Nielsen zeta function, 3-manifolds and asymptotic expansions in Nielsen theory

Alexander Fel’shtyn*

0 Introduction

Before moving on the results of the paper, we briefly describe the few basic notions of Nielsen fixed point theory which will be used. We assume $X$ to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \tilde{X} \to X$ be the universal cover of $X$ and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of $f$, ie. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called conjugate if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(Fix(\tilde{f})) \subset Fix(f)$ is called the fixed point class of $f$ determined by the lifting class $[\tilde{f}]$. Two fixed points $x_0$ and $x_1$ of $f$ belong to the same fixed point class iff there is a path $c$ from $x_0$ to $x_1$ such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map $f$ has only finitely many non-empty fixed point classes, each a compact subset of $X$.

A fixed point class is called essential if its index is nonzero. The number of lifting classes of $f$ (and hence the number of fixed point classes, empty or not) is called the Reidemeister Number of $f$, denoted $R(f)$. This is a positive integer or infinity. The number of essential fixed point classes is called the Nielsen number of $f$, denoted by $N(f)$.

The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy invariants. In the category of compact, connected polyhedra the Nielsen number of a map is, apart from in certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as $f$.

Taking a dynamical point of view, we consider the iterates of $f$, and we may define several zeta functions connected with Nielsen fixed point theory (see [4, 5, 7]). The Nielsen zeta function of $f$ is defined as power series:

$$N_f(z) := \exp \left( \sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n \right).$$

*Part of this work was conducted during author stay in Max-Planck-Institut für Mathematik in Bonn
The Nielsen zeta function $N_f(z)$ is homotopy invariant. The function $N_f(z)$ has a positive radius of convergence which has a sharp estimate in terms of the topological entropy of the map $f$.

We begin the article by proving in Section 1 that the Nielsen zeta function is a rational function or a radical of a rational function for orientation preserving homeomorphisms of special Haken or special Seifert 3-manifolds.

In Section 2, we obtain an asymptotic expansion for the number of twisted conjugacy classes or for the number of Nielsen fixed point classes whose norm is at most $x$ in the case of pseudo-Anosov homeomorphism of surface.

The author would like to thank M. Gromov, Ch. Epstein, R. Hill, L. Potyagailo, R. Sharp, V.G. Turaev for stimulating discussions. In particular, Turaev proposed the author in 1987 the conjecture that the Nielsen zeta function is a rational function or a radical of a rational function for homeomorphisms of Haken or Seifert manifolds.

1 Nielsen zeta function and homeomorphisms of 3-manifolds

1.1 Periodic maps and homeomorphisms of hyperbolic manifolds

We prove in corollary 1 of this subsection that the Nielsen zeta function is a radical of rational function for any homeomorphism of a compact hyperbolic 3-manifold. Lemma 1 and corollary 1 play also important role in the proof of the main theorem of this section.

We denote $N(f^n)$ by $N_n$. We shall say that $f : X \to X$ is a periodic map of period $m$, if $f^m$ is the identity map $id_X : X \to X$. Let $\mu(d), d \in N$, be the Möbius function of number theory. As is known, it is given by the following equations: $\mu(d) = 0$ if $d$ is divisible by a square different from one; $\mu(d) = (-1)^k$ if $d$ is not divisible by a square different from one, where $k$ denotes the number of prime divisors of $d$; $\mu(1) = 1$.

We give the proof of the following key lemma for the completeness.

**Lemma 1** Let $f$ be a periodic map of least period $m$ of the connected compact polyhedron $X$. Then the Nielsen zeta function is equal to

$$N_f(z) = \prod_{d|m} \sqrt[|d|]{1 - z^d}^{-P(d)},$$

where the product is taken over all divisors $d$ of the period $m$, and $P(d)$ is the integer

$$P(d) = \sum_{d_1|d} \mu(d_1)N_{d|d_1}.$$

**Proof** Since $f^m = id$, for each $j, N_j = N_{m+j}$. If $(k, m) = 1$, then there exist positive integers $t$ and $q$ such that $kt = mq + 1$. So $(f^k)^t = f^{kt} = f^{mq+1} = f^{mq}f = (f^m)^qf = f$. Consequently, $N((f^k)^t) = N(f)$. Let two fixed point $x_0$ and $x_1$ belong
to the same fixed point class. Then there exists a path \( \alpha \) from \( x_0 \) to \( x_1 \) such that 
\[ \alpha \circ (f \circ \alpha)^{-1} \approx 0. \]
We have 
\[ f(\alpha \circ f \circ \alpha)^{-1} = (f \circ \alpha) \circ (f^2 \circ \alpha)^{-1} \approx 0 \]
and a product 
\[ \alpha \circ (f \circ \alpha)^{-1} \circ (f \circ \alpha) \circ (f^2 \circ \alpha)^{-1} = \alpha \circ (f^2 \circ \alpha)^{-1} \approx 0. \]
It follows that \( \alpha \circ (f^k \circ \alpha)^{-1} \approx 0 \)
is derived by the iteration of this process. So \( x_0 \) and \( x_1 \) belong to the same fixed point class of \( f^k \).
If two points belong to the different fixed point classes \( f \), then they belong to the different fixed point classes of \( f^k \). So, each essential class (class with nonzero index) for \( f \) is an essential class for \( f^k \); in addition, different essential classes for \( f \) are different essential classes for \( f^k \). So \( N(f^k) \geq N(f) \). Analogously, \( N(f) = N((f^k)^l) \geq N(f^k) \).
Consequently, \( N(f) = N(f^k) \).
One can prove completely analogously that \( N_d = N_{d_l} \), if \( l, m/d = 1 \), where \( d \) is a divisor of \( m \).
Using these series of equal Nielsen numbers, one can regroup the terms of the series in the exponential of the Nielsen zeta function so as to get logarithmic functions by adding and subtracting missing terms with necessary coefficient. We show how to do this first for period \( m = p^l \), where \( p \) is a prime number.

