Change of measure up to a random time: Details

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Abstract

This paper extends results from Mortimer and Williams (1991) about changes of probability measure up to random times. Many new classes of examples involving honest times and pseudo-stopping times are provided. Furthermore, we discuss the question of market viability up to a random time.

1 Introduction

Motivated by models from physics and chemistry Mortimer and Williams (1991) study how to perform a change of measure up to a random time $\sigma : \Omega \to [0, \infty]$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. More precisely, in their paper titled "Change of measure up to a random time: Theory" they derive the semi-martingale decomposition of continuous $(P, \mathcal{F}_t)$-martingales up to time $\sigma$ in the progressively enlarged filtration

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\mathbb{1}_{\{s > t\}}; s \leq t)$$

under an equivalent probability measure $Q$ and they give the expression of the $(Q, \mathcal{G}_t)$-hazard function of $\sigma$. To prove their results they use elementary methods and do no rely on the theory of enlargement of filtrations. Besides, Mortimer and Williams (1991) claim in their paper that "it is the examples which make this topic of some interest", but the only examples they provide deal with the well-known path decomposition of the standard Brownian motion.

In this paper we extend their observations in numerous ways and point out their relevance for applications in mathematical finance. Working under the standing assumptions that $\sigma$ avoids stopping times and that all $(\mathcal{F}_t)$-martingales are continuous we are able to give more general examples involving honest times and pseudo-stopping times, especially we generalize the example of the Brownian path decomposition given in Mortimer and Williams (1991). While honest times are known to be well-suited for a progressive enlargement of filtration since the seminal work of Barlow (1978), pseudo-stopping times were only recently introduced by Nikeghbali and Yor (2005). As opposed to Mortimer and Williams (1991) who provide a Markovian study of their example our analysis is based on semi-martingale calculus only.

As honest times are ends of predictable sets their definition is independent of the underlying probability measure. This however is not true for pseudo-stopping times. We therefore investigate the question of whether there exist equivalent probability measures which leave the pseudo-stopping time property unchanged.

Furthermore, because the progressive enlargement of a filtration with an honest time ensures the stability of the semi-martingale property also after time $\sigma$, we are able to extend the Girsanov-type theorem from Mortimer and Williams (1991) to the whole time horizon in this case. While the result itself is not very surprising and actually already known in greater generality, cf. [5], the way we prove it is interesting because as in Mortimer and Williams (1991) we solely use elementary

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methods and do not assume any prior knowledge of the enlargement of filtration theory. Actually, as it turns out there is a nice link to the so called relative martingales which were studied by Azéma, Meyer and Yor (1992).

Change of measures are ubiquitous in mathematical finance. This is mainly due to the fundamental theorem of asset pricing which states in one form or the other that a market is arbitrage free if and only if there exists an equivalent martingale measure. A rigorous version of this statement involving the acronym NFLVR can be found in Delbaen and Schachermayer (1994). On the other hand, the technique of enlargements of filtrations is a standard method in mathematical finance to model credit risk and insider trading. This led us to the question of market viability up to a random time $\sigma$: If we assume NFLVR with respect to the filtration $(\mathcal{F}_t)$, under which conditions is the market then also arbitrage free with respect to $(\mathcal{G}_t)$ until time $\sigma$? This question is of great interest. Especially, it is known that honest times allow for arbitrage on the time horizon $[0, \sigma]$ in the progressively enlarged filtration. This was recently studied in detail by Fontana, Jeanblanc and Song (2013). We treat the general case here. Even though our results are not as complete as the ones of Fontana et al. (2013), we are able to give sufficient criteria for the absence of arbitrage on the time horizon $[0, \sigma]$ for general $\sigma$.

The paper is organized as follows: In the next section we introduce the general setup and notation before we recall the result from [12] and give some first corollaries and slightly extended versions of their theorem. Applications to honest times and pseudo-stopping times can be found in Section 3. In Subsection 3.3 we generalize the example from [12]. In order to understand the relationship between the $P$-and $Q$-Azéma supermartingale we deal with their multiplicative decomposition in Section 4. In Section 5 we study the stability of the pseudo-stopping time property with respect to certain measure changes. Financial applications can be found in Section 6 where we try to answer the question of market viability up to a random time. Section 7 deals with locally absolutely continuous measure changes and in the last section we study changes of measure after time $\sigma$ for honest times.

## 2 General theory

### 2.1 Setup and notation

Throughout the paper we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, where $\mathcal{F} := \bigvee_{t \geq 0} \mathcal{F}_t$ and $(\mathcal{F}_t)$ is assumed to satisfy the natural conditions, i.e. $(\mathcal{F}_t)$ is right-continuous and $\mathcal{F}_0$ contains all $\mathcal{F}_t$-negligible sets for all $t \in [0, \infty)$. By $\sigma : \Omega \to [0, \infty]$ we denote a random time, which gives rise to the progressively enlarged filtration

$$\mathcal{G}_t := \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(\mathbb{1}_{\{\sigma > r\}}; r \leq s)).$$

For any $(\mathcal{G}_t)$-adapted process $(X_t)$ we denote by $T_a^X := \inf\{t > 0 : X_t = a\}$ the first hitting time of the level $a \in \mathbb{R}$. If $(X_t)$ is a real-valued stochastic process we denote by $(\overline{X}_t)$ and $(\underline{X}_t)$ its supremum resp. infimum process. $\mathcal{M}(P, \mathcal{F}_t)$ denotes the set of $(P, \mathcal{F}_t)$-martingales and $\mathcal{M}_{loc}(P, \mathcal{F}_t)$ resp. $\mathcal{M}_{u.i.}(P, \mathcal{F}_t)$ the set of local resp. uniformly integrable $(P, \mathcal{F}_t)$-martingales.

Throughout the paper we will assume that the following two assumptions are satisfied:

1. **(A)** $\sigma$ avoids any $(\mathcal{F}_t)$-stopping time: $P(\sigma = T) = 0$ for any $(\mathcal{F}_t)$-stopping time $T$.
2. **(C)** All $(\mathcal{F}_t)$-martingales are continuous.

We denote by $Z_t^P := P(\sigma > t | \mathcal{F}_t)$ the Azéma supermartingale of $\sigma$. It decomposes as $Z_t^P = m_t^P - A_t^P$ with $m_t^P = \mathbb{E}^P(A_{\infty}^P | \mathcal{F}_t)$ being a uniformly integrable martingale and $(A_t^P)$ being the $(\mathcal{F}_t)$-dual predictable projection of the process $(\mathbb{1}_{\{\sigma \leq t\}})_{t \geq 0}$. Under the assumptions **(AC)** the Azéma supermartingale is continuous and $Z_t^P = m_t^P - A_t^P$ is thus its Doob-Meyer decomposition.
Let $\rho$ be a non-negative random variable with expectation one. Then $Q = \rho \cdot P$ defines a new probability measure which is absolutely continuous to $P$. We denote by $(\rho_t)$ resp. $(\hat{\rho}_t)$ the optional projection of $\rho$ on $(F_t)$ resp. $(G_t)$, i.e.

$$
\rho_t := \mathbb{E}^P(\rho|F_t), \quad \hat{\rho}_t := \mathbb{E}^P(\rho|G_t),
$$

where $(\hat{\rho}_t)$ is chosen to be càdlàg and $(\rho_t)$ is continuous due to $(C)$. Furthermore, we define the $(P,F_t)$-supermartingale

$$
h_t = \mathbb{E}^P(\rho 1_{\{\sigma > t\}}|F_t).
$$

By Bayes’ formula one has

$$
h_t = \rho_t \cdot Q(\sigma > t|F_t) = \rho_t Z_t^Q.
$$

Since $\sigma$ avoids stopping times, $P(\sigma = \infty) = 0$ and $\sigma$ is finite $P$-almost surely. Therefore, $Z_t^P$ and $h_t$ both converge towards zero almost surely.

If $h$ is strictly positive, we denote by $\mu$ the stochastic logarithm of $h$, i.e. $h_t = \mathcal{E}(\mu)_t$. The process $\mu$ is again a $(P,F_t)$-supermartingale with Doob-Meyer decomposition $\mu = \mu^L - \mu^F$, where $\mu^L \in \mathcal{M}_{loc}(P,F_t)$ and $\mu^F$ is increasing. In general the process $\mu$ is only well-defined on the stochastic interval $[0,T^0]$.

### 2.2 Girsanov-type theorems

We are now ready to state the result of [12], Lemma 2.

**Theorem 2.1.** Assume that $h$ is strictly positive and let $U = (U_t)_{t \geq 0}$ be a local $(P,F_t)$-martingale. Then the process $(\mathbb{1}_{\{\sigma > t\}} V_t \exp(\mu^F_t))_{t \geq 0}$ is a local $(Q,G_t)$-martingale, where $V := U - \langle U, \mu \rangle_t$.

Moreover, the process $(\mu^F_{t\wedge \sigma})_{t \geq 0}$ is the $(Q,G_t)$-dual predictable projection of $(\mathbb{1}_{\{\sigma \leq t\}})_{t \geq 0}$.

As an immediate consequence of the above result we deduce

**Corollary 2.2.** Assume that $h$ is strictly positive. If $U \in \mathcal{M}_{loc}(P,F_t)$, then

$$
U_{t\wedge \sigma} - \langle U, \mu \rangle_{t\wedge \sigma} \in \mathcal{M}_{loc}(Q,G_t).
$$

**Proof.** Taking $U \equiv 1$ in Theorem 2.1 yields that

$$
H_t := \mathbb{1}_{\{\sigma > t\}} \exp(\mu^F_t) \in \mathcal{M}(Q,G_t).
$$

Since $V$ is continuous, $H$ and $V$ are orthogonal to each other. Hence, their product is a local $(Q,G_t)$-martingale if and only if $V$ is also a local $(Q,G_t)$-martingale as long as $H_{t-} > 0$, i.e. on the interval $[0,\sigma]$.

**Remark 2.3.** If we choose $\rho \equiv 1$ in the above corollary, we recover the well-known enlargement formula up to time $\sigma$: For any $M \in \mathcal{M}_{loc}(P,F_t)$ we have

$$
M_{t\wedge \sigma} - \int_0^{t\wedge \sigma} d\langle M, Z_t^{\mathbb{1}_{\{\sigma > t\}} \cdot} \rangle_s \in \mathcal{M}_{loc}(P,G_t).
$$

**Remark 2.4.** In [12] the authors prove their result without applying any results from the theory of progressive enlargement of filtrations. Of course, Corollary 2.2 can also be proven by applying first Girsanov’s theorem and afterwards the enlargement formula under $Q$. For so called honest times this is done in paragraph 81 of [3], where the same result is proven in greater generality, i.e. without assuming the continuity of the processes involved.

Next we show that Theorem 2.1 also holds if $h$ is not necessarily strictly positive.

**Theorem 2.5.** If $U = (U_t)_{t \geq 0}$ is a local $(P,F_t)$-martingale, then $X_t := \mathbb{1}_{\{\sigma > t\}} V_t \exp(\mu^F_t)$ and $V_t$ are local $(Q,G_t)$-martingales, where $V_t := U_{t\wedge \sigma} - \langle U, \mu \rangle_{t\wedge \sigma}$.
Proof. First we show that $Q(\sigma < T^h_0) = 1$. For this note that $T^h_0 = T^0_0 \lor T^Q_0$ because $h_t = \rho_t Z^Q_t$. But we have
\[ Q(T^0_0 < \infty) = \mathbb{E}^P \left( \rho_1 \mathbb{1}_{(T^0_0 < \infty)} \right) = \mathbb{E}^P \left( \rho_\infty \mathbb{1}_{(T^0_0 < \infty)} \right) = \mathbb{E}^P \left( 0 \cdot \mathbb{1}_{(T^0_0 < \infty)} \right) = 0. \]
Since $\sigma$ avoids stopping times under $P$ and $Q$ is absolutely continuous to $P$, $Q(\sigma = T^h_0) = 0$ and $\sigma$ is also $Q$-almost surely finite. Hence,
\[ Q(\sigma \geq T^h_0) = Q(\sigma > T^h_0) = Q(\sigma > T^Q_0) = \mathbb{E}^Q Z^Q_{T^h_0} = 0. \]
Especially, this means that $X$ is $Q$-a.s. well-defined since $\mu$ is well-defined on the interval $[0, T^h_0]$. Second for every $n \in \mathbb{N}$ we write $U^n_t := U_{t \lor T^h_{1/n}}$, $t \geq 0$. According to Theorem 2.1, the process $X^n_t := X_{t \lor T^h_{1/n}}$ is a local $(Q, G_t)$-martingale for every $n \in \mathbb{N}$. Therefore, $X$ is a local $(Q, G_t)$-martingale on the interval $[0, T^h_0) = \bigcup_{n \in \mathbb{N}} [0, T^h_{1/n}]$ and since $[0, T^h_0) \supset [0, \sigma)$ $Q$-almost surely, this implies that
\[ X_t = \mathbb{1}_{\{\sigma > t\}} V_t \exp(\mu^P_t) \in \mathcal{M}(Q, G_t). \]
Finally, $V$ is a local $(Q, G_t)$-martingale by the same reasoning as in the proof of Corollary 2.2. \qed

3 Special cases

In this section we specify the above setting further. Some of the examples are chosen having financial applications in mind, while others are motivated by purely mathematical considerations. The following well-known Lemma, cf. e.g. [3], will be very useful.

