A Note on Cliques in Multipartite Graphs

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Abstract

We consider a set of cliques in any multipartite graph with two vertices in each part. Moreover, we construct a class of peculiar polytopes.

Key words: multipartite graph, clique, polytope.

Introduction. $n$-cliques in any $n$-partite graph with two vertices in each part correspond to models for a conjunctive normal form with two literals in each disjunction (2CNF). Thus, the clique can be found in polynomial time [1]. On the other hand, some properties of a set of such cliques are hard for computing [2]. Our article is a contribution to this collection.

Terminology. A graph $\Gamma$ is said to be multipartite if the vertex set is partitioned into parts such that there is no edge between any pair of vertices from the same part. The multipartite graph $\Gamma$ is complete if every two vertices from different parts are joined. A complete multipartite graph $\Gamma$ is a complete graph if every part of $\Gamma$ consists of a single vertex. A complete subgraph with exactly $n$ vertices is called a $n$-clique.

There is a complete 3-partite graph:

A closed and bounded subset $\Pi$ of the $d$-dimensional rational affine space $\mathbb{Q}^d$ is called a convex polytope if it is the set of solutions to a finite system of linear inequalities. We shall omit convex for convex polytopes, and call them simply polytopes. When the polytope is full-dimensional, each nonredundant inequality corresponds to a facet.

Let $\Pi$ be a convex $d$-dimensional polytope in $\mathbb{Q}^d$. To define faces geometrically, it is convenient to use supporting hyperplanes. A hyperplane $H$ of $\mathbb{Q}^d$...
is supporting the polytope $\Pi$ if one of the closed half-spaces of $H$ contains the polytope $\Pi$. A subset $\Phi \subseteq \Pi$ is called a face of $\Pi$ if it is either empty, $\Pi$ itself or the intersection of $\Pi$ with a supporting hyperplane.

The faces of dimension 0, 1, $d-2$ and $d-1$ are called the vertices, edges, ridges and facets, respectively. The vertices coincide with the extreme points of $\Pi$ which are defined as points which cannot be represented as convex combinations of two other points in the polytope $\Pi$. For a subset $S \subseteq \mathbb{Q}^d$, the convex hull is defined as the smallest convex set in $\mathbb{Q}^d$ containing $S$. See also [3].

**Graphs and Polytopes.** Let the indices $p, q, r$ are equal to either 1 or 2, and the indices $i, j, k$ range over the segment $\{1, 2, \ldots, n\}$.

For any $n \geq 2$ we shall define a polytope $\Omega_n$ in the $4n^2$-dimensional rational affine space as a convex hull of points $X$ with coordinates

$$X_{ijpq} = \begin{cases} 1, & \text{if } p = \rho(i) \text{ and } q = \rho(j) \\ 0, & \text{otherwise}. \end{cases}$$

for any function $\rho : \{1, 2, \ldots, n\} \to \{1, 2\}$.

**Theorem 1** Each vertex $X$ of the polytope $\Omega_n$ satisfy to the equalities

$$\forall i, j \forall p, q \quad X_{ijpq} = X_{jiqp}$$
$$\forall i \quad X_{i11} + X_{i22} = 1$$
$$\forall i \quad X_{i12} = 0$$
$$\forall i, j \forall p \quad X_{ijp1} + X_{ijp2} = X_{iipp}.$$  \hfill (4)

**Proof.** Any coordinate of a vertex $X$ is equal to either 0 or 1. Thus,

$$X_{ijpq} = 1 \iff X_{iipp} = 1 \text{ and } X_{jjqq} = 1;$$

if $i \neq j$ and $X_{ijpq} = 0$, then $X_{iipp} = 0$ or $X_{jjqq} = 0$.

The equalities (1) as well as the equalities (4) are obvious. The equalities (2) are satisfied because in each sum one member is equal to 1 and others are equal to zero. The equalities (3) are satisfied because in each sum no more than one member is equal to 1 and others are equal to zero. \hfill \square

With any point $X \in \Omega_n$ we associate a $n$-partite graph with two vertices in each part, where for any pair $j \neq i$ of indices the $p$-th vertex of the $i$-th part is joined by an edge with the $q$-th vertex of the $j$-th part iff $X_{ijpq} > 0$. Any positive coordinate of a point $X \in \Omega_n$ is convenient to consider as the weight of a vertex or an edge.
Any vertex of the polytope $\Omega_n$ is a $n$-clique of the complete $n$-partite graph with two vertices in each part.

