \text{Ad}(\ell^{-1}) \cdot v_w).$$

- **The definition (3) is smooth.** Consider $x \in M$. From Proposition 2.1 we know that there exist a neighborhood $V \subset M$ and a smooth map $f: V \to K$ such that $(G_{\chi})_0 = f(y)f(y)^{-1}$ for each $y \in V$. The previous point says that we can choose $\ell = f(y)$ in definition (3). So in this neighborhood we have

$$\overline{\eta}(V_{\chi}(y), \ldots, V_{\psi}(y)) = \tau(\text{Ad}(f(y)^{-1}) \cdot v_1, \ldots, \text{Ad}(f(y)^{-1}) \cdot v_w),$$

which is smooth.

- **The form $\overline{\eta}$ is $K$-invariant.** If $k \in K$ we get $(G_{k\chi})_0 = k \ell L \ell^{-1} k^{-1}$ and

$$(k^* \overline{\eta})(V_{\chi}(x), \ldots, V_{\psi}(x)) = \overline{\eta}(k_* V_{\chi}(x), \ldots, k_* V_{\psi}(x)).$$

By induction hypothesis there exists a $K$-invariant differential form $\overline{\eta}_0 \in \Pi_*^\prime_\mathcal{F}(\mathcal{F} \times [0, 1]^p)$ verifying (3). Associated to $\mathcal{F}$, we have the $K$-equivariant imbedding $\sigma: M \setminus S_{\min} \to \mathcal{L}^\prime_\mathcal{F}(M \setminus S_{\min})$, defined by $\sigma(z) = (z, 1)$. The differential form $\overline{\eta} = (\sigma \times \text{identity}_{[0, 1]^p}) \overline{\eta}_0$ belongs to $\Omega^1(R_F \times [0, 1]^p)$, it is $K$-invariant and verifies (3). It remains to prove that $\overline{\eta} \in \Pi_*^\prime_\mathcal{F,D}(M \times [0, 1]^p)$, which is a local property.

So we can assume that $M = T_{\min}$ and prove $(\nabla_{\min} \times \text{identity}_{[0, 1]^p}) \overline{\eta} \in \Pi_*^\prime_\mathcal{F,D}(D_{\min} \times [0, 1]^{p+1})$ (cf. [30, 3.1.1 (e)]). This is the case since the map $\sigma_0 \nabla_{\min}: D_{\min} \times [0, 1] \to D_{\min} \times [1, 1]$ is just the inclusion and we have $\overline{\eta}_0 \in \Pi_*^\prime_\mathcal{F,D}(D_{\min} \times [1, 1] \times [0, 1]^p).

In this Proposition the density of $G$ in $K$ is used to ensure that $G$ is a normal subgroup of $K$. \hfill \blacktriangle
Proposition 5.3 An invariant tangent volume form \( \eta \) verifies:

(a) For each \( \omega \in \Omega^{m-w-1}_r(M/F) \) the product \( \omega \wedge d\eta \) is 0.

(b) For each \( \omega \in \Omega^{m-w}_r(M/F) \) the integral \( \int_{R_F} \omega \wedge \eta \) is finite and it does not depend on the choice of \( \eta \).

(c) For each \( \omega \in \Omega^{m-w-1}_r(M/F) \) the integral \( \int_{R_F} d(\omega \wedge \eta) \) is 0.

Proof.

(a) Since the question is a local one, it is enough to prove \( \omega \wedge d\eta = 0 \) on an open subset \( V \subset R_F \) (cf. proof of Proposition 2.1).

For each \( y \in V \) we have \( (G_y)_0 = f(y)L_f(y)^{-1} \). Then \( \{V_{\text{Ad}(f(y))}\}_{1} \), \( \ldots \), \( V_{\text{Ad}(f(y))}\) is a basis of \( T_y G(y) \). For degree reasons it suffices to prove that we have \( i_{V_{\text{Ad}(f(y))}}(\omega) \cdots i_{V_{\text{Ad}(f(y))}}(\omega) \wedge d\eta = 0 \). Since \( \omega \) is a basic form and \( \eta \) is a \( K \)-invariant form, we can write

\[ i_{V_{\text{Ad}(f(y))}}(\omega) \cdots i_{V_{\text{Ad}(f(y))}}(\omega) \wedge d\eta = (-1)^w \omega \wedge i_{V_{\text{Ad}(f(y))}}(\omega) \cdots i_{V_{\text{Ad}(f(y))}}(\omega) d\eta = \omega \wedge d \left( \eta \left( V_{\text{Ad}(f(y))} \right) \right) \]

\[ = \omega \wedge d \left( \eta \left( V_{\text{Ad}(f(y))} \right) \right) = \omega \wedge d (\tau (v_1, \ldots, v_w)). \]

Notice that this property holds for any \( \omega \in \Pi^{m-w-1}_r(M) \).

(b) To demonstrate the finiteness, it suffices to prove that \( \int_{R_F \times [0,1]^p} \gamma < \infty \) where \( \gamma \in \Pi^{w}_r(M \times \{0,1\}^p) \) is of compact support. We proceed by induction on the depth of \( S_{k,M} \). When depth \( S_{k,M} = 0 \) the result is clear since \( M = R_F \) (see Appendix).

In the general case we know that the result is true for \( M \setminus S_{\text{min}} \times [0,1]^p \) and \( (T_{\text{min}} \setminus S_{\text{min}}) \times [0,1]^p \) by induction. It remains to consider \( T_{\text{min}} \times [0,1]^p \). We have seen that we can identify the perverse forms of \( T_{\text{min}} \times [0,1]^p \) with the perverse forms of \( D_{\text{min}} \times [0,1]^{p+1} \) through the map

\[ \nabla_{\text{min}} \times \text{Identity}_{[0,1]^p} : D_{\text{min}} \times [0,1] \times [0,1]^p \equiv D_{\text{min}} \times [0,1]^{p+1} \longrightarrow T_{\text{min}} \times [0,1]^p \]

(this is a general result for basic intersection cohomology proved in [30, 3.4.1 (d)]). Since this map is a diffeomorphism between \( D_{\text{min}} \times [0,1] \times [0,1]^p \) and \( (T_{\text{min}} \setminus S_{\text{min}}) \times [0,1]^p \), then we have

\[ \int_{R_{DF} \times [0,1]^{p+1}} \gamma = \int_{R_D \times [0,1]^{p+1}} \gamma = \int_{R_{TF} \times [0,1]^{p+1}} \gamma = \int_{R_{DF} \times [0,1]^{p+1}} \gamma. \]

The induction hypothesis gives that this integral is finite.

Let \( N \mathcal{F} \oplus T \mathcal{F} \) be a sub-bundle decomposition of \( TR_F \). Since \( \omega \) is a basic form on \( R_F \) then it vanishes on \( T \mathcal{F} \). It is a top degree basic form, so \( \omega \wedge \eta (a_1, \ldots, a_{m-w}, b_1, \ldots, b_w) = \omega (a_1, \ldots, a_{m-w}) \cdot \eta (b_1, \ldots, b_w) \) and \( \eta (b_1, \ldots, b_w) \) does not depend on \( \eta \) (see (2)).

