A NOTE ON HERMITE POLYNOMIALS

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Abstract. In this paper, we consider linear differential equations satisfied by
the generating function for Hermite polynomials and derive some new identities
involving those polynomials.

1. Introduction

The Hermite polynomials form a Sheffer sequence and are given by the generating
function
\[ e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n, \]  
(see \[1–8, 10, 13, 14\]).

By using Taylor series, we get
\[ H_n(x) = \left( \frac{\partial}{\partial t} \right)^n e^{2xt-t^2} \bigg|_{t=0} \]
\[ = \left( \frac{e^{x^2}}{\partial t} \right)^n e^{-(x-t)^2} \bigg|_{t=0} \]
\[ = (-1)^n x^n \left( \frac{\partial}{\partial x} \right)^n e^{-(x-t)^2} \bigg|_{t=0} \]
\[ = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad (n \geq 0), \quad \text{(see \[1–15, 18\]).} \]

The probabilists’ Hermite polynomials are given by the generating function
\[ H'_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \]
\[ = \left( x - \frac{d}{dx} \right)^n \cdot 1, \quad \text{(see \[10\]).} \]

The physicists’ Hermite polynomials are also given by
\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \]
\[ = \left( 2x - \frac{d}{dx} \right)^n \cdot 1 \quad \text{(see \[20\]).} \]

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Thus, by (1.3) and (1.4), we get
\[(1.5)\]  
\[H_n(x) = 2^n H_n^\ast\left(\sqrt{2}x\right), \quad H_n^\ast(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right),\]
where \(n \geq 0\) (see [9, 11, 12, 15, 18]).

The first several Hermite polynomials are
\[H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^2 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12, \quad H_5(x) = 32x^5 - 160x^3 + 120x, \quad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120, \ldots\]

The probabilists’ Hermite polynomials are solutions of the differential equation:
\[(e^{-x^2/2} u')' + \lambda e^{-1/2} x^2 u = 0,\]
where \(\lambda\) is a constant, with the boundary conditions that \(u\) should be polynomially bounded at infinity.

The generating function of the probabilists’ Hermite polynomials is given by
\[(1.6)\]  
\[e^{xt - \frac{t^2}{2}} = \sum_{n=0}^{\infty} H_n^\ast(x) \frac{t^n}{n!}, \quad (\text{see } [12, 15, 18]).\]

The Hermite polynomials \(H_n^{(\nu)}(x)\) of variance \(\nu\) form an Appell sequence and are defined by the generating function
\[(1.7)\]  
\[\sum_{k=0}^{\infty} \frac{H_k^{(\nu)}(x)}{k!} t^k = e^{xt - \frac{\nu t^2}{2}}, \quad (\text{see } [12]).\]

Thus, by (1.7), we get
\[(1.8)\]  
\[x^{2m+1} = \sum_{l=0}^{m} \binom{2m+1}{2l+1} \frac{(2m-2l)!}{(m-l)!} \left(\frac{\nu}{2}\right)^{m-l} H_{2l+1}^{(\nu)}(x),\]
and
\[(1.9)\]  
\[x^{2m} = \sum_{l=0}^{m} \binom{2m}{2l} \frac{(2m-2l)!}{(m-l)!} \left(\frac{\nu}{2}\right)^{m-l} H_{2l}^{(\nu)}(x), \quad (\text{see } [12]).\]

The Hermite polynomials have been studied in probability, combinatorics, numerical analysis, finite element methods, physics and system theory (see [1–15, 18]).

Recently, Kim has studied nonlinear differential equations arising from Frobenius-Euler numbers and polynomials.

In this paper, we consider linear differential equations arising from Hermite polynomials of variance \(\nu\) and give some new and explicit identities for those polynomials.

2. HERMITE POLYNOMIALS OF VARIANCE \(\nu\)

Let
\[(2.1)\]  
\[F = F(t : x, \nu) = e^{xt - \frac{\nu t^2}{2}}.\]

From (2.1), we note that
\[(2.2)\]  
\[F^{(1)} = \frac{d}{dt} F(t : x, \nu) = (x - \nu t) e^{xt - \frac{\nu t^2}{2}} = (x - \nu t) F,\]
(2.3) \[ F^{(2)} = \frac{d}{dt} F^{(1)} = \left(-\nu + (x - t\nu)^2\right) F, \]
(2.4) \[ F^{(3)} = \frac{d}{dt} F^{(2)} = \left(-3\nu (x - t\nu) + (x - t\nu)^3\right) F, \]
and
(2.5) \[ F^{(4)} = \frac{d}{dt} F^{(3)} = \left(3\nu^2 - 6\nu (x - t\nu)^2 + (x - t\nu)^4\right) F. \]

Continuing this process, we set
(2.6) \[ F^{(N)} = \left(\frac{d}{dt}\right)^N F(t : x, \nu) \]
\[ = \left(\sum_{i=0}^{N} a_i (N, \nu) (x - t\nu)^i\right) F, \]
where \(N \in \mathbb{N} \cup \{0\} \).

