The flat FRW model in LQC: self-adjointness

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Abstract
The flat Friedman–Robertson–Walker (FRW) model coupled to the massless scalar field according to the improved, background scale-independent version of Ashtekar, Pawłowski and Singh [1] is considered. The core of the theory is addressed directly: the APS construction of the quantum Hamiltonian is analyzed under the assumption that the cosmological constant \( \Lambda \leq 0 \). We prove the essential self-adjointness of the operator whose square-root defines in [1] the quantum Hamiltonian operator and therefore provide the explicit definition. If \( \Lambda < 0 \), then the spectrum is discrete. In the \( \Lambda = 0 \) case, the essential and absolutely continuous spectra of the operator are derived. The latter operator is related in the unitary way to the absolutely continuous part of the quantum mechanics operator with \( (a, b > 0 \) being some constants) plus a trace class operator.

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1. Introduction

Loop quantum cosmology [1, 2] is a developing topic, growing into a well-established theory and provides both qualitative and quantitative results which change our beliefs about the origins of the universe [1]. The theory has achieved the point when it deserves the study of the exact properties of the quantum operators describing the evolution of the quantum spacetime geometry. The work in that direction has already been initiated in [6] (see SKL), on the occasion of deriving the gravitational part of the quantum scalar constraint operator in the case of the closed Friedman–Robertson–Walker (FRW) model.

In this paper, we consider the flat FRW model coupled to the massless scalar field according to the improved, background scale-independent version of Ashtekar, Pawłowski and Singh [1]. We are assuming that the cosmological constant

\[
\Lambda \leq 0.
\]
We go directly to the core of the theory and analyze the APS construction of the quantum Hamiltonian. We study the operator whose square-root defines in [1] the quantum Hamiltonian operator and show it is essentially self-adjoint. That result provides an explicit definition of the quantum Hamiltonian. The same definition was correctly anticipated in the numerical calculations in [1] in the case $\Lambda_1 = 0$. Here, we provide an exact proof. We also derive the essential and absolutely continuous spectra of the operator. Moreover, we construct a transform that maps the operator in a unitary fashion into the very well-known quantum mechanics operator $a(-\frac{d^2}{dy^2} - \frac{b}{\cosh^2y})$ ($a, b > 0$ being some constants) plus a trace class operator. In the case $\Lambda_1 < 0$, we show that the operator has a purely discrete spectrum.

Below, we introduce all those elements of the APS model which will be needed to understand our results and derivation. For the physics of the model and all the technical details which are not relevant in what follows, the reader is referred to the original paper [1].

The kinematical Hilbert space for the gravitational degrees of freedom in the FRW model is
\[ H_{\text{gr}} = \text{span}\{ |v\rangle : v \in \mathbb{R} \}, \]
with the scalar product
\[ \langle v | v' \rangle = \begin{cases} 1, & \text{if } v = v' \\ 0, & \text{otherwise.} \end{cases} \]

The gravitational degrees of freedom are represented by the quantum volume operator $\hat{v}$,
\[ \hat{v} |v\rangle = v |v\rangle, \]
defined in the domain
\[ D = \text{span}\{ |v\rangle : v \in \mathbb{R} \} \]
and by the improved holonomy operator
\[ \hat{h}_v |v\rangle = |v + v\rangle, \]
defined in the entire Hilbert space $H_{\text{gr}}$.

Given a function $Q : \mathbb{R} \rightarrow \mathbb{R}$, the corresponding multiplication operator will be denoted by $Q(\hat{v})$,
\[ Q(\hat{v}) |v\rangle = Q(v) |v\rangle. \]

The quantum volume operator measures (modulo a constant factor [1]) the physical volume of some cubic cell fixed in a homogenous 3-slice of spacetime, whereas the improved holonomy operator contains the information about the extrinsic curvature of the 3-slice.

