Flux-tubes in confining gauge theories with gravitational dual

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Abstract

The emergence of flux-tubes as the distance is increased between a quark and an antiquark is explored in a three-dimensional confining gauge theory using the gauge/gravity duality. We delineate the shape of the flux-tube corresponding to the classical open string in the bulk and further explore the fluctuation in its thickness induced by the fluctuation of the open string along the radial direction in the bulk. The relationship between the intrinsic shape of a flux-tube and its effect on the heavy quark potential is also discussed. Python notebooks used for the numerical calculations are included as supplemental material.
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1 Introduction

Various investigations of confining theories like QCD suggest that a fluctuating flux-tube is formed between a quark and an anti-quark [1, 2, 3]. We do not completely understand the physics behind the formation of these flux-tubes, but we expect on the basis of models like the dual-superconductor model of confinement [4, 5, 6] and on fairly general grounds [7, 8], that these flux-tubes are not one-dimensional objects like strings but have a finite intrinsic thickness. Therefore it would be nice to delineate these flux-tubes and measure their intrinsic thickness devoid of the quantum broadening, and hopefully obtain a better understanding of the phenomenon of confinement. One might think that this is the kind of calculation that can be easily done using the techniques of lattice gauge theories, unfortunately that is not the case. For the expectation value of any gauge invariant operator, like the action density, which we could use to delineate the shape of the flux-tube would not distinguish between the contribution due to quantum fluctuations of the flux-tube and the contribution due to the flux-tube itself, and it requires a considerable effort to disentangle the two effects [9, 10, 11, 12]. We will refer to the profile of the flux distribution delineated by a gauge invariant operator as the flux-profile.

It is here that the gauge/gravity duality [13, 14, 15] can provide us with some additional guidance and intuition. For there are quantum field theories that exhibit confinement and also have a dual classical gravitational description [15, 16] (see [17] for a brief review of the gauge/gravity duality that is relevant for the present work.) In such theories the gravitational description of a pair of quark and an anti-quark is an open string living in a higher dimensional curved space time. This string starts from the quark and ends at the anti-quark. The open string is in a quantum state which is a superposition of different configurations, all terminating at the quark and the antiquark, and weighted by the action of the open-string (see for e.g [18, 19].) This string is also a source of dilaton field which, through the dictionary established in [14, 15], induces in the boundary theory a flux-tube connecting the quark and the anti-quark [20, 21, 22]. As a result for every configuration of the fluctuating open sting in the bulk there is a holographic projection in the form of a fluctuating flux-tube of varying thickness, the thickness of the flux-tube depending on the radial position of the corresponding open string in the bulk [23]. The flux-profile produced in the confining theory by a quark and an anti-quark can be thought of as superposition of these flux-tubes.

The main aim of the present work is to substantiate these remarks by explicitly calculating the shape of the flux-tube induced by some physically motivated configurations of the open string. We will do this for a confining theory in three dimensions which has a dual classical gravitational

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1 For a visual representation of the flux distribution in QCD see Visualizations of Quantum Chromodynamics
2 I would like to thank Gunnar Bali for pointing this out to me.
3 By quark we simply mean a source in the fundamental representation of the gauge group.
description [15]. The reason for restricting to three dimensions is simply that in three dimensions the required numerical calculations can be done with reasonable precision using a modest personal computer. All the techniques that we develop can be easily extended to the the four dimensional case, but would correspondingly make a larger demand on computational resources.

The outline of the paper is as follow, in section (2) we review the relationship between the dilaton field sourced by an open string and the resulting flux-tube in the boundary theory. Some aspects of gauge/gravity duality that we will use are reviewed in the appendix (A). In section (3) we develop the framework for numerically calculating the shape of a flux-tube using gauge/gravity duality. For the model confining theory that we are studying, namely the Witten’s model, we can contrive a situation where the open string path-integral is dominated by the classical solution. With this as a motivation, in section (4) we study the flux-tubes which are holographic projections of classical open strings. One of the salient features of the gravitational description of a flux-tube is that the intrinsic thickness of the flux-tube fluctuates due to the fluctuation of the open string in the radial direction, we explore this in section (4.4). In studying confining theories using gauge/gravity duality it is natural to ask if there are any insights that can be carried over to QCD, since for QCD there is no known dual gravitational description. We discuss this question in the final section (5) of the paper in the context of effective string descriptions of flux-tubes and the heavy quark potential.

2 From dilaton fields to flux-profile

2.1 The confining theory

The model confining theory that we will study is $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^3 \times S^1$. This is an interacting theory of gluons, four Weyl fermions and six scalar fields (see for e.g. [17, 24, 25, 26]). All these fields are charged, they belong to the adjoint representation of $SU(N)$, and thus they interact with each other and themselves. The gravitational dual of this theory is given by Witten’s model [15] (We review the necessary aspects of this model in appendix A.1.) This confining three dimensional theory is not the large $N$ limit of QCD in three dimensions, for example the glue-balls in our confining theory are not made purely of gluons but will also get contributions from the fermions and the scalars. This is because the confinement scale is comparable to the radius $r_y$ of $S^1$ (see for e.g. [17]). For convenience we will refer to our confining theory as $\text{QCD}^\text{IR}_3$.

To study the formation of flux-tubes we introduce a pair of quark and an anti-quark into the ground state of this theory. For the purpose of probing the ground state of the theory we can restrict to massive quarks,

$$m_q >> r_y^{-1},$$

(2.1)
and ignore the translational and spin degrees of freedom. The fluctuating color degrees of freedom of the original ground state, $|\Omega\rangle$, interact with the fluctuating color degrees of freedom of the test quarks, and the ground-state get modified [27]. We will refer to the new ground state as

$$|\Omega >_{q\bar q}. \quad (2.2)$$

One possible way of delineating the flux profile is to plot the expectation value of $\text{Tr}F^2$, where $F$ is the Yang-Mills field strength tensor, as a function of the two spatial coordinates, after averaging over the compact $y$ direction,

$$\bar{x} = (x_1, x_2). \quad (2.3)$$

In the context of gauge/gravity duality a more convenient operator is the generalization of $\text{Tr}F^2$ to the supersymmetric case, which is the Lagrangian of the $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^3 \times S^1$. This operator can be written schematically as (see for e.g. [26] for the details)

$$O_{F^2}(x) = \frac{1}{4g_Y^2} \text{Tr} \left\{ F^2 + \text{scalar} + \text{fermion} \right\}, \quad (2.4)$$

where scalar and fermion represent the kinetic energy term for the scalar and the fermionic degrees of freedom. With this in mind we define the profile of the flux-tube as

$$P(\bar{x}) = \langle \Omega | \hat{O}_{F^2}(\bar{x}) | \Omega >_{q\bar q}, \quad (2.5)$$

where we have disregard the dependence of this expectation value on the $y$ coordinate of the compact direction, or more accurately we will effectively average over the compact $y$ direction.

### 2.2 Flux profile using the gauge/gravity duality

We would like to calculate the flux-profile as defined by (2.5) using the gauge/gravity duality. In the classical gravitational description, the boundary state $|\Omega >_{q\bar q}$ is described by a quantum open string living in a specific curved space-time (see for e.g [17].) A given configuration of an open string is a source of a dilaton field which induces a non-zero expectation value for the operator $\hat{O}_{F^2}$ in the boundary theory [20, 21, 22, 28]. Let us first consider the case when the open string wave function is dominated by the classical configuration (for the model that we are considering this corresponds to the situation where the inverse string tension $l_s$ is much smaller than the size of the compact direction $r_y$.) We will refer to the flux-tube induced by the classical open string as the classical flux-tube. In this section we will review the formalism for calculating classical flux-tube and state a formalism to calculate the profile due to the full open string wave function.
2.2.1 Open string as a source of dilaton field

To see how the open string that we have introduced in the bulk generates a dilaton field, we consider the Nambu-Goto action for the string

\[ S_{NG} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{g}, \tag{2.6} \]

where \( g \) is the determinant of the two-dimensional metric induced by the five-dimensional bulk metric, \( G_{mn} \). The metric \( G_{mn} \) experienced by the string is often referred to as the metric in the string frame while the bulk metric (A.12) that provides the gravitation description of QCD_{IR}^{5} is the metric \( G_{mn}^{E} \) in the Einstein frame (see for e.g. \[29\])

\[ ds^2 = G_{mn}^{E} dx^m dx^n = \frac{R^2}{z^2} \left( dt_E^2 + d\bar{x}^2 + (1 - \frac{z^4}{z_0^4}) dy^2 + (1 - \frac{z^4}{z_0^4})^{-1} dz^2 \right), \tag{2.7} \]

and the two are related by

\[ G_{mn}^{E} = e^{-\frac{\Phi}{2}} G_{mn}, \tag{2.8} \]

where \( \Phi (z, t_E, \bar{x}, y) \) is the dilaton field and \( z \) is the coordinate along the fifth dimension, which we will often refer to as the radial direction.

