Tau Functions and Virasoro Symmetries for Drinfeld-Sokolov Hierarchies

Chao-Zhong Wu
School of Mathematics and Computational Science, Sun Yat-sen University
Guangzhou 510275, P.R. China.

Abstract

For each Drinfeld-Sokolov integrable hierarchy associated to affine Kac-Moody algebra, we obtain a uniform construction of tau function by using tau-symmetric Hamiltonian densities, moreover, we represent its Virasoro symmetries as linear/nonlinear actions on the tau function. The relations between the tau function constructed in this paper and those defined for particular cases of Drinfeld-Sokolov hierarchies in the literature are clarified. We also show that, whenever the affine Kac-Moody algebra is simply-laced or twisted, the tau functions of the Drinfeld-Sokolov hierarchy coincide with the solutions of the corresponding Kac-Wakimoto hierarchy constructed from the principal vertex operator realization of the affine algebra.

Key words: Drinfeld-Sokolov hierarchy; tau function; Virasoro symmetry; Kac-Moody algebra

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1 Introduction

For every affine Kac-Moody algebra \( \mathfrak{g} \) with an arbitrary vertex of its Dynkin diagram marked, Drinfeld and Sokolov \[7\] constructed a hierarchy of integrable systems that generalize the celebrated Korteweg-de Vries (KdV) equation. These integrable hierarchies have very important applications in various areas of mathematical physics like 2D topological field theory and Gromov-Witten invariants \[6, 8, 10, 12, 13, 14, 31, 44\]. For instance, the partition function of a topological minimal model of ADE type is given by the logarithm of tau function of the Drinfeld-Sokolov hierarchy associated to the corresponding simply-laced affine Kac-Moody algebra. Such a tau function is selected by the string equation, or equivalently, it must have trivial evolution along the flow generated by the Virasoro symmetry of level \(-1\). In this paper we study tau functions of Drinfeld-Sokolov hierarchies and their Virasoro symmetries.

In the literature there are several methods to define tau functions for Drinfeld-Sokolov hierarchies (or their generalizations \[23\]). One of them is based on certain integrable highest weight representation of the affine Kac-Moody algebra \( \mathfrak{g} \), see \[24, 26, 36\]. In this way, Drinfeld-Sokolov hierarchies are closely related to the systems of Hirota bilinear equations constructed by Date, Jimbo, Kashiwara and Miwa \[5, 27\], and by Kac and Wakimoto \[28, 29\], meanwhile the tau functions are identified with elements of the orbit space of the highest weight vector acted by the affine Lie group. A shortcoming of this method is that, it relies on the representation theory of \( \mathfrak{g} \), and usually involves some dressing operators, given implicitly in a sense, of the hierarchies written in zero-curvature form.

The second method is to define tau function via a family of appropriate densities of Hamiltonians that are called to be tau-symmetric in \[10\]. Such Hamiltonian densities correspond to some special two-point correlation functions whenever the logarithm of the tau function gives a partition function in topological field theory. This method, which does not depend on the representation theory of Lie algebras, works well for the Drinfeld-Sokolov hierarchies associated to affine algebras \( A_n^{(1)} \). For example, the \( A_1^{(1)} \)-type hierarchy is equivalent to the KdV hierarchy:

\[
\frac{\partial L}{\partial t_j} = [(L^{j/2})_+, L], \quad j \in \mathbb{Z}_{\text{odd}},
\]  

(1.1)
where \( L = D^2 + u \) with \( D = d/dx \) and \( u \) being a function of the spatial variable \( x \) and time variables \( t_j \). The hierarchy (1.1) has the following Hamiltonian representation

\[
\frac{\partial u}{\partial t_j} = \{ u(x), H_j \},
\]

in which the Poisson bracket reads

\[
\{ u(x), u(y) \} = 2u(x)\delta'(x - y) + u'(x)\delta(x - y) + \frac{1}{2}\delta'''(x - y)
\]

and the Hamiltonian functionals are

\[
H_j = \int h_j(u; \partial_x u, \partial_x^2 u, \ldots) \, dx, \quad h_j = \frac{2}{j} \text{res } L^{j/2}.
\]

The densities \( h_j \) satisfy the tau-symmetry condition

\[
\frac{i}{2} \frac{\partial h_i}{\partial t_j} = j \frac{\partial h_j}{\partial t_i}, \quad i, j \in \mathbb{Z}^{\text{odd}}_+,
\]

hence they define locally a tau function \( \tau \) by

\[
\frac{\partial^2 \log \tau}{\partial x \partial t_j} = \frac{j}{2} h_j, \quad j \in \mathbb{Z}^{\text{odd}}_+.
\]

In a similar way, we pushed forward the construction of tau function to all \( D^{(1)}_n \)-hierarchies, see [35] or Example 5.3 below.

For an arbitrary Drinfeld-Sokolov hierarchy, however, it was unknown how to choose such tau-symmetric Hamiltonian densities. The original motivation of this paper is to resolve this problem universally, rather than in a case-by-case way. Our first main result is a uniform construction of tau function for each Drinfeld-Sokolov hierarchy by

\[
\frac{\partial^2 \log \tau}{\partial x \partial t_j} = -j (\Lambda_j | H) (\Lambda_j | \Lambda_{-j})^t, \quad j \in E_+.
\]

Here \( H \) is certain generating function of Hamiltonian densities of the hierarchy, \( t_j \) are the time variables corresponding to the generators \( \Lambda_j \) of the principal Heisenberg algebra of \( g \), for which \( E_+ \) is the set of positive exponents and \( (\cdot | \cdot) \) is a nondegenerate invariant symmetric bilinear form (see Section 3 for details). In particular, the tau function given in (1.5) is consistent with that defined in the literature via Hamiltonian densities for each hierarchy of type \( A^{(1)}_n \) or \( D^{(1)}_n \).

Besides the above two methods, the third important approach to defined tau functions is to use certain line bundle on infinite-dimensional Grassmannians. This approach dates back to Sato and Segal-Wilson, who introduced tau functions for the \( A^{(1)}_n \) case, see [39] and references therein. Based on Ben-Zvi and Frenkel’s geometric description [2] of (generalized) Drinfeld-Sokolov hierarchies with smooth projective curves and Lie groups, recently Safronov [37] defined tau functions of these hierarchies on a section of line bundle to the so-called Drinfeld-Sokolov Grassmannians. He also pointed out that his tau function coincides with \( \tau \) in (1.5) for the original Drinfeld-Sokolov hierarchies. Following closely
the approach in [39], we introduced in [4] tau functions of (generalized) Drinfeld-Sokolov hierarchies starting from a reformulation of the hierarchies with dressing operators, and showed that these tau functions are equivalent to those given in (1.5). Here we will not get into details of [37, 4], for they do not concern the present paper.

We continue to consider symmetries for Drinfeld-Sokolov hierarchies. As the most simple case, the KdV hierarchy is known to possess a family of so-called additional symmetries. These symmetries commute with each flow in the hierarchy (1.1), but do not commute among themselves. Instead, they obey a Virasoro commutation relation; that is why such additional symmetries are also called Virasoro symmetries. More precisely, the Virasoro symmetries for the KdV hierarchy are generated by the infinitesimal transformations (see, for example, [41]) of tau function as

$$\tau \mapsto \tilde{\tau} = \tau + \epsilon L_k \tau, \quad k \geq -1,$$

where $\epsilon$ is a small parameter, and the generators $L_k$ reads

$$L_{-1} = \frac{1}{2} \sum_{j \in \mathbb{Z}_{odd}^+} (j + 2)t_{j+2} \frac{\partial}{\partial t_j} + \frac{1}{4} t_1^2,$$

$$L_0 = \frac{1}{2} \sum_{j \in \mathbb{Z}_{odd}^+} j t_j \frac{\partial}{\partial t_j} + \frac{1}{16},$$

$$L_k = \frac{1}{4} \sum_{i=1}^{k} \frac{\partial^2}{\partial t_{2i-1} \partial t_{2k-2i+1}} + \frac{1}{2} \sum_{j \in \mathbb{Z}_{odd}^+} j t_j \frac{\partial}{\partial t_{j+2k}}, \quad k \geq 1.$$

These operators satisfy

$$[L_k, L_l] = (k - l) L_{k+l}, \quad k, l \geq -1.$$

In particular, the first two generators $L_{-1}$ and $L_0$ correspond to the Galilean and the scaling transformations respectively. The string equation of the tau function is

$$\frac{\partial \tau}{\partial t_1} = L_{-1} \tau,$$

which induces a series of constraints to $\tau$ that plays an important role in topological field theory and matrix models [1]. Virasoro symmetries for Drinfeld-Sokolov hierarchy of type $A_n^{(1)}$ or $D_n^{(1)}$ can be constructed by using pseudo-differential operator skills, and they are written as linear actions on tau function like (1.6), see [46] for the cases $D_n^{(1)}$. When the affine Kac-Moody algebra $g$ is simply-laced, similar description of Virasoro symmetries for Drinfeld-Sokolov hierarchies was given by Hollowood, Miramontes and and Sánchez Guillén [25], who used the method of representation theory of $g$.

So far as we know, there are no analogous characterizations of Virasoro symmetries via tau function for arbitrary Drinfeld-Sokolov hierarchies. The reason is probably the lack of an appropriate definition of tau function of them before. As the tau function is defined in (1.5), we will obtain another main result of this paper.
Theorem 1.1 Given an arbitrary affine Kac-Moody algebra $\mathfrak{g}$, the associated Drinfeld-Sokolov hierarchy possesses Virasoro symmetries generated by the following infinitesimal transformations of tau function:

$$\tau \mapsto \tilde{\tau} = \tau + \epsilon (V_k \tau + \tau O_k), \quad k \geq -1,$$

(1.11)

where $V_k$ are Virasoro operators independent of $\tau$, and $O_k$ are differential polynomials in second-order derivatives of $\log \tau$ with respect to the time variables. Moreover, $O_k = 0$ for all $k \geq -1$ if $\mathfrak{g}$ is simply-laced or twisted, while $O_{-1} = O_0 = 0$ if $\mathfrak{g}$ is of type B, C, F or G.

This theorem provides a unified description of Virasoro symmetries for all Drinfeld-Sokolov hierarchies. Firstly, if the affine Kac-Moody algebra $\mathfrak{g}$ is simply-laced, the triviality of $O_k$ shows the linearization of Virasoro symmetries. This agrees with the previous results in [41, 25, 10, 46]; for example, $V_k = L_k$ whenever $\mathfrak{g}$ is of type $A_1^{(1)}$. As an application of the definition of tau functions and the linearization of Virasoro symmetries, Liu, Ruan and Zhang [33] proposed a complete proof of the equivalence between the Drinfeld-Sokolov hierarchies of simply-laced type and Dubrovin and Zhang’s topological hierarchies associated to semisimple Frobenius manifolds for ADE-type simple singularities [10] (see also [9, 35, 46]).

Secondly, in case $\mathfrak{g}$ is of non-ADE type, we conjecture that the functions $O_k$ with $k \geq 1$ may not vanish; namely, the Virasoro symmetries for Drinfeld-Sokolov hierarchies of non-ADE type are not linearizable. This conjecture is partially verified (see Example 5.5 and [4] for the $C_n^{(1)}$-hierarchies), but is still open in general.

Although the Drinfeld-Sokolov hierarchies associated to twisted affine Kac-Moody algebras contain important examples such like the Sawada-Kotera equation [38] (belonging to the $A_2^{(2)}$-hierarchy), they seem not have attracted much attention in a sense. In fact, when the affine Kac-Moody algebra $\mathfrak{g}$ is twisted, the Drinfeld-Sokolov hierarchy is Hamiltonian [7], and probably has only one (local) Hamiltonian structure [34] [38] such that it is not involved in the framework of topological hierarchies in [10]. What is surprising, we now show that this hierarchy has a tau function defined by Hamiltonian densities, as well as linearized Virasoro symmetries acting on the tau function. For such kind of hierarchies, it is unknown whether there is any illustration in topology that is analogous with the case of hierarchies of simply-laced type.

Our proof of linearization of Virasoro symmetries is based on the representation theory of simply-laced or twisted affine Kac-Moody algebras. This naturally leads us to study the relation between Drinfeld-Sokolov hierarchies and Kac-Wakimoto hierarchies of bilinear equations. Recall that Drinfeld-Sokolov hierarchies for simply-laced affine algebras were shown related to the corresponding Kac-Wakimoto hierarchies by Hollowood and Miramontes [24], see their formula (4.53) below. Inspired by their work, we obtain a byproduct of the present paper, that is, if $\mathfrak{g}$ is a simply-laced or twisted affine Kac-Moody algebra, then tau functions given in (1.5) of the Drinfeld-Sokolov hierarchy coincide with solutions of the Kac-Wakimoto hierarchy constructed from the principal vertex operator realization of $\mathfrak{g}$ (see Theorem 4.12 below).

To achieve the above results, we organize the contents of this paper as follows.

In the forthcoming section, we will recall Drinfeld and Sokolov’s original construction of integrable hierarchy on a “loop algebra”, that is, the affine Kac-Moody algebra $\mathfrak{g}$ modulo
the subspace spanned by the central and the scaling elements $c$ and $d$. The hierarchy is
composed of Hamiltonian equations, with Hamiltonian densities being determined up to
addition of total derivatives with respect to the spacial variable.

In Section 3, we reformulate the definition of Drinfeld-Sokolov hierarchies on the derived
algebra $g'$ of $g$. The nontrivial central part helps us to fix the freedom of Hamiltonian
densities to fulfill the tau-symmetry condition, hence a tau function can be defined by
(1.5).

In Section 4, we will review the Kac-Moody-Virasoro algebra consisting of $g$ and a
family of derivations on it. The construction of Virasoro symmetries in [25] will be revised
and extended to all Drinfeld-Sokolov hierarchies. Moreover, the Virasoro symmetries will
be represented via tau function in a unified form (1.11), and they are shown to be linearized
in case $g$ is simply-laced or twisted, which proves Theorem 1.1.

Section 5 is a collection of examples. The first three examples show the consistence
between the tau function in (1.5) and those defined in the literature for Drinfeld-Sokolov
hierarchies of types $A_n^{(1)}$ and $D_n^{(1)}$. The other examples illustrate how to compute $O_k$
in (1.11) that give obstacles when linearizing the Virasoro symmetries. The Virasoro
constraints to tau function will also be derived.

The last section is devoted to the conclusion and some discussions. We try to divide
Drinfeld-Sokolov hierarchies into three classes according to their Hamiltonian structures
and Virasoro symmetries, and discuss possible applications.

In order to make this paper more complete, in the appendix we will consider tau
functions of integrable hierarchies of modified KdV type from Drinfeld and Sokolov’s con-
struction [7]. To avoid lengthy expressions, we call such hierarchies the modified Drinfeld-
Sokolov hierarchies, which are related to the Drinfeld-Sokolov hierarchies (of KdV type) by
certain gauge transformations. Note that for untwisted affine Lie algebras, such modified
hierarchies were also constructed by Kupershmidt and Wilson [32, 43]. Another equivalent
version of these hierarchies was given by Feigin and Frenkel [17]; accordingly Enriquez and
Frenkel [11] introduced tau functions of them with the help of tau-symmetric Hamiltonian
densities.

It will be seen that the right hand side of (1.5) is invariant with respect to gauge trans-
formations, hence $\tau$ also serves as a tau function of the corresponding modified Drinfeld-
Sokolov hierarchy. This tau function will be shown different from the one defined by
Enriquez and Frenkel. As a matter of fact, in the modified case, these two tau functions
coincide with two special cases of the tau function obtained by Miramontes [36] based on
the representation theory of $g$, see Proposition A.2 below.

2 Definition of Drinfeld-Sokolov hierarchies

Given an affine Kac-Moody algebra, Drinfeld and Sokolov’s hierarchies associated to dif-
ferent marked vertices of the Dynkin diagram are related by Miura-type transformations.
For convenience, in this paper we only consider the case that the vertex is chosen to be the
zeroth one, which is the special vertex added to the Dynkin diagram of the corresponding
simple Lie algebra.
2.1 Properties of affine Kac-Moody algebras

Let \( A = (a_{ij})_{0 \leq i, j \leq n} \) be a generalized Cartan matrix of affine type, and \( \mathfrak{g}(A) \) be the corresponding Kac-Moody algebra. Recall that \( \mathfrak{g}(A) \) is generated by a set of Weyl generators

\[
\{ e_i, f_i, \alpha_i^\vee \mid i = 0, 1, 2, \ldots, n \}
\]

and a scaling element \( d \). One has the decomposition \( \mathfrak{g}(A) = \mathfrak{g}'(A) \oplus \mathbb{C}d \), with \( \mathfrak{g}'(A) \) being the derived algebra. The center of \( \mathfrak{g}(A) \) (or of \( \mathfrak{g}'(A) \)) is spanned by the canonical central element, say, \( c \), which satisfies

\[
c = \sum_{i=0}^{n} k_i^\vee \alpha_i^\vee.
\]

(2.1)

Here \( k_i^\vee \) are the dual Kac labels of \( \mathfrak{g}(A) \), i.e., the lowest positive integers that solve the linear equation \( \sum_{i=0}^{n} k_i^\vee a_{ij} = 0 \). For the sake of simplifying notations, we will write \( \mathfrak{g} = \mathfrak{g}(A) \) and \( \mathfrak{g}' = \mathfrak{g}'(A) \) below.

