Rigid Particle and its Spin Revisited

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ABSTRACT

The arguments by Pandres that the double valued spherical harmonics provide a basis for the irreducible spinor representation of the three dimensional rotation group are further developed and justified. The usual arguments against the inadmissibility of such functions, concerning hermiticity, orthogonality, behavior under rotations, etc., are all shown to be related to the unsuitable choice of functions representing the states with opposite projections of angular momentum. By a correct choice of functions and definition of inner product those difficulties do not occur. And yet the orbital angular momentum in the ordinary configuration space can have integer eigenvalues only, for the reason which have roots in the nature of quantum mechanics in such space. The situation is different in the velocity space of the rigid particle, whose action contains a term with the extrinsic curvature.

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1 Introduction

The theory of point particle whose action contains not only the length, but also the extrinsic curvature of the worldline has attracted much attention [1]–[5]. Such particle, commonly called “rigid particle”, is a particular case of rigid membranes of any dimension (called “branes”). The rigid particle behaves in all respects as a particle with spin. The spin occurs because, even if free, the particle traces a worldline which deviates from a straight line. In particular, it can be a helical worldline [3]. In the absence of an external field, the constants of motion are the linear momentum \( p_\mu \) and the total angular momentum \( J_{\mu\nu} \) which is the sum of the orbital angular momentum \( L_{\mu\nu} \) and the spin \( S_{\mu\nu} \). In the presence of a gravitational field, the equation of motion for rigid particle was shown [3] to be just the Papapetrou equation [6]. The algebra of the (classical) Poisson brackets and the (quantum) commutators resembles that of a spinning particle and it was concluded that the rigid particle leads to the Dirac equation [4]. In refs. [7, 8] a counter argument occurred, namely that the spin of the rigid particle is formally like the orbital momentum, with the only difference that it acts not in the ordinary configuration space, but in the space of velocities. Since orbital momentum is well known to possess integer values only, it was concluded that the rigid particle cannot have half-integer spin values. In the present paper we will challenge that conclusion.

A theoretical justification of why orbital angular momentum is allowed to have integer values only, and not half-integer, had turned out to be not so straightforward, and the arguments had changed during the course of investigation. Initially [9] it was taken for granted that the wave function had to be single valued. Then it was realized [10] that only experimental results needed to be unique, but the wave function itself did not need to be single valued. So Pauli [11] found another argument, namely that the appropriate set of basis functions has to provide a representation of the rotation group. He argued that the spherical functions \( Y_{lm} \) with half-integer \( l \) fail to provide such representation.

Amongst many subsequent papers [12, 16–18] on the subject there are those by Pandres [13, 14] who demonstrated that the above assertion by Pauli was not correct. Pandres conclusion was that the functions \( Y_{lm} \) with half-integer \( l \) do provide the basis for an irreducible representation of the rotation group. Pandres explicitly stressed that he had no quarrel with Pauli’s conclusion concerning the inadmissibility of multivalued quantum mechanical wave functions in descriptions of the ordinary orbital angular momentum,
although he took issue with the argument through which Pauli had reached that conclusion. In the following I am going to clarify and further develop Pandre’s arguments. In particular, I will show that although the usual orbital angular momentum in coordinate space indeed cannot have half-integer values, the situation is different in the velocity space of the rigid particle. In the velocity space the functions $Y_{lm}$ with half integer $l$ and $m$ are acceptable not only because they do provide a basis for representation of the rotation group, but also because the dynamics of the rigid particle, its equations of motion and constants of motion, are different from those of a usual quantum mechanical particle. So the linear momentum $\pi_\mu$ in velocity space is not a constant of motion and the eigenfunctions of the operator $\hat{\pi}_\mu$ are not solutions of the wave equation for the quantized rigid particle. Since it has turned out [8] much more convenient to formulate the theory not in the velocity, but in the acceleration space, I will explore the ‘orbital angular momentum’ operator in the latter space, and show that its eigenvalues can be half-integers.

2 The Schrödinger basis for spinor representation of the three-dimensional rotation group

Amongst many papers [10]–[12], [16]–[18] on angular momentum and its representation the paper by Pandres [13] —with the above title— is distinct in claiming that the rotation group can be represented by means of double valued spherical harmonics. I will re-examine his arguments and confirm that Pandres’s understanding of the problem was deeper from that of other researchers. Half-integer spin is special —in comparison with the integer spin— in several respects, the most notorious being its property that a $2\pi$ rotation does not bring the system in its original state: the additional $2\pi$ rotation is necessary if one wishes to arrive at the initial situation. A spin $\frac{1}{2}$ system has an orientation–entanglement with its environment. This has consequences if one tries to describe the system by employing the Schrödinger representation. One immediately finds out that this cannot be done in the same way as in the case of a system with an integer value of angular momentum. The spherical harmonics with half-integer values do provide a basis for the irreducible spinor representation of the three-dimensional rotation group, provided that one imposes certain “amendments” to what is meant by “forming a representation”. Such amendments should not be considered as unusual for spinors—which are themselves unusual objects in comparison with the more “usual” objects—and are in close relation
to orientation–entanglement of a spinor object with its environment, which is illustrated in the well-known example of a classical object attached to its surroundings by elastic threads. Evidently, as stated by Misner et al. [15], in the case of spinors there is something about the geometry of orientation that is not fully taken into account in the usual concept of orientation.

Our first amendment is related to the fact that under rotations the spinorial wave functions do not transform as scalar functions. The failure of half-integer spherical harmonics to behave as scalars has been taken as one of the crucial reasons to reject them in the description of orbital angular momentum, which was indeed reasonable. But if we want to use them in order to represent spinors, then obviously they should not behave as scalars under rotations.

Our second amendment is related to the fact that half-integer spherical harmonics do not form a fully irreducible representation of the rotation group. The total space splits into an infinite dimensional and a finite dimensional subspace. The latter subspace, denoted \( S_l \), is not invariant; if we start from a state, initially in \( S_l \), after a rotation we obtain a state with components not only in \( S_l \), but also outside \( S_l \). But it turns out that the projection of the state onto \( S_l \) transforms just as a spinor, its norm being preserved. The occurrence of the states outside \( S_l \) has no influence on the behavior of the projected state. Therefore we can say that the basis states of \( S_l \) do form a representation of the rotation group in such a generalized, “amended”, sense. This makes sense, because the states outside \( S_l \) are unphysical, due to the fact that with respect to the latter states the expectation value of the nonnegative definite operator \( L_x^2 + L_y^2 \) is negative. So we have to disregard those “ghost-like” states, and we do this by performing the projection of a generic state onto the physical space \( S_l \). We also show that an alternative way of eliminating the unphysical states from the game is in adopting a suitably renormalized inner product with a consequence that norms of the unphysical states are zero.
2.1 Choice of functions

Let $L_i$ be a set of Schrödinger-type operators

\[ L_x = i (\cot \vartheta \cos \varphi \frac{\partial}{\partial \varphi} + \sin \varphi \frac{\partial}{\partial \vartheta}) \]
\[ L_y = i (\cot \vartheta \sin \varphi \frac{\partial}{\partial \varphi} - \cos \varphi \frac{\partial}{\partial \vartheta}) \]
\[ L_z = -i \frac{\partial}{\partial \varphi} \]

where $\vartheta$ and $\varphi$ are the usual polar coordinates.

Let us consider the functions $Y_{lm}(\vartheta, \varphi)$ which satisfy the equation

\[ L^2 Y_{lm} = l(l+1) Y_{lm} \]  
(2)

\[ L_z Y_{lm} = m Y_{lm} \]  
(3)

where

\[ L^2 \equiv L_x^2 + L_y^2 + L_z^2 = -\frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} - \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) \]  
(4)

For integer values of $l$ the $Y_{lm}$ are the familiar single-valued spherical harmonics, whilst for half-integer values of $l$ the $Y_{lm}$ are double valued functions.

In general, for any integer or half-integer values of $l$ the functions that satisfy eqs. (2), (3) are given by

\[ Y_{lm} = \frac{1}{\sqrt{2\pi}} e^{im\varphi} (-1)^l \sqrt{\frac{2l+1}{2}} \sqrt{\frac{\Pi(l+m)}{\Pi(l-m)}} \frac{1}{2^{l+m}} \frac{1}{\sin^{l+m} \vartheta} \frac{d^{l-m}}{\sin^{2l} \vartheta} \]  
(5)

\[ m = \begin{cases} l, l-1, \ldots, -l & \text{if } l \text{ integer} \\ l, l-1, \ldots, -l, -l-1, -l-2, \ldots & \text{if } l \text{ half-integer} \end{cases} \]  
(6)

where $\Pi(l) \equiv \Gamma(l+1)$ is a generalization of $l!$ to non integer values of $l$.

Besides (5) there is another set of functions which solves the system (2), (3):

\[ Z_{lm} = \frac{1}{\sqrt{2\pi}} e^{im\varphi} (-1)^{(l+m)} \sqrt{\frac{2l+1}{2}} \sqrt{\frac{\Pi(l-m)}{\Pi(l+m)}} \frac{1}{2^{l+m}} \frac{1}{\sin^{l+m} \vartheta} \frac{d^{l-m}}{\sin^{2l} \vartheta} \]  
(7)

\[ m = \begin{cases} -l, -l+1, \ldots, l & \text{if } l \text{ integer} \\ -l, l+1, \ldots, l+1, l+2, \ldots & \text{if } l \text{ half-integer} \end{cases} \]  
(8)
The function $Z_{lm}$ coincide with $Y_{l,m}$ for integer values of $l$ only. In the case of half-integer $l$-values, they are different.

If we define the raising and lowering operators as usually

$$L_x + iL_y \equiv L_+ = e^{i\varphi} \left( \frac{\partial}{\partial \vartheta} + i \cot \vartheta \frac{\partial}{\partial \varphi} \right)$$ \hspace{1cm} (9)

$$L_x - iL_y \equiv L_- = e^{-i\varphi} \left( - \frac{\partial}{\partial \vartheta} + i \cot \vartheta \frac{\partial}{\partial \varphi} \right)$$ \hspace{1cm} (10)

we find [13] for half-integer values of $l$

$$L_+ Y_{lm} \propto \begin{cases} Y_{l,m+1}, & m = l - 1, l - 2, ..., -l; \ -l - 2, -l - 3, ... \\ 0, & m = l, m = -l - 1 \end{cases}$$ \hspace{1cm} (11)

$$L_- Y_{lm} \propto Y_{l,m-1}, \quad m = l, l - 1, ..., -l, -l - 1, ...$$ \hspace{1cm} (12)

$$L_+ Z_{lm} \propto Z_{l,m+1}, \quad m = -l, -l + 1, ..., l, l + 1, ...$$ \hspace{1cm} (13)

$$L_- Z_{lm} \propto \begin{cases} Z_{l,m-1}, & m = -l + 1, -l + 2, ..., l; \ l + 2, l + 3, ... \\ 0, & m = -l, m = l + 1 \end{cases}$$ \hspace{1cm} (14)

Now let $S_l$ be a function space which is spanned by the basis functions $Y_{lm}$ for a given value of $l$ and for $m = -l, ..., l$. Further, let $O_l$ be a space spanned by $Y_{lm}$ for a given value of $l$ and for $m = -l - 1, -l - 2, ...$. Analogous we have for functions $Z_{lm}$.

From the relations (11)–(14) we see that although the repeated application of $L_-$ to $Y_{l,-l}$ does not give zero, but gives $Y_{l,-l-1}, Y_{l,-l-2}, ...$, i.e., brings us out of $S_l$ into $O_l$, the reverse is not true. If $L_+$ is applied to $Y_{l,-l-1} \in O_l$ the result is zero. This comes directly from the identity

$$L_{\pm} L_{\mp} Y_{lm} = (l \pm m)(l \mp m + 1)Y_{lm}$$ \hspace{1cm} (15)

from which we find

$$L_+ L_- Y_{l,-l} = 0$$ \hspace{1cm} (16)

Since $L_- Y_{l,-l} \propto Y_{l,-l-1}$ we have from eq. (16) that

$$L_+ Y_{l,-l-1} = 0$$ \hspace{1cm} (17)

Analogously, from $L_- L_+ Z_{l,l} = 0$ we obtain the relation

$$L_- Z_{l,l+1} = 0$$ \hspace{1cm} (18)
In particular we have,

\[ L_- Y_{\frac{1}{2}, -\frac{3}{2}} \propto Y_{\frac{1}{2}, -\frac{3}{2}} \]  \hspace{1cm} (19)
\[ L_+ Y_{\frac{1}{2}, -\frac{3}{2}} = 0 \]  \hspace{1cm} (20)

This can be verified by direct calculations using the differential operators (9), (10) and functions (5), (7). For instance, taking

\[ Y_{\frac{1}{2}, -\frac{3}{2}} = L_- Y_{\frac{1}{2}, -\frac{3}{2}} = -i \frac{\pi}{\sin^{-3/2} \vartheta} e^{-i\frac{3\varphi}{2}} \]  \hspace{1cm} (21)

we find

\[ L_+ Y_{\frac{1}{2}, -\frac{3}{2}} = e^{i\varphi} \left( \frac{\partial}{\partial \vartheta} + i \cot \vartheta \frac{\partial}{\partial \varphi} \right) \left( -i \frac{\pi}{\sin^{-3/2} \vartheta} e^{-i\frac{3\varphi}{2}} \right) = 0 \]  \hspace{1cm} (22)

Similar is true for the functions \( Z_{\ell m} \), with the role of \( L_+ \) and \( L_- \) interchanged.

