COMPRESSIBLE FLOW IN FRONT OF AN AXISYMMETRIC BLUNT OBJECT: ANALYTIC APPROXIMATION AND ASTROPHYSICAL IMPLICATIONS

Uri Keshtet and Yossi Naor
Physics Department, Ben-Gurion University of the Negev, P.O. Box 653, Be’er-Sheva 84105, Israel; ukeshet@bgu.ac.il
Received 2015 September 10; revised 2016 June 4; accepted 2016 July 20; published 2016 October 18

ABSTRACT

Compressible flows around blunt objects have diverse applications, but current analytic treatments are inaccurate and limited to narrow parameter regimes. We show that the gas-dynamic flow in front of an axisymmetric blunt body is accurately derived analytically using a low order expansion of the perpendicular gradients in terms of the parallel velocity. This reproduces both subsonic and supersonic flows measured and simulated for a sphere, including the transonic regime and the bow shock properties. Some astrophysical implications are outlined, in particular for planets in the solar wind and for clumps and bubbles in the intergalactic medium. The bow shock standoff distance normalized by the obstacle curvature is \( \sim 2/(3g) \) in the strong shock limit, where \( g \) is the compression ratio. For a subsonic Mach number \( M \) approaching unity, the thickness \( \delta \) of an initially weak, draped magnetic layer is a few times larger than in the incompressible limit, with amplification \( \sim (1 + 1.3M^2\delta)/(3\delta) \).

Key words: hydrodynamics – intergalactic medium – interplanetary medium – methods: analytical – shock waves

1. INTRODUCTION

Compressible flows around blunt objects play an important role in diverse fields of science and engineering, ranging from fluid mechanics (e.g., Landau & Lifshitz 1959; Park et al. 2006; Mack & Schmid 2011; Grandemange et al. 2013, 2014; Tutty et al. 2013), space physics (e.g., Spreiter & Alksne 1970; Baranov & Lebedev 1988; Cairns & Grabbage 1994; Spreiter & Stahara 1995; Petrinec & Russell 1997; Petrinec 2002), and astrophysics (Lea & De Young 1976; Shaviv & Sápeter 1982; Canto & Raga 1998; Schulreich & Breitschwerdt 2011), to computational physics and applied mathematics (Hejranfar et al. 2009; Gollan & Jacobs 2013; Marrone et al. 2013; Wilson 2013), aeronautical and civil engineering (Nakanishi & Kame moto 1993; Baker 2010; Aulchenko et al. 2012), and aerodynamics (Asanaliev et al. 1988; Liou & Takayama 2005; Pilyugin & Khlebnikov 2006; Volkov 2009). Yet, even for the simple case of an inviscid flow around a sphere, the problem has resisted a general or accurate analytic treatment due to its nonlinear nature.

In particular, in space physics and astrophysics, the interaction of an ambient medium with much denser, in comparison approximately solid, bodies such as comets (e.g., Baranov & Lebedev 1988), planets (Spreiter & Alksne 1970; Cairns & Grabbage 1994; Petrinec & Russell 1997), binary companions (Canto & Raga 1998), galaxies (Shaviv & Sápeter 1982; Schulreich & Breitschwerdt 2011), or large-scale clumps and bubbles (Lea & De Young 1976; Vikhlinin et al. 2001; Lyutikov 2006; Markovitch & Vikhlinin 2007), is important for modeling these systems and understanding their observational signature. This is particularly true for the shocks formed in supersonic flows, due to their rich non-thermal effects (e.g., Spreiter & Stahara 1995; Vikhlinin et al. 2001; Petrinec 2002; Markovitch & Vikhlinin 2007).

Although these fairly complicated systems can be approximately solved numerically, they are often modeled as an idealized, inviscid flow around a simple blunt object, often approximated as axisymmetric or even spherical, with some simplified analytic description employed in order to gain a deeper, more general understanding of the system. Consequently, this fundamental problem of fluid mechanics has received considerable attention. The small Mach number \( M \) regime was studied as an asymptotic series about \( M = 0 \) (Lord Rayleigh 1916; Tamada 1939; Kaplan 1940; Stangeby & Allen 1971; Allen 1973), and solved in the incompressible potential flow limit. Some hodograph plane results and series approximations were found in the transonic and supersonic cases (Hida 1955; Liepmann & Roshko 1957; Guderley 1962). In particular, approximations for the standoff distance of the bow shock (e.g., Moeckel 1949; Hida 1953; Lighthill 1957; Hayes & Probstein 1966; Spreiter et al. 1966; Guy 1974; Corona-Romero & Gonzalez-Esparza 2013) partly agree with experiments (Heberle et al. 1950; Schwartz & Eckerman 1956), spacecraft data (Farris & Russell 1994; Spreiter & Stahara 1995; Verigin et al. 1999) and numerical computations (Chapman & Cairns 2003; Igra & Falcovizt 2010).

However, these analytic results are typically based on ad hoc, unjustified assumptions, such as negligible compressibility effects, a predetermined shock geometry (Lighthill 1957; Guy 1974), or an incompressible (Hida 1953) or irrotational (Kawamura 1950; Hida 1955) flow downstream of the shock. Other approaches use slowly converging, or impractically complicated, expansion series (Lord Rayleigh 1916; Hida 1955; Van Dyke 1958a, 1975). In all cases, the results are inaccurate or limited to a narrow parameter regime. A generic yet accurate analytic approach is needed.

We adopt the conventional assumptions of (i) an ideal, polytropic gas with an adiabatic index \( \gamma \); (ii) negligible viscosity and heat conduction (ideal fluid); (iii) a steady, laminar, non-relativistic flow; and (iv) negligible electromagnetic fields. Typically, these assumptions hold in front of the object, but break down behind it and in its close vicinity. We thus analyze the flow ahead of the object.