The Nambu-Goto action when written in the Einstein frame becomes

\[ S_{NG} = \frac{1}{2\pi\alpha'} \int d^2\sigma e^{\frac{\Phi}{2}} \sqrt{g_E}, \tag{2.9} \]

where \( g_E \) is the determinant of the world-sheet metric induced by \( G_{MN}^{E} \). The complete dynamics of the dilaton field in the presence of an open string is given by the action

\[ S[\Phi] = S_{dilaton} + S_{NG} \tag{2.10} \]

with \( S_{dilaton} \) being the action of a free massless scalar field in the bulk. This action, after Kaluza-Kline reduction over \( S_5 \), takes the following form

\[ S_{dilaton} = \frac{R^5 \Omega_5}{4\kappa^2} \int d^5x \sqrt{g_E} G_{mn}^{E} \partial_m \Phi \partial_n \Phi, \tag{2.11} \]

where \( R \) is the radius of \( S_5 \) and \( \Omega_5 \) is the volume of a unit sphere in five dimensions. In the above equation we have defined

\[ \kappa^2 = 4\pi G_{10}. \tag{2.12} \]

The complete linearized action for the dilaton field in the presence of an open string is then given
by
\[ S_B[Φ] = \frac{1}{4\kappa_5^2} \int d^5x \sqrt{G_E} G_E^{mn} \partial_m Φ \partial_n Φ + \frac{1}{2\pi α'} \int d^2σ \sqrt{g_E} (X) + \frac{1}{4\pi α'} \int d^2σΦ [X] \sqrt{g_E} (X), \] (2.13)

where
\[ \kappa_5^2 = \frac{κ^2}{R^5 Ω_5} \] (2.14)

and \( X(σ) \) are the coordinates of the string world-sheet.

Let us next write the dilaton action (2.13) for a static field configuration and after doing the Kaluza-Kline reduction over the compact \( y \) direction. The background five-dimensional metric is the euclidean version of (A.12)
\[ G_E^{mn} := \text{diag} \left\{ \left( \frac{R}{z} \right)^2 (1, 1, 1, f(z), \frac{1}{f(z)}) \right\}, \] (2.15)

where we have defined
\[ f(z) = \left( 1 - \frac{z^4}{z_0^4} \right). \] (2.16)

From this we obtain
\[ G_E^{mn} := \text{diag} \left\{ \left( \frac{z}{R} \right)^2 (1, 1, 1, \frac{1}{f(z)}, f(z)) \right\}, \] (2.17)

and
\[ \det (G_E^{mn}) = G_E = \left( \frac{R}{z} \right)^{10}. \] (2.18)

The terms in the action involving dilaton field then takes the following form:
\[ S_B[Φ] = \frac{2πr_y}{4κ_5^2} \int dt d^2x dz \left\{ \left( \frac{R}{z} \right)^3 \left( (∇_i Φ)^2 + f(z) (\partial_z Φ)^2 \right) \right\} + \frac{1}{4\pi α'} \int d^2σΦ [X(σ)] \sqrt{g_E} (X(σ)), \] (2.19)

we simplify the notations by using \( 2r_y = z_0 \) and \( α' = \frac{l_s^2}{s} \)
\[ S_B[Φ] = \pi z_0 R^3 \frac{1}{4κ_5^2} \int dt d^2x dz \left\{ \left( \frac{1}{z} \right)^3 \left( (∇_i Φ)^2 + f(z) (\partial_z Φ)^2 \right) \right\} + \frac{1}{4\pi l_s^2} \int d^2σΦ [X(σ)] \sqrt{g_E} (X(σ)). \] (2.20)

2.2.2 Dilaton field due to the classical open string

The dilaton field is sourced by an open string \( X(σ) \), which we take to be lying in the \((x_1, z)\) plane. For numerical purposes it is convenient to parameterize the open string, see Fig. (2.1), as
Figure 2.1: Confining Strings for different values of $z_m = Z(0)$

$$X(\sigma_1, \sigma_2) = X(t, x_1) := (t, x_1, 0, 0, Z(x_1)), \quad (2.21)$$

with this parameterization the action becomes

$$S_B[\Phi] = \frac{\pi z_0 R^3}{4\kappa_5^2} \int dtd^2\bar{x}dz \left\{ \left( \frac{1}{z} \right)^3 \left( (\nabla_i \Phi)^2 + f(z) (\partial_z \Phi)^2 \right) \right\} + \frac{1}{4\pi l_s^2} \int dtdx_1 \Phi [X(x_1)] \sqrt{g_E(X(x_1))}. \quad (2.22)$$

For a classical open string, using (A.37), the source term simplifies to

$$S_B[\Phi] = \frac{\pi z_0 R^3}{4\kappa_5^2} \int dtd^2\bar{x}dz \left\{ \left( \frac{1}{z} \right)^3 \left( (\nabla_i \Phi)^2 + f(z) (\partial_z \Phi)^2 \right) \right\} + \frac{z_m^2 R^2}{4\pi l_s^2} \int dtdx_1 \Phi [X(x_1)] \frac{1}{Z_c^4(x_1)}, \quad (2.23)$$

where

$$z_m = Z_c(0). \quad (2.24)$$

We rescale the dilaton field by

$$\bar{\Phi} = \left( \frac{\pi R^3}{4\kappa_5^2} \right) \Phi, \quad (2.25)$$
and use (A.18) to write the action as

\[ S_B[\Phi] = \left( \frac{4\kappa^2}{\pi R^3} \right) z_0 \int dt d^2\bar{x} dz \left\{ \left( \frac{1}{z} \right)^3 \left( (\nabla_i \Phi)^2 + f(z) (\partial_x \Phi)^2 \right) \right\} + \frac{z^2 m^2}{4\pi} \int dtdx_1 \Phi [X(x_1)] \frac{1}{Z_c^4(x_1)}. \]  

(2.26)

It is natural to measure the spatial coordinates in the units of \( z_0 \),

\[ x_i \to x_i z_0; \quad z \to zz_0; \quad Z_c \to Z_c z_0; \quad z_m \to z_m z_0. \]  

(2.28)

after absorbing an overall scale factor of \( \left( \frac{4\kappa^2}{\pi R^3} \right) \frac{1}{z_0} \) we write the dilaton action as

\[ S_B[\Phi] = \int dtd^2\bar{x} dz \left\{ \left( \frac{1}{z} \right)^3 \left( (\nabla_i \Phi)^2 + f(z) (\partial_x \Phi)^2 \right) \right\} + \frac{z^2 m^2}{4\pi} \int dtdx_1 \Phi [X(x_1)] \frac{1}{Z_c^4(x_1)}. \]  

(2.29)

It will also be useful to write the bulk action as,

\[ S_B[\Phi] = \int dtd^2\bar{x} dz \left\{ \left( \frac{1}{z} \right)^3 \left( (\nabla_i \Phi)^2 + f(z) (\partial_x \Phi)^2 \right) \right\} + \Phi J \]  

(2.30)

where

\[ J = \frac{z^2 m^2 \lambda^{1/2}}{4\pi} \delta(x_2) \delta(z - Z_c(x_1)) \frac{1}{z^4}. \]  

(2.31)

### 2.3 Classical flux-tube in QCD\(_3\)\(^{\text{IR}}\)

The partition function for QCD\(_3\)\(^{\text{IR}}\) in the presence of a \( q\bar{q} \) pair and a source term for \( \hat{O}_{F2} \) can be schematically written as

\[ Z_{\text{QCD3}}[\phi] = \int [dA] \exp \left\{ -S_{\text{QCD3}}^{\phi\bar{q}} [A] + 2\pi r_y \int dtd^2\bar{x} \hat{O}_{F2} (t, \bar{x}) \phi(t, \bar{x}) \right\}. \]  

(2.32)

According to the dictionary for the gauge/gravity duality the source term \( \phi(\bar{x}) \) is related to the asymptotic value of the dilaton field, again after the Kaluza-Klein reduction over the compact \( y \).
direction,
\[
\phi(t, \bar{x}) = \lim_{z \to 0} \Phi_c (z, t, \bar{x}) = \lim_{z \to 0} (\Phi_H (z, t, \bar{x}) + \Phi_s (z, t, \bar{x})) ,
\]  
(2.33)
\[
\lim_{z \to 0} \Phi_H (z, t, \bar{x}) = \phi (t, \bar{x}) ,
\]  
(2.34)
\[
\lim_{z \to 0} \Phi_s (z, t, \bar{x}) = 0 ,
\]  
(2.35)

where \( \Phi_H (z, t, \bar{x}) \) is the solution of the homogenous dilaton field equation with the boundary condition given by (2.34), while \( \Phi_s (z, \bar{x}) \) is the classical dilaton field produced by the fundamental open string in the bulk. \( \Phi_H \) is the so called non-normalizable mode while \( \Phi_s \) is the normalizable mode. It will be convenient to write the source term in terms of the rescaled bulk field (2.25) and to measure the spatial coordinates \( \bar{x} \) at the boundary in the units of \( z_0 = 1 \) (2.28), but we will continue to measure \( \hat{O}_{F^2} \) in normal units in which \( z_0 \neq 1 \),

\[
Z_{\text{QCD}^3} [\phi] = \int [dA] \exp \left\{ -S_{\text{QCD}^3} [A] + \pi z_0 \left( \frac{4 \kappa_5^2}{\pi R^3} \right) \frac{z_0^2}{a} \int dt d^2 \bar{x} \hat{O}_{F^2} (t, \bar{x}) \phi (t, \bar{x}) \right\} .
\]  
(2.36)

