Contextual bandits with surrogate losses: Margin bounds and efficient algorithms

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Abstract

We use surrogate losses to obtain several new regret bounds and new algorithms for contextual bandit learning. Using the ramp loss, we derive new margin-based regret bounds in terms of standard sequential complexity measures of a benchmark class of real-valued regression functions. Using the hinge loss, we derive an efficient algorithm with a $\sqrt{dT}$-type mistake bound against benchmark policies induced by $d$-dimensional regressors. Under realizability assumptions, our results also yield classical regret bounds.

1 Introduction

We study sequential prediction problems with partial feedback, mathematically modeled as contextual bandits (Langford and Zhang, 2008). In this formalism, a learner repeatedly (a) observes a context, (b) selects an action, and (c) receives a loss for the chosen action. The objective is to learn a policy for selecting actions with low loss, formally measured via regret with respect to a class of benchmark policies. Contextual bandit algorithms have been successfully deployed in online recommendation systems (Agarwal et al., 2016), mobile health platforms (Tewari and Murphy, 2017), and elsewhere.

In this paper, we use surrogate loss functions to derive new margin-based algorithms and regret bounds for contextual bandits. Surrogate loss functions are ubiquitous in supervised learning (cf. Zhang (2004); Bartlett et al. (2006); Schapire and Freund (2012)). Computationally, they are used to replace NP-hard optimization problems with tractable ones, e.g., the hinge loss makes binary classification amenable to convex programming techniques. Statistically, they also enable sharper generalization analysis for models including boosting, SVMs, and neural networks (Schapire and Freund, 2012; Anthony and Bartlett, 2009), by replacing dependence on dimension in VC-type bounds with distribution-dependent quantities. For example, to agnostically learn $d$-dimensional halfspaces the optimal rates for excess risk are $\sqrt{d/n}$ for the 0/1 loss benchmark and $\frac{1}{\gamma} \cdot \sqrt{d/n}$ for the $\gamma$-margin loss benchmark (Kakade et al., 2009), meaning the margin bound removes explicit dependence on dimension. Curiously, surrogate losses have seen limited use in partial information settings (some exceptions are discussed below). This paper demonstrates that these desirable computational and statistical properties indeed extend to contextual bandits.

In the first part of the paper we focus on statistical issues, namely whether any algorithm can achieve a generalization of the classical margin bound from statistical learning (Boucheron et al., 2005) in the
Technically, these results build on the non-constructive minimax analysis of Rakhlin et al. (2015b), which, for the online adversarial setting, prescribes a recipe for characterizing statistical behavior of arbitrary classes, and thus provides a counterpart to empirical risk minimization in statistical learning. Indeed, for full-information problems, this approach yields regret bounds in terms of sequential analogues of standard complexity measures including Rademacher complexity and metric entropy. However, since we work in the contextual bandit setting, we must extend these arguments to incorporate partial information. To do so, we leverage the adaptive minimax framework of Foster et al. (2015) along with a careful “adaptive” chaining argument.

In the second part of the paper, we focus on computational issues and derive two new algorithms using the hinge loss as a convex surrogate. The first algorithm, HINGE-LMC, provably runs in polynomial time and achieves a \( \sqrt{dT} \)-mistake bound against \( d \)-dimensional benchmark regressors with convexity properties. HINGE-LMC is the first efficient algorithm with \( \sqrt{dT} \)-mistake bound for bandit multiclass prediction using a surrogate loss without curvature, and so it provides a new resolution to the open problem of Abernethy and Rakhlin (2009). This algorithm is based on the exponential weights update, along with Langevin Monte Carlo for efficient sampling and a careful action selection scheme. The second algorithm is much simpler: in the stochastic setting, Follow-The-Leader with appropriate smoothing matches our information-theoretic results for sufficiently large classes.

1.1 Preliminaries

Let \( \mathcal{X} \) denote a context space and \( \mathcal{A} = \{1, \ldots, K\} \) a discrete action space. In the adversarial contextual bandits problem, for each of \( T \) rounds, an adversary chooses a pair \((x_t, \ell_t)\) where \( x_t \in \mathcal{X} \) is the context and \( \ell_t \in [0, 1]^K \equiv \mathcal{L} \) is a loss vector. The learner observes the context \( x_t \), chooses an action \( a_t \), and incurs loss \( \ell_t(a_t) \in [0, 1] \), which is also observed. The goal of the learner is to minimize the cumulative loss over the \( T \) rounds, and, in particular, we would like to design learning algorithms that achieve low regret against a class \( \Pi \subset (\mathcal{X} \to \mathcal{A}) \) of benchmark policies:

\[
\text{Regret}(T, \Pi) \triangleq \sum_{t=1}^{T} \mathbb{E}[\ell_t(a_t)] - \inf_{\pi \in \Pi} \sum_{t=1}^{T} \mathbb{E}[\ell_t(\pi(x_t))].
\]

In this paper, we always identify \( \Pi \) with a class of vector-valued regression functions \( \mathcal{F} \subset (\mathcal{X} \to \mathbb{R}^K) \), where \( \mathbb{R}^K_{=0} \triangleq \{ s \in \mathbb{R}^K : \sum a s_a = 0 \} \). We use the notation \( f(x) \in \mathbb{R}^K \) to denote the vector-valued output and \( f(x)_a \) to denote the \( a \)-th component. Note that we are assuming \( \sum_a f(x)_a = 0 \), which is a natural generalization of the standard formulation for binary classification (Bartlett et al., 2006) and appears in Pires et al. (2013). Define \( B \triangleq \sup_{f \in \mathcal{F}} \sup_{x \in \mathcal{X}} \|f(x)\|_\infty \) to be the maximum value predicted by any regressor.

Our algorithms use importance weighting to form unbiased loss estimates. If at round \( t \), the algorithm chooses action \( a_t \) by sampling from a distribution \( p_t \in \Delta(\mathcal{A}) \), the loss estimate is defined as \( \hat{\ell}_t(a) \triangleq \ell_t(a) I\{a_t =
We introduce two surrogate loss functions, the ramp loss and the hinge loss, whose scalar versions are defined as $\phi^\gamma(s) \triangleq \min(\max(1 + s/\gamma, 0), 1)$ and $\psi^\gamma(s) \triangleq \max(1 + s/\gamma, 0)$ respectively, for $\gamma > 0$. For $s \in \mathbb{R}^K$, $\phi^\gamma(s)$ and $\psi^\gamma(s)$ are defined coordinate-wise. We start with a simple lemma, demonstrating how $\phi^\gamma, \psi^\gamma$ act as surrogates for cost-sensitive multiclass losses.

**Lemma 1 (Surrogate Loss Translation).** For $s \in \mathbb{R}^K_{+}$, define $\pi_{ramp}(s)$, $\pi_{hinge}(s) \in \Delta(A)$ by $\pi_{ramp}(s)_a \propto \phi^\gamma(s_a)$ and $\pi_{hinge}(s)_a \propto \psi^\gamma(s_a)$. For any vector $\ell \in \mathbb{R}^K_{+}$, we have

$$
\langle \pi_{ramp}(s), \ell \rangle \leq \langle \ell, \phi^\gamma(s) \rangle \leq \sum_{a \in A} \ell(a) \mathbf{1}\{s_a \geq -\gamma\}, \quad \text{and} \quad \langle \pi_{hinge}(s), \ell \rangle \leq K^{-1} \langle \ell, \psi^\gamma(s) \rangle.
$$

Based on this lemma, it will be convenient to define $L_T^\gamma(f) \triangleq \sum_{t=1}^T \sum_{a \in A} \ell_t(a) \mathbf{1}\{f(x_t)_a \geq -\gamma\}$, which is the **margin-based cumulative loss** for the regressor $f$. $L_T^\gamma$ should be seen as a cost-sensitive multiclass analogue of the classical margin loss in statistical learning (Boucheron et al., 2005). We use the term “surrogate loss” here because these quantities upper bound the cost-sensitive loss: $\ell(\arg\max_a s_a) \leq \langle \ell, \phi^\gamma(s) \rangle \leq \langle \ell, \psi^\gamma(s) \rangle$.\footnote{On a related note, the information-theoretic results we present are also compatible with the surrogate function $\theta^\gamma(s)_a := \max\{1 + (s_a - \max_{a'} s_{a'})/\gamma, 0\}$, which also satisfies $\ell(\arg\max_a s_a) \leq \langle \ell, \theta^\gamma(s) \rangle$. This leads to a perhaps more standard notion of multiclass margin bound but does not lead to efficient algorithms.}

In the sequel, $\pi_{ramp}$ and $\pi_{hinge}$ are used by our algorithms, but do not define the benchmark policy class, since we compare directly to $L_T^\gamma$ or the surrogate loss.

**Related work.** Contextual bandit learning has been the subject of intense investigation over the past decade. The most natural categorization of these works is between parametric, realizability-based, and agnostic approaches. Parametric methods (e.g., Abbasi-Yadkori et al. (2011); Chu et al. (2011)) assume a (generalized) linear relationship between the losses and the contexts/actions. Realizability-based methods generalize parametric ones by assuming the losses are predictable by some abstract regression class (Agarwal et al., 2012; Foster et al., 2018a). Agnostic approaches (e.g., Auer et al. (2002); Langford and Zhang (2008); Agarwal et al. (2014); Rakhlin and Sridharan (2016); Syrgkanis et al. (2016a,b)) avoid realizability assumptions and instead compete with VC-type policy classes for statistical tractability. Our work contributes to all of these directions, as our margin bounds apply to the agnostic adversarial setting and yield true regret bounds under realizability assumptions.

A special case of contextual bandits is **bandit multiclass prediction**, where the loss vector is zero for one action and one for all others (Kakade et al., 2008). Several recent papers obtain surrogate regret bounds and efficient algorithms for this setting when the benchmark regressor class $\mathcal{F}$ consists of linear functions (Kakade et al., 2008; Hazan and Kale, 2011; Beygelzimer et al., 2017; Foster et al., 2018b). Our work contributes to this line in two ways: our bounds and algorithms extend beyond linear/parametric classes, and we consider the more general contextual bandit setting.

Our information-theoretic results on achievability are similar in spirit those of Daniely and Halbertal (2013), who derive tight generic bounds for bandit multiclass prediction in terms of the Littlestone dimension. This result is incomparable to our own: their bounds are on the 0/1 loss regret directly rather than surrogate regret, but the Littlestone dimension is not a tight complexity measure for real-valued function classes in agnostic settings, which is our focus.

At a technical level, our work builds on several recent results. To derive achievable regret bounds, we use the adaptive minimax framework of Foster et al. (2015), along with a new adaptive chaining argument to
control the supremum of a martingale process (Rakhlin et al., 2015b). Our HINGE-LMC algorithm is based on log-concave sampling (Bubeck et al., 2018), and it uses randomized smoothing (Duchi et al., 2012) and the geometric resampling trick of Neu and Bartók (2013). We also use several ideas from classification calibration (Zhang, 2004; Bartlett et al., 2006), and, in particular, the surrogate hinge loss we work with is studied by Pires et al. (2013).

2 Achievable regret bounds

This section provides generic surrogate regret bounds for contextual bandits in terms of the sequential metric entropy (Rakhlin et al., 2015a) of the regressor class $\mathcal{F}$. Notably, our general techniques apply when the ramp loss is used as a surrogate, and so, via Lemma 1, they yield the main result of the section—a margin-based regret guarantee—as a special case.

To motivate our approach, consider a well-known reduction from bandits to full information online learning: If a full information algorithm achieves a regret bound in terms of the so-called local norms $\sum_t (p_t, \ell_t^2)$, then running the full information algorithm on importance-weighted losses $\hat{\ell}_t(a)$ yields an expected regret bound for the bandit setting. For example, when $\Pi$ is finite, EXP4 (Auer et al., 2002) uses HEDGE (Freund and Schapire, 1997) as the full information algorithm, and obtains a deterministic regret bound of

$$\text{Regret}(T, \Pi) \leq \frac{\eta}{2} \sum_{t=1}^T \mathbb{E}_{\pi \sim p_t} \left( \hat{\ell}_t \right)^2 + \frac{\log(|\Pi|)}{\eta},$$

(1)

where $\eta > 0$ is the learning rate and $p_t$ is the distribution over policies in $\Pi$ (inducing an action distribution) for round $t$. Evaluating conditional expectations and optimizing $\eta$ yields a regret bound of $O(\sqrt{KT \log(|\Pi|)})$, which is optimal for contextual bandits with a finite policy class.

To use this reduction beyond the finite class case and with surrogate losses we face two challenges:

1. **Infinite classes.** The natural approach of using a pointwise (or sup-norm) cover for $\mathcal{F}$ is insufficient—not only because there are classes that have infinite pointwise covers yet are online-learnable, but also because it yields sub-optimal rates even when a finite pointwise cover is available. Instead, we establish existence of a full-information algorithm for large nonparametric classes that has 1) strong adaptivity to loss scaling as in (1) and 2) regret scaling with the sequential covering number for $\mathcal{F}$, which is the correct generalization of the empirical covering number in statistical learning to the adversarial online setting. This is achieved via non-constructive methods.

2. **Variance control.** With surrogate losses, controlling the variance/local norm term $\mathbb{E}_{\pi \sim p_t} (\hat{\ell}_t) \leq \eta$ in the reduction from bandit to full information is more challenging, since the surrogate loss of a policy depends on the scale of the underlying regressor, not just the action it selects. To address this, we develop a new sampling scheme tailored to scale-sensitive losses.

**Full-information regret bound.** We consider the following full information protocol, which in the sequel will be instantiated via reduction from contextual bandits. Let the context space $\mathcal{X}$ and $\mathcal{A}$ be fixed as in Subsection 1.1, and consider a function class $\mathcal{G} \subset (\mathcal{X} \to \mathcal{S})$, where $\mathcal{S} \subseteq \mathbb{R}^K_+$. The reader may think of $\mathcal{G}$ as representing $\phi \circ \mathcal{F}$ or $\psi \circ \mathcal{F}$, i.e. the surrogate loss composed with the regressor class, so that $\mathcal{S}$ (which is not necessarily convex) represents the image of the surrogate loss over $\mathcal{F}$.

The online learning protocol is: For time $t = 1, \ldots, T$, (1) the learner observes $x_t$ and chooses a distribution $p_t \in \Delta(S)$, (2) the adversary picks a loss vector $\ell_t \in \mathcal{L} \subset \mathbb{R}^K_+$, (3) the learner samples outcome $s_t \sim p_t$ and
experiences loss \((s_t, \ell_t)\). Regret against the benchmark class \(\mathcal{G}\) is given by

\[
\sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_t} (s_t, \ell_t) - \inf_{g \in \mathcal{G}} \sum_{t=1}^{T} g(x_t), \ell_t).
\]

As our complexity measure, we use a multi-output generalization of sequential covering numbers introduced by Rakhlin et al. (2015a). Define a \(\mathcal{Z}\)-valued tree \(z\) to be a sequence of mappings \(z_t : \{\pm 1\}^{t-1} \rightarrow \mathcal{Z}\). The tree \(z\) is a complete rooted binary tree with nodes labeled by elements of \(\mathcal{Z}\), where for any “path” \(\epsilon \in \{\pm 1\}^{T}\), \(z_t(\epsilon) \triangleq z_t(\epsilon_{1:t-1})\) is the value of the node at level \(t\) on the path \(\epsilon\).

**Definition 1.** For a function class \(\mathcal{G} \subset (X \rightarrow \mathbb{R}^K)\) and \(\mathcal{X}\)-valued tree of length \(T\), the \(L_\infty/\ell_\infty\) sequential covering number\(^2\) for \(\mathcal{G}\) on \(x\) at scale \(\varepsilon\), denoted by \(N_{\infty,\infty}(\varepsilon, \mathcal{G}, x)\), is the cardinality of the smallest set \(\mathcal{V}\) of \(\mathbb{R}^K\)-valued trees for which

\[
\forall g \in \mathcal{G} \forall \epsilon \in \{\pm 1\}^T \exists v \in \mathcal{V} \text{ s.t. } \max_{t \in [T]} \|g(x_t(\epsilon)) - v_t(\epsilon)\|_\infty \leq \varepsilon.
\]

Define \(N_{\infty,\infty}(\varepsilon, \mathcal{G}, T) \triangleq \sup_{x: \text{length}(x) = T} N_{\infty,\infty}(\varepsilon, \mathcal{G}, x)\).

We refer to \(\log N_{\infty,\infty}\) as the sequential metric entropy. Note that in the binary case, for learning unit \(\ell_2\) norm linear functions in \(d\) dimensions, the pointwise metric entropy is \(O(d \log(1/\varepsilon))\), whereas the sequential metric entropy is \(O(d \log(1/\varepsilon) \wedge \varepsilon^{-2} \log(d))\), leading to improved rates in high dimension.

With this definition, we can now state our main theorem for full information.

**Theorem 2.** Assume\(^3\) \(\sup_{t, x} \|s\|_1 \leq R\) and \(\sup_{s \in \mathcal{G}} \|s\|_\infty \leq B\). Fix any constants \(\eta \in (0, 1], \lambda > 0\), and \(\beta > \alpha > 0\). Then there exists an algorithm with the following deterministic regret guarantee:

\[
\sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_t} (s_t, \ell_t) - \inf_{g \in \mathcal{G}} \sum_{t=1}^{T} g(x_t), \ell_t) \leq \frac{2R}{\eta} \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_t} (s_t, \ell_t)^2 + \frac{4RB}{\eta} \log N_{\infty,\infty}(\beta/2, \mathcal{G}, T) + 3e^2 \alpha \sum_{t=1}^{T} \|\ell_t\|_1
\]

\[
+ 24e \left( \frac{\lambda}{4R} \sum_{t=1}^{T} \|\ell_t\|_1^2 + \frac{R}{\lambda} \right) \int_{\alpha}^{\beta} \sqrt{\log N_{\infty,\infty}(\varepsilon, \mathcal{G}, T)} d\varepsilon.
\]

Observe that the bound involves the variance/local norms \(\mathbb{E}_{s_t \sim p_t} (s_t, \ell_t)^2\), and has a very mild explicit dependence on the loss range \(R\); this can be verified by optimizing over \(\eta\) and \(\lambda\). This adaptivity to the loss range is crucial for our bandit reduction. Further observe that the bound contains a Dudley-type entropy integral, which is essential for obtaining sharp rates for complex nonparametric classes.

**Bandit reduction and variance control.** To lift Theorem 2 to contextual bandits we use the following reduction: First, initialize the full information algorithm from Theorem 2 with \(\mathcal{G} = \phi^\gamma \circ \mathcal{F}\). For each round \(t\), receive \(x_t\), and define \(P_t(a) \triangleq \mathbb{E}_{s_t \sim p_t} \frac{s_t(a)}{\sum_{a' \in \mathcal{K}} s_t(a')}\) where \(p_t\) is the full information algorithm’s distribution.

Then sample \(a_t \sim P_t^\mu\), observe \(\ell_t(a_t)\), and pass the importance-weighted loss \(\hat{\ell}_t(a)\) back to the algorithm. For the hinge loss we use the same strategy, but with \(\mathcal{G} = \psi^\gamma \circ \mathcal{F}\).

The following lemma shows that this strategy leads to sufficiently small variance in the loss estimates. The definition of the action distribution \(P_t^\mu(a)\) in terms of the real-valued predictions is crucial here.

\(^2\)Sequential coverings for \(L_\infty/\ell_\infty\) can be defined similarly, but do not appear in the present paper.

\(^3\)Measuring loss in \(\ell_1\) may seem restrictive, but it is natural when working with the 1-sparse importance-weighted losses, and it enables us to cover the output space in \(\ell_\infty\) norm.
Lemma 3. Define a filtration \( \mathcal{J}_t = \sigma((x_1, \ell_1, a_1), \ldots, (x_{t-1}, \ell_{t-1}, a_{t-1}), x_t, \ell_t) \). Then for any \( \mu \in [0, 1/K] \) the importance weighting strategy above guarantees

\[
\mathbb{E}_{a_t \sim P_t}[\mathbb{E}_{s_t \sim P_t}(s_t, \ell_t)^2 | \mathcal{J}_t] \leq \begin{cases} 
K^2, & \text{for } S \in \Delta(A), \\
K^2, & \text{for } S = \phi^\gamma \circ \mathcal{F}, \\
(1 + \mathbb{E})^2 K^2, & \text{for } S = \psi^\gamma \circ \mathcal{F}.
\end{cases}
\]

Theorem 2 and Lemma 3 together imply our central theorem: a chaining-based margin bound for contextual bandits, generalizing classical results in statistical learning (cf. (Boucheron et al., 2005)).

Theorem 4 (Contextual bandit margin bound). For any fixed constants \( \beta > \alpha > 0 \), smoothing parameter \( \mu \in (0, 1) \) and margin loss parameter \( \gamma > 0 \) there exists an adversarial contextual bandit strategy with expected regret against the \( \gamma \)-margin benchmark bounded as

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(a_t) \right] \leq \inf_{f \in \mathcal{F}} \mathbb{E} \left[ L^\gamma_T(f) \right] + 4\sqrt{2K^2 T \log N_{\infty, \infty}(\beta/2, \mathcal{F}, T)} + \mu KT \\
+ \frac{8}{\mu} \log N_{\infty, \infty}(\beta/2, \mathcal{F}, T) + 1 \left( 3c^2 \alpha KT + 24e \sqrt{\frac{KT}{\mu}} \int_0^\alpha \sqrt{\log N_{\infty, \infty}(\varepsilon, \mathcal{F}, T)} d\varepsilon \right). \tag{3}
\]

We derive an analogous bound for the hinge loss in Appendix C. The hinge loss bound differs only through stronger dependence on scale parameters.

Before showing the implications of Theorem 4 for specific classes \( \mathcal{F} \) we state a coarse upper bound in terms of the growth rate for the sequential metric entropy.

Proposition 5. Suppose that \( \mathcal{F} \) has sequential metric entropy growth \( \log N_{\infty, \infty}(\varepsilon, \mathcal{F}, T) \propto \varepsilon^{-p} \) for some \( p > 0 \) (nonparametric case), or that \( \log N_{\infty, \infty}(\varepsilon, \mathcal{F}, T) \propto d \log(1/\varepsilon) \) (parametric case). Then there exists a contextual bandit strategy with the following regret guarantee:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(a_t) \right] \leq \inf_{f \in \mathcal{F}} \mathbb{E} \left[ L^\gamma_T(f) \right] + \begin{cases} 
O(K \sqrt{dT \log (KT/\gamma)}), & \text{parametric case.} \\
O((KT)^{\frac{p^2}{p^2 + 1} \gamma^2 - \frac{2p}{p^2 + 1}}), & \text{nonparametric w/ } p \leq 2. \\
O((KT)^{\frac{p^2}{p^2 + 2}}} - \frac{2p}{p^2 + 2})}, & \text{nonparametric w/ } p \geq 2.
\end{cases} \tag{4}
\]

Proposition 5 recovers the parametric rate of \( \sqrt{dT} \) seen with e.g., LINUCB (Chu et al., 2011) but is most interesting for complex classes. The rate exhibits a phase change between the “moderate complexity” regime of \( p \in (0, 2] \) and the “high complexity” regime of \( p \geq 2 \). This is visualized in Figure 1.
Remark 1. Under i.i.d. losses and hinge/ramp loss realizability, the standard tools of classification calibration (Bartlett et al., 2006) can be used to deduce a proper policy regret bound from (3). However, these realizability assumptions are somewhat non-standard, and moreover if one imposes the stronger assumption of a hard margin it is possible to derive improved rates (Daniely and Halbertal, 2013). See also Appendix B.

Remark 2. Classical margin bounds typically hold for all values of $\gamma$ simultaneously, but Theorem 4 requires that $\gamma$ is chosen in advance. Learning the best value of $\gamma$ online appears challenging.

Rates for specific classes. We now instantiate our results for concrete classes of interest.

Example 1 (Finite classes). In the finite class case there is an algorithm with $O\left(K \sqrt{T \log |F|}\right)$ margin regret. When $\Pi \subset (X \rightarrow A)$ is a finite policy class, directly reducing to Theorem 2 yields the optimal $O\left(\sqrt{KT \log |\Pi|}\right)$ policy regret, hinting at the optimality of our approach.

