THE LANGUAGE OF STRATIFIED SETS IS CONFLUENT AND STRONGLY NORMALISING

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Abstract: We study the properties of the language of Stratified Sets (first-order logic with ∈ and a stratification condition) as used in TST, TZT, and (with stratifiability instead of stratification) in Quine’s NF. We find that the syntax forms a nominal algebra for substitution and that stratification and stratifiability imply confluence and strong normalisation under rewrites corresponding naturally to β-conversion.

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1. Introduction

1.1. About Stratified Sets. Consider Russell’s paradox, that if \( s = \{ a \mid a \not\in a \} \) then \( s \in s \) if and only if \( s \not\in s \). One way to avoid the term \( s \) is to restrict to the language of Stratified Sets. This is first-order logic with sets epsilon \( \in \) and:

- Variable symbols \( a \) (called atoms in this paper) are assigned levels, which are typically integers or natural numbers.
- We impose a stratification typing condition that we may only form \( t \in s \) if the level of \( s \) is one plus the level of \( t \).

\[ \text{level}(s) = \text{level}(t)+1. \]

See Definition 5.2 for full details. Then \( s = \{ a \mid a \not\in a \} \) cannot be stratified, since whatever level we assign to \( a \) in \( a \in a \) we cannot make \( \text{level}(a) = \text{level}(a)+1. \)

Stratified Sets are one of a family of syntaxes designed to exclude Russell’s paradox:

- The language of ZF set theory restricts sets comprehension to bounded comprehension \( \{ a \mid X \phi \} \).
- Type Theories (such as Higher-Logic) impose more or less elaborate type systems. The canonical example of this is simple types \( \tau ::= \iota \mid \tau \rightarrow \tau. \)
- Stratified Sets stratifies terms as described.

One feature of Stratified Sets is that we can write a term representing the universal set:

\[ \text{univ} = \{ a \mid T \} \]

is easily stratified by giving \( a \) any level we like. Likewise we can write definitions such as ‘the number 2’ to be ‘the set of all two-element sets’:

\[ 2 = \{ a \mid \exists b, c. (a = \{ b, c \} \land b \neq c) \} \]

(Here we freely use syntactic sugar for readability; this can all be made fully formal.)
This feels liberating: we have the pleasure of full unbounded sets comprehension\(^1\) and we have the pleasure of more sweeping types than are possible in the usual type theories such as Higher-Order Logic and its elaborations.\(^2\)

1.2. **What this paper does.** The published literature using Stratified Sets does not view the basic syntax from the point of view of rewriting. On this topic, this paper makes three observations:

1. The stratification condition implies that the syntax is confluent and strongly normalising under the natural rewrite

   \[ t \in \{ a \mid \phi \} \Rightarrow \phi[a:=t]. \]

   We can write: *stratification \(\Rightarrow\) confluence and strong normalisation*. Similarly for stratifiability.

   See Theorems 5.26 and 5.28.

2. The syntax of normal forms becomes an algebra for substitution in a sense that will be made formal using nominal algebra.

   See Theorem 4.18; in fact the proof of Theorem 5.26 uses this.

3. Our proof is constructed using nominal techniques. The proofs in this paper should be fairly directly implementable in a nominal theorem-prover, such as Nominal Isabelle \(^{Urb08}\).

   In some senses, this paper is deliberately conventional, even simple: we write down a syntax and a rewrite relation and prove some nice properties. But the simplicity is deceptive. TST, TZT, and NF as usually presented do not include sets comprehension in their syntax, if that syntax is even made fully formal; so just noting that there might be rewrite relations here that might be useful to look at, seems to be a new observation. And the proofs are not trivial: it is easy to give a handwaving argument (see for instance Remark 1.6 below), but it is surprisingly difficult to give a rigorous proof with all details. More on this in Section 6.

   We use nominal techniques (see the material in Section 2) to manage the \(\alpha\)-binding in the syntax for universal quantification and sets comprehension. If the reader is unfamiliar with nominal techniques then they can just ignore this aspect: wherever we see reference to a nominal theorem, we can replace it with ‘by \(\alpha\)-conversion’ or with ‘it is a fact of syntax that’. The result should then be close to the kind of argument that might normally pass without comment or challenge.

1.3. **Some remarks.**

**Remark 1.1.** ‘The language of Stratified Sets’ is a description specific to this paper. In the literature, this syntax is unnamed and presented along with the theory we express using it:

1. TST (which stands for Typed Set Theory) is typically taken to be first-order logic with \(\in\) and variables stratified as \(N = \{0, 1, 2, \ldots \}\), along with reasonable axioms for first-order logic and extensional sets equality.

2. TZT is typically taken to be as the syntax and axioms of TST but with variables stratified as \(Z = \{0, 1, -1, 2, -2, \ldots \}\).

3. Quine’s New Foundations (NF) uses the language of first-order logic with \(\in\), and reasonable axioms, and a *stratifiability condition* that variables could be stratified.

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\(^1\)It still needs to be stratified, of course, but by Russell’s paradox we must expect our party to be spoiled. Our choices are only: how, and where?

\(^2\)Two attitudes are possible with types: embrace and enrich them, which leads us in the direction of (for instance) dependent types, or minimise type structure. Stratification minimises types all the way down to just being ‘\(i \in \mathbb{Z}\)’.
So in TST we would have to write (say) \( a^1 \in b^2 \) (choosing a level 1 variable symbol \( a \) and a level 2 variable symbol \( b \)), whereas in NF we could write just \( a \in b \) and say “we could assign \( a \) level 1 and \( b \) level 2”.

**Remark 1.2.** There is a slight ambiguity when we talk about stratification whether we insist that the syntax come delivered with an assignment of levels to all terms, or whether we insist on the weaker condition that an assignment could be made, but this assignment need not be a structural part of the formula or term. This distinguishes the languages of TST and TZT from that of NF: TST and TZT insist on stratification, and NF insists on stratifiability.

Our results will be agnostic in this choice (see for instance Theorems 5.26 and 5.28). So when we write Stratified Sets we could just as well write Stratifiable Sets and everything would still work with only minor bookkeeping changes.

**Remark 1.3.** The reader with a background in TST, TZT, and NF should note that this is not a paper about logical theories: it is a paper about their syntax. This is why we talk about ‘Stratified Sets’ in this paper, and not e.g. ‘Typed Set Theories’. Our protagonist is a language, not a logic.

**Remark 1.4** (Some words on terminology). Some authors expand TST as ‘Theory of Simple Types’. I think this terminology invites confusion with Simple Type Theory, so I prefer the alternative ‘Typed Set Theory’. I would also like to write that TZT stands for ‘Typed Zet Theory’, but really TZT just stands for itself.

**Remark 1.5** (References). For the reader interested in the logical motivations for these syntaxes we provide references:

- A historical account of Russell’s paradox is in [Gri04].
- For ZF set theory, see e.g. [Jec06].
- Excellent discussions of TST, TZT, and NF are in [For95] and [Hol98], and a clear summary with a brief but well-chosen bibliography is in [For97].

**Remark 1.6** (Connection to the \( \lambda \)-calculus). One way to see that something like this paper should work, fingers crossed, is by an analogy:

- The rewrite \( t \in \{ a \mid \phi \} \rightarrow \phi[a:=t] \) can be rewritten as \( (\lambda a.\phi)t \rightarrow \phi[a:=t] \).
- Extensionality is \( s = \{ b \mid b \in s \} \), and we can rewrite this as \( s = \lambda b.(sb) \).

These are of course familiar as \( \beta \)-reduction and \( \eta \)-expansion. The ‘typing system’ of the language of Stratified Sets is more liberal than simple types, and more restrictive than the untyped \( \lambda \)-calculus. The proofs need to be checked. But the analogy above invites an analysis of the kind that we will now carry out.

## 2. Background on Nominal Techniques

Intuitively, a nominal set is “a set \( X \) whose elements \( x \in X \) may ‘contain’ finitely many names \( a, b, c \in \Lambda \)”. We may call names atoms. The notion of ‘contain’ used here is not the obvious notion of ‘is a set element of’: formally, we say that \( x \) has finite support (Definition 2.10).

Nominal sets are formally defined in Subsection 2.1. Examples are in Subsection 2.2. The reader might prefer to read this section only briefly at first, and then use it as a reference for the later sections where these underlying ideas get applied. More detailed expositions are also in [GP01, Gab11, DG12, Pit13].

In the context of the broader literature, the message of this section is as follows:
• The reader with a category-theory background can read this section as exploring the category of nominal sets, or equivalently the Schanuel topos (more on this in [MM92, Section III.9], [Joh03, A.21, page 79], or [Gab11, Theorem 9.14]).
• The reader with a sets background can read this section as stating that we use Fraenkel-Mostowski set theory (FM sets).
  A discussion of this sets foundation, tailored to nominal techniques, can be found in [Gab11, Section 10]). FM sets add urelemente or atoms to the sets universe.
• The reader uninterested in foundations can note that previous work [GP01, Gab11, DG12] has shown that just assuming names as primitive entities in Definition 2.1 yields a remarkable clutch of definitions and results, including Theorem 2.12, Corollary 2.13, and Theorem 2.25.

2.1. Basic definitions.

2.1.1. Atoms and permutations.

**Definition 2.1.** For each \(i \in \mathbb{Z}\) fix a disjoint countably infinite set \(A_i\) of atoms.\(^3\)

• Write \(A = \bigcup_{i \in \mathbb{Z}} A_i\).
• If \(a \in A_i\) (so \(a\) is an atom) write \(\text{level}(a)\) for the unique number such that \(a \in A_{\text{level}(a)}\).
• We use a permutative convention that \(a, b, c, \ldots\) range over distinct atoms.
  If we do not wish to use the permutative convention then we will refer to the atom using \(n\) (see for instance (\(\sigmaeltatm\)) of Figure 2).

2.1.2. Permutation actions on sets.

**Definition 2.2.** Suppose \(\pi : A \cong A\) is a bijection on atoms.

1. If \(\text{nontriv}(\pi) = \{a \mid \pi(a) \neq a\}\) is finite then we call \(\pi\) finite.
2. If \(\pi(a) \in A \Leftrightarrow a \in A\) then call \(\pi\) sort-respecting.
3. A permutation \(\pi\) is a finite sort-respecting bijection on atoms.

Henceforth \(\pi\) will range over permutations.

We will use the following notations in the rest of this paper:

**Notation 2.3.** (1) Write \(\text{id}\) for the identity permutation such that \(\text{id}(a) = a\) for all \(a\).
(2) Write \(\pi' \circ \pi\) for composition, so that \((\pi' \circ \pi)(a) = \pi'(\pi(a))\).
(3) If \(i \in \mathbb{Z}\) and \(a, b \in A_i\) then write \((a \ b)\) for the swapping (terminology from [GP01]) mapping \(a\) to \(b\), \(b\) to \(a\), and all other \(c\) to themselves, and take \((a \ a) = \text{id}\).
(4) Write \(\pi^{-1}\) for the inverse of \(\pi\), so that \(\pi^{-1} \circ \pi = \text{id} = \pi \circ \pi^{-1}\).

\(^3\)These will serve as variable symbols in Definition 3.2.
2.1.3. Sets with a permutation action.

**Notation 2.4.** If $A \subseteq \mathbb{A}$ write
$$\text{fix}(A) = \{ \pi \mid \forall a \in A. \pi(a) = a \}.$$

**Definition 2.5.** A set with a permutation action $X$ is a pair $(|X|, \cdot)$ of an underlying set $|X|$ and a permutation action written $\pi \cdot x$ which is a group action on $|X|$, so that $\text{id} \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $x \in X$ and permutations $\pi$ and $\pi'$.

**Definition 2.6.** (1) Say that $A \subseteq \mathbb{A}$ supports $x \in X$ when $\forall \pi. \pi \in \text{fix}(A) \Rightarrow \pi \cdot x = x$. 
(2) If a finite $A \subseteq \mathbb{A}$ supporting $x$ exists, call $x$ finitely supported (by $A$) and say that $x$ has finite support.

**Notation 2.7.** If $X$ is a set with a permutation action then we may write
- $x \in X$ as shorthand for $x \in |X|$, and
- $X \subseteq X$ as shorthand for $X \subseteq |X|$.

2.1.4. Nominal sets.

**Definition 2.8.** Call a set with a permutation action $X$ a nominal set when every $x \in X$ has finite support. $X, Y, Z$ will range over nominal sets.

**Definition 2.9.** Call a function $f \in X \Rightarrow Y$ equivariant when $\pi \cdot (f(x)) = f(\pi \cdot x)$ for all permutations $\pi$ and $x \in X$. In this case write $f : X \Rightarrow Y$.

The category of nominal sets and equivariant functions between them is usually called the category of nominal sets.

**Definition 2.10.** Suppose $X$ is a nominal set and $x \in X$. Define the support of $x$ by
$$\text{supp}(x) = \bigcap \{ A \subseteq \mathbb{A} \mid A \text{ is small and supports } x \}.$$

**Notation 2.11.**
- Write $a \# x$ as shorthand for $a \notin \text{supp}(x)$ and read this as $a$ is fresh for $x$.
- If $T \subseteq \mathbb{A}$ write $T \# x$ as shorthand for $\forall a \in T. a \# x$.
- Given atoms $a_1, \ldots, a_n$ and elements $x_1, \ldots, x_m$ write $a_1, \ldots, a_n \# x_1, \ldots, x_m$ as shorthand for $\forall 1 \leq j \leq m. \{ a_1, \ldots, a_n \} \# x_j$. That is: $a_i \# x_j$ for every $i$ and $j$.

