LOCAL REGULARITY OF THE BERGMAN PROJECTION ON A CLASS OF PSEUDOCONVEX DOMAINS OF FINITE TYPE

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Abstract. The purpose of this paper is to prove that if a pseudoconvex domains $\Omega \subset \mathbb{C}^n$ satisfies Bell-Ligocka’s Condition R and admits a “good” dilation, then the Bergman projection has local $L^p$-Sobolev and Hölder estimates. The good dilation structure is phrased in terms of uniform $L^2$ pseudolocal estimates for the Bergman projection on a family of anisotropic scalings. We conclude the paper by showing that $h$-extendible domains satisfy our hypotheses.

1. Introduction

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary $\partial \Omega$. The Bergman projection $B = B_\Omega$ is one of the fundamental objects associated to $\Omega$; it is the orthogonal projection of $L^2(\Omega)$ onto the closed subspace of square-integrable holomorphic functions on $\Omega$. We can express the Bergman projection via the integral representation

$$Bv(z) = \int_\Omega B(z, w)v(w)\, dw,$$

where $dw$ is the Lebesgue measure on $\Omega$, and the integral kernel $B$ is called the Bergman kernel.

Since the Bergman projection is defined abstractly on $L^2(\Omega)$, basic questions about $B$ include the local and global regularity and estimates in other spaces, namely

1. $C^\infty$ and $L^2$,
2. $L^p$ ($p \neq 2$) and the spaces of Hölder continuous functions $\Lambda_s$.

When $\Omega$ is of finite type (see [D’A82]), Question 1 has been completely answered [Cat83, Cat87, KN65, FK72], and we therefore focus on aspects of Question 2 that relate directly to the Bergman projection and tools that we can apply in $L^2(\Omega)$ and Hölder spaces. Condition R is a well known property introduced by Bell and Ligocka [BL80] to study the smoothness of biholomorphic mappings and is intimately connected with Question 1. We will introduce a local version and refer to the original as global Condition R. Specifically, for a domain $\Omega \subset \mathbb{C}^n$, we say that $\Omega$ satisfies global Condition R if for every $s \geq 0$ there is $M = M_s$ such that

$$\|Bu\|_{L^2_s(\Omega)} \leq c_{s, \Omega}\|u\|_{L^2_{s+M}(\Omega)}$$

for all $u \in L^2_{s+M}(\Omega)$.

Global Condition R suggests the following local version. For a domain $\Omega \subset \mathbb{C}^n$ and an open set $U \subset \mathbb{C}^n$, we say that $\Omega$ satisfies $L^2$ pseudolocal estimates for the Bergman projection in $U$ if for

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every $s, m \geq 0$ there is $M = M_{s,m}$ and a constant $c = c_{s,M,U,\Omega} > 0$ such that

$$\|\chi_1 Bu\|_{L^2(\Omega)}^2 \leq c \left( \|\chi_2 u\|_{L^2_{s+M}(\Omega)}^2 + \|\chi_3 Bu\|_{L^2_{-M}(\Omega)}^2 \right)$$

for all $u \in L^2_{s+M}(U \cap \Omega) \cap L^2(\Omega)$, where $\chi_j \in C^\infty_c(U), j = 1, 2, 3$ and $\chi_j \prec \chi_j+1$.

We will use the following notation throughout this paper. For cutoff functions $\chi, \chi' \in C^\infty_c(U)$, we write $\chi \prec \chi'$ if $\chi' = 1$ on supp($\chi$). We use the notation $a \lesssim b$ (respectively, $a \gtrsim b$) if there exists a global constant $c > 0$ so that $a \leq cb$ (respectively, $a \geq cb$). Moreover, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$. Also, $L^s_k(\Omega)$ are the usual $L^p$-Sobolev spaces of order $s$ on $\Omega$. The space $L^p_{s+}(\Omega)$ is the dual space of $(L^p_s(\Omega))_0$, which is the closure of $C^\infty_c(\Omega)$ in $L^p_s(\Omega)$. Here, $p$ and $p'$ are Hölder conjugates.

Global Condition $R$ often arises as a consequence of estimates used to prove global regularity for the $\bar{\partial}$-Neumann operator. In particular, compactness estimates (which themselves are a consequence of Catlin’s Property (P) or McNeal’s Property ($\tilde{P}$)) or the existence of a plurisubharmonic defining function both imply the global regularity of the $\bar{\partial}$-Neumann operator [Cat84] [McN02] [BS91]. See [Str08] [Har11] for more general sufficient conditions for global regularity.

Similarly, pseudolocal estimates for the Bergman projection are a consequence of the local regularity theory for the $\bar{\partial}$-Neumann problem. It is classical that both (interior) elliptic and subelliptic estimates for the $\bar{\partial}$-Neumann problem implies this local property [KN65] [FK72]. Ellipticity only holds for interior sets of domains. Subellipticity itself is equivalent to a finite type condition on the boundary [Cat83] [Cat87]. Moreover, there are several classes of pseudoconvex domains of infinite type for which this local property holds [Koh02] [KZ12] [BKZ14] [BPZ15].

A positive answer to Question 2 has been obtained when $\Omega$ is both of finite type and satisfies one of the following hypotheses:

1. strict pseudoconvexity [FS74] [PS77],
2. pseudoconvexity in $C^2$ [Chr88] [FKK88a] [FKK88b] [McN89] [NRSW89] [CNS92],
3. pseudoconvexity in $C^n$ and a Levi-form with comparable eigenvalues [Koo02], or one degenerate eigenvalue [Mac88].
4. decoupled [FKM90] [CD06] [NS06],
5. convexity in $C^n$ [McN94] [MS94] [MS97].

The purpose of this paper is to give a full answer to Question 2 for a class of pseudoconvex domains of finite type that admit a good anisotropic dilation, and these scale turn out to be closely related to Catlin’s multitype. This class of domains includes $h$-extendible domains (defined below) as well as types 1-5 above [Yu94] [Yu95].

Recall that a defining function $\rho$ for a domain $\Omega \subset C^n$ is a $C^1$ function defined on a neighborhood of $\Omega$ so that $\Omega = \{ z \in C^n : \rho(z) < 0 \}$, $b\Omega = \{ z \in C^n : \rho(z) = 0 \}$, and $\nabla \rho \neq 0$ on $b\Omega$. In this paper, we reserve $r = r_\Omega$ for the signed distance to the boundary function.

**Definition 1.1.** Let $\Omega$ be a pseudoconvex domain in $C^n$ with smooth boundary $b\Omega$. Let $p \in b\Omega$ and $z = (z_1, \ldots, z_n)$ be coordinates so that $p$ is the origin and Re $z_1$ is the real normal direction to $b\Omega$ at $p$. We say that $\Omega$ has a **good anisotropic dilation at** $p$ if there exist smooth, increasing functions $\phi_j : (0, 1) \to \mathbb{R}^+$, $j = 1, \ldots, n$, so that $\phi_j(\delta)/\delta$ is decreasing, $\phi_j(\delta) := \delta$, and $\phi_j(1) = 1$ for $j = 1, \ldots, n$. Additionally, for every small $\delta > 0$, the anisotropic dilation

$$\hat{z} = \Phi_\delta(z) = \left( \frac{z_1}{\phi_1(\delta)}, \ldots, \frac{z_n}{\phi_n(\delta)} \right) : C^n \to C^n$$

satisfies two conditions:
Theorem 1.2. Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$ and $(p, q) \in (\Omega \times \Omega) \setminus \{\text{Diagonal of } \Omega \times \Omega\}$. Assume that either $\pi(p)$ or $\pi(q)$ admits a good anisotropic dilation $\Phi_j(\delta) = (\frac{\delta z_1}{\delta \phi_1(\delta)}, \ldots, \frac{\delta z_n}{\delta \phi_n(\delta)})$. Then

$$\left| \prod_{j=1}^{n} \frac{\partial^{\alpha_j + \beta_j}}{\partial p_j^{\alpha_j} \partial \overline{q}_j^{\beta_j}} \right| B(p, q) \leq C_{\alpha, \beta} \prod_{j=1}^{n} \left( \phi_j \left( |r(p)| + |r(q)| + \sum_{k=1}^{n} \phi_j^*(|p_j - q_j|) \right) \right)^{-2 - \alpha_j - \beta_j}$$

for nonnegative integers $\alpha_j, \beta_j$. The constant $C_{\alpha, \beta}$ is independent of $p, q$ and $*$ denotes the function inversion operator, i.e., $\phi^*(\phi(\delta)) = \delta$.

Our first result contains pointwise estimates for derivatives of the Bergman kernel. Also, the function $\pi$ maps points in $\Omega$ that are near $\partial \Omega$ to the closest point of $\partial \Omega$.

Theorem 1.3. Let $\Omega$ be a smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$ satisfying global Condition $R$. Let $U$ be an open set so that both $U \subset \subset \Omega$ and $\Omega \cap U$ is a set of good anisotropic dilation points. Then the Bergman projection $B$ is locally regular on the set $U$ in both $L^p_s$ with $s \geq 0$, $p \in (1, \infty)$ and $\Lambda_s$ with $s > 0$.

Namely, whenever $\chi_0, \chi_1 \in C^\infty_c(U)$ with $\chi_0 < \chi_1$, there exists constants $c_s, c_{s, p} > 0$ so that

$$\| \chi_0 Bv \|_{L^p_s(\Omega)} \leq c_{s, p} \left( \| \chi_1 v \|_{L^p_{s, p}(\Omega)} + \| v \|_{L^p_{s, p}(\Omega)} \right)$$

for $v \in L^p_s(\Omega \cap U) \cap L^p(\Omega)$, $s \geq 0$ and $p \in (1, \infty)$; and

$$\| \chi_0 Bv \|_{\Lambda_s(\Omega)} \leq c_s \left( \| \chi_1 v \|_{\Lambda_s(\Omega)} + \| v \|_{L^\infty(\Omega)} \right)$$

for $v \in \Lambda_s(\Omega \cap U) \cap L^\infty(\Omega)$ and $s > 0$.

We remind the reader of the definition of the H"older spaces $\Lambda_s(\Omega)$ below (Definition 4.4).

Theorem 1.3 is only useful if there exist domains which satisfy the hypotheses, and we now show there are large classes of domains which do so. Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$ and $p$ be a boundary point. There are several notions of the "type" of a point that aim to measure the curvature of $b\Omega$ at $p$. Two of the most widely known are the

- D’Angelo (multi)-type, $\Delta(p) = (\Delta_n(p), \ldots, \Delta_1(p))$ where $\Delta_k(p)$ is the $k$-type, which measures the maximal order of contact of $k$-dimensional varieties with $b\Omega$ at $p$; and
- Catlin multitype, $\mathcal{M}(p) = (m_1(p), \ldots, m_n(p))$, where $m_k(p)$ is the optimal weight assigned to the coordinate direction $z_k$.

With these definitions, $\Delta_n(p) = m_1(p) = 1$. In [Cat87], Catlin proved that $\mathcal{M}(p) \leq \Delta(p)$ in the sense that $m_{n-k+1}(p) \leq \Delta_k(p) < \infty$ for $1 \leq k \leq n$. The following definition is given by Yu:
Definition 1.4. A pseudoconvex domain is called $h$-extendible at $p$ if $\Delta(p) = M(p)$. If $\Omega$ is $h$-extendible at $p$, $M(p)$ is called the multitype at $p$. A pseudoconvex domain is called $h$-extendible if every boundary point is $h$-extendible.

In [Yu94], Yu proves $h$-extendibility at $p$ is equivalent to the existence of coordinates $z = (z_1, z')$ centered at $p$ and a defining function $\rho$ that can be expanded near 0 as follows:

$$\rho(z) = \Re z_1 + P(z') + R(z).$$

Here $P$ is a $(\frac{1}{m_2}, \ldots, \frac{1}{m_n})$-homogeneous plurisubharmonic polynomial, i.e.,

$$P(\delta^{1/m_2} z_2, \ldots, \delta^{1/m_n} z_n) = \delta P(z_2, \ldots, z_n)$$

and contains no pluriharmonic terms. The function $R$ is smooth and satisfies

$$R(z) = o\left(\sum_{j=1}^n |z_j|^{m_j + \alpha}\right)$$

for some $\alpha > 0$.

The $h$-extendible property allows for a pseudoconvex domain $\Omega$ to be approximated by a pseudoconvex domain from the outside. See [BSY95, Yu94, Yu95] for a discussion.

