Nonlinear bending theories for non-Euclidean plates

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Abstract

Thin growing tissues (such as plant leaves) can be modelled by a bounded domain $S \subset \mathbb{R}^2$ endowed with a Riemannian metric $g$, which models the internal strains caused by the differential growth of the tissue. The elastic energy is given by a nonlinear isometry-constrained bending energy functional which is a natural generalization of Kirchhoff’s plate functional. We introduce and discuss a natural notion of (possibly non-minimising) stationarity points. We show that rotationally symmetric immersions of the unit disk are stationary, and we give examples of metrics $g$ leading to functionals with infinitely many stationary points.

1 Introduction

Thin growing tissues in biology, such as plant leaves, display bending patterns even if no external forces are applied. A similar pattern formation is observed in thin elastic sheets that have undergone plastic deformation, cf. [31]. What garbage bags and growing tissues have in common is that their reference configuration is not stress-free. In the case of growing tissues, the internal stresses arise from the differential growth of the leaf: typically, leaves do not grow much near their midrib, but their cells keep dividing and growing to their target size near the edges of the leaf. The excess length created by this growth near the leaf edge creates internal stresses in the planar reference configuration. These stresses can be relaxed by bending the leaf out of the plane.

Such questions have been studied in [3, 8] and elsewhere. Two very different scenarios can arise: either there is a way of fully relaxing the internal stresses

*Work supported by the DFG.
by deforming the leaf (with finite bending energy), or this is not possible. In this paper we are interested in the former case.

The reference configuration of the leaf is modelled by a bounded domain $S \subset \mathbb{R}^2$ and the inhomogeneous and anisotropic local growth pattern is modelled by a Riemannian ‘target’ metric $g$ on $S$. The fact that there is a way of fully relaxing the internal stresses by deforming the leaf with finite bending energy means precisely that the set

$$W^{2,2}_g(S) = \left\{ u \in W^{2,2}(S, \mathbb{R}^3) : (\nabla u)^T (\nabla u) = g \text{ almost everywhere in } S \right\}$$

of $W^{2,2}$ isometric immersions of $(S, g)$ into $\mathbb{R}^3$ is nonempty. If this is the case, then the asymptotic behaviour of the three dimensional elastic energy of the leaf with small thickness is captured by the natural generalization of Kirchhoff’s nonlinear bending theory of plates which we introduce next.

Firstly, for any regular immersion $u : S \to \mathbb{R}^3$ we introduce the Willmore functional (cf. e.g. [36, 29])

$$W(u) = \frac{1}{4} \int_S H^2 d\mu_g + \int_{\partial S} \kappa_g d\mu_{g\partial},$$

(1)

where $\kappa_g$ is the geodesic curvature of $\partial S$ and $g$ is the metric induced by $u$, the induced area measure on $S$ is $\mu_g$ and the induced boundary measure on $\partial S$ is $\mu_{g\partial}$. The Willmore functional has been studied extensively in the literature, cf. [37, 32, 26] and the references cited therein. Its analytical properties have been systematically studied in [22, 1, 23, 27]. More recently, there has been growing interest in constrained versions of the Willmore functional. The typical constraints include prescribed conformal class (cf. [2, 24, 28]) or fixed area and enclosed volume, cf. [9, 30].

The relevant case for thin film elasticity is the restriction of the Willmore functional (1) to isometric immersions of the given Riemannian manifold $(S, g)$ into $\mathbb{R}^3$. More precisely, from now on $S \subset \mathbb{R}^2$ will denote a bounded simply connected domain with smooth boundary, and $g : \overline{S} \to \mathbb{R}^{2\times 2}$ will be a given smooth Riemannian metric on $\overline{S}$. We will study the restriction of the Willmore functional to the class $W^{2,2}_g(S)$. That is, we will study the generalized Kirchhoff plate functionals

$$\widetilde{W}_g(u) = \begin{cases} W(u) & \text{if } u \in W^{2,2}_g(S) \\ +\infty & \text{otherwise.} \end{cases}$$

In addition to their key role in the modelling of thin films in nonlinear elasticity, these functionals are also entirely natural from a geometric viewpoint, as they are the simplest purely extrinsic functionals on surfaces. It is a key
feature of thin films in nonlinear elasticity that they undergo large deformations with low energy. In contrast to von-Kármán theories (cf., e.g., [14]), the above functionals admit such large deformations. In fact, they arise naturally as (rigorous) asymptotic theories from fully nonlinear three-dimensional elasticity in a bending energy regime, cf. [6, 21].

In the case when $g$ is the standard flat metric, i.e. $g_{ij} = \delta_{ij}$, the corresponding constrained Willmore functional $\widetilde{W}_g$ agrees with the energy functional arising Kirchhoff’s nonlinear bending theory for thin elastic plates, cf. [5]. It was studied in [10]. However, the arguments used in that paper heavily depend upon the special structure (developability) of intrinsically flat surfaces, so they do not carry over to other metrics. Moreover, they are not suited to formulate the concept of ‘stationary point’ for functionals such as $\mathcal{W}_g$. A possible notion was introduced in [10] by regarding solutions to the Euler-Lagrange equations derived in that paper as stationary points. But this definition does not help in defining stationarity for $\widetilde{W}_g$ when $g$ is Riemannian metric other than the standard flat metric. The need for a notion of (possibly non-minimising) stationary points was pointed out e.g. in [7].

Moreover, from a conceptual viewpoint, a major drawback of the derivation of the Euler-Lagrange equation in [10] (which is a system of ordinary differential equations) is that the relation to the classical Willmore equation remains unclear. So does the relation to a formal Lagrange multiplier rule.

In this paper we provide a framework for the analysis of isometry-constrained functionals such as $\widetilde{W}_g$, which (as far as it goes) works for arbitrary Riemannian metrics $g$. This approach overcomes the problems mentioned earlier: It leads to a natural notion of stationarity, the relation to the classical Willmore equation is clear, and it leads to a natural formulation in terms of Lagrange multipliers.

Our notion of stationarity is based on the geometric concept of bendings. Bendings are deformations of a given surface which preserve the metric. The velocity field $\tau$ of such a deformation is a solution of the system

$$\partial_i u \cdot \partial_j \tau + \partial_j u \cdot \partial_i \tau = 0 \text{ for all } i = 1, 2,$$

which is obtained by linearizing the system $\partial_i u \cdot \partial_j u = g_{ij}$. Solutions $\tau$ to (2) are called infinitesimal bendings of $u$.

Our abstract Euler-Lagrange equations will be formulated in terms of infinitesimal bendings. We give various formulations of these Euler-Lagrange equations, one of which shows very clearly the relation to the classical Willmore equation (cf. [36]). In particular, it explains how Euler-Lagrange ordinary differential equations such as those derived in [10] (or the particular version stated in [33]) relate to the Willmore equation.
After introducing the general framework, we study rotationally symmetric surfaces and show that rotationally symmetric surfaces with finite energy are always stationary. The simple proof illustrates the use of the stationarity condition introduced here. (We refer to [13] for the analogous result in the context of von Kármán theories.) Moreover, that result allows us to construct a class of smooth metrics $g$ on the unit disk for which $\mathcal{W}_g$ admits infinitely many distinct stationary points.