The spherical harmonics for half-integer \( \ell \)-values differ from those for integer \( \ell \)-values in the properties such as those expressed in eqs. (11), (12), i.e.,

\[ L_- Y_{\ell, -\ell} \neq 0, \text{ and } L_+ Z_{\ell \ell} \neq 0 \]  \hspace{1cm} (23)

which imply the existence of functions outside the space \( S_{\ell} \). Those functions have negative eigenvalues of the operator \( L^2 - L_z^2 = L_x^2 + L_y^2 \), and therefore, according to the ordinary criteria cannot be considered as representing physical states\(^2\). Physical states are obtained by projecting an arbitrary state into the subspace \( S_{\ell} \).

In the following I am going to show that using the functions (5), (7) with the properties (11)–(18) the usual arguments against such functions as representing states with half-integer angular momentum do not hold. The confusion has been spread into several directions. Besides Pauli’s very sound argument concerning the behavior of the system under rotations there are other claims such as :

- functions \( Y_{\ell m} \) and \( Y_{\ell' m} \) are not orthogonal,
- they may have infinite norms,
- \( L_i \) are not Hermitian,
- other problems [12].

\(^2\)Analogously, negative norm states in gauge theories are unphysical, and yet they can be consistently employed in the formulation of the theories.
I will now discuss those claims.

**Orthogonality** - One finds that not only the functions $Y_{lm}$ and $Y_{lm'}$, but also $Y_{lm}$ and $Y_{l'm}$, belonging to the set $\{5\}, \{7\}$ are orthogonal. This is not the case for the set of functions used by Merzbacher and Van Winter who start formally from the same expression $\{5\}$, but restrict the range of allowed $l,m$ values in a way different from $\{6\}$, so that for positive $m$-values they take functions $\{5\}$, whilst for negative $m$-values they take functions $\{7\}$\textsuperscript{3}.

**Infinite norms** - However, a problem remains even with our choice of functions. Certain states of $S_l$, namely those with $m = -\frac{3}{2}, -\frac{5}{2}, ..., -l$, have infinite norms, i.e., $\langle lm|lm \rangle$ is infinite. But, as stated by Pandres, it is a well known fact that the inner product can be redefined so to obtain finite norms, normalized to unity.

Let us consider the quantities

$$G_{mm'}(\epsilon) = \int_{\Omega-\epsilon} d\Omega Y_{lm}^* Y_{lm'}$$

where we have performed a cut off in the integration domain. Instead of integrating over the domain

$$\Omega = \{ (\varphi, \vartheta | \varphi \in [0,2\pi], \vartheta \in [0,\pi] \}$$ (25)

we integrate over a truncated domain

$$\Omega - \epsilon = \{ (\varphi, \vartheta | \varphi \in [0,2\pi], \vartheta \in [\epsilon,\pi - \epsilon] \}$$ (26)

The quantities $G_{mm'}(\epsilon)$ are zero, if $m \neq m'$, and different from zero and finite, if $m = m'$. This is so also if $m = -\frac{3}{2}, -\frac{5}{2}, ..., -l$.

Now let $G_{mm''}(\epsilon)$ be the inverse matrix to $G_{mm'}$. So we have

$$\int_{\Omega-\epsilon} d\Omega Y_{lm}^* Y_{lm''} G_{mm''}(\epsilon) = G_{mm'}(\epsilon) G_{mm''}(\epsilon) = \delta_{m,m'}$$ (27)

**Definition I of inner product** - The latter relation is valid for any value of $\epsilon$, whatever small. Using eq. (27), we define the inner product between two functions according to:

$$\langle Y_{lm}, Y_{lm'} \rangle = \lim_{\epsilon \to 0} \int_{\Omega-\epsilon} d\Omega Y_{lm}^* Y_{lm''} G_{mm''}(\epsilon) = \delta_{m,m'}$$ (28)

If $m = m'$, this can be written as

$$\langle Y_{lm}, Y_{lm} \rangle = \lim_{\epsilon \to 0} \frac{(Y_{lm}, Y_{lm})_\epsilon}{N_{lm}(\epsilon)}, \quad N_{lm}(\epsilon) = (Y_{lm}, Y_{lm})_\epsilon$$ (29)

\textsuperscript{3}Such unsuitable set of functions has been recently used by Hunter et al. \[19\], who otherwise correctly argued that half integer spherical harmonics can represent spin.
For values of \( m \) and \( m' \) other than \( \{-\frac{3}{2}, -\frac{5}{2}, \ldots, -l\} \), we have \( \lim_{\epsilon \to 0} G_{mm'}(\epsilon) = \delta_{mm'} \) and \( \lim_{\epsilon \to 0} G^{m'm'}(\epsilon) = \delta^{m'm'} \), so that in this particular case the inner product coincides with the usual inner product.

**Hermiticity** - By using the set of functions (5),(7) and the relations (11)–(14) one finds that the operators \( L_i, L^2 \) are Hermitian with respect to \( S_l \). This is not the case if one uses a different set of functions—as Merzbacher [17] and Van Winter [12] did—such that, e.g., \( S_l \) for \( l = \frac{1}{2} \) consists of

\[
Y_{\frac{1}{2}, \frac{1}{2}} \propto \sin^{1/2} \vartheta e^{i \varphi/2} \\
Y_{\frac{1}{2}, -\frac{1}{2}} \propto \sin^{1/2} \vartheta e^{-i \varphi/2}
\]

With respect to the above set of functions (30,31) the angular momentum operator is indeed not Hermitian. Hence, the set of functions as used by Merzbacher and Van Winter is indeed not suitable for the representation of angular operator. But the set (5),(7) used in the present paper (and also by Pandres) is free of such a difficulty and/or inconsistency as discussed by Merzbacher and Van Winter.

Let me illustrate this on an example. From (5) we have for \( l = \frac{1}{2} \) the following subset \( S_{\frac{1}{2}} \) of normalized functions:

\[
Y_{\frac{1}{2}, \frac{1}{2}} = \frac{i}{\pi} \sin^{1/2} \vartheta e^{i \varphi/2} \\
Y_{\frac{1}{2}, -\frac{1}{2}} = -\frac{i}{\pi} \cos \vartheta \sin^{-1/2} \vartheta e^{-i \varphi/2}
\]

If, using (9),(10), we write

\[
L_x = \frac{1}{2} (L_+ + L_-) \\
L_y = \frac{1}{2i} (L_+ - L_-)
\]

we find after taking into account

\[
L_+ Y_{\frac{1}{2}, \frac{1}{2}} = 0 \\
L_+ Y_{\frac{1}{2}, -\frac{1}{2}} = Y_{\frac{1}{2}, \frac{1}{2}} \\
L_- Y_{\frac{1}{2}, \frac{1}{2}} = Y_{\frac{1}{2}, -\frac{1}{2}} \\
L_- Y_{\frac{1}{2}, -\frac{1}{2}} = Y_{\frac{1}{2}, -\frac{1}{2}} \\
L_+ Y_{\frac{1}{2}, -\frac{1}{2}} = 0
\]
that the matrix elements of angular momentum operator satisfy:

\[ \langle \frac{1}{2} \frac{1}{2} | L_x | \frac{1}{2} - \frac{1}{2} \rangle = \frac{1}{2} = \langle \frac{1}{2}, -\frac{1}{2} | L_x | \frac{1}{2} \frac{1}{2} \rangle^* \]  

\[ \langle \frac{1}{2} \frac{1}{2} | L_y | \frac{1}{2}, -\frac{1}{2} \rangle = -i = \langle \frac{1}{2}, -\frac{1}{2} | L_x | \frac{1}{2} \frac{1}{2} \rangle^* \]  

where

\[ \langle lm | L_i | lm' \rangle = \int Y_{lm}^* L_i Y_{lm'} \, d\Omega \quad , \quad d\Omega \equiv \sin \vartheta \, d\vartheta \, d\varphi \]  

Here we have also taken into account that the states with the same \( l \) but different \( m \) values are orthogonal. The matrix values (41)–(42) are just the standard ones. The fact that \( L_{-Y} \neq 0 \) has no influence on the values of matrix elements of angular momentum operator, calculated with respect to the basis states of \( S_l \). This is so because of eq. (40).

In eqs. (41)–(42) we have just the property that the matrix elements of a Hermitian operator have to satisfy.

Let us now check by explicit integration whether the operators \( L_i, i = 1, 2, 3 \), satisfy the requirement for self-adjointness

\[ (\phi, L_i \psi) = (L_i \phi, \psi) \quad \text{for all } \phi, \psi \in S_l \]  

with the inner product being defined according to eq. (28). Since any physically admissible \( \phi, \psi \) is by definition a superposition of \( Y_{lm} \in S_l \), it is sufficient to show the relation (44) for functions \( Y_{lm} \in S_l \) only. Taking into account the relations (34), (35) we find that the condition for self-adjointness of the operators \( L_i \) becomes

\[ (Y_{lm}, L_+ Y_{lm'}) = (L_- Y_{lm}, Y_{lm'}) \quad \text{if } m = m' + 1 \]  

\[ (Y_{lm}, L_- Y_{lm'}) = (L_+ Y_{lm}, Y_{lm'}) \quad \text{if } m = m' - 1 \]  

If we calculate the matrix elements by adopting the usual definition of the inner product, we have \((m = m' + 1)\):

\[ \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin \vartheta Y_{lm}^* e^{-i\varphi} \left( -\frac{\partial}{\partial \vartheta} + i \cot \vartheta \frac{\partial}{\partial \varphi} \right) Y_{lm'} \]

\[ = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \left[ e^{-i\varphi} \frac{\partial}{\partial \vartheta} \left( Y_{lm}^* \sin \vartheta - i \cos \vartheta \frac{\partial}{\partial \varphi} \right) Y_{lm'} \right] B_{mm'} \]

\[ = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin \vartheta \left[ e^{i\varphi} \left( \frac{\partial}{\partial \vartheta} + i \cot \vartheta \frac{\partial}{\partial \varphi} \right) Y_{lm} \right]^* Y_{lm'} + B_{mm'} \]  

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The boundary term
\[ B_{mm'} = \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \left[ -\frac{\partial}{\partial \vartheta} (Y_{lm}^* \sin \vartheta Y_{lm'}) e^{-i\varphi} + \frac{\partial}{\partial \varphi} (Y_{lm}^* e^{-i\varphi} Y_{lm'}) \cos \vartheta \right] \] (48)
vanishes if \( m > -\frac{3}{2} \). For instance, if \( l = \frac{1}{2}, \ m = \frac{1}{2}, \ m' = -\frac{1}{2} \) the boundary term is equal to
\[ -\int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \left( -\frac{i}{\pi} \right)^2 \frac{\partial}{\partial \vartheta} \left( \sin^{1/2} \vartheta \sin \vartheta \sin^{-1/2} \vartheta \cos \vartheta \right) = 2\pi \int_0^\pi d(\sin \vartheta \cos \vartheta) = 0 \] (49)
In spite of the fact that the \( Y_{\frac{1}{2}, \frac{1}{2}}, -\frac{1}{2} \) due to \( \sin^{-1/2} \vartheta \) is singular at the boundaries \( \vartheta = 0, \ \vartheta = \pi \), this is compensated by \( \sin^{1/2} \vartheta \) occurring in \( Y_{\frac{1}{2}, \frac{1}{2}}^* \), and so the boundary term is zero. But if \( m \leq -\frac{3}{2} \), the boundary term does not vanish, because there occurs the product \( \sin^m \vartheta \sin^{(m+1)} \vartheta \) which brings a singularity.

We are going to show that the two illnesses, namely the singularity in \( \int d\Omega Y_{lm}^* Y_{lm} \), \( m \leq -\frac{3}{2} \), and the singularity of the boundary term \( B_{mm'} \) compensate each other, so that by using the redefined inner product, the self-adjointness condition (45), (46) are fulfilled for arbitrary \( Y_{lm} \in \mathcal{S}_l \).