Lemma 3.1.

1. If $G$ is a $(G_t)$-predictable process, then there exists an $(\mathcal{F}_t)$-predictable process $F$ such that for all $t \geq 0$,
   \[ G_t \mathbb{1}_{\{t \leq \sigma\}} = F_t \mathbb{1}_{\{t \leq \sigma\}}. \]

2. If $\xi$ is a $P$-integrable variable, then
   \[ \mathbb{E}^P(\xi \mathbb{1}_{\{\sigma > t\}} | \mathcal{G}_t) = \mathbb{1}_{\{\sigma > t\}} \frac{\mathbb{E}^P(\xi \mathbb{1}_{\{\sigma > t\}} | \mathcal{F}_t)}{Z^P_t}. \]

3. If $T$ is a $(G_t)$-stopping time, then there exists an $(\mathcal{F}_t)$-stopping time $S$ such that
   \[ T \lor \sigma = S \lor \sigma. \]

3.1 The case of pseudo-stopping times

In this section we want to perform the change of measure up to a pseudo-stopping time. Pseudo-stopping times were introduced in [4] as follows:

Definition 3.2. A positive random variable $\sigma : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ is called a $(P, \mathcal{F}_t)$-pseudo-stopping time if for every bounded $(P, \mathcal{F}_t)$-martingale $M$ we have $\mathbb{E}^P M_\sigma = \mathbb{E}^P M_0$.

In [4] it is shown that pseudo-stopping times can be characterized in many different ways, which we recall in

Theorem 3.3. The following are equivalent:

(1) $\sigma$ is a $(P, \mathcal{F}_t)$-pseudo stopping time.

(2) $A^P_\infty \equiv 1$ almost surely.
(3) \( A^P_\sigma \sim U[0,1] \)

(4) For any local \((P, \mathcal{F}_t)\)-martingale \(M\), the process \((M_{t \wedge \sigma})_{t \geq 0}\) is a local \((P, \mathcal{G}_t)\)-martingale.

(5) \( Z^P = 1 - A^P \) is a decreasing \((\mathcal{F}_t)\)-predictable process.

**Proof.** The equivalence between (1), (2), (4) and (5) is shown in Theorem 1 of [14]. The relation (2) \(\Leftrightarrow\) (3) follows from the general relation between the Laplace transforms of \(A^P_\sigma\) and \(A^P_\infty\),

\[
\lambda \cdot \mathbb{E}^P \left( e^{-\lambda A^P_\sigma} \right) = \lambda \cdot \mathbb{E}^P \left( \int_0^\infty e^{-\lambda A^P_u} dA^P_u \right) = 1 - \mathbb{E}^P \left( e^{-\lambda A^P_\infty} \right), \quad \lambda > 0.
\]

\[\square\]

In the following two examples \(\sigma\) is assumed to be a \((P, \mathcal{F}_t)\)-pseudo-stopping time.

**Example 3.4.** In this example we make use of part (3) of Theorem 3.3. Since \(A^P_\sigma\) is uniformly distributed on \([0,1]\), we can choose \(\rho = f(A^P_\sigma)\) with \(f > 0\) an integrable function such that \(\int_0^1 f(x) dx = 1\). Since \((A^P_\sigma)\) is the dual optional projection of \(\mathbf{1}_{\{\sigma \leq t\}}\) we have for any \(\mathcal{F}_t\)-measurable random variable \(F_t\),

\[
\mathbb{E}^P \left( f(A^P_\sigma) \mathbf{1}_{\{\sigma > t\}} F_t \right) = \mathbb{E}^P \left( \int_0^\infty f(A^P_u) \mathbf{1}_{\{u > t\}} F_t dA^P_u \right) = \mathbb{E}^P \left( F_t \int_{A^P_\sigma}^\infty f(x) dx \right),
\]

which allows us to compute

\[
h_t = \mathbb{E}^P \left( f(A^P_\sigma) \mathbf{1}_{\{\sigma > t\}} | \mathcal{F}_t \right) = \mathbb{E}^P \left( \int_t^\infty f(A^P_u) dA^P_u | \mathcal{F}_t \right) = \int_t^1 f(x) dx,
\]

\[
dh_t = -f(A^P_\sigma) dA^P_\sigma,
\]

\[
d\mu_t = \frac{dh_t}{h_t} = -\frac{f(A^P_\sigma) dA^P_\sigma}{\int_0^1 f(y) dy} = -d\mu_t^F.
\]

Therefore, for every continuous local \((P, (\mathcal{F}_t))\)-martingale \(U\) the process \((U_{t \wedge \sigma})_{t \geq 0}\) is a local \((Q, \mathcal{G}_t)\)-martingale. Moreover, the dual predictable projection of \(\mathbf{1}_{\{\sigma \leq t\}}\) with respect to \((Q, \mathcal{G}_t)\) is given by \(\mu_{t \wedge \sigma}^F = -\log \left( \int_{A^P_\sigma}^1 f(y) dy \right)\). Note that this particular choice of \(\rho\) does not have any effect on continuous \((\mathcal{G}_t)\)-martingales until time \(\sigma\): \((U_{t \wedge \sigma})\) is a local \((P, \mathcal{F}_t)\)-martingale and a local \((Q, \mathcal{G}_t)\)-martingale. This generalizes Example 2 in [12].

**Example 3.5.** Let \(M\) be a uniformly integrable \((P, \mathcal{F}_t)\)-martingale starting from \(M_0 = 1\). Then we may choose \(\rho = M_\sigma\) since \(\mathbb{E}^P M_\sigma = \mathbb{E}^P M_0 = 1\). We have

\[
h_t = \mathbb{E}^P \left( M_\sigma \mathbf{1}_{\{\sigma > t\}} | \mathcal{F}_t \right) = \mathbb{E}^P \left( \mathbb{E}^P (M_\sigma | \mathcal{G}_t) \mathbf{1}_{\{\sigma > t\}} | \mathcal{F}_t \right) = \mathbb{E}^P \left( M_{\sigma \wedge t} \mathbf{1}_{\{\sigma > t\}} | \mathcal{F}_t \right)
\]

\[= M_t \cdot P(\sigma > t | \mathcal{F}_t) = M_t Z^P_t
\]

\[
dh_t = M_t dZ^P_t + Z^P_t dM_t + d\langle M, Z^P \rangle_t = -M_t dA^P_t + (1 - A^P_t) dM_t
\]

\[
d\mu_t = \frac{dh_t}{h_t} = -\frac{dA^P_t}{1 - A^P_t} + \frac{dM_t}{M_t} = \frac{dM_t}{M_t} + d\log(1 - A^P_t)
\]

Thus, in this case the dual predictable projection of \(\mathbf{1}_{\{\sigma \leq t\}}\) equals \(\mu_t^F = -\log(1 - A^P_t) = -\log(Z^P_t)\). Applying Corollary 2.2 we see that given a continuous local \((P, \mathcal{F}_t)\)-martingale \(U\) the process

\[
V_t := U_{t \wedge \sigma} - \int_0^{t \wedge \sigma} \frac{d\langle M, U \rangle_s}{M_s}
\]

is a local \((Q, \mathcal{G}_t)\)-martingale. Therefore, \(\sigma\) is also a \(Q\)-pseudo-stopping time.

**Remark 3.6.** Note that we cannot choose \(\rho = M_\infty\) instead, because in general \(\mathbb{E}^P(M_\infty | \mathcal{G}_t) \neq M_\sigma\) unless \(\sigma\) is a stopping time.
3.2 The case of honest times

Definition 3.7. A random time \( \sigma \) on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) is called honest if \( \sigma \) is equal to an \( \mathcal{F}_t \)-measurable random variable on \( \{ \sigma < t \} \).

Remark 3.8. Note that the definition of an honest time does not depend on the probability measure, while the definition of a pseudo-stopping time does.

The following result from [15], Theorem 4.1, will be used frequently in the next sections.

Lemma 3.9. For an honest time \( \sigma \) there exists a non-negative local \((\mathbb{P}, \mathcal{F}_t)\)-martingale \((N^\mathbb{P}_t)_{t \geq 0}\) with \(N^\mathbb{P}_0 = 1\) and \(N^\mathbb{P}_t \to 0\) \(\mathbb{P}\)-a.s. such that
\[
Z^\mathbb{P}_t = \mathbb{P}(\sigma > t | \mathcal{F}_t) = \frac{N^\mathbb{P}_t}{N^\mathbb{P}_t}, \quad \text{where } N^\mathbb{P}_t := \sup_{s \leq t} N^\mathbb{P}_s.
\]

Lemma 3.10. Let \( \sigma \) be an honest time and denote by \( Z^\mathbb{P}_t = N^\mathbb{P}_t / N^\mathbb{P}_t \) the multiplicative decomposition of \( Z^\mathbb{P}_t = \mathbb{P}(\sigma > t | \mathcal{F}_t) \) given in Lemma 3.9. Then for all \( x > A^\mathbb{P}_t \),
\[
\mathbb{P}(A^\mathbb{P}_\sigma \in dx | \mathcal{F}_t) = N^\mathbb{P}_t e^{-x}.
\]

Proof. From Lemma 2.1 in [14] we know that for \( x > 0 \),
\[
\mathbb{P} \left( \sup_{s \geq t} N^\mathbb{P}_s > x \right | \mathcal{F}_t) = \left( \frac{N^\mathbb{P}_t}{x} \right) \wedge 1.
\]
It then follows from \( A^\mathbb{P}_t = \log(N^\mathbb{P}_t) \) that
\[
\mathbb{P}(A^\mathbb{P}_\sigma > x | \mathcal{F}_t) = \mathbb{P}(A^\mathbb{P}_\infty > x | \mathcal{F}_t) = \mathbb{P}(N^\mathbb{P}_\infty > e^x | \mathcal{F}_t) = 1_{\{N^\mathbb{P}_t > e^x\}} + 1_{\{N^\mathbb{P}_t \leq e^x\}} N^\mathbb{P}_t e^{-x}.
\]

Example 3.11. Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be any measurable function such that \( \int_0^\infty f(x) e^{-x} dx = 1 \).

\[
h_t = \mathbb{E}^\mathbb{P} \left( f(A^\mathbb{P}_t) 1_{\{\sigma > t\}} | \mathcal{F}_t \right) = \mathbb{E}^\mathbb{P} \left( \int_t^\infty f(A^\mathbb{P}_u) dA^\mathbb{P}_u | \mathcal{F}_t \right) = \mathbb{E}^\mathbb{P} \left( \int_{A^\mathbb{P}_t}^\infty f(x) dx | \mathcal{F}_t \right)
\]
\[
= \mathbb{E}^\mathbb{P} \left( \int_{A^\mathbb{P}_t}^\infty f(x) dx | \mathcal{F}_t \right) = N^\mathbb{P}_t \int_{A^\mathbb{P}_t}^\infty f(x) dx e^{-y}dy = N^\mathbb{P}_t \int_{A^\mathbb{P}_t}^\infty e^{-y}dyf(x)dx
\]
\[
dh_t = \int_{A^\mathbb{P}_t}^\infty f(x) e^{-x} dx \, dN^\mathbb{P}_t - N^\mathbb{P}_t f(A^\mathbb{P}_t) e^{-A^\mathbb{P}_t} dA^\mathbb{P}_t
\]
\[
d\mu_t = \frac{dh_t}{h_t} = \frac{dN^\mathbb{P}_t}{N^\mathbb{P}_t} - \frac{f(A^\mathbb{P}_t) e^{-A^\mathbb{P}_t} dA^\mathbb{P}_t}{\int_{A^\mathbb{P}_t}^\infty f(x) e^{-x} dx} = \frac{dN^\mathbb{P}_t}{N^\mathbb{P}_t} + d\log \left( \int_{A^\mathbb{P}_t}^\infty f(x) e^{-x} dx \right)
\]
Thus, in this case the dual predictable projection of \( 1_{\{\sigma \leq t\}} \) is given by \( \mu_t^F = - \log \left( \int_{A^\mathbb{P}_t}^\infty f(x) e^{-x} dx \right) \).