An arbitrary integer vector \( s = (s_0, s_1, \ldots, s_n) \in \mathbb{Z}^{n+1} \), with \( s_i \geq 0 \) but not all equal to 0, induces a gradation on \( \mathfrak{g}' \) by setting

\[
\deg e_i = s_i, \quad \deg f_i = -s_i, \quad \deg \alpha_i^\vee = 0.
\]

(2.2)

The following two gradations are of particular importance [7, 28]:

(i) the homogeneous/standard gradation

\[
\mathfrak{g}' = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}'_j \text{ induced by } s^0 = (1, 0, \ldots, 0);
\]

(2.3)

(ii) the principal/canonical gradation

\[
\mathfrak{g}' = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}'_j \text{ induced by } s^1 = (1, 1, \ldots, 1).
\]

(2.4)

Conventions like \( \mathfrak{g}'_{\geq 0} = \sum_{i \geq 0} \mathfrak{g}'_i \) and \( \mathfrak{g}'_{< 0} = \sum_{i < 0} \mathfrak{g}'_i \) will be used below.

Let \( E \) be the set of exponents of \( \mathfrak{g}' \). In \( \mathfrak{g}' \) there is a so-called principal Heisenberg subalgebra \( \mathfrak{s} \), which has a basis \( \{ c, \Lambda_j \in \mathfrak{g}'_j \mid j \in E \} \) such that

\[
[\Lambda_i, \Lambda_j] = \delta_{i,-j} i \cdot c.
\]

(2.5)

In particular, 1 is always an exponent, and \( \Lambda_1 = \nu \Lambda \) for some nonzero constant \( \nu \), where \( \Lambda = \sum_{i=0}^{n} e_i \). The element \( \Lambda \) induces the following decomposition of subspaces:

\[
\mathfrak{g}' = \mathfrak{s} + \text{Im ad}_\Lambda, \quad \mathfrak{s} \cap \text{Im ad}_\Lambda = \mathbb{C} c.
\]

(2.6)

This property is crucial in the construction of Drinfeld-Sokolov hierarchies.
2.2 Drinfeld-Sokolov hierarchies

Drinfeld and Sokolov’s original construction works on the centerless affine Lie algebra \( \bar{\mathfrak{g}} = \mathfrak{g}' / \mathbb{C} c \). This algebra is graded in the same way as for \( \mathfrak{g}' \). Similarly, the homogeneous and principal gradations are written respectively as

\[
\bar{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \bar{\mathfrak{g}}_j, \quad \bar{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}} \bar{\mathfrak{g}}^j.
\]

Clearly \( \bar{\mathfrak{g}}_0 = \check{\mathfrak{g}} \), which denotes the simple Lie algebra for the Cartan matrix \( \check{A} = (a_{ij})_{1 \leq i, j \leq n} \) of finite type.

According to the decomposition (2.6), the centralizer of \( \Lambda = \sum_{i=0}^{n} e_i \) in \( \bar{\mathfrak{g}} \) is the principal Heisenberg subalgebra \( \bar{s} = \mathfrak{s} / \mathbb{C} c \) of trivial center. This subalgebra contains a basis \( \{ \Lambda_j \in \bar{\mathfrak{g}}^j \mid j \in E \} \) chosen as before.

We use \( C^\infty(\mathbb{R}, W) \) to denote the set of smooth functions from \( \mathbb{R} \) to some linear space \( W \) (the space \( \mathbb{R} \) is not essential; it can be replaced by other 1-dimensional spaces such like the unit circle \( S^1 \)). Consider operators of the form

\[
\bar{L} = D + \Lambda + q, \quad q \in C^\infty(\mathbb{R}, \check{\mathfrak{g}} \cap \bar{\mathfrak{g}}^{\leq 0}),
\]

(2.7)

where \( D = d/dx \) with \( x \) being the coordinate of \( \mathbb{R} \). Observe that \( q \) is a smooth function taking value in the Borel subalgebra of the simple Lie algebra \( \check{\mathfrak{g}} \) generated by \( \alpha_i^\vee \) and \( f_i \) with \( i = 1, \ldots, n \). For operators of the form (2.7), there are gauge transformations defined by

\[
\bar{L} \mapsto e^{\text{ad}_N} \bar{L}, \quad N \in C^\infty(\mathbb{R}, \check{\mathfrak{g}} \cap \bar{\mathfrak{g}}^{\leq 0}).
\]

(2.8)

In other words, this is an action of the Lie group of the nilpotent subalgebra of \( \check{\mathfrak{g}} \), which has an \( n \)-dimensional orbit space.

The following proposition plays a fundamental role in the construction of Drinfeld-Sokolov hierarchies.

**Proposition 2.1** ([7]) There exists a function \( U \in C^\infty(\mathbb{R}, \check{\mathfrak{g}}^{< 0}) \) such that the operator

\[
\bar{L} = e^{-\text{ad}_U} \bar{L} \text{ has the form}
\]

\[
\bar{L} = D + \Lambda + H, \quad H \in C^\infty(\mathbb{R}, \check{\mathfrak{g}} \cap \bar{\mathfrak{g}}^{< 0}).
\]

(2.9)

Suppose \( \bar{U} \) also satisfies the above condition, then \( e^{-\text{ad}_U} e^{\text{ad}_U} = e^{\text{ad}_S} \) with some \( S \in C^\infty(\mathbb{R}, \check{\mathfrak{g}} \cap \bar{\mathfrak{g}}^{< 0}) \). Moreover, for different choices of \( U \), the function \( H \) differs by adding the total derivative of a differential polynomial in (components of) \( q \).

According to the above proposition, one can choose a function \( U \) and introduce a map

\[
\varphi : C^\infty(\mathbb{R}, \check{\mathfrak{g}}) \rightarrow C^\infty(\mathbb{R}, \check{\mathfrak{g}}),
\]

\[
X \mapsto e^{\text{ad}_U} X.
\]

(2.10)

**Definition 2.2** ([7]) The Drinfeld-Sokolov hierarchy associated to \( \check{\mathfrak{g}} \) and the zeroth vertex of its Dynkin diagram is the following family of partial differential equations

\[
\frac{\partial \mathcal{L}}{\partial t_j} = [-\varphi(\Lambda_j)_{\geq 0}, \mathcal{L}], \quad j \in E_+
\]

(2.11)
restricted to some equivalence class of $\mathcal{L}$ with respect to the gauge transformations (2.8). Here the subscript $\geq 0$ means the projection to $\mathfrak{g}_{\geq 0}$, and $E_+$ is the set of positive exponents.

Drinfeld-Sokolov hierarchies restricted to different gauge slices of $\mathcal{L}$ are equivalent up to a gauge transformation of the form (2.8). In particular, if the gauge slice is given by $q$ taking value in the Cartan subalgebra of $\mathfrak{g}$, then we call the corresponding hierarchy the modified Drinfeld-Sokolov hierarchy. This name is from the fact that in the modified case the first nontrivial equation in the hierarchy for $A_1^{(1)}$ is the modified KdV equation, which is related to the KdV equation by the Miura transformation, see Example A.1 below. One can refer to [32, 43, 17, 11] for equivalent versions of such modified hierarchies associated to untwisted affine Kac-Moody algebras.

Consider formal functionals of the form

$$\mathcal{F} = \int f(q, \partial_x q, \partial_x^2 q, \ldots) \, dx$$

that are invariant under the gauge transformations (2.8). Note that such a functional is not really an integral but formally defined up to addition of total derivatives to the density $f$. The gradient of a functional $\mathcal{F}$ with respect to $q$ is defined to be $\text{grad}_q \mathcal{F} \in C^\infty(\mathbb{R}, \mathfrak{g} \cap \mathfrak{g}_{\geq 0})$ such that

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \mathcal{F}(q+\epsilon \tilde{q}) = \int (\text{grad}_q \mathcal{F} | \tilde{q}) \, dx$$

for arbitrary $\tilde{q} \in C^\infty(\mathbb{R}, \mathfrak{g} \cap \mathfrak{g}_{\leq 0})$, where $(\cdot | \cdot)$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$.

There is a Poisson bracket between the gauge invariant functionals:

$$\{ \mathcal{F}, \mathcal{G} \}(q) = \int \left( \text{grad}_q \mathcal{F} | \left[ \text{grad}_q \mathcal{G}, D + \sum_{i=1}^n e_i + q \right] \right) \, dx. \quad (2.12)$$

**Theorem 2.3 ([7])** The Drinfeld-Sokolov hierarchy (2.11) can be written in a Hamiltonian form as

$$\frac{\partial \mathcal{F}}{\partial t_j} = \{ \mathcal{F}, \mathcal{H}_j \}, \quad j \in E_+, \quad (2.13)$$

where the Hamiltonians are

$$\mathcal{H}_j = \int (-\Lambda_j | H) \, dx \quad (2.14)$$

with $H$ given in Proposition [2.4].

**Remark 2.4** When the affine Lie algebra $\mathfrak{g}$ is untwisted, the Drinfeld-Sokolov hierarchy possesses another Hamiltonian structure that is compatible with (2.13). In other words, it is a hierarchy of bi-Hamiltonian systems. The bi-Hamiltonian structure was shown to be characterized by a semisimple Frobenius manifold [8] on the orbit space of the corresponding Weyl group together with a class of constant central invariants [9]. □
In [7], Drinfeld and Sokolov did not consider tau functions of their hierarchies. Instead, they proposed a scheme to represent their hierarchies into Lax equations of scalar pseudo-differential operators. For example, the hierarchy (2.11) for the affine Kac-Moody algebra of type $A^{(1)}_1$ can be written equivalently to the KdV hierarchy (1.1). In summary, Drinfeld and Sokolov obtained the Lax representations for the hierarchies (2.11) associated to the affine Kac-Moody algebras of types $A^{(1)}_n$, $B^{(1)}_n$, $C^{(1)}_n$, $A^{(2)}_{2n}$, $A^{(2)}_{2n-1}$ and $D^{(2)}_{n+1}$; for the hierarchy of type $D^{(1)}_n$, a Lax representation was partially given in [7], and completed by us in [35] with the help of certain extended pseudo-differential operators. Generally speaking, based on the Lax representations, tau function of such hierarchies can be introduced via Hamiltonian densities chosen similarly as for the KdV hierarchy, see, for example, [35, 4] and Examples 5.2–5.3 below. However, this is a case-by-case method and may fail to work in general. In the next section we will propose a way to construct tau functions for all Drinfeld-Sokolov hierarchies.

### 3 Tau function of Drinfeld-Sokolov hierarchies

Given an operator $\mathcal{L}$ in (2.7), the Hamiltonian densities in (2.14) are defined up to addition of the total derivative of differential polynomials in $q$, which depend on the function $U$ in Proposition 2.1. Our idea is to fix the function $U$ appropriately such that the Hamiltonian densities are tau-symmetric, hence a tau function can be defined. To this end, we will first reformulate the Drinfeld-Sokolov construction on the derived algebra $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}c$, as inspired by [24].

#### 3.1 Reformulation of Drinfeld-Sokolov hierarchies

Recall that the operator in (2.7) is just

$$\mathcal{L} = D + \Lambda + q, \quad q \in C^\infty(\mathbb{R}, \mathfrak{g} \cap \mathfrak{g}^{<0}).$$

(3.1)

In comparison with Proposition 2.1, we have the following

**Proposition 3.1** Given an operator $\mathcal{L}$ as (3.1), there is a unique function $U \in C^\infty(\mathbb{R}, \mathfrak{g}^{<0})$ satisfying the following two conditions

**(i)** The operator $\tilde{\mathcal{L}} = e^{-\text{ad}U} \mathcal{L}$ has the form

$$\tilde{\mathcal{L}} = D + \Lambda + H, \quad H \in C^\infty(\mathbb{R}, \mathfrak{s} \cap \mathfrak{g}^{<0});$$

(3.2)

**(ii)** For every positive exponent $j \in E_+$, the central part of $e^{\text{ad}_c \Lambda_j}$ vanishes, namely

$$(e^{\text{ad}_c \Lambda_j})_c = 0.$$

(3.3)

Here the subscript “$c$” means to take the coefficient of the center $c$ with respect to the decomposition $\mathfrak{C} \alpha_1^\vee \oplus \cdots \oplus \mathfrak{C} \alpha_n^\vee \oplus \mathbb{C}c$ of the Cartan subalgebra of $\mathfrak{g}'$.

Moreover, both $U$ and $H$ are differential polynomials in $q$. 

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Proof. One writes $e^{ad_{\mathcal{L}}} = \mathcal{L}$ to

$$e^{\sum_{k \leq -1} U_k \left( D + \Lambda + \sum_{k \leq -1} H_k \right)} = D + \Lambda + \sum_{k \leq 0} q_k,$$

where $q_k, U_k, H_k$ take value in $\mathfrak{g}^{\leq k}$. By comparing the homogeneous terms we have

$$[U_{-1}, \Lambda] = q_0, \quad (3.5)$$

$$H_{k+1} + [U_k, \Lambda] = *, \quad k = -2, -3, -4, \ldots. \quad (3.6)$$

Here for every $k$ the right hand side of (3.6) depends on $q_k, H_i, U_i$ ($i > k + 1$) and $U_i$ ($i > k$).

First of all, equation (3.5) has a unique solution $U_{-1}$, for which equally (3.3) with $j = 1$ is valid automatically. When $k < -1$, by virtue of the decomposition (2.6), the functions $H_{k+1}$ and $U_k$ can be solved recursively from (3.6). In more details, suppose $H_i$ ($i > k + 1$) and $U_i$ ($i > k$) are given, then $H_{k+1}$ is determined uniquely due to the decomposition (2.6). In finding $U_k$ there are two cases: first, the function $U_k$ is unique whenever $k \not\in E$; second, if $k \in E$, then $U_k$ is determined up to addition of a multiple of $\Lambda_k$. But the freedom in the latter case is fixed precisely by the condition (3.3) with $j = -k$. Therefore the proposition is proved.

Remark 3.2 In [24] Hollowood and Miramontes fix the function $U$ in a different way. They let $U$ take value in $s^\perp \cap \mathfrak{g}^{<0}$, where $s^\perp$ is the orthogonal complement of $s$ with respect to the standard bilinear form on $\mathfrak{g}'$, see Proposition 2.1 in [24]. For such a $U$, the corresponding Hamiltonian densities in (2.14) are not what we look for.

Lemma 3.3 The function $H$ in Proposition 3.1 is invariant with respect to the gauge transformations (2.8).

Proof. Assuming $\mathcal{L}$ to be of the form (3.1), we have $U$ and $H$ determined by Proposition 3.1. For any $\mathcal{L} = e^{ad_{\mathcal{L}}}$ with $N \in C^\infty(\mathbb{R}, \mathfrak{g} \cap \mathfrak{g}^{<0})$, one has

$$\tilde{\mathcal{L}} = e^{ad_{\tilde{U}}} (D + \Lambda + H), \quad e^{ad_{\tilde{U}}} = e^{ad_{\mathcal{L}}} e^{ad_{\tilde{U}}},$$

where $\tilde{U} \in C^\infty(\mathbb{R}, \mathfrak{g}^{<0})$. We need to show that $\tilde{U}$ also satisfies the condition (3.3). In fact, note $[N, X]_c = 0$ for any $X \in \mathfrak{g}'$, hence

$$(e^{ad_{\tilde{U}}} \Lambda_j)_c = e^{ad_{\mathcal{L}}} (e^{ad_{\tilde{U}}} \Lambda_j)_c = 0, \quad j \in E_+.$$ The lemma is proved.