Using the dictionary provided by the gauge/gravity duality in the limit \( N \to \infty \) and \( \lambda \gg 1 \)

\[
- \frac{1}{z_0^4} \left( \frac{\delta \tilde{S}}{\delta \Phi_s} \right)_{z=0} = \langle \Omega| \hat{O}_{F^2} (t, \bar{x}) |\Omega \rangle >_q .
\]  
(2.37)

Evaluating the variation about the classical solution we find that the only non-vanishing contribution comes from

\[
\delta S [\Phi_c] = \int d^3 x \left[ \frac{2 f(z)}{z^3} \left( \partial_2 \Phi_s \right) \delta \Phi \right]_{z=z_0} .
\]  
(2.38)

Since the bulk ends smoothly at \( z_0 \), we have the following boundary condition at \( z_0 \)

\[
\left( \frac{\partial \Phi}{\partial z} \right)_{z=z_0} = 0 ,
\]  
(2.39)
we obtain

\[
\langle \Omega| \hat{O}_{F^2} (t, \bar{x}) |\Omega \rangle >_{q_\sigma} = \lim_{a \to 0} \left( - \frac{1}{z_0^4} \frac{2 f(a)}{a^3} \left( \partial_2 \Phi_s \right)_{z=a} \right) ,
\]  
(2.40)

where the superscript \( X (\sigma) \) on the left hand side is a reminder that this is the expectation value due to a particular string configuration, \( X (\sigma) \), in our case the classical string configuration.
2.4 Prescription for calculating flux-profile in QCD$_3^{\text{IR}}$

Up till now we have considered a classical open string joining the quark and the antiquark as the source of the dilaton field, which in turn via (2.40), describes a corresponding flux-tube in the boundary theory. We will refer to the flux-tube induced by the classical open string configuration as the classical flux-tube. As was noted in the introduction, the flux-profile in the boundary theory can be thought of as a superposition of flux-tubes induced by the various string configurations making up the open string wave-function. We can formally incorporate the fluctuations of the open string by writing the partition function for the dilaton field as

\[ Z_{\text{Gravity}}[\phi(\bar{x})] = \int [dX] \exp \{-S_{NG}[X]\} \int [d\Phi] \exp \{-S_B[\Phi;X]\}, \quad (2.41) \]

\[ = \int [dX] \exp \{-S_{NG}[X]\} \exp \{-S_B[\Phi_c[X]\}, \quad (2.42) \]

where \( X(\sigma) \) are the open string world sheets corresponding to strings that start at a quark and end at an antiquark, \( \Phi \) represent the dilaton configurations that satisfy the boundary conditions given by (2.33,2.34,2.35), \( S_{NG} \) is the Nambu-Goto action for an open string (2.6) and \( S_B[\Phi;X] \) is the dilaton action in the presence of an open string (2.29). The functional integral over the dilaton field has been approximated by its classical value for \( \lambda >> 1 \) and \( N \to \infty \). Formally the flux-profile in the boundary theory can be written as

\[ \langle \Omega|\hat{O}_{F^2}(t,\bar{\bar{x}})|\Omega >_{\bar{q}\bar{q}} = \int [dX] \exp \{-S_{NG}[X]\} \left( -\frac{1}{z_0^4} \left( \frac{\delta S}{\delta \Phi[X]} \right)_{\Phi_c(z=0,\bar{x})} \right) \]

\[ = \int [dX] \exp \{-S_{NG}[X]\} < \Omega|\hat{O}_{F^2}(t,\bar{\bar{x}})|\Omega >_{\bar{q}\bar{q}}^{X(\sigma)} \quad (2.43) \]

where

\[ Z_{NG} = \int [dX] \exp \{-S_{NG}[X]\}, \quad (2.44) \]

is the partition function of the Nambu-Goto string.

We will not make an attempt to evaluate the above functional integral but will try and get a qualitatively understanding by evaluating the shape of the flux-tubes for a class of fluctuating open strings.
3 Framework for numerical calculations

To calculate the profile of the flux-tube using (2.40) we need to solve for the classical dilaton field, \( \Phi_c(x, z) \), sourced by an open string. We will solve for the classical dilaton field numerically, to do so we need a discrete version of the dilaton field equation. A convenient way of doing that is by first discretizing the dilaton action [30, 31].

3.1 Discretized dilaton action

Since we are working with static fields therefore it is more convenient to discretize

\[
L[\Phi] = \int d^2\bar{x}dz \left\{ \left( \frac{1}{z} \right)^3 (\nabla_i \Phi)^2 + f(z) (\partial_z \Phi)^2 \right\} + \frac{z^2 m \lambda^{1/2}}{4\pi} \int dx_1 \Phi [X(x_1)] \frac{1}{Z^4_c(x_1)}. \tag{3.1}
\]

Consider first the “kinetic” part of the action

\[
L_1[\Phi] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_0^1 dz \left\{ A(z) \left( \frac{\partial \Phi}{\partial x_1} \right)^2 + A(z) \left( \frac{\partial \Phi}{\partial x_2} \right)^2 + B(z) (\partial_z \Phi)^2 \right\}, \tag{3.2}
\]

where we have defined

\[
A(z) = \frac{1}{z^3}; \quad B(z) = \frac{f(z)}{z^3} = \frac{1 - z^4}{z^3}. \tag{3.3}
\]

We put the system in a three-dimensional box:

\[
-\frac{L_1}{2} \leq x_1 \leq \frac{L_1}{2}, \quad -\frac{L_2}{2} \leq x_2 \leq \frac{L_2}{2}, \quad 0 \leq z \leq 1, \tag{3.4}
\]

\( L_i \) acting as an infrared cut-off. Next we discretize the spatial coordinates by introducing an anisotropic lattice

\[
x_i = n_i b, \quad n_i \in (-N_i, N_i), \quad L_i = 2N_i b, \\
z = n_3 a, \quad n_3 \in (0, N_3), \quad 1 = N_3 a, \tag{3.5} \\
b = \sigma a, \quad \sigma > 0. \tag{3.6}
\]

Having an anisotropic lattice will be convenient for numerical solution of the dilaton field equation.
We denote the value of the dilaton field on the points of the lattice as
\[ \Phi(x_1, x_2, x_3) = \Phi(n_1b, n_2b, n_3a) = \Phi(n_1, n_2, n_3). \] (3.7)

Now we can discretize the action (3.2) by calculating the integrand at the centre of the elementary cubes
\[
L_{\text{lattice}}[\Phi] = \sum_{(n_1, n_2, n_3)} b^2 a \left\{ A (n_3 + 1/2) \left( \Phi(n_1 + 1, n_2, n_3) - \frac{\Phi(n_1, n_2, n_3)}{b} \right)^2 \\
+ A (n_3 + 1/2) \left( \Phi(n_1, n_2 + 1, n_3) - \frac{\Phi(n_1, n_2, n_3)}{b} \right)^2 \\
+ B (n_3 + 1/2) \left( \Phi(n_1, n_2, n_3 + 1) - \frac{\Phi(n_1, n_2, n_3)}{a} \right)^2 \right\}. \] (3.8)

In terms of the scale factor \( \sigma \), given by (3.6), the action becomes
\[
L_{\text{lattice}}[\Phi] = \sum_{(n_1, n_2, n_3)} a \left\{ A (n_3 + 1/2) (\Phi(n_1 + 1, n_2, n_3) - \Phi(n_1, n_2, n_3))^2 \\
+ A (n_3 + 1/2) (\Phi(n_1, n_2 + 1, n_3) - \Phi(n_1, n_2, n_3))^2 \\
+ \sigma^2 B (n_3 + 1/2) (\Phi(n_1, n_2, n_3 + 1) - \Phi(n_1, n_2, n_3))^2 \right\}. \] (3.9)

The source part of the action (3.1) for the classical open string is
\[
L_s = \frac{z^2 m \lambda^{1/2}}{4\pi} \int dx_1 \Phi(X(x_1)) \frac{1}{Z^4_c(x_1)}. \] (3.10)

and can be discretized as
\[
L_{\text{lattice}}[\Phi] = \frac{z^2 m \lambda^{1/2}}{4\pi} \sum_{n_1} \sigma a \left\{ \Phi(n_1, 0, Z_c(n_1)) \frac{1}{Z^4_c(n_1)} \right\}. \] (3.11)