(c) Since supp \( \omega \) is compact then it suffices to prove \( \int_{U \cap R_F} d(\omega \wedge \eta) = 0 \) where \( (U, \varphi) \) is a conical chart of \( \mathcal{F} \) and \( \omega \in \Omega^{m-w-1}_r(U/F) \) with supp \( \omega \subset U \). Recall that the restriction \( P_\varphi : \mathbb{R}^{m-w} \times \mathbb{S}^{n-1} \times [0,1] \longrightarrow U \cap R_F \) is a
diffeomorphism. Also, the pull-back $P^*_\varphi : \Pi_\varphi^*(U) \to \Pi_\varphi^*(\mathbb{R}^{m-n} \times \mathbb{S}^{n-1} \times [0,1])$ is a dgca isomorphism (this is a general result for basic intersection cohomology, see for example [30, Section 3.1]). So, we have:

$$\int_{U \cap R_\varphi} d(\omega \wedge \eta) = \int_{\mathbb{R}^{m-n} \times R_G \times [0,1]} d(P^*_\varphi \omega \wedge P^*_\varphi \eta) = \int_{\mathbb{R}^{m-n} \times R_G \times [0,1]} d(\omega \varphi \wedge \eta \varphi) \overset{\text{Stokes}}{\Rightarrow} \int_{\mathbb{R}^{m-n} \times R_G \times [0]} \omega \varphi \wedge \eta \varphi = 0$$

since, for any $\omega \in \Omega^{m-n-1}_\varphi(M/F)$, the restriction of $\omega \varphi$ to $\mathbb{R}^{m-n} \times R_G \times \{0\}$ vanishes (this is a general result for basic intersection cohomology, see for example [30, 3.4.1 (c)]).

6. The pairing.

In Section 9 we will prove the Poincaré duality property : $\mathbb{H}_\varphi^*(M/F) \cong \text{Hom}(\mathbb{H}^m_{\tau, e}(M/F), \mathbb{R})$, when $F$ is orientable and the two perversities $\overline{\varphi}$ and $\overline{\varphi}$ are complementary. This isomorphism comes from the pairing $P_M$ constructed from the a tangent volume form $\eta$ (cf. Proposition 5.2) in the following way:

$$P_M : \Omega^*,(M/F) \times \Omega^{m-n-\ast}_{\tau, e}(M/F) \to \mathbb{R} \quad : \quad (\alpha, \beta) \sim \int_{R_\varphi} \alpha \wedge \beta \wedge \eta.$$  

Proposition 5.3 implies that this operator is well defined and that it induces the pairing

$$P_M : \mathbb{H}_\varphi^*(M/F) \times \mathbb{H}^{m-n-\ast}_{\tau, e}(M/F) \to \mathbb{R},$$

defined by $P_M([\alpha], [\beta]) = P_M(\alpha, \beta)$. Moreover, it does not depend on the choice of the tangent volume form $\eta$. The Poincaré duality property asserts that $P_M$ is a non degenerate pairing, that is, the operator

$$P_M : \mathbb{H}_\varphi^*(M/F) \to \text{Hom}(\mathbb{H}^{m-n-\ast}_{\tau, e}(M/F), \mathbb{R})$$

defined by $P_M([\alpha])([\beta]) = \int_{R_\varphi} \alpha \wedge \beta \wedge \eta$ is an isomorphism.

7. Twisted product and Poincaré duality .

We first get the Poincaré duality property in the framework of a twisted product $K \times_H N$ (see Section 4). We fix a bi- invariant metric on the Lie algebra $\mathfrak{t}$ and we choose

$$B = \{u_1, \ldots, u_a, u_{a+1}, \ldots, u_w, u_{w+1}, \ldots, u_b, u_{b+1}, \ldots, u_c, u_{c+1}, \ldots, u_f\}, \text{ with } 0 \leq a \leq w \leq b \leq c \leq f$$

an orthonormal basis of $\mathfrak{t}$ with $\{u_1, \ldots, u_b\}$ basis of $\mathfrak{g}$, $\{u_{a+1}, \ldots, u_c\}$ basis of the Lie algebra $\mathfrak{h}$ of $H$ and $\{u_{w+1}, \ldots, u_b\}$ basis of $\mathfrak{l}$. Notice that $b = w$ when $G$ is abelian.

Let $\{\gamma_1, \ldots, \gamma_f\}$ (resp. $\{\zeta_{a+1}, \ldots, \zeta_c\}$) be the dual forms associated to a basis $\{u_1, \ldots, u_f\}$ (resp. $\{u_{a+1}, \ldots, u_c\}$) relatively to a bi -invariant metric $\nu$ on $K$ (resp. $H$-invariant Riemannian metric on $N$).

We shall use the following notation.

- $X_\bullet$ the fundamental vector fields of the right action of $G$ on $K$,
- $X^\ast$ the fundamental vector fields of the left action of $G$ on $K$. 

They are related \( X_u(k) + X^{\Ad(k)}u(k) = 0 \) if \( k \in K \) and \( u \in \mathfrak{g} \).

- \( W_\ast \) are the fundamental vector fields of the action \( \Theta \).
- \( V_\ast = \bigcap_\ast \langle X_\ast, 0 \rangle \) are the fundamental vector fields of the action \( \Phi \).

First of all, we establish a relationship between the tangent volume forms of \( \Phi \) and \( \Theta_0 \).

**Lemma 7.1** If the action \( \Phi : G \times (K \times_H N) \to (K \times_H N) \) is orientable then the action \( \Theta_0 : (G \cap H)_0 \times N \to N \) is also orientable. We denote by \( \eta \) a tangent volume form of \( \Phi \) (resp. \( \eta_0 \) of \( \Theta_0 \)) associated to the metric \( \nu \) (resp. of \( \nu_{|G \cap H} \)). Then

\[
(5) \quad (-1)^{ab} \gamma_{a+1} \wedge \cdots \wedge \gamma_b \wedge \bigcap_\ast \eta = \gamma_1 \wedge \cdots \wedge \gamma_b \wedge \eta_0 \text{ on the tangent bundle of } \mathcal{K} \times N.
\]

**Proof.** We fix a base point \( v_0 \in R_N \). Since \( G_{<\nu,\nu>} = (G \cap H)_0 \) then \( <e,v_0> \in R_W \). We fix these two base points. So, we have the same \( L \) for \( \mathcal{N} \) and \( \mathcal{W} \). The orientability of \( \Theta_0 \) comes now from the inclusion \( N_{H'}(L) \subset N_K(L) \).

