Finite-time fuzzy output-feedback control for $p$-norm stochastic nonlinear systems with output constraints

Liandi Fang$^{1,2}$, Li Ma$^{1,*}$ and Shihong Ding$^1$

1 School of Electrical and Information Engineering, Jiangsu University, Zhenjiang, 212013, China
2 College of Mathematics and Computer Science, Tongling University, Tongling, 244000, China

* Correspondence: Email: mali@mail.ujs.edu.cn; Tel: 0511-88787826; Fax: 0511-88787826.

Abstract: This paper investigates the finite-time control problem of $p$-norm stochastic nonlinear systems subject to output constraint. Combining a tan-type barrier Lyapunov function (BLF) with the adding a power integrator technique, a fuzzy state-feedback controller is constructed. Then, an output-feedback controller design scheme is developed by the constructed state-feedback controller and a reduce-order observer. Finally, both the rigorous analysis and the simulation results demonstrate that the designed output-feedback controller not only guarantees that the output constraint is not violated, but also ensures that the system is semi-global finite-time stable in probability (SGFSP).

Keywords: finite-time control; stochastic nonlinear systems; output constraint; output-feedback controller; barrier Lyapunov function

Mathematics Subject Classification: 37F15, 34D09

1. Introduction

Over the past decades, a variety of control design strategies have been proposed for different nonlinear systems [1–7]. Especially, many approximated-based control schemes have been developed for uncertain nonlinear systems by using neural networks (NNs) or the fuzzy logic systems (FLSs) [8–19]. Among these studies, the research of stochastic systems is much more attracted (see, e.g. [16–19] and the references therein), due to their wide application. It is worth noting that the aforementioned NNs-based or FLSs-based control strategies haven’t taken output constraint into account. In fact, many practical systems are usually required to satisfy an output constraint in the operation for considering the performance specifications or safety [20, 21]. It is well known that, the BLF-based approaches are useful tools to settle controller design problems of output-constrained nonlinear systems, see references [22–27] for instances. In the latest research progress of constrained control, many kinds of adaptive neural or fuzzy control design methods have been presented by
combining the different BLFs with NNs or FLSs approximators for various stochastic nonlinear systems subject to output constraint and unknown nonlinearities in [28–32]. Nevertheless, the above-mentioned works have mainly considered strict-feedback stochastic systems whose fractional powers are all equal to one, rather than $p$-norm stochastic nonlinear systems in which the fractional powers are the positive odd rational numbers and at least one of the fractional powers is greater than one.

As an important class of nonlinear systems, the $p$-norm stochastic nonlinear systems which are more general and complex, have received increased attention in recent years. For such a kind of systems, many control design problems have been well solved by utilizing the adding a power integrator technique under some strong or weaker growth conditions [33–37]. Subsequently, references [38–40] have taken these growth conditions away and considered the adaptive NNs control for switched and large-scale stochastic high-order nonlinear systems. Meanwhile, the control problems have been investigated for $p$-norm nonlinear systems with output/states constraints under some nonlinear growth conditions in a few literatures, such as [41–44]. In these works, different state-feedback controllers have been mainly constructed and finite-time stability has been obtained. Worth noting that an output-feedback controller has been designed for high-order planar deterministic nonlinear systems in [44]. On these basics, references [45–48] have further taken both unknown nonlinearities and constraints into accounts, and designed some fuzzy controllers for $p$-norm stochastic nonlinear systems with output/states constraints. However, it should be pointed out that the above-mentioned results of $p$-norm stochastic nonlinear systems have mainly focused on addressing the asymptotical convergence rather than the finite-time convergence in the case of unknown nonlinearities. As a matter of fact, the finite-time convergence has only been investigated for $p$-norm stochastic nonlinear systems with known nonlinearities satisfying some nonlinear growth conditions. On the other hand, it can be observed that the existing constrained controllers of $p$-norm stochastic nonlinear systems are mainly based on the assumption that full-state measurements are available. In other words, when only the system output can be accurately measured, the problem of finite-time output-feedback control for $p$-norm stochastic nonlinear systems with unknown nonlinearities and output constraints, to our best knowledge, has never been considered in the literature.

Motivated by above discussions, this paper will investigate how to design the finite-time output-feedback controller for a kind of $p$-norm stochastic nonlinear systems subject to output constraint. The main contributions can be summarized as follows: 1) It is first time to consider $p$-norm stochastic nonlinear systems with output constraints and unmeasurable states. Note that the existing studies have mainly addressed the constrained control problems under the the assumption that all the states are measurable (e.g., [33–37, 41, 43]), while this paper investigates the constrained control design in the case of that the states are all unmeasurable except the system output. What’s more, the system nonlinearities are completely unknown. Thus, this work will extend and develop the existing control design theory for $p$-norm stochastic nonlinear systems; 2) A finite-time controller is designed. The developed scheme not only ensures the system output is constrained in a given compact set, but also enables the closed-loop system is semi-global finite-time stable in probability (SGFSP).

### 2. Problem and Preliminaries

In this paper, we consider the following class of $p$-norm stochastic nonlinear systems
\[ dx_i = x_{i+1}^{T} dt + \phi_i(\tilde{x}_i) dt + g_i^T(\tilde{x}_i) d\omega, \quad i = 1, \cdots, n-1, \]
\[ dx_n = u^T dt + \phi_n(x) dt + g_n^T(x) d\omega, \]
\[ y = x_1, \]

where \( \omega \) is a \( r \)-dimension standard Wiener process; \( x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n \) is system state vector; \( u \in \mathbb{R} \) and \( y \in \mathbb{R} \) are respectively control input and output; the fractional power \( \frac{1}{k} \in \mathbb{R}_{odd}^{\geq k, m} \); for \( i = 1, \cdots, n, \tilde{x}_i = (x_1, \cdots, x_i)^T \in \mathbb{R}^i \); \( \phi_i : \mathbb{R}^i \rightarrow \mathbb{R} \) and \( g_i : \mathbb{R}^i \rightarrow \mathbb{R}^r \) are unknown continuous functions satisfying \( \phi_i(0) = 0, g_i(0) = 0 \). The system output \( y = x_1 \) is measurable and constrained in \( \Pi_1 = \{ y(t) \in \mathbb{R}, |y(t)| < \varepsilon \} \) with a constant \( \varepsilon > 0 \), while the other states \( x_2, \cdots, x_n \) are all unmeasurable.

The objective of this paper is to design a finite-time fuzzy output-feedback controller for system (2.1) such that: 1) the output don’t violate the given constrained boundary; 2) all the signals of the closed-loop system converge to a small compact of the original point in finite-time in probability in presence of unknown nonlinearities and unmeasured states \( x_i (i = 2, \cdots, n) \).

Firstly, some concepts and lemmas are presented for preliminaries. Consider the following stochastic system
\[ dx = \phi(x) dt + g(x) d\omega, \quad (2.2) \]
where \( \phi(x) \) and \( g(x) \) are continuous functions with satisfying \( \phi(0) = g(0) = 0 \).

**Definition 1.** [1] For any given \( V(x) \in C^2(\mathbb{R}^n) \), associated with system (2.2), the second-order differential operator \( \ell \) is defined as follows:
\[ \ell V = \frac{\partial V}{\partial x} \phi(x) + \frac{1}{2} \text{tr} \left\{ g^T(x) \frac{\partial^2 V}{\partial x^2} g(x) \right\}. \]

**Definition 2.** [17] The equilibrium \( x = 0 \) of stochastic nonlinear system (2.2) is semi-global finite-time stable in probability (SGFSP) if for all \( x(t_0, \omega) = x(0) \), there exist a constant \( c > 0 \) and a settling time \( T^*(c, x_0, \omega) < \infty \) to make \( E[|x(t, \omega)|] \leq c \), for all \( t \geq t_0 + T^* \).

**Lemma 1.** [17] Consider the stochastic system (2.2) and assume that \( f(0), h(0) \) are bounded uniformly in \( t \). If there exist a \( C^2 \) Lyapunov function \( V : \mathbb{R}^n \rightarrow \mathbb{R}^+ \), functions \( \varrho_1, \varrho_2 \in \mathcal{K}_\infty \), and constants \( \mu_0 > 0, 0 < \tilde{\varsigma}_0 < 1 \) and \( \tilde{\nu}_0 > 0 \) such that
\[ \begin{cases} \varrho_1(|x|) \leq V(x) \leq \varrho_2(|x|) \\ \ell V(x) \leq -\mu_0 V(x)^{\tilde{\varsigma}_0} + \tilde{\nu}_0, \end{cases} \quad \forall x \in \mathbb{R}^n, \quad (2.4) \]

Then, the stochastic nonlinear system (2.2) is SGFSP.

**Remark 1.** As stated in [17], the Eq (2.4) implies that there exists the stochastic setting time function
\[ T^*(x, \omega) = \frac{1}{\mu_0^{1/\tilde{\varsigma}_0}} \left[ E[V^{1-\tilde{\varsigma}_0}(x(0))] - \left( \frac{\tilde{\nu}_0}{(1-\mu_0^{1/\tilde{\varsigma}_0})} \right)^{1/\tilde{\varsigma}_0} \right], \]
such that \( E[V^{\tilde{\varsigma}_0}(x)] \leq \frac{\tilde{\nu}_0}{(1-\mu_0^{1/\tilde{\varsigma}_0})} \), for all \( t \geq t_0 + T^*(x, \omega) \), where \( 0 < \tilde{\nu}_0 \leq 1 \) is an arbitrary constant.

**Lemma 2.** [7] Let \( a, b \in \mathbb{R}^+ \) with \( a \geq 1 \). For any \( \zeta, \eta \in \mathbb{R} \), the following inequalities hold:
\[ (i) \quad |\zeta^a - \eta^a| \leq a(2^{a-2} + 2)|\zeta - \eta|(|\zeta - \eta|^{a-1} + \eta^{a-1}), \]

\textit{AIMS Mathematics} Volume 6, Issue 3, 2244–2267.
(ii) \(|\zeta^k - \eta^k| \leq 2^{1-\frac{1}{p}} |\zeta^b - \eta^b|^\frac{1}{k}\),
(iii) \((|\zeta| + |\eta|)^\frac{1}{2} \leq |\zeta|^\frac{1}{2} + |\eta|^\frac{1}{2} \leq 2^{1-\frac{1}{2}} (|\zeta| + |\eta|)^\frac{1}{2}\).