### 3.2 The discrete field equation

We obtain the discrete field equation by minimizing the discrete Lagrangian
\[
\frac{\partial}{\partial \Phi(m_1, m_2, m_3)} \{ L_{\text{lattice}}^1 + L_{\text{lattice}}^s \} = 0, \] (3.12)
which leads to the following discrete field equation for the dilaton field

\[
(4A(m_3 + 1/2) + \sigma^2 B(m_3 - 1/2) + \sigma^2 B(m_3 + 1/2)) \Phi (m_1, m_2, m_3) \\
- A(m_3 + 1/2) (\Phi (m_1 - 1, m_2, m_3) + \Phi (m_1 + 1, m_2, m_3)) \\
- A(m_3 + 1/2) (\Phi (m_1, m_2 - 1, m_3) + \Phi (m_1, m_2 + 1, m_3)) \\
- (\sigma^2 B(m_3 - 1/2) \Phi (m_1, m_2, m_3 - 1) + \sigma^2 B(m_3 + 1/2) \Phi (m_1, m_2, m_3 + 1)) \\
+ \frac{\sigma_{\lambda}^2}{8\pi} \left( \delta (0, m_2) \delta (Z_c (m_1), m_3) \frac{1}{Z_c^4 (m_1)} \right) = 0
\] (3.13)

The dilaton field equation on the lattice, (3.13), has to be supplemented by the boundary conditions on our box (3.4,3.5). We are interested only in the dilaton field which is sourced by the open string connecting a quark and an anti-quark, for such field we expect

\[
\lim_{|\bar{x}| \to \infty} \Phi (\bar{x}, z) = 0.
\] (3.14)

On our lattice (3.5) this translates into

\[
\Phi (\pm N_1, n_2, n_3) = 0, \quad \Phi (n_1, \pm N_2, n_3) = 0.
\] (3.15)

The bulk smoothly ends at \( z_0 \) that results in

\[
\left( \frac{\partial \Phi}{\partial z} \right)_{z = z_0} = 0,
\]

which translates on our lattice as

\[
\Phi (\bar{n}, N_3) = \Phi (\bar{n}, N_3 - 1).
\] (3.16)

Next we have to impose the boundary condition at the conformal infinity \( z = 0 \) (2.35) which translates on the lattice as

\[
\Phi (\bar{n}, 0) = 0.
\] (3.17)

For convenience we collect the lattice boundary conditions.
\[ \Phi (\pm N_1, n_2, n_3) = 0 \]
\[ \Phi (n_1, \pm N_2, n_3) = 0 \]
\[ \Phi (\bar{n}, N_3) = \Phi (\bar{n}, N_3 - 1) \]
\[ \Phi (\bar{n}, 0) = 0 \]

### 3.3 Numerical definition of the flux-tube

With all these preliminaries in place, we can translate the formula (2.40) on to the lattice

\[ < \Omega | \hat{O}_{F^2} (\bar{n}) | \Omega >_{\bar{q}\bar{q}}^{x(\sigma)} = - \frac{1}{z_0^4} B (1) \left( \frac{\Phi (\bar{n}, 1) - \Phi (\bar{n}, 0)}{a} \right), \]  

(3.18)

further using the boundary condition (3.17) we get

\[ < \Omega | \hat{O}_{F^2} (\bar{n}; ) | \Omega >_{\bar{q}\bar{q}}^{x(\sigma)} = - \frac{1}{z_0^4} B (1) \left( \frac{\Phi (\bar{n}, 1)}{a} \right). \]  

(3.19)

### 4 Shapes of confining flux-tubes

Now we are ready to numerically explore the formation of the flux-tubes in our theory. We will consider the case when the inverse string tension \( l_s \) is much smaller than \( r_y \sim z_0 \) so that the flux-profile is dominated by the flux-tube corresponding to the classical open string configuration. In solving (3.13) the string-dilaton coupling

\[ g_{sd} = \frac{\lambda^{1/2}}{8\pi} \]  

(4.1)

appears only as a multiplying constant in the source term, therefore without any loss of generality we can solve the equation for an arbitrary value of \( g_{sd} \). The shape of the flux-tube will be independent of the value of \( g_{sd} \), the actual value of the action density will depend on \( g_{sd} \) but knowing its value for a given \( g_{sd} \) we can easily obtain the value for any other \( g_{sd} \) by simple rescaling. In numerically solving (3.13) to desired precision, one faces certain number of difficulties which we point out in the appendix (B) and there we also describe the strategies we have used to overcome them.
4.1 Emergence of a classical flux-tube

We will solve the dilaton equation (3.13) on an anisotropic lattice with the lattice constant along the bulk direction \( z \) equal to

\[
a = \frac{1.0}{128z_0}
\]

(4.2)

and with the lattice constant along the boundary directions \((x_1, x_2)\)

\[
b = 5a = 0.0390z_0.
\]

(4.3)

The size of the resulting lattice is

\[
V_{\text{lattice}} = 128a \times 128b \times 128b = z_0 \times 5z_0 \times 5z_0
\]

(4.4)

With such a lattice we can explore a maximum quark - antiquark distance \(L_{q\bar{q}} = 2.34z_0\), beyond that the boundary of the lattices start distorting the profile. For each value of \(L_{q\bar{q}}\) the equation (3.13) was solved using the full multi-grid algorithm till the norm of the residue (B.5) was of the order \(10^{-2}\) (see the accompanying python notebook, which were used to obtain the numerical results, for the details.) In delineating the shape of a flux-tube one faces the difficulty that the value of the action density diverges at the position of the quark and the anti-quark which completely masks the shape of the flux-tube. Since our interest is just to delineate and display the shape of the flux-tube we will normalize the flux-tube profile,

\[
P^{X(\sigma)}(x_1, x_2) = \langle \Omega | \hat{O}_{F2}(x_1, x_2) | \Omega \rangle_{q\bar{q}}^{X(\sigma)}
\]

(4.5)

in the following manner

If \(P^{X(\sigma)}(x_1, x_2) > P^{X(\sigma)}(x_{Lc}, x_{Tc})\) then \(P^{X(\sigma)}(x_1, x_2) = P^{X(\sigma)}(x_{Lc}, x_{Tc})\),

(4.6)

where \(P^{X(\sigma)}(x_{Lc}, x_{Tc})\) is the value of the flux-tube profile at the central point along the line joining the quark and the anti-quark when \(L_{q\bar{q}} = 2.34z_0\)\(^4\). We present the results in Figs. (4.1). It is important to note that our results for these flux-tube profiles are only qualitative in nature as the inter quark distance is comparable to the size \(r_y\) of the compact \(y\) direction and we are averaging dilaton field over the \(y\) direction. Even with this smearing of profile what we can glean is that a flux-tube starts appearing when the inter-quark distance is of the order of \(3z_0\).

\(^4\)A reader wishing to explore the profile in a finer detail can do so by changing the normalization condition (4.6) in the accompanying python notebook - 1.
4.2 Approximate shape of a long but finite classical flux-tube

As one increases the distance between a quark and an antiquark the corresponding open string moves closer and closer to $z_0$, asymptotically approaching $z_0 = 1$. To represent such an open string on a lattice requires that the lattice constant in the $z$ direction should tend to zero or equivalently the number of points along the $z$ direction tend to infinity, making the problem computationally prohibitive. We circumvent the problem approximately by replacing the open string by a rectangular source of dilaton whose top is placed at

$$z_{Top} = z_{0} - a,$$  \hspace{1cm} (4.7)

where $a$ is the lattice constant along the $z$ direction (see Fig. 4.2.)
In Fig (4.3) and Fig. (4.4) we exhibit the flux-tube obtained using a rectangular dilaton source on an anisotropic lattice with the lattice constant $a$ along $z$ direction is

$$a = \frac{1.0}{128}z_0$$  \hspace{1cm} (4.8)

and with lattice constant $b$ along the boundary directions $(x_1, x_2)$

$$b = 20.0a.$$  \hspace{1cm} (4.9)

The source term for rectangular open string configuration can be easily obtained using (2.29) and is noted in (A.3). As in the previous case, the dilaton field equation was solved using the full multigrid algorithm till the norm of the residue (B.5) was of the order $10^{-2}$. With in the approximations that we have made, one can clearly see formation of a flux-tube with an intrinsic thickness.
4.3 Intrinsic thickness of an infinitely long classical flux-tube

We will next consider the situation where the quark and the anti-quark are separated by a distance much greater than $r_y$, or formally when

$$L_{q\bar{q}} \to \infty. \quad (4.10)$$

Gravitationally such a situation is described by an infinitely long open string placed at $z = z_0$. An approximate analytic solution for this case was already obtained in [22].

On a lattice the profile of infinitely long open string is given by

$$Z_{\text{lattice}} (m_1) = Z_c (0) = z_m = a N_m. \quad (4.11)$$

The dilaton field sourced by such a long open string depends only on the transverse coordinate $x_2 = am_2$ and on the radial coordinate $z = am_3$,

$$Z^2_{\text{lattice}} (0) = 0. \quad (4.12)$$

We will solve the above equation numerically and use (3.19) to plot the intrinsic shape of the flux tube.