### The setting

Throughout the paper $S \subset \mathbb{R}^2$ is a smoothly bounded simply connected domain, and $g \in C^\infty(S, \mathbb{R}^{2\times2})$ is a Riemannian metric on $S$. We write $M$ to denote the Riemannian manifold $(S, g)$. By $\langle \cdot, \cdot \rangle$ resp. $|\cdot|$ we denote the natural scalar product on bundles inherited from the metric $g$. We set $|g| = \det g$ and $\sqrt{g} = \sqrt{|g|}$. We use the common convention regarding the raising and lowering of indices. In particular, $g^{kl}$ denotes the $(kl)$-entry of the matrix $g^{-1}$. The Christoffel symbols of (the connection associated with) $g$ are denoted by $\Gamma^k_{ij}$, that is, $\Gamma^k_{ij} := \frac{1}{2} g^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij})$.

By $TM$ we denote the tangent bundle along $M$; tensor bundles are denoted in the usual way. By $\mathcal{S}M$ we denote the bundle $T^*M \otimes T^*M$ of symmetric tensors in $T^*M \otimes T^*M$. The metric connection is denoted by $D$, and the same letter denotes the natural connection on tensor bundles. The class of all $L^2$ sections of a bundle $\Gamma$ are denoted by $L^2(\Gamma)$, and similar notation is used for other function spaces. For $Y \in C^\infty(TM)$ we set define $\text{Def} Y = \mathcal{L}_Y g$ (where $\mathcal{L}_Y$ is the Lie-derivative), which is the section of $\mathcal{S}M$ given in coordinates by

$$
(\text{Def} Y)_{\alpha\beta} = \frac{1}{2} \left((D_\alpha Y)_\beta + (D_\beta Y)_\alpha\right).
$$

The standard connection on $\mathbb{R}^3$ is denoted by $\nabla$. Let $u : M \to \mathbb{R}^3$ be an isometric immersion. Its normal is denoted by $n$, so

$$
n = \frac{\partial_1 u \times \partial_2 u}{|\partial_1 u \times \partial_2 u|}.
$$

Here and in what follows we denote by $\times$ the cross product in $\mathbb{R}^3$. By $A$ we denote the second fundamental form of $u$; following common convention its coordinates are denoted by $h_{ij}$, so

$$
h_{ij} = -\partial_i n \cdot \partial_j u = n \cdot \partial_i \partial_j u.
$$

The section $B$ of $\text{End}(TM)$ associated with a section $b$ of $\mathcal{S}M$ is defined via $\langle BX, Y \rangle = b(X, Y)$, and viceversa. By $S$ we denote the Weingarten map. It
is the negation of the section of $\text{End}(TM)$ associated with $A$.

By $J$ we denote the natural almost complex structure on $M$, i.e., the section of $\text{End}(TM)$ determined by the condition that

$$\nabla_{JX}u = n \times \nabla_X u \text{ for all } X \in \mathcal{C}^\infty(TM).$$

In coordinates we have

$$J_{\alpha\beta} = \begin{cases} 
-\sqrt{g} & \text{if } (i,j) = (1,2) \\
\sqrt{g} & \text{if } (i,j) = (2,1) \\
0 & \text{otherwise.}
\end{cases}$$

We define an associated section $J$ of $\text{End}(\mathcal{S}M)$, also denoted $J$, by setting

$$(Jq)(X,Y) := q(JX,JY)$$

for every section $q$ of $\mathcal{S}M$. Here and in what follows, unless specified otherwise, $X$ and $Y$ always denote smooth tangent vector fields. We observe that, in coordinates,

$$(Jq)^{\alpha\beta} = \frac{1}{|g|} (\text{cof } q)_{\alpha\beta}.$$ As usual, for sections $b$ of $\mathcal{S}M$ we define the 1-form $\text{div } b$ by

$$(\text{div } b)(X) = (D_{\alpha}b)(\partial^\alpha, X).$$

Here and elsewhere we use the summation convention.

A Codazzi tensor is a section $b$ of $\mathcal{S}M$ satisfying the Codazzi-Mainardi equations, i.e.,

$$\text{div } (Jb) = 0.$$ The formal adjoint of $\text{Def}$ is $-\text{div }$. So $b \in L^1_{\text{loc}}(\mathcal{S}M)$ is (weakly) Codazzi if

$$\int_M \langle Jb, \text{Def } Y \rangle = 0 \text{ for all } Y \in C^\infty_0(TM).$$

For a given immersion $u : M \to \mathbb{R}^3$ and for displacements $\tau, \rho : M \to \mathbb{R}^3$ we define the section $d\tau \cdot d\rho$ of $T^*M \otimes T^*M$ by

$$(d\tau \cdot d\rho)(X,Y) = \nabla_X \tau \cdot \nabla_Y \rho.$$ On the right-hand side a simple dot denotes the standard scalar product in the ambient space $\mathbb{R}^3$, so

$$\nabla_X \tau \cdot \nabla_Y \rho = \sum_{i=1}^3 (\nabla_X \tau_i)(\nabla_X \rho_i).$$

The expression $d\tau \times d\rho$ is defined similarly.

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By $d\tau \odot d\rho = d\rho \odot d\tau$ we denote the symmetrisation of $d\tau \cdot d\rho$, i.e., the section of $SM$ given by
\[(d\rho \odot d\tau) = \frac{1}{2} (d\tau \cdot d\rho + d\rho \cdot d\tau).\]

A displacement $\tau$ is called an infinitesimal bending of an immersion $u : M \to \mathbb{R}^3$ provided that $du \odot d\tau = 0$.

The Hessian of a scalar function $f$ on $M$ is denoted by $\text{Hess } f$. By definition, it is the section of $\text{End}(TM)$ (resp. of $SM$) given by
\[
\text{Hess } f = Ddf.
\]

Here and elsewhere we abuse notation by writing $df$ to denote both the gradient vector field of $f$ and its associated one-form. We write $u_* V = \nabla_V u$ to denote the usual push-forward.

If $F$ and $G$ are matrices of the same order, we will write $F : G = \text{Tr}(F^T G)$.

2 Minimisers of $\mathcal{W}_g$

As compared to the un-constrained Willmore functional, the isometry constraint simplifies matters drastically when it comes to proving existence of minimizers. This is mainly because it breaks the invariance under diffeomorphisms. Therefore, proving existence of minimizers of $\tilde{\mathcal{W}}_g$ will turn out to be straightforward. In fact, we will see that the stationary points of $\tilde{\mathcal{W}}_g$ agree with those of the functional $\mathcal{W}_g$ defined by

$$
\mathcal{W}_g(u) = \begin{cases} 
\frac{1}{2} \int_M |A|^2 & \text{if } u \in W^{2,2}_g(S, \mathbb{R}^3) \\
+\infty & \text{otherwise.} 
\end{cases} 
$$

(3)

For all $u \in W^{2,2}_g(S)$ we clearly have
\[
|A|^2 = 4H^2 - 2K,
\]

where $K$ is the Gauss curvature of the metric $g$. As $K$ is an intrinsic quantity it is the same for all $u \in W^{2,2}_g(S)$. This is also true for the boundary integral $\int_{\partial S} \kappa_g$. Hence the stationary points of $\tilde{\mathcal{W}}_g$ agree with those of the functional $\mathcal{W}_g$ defined by (3).

Writing $A^0 = A - Hg$ for the trace-free part of $A$ and using (4), we see
that, up to a constant prefactor and an additive constant, $W_g$ agrees with the functional

$$W^{(0)}_g(u) = \begin{cases} \frac{1}{2} \int_M |A^0|^2 & \text{if } u \in W^{2,2}_g(S, \mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases} \quad (5)$$

**Remark.** The functional $W_g$ is coercive with respect to the $W^{2,2}$ seminorm.

This follows from the equality $|\nabla_X \nabla_Y u|^2 = |A(X,Y)|^2 + |D_X Y|^2$ and the fact that the second term is intrinsic and therefore the same for all $u \in W^{2,2}_g(S)$.