Lemma 3. [42] For any constants \(k_1, k_2, \vartheta, \zeta \in \mathbb{R}^+\) and any variables \(\zeta_1, \zeta_2 \in \mathbb{R}\), we have the following inequality
\[
\vartheta |\zeta_1|^{k_1} |\zeta_2|^{k_2} \leq \frac{k_1}{k_1 + k_2} |\zeta_1|^{k_1 + k_2} + \frac{k_2}{k_1 + k_2} \vartheta |\zeta_2|^{k_1 + k_2}.
\]

Lemma 4. [8] Let \(p \in (0, \infty)\), for any \(\zeta_i \in \mathbb{R}, i = 1, \cdots, n\), one has
\[
(|\zeta_1| + \cdots + |\zeta_n|)^p \leq b(|\zeta_1|^p + \cdots + |\zeta_n|^p),
\]
where \(b = \max\{n^{p-1}, 1\}\).

Lemma 5. [36] If \(\zeta, \eta \in \mathbb{R}\) and \(p > 1\) is an odd number, then
\[
-(\zeta - \eta)(\zeta^p - \eta^p) \leq -\frac{1}{2p-1}(\zeta - \eta)^{p+1}.
\]

Lemma 6. [12] With any \(a_0 > 0, c_0 > 0, \) and \(\zeta(t) > 0\), for \(\eta(t) = a_0 \zeta(t) - c_0 \eta(t)\), if \(\eta(0) \geq 0\) can be satisfied, then one has \(\eta(t) \geq 0\) for \(\forall t \geq 0\).

In this paper, the nonlinear functions \(\phi_i(\cdot)\) and \(g_i(\cdot)\) are all unknown. The unknown functions will be approximated by the FLSs based on the following presented lemma.

Lemma 7. [49] Let \(F(X)\) be a continuous function defined on a compact set \(\Pi_0\). Then, for a given desired level of accuracy \(\delta > 0\), there exists a fuzzy logic system \(\Upsilon^T \Psi (X)\) such that
\[
\sup_{X \in \Pi_0} |F(X) - \Upsilon^T \Psi (X)| \leq \delta,
\]
\(\Upsilon = (\upsilon_1, \cdots, \upsilon_N)^T\) is the ideal constant weight vector, and \(\Psi (X) = \frac{(\varphi_1(X), \cdots, \varphi_N(X))^T}{\sum_{j=1}^{N} \varphi_j(X)}\) is the basis function vector, with \(N > 1\) being the number of the fuzzy rules and \(\varphi_j(X)\) being chosen as Gaussian functions, i.e., for \(j = 1, \cdots, N\)
\[
\varphi_j(X) = \exp \left( -\frac{(X - \lambda_j)^T (X - \lambda_j)}{\vartheta_j^2} \right)
\]
where \(\lambda_j = (\lambda_{j1}, \cdots, \lambda_{jn})^T\) and \(\vartheta_j\) respectively denote the center vector and the width of the Gaussian function.

Remark 2. In view of Lemma 7, any function \(F(X)\) which is defined and continuous on a compact set \(\Pi_0\), can be approximated by
\[
F(X) = \Upsilon^T \Psi (X) + \epsilon(X),
\]
where \(\epsilon(X)\) is the FLS approximation error satisfying \(|\epsilon(X)| < \delta\).
3. Main results

3.1. State-feedback controller design

In this section, a fuzzy state-feedback controller will be explicitly designed for system (2.1) by combining a tan-type BLF and the FLSs into the adding a power integrator technique.

First of all, we introduce a coordinate transformation as follow

\[
\chi_1 = x_1, \chi_i = \frac{x_i}{H_i}, i = 2, \cdots, n, \nu = \frac{u}{H_{n+1}}.
\] (3.1)

where \(q_1 = 0, q_j = \frac{q_{j-1} + 1}{p} (j = 2, \cdots, n + 1),\) and \(H > 1\) is a constant to be determined later. Based on (3.1), system (2.1) turns into

\[
\begin{align*}
\frac{d\chi_i}{dt} &= H\chi_i^p dt + f_i(\bar{\chi}_i)dt + h_i^T(\bar{\chi}_i)d\omega, i = 1, \cdots, n - 1, \\
\frac{d\chi_n}{dt} &= Hv^p dt + f_n(\chi)dt + h_n^T(\chi)d\omega, \\
y &= \chi_1.
\end{align*}
\] (3.2)

where \(\bar{\chi}_i = (\chi_1, \cdots, \chi_i), f_1 = \phi_1, h_1 = g_1, f_j = \frac{\phi_j}{H_j^p}, \) and \(h_j = \frac{g_j}{H_j^p}, (j = 2, \cdots, n).\)

In what follows, a fuzzy state-feedback controller will be designed through \(n\) steps based on the equivalent system (3.2).

Define

\[
\xi_1 = \chi_1, \xi_i = \chi_i - \beta_{i-1}, i = 2, \ldots, n,
\] (3.3)

where \(\beta_i\)'s are the virtual signals being constructed later.

**Step 1.** From (3.1), we can get

\[
\frac{d\xi_1}{dt} = \frac{d\chi_1}{dt} = (H\chi_2^p + f_1)dt + h_1^T d\omega.
\] (3.4)

Choose the first Lyapunov function

\[
V_1 = \frac{\xi_1^4}{2\pi} \tan \left( \frac{\pi \xi_1^4}{2 \xi_1^2} \right) + \frac{1}{2b_1} \tilde{\alpha}_1^2 \pm V_B(\xi_1) + \frac{1}{2b_1} \bar{\alpha}_1^2,
\]

where \(b_1 > 0\) is an adjustment parameter, \(\tilde{\alpha}_1 = \alpha_1 - \hat{\alpha}_1\) is the estimate error and \(\hat{\alpha}_1\) is the estimator of the parameter \(\alpha_1\).

**Remark 3.** Clearly, \(V_B(\xi_1) = \frac{\xi_1^4}{2\pi} \tan \left( \frac{\pi \xi_1^4}{2 \xi_1^2} \right)\) is a tan-type BLF adopted to deal with the system output constraint. Compared to the log-type BLF, \(V_B(\xi_1)\) possesses the following characteristic:

\[
\lim_{\xi_1 \to \infty} V_B(\xi_1) = \frac{\xi_1^4}{2\pi} \tan \left( \frac{\pi \xi_1^4}{2 \xi_1^2} \right) = \frac{\xi_1^4}{4},
\]

which implies that the proposed method is also applicable to the system without output constraints.
Then, one can easily get from the definition of $V_B(\xi_1)$ that

$$\frac{\partial V_B}{\partial \xi_1} = S_1(\xi_1)\xi_1^3,$$

(3.5)

$$\frac{\partial^2 V_B}{\partial \xi_1^2} = 3S_1(\xi_1)\xi_1^2 + \frac{4\pi}{\varepsilon^4} \tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right)S_1(\xi_1)\xi_1^6,$$

(3.6)

where $S_1(\xi_1) = \sec^2\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right)$.

In view of (2.3), (3.5) and (3.6), it is not hard to gain

$$\ell V_1 = \frac{\partial V_B}{\partial \xi_1}(H\chi_2^p + f_1) + \frac{1}{2} \frac{\partial^2 V_B}{\partial \xi_1^2} h_1^T h_1 - \frac{1}{b_1} \vec{\alpha}_1 \dot{\vec{a}}_1$$

$$= S_1(\xi_1)\xi_1^3(H\chi_2^p + f_1) - \frac{1}{b_1} \vec{\alpha}_1 \dot{\vec{a}}_1 + \frac{3}{2} S_1(\xi_1)||h_1||^2 \xi_1^2$$

$$+ \frac{2\pi}{\varepsilon^4} \tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right)S_1(\xi_1) ||h_1||^2 \xi_1^6,$$

From Lemma 3, one obtains

$$\frac{3}{2} S_1(\xi_1) ||h_1||^2 \xi_1^2 \leq \frac{3}{4} S_1(\xi_1)^2 ||h_1||^4 \xi_1^4 + \frac{3}{4}.$$

Then, we have

$$\ell V_1 \leq H S_1(\xi_1)\xi_1^3 \chi_2^p + H S_1(\xi_1)\xi_1^3 F_1(Z_1) - \frac{\varrho_1 \varepsilon^4}{2\pi} \tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right)$$

(3.7)

$$- \frac{1}{b_1} \vec{\alpha}_1 \dot{\vec{a}}_1 + \frac{3}{4},$$

where $Z_1 = \chi_1$.

$$F_1(Z_1) = \frac{1}{H} \left[ f_1 + \frac{3}{4} S_1(\xi_1)^2 ||h_1||^4 \xi_1 \right] + \frac{1}{2\pi} \frac{2\pi}{\varepsilon^4} \tan\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) S_1(\xi_1) ||h_1||^2 \xi_1^3$$

$$+ \frac{\varrho_1 \varepsilon^4}{2H\pi \xi_1^4} \sin\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right) \cdot \cos\left(\frac{\pi \xi_1^4}{2\varepsilon^4}\right).$$

and $\varrho_1 > 0$ is an adjustment parameter.

Thus, by Lemma 7, one can approximate $F_1(Z_1)$ by

$$F_1(Z_1) = \Psi_1^T \Psi_1(Z_1) + \epsilon_1(Z_1),$$

(3.8)

where $|\epsilon_1(Z_1)| \leq \delta_1$ and $\delta_1 > 0$ is a given constant.

Since $\Psi_1^T(\cdot) \Psi_1(\cdot) \leq 1$, it is easily obtained from Lemma 3 that

$$S_1(\xi_1)\xi_1^3 F_1(Z_1) \leq S_1(\xi_1) ||\chi_1||^2 ||\Psi_1|| + \delta_1$$

(3.9)

$$\leq \frac{3\varrho_{11} \alpha_1}{p + 3} S_1(\xi_1) ||\Psi_1||^2 + \frac{3}{p + 3} S_1(\xi_1) \xi_1^{p + 3}$$

$$+ \frac{p}{p + 3} \frac{\varrho_{11}^3}{p + 3} \xi_1^{p + 3} + \frac{p}{p + 3} \delta_1^{p + 3},$$

AIMS Mathematics
Volume 6, Issue 3, 2244–2267.
where $\alpha_1 = \|\Gamma_1\|^{\frac{\mu_1}{2}}$ and $\sigma_{11} > 0$ is an adjustment parameter.