Let us first illustrate this in the case \( l = \frac{3}{2} \). Using eq. (5) we obtain the following four top functions, (i.e., those of \( \mathcal{S}_l \)):
\[ Y_{\frac{3}{2}, \frac{3}{2}} = K_{\frac{3}{2}} \sqrt{6} e^{3i\varphi/2} \sin^{3/2} \vartheta \]
\[ Y_{\frac{3}{2}, \frac{1}{2}} = K_{\frac{3}{2}} \sqrt{2} (-3) e^{i\varphi/2} \sin^{1/2} \vartheta \cos \vartheta \]
\[ Y_{\frac{3}{2}, -\frac{1}{2}} = K_{\frac{3}{2}} \sqrt{\frac{1}{2} \frac{3}{6}} e^{-3i\varphi/2} \sin^{-1/2} \vartheta (2\cos^2 \vartheta - 1) \]
\[ Y_{\frac{3}{2}, -\frac{3}{2}} = K_{\frac{3}{2}} \sqrt{\frac{1}{6} \frac{3}{2}} (-3) e^{-3i\varphi/2} \sin^{-3/2} \vartheta (2\cos^2 \vartheta - 3) \] (50)
where
\[ K_{\frac{3}{2}} = \frac{(-1)^l}{\sqrt{2\pi}} \frac{\sqrt{2l+1}}{2} \frac{1}{2^l \Pi(l)} \quad \text{and} \quad K_{\frac{3}{2}} = -\frac{i\sqrt{2}}{3\pi} \] (51)
Through a direct computation of the explicit action of the operators \( L_+ \) and \( L_- \) on functions \( Y_{lm} \) defined in eq. (5), one finds that the following relations are satisfied
\[ L_+ Y_{lm} = \sqrt{(l-m)(l+m+1)} Y_{l,m+1}, \quad m = l, \ldots, -l, -l-1, l-2, \ldots \] (52)
\[ L_- Y_{lm} = \sqrt{(l+m)(l-m+1)} Y_{l,m-1}, \quad m = l, \ldots, -l, \quad m \neq -l, \quad m = -l-1, -l-2, \ldots \] (53)
Using the abbreviation $Y_{\frac{3}{2}m} \equiv Y_m$ we now calculate the terms in the relation

$$(Y_{-\frac{3}{2}}, L_{-} Y_{-\frac{3}{2}})_{\epsilon} = (L_{+} Y_{-\frac{3}{2}}, Y_{-\frac{3}{2}})_{\epsilon} + B_{-\frac{3}{2}, -\frac{3}{2}}(\epsilon)$$

We obtain

$$(Y_{-\frac{3}{2}}, L_{-} Y_{-\frac{3}{2}})_{\epsilon} = \sqrt{3} (Y_{-\frac{3}{2}}, Y_{-\frac{3}{2}})_{\epsilon} = \sqrt{3} \int_{\epsilon}^{\pi - \epsilon} 2\pi d\vartheta \sin \vartheta Y_{\frac{3}{2}}^* Y_{-\frac{3}{2}}$$

$$= \sqrt{3} |K_{\frac{3}{2}}|^2 2\pi \frac{3}{2} \left(\frac{3\pi}{2} - 3\epsilon - 2\sin 2\epsilon + \frac{1}{4} \sin 4\epsilon + 2\cot \epsilon\right)$$

$$(L_{+} Y_{-\frac{3}{2}}, Y_{-\frac{3}{2}})_{\epsilon} = \sqrt{3} (Y_{-\frac{3}{2}}, Y_{-\frac{3}{2}})_{\epsilon} = \sqrt{3} \int_{\epsilon}^{\pi - \epsilon} 2\pi d\vartheta \sin \vartheta Y_{\frac{3}{2}} Y_{-\frac{3}{2}}$$

$$= \sqrt{3} |K_{\frac{3}{2}}|^2 2\pi \left(\frac{3\pi}{2} - \epsilon - \frac{1}{4} \sin 4\epsilon\right)$$

$$B_{-\frac{3}{2}, -\frac{3}{2}}(\epsilon) = -2\pi Y_{\frac{3}{2}}^* Y_{-\frac{3}{2}} \sin \vartheta \left|\int_{\epsilon}^{\pi - \epsilon} e^{-i\varphi} \right|$$

$$= \sqrt{3} |K_{\frac{3}{2}}|^2 2\pi \left(-\frac{3}{2}\right) \left(1 - 4 \cos 2\epsilon + \cos 4\epsilon\right) \cot \epsilon$$

from which we can verify that the relation (54) is indeed satisfied for any $\epsilon$. A check is straightforward, if we use the symbolic package Mathematica. An easy check by hand can be done for small $\epsilon$. We have:

$$(Y_{-\frac{3}{2}}, L_{-} Y_{-\frac{3}{2}})_{\epsilon} = \sqrt{3} |K_{\frac{3}{2}}|^2 2\pi \frac{3}{2} \left(\frac{3\pi}{2} + 2\cot \epsilon\right) + O_1(\epsilon)$$

$$(L_{+} Y_{-\frac{3}{2}}, Y_{-\frac{3}{2}})_{\epsilon} = \sqrt{3} |K_{\frac{3}{2}}|^2 2\pi \frac{3}{2} \left(\frac{3\pi}{2}\right) + O_2(\epsilon)$$

$$B_{-\frac{3}{2}, -\frac{3}{2}}(\epsilon) = \sqrt{3} |K_{\frac{3}{2}}|^2 2\pi \frac{3}{2} \left(-2\cot \epsilon\right) + O_3(\epsilon)$$

where $O_1$, $O_2$ and $O_3$ are small $\epsilon$-dependent terms that go to zero with vanishing $\epsilon$. We see that the boundary term which grows to infinity, exactly matches the infinite term in the matrix element $(Y_{-\frac{3}{2}}, L_{-} Y_{-\frac{3}{2}})_{\epsilon \to 0}$. This enables us to adopt a suitable renormalization procedure. One possibility is in modifying the inner product according to (28) Then eqs. (45), (46) written in the form

$$(Y_{-\frac{3}{2}}, L_{-} Y_{-\frac{3}{2}})_{\epsilon} = \frac{(L_{+} Y_{-\frac{3}{2}}, Y_{-\frac{3}{2}})_{\epsilon}}{(Y_{-\frac{3}{2}}, Y_{-\frac{3}{2}})_{\epsilon}}$$

are indeed satisfied, as can be straightforwardly verified.

12
For arbitrary \( l, m \) we have
\[
(Y_{lm}, L_- Y_{l,m+1})_\epsilon = (L_+ Y_{lm}, Y_{l,m+1})_\epsilon + B_{lm,m+1}(\epsilon) \tag{62}
\]
\[
(Y_{lm}, L_- Y_{l,m+1})_\epsilon = (Y_{lm}, L_- Y_{l,m+1})_R + A_{lm}(\epsilon) = \sqrt{(l - m)(l + m + 1)} N_{lm}(\epsilon) \tag{63}
\]
\[
(L_+ Y_{lm}, Y_{l,m+1})_\epsilon = (L_+ Y_{lm}, Y_{l,m+1})_R + A_{l,m+1}(\epsilon) = \sqrt{(l + m)(l - m + 1)} N_{l,m+1}(\epsilon) \tag{64}
\]
Here \( R \) denotes the \( \epsilon \)-independent part which satisfies
\[
(Y_{lm}, L_- Y_{l,m+1})_R = (L_+ Y_{lm}, Y_{l,m+1})_R = \sqrt{(l - m)(l + m + 1)} \tag{65}
\]
In the case \( l = \frac{3}{2} \) we read from eqs. (65), (64) that
\[
(Y_{-\frac{3}{2}, Y_{-\frac{1}{2}}})_R = (L_+ Y_{-\frac{3}{2}}, Y_{-\frac{1}{2}})_R = \sqrt{3} |K_{\frac{3}{2}}|^2 \frac{9\pi}{4} = \sqrt{3} \tag{66}
\]
From eqs. (62)–(65) it follows that the \( \epsilon \)-dependent terms satisfy the relation
\[
A_{lm}(\epsilon) = A_{l,m+1}(\epsilon) + B_{lm,m+1}(\epsilon) \tag{67}
\]
Let us now consider the condition for self-adjointness in which the scalar products are defined according to (28). For arbitrary finite \( \epsilon \) we have
\[
\frac{(Y_{lm}, L_- Y_{l,m+1})_\epsilon}{N_{lm}(\epsilon)} = \frac{(L_+ Y_{lm}, Y_{l,m+1})_\epsilon}{N_{l,m+1}(\epsilon)} \tag{68}
\]
Performing the partial integration in the l.h.s. of eq. (68), i.e., by using eq. (62), we obtain
\[
\frac{(L_+ Y_{lm}, Y_{l,m+1})_\epsilon + B_{l,m+1}(\epsilon)}{N_{lm}(\epsilon)} = \frac{(L_+ Y_{lm}, Y_{l,m+1})_\epsilon}{N_{l,m+1}(\epsilon)} \tag{69}
\]
which in view of eqs. (63), (64) becomes
\[
\sqrt{(l - m)(l + m + 1)} N_{l,m+1}(\epsilon) + B_{lm,m+1}(\epsilon) = \sqrt{(l - m)(l + m + 1)} N_{lm}(\epsilon) \tag{70}
\]
Using again (63), (64) we find that the latter relation is identical to (67). So we have verified that the condition for self-adjointness (68) holds for arbitrary integration domain, determined by \( \epsilon \), and hence also in the limit \( \epsilon \to 0 \), regardless of which \( Y_{lm} \in \mathcal{S}_l \) we take.
Thus in the problematic case of \( m \leq -\frac{3}{2} \), the infinities are regularized with our definition of the inner product.

This procedure works straightforwardly for physical wave functions with \( m \)-values in the range between \( l \) and \(-l\). It also works for unphysical wave functions with \( m < -l-1 \). But there is a problem at \( m = -l-1 \). Inserting the latter \( m \)-value in the self-adjointness condition \( \text{(68)} \) we obtain that the integral \((L+Y_{l,-l-1},Y_{l,-l})_\varepsilon\) on the right hand side does not vary with \( \varepsilon \), but is exactly zero, because of the relation \( L+Y_{l,-l-1} = 0 \) (see eq. \( \text{(18)} \)). So the validity of the condition \( \text{(68)} \) breaks down in this particular case of the matrix element between physical and unphysical states.

We have thus shown that angular momentum operator is self-adjoint with respect to the domain \( S_l \) of physical half-integer spin wave functions, but in general it is not self-adjoint with respect to the space of all wave functions entering the game. Since the extra wave functions are unphysical, we need to project them out. In the following we will provide an alternative definition of inner product by which such complications with unphysical wave functions will be eliminated.

**Definition II of inner product** - The fact that the infinity in a matrix element for \( m \leq -\frac{3}{2} \) matches the infinity in the boundary term suggests us to define a modified inner product in which the infinities are eliminated \( \text{[14]} \):

\[
(\psi, \psi') = \int_0^\pi \sin \vartheta \left[ \int_0^{2\pi} d\varphi \, \psi^* \psi' - f(\vartheta) \right]
\]  

(71)

Here \( f(\vartheta) \) is a singular function chosen so that the term in the bracket becomes integrable. For instance, in the case considered in eq. \( \text{(63)} \), the modified inner product is

\[
(Y_{lm}, L_-Y_{l,m+1}) = \lim_{\varepsilon \to 0} \left[ \int_\varepsilon^{\pi-\varepsilon} d\vartheta \int_0^{2\pi} d\varphi \, Y_{lm}^* L_-Y_{l,m+1} - A_{lm}(\varepsilon) \right] = (L+Y_{lm}, Y_{l,m+1})
\]

\[
A_{lm}(\varepsilon) = 2\pi \int_\varepsilon^{\pi-\varepsilon} d\vartheta \, f(\vartheta)
\]

(72)
In particular (see eq. (53)) we have

\[
(Y_{\frac{3}{2}}, L_{-\frac{1}{2}}) = \lim_{\epsilon \to 0} \left[ \int_{\epsilon}^{\pi-\epsilon} d\vartheta \int_{0}^{2\pi} d\phi \ Y_{\frac{3}{2}}^* L_{-\frac{1}{2}} Y_{\frac{3}{2}}^* \right.
\]

\[
- \sqrt{3} |K_{\frac{3}{2}}|^2 2\pi \frac{3}{2} (-3\epsilon - 2\sin 2\epsilon + \frac{1}{4}\sin 4\epsilon + 2\cot \epsilon)
\]

\[
= \sqrt{3} |K_{\frac{3}{2}}|^2 2\pi \frac{9\pi}{4} = \sqrt{3}
\]

\[
= \int_{0}^{\pi} d\vartheta \int_{0}^{2\pi} d\phi \ (L+Y_{\frac{3}{2}})^* Y_{\frac{1}{2}} = (L+Y_{\frac{3}{2}}, Y_{\frac{1}{2}})
\]

\[(73)\]

The term \(A_{lm}(\epsilon)\) that we added to the integral on the left hand side of eq. (73) is just equal, in the limit \(\epsilon \to 0\), to the boundary term that we obtain after performing the integration per partes of the right hand side integral.

In fact, the above procedure is a sort of renormalization. The necessity for such a procedure can be seen from considering, e.g, the matrix elements \((L+Y_{\frac{3}{2}}, Y_{\frac{1}{2}})\) or \((Y_{\frac{3}{2}}, L+Y_{\frac{1}{2}})\) which are finite according to the usual definition of inner product (without a renormalization). After performing the integration per partes one obtains two infinite terms which cancel each other, so that the result is still finite, as it should be. So it makes sense to redefine the matrix elements \((Y_{\frac{3}{2}}, L_{-\frac{1}{2}})\) and \((L_{-\frac{1}{2}}, Y_{\frac{3}{2}})\) by including into their definition the corresponding boundary terms.