We have the following series of equal Nielsen numbers:

\[ N_1 = N_{k^l}, \quad (k, p^l) = 1 \text{ (i.e., no } N_{ip^l}, N_{ip^{2l}}, \ldots, N_{ip^{il}}, i = 1, 2, 3, \ldots), \]
\[ N_p = N_{2p} = N_{3p} = \ldots = N_{(p-1)p} = N_{(p+1)p} = \ldots (\text{no } N_{ip^2}, N_{ip^3}, \ldots, N_{ip^l}) \]
etc.; finally,
\[ N_{p^{l-1}} = N_{2p^{l-1}} = \ldots (\text{no } N_{ip^l}) \]
and separately the number \( N_{ip^l} \).

Further,
\[
\sum_{i=1}^{\infty} \frac{N_i}{i} z^i = \sum_{i=1}^{\infty} \frac{N_1}{i} z^i + \sum_{i=1}^{\infty} \frac{(N_p - N_1) z^{pi}}{p^i} + \]
\[
+ \sum_{i=1}^{\infty} \frac{(N_{p^2} - (N_p - N_1) - N_1) z^{p^2 i}}{p^{2i}} + \ldots \]
\[
+ \sum_{i=1}^{\infty} \frac{(N_{p^l} - \ldots - (N_p - N_1) - N_1) z^{p^l i}}{p^{li}} \]
\[
= -N_1 \cdot \log(1 - z) + \frac{N_1 - N_p}{p} \cdot \log(1 - z^p) + \]
\[
+ \frac{N_p - N_{p^2}}{p^2} \cdot \log(1 - z^{p^2}) + \ldots \]
\[
+ \frac{N_{p^{l-1}} - N_{p^l}}{p^{li}} \cdot \log(1 - z^{p^l}). \]

For an arbitrary period \( m \), we get completely analogously,

\[
N_f(z) = \exp \left( \sum_{i=1}^{\infty} \frac{N(f^i)}{i} z^i \right) \]
\[
= \exp \left( \sum_{d|m} \sum_{i=1}^{\infty} \frac{P(d)}{d} \cdot \frac{z^{di}}{i} \right) . \]
\[
\exp \left( \sum_{d|m} \frac{P(d)}{d} \cdot \log(1 - z^d) \right) = \prod_{d|m} \sqrt[1-P(d)]{(1 - z^d)}
\]

where the integers \( P(d) \) are calculated recursively by the formula

\[
P(d) = N_d - \sum_{d_1 | d, d_1 \neq d} P(d_1).
\]

Moreover, if the last formula is rewritten in the form

\[
N_d = \sum_{d_1 | d} \mu(d_1) \cdot P(d_1)
\]

and one uses the Möbius Inversion law for real function in number theory, then

\[
P(d) = \sum_{d_1 | d} \mu(d_1) \cdot N_{d/d_1},
\]

where \( \mu(d_1) \) is the Möbius function in number theory. The lemma is proved.

**Corollary 1** Let \( f : M^n \to M^n, n \geq 3 \) be a homeomorphism of a compact hyperbolic manifold. Then by Mostow rigidity theorem \( f \) is homotopic to periodic homeomorphism \( g \). So lemma 1 applies and the Nielsen zeta function \( N_f(z) \) is equal to

\[
N_f(z) = N_g(z) = \prod_{d|m} \sqrt[1-P(d)]{(1 - z^d)},
\]

where the product is taken over all divisors \( d \) of the least period \( m \) of \( g \), and \( P(d) \) is the integer \( P(d) = \sum_{d_1 | d} \mu(d_1) N_{g^d/d_1} \).

The proof of the following lemma is based on Thurston’s theory of homeomorphisms of surfaces [18].

**Lemma 2** [7]

The Nielsen zeta function of a homeomorphism \( f \) of a compact surface \( F \) is either a rational function or the radical of a rational function.

**Proof** The case of a surface with \( \chi(F) > 0 \) and case of torus were considered in [4]. If surface has \( \chi(F) = 0 \) and \( F \) is not a torus then any homeomorphism is isotopic to periodic one (see [11]) and Nielsen zeta function is a radical of rational function by lemma 1. In the case of a hyperbolic (\( \chi(F) < 0 \)) surface, according to Thurston’s classification theorem, the homeomorphism \( f \) is isotopic either to a periodic or a pseudo-Anosov, or a reducible homeomorphism. In the first case the assertion of the lemma follows from lemma 1. If \( f \) is a pseudo-Anosov homeomorphism of a compact surface then for each \( n > 0 \), \( N(f^n) = F(f^n) \) [22]. Consequently, in this case the Nielsen zeta function coincides with the Artin-Mazur zeta function: \( N_f(z) = F_f(z) \).
Since in [3] Markov partitions are constructed for a pseudo-Anosov homeomorphism, Manning’s proof [13] of the rationality of the Artin-Mazur zeta function for diffeomorphisms satisfying Smale’s axiom A carries over to the case of pseudo-Anosov homeomorphisms. Thus, the Nielsen zeta function $N_f(z)$ is also rational. Now if $f$ is isotopic to a reduced homeomorphism $\phi$, then there exists a reducing system $S$ of disjoint circles $S_1, S_2, \ldots, S_m$ on $intF$ such that

1) each circle $S_i$ does not bound a disk in $F$;

2) $S_i$ is not isotopic to $S_j$, $i \neq j$;

3) the system of circles $S$ is invariant with respect to $\phi$;

4) the system $S$ has an open $\phi$-invariant tubular neighborhood $\eta(S)$ such that each $\phi$-component $\Gamma_j$ of the set $F - \eta(S)$ is mapped into itself by some iterate $\phi^{n_j}$, $n_j > 0$ of the map $\phi$; here $\phi^{n_j}$ on $\Gamma_j$ is either a pseudo-Anosov or a periodic homeomorphism;

5) each band $\eta(S_i)$ is mapped into itself by some iterate $\phi^{m_i}$, $m_i > 0$; here $\phi^{m_i}$ on $\eta(S_i)$ is a generalized twist (possibly trivial).