Substituting (3.9) into (3.7) gets

$$
lV_1 \leq HS_1(\xi_1)\xi_1^p(\chi_2^p - \beta_1^p) + HS_1(\xi_1)\xi_1^3\beta_1^p - \frac{\varrho_1 e^4}{2\pi} \tan \left(\frac{\pi \xi_1^4}{2e^4}\right)
+ \frac{3H}{p + 3} \left[\sigma_{11}^p \xi_1^p + \frac{pH_s}{p + 3} \rho_1^{\frac{-3}{p}}\right]
+ H\tilde{\alpha}_1 \left[\frac{3H_s}{p + 3} (\xi_1)p^p + \frac{1}{b_1H} \tilde{\alpha}_1\right] + \frac{3}{4} + \frac{pH_s}{p + 3} \delta_1^{\frac{\mu_3}{p}}.
$$

(3.10)

Then, one could design

$$
\beta_1 = -M_1^{\frac{1}{p}} \xi_1 \equiv -\varphi_1 \xi_1
$$

(3.11)

and

$$
\tilde{\alpha}_1 = \frac{3Hb_1\sigma_{11}}{p + 3} (S_1(\xi_1))^{\frac{\mu_3}{p}} \xi_1^{p+3} - d_1\tilde{\alpha}_1.
$$

(3.12)

where $M_1 \geq \frac{\sigma_{11}}{p + 3} (S_1(\xi_1))^\frac{\mu_3}{p} \sigma_{11}^p + \frac{\varrho_1}{S_1(\xi_1)} + \varrho_1 > 0$; $\varrho_1, d_1 > 0$ are adjustment parameters; and the value of $\varrho_1 > 0$ will be given in the next step.

Substituting (3.11) and (3.12) into (3.10), gets

$$
lV_1 \leq -\frac{\varrho_1 e^4}{2\pi} \tan \left(\frac{\pi \xi_1^4}{2e^4}\right) - H(\rho_1 + \varrho_1)\xi_1^{p+3} + HS_1(\xi_1)\xi_1^3(\chi_2^p - \beta_1^p)
+ \frac{d_1}{b_1} \tilde{\alpha}_1\tilde{\alpha}_1 + \frac{3}{4} + \frac{pH_s}{p + 3} \sigma_{11}^p + \frac{pH}{p + 3} \delta_1^{\frac{\mu_3}{p}}.
$$

(3.13)

In addition, it is easily obtained that

$$
\frac{d_1}{b_1} \tilde{\alpha}_1\tilde{\alpha}_1 = \frac{d_1}{b_1} (\alpha_1 - \tilde{\alpha}_1)\tilde{\alpha}_1 \leq \frac{d_1\tilde{\alpha}_1^2}{2b_1} + \frac{d_1\alpha_1^2}{2b_1}.
$$

Therefore, we can get

$$
lV_1 - \frac{\varrho_1 e^4}{2\pi} \tan \left(\frac{\pi \xi_1^4}{2e^4}\right) - \frac{d_1\tilde{\alpha}_1^2}{2b_1} - H(\rho_1 + \varrho_1)\xi_1^{p+3}
+ Q_1 + HS_1(\xi_1)\xi_1^3(\chi_2^p - \beta_1^p),
$$

(3.13)

where $Q_1 = \frac{\varrho_1}{p + 3} \sigma_{11}^p + \frac{pH_s}{p + 3} \delta_1^{\frac{\mu_3}{p}} + \frac{d_1\alpha_1^2}{2b_1}$.

**Remark 4.** Notice that

$$
\lim_{\xi_1 \to 0} \frac{e^4 \sin \left(\frac{\pi \xi_1^4}{2e^4}\right) \cos \left(\frac{\pi \xi_1^4}{2e^4}\right)}{2\pi \epsilon_1^3} = \lim_{\xi_1 \to 0} \frac{e^4 \pi \xi_1^4 \cos \left(\frac{\pi \xi_1^4}{2e^4}\right)}{2\pi \epsilon_1^3} = 0,
$$

which means that the continuity of $F_1(Z_1)$ is ensured.
Remark 5. According to Lemma 6, one gets $\hat{\alpha}_1 \geq 0$, for $\forall t \geq 0$. In each design step, this characteristic will be always applied.

Step 2. From (3.3) and Itô’s formula, we have

$$d\xi_2 = (H\chi_3^p + f_2 - \ell \beta_1)dt + \left(h_2 - \frac{\partial \beta_1}{\partial \chi_1} h_1\right)^T d\omega,$$  \hspace{1cm} (3.14)

where $\ell \beta_1 = \frac{\partial \beta_1}{\partial \chi_1}(H\chi_3^p + f_1) + \frac{\partial \beta_1}{\partial \chi_1} \hat{\alpha}_1 + \frac{1}{2} \frac{\partial^2 \beta_1}{\partial \chi_1^2} h_1^T h_1$. Combining the definition of $\beta_1$ with the properties of $f_1(\chi_1)$ and $h_1(\chi_1)$, implies that $\ell \beta_1$ is valid and continuous.

Choose the second Lyapunov function as

$$V_2 = V_1 + \Lambda_2$$ \hspace{1cm} (3.15)

with

$$\Lambda_2 = \frac{1}{4} \xi_2^4 + \frac{1}{2b_2} \tilde{a}_2^2,$$ \hspace{1cm} (3.16)

where $b_2 > 0$ is an adjustment parameter, $\tilde{a}_2 = \alpha_2 - \hat{\alpha}_2$ is the estimate error and $\hat{\alpha}_2$ is the estimator of the parameter $a_2$.

Applying (2.3), (3.14) and (3.16), it can be gotten that

$$\ell \Lambda_2 = \xi_2^3 (H\chi_3^p + f_2 - \ell \beta_1) - \frac{1}{b_2} \tilde{a}_2 \dot{\tilde{a}}_2 + \frac{3}{2} \left\| h_2 - \frac{\partial \beta_1}{\partial \chi_1} h_1 \right\|^2 \xi_2^2.$$ \hspace{1cm} (3.17)

Besides, applying Lemma 3 renders

$$\frac{3}{2} \left\| h_2 - \frac{\partial \beta_1}{\partial \chi_1} h_1 \right\|^2 \xi_2^2 \leq \frac{3}{4} \left\| h_2 - \frac{\partial \beta_1}{\partial \chi_1} h_1 \right\|^4 \xi_2^4 + \frac{3}{4}.$$ \hspace{1cm} (3.18)

On the other hand, we get

$$S_1(\xi_1) \xi_1^3 (\chi_2^p - \beta_2^p) \leq S_1(\xi_1) \xi_1^3 |\xi_2^p - \beta_2^p|$$

$$\leq \text{DS}_1(\xi_1) |\xi_2^p| \left( |\xi_2^p + \varphi_1^p| |\xi_1|^{p-1} \cdot |\xi_2| \right)$$

$$\leq \rho_1 \xi_2^{p+3} + \tau_2 \xi_2^{p+3} + \frac{p \sigma_{12}}{p + 3} \xi_2^{p+3},$$ \hspace{1cm} (3.18)

where $D = (2p^2 + 2)p; \rho_1 = \frac{1}{p+3} \left[ 3(\text{DS}_1(\xi_1)) \right]^{\frac{p+1}{p+3}} \sigma_{12}^{\frac{6}{p+3}} + (p + 2)S_1(\xi_1) \right]$, $\tau_2 = \frac{S_1(\xi_1) \xi_2^{p+3}}{p+3}$, and $\sigma_{12} > 0$ is an adjustment parameter.

Thus, from (3.13), (3.17) and (3.18), we have

$$\ell V_2 = \ell V_1 + \ell \Lambda_2$$

$$\leq -\frac{a_1}{2} \frac{a_1^2}{2} \tan \left( \frac{\pi a_1^2}{2} \right) - \frac{d_1}{2b_1} \tilde{a}_1^2 - H \xi_2 \xi_2^{p+3} - \frac{1}{b_2} \tilde{a}_2 \dot{\tilde{a}}_2$$

$$+ H \xi_2^3 (\chi_3^p - \beta_2^p) + H \xi_2^3 \beta_2^p + H \xi_2^3 F_2(Z_2) + \frac{H \sigma_{12}}{p + 3} \xi_2^{p+3} + Q_1 + \frac{3}{4},$$ \hspace{1cm} (3.19)
where $Z_2 = (\bar{x}_2, \hat{a}_1)^T$,

$$F_2(Z_2) = \frac{1}{H} [f_2 - \ell \beta_1] + \tau_2 \xi_2^p + \frac{3}{4H} \left\| h_2 - \frac{\partial \beta_1}{\partial \chi_1} h_1 \right\|^4 \xi_2.$$

Obviously, the continuity of $F_2(Z_2)$ can be directly proved by the fact that the functions $f_2(\bar{x}_2)$, $h_2(\bar{x}_2)$ and $\ell \beta_1$ are all continuous. Thus, $F_2(Z_2)$ can be approximated as

$$F_2(Z_2) = \vartheta_2^T \Psi_2(Z_2) + \epsilon_2(Z_2),$$

(3.20)

where $|\epsilon_2(Z_2)| \leq \delta_2$ and $\delta_2 > 0$ is a given constant.

In view of the fact $\Psi_2^T(\cdot) \Psi_2(\cdot) \leq 1$ and Lemma 3, we obtain

$$\xi_2^3 F_2(Z_2) \leq |\xi_2|^3 (||\Psi_2|| \|\Psi_2\| + \delta_2)
\leq 3 \sigma_{21} \alpha^p + \frac{p}{p + 3} \xi_2^p + \frac{3}{p + 3} \xi_2^{p+3} + \frac{p}{p + 3} \xi_2^p,$$

(3.21)

where $\alpha = ||\Psi_2||^{\frac{p}{p+3}}$ and $\sigma_{21} > 0$ is an adjustment parameter.