Applying Corollary 2.2 we see that given a continuous local \((\mathbb{P}, \mathcal{F}_t)\)-martingale \( U \) the process
\[
V_t := U_{t \wedge \sigma} - \int_0^{t \wedge \sigma} \frac{d(N^\mathbb{P}_s, U)_s}{N^\mathbb{P}_s}
\]
is a local \((\mathbb{Q}, \mathcal{G}_t)\)-martingale. Therefore, as in Example 3.9 this particular choice of \( \rho \) allows continuous local \((\mathbb{P}, \mathcal{G}_t)\)-martingales to stay local \((\mathbb{Q}, \mathcal{G}_t)\)-martingales until time \( \sigma \).
3.3 Generalization of Example 1 from [12]

Let $\sigma$ be an honest time whose $\mathcal{P}$-Azéma supermartingale will be denoted by $Z_t^\sigma = \mathcal{P}(\sigma > t|\mathcal{F}_t)$ (instead of $Z_t^\mathcal{P}$) in this subsection. It is shown in [13] that

$$\pi = \sup \left\{ t < \sigma : Z_t^\sigma = \inf_{u\leq \sigma} Z_u^\sigma \right\}$$

is a $\mathcal{P}$-pseudo-stopping time. From Proposition 5 in [14] we know that $\inf_{u\leq \sigma} Z_u^\sigma$ is uniformly distributed and that the supermartingale $Z_t^\sigma = \mathcal{P}(\pi > t|\mathcal{F}_t)$ equals $Z_t^\sigma = \inf_{u\leq t} Z_u^\sigma$ for all $t \geq 0$. We define

$$\rho := f \left( 1 - \inf_{u\leq \sigma} Z_u^\sigma \right) = f (1 - Z_t^\sigma) = f (1 - Z_t^\pi) = f (A_t^\pi)$$

for some $f \in C^1[0,1]$, $f > 0$ with $\int_0^1 f(x)dx = 1$, i.e. $\mathbb{E}\rho = 1$. We have

$$h_t = \mathbb{E} \left( \rho \mathbbm{1}_{\{\sigma > t\}} | \mathcal{F}_t \right) = \mathbb{E} \left( f(A_t^\sigma) \mathbbm{1}_{\{\sigma > t\}} | \mathcal{F}_t \right) = \mathbb{E} \left( f(A_t^\pi) \mathbbm{1}_{\{\sigma > t\}} | \mathcal{F}_t \right) + \mathbb{E} \left( f(A_t^\pi) \mathbbm{1}_{\{\pi > t\}} | \mathcal{F}_t \right)$$

The second term on the RHS has already been computed in Example 3.4 as

$$\mathbb{E} \left( f(A_t^\pi) \mathbbm{1}_{\{\pi > t\}} | \mathcal{F}_t \right) = \int_{A_t^\pi}^1 f(x)dx.$$

Concerning the first term we have

$$\mathcal{P}(\sigma > t \geq \pi | \mathcal{F}_t) = \mathcal{P}(\sigma > t | \mathcal{F}_t) - \mathcal{P}(\pi > t | \mathcal{F}_t) = Z_t^\sigma - Z_t^\pi$$

and

$$\mathbb{E} \left( f(A_t^\pi) \mathbbm{1}_{\{\sigma > t \geq \pi\}} | \mathcal{F}_t \right) = \mathbb{E} \left( f(1 - Z_t^\sigma) \mathbbm{1}_{\{\sigma > t \geq \pi\}} | \mathcal{F}_t \right) = \mathbb{E} \left( f(1 - Z_t^\pi) \mathbbm{1}_{\{\sigma > t \geq \pi\}} | \mathcal{F}_t \right) = f(A_t^\sigma) \cdot (Z_t^\sigma - Z_t^\pi).$$

Hence,

$$h_t = \int_{A_t^\pi}^1 f(x)dx + f(A_t^\pi)(Z_t^\sigma - Z_t^\pi)$$

$$dh_t = -f(A_t^\sigma)dA_t^\sigma + f'(A_t^\sigma)(Z_t^\sigma - Z_t^\pi)dA_t^\pi + f(A_t^\pi)(dZ_t^\sigma - dZ_t^\pi).$$

Since $1 - A_t^\sigma = Z_t^\pi = Z_t^\mathcal{P}$, we have $\text{supp}(A_t^\sigma) = \{ Z_t^\sigma = Z_t^\pi \}$, which implies that

$$dh_t = f(A_t^\sigma)(dZ_t^\sigma - dZ_t^\pi - dA_t^\sigma) = f(A_t^\sigma)dZ_t^\sigma.$$

Therefore,

$$d\mu_t = \frac{dh_t}{h_t} = \frac{f(A_t^\sigma)dZ_t^\sigma}{\int_{A_t^\pi}^1 f(x)dx + f(A_t^\sigma)(Z_t^\sigma - Z_t^\pi)}, \quad d\mu_t^F = \frac{f(A_t^\pi)dA_t^\sigma}{\int_{A_t^\pi}^1 f(x)dx + f(A_t^\pi)A_t^\pi},$$

where we used that $\text{supp}(dA_t^\sigma) \subset \{ Z_t^\sigma = 1 \}$. Thus, given a continuous local $(\mathcal{P}, \mathcal{F}_t)$-martingale $U$ the process

$$V_t := U_{t \land \sigma} - \int_0^{t \land \sigma} \frac{f(A_s^\sigma)d\langle m^\sigma, U \rangle_s}{\int_{A_t^\pi}^1 f(x)dx + f(A_t^\sigma)(Z_t^\sigma - Z_s^\pi)}$$

is a local $(\mathcal{Q}, \mathcal{G}_t)$-martingale by Corollary 2.2.

We briefly recall Example 1 from [12], which deals with the path decomposition of the Brownian motion, to see how it fits in the above framework.
Example 3.12. For a standard Brownian motion $B$ one defines the random times

$$\sigma = \sup\{ t < T^B_1 : B_t = 0 \}, \quad \pi = \sup\{ t < \sigma : B_t = \overline{B}_1 \},$$

i.e. $\sigma$ is the time of the last zero of $B$ before it first hits one, and $\pi$ is the last time at which $B$ reaches its supremum before $\sigma$. Clearly, $\sigma$ is an honest time and $Z^\sigma_t = \mathbb{P}(\sigma > t | \mathcal{F}_t) = 1 - B_{\pi T}^+$. Since

$$\pi = \sup\{ t < \sigma : B_t = \overline{B}_1 \} = \sup\{ t < \sigma : Z^\sigma_t = Z^\sigma_1 \},$$

$\pi$ is a pseudo-stopping time and $Z^\pi_t = 1 - \overline{B}_{\pi T}^+$, cf. Proposition 5 in [14]. In this case $A^\pi_t = \overline{B}_\sigma$ and

$$h_t = \int_{B_{\pi T}^+}^1 f(y) dy + f(\overline{B}_1)(\overline{B}_{\pi T}^+ - B_{\pi T}^+)$$

$$dh_t = -f(\overline{B}_1) \left( \mathbb{1}_{\{B_t > 0\}} dB_{\pi T}^+ + \frac{dL_{\pi T}}{2} \right),$$

where $L$ denotes the local time of $B$ at level zero. Hence, up to time $\sigma$ the $(\mathbb{P}, \mathcal{F}_t)$-Brownian motion $B$ follows the dynamics

$$dB_t = dW_t - \frac{\mathbb{1}_{\{B_t > 0\}} f(\overline{B}_1) dt}{\int_{B_1}^y f(y) dy + f(\overline{B}_1)(\overline{B}_1 - B_1^+)}.$$ 

where $(W_t)$ is a $(\mathbb{Q}, \mathcal{G}_t)$-Brownian motion. Especially, if we choose $f \equiv 1$, we see that $B$ behaves like a reflected Brownian motion until time $\sigma$. This result is part of the well-known path decomposition of the standard Brownian motion due to Williams.

4 Multiplicative decompositions

When performing a change of measure up to a random time one needs to compute the process

$$h_t = \mathbb{E}^P \left( \rho \mathbb{1}_{\{\sigma > t\}} | \mathcal{F}_t \right) = \mathbb{E}^P \left( \hat{\rho}_t \mathbb{1}_{\{\sigma > t\}} | \mathcal{F}_t \right).$$

Therefore, the behaviour of the process $(\hat{\rho}_t)$ before time $\sigma$ is of particular interest. In this section we will repeatedly make use of the following result, which is an immediate consequence of the proof of Theorem 3.1 in [9], cf. also Théorème 5.12 and Lemme 5.15 of [10].

Theorem 4.1. For any bounded $\zeta \in \mathcal{G}_\sigma$ there exists a local $(\mathbb{P}, \mathcal{F}_t)$-martingale $M$ and a bounded $(\mathcal{F}_t)$-predictable process $K$ such that

$$\mathbb{E}^P(\zeta | \mathcal{G}_t) = M_t - \int_0^t \frac{d(M, Z^P)_s}{Z^P_s} - \int_0^t \frac{K_s}{Z^P_s} dA^P_s \quad \text{on } \{ \sigma > t \}.$$

Furthermore, if $t \mapsto \mathbb{E}^P(\zeta | \mathcal{G}_t)$ is continuous almost surely (i.e. it does not jump at $\sigma$), then $K \equiv 0$.

Proof. To prove the theorem one can do exactly the same computations as in the proof of Theorem 3.1 in [9] without using any martingale representation property. Since we are only interested in the behaviour before time $\sigma$, we do not need the ($\mathcal{H}'$) hypothesis.

Remark 4.2. The assumption $(AC)$ is not needed to obtain a characterization of any bounded $(\mathcal{G}_t)$-martingale before time $\sigma$, cf. [9]. The above formulation is however sufficient for our purposes. We have the following corollary which follows by localization.
Corollary 4.3. Let \((\tilde{\rho}_t)_{t \geq 0}\) be a non-negative continuous local \((P, \mathcal{G}_t)\)-martingale. Then there exists a local \((P, \mathcal{F}_t)\)-martingale \(M\) such that
\[
\tilde{\rho}_{t \land \sigma} = M_{t \land \sigma} - \int_0^{t \land \sigma} \frac{d(M, Z^P)_s}{Z^P_s}.
\]

In the following we will repeatedly use the so called Itô-Watanabe decomposition of the Azéma supermartingale. Since it is less known than the additive Doob-Meyer decomposition, we briefly recall a continuous version of the result from [7] here.

Theorem 4.4. Let \(Z\) be a continuous non-negative supermartingale with Doob-Meyer decomposition \(Z = m - A\). Then \(Z\) factorises uniquely as \(Z = DN\), where \(N\) is a positive local martingale starting from \(N_0 = 1\) and \(D\) is a decreasing process such that both \(N\) and \(D\) are constant on the set \(\{Z = 0\}\). Moreover, \(N\) and \(D\) are given by
\[
D_t = Z_0 \exp \left( - \int_0^{t \land T_0} \frac{dA_s}{Z_s} \right), \quad N_t = \mathcal{E} \left( \int_0^{t \land T_0} \frac{dm_s}{Z_s} \right).
\]

Remark 4.5. If \(Z = Z^P\) is the Azéma supermartingale of \(\sigma\), then
\[
Z^P_t = 0 \iff m^P_t - A^P_t = \mathbb{E}(A^P_\infty - A^P_t \mid \mathcal{F}_t) = 0 \iff A^P_t = A^P_s \forall s \geq t,
\]
since \(A^P\) is an increasing process. Therefore, supp\((dA^P)\) \(\subset \{Z^P > 0\}\) and supp\((dm^P)\) \(\subset \{Z^P > 0\}\).

Example 4.6. If \(\sigma\) is an honest time and the assumptions \((AC)\) are satisfied, then the Azéma supermartingale of \(\sigma\) decomposes as
\[
Z^P_t = P(\sigma > t \mid \mathcal{F}_t) = \frac{N^P_t}{N^t}, \quad \text{i.e. } D^P_t = \frac{1}{N^P_t},
\]
where \(N^P\) is a non-negative local martingale converging to zero almost surely, cf. Lemma 3.39.

Theorem 4.7. Assume that \(\rho > 0\) almost surely. If \(Z^P = N^P D^P\) and \(Z^Q = N^Q D^Q\) denote the Itô-Watanabe decompositions of the Azéma supermartingales of \(\sigma\) under \(P\) resp. \(Q\), then \(D^P_t = D^Q_t\) for all \(t \geq 0\) almost surely on the set \(\{Z^P > 0\} = \{Z^Q > 0\}\).