The kinematical Hilbert space of the homogeneous scalar field is just
\[ H_{\text{sc}} = L^2(\mathbb{R}), \]
the measure being the Lebesgue one. The field operator $\hat{\phi}$ and the canonically conjugate momentum $\hat{p}_\phi$ are defined as in the Schrödinger quantum mechanics,
\[ (\hat{\phi}\psi)(\phi) = \phi \psi(\phi), \quad (\hat{p}_\phi\psi)(\phi) = -i \frac{\partial}{\partial \phi} \psi(\phi). \]

The total kinematical Hilbert space is
\[ H = H_{\text{gr}} \otimes H_{\text{sc}}. \]
In this model, all the Einstein–Klein–Gordon constraints are solved classically except the scalar constraint which takes the form of the following quantum scalar constraint operator
\[ \hat{C} = B(\hat{v}) \otimes \hat{p}_\phi^2 - \hat{C}_{\text{gr}} \otimes \text{id}, \]
defined in the domain

$$D(\hat{C}) = \text{span}\{|v\rangle : v \in \mathbb{R}\} \otimes C_0^\infty(\mathbb{R}),$$

where $B(\hat{v})$ is proportional to the quantum operator coming from the quantization of the classical inverse volume, namely

$$B(v) = \frac{27}{8} |v||v+1|^{1/3} - |v-1|^{1/3}, \quad (11)$$

and $\hat{C}_{\text{gr}}$ is the gravitational field part of the scalar constraint, namely

$$\hat{C}_{\text{gr}} = -\hat{h}^2 A(\hat{\phi})\hat{h}^2 - \hat{h}_{-2}A(\hat{\phi})\hat{h}_{-2} + A(\hat{\phi} + 2) + A(\hat{\phi} - 2) + \tilde{\Lambda} |\hat{\phi}|, \quad (12)$$

$$A(v) = \frac{3\pi G}{8} |v|\cdot |(|v+1| - |v-1|)|, \quad (13)$$

where $\tilde{\Lambda}$ is proportional to the cosmological constant $\Lambda$ and the cube of the Planck length $\ell$,

$$\tilde{\Lambda} = -\left(\frac{4\pi \gamma}{3}\right)^{\frac{1}{2}} \sqrt{\frac{3}{2}} \sqrt{\frac{3}{2}} \ell^3 \Lambda. \quad (14)$$

In this paper, we are considering the case

$$\Lambda \leq 0. \quad (15)$$

The scalar constraint operator $\hat{C}$ preserves every subspace $D_{\epsilon} \otimes C_0^\infty(\mathbb{R})$ defined by

$$D_{\epsilon} = \text{span}\{|v\rangle : v = 4n + \epsilon, n \in \mathbb{Z}\}, \quad (16)$$

where $\epsilon \in [0, 4)$. There is an orthogonal decomposition

$$\mathcal{H}_{\text{gr}} = \bigoplus_{\epsilon \in [0,4]} D_{\epsilon}. \quad (17)$$

The subspaces $\mathcal{H}_{\epsilon} \otimes \mathcal{H}_{\text{sc}}$, where

$$\mathcal{H}_{\epsilon} = D_{\epsilon}, \quad (18)$$

are sometimes considered analogs of ‘super-selection sectors’, meaning that the representation of the given quantum theory is reducible and should be reduced to one of the subspaces.

In order to ‘solve’ the quantum scalar constraint, and use the solutions to define the physical Hilbert space, one considers the restriction of the operator $\hat{C}$ to any of the subspaces $\mathcal{H}_{\epsilon} \otimes \mathcal{H}_{\text{sc}}$ and tries to give a meaning to the quantum counterpart of the scalar constraint equation, namely

$$\hat{C}|\psi\rangle = 0. \quad (19)$$

The problem is that there are not sufficiently many solutions of that equation in $\mathcal{H}_{\text{sc}} \otimes \mathcal{H}_{\epsilon}$. Moreover, our experience in the standard quantization shows that unless the gauge group generated by the constraints is compact, the physical solutions to the quantum constraint are ‘non-normalizable’, that is they belong to a new Hilbert space, constructed from the starting kinematical Hilbert space.

In this paper, we focus on the quantization scheme introduced and successfully applied by [1].

The starting point is replacing the heuristic formula (19) by another, also heuristic formula

$$(\text{id} \otimes \hat{p}_{\phi} - \sqrt{B(\hat{\phi})^{-1}\hat{C}_{\text{gr}} \otimes \text{id}})|\psi\rangle = 0. \quad (20)$$

The difference between (19) and (20) consists of the following two steps:

(i) going from the operator $\hat{C}$ to $B(\hat{\phi})^{-1}\hat{C}$;
(ii) ‘selecting the positive frequency modes’ of the $\hat{p}_\phi$ operator, meaning using the spectral decomposition of the Hilbert space defined by $\hat{p}_\phi$ and restricting to the non-negative part of the spectrum.