Example 2 (Lipschitz CB). The class of Lipschitz functions over $[0,1]^p$ admits a sequential cover with metric entropy $\tilde{O}(\varepsilon^{-p})$, so Proposition 5 implies an $\tilde{O}(T^{p/2} + \varepsilon^{-p})$ regret bound. Since our proof goes through Lemma 1, it also yields a policy regret bound against the $\pi_{\text{ramp}}(\cdot)$ class. Therefore, this result is directly comparable to the $\tilde{O}(T^{p/2})$ bound of Cesa-Bianchi et al. (2017), applied to the $\pi_{\text{ramp}}$ policy class. Our bound achieves a smaller exponent for all values of $p$ (see Figure 1).

Learnability in full information online learning is known to be characterized entirely by the sequential Rademacher complexity of the hypothesis class (Rakhlin et al., 2015a), and tight bounds on this quantity are known for standard classes including linear predictors, decision trees, and neural networks. The next example, a corollary of Theorem 4, bounds contextual bandit margin regret in terms of sequential Rademacher complexity, which is defined for any scalar-valued function class $G \subseteq (X \rightarrow \mathbb{R})$ as:

$$\mathcal{R}(G,T) \doteq \sup_{x} \mathbb{E}_{x} \sup_{g \in G} \sum_{t=1}^{T} \epsilon_{t} g(x_{t}(\cdot)).$$

Example 3. Let $F_{|a\rangle} \doteq \{x \mapsto f(x)_{a} \mid f \in F\}$ be the scalar restriction of $F$ to output coordinate $a$ and suppose that $\max_{a \in [K]} \mathcal{R}(F_{|a\rangle},T) \geq 1$ and $B \leq 1.4$. Then there exists an adversarial contextual bandit algorithm with margin regret bound $\tilde{O}\left(\max_{a} K(\mathcal{R}(F_{|a\rangle},T)/\gamma)^{2/3} T^{1/3}\right)$. Thus, for margin-based contextual bandits, full information learnability is equivalent to bandit learnability. Since the optimal regret in full information is $\Omega(\max_{a} \mathcal{R}(F_{|a\rangle},T))$, it further shows that the price of bandit information is at most $\tilde{O}\left(\max_{a} K(T/\mathcal{R}(F_{|a\rangle},T))^{1/3}\right)$. Note that while this bound is fairly user-friendly, it yields worse rates than Proposition 5 when translated to sequential metric entropy, except when $p = 2$ (Rakhlin et al., 2010). For comparison, Rakhlin et al. (2015a) obtain $\tilde{O}(\mathcal{R}(F,T)/\gamma)$ margin regret in full information for binary classification. For partial information, BISTRO (Rakhlin and Sridharan, 2016) has an $O\left(\sqrt{KT\mathcal{R}(\Pi,T)}\right)$ policy regret bound, which involves the policy complexity and a worse $T$ dependence than our bound, but our bound (in terms of $F$) applies only to the margin regret. A similar discussion applies to Theorem 4.4 of Lykouris et al. (2018).

Instantiating Example 3 with linear classes generalizes the $O(T^{2/3})$ dimension-independent guarantee of BANDITRON (Kakade et al., 2008) from Euclidean geometry to arbitrary uniformly convex Banach spaces, essentially the largest linear class for which online learning is possible (Srebro et al., 2011). The result also generalizes BANDITRON from multiclass to general contextual bandits and strengthens it from hinge loss to ramp loss. Note that many subsequent works (Abernethy and Rakhlin, 2009; Beygelzimer et al., 2017; Foster et al., 2018b) obtain dimension-dependent $O(\sqrt{KT})$ bounds for bandit multiclass prediction, as we
will in the next section, but, none have explored dimension-independent $O(T^{2/3})$-type rates, which are more appropriate for high-dimensional settings.

**Example 4.** Let $\mathcal{X}$ be the unit ball in a Banach space $(\mathcal{B}, \|\cdot\|)$, and let $\mathcal{F}$ be induced by stacking $K - 1$ linear predictors each in the unit ball of the dual space $(\mathcal{B}^*, \|\cdot\|_\ast)$. Suppose that $\|\cdot\|$ has martingale type $2$ (Pisier, 1975), which means there exists $\Psi : \mathcal{B} \to \mathbb{R}$ such that $\frac{1}{2} \|x\|^2 \leq \Psi(x)$ and $\Psi$ is $\beta$-smooth w.r.t. $\|\cdot\|$. Then there exists a contextual bandit strategy with margin regret $O(K(T/\gamma)^{2/3})$.

Beyond linear classes, we also obtain $\tilde{O}(K(T/\gamma)^{2/3})$ margin regret when each $\mathcal{F}_a$ is a class of neural networks with weights in each layer bounded in the $(1, \infty)$ group norm, or when each $\mathcal{F}_a$ is a class of bounded depth decision trees on finitely many decision functions. These results follow by appealing to the existing sequential Rademacher complexity bounds derived in Rakhlin et al. (2015a).

As our last example, we consider $\ell_p$ spaces for $p < 2$. These spaces fail to satisfy martingale type $2$ in a dimension-independent fashion, but they do satisfy martingale type $p$ without dimension dependence, and so have sequential metric entropy of order $\varepsilon^{-\frac{p}{p-1}}$ (Rakhlin and Sridharan, 2017). Moreover, in $\mathbb{R}^d$ the $\ell_p$ spaces admit a pointwise cover with metric entropy $O(d \log(1/\varepsilon))$, leading to the following dichotomy.

**Example 5.** Consider the setting of Example 4, with $(\mathcal{B}, \|\cdot\|) = (\mathbb{R}^d, \|\cdot\|_p)$ for $p \leq 2$. Then there exists a contextual bandit strategy with margin regret $\tilde{O}(K(T/\gamma)^{p-1} \wedge K \sqrt{dT \log(KT/\gamma)})$.

## 3 Efficient algorithms

We derive two new algorithms for contextual bandits using the hinge loss $\psi^\gamma$. The first algorithm, HINGE-LMC, focuses on the parametric setting; it is based on a continuous version of exponential weights using a log-concave sampler. The second, SMOOTHFTL, is simply Follow-The-Leader with uniform smoothing. SMOOTHFTL applies to the stochastic setting with classes that have “high complexity” in the sense of Proposition 5.

### 3.1 Hinge-LMC

For this section, we identify $\mathcal{F}$ with a compact convex set $\Theta \subset \mathbb{R}^d$, using the notation $f(x; \theta) \in \mathbb{R}^K_{\geq 0}$ to describe the parametrized function. We assume that $\psi^\gamma(f(x; \theta)_a)$ is convex in $\theta$ for each $(x, a)$ pair, $\sup_{x, \theta} \|f(x; \theta)\|_\infty \leq B$, $f(x; \cdot)_a$ is $L$-Lipschitz in $\theta$ with respect to the $\ell_2$ norm, and that $\Theta$ contains the centered Euclidean ball of radius $1$ and is contained within a Euclidean ball of radius $R$. These assumptions are satisfied when $\mathcal{F}$ is a linear class, under appropriate boundedness conditions.

The pseudocode for HINGE-LMC is displayed in Algorithm 1, and all parameters settings are given in Appendix D. The algorithm is a continuous variant of exponential weights (Auer et al., 2002), where at round $t$, we define the exponential weights distribution via its density (w.r.t. the Lebesgue measure over $\Theta$):

$$P_t(\theta) \propto \exp(-\eta w_{t-1}(\theta)), \quad w_{t-1}(\theta) \triangleq \sum_{s=1}^{t-1} \ell_s, \psi^\gamma(f(x_s; \theta)).$$

---

5Only $K - 1$ predictors are needed due to the sum-to-zero constraint of $\mathbb{R}^K_{\geq 0}$.

6Norms that satisfy this property with dimension-independent or logarithmic constants include $\ell_p$ for all $p \geq 2$, Schatten $S_p$ norms for $p \geq 2$ (including the spectral norm), and $(2, p)$ group norms for $p \geq 2$ (Kakade et al., 2009, 2012).
Algorithm 1 HINGE-LMC

Input: Class $\Theta$, learning rate $\eta$, margin parameter $\gamma$.
Define $w_0(\theta) \triangleq 1$ for all $\theta \in \Theta$.

for $t = 1, \ldots, T$ do

Receive $x_t$, set $\theta_t \leftarrow \text{LMC}(\eta w_{t-1})$.
Set $p_t(\cdot; \theta_t) \propto \psi^\gamma(f(x_t; \theta_t))$.
Set $p_t^\mu(\cdot; \theta_t) \propto (1 - K \mu)p_t + \mu$.
Play $a_t \sim p_t^\mu(\cdot; \theta_t)$, observe $\ell_t(a_t)$.

// Geometric resampling.
for $m = 1, \ldots, M$ do

Sample $\tilde{a}_t \sim p_t^\mu(\cdot; \theta_t)$, if $\tilde{a}_t = a_t$, break.
end for

Set $m_t = m$, and $\tilde{\ell}_t(a) \triangleq \ell_t(a_t) \cdot m_t \{a_t = a\}$.
Update $w_t(\theta) \leftarrow w_{t-1}(\theta) + \{\ell_t, \psi^\gamma(f(x_t; \theta))\}$.
end for

$\tilde{\theta}_n$ is a loss vector estimate. At a high level, at each iteration the algorithm samples $\tilde{a}_t \sim P_t$, then samples the action $a_t$ from the induced policy distribution $p_t(\cdot; \theta_t) = \pi_{\text{hinge}}(f(x_t; \theta_t))$, appropriately smoothed. The algorithm plays $a_t$ and constructs a loss estimate $\tilde{\ell}_t \triangleq m_t \cdot \ell_t(a)1\{a_t = a\}$, where $m_t$ is an approximate importance weight computed by repeatedly sampling from $P_t$. This vector $\tilde{\ell}_t$ is passed to exponential weights to define the distribution at the next round. To sample from $P_t$ we use Projected Langevin Monte Carlo (LMC), displayed in Algorithm 2.

The algorithm has many important subtleties. Apart from passing to the hinge surrogate loss to obtain a tractable log-concave sampling problem, by using the induced policy distribution $\pi_{\text{hinge}}(\cdot)$, we are also able to control the local norm term in the exponential weights regret bound. Then, the analysis for Projected LMC Bubeck et al. (2018) requires a smooth potential function, which we obtain by convolving with the gaussian density, also known as randomized smoothing (Duchi et al., 2012). We also use $\ell_2$ regularization for strong convexity and to overcome sampling errors introduced by randomized smoothing. Finally, we use the geometric resampling technique (Neu and Bartók, 2013) to approximate the importance weight by repeated sampling.

Here, we state the main guarantee and its consequences. A more complete theorem statement, with exact parameter specifications and the precise running time is provided in Appendix D as Theorem 18.

**Theorem 6** (Informal). Under the assumptions of Subsection 3.1, HINGE-LMC with appropriate parameter settings runs in time poly$(T, d, B, K, \frac{1}{\gamma}, R, L)$ and guarantees

$$
\mathbb{E} \sum_{t=1}^{T} \ell_t(a_t) \leq \inf_{\theta \in \Theta} \frac{1}{K} \mathbb{E} \sum_{t=1}^{T} \{\ell_t, \psi^\gamma(f(x_t; \theta))\} + \tilde{O}\left(\frac{B}{\gamma} \sqrt{dT}\right).
$$

Since bandit multiclass prediction is a special case of contextual bandits, Theorem 6 immediately implies a $\sqrt{dT}$-mistake bound for this setting. See Appendix B for more discussion.

**Corollary 7** (Bandit multiclass). In the bandit multiclass setting, Algorithm 1 enjoys a mistake bound of $\tilde{O}\left((B/\gamma) \sqrt{dT}\right)$ against the cost-sensitive $\gamma$-hinge loss and runs in polynomial time.

---

Algorithm 2 Langevin Monte Carlo (LMC)

Input: $F(\cdot)$, parameters $m, u, \lambda, N, \alpha$.

// Parameter choices are in Appendix D.

Set $\tilde{\theta}_0 \leftarrow 0 \in \mathbb{R}^d$

for $k = 1, \ldots, N$ do

Draw $z_1, \ldots, z_m \mu \sim \mathcal{N}(0, u^2 I_d)$. Define

$$
\tilde{\theta}_k \mu \sim \mathcal{N}(0, I_d) \quad \text{and update}
$$

$$
\tilde{\theta}_k \leftarrow \mathcal{P}_\Theta\left(\tilde{\theta}_{k-1} - \frac{\alpha}{2} \nabla F(\theta_{k-1}) + \sqrt{\alpha} \xi_k\right).
$$

end for

Return $\hat{\theta}_N$.

---

This seems specialized to surrogates that can be expressed as an inner product between the loss vector and (a transformation of) the prediction, so it does not apply to standard loss functions in bandit multiclass prediction.
Additionally, under a realizability condition for the hinge loss, we obtain a standard regret bound. For simplicity in defining the condition, assume that for every \((x, \ell)\) pair, \(\ell\) is a random variable with conditional mean \(\tilde{\ell}\) (chosen by the adversary) and \(\tilde{\ell}\) has a unique action with minimal loss.

**Corollary 8** (Realizable bound). In addition to the conditions above, assume that there exists \(\theta^* \in \Theta\) such that for every \((x, \ell)\) pair and for all \(a \in A\), we have \(f(x; \theta^*)_a \equiv K \gamma \mathbf{1}_\{\tilde{\ell}(a) \leq \min_a \tilde{\ell}(a')\} - \gamma\). Then \(\text{HINGE-LMC}\) runs in polynomial time and guarantees

\[
\sum_{t=1}^{T} \mathbb{E} \tilde{\ell}_t(a_t) \leq \sum_{t=1}^{T} \mathbb{E} \min_a \tilde{\ell}(a) + \tilde{O}\left(\frac{B}{\gamma} \sqrt{dT}\right).
\]

A few comments are in order:

1. The use of LMC for sampling is not strictly necessary. Other log-concave samplers do exist for nonsmooth potentials (Lovász and Vempala, 2007), which will remove the parameters \(m, u, \lambda\), significantly simplify the algorithm, and even lead to a better run-time guarantee using current theory. However, we prefer to use LMC due to its success in Bayesian inference and deep learning, and its connections to incremental optimization methods. Note that more recent results in slightly different settings (Raginsky et al., 2017; Dalalyan and Karagulyan, 2017; Cheng et al., 2018) suggest that it may be possible to substantially improve upon the LMC analysis that we use and even extend it to non-convex settings. We are hopeful that the LMC approach will lead to a practically useful contextual bandit algorithm and plan to explore this direction further.

2. **Corollary 7** provides a new solution to the open problem of Abernethy and Rakhlin (2009). In fact, it is the first efficient \(\sqrt{dT}\)-type regret bound against a hinge loss benchmark, although our loss is slightly different from the multiclass hinge loss used by Kakade et al. (2008) in their \(T^{2/3}\)-regret BanditRON algorithm (which motivated the open problem). All prior \(\sqrt{dT}\)-regret algorithms (Hazan and Kale, 2011; Beygelzimer et al., 2017; Foster et al., 2018b) use losses with curvature such as the multiclass logistic loss or the squared hinge loss. See Appendix B for a comparison between cost-sensitive and multiclass hinge losses.

3. In **Corollary 8**, regret is measured relative to the policy that chooses the best action (in expectation) on every round. As in prior results (Abbasi-Yadkori et al., 2011; Agarwal et al., 2012), this is possible because the realizability condition ensures that this policy is in our class. Note that here, a requirement for realizability is that \(B \geq K \gamma\), and hence the dependence on \(K\) is implicit and in fact slightly worse than the optimal rate (Chu et al., 2011).

4. For **Corollary 8**, the best points of comparisons are methods based on square-loss realizability (Agarwal et al., 2012; Foster et al., 2018a), although our condition is different. Compared with LinUCB and variants (Chu et al., 2011; Abbasi-Yadkori et al., 2011) specialized to \(\ell_2/\ell_2\) geometry, our assumptions are somewhat weaker but these methods have slightly better guarantees for linear classes.\(^8\) Compared with Foster et al. (2018a), which is the only other efficient approach at a comparable level of generality, our assumptions on the regressor class are stronger, but we obtain better guarantees, in particular removing distribution-dependent parameters.

To summarize, **HINGE-LMC** is the first efficient \(\sqrt{dT}\)-regret algorithm for bandit multiclass prediction using the hinge loss. It also represents a new approach to adversarial contextual bandits, yielding \(\sqrt{dT}\) policy regret under hinge-based realizability. Finally, while we lose the theoretical guarantees, the algorithm easily extends to non-convex classes, which we expect to be practically effective.

\(^8\)In the abstract linear setting we take \(\mathcal{F}\) to be the set of linear functions in the ball for some norm \(\|\cdot\|\) and contexts to be bounded in the dual norm \(\|\cdot\|_\infty\). The runtime of **HINGE-LMC** will degrade (polynomially) with the ratio \(\|\theta\|/\|\theta\|_2\), but the regret bound is the same for any such norm pair.
3.2 SmoothFTL

A drawback of HINGE-LMC is that it only applies in the parametric regime. We now introduce an efficient (in terms of queries to a hinge loss minimization oracle) algorithm with a regret bound similar to Theorem 4, but in the stochastic setting, where \( \{(x_t, \ell_t)\}_{t=1}^T \) are drawn i.i.d. from some joint distribution \( D \) over \( \mathcal{X} \times \mathbb{R}^K \). Here we return to the abstract setting with regression class \( \mathcal{F} \), and for simplicity, we assume \( B = 1 \).

The algorithm we analyze is simply Follow-The-Leader with uniform smoothing and epoching, which we refer to as SmoothFTL. We use an epoch schedule where the \( m \)th epoch lasts for \( n_m = 2^m \) rounds (starting with \( m = 0 \)). At the beginning of the \( m \)th epoch, we compute the empirical importance weighted hinge-loss minimizer \( \hat{f}_{m-1} \) using only the data from the previous epoch. That is, we set
\[
\hat{f}_{m-1} = \arg\min_{f \in \mathcal{F}} \frac{1}{n_m-1} \sum_{\tau=n_m-1}^{n_m-1} \langle \ell_{\tau}, \psi^\gamma(f(x_\tau)) \rangle.
\]
Then, for each round \( t \) in the \( m \)th epoch, we sample \( a_t \) from \( p_t = (1 - K\mu)\pi_{\text{hinge}}(\hat{f}_{m-1}(x_t)) + \mu \). The parameter \( \mu \in (0, 1/\sqrt{K}] \) controls the smoothing. At time \( t = 1 \) we simply take \( p_t \) to be uniform.

**Theorem 9.** Suppose that \( \mathcal{F} \) satisfies \( \log N_{\infty, \infty}(\varepsilon, \mathcal{F}, T) \leq \varepsilon^{-p} \) for some \( p > 2 \). Then in the stochastic setting, with \( \mu = K^{-1}T^{\frac{1}{p+1}} \), SmoothFTL enjoys the following expected regret guarantee\(^9\)
\[
\sum_{t=1}^{T} \mathbb{E} \ell_t(a_t) \leq \inf_{f \in \mathcal{F}} \frac{T}{K} \mathbb{E} \ell(f) + \tilde{O}(T^{\frac{p}{p+1}}).
\]

This provides an algorithmic counterpart to Proposition 5 in the \( p \geq 2 \) regime. The algorithm is quite similar to Epoch-Greedy (Langford and Zhang, 2008), and the main contribution here is to provide a careful analysis for large function classes. We leave obtaining an oracle-efficient algorithm that matches Proposition 5 in the regime \( p \in (0, 2) \) as an open problem.

A similar bound can be obtained for the ramp loss by simply replacing the hinge loss ERM. We analyze the hinge loss version because standard (e.g. linear) classes admit efficient hinge loss minimization oracles. Interestingly, the bound in Theorem 9 actually improves on Proposition 5, in that it is independent of \( K \). This is due to the scaling of the hinge loss in Lemma 1.

In Appendix F, we extend the analysis to the stochastic Lipschitz contextual bandit setting. Here, instead of measuring regret against the benchmark \( \psi^\gamma \circ \mathcal{F} \) we compare to the class of all \( 1 \)-Lipschitz functions from \( \mathcal{X} \) to \( \Delta(\mathcal{A}) \), where \( \mathcal{X} \) is a metric space of bounded covering dimension. We show that SmoothFTL achieves \( T^{\frac{p}{p+1}} \) regret with a \( p \)-dimensional context space and finite action space. This improves on the \( T^{\frac{p+1}{p+2}} \) bound of Cesa-Bianchi et al. (2017), as in Example 2, yet the best available lower bound is \( T^{\frac{p+1}{p}} \) (Hazan and Megiddo, 2007). Closing this gap remains an intriguing open problem.

4 Discussion

This paper initiates a study of the utility of surrogate losses in contextual bandit learning. We obtain new margin-based regret bounds in terms of sequential complexity notions on the benchmark class, improving on the best known rates for Lipschitz contextual bandits and providing dimension-independent bounds for linear
\(^9\)This result is stated in terms of the sequential cover \( N_{\infty, \infty} \) to avoid additional definitions, but can easily be improved to depend on the classical (worst-case) covering number seen in statistical learning.
classes. On the algorithmic side, we provide the first solution to the open problem of Abernethy and Rakhlin (2009) with a non-curved loss and we also show that Follow-the-Leader with uniform smoothing performs well in nonparametric settings.

Yet, several open problems remain. First, our bounds in Section 2 are likely suboptimal in the dependence on $K$, and improving this is a natural direction. Other questions involve deriving stronger lower bounds (e.g., for the non-parametric setting) and adapting to the margin parameter. We also hope to experiment with Hinge-LMC, and develop a better understanding of computational-statistical tradeoffs with surrogate losses. We look forward to studying these questions in future work.

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A Calibration lemmas

**Proof of Lemma 1.** We start with the ramp loss. First since $s \in \mathbb{R}_{\geq 0}^K$, we know that the normalization term in $\pi_{\text{ramp}}(s)$ is

$$\sum_{a \in A} \phi^\gamma(s_a) \geq 1,$$

from which the first inequality follows. The second inequality follows from the fact that $s_a \leq -\gamma$ implies that $\pi_{\text{ramp}}(s)_a = 0$, along with the trivial fact that $\pi_{\text{ramp}}(s)_a \leq 1$.

The hinge loss claim is also straightforward, since here the normalization is

$$\sum_{a \in A} \psi^\gamma(s_a) = \sum_{a \in A} \max\{1 + s_a/\gamma, 0\} \geq \sum_a 1 + s_a/\gamma \geq K.$$

**Lemma 10 (Hinge loss realizability).** Let $\ell \in \mathbb{R}^K_+$ and let $a^* = \arg\min_{a \in A} \ell_a$. Define $s \in \mathbb{R}_{\leq 0}^K$ via $s_a = K\gamma I\{a = a^*\} - \gamma$. Then we have

$$\langle \ell, \psi^\gamma(s) \rangle = K \langle \ell, \pi_{\text{hinge}}(s) \rangle = K \ell_{a^*}.$$

**Proof.** For this particular $s$, the normalizing constant in the definition of $\pi_{\text{hinge}}$ is

$$\sum_{a \in A} \max\left(1 + \frac{K\gamma I\{a = a^*\} - \gamma}{\gamma}, 0\right) = K,$$

and so the first equality follows. The second equality is also straightforward since the score for every action except $a^*$ is clamped to zero.

**Proof of Lemma 3.**

For the case when $S \subset \Delta(A)$, this claim is a well-known property of importance weighting:

$$\mathbb{E}\left[\mathbb{E}_{s_\ell \sim p_\ell}(s_\ell, \hat{\ell}_\ell)^2 \mid J_\ell\right] = \sum_{a \in [K]} P_\ell^\mu(a) \mathbb{E}_{s_\ell \sim p_\ell} \ell^2_\ell(a) s^2_\ell(a) (P_\ell^\mu(a))^2 \leq \sum_{a \in [K]} \mathbb{E}_{s_\ell \sim p_\ell} s^2_\ell(a) = \sum_{a \in [K]} \frac{P_\ell(a)}{1 - K\mu} \frac{P_\ell(a) + \mu}{P_\ell(a)}.$$

Here we use Hölder’s inequality twice, using that $\|\ell\|_\infty \leq 1$ and $s \in \Delta(A)$. Now, since the function $x \mapsto 1/(1 - K\mu + \mu/x)$ is concave in $x$, it follows that

$$\sum_{a \in [K]} \frac{P_\ell(a)}{(1 - K\mu) P_\ell(a) + \mu} = \sum_{a \in [K]} \frac{1}{(1 - K\mu) + \mu P_\ell(a)} \leq K \frac{1}{(1 - K\mu) + K\mu \sum_{a \in [K]} P_\ell(a)} = K,$$

which proves the claim for $S \subset \Delta(A)$.