**Theorem 2.12.** Suppose $X$ is a nominal set and $x \in X$. Then $\text{supp}(x)$ is the unique least small set of atoms that supports $x$.

**Proof.** Consider a permutation $\pi \in \text{fix}(\text{supp}(x))$. Write $\{ a_1, \ldots, a_n \} = \text{nontriv}(\pi)$ and choose any small $A \subseteq \mathbb{A}$ that supports $x$, so by construction $\text{supp}(x) \subseteq A$.

Let $\{ b_1, \ldots, b_n \}$ be a choice of fresh atoms; so $b_i \notin A \cup \{ a_1, \ldots, a_n \}$ for $1 \leq i \leq n$. Write $\tau = (b_1 a_1) \circ \cdots \circ (b_n a_n)$. It is a fact that $(\tau \circ \pi \circ \tau)(a) = a$ for every $a \in A$ so $\tau \circ \pi \circ \tau \in \text{fix}(A)$. Also by the group action $(\tau \circ \pi \circ \tau) \cdot x = \tau \cdot (\pi \cdot x)$. Since $A$ supports $x$, we have $\tau \cdot (\pi \cdot x) = x$. We apply $\tau$ to both sides and note that $\tau \cdot x = x$, and it follows that $\pi \cdot x = x$. \hfill $\Box$

**Corollary 2.13.** (1) If $\pi(a) = a$ for all $a \in \text{supp}(x)$ then $\pi \cdot x = x$. Equivalently:

(a) If $\pi \in \text{fix}(\text{supp}(x))$ then $\pi \cdot x = x$.
(b) If $\forall a \in \mathbb{A}. (\pi(a) \neq a) \Rightarrow a \# x$ then $\pi \cdot x = x$ (see Notation 2.11).

(2) If $\pi(a) = \pi'(a)$ for every $a \in \text{supp}(x)$ then $\pi \cdot x = \pi' \cdot x$.

(3) $a \# x$ if and only if $\exists b. (b \# x \land (a \cdot b) = x)$.

**Proof.** By routine calculations from the definitions and from Theorem 2.12 (see also [Gab11, Theorem 2.21]).
2.2. Examples. Suppose $X$ and $Y$ are nominal sets. We consider some examples, some of which will be useful later.

2.2.1. Atoms. $A$ is a nominal set with the natural permutation action $\pi \cdot a = \pi(a)$.

2.2.2. Cartesian product. $X \times Y$ is a nominal set with underlying set $\{(x, y) \mid x \in X, y \in Y\}$ and the pointwise action $\pi \cdot (x, y) = (\pi \cdot x, \pi \cdot y)$.

It is routine to check that $\text{supp}((x, y)) = \text{supp}(x) \cup \text{supp}(y)$.

2.2.3. Full function space. $X \to Y$ is a set with a permutation action with underlying set all functions from $|X|$ to $|Y|$, and the conjugation permutation action

$$(\pi \cdot f)(x) = \pi \cdot (f(\pi^{-1} \cdot x)).$$

2.2.4. Small-supported function space. $X \Rightarrow Y$ is a nominal set with underlying set the functions from $|X|$ to $|Y|$ with small support under the conjugation action, and the conjugation permutation action.

2.2.5. Full powerset.

**Definition 2.14.** Suppose $Z$ is a set with a permutation action. Give subsets $Z \subseteq Z$ the pointwise permutation action

$$\pi \cdot Z = \{ \pi \cdot z \mid z \in Z \}.$$  

Then $\text{pset}(Z)$ (the full powerset of $Z$) is a set with a permutation action with

- underlying set $\{ Z \mid Z \subseteq Z \}$ (the set of all subsets of $|Z|$), and
- the pointwise action $\pi \cdot Z = \{ \pi \cdot z \mid z \in Z \}$.

A particularly useful instance of the pointwise action is for sets of atoms. As discussed in Subsection 2.2.1 above, if $a \in A$ then $\pi \cdot a = \pi(a)$. Thus if $A \subseteq A$ then

$$\pi \cdot A = \{ \pi(a) \mid a \in A \}.$$  

**Lemma 2.15.** Even if $Z$ is a nominal set, $\text{pset}(Z)$ need not be a nominal set.

**Proof.** Take $Z = A$ which we enumerate as $\{a_0, a_1, a_2, \ldots\}$ and we take $Z \in \text{pset}(Z)$ to be equal to $\text{comb}$ defined by

$$\text{comb} = \{a_0, a_2, a_4, \ldots\}.$$  

This does not have finite support (see also [Gab11, Remark 2.18]).
2.2.6. Finite powerset. For this subsection, fix a nominal set \( X \).

**Definition 2.16.** Write \( \text{FinPow}(X) \) for the nominal set with
- underlying set the set of all finite subsets of \( X \),
- with the pointwise action from Definition 2.14.

**Notation 2.17.** We might write \( X \subseteq_{\text{fin}} X \) for \( X \in \text{FinPow}(X) \).

**Lemma 2.18.** If \( X \subseteq_{\text{fin}} X \) then:
1. \( \bigcup\{\text{supp}(x) \mid x \in X\} \) is finite.
2. \( \bigcup\{\text{supp}(x) \mid x \in X\} = \text{supp}(X) \).
3. \( x \in X \) implies \( \text{supp}(x) \subseteq \text{supp}(X) \).

Rewriting this using Notation 2.11: if \( X \) is finite and \( x \in X \) then \( a \# X \) implies \( a \# x \).

**Proof.** The first part is immediate since by assumption there is some finite \( A \subseteq \mathbb{A} \) that bounds \( \text{supp}(x) \) for all \( x \in X \). The second part follows by an easy calculation using part 3 of Corollary 2.13; full details are in [Gab11, Theorem 2.29], of which Lemma 2.18 is a special case. Part 3 follows from the first and second parts.

2.2.7. Atoms-abstraction. Atoms-abstraction was the first real application of nominal techniques; it was used to build inductive datatypes of syntax-with-binding. Nominal atoms-abstraction captures the essence of \( \alpha \)-binding. In this paper we use it to model the binding in universal quantification and sets comprehension (see Definition 3.2). The maths here goes back to [Gab01, GP01]; we give references to proofs in a more recent presentation [Gab11].

Assume a nominal set \( X \) and an \( i \in \mathbb{Z} \).

**Definition 2.19.** Let the atoms-abstraction set \( [\mathbb{A}]X \) have
- Underlying set \( \{[a]x \mid a \in \mathbb{A}, x \in X\} \) where \( [a]x = \{\pi(a), \pi \cdot x \mid \pi \in \text{fix}(\text{supp}(x) \setminus \{a\})\} \).
- Permutation action \( \pi \cdot [a]x = [\pi \cdot a]\pi \cdot x \).

**Lemma 2.20.** If \( x \in X \) and \( a \in \mathbb{A} \) then \( \text{supp}([a]x) = \text{supp}(x) \setminus \{a\} \). In particular \( a \# [a]x \) (Notation 2.11).

**Proof.** See [Gab11, Theorem 3.11].

**Lemma 2.21.** Suppose \( x \in X \) and \( a, b \in \mathbb{A} \). Then if \( b \# x \) then \( [a]x = [b](b \cdot a) \cdot x \).

**Proof.** See [Gab11, Lemma 3.12].

**Definition 2.22.** Suppose \( z \in [\mathbb{A}]X \) and \( b \in \mathbb{A} \). Write \( z \odot b \) for the unique \( x \in X \) such that \( z = [b]x \), if this exists.

**Lemma 2.23.** Suppose \( b \in \mathbb{A} \) and \( z \in [\mathbb{A}]X \). Then \( b \# z \) implies \( z \odot b \in X \) is well-defined.

**Proof.** See [Gab11, Theorem 3.19].

**Lemma 2.24.** Suppose \( a \in \mathbb{A} \) and \( x \in X \). Then:
1. \( ([a]x) \odot a = x \) and if \( b \# x \) then \( ([a]x) \odot b = (b \cdot a) \cdot x \).
2. If \( a \# z \) then \( [a](z \odot a) = z \).

**Proof.** See [Gab11, Theorem 3.19].

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\(^4\)This is not necessarily true if \( X \) is infinite. For instance if we take \( X = \mathbb{A} = X \) then the reader can verify that \( a \# X \) for every \( a \), but \( a \# a \) does not hold for any \( a \in \mathbb{A} \). This is a feature of nominal techniques, not a bug; but for the case of finite sets, things are simpler.
2.3. The principle of equivariance. Names are by definition symmetric (i.e. can be permuted). Taking names and permutations as primitive implies that permutations propagate to the things we build using them. This is the principle of equivariance (Theorem 2.25 below; see also [Gab11, Subsection 4.2] and [GP01, Lemma 4.7]). It enables a particularly efficient management of renaming and α-conversion in syntax and semantics and captures why it is so useful to use names to model them instead of, for instance, numbers.

The principle of equivariance implies that, provided we permute names uniformly in all the parameters of our definitions and theorems, we then get another valid set of definitions and theorems.

**Theorem 2.25.** Suppose \( \pi \) is a list \( x_1, \ldots, x_n \). Suppose \( \pi \) is a (not necessarily finite) permutation and write \( \pi \cdot x \) for \( \pi \cdot x_1, \ldots, \pi \cdot x_n \). Suppose \( \Phi(\pi) \) is a first-order logic predicate with free variables \( \pi \). Suppose \( \Upsilon(\pi) \) is a function specified using a first-order predicate with free variables \( \pi \).

Then we have the following principles:

1. **Equivariance of predicates.** \( \Phi(\pi) \iff \Phi(\pi \cdot x) \).
2. **Equivariance of functions.** \( \pi \cdot \Upsilon(\pi) = \Upsilon(\pi \cdot x) \).
3. **Conservation of support.** If \( \pi \) denotes elements with small support then \( \text{supp}(\Upsilon(\pi)) \subseteq \text{supp}(\pi(x_1) \cup \cdots \cup \text{supp}(x_n)) \).

**Proof.** See Theorem 4.4, Corollary 4.6, and Theorem 4.7 from [Gab11].

**Remark 2.26.** Theorem 2.25 is three fancy ways of observing that if a specification is symmetric in atoms, then the result must be at least as symmetric as the inputs. We will use Theorem 2.25 frequently in this paper, either to move permutations around (parts 1 and 2) or to get ‘free’ bounds on the support of elements (part 3).

‘Free’ here means ‘we know it from the form of the definition, without having to verify it by concrete calculations’. Theorem 2.25 is ‘free’ in the spirit of Wadler’s marvellously titled *Theorems for free!* [Wad89].

**Proposition 2.27.** \( \text{supp}(\pi \cdot x) = \pi \cdot \text{supp}(x) \) (which means \( \{ \pi(a) \mid a \in \text{supp}(x) \} \)).

Using Notation 2.11, \( a \# \pi \cdot x \) if and only if \( \pi^{-1}(a) \# x \), and \( a \# x \) if and only if \( \pi(a) \# \pi \cdot x \).

**Proof.** Immediate consequence of part 2 of Theorem 2.25 (for the ‘not-free’ proof by concrete calculations see [Gab11, Theorem 2.19]).

3. Internal syntax

3.1. Basic definition.

**Notation 3.1.** Write \( \mathbb{Z} \) for the **integers**, so \( \mathbb{Z} = \{0, 1, -1, 2, -2, \ldots \} \) and \( \mathbb{N} \) for the **natural numbers**, which we start at 0, so \( \mathbb{N} = \{0, 1, 2, \ldots \} \).

**Definition 3.2.** (1) Define datatypes \( \text{Pred} \) of **internal predicates** and \( \text{Set}^i \) for \( i \in \mathbb{Z} \) of **internal (level \( i \)) sets** inductively by the rules in Figure 1, where \( \kappa \) ranges over finite ordinals.

(2) Define \( \text{Pred} = \bigcup_\kappa \text{Pred}(\kappa) \) and \( \text{Set}^i = \bigcup_\kappa \text{Set}^i(\kappa) \).

(3) Write \( \text{age}(X) \) for the least \( \kappa \) such that \( X \in \text{Pred}_\kappa \).

(4) Write \( \text{age}(x) \) for the least \( \kappa \) such that \( x \in \text{Set}^i(\kappa) \).

\(^5\) It is important to realise here that \( \pi \) must contain all the variables mentioned in the predicate. It is not the case that \( a = a \) if and only if \( a = b \)—but it is the case that \( a = b \) if and only if \( b = a \) (both are false).
We can alpha-convert.

