Algebraic-geometrical n-orthogonal curvilinear coordinate systems and solutions to the associativity equations

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Abstract

Algebraic-geometrical n-orthogonal curvilinear coordinate systems in a flat space are constructed. They are expressed in terms of the Riemann theta function of auxiliary algebraic curves. The exact formulae for the potentials of algebraic geometrical Egoroff metrics and the partition functions of the corresponding topological field theories are obtained.

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1 Introduction

The problem of constructing of \( n \)-orthogonal curvilinear coordinate systems or flat diagonal metrics

\[
ds^2 = \sum_{i=1}^{n} H_i^2(u)(du^i)^2, \quad u = (u^1, \ldots, u^n),
\]

for more than a century starting from the work of Dupin and Binet published in 1810 was among the first rate problems of Differential Geometry. As a classification problem it was mainly solved at the turn of this century. A milestone in a history of this problem was a fundamental monograph by G. Darboux [1].

At the beginning of the 80-s it was found that this very classical and old problem has deep connections and applications to the modern theory of integrable quasi-linear hydrodynamic type systems in \((1+1)\)-dimensions [2], [3], [4]. This theory was proposed by B. Dubrovin and S. Novikov as the Hamiltonian theory for the averaging (Whitham) equations for periodic solutions of integrable soliton equations in \((1+1)\)-dimensions. Later it was noticed (5) that the classification of Egoroff metrics, i.e. flat diagonal metrics such that

\[
\partial_j H_i^2 = \partial_i H_j^2, \quad \partial_i = \frac{\partial}{\partial u^i}, \tag{1.2}
\]

solves the classification problem of the massive topological field theories. Note that (1.2) implies that there exists a function \( \Phi(u) \), called a potential, such that

\[
H_i^2(u) = \partial_i \Phi(u) \tag{1.3}
\]

It should be emphasized, that the "classical" results are mainly that of the classification nature. It was shown that locally a general solution of the Lamé equations (1.4), (1.5)

\[
\partial_k \beta_{ij} = \beta_{ik} \beta_{kj}, \quad i \neq j \neq k, \tag{1.4}
\]

\[
\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{m \neq i,j} \beta_{mi} \beta_{mj} = 0, \quad i \neq j, \tag{1.5}
\]

for, so-called, rotation coefficients

\[
\beta_{ij} = \frac{\partial_j H_i}{H_i}, \quad i \neq j, \tag{1.6}
\]

depends on \( n(n - 1)/2 \) arbitrary functions of two variables. Equations (1.4), (1.3) are equivalent to the vanishing conditions of all a’priory non-trivial components of the curvature tensor. (Equations (1.4) imply that \( R_{ij,ik} = 0 \) and (1.3) imply \( R_{ij,ij} = 0 \.)

If a solution for (1.4), (1.5) is known, then the Lamé coefficients \( H_i \) can be found from the linear equations (1.6) that are compatible due to (1.4). They depend on \( n \) functions of one variable that are the initial data

\[
f_i(u') = H_i(0, \ldots, 0, u', 0, \ldots, 0) \tag{1.7}
\]
After that flat coordinates $x^k(u)$ can be found from a set of the linear equations
\[ \partial^2_{ij} x^k = \Gamma^i_{ij} \partial_i x^k + \Gamma^j_{ji} \partial_j x^k, \quad (1.8) \]
\[ \partial^2_{ii} x^k = \sum_{j=1}^{n} \Gamma^j_{ii} \partial_j x^k, \quad (1.9) \]
where $\Gamma^k_{ij}$ are the Christoffel coefficients of the metric (1.1):
\[ \Gamma^i_{ik} = \frac{\partial_k H_i}{H_i}, \quad \Gamma^i_{ii} = -\frac{H_i \partial_i H_i}{H_j^2}, \quad i \neq j. \quad (1.10) \]

Note that the compatibility of (1.8) and (1.9) requires (1.4) and (1.5), as well.

Though this way one gets the complete description of n-th orthogonal systems, the list of known exact examples was relatively short. A number of new examples were obtained from the Whitham theory. In particular, in [1] it was shown that the moduli spaces of algebraic curves with jets of local coordinates at punctures generate a wide class of flat diagonal metrics.

Recently solutions of (1.4) and (1.5) have been constructed by V. Zakharov [7] with the help of the ”dressing procedure” within the framework of the inverse problem method. The equations (1.4) are equivalent to the compatibility conditions for the auxiliary linear system
\[ \partial_i \Psi_j = \beta_{ij} \Psi_i, \quad i \neq k. \quad (1.11) \]

Therefore, any known inverse method scheme may be relatively easily adopted for a construction of various classes of its exact solutions. That can be the dressing scheme or algebraic-geometrical one, as well. The next step is to find the way to satisfy the constraints (1.3). As it was shown in [8] the differential reduction proposed in [9] and solving this problem in the case of dressing scheme becomes very natural in terms of the, so-called, $\bar{\partial}$-problem.

The main goal of this paper is not merely to construct the finite-gap or algebraic-geometrical solutions to the Lamé equations (1.4), (1.5) but to propose a scheme that solves simultaneously the whole system (1.4)-(1.10), i.e. gives Lamé coefficients $H_i$ and flat coordinates $x^k(u)$, as well.

At first sight it seems that our approach is completely different from that proposed in [7] and [8]. We consider the basic multi-point Baker-Akhiezer functions $\psi(u,Q)$ which are defined by their analytical properties on auxiliary Riemann surface $\Gamma$, $Q \in \Gamma$, and directly prove (without any use of differential equations !) that under certain constraints on the corresponding set of algebraic-geometrical data the evaluations $x^k(u) = \psi(u,Q_k)$ of $\psi$ at a set of punctures on $\Gamma$ satisfy the equations
\[ \sum_{k,l} \eta_{kl} \partial_i x^k(u) \partial_j x^l(u) = 0, \quad i \neq j, \quad (1.12) \]
where $\eta_{kl}$ is a constant matrix. Therefore, $x^k(u)$ are flat coordinates for the diagonal metric (1.1) with the coefficients
\[ H_i^2(u) = \sum_{k,l} \eta_{kl} \partial_i x^k(u) \partial_l x^l(u) \quad (1.13) \]
It turns out that up to constant factors the Lamé coefficients $H_i(u)$ are equal to the leading terms of the expansion of the same function $\psi$ at the punctures $P_i$ on $\Gamma$ where $\psi$ has exponential type singularities. We would like to mention that our constraints on the algebraic geometrical data that lead to solutions of (1.12), (1.13) are the generalization of conditions proposed in (13) for the description of the potential two-dimensional Schrödinger operators (see 16, also).

In the third section we link our results to the approach of [7], [8] and show that $\psi$ is a generating function

$$\partial_i \psi(u, Q) = h_i(u) \Psi^0_i(u, Q), \quad H_i = \varepsilon_i h_i(u), \quad \varepsilon_i = \text{const},$$

(1.14)

for solutions of the system

$$\begin{align*}
\partial_i \Psi^0_j & = \beta_{ji} \Psi^0_i, \\
\partial_j \Psi^0_j & = \Psi^1_j - \sum_{m \neq j} \beta_{mj} \Psi^0_m
\end{align*}$$

(1.15)

Note that the compatibility conditions of this extended auxiliary linear system are equivalent to both the sets of the equations (1.4) and (1.3).

In the forth section of the paper we specify the algebraic geometrical data corresponding to Egoroff metrics and obtain the exact formula in terms of Riemann theta functions for the potentials $\Phi(u)$ of such metrics.

As it was mentioned above, the connection of the classification problem for Egoroff metrics and the classification problem of the topological field theories was found in [5]. The last problem for the theory with $n$ primary fields $\phi_1, \ldots, \phi_n$ may be formulated in terms of the associativity equations for the partition function $F(x_1, \ldots, x_n)$ of the deformed theories [9], [10]. These equations are the conditions that the commutative algebra with generators $\phi_k$ and the structure constants defined by the third derivatives of $F$:

$$c_{klm}(x) = \frac{\partial^3 F(x)}{\partial x^k \partial x^l \partial x^m},$$

(1.16)

$$\phi_k \phi_l = c^m_{kl}(x) \phi_m; \quad c^m_{kl} = c_{klm} \eta^{jm}; \quad \eta_{ki} \eta^{jm} = \delta^m_k,$$

(1.17)

is an associative algebra, i.e.

$$c^k_{ij}(x)c^m_{km}(x) = c^k_{jm}(x)c^l_{ik}(x)$$

(1.18)

In addition, it is required that there exist constants $r^m$ such that the constant metric $\eta$ in (1.17) is equal to

$$\eta_{kl} = r^m c_{klm}(x)$$

(1.19)

The conditions (1.18) are a set of over-determined non-linear equations for the function $F$. It turns out that for any solution to the system (1.18), (1.19) in case when the algebra (1.17) is semisimple there exists Egoroff metric such that the third derivatives of the partition function can be written in the form

$$c_{klm} = \sum_{i=1}^n H_i^2 \frac{\partial u^i}{\partial x^k} \frac{\partial u^i}{\partial x^l} \frac{\partial u^i}{\partial x^m},$$

(1.20)
where \( u^i \) and \( x^k \) are the corresponding n-th orthogonal curvilinear and flat coordinates. Moreover, it turns out that for any set of rotation coefficients \( \beta_{ij} = \beta_{ji} \) satisfying (1.4), (1.3) there exists \( n \)-parametric family of Egoroff metrics such that the functions defined by (1.20) are the third derivatives of a certain function \( F \). (Recall, that for the given rotation coefficients there are infinitely many corresponding metrics.)

