RECURRENCE OF PRODUCT OF LINEAR RECURSIVE FUNCTIONS

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Abstract. We study the recurrence of the product of \( n \) functions, each of which satisfies the recurrence relation \( x(m) = A_1 x(m-1) + A_2 x(m-2) + \cdots + A_s x(m-s) \).

1. Introduction

Let \( \{ x(m) : m \in \mathbb{Z} \} \) be a function defined by the following linear recurrence relation

\[ x(m) = A_1 x(m-1) + A_2 x(m-2) + \cdots + A_s x(m-s), \tag{1.1} \]

where \( A_1, A_2, \cdots, A_s \) are constants and \( A_i \neq 0 \) for some \( i \). Denoted by \( \{ u(m) : m \in \mathbb{Z} \} \) the function that satisfies (1.1) and that \( u(1) = u(2) = \cdots = u(s-1) = 0, u(s) = 1 \). The main purpose of this article is to determine a recurrence relation for \( X(m) \), where \( X(m) \) is a product of \( n \) functions, each of which satisfies (1.1). The closed form (in terms of matrix) of such a recurrence relation for \( X(m) \) can be found in Section 2. This fact answers a question (and its generalisation) raised by Cooper and Kennedy [CK] which was partially solved by Stinchcombe [S] (Partially solved is the sense that the characteristic polynomial of his \( x_m \) must admit distinct roots). Two amazing recurrences can be found in Section 5 of [S]. The main result of our study can be found in Proposition 2.3. Our method is elementary and uses linear algebra only.

Section 3 studies the recurrence relation of the product of \( n \) recursive functions, each of which satisfies \( W_m = pW_{m-1} - qW_{m-2} \). Note that such recurrence has been studied by Jarden [J], Brenann [B] and Cooper and Kennedy [CK]. The main results can be found in Proposition 3.2 and 3.2. Our proof is direct and is slightly different from [CK]. Product of two recursive sequences each of which satisfies \( W_m = pW_{m-1} - qW_{m-2} \) is the most interesting case. The recurrence for such functions can be used to verify various identities concerning Fibonacci and Lucas numbers. See [LL1] and [LL2] for more detail.

A rather interesting connection (from our point of view) between the identity \( L_n^2 - 5F_n^2 = 4(-1)^n \) and the Galois field of the characteristic polynomial of the recurrence of \( X(m) = F_m \) can be found in Section 4.

2. The Matrix Form of the Recurrence of \( X(m) \)

The main purpose of this section is to give a recurrence relation for \( X(m) \) (a product of \( n \) functions \( x_i(m) \), each of which satisfies (1.1)). We shall start with an example which best reveals our idea and strategy.

Example 2.1. Suppose that \( x(m) \) satisfies the recurrence \( x(m+2) = px(m+1) - qx(m) \). Let \( X(m) = x(m)^2 \). Then \( X(m) \) satisfies the following recurrence relation.

\[ X(m+3) = (p^2 - q)X(m+2) + (q^2 - p^2q)X(m+1) + q^3X(m). \tag{2.1} \]

Proof. Denoted by \( u(m) \) the recursive function that satisfies \( u(m+2) = pu(m+1) - qu(m) \) and \( u(0) = 0, u(1) = 1 \). One sees easily that
\[ x(m + r + 1) = u(m + 1)x(r + 1) - qu(m)x(r). \]  
(2.2)

Take the square of the left and right hand side of (2.2) and shift all the squares to the left of the equation, one has

\[ x(m + r + 1)^2 - u(m + 1)^2x(r + 1)^2 - q^2u(m)^2x(r)^2 = 2qu(m + 1)u(m)x(r)x(r + 1). \]  
(2.3)

We note that on the right hand side of (2.3), the function \( A(r) = 2q x(r)x(r + 1) \) is independent of \( m \). The cases for \( m = 1 \) and \( 2 \) are given as follows.

\[
\begin{align*}
x(r + 2)^2 &- u(2)^2x(r + 1)^2 - q^2u(1)^2x(r)^2 = 2qu(2)u(1)x(r)x(r + 1), \quad (2.4) \\
x(r + 3)^2 &- u(3)^2x(r + 1)^2 - q^2u(2)^2x(r)^2 = 2qu(3)u(2)x(r)x(r + 1). \quad (2.5)
\end{align*}
\]

We now put these two equations into the following \( 2 \times 2 \) matrix.

\[
Z = \begin{bmatrix}
x(r + 2)^2 - u(2)^2x(r + 1)^2 - q^2u(1)^2x(r)^2 & u(2)u(1) \\
x(r + 3)^2 - u(3)^2x(r + 1)^2 - q^2u(2)^2x(r)^2 & u(3)u(2)
\end{bmatrix}
\]  
(2.6)

Denoted by \( C_i \) the columns of \( Z \). Applying (2.4) and (2.5), the first column is a multiple of the second column. To be more accurate, \( C_1 = 2q x(r + 1)x(r)C_2 = C_1 - A(r)C_2 = 0 \). Hence the determinant of the above matrix \( Z \) is zero. Note that this holds for all \( r \) and that \( u(1) = 1, u(2) = p \) and \( u(3) = p^2 - q \). This completes the study of our example.