For the second part we use the foliated blow up \( \mathcal{P} : (K \times N, \mathcal{K} \times N) \to (K \times_H N, \mathcal{W}) \). We also recall that \( \tau = u_1^* \wedge \cdots \wedge u_w^* \) and \( \tau_0 = u_{a+1}^* \wedge \cdots \wedge u_w^* \) are the associated volume forms of \( \mathcal{L}^\ast \) on \( \mathfrak{g} \) and \( \mathfrak{g} \cap \mathfrak{h} \), respectively.

The leaf of \( \mathcal{K} \times \mathcal{N} \) at the point \((k, z) \in K \times N \) is generated by

\[
\mathfrak{B} = \{ X_{a_1}(k), \ldots, X_{a_b}(k), W_\Ad(k, u_{a+1})(z), \ldots, W_\Ad(k, u_w)(z) \},
\]

where \( \ell \in K \) with \((H \cap G)_0 = k((G \cap H)_0)k^{-1} = k\ell(k\ell)^{-1} \).

The RHS of (5) applied to \( \mathfrak{B} \) gives:

\[
\eta_0(W_\Ad(k, u_{a+1})(z), \ldots, W_\Ad(k, u_w)(z)) = \tau_0(u_{a+1}, \ldots, u_w) = (u_{a+1}^* \wedge \cdots \wedge u_w^*)(u_{a+1}, \ldots, u_w) = 1.
\]

Using the fact that \( \bigcap_\ast X_u(k) = -\bigcap_\ast W_u(z) \) if \( u \in \mathfrak{g} \cap \mathfrak{h} \) (cf. [29, Proposition 5 (5)]) the LHS of (5) applied to \( \mathfrak{B} \) gives:

\[
\bigcap_\ast \eta(X_{a_1}(k), \ldots, X_{a_b}(k), W_\Ad(k, u_{a+1})(z), \ldots, W_\Ad(k, u_w)(z)) = (-1)^{w-d} \eta\left( \bigcap_\ast X_{a_1}(k), \ldots, \bigcap_\ast X_{a_b}(k), \bigcap_\ast X_\Ad(k, u_{a+1})(k), \ldots, \bigcap_\ast X_\Ad(k, u_w)(k) \right) = (-1)^d \eta\left( \bigcap_\ast X^\Ad(k, u_{a+1})(k), \ldots, \bigcap_\ast X^\Ad(k, u_w)(k) \right) = (-1)^d \tau(\Ad(\ell^{-1}) \cdot u_1, \ldots, \Ad(\ell^{-1}) \cdot u_{a+1}, u_{a+1}, \ldots, u_w) = (-1)^d(u_1^* \wedge \cdots \wedge u_w^*)(\Ad(\ell^{-1}) \cdot u_1, \ldots, \Ad(\ell^{-1}) \cdot u_a),
\]

up to the sign \((-1)^{a(b-d)}\), coming form the reordering: \((X_{a_1}, \ldots, X_{a_b}, X_{u_{a+1}}, \ldots, X_{u_w}) \mapsto (X_{u_{a+1}}, \ldots, X_{u_w}, X_{a_1}, \ldots, X_{a_b}) \). Since \( \ell \in H' \), the closure of \((G \cap H)_0 \) on \( H \), then \( \Ad(\ell) \) preserves \( \mathfrak{g} \) and \( \mathfrak{h} \). The connectedness of \( H' \) gives that the operator \( \Ad(\ell) : (\mathfrak{g} \cap \mathfrak{h})^{\perp_\delta} \to (\mathfrak{g} \cap \mathfrak{h})^{\perp_\delta} \) is an orthogonal map preserving orientation. So

\[
(u_1^* \wedge \cdots \wedge u_a^*)(\Ad(\ell^{-1})(u_1), \ldots, \Ad(\ell^{-1})(u_a)) = \det(\Ad(\ell^{-1}))(u_1^* \wedge \cdots \wedge u_a^*)(u_1, \ldots, u_a) = 1,
\]

We obtain (5).

**Proposition 7.2** If \( \mathcal{N} \) verifies the Poincaré duality property then \( \mathcal{W} \) also verifies the Poincaré duality property.
Proof. Let $\overline{p}$ and $\overline{q}$ two complementary perversities on $N$. We have $\dim N = m + c - a - f$ and $\dim N = w - a$, where $w = \dim W$ and $m = \dim K \times p N$. By hypothesis, the pairing $P_N : H^\prime_p (N/N) \times H^{\dim w - f \to} (N/N) \to \mathbb{R}$ is non degenerate.

The foliation $\mathcal{E}$ of $K$ is generated by the vector fields $\{X_1, \ldots, X_c\}$. This gives $H^\prime (K/\mathcal{E}) = \bigwedge (\gamma_{c+1}, \ldots, \gamma_f)$. It is clear that the pairing $P : H^\prime (K/\mathcal{E}) \times H^{\dim w - f \to} (K/\mathcal{E}) \to \mathbb{R}$, defined by $P([\xi], [\gamma]) = \int_K \xi \wedge \gamma_1 \wedge \cdots \wedge \gamma_c$ is also non degenerate. Notice that the group $\Phi$ preserves $\mathcal{E}$ and therefore $h^\prime \gamma_1 \wedge \cdots \wedge \gamma_c = \pm \gamma_1 \wedge \cdots \wedge \gamma_c$ for each $h \in H$. We conclude that the induced pairing

$$P \otimes P_N : (H^\prime (K/\mathcal{E}) \otimes H^\prime_p (N/N))^{H/0} \times (H^{\dim w - f \to} (K/\mathcal{E}) \otimes H^{\dim w - f \to} (N/N))^{H/0} \to \mathbb{R}$$

is non degenerate. Notice that the group $H/H_0$ is finite. The group $H \cap G$ is normal on $H$, so the subgroup $(H \cap G)_0$ is also normal on $H$. The remark following Proposition 5.2 implies that we can suppose that the tangent volume form defining $P_N$ can be chosen $H$-invariant. On the other hand, the right-action of $H$ on $K$ preserves $\mathcal{E}$ and therefore $h^\prime \gamma_1 \wedge \cdots \wedge \gamma_c$ for each $h \in H$. We conclude that the induced pairing

$$P \otimes P_N : (H^\prime (K/\mathcal{E}) \otimes H^\prime_p (N/N))^{H/0} \times (H^{\dim w - f \to} (K/\mathcal{E}) \otimes H^{\dim w - f \to} (N/N))^{H/0} \to \mathbb{R}$$

is also non degenerate.

We also denote by $\overline{p}$ and $\overline{q}$ the associated perversities on $K \times_H N$, which also are two complementary perversities. Recall that the isomorphisms

$$\nabla_{\min} : (H^\prime (K/\mathcal{E}) \otimes H^\prime_p (N/N))^{H/0} \to H^\prime_p (K \times_H N),$$

$$\nabla_{\min} : (H^\prime (K/\mathcal{E}) \otimes H^\prime_q (N/N))^{H/0} \to H^\prime_q (K \times_H N)$$

are characterized by

$$\prod^\prime \nabla_{\min} ([\xi] \otimes [\alpha]) = \left[ \xi \wedge \alpha + \sum_{b < i_1 < \cdots < i_t < c} (-1)^t \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_t} \wedge (i_{w_{i_1}} \cdots i_{w_{i_t}} \alpha) \right]$$

(cf. [31, Proposition 5] and Proposition 4.1).

