Unitary Subharmonic Response and Floquet Majorana Modes

Oles Shtanko\(^1\),\(^2\) and Ramis Movassagh\(^3\)

\(^1\)Joint Quantum Institute, NIST/University of Maryland, College Park, MD 20742, USA
\(^2\)Joint Center for Quantum Information and Computer Science, NIST/University of Maryland, College Park, MD 20742, USA
\(^3\)IBM Research, MIT-IBM AI lab, Cambridge MA, 02142, USA

Detection and manipulation of excitations with non-Abelian statistics, such as Majorana fermions, are essential for creating topological quantum computers. To this end, we show the connection between the existence of such localized particles and the phenomenon of unitary subharmonic response (SR) in periodically driven systems. In particular, starting from highly non-equilibrium initial states, the unpaired Majorana modes exhibit spin oscillations with twice the driving period, are localized, and can have exponentially long lifetimes in clean systems. While the lifetime of SR is limited in translationally invariant systems, we show that disorder can be engineered to stabilize the subharmonic response of Majorana modes. A viable observation of this phenomenon can be achieved using modern multi-qubit hardware, such as superconducting circuits and cold atomic systems.

The recent experimental frontiers have succeeded in the creation and manipulation of systems consisting of many well-isolated controllable qubits [1, 2]. Such devices promise to have applications from quantum computing to simulating quantum many-body systems out of equilibrium [3–6]. In the context of periodically driven systems, the prominent examples of such non-equilibrium systems are ones exhibiting persistent oscillations with a period equal to multiple initial driving periods. This phenomenon was recently studied in the context of discrete time crystals [7–15] and reported in several experimental settings [16–19].

In this work, we study the oscillations similar to time crystals but localized only at the boundaries of symmetry-protected topological (SPT) phases [20–22]. In equilibrium, one-dimensional SPT phases are widely studied due to the emergence of topologically protected Majorana zero modes (MZM) at the boundaries [23]. This phenomenon is of interest for fundamental physics perspective and the potential for realization of robust quantum computing [24]. In a driven setting, SPT phases have even richer phenomenology and may exhibit, additionally to MZM, a pair of Majorana \(\pi\) modes (MPM) [25–28]. Here we study in detail how the emergence of MPM, in connection with MZM, leads to robust oscillations at the boundaries [29]. We also show that double-period boundary oscillations may exhibit a sufficiently long equilibration time resisting thermalization [9, 30–33] and can be reliably protected by a mechanism of many-body localization (MBL) [34–36]. We also propose the boundary double-period oscillations as an alternative probe of Floquet Majorana modes in quantum systems. In particular, the observation of local persistent two-period oscillations can be used to establish both the presence of Majorana modes, their physical location, and localization length.

We consider the Majorana modes oscillations in a broader context of subharmonic response (SR) defined in the following way. Consider a Floquet system defined by a time-dependent Hamiltonian \(H(t) = H(t + T)\), where \(T\) is a fixed period. The periodic field affects the time dependence of expectations for local observables operators \(O_\mu(t)\). We define SR as a phenomenon when one or several of these observables permanently oscillate with a period \(kT\) for integer \(k > 1\), i.e., \(\langle O_\mu(t) \rangle = \langle O_\mu(t + kT) \rangle\), where \(\langle \ldots \rangle\) is the expectation value in the initial state. At the same time, the SR oscillations must persist regardless of the choice of the initial state and in the presence of small but finite perturbations. Therefore, by definition, SR does not include fine-tuned systems, for example, synchronized uncoupled qubits and integrable systems, due to lack of robustness to factors such as disorder, qubit coupling, or initial conditions. Also, SR is more broadly defined than the discrete time crystal because it does not require long-range spatial correlations across the system.

Under what conditions does SR happen? To address this question, it is convenient to limit our consideration from continuous time \(t\) to the discrete stroboscopic time \(t_n = nT\). The discrete time dynamics is generated by the unitary Floquet operator \(U_F = T \exp(-i \int_0^T H(t) \, dt)\) describing time evolution between discrete times \(t_n\) and \(t_{n+1}\). Then, a sufficient condition for existence of SR for arbitrary local \(O_\mu\) is the existence of a set of local oscillating integrals of motion \(\tau_i\) such that \(\text{Tr}(O_\mu \tau_i) > 0\) and \(U_F \tau_i = e^{2\pi i/k_i} U_F\). The appearance of such integrals of motion in generic systems requires a presence of specific symmetries.

Floquet SPT phases represent an example where the presence of symmetries leads to the emergence of oscillating local integrals associated with Majorana modes at the boundaries. In particular, for free fermion systems the Floquet operator \(U_F\) satisfies the relation \(U_F \Gamma_\alpha = e^{i\alpha} \Gamma_\alpha U_F\), where \(\alpha = 0, \pi\), where operator \(\Gamma_0\) creates MZM, and operator \(\Gamma_\pi\) creates MPM. The Majorana operators satisfy \(\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij} \delta_{\alpha\beta}\), and localized on opposite boundaries of the 1D system. Also, the Floquet operator of the SPT system must commute with the par-
ity operator $\mathcal{P}$ ensuring $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry [27]. As a result, one may construct local integrals of motion using operators $\Gamma_0^\alpha$, $\mathcal{P}\Gamma_0^\alpha$, and $\Gamma_0^\alpha\Gamma_0^\beta$, all of which anticommutes with $U_F$. Following the considerations above, any local operator with a non-vanishing overlap with these integrals of motion exhibits double-period SR. Because the unpaired Majoranas are localized at the boundaries, the SR is limited to spins at the system boundaries for general settings, while the bulk double-period oscillations decay to zero in the infinite time limit (see Fig. 1). For ergodic systems that cannot be presented as non-interacting fermions, the boundary double-period oscillations exhibit a finite lifetime. Nevertheless, it is possible to show that in specific settings, the lifetime can be exponentially large with respect to the inverse strength of the correction. Moreover, a specially constructed disorder may stabilize the boundary oscillations, as we show below.

**Model.** We study a prototypical example of a driven one-dimensional topological system described by a time-dependent Hamiltonian

$$H(t) = A(t)\left(J \sum_{i=1}^{L-1} \sigma_i^x \sigma_{i+1}^x + \sum_{i=1}^{L} h_i^x \sigma_i^x\right) + B(t) h_z \sum_{i=1}^{L} \sigma_i^z,$$

where $\sigma_i^\alpha$ are $2 \times 2$ Pauli matrices, $J$ is a coupling constant, $h_i^x$ and $h_z$ are local fields, and $A(t) = A(t + T)$ and $B(t) = B(t + T)$ are periodic control parameters. The parity operator of this system is $\mathcal{P} = \prod_{i=1}^{L} \sigma_i^z$. Below we consider a two-pulse dynamics setting $A(t) = 1$, $B(t) = 0$ for $0 \leq t < \tau$, and $A(t) = 0$, $B(t) = 1$ for $t \geq \tau$.

To quantitatively characterize double-period SR for Majorana fermion, one needs a physical parameter that reflects the persistent local oscillations with twice the period. Let

$$C(O_\mu) = \frac{1}{2} \left| \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left( \langle O_\mu(t_{2n}) \rangle - \langle O_\mu(t_{2n+1}) \rangle \right) \right|$$

be the response function that quantifies the double-periodicity in analytical calculations; for numerical and experimental analysis, $C(O_\mu)$ can also be used for finite but large number of periods $N$ without taking the limit.