As observed by Pandres [14], the functional \((\psi, \psi')\) satisfies the identities necessary for an inner product:

\[
(\psi, \psi') \geq 0
\]

\[
(c\psi, \psi') = c^* (\psi, \psi')
\]

\[
(\psi, \psi')^* = (\psi', \psi)
\]

\[
(\psi + \psi', \psi'') = (\psi, \psi'') + (\psi', \psi'')
\]

\[(74)\]

for all \(\psi, \psi'\) spanned by \(Y_{lm}\) of eqs. (5), (6) with \(c\) an arbitrary complex constant.

The modified inner product has the following important properties:

(i) The conditions for self-adjointness of angular momentum operator are satisfied for all half-integers \(m \leq l\).

(ii) The basis functions \(Y_{lm} \in S_l\), i.e., the physical ones are orthonormal:

\[
(Y_{lm}, Y_{l'm'}) = \delta_{ll'} \delta_{mm'}, \quad m, m' = l, ..., -l
\]

\[(75)\]
(iii) The basis functions \( Y_{lm} \in \mathcal{O}_l \), i.e., the unphysical ones, have zero norm:

\[
(Y_{lm}, Y_{lm'}) = 0, \quad m, m' = -l - 1, -l - 2, -l - 3, ...
\]  

(76)

Property (iii) comes from considering the inner product

\[
(Y_{l,-l-1}, Y_{l,-l-1}) \propto (Y_{l,-l-1}, L^{-} Y_{l,-l}) = (L^{+} Y_{l,-l-1}, Y_{l,-l}) = 0
\]  

(77)

where use has been made of Property (i) and eq. (17). In general

\[
(Y_{l,-l-k}, Y_{l,-l-k}) \propto (Y_{l,-l-k}, L^{-} Y_{l,-l-k+1}) \propto (L^{+} Y_{l,-l-k}, Y_{l,-l-k+1})
\]

\[
\propto (Y_{l,-l-k+1}, Y_{l,-l-k+1}) \propto ... \propto (Y_{l,-l-1}, Y_{l,-l-1}) = 0
\]  

(78)

which proves Property (iii).

**Other problems** - Van Winter pointed to a number of problems and inconsistencies that all can be shown as resulting from his choice of functions. Such problems do not arise with our and Pandres’s choice of functions (5),(7). Namely, for any function \( \psi \in \mathcal{S}_l \) the relations such as

\[
[L_x, L_y] \psi = iL_z \psi
\]

(79)

\[
[L_i, L^2] \psi = 0, \quad L_i = L_x, L_y, L_z
\]

(80)

\[
L^2 l \psi = l(l + 1) L_i \psi
\]

(81)

are valid\(^4\). This is not so for Van Winter’s choice of functions.

However, one problem—discussed by Pauli and Van Winter—remains even with our choice of functions, if we take Definition I of inner product. Namely, a rotation applied to a function \( Y_{lm} \) belonging to \( \mathcal{S}_l \) will give a function outside \( \mathcal{S}_l \). At first sight this seems as an evidence that functions of \( \mathcal{S}_l \) cannot form a representation of rotations and angular momentum. Following Pandres we will show that this is not the case, provided that we suitably generalize the concept of representation space. With Definition II of inner product no such complication arises, because the norms of the unphysical states vanish, and consequently the subspace \( \mathcal{S}_l \) is invariant with respect to rotations.

\(^4\)Crutial here is the relation \( L^+ Y_{l,-l-1} = 0 \).
2.2 Behaviour of the spherical harmonics with half-integer \( l \) values under rotations

**Behaviour in the presence of Definition I of inner product**

We will now first explore how the spherical functions for half-integer \( l \) values change under infinitesimal rotations. Let a state\(^5\) \( |\Psi\rangle \) with a half-integer of \( l \) be a superposition of the states \( |lm\rangle \equiv |m\rangle \) with different values of \( m \):

\[
|\Psi\rangle = \sum_{lm'} |lm'\rangle \langle lm'|\Psi\rangle , \quad m' = l, l-1, ..., -l, -l-1, -l-2, ...
\]  

(82)

In general the expansion coefficients \( \langle lm'|\Psi\rangle \) are arbitrary. Let us consider a particular case in which the coefficients are zero for the values of \( m' \) outside \( S_l \):

\[
\langle lm'|\Psi\rangle = \begin{cases} 
\text{nonzero} & \text{if} \quad m' = -l, ..., l \\
0 & \text{if} \quad m' = -l-1, -l-2, ...
\end{cases}
\]  

(83)

Under a rotation around an axis, say \( x \)-axis, the state changes as

\[
|\Psi\rangle \rightarrow e^{i\epsilon Lx} |\Psi\rangle
\]  

(84)

where \( \epsilon \) is an angle of rotation. For an infinitesimal rotation we have

\[
|\Psi\rangle \rightarrow (1 + i\epsilon Lx)|\Psi\rangle
\]

\[
\delta |\Psi\rangle = i\epsilon Lx |\Psi\rangle
\]  

(85)

The projection \( \langle m|\Psi\rangle \) changes according to

\[
\langle lm|\delta |\Psi\rangle \equiv \delta \langle lm|\Psi\rangle = i\epsilon \sum_{m'} \langle lm|Lx|m'\rangle \langle lm'|\Psi\rangle
\]  

(86)

Let us consider the example in which \( l = \frac{1}{2}, m = \frac{1}{2}, -\frac{1}{2} \). Then (82) and (86) read

\[
|\Psi\rangle = |\frac{1}{2}\frac{1}{2}\rangle \langle \frac{1}{2}\frac{1}{2}|\Psi\rangle + |\frac{1}{2}\frac{1}{2}\rangle \langle \frac{1}{2}\frac{1}{2}|\Psi\rangle \equiv |\frac{1}{2}\frac{1}{2}\rangle \alpha + |\frac{1}{2}\frac{1}{2}\rangle \beta
\]  

(87)

\[
\delta \langle lm|\Psi\rangle = i\epsilon \langle lm|Lx|\frac{1}{2}\frac{1}{2}\rangle \langle \frac{1}{2}\frac{1}{2}|\Psi\rangle + i\epsilon \langle lm|Lx|\frac{1}{2}\frac{1}{2}\rangle \langle \frac{1}{2}\frac{1}{2}|\Psi\rangle , \quad m = \frac{1}{2}, -\frac{1}{2}
\]  

(88)

\(^5\)In order to simplify the notation we now use the ket notation \( |lm\rangle, |\Psi\rangle, Lx|\Psi\rangle \), etc., with the understanding that \( \langle \Omega|lm\rangle \equiv Y_{lm}, \langle \Omega|\Psi\rangle \equiv \Psi(\Omega), \langle \Omega|Lx|\Psi\rangle = Lx \Psi(\Omega) \), etc.. In fact, we should have used different symbols for the abstract operator and its representation in the basis \( |\Omega\rangle \). But for simplicity reasons we avoid such complication.
or explicitly
\[
\delta\langle\frac{1}{2},\frac{1}{2}|\Psi\rangle \equiv \delta\alpha = i\epsilon\langle\frac{1}{2}\frac{1}{2}|L_x|\frac{1}{2},\frac{1}{2}\rangle = \frac{i\epsilon}{2}\beta \tag{89}
\]

\[
\delta\langle\frac{1}{2},\frac{1}{2}|\Psi\rangle \equiv \delta\beta = i\epsilon\langle\frac{1}{2},\frac{1}{2}|L_x|\frac{1}{2}\rangle \alpha = \frac{i\epsilon}{2}\alpha \tag{90}
\]
where we have taken into account (34)-(42).

Working directly with the functions we have:
\[
\psi(\Omega) \equiv \langle\Omega|\Psi\rangle = \alpha\psi_{\frac{1}{2}} + \beta\psi_{\frac{1}{2}} \tag{91}
\]

\[
\delta\psi = i\epsilon L_x\psi = i\epsilon L_x\psi_{\frac{1}{2}}\alpha + i\epsilon L_x\psi_{-\frac{1}{2}}\beta \tag{92}
\]
where \(\psi_{\frac{1}{2}} \equiv Y_{\frac{1}{2} \frac{1}{2}}\) and \(\psi_{-\frac{1}{2}} \equiv Y_{\frac{1}{2} -\frac{1}{2}}\). Multiplying (92) with \(\psi_{-\frac{1}{2}}^*\), and \(\psi_{\frac{1}{2}}^*\) respectively, and integrating over \(d\Omega = \sin\theta\ d\theta\ d\phi\) we find after taking into account (34)-(42) that
\[
\int d\Omega\psi_{-\frac{1}{2}}^*\delta\psi = \delta\beta = i\epsilon \int d\Omega\psi_{-\frac{1}{2}}^*L_x\psi = \int d\Omega\psi_{-\frac{1}{2}}^*L_x\psi_{\frac{1}{2}}\alpha = \frac{i\epsilon}{2}\alpha \tag{93}
\]

\[
\int d\Omega\psi_{\frac{1}{2}}^*\delta\psi = \delta\alpha = i\epsilon \int d\Omega\psi_{\frac{1}{2}}^*L_x\psi = \int d\Omega\psi_{\frac{1}{2}}^*L_x\psi_{-\frac{1}{2}}\beta = \frac{i\epsilon}{2}\beta \tag{94}
\]
which is the same result as in eqs. (89), (90).

The above result demonstrates that under an infinitesimal rotation the expansion coefficients \(\langle m l'|\Psi\rangle\) for \(l = \frac{1}{2}\), change precisely in the same way as in the usual theory of spin \(\frac{1}{2}\) state. From eq. (86) we find that this is so in the case of an arbitrary \(l\) and \(m' = -l, -l+1,..., l-1, l\) as well.

In a state (82) the coefficients \(\langle m|\Psi\rangle\) are zero for the values of \(m\) outside \(S_l\), i.e., for \(m > l\) and \(m < -l\). We will now explore how those coefficients change under an infinitesimal rotation. For the sake of definiteness let us again consider the special case of \(l = \frac{1}{2}\) and the state given in eq. (87) in which the coefficients \(\langle\frac{1}{2}, -\frac{3}{2}|\Psi\rangle, \langle\frac{1}{2}, -\frac{5}{2}|\Psi\rangle\), etc., are zero. The change of the coefficient \(\langle -\frac{3}{2}|\Psi\rangle\) under the transformation (85) as given by (86) and (88) reads
\[
\delta\langle\frac{1}{2}, -\frac{3}{2}|\Psi\rangle = i\epsilon\langle\frac{1}{2}, -\frac{3}{2}|L_x|\frac{1}{2}, -\frac{3}{2}\rangle = \frac{i\epsilon}{2}\langle\frac{1}{2}, -\frac{3}{2}|\frac{1}{2}, -\frac{3}{2}\rangle \beta \tag{95}
\]
By rotation we thus obtain a state which is no longer of the form (87), but of the form
\[
|\Psi'\rangle = |\frac{1}{2}\frac{1}{2}\rangle|\frac{1}{2}\frac{1}{2}|\Psi'\rangle + |\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, -\frac{1}{2}|\psi'\rangle + |\frac{1}{2}, -\frac{3}{2}\rangle|\frac{1}{2}, -\frac{3}{2}|\Psi'\rangle \tag{96}
\]
where

\[
\langle \frac{1}{2}, \frac{1}{2} | \Psi' \rangle = \alpha + \frac{i}{2} \beta
\]

\[
\langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle = \beta + \frac{i}{2} \alpha
\]

\[
\langle \frac{1}{2}, -\frac{3}{2} | \Psi' \rangle = \frac{i}{2} \langle -\frac{3}{2} | \frac{1}{2}, -\frac{3}{2} \rangle \beta
\]

(97)

In other words, by a rotation we obtain a state which is outside \( S_{\frac{1}{2}} \).