To give (20) the exact meaning, APS provide the vector space $D_\epsilon$ with a new scalar product $\langle \cdot, \cdot \rangle'$ such that

$$\langle v_1 | v_2 \rangle' := \begin{cases} B(v_1), & \text{if } v_1 = v_2 \\ 0, & \text{otherwise.} \end{cases}$$

(21)

The resulting Hilbert space $\mathcal{H}_\epsilon'$, that is the completion of $D_\epsilon$ in the new scalar product, is promoted to be the physical Hilbert space. We will also be assuming the generic case $\epsilon \neq 0$ when the restriction of the $B(\hat{v})$ operator is invertible.

The operator $B(\hat{v})^{-1} \hat{C}_{gr}$ preserves $D_\epsilon$, can be written in a manifestly positive definite form and is symmetric (due to the modification in the scalar product). It follows that it admits a self-adjoint extension, say $\hat{\theta}$. It is used in [1] to give the exact meaning to (20). Solutions of (20) are defined to be all the maps $\mathbb{R} \ni \phi \mapsto U_\phi \Psi \in \mathcal{H}_\epsilon'$, such that

$$U_\phi = \exp(i \phi \sqrt{\hat{\theta}}) \quad \text{and} \quad \Psi \in \mathcal{H}_\epsilon'.$$

However, the unitary map $U_\phi$ depends on the choice of the self-adjoint extension $\hat{\theta}$, unless the extension is unique. APS circumvent that problem by declaring a choice of the so-called Friedrichs extension, a mathematically distinguished extension which exists and is unique. That is a natural choice and makes the APS model complete. The open question was, what form that extension took in the case at hand, and whether or not there were possible other self-adjoint extensions that could a priori have different physical properties. Those issues gave the motivation for our current paper.

2. The results

In this section we present our results. The case $\Lambda < 0$ is simple enough to outline the proof. In the case $\Lambda = 0$, the sketch of the proof presented in this section will be followed by the detailed derivation in the subsequent sections.

The starting point for our work is the operator

$$B(\hat{v})^{-1} \hat{C}_{gr}$$

introduced above (see (12), (11)), defined in the domain

$$D(B(\hat{v})^{-1} \hat{C}_{gr}) = D_\epsilon$$

in the Hilbert space $\mathcal{H}_\epsilon$ (see (16), (21)). Our first step is using a unitary map

$$B(\hat{v})^\perp : \mathcal{H}_\epsilon' \to \mathcal{H}_\epsilon'$$

$$|v\rangle \mapsto \sqrt{B(v)}|v\rangle,$$

(22)

which maps the physical Hilbert space back into the Hilbert space (18). The operator in question is carried into the following operator

$$B(\hat{v})^{-1} \circ \hat{C}_{gr} \circ B(\hat{v})^{-\perp}$$

(23)

defined in the domain $D_\epsilon$. 
In order to show to what extent our results do not depend on the definition of the functions $A$ and $B$, we consider in this work the following generalization of the operator $\hat{C}_{\text{gr}}$:

$$H_{\text{APS}} = \tilde{B}(\hat{v})^{-\frac{1}{2}} (-\hat{h}_2 \tilde{A}(\hat{v}) \hat{h}_2 - \hat{h}_2 \tilde{A}(\hat{v}) \hat{h}_2 + \tilde{A}(\hat{v} + 2) + \tilde{A}(\hat{v} - 2) + \tilde{A} |\hat{v}|) \tilde{B}(\hat{v})^{-\frac{1}{2}},$$  \hspace{1cm} (24)

defined in the domain

$$D(H_{\text{APS}}) = D_{\epsilon}$$

(16) in the Hilbert space $\mathcal{H}_\epsilon$ (18), where the operator $\hat{h}_2$ is defined in (5), and about the functions $\tilde{A}, \tilde{B}$:

$$\tilde{A}(v) = \frac{3}{4} \pi G |v| + o_0(v),$$  \hspace{1cm} (25)