We proceed in the same fashion for both the ramp and hinge loss. Recall the definition $P_\ell^\mu(a) = (1 -
While our surrogate loss functions apply to general cost-sensitive classification, when specialized to the hinge loss that we use here is:

\[ \max_{a \in [K]} \max_{s \in S} s(a) \leq K \max_{a \in [K]} \max_{s \in S} s(a) \leq \frac{1}{1 - K \mu} \leq 1. \]

For the set \( S \) induced by the ramp loss we have \( \max_{a \in [K]} \max_{s \in S} s(a) \leq 1 \), and for the set \( S \) induced by the hinge loss we have \( \max_{a \in [K]} \max_{s \in S} s(a) \leq (1 + \frac{B}{2}) \). \( \square \)

### B Comparing Multiclass Loss Functions and Notions of Realizability

While our surrogate loss functions apply to general cost-sensitive classification, when specialized to the multiclass zero-one feedback, as in bandit multiclass prediction, they are somewhat non-standard. In this appendix we provide a discussion of the differences, focusing on the hinge loss.

Let us detail the multiclass setting: On each round, the adversary chooses a pair \((x, y^*)\) where \(x \in \mathcal{X}, y^* \in \mathcal{A}\) and shows \(x\) to the learner. The learner then makes a prediction \(\hat{y}\). The 0/1-loss for the learner is \(1_{\hat{y} \neq y}\).

Using a class of regression functions \(\mathcal{G} \subset (\mathcal{X} \to \mathbb{R}^K)\), the standard multiclass hinge loss for a regressor \(g \in \mathcal{G}\) is:

\[ \ell_{\text{MC-hinge}}^\gamma(g, (x, y^*)) = \max\{1 - \gamma^{-1}(g(x)y^* - \max_{y \neq y^*} g(x)y), 0\}. \]

On the other hand, for our results we assume that the regressor class \(\mathcal{F} \subset (\mathcal{X} \to \mathbb{R}_{=0}^K)\), and the cost-sensitive hinge loss that we use here is:

\[ \ell_{\text{CC-hinge}}^\gamma(f, (x, y^*)) = \sum_{y \neq y^*} \psi\gamma(f(x)y) = \sum_{y \neq y^*} \max\{1 + \gamma^{-1} f(x)y, 0\}. \]

More precisely in Corollary 7, we are measuring the benchmark using \(\ell_{\text{CC-hinge}}^\gamma\) and our bound is

\[ \mathbb{E} \sum_{t=1}^T \ell_t(a_t) \leq \inf_{\theta \in \Theta} \frac{1}{K} \sum_{t=1}^T \ell_{\text{CC-hinge}}^\gamma(f(\cdot; \theta), (x_t, y_t^*)) + O\left(\frac{B}{\gamma} \sqrt{dT}\right). \]
On the other hand, the open problem of Abernethy and Rakhlin (2009) asks for a $\sqrt{dT}$ bound when the benchmark is measured using $\ell_{MC\text{-}hinge}$. As we will see, the two loss functions are somewhat different.

Let us first standardize the function classes. By rebinding $f_g(x)_y \triangleq g(x)_y - K^{-1} \sum_{y'} g(x)_{y'}$ we can easily construct a “sum-to-zero” class from an unconstrained class $G \subset (X \to \mathbb{R})$, and with this definition, the cost-sensitive hinge loss for any function $g \in G$ is:

$$\ell_{CC\text{-}hinge}(g, (x, y^\star)) \triangleq \sum_{y \neq y^\star} \psi_\gamma(f_g(x)_y) = \sum_{y \neq y^\star} \max\{1 - \gamma^{-1}(K^{-1} \sum_{y'} g(x)_{y'} - g(x)_y), 0\}.$$ 

The main proposition in this appendix is that if the cost-sensitive hinge loss is zero, then so is the multiclass hinge loss, while the converse is not true.

**Proposition 11.** We have the following implication

$$\ell_{CC\text{-}hinge}(g, (x, y^\star)) = 0 \Rightarrow \ell_{MC\text{-}hinge}(g, (x, y^\star)) = 0.$$

The converse does not hold: For any $\gamma, \tilde{\gamma} > 0$ and $K \geq 0$ there exists a function $g$ and an $(x, y^\star)$ pair such that $\ell_{MC\text{-}hinge}(g, (x, y^\star)) = 0$ but $\ell_{CC\text{-}hinge}(g, (x, y^\star)) \geq 1$.

Note that $\ell_{MC\text{-}hinge} \geq \ell_{MC\text{-}hinge}^{12}$ whenever $\gamma_1 \geq \gamma_2$, so the first implication also holds when the RHS is replaced with $\ell_{MC\text{-}hinge}$.

The proposition implies that cost-sensitive hinge realizability — that there exists a predictor $g^\star$ such that $\ell_{CC\text{-}hinge}(g^\star, (x, y^\star)) = 0$ for all rounds — is a strictly stronger condition than multiclass hinge realizability. Under multiclass separability assumptions (Kakade et al., 2008), the bandit multiclass surrogate benchmark is zero, while our cost-sensitive benchmark may still be large, and so the cost-sensitive translation approach cannot be used to obtain a sublinear upper bound on the number of mistakes made by the learner. In this vein, Hinge-LMC does not completely resolve the open problem of Abernethy and Rakhlin (2009), since the loss function we use can result in a weaker bound than desired. Note that the cost-sensitive surrogate loss may also prevent us from exploiting small-loss structure in the multiclass surrogate to obtain fast rates.

On the other hand, our cost-sensitive surrogate losses are applicable in a much wider range of problems, and cost-sensitive structure is common in contextual bandit settings. As such, we believe that designing algorithms for this more general setting is valuable.

**Proof of Proposition 11.** If $\ell_{CC\text{-}hinge}(g, (x, y^\star)) = 0$ then we know that for all $y \neq y^\star$, we must have

$$\forall y \neq y^\star: \frac{1}{K} \sum_{y'} g(x)_{y'} - g(x)_y \geq \gamma.$$

Adding these inequalities together for $y \neq y^\star$ and subtracting $g(x)_{y^\star}$ from both sides we get

$$\frac{K - 1}{K} \sum_{y'} g(x)_{y'} - \sum_{y \neq y^\star} g(x)_y - g(x)_{y^\star} \geq (K - 1)\gamma - g(x)_{y^\star},$$

$$\Rightarrow g(x)_{y^\star} \geq (K - 1)\gamma + \frac{1}{K} \sum_y g(x)_y$$

$$\Rightarrow g(x)_{y^\star} \geq K\gamma + g(x)_y \forall y \neq y^\star.$$ 

The last line follows from re-using the original inequality and proves the desired implication.

On the other hand, if $\ell_{MC\text{-}hinge}^{K\gamma}(g, (x, y^\star)) = 0$ it does not imply that $\ell_{CC\text{-}hinge}(g, (x, y^\star)) = 0$. To see why, suppose $K = 3$ and assume $y^\star = y_1$ with $y_2, y_3$ as the other labels. Set $g(x)_{y_1} = 3\gamma$, $g(x)_{y_2} = 0$ and
For a collection \( g(x)_{y_1} = -3\gamma \). With these predictions, we have \( \sum_y g(x)_y = 0 \) and also \( \ell_{\text{MC}}^{2\gamma} g_1, g_2, g_y ) = 0 \). On the other hand since \( g(x)_{y_2} = \sum_y g(x)_y = 0 \), we get:

\[
\ell_{\text{CC}}^{\tilde{\gamma}} g, (x, y) \geq \max \{ 1 - \tilde{\gamma}^{-1}(\sum_y g(x)_y - g(x)_{y_2}), 0 \} = 1,
\]

for any \( \tilde{\gamma} \). This proves that the converse cannot be true.

\[\square\]

C Proofs from Section 2

Let us start with an intermediate result, which will simplify the proof of Theorem 2.

**Theorem 12.** Assume \( \|f\|_1 \leq 1 \) for all \( f \in \mathcal{L} \) and \( \sup_{s \in S} \|s\|_\infty \leq 1 \). Further assume that \( S \) and \( \mathcal{L} \) are compact. Fix any constants \( \eta \in (0, 1], \lambda > 0 \), and \( \beta > \alpha > 0 \). Then there exists an algorithm with the following deterministic regret guarantee:

\[
\sum_{t=1}^{T} \mathbb{E}_{s_{t-1}}(s_t, \ell_t) - \inf_{f \in \mathcal{F}} \sum_{t=1}^{T} f(s_t, \ell_t) \leq 2\eta \sum_{t=1}^{T} \mathbb{E}_{s_{t-1}}(s_t, \ell_t)^2 + \frac{4}{\eta} \log \mathcal{N}_{\infty, \infty}(\beta/2, \mathcal{G}, T) + 3e^2 \alpha \int_{\alpha}^{\beta} \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \mathcal{G}, T)} d\varepsilon.
\]

The difference here is that have set \( R, B = 1 \). The first part of this section will be devoted to proving this theorem, and Theorem 2 will follow from this result via Corollary 16.

C.1 Preliminaries

**Definition 2** (Cover for a collection of trees). For a collection of \( \mathbb{R}^K \)-valued trees \( U \) of length \( T \), we let \( \mathcal{N}_{\infty, \infty}(\varepsilon, U) \), denote the cardinality of the smallest set \( V \) of \( \mathbb{R}^K \) valued trees for which

\[
\forall u \in U \forall \varepsilon \in \{ \pm 1 \}^T \exists v \in V \text{ s.t. } \max_{t \in [T]} \| u_t(\varepsilon) - v_t(\varepsilon) \|_{\infty} \leq \varepsilon.
\]

**Definition 3** (\( L_\infty \)-\( \ell_\infty \) radius). For a function class \( \mathcal{F} \), define

\[
\text{rad}_{\infty, \infty}(\mathcal{F}, T) = \min \{ \varepsilon \mid \log \mathcal{N}_{\infty, \infty}(\varepsilon, \mathcal{F}, T) = 0 \}.
\]

For a collection \( U \) of trees, define \( \text{rad}_{\infty, \infty}(U) = \min \{ \varepsilon \mid \log \mathcal{N}_{\infty, \infty}(\varepsilon, U) = 0 \} \).

The following two lemmas are Freedman-type inequalities for Rademacher tree processes that we will use in the sequel. The first has an explicit dependence on the range, while the second does not.

**Lemma 13.** For any collection of \([-R, +R] \)-valued trees \( V \) of length \( T \), for any \( \eta > 0 \) and \( \alpha > 0 \),

\[
\mathbb{E}_\varepsilon \sup_{v \in V} \left[ \sum_{t=1}^{T} \epsilon_t (v_t(\varepsilon) - \eta v_t^2(\varepsilon)) - \alpha \eta v_t^2(\varepsilon) \right] \leq 2 \log |V| \cdot \left( \frac{1}{\alpha \eta} + \frac{\eta R^2}{\alpha} \right).
\]

\(^{10}\) Measuring loss in \( \ell_1 \) may seem restrictive, but this is natural when working with importance-weighted losses since these are 1-sparse, and by duality this enables us to cover in \( \ell_\infty \) norm on the output space.
Applying the standard Rademacher mgf bound

The exponent at time $t$

Since $v$ takes values in $[-R, +R]$, the exponent at time $t$ can be upper bounded as

$$\frac{1}{2} \lambda^2 (v_t(\epsilon) - \eta v_t^2(\epsilon))^2 - \lambda \alpha \eta v_t^2(\epsilon) \leq \lambda^2 (1 + \eta^2 R^2)v_t^2(\epsilon) - \lambda \alpha \eta v_t^2(\epsilon).$$

By setting $\lambda = \frac{1}{2} \min\{\alpha, \alpha/(\eta R^2)\}$, this is bounded by zero, which leads to a final bound of $\log|V|/\lambda$.

**Lemma 14.** For any collection of trees $V$ of length $T$, for any $\eta > 0$,

$$\mathbb{E}_\epsilon \sup_{v \in V} \left[ \sum_{t=1}^T \epsilon_t v_t(\epsilon) - \eta v_t^2(\epsilon) \right] \leq \frac{\log |V|}{2\eta}.$$

**Proof of Lemma 14.** Take $V$ to be finite without loss of generality. As in the proof of Lemma 13, using the standard Rademacher mgf bound and working backward from $T$, for any $\lambda > 0$ we have

$$\mathbb{E}_\epsilon \sup_{v \in V} \left[ \sum_{t=1}^n \epsilon_t v_t(\epsilon) - \eta v_t^2(\epsilon) \right] \leq \frac{1}{\lambda} \log \left( \sum_{v \in V} \mathbb{E}_\epsilon \exp \left( \sum_{t=1}^n \epsilon_t \lambda v_t(\epsilon) - \eta \lambda v_t^2(\epsilon) \right) \right) \leq \frac{1}{\lambda} \log \left( \sum_{v \in V} \max_{\epsilon} \exp \left( \sum_{t=1}^n \frac{1}{2} \lambda^2 v_t(\epsilon)^2 - \eta \lambda v_t^2(\epsilon) \right) \right).$$

The exponent at time $t$ is

$$\frac{1}{2} \lambda^2 v_t^2(\epsilon) - \eta \lambda v_t^2(\epsilon).$$

By setting $\lambda = 2\eta$, this is exactly zero, which leads to a final bound of $\log|V|/\lambda$.

**Lemma 15.** Let $\mathcal{Z}, \mathcal{W}$, and $\mathcal{G}$ be abstract sets and let functions $A_g : \mathcal{W} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ and $B_g : \mathcal{W} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ be given for each element $g \in \mathcal{G}$. Suppose that for any $z, z' \in \mathcal{Z}$ and $w \in \mathcal{W}$ it holds that $A(w, z, z') = -A(w, z', z)$ and $B(w, z, z') = B(w, z', z)$. Then

$$\mathbb{E}_\epsilon \left[ \sup_{v_1 \in \mathcal{W}} \left\{ \sup_{q_1, q_2 \in \Delta(\mathcal{Z})} \mathbb{E}_{z_1, z'_1 \sim q_1} \right\} \right] \leq \mathbb{E}_\epsilon \left[ \sup_{v_1 \in \mathcal{W}} \left\{ \mathbb{E}_{z_1, z'_1 \sim q_1} \right\} \right] \leq \sum_{g \in \mathcal{G}} \mathbb{E}_\epsilon \left[ \sup_{v_1 \in \mathcal{W}} \left( \sum_{t=1}^T \epsilon_t A_g(w_t, z_t, z'_t) + B_g(w_t, z_t, z'_t) \right) \right] (5)$$

where $\epsilon$ is a sequence of independent Rademacher random variables.

See Subsection C.2 for a discussion of the $\langle \bullet \rangle$ notation used in the above lemma statement.

**Proof of Lemma 15.** See proof of Lemma 3 in Rakhlin et al. (2010).
C.2 Proof of Theorem 12

Before proceeding, we note that this proof uses a number of techniques which are now somewhat standard in minimax analysis of online learning, and the reader may wish to refer to, e.g., Rakhlin et al. (2015a) for a comprehensive introduction to this type of analysis.

Let $\eta_1, \eta_2, \eta_3 > 0$ be fixed constants to be chosen later in the proof, and define

$$B(p_{1:T}, \ell_{1:T}) \triangleq \eta_1 \sum_{t=1}^{T} \|\ell_t\|_1 + \eta_2 \sum_{t=1}^{T} \|\ell_t\|^2 + 2\eta_3 \sum_{t=1}^{T} E_{s_{t-1}}(s, \ell_t)^2.$$

We consider a game where the goal of the learner is to achieve regret bounded by $B$, plus some additive constant that will depend on $\eta_1, \eta_2, \eta_3$, and the complexity of the class $\mathcal{F}$. The value of the game is given by:

$$\mathcal{V} \triangleq \left\{ \sup_{x_t \in \mathcal{X}} \inf_{p_t \in \Delta(S)} \sup_{\ell_t \in \mathcal{L}} \left[ \sum_{t=1}^{T} (s, \ell_t) - \inf_{g \in G} \sum_{t=1}^{T} g(x_t), \ell_t) - B(p_{1:T}, \ell_{1:T}) \right] \right\}.$$

Here we are using the notation $\langle \circ \rangle_{t=1}^{T}$ to denote sequential application of the operator $\circ$ (indexed by $t$) from time $t = 1, \ldots, T$, following e.g. (Rakhlin et al., 2015a). This notation means that first the adversary chooses $x_1$, then the learner chooses $p_1$, and then the adversary chooses $\ell_1$ while the learner samples $s_1$ and suffers the loss $(s_1, \ell_1)$. Then we proceed to round 2 and so on, so that the learner is trying to minimize the (offset) regret after $T$ rounds while the adversary is trying to maximize it. If we show that $\mathcal{V} \leq C$ for some constant $C$ then we have established existence of a randomized strategy that achieves an adaptive regret bound of $B(\cdot) + C$. See Foster et al. (2015) for a more extensive discussion of this principle.

C.2.1 Minimax swap

At time $t$ the value to go is given by

$$\sup_{x_t \in \mathcal{X}} \inf_{p_t \in \Delta(S)} \sup_{\ell_t \in \mathcal{L}} E_{\ell_{t-1}} \left[ E_{s_{t-1}}(s, \ell_t) - 2\eta_3 E_{s_{t-1}}(s, \ell_t)^2 - \eta_1 \|\ell_t\|_1 - \eta_2 \|\ell_t\|^2 + \langle \sup_{x_t \in \mathcal{X}} \inf_{p_t \in \Delta(S)} \sup_{\ell_t \in \mathcal{L}} \left[ \sum_{t=1}^{T} E_{s_{t-1}}(s, \ell_t) - \inf_{g \in G} \sum_{t=1}^{T} g(x_t), \ell_t) - B(p_{t+1:T}, \ell_{t+1:T}) \right] \right].$$

Note that the benchmark’s loss is only evaluated at the end, while we are incorporating the adaptive term into the instantaneous value. Convexifying the $\ell_t$ player by allowing them to select a randomized strategy $q_t$, this is equal to

$$\sup_{x_t \in \mathcal{X}} \inf_{p_t \in \Delta(S)} \sup_{q_t \in \Delta(L)} E_{\ell_{t-1}} \left[ E_{s_{t-1}}(s, \ell_t) - 2\eta_3 E_{s_{t-1}}(s, \ell_t)^2 - \eta_1 \|\ell_t\|_1 - \eta_2 \|\ell_t\|^2 + \langle \sup_{x_t \in \mathcal{X}} \inf_{p_t \in \Delta(S)} \sup_{q_t \in \Delta(L)} \left[ \sum_{t=1}^{T} E_{s_{t-1}}(s, \ell_t) - \inf_{g \in G} \sum_{t=1}^{T} g(x_t), \ell_t) - B(p_{t+1:T}, \ell_{t+1:T}) \right] \right].$$

This quantity is convex in $p_t$ and linear in $q_t$ so, under the compactness assumption on $S$ and $L$, the minimax theorem implies that this is equal to

$$\sup_{x_t \in \mathcal{X}} \inf_{q_t \in \Delta(L)} \sup_{p_t \in \Delta(S)} E_{\ell_{t-1}} \left[ E_{s_{t-1}}(s, \ell_t) - 2\eta_3 E_{s_{t-1}}(s, \ell_t)^2 - \eta_1 \|\ell_t\|_1 - \eta_2 \|\ell_t\|^2 + \langle \sup_{x_t \in \mathcal{X}} \inf_{p_t \in \Delta(S)} \sup_{q_t \in \Delta(L)} \left[ \sum_{t=1}^{T} E_{s_{t-1}}(s, \ell_t) - \inf_{g \in G} \sum_{t=1}^{T} g(x_t), \ell_t) - B(p_{t+1:T}, \ell_{t+1:T}) \right] \right].$$
We now use a standard “rearrangement” trick (see (Rakhlin et al., 2015a), Theorem 1) to show that

\[
\mathcal{V} = \left\langle \sup_{x_t \in \mathcal{X}} \sup_{q_t \in \Delta(\mathcal{L})} \left( \sup_{t=1}^{T} \inf_{p_t \in \Delta(S)} \mathbb{E}_{\ell_t \sim q_t} \right) \right\rangle \left[ \sum_{t=1}^{T} \mathbb{E}_{s \sim p_t} \left( \left\langle s, \ell_t \right\rangle - 2\eta_3(s, \ell_t)^2 \right) - \inf_{g \in \mathcal{G}} \sum_{i=1}^{T} \left( g(x_i), \ell_t \right) - B_1(\ell_{1:T}) \right].
\]

Repeating this analysis at each timestep and expanding the terms from \(B_2\), we arrive at the expression

\[
\mathcal{V} = \left\langle \sup_{x_t \in \mathcal{X}} \sup_{q_t \in \Delta(\mathcal{L})} \left( \sup_{t=1}^{T} \inf_{p_t \in \Delta(S)} \mathbb{E}_{\ell_t \sim q_t} \right) \right\rangle \left[ \sum_{t=1}^{T} \mathbb{E}_{s \sim p_t} \left( \left\langle s, \ell_t \right\rangle - 2\eta_3(s, \ell_t)^2 \right) - \inf_{g \in \mathcal{G}} \sum_{i=1}^{T} \left( g(x_i), \ell_t \right) - B_1(\ell_{1:T}) \right].
\]

C.2.2 Upper bound by martingale process

We now use a standard “rearrangement” trick (see (Rakhlin et al., 2015a), Theorem 1) to show that

\[
\mathcal{V} = \left\langle \sup_{x_t \in \mathcal{X}} \sup_{q_t \in \Delta(\mathcal{L})} \left( \sup_{t=1}^{T} \inf_{p_t \in \Delta(S)} \mathbb{E}_{\ell_t \sim q_t} \right) \right\rangle \left[ \sum_{t=1}^{T} \mathbb{E}_{s \sim p_t} \mathbb{E}_{\ell_t' \sim q_t} \left( \left\langle s, \ell_t' \right\rangle - 2\eta_3(s, \ell_t')^2 \right) - \sum_{t=1}^{T} \left( f(x_t), \ell_t \right) - B_1(\ell_{1:T}) \right],
\]

where \(\ell_t'\) is a sequence of “tangent” samples, where \(\ell_t'\) is an independent copy of \(\ell_t\) conditioned on \(\ell_{1:t-1}\). This can be seen by working backwards from time \(T\). Indeed, at time \(T\), expanding the \(\left\langle \ast \right\rangle_{t=1}^{T}\) operator, we have

\[
\mathcal{V} = \left\langle \cdots \right\rangle_{t=1}^{T-1} \sup_{x_t \in \mathcal{X}} \sup_{q_t \in \Delta(\mathcal{L})} \left( \sup_{t=1}^{T} \inf_{p_t \in \Delta(S)} \mathbb{E}_{\ell_t \sim q_t} \right) \mathbb{E}_{\ell_{T-1} \sim q_{T-1}} \left[ \sum_{t=1}^{T-1} \mathbb{E}_{s \sim p_t} \left( \left\langle s, \ell_t \right\rangle - 2\eta_3(s, \ell_t)^2 \right) + \mathbb{E}_{s \sim p_{T-1}} \left( \left\langle s, \ell_{T-1} \right\rangle - 2\eta_3(s, \ell_{T-1})^2 \right) - \inf_{g \in \mathcal{G}} \sum_{i=1}^{T} \left( g(x_i), \ell_t \right) - B_1(\ell_{1:T}) \right].
\]

Using linearity of expectation:

\[
\mathcal{V} = \left\langle \cdots \right\rangle_{t=1}^{T-1} \sup_{x_t \in \mathcal{X}} \sup_{q_t \in \Delta(\mathcal{L})} \left( \sup_{t=1}^{T} \inf_{p_t \in \Delta(S)} \mathbb{E}_{\ell_t \sim q_t} \right) \mathbb{E}_{\ell_{T-1} \sim q_{T-1}} \left[ \sum_{t=1}^{T-1} \mathbb{E}_{s \sim p_t} \left( \left\langle s, \ell_t \right\rangle - 2\eta_3(s, \ell_t)^2 \right) + \mathbb{E}_{s \sim p_{T-1}} \mathbb{E}_{\ell_{T-1} \sim p_{T-1}} \left( \left\langle s, \ell_{T-1} \right\rangle - 2\eta_3(s, \ell_{T-1})^2 \right) - \inf_{g \in \mathcal{G}} \sum_{i=1}^{T} \left( g(x_i), \ell_t \right) - B_1(\ell_{1:T}) \right].
\]