Proof. Corollary 3.3 implies the existence of a local \((P, \mathcal{F}_t)\)-martingale \(M\) such that
\[
h_t = \mathbb{E}(\rho_t 1_{\{\sigma > t\}} \mid \mathcal{F}_t) = \mathbb{E}(\left( M_t - \int_0^t \frac{d(N^P, M)_s}{N^P_s} 1_{\{\sigma > t\}} \right) \mid \mathcal{F}_t) = \left( M_t - \int_0^t \frac{d(N^P, M)_s}{N^P_s} \right) Z^P_t.
\]
Hence,
\[
\rho_t N^Q_t D^Q_t = \rho_t Z^Q_t = h_t = \left( M_t - \int_0^t \frac{d(N^P, M)_s}{N^P_s} \right) N^P_t D^P_t.
\]

Obviously, we have \(\{Z^P > 0\} = \{h > 0\} = \{Z^Q > 0\}\). Moreover, the process
\[
\left( M_t - \int_0^t \frac{d(N^P, M)_s}{N^P_s} \right) N^P_t = M_0 N^P_0 + \int_0^t N^P_s dM_s + \int_0^t \left( M_s - \int_0^s \frac{d(N^P, M)_u}{N^P_u} \right) dN^P_s
\]
is a non-negative local \((P, \mathcal{F}_t)\)-martingale. Since \(\rho N^Q\) is also a non-negative local \(P\)-martingale, the uniqueness of the Itô-Watanabe decomposition yields that
\[
\left( M_t - \int_0^t \frac{d(N^P, M)_s}{N^P_s} \right) N^P_t = \rho_t N^Q_t \quad \text{and} \quad D^P_t = D^Q_t
\]
for all \(t \geq 0\) almost surely on \(\{Z^P > 0\} = \{Z^Q > 0\}\).
The following counterexample shows that the assumption that \( \tilde{\rho} \) is continuous cannot be dropped in the above theorem.

**Counterexample 4.8.** Let \( \sigma \) be an honest time. We then have according to Lemma 3.11

\[
\sigma = \sup \{ t > 0 : N^P_t = \overline{N}^P_t \}, \quad Z^P_t = \frac{N^P_t}{\overline{N}^P_t}
\]

for some non-negative local martingale \( N^P \) converging to zero. Take \( \rho = \log(\overline{N}^P_\infty) = \log(N^P_\infty) = A^P_\infty \).

\[
\tilde{\rho}_t = \mathbb{E}^P(\rho|\mathcal{G}_t) = 1_{\{\sigma \leq t\}} \log(\overline{N}^P_t) + 1_{\{\sigma > t\}} \mathbb{E}^P\left( \left. \frac{\log(\overline{N}^P_\infty) 1_{\{\sigma > t\}}}{Z^P_t} \right| \mathcal{F}_t \right)
\]

\[
= 1_{\{\sigma \leq t\}} \log(\overline{N}^P_t) + 1_{\{\sigma > t\}} \overline{N}^P_t \cdot h_t
\]

\[
= 1_{\{\sigma \leq t\}} \log(\overline{N}^P_t) + 1_{\{\sigma > t\}} \frac{1 + \log(\overline{N}^P_t)}{N^P_t} \cdot N^P_t \int_0^\infty xe^{-x} dx
\]

\[
= 1_{\{\sigma \leq t\}} \log(\overline{N}^P_t) + 1_{\{\sigma > t\}} (1 + \log(\overline{N}^P_t)) = \log(\overline{N}^P_t) + 1_{\{\sigma > t\}},
\]

where \( h_t \) has already been computed in Example 3.11. Hence, \( \tilde{\rho} \) is a purely discontinuous \((P, \mathcal{G}_t)\)-martingale and

\[
\rho_t = \log(N^P_t) + \frac{N^P_t}{\overline{N}^P_t}.
\]

Therefore,

\[
Z^Q_t = \frac{h_t}{\rho_t} = \frac{N^P_t (1 + \log(\overline{N}^P_t))}{\overline{N}^P_t} \cdot \frac{1}{\rho_t} = \frac{1 + \log(\overline{N}^P_t)}{N^P_t} \frac{N^P_t}{N^P_t + \overline{N}^P_t \log(\overline{N}^P_t)} = \frac{N^P_t + \overline{N}^P_t \log(\overline{N}^P_t)}{N^P_t + \overline{N}^P_t \log(\overline{N}^P_t)}.
\]

And since \( N^P/\rho \in \mathcal{M}_{loc}(P, \mathcal{F}_t) \), the Itô-Watanabe decomposition of \( Z^Q \) takes the form

\[
Z^Q_t = \frac{N^P_t}{\rho_t} \cdot \frac{1 + \log(\overline{N}^P_t)}{N^P_t} = N^P_t \mathcal{D}^Q_t
\]

with

\[
\mathcal{D}^Q_t = \frac{1 + \log(\overline{N}^P_t)}{N^P_t} \neq \frac{1}{\overline{N}^P_t} = \mathcal{D}^P_t \quad \forall \ t > 0.
\]

**Remark 4.9.** In view of Examples 3.3 and 3.11 one may wonder whether for every random time \( \sigma \) the measure change \( \rho = f(A^P_\sigma) \) implies that \( \mu^L = N^P \), where \( f(\cdot) > 0 \) is chosen such that \( \mathbb{E}^P \rho = 1 \). Because of

\[
h_t = \mathbb{E}^P \left( \left. f(A^P_\sigma) 1_{\{\sigma > t\}} \right| \mathcal{F}_t \right) = \mathbb{E}^P \left( \int_t^\infty \left. f(A^P_u) dA^P_u \right| \mathcal{F}_t \right) = \int_t^\infty \mathbb{E}^P \left( \left. A^P_u \in dx \right| \mathcal{F}_t \right) \int_y^\infty f(y) dy
\]

\[
= \int_y A^P_u \int_y^\infty \mathbb{P}(A^P_\infty \in dx|\mathcal{F}_t) dy = \int_0^\infty f(y + A^P_t) \cdot \mathbb{P}(A^P_\infty - A^P_t \geq y|\mathcal{F}_t) dy,
\]

this is the case if and only if

\[
\mathbb{P}(A^P_\infty - A^P_t \geq y|\mathcal{F}_t) = N^P_t \cdot F_t(y)
\]

for some measurable function \( F_t(y)(\omega) : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) such that for all \( t \geq 0 \),

\[
\mathcal{D}^P_t = \int_0^\infty F_t(y) dy.
\]
5 Invariance of pseudo-stopping times

Since the definition of a pseudo-stopping time depends on the underlying probability measure, one may wonder whether there exist equivalent changes of probability measure which preserve the pseudo-stopping time property. Let us look at an example.

Example 5.1. For a standard \((\mathbb{P}, \mathcal{F}_t)\)-Brownian motion \(B\) define for all \(a \in \mathbb{R}\) and \(s \geq 0\) the stopping time
\[
\tau^a_s := \inf\{t > s : B_t = a\}
\]
as well as
\[
L := \sup\{t < \tau^1_0 : B_t = 0\}, \quad \sigma := \sup\{t < L : \overline{B}_t = B_t\} = \sup\{t < L : \overline{B}_L = B_t\}.
\]
It is well-known that \(\sigma\) is a \((\mathbb{P}, \mathcal{F}_t)\)-pseudo-stopping time. Let \(b : \mathbb{R} \to \mathbb{R}\) be a bounded function and set
\[
\rho_t = \mathcal{E} \left( \int_0^t b(B_s) dB_s \right).
\]
Then \((\rho_t)_{t \geq 0}\) is a positive \((\mathbb{P}, \mathcal{F}_t)\)-martingale which under some technical conditions on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) defines a measure \(Q\) on \(\mathcal{F}_{\infty}\) such that
\[
\frac{dQ}{d\mathbb{P}}_{\mathcal{F}_t} = \rho_t, \quad \forall \ t \geq 0.
\]
Note that in general \(Q\) is only locally equivalent to \(\mathbb{P}\), i.e. it may be singular to \(\mathbb{P}\) on \(\mathcal{F}_{\infty}\). By Girsanov’s theorem the process
\[
W_t := B_t - \int_0^t b(B_s) ds
\]
is a \(Q\)-Brownian motion and \(B\) is an Itô-diffusion. We denote its \(Q\)-scale function by \(s(\cdot)\). Using the Markov property of \(B\) we compute the \(Q\)-Azéma supermartingale of \(L\) as
\[
Z_t^{L,Q} := Q(L > t | \mathcal{F}_t) = Q(\tau^1_t > \tau^0_t | \mathcal{F}_t) = \frac{s(1) - s(B_t)}{s(1) - s(0)} \wedge 1.
\]
Since \(s\) is an increasing function,
\[
\sigma = \sup\{t < L : \overline{B}_L = B_t\} = \sup\{t < L : s(\overline{B}_L) = s(B_t)\} = \sup\{t < L : Z_t^{L,Q} = Z_L^{L,Q}\}.
\]
According to Proposition 5 in [11], \(\sigma\) is thus also a \(Q\)-pseudo-stopping time and
\[
Z_t^{\sigma,Q} := Q(\sigma > t | \mathcal{F}_t) = Z_t^{L,Q}.
\]
The previous example suggests that pseudo-stopping times can be robust with respect to (locally) equivalent changes of probability. Of course, this is not a universal result as the following counterexample shows.

Counterexample 5.2. Let \(\sigma\) be a \((\mathbb{P}, \mathcal{F}_t)\)-pseudo-stopping time and define the \(\mathcal{F}_\infty\)-measurable random variable \(\rho = 2Z_\sigma^{P}\). Since \(Z_\sigma^{P} \sim \mathcal{U}[0,1]\), the measure \(Q = \rho \mathbb{P}\) is well-defined and equivalent to \(\mathbb{P}\).

\[
\hat{\rho}_t = 2\mathbb{E}^\rho[Z_\sigma^{P} | \mathcal{G}_t] = 2\mathbbm{1}_{\{\sigma \leq t\}} Z_\sigma^{P} + 2\mathbbm{1}_{\{\sigma > t\}} \frac{\mathbb{E}^\rho[Z_\sigma^P \mathbbm{1}_{\{\sigma > t\}} | \mathcal{F}_t]}{Z_t^{P}}
\]
\[
= 2\mathbbm{1}_{\{\sigma \leq t\}} Z_\sigma^{P} + 2\mathbbm{1}_{\{\sigma > t\}} \frac{\mathbb{E}^\rho \left( \int_0^\infty (1 - A_u^\rho) dA_u^\rho | \mathcal{F}_t \right)}{Z_t^{P}}
\]
\[
= 2\mathbbm{1}_{\{\sigma \leq t\}} Z_\sigma^{P} + \mathbbm{1}_{\{\sigma > t\}} \frac{(1 - A_t^\rho)^2}{Z_t^{P}} = Z_{t \wedge \sigma} + \mathbbm{1}_{\{\sigma \leq t\}} Z_\sigma^{P}.
\]
which jumps at time $\sigma$. Moreover,
\[ h_t = \mathbb{E}^P(\tilde{\rho}_t \mathbf{1}_{(\sigma > t)}|\mathcal{F}_t) = Z_t^P \cdot \mathbb{E}^P(\mathbf{1}_{(\sigma > t)}|\mathcal{F}_t) = (Z_t^P)^2. \]

Since $\rho = \mathbb{E}^P(\rho|\mathcal{F}_\infty) = \rho_\infty \neq 1$ almost surely, the continuous uniformly integrable martingale $(\rho_t)$ is not identical to one. Therefore, having in mind that $(Z_t^P)$ is of finite variation,
\[ Z_t^Q = \frac{h_t}{\rho_t} = \frac{(Z_t^P)^2}{\rho_t} \]
cannot be of finite variation, which implies that $\sigma$ is not a $Q$-pseudo-stopping time.

On the other hand, suppose that there exists a measure $Q$ such that $\sigma$ is a $P$- and $Q$-pseudo-stopping time. In this case, if $\tilde{\rho}_t$ is strictly positive and continuous, Theorem 4.7 would imply that $Z_t^Q = Z_t^P$ almost surely for all $t \geq 0$. This observation led us to look for possible measure changes $\rho$ which preserve the pseudo-stopping time property for the class of $P$-pseudo-stopping times we dealt with in Subsection 3.3.