In the last section for each algebraic geometrical Egoroff metric we define a function \( F \) and show that its third derivatives have the form (1.20) and satisfy (1.18), that are the truncated set of the associativity conditions. We specify also the set of algebraic-geometrical Egoroff metrics such that (1.19) is fulfilled, also, and obtain for the corresponding partition functions \( F \) an exact formula in terms of the theta functions.

2 Bilinear relations for the Baker-Akhiezer functions and flat diagonal metrics

To begin with let us present some necessary facts from the general algebraic-geometrical scheme proposed by the author \[1\], \[2\]. The core-stone of this scheme is a notion of the Baker-Akhiezer functions that are defined by their analytical properties on the auxiliary Riemann surfaces.

Let \( \Gamma \) be a smooth genus \( g \) algebraic curve with fixed local coordinates \( w_i(Q) \) in neighborhoods of \( n \) punctures \( P_i, \ i = 1, \ldots, n \), on \( \Gamma, w_i(P_i) = 0 \). Then for any set of \( l \) points \( R_\alpha, \ \alpha = 1, \ldots, l \), and for any set of \( g + l - 1 \) points \( \gamma_1, \ldots, \gamma_{g+l-1} \) in a general position there exists a unique function \( \psi(u, Q|D, R), \ u = (u_1, \ldots, u_n), Q \in \Gamma, \) such that:

1. \( \psi(u, Q|D, R) \) as a function of the variable \( Q \in \Gamma \) is meromorphic outside the punctures \( P_j \) and at most has simple poles at the points \( \gamma_s \) (if all of them are distinct);

2. in the neighborhood of the puncture \( P_j \) the function \( \psi \) has the form:

\[
\psi = e^{w_j w_j^{-1} \left( \sum_{s=0}^{\infty} \xi_s^j(u) w_j^s \right)}, \ w_j = w_j(Q); \tag{2.1}
\]

3. \( \psi \) satisfies the conditions

\[
\psi(u, R_\alpha) = 1 \tag{2.2}
\]

Below we shall often denote the Baker-Akhiezer function by \( \psi(u, Q) \) without explicit indication on the defining divisors \( D = \gamma_1 + \cdots + \gamma_{g+l-1} \) and \( R = R_1 + \cdots + R_l \).

The exact theta-functional formula for the Baker-Akhiezer functions in terms of the Riemann theta-functions was proposed (\[2\]) as a generalization of the formula found in \[3\] for the Bloch solutions of the finite-gap ordinary Schrödinger operators.

The Riemann theta-function corresponding to an algebraic genus \( g \) curve \( \Gamma \) is an entire function of \( g \) complex variables \( z = (z_1, \ldots, z_g) \) defined by the formula

\[
\theta(z_1, \ldots, z_g) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i (m, z) + \pi i (Bm, m)}, \tag{2.3}
\]
where the matrix $B = B_{ij}$ is a matrix of $b$-periods

$$B_{ij} = \oint_{b_i} \omega_j$$

(2.4)
of the normalized holomorphic differentials $\omega_j(P)$

$$\oint_{a_j} \omega_i = \delta_{ij}$$

(2.5)
on $\Gamma$. Here $a_i, b_i$ is a basis of cycles on $\Gamma$ with canonical matrix of intersections $a_i \cdot a_j = b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$.

The theta function has remarkable automorphy properties with respect to the lattice $B$ generated by the basic vectors $e_i \in C^g$ and by the vectors $B_j \in C^g$ with coordinates $B_{ij}$: for any $l \in \mathbb{Z}^g$ and $z \in C^g$

$$\theta(z + l) = \theta(z), \quad \theta(z + Bl) = \exp[-i\pi(Bl, l) - 2i\pi(l, z)]\theta(z)$$

(2.6)
The torus $J(\Gamma)$

$$J(\Gamma) = C^g/B$$

(2.7)
is called the Jacobian variety of the algebraic curve $\Gamma$.

The vector $A(P)$ with coordinates

$$A_k(Q) = \int_{Q_0}^Q \omega_k$$

(2.8)
defines the, the so-called, Abel map.

According to the Riemann-Roch theorem for any divisors $D = \gamma_1 + \cdots + \gamma_{g+l-1}$ and $R = R_1 + \cdots + R_l$ in the general position there exists a unique meromorphic function $r_\alpha(Q)$ such that the divisor of its poles coincides with $D$ and such that

$$r_\alpha(R_\beta) = \delta_{\alpha,\beta}$$

(2.9)
This function may be written as follows (see [14]):

$$r_\alpha(Q) = \frac{f_\alpha(Q)}{f_\alpha(R_\alpha)}, \quad f_\alpha(Q) = \theta(A(Q) + Z_\alpha) \prod_{\beta \neq \alpha} \theta(A(Q) + F_\beta) \prod_{m=1}^{g} \theta(A(Q) + S_m),$$

(2.10)
where

$$F_\beta = -K - A(R_\beta) - \sum_{s=1}^{g-1} A(\gamma_s),$$

(2.11)

$$S_m = -K - A(\gamma_{g-1} + m) - \sum_{s=1}^{g-1} A(\gamma_s),$$

(2.12)

$$Z_\alpha = Z_0 - A(R_\alpha), \quad Z_0 = -K - \sum_{s=1}^{g+l-1} A(\gamma_s) + \sum_{\alpha=1}^{l} A(R_\alpha).$$

(2.13)
and $K$ is a vector of the Riemann constants.

Let $d\Omega_j$ be the unique meromorphic differential holomorphic on $\Gamma$ outside the puncture $P_j$, which has the form

$$d\Omega_j = d(w_j^{-1} + O(w_j))$$

near the puncture and is normalized by the conditions

$$\oint_{a_k} d\Omega_j = 0$$

It defines a vector $V^{(j)}$ with coordinates

$$V_k^{(j)} = \frac{1}{2\pi i} \oint_{b_k} d\Omega_j$$

**Theorem 2.1** The Baker-Akhiezer function $\psi(u, Q|D, R)$ has the form:

$$\psi = \sum_{\alpha=1}^{l} r_\alpha(Q) \frac{\theta(A(Q) + \sum_{i=1}^{n}(u_i V^{(i)}) + Z_0)\theta(Z_0)}{\theta(A(Q) + Z_0)} \theta(\sum_{i=1}^{n}(u_i V^{(i)}) + Z_0) \exp\left(\sum_{i=1}^{n} u_i \int_{R_\alpha} d\Omega_i\right)$$

Admissible curves.

Now in order to get the algebraic-geometrical flat diagonal metrics and their flat coordinates we are going to specify the algebraic-geometrical data defining the Baker-Akhiezer functions. The corresponding algebraic curve $\Gamma$ should be a curve with a holomorphic involution

$$\sigma : \Gamma \to \Gamma,$$

with $2m \geq n$ fixed points $P_1, \ldots, P_n, Q_1, \ldots, Q_{2m-n}, m \leq n$. The local coordinates $w_j(Q)$ in the neighborhoods of $P_1, \ldots, P_n$ should be odd

$$w_j(Q) = -w_j(\sigma(Q)).$$

The factor curve $\Gamma_0 = \Gamma/\sigma$ is a smooth algebraic curve. The projection

$$\pi : \Gamma \to \Gamma_0 = \Gamma/\sigma$$

represents $\Gamma$ as a two-sheet covering of $\Gamma_0$ with $2m$ branching points $P_j, Q_s$. In this realization the involution $\sigma$ is a permutation of the sheets. For $Q \in \Gamma$ we denote the point $\sigma(Q)$ by $Q^\sigma$.

From the Riemann-Hurwitz formula it follows that

$$g = 2g_0 - 1 + m,$$

where $g_0$ is genus of $\Gamma_0$.