From (2.6), we have
(2.7) \[ F^{(N+1)} = \frac{d}{dt} F^{(N)} \]
\[ = \sum_{i=0}^{N} a_i (N, \nu) i (x - t\nu)^{i-1} (-\nu) F \]
\[ + \sum_{i=0}^{N} a_i (N, \nu) (x - t\nu)^i F^{(1)}. \]

By (2.2) and (2.7), we easily get
(2.8) \[ F^{(N+1)} = \left\{-\nu a_1 (N, \nu) + a_N (N, \nu) (x - t\nu)^N + a_{N-1} (N, \nu) (x - t\nu)^{N-1} \right. \]
\[ + \sum_{i=1}^{N-1} (-i + 1) \nu a_{i+1} (N, \nu) + a_{i-1} (N, \nu) \left(x - t\nu\right)^i \} F. \]

By replacing \(N\) by \((N + 1)\) in (2.6), we get
(2.9) \[ F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i (N + 1, \nu) (x - t\nu)^i\right) F. \]

From (2.8) and (2.9), we can derive the following equations:
(2.10) \[ a_0 (N + 1, \nu) = -\nu a_1 (N, \nu), \]
(2.11) \[ a_N (N + 1, \nu) = a_{N-1} (N, \nu), \]
(2.12) \[ a_{N+1} (N + 1, \nu) = a_N (N, \nu) \]
and
(2.13) \[ a_i (N + 1, \nu) = -(i + 1) \nu a_{i+1} (N, \nu) + a_{i-1} (N, \nu), \]
where \(1 \leq i \leq N - 1.\)

It is not difficult to show that
(2.14) \[ F = F^{(0)} = a_0 (0, \nu) F. \]

Thus, by (2.14), we get
(2.15) \[ a_0 (0, \nu) = 1. \]
From (2.2) and (2.6), we note that
\[(2.16) \quad (x - \nu t) F = F^{(1)} = (a_0 (1, \nu) + a_1 (1, \nu) (x - \nu t)) F.\]

Thus, by comparing the coefficients on both sides of (2.16), we get
\[(2.17) \quad a_0 (1, \nu) = 0, \quad a_1 (1, \nu) = 1.\]

From (2.11), (2.12), (2.15) and (2.17), we have
\[(2.18) \quad a_N (N + 1, \nu) = a_{N-1} (N, \nu) = \cdots = a_0 (1, \nu) = 0,\]
and
\[(2.19) \quad a_{N+1} (N + 1, \nu) = a_N (N, \nu) = \cdots = a_1 (1, \nu) = 1.\]

Therefore, we obtain the following theorem.

**Theorem 1.** The linear differential equations
\[F^{(N)} = \left( \frac{d}{dt} \right)^N F (t : x, \nu) = \left( \sum_{i=0}^{N} a_i (N, \nu) (x - \nu t)^i \right) F, \quad (N \in \mathbb{N} \cup \{0\})\]
has a solution
\[F = F (t : x, \nu) = e^{xt - \frac{\nu t^2}{2}},\]
where
\[a_0 (N, \nu) = -\nu a_1 (N - 1, \nu),\]
\[a_{N-1} (N, \nu) = a_{N-2} (N - 1, \nu) = \cdots = a_1 (2, \nu) = a_0 (1, \nu) = 0,\]
\[a_N (N, \nu) = a_{N-1} (N - 1, \nu) = \cdots = a_1 (1, \nu) = a_0 (0, \nu) = 1,\]
and
\[a_i (N, \nu) = -(i + 1) \nu a_{i+1} (N - 1, \nu) + a_{i-1} (N - 1, \nu), \quad (1 \leq i \leq N - 2).\]

