Derived equivalence of symmetric special biserial algebras

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Abstract

We introduce Brauer complex of symmetric SB-algebra, and reformulate in terms of
Brauer complex the so far known invariants of stable and derived equivalence of symmetric
SB-algebras. In particular, the genus of Brauer complex turns out to be invariant under
derived equivalence. We study transformations of Brauer complexes which preserve class
derived equivalence. Additionally, we establish a new invariant of derived equivalence of
symmetric SB-algebras. As a consequence, symmetric SB-algebras with Brauer complex
of genus 0 are classified.

Keywords: Brauer tree algebras, special biserial algebras, tilting complex

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1 Introduction

The present paper lies within a series of papers, devoted to classification of symmetric special
biserial algebras up to derived equivalence (i.e., up to equivalence of derived categories). Recall
that a symmetric SB-algebra $\Lambda$ is uniquely determined by a pair $(\Gamma(\Lambda), f)$, where $\Gamma(\Lambda)$ is the

Brauer graph of $\Lambda$ and $f : V(\Gamma(\Lambda)) \rightarrow \mathbb{N}$ maps vertices of $\Gamma(\Lambda)$ to their multiplicities (see, e.g., [1] and Proposition 3.9).

- We show that the multiset of multiplicities of vertices of $\Gamma(\Lambda)$ is invariant under derived equivalence (Proposition 2.1). In order to prove this, we determine the center $Z(\Lambda)$.

- In section 3 we introduce Brauer CW-complex $C(\Lambda)$ — a relevant tool for studying derived equivalence. Topologically, $C(\Lambda)$ is a sphere with handles. We reformulate in terms of $C(\Lambda)$ the basic notions related to algebra $\Lambda$ and the invariants of stable equivalence, which appeared in [2]. In particular, the genus of $C(\Lambda)$ turns out to be invariant under stable equivalence. By a celebrated theorem of Rickard [7], these invariants are invariants of derived equivalence, too.

- We introduce elementary tilting complexes over symmetric special biserial algebras — a generalization of tilting complexes, which were treated in [3] (section 4). Equivalences of algebras, corresponding to elementary tilting complexes, can be reformulated in terms of ‘elementary transformations’ of Brauer CW-complexes of these algebras (Proposition 4.4). One sees that the algebra, which corresponds to the CW-complex obtained from $C(\Lambda)$ by an elementary transformation, is derived equivalent to $\Lambda$. Thus we obtain a direct graphic way of proving derived equivalence.

- In the last section we show that if the geometric realization of $C(\Lambda)$ is a sphere, then the invariants which we discuss in this paper determine $\Lambda$ up to derived equivalence.

2 The center $Z(\Lambda)$ and the multiplicities of $A$-cycles

Let $\Lambda$ be a symmetric SB-algebra over field $K$. Consider an extended quiver $Q_e = Q_e(\Lambda)$. Consider the partitions of its arrow set into $A$-cycles and into $G$-cycles (see [1]). Recall that $A$-cycles (and their multiplicities) correspond to the vertices of Brauer graph $\Gamma(\Lambda)$. We denote $A$-cycles by lower-case latine letters and denote vertices of $\Gamma(\Lambda)$ by the correspondent upper-case latine letters.

Let $\{c_1, c_2, \ldots, c_k\}$ be the set of $A$-cycles. For each $i = 1, \ldots, k$ consider a cyclic sequence $(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,l_i})$ of arrows of the cycle $c_i$. Let $f(c_1), f(c_2), \ldots, f(c_k) \in \mathbb{N}$ denote the multiplicities of $A$-cycles. For each loop $\alpha = \alpha_{i,k}$ which is not formal, set

$$q_\alpha = (\alpha_{i,k+1}\alpha_{i,k+2} \ldots, \alpha_{i,l_i} \ldots \alpha_{i,k})^{f(c_i)-1}\alpha_{i,k+1}\alpha_{i,k+2} \ldots, \alpha_{i,l_i} \ldots \alpha_{i,k-1}.$$

Proposition 2.1. 1. The center $Z(\Lambda)$ is generated as a vector space over $K$ by 1 and by the elements of the following three forms:

a. Elements $m_{i,t} = (\alpha_{i,1}\alpha_{i,2} \ldots, \alpha_{i,l_i})^t + (\alpha_{i,2}\alpha_{i,3} \ldots, \alpha_{i,1})^t + \cdots + (\alpha_{i,l_i}\alpha_{i,1} \ldots, \alpha_{i,l_i-1})^t$ for all $i = 1, 2, \ldots, k$ and $t = 1, \ldots, f(c_i) - 1$.

b. Elements $q_\alpha$ for each non-formal loop $\alpha$.

c. Elements $s_r = (\alpha_{r,1}\alpha_{r,2} \ldots, \alpha_{r,l_r})^{f(c_r)}$ for each vertex $r$ of $Q_e$, where $c_r$ is one of the two $A$-cycles, passing through $r$. 
2. \(Z/(\text{Soc } Z) \cong K[x_1, x_2, \ldots, x_k]/\langle x_i^{f(c_i)}(x_i x_j)^{f(c_j)} \rangle\), where \(i, j \in 1, \ldots, k\).

3. The multiset \((f(c_1), f(c_2), \ldots, f(c_k))\) is invariant under derived equivalence.

Proof. Recall that the value of \(s_r\) doesn’t depend on the choice of an \(A\)-cycle \(c_i\), and that the elements \(s_1, s_2, \ldots, s_n\) form a \(K\)-basis of \(\text{Soc}(\Lambda)\) (see, e.g., [1]). Since \(\Lambda\) is a symmetric algebra, the socle \(\text{Soc}(\Lambda)\) is contained in \(Z\), so \(s_r \in Z\). Moreover, for a non-formal loop \(\alpha\) at vertex \(r\) and for the corresponding idempotent \(e_r\) and path \(p \notin \{e_r, \alpha\}\) we get \(e_r q_\alpha = q_\alpha e_r, \alpha q_\alpha = s_r = q_\alpha \alpha, q_\alpha p = 0 = p q_\alpha\). Thus \(q_\alpha \in Z\). Similarly, for all \(i, t, r\) we get \(e_r m_{i,t} = m_{i,t} e_r\), since the summands in \(m_{i,t}\) are circuits. Furthermore, for all \((l_1, l_2, l_3)\)

\[
(a_{i,l_1} a_{i,l_1+1} \ldots a_{i,l_1-n})^{t_1} a_{i,l_1} a_{i,l_1+1} \ldots a_{i,l_2} \alpha_{i,t} m_{i,t} =
(a_{i,l_1} a_{i,l_1+1} \ldots a_{i,l_1-n})^{t_1} a_{i,l_1} a_{i,l_1+1} \ldots a_{i,l_2} =
\alpha_{i,t} m_{i,t} (a_{i,l_1} a_{i,l_1+1} \ldots a_{i,l_1-n})^{t_1} a_{i,l_1} a_{i,l_1+1} \ldots a_{i,l_2}
\]

Since for the rest paths \(p\) (subpaths of other \(A\)-cycles) \(m_{i,t} p = pm_{i,t} = 0\), we get \(m_{i,t} \in Z\).