Given an operator $\mathcal{L}$ in (3.1), henceforth we fix the functions $U$ and $H$ as in Proposition 3.1. Similar to (2.10) we now have

$$\varphi : C^\infty(\mathbb{R}, \mathfrak{g}') \to C^\infty(\mathbb{R}, \mathfrak{g}'),$$

$$X \mapsto e^{ad_{X}},$$

(3.7)

By virtue of the condition (3.3), the evolutionary equations (2.11) are still well defined with $\tilde{\mathcal{L}}$ replaced by $\mathfrak{g}'$ and simultaneously the subscript “$\geq 0$” becomes the projection $\mathfrak{g}' \to \mathfrak{g}'_{\geq 0}$. These equations compose the Drinfeld-Sokolov hierarchy when restricted to the gauge equivalence class of $\mathcal{L}$ with respect to the transformations (2.8).
3.2 Definition of tau function

In order to define tau function of the Drinfeld-Sokolov hierarchy (2.11), we introduce a nonlocal function
\[ \Omega = - \int x H(q; \partial_x q, \partial_x^2 q, \ldots) \, dx. \] (3.8)
Namely, \( \Omega \) satisfies
\[ \frac{\partial \Omega}{\partial x} = -H, \] (3.9)
and takes the form
\[ \Omega = \sum_{j \in E_+} \omega_j \Lambda_{-j}, \] (3.10)
where \( \omega_j \) are scalar functions determined up to addition of constants.

**Lemma 3.4** For the scalar functions \( \omega_j \) given above, all derivatives \( \partial \omega_j / \partial t_i \) are differential polynomials in \( q \), and they satisfy
\[ \frac{\partial \omega_j}{\partial t_i} = \frac{\partial \omega_i}{\partial t_j}, \quad i, j \in E_+. \] (3.11)

**Proof** Recalling \( L = e^{ad_U} (D + \Lambda + H) \), the following identity can be verified straightforwardly (see, for example, Lemma A.1 in [40]):
\[ \frac{\partial L}{\partial t_j} = e^{ad_U} \frac{\partial H}{\partial t_j} + [\nabla_{t_j, U} U, L] \] (3.12)
where
\[ \nabla_{t_j, U} U = \sum_{m \geq 0} \frac{1}{(m + 1)!} (ad_U)^m \frac{\partial U}{\partial t_j}. \]
This together with (2.11) leads to
\[ e^{ad_U} \frac{\partial H}{\partial t_j} + [\nabla_{t_j, U} U - \varphi(\Lambda_j)_{<0}, L] = [-\varphi(\Lambda_j), L], \]
namely,
\[ \frac{\partial H}{\partial t_j} + [e^{-ad_U} (\nabla_{t_j, U} U - \varphi(\Lambda_j)_{<0}), D + \Lambda + H] = \frac{\partial \omega_j}{\partial x} \cdot c. \] (3.13)
Here on the right hand side we have used the condition (3.3).

According to the decomposition (2.6), we write
\[ e^{-ad_U} (\nabla_{t_j, U} U - \varphi(\Lambda_j)_{<0}) = G + \tilde{G} \]
with \( G \) and \( \tilde{G} \) taking value in \( s \cap g^{<0} \) and in \( \text{Im} \, \text{ad}_\Lambda \cap g^{<0} \) respectively. Equation (3.13) splits into three parts:
\[ \frac{\partial H}{\partial t_j} - \frac{\partial G}{\partial x} = 0, \] (3.14)
\[ [G, \Lambda]_c = \frac{\partial \omega_j}{\partial x}, \quad (3.15) \]

\[- \frac{\partial \tilde{G}}{\partial x} + [\tilde{G}, \Lambda + H] = 0. \quad (3.16)\]

Firstly, equation (3.14) together with (3.9) implies
\[ \frac{\partial \Omega}{\partial t_j} = G, \quad j \in E_+. \quad (3.17) \]
Hence
\[ \frac{\partial \omega_i}{\partial t_j} = \left[ \Lambda_i, \frac{\partial \Omega}{\partial t_j} \right]_c = [\Lambda_i, G]_c \quad (3.18) \]
are differential polynomials in \( q \).

In particular, taking \( i = 1 \) in (3.18), from \( \Lambda_1 = \nu \Lambda \) and (3.15) it follows that
\[ \frac{\partial \omega_1}{\partial t_j} = \nu \frac{\partial \omega_j}{\partial x}, \quad j \in E_+. \quad (3.19) \]
Hence for \( i, j \in E_+ \), we obtain
\[ \frac{\partial^2 \omega_j}{\partial t_i \partial x} = \frac{\partial^2 \omega_i}{\partial t_j \partial x}. \]
Both sides of the equality are total derivatives of differential polynomials in \( q \) with respect to \( x \), hence it leads to (3.11) by integration. The lemma is proved. \( \square \)

Given a solution of the hierarchy (2.11), there locally exists a smooth function \( \tau \) of \( t = (t_j)_{j \in E_+} \) such that
\[ \omega_j = \frac{\partial \log \tau}{\partial t_j}, \quad j \in E_. \quad (3.20) \]
It follows from Lemma 3.3 that \( \log \tau \) is independent of the choice of gauge equivalence slice of \( L \).

**Definition 3.5** The function \( \tau \) satisfying (3.20) is called tau function of the Drinfeld-Sokolov hierarchy (2.11).

Let us consider how the tau function is related to the densities of Hamiltonians in Theorem 2.3. The Hamiltonian densities are
\[ h_j = (-\Lambda_j | H), \quad j \in E_+. \quad (3.21) \]
with \( H \) given in Proposition 3.1, and they are invariant with respect to gauge transformations. Recall the definition of \( \omega_j \) in (3.10), one has
\[ \frac{1}{j} \frac{\partial \omega_j}{\partial x} = (\Lambda_j | -H) (\Lambda_j | \Lambda_{-j}) = \frac{h_j}{(\Lambda_j | \Lambda_{-j})}. \quad (3.22) \]
This together with (3.11) leads to
\[ \frac{j}{(\Lambda_j | \Lambda_{-j})} \frac{\partial h_j}{\partial t_i} = \frac{i}{(\Lambda_i | \Lambda_{-i})} \frac{\partial h_i}{\partial t_j}, \quad i, j \in E_. \quad (3.23) \]
It means that a family of tau-symmetric Hamiltonian densities of the Drinfeld-Sokolov hierarchy is found.

Equation (3.22) can be written as

$$\frac{\partial^2 \log \tau}{\partial x \partial t_j} = \frac{j}{(\Lambda_j | \Lambda_{-j})} h_j, \quad j \in E_+,$$

which is just (1.5). Hence the tau function of the Drinfeld-Sokolov hierarchy can be defined equivalently by

$$\frac{\partial^2 \log \tau}{\partial t_i \partial t_j} = \frac{j}{(\Lambda_j | \Lambda_{-j})} \partial^{-1} \left(-\Lambda_j \left| \frac{\partial H}{\partial t_i} \right) \right), \quad i, j \in E_+. \tag{3.25}$$

The right hand side is independent of the choice of nondegenerate invariant symmetric bilinear form, in which $\partial H/\partial t_i$ is a total derivative (see (3.14)), and the integral constant is assumed to be zero. Thus $\log \tau$ is determined up to the addition of a linear function of the time variables.

It is natural to ask what is the relation between the above tau function and those tau functions given in the literature. This question will be dealt with below in Section 5 and the appendix, for the reason that more notations are needed there.

### 3.3 Zero-curvature representation for Drinfeld-Sokolov hierarchies

We want to represent the Drinfeld-Sokolov hierarchies in a zero-curvature form, which will be applied in the next section.

Given an operator $\mathcal{L}$ as (3.1), the functions $U$ in Proposition 3.1 and $\Omega$ in (3.10) take value in $g^{<0}$, then the following element of the Lie group of $g^{<0}$ is well defined

$$\Theta = e^U e^\Omega. \tag{3.26}$$

This can be considered as a formal series that converges with respect to a topology induced by the principal gradation on $g$. Note that generally the function $\Theta$ may not be a differential polynomial in $q$.

Recalling $\Lambda_1 = \nu \Lambda$ with constant $\nu$, we have

$$\mathcal{L} = e^{adv} e^{adn} \left(D + \Lambda + \frac{\omega_1}{\nu} c\right) = \Theta \left(D + \Lambda + \frac{\omega_1}{\nu} c\right) \Theta^{-1}. \tag{3.27}$$

Note that $\Theta$ is determined by $\mathcal{L}$ up to multiplication to the right by $\exp \left(\sum_{j \in E_+} c_j \Lambda_{-j}\right)$ with constants $c_j$. The gauge transformation (2.3) of $\mathcal{L}$ induces a transformation of the dressing operator as $\Theta \mapsto e^N \Theta$. Hence there is a gauge slice of $\mathcal{L}$ such that the dressing operator takes the form (see [24])

$$\Theta = e^V, \quad V \in C^\infty(\mathbb{R}, g_{<0}). \tag{3.28}$$

In the sequel we will fix such a gauge slice.
Note ∂/∂t_1 = ν ∂/∂x. We let

\[ L_1 = \Theta \left( \frac{\partial}{\partial t_1} + \Lambda_1 \right) \Theta^{-1}, \]

namely, \( L_1 = \nu L - \omega_1 c \).

**Lemma 3.6** The evolutionary equations in (2.11) are equivalent to

\[ \frac{\partial L_1}{\partial t_j} = [- (\Theta \Lambda_j \Theta^{-1})_{\geq 0}, L_1], \quad j \in E_. \]  

(3.30)

**Proof** Since \( \Theta \Lambda_j \Theta^{-1} = \varphi(\Lambda_j) - \omega_j c \), then the off-center part of (3.30) coincides with (2.11), while the center part is just (3.19) that can be derived from (2.11). Thus the lemma is proved. \( \square \)

The result of the following lemma have existed in [25, 36].

**Lemma 3.7** The dressing operator \( \Theta \) in (3.28) satisfies

\[ \frac{\partial \Theta}{\partial t_j} = (\Theta \Lambda_j \Theta^{-1})_{< 0} \Theta, \quad j \in E_. \]  

(3.31)

**Proof** Equations (3.30) can be written as

\[ \frac{\partial L_1}{\partial t_j} = [(\Theta \Lambda_j \Theta^{-1})_{< 0}, L_1], \quad j \in E_. \]  

(3.32)

Substitute into it with (3.29), then one has

\[ \left[ \frac{\partial \Theta}{\partial t_j} \Theta^{-1} - (\Theta \Lambda_j \Theta^{-1})_{< 0}, L_1 \right] = 0, \quad j \in E_. \]

For \( j \in E_+ \), denote

\[ \Delta(j) = \Theta^{-1} \frac{\partial \Theta}{\partial t_j} - \Theta^{-1} (\Theta \Lambda_j \Theta^{-1})_{< 0} \Theta, \]

(3.33)

then \( \Delta(j) \in C^\infty(\mathbb{R}, g'_{< 0}) \) and

\[ \left[ \Delta(j), \frac{\partial}{\partial t_1} + \Lambda_1 \right] = 0. \]

(3.34)

According to the decomposition (2.6), we derive

\[ \Delta(j) = \sum_{i \in E_+} c_i \Lambda_{-i} \]

with some constants \( c_i \). But equation (3.34) is independent of \( q \), hence by letting \( q = 0 \) (which implies \( U = 0 \) and \( \Omega = 0 \)) one obtains \( \Delta(j) = 0 \). Thus we arrive at (3.31) and conclude the lemma. \( \square \)

Conversely, starting from (3.31) it is easy to derive equations (3.30), which yields the Drinfeld-Sokolov hierarchy (2.11).
With the operator $\Theta$ in (3.28), let us introduce

$$L_j = \Theta \left( \frac{\partial}{\partial t_j} + \Lambda_j \right) \Theta^{-1}, \quad j \in E_+.$$  

(3.35)

By using Lemma 3.7 one can write these operators as

$$L_j = \frac{\partial}{\partial t_j} + \Lambda_j + q(j), \quad j \in E_+,$$

(3.36)

where $q(j) = (\Theta \Lambda_j \Theta^{-1})_{\geq 0} - \Lambda_j$. Thanks to (3.26) and (3.3), one sees

$$q(j)c = -\omega_j, \quad j \in E_+.$$  

(3.37)

Clearly, the operators (3.12) satisfy

$$[L_i, L_j] = 0, \quad i, j \in E_+.  

(3.38)

This gives the zero-curvature representation for the Drinfeld-Sokolov hierarchy (3.30). In fact, it also confirms the commutativity between the flows in (3.30).

4 Virasoro symmetries

It was shown [25] that the (generalized) Drinfeld-Sokolov hierarchy associated to an untwisted affine Kac-Moody algebra possesses certain additional symmetries that obey a Virasoro commutation relation. Now we want to revise the construction in [25] and derive such Virasoro symmetries for Drinfeld-Sokolov hierarchy associated to an arbitrary affine Kac-Moody algebra. Our aim is to represent these Virasoro symmetries via the tau function defined in the previous section.

4.1 Kac-Moody-Virasoro algebras and their representations

Based on [28, 42], we review the extension of the affine algebra $\mathfrak{g}(A)$ to a Kac-Moody-Virasoro algebra, as well as some properties of their representations. Suppose the Cartan matrix $A = (a_{ij})_{0 \leq i,j \leq n}$ is of affine type $X_N^{(r)}$; the lowest positive integers $k_i$ satisfying $\sum_{j=0}^n a_{ij} k_j = 0$ are called the Kac labels of $\mathfrak{g}(A)$. The set of gradations on $\mathfrak{g}(A)$ is

$$\Gamma = \{(s_0, s_1, \ldots, s_n) \in \mathbb{Z}^{n+1} \mid s_i \geq 0, s_0 + s_1 + \cdots + s_n > 0\}.$$  

(4.1)

For every $s = (s_0, s_1, \ldots, s_n) \in \Gamma$, denote

$$N_s = \sum_{i=0}^n k_i s_i.$$  

(4.2)

In particular, recalling the homogeneous and the principal gradations (2.3)–(2.4), we have $N_{s_0} = k_0$, and $N_{s_1} = h$ being the Coxeter number of $\mathfrak{g}(A)$.

Let $\mathcal{G}$ be the simple Lie algebra of type $X_N$, on which there is a diagram automorphism of order $r$. Given an integer vector $s = (s_0, s_1, \ldots, s_n) \in \Gamma$, it induces a $\mathbb{Z}/rN_s \mathbb{Z}$-gradation
\[ \mathcal{G} = \bigoplus_{k=0}^{rN_s-1} \mathcal{G}_k \] (see §8.6 of [28] for details). The Kac-Moody algebra \( \mathfrak{g}(A) \) graded by \( s \) can be realized as

\[
\mathfrak{g}(A; s) = \bigoplus_{k \in \mathbb{Z}} (\lambda^k \otimes \mathcal{G}_{k \mod rN_s}) \oplus \mathbb{C} \rightarrow c \oplus \mathbb{C}d_0(s),
\]

in which \( c \) is the canonical central element, and the Lie bracket between \( X(k), Y(k) \in \lambda^k \otimes \mathcal{G}_{k \mod rN_s} \) and \( d_0(s) \) is defined by

\[
[X(k), Y(l)] = kX(k + rN_sl) + \delta_{k, l} k \mathbb{C}c,
\]

\[
[d_0(s), X(k)] = kX(k).
\]

Here \((\cdot \mid \cdot)_0\) is the standard invariant symmetric bilinear form on \( \mathcal{G} \). An element \( X(k) \) will be written more precisely as \( X(k; s) \) whenever it is necessary to distinguish the gradation \( s \) from others.

On the derived algebra \( \mathfrak{g}'(A; s) \) of \( \mathfrak{g}(A; s) \) one introduces a family of derivations \( d_l^{(s)}(l \in \mathbb{Z}) \) such that

\[
[d_l^{(s)}, X(k)] = kX(k + rN_sl), \quad [d_l^{(s)}, c] = 0,
\]

\[
[d_k^{(s)}, d_l^{(s)}] = rN_sd_{k+l}^{(s)},
\]

These derivations generate an infinite-dimensional Lie algebra, say, \( \mathfrak{d}^{(s)} \), of Virasoro type. Thus a Kac-Moody-Virasoro algebra \( \mathfrak{g}^{(s)} \simeq \mathfrak{g}'(A; s) \) is constructed.

Following the notations in §8.3 of [28], the simple Lie algebra \( \mathcal{G} \) contains certain elements written as \( E_i, F_i \) and \( H_i \) with \( i = 0, 1, \ldots, n \). These elements give a set of Weyl generators of \( \mathfrak{g}'(A; s) \) as follows: for \( i = 0, 1, \ldots, n \),

\[
e_i^{(s)} = E_i(s_i), \quad f_i^{(s)} = F_i(-s_i), \quad \alpha_i^{(s)} = H_i(0) + \frac{k_is_i}{k_i^\vee N_sc},
\]

with \( k_i \) and \( k_i^\vee \) being the Kac labels and the dual Kac labels respectively. Hence \( d_k^{(s)} \) can be considered as derivations on \( \mathfrak{g}'(A) \).