$$\tilde{B}(v) = \frac{1}{|v|} \left( 1 + \frac{\alpha}{|v|^2} + O \left( \frac{1}{|v|^4} \right) \right),$$  \hspace{1cm} (26)

where $o_0$ has a compact support. Functions (13) and (11) are a special case of (25) and (26) with

$$\alpha = \frac{10}{18}. \hspace{1cm} (27)$$

The operator $H_{\text{APS}}$ is manifestly symmetric. It is also positive definite provided $A(v) \geq 0$ for every $v = 4n + \epsilon, n \in \mathbb{Z}$ and given $\epsilon$. To see the latter property, it is enough to write the operator in the following form:

$$H_{\text{APS}} = \tilde{B}(\hat{v})^{-\frac{1}{2}} \left( \hat{h}_2 - \hat{h}_2 \right) \tilde{A}(\hat{v}) \hat{h}_2 - \hat{h}_2 \tilde{A}(\hat{v}) \tilde{B}(\hat{v})^{-\frac{1}{2}} + \tilde{A} |\hat{v}| \tilde{B}(\hat{v})^{-\frac{1}{2}}.$$  \hspace{1cm} (29)

We study separately the cases $\tilde{A} > 0$ (that is the cosmological constant $\Lambda > 0$) and, respectively, $\tilde{A} = 0$ (that is $\Lambda = 0$).

**Theorem 1.** Suppose $\tilde{A} > 0$. \hspace{1cm} (28)

Then, for arbitrary $\epsilon \in (0, 4)$, the operator $H_{\text{APS}}$ (24) has the following properties:

(i) it is essentially self-adjoint;

(ii) it has a discrete spectrum;

(iii) the spectrum is contained in $(0, \infty)$;

(iv) for arbitrary $E > 0$, the dimension of the Hilbert subspace spanned by the eigenvectors such that the corresponding eigenvalue $\lambda$ satisfies

$$\lambda < E$$

is less than or equal to the number of elements of the set

$$\left\{ n \in \mathbb{Z} : \tilde{A} \frac{|4n + \epsilon|}{B(4n + \epsilon)} < E \right\}.$$  \hspace{1cm} (30)

Technically, the theorem and proof are similar to those presented in [6] (see SKL) concerning the operator $\hat{C}_{\text{gr}}$ in the $k = 1$ case. Therefore, we just outline the proof. To begin with, we write the operator $H_{\text{APS}}$ in the following form:

$$H_{\text{APS}} = -\hat{h}_2 \frac{\tilde{A}(\hat{v})}{\sqrt{B(\hat{v} + 2)\tilde{B}(\hat{v} - 2)}} \hat{h}_2 - \hat{h}_2 \frac{\tilde{A}(\hat{v})}{\sqrt{B(\hat{v} + 2)\tilde{B}(\hat{v} - 2)}} \hat{h}_2 + \frac{\tilde{A}(\hat{v} + 2) + \tilde{A}(\hat{v} - 2) + \tilde{A} |\hat{v}|}{B(\hat{v})}. \hspace{1cm} (29)$$
Then we split the operator into two operators $H_0$ and $H_1$ defined in the same domain, namely

$$H_{APS} = H_1 + H_0,$$

(30)

$$H_1 = -\hat{h}^2 \frac{\tilde{A}(\hat{v})}{\sqrt{B(\hat{v} + 2)B(\hat{v} - 2)}} \frac{\tilde{A}(\hat{v})}{\sqrt{B(\hat{v} + 2)B(\hat{v} - 2)}} \hat{h}_{-2},$$

(31)

$$H_0 = \frac{\tilde{A}(\hat{v} + 2) + \tilde{A}(\hat{v} - 2) + \tilde{A} |\hat{v}|}{B(\hat{v})}.$$

(32)

Note that the operator $H_0$ is essentially self-adjoint. It is not hard to derive the following inequality for some constant $\beta' > 0$ and arbitrary $\psi \in D_\epsilon$,

$$\|H_1 \psi\|^2 \leq \|H_0 \psi\|^2 + \beta' \|\psi\|^2.$$  

(33)