**Remark 3.4.** We read through and comment on Definition 3.2:
- $\kappa$ measures the age or stage of an element; at what point in the induction it is introduced into the datatype. This is an inductive measure.
- If we elide $\kappa$ and levels and simplify, we can rewrite Definition 3.2 semi-formally as follows:

  $$
  \begin{align*}
  x \in \text{Set} & := \text{atm}(a) \cup \text{st}([a]X) \\
  X \in \text{Pred} & := \text{and}(X) \cup \text{neg}(X) \cup \text{all}([a]X) \cup \text{elt}(x, a)
  \end{align*}
  $$

- neg represents negation, and represents logical conjunction.
- and takes a finite set rather than a pair of terms. This is a nonessential eccentricity that cuts down on cases later on. Truth is represented as $\text{and}(\varnothing)$. See Example 3.8.
- all represents universal quantification; read $\text{all}([a]X)$ as ‘for all $a$, $X$’ or in symbols: ‘$\forall a. \phi$’. In $\text{all}([a]X)$, $[a]X$ is the nominal atoms-abstraction from Definition 2.19. It implements the binding of the universal quantifier by the standard nominal method.
- $\in$ represents a sets membership; read $\text{elt}(x, a)$ as ‘$x$ is an element of $a$’. Note here that $a$ is an atom; it does not literally have any elements. $\text{elt}(x, a)$ represents the predicate ‘we believe that $x$ is an element of the variable $a$’, or in symbols: ‘$x \in a$’.
- $\text{st}([a]X)$ represents sets comprehension; read $\text{st}([a]X)$ as ‘the set of $a$ such that $\phi$’ or in symbols: ‘$\{a \mid \phi\}$’. Again, as standard in nominal techniques, nominal atoms-abstraction is used to represent the binding.
  
  If $a \in \mathbb{A}$ then $\text{st}([a]X) \in \text{Set}^{i+1}$.
- $\text{atm}(a)$ is a copy of $a \in \mathbb{A}$ wrapped in some formal syntax $\text{atm}$.

**Lemma 3.5.** Suppose $X \in \text{Pred}$ and $i \in \mathbb{Z}$ and $a, a' \in \mathbb{A}$ and $a' \# X$. Then:

1. $\text{st}([a]X) = \text{st}([a'](a' \cdot a) \cdot X)$ and $\text{all}([a]X) = \text{all}([a'](a' \cdot a) \cdot X)$.
2. $a \# \text{st}([a]X)$ and $a \# \text{all}([a]X)$, and $\text{supp}(\text{st}([a]X)) \subseteq \text{supp}(X) \setminus \{a\}$.

**Proof.** Immediate from Lemma 2.21, and Lemma 2.20 with Theorem 2.25.

**Remark 3.6.** Figure 1 defines a nominal datatype, in which atoms-abstraction is used to manage binding. This gives us Lemma 3.5.

(1) Part 1 of Lemma 3.5 says “We can alpha-convert”. 

---

6So every internal comprehension or internal atom is an internal set. Another choice of terminology would be to call $\text{atm}(a)$ an internal atom, $\text{st}([a]X)$ an internal set, and $\text{atm}(a)$ or $\text{st}([a]X)$ internal elements.

However, note that $\text{st}([a]X)$ is not a set and neither is $\text{atm}(a)$; they are both syntax and we can call them what we like.
(2) In part 2 of Lemma 3.5, \(\text{supp}\) corresponds exactly to the notion that would normally be written “Free variables of”, and \(a\#X\) corresponds to “\(a\) is not free in \(X\)”. So why not just write that? Nominal techniques are a general basket of ideas with implications that go well beyond modelling syntax, but the specific benefit of using nominal techniques to model syntax is that we get alpha-conversion for free from the ambient nominal theory (see \([GP01]\) and Section 2). We do not have to define \(\alpha\)-conversion and free variables of by induction, and then prove their properties (which is actually a more subtle undertaking than is often realised; cf. Remark 4.13).

The reader does not expect to see notions of ordered pairs, trees, numbers, functions, and function application developed from first principles every time we want to write abstract syntax and write a function on a syntax tree. It is assumed that these things have been worked out. Nominal techniques do that for binding (and more).

3.2. Some notation.

**Notation 3.7.** Suppose \(X,Y \in \text{Pred}\) and \(\mathcal{X} \subseteq_{\text{fin}} \text{Pred}\). Define syntactic sugar \(\text{or}(\mathcal{X})\), \(\text{imp}(X,Y)\) and \(\text{iff}(X,Y)\) by

\[
\text{or}(\mathcal{X}) = \neg((\text{and}((\neg(X) \mid X \in \mathcal{X})))) \\
\text{imp}(X,Y) = \text{or}((\neg(X),Y)) \\
\text{iff}(X,Y) = \text{and}((\text{imp}(X,Y),\text{imp}(Y,X))).
\]

**Example 3.8.** Define \(F \in \text{Pred}\) and \(T \in \text{Pred}\) by

\[
F = \text{or}(\emptyset) \quad \text{and} \quad T = \text{and}(\emptyset).
\]

Intuitively, \(F\) represents the empty disjunction, and \(T\) represents the empty conjunction.

**Lemma 3.9.** Suppose \(i \in \mathbb{Z}\) and \(a' \in \mathbb{A}^i\). Suppose \(x \in \text{Set}^i\) and \(a'\#x\). Then \(x@a' \in \text{Pred}\).

**Proof.** By Definition 3.2 \(\text{Set}^i = [\mathbb{A}^{i-1}]\text{Pred}\). So this result just repeats Lemma 2.23.

Recall \(F = \text{or}(\emptyset)\) from Example 3.8.

**Definition 3.10.** Suppose \(i \in \mathbb{Z}\) and \(a \in \mathbb{A}^i\). Define \(\text{empt}^i\) and \(\text{set}^i\) by

\[
\text{empt}^i = \text{st}([a]F) = \text{st}([a]\text{or}(\emptyset)) \\
\text{set}^i = \text{st}([a]T) = \text{st}([a]\text{and}(\emptyset)).
\]

**Remark 3.11.** Note that by Theorem 2.25 \(a\#F\) and \(a\#T\) for any \(a \in \mathbb{A}^i\), and it follows using Corollary 2.13 that Definition 3.10 does not depend on the choice of \(a\).

We conclude with an easy lemma:

**Lemma 3.12.** Suppose \(i \in \mathbb{Z}\) and \(a \in \mathbb{A}^i\). Then:

1. \(\text{empt}^i@a = F\) and \(\text{st}([a]F) = \text{empt}^i,\) and similarly \(\text{set}^i@a = T\) and \(\text{st}([a]T) = \text{set}^i\).
2. \(a\#F\) and \(a\#T\).

**Proof.** (1) From Definition 3.10 and Lemma 2.24(1).

(2) From part 1 of this result, since \(a\#F\) by Theorem 2.25.
4.1. Basic definition and well-definedness. Intuitively, Definition 4.1 defines a substitution action. It is slightly elaborate, especially because of \((\sigma\text{elt}\alpha)\) of Figure 2, so it gets a fancy name (‘\(\sigma\text{-}\)action’) and we need to make formal and verify that it behaves as a substitution action should; see Remark 4.7.

Definition 4.1. Suppose \(i \in \mathbb{Z}\) and \(a \in A^i\) and \(x \in \text{Set}^i\). Then define a \(\sigma\text{-}\)\text{action} (sigma-action) inductively by the rules in Figure 2. In that figure:

- In rule (\(\sigma\text{and}\)), \(X \subseteq \text{fin}\text{Pred}\).
- In rule (\(\sigma\text{neg}\)), \(X \in \text{Pred}\).
- In rule (\(\sigma\text{all}\)), \(X \in \text{Pred}\) and \(b \in A^j\) for some \(j \in \mathbb{Z}\).
- In rule (\(\sigma\text{elt}\alpha\)), \(a' \in A^{i-1}\).
- In rule (\(\sigma\text{elt}\text{atm}\)), \(n\) ranges over all atoms in \(A^i\) (not just those distinct from \(a\)).
- In rule (\(\sigma\text{eltb}\)), \(b \in A^j\) for some \(j \in \mathbb{Z}\).
- In rule (\(\sigma\text{st}\)), \(X \in \text{Pred}\) and \(c \in A^k\) for some \(k \in \mathbb{Z}\).

Remark 4.2. Figure 2 slips in no fewer than three abuses of the mathematics:

1. We do not know that \(X \in \text{Pred}\) implies \(X[a \rightarrow x] \in \text{Pred}\), so we should not write \(\text{and} (\{X[a \rightarrow x] \mid \ldots\})\) on the right-hand side of (\(\sigma\text{and}\)), or indeed \(X[a \rightarrow x]\) on the right-hand side of (\(\sigma\text{neg}\)), and so on.
   
   In fact, all right-hand sides of Figure 2 are suspect except those of (\(\sigma\alpha\)) and (\(\sigma\beta\)).
2. We do not know whether the choice of fresh \(a' \in A^{i-1}\) in (\(\sigma\text{elt}\alpha\)) matters, so we do not know that (\(\sigma\text{elt}\alpha\)) is well-defined.
3. The definition looks inductive at first glance, however in the case of (\(\sigma\text{elt}\alpha\)) there is no guarantee that \(X\) (on the right-hand side) is smaller than \(\text{elt}(y, a)\) (on the left-hand side). The level of \(a'\) is strictly lower than the level of \(a\), however levels are taken from \(\mathbb{Z}\) which is totally ordered but not well-ordered by \(\leq\).

In fact:

- \(X \in \text{Pred}\) does indeed imply \(X[a \rightarrow x] \in \text{Pred}\).
- The choice of fresh \(a'\) in (\(\sigma\text{elt}\alpha\)) is immaterial.
- The levels of atoms involved are bounded below (see Definition 4.4) so we only ever work on a well-founded fragment of \(\mathbb{Z}\).

For proofs see Proposition 4.6 and Lemma 4.8.
Would it be more rigorous to interleave the proofs of these lemmas with the definition, so that at each stage we are confident that what we are writing actually makes sense? Certainly we could; the reader inclined to worry about this need only read Definition 4.1 alongside Proposition 4.6 and Lemma 4.8 as a simultaneous inductive argument on \((\text{level}(a), \text{age}(X))\) lexicographically ordered.

Remark 4.3 (Why ‘minimum level’). Levels are in \(\mathbb{Z}\) and are totally ordered by \(\leq\) but not well-founded (since integers can ‘go downwards forever’).

However, any (finite) internal predicate or internal set can mention only finitely many levels, so we can calculate the minimum level of a predicate or set, which is lower bound on the levels of atoms appearing in that predicate or set. We will use this lower bound to reason inductively on levels in Proposition 4.6 and Lemma 4.12.

Definition 4.4. Define \(\text{minlevel}(Z)\) and \(\text{minlevel}(z)\) the minimum level of \(Z\) or \(z\), inductively on \(Z \in \text{Pred}\) and \(z \in \text{Set}^i\) for \(i \in \mathbb{Z}\) as follows:

- \(\text{minlevel}(\text{atm}(a)) = \text{level}(a)\)
- \(\text{minlevel}(\text{and}(X)) = \min\{\{0\} \cup \{\text{minlevel}(X) \mid X \in \mathcal{X}\}\}\)
- \(\text{minlevel}(\text{neg}(X)) = \text{minlevel}(X)\)
- \(\text{minlevel}(\text{all}(a)X) = \min\{\{\text{level}(a)\}, \text{minlevel}(X)\}\)
- \(\text{minlevel}(\text{elt}(x, a)) = \min\{\{\text{minlevel}(x)\}, \text{level}(a)\}\)
- \(\text{minlevel}(\text{st}([a]X)) = \min\{\{\text{level}(a)\}, \text{minlevel}(X)\}\)

Above, \(\text{min}(\mathcal{I})\) is the least element of \(\mathcal{I} \subseteq_{\text{fin}} \mathbb{Z}\). We add 0 in the clause for \(\text{and}\) as a ‘default value’ to exclude calculating a minimum for the empty set; any other fixed integer element would do as well or, if we do not want to make this choice, we can index \(\text{minlevel}\) over a fixed but arbitrary choice. The proofs to follow will not care.

It will be convenient to apply \(\text{minlevel}\) to a mixed list of internal predicates, atoms, and internal sets:

Notation 4.5. • Define \(\text{minlevel}(a) = \text{level}(a)\).
• If \(l = (l_1, l_2, \ldots, l_n)\) is a list of elements from \(\text{Pred} \cup \bigcup_{i \in \mathbb{Z}} \text{Set}^i \cup A\) then we write \(\text{minlevel}(l)\) for the least element of \(\{\text{minlevel}(l_1), \ldots, \text{minlevel}(l_n)\}\).

Proposition 4.6. Suppose \(i \in \mathbb{Z}\) and \(a \in \mathcal{A}^i\) and \(x \in \text{Set}^i\).

1. If \(Z \in \text{Pred}\) then
   • \(Z[a \rightarrow x]\) is well-defined,
   • \(\text{minlevel}(Z[a \rightarrow x]) \geq \text{minlevel}(Z, a, x)\), and
   • \(Z[a \rightarrow x] \in \text{Pred}\).
2. If \(k \in \mathbb{Z}\) and \(z \in \text{Set}^k\) then
   • \(z[a \rightarrow x]\) is well-defined,
   • \(\text{minlevel}(z[a \rightarrow x]) \geq \text{minlevel}(z, a, x)\), and
   • \(z[a \rightarrow x] \in \text{Set}^k\).

