How random are random nuclei?
Shapes, triangles and kites

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The energy systematics of medium and heavy even-even nuclei shows very regular features, such as for example the tripartite classification of nuclear structure into seniority, anharmonic vibrator and rotor regions [1,2]. Traditionally, this regular behavior has been interpreted as a consequence of particular interactions, such as an attractive pairing force in semimagic nuclei and an attractive neutron-proton quadrupole-quadrupole interaction for deformed nuclei.

It came as a surprise, therefore, that recent studies of even-even nuclei in the nuclear shell model [3–5] and in the interacting boson model (IBM) [6–8] with random interactions also displayed a high degree of order. Both models showed a statistical preference for $L = 0$ ground states, despite the random nature of the interactions. In addition, in the shell model evidence was found for the occurrence of pairing properties [5], and in the IBM for vibrational and rotational bands [6,7].

These unexpected results have sparked a large number of investigations to try to understand their origin [9]. In this contribution, we discuss the phenomenon of emerging regular spectral features from the IBM with random interactions, and its relation with the underlying geometric shapes and critical points.

In order to study the geometric shapes associated with the IBM [10–12], we first consider the schematic Hamiltonian of the consistent-Q formulation (CQF) [13]

$$H = \epsilon n_d - \kappa Q(\chi) \cdot Q(\chi).$$

The parameters are restricted to the ‘physically’ allowed region, i.e. $\epsilon > 0$, $\kappa > 0$ and $-\sqrt{7}/2 \leq \chi \leq \sqrt{7}/2$. The properties of the CQF Hamiltonian are investigated by taking the scaled parameters $\eta = \epsilon/[\epsilon + 4\kappa(N-1)]$ and $\bar{\chi} = 2\chi/\sqrt{7}$ randomly on the intervals $0 \leq \eta \leq 1$ and $-1 \leq \bar{\chi} \leq 1$. In Fig 1 we present the results in a shape phase diagram as a function of the coefficients $r_1$ and $r_2$ which were introduced in [14] as the essential control parameters to classify the equilibrium configurations of the IBM Hamiltonian. $r_1$ and $r_2$ are determined by particular combinations of the interaction parameters. In Fig. 1 we have identified each of the dynamical symmetries of the IBM: $U(5)$, $SU(3)$ with prolate/oblate symmetry and $SO(6)$. The transitions between the dynamical symmetries are indicated by the solid lines.
FIGURE 1. Shape phase diagram for the random CQF Hamiltonian obtained for \( N = 16 \) and 1000 runs. The dashed line separates the spherical from the deformed shape.

The resulting figure is that of a kite. The so-called critical point symmetries \( E(5) \) and \( X(5) \) [15] are related to the points at the intersections of the solid lines and the separatrix (dashed line) which separates the spherical and deformed shapes. The prolate and oblate deformed shapes are separated by \( r_2 = 0, \; r_1 < 0 \). The associated critical point symmetry coincides with the \( SO(6) \) limit [16].

Each CQF Hamiltonian corresponds to a point in the \( r_2r_1 \) plane and is labeled by a + sign in Fig. 1. The random ensemble of CQF Hamiltonians covers the interior part of the kite: 50\% for the spherical shape, and 25\% each for the prolate and oblate deformed shapes. For all cases, the ground state has angular momentum \( L = 0 \). The existence of two definite geometric shapes, a spherical and an axially symmetric deformed one, is also evident from a plot of the probability distribution \( P(R) \) of the energy ratio \( R = \frac{E(4_1) - E(0_1)}{E(2_1) - E(0_1)} \). Fig. 2 shows that for the CQF there are two characteristic peaks, one at the vibrator value \( R = 2 \) and one at the rotor value \( R = 10/3 \) (solid line).