We shall now study the general case by a simple generalisation of the techniques we presented in Example 2.1.

**Lemma 2.2.** Let \( x(m) \) be a function satisfies (1, 1). Then

\[ x(m + r + 1) = a_1(m)x(r + 1) + a_2(m)x(r) + \cdots + a_s(m)x(r + 2 - s), \]  
(2.7)

where \( a_1(m) = u(m + 1), a_i(m) = A_i u(m) + A_{i+1} u(m - 1) + \cdots + A_s u(m - s + i) \) for \( i \geq 2 \).

**In particular,** \( a_i(m) \) **satisfies** (1, 1) **for every** \( i \).

**Proof.** Apply mathematical induction. \( \square \)

Let \( x_1(m), x_2(m), \ldots, x_n(m) \) be functions, each of which satisfies (1.1). It follows from Lemma 2.2 that each one of them satisfies (2.7). Let \( X(m) = x_1(m)x_2(m) \cdots x_n(m) \). Applying (2.7), one has the following.

\[ X(m + r + 1) - \sum_{i=1}^s a_i(m)^n X(r + 2 - i) = \sum_{\Delta} A(e_1, e_2, \ldots, e_s) \prod_{i} a_i(m)^{e_i}, \]  
(2.8)

where \( \Delta \) is the set of \( (e_1, e_2, \ldots, e_s) \)'s such that \( 0 \leq e_i \) for all \( i \), \( e_1 + e_2 + \cdots + e_s = n \) and \( e_i \neq 0 \) for at least two \( i \)'s as we have shifted all the \( X(r + 2 - i) \)'s to the left of the equation. \( \Delta \) has \( k \) members (see Remark 2.4). \( A(e_1, e_2, \ldots, e_s) \) is a function in \( x_i(r + j) \)'s, where \( 1 \leq i \leq n, 2 - s \leq j \leq 1 \) (take \( X(m) = x(m)^2 \) in Example 2.1 for example, \( A(r) = 2q x(r + 1)x(r) \)). It is important to note that \( A(e_1, e_2, \ldots, e_s) \) is independent of \( m \). For instance, in (2.8), regardless the value of \( m \), the coefficient of \( a_{s-1}(m)a_s(m)^{n-1} \) is given by \( x(r + 2 - (s - 1))x(r + 2 - s)^{n-1} \).

Identity (2.8) for \( m = 1, 2, 3, \cdots \) take the following forms
\[
X(1 + r + 1) - \sum_{i=1}^{s} a_i(1)^n X(r + 2 - i) = \sum_{\Delta} A(e_1, e_2, \cdots, e_s) \prod_{i=1}^{s} a_i(1)^{e_i} \tag{2.8_1}
\]
\[
X(2 + r + 1) - \sum_{i=1}^{s} a_i(2)^n X(r + 2 - i) = \sum_{\Delta} A(e_1, e_2, \cdots, e_s) \prod_{i=1}^{s} a_i(2)^{e_i} \tag{2.8_2}
\]
\[
X(3 + r + 1) - \sum_{i=1}^{s} a_i(3)^n X(r + 2 - i) = \sum_{\Delta} A(e_1, e_2, \cdots, e_s) \prod_{i=1}^{s} a_i(3)^{e_i} \tag{2.8_3}
\]

Set
\[
c(m) = X(m + r + 1) - \sum_{i=1}^{s} a_i(m)^n X(r + 2 - i). \tag{2.9}
\]

Associate to (2.8\_m) a row vector of length \(k + 1\) of the following form (the terms \(\prod_i a_i(m)^{e_i}\) are ordered according to the lexicographical order of \(\Delta\))
\[
v_m = (c(m), a_1(m)^{n-1}a_2(m), \cdots, a_{s-2}(m)a_s(m)^{n-1}, a_{s-1}(m)a_s(m)^{n-1}). \tag{2.10}
\]

Similar to how we form matrix (2.6) from (2.4) and (2.5), we now put these \(k + 1\) vectors \(v_1, v_{1+k}, \cdots, v_{1+2k}\) (in this order) into the following \((k + 1) \times (k + 1)\) square matrix which we call it the recurrence matrix associated to \(X(m)\).

\[
Z = \begin{bmatrix}
  c(1) & \cdots & a_{s-2}(1)a_s(1)^{n-1} & a_{s-1}(1)a_s(1)^{n-1} \\
  c(1+1) & \cdots & a_{s-2}(1+1)a_s(1+1)^{n-1} & a_{s-1}(1+1)a_s(1+1)^{n-1} \\
  \vdots & \ddots & \vdots & \vdots \\
  c(1+k) & \cdots & a_{s-2}(1+k)a_s(1+k)^{n-1} & a_{s-1}(1+k)a_s(1+k)^{n-1}
\end{bmatrix} \tag{2.11}
\]

**Proposition 2.3.** Let \(X(m)\) be a product of \(n\) functions, each of which satisfies (1.1). Then \(X(m)\) admits a recurrence relation. Let \(Y(m)\) be another function which is also a product of \(n\) functions, each of which satisfies (1.1). Then \(X(m)\) and \(Y(m)\) satisfy the same recurrence relation.