**Theorem 2** The dimension

$$\dim \Omega_n = \frac{n(n+1)}{2}.$$ 

**Proof.** The coordinates of a kind $X_{ij11}$ univocally determine a point in the polytope $\Omega_n$. Thus, its dimension

$$\dim \Omega_n \leq \frac{n(n+1)}{2}.$$ 

On the other hand, the polytope $\Omega_n$ contains a set of $\frac{n(n+1)}{2} + 1$ affine independent points. In more detail,

- there is a point with every coordinate $X_{ii11} = 0$;
- there are $n$ points with one unit and $n-1$ zeroes among the coordinates $X_{ii11}$;
- there are $\frac{n(n-1)}{2}$ points with two units and $n-2$ zeroes among the coordinates $X_{ii11}$.

Therefore, the dimension $\dim \Omega_n \geq \frac{n(n+1)}{2}$. As a conclusion, we have the desired equality. $\square$

**Theorem 3** A convex hull of any vertex pair in the polytope $\Omega_n$ is an edge.

**Proof.** Let us consider two vertices $X$ and $Y$ in the polytope $\Omega_n$. They correspond to the pair of $n$-cliques denoted $X$ and $Y$ too. Let us mark one edge in each clique that does not belong to the other clique. Define a linear form

$$F(X) = \sum_{i>j} \alpha_{ijpq} X_{ijpq},$$

where $\alpha_{ijpq}$ is defined in this way. Let us consider an edge between the $p$-th vertex in the $i$-th part and the $q$-th vertex in the $j$-th part. Then

$$\alpha_{ijpq} = \begin{cases} 2, & \text{if this edge belongs to neither the } n\text{-clique } X \text{ nor the } n\text{-clique } Y \\ 1, & \text{if this edge is marked} \\ 0, & \text{otherwise} \end{cases}$$

The equations $F(X) = 1$ and $F(Y) = 1$ are satisfied on these vertices. And for any other vertex $Z$ we have $F(Z) \geq 2$. As a consequence, these vertices $X$ and $Y$ are end points of an edge in $\Omega_n$. $\square$
Geometry of $\Omega_3$. Let us consider the polytope $\Omega_3$. The dimension $\dim \Omega_3 = 6$. Any facet is determined by six of eight its vertices, i.e. all except for two. These excluded vertices correspond to a pair of triangles in a complete 3-partite graph. There are three cases.

*Excluded triangles have no common vertex.* Without loss of generality we assume that the first triangle consists of the first vertices and the second triangle consists of the second vertices in each part.

Other six vertices of the polytope $\Omega_3$ belong to a facet defined by the equation

$$X_{1211} + X_{1311} + X_{2311} + X_{1222} + X_{1322} + X_{2322} = 1.$$

*Excluded triangles have a common edge.* Without loss of generality we assume that first triangle consists of the first vertices and the common edge joints the first part with the second part.

A convex hull of other six vertices of the polytope $\Omega_3$ is a face defined by the equation $X_{1211} = 0$.

*Excluded triangles have a sole common vertex.* Without loss of generality we assume that the first triangle consists of the first vertices and the common vertex belongs to the first part.

A convex hull of other six vertices of the polytope $\Omega_3$ can not be a face. For example, the form

$$X_{1222} + X_{1312} + X_{1212} + X_{1221}$$

is evaluated as 0 and 2 on the excluded vertices of the polytope $\Omega_3$, but it is evaluated as 1 on other six vertices.

**Some remarks.** The 3-dimensional polytope $\Omega_2$ has four vertices, i.e. $\Omega_2$ is a simplex.

Obviously, the polytope $\Omega_n$ is symmetrical. Thus, each vertex belongs to the same number of facets. The important question is still open: how many facets are in the polytope $\Omega_n$?

Note that in high dimensions, there are three regular polytopes.
- The $d$-dimensional simplex has $d + 1$ vertices and $d + 1$ facets.
- The $d$-dimensional cube has $2^d$ vertices and $2d$ facets.
- The $d$-dimensional cross-polytope has $2d$ vertices and $2^d$ facets.

The dual of a regular polytope is another polytope, also regular, having one vertex in the center of each facet of the polytope we started with. The simplex is self-dual, and the dual of the cube is the cross-polytope. So there is not a polynomial upper bound for the number of facets intersecting at one vertex.

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