Lyapunov Exponents for Burgers’ Equation

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Abstract

We establish the existence, uniqueness, and stability of the stationary solution of the one-dimensional viscous Burgers equation with the Dirichlet boundary conditions on a finite interval. We obtain explicit formulas for solutions and analytically determine the Lyapunov exponents characterizing the asymptotic behavior of arbitrary solutions approaching the stationary one.

Keywords: nonlinear PDE, Burgers equation, boundary value problem, Dirichlet boundary conditions, Lyapunov exponent.

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Introduction

Burgers’ equation has the same nonlinearity form as the Navier-Stokes equations [1]. It is often used as a model equation in studying computational methods for solving partial differential equations (PDEs) [2]. In this paper we establish the existence, uniqueness, and stability of the stationary solution of the one-dimensional viscous Burgers equation (1) on a finite interval with the Dirichlet boundary conditions (4). We use the Cole-Hopf transformation to give the result for any combination of A and B in the boundary conditions (4). Using a different method (linearization) H.-O. Kreiss and G. Kreiss (1985) gave a similar result for a subset of cases: A ≥ |B|, B ≤ 0 < A, as well as for Burgers’ equation with forcing [8]. We obtain explicit formulas for solutions and analytically determine the Lyapunov exponents characterizing the asymptotic behavior of arbitrary solutions approaching the stationary solution with the same boundary conditions (4).

1 Explicit formulas for stationary solutions

The viscous Burgers equation is the nonlinear partial differential equation

\[ u_t + uu_x = \nu u_{xx} \]  

(1)

with \( \nu = \text{const} > 0 \). If we set \( u_t \) to zero, for the stationary solution \( u = u^S(x) \) we obtain

\[ uu_x = \nu u_{xx}. \]  

(2)

We note that \( uu_x = \frac{1}{2}(u^2)'_x \), therefore (2) gives

\[ 2\nu u_x = u^2 + C_0. \]  

(3)

First, assume that \( u_x \neq 0 \) and \( C_0 \) is negative, \( C_0 = -a^2 < 0 \) (i.e. \( 2\nu u_x < u^2 \)). We have \( dx = 2 \nu du/(u^2 - a^2) \),

\[ \frac{ax}{\nu} = \ln \left( C_1 \left| \frac{a - u}{a + u} \right| \right), \]  

where \( C_1 = \left| \frac{a + u(0)}{a - u(0)} \right| \).
If, in addition, $|u| < a$ (i.e. $u_x < 0$), then

$$u = a \frac{C_1 - \exp(ax/v)}{C_1 + \exp(ax/v)} = -2v/k_0 \tanh(k_0(x - x_0)), \quad \text{where } k_0 = \frac{a}{2v}, \quad x_0 = \frac{1}{k_0} \arctan \frac{u(0)}{2vk_0},$$

while if $|u| > a$ (i.e. $0 < 2vu_x < u^2$), then

$$u = a \frac{C_1 + \exp(ax/v)}{C_1 - \exp(ax/v)} = -2v/k_0 \coth(k_0(x - x_0)), \quad \text{where } k_0 = \frac{a}{2v}, \quad x_0 = \frac{1}{k_0} \arccot \frac{u(0)}{2vk_0}.$$ 

Now assume that $C_0$ is positive, $C_0 = a^2 > 0$ (i.e. $2vu_x > u^2$). Then $dx = 2vu(1/u^2 + a^2)$,

$$\frac{ax}{2v} = \arctan \frac{u}{a} + C_1, \quad \text{where } C_1 = -\arctan \frac{u(0)}{a}; \quad \text{hence } u = a \tan \left( \frac{ax}{2v} - C_1 \right);$$

or, equivalently,

$$u = -2v/k_0 \cot(k_0(x - x_0)), \quad \text{where } k_0 = \frac{a}{2v}, \quad x_0 = \frac{1}{k_0} \arccot \frac{u(0)}{2vk_0}.$$ 

Finally, if $C_0 = 0$, then $u = -2v/(x - x_0)$; if $u_x = 0$, then $u = \text{const}$ and $u^2 = |C_0| = \text{const}$. For convenience, all explicit formulas for stationary solutions are listed together in Table 1 (left column).