Let us show how to use a free fermion representation of the Hamiltonian in Eq.(1) and construct the Majorana fermion operators. First, consider the Jordan-Wigner transformation [37] $c_i = \frac{1}{2} P_{z,i} (\sigma_i^x - \text{i} \sigma_i^y)$, and $c_i^\dagger = \frac{1}{2} P_{z,i} (\sigma_i^x + \text{i} \sigma_i^y)$, where $P_{z,i} = \prod_{k=1}^{i-1} (\sigma_k^z)$ is a string operator, $c_i$ and $c_i^\dagger$ are spinless fermion creation and annihilation operators, $\{c_i, c_j\} = 0$, $\{c_i^\dagger, c_j\} = \delta_{ij}$. Under the assumption $h_i^{x,\tau} = n \pi/2$, $n \in \mathbb{Z}$, the system is described by stationary orthogonal single-fermion modes

$$\psi_k(t_{n+1}) = e^{-\text{i} \theta_k} \psi_k(t_n),$$

where $\psi_k = \sum u_{ki} c_i + v_{ki} c_i^\dagger$, $\theta_k \in [-\pi, \pi]$ are single mode quasienergies, $u_{ki}$ and $v_{ki}$ are complex-valued coefficients satisfying the normalization condition $\sum |u_{ki}|^2 + |v_{ki}|^2 = 1$ (see Supplementary Information, SI). The quasienergy spectrum of single-particle modes is shown in Fig. 2b for two homogeneous signature cases $h_i^x = 0$ (top) and $h_i^x = \pi/2$ (bottom). In both cases, the Majorana modes $\Gamma_0^\alpha$ can be seen as unique self-adjoint modes with quasienergies $\theta_k = 0$ for MZM and $\theta_k = \pm \pi$ for MPM confined to the boundaries, i.e., $|v_{ki}|, |v_{ki}|$ are evanescent in the bulk and essentially non-zero only at a boundary. The phase diagram in Fig. 2a shows the appearance of MZM and MPM depending on parameters of the Hamiltonian in Eq.(1).

After we introduced the Majorana modes, let us study the double-period oscillations for a local observable operator $O_\mu$. For generic initial state $|\Psi\rangle$ which has exponentially small overlap with any eigenstate of the Floquet operator $U_F$, the expression for SR parameters in Eq.(2)
is equal to

$$C(O_\mu) = \left| \sum_i (\Gamma_i^\dagger \Gamma_i O_\mu) + \langle \mathcal{P} \Gamma_i^\dagger \mathcal{P} \Gamma_i^\dagger O_\mu \rangle \right|,$$

where \( \langle \ldots \rangle = \langle \Psi | \ldots | \Psi \rangle \).

Due to localization of Majorana modes, the double period oscillations are from zero only for the spins near the boundary. As an example, consider the oscillations of expectation \( \langle \sigma_i^z \rangle \) for the system initialized randomly in \( z \)-basis. Because the Majorana operators are linear combinations of fermion operators, their overlap with \( \sigma_i^z \) is distinct from zero only for the very first and the very last qubit of the system, i.e., \( C(\sigma_i^z) = 0 \) for \( r \neq 1, L \). This property is illustrated in Fig. 1a showing the evolution of local \( x \)-polarizations for a system initiated in a random product state of \( x \)-polarized spins. As seen from Fig. 1a and c, the bulk oscillations quickly decay as a result of a partial equilibration process, while the boundary oscillations persist without any decay. Remarkably, for spin product states \( |\Psi\rangle \) the particular values of the parameter \( C(\sigma_i^z) \) is insensitive to initial state of all other spins except the boundary ones.

Let us consider another example and study the parameter \( C(\sigma_i^z) \) for the system initialized as a random state in \( z \)-basis. The contribution to this quantity is given by the third term in Eq.(4) and thus tied to the presence of both MZM and MPM modes. The spatial distribution of this type of oscillations is

$$C(\sigma_i^z) = \left| \sum_i (\Gamma_i^\dagger \Gamma_i^\dagger O_\mu) \right| \sim e^{-r/\xi_0}$$

where \( \xi_0 \) is characteristic confinement length of \( \alpha \) Majorana fermion mode. As seen from this example, this type of oscillation is also confined to the boundary and suppressed by the double factor corresponding to the decay of both modes. Measuring the profile in Eq.(5), one can estimate the localization length of the overlap of MZM and MPM. Analysis for the more general setting is shown in Fig. 1a and c for the range of parameters.

The picture described above can be different if the...
state $|\Psi\rangle$ has non-vanishing overlap with the eigenstates of Floquet operators. The examples of these settings include fine-tuned parameter $h_z(T - \tau) = \pi/2$ or using a homogeneous initial state $|\Psi\rangle$. For the latter, the oscillations are exponentially tied to the boundary, but localization length may differ. A brief analysis of these non-generic cases can be found in SI.

**Effects of interactions.** Let us discuss the effect of weak interactions added to the free fermion Hamiltonian. Strictly speaking, exact strong Majorana modes no longer exist in the presence of generic interactions [33]. Therefore, the local SR can only be observed within a certain time limited by the timescale of equilibration $\tau$, in Floquet systems [9, 30, 31]. To ensure the observation of the phenomenon, we describe below the strategies to make the equilibration time sufficiently long or even infinite in isolated systems.

The first strategy assumes setting the system such that the bandwidth of the single-particle quasienergies is as small as possible. In the latter case, the interaction-induced transitions between single-particle levels as the main mechanism of energy absorption are suppressed by the mismatch between the driving frequency and transition energies. The following theorem formalizes this more rigorously:

**Theorem.** Let $\Gamma^\alpha_1$ be a pair of unitary operators, $\mathcal{P}$ is the parity operator, $\mathcal{P}^2 = 1$, and $U_F$ the Floquet operator for the Hamiltonian $H(t) = H(t + T)$ such that

1. $(\Gamma^\alpha_1)^2 = I$ and $U_F \Gamma^\alpha_1 = e^{i\alpha} \Gamma^\alpha_1 U_F$,
2. $\| [O \Gamma^\alpha_1, \Gamma^\alpha_2] \| < 2^{-\mu L}$ for any local operator $O$,
3. There exists $N \in \mathbb{Z}$ such that $U^N_F = \mathcal{P} \Gamma^\alpha_1 \Gamma^\alpha_2$,

where $L \to \infty$ is system size. Consider the dynamics generated by the new Hamiltonian $H'(t) = H(t) + V(t)$ such that $V(t) = V(t + T)$ is a sum of $S$-local terms, and let $\eta = S \int_0^T dt |V(t)| < 1$ be a small parameter. If the correction preserves parity $[V(t), \mathcal{P}] = 0$, then there exists a unitary transformation $U$ such that the operators $\hat{\Gamma}^\alpha_1 = U \Gamma^\alpha_1 \Gamma^\alpha_2 U$ satisfy

$$\| \hat{\Gamma}^\alpha_1 (t_n) - e^{i\alpha} \hat{\Gamma}^\alpha_1 \| = O(2^{-c/\eta} n),$$

otherwise, if $[V(t), \mathcal{P}] \neq 0$, then

$$\| \hat{\Gamma}^\alpha_1 (t_n) - e^{i\alpha} \hat{\Gamma}^\alpha_1 \| = O(\eta n),$$

where $c = [S(2N + 3)]^{-1}$ is a constant.