Theorem 1.5. Let $\Omega$ be an $h$-extendible, bounded domain in $\mathbb{C}^n$. Then $\Omega$ satisfies global condition $R$ and $b\Omega$ is a set of good anisotropic dilation points. Consequently, the Bergman projection is locally regular in the spaces $L^p_s(\Omega)$ with $1 < p < \infty$, $s \geq 0$ and $\Lambda_s(\Omega)$ with $s > 0$.

The proof of Theorem 1.5 reveals a new property of $h$-extendible points which we record as our final theorem.

Theorem 1.6. Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$. Assume that the open set $S \subset b\Omega$ is $h$-extendible. Then the function $T$ defined by

$$T(p) = \sum_{k=1}^n \frac{1}{m_k(p)}, \quad \text{for } p \in S,$$

is lower semicontinuous.

Remark 1.7. Since the Catlin multitype takes on a finite number of values on $S$, the lower semicontinuity is equivalent to the following maximality property for $T$: For every point $p \in S$ there exists a neighborhood $V \subset S$ of $p$ such that for every $q \in V$, $T(p) \leq T(q)$.

The paper is organized as follows. In Section 2 we recall results on local $L^2_s$ estimates and $C^\infty$-regularity of the $\bar{\partial}$-Neumann operator and the Bergman projection. In Section 3 we give a proof of Theorem 1.2. In Section 4 we prove Theorem 1.3. In Section 5 we prove Theorem 1.5 and Theorem 1.6.

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2. Uniform estimates on the Bergman kernel

2.1. The smoothness of kernels: local behavior. In this subsection, $\Omega$ is a smooth, bounded pseudoconvex domain and $U$ is an open set in which $L^2$ pseudolocal estimates for the Bergman projection hold. We start our estimate of the Bergman kernel by proving that $B(z,w)$ is smooth near the diagonal and satisfies uniform estimates when the points $z$ and $w$ are a uniform distance apart. Throughout the paper we use the notation that if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple of nonnegative integers, then $D^n = \prod_{j=1}^n \frac{\partial^{\alpha_j}}{\partial z_j^{\alpha_j}}$.

Theorem 2.1. Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded pseudoconvex domain and $U$ be an open set in $\mathbb{C}^n$. Suppose that $L^2$ pseudolocal estimates for the Bergman projection hold on $U$. Then the Bergman kernel is smooth on $((\Omega \cap U) \times (\Omega \cap U)) \setminus \{\text{Diagonal of } \Omega \cap U\}$. Moreover, for every $c > 0$ and multi-indices $\alpha$ and $\beta$, there exists a positive constant $c_{\alpha,\beta,U}$ so that for every $(z,w) \in ((\Omega \cap U) \times (\Omega \cap U))$

$$\delta_I(z,w) := |r(z)| + |r(w)| + |z - w| \geq c,$$

then

$$|D_p^\alpha D_q^\beta B(z,w)| \leq c_{\alpha,\beta,U}$$

and $c_{\alpha,\beta,U}$ is independent of $z$, $w$, and $\Omega$.

We refer to $\delta_I(p,q)$ as the isotropic distance of $\Omega$, though we recognize that $\delta_I(\cdot,\cdot)$ is usually neither isotropic nor a distance function of $\mathbb{C}^n$. We introduce a “nonistropic distance” in Lemma 4.1 below.

Proof. We wish to apply $B$ to an approximation of the identity, so we let $\psi \in C_0^\infty(B(0,1))$ where $\psi \geq 0$, radial, and $\int_{\mathbb{C}^n} \psi \, dw = 1$. Let $w \in \Omega \cap U$ and set $\psi_t(\zeta) = t^{-2n} \psi((\zeta - w)/t)$.

When $z \neq w$, the fact that $B(z,w)$ is harmonic in $w$ means that for $t$ small enough

$$D_w^\beta B(z,w) = \int_\Omega B(z,\zeta) D_w^\beta \psi_t(\zeta) \, d\zeta = (-1)^{|eta|} \int_\Omega B(z,\zeta) D_\zeta^\beta \psi_t(\zeta) \, d\zeta = (-1)^{|eta|} (BD^\beta \psi_t)(z).$$

Since the Bergman operator is locally regular in $C^\infty$,

$$D_z^\alpha D_w^\beta B(z,w) = (-1)^{|eta|} (D^\alpha BD^\beta \psi_t)(z) \in C^\infty(\Omega \cap U)$$

The hypothesis $\delta_I(z,w) \geq c$ implies that $|z - w| \geq \frac{c}{\|B\|}$, $|r(z)| \geq \frac{c}{r}$, or $|r(w)| \geq \frac{c}{r}$.

Case 1: $|z - w| \geq \frac{c}{r}$. We choose $\epsilon$ sufficiently small such that $B(z,2\epsilon) \cap B(w,2\epsilon) = \emptyset$ and $B(z,2\epsilon), B(w,2\epsilon) \subset U$. Let $\chi_1 < \chi_2 < \chi_3$ such that $\chi_1 = 1$ on $B(z,\epsilon)$ and $\supp(\chi_3) \subset B(z,2\epsilon)$. By the Sobolev Lemma, we have (for $t < \epsilon/2$)

$$|D_z^\alpha D_w^\beta B(z,w)| \leq \sup_{\xi \in B(z,\epsilon) \cap \Omega} |D_z^\alpha D_w^\beta B(\xi,w)| \leq c_\alpha \|\chi_1 BD^\beta \psi_t\|_{L^2_{n+1+|\alpha|}(\Omega)}.$$

Using (1.1) with $u = D^\beta \psi_t$, we obtain

$$\|\chi_1 BD^\beta \psi_t\|_{L^2_{n+1+|\alpha|}(\Omega)} \leq c_\alpha \left( \|\chi_2 D^\beta \psi_t\|_{L^2_2(\Omega)} + \|\chi_3 BD^\beta \psi_t\|_{L^2(\Omega)} \right)$$

where $c$ depends on $|\alpha|$ and $n$ but not on $\Omega$. Here the equality follows from the fact that $\supp(\chi_j) \cap \supp(\psi_t) = \emptyset$. On the other hand, by the density of smooth, compactly supported functions in $L^2(\Omega)$,

$$\|\chi_3 BD^\beta \psi_t\|_{L^2(\Omega)} = \sup_{v \in C_c^\infty(\Omega), \|v\|_{L^2(\Omega)} \leq 1} |(\chi_3 BD^\beta \psi_t,v)|_{L^2(\Omega)}.$$
Using the self-adjointness of $B$ and the pairing of $(L^2_{n+1}(\Omega))_0$ with its dual $L^2_{-n}(\Omega)$, we have

$$|(\chi_3 B D^\beta \psi_t, v)_{L^2(\Omega)}| = |(\psi_t, D^\beta B \chi_3 v)_{L^2(\Omega)}|$$

$$= |(\psi_t, \tilde{\chi}_1 D^\beta B \chi_3 v)_{L^2(\Omega)}|$$

$$= \|\psi_t\|_{L^2_{-n}(\Omega)} \|\tilde{\chi}_1 D^\beta B \chi_3 v\|_{L^2_{n+1}(\Omega)},$$

where $\tilde{\chi}_1$ is chosen such that $\tilde{\chi}_1 = 1$ on $B(w, \epsilon)$. Additionally, choose $\tilde{\chi}_j \in C^\infty_0(\mathbb{C}^n)$, $j = 2, 3$ so that $\tilde{\chi}_1 < \tilde{\chi}_2 < \tilde{\chi}_3$ and supp$(\tilde{\chi}_3) \subset B(w, 2\epsilon)$. Note that this forces supp$(\tilde{\chi}_2) \cap$ supp$(\tilde{\chi}_3) = \emptyset$. Since $\psi_t \to \delta_w$ in $(C^0(\mathbb{C}^n))^*$ and $L^2_{n+1}(\mathbb{C}^n) \subset C^0_0(\mathbb{C}^n)$ by Sobolev’s Lemma, it follows from duality that

$$\|\psi_t\|_{L^2_{-n}(\Omega)} < c.$$

for some $c > 0$ that is independent of $t$ and $\Omega$. By a second application of the inequality (1.1) for cut-off functions $\tilde{\chi}_1, \tilde{\chi}_2$ and $\tilde{\chi}_3$ and the fact that $\tilde{\chi}_2 \tilde{\chi}_3 = 0$ by support considerations, we obtain

$$\|\tilde{\chi}_1 D^\beta B \chi_3 v\|_{L^2_{n+1}(\Omega)} \leq c_\beta (\|\tilde{\chi}_2 \chi_3 v\|_{L^2_{n+1}(\Omega)} + \|\tilde{\chi}_3 B \chi_3 v\|_{L^2(\Omega)})$$

$$= c_\beta \|\tilde{\chi}_3 B \chi_3 v\|_{L^2(\Omega)} \leq c_\beta \|B \chi_3 v\|_{L^2(\Omega)} \leq c_\beta \|\chi_3 v\|_{L^2(\Omega)} \leq c_\beta,$$

since $B$ is an orthogonal projection on $L^2(\Omega)$. Here, $c_\beta$ depends on $\beta$ and $n$ but does not depend on $\Omega$.

**Case 2:** $|r(z)| \geq \frac{\epsilon}{3}$ or $|r(w)| \geq \frac{\epsilon}{3}$. Assume $|r(z)| \geq c$. If $w$ is near the boundary then $|z - w| \geq \frac{\epsilon}{2}$, and the conclusion follows from Case 1. Otherwise $z$ is near $w$, and we can use the interior elliptic regularity of the $\bar{\partial}$-Neumann problem and (1.1) (and Sobolev’s Lemma, as above) to obtain

$$|D^\alpha_z D^\beta_w B(z, w)| \leq c_{\alpha, \beta}$$

where $c_{\alpha, \beta}$ is independent of both $z, w$ and the diameter of $\Omega$ when $|r(z)|, |r(w)| \geq \frac{\epsilon}{2}$.

In both cases, we have proven that

$$|D^\alpha_z D^\beta_w B(z, w)| \leq c_{\alpha, \beta}$$

uniformly for $w \in \Omega \cap U$. As a consequence of the $L^2$-Sobolev regularity of the Bergman projection on finite type domains and the Sobolev Embedding Theorem, this inequality still holds for $w \in \Omega \cap U$. This completes the proof of Theorem 2.1.

2.2. **The smoothness of kernels: local/nonlocal.** In this subsection we establish smoothness of the Bergman kernel in the case that one point is in a set for which $L^2$ pseudolocal estimates for the Bergman projection hold and the other is arbitrary. We observe that our estimates may depend on diameter of $\Omega$, however, we will only apply these estimates in a fixed domain.

**Theorem 2.2.** Let $\Omega \subset \mathbb{C}^n$ be a smooth, bounded pseudoconvex domain and $U$ be an open set in $\mathbb{C}^n$. Suppose that $L^2$ pseudolocal estimates for the Bergman projection hold on $U$ and global Condition $R$ holds for $\Omega$. Then the Bergman kernel is smooth on $((\Omega \cap U) \times \bar{\Omega}) \setminus \{\text{Diagonal of } b\Omega \cap U\}$. Moreover, for fixed $c > 0$ and multi-indices $\alpha$ and $\beta$, whenever there exists $c_{\alpha, \beta} > 0$ so that for every $(z, w) \in ((\Omega \cap U) \times \bar{\Omega})$ satisfying

$$|z - w| \geq c,$$

it follows that

$$|D^\alpha_z D^\beta_w B(z, w)| \leq c_{\alpha, \beta}.$$
Proof. Adopting the notation and argument from the first part of the proof of Theorem 2.1, we have

\[ |D^\alpha D^\beta B(z, w)| \leq \|\chi_1 BD^\beta \psi_t\|_{L^2_{\alpha+|\alpha|}(\Omega)} \]

\[ \leq c_{\alpha,m}(\chi_2 B^2 \psi_t\|_{L^2_\alpha(\Omega)} + \|\chi_3 BD^\beta \psi_t\|_{L^2_{\alpha+m}(\Omega)}) \]

\[ = c_{\alpha,m}\|\chi_3 BD^\beta \psi_t\|_{L^2_{\alpha+m}(\Omega)} \]

where \( m \geq 0 \) will be chosen later and \( c_{\alpha,m} \) depends on \( \alpha \) and \( m \). However,

\[ \|BD^\beta \psi_t\|_{L^2_{\alpha+m}(\Omega)} = \sup\{(|BD^\beta \psi_t, v|_{L^2(\Omega)} : \|v\|_{L^2_{\alpha+m}(\Omega)} \leq 1\} \]

\[ = \sup\{(|\psi_t, D^\beta Bv|_{L^2(\Omega)} : \|v\|_{L^2_{\alpha+m}(\Omega)} \leq 1\} \]

\[ \leq \sup\{(\|\psi_t\|_{L^2_{(\alpha+1)}(\Omega)} \|Bv\|_{L^2_{\alpha+|\beta|+M}(\Omega)} : \|v\|_{L^2_{\alpha+m}(\Omega)} \leq 1\} \]

\[ \leq c_{\beta}\|v\|_{L^2_{\alpha+m}(\Omega)} \|\psi_t\|_{L^2_{\alpha+m}(\Omega)} \]

where the second inequality follows from the facts that \( \|\psi_t\|_{L^2_{\alpha+m}(\Omega)} \leq C \) for a constant \( c > 0 \) that is independent of \( t \) and the global Condition \( R \) with the choice \( m \geq n+1+|\beta|+M \). \( \square \)

Remark 2.3. In [Boa87], Boas proved a result similar to Theorem 2.2 with the stronger hypothesis that \( z \in \partial \Omega \) is a point of finite type and Catlin’s Property \( (P) \) holds.