**Table 1. Stationary solutions $u^S$ of Burgers equation and the corresponding solutions $\varphi^S$ of the heat equation (6).**

| Solution $u^S(x)$ of (1) | Conditions on $u, u_x$ | Conditions on $A, B, H$ | Solution $\varphi^S(x, t)$ of (6) |
|--------------------------|------------------------|------------------------|----------------------------------|
| (a) $-2v/k_0 \cot(k_0(x - x_0))$ | $0 \leq u^2 < 2vu_x$ | $A < B, H > 0$ | $C \sin(k_0(x - x_0)) \exp(-vk^2_0t)$ |
| $2v/k_0 \tan(k_0(x - x_0))$ | (the same conditions and same solution as above) | | $C \cos(k_0(x - x_0)) \exp(-vk^2_0t)$ |
| (b) $-2v/(x - x_0)$ | $0 < u^2 = 2vu_x$ | $A < B, H = 0$ | $C(x - x_0)$ |
| (c) $-2v/k_0 \coth(k_0(x - x_0))$ | $0 < 2vu_x < u^2$ | $A < B, H < 0$ | $C \sinh(k_0(x - x_0)) \exp(vk^2_0t)$ |
| (d) $\pm 2v/k_0 = \text{const}$ | $u_x = 0$ | $A = B$ | $C \exp(vk^2_0t + k_0x)$ |
| (e) $-2v/k_0 \tanh(k_0(x - x_0))$ | $u_x < 0$ | $A > B$ | $C \cosh(k_0(x - x_0)) \exp(vk^2_0t)$ |

We will now consider Burgers equation (1) with the Dirichlet boundary conditions on the interval $x \in [0, l]$: 

$$u(0, t) = A, \quad u(l, t) = B,$$

where $A$ and $B$ are constants. Let us find out which explicit formulas (Table 1) can represent the stationary solution $u^S$ of equation (1) with boundary conditions (4). Here we are concerned exclusively with solutions that are continuous, bounded, and sufficiently smooth everywhere on the interval $x \in [0, l]$.

Clearly, when $A > B$, the stationary solution $u^S$ can only have form (e) $-2v/k_0 \tanh(k_0(x - x_0))$ which is the only decreasing function in the left column of Table 1. When $A = B$, the stationary solution $u^S$ can only have form (d); all other explicit formulas for $u^S$ defined on $[0, l]$ are either strictly decreasing or strictly increasing functions of $x$.

To examine the stationary solution $u^S(x)$ for the trickiest case, $A < B$, we introduce the quantity

$$H = 2v(B - A) - lAB.$$

An elementary calculation shows that $u^S$ has form (b) if and only if $H = 0$, $A < B$. It remains to analyze the situations that yield solutions (a) and (c). We note that, at any given point $(x, u(x))$, any graph $u^S$ of form (a) is steeper than (b), while any graph of form (c) is less steep than (b). Indeed, for any stationary solution $u^S$ we have a constant value of $C_0 = 2vu_x - u^2$: solutions (a) are obtained from (3) when $2vu_x > u^2$ (steeper graphs, $C_0 > 0, H > 0$), while solutions (c) are obtained from (3) when $2vu_x < u^2$ (less steep graphs, $C_0 < 0, H < 0$). Thus when $A < B$ and $H > 0$, we can only have $u^S$ given by formula (a); when $A < B$ and $H < 0$ we can only have $u^S$ given by formula (c).

Note also that we have not yet proved that a stationary solution satisfying boundary conditions (4) exists for an arbitrary combination of $A$ and $B$. (We will prove this in Section 3.) Still, in the simple cases (b) $A < B, H = 0$ and (d) $A = B$, it is already obvious that such stationary solutions do exist.
2 The Cole-Hopf transformation

Burgers equation (1) is a rare example of a nonlinear PDE that can be linearized using a simple transformation. Specifically, if in equation (1) we substitute
\[ u(x, t) = -2\nu(\ln|\varphi(x, t)|)_x' \] (5)
then for the unknown function \( \varphi(x, t) \) we obtain the heat equation
\[ \varphi_t = \nu \varphi_{xx}. \] (6)

The substitution (5) is known as the Cole-Hopf transformation [1, 2, 5, 6]. Let us discuss some interesting properties of this transformation.