While spatial series expansions and hodograph plane analyses, when employed separately, are of limited use (for reviews, see Van Dyke 1958b, 1975), we find that their combination gives good results over the full parameter range. In particular, we expand the axial flow in terms of the parallel velocity, rather than distance. This yields an accurate, fully analytic description of the gas-dynamic flow in both the
subsonic and supersonic regimes, already in a second or third order expansion, as shown in Figure 1.

After introducing the flow equations in Section 2, in particular along the axis of symmetry, we derive the expansion series for the subsonic regime in Section 3, and for the supersonic regime in Section 4. Some astrophysical implications are demonstrated in Section 5, in particular for planetary bow shocks and for clumps and bubbles in the intergalactic medium (IGM). We begin the analysis with a sphere and outline the generalization for arbitrary blunt axisymmetric objects in Section 6, where the results are summarized and discussed. For convenience, the full results are given explicitly in the Appendix.

2. FLOW EQUATIONS

Under the above assumptions, the flow is governed by the stationary continuity, Euler, and energy equations,

\[ \nabla \cdot (\rho \mathbf{v}) = 0; \quad (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{\rho}; \quad \nabla \cdot \left( \frac{\mathbf{P}}{\rho} \right) = 0, \]

where \( \mathbf{v}, P, \) and \( \rho \) are the velocity, pressure, and mass density. At a shock, downstream (subscript \( d \)) and upstream (subscript \( u \)) quantities are related by the shock adiabat (e.g., Landau & Lifshitz 1959),

\[ \frac{\rho_d}{\rho_u} = (\gamma + 1) \frac{M_u^2}{(\gamma - 1) M_u^2 + 2}; \quad P_d = 2 \gamma M_u^2 + 1 - \gamma, \]

with \( M = \mathbf{v}/c, \) and \( c = (\gamma P/\rho)^{1/2} \) being the sound speed.

Along streamlines, Bernoulli’s equation implies that

\[ w + \frac{v^2}{2} = \bar{\sigma} = \text{const.}, \]

where \( w = \gamma P/[(\gamma - 1) \rho] \) is the enthalpy, and a bar denotes (henceforth) a putative stagnation (\( v = 0 \)) point. The far incident flow is assumed to be uniform and unidirectional, so \( \bar{\sigma} \) is the same constant for all streamlines. Equation (3) remains valid across shocks, as \( w + \frac{v^2}{2} \) is the ratio between the normal fluxes of energy and of mass, each conserved separately across a shock.

Bernoulli’s Equation (3) relates the local Mach number

\[ M = \mathbf{v}/c = (M_0^2 - S^{-2})^{-\frac{1}{2}} = (\Pi^{-\frac{1}{2}} - 1)^{\frac{1}{2}}, \]

to the Mach number with respect to stagnation sound, \( M_0 \equiv v/c, \) and to the normalized pressure, \( \Pi \equiv P/P. \) We define \( S^2 \equiv 2/(\gamma - 1) \) and \( W^2 \equiv 2/(\gamma + 1) \) as the strong and weak shock limits of \( M_0^2, \) so the subsonic (supersonic) regime becomes \( 0 < M_0 < W \) (\( W < M_0 < S \)). Figure 1 illustrates these definitions, and shows the shock adiabat Equation (2) (as horizontal jumps at fixed \( r \) for \( \gamma = 7/5 \)).

Consider the flow ahead of a sphere along the symmetry axis, \( \theta = 0 \) in spherical coordinates \{r, \theta, \phi\}. Here, the flow monotonically slows with decreasing \( r, \) down to \( v = 0 \) at the stagnation point, which we normalize as \( \mathbf{r} = \{1, 0, 0\}. \) Symmetry implies that along the axis \( \mathbf{v} = -u(r)\hat{r}, \) where \( u \geq 0. \) Here, Equations (1) become

\[ \frac{\partial \ln(\rho u)}{\partial \ln r^2} = \frac{q - u}{u}; \quad \partial_r P = -\rho u \partial_r u; \quad \partial_r P = 0, \]

along with Bernoulli’s Equation (3), where we defined \( q \equiv (\partial_\theta \theta)_\theta=0 \) as a measure of the perpendicular velocity.

Hence,

\[ \partial_r u = \frac{2}{r} \left( \frac{q - u}{1 - M_0^2/S^2} - \frac{M_0^2}{1 - M_0^2/W^2} \right) \]

Our analysis relies on \( u(r) \) being a monotonic function. This allows us to write \( q = q(u) \) as a function of \( u \) and not of \( r. \)

Integrating Equation (6) thus yields

\[ 2 \ln r = \int_0^{u(r)} \frac{1 - M_0(u')^2/W^2}{1 - M_0(u')^2/S^2} \frac{du'}{q(u') - u'}, \]

so given \( q(u), \) the near-axis flow is directly determined.

Unlike \( u(r), \) or other expansion parameters used previously, the \( q(u) \) profile for typical bodies varies little, and nowhere vanishes. It is well approximated by a few terms in a power expansion of the form

\[ q(u) = q_0 + q_1(u - U) + q_2(u - U)^2 + \ldots, \]

where \( U \) is a reference velocity, so the integral in Equation (7) can be analytically carried out to any order (see the Appendix). Moreover, we next show that the boundary conditions tightly fix \( q(u), \) giving a good approximation for the near axial flow.