**Theorem 5.3.** Let $L$ be an honest time. Then $Z_t^L := P(L > t|\mathcal{F}_t) = M_t/M_t$ for some non-negative local $P$-martingale $M$ converging to zero. Define
\[ \sigma := \sup \left\{ t < L : Z_t^L = \inf_{u \leq L} Z_u^L \right\}. \]

If for a function $g : [0, 1] \to \mathbb{R}$ which satisfies
\[ \int_0^1 \exp \left( \int_z^1 g(y) dy \right) dz = 1, \]
the process
\[ \rho_t := \mathbb{E} \left( \int_0^t g \left( \frac{M_s}{M_s} \right) \frac{dM_s}{2M_s} \right) \]
is a uniformly integrable $(P, \mathcal{F}_t)$-martingale, then there exists a measure $Q \sim P$ such that $\sigma$ is a pseudo-stopping time with respect to $P$ and $Q$. Moreover, $Z_t^P = P(\sigma > t|\mathcal{F}_t) = Q(\sigma > t|\mathcal{F}_t) = Z_t^Q$ almost surely.

**Proof.** From [14], Proposition 5, it is known that $\sigma$ is a $P$-pseudo-stopping time. As usual set $Q = \rho_\infty \cdot P$ and define
\[ N_t := h \left( \frac{M_t}{\overline{M}_t} \right) \cdot \overline{M}_t, \]
where $h : [0, 1] \to \mathbb{R}_+$ is the function
\[ h(x) = \int_0^x \exp \left( \int_z^1 g(y) dy \right) dz. \]

Note that $h$ satisfies
\[ g(x)h'(x) + h''(x) = 0, \quad h(1) = h'(1) = 1, \quad h(0) = 0. \]

This implies that $N$ is a local $(Q, \mathcal{F}_t)$-martingale. Indeed,
\[ \overline{M}_t := M_t - \int_0^t g \left( \frac{M_s}{M_s} \right) \frac{d(M)_s}{2M_s} \]
is a local $Q$-martingale and
\[ dN_t = h \left( \frac{M_t}{\overline{M}_t} \right) d\overline{M}_t + \overline{M}_t h' \left( \frac{M_t}{\overline{M}_t} \right) \left[ \frac{dM_t}{\overline{M}_t} - \frac{dM_t}{M_t} \right] + \frac{1}{2} h'' \left( \frac{M_t}{\overline{M}_t} \right) \frac{d(M)_{t}}{\overline{M}_t} \]
\[ = [h(1) - h'(1)]d\overline{M}_t + h' \left( \frac{M_t}{\overline{M}_t} \right) d\overline{M}_t. \]
Furthermore, $h$ is strictly increasing. Therefore, $\overline{M} = \overline{N}$ and

$$L = \sup \{ t > 0 : M_t = \overline{M}_t \} = \sup \{ t > 0 : N_t = \overline{N}_t \}.$$

Since $N \to 0$ almost surely,

$$Z^{Q,L}_t := Q(L > t | \mathcal{F}_t) = \frac{N_t}{\overline{N}_t} = h \left( \frac{M_t}{\overline{M}_t} \right) = h \left( \frac{\overline{M}_t}{M_t} \right).$$

But then

$$\sigma = \sup \left\{ t < L : Z^L_u = \inf_{u \leq L} Z^L_u \right\} = \sup \left\{ t < L : Z^{Q,L}_u = \inf_{u \leq L} Z^{Q,L}_u \right\}$$

and $\sigma$ is a $Q$-pseudo stopping time by Proposition 5 of [14].

**Remark 5.4.** Even if $(\rho_t)$ is not a uniformly integrable martingale, there exists a measure $Q$ (under some technical conditions on the probability space) such that $\sigma$ is not only a $P$- but also a $Q$-pseudo-stopping time. However, the measure $Q$ will only be a dominating measure in general which is known as the so called Föllmer measure associated to $(\rho_t)$. Nevertheless, it is also not hard to find an example which satisfies the integrability assumption as one can see below.

**Example 5.5.** Consider $g(x) = x - c$, where $c > 0$ is chosen such that

$$\int_0^1 \exp \left( \frac{1 - z^2}{2} - c(1 - z) \right) dz = 1.$$

Using product integration,

$$\int_0^t \frac{M_t}{\overline{M}_t} dM_t = \frac{M_t^2}{\overline{M}_t} - 1 - \int_0^t M_t \left( \frac{dM_t}{\overline{M}_t} - \frac{2M_t}{\overline{M}_t} d\overline{M}_t \right) - \int_0^t \frac{d(\overline{M}_t)}{\overline{M}_t} \leq - \int_0^t \frac{M_t}{\overline{M}_t} dM_t + 2 \int_0^t \frac{d\overline{M}_t}{\overline{M}_t}$$

$$\Rightarrow \int_0^t \frac{M_t}{\overline{M}_t} dM_t \leq \log(\overline{M}_t)$$

$$X_t := \int_0^t \frac{dM_t}{\overline{M}_t} = \frac{M_t}{\overline{M}_t} - 1 + \int_0^t \frac{M_t}{\overline{M}_t} d\overline{M}_t = \frac{M_t}{\overline{M}_t} - 1 + \log(\overline{M}_t) \geq -1$$

$$Y_t := \int_0^t g \left( \frac{M_t}{\overline{M}_t} \right) d\overline{M}_t = \int_0^t \frac{M_t}{\overline{M}_t} d\overline{M}_t - c \int_0^t \frac{d\overline{M}_t}{\overline{M}_t} \leq c + \log(\overline{M}_t) \leq c + \log(\overline{M}_\infty).$$

First note that $(X_t)$ is a uniformly integrable martingale bounded from below, since

$$\mathbb{E}^P X_{\infty} = 0 - 1 + \mathbb{E}^P \log(\overline{M}_\infty) = 0 - 1 + 1 = 0,$$

where we have used the fact that $\log(\overline{M}_\infty) \sim \text{Exp}(1)$, cf. Lemma 3.10. Moreover,

$$\sup_{t \geq 0} \mathbb{E}^P X^2_t \leq \mathbb{E}^P \left( 1 + \log(\overline{M}_\infty) \right)^2 = \int_0^\infty (1 + x)^2 e^{-x} dx = 5.$$

Therefore $X$ is square-integrable and

$$\mathbb{E}^P \langle Y \rangle_{\infty} = \mathbb{E}^P \int_0^\infty g^2 \left( \frac{M_t}{\overline{M}_t} \right) d\langle X \rangle_t \leq (1 + c)^2 \cdot \mathbb{E}^P \langle X \rangle_{\infty} < \infty.$$

By the BDG inequality thus $\mathbb{E}^P \sup_{t \geq 0} |Y_t| < \infty$ and the dominated convergence theorem yields the martingality of $(Y_t)$. Moreover for all $t \geq 0$,

$$\mathbb{E}^P \exp \left( \frac{Y_t}{2} \right) \leq e^{c/2} \cdot \mathbb{E}^P \exp \left( \frac{\log(\overline{M}_\infty)}{2} \right) = e^{c/2} \cdot \mathbb{E}^P \sqrt{\overline{M}_\infty} = e^{c/2} \cdot \int_0^1 \frac{dx}{\sqrt{x}} = 2e^{c/2}.$$

Hence, by Jensen’s inequality $(\exp(Y_t/2))$ is a uniformly integrable submartingale and Kazamaki’s criterion implies the uniform integrability of $(\rho_t)$. 

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6 Financial Applications: No arbitrage up to a random time?

In the following we will work with a financial market model consisting of one risky security $S$ and a risk free bond. For simplicity, we assume that the interest rate is equal to zero. We suppose that the market satisfies NFLVR and w.l.o.g. $P$ is assumed to be the risk-neutral measure, i.e. $S$ is a positive local $(P, \mathcal{F}_t)$-martingale. A natural question is now, if the market is still arbitrage free after adding new information by enlarging the filtration progressively with $\sigma$.

In the case where $\sigma$ is an honest time this question has been discussed in details by [6]. Furthermore, it is known that NFLVR fails if $S$ does not remain a semimartingale in the enlarged filtration. Since this is only clear until time $\sigma$, we will in the following restrict ourselves to the question whether the market $(S_t \land \sigma, \mathcal{G}_t \land \sigma, P)$ is arbitrage-free. Note also that the question of market viability on the whole time horizon $[0, \infty)$ has previously been addressed in [3], where its connection to the so called $(\mathcal{H'})$-hypothesis has been pointed out.

For the reader’s convenience we first repeat some notions commonly used in finance: An $a$-admissible trading strategy for the $(\mathcal{F}_t)$-adapted price process $(S_t)$ is any $(\mathcal{F}_t)$-predictable process $(\theta_t)$, which is $(S_t)$ integrable such that the value process $V(x, \theta)_{t} := x + \int_{0}^{t} \theta_s dS_s$ satisfies $V(0, \theta) \geq -a$ $P$-almost surely. A trading strategy is admissible if it is $a$-admissible for some $a \in \mathbb{R}_+$. The notion of admissibility allows us to define two different concepts of market viability.

**Definition 6.1.** In the market model $(S_t, \mathcal{F}_t, P)$ there is

- an Arbitrage of the First Kind if and only if there exists a non-negative $\mathcal{F}_\infty$-measurable random variable $\xi$ with $P(\xi > 0) > 0$ such that for all $a > 0$ there exists an $a$-admissible trading strategy $\theta$ such that $V(a, \theta)_\infty \geq \xi$ almost surely. Otherwise we say that the market satisfies the NA1 (No Arbitrage of the First Kind) condition.

- a Free Lunch with Vanishing Risk (FLVR) if and only if there exists an $\varepsilon > 0$ and a sequence $(\theta^n)$ of $(\mathcal{F}_t)$-admissible strategies together with an increasing sequence $(\delta_n)$ of positive numbers converging to one such that $P(V(0, \theta^n)_\infty \geq -1 + \delta_n) = 1$ and $P(V(0, \theta^n)_\infty > \varepsilon) \geq \varepsilon$. Otherwise we say that the market satisfies the NFLVR (No Free Lunch with Vanishing Risk) condition.

Theorem 6.4 below gives the connection between the above no arbitrage criteria with the dual variables defined in

**Definition 6.2.** In the market model $(S_t, \mathcal{F}_t, P)$ we call

- a strictly positive local $(\mathcal{F}_t)$-martingale $(L_t)$ with $L_0 = 1$ and $L_\infty > 0$ almost surely a local martingale deflator, if the process $(L_t S_t)$ is a local $(\mathcal{F}_t)$-martingale.

- $\tilde{P} := L_\infty P$ an Equivalent Local Martingale Measure (ELMM), if there exists a local martingale deflator $(L_t)$ which is a uniformly integrable martingale closed by $L_\infty$.

**Remark 6.3.** NA1 is also known under the acronym NUPBR (No Unbounded Profit with Bounded Risk).

The following theorem contains results which are non-trivial but well-known. Their proofs can be found in [1] and [11].

**Theorem 6.4.** In the financial market model $(S_t, \mathcal{F}_t, P)$

- the NA1 condition is equivalent to the existence of a local martingale deflator.
the NFLVR condition is equivalent to the existence of an ELMM.

For the enlarged market model \((S_{t\leq\sigma}, \mathcal{G}_{t\wedge\sigma}, P)\) things can be defined in an analogous way. For notational convenience we will write \(Z = m - A\) instead of \(Z^P = m^P - A^P\) for the Azéma supermartingale of \(\sigma\) under \(P\) for the rest of this section.

The following theorem gives a necessary criterion to have NFLVR on the time horizon \([0, T \wedge \sigma]\), where \(T\) is a \((\mathcal{G}_t)\)-stopping time. In the case of \(\sigma\) being an honest time the following statement can be found in [6] together with a long technical proof. However, we will give an apparently new proof of the statement, valid for all random times, which appeals to purely intuitive reasoning.

**Theorem 6.5.** Let \(T\) be a \((\mathcal{G}_t)\)-stopping time. If \(P(T^\mathbb{Z}_0 \leq T) = 0\), then NFLVR also holds in the enlarged financial market on the time horizon \([0, \sigma \wedge T]\).

The idea of the proof is that even at time \(T\) we cannot be sure that \(\sigma\) has already occurred. In fact \(\sigma\) may still happen only after the stopping time \(T\) because \(P(T^\mathbb{Z}_0 \leq T) = 0\).