**Proof.** Denoted by \(C_1\) the first column of \(Z\) and denoted by \(C_{e_1, e_2, \cdots, e_s}\) the remaining columns. Applying identities (2.8\_1)-(2.8\_k+1) and (2.9), the first column of the above matrix \(Z\) is a linear combination of the remaining columns. To be more accurate,
\[
C_1 - \sum_{\Delta} A(e_1, e_2, \cdots, e_s) C_{e_1, e_2, \cdots, e_s} = 0. \tag{2.12}
\]
As a consequence, the determinant of \(Z\) is zero. Hence the set of row vectors \(\{R_1, R_2, \cdots, R_{k+1}\}\) of \(Z\) is a linearly dependent set. Hence there exists \(\tau_j\) (\(0 \leq j \leq k\)), not all zero, such that \(\sum_i \tau_i R_i\) takes the form \(\sum_{j=0}^{k} \tau_j c(1+j), 0, 0, \cdots, 0\). Since the fact that the first column is a linear combination of the remaining columns remains valid under row operation, we conclude that \(\sum_j \tau_j c(1+j) = 0\). We note that in (2.7)-(2.9), \(r\) is independent of \(m\). This implies that \(0 = \sum \tau_j c(1+j) = \sum \tau_j [X(1 + r + 1) - \sum_{i=1}^{s} a_i(1)^n X(r + 2 - i)]\) for all \(r\). As a consequence, this gives a recurrence relation for \(X(m)\).

Since \(X(m)\) and \(Y(m)\) admit the same recurrence matrix \(Z\), they satisfy the same recurrence relation.

\[
\square
\]
Remark 2.4. Apply the Multinomial Theorem for \((y_1 + y_2 + \cdots + y_s)^n, k = \binom{n + s - 1}{n - s} - s\).

Remark 2.5. (i) Denoted by \(C_1, C_2, \cdots, C_{k+1}\) the column vectors of \(Z\). Let \(V = [C_2, C_3, \cdots, C_{k+1}]\). In the case the matrix \(V = [C_2, C_3, \cdots, C_{k+1}]\) is of rank \(k\), the cofactor expansion along the first column of \(Z\) gives a recurrence relation of \(X(m)\). In the case the rank of \(V\) is \(r\) and \(r \leq k - 1\), one can still apply cofactor expansion to an \((r + 1) \times (r + 1)\) submatrix of \(Z\) to get a recurrence relation of \(X(m)\).

(ii) In [S], Stinchcombe [S] studies the characteristic polynomial of \((1.1)\) and draws the conclusion about the existence of a recurrence relation for \(X(m) = x(m)^n\). Note that his method is different from ours where his calculation must be carried through over the complex field \(\mathbb{C}\) and the roots of the characteristic polynomial of \(x(m)\) must be distinct. Two amazing recurrences can be found in Section 5 of [S].

2.1. Discussion. (i) Let \(X(m)\) be given as in Proposition 2.3. Suppose that the recurrence of \(X(m)\) involves \(t\) terms. Since \(\{X(1+k), \cdots, X(t+k)\}\) satisfies the same recurrence relation for arbitrary \(k\), determinant of the following matrix is zero, which can be viewed as a recurrence relation for \(X(m)\) as well. This generalises a result stated in [W].

\[
\begin{bmatrix}
X(k+1) & X(k+2) & \cdots & X(k+t) \\
X(k+t+1) & X(k+t+2) & \cdots & X(k+2t) \\
\vdots & \vdots & \ddots & \vdots \\
X(k+t^2-t+1) & X(k+t^2-t+2) & \cdots & X(k+t^2)
\end{bmatrix}.
\]  

(ii) As one recurrence function may satisfy more than one recurrence relation, one would like to know whether Proposition 2.3 offers the best recurrence relation for \(X(m)\). The answer is No. Take \(x(m) = 2x(m-1) - x(m-2)\) \((x(0) = x(1) = 1)\) for instance, applying Proposition 2.3, the recurrence for \(X(m) = x(m)^2\) is \(X(m) = 3X(m-1) - 3X(m-2) + X(m-3)\). This is obviously not the best recurrence for \(X(m)\) as \(X(m)\) satisfies the recurrence relation \(X(m) = 2X(m-1) - X(m-2)\) as well. Is this because the characteristic polynomial of \(x(m) = 2x(m-1) - x(m-2)\) has repeated root? Again, the answer is No. Take \(y(m) = 2y(m-1) - y(m-2)\) \((y(0) = 0, y(1) = 1)\) for instance, \(Y(m) = y(m)^2\) satisfies the relation \(Y(m) = 3Y(m-1) - 3Y(m-2) + Y(m-3)\) but \(Y(m) \neq 2Y(m) - Y(m-1)\). Hence the initial values also play some roles.