In view of this remark, the existence of minimizers under typical boundary conditions follows at once from the direct method of the calculus of variations.

**Proposition 2.1.** The restriction of the functional $W_g$ to the space

$$\mathcal{A}_0 = \{ u \in W^{2,2}_g(S) : \int_M u = 0 \}$$

attains a global minimum on this space.

If $\Lambda \subset \partial M$ has positive length and $u_0 \in W^{2,2}_g(S)$ is given then similar statements are true with $\mathcal{A}_0$ replaced by

$$\mathcal{A}_{\Lambda, u_0} = \{ u \in W^{2,2}_g(S) : (u, du) = (u_0, du_0) \text{ on } \Lambda \}$$

or by

$$\tilde{\mathcal{A}}_{\Lambda, u_0} = \{ u \in W^{2,2}_g(S) : u = u_0 \text{ on } \Lambda \}.$$ 

**Proof.** Sequences in $\mathcal{A}_0$ with bounded $W_g$-energy subconverge weakly in $W^{2,2}$. The second derivatives are controlled because $W_g$ is coercive with respect to the $W^{2,2}$ seminorm, the first derivatives are uniformly bounded by the isometry constraint, and by the normalization (resp. boundary conditions) together with Poincaré’s inequality, $u$ itself is controlled as well.

Moreover, $W_g$ is obviously lower semicontinuous with respect to weak $W^{2,2}$-convergence. Finally, notice that the isometry constraint and the subsidiary conditions are continuous with respect to this convergence.

3 Infinitesimal bendings

We begin with a definition of infinitesimal bendings which makes sense under minimal regularity assumptions:
Definition 3.1. A vector field $\tau \in L^2(S, \mathbb{R}^3)$ is called an infinitesimal bending of an immersion $u \in W^{2,2}(S, \mathbb{R}^3)$ if it satisfies

$$\partial_i(\tau \cdot \partial_j u) + \partial_j(\tau \cdot \partial_i u) = 2\tau \cdot \partial_i \partial_j u$$

in $\mathcal{D}'(S)$ for $i, j = 1, 2$.

(6)

An infinitesimal bending is said to be trivial if it is the velocity field of a rigid motion.

Observe that if $\tau \in W^{1,2}(S, \mathbb{R}^3)$ then by the Leibniz rule (6) is equivalent to $du \odot d\tau = 0$. In fact, we will mainly encounter infinitesimal bendings of $u$ which belong to the space $W^{2,2}$, so we will mostly use this condition as a definition of infinitesimal bending.

For a given infinitesimal bending $\tau \in W^{1,2}(M, \mathbb{R}^3)$ of $u$, as in [15] we define its linearised second fundamental form $b$ by $b = n \cdot \text{Hess} \tau$. In coordinates, this reads

$$b_{ij} = n \cdot (\partial_i \partial_j \tau - \Gamma^k_{ij} \partial_k \tau).$$

Observe that $b$ as defined here is well-defined as a distribution, because $n \in W^{1,2}$ and Hess $\tau \in H^{-1}$. The infinitesimal bendings $\tau$ considered in this paper will mostly belong at least to $W^{2,1}$, in which case the definition of $b$ makes sense pointwise almost everywhere and $b \in L^1(SM)$.

There are various ways of representing infinitesimal bendings. Here we will use three of them. Among geometers, these are well-known in the smooth setting, cf. e.g. [35]. Our approach here differs from the usual one, as we need to display the links between these representations. For later use, we formulate these facts in Sobolev spaces and derive formulae which allow to switch representations with minimal loss of integrability.

### 3.1 Representation via the displacement field

Decomposing the vector field $\tau : S \rightarrow \mathbb{R}^3$ into its tangential part and its normal part,

$$\tau = \nabla_V u + \Phi n,$$

we have $du \odot d\tau = \text{Def} V - \Phi A$. Hence $\tau$ of the form (7) is an infinitesimal bending of $u$ precisely if

$$\text{Def} V = \Phi A.$$

(8)

As an immediate consequence of these observations and of Definition 3.1, we note the following result:

**Lemma 3.2.** Let $\tau \in L^2(M, \mathbb{R}^3)$, set $\Phi = \tau \cdot n$ and define $V$ by $\nabla_V u = \tau - \Phi n$. Then $\tau$ is an infinitesimal bending of $u$ if and only if $V$ and $\Phi$ satisfy $\text{Def} V = \Phi A$ in the distributional sense.
Whenever \( f \in W^{1,2} \) and \( \mu \in H^{-1} \) then we interpret \( f\mu \) as the distribution acting on test functions \( \varphi \) via \( (f\mu)(\varphi) = \mu(\varphi f) \). The proof of the following two lemmas is straightforward.

**Lemma 3.3.** If \( \rho, f \in W^{1,2}(M) \) then, as distributions,
\[
\text{Hess}(f \rho) = f \text{Hess} \rho + \rho \text{Hess} f + 2d\rho \odot df.
\]

**Lemma 3.4.** If \( u \in W^{2,2}(M, \mathbb{R}^3) \) then, as distributions,
\[
(Hess n)(X,Y) = u_*( (D_X S)(Y) ) - \langle S X, SY \rangle n.
\]

For a vector field \( V \) we regard \( DV \) as a section of \( \text{End}(TM) \). For two sections \( \Psi \) and \( \Xi \) of \( \text{End}(TM) \) we define a section \( \Psi \cdot \Xi \) of \( T^*M \otimes T^*M \) by
\[
(\Psi \cdot \Xi)(X,Y) = \langle \Psi(X), \Xi(Y) \rangle.
\]

By \( \Psi \odot \Xi \) we denote the symmetrisation of this.

**Lemma 3.5.** Let \( u \in W^{2,2}(M, \mathbb{R}^3) \) and \( \Phi \in W^{1,2}(M) \) and \( V \in W^{1,2}(TM) \). Then, as distributions,
\[
\begin{align*}
    n \cdot \text{Hess} \nabla_V u &= D_V A - 2S \odot DV \\
    n \cdot \text{Hess}(\Phi n) &= \text{Hess} \Phi - \Phi S \odot S.
\end{align*}\]
(9) (10)

In particular, the map \( \tau = u_* V + \Phi n \) satisfies
\[
\begin{align*}
    d\tau &= u_*(DV + \Phi S) + (d\Phi - SV) \odot n \\
    n \cdot \text{Hess} \tau &= D_V A - 2S \odot DV + \text{Hess} \Phi - \Phi S \odot S.
\end{align*}\]
(11) (12)

**Proof.** Applying Lemma 3.3 to \( 0 = \text{Hess}(n \cdot \nabla_V u) \), we have
\[
0 = \nabla_V u \cdot \text{Hess} n + n \cdot \text{Hess} \nabla_V u + 2dn \odot d(\nabla_V u).
\]

Since by Lemma 3.4
\[
(\nabla_V u \cdot \text{Hess} n)(X,Y) = \langle V, (D_X S)(Y) \rangle
\]
\[
= X(\langle V, SY \rangle) - \langle D_X V, SY \rangle + A(D_X Y, V)
\]
\[
= -(D_X A)(V, Y) = -(D_V A)(X,Y).
\]

In the last step we used that \( A \) is Codazzi. Since clearly \( dn \odot d(\nabla_V u) = S \odot DV \), equation (9) follows.