**Proof.** W.l.o.g. we may assume that \(T\) is actually an \((\mathcal{F}_t)\)-stopping time, cf. Lemma [3.1]. Note that the condition \(P(T^\mathbb{Z}_0 \leq T) = 0\) is in fact equivalent to

\[
P(\forall t \in [0, T]: Z_t > 0) = 1.
\]

We proceed by contradiction: Assume that there is a FLVR in the enlarged market on the time horizon \([0, \sigma \wedge T]\). Then there exists a sequence of \((\mathcal{G}_t)\)-admissible trading strategies \((\theta^n)_{n \in N}\) and an increasing deterministic sequence \((\delta_n)\) converging towards 1 such that for some \(\varepsilon > 0\) and all \(n \in N\),

\[
P(V(0, \theta^n)_{\sigma \wedge T} > -1 + \delta_n) = 1, \quad P(V(0, \theta^n)_{\sigma \wedge T} > \varepsilon) \geq \varepsilon.
\]

With the help of Lemma [3.1] we can find for every \(n \in N\) an \((\mathcal{F}_t)\)-predictable process \((y^n_t)\) such that

\[
\theta^n_t \mathbb{1}_{\{t \leq \sigma\}} = y^n_t \mathbb{1}_{\{t \leq \sigma\}}.
\]

We will prove that

\[
P(V(0, y^n)_T > -1 + \delta_n) = 1.
\]

Assume that this was not the case, i.e.

\[
P(V(0, y^n)_T \leq -1 + \delta_n) > 0.
\]

Since \(Z_T > 0\) almost surely, this would imply that

\[
0 < \mathbb{E}^P \left( \mathbb{1}_{\{V(0, y^n)_T \leq -1 + \delta_n\}} Z_T \right) = P(V(0, y^n)_T \leq -1 + \delta_n; \sigma > T)
\]

\[
= P(V(0, \theta^n)_{\sigma \wedge T} \leq -1 + \delta_n; \sigma > T) \leq P(V(0, \theta^n)_{\sigma \wedge T} \leq -1 + \delta_n),
\]

a contradiction. Similarly one shows that for all \(r \in \mathbb{Q} \cap [0, T]\),

\[
P(V(0, y^n)_r > -a_n) = 1,
\]

for some \(a_n \in \mathbb{R}_+\), because each \(\theta^n\) is assumed to be admissible. Since \(V(0, y^n)\) is continuous, this implies that

\[
P(\forall t \leq T: V(0, y^n)_t \geq -a_n) = 1
\]

and thus \(y^n\) is admissible. For every \(n \in N\) we define the \((\mathcal{F}_t)\)-trading strategy

\[
y^n_t := y^n_t \mathbb{1}_{\{0 \leq t \leq T^n\}},
\]

where

\[
T^n_\varepsilon := \inf\{t \geq 0: V(0, y^n)_t = \varepsilon\}.
\]
Obviously, $\vartheta^n$ is admissible and
$$P(V(0, \vartheta^n)_T > -1 + \delta_n) \geq P(V(0, y^n)_T > -1 + \delta_n) = 1.$$ 

Moreover,
$$P\left(V(0, \vartheta^n)_T > \frac{\varepsilon}{2}\right) \geq P(\exists u \leq T : V(0, y^n)_u \geq \varepsilon) \geq P(\exists u \leq \sigma \wedge T : V(0, \theta^n)_u \geq \varepsilon) \geq P(V(0, \theta^n)_{\sigma \wedge T} \geq \varepsilon) > \varepsilon.$$

Choosing $\tilde{\varepsilon} := \varepsilon/2$, this would give a FLVR with respect to $(\mathcal{F}_t)$ and thus a contradiction because $S$ is assumed to be a local $(\mathcal{P}, \mathcal{F}_t)$-martingale. $\square$

In [6] it is moreover shown that the condition $P(T^2_0 \leq T) = 0$ is not only sufficient but also necessary to have NFLVR on $[0, T \wedge \sigma]$, if $\sigma$ is honest and the market is complete. However, the condition $P(T^2_0 \leq T) = 0$ is not in general necessary, even in a complete market, as the following example shows.

**Example 6.6.** Let $\sigma$ be a pseudo-stopping time bounded by one. Then $1 - Z_1 = A_1 = 1$ and therefore $P(T^2_0 \leq 1) = 1$. However, since $\sigma$ is a pseudo-stopping time $(S_{t \wedge \sigma})$ is a local $(\mathcal{G}_t)$-martingale and therefore NFLVR holds in the enlarged market on the interval $[0, \sigma] = [0, \sigma \wedge 1]$.

The following Lemma was proven in [6] in the case of honest times, where it was remarked that it also holds in more generality. For completeness we provide a proof as well.

**Lemma 6.7.** The process $(1/N_{t \wedge \sigma})_{t \geq 0}$ is a local martingale deflator for $(S_{t \wedge \sigma})$ in the filtration $(\mathcal{G}_t)$, i.e. $\text{NA1}$ holds with respect to $(\mathcal{G}_t)$ on the time horizon $[0, \sigma]$.

**Proof.** First note that the process $1/N_{t \wedge \sigma}$ is well-defined, since $Z_\sigma = Z_{\sigma-} > 0$, cf. [10]. From the enlargement formula the processes
$$\tilde{S}_t = S_{t \wedge \sigma} - \int_0^{t \wedge \sigma} \frac{d(S,N)_s}{N_s}$$

and
$$\tilde{N}_t = N_{t \wedge \sigma} - \int_0^{t \wedge \sigma} \frac{d(N)_s}{N_s}$$

are local $(\mathcal{P}, \mathcal{G}_t)$-martingales. With Itô’s formula we then have on $[0, \sigma]$,
$$d\left(\frac{S}{N}\right) = \frac{dS}{N} - \frac{S}{N^2}dN + \frac{S}{N^3}d\langle N \rangle - \frac{d\langle S,N \rangle}{N^2} = \frac{S}{N} \left(\frac{d\tilde{S}}{S} + \frac{d\langle S,N \rangle}{SN} - \frac{d\tilde{N}}{N} - \frac{d\langle N \rangle}{N^2} + \frac{d\langle N \rangle}{N^2} - \frac{d\langle S,N \rangle}{SN}\right) = \frac{S}{N} \left(\frac{d\tilde{S}}{S} - \frac{d\tilde{N}}{N}\right).$$

Especially, taking $S \equiv 1$ yields that $1/N_{t \wedge \sigma} \in \mathcal{M}_{loc}(\mathcal{P}, \mathcal{G}_t)$. $\square$

Next we prove a sufficient and necessary criterion such that $1/N_{t \wedge \sigma}$ is a uniformly integrable martingale on the time interval $[0, \sigma \wedge T]$, where $T$ is a $(\mathcal{G}_t)$-stopping time. For this we need the Itô-Watanabe decomposition of the Azéma supermartingale of $\sigma$ which we denote as before by $Z = ND$, where $N$ and $D$ are defined as in Remark [4.5]

**Theorem 6.8.** Let $T$ be a $(\mathcal{G}_t)$-stopping time. Then,
$$\left(\frac{1}{N_{t \wedge \sigma \wedge T}}\right) \in \mathcal{M}_{u.i.}(\mathcal{P}, \mathcal{G}_t) \iff P(T \geq T^N_0 < T^D_0) = 0.$$
Proof. The local \((\mathcal{G}_t)\)-martingale \((1/N_{t \land \sigma \land T})_{t \geq 0}\) is a uniformly integrable martingale if and only if 
\[ \mathbb{E}^P(1/N_{\sigma \land T}) = 1. \] Again, we may assume w.l.o.g. that \(T\) is actually an \((\mathcal{F}_t)\)-stopping time, cf. Lemma 3.1. Since \(\sigma < T^D_0 = T^N_0 \land T^D_0\) almost surely,

\[
\mathbb{E}^P \left( \frac{1}{N_{\sigma \land T}} \right) = \mathbb{E}^P \left( \frac{1_{\{T^Z_T > \sigma\}}}{N_{\sigma \land T}} \right) = \mathbb{E}^P \left( \int_0^{T^Z_T} \frac{dA_u}{N_{u \land T}} \right) = \mathbb{E}^P \left( \int_0^{T^Z_T} \frac{dA_u}{N_u} + 1_{\{T^Z_T > \sigma\}} \frac{A_{T^Z_T} - A_T}{N_T} \right) 
\]

\[
= \mathbb{E}^P \left( \int_0^{T^Z_T} \frac{dA_u}{Z_u} + 1_{\{T^Z_T > \sigma\}} \frac{Z_T}{N_T} \right) = \mathbb{E}^P \left( - \int_0^{T^Z_T} \frac{dD_u}{Z_u} + \frac{D_u}{N_u} + 1_{\{T^Z_T > \sigma\}} D_T \right) 
\]

\[
= \mathbb{E}^P \left( 1 - D_{T^Z_T} + 1_{\{T^Z_T > \sigma\}} D_T \right) = 1 - \mathbb{E}^P \left( D_{T^Z_T} \mathbbm{1}_{\{T^Z_T \leq T\}} \right) 
\]

where in the last equality we have used that \(\text{supp}(dD) \subset \{Z > 0\}\), cf. Remark 4.5. Finally note that

\[
\mathbb{E}^P \left( D_{T^Z_T} \mathbbm{1}_{\{T^Z_T \leq T\}} \right) = 0 \iff \mathbb{P}(T > T^N_0 < T^D_0) = 0. 
\]

By taking \(T = \infty\) in Theorem 6.8 we get

Corollary 6.9. If \(\mathbb{P}(T^N_0 < T^D_0) = 0\), then NFLVR holds on the interval \([0, \sigma]\) with respect to the filtration \((\mathcal{G}_t)\).

Remark 6.10. For an honest time \(\sigma\) the multiplicative decomposition of \(Z\) is given by \(Z_\sigma = N_\sigma / N_t\), where \(N\) is a non-negative local martingale converging to zero. And since a non-negative local martingale does not explode almost surely,

\[
D_\infty = \frac{1}{N_\infty} > 0 \quad \text{a.s.}
\]

Therefore, \(\mathbb{P}(T^D_0 = \infty) = 1\) and \(1/N_{t \land \sigma \land T}\) is a uniformly integrable martingale if and only if \(\mathbb{P}(T^N_0 \leq T) = \mathbb{P}(T^Z_T \leq T) = 0\). Therefore, if \(T^N_0 = \infty\) almost surely, \((1/N_{t \land \sigma \land T})_{t \geq 0}\) is actually a true martingale and not a strict local martingale. In this case the fact that it is not uniformly integrable is already evident from the fact that \(N_\sigma = N_\infty > 1\) almost surely. Moreover, an application of Doob’s maximal identity (cf. Lemma 2.1 in [15]) gives

\[
\mathbb{E}^P \left( \frac{1}{N_{\sigma}} \right) = \mathbb{E}^P \left( \frac{1}{N_{\infty}} \right) = \int_0^\infty \mathbb{P} \left( \frac{1}{N_{\sigma}} > x \right) dx = \int_0^1 \mathbb{P} \left( \frac{1}{x} > N_{\infty} \right) dx = \int_0^1 (1-x) dx = \frac{1}{2}.
\]

Corollary 6.11. Let \(T\) be a \((\mathcal{G}_t)\)-stopping time. If either \(\mathbb{P}(T^Z_T \leq T) = 0\) or \(D_\infty = 0\) almost surely, then NFLVR holds in the enlarged market on the time interval \([0, T \land \sigma]\).

Proof. Note that the first claim is actually Theorem 6.3, but we can also derive it directly from Theorem 6.8. If \(\mathbb{P}(T^Z_T \leq T) = 0\), then \(\mathbb{P}(T \geq T^N_0) = \mathbb{P}(T \geq T^N_0 \land T^Z_T) = 0\).

Moreover, if \(D_\infty = D_{T^Z_T} = 0\) almost surely, then \(\mathbb{P}(T^D_0 \leq T^N_0) = 1\).

Hence, the claim follows from a combination of Theorem 6.8, Lemma 6.7 and Theorem 6.3. □

Of course, every pseudo-stopping time fulfills \(D_\infty = 1 - A_\infty = 1 - 1 = 0\). The following example shows that there are also other random times, which allow for an equivalent local martingale measure up to time \(\sigma\), even in a complete market.