Since the band $\eta(S_i)$ is homotopically equivalent to the circle $S^1$ the Nielsen zeta function $N_{\phi^{m_i}}(z)$ is rational (see [7]). The zeta functions $N_{\phi}(z)$ and $N_{\phi^{m_i}}(z)$ are connected on the $\phi$-component $\Gamma_j$ by the formula $N_{\phi}(z) = \sqrt[N_{\phi^{n_j}}(z^{m_j})]{N_{\phi^{n_j}}(z^{m_j})}$; analogously, on the band $\eta(S_i)$, $N_{\phi}(z) = \sqrt[N_{\phi^{n_j}}(z^{m_j})]{N_{\phi^{n_j}}(z^{m_j})}$. The fixed points of $\phi^n$, belonging to different components $\Gamma_j$ and bands $\eta(S_i)$ are nonequivalent [1], so the Nielsen number $N(\phi^n)$ is equal to the sum of the Nielsen numbers $N(\phi^n/\Gamma_j)$ and $N(\phi^n/\eta(S_i))$ of $\phi$-components and bands. Consequently, by the properties of the exponential, the Nielsen zeta function $N_{\phi}(z) = N_{\phi^n}(z)$ is equal to the product of the Nielsen zeta functions of the $\phi$-components $\Gamma_j$ and the bands $\eta(S_i)$, i.e. is the radical of a rational function.

### 1.2 Homeomorphisms of Seifert fibre spaces

Let $M$ be a compact 3-dimensional Seifert fibre space. That is a space which is foliated by simple closed curves, called fibres, such that a fibre $L$ has a neighborhood which is either a solid Klein bottle or a fibred solid torus $T_r$, where $r$ denotes the number of times a fibre near $L$ wraps around $L$. A fibre is regular if it has neighborhoods fibre equivalent to solid torus $S^1 \times D^2$, otherwise it is called a critical fibre. There is a natural quotient map $p : M \to F$ where $F$ is a 2-dimensional orbifold, and hence, topologically a compact surface. The projection of the critical fibres, which we denote by $S$, consists of a finite set of points in the interior of $F$ together with a finite subcollection of the boundary components. The reader is referred to [17] for details, for definitions about 2-dimensional orbifolds, their Euler characteristics and other properties of Seifert fibre spaces. An orbifold is hyperbolic if it has negative Euler characteristic. Hyperbolic orbifold admits hyperbolic structure with totally geodesic
boundary.

Any fibre-preserving homeomorphism \( f : M \to M \) naturally induces a relative surface homeomorphism of the pair \((F, S)\), which we will denote by \( \hat{f} \). Recall from \([17]\) that there is a unique orientable Seifert fibre space with orbifold \( P(2, 2) \)-projective plane with cone singular points of order 2, 2. We denote this manifold by \( M_{P(2,2)} \).

**Lemma 3** \([17, 11]\) Suppose \( M \) is a compact orientable Seifert fiber space which is not \( T^3, S^1 \times D^2, T^2 \times I \) or \( M_{P(2,2)} \). Then there is a Seifert fibration \( p : M \to F \), so that any orientation preserving homeomorphism on \( M \) is isotopic to a fiber preserving homeomorphism with respect to this fibration.

Observe that if \( M \) and \( F \) are both orientable, then \( M \) admits a coherent orientation of all of its fibres and the homeomorphism \( f \) either preserves fibre orientation of all the fibres or it reverses fibre orientation.

**Lemma 4** \([12]\) Let \( M \) be a compact, orientable aspherical, 3-dimensional Seifert fibre space such that the quotient orbifold \( F \) is orientable and all fibres have neighborhoods of type \( T_1 \). Let \( f : M \to M \) be a fibre-preserving homeomorphism inducing \( \hat{f} : F \to F \). If \( f \) preserves fibre orientation, then Nielsen number \( N(f) = 0 \). If \( f \) reverses fibre orientation, then \( N(f) = 2N(\hat{f}) \).

**Proof** First, by small isotopy, arrange that \( f \) has a finite number of fibres which are mapped to themselves. If \( f \) preserves fibre orientation a further isotopy, which leaves \( \hat{f} \) unchanged, ensures that none of these fibres contains a fixed point. Thus, \( \text{Fix} (f) = \emptyset \) and so, \( N(f) = 0 \). If \( f \) reverses fibre orientation there are exactly 2 fixed points on each invariant fibre. Since \( M \) is aspherical lemma 3.2 in \([17]\) ensures that for any invariant fibre the 2 fixed points on that fibre are in distinct fixed point classes. On the other hand, the restriction on the fibre types in the hypothesis implies that a Nielsen path in \( F \) can always be lifted to a Nielsen path path in \( M \). In fact, there will be two distinct lifts of each path in \( F \). As a result, \( f \) has two fixed point classes covering each fixed point class of \( \hat{f} \). Since the index of a fixed point class of \( f \) is the same as that of its projection under \( p \) the result follows.

**Remark 1** As one can see lemma deals with a restricted class of Seifert fibre space. M. Kelly gave some examples in \([12]\) which indicate how critical fibres of Seifert fibre space effect Nielsen classes in the case when \( r > 1 \) in \( T_r \) and which difficulties arise in this case.

**Definition 1** A special Seifert fibre space is a Seifert fiber space such that the quotient orbifold \( F \) is orientable and all fibres have neighborhoods of type \( T_1 \).