Using that only a single term has functional dependence on \(p_{T-1}\):

\[
\mathcal{V} = \left\langle \cdots \right\rangle_{t=1}^{T-1} \sup_{x_t \in \mathcal{X}} \sup_{q_t \in \Delta(\mathcal{L})} \mathbb{E}_{\ell_{T-1} \sim q_{T-1}} \left[ \sum_{t=1}^{T-1} \mathbb{E}_{s \sim p_t} \left( \left\langle s, \ell_t \right\rangle - 2\eta_3(s, \ell_t)^2 \right) + \inf_{p_{T-1} \in \Delta(S)} \mathbb{E}_{\ell_{T-1} \sim p_{T-1}} \mathbb{E}_{s \sim p_{T-1}} \left( \left\langle s, \ell_{T-1} \right\rangle - 2\eta_3(s, \ell_{T-1})^2 \right) - \inf_{g \in \mathcal{G}} \sum_{i=1}^{T} \left( g(x_i), \ell_t \right) - B_1(\ell_{1:T}) \right].
\]

Expanding the infimum over \(g \in \mathcal{G}\):

\[
\mathcal{V} = \left\langle \cdots \right\rangle_{t=1}^{T-1} \sup_{x_t \in \mathcal{X}} \sup_{q_t \in \Delta(\mathcal{L})} \mathbb{E}_{\ell_{T-1} \sim q_{T-1}} \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T-1} \mathbb{E}_{s \sim p_t} \left( \left\langle s, \ell_t \right\rangle - 2\eta_3(s, \ell_t)^2 \right) + \inf_{p_{T-1} \in \Delta(S)} \mathbb{E}_{\ell_{T-1} \sim p_{T-1}} \mathbb{E}_{s \sim p_{T-1}} \left( \left\langle s, \ell_{T-1} \right\rangle - 2\eta_3(s, \ell_{T-1})^2 \right) - \sum_{i=1}^{T} \left( g(x_i), \ell_t \right) - B_1(\ell_{1:T}) \right].
\]

We handle time \(T - 1\) in a similar fashion by first splitting the \(\left\langle \ast \right\rangle_{t=1}^{T-1}\) operator:

\[
\mathcal{V} = \left\langle \cdots \right\rangle_{t=1}^{T-2} \sup_{x_t \in \mathcal{X}} \sup_{q_t \in \Delta(\mathcal{L})} \left( \sup_{t=1}^{T-1} \inf_{p_t \in \Delta(S)} \mathbb{E}_{\ell_{T-1} \sim q_{T-1}} \right) \mathbb{E}_{\ell_{T-2} \sim q_{T-2}} \left[ \sum_{t=1}^{T-2} \mathbb{E}_{s \sim p_t} \left( \left\langle s, \ell_t \right\rangle - 2\eta_3(s, \ell_t)^2 \right) + \mathbb{E}_{s \sim p_{T-2}} \left( \left\langle s, \ell_{T-2} \right\rangle - 2\eta_3(s, \ell_{T-2})^2 \right) - \sum_{i=1}^{T} \left( g(x_i), \ell_t \right) - B_1(\ell_{1:T}) \right].
\]
Rearranging the supremums to make dependence on terms from time $T - 1$ clear:

$$\sum_{t=1}^{T-2} \sum_{x_T \in \mathcal{X}} \sum_{q_{t-1} \in \Delta(S)} \sum_{t' \in \Delta(S)} [\sum_{t=1}^{T-2} E_{s-p_t} \left( s, \ell_t \right) - 2 \eta_t (s, \ell_t)^2 + E_{s-p_{t-1}} \left( s, \ell_{t-1} \right) - 2 \eta_{t-1} (s, \ell_{t-1})^2] + \sum_{x_t \in \mathcal{X}} \sum_{q_t \in \Delta(S)} \inf_{g \in \mathcal{G}} \sum_{t=1}^{T-1} \sum_{t' \in \Delta(S)} \left( s, \ell_t \right) - 2 \eta_t (s, \ell_t)^2 - \sum_{t=1}^{T} (g(x_t), \ell_t) - B_1(\ell_{1:T})].$$

Using linearity of expectation and moving the infimum over $q_{T-1}$:

$$\sum_{t=1}^{T-2} \sum_{x_T \in \mathcal{X}} \sum_{q_{t-1} \in \Delta(S)} \sum_{t' \in \Delta(S)} [\sum_{t=1}^{T-2} E_{s-p_t} \left( s, \ell_t \right) - 2 \eta_t (s, \ell_t)^2 + E_{s-p_{t-1}} \left( s, \ell_{t-1} \right) - 2 \eta_{t-1} (s, \ell_{t-1})^2] + \sum_{x_t \in \mathcal{X}} \sum_{q_t \in \Delta(S)} \inf_{g \in \mathcal{G}} \sum_{t=1}^{T-1} \sum_{t' \in \Delta(S)} \left( s, \ell_t \right) - 2 \eta_t (s, \ell_t)^2 - \sum_{t=1}^{T} (g(x_t), \ell_t) - B_1(\ell_{1:T})].$$

The last step is to move the supremums from time $t = T$ and the supremum over $g \in \mathcal{G}$ outside the entire expression, similar to what was done at time $t = T$.

$$\sum_{t=1}^{T-2} \sum_{x_T \in \mathcal{X}} \sum_{q_{t-1} \in \Delta(S)} \sum_{t' \in \Delta(S)} [\sum_{t=1}^{T-2} E_{s-p_t} \left( s, \ell_t \right) - 2 \eta_t (s, \ell_t)^2 + E_{s-p_{t-1}} \left( s, \ell_{t-1} \right) - 2 \eta_{t-1} (s, \ell_{t-1})^2] + \sum_{x_t \in \mathcal{X}} \sum_{q_t \in \Delta(S)} \inf_{g \in \mathcal{G}} \sum_{t=1}^{T-1} \sum_{t' \in \Delta(S)} \left( s, \ell_t \right) - 2 \eta_t (s, \ell_t)^2 - \sum_{t=1}^{T} (g(x_t), \ell_t) - B_1(\ell_{1:T})].$$

Repeating this argument down from time $t = T - 2$ to time $t = 1$ yields the result.

To conclude this portion of the proof, we move to an upper bound by choosing the infimum over $p_t$ at each timestep $t$ to match $g$, which is possible because each infimum now occurs inside the expression for which the supremum over $g \in \mathcal{G}$ is taken:

$$\mathcal{V} = \left\{ \sup_{x_T \in \mathcal{X}} \sup_{q_{t-1} \in \Delta(S)} E_{\ell_{t-1}q_t} \right\} \sum_{t=1}^{T} \inf_{g \in \mathcal{G}} \sum_{t=1}^{T} \sum_{t' \in \Delta(S)} \left( s, \ell_t \right) - 2 \eta_t (s, \ell_t)^2 - \sum_{t=1}^{T} (g(x_t), \ell_t) - B_1(\ell_{1:T})].$$

(7)

C.2.3 Symmetrization

Introduce the notation $H(x) = x - \eta_3 x^2$. We now claim that the quantity appearing in (7) is bounded by

$$2 \cdot \sup_{x, \ell} \sup_{g \in \mathcal{G}} \left\{ \sup_{t=1}^{T} \sum_{t=1}^{T} \sum_{t' \in \Delta(S)} \left( s, \ell_t \right) - 2 \eta_t (s, \ell_t)^2 - \sum_{t=1}^{T} (g(x_t), \ell_t) - B_1(\ell_{1:T})].$$

(8)
where the supremum ranges over all $\mathcal{X}$-valued trees $x$ and $\mathcal{L}$-valued trees $\ell$, both of length $T$.

The value

$$
\left\langle \sup_{x \in \mathcal{X}} \sup_{q \in \Delta(\mathcal{L})} \mathbb{E}_{\ell, q} \left[ \sup_{t=1}^{T} \sum_{i=1}^{T} \mathbb{E}_{\ell, q} \left[ (g(x_t), \ell'_t) - (g(x_t), \ell_t) - 2\eta_3 \sum_{i=1}^{T} \mathbb{E}_{\ell, q} \left[ (g(x_t), \ell'_t)^2 \right] - B_1(\ell_{1:T}) \right] \right] \right\rangle
$$
by adding and subtracting the same term, is equal to

$$
\left\langle \sup_{x \in \mathcal{X}} \sup_{q \in \Delta(\mathcal{L})} \mathbb{E}_{\ell, q} \left[ \sup_{t=1}^{T} \sum_{i=1}^{T} \mathbb{E}_{\ell, q} \left[ (g(x_t), \ell'_t) - \eta_3 (g(x_t), \ell'_t)^2 \right] - (g(x_t), \ell_t) - \eta_3 (g(x_t), \ell_t)^2 \right] - B_1(\ell_{1:T}) \right\rangle
$$

Using Jensen’s inequality, this is upper bounded by

$$
\left\langle \sup_{x \in \mathcal{X}} \sup_{q \in \Delta(\mathcal{L})} \mathbb{E}_{\ell, \ell'_{1:T}} \left[ \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T} H((g(x_t), \ell'_t)) - H((g(x_t), \ell_t)) \right] - \eta_3 \sum_{t=1}^{T} (g(x_t), \ell'_t)^2 + (g(x_t), \ell_t)^2 \right] - B_1(\ell_{1:T}) \right\rangle.
$$

where $\ell'_{1:T}$ is a tangent sequence. We now claim that this is equal to

$$
\left\langle \sup_{x \in \mathcal{X}} \sup_{q \in \Delta(\mathcal{L})} \mathbb{E}_{\ell, \ell'_{1:T}} \left[ \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T} H((g(x_t), \ell'_t)) - H((g(x_t), \ell_t)) \right] - \eta_3 \sum_{t=1}^{T} (g(x_t), \ell'_t)^2 + (g(x_t), \ell_t)^2 \right] - \frac{1}{2} B_1(\ell_{1:T}) - \frac{1}{2} B_1(\ell'_{1:T}) \right\rangle.
$$

This can be seen as follows: Let $Q$ be the joint distribution over $\ell_1, \ldots, \ell_T$ obtaining the supremum above, or if the supremum is not obtained let it be any point in a limit sequence approaching the supremum. Then the value of the $B_1$ term in (9) is equal to (respectively, $\varepsilon$-close to)

$$
\mathbb{E}_Q B_1(\ell_{1:T}) = \eta_1 \sum_{t=1}^{T} \mathbb{E}_Q \| \ell_t \|_1 + \eta_2 \sum_{t=1}^{T} \mathbb{E}_Q \| \ell_t \|_1^2
$$

Replacing $\ell_t$ with $\ell'_t$ follows from the definition of the tangent sequence, since $\ell'_t$ and $\ell_t$ are identically distributed, conditioned on $\ell_{1:t-1}$. This shows that we can replace $B_1(\ell_{1:T})$ with $B_1(\ell_{1:T})/2 + B_1(\ell'_{1:T})/2$ above, since we are working with the expectation.
We will focus on the supremum for now. We begin by adapting a trick from Rakhlin and Sridharan (2015) to introduce a coarse sequential cover at scale $\ell_{\infty}$. Let $\mathcal{G}$ be a cover for $\mathcal{X}$ on the tree $\mathcal{A}$ with respect to $L_{\infty}/\ell_{\infty}$ at scale $\beta/2$. Then the size of $V'$ is $N_{\infty,\infty}(\beta/2, \mathcal{G}, \mathcal{A})$, and

$$
\max_{g \in \mathcal{G}} \max_{\epsilon \in \mathcal{A}} \min_{\epsilon' \in V'} \|g(\mathcal{A}(\epsilon)) - \mathcal{A}(\epsilon')\|_{\infty} \leq \beta/2.
$$

Recall that since $g(x) \in \mathbb{R}_+^K$ for all $g \in \mathcal{G}$, we may take each $v' \in V'$ to have non-negative coordinates without loss of generality. Likewise, it follows that we may take each $v' \in V'$ to have $\|v'(\epsilon)\|_{\infty} \leq \sup_{x \in \mathcal{X}} \sup_{g \in \mathcal{G}} \|g(x)\|_{\infty}$ without loss of generality.
We construct a new \( \beta \)-cover \( V^1 \) from \( V' \) by defining for each tree \( v' \in V' \) a new tree \( v \) as follows:

\[
\forall \epsilon \in \{ \pm 1 \}^T \forall t \in [T] \forall a \in [K] : \quad v_t(\epsilon)_a = \max\{ v'_t(\epsilon)_a - \beta/2, 0 \}.
\]

It is easy to verify that for each time \( t \) and path \( \epsilon \) we have \( \| v_t(\epsilon) - v'_t(\epsilon) \|_{\infty} \leq \beta/2 \), so \( V^1 \) is indeed a \( \beta \)-cover with respect to \( L_\infty/\ell_\infty \). More importantly, for each \( g \in \mathcal{G} \) and path \( \epsilon \), there exists a tree \( v \in V' \) that is \( \beta \)-close in the \( L_\infty/\ell_\infty \) sense and has \( v_t(\epsilon)_a \leq g(x_t(\epsilon))_a \) coordinate-wise. We will let \( v_1^1[\epsilon, g] \) denote this tree, and it is constructed by taking the \( \beta/2 \)-close tree \( v' \) promised by the definition of \( V' \), then performing the clipping operation above to get the corresponding \( \beta \)-close element of \( V^1 \). The clipping operation and \( \beta/2 \) closeness of \( v' \) imply that for each time \( t \in [T] \) and coordinate \( a \in [K] \),

\[
v_t^1[\epsilon, g]_a - g(x_t(\epsilon))_a = \max\{ v_t'_a(\epsilon)_a - \beta/2, 0 \} - g(x_t(\epsilon))_a \\
\leq \max\{ \| v_t'(\epsilon) - g(x_t(\epsilon)) \|_{\infty} + g(x_t(\epsilon))_a - \beta/2, 0 \} - g(x_t(\epsilon))_a \\
\leq \max\{ g(x_t(\epsilon))_a, 0 \} - g(x_t(\epsilon))_a = 0.
\]

This establishes the desired ordering on coordinates. Returning to the process at hand, we have

\[
\mathbb{E}_\epsilon \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T} \epsilon_t H(g(x_t(\epsilon), \ell_t(\epsilon))) - \eta_3 \sum_{t=1}^{T} g(x_t(\epsilon), \ell_t(\epsilon))^2 \right].
\]

Now we add and subtract terms involving the covering element \( v^1_t(\epsilon, g) \):

\[
= \mathbb{E}_\epsilon \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T} \epsilon_t H((v_t^1[\epsilon, g], \ell_t(\epsilon))) - \eta_3 \sum_{t=1}^{T} g(x_t(\epsilon), \ell_t(\epsilon))^2 \right. \\
\left. + \sum_{t=1}^{T} \epsilon_t H(g(x_t(\epsilon), \ell_t(\epsilon))) - \epsilon_t H((v_t^1[\epsilon, g], \ell_t(\epsilon))) \right].
\]

We now invoke the coordinate domination property of \( v^1_t[\epsilon, g] \) described above. Observe that since \( g(x_t(\epsilon)) \), \( v_t^1[\epsilon, g] \), and \( \ell_t(\epsilon) \) are all nonnegative coordinate-wise, it holds that \( \{ v_t^1[\epsilon, g], \ell_t(\epsilon) \}^2 \leq g(x_t(\epsilon), \ell_t(\epsilon))^2 \). Consequently, we can replace the offset term (not involving \( \epsilon_t \)) with a similar term involving \( v_t^1[\epsilon, g] \)

\[
\leq \mathbb{E}_\epsilon \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T} \epsilon_t H((v_t^1[\epsilon, f], \ell_t(\epsilon))) - \eta_3 \sum_{t=1}^{T} v_t^1[\epsilon, g], \ell_t(\epsilon))^2 \right. \\
\left. + \sum_{t=1}^{T} \epsilon_t H(g(x_t(\epsilon), \ell_t(\epsilon))) - \epsilon_t H((v_t^1[\epsilon, f], \ell_t(\epsilon))) \right].
\]

Splitting the supremum and gathering terms, this implies that \( \mathcal{V} \) is upper bounded by

\[
\mathbb{E}_\epsilon \sup_{v^1 \in V^1} \left[ \sum_{t=1}^{T} \epsilon_t H((v_t^1(\epsilon), \ell_t(\epsilon))) - \eta_3 \sum_{t=1}^{T} (v_t^1(\epsilon), \ell_t(\epsilon))^2 \right] \quad (\ast) \\
+ \mathbb{E}_\epsilon \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T} \epsilon_t H(g(x_t(\epsilon), \ell_t(\epsilon))) - \epsilon_t H((v_t^1[\epsilon, g], \ell_t(\epsilon))) \right] - \mathbb{E}_\epsilon B_1(\ell_{1:T}(\epsilon)) \quad (\ast\ast).
C.2.5 Bounding (\(\ast\))

We appeal to Lemma 13 with a class of real-valued trees \(U \doteq \{ \epsilon \mapsto (\{v^1_t(\epsilon), \ell_t(\epsilon)\})_{t \leq T} \mid v^1 \in V^1 \}\). The class \(U\) has range contained in \([-1, +1]\), since \(|\{v^1_t(\epsilon), \ell_t(\epsilon)\}| \leq \|v^1_t(\epsilon)\|_\infty \|\ell_t(\epsilon)\|_1 \leq 1\), where these norm bounds are by assumption on \(G\) and \(L\). Recall that \(H(x) = x - \eta_3 x^2\). We therefore conclude that

\[
(* \ast) = \mathbb{E}_\epsilon \sup_{v^1 \in V^1} \left[ \sum_{t=1}^T \epsilon_t H(\{v^1_t(\epsilon), \ell_t(\epsilon)\}) - \eta_3 \sum_{t=1}^T \|v^1_t(\epsilon)\|_1 \right] \\
\leq 2 \frac{1 + \eta_3^2}{\eta_3} \log |V^1| = 2 \frac{1 + \eta_3^2}{\eta_3} \log N_{\infty, \infty}(\beta/2, G, x).
\]

C.2.6 Bounding (\(\ast \ast\))

Fix \(\alpha > 0\) and let \(N = \lfloor \log(\beta/\alpha) \rfloor - 1\). For each \(i \geq 1\) define \(\epsilon_i = \beta e^{-(i-1)}\), and for each \(i > 1\) let \(V^i\) be a sequential cover of \(G\) on \(x\) at scale \(\epsilon_i\) with respect to \(L_\infty/\ell_\infty\) (keeping in mind that \(V^1\) is defined as in the preceding section). For a given path \(\epsilon \in \{\pm 1\}^T\) and \(g \in G\), let \(v^i(\epsilon, g)\) denote the \(\epsilon_i\)-close element of \(V^i\).

Below, we will only evaluate \(H(x) = x - \eta_3 x^2\) over the domain \([-1, +1]\); it is \((1 + 2\eta_3)\)-Lipschitz over this domain. Then the leading term of (\(\ast \ast\)) is equal to

\[
\mathbb{E}_\epsilon \sup_{g \in G} \left[ \sum_{t=1}^T \epsilon_t \left( H(\langle g(x_t(\epsilon)), \ell_t(\epsilon)\rangle) - H(\langle v^1_t[\epsilon, g], \ell_t(\epsilon)\rangle) \right) \right].
\]

Introducing the covering elements defined above to this expression, we have the equality

\[
= \mathbb{E}_\epsilon \sup_{g \in G} \left[ \sum_{t=1}^T \epsilon_t \left( H(\langle g(x_t(\epsilon)), \ell_t(\epsilon)\rangle) - H(\langle v^1_t[\epsilon, g], \ell_t(\epsilon)\rangle) \right) + \sum_{t=1}^{N-1} \sum_{i=1}^T \epsilon_t \left( H(\langle v^{i+1}_t[\epsilon, g], \ell_t(\epsilon)\rangle) - H(\langle v^i_t[\epsilon, g], \ell_t(\epsilon)\rangle) \right) \right] \\
\leq \mathbb{E}_\epsilon \sup_{g \in G} \left[ \sum_{t=1}^T \epsilon_t \left( H(\langle g(x_t(\epsilon)), \ell_t(\epsilon)\rangle) - H(\langle v^1_t[\epsilon, g], \ell_t(\epsilon)\rangle) \right) \right] \\
\leq \sum_{i=1}^{N-1} \mathbb{E}_\epsilon \sup_{g \in G} \left[ \sum_{t=1}^T \epsilon_t \left( H(\langle v^{i+1}_t[\epsilon, g], \ell_t(\epsilon)\rangle) - H(\langle v^i_t[\epsilon, g], \ell_t(\epsilon)\rangle) \right) \right].
\]
C.2.7 Bounding $C_N$

We first bound $C_N$ in terms of one of the terms appearing in $B_1$.

$$C_N = \mathbb{E}_\epsilon \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T} \epsilon_t \left( H \left( \langle g(x_t(\epsilon)), \ell_t(\epsilon) \rangle \right) - H \left( \langle v_t^N[\epsilon, g], \ell_t(\epsilon) \rangle \right) \right) \right]$$

$$\leq \mathbb{E}_\epsilon \left[ \sum_{t=1}^{T} \sup_{g \in \mathcal{G}} H \left( \langle g(x_t(\epsilon)), \ell_t(\epsilon) \rangle \right) - H \left( \langle v_t^N[\epsilon, g], \ell_t(\epsilon) \rangle \right) \right]$$

$$\leq (1 + 2\eta_B) \mathbb{E}_\epsilon \left[ \sum_{t=1}^{T} \sup_{g \in \mathcal{G}} \|g(x_t(\epsilon)) - v_t^N[\epsilon, g]\|_\infty \right]$$

$$\leq (1 + 2\eta_B) \max_{\epsilon} \sup_{g \in \mathcal{G}} \max_{t \in [T]} \|g(x_t(\epsilon)) - v_t^N[\epsilon, g]\|_\infty \cdot \mathbb{E}_\epsilon \left[ \sum_{t=1}^{T} \|\ell_t(\epsilon)\|_1 \right]$$

$$\leq (1 + 2\eta_B) e^{2\alpha} \cdot \mathbb{E}_\epsilon \left[ \sum_{t=1}^{T} \|\ell_t(\epsilon)\|_1 \right].$$

The first inequality uses that $\epsilon_t \in \{\pm 1\}$, while the second uses the Lipschitzness of $H$ over $[-1, 1]$. The third and fourth are both applications of Hölder’s inequality, first to the $\ell_1/\ell_\infty$ dual pairing, and then to for the distributions over $L_1/L_\infty$. Finally, the definition of the covering element $v_t^N$—in particular, that it is an $L_\infty/\ell_\infty$-cover—implies that the supremum term is bounded by $\epsilon_N \leq e^2 \cdot \alpha$, which yields the final bound.