Let us briefly analyze the conditions of the theorem. The first condition just establishes the nature of Majorana operators $\Gamma_\alpha$, as we also described earlier. The second condition ensures that Majorana fermion are localized and spatially separated by the distance $L$. The third condition defines the class of Hamiltonians for which prethermalization of Majorana fermion occurs. As we show below, this condition is naturally satisfied for the class of Hamiltonians studied in this work.

The main result of the theorem is a rigorous proof of existence of prethermal Majorana fermion modes $\check{\Gamma}^\alpha_i$ as approximate integrals of motion for exponentially long time $\tau_i \sim 2^{c/\eta} n T$, and apply for both MZM and MPM. To illustrate this general result, let us consider how it applies to the Hamiltonian in Eq.(1). First, let us start from a static limit $h_x = h_z = 0$ and assume $J\tau = \pi/2$. In this case $U_F = \sigma^z \Gamma^\alpha_{12}$. According to the theorem, the system is characterized by the presence of stable MZM $\check{\Gamma}^\alpha_i$. This result is in full agreement with the previous works studying stability of (Floquet) Majorana fermion modes [28, 33]. In the driven setting, the result of the theorem provides the evidence of stability of double-period oscillations. For example, consider $h_z(T - \tau) = \pi/2, J\tau = \pi/(4m + 2)$ and $h_x = 0$. In this case $U_F^{2m+1} = \sigma^z \sigma^x \Gamma^\alpha_{12}^z$. Thus, the system exhibits existence of approximate integrals of motion $\check{\Gamma}^\alpha_i$. Although the values of $J\tau$ in this case are fine-tuned, the result may also apply to generic $J$ if we assume that deviations from the fine-tuned case can be incorporated into $V(t)$. As a result, all the region near values $h_z(T - \tau) = \pi/2$ should exhibit stable oscillations.

Prethermalization provides a way to make oscillations long-living but does not extend its lifetime to infinity. At the same time, the thermalization can be prevented completely by introducing strong local disorder into the system [10, 14, 38]. Unfortunately, adding strong generic disorder, even if it keeps the system in a topological MBL phase, “dilutes” the effect of local SR in the presence of bulk oscillations associated with localized states. One can find a solution by applying a special discrete disorder. Let us set the local $x$-fields by $h^+ = k_i \pi/2$, where $k_i$ are randomly sampled odd integers, $k_i \in [1, A]$, and $A \geq 3$. This integer-valued disorder has no effect on the initial system without corrections because for any odd $k_i$ the single particle unitary reduces as $\exp(-i\pi k_i \sigma^x_i/2) = \pm \exp(-i\pi \sigma^x_i/2)$. For simplicity, let us choose a particular simple model for the correction, $H' = H(t) + \lambda \sum_k \sigma^z_i \sigma^z_{i+1}$, where $\lambda$ is a small coupling constant, $\Delta T \ll 1$. In the absence of the discrete disorder as above, such a term would turn the state of the system into an ergodic phase [34]. Let us illustrate the effect of disorder in the simultaneous limit $J\tau = h_z(T - \tau) = \pi/2$, and $\tau/T \to 0$. Neglecting boundary effects, the double period evolution $U_F^{(2)} = \exp(-iH_-)\exp(-iH_+)$, where

$$H_\pm = \lambda T \sum_i \sigma^z_i \sigma^z_{i+1} \pm \frac{\pi}{2} \sum_i k_i \sigma^z_i + O(\tau/T)$$

This double-period Floquet operator, as the previous one in Eq. (1), can be studied using Jordan-Wigner transformation upon a preliminary transformation $\sigma^x \mapsto \sigma^z, \sigma^y \mapsto -\sigma^x$. After the mapping to the free fermion modes as in Eq.(3), we study the inverse participation ratio $I_k = \sum \{|u_k|^4 + |v_k|^4|^4$ for the single fermion modes $\psi_k$ of the Floquet operator in Eq.(8). For large system sizes, the values of $I_k$ remain finite for finite $\lambda > 0$ pointing to strong Anderson localization (see Fig. 2d
for size $L = 500$). Further, deviation of the parameters $J$ and $h_z$ from the fine-tuned values induces interaction between fermion modes and converts the Anderson localized model into MBL phase. To illustrate the stability of this phase, we study the level spacing parameter $r = \mathbb{E}[\min(d\Theta_k, d\Theta_{k+1})/\max(d\Theta_k, d\Theta_{k+1})]$, where $d\Theta_k = \Theta_{k+1} - \Theta_k$ as function of $\lambda$ at a non-integrable point, where $\Theta_k$ are arguments of eigenvalues of Floquet operator (many-body quasienegries) and the expectation is taken respect to discrete disorder realizations. Numerical simulations for increasing system sizes point on the existence of regions where $r$ is close to expected localized values $r \approx 0.386$ (see Fig. 2e). The MBL systems can be considered in the context of prethermalization as the system with thermalization time $\tau_r \to \infty$ preserving the SR oscillations indefinitely long in ideally isolated systems.

Finally, we address the problem of the presence of the gap protecting the Majorana modes $\tilde{\Gamma}_0$ from mixing with bulk degrees of freedom as well as suppressing quasiparticle excitations induced by the environment. For weak interactions $\lambda$ the system can be understood in terms of quasiparticle modes $\tilde{\psi}_k \approx \sum_{k'} \gamma^\lambda_{kk'} \psi_{k'}$ for some unitary $\gamma^\lambda$ depending on $\lambda$. A qualitative random matrix theory analysis [39] shows that the transition happens for finite $\lambda \sim \sqrt{G\Delta}$, where $G$ is the quasieenergy bandwidth, and $\Delta$ is the gap of non-interacting system (see SI).

**Discussions.** We studied the effect of local unitary subharmonic response (SR) in isolated periodically driven systems. We relate this phenomenon to the existence of unpaired MPM and MZM at the boundaries of 1D topological systems. We have shown the long-living nature of the SR oscillations in several settings and developed a way to protect it using discrete disorder.

The local SR effect can find applications in topological quantum computing. In particular, it was recently shown that MZM and MPM could be used both for encoding the quantum information as well as for braiding in 1D systems [40]. In this case, SR provides a way to probe the MPM and MZM as well as establish its localized spatial positions. At the same time, the Hamiltonian in Eq.(1) and its variations have various implementations in the trapped ion chains [1-3] and arrays of superconducting qubits such as an array of charge qubits coupled by Josephson junctions [41–43].

Future research in this area could be directed towards the generalization of SR to the topological systems with parafermions [44], which potentially show the $k$-periodic oscillations for $k > 2$. Also, we emphasize here the role of the discrete disorder as a tool for creating Floquet MBL phases without diluting boundary effects. Another important future aspect of the future studies is the robustness of the SR oscillations against the influence of generic environment.

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Supplementary Information for
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Derivation of SR parameter in Eq.(4)

In this section we derive the connection between SR oscillation of local observables and, in particular, the unpaired Majorana modes. First, let us consider the spectral decomposition for the Floquet operator

\[ U_F = \sum_{\nu} e^{-i\Theta_{\nu}} |\Phi_{\nu}\rangle \langle \Phi_{\nu}|, \]  

(S.1)

where \(\Theta_{\nu} \in [-\pi, \pi]\) are the many-body quasienergies of the system, and \(|\Phi_{\nu}\rangle\) are corresponding eigenvectors.