**Example.**
1. \(N = 3, i = 1.\) By (2.13), we get
   \[a_1 (3, \nu) = -2\nu a_2 (2, \nu) + a_0 (2, \nu) = -2\nu - \nu = -3\nu.\]
2. \(N = 4, 1 \leq i \leq 2.\) By (2.13), we have
   \[a_1 (4, \nu) = 0, \quad a_2 (4, \nu) = -6\nu.\]
3. \(N = 5, 1 \leq i \leq 3.\) By (2.13), we get
   \[a_1 (5, \nu) = 15\nu^2, \quad a_2 (5, \nu) = 0, \quad a_3 (5, \nu) = -10\nu.\]
4. \(N = 6, 1 \leq i \leq 4.\) From (2.13), we have
   \[a_1 (6, \nu) = 0, \quad a_2 (6, \nu) = 45\nu^2, \quad a_3 (6, \nu) = 0, \quad a_4 (6, \nu) = -15\nu.\]

Thus, we obtain the following result.
Remark. The matrix \((a_{i,j}(j, \nu))_{0 \leq i,j \leq 6}\) is given by

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & -\nu & 0 & 3\nu^2 & 0 & -15\nu^3 \\
1 & 0 & -3\nu & 0 & 15\nu^2 & 0 & \\
2 & 1 & 0 & -6\nu & 0 & 45\nu^2 & 0 \\
3 & 1 & 0 & -10\nu & 0 & & \\
4 & 1 & 0 & & -15\nu & & \\
5 & 0 & 1 & 0 & & & \\
6 & & & & & & 1 \\
\end{bmatrix}.
\]

From (1.7), we note that

\[
F = F(t : x, \nu) = e^{xt - \nu t^2} = \sum_{k=0}^{\infty} H_{k}^{(\nu)}(x) \frac{t^k}{k!}.
\]

Thus, by (2.20), we get

\[
F^{(N)}(t : x, \nu) = \left(\frac{d}{dt}\right)^N F(t : x, \nu) = \sum_{k=0}^{\infty} H_{k}^{(\nu)}(x) (k)_N \frac{t^{k-N}}{k!}.
\]

By Theorem 1, we easily get

\[
F^{(N)}(t : x, \nu) = \left(\sum_{i=0}^{N} a_i(N, \nu) (x - \nu t)^i\right) F
\]

\[
= \sum_{k=0}^{\infty} H_{k+N}^{(\nu)}(x) k! \frac{t^k}{(n+k)!}
\]

Therefore, by (2.21) and (2.22), we obtain the following theorem.
Theorem 2. For \( k, N \in \mathbb{N} \cup \{0\} \), we have

\[
H_{k+N}^{(\nu)}(x)
= \sum_{i=0}^{N} a_i(N, \nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l} (-\nu)^{k-l} x^{i+k-l} H_{i}^{(\nu)}(x).
\]

It is easy to show that

\[
(2.23) \quad H_{k+1}^{(\nu)}(x) = \left(x - \nu \frac{\partial}{\partial x}\right) H_{k}^{(\nu)}(x).
\]

Thus, by (2.23), we have

\[
(2.24) \quad H_{k+N}^{(\nu)}(x) = \left(x - \nu \frac{\partial}{\partial x}\right)^N H_{k}^{(\nu)}(x), \quad (N \in \mathbb{N} \cup \{0\}).
\]

From Theorem 2 we note that

\[
(2.25) \quad \left(x - \nu \frac{\partial}{\partial x}\right)^N H_{k}^{(\nu)}(x)
= \sum_{i=0}^{N} a_i(N, \nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l} (-\nu)^{k-l} x^{i+k-l} H_{i}^{(\nu)}(x),
\]

where \( \frac{\partial}{\partial x}x - x \frac{\partial}{\partial x} = \text{identity} \).

Now, we observe explicit determination of \( a_i(j, \nu) \).

From (2.12) and (2.13), we can derive the following equations:

\[
(2.26) \quad a_N(N, \nu) = 1,
\]

\[
(2.27) \quad a_{N-2}(N, \nu) = -(N-1) \nu a_{N-1}(N-1, \nu) + a_{N-3}(N-1, \nu)
= -(N-1) \nu a_{N-1}(N-1, \nu) - (N-2) \nu a_{N-2}(N-2, \nu) + a_{N-4}(N-2, \nu)
\]

\[
\vdots
= -(N-1) \nu a_{N-1}(N-1, \nu) - (N-2) \nu a_{N-2}(N-2, \nu) - 2 \nu a_2(2, \nu) + a_0(2, \nu)
= -(N-1) \nu a_{N-1}(N-1, \nu) - (N-2) \nu a_{N-2}(N-2, \nu) - 2 \nu a_2(2, \nu) - \nu a_1(1, \nu)
\]

\[
= -\nu \sum_{i=1}^{N-1} ia_i(i, \nu),
\]