C.2.8 Bounding $C_i$

Our goal is to bound

$$C_i = \mathbb{E}_\epsilon \sup_{g \in \mathcal{G}} \left[ \sum_{t=1}^{T} \epsilon_t \left( H \left( \langle v_t^{i+1}[\epsilon, g], \ell_t(\epsilon) \rangle \right) - H \left( \langle v_t^i[\epsilon, g], \ell_t(\epsilon) \rangle \right) \right) \right].$$

We define a class $W$ of real-valued trees as follows. Let $1 \leq a \leq |V^i|$ and $1 \leq b \leq |V^{i+1}|$, and fix an arbitrary ordering $v^a \in V^i$ and $v^b \in V^{i+1}$ of the elements of $V^i/V^{i+1}$. For each pair $(a, b)$ define a tree $w^{(a,b)}$ via

$$w^{(a,b)}_t(\epsilon) = \left\{ \begin{array}{ll} H \left( \langle v^b_t(\epsilon), \ell_t(\epsilon) \rangle \right) - H \left( \langle v^a_t(\epsilon), \ell_t(\epsilon) \rangle \right), & \exists g \in \mathcal{G} \text{ s.t. } v^a = v[\epsilon, g]^{i}, v^b = v[\epsilon, g]^{i+1}, \\ 0, & \text{otherwise.} \end{array} \right. $$

Then $C_i$ is bounded by

$$\mathbb{E}_\epsilon \sup_{w \in W} \sum_{t=1}^{T} \epsilon_t w_t(\epsilon).$$

Then Lemma 14 implies that for any fixed $\eta > 0$,

$$\mathbb{E}_\epsilon \sup_{w \in W} \left[ \sum_{t=1}^{T} \epsilon_t w_t(\epsilon) - \eta w_t^2(\epsilon) \right] \leq \frac{\log |W|}{2\eta}. $$

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Rearranging and applying subadditivity of the supremum, this implies
\[
\mathbb{E}_\epsilon \sup_{w \in W} \sum_{t=1}^T \epsilon_t w_t(\epsilon) \leq \eta \cdot \mathbb{E}_\epsilon \sup_{w \in W} \sum_{t=1}^T w_t^2(\epsilon) + \frac{\log|W|}{2\eta}.
\]

Optimizing over \(\eta\) (which is admissible because the statement above is a deterministic inequality) leads to a further bound of
\[
\mathbb{E}_\epsilon \sup_{w \in W} \sum_{t=1}^T \epsilon_t w_t(\epsilon) \leq 2 \mathbb{E}_\epsilon \sup_{w \in W} \sum_{t=1}^T w_t^2(\epsilon) \cdot \log|W|.
\]

We proceed to bound each term in the square root. For the logarithmic term, by construction we have
\[
|\log|W| | \leq |V_i| V^{i+1}| \leq |V^{i+1}|^2 = \mathcal{N}_{\infty,\infty}(\epsilon_{i+1}, \mathcal{G}, \mathbf{x})^2.
\]

For the variance, let \(w^{(a,b)} \in W\) and the path \(\epsilon\) be fixed. There are two cases: Either \(w(\epsilon) = 0\), or there exists \(g \in \mathcal{G}\), such that \(v^a = v[\epsilon, g]^1\) and \(v^b = v[\epsilon, g]^{i+1}\). The former case is trivial while for the latter, in a similar way to the bound for \(C_N\), we get
\[
\sum_{t=1}^T w_t^{(a,b)}(\epsilon)^2 = \sum_{t=1}^T (H((v_t^{i+1}[\epsilon, g], \ell_t(\epsilon))) - H((v_t^i[\epsilon, g], \ell_t(\epsilon))))^2
\]
\[
\leq (1 + 2\eta_3)^2 \sum_{t=1}^T (v_t^{i+1}[\epsilon, g], \ell_t(\epsilon)) - (v_t^i[\epsilon, g], \ell_t(\epsilon))^2
\]
\[
\leq (1 + 2\eta_3)^2 \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2 ||v_t^{i+1}[\epsilon, g] - v_t^i[\epsilon, g]|^2
\]
\[
\leq (1 + 2\eta_3)^2 \max_{v \in T} \max_{\epsilon} \|v_t^{i+1}[\epsilon', g] - v_t^i[\epsilon', g]\|^2 \cdot \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2.
\]

Where we have used Lipschitzness of \(H\) in the first inequality and Hölder’s inequality in the second and third.

Finally, using the \(L_\infty/\ell_\infty\) cover property of \(v^i[\epsilon, g]\) and \(v^{i+1}[\epsilon, g]\) and the triangle inequality, we have
\[
\max_{\epsilon} \max_{t \in T} \|v_t^{i+1}[\epsilon, g] - v_t^i[\epsilon, g]\|_\infty
\]
\[
\leq \max_{\epsilon} \max_{t \in T} \|v_t^{i+1}[\epsilon, g] - g(x_t(\epsilon))\|_\infty + \max_{\epsilon} \max_{t \in T} \|g(x_t(\epsilon)) - v_t^i[\epsilon, g]\|_\infty
\]
\[
\leq \epsilon_i + \epsilon_{i+1} \leq 2\epsilon_i.
\]

We have just shown that for every sequence \(\epsilon\) and every \(w^{(a,b)} \in W\), \(\sum_{t=1}^T w_t^{(a,b)}(\epsilon)^2 \leq 4(1 + 2\eta_3)^2 \epsilon_i^2 \cdot \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2\). It follows that
\[
\mathbb{E}_\epsilon \sup_{w \in W} \sum_{t=1}^T w_t(\epsilon)^2 \leq 4(1 + 2\eta_3)^2 \epsilon_i^2 \cdot \mathbb{E}_\epsilon \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2.
\]

Plugging this bound back into the main inequality, we have shown
\[
\mathbb{E}_\epsilon \sup_{w \in W} \sum_{t=1}^T \epsilon_t w_t(\epsilon) \leq 4\epsilon(1 + 2\eta_3) \epsilon_{i+1} \sqrt{\mathbb{E}_\epsilon \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2 \cdot \log \mathcal{N}_{\infty,\infty}(\epsilon_{i+1}, \mathcal{G}, \mathbf{x})}.
\]
C.2.9 Final bound on \((**\))

Collecting terms, we have shown that

\[(**\) \leq (1 + 2\eta_3)e^2\alpha \cdot \mathbb{E}_x \left[ \sum_{t=1}^T \|\ell_t(\epsilon)\|_1 \right] + 4e(1 + 2\eta_3)\sqrt{\mathbb{E}_x \left[ \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2 \right.} \sum_{i=1}^{N-1} \epsilon_{i+1} \log N_{oo,oo}(\epsilon_i, \mathcal{G}, x) + \mathbb{E}_x B_1(\ell_{1:T}(\epsilon)). \]

Following the standard Dudley chaining proof, we have

\[
\sum_{i=1}^{N-1} \epsilon_{i+1} \log N_{oo,oo}(\epsilon_i, \mathcal{G}, x) \leq \sum_{i=1}^N \epsilon_i \log N_{oo,oo}(\epsilon, \mathcal{G}, x) \leq 2 \sum_{i=1}^N (\epsilon_i - \epsilon_{i+1}) \log N_{oo,oo}(\epsilon_i, \mathcal{G}, x) \\
\leq 2 \int_{\epsilon_{N+1}}^\beta \log N_{oo,oo}(\epsilon, \mathcal{G}, x) d\epsilon \leq 2 \int_{\alpha}^\beta \log N_{oo,oo}(\epsilon, \mathcal{G}, x) d\epsilon \\
\leq 2 \int_{\alpha}^\beta \log N_{oo,oo}(\epsilon, \mathcal{G}, T) d\epsilon.
\]

Where we are using the definition of \(N\), which implies that \(\alpha \leq \epsilon_{N+1}\).

Now recall the definition of \(B_1(\ell_{1:T}(\epsilon))\):

\[B_1(\ell_{1:T}(\epsilon)) = \eta_1 \sum_{t=1}^T \|\ell_t(\epsilon)\|_1 + \eta_2 \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2\]

Taking \(\eta_1 \geq (1 + 2\eta_3)e^2\alpha\), the first term in \(B_1\) cancels out the first term in \((10)\), leaving us with

\[(**) \leq 8e(1 + 2\eta_3)\sqrt{\mathbb{E}_x \left[ \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2 \right.} \int_{\alpha}^\beta \log N_{oo,oo}(\epsilon, \mathcal{G}, T) d\epsilon - \eta_2 \mathbb{E}_x \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2
\]

\[\leq 8e(1 + 2\eta_3) \left(\eta_4/4 \mathbb{E}_x \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2 + 1/\eta_4\right) \int_{\alpha}^\beta \log N_{oo,oo}(\epsilon, \mathcal{G}, T) d\epsilon - \eta_2 \mathbb{E}_x \sum_{t=1}^T \|\ell_t(\epsilon)\|_1^2.
\]

Where the last step applies for any \(\eta_4 > 0\) by the AM-GM inequality. For any \(\eta_2 \geq 8e(1 + 2\eta_3)\eta_4 \cdot \int_{\alpha}^\beta \log N_{oo,oo}(\epsilon, \mathcal{G}, T) d\epsilon\), the first and third terms cancel, leaving us with an upper bound of

\[(**\) \leq \frac{8e(1 + 2\eta_3)}{\eta_4} \int_{\alpha}^\beta \log N_{oo,oo}(\epsilon, \mathcal{G}, T) d\epsilon.
\]

This term does not depend on the trees \(x\) or \(\ell\), so we are done with (**).

C.2.10 Final bound

Under the assumptions on \(\eta_1, \eta_2, \eta_3, \eta_4, \alpha,\) and \(\beta,\) the bounds on (*) and (***) we have established imply

\[V \leq \frac{1 + \eta_3^2}{\eta_3} \log N_{oo,oo}(\beta/2, \mathcal{G}, T) + \frac{8e(1 + 2\eta_3)}{\eta_4} \int_{\alpha}^\beta \log N_{oo,oo}(\epsilon, \mathcal{G}, T) d\epsilon.
\]
The definition of $\mathcal{V}$ implies that there exists an algorithm with regret bounded by $\mathcal{V} + B(p_{1:T}, \ell_{1:T})$ on every sequence. The final regret inequality is

$$
\sum_{t=1}^{T} E_{s \sim p_t}(s, \ell_t) - \inf_{g \in \mathcal{G}} \sum_{t=1}^{T} g(x_t), \ell_t
\leq 2\eta_3 \sum_{t=1}^{T} E_{s \sim p_t}(s, \ell_t)^2 + 2 \frac{1 + \eta_3^2}{\eta_3} \log N_{\infty, \infty}(\beta/2, \mathcal{G}, T)
+ 8e(1 + 2\eta_3) \left( \frac{\eta_4}{4} \sum_{t=1}^{T} \|\ell_t\|_1^2 + \frac{1}{\eta_4} \right) \int_{\alpha}^{\beta} \sqrt{\log N_{\infty, \infty}(\epsilon, \mathcal{G}, T)} d\epsilon + (1 + 2\eta_3)e^2 \alpha \sum_{t=1}^{T} \|\ell_t\|_1.
$$

To obtain the bound in the theorem statement, we rebind $\eta = \eta_3$, $\lambda = \eta_4$ and use the assumption $\eta \leq 1$.

### C.3 Proofs for remaining results

Our bandit results require a generalization of Theorem 12 to the case where losses and the class $\mathcal{G}$ may not be bounded by 1.

**Corollary 16.** Suppose we are in the setting of Theorem 12, but with the bounds $\|\ell\|_1 \leq R$ for all $\ell \in \mathcal{L}$ and $\|s\|_{\infty} \leq B$ for all $s \in S$. For any constants $\eta \in (0, 1]$, $\lambda > 0$, and $\beta > \alpha > 0$, there exists an algorithm making predictions in $\mathcal{S}$ that attains a regret guarantee of

$$
\sum_{t=1}^{T} E_{s \sim p_t}(s, \ell_t) - \inf_{g \in \mathcal{G}} \sum_{t=1}^{T} g(x_t), \ell_t
\leq \frac{2\eta}{RB} \sum_{t=1}^{T} E_{s \sim p_t}(s, \ell_t)^2 + \frac{4RB}{\eta} \log N_{\infty, \infty}(\beta/2, \mathcal{G}, T) + 3e^2 \alpha \sum_{t=1}^{T} \|\ell_t\|_1
+ 24e(\frac{\lambda R}{4R} \sum_{t=1}^{T} \|\ell_t\|_1^2 + \frac{R}{\lambda} \int_{\alpha}^{\beta} \sqrt{\log N_{\infty, \infty}(\epsilon, \mathcal{G}, T)} d\epsilon.
$$

Furthermore, if upper bounds $\sum_{t=1}^{T} \|\ell_t\|_1^2 \leq V$ and $\sum_{t=1}^{T} E_{s \sim p_t}(s_t, \ell_t)^2 \leq \tilde{V}$ are known in advance, $\eta$ and $\lambda$ can be selected to guarantee regret

$$
\sum_{t=1}^{T} E_{s \sim p_t}(s_t, \ell_t) - \inf_{g \in \mathcal{G}} \sum_{t=1}^{T} g(x_t), \ell_t
\leq 8\sqrt{\tilde{V}} \cdot \log N_{\infty, \infty}(\beta/2, T, T) + 8RB \log N_{\infty, \infty}(\beta/2, \mathcal{G}, T)
+ 24e\sqrt{\tilde{V}} \int_{\alpha}^{\beta} \sqrt{\log N_{\infty, \infty}(\epsilon, \mathcal{G}, T)} d\epsilon + 3e\alpha \sum_{t=1}^{T} \|\ell_t\|_1.
$$

**Proof of Corollary 16.** Apply Theorem 12 with losses $\ell_t/R$ and class $\mathcal{G}/B$. The preconditions of the theorem are satisfied, so it implies existence of an algorithm making predictions in $\mathcal{S}/B$ with regret bound

$$
\frac{1}{R} \sum_{t=1}^{T} E_{s \sim p_t}(s_t, \ell_t) - \frac{1}{R} \inf_{g \in \mathcal{G}/B} \sum_{t=1}^{T} g(x_t), \ell_t
\leq \frac{2\eta}{R^2} \sum_{t=1}^{T} E_{s \sim p_t}(s_t, \ell_t)^2 + \frac{4}{\eta} \log N_{\infty, \infty}(\beta/2, \mathcal{G}/B, T) + 3e^2 \alpha \sum_{t=1}^{T} \|\ell_t\|_1
+ 24e(\frac{\lambda R}{4R^2} \sum_{t=1}^{T} \|\ell_t\|_1^2 + \frac{1}{\lambda} \int_{\alpha}^{\beta} \sqrt{\log N_{\infty, \infty}(\epsilon, \mathcal{G}/B, T)} d\epsilon.
$$
Rescaling both sides by $BR$ and letting $\hat{s}_t = s_t \cdot B$ (so $\hat{s}_t \in S$), this implies
\[
\sum_{t=1}^{T} \mathbb{E}_{\hat{s}_t \sim p_t}(\hat{s}_t, \ell_t) - \inf_{g \in \mathcal{G}} \sum_{t=1}^{T} (g(x_t), \ell_t) \leq \frac{2\eta}{RB} \sum_{t=1}^{T} \mathbb{E}_{\hat{s}_t \sim p_t}(\hat{s}_t, \ell_t)^2 + \frac{4RB}{\eta} \log \mathcal{N}_{\infty, \infty}(\beta/2, \mathcal{G}/B, T) + 3e^2 \alpha B \sum_{t=1}^{T} \|\ell_t\|_1 \\
+ 24e \left( \frac{\lambda B}{4R} \sum_{t=1}^{T} \|\ell_t\|_1 + \frac{RB}{\lambda} \right) \int_{\alpha}^{\beta} \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \mathcal{G}, T)} d\varepsilon.
\]

Using a change of variables in the Dudley integral, we get
\[
\leq \frac{2\eta}{RB} \sum_{t=1}^{T} \mathbb{E}_{\hat{s}_t \sim p_t}(\hat{s}_t, \ell_t)^2 + \frac{4RB}{\eta} \log \mathcal{N}_{\infty, \infty}(\beta B/2, \mathcal{G}, T) + 3e^2 \alpha B \sum_{t=1}^{T} \|\ell_t\|_1 \\
+ 24e \left( \frac{\lambda B}{4R} \sum_{t=1}^{T} \|\ell_t\|_1 + \frac{RB}{\lambda} \right) \int_{\alpha}^{\beta} \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \mathcal{G}, T)} d\varepsilon.
\]

The final result follows by rebinding $\alpha' = \alpha B$ and $\beta' = \beta B$.

For the second claim, apply the upper bounds to obtain
\[
\frac{2\eta}{RB} \tilde{V} + \frac{4RB}{\eta} \log \mathcal{N}_{\infty, \infty}(\beta/2, \mathcal{G}, T) + 3e^2 \alpha B \sum_{t=1}^{T} \|\ell_t\|_1 \\
+ 24e \left( \frac{\lambda}{4R} \tilde{V} + \frac{R}{\lambda} \right) \int_{\alpha}^{\beta} \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \mathcal{G}, T)} d\varepsilon.
\]

Now set $\lambda = 2R/\sqrt{\tilde{V}}$ and $\eta = \sqrt{2RB} \sqrt{\log \mathcal{N}_{\infty, \infty}(\beta/2, \mathcal{G}, T)/\tilde{V}} \wedge 1$ to obtain the claimed bound. Note that the range term arises from the constraint that $\eta \in (0, 1]$.

**Proof of Theorem 4.** Recall that we use the reduction:

- Initialize full information algorithm whose existence is guaranteed by Theorem 12 with $\mathcal{G} = \phi^\gamma \circ \mathcal{F}$:

- For time $t = 1, \ldots, T$:
  - Receive $x_t$ and define $P_t(a) = \mathbb{E}_{s_t \sim p_t} \sum_{a' \in [K]} s_t(a')$, where $p_t$ is the output of the full information algorithm at time $t$.
  - Sample action $a_t \sim P_t^\mu$ and feed importance-weighted loss $\hat{\ell}_t(a) = 1\{a_t = a\} \ell_t(a)/P_t^\mu(a)$ into the full information algorithm.

With this setup, Corollary 16 guarantees that the following deterministic regret inequality holds for every sequence of outcomes (i.e. for every sequence $a_1, \ldots, a_T$ sampled by the algorithm):
\[
\sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_t}(s_t, \hat{\ell}_t) - \inf_{f \in \mathcal{F}} \sum_{t=1}^{T} \phi^\gamma(f(x_t)), \hat{\ell}_t) \\
\leq \frac{2\eta}{RB} \sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_t}(s_t, \ell_t)^2 + \frac{4RB}{\eta} \log \mathcal{N}_{\infty, \infty}(\beta/2, \phi^\gamma \circ \mathcal{F}, T) + 3e^2 \alpha \sum_{t=1}^{T} \|\ell_t\|_1 \\
+ 24e \left( \frac{\lambda}{4R} \sum_{t=1}^{T} \|\ell_t\|_1 + \frac{R}{\lambda} \right) \int_{\alpha}^{\beta} \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \phi^\gamma \circ \mathcal{F}, T)} d\varepsilon.
\]
where the boundedness of the ramp loss implies $B \leq 1$ and the smoothing factor $\mu$ in $P_t^\mu$ guarantees $R \leq 1/\mu$. Taking expectation over the draw of $a_1, \ldots, a_T$, for any fixed $f \in \mathcal{F}$ we obtain the inequality

$$
\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}_{s_{t-1}} \langle s_t, \hat{\ell}_t \rangle - \sum_{t=1}^{T} \langle \phi^\gamma(f(x_t)), \hat{\ell}_t \rangle\right] 
\leq \mathbb{E}\left[\frac{2\eta}{1/\mu} \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{E}_{s_{t-1}} \langle s_t, \hat{\ell}_t \rangle^2 \mid \mathcal{J}_t\right] + \frac{4}{\lambda\mu} \log \mathcal{N}_{\infty, \infty}(\beta/2, \phi^\gamma \circ \mathcal{F}, T) + 3e^2 \alpha \mathbb{E}\left[\|\hat{\ell}_t\|_1 \mid \mathcal{J}_t\right] + 24e \left(\frac{\lambda KT}{4} + \frac{1}{\lambda\mu}\right) \int_0^\beta \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \phi^\gamma \circ \mathcal{F}, T) d\varepsilon}\right],
$$

where the filtration $\mathcal{J}_t$ is defined as in Lemma 3. Using that the importance weighted losses are unbiased, we have that the left-hand side is equal to

$$
\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}_{s_{t-1}} \langle s_t, \hat{\ell}_t \rangle - \sum_{t=1}^{T} \langle \phi^\gamma(f(x_t)), \hat{\ell}_t \rangle\right].
$$

We also have the following three properties, where the first two use that $\hat{\ell}_t$ is 1-sparse, and the last follows from Lemma 3:

1. $\mathbb{E}\left[\|\hat{\ell}_t\|_1 \mid \mathcal{J}_t\] = \sum_{a \in [K]} P_t^\mu(a) \hat{\ell}_t(a) = \sum_{a \in [K]} \ell_t(a) \leq K$.

2. $\mathbb{E}\left[\|\hat{\ell}_t\|_2^2 \mid \mathcal{J}_t\] = \sum_{a \in [K]} P_t^\mu(a) \hat{\ell}_t^2(a) = \sum_{a \in [K]} \frac{\ell_t(a)}{P_t^\mu(a)} \leq K \mu$.

3. $\mathbb{E}\left[\mathbb{E}_{s_{t-1}} \langle s_t, \hat{\ell}_t \rangle^2 \mid \mathcal{J}_t\] \leq K^2$.

Together, these facts yield the bound

$$
\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}_{s_{t-1}} \langle s_t, \hat{\ell}_t \rangle - \sum_{t=1}^{T} \langle \phi^\gamma(f(x_t)), \hat{\ell}_t \rangle\right] \leq \frac{2\eta}{1/\mu} K^2 T + \frac{4}{\lambda\mu} \log \mathcal{N}_{\infty, \infty}(\beta/2, \phi^\gamma \circ \mathcal{F}, T) + 3e^2 \alpha KT \left(\frac{\lambda KT}{4} + \frac{1}{\lambda\mu}\right) \int_0^\beta \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \phi^\gamma \circ \mathcal{F}, T) d\varepsilon}\right].
$$

Optimizing $\eta$ and $\lambda$ (as in the proof of the second claim of Corollary 16) leads to a bound of

$$
\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}_{s_{t-1}} \langle s_t, \ell_t \rangle - \sum_{t=1}^{T} \langle \phi^\gamma(f(x_t)), \ell_t \rangle\right] \leq 4\sqrt{2K^2 T \log \mathcal{N}_{\infty, \infty}(\beta/2, \phi^\gamma \circ \mathcal{F}, T)} + \frac{8}{\mu} \log \mathcal{N}_{\infty, \infty}(\beta/2, \phi^\gamma \circ \mathcal{F}, T) + 3e^2 \alpha KT + 24e \sqrt{\frac{KT}{\mu}} \int_0^\beta \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \phi^\gamma \circ \mathcal{F}, T) d\varepsilon}.
$$

Since $\phi^\gamma$ is $1/\gamma$-Lipschitz with respect to the $\ell_\infty$ norm (as a coordinate-wise mapping from $\mathbb{R}^K$ to $\mathbb{R}^K$), we can upper bound in terms of the covering numbers for the original class:

$$
\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{E}_{s_{t-1}} \langle s_t, \ell_t \rangle - \sum_{t=1}^{T} \langle \phi^\gamma(f(x_t)), \ell_t \rangle\right] \leq 4\sqrt{2K^2 T \log \mathcal{N}_{\infty, \infty}(\gamma\beta/2, \mathcal{F}, T)} + \frac{8}{\mu} \log \mathcal{N}_{\infty, \infty}(\gamma\beta/2, \mathcal{F}, T) + 3e^2 \alpha KT + 24e \sqrt{\frac{KT}{\mu}} \int_0^\beta \sqrt{\log \mathcal{N}_{\infty, \infty}(\varepsilon, \mathcal{F}, T) d\varepsilon}.
$$
Using a change of variables and the reparameterization $\alpha' = \alpha \gamma$, $\beta' = \beta \gamma$, the right hand side equals

$$4\sqrt{2K^2T \log N_{\infty, \infty}(\beta'/2, \mathcal{F}, T)} + \frac{S}{\mu} \log N_{\infty, \infty}(\beta'/2, \mathcal{F}, T) + \frac{1}{\gamma}\left(3e^2\alpha KT + 24e^2\sqrt{\frac{KT}{\mu}} \int_{\alpha'}^{\beta'} \sqrt{\log N_{\infty, \infty}(\varepsilon, \mathcal{F}, T)} d\varepsilon \right).$$

Lastly, via Lemma 1, we have

$$\sum_{t=1}^{T} \mathbb{E}_{s_t \sim p_t} (s_t, \ell_t) \geq \frac{\sum_{a \in [K]} s_t(a) \ell_t(a)}{\sum_{a \in [K]} s_t(a)} = \sum_{t=1}^{T} \mathbb{E}_{a_t \sim P_t} \ell_t(a_t).$$

Finally, the definition of the smoothed distribution $P_t^\mu$ and boundedness of $\ell$ immediately implies

$$\sum_{t=1}^{T} \mathbb{E}_{a_t \sim P_t} \ell_t(a_t) = \sum_{t=1}^{T} \mathbb{E}_{a_t \sim P_t^\mu} \ell_t(a_t) - \mu KT. \quad \square$$

**Proof of Proposition 5.** Suppose $\log N_{\infty, \infty}(\varepsilon, \mathcal{F}, T) \propto \varepsilon^{-p}$.

- When $p \geq 2$, it suffices to set $\beta = \operatorname{rad}_{\infty, \infty}(\mathcal{F}, T)$, $\mu = (KT)^{-1/(p+1)} \gamma^{-p/(p+1)}$, and $\alpha = 1/(KT)^{1/p}$ in Theorem 4 to obtain $\mathcal{O}\left((KT\gamma)^{p/(p+1)}\right)$.