3. Recurrence Relation for powers of \(W_m = pW_{m-1} - qW_{m-2}\)

Let \(a, b, p, q \in \mathbb{C}, q \neq 0\). Following the notations of [Ho], we define the generalised Fibonacci sequence \(\{W_m\} = \{W_m(a, b; p, q)\}\) by \(W_0 = a, W_1 = b,\)

\[W_m = pW_{m-1} - qW_{m-2}.
\]  

Obviously the definition can be extended to negative subscripts; that is, for \(n = 1, 2, 3, \cdots\), define

\[W_{-m} = (pW_{-m+1} - W_{-m+2})/q.
\]  

In the case \(a = 0, b = 1\), we shall denote the sequence \(\{W_m(0, 1; p, q)\}\) by \(\{u_m\}\). Equivalently,

\[u_m = W_m(0, 1; p, q).
\]
Note that \( u_m = -q^{-m}u_m \) (this simple fact can be proved by mathematical induction) and that
\[
W_{m+r+1} = u_{m+1}W_{r+1} - qu_m W_r. \tag{3.4}
\]
The purpose of this section is to give a recurrence relation of \( X(m) \), a function which is a product of \( n \) functions, each of which satisfies the recurrence (3.1). Applying our results in Proposition 2.3, the recurrence matrix \( Z \) (see (2.11)) is an \( n \times n \) matrix of the following form and \( \det Z = 0 \) gives us a recurrence for \( X(m) \). A detailed study of the recurrence \( \det Z = 0 \) can be found in sections 3.1 and 3.2.

\[
Z = \begin{bmatrix}
X(r + 2) - X(r + 1)u^n_3 - (-q)^nX(r)u^n_1 & u^n_2 - 2u^n_1 & u^n_3 - 2u^n_2 & \cdots & u^n_1 - 1 \\
X(r + 3) - X(r + 1)u^n_3 - (-q)^nX(r)u^n_2 & u^n_2 - u^n_1 & u^n_3 - u^n_2 & \cdots & u^n_1 - 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
X(n + r + 1) - X(r + 1)u^n_{n+1} - (-q)^nX(r)u^n_n & u^n_{n+1} - u^n_n & u^n_{n+2} - u^n_n & \cdots & u^n_{n+1} - 1 \\
\end{bmatrix} \tag{3.5}
\]

3.1 The nontrivial case. Throughout this subsection, we shall assume that \( u_1 u_2 \cdots u_{n+1} \neq 0 \). Denoted by \( c_i \), the \( i \)-th column of the above matrix. The determinant of \( Z \) can be determined as follows. Let \( R = (u^n_2, u^n_3, \ldots, u^n_{n+1})^t \), \( S = (u^n_1, u^n_2, \ldots, u^n_n)^t \), \( T = (X(r + 2), X(r + 3), \ldots, X(n + r + 1))^t \) (\( t \) for transpose). Set \( A = (R, c_2, c_3, \ldots, c_n) \), \( B = (S, c_2, c_3, \ldots, c_n) \), \( C = (T, c_2, c_3, \ldots, c_n) \). Since the determinant function is a linear function, one has
\[
\det Z = \det C - X(r + 1) \cdot \det A - (-q)^nX(r) \cdot \det B = 0. \tag{3.6}
\]

Applying (A3) of Appendix A, Appendix B and (C2), (C5) of Appendix C, one has the following.

**Lemma 3.1.** Let \( n \in \mathbb{N} \) be fixed and let \( A, B \) and \( C \) be \( n \times n \) matrices given as above (see (3.5) and (3.6)). Suppose that \( u_1 u_2 \cdots u_{n+1} \neq 0 \). Then
\[
\det C = \sum_{i=1}^{n} C_i, \quad \det B = (-1)^{n-1} \sigma(n-1) \det A_1, \quad \det A = \sigma(n) \det A_1, \tag{3.7}
\]
where \( A_1 \) is a nonsingular \( n \times n \) matrix, \( \sigma(n) = u_{n+1}u_n \cdots u_3u_2 \) and the quantity \( C_i \) is given by the following.
\[
C_i = \frac{X(r + i + 1)\sigma(n)\sigma(n-1) \det A_1 \det Q^i_{n-1}}{u_{i+1}u_i(u_1 - i - u_{i-1})(u_1 u_2 \cdots u_{n-1}) \det Q^i_{n}}, \tag{3.8}
\]
where \( 1 \leq i \leq n, u_{-m} = -q^{-m}u_m \) (\( q \) is the fixed constant given in (3.1)) and \( Q_m \) is an \( m \times m \) matrix of determinant \( q^{m(m-1)/2} \).

**Proof.** See Appendix A, B and C.

\[ \Box \]

Note that \( \det A_1 \) appears in \( C_i \), \( \det B \) and \( \det A \) and that \( \det A_1 \) is nonzero. Applying (3.7), (3.8) and \( u_{-m} = -q^{-m}u_m \), one may now simplify (3.6) and conclude that
\[
\sum_{i=0}^{n+1} (-1)^i q^{i(i-1)/2}(n + 1)_{n} X(n + r + 1 - i) = 0, \tag{3.9}
\]
where \( (m|k)_u = 1 \) if \( k = 0 \) and \( (m|k)_u = u_m u_{m-1} \cdots u_{m-k-1}/u_k u_{k-1} \cdots u_1 \) if \( 1 \leq k \leq m \). Note that \( (m|k)_u \) is known as the generalised binomial coefficient. In summary, the following proposition holds.
**Proposition 3.2.** Let \( n \in \mathbb{N} \) be fixed and let \( X(m) \) be a product of \( n \) functions, each of which satisfies (3.1). Suppose that \( u_1u_2\cdots u_{n+1} \neq 0 \). Then \( X(m) \) satisfies the following recurrence.