Example 6.12. Let \(W\) be a \((\mathbb{P}, \mathcal{F}_t)\)-Brownian motion and set \(\sigma = \sup\{t \leq 1 : 2W_t = W_1\}\). The corresponding Azéma supermartingale is

\[
Z_t = \sqrt{\frac{2}{\pi}} \int_{\frac{|W_t|}{\sqrt{t}}}^\infty x^2 e^{-x^2/2} dx = m_t - \sqrt{\frac{2}{\pi}} \int_0^t \frac{|W_u|}{(1-u)^{3/2}} \exp \left( -\frac{W_u^2}{2(1-u)} \right) du,
\]

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For every $n \in \mathbb{N}$ define the set
\[ B_n = \left\{ |W_u| > \sqrt{\frac{2}{n}} \forall u \in \left[ 1 - \frac{1}{n}, 1 \right] \right\} \]
and note that
\[ 1 = \mathbb{P}(W_1 \neq 0) = \lim_{n \to \infty} \mathbb{P}(B_n). \]
On the set $B_n$ we have for all $u \in \left[ 1 - \frac{1}{n}, 1 \right]$:
\[ \frac{|W_u|}{\sqrt{1 - u}} > \sqrt{2} \Rightarrow \frac{1}{2} \int_{1-\frac{1}{n}}^1 \frac{dA_t}{\sqrt{Z_t}} = \frac{1}{2} \int_{1-\frac{1}{n}}^1 \frac{dA_t}{\sqrt{\frac{2}{n} \exp(-\frac{W_u^2}{2(1-u)})}} = \frac{1}{2} \int_{1-\frac{1}{n}}^1 \frac{dt}{1-t} = \infty. \]
Therefore, on each $B_n$ we have
\[ D_\infty = D_1 = \exp\left(-\int_0^1 \frac{dA_t}{Z_t}\right) = 0, \]
and by monotone convergence
\[ \mathbb{E}^\mathbb{P}(D_\infty) = \lim_{n \to \infty} \mathbb{E}^\mathbb{P}(D_\infty \mathbb{1}_{B_n}) = 0 \iff D_\infty \equiv 0. \]
Unfortunately, the above results do only provide sufficient but not necessary conditions for NFLVR until time $\sigma$. In fact there may exist other local martingale deflators than $(1/N_{\sigma \wedge t})$ which could be uniformly integrable martingales. Even though the structure of all local martingale deflators can be derived as in [6], we cannot prove that they fail to be uniformly integrable martingales in general unless $N_{\sigma} > 1$ almost surely, which is e.g. the case for honest times, cf. Lemma 3.3 in [6].

### 7 Locally absolutely continuous change of measure

In this section we slightly change the general setup introduced in section 2.1. We will no longer rely on the existence of a random variable $\rho \geq 0$ to define $Q$, but instead we will only assume the existence of some non-negative $(P, \mathcal{G}_t)$-martingale $(\tilde{\rho}_t)$ with expectation one. As before $(\rho_t)$ is the $(\mathcal{F}_t)$-optional projection of $(\tilde{\rho}_t)$. Moreover, we will assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is the natural augmentation of a probability space satisfying condition $(P)$ introduced in [13].

For every $t \geq 0$ we now define a probability measure $Q_t$ on $\mathcal{G}_t$ via $Q_t = \tilde{\rho}_t P|_{\mathcal{G}_t}$. This family of probability measures is consistent and since we assume our probability space to satisfy the natural (but not the usual!) assumptions, Corollary 4.9 of [13] yields the existence of a measure $Q$ on $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{G}_t$ such that $Q|_{\mathcal{G}_t} = Q_t$ for all $t \geq 0$. Note that $Q$ is only locally absolutely continuous to $P$ which we denote by $Q \ll P$. We define the process $h$ in this case by $h_t = \mathbb{E}^P(\tilde{\rho}_t \mathbb{1}_{\{\sigma > t\}}|\mathcal{F}_t)$. If $Q \ll P$, this definition coincides with the one in section 2.1 $\mu$ can now be defined as before.

In this setting the following slightly extended version of Theorem 2.1 holds.

**Theorem 7.1.** Assume that $Q \ll P$. If $U = (U_t)_{t \geq 0}$ is a local $(P, \mathcal{F}_t)$-martingale, then the processes $X_t := \mathbb{1}_{\{\sigma > t\}} V_t \exp(\mu_t^P)$ and $V_t$ are local $(Q, \mathcal{G}_t)$-martingales, where $V_t := U_{t \wedge \sigma} - \langle U, \mu \rangle_{t \wedge \sigma}$. 


Proof. Since \( Q|_{\mathcal{G}_n} \ll P|_{\mathcal{G}_n} \) the claim holds for every \( U^n := U_{t\wedge n} \) according to Theorem 2.5. Especially, all processes are well-defined on \( \bigcup_{n\in\mathbb{N}}[0,n] = \mathbb{R}_+ \). But every process which is locally in \( \mathcal{M}_{loc}(Q,\mathcal{G}_t) \) is actually a local martingale on the whole time interval.

The motivation to study locally absolutely continuous changes of measures comes from the fact that it may allow us to get rid off the random time \( \sigma \) by pushing it to infinity as the following example demonstrates.

Example 7.2. Consider

\[
\tilde{\rho}_t = \frac{1_{\{\sigma > t\}}}{Z^P_t}.
\]

This does indeed define a \((\mathcal{G}_t)\)-martingale: For \( s \leq t \),

\[
\mathbb{E}^P \left( \frac{1_{\{\sigma > t\}}}{Z^P_s} \middle| \mathcal{G}_s \right) = \frac{1_{\{\sigma > s\}}}{Z^P_s} \cdot \mathbb{E}^P \left( \frac{1_{\{\sigma > t\}}}{Z^P_t} \middle| \mathcal{F}_s \right) = \frac{1_{\{\sigma > s\}}}{Z^P_s}.
\]

Under the measure \( Q \) defined as above \( \sigma \) is pushed to infinity since

\[
Q(\sigma \leq t) = \mathbb{E}^P \left( \tilde{\rho}_t 1_{\{\sigma \leq t\}} \right) = \mathbb{E}^P \left( \frac{1_{\{\sigma > t\}}}{Z^P_t} 1_{\{\sigma \leq t\}} \right) = 0 \quad \forall \ t \geq 0.
\]

This is possible because \( \tilde{\rho}_t \to 0 \) and therefore \( Q \) is not absolutely continuous to \( P \) on \( \mathcal{F} \). Therefore, \( Q \) puts only positive weight on those events taking place before \( \sigma \).

The fact that \( \rho_t = 1 \) for all \( t \geq 0 \) implies that all \( \mathcal{F}_t \)-events do not ”feel” the change of measure. Especially, any \((P,\mathcal{F}_t)\)-martingale is also a \((Q,\mathcal{F}_t)\)-martingale and by Theorem 7.1 also a \((Q,\mathcal{G}_t)\)-martingale because \( h_t = \rho_t \equiv 1 \).

Note that in computations of pre-\( \sigma \) events this measure change has the same impact as simply projecting down on \((\mathcal{F}_t)\). Indeed, every \( \mathcal{G}_t \)-measurable random variable is equal to an \( \mathcal{F}_t \)-measurable random variable before time \( \sigma \), and for every \( F_t \in \mathcal{F}_t \) one has

\[
\mathbb{E}^P(F_t 1_{\{\sigma > t\}}) = \mathbb{E}^P \left( \frac{1_{\{\sigma > t\}}}{Z^P_t} \cdot F_t Z^P_t \right) = \mathbb{E}^Q(F_t Z^P_t) = \mathbb{E}^P(F_t Z^P_t).
\]

7.1 A measure change which is equivalent to the enlargement formula

As before we denote by \( Z^P = N^P D^P \) the Itô-Watanabe decomposition of the Azéma supermartingale of \( \sigma \). Under the assumption that \( N^P \) is a true martingale, we may define

\[
\hat{\rho}_t = \frac{1_{\{\sigma > t\}}}{D^P_t}.
\]

One easily checks that this indeed gives a \((\mathcal{G}_t)\)-martingale: For \( s \leq t \),

\[
\mathbb{E}^P \left( \frac{1_{\{\sigma > t\}}}{D^P_s} \middle| \mathcal{G}_s \right) = \frac{1_{\{\sigma > s\}}}{Z^P_s} \cdot \mathbb{E}^P \left( \frac{1_{\{\sigma > t\}}}{D^P_t} \middle| \mathcal{F}_s \right) = \frac{1_{\{\sigma > s\}}}{Z^P_s} \cdot \mathbb{E}^P(N^P_t \middle| \mathcal{F}_s) = \frac{1_{\{\sigma > s\}}}{D^P_s}.
\]

As in Example 7.2 we have \( Q(\sigma < \infty) = 0 \) and hence any local \((Q,\mathcal{F}_t)\)-martingale is also a local \((Q,\mathcal{G}_t)\)-martingale. However,

\[
h_t = \rho_t = N^P_t
\]

is non-trivial and therefore the measure change will affect \((P,\mathcal{F}_t)\)-martingales according to the usual Girsanov theorem. Indeed, changing the measure in this way has the same effect as an application of the enlargement formula under \( P \). This can be compared to [10], where the enlargement formula was derived by passing to the so called Föllmer measure associated with \( Z^P \).

In our setup we have for any \( \mathcal{F}_t \)-measurable random variable \( F_t \),

\[
\mathbb{E}^P(F_t 1_{\{\sigma > t\}}) = \mathbb{E}^Q(F_t D^P_t).
\]
Since $D^P$ is decreasing, one can interpret $D^P_t$ as a discount factor in the above formula.

The following example provides some intuition why the above measure change pushes $\sigma$ to infinity.

**Example 7.3.** Consider the honest time

$$\sigma = \sup \left\{ t \geq 0 : N_t = \sup_{s \leq t} N_s \right\} = \sup \left\{ t \geq 0 : \frac{1}{N_t} = \inf_{s \leq t} \frac{1}{N_s} \right\},$$

where $N$ is supposed to be a non-negative $(P, F_t)$-martingale with $N_0 = 1$, converging towards zero almost surely. If we take $\tilde{\rho}_t$ as above, the reciprocal of $N$ becomes a $Q$-martingale: For $s \leq t$,

$$E^Q \left( \frac{1}{N_t} \big| F_s \right) = \frac{1}{\rho_s} E^P \left( \frac{\rho_t}{N_t} \big| F_s \right) = \frac{1}{N_s}.$$

However, $1/N$ does not converge to infinity but to zero under $Q$ because $Q$ is singular to $P$ on $F_\infty$. For all $\varepsilon > 0$ we have by dominated convergence as $t \to \infty$,

$$Q \left( \frac{1}{N_t} > \varepsilon \right) = E^P (N_t \mathbb{1}_{\{1/\varepsilon > N_t\}}) \to 0.$$

Therefore, $\sigma$ equals infinity almost surely under $Q$.

**Remark 7.4.** In the above computations we have assumed that $N^P$ is a true martingale. If $N^P$ is only a local $(P, F_t)$-martingale, analogous computations can be done if one defines $Q$ as the Föllmer measure associated to $(\tilde{\rho}_t)$. In this case the random time $\sigma$ is "replaced" under $Q$ by the explosion time of $(\tilde{\rho}_t)$, which equals the $(F_t)$-stopping time $T^{P^0}_{\tilde{\rho}}$ $Q$-almost surely.

## 8 An extension for honest times

So far we were only concerned with the time horizon $[0, \sigma]$. Of course, we cannot expect an analogue of Theorem 2.1 to hold after time $\sigma$ because in general $(F_t)$-semi-martingales are not necessarily $(G_t)$-semi-martingales after time $\sigma$. Therefore, in this section we will assume that $\sigma$ is an honest time. In this case it is well-known that the semi-martingale property is preserved when passing from $(F_t)$ to $(G_t)$. Our goal is to proceed similarly to [12] in that we do not apply any results from the theory of enlargements of filtrations.

### 8.1 Change of measure after time $\sigma$

As before we assume that there exists a non-negative random variable $\rho$ with expectation one and we set $Q = \rho \cdot P$. We define the $(P, F_t)$-submartingale $k$ via

$$k_t = E^P (\rho \big| F_t) - h_t = E^P (\rho \mathbb{1}_{\{s \leq t\}} \big| F_t).$$

In the following we will use for fixed $u \geq 0$ the notation

$$\mathcal{M}_u^\alpha (P, F_t)$$

to denote the class of processes which are $(P, F_t)$-martingales on the interval $[u, \infty)$. Moreover, for each $t \geq 0$ we choose an $F_t$-measurable random variable $\sigma_t$ which satisfies the requirement of Definition 3.7, i.e. $\mathbb{1}_{\{\sigma_t < t\}} \sigma = \mathbb{1}_{\{\sigma_t < t\}} \sigma_t$.

**Lemma 8.1.** Let $Y$ be an $(F_t)$-adapted process such that $(\mathbb{1}_{\{\sigma_t < u\}} k_t Y_t)_{t \geq u} \in \mathcal{M}_u^\alpha (P, F_t)$ for some fixed $u \geq 0$. Then $Y_t \mathbb{1}_{\{\sigma_t < u\}} \in \mathcal{M}_u^\alpha (Q, \mathcal{G}_t)$.

**Remark 8.2.** As can be seen from the proof the reverse implication of the above statement holds as well.
Lemma 8.4. \(G\) then it is also a martingale on any \(Y\).