What happens if we perform another infinitesimal rotation \( \delta \) on the state \( |\psi'\rangle \) given in eq. (96). The coefficients change according to eq. (86) which now read

\[
\delta \langle \frac{1}{2}, \frac{1}{2} | \Psi' \rangle = i \epsilon \langle \frac{1}{2}, \frac{1}{2} | L_+ \frac{1}{2}, -\frac{1}{2} \rangle \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle + i \epsilon \langle \frac{1}{2}, \frac{1}{2} | L_\times \frac{1}{2}, -\frac{3}{2} \rangle \langle \frac{1}{2}, -\frac{3}{2} | \Psi' \rangle
\]

(98)

\[
\delta \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle = i \epsilon \langle \frac{1}{2}, -\frac{1}{2} | L_+ \frac{1}{2}, -\frac{1}{2} \rangle \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle + i \epsilon \langle \frac{1}{2}, -\frac{1}{2} | L_\times \frac{1}{2}, -\frac{3}{2} \rangle \langle \frac{1}{2}, -\frac{3}{2} | \Psi' \rangle
\]

(99)

\[
\delta \langle \frac{1}{2}, -\frac{3}{2} | \Psi' \rangle = i \epsilon \langle \frac{1}{2}, -\frac{3}{2} | L_+ \frac{1}{2}, -\frac{1}{2} \rangle \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle = i \epsilon \langle \frac{1}{2}, -\frac{3}{2} | \frac{1}{2}, -\frac{3}{2} \rangle \langle \frac{1}{2}, -\frac{3}{2} \rangle \beta
\]

(100)

\[
\delta \langle \frac{1}{2}, \frac{5}{2} | \Psi' \rangle = i \epsilon \langle \frac{1}{2}, \frac{5}{2} | L_+ \frac{1}{2}, \frac{3}{2} \rangle \langle \frac{1}{2}, \frac{3}{2} | \Psi' \rangle = i \epsilon \langle \frac{1}{2}, \frac{5}{2} | \frac{1}{2}, \frac{3}{2} \rangle \langle \frac{1}{2}, -\frac{3}{2} \rangle \beta
\]

(101)

Writing \( L_x \) in terms of \( L_+ \) and \( L_- \) (eq. 83) and taking into account the relation \( 20 \) which implies

\[
L_+ |\frac{1}{2}, -\frac{3}{2} \rangle = 0
\]

(102)

we have

\[
\langle \frac{1}{2}, \frac{1}{2} | L_+ \frac{1}{2}, -\frac{3}{2} \rangle = 0 , \quad \langle \frac{1}{2}, -\frac{1}{2} | L_+ \frac{1}{2}, -\frac{3}{2} \rangle = 0
\]

(103)

Therefore \( \delta \langle \frac{1}{2}, \frac{1}{2} | \Psi' \rangle \) and \( \delta \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle \) are again of the same form as in (83), (30). Becasue of the relation (102), the presence of a non vanishing coefficients \( \langle \frac{1}{2}, -\frac{3}{2} | \Psi' \rangle \) has no influence on the transformations of \( \langle \frac{1}{2}, \frac{1}{2} | \Psi' \rangle \) and \( \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle \Psi' \).

Altogether, our new state after the second rotation is

\[
|\Psi''\rangle = |\frac{1}{2}, \frac{1}{2} \rangle \langle \frac{1}{2}, \frac{1}{2} | \Psi''\rangle + |\frac{1}{2}, \frac{3}{2} \rangle \langle \frac{1}{2}, \frac{3}{2} | \Psi''\rangle + |\frac{1}{2}, -\frac{1}{2} \rangle \langle \frac{1}{2}, -\frac{1}{2} | \Psi''\rangle + |\frac{1}{2}, -\frac{3}{2} \rangle \langle \frac{1}{2}, -\frac{3}{2} | \Psi''\rangle
\]

(104)

where

\[
\langle \frac{1}{2}, \frac{1}{2} | \Psi''\rangle = \langle \frac{1}{2}, \frac{1}{2} | \Psi' \rangle + \frac{i}{2} \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle
\]

(105)

\[
\langle \frac{1}{2}, -\frac{1}{2} | \Psi''\rangle = \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle + \frac{i}{2} \langle \frac{1}{2}, \frac{1}{2} | \Psi' \rangle
\]

(106)

\[
\langle \frac{1}{2}, -\frac{3}{2} | \Psi''\rangle = \frac{i}{2} \langle \frac{1}{2}, -\frac{1}{2} | \Psi' \rangle = \frac{i}{2} \langle \frac{1}{2}, \frac{1}{2} | \Psi' \rangle
\]

(107)

\[
\langle \frac{1}{2}, -\frac{5}{2} | \Psi''\rangle = \frac{i}{2} \langle \frac{1}{2}, -\frac{3}{2} | \Psi' \rangle = \frac{i}{2} \langle \frac{1}{2}, \frac{3}{2} | \Psi' \rangle
\]

(108)

\[
\langle \frac{1}{2}, -\frac{7}{2} | \Psi''\rangle = \frac{i}{2} \langle \frac{1}{2}, -\frac{5}{2} | \Psi' \rangle = \frac{i}{2} \langle \frac{1}{2}, \frac{5}{2} | \Psi' \rangle
\]

(109)
Applying now an infinitesimal rotation on $|\Psi''\rangle$ we find that

\[
\delta\langle \frac{1}{2}, \frac{1}{2} | \Psi'' \rangle = i\epsilon\langle \frac{1}{2}, \frac{1}{2} | L_x | \frac{1}{2}, -\frac{1}{2} | \Psi'' \rangle = i\epsilon\langle \frac{1}{2}, -\frac{1}{2} | \Psi'' \rangle
\]

(110)

\[
\delta\langle \frac{1}{2}, -\frac{1}{2} | \Psi'' \rangle = i\epsilon\langle \frac{1}{2}, -\frac{1}{2} | L_x | \frac{1}{2}, -\frac{1}{2} | \Psi'' \rangle = i\epsilon\langle \frac{1}{2}, \frac{1}{2} | \Psi'' \rangle
\]

(111)

which is again a relation of the same form (89), (90) as in the first and the second infinitesimal rotation.

A rotation brings a state $|\Psi\rangle$ into a state $|\Psi'\rangle$ which lies outside the space spanned, e.g., in the case $l = \frac{1}{2}$, by the basis vectors $|\frac{1}{2}, \frac{1}{2}\rangle$, $|\frac{1}{2}, -\frac{1}{2}\rangle$, but the projection $|\bar{\Psi}\rangle$ onto that space behaves as the usual spinor. The coefficients $\alpha \equiv \langle \frac{1}{2}, \frac{1}{2} | \Psi \rangle$, $\beta \equiv \langle \frac{1}{2}, -\frac{1}{2} | \Psi \rangle$ transform into $\alpha' = \langle \frac{1}{2}, \frac{1}{2} | \bar{\Psi}' \rangle = \langle \frac{1}{2}, \frac{1}{2} | \Psi' \rangle$, $\beta' \equiv \langle \frac{1}{2}, -\frac{1}{2} | \bar{\Psi}' \rangle = \langle \frac{1}{2}, -\frac{1}{2} | \psi' \rangle$ in the same way as those of the usual spinors and their norm is preserved: $|\alpha|^2 + |\beta|^2 = |\alpha'|^2 + |\beta'|^2 = 1$.

It is important that under rotation

\[
|\alpha|^2 + |\beta|^2 = |\alpha'|^2 + |\beta'|^2 = 1
\]

(112)

This is essential. In the full space spanned by $|\frac{1}{2}, \frac{1}{2}\rangle$, $|\frac{1}{2}, -\frac{1}{2}\rangle$, $|\frac{1}{2}, -\frac{3}{2}\rangle$, $|\frac{1}{2}, -\frac{5}{2}\rangle$, ..., a vector $|\Psi\rangle$ transforms in a peculiar way. But its projection $|\bar{\Psi}\rangle$ onto the space spanned by $|\frac{1}{2}, \frac{1}{2}\rangle$, $|\frac{1}{2}, -\frac{1}{2}\rangle$ behaves as a usual spinor. The matrix elements of $L_x$, $L_y$, $L_z$ in the states $|lm\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$, $|\frac{1}{2}, -\frac{1}{2}\rangle$ are the same as those in the usual theory of spinors.

Analogous results hold in the case of an arbitrary $l$. If we perform an arbitrary succession of infinitesimal rotations we find that the coefficients $\langle lm | \Psi \rangle$, $m = -l, ..., l$, change under rotations in the same manner as in the case of spinors.; this is so because of the relation (17) which has for a consequence that for $m = -l, ..., l$ the matrix elements $\langle lm | L_x | l, -l - 1 \rangle$ vanish. The presence of the non vanishing coefficients $\langle lm | \psi \rangle$, $m < -l$, has no influence. The latter coefficients behave in this respect like “ghosts”. The same is true for a finite rotation as well, since a finite rotation can be considered as an infinite sequence of infinitesimal rotations.

Analogous transformations properties hold if we represent states $|lm\rangle$ by the functions $Z_{lm} = \langle \Omega | lm \rangle$ defined in eq.(7). Since there is no reason why just one set of the functions, say $Y_{lm}$, should represent spinors, we shall later consider both sets of functions at once. At the moment let us still keep on considering the functions $Y_{lm}$ only.

For a finite rotation $D_R$ a state $|\Psi\rangle$ of $S_l$

\[
|\Psi\rangle = \sum_{m=-l}^{l} C_m |lm\rangle , \quad C_m \equiv \langle lm | \Psi \rangle
\]

(113)
transforms into another state

\[ |\Psi'\rangle = D_R|\Psi\rangle = D_R \sum_{m=-l}^{l} C_m|lm\rangle \]

(114)

which does no longer belong to \( S_l \). We can decompose (114) according to [13]

\[ |\Psi'\rangle = D_R|\Psi\rangle = |\bar{\Psi}'\rangle + |O\rangle \]

(115)

where

\[ |\bar{\Psi}'\rangle = \sum_{m'=-l}^{l} C'_{m'}|lm'\rangle \]

(116)

\[ |O\rangle = \sum_{m'=-l-1,-l-2,...}^{l} C'_{m'}|lm'\rangle \]

(117)

and

\[ C'_{m'} = \sum_{m=-l}^{l} \langle lm'|D_R|lm\rangle C_m \]

(118)

It is important to bear in mind that \( |O\rangle \) is orthogonal to \( |\bar{\Psi}'\rangle \):

\[ \langle \bar{\Psi}'|O\rangle = 0 \]

(119)

and that

\[ \sum_{m'=-l}^{l} |C'_{m'}|^2 = \sum_{m=-l}^{l} |C_m|^2 \]

(120)

Eq. (116) can be rewritten as

\[ |\bar{\Psi}'\rangle = U|\Psi\rangle \]

(121)

where \( U \) is just the usual unitary operator for a rotation of a spinor, represented by the matrix whose elements are \( \langle lm'|D_R|lm\rangle \):

\[ U \rightarrow \langle lm'|D_R|lm\rangle \quad m, m' = -l, ..., l \]

(122)

Unitarity is assured for all states of \( S_l \), if one uses either Definition I, or Definition II of inner product.

A state \( |\Psi\rangle \) as given in eq. (113) thus transforms under a finite rotation \( D_R \) in such a way that the projection onto the subspace \( S_l \) spanned by the basis vectors \( |lm\rangle, m = -l, ..., l \) is transformed in the same manner as an ordinary state with half-integer \( l \).
The above considerations in eqs. (113)–(122) can be rephrased by saying that a matrix $D_R$ representing a rotation $R$, calculated in the basis of functions $Y_{lm}$, has the form

$$
\begin{pmatrix}
D_R^{(S)} & 0 \\
\vdots & \vdots \\
F_R & D_R^{(O)}
\end{pmatrix}
$$

(123)

where $D_R^{(S)}$ is just the usual rotation matrix. Whilst the submatrix $D_R^{(S)}$ is hermitian, the total matrix $D_R$ is not hermitian. The product

$$
D_{RS} = D_R D_S =
\begin{pmatrix}
D_R^{(S)} D_S^{(S)} & 0 \\
\vdots & \vdots \\
F_R D_S^{(S)} + D_R^{(O)} F_S & D_R^{(O)} D_S^{(O)}
\end{pmatrix}
$$

(124)

has the same form as (123). The matrices $D_{RS}^{(S)} = D_R^{(S)} D_S^{(S)}$ and $D_{RS}^{(O)} = D_R^{(O)} D_S^{(O)}$ provide us, respectively, with an $l(l+1)$-dimensional and infinite dimensional representation of the rotation group. The original representation $D_R$ is thus reducible. The representation space is split into two subspaces: the $l(l+1)$-dimensional space $S_l$ and the infinite dimensional space $O_l$. From the form of the matrix (123) we find that the subspace $O_l$ is invariant, whilst the subspace $S_l$ is not invariant.

A representation is said to be fully reducible, if both subspaces are invariant, i.e., if it is possible to find a basis in which $F_R = 0$. In many important cases this happens to be the case. But the representation with the basis given in terms of spherical harmonics is not fully reducible (in the above sense) for half-integer $l$-values. And yet, according to the ordinary representation theory, it is reducible, since $D_R^{(S)}$ and $D_R^{(O)}$ in eq. (123) are in themselves representations of the 3-dimensional rotation group. We thus see that the spherical harmonics with $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$, do fit into the theory of group representations, only the subspace $S_l$ is not invariant. If initially we have a state

$$
\begin{pmatrix}
C \\
\vdots \\
0
\end{pmatrix} \in S_l, \quad C \equiv C_m = \langle lm|\psi \rangle
$$

(125)

then after applying a rotation, e.g., once and twice, we have respectively

$$
D_S \begin{pmatrix}
C \\
\vdots \\
0
\end{pmatrix} = \begin{pmatrix}
D_S^{(S)} C \\
\vdots \\
F_S C
\end{pmatrix}, \quad D_R D_S \begin{pmatrix}
C \\
\vdots \\
0
\end{pmatrix} = \begin{pmatrix}
D_R^{(S)} D_S^{(S)} C \\
\vdots \\
(F_S D_S^{(S)} + D_R^{(O)} F_S) C
\end{pmatrix}
$$

(126)

\footnote{See, e.g., ref. [20]}

22
Since $D^{(S)}_R$, $D^{(S)}_S$ are just the ordinary rotation matrices, the states $C$, $D^{(S)}_SC$, $D^{(S)}_RD^{(S)}_S$ in the subspace $S_l$ are normalized according to (120) and they behave as ordinary half-integer $l$ states. Analogous considerations hold for the basis functions $Z_{lm}$.