Substituting (3.21) into (3.19) yields

$$\ell V_2 \leq -\frac{\varrho_1 \xi_2}{2\pi} \tan \left( \frac{\varpi_1 \xi_2}{2\varpi} \right) - \frac{d_1}{2b_1} \bar{\alpha}_1^2 - H \varrho_1 e_1^{p+3}
+ H \xi_2^3 (\bar{\chi}_3^p - \beta_2^p) + \xi_2^3 \beta_2^p + \frac{3H}{p + 3} [\sigma_{21} \hat{a}_2 + 1] \xi_2^{p+3} + \frac{H \varrho_1 \xi_2^{p+1}}{p + 3} \xi_2
+ H \hat{a}_2 \left[ \frac{3 \sigma_{21} \alpha_1}{p + 3} \xi_2^p - \frac{1}{b_2 H} \hat{a}_2 \right] + Q_1 + \frac{3}{4} + \frac{pH}{p + 3} \xi_2^{p+3} + \frac{pH}{p + 3} \xi_2^p.$$

(3.22)

Then, one can design

$$\beta_2 = -M_2 \xi_2^p \hat{\alpha}_2 \Rightarrow -\varphi_2 \xi_2$$

(3.23)

and

$$\hat{\alpha}_2 = \frac{3 H b_2 \sigma_{21} \xi_2^{p+3}}{p + 3} \hat{\alpha}_2 - d_2 \hat{\alpha}_2,$$

(3.24)

where $M_2 \geq \frac{3}{p + 3} [\sigma_{21} \hat{a}_2 + 1] + \frac{p \varrho_1 \xi_2^{p+1}}{p + 3} + \rho_2 + \varrho_2 > 0; \varrho_2 > 0, \ \rho_2 > 0$ are adjustment parameters. The value of $\rho_2 > 0$ will be given in the next step.

In addition, it is evident that

$$\frac{d_2}{b_2} \bar{\alpha}_2 \hat{\alpha}_2 \leq - \frac{d_2}{2b_2} \bar{\alpha}_2^2 + \frac{d_2}{2b_2} \hat{\alpha}_2^2.$$

(3.25)

Substituting (3.25) into (3.22), one gets

$$\ell V_2 \leq -\frac{\varrho_1 \xi_2}{2\pi} \tan \left( \frac{\varpi_1 \xi_2}{2\varpi} \right) - \sum_{j=1}^{2} \frac{d_j}{2b_j} \bar{\alpha}_1^2 - H \sum_{j=1}^{2} \varrho_j \xi_j^{p+3}
+ H \xi_2^3 (\bar{\chi}_3^p - \beta_2^p) + \sum_{j=1}^{2} Q_j,$$

(3.26)
where \( Q_2 = \frac{3}{4} + \frac{pH}{p+3} \sigma_1 \sigma_2 + \frac{pH}{p+3} \sigma_2^2 + \frac{d_{\text{out}}^2}{2p^2} \).

**Inductive Step** \((3 \leq k \leq n)\). In view of above two steps, we can deduce the following similar property whose proof can be found in the Appendix.

**Proposition 1.** For the \( k \)th Lyapunov function \( V_k : \Pi_k \rightarrow \mathbb{R}^+ \) as

\[
V_k = V_{k-1} + \Lambda_k
\]

with

\[
\Lambda_k = \frac{1}{4} \xi_k^4 + \frac{1}{2b_k} \hat{\alpha}_k^2,
\]

there exists a virtual controller \( \beta_k \) and the adaptive law of \( \hat{\alpha}_k \) of the following forms

\[
\beta_k = -M_k ^{1/2} \hat{\xi}_k \triangleq -\varphi_k \xi_k,
\]

\[
\dot{\hat{\alpha}}_k = \frac{3Hb_k \sigma_k \xi_k^{p+3}}{p+3} - d_k \hat{\alpha}_k,
\]

such that

\[
\ell V_k \leq -\frac{\rho_1 \xi_k^4}{2\pi} \tan \left( \frac{\pi \xi_k^4}{2\xi_k^4} \right) - \sum_{j=1}^k \frac{d_j \hat{\alpha}_j^2}{2b_j} - H \rho_k \xi_k^{p+3} - H \sum_{j=1}^k Q_j \xi_j^{p+3} + \sum_{j=1}^k Q_j + H \xi_n^{3}(\chi_{k+1} - \beta_k^p),
\]

where \( M_k \geq \frac{3}{p+3} [\sigma_k \hat{\alpha}_k + 1] + \frac{p\sigma_k}{p+3} + \rho_k + \varrho_k > 0; b_k, \varphi_k, d_k > 0 \) are adjustment parameters; and the value of \( \rho_k > 0 \) will be given in the next step.

**Step n** According to above steps, there exist a series of virtual controllers and adaptive parameter laws \( (\beta_k, \hat{\alpha}_k)(k = 1, \cdots, n) \) make that Eq (3.31) holds when \( k = n \) with \( \chi_{n+1} = \nu \). Therefore, the fuzzy adaptive control law can be designed as

\[
\beta_n = -M_n ^{1/2} \hat{\xi}_n \triangleq -\varphi_n \xi_n,
\]

\[
\dot{\hat{\alpha}}_n = \frac{3Hb_n \sigma_n \xi_n^{p+3}}{p+3} - d_n \hat{\alpha}_n.
\]

Apparently, one can further get

\[
\ell V_n \leq -\frac{\rho_1 \xi_n^4}{2\pi} \tan \left( \frac{\pi \xi_n^4}{2\xi_n^4} \right) - \sum_{j=1}^n \frac{d_j \hat{\alpha}_j^2}{2b_j} - H \sum_{j=1}^n Q_j \xi_j^{p+3} + \sum_{j=1}^n Q_j + H \xi_n^{3} (\nu - \beta_n^p),
\]

By the definitions of \( \xi_k(k = 1, \cdots, n) \), one gets the fuzzy state-feedback controller

\[
\beta_n = -(\varphi_1 \chi_1 + \cdots + \varphi_n \chi_n),
\]

where \( \varphi_k = \prod_{j=k}^n \varphi_j \) for \( k = 1, \ldots, n \).
3.2. Output-feedback controller design

In this subsection, a fuzzy output-feedback controller will be designed by combining the above-constructed state-feedback controller with the state-observer constructed later. Since $\chi_2, \ldots, \chi_n$ are unmeasurable, a reduced-order observer is required. Firstly, define a series of new variables as follows:

$$z_i = \chi_i - \gamma_i \gamma_2 \chi_1, \ i = 2, \ldots, n.$$ 

Then, it directly gets

$$dz_i = H(\chi_{i+1}^p - \gamma_i \gamma_2 \chi_1^p) dt + (f_i - \gamma_i \gamma_2 f_1) dt$$
$$+ [h_i - \gamma_i \gamma_2 h_1] d\omega, i = 2, \ldots, n - 1,$$
$$dz_n = H(v^n - \gamma_n \gamma_2 \chi_1^p) dt + (f_n - \gamma_n \gamma_2 f_1) dt$$
$$+ [h_n - \gamma_n \gamma_2 h_1] d\omega,$$

where $\gamma_i \geq 1 (i = 2, \ldots, n)$ are gain parameters to be determined. Hence, the $(n - 1)$-dimensional observer can be constructed as [36]

$$\hat{z}_i = H(\hat{z}_i + \gamma_i \gamma_2 \chi_1^p) - H\gamma_i \gamma_2(\hat{z}_2 + \gamma_2 \chi_1^p), i = 2, \ldots, n - 1,$$
$$\hat{z}_n = Hv^n - H\gamma_n \gamma_2(\hat{z}_2 + \gamma_2 \chi_1^p).$$

According to (3.36), the estimate $\hat{\chi}_i$ of $\chi_i$ can be gotten by

$$\hat{\chi}_i = \hat{z}_i + \gamma_i \gamma_2 \chi_1, i = 2, \ldots, n.$$ 

Thus, one constructs the implementable controller of system (2.4) by using Eq (3.35) and the certainty equivalence principle as below:

$$v = -(\phi_1 \hat{\chi}_1 + \cdots + \phi_n \hat{\chi}_n).$$

Therefore, the output-feedback controller of origin system (2.1) is

$$u = H^{s+1}v = -H^{s+1}(\phi_1 \hat{\chi}_1 + \cdots + \phi_n \hat{\chi}_n).$$

3.3. Selection of the observer gains

In this section, we will analyze the appropriate values of the gains $\gamma_i (i = 2, \ldots, n)$ and some constant parameters in output-feedback controller.

To determine the observer gains $\gamma_2, \ldots, \gamma_n$, we first define the error dynamics

$$e_i \triangleq \chi_i - \hat{\chi}_i, \ i = 2, \ldots, n.$$ 

Further, the following coordinate transformation is introduced

$$\tilde{e}_2 = e_2, \tilde{e}_3 = e_3 - \gamma_3 e_2, \ldots, \tilde{e}_n = e_n - \gamma_n e_{n-1}.$$ 

It can easily infer from (3.40) and (3.41) that

$$d\tilde{e}_i = H[(\chi_{i+1}^p - \hat{\chi}_{i+1}^p) - \gamma_i (\chi_i^p - \hat{\chi}_i^p)] dt$$
$$+ [f_i - \gamma_i f_{i-1}] dt + [h_i - \gamma_i h_{i-1}] d\omega, i = 2, \ldots, n - 1$$
$$d\tilde{e}_n = -H\gamma_n (\chi_n^p - \hat{\chi}_n^p) dt + [f_n - \gamma_n f_{n-1}] dt + [h_n - \gamma_n h_{n-1}] d\omega.$$
Proposition 2. For the Lyapunov function

\[ U_n = \frac{1}{4} \gamma \sum_{i=2}^{n} \tilde{e}_i^4, \]

by utilizing Lemmas 2-5, Eqs (2.3) and (3.42), it is not difficult to obtain

\[ \ell U_n \leq -H \left[ \sum_{i=2}^{n} \frac{\gamma \gamma_i}{2p-1} e_i^{p+3} + \sum_{i=2}^{n} \tilde{c}_i e_i^{p+3} + \sum_{i=2}^{n} \frac{6\gamma \gamma_i^{p+3}}{(p+3)H} e_i^{p+3} \right] \]

\[ + \frac{(p-1)H}{p+3} + \bar{F}_1(\chi), \]

where \( \gamma > 0 \) is an adjustment constant; \( \tilde{c}_i = \tilde{c}_i(\gamma_i, \cdots, \gamma_n) > 0, (i = 2, \cdots, n-1) \) are constants independent of \( H \); and \( \tilde{c}_n \) is a constant independent of \( H \) and \( \gamma_i(i = 2, \cdots, n) \); and \( \bar{F}_1(\chi) = H \sum_{i=2}^{n} \left( \frac{1}{2} (f_i - \gamma_i f_i^{(i-1)})^4 + \frac{3\gamma_i}{4h_i^2} ||h_i - \gamma_i h_i^{(i-1)}||^4 + \frac{2p-1}{p+3} \chi_i^{p+3} \right) \) is an unknown continuous function.