Using the orthogonality of the operators \(\Gamma_i^+\) and \(\Gamma_i^0\), we consider the decomposition

\[ O_{\mu} = \sum_i A_i^\mu \Gamma_i^+ + \sum_i A_i^\mu \Gamma_i^0 + \sum_i C_i^\mu \Gamma_i^+ \Gamma_i^0 + \sum_i B_i^\mu \Gamma_i^0 + \sum_i B_i^\mu \Gamma_i^0 + \hat{O}_\mu, \]  

(S.2)

where \(\hat{O}_\mu\) represent the rest of the basis decomposition. Then, we can express the local observable operator \(O_{\mu}\) at discrete times \(t_n\) as follows

\[ U_F^n O_{\mu} U_F^{-n} = (-1)^n \sum_i A_i^\mu \Gamma_i^+ + (-1)^n \sum_i A_i^\mu \Gamma_i^0 + \sum_i C_i^\mu \Gamma_i^+ \Gamma_i^0 + \sum_i B_i^\mu \Gamma_i^0 + \sum_i B_i^\mu \Gamma_i^0 + \hat{O}_\mu, \]  

(S.3)

Let us focus on the last term and show that its even-times averaged expectation value is

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} e^{-2\pi n (\Theta_{\nu} - \Theta_{\omega})} \langle \Psi | \Phi_{\omega} \rangle \langle \Phi_{\omega} | \hat{O}_{\mu} | \Phi_{\nu} \rangle \langle \Phi_{\nu} | \Psi \rangle = \sum_{\nu, \omega} \delta\Theta_{\nu} - \Theta_{\omega} \langle \Psi | \Phi_{\omega} \rangle \langle \Phi_{\omega} | \hat{O}_{\mu} | \Phi_{\nu} \rangle \langle \Phi_{\nu} | \Psi \rangle + \sum_{\nu, \omega} \delta\Theta_{\nu} - \Theta_{\omega} \langle \Psi | \Phi_{\omega} \rangle \langle \Phi_{\omega} | \hat{O}_{\mu} | \Phi_{\nu} \rangle \langle \Phi_{\nu} | \Psi \rangle, \]  

(S.4)

where we used the identity

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{-i\eta x} = \delta_{x,0} + (-1)^n \delta_{x,\pi}, \]  

(S.5)

and \(\delta_{a,b}\) is a Kronecker delta.

One may compare the expression in Eq.(S.4) with the odd-times average expectation value which differs by the sign of the second term,

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} e^{-i(2n+1)(\Theta_{\nu} - \Theta_{\omega})} \langle \Psi | \Phi_{\omega} \rangle \langle \Phi_{\omega} | \hat{O}_{\mu} | \Phi_{\nu} \rangle \langle \Phi_{\nu} | \Psi \rangle = \sum_{\nu, \omega} \delta\Theta_{\nu} - \Theta_{\omega} \langle \Psi | \Phi_{\omega} \rangle \langle \Phi_{\omega} | \hat{O}_{\mu} | \Phi_{\nu} \rangle \langle \Phi_{\nu} | \Psi \rangle - \sum_{\nu, \omega} \delta\Theta_{\nu} - \Theta_{\omega} \langle \Psi | \Phi_{\omega} \rangle \langle \Phi_{\omega} | \hat{O}_{\mu} | \Phi_{\nu} \rangle \langle \Phi_{\nu} | \Psi \rangle \]  

(S.6)

Using the orthogonality condition, one may express \(A_i^\mu = \text{Tr}(\Gamma_i^\mu O_{\mu})\) and \(B_i^\mu = \text{Tr}(\Gamma_i^0 \Gamma_i^\mu O_{\mu})\). Combining this result with Eq.(S.4) and Eq.(S.6), we arrive at the expression for SR that we used in the main text

\[ C_{\mu} = \frac{1}{N} \sum_i \langle \Gamma_i^\mu \rangle \text{Tr}(\Gamma_i^\mu O_{\mu}) + \langle \mathcal{P} \Gamma_i^\mu \rangle \text{Tr}(\mathcal{P} \Gamma_i^\mu O_{\mu}) + \sum_{i,j} \langle \Gamma_i^\mu \Gamma_j^0 \rangle \text{Tr}(\Gamma_i^\mu \Gamma_j^0 O_{\mu}) + \sum_{\nu, \omega} \delta\Theta_{\nu} - \Theta_{\omega} \langle \Psi | \Phi_{\omega} \rangle \langle \Phi_{\omega} | \hat{O}_{\mu} | \Phi_{\nu} \rangle \langle \Phi_{\nu} | \Psi \rangle \]  

(S.7)

where \(\langle \ldots \rangle = \langle \Psi | \ldots | \Psi \rangle\).

We now study the role of the last term in Eq.(S.7). First, we focus on the case where this term is non-negligible for all spins, for example \(h_z(T - \tau) = \pi/2\) and \(|\Psi⟩ = 2^{-L/2}|+\rangle\), where we use a notation \(|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\). Then, for any \(J\) the Floquet Hamiltonian has a pair of eigenstates \(|\Phi_1\rangle = 2^{-L/2}(|+\rangle + |−\rangle)\) with quasienergy \(\Theta_1 = \pi\) and \(|\Phi_2\rangle = 2^{-L/2}(|+\rangle - |−\rangle)\) with quasienergy \(\Theta_2 = \pi\). Simultaneously, \(|\Phi_1\rangle\) and \(|\Phi_2\rangle\) are the only eigenstates with non-zero overlaps with \(|\Psi⟩\). Let us consider the observable \(\sigma_z^i\), \(i \neq 1, L\). The SR for these observables is given by the last term equal to 1. Therefore, all spins oscillate without a decay (see Fig. S1c).

The intermediate case is possible if the parameters are not fine-tuned but \(|\Psi⟩\) is a homogeneous states, e.g., \(|\Psi⟩ = 2^{-L/2}|+\rangle\) as above. In this case the oscillations decay into the bulk with a characteristic length much larger that Majorana fermion lengthscale \(\xi_a\) (see Fig. S1b).

Finally, if we assume that \(|\langle \Psi | \Phi_\nu \rangle|^2 \sim 2^{-L}\), the last term has exponentially vanishing contribution for in Eq.(S.7). Therefore, the oscillations in the bulk vanish (see Fig. S1a).
Proof of the main theorem

For each Hermitian operator $O(t)$ we consider a decomposition

$$O(t) = \sum_{\alpha=1}^{2L} \sum_{i=1}^{L} \frac{1}{S_\alpha} \xi_{i,\alpha}(t) P_\alpha$$

(S.8)

where $\xi_{i,\alpha} = s_{i,\alpha} \text{Tr}[O(t)P_\alpha]$ are real-valued coefficients, $P_\alpha$ are generalized Pauli matrices, $s_{i,\alpha} = 1$ if $i \in \text{supp}(P_\alpha)$ and $s_{i,\alpha} = 0$ otherwise, and $S_\alpha \equiv |\text{supp}(P_\alpha)|$ is the size of the support of operator $P_\alpha$. The normalization is such that

$$\sum_{\alpha} \frac{1}{S_\alpha^2} \left( \sum_i \xi_{i,\alpha}(t) \right)^2 = \frac{1}{2L} \text{Tr}[O^2(t)]$$

(S.9)

Then, we define a parametrized family of norms

$$||O||_\kappa = \sup_{\alpha} \sum_i |\xi_{i,\alpha}(t)| e^{\kappa S_\alpha},$$

(S.10)

where $\kappa > 0$ is a real parameter and $\bar{x}(t)$ denotes the time average of the function $x(t)$ over period $T$. We refer to operators $||O||_\kappa < \infty$ for some $\kappa > 0$ as quasi-local operators.