So far, we have discussed the properties of a random ensemble of IBM Hamiltonians with minimal constraints to ‘realistic’ interactions. Surprisingly enough, the results for a general IBM Hamiltonian with random one- and two-body interactions chosen independently from a Gaussian distribution of random numbers with zero mean and width \( \sigma \) are very similar [6]. Also in this case, the probability distribution \( P(R) \) exhibits peaks at \( R \sim 1.9 \) and \( R \sim 3.3 \) (dashed line). The vibrational and rotational nature of these peaks has been confirmed by a simultaneous study of the quadrupole transitions between the levels [6]. Despite the random nature of
FIGURE 2. Probability distribution $P(R)$ of the energy ratio $R$ in the IBM for the random CQF (solid line) and for the general case of random one- and two-body interactions (dashed line), obtained for $N = 16$ and 10000 runs.

the interaction strengths both in relative size and sign, the ground state still has $L = 0$ in $\sim 63\%$ of the cases. In Fig. 3 we show the percentages of ground states with $L = 0$ and $L = 2$ as a function of the boson number $N$ (solid line). We see a clear dominance of ground states with $L = 0$ with $\sim 60-75\%$. For $N = 3k$ (a multiple of 3) we see an enhancement for $L = 0$ and a decrease for $L = 2$. The sum of the two hardly depends on the number of bosons.

These are surprising results in the sense that, according to the conventional ideas in the field, the occurrence of $L = 0$ ground states and the existence of vibrational and rotational bands are due to very specific forms of the interactions. The basic ingredients of the numerical simulations, both for the nuclear shell model and for the IBM, are the structure of the model space, the ensemble of random Hamiltonians, the order of the interactions (one- and two-body), and the global symmetries, i.e. time-reversal, hermiticity and rotation and reflection symmetry. The latter three symmetries cannot be modified, since we are studying many-body systems whose eigenstates have real energies and good angular momentum and parity. It has been shown that the observed spectral order is a robust property that does not depend on the specific choice of the ensemble of random interactions [3,4], the time-reversal symmetry [4], or the restriction of the Hamiltonian to one- and two-body interactions [7]. These results suggest that an explanation of the origin of the observed regular features has to be sought in the many-body dynamics of the model space and/or the general statistical properties of random interactions.
For the IBM, the emergence of regular features from random interactions can be explained in a Hartree-Bose mean-field analysis of the random ensemble of Hamiltonians, in which different regions of the parameter space are associated with particular intrinsic states, which in turn correspond to definite geometric shapes [17]. There are three solutions: a spherical shape carried by a single state with $L = 0$, a deformed shape which corresponds to a rotational band with $L = 0, 2, \ldots, 2N$, and a condensate of quadrupole bosons which has a more complicated angular momentum content. We note, that the latter solution does not occur for the random ensemble of CQF Hamiltonians. The ordering of rotational energy levels depends on the sign of the corresponding moments of inertia, which have been evaluated with the Thouless-Valatin formula. In Fig. 3 we show the percentages of ground states with $L = 0$ and $L = 2$ as a function of the number of bosons $N$. A comparison of the results of the mean-field analysis (dashed lines) and the exact ones (solid lines) shows excellent agreement. The oscillations with $N$ are entirely due to the contribution of the condensate of quadrupole bosons. The mean-field analysis explains both the distribution of ground state angular momenta and the occurrence of vibrational and rotational bands. The same conclusions hold for the vibron model for which a large part of the results has been obtained analytically [17].

In this contribution, we addressed the origin of the regular features obtained in numerical studies of the IBM with random interactions, in particular the dominance
of $L = 0$ ground states and the occurrence of vibrational and rotational band structures. It was shown that the geometric shapes associated with IBM Hamiltonians play a crucial role in understanding these regular properties. Different regions of the parameter space are associated with definite geometric shapes, such as spherical and deformed shapes and a condensate of quadrupole bosons. For a random ensemble of CQF Hamiltonians the latter solution is absent, and the shape phase diagram assumes the simple form of a kite or a double triangle.

For the nuclear shell model the situation is less clear. Although a large number of investigations to explain and further explore the properties of random nuclei have shed light on various aspects of the original problem, i.e. the dominance of $L = 0$ ground states, in our opinion, no definite answer is yet available, and the full implications for nuclear structure physics are still to be clarified.

It is a great pleasure to dedicate this contribution to Rick Casten on the occasion of his 60th birthday in appreciation of the numerous occasions where his profound and stimulating comments have had a strong impact on our work. In particular, we gratefully acknowledge his enthusiastic support of the random ideas of his theoretician friends.

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