First expand \( q \approx \bar{q} \) near stagnation, with \( U = \bar{u} = 0. \) An initially homogeneous subsonic or even mildly supersonic (Landau & Lifshitz 1959) flow remains irrotational, \( \nabla \times \mathbf{v} = 0, \) in which case the lowest-order constraint is

\[ \bar{q}_i = -1/2, \]

whereas for a supersonic, rotational flow, it becomes

\[ 3\epsilon^2 \bar{q}_1 + 7\epsilon \bar{q}_2 = 2\bar{q}_1 + 6 \frac{\bar{q}_0}{\epsilon} + \bar{q}_1 \left( \frac{\bar{q}_0}{\epsilon} \right)^2 + \left( \frac{\bar{q}_0}{\epsilon} \right)^3, \]

as seen by expanding Equations (1) to order \( \theta^2(r - 1)^3. \) The generalization for non-spherical objects is discussed in Section 6. Next, we estimate \( q \) far from the body, and use it to approximate the flow in both the subsonic (Section 3) and supersonic (Section 4) regimes.

3. SUBSONIC FLOW

In the subsonic, \( \bar{M} < 1 \) case, we derive the incoming axial flow out to \( r \to \infty. \) Using the incident flow (henceforth labeled by a tilde) boundary condition \( \bar{\hat{v}} = \bar{u}(-\cos \theta, \sin \theta, 0), \) we may expand \( \bar{q} \) with \( \bar{U} = \bar{u}, \) such that

\[ \bar{q}_0 = \left( \partial_\theta \bar{v}_\theta \right)_{\theta=\bar{\theta}} = \bar{u}. \]

Additional terms can be derived using \( \bar{M} \ll 1 \) or \( r \gg 1 \) expansions appropriate for the relevant object. Here, it will suffice to consider the leading, \((u - \bar{u}) \propto r^{-\alpha} \) behavior at large radii, such that Equation (6) yields

\[ \bar{q}_1 = 1 - \frac{\alpha}{2} \frac{1 - \bar{M}_0^2/W^2}{1 - \bar{M}_0^2/S^2}. \]

In the incompressible limit, \( \alpha = 3 \) for any object (e.g., Landau & Lifshitz 1959). This also holds for general forward–backward symmetric objects in any potential flow. To see the latter, expand the potential \( \Phi, \) defined by \( \mathbf{v} = \bar{u} \nabla \Phi, \) as a power series in \( r. \) Imposing the \( r \to \infty \) boundary conditions and...
regularity across $\theta = 0$ yields

$$\Phi = -r \cos \theta + \frac{\varphi_1}{r^{\Theta}} + \frac{\varphi_2 \cos \theta}{r^{2\Theta}} + \ldots,$$

(13)

where $\Theta \equiv [1 - M^2(S^2 + \sin^2 \theta)]^{1/2}$. The constants $\varphi_k$ are determined by the boundary conditions on the specific body. Symmetry under forward–backward inversion, $\Phi \to -\Phi$ if $\theta \to \pi - \theta$, requires that $\varphi_1 = 0$. In general $\varphi_2 \neq 0$, implying that indeed $\alpha = 3$. Such behavior is demonstrated for an arbitrary compressible flow around a sphere by the Janzen–Rayleigh series (e.g., Tamada 1939; Kaplan 1940).

Finally, the $\tilde{q}$ expansion at $r \to \infty$ is matched to the $\tilde{q}$ expansion at stagnation for a potential flow. In the limit of an incompressible flow around a sphere, Equations (9), (11), and (12) yield $q(u) = \tilde{u} - (u - \tilde{u})/2 + O(u - \tilde{u})^2 = 3\tilde{u}/2 - u/2$, which is indeed the exact solution.

This procedure reasonably approximates arbitrary compressible, subsonic flows. Better results are obtained by noting that the constraint (9) holds also before stagnation, as long as $\partial_w v_r$ is negligible, implying that $\tilde{q}_r \approx 0$. Combining this with constraints (9), (11), and (12) yields an accurate, third order approximation, shown in Figure 1 as dotted–dashed curves. See Section A.1 for an explicit solution.

4. SUPersonic FLOW

In the supersonic, $\tilde{M} > 1$ case, a detached bow shock forms in front of the object, at the so-called standoff distance $\Delta$ from its nose. The transition between subsonic and supersonic regimes is continuous, so $\Delta \to \infty$ as $\tilde{M} \to 1$, or equivalently as $M_0 \to W$. The unperturbed upstream flow and the shock transition are shown on the right side of Figure 1.

Consider the flow between the shock and stagnation along the axis of symmetry. The $q(u)$ profile is strongly constrained if the normalized shock curvature $\xi^{-1} \equiv (R/r_s)_{\theta=0}$ is known. Here, $r_s$ is the shock radius, such that $r_s/(1 - r_s/(\theta))$ is its local radius of curvature.

Expanding the flow Equations (1) using Equations (2) as boundary conditions, yields the $q^{(d)}$ expansion coefficients around $U = u_d$, just downstream of the shock,

$$q_0^{(d)} = (1 + g\xi - \xi)g^{-1}\tilde{u};$$

(14)

$$q_1^{(d)} = \frac{3 + (g - 3)\xi}{2} - \frac{1 + (3g - 1)\xi}{1 + g + (g - 1)\gamma};$$

(15)
where \( g \equiv (M_0/W)^2 \geq 1 \) is the axial compression ratio.