To prove (10) apply Lemma 3.3 to find
\[
n \cdot \text{Hess}(\Phi n) = \text{Hess} \Phi + \Phi n \cdot \text{Hess} n,
\]

because \( n \cdot (d\Phi \odot dn) = 0 \). Hence (10) follows from Lemma 3.4. \( \square \)
3.2 Representation via the bending field

The field $\Omega$ in the following lemma is called the bending field of $\tau$.

**Lemma 3.6.** Let $\tau \in W^{1,2}(M, \mathbb{R}^3)$ be an infinitesimal bending of $u$. Then there exists a unique $\Omega \in L^2(M, \mathbb{R}^3)$ such that

$$d\tau = \Omega \times du \quad \text{almost everywhere.}$$

(13)

It is given by $\Omega = \nabla \omega + \psi n$, where $\omega \in L^2(TM)$, $\psi \in L^2(M)$ are defined by

$$J\omega = n \cdot d\tau$$

(14)

$$\psi \langle X, Y \rangle = \nabla_X \tau \cdot \nabla_Y u.$$  

(15)

Moreover, $\Omega$ satisfies $d(\Omega \times du) = 0$ in the distributional sense. Conversely, if $\Omega \in L^2(M, \mathbb{R}^3)$ satisfies $d(\Omega \times du) = 0$, then (13) admits a solution $\tau \in W^{1,2}(M, \mathbb{R}^3)$, and $\tau$ is an infinitesimal bending of $u$.

**Proof.** The last statement is immediate, because $M$ is simply connected, and because for $\tau$ satisfying (13) clearly $du \otimes d\tau = 0$.

In order to prove the first part of the lemma, observe that uniqueness follows at once from (13) because $u$ is an immersion. In order to construct $\Omega$, define $\psi$ and $\omega$ by (15), (14), and set $\Omega = u^* \omega + \psi n$. Then clearly $\Omega \in L^2$ since so are $\psi$ and $\omega$, and (13) is easily seen to be satisfied.

Finally note that indeed there exists a (clearly unique) solution $\psi$ to (15). In fact, we can define $\psi$ by

$$d\tau \wedge du = 2\sqrt{g} \psi dx^1 \wedge dx^2;$$

on the left we contract in $\mathbb{R}^3$. Now (15) follows from $d\tau \cdot du = \frac{1}{2} d\tau \wedge du$, which is true because $\tau$ is an infinitesimal bending.

**Lemma 3.7.** Let $\psi \in W^{1,1}(M)$, $\omega \in W^{1,1}(TM)$ and set $\Omega = u^* \omega + \psi n$. Then $\Omega \in W^{1,1}(M, \mathbb{R}^3)$, and the following assertions are equivalent:

(i) We have $d(\Omega \times du) = 0$ in distributions.

(ii) The section $d\Omega \times du$ of $T^*M \otimes T^*M \otimes \mathbb{R}^3$ is symmetric.

(iii) We have almost everywhere $n \cdot d\Omega = 0$ and $\text{Tr} d\Omega = 0$, where we view $d\Omega$ as a section of $\text{End}(TM)$, which we may do by the first equality.

(iv) We have, pointwise almost everywhere,

$$d\psi = S\omega$$

$$\text{div } \omega = 2H \psi.$$  

(16)

(17)
There exists an infinitesimal bending \( \tau \in W^{2,1}(S, \mathbb{R}^3) \) of \( u \) satisfying \( d\tau = \Omega \times du \).

If any of the above assertions is satisfied, then

\[
d\Omega = u_*(D\omega + \psi S). \tag{18}
\]

**Proof.** Observe that by the Leibniz rule and since \( W^{1,1} \) embeds into \( L^2 \), we have \( \Omega \in W^{1,1} \). Clearly

\[
d\Omega = d(u_*\omega + \psi n) = u_*(D\omega + \psi S) + (d\psi - S\omega)n \text{ a.e. on } M. \tag{19}
\]

The Leibniz rule shows that (i) is equivalent to (ii). The tangential part of (ii) is equivalent to \( n \cdot d\Omega = 0 \). By (19), this is just \( d\psi = S\omega \), and (18) follows as well. Next we multiply (ii) by \( n \) to find \( \text{Tr } d\Omega = 0 \), which is equivalent to (17) in view of (18). The existence of an infinitesimal bending \( \tau \in W^{1,2} \) solving \( d\tau = \Omega \times du \) is ensured by Lemma 3.6, and it follows from the Leibniz rule that \( \Omega \in W^{1,1} \) implies \( \tau \in W^{2,1} \).

**Remark.** If the Gauss curvature \( K \) differs from zero on \( M \), then the Weingarten map \( S \) is invertible. Denoting by \( S^{-1} \in L^2(\text{End}(TM)) \) its fibrewise inverse, we see that the system (16), (17) is equivalent to the conjunction of

\[
\text{div } (S^{-1}d\psi) - 2H\psi = 0 \tag{20}
\]

with the algebraic equation

\[
\omega = S^{-1}(d\psi). \tag{21}
\]

**Lemma 3.8.** Let \( \tau \in W^{1,2}(M, \mathbb{R}^3) \) be an infinitesimal bending of \( u \in W^{2,2}(S) \), denote by \( \Omega \in L^2(M, \mathbb{R}^3) \) its bending field, by \( b = n \cdot \text{Hess } \tau \) its linearised second fundamental form and by \( B \) the section of \( \text{End}(TM) \) associated with \( b \). Then we have

\[
\tau \in W^{2,1}(M, \mathbb{R}^3) \iff \Omega \in W^{1,1}(M, \mathbb{R}^3). \tag{22}
\]

If these are satisfied, then

\[
(\text{Hess } \tau)(X,Y) = \nabla_X\Omega \times \nabla_Y u + A(X,Y) \quad (\Omega \times n) \tag{23}
\]

almost everywhere. In particular (since \( \nabla_X\Omega \) is tangential),

\[
\nabla_X\Omega = -u_*(JBX) \tag{24}
\]

\[
b(X,Y)n = \nabla_X\Omega \times \nabla_Y u, \text{ i.e., } b(X,JY) = \nabla_X\Omega \cdot \nabla_Y u. \tag{25}
\]

Moreover, writing \( \Omega = u_*\omega + \psi n \), we have \( \omega, \psi \in W^{1,1} \) and

\[
B = D(J\omega) + \psi J \circ S. \tag{26}
\]
Proof. By Lemma 3.7 we know that $d\Omega$ is tangential. Formula (22) follows from $d\tau = \Omega \times du$. From this we also deduce (23). By the definition of $b$, this implies (25), which in turn is just (24).

If $\Omega \in W^{1,1}$, then the Leibniz rule shows that $\psi = \Omega \cdot n$ is $W^{1,1}$ and that $\omega \in W^{1,1}$. Formula (26) follows from (25). □

**Corollary 3.9.** If $\tau \in W^{2,1}(S, \mathbb{R}^3)$ is an infinitesimal bending of $u$ then we have $n \cdot \text{Hess} \tau = 0$ almost everywhere if and only if there exist $c_0, c_1 \in \mathbb{R}^3$ such that $\tau = c_0 + c_1 \times u$.

Proof. If $\tau = c_0 + c_1 \times u$, then clearly $n \cdot \text{Hess} \tau = 0$. Conversely, if $b = n \cdot \text{Hess} \tau = 0$, then (25) implies that the tangential component of $d\Omega$ is zero. But $d\Omega$ is tangential, so $\Omega$ is constant. □

### 3.3 The linearised Gauss-Codazzi-Mainardi system

The infinitesimal change $b$ of the second fundamental form of an immersion $u$ under a bending clearly satisfies the linearisation of the Gauss-Codazzi-Mainardi system: The Codazzi-Mainardi equations are linear, so $b$ is Codazzi.