In other words to construct an admissible algebraic curve $\Gamma$ one has to start with an algebraic curve $\Gamma_0$, take a meromorphic function $E(P), P \in \Gamma_0$, with $2m$ simple zeros or
poles on it and take $\Gamma$ as a Riemann surface of the function $\sqrt{E(P)}$. If $w_j^0(P)$ are local coordinates on $\Gamma_0$ at the points $P_j$ then we choose $w_j = \sqrt{w_j^0(P)}$ as the local coordinates on $\Gamma$ at the branching points $P_j$.

**Admissible divisors.**

Let us fix on $\Gamma_0$ a set of $n - m$ punctures $\hat{Q}_1, \ldots, \hat{Q}_{n-m}$. A pair of the divisors $D$ and $R$ on $\Gamma$ is called admissible if there exists a meromorphic differential $d\Omega_0$ on $\Gamma_0$ such that:

a) $d\Omega_0(P), \ P \in \Gamma_0$ has $m + l$ simple poles at the points $Q_1, \ldots, Q_{2m-n}, \hat{Q}_1, \ldots, \hat{Q}_{n-m}$ and at the points $\hat{R}_\alpha = \pi(R_\alpha)$;

b) the differential $d\Omega_0$ is equal to zero at the projections $\hat{\gamma}_s$ of the points of the divisor $D$,

$$d\Omega_0(\hat{\gamma}_s) = 0, \ \hat{\gamma}_s = \pi(\gamma_s) \ (2.22)$$

The differential $d\Omega_0$ can be considered an even (with respect to involution $\sigma$) meromorphic differential on $\Gamma$ where it has $n + 2l$ simple poles at the branching points $Q_1, \ldots, Q_{2m-n}$ and at the preimages of its other poles on $\Gamma_0$. Let us denote the preimages of the points $\hat{Q}_k$ by $Q_{2m-n+1}, \ldots, Q_{2m}$:

$$\pi(Q_{2m-n+i}) = \pi(Q_{n-i+1}) = \hat{Q}_i, \ i = 1, \ldots, n - m. \ (2.23)$$

The involution $\sigma$ induces the involution $\sigma(k)$ of the indices of the punctures $Q_k$,

$$\sigma(Q_k) = Q_{\sigma(k)}): \ \sigma(k) = k, \ k = 1, \ldots, 2m - n; \ \sigma(k) = 2m - k + 1, \ k = 2m - n + 1, \ldots, n \ (2.24)$$

In terms of equivalence classes the admissible pairs of the divisors $D$ and $R$ can be described as that satisfying the condition:

$$D + D^\sigma - R - R^\sigma = K + \sum_{j=1}^{n} (Q_j - P_j). \ (2.25)$$

**Example. Hyperelliptic curves.**

The simplest example of the admissible curve is the hyperelliptic curve $\Gamma$ defined by the equation

$$\chi^2 = \prod_{j=1}^{2m-n}(E - Q_j) \prod_{k=1}^{n-m}(E - \hat{Q}_k) \prod_{i=1}^{m}(E - P_i), \ m \leq n \ (2.26)$$

Here $P_i, Q_j, \hat{Q}_k$ are complex numbers. A point $\gamma$ on $\Gamma$ is a complex number and a sign of the square root from the right hand side of $(2.26)$. The curve $\Gamma$ has genus $g = m - 1$. Any set of $m + l - 2$ points $\gamma_s, \ \gamma_s \neq \gamma_s'$ and any set of $l$ points are a admissible pair of the divisors. The corresponding differential $d\Omega_0$ is equal to

$$d\Omega_0 = \frac{\prod_{s=1}^{m+l-2}(E - \gamma_s)}{\prod_{j=1}^{2m-n}(E - Q_j) \prod_{k=1}^{n-m}(E - \hat{Q}_k) \prod_{\alpha=1}^{l}(E - R_\alpha)} dE \ (2.27)$$
As we shall see in the last section of the paper the flat diagonal metrics corresponding to the hyperelliptic curves are Egoroff metrics. Moreover, this set of data gives also the simplest examples of algebraic-geometrical solutions to the associativity equations.

**Important remark.** In the following main body of the paper we assume for the simplicity of the formulae, only, that the divisors $R$ and the punctures $Q_j$ are in the general position and do not intersect with each other. We return to the special case when they coincide at the end of the last section.

**Theorem 2.2** Let $\psi(u,Q|D,R)$ be the Baker-Akhiezer function corresponding to an admissible algebraic curve and an admissible pair of the divisors $D$ and $R$. Then the functions $x^j(u)$:

$$x^j(u) = \psi(u,Q_j), \quad j = 1, \ldots, n,$$

(2.28)

satisfy the equations

$$\sum_{k,l} \eta_{kl} \partial_i x^k \partial_j x^l = \varepsilon_i^2 h_i^2 \delta_{ij},$$

(2.29)

where

$$h_i = \zeta_i^0(u)$$

(2.30)

are the first coefficients of the expansions (2.24) of $\psi$ at the punctures $P_i$; the constants $\varepsilon_i^2$ are defined by the expansion of $d\Omega_0$ at the punctures $P_i$

$$d\Omega_0 = \frac{1}{2}(\varepsilon_i^2 + O(w_1^i))dw_i = \varepsilon_i^2 dw_i$$

(2.31)

and at last the constants $\eta_{kl}$ are equal to

$$\eta_{kl} = \eta_k \delta_{k,\sigma(l)}, \quad \eta_k = \text{res}_{Q_k} d\Omega_0,$$

(2.32)

where $\sigma(k)$ is the involution of indices defined in (2.24).

**Proof.** Let us consider the differential

$$d\Omega_{ij}^{(1)}(u,Q) = \partial_i \psi(u,Q) \partial_j \psi(u,\sigma(Q))d\Omega_0(\pi(Q))$$

(2.33)

From the definition of the admissible data, it follows that this differential for $i \neq j$ is a meromorphic differential with poles at the points $Q_1, \ldots, Q_n$, only. Indeed, the poles of the first two factors $\partial_i \psi(u,Q)$ and $\partial_j \psi(u,\sigma(Q))$ at the points $\gamma_s$ and $\sigma(\gamma_s)$ cancel with zeros of $d\Omega_0$. At the punctures $P_k$ the essential singularities of the same factors cancel each other. The product of these factors has simple poles at the points $P_i$ and $P_j$ that cancel with zeros of $d\Omega_0$ considered as a differential on $\Gamma$, (2.31). At last $d\Omega^{(1)}$ has no poles at the points $R_\alpha$ and $R_\sigma$ due to the condition (2.2). A sum of all the residues of a meromorphic differential on a compact Riemann surface is equal to zero. Therefore,

$$\sum_{k=1}^n \text{res}_{Q_k} d\Omega_{ij}^{(1)} = 0, \quad i \neq j$$

(2.34)
The left hand side of this equality coincides with the left hand side of (2.29).

In the case \( i = j \), the differential \( d\Omega_{ii}^{(1)} \) has an additional pole at the point \( P_i \) with the residue
\[
\text{res}_{P_i} d\Omega_{ii}^{(1)} = -\varepsilon_i^2 h_i^2
\]
That implies (2.29) for \( i = j \) and completes the proof of the theorem.

**Corollary 2.1** Let \( \{ \Gamma, P_i, Q_j, D, R \} \) be a set of the admissible data. Then the formula
\[
H_i(u) = \varepsilon_i \sum_{\alpha=1}^{l} r_\alpha(P_i) \frac{\theta(A(P_i) + \sum_{i=1}^{n}(u^i V^{(i)}) + Z_\alpha)\theta(Z_0)}{\theta(A(P_i) + Z_\alpha)\theta(\sum_{i=1}^{n}(u^i V^{(i)}) + Z_0)} \exp \left( \sum_{j=1}^{n} \omega_{ij}^\alpha u^j \right),
\]
where \( r_\alpha(Q) \) is the function defined by (2.10),
\[
\omega_{ij}^\alpha = \int_{R_\alpha}^{P_i} d\Omega_j, \quad i \neq j,
\]
\[
\omega_{ii}^\alpha = \lim_{Q \to P_i} \left( \int_{Q}^{P_i} d\Omega_i - w_i^{-1}(Q) \right),
\]
defines the coefficients of the flat diagonal metric. The flat coordinates (not orthonormal) corresponding to this metric are given by the formulae:
\[
x^k(u) = \sum_{\alpha=1}^{l} r_\alpha(Q_k) x^k_\alpha(u),
\]
\[
x^k_\alpha(u) = \frac{\theta(A(Q_k) + \sum_{i=1}^{n}(u^i V^{(i)}) + Z_\alpha)\theta(Z_0)}{\theta(A(Q_k) + Z_\alpha)\theta(\sum_{i=1}^{n}(u^i V^{(i)}) + Z_0)} \exp \left( \sum_{j=1}^{n} u^i \int_{R_\alpha}^{Q_k} d\Omega_j \right)
\]

**Important remark.** Note that the theta-function and the vectors \( V^{(j)} \) are defined by the curve \( \Gamma_0 \), the punctures \( \{ P_1, \ldots, P_n, Q_1, \ldots, Q_{2m-n} \} \) and the first jets of local coordinates at \( P_j \). A space of these parameters for given genus \( g_0 \) of \( \Gamma_0 \) is equal to \( 3g_0 - 3 + 2m + n \). The vectors \( Z_\alpha \) are defined in (2.13), where the vector \( Z_0 \) depends on the points \( Q_k, \ k = 1, \ldots, n - m \) and the divisors \( D, R \) also. It belongs to the affine space defined by the relation (2.25). We would like to stress that in the formula (2.36) the last set of parameters enters through \( \varepsilon_i \) and \( Z_0 \), only. Therefore, for \( (n - m) > g_0 \) formula (2.36) with an arbitrary vector \( Z_0 \) and with arbitrary constants \( \varepsilon_i \) defines the flat diagonal metric.