4.3.1 Numerical Results: Shape of a very long classical flux-tube

The transverse profile of a long classical flux-tube, $L_{q\bar{q}} \to \infty$, can be defined as

$$P_T^{X_c (\sigma)} (x_T) = \lim_{{L_{q\bar{q}} \to \infty}} < \Omega | \hat{O}_{F^2} (x_1 = 0; x_2 = x_T) | \Omega >_{q\bar{q}}^{X_c (\sigma)}. \quad (4.13)$$

If the flux-tube is formed because of the finite correlation-length of flux-lines, as suggested in [7], then we expect that the transverse profile should take the following form

$$P_T (x_T) = C_0 \exp \left( - \frac{|x_T|}{\xi_{FT}} \right) = C_0 \exp \left( - |x_T| M^{-1}_{\text{glueball}} \right), \quad (4.14)$$

where $C_0$ is a constant of mass dimension four and $\xi_{FT}$ is the correlation length of the flux-lines (see the discussion in section 2 of [7]). We will refer to $\xi_{FT}$ as the intrinsic thickness of the flux-
Figure 4.5: Dilaton field due to a infinitely long open string along the $x_1$ axis and the resulting confining flux-tube on the boundary.

Figure 4.6: Intrinsic shape of a infinitely long confining flux-tube and its fit to an exponential and Gaussian function

tube, this is in contrast to the additional widening of the flux-tube due to quantum fluctuations of the flux-tube, or in the gravitational description due to quantum fluctuations of the open string connecting $q$ and $\bar{q}$.

The results of numerical calculations are presented in Fig. (4.5) and more detailed description of fitting of the data with a Gaussian and an exponential function is provided in Fig. (4.6). Because of the two dimensional nature of the problem the dilaton field equation could be solved with great precision, norm of the residue (B.6) being of the order of $10^{-6}$.

These results indicate that there are two related length scale that describes the intrinsic shape of the confining flux-tube, one is the scale that sets the exponential decay away from the centre of the flux-tube, and other is the scale that describe near Gaussian behavior close to the centre of the flux-tube. It is interesting to note that existence of two length scales to describe a flux-tube is a characteristic of the dual-superconductor model of confinement in QCD (see for e.g. [10].)

In Table (1) we summarize the relationship between the thickness and length of a classical flux-tube.
| $L_{q\bar{q}}$ | Half width along the line $x_1 = L_{q\bar{q}}/2$ |
|-------------|-----------------------------------|
| 1.56        | 0.65                              |
| 1.95        | 0.61                              |
| 2.34        | 0.58                              |
| 16.00 rect. string | 0.57                      |
| $\infty$    | 0.57                              |

Table 1: Half width of the normalized profile in units of $z_0$ for different values of $L_{q\bar{q}}$.

### 4.4 Flux-tube with fluctuating thickness

From the very inception of the gauge/gravity duality it was recognized that the position of an object along the extra radial direction is related to the size of it’s holographic projection in the boundary theory [32]. For a confining theories this suggest that the size of the flux-tube must be related to the position of the corresponding open string in bulk. Further an open string with a fluctuating radial coordinate along the extra dimension will correspond to a flux-tube with varying thickness. We will verify that this is indeed the case for QCD$_3^{IR}$. We do so by considering an open rectangular string, like the one we considered in section (4.2), on which we superimpose random fluctuations along the $z$ direction and then calculate the profile of the corresponding flux-tube. We present the result in Fig(4.7).

![Figure 4.7: Static radially fluctuating open string and the corresponding flux-tube](image)

5 Discussion

It is natural to ask, is there any thing that we can learn about real QCD from the toy example considered in the present work. There are few ways in which the phenomenas that we have
explored can get reflected in QCD. Firstly QCD flux-tubes should have an intrinsic thickness which can fluctuate. These radial fluctuations should lead to an attractive Yukawa like potential [19]. Further, the shape of the flux-tubes (4.1,4.3) and their width (1), suggests that in an effective string description of QCD flux-tube the string tension should depend on the distance between the quark and the anti-quark and approaches its constant value only as that distance tends to infinity [33, 34]. This would imply that when the distance between the quark and the anti-quark is of the order of the confinement scale then there should be deviations from the heavy quark potential calculated using an infinitely long open string [35, 36]. For in this regime the open string in the gravitational description is not an infinitely long open string placed at the confinement scale but has to descent to the conformal boundary. In the boundary theory this is reflected in the fact that when the quark anti-quark distance is of the order of the confinement scale then the size of the blob of action density surrounding the quark and the antiquark is comparable to the distance between the quark and the anti-quark. A preliminary estimate of this effect was made in [37] but a more precise calculation and its comparison with a high precision lattice simulations like [38] should be revealing. More speculatively, it might be interesting to incorporate the shape of the flux-tubes and their radial fluctuations as suggested by the gauge/gravity duality in string models of fragmentations of hadrons [39] and in the holographic stringy description of hadrons [40].

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A Appendix: Review of the gauge/gravity duality

A.1 Review of Witten’s model

The model confining theory that we will study is the $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^3 \times S^1$ whose gravitational dual was provided in [15, 16] which we briefly review here.
A.1.1 $\mathcal{N} = 4$ Super Yang-Mills gauge theory

$\mathcal{N} = 4$ super Yang-Mills gauge theory, with the gauge group $SU(N)$ defined on a four-dimensional Minkowskian space-time is the prototype theory for which there is a dual description in terms of closed super-strings of type IIB propagating on $AdS_5 \times S_5$. The $\mathcal{N} = 4$ SYM theory is an interacting theory of gluons, four Weyl fermions and six scalar fields (see for e.g. [17, 24, 25, 26]). All these fields are charged, they belong to the adjoint representation of $SU(N)$, and thus they interact with each other and themselves. Though the interaction of gluons among themselves is exactly as in QCD but unlike QCD there are additional interactions of gluons with fermions and the scalars. The theory is completely characterized by the rank of the gauge group $N$ and the Yang-Mills coupling constant $g_{YM}$. Most importantly the coupling constant $g_{YM}$, unlike in QCD, does not run and is truly a constant. There is no intrinsic length scale in the theory. The Lagrangian of $\mathcal{N} = 4$ SYM theory can be written schematically as

$$\mathcal{L}(x) \equiv \mathcal{O}_{F^2}(x) = \frac{1}{4g_{YM}^2} \text{Tr} \{ F^2 + \text{scalar} + \text{fermion} \}, \quad (A.1)$$

where $F$ is the Yang-Mills field strength. Informally, we can think of $\hat{\mathcal{O}}_{F^2}$ as an operator that creates a scalar massless “glueball” made of gluons and its super-partners.

Since the interaction of gluons among themselves in $\mathcal{N} = 4$ super Yang-Mills is identical to the interaction of gluons with themselves in QCD, a natural question to ask is can gluon-fermion and gluon-scalar interaction be suppressed, or made irrelevant, but in a manner that the modified theory still has a useful dual gravitational description. One such possible modification was suggested by Witten [16] and the resulting theory, in its simplest implementation, is a confining theory in three space-time dimensions.

A.1.2 $\mathcal{N} = 4$ Super Yang-Mills gauge theory on $\mathbb{R}^{1+2} \times S^1$

The first step in Witten’s construction is to give fermion’s mass, if fermions have mass $m_\lambda$, then it is plausible that on length scales greater than $m_\lambda^{-1}$ their interactions with gluons will be irrelevant. To do so one defines $\mathcal{N} = 4$ SYM theory on a four dimensional Minkowskian spacetime with one of the spatial direction compactified to a circle. A point on such a space can be given following coordinates $(t, x^1, x^2, y)$, where

$$y \sim y + 2\pi r_y. \quad (A.2)$$

Further we impose periodic boundary conditions on the gauge fields, $A$, and the scalar field, $X$, while an anti-periodic boundary condition is imposed on the fermionic fields $\lambda^i$

$$\lambda^i(t, x^1, x^2, y) = -\lambda^i(t, x^1, x^2, y + 2\pi r_y); \quad i = 1 \cdots 4. \quad (A.3)$$
Now we analytically continue the theory to imaginary time and obtain $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^3 \times S^1$. The generating function of this theory is identical to the partition function of the $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^{1+3}$ at finite temperature, $\beta = 2\pi r_y$,

$$Z^{1+3}_{SYM} = \sum_n <n| \exp\{-\beta \hat{H}_{SYM}\}|n> = \int_{A,X_{PBC}} \int_{\lambda_{APBC}} \exp\{-S^E[A, \lambda, X]\}. \quad (A.4)$$

The Euclidean action in the above path-integral has no mass term for the gluons, fermions and the scalars but because of the anti-periodic boundary condition

$$\lambda (x, y) = \sum_{n=1/2, 3/2, \ldots} \lambda_n(x) \exp\left\{\frac{iny}{r_y}\right\} = -\lambda (x, y + 2\pi r_y), \quad (A.5)$$

we find that

$$k_y = \frac{n}{r_y}, \quad n = 1/2, 3/2, \ldots \quad (A.6)$$

which is to say there are no fermionic zero modes. In other words just because of the anti-periodic boundary condition fermions acquire mass of the order of $r_y^{-1}$. The bosonic fields satisfy a periodic boundary condition along the $y$ direction so in the action there is a zero mode, but for the scalar field there are quantum corrections which can give it a mass of the order of $g_{YM}^2 r_y^{-1}$, note these correction vanish in the limit $r_y \rightarrow \infty$. The gauge bosons don’t acquire mass, as boundary condition cannot change the number of degrees of freedom. The above statements can be summarized by saying that the compact $y$ direction breaks super-symmetry which leads to fermion acquiring mass at the tree-level and scalars acquiring mass at one-loop level, while the gluons remain massless because boundary condition does not break gauge invariance.