These coefficients depend on the shock profile only through \( \xi \); higher-order terms are sensitive to deviations of the profile from a sphere of radius \( R \). In the weak shock limit \( g \to 1 \), so \( \xi \) must vanish to avoid the divergence of \( q_2^{(d)} \). This implies that \( R \) diverges faster than \( \Delta \), and \( q_2^{(d)} \to (1 - 2\xi) \) asymptotes to unity, consistent with a smooth transition to the subsonic regime. Moreover, if we require that \( q_2^{(d)} \to q_2 \to 3/(2\bar{c}W) \) in this limit (see Section A.1), then

\[
\xi (M_0 \approx W) \to (4 + \gamma)(-1 + \tilde{M}_0/W),
\]

so \( R/r_\xi \) diverges as \((M_0 - W)^{-1}\), consistent with Hida (1953, 1955; as expected in the irrotational limit).

Equations (14)–(16) yield a good, second order approximation to the flow, as shown in Figure 1 (dashed curves), once \( \xi \) or any of the \( q_2^{(d)} \) coefficients are determined. This can be done using the stagnation boundary conditions, such as Equation (10), but is laborious and body-specific due to the high order involved. A simpler approach is to estimate \( \xi (M) \), using the weak and strong shock limits.

In the strong shock, \( \tilde{M}_0 \to S \) limit, the curvature of the shock approaches that of the object (e.g., Guy 1974); \( \xi \to 1 \) in the case of the sphere. We find that Equation (19) nicely fits the measured flow for Mach numbers that are not too small, with \( \beta \approx 1/2 \).

The \( \tilde{M}_0/\xi \) relation may be derived as a power series, using this and the constraint Equation (17). A second order expansion in \( \xi \) gives a good approximation, valid throughout the supersonic regime,

\[
\tilde{M}_0/W - 1 \approx \frac{\xi}{4 + \gamma} + \left( \frac{S}{W} - \frac{5 + \gamma}{4 + \gamma} \right) \xi^2,
\]

with no free parameters. The good fit suggests that higher-order terms in \( \xi \) are negligible or absent.

Alternatively, the result \( \xi (\tilde{M}_0 \to S) = 1 \) and direct measurements of \( \xi \) (Heberle et al. 1950), motivate a power-law approximation of the form

\[
\xi \approx [(M_0 - W)/(S - W)]^{\beta}.
\]

The standoff distance may now be found by solving Equation (7) for \( r_\xi = 1 + \Delta \), taking \( u = u_\xi \) or equivalently \( M_0 = \tilde{M}_0/r_\xi = W^2/\tilde{M}_0 \), using the expansion (8) with coefficients (14–16) fixed by the \( \xi (\tilde{M}_0) \) relation. The figure inset shows that Equation (18) provides a good fit to the standoff distance throughout the supersonic range, for two equations of state. It also shows that a single \( \beta \approx 1/2 \) power-law in Equation (19) reproduces \( \Delta \) away from the transonic regime. Indeed, \( \Delta \) is sensitive to the precise value of \( \beta \) only in the \( M \approx 1 \) limit; best results are obtained with \( \beta = 0.48 \) (\( \beta = 0.52 \)) for \( \gamma = 7/5 \) (\( \gamma = 5/3 \)).

5. ASTROPHYSICAL IMPLICATIONS

The above prescription for the flow in front of a blunt object is useful in a wide range of astrophysical circumstances, as the low-density medium can often be approximated as ideal and inviscid, the body as impenetrable, and the motion as steady and non-relativistic.

Consider for example the standoff distance \( \Delta \) in front of a supersonic astronomical object. It is useful to plot \( \Delta \) as a function of the compression ratio \( g \), rather than of the Mach number, because it is typically easier to measure \( g \). As Figure 2 shows, \( \Delta (g) \) at a given \( \gamma \) approximately follows a power-law, for example \( \Delta (\gamma = 7/5) \approx 1.6g^{-1.5} \) and \( \Delta (\gamma = 5/3) \approx 1.5g^{-1.6} \). For high \( M \), the standoff distance approaches the strong shock limit, approximately given by (see Appendix A.3.3)

\[
\Delta (M \to \infty) \approx \frac{2}{3g}.
\]

For an arbitrary axisymmetric blunt body, the above results for a sphere are trivially generalized, if \( \Delta \) is defined as the distance from the nose of the body, normalized by its radius of curvature (for additional corrections see Section 6). One may thus superimpose \( \Delta (g) \) estimates of astronomical bow shocks on Figure 2, even for non-spherical bodies.

Consider for example the bow shock of a planet, moving supersonically through the solar wind. Although the magnetic Mach number and the ratio \( \Delta/\lambda_i \) (\( \lambda_i \) being the ion gyroradius) are not very high in such systems, a gas-dynamic approach remains useful as a first approximation, provided that \( M \) is replaced by the fast magnetosonic Mach number upstream (Stahara 1984; Spreiter & Stahara 1995; Fairfield et al. 2001).

Here, we define \( \Delta \) as the distance between the bow shock and the nose of the obstacle, namely the planetary magnetosphere or ionosphere, normalized by the radius of curvature of this obstacle’s nose. For a discussion of planetary bow shocks, and a compilation of \( \Delta \) estimates based on analytic arguments and numerical simulations, see Verigin et al. (2003, and references therein). Note that our analysis directly provides not only \( \Delta (g) \), but also the flow profile and the shock radius of curvature.

Estimates of \( \Delta (g) \) for the solar system planets are shown in Figure 2, with references provided in the caption. Interestingly, some planetary data seem to suggest a soft equation of state with \( \gamma < 5/3 \). However, such an interpretation is hindered by the substantial simplifying assumptions, in particular the neglected MHD effects, kinetic effects, variable solar wind conditions, and non-axisymmetric corrections to the obstacles. The positions and shapes of the obstacles are in some cases highly uncertain; indeed, the results suggest a significant flattening of the magnetospheres of Saturn and Uranus.