**Reality conditions.**

In a general case the above-constructed flat diagonal metrics \( H_i(u) \) and their flat coordinates are complex meromorphic functions of the variables \( u^i \). Let us specify the algebraic geometrical data such that the coefficients of the corresponding metrics are real functions of the real variables \( u^i \).

Let \( \Gamma_0 \) be a real algebraic curve, i.e. a curve with an antiholomorphic involution
\[
\tau_0 : \Gamma_0 \mapsto \Gamma_0
\]
and let the punctures \( \{ P_1, \ldots, P_n \} \) and \( \{ Q_1, \ldots, Q_{2m-n} \} \) be fixed points of \( \tau_0 \). Then there exists an antiholomorphic involution \( \tau \) on \( \Gamma \). We assume that the local coordinates \( w_j \) at the punctures \( P_j \) satisfy the condition
\[
w_j(\tau(Q)) = \overline{w_j(Q)}
\] (2.42)

Let us assume that the set of the points \( \hat{Q}_k \) and the divisors \( D, R \) are invariant with respect to \( \tau \), also:
\[
\tau(Q_j) = Q_{\kappa(j)}, \quad \tau(R_\alpha) = R_{\kappa_1(\alpha)}, \quad \tau(\gamma_s) = \gamma_{\kappa_2(s)},
\] (2.43)
where \( \kappa_i(.) \) are the corresponding permutations of indices. The conditions (2.43) imply that the differentials \( d\Omega_0 \) corresponding to the admissible pair \( D, R \) satisfies the condition
\[
\tau_0^* d\Omega_0 = \overline{d\Omega_0}
\] (2.44)

**Theorem 2.3** Let a set of the admissible data be real (i.e. satisfies the conditions (2.43)). Then the Baker-Akhiezer function \( \psi(u, Q|D, R) \) satisfies the relation
\[
\psi(u, Q|D, R) = \overline{\psi(u, \tau(Q)|D, R)}
\] (2.45)
and the formula (2.36) defines a real flat diagonal metric.

The signature of the corresponding metrics depends on the involutions \( \kappa(j) \) and \( \kappa_1(\alpha) \). For various choices of the initial data one may get the flat diagonal metrics in all the pseudo-euclidean spaces \( \mathbb{R}^{p,q} \). In general, these metrics are singular at some values of the variables \( u^i \). In order to get smooth metrics for all the values of \( u \) it is necessary to further specify the initial data. The way to do that is relatively standard in the finite-gap theory and we address this problem somewhere else.

### 3 Differential equations for the Baker-Akhiezer function

In this section we are going to clarify the meaning of our constraints in terms of differential equations for the Baker-Akhiezer functions. First of all, we discuss the equations for an arbitrary set of algebraic geometrical data.

The following statement is a simple generalization of the results [17] where in the case \( n = 2 \) it was shown that the corresponding Baker-Akhiezer function satisfies the two-dimensional Schrödinger type equation.

**Lemma 3.1** Let \( \psi(u, Q|D, R) \) be the Baker-Akhiezer function. Then it satisfies the equation:
\[
\partial_i \partial_j \psi = c_{ij}^i \partial_i \psi + c_{ij}^j \partial_j \psi, \quad i \neq j,
\] (3.1)
where
\[ c^i_{ij}(u) = \frac{\partial_j h_i}{h_i}, \quad \frac{\partial_i h_j}{h_j} \] (3.2)
and
\[ h_i(u) = \xi^i_0(u), \] (3.3)
are the first coefficients of the expansion (2.1).

The equations (3.1) have the form (1.8) that are the first set of the equations defining flat coordinates for the diagonal metric with the coefficients \( H_i(u) = \varepsilon_i h_i(u) \), where \( \varepsilon_i \) are constants. Now we are going to obtain the equations that in the case of admissible algebraic-geometrical data provide the second set (1.9) of the equations for the flat coordinates.

Let \( \{\Gamma, P_j, w_j, \gamma_s, R_\alpha\} \) be a set of the data that defines the Baker-Akhiezer functions \( \psi(u,Q|D,R) \). Let us fix a set of additional \( n \) points \( Q_1, \ldots, Q_n \). Then their exists a unique function \( \psi^1 = \psi^1(u,Q|D,R) \) such that:

1. \( \psi^1(u,Q) \), as a function of the variable \( Q \in \Gamma \), is meromorphic outside the punctures \( P_j \), at most has simple poles at the points \( \gamma_s \) and equals to zero at the punctures \( Q_1, \ldots, Q_n \):  
   \[ \psi^1(u,Q_k) = 0; \] (3.4)  
2. in the neighborhood of the puncture \( P_j \) the function \( \psi^1 \) has the form:  
   \[ \psi^1 = w_j^{-1} e^{w_j w_j^{-1}} \left( \sum_{s=0}^{\infty} \xi^s_{1,j}(u) w_j^s \right), \quad w_j = w_j(Q); \] (3.5)  
3. at the points \( R_\alpha \) the function \( \psi^1 \) equals 1,  
   \[ \psi^1(u,R_\alpha|D,R) = 1 \] (3.6)

**Lemma 3.2** Let \( \psi(u,Q|D,R) \) and \( \psi^1(u,Q|D,R) \) be the Baker-Akhiezer functions corresponding to an arbitrary set of the algebraic-geometric data. Then they satisfy the equations  
\[ \partial^2_i \psi - c^1_i \partial_i \psi^1 + \sum_{j=1}^{n} v_{ij} \partial_j \psi = 0, \] (3.7)
where
\[ c^1_i = \frac{h_i}{h^1_i}, \quad v_{ii} = \frac{\partial_i h^1_i}{h^1_i} - 2 \frac{\partial_i h_i}{h_i} + \frac{g^1_i}{h^1_i}, \quad v_{i j} = \frac{h_i}{h_j} \frac{\partial_j h^1_i}{h^1_i}, \quad i \neq j, \] (3.8)
and the functions
\[ h_i = \xi^i_0; \quad h^1_i = \xi^i_{0,0}; \quad g_i = \xi^i_1; \quad g^1_i = \xi^i_{1,1} \] (3.10)
are the first coefficients of the expansions (2.4) and (2.3).
Again the proof is standard. Consider the function that is defined by the left hand side of (3.4). The formulae (3.8) and (3.9) imply that this function satisfies all the defining conditions of the function \( \psi \) and is regular at all the punctures \( R_\alpha \). Therefore, it is equal to zero.

Now consider the case of the admissible algebraic-geometrical data. (In that case the set of the punctures in the definition of \( \psi^1 \) is the same set as in the definition of the admissible curves and divisors, i.e. \( Q_1, \ldots, Q_{2m-n} \) are the branching points and \( Q_{2m-n+1}, \ldots, Q_{2n} \) are the preimages of \( \hat{Q}_k \).)

**Theorem 3.1** Let \( \psi(u, Q|D, R) \) and \( \psi^1(u, Q|D, R) \) be the Baker-Akhiezer functions corresponding to an admissible set of algebraic-geometrical data. Then the equations (3.1) and the equations

\[
\partial^2_i \psi = c_i \partial_i \psi^1 + \sum_{j=1}^n \Gamma_{ii}^j \partial_j \psi
\]

(3.11)

are fulfilled. Here \( \Gamma_{ii}^j \) are the Cristoffel’s coefficients (1.10) of the metric \( H_i(u) = \varepsilon_i h_i(u) \).