Next step in Witten’s construction is to consider the physics at length scales much greater than $r_y$. This physics can be described by an effective Euclidean theory in three dimensions consisting only of massless gluons.

### A.1.3 Gravity dual to $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R}^{1+3}$

$\mathcal{N} = 4$ SYM theory on four-dimensional Minkowskian spacetime is conjectured to be dual to type IIB string theory on $\text{AdS}_5 \times S^5$. The field theory is supposed to be living on the conformal boundary of $\text{AdS}_5$, and indeed the conformal boundary of $\text{AdS}_5$ is a four-dimensional Minkowskian spacetime (??). The $\mathcal{N} = 4$ SYM theory is characterized by two parameters $N$, the rank of the $SU(N)$ gauge group, and by the Yang-Mills coupling constant $g_{YM}$, while the type IIB string theory on $\text{AdS}_5 \times S^5$ is characterized by three parameters, the string tension, $\alpha'^{-1}$,

$$\alpha' = l_s^2, \quad (A.7)$$
the string coupling constant \( g_s \) and the radius of curvature of \( AdS_5 \), \( R \), which is also the radius of \( S^5 \). The relevant dimensionless parameters on the gravity side are \( g_s \) and \( R/l_s \). The duality requires that the dimensionless parameters of the IIB string theory are related to the dimensionless parameters of \( \mathcal{N} = 4 \) SYM theory as shown in box (A.1.6).

### A.1.4 Gravity dual to SYM on \( \mathbb{R}^3 \times S^1 \)

The gravitational dual of SYM theory on \( \mathbb{R}^3 \times S^1 \) must be formulated on a space-time whose conformal boundary is \( \mathbb{R}^3 \times S^1 \) [16]. Again we recall that the Euclidean generating function, vacuum to vacuum amplitude, of \( \mathcal{N} = 4 \) SYM on \( \mathbb{R}^3 \times S^1 \) with periodic boundary conditions for the bosonic fields and anti-periodic boundary condition for the fermionic field on \( S^1 \) of radius \( r_y \) is same as the partition function of \( \mathcal{N} = 4 \) SYM on \( \mathbb{R}^{1+3} \) at finite temperature

\[
T = \beta^{-1} = (2\pi r_y)^{-1}. \tag{A.8}
\]

To find the required gravitational dual to SYM on \( \mathbb{R}^3 \times S^1 \) we should ask what happens in \( AdS_5 \) spacetime when the dual boundary theory is in thermal equilibrium at some temperature \( T \). Image the boundary field theory on \( \mathbb{R} \times S^3 \), so that the theory lives on a finite box, but with no boundary, now bring \( S^3 \) in contact with a “heat-bath” at some non-zero temperature, \( \beta^{-1} \). So the theory on \( S^3 \) is in a mixed state, or in “thermal-vacuum” (see for e.g. [41]), the only purely geometrical objects to which can relate temperature are the black-holes, this suggests that the gravitational dual to finite temperature \( \mathcal{N} = 4 \) SYM in four-dimensions could be \( AdS_5 \) but with a black-hole, or black-brane, since the field theory is in translationally invariant state\(^5 \). Such a black-hole in \( AdS_5 \) is described by the metric

\[
ds^2 = \frac{R^2}{z^2} \left( -(1 - \frac{z^4}{z_0^4}) dt^2 + dx^2 + (1 - \frac{z^4}{z_0^4})^{-1} dz^2 \right). \tag{A.9}\]

the Hawking temperature corresponding to this metric is

\[
T^{-1} = \beta = \pi z_0. \tag{A.10}\]

Note that in the limit \( z \to 0 \) the metric (A.9) approaches \( AdS_5 \) metric,

\[
ds^2 = \frac{R^2}{z^2} \left( -dt^2 + dx^2 + dz^2 \right). \tag{A.11}\]

\(^5\)We are ignoring \( S^5 \) - why? Is it because the supersymmetry is broken so we don’t have to worry about the scalar fields in the boundary theory?
The duality implies that if we excite $\mathcal{N} = 4$ SYM in four-dimensional Minkowski spacetime to a thermal state then in the dual description we create a black-hole described by (A.9). The partition function of $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^{1+3}$, as we have noted before, is also given by the generating function of a Euclidean QFT $\mathcal{N} = 4$ SYM on $\mathbb{R}^3 \times S^1$ this theory then is dual to the Euclidean version of type IIB string theory in the background of the metric

$$ds^2_E = \frac{R^2}{z^2} \left( (1 - \frac{z^4}{z_0^4})dt_E^2 + dx_1^2 + dx_2^2 + dx_3^2 + (1 - \frac{z^4}{z_0^4})^{-1}dz^2 \right).$$

Now we go back to Lorentzian signature but not by analytically continuing $t_E$ rather we analytically continue one of the non-compact directions, say $x_3$, while $t_E$ is identified with the coordinate $y$ of the compact direction $S^1$. Finally we obtain a gravitational description of $\mathcal{N} = 4$ SYM on $\mathbb{R}^{1+2} \times S^1$ as a string theory on the spacetime

$$ds^2 = \frac{R^2}{z^2} \left( -dt^2 + d\bar{x}^2 + (1 - \frac{z^4}{z_0^4})dy^2 + (1 - \frac{z^4}{z_0^4})^{-1}dz^2 \right), \quad (A.12)$$

where

$$y \sim y + 2\pi r_y; \quad z_0 = 2r_y, \quad (A.13)$$

and

$$\bar{x} = (x_1, x_2). \quad (A.14)$$

### A.1.5 Gravitational description of a heavy quark anti-quark pair

Based on our preliminary discussion, the $\mathcal{N} = 4$ SYM on $\mathbb{R}^3 \times S^1$ is expected to be a confining theory, we will refer to it at as $\text{QCD}^{\text{IR}}_3$. This implies that it must have a mass-gap in its spectrum, and the Wilson-loop for a test charge in the fundamental representation must show area law, equivalently there must be a linear potential between a heavy quark and an anti-quark. If we introduce a pair of quark-antiquark into the vacuum of $\text{QCD}^{\text{IR}}_3$ then the quantum state of theory will get modified, there will be color electric field-lines emerging from the quark and ending on the antiquark. These field lines are expected to be collimated, forming a flux-tube. Our aim is to calculate the shape of this flux-tube using dual gravity description. For this first we need to find a way of introducing a quark-antiquark pair in the dual gravity description. The gravitational description of $\text{QCD}^{\text{IR}}_3$ is given by closed strings propagating in the confining background (A.12), the addition of a pair of test quark-antiquark in $\text{QCD}^{\text{IR}}_3$ can be incorporated in this description by introducing an open string whose endpoints terminate on the coordinates of the test pair at the conformal boundary [42, 43, 44]. The open strings in type IIB theory are introduced via D-branes. One introduces an open string which starts on one of the $N$ D-Branes, where $N$ is the number of
colors in the dual field theory, and stretches the other end of the open string to the radial infinity (presumably terminating on a flavor brane at \( r = \infty \)). From the point of view of the low energy excitation of \( N \) color branes, the starting point of the open string appears as a massive particle located where the open string starts and carries fundamental charge. More precisely it acts like a massive vector boson in the fundamental representation of \( SU(N) \) \(^{43}\) the single index of the fundamental representation corresponding to which of the \( N \) color brane does the open string start from, while the second index of the open string is a fixed \( U(1) \) index. Similarly one introduces an anti-quark by an open string of opposite orientation. These two open strings can lower their energy by forming a single open string whose end points are on the \( z = 0 \) plane.

This suggests our working hypothesis, namely the gravitational dual to the state \( |\Omega > | q \bar{q} \rangle \) is an open string fluctuating in the geometry given by (A.12). The end point of this open string are on the \( z = 0 \) plane. At this stage a natural question arises, since we have an infinitely long open string with finite string tension, will not such a string produce its own gravitational field and modify the geometry given by (A.12)? To answer this let us estimate the right hand side of the Einsteins’s field equation with a string as the source term, working in the units \( R = 1 \) (see A.1.6)

\[
G_{10} T_{ab} \sim G_{10} \frac{1}{l_s^2} \sim \frac{1}{N^2} \lambda^{1/2} \sim \frac{\sqrt{g_{YM}} N}{N^2} \sim \frac{g_s^{1/2}}{N^{3/2}}. \tag{A.15}
\]

if we consider the limit where \( g_s \) is finite and \( N \to \infty \) then we can indeed neglect the modification of A.12 by the presence of an open-string.