Each \(z \in Z\) can be uniquely represented as

\[
z = \sum_{j=1}^{N} a_j p_j + s, \tag{1}
\]

where \(0 \neq a_j \in K\), paths \(p_j\) are distinct nonzero paths in the quiver \(Q\) which are not contained in the socle, \(s \in \text{Soc}(\Lambda)\).

By induction on the number of summands in the sum [1] we show, that \(z\) can be represented as a linear combination of elements \(m_{i,t}\) and \(q_\alpha\). Fix \(i \in \{1, \ldots, N\}\) and write

\[
p_i = \alpha_1 \alpha_2 \ldots \alpha_m,
\]

where \(\alpha_1, \alpha_2, \ldots, \alpha_m\) are consequent arrows of an \(A\)-cycle \(c_i\). Let \(\alpha_{m+1}\) be the next arrow of \(c_i\). There are two cases:

- Case 1: \(\alpha_1 \alpha_2 \ldots \alpha_m \alpha_{m+1} \notin \text{Soc}(\Lambda)\). In this case the path \(\alpha_{m+1} \alpha_1 \alpha_2 \ldots \alpha_m\) has coefficient \(a_i\) in the sum \(\sum a_j \alpha_{m+1} p_j\). Since \(z \alpha_{m+1} = \alpha_{m+1} z\), we obtain \(\alpha_{m+1} = \alpha_1\), i.e. \(p_i = (\alpha_{u,1} \alpha_{u,2} \ldots \alpha_{u,l_u})^t\) for some \(A\)-cycle \(c_u, t < f(c_u)\). Moreover, the other summands of \(m_{u,t}\) also have coefficient \(a_i\) in the sum [1]. We see that the sum representing element \(z - a_i m_{u,t} \in Z\) has less summands than the sum representing \(z\), so the inductive hypothesis is applied.

- Case 2: \(\alpha_1 \alpha_2 \ldots \alpha_m \alpha_{m+1} = s_l \in \text{Soc}(\Lambda)\) for some \(l\). Consider an idempotent \(e_r\) such that \(\alpha_m e_r \alpha_{m+1} \neq 0\). The expressions for \(ze_r\) and \(e_r z\) must contain \(p_i\) as a summand. Therefore \(p_i\) is a closed path. It follows that \(\alpha_{m+1}\) is a loop and \(p_i = q_\alpha_{m+1}\), and we apply the inductive hypothesis to \(z - a_i p_i\).

2. Observe that \(\text{Soc}(Z)\) is generated by the elements \(s_r\) and \(q_\alpha\), for all loops \(\alpha\) which are not separate \(A\)-cycles (i.e., \(\alpha q_\alpha = 0\)). Moreover, \(m_{i,t} m_{j,t_1} = \delta_{ij} m_{i,t+t_1}\) and \(m_{i,t}^{f(c_i)} \in \text{Soc}(Z)\). These two observations imply the claim.

3. The claim follows directly from p.2, since \(Z(\Lambda)\) is invariant under derived equivalence (see [3]). The maximal element \(f(c_i)\) equals the maximal index of nilpotency of nilpotents in \(Z/\text{Soc}(Z)\); the remaining proof is by induction.

\[\square\]
3 Brauer complex

3.1 Definitions and constructions

In this section we define a 2-dimensional CW-complex corresponding to a symmetric SB-algebra \( \Lambda \). Associate with each \( G \)-cycle \( z \) of length \( k \) a \( k \)-gon \( F_z \) with an oriented border. The sides of \( F_z \) are labeled with the vertices of \( Q_e \) which lie on \( z \) (in the counter-clockwise order in the orientation of \( F_z \)). Consider a CW-complex \( C = C(\Lambda) \) which is obtained from the resulting set of polygons by identifying oppositely oriented edges labeled by the same vertex. Since each vertex of \( Q_e \) belongs to exactly two \( G \)-cycles, \( C \) is an oriented manifold (without boundary).

**Definition 3.1.** CW-complex \( C(\Lambda) \) is called Brauer complex of \( \Lambda \).

Denote by \( \Gamma = \Gamma(\Lambda) \) the Brauer graph of \( \Lambda \). For a vertex \( V \in V(\Gamma) \), consider a cyclic permutation \( \pi_V \) of half-edges, incident with \( V \), which is defined by passing along the corresponding \( A \)-cycle. A 'picture' of a graph \( \Gamma \) on an oriented surface also determines, for any vertex of the graph, a cyclic permutation on the set of incident half-edges, which agrees with orientation. There exists an embedding \( i_\Gamma \) of \( \Gamma \) into an oriented surface \( M \), which preserves the cyclic permutations (\( i_\Gamma \) and \( M \) are uniquely defined up to a homeomorphism). Note that we consider strict embeddings, i.e. such embeddings that each connectivity component of \( M \setminus \Gamma \) is homeomorphic to an open disk). See [4] for the construction of embedding. It follows from the construction of embedding that the connectivity components of \( M \setminus \Gamma \) correspond to the \( G \)-cycles of \( \Lambda \). Now it is clear that \( M \) is a geometric realization of \( C(\Lambda) \) and that the 1-skeleton \( S_M \) of \( C(\Lambda) \) is isomorphic as a graph to \( \Gamma \) (we will refer to \( S_M \) as \( \Gamma \)). In particular, the vertices (edges) of \( C(\Lambda) \) are in one-to-one correspondence with the \( A \)-cycles (resp., vertices) of \( Q_e \). It is to be mentioned that the arrows of \( Q_e \) are in one-to-one correspondence with the angles of the 2-dimensional faces of \( C(\Lambda) \).

**Definition 3.2.** Perimeter of a 2-dimensional face of \( C(\Lambda) \) is the number of its edges, taking multiplicities into account (i.e., perimeter is the length of the corresponding \( G \)-cycles).

3.2 Invariants of stable equivalence

Observe that \( C(\Lambda) \) is an oriented surface. The following statement holds since the Euler characteristic of an oriented surface is even.

**Proposition 3.3.** If in the extended quiver \( Q_e \) of \( \Lambda \) the number of \( A \)-cycles is \( k \), the number of \( G \)-cycles is \( g \) and the number of vertices is \( n \), then \( k + g - n \) is even.

**Remark 3.4.** This statement was proved in [2] without topological arguments (Lemma 3.2).

**Definition 3.5.** The value \( k + g - n \) is called the genus of \( \Lambda \) (and of \( C(\Lambda) \)).

In [2] it is proved that the multiset of lengths of \( G \)-cycles, as well as the number of \( A \)-cycles, is invariant under stable equivalence. By Rickard’s Theorem, the derived equivalence of self-injective algebras implies stable equivalence (See [7]). The number of isomorphism classes of simple modules (i.e., the number of vertices of \( Q_e \)) is also stable invariant (See [5]). Therefore we get
Proposition 3.6. The multiset of perimeters of faces, the number of vertices and the genus of $C(\Lambda)$ are invariant under derived equivalence.

