Double stochastic resonance peaks in systems with dynamic phase transitions

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Abstract. – To probe the connection between the dynamic phase transition and stochastic resonance, we study the mean-field kinetic Ising model and the two-dimensional Josephson-junction array in the presence of appropriate oscillating magnetic fields. Observed in both systems are double stochastic resonance peaks, one below and the other above the dynamic transition temperature, the appearance of which is argued to be a generic property of the system with a continuous dynamic phase transition. In particular, the frequency matching condition around the dynamic phase transition between the external drive frequency and the internal characteristic frequency of the system is identified as the origin of such double peaks.

When a system with an energetic activation barrier is weakly coupled to a temporally periodic driving, the inherent thermal stochastic noise can enhance the signal out of the system rather than weaken it. This behavior, called stochastic resonance (SR)\(^(^*)\), has been studied extensively in systems with a few degrees of freedom; the appearance of a resonance peak at a finite temperature \(T_{SR}\), characteristic of SR, is conveniently explained by matching the two time scales: the deterministic one due to the external driving and the stochastic one which is inversely proportional to the thermal activation rate (Kramer rate)\(^(\pi)\). Recently, there have also been attempts to investigate the SR in more complex systems\(^(\theta)\). Here one should note that such a system with many degrees of freedom may undergo a phase transition at a nonzero temperature and have the characteristic time scale not described by the simple Kramer rate\(^(\theta)\). Whereas the Kramer rate gives a diverging time scale only at zero temperature, the system with a finite-temperature continuous transition possesses a diverging characteristic time scale at a nonzero critical temperature \(T_c\), yielding finite values of the characteristic time both below and above \(T_c\).

In this Letter we propose to extend the time-scale matching condition in such a way that the matching between the time scale of the external periodic driving and the characteristic

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time scale associated with the phase transition determines the position of the SR peak. Since
the time scales then match at two distinct temperatures, one below and the other above \( T_c \), it
follows that the system with a continuous phase transition should show double SR peaks, one
below and the other above the transition. In order to elucidate this scenario, we investigate
the interplay between the dynamic phase transition and the SR behavior in the infinite-ranged
kinetic Ising model and in the two-dimensional (2D) fully frustrated Josephson-junction array
(FFJJA), driven by appropriate oscillating magnetic fields. Unlike the latter system, where
the SR behavior has not been examined, the kinetic Ising model in an oscillating magnetic
field has been studied previously via the mean-field approximation and numerical simulations,
with regard to both the dynamic phase transition \[7–9\] and the SR behavior \[3,4\], and double
SR peaks have been found and discussed by Leung and Nédé in \[4\]. However, the physics
behind the emergence of double peaks has not been addressed per se: To our knowledge,
neither the connection between the origin of the double peaks and the existence of a dynamic
phase transition nor the generality of this connection has been recognized and addressed at all.

We begin with the infinite-ranged kinetic Ising model described by the Hamiltonian

\[
H = -\frac{J}{N} \sum_{i<j} \sigma_i \sigma_j - h(t) \sum_{i=1}^{N} \sigma_i, \tag{1}
\]

where the Ising spin \( \sigma_i \) at site \( i \) can have values \( \pm 1 \), \( J \) is the coupling strength, and the
external oscillating magnetic field is given by \( h(t) = h_0 \cos \Omega t \). Henceforth we set \( J \equiv 1 \) and
measure the temperature \( T \) in units of \( J/k_B \). For the above infinite-ranged Hamiltonian,
the mean-field approximation is exact and yields the equation of motion under the Glauber
dynamics \[4,8\]

\[
\frac{dm(t)}{dt} = -m(t) + \tanh \left( \frac{m(t) + h(t)}{T} \right), \tag{2}
\]

where the magnetization \( m(t) \) is given by the ensemble average of the spin at time \( t \).

We first recapitulate the dynamic phase transition present in the system described by
Eq. (2). In the absence of the external field, the system undergoes a thermodynamic phase
transition at the transition temperature \( T_0 = 1 \) between the low-temperature ferromagnetic
phase \((m_0 \neq 0)\) and the high-temperature paramagnetic phase \((m_0 = 0)\), where \( m_0 \) is the
spontaneous magnetization in equilibrium. In the presence of the external oscillating field, on
the other hand, the system displays a dynamic phase transition \[4,8\], for which the dynamic
order parameter \( Q \) can be defined to be

\[
Q = \lim_{n \to \infty} Q_n, \tag{3}
\]

with \( Q_n \) denoting the time average during the \( n \)th period of external driving:

\[
Q_n = \frac{\Omega}{2\pi} \int_{2\pi n/\Omega}^{2\pi(n+1)/\Omega} dt \, m(t). \tag{4}
\]