**A.1.6 Matching the parameters of \( \mathcal{N} = 4 \) SYM and IIB string theory on \( \text{AdS}_5 \times S^5 \)**

The duality requires that

\[
g_{YM}^2 = 4\pi g_s; \quad \left( \frac{R}{l_s} \right)^4 = g_{YM}^2 N = \lambda. \tag{A.16}
\]

Often it is convenient to trade \( g_s \) for the Newton’s constant in ten-dimensions, \( G_{10} \), using the relation

\[
16\pi G_{10} = (2\pi)^7 g_s^2 l_s^8, \tag{A.17}
\]

to obtain

\[
\frac{G_{10}}{R^8} = \frac{\pi^4}{2N^2}, \quad \left( \frac{R}{l_s} \right)^4 = N g_{YM}^2 = \lambda. \tag{A.18}
\]

(For a pedagogical explanation of the reasoning leading to the above relationships see, for e.g. \([17, 45]\).) Often it will be useful to work in the units where

\[
R = 1, \quad (A.19)
\]
then we have
\[ \alpha' = t_s^2 = \frac{1}{\sqrt{Ng_Y M}} = \frac{1}{\lambda^{1/2}}, \] (A.20)

and
\[ G_{10} = \frac{1}{N^2} \left( \frac{\pi^4}{2} \right). \] (A.21)

### A.2 Shape of the open string sourcing the dilaton field

The shape of the classical open string connecting the quark to the anti-quark can be obtained from
\[ S_{NG}[X (\sigma)] = \frac{1}{2\pi \alpha'} \int d^2 \sigma \sqrt{g_E (X)}. \] (A.22)

We parameterize the world-sheet as
\[ X (t, x) := (t, x_1 = x, x_2, y, Z (x)). \] (A.23)

Using the metric (2.15) we obtain the induced world-sheet metric for an open string lying in the \(x_1 - z\) plane as
\[ g_{ab} = \text{diag} \begin{Bmatrix} \left( \frac{R}{Z} \right)^2, \left( \frac{R}{Z} \right)^2 \left( 1 + \frac{1}{f (z)} \left( \frac{dZ}{d x_1} \right)^2 \right) \end{Bmatrix}, \] (A.24)

where
\[ f (z) = \left( 1 - \left( \frac{z}{z_0} \right)^4 \right). \] (A.25)

The parametrized Nambu-Goto action then takes the form
\[ S_{NG} [X] = \frac{R^2}{2\pi \alpha'} \int dt dx \left\{ \frac{1}{Z (x)^2} \left( 1 + \frac{1}{f (z)} \left( \frac{dZ}{d x_1} \right)^2 \right)^{1/2} \right\}. \] (A.26)

Noting that the action does not depend explicitly on \( t \) therefore for the time interval \((-T/2, T/2)\) it can be written as
\[ S_{NG}^0 = \int_{-L/2}^{L/2} dx_1 L (Z, Z'), \]

where
\[ L (Z, Z') = C_0 \frac{1}{Z^2} \left( 1 + \frac{1}{f (z)} \left( \frac{dZ}{d x_1} \right)^2 \right)^{1/2} \] (A.27)

and we have defined the constant
\[ C_0 = \frac{TR^2}{2\pi \alpha'}. \] (A.28)
describes a one-dimensional problem with $x_1$ playing the role of time parameter, since $L$ does not depend explicitly on $x_1$ there is a conserved quantity

$$H = P \frac{dZ}{dx_1} - L = PZ' - L, \quad (A.29)$$

where the momentum conjugate to $Z$ is given by

$$P = \frac{\partial L}{\partial \left( \frac{dZ}{dx_1} \right)} = \frac{\partial L}{\partial Z'}, \quad (A.30)$$

Using the above equations we obtain

$$H = \frac{-C_0}{Z^2 \left( 1 + \frac{1}{f(Z)} \left( \frac{dZ}{dx_1} \right)^2 \right)^{1/2}}, \quad (A.31)$$

which is conserved

$$\frac{dH}{dx_1} = 0. \quad (A.32)$$

By symmetry

$$\left( \frac{dZ}{dx_1} \right)_{x_1=0} = 0, \quad (A.33)$$

let

$$Z(0) = z_m. \quad (A.34)$$

Conservation of $H$ allows us to write

$$\frac{1}{Z^2 \left( 1 + \frac{1}{f(z)} \left( \frac{dZ}{dx_1} \right)^2 \right)^{1/2}} = \frac{1}{z_m^2}, \quad (A.35)$$

and we can obtain an equation for $Z(x_1)$

$$\frac{dZ}{dx_1} = \pm \sqrt{f(z) \left( \frac{z_m^4}{Z} \right)^4 - 1} \quad (A.36)$$

the $\pm$ signs corresponding to the left and the right part of the string that connects a quark at $-L/2$ to an anti-quark at $L/2$. Using above result we can calculate

$$\sqrt{g_E(Z_c)} = \frac{1}{Z(x)^2} \left( 1 + \frac{1}{f(z)} \left( \frac{dZ}{dx} \right)^2 \right)^{1/2} = R^2 \frac{z_m^2}{Z(x)^4} \quad (A.37)$$
Next consider the positive branch of (A.36), and noting that $Z(x_1)$ is an invertible function, we obtain
\begin{equation}
\frac{dX_1}{dz} = \left(f(z) \left(\left(\frac{z_m}{Z}\right)^4 - 1\right)^{-1/2}\right),
\end{equation}
(A.38)
which is more convenient for numerical calculations.

**A.3 Dilaton source term due to a rectangular string**

The dilaton string coupling for a static string configuration is given by
\begin{equation}
S_{ds}[X(\sigma)] = \frac{1}{4\pi l_s^2} \int dt d\sigma \Phi[X(\sigma)] \sqrt{g_E(X(\sigma))}.
\end{equation}
(A.39)

For a rectangular source of a dilaton field (4.2) we will parametrize the two sides of the rectangle by $\sigma = z$, and obtain
\begin{equation}
S_{\text{Rect. side}} = \frac{1}{4\pi} \int dt dz \Phi[X(z)] \frac{1}{z^2 (1 - z^4)^{1/2}},
\end{equation}
(A.40)
with
\begin{equation}
X(\sigma) := (t, x_{1q}, 0, 0, z).
\end{equation}
(A.41)

For the top of the rectangle we use $\sigma = x_1$ as the parameter,
\begin{equation}
S_{\text{Rect. top}} = \frac{1}{4\pi} \int dt dx_1 \Phi[X(x_1)] \frac{1}{Z^2} \left(1 + \frac{1}{(1 - Z^4)} \left(\frac{dZ}{dx_1}\right)^2\right)^{1/2},
\end{equation}
(A.42)
where the string profile is given by
\begin{equation}
X(\sigma) := (t, x_1, 0, 0, Z(x_1)).
\end{equation}
(A.43)

**B Appendix: Numerical solution of the dilaton field equation**

**B.1 Relaxation Algorithm**

**B.1.1 Three dimensional case**

The discrete version of the dilaton field equation (3.13) can be solved using relaxation algorithm [31]
\[ \Phi(m_1, m_2, m_3) \rightarrow (1 - \omega) \Phi(m_1, m_2, m_3) \]
\[ + \frac{\omega}{C(m_3)} \left( A_{m_3+1/2} (\Phi(m_1 - 1, m_2, m_3) + \Phi(m_1 + 1, m_2, m_3)) \right) \]
\[ + \frac{\omega}{C(m_3)} \left( A_{m_3+1/2} (\Phi(m_2 - 1, m_3) + \Phi(m_2 + 1, m_3)) \right) \]
\[ + \frac{\omega}{C(m_3)} \left( B_{m_3-1/2} \Phi(m_2, m_3 - 1) + B_{m_3+1/2} \Phi(m_2, m_3 + 1) \right) \]
\[ - \frac{\omega}{C(m_3)} \left( \sigma^2 \frac{\lambda^{1/2}}{8\pi} \left( \delta(0, m_2) \delta(Z_c(m_1), m_3) \frac{1}{Z_c^4(m_1)} \right) \right) , \quad (B.1) \]

with \( C(m_3) \) defined as
\[ C(m_3) = 4A + \sigma^2 B_{m_3-1/2} + \sigma^2 B_{m_3+1/2} , \]
and \( \omega \) is the relaxation parameter.

**B.1.2 Two dimensional case**

For the two dimensional case discussed in sec.(4.3) the discrete version of the dilaton field equation (4.12) can be solved by the following relaxation procedure [31], setting \( \sigma = 1 \),
\[ \Phi(m_2, m_3) = (1 - \omega) \Phi(m_2, m_3) \]
\[ + \frac{\omega}{C(m_3)} \left( A_{m_3+1/2} (\Phi(m_2 - 1, m_3) + \Phi(m_2 + 1, m_3)) \right) \]
\[ + \frac{\omega}{C(m_3)} \left( B_{m_3-1/2} \Phi(m_2, m_3 - 1) + B_{m_3+1/2} \Phi(m_2, m_3 + 1) \right) \]
\[ - \frac{\omega}{C(m_3)} \left( \lambda^{1/2} \delta_0(m_2) \delta(Z_c(m_1), m_3) \frac{1}{Z_c^4(m_1)} \right) , \quad (B.2) \]

where
\[ C(m_3) = 2A_{m_3+1/2} + B_{m_3-1/2} + B_{m_3+1/2} \]
and \( \omega \) is again the relaxation parameter.