**Theorem 1** Suppose \( M \) is a closed orientable aspherical manifold which is a special Seifert fibre space and \( f : M \to M \) is an orientation preserving homeomorphism. Then the Nielsen zeta function \( N_f(z) \) is a rational function or a radical of a rational function.
If $f$ preserves fibre orientation, then $f^n$ also preserves fibre orientation and so, by lemma 4 Nielsen numbers $N(f^n) = 0$ for all $n$ and the Nielsen zeta function $N_f(z) = 1$. If $f$ reverses fibre orientation, then $f^2$ preserves fibre orientation, $f^3$ reverses fibre orientation and so on. Thus, we have $N(f^{2k+1}) = 2N(f^{2k+1})$ and $N(f^{2k}) = 0$ for $k = 0, 1, 2, \ldots$. As result, the Nielsen zeta function $N_f(z)$ equals

$$
N_f(z) = \exp \left( \sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n \right) \\
= \exp \left( \sum_{k=0}^{\infty} \frac{N(f^{2k+1})}{2k+1} z^{2k+1} \right) \\
= \exp \left( \sum_{k=0}^{\infty} \frac{2N(f^{2k+1})}{2k+1} z^{2k+1} \right) \\
= \exp \left( 2 \sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n - 2 \sum_{k=1}^{\infty} \frac{N(f^{2k})}{2k} z^{2k} \right) \\
= (N_f(z))^2 \cdot \exp - \left( \sum_{k=1}^{\infty} \frac{N(f^2)^k}{k} (z^2)^k \right) \\
= (N_f(z))^2 / N_{\hat{f}^2}(z^2).
$$

From lemma 2 it follows that Nielsen zeta functions $N_f(z)$ and $N_{\hat{f}^2}(z^2)$ are either rational functions or the radicals of rational functions. Consequently, the Nielsen zeta function $N_f(z)$ is a rational function or a radical of a rational function.

### 1.3 Main theorem

Basic concepts about 3-manifold topology can be found in [10], in particular the Jaco-Shalen-Johannson decomposition of Haken manifolds and Seifert fiber spaces. See [18] for discussions on hyperbolic 3-manifolds. We recall some basic facts about compact connected orientable 3-manifolds. A 3-manifold $M$ is irreducible if every embedded 2-sphere bounds an embedded 3-disk. By the sphere theorem [8], an irreducible 3-manifolds is a $K(\pi, 1)$ Eilenberg-MacLane space if and only if it is a 3-disk or has infinite fundamental group.

A properly-embedded orientable connected surface in a 3-manifold is incompressible if it is not a 2-sphere and the inclusion induces a injection on the fundamental groups. An irreducible 3-manifold is Haken if it contains an embedded orientable incompressible surface. We use notation $N(X)$ to denote a regular neighborhood of set $X$. A Haken 3-manifold $M$ can be decomposed along a canonical set $T$ of incompressible tori into pieces such that each component of $M - N(T)$ is either hyperbolic manifold, or twisted $I$-bundle over Klein bottle, or a Seifert fiber space with hyperbolic orbifold. This decomposition is called Jaco-Shalen-Johannson decomposition.

**Definition 2** A special Haken manifold is a Haken manifold $M$ such that each component of $M - N(T)$ in JSJ decomposition of $M$ is either hyperbolic manifold, or...
We need some definitions from the paper [1].

**Definition 3** Suppose \( f : M \to M \) is a map, and \( A, B \) are \( f \)-invariant sets of \( M \). If there is a path \( \gamma \) from \( A \) to \( B \) such that \( \gamma \sim f \circ \gamma \) rel \((A, B)\), then we say that \( A, B \) are \( f \)-related.

**Definition 4** Suppose \( M \) is a compact 3-manifold with torus boundaries. A map \( f : M \to M \) is standard on boundary if for any component \( T \) of \( \partial M \), the map \( f/T \) is one of the following types: (1) a fixed point free map; (2) a periodic map with isolated fixed points; (3) a fiber preserving, fiber orientation reversing map with respect to some \( S^1 \) fibration of \( T \).

**Definition 5** A map \( f \) on a compact 3-manifold \( M \) is said to have \( FR \)-property (fixed-point relating property) if the following is true: if \( A \in \text{Fix}(f) \) and \( B \) is either a fixed point of \( f \) or an \( f \)-invariant component of \( \partial M \), and \( A, B \) are \( f \)-related by a path \( \gamma \), then \( \gamma \) is \( A, B \) homotopic to a path in \( \text{Fix}(f) \).

**Definition 6** A map \( f \) is a type I standard map if (1) \( f \) has \( FR \)-property, (2) \( f \) is standard on boundary, (3) \( \text{Fix}(f) \) consists of isolated points, and (4) \( f \) is of flipped pseudo-Anosov type at each fixed point. A map \( f \) is type II standard map if it satisfies (1), (2) above, as well as (3) \( \text{Fix}(f) \) is a properly embedded 1-dimensional submanifold and (4) \( f \) preserves a normal structure on \( \text{Fix}(f) \).

The main result of this section is the following theorem.

**Theorem 2** Suppose \( M \) is a closed orientable manifold which is special Haken manifold, and \( f : M \to M \) is an orientation preserving homeomorphism. Then the Nielsen zeta function \( N_f(z) \) is a rational function or a radical from rational function.