Proof. Fix some \(k \in \mathbb{Z}\). We prove the Proposition for all \(Z, a, x\) and \(z, a, x\) such that \(\text{minlevel}(Z, a, x) \geq k\) and \(\text{minlevel}(z, a, x) \geq k\), by induction on \((\text{level}(a), \text{age}(Z))\) and \((\text{level}(a), \text{age}(z))\) lexicographically ordered. Since \(k\) was arbitrary, this suffices to prove it for all \(Z, a, x\) and \(z, a, x\).

We consider the possibilities for \(Z \in \text{Pred}\):

• The case of \(\text{and}(X)\) for \(X \subseteq_{\text{fin}} \text{Pred}\).
  By Figure 2 (\text{and}) \(Z[a \rightarrow x] = \text{and}(\{X'[a \rightarrow x] \mid X' \in \mathcal{X}\})\). We use the inductive hypothesis on each \(X'[a \rightarrow x]\) and some easy arithmetic calculations.
• The case of \text{neg}(X') for \(X' \in \text{Pred}\).
  By Figure 2 (\(\text{neg}\) \(Z[a \rightarrow x] = \text{neg}(X'[a \rightarrow x])\)). We use the inductive hypothesis on \(X'[a \rightarrow x]\).

• The case of \(\text{all}(b' X')\) for \(X' \in \text{Pred}\) and \(b \in \mathbb{A}^j\) for some \(j \in \mathbb{Z}\).
  Using Lemma 3.5(1) we may assume without loss of generality that \(b \# x\). By Figure 2 (\(\text{all}\) (\(\text{all}(b' X')[a \rightarrow x] = \text{all}(b' f(X[a \rightarrow x])\)). We use the inductive hypothesis on \(X'[a \rightarrow x]\).

• The case of \(\text{elt}(z, a)\) for \(z \in \text{Set}^{i-1}\). There are two sub-cases:
  - Suppose \(x = \text{atm}(n)\) for some \(n \in \mathbb{A}^k\).
    By Figure 2 (\(\sigma \text{eltatm}\) (\(\text{elt}(z, a))[a \rightarrow x] = \text{elt}(z[a \rightarrow \text{atm}(n)], n)\). We use the inductive hypothesis on \(z[a \rightarrow \text{atm}(n)]\).
  - Suppose \(x = \text{st}([a' x] @ a')\) for some fresh \(a' \in \mathbb{A}^{i-1}\) (so \(a' \# x, z\).
    By Figure 2 (\(\sigma \text{elt}\) (\(\text{elt}(z, a))[a \rightarrow x] = (x @ a')[a' \rightarrow z[a \rightarrow x]]\). We have the inductive hypothesis on \(z[a \rightarrow x]\). We chose \(a' \# x\) so by Lemma 3.9 \(x @ a' \in \text{Pred}\). We also have the inductive hypothesis (since \(k \leq \text{level}(a') = i-1 \leq i = \text{level}(a)\)) on \((x @ a')[a' \rightarrow z[a \rightarrow x]]\), and this suffices.

• The case of \(\text{elt}(z, c)\) where \(c \in \mathbb{A}^k\) and \(z \in \text{Set}^{k-1}\) and \(k \in \mathbb{Z}\).
  By Figure 2 (\(\sigma \text{eltb}\)) and the inductive hypothesis.

We consider the possibilities for \(z \in \text{Set}^k\):

• The case that \(z\) is an internal atom.
  We use \((\alpha a)\) or \((\alpha b)\) of Figure 2.

• The case that \(z\) is an internal comprehension.
  Choose fresh \(c \in \mathbb{A}^{k-1}\) (so \(c \# x, z\)), so that by Lemma 2.24(2) \(z = \text{st}([c] z @ c)\). We use the first part of this result and Figure 2 (\(\sigma \text{st}\)).

\(\square\)

4.2. Nominal algebraic properties of the \(\sigma\)-action.

Remark 4.7. Several useful properties of the \(\sigma\)-action are naturally expressed as nominal algebra judgements—equalities subject to freshness conditions [GM09]. Some are listed for the reader’s convenience in Figure 3, which goes back to nominal axiomatic studies of substitution from [GM06, GM08].

In this paper we are dealing with a concrete model, so the judgements in Figure 3 not assumed and are not axioms. Instead they must be proved; they are propositions and lemmas:

• \((\sigma \alpha)\) is Proposition 4.6.
• \((\sigma \#)\) is Lemma 4.9.
• \((\sigma \sigma)\) is Lemma 4.12.
• \((\sigma \text{swp})\) and \((\sigma \text{case})\) are Corollaries 4.14 and 4.15.
• \((\sigma \text{id})\) is Lemma 4.16.
• \((\sigma \text{ren})\) is Lemma 4.17.
• \((\sigma \oplus)\) is Lemma 4.11.

These are familiar properties of substitution on syntax: for instance

• \((\sigma \alpha)\) looks like an \(\alpha\)-equivalence property—and indeed that is what it is—and
• \((\sigma \#)\) (Lemma 4.9) is sometimes called \textit{garbage collection} and corresponds to the property “if \(a\) is not free in \(t\) then \(t[a \rightarrow s] = t'\)”, and
• \((\sigma \sigma)\) (Lemma 4.12) is often called the \textit{substitution lemma}. See the discussion in Remark 4.13.

But, the proofs of these properties that we see in this paper are not replays of the familiar syntactic properties.
This is because the \( \sigma \)-action on \( \text{Pred} \) is not a simple ‘tree-grafting’ operation—not even a capture-avoiding one—because of \((\sigma \text{elta})\) in Figure 2. The proofs work, but we cannot take this for granted, and they require checking.

4.2.1. Alpha-equivalence of the \( \sigma \)-action.

\begin{figure}
\begin{align}
(\sigma \alpha) & \quad b' \# Z \Rightarrow Z[b \rightarrow y] = ((b' \ b) \cdot Z)[b' \rightarrow y] \\
(\sigma \#) & \quad b' \# Z \Rightarrow Z[b \rightarrow y] = Z \\
(\sigma \sigma) & \quad a' \# Z \Rightarrow Z[a \rightarrow x] = Z[b \rightarrow y] = Z[\sigma[b \rightarrow y][a \rightarrow x]]
\end{align}
\end{figure}

\begin{figure}
\begin{align}
(\sigma \text{swp}) & \quad a' \# y, b' \# x \Rightarrow Z[a \rightarrow x[b \rightarrow y]] = Z[\sigma[b \rightarrow y][a \rightarrow x]] \\
(\sigma \text{asc}) & \quad a' \# Z \Rightarrow Z[a \rightarrow x[b \rightarrow y]] = Z[\sigma[b \rightarrow y][a \rightarrow x]] \\
(\sigma \text{id}) & \quad Z[a \rightarrow \text{nat}(\sigma(a))] = Z \\
(\sigma \text{ren}) & \quad a' \# Z \Rightarrow Z[a \rightarrow \text{nat}(\sigma(a))] = (\sigma \ a' \ Z) \\
(\sigma @) & \quad c \# x \Rightarrow (z \ @ c)[a \rightarrow x] = z[a \rightarrow x] @ c
\end{align}
\end{figure}

\textbf{Figure 3:} Further nominal algebra properties of the \( \sigma \)-action

\begin{enumerate}
\item The case of \( \sigma \)-action. \( \bullet \) \ The case of \( \text{neg}(X) \) for \( X \in \text{Pred} \). \( \bullet \) \ The case of \( \text{all}(b, X) \) for \( X \in \text{Pred} \) and \( b \in \mathbb{K} \) for some \( j \in \mathbb{Z} \). \( \bullet \) \ The case of \( \text{elt}(y, a) \) for some \( y \in \text{Set}^{i-1} \).
\end{enumerate}

\begin{itemize}
\item Suppose \( x = \text{atm}(n) \) for some \( n \in \mathbb{K} \). We reason as follows:

\begin{align}
(\text{elt}(y, a))[a \rightarrow \text{atm}(n)] & = \text{elt}(y)[a \rightarrow \text{atm}(n), n] \quad \text{Figure 2}(\sigma \text{elta}) \\
& = \text{elt}(((a' \ a) \cdot y)[a' \rightarrow \text{atm}(n)]), n) \quad \text{Ind hyp for } y \\
& = (\text{elt}(((a' \ a) \cdot y)[a' \rightarrow \text{atm}(n)]), a' \rightarrow \text{atm}(n)) \quad \text{Figure 2}(\sigma \text{elta}) \\
& = ((a' \ a) \cdot (\text{elt}(y, a))[a' \rightarrow x]) \quad \text{Theorem 2.25}
\end{align}

\item Suppose \( x = [b')(x @ b') \) for some fresh \( b' \in \mathbb{K}^{-1} \) (so \( b' \# x, y, z \) and \( k \leq \text{level}(b') \)). We reason as follows:

\begin{align}
(\text{elt}(y, a))[a \rightarrow x] & = (x @ b')'[b' \rightarrow y[a \rightarrow x]] \quad \text{Figure 2}(\sigma \text{elta}) \\
& = (x @ b')'[b' \rightarrow ((a' \ a) \cdot y)[a' \rightarrow x]] \quad \text{Ind hyp for } y \\
& = ((a' \ a) \cdot (\text{elt}(y, b))[a' \rightarrow x]) \quad \text{Figure 2}(\sigma \text{elta})
\end{align}

\item The case \( \text{elt}(y, b) \) for \( j \in \mathbb{Z} \) and \( b \in \mathbb{K} \) and \( y \in \text{Set}^{i-1} \). We reason as follows:

\begin{align}
(\text{elt}(y, b))[a \rightarrow x] & = \text{elt}(y)[a \rightarrow x, b] \quad \text{Figure 2}(\sigma \text{eltb}) \\
& = \text{elt}(((a' \ a) \cdot y)[a' \rightarrow x], b) \quad \text{Ind hyp for } y \\
& = (\text{elt}(((a' \ a) \cdot y)[a' \rightarrow x], b)) \quad \text{Figure 2}(\sigma \text{eltb}) \\
& = ((a' \ a) \cdot (\text{elt}(y, b))[a' \rightarrow x]) \quad \text{Theorem 2.25}
\end{align}
\end{itemize}
We consider the possibilities for \(z \in \text{Set}^k\):

- **The case that \(z\) is an internal atom.** We use \((\sigma a)\) or \((\sigma b)\) of Figure 2.

- **The case that \(z\) is an internal comprehension.** We use Lemma 2.24(2) for a fresh \(c \in \text{Set}^{k-1}\) (so \(c \# z\)), \((\sigma \text{st})\), and the inductive hypothesis on \(z@c\).

For part 2, we note that by Theorem 2.25 and Proposition 2.27

\[
supp(Z[a\rightarrow x]) \subseteq supp(Z) \cup \{a\} \cup supp(x) \quad \text{and} \quad supp(((a' a) \cdot Z)[a' \rightarrow x]) \subseteq (a' a) \cdot supp(Z) \cup \{a'\} \cup supp(x).
\]

We take a sets intersection. The case of \(z\) is similar.

Part 3 follows, recalling from Notation 2.11 that \(a#x\) means \(a \notin supp(x)\).

4.2.2. Property \((\sigma\#)\) (garbage collection).

**Lemma 4.9** ((\(\sigma\#\)). Suppose \(i \in \text{Z}\) and \(a \in \text{Set}^i\) and \(x \in \text{Set}^i\) and \(Z \in \text{Pred}\) and \(z \in \text{Set}^k\) for \(k \in \text{Z}\). Then

\[
a#Z \Rightarrow Z[a \rightarrow x] = Z \\
\]

\[
a#z \Rightarrow z[a \rightarrow x] = z.
\]

**Proof.** By induction on \(Z\) and \(z\). We consider the possibilities for \(Z \in \text{Pred}\):

- **The case of \((\text{and}(\mathcal{X}'))\) for \(\mathcal{X}' \subseteq \text{fin} \text{ Pred} \).**

  By Figure 2 \((\sigma \text{and}) (\text{and}(\mathcal{X}))[a \rightarrow x] = \text{and} ([X[a \rightarrow x] \mid X \in \mathcal{X}'])\). By Lemma 2.18(3) \(a#X\) for every \(X \in \mathcal{X}'\). We use the inductive hypothesis on each \(X\).

- **The case of \(\text{neg}(X)\) for \(X \in \text{Pred}\).**

  By Figure 2 \((\sigma \text{neg}) \text{neg}(X)[a \rightarrow x] = \text{neg}(X[a \rightarrow x])\). We use the inductive hypothesis on \(X\).

- **The case of \(\text{all}([b]X)\) for \(X \in \text{Pred}\) and \(b \in \text{Set}^j\) for some \(j \in \text{Z}\).**

  Using Lemma 3.5(1) we may assume without loss of generality that \(b \# x\). By Figure 2 \((\sigma \text{all}) (\text{all}([b]X))[a \rightarrow x] = \text{all}([b](X[a \rightarrow x]))\). We use the inductive hypothesis on \(X\).

- **The case of \(\text{elt}(y, a)\) for \(i \in \text{Z}\) and \(y \in \text{Set}^{i-1}\).**

  This is impossible because we assumed \(a#Z\).