It was shown in [2] that the free rank of the Grothendieck group of the stable category $\text{stmod-}\Lambda$ equals $n - k$ if and only if $\Gamma(\Lambda)$ is not bipartite. Therefore, we have

Proposition 3.7. Derived (stable) equivalence preserves the property of the Brauer graph to be bipartite.

It should be mentioned that for algebras of genus 0 this invariant gives nothing new, since an embedded into a sphere graph is bipartite if and only if the perimeters of all its faces are even. But there are algebras of genus 1, the derived categories of which are not distinguished by the previously discussed invariants, but which are not equivalent by Proposition 3.7.

Example 3.8. Consider the following symmetric SB-algebras $\Lambda_1$ and $\Lambda_2$:

The quiver $Q_1$ of $\Lambda_1$ consists of vertices 1, 2, 3 and arrows

\[ \alpha, \delta : 1 \to 2, \beta, \varepsilon : 2 \to 3, \gamma, \eta : 3 \to 1. \]

Ideal $I_1$ of relations of $\Lambda_1$ is generated by the elements

\[ \alpha \beta, \beta \gamma, \gamma \delta, \delta \varepsilon, \varepsilon \eta, \eta \alpha, \alpha \varepsilon \gamma - \delta \beta \eta, \varepsilon \gamma \alpha - \beta \eta \delta, \gamma \alpha \varepsilon - \eta \delta \beta. \]

The quiver $Q_2$ of $\Lambda_2$ consists of vertices 1, 2, 3 and arrows

\[ \alpha_1 : 1 \to 2, \beta_1 : 2 \to 3, \gamma_1 : 3 \to 1, \delta_1 : 1 \to 3, \epsilon_1 : 3 \to 2, \eta_1 : 2 \to 1. \]

Ideal $I_2$ of relations of $\Lambda_2$ is generated by the elements

\[ \alpha_1 \beta_1, \beta_1 \gamma_1, \gamma_1 \delta_1, \delta_1 \epsilon_1, \epsilon_1 \eta_1, \eta_1 \alpha_1, \alpha_1 \eta_1 \delta_1 \gamma_1 - \delta_1 \gamma_1 \alpha_1 \eta_1, \beta_1 \epsilon_1 - \eta_1 \delta_1 \gamma_1 \alpha_1, \epsilon_1 \beta_1 - \gamma_1 \alpha_1 \eta_1 \delta_1. \]

It is easy to see that $\Lambda_1$ and $\Lambda_2$ are algebras with 3 simple modules, with one $G$-cycle of length 6 (\((\alpha \beta \gamma \delta \varepsilon \eta)\) and \((\alpha_1 \beta_1 \gamma_1 \delta_1 \epsilon_1 \eta_1)\), respectively) and with 2 $A$-cycles of multiplicities 1 (\(c_1^1 = (\alpha \varepsilon \gamma)\), \(c_2^1 = (\delta \beta \eta)\) and \(c_1^2 = (\epsilon_1 \beta_1)\), \(c_2^2 = (\gamma_1 \alpha_1 \eta_1 \delta_1)\)). In particular, $\Lambda_1$ and $\Lambda_2$ have genus 1. But $\Gamma(\Lambda_1)$ is bipartite (it consists of 2 vertices, connected by 3 edges) whereas $\Gamma(\Lambda_2)$ is not (the edge, corresponding the vertex 1 of $Q_2$ is a loop). Therefore, $\Lambda_1$ and $\Lambda_2$ are not derived equivalent.

Despite existence of an 'additional' invariant, the invariants and equivalences which are discussed in this paper are not enough to classify algebras of positive genus, in contrast to the 'spherical' case, which is treated in section 5 (see also example 4.7).

Proposition 3.9. Correspondence $\Lambda \mapsto C(\Lambda)$ gives a bijection from the set of (pairly non-isomorphic) indecomposable symmetric SB-algebras to the set of (pairly non-isomorphic) pairs $(C, f)$, where

1. $C$ is a CW-complex homeomorphic to 2-dimensional oriented manifold with fixed orien-
2. \(f\) is an arbitrary map from the 0-skeleton of \(C\) to \(\mathbb{N}\).

\[\text{Proof.}\] It remains to show that a Brauer complex uniquely determines a symmetric SB-algebra. It follows from the fact the 1-skeleton of Brauer complex has a structure of Brauer graph, which uniquely determines a symmetric SB-algebra (see \([1]\)). \(\square\)

4 \hspace{1em} \text{Elementary tilting complexes}

4.1 \hspace{1em} \text{Definition of elementary tilting complex}

Fix an edge \(i\) of \(C\) (equivalently, fix a vertex \(i\) in quiver \(Q_e\)), and suppose that there are other edges in \(C\). We distinguish three cases.

1. \(i\) is a leaf of \(\Gamma\). Equivalently, in the quiver \(Q_e\) there is a loop \(\alpha_i\) at vertex \(i\) and this loop is an \(A\)-cycle (i.e., it annihilates all other arrows of \(Q_e\)).

2. \(i\) is a loop, which bounds some face of \(C\). Equivalently, in the quiver \(Q_e\) there is a loop \(\alpha_i\) at vertex \(i\) and this loop is a \(G\)-cycle. In this case there is a unique \(A\)-cycle passing through \(i\) (this cycle contains at least 3 arrows, one of which is \(\alpha_i\)).

3. For \(r = 1, 2\) the end \(C_{i,r}\) of the edge \(i\) is incident with an edge \(i, r \neq i\), such that \(\pi_{C_{i,r}}(i_r) = i\). We permit \(i_1 = i_2\) and we permit \(i\) to be a loop (i.e., \(C_{i,1} = C_{i,2}\)). Equivalently, there is no loop at vertex \(i\) of \(Q_e\), i.e. the vertices \(i_1, i_2\) which precede \(i\) on both \(A\)-cycles passing through \(A\) (\(c_{i,1}\) and \(c_{i,2}\)) are different from \(i\).

In each of these cases, to the edge \(i\) we put in correspondence a complex \(T_i\) as follows. For a vertex \(j \in V(Q_e)\) we denote by \(P_j\) the indecomposable left projective \(A\)-module, which corresponds to \(j\). For \(i \neq j\), denote by \(T_{ij}\) the complex \(\cdots \rightarrow 0 \rightarrow P_j \rightarrow 0 \rightarrow \cdots\) concentrated in degree 0. If \(i\) is a leaf of \(\Gamma\), define complex \(T_i\) by

\[T_i : \cdots \rightarrow 0 \rightarrow P_j \xrightarrow{\beta_i} P_1 \rightarrow 0 \rightarrow \cdots\]

where \(j \in V(Q_e), j \neq i\) is the vertex preceding vertex \(i\) on the (unique) \(G\)-cycle, which contains \(i\); \(\beta_i \neq \alpha_i\) is the arrow preceding \(\alpha_i\) on the same \(G\)-cycle.