**Proof** Let \( \mathcal{T} \) be (possibly empty) set of invariant tori of the JSJ decomposition of \( M \). Then each component of \( M - \text{Int}N(\mathcal{T}) \) is either hyperbolic manifold, or a twisted \( I \)-bundle over Klein bottle, or a spherical special Seifert fiber space with hyperbolic orbifold. Isotop \( f \) so that it maps \( N(\mathcal{T}) \) homeomorphically to itself. Suppose that \( P \) is a Seifert fibered component of \( M - \text{Int}N(\mathcal{T}) \) such that \( f(P) = P \). By [1], \( f/P \) is isotopic to a fiber preserving map. Recall that a torus \( T \) in \( M \) is a vertical torus if it is union of fibers in \( M \). By [1] lemma 1.10, we can find a set of vertical tori \( \mathcal{T}^* \) in \( P \), cutting \( P \) into pieces which are either a twisted \( I \)-bundle over Klein bottle, or have hyperbolic orbifold, and a fiber preserving isotopy of \( f \), so that after isotopy, the restriction of \( f \) on each invariant piece has periodic or pseudo-Anosov orbifold map. Adding all such \( \mathcal{T}^* \) to \( \mathcal{T} \), we get a collection of tori \( \mathcal{T}' \), such that: (1) \( f(N(\mathcal{T}')) = N(\mathcal{T}') \); (2) each component \( M_i \) of \( M - \text{Int}N(\mathcal{T}') \) either is a twisted \( I \)-bundle over Klein bottle or has hyperbolic orbifold; (3) If \( f \) maps a Seifert fibered
component $M_i$ to itself, and if $M_i$ has hyperbolic orbifold, then $f$ is fiber preserving, and the orbifold map $\hat{f}$ on $F_i$ is either periodic or pseudo-Anosov. By [13] we can isotop $f$ so that restriction $f/M_i$ is a standard map for all $M_i$ and after that we can further isotop $f$, rel $\partial N_j$, on each component $N_j$ of $N(T')$, so that it is a standard map on $N_j$. By the definition of standard maps, Fix($f$) intersects each of $M_i$ and $N_j$ in points and 1-manifolds, so Fix($f$) is a disjoint union of points, arcs, and circles. Jiang, Wang and Wu proved [11] theorem 9.1 that different components of Fix($f$) are not equivalent in Nielsen sense. We repeat their proof here for the completeness. If exist two different components $C_0, C_1$ of Fix($f$) which are equivalent, then there is a path $\alpha$ connecting $C_0, C_1$ such that $f \circ \alpha \sim \alpha$ rel $\partial$. Denote by $T''$ the set of tori $\partial N(T')$. Among all such $\alpha$, choose one such that the number of components $\sharp(\alpha - T'')$ of the set $\alpha - T''$ is minimal. In below we will find another such curve $\alpha''$ with $\sharp(\alpha'' - T'') < \sharp(\alpha - T'')$, which would contradict the choice of $\alpha$. Let $D$ be a disk, and let $h : D \rightarrow M$ be a homotopy $f \circ \alpha \sim \alpha$ rel $\partial$. We may assume that $h$ is transverse to $T''$, and $\sharp h^{-1}(T'')$ is minimal among all such $h$. Then $h^{-1}(T'')$ consists of a properly embedded 1-manifold on $D$, together with possibly one or two isolated points mapped to the ends of $\alpha$. $T''$ is $\pi_1$-injective in $M$, so one can modify $h$ to remove all circles in $h^{-1}(T'')$. Note that $h^{-1}(T'')$ must contain some arcs, otherwise $\alpha$ would lie in some $M_i$ or $N(T_j)$, which is impossible because the restriction of $f$ in each piece has FR-property. Now consider an outermost arc $b$ in $h^{-1}(T'')$. Let $\beta = h(b)$. The ends of $\beta$ can not both be on $\alpha$, otherwise we can use the outermost disk to homotope $\alpha$ and reduce $\sharp(\alpha - T'')$, contradicting the choice of $\alpha$. Since $f$ is a homeomorphism, the same thing is true for $f \circ \alpha$. Therefore, $\beta$ has one end on each of $\alpha$ and $f \cdot \alpha$. The arc $b$ cut off a disk $\Delta$ on $D$ whose interior is disjoint from $h^{-1}(T'')$. The boundary of $\Delta$ gives rise to a loop $h(\partial\Delta) = \alpha_1 \cup \beta \cup (f \circ \alpha_1)^{-1}$, where $\alpha_1$ is subpath of $\alpha$ starting from an end point $x$ of $\alpha$. Let $T$ be the torus in $T''$ which contains $\beta$. Then the restriction of $h$ on $\Delta$ gives a homotopy $\alpha_1 \sim f \cdot \alpha_1$ rel $(x, T)$. Since $f$ has FR-property on each component of $M - Int N(T)$ and $N(T)$, by definition there is a path $\gamma$ in Fix($f$) such that $\gamma \sim \alpha_1$ rel $(x, T)$. Since $\gamma$ is in Fix($f$), the path $\alpha' = \gamma^{-1} \cdot \alpha$ has the property that $f \circ \alpha' = (f \circ \gamma^{-1}) \cdot (f \circ \alpha) = \gamma^{-1} \cdot (f \circ \alpha) \sim \gamma^{-1} \cdot \alpha = \alpha'$ rel $\partial$. Since $\alpha_1 \sim \gamma$ rel $(x, T)$, the path $\gamma^{-1} \cdot \alpha_1$ is rel $\partial$ homotopic to a path $\delta$ on $T$. Write $\alpha = \alpha_1 \cdot \alpha_2$. Then $\alpha' = \gamma^{-1} \cdot \alpha = (\gamma^{-1} \cdot \alpha_1) \cdot \alpha_2 \sim \delta \cdot \alpha_2$ rel $\partial$. By a small perturbation on $\delta$, we get a path $\alpha'' \sim \alpha'$ rel $\partial$ such that $\sharp(\alpha'' - T'') < \sharp(\alpha - T'')$. Since $f \circ \alpha'' \sim f \circ \alpha' \sim \alpha' \sim \alpha''$ rel $\partial$, this contradicts the minimality of $\sharp(\alpha - T'')$.

We can prove in the same way that the fixed points of an each iteration $f^n$, belonging to different components $M_i$ and $N_j$ are nonequivalent, so the Nielsen number $N(f^n)$ equals to the sum of the Nielsen numbers $N(f^n/M_i)$ and $N(f^n/N_j)$ of components. Consequently, by the properties of the exponential, the Nielsen zeta function $N_f(z)$ is equal to the product of the Nielsen zeta functions for the induced homeomorphisms of the components $M_i$ and $N_j$. By corollary 1 the Nielsen zeta function of the hyperbolic component $M_i$ is a radical of a rational function. The Nielsen zeta function is a radical of a rational function for the component $M_i$ which is a aspherical special Seifert fibre space by the theorem 1. The Nielsen zeta function is a rational function or a radical of a rational function for a homeomorphism of torus or Klein bottle by lemma 2. This implies that the Nielsen zeta function is a rational func-
tion or a radical of a rational function for the induced homeomorphism of component $N_j$ which is $I$-bundle over torus and for the induced homeomorphism of component $M_i$ which is twisted $I$-bundle over Klein bottle. So, the Nielsen zeta function $N_f(z)$ of the homeomorphism $f$ of the whole manifold $M$ is a rational function or a radical of a rational function as a product of the Nielsen zeta functions for the induced homeomorphisms of the components $M_i$ and $N_j$.