- **The case of \(\text{elt}(y, b)\) for \(j \in \text{Z}\) and \(b \in \text{Set}^j\) and \(y \in \text{Set}^{j-1}\).**

  By Figure 2 \((\sigma \text{eltb}) (\text{elt}(y, b))[a \rightarrow x] = \text{elt}(y[a \rightarrow x], b)\). We use the inductive hypothesis on \(y\).

We consider the possibilities for \(z \in \text{Set}^k\):

- If \(z\) is an internal atom then we reason using \((\sigma a)\) or \((\sigma b)\) of Figure 2.

- If \(z\) is an internal comprehension then we use Lemma 2.24(2) for a fresh \(c \in \text{Set}^{k-1}\) (so \(c#z\)), \((\sigma \text{st})\), and the inductive hypothesis on \(z@c\).

Recall \(F = \text{or}(\emptyset)\) and \(T = \text{and}(\emptyset)\) from Example 3.8. Corollary 4.10 is an easy consequence of Lemma 4.9 and will be useful later:

**Corollary 4.10.** Suppose \(i \in \text{Z}\) and \(a \in \text{Set}^i\) and \(x \in \text{Set}^i\). Then

\[
F[a \rightarrow x] = F \quad \text{and} \quad T[a \rightarrow x] = T.
\]

**Proof.** By Theorem 2.25 \(supp(F) = \emptyset\) so that \(a#x\). We use Lemma 4.9. Similarly for \(T\).
4.2.3. \( \sigma \) commutes with atoms-concretion. Lemma 4.11 will be useful later, starting with Lemma 4.12:

**Lemma 4.11** ((\( \sigma@ \))). Suppose \( i \in \mathbb{Z} \) and \( a \in h^i \) and \( x \in \text{Set}^i \). Suppose \( k \in \mathbb{Z} \) and \( z \in \text{Set}^k \) and \( c \in h^{k-1} \) and \( c \# z, x \). Then
\[
(z@c)[a\rightarrow x] = z[a\rightarrow x]@c.
\]

**Proof.** Note that by Lemma 2.23 (since \( c \# z \)) \( z@c \) exists. We reason as follows:
\[
(z@c)[a\rightarrow x] = \text{st}(c(z@c)(a\rightarrow x))@c \quad \text{Lemma 2.24(1)}
\]
\[
= \text{st}(c(z@c)(a\rightarrow x))@c \quad \text{Figure 2}(\sigma \text{st}), \ c\#x
\]
\[
= z[a\rightarrow x]@c \quad \text{Lemma 2.24(2), } c\#z
\]

4.2.4. \( \sigma \) commutes with itself: the 'substitution lemma'. The inductive quantity we use in Lemma 4.12 will be \((\text{level}(a), \text{age}(Z))\), lexicographically ordered. It will become clear in the proof how this works:

**Lemma 4.12.** Suppose \( Z \in \text{Pred} \) and \( k \in \mathbb{Z} \) and \( z \in \text{Set}^k \). Suppose \( i \in \mathbb{Z} \) and \( a \in h^i \) and \( x \in \text{Set}^i \) and suppose \( j \in \mathbb{Z} \) and \( b \in h^j \) and \( y \in \text{Set}^j \) and \( a\# y \). Then
\[
Z[a\rightarrow x][b\rightarrow y] = Z[\text{atm}(n)]\]
\[
z[a\rightarrow x][b\rightarrow y] = z[a\rightarrow x][b\rightarrow y].
\]

**Proof.** For brevity we may write \( \sigma \) for \([a\rightarrow x][b\rightarrow y]\) and \( \sigma' \) for \([b\rightarrow y][a\rightarrow x]\).

Fix some \( k \in \mathbb{Z} \). We prove the Lemma for all \( Z, a, x, b, y \) and \( z, a, x, b, y \) such that \( \text{minlevel}(Z, a, x, b, y) \geq k \) and \( \text{minlevel}(z, a, x, b, y) \geq k \) (Definition 4.4), reasoning by induction on \((\text{level}(a) + \text{level}(b), \text{age}(Z))\) and \((\text{level}(a) + \text{level}(b), \text{age}(z))\) lexicographically ordered. Since \( k \) was arbitrary, this suffices to prove it for all \( Z, a, x, b, y \) and \( z, a, x, b, y \).

We consider the possibilities for \( Z \in \text{Pred} \):

- **The case of \( \text{and}(X) \) for \( X \subseteq \text{fin} \)**. We use rule \((\sigma \text{and})\) of Figure 2 and the inductive hypothesis.
- **The case of \( \neg(X) \) for \( X \subseteq \text{Pred} \)**. We use \((\sigma \neg)\) of Figure 2 and the inductive hypothesis.
- **The case of \( \forall \text{all}(a'[X]) \) for \( X \subseteq \text{Pred} \) and \( a' \in h^k \) for some \( i \in \mathbb{Z} \)**. We use Lemma 3.5(1) to assume without loss of generality that \( a' \# x, y \), and then we use \((\sigma \text{all})\) of Figure 2 and the inductive hypothesis.
- **The case of \( \text{elt}(z, b) \) for \( z \in \text{Set}^{i-1} \) where \( j \in \mathbb{Z} \).** There are two sub-cases:
  - **Suppose \( y = \text{atm}(n) \) for some \( n \in h^i \) other than \( a \) (we assumed \( a \# y \) so \( n=a \) is impossible).**
    We reason as follows:
    \[
    (\text{elt}(z, b)) \ [a\rightarrow x][b\rightarrow \text{atm}(n)]
    = (\text{elt}(z[a\rightarrow x], b))[b\rightarrow \text{atm}(n)]
    = \text{elt}(z, a, x, b, n)[b\rightarrow \text{atm}(n)]
    = (\text{elt}(z, b))[b\rightarrow \text{atm}(n)][a\rightarrow x][b\rightarrow \text{atm}(n)]
    \]
    - **Suppose \( y = [b'](y@b') \) for some fresh \( b' \in h^{i+1} \) (so \( b' \# z, x, y \) and \( k \leq \text{level}(b') \)).**
Note by Theorem 2.25 that $a\#y@b'$ and $b'\#x[b\rightarrow y]$. We reason as follows:

$$\text{elt}(z,b)[a\rightarrow x][b\rightarrow y] = (\text{elt}(z[a\rightarrow x],b))[b\rightarrow y]$$  \hspace{1cm} \text{Figure 2(eltb)}

$$= (y@b')[b\rightarrow z\sigma]$$  \hspace{1cm} \text{Figure 2(elt)}

$$= (y@b')[b\rightarrow z\sigma']$$  \hspace{1cm} \text{IH age}(z) < \text{age}(\text{elt}(z,b)), a\#y

$$= (y@b')[a\rightarrow x[b\rightarrow y]]$$  \hspace{1cm} \text{Lemma 4.9, a\#y@b'}

$$= (y@b')[b'\rightarrow z[b\rightarrow y]][a\rightarrow x[b\rightarrow y]]$$  \hspace{1cm} \text{IH level}(b') < \text{level}(b), b'\#x[b\rightarrow y]

$$= (\text{elt}(z,b))[b\rightarrow y][a\rightarrow x[b\rightarrow y]]$$  \hspace{1cm} \text{Figure 2(elt)}

- **The case of elt(z,a) for $z\in \mathbb{Z}$**

  There are two sub-cases:

  - **Suppose $x=\text{atm}(n)$ for some $n\in \mathbb{Z}$**

    If $n\neq b$ then we reason as follows:

    $$(\text{elt}(z,a))[a\rightarrow \text{atm}(n)][b\rightarrow y] = (\text{elt}(z[a\rightarrow \text{atm}(n)],n))[b\rightarrow y]$$  \hspace{1cm} \text{Figure 2(elt)atm}

    $$= \text{elt}(z\sigma,n)$$  \hspace{1cm} \text{Figure 2(eltb)}

    $$= \text{elt}(z\sigma',n)$$  \hspace{1cm} \text{IH age}(z) < \text{age}(\text{elt}(z,a)), a\#y

    $$= \text{elt}(z[b\rightarrow y][a\rightarrow \text{atm}(n)],n)$$  \hspace{1cm} \text{(sb) n\neq b}

    $$= (\text{elt}(z[b\rightarrow y],a))[a\rightarrow \text{atm}(n)]$$  \hspace{1cm} \text{Figure 2(eltatm)}

    $$= (\text{elt}(z[b\rightarrow y],a))[a\rightarrow \text{atm}(n)]$$  \hspace{1cm} \text{(sb) n\neq b}

    $$= (\text{elt}(z,a))[b\rightarrow y][a\rightarrow \text{atm}(n)]$$  \hspace{1cm} \text{Figure 2(elt)}

  - **Suppose $x=\text{st}([a',x]@a')$ for some fresh $a' \in \mathbb{A}^{i-1}$ (so $a'\#z, x, y$ and $k \leq \text{level}(b')$)**

    Then we reason as follows (note by Theorem 2.25 that $a\#y@b'$ and $b'\#x[b\rightarrow y]$):

    $$(\text{elt}(z,a))[a\rightarrow \text{atm}(b)][b\rightarrow y] = (\text{elt}(z[a\rightarrow \text{atm}(b)],b))[b\rightarrow y]$$  \hspace{1cm} \text{Figure 2(eltatm)}

    $$= (y@b')[b\rightarrow z\sigma]$$  \hspace{1cm} \text{Figure 2(elt)}

    $$= (y@b')[b\rightarrow z\sigma']$$  \hspace{1cm} \text{IH age}(z) < \text{age}(\text{elt}(z,a)), a\#y

    $$= (y@b')[b\rightarrow z[b\rightarrow y]][a\rightarrow \text{atm}(b)]$$  \hspace{1cm} \text{IH level}(b') < \text{level}(b), b'\#x[b\rightarrow y]

    $$= (\text{elt}(z,a))[b\rightarrow y][a\rightarrow \text{atm}(b)]$$  \hspace{1cm} \text{Figure 2(eltb)}

- **Suppose $x=\text{atm}(n)$ for some $n\in \mathbb{A}^{i-1}$**

  (so $a'\#z, x, y$ and $k \leq \text{level}(a')$).

  We reason as follows:

  $$(\text{elt}(z,a))[a\rightarrow x][b\rightarrow y] = (x@a')[a'\rightarrow z[a\rightarrow x]]$$  \hspace{1cm} \text{Figure 2(elt)}

  $$= (x@a')[b\rightarrow y][a'\rightarrow z\sigma]$$  \hspace{1cm} \text{IH $k \leq \text{level}(a')=\text{level}(a)-1$; $a'\#y$

  $$= (x@a')[b\rightarrow y][a'\rightarrow z\sigma']$$  \hspace{1cm} \text{IH age}(z) < \text{age}(\text{elt}(z,a)), a\#y

  $$= (x[b\rightarrow y]@a')[a'\rightarrow z\sigma']$$  \hspace{1cm} \text{Lemma 4.11, a'\#y

  $$= (\text{elt}(z[b\rightarrow y],a))[a\rightarrow x[b\rightarrow y]]$$  \hspace{1cm} \text{Figure 2(elt)}

  $$= (\text{elt}(z,a))[b\rightarrow y][a\rightarrow x[b\rightarrow y]]$$  \hspace{1cm} \text{Figure 2(elt), a\#y
• The case of $\text{elt}(z,c)$ for $k \in \mathbb{Z}$ and $c \in A^k$ and $z \in \text{Set}^{k-1}$. We reason as follows:

$$(\text{elt}(z,c))[a \mapsto x][b \mapsto y] = \text{elt}(z,\sigma(c))$$  

Figure 2 ($\sigma \text{elt} b$), twice

$$= \text{elt}(\sigma(c),c)$$  

IH $\text{age}(z) < \text{age}(\text{elt}(z,a))$, $a \# y$

$$= (\text{elt}(z,c))[b \mapsto y][a \mapsto x[b \mapsto y]]$$  

Figure 2 ($\sigma \text{elt} b$), twice

We consider the possibilities for $z \in \text{Set}^k$:

• If $z$ is an internal atom then we reason using $(\sigma a)$ and $(\sigma b)$ of Figure 2.

• If $z$ is an internal comprehension then we use Lemma 2.24(2) for a fresh $c \in A^{k-1}$ (so $c \# z$), $(\sigma \text{st})$, and the inductive hypothesis on $z \in \text{c}$. 

\[\square\]

Remark 4.13. Were Lemma 4.12 about the syntax of first-order logic or the $\lambda$-calculus, then it could be called the substitution lemma, and the proof would be a routine induction on syntax.

In fact, even in the case of first-order logic or the $\lambda$-calculus, the proof is not routine. Issues with binders (Figure 2 ($\sigma \text{elt} a$), and one explicit in $(\sigma \text{st})$) were the original motivation for my thesis [Gab01] and for nominal techniques in general.

For a standard non-rigorous non-nominal proof of the substitution lemma see [Bar84]; for a detailed discussion of the lemma in the context of Nominal Isabelle, see [Bar14] which includes many further references.

But the proof of Lemma 4.12 is not just a replay of the proofs; neither in the ‘classic’ sense of [Bar84] nor in the ‘nominal’ sense of [Gab01, Bar14]. This is because of the interaction of $\text{elt}$ with the $\sigma$-action, mostly because of $(\sigma \text{elt} a)$ (to a lesser extent also because of the nominal binder $(\sigma \text{st})$).