For two arbitrary gradations \( s, s' \in \Gamma \), there is a natural isomorphism between \( \mathfrak{g}'(A; s) \) and \( \mathfrak{g}'(A; s') \) induced by

\[
e_i^{(s)} \mapsto e_i^{(s')}, \quad f_i^{(s)} \mapsto f_i^{(s')}, \quad i = 0, 1, \ldots, n.
\]

Up to such an isomorphism, one has the following lemma.

**Lemma 4.1 ([42])** Given two gradations \( s, s' \in \Gamma \), the corresponding derivations on \( \mathfrak{g}'(A) \) satisfy

\[
d_k^{(s+s')} = d_k^{(s)} + d_k^{(s')},
\]

\[
[d_k^{(s)}, d_l^{(s')}] = rN_sd_{k+l}^{(s)} - rN_sd_{k+l}^{(s')},
\]

for all integers \( k \) and \( l \).
In the Cartan subalgebra of $g'(A; s)$, one introduces the following elements

$$h_s = \left(\alpha_1^{\vee(s)}, \ldots, \alpha_n^{\vee(s)}\right)^T \hat{A}^{-1} (s_1, \ldots, s_n)^T,$$

$$H_s = \left(H_1(0), \ldots, H_n(0)\right)^T \hat{A}^{-1} (s_1, \ldots, s_n)^T,$$

where $\hat{A} = (a_{ij})_{1 \leq i,j \leq n}$, and the superscript “$T$” means the transpose of matrices.

**Lemma 4.2 (Lemma 2.4 in [42])** Let $s^0 = (1, 0, \ldots, 0) \in \Gamma$ be the homogeneous gradation. For any $s \in \Gamma$ it holds that

$$d_k^s = \begin{cases} N_s d_k^{s^0} + h_s, & k = 0; \\ N_s d_k^{s^0} + H_s (rk_0; s^0), & k \neq 0. \end{cases}$$

(4.13)

Consider highest weight representations of affine Kac-Moody algebras. Denote $x = (x_j)_{j \in E_+}$, then on the Fock space $\mathbb{C}[[x]]$ one defines an action of the principal Heisenberg subalgebra $\mathfrak{s}$ of $\mathfrak{g}'(A)$ by

$$c \mapsto 1; \quad \Lambda_j \mapsto -\frac{\partial}{\partial x_j}, \quad \Lambda_{-j} \mapsto -j x_j, \quad j \in E_+.$$  

(4.14)

This action generates a highest weight representation of $\mathfrak{s}$.

**Theorem 4.3 (Theorem 14.6 in [28])** Suppose the Dynkin diagram of the affine Lie algebra $\mathfrak{g}'(A)$ is simply-laced or twisted, then the action of the Heisenberg subalgebra $\mathfrak{s}$ on the Fock space $\mathbb{C}[[x]]$ given by (4.14) can be lifted to a basic representation $L(\Lambda_0)$ of $\mathfrak{g}'(A)$ on $\mathbb{C}[[x]]$, and, the highest weight vector is 1.

Recall the principal gradation $s^1 = (1, 1, \ldots, 1) \in \Gamma$, and realize $\mathfrak{g}(A)$ as $\mathfrak{g}(A; s^1)$. The following result is implied by Theorems 3.2 and 5.1 in [42].

**Theorem 4.4** Under the same assumption as in Theorem 4.3, the action of $\mathfrak{g}'(A)$ on $\mathbb{C}[[x]]$ given there can be uniquely extended to a $\mathfrak{d}^{(s^1)} \ltimes \mathfrak{g}'(A; s^1)$-action by setting

$$d_k^{s^1} \mapsto -L_k(\partial/\partial x; x), \quad k \in \mathbb{Z},$$

(4.15)

where

$$L_k(\partial/\partial x; x) = \frac{1}{2} \sum_{j \in E_+} \left(p_{rk_j} - p_j + p_{-j} p_{rk_j + j}\right)$$

(4.16)

with $h = N_{s^1}$ being the Coxeter number, and

$$p_j = \frac{\partial}{\partial x_j}, \quad p_{-j} = j x_j, \quad j \in E_+.$$
4.2 Virasoro symmetries represented via tau function

We proceed to construct Virasoro symmetries for the Drinfeld-Sokolov hierarchy, and consider their action on the tau function defined in the previous section.

Let \( g \) be an arbitrary affine Kac-Moody algebra. For the homogeneous and the principal gradations \( s^0 \) and \( s^1 \), recalling \( N_{s^0} = k_0 \) and \( N_{s^1} = h \), we normalize the derivations on \( g' \) as:

\[
d_k = -\frac{1}{r N_{s^0}} d_k^{(s^0)} = -\frac{1}{rk_0} d_k^{(s^0)},
\]

\[
d'_k = -\frac{1}{r N_{s^1}} d_k^{(s^1)} = -\frac{1}{rh} d_k^{(s^1)}.
\]

(4.17)

(4.18)

Thanks to (4.7), these normalized derivations satisfy

\[
[d_k, d_l] = (k - l) d_{k+l}, \quad [d'_k, d'_l] = (k - l) d'_{k+l}, \quad k, l \in \mathbb{Z}.
\]

(4.19)

From Lemma 4.2 it follows that

\[
d_k - d'_k = \frac{1}{r h} \left( d_k^{(s^1)} - \frac{h}{k_0} d_k^{(s^0)} \right) = \begin{cases} 
\frac{1}{rh} h s^1, & k = 0; \\
\frac{1}{rh} H_{s^1}(rk_0k; s^0), & k \neq 0.
\end{cases}
\]

(4.20)

Clearly \( d_k - d'_k \in g'_{r k_0 k} \) in the notations for the homogeneous gradation (2.3).

We fix the generators \( \Lambda_j \in g' \) with \( j \in E \) of the principle Heisenberg algebra \( s \) such that

\[
[\Lambda_i, \Lambda_j] = \delta_{i, -j} i \cdot c, \quad [d'_k, \Lambda_j] = -\frac{j}{rh} \Lambda_{j+rhk} \text{ for all } k \in \mathbb{Z}.
\]

(4.21)

Given any integer \( k \), introduce

\[
\hat{B}_k = d'_k - \sum_{i \in E_+} \frac{ij}{rh} \Lambda_{i+rhk} + \frac{1}{2rh} \sum_{i, j \in E_+ \atop i+j = -rkh} ij t_i t_j \cdot c,
\]

(4.22)

where the third term on the right hand side exists only if \( k < 0 \). It is straightforward to check

\[
\left[ \hat{B}_k, \frac{\partial}{\partial t_j} + \Lambda_j \right] = 0, \quad j \in E_+, \quad k \in \mathbb{Z}.
\]

(4.23)

Recalling the dressing operator \( \Theta \) in (3.28), we let

\[
B_k = \Theta \hat{B}_k \Theta^{-1} - d_k, \quad k \in \mathbb{Z}.
\]

(4.24)

Note \( B_k \in g' \) for all \( k \), and that (4.23) leads to

\[
[B_k + d_k, \mathcal{L}_j] = 0, \quad j \in E_+, \quad k \in \mathbb{Z}.
\]

(4.25)

For \( k \geq -1 \), we define a class of evolutionary equations as follows

\[
\frac{\partial \mathcal{L}_1}{\partial \beta_k} = [-(B_k)_{<0}, \mathcal{L}_1].
\]

(4.26)
The right hand side can also be rewritten as \( \{(B_k)_{\geq 0} + d_k, \mathcal{L}_1\} \), hence the equations (4.26) are well defined by comparing the degrees with respect to the homogeneous and the principal gradations, as well as by an analysis like in the proof of Lemma 3.6 for the central part. The flows (4.26) are assumed to commute with \( \partial/\partial t = \nu \partial/\partial x \).

With the same method as to prove Lemma 3.7, we have the following useful formulae

\[
\frac{\partial \Theta}{\partial \beta_k} = -(B_k)_{< 0} \Theta, \quad k \geq -1. \tag{4.27}
\]

This formulae lead immediately to

\[
\frac{\partial L_j}{\partial \beta_k} = \left[(B_k)_{< 0} + d_k, L_j\right], \quad k \geq -1, \quad j \in E_. \tag{4.28}
\]

**Proposition 4.5** The following assertions are true (cf. Propositions 3.3 and 3.4 in [25]):

(i) The flows (4.26) commute with those in (3.30), or equivalently,

\[
\left[ \frac{\partial}{\partial \beta_k}, \frac{\partial}{\partial t_j} \right] \Theta = 0, \quad k \geq -1, \quad j \in E_. \tag{4.29}
\]

(ii) For \( k, l \geq -1 \),

\[
\left[ \frac{\partial}{\partial \beta_k}, \frac{\partial}{\partial \beta_l} \right] \Theta = (l - k) \frac{\partial \Theta}{\partial \beta_{k+l}}. \tag{4.30}
\]

**Proof** The first assertion follows from a straightforward calculation by using (4.27) and (3.31). Let us check the second assertion, which is more nontrivial.

Using (4.19) and (4.21), one can verify

\[
[\hat{B}_k, \hat{B}_l] = (k - l)\hat{B}_{k+l}.
\]

Since

\[
\frac{\partial}{\partial \beta_k} \frac{\partial}{\partial \beta_l} \Theta = - \frac{\partial}{\partial \beta_k} (B_l)_{< 0} \Theta
\]

\[
= (B_l)_{< 0} (B_k)_{< 0} \Theta + [(B_k)_{< 0}, \Theta \hat{B}_l \Theta^{-1}]_{< 0} \Theta
\]

\[
= (B_l)_{< 0} (B_k)_{< 0} \Theta + [(B_k)_{< 0}, B_l + d_l]_{< 0} \Theta,
\]

then

\[
\left[ \frac{\partial}{\partial \beta_k}, \frac{\partial}{\partial \beta_l} \right] \Theta \cdot \Theta^{-1}
\]

\[
= [(B_l)_{< 0}, (B_k)_{< 0}] + [(B_k)_{< 0}, B_l + d_l]_{< 0} - [(B_l)_{< 0}, B_k + d_k]_{< 0}
\]

\[
= [B_k, B_l]_{< 0} + [(B_k)_{< 0}, d_l]_{< 0} + [d_k, (B_l)_{< 0}]_{< 0}
\]

\[
= ([B_k + d_k, B_l + d_l] - [d_k, d_l])_{< 0}
\]

\[
= ((k - l)(B_{k+l} + d_{k+l}) - (k - l)d_{k+l})_{< 0}
\]

\[
= (k - l)(B_{k+l})_{< 0}
\]
\[(l - k) \frac{\partial \Theta}{\partial \beta_{k+l}} \Theta^{-1}. \quad (4.31)\]

Note that in the third equality we have used \([d_k, B_{\geq 0}]_{\leq 0} = 0\) for any element \(B \in \mathfrak{g}'\) and integer \(k \geq -1\). The proposition is proved. \(\square\)

The commutativity (4.29) of flows means that equations (4.23) define symmetries for the Drinfeld-Sokolov hierarchy (2.11). Such symmetries are called the Virasoro symmetries due to the commutation relation (4.30).

Now we arrive at the main result of the present section.

**Theorem 4.6** For \(k \geq -1\), the tau function in (3.20) of the Drinfeld-Sokolov hierarchy satisfies

\[
\frac{\partial \log \tau}{\partial \beta_k} = \frac{1}{r h} O'_k + \frac{1}{2 r h} \sum_{i+j=rhk} \frac{\partial \log \tau}{\partial t_i} \frac{\partial \log \tau}{\partial t_j} + \frac{1}{r h} \sum_{i \geq j} t_i t_j + \delta_{k,0} C_A. \quad (4.32)
\]

Here \(1_{k \geq 1}\) equals 1 whenever \(k \geq 1\) and vanishes otherwise,

\[O'_k = r h (e^{ad_U} d_k - d_k) c - (e^{ad_H} s^0)) c \quad (4.33)\]

with \(U\) given in Proposition (7.1) and \(H_s^i(r_k^0k; s^0)\) introduced from (4.12), \(C_A\) is a constant depending on the Cartan matrix \(A\), and all indices \(i, j \in E_+\). Note that \(O'_k\) are independent of the gauge transformations (2.8).

**Proof** Recall (3.36). The equations in (4.28) yield

\[
\frac{\partial q(j)_c}{\partial \beta_k} = -\frac{\partial (B_k)_c}{\partial t_j}, \quad k \geq -1, j \in E_+. \quad (4.34)
\]

The left hand side is

\[
-\frac{\partial \omega_j}{\partial \beta_k} = -\frac{\partial^2 \log \tau}{\partial \beta_k \partial t_j}; \quad (4.35)
\]

let us compute \((B_k)_c\) on the right hand side of (4.31).

Recalling (3.10) and (4.21), it is straightforward to calculate

\[
e^{ad_U} \tilde{B}_k e^{ad_U} = e^{ad_U} \left( d'_k - \sum_{i \in E_+} \frac{it_i}{r h} \Lambda_{-i+rhk} + \frac{1}{2 r h} \sum_{i+j=rhk} ijt_i t_j \cdot c \right) = d'_k - \sum_{i \in E_+} \frac{it_i}{r h} \Lambda_{-i+rhk} + \frac{1}{2 r h} \sum_{i+j=rhk} \omega_{ijt} \cdot c - \sum_{i \in E_+} \frac{it_i}{r h} \Lambda_{i+rhk} - \sum_{i \geq j} \frac{it_i}{r h} \omega_{ijt} \cdot c + \frac{1}{2 r h} \sum_{i+j=rhk} ijt_i t_j \cdot c.
\]

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\[
\begin{align*}
&= d'_k - \frac{1}{r} \sum_{i \in E_k} \left( \omega_i \Lambda_{i-rh} + it_i \Lambda_{i+rh} \right) + \frac{1}{2r} \sum_{i+j=rh} \omega_i \omega_j \cdot c \\
&\quad + \frac{1}{r} \sum_{i>j=-rhk} \omega_i \omega_j \cdot c + \frac{1}{2r} \sum_{i+j=-rhk} \omega_i \omega_j \cdot c.
\end{align*}
\] (4.36)

By virtue of the property of \( \Lambda \) given in Proposition 3.1, we arrive at
\[
(B_k)_c = \left( e^\text{ad}_V e^\text{ad}_\alpha B_k - d_k \right)_c
\]
\[
= \left( e^\text{ad}_V d'_k - d_k \right)_c + \frac{1}{2} \sum_{i+j=rh} \omega_i \omega_j
\]
\[
+ \frac{1}{r} \sum_{i>j=-rhk} \omega_i \omega_j \cdot c + \frac{1}{2r} \sum_{i+j=-rhk} \omega_i \omega_j \cdot c.
\] (4.37)

Here we have used the fact that \((B_k)_c\) is independent of the gauge transformations (2.8), for the same reason as in the proof of Lemma 3.3.

Denote \(O'_k = rh \left( e^\text{ad}_V d'_k - d_k \right)_c\). Clearly,
\[
O'_{-1} = 0, \quad (O'_0)_c = -(h_s^0)_c = 0.
\] (4.38)

When \(k \geq 1\), by using (4.35), (4.37)–(4.39) we have
\[
O'_k = rh \left( e^\text{ad}_V \left( d_k - \frac{1}{r} \sum_{i+j=rh} H_s^1(rk_0 k; s^0) \right) \right)_c
\]
\[= rh \left( e^\text{ad}_V d'_k - d_k \right)_c - \left( e^\text{ad}_V H_s^1(rk_0 k; s^0) \right)_c.
\] (4.39)

We substitute (4.35), (4.37) –(4.39) into (4.34), then obtain (4.32) by integration with respect to \(t_j\). The constant \(C_A\) is chosen such that \(\partial/\partial \beta_k\) acting on \(\tau\) obey the Virasoro commutation relation. The theorem is proved. \(\square\)

In general, we only know that \(O'_k\) are differential polynomials in second-order derivatives of log \(\tau\) with respect to the time variables \(t_j\). The reason is that in Proposition 3.1 the function \(U\) is a differential polynomial in \(q\), which can be chosen canonically as a differential polynomial in the first \(n\) Hamiltonian densities \(h_j = (-\Lambda_j | H)\) with exponents \(j\) such that \(0 < j < rh\). How to compute \(O'_k\) will be illustrated by examples in next section.