If \(i\) is a loop which bounds some face of \(C\), define \(T_i\) by

\[T_i : \cdots \rightarrow 0 \rightarrow P_j \bigoplus P_j \xrightarrow{(\beta_i, \beta_i \alpha_i)} P_1 \rightarrow 0 \rightarrow \cdots\]

where \(j \in V(Q_e), j \neq i\) is the vertex preceding vertex \(i\) on the (unique) \(A\)-cycle, which contains \(i\); \(\beta_i \neq \alpha_i\) is the arrow preceding \(\alpha_i\) on the same \(A\)-cycle.

Otherwise, define \(T_i\) by

\[T_i : \cdots \rightarrow 0 \rightarrow P_{i_1} \bigoplus P_{i_2} \xrightarrow{(\beta_i^1, \beta_i^2)} P_i \rightarrow 0 \rightarrow \cdots\]

where \(i_1, i_2\) are the vertices preceding \(i\) on the \(A\)-cycles \(c_{i,1}\) and \(c_{i,2}\), respectively; \(\beta_i^1, \beta_i^2\) are the respective arrows preceding \(\alpha_i\). Finally, set \(T_i = \bigoplus_{j=1}^n T_{ij}\).

\[\text{1. In} \ [2]\text{ it was shown that a symmetric SB-algebra is uniquely determined by the (labeled) Brauer graph and certain parameters. It can be easily shown that these parameters are excessive and can be eliminated.}\]
Proposition 4.1. \( T_i \) is a tilting complex over \( \Lambda \).

Proof. We verify that \( T_i \) satisfies the two conditions from the definition of tilting complex. In the definition of \( T_i \) we distinguished three cases. We show verification only for the third case, the other cases are treated in the same way.

First, we must verify that \( D^b(\Lambda) = \text{Add}(T_i) \), where \( \text{Add}(T_i) \) is the smallest triangulated subcategory, which contains all direct summands of object \( T_i \). It is enough to verify that all objects of the form \( 0 \to P_j \to 0 \) belong to \( \text{Add}(T_i) \). For \( i \neq j \) this is by definition of \( T_i \). For \( i = j \) it is easy to see that \( P_i[-1] \) is the third term of the triangle, which corresponds to the natural embedding of \( T_{ii_1} \oplus T_{ii_2} \) into \( T_{ii} \). It follows that \( T_i \) satisfies the first condition.

Now we verify that \( \text{Hom}_{D^b(\Lambda)}(T_i, T_i[r]) = 0 \) for \( r \in \mathbb{Z} \setminus 0 \). It is enough to proof that for each \( j \in V(Q_\Lambda) \) \( \text{Hom}_{D^b(\Lambda)}(T_{ii}, T_{ij}[-1]) = \text{Hom}_{D^b(\Lambda)}(T_{ij}[-1], T_{ii}) = 0 \). Each morphism from \( T_{ij} \) to \( T_{ii} \) is determined by a morphism \( f : P_j \to P_i \), where \( f \) is a multiplication by a linear combination of paths with starting point \( j \) and endpoint \( i \). Each of these paths ends either with \( \beta_i^1 \) or with \( \beta_i^2 \). Therefore \( f \) factors through \( (\beta_i^1, \beta_i^2) : P_{ij} \oplus P_{ij} \to P_i \). It follows that \( f \) is homotopic to zero. Similarly, each morphism from \( T_{ii} \) to \( T_{ij} \) is determined by a morphism \( f : P_i \to P_j \), where \( f \) is a multiplication by a linear combination \( S \) of paths with starting point \( i \) and endpoint \( j \). Suppose that \( S \) has nonzero summands. Since \( i \neq j \), the underlying paths are not maximal. Multiplying \( S \) by \( \beta_i^1 \) or by \( \beta_i^2 \) from the left, we again get a nontrivial sum of linearly independent summands. This contradicts the definition of morphism of complexes. Therefore \( f = 0 \) and \( \text{Hom}_{D^b(\Lambda)}(T_{ii}, T_{ij}[-1]) = 0 \). \( \square \)

4.2 Elementary transformations of Brauer complexes

Now we define elementary transformations of Brauer complexes. We will prove below that in terms of algebras, an elementary transformation puts an algebra \( \Lambda \) to the endomorphism algebra of one of the above defined tilting complexes over \( \Lambda \). We fix convention that under elementary transformation the vertices are fixed, the configuration of edges (labeled with vertices of a quiver) — and therefore the configuration of faces (labeled with \( G \)-cycles) — is changed. In other words, we identify the edges (and faces) by their labels, not by the vertices incident to them. The pictures below illustrate the simplest cases, in general they can be quite different.

Definition 4.2. Let \( C \) be a Brauer complex, let \( Q_e \) be the corresponding extended quiver. Let \( a \in E(C) \), \( V \in V(C) \), let \( F \) be a face of \( C \). Permutations \( \text{Next}_{V} : V(C) \to V(C) \) and \( E(C) \to E(C) \) are induced by the counter-clockwise order of vertices and edges in the orientation of \( F \). Recall that \( \pi_V \) denotes the permutation of half-edges incident with vertex \( V \in V(C) \), which is defined by passing along the corresponding \( A \)-cycle \( v \). By abuse of language, we will name half-edges after correspondent edges. Thus by abuse of language for a loop \( a \) both situations \( \pi_V(a) = a \) and \( \pi_V(a) \neq a \) can happen. However, from the context it will always be clear which half-edge is meant.

4.2.1 Transformation of type 1: shift of a leaf

Let \( V \in V(C) \) be a dangling vertex. Suppose that the edge (the face) incident with \( V \) is labeled by \( a \) (resp., by \( F \)). Let \( V_1 \) be the second vertex incident with \( a \). Put \( V_2 = \text{Next}_{F}(V_1), a_1 = \text{Next}_{F}(a). \) Now shift edge \( a \), so that \( a \) becomes incident with \( V \) and \( V_2 \) and \( a = \text{Next}_{F}(a_1). \)
4.2.2 Transformation of type 2: shift of a loop

Let $a$ be a loop at vertex $V_1$, bounding some face $F_1$. Let $F_2$ be the second face, incident with $a$, put $V_2 = \text{Next}_{F_2}(V_1)$, $a_1 = \text{Next}_{F_2}(a)$. Replace loop $a$ with a loop at vertex $V_2$, which lies inside $F_2$ after $a_1$. Note that $F_1$ is again bounded by a loop, which separates it from $F_2$.