The linearisation of the Gauss equation under bendings is $\langle JA, b \rangle = 0$. In coordinates, the linearised Gauss-Codazzi-Mainardi system is this:

\[
\begin{align*}
    b_{11} h_{22} + h_{11} b_{22} - 2 h_{12} b_{12} &= 0 \\
    \partial_2 b_{11} - \partial_1 b_{12} - \Gamma_{12}^k b_{k1} + \Gamma_{11}^k b_{k2} &= 0 \\
    \partial_2 b_{12} - \partial_1 b_{22} - \Gamma_{22}^k b_{k1} + \Gamma_{12}^k b_{k2} &= 0.
\end{align*}
\]

**Proposition 3.10.** Let $u \in W^{2,2}_g(S)$. Then the following are true:

(i) If $\tau \in W^{2,1}(M, \mathbb{R}^3)$ is an infinitesimal bending of $u$ and $b = n \cdot \text{Hess} \tau \in L^2(SM)$ then $b$ is Codazzi and satisfies $\langle JA, b \rangle = 0$.

(ii) If $b \in L^2(SM)$ is Codazzi and satisfies $\langle JA, b \rangle = 0$, then there exists a solution $\tau \in W^{2,1}(M, \mathbb{R}^3)$ of $n \cdot \text{Hess} \tau = b$. The map $\tau$ is an infinitesimal bending of $u$, and it is unique up to trivial infinitesimal bendings.

The proof uses the following key lemma. In its statement, $d\rho_B$ is the exterior derivative of $\rho_B$ and $d^D$ denotes the covariant-exterior derivative acting on $\text{End}(TM)$.

**Lemma 3.11.** Let $u \in W^{2,2}_g(S)$, let $B \in L^2(\text{End}(TM))$ be symmetric and define the $\mathbb{R}^3$-valued 1-form $\rho_B$ by setting $\rho_B(X) = u_*(JBX)$. Then

\[
(d\rho_B)(X,Y) = u_*(J(d^D B)(X,Y)) + (A(X,JBY) - A(Y,JBX)) n
\]
as distributions. In particular, $\rho_B$ is closed if and only if the quadratic form $b \in L^2(SM)$ associated with $B$ is (weakly) Codazzi and satisfies $\langle JA, b \rangle = 0$ almost everywhere.

Proof. We compute

$$d\rho_B(X,Y) = \nabla_X JBYu - \nabla_Y JBXu + u_*(JBD_Y X - JBD_X Y)$$

$$= (A(X, JBY) - A(Y, JBX)) n$$

$$+ u_*(D_X JBY - D_Y JBX - JB[X, Y]),$$

which is the claim because $D_X J = JD_X$.

Proof of Proposition 3.10. To prove (i) note that Lemma 3.8 implies that $\Omega \in W^{1,1}$ and $\nabla X \Omega = -u_*(JBX)$. Hence, with $\rho_B$ as in Lemma 3.11 we have $\rho_B = -d\Omega$. Hence $\rho_B$ is closed, so Lemma 3.11 implies that $b$ is Codazzi and satisfies $\langle JA, b \rangle = 0$.

To prove (ii) let $B \in L^2(\text{End}(TM))$ be induced by $b$, and let $\rho_B$ as in Lemma 3.11. Then the lemma implies that $\rho_B$ is closed, so since $M$ is simply connected and since $\rho_B \in L^2$ (because $du \in L^\infty$ and $B \in L^2$), there exists $\Omega \in W^{1,2}(M, \mathbb{R}^3)$ such that $d\Omega = -\rho_B$, i.e., $\nabla X \Omega = -u_*(JBX)$. Clearly $\Omega$ is unique up to a constant vector, and it satisfies Lemma 3.7 (iii). Hence that lemma shows that there exists $\tau \in W^{2,1}$ such that $d\tau = \Omega \times du$, and $n \cdot \text{Hess} \tau = b$ e.g. by Lemma 3.8.

Clearly, for given $\Omega$, the map $\tau$ is unique up to a constants; since the same is true for $\Omega$ itself, this implies that $\tau$ is unique up to trivial infinitesimal bendings.

Remark. For completeness, we note that the infinitesimal change of the normal vector $n$ under an infinitesimal bending $\tau$ of $u$ with bending field $\Omega$ is given by $\mu = \Omega \times n$. In fact, denoting by a dot the infinitesimal change of a quantity under the displacement $\tau$, we have $0 = (n \cdot du) = \dot{n} \cdot du + n \cdot d\tau$. This conditions and $n \cdot \dot{n} = 0$ determine $\dot{n}$ uniquely. It is easy to check that $\mu$ satisfies these two conditions. By linearisation of the Gauss and Weingarten equations for $u$ it is easy to see that the velocity field $\tau$ of a bending of $u$ satisfies

$$\text{Hess} \tau = A \otimes \mu + b \otimes n \quad (30)$$

$$\nabla_X \mu = \nabla_X \tau - \nabla_{BX} u, \quad (31)$$

This remains true if $\tau$ is an arbitrary infinitesimal bending; for (30) cf. Lemma 3.8, and for (31) refer, e.g., to [11].
4 Stationary points of $W_g$

The following definitions are central, as they provide a natural notion of (possibly non-minimising) stationary points of $W_g$.

**Definition 4.1.** For a given immersion $u \in W_g^{2,2}(S)$ we make the following definitions:

- A bending of $u$ is a strongly continuous one-parameter family $\{u_t\}_{t \in (-1,1)} \subset W_g^{2,2}(S)$ with $u_0 = u$, and which is such that the weak $L^2$-limit
  \[
  b = \lim_{t \to 0} \frac{1}{t}(A_t - A)
  \]
  exists. Here $A$ denotes the second fundamental form of $u$ and $A_t$ that of $u_t$.

- The section $b \in L^2(SM)$ is called the linearised second fundamental form induced by the bending $\{u_t\}_{t \in (-1,1)}$.

- Any $b \in L^2(SM)$ induced as in (32) by some strongly continuous family $\{u_t\}_{t \in (-1,1)} \subset W_g^{2,2}(S)$ is called a continuable linearised second fundamental form for $u$.

Another natural but (without further regularity hypotheses on $u$) slightly more restrictive notion would be to regard $\{u_t\}$ as a bending of $u$ if the weak $W_g^{2,2}$ limit
\[
\tau = \lim_{t \to 0} t^{-1}(u_t - u_0)
\]
exists. In this case, $\tau$ is called the infinitesimal bending induced by the bending $\{u_t\}_{t \in (-1,1)}$. Any vector field $\tau$ induced in this manner by a $W_g^{2,2}$-bending of $u$ is called a continuable infinitesimal $W_g^{2,2}$-bending of $u$.

Of course the regularity hypotheses chosen in these definitions are somewhat arbitrary; one may well wish to consider more regular maps. One may also impose boundary conditions on $u$ and on the admissible bendings (and hence on their induced infinitesimal bendings).

Observe that in the case without boundary conditions considered here, a trivial class of $W_g^{2,2}$ bendings is given by the rigid motions. The corresponding continuable infinitesimal bending fields are precisely those whose gradient is of the form $\Omega \times du$ for some constant $\Omega \in \mathbb{R}^3$. Immersions $u$ which only permit such trivial bendings are called rigid. They are clearly stationary.

Remarks.
(i) We adopt the term ‘continuable’ from the review paper [17] and other papers on this subject (cf. e.g. [20, 16, 19]), in order to highlight the connection to this large body of literature.