**Proof.** Let us consider the differential

\[
d\Omega_i^{(2)} = \partial_i \psi^1(u, Q) \partial_j \psi(u, Q^\sigma) d\Omega_0(\pi(Q))
\]

(3.12)

This differential is holomorphic everywhere except for the points \( P_i \) and \( P_j \). The residues at these points are equal to,

\[
\text{res}_{P_j} d\Omega_i^{(2)} = \varepsilon^2_i h_i^1 \partial_j h_i, \quad \text{res}_{P_j} d\Omega_i^{(2)} = -\varepsilon^2_j h_j^1 \partial_i h_j.
\]

(3.13)

Therefore,

\[
\varepsilon^2_i h_i^1 \partial_j h_i = \varepsilon^2_j h_j^1 \partial_i h_j.
\]

(3.14)

The last formula implies that the coefficients \( v_{ij} \) (3.9) are equal to \( \Gamma_{ii}^j \) for \( i \neq j \).

The differential \( d\Omega_i^{(1)} \) has the only pole at the point \( P_i \). Therefore, its residue at this point is equal to zero,

\[
\text{res}_{P_i} d\Omega_i^{(1)} = h_i^1 (g_i + \partial_i h_i) - h_i (g_i^1 + \partial_i h_i^1) = 0.
\]

(3.15) implies that \( v_{ii} \) given by (3.8) are equal to \( \Gamma_{ii}^j \).

Note once again, that at the points \( Q_j \) (where \( \psi^1 \) is equal to zero) the equations (3.11) coincide with (1.9).

**Corollary 3.1** The functions \( \Psi^0_i(u, Q) \) and \( \Psi^1_i(u, Q) \)

\[
\Psi^0_i(u, Q) = \frac{1}{h_i(u)} \partial_i \psi(u, Q), \quad \Psi^1_i(u, Q) = \frac{1}{h_i^1(u)} \partial_i \psi^1(u, Q).
\]

(3.16)

satisfy the equations (1.13), where \( \beta_{ij}(u) \) are the rotation coefficients (1.6) of the metric \( H_i(u) \).
The proof of the corollary follows from the direct substitution of (3.16) into (3.1) and (3.11).

Consider the analytical properties of $\Psi_i^0(u, Q)$ and $\Psi_i^1(u, Q)$ as functions on the algebraic curve $\Gamma$. The defining analytical properties of the Baker-Akhiezer function imply that:

1*. $\Psi_i^N(u, Q)$, $N = 0, 1$ is meromorphic outside the punctures $P_j$ and at most has simple poles at the points $\gamma_1, \ldots, \gamma_{g+l-1}$;

2*. in the neighborhood of the puncture $P_j$ the function $\Psi_i^N$ has the form:

$$
\Psi_i^N = w_j^{-N-1}e^{u_j}w_j^{-1}(\delta_{ij} + \sum_{s=1}^{\infty} \epsilon_{s,N}^j(u)w_j^s), \ w_j = w_j(Q);
$$  \hspace{1cm} (3.17)

3*. the functions $\Psi_i^N$ are equal to zero at the punctures $R_\alpha$:

$$
\Psi_i^N(u, R_\alpha) = 0;
$$  \hspace{1cm} (3.18)

4*. the functions $\Psi_i^1$ are equal to zero at the punctures $Q_j$:

$$
\Psi_i^1(u, Q_j) = 0
$$  \hspace{1cm} (3.19)

**Lemma 3.3** Let $\Gamma$ be a smooth genus $g$ algebraic curve with $2n$ punctures $P_j$, $Q_j$ and with fixed local coordinates $w_j(Q)$ in the neighborhoods of the punctures $P_j$. Then for any set of $(g+l-1)$ point $\gamma_s$ in a general position there exist the unique functions $\Psi_i^0(u, Q)$ and $\Psi_i^1(u, Q)$ satisfying the above formulated conditions 1*-3*.

For a given admissible curve $\Gamma$ with punctures $P_i$, $Q_i$ and local coordinates $w_i$ the Baker-Akhiezer functions and therefore, the coefficients $H_i(u|D,R)$ of the corresponding diagonal flat metric depend on the admissible pair of the divisors $D$ and $R$. Two pairs of the divisors $D, R$ and $D', R'$ are called equivalent if there exists a meromorphic function $f(Q)$ on $\Gamma$ such the divisor $(f)_{\infty}$ of its pooles and the divisor $(f)_0$ of its zeros are equal to

$$(f)_{\infty} = D + R', \ (f)_0 = D' + R
$$  \hspace{1cm} (3.20)

In terms of the Abelian map $D, R$ are equivalent to $D', R'$ iff

$$
\sum_{s=1}^{g+l-1} A(\gamma_s) - \sum_{\alpha=1}^{l} A(R_\alpha) = \sum_{s=1}^{g+l'-1} A(\gamma_s) - \sum_{\alpha=1}^{l'} A(R'_\alpha)
$$  \hspace{1cm} (3.21)

Lemma 3.3 implies that the following statement is valid.

**Corollary 3.2** The rotation coefficients $\beta_{ij}(u|D, R)$ and $\beta_{ij}(u|D'R')$ corresponding to the equivalent pairs of the divisors satisfy the relation

$$
f(R_i)\beta_{ij}(u|D, R) = f(P_j)\beta_{ij}(u|D'R'),
$$  \hspace{1cm} (3.22)

where $f(Q)$ is the function such that (3.20) are fulfilled.
From the uniqueness of $\Psi^N_i$ and the definition of the equivalent pairs it follows that

$$f(P_i)\Psi^N_i(u, Q|D, R) = f(Q)\Psi^N_i(u, Q|D', R')$$  \hspace{1cm} (3.23)

and, therefore, (3.22) are valid.

Let us express the rotation coefficients and the Lamé coefficients in terms of the function $\Psi^1_i(u, Q|D, R)$, only.

**Theorem 3.2** The rotation coefficients $\beta_{ij}(u)$ of the algebraic-geometrical flat diagonal metric with the coefficients $H_i(u|D, R)$ corresponding to the Baker-Akhiezer function $\psi(u, Q|D, R)$ are equal to

$$\beta_{ij} = \beta_{ij}(u|D, R) = \zeta_{ji}^1(u|D, R),$$  \hspace{1cm} (3.24)

where $\zeta_{ji}^1$ are the first coefficients of the expansion (3.17) for the functions $\Psi^1_i(u, Q|D, R)$. The Lamé coefficients $H_i(u|D, R, r')$ are equal to

$$H_i(u|D, R) = -\sum_{\alpha} d_\alpha \Psi^1_i(u, R'_{\alpha}|D, R),$$  \hspace{1cm} (3.25)

where

$$d_\alpha = \text{res}_{R_{\alpha}} d\Omega_0$$  \hspace{1cm} (3.26)

**Proof.** The equations (1.13) imply that the functions $\Psi^1_i$ satisfy the equation

$$\partial_i \Psi^1_j = \beta_{ij} \Psi^1_i, \hspace{0.5cm} i \neq j$$  \hspace{1cm} (3.27)

The formula (3.24) is a result of the direct substitution of the expansion (3.17) into (3.27). For the proof of (3.25) let us consider the differential

$$d\Omega^{(3)}_i(u, Q) = \Psi^1_i(u, Q)\psi(u, Q')d\Omega_0$$  \hspace{1cm} (3.28)

This differential is meromorphic with the poles at the point $P_i$ and the punctures $R'_{\alpha}$. The residue of $d\Omega^{(3)}_i$ at the point $P_i$ is equal to

$$\text{res}_{P_i} d\Omega^{(3)}_i = H_i(u).$$  \hspace{1cm} (3.29)

The residues of this differential at the points $R'_{\alpha}$ are equal to the corresponding terms of the sum in the right hand side of (3.25). The sum of all these residues is equal to zero. Theorem is proved.

**Important remark.** Theorem 3.2 allows one to give another interpretation of our basic construction. The rotation coefficients $\beta_{ij}$ up to gauge transformation (3.22), depend on the equivalence classes of the divisors $D, R$, only. The functions $\Psi^1_i(u, Q|D, R)$ satisfy (3.27). Therefore, any linear combination of the form (3.25) defines the coefficients of the flat diagonal metrics. Therefore, the formulae (2.40) may be considered as the solution for the problem of the construction of the corresponding flat coordinates.
4 Egoroff metrics and the partition functions of the topological field theories

In this section we describe the algebraic geometrical data corresponding to Egoroff metric, i.e. the metrics with symmetric rotation coefficients $\beta_{ij} = \beta_{ji}$.