- When $p \in (0, 2]$, it suffices to set $\alpha = 1/(KT)$, $\mu = (KT)^{-2/(p+4)} \gamma^{-2p/(p+4)}$, and $\beta = \mathcal{O}\left((KT\mu)^{1/(2+p)}\right)$ in Theorem 4 to obtain $\mathcal{O}\left((KT\mu)^{1/(2+p)}\right)$.

For the parametric case, set $\alpha = \beta = \gamma/\mu$ and $\mu = \sqrt{d \log(KT/\gamma)/KT}$ to conclude the bound. Similarly, in the finite class case, set $\alpha = \beta = 0$ and $\mu = \sqrt{\log|\Pi|/KT}$. \quad \square

**Proof of Example 3.** Let $\mathcal{F}|_a = \{x \mapsto f(x)_a \mid f \in \mathcal{F}\}$. Then clearly it holds that

$$\log N_{\infty, \infty}(\varepsilon, \mathcal{F}, T) \leq \sum_{a \in [K]} \log N_{\infty}(\varepsilon, \mathcal{F}|_a, T) \leq K \max_{a \in [K]} \log N_{\infty}(\varepsilon, \mathcal{F}|_a, T),$$

where have dropped the second “$\infty$” subscript on the right-hand side to denote that this is the covering number for a scalar-valued class. Let $a^*$ be the action that obtains the maximum in this expression. Returning to the integral expression in Theorem 4, we have just shown an upper bound of

$$3e^2\alpha KT + 24e^2K \sqrt{\frac{T}{\mu}} \int_{\alpha}^{\beta} \sqrt{\log N_{\infty}(\varepsilon, \mathcal{F}|_{a^*}, T)} d\varepsilon.$$

For any scalar-value function class $\mathcal{G} \subseteq (\mathcal{X} \rightarrow [0, 1])$, define

$$\mathfrak{R}(\mathcal{G}, T) = \sup_x \mathbb{E}_x \sup_{g \in \mathcal{G}} \sum_{t=1}^{T} \ell_t g(x_t(\varepsilon)).$$

Following the proof of Lemma 9 in Rakhlin et al. (2015b), by choosing $\beta = 1$ and $\alpha = 2\mathfrak{R}(\mathcal{F}|_{a^*}, T)/T$, we may upper bound the $L_{\infty}$ covering number by the sequential Rademacher complexity (via fat-shattering), to obtain

$$6eK\mathfrak{R}(\mathcal{F}|_{a^*}, T) + 96\sqrt{2eK} \sqrt{\frac{T}{\mu}} \mathfrak{R}(\mathcal{F}|_{a^*}, T) \int_{2\mathfrak{R}(\mathcal{F}|_{a^*}, T)/T}^{1} \frac{1}{\varepsilon} \sqrt{\log(2eT/\varepsilon)} d\varepsilon.$$
Using straightforward calculation from the proof of Lemma 9 in Rakhlin et al. (2015b), this is upper bounded by

\[ O\left(\frac{K}{\sqrt{\mu}} \mathcal{R}(\mathcal{F}|a^*, T) \log^{3/2}(T/\mathcal{R}(\mathcal{F}|a^*, T))\right) \]

Returning to the regret bound in Theorem 4, we have shown an upper bound of

\[ O\left(\frac{K}{\gamma \sqrt{\mu}} \mathcal{R}(\mathcal{F}|a^*, T) \log^{3/2}(T/\mathcal{R}(\mathcal{F}|a^*, T)) + \mu KT\right) \]

where we have used that \( \log N_\infty(1, \mathcal{F}|a^*, T) = 0 \) under the boundedness assumption on \( \mathcal{F} \). Setting \( \mu \propto \left(\frac{\mathcal{R}(\mathcal{F}|a^*, T)}{(T\gamma)}\right)^{2/3} \) yields the result.

**Proof of Example 4.** This is an immediate consequence of Example 3 and that Banach spaces for which the martingale type property holds with constant \( \beta \) have sequential Rademacher complexity \( O(\sqrt{\beta T}) \) (Srebro et al., 2011).

### C.4 Additional results

Here we briefly state an analogue of Theorem 4 for the hinge loss. Note that this bound leads to the same exponents for \( T \) as Theorem 4, but has worse dependence on the margin \( \gamma \) and depends on the scale parameter \( B \) explicitly.

**Theorem 17** (Contextual bandit chaining bound for hinge loss). For any fixed constants \( \beta > \alpha > 0 \), hinge loss parameter \( \gamma > 0 \), and smoothing parameter \( \mu \in (0, 1/K] \) there exists an adversarial contextual bandit strategy \( (P_t)_{t \leq T} \) with expected regret bounded as

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \ell_t(a_t) \right] \leq \frac{1}{K} \left\{ \inf_{f \in \mathcal{F}} \mathbb{E} \left[ \sum_{t=1}^{T} \left\langle \psi^\gamma(f(x_t)), \ell_t \right\rangle \right] + \frac{1}{\gamma} \sqrt{2K^2B^2T \log N_{\infty, T}(\beta/2, \mathcal{F}, T)} + \mu K^2T \\
+ \frac{8B}{\gamma \mu} \log N_{\infty, T}(\beta/2, \mathcal{F}, T) + \frac{1}{\gamma} \left( 3\epsilon \alpha KT + 24\epsilon \sqrt{\frac{KT}{\mu}} \int_0^\beta \sqrt{\log N_{\infty, T}(\epsilon, \mathcal{F}, T)} d\epsilon \right) \right\},
\]

where we recall \( B = \sup_{f \in \mathcal{F}} \sup_{x \in X} \| f(x) \|_\infty \).

### D Analysis of HINGE-LMC

This appendix contains the proofs of Theorem 6 and the corresponding corollaries. The proof has many ingredients which we compartmentalize into subsections. First, in Subsection D.1, we analyze the sampling routine, showing that Langevin Monte Carlo can be used to generate a sample from an approximation of the exponential weights distribution. Then, in Subsection D.2, we derive the regret bound for the continuous version of exponential weights. Finally, we put the components together together, instantiate all parameters, and compute the final regret and running time in Subsection D.3. The corollaries are straightforward and proved in Subsection D.4

To begin, we restate the main theorem, with all the assumptions and the precise parameter settings.

**Theorem 18.** Let \( \mathcal{F} \) be a set of functions parameterized by a compact convex set \( \Theta \subset \mathbb{R}^d \) that contains the origin-centered Euclidean ball of radius 1 and is contained within a Euclidean ball of radius \( R \). Assume that
Therefore, the algorithm runs in polynomial time.

Let us define the Wasserstein distance. For random variables \(X, Y\), if \(f(x; \theta)\) is convex in \(\theta\) for each \(x \in \mathcal{X}\), and that \(\sup_{x, \theta} \|f(x; \theta)\|_{\infty} \leq B\), that \(f(x, a; \theta)\) is \(L\)-Lipschitz as a function of \(\theta\) with respect to the \(\ell_2\) norm for each \(x, a\). For any \(\gamma\), if we set

\[
\eta = \sqrt{\frac{d\gamma^2 \log(RLTK/\gamma)}{5K^2B^2T}}, \quad \mu = \sqrt{\frac{1}{K^2T}}, \quad M = \sqrt{T},
\]

in Hinge-LMC, and further set

\[
u = \frac{1}{T^{3/2}LB_tR_\eta\sqrt{d}}, \quad \lambda = \frac{1}{8T^{1/2}R_3}, \quad \alpha = \frac{R^2}{N},
\]

\[
N = \tilde{O} \left( R^{18}L^{12}T^6d^{12} + \frac{R^{24}L^{48}d^{12}}{K^{24}} \right), \quad m = \tilde{O} \left( T^3dR^4L^2B_t^2(K\gamma)^{-2} \right),
\]

in each call to Projected LMC, then Hinge-LMC guarantees

\[
\sum_{t=1}^{T} \mathbb{E}_{\ell_t}(a_t) \leq \min_{\theta \in \Theta} \sum_{t=1}^{T} \mathbb{E}_{\ell_t}(\psi^\gamma(f(x_t; \theta))) + \sqrt{\frac{T}{\gamma}} + \frac{2d}{K\eta} \log(RLTK/\gamma) + \frac{10\eta}{\gamma^2}B^2KT
\]

\[
\leq \min_{\theta \in \Theta} \sum_{t=1}^{T} \mathbb{E}_{\ell_t}(\psi^\gamma(f(x_t; \theta))) + \tilde{O} \left( \frac{B}{\gamma} \sqrt{dT} \right).
\]

Moreover, the running time of Hinge-LMC is \(\tilde{O} \left( \frac{R^{22}L^{14}d^{14}B_t^2T^{10}}{K^2\gamma^2} + \frac{R^{28}L^{50}d^{14}B_t^2T^4}{K^2\gamma^2} \right)\).

### D.1 Analysis of the sampling routine

In this section, we show how Projected LMC can be used to generate a sample from a distribution that is close to the exponential weights distribution. Define

\[
F(\theta) = \eta \sum_{\tau=1}^{t} \langle \tilde{\ell}_\tau, \psi^\gamma(f(x_\tau; \theta)) \rangle, \quad P(\theta) = \exp(-F(\theta)).
\]

(11)

We are interested in sampling from \(P(\theta)\).

Let us define the Wasserstein distance. For random variables \(X, Y\) with density \(\mu, \nu\) respectively

\[
\mathcal{W}_1(\mu, \nu) \triangleq \inf_{\pi \in \Gamma(\mu, \nu)} \int \|X - Y\|_2d\pi(X, Y) = \sup_{f \in \text{Lip}(f)} \left| \int f(d\mu(X) - d\nu(Y)) \right|.
\]

Here \(\Gamma(\mu, \nu)\) is the set of couplings between the two densities, that is the set of joint distributions with marginals equal to \(\mu, \nu\). \(\text{Lip}(f)\) is the set of all functions that are \(1\)-Lipschitz with respect to \(\ell_2\).

**Theorem 19.** Let \(\Theta \subset \mathbb{R}^d\) be a convex set containing a Euclidean ball of radius \(r = 1\) with center \(0\), and contained within a Euclidean ball of radius \(R\). Let \(f : \mathcal{X} \times \Theta \to \mathbb{R}^K\) be convex in \(\theta\) with \(f_\alpha(x; \cdot)\) being \(L\)-Lipschitz w.r.t. \(\ell_2\) norm for each \(\alpha \in \mathcal{A}\). Assume \(\|\tilde{\ell}_r\|_1 \leq B_t\) and define \(F\) and \(P\) as in (11). Let a target accuracy \(\tau > 0\) be fixed. Then Algorithm 3 with parameters \(m, N, \lambda, \mu, \alpha \in \text{poly}(1/\tau, d, R, \eta, B_t, L)\) generates a sample from a distribution \(\tilde{P}\) satisfying

\[
\mathcal{W}_1(\tilde{P}, P) \leq \tau.
\]

Therefore, the algorithm runs in polynomial time.
Algorithm 3 Smoothed Projected Langevin Monte Carlo for (11)

Input: Parameters $m, u, \lambda, N, \alpha$.

Set $\tilde{\theta}_0 \leftarrow 0 \in \mathbb{R}^d$

for $k = 1, \ldots, N$ do

Sample $z_1, \ldots, z_m \sim \mathcal{N}(0, u^2 I_d)$ and form the function

$$F_k(\theta) = \frac{1}{m} \sum_{i=1}^m F(\theta + z_i) + \frac{\lambda}{2} \|\theta\|_2^2.$$ 

Sample $\xi_k \sim \mathcal{N}(0, I_d)$ and update

$$\tilde{\theta}_k \leftarrow \mathcal{P}_\Theta \left( \tilde{\theta}_{k-1} - \frac{\alpha}{2} \nabla \tilde{F}_k(\tilde{\theta}_{k-1}) + \sqrt{\xi_k} \right).$$

end for

Return $\tilde{\theta}_N$.

The precise values for each of the parameters $m, N, u, \lambda, \alpha$ can be found at the end of the proof, which will lead to a setting of $\tau$ in application of the theorem.

Towards the proof, we will introduce the intermediate function $\tilde{F}(\theta) = \mathbb{E}_Z F(\theta + Z) + \frac{\lambda}{2} \|\theta\|_2^2$, where $Z$ is a random variable with distribution $\mathcal{N}(0, u^2 I_d)$. This is the randomized smoothing technique studied by Duchi, Bartlett and Wainwright (Duchi et al., 2012). The critical properties of this function are

**Proposition 20** (Properties of $\tilde{F}$). Under the assumptions of Theorem 19, The function $\tilde{F}$ satisfies

1. $F(\theta) \leq \tilde{F}(\theta) \leq F(\theta) + \eta TB_L u \sqrt{d}/\gamma + \frac{\lambda}{2} R^2$.
2. $\tilde{F}(\theta)$ is $\eta TB_L u \sqrt{d}/\gamma + \lambda R$-Lipschitz with respect to the $\ell_2$ norm.
3. $\tilde{F}(\theta)$ is continuously differentiable and its gradient is $\frac{\eta TB_L u \sqrt{d}}{\gamma} + \lambda$-Lipschitz continuous with respect to the $\ell_2$ norm.
4. $\tilde{F}(\theta)$ is $\lambda$-strongly convex with respect to the $\ell_2$ norm.
5. $\mathbb{E} \nabla F(\theta + Z) = \nabla \tilde{F}(\theta)$.

**Proof.** See Duchi et al. (2012, Lemma E.3) for the proof of all claims, except for claim 4, which is an immediate consequence of the $\ell_2$ regularization term.

Using property 1 in Proposition 20 and setting $\varepsilon_1 \triangleq \eta TB_L u \sqrt{d}/\gamma + \lambda R^2$, we know that

$$e^{-\varepsilon_1} \exp(-F(\theta)) \leq \exp(-\tilde{F}(\theta)) \leq \exp(-F(\theta)),$$

pointwise. Therefore, defining $\hat{P}$ to be the distribution with density $\hat{p}(\theta) = \exp(-\tilde{F}(\theta))/\hat{Z}$, where $\hat{Z} = \int \exp(-\tilde{F}(\theta)) d\theta$, we have

$$TV(P \parallel \hat{P}) = \int \frac{e^{-F(\theta)}}{\hat{Z}} \left| \frac{e^{-\tilde{F}(\theta) + F(\theta)}}{\hat{Z}/\hat{Z}} - 1 \right| d\theta \leq e^{\varepsilon_1} - 1 \leq 2\varepsilon_1,$$

for $\varepsilon_1 \leq 1$. This shows that $\hat{P}$ approximates $P$ well when $u$ and $\lambda$ are sufficiently small. The next lemma further shows that the $\tilde{F}_k$ functions themselves approximate $\tilde{F}$ well.
Lemma 21 (Properties of $\tilde{F}_k$). For any fixed $\theta$, $k \in [N]$, and constant $\varepsilon_2 > 0$,
\[
\mathbb{P}\left[\|\nabla \tilde{F}(\theta) - \nabla \tilde{F}_k(\theta)\|_2 \geq \varepsilon_2 + \frac{2}{\sqrt{m}} \frac{\eta TB_t L}{\gamma} \right] \leq \exp\left(\frac{-4\varepsilon_2^2 \gamma^2 m}{(\eta T LB_t)^2}\right).
\]

Proof of Lemma 21. Let $k$ be fixed. Since $\tilde{F}_k$ are identically distributed for all $k$ we will henceforth abbreviate to $\tilde{F}$.

We proceed using a crude concentration argument. Observe that by Proposition 20, $\mathbb{E}\nabla \tilde{F}(\theta) = \nabla \tilde{F}(\theta)$ and moreover $\nabla \tilde{F}(\theta)$ is a sum of $m$ i.i.d., vector-valued random variables (plus the deterministic regularization term).

Via the Chernoff method, for any fixed $\theta$, we have
\[
\mathbb{P}\left[\|\nabla \tilde{F}(\theta) - \nabla \tilde{F}(\theta)\|_2 \geq t\right] \leq \inf_{\beta > 0} \exp(-t \beta) \mathbb{E} \exp(\beta \|\nabla \tilde{F}(\theta) - \nabla \tilde{F}(\theta)\|_2)
\]

Using the sum structure and symmetrizing:
\[
\leq \inf_{\beta > 0} \exp(-t \beta) \mathbb{E}_{z_1:m} \mathbb{E}_{\epsilon} \exp\left(2\beta \left\| \frac{1}{m} \sum_{i=1}^m \epsilon_i \nabla G(\theta + z_i) \right\|_2 \right),
\]

where $G(\theta) = \eta \sum_{\tau=1}^k \{\tilde{\ell}_\tau, \psi^\gamma(f(x_\tau; \theta))\}$. Condition on $z_1:m$ and let $W(\epsilon) = \left\| \frac{1}{m} \sum_{i=1}^m \epsilon_i \nabla G(\theta + z_i) \right\|_2$. Then for any $i$,
\[
|W(\epsilon_1, \ldots, \epsilon_i, \ldots, \epsilon_m) - W(\epsilon_1, \ldots, -\epsilon_i, \ldots, \epsilon_m)| \leq \frac{1}{m} \left\| \nabla G(\theta + z_i) \right\|_2
\]
\[
\leq \frac{\eta}{m} \sum_{\tau=1}^k \left\| \tilde{\ell}_\tau \right\|_1 \left\| \nabla \psi^\gamma(f(x_\tau; \theta + z_i)) \right\|_2
\]
\[
\leq \frac{\eta T B_t L}{m \gamma}.
\]

By the standard bounded differences argument (e.g. (Boucheron et al., 2013)), this implies that $W - \mathbb{E} W$ is subgaussian with variance proxy $\sigma^2 = \frac{1}{4m} \left(\frac{\eta T B_t L}{\gamma}\right)^2$. Furthermore, the standard application of Jensen’s inequality implies that $\mathbb{E} W \leq 2\sigma$.

Returning to the upper bound, these facts together imply
\[
\mathbb{E}_{\epsilon} \exp\left(2\beta \left\| \frac{1}{m} \sum_{i=1}^m \epsilon_i \nabla G(\theta + z_i) \right\|_2 \right) \leq \exp(2\beta^2 \sigma^2 + 4\beta \sigma).
\]

The final bound is therefore,
\[
\mathbb{P}\left[\|\nabla \tilde{F}(\theta) - \nabla \tilde{F}(\theta)\|_2 \geq t\right] \leq \inf_{\beta > 0} \exp(-t \beta + 2\beta^2 \sigma^2 + 4\beta \sigma).
\]

Rebinding $t = t' + 4\sigma$ for $t' \geq 0$, we have
\[
\mathbb{P}\left[\|\nabla \tilde{F}(\theta) - \nabla \tilde{F}(\theta)\|_2 \geq t' + 4\sigma\right] \leq \inf_{\beta > 0} \exp(-t' \beta + 2\beta^2 \sigma^2) = \exp(-(t')^2/8\sigma^2).
\]

\[\square\]
Now, for the purposes of the proof, suppose we run the Projected LMC algorithm on the function \( \hat{F} \), which generates the iterate sequence \( \hat{\theta}_0 = 0 \)
\[
\hat{\theta}_k \leftarrow \mathcal{P}_\Theta \left( \hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1}) + \sqrt{\alpha} \xi_k \right).
\]

Owing to the smoothness of \( \hat{F} \), we may apply the analysis of Projected LMC due to Bubeck, Eldan, and Lehec (Bubeck et al., 2018) to bound the total variation distance between the random variable \( \hat{\theta}_N \) and the distribution with density proportional to \( \exp(-\hat{F}(\theta)) \).

**Theorem 22 (Bubeck et al. (2018)).** Let \( \hat{P} \) be the distribution on \( \Theta \) with density proportional to \( \exp(-\hat{F}(\theta)) \). For any \( \varepsilon > 0 \) and with \( \alpha = \Theta(R^2/N) \), we have \( TV(\hat{\theta}_N, \hat{P}) \leq \varepsilon \) with
\[
N \geq \tilde{\Omega} \left( \frac{R^6 \max\{d, R\eta TB_\ell L/\gamma + R^2 \lambda, R(\eta TB_\ell L/(\eta \gamma) + \lambda)\}}{\varepsilon^2} \right)^{12}.
\]
This specializes the result of Bubeck et al. (2018) to our setting, using the Lipschitz and smoothness constants from Proposition 20.

Unfortunately, since we do not have access to \( \hat{F} \) in closed form, we cannot run the Projected LMC algorithm on it exactly. Instead, **Algorithm 3** runs LMC on the sequence of approximations \( \hat{F}_k \) and generates the iterate sequence \( \hat{\theta}_k \). The last step in the proof is to relate our iterate sequence \( \hat{\theta}_k \) to a hypothetical iterate sequence \( \tilde{\theta}_k \) formed by running Projected LMC on the function \( \hat{F} \).

**Lemma 23.** Let \( \varepsilon_2 \) be fixed. Assume the conditions of Theorem 19—in particular that
\[
m \geq 16(\eta TLB_\ell/\gamma)^2 \log(4R/\alpha \varepsilon_2)/\varepsilon_2^2, \quad \alpha \leq 2(\eta TLB_\ell/(\eta \gamma) + \lambda)^{-1}.
\]
Then for any \( k \in [N] \) we have
\[
\mathcal{W}_1(\hat{\theta}_k, \tilde{\theta}_k) \leq k \alpha \varepsilon_2.
\]

**Proof of Lemma 23.** The proof is by induction, where the base case is obvious, since \( \hat{\theta}_0 = \tilde{\theta}_0 \). Now, let \( \pi_{k-1}^* \) denote the optimal coupling for \( \hat{\theta}_{k-1}, \tilde{\theta}_{k-1} \) and extend this coupling in the obvious way by sampling \( z_1, \ldots, z_m \) i.i.d. and by using the same gaussian random variable \( \xi_k \) in both LMC updates. Let \( \xi_k = \{ z_1, \ldots, z_m : \|\nabla \hat{F}(\tilde{\theta}_{k-1}) - \nabla \hat{F}(\hat{\theta}_{k-1})\| \leq \varepsilon_2 + \varepsilon' \} \), where \( \varepsilon' = \frac{2}{\sqrt{m}} \cdot \frac{\eta TLB_\ell}{\gamma} \), this is the “good” event in which the samples provide a high-quality approximation to the gradient at \( \hat{\theta}_{k-1} \). We then have
\[
\mathcal{W}_1(\hat{\theta}_k, \tilde{\theta}_k) = \inf_{\pi \in \Pi(\hat{\theta}_k, \tilde{\theta}_k)} \int \|\hat{\theta}_k - \tilde{\theta}_k\|_2 d\pi
\leq \int \mathbb{E}_{z_1, \ldots, z_m} \{ \mathcal{E}_k \} \|\mathcal{P}_\Theta(\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1}) + \sqrt{\alpha} \xi_k) - \mathcal{P}_\Theta(\tilde{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\tilde{\theta}_{k-1}) + \sqrt{\alpha} \xi_k)\|_2 d\pi_{k-1}^*
\leq \int \mathbb{E}_{z_1, \ldots, z_m} \{ \mathcal{E}_k \} \|\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1}) - (\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1}))\|_2 d\pi_{k-1}^* + 2R \int \mathbb{P}[\mathcal{E}_k^C] d\pi_{k-1}^*
\leq \int \mathbb{E}_{z_1, \ldots, z_m} \{ \mathcal{E}_k \} \|\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1}) - (\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1}))\|_2 d\pi_{k-1}^* + 2R \exp \left( \frac{-\varepsilon_2^2 \gamma^2 m}{(\eta TLB_\ell)^2} \right).
\]
The first inequality introduces the potentially suboptimal coupling \( \pi_{k-1}^* \). In the second inequality we first use that the projection operator is contractive, and we also use that the domain is contained in a Euclidean ball of radius \( R \), providing a coarse upper bound on the second term. For the third inequality, we apply the
concentration argument in Lemma 21. Working just with the first term, using the event in the indicator, we have
\[
\int \mathbb{E}_{\varepsilon k} \mathbb{1}\{E_k\} \|\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1}) - \left(\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1})\right)\|_{2d\pi_{k-1}^*}^2 \\
\leq \int \|\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1}) - \left(\hat{\theta}_{k-1} - \frac{\alpha}{2} \nabla \hat{F}(\hat{\theta}_{k-1})\right)\|_{2d\pi_{k-1}^*}^2 + \frac{\alpha (\varepsilon_2 + \varepsilon')}{2}.
\]
Now, observe that we are performing one step of gradient descent on \(\hat{F}\) from two different starting points, \(\hat{\theta}_{k-1}\) and \(\tilde{\theta}_{k-1}\). Moreover, we know that \(\hat{F}\) is smooth and strongly convex, which implies that the gradient descent update is contractive. Thus we will be able to upper bound the first term by \(\mathcal{W}_1(\hat{\theta}_{k-1}, \tilde{\theta}_{k-1})\), which will lead to the result.