As another astronomical system, consider the large-scale extreme, namely the IGM of a galaxy group or cluster. Here, hot bubbles inflated by the active galactic nucleus (AGN) rise buoyantly through the IGM (e.g., Fabian et al. 2000; Nulsen et al. 2005), and the subsonic motion of the plasma in front of the bubbles (e.g., Churazov et al. 2001) and the draping of magnetic fields around them (Lyutikov 2006; Dursi & Pfrommer 2008; Naor & Keshet 2015). Large-scale structure mergers lead to dense clumps moving subsonically or supersonically through the IGM, giving rise to dramatic effects such as shocks, cold fronts, and even a spatial separation between baryonic and dark matter components (Vikhlinin et al. 2001;
Details such as the bow shock location and the downstream flow pattern are important for correctly interpreting the underlying dynamics.

Consider first a supersonic clump moving through the AGN. A well known example is the 1E0657-56, so-called bullet, cluster at redshift $z = 0.296$, showing a merger nearly in the plane of the sky (Barrena et al. 2002; Markevitch et al. 2002). The moving clump is seen as a bullet-shaped discontinuity, preceded by a bow shock with $g \simeq 3.0$ (Markevitch 2006) and $\Delta \simeq 2.4 \pm 0.2$. Our analysis indicates that the large $\Delta$ corresponds to a weak shock, with $\dot{M} \simeq 1.1$ (for $\gamma \simeq 5/3$, used henceforth). This is consistent with the $\sim 65^\circ$ asymptotic shock angle far from the nose, which also suggests a $\dot{M} \simeq 1.1$ shock. However, the high compression ratio corresponds to a much stronger shock with $\dot{M} \simeq 3$, indicating that the system is not in a steady state. Indeed, plotting the corresponding $\Delta(g)$ on Figure 2 would suggest an unrealistically soft equation of state. Simulations indicate that the shock velocity can be higher by a factor of 1.7 (Springel & Farrar 2007) or even 6 (Milosavljević et al. 2007) than expected from the clump velocity, because the shock (i) moves faster than the clump; and (ii) plows through gas that is infalling toward the clump (Springel & Farrar 2007). Evidently, Figure 2 provides a simple way to gauge the relaxation level of a system.

Next consider the subsonic IGM flow in front of an AGN bubble or a slow clump. While previous studies (e.g., Lyutikov 2006; Dursi & Pfommer 2008) have approximated the motion as incompressible, the inferred velocities are often nearly sonic (Churazov et al. 2001; Markevitch & Vikhlinin 2007), implying considerable compressibility effects. To illustrate this, we compute the magnetization caused by the draping of a weak upstream magnetic field around the moving object. The results are applicable only to weak magnetic fields, where Equations (1)–(2) remain a good approximation.

The magnetic field generally evolves as $B \propto \rho I$, where $I$ is a length element attached to the flow. Hence, the magnetic components initially perpendicular or parallel to the flow evolve along the axis of symmetry according to

$$\frac{B_\perp}{B_\parallel} = \left(\frac{\rho / v}{\rho / \dot{v}}\right)^{1/2} = \left(\frac{M_0}{\dot{M}_0}\right)^{1/2} \left(\frac{S^2 - M_0^2}{S^2 - \dot{M}_0^2}\right)^{5/4}; \quad (21)$$

or

$$\frac{B_\parallel}{B_\perp} = \frac{\rho \dot{v}}{\rho v} = \frac{M_0}{\dot{M}_0} \left(\frac{S^2 - M_0^2}{S^2 - \dot{M}_0^2}\right)^{5/2}; \quad (22)$$

for a detailed discussion, see Naor & Keshet (2015). The resulting magnetic energy amplification is shown in Figure 3, for $\gamma = 5/3$, as a function of $\dot{M}$ and of the normalized distance from the body, $\delta = (r - 1)$. Near the object, the magnetization is predominantly
perpendicular, and approximately given by
\[ \frac{B_y}{B_z} \simeq \frac{1 + 1.3\tilde{M}^{2.6}}{3\epsilon}, \]  
(23)
as illustrated in the figure.

As the figure shows, the magnetized layer is typically a few times thicker for \( \tilde{M} \approx 1 \) than it would appear in the incompressible limit. Such thick layers may have observational implications, through their non-thermal pressure and as synchrotron emission in front of nearly sonic objects. Such a synchrotron signal may contribute to the radio bright edges seen above AGN bubbles, for example in the Virgo cluster (Owen et al. 2000).

6. DISCUSSION

The compressible, inviscid flow in front of a blunt object is approximated analytically, using a hodograph-like, \( \mathbf{v} \simeq (-u, q(u)\theta, 0) \) transformation. The velocity (Equation (7)) and pressure (Equation (4)) profiles are derived by expanding \( q \) as a (rapidly converging) power series in \( u \) (Equation (8)), using the constraints imposed by the object (Equations (9) or (10) for a sphere) and by the far upstream subsonic (Equations (11)–(12)) or shocked supersonic (Equations (14)–(16)) flow. In the latter case, the weak (Equation 17) and strong shock limits approximately fix the shock curvature (Equation 18) and consequently the flow, independent of the object shape.

Figure 1 shows that a low order \( q(u) \) expansion suffices to recover the measured flow in front of a sphere. The supersonic results also reproduce the measured standoff distance (solid curve and figure inset) of the shock, and constrain its curvature (Equation 18 or the fit Equation (19)). Higher-order constraints can be used to improve the approximation further; here we used only the lowest-order constraint at stagnation, and only in the subsonic case.