3. Proof of Theorem 1.2

The following lemma follows easily by the definitions.

Lemma 3.1. Let \( u \) be a smooth function on \( \Omega \). For \( z \in \Omega \), denote \( \hat{z} := \Phi_\delta(z) \) and \( \hat{u}(\hat{z}) := u(z) \).

Then

\[ \left( \prod_{j=1}^{n} D^\alpha_j \right) u(z) = \left( \prod_{j=1}^{n} (\phi_j(\delta))^{-\alpha_j} \right) \left( \prod_{j=1}^{n} D^\alpha_j \right) \hat{u}(\hat{z}) \]

Proof of Theorem 1.2. The proof has three steps.

Step 1. We observe that the result is only in question for points close to \( \partial \Omega \), so we fix \( \sigma > 0 \) and focus on points of distance at most \( \sigma \) from \( \partial \Omega \). Therefore, we fix a point \( p \in \Omega \) with \( r(p) > -\sigma \), translate and rotate (unitarily) the domain so that \( \pi(p) = 0 \) and \( p \) is on the \( \text{Re } z_1 \) axis. Next, we fix a second point \( q \) in \( B(0, \sigma) \cap \Omega \).

Step 2. We employ a nonisotropic scaling based on the good anisotropic dilation functions \( \phi_j \) and a scaling constant \( A \geq 1 \) that we determine later but will depend only on \( \sigma \). We then observe how the Bergman kernel behaves under the scaling.

Step 3. We conclude by showing that if \( A > \sqrt{n+1}/\sigma \), then \( \hat{p} \) and \( \hat{q} \) are \( \hat{\Omega} \cap B(0, \sigma) \). In this case, the (scaled, isotropic) distance between them is bounded away from 0, independently of \( p \) and \( q \). We can therefore apply Theorem 2.1 because the constant in Theorem 2.1 depends only on \( \alpha, \beta \), and \( B(0, \sigma) \) and NOT on \( \Omega \). We now turn to the detailed arguments of Steps 1-3.

Step 1. By Theorem 2.1, we only need to work on the case that \( \delta_{L, \Omega}(p, q) \) is sufficiently small, say, \( \delta_{L, \Omega}(p, q) \leq \sigma \) for some fixed \( \sigma > 0 \). Without loss of generality, we can assume that \( \pi(p) \) is a point with a good anisotropic dilation

\[ \Phi_\delta(z) = \left( z_1/\phi_1(\delta), \ldots, z_n/\phi_n(\delta) \right) \]

with associated coordinates \( z \) and a fixed neighborhood \( \hat{U} = B(0, \sigma) \) of the origin \( \pi(p) \) such that \( p \in \text{Re } z_1 \) and \( p, q \in \hat{U} \). Denote \( \hat{p} = \Phi_\delta(p), \hat{q} = \Phi_\delta(q) \), and \( \Omega_\delta = \Phi_\delta(\Omega) \). Define \( \hat{r}(\hat{z}) := \sigma r(\Phi^{-1}_\delta(\hat{z})) \)
for \( \hat{z} \in \mathbb{C}^n \). Then the function \( \hat{r}_\delta \) is a defining function of \( \Omega_\delta \). Moreover, for all \( j = 1, \ldots, n \) we have
\[
\left| \frac{\partial \hat{r}_\delta}{\partial \hat{z}_j} \right| = \left| \frac{\phi_j(\delta) \partial r(\Phi_\delta^{-1}(\hat{z}))}{\delta} \right| \lesssim 1, \quad \text{for all } \hat{z} \in \hat{U},
\]
where the inequality follows by Definition \( \text{[1.1]} \) part 1. In fact, when \( j = 1 \), the inequality \( \lesssim \) can be replaced by the equality \( \approx \) since Re \( z_1 \) is the normal direction to \( b\Omega \) at \( \pi(p) \) (see Definition \( \text{[1.1]} \)). Thus
\[
|\nabla_{\hat{z}} \hat{r}_\delta(\hat{z})| \approx 1, \quad \text{for } \hat{z} \in \hat{U},
\]
uniformly in \( \delta \). This means that \( \hat{r}_\delta(\hat{z}) \) can be considered as a distance function from \( \Omega_\delta \cap \hat{U} \) to \( b\Omega_\delta \).

Step 2. By the transformation law for the Bergman kernel under biholomorphic mappings, we have
\[
\text{B}_{\Omega}(p, q) = \text{det} J_{\Phi}(p) B_{\Omega_\delta}(\hat{p}, \hat{q}) \text{det} J_{\Phi}(q) = \prod_{j=1}^n (\phi_j(\delta))^{-2} B_{\Omega_\delta}(\hat{p}, \hat{q}).
\]
Combining (3.1) with Lemma \( \text{[3.1]} \) we obtain
\[
(3.2) \quad \left( \prod_{j=1}^n \frac{\partial^\alpha_j + \beta_j}{\partial p_j^{\alpha_j} \partial q_j^{\beta_j}} \right) \text{B}_{\Omega}(p, q) = \prod_{j=1}^n (\phi_j(\delta))^{-2-\alpha_j-\beta_j} \left( \prod_{j=1}^n \frac{\partial^\alpha_j + \beta_j}{\partial p_j^{\alpha_j} \partial q_j^{\beta_j}} \right) B_{\Omega_\delta}(\hat{p}, \hat{q}).
\]

For \( A \geq 1 \) to be determined later and \( \sigma \) suitably small (so the expressions below are defined), we set
\[
\delta = \left( A|r(p)| + A|r(q)| + \sum_{j=1}^n \phi_j^*(A|p_j - q_j|) \right) \geq |r(p)| + |r(q)| + \sum_{j=1}^n \phi_j^*(|p_j - q_j|).
\]
Since the \( \phi_j^* \)'s are increasing,
\[
\prod_{j=1}^n (\phi_j(\delta))^{-2-\alpha_j-\beta_j} \leq \prod_{j=1}^n \left( \phi_j \left( |r(p)| + |r(q)| + \sum_{j=1}^n \phi_j^*(|p_j - q_j|) \right) \right)^{-2-\alpha_j-\beta_j}.
\]
Thus the proof of this theorem is complete if we show that there exists \( C_{\alpha, \beta} > 0 \) so that
\[
(3.3) \quad \left( \prod_{j=1}^n \frac{\partial^\alpha_j + \beta_j}{\partial p_j^{\alpha_j} \partial q_j^{\beta_j}} \right) B_{\Omega_\delta}(\hat{p}, \hat{q}) \leq C_{\alpha, \beta}
\]
uniformly in \( \hat{p} \) and \( \hat{q} \).

Step 3. We are going to apply Theorem \( \text{[2.1]} \) to prove \( \text{[3.3]} \). In order to do it, we must to check that \( \hat{p}, \hat{q} \in \hat{U} \) and with our choice of \( \delta \) that \( \delta_{\Omega_\delta}(\hat{p}, \hat{q}) \geq c \) independently of \( p \) and \( q \) (our choice of \( \delta \) will ensure that \( \hat{p} \) and \( \hat{q} \) are sufficiently far apart.) We have
\[
|\hat{p}|^2 = |\hat{p}_1|^2 = \left| \frac{\text{Re} \ p_1}{\delta} \right|^2 \leq \left| \frac{r(p)}{A|r(p)|} \right|^2 = \frac{1}{A^2};
\]
and
\[
|\hat{p} - \hat{q}|^2 = \sum_{j=1}^n |\hat{p}_j - \hat{q}_j|^2 = \sum_{j=1}^n \left( \frac{|p_j - q_j|}{\phi_j(\delta)} \right)^2 \leq \sum_{j=1}^n \left( \frac{|p_j - q_j|}{\phi_j^*(A|p_j - q_j|)} \right)^2 = \frac{n}{A^2}.
\]
This implies \( \hat{p}, \hat{q} \in B(0, \sqrt{\frac{n+1}{4}}). \) Choosing \( A > \sqrt{\frac{n+1}{4}} \), we note that \( \hat{p}, \hat{q} \in \hat{U}. \) Since \( \frac{\phi_j(\delta)}{\delta} \) is decreasing and \( \delta \geq \phi_j^*(A|p_j - q_j|) \) for \( j = 1, \ldots, n \), it follows

\[
\frac{\phi_j(\delta)}{\delta} \leq \frac{\phi_j^*(A|p_j - q_j|)}{\phi_j^*(A|p_j - q_j|)} = \frac{A|p_j - q_j|}{\phi_j^*(A|p_j - q_j|)}.
\]

This is the same as

\[
|p_j - q_j| \geq \frac{\phi_j^*(A|p_j - q_j|)}{A\delta}.
\]

Therefore, the isotropic distance \( \delta_{I,\Omega_\delta}(\hat{p}, \hat{q}) \) satisfies

\[
\delta_{I,\Omega_\delta}(\hat{p}, \hat{q}) = |\hat{r}_s(\hat{p})| + |\hat{r}_s(\hat{q})| + |\hat{p} - \hat{q}|
\]

\[
= \frac{|r(p)|}{\delta} + \frac{|r(q)|}{\delta} + \sum_{j=1}^n \frac{|p_j - q_j|^2}{\phi_j^2(\delta)}
\]

\[
\geq \frac{|r(p)|}{\delta} + \frac{|r(q)|}{\delta} + \sum_{j=1}^n \frac{\phi_j^*(A|p_j - q_j|)}{A\sqrt{n}\delta}
\]

\[
\geq \frac{|r(p)|}{A\sqrt{n}\delta} + \frac{|r(q)|}{A\sqrt{n}\delta} + \sum_{j=1}^n \frac{\phi_j^*(A|p_j - q_j|)}{A\sqrt{n}\delta} = \frac{1}{A\sqrt{n}}.
\]

This completes the proof of Theorem 1.2 for the Bergman kernel. \( \square \)

4. Proof of Theorem 1.3

We first consider the case when \( \hat{U} \) is a compact subset of \( \Omega \). It is well known that elliptic estimates for the \( \bar{\partial} \)-Neumann problem hold for forms with compact support in \( U \) and hence \( L^2 \) pseudolocal estimates for the Bergman projection hold on \( U \). Theorem 2.2 therefore implies that

\[
D^\alpha_z (\chi_0(z)B(z,w)) \leq c_{\alpha,\chi_0}(b\Omega,bU)
\]

for every cut off function \( \chi_0 \) such that \( \text{supp}(\chi_0) \subset U \). Thus the operator \( D^\alpha \chi_0B \) is continuous in \( L^p(\Omega) \) for \( 0 < p \leq \infty \). Namely, we get the desired inequality

\[
\|\chi_0Bv\|_{L^p(\Omega)} \lesssim \|v\|_{L^p(\Omega)}
\]

for every \( s \geq 0, p \in (1,\infty], \) and \( v \in L^p(\Omega) \). For the case that \( b\Omega \cap U = S \) is a set of good anisotropic dilation points, we have the following lemma.