Besides, we can obtain from Lemmas 2-4 that

\[ |\xi_\xi^3|^{\gamma - \beta_n^p} \leq D|\xi_i|^{3} |v - \beta_n|^{\gamma} + |\beta_n|^{p} \]

\[ \leq D(n-1)^{p} |\xi_i|^3 \cdot \sum_{i=2}^{n} \tilde{e}_i^{p+3} + \gamma \tilde{e}_i^{p+3} \]

\[ \leq \frac{p+1}{p+3} \sum_{i=2}^{n} \tilde{e}_i^{p+3} + \tau_n \tilde{e}_i^{p+3} \]

\[ \leq \bar{e}_i \sum_{i=2}^{n} \tilde{e}_i^{p+3} + \bar{F}_2(\xi_i), \]

where \( \bar{F}_2(\xi_i) = \tau_n \tilde{e}_i^{p+3} \); \( \tau_n = \frac{3}{p+3} \sum_{i=2}^{n} (D(n-1)^{p} \tilde{e}_i^{p+3})^{\frac{p+3}{p+1}} + \frac{p+2}{p+3} \sum_{i=2}^{n} (D(n-1)^{p} \tilde{e}_i^{p+3})^{\frac{p+3}{p+1}} \); \( \bar{e}_i = \bar{e}_i(\gamma_i, \cdots, \gamma_n) \) are constants independent of \( H \) and \( \gamma_i \) and \( \bar{e}_i = \bar{e}_i(\gamma_i, \cdots, \gamma_n) \) are constants independent of \( H \) and \( \gamma_i \) and \( \gamma_i(i = 2, \cdots, n-1) \) are positive constants.

Then, substituting (3.44) into (3.34) yields

\[ \ell V_n \leq -\frac{\vartheta \pi}{2\pi} \tan \left( \frac{\pi \xi_i^{(i)}^{\frac{1}{2}}}{2\tilde{e}_i} \right) - \sum_{j=1}^{n} \frac{d\bar{\alpha}_j^2}{2b_j} \]

\[ - H \sum_{j=1}^{n} \vartheta_j \tilde{e}_j^{p+3} + \sum_{j=1}^{n} Q_j + H \sum_{i=2}^{n} \bar{e}_i^{p+3} + \bar{F}_2(\xi_i). \]

Then, define the entire Lyapunov function \( V = V_n + U_n \). Apparently, it directly infers from Proposition 2 and Eq (3.45) that

\[ \ell V = \ell V_n + \ell U_n \]

\[ \leq -\frac{\vartheta \pi}{2\pi} \tan \left( \frac{\pi \xi_i^{(i)}^{\frac{1}{2}}}{2\tilde{e}_i} \right) - \sum_{j=1}^{n} \frac{d\bar{\alpha}_j^2}{2b_j} - H \sum_{j=1}^{n} \vartheta_j \tilde{e}_j^{p+3} + \bar{F}(\cdot) + \sum_{j=1}^{n} Q_j \]

\[ - H \sum_{i=2}^{n} \left[ \frac{\gamma \gamma_i}{2p-1} - \bar{e}_i - \bar{\gamma}_i \right] \tilde{e}_i^{p+3} + H \sum_{i=2}^{n} \frac{6\gamma \gamma_i^{p+3}}{(p+3)H} \tilde{e}_i^{p+3} + \frac{(p-1)H}{p+3}, \]

\[ AIMS Mathematics \]

Volume 6, Issue 3, 2244–2267.
where \( \bar{F}(\cdot) = \bar{F}_1(\cdot) + \bar{F}_2(\cdot) \).

Using FLS to deal with the unknown function \( \tilde{F}(\cdot) \), one deduces from Lemma 7 that 
\( \tilde{F}(\cdot) = \gamma_0^p \Psi_0 + \epsilon_0(\cdot) \leq ||\Psi_0|| ||\Psi_0||_0 + \delta_0 \leq \frac{3}{p+3} \alpha_0 + \frac{\rho}{p+3} + \delta_0 \) where \( \alpha_0 = ||\Psi_0|| ||\omega|| \).

Therefore, the observer gains \( \gamma_2, \ldots, \gamma_n \) and constant \( H \) can be chosen in the following recursive manner

\[
\begin{align*}
\gamma_n & \geq \max \left\{ \frac{2^{p-1}}{\gamma} (\bar{c}_n + \bar{i}_n + 1 + \theta_n), 1 \right\}, \\
\gamma_{n-1} & \geq \max \left\{ \frac{2^{p-1}}{\gamma} (\bar{c}_{n-1}(\gamma_n) + \bar{i}_{n-1}(\gamma_n) + 1 + \theta_{n-1}), 1 \right\}, \\
& \vdots \\
\gamma_2 & \geq \max \left\{ \frac{2^{p-1}}{\gamma} (\bar{c}_2(\gamma_n, \ldots, \gamma_3) + \bar{i}_2(\gamma_n, \ldots, \gamma_3) + 1 + \theta_2), 1 \right\}, \\
H & \geq \max \left\{ 1, \left( \frac{6}{p+3} \right)^{\frac{1}{p+3}} \gamma \right\}.
\end{align*}
\]

(3.47)

Then, Eq (3.46) turns into

\[
\ell V \leq -\frac{\theta_1 \bar{e}_1^4}{2\pi} \tan \left( \frac{\pi \bar{e}_1^4}{2\bar{e}_1} \right) - \sum_{j=1}^{n} \frac{d_j \bar{\alpha}_j^2}{2b_j} - H \sum_{i=2}^{n} \theta_i \bar{e}_i^{p+3} + Q,
\]

(3.48)

where \( \theta_i(i = 2, \ldots, n) \) are positive constants, and 
\( Q = \sum_{i=1}^{n} Q_j + \frac{(p-1)H}{p+3} + \frac{3}{p+3} \alpha_0 + \frac{\rho}{p+3} + \delta_0. \)

On the other hand, it can be verified from Lemma 3 that

\[
\begin{align*}
\bar{e}_i^4 & \leq \frac{4}{p+3} \bar{e}_i^{p+3} + \frac{p-1}{p+3}, \\
\bar{\xi}_i^4 & \leq \frac{4}{p+3} \bar{\xi}_i^{p+3} + \frac{p-1}{p+3},
\end{align*}
\]

(3.49)

which renders

\[
\begin{align*}
-H \sum_{i=2}^{n} \theta_i \bar{e}_i^{p+3} & \leq - \sum_{i=2}^{n} \frac{\bar{\theta}_i}{4} \bar{e}_i^4 + \frac{p-1}{4} H \sum_{i=2}^{n} \theta_i^1, \\
-H \sum_{i=2}^{n} \bar{q}_i \bar{\xi}_i^{p+3} & \leq - \sum_{i=2}^{n} \frac{\bar{\bar{q}}_i}{4} \bar{\xi}_i^4 + \frac{p-1}{4} H \sum_{i=2}^{n} \bar{q}_i.
\end{align*}
\]

(3.50)

where \( i = 2, \ldots, n, \bar{\theta}_i = (p+3)H \theta_i \) and \( \bar{\bar{q}}_i = (p+3)H \bar{q}_i \). Further, we get

\[
\ell V \leq -\frac{\theta_1 \bar{e}_1^4}{2\pi} \tan \left( \frac{\pi \bar{e}_1^4}{2\bar{e}_1} \right) - \sum_{j=1}^{n} \frac{d_j \bar{\alpha}_j^2}{2b_j} - \sum_{i=2}^{n} \frac{\bar{\theta}_i}{4} \bar{e}_i^4 - \sum_{i=2}^{n} \frac{\bar{\bar{q}}_i}{4} \bar{\xi}_i^4 - \sum_{i=2}^{n} \frac{\bar{\theta}_i}{4} \bar{e}_i^4 + \bar{Q},
\]

(3.51)

where \( \bar{Q} = Q + \frac{p-1}{4} H \sum_{i=2}^{n} \theta_i + \frac{p-1}{4} H \sum_{i=2}^{n} \bar{q}_i. \)
3.4. Stability analysis

To state the main result, the following theorem is presented.

**Theorem 1.** For the $p$-norm stochastic nonlinear system (2.1) and a given constant, there exists a finite-time fuzzy output-feedback controller (3.39) together with the parameter adaptive laws (3.12), (3.24), (3.30), and (3.33) such that

i) the system output isn’t violated in the sense of probability, i.e., $P(|y(t)| < \varepsilon) = 1$.

ii) all the signals in the closed-loop stochastic nonlinear system (2.1) are SGFSP.

**Proof.** i) Let $\mu_0 = \min\{q_1, d_1, \cdots, d_n, \bar{q}_2, \cdots, \bar{q}_n, \bar{\sigma}, \cdots, \bar{\sigma}_n\}$ and $\pi_0 = \bar{Q}$. Then, Eq (3.51) can be expressed as

$$\ell V \leq -\mu_0 V + \pi_0. \tag{3.52}$$

We can easily get from Eq (3.52) that

$$EV(t) \leq V(t_0)e^{-\mu_0t} + \frac{\pi_0}{\mu_0}. \tag{3.53}$$

For $x(0) = (x_1(t_0), \cdots, x_n(t_0))^T$ satisfying $x_1(t_0) \in \Pi$, it easily obtains that the mean of $V(t)$ is bounded, which implies that $V$ is bounded in probability. It can be directly deduced from the definition of $V$ that

$$P[V_B(\xi_1) < \infty] = 1. \tag{3.54}$$

Consequently, it is clear that $P[|y(t)| < \varepsilon] = P[|\xi_1(t)| < \varepsilon] = 1$, which demonstrates that the output constraint of system (2.1) is not violated in the sense of probability.

ii) For $\forall 0 < \bar{\xi}_0 < 1$, it is easy to get from Lemma 3 that

$$V_{\bar{\xi}_0} \leq \bar{\xi}_0 V + (1 - \bar{\xi}_0).$$

Further, one has

$$-\mu_0 V \leq -\frac{\mu_0}{\bar{\xi}_0} V_{\bar{\xi}_0} + \frac{(1 - \bar{\xi}_0)\mu_0}{\bar{\xi}_0}. \tag{3.55}$$

Then, substituting (3.55) into (3.52) drives

$$\ell V \leq -\bar{\mu}_0 V_{\bar{\xi}_0} + \bar{\pi}_0, \tag{3.56}$$

where $\bar{\mu}_0 = \frac{t_0}{\bar{\xi}_0}$ and $\bar{\pi}_0 = \frac{(1 - \bar{\xi}_0)t_0}{\bar{\xi}_0} + \pi_0$.

