P-Regular Nearrings Characterized by Their Bi-ideals

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Abstract

Using the idea of quasi-ideals of P-regular nearrings, the concept of bi-ideals of P-regular nearrings is generalized, which is an extension of the concept of quasi-ideals of P-regular nearrings and some interesting characterizations of bi-ideals are obtained. As a result, we prove that every element of a bi-ideal B of a P-regular nearring can be represented as the sum of two elements of P and Q. Moreover, every element of the finite intersection \( \bigcap_{i=1}^{n} B_i \) of bi-ideals of a P-regular distributive nearring \( N \) can be represented as the sum of two elements of \( P \) and \( B_1 NB_2 N \ldots NB_{n-1} NB_n \).

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1 Introduction and Preliminaries

The notion of nearrings is first defined by Pilz [8] in 1977 and that of bi-ideals by Chelvam and Ganesan [3] in 1987. As we know, nearrings are a generalization of rings, and bi-ideals are a generalization of quasi-ideals and ideals in nearrings. Many types of ideals on the algebraic structures were characterized by several authors such as: In 1983, Yakabe [10] introduced and characterized the notion of quasi-ideals of nearrings. In 1987, Chelvam and Ganesan [3] introduced and generalized the notion of quasi-ideals of nearrings which was introduced by [10] to bi-ideals. In 1989, Yakabe [11] characterized regular zero-symmetric nearrings without nonzero nilpotent elements in terms of quasi-ideals. In 1990, Andrunakievich [2] introduced P-regular rings. In 1991, Choi [4] extended the P-regularity of rings which was introduced by [2] to the P-regularity of nearrings. In 2005, Kim, Jun and Yon [7] introduced the notion of anti fuzzy ideals of near-rings and investigated some related properties. In 2008, Abbasi and Rizvi [1] studied prime ideals in near-rings. In 2009,
Zhan and B. Davvaz [12] introduced the concept of \((\xi, \xi \lor \eta)\)-fuzzy subnear-rings (ideals) of near-rings and obtain some of its related properties. In 2010, Choi [5] gave some characterizations of quasi-ideals of \(P\)-regular nearrings and proved that every element of a quasi-ideal \(Q\) of a \(P\)-regular nearring can be represented as the sum of two elements of \(P\) and \(Q\). In 2011, Dheena and Manivasan [6] gave some characterizations of quasi-ideals of \(P\)-regular nearrings in the same way as of Choi [5]. In 2012, Sharma [9] studied the properties of intuitionistic fuzzy ideals of nearring with the help of their \((\alpha, \beta)\)-cut sets. The concept of quasi-ideals play an important role in studying the structure of nearrings. Now, the notion of bi-ideals is an important and useful generalization of quasi-ideals of nearrings. Therefore, we will study bi-ideals of nearrings in the same way as of quasi-ideals of nearrings which was studied by Choi [5].

To present the main results we discuss some elementary definitions that we use later.

**Definition 1.1.** [8] A nearring is a system consisting of a nonempty set \(N\) together with two binary operations on \(N\) called addition and multiplication such that

1. \(N\) together with addition is a group,
2. \(N\) together with multiplication is a semigroup, and
3. \((a + b)c = ac + bc\) for all \(a, b, c \in N\).

For two nonempty subsets \(A\) and \(B\) of a nearring \(N\), let

\[A + B := \{a + b \mid a \in A \text{ and } b \in B\}\]

and

\[AB := \{ab \mid a \in A \text{ and } b \in B\}.

If \(A = \{a\}\), then we also write \(a + B\) as \(a + B\), and \(\{a\}B\) as \(aB\), and similarly if \(B = \{b\}\).

**Definition 1.2.** A nonempty subset \(S\) of a nearring \(N\) is called a left (right) \(N\)-subgroup of \(N\) if

1. \(S\) together with addition is a subgroup of \(N\), and
2. \(SN \subseteq S\) (\(SN \subseteq S\)).

**Definition 1.3.** A nonempty subset \(S\) of a nearring \(N\) is called an ideal of \(N\) if

1. \(S\) together with addition is a normal subgroup of \(N\),
2. \(SN \subseteq S\),
3. \(NS \subseteq S\), and
4. \(n_1(n_2 + s) - n_1n_2 \in S\) for all \(s \in S\) and \(n_1, n_2 \in N\).
Note that $S$ is a left ideal of $N$ if $S$ satisfies (1), (3) and (4), and $S$ is a right ideal of $N$ if $S$ satisfies (1) and (2).