We continue to write (4.32) into a more concise form. Observe that the Sugawara construction of \(L_k\) in (4.16) can be extended to an arbitrary affine Kac-moody algebra \(g\). Let
\[
V_k = \frac{1}{r} L_k(\partial/\partial t; t) + \delta_{k,0} \cdot C_A,
\] (4.40)

and they satisfy
\[
[V_k, V_l] = (k-l) V_{k+l}, \quad k, l \geq -1.
\]

Theorem 4.6 leads us to

**Corollary 4.7** The Virasoro symmetries (4.26) for the Drinfeld-Sokolov hierarchies can be represented via tau function as
\[
\frac{\partial \tau}{\partial \beta_k} = V_k \tau + \tau O_k, \quad k \geq -1,
\] (4.41)

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where

\[
O_k = \begin{cases} 
0, & k = -1, 0; \\
\frac{1}{r_h} \left( O'_k - \frac{1}{2} \sum_{i+j=kh} \frac{\partial^2 \log \tau}{\partial t_i \partial t_j} \right), & k \geq 1. 
\end{cases}
\]

In particular, the symmetries generated by \( \partial/\partial \beta_{-1} \) and \( \partial/\partial \beta_0 \) are linearized when acting on the tau function. In the next subsection we will show that, when the Drinfeld-Sokolov hierarchies are associated to ADE-type affine Kac-Moody algebras, the symmetries generated by \( \partial/\partial \beta_k \) with all \( k \geq -1 \) are linearized. However, such kind of linearization of Virasoro symmetries is not valid for all Drinfeld-Sokolov hierarchies due to the functions \( O_k \) may not vanish.

**Definition 4.8** The functions \( O_k \) in (4.41) are called obstacles in linearizing Virasoro symmetries.

### 4.3 Linearization of Virasoro symmetries for ADE-type hierarchies

In this subsection we only consider the Drinfeld-Sokolov hierarchies associated to simply-laced or twisted affine Kac-Moody algebras, for which the Virasoro symmetries acting on the tau function will be seen linearized.

**Lemma 4.9** Let \( g' \) be a simply-laced or twisted affine Lie algebra. The basic representation given in Theorem 4.3 induces an action of the Lie group on the Fock space \( \mathbb{C}[[x]] \), then the operator \( \Theta \) introduced in (3.28) satisfies

\[
\Theta^{-1} \cdot 1 = \frac{\tau(t + x)}{\tau(t)},
\]

where \( \tau \) is the tau function of the corresponding Drinfeld-Sokolov hierarchy.

**Proof** In the basic representation \( \mathbb{C}[[x]] \) of \( g' \), every element \( X \in g'_{\geq 0} \) satisfies \( X \cdot 1 = X_c \). For each positive exponent \( j \in E_+ \),

\[
\left( \frac{\partial}{\partial t_j} + \Lambda_j \right) \Theta^{-1} \cdot 1 = \Theta^{-1} \mathcal{L}_j \cdot 1 = \Theta^{-1} \left( \frac{\partial}{\partial t_j} + q(j) + \Lambda_j \right) \cdot 1 = \Theta^{-1} q(j) c \cdot 1 = -\omega_j \Theta^{-1} \cdot 1.
\]

Denote \( \Theta^{-1} \cdot 1 = f(t, x) \), then the above equation is just

\[
\left( \frac{\partial}{\partial t_j} - \frac{\partial}{\partial x_j} \right) f(t, x) = -\frac{\partial \log \tau(t)}{\partial t_j} f(t, x).
\]

This equation together with the “boundary” condition \( f(t, 0) = (\Theta^{-1} \cdot 1)|_{x=0} = 1 \) implies \( f(t, x) = \tau(t + x)/\tau(t) \). The lemma is proved. \( \square \)

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Proposition 4.10 Suppose the affine Lie algebra $g'$ is simply-laced or twisted, then the Virasoro symmetries (4.26) for the Drinfeld-Sokolov hierarchy act linearly on the tau function. More precisely,
\[
\frac{\partial \tau}{\partial \beta_k} = V_k \tau, \quad k \geq -1,
\] (4.46)
where the operators $V_k$ are given in (4.40).

Proof We only need to show the cases $k \geq 1$. The proof is almost the same as that of Proposition 4.2 in [25] for simply-laced affine Lie algebras.

Consider the basic representation $C[[x]]$ of $g'$ as in Theorem 4.3. It follows from the formula (4.43) that
\[
\frac{\partial}{\partial \beta_k} \Theta^{-1} \cdot 1 = \frac{\partial}{\partial \beta_k} \frac{\tau(t + x)}{\tau(t)} = \frac{1}{\tau(t)} \frac{\partial \tau(t + x)}{\partial \beta_k} - \frac{\tau(t + x) \partial \tau(t)}{\tau(t)^2} \frac{\partial \tau(t)}{\partial \beta_k}
\] (4.47)
On the other hand, let us extend the basic representation $C[[x]]$ to a module of the Kac-Moody-Virasoro algebra as in Theorem 4.4. Recall $p_i$ given in (4.14), and set
\[
\Lambda_j \mapsto -p_j, \quad j \in E; \quad d'_k \mapsto \frac{1}{\tau h} L_k(\partial/\partial x; x), \quad k \in \mathbb{Z}.
\]
When $k \geq 1$, by using (4.27) and (4.34) we have
\[
\frac{\partial}{\partial \beta_k} \Theta^{-1} \cdot 1 = -\Theta^{-1} \frac{\partial \Theta}{\partial \beta_k} \Theta^{-1} \cdot 1 = \Theta^{-1} (B_k)_{<0} \cdot 1
= \Theta^{-1} B_k \cdot 1 - \Theta^{-1} (B_k)_{\geq 0} \cdot 1
= B_k \Theta^{-1} \cdot 1 - \Theta^{-1} d_k \cdot 1 - (B_k)_c \Theta^{-1} \cdot 1
= \left( \frac{1}{\tau h} L_k(\partial/\partial x; x) + \frac{1}{\tau h} \sum_{j \in E_+} j t_j p_{j+r h k} \right) \frac{\tau(t + x)}{\tau(t)} - 0
- \frac{\partial \log \tau(t)}{\partial \beta_k} \frac{\tau(t + x)}{\tau(t)}
= \frac{1}{\tau(t) \tau h} L_k(\partial/\partial x; t + x) \tau(t + x) - \frac{\tau(t + x) \partial \tau(t)}{\tau(t)^2} \frac{\partial \tau(t)}{\partial \beta_k}.
\] (4.48)
Taking (4.47) and (4.48) together, we obtain
\[
\frac{\partial \tau(t + x)}{\partial \beta_k} = \frac{1}{\tau h} L_k(\partial/\partial x; t + x) \tau(t + x), \quad k \geq 1,
\]
which is recast to (4.46) by a translation of variables: $t + x \mapsto t$. The proposition is proved. □

Comparing equations (4.32) and (4.46), we have immediately
Corollary 4.11 For any simply-laced or twisted affine Lie algebra, the obstacles $O_k$ vanish for all $k \geq -1$. In other words, the functions $O_k'$ defined in (4.33) can be written as
\[ O_k' = \frac{1}{2} \sum_{i+j=r} \frac{\partial^2 \log \tau}{\partial t_i \partial t_j}, \quad k \geq 1. \quad (4.49) \]

Proof of Theorem 1.1 The theorem follows from Corollary 4.7 and Proposition 4.10. \(\square\)

At the end this section, we digress to consider another important application of Lemma 4.9. Recall that Kac and Wakimoto [29] constructed a hierarchy of Hirota bilinear equations based on the principal vertex operator realization of the basic representation of $g'$. Every solution of these equations is a point of the orbit space of the highest weight vector acted by the Lie group. More exactly, let $L(\Lambda_0) = \mathbb{C}[x]$ be the basic representation of $g'$ given in Theorem 4.3, and $G$ be the Lie group of $g'$, then $\tau \in L(\Lambda_0)$ lies in the orbit $G \cdot 1$ if and only if it satisfies a hierarchy of Hirota bilinear equations of the form
\[ R_{\sigma} \tau \cdot \tau = 0, \quad \sigma \in SA, \quad (4.50) \]
where $R_{\sigma}$ are certain constant-coefficient even polynomials in $\{D_j \mid j \in E_+\}$ with Hirota differential operators $D_j$ given by
\[ D_j f \cdot g = \left. \frac{\partial}{\partial y} \right|_{y=0} f(x_j + y)g(x_j - y). \]
See [28, 29] for more details.

According to [7, 27, 29], it is known that the Kac-Wakimoto hierarchies of bilinear equations for $g'$ of type $A^{(1)}_n$ is equivalent with the corresponding Drinfeld-Sokolov hierarchies. Such an equivalence was proved by us in [35] for the case of type $D^{(1)}_n$, see also [15]. In general, we have the following

Theorem 4.12 For any simply-laced or twisted affine Lie algebra $g'$, the tau functions defined in (3.20) of the Drinfeld-Sokolov hierarchy coincide with the corresponding Kac-Wakimoto hierarchy. The variables of these two hierarchies are related via a basis of the principal subalgebra $\mathfrak{s}$ according to the relations (2.11) and (4.14).

Proof Given a tau function $\tau(t)$ of the Drinfeld-Sokolov hierarchy, the formula (4.43) implies that $\tau(t+x)/\tau(t)$ is a solution of the Kac-Wakimoto hierarchy of Hirota equations with variable $x$. By setting $t \to 0$ one sees that $\tau(x)$ also solves the Hirota equations due to their bilinearity and translation invariance with respect to $x$.

Conversely, suppose $\tau(x)$ is a solution of the Kac-Wakimoto hierarchy. The substitution of $\tau$ into the right hand side of (4.43) determines an element $\Theta = e^{V(t)}$ with $V(t)$ being a smooth function that takes value in $g'_{<0}$. For every $j \in E_+$, introduce $\mathcal{L}_j = \Theta (\partial / \partial t_j + \Lambda_j) \Theta^{-1}$. They act on the highest weight vector as
\[ \mathcal{L}_j \cdot 1 = \Theta \left( \frac{\partial}{\partial t_j} - \frac{\partial}{\partial x_j} \right) \frac{\tau(t+x)}{\tau(t)} \cdot 1 = - \frac{\partial \log \tau(t)}{\partial t_j} \frac{\tau(t+x)}{\tau(t)} \Theta \cdot 1 \]

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\begin{align}
\frac{\partial \log \tau(t)}{\partial t_j},
\end{align}

which is independent of \(x\). It implies that \(\mathcal{L}_j = \partial/\partial t_j + \Lambda_j + q(j)\), where \(q(j)\) lies in \(g_{\geq 0}'\) and

\begin{align}
q(j)_c = - \frac{\partial \log \tau(t)}{\partial t_j}.
\end{align}

The following zero-curvature equations

\begin{align}
[\mathcal{L}_i, \mathcal{L}_j] = 0, \quad i, j \in E_+
\end{align}

are well defined. These equations recover the Drinfeld-Sokolov hierarchy, of which the tau function defined by (3.20) is just \(\tau(t)\). The theorem is proved. \(\square\)

**Remark 4.13** The results in the present subsection rely on the property of the basic representation of the affine Lie algebra \(g'\); they may not be valid for Drinfeld-Sokolov hierarchies associated to other than the zeroth vertex of the Dynkin diagram of \(g'\). For example, the Virasoro symmetries for the \(A^{(2)}_2\)-hierarchy associated to the vertex labeled 1 of the Dynkin diagram are not linearized; note that the first non-trivial equation in this hierarchy is also known as the Kaup-Kupershmidt equation \([30]\). \(\square\)

Finally, we shall emphasize our inspiration from \([24, 25]\). In fact, for each simply-laced affine Kac-Moody algebra \(g'\), Hollowood and Miramontes proved the following equality (see equation (5.1) in \([24]\)):

\begin{align}
\Theta^{-1} \cdot v_s = \frac{\tau_s(t + x)}{\tau_s^{(0)}(t)},
\end{align}

with a method of “big cell” factorization of the Lie group of \(g'\). Here \(\Theta\) is a dressing operator lying in the Lie subgroup \(U_-(s)\), \(v_s\) is the highest weight vector in an integrable highest weight representation \(L(s)\), and \(\tau_s\) is the tau function of Hirota equations from Kac and Wakimoto’s construction \([29]\). The formula (4.53) provides a map from solutions of a Kac-Wakimoto hierarchy to those of the zero-curvature hierarchy (4.52) of Drinfeld-Sokolov type. This formula was further employed in \([25]\) to obtain the linearized Virasoro symmetries for generalized Drinfeld-Sokolov hierarchies associated to simply-laced affine Lie algebras, whose tau function is considered to be \(\tau_s\).

Inspired by Hollowood, Miramontes and Sánchez Guillén \([24, 25]\), we now obtain, in a more straightforward way, the formula (4.53) of the form (4.53) (note the different definitions of tau functions) whenever the affine algebra \(g'\) is simply-laced or twisted. For such cases, moreover, we clarify in Theorem 4.12 the equivalence between the Drinfeld-Sokolov and the Kac-Wakimoto hierarchies.

### 4.4 Virasoro constraints to tau function

Generally speaking, the tau function that gives partition function in topological field theory is selected by the string equation. Based on the Lax representation in pseudo-differential operators for Drinfeld-Sokolov hierarchies of type \(A_n^{(1)}\) or \(D_n^{(1)}\), it was shown that the string equation induces a series of Virasoro constraints to the tau function \([1, 46]\). Such kind of
constraints to the tau function \( \tau \) in (4.53) of any Drinfeld-Sokolov hierarchy associated to simply-laced affine Kac-Moody algebra \( g \) were derived in [25]. It is not hard to generalize the deduction in [25] to all Drinfeld-Sokolov hierarchies and derive Virasoro constraints to the tau function \( \tau \) in (3.24).

For the Drinfeld-Sokolov hierarchy associated to an arbitrary affine Kac-Moody algebra \( g \), we assume its tau function \( \tau \) satisfies the following string equation

\[
\frac{\partial \tau}{\partial t_1} = V_{-1}\tau. \tag{4.54}
\]

This equation is just \( \partial \tau / \partial t_1 = \partial \tau / \partial \beta_{-1} \), hence it follows from (3.31) and (4.27) that

\[
(\Theta \Lambda_1 \Theta^{-1})_{<0} = -(B_{-1})_{<0}. \tag{4.55}
\]

Denote

\[
P_k = \Theta \Lambda_{1+r(h+1)k}^{-1} + B_k, \quad k \geq -1. \tag{4.56}
\]

First of all, one has \((P_{-1})_{<0} = 0\).

Consider the realization of \( g' \) graded by the homogeneous gradation \( s^0 \), which is acted by a series of derivations \( d_k^{(s^0)} = \lambda^{1+r_k} \cdot d/d\lambda \). Note that the generators of the principal Heisenberg subalgebra \( s \) satisfy

\[
\Lambda_j + rh(h+1) = \lambda^{rk} \Lambda_j + rhk,
\]

then recalling (4.22) and (4.24) we have \( P_{k+1} = \lambda^{rk} P_k \) modulo the central part. Hence we obtain

\[
(P_k)_{<0} = (\lambda^{(k+1)k_0} P_{-1})_{<0} = \lambda^{(k+1)k_0} (P_{-1})_{<-(k+1)k_0} = 0, \quad k \geq -1. \tag{4.57}
\]

Therefore \( \partial \tau / \partial t_{1 + rh(k+1)} = \partial \tau / \partial \beta_k \), namely,

\[
\frac{\partial \tau}{\partial t_{1 + rh(k+1)}} = V_k \tau + \tau O_k, \quad k \geq -1. \tag{4.58}
\]

They are the Virasoro constraints to the tau function of the Drinfeld-Sokolov hierarchy. We plan to study solutions to these constraints elsewhere.

**Remark 4.14** In the recent paper [37], Safronov proposed a set of linear Virasoro constraints to his tau function on Drinfeld-Sokolov Grassmannians for the case of simply-laced semisimple Lie groups. His Virasoro operators are also from the Sugawara construction (4.16) for Heisenberg subalgebras, and the string solutions are described geometrically by principal bundles possessing connections compatible with the Higgs field near infinity. \( \square \)

## 5 Examples

Let us present some examples to illustrate our construction of tau function of Drinfeld-Sokolov hierarchies and the obstacles in linearizing their Virasoro symmetries.

In the examples below we will use a matrix realization of \( g \) of affine type \( X_N^{(r)} \) corresponding to the homogeneous gradation \( s^0 \) as in (1.3)–(1.5) (see [28] or the appendix
In (4.4) the standard invariant symmetric bilinear form \((\cdot \mid \cdot)_0\) on \(G\) is the Killing form for special linear algebras, and is half the Killing form for special orthogonal algebras. It induces the standard invariant symmetric bilinear form on the derived algebra \(g'\) as
\[
(X \otimes \lambda^k + \alpha c \mid Y \otimes \lambda^l + \beta c) = \delta_{k,-l} \frac{1}{r} (X \mid Y)_0, \quad X, Y \in G.
\]
(5.1)

On \(g'\) the derivations \(d_k\) reads
\[
d_k = -\frac{1}{r k_0} \lambda^l + r k_0 \frac{d}{d\lambda}, \quad k \in \mathbb{Z},
\]
(5.2)
where \(k_0 = 2\) whenever \(g\) is of type \(A_2^{(2)}\) and \(k_0 = 1\) otherwise.