Corollary 4.14 ((\text{swp}). Suppose $Z \in \text{Pred}$ and $k \in \mathbb{Z}$ and $z \in \text{Set}^k$. Suppose $i \in \mathbb{Z}$ and $a \in A^i$ and $x \in \text{Set}^i$ and suppose $j \in \mathbb{Z}$ and $b \in A^j$ and $y \in \text{Set}^j$. Suppose $a \# y$ and $b \# x$. Then

$$Z[a \mapsto x][b \mapsto y] = Z[b \mapsto y][a \mapsto x].$$

Proof. From Lemmas 4.12 and 4.9. \[\square\]

Corollary 4.15 ((\text{rasc}). Suppose $Z \in \text{Pred}$ and $k \in \mathbb{Z}$ and $z \in \text{Set}^k$. Suppose $i \in \mathbb{Z}$ and $a \in A^i$ and $x \in \text{Set}^i$ and suppose $j \in \mathbb{Z}$ and $b \in A^j$ and $y \in \text{Set}^j$. Suppose $a \# y$ and $b \# Z, z.$ Then

$$Z[a \mapsto x][b \mapsto y] = Z[a \mapsto x][b \mapsto y].$$

Proof. From Lemmas 4.12 and 4.9. \[\square\]

4.2.5. $(\sigma \text{id})$: substitution for atoms and its corollaries. We called $\text{atm}(a)$ in Definition 3.2 an internal atom. Atoms in nominal techniques interpret variables, so if we call $\text{atm}(a)$ an internal atom this should suggest that $\text{atm}(a)$ should behave like a variable (or a variable symbol). Rules $(\sigma a)$ and $(\sigma b)$ from Figure 2 are consistent with that, and Lemma 4.16 makes formal more of this intuition:

Lemma 4.16 ((\sigma \text{id}). Suppose $i \in \mathbb{Z}$ and $a \in A^i$. Then:

1. If $Z \in \text{Pred}$ then $Z[a \mapsto \text{atm}(a)] = Z$.
2. If $k \in \mathbb{Z}$ and $z \in \text{Set}^k$ then $z[a \mapsto \text{atm}(a)] = z$.

\[\square\]
Proof. We reason by induction on \(\text{age}(Z)\) and \(\text{age}(z)\). We consider the possibilities for \(Z \in \text{Pred}\):

- If \(Z = \text{and}(Z')\) for \(Z' \in \text{Pred}\) or \(Z = \text{neg}(Z')\) for \(Z' \in \text{Pred}\) then we use rules \((\sigma \text{and})\) and \((\sigma \text{neg})\) of Figure 2 and the inductive hypothesis.
- If \(Z = \text{all}(\{a'|Z'\})\) for \(Z' \in \text{Pred}\) and \(a' \in \mathcal{A}^k\) for some \(i \in \mathbb{Z}\) then we use \((\sigma \text{all})\) of Figure 2 and the inductive hypothesis.
- If \(Z = (\text{elt}(z, b))\) for \(j \in \mathbb{Z}\) and \(b \in \mathcal{A}^j\) and \(z \in \text{Set}^{i-1}\) then we use \((\sigma \text{elt} b)\) of Figure 2 and the inductive hypothesis.
- If \(Z = (\text{elt}(z, a))\) for \(z \in \text{Set}^{i-1}\) then we use \((\sigma \text{elt} \text{atm})\) of Figure 2 and the inductive hypothesis for \(z\).

We consider the possibilities for \(z \in \text{Set}^k\):

- If \(z\) is an atom then we reason using \((\sigma \text{a})\) or \((\sigma \text{b})\) of Figure 2.
- If \(z\) is an internal comprehension then we use Lemma 2.24(2) for a fresh \(c \in \mathcal{A}^{k-1}\) (so \(c#z\)), \((\sigma \text{st})\), and the inductive hypothesis on \(z@c\).

Given what we have so far, Lemma 4.17 is not hard to prove.

Lemma 4.17 \((\sigma \text{ren})\). Suppose \(i \in \mathbb{Z}\) and \(a, a' \in \mathcal{A}^i\). Then:

- If \(Z \in \text{Pred}\) and \(a' \# Z\) then \(Z[a \mapsto \text{atm}(a')] = (a' \cdot a) \cdot Z\).
- If \(k \in \mathbb{Z}\) and \(z \in \text{Set}^k\) and \(a' \# z\) then \(z[a \mapsto \text{atm}(a')] = (a' \cdot a) \cdot z\).

Proof. Suppose \(Z \in \text{Pred}\) and \(a' \# Z\). We note by Lemma 4.8(1) (since \(a' \# Z\)) that \(Z[a \mapsto \text{atm}(a')] = ((a' \cdot a) \cdot Z)[a' \mapsto \text{atm}(a')]\) and use Lemma 4.16(1). The case of \(z \in \text{Set}^k\) is exactly similar.

\[\square\]

4.3. Sigma-algebras and SUB. We can now observe that our sigma-action is consistent with the nominal algebra literature in the following sense:

Theorem 4.18. The syntaxes of internal predicates and internal terms, with the sigma-action from Definition 4.1, are sigma-algebras in the sense of [Gab16, GG16], and models of SUB in the sense of [GM08].

Concretely, this means that the sigma-action from Definition 4.1 and Figure 2 should

- distribute through and, neg, and
- distribute in a capture-avoiding manner through all, and \(\text{st}\), and
- should act on atm by direct substitution (see \((\sigma \text{a})\) and \((\sigma \text{b})\) in Figure 2), and
- should satisfy the equalities in Figure 3.

Proof. Immediate from the definitions and lemmas thus far, which were designed to verify these properties.

\[\square\]

Remark 4.19. There is redundancy in Figure 3. For instance, a nominal algebra that satisfies \((\sigma \text{a})\) satisfies \((\sigma \text{id})\) if and only if it satisfies \((\sigma \text{ren})\). One half of this implication is implicit in the proof of Lemma 4.17, which derives \((\sigma \text{ren})\) from (the lemmas corresponding to) \((\sigma \text{a})\) and \((\sigma \text{id})\); going in the other direction is no harder.

Likewise \((\sigma \text{swp})\) can be derived from \((\sigma \text{a})\) and \((\sigma \#)\). This does no harm: in this paper we are interested in exploring the good properties of Definition 4.1, rather than studying minimal sets of axioms for their own sake (for which see [GM08]).

Remark 4.20. We do not demand that the sigma-action should distribute through elt but this is because this is syntactically impossible: \(\text{elt}(y, a)[a \mapsto x]\) cannot be equal to \(\text{elt}(y[a \mapsto x], x)\) because \(\text{elt}(y[a \mapsto x], x)\) is not syntax according to Figure 1.
We shall see in Subsection 4.4, however, that this all works after all, in a suitable sense, and this will become an important observation when interpreting TST in internal syntax in Section 5.

**Remark 4.21.** Another way to approach the proofs in this paper would be to admit \( \text{elt}(y, x) \) and an explicit substitution term-former, and orient Figure 2 as rewrite rules. We would obtain a nominal rewrite system [FG07]. Essentially this would amount to converting Figure 2 (and the proofs that use it) to a ‘small-step’ presentation, from the current ‘big-step’ form.

### 4.4. The sugar \( y \in x \) and its properties.

Figure 1 only permits the syntax \( \text{elt}(y, a) \), not the syntax \( \text{elt}(y, x) \). We can obtain the power of \( \text{elt}(y, x) \) via a more sophisticated operation which we construct out of components already available:

**Notation 4.22.**

• Suppose \( i \in \mathbb{Z} \) and \( x \in \text{Set}^i \) is an internal comprehension\(^8\) and \( y \in \text{Set}^{i-1} \). Then define \( y \in x \) by

\[
y \in x = (x @ b)[b \mapsto y]
\]

where we choose \( b \in \mathcal{A}^{i-1} \) fresh (so \( b \# x, y \)).

• Suppose \( i \in \mathbb{Z} \) and \( a \in \mathcal{A}^i \) and \( y \in \text{Set}^{i-1} \). Then define \( y \in \text{atm}(a) \) by

\[
y \in \text{atm}(a) = \text{elt}(y, a).
\]

**Remark 4.23.** There are two obvious sanity properties for Notation 4.22: it should interact well with sets comprehension on the right-hand side, and it should interact well with the sigma-action substituting variables for terms. This is Lemmas 4.24 and 4.25.

**Lemma 4.24.** Suppose \( X \in \text{Pred} \) and \( i \in \mathbb{Z} \) and \( a \in \mathcal{A}^i \) and \( x \in \text{Set}^i \) and \( a \# x \). Then (using Notation 4.22)

\[
x \in \text{st}([a]X) = X[a \mapsto x].
\]

**Proof.** Note by Lemma 2.20 that \( a \# \text{st}([a]X) \). By Notation 4.22 (since \( a \# x \), \( \text{st}([a]X) \)) \( x \in \text{st}([a]X) \) is equal to \( ((\text{st}([a]X)) \circ a)[a \mapsto x] \) and by Lemma 2.24(1) this is equal to \( X[a \mapsto x]. \)

**Lemma 4.25.** Suppose \( i, j \in \mathbb{Z} \) and \( x \in \text{Set}^{i+1} \) and \( y \in \text{Set}^i \) and \( a \in \mathcal{A}^j \) and \( u \in \text{Set}^j \). Then:

1. \( (y \in x)[a \mapsto u] = y[a \mapsto u] \in x[a \mapsto u] \).
2. \( (y \in a)[a \mapsto u] = y[a \mapsto u] \in u \), where \( j = i+1 \).
3. \( (y \in b)[a \mapsto u] = y[a \mapsto u] \in b \), where \( b \in \mathcal{A}^{i+1} \).

**Proof.** First, suppose we have proved part 1 of this result. Then part 2 follows using Figure 2 (\( \sigma a \)) and part 3 follows using Figure 2 (\( \sigma b \)).

To prove part 1 there are three cases:

• **Suppose \( x = \text{atm}(a') \) for some \( a' \in \mathcal{A}^{i+1} \) not equal to \( a \).**

We reason as follows:

\[
(y \in \text{atm}(a'))[a \mapsto u] = (\text{elt}(y, a'))[a \mapsto u] \quad \text{Notation 4.22}
\]

\[
= \text{elt}(y[a \mapsto u], a') \quad \text{Figure 2(\( \sigma \text{elt}b \))}
\]

\[
= y[a \mapsto u] \in \text{atm}(a') \quad \text{Notation 4.22}
\]

\[
= y[a \mapsto u] \in (\text{atm}(a')[a \mapsto u]) \quad \text{Figure 2(\( \sigma b \))}
\]

• **Suppose \( x = \text{atm}(a) \) (so that \( j = i+1 \)).**

There are two sub-cases:

\(^8\)Terminology from Notation 3.3. So \( x \) has the form \( \text{st}([b]x @ b) \) for some \( b \in \mathcal{A}^{i-1} \) with \( b \# x, \) and \( x \) does not have the form \( \text{atm}(a) \) for any \( a \in \mathcal{A}^{i-1} \).
\[ \phi, \psi ::= \bot \mid \lnot \phi \mid \phi \land \phi \mid \forall \alpha. \phi \mid s \in s \]

\[ s, t ::= a \mid \{a \mid \phi\} \]

**Figure 4:** The syntax of Stratified Sets

- Suppose \( u = \text{atm}(n) \) for some \( n \in \mathbb{N} \). We reason as follows:
  \[ (y \in \text{atm}(a))[a \rightarrow \text{atm}(n)] = (\text{elt}(y, a))[a \rightarrow \text{atm}(n)] \quad \text{Notation 4.22} \]
  \[ = \text{elt}(y[a \rightarrow \text{atm}(n)], n) \quad \text{Figure 2 (eltatm)} \]
  \[ = y[a \rightarrow \text{atm}(n)] \in \text{atm}(n) \quad \text{Notation 4.22} \]

- Suppose \( u = \text{st}([a']u @ a') \) for some fresh \( a' \in \mathbb{N}^{-1} \) (so \( a' \# u, y \)). We reason as follows:
  \[ (y \in \text{atm}(a))[a \rightarrow u] = (\text{elt}(y, a))[a \rightarrow u] \quad \text{Notation 4.22} \]
  \[ = (u @ a')[a' \rightarrow y[a \rightarrow u]] \quad \text{Figure 2 (elta)} \]
  \[ = y[a \rightarrow u] \in u \quad \text{Notation 4.22} \]

- Suppose \( x \) is an internal comprehension (not an internal atom).
  Choose \( b \in \mathbb{N} \) fresh (so \( b \# x, y, u \)). We reason as follows:
  \[ (y \in x)[a \rightarrow u] = (x @ b)[b \rightarrow y][a \rightarrow u] \quad \text{Notation 4.22} \]
  \[ = (x @ b)[a \rightarrow u][b \rightarrow y[a \rightarrow u]] \quad \text{Lemma 4.12 b#u} \]
  \[ = (x[a \rightarrow u] @ b)[b \rightarrow y[a \rightarrow u]] \quad \text{Lemma 4.11 b#x, u} \]
  \[ = y[a \rightarrow u] \in x[a \rightarrow u] \quad \text{Notation 4.22} \]

**Corollary 4.26.** Suppose \( k \in \mathbb{Z} \) and \( c \in \mathbb{N}^k \) and \( x \in \text{Set}^k \) and \( y \in \text{Set}^{k-1} \) and \( c \# y \). Then if \( z = \text{st}([c](y \in c)) \in \text{Set}^{k+1} \) then \( x \in z = y \in x \).