Let $E(P)$ be a meromorphic function on a smooth genus $g_0$ algebraic curve $G_0$ with $n$ simple poles at points $P_i$ having $2m - n$ simple zeros at points $Q_1, \ldots, Q_{2m-n}$ and $n - m$ double zeros at points $\hat{Q}_1, \ldots, \hat{Q}_{n-m}$. The Riemann surface $\Gamma$ of the function

$$\lambda = \sqrt{E(P)} \quad (4.1)$$

is an admissible curve in the sense of our definition. The point $Q \in G$ is a pair $P$ and a sign $\pm$ of the square root. The function $\lambda = \lambda(Q)$ is an odd function with respect to the involution of $\Gamma$. As a function on $\Gamma$ it has simple poles at the points $P_i$ and simple zeros at the points $Q_j, j = 1, \ldots, n$. The function $\lambda^{-1}$ defines local coordinates

$$w_j(Q) = \lambda^{-1}(Q) \quad (4.2)$$

in the neighborhoods of the punctures $P_i$.

**Example.** Let $\Gamma_0$ be a plane curve defined by the equation

$$L(y, E) = 0 \quad (4.3)$$

where $L$ is degree $n$ polynomial in two variables. Let $\Gamma_0$ has $n$ smooth points $P_j$ and $n$ smooth points $Q_j$ as preimages of $E = \infty$ and $E = 0$, respectively. Then the admissible curve $\Gamma$ is defined by the equation

$$L(y, \lambda^2) = 0 \quad (4.4)$$

**Theorem 4.1** Let $D, R$ be an admissible pair of the divisors on the Riemann surface $\Gamma$ of the function $\lambda(Q)$. Then the rotation coefficients defined by $D, R$ are symmetric

$$\beta_{ij}(u|D, R) = \beta_{ji}(u|D, R) \quad (4.5)$$

The potential $\Phi(u|D, R)$ of the Egoroff metric $H_i(u|D, R, r')$ is equal to

$$\Phi = \sum_{\alpha=1}^n \lambda(R_\alpha) \ d_\alpha \ \psi(u, R_\alpha^\sigma), \quad (4.6)$$

where $d_\alpha$ are residues of $d\Omega_0$ at $R_\alpha \quad (3.26)$.

**Proof.** The equality (4.5) can be obtained by the consideration of the differential

$$\lambda(Q) \Psi_i^0(u, Q) \Psi_j^0(u, \sigma(Q)) d\Omega_0 \quad (4.7)$$
that has poles at the points $P_i$ and $P_j$, only. The residues at these points are equal to $\beta_{ji}$ and $-\beta_{ij}$, respectively.

We would like to mention the other way to prove (4.5), as well. Let $\Psi_0^i$ and $\Psi_1^i$ be the Baker-Akhiezer functions defined by the conditions 1*-4*. Then

$$\Psi_1^i(u, Q) = \lambda(Q)\Psi_0^i(u, Q)$$

(4.8)

Indeed, the function $\lambda\Psi_0^i$ satisfies all the defining conditions for the function $\Psi_1^i$. Therefore, (4.8) is a corollary of the uniqueness of the Baker-Akhiezer functions. From (4.8) and (1.15) it follows that the functions $\Psi_0^i$ satisfy the equations

$$\partial_i\Psi_0^j = \beta_{ji}\Psi_0^i,$$
$$\partial_j\Psi_0^j = \lambda\Psi_1^i - \sum_{m \neq j} \beta_{mj}\Psi_0^m$$

(4.9)

The compatibility conditions of this system are equations (1.4) and (1.5) together with the symmetry condition (4.5). Note that (4.9) as the auxiliary linear system for the full set of equations describing Egoroff metrics was proposed in [5].

For the proof of (4.6) let us consider the differential

$$d\Omega_i^{(4)} = \lambda(Q)\psi(u, Q)\partial_i\psi(u, \sigma(Q))d\Omega_0$$

(4.10)

This differential has poles at the points $P_i$ and $R_\alpha$ with the residues

$$\text{res}_{P_i}d\Omega_i^{(4)} = -H_i, \quad \text{res}_{R_\alpha}d\Omega_i^{(3)} = d_\alpha\lambda(R_\alpha)\partial_i\psi(u, R_\alpha)$$

(4.11)

The sum of these residues is equal to zero. Therefore, (4.6) is proved.

**Corollary 4.1** The formula

$$\Phi = \sum_{\beta=1}^l \lambda(R_\beta)\Phi_\beta(u),$$

(4.12)

where

$$\Phi_\beta = \sum_{\alpha=1}^l r_\alpha(R_\beta^\alpha) \frac{\theta(A(R_\beta^\alpha) + \sum_{i=1}^n (u^iV^{(i)}) + Z_\alpha)\theta(Z_0)}{\theta(A(R_\beta^\alpha) + Z_\alpha)\theta(\sum_{i=1}^n (u^iV^{(i)}) + Z_0)} \exp \left(\sum_{i=1}^n u^i \int_{R_\alpha}^{\sigma(R_\alpha)} d\Omega_i\right),$$

(4.13)

defines the potentials of the algebraic-geometric Egoroff metrics.

**5 Solutions to the associativity equations**

The equivalence of the classification problem for the rotation coefficients of Egoroff metrics and the classification problem for the massive topological fields found in [5] does not provide
the exact expression for the partition functions of these models. Now for each of the above-constructed algebraic-geometrical Egoroff metrics we are going to define a function $F$ such that its third derivatives have the form

$$c_{klm} = \sum_{i=1}^{n} H_i^2 \frac{\partial u^i \partial u^j \partial u^k}{\partial x^k \partial x^l \partial x^m}$$  \quad (5.1)$$

and the functions

$$c_{kl}^m = \sum_{i=1}^{n} \frac{\partial u^i \partial u^i \partial x^m}{\partial x^k \partial x^l \partial u^j}$$  \quad (5.2)$$
define an associate algebra.

**Theorem 5.1** Let $\psi(u, Q|D, R)$ be the Baker-Akhiezer function defined on the Riemann surface $\Gamma$ of the function $\lambda(Q)$ and corresponding to an admissible pair of the divisor $D, R$. Then the function $F(x) = F(u(x))$ defined by the formula

$$F(u) = \frac{1}{2} \left( \sum_{k,l=1}^{n} \eta_{kl} x^k(y^l(u) - \sum_{\alpha} d_\alpha \psi(u, R_\alpha^\sigma)) \right),$$

where $\eta_{kl}$ is a constant matrix defined by (2.33), $d_\alpha$ are equal to (3.26) and

$$x^k(u) = \psi(u, Q_k), \quad y^k = \frac{d\psi}{d\lambda}(u, Q_k),$$

satisfies the equation

$$\frac{\partial^3 F(x)}{\partial x^k \partial x^l \partial x^m} = c_{klm}$$  \quad (5.5)$$

where $c_{klm}$ are defined by (5.1). The functions $c_{kl}^m$ defined by (5.2) satisfy the associativity equations (1.18).

**Proof.** Let us consider the functions

$$\phi_k = \frac{\partial \psi}{\partial x^k}, \quad \phi_{kl} = \frac{\partial^2 \psi}{\partial x^k \partial x^l}$$

At the puncture $P_i$ they have the form

$$\phi_k = \frac{\partial u^i}{\partial x^k} \lambda e^{\lambda u^i} (h_i + O(\lambda^{-1})), \quad \phi_{kl} = \frac{\partial u^i \partial u^j}{\partial x^k \partial x^l} \lambda^2 e^{\lambda u^i} (h_i + O(\lambda^{-1})).$$

Therefore,

$$c_{klm} = \sum_{i=1}^{n} \text{res}_{P_i} d\Omega_{klm},$$

where

$$d\Omega_{klm} = \phi_k(u, Q)\phi_{lm}(u, Q^\sigma) \frac{d\Omega_0}{\lambda(Q)}$$

18
By the definition of the flat coordinates $x^k$ we have

$$\phi_k(u, Q_m) = \delta_{km}, \quad \phi_{kl}(u, Q_m) = 0$$

Therefore, the differential $d\Omega_{k;lm}$ outside the punctures $P_i$ has the pole at $Q_k$, only. Hence,

$$c_{klm} = - \text{res}_{Q_k} d\Omega_{k;lm} = -\text{res}_{Q_k} \phi_{lm}(u, \sigma(Q_k)) \frac{d\Omega_0}{\lambda(Q)}$$

(5.12)

At the puncture $Q_k$ the function $\psi(u, \sigma(Q))$ has the form

$$\psi(u, \sigma(Q)) = x^{\sigma(k)} - y^{\sigma(k)}\lambda + O(\lambda^2)$$

(5.14)

and

$$d\Omega_0 = \frac{d\lambda}{\lambda}(\eta_k + \eta_1^k\lambda + O(\lambda^2))$$

(5.15)