4.2.3 Transformation of type 3: the general case

Let $a$ be an edge. Suppose that the vertices (faces) incident with $a$ are labeled by $V_1, V_2$ (resp., by $F_1$ and $F_2$; we permit $F_1 = F_2$). For $i = 1, 2$ put $V_i' = \text{Next}_{F_i}^{-1}(V_i)$, $a_i = \text{Next}_{F_i}^{-1}(a)$. Shift $a$ so that it becomes incident with $V_1'$ and $V_2'$, separates $F_1$ from $F_2$ and lies after $a_i$ on the new boundary of $F_{3-i}$.

**Definition 4.3.** We call the transformations of types 1-3 tilting transformations. The resulting complex is denoted by $C(a)$.

4.3 Correspondence

**Proposition 4.4.** Let $\Lambda$ be an SB-algebra, $C = C(\Lambda)$, $a \in E(C)$. Let $T_a$ be the tilting complex which corresponds to $a$. Then $\text{End}_{D^b(\Lambda)}T_a$ is a symmetric SB-algebra with Brauer complex $C(a)$.
(C and C(a) have the same multiplicities of vertices).

Proof. Denote by Q_e the extended quiver of Λ. By Rickard’s theorem [5], Λ_a = End_{D^n(Λ)} T_a is derived equivalent to Λ. Since Λ is a symmetric algebra, Λ_a is a symmetric algebra, too. By Pogorjaly’s result, an algebra, which is stable equivalent to an SB-algebra, is an SB-algebra, too [5]. Therefore, by another Rickard’s theorem [7] Λ_a is an SB-algebra. Let e = ∑i η_i be the decomposition of unity of Λ_a, which corresponds to the decomposition T_a = ∑i T_{ai}. Since the number of simple modules is invariant under derived equivalence, Λ_a is an algebra with n simple modules and therefore {e_i} is a set of primitive orthogonal idempotents. Set f_a = 1 − e_a ∈ Λ, and denote Λ_a = f_a Λ f_a. Since for i ≠ a the complexes T_{ai} are concentrated in degree 0, we have Λ_a = End_Λ ∑i T_{ai} = End_{D^n(Λ)} ∑i T_{ai}. Consider Brauer complex C_a, obtained from C by deletion of an edge a (if a is a leaf, we delete it with the incident dangling vertex). The marks on the remaining vertices are preserved. We need the following lemma.

Lemma 4.5. The symmetric SB-algebra which corresponds to C_a is isomorphic to Λ_a.

Proof. We consider the case when (C') a is obtained from C by a transformation of type 3 (i.e., a is not a loop which bounds a face and not a leaf). The other cases are treated in the same way. Denote the arrows of Q incident with a by α, β, γ, δ, so that αβ ≠ 0 and γδ ≠ 0. The elements of Λ_a are linear combinations of paths whose starting points and endpoints differ from a. It is clear that Λ_a is generated as algebra by idempotents e_i, where i ≠ a, by arrows of Q different from α, β, γ, δ and by the elements αβ, γδ. Observe that in terms of quivers Λ_a can be obtained from Λ in the following way: the arrows α and β, lying on a common A-cycle, are replaced with an arrow αβ on the same A-cycle (respectively, the arrows α and β are replaced with an arrow γδ). This implies the claim. □

We return to the proof of proposition [4.2.2]. Observe that the symmetric SB-algebra which corresponds to C(a)_a = C_a is isomorphic to Λ_a. To obtain the Brauer complex of Λ_a from C_a we need to add an edge on some face of C_a (the multiplicities of vertices are preserved).

It should be noted that all arrows of the quiver of Λ_a except at most two coincide with the respective arrows of the quiver of Λ_a. The arrows which don’t coincide, are products of two or three arrows of the quiver of Λ_a. Again, we finish the proof only for the case when C(a) is obtained from C by tilting transformation of type 3; the other cases are treated in the same way. For i = 1, 2 denote by b_i the edge, which precedes a_i on F_i in counter-clockwise order, i.e. b_i precedes a_i on a G-cycle (see notations in [4.2.3]). Denote by μ (by ρ) the arrow in Q_e which corresponds to the angle at vertex V'_1 included between a_1 and b_1 (resp., to the angle at V'_2 included between a_2 and b_2). Define elements α_1, β_1, γ_1, δ_1 ∈ End_{D^n(Λ)} T_a such that α_1 β_1 = μ, γ_1 δ_1 = ρ in End_{D^n(Λ)} T_a. Each of these elements is induced by a morphism between two indecomposable summands of T_a:

α_1: 
\[ \cdots \rightarrow 0 \rightarrow P_{b_1} \rightarrow 0 \rightarrow \cdots \]
\[ \downarrow \quad \downarrow^{(\rho)} \quad \downarrow \]
\[ \cdots \rightarrow 0 \rightarrow P_{a_1} \oplus P_{a_2} \underset{(\alpha, \gamma)}{\rightarrow} P_a \rightarrow 0 \rightarrow \cdots \]
The elements $\alpha_1, \beta_1, \gamma_1, \delta_1$ are not invertible, since for $i \neq a$ $H^*(T_{aa}) \neq H^*(T_{ai})$. Therefore these are the arrows $\mu$ and $\rho$ (in $\Lambda_a$) which are products of two arrows of $\Lambda_a$. Now observe that in terms of Brauer complexes, transformation of the quiver of $\Lambda_a$ to $\Lambda_a$ is insertion of edge labeled by $a$, incident with $V'_1$ and $V'_2$ into the union of faces $F_1$ and $F_2$.

**Corollary 4.6.** Let $\Lambda_1$ and $\Lambda_2$ be symmetric SB-algebras, let $C_1$ and $C_2$ be their Brauer complexes. Suppose that $C_2$ can be obtained from $C_1$ by a sequence of tilting transformations. Then $\Lambda_1$ and $\Lambda_2$ are derived equivalent.

**Proof.** The statement follows from Proposition 4.4, Lemma 4.5 and the Rickard’s Theorem.

**Example 4.7.** Consider decagons $D_1$ and $D_2$. Fix an orientation on each of decagons. Mark the edges of $D_1$ (of $D_2$) with letters $a, b, c, d, e$ so that they form a word $abcdeabcde$ (resp., $abcdeadebc$) in counter-clockwise order. In each decagon, identify the edges which are marked by the same letter in such way that the resulting manifolds are oriented. It’s easy to see that both complexes (we call them $C_1$ and $C_2$) have 2 vertices, 5 edges, one face, i.e. they are homeomorphic to a sphere with two handles. Moreover, the 1-skeletons of $C_1$ and $C_2$ are bipartite graphs. But these complexes cannot be obtained from each other by tilting transformations: any complex $C'$, obtained from the complex $C_1$, is isomorphic to $C_1$. This construction gives pairs of symmetric SB-algebras of genus 2, for which the methods given in present paper are not enough to determine whether they are derived equivalent or not.