\[
\sum_{i=0}^{n+1} (-1)^i q^{(i-1)/2} (n + 1 | i) u X(m - i) = 0.
\] (3.10)

**Proof. Alternative Proof of Proposition 3.2.** One sees easily that the \( X(m) \) in Proposition 3.2 and \( u_m^n \) admit the same recurrence matrix (see (2.11) for the definition and (3.5) for the actual matrix). Hence \( X(m) \) and \( u_m^n \) satisfy the same recurrence relation (see Proposition 2.3). One may now prove the proposition by applying Jarden’s Theorem which gives the recurrence for \( u_m^n \) (see Theorem 4.1 of [CK]). \( \square \)

3.2. **The trivial case.** Throughout this section, \( u_1u_2\cdots u_{n+1} = 0 \). This implies that \( u_k = 0 \), for some \( k \), where \( 1 \leq k \leq n+1 \). Identity (3.4) now becomes \( W_{k+r+1} = u_{k+1}W_{r+1} \). Since \( X(m) \) is a product of \( n \) function, each of which satisfies \( W_{k+r+1} = u_{k+1}W_{r+1} \), one has \( X(k+r+1) = u_{k+1}X(r+1) \). Since this holds for every \( r \in \mathbb{Z} \) and \( k \) is fixed, the following proposition is clear.

**Proposition 3.3.** Let \( n \in \mathbb{N} \) be fixed and let \( X(m) \) be a product of \( n \) functions, each of which satisfies (3.1). Suppose that \( u_k = 0 \), for some \( k \), where \( 1 \leq k \leq n+1 \). Then \( X(m) \) satisfies the following recurrence.

\[
X(k + r + 1) = u_{k+1}X(r+1).
\] (3.11)

**Example 3.4.** Let \( u_0 = 1, u_1 = 1 \) and \( u_n = 2u_{n-1} - 4u_{n-2} \). Then \( u_2 = 2, u_3 = 0, u_4 = -8 \). By Proposition 3.3, \( u_m^n \) satisfies the following recurrence

\[
u^{3}_{r+4} + 8^{3}u^{3}_{r+1} = 0.
\] (3.12)

Note that \( u_m^n \) also satisfies the recurrence relation \( u^{3}_{r+4} - 8u^{3}_{r+3} - 512u^{3}_{r+1} + 4096u^{3}_{r} = 0 \).

3.3. **Application.** The following corollary of Proposition 3.2 has been applied by Lang and Lang ([LL1] and [LL2]) to prove various identities concerning the generalised Fibonacci sequence.

**Corollary 3.5.** \( A(m) = W_{2m}, B(m) = W_{m}W_{m+r} \) and \( C(m) = q^{m} \) satisfy the following recurrence relation

\[
X(m + 3) = (p^2 - q)X(m + 2) + (q^2 - p^2q)X(m + 1) + q^{3}X(m).
\] (3.13)

The following corollary of Proposition 3.2 shows that the recurrence (3.10) can be used to describe the characteristic polynomial of \( Q_v \) (see (A4) of Appendix A) which reveals the fact that the recurrence relation carries a lot of information about the generalised Fibonacci numbers. Proof of Corollary 3.6 can be found in Appendix D.

**Corollary 3.6.** Let \( Q_v \) be given as in (A4) of Appendix A. Then the characteristic polynomial of \( Q_v \) is \( \sum_{i=0}^{n} (-1)^i q^{(i-1)/2} (v | i) u x^i \). Note that \( f(x) \) is the polynomial that characterises the recurrence (3.10).
4. The Galois Group of the Recurrence Relation

Let \( p \) and \( q \) be given as in (3.1) and let

\[
\phi_n(p, q, x) = \sum_{i=0}^{n} (-1)^i q^{i(i-1)/2} (n|i) u^i x^i.
\]

We call the polynomial in (4.1) (see Corollary 3.6 also) the Galois polynomial of the recurrence relation (3.10). Similarly we call the Galois group \( G_n(p, q) \) of \( \phi_n(p, q, x) \) over \( \mathbb{Q} \) the Galois group of the recurrence relation (3.10).

Proposition 4.1. \( G_n(1, -1) \cong \mathbb{Z}_2 \), a cyclic group of order 2, for all \( n \geq 2 \).

Proof. Let \( a \) and \( b \) be roots of \( \phi_2(x) = x^2 - x - 1 = 0 \). One can show by induction that (see [B])

\[
\phi_n(1, -1, x) = (ab)^{n-1}(x-a^n)(x-b^n)\phi_{n-2}(1, -1, x/ab)
\]

for all \( n \geq 2 \) (\( \phi_0(x) \) is defined to be 1). Hence \( \phi_n(1, -1, x) \) splits completely in the field \( \mathbb{Q}(\sqrt{5}) \) and \( G_n(1, -1) \cong \mathbb{Z}_2 \). \( \square \)

4.1. Discussion. Let \( F_n \) and \( L_n \) be the \( n \)-th Fibonacci and Lucas numbers respectively. It is a well known fact that

\[
L_n^2 - 5F_n^2 = 4(-1)^n.
\]

Identity (4.3), from our point of view, is a must rather than an interesting connection between Fibonacci and Lucas numbers as the following suggested. Recall another well known fact about the factorisation of \( \phi_n(1, -1, x) \).