**Remark 1.4.** By Definition 1.3, we have that

1. $S$ is a left ideal of $N$ if and only if $S$ is a normal left $N$-subgroup of $N$ and $n_1(n_2 + s) - n_1n_2 \subseteq S$ for all $s \in S$ and $n_1, n_2 \in N$.
2. $S$ is a right ideal of $N$ if and only if $S$ is a normal right $N$-subgroup of $N$.

**Definition 1.5.** A nearring $N$ is called a distributive nearring if $a(b + c) = ab + ac$ for all $a, b, c \in N$.

**Definition 1.6.** A nonempty subset $Q$ of a nearring $N$ is called a quasi-ideal of $N$ if

1. $Q$ together with addition is a subgroup of $N$, and
2. $QN \cap NQ \subseteq Q$.

**Definition 1.7.** A nonempty subset $B$ of a nearring $N$ is called a bi-ideal of $N$ if

1. $B$ together with addition is a subgroup of $N$, and
2. $BNB \subseteq B$.

**Definition 1.8.** A nearring $N$ is called regular nearring if for each $x \in N$ there exists $y \in N$ such that $xyx = x$.

**Definition 1.9.** Let $N$ be a nearring with unity and $P$ an ideal of $N$. Then $N$ is said to be $P$-regular nearring if for each $x \in N$ there exists $y \in N$ such that $xyx - x \in P$.

## 2 Lemmas

Before the characterizations of bi-ideals of nearrings for the main results, we give some auxiliary results which are necessary in what follows.

**Lemma 2.1.** [5] Let $N$ be a nearring and $P = \{0\}$. If $N$ is a $P$-regular nearring, then $N$ is a regular nearring.

**Lemma 2.2.** Let $\mathcal{B}$ be a nonempty family of bi-ideals of a nearring $N$. Then $\bigcap \mathcal{B}$ is a bi-ideal of $N$.

**Proof.** Clearly, $\bigcap \mathcal{B}$ together with addition is a subgroup of $N$. Now, for all $B \in \mathcal{B}$, we have

$$\bigcap BN \bigcap B \subseteq BNB \subseteq B.$$

Thus $\bigcap BN \bigcap B \subseteq \bigcap B$. Hence $\bigcap \mathcal{B}$ is a bi-ideal of $N$. $\square$

**Corollary 2.3.** Any finite intersection of bi-ideals of a nearring is a bi-ideal.

**Lemma 2.4.** Every quasi-ideal of a nearring is a bi-ideal.

**Proof.** Let $Q$ be a quasi-ideal of a nearring $N$. Then $Q$ together with addition is a subgroup of $N$. Thus $QNQ \subseteq QN$ and $QNQ \subseteq NQ$, so $QNQ \subseteq QN \cap NQ \subseteq Q$. Hence $Q$ is a bi-ideal of $N$. $\square$
3 Main Results

In this section, give some characterizations of bi-ideals of nearrings. Finally, we prove that every element of a bi-ideal $B$ of a $P$-regular nearring can be represented as the sum of two elements of $P$ and $Q$. Moreover, every element of the finite intersection $\bigcap_{i=1}^{n} B_i$ of bi-ideals of a $P$-regular distributive nearring $N$ can be represented as the sum of two elements of $P$ and $B_1NB_2N \ldots NB_{n-1}NB_n$.

**Theorem 3.1.** Let $N$ be a $P$-regular nearring. Then for each $n \in N$ there exists $n' \in N$ such that $n'n \in P$.

**Theorem 3.2.** Let $N$ be a $P$-regular distributive nearring. Then for every right ideal $R$ and every left ideal $L$ of $N$,

$$(P + R) \cap (P + L) = P + RL.$$ 

**Theorem 3.3.** Let $N$ be a $P$-regular nearring and $B$ a bi-ideal of $N$. Then every $x \in B$ there exist $p' \in P$ and $b' \in B$ such that $x = p' + b'$.

**Proof.** Let $x \in B$. Since $N$ is a $P$-regular nearring and $x \in B \subseteq N$, there exists $y \in N$ such that $xy - x = p$ for some $p \in P$. Thus $x = -p + xy$. Since $B$ is a bi-ideal of $N$, we have $xy \in BNB \subseteq B$. Since $p \in P$ and $P$ together with addition is a subgroup of $N$, we have $-p \in P$. Put $p' = -p$ and $b' = xy$. Thus

$$x = -p + xy = p' + b' \in P + B.$$ 

**Theorem 3.4.** Let $N$ be a $P$-regular distributive nearring and $B_1$ and $B_2$ bi-ideals of $N$. If $b \in B_1 \cap B_2$ and $x \in N$, then the element $b$ can be represented as

$$b = p + b_1x_1b_2 \text{ and } b_1x_1b_2xP \subseteq P$$

for some $p \in P, x_1 \in N, b_1 \in B_1$ and $b_2 \in B_2$.