\[
(2.28) \quad a_{N-4}(N, \nu) = -(N-3) \nu a_{N-3}(N-1, \nu) + a_{N-5}(N-1, \nu)
= -(N-3) \nu a_{N-3}(N-1, \nu) - (N-4) \nu a_{N-4}(N-2, \nu) + a_{N-6}(N-2, \nu)
\]

\[
\vdots
= -(N-3) \nu a_{N-3}(N-1, \nu) - (N-4) \nu a_{N-4}(N-2, \nu)
\]
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\[ - \cdots - 2\nu a_2 (4, \nu) + a_0 (4, \nu) \]
\[ = - (N - 3) \nu a_{N-3} (N - 1, \nu) - (N - 4) \nu a_{N-4} (N - 2, \nu) \]
\[ - \cdots - 2\nu a_2 (4, \nu) - \nu a_1 (3, \nu) \]
\[ = - \nu \sum_{i=0}^{N-3} ia_i (i + 2, \nu), \]

and

\[ a_{N-6} (N, \nu) = - (N - 5) \nu a_{N-5} (N - 1, \nu) + a_{N-7} (N - 1, \nu) \]
\[ = - (N - 5) \nu a_{N-5} (N - 1, \nu) - (N - 6) \nu a_{N-6} (N - 2, \nu) \]
\[ + a_{N-8} (N - 2, \nu) \]
\[ \vdots \]
\[ = - (N - 5) \nu a_{N-5} (N - 1, \nu) - (N - 6) \nu a_{N-6} (N - 2, \nu) \]
\[ - \cdots - 2\nu a_2 (6, \nu) - \nu a_1 (5, \nu) \]
\[ = - \nu \sum_{i=1}^{N-5} ia_i (i + 4, \nu). \]

Continuing in this fashion, for \( l \) with \( 1 \leq l \leq \left\lfloor \frac{N-1}{2} \right\rfloor \),

\[ a_{N-2l} (N, \nu) = - \nu \sum_{i=1}^{N-2l+1} ia_i (i + 2l - 2, \nu). \]

By (2.26), (2.27), (2.28), (2.29) and (2.30), we get

\[ a_{N-2l} (N, \nu) = (-\nu)^l \sum_{i_1=1}^{N-2l+1} \sum_{i_2=1}^{i_1+1} \cdots \sum_{i_l=1}^{i_{l-1}+1} i_1 \cdot i_2 \cdot \cdots \cdot i_l, \]

where \( 1 \leq l \leq \left\lfloor \frac{N-1}{2} \right\rfloor \).
By (2.11) and (2.13), we easily get
\begin{align}
\tag{2.35}
a_{N-1}(N, \nu) &= a_{N-2}(N-1, \nu) = a_{N-3}(N-2, \nu) = \cdots = a_0(1, \nu) = 0, \\
\tag{2.36}
a_{N-3}(N, \nu) &= -(N-2)\nu a_{N-2}(N-1, \nu) + a_{N-4}(N-1, \nu) \\
&= a_{N-4}(N-1, \nu) \\
&\vdots \\
&= a_0(3, \nu) = -\nu a_1(2, \nu) = -\nu a_0(1, \nu) = 0, \\
\tag{2.37}
a_{N-5}(N, \nu) &= -(N-4)\nu a_{N-4}(N-1, \nu) + a_{N-6}(N-1, \nu) = a_{N-6}(N-1, \nu) \\
&\vdots \\
&= a_0(5, \nu) = -\nu a_1(4, \nu) = 0, \\
\tag{2.38}
a_{N-7}(N, \nu) &= -(N-6)\nu a_{N-6}(N-1, \nu) + a_{N-8}(N-1, \nu) \\
&\vdots \\
&= a_0(7, \nu) = -\nu a_1(6, \nu) = 0, \\
\end{align}
and
\begin{align}
\tag{2.39}
a_{N-(2l-1)}(N, \nu) &= 0, \quad (1 \leq l \leq \left\lfloor \frac{N}{2} \right\rfloor).
\end{align}

Therefore, we obtain the following theorem.

**Theorem 3.** For \( N \in \mathbb{N} \cup \{0\} \), we have
\[ a_{N-2l}(N, \nu) = (-\nu)^l \sum_{i_1=1}^{N-2l+1} \sum_{i_{l-1}=1}^{i_{l-1}+1} \cdots \sum_{i_1=1}^{i_1+i_{l-1}} \cdot i_1, \]
where \( 1 \leq l \leq \left\lfloor \frac{N-1}{2} \right\rfloor. \)

Also,
\[ a_{N-(2l-1)}(N, \nu) = 0, \quad \text{if} \ 1 \leq l \leq \left\lfloor \frac{N}{2} \right\rfloor. \]

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