**Lemma 4.1.** Let \( V_\epsilon \) be a compact set of \( U \) such that \( d(bU,bV_\epsilon) \geq \epsilon \). Then there exists \( c_\epsilon > 0 \) such that

\[
\left| \left( \prod_{j=1}^n \frac{\partial^{\alpha_j}}{\partial z_j^{\alpha_j}} \right) B(z,w) \right| \leq c_\epsilon \alpha \prod_{j=1}^n \phi_j^{-2-\alpha_j}(\delta_{NI}(z,w))
\]

for \( z \in V_\epsilon \cap \bar{\Omega} \) and \( w \in \bar{\Omega} \), where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and

\[
\delta_{NI}(z,w) := |r(z)| + |r(w)| + \sum_{j=1}^n \phi_j^*(|z_j - w_j|).
\]

**Proof.** Denote \( S_\epsilon = \{ z \in \Omega : d(z,b\Omega) < \epsilon \} \). If \( z \in V_\epsilon \cap S_\epsilon \), then \( \pi(z) \in b\Omega \cap U \) is a good anisotropic dilation point by hypothesis. By Theorem 1.2, we have

\[
(4.1) \quad \left| \left( \prod_{j=1}^n \frac{\partial^{\alpha_j}}{\partial z_j^{\alpha_j}} \right) \right| \leq c \prod_{j=1}^n \phi_j^{-2-\alpha_j}(\delta_{NI}(z,w)), \quad \text{for } w \in \Omega
\]
Otherwise if \( z \in (\Omega \cap V_\varepsilon) \setminus S_\varepsilon \) then \(|r(z)| \geq \varepsilon \). By Theorem 2.2, we have

\[
(4.2) \quad \left| \left( \prod_{j=1}^{n} \frac{\partial^{\alpha_j}}{\partial z_j^{\alpha_j}} \right) B(z, w) \right| \leq c_{\varepsilon, \alpha} \quad \text{for } w \in \Omega.
\]

The proof follows from (4.1) and (4.2).

The remainder of the proof of Theorem 1.3 uses the ideas of McNeal and Stein [MS94], though their hypotheses on the type are global while ours are local. We use Lemma 4.1 to overcome this problem.

4.1. Local \( L^p \) estimates. Let \( s \geq 0 \) be an integer. Let \( \{ \zeta_m : m = 0, 1, \ldots, s \} \) be a sequence of cutoff functions in \( C^\infty_{c}(U) \) so that \( \zeta_0 = \chi_1, \zeta_s = \chi_0 \), and \( \zeta_m < \zeta_{m-1} \) for all \( m = 1, \ldots, s \). For \( \varepsilon > 0 \), we define \( \psi_\varepsilon \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n) \) so that

\[
\psi_\varepsilon(z, w) = \begin{cases} 
1 & \text{if } |z - w| < \varepsilon, \\
0 & \text{if } |z - w| > 2\varepsilon.
\end{cases}
\]

We may choose \( \varepsilon \) sufficiently small such that

\[
(4.3) \quad \zeta_0(w) = 1 \text{ if there exists } 1 \leq m \leq s \text{ and } z \in \text{supp}\zeta_m \text{ so that } w \in \text{supp}(\psi_\varepsilon(z, \cdot)).
\]

We observe that

\[
\|\zeta_m Bv\|_{L^p_{\Delta_{m}}} \lesssim \sum_{|\alpha| = m} \|\zeta_m D^\alpha Bv\|_{L^p_0} + \|\zeta_{m-1} Bv\|_{L^p_{m-1}}
\]

\[
= \sum_{|\alpha| = m} \int_{\Omega} \left| \int_{\Omega} \zeta_m(z)D^\alpha B(z, w)v(w) \, dw \right|^p \, dz + \|\zeta_{m-1} Bv\|_{L^p_{m-1}}
\]

\[
\lesssim \sum_{|\alpha| = m} \left[ \int_{\Omega} \left| \int_{\Omega} \zeta_m(z)D^\alpha B(z, w)\psi_\varepsilon(z, w)v(w) \, dw \right|^p \, dz
\]

\[
+ \int_{\Omega} \left| \int_{\Omega \setminus \{|z - w| > \varepsilon\}} \zeta_m(z)D^\alpha B(z, w)|v(w)| \, dw \right|^p \, dz \right] + \|\zeta_{m-1} Bv\|_{L^p_{m-1}}
\]

\[
\lesssim \sum_{|\alpha| = m} \|B^\alpha v\|_{L^p_0} + \|v\|_{L^p_0} + \|\zeta_{m-1} Bv\|_{L^p_{m-1}}
\]

where \( B^\alpha \) is the operator with integral kernel \( \zeta_m(z)(D^\alpha B(z, w))\psi_\varepsilon(z, w) \). Here the last inequality follows by Theorem 2.2 and consequently the constant hidden in the final \( \lesssim \) depends on \( \varepsilon \). To complete the proof of Theorem 1.3 for continuity in \( L^p \)-Sobolev spaces, we need to show that for every multiindex \( \alpha \) with \( |\alpha| = m \),

\[
(4.5) \quad \|B^\alpha v\|_{L^p_0} \lesssim \|\zeta_{\alpha v}\|_{L^p_{\Delta_{m}}}.
\]

Let \( B_0 \) be the operator with associated integral kernel

\[
B_0(z, w) = \zeta_0(z) \prod_{j=1}^{n} \phi_j(\delta_{N_1}(z, w))^{-2} \zeta_0(w).
\]

The proof of (4.5) will follow immediately from Lemma 4.2 and Lemma 4.3.

**Lemma 4.2.** Let \( \alpha \) be a multiindex of length \( m \). Then for \( z \in \Omega \),

\[
|(B^\alpha_0 v)(z)| \lesssim \sum_{j=0}^{m} (B_0(|D^j \zeta_0 v|))(z).
\]
Lemma 4.3. The operator

\[ B_0 : L^p_0(\Omega) \rightarrow L^p_0(\Omega) \]

for every \( 1 < p < \infty \).

Proof of Lemma 4.3. Without loss of generality, we translate and rotate (unitarily) \( \Omega \) so that \( U \) is a neighborhood of the origin, and \( \text{Re} \frac{\partial}{\partial w_1} \) is the (outward) unit normal to \( b\Omega \) at the origin. Also, denote \( w' = (w_2, \ldots, w_n) \). We can write

\[
B_0^a v(z) = \int_{\Omega} (\zeta_m(z) D^a_z B(z, w)) \psi_{\epsilon}(z, w) v(w) \, dw
\]

where

\[
I = (-1)^m \int_{\Omega} \int_0^{3\epsilon} \cdots \int_0^{3\epsilon} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_m} \left( \zeta_m(z) D^a_z B(z, (w_1 - (t_1 + \cdots + t_m), w')) \right) \psi_{\epsilon}(z, w) v(w) \, dt_1 \cdots dt_m \, dw
\]

and

\[
II = \sum_{j=1}^{m} \int_{\Omega} \zeta_m(z) D^a_z B(z, (w_1 - 3\epsilon_j, w')) \psi_{\epsilon}(z, w) v(w) \, dw.
\]

For \( II \), since \( |z - (w_1 - 3\epsilon_j, w')| \geq 3\epsilon_j - |z - w| \geq \epsilon \) for \( j \geq 1 \) and \( w \in \text{supp} \psi_{\epsilon}(z, \cdot) \), we can use Theorem 2.1 to obtain

\[
|II| \lesssim \int_{\Omega} |\zeta_m(z) \psi_{\epsilon}(z, w) v(w)| \, dw \lesssim \int_{\text{supp} \psi_{\epsilon}(z, \cdot)} |\zeta_0(w) v(w)| \, dw \lesssim (B_0 |\zeta_0 v|)(z),
\]

where the second inequality follows by (4.3) and the last one by the bound \( 1 \lesssim |B_0(z, w)| \) which follows from the support condition on \( \psi_{\epsilon} \).

To estimate \( I \), we notice that

\[
\frac{\partial}{\partial t_m} \cdots \frac{\partial}{\partial t_1} B(z, w_t) = (-1)^m \frac{\partial^m}{\partial (\text{Re} \, w_1)^m} B(z, w_t)
\]

where \( w_t = (w_1 - \sum_{j=1}^{m} t_j, w') \). We can write

\[
\frac{\partial}{\partial \text{Re} \, w_1} = T + aL_1,
\]

where \( a \in C^{\infty} \) and \( T \) is a tangent to \( b\Omega \) acting in \( w \). On other hand we know that \( B(z, w) \) is anti-holomorphic in \( w \), so \( L_1 B(z, w_1) = 0 \) (here \( L_1 \) acts \( w \)). Thus, we have

\[
(-1)^m \frac{\partial^m}{\partial (\text{Re} \, w_1)^m} B(z, w_t) = \sum_{j=0}^{m} a_j T^j B(z, w_t)
\]

where each \( a_j \) is a \( C^{\infty} \)-function in \( w \). Using integration by parts, we obtain

\[
I = \sum_{j=0}^{m} \int_{\Omega} \int_0^{3\epsilon} \cdots \int_0^{3\epsilon} (D^a_z B(z, w_t)) \left( \zeta_m(z) (T^*)^j (a_j(w) \psi_{\epsilon}(z, w) v(w)) \right) \, dt_1 \cdots dt_m \, dw
\]

where \( T^* \) is the \( L^2(\Omega) \)-adjoint of \( T \).

To start the estimate of the integrand on \( I \), we use Taylor’s theorem and observe

\[
r(w_t) = r(w_1 - t, w') = r(w) - \frac{\partial r(w)}{\partial (\text{Re} \, w_1)} t + \frac{\partial^2 r(\tilde{w})}{\partial^2 (\text{Re} \, w_1)} t^2
\]

where \( \tilde{w} \) lies in the segment \([w, w_t]\). Since \( \frac{\partial r(w)}{\partial (\text{Re} \, w_1)} > 0 \) and \( t \in [0, 3\epsilon] \), for small \( \epsilon \), it follows

\[
|r(w_t)| \approx |r(w)| + t.
\]
Since $\phi_1(\delta) = \delta$ and $\delta \leq \phi(\delta)$ for $j = 2, \ldots, n$ and any small $\delta \leq 1$,

$$\left| D_z^\alpha B(z, w_t) \right| \leq c_\epsilon (\delta_{NI}(z, w_t))^{-2-m} \prod_{j=2}^{n} \phi_j(\delta_{NI}(z, w_t))^{-2}.$$ 

for $z \in \text{supp}(\zeta_m)$ by Lemma 4.1. By the definition of $\delta_{NI}(z, w_t)$ and the fact that $(w_t)_j = w_j$ for $j = 2, \ldots, n$, we have

$$\delta_{NI}(z, w_t) \approx |r(z)| + |r(w_t)| + |z_1 - (w_t)_1| + \sum_{j=2}^{n} \phi_j^*(|z_j - (w_t)_j|)$$

$$\approx |r(z)| + |r(w)| + t + |z_1 - (w_t)_1| + \sum_{j=2}^{n} \phi_j^*(|z_j - w_j|)$$

$$\approx |r(z)| + |r(w)| + t + |z_1 - w_1| + \sum_{j=2}^{n} \phi_j^*(|z_j - w_j|)$$

$$\approx \delta_{NI}(z, w) + t.$$ 

Hence, $\phi_j(\delta_{NI}(z, w_t)) \geq \phi_j(\delta_{NI}(z, w))$ for $j = 2, \ldots, n$.