We employ the following notation

$$||O||_n \equiv ||O||_{\kappa_n}, \quad \kappa_n = \frac{\kappa_0}{1 + \log(n + 1)}$$

(S.11)

for some $\kappa_0 > 0$ and the positive integer $n$.

We also define a $2N$-dimensional group $\mathbb{G}_{2N,\Delta} = \{X^k\}$ of local unitary transformations generated by a unitary

$$X^{2N} = I, \quad |\text{supp}(XP_\alpha X^\dagger)| \leq S_\alpha + \Delta$$

(S.12)

Because $(PT_i^\alpha T_i^\alpha)^2 = I$, under the condition of the theorem $U_F \in \mathbb{G}_{2N,\Delta}$.

Consider the full Floquet operator

$$U_F' = T \exp\left(-i \int_0^T (H(t) + V(t))dt\right),$$

(S.13)

Let us prove the following theorem that connects $U_F'$ and $U_F$; it serves as a generalization of Theorem 1 in Ref. [9].

**Theorem S1.** Assume $U_F \in \mathbb{G}_{2N,\Delta}$ and $V(t)$ satisfies $\eta = ||V||_{\kappa_0} T / \kappa_0 \ll 1$ for some $\kappa_0 < \infty$. Then there exists a unitary operator $\mathcal{U}$ such that

$$\mathcal{U} U_F' \mathcal{U}^\dagger = U_F U_{\text{corr}}$$

(S.14)

where

$$U_{\text{corr}} = T \exp\left(-i \int_0^T (D + V(t))dt\right)$$

(S.15)

satisfying $[D, U_F] = 0$ and

$$||D||_{n_\alpha} \leq 2e^{2\kappa_0 \Delta N}, \quad ||V||_{n_\alpha} \leq O(2^{-n_\alpha})$$

(S.16)
where $n_*=O\left(\kappa_0/2\eta(N+3)\right)$.

**Theorem S2.** (Abain, De Roeck, Ho, Huveniers [32]) Consider the operator $O$ that has a finite support $S$ and unitary transformation in the form $U_{corr}$ in Eq.(15) such that $|D|_\kappa < \infty$ and $||V||_\kappa' < 1$ for some $\kappa$ and $\kappa'$. Then

$$||U_{corr}^\dagger O U_{corr} - e^{-iDT} O e^{iDT}|| \leq c_1 ||O|| ||V||_\kappa(T + c_2)$$

(S.17)

for some $c_1 > 0$ and $c_2 > 0$ independent of $T$.

The proof of Theorem S1 is provided below in this section and is a $\Delta > 0$ generalization of Theorem 1 from Ref. [9]. First let us note that

$$||(U_F)^n \tilde{\Gamma}_i^n (U_F^\dagger)^n - e^{i\alpha\tilde{\Gamma}_i^n}||$$

$$\leq \sum_{k=1}^n \left| e^{-i(k+1)\alpha} (U_F^\dagger)^k \Gamma_i (U_F)^k - e^{-i\alpha} (U_F^\dagger)^k \Gamma_i (U_F)^k \right||$$

$$= n\| (U_F^\dagger)^k \Gamma_i U_F - e^{i\alpha} \Gamma_i \|$$

(S.18)

Using Cauchy-Schwarz inequality, we obtain

$$||U_{corr}^\dagger \Gamma_i^{\alpha} U_{corr} - \Gamma_i^{\alpha}|| \leq ||U_{corr}^\dagger \Gamma_i^{\alpha} U_{corr} - e^{-iDT} \Gamma_{\alpha} e^{iDT}||$$

$$+ ||e^{-iDT} \Gamma_{\alpha} e^{iDT} - \Gamma_{\alpha}||$$

(S.19)

If $[V(t), P] = 0$, then also $[D, P] = 0$. Because $[D, U_F] = 0$, therefore $[D, U_F^\dagger] = 0$ and

$$[D, \Gamma_1^{\alpha}\Gamma_2^{\alpha}] = \Gamma_1^{\alpha}[D, \Gamma_2] + [D, \Gamma_1^{\alpha}]\Gamma_2 = 0.$$  

(S.20)

From this expression we find

$$[D, \Gamma_1^{\alpha}] = \Gamma_2^{\alpha}[D, \Gamma_2] = \Gamma_2^{\alpha}[D, \Gamma_2].$$  

(S.21)

Then according to the condition 2 of the theorem,

$$||[D, \Gamma_1^{\alpha}]|| \leq ||[D, \Gamma_1^{\alpha}]|| \sim e^{-\mu L} \rightarrow 0$$

(S.22)

where $\kappa = \max(\kappa_\alpha, \mu)$.

As a result, we get the expression

$$||(U_F)^n \tilde{\Gamma}_i^n (U_F^\dagger)^n - e^{i\alpha\tilde{\Gamma}_i^n}|| \leq O(2^{-n_+ n})$$

(S.23)

where $n_+$ is given by Theorem S1. If the operator $V(t)$ has finite support $S$, then $||V||_\kappa \leq e^{cS}$. In this case $||V||_\kappa/\kappa$ has minimum at $\kappa = S^{-1}$. This proves the first part of the main theorem in the main text.

If $[V(t), P] \neq 0$, the last term in Eq.(19) is dominant, therefore

$$||U_{corr}^\dagger \Gamma_i^{\alpha} U_{corr} - \Gamma_i^{\alpha}|| \leq O(||D||_{n_+})$$

(S.24)

Using Theorem S1, we can also bound this expression.

**Proof of Theorem S1.** Following steps from Theorem 1 in Ref. [9] we construct a sequence of operators $U_n$ such that

$$U_{n+1} = U_n^\dagger U_0 U_n, \quad U_n = \prod_{k=0}^{n-1} e^{iA_k},$$

(S.25)

where $U_0 \equiv U_F^\dagger$ and $A_k$ are Hermitian operators we define below. For each $U_n$ we consider a decomposition

$$U_n = U_F T \exp(-i \int_0^T H_n(t) dt)$$

(S.26)

for a non-unique choice of the time-dependent operator $H_n(t)$. Our goal is to show the existence of an optimal choice for the sequence $A_n$ and the operators $H_n(t) = D_n + V_n(t)$, such that

$$D_n = \langle \overline{H}_n \rangle U_F \equiv \frac{1}{2N^2} \sum_{k=0}^{2N-1} U_F^k \overline{H}_n U_F^k = 0$$

(S.27)

where overbar denotes the time average, and the norm of operator $V_n$ decreases exponentially with $n$ if $n \leq n_+$.

Assume that we found the sequence $H_n(t)$ for $k \leq n$. Let us show the procedure for $H_{n+1}(t)$. For this, we rewrite

$$U_{n+1} = e^{-iA_n} U_n e^{iA_n}$$

$$= U_F \left[ U_F^\dagger e^{-iA_n} U_F T \exp(-i \int_0^T H_n(t) dt) e^{iA_n} \right]$$

$$= T \exp(-i \int_0^T H_{n+1}(t) dt)$$

(S.28)

where $H_{n+1}(t)$ represents a (suboptimal) decomposition which easily follows from Eq.(S.28) by

$$H_{n+1}(t) = \begin{cases} 
\tau^{-1} A_n, & 0 < t \leq \tau \\
(T - 2\tau)^{-1} H_n(t'), & \tau < t \leq T - \tau \\
-\tau^{-1} U_F^\dagger A_n U_F, & T - \tau < t \leq T 
\end{cases}$$

(S.29)

where $0 < \tau < T/2$ is an arbitrary real parameter and $t' = T(t - \tau)/(T - 2\tau)$.