Let us consider the following diagram

$$\begin{array}{ccc}
(H^\prime (K/\mathcal{E}) \otimes H^\prime_p (N/N))^{H/0} \times (H^{\dim w - f \to} (K/\mathcal{E}) \otimes H^{\dim w - f \to} (N/N))^{H/0} & \to & \mathbb{R} \\
\nabla_{\min} \times \nabla_{\min} \downarrow & & \\
H^\prime_p (K \times_H N) \times H^\prime_q (K \times_H N) & \overset{P_{K \times_H N}}{\longrightarrow} & \mathbb{R}
\end{array}$$

where a suitable rearrangement in the top left term is necessary in order to apply $P \otimes P_N$. We will complete the proof if we show that this diagram commutes up to a non-zero constant depending on $\Phi$. In effect, vertical arrows are isomorphisms and top arrow is a non degenerate pairing.

Write $\nabla_{\min} ([\xi] \otimes [\alpha]) = [\xi \bullet \alpha]$. We have,

$$P_{K \times_H N} (\nabla_{\min} \times \nabla_{\min}) ([\xi] \otimes [\alpha], [\gamma] \otimes [\beta]) = \int_{K \times_H N} \xi \bullet \alpha \wedge \chi \bullet \beta \wedge \eta.$$
Recall that we have denoted by \( W_{u_{a+1}} = W_{a+1}, \ldots, W_{u_{c}} = W_{c} \) the fundamental vector fields of the action \( \Theta: H \times N \to N \) associated to the basis \( \{u_{a+1}, \ldots, u_{c}\} \). Let \( \{\xi_{a+1}, \ldots, \xi_{c}\} \) be the associated dual forms relatively to an \( H \)-invariant Riemannian metric on \( R_N \). So \( \frac{1}{2^a}(\gamma_{a+1} + \zeta_{a+1}) \wedge \cdots \wedge (\gamma_{c} + \zeta_{c}) \) is a differential form of \( K \times R_N \) giving a volume form on each fiber of \( \Pi \). Thus

\[
P_{K \times H^N}(\nabla_{\min} \times \nabla_{\min})([\xi] \otimes [\alpha], [\chi] \otimes [\beta]) = \frac{1}{2^{a-c}} \int_{K \times R_N} \prod_{i=1}^{c} \left( \xi \cdot \alpha \wedge \chi \cdot \beta \wedge \eta \right) \wedge (\gamma_{a+1} + \zeta_{a+1}) \wedge \cdots \wedge (\gamma_{c} + \zeta_{c}).
\]

We claim that the integrand is the differential form \( \xi \wedge \alpha \wedge \chi \wedge \beta \wedge \eta_0 \wedge \gamma_1 \wedge \cdots \wedge \gamma_c \), up to a non-zero constant \( C \) depending on \( \Phi \). We get

\[
P_{K \times H^N}(\nabla_{\min} \times \nabla_{\min})([\xi] \otimes [\alpha], [\chi] \otimes [\beta]) = \frac{(-1)^{|\alpha|+|\beta|+\dim N} C}{2^{c-a}} \int_K \xi \wedge \chi \wedge \gamma_1 \wedge \cdots \wedge \gamma_c \cdot \int_{R_N} \alpha \wedge \beta \wedge \eta_0 = \frac{(-1)^{|\alpha|+|\beta|+\dim N} C}{2^{c-a}} P([\xi], [\chi]) \cdot P_N([\alpha], [\beta]) = \frac{(-1)^{|\alpha|+|\beta|+\dim N} C}{2^{c-a}} (P \otimes P_N)([\xi] \otimes [\alpha], [\chi] \otimes [\beta]).
\]

This would end the proof after the verification of the claim, which is a local question.

We denote by \( \mathcal{H} \) the foliation defined by the action of \( H \) on \( R_N \). It is a regular foliation. Since \( R_{\mathcal{H}} \) is dense in \( R_N \), it suffices to prove the claim on \( K \times R_{\mathcal{H}} \). Let us consider a point \( (k, z) \in K \times R_{\mathcal{H}} \). We need to find a non-zero constant \( C \) depending on \( \Phi \) such that

\[
\xi \wedge (\alpha + \overline{\alpha}) \wedge \chi \wedge (\beta + \overline{\beta}) \wedge \prod_{i=1}^{c} \eta \wedge (\gamma_{a+1} + \zeta_{a+1}) \wedge \cdots \wedge (\gamma_{c} + \zeta_{c}) = C \xi \wedge \alpha \wedge \chi \wedge \beta \wedge \eta_0 \wedge \gamma_1 \wedge \cdots \wedge \gamma_c,
\]

on the tangent space \( T_k K \times T_z N \). Without loss of generality, we can suppose that this vector space is generated by the family

\[
\{X_1(k), \ldots, X_f(k), W_{a+1}(z), \ldots, W_w(z), W_{b+1}(z), \ldots, W_v(z), V_1, \ldots, V_{\ell}\},
\]

where \( \{W_{a+1}(z), \ldots, W_w(z)\} \) is a basis of \( T_z N \), \( \{W_{a+1}(z), \ldots, W_w(z), W_{b+1}(z), \ldots, W_v(z)\} \) is a basis of \( T_k K \times T_z N \).

The following differential forms vanish on the basis \( \{X_1(k), \ldots, X_f(k), W_{a+1}(z), \ldots, W_w(z)\} \) of \( T_k K \times T_z N \):

\[
\xi \wedge \chi \in \bigwedge^\ast \{\gamma_{c+1}, \ldots, \gamma_f, \{\alpha, \beta, \overline{\alpha}, \overline{\beta}\} \subset \bigwedge^\ast \{\gamma_{b+1}, \ldots, \gamma_c\} \otimes \Omega^\ast (R_{\mathcal{H}}/N) \text{ and } \{\gamma_{b+1}, \ldots, \gamma_c\}.
\]

Moreover, the forms \( \{\gamma_{w+1} + \xi_{w+1}, \ldots, \gamma_{c} + \xi_{c}\} \) vanish on \( \{X_{a+1}(k) + W_{a+1}(z), \ldots, X_w(k) + W_w(z)\} \).

Applying Lemma 7.1 we conclude that it suffices to find a non-zero constant \( C \) depending on \( \Phi \) such that

\[
\xi \wedge \chi \wedge (\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) \wedge (\gamma_{b+1} + \zeta_{b+1}) \wedge \cdots \wedge (\gamma_{c} + \zeta_{c}) = C \frac{(-1)^{w} C}{2^{c-a}} \xi \wedge \chi \wedge \alpha \wedge \beta \wedge \gamma_{b+1} \wedge \cdots \wedge \gamma_c,
\]

on the family \( \{X_{b+1}(k), \ldots, X_f(k), W_{b+1}(z), \ldots, W_v(z), V_1, \ldots, V_{\ell}\} \).