2 Asymptotic expansions for fixed point classes and twisted conjugacy classes of pseudo-Anosov homeomorphism

2.1 Twisted conjugacy and Reidemeister Numbers.

Let $\Gamma$ be a group and $\phi : \Gamma \rightarrow \Gamma$ an endomorphism. Two elements $\alpha, \alpha' \in \Gamma$ are said to be $\phi$–conjugate iff there exists $\gamma \in \Gamma$ with

$$\alpha' = \gamma \alpha \phi(\gamma)^{-1}.$$ 

We shall write $\{x\}_\phi$ for the $\phi$-conjugacy class of the element $x \in \Gamma$. The number of $\phi$-conjugacy classes is called the Reidemeister number of $\phi$, denoted by $R(\phi)$. If $\phi$ is the identity map then the $\phi$-conjugacy classes are the usual conjugacy classes in the group $\Gamma$.

In [6] we have conjectured that the Reidemeister number to be infinite as long as the endomorphism is injective and group has exponential growth. Below we prove this conjecture for surface groups and pseudo-Anosov maps and, in fact, we obtain an asymptotic expansion for the number of twisted conjugacy classes whose norm is at most $x$.

Let $f : X \rightarrow X$ be given, and let a specific lifting $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$ be chosen as reference. Let $\Gamma$ be the group of covering translations of $\tilde{X}$ over $X$. Then every lifting of $f$ can be written uniquely as $\gamma \circ \tilde{f}$, with $\gamma \in \Gamma$. So elements of $\Gamma$ serve as coordinates of liftings with respect to the reference $\tilde{f}$. Now for every $\gamma \in \Gamma$ the composition $\tilde{f} \circ \gamma$ is a lifting of $f$ so there is a unique $\gamma' \in \Gamma$ such that $\gamma' \circ \tilde{f} = \tilde{f} \circ \gamma$. This correspondence $\gamma \rightarrow \gamma'$ is determined by the reference $\tilde{f}$, and is obviously a homomorphism.

**Definition 7** The endomorphism $\tilde{f}_* : \Gamma \rightarrow \Gamma$ determined by the lifting $\tilde{f}$ of $f$ is defined by

$$\tilde{f}_*(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma.$$

It is well known that $\Gamma \cong \pi_1(X)$. We shall identify $\pi = \pi_1(X, x_0)$ and $\Gamma$ in the following way. Pick base points $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$ once and for all. Now points of $\tilde{X}$ are in 1-1 correspondence with homotopy classes of paths in $X$ which start at $x_0$: for $\tilde{x} \in \tilde{X}$ take any path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$ and project it onto $X$; conversely for a path $c$ starting at $x_0$, lift it to a path in $\tilde{X}$ which starts at $\tilde{x}_0$, and then take its endpoint. In this way, we identify a point of $\tilde{X}$ with a path class $<c>$ in $X$ starting
from $x_0$. Under this identification, $\tilde{x}_0 = < e >$ is the unit element in $\pi_1(X, x_0)$. The action of the loop class $\alpha = < a > \in \pi_1(X, x_0)$ on $\tilde{X}$ is then given by

$$\alpha = < a > : < c > \mapsto \alpha \cdot c = < a \cdot c > .$$

Now we have the following relationship between $\tilde{f}_* : \pi \to \pi$ and $f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$.

**Lemma 5** Suppose $\tilde{f}(\tilde{x}_0) = < w >$. Then the following diagram commutes:

$$\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(X, f(x_0)) \\
\tilde{f}_* & \downarrow \cong & \downarrow w_* \\
\pi_1(X, x_0) & \end{array}$$

where $w_*$ is the isomorphism induced by the path $w$.

In other words, for every $\alpha = < a > \in \pi_1(X, x_0)$, we have

$$\tilde{f}_*(< a >) = < w(f \circ a)w^{-1} >$$

**Remark 2** In particular, if $x_0 \in p(Fix(\tilde{f}))$ and $\tilde{x}_0 \in Fix(\tilde{f})$, then $\tilde{f}_* = f_*$.

**Lemma 6** Lifting classes of $f$ (and hence fixed point classes, empty or not) are in 1-1 correspondence with $\tilde{f}_*$-conjugacy classes in $\pi$, the lifting class $[\gamma \circ \tilde{f}]$ corresponding to the $\tilde{f}_*$-conjugacy class of $\gamma$. We therefore have $R(f) = R(\tilde{f}_*)$.

We shall say that the fixed point class $p(Fix(\gamma \circ \tilde{f}))$, which is labeled with the lifting class $[\gamma \circ \tilde{f}]$, corresponds to the $\tilde{f}_*$-conjugacy class of $\gamma$. Thus $\tilde{f}_*$-conjugacy classes in $\pi$ serve as coordinates for fixed point classes of $f$, once a reference lifting $\tilde{f}$ is chosen.

### 2.2 Asymptotic expansions

We assume $X$ to be a compact surface of negative Euler characteristic and $f : X \to X$ is a pseudo-Anosov homeomorphism, i.e. there is a number $\lambda > 1$ and a pair of transverse measured foliations $(F^s, \mu^s)$ and $(F^u, \mu^u)$ such that $f(F^s, \mu^s) = (F^s, \lambda^{-1}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda \mu^u)$. The mapping torus $T_f$ of $f : X \to X$ is the space obtained from $X \times [0, 1]$ by identifying $(x, 1)$ with $(f(x), 0)$ for all $x \in X$. It is often more convenient to regard $T_f$ as the space obtained from $X \times [0, \infty)$ by identifying $(x, s + 1)$ with $(f(x), s)$ for all $x \in X, s \in [0, \infty)$. On $T_f$ there is a natural semi-flow $\phi : T_f \times [0, \infty) \to T_f, \phi_t(x, s) = (x, s + t)$ for all $t \geq 0$. Then the map $f : X \to X$ is the return map of the semi-flow $\phi$. A point $x \in X$ and a positive number $\tau > 0$ determine the orbit curve $\phi(x, \tau) := \phi_t(x)_{0 \leq t \leq \tau}$ in $T_f$. The fixed points and periodic
points of $f$ then correspond to closed orbits of various periods. Take the base point $x_0$ of $X$ as the base point of $T_f$. According to van Kampen Theorem the fundamental group $G := \pi_1(T_f, x_0)$ is obtained from $\pi$ by adding a new generator $z$ and adding the relations $z^{-1}gz = \tilde{f}_s(g)$ for all $g \in \pi = \pi_1(X, x_0)$, where $z$ is the generator of $\pi_1(S^1, x_0)$. This means that $G$ is a semi-direct product $G = \pi \rtimes Z$ of $\pi$ with $Z$.