Note that \( Q \) separates the ferromagnetic phase \((Q \neq 0)\) from the paramagnetic phase \((Q = 0)\)
and determines the dynamic phase transition temperature \( T_c \). The phase boundary, obtained
numerically by solving Eqs. (2)-(4) \[8\] and analytically via the adiabatic approximation valid
for sufficiently small \( \Omega \) \[10\], in general connects the points \((T, h_0) = (1, 0)\) and \((0, 1)\) on the
\( T-h_0 \) plane. For given frequency \( \Omega \), as the amplitude \( h_0 \) is increased from zero, the transition
not only occurs at a lower temperature, lowering \( T_c \), but also changes its nature from a
A key quantity in our argument for the double SR peaks is the relaxation time $\tau$ or equivalently, the intrinsic frequency scale $1/\tau$ of the system, which describes the decay of the average magnetization: $Q_n - Q \propto e^{-t/\tau}$. Thus one may start from, e.g., the ordered state $m(t=0) = 1$, and extract the exponential decay of $Q_n$ to obtain numerically the relaxation time [9]. At weak driving fields, the relaxation time can also be calculated analytically to the linear order in the amplitude $h_0$ [4, 10]:

$$\frac{1}{\tau} = 1 - \frac{1}{T \cosh^2(m_0/T)},$$

where the spontaneous magnetization $m_0$ is determined by the equation of state: $m_0 = \tanh(m_0/T)$. Figure [3] displays the relaxation time for $h_0 = 0.1$ and $\Omega/2\pi = 0.1$, determined by these two methods. It is shown that at the dynamical phase transition the characteristic frequency $1/\tau$ vanishes or equivalently, the relaxation time $\tau$ diverges. Note also the good agreement between the analytical and numerical results, implying that Eq. (3) gives an excellent approximation, except for the discrepancy in the temperature at which $\tau$ diverges:

Equation (3) leads to the divergence at the equilibrium transition temperature $T_0 = 1$, whereas the actual relaxation time diverges at the dynamical transition temperature $T_c \approx 0.98$, slightly

Fig. 1 – Inverse of the relaxation time versus temperature, computed numerically from the behavior of $Q_n$ for $h_0 = 0.1$ and $\Omega/2\pi = 0.1$ (full line) and analytically via the linear response theory (dashed line). The relaxation time is shown to diverge, as the transition temperature is approached either from above or from below.
Fig. 2 – Occupancy ratio versus temperature in the kinetic Ising model for $h_0 = 0.1$ and $\Omega/2\pi = 0.1$. Data directly from Eq. (6) together with numerical integration of Eq. (2) are represented by small circles whereas the analytical expression given by Eq. (7) with $\tau$ obtained numerically in Fig. 1 is plotted by the solid line. The dashed line corresponds to the linear response calculation [Eq. (7) with $\tau$ given by Eq. (5)]. Inset: the time scale matching condition with the solid line corresponding to the inverse relaxation time obtained numerically in Fig. 1 and the horizontal dashed line to the right-hand side in Eq. (8). The two crossing points below and above the transition temperature determine the positions of the two SR peaks in the main body.

...lowered by the external driving field. It is important here that a continuous dynamical transition as well as a continuous equilibrium transition is associated with a divergent relaxation time. In particular the relaxation time can assume the same (finite) value at two temperatures, one above and the other below the transition.

It is convenient to study the SR behavior in the system by means of the occupancy ratio (OR), defined to be the average fraction of the spins in the direction of the external field $h$: \[ OR = \frac{\text{number of spins in the direction of } h(t)}{\text{total number of spins}}. \] (6)

At sufficiently low temperatures most spins are stuck to one of two spin directions ($\pm 1$) and do not follow the direction preferred by the field $h(t)$. Since $h(t)$ oscillates with period $2\pi/\Omega$, most spins point in the same direction as $h(t)$ during one half of the period while almost no spin follows $h(t)$ during the other half; this results in OR $\approx 1/2$. At very high temperatures, on the other hand, each spin can take the values 1 and $-1$ with equal probability, again resulting in the value of OR close to 1/2, since about half the spins point in the same direction as $h(t)$ at a given instant. Somewhere in between the spins may follow closely the external driving field, giving rise to a peak in the OR as a manifestation of SR. Within the linear response,
the OR takes the analytic form \[ \text{OR} = \frac{1}{2} + \frac{\Omega}{4\pi} \int dt \, m(t) \text{sgn}(\cos\Omega t) \]

\[ = \frac{1}{2} + \frac{h_0}{\pi} \frac{\tau - 1}{1 + \Omega^2 \tau^2}, \tag{7} \]

where \( \int \) represents the integration over a full period of the driving field in the stationary state and \( \tau \) is given by Eq. \((5)\). The resulting behavior of the OR with the temperature is shown in Fig. 2, which reveals most clearly the double-peak structure of SR. Figure 2, where the numerical result directly from the equation of motion \((2)\) has also been displayed, demonstrates that the analytical result in Eq. \((7)\) is rather accurate. Further shown in Fig. 2 is that the analytical result can even be improved by adopting in Eq. \((7)\) the (more precise) numerical values of \( \tau \) shown in Fig. 1 instead of those given by Eq. \((5)\).