(ii) Clearly, the limit $\tau \in W^{2,2}(S, \mathbb{R}^3)$ in (33) necessarily is an infinitesimal bending of $u$. Similarly, every continuable linearised second fundamental form $b \in L^2(\mathcal{S}M)$ is Codazzi and satisfies $\langle JA, b \rangle = 0$ almost everywhere on $M$ (cf. Section 3 for details).

However, it is well-known that the converse is false in general, i.e., the class of infinitesimal bendings can be strictly larger than the class of continuable infinitesimal bendings. Nevertheless, when $K > 0$ then the two classes are known to agree (in the presence of enough regularity), cf. [19, 18, 25]. A similar result has recently been obtained in [12] for nondegenerate intrinsically flat (i.e. $K = 0$) surfaces $u$; it is false in the degenerate case when $u$ contains a planar region.

**Definition 4.2.** For a given immersion $u \in W^{2,2}_g(S)$ we say that

- $u$ is stationary for $\mathcal{W}_g$ provided that
  $$ \left. \frac{d}{dt} \right|_{t=0} \mathcal{W}_g(u_t) = 0 \text{ for all bendings } \{u_t\}_{t \in (-1,1)} \text{ of } u. $$

- $u$ is formally stationary for $\mathcal{W}_g$ provided that
  $$ \int_M \langle A, b \rangle = 0 \text{ for all Codazzi tensors } b \in L^2(\mathcal{S}M) \text{ satisfying } \langle JA, b \rangle = 0 \text{ almost everywhere on } M. $$

The notion of formal stationarity is motivated by the following remark:

**Proposition 4.3.** Let $u \in W^{2,2}_g(S)$ and let $\{u_t\}_{t \in (-1,1)}$ be a bending of $u$ inducing the linearised second fundamental form $b$. Then

$$ \left. \frac{d}{dt} \right|_{t=0} \mathcal{W}_g(u_t) = \int_M \langle A, b \rangle. $$ \hspace{1cm} (34)

In particular, $u$ is stationary for $\mathcal{W}_g$ if and only if $\int_M \langle A, b \rangle = 0$ for all continuable linearised second fundamental forms $b$. So every formally stationary immersion is stationary.

**Proof.** By the weak $L^2$-convergence (32) we have $\|A_t - A\|_{L^2} \leq Ct$, so $A_t \to A$ strongly in $L^2$, hence $A + A_t \to 2A$ strongly in $L^2$. Hence using (32), we conclude that, as $t \to 0$,

$$ \frac{1}{t} \int_M |A_t|^2 - |A|^2 = \frac{1}{t} \int_M \langle A_t + A, A_t - A \rangle \to 2 \int_M \langle A, b \rangle. $$

$\square$
The next lemma follows from a simple computation.

**Lemma 4.4.** If \( q, b \in L^2(SM) \) and \( \langle Jq, b \rangle = 0 \) then \( \langle q, b \rangle = (\text{Tr} q)(\text{Tr} b) \).

**Theorem 4.5.** If \( u \in W^{2,2}_g(S) \) then the following are equivalent:

(i) The immersion \( u \) is formally stationary.

(ii) We have \( \int_M H \text{Tr} b = 0 \) for all Codazzi tensors \( b \in L^2(SM) \) satisfying \( \langle JA, b \rangle = 0 \).

(iii) We have \( \int_M H \text{Tr} b = 0 \) for all \( b \in L^2(SM) \) with \( \text{div} b = 0 \) and \( \langle A, b \rangle = 0 \).

(iv) There exist sequences of Lagrange multipliers \( \lambda^{(n)} \in C_0^\infty(M) \) and \( Y^{(n)} \in C_0^\infty(TM) \) such that

\[
\lambda^{(n)} A + \text{Def} Y^{(n)} \rightharpoonup Hg
\]

weakly in \( L^2(SM) \).

**Proof.** The equivalence of the first three items follows from Lemma 4.4 and the fact that \( \text{Tr} b = \text{Tr}(Jb) \). The equivalence of (iii) and (iv) follows by standard functional analysis from the fact that \( -\text{Def} \) is the formal adjoint of the divergence operator on sections of \( SM \), cf. e.g. [14, 11] for details. \qed

**Corollary 4.6.** Let \( u \in W^{3,1}_g(S) \) be formally stationary. Then the following are true:

(i) We have \( \int_M \nabla_V (H^2) + \Phi H(4H^2 - 2K) + H \Delta \Phi = 0 \) for all \( V \in W^{2,2}(TM) \), \( \Phi \in W^{2,2}(M) \) satisfying \( \text{Def} V = \Phi A \).

(ii) We have \( \int_M H \text{div} (J\omega) = 0 \) for all \( \omega \in W^{1,2}(TM) \) such that there exists \( \psi \in W^{1,2}(M) \) with

\[
d\psi = S\omega,
\]

\[
\text{div} \omega = 2H\psi.
\]

(iii) If \( K \neq 0 \) then \( \int H \text{div} (JS^{-1}d\psi) = 0 \) whenever \( \psi \in W^{1,2}(M) \) satisfies \( \text{div} (S^{-1}d\psi) = 2H\psi \). Here \( S^{-1} \) is the section of \( \text{End}(TM) \) obtained by inverting \( S \) fibrewise.
Proof. If \( u, V \) and \( \Phi \) are as in the hypotheses then the map \( \tau = u_n V + \Phi n \) is an infinitesimal bending of \( u \), cf. Lemma 3.2. Moreover, \( b = n \cdot \text{Hess} \tau \in L^2 \), according to Lemma 3.5. Proposition 3.10 (i) shows that \( b = n \cdot \text{Hess} \tau \) is Codazzi with \( \langle JA, b \rangle = 0 \). Hence statement (i) follows from Theorem 4.5 and because

\[
\text{Tr} b = 2\nabla_V H + \Phi |A|^2 + \Delta \Phi,
\]
due to (12).

If \( u, \omega \) and \( \psi \) are as in the hypotheses then the map \( \Omega = u_n \omega + \psi n \) belongs to \( W^{1,1} \). Moreover, \( \Omega \) is the bending field of some infinitesimal bending \( \tau \) of \( u \), cf. Lemma 3.7. Proposition 3.10 (i) shows that \( b = n \cdot \text{Hess} \tau \) is Codazzi with \( \langle JA, b \rangle = 0 \). Note that (26) implies that \( b \in L^2 \). Hence statement (ii) follows from Theorem 4.5 and because \( \text{Tr} b = \text{div} (J\omega) \), due to (26).

Statement (iii) follows from (ii).

Observe that the conclusions of Corollary 4.6 are not merely consequences of formal stationarity of \( u \), but in fact they are roughly equivalent to it. Moreover, Lagrange multiplier rules as in (35) can be easily derived from Corollary (4.6)

Clearly, the absolute minimisers of \( W_g \) when \( K = 0 \) are the affine maps. This remains true for formally stationary points:

**Remark 4.7.** Assume that \( g \) has constant Gauss curvature \( K = 0 \). Then \( u \in W^{2,2}_g(S) \) is formally stationary if and only if \( u \) is affine.

**Proof.** If \( K = 0 \) then \( \langle JA, A \rangle = 0 \). Since, moreover, \( A \in L^2 \) is Codazzi, it is an admissible test tensor in the definition of formal stationarity, which therefore implies that \( \int_M |A|^2 = 0 \). \( \square \)

**Remark 4.8.** Assume that \( g \) has constant Gauss curvature \( K = K_0 > 0 \). Then the absolute minimiser of \( W_g \) is the standard immersion of \( (S, g) \) as a subset of the sphere of radius \( K_0^{-1} \).