We now describe some known results.

**Lemma 7** If $\Gamma$ is a group and $\phi$ is an endomorphism of $\Gamma$ then an element $x \in \Gamma$ is always $\phi$-conjugate to its image $\phi(x)$.

**Proof.** If $\gamma = x^{-1}$ then one has immediately $\gamma x = \phi(x)\phi(\gamma)$. The existence of a $\gamma$ satisfying this equation implies that $x$ and $\phi(x)$ are $\phi$-conjugate.

**Lemma 8** Two elements $x, y$ of $\pi$ are $\tilde{f}_s$-conjugate iff $xz$ and $yz$ are conjugate in the usual sense in $G$. Therefore $R(f) = R(\tilde{f}_s)$ is the number of usual conjugacy classes in the coset $\pi \cdot z$ of $\pi$ in $G$.

**Proof.** if $x$ and $y$ are $\tilde{f}_s$-conjugate then there is a $\gamma \in \pi$ such that $\gamma x = y\tilde{f}_s(\gamma)$. This implies $\gamma x = yz\gamma z^{-1}$ and therefore $\gamma(xz) = (yz)\gamma$ so $xz$ and $yz$ are conjugate in the usual sense in $G$. Conversely suppose $xz$ and $yz$ are conjugate in $G$. Then there is a $\gamma z^n \in G$ with $\gamma z^n xz = yz\gamma z^n$. From the relation $\gamma z^n xz = \gamma z^n xz$ we obtain $\gamma z^n xz = \gamma z^n xz$ and therefore $\gamma z^n xz = \gamma z^n xz$. This shows that $\tilde{f}_s^1(x)$ and $y$ are $\tilde{f}_s$-conjugate. However by lemma 7 $x$ and $\tilde{f}_s^1(x)$ are $\tilde{f}_s$-conjugate, so $x$ and $y$ must be $\tilde{f}_s$-conjugate.

There is a canonical projection $\tau : T_f \to R/Z$ given by $(x, s) \mapsto s$. This induces a map $\pi_1(\tau) : G = \pi_1(T_f, x_0) \to Z$.

We see that the Reidemeister number $R(f)$ is equal to the number of homotopy classes of closed paths $\gamma$ in $T_f$ whose projections onto $R/Z$ are homotopic to the path

$$
\sigma : [0, 1] \to R/Z
$$

$$
s \mapsto s.
$$

Corresponding to this there is a group theoretical interpretation of $R(f)$ as the number of usual conjugacy classes of elements $\gamma \in \pi_1(T_f)$ satisfying $\pi_1(\tau)(\gamma) = z$.

**Lemma 9** [19, 14] Interior of the mapping torus $\text{Int}(T_f)$ admits a hyperbolic structure of finite volume if and only if $f$ is isotopic to pseudo-Anosov homeomorphism.

So, if the surface $X$ is closed and $f$ is isotopic to pseudo-Anosov homeomorphism the mapping torus $T_f$ can be realised as a hyperbolic 3-manifold, $H^3/G$, where $H^3$ is the Poincare upper half space $\{(x, y, z) : z > 0, (x, y) \in R^2\}$ with the metric $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$. The closed geodesics on a hyperbolic manifold are in one-to-one correspondence with the free homotopy classes of loops. These classes of loops are in one-to-one correspondence with the conjugacy classes of loxodromic elements in the fundamental group of the hyperbolic manifold. This correspondences allow Ch. Epstein (see [2], p.127) to study the asymptotics of such functions as $p_n(x) = \# \{\text{primitive closed geodesics of length less than } x \text{ represented by an element of}$
the form $gz^n \}$ using the Selberg trace formula. A primitive closed geodesic is one which is not an iterate of another closed geodesic. Later, Phillips and Sarnak [13] generalised results of Epstein and obtained for $n$-dimensional hyperbolic manifold the asymptotic of the number of primitive closed geodesics of length at most $x$ lying in fixed homology class. The proof of this result makes routine use of the Selberg trace formula. In the more general case of variable negative curvature such asymptotic was obtained by Pollicott and Sharp [14]. They used dynamical approach based on the geodesic flow. We will only need an asymptotic for $p_1(x)$. Note, that closed geodesics represented by an element of the form $gz$ are automatically primitive, because they wrap exactly once around the mapping torus (once around generator $z$). We have following asymptotic expansion [4, 13, 14]

\[
  p_1(x) = \frac{e^{hx}}{x^{3/2}} \left( \sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + O\left( \frac{1}{x^{N/2}} \right) \right),
\]

for any $N > 0$, where $h = \dim T_f - 1 = 2$ is the topological entropy of the geodesic flow on the unit-tangent bundle $ST_f$, the constant $C_0 > 0$ depend on the volume of hyperbolic 3-manifold $T_f$. N. Anantharaman [1] has shown that constants $C_n$ vanish if $n$ is odd. So, we have following asymptotics

\[
  p_1(x) \sim C_0 \frac{e^{hx}}{x^{3/2}}, \quad x \to \infty
\]

Notation. We write $f(x) = g(x) + O(h(x))$ if there exists $C > 0$ such that $|f(x) - g(x)| \leq C h(x)$. We write $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$.