The axial approximation directly constrains the flow beyond the axis and along the body, as it determines the perpendicular derivatives. For example, one can use it to estimate \( \partial_{\theta\theta} P = -\rho_0 (q^2 - u\partial_r (rq)) (1 - M_0^2 / S^2)^{\gamma - 1} \), found by expanding Equations (1) to \( \theta^4 \) order. Extrapolation beyond the axis is simpler in the potential flow regime, where, in particular, \( \partial_{\theta\theta} v_\theta = \partial_r (rq) \).

The axial analysis is generalized for any blunt, axisymmetric object, by modifying the \( q \) boundary conditions. For a body with radius of curvature \( R_c > 0 \) at a stagnation radius \( r_b \), take \( \{ z \equiv r \cos \theta = R_c - r_b, \theta \equiv r \sin \theta = 0 \} \) as the origin, and rescale lengths by \( R_c \). This maps the stagnation region of the body onto that of the unit sphere, so Equations (3)–(9), (11)–(16) remain valid. The subsonic analysis is unchanged; for an asymmetric body, \( \alpha \) may need to be altered, e.g., using the Janzen–Rayleigh series. The supersonic analysis is also unchanged, if Equation (10) is used and adapted for the specific body. The alternative use of Equation (18) or Equation (19) is still expected to hold, although higher order terms or a tuned \( \beta \) may be needed if an aspherical body modifies the weak or strong shock limits.

It may be possible to generalize our hodograph-like analysis even for a non-axisymmetric object, using the stagnant streamline instead of the symmetry axis, as long as the corresponding \( u \) profile remains monotonic.

Our analysis is applicable to a wide range of subsonic and supersonic astronomical bodies. Illustrative examples are discussed (in Section 5), on both small, planetary scales, and large, galaxy cluster scales. In particular, plotting the standoff distance as a function of the compression ratio (Figure 2) can be used to gauge the equation of state and the relaxation level of the system. The results are particularly useful for nearly sonic flows, where compressibility effects play an important role; this is seen for example in the thicker magnetically draped
layers that form in front of a moving body (Figure 3), such as a large-scale clump or an AGN bubble.

We thank Ephim Golbraikh and Yuri Lyubarsky for helpful advice. This research has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n° 293975, from an AEC-UPBC joint research foundation grant, from an ISF-UGC grant, and from an individual ISF grant.

APPENDIX

EXPLICIT DESCRIPTION OF THE AXIAL FLOW

Here we present the flow in front of a sphere in explicit form, for both subsonic and supersonic regimes. Fully analytic expressions are provided to second order in the supersonic case, and to third order in the subsonic case. The generalization to an arbitrary axisymmetric blunt object is discussed in Section 6.

A.1. Subsonic Regime

In the subsonic regime, \(q(u)\) is constrained at stagnation and at infinity; we may expand it equivalently either around the stagnation point \((u = 0)\) or around infinity \((u = \bar{u})\). Here we arbitrarily choose to write \(q(u)\) using an expansion at stagnation, namely

\[
q = \bar{q}_0 - \frac{1}{2}u + \bar{q}_2 u^3 + O(u^4),
\]

where we used the stagnation boundary conditions \(\bar{q}_1 = (-1/2)\) and \(\bar{q}_2 = 0\) derived in the main text. This expression is matched with the boundary conditions at \(r \to \infty\), in order to find the remaining coefficients,

\[
\bar{q}_0 = \frac{3}{2} \bar{u} - \bar{q}_3 \bar{u}^3 \quad \text{and} \quad \bar{q}_3 = \frac{(S/W)^2 - 1}{2(S^2 \bar{u}^2 - \bar{u}^2)}.
\]

A.2. Supersonic Regime

In the supersonic regime, we may write \(q(u)\) using the downstream expansion at the shock,

\[
q = q_0^{(d)} + q_1^{(d)} (u - u_d) + q_2^{(d)} (u - u_d)^2,
\]

where the coefficients are given in the main text.

A.3. Explicit Analytic Results

By plugging the relevant expressions for \(q(u)\) into Equation (7), \(r(u)\) can be computed in both subsonic and supersonic cases. In the latter, the shock radius is then determined as \(r_s = 1 + \Delta = r(u_d)\). The integral in Equation (7) is easily evaluated numerically, using either Equations (24) or (26) for \(q\). However, the integral can also be carried out analytically, as follows.

A.3.1. Subsonic Regime

In the subsonic case, plugging \(q(u)\) from Equation (24) into Equation (7), and carrying out the integral, yields

\[
\ln(r) = \frac{S^2}{2W^2} \left[ \frac{1}{2} \bar{z} \left( -\frac{1}{2} \bar{z} - \bar{q}_3 \bar{z}^3 \right) \ln \left( 1 - \frac{u^2}{S^2 \bar{z}^2} \right) - \frac{\bar{q}_1 \coth^{-1} \left( \frac{S}{S} \right)}{s} \right] - F(u) + F(0),
\]

where we defined

\[
F(u) = H \left( x^3 \bar{q}_3 + \bar{q}_0 - \frac{3x}{2}, \frac{\ln(u - x)(S^2 \bar{z}^2 \bar{q}_3 - 3)(W^2 (2q_1^2 \bar{z}^2 + x^2) - 3 - 2S^2 x^2 \bar{q}_3) + 4x^2 \bar{q}_0 \bar{q}_3 (S^2 - W^2) - 4 \bar{q}_0^2}{12x^2 \bar{q}_3 - 6} \right).
\]