## 5 Algebras of genus 0

Now we prove that if Brauer complex of $\Lambda$ is homeomorphic to a sphere, then the multiset of perimeters of its faces and the multiset of multiplicities of vertices determine the class of derived equivalence of $\Lambda$. For a start, we don’t take into consideration the multiplicities of vertices, i.e. we consider graphs with non-labeled vertices. We fix plane graphs $\Gamma_1$ and $\Gamma_2$ with the same multisets of perimeters of faces and show that $\Gamma_2$ can be obtained from $\Gamma_1$ by a sequence of tilting transformations (statements from Lemma 5.2 to Proposition 5.18).
**Definition 5.1.** Graphs which can be obtained from each other by a sequence of tilting transformations will be called *chain equivalent* graphs.

**Lemma 5.2.** Let $\Gamma$ be a plane graph, $A \in V(\Gamma)$. There exists a plane graph $\Gamma'$, chain equivalent to $\Gamma$, in which the vertex $A$ is incident with all edges and one of the following conditions holds:

1. $\Gamma'$ has no loops
2. Each edge of $\Gamma'$ is either a leaf or a loop at vertex $A$ (i.e., there are no multiedges in $\Gamma'$ except for loops).

**Definition 5.3.** Plane graph of this form is called a *reduced graph*.

**Proof.** Consider among graphs, which are chain equivalent to $\Gamma$, a graph $\Gamma'$ with a maximal degree of $A$. Observe that all edges of $\Gamma'$ are incident with $A$. Indeed, otherwise there are vertices $B, C \neq A$ and an edge $e \in E(B, C)$ such that either $B$ or $C$ is incident with $A$ (without loss of generality, $B$) and such that the edge $\pi_B(e) \in E(A, B)$. If $B \neq C$, we apply to $e$ a transformation of type 3. If $B = C$, we apply to $e$ a transformation of type 2 so that $e$ shifts from $B$ to $A$. Thus the degree of $A$ can be increased, a contradiction. It follows that there are three types of edges in $\Gamma'$:

a) a loop at vertex $A$;

b) edges which form a multiedge incident with $A$;

c) a leaf $(A, X)$.

For further convenience, elements of type a) don’t belong to type b). We show that in $\Gamma'$ edges of types a) and b) cannot exist simultaneously. Suppose that there is a loop $a$, leaves $a_1 = \pi_A(a)$, $a_2 = \pi_A(a_1)$, $\ldots$, $a_s = \pi_A(a_{s-1})$ and an edge $b = \pi_A(a_s)$ of type b). Consider the edge $c = \pi_B(b)$. By transformations of type 1, we shift $a_1, \ldots, a_s$ along $a$. Now there are no edges between $a$ and $b$ around $A$, and we can apply a transformation of type 3 to the edge $b$ and $b$ becomes a loop. This increases the degree of $A$, a contradiction. □

**Definition 5.4.** A reduced graph which has no loops is called a *reduced graph of type 1*.

Observe that the border of any face of a reduced graph of type 1 is formed by several pairs of edges $(A, B_1), \ldots, (A, B_k)$ and by several leaves (any leaf is counted in the perimeter of the face twice). Observe that a reduced graph of type 1 is bipartite.

**Definition 5.5.** A reduced graph which has loops is called a *reduced graph of type 2*.

In a reduced graph of type 2, any edge which is not a leaf is a loop. Observe that a reduced graph of type 2 is not bipartite.

Let $\Gamma'_1$ and $\Gamma'_2$ be reduced graphs, chain equivalent to $\Gamma_1$ and to $\Gamma_2$, respectively. By Proposition 3.7, $\Gamma'_1$ and $\Gamma'_2$ are of the same type. We will show that all reduced graphs of the same type, with the same multisets of perimeters of faces, are chain equivalent:

I. **Reduced graphs of type 1.** Fix a graph $\Gamma$ of type 1.

**Lemma 5.6.** Each reduced graph $\Gamma$ of type 1 is chain equivalent to a reduced graph $\Gamma'$ of type 1, which has at most two non-dangling vertices.
Proof. Consider among reduced graphs, which are chain equivalent to \( \Gamma \), a graph \( \Gamma' \) with maximal number of dangling vertices. Let \( A \) be the vertex of \( \Gamma' \), which is incident with all edges. We show that \( \Gamma' \) has at most two non-dangling vertices (including \( A \)). Indeed, let \( b \in E(A, B) \) and \( c \in E(A, C) \) be two edges of type 2 \( (B \neq C) \) such that there are only leaves between \( b \) and \( c \) in clockwise order around \( A \). As above, by transformations of type 1 we obtain a graph, in which there are no leaves between \( b \) and \( c \) (around \( A \)). Suppose that \( C \) has degree 2. Applying the transformation of type 3 to \( c \) (shift along \( b \)), we get a reduced graph with a greater number of leaves, since \( C \) becomes a leaf. In order to transform \( C \) to a leaf when \( \deg(C) = r \), we need to carry out the same operations with \( r - 1 \) edges, which are incident with \( C \).

Consider a reduced graph \( \Gamma' \) which was obtained in lemma 5.6. It is easy to see that the faces of \( \Gamma' \) and the edges of \( \Gamma' \) which are not leaves can be cyclically numbered by 1, 2, \ldots, \( g \) so that the border of the face number \( i \) consists of the edges number \( i \) and \( i + 1 \) and several inner leaves. It should be mentioned that if \( g = 1 \) then \( \Gamma' \) is a tree in a form of star, and we get Brauer trees, which were studied by Rickard in [7], as a first application of the criterion of derived equivalence.

**Lemma 5.7.** Graph \( \Gamma' \) is chain equivalent to a graph of the same form (i.e., as in lemma 5.6), in which the perimeters of faces are in ascending ordering.

**Proof.** It’s enough to show how to ‘transpose’ two faces, see Figure 4.

![Figure 4](image)

Figure 4: to Lemma 5.7

We see that any bipartite plane graph is chain equivalent to a (unique) canonical representative (we will also say “a graph in canonical form”) — a graph in which the perimeters of faces are in ascending ordering. Two graphs with the same multisets of perimeters are chain equivalent to the same canonical representative, and therefore they are chain equivalent to each other.

**II. Reduced graphs of type 2.**

Consider a reduced graph \( \Gamma \) of type 2. First suppose that \( A \) is the only vertex of \( \Gamma \), i.e. all edges of \( \Gamma \) are loops and \( n = g - 1 \), where \( n \) is the number of vertices of \( Q_e \) and \( g \) is the number of \( G \)-cycles. Consider a graph \( T = T(\Gamma) \), which is plane dual to \( \Gamma \). \( T \) is a tree with \( g - 1 \) edges and \( g \) vertices. Observe that the transformations of type 1 cannot be applied to \( \Gamma \). The transformations of types 2 and 3 can be described in terms of \( T \) as follows.
• Transformation of type 2. A leaf $V_1V_2$ of $T$ (with dangling vertex $V_1$) is shifted around $V_2$ in arbitrary way. This transformation of a plane labeled tree will be called a flip-over.