### 5.1 Tau functions of hierarchies of types \(A_n^{(1)}\) and \(D_n^{(1)}\)

**Example 5.1** We realize the affine Kac-Moody algebra \(g'\) of type \(A_1^{(1)}\) by taking a set of Weyl generators as
\[
e_0 = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 & 0 \\ 1/\lambda & 0 \end{pmatrix}, \quad \alpha_0^\vee = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c,
\]
(5.3)
\[
e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \alpha_1^\vee = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
(5.4)
Let \(\Lambda = e_0 + e_1\). The set of exponents is \(E = \mathbb{Z}^{\text{odd}}\), and the principal Heisenberg subalgebra \(s\) has a basis
\[
\{c, \Lambda_j = \Lambda^j \in g^j \mid j \in \mathbb{Z}^{\text{odd}}\}.
\]

The operator \(\mathcal{L}\) in (2.7) is gauge equivalent to the following canonical form
\[
\mathcal{L}^\text{can} = D + \Lambda + q^\text{can} = D + \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -u \\ 0 & 0 \end{pmatrix},
\]
(5.5)
let us compute the functions \(U\) and \(H\) determined as in Proposition 3.1 by
\[
D + \Lambda + q^\text{can} = e^\text{adv}(D + \Lambda + H),
\]
(5.6)
\[
(e^\text{adv} \Lambda_j)_c = 0, \quad j = 1, 3, 5, \ldots.
\]
(5.7)
First of all, we decompose \(U\) and \(H\) according to the principal gradation \(g' = \bigoplus_{k \in \mathbb{Z}} g^k\) as
\[
U = U_{-1} + U_{-2} + U_{-3} + U_{-4} + \ldots
\]
\[
= \begin{pmatrix} 0 & a_1 \\ b_1/\lambda & 0 \end{pmatrix} + \begin{pmatrix} -a_2/\lambda & 0 \\ 0 & a_3/\lambda \end{pmatrix} + \begin{pmatrix} 0 & a_3/\lambda \\ b_3/\lambda^2 & 0 \end{pmatrix} + \begin{pmatrix} -a_4/\lambda^2 & 0 \\ 0 & a_4/\lambda^2 \end{pmatrix} + \ldots,
\]
(5.8)
\[ H = -\frac{h_1}{2} \Lambda_{-1} - \frac{h_3}{2} \Lambda_{-3} - \cdots, \]  
(5.9)

where \( a_i, b_i \) and \( h_i \) are scalar functions. Now substitute them into (5.6) and compare terms with equal principal degrees. Clearly, the case of degree 1 is trivial. The first nontrivial equation is

\[ \text{degree 0: } 0 = [U_{-1}, \Lambda] = (a_1 - b_1) \text{diag}(1, -1) - b_1 \cdot c. \]  
(5.10)

Hence \( a_1 = b_1 = 0 \), i.e., \( U_{-1} = 0 \). Note that the equality (5.7) with \( j = 1 \) holds automatically. Secondly, we have

\[ \text{degree } -1 : \begin{pmatrix} 0 & -u \\ 0 & 0 \end{pmatrix} = [U_{-2}, \Lambda] - \frac{h_1}{2} \Lambda_{-1} \]
\[ = \begin{pmatrix} 0 & -2a_2 \\ 2a_2/\lambda & 0 \end{pmatrix} - \frac{h_1}{2} \begin{pmatrix} 0 & 1 \\ 1/\lambda & 0 \end{pmatrix}, \]  
(5.11)

which implies

\[ a_2 = \frac{u}{4}, \quad h_1 = u. \]

Then it comes

\[ \text{degree } -2 : \begin{pmatrix} 0 \\ 0 \end{pmatrix} = [U_{-3}, \Lambda] - \partial_x U_{-2} \]
\[ = (a_3 - b_3) \text{diag}(1/\lambda, -1/\lambda) - \frac{u_x}{4} \text{diag}(-1/\lambda, 1/\lambda). \]  
(5.12)

Here and below the subscript “\( x \)” stands for the partial derivative with respective to it, for instance, \( u_x = \partial_x u \) and \( u_{xx} = \partial_x^2 u \). The condition (5.7) with \( j = 3 \) reads

\[ 0 = [U_{-3}, \Lambda_3]_{c} = -a_3 - 2b_3. \]  
(5.13)

Equations (5.12) and (5.13) determine \( a_3 \) and \( b_3 \) uniquely:

\[ a_3 = -\frac{u_x}{6}, \quad b_3 = \frac{u_x}{12}. \]

Subsequently,\n
\[ \text{degree } -3 : \begin{pmatrix} 0 \\ 0 \end{pmatrix} = [U_{-4}, \Lambda] + \frac{1}{2} [U_{-2}, [U_{-2}, \Lambda]] - \partial_x U_{-3} + [U_{-2}, -\frac{h_1}{2} \Lambda_{-1}] - \frac{h_3}{2} \Lambda_{-3}. \]  
(5.14)

implies

\[ a_4 = \frac{u^2}{8} + \frac{u_{xx}}{16}, \quad h_3 = \frac{u^2}{4} + \frac{u_{xx}}{12}. \]

Lemma 3.3 shows that the Hamiltonian densities \( h_j \) are independent of the choice of gauge slice of \( \mathcal{L} \). By using (3.24), the tau function \( \tau \) of the Drinfeld-Sokolov hierarchy satisfies (note \( \partial_{t_1} = \partial_x \))

\[ \frac{\partial^2 \log \tau}{\partial x^2} = \frac{1}{2} h_1 = \frac{1}{2} u, \]  
(5.15)

\[ \frac{\partial^2 \log \tau}{\partial x \partial t_3} = \frac{3}{2} h_3 = \frac{3}{8} u^2 + \frac{1}{8} u_{xx}. \]  
(5.16)
They imply the KdV equation:

\[
\frac{\partial u}{\partial t_3} = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}
\]  

(5.17)

One can also fix the gauge slice of \(\mathcal{L}\) such that the dressing operator \(\Theta\) takes the form (3.28). In fact,

\[
\Theta_0 = e^{U}e^{\Omega} = e^{X_0 + \lambda^{-1}X_1 + \lambda^{-2}X_2 + \cdots},
\]  

(5.18)

where \(\lambda^{-k}X_k\) lies in \(\mathfrak{g}^{-k}\) and particularly

\[
X_0 = \begin{pmatrix} 0 & \omega_1 \\ 0 & 0 \end{pmatrix}.
\]

Let \(N = -X_0\), then \(\Theta = e^{N}\Theta_0\) has the form (3.28). Noting \(u = 2\partial_x\omega_1\), it is straightforward to calculate the corresponding gauge slice:

\[
\mathcal{L} = e^{-\text{ad}X_0}\mathcal{L}^\text{can} = D + \Lambda + \begin{pmatrix} -\omega & -\omega^2 - \omega_x \\ 0 & \omega \end{pmatrix},
\]  

(5.19)

in which we write \(\omega = \omega_1\) to simply notations.

**Example 5.2** Let \(\mathfrak{g}'\) be an affine Lie algebra of type \(A_n^{(1)}\). It can be realized by choosing Weyl generators as follows [7, 28]:

\[
e_0 = \lambda e_{1,n+1}, \quad e_i = e_{i+1,i} \quad (1 \leq i \leq n),
\]  

(5.20)

\[
f_0 = \frac{1}{\lambda}e_{n+1,i}, \quad f_i = e_{i,i+1} \quad (1 \leq i \leq n),
\]  

(5.21)

\[
\alpha_i^\vee = [e_i, f_i] \quad (0 \leq i \leq n),
\]  

(5.22)

where \(e_{i,j}\) is the \((n+1) \times (n+1)\) matrix with its \((i,j)\)-component being 1 and the others being zero. One has \(\Lambda = e_0 + e_1 + \cdots + e_n\). A basis of the principal Heisenberg subalgebra \(\mathfrak{s}\) is \(\{e, \Lambda_j = \Lambda^j \in \mathfrak{g}' \mid j \in E\}\) with \(E = \mathbb{Z} \setminus (n+1)\mathbb{Z}\) being the set of of exponents of \(\mathfrak{g}'\).

Take the operator \(\mathcal{L}\) in (3.1) as the following canonical form

\[
\mathcal{L}^\text{can} = D + \Lambda + q^\text{can}, \quad q^\text{can} = \begin{pmatrix} 0 & \cdots & 0 & 0 & -u_n \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -u_2 & 0 \\ 0 & 0 & -u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]  

(5.23)

According to [7], the equations in (2.11) can be represented equivalently as

\[
\frac{\partial L}{\partial t_j} = [(L^{j/(n+1)}), L], \quad j \in E_+,
\]  

(5.24)

where

\[
L = D^{n+1} + u_1 D^{n-1} + \cdots + u_{n-1} D + u_n,
\]  

(5.25)
\[ L^{1/(n+1)} = D + v_1 D^{-1} + v_2 D^{-2} + \cdots, \]

\((L^{j/(n+1)})_+\) means the differential part of the operator \(L^{j/(n+1)}\), and the multiplication of two pseudo-differential operators is defined by

\[ uD^k \cdot vD^l = \sum_{m \geq 0} uD^m(v)D^{k+l-m}. \]

Clearly \(\partial/\partial t_1 = \partial/\partial x\). The hierarchy (5.24) is just the Gelfand-Dickey hierarchy \([20]\), or the \(n\)th KdV hierarchy.

It is known that the Gelfand-Dickey hierarchy (5.24) carries two compatible Hamiltonian structures, with (2.12) usually being called the second one. The densities of the Hamiltonian functionals are

\[ \hat{h}_j = \frac{n+1}{j} \text{res } L^{j/(n+1)}, \quad j \in E+. \]  

(5.26)

Note that the residue of a pseudo-differential operator is defined by \(\text{res } \sum_{i \in \mathbb{Z}} a_i D^i = a_{-1}\).

It is easy to check

\[ \frac{j}{n+1} \frac{\partial \hat{h}_j}{\partial t_i} = \frac{i}{n+1} \frac{\partial \hat{h}_i}{\partial t_j}, \quad i, j \in E+. \]  

(5.27)

Given any solution of the hierarchy (5.24), there locally exits a tau function \(\hat{\tau}\) such that

\[ \frac{\partial^2 \log \hat{\tau}}{\partial t_i \partial t_j} = \frac{j}{n+1} \frac{\partial \hat{h}_j}{\partial t_i}, \quad i, j \in E+. \]  

(5.28)

When \(n = 1\), this is just the case of the KdV hierarchy reviewed in Section 1.

**Proposition 5.1** For the Drinfeld-Sokolov hierarchy of type \(A_n^{(1)}\), the tau function \(\hat{\tau}\) in (5.28) coincides\(^1\) with \(\tau\) defined in (3.24).

**Proof** It is sufficient to identify the second-order derivatives of \(\log \hat{\tau}\) and \(\log \tau\) with respect to the time variables.

On the one hand,

\[ \frac{\partial^2 \log \hat{\tau}}{\partial x^2} = \frac{1}{n+1} \hat{h}_1 = \text{res } L^{1/(n+1)} = \frac{u_1}{n+1}. \]  

(5.29)

On the other hand, for \(\mathcal{L} = \mathcal{L}^{\text{can}}\) we do the calculation as in the proof of Proposition 3.1

Equation (3.5) now reads

\[ [U_{-1}, \Lambda] = q_0^{\text{can}} = 0, \]

which implies \(U_{-1} = 0\). The first equation in (3.6) is

\[ H_{-1} + [U_{-2}, \Lambda] = q_1^{\text{can}} = -u_1 e_{n,n+1}, \]

\(^1\)In the present paper we say that two tau functions are the same if their logarithms differ by addition of a linear function of the time variables.
hence
\[
\frac{\partial^2 \log \tau}{\partial x^2} = (-\Lambda_1 | H) (\Lambda_1 | \Lambda_{-1})^{-1} (-\Lambda_1 | [\Lambda, U_{-2}] + q^\text{can}_1) = \frac{1}{\Lambda_1 | \Lambda_{-1}} (-\Lambda_1 | -u_1 e_{n,n+1}) = \frac{u_1}{n+1}.
\]
(5.30)

It follows from (5.29) and (5.30) that
\[
\frac{\partial}{\partial x} \left( \frac{\partial^2 \log \tau}{\partial x \partial t_j} - \frac{\partial^2 \log \hat{\tau}}{\partial x \partial t_j} \right) = 0, \quad j \in E_+.
\]
The difference between the parentheses is a differential polynomial in \(u_1, \ldots, u_n\) without free term, thus it vanishes indeed. In the same way, we derive
\[
\frac{\partial^2 \log \tau}{\partial t_i \partial t_j} - \frac{\partial^2 \log \hat{\tau}}{\partial t_i \partial t_j} = 0, \quad i, j \in E_+.
\]
The proposition is proved. □

Example 5.3 Assume the affine Lie algebra \(g'\) is of type \(D_n^{(1)}\), and it is realized as in [35], see also [7, 45]. We choose a set of generators of the principal Heisenberg algebra \(s\) as
\[
\Lambda_k = \sqrt{2} \Lambda^k, \quad \Lambda_{-k} = \sqrt{2} \Lambda^{-k},
\]
\[
\Lambda_{k(n-1)} = -\sqrt{2n-2} \Gamma^k, \quad \Lambda_{-k(n-1)} = -\sqrt{2n-2} \Gamma^{-k}
\]
(5.31) (5.32)
for \(k \in \mathbb{Z}^\text{odd}_+\), where \(\Lambda\) and \(\Gamma\) are two certain \(2n \times 2n\) matrices given in §4.2 of [35]. Note that the constant \(\nu = \sqrt{2}\), and that \(k(n-1)\) are double exponents of \(g'\) whenever \(n\) is even.

Introduce a pseudo-differential operator
\[
L = D^{2n-2} + \frac{1}{2} \sum_{i=1}^{n-1} D^{-1} (u_i D^{2i-1} + D^{2i-1} u_i) + D^{-1} \rho D^{-1} \rho,
\]
(5.33)
There are uniquely two operators
\[
P = D + v_1 D^{-1} + v_2 D^{-2} + \cdots, \quad Q = D^{-1} \rho + \hat{v}_1 D + \hat{v}_2 D^2 + \cdots
\]
such that \(P^{2n-2} = L = Q^2\). The following integrable hierarchy are well defined [7, 35]:
\[
\frac{\partial L}{\partial t_k} = \left[ \sqrt{2} (P^k)_+, L \right], \quad \frac{\partial L}{\partial t_{k(n-1)}} = \left[ -\sqrt{2n-2} (Q^k)_-, L \right], \quad k \in \mathbb{Z}^\text{odd}_+,
\]
(5.34)
which is equivalent to the Drinfeld-Sokolov hierarchy (2.11) of type \(D_n^{(1)}\) with \(q^\text{can}\) chosen in [35].
The hierarchy (5.34) carries a bi-Hamiltonian structure, with Hamiltonian densities being \( \tau \)-symmetric. Hence a \( \hat{\tau} \) function is defined by

\[
d (2 \partial_x \log \hat{\tau} ) = \sum_{k \in \mathbb{Z}^{odd}+} \left( \sqrt{2} \mathrm{res} P^k dt_k + \sqrt{2} n - 2 \mathrm{res} Q^k dt_{k(n-1)} \right).
\]

(5.35)

With the same method as for Proposition 5.1, we obtain the following

**Proposition 5.2** For the Drinfeld-Sokolov hierarchy of type \( D_n^{(1)} \), the tau functions \( \hat{\tau} \) in (5.35) and \( \tau \) in (3.24) coincide.

Based on this proposition, it is easy to see that the Virasoro symmetries (4.46) are consistent with those derived in [46] with skills of pseudo-differential operators.