**Proof.** Let $b \in B_1 \cap B_2$. Since $N$ is a $P$-regular nearring, there exists $x_1 \in N$ such that $bx_1b - b \in P$. By Lemma 2.2, we have $B_1 \cap B_2$ is a bi-ideal of $N$. Since $b \in B_1 \cap B_2$, we have $b \in B_1$ and $b \in B_2$. By Theorem 8.3 we have $b = p_1 + b_1$ for some $p_1 \in P$ and $b_1 \in B_1$, and $b = p_2 + b_2$ for some $p_2 \in P$ and $b_2 \in B_2$. Since $bx_1b - b \in P$, we have $bx_1b - b = p_3$ for some $p_3 \in P$. Thus $b = p_3 + bx_1b$. Hence

$$b = p_3 + bx_1b$$

$$= p_3 + (p_1 + b_1)x_1(p_2 + b_2)$$

$$= p_3 + p_1x_1p_2 + p_1x_1b_2 + b_1x_1p_2 + b_1x_1b_2.$$ 

Since $P$ is an ideal of $N$, we have $-p_3, p_1x_1p_2, p_1x_1b_2, b_1x_1p_2 \in P$. Then $-p_3 + p_1x_1p_2 + p_1x_1b_2 + b_1x_1p_2 = p_4$ for some $p_4 \in P$. Thus $b = p_4 + b_1x_1b_2$, so $b_1x_1b_2 = -p_4 + b$. Hence
\[ b_1x_1b_2xP = (-p_4 + b)xP \subseteq -p_4xP + bxP \subseteq P + P \subseteq P. \]

\[ \square \]

**Theorem 3.5.** Let \( N \) be a \( P \)-regular distributive nearring and \( \{ B_i \mid i \in \mathbb{Z} \text{ and } 1 \leq i \leq n \} \) a nonempty family of bi-ideals of \( N \). If \( b \in \bigcap_{i=1}^{n} B_i \) and \( x \in N \), then the element \( b \) can be represented as

\[ b = p + b_1x_1b_2x_2 \ldots b_{n-1}x_{n-1}b_n \text{ and } b_1x_1b_2x_2 \ldots b_{n-1}x_{n-1}b_nP \subseteq P \]

for some \( p \in P, x_1, x_2, \ldots, x_{n-1} \in N \) and \( b_i \in B_i \) for all \( 1 \leq i \leq n \).

**Proof.** If \( b \in B_1 \), then by Theorem 3.3 we have \( b = p + b_1 \) for some \( p \in P \) and \( b_1 \in B_1 \). Thus

\[ b_1xP = (-p + b)xP \subseteq -pxP + bxP \subseteq P + P \subseteq P. \]

Assume that the theorem is true for integer \( n - 1 \). Let \( b \in \bigcap_{i=1}^{n} B_i \). Since \( \bigcap_{i=1}^{n-1} B_i \subseteq \bigcap_{i=1}^{n} B_i \) and \( \bigcap_{i=1}^{n} B_i \subseteq B_n \), we have \( b \in \bigcap_{i=1}^{n-1} B_i \) and \( b \in B_n \). By assumption, we have

\[ b = p_1 + b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1} \]

and \( b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}xP \subseteq P \) for some \( p_1 \in P, x_1, x_2, \ldots, x_{n-2} \in N \) and \( b_i \in B_i \) for all \( 1 \leq i \leq n - 1 \). By Theorem 3.3 we have

\[ b = p_2 + b_n \]

for some \( p_2 \in P \) and \( b_n \in B_n \). Since \( N \) is a \( P \)-regular nearring, there exists \( x_{n-1} \in N \) such that \( bx_{n-1}b - b \in P \). Thus \( bx_{n-1}b = p_3 \) for some \( p_3 \in P \), so \( b = -p_3 + bx_{n-1}b \).