Firstly, transformation (5) can produce the same solution \( u(x, t) \) of (1) from many different solutions \( \varphi(x, t) \) of (6); these \( \varphi(x, t) \) may differ from each other by an arbitrary nonzero multiplier \( C \). Indeed, \( (\ln|\varphi|)_x' = (\ln|C\varphi|)_x' \) for any constant \( C \neq 0 \).

Secondly, zero values of \( \varphi(x, t) \) are mapped by (5) into discontinuities of \( u(x, t) \). Therefore, to get a continuous \( u(x, t) \), it is not enough to start from a continuous solution \( \varphi(x, t) \) of (6). We, moreover, need to restrict ourselves to those solutions \( \varphi(x, t) \) that are nonzero everywhere on \([0, l] \) for all \( t \geq 0 \).

Further, stationary solutions \( u^S(x) \) of (1) correspond to solutions \( \varphi^S(x, t) \) of (6) that may or may not be stationary. Explicit formulas for those \( \varphi^S(x, t) \) that yield stationary solutions \( u^S(x) \) are listed in the right column of Table 1. Interestingly, among these \( \varphi^S(x, t) \) we find “non-physical” solutions of the heat equation that grow infinitely large when \( t \to \infty \).

3 Existence and uniqueness of the stationary solution

Using the Cole-Hopf transformation (5), we will now establish the existence and uniqueness of the stationary solution of (1), (4) for any \( A \) and \( B \). Note that (5) transforms the problem (1), (4) into the following problem for heat equation (6) with the Robin boundary conditions:
\[ \varphi_t = \nu \varphi_{xx}, \]
\[ \varphi_x(0, t) + \frac{A}{2\nu} \varphi(0, t) = 0, \quad \varphi_x(l, t) + \frac{B}{2\nu} \varphi(l, t) = 0. \] (7)

Denote by \( \varphi^S \) the solution of (6) that under transformation (5) yields the stationary solution \( u^S \) of (1). Our \( \varphi^S \) must have the form \( \varphi^S(x, t) = X(x) \cdot T(t) \). (This can be checked directly by substituting \( \varphi^S \) into (5), or simply by inspection of the right column in Table 1.) Here \( X(x) \) is a function of the \( x \) coordinate only, and \( T(t) \) is a function of time \( t \) only. Substituting this \( \varphi^S \) into the heat equation (6) and dividing through by \( \nu T X \), we get
\[ \frac{T'}{\nu T} = \frac{X''}{X} = -\lambda. \]

(One ratio is a function of \( t \) only, while the other ratio is a function of \( x \) only. In order for these two ratios to be equal, they both must be equal to a constant which we denote \(-\lambda \).)

For the function \( X(x) \), problem (6), (7) translates into an eigenvalue problem (a Sturm-Liouville problem) with Robin boundary conditions:
\[ -X''(x) = \lambda X(x) \] (8)
\[ X'(0) + \frac{A}{2\nu} X(0) = 0, \quad X'(l) + \frac{B}{2\nu} X(l) = 0; \] (9)

and for the function \( T(t) \) we readily obtain
\[ T(t) = C \exp(-\nu \lambda t). \] (10)
For \( u^S \) to be continuous, \( \varphi^S \) must be nonzero everywhere on the interval \([0, l] \). So the question now is: how many eigenfunctions of (8), (9) are nonzero everywhere on \([0, l] \)? The answer is well known: for any \( A \) and \( B \), there is one and only one such eigenfunction. This follows from the familiar fact that, for any \( A \) and \( B \) in problem (8), (9), all eigenvalues \( \lambda_i \) \( (\lambda_0 < \lambda_1 < \ldots) \) have multiplicity 1, and the respective eigenfunction \( X_i(x) \) has exactly \( i \) zeros inside the interval \((0, l) \); see [3, pp. 14-18]. Thus, in problem (8), (9) we are interested in the eigenfunction \( X_0(x) \) that has no zeros for \( x \in [0, l] \) and corresponds to the least eigenvalue \( \lambda_0 \). For \( \varphi^S \) we find, up to a nonzero multiplier \( C \),

\[
\varphi^S(x, t) = CX_0(x) \exp(-\nu \lambda_0 t) \quad (\varphi^S \text{ has no zeros for } x \in [0, l]).
\]