First, let us decompose the correction time-dependent potential into static and zero-average components,

$$V_n(t) = E_n + \delta V_n(t)$$

(S.30)

such that $\overline{\delta V_n} = 0$. Then the time-averaged value of the Hamiltonian is

$$\overline{H}_{n+1} = D_n + E_n + A_n - U_F^\dagger A_n U_F$$

(S.31)

The time-dependent part is bounded as

$$||\overline{\delta V'}_{n+1}||_{\kappa_n} = ||H_{n+1} - \overline{H}_{n+1}||_{\kappa_n}$$

$$\leq 2 ||A_n||_{\kappa_n} + ||E_n||_{\kappa_n} + ||\delta V||_{\kappa_n} + 4\tau ||D_n||_{\kappa_n}$$

(S.32)
Following Ref. [9], we choose
\[ A_n = \frac{1}{2N} \sum_{k=0}^{2N-1} \sum_{p=0}^{k} U_p^n E_n U_p^{-p} \] (S.33)

With this choice
\[ \overline{H}_{n+1} = D_n, \quad \|A_n\|_\infty \leq \frac{1}{2} \alpha(\kappa) \|E_n\|_\infty \] (S.34)

where \( \alpha(\kappa) = (2N+1)e^{2\kappa N^2} \).

As a result,
\[ \|\delta Y'_{n+1}\|_\kappa \leq (\alpha(\kappa) + 1)\|E_n\|_\kappa + \|\delta V_n\|_\kappa \] (S.35)

According to Theorem 1 in Ref. [32], there exists a unitary \( Y(t) = Y(t + T) \), with \( Y(0) = I \) such that
\[ H_{n+1}(t) = Y_n(t) H'_{n+1}(t) Y_n^\dagger(t) - i Y_n(t) \partial_t Y_n^\dagger(t) \] (S.36)

and, under the condition \( 3\|\delta V_n\|_n \leq \kappa_n - \kappa_{n+1} \), the transformed Hamiltonian satisfies
\[ \|H_{n+1} - \overline{H}_{n+1}\|_{n+1} \leq \epsilon_n/2 \] (S.37)

where
\[ \epsilon_n = Tm_n\|\delta Y'_{n+1}\|_n (\|\overline{H}_{n+1}\|_n + 2\|\delta Y'_{n+1}\|_n) \] (S.38)

and
\[ m_n = \frac{18}{\kappa_{n+1}(\kappa_n - \kappa_{n+1})} \] (S.39)

Using this result, we obtain the optimal Hamiltonian \( H_{n+1}(t) \) from suboptimal \( H'_{n+1}(t) \). The parameters of optimal Hamiltonian satisfy the following bounds
\[ \|D_{n+1} - D_n\|_{n+1} = \|(D_{n+1} + E_{n+1} - D_n)U_p\|_{n+1} \leq \beta(\kappa_{n+1})\epsilon_n/2 \] (S.40)

where \( \beta(\kappa) = e^{2N\kappa N^2} \). Also
\[ \|E_{n+1}\|_{n+1} \leq \|D_{n+1} + E_{n+1} - D_n\|_{n+1} \]
\[ + \|D_{n+1} - D_n\|_{n+1} \leq \gamma(\kappa_{n+1})\epsilon_n \] (S.41)

where \( \gamma(\kappa) = (1 + \beta(\kappa))/2 \).

Now, let us use the induction. Assume that for nth step the operators obey
\[ \|E_n\|_n \leq 2^{-n}\gamma(\kappa_n)\lambda \] (S.42)
\[ \|\delta V_n\|_n \leq 2^{-n}\lambda \]

as well as
\[ \|D_{n+1} - D_n\|_{n+1} \leq 2^{-n-1}\beta(\kappa_{n+1})\lambda \] (S.43)

where we denote \( \lambda = 2\|V\|_0 \).

First, we need to verify Eq. (S.42) for \( n = 0 \). Let us set
\[ U_0 = U_F' = U_F \exp \left(-i \int_0^T dt H_0(t)\right) \] (S.44)

where \( H_0(t) = U_t V(t)U_t^\dagger \) and
\[ U_t = T \exp \left(-i \int_0^t dt' H(t')\right) \] (S.45)

We derive \( \|D_0\|_0 = \|\overline{H}_0\|_{U_0} \leq \beta(\kappa_0)\lambda/2 \) as well as
\[ \|E_0\|_0 = \|\overline{H}_0 - D_0\|_0 \leq \gamma(\kappa_0)\lambda \]
\[ \|\delta V_0\|_0 = \|H_0 - \overline{H}_0\|_0 \leq \lambda \] (S.46)

Now, for \( n \geq 1 \) we substitute Eq.(S.42) into Eq.(S.35) and, in turn, using this expression in Eq.(S.38) leads to
\[ \epsilon_n \leq 2^{-n}\xi_n m_n \lambda T \|D_n\|_n \] (S.47)

where \( \xi_n \equiv (\alpha(\kappa_n) + 1)\gamma(\kappa_n) + 1 \).

Using Cauchy-Schwarz inequality, we can estimate that
\[ \|D_n\|_n = \left\|D_0 + \sum_{k=0}^{n-1} D_{k+1} - D_k\right\|_n \]
\[ \leq \|D_0\|_n + \sum_{k=0}^{n-1} \|D_{k+1} - D_k\|_n \]
\[ \leq \beta(\kappa_0)\lambda/2 + \sum_{k=0}^{n-1} \|D_{k+1} - D_k\|_{k+1} \]
\[ \leq \beta(\kappa_0)\lambda + O(2^{-n}\lambda) \] (S.48)

where we used that for \( m < n \) norms satisfy \( \|O\|_m \geq \|O\|_n \), as well as the bounds from Eq.(S.43). As the result we obtain
\[ \epsilon_n \leq 2^{-n}\lambda^2 T \xi_n m_n \beta(\kappa_0) + O(4^{-n}) \] (S.49)

Taking into account Eqs.(S.37), (S.40), and (S.41), the step \( n + 1 \) is satisfied if
\[ \xi_n m_n \beta(\kappa_0) \lambda T \leq \frac{1}{2} \] (S.50)

Assuming that \( \lambda T \ll 1 \), this expression is valid for \( n \leq n_* \), where
\[ n \leq n_* = O\left(\kappa_0^2/(2N + 3)\lambda T\right) \] (S.51)

Because \( \|\delta V_0\|_0 \leq \|\delta V_n\|_0 = \lambda \), the condition
\[ 3\|\delta V_n\|_n \leq \kappa_n - \kappa_{n+1} \] (see paragraph after Eq.(S.36)) is satisfied for the conditions of the theorem, \( \lambda T/\kappa_0 \ll 1 \).