For that purpose we use the expansion and expand the functions

$$\frac{\tilde{A}(v)}{\sqrt{B(v + 2)B(v - 2)}} = \frac{3\pi G}{4} (v^2 - 2 - \alpha) + O \left( \frac{1}{v^2} \right),$$

(34)

$$\frac{|v|}{B(v)} = (v^2 - \alpha) + O \left( \frac{1}{v^2} \right),$$

(35)

and the fact that the operator $f(\hat{v})$ for every function $f : \{v : v = 4n + \epsilon, n \in \mathbb{Z}\} \to \mathbb{R}$ of a compact support is bounded. The inequality shows the self-adjointness. The properties of the spectrum follow from the inequality

$$H_{APS} \geq \tilde{A} |\hat{v}| \tilde{B}(\hat{v})^{-1}$$

(see [6], SKL).

**Theorem 2.** Suppose

$$\tilde{A} = 0.$$  

(36)

Then, for arbitrary $\epsilon \in (0, 4)$, the operator $H_{APS}$ (24) has the following properties:

(i) it is essentially self-adjoint;
(ii) the absolutely continuous spectrum is $[0, \infty)$;
(iii) the essential spectrum is $[0, \infty)$;
(iv) the absolutely continuous part of the operator $H_{APS}$ is unitarily equivalent to the operator

$$C^\infty_0(\mathbb{R}) \ni f \mapsto -f'' \in C^\infty_0(\mathbb{R})$$

in the Hilbert space $L^2(\mathbb{R})$.

For those readers who do not have much experience with the terms used above\(^1\), we now describe our result without referring to sophisticated elements of the Hilbert space theory. What we do, and what is presented in detail in the following sections, is we just construct a unitary transformation which significantly simplifies the operator in question modulo a trace class operator (that is an operator of a finite trace). Then, we refer to the known properties of the simpler operator.

More specifically, consider the operator given by the first two terms of the expansion (34), (35), namely

$$H_{APS} = \frac{3\pi G}{4} (-\hat{h}^2(\hat{v}^2 - 2 - \bar{\alpha})\hat{h}_{-2} - \hat{h}_{-2}(\hat{v}^2 - 2 - \bar{\alpha})\hat{h}_{-2} + 2\bar{\alpha} - 2\alpha).$$

(37)

\(^1\) The exact definitions can be found in [3]. We quickly recall that the essential spectrum of a self-adjoint operator $H$ is a complement of $\lambda \in \mathbb{C}$, such that $H - \lambda$ is invertible up to a compact operator. The absolutely continuous part of the operator is, vaguely speaking, restriction to the subspace of vectors whose spectral measure is absolutely continuous with respect to the Lebesgue measure. The absolutely continuous spectrum is the spectrum of this restricted operator. This notion is strongly related to the scattering theory.
It is not hard to check that the difference between the operators $H_{\text{APS}}$ and $H'_{\text{APS}}$ is the following finite sum

$$\hat{h}_2 g_1(\hat{v})\hat{h}_2 + g_2(\hat{v}) + \hat{h}_2^{-2} g_1(\hat{v})\hat{h}_2^{-2},$$

where each of the functions $g_i, i = 1, 2,$ vanishes in infinity at least as fast as $v^{-2}$. Hence the sum is a compact, trace class operator.

In the consequence [4], the operator $H_{\text{APS}}$ is essentially self-adjoint if and only if $H'_{\text{APS}}$ is. If the operators are self-adjoint, then their spectra mentioned in the theorem (the essential and, respectively, absolutely continuous spectrum) are the same, and also the absolutely continuous parts of the operators are unitarily equivalent.

Therefore, we continue by studying the operator $H'_{\text{APS}}$. As mentioned in section 1, it turns out that there is a unitary map which carries some very well-known operator in the standard quantum mechanics into our $H'_{\text{APS}}$. Before discussing the properties of that map, we need the following short preparation. In order to compare a given symmetric operator $X$ defined in a domain $D(X)$, with another, self-adjoint operator, one provides $D(X)$ with the graph norm $\|\cdot\|_X$ defined as follows:

$$\|\cdot\|_X^2 = \|\cdot\|^2 + \|X\cdot\|^2.$$ 

Next, one considers the completion $\hat{D}(X)$ of $D(X)$ in the graph norm, and finally the continuous extension $\hat{X}$ of $X$ in $D(\hat{X})$ with respect to the graph norm, so-called closure of $X$.