**Behaviour in the presence of the Definition II of scalar product**

If we adopt the Definition II of inner product, then the situation simplifies significantly because of the validity of eqs. (75), (76) which say that the physical functions are orthonormal, whereas the unphysical functions have vanishing norms. Consequently, the matrix elements of the angular momentum operator between such unphysical states vanish. The complications described in eqs. (95)–(101) do not arise. A rotation does not bring a state, initially in $S_l$, into a state with components outside $S_l$. The latter space is now invariant. Therefore, a residual vector $|O\rangle$ that occurred under a rotation in eq. (115), does no longer occur. A matrix $D_R$ representing a rotation has nonvanishing elements only between the physical functions. That is, in eq. (123) the submatrix $D^{(S)}_R$ is different from zero, whilst $F_R$ and $D^{(O)}_R$ are zero. Thus the the spherical harmonics with half-integer spin values form a fully reducible representation of the 3-dimensional rotation group.

### 2.3 Inclusion of the functions $Z_{lm}$ into the description of half-integer spin

If one looks at the functions $Y_{\frac{1}{2}, \frac{1}{2}}$ and $Y_{\frac{1}{2}, -\frac{1}{2}}$ (eqs. (32), (33)) one finds that they have completely different forms, so that they cannot be related by a transformation such as a rotation, space reflection or time reversal. On the other hand, we would expect that a state which has only its spin direction reversed should be obtained from the original state by any of those transformations. Does it means that $Y_{lm}$ for half integer $l$ are not suitable for the description of spin half states after all? Such a conclusion would be too hasty, since besides the functions $Y_{lm}$ there are also the functions $Z_{lm}$ given in eq. (7). In particular, for $l = \frac{1}{2}$, we have

$$Z_{\frac{1}{2}, \frac{1}{2}} = \frac{1}{\pi} \sin^{-1/2} \vartheta \cos \vartheta e^{i\varphi}$$  \hspace{1cm} (127)

$$Z_{\frac{1}{2}, -\frac{1}{2}} = \frac{1}{\pi} \sin^{1/2} \vartheta e^{-i\varphi}$$  \hspace{1cm} (128)
They satisfy the following relations

\[ L_+ Z_{\frac{1}{2}, \frac{1}{2}} = Z_{\frac{1}{2}, -\frac{1}{2}} \]  

(129)

\[ L_+ Z_{\frac{1}{2}, -\frac{1}{2}} = Z_{\frac{1}{2}, \frac{1}{2}} \]  

(130)

A state \(|lm\rangle\) with half integer \(l\) can be represented either by functions \(Y_{lm}\) or \(Z_{lm}\), or, in general, by a superposition

\[ |lm\rangle \rightarrow \Psi_{lm} = aY_{lm} + bZ_{lm} \]  

(131)

where \(a, b\) are complex constants, such that \(|a|^2 + |b|^2 = 1\).

For \(l = \frac{1}{2}\) eq. (131) becomes

\[ \left| \frac{1}{2}, \frac{1}{2} \right\rangle \rightarrow \psi_{\frac{1}{2}, \frac{1}{2}} = \left( a \frac{i}{\pi} \sin^{1/2} \vartheta + b \frac{1}{\pi} \sin^{-1/2} \vartheta \cos \vartheta \right) e^{i \phi} \]  

(132)

\[ \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \rightarrow \psi_{\frac{1}{2}, -\frac{1}{2}} = L_- \psi_{\frac{1}{2}, \frac{1}{2}} = \left( -a \frac{i}{\pi} \sin^{-1/2} \vartheta \cos \vartheta - b \frac{1}{\pi} \sin^{1/2} \vartheta \right) e^{-i \phi} \]  

(133)

The preceding expressions demonstrate that functions \(\psi_{\frac{1}{2}, \frac{1}{2}}\) and \(\psi_{\frac{1}{2}, -\frac{1}{2}}\) are given by similar expressions. There is no longer such a drastic difference between the \(m = 1/2\) and \(m = -1/2\) functions. Functions \(\psi_{\frac{1}{2}, \frac{1}{2}}, \psi_{\frac{1}{2}, -\frac{1}{2}}\) are eigenfunctions of \(L^2\) and \(L_z\) with the eigenvalues \(l(l + 1) = \frac{3}{4}\) and \(m = \frac{1}{2}, -\frac{1}{2}\), respectively.

Let us now study the behaviour of the functions \(\psi_{\frac{1}{2}, \frac{1}{2}}, \psi_{\frac{1}{2}, -\frac{1}{2}}\) under some transformations of particular interest.

a) 180° rotation around the \(y\)-axis. The polar coordinates change according to

\[ r \rightarrow r \]

\[ \vartheta \rightarrow \pi - \vartheta \]

\[ \varphi \rightarrow \pi - \varphi \]  

(134)

This gives

\[ x = r \sin \vartheta \cos \varphi \rightarrow r \sin (\pi - \vartheta) \cos (\pi - \varphi) = -x \]

\[ y = r \sin \vartheta \sin \varphi \rightarrow r \sin (\pi - \vartheta) \sin(\pi - \varphi) = y \]  

(135)

\[ z = r \cos \vartheta \rightarrow r \cos (\pi - \vartheta) = -z \]

where we have taken into account \(\sin(\pi - \vartheta) = \sin \vartheta, \cos(\pi - \varphi) = -\cos \varphi\).
Under the change of coordinates $(134), (135)$, the basis functions transform as

\[
Y_{\frac{1}{2}, \frac{1}{2}}(\vartheta, \varphi) \rightarrow \frac{i}{\pi} \sin^{1/2}(\pi - \vartheta) e^{i\frac{\pi}{2}(\pi - \varphi)} = -Z_{\frac{1}{2}, -\frac{1}{2}}(\vartheta, \varphi)
\]

\[
Z_{\frac{1}{2}, \frac{1}{2}}(\vartheta, \varphi) \rightarrow \frac{1}{\pi} \sin^{-1/2}(\pi - \vartheta) \cos(\pi - \vartheta) e^{i\frac{\pi}{2}(\pi - \varphi)} = Y_{\frac{1}{2}, -\frac{1}{2}}(\vartheta, \varphi)
\]

so that

\[
\psi_{\frac{1}{2}, \frac{1}{2}} = aY_{\frac{1}{2}, \frac{1}{2}} + bZ_{\frac{1}{2}, \frac{1}{2}} \rightarrow \psi'_{\frac{1}{2}, \frac{1}{2}} = -aZ_{\frac{1}{2}, -\frac{1}{2}} + bY_{\frac{1}{2}, -\frac{1}{2}}
\]

Comparing the transformed wave function $\psi'_{\frac{1}{2}, \frac{1}{2}}$ with

\[
\psi_{\frac{1}{2}, -\frac{1}{2}} = aY_{\frac{1}{2}, -\frac{1}{2}} + bZ_{\frac{1}{2}, -\frac{1}{2}}
\]

we find that they are related according to

\[
\psi'_{\frac{1}{2}, \frac{1}{2}} = A\psi_{\frac{1}{2}, -\frac{1}{2}} = \tilde{\psi}_{\frac{1}{2}, -\frac{1}{2}}
\]

Here $A$ denotes the transformations which changes $a$ into $b$ and $b$ into $-a$. We see that

\[
\text{under the 180° rotation around the $y$-axis the function } \psi_{\frac{1}{2}, \frac{1}{2}}(\vartheta, \varphi) \text{ becomes the function } \psi_{\frac{1}{2}, -\frac{1}{2}}(\vartheta, \varphi), \text{ apart from an active SU(2) “rotation” in the space spanned by the basis functions } Y_{\frac{1}{2}, -\frac{1}{2}} \text{ and } Z_{\frac{1}{2}, -\frac{1}{2}}. \text{ In other words, the 180° rotation of the coordinate axes transforms the function } \psi_{\frac{1}{2}, \frac{1}{2}} \text{ into the function which is of the same form as the function } \psi_{\frac{1}{2}, -\frac{1}{2}} \text{ (see eq. (133)), only the coefficients are different. They are changed by an SU(2) transformation which in matrix form reads}
\]

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= 
\begin{pmatrix}
b \\
-a
\end{pmatrix}
\]

where

\[
\psi_{\frac{1}{2}, -\frac{1}{2}} \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}, \quad \tilde{\psi}_{\frac{1}{2}, -\frac{1}{2}} \rightarrow \begin{pmatrix} b \\ -a \end{pmatrix}
\]

The function $\psi'_{\frac{1}{2}, \frac{1}{2}}$ which we obtain from $\psi_{\frac{1}{2}, \frac{1}{2}}$ by the change of coordinates is an eigenfunction of $L^2$ and $L_z$ with the eigenvalues $l(l + 1) = 3/4$ and $m = -1/2$, respectively. Therefore we may write $\tilde{\psi}_{\frac{1}{2}, -\frac{1}{2}}$ instead of $\psi'_{\frac{1}{2}, \frac{1}{2}}$.

Let us consider two particular cases of special interest:

**Case I**

\[
a = \frac{1}{\sqrt{2}}, \quad b = \frac{i}{\sqrt{2}}
\]
Then eq. (140) gives
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}
\]
(143)
i.e.,
\[
\tilde{\psi}_{1,-\frac{1}{2}} = i\psi_{1,-\frac{1}{2}}
\]
(144)
We see that the particular wave function
\[
\chi_{1\frac{1}{2}} = \frac{1}{\sqrt{2}} (Y_{1\frac{1}{2}} + iZ_{1\frac{1}{2}})
\]
(145)
for which we introduce the new symbol \( \chi \) transforms under the 180° rotation around \( y \)-axis into the wave function
\[
\chi'_{1\frac{1}{2}} = \tilde{\chi}_{1,-\frac{1}{2}} = \frac{i}{\sqrt{2}} (Y_{1\frac{1}{2}} + iZ_{1\frac{1}{2}}) = i\chi_{1\frac{1}{2}}
\]
(146)
which is equal to the wave function \( \chi_{1,-\frac{1}{2}} \) multiplied by \( i \).

Case II
\[
a = \frac{i}{\sqrt{2}}, \quad b = \frac{1}{\sqrt{2}}
\]
(147)
Then the particular function is
\[
\theta_{1\frac{1}{2}} = \frac{1}{\sqrt{2}} (iY_{1\frac{1}{2}} + Z_{1\frac{1}{2}})
\]
(148)
and it transforms under the rotation (134) into
\[
\theta'_{1\frac{1}{2}} = \tilde{\theta}_{1,-\frac{1}{2}} = \frac{i}{\sqrt{2}} (Y_{1\frac{1}{2}} + iZ_{1\frac{1}{2}}) = -i\theta_{1\frac{1}{2}}
\]
(149)

To sum up, for the particular choice of coefficients (142) and (147) (Case I and Case II) the wave functions transform under the 180° rotation (134) according to
\[
\chi_{1\frac{1}{2}} \rightarrow i\chi_{1\frac{1}{2}}
\]
(150)
\[
\theta_{1\frac{1}{2}} \rightarrow -i\theta_{1\frac{1}{2}}
\]
(151)

b) Reflection of \( x \)-axis
\[
x \rightarrow -x \quad \text{or} \quad r \rightarrow r
\]
\[
y \rightarrow y \quad \text{or} \quad \vartheta \rightarrow \vartheta
\]
\[
z \rightarrow z \quad \varphi \rightarrow \pi - \varphi
\]
(152)
This gives

\[
Y_{\frac{1}{2}, \frac{1}{2}}(\theta, \varphi) \rightarrow \frac{i}{\pi} \sin^{1/2} \theta e^{i(\pi - \varphi)} = -Z_{\frac{1}{2}, -\frac{1}{2}}(\theta, \varphi)
\]

\[
Z_{\frac{1}{2}, \frac{1}{2}}(\theta, \varphi) \rightarrow \frac{1}{\pi} \sin^{-1/2} \theta \cos \theta e^{i(\pi - \varphi)} = -Y_{\frac{1}{2}, -\frac{1}{2}}(\theta, \varphi)
\]  

(153)

The transformation of a generic wave function reads

\[
\psi_{\frac{1}{2}, \frac{1}{2}} = aY_{\frac{1}{2}, \frac{1}{2}} + bZ_{\frac{1}{2}, \frac{1}{2}} \rightarrow \psi'_{\frac{1}{2}, \frac{1}{2}} = -aZ_{\frac{1}{2}, -\frac{1}{2}} - bY_{\frac{1}{2}, -\frac{1}{2}} = \tilde{\psi}_{\frac{1}{2}, -\frac{1}{2}}
\]  