Therefore,

$$\text{res}_{Q_k} \psi(u, \sigma(Q)) \frac{d\Omega_0}{\lambda} = \eta_1^k x^{\sigma(k)} - \eta_k y^{\sigma(k)}$$

(5.16)

From the definition of $F$ it follows that

$$2 \frac{\partial}{\partial x^k} F = \eta_k y^{\sigma(k)} + \sum_{l=1}^{n} \eta_l x^{l} \frac{\partial y^{\sigma(l)}}{\partial x^k} - \frac{d_\alpha}{\lambda(R_\alpha)} \frac{\partial \psi(u, R_\alpha)}{\partial x^k}$$

(5.17)

Consider the differential

$$d\Omega_k^{(5)} = \frac{\partial \psi(u, Q)}{\partial x^k} \psi(u, Q^*) \frac{d\Omega_0}{\lambda(Q)}$$

(5.18)

That meromorphic differential has poles at the punctures $Q_i$ and at the points $R_\alpha^*$, only. Its residues at these points are equal to

$$\text{res}_{Q_i} d\Omega_k^{(5)} = \eta_l x^{\sigma(l)} \frac{\partial y^l}{\partial x^k} + \delta_{l,\sigma(k)} \left( \eta_1^k x^{\sigma(k)} - \eta_k y^{\sigma(k)} \right)$$

(5.19)

$$\text{res}_{R_\alpha^*} d\Omega_k^{(5)} = - \frac{d_\alpha}{\lambda(R_\alpha)} \frac{\partial \psi(u, R_\alpha^*)}{\partial x^k}$$

(5.20)

Therefore,

$$\sum_{l=1}^{n} \eta_l x^{l} \frac{\partial y^{\sigma(l)}}{\partial x^k} - \sum_{\alpha} \frac{d_\alpha}{\lambda(R_\alpha)} \frac{\partial \psi(u, R_\alpha)}{\partial x^k} = \eta_k y^{\sigma(k)} - \eta_1^k x^{\sigma(k)}$$

(5.21)

Finally,

$$\frac{\partial}{\partial x^k} F = \eta_k y^{\sigma(k)} - \frac{1}{2} \eta_1^k x^{\sigma(k)}$$

(5.22)

The last equality and equalities (5.13), (5.16) imply (5.5).
Lemma 5.1 Let $\psi(u, Q|D, R)$ be the Baker-Akhiezer function defined on the Riemann surface $\Gamma$ of the function $\lambda(Q)$ and corresponding to an admissible pair of the divisors $D, R$. Then it satisfies the equations

$$\frac{\partial^2}{\partial x^k \partial x^l} \psi - \lambda \sum_{m=1}^{n} c_{kl}^m \frac{\partial}{\partial x^m} \psi_m = 0 \quad (5.23)$$

Proof. Consider the function $\tilde{\psi}$ defined by the left hand side of (5.23). Outside of the punctures $P_j$ it has poles at the points the divisor $D$, only, and is equal to zero at the points $Q_j$. From the definition of $c_{kl}^m$ it follows that in the expansion of $\tilde{\psi}$ at the points $P_i$ the factor in front of the exponent has the form $O(\lambda^{-1})$. Therefore, from the uniqueness of the Baker-Akhiezer function it follows that $\tilde{\psi} = 0$.

The associativity equations (1.18) for the functions $c_{kl}^m$ are the compatibility conditions for the system (5.23). Theorem is proved.

Remark. The equations (5.23) can be written in the vector form

$$\frac{\partial}{\partial x^k} \tilde{\Psi}_l = \lambda \sum_{m=1}^{n} c_{kl}^m \tilde{\Psi}_m, \quad (5.24)$$

where

$$\tilde{\Psi}_k = \frac{\partial \psi}{\partial x^k} \quad (5.25)$$

The system (5.24) with symmetric coefficients $c_{kl}^m = c_{lk}^m$ was introduced in [5] as an auxiliary linear system for the associativity equation (1.18).

Now we are going to consider the special case of our construction when the divisor $R$ coincides with the divisor $Q$ of the punctures $Q_j$. As it was mentioned in the remark before the theorem 2.2 the assumption that $R$ does not intersect with $Q_j$ we take for the simplicity of the formulae, only.

In the case when $R = Q$ of the admissible divisors $D$ are defined as follows. The divisor $D = \gamma_1 + \cdots + \gamma_{g+n-1}$ is called admissible if there exists a meromorphic differential $d\Omega_0$ on $\Gamma_0$ with poles of the order 2 at the points $Q_{2m-\gamma+n+1}, \ldots, Q_{2m}$ and poles of the order 3 at the double zeros of $E(P)$ and such that

$$d\Omega_0(\pi(\gamma_s)) = 0 \quad (5.26)$$

The differential $d\Omega_0$ considered as differential on $\Gamma$ has the form

$$d\Omega_0 = \frac{d\lambda}{\lambda^3(P)}(\eta_k + O(\lambda)) \quad (5.27)$$

at the punctures $Q_k, \ k = 1 \ldots, n$. In the special case under consideration, the flat coordinates are defined not by the evaluation the Baker-Akhiezer function $\psi$ at the punctures $Q^k$ (where it equals 1 now) but by taking the next term in the expansions.
**Theorem 5.2** Let \( \psi(x, Q|D, \lambda) \) be the Baker-Akhiezer function defined by an admissible divisor \( D \) on the Riemann surface of the function \( \lambda(Q) \). Then the function \( F(x) = F(u(x)) \)

\[
F(u) = \frac{1}{2} \sum_{k=1}^{n} \eta_k x^k(u) y^\sigma(u), \tag{5.28}
\]

where

\[
\eta_k = \text{res}_Q \lambda^2 d\Omega_0 \tag{5.29}
\]

and \( x^k(u) \) and \( y^k(u) \) are defined from the expansion

\[
\psi = 1 + x^k(u) \lambda + y^k(u) \lambda^2 + O(\lambda^3) \tag{5.30}
\]

is a solution of the associativity equations (1.16)-(1.19), i.e. it satisfies the equation (5.5),

where \( c_{klm} \) are given by (5.1); the functions \( c_{mkl} \) defined by (5.2) satisfy (1.18) and at last the relation

\[
\sum_{m=1}^{n} c_{klm}(u) = \eta_{kl} \tag{5.31}
\]

is fulfilled.

**Proof.** The proof that the functions \( x^k \) are flat coordinates for the diagonal metric with the Lamé coefficients \( H_i = \varepsilon_i h_i(u) \), where \( h_i(u) \) is the leading term in the expansion of the corresponding Baker-Akhiezer function at the puncture \( P_i \) is just the same as in the general case. The proof of all the other statements of the theorem but the last one is also almost identical to the their proof in Theorem 5.1 The last statement of the theorem (5.32) follows from:

**Lemma 5.2** Let \( \psi \) be the Baker-Akhiezer function corresponding to the data specified by the assumptions of Theorem 4.3. Then it satisfies the equation

\[
\sum_{s=1}^{n} \frac{\partial}{\partial u^k} \psi = \lambda \psi \tag{5.32}
\]

The left and the right hand sides of (5.32) are regular outside the punctures \( P_k \) and have the same leading terms in their expansions at \( P_k \). Therefore, they are identically equal to each other.

The evaluation of (5.32) at \( Q_m \) gives the equality

\[
\sum_{k=1}^{n} \frac{\partial x^m}{\partial u^k} = 1 \tag{5.33}
\]

Hence,

\[
\sum_{m=1}^{n} c_{klm}(u) = \sum_{i=1}^{n} H_i^2 \frac{\partial u^i}{\partial x^k} \frac{\partial u^i}{\partial x^l} = \eta_{kl} \tag{5.34}
\]

Theorem is proved.
The exact theta functional formula for the partition function $F$ can be obtained by the simple substitution of the corresponding expressions for the Baker-Akhiezer function. The formulae for the functions $x^k(u)$ in (5.28) are slightly different from (2.39), (2.40). Namely,

$$x^k(u) = \frac{\partial}{\partial A(Q_k)} \sum_{\alpha=1}^l r_m(A(Q_k)) \ x^k_m(u),$$

(5.35)

where

$$x^k_m(u) = \frac{\theta(A(Q_k) + \sum_{i=1}^n u^i V^{(i)} + Z_m) \theta(Z_0)}{\theta(A(Q_k) + Z_m) \theta(\sum_{i=1}^n u^i V^{(i)} + Z_0)} \ \exp \left( \sum_{i=1}^n u^i \int_{Q_m} Q_k \ d\Omega_i \right).$$

(5.36)

The formulae for $y^k(u)$ can be obtained by taking the derivatives of the formulae in the right hand sides of (5.35) with respect to the variables $A(Q_k)$.