Therefore, we have the closure $H'_{\text{APS}}$ of the operator $H'_{\text{APS}}$ defined in the domain $D(H'_{\text{APS}})$.

Subject to the comparison, the second operator is $H_{\text{ch}} : C_0^\infty(\mathbb{R}) \rightarrow C_0^\infty(\mathbb{R})$ defined in the Hilbert space $L^2(\mathbb{R})$, whose action on every $f \in C_0^\infty(\mathbb{R})$ is defined in the following way:

$$(H_{\text{ch}}f)(y) = \left(-\frac{d^2}{dy^2} - \frac{\alpha + 1}{\text{ch}^2(2y)}\right)f(y).$$ (38)

We construct a unitary map $U : L^2(\mathbb{R}) \rightarrow \mathcal{H}_e$ which has the following two properties:

(i) $U$ carries $C_0^\infty(\mathbb{R})$ into $D(H_{\text{APS}}') \subset \mathcal{H}_e$;

(ii) The operator $H'_{\text{APS}}$ restricted to the image $U(C_0^\infty(\mathbb{R}))$ satisfies

$$U H_{\text{ch}} U^{-1} = \frac{1}{3\pi G} H'_{\text{APS}}.$$ (39)

Now, the properties of operators such as $H_{\text{ch}}$ are very well known [3]. In particular, the operator is essentially self-adjoint. This is sufficient to conclude that the operator $H'_{\text{APS}}$ and in the consequence $H_{\text{APS}}$ are essentially self-adjoint. The essential spectrum of $H_{\text{ch}}$ is $[0, \infty)$ as well as the absolutely continuous spectrum. Those features are preserved by every unitary map and trace class perturbation. Finally, it is known that the absolutely continuous part of the operator $H_{\text{ch}}$ is unitarily equivalent to the part $-\frac{d^2}{dy^2}$ of the operator. However, that does not exclude the existence of a point spectrum or singular continuous one. The whole spectrum is also equal to $[0, \infty)$ due to the positive definiteness.

What we are left with is the construction of the unitary map (39). It will be performed in the following two sections.

\textsuperscript{2} It is worth noting that if two operators are unitarily equivalent the same is true for their closures, whereas the reverse statement is incorrect.
3. The Fourier transform

We now define a unitary map
\[ \mathcal{F} : \mathcal{H}_\epsilon \to L^2([0, 2\pi]), \]
by the following formula:
\[ \psi = \sum_{n \in \mathbb{Z}} \psi(4n + \epsilon)(4n + \epsilon) \mapsto \mathcal{F} \psi, \tag{40} \]
\[ \mathcal{F} \psi(x) = \sum_{n \in \mathbb{Z}} \psi(4n + \epsilon) e^{i(n + \epsilon)\frac{\pi}{4} x}. \tag{41} \]

The operator \( \hat{h}_4 \) is transformed into the multiplication operator \( e^{ix} \) (we use the convention analogous to (6)),
\[ (e^{ix} f)(x) = e^{ix} f(x). \tag{42} \]

The operator \( \frac{1}{i} \frac{\partial}{\partial x} \) is transformed into the derivative \( -i \frac{\partial}{\partial x} \) of a domain, however, different than the usual, namely
\[ D \left( -i \frac{\partial}{\partial x} \right) = \{ f \in L^2([0, 2\pi]) : f \text{ is absolutely continuous, } f(2\pi) = e^{\frac{1}{2}i\pi\epsilon} f(0) \}. \]

The operator generates a unitary group
\[ U_t(f)(x) = \begin{cases} f(x + t), & x + t \leq 2\pi \\ f(x + t - 2\pi) e^{\frac{i}{4} \pi \epsilon}, & x + t > 2\pi. \end{cases} \]
In this section, it is convenient to write the operator \( H'_{\text{APS}} \) in the following form:
\[ H'_{\text{APS}} = \frac{3\pi G}{4} [-(\hat{h}_4 + \hat{h}_{-4} - 2) \hat{\beta}^2 - 4(\hat{h}_4 - \hat{h}_{-4}) \hat{\beta} + (\alpha - 2)(\hat{h}_4 + \hat{h}_{-4}) - 2\alpha]. \tag{43} \]