(154)

Comparing \(\psi'_{\frac{1}{2}, \frac{1}{2}} = \tilde{\psi}_{\frac{1}{2}, -\frac{1}{2}}\) with \(\psi_{\frac{1}{2}, -\frac{1}{2}}\) we have

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}
-b \\
-a
\end{pmatrix}
\]  

(155)

where

\[
\psi_{\frac{1}{2}, -\frac{1}{2}} \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}, \quad \tilde{\psi}_{\frac{1}{2}, -\frac{1}{2}} \rightarrow \begin{pmatrix} -b \\ -a \end{pmatrix}
\]  

(156)

For the particular choice of coefficients, (140) and (147) (Case I and Case II), we find that under the reflection (152) the corresponding wave functions transform according to

\[
\chi_{\frac{1}{2}, \frac{1}{2}} \rightarrow -\chi_{\frac{1}{2}, -\frac{1}{2}}
\]  

(157)

\[
\theta_{\frac{1}{2}, \frac{1}{2}} \rightarrow -\chi_{\frac{1}{2}, -\frac{1}{2}}
\]  

(158)

We see that the reflection interchanges functions \(\chi\) and \(\theta\).

c) \textit{Space inversion}

\[
x \rightarrow -x \quad r \rightarrow r
\]

\[
y \rightarrow -y \quad \text{or} \quad \vartheta \rightarrow \vartheta - \pi
\]

\[
z \rightarrow -z \quad \varphi \rightarrow \varphi + \pi
\]  

(159)

We then find

\[
\chi_{\frac{1}{2}, \frac{1}{2}} \rightarrow \theta_{\frac{1}{2}, \frac{1}{2}}
\]  

(160)

\[
\theta_{\frac{1}{2}, \frac{1}{2}} \rightarrow -\chi_{\frac{1}{2}, \frac{1}{2}}
\]  

(161)

The particular wave functions \(\chi_{\frac{1}{2}, \frac{1}{2}}\) and \(\theta_{\frac{1}{2}, \frac{1}{2}}\) have good behaviour under the 180° rotation (134), because \(\chi_{\frac{1}{2}, \frac{1}{2}}\) transforms into \(\chi_{\frac{1}{2}, -\frac{1}{2}}\) and \(\theta_{\frac{1}{2}, \frac{1}{2}}\) into \(\theta_{\frac{1}{2}, -\frac{1}{2}}\), apart from a factor \(i\) or \(-i\). Neglecting the latter factor, the rotated state is distinguished from the original
state in the sign of the quantum number $m = \pm \frac{1}{2}$. The situation is different in the case of space reflection and space inversion. The latter transformations interchange the type of the wave function.

Inspecting now *rotations of coordinates axes* for other angles, e.g., the $2\pi$ rotation in the $(x, z)$-plane, we find that it brings $\chi_{\frac{3}{2}} \rightarrow \chi_{\frac{5}{2}}$ and $\theta_{\frac{3}{2}} \rightarrow \theta_{\frac{5}{2}}$. That is, a $2\pi$ rotation of coordinates axes transforms a wave function $\chi_{\frac{3}{2}}$ or $\theta_{\frac{3}{2}}$ back into the original wave function. This is so because a rotation of a coordinate frame does not affect a physical system in question; from the point of view of the physical system it is a *passive transformation*. Consider now the popular illustration of spinor by means of a ball connected to a box with elastic threads$^8$. The *active transformation* of the latter system, corresponding to the rotation of a coordinate frame, is a rotation of the box$^9$ that keeps the relative orientation of the ball unchanged. The box together with the ball is rotated. On the contrary, rotations by which one illustrates the spinor properties affect the ball only. A $2\pi$ rotation of the ball then entangles the ball and the box in such a way that the transformed system is not equivalent to the original system. A $4\pi$ rotation is needed in order to bring the system back into its original state.

The above considerations demonstrate the general rule that for wave functions which represent spinors, a relation such as $D_R\psi(x) = \psi(R^{-1}x)$, valid for *scalars*, does not hold. *Wave functions representing spinors do not transform as scalars*. Here $D_R$ is a linear operator which acts on functions $\psi(x)$, whilst $R$ is a rotation which acts on coordinates $x$. In particular, $R$ can be a rotation around $y$-axis and $D_R = \exp(\imath \alpha L_y)$, i.e., the operator analogous to the one considered in eq.(84). The case of $R$ for $\alpha = \pi$ (i.e., $180^0$) has been considered in eqs.(134)–146).

Inclusion of wave functions which do not behave as scalars under rotations, is one amendment to the notion of representation space. Functions that can form a representation of the 3-dimensional rotation group need not be scalars. This is in agreement with the fact that spinors are indeed not scalars.

If we take into account also the states represented by functions $Z_{lm}$, so that the basis is given in terms of functions

$$
\psi_{lm} = aY_{lm} + bZ_{lm} = A\chi_{lm} + B\theta_{lm}
$$

(162)

$^8$The box may represent the entire environment.

$^9$If the box represents the environment, then the environment together with the attached ball is rotated.
spanning a space $\mathcal{S}_l$, and compute the matrix elements
\[
\langle lm|D_R|lm'\rangle = \int d\Omega \psi_{lm}^* e^{i\mathbf{L}\mathbf{a}\psi_{lm'}}
\]
we find that the transformation matrix representing a rotation $R$ has the form
\[
D_R = \begin{pmatrix}
D_R^{(O_l^+)} & G_R & 0 \\
0 & D_R^{(S_l)} & 0 \\
0 & F_R & D_R^{(O_l^-)}
\end{pmatrix}
\]
Now $\mathcal{S}_l$ denotes a space spanned by the functions $\psi_{lm}$ for $m = -l, \ldots, l$ (i.e., superpositions given in eq. (162)), while $O_l^-$ and $O_l^+$ are the corresponding spaces for $m < -l$ and $m > l$, respectively.

The product of two rotations gives
\[
D_{RS} = D_R D_S = \begin{pmatrix}
D_R^{(O_l^+)} & D_S^{(O_l^+)} & G_R + G_R D_S^{(S_l)} & 0 \\
0 & D_S^{(S_l)} & D_S^{(S_l)} & 0 \\
0 & F_R D_S^{(S_l)} + D_R^{(S_l)} F_S & D_R^{(O_l^-)} & D_S^{(O_l^-)}
\end{pmatrix}
\]
which is of the same form as (164): $D_R^{(O_l^+)}$, $D_S^{(S_l)}$ and $D_R^{(O_l^-)}$ are in themselves representations of rotations.

Suppose that initially we have a state vector
\[
\begin{pmatrix}
0 \\
\vdots \\
C \\
0
\end{pmatrix} \in \mathcal{S}_l, \quad C = C_{lm} = \langle lm|\psi \rangle, \quad m = -l, \ldots, l
\]
then after applying a rotation, e.g., once and twice, we have
\[
D_S \begin{pmatrix}
0 \\
\vdots \\
C \\
0
\end{pmatrix} = \begin{pmatrix}
G_S C \\
D_S^{(S_l)} C \\
F_S C
\end{pmatrix}, \quad D_R D_S \begin{pmatrix}
0 \\
\vdots \\
C \\
0
\end{pmatrix} = \begin{pmatrix}
(D_R^{(O_l^+)} G_S + G_R D_S^{(S_l)}) C \\
(D_R^{(S_l)} D_S^{(S_l)} C \\
(F_R D_S^{(S_l)} + D_R^{(O_l^-)} F_S) C
\end{pmatrix}
\]
Again $D_R^{(S_l)}$ and $D_S^{(S_l)}$ are the ordinary, unitary, rotation matrices for half integer values of $l$. They act in the subspace $\mathcal{S}_l$. Although the latter subspace is not invariant under
rotations, it holds that the norms of the states $C \in S_l$, and the corresponding rotated states $D_S^{(S_l)} C$, $D_R^{(S_l)} D_S^{(S_l)} C$ are invariant.

This is another amendment to the notion of “forming a representation”: a representation (sub)space $S_l$ need not be invariant, provided that the norms of the states projected into $S_l$ are preserved under the action of the group elements.

The above considerations are valid for Definition I of inner product. With Definition II the situation again simplifies significantly, since in matrix $D_R$ of eq. (164) only the piece $D_R^{S_l}$ is different from zero, whilst other pieces all vanish.

We have seen that the spherical harmonics with half-integer values of $l$ can represent the states with half-integer values of angular momentum. But this cannot be the states of orbital angular momentum, since it is well known experimentally that orbital angular momentum can have integer values only\(^{10}\). Hence, our Schrödinger basis for spinor representation of the 3-dimensional rotation group cannot refer to the ordinary configuration space of positions, but to an internal space associated with every point of the ordinary space. A possible internal space is the space of particle’s velocities, or equivalently, of accelerations. This will be discussed in Sec. 3.

On the SU(2) in the space spanned by functions $\chi_{lm}$ and $\theta_{lm}$. Functions $\chi_{lm}$ and $\theta_{lm}$ are linearly independent. Let us assume that for fixed $l$, $m$ they represent two distinct quantum states classified by eigenvalues of an operator $T_3$. Let us denote those states as

$$|lm\Lambda\rangle, \quad \Lambda = \frac{1}{2}, -\frac{1}{2} \tag{168}$$

so that

$$\langle \Omega |lm, \frac{1}{2}\rangle = \chi_{lm} \quad \text{and} \quad \langle \Omega |lm, -\frac{1}{2}\rangle = \theta_{lm} \tag{169}$$

The operator $T_3$ is defined by

$$T_3|lm\Lambda\rangle = \Lambda|lm\Lambda\rangle \tag{170}$$

\(^{10}\)A reason of why to reject $Y_{lm}$ and $Z_{lm}$ with half-integer $l$-values in the description of orbital angular momentum was given correctly by Dirac [22]. In the free case, a complete set of solutions to the Schrödinger equation consists of plane waves, which are single valued. The latter property has to be preserved when we use another representation, i.e., one with spherical harmonics. (See also Sec. 3.2.)
We can also define the operators $T_1$ and $T_2$ so that $T^\pm = T_1 \pm iT_2$ connect the states with different values of $\Lambda$

$$
T^+|lm, -\frac{1}{2}\rangle = |lm, \frac{1}{2}\rangle \\
T^-|lm, \frac{1}{2}\rangle = |lm, -\frac{1}{2}\rangle \\
T^+|lm, \frac{1}{2}\rangle = 0, \quad T^-|lm, -\frac{1}{2}\rangle = 0
$$

(171)

The matrices which represent $T_\alpha, \alpha = 1, 2, 3$ on the basis $|lm\Lambda\rangle$ are just the Pauli matrices. $T_\alpha$ are the generators of the group SU(2) and they commute with the generators $L_x, L_y, L_z$ of O(3). If we have a state $|\psi\rangle$ which is a superposition of the states with $\Lambda = \frac{1}{2}$ and $\Lambda = -\frac{1}{2}$,

$$
|\psi\rangle = A|lm, \frac{1}{2}\rangle + B|lm, -\frac{1}{2}\rangle
$$

(172)

then an element $S$ of the group SU(2) changes the coefficients $A, B$ into new coefficients $A', B'$, so that the new state is

$$
|\psi'\rangle = S|\psi\rangle = A'|lm, \frac{1}{2}\rangle + B'|lm, -\frac{1}{2}\rangle
$$

(173)

That an extra SU(2) group is present in our representation of spin $\frac{1}{2}$ states is very interesting. It would be challenging to investigate whether the group SU(2) generated by $T_\alpha$ has any relation with weak interactions and whether the states $\chi_{lm}$ and $\theta_{lm}$, $l = \frac{1}{2}, m = \pm \frac{1}{2}$ could represent the weak interaction doublet, with the difference that they cannot be directly identified with electron $e$ and neutrino $\nu_e$. Wave functions for the realistic electron and neutrino would take place in a full relativistic theory. A step into this direction is provided in next section.

### 3 Rigid Particle

The so called “rigid particle” which is described by the action containing second order derivatives (extrinsic curvature) has attracted much attention. [1]–[5, 21]. Such particle follows in general a worldline which deviates from a straight line. According to the terminology used in a recent review [21] it exhibits non Galilean motion which manifests itself as Zitterbewegung responsible for particle’s spin. Hence, although the particle is point like it possesses spin.
We are now going to present a revisited review of the rigid particle with the square of the extrinsic curvature in the action and show that according to the findings of Sec. 2 the rigid particle can have integer and half integer spin values.