• Transformation of type 3. Suppose that $\pi^{-1}_V(V_1V_2) = V_3V_4$ and that $\pi^{-1}_V(V_1V_2) = V_1V_3$. Replace edges $V_1V_3$ and $V_2V_4$ with edges $V_1V_4$ and $V_2V_3$ in a way that $\pi_1(V_1V_2) = V_1V_4$ and $\pi_2(V_1V_2) = V_2V_3$. This transformation of a plane labeled tree will be called a flip (see Figure 5; an arc between two edges in the pictures denotes absence of other edges).

Figure 5: Flip

Definition 5.8. Plane trees with labeled vertices, which can be obtained from each other by flips and flip-overs, are called equivalent. Clearly, equivalent trees are dual to chain-equivalent graphs.

Proposition 5.9. Two plane trees with the same multisets of labeled vertices and the same degrees of correspondent vertices are equivalent.

We need the following lemma.

Lemma 5.10. Let $V_1V_2$ be a leaf in a plane tree $T$ with dangling vertex $V_1$. Let $V_1, V_2, \ldots, V_r$ be a path in $T$ such that $V_r$ is an non-dangling vertex. Then $T$ is equivalent to a tree, in which $V_1$ is adjacent with $V_r$.

Proof. The proof is by induction on $r$. For $r = 2$ the claim is trivial. Suppose that there is a number $i \in \{1, \ldots, r\}$ such that $\deg(V_i) \geq 3$. Consider the minimal such $i$. Without loss of generality we assume that $V_{i+1} \neq V$, where $V$ is such vertex that $\pi_{V_i}(V_{V_{i-1}}) = V_iV$. If $i \neq 2$, replace edges $V_{i-1}V_{i-2}$ and $V_iV$ with $V_{i-1}V_{i-2}$ and $V_{i-1}V$ by a flip. Otherwise, we make $V_2V_3$ follow $V_2V_1$ by several flip-overs, and then make the above flip. The distance between $V_1$ and $V_r$ decreases, and we apply the inductive hypothesis.

Now we prove Proposition 5.9.

Proof. The proof is by induction on the number of vertices. For $g = 1$ the claim is trivial. Let $T_1$ and $T_2$ be two plane trees with $g$ vertices. Let a dangling vertex $V$ be adjacent with $V_1$ in $T_1$ and with $V_2$ in $T_2$. By Lemma 5.10 we can replace $T_1$ with an equivalent tree $T_3$ in which $V$ is adjacent with $V_2$. Let $T_3^1$ and $T_2^1$ be the trees, obtained from $T_3$ and $T_2$ by removing $V$ with
the corresponding edge. They have the same degrees of correspondent vertices, and therefore
they are equivalent by inductive hypothesis. It remains to show that it is still possible to carry
out the sequence of transformations, which puts $T^3_3$ to $T^3_2$, when edge $V_2V$ is not deleted. After
these transformation we will be able to flip-over the edge $V_2V$ to the required place.

Start to apply the above sequence of transformations to $T_3$. We can encounter difficulties in
the following cases:

- When in $T_3$ the edge $V_2V$ is between two subsequent edges (around $V_2$) of $T^3_3$ and doesn’t
  allow to make a flip. We cope with this by an arbitrary flip-over of $V_2V$.

- If $V_2$ is a dangling vertex in $T^3_3$, incident with an edge $V_2V_3$, and in $T^3_3$ it is possible to
  make a flip-over of $V_2V_3$. In $T_3$ instead of this flip-over we make the following sequence of
  transformations (Figure 6).

Figure 6: to Proposition 5.9

This finishes the proof.

Now suppose that there are dangling vertices in $\Gamma$.

**Definition 5.11.** External perimeter of a face is the number of its edges, which separate it
from other faces (in our case, these are loops).

**Definition 5.12.** Reduction of graph $\Gamma$ is a graph $R(\Gamma)$ which is obtained from $\Gamma$ by removing
all dangling vertices.

**Proposition 5.13.** Let $\Gamma_1$ and $\Gamma_2$ be reduced graphs of type 2. Suppose that there is a tilting
transformation $p$ which puts $R(\Gamma_1)$ to $R(\Gamma_2)$. Suppose also that the correspondent labeled faces
of $\Gamma_1$ and of $\Gamma_2$ have the same number of edges. Then graphs $\Gamma_1$ and $\Gamma_2$ are chain equivalent.

**Proof.** Let $l$ be the loop, which is shifted by $p$ and let $F_1$ and $F_2$ be the faces separated by $l$.
We need to obtain a sequence of transformations which would serve as an analogue of $p$ for $\Gamma_1$.
Figure 7 illustrates the case when $F_1$ has inner leaves and $l$ is the only loop on the border of
$F_1$. The case when there are other loops on the border of $F_1$ is even easier: the analogue of $p$ is
a transformation of type 3, made after necessary flip-overs of leaves.
Remark 5.14. It follows from Propositions 5.9 and 5.13 that the class of chain equivalence of a reduced graph of type 2 is determined by the multiset of pairs \((P(F_i), p(F_i))\), where \(P(F_i)\) is the perimeter and \(p(F_i)\) is the external perimeter of the face \(F_i\).

Definition 5.15. The multiset of pairs \((P(F_i), p(F_i))\) will be called a multiset of double perimeters of graph \(\Gamma\).

Proposition 5.16. Let \(\Gamma_1\) and \(\Gamma_2\) be reduced graphs of type 2 with the same multisets of perimeters of faces. Then there exists a reduced graph \(\Gamma_3\) of type 2, chain equivalent to \(\Gamma_1\), such that the multisets of double perimeters of \(\Gamma_2\) and \(\Gamma_3\) are the same.

Proof. Let \(\{(P_i, p_i)\}\) be the multiset of double perimeters of \(\Gamma_2\), let \(\{(P_i, p_i^1)\}\) be the multiset of double perimeters of \(\Gamma_1\), for \(i = 1, \ldots, g\). Observe that \(p_i \equiv P_i \equiv p_i^1 \pmod{2}\) for each \(i \in \{1, \ldots g\}\) and that \(\sum_i p_i = 2g - 2 = \sum_i p_i^1\). Set \(q_i = p_i^1\) for each \(i\). Consider the following algorithm of 'transformation' of the multiset \(\{q_i\}\) to the multiset \(\{p_i\}\). Below we will show that for each step of this algorithm there is a chain equivalence of graphs, which properly changes their external perimeters.

Consider maximal \(k\) such that \(q_i = p_i\) for all \(i < k\).

1. If \(q_k < p_k\) then \(q_j > p_j\) for some \(j > k\). Replace \(q_k\) with \(q_k + 2\) and replace \(q_j\) with \(q_j - 2\).
2. Otherwise \(q_k > p_k \geq 1\) and \(q_j < p_j\) for some \(j > k\). In this case we replace \(q_k\) with \(q_k - 2\) and replace \(q_j\) with \(q_j + 2\).