The following differential forms vanish on \( \{X_{c+1}(k), \ldots, X_f(k)\} \):

\[
\{\alpha, \beta, \overline{\alpha}, \overline{\beta}\} \subset \bigwedge^\ast \{\gamma_{b+1}, \ldots, \gamma_c\} \otimes \Omega^\ast (R_{\mathcal{H}}/N), \{\gamma_{b+1}, \ldots, \gamma_c\} \text{ and } \{\zeta_{b+1}, \ldots, \zeta_c\}.
\]

We conclude that it suffices to find a non-zero constant \( C \) depending on \( \Phi \) such that

\[
(\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) \wedge (\gamma_{b+1} + \xi_{b+1}) \wedge \cdots \wedge (\gamma_{c} + \xi_{c}) = C \frac{(-1)^{w} C}{2^{c-a}} \alpha \wedge \beta \wedge \gamma_{b+1} \wedge \cdots \wedge \gamma_c,
\]
on the family \( \{X_{b+1}(k), \ldots, X_v(k), W_{b+1}(z), \ldots, W_v(z), V_1, \ldots, V'_v\} \).

Since \( W_{i+1}(z) = \cdots = W_c(z) = 0 \), the following differential forms vanish on \( \{X_{i+1}(k), \ldots, X_v(k)\} \):

\[
\{\alpha, \beta, \alpha, \beta\} \subseteq \bigwedge^\cdot (Y_{b+1}, \ldots, Y_v) \otimes \Omega^*(R_{H/N}), \{\zeta_{b+1}, \ldots, \zeta_c\} \text{ and } \{\gamma_{b+1}, \ldots, \gamma_v\}.
\]

(cf. (6)). We conclude that it suffices to find a non-zero constant \( C \) depending on \( \Phi \) such that

\[
(\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) \wedge (\gamma_{b+1} + \zeta_{b+1}) \wedge \cdots \wedge (\gamma_v + \zeta_v) = C \frac{(-1)^{w+b}}{2^{w-a-\delta}} \alpha \wedge \beta \wedge \gamma_{b+1} \wedge \cdots \wedge \gamma_v
\]
on the family \( \{X_{b+1}(k), \ldots, X_v(k), W_{b+1}(z), \ldots, W_v(z), V_1, \ldots, V'_v\} \).

We consider the canonical representation of \( \alpha \):

\[
\alpha = \alpha_0 + \sum_{b<i_1<\cdots<i_{\ell} \leq v} \zeta_{i_1} \wedge \cdots \wedge \zeta_{i_{\ell}} \wedge \alpha_{i_1, \ldots, i_{\ell}},
\]

where \( \alpha_{i_1, \ldots, i_{\ell}} \in \Omega^*(R_{H/N}) \) and \( i_w \alpha_{i_1, \ldots, i_{\ell}} = 0 \) for any \( j \in \{b + 1, \ldots, v\} \). A straightforward calculation gives

\[
\alpha + \overline{\alpha} = \alpha_0 + \sum_{b<i_1<\cdots<i_{\ell} \leq v} (\zeta_{i_1} - \gamma_{i_1}) \wedge \cdots \wedge (\zeta_{i_{\ell}} - \gamma_{i_{\ell}}) \wedge \alpha_{i_1, \ldots, i_{\ell}},
\]

Analogously for \( \beta \).

Differential forms \( \alpha \wedge \beta \) and \( (\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) \) vanish respectively on \( \{X_{b+1}(k), \ldots, X_v(k)\} \) and \( \{X_{b+1}(k) + W_{b+1}(z), \ldots, X_v(k) + W_v(z)\} \). On the other hand, we have \( (\gamma_j + \zeta_j)(X_j(k) + W_j(z)) = 2 \) for \( j \in \{b + 1, \ldots, v\} \). Then it suffices to find a non-zero constant \( C \) depending on \( \Phi \) such that

\[
(\alpha + \overline{\alpha}) \wedge (\beta + \overline{\beta}) = C \frac{(-1)^{w+b}}{2^{w-v-a-b}} \alpha \wedge \beta
\]
on the family \( \{W_{b+1}(z), \ldots, W_v(z), V_1, \ldots, V'_v\} \).

Without loss of generality we can suppose \( \alpha = \zeta_{b+1} \wedge \cdots \wedge \zeta_{\ell} \wedge \alpha_{b+1, \ldots, \ell} \) and \( \beta = \zeta_{\ell+1} \wedge \cdots \wedge \zeta_v \wedge \beta_{\ell+1, \ldots, v} \) where \( b < \ell \leq v \). By orthogonality, differential forms \( \{\zeta_{b+1}, \ldots, \zeta_v, \gamma_{b+1}, \ldots, \gamma_v\} \) vanish on \( \{V_1, \ldots, V'_v\} \). Then it suffices to find a non-zero constant \( C \) depending on \( \Phi \) such that

\[
(\zeta_{b+1} - \gamma_{b+1}) \wedge \cdots \wedge (\zeta_v - \gamma_v) = C \frac{(-1)^{w+b}}{2^{w+v-a-b}} \zeta_{b+1} \wedge \cdots \wedge \zeta_v
\]
on the family \( \{W_{b+1}(z), \ldots, W_v(z)\} \). As \( \gamma_j(W_j(x)) = 0 \) for any \( j, j' \in \{b + 1, \ldots, v\} \), just take \( C = (-1)^{w+b}2^{w+v-a-b} \). Notice that \( b = \dim G, a = b - \dim G \cap H, w = \dim W \) and \( v = b + \dim H - \dim N \) do not depend on the choice of the point \( (k, v) \) but on \( \Phi \).

\[\blacklozenge\]

8. Tubular neighborhoods and Poincaré duality

In this section we consider a manifold \( M \) endowed with a Killing foliation \( F \) induced by an orientable action \( \Phi: G \times M \to M \), \( G \) connected. We fix a tamer group \( K \) and a tangent volume form \( \eta \). We also fix a couple of complementary perversities \( \overline{p} \) and \( \overline{q} \).