Denoting \( U \equiv U_{n_*}, D \equiv D_n, \) and \( V(t) = E_{n_*} + \delta V_{n_*} \), we prove statement of the theorem.
Free fermion solution (Fig. 1 and Fig. 2a-c)

Let us consider the time-periodic Hamiltonian in Eq.(1) in the main text for discrete $x$-field values,
\[
H(t) = A(t) \left( \sum_i \sigma_i^x \sigma_{i+1}^x + \frac{\pi}{2\tau} \sum_i k_i \sigma_i^x \right) + B(t) h_z \sum_i \sigma_i^z
\]
where $k_i \in \mathbb{Z}$ are integer variables, $A(t) = 1, B(t) = 0$ for $0 \leq t < \tau$, and $A(t) = 0, B(t) = 1$ for $t \geq \tau$.

The Floquet Hamiltonian corresponding to the Hamiltonian $H(t)$ is
\[
U_F = U_Z U_X U_P,
\]
where the unitary operators are defined as follows
\[
U_Z = \exp \left( -iJ \tau \sum_i \sigma_i^z \right), \\
U_X = \exp \left( -i h_z (T - \tau) \sum_i \sigma_i^x \sigma_{i+1}^x \right), \\
U_P = \prod_i (\sigma_i^x)^{[k_i/2]},
\]
where $[x/2]$ is modulo operation acting on integer $x$, it returns 0 if $x$ is even and 1 if $x$ is odd.

First, the summands in the Hamiltonian are quadratic in the fermion operators
\[
\sum_i \sigma_i^z = \sum_i c_i^\dagger c_i - c_i c_i^\dagger, \\
\sum_i \sigma_i^x \sigma_{i+1}^x = \frac{1}{2} \sum_i c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + c_{i+1} c_i
\]

Also, the action of the unitary operator $U_P$ is
\[
U_P c_m U_P^\dagger = \prod_{i=1}^L (\sigma_i^x)^{[k_i/2]} \prod_{k=1}^m (-\sigma_k^z) \prod_{j=1}^L (\sigma_j^x)^{[k_j/2]} = (-1)^{R_m} c_m^\dagger
\]

where $R_m = \sum_{i}^{m-1} [k_i/2]$.

Let us introduce the vector $\psi = (c_1 \ldots c_L, c_1^\dagger \ldots c_L^\dagger)$. The resulting transformation can be written as
\[
U_F \psi_1 U_F^\dagger = \sum_i V_{ij} \psi_j
\]

where the $2L \times 2L$ unitary matrix $V$ is a single-fermion Floquet operator. For the problem above, it can be presented in the form of the product
\[
V = V_Z V_X V_P
\]

where
\[
V_Z = \exp \left( -i \theta_1 \left( \tau_z + i \tau_y \right) \sum_i |i\rangle \langle i+1| + \text{h.c.} \right), \\
V_X = \exp \left( -2i \theta_2 \tau_z \sum_i |i\rangle \langle i| \right), \\
P = \sum_i (-1)^{R_i} \tau_z^{[k_i/2]} |i\rangle \langle i|, \\
R_i = \sum_{j=1}^{i-1} [\xi_j/2],
\]

where $\tau_i$ are Pauli matrices associated to creation and annihilation operators (Nambu space). The $V$ has eigenvalues $\exp(\pm i \theta_k)$ and corresponding eigenvectors $\psi_k, \psi_k^\dagger = (u_{ki}, v_{ki}), (v_{ki}^\dagger, u_{ki}^\dagger)$ setting the free fermion representation discussed in the main text. The spectrum calculated using operators $V$ in Eq.(S.58) is shown in Fig. 2b in the main text.

The system dynamics can be characterized by one- and two-point correlator functions,
\[
\phi_i(t) = \langle \Psi_t | \psi_i | \Psi_t \rangle, \quad \rho_{ij}(t) = \langle \Psi_t | \psi_i^\dagger \psi_j | \Psi_t \rangle
\]

where we call $\phi(t)$ a vector of operator expectations and $\rho(t)$ a single-particle density matrix.

The evolution of the correlators in Eq. (S.60) is given by
\[
\phi(t_{n+1}) = V \phi(t_n), \quad \rho(t_{n+1}) = V \rho(t_n) V^\dagger
\]

To evaluate the evolution of $\phi(t_n)$ and $\rho(t_n)$ using the equations above, we need to know the initial conditions, $\phi(0)$ and $\rho(0)$. Below, we provide the initial conditions for several relevant spin configurations.

Assume initially all the qubits are polarized in $x$-direction, $|\Psi \rangle = \bigotimes_{i=1}^{L} |s_i \rangle_x$, where the coefficients $s_i = \pm 1$ represent a binary vector and $|k\rangle_\alpha$ are eigenvalues of the operator $\sigma^\alpha$ with corresponding eigenvectors $k = \pm 1$. Then, the initial values of the correlators are
\[
\phi(0) = |\phi_x\rangle = \left( \frac{s_1}{2}, 0, \ldots, \frac{s_1}{2}, 0 \right), \\
\rho(0) = \rho_x(s) \equiv \frac{1}{4} \begin{pmatrix}
\text{diag}_3(d, 2, d) & \text{diag}_3(-d, 0, d) \\
\text{diag}_3(d, 0, -d) & \text{diag}_3(-d, 2, -d)
\end{pmatrix}
\]

where $d_i = s_{i+1} - s_i$, $\text{diag}(x)$ is a diagonal matrix with elements $x_i$ on the diagonal, $\text{diag}_3(x, n, y)$ is a tridiagonal matrix with all diagonal elements equal to $n$, and $x_i$ and $y_i$ on lower and upper diagonals respectively.

Similar expression can be obtained for the product of $y$-spins, $|\Psi \rangle = \bigotimes_{i=1}^{L} |s_i \rangle_y$.
\[
\phi(0) = |\phi_y\rangle = \left( -\frac{i s_1}{2}, 0, \ldots, \frac{i s_1}{2}, 0 \right), \\
\rho(0) = \rho_y(s) \equiv \frac{1}{4} \begin{pmatrix}
\text{diag}_3(-d, 2, d) & \text{diag}_3(d, 0, -d) \\
\text{diag}_3(-d, 0, d) & \text{diag}_3(d, 2, -d)
\end{pmatrix}
\]
Finally the expression for system initially polarized in z basis $|\Psi\rangle = \bigotimes_{i=1}^{L} |s_{i}\rangle^{\otimes L}$, is

\[
\phi(0) = 0, \\
\rho_{z}(s,0) = \frac{1}{2} \begin{pmatrix} 1 + \text{diag}(s) & 0 \\ 0 & 1 - \text{diag}(s) \end{pmatrix}, \quad (S.64)
\]

Using these initial values and the operator in Eq. (S.58), it is possible to compute the values of $\phi(t_{n})$ and $\rho(t_{n})$ at any given time $t_{n}$. Then, these values can be used to find the SR parameters.

For example, we derive the value of SR parameter for $\alpha$-polarization of the first spin given initially it is $\beta$-polarized, $\alpha, \beta = x, y, z$

\[
C_{\alpha\beta,1} = \left| \sum_{i} \langle \Gamma_{i}^{\pi} \rangle \text{Tr} (\Gamma_{i}^{\pi} \sigma_{i}^{\pi}) \right| = \sum_{i=1,2} \langle \phi_{i} | \phi_{i}^{\pi} \rangle \langle \phi_{i} | \phi_{i}^{\pi} \rangle, \quad (S.65)
\]

where $|\phi_{i}^{\pi}\rangle$ are $\pi$ quasienergy eigensates of the single-fermion Floquet unitary in Eq. (S.57), and $|\phi_{i}\rangle$ are defined in Eqs. (S.63)-(S.64).