Observe that at each step the number which is decreased is greater than two, so the resulting numbers are positive. Moreover, since \(q_i \leq \max(p_i, p_i^1)\), at each step \(q_i \leq P_i\) for all \(i\). Clearly, the multiset of numbers \(q_i\) can be transformed to the multiset of numbers \(p_i\) by these operations. To find the chain equivalences which correspond to these operations, we need the following lemma.

Lemma 5.17. Let \(T\) be a tree, let \(V_1, V_2 \in V(T)\). If \(V_1\) and \(V_2\) are not both dangling vertices, then there exists a tree in which the degrees of all vertices are the same and the vertices \(V_1\) and \(V_2\) are adjacent.

Proof. The proof is by induction on the number of vertices in \(T\).

We return to the proof of Proposition 5.16. We need a sequence of tilting transformations under which the multiset of external perimeters changes in accordance to the above algorithm.
Suppose that we are to change the external perimeters $q_i$ and $q_k$ of faces $F_i$ and $F_k$, respectively, in a graph $\Gamma$. By Lemma 5.17 and Remark 5.14, $\Gamma$ can be transformed to a chain equivalent graph $\Gamma'$ with the same multiset of double perimeters, such that in the dual tree $T(\Gamma')$ the vertices of degrees $q_i$ and $q_k$ are adjacent. Without loss of generality, we are to increase $q_i$. In this case $q_i < P_i$ and $q_k \geq 3$. Consider faces $F_1$ and $F_2$ of $\Gamma'$ which can be described in terms of dual tree $T(\Gamma')$ as follows: $F_1 = \pi_{F_k}^{-1}F_i$, $F_2 = \pi_{F_k}^{-1}F_1$ (all faces $F_1$, $F_2$ and $F_i$ are different, since $q_k \geq 3$). Since $q_i < P_i$, there is at least one leaf in $F_i$. The following sequence of transformations finishes the proof (see Figure 8 in the picture the shifts of leaves are omitted). Thus $q_i$ is increased by 2 and $q_k$ is decreased by 2, which was required.

Altogether, we get

Proposition 5.18. Two plane graphs with the same multiset of perimeters of faces are chain equivalent.

Now we again consider graphs with labeled vertices, i.e., we return the multiplicities of vertices into consideration. In statements from Lemma 5.19 to Theorem 5.22 we prove that if two plane Brauer graphs with the same multisets of labels of vertices are isomorphic as non-labeled graphs, then they are chain equivalent as labeled graphs. In view of the above arguments, it’s enough to prove this for reduced graphs. Moreover, in the case of bipartite graphs we may restrict ourselves to considering graphs in canonical form. Recall that the process of putting a graph to reduced form (and to canonical form, for graphs of type 1) started with choosing an arbitrary vertex $A$. Recall also that we can arbitrarily shift leaves in a face, by tilting

I. Reduced graphs of type 1. For reduced graphs of type 1, it suffices to prove the following lemmas:

Lemma 5.19. Let $\Gamma$ be a graph in canonical form, let $B \neq A$ be the second non-dangling vertex of $\Gamma$, let $F$ be a face. Then $\Gamma$ is chain equivalent to a graph in canonical form, in which

1. $B$ is a dangling vertex in the face $F$.

2. Some vertex $C \neq A$ which belongs in $\Gamma$ to $F$ is a non-dangling vertex.

3. The other dangling vertices belong in $\Gamma$ and in $\Gamma_1$ to the same faces.
Lemma 5.20. Let $\Gamma$ be a graph in canonical form, let $B \neq A$ be the second non-dangling vertex of $\Gamma$, let faces $F_1$ and $F_2$ be adjacent. Then $\Gamma$ is chain equivalent to a graph in canonical form $\Gamma_1$, in which

1. There is a dangling vertex which belongs to $\Gamma$ to $F_1$ and belongs to $\Gamma_1$ to $F_1$, and there is another dangling vertex which belongs to $\Gamma$ to $F_2$ and in $\Gamma_1$ to $F_2$.

2. The other dangling vertices belong in $\Gamma$ an in $\Gamma_1$ to the same faces.

For the proof of Lemma 5.19 see Figure 9. For the proof of Lemma 5.20 see Figure 10.

![Figure 9: Proof of Lemma 5.19](image1)

![Figure 10: Proof of Lemma 5.20](image2)

II. Reduced graphs of type 2. Since the dangling vertices in a face can be shifted in arbitrary way, it’s enough to show how to interchange dangling vertices belonging to different faces (say, to $F_1$ and $F_2$). First consider the case when the external perimeter of $F_1$ or $F_2$ is greater then 1. Then by Lemma 5.17 and Remark 5.14, there is a sequence of tilting transformations making $F_1$ and $F_2$ adjacent. Moreover, this sequence preserves the faces to which belong the dangling vertices (see Figure 7). Therefore, in this case it’s enough to show how to interchange dangling vertices which belong to adjacent faces: see Figure 11. Now consider the case when the dangling vertices which we want to interchange belong to faces, which correspond to dangling vertices of the dual tree.

Lemma 5.21. Let $T$ be a plane tree with labeled vertices, let $V_1$ and $V_2$ be dangling vertices of $T$. Suppose that $T$ is not a chain. Then $T$ is equivalent to a tree, in which the edges which are incident with $V_1$ and $V_2$ are incident to a common vertex $V$. Moreover, $\pi_T(VV_1) = VV_2$. 
Proof. By Remark 5.14 it is enough to find a tree with the same multiset of degrees as in $T$, in which some two leaves are adjacent to a common vertex. Denote the degrees of $T$ by $r_1, \ldots, r_g$ in such way that $r_1 = r_2 = 1, r_3 \geq 3$. Observe that the sum of numbers $d_3 - 2, d_4, \ldots, d_g$ equals $2g - 6$. It can be shown by induction on $g$ that there is a tree $T'$, in which these numbers are the degrees of vertices. To obtain the needed tree, we add two leaves to the vertex of $T'$ of degree $r_3 - 2$.

We see that if $T$ is not a chain, then it suffices to show how to interchange dangling vertices between two "dangling" faces, which have a common adjacent face: see Figure 12. It remains to examine the case when $T(\Gamma)$ is a chain, and $F_1$ and $F_2$ correspond to the two dangling vertices of $T(\Gamma)$. If some other face of $\Gamma$ contains a dangling vertex, the needed interchange comes to three interchanges of the above form. In Figure 13 is is shown how to interchange leafs in case when the rest faces have perimeter 2. (For the graph in the picture $g = 4$, and this case fully represents the general case.)

This finishes the proof of the main theorem in this section:

**Theorem 5.22.** Let $\Lambda_1$ and $\Lambda_2$ be symmetric SB-algebras of genus 0. Then $\Lambda_1$ and $\Lambda_2$ are derived equivalent if and only if their Brauer complexes have the same multisets of perimeters of faces and the same multisets of labels on vertices.
Figure 13: Chain case

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