By (3.1) and (3.2), we have

\[ bx_{n-1}b = (p_1 + b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1})x_{n-1}(p_2 + b_n). \] (3.3)

By (3.3), we have

\[
\begin{align*}
 b &= -p_3 + bx_{n-1}b \\
 &= -p_3 + (p_1 + b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1})x_{n-1}(p_2 + b_n) \\
 &= -p_3 + p_1x_{n-1}p_2 + p_1x_{n-1}b_n + \\
 &\quad b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}p_2 + \\
 &\quad b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}b_n.
\end{align*}
\]

Put \( -p_3 + p_1x_{n-1}p_2 + p_1x_{n-1}b_n + b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}p_2 = p_4 \) for some \( p_4 \in P \). Thus

\[ b = p_4 + b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}b_n. \]
That is $b_1 x_1 b_2 x_2 \ldots b_{n-2} x_{n-2} b_{n-1} x_{n-1} b_n = -p_4 + b$. Hence

$$b_1 x_1 b_2 x_2 \ldots b_{n-2} x_{n-2} b_{n-1} x_{n-1} b_n x P = (-p_4 + b)x P \subseteq -p_4 x P + b x P \subseteq P + P \subseteq P.$$ \hfill \Box

**Theorem 3.6.** Let $N$ be a $P$-regular nearring and $B$ a bi-ideal of $N$. Then

$$P + B = P + BN B.$$

**Proof.** Since $B$ is a bi-ideal of $N$, we have $BN B \subseteq B$. Thus

$$P + BN B \subseteq P + B. \quad (3.4)$$

On the other hand, let $n \in P + B$. Then $n = p' + b'$ for some $p' \in P$ and $b' \in B$. Since $N$ is a $P$-regular nearring, there exists $x \in N$ such that $b' x b' - b' \in P$. Thus $b' x b' - b' = p''$ for some $p'' \in P$, so $b' = -p'' + b' x b'$. Therefore

$$n = p' + b' = p' + (-p'' + b' x b') = (p' - p'') + b' x b' \in P + BN B.$$

Hence

$$P + B \subseteq P + BN B. \quad (3.5)$$

By (3.4) and (3.5), we have $P + B = P + BN B$. \hfill \Box

**Theorem 3.7.** Let $N$ be a $P$-regular nearring, and $B_1$ and $B_2$ bi-ideals of $N$. Then

$$P + (B_1 \cap B_2) \subseteq P + (B_1 NB_2 \cap B_2 NB_1).$$

**Proof.** Let $b \in P + (B_1 \cap B_2)$. Then $b = p + b'$ for some $p \in P$ and $b' \in B_1 \cap B_2$. Thus $b' \in B_1$ and $b' \in B_2$. Since $N$ is a $P$-regular nearring, there exists $x \in N$ such that $b' x b' - b' \in P$. Thus $b' x b' - b' = p'$ for some $p' \in P$, so $b' = -p' + b' x b'$. Hence

$$b = p + b' = p - p' + b' x b' = p'' + b' x b' \in P + (B_1 NB_2 \cap B_2 NB_1)$$

where $p'' = p - p'$. Therefore

$$P + (B_1 \cap B_2) \subseteq P + (B_1 NB_2 \cap B_2 NB_1). \quad (3.6)$$ \hfill \Box

**Theorem 3.8.** Let $N$ be a $P$-regular nearring, and $\{B_i \mid i \in \mathbb{Z} \text{ and } 1 \leq i \leq n\}$ a nonempty family of bi-ideals of $N$. Then

$$P + (\bigcap_{i=1}^{n} B_i) \subseteq P + (B_1 NB_2 \cap B_2 NB_n \cap \ldots \cap B_{n-1} NB_n \cap B_n NB_1 \cap B_n NB_2 \cap \ldots \cap B_n NB_{n-1}).$$
Proof. By Theorem 3.6, we have $P + B_1 = P + B_1 NB_1$. That is $P + B_1 \subseteq P + B_1 NB_1$. Assume that the theorem is true for integer $n - 1$. By Theorem 3.7, we have

$$P + (\bigcap_{i=1}^{n} B_i) = P + \left( \bigcap_{i=1}^{n-1} \left( B_i \cap B_n \right) \right) \subseteq P + (\bigcap_{i=1}^{n-1} B_i)NB_n \cap B_n N(\bigcap_{i=1}^{n-1} B_i) \subseteq P + (B_1 NB_n \cap B_n N(B_1 \cap B_2 \cap \ldots \cap B_{n-1})) \subseteq P + (B_1 NB_n \cap B_2 NB_n \cap \ldots \cap B_{n-1} NB_n \cap B_n NB_1 \cap B_n NB_2 \cap \ldots \cap B_n NB_{n-1})$$

\[ \square \]

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