The expression for $z$-polarization is different. It can be found as

\[
C_{zz,1} = \sum_{i,j=1,2} \langle i | \phi_{i}^{x} \rangle \langle \phi_{j}^{x} | \rho_{z}(s) | \phi_{j}^{x} \rangle \langle \phi_{i}^{x} | 1 \rangle + \text{h.c.} \quad (S.66)
\]

while the expression for $C_{xx,1}$ and $C_{yy,1}$ vanish. The expressions from Eqs. (S.66) are plotted in Fig. 2c.

To obtain the expectation values for the rest of the qubits, one can use the Majorana basis $|\gamma_{i} = c_{i} + c_{i}^{\dagger}\rangle$, $\gamma_{i+1} = -i(c_{i} - c_{i}^{\dagger})$ and the two-point correlation function

\[
K_{ij}(t) = i\langle \Psi | \gamma_{i}(t) \gamma_{j}(t) | \Psi \rangle, \quad (S.67)
\]

The evolution of the matrix $K$ can be connected to the evolution of SPDM by

\[
K(t) = R \rho(t) R^{T}, \quad R = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \otimes I \quad (S.68)
\]

The matrix $K$ can be used to connect single-particle excitations with spin observables. Let $I$ and $J$ be two subsets of indices with increasing order, then one defines $A_{IJ}$ as the matrix whose elements are $A_{IJ}$ with $i \in I, j \in J$. Then the time-dependent expectation of $x$-polarizations writes

\[
\langle \sigma_{x}^{I} \rangle = \langle \Psi | \mathcal{P}_{I}^{x}(t) \gamma_{2i}(t) | \Psi \rangle = \text{Pf} K_{II}, \quad I = \{1, \ldots, 2i\} \quad (S.69)
\]

where $\mathcal{P}_{I}^{x} = \prod_{i=1}^{2i-1}(-\sigma_{x}^{k})$ is a string operator. Similarly, one calculates

\[
\langle \sigma_{y}^{I} \rangle = \langle \Psi | \mathcal{P}_{I}^{y}(t) \gamma_{2i+1}(t) | \Psi \rangle = \text{Pf} K_{II}, \quad (S.70)
\]

where $I = \{1, \ldots, 2i - 1, 2i + 1\}$, and

\[
\langle \sigma_{z}^{I} \rangle = i\langle \Psi | \gamma_{2i}(t) \gamma_{2i+1}(t) | \Psi \rangle = K_{2i,2i+1}(t). \quad (S.71)
\]

Using these time-dependent expression, we derive the dynamics shown in Fig. 1 panels a-c.

**Stability of the gap (p.4)**

The Floquet Hamiltonian in presence of weak interactions preserves the modes $\psi_{k} = \sum_{k} \gamma_{kk'} \psi_{k'} + O(\lambda^{2})$, where $U_{kk'}$ are parameters depending on $\lambda$ and the type of discrete disorder, if present. The modes operators $\psi_{k}$ in this linear approximation, in contrast to $\psi_{k}$ in Eq. (S.3), represent not real particles but quasiparticles with lifetime depending on the neglected $O(\lambda^{2})$ part. Then the corresponding Floquet operator is characterized by a single fermion unitary matrix $V'$ (see Eq. (S.57)), which obeys

\[
V' = UV, \quad (S.72)
\]

where $U$ is a interaction correction unitary operator and $V$ is a single fermion unitary corresponding to the non-interacting system.

The structure of the unitary $U$ is unknown, therefore we approximate its eigenvectors as Haar and restrict its eigenvalues $(mod\ 2)$ to be such that $|\log U| \leq \theta$, where $\theta$ is a maximum mixing angle. This makes $U$ and $V$ free independent and we can use an imaginary time version of the S-transform in free probability theory.

As an example, let us consider the normalized density of states as a function of quasienergy $\varepsilon \in [-\pi, \pi]$ for the original non-interacting system unitary operator $V$ to be equal to

\[
\rho_{V}(\varepsilon) = \begin{cases} (G - \Delta)^{-1}, & \Delta/2 \leq |\varepsilon| \leq G/2 \\ 0, & \text{otherwise} \end{cases} \quad (S.73)
\]

This expression is simplification band structure for one shown in Fig. 2c in the main text. Despite being not exact, it allows us understand qualitatively the effect of random unitary rotation in Eq. (S.72).

Let us also assume that the density of states for the unitary $U$ is

\[
\rho_{U}(\varepsilon) = \begin{cases} \theta^{-1}, & |\varepsilon| < \theta/2 \\ 0, & \text{otherwise} \end{cases} \quad (S.74)
\]

The parameter $\theta \to 0$ represents the case $U = I$, while $\theta = 2\pi$ corresponds to $U$ being a random unitary by Haar measure.

The density of states can be obtained from Herglotz transform:

\[
\rho(\varepsilon) = \frac{1}{2\pi} \lim_{\xi \to +0} \text{Re} h(e^{-i\xi - \xi}), \quad (S.75)
\]

This can be inverted to obtain the density of states:

\[
\rho(\varepsilon) = \frac{1}{2\pi} \lim_{\xi \to +0} \text{Re} \int e^{\xi z} h(e^{-i\xi - \xi}) \rho(\varepsilon) d\varepsilon \quad (S.76)
\]

In particular, the Herglotz transform for the product in Eq. (S.72) can be obtained by solving simultaneously the
**Floquet band structure.** a. The two-band spectral structure of the unitary operator \( V' \) in Eq. (S.72) given the density of states of \( V \) and \( U \) as in Eqs. (S.73)-(S.74). The gap closes at critical \( \theta_c \) which depends on \( G \) and \( \Delta \). b. The logarithmic scale plot of \( \theta_c \) along with linear approximation (dashed lines). The linear approximation for most of curves is given by \( \theta_c = c \sqrt{G\Delta} \) almost for all values of gap \( \Delta \) and the bandwidth \( T \), where \( c = \sqrt{5/2} \) for the case we study here.

The Hertglothz transformations \( h_V(z) \) and \( h_U(z) \) can be calculated analytically, which yields

\[
h_V(z) = -1 + \frac{2i}{G - \Delta} \log\left(\frac{z^2 - az + b}{z^2 - a^*z + b^*}\right), \quad (S.81)
\]

and

\[
h_U(z) = -1 + \frac{2i}{\theta} \log\left(\frac{e^{-i\theta/2} - z}{e^{i\theta/2} - z}\right), \quad (S.82)
\]

where \( a = e^{-i(G-\Delta)/2} + e^{i\Delta/2}, \ b = e^{-i(G-\Delta)/2}. \)

Functional inversion of the transform \( h_U(z) \) can be written in the compact form

\[
h_U^{-1}(w) = \frac{\sin[(w-1)\theta/4]}{\sin[(w+1)\theta/4]} \quad (S.83)
\]

Combining Eqs. (S.81) and (S.83) with Eq. (S.80), we obtain the numerical solution for the density of states as well as the gap. As seen from Fig. S2, for given bandwidth \( G \) and \( \Delta \) the bandgap closes at a particular maximum mixing angle scaling as \( \theta_c = c\sqrt{G\Delta} \), for some \( c \). Assuming that for a small coupling term \( \lambda \) the phase in Eq.(S.74) corresponds to \( \theta \sim \lambda \), we conclude that the critical disorder closing the gap is \( \lambda_c \sim \sqrt{GT} \).