Let $T^* = \frac{1}{\bar{\mu}_0(1 - \bar{\xi}_0)} \left[ E\left(V^{1-\bar{\xi}_0}(\chi(0), \varepsilon(0), \hat{\alpha}(0))\right) - \left(\frac{\bar{\xi}_0}{\bar{\mu}_0(1 - \bar{\xi}_0)}\right)^{1-\bar{\xi}_0} \right]$ where $\chi(0) = (\chi_1(t_0), \cdots, \chi_n(t_0))^T$, $\varepsilon(0) = (\varepsilon_2(t_0), \cdots, \varepsilon_n(t_0))^T$, $\hat{\alpha}(0) = (\hat{\alpha}_1(t_0), \cdots, \hat{\alpha}_n(t_0))$, $0 < t_0 < 1$ is a constant. Then it follows from Lemma 1 that for $\forall t \geq t_0 + T^*$, $E\left(V^{1-\varepsilon}(\chi, \varepsilon, \hat{\alpha})\right) \leq \frac{\bar{\xi}_0}{\bar{\mu}_0(1 - \bar{\xi}_0)}$, which means that all the signals in the closed-loop systems are semi-global finite-time stable in probability.
Remark 6. In this paper, we construct an output-feedback controller rather than the designed state-feedback controllers in existing results about output constraints. On the other hand, it should be pointed out the considered constraint is symmetric rather than asymmetric, which leads that the proposed scheme can not be directly employed or further extended to the case of asymmetric constraints. However, a control scheme based on a new BLF can be developed for asymmetric output constraints in a similar way to this paper. In addition, another limitation is that all of the fractional powers are equal to p. If p’s are taken different values, the proposed strategy seems not applicable. In the future, we will address the two issues.

4. Simulation example

The validation of the proposed strategy will be testified by the following system.

\[
\begin{aligned}
    dx_1 &= \frac{12}{7} x_1^\frac{11}{7} dt + 2 \ln(1 + x_1^2)dt + 4 x_1^2 d\omega, \\
    dx_2 &= u^\frac{13}{7} dt + x_1 x_2^2 dt + x_2^2 d\omega, \\
    y &= x_1,
\end{aligned}
\]

where the output \( y = x_1 \) is measurable and constrained by \( \Pi_1 = \{y(t) \in R, |y(t)| < 1\} \), and the state \( x_2 \) is unmeasurable.

![Figure 1. Trajectory of \( x_1(t) \) with \( \varepsilon = 1 \).](image1)

![Figure 2. Trajectory of \( x_2(t) \) with \( \varepsilon = 1 \).](image2)
According to the controller design procedure, we can respectively design the finite-time output-feedback controller, the adaptive laws and the observer as follows:

\[ u = -H \frac{15}{28} (M_1^5 x_1 + M_2^5 \dot{x}_2), \]

\[ \dot{\alpha}_1 = \frac{15}{28} H b_1 \sigma_{11} (S_1(x_1)) \frac{\dot{\alpha}_1}{28} - d_1 \dot{\alpha}_1, \]

\[ \dot{\alpha}_2 = \frac{15}{28} H b_2 \sigma_{21} \frac{\dot{\alpha}_2}{28} - d_2 \dot{\alpha}_2, \]

\[ \dot{z}_2 = H \nu \frac{15}{28} - H \gamma_2 (\dot{z}_2 + \gamma_2 x_1), \]

\[ \hat{\chi}_2 = \dot{\hat{z}}_2 + \gamma_2 x_1, \]

where \( M_1 = \frac{15}{28} (S_1(x_1)) \frac{\dot{\alpha}_1}{28} + \frac{\rho_1}{28} x_1 + \varrho_1 \) and \( M_2 = \frac{15}{28} (S_2(x_1)) \frac{\dot{\alpha}_2}{28} + \frac{13 \rho_1}{28} + \varrho_2. \)

Now, we choose the related parameters as \( \gamma = 1.5, \sigma_{11} = 22, \sigma_{21} = 20, \sigma_{12} = 1, \theta_2 = 3, b_1 = b_2 = 40, d_1 = d_2 = 2, \varrho_1 = \varrho_2 = 1 \) and infer \( H = 1.6, \gamma_2 = 8.1. \) Next, the initial states and adaptive parameters are selected as \([x_1(0), x_2(0), \dot{x}_2(0), \dot{\alpha}_1, \dot{\alpha}_2]^T = [0.4, 8, 8, 10, 10]^T\) the simulation results are displayed in Figures 1–4.

Figure 3. Trajectory of \( u. \)

Figure 4. Trajectory of \( \dot{\alpha}. \)

Figure 1 provides the trajectory of \( x_1(t), \) which indicates that the system output constraint is not violated under controller (4.2). Meanwhile, the trajectories of \( x_2(t) \) and \( \dot{x}_2(t) \) are given in Figure 2,
which shows that \( x_2(t) \) is well estimated by \( \hat{x}_2(t) \). Moreover, the trajectory of the controller \( u \) is displayed in Figure 3. Finally, Figure 4 expresses the curves of the adaptive parameter vector under the developed strategy. Also, one could evidently observe from these figures that all the signals of system (4.1) are semi-global finite-time stable in probability under controller (4.2).

5. Conclusion

In this paper, the output-feedback controller design problem is investigated for a class of \( p \)-norm stochastic nonlinear systems with output constraints. Through using a tan-type BLF, an adaptive fuzzy state-feedback controller is proposed by the adding a power integrator technique. Then, a finite-time fuzzy output-feedback controller is constructed by combining the proposed state-feedback controller and a reduced-order observer. Both rigorous proof and the simulation example verify that the designed controller can ensure the achievement of the system output constraint and semi-global finite-time stability of all the signals in probability. In the future, we will consider the situations of asymmetric constraints, different fractional powers, or multi-input multi-output stochastic nonlinear systems.

Acknowledgments

This work was supported by supported by the National Science Foundation of China under Grant 61973142, the Jiangsu Natural Science Foundation for Distinguished Young Scholars under Grant BK20180045, the Priority Academic Program Development of Jiangsu Higher Education Institutions, and the Project of Anhui Province Outstanding Young Talent Support Program under Grant gxyq2018089.