**Proof.** We reason as follows:
\[
\begin{align*}
x \in z &= x \in \text{st}([c]y \in c) \quad \text{Assumption} \\
&= (y \in c)[c \rightarrow x] \quad \text{Lemma 4.24} \\
&= y[c \rightarrow x] \in x \quad \text{Lemma 4.25(1)} \\
&= y \in x \quad \text{Lemma 4.9 c#y}
\end{align*}
\]

5. The Language of Typed Sets

We now have everything we need to develop the syntax of Typed Set Theory.

5.1. Syntax of Stratified Sets.

**Definition 5.1.** Let formulae and terms be inductively defined as in Figure 4. In that figure, \( a \) ranges over atoms (Definition 2.1).

Definition 5.2 is standard:

**Definition 5.2.** Suppose \( t \) is a term (Definition 5.1). Then extend level \( (a) \) from Definition 2.1 from atoms to all terms by:
\[ \text{level}(\{a \mid \phi\}) = \text{level}(a) + 1 \]

Call a formula \( \phi \) or term \( t \) **stratified** when:
\[ \langle \bot \rangle = F \]
\[ \langle \neg \phi \rangle = \text{neg}(\langle \phi \rangle) \]
\[ \langle \phi \land \psi \rangle = \text{and}(\{\langle \phi \rangle, \langle \psi \rangle\}) \]
\[ \langle \forall a. \phi \rangle = \text{all}(\{\langle a \rangle \langle \phi \rangle\}) \]
\[ \langle t \in s \rangle = \langle \{t\} \rangle \in \langle s \rangle \]
\[ \langle \{\{a\}||\phi\} \rangle = \text{st}(\{\langle a \rangle \langle \phi \rangle\}) \]
\[ \langle a \rangle = \text{atm}(a) \]

**Figure 5:** Interpretation of formulae and terms

if \( s' \in s \) is a subterm of \( \phi \) or \( t \) then \( \text{level}(s) = \text{level}(s') + 1 \).

**Example 5.3.** Suppose \( a \in A^2 \), \( b \in A^3 \), and \( c \in A^4 \). Then \( a \in b \) and \( b \in c \) are stratified, and \( a \in c \), \( b \in a \), and \( a \in a \) are not stratified.

**Definition 5.4.** The language of *Stratified Sets* consists of stratified formulae and terms.

In that figure and henceforth, we write \( \phi[a := s] \) and \( t[a := s] \) for the usual capture-avoiding substitution on syntax.

We assume that levels are arranged to respect stratification, so that when we write \( [a := s] \) it is understood that we assume \( a \in A^{\text{level}(s)} \).

**Remark 5.5.** We only care about stratified formulae and terms henceforth—that is, we restrict attention from all (raw) formulae and terms of Definition 5.1, to those that are stratified.

So for all terms and formulae considered from now on, the reader should assume they are stratified, where:

- \( \in \) polymorphically takes two terms of type \( i \) and \( i + 1 \) to a formula for each \( i \in \mathbb{Z} \), and
- sets comprehension \( \{a \mid \phi\} \) takes an atom of type \( i \in \mathbb{Z} \) and a formula \( \phi \) to a term of type \( i + 1 \).

**Remark 5.6.** If we wished to add equality \( s \equiv t \) to our syntax in Figure 4, then we could do so, just at some modest cost in extra cases in inductive arguments. The pertinent stratification condition would be the natural one that if \( s \equiv t \) is a subterm then \( \text{level}(s) = \text{level}(t) \). Our results extend without issues to the syntax with equality.

### 5.2. Interpretation for formulae and terms.

**Definition 5.7.** Define an interpretation of stratified formulae \( \phi \) and terms \( s \) as in Figure 5, mapping \( \phi \) to \( \langle \phi \rangle \in \text{Pred} \) and \( s \) of level \( i \in \mathbb{Z} \) to \( \langle s \rangle \in \text{Set}^i \).

**Remark 5.8.** For the reader’s convenience we give pointers for the notation used in the right-hand sides of the equalities in Figure 5:

- **F** is from Example 3.8.
- **neg** is from Definition 3.2.
- \( \langle t \rangle \in \langle s \rangle \) is from Notation 4.22.
- \( \text{st}(\{a\} \langle \phi \rangle) \) is from Definitions 2.19 and 3.2.
- **atm** is from Definition 3.2.

Note that the translation in Figure 5 from the syntax of formulae \( \phi \) and terms \( s \) from Figure 4 to the syntax of internal predicates and internal sets from Definition 3.2 is not entirely direct: \( t \in s \) is primitive in formulae but only primitive in internal predicates if \( s \) is an atom.

**Lemma 5.9.** Suppose \( \phi \) is a stratified formula and \( s \) is a stratified term with \( \text{level}(s) = i \in \mathbb{Z} \). Then
\[ \langle \phi \rangle \in \text{Pred} \quad \text{and} \quad \langle s \rangle \in \text{Set}^i. \]
Proof. By induction on \( \phi \) and \( s \):

- **The case of \( a \).** By Figure 5 (a) = atm(a). By Definition 3.2 atm(a) \( \in \text{Set}^i \).
- **The case of \( \{a|\phi\} \) for \( j \geq 1 \) and \( b \in \mathbb{H}^j \).** By Figure 5 \( \langle b \rangle \phi \rangle \text{st}([b] \langle \phi \rangle) \). By Definition 5.2 level\( \{b|\phi\} \) = \( j+1 \). By inductive hypothesis \( \langle \phi \rangle \in \text{Pred} \) and by Definition 3.2 st([b] \langle \phi \rangle) \in \text{Set}^{j+1} \).
- **The case of \( \bot \).** By Figure 5 \( \langle \bot \rangle = F \in \text{Pred} \).
- **The case of \( \neg \phi \).** From Figure 5 and Definition 3.2 using the inductive hypothesis.
- **The case of \( \psi \).** From Figure 5 and Definition 3.2 using the inductive hypothesis.
- **The case of \( \forall a. \phi \).** From Figure 5 and Definition 3.2 using the inductive hypothesis.
- **The case of \( t \in s \).** We refer to Notation 4.22 and use Lemma 3.9 and Proposition 4.6. \( \square \)

5.3. Properties of the interpretation.

**Definition 5.10.** Define the **size** of a stratified formula \( \phi \) and stratified term \( t \) inductively as follows:

\[
\begin{align*}
\text{size}(a) & = 1 \\
\text{size}(\{a|\phi\}) & = \text{size}(\phi)+1 \\
\text{size}(\bot) & = 1 \\
\text{size}(\phi \wedge \psi) & = \text{size}(\phi) + \text{size}(\psi) + 1 \\
\text{size}(\neg \phi) & = \text{size}(\phi)+1 \\
\text{size}(t \in s) & = \text{size}(t) + \text{size}(s) + 1 
\end{align*}
\]

**Proposition 5.11.** Suppose \( \phi \) is a stratified formula and \( t \), and \( r \) are stratified terms and \( b \in \mathbb{A}^{\text{level}(t)} \). Then:

\[
\begin{align*}
\langle \phi \rangle [b \to \langle t \rangle] & = \langle \phi[b:=t] \rangle \\
\langle r \rangle [b \to \langle t \rangle] & = \langle r[b:=t] \rangle
\end{align*}
\]

Note by Lemma 5.9 that \( \langle t \rangle \in \text{Set}^{\text{level}(t)} \) so that the \( \sigma \)-action \( [b \to \langle t \rangle] \) above is well-defined (Definition 4.1).

**Proof.** By induction on size\( (\phi) \) and size\( (r) \). We consider each case in turn; the interesting case is for \( \epsilon \), where we use Lemma 4.25:

- **The case of \( \bot \).** We reason as follows:

\[
\begin{align*}
\langle \bot \rangle [b \to \langle t \rangle] & = F[\bot \to \langle t \rangle] & \text{Figure 5} \\
& = F & \text{Corollary 4.10} \\
\langle \bot \rangle [b:=\epsilon]\rangle & = \langle \bot \rangle & \text{Fact of syntax} \\
& = F & \text{Figure 5}
\end{align*}
\]

- **The case of \( \neg \phi \).** We reason as follows:

\[
\begin{align*}
\langle \neg \phi \rangle [b \to \langle t \rangle] & = \langle \neg \phi[b:=\epsilon]\rangle & \text{Figure 5} \\
& = \langle \neg \phi[b:=\epsilon]\rangle & \text{Figure 2} \\
& = \langle \neg \phi[b:=\epsilon]\rangle & \text{Fact of syntax}
\end{align*}
\]

- **The case of \( \psi \).** We reason as follows:

\[
\begin{align*}
\langle \psi \rangle [b \to \langle t \rangle] & = \langle \psi[b:=\epsilon]\rangle & \text{Figure 5} \\
& = \langle \psi[b:=\epsilon]\rangle & \text{Figure 2} \\
& = \langle \psi[b:=\epsilon]\rangle & \text{Fact of syntax}
\end{align*}
\]

- **The case of \( \phi \wedge \psi \).** We reason as follows:

\[
\begin{align*}
\langle \phi \wedge \psi \rangle [b \to \langle t \rangle] & = \langle \phi \wedge \psi[b:=\epsilon]\rangle & \text{Figure 5} \\
& = \langle \phi \wedge \psi \rangle & \text{Fact of syntax}
\end{align*}
\]

- **The case of \( \phi \vee \psi \).** We reason as follows:

\[
\begin{align*}
\langle \phi \vee \psi \rangle [b \to \langle t \rangle] & = \langle \phi \vee \psi \rangle & \text{Figure 5} \\
& = \langle \phi \vee \psi \rangle & \text{Fact of syntax}
\end{align*}
\]

- **The case of \( \forall a. \phi \).** We reason as follows:

\[
\begin{align*}
\langle \forall a. \phi \rangle [b \to \langle t \rangle] & = \langle \forall a. \phi \rangle & \text{Figure 5} \\
& = \langle \forall a. \phi \rangle & \text{Fact of syntax}
\end{align*}
\]

- **The case of \( \neg \phi \).** We reason as follows:

\[
\begin{align*}
\langle \neg \phi \rangle [b \to \langle t \rangle] & = \langle \neg \phi \rangle & \text{Figure 5} \\
& = \langle \neg \phi \rangle & \text{Fact of syntax}
\end{align*}
\]
• The case of $\forall a. \phi$. We reason as follows, where we $\alpha$-rename if necessary to assume $a \# t$ (from which it follows by Theorem 2.25 that $a \#(t)$):

$$\begin{align*}
(\forall a. \phi)[b \mapsto (t)] &= \text{all}([a](\phi))[b \mapsto (t)] & \text{Figure 5} \\
&= \text{all}([a](\forall a. \phi)[b \mapsto (t)]) & \text{Lemma 5.13} \\
&= \text{all}([a](\phi[b:=t])) & \text{IH size}(\phi) < \text{size}(\forall a. \phi) \\
&= (\forall a. (\phi[b:=t])) & \text{Figure 5} \\
&= \langle \forall a. (\phi[b:=t]) \rangle & \text{Fact of syntax, } a \# t
\end{align*}$$

• The case of $b$. By Figure 5 $\langle b \rangle = \text{atm}(b)$. By assumption $\langle t \rangle \in \text{Set}_{\text{level}(b)}$ so by Figure 2 $(\sigma a)$

$$\text{atm}(b)[b \mapsto (t)]= \langle t \rangle.$$  

• The case of $a$ (any atom other than $b$). By Figure 5 $\langle a \rangle = \text{atm}(a)$. We use rule $(\sigma b)$ of Figure 2.