**Proof.** This follows from the fact that the absolute minimiser of \( W_g \) agrees with that of the functional (5), and that surfaces consisting of umbilical points are (subsets of) round spheres. \( \square \)

**Remark about related variational problems**

As seen earlier, up to addition of a term depending only on the metric \( g \), the functional \( W_g \) agrees with the Willmore functional \( \int H^2 \), restricted to \( W^{2,2}_g \). Hence, as observed e.g. in [4], minimal immersions are absolute minimisers.
So if the metric $g$ is induced by some minimal surface, then this minimal surface is an absolute minimiser of $\mathcal{W}_g$. More generally, if $u \in W^{2,2}_g(S)$ is a stationary point of the classical Willmore functional (i.e., without the isometry constraint), then it clearly is also stationary for $\mathcal{W}_g$. In this sense, Willmore surfaces are critical points of $\mathcal{W}_g$. Note, however, that a ‘Willmore surface’ in this context has a boundary and must satisfy certain boundary conditions.

5 Symmetric immersions

The notion of symmetry used in this chapter is common in the context of nonlinear wave maps, cf. [34]. We only use it as a convenient way to express rotational symmetry.

5.1 Principle of symmetric stationarity

Let $G$ be a Lie group acting transitively on $M$ and isometrically on $\mathbb{R}^3$. For each $\rho \in G$ define $\lambda_\rho : M \to M$ by $\lambda_\rho(x) = \rho x$. The action of $\rho$ in $\mathbb{R}^3$ is denoted by $L_\rho \in SO(3)$. An immersion $u : M \to \mathbb{R}^3$ is said to be symmetric if

$$u \circ \lambda_\rho = L_\rho u \text{ for all } \rho \in G.$$ 

We are only interested in the case when $S = B_1$ is the unit ball and $\nabla \lambda_\rho \in SO(2)$.

Denote by $\mu_G$ the Haar measure on $G$ normalised such that $\mu_G(G) = 1$. For an immersion $u : M \to \mathbb{R}^3$ define $(u)_G : M \to \mathbb{R}$ by

$$(u)_G(x) = \int_G L_\rho^{-1} u(\lambda_\rho(x)) \, d\mu_G(\rho).$$

Clearly, $u$ is symmetric precisely if $u = (u)_G$.

A section $q$ of $SM$ is said to be invariant if $\lambda_\rho^* q = q$ for all $\rho \in G$; here $\lambda_\rho^* q$ denotes the pullback under the diffeomorphism $\lambda_\rho^*$. We define

$$(q)_G = \int_G \lambda_\rho^* q \, d\mu_G.$$ 

The following lemmas are readily verified:

**Lemma 5.1.** If $u$ is symmetric then $A$ and $g$ are invariant in the sense that $\lambda_\rho^* A = A$ and $\lambda_\rho^* g = g$ (i.e., $\lambda_\rho$ is an isometry) for all $\rho \in G$. In particular, if $b \in L^2(SM)$ is Codazzi then so is $\lambda_\rho^* b$. 

Lemma 5.2. Let $q \in L^2(T^*M \otimes T^*M)$ and let $\lambda : M \to M$ be an isometry. Then we have $J(\lambda^*q) = \lambda^*(Jq)$. In particular, if $q \in L^2(SM)$ is invariant then so is $Jq$.

Lemma 5.3. Let $q, b \in L^2(SM)$ and suppose that $q$ is invariant. Then

$$\langle q, (b)_G \rangle = \int_G \langle q, b \rangle \circ \lambda_\rho \, d\mu_G(\rho).$$

Proof. Since $\lambda_\rho$ is an isometry, we have $\langle q, b \rangle \circ \lambda_\rho = \langle \lambda_\rho^*q, \lambda_\rho^*b \rangle = \langle q, \lambda_\rho^*b \rangle$, because $\lambda_\rho^*q = q$. Integration over $\rho$ yields the claim. \qed

Proposition 5.4. Assume that $u \in W^{2,2}_g(S)$ is symmetric and satisfies $\int H \text{Tr} b = 0$ for all invariant Codazzi tensors $b \in L^2(SM)$ with $\langle JA, b \rangle = 0$. Then $\int H \text{Tr} b = 0$ for all Codazzi tensors $b \in L^2(SM)$ with $\langle JA, b \rangle = 0$.

Proof. Let $b \in L^2(SM)$ be a (possibly non-invariant) Codazzi tensor satisfying $\langle JA, b \rangle = 0$. Since $A$ is invariant, Lemma 5.2 implies that so is $JA$. Hence Lemma 5.3 implies that $\langle JA, (b)_G \rangle = 0$. And Lemma 5.3 ensures that $(b)_G$ is still Codazzi. Hence the hypotheses imply that $\int_M H \text{Tr}(b)_G = 0$. On the other hand, since each $\lambda_\rho$ is an isometry,

$$\text{Tr}(b)_G = \int_G \text{Tr}(\lambda_\rho^*b) \, d\mu_G(\rho) = \int_G (\text{Tr} b) \circ \lambda_\rho \, d\mu_G(\rho).$$

Hence by Fubini and since (by invariance of $A$) we have $H = H \circ \lambda_\rho$,

$$0 = \int_M H \text{Tr}(b)_G = \int_M H \left( \int_G (\text{Tr} b) \circ \lambda_\rho \, d\mu_G(\rho) \right)$$

$$= \int_G \left( \int_M (H \text{Tr} b) \circ \lambda_\rho \right) \, d\mu_G(\rho).$$

Since $\lambda_\rho$ is an isometry, the inner integral equals $\int_M H \text{Tr} b$ for all $\rho \in G$. So indeed $\int_M H \text{Tr} b = 0$. \qed

5.2 Radially symmetric surfaces

Lemma 5.5. Let $T > 0$ and let $R \in W^{1,\infty}(0, T)$ be positive on $[0, T]$, and let

$$g(t, \varphi) = (dt)^2 + R^2(t)(d\varphi)^2$$

be a Riemannian metric on $U = [0, T] \times [0, 2\pi]$; set $M = (U, g)$. Let $b_{ij} \in L^2(0, T)$ and let $b \in L^2(SM)$ be given by

$$b(t, \varphi) = b_{11}(t)(dt)^2 + b_{22}(t)(d\varphi)^2 + 2b_{12}(t)dt \otimes d\varphi.$$
Then \( b \) is Codazzi if and only if there exists a constant \( C \in \mathbb{R} \) such that
\[
\left( \frac{b_{22}}{R} \right)' = R' b_{11} \quad (36)
\]
\[
b_{12} = \frac{C}{R} \quad (37)
\]

**Proof.** All Christoffel symbols of \( g \) are zero, except
\[
\Gamma_{22}^1(t) = -R(t)R'(t) \quad \text{and} \quad \Gamma_{12}^2(t) = (\log R)'(t);
\]
here we set \( x_1 = t \) and \( x_2 = \varphi \). Since, moreover, \( b_{ij} \) are independent of \( \varphi \), the Codazzi equations read
\[
b_1'_{12} + (\log R)' b_{12} = 0
\]
\[
b_2'_{22} - RR' b_{11} - (\log R)' b_{22} = 0.
\]
The first of these equations is clearly equivalent to (37). Dividing the second equation by \( R \), we see that it is equivalent to (36).