Next, by Theorem 1.2,

$$\int_0^{3\epsilon} \cdots \int_0^{3\epsilon} |D_z^\alpha B(z, w_t)| dt_1 \cdots dt_m \lesssim \prod_{j=2}^{n} \phi_j(\delta_{NI}(z, w))^{-2} \int_0^{3\epsilon} \cdots \int_0^{3\epsilon} \frac{dt_1 \cdots dt_m}{(\delta_{NI}(z, w) + \sum_{j=1}^{m} t_j)^m}$$

(4.6) 

$$\lesssim (\delta_{NI}(z, w))^{-2} \prod_{j=2}^{n} \phi_j(\delta_{NI}(z, w))^{-2} = \prod_{j=1}^{n} \phi_j(\delta_{NI}(z, w))^{-2}.$$ 

Moreover, from (1.3) we have

$$\sum_{j=0}^{m} |\zeta_m(z)(T^*)^j (a_j(w)\psi_t(z, w)v(w))| \lesssim \sum_{j=0}^{m} |D_w^j(\zeta_0(w)v(w))|.$$ 

Therefore,

$$|I| \lesssim \int_{\Omega} \sum_{j=0}^{m} B_0(z, w)|D^j \zeta_0 v(w)| dw = \sum_{j=0}^{m} (B_0|D^j \zeta_0 v|)(z).$$

□

**Proof of Lemma 4.3.** That $\phi_k''(\delta) < 0$ is a consequence of the fact that $\frac{\phi_k'(\delta)}{\delta}$ is decreasing. Therefore, $\phi_j(a + b) \geq \frac{1}{j}(\phi_j(a) + \phi_j(b))$ which yields

$$\phi_j(\delta_{NI}(z, w)) \geq |z_j - w_j| + \phi_j \left( |r(z)| + |r(w)| + \sum_{k=2}^{j-1} \phi_k^*(|z_k - w_k|) \right).$$
for $j = 2, \ldots, n$. Thus, for $0 \leq \eta < 1$ we have

$$I_{\eta}(z) = \int_{\Omega} |B_{0}(z,w)||r(w)|^{-\eta} dw$$

\[
\leq \int_{\Omega} |r(w)|^{\eta} |\delta^{2}_{N}(z,w)| \prod_{j=2}^{n} (\phi_{j}(\delta_{N}(z,w)))^{2} dw \\
\leq \int_{0}^{\delta_{0}} \cdots \int_{0}^{\delta_{0}} \frac{\rho_{2} \cdots \rho_{n} dr \, d\rho_{2} \cdots d\rho_{n} dy_{1}}{r^{\eta}(y_{1} + r + |r(z)| + \sum_{j=2}^{n} \phi_{j}(\rho_{j})^{2} \prod_{j=2}^{n} (\rho_{j} + \phi_{j}(r + |r(z)| + \sum_{k=2}^{j-1} \phi_{k}(\rho_{k}))^{2}} \\
\leq \int_{0}^{\delta_{0}} \cdots \int_{0}^{\delta_{0}} \frac{dr \, d\rho_{2} \cdots d\rho_{n}}{r^{\eta}(r + |r(z)| + \sum_{j=2}^{n} \phi_{j}(\rho_{j})^{2} \prod_{j=2}^{n} \phi_{j}(r + |r(z)| + \sum_{k=2}^{j-1} \phi_{k}(\rho_{k})}} \\
\leq \cdots \leq \frac{1}{|r(z)|^{|\eta|}}.
\]

Let $q$ be the conjugate exponent of $p$ and $v \in L^{p}(\Omega)$. An application of Hölder’s inequality establishes

\[
|(B_{0}v)(z)|^{p} = \left( \int_{\Omega} |B_{0}(z,w)||v(w)|^{p} dw \right)^{p} \\
\leq \left( \int_{\Omega} |B_{0}(z,w)||v(w)|^{p} |r(w)|^{\eta p/q} dw \right) \left( \int_{\Omega} |B_{0}(z,w)||r(w)|^{-\eta} dw \right)^{p/q} \\
\leq \left( \int_{\Omega} |B_{0}(z,w)||v(w)|^{p} |r(w)|^{\eta p/q} |r(z)|^{-\eta p/q} dw \right) \left( \int_{\Omega} |v(w)|^{p} |r(w)|^{-\eta p/q} dw \right)^{\eta p/q}.
\]

Therefore,

\[
\|B_{0}v\|_{p}^{p} \leq \int_{\Omega} \int_{\Omega} |B_{0}(z,w)||v(w)|^{p} |r(w)|^{\eta p/q} |r(z)|^{-\eta p/q} \, dw \, dz. \\
\leq \int_{\Omega} I_{\eta p/q}(w)|v(w)|^{p} |r(w)|^{\eta p/q} dw \\
\leq \int_{\Omega} |v(w)|^{p} dw = \|v\|_{p}^{p}
\]

if $0 < \eta < q/p$. This completes the proof of this lemma. 

\[\Box\]

4.2. Local Hölder estimates. We consider the classical Hölder spaces.

**Definition 4.4.** The space $\Lambda_{s}(\Omega)$ is defined by:
1. For $0 < s < 1$,

$$\Lambda_s(\Omega) = \left\{ u : \|u\|_{\Lambda_s} := \|u\|_{L^\infty} + \sup_{z, z+h \in \Omega} \frac{|u(z+h) - u(z)|}{|h|^s} < \infty \right\}.$$ 

2. For $s > 1$ and non-integer,

$$\Lambda_s(\Omega) = \left\{ u : \|u\|_{\Lambda_s} := \|D^\alpha u\|_{\Lambda_{s-\lfloor\alpha\rfloor}} < \infty, \text{ for all } \alpha \text{ such that } |\alpha| \leq [s] \right\}.$$ 

Here $[s]$ is the greatest integer less than $s$.

3. For $s = 1$,

$$\Lambda_1(\Omega) = \left\{ u : \|u\|_{\Lambda_1} := \|u\|_{L^\infty} + \sup_{z, z+h \in \Omega} \frac{|u(z+h) + u(z-h) - 2u(z)|}{|h|} < \infty \right\}.$$ 

4. For $s > 1$ and integer,

$$\Lambda_s(\Omega) = \left\{ u : \|u\|_{\Lambda_s} := \max_{0 \leq |\alpha| \leq [s]} \|D^\alpha u\|_{\Lambda_1} < \infty, \text{ for all } \alpha \text{ such that } |\alpha| \leq s-1 \right\}.$$ 

From [MIS94] §3, we have the following equivalent formulation of the Hölder spaces.

**Proposition 4.5.** Let $s > 0$. A function $u \in \Lambda_s$ if and only if for every $k \in \mathbb{N}$ with $k > s$, there are functions $u_k$ such that $u = \sum_{k=1}^\infty u_k$ and

(i) $\|u_k\|_{L^\infty(\Omega)} \lesssim 2^{-ks}\|u\|_{\Lambda_s}$

(ii) $\|D^m u_k\|_{L^\infty(\Omega)} \lesssim 2^{mk}2^{-ks}\|u\|_{\Lambda_s}$.

The existence of $\{u_k\}$ is equivalent to the decomposition $u = g_k + b_k$ where

(1) $\|b_k\|_{L^\infty(\Omega)} \lesssim 2^{-ks}\|u\|_{\Lambda_s}$

(2) $\|D^j g_k\|_{L^\infty(\Omega)} \lesssim 2^{k(j-s)}\|u\|_{\Lambda_s}$, for $j \leq m$.

**Proof.** In the case that $\Omega = \mathbb{R}^d$ for some $d \in \mathbb{N}$, Stein proves the equivalence of $u \in \Lambda_s$ with properties (i) and (ii) holding as a consequence of the pseudodifferential calculus [Ste93] §VI.5]. Essentially, $u$ is decomposed into $\sum u_k$ using the standard dyadic difference operators. When $\Omega \subset \mathbb{R}^d$, McNeal and Stein point out that the extension theorems in Stein [Ste70] Chapter VI allow us to pass from $\Omega$ to $\mathbb{R}^n$.

The equivalence of (i) and (ii) with (1) and (2) is straightforward. Given $u = \sum_{k=1}^\infty u_k$, take $b_k = \sum_{\ell=k}^\infty u_k$ and $g_k = \sum_{\ell=1}^{k-1} u_k$. Conversely, given $u = g_k + b_k$, observe that $g_k - g_{k+1} = b_{k+1} - b_k$. Consequently, if we take $u_k = g_k - g_{k+1}$, then $u_k$ satisfies the desired estimates. \(\square\)

The following proposition is essentially due to Hardy and Littlewood [MIS94].

**Proposition 4.6.** Let $s > 0$. If $u \in C^\infty(\Omega) \cap L^\infty(\Omega)$ satisfies

$$|\nabla^m u(z)| \leq A|r(z)|^{-(m-s)} \text{ for every } z \in \Omega$$

for every $m > s$, then $u \in \Lambda_s(\Omega)$ and $\|u\|_{\Lambda_s(\Omega)} \lesssim A + \|u\|_{L^\infty(\Omega)}$.

**Proof of Theorem 1.3 for local Hölder estimates.** Our goal is to establish the estimate

$$(4.8) \quad \|\chi_0 Bv\|_{\Lambda_s(\Omega)} \lesssim \|\chi v\|_{\Lambda_s(\Omega)} + \|v\|_{L^\infty(\Omega)}.$$

Let $m = [s] + 1$. An application of Proposition 4.6 reduces the proof of (4.8) to showing

$$|\nabla^m \chi_0 Bv(z)| \lesssim |r(z)|^{-(m-s)} (\|\chi v\|_{\Lambda_s(\Omega)} + \|v\|_{L^\infty(\Omega)}).$$
We let \( \{\zeta_j\}_{j=0}^m \) and \( \psi_\epsilon(z,w) \) be as Section 4.1 and choose \( \epsilon \) sufficiently small such that
\[
\zeta_0 = 1 \quad \text{on} \quad \bigcup_{z \in \text{supp}(\zeta)} \text{supp}(\psi_\epsilon(z,\cdot)), \quad \text{for } 1 \leq j \leq m.
\]
Then similarly to (4.4), by applying Theorem 2.2
\[
|\nabla^m \zeta_m Bv(z)| \lesssim \sum_{|\alpha|=m} |\zeta_m D^\alpha Bv(z)| + |\nabla^{m-1} \zeta_{m-1} Bv(z)|
\]
\[
\lesssim \sum_{|\alpha|=m} \left| \int_\Omega \zeta_m(z) D_z^\alpha B(z,w) \psi_\epsilon(z,w) v(w) \, dw \right| + \int_\Omega |v(w)| \, dw + |\nabla^{m-1} \zeta_{m-1} Bv(z)|
\]
\[
\lesssim \sum_{|\alpha|=m} |B_\epsilon^\alpha v(z)| + |\nabla^{m-1} \zeta_{m-1} Bv(z)| + \|v\|_{L^\infty}.
\]
To estimate \( |B_\epsilon^\alpha v(z)| \), we use the following lemmas.

**Lemma 4.7.** For every \( z \in \Omega \) and multiindex \( \alpha \) of length \( m \), we have
\[
|B_\epsilon^\alpha v(z)| \lesssim |r(z)|^{-m} \|\zeta_0 v\|_{L^\infty(\Omega)}.
\]

**Proof.** It follows from the definition of \( B_\epsilon^\alpha \), the fact that \( \zeta_0 \equiv 1 \) on \( \text{supp} \zeta_m \), and (4.3) that
\[
|B_\epsilon^\alpha v(z)| \lesssim \|\zeta_0 v\|_{L^\infty(\Omega)} \int_\Omega \zeta_0(z) |D_z^\alpha B(z,w)| \zeta_0(w) \, dw, \quad \text{for } z \in \Omega.
\]
Since \( z, w \in \text{supp}(\zeta_0) \subset U \), Theorem 1.2 yields
\[
|D_z^\alpha B(z,w)| \lesssim (\delta_{NI}(z,w))^{-m-2} \prod_{j=2}^n \phi_j(\delta_{NI}(z,w))^{-2}
\]
\[
\lesssim |r(z)|^{-m+\eta} |r(w)|^{-\eta} (\delta_{NI}(z,w))^{-2} \prod_{j=2}^n \phi_j(\delta_{NI}(z,w))^{-2}
\]
for \( z, w \in \Omega \cap U \), where \( 0 < \eta < 1 \). Thus,
\[
\int_\Omega \zeta_0(z) |D_z^\alpha B(z,w)| \zeta_0(w) \, dw \lesssim |r(z)|^{-m+\eta} I_\eta(z) \lesssim |r(z)|^{-m}, \quad \text{for } z \in \Omega.
\]
Here the last inequality follows the estimate of \( I_\eta \) in the proof of Lemma 4.3. \( \square \)

**Lemma 4.8.** For every \( z \in \Omega \) and multiindex \( \alpha \) of length \( m \), we have
\[
|B_\epsilon^\alpha v(z)| \lesssim |r(z)|^{-1} \sum_{j=0}^{m-1} \|D^j \zeta_0 v\|_{L^\infty(\Omega)}.
\]