From Eq. \((7)\), it is straightforward to obtain the peak position \( T_{\text{SR}} \) of the OR, which is determined by the condition \( \frac{d}{dT} \text{OR} = 0 \). This leads directly to the matching condition between the external and internal (intrinsic) time scales:

\[ \tau = 1 + \sqrt{1 + \Omega^{-2}}, \tag{8} \]

which reduces in the limit \( \Omega \to 0 \) to the direct matching condition between the frequency scales:

\[ \tau^{-1} \approx \Omega. \tag{9} \]

It is thus concluded that the resonance becomes strong when the external time scale \( 1/\Omega \) matches the intrinsic time scale \( \tau \) according to the matching condition in Eq. \((8)\). Since \( \tau \), diverging at \( T_c \), decreases away from \( T_c \), there in general exist two temperatures, one above and the other below the transition, at which Eq. \((8)\) is satisfied for given \( \Omega \); obviously, this gives rise to the double peaks. The inset of Fig. 2 displays the matching condition in Eq. \((8)\), the right-hand side of which is, for \( \Omega/2\pi = 0.1 \), depicted by the horizontal dashed line: The two intersections determine precisely the positions of the two SR peaks in the main body of Fig. 2, demonstrating explicitly the proposed matching condition for the existence of the double SR peaks.

To disclose ubiquity of the double-peak structure associate with a dynamic phase transition, we next consider the 2D \( L \times L \) FFJJA, driven by a weak magnetic field, staggered in space and periodic in time, in addition to the uniform field introducing full frustration [see Fig. 3(a)]. Such a system is described by the Hamiltonian

\[ H = -J \sum_{(i,j)} \cos(\phi_i - \phi_j - A_{ij}), \tag{10} \]

where \( J \) is the Josephson coupling strength, \( \phi_i \) denotes the phase of the superconducting order parameter at site \( i \), and the summation is over all nearest neighboring pairs. The bond angle is given by the line integral of the vector potential and consists of two parts: \( A_{ij} = A_{ij}^0 + \delta A_{ij}(t) \), where \( A_{ij}^0 \) describes the uniform magnetic field corresponding to half a flux quantum per plaquette \( (f = 1/2) \) and \( \delta A_{ij}(t) \) takes into account the additional oscillating magnetic field of frequency \( \Omega \). When the oscillating field is absent \( (\delta A_{ij} = 0) \), the ground state possesses the two-fold degeneracy and vortices form a \( 2 \times 2 \) superlattice at zero temperature \[ \text{[12]} \]. On the other hand, the presence of the oscillating field, which leads to the oscillating flux \( \delta f(t) = f_0 \cos\Omega t \), lifts the two-fold degeneracy. For \( \Omega \ll 1 \), the energy splitting \( \Delta E \) between the
two states [each of which has the staggered vorticity ±1; see Fig. 3(b)] is given by \( \Delta E = 2J\sqrt{2}\sin(\pi f_0 \cos \Omega t/2) \). It is thus clear that this FFJJA subject to the oscillating staggered magnetic field contains all the basic ingredients required for SR behavior. Figure 3(c) suggests one possible realization of this system in experiment. Periodic arrays of magnetic particles are already available and used frequently in experiment \([13]\), and we believe that the relevant SR behavior can be observed through the use of various vortex imaging techniques.

The dynamics of the system is described by the resistively-shunted junction model, stating the current conservation at each site:

\[
\sum_j' \left[ \dot{\phi}_i - \dot{\phi}_j - \dot{A}_{ij} + \sin(\phi_i - \phi_j - A_{ij}) + \zeta_{ij} \right] = 0, \tag{11}
\]

where the prime restricts the summation to the neighbors of site \( i \), and \( \zeta_{ij}(t) \) is the random noise current across the junction with zero mean and correlation \( \langle \zeta_{ij}(t)\zeta_{kl}(t') \rangle = 2T(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\delta(t-t') \). Note that energy (and temperature) has been measured in units of \( J \) and time in units of \( \hbar/4e^2JR \) with the shunt resistance \( R \).

To obtain the dynamic behavior, we integrate directly the set of equations of motion (11) and measure the staggered vorticity at each plaquette. The OR of the staggered vorticity, measuring how many plaquettes have the staggered vorticity preferred by the external driving, is then computed according to Eq. (6), with the oscillating flux \( f(t) \) taking the role of \( \hbar(t) \).
Figure 4 displays the obtained OR as a function of the temperature $T$ in the array of size $L = 16$ for $f_0 = 0.02$ and $\Omega = 0.01$. Observed clearly is the emergence of double SR peaks, one below and the other above the (dynamic) transition temperature $T_c \approx 0.443$ \[10\], although the peak below the transition appears much weaker than the one above. Note that sufficient decrease of $\tau$ is essential for the time-scale matching condition to be satisfied. In some systems, $\tau$ may become large at low temperatures as well as near $T_c$ and the SR peak below $T_c$ may not be observed \[14\].

In summary, we have proposed the extended time-scale matching condition for the SR, between the external periodic driving and the relaxation time associated with a phase transition, thus elucidating the link between the appearance of double SR peaks and the existence of a dynamic phase transition. This has been demonstrated in the 2D fully frustrated Josephson-junction array as well as in the mean-field kinetic Ising model. The argument presented here is rather general, and we expect that the double-peak structure should appear ubiquitously in systems undergoing dynamic phase transitions with diverging relaxation time at the transition.

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