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. S. Y. Khoo, J. L. Yin, Z. Man, X. Yu, Finite-time stabilization of stochastic nonlinear systems in strict-feedback form, *Automatica*, 47 (2013), 1403–1410.
2. S. H. Ding, W. H. Chen, K. Q. Mei, D. Murray-Smith, Disturbance observer design for nonlinear systems represented by input-output models, *IEEE Trans. Ind. Electron*, 67 (2020), 1222–1232.
3. X. D. Li, X. Y. Yang, T. W. Huang, Persistence of delayed cooperative models: Impulsive control method, *Appl. Math. Comput.*, 342 (2019), 130–146.
4. H. Shen, M. S. Chen, Z. G. Wu, J. D. Cao, J. H. Park, Reliable event-triggered asynchronous passive control for semi-Markov jump fuzzy systems and its application, *IEEE T. Fuzzy Syst.*, 28 (2020), 1708–1722.
5. S. H. Ding, A. Levant, S. H. Li, Simple homogeneous sliding-mode controller, *Automatica*, 67 (2016), 22–32.
6. X. Y. Yang, X. D. Li, Q. Xi, P. Y. Duan, Review of stability and stabilization for impulsive delayed systems, *Math. Biosci. Eng.*, 15 (2018), 1495–1515.
7. S. H. Ding, S. H. Li, Second-order sliding mode controller design subject to mismatched term, *Automatica*, 77 (2017), 388–392.
8. K. Q. Mei, S. H. Ding, Second-order sliding mode controller design subject to an upper-triangular structure, *IEEE T. Syst. Man Cybern. Syst.*, (2018), 1–11.
9. S. C. Tong, Y. M. Li, Observer-based adaptive fuzzy backstepping control of uncertain nonlinear pure-feedback systems, *Sci. China Inform. Sci.*, 57 (2014), 1–14.
10. J. T. Hu, G. X. Sui, X. X. Lv, X. D. Li, Fixed-time control of delayed neural networks with impulsive perturbations, *Nonlinear Anal. Model. Control*, 23 (2018), 904–920.
11. X. D. Zhao, X. Y. Wang, L. Ma, G. D. Zong, Fuzzy approximation based asymptotic tracking control for a class of uncertain switched nonlinear systems, *IEEE T. Fuzzy Syst.*, 28 (2020), 632–644.
12. F. Wang, B. Chen, Y. Sun, Y. Gao, C. Lin, Finite-time fuzzy control of stochastic nonlinear systems, *IEEE T. Syst. Man Cybern.*, 50 (2020), 2617–2626.
13. L. Liu, W. Zheng, S. H. Ding, An adaptive SOSM controller design by using a sliding-mode-based filter and its application to buck converter, *IEEE T. Circ. Syst. I.*, 67 (2020), 2409–2418.
14. H. Y. Li, Y. Wu, M. Chen, Adaptive fault-tolerant tracking control for discrete-time multi-agent systems via reinforcement learning algorithm, *IEEE T. Syst. Man Cybern.*, (2020), 1–12.
15. Q. Zhou, W. Wang, H. Liang, M. Basin, B. Wang, Observer-based event-triggered fuzzy adaptive bipartite containment control of multi-agent systems with input quantization, *IEEE T. Fuzzy Syst.*, (2019), 1–1.
16. Z. F. Li, T. S. Li, G. Feng, R. Zhao, Q. H. Shan, Neural network-based adaptive control for pure-feedback stochastic nonlinear systems with time-varying delays and dead-zone input, *IEEE T. Syst. Man Cybern. Syst.*, 50 (2020), 5317–5329.
17. S. Sui, C. L. P. Chen, S. C. Tong, Fuzzy adaptive finite-time control design for non-triangular stochastic nonlinear systems, *IEEE T. Fuzzy Syst.*, 27 (2019), 172–184.
18. B. Niu, Y. J. Liu, W. L. Zhou, H. T. Li, P. Y. Duan, J. Q. Li, Multiple Lyapunov functions for adaptive neural tracking control of switched nonlinear non-lower-triangular systems, *IEEE T. Cybernetics*, 50 (2020), 1877–1886.
19. Y. M. Li, S. C. Tong, T. S. Li, Observer-based adaptive fuzzy tracking control of MIMO stochastic nonlinear systems with unknown control directions and unknown dead zones, *IEEE T. Fuzzy Syst.*, 23 (2015), 1228–1241.
20. X. D. Li, J. H. Shen, R. Rakkiyappan, Persistent impulsive effects on stability of functional differential equations with finite or infinite delay, *Appl. Math. Comput.*, 329 (2018), 14–22.
21. J. H. Park, H. Shen, X. H. Chang, T. H. Lee, *Recent advances in control and filtering of dynamic systems with constrained signals*, Switzerland: Springer, 2019.
22. L. D. Fang, L. Ma, S. H. Ding, J. H. Park, Finite-time stabilization of high-order stochastic nonlinear systems with asymmetric output constraints, *IEEE T. Syst. Man Cybern. Syst.*, (2020), 1–13.
23. C. C. Chen, A unified approach to finite-time stabilization of high-order nonlinear systems with and without an output constraint, *Int. J. Robust Nonlin. Control*, 29 (2019), 393–407.
24. S. H. Ding, J. H. Park, C. C. Chen, Second-order sliding mode controller design with output constraint, *Automatica*, 112 (2020), 108704.
25. L. B. Wu, J. H. Park, Adaptive fault-tolerant control of uncertain switched nonaffine nonlinear systems with actuator faults and time delays, *IEEE T. Syst., Man, Cybern., Syst.*, **50** (2020), 3470–3480.
26. X. Jin, Adaptive fault tolerant tracking control for a class of stochastic nonlinear systems with output constraint and actuator faults, *Syst. Control Lett.*, **107** (2017), 100–109.
27. S. H. Ding, K. Q. Mei, S. H. Li, A new second-order sliding mode and its application to nonlinear constrained systems, *IEEE T. Automat. Contr.*, **64** (2019), 2545–2552.
28. Y. M. Li, S. C. Tong, Adaptive fuzzy output constrained control design for multi-input multioutput stochastic nonstrict-feedback nonlinear systems, *IEEE T. Syst. Man Cybern.*, **47** (2017), 4086–4095.
29. B. Niu, W. Ding, H. Li, X. Xie, A novel neural-network-based adaptive control scheme for output-constrained stochastic switched nonlinear systems, *IEEE T. Syst. Man Cybern.*, **49** (2017), 418–432.
30. S. Yin, H. Yu, R. Shahnazi, A. Haghani, Fuzzy adaptive tracking control of constrained nonlinear switched stochastic pure-feedback systems, *IEEE T. Syst. Man Cybern.*, **47** (2017), 579–588.
31. Q. K. Hou, S. H. Ding, X. H. Yu, Composite super-twisting sliding mode control design for PMSM speed regulation problem based on a novel disturbance observer, *IEEE T. Energy Conver.*, (2020), 1–1.
32. D. Yang, X. D. Li, J. L. Qiu, Output tracking control of delayed switched systems via state-dependent switching and dynamic output feedback, *Nonlinear Anal. Hybri. Syst.*, **32** (2019), 294–305.
33. H. Wang, Q. Zhu, Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form, *Automatica*, **54** (2015), 284–291.
34. W. T. Zha, J. Y. Zhai, S. M. Fei, Output feedback control for a class of stochastic high-order nonlinear systems with time-varying delays, *Int. J. Robust Nonlin. Control*, **24** (2015), 2243–2260.
35. H. Wang, Q. Zhu, Global stabilization of stochastic nonlinear systems via $C^1$ and $C^\infty$ controllers, *IEEE T. Automat. Contr.*, **62** (2017), 5880–5887.
36. W. Q. Li, X. J. Xie, S. Y. Zhang, Output-feedback stabilization of stochastic high-order nonlinear systems under weaker conditions, *SIAM J. Control Optim.*, **49** (2011), 1262–1282.
37. M. Jiang, X. Xie, K. Zhang, Finite-time stabilization of stochastic high-order nonlinear systems with FT-SISS inverse dynamics, *IEEE T. Automat. Contr.*, **64** (2019), 313–320.
38. W. J. Si, X. D. Dong, F. F. Yang, Decentralized adaptive neural control for high-order interconnected stochastic nonlinear time-delay systems with unknown system dynamics, *Neural Netw.*, **99** (2018), 123–133.
39. N. Duan, H. F. Min, Decentralized adaptive NN state-feedback control for large-scale stochastic high-order nonlinear systems, *Neurocomputing*, **173** (2016), 1412–1421.
40. X. D. Zhao, X. Y. Wang, G. D. Zong, X. L. Zheng, Adaptive neural tracking control for switched high-order stochastic nonlinear systems, *IEEE T. Syst. Man Cybern.*, **47** (2017), 3088–3099.
41. L. D. Fang, L. Ma, S. H. Ding, D. A. Zhao, Finite-time stabilization for a class of high-order stochastic nonlinear systems with an output constraint, *Appl. Math. Comput.*, **358** (2019), 63–79.
42. C. C. Chen, Z. Y. Sun, A unified approach to finite-time stabilization of high-order nonlinear systems with an asymmetric output constraint, *Automatica*, **111** (2020), 108581.
Appendix

Proof of Proposition 1. Firstly, suppose there exist $\beta_{k-1}(3 \leq k \leq n)$ such that

$$
\ell V_{k-1} \leq -\frac{Q_1 \xi^4}{2\pi} \tan\left(\frac{\pi \xi^4}{2\varepsilon^4}\right) - \sum_{j=1}^{k-1} d_j \bar{\alpha}_j^2 - H \rho_{k-1} \xi^p_{k-1} - H \sum_{j=1}^{k-1} \rho_j \xi^p_j + \sum_{j=1}^{k-1} Q_j + H \xi^3_{k-1} (\alpha^p_k - \beta^p_{k-1}).
$$

(A.1)

At the same time, from Itô’s formula and (3.1), one gets

$$
d\xi_k = (H \chi^p_{k-1} + f_k - \ell \beta_{k-1}) dt + \left(h_k - \sum_{j=1}^{k-1} \frac{\partial \beta_{k-1}}{\partial \chi_j} \dot{h}_j\right) d\omega,
$$

(A.2)

where $\ell \beta_{k-1} = \sum_{j=1}^{k-1} \frac{\partial \beta_{k-1}}{\partial \chi_j} (H \chi^p_{j+1} + f_j) + \sum_{j=1}^{k-1} \frac{\partial \beta_{k-1}}{\partial \alpha_j} \dot{\alpha}_j + \frac{1}{2} \sum_{j=1}^{k-1} \frac{\partial^2 \beta_{k-1}}{\partial \chi_j^2} \dot{h}_j^T \dot{h}_j$. Clearly, $\ell \beta_{k-1}$ is valid and continuous, which can be illustrated in a similar way by following the lines to obtain $\ell \beta_1$.

We choose the Lyapunov function as

$$
V_k = V_{k-1} + \Lambda_k
$$

(A.3)

with

$$
\Lambda_k = \frac{1}{4} \xi^4_k + \frac{1}{2b_k} \bar{\alpha}_k^2,
$$

(A.4)

where $b_k > 0$ is an adjustment parameter, $\bar{\alpha}_k = \alpha_k - \hat{\alpha}_k$ is the parameter error, and $\hat{\alpha}_k$ is the estimation of the unknown parameter $\alpha_k$. 

Aims Mathematics Volume 6, Issue 3, 2244–2267.
Applying (2.3), (A.2) and (A.4), one can obtain
\[
\ell \Lambda_k = \xi_k^3 (H \xi_{k+1}^p + f_k - \ell \beta_{k-1}) - \frac{1}{b_k} \tilde{\alpha}_k \dot{\alpha}_k
\]
\[+ \frac{3}{2} \left\| h_k - \sum_{j=1}^{k-1} \frac{\partial \beta_{k-j}}{\partial \chi_j} h_j \right\|^2 \xi_k^2.
\] (A.5)

It is not difficult to get from Lemma 3 that
\[
\frac{3}{2} \left\| h_k - \sum_{j=1}^{k-1} \frac{\partial \beta_{k-j}}{\partial \chi_j} h_j \right\|^2 \xi_k^2 \leq \frac{3}{4} \left\| h_k - \sum_{j=1}^{k-1} \frac{\partial \beta_{k-j}}{\partial \chi_j} h_j \right\|^4 \xi_k^4 + \frac{3}{4}.
\] (A.6)

On the other hand, from Lemmas 2 and 3, one can verify
\[
\xi_k^{p-1} (\chi_p^k - \beta_{k-1}) \leq D|\xi_{k-1}|^3 \left[ |\xi_k|^p + \varphi_{p-1}^{-1} |\xi_{k-1}|^{p-1} \cdot |\xi_k| \right] - \rho_k \xi_k^{p+3} + \tau_k \xi_k^{p+3} + \frac{p \sigma_{k-1}^3 + \rho_k \xi_k^{p+3}}{p + 3} \xi_k^{p+3},
\] (A.7)

where \( \rho_{k-1} = \frac{1}{p+3} \left[ 3D^{p+3} \sigma_{1,2}^3 + p + 2 \right] \), \( \tau_k = \frac{\rho_k (p-1)^{p+3}}{p+3} \), and \( \sigma_{k-1} > 0 \) is an adjustment parameter.

From (A.1)-(A.7), it can be deduced that
\[
\ell V_k = \ell \Lambda_k + \ell \Lambda_k
\]
\[
\leq -\frac{g_1 e^4}{2\pi} \tan \left( \frac{\pi \xi_k^4}{2e^4} \right) - \sum_{j=1}^{k-1} \frac{d_j \dot{\alpha}_j}{2b_j} - H \sum_{j=1}^{k-1} \sigma_{k-1}^{p+3} + \sum_{j=1}^{k-1} Q_j + \frac{3}{4}
\]
\[+ H \frac{p \sigma_{k-1}^3 + \rho_k \xi_k^{p+3}}{p + 3} \xi_k^{p+3} + H \xi_k^3 F_k(Z_k) + H \xi_k^3 (\chi_{k+1}^p - \beta_{k}^0) + H \xi_k^3 \beta_{k}^0
\] (A.8)

where \( Z_k = (\chi_{k+1}^T, \tilde{\alpha}_{k-1}^T)^T, \tilde{\alpha}_{k-1} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{k-1})^T \) and
\[
F_k(Z_k) = \frac{1}{H} (f_k - \ell \beta_{k-1}) + \tau_k \xi_k^p + \frac{3}{4} \left\| h_k - \sum_{j=1}^{k-1} \frac{\partial \beta_{k-j}}{\partial \chi_j} h_j \right\|^4 \xi_k.
\]

Similar to the first two steps, \( F_k(Z_k) \) can also be approximated as
\[
F_k(Z_k) = ||\bar{\Psi}_k(Z_k)|| + \epsilon_k(Z_k),
\] (A.9)

where \( \epsilon_k(Z_k) \leq \delta_k \) and \( \delta_k > 0 \) is a given constant.