The operator \( H'_{\text{APS}} \) is mapped by \( \mathcal{F} \) into the following operator
\[ H''_{\text{APS}} = 3\pi G \left[ 4(\hat{h}_4 + \hat{h}_{-4} - 2) \frac{d^2}{dx^2} + 4i(\hat{h}_4 - \hat{h}_{-4}) \frac{d}{dx} + \frac{\alpha - 2}{4}(\hat{h}_4 + \hat{h}_{-4}) - \frac{\alpha}{2} \right], \tag{44} \]
whose natural domain \( D(H''_{\text{APS}}) \) is the image of the domain of the closure of the operator \( H'_{\text{APS}} \). We will now characterize a part of \( D(H''_{\text{APS}}) \) that will be particularly useful in our paper. From the very definition, each function of the form
\[ g(x) = \sum_{k=n_0}^{n_1} a_k e^{i(k + \frac{1}{2})x} \]
belongs to the domain \( D(H''_{\text{APS}}) \). The set of function interesting from the point of view of this paper is
\[ C^\infty_{0, 2\pi} ([0, 2\pi]) := \{ f \in C^\infty([0, 2\pi]) : f \text{ vanishes in a neighborhood of } 0 \text{ and } 2\pi \}. \tag{45} \]
The key result of this subsection is the following.

**Lemma 3.**
\[ C^\infty_{0, 2\pi} ([0, 2\pi]) \subset D(H''_{\text{APS}}). \]

**Proof.** Writing the operator \( H''_{\text{APS}} \) in the following form
\[ H''_{\text{APS}} = 3\pi G \left[ -16 \sin^2 \left( \frac{\hat{x}}{2} \right) \frac{d^2}{dx^2} - 8 \sin(\hat{x}) \frac{d}{dx} + \frac{\alpha - 2}{2} \cos(\hat{x}) - \frac{\alpha}{2} \right], \]

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we can see an estimate
\[
| (H''_{APS} f)(x) | \leq 3\pi G \left[ 16 \left| \frac{d^2}{dx^2} f(x) \right| + 4 \left| \frac{d}{dx} f(x) \right| + \max \left( \left| \alpha - \frac{2}{2} + \alpha \right|, \left| \alpha - \frac{2}{2} - \alpha \right| \right) |f(x)| \right].
\] (46)

That gives us an estimate on the norms of the form
\[
\| f \|_2 + \| H''_{APS} f \|_2 \leq \beta \| f'' \|_2 + \gamma \| f' \|_2 + \chi \| f \|_2,
\] (47)
where \( \beta, \gamma, \chi \) are some constants one can calculate from (46) applying the inequality \( (|a| + |b| + |c|)^2 \leq 3 (|a|^2 + |b|^2 + |c|^2) \).

Now, we use the following facts:

- every \( f \in C^{\infty}_{0,2\pi}([0,2\pi]) \) satisfies the condition \( f(2\pi) = e^{i\pi/2} f(0) \) and therefore belongs in the domain of each of the operators \(-i\frac{d}{dx}\);
- the set of functions
  \[
  \sum_{k=n_0}^{n_1} a_k e^{i(k+\epsilon)x}, \quad n_0, n_1 \in \mathbb{Z}, a_k \in \mathbb{C}
  \] (48)
is dense in the domain of each of the operators \(-i\frac{d}{dx}\) in the graph norm (the set is spanned by the eigenfunctions);
- in the consequence, every \( f \in C^{\infty}_{0,2\pi}([0,2\pi]) \) belongs in the closure of space (48) in the right-hand-side norm (47);
- the graph norm of the operator \( H''_{APS} \) (the left-hand-side of (47)) is weaker than the right-hand-side norm, hence every \( f \in C^{\infty}_{0,2\pi}([0,2\pi]) \) belongs in the closure of space (48) in the right-hand-side norm (47).

That completes the proof.

In the following section, we will see that the set of functions \( C^{\infty}_{0,2\pi}([0,2\pi]) \) is in fact sufficient for our purposes because the restriction of the operator \( H''_{APS} \) to that set is an essentially self-adjoint operator. (It follows, in particular, that \( D(H''_{APS}) \) is the closure of \( C^{\infty}_{0,2\pi}([0,2\pi]) \) in the graph norm defined by \( H''_{APS} \).)