A.3.2. Supersonic Regime

For the supersonic regime, the same procedure using Equation (26) yields

\[
\ln(r) = \frac{S^2}{2W^2} \left[ \frac{1}{2} \bar{z} \left( -\frac{1}{2} \bar{z} - \bar{q}_3 \bar{z}^3 \right) \ln \left( 1 - \frac{u^2}{S^2 \bar{z}^2} \right) + \frac{\coth^{-1} \left( \frac{S}{S} \right) (q_1 u_d - q_1 S^2 \bar{z}^2 - q_1 u_d^2 - q_1)}{s} \right] + G(u) - G(0),
\]

where

\[
F(u) = H \left( x^3 \bar{q}_3 + \bar{q}_0 - \frac{3x}{2}, \frac{\ln(u - x)(S^2 \bar{z}^2 \bar{q}_3 - 3)(W^2 (2q_1^2 \bar{z}^2 + x^2) - 3 - 2S^2 x^2 \bar{q}_3) + 4x^2 \bar{q}_0 \bar{q}_3 (S^2 - W^2) - 4 \bar{q}_0^2}{12x^2 \bar{q}_3 - 6} \right).
\]

A.3.3. Explicit Analytic Results

In the subsonic case, plugging \(q(u)\) from Equation (24) into Equation (7), and carrying out the integral, yields

\[
\ln(r) = \frac{S^2}{2W^2} \left[ \frac{1}{2} \bar{z} \left( -\frac{1}{2} \bar{z} - \bar{q}_3 \bar{z}^3 \right) \ln \left( 1 - \frac{u^2}{S^2 \bar{z}^2} \right) - \frac{\bar{q}_1 \coth^{-1} \left( \frac{S}{S} \right)}{s} \right] - F(u) + F(0),
\]

where we defined

\[
F(u) = H \left( x^3 \bar{q}_3 + \bar{q}_0 - \frac{3x}{2}, \frac{\ln(u - x)(S^2 \bar{z}^2 \bar{q}_3 - 3)(W^2 (2q_1^2 \bar{z}^2 + x^2) - 3 - 2S^2 x^2 \bar{q}_3) + 4x^2 \bar{q}_0 \bar{q}_3 (S^2 - W^2) - 4 \bar{q}_0^2}{12x^2 \bar{q}_3 - 6} \right).
\]

A.3.4. Explicit Analytic Results

In the subsonic case, plugging \(q(u)\) from Equation (24) into Equation (7), and carrying out the integral, yields

\[
\ln(r) = \frac{S^2}{2W^2} \left[ \frac{1}{2} \bar{z} \left( -\frac{1}{2} \bar{z} - \bar{q}_3 \bar{z}^3 \right) \ln \left( 1 - \frac{u^2}{S^2 \bar{z}^2} \right) - \frac{\bar{q}_1 \coth^{-1} \left( \frac{S}{S} \right)}{s} \right] - F(u) + F(0),
\]

where we defined

\[
F(u) = H \left( x^3 \bar{q}_3 + \bar{q}_0 - \frac{3x}{2}, \frac{\ln(u - x)(S^2 \bar{z}^2 \bar{q}_3 - 3)(W^2 (2q_1^2 \bar{z}^2 + x^2) - 3 - 2S^2 x^2 \bar{q}_3) + 4x^2 \bar{q}_0 \bar{q}_3 (S^2 - W^2) - 4 \bar{q}_0^2}{12x^2 \bar{q}_3 - 6} \right).
\]
where we omitted the \((d)\) superscripts on the coefficients \(q_i\), and defined

\[
G(u) \equiv H \{ q_2 (u_d - x)^2 + q_1 (x - u_d) + q_0 - x, \frac{\ln(u - x)}{2q_2 (x - u_d) + q_1 - 1} [q_3 q_2^2 (S^2 + W^2) + 2u_d (q_2 u_d - q_1)] + \frac{q_2^2 S^2 W^2 \varepsilon^4 + u_7^2 (q_1 - q_2 u_d)^2 + q_0^2 + c^2 (q_2 x (S^2 - W^2) (-2q_2 u_d + q_1 - 1) + q_2 S^2 u_d (q_2 u_d - q_1) + W^2 (-3q_2 u_d^2 + (3q_1 - 4) q_2 u_d - (q_1 - 1)^2)) / 1] \},
\]

\(A.3.3.\) Shock Standoff Distance

The standoff distance \(\Delta\) may be found by computing \(r_s = (1 + \Delta)\) from Equation (29) with \(u = u_d\). In the strong shock limit, the downstream Mach number is given by \(M_0 = S/g = W^2/S\), and one obtains (after considerable algebra)

\[
r_s = \gamma^G \{ 4 (2 + \gamma + \gamma^{-1}) \}^G A_+^B A_-^B, \tag{31}
\]

where

\[
A_+ = \frac{\gamma (\gamma (5 \gamma - 6) - 3 \pm C)}{4 + \gamma (\gamma (-37 + \gamma (19 + 5 \gamma)) - 15 \pm 4 C)}, \tag{32}
\]

\[
B_\pm = \frac{(-1 + \gamma)^{(\pm 40 - 2C (-1 + \gamma) (4 + \gamma (-15 + \gamma (-37 + \gamma (19 + 5 \gamma))))}) \pm (\gamma (-266 + \gamma (529 + \gamma (27 + \gamma (-174 + \gamma (-292 + \gamma (19 + 5 \gamma))))) / (32 + \gamma (5 + \gamma (-11 + 5 \gamma)) (4 + \gamma (-19 + \gamma (10 + \gamma))))}, \tag{33}
\]

\[
C = \sqrt{\gamma (\gamma (5 \gamma - 56) + 58) + 48 - 7}, \tag{34}
\]

and

\[
G_\pm = -2 \pm \frac{2 - 2\gamma}{32 + \gamma (5 + \gamma (-11 + 5 \gamma))} + \frac{8 - 38 \gamma + 22 \gamma^2}{4 + \gamma (-19 + \gamma (10 + \gamma))}. \tag{35}
\]

This roughly gives \(\Delta \sim 2/(3g)\). Other fits in the range \(1 < \gamma < 2\) include \(\Delta \approx 0.61 g^{-0.94}\) and \(\Delta \approx 0.37 (g - 1)^{-0.75}\).