One directly obtains from Eq (A.9) and Lemma 3 that
\[
\xi_k^3 F_k(Z_k) \leq ||\bar{\Psi}_k|| ||\bar{\Psi}_k|| + \delta_k
\]
\[\leq \frac{3 \sigma_{k1} \alpha_k \xi_k^{p+3}}{p + 3} + \frac{p}{p + 3} \sigma_{k1}^{-1} \xi_k^{p+3} + \frac{3}{p + 3} \xi_k^{p+3} + \frac{p \sigma_{k1}^3 + \rho_k \xi_k^{p+3}}{p + 3} \xi_k^{p+3},
\] (A.10)

where \( \alpha_k = ||\bar{\Psi}_k||^{\frac{p+3}{2}} \) and \( \sigma_{k1} > 0 \) is an adjustment parameter.
Substituting (A.10) into (A.8) renders

\[
\ell V_k \leq \frac{q_1 e^4}{2\pi} \tan \left( \frac{\pi \xi_k^1}{2e^4} \right) - \sum_{j=1}^{k-1} \frac{d_j \hat{\alpha}_j^2}{2b_j} - H \sum_{j=1}^{k-1} \varrho_j \xi_j^{p+3} + H \xi_k^3 (\chi_{k+1} - \beta_k^p) \\
+ \frac{3H}{p + 3} \left[ \sigma_{k1} \hat{\alpha}_k + 1 \right] \xi_k^{p+3} + H \frac{p \alpha_{k1} - 1}{p + 3} \xi_k^{p+3} + H \xi_k^3 k + \sum_{j=1}^{k-1} Q_j
\]

(A.11)

Then, we can design

\[
\beta_k = -M_k^\frac{1}{2} \xi_k - \varrho_k \xi_k,
\]

\[
\hat{\alpha}_k = \frac{3Hb_k \sigma_{k1} \xi_k^{p+3}}{p + 3} - d_k \hat{\alpha}_k,
\]

(A.12)

where \( M_k \geq \frac{3}{p + 3} \left[ \sigma_{k1} \hat{\alpha}_k + 1 \right] + \frac{p \alpha_{k1} - 1}{p + 3} + \varrho_k \); \( \varrho_k, d_k > 0 \) are adjustment parameters; and the value of \( \varrho_k > 0 \) will be given in \( (k + 1) \)th step.

Moreover, one has

\[
\frac{d_k}{b_k} \hat{\alpha}_k \hat{\alpha}_k \leq - \frac{d_k}{2b_k} \hat{\alpha}_k^2 + \frac{d_k}{2b_k} \hat{\alpha}_k^2.
\]

(A.13)

From (A.1)-(A.13), it can be deduced that

\[
\ell V_k \leq -\frac{q_1 e^4}{2\pi} \tan \left( \frac{\pi \xi_k^1}{2e^4} \right) - \sum_{j=1}^{k-1} \frac{d_j \hat{\alpha}_j^2}{2b_j} - H \sum_{j=1}^{k} \varrho_j \xi_j^{p+3} - H \tau_k \xi_k^{p+3} + \sum_{j=1}^{k} Q_j + H \xi_k^3 (\chi_{k+1} - \beta_k^p),
\]

(A.14)

where \( Q_k = \frac{3}{4} + \frac{pH}{p + 3} \sigma_{k1}^\frac{1}{2} + \frac{pH}{p + 3} \delta_k^p + \frac{d_1 \alpha_1^2}{2b_1^2} \). The proof of Proposition 1 is completed.

**Proof of Proposition 2.** By the definition of \( U_n \), one has

\[
\ell U_n = H \gamma \sum_{i=2}^{n-1} \hat{e}_i^3 (\chi_i^p - \hat{\chi}_i^p) - H \gamma \sum_{i=2}^{n} \hat{e}_i^3 (\chi_i^p - \hat{\chi}_i^p) + \frac{3}{2} \gamma \sum_{i=2}^{n} \hat{e}_i^3 ||h_i - \gamma_i h_{i-1}||^2
\]

(A.15)

\[
= -H \sum_{i=2}^{n} \gamma \hat{e}_i^3 [(\hat{\chi}_i + \hat{e}_i) - \chi_i^p] - H \sum_{i=2}^{n} \gamma \hat{e}_i^3 [\chi_i^p - (\hat{\chi}_i + \hat{e}_i)]
\]

\[
+ H \sum_{i=3}^{n} \gamma \hat{e}_{i-1}^3 (\chi_i^p - \hat{\chi}_i^p) + H \sum_{i=2}^{n} \gamma \hat{e}_i^3 (f_i - \chi_i h_{i-1}) + H \sum_{i=2}^{n} \frac{3\gamma}{2H} \hat{e}_i^3 ||h_i - \gamma_i h_{i-1}||^2.
\]
By Lemma 5, one can infer

$$\nabla \gamma_i \hat{e}_i^3[(\hat{x}_i + \tilde{e}_i)^p - \hat{x}_i^p] = \nabla \gamma_i \hat{e}_i^3[\hat{x}_i + \tilde{e}_i - \hat{x}_i][(\hat{x}_i + \tilde{e}_i)^p - \hat{x}_i^p]$$

$$\leq \frac{\gamma_i}{2p-1} \hat{e}_i^{p+3}. \quad (A.16)$$

Since \(x_i - \hat{x}_i - \tilde{e}_i = \sum_{j=2}^{i-1} \gamma_i \cdots \gamma_{j+1} \tilde{e}_j\), we can get that through applying Lemmas 2-5

$$| - \nabla \gamma_i \hat{e}_i^3[\hat{x}_i^p - (\hat{x}_i + \tilde{e}_i)^p]|$$

$$\leq \gamma_i |\tilde{e}_i| (|x_i - \hat{x}_i - \tilde{e}_i|) (|x_i - \hat{x}_i - \tilde{e}_i|^{p-1} + \hat{x}_i^{p-1})$$

$$\leq \gamma_i |\tilde{e}_i|^3 \left( \sum_{j=2}^{i-1} \gamma_i \cdots \gamma_{j+1} |\tilde{e}_j| \right)^p$$

$$+ \gamma_i |\tilde{e}_i|^3 \left( \sum_{j=2}^{i-1} \gamma_i \cdots \gamma_{j+2} |\tilde{e}_j| \right)^p + \frac{p-1}{p} |\hat{x}_i|^p$$

$$\leq \frac{3 \gamma_i}{p+3} \hat{e}_i^{p+3} + \frac{p \gamma_i}{p+3} \left( \sum_{j=2}^{i-1} \gamma_i \cdots \gamma_{j+1} |\tilde{e}_j| \right)^{p+3} + \frac{3 \gamma_i}{p(p+3)} \hat{e}_i^{p+3}$$

$$+ \frac{3(p-1) \gamma_i}{p(p+3)} \hat{e}_i^{p+3} + \frac{p-1}{p} x_i^{p+3}$$

$$\leq \sum_{j=2}^{i} \tilde{a}_{ij} \hat{e}_j^{p+3} + \frac{p-1}{p+3} x_i^{p+3}, \quad (A.17)$$

where \(\tilde{a}_{ij} = \tilde{a}_{ij}(\gamma_i, \cdots, \gamma_{j+1})\) is a constant independent of \(H\).

Noting that \(e_i = \sum_{j=2}^{i-1} \gamma_i \cdots \gamma_{j+1} \tilde{e}_j + \tilde{e}_i\), one has

$$|\gamma \hat{e}_i^3[\hat{x}_i^p - \hat{x}_i^p]| = \gamma \hat{e}_i^3[|\hat{x}_i^p - (x_i - e_i)^p|]$$

$$\leq 2^p \gamma \hat{e}_i^3[|\hat{x}_i^p| + |e_i|^p]$$

$$\leq \frac{3(2p) \gamma}{p+3} \hat{e}_i^{p+3} + \frac{p}{p+3} \hat{e}_i^{p+3} + \frac{3(2p) \gamma}{p+3} \hat{e}_i^{p+3}$$

$$+ \frac{p}{p+3} \hat{e}_i^{p+3} + \frac{3(p-1)(2p) \gamma}{p(p+3)} \hat{e}_i^{p+3} + \frac{p-1}{p+3} x_i^{p+3}$$

$$\leq \sum_{j=2}^{i} \tilde{c}_{ij}(\gamma_i, \cdots, \gamma_{j+1}) \hat{e}_j^{p+3} + \frac{p}{p+3} x_i^{p+3}, \quad (A.18)$$

where \(\hat{c}_{ij} = \hat{c}_{ij}(\gamma_i, \cdots, \gamma_{j+1})\) is a constant independent of \(H\).
In addition, we have
\[
\frac{\gamma}{H} \tilde{e}_i^3 (f_i - \gamma_i f_{i-1}) + \frac{3}{2} \frac{\gamma}{H} \tilde{e}_i^2 ||h_i - \gamma_i h_{i-1}||^2 \\
\leq \frac{3\gamma^4}{2H^3} \tilde{e}_i^4 + \frac{1}{4} (f_i - \gamma_i f_{i-1})^4 + \frac{\gamma^3}{4H^3} ||h_i - \gamma_i h_{i-1}||^4 \\
\leq \frac{6\gamma^{n+3}}{(p+3)H^{n+2}} \tilde{e}_i^{p+3} + \frac{p-1}{p+3} + \frac{1}{4} (f_i - \gamma_i f_{i-1})^4 + \frac{3\gamma^2}{4H^3} ||h_i - \gamma_i h_{i-1}||^4.
\]
(A.19)

Substituting (A.16)-(A.19) into (A.15) yields
\[
\ell U_n \leq -H \left[ \sum_{i=2}^{n} \frac{\gamma \gamma_i}{2^{p-1}} \tilde{e}_i^{p+3} - \sum_{i=2}^{n} \tilde{c}_i \tilde{e}_i^{p+3} - \frac{6\gamma^{n+3}}{(p+3)H^{n+2}} \tilde{e}_i^{p+3} \right] + \frac{(p-1)H}{p+3} \\
+ \sum_{i=2}^{n} \frac{(2p-1)H}{p+3} \chi_i^{p+3} + \sum_{i=2}^{n} \frac{1}{4} (f_i - \gamma_i f_{i-1})^4 + \frac{3\gamma^2}{4H^3} ||h_i - \gamma_i h_{i-1}||^4 \right],
\]
where \(\tilde{c}_i = \sum_{j=2}^{i} (\tilde{a}_{ji} + \tilde{c}_{ji})\). The proof of Proposition 2 is completed.