3.1 Classical rigid particle

3.1.1 The action and equations of motion

We shall consider the free rigid particle in Minkowski spacetime with the metric $g_{\mu\nu} = \text{diag}(+, -, -, -)$. The action is \[ I = \int d\tau \gamma^{1/2} \left( m - \mu H^2 \right), \quad \gamma \equiv \dot{x}^\mu \dot{x}_\mu \] (174)

where $m$ and $\mu$ are constants, the bare mass and rigidity, respectively; $\tau$ is an arbitrary monotonically increasing parameter on the worldline, $\dot{x}^\mu \equiv dx^\mu / d\tau$, $H^2 \equiv g_{\mu\nu} H^\mu H^\nu$, and

$$H^\mu \equiv \frac{D^2 x^\mu}{D\tau^2} \equiv \frac{1}{\gamma^{1/2}} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\gamma^{1/2}} \right) \equiv \frac{d^2 x^\mu}{ds^2}, \quad ds = \gamma^{1/2} d\tau$$ (175)

From the action (174) one can derive, besides the usual pair of canonically conjugate variables $(x^\mu, p_\mu)$ also the pair $(\dot{x}^\mu, \pi_\mu)$, where $\pi_\mu = -\left( \frac{2}{\gamma^{1/2}} \right) H^\mu$. The “internal” space here consists of velocities and the corresponding conjugate momenta $\pi_\mu$.

A classically equivalent action that was considered by Lindström \[ I[x^\mu, y^\mu] = \int d\tau \gamma^{1/2} \left[ m - \mu \left( y^\mu y_\mu - 2 \frac{\dot{x}^\mu \dot{y}_\mu}{\gamma} - \frac{(\dot{x}^\mu y_\mu)^2}{\gamma} \right) \right] \] (176)

The latter action is invariant under reparametrizations of $\tau$ and also under an extra gauge symmetry discussed by Lindstroëm \[ y^\mu \rightarrow y^\mu + v(\tau) \dot{x}^\mu \] (177)

where $v(\tau)$ is an arbitrary function.

Varying the action (176) with respect to $x^\mu$ and $y^\mu$ we obtain ($\gamma \equiv \dot{x}^2$):

$$\delta x^\mu : \quad \dot{p}_\mu = 0$$ (178)

$$\delta y^\mu : \quad \dot{P}_\mu = 2\mu \gamma^{1/2} \left( y_\mu - \frac{1}{\gamma} (\dot{x}^\nu y_\nu) \dot{x}^\mu \right)$$ (179)

where

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m \dot{x}_\mu}{\gamma^{1/2}} - \frac{\mu}{2\gamma^{3/2}} \left[ \gamma y^2 \dot{x}_\mu + 2 \dot{x}^\nu \dot{y}_\nu \dot{x}_\mu + 2 \gamma \dot{y}_\mu + (\dot{x}^\nu y_\nu)^2 \dot{x}_\mu - 2 \gamma \dot{x}^\nu y_\nu y_\mu \right]$$ (180)
\[ P_\mu = \frac{\partial L}{\partial \dot{y}^\mu} = -\frac{2\mu \ddot{x}_\mu}{\gamma^{1/2}} \]  

(181)

From the equation of motion (179) we find the relation

\[ y^\mu = \frac{\ddot{x}_\mu}{\dot{x}^2} \]  

(182)

We see that \( y^\mu \) is proportional to the acceleration \( \ddot{x}_\mu \), whilst \( P_\mu \) is proportional to the velocity \( \dot{x}_\mu \).

The pairs of canonically conjugate variables are

\[ (x^\mu, p_\mu) \quad \text{and} \quad (y^\mu, P_\mu) \]  

(183)

The generators of infinitesimal translations \( x^\mu \to x^\mu + \epsilon^\mu \) and rotations \( x^\mu \to x^\mu + \epsilon_\mu x^\nu \), \( y^\mu \to y^\mu + \epsilon^\mu y^\nu \), \( \epsilon_\mu\nu = -\epsilon_\nu\mu \) are \( p_\mu \) and \( J_\mu\nu = M_\mu\nu + S_\mu\nu \), respectively, where

\[ M_\mu\nu = x_\mu p_\nu - x_\nu p_\mu \]  

(184)

\[ S_\mu\nu = y_\mu P_\nu - y_\nu P_\mu \]  

(185)

are orbital angular momentum and spin tensor. Occurrence of spin in our dynamical system results from the curvature term in the action (174), or equivalently, from the terms with \( y^\mu \) in the action (176). A result is that the particle does not follow a straight world line, but performs a Zitterbewegung.

The canonical momenta \( p_\mu \) and \( P_\mu \) satisfy the following two constraints [8]:

\[ \phi_2 \equiv p_\mu P^\mu - \frac{\mu m}{2} - S_\mu\nu S^{\mu\nu} = 0 \]  

(186)

\[ \phi_1 \equiv P_\mu P^\mu - \mu^2 = 0 \]  

(187)

which are due to the invariance of the action (176) under reparametrisations of \( \tau \), and under the transformation (177). The constraints (186), (187) are first class, because their Poisson bracket is strongly zero:

\[ \{\phi_1, \phi_2\} = 0 \]  

(188)

The Hamiltonian is a linear combination of constraints:

\[ H = v_1 \phi_1 + v_2 \phi_2 \]  

(189)
and it generates the $\tau$-evolution of an arbitrary quantity $A(x^\mu, p_\mu, y^\mu, P_\mu)$ of the canonically conjugate variables $x^\mu, p_\mu, y^\mu, P_\mu$. So we obtain that the total angular momentum $J_{\mu\nu}$ is a constant of motion:

$$\dot{J}_{\mu\nu} = \{J_{\mu\nu}, H\} = 0 \quad (190)$$

where the dot denotes the derivative with respect to $\tau$.

Another quantities which are also conserved are the Pauli-Lubanski pseudo vector

$$S^\mu = \frac{1}{\sqrt{p^2}} \epsilon^{\mu\nu\alpha\beta} p_\nu J_{\alpha\beta} = \frac{1}{\sqrt{p^2}} \epsilon^{\mu\nu\alpha\beta} p_\nu S_{\alpha\beta} \quad (191)$$

and the momentum $p_\mu$ (defined in (180)). Thus

$$\dot{p}_\mu = \{p_\mu, H\} = 0 \quad (192)$$

$$\dot{S}^\mu = \{S^\mu, H\} = 0 \quad (193)$$

But the momentum $P_\mu$, conjugate to $y^\mu$, is not conserved:

$$\dot{P}_\mu = \{P_\mu, H\} \neq 0 \quad (194)$$

where the right hand side of the latter equation is given in eq.(179).

### 3.2 Quantization

The system can be quantized by replacing the canonically conjugate pairs of variables $(x^\mu, p_\mu)$ and $(y^\mu, P_\mu)$ by operators satisfying the following commutation relations\textsuperscript{11}

$$[x^\mu, p_\nu] = i\delta^\mu_\nu, \quad [y^\mu, P_\nu] = i\delta^\mu_\nu \quad (195)$$

The constraints (186), (187) become the conditions a physical state has to satisfy:

$$\phi_1 \psi = (P^\mu P_\mu - \mu^2) \psi = 0 \quad (196)$$

$$\phi_2 \psi = \left( P_\mu P^\mu - \frac{\mu m}{2} - S_{\mu\nu} S^{\mu\nu} \right) \psi = 0 \quad (197)$$

We find $[\phi_1, \phi_2] = 0$ which assures that the conditions (196), (197) are consistent.

The momentum $p_\mu$ and the Pauli-Lubanski operator $S^\mu$ commute with the operators $\phi_1$ and $\phi_2$:

$$[p_\mu, \phi_1] = 0, \quad [p_\mu, \phi_2] = 0 \quad (198)$$

\textsuperscript{11}We use the units in which $\hbar = c = 1$. 

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\[ [S^\mu, \phi_1] = 0, \quad [S^\mu, \phi_2] = 0 \]  

(199)

The set of mutually commuting operators is \( \{ p_\mu, S^\mu, \phi_1, \phi_2 \} \). They can thus have simultaneous eigenstates and eigenvalues. The physical states can be classified by the eigenvalues of the mass squared operator \( p^\mu p_\mu \) and spin \( S^\mu S_\mu \). Eigenvalues of the spin operator \( S^\mu S_\mu \) are \( s(s + 1) \). Choosing a representation in which \( x^\mu \) and \( y^\mu \) are diagonal, the corresponding momenta and spin are differential operators

\[
p_\mu = -i \partial/\partial x^\mu \quad \text{and} \quad P_\mu = -i \partial/\partial y^\mu
\]  

(200)

\[
S_{\mu\nu} = P_\mu y_\nu - P_\nu y_\mu
\]  

(201)

Assuming that \( \psi \) are eigenfunctions of the momentum \( p^\mu \) and that a reference frame exists in which \( p^\mu = (p^0, 0, 0, 0) \), we find that the equations

\[
S^\mu S_\mu \psi = s(s + 1) \psi
\]  

(202)

\[
S_z \psi = s_z \psi
\]  

(203)

become differential equations equivalent to the equations (2),(3).

Eq. (196) becomes the differential equation and can be reduced to a form which is mathematically equivalent to the static Schrödinger equation and which in spherical coordinates leads to the equation for the eigenfunctions of the angular momentum operator.

The formalism describing the rigid particle thus becomes equivalent to the formalism of Sec. 2, where we considered the Schrödinger basis for spinor representation of the rotation group. In rigid particle we have a concrete physical realization the Schrödinger basis for spin which, as we have shown in Sec. 2, allows for integer and half-integer spin values.

Although the spin angular momentum \( S_{\mu\nu} \) formally looks like the orbital momentum operator, there is a big difference.

In the case of a free point particle, its momentum \( p_\mu \) is a constant of motion, and so are its orbital angular momentum squared and \( L_z \). Therefore, a state of a free particle can be expanded either in terms of the momentum eigenfunctions or equivalently, in terms of the orbital angular momentum eigenfunctions. Momentum eigenfunctions form a complete set of states, and they are single valued. Therefore, when using the orbital angular
momentum eigenfunctions one has to take into account only *single valued* functions. The orbital angular momentum of a point particle has thus integer values only. Such argument was provided in Dirac’s book on quantum mechanics \[22\].

In the case of *rigid particle* the role that \(p_\mu, x^\mu\) had in Sec.2 is assumed by \(P_\mu, y^\mu\). But \(P_\mu\), unlike \(p_\mu\), is *not* a constant of motion\[12\]. A state of the rigid particle cannot be described as a superposition of the eigenstates of \(P_\mu\). However, it can be described as a superposition of the eigenstates of \(S^\mu S_\mu\) and \(S_z\), which are constants of motion, and which, as shown in sec. 2, can have eigenvalues either for integer or half-integer \(l\). In other words, the linear momentum \(P_\mu\) (which is conjugate to the acceleration), does not commute with the Hamiltonian operator \(H = v_1 \phi_1 + v_2 \phi_2\), hence it is not a constant of motion, and therefore the eigenfunctions of \(P_\mu\), which are single-valued only, cannot serve for a description of the rigid particle. On the other hand, the mutually commuting operators, \(S^\mu S_\mu\), \(S_z\) and \(H\) do have the simultaneous eigenfunctions. The latter eigenfunctions do provide a description of the rigid particle, and they may be single or double valued.

### 4 Conclusion

We have clarified a long standing problem concerning the admissibility of double valued spherical harmonics in providing a spinor representation of the three dimensional rotation group. The usual arguments against the inadmissibility of such functions, concerning hermiticity, orthogonality, behaviour under rotations, etc., are all related to the unsuitable choice of functions representing the states with positive and negative values of the quantum number \(m\), and to an inappropriate definition of inner product. By the correct choice of functions such problems do not occur, provided that we modify the inner product as well. We have considered two different definitions of inner product. By using Definition I the spherical harmonics with half-integer spin values do form a reducible representation of the 3-dimensional rotation group. But the latter representation is not fully reducible, because the physical space \(S_l\) with \(m\)-values in the range between \(l\) and \(-l\) is not invariant.

---

\[12\] If we switch off the rigidity by setting the rigidity constant \(\mu\) equal to zero, then, according to eq. (181), \(P_\mu = 0\), which is a trivial constant. For non vanishing rigidity constant \(\mu\) we have that in general \(P_\mu\) differs from zero; and if it differs from zero, then automatically it cannot be a constant of motion. In this respect rigid particle is drastically different from the ordinary particle. If an ordinary particle moves in the presence of a spherically symmetric potential, then of course, its linear momentum \(p_\mu\) is not a constant of motion, while its angular momentum is constant. But if one switches off the potential, \(p_\mu\) becomes constant that can differ from zero.
under rotations. But because the states projected onto $S_l$ transform in the correct way with their norms preserved, such representation makes sense. This is explicitly illustrated on an example for $m = \pm \frac{1}{2}, -\frac{1}{2}$. With Definition II of inner product spherical harmonics form a fully reducible representation of rotation group even for half-integer spin values.

Double valued spherical harmonics are admissible, if they do not refer to the ordinary configuration space in which the usual quantum mechanical orbital angular momentum is defined, but if they refer to an internal space in which a spin angular momentum is defined. An example of such an internal space is the space of velocities, or, equivalently, the space of accelerations, associated with the so called rigid particle whose action contains the square of the extrinsic curvature of a particle’s world line. If one considers the action (174), then one has the space of velocities. But in several respects it is more convenient to consider an alternative, although classically equivalent action (176), in which case the internal space consists of accelerations.

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