Now, using one-to-one correspondences in lemma 6 and lemma 8 we define a norm of fixed point class, or corresponding to him lifting class, or corresponding to them twisted conjugacy class $\{g\}_{f_*}$ in the fundamental group of surface $\pi = \pi_1(X, x_0)$ as the length of the primitive closed geodesic $\gamma$ on $T_f$, which represented by an element of the form $gz$. So, for example, the norm function $l^*$ on the set of twisted conjugacy classes equals $l^* = l \circ B$, where $l$ is length function on geodesics($l(\gamma)$ is the length of the primitive closed geodesic $\gamma$) and $B$ is bijection between the set of twisted conjugacy class $\{g\}_{f_*}$ in the fundamental group of surface $\pi = \pi_1(X, x_0)$ and the set of closed geodesics represented by an elements of the form $gz$ in the fundamental group $G := \pi_1(T_f, x_0)$. We introduce following counting functions

\[
  \text{FPC}(x) = \# \{ \text{fixed point classes of } f \text{ of norm less than } x \},
\]

\[
  \text{L}(x) = \# \{ \text{lifting classes of } f \text{ of norm less than } x \},
\]

\[
  \text{Tw}(x) = \# \{ \text{twisted conjugacy classes for } f_* \text{ in the fundamental group of surface of norm less than } x \}
\]

**Theorem 3** Let $X$ be a closed surface of negative Euler characteristic and $f : X \to X$ is a pseudo-Anosov homeomorphism. Then

\[
  \text{FPC}(x) = \text{L}(x) = \text{Tw}(x) = \frac{e^{2x}}{x^{3/2}} \left( \sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + O\left( \frac{1}{x^{N/2}} \right) \right),
\]

where the constant $C_0 > 0$ depend on the volume of hyperbolic 3-manifold $T_f$, constants $C_n$ vanish if $n$ is odd.
Proof The proof follows from lemmas 8 and 9 and asymptotic expansion (1).

Corollary 2 For pseudo-Anosov homeomorphism of surface the Reidemeister number is infinite.

We can generalise Theorem 3 in following way

Theorem 4 Let $M$ be a compact manifold and $f : M \to M$ is a homeomorphism. Suppose that the mapping torus $T_f$ admits a Riemannian metric of negative sectional curvatures . Then there exist constants $C_0, C_1, C_2, \ldots$ with $C_0 > 0$ such that

$$FPC(x) = L(x) = Tw(x) = e^{hx} \frac{x^{3/2}}{N} \left( \sum_{n=0}^{N} \frac{C_n}{x^{n/2}} + O\left( \frac{1}{x^{N/2}} \right) \right)$$

for any $N > 0$, where $h > 0$ is the topological entropy of the geodesic flow on the unit-tangent bundle $ST_f$ and constants $C_n$ vanish if $n$ is odd.

Question 1 For which compact manifolds $M$ and homeomorphisms $f : M \to M$ the mapping torus $T_f$ admits a Riemannian metric of negative curvature or negative sectional curvatures?

Question 2 How to define the norm of the fixed point class or twisted conjugacy class in general case? Is it exist twisted Selberg trace formula for the discrete group? Such formula can give asymptotic expansion of counting function for twisted conjugacy classes.

References

[1] N. Anantharaman, Precise counting results for closed orbits of Anosov flows. Preprint, 1998.

[2] C. Epstein, The spectral theory of geometrically periodic hyperbolic 3-manifolds. Memoirs of the AMS, vol. 58, number 335, 1985.

[3] A. Fathi and M. Shub, Some dynamics of pseudo-Anosov diffeomorphisms. Asterisque 66-67 (1979), 181-207.

[4] A.L. Fel’shtyn, New zeta functions for dynamical systems and Nielsen fixed point theory. Lecture Notes in Math. 1346, Springer, 1988, 33-55.

[5] A.L.Fel’shtyn, Dynamical zeta functions, Nielsen theory and Reidemeister torsion. Memoirs of the AMS, 150 pages, to appear.

[6] Fel’shtyn A.L.,Hill R, Trace formulae, zeta functions, congruences and Reidemeister torsion in Nielsen theory. Forum Mathematicum, v. 10, n. 6, 1998, 641-663.
[7] Fel’shtyn A.L, Pilyugina V.B. The Nielsen zeta function. Funct. Anal. Appl., v. 19, n. 4, 1985, 61 -67.

[8] J. Hempel, 3-manifolds. Annals of math. studies 86, Princeton, 1976.

[9] N. V. Ivanov, Nielsen numbers of maps of surfaces. Journal Sov. Math., 26 (1984), 1636-1641.

[10] W. Jaco, Lectures on three-manifold topology. Regional conference series in mathematics, AMS, v. 43, 1977.

[11] B. Jiang, S. Wang, Y. Wu, Homeomorphisms of 3-manifolds and realisation of Nielsen number. Preprint n. 18 of Institute of mathematics, Peking University, 1997.

[12] M. Kelly, Nielsen numbers and homeomorphisms of geometric 3-manifolds. Topology Proceedings, vol. 19, 1994.

[13] A. Manning, Axiom A diffeomorphisms have rational zeta function, Bull. London Math. Soc. 3 (1971), 215-220.

[14] J.-P. Otal, Le theoreme d’hyperbolisation pour les varietes fibrees de dimension 3. Asterisque vol. 235, 1996.

[15] R. Phillips, P. Sarnak, Geodesics in homology classes. Duke math. journal, v.55, n.2, 1987, p. 287-297.

[16] M. Pollicott, R. Sharp, Asymptotic expansions for closed orbits in homology classes. Preprint ESI 594, 1998, Vienna.

[17] P. Scott, The geometry of 3-manifolds. Bull. London Math. Soc. v.15, n.56, p.401-487.

[18] W. Thurston, The geometry and topology of 3-manifolds. Princeton University, 1978.

[19] W. Thurston, Hyperbolic structures on 3-manifolds, II: surface groups and 3-manifolds which fibers over the circle. Preprint.

[20] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces. Bull. AMS 19 (1988), 417-431.

Institut für Mathematik,
Ernst-Moritz-Arndt-Universität Greifswald
Jahn-strasse 15a, D-17489 Greifswald, Germany.
E-mail address: felshtyn@mail.uni-greifswald.de