**B.2 Critical slowing down and multi-grid algorithm**

In using relaxation algorithm one encounters the problem of critical slowing down, namely that the long wavelength part of the solution takes very large number of iterations to relax to its true value, the number of iterations diverging as the size of the lattice increases. For example if we consider a linear problem
\[ Lu = s , \quad (B.3) \]
with a given source vector \( s \) and tries to solve for the vector \( u \) using a relaxation algorithm then one finds that the norm of the \textit{residue} vector

\[
  r = s - Lu,
\]  

initially decreases with the number of iterations but after a while it stops reducing. To mitigate the problem of critical slowing down we used multigrid algorithm [46, 47].

For the case of three dimensional anisotropic lattice we used the following definition of the norm

\[
  \|r\| = \sqrt{(a \times b^2 r \cdot r)},
\]

where \( a \) is the lattice constant along the \( z \) direction while \( b \) is the lattice constant along \( x_1 \) and \( x_2 \) direction. For the two dimensional case with an isotropic lattice constant \( a \) the norm was taken to be

\[
  \|r\| = \sqrt{(a^2 r \cdot r)}
\]

\[\text{B.3 Python notebooks}\]

All the numerical calculations involved in this work were done using python 3.0 JupyterLab notebooks which are included as supplemental material. Easiest way to run these notebooks is to install free and open-source python distribution like Anaconda. The four notebooks included in the supplemental material are

1. \textit{ShapeOfFluxTube-1.ipynb} which was used for the calculations reported in section (4.1). This notebook depends on two python files

   (a) \textit{utility\_3DimMG.py}

   (b) \textit{AiStringSource.py}

2. \textit{RectangularString.ipynb} which was used for calculations reported in section (4.2). This notebook depends on the python file

   (a) \textit{utility\_3DimMG.py}

3. \textit{ShapeOfFluxTube-3.ipynb} which was used for calculations reported in section (4.3.1). This notebook depends on the python file

   (a) \textit{utility\_2DimMG.py}
4. *flucRectString.ipynb* which was used for the calculations reported in section (4.4). This notebook depends on the python file

(a) *utility_2DimMG.py*

**References**

[1] V. Singh, D. Browne, and R. Haymaker, *Structure of abrikosov vortices in su(2) lattice gauge theory*, Physics Letters B 306 (May, 1993) 115–119, [hep-lat/9301004](https://arxiv.org/abs/hep-lat/9301004).

[2] G. S. Bali, K. Schilling, and C. Schlichter, *Observing long color flux tubes in su(2) lattice gauge theory*, Phys. Rev. D51 (1995) 5165–5198, [hep-lat/9409005](https://arxiv.org/abs/hep-lat/9409005).

[3] F. Bissey, A. I. Signal, and D. B. Leinweber, *Comparison of gluon flux-tube distributions for quark-diquark and quark-antiquark hadrons*, Phys.Rev.D 80 (2009) 114506, [arXiv:0910.0958](https://arxiv.org/abs/0910.0958).

[4] Y. Nambu, *Strings, monopoles, and gauge fields*, Phys. Rev. D 10 (Dec, 1974) 4262–4268.

[5] G. ’t Hooft, *A Property of Electric and Magnetic Flux in Nonabelian Gauge Theories*, Nucl. Phys. B153 (1979) 141–160.

[6] S. Mandelstam, *General introduction to confinement*, Physics Reports 67 (1980), no. 1 109 – 121.

[7] K. G. Wilson, *Quantum chromodynamics on a lattice*, 1977. Presented at Cargese Summer Institute, Cargese, France, Jul 12- 31, 1976.

[8] R. P. Feynman, *The qualitative behavior of yang-mills theory in 2 + 1 dimensions*, Nuclear Physics B 188 (1981), no. 3 479 – 512.

[9] P. Cea, L. Cosmai, and A. Papa, *Chromoelectric flux tubes and coherence length in QCD*, Phys. Rev. D86 (2012) 054501, [arXiv:1208.1362](https://arxiv.org/abs/1208.1362).

[10] N. Cardoso, M. Cardoso, and P. Bicudo, *Inside the su(3) quark-antiquark qcd flux tube: screening versus quantum widening*, [arXiv:1302.3633](https://arxiv.org/abs/1302.3633).

[11] M. Caselle, M. Panero, and D. Vadacchino, *Width of the flux tube in compact u(1) gauge theory in three dimensions*, [arXiv:1601.0745](https://arxiv.org/abs/1601.0745).
[12] P. Cea, L. Cosmai, F. Cuteri, and A. Papa, *Flux tubes in the QCD vacuum*, Phys. Rev. D95 (2017), no. 11 114511, [arXiv:1702.0643].

[13] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998) 231, [hep-th/9711200].

[14] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B428 (1998) 105–114, [hep-th/9802109].

[15] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2 (1998) 253–291, [hep-th/9802150].

[16] E. Witten, *Anti-de sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. 2 (1998) 505–532, [hep-th/9803131].

[17] J. M. Maldacena, *Tasi 2003 lectures on ads/cft*, hep-th/0309246.

[18] Y. Kinar, E. Schreiber, and J. Sonnenschein, *Q anti-q potential from strings in curved spacetime: Classical results*, Nucl. Phys. B566 (2000) 103, [hep-th/9811192].

[19] Y. Kinar, E. Schreiber, J. Sonnenschein, and N. Weiss, *Quantum fluctuations of Wilson loops from string models*, Nucl. Phys. B583 (2000) 76, [hep-th/9911123].

[20] V. Balasubramanian, P. Kraus, and A. Lawrence, *Bulk vs. boundary dynamics in anti-de sitter spacetime*, hep-th/9805171.

[21] V. Balasubramanian, P. Kraus, A. E. Lawrence, and S. P. Trivedi, *Holographic probes of anti-de Sitter space-times*, Phys. Rev. D59 (1999) 104021, [hep-th/9808017].

[22] U. H. Danielsson, E. K. Vakkuri, and E. Kruczenski, *Vacua, Propagators, and Holographic Probes in AdS/CFT*, JHEP 01 (1999) 002, [hep-th/9812007].

[23] J. Polchinski and L. Susskind, *String theory and the size of hadrons*, hep-th/0112204.

[24] K. Peeters and M. Zamaklar, *The string/gauge theory correspondence in qcd*, 0708.1502.

[25] S. S. Gubser and A. Karch, *From gauge-string duality to strong interactions: a Pedestrian’s Guide*, arXiv:0901.0935.

[26] M. Ammon and J. Erdmenger, *Gauge/Gravity Duality*. Cambridge University Press, 2015.

[27] A. M. Polyakov, *Gauge Fields and Strings*. Contemporary concepts in physics. Taylor & Francis, 1987.
[28] C. G. Callan, Jr. and A. Gâţijosa, *Undulating strings and gauge theory waves*, *Nuclear Physics B* **565** (2000), no. 1-2 157–175.

[29] D. Tong, *Lectures on string theory*, arXiv:0908.0333.

[30] S. L. Adler and T. Piran, *Relaxation methods for gauge field equilibrium equations*, *Rev. Mod. Phys.* **56** (Jan, 1984) 1–40.

[31] S. Koonin and D. Meredith, *Computational Physics: Fortran Version*. Addison-Wesley, 1998.

[32] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, hep-th/9805114.

[33] H. B. Nielsen and P. Olesen, *Vortex-line models for dual strings*, *Nucl. Phys.* **B61** (1973) 45.

[34] V. Vyas, *Intrinsic thickness of qcd flux-tubes*, arXiv:1004.2679.

[35] O. Aharony and N. Klinghoffer, * Corrections to Nambu-Goto energy levels from the effective string action*, *JHEP* **1012** (2010) 058, [arXiv:1008.2648].

[36] O. Aharony and M. Field, *On the effective theory of long open strings*, *JHEP* **1101** (2011) 065, [arXiv:1008.2636].

[37] V. Vyas, *Heavy quark potential from gauge/gravity duality: A large d analysis*, *Phy. Rev. D* **87** (09, 2013) 045026, [arXiv:1209.0883].

[38] B. B. Brandt, *Spectrum of the open qcd flux tube and its effective string description i: 3d static potential in su(n=2,3)*, arXiv:1705.0382.

[39] D. Berényi, S. Varró, V. V. Skokov, and P. Lévai, *Pair production at the edge of the qcd flux tube*, *Physics Letters B* **749** (2015) 210 – 214.

[40] J. Sonnenschein, *Holography inspired stringy hadrons*, arXiv:1602.0070.

[41] A. K. Das, *Topics in finite temperature field theory*, hep-ph/0004125.

[42] A. M. Polyakov, *String theory and quark confinement*, *Nucl. Phys. Proc. Suppl.* **68** (1998) 1, [hep-th/9711002].

[43] J. M. Maldacena, *Wilson loops in large n field theories*, *Phys. Rev. Lett.* **80** (1998) 4859, [hep-th/9803002].

[44] S. J. Rey and J. T. Yee, *Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C22** (2001) 379, [hep-th/9803001].
[45] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal, and U. Wiedemann, *Gauge/String Duality, Hot QCD and Heavy Ion Collisions*. Cambridge University Press, 2014.

[46] U. Trottenberg, C. Oosterlee, and A. Schuller, *Multigrid*. Elsevier Science, 2001.

[47] J. Stewart, *Python for Scientists*. Cambridge University Press, 2017.