**Proof.** By repeating the argument of Lemma 4.2 and the estimate leading to (4.6) but integrating by parts only \( (m-1) \)-times, we are led to the inequality
\[
|B_\epsilon^\alpha v(z)| \lesssim \sum_{j=0}^{m-1} \|D^j \zeta_0 v\|_{L^\infty(\Omega)} \int_\Omega \frac{\zeta_0(z) \zeta_0(w) \, dw}{(\delta_{NI}(z,w))^3 \prod_{j=2}^n (\phi_j(\delta_{NI}(z,w)))^2}.
\]
Also, the estimate of \( I_\eta \) with \( \eta = 0 \) in Lemma 4.3 immediately yields
\[
\int_\Omega \frac{\zeta_0(z) \zeta_0(w) \, dw}{(\delta_{NI}(z,w))^3 \prod_{j=2}^n (\phi_j(\delta_{NI}(z,w)))^2} \lesssim |r(z)|^{-1}.
\]
\( \square \)
We now return to the proof of Theorem 1.3. Choose \( k \) such that \( 2^{-k} \approx |r(z)| \). Since \( \zeta_0 v \in \Lambda^s(\Omega) \), by Proposition 4.5 there exists \( g_k \) and \( b_k \) such that
\[
\zeta_0 v = g_k + b_k, \quad \text{on } \Omega,
\]
where
\[
\|b_k\|_{L^\infty(\Omega)} \lesssim 2^{-ks}\|\zeta_0 v\|_{\Lambda^s(\Omega)} = |r(z)|^s\|\zeta_0 v\|_{\Lambda^s(\Omega)}
\]
and
\[
\|D^j g_k\|_{L^\infty(\Omega)} \lesssim 2^{k(j-s)}\|\zeta_0 v\|_{\Lambda^s(\Omega)} = |r(z)|^{-(j-s)}\|\zeta_0 v\|_{\Lambda^s(\Omega)}, \quad \text{for } j \leq m.
\]
Then
\[
|B^\alpha r(z)| \leq |B^\alpha \zeta_0^{-1} b_k(z)| + |B^\alpha \zeta_0^{-1} g_k(z)|
\]
\[
\lesssim |r(z)|^{-m}\|b_k\|_{L^\infty} + |r(z)|^{-1}\sum_{j=0}^{m-1}\|D^j g_k\|_{L^\infty}
\]
\[
\lesssim \|\zeta_0 v\|_{\Lambda^s} \left( |r(z)|^{-m}|r(z)|^s + |r(z)|^{-1}\sum_{j=0}^{m-1}|r(z)|^{-(j-s)} \right)
\]
\[
\lesssim \|\zeta_0 v\|_{\Lambda^s}|r(z)|^{-(m-s)}.
\]

An application of Proposition 4.6 completes the proof. \(\square\)

5. Proof of Theorem 1.3

Our main theorem in this subsection is

**Theorem 5.1.** The boundary of an bounded \( h \)-extendible domain is a set of good anisotropic dilation points.

The proof of this theorem is divided in following four lemmas. In Lemma 5.2 we prove the condition (1) in Definition 1.1. The proof of the condition (2) in Definition 1.1 is divided into Lemma 5.3, Lemma 5.5 and Lemma 5.6.

Throughout this section, \( U_o \) is a neighborhood of the origin and \( \Omega \) is a bounded domain with smooth boundary \( \partial \Omega \) in which every boundary point is \( h \)-extendible. As discussed in Yu [Yu94, Yu95], \( p \in \partial \Omega \) is \( h \)-extendible if there is a multitype \( \mathcal{M}(p) = (m_{p,1}, m_{p,2}, \ldots, m_{p,n}) \) with \( m_{p,1} = 1 \), a neighborhood \( U_p \) of \( p \), a defining function \( r_p \) defined in \( U_p \), a biholomorphism \( H_p : U_p \to U_o \), (that is, local coordinates associated to \( p \)) so that \( H_p(p) = 0 \) and \( r_{p,1}(z) := r_p(H_p^{-1}(z)) \) has the expansion
\[
r_{p,1}(z) := \text{Re } z_1 + P_p(z') + R_p(z) \quad \text{for } z = (z_1, z') \in U_o
\]
where \( P_p(z') \) is a \((1/m_{p,2}, \ldots, 1/m_{p,n})\)-homogeneous pluriharmonic polynomial that contains no pluriharmonic terms and \( R_q(z) = o(\sigma_p(z)) \). Here,
\[
\sigma_p(z) := \sum_{j=1}^n |z_j|^{m_{p,j}}.
\]
Thus, there exist constants \( C > 0 \) and \( \gamma_p > 1 \) so that the smooth function \( R \) satisfies
\[
|R_p(z)| \leq C\sigma_p(z)^{\gamma_p}
\]
(see [Yu94, Definition 1.4] and the following discussion). Recall that if \( f(x) = o(g(x)) \) and both functions are smooth, then it follows that \( |\nabla f| = o(|\nabla g|) \).
We show that for small $\delta > 0$, the map
\begin{equation}
\Phi_{p,\delta}(z) = \left(\frac{z_1}{\delta^1/m_{p,j}}, \frac{z_2}{\delta^1/m_{p,j}}, \ldots, \frac{z_n}{\delta^1/m_{p,j}}\right)
\end{equation}
is a good anisotropic dilation at $p$. Note that the homogeneity of $P_p$ means
\begin{equation}
P_p \left(\frac{z_2}{\delta^1/m_{p,j}}, \ldots, \frac{z_n}{\delta^1/m_{p,j}}\right) = \delta^{-1}P_p(z_2, \ldots, z_n)
\end{equation}
for $z' = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$ and $\delta > 0$.

**Lemma 5.2.** The dilation $\Phi_{p,\delta}$ satisfies the condition (1) in Definition 1.1.

**Proof.** Since $\left|\frac{\partial P_p}{\partial z_j}(z)\right| \approx 1 = \frac{\delta}{\delta}$ for $z \in U_o$, we only need to check the first condition in Definition 1.1 for $j = 2, \ldots, n$. For $\delta > 0$ sufficiently small, $\Phi_{p,\delta}^{-1}(B(0,1)) \subset U_o$. Fix such a $\delta$, and suppose that $z \in \Phi_{p,\delta}^{-1}(B(0,1))$. Then there exists $\tilde{z} = (\tilde{z}_1, \tilde{z}') \in B(0,1)$ such that $z = (z_1, z') = \Phi_{p,\delta}^{-1}(\tilde{z})$. Since $\hat{z}_j = \frac{\tilde{z}_j}{\delta^1/m_{p,j}}$, (5.2) and (5.3) imply that
\[
\frac{\partial P_p(z')}{\partial z_j} = \frac{1}{\delta} \frac{\partial P_p(z')}{\partial z_j} = \frac{\delta^1/m_j}{\delta} \frac{\partial P_p(z')}{\partial z_j}
\]
from which it follows that $\left|\frac{\partial P_p(z')}{\partial z_j}\right| \lesssim \delta^{-1/m_{p,j}}$. Since $\gamma > 1$, it follows that
\[
\left|\frac{\partial R_p(z)}{\partial z_j}(z)\right| = \left|\frac{\partial}{\partial z_j} \left(\sigma_p(\Phi_{p,\delta}^{-1}(\tilde{z}))\right)\right| \lesssim o(\delta^{-1/m_{p,j}}).
\]
Therefore
\[
\left|\frac{\partial R_p}{\partial z_j}(z)\right| \leq \left|\frac{\partial P_p}{\partial z_j}(z')\right| + \left|\frac{\partial R_p}{\partial z_j}(z)\right| \lesssim \delta^{-1/m_{p,j}}
\]
for $z \in \Phi_{p,\delta}^{-1}(B(0,1))$. We have now established the condition (1) in Definition 1.1. \hfill $\square$

Denote $E_{p,\delta} = \{z \in \mathbb{C}^n : \sigma_p(z) < \delta\}$ the ellipsoid associated with the multitype $\mathcal{M}(p)$ with radius $\delta$ and centered at the origin. Let $q \in H_{p}^{-1}(E_{p,\delta}) \cap b\Omega$ and
\begin{equation}
\gamma = \min \{\gamma_q : q \in H_{p}^{-1}(E_{p,\delta}) \cap b\Omega\}.
\end{equation}
Let
\[
\Psi_{q \rightarrow p,\delta} = \Phi_{p,\delta}H_pH_q^{-1}\Phi_{q,\delta}^{-1}.
\]
The key point of the second condition in Definition 1.1 is in the following lemma.

**Lemma 5.3.** For every $t > 0$ sufficiently small, there exist positive constants $C$ and $\delta(t)$ such that
\begin{equation}
|\det J \Psi_{q \rightarrow p,\delta}(\cdot, t, \cdot)| \leq C
\end{equation}
holds uniformly for $0 < \delta \leq \delta(t)$.

The proof of this lemma is inspired by the proof of the main theorem in [Nik02] (See Theorem 6.5 below).

**Proof of Lemma 5.3.** Recall that $H_p$ is a biholomorphism from $U_p$ to $U_o$, with $H_p(p) = 0$, so it can be extended to be a $C^\infty$ diffeomorphism from $\mathbb{C}^n$ to $\mathbb{C}^n$. Define $\Omega_{p,\delta} := \Phi_{p,\delta}H_p(\Omega)$ and
\[
r_{p,\delta}(z) := \frac{1}{\delta} r_{p,1}(\Phi_{p,\delta}^{-1}(z)) = \text{Re} z_1 + P_p(z') + O(\delta^{-\gamma_p - 1}\sigma_p(z)\gamma_p),
\]
for $z \in \Phi_{p,\delta}(U_o)$ and $\gamma_p > 1$. Thus, $r_{p,\delta}(z)$ is a defining function for $\Omega_{p,\delta}$ in $\Phi_{p,\delta}(U_o)$. When $\delta \to 0$,
\[
\Omega_{p,\delta} \to \Omega_{p,0} := \{z \in \mathbb{C}^n : r_{p,0}(z) := \text{Re} z_1 + P_p(z') < 0\},
\]
where $\Omega_{p,0}$ is an associated model for $\Omega$ at $p$. It is obvious that $\Omega_{p,1} \cap U_o \equiv H_p(\Omega) \cap U_o$. For $\alpha, \beta > 0$, we define perturbations of $\Omega_{p,0}$ and $\Omega_{p,1}$ by

$$\Omega_{p,0}^\alpha = \{ z \in \mathbb{C}^n : r_{p,0}^\alpha(z) := r_{p,0}(z) - \alpha a_p(z) < 0 \}$$

where $a_p$ is the bumping function from Yu [Yu95, Definition 3.3] so that $\Omega_{p,0}^\alpha$ is pseudoconvex, and

$$\Omega_{p,1}^\beta = \{ z \in \mathbb{C}^n : r_{p,1}^\beta(z) := r_{p,1}(z) + \beta < 0 \}.$$

Let

$$\Theta_{p,q}^\delta := \Psi_{q \to p,\delta} \Psi_{p \to q,1} = \Phi_{p,\delta} H_p^{-1} \Phi_{q,\delta} H_q H_p^{-1}.$$

Then $\Theta_{p,q}^\delta$ is a biholomorphism from $U_o$ to its map $\Theta_{p,q}^\delta(U_o)$ since we may choose $U_p$ and $U_q$ such that $H_q$ is holomorphic on $U_p$ and $H_p$ is holomorphic on $U_q$.

The proof of (5.5) is divided into three steps.

- Step 1. We construct the open set $X$ and $Y$ such that $\{ \Theta_{p,q}^\delta \} \in \text{Hol}(X,Y)$ and $Y$ is a taut manifold.
- Step 2. Since $Y$ is a taut manifold, every subsequence of $\{ \Theta_{p,q}^\delta \}$ either converges normally or diverges compactly. In this step, we prove it is NOT compact divergence.
- Step 3. Using the conclusion in Step 2, we prove that (5.5) holds.

**Proof of Step 1.** In this step we prove that for sufficiently small $\alpha, \beta > 0$ there exists $\delta_0 = \delta(\alpha, \beta)$ such that if $X := \Omega_{p,1}^\beta \cap B(0, \beta^1/\gamma)$ with $\gamma$ as in (5.4) and $Y := \Omega_{p,0}^\alpha$ then $\Theta_{p,q}^\delta \in \text{Hol}(X,Y)$ for $q \in H_p^{-1}(E_{p,\delta})$ and $0 < \delta \leq \delta_0$.

First, we fix $z_{p,1} \in X$. Then

$$\left\{ \begin{array}{l}
|z_{p,1}| \leq \beta^{\frac{1}{\gamma}}, \\
r_{p,1}(z_{p,1}) + \beta < 0.
\end{array} \right.$$ 

Let $z_{q,1} := H_q H_p^{-1}(z_{p,1})$. We have

$$|z_{q,1}| \leq |H_q H_p^{-1}(z_{p,1}) - H_q H_p^{-1}(0)| + |H_q H_p^{-1}(0) - H_q H_p^{-1} H_p(q)| \quad (\text{since } H_q H_p^{-1} H_p(q) = H_q(q) = 0)$$

$$\leq c \left( |z_{p,1}| + |H_p(q)| \right)$$

$$\leq c \left( |\beta|^{\frac{1}{\gamma}} + \delta^{\alpha_{p,q}} \right)$$

where the last inequality follows by the first inequality of (5.5) and the inclusion $H_p(q) \in E_{p,\delta}$.