\[
\phi_n(1, -1, x) = (-1)^n(x^2 - L_n x + (-1)^n)\phi_{n-2}(1, -1, -x).
\]

Since \( \phi_n(1, -1, x) \) splits in \( \mathbb{Q}(\sqrt{5}) \) and \( x^2 - L_n x + (-1)^n = 0 \) splits in \( \mathbb{Q}(\sqrt{L_n^2 - 4(-1)^n}) \), one must have

\[
L_n^2 - 4(-1)^n = 5A_n^2,
\]

for some \( A_n \in \mathbb{N} \). This tells us that the difference between \( 4(-1)^n \) and the square of the Lucas number \( L_n \) must be five times a square \( A_n^2 \). As for why \( A_n \) must be \( F_n \), we note that both \( F_n^2 \) and \( L_n^2 \) satisfy the recurrence relation (3.13) (to be more accurate, the recurrence \( X(n+3) = 2X(n+2) + 2X(n+2) - X(n) \) and that \( L_n^2 - 4(-1)^n \) and \( F_n^2 \) have the same initial values. In general, one has (Lemma 3.3 of Cooper and Kennedy [CK]),

\[
\phi_n(p, q, x) = \prod_{j=0}^{n} (x - \alpha^j \beta^{n-j}),
\]

where \( \alpha \) and \( \beta \) are roots of \( x^2 - px + q = 0 \). Hence \( G(p, q) \cong \mathbb{Z}_2 \) for all \( n \geq 2 \). This fact will give an identity similar to (4.5).
5. Appendix A

Throughout the appendix $u_1 u_2 \cdots u_{n+1} \neq 0$. We shall give full detail of how the determinant of the matrix $A$ is evaluated ($A$ is given as follows). Applying our technique given in this appendix, the determinants of $B$ and $C$ can be calculated similarly (see Appendix $B$ and $C$).

$$A = \begin{bmatrix}
  u_2^n & u_2^{n-1} u_1 & u_2^{n-2} u_1^2 & \cdots & u_2 u_1^{n-1} \\
  u_3^n & u_3^{n-1} u_2 & u_3^{n-2} u_2^2 & \cdots & u_3 u_2^{n-1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  u_{n+1}^{n+1} & u_{n+1}^n u_n & u_{n+1}^{n-2} u_n^2 & \cdots & u_{n+1} u_n^{n-1}
\end{bmatrix} \quad (A1)$$

Note that $u_{i+1}$ is a common factor of the entries of the $i$-th row. Hence det $A$ can be written as the product $u_{n+1} u_n \cdots u_2 \det A_1$, where

$$A_1 = \begin{bmatrix}
  u_2^{n-1} & u_2^{n-2} u_1 & u_2^{n-3} u_1^2 & \cdots & u_2 u_1^{n-1} \\
  u_3^{n-1} & u_3^{n-2} u_2 & u_3^{n-3} u_2^2 & \cdots & u_3 u_2^{n-1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  u_{n+1}^{n-1} & u_{n+1}^{n-2} u_n & u_{n+1}^{n-2} u_n^2 & \cdots & u_{n+1} u_n^{n-1}
\end{bmatrix} \quad (A2)$$

For our convenience, we shall define $\sigma(n)$, $\tau(n)$ as follows which will be used in the following discussion.

$$\sigma(n) = \prod_{i=2}^{n+1} u_i, \quad \tau(n) = q^{(n-1)/2}, \quad \det A = \sigma(n) \det A_1. \quad (A3)$$

The rest of this section is devoted to the determination of the determinant of $A_1$. To save space, we denote the binomial coefficient $\left( \begin{array}{c} n \\ k \end{array} \right)$ by $(n/k)$. Consider the following matrix.

$$Q_n = \begin{bmatrix}
  (n-1|0)p^{n-1} & \cdots & (n-2|0)p^{n-2} & \cdots & p & 1 \\
  (n-1|1)p^{n-2}(-q) & \cdots & (n-2|1)p^{n-3}(-q) & \cdots & -q & 0 \\
  (n-1|2)p^{n-3}(-q)^2 & \cdots & (n-2|2)p^{n-4}(-q)^2 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  (n-1|n-2)p(-q)^{n-2} & \cdots & (n-2|n-2)(-q)^{n-2} & \cdots & 0 & 0 \\
  (n-1|n-1)(-q)^{n-1} & \cdots & \cdots & \cdots & 0 & 0
\end{bmatrix} \quad (A4)$$