Therefore, for any \( A \) and \( B \), there exists a unique stationary solution \( u^S(x) \) of Burgers equation (1) with boundary conditions (4):

\[
u^S(x) = -2\nu(\ln |\varphi^S(x, t)|)_x = -2\nu(\ln |X_0(x)|)_x.
\]

4 Stability and Lyapunov exponents

Now let us study the evolution of the absolute value \(|u - u^S|\) for an arbitrary non-stationary solution

\[
u(x, t) = -2\nu(\ln |\varphi(x, t)|)_x = -2\nu \frac{\varphi(x, t)}{\varphi(x, t)},
\]

where both \( u(x, t) \) and \( u^S(x) \) satisfy the Burgers equation (1) with boundary conditions (4), and \( \varphi(x, t) \) is a suitable positive solution of (6). It is known that the solution \( u(x, t) \) exists for "reasonable" combinations of the boundary conditions (4) and initial condition \( u(x, 0) \) [9]. We say that \( u^S \) is stable if \(|u - u^S| \to 0\) as \( t \to \infty \), for an arbitrary \( u \) obeying (1), (4). We have

\[
[u - u^S] = 2\nu \left| \frac{\varphi_x^S - \varphi}{\varphi} \right| = 2\nu \left| \frac{\varphi_x^S - \varphi^S_x}{\varphi^S} \right| = 2\nu \left| \frac{\varphi_x^S - \varphi^S_x + \varphi^S_x(\varphi - \varphi^S)}{\varphi^S} \right| = 2\nu \left| \varphi_x^S - \varphi^S_x \right|.
\]

Here we have introduced the notation \( \tilde{\varphi} = \varphi - \varphi^S \). Taking into account that \( u = -2\nu \varphi_x / \varphi \), for all \( x \in [0, l] \) and all \( t \geq 0 \) we obtain the estimate

\[
[u - u^S] \leq [u] \cdot \left| \frac{\tilde{\varphi}}{\varphi^S} \right| + 2\nu \left| \frac{\tilde{\varphi}}{\varphi^S} \right| \leq \max_{x \in [0, l]} [u(x, 0)] \cdot \left| \frac{\tilde{\varphi}}{\varphi} \right| + 2\nu \left| \frac{\tilde{\varphi}}{\varphi^S} \right|. \tag{11}
\]

In inequality (11) we have used the maximum principle for Burgers equation: the solution \( u(x, t) \) attains its maximum either in the initial value \( u(x, 0) \) or at the boundary of the interval \([0, l] \). (A discussion of maximum principles for PDEs can be found in [4, 7, 9]. The proof of the maximum principle for Burgers equation is similar to that for linear parabolic PDEs.)

Expand \( \varphi(x, t) \) in a series over the system of eigenfunctions \( X_i(x) \) of (8), (9):

\[
\varphi(x, t) = \sum_{i=0}^{\infty} \alpha_i X_i(x) T_i(t) = \sum_{i=0}^{\infty} \varphi_i(x, t), \quad T_i(t) = \exp(-\nu \lambda_i t), \quad T_i(0) = 1. \tag{12}
\]

In this series, the term \( \varphi_0(x, t) = \alpha_0 X_0(x) T_0(t) \) is the same as \( \varphi^S \) (Table 1) up to a constant nonzero multiplier. Let us choose \( C \) in the expression of \( \varphi^S \) (Table 1) so that \( \varphi_0 = \varphi^S \). If we now compute the difference \( \varphi - \varphi^S \), the term \( \varphi_0(x, t) \) will cancel out, and we get

\[
\tilde{\varphi} = \varphi - \varphi^S = \sum_{i=1}^{\infty} \varphi_i(x, t). \tag{13}
\]