**Remark 5.3** In not so straightforward a way, Propositions 5.1 and 5.2 can be considered as corollaries of Theorem 4.12 by using relevant results in [29, 35] that both \( \tau \) and \( \hat{\tau} \) are solutions of the corresponding Kac-Wakimoto hierarchies. □

### 5.2 Obstacles in linearizing Virasoro symmetries

**Example 5.4** We realize the affine Kac-Moody algebra \( g' \) of type \( A^{(1)}_1 \) as in Example 5.1. The simple Lie algebra \( \tilde{g} = \mathfrak{sl}_2 \) contains the following elements of \( 2 \times 2 \) matrices:

\[
\begin{align*}
E_0 &= e_{1,2}, & F_0 &= e_{2,1}, & H_0 &= e_{1,1} - e_{2,2}, \\
E_1 &= e_{2,1}, & F_1 &= e_{1,2}, & H_1 &= -e_{1,1} + e_{2,2},
\end{align*}
\]

which correspond to the Weyl generators (5.3)–(5.4). The derivations on \( g' \) are given by (5.2) with \( r = k_0 = 1 \), i.e.,

\[
d_k = -\lambda^{k+1} \frac{d}{d\lambda}, \quad k \in \mathbb{Z}.
\]

According to (4.12), one has \( H_{s^1} = \frac{1}{2} H_1 \), and then

\[
H_{s^1}(k; s^0) = \frac{\lambda^k}{2} H_1.
\]

(5.36)

Now we use the data in Example 5.1 to compute the obstacles of Virasoro symmetries for the Drinfeld-Sokolov hierarchy associated to \( g' \). After a straightforward calculation, we have

\[
O'_1 = 2 \left( e^\text{ad}_{d_1} d_1 - d_1 \right)_c - \left( e^\text{ad}_H s^1(1; s^0) \right)_c = 0 + \frac{1}{4} h_1 = \frac{1}{2} \frac{\partial^2 \log \tau}{\partial t_1^2},
\]

(5.37)

\[
O'_2 = 2 \left( e^\text{ad}_{d_2} d_2 - d_2 \right)_c - \left( e^\text{ad}_H s^1(2; s^0) \right)_c = \frac{3}{2} h_3 = \frac{\partial^2 \log \tau}{\partial t_1 \partial t_3}.
\]

(5.38)

Hence \( O_1 = O_2 = 0 \); furthermore, by using the Virasoro commutation relation we obtain \( O_k = 0 \) for all \( k \geq 3 \). This fact agrees with Corollary 4.11 which confirms directly the linearization of Virasoro symmetries for the KdV hierarchy.
Example 5.5 Let $\mathfrak{g}$ be the Kac-Moody algebra of type $B_2^{(1)}$ (isomorphic to $C_2^{(1)}$), whose Cartan matrix is

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -2 & -2 & 2 \end{pmatrix}. $$

The elements $E_i$, $F_i$ and $H_i$ in subsection 4.1 can be realized by $5 \times 5$ matrices as

$$E_0 = \frac{1}{2}(e_{1,1} + e_{2,5}), \quad E_1 = e_{2,1} + e_{5,4}, \quad E_2 = e_{3,2} + e_{4,3}, $$

(5.39)

$$F_0 = 2(e_{1,1} + e_{5,2}), \quad F_1 = e_{1,2} + e_{4,5}, \quad F_2 = 2(e_{2,3} + e_{3,4}), $$

(5.40)

$$H_i = [E_i, F_i] \quad (i = 0, 1, 2). $$

(5.41)

These elements give a set of Weyl generators according to [1.8] for the homogeneous gradation $s^0$, hence $\mathfrak{g}(A; s^0)$ is realized. The derivations in (5.2) are

$$d_k = -\lambda^{k+1} \frac{d}{d\lambda}, \quad k \in \mathbb{Z}. $$

Note $\Lambda = E_1 + E_2 + \lambda E_0$, and $h = 4$ is the Coxeter number. The set of exponents of $\mathfrak{g}$ is $E = \mathbb{Z}_{\text{odd}}$, and the generators normalized by (4.21) of the principal Heisenberg subalgebra $\mathfrak{s}$ are

$$\Lambda_k = \sqrt{2}\Lambda^k, \quad \Lambda_{-k} = \sqrt{2}(\lambda^{-1}\Lambda^3)^k, \quad k \in \mathbb{Z}_{\text{odd}}. $$

(5.42)

The operator $\mathcal{L}^\text{can}$ is gauge equivalent to the canonical form

$$\mathcal{L}^\text{can} = D + \Lambda + \mathcal{q}^\text{can}, \quad \mathcal{q}^\text{can} = -u(e_{1,2} + e_{4,5}) - v(e_{1,4} + e_{2,5}). $$

(5.43)

In the same way as before, we compute the functions $U = U_{-1} + U_{-2} + \ldots$ and $H$ given by Proposition 3.1. As a result,

$$U_{-1} = 0, $$

(5.44)

$$U_{-2} = \frac{3u}{4}(-e_{1,3} + e_{3,5}) + \frac{u}{2\lambda}(-e_{3,1} + e_{5,3}), $$

(5.45)

$$U_{-3} = -\frac{5ux}{6}(e_{1,4} + e_{2,5}) + \frac{ux}{3\lambda}(e_{2,1} + e_{5,4}) - \frac{ux}{6\lambda}(e_{3,2} + e_{4,3}), $$

(5.46)

$$U_{-4} = \frac{u^2 + 4(v + uxx)}{4\lambda}(-e_{1,1} + e_{5,5}) + \frac{2v + uxx}{4\lambda}(-e_{2,2} + e_{4,4}); $$

(5.47)

$$H = -\frac{u}{2\sqrt{2}}\Lambda_{-1} - \frac{3u^2 + 12v + 10uxx}{24\sqrt{2}}\Lambda_{-3} + \ldots. $$

(5.48)

Note $\partial/\partial t_1 = \sqrt{2}\partial/\partial x$, hence

$$\frac{\partial^2 \log \tau}{\partial t_1^2} = \sqrt{2}\left(\frac{-\Lambda_1 \mid H}{(\Lambda_1 \mid \Lambda_{-1})}\right) = \frac{u}{2}, $$

(5.49)

$$\frac{\partial^2 \log \tau}{\partial t_1 \partial t_3} = \sqrt{2}\left(\frac{-\Lambda_3 \mid H}{(\Lambda_3 \mid \Lambda_{-3})}\right) = \frac{1}{8} (3u^2 + 12v + 10uxx). $$

(5.50)

Clearly,

$$H_{s_1} = (H_1, H_2) \left(\begin{pmatrix} A^T \end{pmatrix}^{-1} \right) (1, 1)^T = 2H_1 + \frac{3}{2}H_2, $$

(5.51)

34
\[ H_{s^1}(k; s^0) = \lambda^k H_{s^1}, \quad k \in \mathbb{Z}. \] (5.52)

A straightforward calculation leads to
\[ O'_1 = 4(e^{adU} - d_1)c - (e^{adU} H_{s^1}(1; s^0))c 
= \frac{7u^2}{8} + \frac{5v}{2} + \frac{9u_{xx}}{4} 
= \frac{5}{3} \frac{\partial^2 \log \tau}{\partial t_1 \partial t_3} + \frac{1}{6} \frac{\partial^4 \log \tau}{\partial t_1^4} + \left( \frac{\partial^2 \log \tau}{\partial t_1^2} \right)^2. \] (5.53)

It implies \( O_1 \neq 0 \) in (4.41), namely, the symmetry \( \partial/\partial \beta_1 \) acting on the tau function cannot be linearized.

**Example 5.6** Based on the matrix realization of the simple Lie algebra \( A_2 \) in Example 5.2, one chooses Weyl generators of the twisted affine Lie algebra \( g'_1 \) of type \( A_2^{(2)} \) as the following 3 \times 3 matrices:

\[
\begin{align*}
e_0 &= \lambda(e_{2,1} + e_{3,2}), \quad e_1 = e_{1,3}, \\
f_0 &= \frac{2}{\lambda}(e_{1,2} + e_{2,3}), \quad f_1 = e_{3,1}, \\
\alpha^\vee_0 &= H_0 + c = -2e_{1,1} + 2e_{3,3} + c, \\
\alpha^\vee_1 &= H_1 = e_{1,1} - e_{3,3}.
\end{align*}
\]

The derivations (5.2) corresponding to the homogeneous realization are
\[ d_k = -\frac{1}{4} \lambda^{4k+1} \frac{d}{d\lambda}, \quad k \in \mathbb{Z}. \]

Let \( \Lambda = e_0 + e_1 \). The Coxeter number is 3, and the principal Heisenberg subalgebra \( \mathfrak{s} \) contains a set of generators
\[ \Lambda_j = \sqrt{2} \Lambda^j, \quad j \equiv \pm 1 \mod 6, \] (5.54)

which satisfy the normalization condition (4.21).

Take the canonical form of the operator \( \mathcal{L} \) as
\[ \mathcal{L}^\text{can} = D + \Lambda + q^\text{can}, \quad q^\text{can} = -ue_{3,1}. \]

We compute the matrix-value functions \( U \) and \( H \) according to Proposition 3.1 and have
\[
\begin{align*}
U &= U_{-2} + U_{-3} + U_{-4} + U_{-5} + U_{-6} + \ldots \\
&= \frac{u}{3\lambda}(e_{2,1} - e_{3,2}) + \frac{u_x}{9\lambda^2}(-e_{1,1} + 2e_{2,2} - e_{3,3}) + \frac{u^2 + 2u_{xx}}{18\lambda^3}(-e_{1,2} + e_{2,3}) \\
&\quad + \left( -\frac{5uu_x + 3u_{xxx}}{45\lambda^4} e_{1,3} + \frac{5uu_x + 4u_{xxx}}{90\lambda^3} (e_{2,1} + e_{3,2}) \right) \\
&\quad + \frac{5u^3 + 18u_x^2 + 30uu_{xx} + 12u_{xxx}}{324\lambda^4}(-e_{1,1} + e_{3,3}) + \ldots ,
\end{align*}
\] (5.55)
\[
H = -\frac{h_1}{3} \Lambda_{-1} - \frac{h_5}{3} \Lambda_{-5} + \ldots
\]

\[
= -\frac{u}{3\sqrt{2}} \Lambda_{-1} + \frac{5u^3 + 15uu_{xx} + 3u_{xxxx}}{405\sqrt{2}} \Lambda_{-5} + \ldots.
\]

(5.56)

Hence by using \(\partial/\partial t_1 = \sqrt{2} \partial/\partial x\) we obtain

\[
\frac{\partial^2 \log \tau}{\partial t_1^2} = \frac{u}{3}, \quad \frac{\partial^2 \log \tau}{\partial t_1 \partial t_5} = -\frac{1}{81} \left(5u^3 + 15uu_{xx} + 3u_{xxxx}\right).
\]

(5.57)

On the other hand, one has

\[
H_s(4; s^0) = \frac{\lambda^4}{2} H_1, \quad k \in \mathbb{Z},
\]

(5.58)

then

\[
O'_1 = 6(e^{adv} d_1 - d_1)_c - (e^{adv} H_s(4; s^0))_c = \frac{5\sqrt{2}}{3} h_5 = \frac{\partial^2 \log \tau}{\partial t_1 \partial t_5}.
\]

(5.59)

Thus we obtain \(O_1 = 0\), which agrees with Corollary 4.11.

Remark 5.4 As an application of (5.57), one has the first nontrivial equation of the \(A^{(2)}_2\)-hierarchy:

\[
\frac{\partial u}{\partial t_5} = \frac{1}{108} \frac{\partial}{\partial t_1} \left(20u^3 + 30u \frac{\partial^2 u}{\partial t_1^2} + 3 \frac{\partial^4 u}{\partial t_1^4}\right),
\]

(5.60)

which is also known as the Sawada-Kotera equation \([38]\). This equation carries a generalized bi-Hamiltonian structure consisting of a local and a nonlocal operators, as was shown by Fuchssteiner and Oevel \([19]\). In fact, by using a perturbation approach we proved that equation (5.60) possesses exactly one local Hamiltonian structure \([34]\) that coincides with the first Hamiltonian structure in \([19]\). \(\square\)

6 Conclusion and remark

We have obtained a unified characterization of tau function and Virasoro symmetries for Drinfeld-Sokolov hierarchy associated to any affine Kac-Moody algebra and the zeroth vertex of its Dynkin diagram. This together with the results in \([7, 9]\) suggests us, although a complete proof is still missing, to divide these Drinfeld-Sokolov hierarchies into three classes as in Table 1.

The hierarchies in Class I are bi-Hamiltonian and have linearized Virasoro symmetries acting on the tau function. They coincide, up to a rescaling of the time variables, with the topological hierarchies constructed by Dubrovin and Zhang \([10, 33]\) associated to semisimple Frobenius manifolds for ADE-type Weyl groups. Their tau functions defined in \([3, 24]\) that admit the Virasoro constraints give partition functions of 2D topological minimal models, as well as Givental’s total descendant potentials for simple singularities, see \([6, 21, 22, 18, 45]\) and references therein.

Each hierarchy in Class II also carries a bi-Hamiltonian structure whose leading term is associated to a semisimple Frobenius manifold. However, its Virasoro symmetries are
Table 1:

| Class | Affine algebra \( X_N^{(r)} \) | Hamiltonian structure | Virasoro symmetries |
|-------|---------------------------------|----------------------|---------------------|
| I     | \( A_n^{(1)} \), \( D_n^{(1)} \), \( E_{6,7,8}^{(1)} \) | bi-Hamiltonian       | linearizable        |
| II    | \( B_n^{(1)} \), \( C_n^{(1)} \), \( F_4^{(1)} \), \( G_2^{(1)} \) | bi-Hamiltonian       | non-linearizable    |
| III   | \( A_{2n}^{(2)} \), \( A_{2n-1}^{(2)} \), \( D_{n+1}^{(2)} \), \( E_6^{(2)} \), \( D_4^{(3)} \) | Hamiltonian          | linearizable        |

expected to be non-linearizable. In fact, the non-linearizability of Virasoro symmetries for the \( C_n^{(1)} \)-hierarchies has been verified by us [11] based on a generating function for \( O_k \); for the \( B_n^{(1)} \)-hierarchies it can be implied by the failure in reducing the additional symmetries of the BKP hierarchy to symmetries of \( B_n^{(1)} \)-hierarchies with the method of [16], which and the exceptional cases will be considered elsewhere. The meaning of the obstacles in linearizing Virasoro symmetries is still far from being well understood, which probably illustrates the observation of the absence of a consistent higher-genus expansion beyond genus one in topological minimal models associated to Lie algebras other than ADE type [12].

From the viewpoint of tau functions and Virasoro symmetries, the hierarchies in Class III look very similar with those in Class I. The main difference between them is that, every hierarchy in Class III, such like the \( A_2^{(2)} \)-hierarchy, may not possess more than one (local) Hamiltonian structure. It is an interesting question whether there is some topological meaning or axiomatic construction for them such like in [10].

As a byproduct, we have shown that the tau function of a hierarchy in Class I or III is a solution of the corresponding Kac-Wakimoto hierarchy of bilinear equations. This is consistent with the corresponding results for the case of types \( A_n^{(1)} \) and \( D_n^{(1)} \) in the literature.

Finally, note that we have concentrated ourselves to the original Drinfeld-Sokolov hierarchies, which is widely applied in some other research areas. Since there are varies of generalizations of the Drinfeld-Sokolov hierarchies, see, for example, [15, 16, 23], for them it is natural to ask whether there is an analogous characterization of tau function and Virasoro symmetries. By now a positive answer can be seen for the so-called generalized hierarchies of type I in [23], in which the principle Heisenberg subalgebra is replaced by an arbitrary Heisenberg subalgebra (see also [3]). However, it is still unclear for the other cases. We will study it on other occasions.

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Appendix

A  Tau functions of modified Drinfeld-Sokolov hierarchies

Let us consider the modified Drinfeld-Sokolov hierarchy associated to an affine Lie algebra \( g' = \mathfrak{g} \oplus \mathbb{C}c \), which is defined by (2.11) with \( \mathcal{L} \) given by a function \( q \) taking value in the Cartan subalgebra of \( \mathfrak{g} \subseteq \mathfrak{g} \). This particular gauge slice of \( \mathcal{L} \) is related to the others by gauge transformations (2.8) that do not change the definition of \( \tau \) in (3.24), hence \( \tau \) also serves as a tau function of the modified Drinfeld-Sokolov hierarchy. We are to compare this \( \tau \) with those tau functions introduced by Enriquez and Frenkel [11] and by Miramontes [36].