We prove the Poincaré property of \( F \) by cutting the manifold \( M \) into nicer pieces where this property is checked and by gluing these facts using Mayer-Vietoris technique and Bredon’s Trick. These pieces are essentially tubular neighborhoods of strata. But we do not use the stratification \( S_F \) (induced by \( G \)) since its properties are not strong enough. We work with the stratification \( S_{K,M} \) (induced by \( K \)). It possesses very useful properties because of the compactness of \( K \).
8.1 Mayer-Vietoris. For any saturated open subset \( U \subset M \) we introduce the following statement:

\[
\Psi(U) = \"The pairing \( P_U : H^\tau_p(U/F) \to \text{Hom} \left( H^{\tau - w*p}_p(U/F), \mathbb{R} \right) \) is an isomorphism.
\]

Any covering \( \{U, V\} \) of an open subset of \( M \), made up of saturated (union of leaves) open subsets of \( M \), possesses a subordinated partition of the unity made up of basic functions (see for example [26, Lemma 2.1.1]), which also are controlled functions [30, Remark 3.2.1 (b)]. This is the main ingredient permitting us to say that the two rows of the following commutative diagram are exact (see for example [4, Propositions 2.3 and 2.7])

\[
\begin{array}{cccccc}
0 & \to & \Omega^\tau_p((U \cup V)/F) & \to & \Omega^\tau_p(U/F) \oplus \Omega^\tau_p(V/F) & \to & \Omega^\tau_p((U \cap V)/F) & \to & 0 \\
\downarrow P_{U,V} & & \downarrow P_U \oplus P_V & & & & \downarrow P_{U,V} & & \\
0 & \to & \text{Hom}(\Omega^\tau_{\tau p}(U/F); \mathbb{R}) & \oplus & \text{Hom}(\Omega^\tau_{\tau p}(V/F); \mathbb{R}) & \to & \text{Hom}(\Omega^\tau_{\tau p}((U \cap V)/F); \mathbb{R}) & \to & 0
\end{array}
\]

The Five Lemma gives the Mayer-Vietoris property:

\[
\Psi(U), \Psi(U) \text{ and } \Psi(U \cap V) \implies \Psi(U \cup V).
\]

8.2 Bredon’s trick. The Mayer-Vietoris property allows to make computations when the manifold is covered by suitable covering. The passage local-global may be done using an adapted version of the Bredon’s trick of [6, pag. 289]:

Let \( X \) be a paracompact topological space and let \( \{U_\alpha\} \) be an open covering, closed for finite intersection. Suppose that \( Q(U) \) is a statement about open subsets of \( X \), satisfying the following three properties:

- (BT1) \( Q(U_\alpha) \) is true for each \( \alpha \);
- (BT2) \( Q(U), Q(z) \) and \( Q(U \cap V) \implies Q(U \cup V) \), where \( U \) and \( V \) are open subsets of \( X \);
- (BT3) \( Q(U_i) \implies Q\left( \bigcup_i U_i \right) \), where \( \{U_i\} \) is a disjoint family of open subsets of \( X \).

Then \( Q(X) \) is true.

Consider a singular closed stratum \( S \) of the stratification \( S_{K,M} \). Since \( S \) is a \( K \)-invariant sub manifold of \( M \) and then it possesses a \( K \)-invariant tubular neighborhood \((T, \tau, S, \mathbb{R}^n)\). Notice the the restriction of the \( G \)-action on \( T \) is still orientable. The associated Killing foliation is \( F_T \). We fix a base point \( x \in S \). The isotropy subgroup \( K_x \) acts orthogonally and effectively on the fiber \( \mathbb{R}^n = \tau^{-1}(x) \). So, the induced action \( \Lambda_y : G_x \times \mathbb{R}^n \to \mathbb{R}^n \) is a tame action and also an orientable action since \( (G_y)_x = G_y \) for each \( y \in \tau^{-1}(x) \). The associated Killing foliation is \( F_{\mathbb{R}^n} \).

We have:

**Proposition 8.3** If \( F_{\mathbb{R}^n} \) verifies the Poincaré duality property then \( F_T \) also verifies the Poincaré duality property.

**Proof.** The canonical projection \( \pi : S \to S/K \) is a \( K/K_{\alpha} \)-homogeneous bundle over the manifold \( S/K \) (cf. [5, Chapter 5, Theorem 5.8]). We can find a good covering \( \{U_\alpha\} \) of it (cf. [4, Theorem 5.1]). For each open subset \( V \subset S/K \) we define the statement: \( Q(z) = \Psi(\tau^{-1}\pi^{-1}(z)) \). We prove (BT1)-(BT3) and the Bredon’s trick gives \( Q(S/K) \), ending the proof.

---

\(^4\text{This fact is a direct consequence of the compactness of } K \text{ and the main reason to introduce the stratification } S_{K,M}.\)
Since $U_\alpha$ is contractible then $\pi^{-1}(U_\alpha) = U_\alpha \times K/K_\xi$ and therefore $\tau^{-1}\pi^{-1}(U_\alpha) = U_\alpha \times (K \times_{K_\xi} \mathbb{R}^n)$. Notice that the action of $G$ on $K \times_{K_\xi} \mathbb{R}^n$ is orientable. The contractibility of $U_\alpha$ implies that we can suppose that $U_\alpha$ is a point. So, property $Q(U_\alpha)$ becomes $\Psi(K \times_{K_\xi} \mathbb{R}^n)$, which is true from Proposition 7.2.

(BT2) Mayer-Vietoris (8).
(BT3) Straightforward.

9. Poincaré duality.

We present the main result of this work.

**Theorem 9.1** Any orientable Killing foliation $\mathcal{F}$ verifies the Poincaré duality property.

**Proof.** We assume that the Killing foliation $\mathcal{F}$ is induced by an orientable action $\Phi: G \times M \to M$ with $G$ connected. We fix a tamer $K$. We prove $\Psi(M)$ by induction on depth $S_{K,M}$.

**First step:** depth $S_{K,M} = 0$.

We know that the foliation $\mathcal{F}$ is a regular Riemannian foliation induced by the action of a group of isometries. So, its central sheaf is trivial (cf. [16, Lemma III]). Since the action $\Phi$ is orientable we get that the manifold $M$ is orientable and the foliation $\mathcal{F}$ is tangentially orientable (existence of tangent volume form $\eta$). When the manifold $M$ is compact then $\Psi(M)$ comes from [2, Theorem 1.1] and [12, Theorem 3.1] (see also [32, Remarque 2.5 (ii), Corollaire I] and [9, Théorème 4.10]).

In the case of non-compact $M$, we notice that the canonical projection $\pi: M \to M/K$ is a $K/K_\xi$-homogeneous bundle over the manifold $M/K$ (cf. [5, Chapter 5, Theorem 5.8]). For each open subset $V \subset M/K$ we formulate the statement: $Q(\pi^{-1}(z)) = \Psi(\pi^{-1}(z))$ (see (8)). We prove (BT1)-(BT3) and the Bredon’s trick gives $Q(S)$, ending the proof.

(BT1) Since $U_\alpha$ is contractible then $\pi^{-1}(U_\alpha) = U_\alpha \times K/K_\xi$. Notice that the action of $G$ on $K/K_\xi$ is orientable. The contractibility of $U_\alpha$ implies that we can suppose that $U_\alpha$ is a point. So, the property $Q(U_\alpha)$ becomes $\Psi(K/K_\xi)$, which is true by the previous argument since $K/K_\xi$ is compact

(BT2) Mayer-Vietoris (8).
(BT3) Straightforward.