Thus there exist $\delta'(\beta) > 0$ such that for every $0 \leq \delta \leq \delta'(\beta)$, one has $|z_{q,1}| \leq c \beta^{1/\gamma}$, and hence,

$$H_q H_p^{-1}(B(0, \beta^{1/\gamma})) \subset B(0, c \beta^{1/\gamma}).$$

By our definitions $z_{q,1} = H_q H_p^{-1}(z_{p,1})$ and $r_p(z) \approx r_q(z)$ for $z \in U_o$, it follows

$$r_{p,1}(z_{p,1}) = r_p(H_p^{-1}(z_{p,1})) = r_p(H_q^{-1}(z_{q,1})) \approx r_q(H_q^{-1}(z_{q,1})) = r_{q,1}(z_{q,1})$$

Thus

$$r_{q,1}(z_{q,1}) + c \beta \leq c(r_{p,1}(z_{p,1}) + \beta).$$
On the other hand, \( \gamma > 1 \) and \( 0 \leq \delta < 1 \) so
\[
    r_{q,\delta}(z_{q,1}) \leq r_{q,1}(z_{q,1}) + |O(\delta^{\gamma-1}\sigma_p(z_{q,1})^\gamma) - O(\sigma_p(z_{q,1})^\gamma)|
\]
\[
    \leq r_{q,1}(z_{q,1}) + c\sigma_p^2(z_{q,1})
\]
\[
    \leq r_{q,1}(z_{q,1}) + c|z_{q,1}|^\gamma
\]
\[
    \leq r_{q,1}(z_{q,1}) + c\beta.
\]

Thus,
\[
    r_{q,\delta}(z_{q,1}) \leq c(r_{p,1}(z_{p,1}) + \beta).
\]

It follows
\[
    H_q H_p^{-1}(\Omega_{p,1}^\beta) \cap B(0, c\beta^{1/\gamma}) \subset \Omega_{q,\delta} \cap B(0, c\beta^{1/\gamma}).
\]

Therefore, we have
\[
    H_q H_p^{-1}(X) \subset \Omega_{q,\delta} \cap B(0, c\beta^{1/\gamma}),
\]
and
\[
    \Phi_{q,\delta} H_q H_p^{-1}(X) \subset \Omega_{q,1} \cap E_{q,\delta} \subset \Omega_{q,1} \cap E_{q,\delta}
\]
by requiring \( \beta \) to be small enough to satisfy \( c\beta^{1/\gamma} \leq 1 \). Thus,
\[
    H_p H_q^{-1} \Phi_{q,\delta} H_q H_p^{-1}(X) \subset \Omega_{p,1} \cap H_p H_q^{-1}(E_{q,\delta}) \subset \Omega_{p,1} \cap A_{\delta}
\]
where
\[
    A_{\delta} := \bigcup_{q \in H_p^{-1}(E_{p,\delta})} H_p H_q^{-1}(E_{q,\delta})
\]

It is easy to see that \( A_{\delta} \) tends to the origin as \( \delta \to 0 \). Thus, for every \( \alpha > 0 \), there exists \( \delta(\alpha) \) such that
\[
    A_{\delta} \subset \left\{ z \in U_o : |O(\sigma_p(z)^\gamma)| \leq \alpha \sigma_p(z) \right\}, \quad \text{for } 0 < \delta \leq \delta(\alpha).
\]

This implies \( r_{p,0}^{-\alpha}(z) \leq r_{p,1}(z) \) for \( z \in A_{\delta} \) and hence
\[
    H_p H_q^{-1} \Phi_{q,\delta} H_q H_p^{-1}(X) \subset \Omega_{p,1} \cap A_{\delta} \subset \Omega_{p,0}^{-\alpha} \cap A_{\delta} \subset \Omega_{p,0}^{-\alpha}
\]
Since \( \Phi_{p,\delta} \) in an automorphism of \( \Omega_{p,0}^{-\alpha} \), we obtain
\[
    \Theta^\delta_q(X) = \Phi_{p,\delta} H_p H_q^{-1} \Phi_{q,\delta} H_q H_p^{-1}(X) \subset \Phi_{p,\delta}(\Omega_{p,0}^{-\alpha}) = \Omega_{p,0}^{-\alpha} = Y
\]
for \( 0 < \delta \leq \delta(\alpha, \beta) \).

**Proof of Step 2.** The family \( \{\Theta^\delta_q\}_{\delta \in (0,\delta_0], q \in H_p^{-1}(E_{p,\delta}) \cap \partial \Omega} \subset H(X, Y) \) is a normal family since \( Y \) is a taut complex manifold by Theorem 6.5. A consequence of tautness is that every subsequence of \( \{\Theta^\delta_q\}_{\delta \in (0,\delta_0], q \in H_p^{-1}(E_{p,\delta}) \cap \partial \Omega} \) either converges normally or diverges compactly. For \( t \in (0, +\infty) \), let \( x_{in} = H_p H_q^{-1}(-t, 0') \) and \( y_{out} = \Psi_{q \to p, \delta}((-t, 0')) \). Then
\[
    y_{out} = \Psi_{q \to p, \delta} H_q H_p^{-1}(x_{in}) = \Theta^\delta_q(x_{in}).
\]

We will show that compact divergence fails by establishing the existence of \( t \) and \( c \) that are independent of \( \delta \) and such that \( x_{in} \subset X \) and \( |y_{out}| \leq M \). We have
\[
    |x_{in}| \leq c_1(|t| + \delta^{1/m_{p,n}})
\]
and
\[
    r_{p,1}(x_{in}) \leq c_2 r_{q,1}(-t, 0) = -c_2 t.
\]
For $c_4 \delta^{1/m_{p,n}} \leq \frac{1}{2}$, in order to force $x_m \in X$, we need
\begin{equation}
\frac{\beta}{c_2} < t < \frac{1}{2c_1} \beta^{1/\gamma}.
\end{equation}
Since $\gamma > 1$, for $\beta < \beta_0 = \left(\frac{c_2}{2c_1}\right)^{1/(\gamma-1)}$, we chose $t$ in the non empty set $\left(\frac{\beta}{c_2}, \frac{1}{2c_1} \beta^{1/\gamma}\right)$.

On the other hand,
\[ \Psi_{q \to p, \delta}(-t, 0') = \Phi_{p, \delta} H_p^{-1}(-\delta t, 0'). \]
Here the equality follows by $\Phi_{q, \delta}(-t, 0') = (-\delta t, 0)$. Thus the length
\[ |H_p H^{-1}(-\delta t, 0') - H_p(q)| = |H_p H^{-1}(-\delta t, 0') - H_p H^{-1}(0)| \leq c\delta t. \]
By the hypothesis $q \in H^{-1}_p(E_{p, \delta})$, it follows
\[ H_p H^{-1}(-\delta t, 0') \in E_{p, \delta(1+ct)} \]
and hence
\[ \Phi_{p, \delta} H_p H^{-1}(-\delta t, 0') \in B(0, 1 + ct) \]
for some $c$. Thus, $\Theta_p^\delta q(x_m) \in \Omega^{-\alpha}_{p,0} \cap B(0, M)$ with $M$ independent of $\delta$. This means no subsequence of the family $\Theta_p^\delta q$ is compactly divergent. Therefore, it converges uniformly on a compact subset of $X$.

**Proof of Step 3.** Let $\{\delta_j\}_{j=0}^\infty \subset (0, \delta_0]$ such that be $\{\delta_j\}_{j=0}^\infty \searrow 0$ and $\{q_j\}_{j=0}^\infty$ be a sequence of points in $\mathbb{C}^n$ such that $q_j \in H^{-1}_p(E_{p, \delta}) \cap b\Omega$. Thus, $\{\Theta_p^{\delta_j q_j}(z)\}_{j=0}^\infty$ is a subsequence of the family $\{\Theta_p^{\delta q}\}_{\delta \in (0, \delta_0], q \in H^{-1}_p(E_{p, \delta}) \cap \partial \Omega}$. As a consequence of Step 2, when $S$ is a compact subset of $X = \Omega^{-\alpha}_{p,1} \cap B(0, \beta^{1/\gamma_p})$, $\{\Theta_p^{\delta_j q_j}(z)\}_{j=0}^\infty$ converges uniformly on $S$. Let
\[ \Theta_p(z) = \lim_{j \to \infty} \Theta_p^{\delta_j q_j}(z), \quad z \in S. \]

It now follows from the uniform convergence of holomorphic functions on compact sets that $\Theta_p(z)$ is holomorphic on $S$ and
\begin{equation}
\det(J \Theta_p) = \lim_{j \to \infty} \det(J \Theta_p^{\delta_j q_j})
\end{equation}
uniformly on $S$. Recall that
\[ \Theta_p^{\delta_j q_j} = \Psi_{q_j \to p, \delta_j} \Psi_{p \to q_j, 1} \]
This means
\begin{equation}
\det \left( J \Theta_p^{\delta_j q_j} \right) = \det \left( J \Psi_{q_j \to p, \delta_j} \right) \det \left( J \Psi_{p \to q_j, 1} \right),
\end{equation}
for $z \in S$. We notice that $\Psi_{p \to q_j, 1} = H_{q_j} H^{-1}_p$ is a transformation of a local coordinates associated to $q_j$ to a local coordinates associated to $p$. Thus,
\begin{equation}
\lim_{j \to \infty} \Psi_{p \to q_j, 1} = \lim_{j \to \infty} H_{q_j} H^{-1}_p = G,
\end{equation}
where $G$ is holomorphic and its Jacobian has a non-zero determinant on $U_p$ (a set that contains $S$). The reason that $G$ may not be the identity map because $H_{q_j}$ may approach another local coordinate choice associated with the $h$-extendible point $p$ since they are not unique. Combining (5.9), (5.10) and (5.11), we obtain
\begin{equation}
\lim_{j \to \infty} \det \left( J \Psi_{q_j \to p, \delta_j} \right) = \det(J \Theta_p) \left( \det(J G) \right)^{-1}, \quad z \in S.
\end{equation}
This implies there exist $N$ and $C$ independent of $j$ such that for all $j \geq N$,
\[
|\det(J\Psi_{q_j \rightarrow p, \delta})| \leq C
\]
holds for $\tilde{z} \in \Psi_{p \rightarrow q_1}(S)$ and $(\delta, q_j) \in \{(\delta_j, q_j) : j \geq N\}$. A consequence of this argument is the existence of $C > 0$ and $\delta_0(\beta) > 0$ so that if $0 < \delta \leq \delta_0(\beta)$ and $q \in H_p^{-1}(E_{p, \delta}) \cap b\Omega$ then
\[
(5.13) \quad |\det(J\Psi_{q \rightarrow p, \delta})| \leq C, \quad \text{for } z \in \Psi_{p \rightarrow q_1}(S)
\]
holds. Moving forward, we assume that $\delta(\beta)$ is small enough that (5.13) holds.

As in the proof of Step 2, for $0 < \beta \leq \beta_0$, $0 < \delta \leq \delta(\beta)$, and $t$ satisfying (5.8), it follows
\[
x_{in} := \Psi_{q \rightarrow p, 1}(-t, 0') = \Psi_{p \rightarrow q_1}^{-1}(-t, 0') \in X.
\]
So if we choose the compact set $S \subset X$ containing $x_{in}$, we obtain for $0 < t < t_0$, there exist $\delta(t) > 0$ such that
\[
(5.14) \quad |\det(J\Psi_{q \rightarrow p, \delta})|_{(-t, 0')} \leq C,
\]
hold uniformly in $0 < \delta \leq \delta(t)$. This proves Step 3 and also Lemma 5.3.