**Proposition 5.6.** Let \( T > 0 \) and let \( R \in C^0([0, T]) \) be positive and let \( L \in C^0([0, T]) \) be such that
\[
u(x) = R(|x|) \frac{x}{|x|} + L(|x|) e_3 \quad (38)
\]
defines a map \( u \in W^{2,2}(B_T) \). Then \( u \) is formally stationary for \( \mathcal{W}_g \), where \( g = u^* g_{\mathbb{R}^3} \).

**Proof.** After possibly reparametrising the curve \( t \mapsto (R(t), L(t)) \), we may assume that \( (R')^2 + (L')^2 = 1 \). Denote by \( M \) the Riemannian manifold \((B_T, g)\). Clearly \( u \) satisfies \( u(Qx) = Qu(x) \) for all \( Q \in SO(2) \) and all \( x \in B_T \); on the right-hand side we regard \( SO(2) \) as a subset of \( SO(3) \) in the obvious way.

Hence by Proposition 5.4 we must prove that \( \int_M \langle A, b \rangle = 0 \) for all \( SO(2) \)-invariant Codazzi tensors \( b \in L^2(SM) \) with \( \langle JA, b \rangle = 0 \). Let \( b \) be such a tensor.

We introduce radial coordinates \((t, \varphi)\) via \( x = te^{i\varphi} \); we identify \( \mathbb{C} \) with \( \mathbb{R}^2 \times \{0\} \). In these coordinates we have (with the usual abuse of notation)
\[
u(t, \varphi) = R(t) e^{i\varphi} + L(t) e_3.
\]
Since \( (R')^2 + (L')^2 = 1 \), we see that
\[
g(t, \varphi) = (dt)^2 + R^2(t)(d\varphi)^2.
\]

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Invariance implies that there exist functions $b_{ij} \in L^2(0,T)$ such that

$$b(t, \varphi) = b_{11}(t)(dt)^2 + b_{22}(t)(d\varphi)^2 + b_{12}(t)(dt \odot d\varphi).$$

Since $b$ is Codazzi, Lemma 5.5 shows that $b_{22} \in W^{1,2}$ and

$$
\left( \frac{b_{22}}{R} \right) ' = R'b_{11}. \quad (39)
$$

On the other hand,

$$
\begin{align*}
    h_{11} &= R'L'' - L'R'' \\
    h_{12} &= 0 \\
    h_{22} &= RL'.
\end{align*}
$$

Hence $\langle JA, b \rangle$ means that

$$b_{11}R'L' = (L'R'' - R'L'')b_{22}. \quad (40)$$

We therefore deduce from (39) that there exists a constant $C_1 \in \mathbb{R}$ such that

$$b_{22}L' = C_1R. \quad (41)$$

However, since $R' \in L^\infty$ and $b_{11} \in L^2$, equation (39) shows, in particular, that $R^{-1}b_{22}$ is bounded. On the other hand, $u \in W^{2,2}$ implies that $L'(0) = 0$. Hence (41) implies that $b_{22} = 0$. But then (39) shows that $b_{11}R' = 0$, and (40) shows that $b_{11}L' = 0$. Hence $b_{11} = 0$. Since both $g$ and $h$ are diagonal, this shows that $\langle A, b \rangle = 0$. \hfill \Box

**Lemma 5.7.** Let $\tilde{u}$ and $u$ be of the form (38). Then $\tilde{u}$ and $u$ are isometric if and only if (with the obvious notation) $\tilde{R} = R$ and $|\tilde{L}'| = |L'|$ almost everywhere on $(0,T)$.

**Proof.** The metric induced by $u$ is

$$((R')^2 + (L')^2) (dt)^2 + R^2 (d\varphi)^2,$$

and similarly for $\tilde{u}$. So keeping in mind that $R, \tilde{R}$ are nonnegative, we see that $u$ and $\tilde{u}$ are isometric if and only if

$$\tilde{R} = R \text{ and } |\tilde{L}'| = |L'| \text{ almost everywhere,}$$

because the former implies that $\tilde{R}' = R'$ almost everywhere. \hfill \Box
Corollary 5.8. Let \( \tilde{u} \) and \( u \) be of the form (38), and assume that they are isometric. Then the modulus of their mean curvatures agrees almost everywhere. In particular, \( W_g(\tilde{u}) = W_g(u) \).

Proof. As before, we assume without loss of generality that \( (L')^2 + (R')^2 = 1 \). Set \( \kappa = R'' - L' \) and denote the corresponding quantities for \( \tilde{u} \) by a tilde. We compute that

\[
2H = \kappa + \frac{(L')^2}{R'}.
\]

Hence

\[
4H^2 = \kappa^2 + \frac{(L')^2}{R^2} + \frac{2\kappa L'}{R}.
\]

By Lemma 5.7, the second term is clearly the same for \( \tilde{u} \). Regarding the last term, we compute

\[
\kappa L' = \frac{1}{2} R' \left( (L')^2 - (L')^2 R'' \right).
\]

Hence Lemma 5.7 shows that the last term in (42) also is the same for \( \tilde{u} \). By (43) so is \( |\kappa||L'| \). Hence \( |\kappa| = |\tilde{\kappa}| \) almost everywhere on \( \{L' \neq 0\} = \{\tilde{L}' \neq 0\} \). But on \( \{L' = 0\} = \{\tilde{L}' = 0\} \) we have \( L'' = \tilde{L}'' = 0 \) almost everywhere, hence \( \kappa = \tilde{\kappa} = 0 \). We conclude that \( |\kappa| = |\tilde{\kappa}| \) almost everywhere. This shows that \( |H| = |\tilde{H}| \) almost everywhere. \( \square \)

A pathological example

In this example we construct a class of Riemannian metrics \( g \in C^\infty(B_1) \) such that the functional \( W_g \), with \( g = u^*g_3 \), admits infinitely many stationary points. This is analogous to the examples in [13].

Let \( \eta \in C^\infty(\mathbb{R}) \) be nonnegative and supported in \((-\frac{1}{2}, \frac{1}{2})\) (but not identically zero). Let \( R \in (0, 1] \) and let \( (t_n)_{n=1}^\infty \subset (0, R) \) be a strictly increasing sequence with \( t_0 = 0 \) and define \( \tilde{L} : [0, 1) \to \mathbb{R} \) by setting

\[
L(t) = \sum_{n=0}^{\infty} (t_{n+1} - t_n)^n \eta \left( \frac{2t - t_n - t_{n+1}}{2(t_{n+1} - t_n)} \right).
\]

Since \( L \in W^{2, \infty}(0, 1) \) vanishes near zero, we see that \( u : B_1 \to \mathbb{R}^3 \) given by \( u(x) = x + L(|x|)e_3 \) belongs to \( W^{2, \infty} \), in fact \( u \in C^\infty(\overline{B_1}) \), because \( L \in C^\infty([0, 1]) \). For each \( n = 1, 2, 3, ... \) define \( u_n : [0, 1) \to \mathbb{R} \) by

\[
L_n(t) = \begin{cases} 
-L(t) & \text{if } t \in (t_n, t_{n+1}) \\
L(t) & \text{otherwise},
\end{cases}
\]

and define \( u_n \in W^{2, \infty}(B_1) \) by setting \( u_n(x) = x + L_n(|x|)e_3 \). Since \( |L'_n| = |L'| \) everywhere, Lemma 5.7 shows that all \( u_n \) are isometric to \( u \). Clearly, the \( u_n \)
are pairwise distinct and of the form (38). Proposition 5.4 shows that each $u_n$ is (even formally) stationary for $W_g$, with $g = u^* g_{\mathbb{R}^3}$.

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