**Second step:** Conical case.

We suppose that $M = \mathbb{R}^m$ and the Killing foliation $\mathcal{F}$ is defined by an orthogonal, orientable action $\Phi: G \times M \to M$ having the origin as unique fixed point. We prove $\Psi(\mathbb{R}^m)$.

We denote by $\mathcal{G}$ the induced Killing foliation on $\mathbb{S}^{m-1}$ defined by the restriction an orthogonal, orientable action $\Phi: G \times \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ having no fixed point. Since depth $S_{K,\mathbb{S}^{m-1}} < \text{depth } S_{K,\mathbb{R}^m}$, we have $\Psi(\mathbb{S}^{m-1})$. 
From [31, Proposition 4] we have

\[ H^i_p(\mathbb{R}^m/\mathcal{F}) = \begin{cases} H^i_{\overline{\tau}}(S^{m-1}/\mathcal{G}) & \text{if } i \leq \overline{\tau}(\theta) \\ 0 & \text{if } i \geq \overline{\tau}(\theta) + 1. \end{cases} \]  

Here, \( \mathbb{R}^{m+1} = S^{m-1} \times [0, \infty[ \times \{0\} \) and \( \{\theta\} \) is the vertex of the cone. Since \( \overline{\tau} \) and \( \overline{\eta} \) are complementary perversities on \( \mathbb{R}^m \) then we have \( \overline{\tau}(\theta) + \overline{\eta}(\theta) = \overline{\tau}(\theta) = m - w - 2 \). From [30, Proposition 3.7.2] we get

\[ H^{m-w-1}_{\overline{\tau}}(\mathbb{R}^m/\mathcal{F}) = \begin{cases} H^{m-w-1}_{\overline{\tau}}(S^{m-1}/\mathcal{G}) & \text{if } i \leq \overline{\tau}(\theta) \\ 0 & \text{if } i \geq \overline{\tau}(\theta) + 1. \end{cases} \]

Now, \( \Psi(\mathbb{R}^m) \) comes from \( \Psi(S^{m-1}) \) and the two following facts

(i) The perversities \( \overline{\tau} \) and \( \overline{\eta} \) are complementary on \( S^{m-1} \) (see definition (1)).

A perversity \( \overline{\tau} \) on \( \mathbb{R}^m \) induces the perversity \( \overline{\tau} \) on \( S^{m-1} \) by: \( \overline{\tau}(S) = \overline{\tau}(S \times [0, \infty[ \times \{0\}) \) for any stratum \( S \in S_{\mathcal{G}} \). The restriction of the foliation \( \mathcal{F} \) to \( S^{m-1} \times [0, \infty[ \) is the product \( \mathcal{G} \times I \) where \( I \) is the point-wise foliation of \( [0, \infty[ \). Then we have

\[ \overline{\tau}(S) + \overline{\eta}(S) = \overline{\tau}(S \times [0, \infty[ \times \{0\}) + \overline{\eta}(S \times [0, \infty[ \times \{0\}) = \overline{\tau}(S \times [0, \infty[ \times \{0\}) = \text{codim}_{\mathbb{R}^m} \mathcal{F} - \text{codim}_{S \times [0, \infty[ \times \{0\}}(\mathcal{G} \times I) - 2 = \text{codim}_{S^{m-1}} \mathcal{G} - \text{codim}_S \mathcal{G} - 2 = \overline{\tau}(S). \]

(ii) The pairing \( P_{\mathbb{R}^m} \) becomes the pairing \( P_{S^{m-1}} \) through the isomorphisms induced by (9) and (10).

First notice that if \( \eta \) is a tangent volume form of \( \mathcal{G} \) then \( \chi^* \eta \) is a tangent volume form for \( \mathcal{F}_{\mathbb{R}^m} \) since \( G_{(x,t)} = G_x \) for each \((x,t) \in \mathcal{F}_{\mathbb{R}^m} = R_G \times [0, \infty[ \). Here \( \chi: R_{\mathcal{F}} \to R_G \) is the canonical projection. The operator \( N: H^i_p(S^{m-1}/\mathcal{G}) \to H^i_{\overline{\tau}}(\mathbb{R}^m/\mathcal{F}) \) defining (9) is \( N([\alpha]) = [\chi^* \alpha] \), (see [30, Proposition 3.5.2]). The operator \( N': H^i_{\overline{\tau}}(S^{m-1}/\mathcal{G}) \to H^i_{\overline{\tau}}(\mathbb{R}^m/\mathcal{F}) \) defining (10) is \( N'([\beta]) = [g dt \wedge \chi^* \beta] \), where \( g \in C^\infty([0, \infty[ \) with \( g \equiv 1 \) on \([0, 1/4[ \), \( g \equiv 0 \) on \([3/4, 1[ \) and \( \int_0^1 g = 1 \) [30, Proposition 3.7.2].

Now, for \([\alpha] \in H^i_{\overline{\tau}}(S^{m-1}/\mathcal{G})\) and \([\beta] \in H^{m-w-1}_{\overline{\tau}}(S^{m-1}/\mathcal{G})\) we have

\[ P_{\mathbb{R}^m}([\alpha], [\beta]) = \int_{R_G \times [0, \infty[} g \chi^* \alpha \wedge dt \wedge \chi^* \beta \wedge \chi^* \eta = \left( \int_{R_G} \alpha \wedge \beta \wedge \eta \right) \left( \int_0^1 g dt \right) = P_{S^{m-1}}([\alpha], [\beta]). \]

Third step: General case.

Let us suppose that depth \( S_{K,M} > 0 \). Since the family \( \{M \setminus S_{\min, T_{\min}}\} \) is a basic covering of \( M \) and then we get: \( \Psi(M \setminus S_{\min}), \Psi(T_{\min}), \Psi(T_{\min}) \setminus S_{\min} \Rightarrow \Psi(M) \) (cf. (8)). The depth of the restriction to \( M \setminus S_{\min} \) and \( T_{\min} \setminus S_{\min} \) is strictly smaller than depth \( S_{T_{\min}} \). So, we get \( \Psi(T_{\min}) \Rightarrow \Psi(M) \). It remains to prove \( \Psi(T_{\min}) \). This result comes from Proposition 8.3 and the Second step.

\[ \blacklozenge \]

10. Appendix: Molino’s blow up [17],[31, Section 4.2].
We consider in this Appendix a manifold \( M \) endowed with a Killing foliation \( \mathcal{F} \) induced by an orientable action \( \Phi: G \times M \to M, G \) connected. We fix a tamer \( K \). The compact Lie group \(^5\) \( K \) acts smoothly on \( M \) defining the isotropy type stratification \( S_{K,M} \). It is given by the equivalence relation

\[
x \sim y \Leftrightarrow K_x \text{ is conjugate to } K_y.
\]

Since \( G \) is a normal subgroup of \( K \) then \( x \sim y \) implies that \( G_x \) is conjugate to \( G_y \) on \( K \) and therefore \( \dim(G_x)_0 = \dim(G_y)_0 \). So, condition depth \( S_{K,M} = 0 \) implies that the foliation \( \mathcal{F} \) is regular foliation and then \( M = R_{\mathcal{F}} \).