**Example. Elliptic solutions**

Let us consider the simplest elliptic curvilinear coordinates and solutions to the associativity equations that correspond to $n = l = 3, m = 2$ in the example of Section 2.

Let us represent the elliptic curve $\Gamma$ as a factor of the complex plane of a variable $z$ with respect to the lattice generated by periods $2\omega, 2\omega'$, $\text{Im}\omega'/\omega > 0$. In that realization we identify the punctures $P_i$ with half periods $\omega_i$, i.e. with the points:

$$P_1 = \omega_1 = \omega, \ P_2 = \omega_2\omega', \ P_3 = \omega_3 = -\omega - \omega'$$

(5.37)

The punctures $Q_j$ are the points

$$Q_1 = 0, \ Q_2 = z_0, \ Q_3 = -z_0$$

(5.38)

In the case $g = 1$ any sets $\gamma_1, \ldots, \gamma_l$ and $R_1, \ldots, R_l$ are the admissible divisors. The corresponding differential $d\Omega_0$ has the form

$$d\Omega_0 = \eta_0 \frac{\sigma(z - \omega)\sigma(z - \omega')\sigma(z + \omega + \omega')}{\sigma(z)\sigma(z + z_0)\sigma(z - z_0)} \ \prod_{s=1}^l \frac{\sigma(z - \gamma_s)\sigma(z + \gamma_s)}{\sigma(z - R_s)\sigma(z + R_s)} \ dz,$$

(5.39)

where $\sigma(z) = \sigma(z|\omega, \omega')$ is classical $\sigma$-Weierstrass function. The residues of this differential

$$\text{res}_{z=0} d\Omega_0 = \eta_1, \ \text{res}_{z=\pm z_0} d\Omega_0 = \eta_2$$

are the coefficients of the flat metric

$$ds^2 = \eta_1(dx^1)^2 + \eta_2(dx^2)(dx^3)$$

(5.41)

The Baker-Akhiezer function corresponding to $\gamma_s$ and $R_s$ has the form:

$$\psi(u, z) = \prod_{s=1}^l \frac{\sigma(z - R_s)}{\sigma(z - \gamma_s)} \ \left[ \sum_{\alpha=1}^l r_{\alpha} \frac{\sigma(z + U - R_{\alpha})}{\sigma(z - R_{\alpha})\sigma(U)} \ \exp(\Omega(u, z) - \Omega(u, R_{\alpha})) \right],$$

(5.42)

where

$$U = u^1 + u^2 + u^3,$$

(5.43)
\[ \Omega(u, z) = u^1(\zeta(z - \omega) + \eta) + u^2(\zeta(z - \omega') + \eta') + u^3(\zeta(z + \omega' - \eta - \eta'), \]  \hspace{1cm} (5.44) \\
\zeta(z) = e^{\sigma'(z)}, \ \eta = \zeta(\omega), \ \eta' = \zeta(\omega') \hspace{1cm} (5.45)

and the constants \( r_\alpha \) are equal to

\[ r_\alpha = \frac{\prod_{s=1}^l \sigma(R_\alpha - \gamma_s)}{\prod_{s \neq \alpha} \sigma(R_\alpha - R_s)} \hspace{1cm} (5.46) \]

In the general case when \( R_\alpha \neq Q_j \) the evaluation of \( \psi \) at \( Q_j \) gives an expression of the flat coordinates through the curvilinear 3-orthogonal coordinates \( u^1, u^2, u^3 \):

\[ x^1 = \psi(u, 0), \ x^2 = \psi(u, z_0), \ x^3 = \psi(u, -z_0) \hspace{1cm} (5.47) \]

The metric \([5.41]\) in the coordinates \( u^i \) is diagonal with the coefficients

\[ H_1(u) = \varepsilon_1 \sum_{\alpha=1}^n r_\alpha \frac{\sigma(U - R_\alpha)}{\sigma(\omega - R_\alpha)\sigma(U)} e^{U\eta} \hspace{1cm} (5.48) \]
\[ H_2(u) = \varepsilon_2 \sum_{\alpha=1}^n r_\alpha \frac{\sigma(U - R_\alpha)}{\sigma(\omega - R_\alpha)\sigma(U)} e^{U\eta'} \hspace{1cm} (5.49) \]
\[ H_3(u) = \varepsilon_3 \sum_{\alpha=1}^n r_\alpha \frac{\sigma(U - R_\alpha)}{\sigma(\omega + \omega' - R_\alpha)\sigma(U)} e^{(-U\eta - U\eta')} \hspace{1cm} (5.50) \]

The constants \( \varepsilon_i \) are equal to:

\[ \varepsilon_1 = \eta_0 F(\omega) \frac{\sigma(\omega - \omega')\sigma(2\omega + \omega')}{\sigma(\omega)\sigma(\omega + z_0)\sigma(\omega - z_0)}, \hspace{1cm} (5.51) \]
\[ \varepsilon_2 = \eta_0 F(\omega') \frac{\sigma(\omega' - \omega)\sigma(2\omega' + \omega)}{\sigma(\omega')\sigma(\omega' + z_0)\sigma(\omega' - z_0)}, \hspace{1cm} (5.52) \]
\[ \varepsilon_3 = -\eta_0 F(\omega + \omega') \frac{\sigma(2\omega + \omega')\sigma(2\omega' + \omega)}{\sigma(\omega + \omega')\sigma(\omega + \omega' + z_0)\sigma(\omega + \omega' - z_0)}, \hspace{1cm} (5.53) \]
\[ F(z) = \prod_{s=1}^l \frac{\sigma(z - \gamma_s)\sigma(z + \gamma_s)}{\sigma(z - R_s)\sigma(z + R_s)} \hspace{1cm} (5.54) \]

The elliptic solutions for the associativity equations correspond to the Baker-Akhiezer functions given by the formula \([5.42]\) with \( l = 3 \) and \( R_1 = 0, \ R_2 = z_0, \ R_3 = -z_0 \). For the simplicity of the formulae let us consider the special case corresponding to the basic elliptic Baker-Akhiezer function

\[ \psi(u, z) = \frac{\sigma(z + s)}{\sigma(z)\sigma(s)} e^\Omega(u, z) \hspace{1cm} (5.55) \]

The coefficients of the expansions of \( \psi \) at the points \( z = 0, \ z_0, \ -z_0 \)

\[ \psi = \frac{1}{z} + x^1(u) + y^1(u)z + O(z^2), \hspace{1cm} (5.56) \]
\[ \psi = x^2 + y^2(u)(z - z_0) + O((z - z_0)^2), \hspace{1cm} (5.57) \]
\[ \psi = x^3 + y^3(u)(z + z_0) + O((z + z_0)^2) \hspace{1cm} (5.58) \]
define the solution
\[ F = x^1 y^1 - \frac{1}{2} (x^2 y^3 + x^3 y^2) \] (5.59)
to the associativity equations. From (5.55) it follows that
\[ x^1 = \zeta(U) - \varphi(\omega) u^1 - \varphi(\omega') u^2 - \varphi(\omega + \omega') u^3, \] (5.60)
\[ x^2 = \frac{\sigma(z_0 + U)}{\sigma(z_0) \sigma(U)} \exp \Omega(u, z_0), \] (5.61)
\[ x^3 = \frac{\sigma(U - z_0)}{\sigma(-z_0) \sigma(U)} \exp \Omega(u, -z_0) \] (5.62)
and
\[ y^1 = \frac{\sigma''(U)}{2 \sigma(U)} - \zeta(U) \sum_{i=1}^{3} (\varphi(\omega_i) u^i) + \frac{1}{2} \left( \sum_{i=1}^{3} (\varphi(\omega_i) u^i) \right)^2, \] (5.63)
\[ y^2 = x^2(u) \left( \zeta(z_0 + U) - \zeta(z_0) - \sum_{i=1}^{3} (\varphi(z_0 - \omega_i) u^i) \right), \] (5.64)
\[ y^3 = x^3(u) \left( \zeta(-z_0 + U) + \zeta(z_0) - \sum_{i=1}^{3} (\varphi(z_0 - \omega_i) u^i) \right) \] (5.65)
The function
\[ \tilde{F} = F - \frac{1}{2} (x^1)^2, \] (5.66)
has the same third derivatives as \( F \). After substitution of the exact expression for \( x^i \) and \( y^i \) into (5.59) we obtain the final formula for the simplest elliptic solution to the associativity equations
\[ \tilde{F} = -\frac{1}{2} \varphi(U) - \frac{1}{2} (\varphi(U) - \varphi(z_0)) \left( \zeta(z_0 - U) - \zeta(z_0 - U) - \sum_{i=1}^{3} (\varphi(z_0 - \omega_i) u^i) \right) \] (5.67)

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