Here is the argument. Consider two arbitrary points \(\theta, \theta' \in \Theta\). Let \(G: \theta \rightarrow \theta - \alpha/2 \nabla \hat{F}(\theta)\) be a vector valued function, and observe that the Jacobian is \(I - \alpha/2 \nabla \hat{F}(\theta)\). By the mean value theorem, there exists \(\theta''\) such that
\[
\|\theta - \frac{\alpha}{2} \nabla \hat{F}(\theta) - (\theta' - \frac{\alpha}{2} \nabla \hat{F}(\theta'))\|_2 \leq \|I - \alpha/2 \nabla \hat{F}(\theta'')\|_q \|\theta - \theta'\|_q.
\]
Now, since \(\hat{F}\) is \(\lambda\)-strongly convex and \(\eta T B_L u + \lambda\) smooth, we know that all eigenvalues of \(\nabla^2 \hat{F}(\theta'')\) are in the interval \([\lambda, \eta T B_L u/(\gamma) + \lambda]\). Therefore, if \(\alpha \leq 2 (\eta T B_L u/(\gamma) + \lambda)^{-1} \leq 1/\lambda\), the spectral norm term here is at most 1, implying that gradient descent is contractive. Thus, we get
\[
\mathcal{W}_1(\hat{\theta}_k, \hat{\theta}_{k-1}) \leq \int \|\hat{\theta}_{k-1} - \hat{\theta}_{k-1}\|_{2d\pi_{k-1}^*} + \frac{\alpha (\varepsilon_2 + \varepsilon')}{2} + 2R \exp \left(\frac{-4\varepsilon_2^2 \gamma^2 m}{(\eta T L B_L)^2}\right)
\]
\[
\leq \mathcal{W}_1(\hat{\theta}_{k-1}, \hat{\theta}_{k-1}) + \frac{\alpha}{2} \varepsilon_2 + \frac{\alpha}{\sqrt{m}} \cdot \frac{\eta T B_L L}{\gamma} + 2R \exp \left(\frac{-4\varepsilon_2^2 \gamma^2 m}{(\eta T L B_L)^2}\right).
\]
The choice of \(m\) ensures that the second and third term together are at most \(\alpha \varepsilon_2\), from which the result follows.

\textbf{Fact 24.} For any two distributions \(\mu, \nu\) on \(\Theta\), we have
\[
\mathcal{W}_1(\mu, \nu) \leq R \cdot TV(\mu, \nu).
\]

\textbf{Proof.} We use the coupling characterization of the total variation distance:
\[
\mathcal{W}_1(\mu, \nu) = \inf_{\pi} \int \|\theta - \theta'\|_{2d\pi} \leq \text{diam}(\Theta) \inf_{\pi} \mathbb{P}_\pi[\theta \neq \theta'] \leq R \cdot TV(\mu, \nu).
\]

\textbf{Proof of Theorem 19.} By the triangle inequality and Fact 24 we have
\[
\mathcal{W}_1(\hat{\theta}_N, P) \leq \mathcal{W}_1(\hat{\theta}_N, \hat{\theta}_N) + R \cdot (TV(\hat{\theta}_N, \hat{P}) + TV(\hat{P}, P)).
\]
The first term here is the Wasserstein distance between our true iterates \(\hat{\theta}_N\) and the idealized iterates from running LMC on \(\hat{F}\), which is controlled by Lemma 23. The second is the total variation distance between the idealized iterates and the smoothed density \(\hat{P}\), which is controlled in Theorem 22. Finally, the third term is the approximation error between the smoothed density \(\hat{P}\) and the true, non-smooth one \(P\). Together, for any choice of \(\varepsilon > 0\) and \(\varepsilon_2 > 0\) we obtain the bound
\[
\mathcal{W}_1(\hat{\theta}_N, P) \leq N \alpha \varepsilon_2 + R \varepsilon + 2R (\eta T B_L u \sqrt{d}/\gamma + \lambda R^2),
\]
(12)
under the requirements
\[ N \geq \frac{c_0 R^6 \max\{d, R\eta T B_t L/\gamma + R^2 \lambda, R(\eta T B_t L/(u\gamma) + \lambda)\}_{12}}{\varepsilon_{12}}, \] (13)
\[ m \geq \frac{16(\eta T L B_t/\gamma)^2 \log(4R/\alpha\varepsilon_2)}{\varepsilon_{2}^2}. \]

There are also two requirements on \( \alpha \), one arising from Theorem 22 and the other from Lemma 23. These are:
\[ \alpha \leq 2(\eta T B_t L/(u\gamma) + \lambda)^{-1}, \quad \text{and} \quad \alpha = c_1 R^2/N, \] (14)
for any constant \( c_1 \).

Returning to the error bound, if we set
\[ u = \frac{\tau}{8R\eta T B_t \sqrt{d}}, \quad \text{and} \quad \lambda = \frac{\tau}{8R^2}, \]
the last term in (12) is at most \( \tau/2 \).

We will make the choice \( \alpha = c_1 R^2/N \). In this case, the values for \( u \) and \( \lambda \) above, combined with the inequality (14) give the constraint
\[ N \geq 2c_1 R^2 \cdot \left( \frac{8(\eta T L B_t)^2 R \sqrt{d}}{\gamma \tau} + \frac{\tau}{8R^2} \right). \] (15)

Now for the first term in (12), plug in the choice \( \alpha = c_1 R^2/N \) and set \( \varepsilon_2 = \tau/(4c_1 R^2) \) so that this term is at most \( \tau/4 \). For the second term, set \( \varepsilon = \tau/(4R) \) so that this term is also at most \( \tau/4 \). With these choices, the requirements on \( m \) and \( N \) become:
\[ m \geq \frac{64c_1^2 R^4(\eta T B_t/\gamma)^2 \log(\tau/(16RN))}{\tau^2}, \quad \text{and} \quad N \geq c_0 R^{18} \max\{d, (\eta T B_t L/\gamma)^2 \sqrt{d}/\tau\}_{12}/\tau_{12}, \]
where we have noted that the first constraint (13) clearly implies the second constraint (15), and this proves the theorem.

\[ \square \]

D.2 Continuous exponential weights.

The focus of this section of the appendix is Lemma 25, which analyzes a continuous version of the Hedge/exponential weights algorithms in the full information setting. This lemma appears in various forms in several places, e.g. Cesa-Bianchi and Lugosi (2006). For the setup, consider an online learning problem with a parametric benchmark class \( \mathcal{F} = \{ f(\cdot; \theta) \mid \theta \in \Theta \} \) where \( f(\cdot; \theta) \in (\mathcal{X} \to \mathbb{R}_+^K) \) and further assume that \( \Theta \in \mathbb{R}^d \) contains the centered Euclidean ball of radius \( r = 1 \) and is contained in the Euclidean ball of radius \( R \). Finally, assume that \( f(x; \cdot) \) is \( L \)-Lipschitz with respect to \( \ell_2 \) norm in \( \theta \) for all \( x \in \mathcal{X} \). On each round \( t \) an adversary chooses a context \( x_t \in \mathcal{X} \) and a loss vector \( \ell_t \in \mathbb{R}_+^K \), the learner then choose a distribution \( p_t \in \Delta(\mathcal{F}) \) and suffers loss:
\[ \mathbb{E}_{f \sim p_t}(\ell_t, \psi^\gamma(f(x_t))). \]

The entire loss vector \( \ell_t \) is then revealed to the learner. Here, performance is measured via regret:
\[ \text{Regret}(T, \mathcal{F}) \triangleq \sum_{t=1}^T \mathbb{E}_{f \sim p_t}(\ell_t, \psi^\gamma(f(x_t))) - \inf_{f \in \mathcal{F}} \sum_{t=1}^T (\ell_t, \psi^\gamma(f(x_t))). \]
Our algorithm is a continuous version of exponential weights. Starting with \( w_0(f) = 0 \), we perform the updates:

\[
p_t(f) = \frac{\exp(-\eta w_t(f))}{\int_{\mathcal{F}} \exp(-\eta w_t(f)) d\lambda(f)}, \quad \text{and} \quad w_{t+1}(f) = w_t(f) + \langle \ell_t, \psi^\gamma(f(x_t)) \rangle.
\]

Here \( \eta \) is the learning rate and \( \lambda \) is the Lebesgue measure on \( \mathcal{F} \) (identifying elements \( f \in \mathcal{F} \) with their representatives \( \theta \in \mathbb{R}^d \)).

With these definitions, the continuous Hedge algorithm enjoys the following guarantee.

**Lemma 25.** Assume that the losses \( \ell_t \) satisfy \( \|\ell_t\|_\infty \leq B, \Theta \subset \mathbb{R}^d \) is contained within the Euclidean ball of radius \( R \), and \( f(x; \cdot) \) is \( L \)-Lipschitz continuous in the third argument with respect to \( \ell_2 \). Let the margin parameter \( \gamma \) be fixed. Then the continuous Hedge algorithm with learning rate \( \eta > 0 \) enjoys the following regret guarantee:

\[
\text{Regret}(T, \mathcal{F}) \leq \inf_{\varepsilon > 0} \left\{ \frac{T KB_\varepsilon}{\gamma} + \frac{d}{\eta} \log(RL/\varepsilon) + \frac{\eta}{2} \sum_{t=1}^{T} \mathbb{E}_{f-p_t}\langle \ell_t, \psi^\gamma(f(x_t)) \rangle^2 \right\} + \frac{1}{\eta} \sum_{t=1}^{T} \text{KL}(p_{t-1} \parallel p_t).
\]

**Proof.** Following the standard analysis for continuous Hedge (e.g., Lemma 10 in Narayanan and Rakhlin (2017)), we know that the regret to some benchmark distribution \( Q \in \Delta(\mathcal{F}) \) is

\[
\sum_{t=1}^{T} (\mathbb{E}_{f-p_t} - \mathbb{E}_{f-Q})\langle \ell_t, \psi^\gamma(f(x_t)) \rangle = \frac{\text{KL}(Q \parallel p_0) - \text{KL}(Q \parallel p_T)}{\eta} + 1\frac{T}{\eta} \sum_{t=1}^{T} \text{KL}(p_{t-1} \parallel p_t).
\]

For the KL terms, using the standard variational representation, we have

\[
\text{KL}(p_{t-1} \parallel p_t) = \log \mathbb{E}_{f-p_{t-1}} \exp\left( -\eta \langle \ell_t, \psi^\gamma(f(x_t)) - \mathbb{E}_{f-p_{t-1}} \psi^\gamma(f(x_t)) \rangle \right)
\]

\[
\leq \log \left( 1 + \frac{\eta^2}{2} \mathbb{E}_{f-p_{t-1}} \langle \ell_t, \psi^\gamma(f(x_t)) - \mathbb{E}_{f-p_{t-1}} \psi^\gamma(f(x_t)) \rangle^2 \right)
\]

\[
\leq \frac{\eta^2}{2} \mathbb{E}_{f-p_{t-1}} \langle \ell_t, \psi^\gamma(f(x_t)) \rangle^2.
\]

Here the first inequality is \( e^{-x} \leq 1 - x + x^2/2 \), using that the term inside the exponential is centered. The second inequality is \( \log(1 + x) \leq x \).

Using non-negativity of KL, we only have to worry about the \( \text{KL}(Q \parallel p_0) \) term. Let \( f^* \) be the minimizer of the cumulative hinge loss. Let \( \theta^* \in \Theta \) be a representative for \( f^* \) and let \( Q \) be the uniform distribution on \( \mathcal{F}_\varepsilon(\theta^*, x; T) \triangleq \{ \theta : \max_{t \in [T]} \| f(x_t; \theta) - f(x_t; \theta^*) \|_\infty \leq \varepsilon \} \), then we have

\[
\text{KL}(Q \parallel p_0) = \int q(f) \log(q(f)/p_0(f)) d\lambda(f) = \int dQ(f) \cdot \log \frac{\int_{\mathcal{F}_\varepsilon} d\lambda(f)}{\int_{\mathcal{F}_\varepsilon} d\lambda(f)} = \log \frac{\text{Vol}(\mathcal{F})}{\text{Vol}(\mathcal{F}_\varepsilon(\theta^*, x; T))},
\]

where \( \text{Vol}(S) \) denotes the Lebesgue integral. We know that \( \text{Vol}(\Theta) \leq c_d R^d \) where \( c_d \) is the Lebesgue volume of the unit Euclidean ball and \( R \) is the radius of the ball containing \( \Theta \), and so we must lower bound the volume of \( \mathcal{F}_\varepsilon(f^*, x; T) \). For this step, observe that by the Lipschitz-property of \( f \),

\[
\sup_{x \in \mathcal{X}} \| f(x; \theta) - f(x; \theta^*) \|_\infty \leq L \| \theta - \theta^* \|_2,
\]

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and hence $\mathcal{F}_\varepsilon(\theta^*, x_{1:T}) \supseteq B_2(\theta^*, \varepsilon/L)$. Thus the volume ratio is

$$\frac{\text{Vol}(\mathcal{F})}{\text{Vol}(\mathcal{F}_\varepsilon(\theta^*, x_{1:T}))} \leq \frac{c_d R^d}{c_d(\varepsilon/L)^d} = (RL/\varepsilon)^d.$$ 

Finally, using the fact that the hinge surrogate is $1/\gamma$-Lipschitz, we know that

$$\sum_{t=1}^{T} \mathbb{E}_{f \sim Q}(\ell_t, \psi^\gamma(f(x_t)) - \psi^\gamma(f^*(x_t))) \leq TB_T \sup_{t \in [T], f \in \text{supp}(Q)} \|\psi^\gamma(f(x_t)) - \psi^\gamma(f^*(x_t))\|_1 \leq \frac{TKB_T \varepsilon}{\gamma}. \qedhere$$

### D.3 From full information to bandits.

We now combine the results of Subsection D.1 and Subsection D.2 to give the final guarantee for Hinge-LMC.

We begin by translating the regret bound in Lemma 25, followed by many steps of approximation. At round $t$, let $P_t$ denote the Hedge distribution on $T$ using the losses $\hat{\ell}_{1:t-1}$. Let $\tilde{P}_t$ denote the distribution from which $\theta_t \in \Theta$ is sampled in Algorithm 3.

Let $p_t \in \Delta(A)$ denote the induced distributions on actions induced by $P_t$, i.e. the distribution induced by the process $\theta \sim P_t, p_t(a) \propto \psi^\gamma(f(x_t; \theta))$. Likewise, let $\theta_t \sim \Delta(A)$ be the distribution induced by $\tilde{P}_t, \tilde{p}_t(a) \propto \psi^\gamma(f(x_t; \theta))$; in this notation $\tilde{p}_t^\mu$ is precisely the distribution from which actions are sampled in Algorithm 1.

Recall that we use $\mu$ in the superscript to denote smoothing (e.g. $p_t^\mu$). Let $m_t$ denote the random variable sampled at round $t$ to approximate the importance weight.

We also let $\hat{\ell}_t(a) = \ell_t(a)/\tilde{p}_t^\mu(a) \mathbf{1}[a_t = a]$ denote estimated losses under the true importance weights, which are not explicitly used by Algorithm 1 but are used in the analysis.

Let $\mathbf{1}_a \in \mathbb{R}^K$ be the vector with 1 at coordinate $a$ and 0 at all other coordinates.

**Proof of Theorem 18.** The thrust of this proof is to show that the full information bound in Lemma 25 does not degrade significantly under importance weighting and under the approximate LMC implementation of continuous exponential weights.

**Variance control.** Controlling the variance term in Lemma 25 requires an application of Lemma 3. After taking conditional expectations, the variance term is

$$\sum_{t=1}^{T} \mathbb{E}_{\theta \sim P_t} \mathbb{E}_{a_t \sim p_t^\mu} \mathbb{E}_{m_t} \left(\hat{\ell}_t, \psi^\gamma(f(x_t; \theta))\right)^2 = \sum_{t=1}^{T} \mathbb{E}_{s \sim P_t} \mathbb{E}_{a_t \sim p_t^\mu} \mathbb{E}_{m_t} m_t^2 (\ell_t(a_t) \mathbf{1}_{a_t}, s)^2.$$

Here we are identifying $s$ with $\psi^\gamma(f(x_t; \theta))$ and marginalizing out $\theta$ in the outermost expectation. Note that this is the same definition of $s$ as in Lemma 3.

First let us handle the $m_t$ random variable. Note that conditional on everything up to round $t$ and $a_t$, $m_t$ is distributed according to a geometric distribution with mean $\tilde{p}_t^\mu(a_t)$, truncated at $M$. It is straightforward (cf.
Neu and Bartók (2013)) to show that $m_t$ is stochastically dominated by a geometric random variable with mean $\frac{1}{\tilde{p}_t(a_t)}$ and hence the second moment of this random variable is at most $\frac{2}{\tilde{p}_t(a_t)^2}$. Thus, we are left with

$$\leq 2 \sum_{t=1}^{T} \mathbb{E}_{s \sim P_t} \mathbb{E}_{a_t \sim p_t^\mu} \frac{1}{\tilde{p}_t^2(a_t)} \ell_t(a_t) \mathbf{1}_{a_t, s}^2$$

$$= 2 \sum_{t=1}^{T} \mathbb{E}_{s \sim P_t} \mathbb{E}_{a_t \sim p_t^\mu} \ell_t(s)^2$$

$$\leq 2 \sum_{t=1}^{T} \left( \mathbb{E}_{s \sim P_t} - \mathbb{E}_{s \sim P_t} \right) \mathbb{E}_{a_t \sim p_t^\mu} \ell_t(s)^2 + \mathbb{E}_{s \sim P_t} \mathbb{E}_{a_t \sim p_t^\mu} \ell_t(s)^2.$$  

We can apply Lemma 3 on the second term, since the only condition for the lemma is that the action distribution is induced from the distribution in the outer expectation. It follows that this term is bounded as

$$\sum_{t=1}^{T} \mathbb{E}_{s \sim P_t} \mathbb{E}_{a_t \sim p_t^\mu} \ell_t(s)^2 \leq T K^2 (1 + B/\gamma)^2.$$  

For the first term, evaluating the inner expectation, using the fact that $\tilde{p}_t^\mu(a) \geq \mu$ and applying the Lipschitz properties of $\psi^\gamma(\cdot), f(x; \cdot)$ (in particular that $f(x; \cdot)$ is $L$-Lipschitz with respect to $\ell_2$ and that the Wasserstein distance we work with is defined relative to $\ell_2$) we have

$$(\mathbb{E}_{s \sim P_t} - \mathbb{E}_{s \sim P_t}) \mathbb{E}_{a_t \sim p_t^\mu} \ell_t(s)^2 = \sum_{a} (\mathbb{E}_{\theta \sim P_t} - \mathbb{E}_{\theta \sim P_t}) \frac{\ell_t^2(a)}{p_t^\mu(a)} \psi^\gamma(f(x_t; \theta)_a^2)$$

$$\leq 2 \frac{(1 + B/\gamma) KL}{\gamma \mu} \sup_{g_1, g_2 \|_{\text{lip}} \leq 1} \left| \int g(\text{d}P_t - \text{d}\tilde{P}_t) \right| = 2 \frac{(1 + B/\gamma) KL}{\gamma \mu} \mathcal{W}_1(P_t, \tilde{P}_t).$$

Finally, using the Wasserstein guarantee $\mathcal{W}_1(P_t, \tilde{P}_t) \leq \tau$ from Theorem 19, we conclude that the cumulative variance term is upper bounded as

$$\sum_{t=1}^{T} \mathbb{E}(\ell_t, \psi^\gamma(f(x_t; \theta)))^2 \leq \frac{4(1 + B/\gamma) KTL \tau}{\gamma \mu} + 2(1 + B/\gamma)^2 K^2 T.$$  

**Bounding regret** We first relate the cumulative loss under Algorithm 1 to the cumulative loss of continuous exponential weights. Observe that

$$\sum_{t=1}^{T} \langle \ell_t, \tilde{p}_t^\mu \rangle \leq \mu KT + \sum_{t=1}^{T} \langle \ell_t, \tilde{p}_t \rangle$$

$$\leq \mu KT + \frac{1}{K} \sum_{t=1}^{T} \mathbb{E}_{\theta \sim P_t} \langle \ell_t, \psi^\gamma(f(x_t; \theta)) \rangle$$

$$\leq \mu KT + \frac{TL \tau}{\gamma} + \frac{1}{K} \sum_{t=1}^{T} \mathbb{E}_{\theta \sim P_t} \langle \ell_t, \psi^\gamma(f(x_t; \theta)) \rangle.$$

This first inequality is a straightforward consequence of smoothing, while the second is a direct application of Lemma 1.

The third inequality is based on the fact that $\langle \ell_t, \psi^\gamma(f(x_t; \theta)) \rangle$ is $KL/\gamma$-Lipschitz in $\theta$ with respect to $\ell_2$ norm under our assumptions. This step also uses the Wasserstein guarantee in Theorem 19 which produces the approximation factor $\tau$. 

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Following the analysis in Neu and Bartók (2013) and using the boundedness of $\psi^\gamma$, the bias introduced due to using geometric resampling with truncation at $M$ instead of exact inverse propensity scores is

$$\sum_{t=1}^{T} \mathbb{E}_{\theta \sim P_t}(\ell_t, \psi^\gamma(f(x_t; \theta))) \leq \mathbb{E}_{\theta \sim P_t} \sum_{t=1}^{T} \mathbb{E}_{\theta \sim P_t}(\ell_t, \psi^\gamma(f(x_t; \theta))) + \frac{T(1 + B/\gamma)}{eM}.$$ 

For the remaining term, we apply Lemma 25 with $\epsilon = \gamma/(TKM)$, since $M$ is an upper bound on the norm $\|\tilde{\ell}_t\|_1$ of the losses to the full information algorithm.

$$\mathbb{E}_{\theta_1:T, \theta_{m:1:T}} \sum_{t=1}^{T} \mathbb{E}_{\theta \sim P_t}(\ell_t, \psi^\gamma(f(x_t; \theta)))$$

$$\leq \mathbb{E} \inf_{\theta \in \Theta} \sum_{t=1}^{T} (\ell_t, \psi^\gamma(f(x_t, \theta))) \leq \frac{d}{\eta} \log(\gamma) + \frac{2(1 + B/\gamma)KTL\tau}{\gamma \mu} + 4(1 + B/\gamma)^2K^2T.$$ 

The first term here is the benchmark we want to compare to, since $\mathbb{E} \inf(\cdot) \leq \mathbb{E}[\cdot]$ and so the regret contains several terms:

$$\mu KT + \frac{TL\tau}{\gamma} + \frac{T(1 + B/\gamma)}{eMK} + \frac{1}{K} + \frac{d}{\eta} \log(\gamma) + \frac{2(1 + B/\gamma)KTL\tau}{\gamma \mu} + 4(1 + B/\gamma)^2K^2T.$$ 

Here we use the assumption $B/\gamma \geq 1$. We will simplify the expression to obtain an $\tilde{O}(\sqrt{dKT})$-type bound, first set $\mu = 1/(K\sqrt{T})$, $M = \sqrt{T}$ and $\tau = \sqrt{1/(TL^2)}$. This gives

$$2\sqrt{T} + \frac{2B}{\gamma} \sqrt{T} + \frac{2d}{\eta K} \log(\gamma) + \frac{2\eta}{K^{\gamma^2}} \left( BK^2T + 4B^2K^2T \right)$$

$$\leq O(B\sqrt{T}/\gamma) + \frac{2d}{\eta K} \log(\gamma) + \frac{10d}{\gamma^2} B^2 K T.$$ 

Finally set $\eta = \sqrt{\frac{d\gamma^2 \log(\gamma)}{5K^2B^2T}}$ to get

$$O(\sqrt{T}/\gamma) + O\left( \frac{B}{\gamma} \sqrt{dT \log(\gamma)} \right) = \tilde{O} \left( \frac{B}{\gamma} \sqrt{dT} \right).$$

This concludes the proof of the regret bound.

**Running time calculation.** At each round make $M + 1$ calls to the LMC sampling routine for a total of $O(T^{3/2})$ calls across all rounds. We now bound the running time for a single call.