The Molino’s blow up is a technical tool we use to desingularize the Killing foliation \( \mathcal{F} \) when depth \( S_{K,M} > 0 \). It is a continuous map \( L: (\hat{M}, \hat{\mathcal{F}}) \to (M, \mathcal{F}) \) verifying:

- the foliation \( \hat{\mathcal{F}} \) is a Killing foliation defined by an orientable action \( \hat{\Phi}: G \times \hat{M} \to \hat{M} \) having also \( K \) as a tamer group,
- depth \( S_{K,\hat{M}} < \text{depth} S_{K,M} \),
- the map \( L \) is \( G \)-equivariant,
- the restriction \( L: \hat{M} \setminus L^{-1}(S_{min}) \to M \setminus S_{min} \), where \( S_{min} \) is the union of closed strata of depth \( S_{K,M} \), is a \( K \)-equivariant smooth trivial 2-covering,
- depth \( S_{K,M \setminus S_{min}} < \text{depth} S_{K,M} \),
- there exists a commutative diagram

\[
\begin{array}{ccc}
  D_{min} \times 1, 1 & \to & \hat{M} \\
\downarrow \nabla_{min} & & \downarrow L \\
  T_{min} & \to & M
\end{array}
\]

where

+ \( T_{min} \) is a \( K \)-equivariant tubular neighborhood of \( S_{min} \),
+ \( D_{min} \subset T_{min} \) are the points whose distance to \( S_{min} \) is \( 1/2 \),
+ \( \nabla_{min}(x,t) = 2|t| \cdot x \).

- depth \( S_{K,D_{min}} < \text{depth} S_{K,M} \).

Hau amaiera da.

References

[1] J. A. Álvarez López. The basic component of the mean curvature of Riemannian foliations. *Ann. Global Anal. Geom.*, 10(2):179–194, 1992.

[2] J. A. Álvarez López. On the first secondary invariant of Molino’s central sheaf. *Ann. Polon. Math.*, 64:253–265, 1996.

\(^5\)For notions related with compact Lie group actions, we refer the reader to [5].
[3] W. M. Boothby. An introduction to differentiable manifolds and Riemannian geometry, volume 120 of Pure and Applied Mathematics. Academic Press, Inc., Orlando, FL, second edition, 1986.

[4] R. Bott and L. W. Tu. Differential forms in algebraic topology, volume 82 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982.

[5] G. E. Bredon. Introduction to compact transformation groups. Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46.

[6] G. E. Bredon. Topology and geometry, volume 139 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.

[7] J.-L. Brylinski. Equivariant intersection cohomology. In Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989), volume 139 of Contemp. Math., pages 5–32. Amer. Math. Soc., Providence, RI, 1992.

[8] Y. Carrière. Flots riemanniens. Astérisque, (116):31–52, 1984. Transversal structure of foliations (Toulouse, 1982).

[9] A. El Kacimi-Alaoui and G. Hector. Décomposition de Hodge basique pour un feuilletage riemannien. Ann. Inst. Fourier (Grenoble), 36(3):207–227, 1986.

[10] M. Goresky and R. MacPherson. Intersection homology theory. Topology, 19(2):135–162, 1980.

[11] M. Goresky and R. MacPherson. Intersection homology. II. Invent. Math., 72(1):77–129, 1983.

[12] F. W. Kamber and P. Tondeur. Duality theorems for foliations. Astérisque, 116:108–116, 1984. Transversal structure of foliations (Toulouse, 1982).

[13] S. Kobayashi. Transformation groups in differential geometry. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.

[14] X. Masa. Duality and minimality in Riemannian foliations. Comment. Math. Helv., 67(1):17–27, 1992.

[15] V. Miquel and R. Wolak. Minimal singular Riemannian foliations. C. R. Math. Acad. Sci. Paris, 342(1):33–36, 2006.

[16] P. Molino. Désingularisation des feuilletages Riemanniens. Amer. J. Math., 106(5):1091–1106, 1984.

[17] P. Molino. Feuilletages riemanniens réguliers et singuliers. In Géométrie différentielle (Paris, 1986), volume 33 of Travaux en Cours, pages 173–201. Hermann, Paris, 1988.

[18] P. Molino. Riemannian foliations, volume 73 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1988. Translated from the French by Grant Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu.

[19] P. Molino. Orbit-like foliations. In Geometric study of foliations (Tokyo, 1993), pages 97–119. World Sci. Publ., River Edge, NJ, 1994.

[20] P. Molino and V. Sergiescu. Deux remarques sur les flots riemanniens. Manuscripta Math., 51(1-3):145–161, 1985.

[21] J.-P. Ortega and T. S. Ratiu. Momentum maps and Hamiltonian reduction, volume 222 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2004.
[22] D. Poguntke. Dense Lie group homomorphisms. *J. Algebra*, 169(2):625–647, 1994.

[23] B. L. Reinhart. Foliated manifolds with bundle-like metrics. *Ann. of Math. (2)*, 69:119–132, 1959.

[24] J. Royo Prieto. *Estudio cohomológo de Flujos Riemannianos*. PhD thesis, University of the Basque Country UPV/EHU, http://www.ehu.eus/joseroyo/pdf/tesis.pdf, 10 2003.

[25] J. Royo Prieto, M. Saralegi-Aranguren, and R. Wolak. Top-dimensional group of the basic intersection cohomology for singular Riemannian foliations. *Bull. Pol. Acad. Sci. Math.*, 53(4):429–440, 2005.

[26] J. Royo Prieto, M. Saralegi-Aranguren, and R. Wolak. Tautness for Riemannian foliations on non-compact manifolds. *Manuscripta Math.*, 126(2):177–200, 2008.

[27] J. Royo Prieto, M. Saralegi-Aranguren, and R. Wolak. Cohomological tautness for Riemannian foliations. *Russ. J. Math. Phys.*, 16(3):450–466, 2009.

[28] M. Saralegi. Homological properties of stratified spaces. *Illinois J. Math.*, 38(1):47–70, 1994.

[29] M. Saralegi-Aranguren and R. Wolak. Basic intersection cohomology of conical fibrations. *Mat. Zametki*, 77(2):235–257, 2005.

[30] M. Saralegi-Aranguren and R. Wolak. The BIC of a singular foliation defined by an abelian group of isometries. *Ann. Polon. Math.*, 89(3):203–246, 2006.

[31] M. Saralegi-Aranguren and R. Wolak. Finiteness of the basic intersection cohomology of a Killing foliation. *Math. Z.*, 272(1-2):443–457, 2012.

[32] V. Sergiescu. Cohomologie basique et dualité des feuilletages riemanniens. *Ann. Inst. Fourier (Grenoble)*, 35(3):137–158, 1985.

[33] P. Tondeur. *Geometry of foliations*, volume 90 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1997.