Since \( T_i(t) = \exp(-\nu \lambda_i t) \), we see that \( \varphi_1(x, t) \) becomes the largest term in (13) when \( t \to \infty \) (assuming \( \alpha_1 \neq 0 \) in (12)). We then have

\[
\max_{x \in [0, l]} |\varphi^S| \asymp \exp(-\nu \lambda_0 t), \quad \max_{x \in [0, l]} |\tilde{\varphi}| \asymp \exp(-\nu \lambda_1 t), \quad \max_{x \in [0, l]} |\varphi_1| \asymp \exp(-\nu \lambda_1 t) \quad \text{as } t \to \infty,
\]

\[
4.
\]
so the estimate (11) results in
\[
\max_{x \in [0, l]} |u - u^S| = \exp(-\nu(\lambda_1 - \lambda_0)t) \quad \text{as } t \to \infty.
\] (14)

This paves the way to proving the stability of the stationary solution \(u^S\). Indeed, the difference \(|u - u^S|\) is an exponentially vanishing quantity when \(t \to \infty\). Nevertheless, the convergence of \(|u - u^S|\) to zero might turn out to be very slow; this is the case when the least two eigenvalues \(\lambda_0\) and \(\lambda_1\) in problem (8), (9) differ only slightly.

We got the estimate (14) under the assumption that \(\alpha_1 \neq 0\) in (12), that is, in the series expansion of \(\varphi\) over the system of eigenfunctions \(X_i(x)\) there is a nonzero term \(\varphi_1\) containing the eigenfunction \(X_1(x)\). However, if it so happens that one or more initial terms in (13) are zero, then the series (13) for \(\tilde{\varphi} = \varphi - \varphi^S\) will start at some \(\varphi_n\) \((n > 1)\). In the general case, therefore, instead of (14) we would have
\[
\max_{x \in [0, l]} |u - u^S| = \exp(-\nu(\lambda_n - \lambda_0)t) \quad \text{as } t \to \infty,
\] (15)

where \(n\) is the number of the first nonzero term in the series expansion of \(\tilde{\varphi} = \varphi - \varphi^S\) (13). We have thus proved that the stationary solution \(u^S\) is stable: \(|u - u^S| \to 0\) as \(t \to \infty\).

Note that the functions \(\varphi_i\) \((i = 1, 2, \ldots)\) in (12) have the same explicit formulas as \(\varphi^S\) (Table 1), except that each \(\varphi_i\) contains its own values in place of \(k_0\) and \(x_0\); let us denote these new constant values by \(k_i\) and \(x_i\), respectively.

All constants \(k_i\) and \(x_i\) can be found if we substitute the general solutions of (8) (trigonometric, exponential or hyperbolic functions) for the eigenfunctions \(X_i(x)\) \((i = 0, 1, 2, \ldots)\) in the boundary conditions (9). In most cases (i.e., cases (a), (c), (e) in Table 1), this substitution yields the following transcendent equations for \(\xi_i = k_i\):
\[
\cot \xi_i = \frac{p}{\xi_i} + q\xi_i \quad \text{for } \varphi_i \text{ of form (a) in Table 1,} \quad \lambda_i = k_i^2 > 0, \quad \text{and}
\]
\[
\coth \xi_i = \frac{p}{\xi_i} - q\xi_i \quad \text{for } \varphi_i \text{ of form (c) or (e) in Table 1,} \quad \lambda_i = -k_i^2 < 0,
\] (16) \hspace{1cm} (17)

where \(\xi_i = k_i > 0, \quad p = \frac{LAB}{2\nu(B - A)}, \quad q = \frac{2\nu}{B - A}\).

The transcendent equation (16), with \(\cot \xi_i\), may correspond to any \(i\), whereas equation (17), with \(\coth \xi_i\), may correspond only to \(i = 0, 1\) (the least two eigenvalues \(\lambda_0, \lambda_1\) because hyperbolic functions cannot have more than one zero value on the interval \([0, l]\).