We always use parameter $\tau = \sqrt{1/(TL^2)}$ and we know $\|\tilde{\ell}_t\|_1 \leq 1/\mu = K\sqrt{T}$ and $\eta = \tilde{O}(\sqrt{\frac{d}{K^2T}})$. Plugging into the parameter choices at the end of the proof of Theorem 19, we must sample

$$m = \tilde{O}(T^3dR^4L^2B^2/(K\gamma)^2)$$

samples from a gaussian distribution on each iteration, and the number of iterations to generate a single sample is:

$$N = \tilde{O} \left( \frac{R^{18}L^{12}T^6d^{12} + R^{24}L^{48}d^{12}}{K^{24}} \right).$$
Therefore, the total running time across all rounds is
\[
\hat{O} \left( \frac{R^{22}L^{14}d^{14}B^2T^{10}}{K^2\gamma^2} + \frac{R^{28}L^{50}d^{14}B^2T^4}{K^{26}\gamma^2} \right).
\]

D.4 Proofs for corollaries

Corollary 7 is an immediate consequence of Theorem 6. For Corollary 8, we apply Lemma 10, since \( \theta^* \in \Theta \) satisfies the conditions of the lemma pointwise. Thus
\[
K^{-1} \mathbb{E} \{ \ell, \psi^\gamma(f(x_i; \theta^*)) \} = K^{-1} \mathbb{E} \{ \ell, \psi^\gamma(f(x_i; \theta^*)) \mid x_i \} = \mathbb{E} \left[ \min_a \ell_t(a) | x_i \right].
\]
Therefore, letting \( a_i^* \) denote the optimal action minimizing \( \ell_t \), we obtain the expected regret bound
\[
\sum_{t=1}^T \mathbb{E} \{ \ell_t - a_i^* \} \leq \hat{O} \left( \frac{B}{\gamma} \sqrt{dT} \right).
\]

E Analysis of SMOOTHFTL

Recall we are in the stochastic setting. Let \( \mathcal{D} \) denote the distribution over \( (\mathcal{X}, \mathbb{R}^K) \).

The bulk of the analysis is the following uniform convergence lemma, which is based on chaining for the function class \( \mathcal{F} \). Recall that \( \mathcal{N}_{\infty, \infty}(\varepsilon, \mathcal{F}) \) is the \( L_\infty/\ell_\infty \) covering number from Definition 1.

**Lemma 26.** Fix a predictor \( \hat{f} \) and let \( \{x_i, a_i, \ell_i(a_i)\}_{i=1}^n \) be a dataset of \( n \) samples, Suppose that \( (x_i, \ell_i) \) are drawn i.i.d. from some distribution \( \mathcal{D} \) and \( a_i \) is sampled from \( p_i \sim (1 - K\mu)\pi_{\text{hinge}} f(x_i) + \mu \). Define \( \hat{R}_n^\psi(f) \equiv \frac{1}{n} \sum_{i=1}^n \ell_i, \psi^\gamma(f(x_i)) \), where \( \ell_i \) is the importance-weighted loss. Then:
\[
\mathbb{E} \sup_{f \in \mathcal{F}} |R_n^\psi(f) - \hat{R}_n^\psi(f)| \leq \frac{1}{\gamma} \inf_{\beta \geq 0} \left\{ 2K \beta + 12 \int_{\beta}^2 \left( \sqrt{\frac{2K}{n\mu \log(nN_{\infty, \infty}(\varepsilon, \mathcal{F}, n))}} + \frac{3\log(nN_{\infty, \infty}(\varepsilon, \mathcal{F}, n))}{n\mu} \right) d\varepsilon \right\}.
\]

**Proof of Lemma 26.** Note that since the data-collection policy \( \hat{f} \) is fixed, and since we are in the stochastic setting with \( (x_i, \ell_i) \sim \mathcal{D} \), the samples \( \{x_i, a_i, \ell_i(a_i)\}_{i=1}^n \) are i.i.d. Consequently, we can apply the standard symmetrization upper bound for uniform convergence. Beginning with
\[
\mathbb{E}_{x_1:n, a_1:n, \ell_1:n} \sup_{f \in \mathcal{F}} \left[ R_n^\psi(f) - \hat{R}_n^\psi(f) \right],
\]
we introduce a second “ghost” dataset of samples \( \tau = n + 1, \ldots, 2n \) via Jensen’s inequality.
\[
\leq \mathbb{E}_{x_1:2n, a_1:2n, \ell_1:2n} \sup_{f \in \mathcal{F}} \left( \frac{1}{n} \sum_{\tau=n+1}^{2n} \ell_{\tau}, \psi^\gamma(f(x_\tau))) - \frac{1}{n} \sum_{\tau=1}^{n} \ell_{\tau}, \psi^\gamma(f(x_\tau))) \right).
\]
Introducing Rademacher random variables and splitting the supremum:
\[
\leq 2\mathbb{E}_{x_1:n, a_1:n, \ell_1:n, \epsilon_{1:n}} \sup_{f \in \mathcal{F}} \left( \frac{1}{n} \sum_{\tau=1}^{n} \epsilon_{\tau} \ell_{\tau}, \psi^\gamma(f(x_\tau))) \right).
\]
Now condition on \(x_{\text{1:n}}\) and define a sequence \(\beta_i = 2^{1-i}\) for \(i \in \{0, 1, 2, \ldots, N\}\), where \(N\) is such that \(\beta_{N+1} \geq \beta \geq \beta_{N+2}\) for the value of \(\beta\) in the lemma statement. For each \(\beta_i\) let \(V_i\) be a (classical) \(L_\infty/\ell_\infty\) cover for \(f\) at scale \(\beta_i\) on \(x_{\text{1:n}}\), that is

\[
\forall f \in \mathcal{F}, \forall i, \exists v \in V_i \text{ s.t. } \max_{t \in [n]} \|f(x_t) - v\|_\infty \leq \beta_i.
\]

We can always ensure \(|V_i| \leq N_{\infty, \infty}(\beta_i, \mathcal{F}, n)\) and since \(|f(x)|_\infty \leq 1\), we know that \(N_{\infty, \infty}(\beta_0, \mathcal{F}, n) \leq 1\).

Now, let \(v^{(i)}(f)\) denote the covering element for \(f\) at scale \(\beta_i\), we have

\[
\mathbb{E}_{a_{1:n}, \ell_{1:n}, \epsilon_{1:n}} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{\tau=1}^{n} \epsilon_{\tau} \tilde{\ell}_{\tau, \psi}(f(x_{\tau})) \\
\leq \mathbb{E}_{a_{1:n}, \ell_{1:n}, \epsilon_{1:n}} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{\tau=1}^{n} \epsilon_{\tau} \tilde{\ell}_{\tau, \psi}(f(x_{\tau})) - \psi'(v^{(N)}(f)) \\
+ \sum_{i=1}^{N} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{\tau=1}^{n} \epsilon_{\tau} \tilde{\ell}_{\tau, \psi}(v^{(i)}(f)) - \psi'(v^{(i-1)}(f)) \\
+ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{\tau=1}^{n} \epsilon_{\tau} \tilde{\ell}_{\tau, \psi}(v^{(0)}(f)).
\]

Since \(|V_0| \leq 1\), the expected value of the third term is zero. The remaining work is to bound the first and second terms.

For the first term note that by Hölder’s inequality, for any \(f \in \mathcal{F}\),

\[
\frac{1}{n} \sum_{\tau=1}^{n} \epsilon_{\tau} \tilde{\ell}_{\tau, \psi}(f(x_{\tau})) - \psi'(v^{(N)}(f)) \leq \frac{1}{n} \sum_{\tau=1}^{n} \|\tilde{\ell}_{\tau}\|_1 \|\psi'(f(x_{\tau})) - \psi'(v^{(N)}(f))\|_\infty \\
\leq \frac{\beta_{N+1}}{\gamma} \frac{1}{n} \sum_{\tau=1}^{n} \|\tilde{\ell}_{\tau}\|_1,
\]

since \(\psi'\) is \(1/\gamma\)-Lipschitz. Thus for the first term, we have

\[
\mathbb{E}_{a_{1:n}, \ell_{1:n}, \epsilon_{1:n}} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{\tau=1}^{n} \epsilon_{\tau} \tilde{\ell}_{\tau, \psi}(f(x_{\tau})) - \psi'(v^{(N)}(x_{\tau}))) \leq \frac{\beta_{N+1}}{\gamma} \mathbb{E}_{a_{1:n}, \ell_{1:n}} \frac{1}{n} \sum_{\tau=1}^{n} \|\tilde{\ell}_{\tau}\|_1 \leq \frac{\beta_{N+1}K}{\gamma}.
\]

Note that there is no dependence on the smoothing parameter \(\mu\) here.

For the second term, let us denote the \(i\)th term in the summation by

\[
\mathbb{E}_{a_{1:n}, \ell_{1:n}, \epsilon_{1:n}} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{\tau=1}^{n} \epsilon_{\tau} \tilde{\ell}_{\tau, \psi}(v^{(i)}(f)) - \psi'(v^{(i-1)}(f))).
\]

We control \(\mathcal{E}_i\) using Bernstein’s inequality and a union bound. First, note that the individual elements in the sum satisfy the deterministic bound

\[
|\epsilon_{\tau} \tilde{\ell}_{\tau, \psi}(v^{(i)}(f)) - \psi'(v^{(i-1)}(f)))| \leq \frac{3\beta_i}{\mu \gamma}, \quad (16)
\]

and the variance bound,

\[
\mathbb{E}(\tilde{\ell}_{\tau, \psi}(v^{(i)}(f)) - \psi'(v^{(i-1)}(f)))^2 \leq \sum_a \mathbb{E}_{a_{\tau}} \frac{1}{\mu} \frac{(a_{\tau} = a_1)^3}{\mu(a)\gamma} (\psi'(v^{(i)}(f)a) - \psi'(v^{(i-1)}(f)a))^2 \\
\leq \sum_a \frac{1}{\mu} (3\beta_i/\gamma)^2 = \frac{9\beta_i^2 K}{\mu \gamma^2}. \quad (17)
\]

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Here we are using that \( v^{(i)}(f) \) and \( v^{(i-1)}(f) \) are the covering elements for \( f \), Lipschitzness of \( \psi^\gamma \), and the definition of the importance weighted loss \( \ell_\tau \).

Using (16) and (17), Bernstein’s inequality (e.g. Boucheron et al. (2013), Theorem 2.9) implies that for any \( \delta \in (0, 1) \),

\[
\frac{1}{n} \sum_{\tau=1}^{n} \epsilon_\tau (\hat{\ell}_\tau, \psi^\gamma (v^{(i)}_\tau (f)) - \psi^\gamma (v^{(i-1)}_\tau (f))) \leq 6 \sqrt{\frac{\beta_i^2 K}{n \mu \gamma^2} \log(1/\delta) + \frac{6\beta_i}{n \mu \gamma} \log(1/\delta)},
\]

with probability at least \( 1 - \delta \). The important point here is that \( 1/(n\mu) \) appears in the square root, as opposed to \( 1/(n\mu^2) \). Via a union bound, for any \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \),

\[
\sup_f \frac{1}{n} \sum_{\tau=1}^{n} \epsilon_\tau (\hat{\ell}_\tau, \psi^\gamma (v^{(i)}_\tau (f)) - \psi^\gamma (v^{(i-1)}_\tau (f))) \leq 6 \sqrt{\frac{\beta_i^2 K}{n \mu \gamma^2} \log(|V_i|/|V_{i-1}|/\delta) + \frac{6\beta_i}{n \mu \gamma} \log(|V_i|/|V_{i-1}|/\delta)}
\]

\[
\leq \frac{6\beta_i}{\gamma} \left( \sqrt{\frac{2K}{n \mu} \log(|V_i|/\delta)} + \frac{2\log(|V_i|/\delta)}{n \mu} \right),
\]

since \( |V_{i-1}| \leq |V_i| \). Now, recalling the shorthand definition \( \mathcal{E}_i \)

\[
\mathbb{E}_x \mathbb{E}_y \mathbb{E}_z \leq \inf_{\zeta} \mathbb{E}_1 \{ \mathcal{E}_i \leq \zeta \} \cdot \mathbb{E}_1 \{ \mathcal{E}_i > \zeta \} \cdot \frac{3\beta_i}{\mu \gamma}
\]

\[
\leq \inf_{\delta \in (0, 1)} \frac{6\beta_i}{\gamma} \left( \sqrt{\frac{2K}{n \mu} \log(|V_i|/\delta)} + \frac{2\log(|V_i|/\delta)}{n \mu} \right) + \frac{3\beta_i}{\mu \gamma}.
\]

Choosing \( \delta = 1/n \):

\[
\leq \frac{6\beta_i}{\gamma} \left( \sqrt{\frac{2K}{n \mu} \log(|V_i|)} + \frac{3\log(n|V_i|)}{n \mu} \right).
\]

Thus, the second term in the chaining decomposition is

\[
\frac{6}{\gamma} \sum_{i=1}^{N} \beta_i \left( \sqrt{\frac{2K}{n \mu} \log(n|V_i|)} + \frac{3\log(n|V_i|)}{n \mu} \right)
\]

\[
= \frac{12}{\gamma} \sum_{i=1}^{N} (\beta_i - \beta_{i+1}) \left( \sqrt{\frac{2K}{n \mu} \log(n|V_i|)} + \frac{3\log(n|V_i|)}{n \mu} \right)
\]

\[
\leq \frac{12}{\gamma} \int_{\beta_{N+1}}^{\beta_0} \left( \sqrt{\frac{2K}{n \mu} \log(nN_{\infty, \infty}(\beta, \mathcal{F}))} + \frac{3\log(nN_{\infty, \infty}(\beta, \mathcal{F}))}{n \mu} \right) d\beta.
\]

This concludes the uniform deviation statement. Exactly the same argument applies to the other tail, so the bound holds on the absolute value. \( \square \)

**Proof of Theorem 9.** Let us denote the right hand side of Lemma 26, when the dataset is size \( n \), as \( \Delta_n \). Define,

\[
f^* = \arg\min_{f \in \mathcal{F}} \mathbb{E}_x (\ell, \psi^\gamma (f(x))),
\]

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Since the \( m \)th epoch proceeds for \( n_m = 2^m \) rounds, and the predictor that we use in the \( m \)th epoch is the ERM on all of the data from the \((m - 1)\)'st epoch, the expected cumulative hinge regret for the \( m \)th epoch is

\[
2^m \cdot \mathbb{E} R^\psi(\hat{f}_{m-1}) - R^\psi(f^*)
\]

Using the optimality guarantee for ERM:

\[
\leq 2^m \cdot \left( \mathbb{E} R^\psi(\hat{f}_{m-1}) - \frac{1}{n_{m-1}} \sum_{\tau=n_{m-1}}^{n_{m-1}} \langle \hat{e}_\tau, \psi^\gamma(\hat{f}_{m-1}(x_\tau)) \rangle + \frac{1}{n_{m-1}} \sum_{\tau=n_{m-1}}^{n_{m-1}} \langle \hat{e}_\tau, \psi^\gamma(f^*(x_\tau)) - R^\psi(f^*) \rangle \right)
\]

\[
\leq 2^{m+1} \mathbb{E} \sup_f |R^\psi(f) - \hat{R}_{n_{m-1}}^\psi(f)|.
\]

Using the guarantee from Lemma 26:

\[
\leq 2^{m+1} \Delta_{n_{m-1}}.
\]

(18)

Summing this bound over all rounds, the cumulative expected regret after the zero-th epoch is \( \sum_{m=1}^{\log_2(T)} 2^{m+1} \Delta_{n_{m-1}} \). The zero-th epoch contributes \( 1/\gamma \) to the regret, which will be lower order. This gives the following upper bound on the cumulative expected hinge loss regret.

\[
\text{Regret}(T, \mathcal{F}) \leq \sum_{m=1}^{\log_2(T)} 2^{m+1} \Delta_{n_{m-1}}
\]

\[
\leq \frac{4}{\gamma} \sum_{m=1}^{\log_2(T)} \inf_{\beta > 0} \left\{ \frac{n_m K \beta + 12 \cdot 2^{m-1} \cdot \int_\beta^{2B} \left( \sqrt{\frac{2K}{n_{m-1} \mu} \log(n_{m-1}N_{\infty, \infty}(\varepsilon, \mathcal{F})) + \frac{3 \log(n_{m-1}N_{\infty, \infty}(\varepsilon, \mathcal{F}))}{n_{m-1} \mu}} \right) d\varepsilon}{\beta} \right\}
\]

\[
\leq \frac{4}{\gamma} \inf_{\beta > 0} \left( K T \beta + 12 \log_2(T) \cdot \int_\beta^{2B} \left( \sqrt{\frac{2K}{\mu} \log(T N_{\infty, \infty}(\varepsilon, \mathcal{F})) + \frac{3 \log(T N_{\infty, \infty}(\varepsilon, \mathcal{F}))}{\mu}} \right) d\varepsilon \right)
\]

\[
\leq C.
\]

Let \( z_t = \hat{f}_{m-1}(x_t) \) for each time \( t \) in epoch \( m \). We have just shown

\[
\sum_{t=1}^{T} \mathbb{E} \langle \ell_t, \psi^\gamma(z_t) \rangle \leq T \cdot \mathbb{E} \langle \ell, \psi^\gamma(f^*(x)) \rangle + C.
\]

Using Lemma 1, this implies

\[
\sum_{t=1}^{T} \mathbb{E} \langle \ell_t, \pi_{\text{hinge}}(z_t) \rangle \leq \frac{T}{K} \cdot \mathbb{E} \langle \ell, \psi^\gamma(f^*(x)) \rangle + \frac{C}{K} + \mu KT.
\]

Finally since \( p_t = (1 - K \mu) \pi_{\text{hinge}}(z_t) + \mu \) and \( \| \ell_t \|_\infty \leq 1 \), this implies the bound

\[
\sum_{t=1}^{T} \mathbb{E} \ell_t(a_t) \leq \frac{T}{K} \cdot \mathbb{E} \langle \ell, \psi^\gamma(f^*(x)) \rangle + \frac{C}{K} + \mu KT.
\]

We proceed to bound the final regret \( C' \) under the specific covering number behavior assumed in the theorem statement. Assume that \( \log(N_{\infty, \infty}(\varepsilon, \mathcal{F})) \leq \varepsilon^{-p} \) for some \( p > 2 \). Omitting the \( \log(T) \) additive terms, which will contribute \( O(B \gamma^{-1} \sqrt{KT \log(T)/\mu} + B \gamma^{-1} \log(T)/\mu) \) to the overall regret, the bound is now

\[
\mu KT + \frac{1}{\gamma K} \left( \inf_{\beta > 0} 4K T \beta + 12 \log_2(T) \cdot \int_\beta^{2B} \sqrt{\frac{2K}{\mu} \log(T N_{\infty, \infty}(\varepsilon, \mathcal{F}))} d\varepsilon + 36 \log_2(T) \cdot \int_\beta^{2} \frac{1}{\mu \varepsilon^p} d\varepsilon \right).
\]
Choosing $\beta = (KT\mu)^{-1/p}$, this bound becomes

$$O\left(\mu KT + \frac{1}{\gamma K} \log(T)(KT)^{1-1/p}\mu^{-1/p}\right).$$

Finally, we choose $\mu = \gamma^{-\frac{p}{p+1}} T^{-\frac{1}{p+1}} K^{-1}$, leading to a final bound of $O\left((T/\gamma)^{\frac{p}{p+1}}\right)$. \hfill \square

**F SMOOTHFTL for Lipschitz CB**

Here we analyze SMOOTHFTL in a stochastic Lipschitz contextual bandit setting. To describe the setting, let $\mathcal{X}$ be a metric space endowed with metric $\rho$ and with covering dimension $p$. This latter fact means that for each $0 < \varepsilon \leq 1$, $\mathcal{X}$ can be covered using at most $C_{\mathcal{X}}\varepsilon^{-p}$ balls of radius $\varepsilon$. Let $\mathcal{A}$ be a finite set of $K$ actions. In this section, we define the benchmark class $\mathcal{G} \subset (\mathcal{X} \to \Delta(\mathcal{A}))$ to be the set of 1-Lipschitz functions, meaning that $\|g(x) - g(x')\|_1 \leq \rho(x, x')$ for all $g, x, x'$ (The choice of $\ell_1$ norm is natural since we are operating over the simplex).

We focus on the stochastic setting where there is a distribution $\mathcal{D}$ over $\mathcal{X} \times [0, 1]^K$. At each round $(x_t, \ell_t) \sim \mathcal{D}$ is drawn and $x_t$ is presented to the learner. The learner chooses a distribution $p_t \in \Delta(\mathcal{A})$, samples an action $a_t \in \mathcal{A}$ from $p_t$, and observes the loss $\ell_t(a_t)$. We measure regret via

$$\text{Regret}(T, \mathcal{G}) = \sum_{t=1}^{T} \mathbb{E}\ell_t(a_t) - \inf_{g \in \mathcal{G}} \sum_{\tau=1}^{T} \mathbb{E}\ell_{\tau}(g(x_{\tau})).$$

In this setting, SMOOTHFTL takes the following form. Before the $m^\text{th}$ epoch, we choose a function $\hat{g}_{m-1}$ by solving the empirical risk minimization (ERM) problem

$$\hat{g}_{m-1} = \arg\min_{g \in \mathcal{G}} \sum_{\tau=1}^{n_{m-1}} \langle \hat{\ell}_\tau, g(x_{\tau}) \rangle,$$

where $\hat{\ell}_\tau$ is the importance weighted loss. Then, we use $\hat{g}_{m-1}$ for all the rounds in the $m^\text{th}$ epoch, which means that after observing $x_t$, we set $p_t(a) = (1 - K\mu)\hat{g}_m(x_t, a) + \mu$. We sample $a_t \sim p_t$, observe $\ell_t(a_t)$ and use the standard importance weighting scheme:

$$\hat{\ell}_t(a) = \frac{\ell_t(a)1\{a = a_t\}}{p_t(a)}.$$

For this algorithm, we have the following guarantee.

**Theorem 27.** SMOOTHFTL in the Lipschitz CB setting enjoys a regret of $\tilde{O}((KT)^{\frac{p}{p+1}})$.

This theorem improves upon the recent result of Cesa-Bianchi et al. (2017), who obtain $\tilde{O}(T^{\frac{p+1}{p+2}})$ in this setting.

**Proof.** We are in a position to apply Lemma 26. The main difference is that there is no margin parameter, since our functions are 1-Lipschitz, instead of $1/\gamma$-Lipschitz after applying the surrogate loss. The $\ell_\infty$-metric entropy at scale $\varepsilon$ is $C_{\mathcal{X}}\varepsilon^{-p}$ up to polynomial factors in $K$ and logarithmic factors, and so in the $m^\text{th}$ epoch the ERM has sub-optimality (see (18)) at most

$$\tilde{O}\left(\inf_{\beta} K\beta + \int_{\beta}^1 \sqrt{\frac{K\beta^{-p}}{n_{m-1}\mu} + \frac{\beta^{-p}}{n_{m-1}\mu}}\right),$$

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where $\tilde{O}$ hides dependence on $C_X$. Following the argument in the proof of Theorem 9, the overall regret is then

$$\text{Regret}(T, \mathcal{G}) = \tilde{O}\left( \mu KT + \inf_{\beta} TK\beta + \int_{\beta}^{1} \sqrt{TK\beta^{-p} \frac{T}{\mu} + \frac{\beta^{-p}}{\mu}} \right).$$

Set $\beta = (TK\mu)^{-1/p}$ and then $\mu = (TK)^{1/p}$ now to obtain the result.

In principle our technique can be further extended to the setting where the action space is also a general metric space, and the losses are Lipschitz, which is the more general setting addressed by Cesa-Bianchi et al. (2017). If the action space has covering dimension $p_A$ then we discretize the action space to resolution $\epsilon$, set $\hat{K} = \epsilon^{-p_A}$ in the above argument, and balance $\epsilon$ with an additional $T\epsilon$ factor that we pay for discretization. This is the approach used in Cesa-Bianchi et al. (2017) to obtain $T^{p_A+1/p + 1}$. Unfortunately, our argument above obtains a somewhat poor dependence on $\hat{K}$ ($K^{p_A}$ as opposed to $K^{p_A+1}$, which is more natural). Consequently, the argument produces a bound of $\tilde{O}(T^{p_A+1/p + 1})$ which only improves on Cesa-Bianchi et al. (2017) when $p_A \leq 1/(p-1)$. 