Proof. We approximate \(\sigma\) by \(u > 0\). The monotone class theorem allows us to conclude that \(Y_t \mathbf{1}_{\{\sigma \leq u\}}\) is a \(Q\)-martingale with respect to \((F_t \vee \sigma(\mathbf{1}_{\{\sigma \leq r\}}; r \leq t))_{t \geq 0}\). Because martingales with respect to some filtration remain martingales with respect to its right-continuous augmentation, we thus conclude that \(Y_t \mathbf{1}_{\{\sigma \leq u\}} \in M^u(G_t, Q)\).

Remark 8.3. Note that if \((Y_t)_{t \geq 0}\) is a martingale with respect to \(Q\) and \((G_t)_{t \geq u}\) on the set \(\{\sigma \leq u\}\), then it is also a martingale on any \(G_u\)-measurable subset of \(\{\sigma \leq u\}\). Thus, for example

\[(Y_t \mathbf{1}_{\{u \leq \sigma < u_j\}})_{\ell \geq u} \in M^u(G_t, Q)\]

for every \(0 \leq u_i < u_j \leq u\).

Lemma 8.4. Let \(Y = (Y_t)_{t \geq 0}\) be a process such that \((\mathbf{1}_{\{\sigma \leq u\}}(Y_{t \vee u} - Y_u))_{t \geq 0} \in M_{loc}(Q, G_t)\) for all \(u > 0\). Then \(Y_{t \vee \sigma} - Y_\sigma \in M_{loc}(Q, G_t)\).

Proof. Let us first assume that \(Y\) is bounded. Then by Remark 8.3, for all \(u > v \geq 0\),

\[(\mathbf{1}_{\{\sigma \leq u\}}(Y_{t \vee u} - Y_u))_{t \geq 0} \in M(Q, G_t)\]

We approximate \(\sigma\) with the decreasing sequence of \((G_t)\)-stopping times

\[s_n := \sum_{k=1}^{n2^n} \mathbf{1}_{\{(k-1)2^{-n} \leq \sigma < k2^{-n}\}} + \infty \mathbf{1}_{\{\sigma \geq n\}}\]

taking only finitely many values. Then for \(s \leq t\) and \(G_s \in G_s\), because \(Y\) is assumed to be bounded and càdlàg,

\[E^Q((Y_{t \vee \sigma} - Y_\sigma) \mathbf{1}_{G_s}) = \lim_{n \to \infty} E^Q((Y_{t \vee s_n} - Y_{s_n}) \mathbf{1}_{G_s})\]

\[= \lim_{n \to \infty} E^Q\left(\sum_{k=1}^{n2^n} \mathbf{1}_{\{s_n=k2^{-n}\}}(Y_{t \vee (k2^{-n})} - Y_{k2^{-n}}) \mathbf{1}_{G_s}\right)\]

\[= \lim_{n \to \infty} \sum_{k=1}^{n2^n} E^Q\left(\mathbf{1}_{\{(k-1)2^{-n} \leq \sigma < k2^{-n}\}}(Y_{t \vee (k2^{-n})} - Y_{k2^{-n}}) \mathbf{1}_{G_s}\right)\in M(Q, G_t)\]

\[= \lim_{n \to \infty} \sum_{k=1}^{n2^n} E^Q\left(\mathbf{1}_{\{(k-1)2^{-n} \leq \sigma < k2^{-n}\}}(Y_{s \vee (k2^{-n})} - Y_{k2^{-n}}) \mathbf{1}_{G_s}\right)\]

\[= \lim_{n \to \infty} E^Q\left(\sum_{k=1}^{n2^n} \mathbf{1}_{\{s_n=k2^{-n}\}}(Y_{s \vee (k2^{-n})} - Y_{k2^{-n}}) \mathbf{1}_{G_s}\right)\]

\[= \lim_{n \to \infty} E^Q((Y_{s \vee s_n} - Y_{s_n}) \mathbf{1}_{G_s}) = E^Q((Y_{s \vee \sigma} - Y_\sigma) \mathbf{1}_{G_s})\]

which proves that \(Y_{t \vee \sigma} - Y_\sigma \in M(Q, G_t)\). Now the general case follows by localization of \(Y\).
Theorem 8.5. Let \( \sigma \) be an honest time and suppose that \((U_t)_{t \geq 0}\) is local \((P, F_t)\)-martingale. Then the process
\[
V_t := U_t + \int_0^{\sigma \wedge t} \frac{d(U,h)_s}{h_s} - \int_{\sigma}^{\sigma \vee t} \frac{d(U,k)_s}{k_s}
\]
is a local \((Q, G_t)\)-martingale.

Proof. From Theorem 2.5 we already know that \((V_t)_{t \geq 0}\) is a local \((Q, G_t)\)-martingale. Therefore, it remains to show that \(V_t - V_\sigma \in M_{\text{loc}}(Q, G_t)\). According to Lemma 8.1 this holds if for all \(u > 0\),

\[
\mathbb{1}_{\{\sigma \leq u\}} (V_t - V_\sigma) \equiv \mathbb{1}_{\{\sigma \leq u\}} V_t^u \in M_{\text{loc}}(Q, G_t)
\]

where we have defined for each \(s \in \mathbb{R}_+\) the \((F_t)\)-adapted process
\[
V_t^s := U_{t \vee s} - U_s - \int_s^{t \wedge s} \frac{d(k,U)_u}{k_u}.
\]

Therefore, an application of Lemma 8.1 will yield the result, if we can show that for all \(u \geq 0\),

\[
(\mathbb{1}_{\{\sigma \leq u\}} k_t V_t^u)_{t \geq u} \in M_{\text{loc}}^u(P, F_t).
\]

First note that
\[
m_t^u := \mathbb{1}_{\{\sigma \leq u\}} k_t = \mathbb{E}^P(\rho \mathbb{1}_{\{\sigma \leq t, \sigma \leq u\}} | F_t) = \mathbb{E}^P(\rho \mathbb{1}_{\{\sigma \leq u \wedge t\}} | F_t)
\]

and hence for every fixed \(u > 0\), \(m^u \in M(P, F_t)\). We apply integration by parts for \(t \geq u\) to get
\[
d(\mathbb{1}_{\{\sigma \leq u\}} k_t V_t^u) = d(m_t^u V_t^u) = V_t^u dm_t^u + m_t^u dV_t^u + d(m^u, V^u)_t
\]

which is an element of \(M_{\text{loc}}^u(P, F_t)\) for every \(u > 0\) as required. \(\Box\)

Remark 8.6. In fact a more general version of Theorem 8.5 is known to hold even without assuming \((AC)\). This can be proven by applying first Girsanov’s theorem and second the enlargement formula for honest times as it is done in paragraph 81 in [3]. Note however, that our proof does not make use of the enlargement formula. It only uses Definition 8.1 of an honest time. Therefore as a byproduct by setting \(\rho \equiv 1\) we do actually recover the enlargement formula after \(\sigma\) for honest times.

Example 8.7. (Continuation of Example 8.12) We set \(\sigma_t = \sup\{u \leq t \wedge T^B_1 : B_u = 0\}\).

\[
k_t = \mathbb{E}^P(\mathbb{1}_{\{\sigma \leq t\}} | F_t) = \mathbb{E}^P(f(B_\sigma) \mathbb{1}_{\{\sigma \leq t\}} | F_t) = \mathbb{E}^P(f(B_\sigma) | F_t)
\]

\[
dk_t = f(B_{\sigma t})(1 - Z_t^\sigma) = f(B_\sigma) B_{t \wedge T^B_1}^+ + \frac{1}{2} dB_{t \wedge T^B_1^+}
\]

where we used the fact that \(\text{supp}(d\sigma_t) \subset \{B = 0\}\). Hence, according to Theorem 8.5 the process
\[
W_t := B_t + \int_0^{t \wedge \sigma} \mathbb{1}_{\{B_t > 0\}} f(B_t) dt + \int_{t \wedge \sigma}^{t \wedge T^B_1} dt
\]
is a \((Q, G_t)\)-Brownian motion. The result is not surprising of course since \(\rho = B_\sigma = B^\sigma \in G_\sigma\). Therefore the measure change has no effect after \(\sigma\) and we do indeed recover the usual term from the enlargement formula under \(P\) on the interval \([\sigma \wedge t, t]\). Note that the same effect will appear when dealing with the generalization of this example from Section 4.3

In the next subsection we will discover more interesting examples.
8.2 Relative martingales

Relative martingales were introduced in \[1\]. We will work with

**Definition 8.8.** Let \( \sigma \) be an honest time and \((Y_t)\) an \((\mathcal{F}_t)\)-adapted right-continuous process such that \(Y_\infty := \lim_{t \to \infty} Y_t\) exists \(\mathbb{P}\)-almost surely and in \(L^1(\mathbb{P})\). Then \((Y_t)\) is called a relative martingale associated with \(\sigma\), if \(Y_t = \mathbb{E}^\mathbb{P}(Y_\infty \mathbb{1}_{[\sigma \leq t]} | \mathcal{F}_t)\) for all \(t \geq 0\).

Note that for an honest time \(\sigma\) the process \(k_t = \mathbb{E}^\mathbb{P}(\rho \mathbb{1}_{[\sigma \leq t]} | \mathcal{F}_t)\) introduced in the last subsection is a relative martingale with final value \(k_\infty = \rho \in \mathcal{F}_\infty = \mathcal{F}\). Therefore, the class of relative martingales associated to \(\sigma\) will provide us with nice non-trivial examples to illustrate Theorem \[8.5\].

The following result from \[1\] is very helpful in finding relative martingales.

**Lemma 8.9.** Let \((Y_t)\) be a continuous non-negative submartingale of class \((D)\) with Doob-Meyer decomposition \(Y = M + F\), where \(M \in \mathcal{M}_{loc}(\mathbb{P}, \mathcal{F}_t)\) and \(F\) is an increasing \((\mathcal{F}_t)\)-adapted process. Assume that \(M_0 = F_0 = 0\), \(\mathbb{P}(Y_\infty = 0) = 0\) and that the measure \((dF_t)\) is carried by the set \(\{t : Y_t = 0\}\). Then \((Y_t)\) is a relative martingale associated with \(\sigma = \sup\{t \geq 0 : Y_t = 0\}\).

**Example 8.10.** Let \(B\) be a standard \((\mathbb{P}, \mathcal{F}_t)\)-Brownian motion with \(L\) denoting its local time at level zero. Set \(\sigma = \sup\{\sigma \leq 1 : B_t = 0\}\). The submartingale

\[ |B_{t\wedge 1}| = \int_0^{t\wedge 1} \text{sgn}(B_u)dB_u + L_{t\wedge 1} \]

fulfills the assumptions of Lemma \[8.9\] and is hence a relative martingale associated with \(\sigma\). Setting \(\rho = |B_1|\) we have for \(t \leq 1\),

\[
\begin{align*}
  k_t &= |B_t| = \int_0^t \text{sgn}(B_u)dB_u + L_t \\
  \rho_t &= \mathbb{E}^\mathbb{P}(\rho | \mathcal{F}_t) = \mathbb{E}^\mathbb{P}(|B_t| | \mathcal{F}_t) = \int_{-\infty}^\infty \frac{|x + B_t|}{\sqrt{2\pi(1-t)}} \exp \left(-\frac{x^2}{2(1-t)}\right) dx \\
  &= |B_t| \cdot \left[ 2\Phi \left( \frac{|B_t|}{\sqrt{1-t}} \right) - 1 \right] + \sqrt{2(1-t)} \cdot \frac{|B_t|^2}{2(1-t)} \\
  h_t &= \rho_t - k_t = 2|B_t| \cdot \left[ \Phi \left( \frac{|B_t|}{\sqrt{1-t}} \right) - 1 \right] + \sqrt{2(1-t)} \cdot \frac{|B_t|^2}{2(1-t)} \\
  dh_t &= 2 \left[ \Phi \left( \frac{|B_t|}{\sqrt{1-t}} \right) - 1 \right] \text{sgn}(B_t)dB_t + \text{finite variation part}.
\end{align*}
\]

Thus according to Theorem \[8.5\] the process

\[ W_t := B_t - \int_0^{t\wedge \sigma} \frac{\text{sgn}(B_s) \left[ \Phi \left( \frac{|B_s|}{\sqrt{1-s}} \right) - 1 \right] ds}{\sqrt{2(1-s)}} + \int_{t\wedge \sigma}^{t\wedge 1} \frac{ds}{B_s} \]

is a \((Q, \mathcal{G}_t)\)-Brownian motion.

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