REFERENCES

Achilleos, N., Bertucci, C., Russell, C. T., et al. 2006, JGRA, 111, 3201
Allen, J. E. 2013, JPlPh, 79, 315
Anderson, B. J., Acuña, M. H., Korth, H., et al. 2008, Sci, 321, 82
Asanaliev, M. K., Zheenbaev, Z. Z., Lelevkin, V. M., Makesheva, K. K., & Pahkomenov, E. P. 1988, TepVT, 26, 527
Aulchenko, S. M., Zumaraev, V. P., & Kalinina, A. P. 2012, IEPT, 85, 1372
Bagental, F., Belcher, J. W., Sittler, E. C., & Lepping, R. P. 1987, JGR, 92, 8603
Baker, C. 2010, Journal of Wind Engineering and Industrial Aerodynamics, 98, 277
Baranov, V. B., & Lebedev, M. G. 1988, A&SS, 147, 69
Barrena, R., Biviano, A., Ramella, M., Falco, E. E., & Seitz, S. 2002, A&A, 386, 816
Bono, G., & Awruch, A. M. 2008, J. of the Braz. Soc. of Mech. Sci. & Eng., 30, 189
Cairns, I. H., & Grabbe, C. L. 1994, GRL, 21, 2781
Canto, J., & Raga. A. 1998, MNRRAS, 297, 383
Chapman, J. F., & Cairns, I. H. 2003, JGRA, 108, 1174
Churazov, E., Breits, M., Kaiser, C. R., Börhringer, H., & Forman, W. 2001, ApJ, 545, 261
Corona-Romero, P., & Gonzalez-Esparza, A. 2013, AdSpR, 51, 1813
Czaszkovska, A., Bauer, T. M., Treumann, R. A., & Baumjohann, W. 2000, arXiv:physics/0009046
Dursi, L. J., & Pfrommer, C. 2008, ApJ, 677, 993
Fabian, A. C., Sanders, J. S., Ettori, S., et al. 2000, MNRRAS, 318, L65
Fairfield, D. H., Cairns, I. H., Desch, M. D., et al. 2001, JGR, 106, 25361
Farris, M. H., & Russell, C. T. 1994, JGR, 99, 17681
Frank, L. A., Paterson, W. R., Ackerson, K. L., Corinoti, F. V., & Vassiliunas, M. V. 1991, Sci, 253, 1528
Gloeckler, G., Geiss, J., & Fisk, L. A. 2004, in AIP Conf. Ser. 719, Physics of the Outer Heliosphere, ed. V. Florinski, N. V. Pogorelov, & G. P. Zank, (Melville, NY: AIP), 201
Gollan, R., & Jacobs, P. 2013, JNMF, 73, 19
Grandemange, M., Gohlike, M., & Cadot, O. 2013, JFM, 722, 51
Grandemange, M., Gohlike, M., & Cadot, O. 2014, JFM, 752, 439
Green, D. A. 2011, BASI, 39, 289
Guderley, K. G. 1962, The Theory of Transonic Flow (Oxford: Pergamon Press)
Guy, T. B. 1974, AIAAJ, 12, 380
Hayes, W. D., & Probst, R. D. 1966, Hypersonic Flow Theory (Mineloa, New York: Dover Publications)
Heberle, J. W., Wood, G. P., & Goederen, P. B. 1950, NACA Technical NOTE, (http://www.dtic.mil/cgi-bin/GetTRDoc?Location=U2&doc=GetTRDoc.pdf&AD=ADA380521)
Hejeanfar, K., Esfahanian, V., & Najafi, M. 2009, JCoPh, 228, 3936
Hida, K. 1953, JPSJ, 8, 740
Hida, K. 1955, JPSJ, 10, 882
Igra, D., & Falcovitz, J. 2010, ShWav, 20, 441
Kaplan, C. 1940, NACA Technical Note Karanjkar, P. V. 2008, PhD thesis, Univ. Florida
Kawamura, T. 1950, Kyoto Univ., Vol. 26, 207
Krause, E. 1975, STIN, 75, 31386
Landau, L. D., & Lifshitz, E. M. 1959, Fluid Mechanics (Oxford: Pergamon Press)
Lea, S. M., & De Young, D. S. 1976, ApJ, 210, 647
Limpmann, H., & Roshko, A. 1957, Elements of Gasdynamics (North Chelmsford, MA: Courier Corporation)
Lighthill, M. J. 1957, JFM, 2, 1
Liu, M.-S., & Takayama, K. 2005, in Aerodynamic Characteristics of High Mach, Low Reynolds Numbers Flow Past Micro Spheres, ed. Z. Jiang, (Berlin: Springer), 1169
Lord Rayleigh 1916, Phil. Mag., Ser. 6, 32, 1
Lytikov, M. 2006, MNRRAS, 373, 73
Mack, C. J., & Schmid, P. J. 2011, JFM, 678, 589
Markevitch, M. 2006, in ESA Special Publication, Vol. 604, The X-ray universe 2005, ed. A. Wilson (Noordwijk: ESA), 723
Markovitch, M., Gonzalez, A. H., David, L., et al. 2002, ApJL, 567, L27
Markovitch, M., & Vikhlinin, A. 2007, PhR, 443, 1
Marrone, S., Colagrossi, A., Antuono, M., Colicchio, G., & Graziani, G. 2013, JCoPh, 245, 436