4. Unitary equivalence with particle on a real line

The last step of our construction is a unitary map
\[
W : L^2((-\infty, \infty)) \to L^2([0, 2\pi])
\]
Consider the following function:
\[
y(x) = \frac{1}{2} \ln \tan \frac{x}{4}.
\]
It transforms bijectively the interval \([0, 2\pi]\) onto \( \mathbb{R} \). Define \( W \) to act on \( f \in C^0_0(\mathbb{R}) \) in the following way:
\[
W(f)(x) = \sqrt{\frac{dy}{dx}} f(y(x)).
\]
We prove the following lemma.
Lemma 4. The operator $H''_{\text{APS}}$ (44) defined in the domain $C^\infty_{0,2\pi}([0, 2\pi])$ (45) in the Hilbert space $L^2([0, 2\pi])$ is mapped by the unitary map $W^{-1}$ into the following operator defined in the domain $C^\infty_0(\mathbb{R})$ in the Hilbert space $L^2(\mathbb{R})$: 

$$W^{-1} H''_{\text{APS}} W = -12\pi G \left( \frac{d^2}{dy^2} + \frac{\alpha + 1}{4 \cosh^2(2y)} \right).$$

Proof. First, we compute the image under $W$ of the operator $\text{id} \frac{dy}{dx}$ defined in the domain $C^\infty_{0,2\pi}(\mathbb{R})$ in the Hilbert space $L^2(\mathbb{R})$, and find the result

$$W_{\text{id}} \frac{dy}{dx} W^{-1} = 2i \sin \left( \frac{\hat{x}}{2} \right) \frac{d}{dx} + i \frac{1}{2} \cos \left( \frac{\hat{x}}{2} \right)$$

defined in the domain $C^\infty_{0,2\pi}([0, 2\pi])$. We have used the formula 

$$\frac{dy}{dx} = \frac{1}{2 \sin \left( \frac{\hat{x}}{2} \right)}.$$

Next, we write $H''_{\text{APS}}$ in the following form:

$$H''_{\text{APS}} = -3\pi G \left( \left( \frac{4 \sin \frac{\hat{x}}{2} \frac{d}{dx} + \cos \left( \frac{\hat{x}}{2} \right) \right)^2 + (\alpha + 1) \sin^2 \left( \frac{\hat{x}}{2} \right) \right).$$

(49)

Finally, we note that

$$\sin(x(y)) = \frac{2 \text{tg}(x/4)}{1 + \text{tg}^2(x/4)} = \frac{2e^{2y}}{1 + e^{4y}} = \frac{1}{\cosh(2y)}. \quad \square$$

5. Remarks and outlook

The results of this paper were presented and discussed in section 2. The established self-adjointness in the $\Lambda \leq 0$ case explains why the numerical computation of the evolution of the quantum state of the universe made in [1, 8] went so well. On the other hand, we were not able to generalize our proof to the case of $\Lambda > 0$. Also, some new results [10] suggest a conjecture that the operator in question may have more than one self-adjoint extensions when $\Lambda > 0$.

Our results apply to the generalization (24) of the operator $B^{-1}(\hat{v})\hat{C}_{\text{gr}}$ derived in [1]. The value $\alpha = 5/9$ corresponds to the operator $B^{-1}(\hat{v})\hat{C}_{\text{gr}}$. However, if $\alpha = 0$, then the corresponding function $\tilde{B}$ is just the inverse volume operator (modulo a constant factor). That case is currently being studied in detail by Ashtekar, Corichi and Singh [8] independently of our work. Our result ensures that in that case the operator $H_{\text{APS}}$ is essentially self-adjoint as well provided $\Lambda \leq 0$.

In this paper we focus on the flat FRW model. However, the arguments used in the $\Lambda < 0$ case can be easily generalized to the $k = -1$ and $k = 1$ cases to prove the self-adjointness and discreteness of the spectrum of the $B(\hat{v})^{-1}\hat{C}_{\text{gr}}$ operator.

The summary and new extensions of the results achieved in [6, 7] and in the current paper, as well as a discussion of the relevance for loop quantum gravity [5] will be discussed in [9].

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