When \(A = B\) in (4) and (9), we have an exceptional case: all \(k_i\) and \(\lambda_i\) can be found in a closed form. Here the interval \([0, l]\) contains a whole number of semiperiods of the eigenfunction \(X_i(x) = \sin(k_i(x - x_i)), \quad i = 1, 2, \ldots, \) which readily yields
\[
k_i = \frac{\pi i}{l} \quad (i = 1, 2, \ldots), \quad \text{while} \quad k_0 = \frac{|A|}{2\nu}, \quad X_0(x) = C \exp(\pm x k_0 x); \quad \text{see Table 1 (d)}.
\]

Therefore, if \(A = B\), we find
\[
\lambda_n = \left(\frac{\pi n}{l}\right)^2 \quad (n \geq 1), \quad \lambda_0 = -\left(\frac{A}{2\nu}\right)^2, \quad \text{and} \quad \lambda_n - \lambda_0 = \left(\frac{\pi n}{l}\right)^2 + \left(\frac{A}{2\nu}\right)^2; \quad \text{cf. (14), (15)}.
\]

Now we will reuse the customary definition of Lyapunov exponents in the context of problem (1), (4) for Burgers equation. Let \(u(x, t)\) be a solution of (1), (4). The Lyapunov exponent \(\mu\) of this solution is defined as
\[
\mu = \limsup_{t \to \infty} \frac{\ln ||u - u^S||}{t}.
\] (18)

This definition, in general, depends on our choice of the norm \(\| \cdot \|\). If \(u(x, t)\) behaves so that \(||u - u^S|| \approx \exp(\delta t)\) as \(t \to \infty\), then it is easy to see that \(\delta\) is the Lyapunov exponent of this \(u(x, t)\).

Let us use the norm defined as the maximum absolute value:
\[
||w(x)|| = \max_{x \in [0, l]} |w(x)|.
\]
Then estimates (14), (15) allow us to determine all Lyapunov exponents for any $u(x, t)$ satisfying (1), (4):

$$\mu_i = -\nu(\lambda_i - \lambda_0), \quad i = 1, 2, \ldots, \tag{19}$$

where, as before, $\lambda_i$ are eigenvalues of (8), (9). Solutions $u(x, t)$ corresponding to the Lyapunov exponents $\mu_i$ can be written simply as

$$u_i(x, t) = -2\nu(\ln|\varphi_i(x, t)|', \quad i = 1, 2, \ldots, \tag{20}$$

where $\varphi_i(x, t)$ is the respective term of (12). For example, when $u^S$ has the form (a) in Table 1, we have

$$\varphi^S(x, t) = C \sin(k_0(x - x_0)) \exp(-\nu k_0 t) \quad (\varphi^S \text{ has no zeros for } x \in [0, l]),$$

$$\varphi_i(x, t) = \alpha_i \sin(k_i(x - x_i)) \exp(-\nu k_i^2 t) \quad (\varphi_i \text{ has } i \text{ zeros for } x \in [0, l]),$$

and we can write a solution $u_i(x, t)$ corresponding to the Lyapunov exponent $\mu_i$ as follows:

$$u_i(x, t) = -2\nu C k_0 \cos(k_0(x - x_0)) + \alpha_i k_i \cos(k_i(x - x_i)) \cdot \exp(-\nu(k_i^2 - k_0^2)t).$$

Because each individual term in series (12) satisfies the Robin boundary conditions (7), each function $u_i(x, t)$ defined as above must satisfy the Dirichlet boundary conditions (4).

We have thus determined the Lyapunov exponents in the nonlinear problem (1), (4) for Burgers equation: we have found that formula (19) relates the Lyapunov exponents $\mu_i$ to the eigenvalues $\lambda_i$ of the linear problem (8), (9). All Lyapunov exponents $\mu_i$ are negative; there are countably many of them; we can write explicit formulas for the corresponding solutions $u_i(x, t)$ of Burgers equation (1). This is an interesting example of a situation where one can analytically determine the Lyapunov exponents for solutions of a nonlinear PDE with Dirichlet boundary conditions.

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