Note that $Q_n$ is an $n \times n$ matrix where the $i$-th column of $Q_n$ gives the coefficients of the binomial expansion of $(px - qy)^{n-i}$ and that det $Q_n = \tau(n) = q^{(n-1)/2}$. Since $u_{r+1} = pu_r - qu_{r-1}$, $u_0 = 0$, $u_1 = 1$ (see (1.3)), the multiplication of $A_1$ by $Q_n$ shifts the indices. That is,

$$A_1 Q_n^{-1} = \begin{bmatrix}
  u_1^{n-1} & u_1^{n-2} u_0 & \cdots & u_1^{n-1} \\
  u_2^{n-1} & u_2^{n-2} u_1 & \cdots & u_1^{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_n^{n-1} & u_n^{n-2} u_{n-1} & \cdots & u_1^{n-1}
\end{bmatrix}, \quad A_1 Q_n = \begin{bmatrix}
  u_3^{n-1} & u_3^{n-2} u_2 & \cdots & u_2^{n-1} \\
  u_4^{n-1} & u_4^{n-2} u_3 & \cdots & u_3^{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{n+2}^{n-1} & u_{n+2}^{n-2} u_{n+1} & \cdots & u_{n+1}^{n-1}
\end{bmatrix} \quad (A5)$$
Since \( u_0 = 0, u_1 = 1 \), the first row of \( A_1 Q_n^{-1} \) is \((1^{n-1}, 0, 0, \ldots, 0)\). Applying the cofactor expansion to the first row of \((A5)\), the determinant of \( A_1 Q_n^{-1} \) is the determinant of the following matrix.

\[
Z = \begin{bmatrix}
  u_2^{n-1} & u_2^{n-2} & \cdots & 0 \\
  u_3^{n-2} & u_2^{n-2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{n-1}^{n-2} & u_{n-2}^{n-2} & \cdots & 0 \\
  u_{n-1}^{n-1} & u_{n-2}^{n-1} & \cdots & 0
\end{bmatrix}
\] \hfill (A6)

Compare the matrices in \((A1)\) (an \(n \times n\) matrix) and \((A6)\) (an \((n - 1) \times (n - 1)\) matrix), we have established a recursive process which enables us to calculate the determinant of the matrix \( A \) as well as \( A_1 \).

\[
\det A_1 = \tau(n) \tau(n-1) \cdots \tau(2) \sigma(n-2) \sigma(n-3) \cdots \sigma(1) \neq 0. \hfill (A7)
\]

Since the multiplication of \( A_1 \) by \( Q_n \) shifts the indices (see \((A5)\)), the following is clear. Note that this matrix plays an important in the study of matrix \( C \) (see Appendix C).

\[
A_1 Q_n^{-i} = \begin{bmatrix}
  u_2^{n-1} & u_2^{n-2} u_{1-i} & u_2^{n-3} u_{1-i}^2 & \cdots & u_2^{n-1} \\
  u_3^{n-2} & u_3^{n-3} u_{2-i} & u_3^{n-4} u_{2-i}^2 & \cdots & u_3^{n-2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  u_{n-1}^{n-2} & u_{n-1}^{n-3} u_{n-i-1} & u_{n-1}^{n-4} u_{n-i-1}^2 & \cdots & u_{n-1}^{n-2} \\
  u_{n-1}^{n-1} & u_{n-2} u_0 & u_{n-3} u_0^2 & \cdots & u_0^{n-1}
\end{bmatrix}
\] \hfill (A8)

Note that the \( i \)-th row of \( A_1 Q_n^{-i} \) takes the form \((1^{n-1}, 0, 0, \ldots, 0)\). The determinant of \( A_1 Q_n^{-i} \) can be determined by the cofactor expansion by the \( i \)-th row (see \((C5)\) of Appendix C).

6. Appendix B

Throughout the appendix \( u_1 u_2 \cdots u_{n+1} \neq 0 \). The matrix \( B \) is given as follows.

\[
B = \begin{bmatrix}
  u_1^{n} & u_2^{n-1} & u_2^{n-2} & u_2^{n-3} & \cdots & u_2 u_1^{n-1} \\
  u_2^{n} & u_3^{n-1} & u_3^{n-2} & u_3^{n-3} & \cdots & u_3 u_2^{n-1} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  u_n^{n} & u_{n+1}^{n-1} & u_{n+1}^{n-2} & u_{n+1}^{n-3} & \cdots & u_{n+1} u_n^{n-1}
\end{bmatrix}
\] \hfill (B1)

After the removal of the common factor \( u_i \) from each entry of the \( i \)-th row, one sees that the resulting matrix is just the matrix \( A_1 \) (see equation \((A2)\)) when the first column is moved to the last. Hence \( \det B = (-1)^{n-1} \sigma(n-1) \det A_1 \) (see \(A(3)\) for the definition of \( \sigma(n-1) \)).
7. Appendix C

Throughout the appendix \( u_1 u_2 \cdots u_{n+1} \neq 0 \). The matrix \( C \) is given as follows. Its determinant can be calculated by the cofactor expansion of the first column. The calculation is tedious but elementary.

\[
C = \begin{bmatrix}
W^n_{r+2} & u_2^{n-1} u_1 & u_2^{n-2} u_1^2 & \cdots & u_2 u_1^{n-1} \\
W^n_{r+3} & u_3^{n-1} u_2 & u_3^{n-2} u_2^2 & \cdots & u_3 u_2^{n-1} \\
& \vdots & \vdots & \ddots & \vdots \\
W^n_{n+r+1} & u_n^{n-1} u_{n-1} & u_n^{n-2} u_{n-1}^2 & \cdots & u_n u_{n-1}^{n-1}
\end{bmatrix}
\]  
(C1)

Denoted by \( C_i \) the \( i \)-th cofactor of the first column of \( C \). The determinant of \( C \) is given by the following.