Proof of Theorem 7.6. Fix $p \in S$ and let $q \in H_p^{-1}(E_{p, \delta}) \cap S$. We first notice that if $\mathcal{M}(p) = (m_{p, 1}, m_{p, 2}, \ldots, m_{p, n})$ and $\mathcal{M}(q) = (m_{q, 1}, m_{q, 2}, \ldots, m_{q, n})$ are multitypes associated to $p$ and $q$, respectively, then
\[
\det(J\Phi_{p, \delta}|_z) = \delta^{\sum_{k=1}^n \frac{1}{m_{p, k}}} \quad \text{and} \quad \det(J\Phi_{q, \delta}|_z) = \delta^{\sum_{k=1}^n \frac{1}{m_{q, k}}}
\]
for all $z \in \mathbb{C}^n$. Since $\Psi_{q \rightarrow p, \delta} = \Phi_{p, \delta} \Phi_{q \rightarrow p, \delta} \Phi_{q, \delta}$ and $|\det(J\Psi_{q \rightarrow p, 1})|$ bounded away from zero, by (5.14) we have
\[
\delta^{\sum_{k=1}^n \frac{1}{m_{p, k}} - \sum_{k=1}^n \frac{1}{m_{q, k}}} \leq C'
\]
for some constant $C'$ for small $\delta > 0$. This implies
\[
(5.15) \quad \sum_{k=1}^n \frac{1}{m_{p, k}} \leq \sum_{k=1}^n \frac{1}{m_{q, k}}.
\]

Remark 5.4. The inequality (5.15) holds for all $h$-extendible domains. For example, say $\Omega$ is the decoupled domain defined by
\[
\Omega = \{z \in \mathbb{C}^n : r(z) = \text{Re} \ z_1 + \sum_{k=2}^n |z_k|^{2m_k}\}
\]
Then $\mathcal{M}(0) = (1, 2m_2, \ldots, 2m_n)$, and it is easy to see that for every $q$ in a neighborhood of 0, the $k$-entry $m_{q, k}$ of $\mathcal{M}(q)$ is always less than or equal $2m_k$. The inequality (5.15) holds.

Denote $\mathfrak{B}_{\Omega, p, \delta}$ be the Bergman metric associated to $\Omega_{p, \delta}$ and $d_{\Omega_{p, \delta}}(z)$ the distance from $z$ to the boundary of $\Omega_{p, \delta}$. Let
\[
\kappa_p = \max \left\{ \frac{1}{m_{q, n}} : q \in H_p^{-1}(E_{p, \delta}) \cap b\Omega \right\}.
\]
Note that $\kappa$ is bounded uniformly in $\delta$.

Lemma 5.5. The Bergman metric associated to the scaled domain $\Omega_{p, \delta}$ has a uniformly lower bound with the rate $d_{\Omega_{p, \delta}}^{-\kappa_p}(z)$. In particular, one has
\[
(5.16) \quad \mathfrak{B}_{\Omega_{p, \delta}}(z, X) \geq c d_{\Omega_{p, \delta}}^{-\kappa_p}(z)|X|
\]
for $z \in U_0$ and $X \in T^1_0\mathbb{C}^n|_{\Omega_{p, \delta}}$, where $c$ is independent of $\delta$. 

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Proof of Lemma 5.5. Fix \( z_{p,\delta} \in \Omega_{p,\delta} \cap B(0,1) \) and \( X_{p,\delta} = \sum_{j=1}^{n} X_{p,\delta}^j \frac{\partial}{\partial z_j} \in T^{1,0}_{\Omega_p} |_{\Omega_{p,\delta}} \) with \( X_{p,\delta}^j \in \mathbb{R} \) for \( j = 1, 2, \ldots, n \). Let \( q \) be the projection of \( H_p^{-1}\Phi_{p,\delta}^{-1}(z_{p,\delta}) \) to the boundary \( b\Omega \). Thus, \( q \in H_p^{-1}(E_{p,\delta}) \cap b\Omega \). Let \( z_{q,\delta} = \Psi_{p\rightarrow q,\delta}(z_{p,\delta}) \). Then \( z_{q,\delta} \) is of the form \( z_{q,\delta} = (-t,0') \) where \( t = d_{\Omega_{q,\delta}}(z_{q,\delta}) \approx d_{\Omega_{p,\delta}}(z_{p,\delta}) \) (as verified in (5.19) below) independently in \( \delta \). By Lemma 5.3

\[
(5.17) \quad | \det J\Psi_{q\rightarrow p,\delta}(z_{q,\delta}) | \leq C
\]

for sufficiently small \( \delta \). Since

\[
J(\Psi_{p\rightarrow q,\delta}(z_{p,\delta})) \cdot J(\Psi_{q\rightarrow p,\delta}(z_{q,\delta})) = I_n,
\]

we conclude that

\[
(5.18) \quad | \det J(\Psi_{p\rightarrow q,\delta}(z_{p,\delta})) | \geq C.
\]

By the invariance property of the Bergman metric under biholomorphic mappings,

\[
\mathcal{B}_{\Omega_{p,\delta}}(z_{p,\delta}, X_{p,\delta}) = \mathcal{B}_{\Omega_{q,\delta}}(z_{q,\delta}, X_{q,\delta}) = \mathcal{B}_{q,1}(z_{q,1}, X_{q,1})
\]

where \( X_{q,\delta} = (J\Psi_{p\rightarrow q,\delta})X_{p,\delta} \) and \( X_{q,1} = (J\Phi_{q,\delta}^{-1})X_{q,\delta} \). By [BSY95, Theorem 2] (see in Appendix below), it follows

\[
\mathcal{B}_{\Omega_{q,1}}(z_{q,1}, X_{q,1}) \geq \frac{1}{2} \left( J\Phi_{q,\eta} \big|_{\eta=d_{\Omega_{p,1}}(z_{q,1})} \right) \cdot X_{q,1} \cdot \mathcal{B}_{\Omega_{q,\delta}}(\omega, \hat{X}) \geq c \left( J\Phi_{q,\eta} \big|_{\eta=d_{\Omega_{p,1}}(z_{q,1})} \right) \cdot X_{q,1}.
\]

where \( \omega = (-1,0') \), \( \hat{X} \) is a unit vector defined in Theorem 6.1 and \( c = \inf_{|\hat{X}|=1} \mathcal{B}_{q,0}(\omega, \hat{X}) > 0 \). Thus \( c \) is independent of \( z_{q,1} \) and \( X_{q,1} \); it depends only on the multitype \( \mathcal{M}(q) \). We also estimate

\[
\left| \left( J\Phi_{q,\eta} \big|_{\eta=d_{\Omega_{p,1}}(z_{q,1})} \right) \cdot X_{q,1} \right| = \left| \left( J\Phi_{q,\eta} \big|_{\eta=d_{\Omega_{p,1}}(z_{q,1})} \right) \cdot (J\Phi_{q,\delta}^{-1})X_{q,\delta} \right|
\]

\[
= \left| \left( J\Phi_{q,\eta} \big|_{\eta=d_{\Omega_{p,1}}(z_{q,1})} \right) \cdot X_{q,\delta} \right|
\]

\[
= \sum_{j=1}^{n} \left| \frac{|X_{q,\delta}|^2}{(\delta^{-1}d_{\Omega_{p,1}}(z_{q,1}))^{2/m_{q,j}} \cdot X_{q,\delta}} \right|^{\frac{1}{2}}
\]

\[
\geq \frac{\delta^{-1}d_{\Omega_{p,1}}(z_{q,1})^{1/m_{q,n}}}{|X_{q,\delta}| \cdot |X_{q,\delta}|}
\]

\[
= \frac{d_{\Omega_{p,\delta}}(z_{p,\delta})}{d_{\Omega_{p,\delta}}(z_{q,\delta})}
\]

\[
\geq c d_{\Omega_{p,\delta}}(z_{p,\delta}) X_{p,\delta}.
\]

where the last inequality follows by (5.18) and

\[
(5.19) \quad d_{\Omega_{p,\delta}}(z_{p,\delta}) \approx |r_{p,\delta}(z_{p,\delta})| \approx \left| \frac{r_{q,1}(z_{q,1})}{\delta} \right| \approx \frac{d_{\Omega_{q,1}}(z_{q,1})}{\delta} \approx d_{\Omega_{q,\delta}}(z_{q,\delta}).
\]

Therefore, we conclude that

\[
B_{\Omega_{p,\delta}}(z, X) \geq c d_{\Omega_{p,\delta}}(z) |X|.
\]

\( \square \)
Lemma 5.6. The second condition in Definition 1.2 satisfies. In particular, one has, for $\chi_2 \in C^\infty_c(U_0)$ such that $\chi_1 < \chi_2 < \chi_3$ and for every $s, m \geq 0$, the estimates
\begin{equation}
\|\chi_1 B_{\Omega_{p, \delta}} u\|_{L^2_\alpha(\Omega_{p, \delta})}^2 \leq c_{s,m} \left( \|\chi_2 u\|_{L^2_\alpha(\Omega_{p, \delta})}^2 + \|\chi_3 B_{\Omega_{p, \delta}} u\|_{L^2_{2-m}(\Omega_{p, \delta})}^2 \right)
\end{equation}
holds for all $u \in L^2_s(U_0 \cap \Omega_{p, \delta}) \cap L^2(\Omega_{p, \delta})$, where the constant $c_{s,m}$ is independent of $\delta$.

Proof of Lemma 5.6. By [KZ12, Section 5], a lower bound of the Bergman metric implies the existence of a family of bounded functions $\{\phi^n\}_{n \geq 0}$ such that
\[ i\partial\bar{\partial}\phi^n(X, X) \geq C\eta^{-2\kappa'}|X|^2 \quad \text{on } S_\eta \cap V, \]
where $S_\eta = \{ \hat{z} \in \Omega_{p, \delta} : -\eta < r_{p, \delta}(\hat{z}) < 0 \}$ and any $\kappa' < \kappa$, $C$ is independent of $\delta$ and $\eta$. Thus, by [Cat87, Theorem 2.1] the subelliptic estimates for $\Omega_{p, \delta}$ hold in a neighborhood of the origin with uniformly in $\delta$. Consequently, the $L^2$ pseudolocal estimates in a neighborhood of the origin hold for the Bergman projection $B_{\Omega_{p, \delta}}$ uniformly in $\delta$. \hfill \Box

6. Appendix

Theorem 6.1 (Theorem 2 in [BSY95]). Let $\Omega_{q,1}$ be an $h$-extendible at the boundary point $q$ with multitype $(1, m_{q,2}, \ldots, m_{q,n})$ and local model $\Omega_{q,0}$. If $\Gamma$ be a nontangential cone in $\Omega_{q,1}$ with vertex at $p$, then
\[ \lim_{z \in \Gamma, z \to q} \frac{\mathcal{B}_{\Omega_{q,1}}(z, X)}{(J\Phi_{q, \eta}|_{\eta = d_{\Omega_{q,1}}(z)}) \cdot (X)} = \mathcal{B}_{\Omega_{q,0}}(\omega, \hat{X}). \]
Here $\hat{X}$ is a unit vector defined by $\hat{X} = \lim_{z \to p} \frac{(J\Phi_{q, \eta}|_{\eta = d_{\Omega_{q,1}}(z)}) \cdot (X)}{(J\Phi_{q, \eta}|_{\eta = d_{\Omega_{q,1}}(z)}) \cdot (X)}$ and $\omega = (-1, 0, \ldots, 0)$.

Let $X$ and $Y$ be two complex manifolds. Denote $\text{Hol}(Y, X)$ the set of holomorphic maps from $Y$ to $X$. Now, we recall the definition of the normal family and taut complex manifold in [Aba89].

Definition 6.2. Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in A}$ be a family in $\text{Hol}(X, Y)$. We say that $\mathcal{F}$ is a normal family if every subsequence $\{f_j\} \subseteq \mathcal{F}$ either
- (normal convergence) has a subsequence that converges uniformly on compact subsets of $X$; or
- (compact divergence) has a subsequence $\{f_{j_k}\}$ such that, for each compact $K \subseteq X$ and each compact $L \subseteq Y$, there is a number $N$ so large that $f_{j_k}(K) \cap L = \emptyset$ whenever $k \geq N$.

Let $\Delta$ be a unit disk in $\mathbb{C}$.

Definition 6.3. A complex manifold $Y$ is taut if $\text{Hol}(\Delta, Y)$ is a normal family.

Theorem 6.4 (Theorem 2.1.2 in [Aba89]). Let $Y$ be a taut complex manifold. Then $\text{Hol}(X, Y)$ is a normal family for every complex manifold $X$.

Theorem 6.5 (Theorem 3.1 in [Yu95]). Every $h$-extendible model is taut.

Theorem 6.6 (The main theorem in [Nik02]). Let $\Omega_{p,1}$ be an $h$-extendible at the boundary point $p$. Then any two models for $\Omega_{p,1}$ at $p$ are biholomorphically equivalent and determinant of its Jacobian mapping is bounded away from zero in a neighborhood of the origin.
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