A.1 Enriquez and Frenkel’s tau function

For the modified Drinfeld-Sokolov hierarchy associated to an untwisted affine Lie algebra \( \mathfrak{g} \), Enriquez and Frenkel [11] found the following Hamiltonian densities:

\[
\tilde{h}_j = (\Lambda_1 | e^{\text{ad}_{\Lambda_j}} \Lambda_j), \quad j \in E_+, \tag{A.1}
\]

where \( U \) is given in Proposition 2.1. These densities satisfy

\[
\frac{\partial \tilde{h}_j}{\partial t_i} = \frac{\partial \tilde{h}_i}{\partial t_j} = \frac{\partial H_{i,j}}{\partial x}
\]

with \( H_{i,j} \) being differential polynomials in \( q \); in other words, they are tau-symmetric. There locally exists a tau function \( \tilde{\tau} \) such that

\[
\frac{1}{(\Lambda_1 | \Lambda_{-1})} \tilde{h}_j = \frac{\partial^2 \log \tilde{\tau}}{\partial t_1 \partial t_j}, \quad j \in E_+. \tag{A.2}
\]

In comparison with those equations in §5.5 of [11], here on the left hand side of (A.2) we assign a coefficient \( 1/(\Lambda_1 | \Lambda_{-1}) \) to make \( \tilde{\tau} \) independent of the choice of the invariant symmetric bilinear form.

According to Proposition 2.1, the freedom of the function \( U \) does not change the densities \( \tilde{h}_j \). Without lose of generality, let us consider the derived algebra \( \mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}c \) instead of \( \mathfrak{g} \), and fix \( U \) as in Proposition 3.1.

Proposition A.1 For the modified Drinfeld-Sokolov hierarchy associated to any affine Lie algebra \( \mathfrak{g}' \), the tau functions defined by (A.2) and by (3.24) satisfy

\[
\log \tilde{\tau} - \log \tau = \frac{1}{(\Lambda_1 | \Lambda_{-1})} \left( \frac{\partial}{\partial t_1} \right)^{-1} (\Lambda_1 | U_{-1}), \tag{A.3}
\]

where \( U_{-1} \in C^\infty(\mathbb{R}, \mathfrak{g}'^{-1}) \) is uniquely determined by \([U_{-1}, \Lambda] = q\). Note that \((\Lambda_1 | U_{-1})\) is a nontrivial linear combination of components of \( q \).
Proof. According to the principal gradation (2.4) on \( g' \), we write \( U = U_{-1} + U_{-2} + \cdots \) with \( U_k \in g'^k \). Since \( q \) takes value in the Cartan subalgebra, then \([U_{-1}, \Lambda] = q\).

Recall
\[
L_1 = e^{ad_U} \left( \frac{\partial}{\partial t_1} + \Lambda_1 - \sum_{j \in E^+} \frac{1}{j} \frac{\partial \omega_i}{\partial t_1} \Lambda_{-j} - \omega_1 \cdot c \right).
\]
In particular, the projection of \( L_1 - \partial / \partial t_1 \) onto \( g'^{-1} \) vanishes, namely,
\[
-\frac{\partial U_{-1}}{\partial t_1} + (e^{ad_U} \Lambda)_1 - \frac{\partial \omega_1}{\partial t_1} \Lambda_{-1} = 0.
\]
Since \( \omega_1 = \partial \log \tau / \partial t_1 \), then
\[
(L_1 | e^{ad_U} \Lambda_1) = \frac{\partial}{\partial t_1} (L_1 | U_{-1}) + (L_1 | \Lambda_{-1}) \frac{\partial^2 \log \tau}{\partial t_1^2}.
\] (A.4)
Substitute (A.1) and (A.2) into the left hand side of (A.4), then the equality (A.3) follows from integrations. The proposition is proved. □

Example A.1 Let \( g' \) be the affine Lie algebra of type \( A_1^{(1)} \). We realize \( g' \) as in Example 5.1, and take
\[
\mathcal{L} = D + \Lambda + q, \quad q = \text{diag}(-v, v).
\] (A.5)
It is straightforward to compute the functions \( U \) and \( H \) in Proposition 3.1. Then it follows from (A.2) and (3.24) that
\[
\frac{\partial^2 \log \tilde{\tau}}{\partial x^2} = \frac{1}{2} \tilde{h}_1 = -\frac{1}{2} v^2,
\] (A.6)
\[
\frac{\partial^2 \log \tau}{\partial x^2} = \frac{1}{2} h_1 = -\frac{1}{2} (v^2 - v_x),
\] (A.7)
where we have used \( \partial / \partial t_1 = \partial / \partial x \). These two tau functions satisfy
\[
v = 2 \partial_x \log \frac{\tau}{\tilde{\tau}},
\] (A.8)
which is an equivalent version of (A.3).

In fact, by solving the equation of gauge transformation
\[
\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -v & 0 \\ 0 & v \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1 & -g \\ 0 & 1 \end{pmatrix} = D + \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -u \\ 0 & 0 \end{pmatrix},
\] (A.9)
(A.10)
one has \( g = v \) and
\[
u = -v^2 + v_x.
\] (A.11)
This is the well-known Miura transformation that relates the KdV and the modified KdV equations. From (A.7) we derive again \( u = 2 \partial_x^2 \log \tau \), see (5.15).

Enriquez and Frenkel’s construction of tau function \( \tilde{\tau} \) can be formally extended to the modified Drinfeld-Sokolov hierarchies associated to twisted affine algebras by (A.3). In fact, the different tau functions \( \tau \) and \( \tilde{\tau} \) can be interpreted by a general result in [36], see Proposition A.2 below.
A.2 Miramontes’ tau function

Tau functions of the (generalized) modified Drinfeld-Sokolov hierarchy were also constructed by Miramontes [36]. His method is based on a highest weight representation of the Kac-Moody group, and the resulting tau function is related to a family of conservation laws for the hierarchy.

Let us recall some statements in [36], following closely the original notations used there. Given a gradation \( s' = (s'_0, s'_1, \ldots, s'_n) \in \Gamma \) on the affine Lie algebra \( g' \) (recall (2.2)):

\[
g' = \bigoplus_{j \in \mathbb{Z}} g'_j[s'],
\]

(A.12)

Similarly as before, we use notations like \( g'_{\leq k}[s'] = \sum_{j \leq k} g'_j[s'] \), and let the subscript “\( \leq k \)” stand for the projection \( g' \rightarrow g'_{\leq k}[s'] \). Assume that an element \( \Lambda \in g'_{k}[s'] \) \((k > 0)\) is fixed, and it induces the following decomposition of subspaces:

\[
g' = \ker \text{ad}_\Lambda + \text{im} \text{ad}_\Lambda,
\]

(A.13)

Here \( \ker \text{ad}_\Lambda \) is a subalgebra consisting of elements that commute with \( \Lambda \) modulo \( \mathbb{C} c \); this subalgebra is generated by \( \Lambda_j \in g'_j[s'] \) \((j \in E_{\Lambda} \subset \mathbb{Z})\) such that \( [\Lambda_j, \Lambda_{-j}] \in \mathbb{C} c \).

Choose another gradation \( s = (s_0, s_1, \ldots, s_n) \preceq s' \) of \( g' \), namely, \( s_i \preceq s'_i \) for \( i = 0, 1, \ldots, n \). With the principal and homogeneous gradations on \( g' \) replaced by the gradations \( s' \) and \( s \) respectively, de Groot, Hollowood and Miramontes [23] defined the generalized Drinfeld-Sokolov hierarchies similarly as in Definition 2.2.

In more details, let

\[
\mathcal{L} = D + \Lambda + q + w \cdot c, \quad q \in C^\infty \left( \mathbb{R}, g'_{0}[s'] \cap g'_{\leq 0}[s'] \right),
\]

(A.14)

where the central part of \( q \) is zero, and \( w \) is an arbitrary smooth function. Similar to Proposition 2.1 there exists a function \( Y \in C^\infty \left( \mathbb{R}, g'_{< 0}[s'] \right) \) such that

\[
\mathcal{L} = e^{\text{ad}_Y} (D + \Lambda + h^\Phi) = \Phi(D + \Lambda + h^\Phi)\Phi^{-1}
\]

(A.15)

with \( \Phi = e^Y \) and \( h^\Phi \in C^\infty \left( \mathbb{R}, \ker \text{ad}_\Lambda \cap g'_{\leq 0}[s'] \right) \). Moreover, the functions \( Y \) and \( h^\Phi \) are supposed to be specified by the constraints (see (2.25) in [36]):

\[
Y \in g'_{< 0}[s], \quad (h^\Phi)_{\geq 0}[s] \in \mathbb{C} c.
\]

(A.16)

Introduce operators

\[
\mathcal{L}_j = \frac{\partial}{\partial t_j} + A_j, \quad A_j = (\Phi \Lambda_j \Phi^{-1})_{\geq 0}[s] + w_j \cdot c, \quad j \in E_{\Lambda,+},
\]

(A.17)

where \( w_j \) are some priori functions. The flows of the generalized Drinfeld-Sokolov hierarchy are defined by the following zero-curvature equations

\[
[\mathcal{L}, \mathcal{L}_j] = 0, \quad j \in (E_{\Lambda})_{\geq 0}.
\]

(A.18)
These equations imply
\[ \Phi^{-1} \mathcal{L}_j \Phi = \frac{\partial}{\partial t_j} + \Lambda_j + h(j) + w_j \cdot c, \quad (A.19) \]
where
\[ h(j) = \Phi^{-1} \frac{\partial \Phi}{\partial t_j} - \Phi^{-1} (\Phi \Lambda_j \Phi^{-1}) <_0 \Phi \in C^\infty \left( \mathbb{R}, \text{Ker ad}_\Lambda \cap g' \right). \]

In [36] the hierarchy (A.18) is restricted by \( t_1 = \nu x \) and \( L_1 = \nu L \) with some normalization constant \( \nu \). Hence one has \( \nu h \Phi = h(1) + w_1 \cdot c \).

For every auxiliary gradation \( m = (m_0, m_1, \ldots, m_n) \preceq s \), there exists a derivation \( d_0^{(m)} \) in the Kac-Moody algebra \( g \) whose derived algebra is \( g' \). For the sake of simplifying notations, we redenote \( d_0^{(m)} \) as \( d_m \); it satisfies (see Section 4)
\[ [d_m, e_i] = m_i e_i, \quad [d_m, f_i] = -m_i f_i, \quad [d_m, \alpha_i^\vee] = 0 \quad (A.20) \]
for \( i = 0, 1, \ldots, n \). The standard bilinear form on \( g' \) is extended to \( g \) by
\[ (d_m | d_m) = 0, \quad (d_m | \alpha_i^\vee) = \frac{k_i}{k_i'} m_i \quad \text{for} \quad i = 0, 1, \ldots, n. \quad (A.21) \]
Note that \( (d_m | c) = N_m \) with \( N_m = \sum_{i=0}^{n} k_i m_i \).

One can define the following conserved densities of the hierarchy (A.18) (see equation (3.8) in [36]):
\[ J_{j,k}[m] = \frac{N'}{k N_m} (d_m | [\Phi \Lambda_k \Phi^{-1}, A_j]), \quad j, k \in (E_\Lambda)_{\geq 0}. \quad (A.22) \]
These conserved densities are related to a tau function \( \tau_m \) as (cf. equation (4.13) in [36])
\[ J_{j,k}[m] = -\frac{N'}{k} \frac{\partial^2 \log \tau_\bar{m}}{\partial t_j \partial t_k}, \quad j, k \in (E_\Lambda)_{\geq 0}. \quad (A.23) \]
Here \( \bar{m} = (m_0 k_0 \alpha_0^\vee, \ldots, m_n k_n \alpha_n^\vee) \), it induces an integrable highest weight representation \( L(\bar{m}) \) of \( g \) with highest weight vector \( |v_\bar{m}\rangle \), and \( \tau_\bar{m} \) is defined by an element of the orbit of \( |v_\bar{m}\rangle \) acted by the Kac-Moody group, see § 4.1 and the appendix in [36] for details.

Now let us compare the tau function in (A.23) with those in (A.3) for the modified Drinfeld-Sokolov hierarchies. Henceforth we fix both \( s \) and \( s' \) to be the principal gradation \( s^1 \). In this case \( E_\Lambda = E \) is the set of exponents of \( g' \), and \( \text{Ker ad}_\Lambda \) is the principal Heisenberg subalgebra \( s \). Choose generators \( \Lambda_j \) normalized by (1.21), with \( \Lambda_1 = \nu \Lambda = \nu \sum_{i=0}^{n} e_i \). Let
\[ \mathcal{L}_j = \mathcal{L}_j \quad \text{for} \quad j \in E_+, \quad Y = U, \quad h(1) = \nu H \]
with \( U \) and \( H \) determined as in Proposition 3.1. Observe that both \( Y \) and \( h^\Phi \) admit the constraint (A.16).

**Proposition A.2** For the modified Drinfeld-Sokolov hierarchy associated to affine Lie algebra \( g' \) with \( s = s' = s^1 \) being the principal gradation, the two tau functions in (A.3) can
be considered as particular cases of tau functions in (A.23). More precisely, the following equalities hold true

\[
\log \tau_m = \begin{cases} 
\log \tau, & m = (1,0,\ldots,0); \\
\log \tilde{\tau}, & m = (1,1,\ldots,1).
\end{cases}
\]  

(A.24)

Proof Since \(N_{s_1}\) is equal to the Coxeter number \(h\), one recasts (A.22) to

\[
J_{j,k}[m] = \frac{h}{kN_m} \left( \frac{d}{\partial t_j} \right) \left( \Phi \Lambda_k \Phi^{-1}, \mathcal{L}_j - \frac{\partial}{\partial t_j} \right)
\]

\[
= \frac{h}{kN_m} \left( \frac{d}{\partial t_j} \right) \left( \Phi \Lambda_k, \frac{\partial}{\partial t_j} + \Lambda_j + h(j) + w_j \cdot c \right) \Phi^{-1}
\]

\[
= \frac{h}{kN_m} \left( \frac{d}{\partial t_j} \right) (\Phi \Lambda_k \Phi^{-1})
\]

\[
\frac{h}{k}[\Lambda_k, h(j)]_c + \frac{h}{kN_m} \frac{\partial}{\partial t_j} (d_m | \Phi \Lambda_k \Phi^{-1}).
\]  

(A.25)

If the auxiliary gradation \(m = (1,0,\ldots,0)\) is the homogeneous one, then \(N_m = k_0\). By virtue of

\[
(d_m | \alpha_i^\vee) = \delta_{i0} \frac{k_i}{k^\vee_i}, \quad (\Phi \Lambda_k \Phi^{-1})_c = (e^{ad^c \Lambda_k})_c = 0,
\]

the second term in (A.25) vanishes. Hence we have

\[
\frac{k}{h} J_{1,k}[m] = [\Lambda_k, h(1)]_c = \frac{k}{(\Lambda_k | \Lambda_{-k})} (\Lambda_k | \nu H),
\]  

(A.26)

that is, thanks to (A.23) and (3.22),

\[
- \frac{\partial^2 \log \tau_m}{\partial t_1 \partial t_k} = - \frac{\partial^2 \log \tau}{\partial t_1 \partial t_k}.
\]  

(A.27)

If \(m = (1,1,\ldots,1)\) is the principal gradation, then \(N_m = h\). By using (A.25) and \((\Lambda_j | \Lambda_k) = \delta_{j,-k} h\) one has

\[
\frac{1}{h} J_{1,1}[m] = [\Lambda_1, h(1)]_c + \frac{1}{h} \frac{\partial}{\partial t_1} (d_m | \Phi \Lambda_1 \Phi^{-1})
\]

\[
= \frac{(\Lambda_1 | \nu H)}{(\Lambda_1 | \Lambda_{-1})} + \frac{1}{h} \frac{\partial}{\partial t_1} (d_m | [U_{-1}, \Lambda_1])
\]

\[
= - \frac{\partial^2 \log \tau}{\partial t_1^2} - \frac{1}{h} \frac{\partial}{\partial t_1} (d_m, \Lambda_1 | U_{-1})
\]

\[
= - \frac{\partial^2 \log \tau}{\partial t_1^2} - \frac{1}{(\Lambda_1 | \Lambda_{-1})} \frac{\partial}{\partial t_1} (\Lambda_1 | U_{-1}).
\]  

(A.28)

This together with (A.23) and (A.3) leads to

\[
\frac{\partial^2 \log \tilde{\tau}_m}{\partial t_1^2} = \frac{\partial^2 \log \tilde{\tau}}{\partial t_1^2}.
\]  

(A.29)
Therefore, the proposition follows from (A.27) and (A.29) by integration in the same way as to show Proposition 5.1.

The proof of this theorem also implies that, whenever $s$ is the principal gradation, one can choose $n+1$ linearly independent auxiliary gradations $m \preceq s$, which correspond to $n+1$ distinct tau functions $\tau_m$ of the modified Drinfeld-Sokolov hierarchy. In particular, suppose $g'$ is of type $A_1^{(1)}$, then such two tau functions of the modified KdV hierarchy are given in Example A.1.

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