\[
\det C = C_1 + C_2 + \cdots + C_n,
\]  
(C2)

where \( C_i = (-1)^{i+1} W^n_{r+i+1} \det X \) and \( X \) is the \((n-1) \times (n-1)\) matrix that takes the following form.

\[
X = \begin{bmatrix}
u_2^{n-1} u_1 & u_2^{n-2} u_1^2 & \cdots & u_2 u_1^{n-1} \\
u_3^{n-1} u_2 & u_3^{n-2} u_2^2 & \cdots & u_3 u_2^{n-1} \\
& \vdots & \vdots & \ddots & \vdots \\
u_{n+1}^{n-1} u_n & u_{n+1}^{n-2} u_n^2 & \cdots & u_{n+1} u_n^{n-1}
\end{bmatrix}
\]  
(C3)

Note that \( u_{j+1} u_j \) is a common factor of the entries of the \( j \)-th row when \( j \leq i - 1 \) and that \( u_{j+2} u_{j+1} \) is a common factor of the entries of the \( j \)-th row when \( j \geq i \). As a consequence, the determinant of \( X \) is the product of \( \sigma(n) \sigma(n-1) / u_{i+1} u_i \) and the determinant of \( Y \), where

\[
Y = \begin{bmatrix}
u_2^{n-2} & u_2^{n-3} u_1 & \cdots & u_2^{n-2} \\
u_3^{n-2} & u_3^{n-3} u_2 & \cdots & u_3^{n-2} \\
& \vdots & \vdots & \ddots & \vdots \\
u_{n+1}^{n-2} & u_{n+1}^{n-3} u_n & \cdots & u_{n+1}^{n-2}
\end{bmatrix}, \quad YQ_{n-1}^i = \begin{bmatrix}
u_2^{n-2} & u_2^{n-3} u_{1-i} & \cdots & u_2^{n-2} \\
u_3^{n-2} & u_3^{n-3} u_{2-i} & \cdots & u_3^{n-2} \\
& \vdots & \vdots & \ddots & \vdots \\
u_{n+1}^{n-2} & u_{n+1}^{n-3} u_{n-i} & \cdots & u_{n+1}^{n-2}
\end{bmatrix}
\]  
(C4)

Similar to (A8), we consider the matrix \( YQ_{n-1}^i \). It is an easy matter to write down the matrix as multiplication by \( Q_{n-1} \) shifts the indices (see (A5) and (A8)). To calculate the determinant of \( Y \), we consider the cofactor expansion of the matrix \( A_1 Q_{n-i}^{-i} \) by the \( i \)-th row (see (A8)), where the \( i \)-th row of \( A_1 Q_{n-i}^{-i} \) takes the form \((1^{n-1}, 0, 0, \cdots, 0)\). An easy observation of the actual forms of the matrices \( A_1 Q_{n-i}^{-i} \) and \( YQ_{n-1}^{-i} \) shows that

\[
\det A_1 Q_{n-i}^{-i} = (-1)^{i+1} (u_{1-i} u_{2-i} \cdots u_{-1}) (u_1 u_2 \cdots u_{n-i}) \det YQ_{n-1}^{-i}.
\]  
(C5)
In summary, 
\[ C_i = \frac{W_{r+i+1}^n \sigma(n) \sigma(n - 1) \det A_1 \det Q_{n-1}^i}{u_{i+1} u_i (u_{i-1} u_{i-2} \cdots u_{i-1}) (u_{i+1} u_i \cdots u_{n-1}) \det Q_n^i}. \]  
(C6)

Recall that \( u_m = -q^m u_m \) (this simple fact can be proved by induction) and that \( \det Q_n = q^{n(n-1)/2} \). This gives the determinant of \( C \) (see (C2)).

8. APPENDIX D

Suppose that \( u_1 u_2 \cdots u_v \neq 0 \). This appendix is devoted to the study of the characteristic polynomial of the matrix \( Q_v \) (see (A4)). For simplicity, we shall denote \( Q_v \) by \( Q \). Let \( E \) be the rational canonical form of \( f(x) = \sum_{i=0}^v a_i x^i = \sum_{i=0}^v (-1)^i q^{i(i-1)/2} (v|i) u_i x^i \). Then

\[
E = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
& & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
a_0 & a_1 & a_2 & a_3 & \cdots & a_{v-2} & a_{v-1}
\end{bmatrix}.
\]  
(D1)

Let \( A_1 \) (set \( n = v \)) be the matrix given as in (A2). Applying Proposition 3.2, the function \( X(m) = u_{m-1}^k u_m^{v-1-k} \) satisfies the recurrence relation

\[
\sum_{i=0}^v (-1)^i q^{i(i-1)/2} (v|i) u_i^k u_{i-1}^{v-1-k} = 0.
\]  
(D2)

Applying (D2), the multiplication of \( A_1 \) by \( E \) (to the left of \( A_1 \)) shifts the indices. Since the multiplication of \( A_1 \) by \( Q \) (to the right of \( A_1 \)) shifts the indices as well (see (A5)), one has

\[
E A_1 = A_1 Q,
\]  
(D3)

which can be verified by direction calculation. Since the determinant of \( A_1 \) is nonzero (see (A7)), \( E \) and \( Q \) are similar to each other. As a consequence, the characteristic polynomial of \( Q \) is \( f(x) \).

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