• The case of $\{a|\phi\}$. $\alpha$-converting if necessary assume $a$ is fresh (so $a \# t$, and by Theorem 2.25 also $a \#(t)$). We reason as follows:

$$\begin{align*}
\langle \{a|\phi\} \rangle[b \mapsto (t)] &= \langle \text{st}([a](\phi))[b \mapsto (t)] \rangle & \text{Figure 5} \\
&= \langle \text{st}([a](\phi))[b \mapsto (t)] \rangle & \text{Lemma 2}(\sigma \text{st}, a \#(t)) \\
&= \langle \text{st}([a](\phi[b:=t])) \rangle & \text{IH size}(\phi) < \text{size}(\{a|\phi\}) \\
&= \langle \{a|\phi[b:=t]\} \rangle & \text{Figure 5, } a \# t \\
&= \langle \{a|\phi[b:=t]\} \rangle & \text{Fact of syntax}
\end{align*}$$

• The case of $t' \in s'$. We reason as follows:

$$\begin{align*}
\langle t' \in s' \rangle[b \mapsto (t)] &= \langle \langle t' \in (s') \rangle[b \mapsto (t)] \rangle & \text{Figure 5} \\
&= \langle \langle t' \in (s') \rangle[b \mapsto (t)] \rangle & \text{Lemma 4.25} \\
&= \langle \langle t' [b:=t] \rangle \in (s'[b:=t]) \rangle & \text{IH size}(t'), \text{size}(s') < \text{size}(t' \in s') \\
&= \langle \langle t' [b:=t] \rangle \in (s'[b:=t]) \rangle & \text{Figure 5}
\end{align*}$$

\[ \square \]

Lemma 5.12. Suppose $\phi$ is a stratified formula and $s$ is a stratified term. Suppose $a \in \mathbb{N}^+ \land \text{level}(s) = i$. Then

$$\langle s \in \{a|\phi\} \rangle = \langle \phi[a:=s] \rangle.$$  

Proof. We reason as follows:

$$\begin{align*}
\langle s \in \{a|\phi\} \rangle &= \langle \text{st}([a](\phi)) \rangle[a \mapsto \langle s \rangle] & \text{Fig 5 & Ntn 4.22} \\
&= \langle \phi \rangle[a \mapsto \langle s \rangle] & \text{Lemma 2.24(1)} \\
&= \langle \phi[a:=s] \rangle & \text{Proposition 5.11}
\end{align*}$$

\[ \square \]

Lemma 5.13. $\langle s \in \{a|\phi\} \rangle = \langle \phi \rangle[a \mapsto \langle s \rangle]$.

Proof. We reason as follows:

$$\begin{align*}
\langle s \in \{a|\phi\} \rangle &= \langle s \rangle \in \{a|\phi\} & \text{Figure 5} \\
&= \langle s \rangle \in \text{st}([a](\phi)) & \text{Figure 5} \\
&= \langle \phi \rangle[a \mapsto \langle s \rangle] & \text{Notation 4.22}
\end{align*}$$

\[ \square \]
5.4. Confluence.

Definition 5.14. (1) Let $\rightarrow$ be a rewrite relation on the language of Stratified Sets (Definition 5.4) defined by the rules in Figure 6.
(2) Write $\rightarrow^*$ for the transitive reflexive closure of $\rightarrow$ (so the least transitive reflexive relation containing $\rightarrow$).

Notation 5.15. It is clear that there is a natural injection of internal predicates and internal terms into stratified formulae and terms. To save on notation, we elide this inclusion, thus effectively treating the syntax of internal predicates and terms from Definition 3.2 as a direct subset of the syntax of stratified formulae and terms from Figure 4.\(^9\)

Notation 5.16. Call a formula of the form $t \in \{a|\phi\}$ a reduct.

Lemma 5.17. $\langle \phi \rangle$ considered as a formula, is a $\rightarrow$-normal form, and similarly for $\langle s \rangle$.

**Proof.** Reducts are impossible because the internal syntax from Figure 1 only allows us to form $y \in x$ (written $\text{elt}(y, x)$ in that figure) when $x$ is an atom and not a comprehension.

We can now state a kind of converse to Lemma 5.17:

Theorem 5.18. (1) $\langle t \in \{a|\phi\} \rangle = \langle \phi[a:=t] \rangle$.
(2) $\phi \rightarrow^* \langle \phi \rangle$ and $s \rightarrow^* \langle s \rangle$.

**Proof.** We reason as follows:

\[
\langle t \in \{a|\phi\} \rangle = \langle \phi[a:=t] \rangle \quad \text{Lemma 5.13}
\]

(2) By induction on syntax. The interesting case is for $t \in \{a|\phi\}$. Suppose $\phi \rightarrow^* \langle \phi \rangle$. Then

\[
t \in \{a|\phi\} \rightarrow^* \langle t\rangle \in \{a|\phi\} \quad \text{IH, Figure 6}
\]

\[
\rightarrow \langle \phi[a:=t] \rangle \quad \text{Figure 6}
\]

\[
= \langle t \in \{a|\phi\} \rangle \quad \text{Lemma 5.13}
\]

(3) By induction on the derivation of the rewrite (that is, on the term-context in which the rewrite takes place). We consider three cases:

- Suppose $t \in \{a|\phi\} \rightarrow \phi[a:=t]$. By part 1 of this result $\langle t \in \{a|\phi\} \rangle = \langle \phi[a:=t] \rangle$.

---

\(^9\)The reader who does not wish to abide with this abuse of notation can easily fill in an explicit injection function $\iota$ as required, to map the former injectively into the latter. Either way, the meaning will be the same.
• Suppose $t \in \{a|\phi\} \rightarrow t \in \{a|\phi'\}$ because $\{a|\phi\} \rightarrow \{a|\phi'\}$ because $\phi \rightarrow \phi'$. By induction hypothesis $\langle \phi \rangle = \langle \phi' \rangle$. Then using Lemma 5.13 we have

$$
\langle t \in \{a|\phi\} \rangle \overset{L5.13}{=} \langle \phi \rangle[a \rightarrow \langle t \rangle] \overset{L5.13}{=} \langle t \in \{a|\phi\} \rangle.
$$

• Suppose $t \in \{a|\phi\} \rightarrow t' \in \{a|\phi\}$ because $t \rightarrow t'$. By induction hypothesis $\langle t \rangle = \langle t' \rangle$. We have

$$
\langle t \in \{a|\phi\} \rangle \overset{L5.13}{=} \langle \phi \rangle[a \rightarrow \langle t \rangle] \overset{L5.13}{=} \langle t' \in \{a|\phi\} \rangle.
$$

(4) Confluence follows combining parts 2 and 3 of this result.

\[\square\]

5.5. **Strong normalisation.** Our use of multiset orderings in this Subsection goes back to [DM79].

**Notation 5.19.** (1) Write multiset($\mathbb{N}$) for the set of finite multisets of numbers. So:

(a) $M \in$ multiset($\mathbb{N}$) can be viewed as an element $M \in \mathbb{N} \rightarrow \mathbb{N}$ that is zero for all but a finite set of elements.
(b) The intended interpretation of $M(m)$ is the number of times that $m$ occurs in $M$.
(c) The empty multiset $\emptyset$ is such that $\emptyset(m) = 0$ for every $m \in \mathbb{N}$.
(d) Multisets union $M \cup M'$ is implemented by $(M \cup M')(m) = M(m) + M'(m)$.

(2) Write $dom(M)$ for

$$
dom(M) = \{m \in \mathbb{N} \mid M(m) > 0\}
$$

and call this the **domain** of $M$.

(3) Define $\max(dom(M))$ to be the greatest element in $dom(M)$ if $dom(M)$ is nonempty, and zero otherwise.

(4) If $M \in$ multiset($\mathbb{N}$) and $m \in \mathbb{N}$ then define $M \setminus m$ to be the multiset such that

- $(M \setminus m)(m') = M(m')$ if $m' \neq m$, and
- $(M \setminus m)(m) = 0$.

**Definition 5.20.** To stratified formulae and terms assign a **measure** $|\phi|, |s| \in$ multiset($\mathbb{N}$) which is a multiset of finite numbers, defined inductively by:

- $|\bot| = \{\}$
- $|\forall a.\phi| = |\phi|$
- $|\exists a.\phi| = |\phi|$
- $|\phi \land \phi'| = |\phi| \cup |\phi'|$
- $|\phi \lor \phi'| = \{\{a\} \mid \exists t.\phi \land \phi'\} \cup |\langle t \rangle| \cup |\phi|$
- $|\{a|\phi\}| = |\phi|$
- $|t \in \{a|\phi\}| = \{\text{size}(\phi) + \text{size}(t)\} \cup |\langle t \rangle| \cup |\phi|$

**Remark 5.21.** Intuitively, Definition 5.20 counts reducts (Notation 5.16), and it also remembers the size of each reduct that it counts. Thus, it returns a multiset of reduct-sizes.

**Definition 5.22.** Give multiset($\mathbb{N}$) the **lexicographic order** $M' \leq M$ defined by:

1. If $\max(dom(M')) < \max(dom(M))$ then $M' \leq M$.
2. If $\max(dom(M)) = \max(dom(M')) = m$ and $M'(m) \leq M(m)$, then $M' \leq M$.
3. If $\max(dom(M)) = \max(dom(M')) = m$ and $M'(m) = M(m)$ and $(M' \setminus m) \leq (M \setminus m)$, then $M' \leq M$.

**Lemma 5.23.** The lexicographic order $\leq$ from Definition 5.22 is well-founded on multiset($\mathbb{N}$) (no infinite descending chains).

**Proof.** Since domains and multiplicities are all finite. For more details see [DM79, Section 2]. \[\square\]
Lemma 5.24. The rewrites from Figure 6 strictly decrease $|\phi|$ and $|s|$. That is, if $\phi \rightarrow \phi'$ then $|\phi'| \leq |\phi|$ and if $s \rightarrow s'$ then $|s'| \leq |s|$.

Proof. It is a fact of syntax that the rewrite $t \in \{a|\phi\} \rightarrow \phi[a:=t]$ replaces a reduct of size $\text{size}(\phi)+\text{size}(t)$ with some finite number of strictly smaller reducts. By the design of the lexicographic order in Definition 5.22, this strictly reduces measure.

Corollary 5.25. The rewrite system from Figure 6 is terminating (no infinite chain of rewrites).

Proof. Immediate from Lemmas 5.23 and 5.24.

Theorem 5.26. Formulae and terms of Stratified Sets, with the rewrites from Figure 6, are confluent and strongly normalising.

Proof. Weak normalisation (every formula/term has some rewrite to a normal form) is a corollary of Theorem 5.18, from which it is easy to prove that if $\phi \rightarrow^* \phi'$ then $\phi' \rightarrow^* \langle \phi \rangle$, and similarly for $s$.

Strong normalisation follows from weak normalisation and termination (Corollary 5.25).

Remark 5.27. So from Theorems 5.26 and 4.18 we see that:
(1) the syntax of formulae and terms has normal forms, and furthermore
(2) normal forms with the natural substitution action given by substitute-then-renormalise, corresponds precisely to the theory of internal predicates and terms from Definition 3.2 and 4.1, and furthermore
(3) this theory of normal forms is an instance of the notion of nominal algebras for substitution, also called sigma-algebras, as used in the previous literature studying $\lambda$-calculus, first-order logic, and pure substitution [GG16, Gab16, GM08].

Recall from Remarks 1.1 and 1.2 that in stratifiable syntax, as used in Quine’s NF, variables do not have predefined levels but we insist on a stratifiability condition that $\phi$ and $s$ are only legal if we could assign levels to their variables to stratify them. We obtain as an easy corollary:

Theorem 5.28. Formulae and terms of stratifiable syntax, with the rewrites from Figure 6, are confluent and strongly normalising.

Proof. The result follows from Theorem 5.26 by taking a stratifiable $\phi$, and stratifying it so that we now have $\phi'$ in the language of Typed Sets. Rewrites on $\phi'$ clearly correspond 1-1 with rewrites on $\phi$, since Figure 6 makes no reference to the levels of variables.

6. Conclusions and future work

Stratified Sets occupy a nice middle ground between ZF sets and simple types. They typically appear used as a foundational syntax. However, we have seen in this paper that Typed-Sets-the-syntax in and of itself forms a well-behaved rewrite system, and a well-behaved nominal algebra. This had not previously been noted, and this paper gives a reasonably full and detailed account of how rewriting and nominal algebra apply to this family of syntaxes. This account is intended to be suitable for

- readers familiar with rewriting who are unfamiliar with stratified sets syntax\(^{10}\)
- readers familiar with stratified sets syntax but unfamiliar with techniques from rewriting and nominal algebra.

\(^{10}\)Stratified sets syntax is not hard to define—but it requires experience to learn what kinds of predicates are and are not stratifiable. In use, stratifiability is a subtle and powerful condition.
We have also tried to smooth a path to implementing these proofs in a machine, hopefully in a nominal context. We have designed the proofs to be friendly to such an implementation as future work, yet without compromising readability for humans. Where we have cut corners (relative to a machine implementation), we tried to signpost this fact (see for instance Remark 4.2 and Notation 5.15).

Concerning other applications, it is often possible to use normal forms to build denotations. In some contexts, the normal form is the denotation of the terms that reduce to it. That will not work for Stratified Sets because we are usually interested in imposing additional axioms. But there are standard things that can be done about that, and this has been investigated in a nominal context in papers like [Gab16, GG16]. These papers build denotations for first-order logic and the \(\lambda\)-calculus using maximally consistent sets, and using nominal techniques to manage binding in denotations (extending how we used nominal techniques in this paper to manage binding in syntax). Having normal forms is useful here and the ideas in this paper can be used to give a denotational analysis of theories in the languages of Stratified and Stratifiable Sets. This will be the subject of another paper.

We can ask about a converse to Theorems 5.26 and 5.28. We have shown that a stratifiable formula rewrites to a normal form. Now if a formula (without levels) rewrites to normal form, is it stratifiable? We see that we cannot hope for a perfect converse by the following easy example: \(\emptyset \in \{a \mid a \notin a\}\) is not stratifiable but it rewrites to \(\emptyset \notin \emptyset\), which is. However there may be special cases in which stratification information can be recovered from normalisation, and this is future work.

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