Scalable Quantitative Verification For Deep Neural Networks

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Abstract
Verifying security properties of deep neural networks (DNNs) is becoming increasingly important. This paper introduces a new quantitative verification framework for DNNs that can decide, with user-specified confidence, whether a given logical property $\psi$ defined over the space of inputs of the given DNN holds for less than a user-specified threshold, $\theta$. We present new algorithms that are scalable to large real-world models as well as proven to be sound. Our approach requires only black-box access to the models. Further, it certifies properties of both deterministic and non-deterministic DNNs. We implement our approach in a tool called PROVERO.

We apply PROVERO to the problem of certifying adversarial robustness. In this context, PROVERO provides an attack-agnostic measure of robustness for a given DNN and a test input. First, we find that this metric has a strong statistical correlation with perturbation bounds reported by 2 of the most prominent white-box attack strategies today. Second, we show that PROVERO can quantitatively certify robustness with high confidence in cases where the state-of-the-art qualitative verification tool (ERAN) fails to produce conclusive results. Thus, quantitative verification scales easily to large DNNs.

1 Introduction

The past few years have witnessed an increasing adoption of deep neural networks (DNNs) in domains such as autonomous vehicles [4,28,42], drones [20] or robotics [15,55], where mis-predictions can have serious long-term consequences. Consequently, robustness, privacy, and fairness have emerged as central concerns to be addressed for safe adoption of DNNs. This has led to a surge of interest in testing and verification of neural networks for properties of interest.

To establish that the resulting DNNs have the desired properties, a large body of prior work has focused on techniques based on empirical testing [43,46,54] or specialized attack vectors [1,29,34,38,49]. While such techniques are useful, they do not rigorously quantify how sure we can be that the desired property is true after testing.

In contrast to prior testing approaches, formal verification seeks to provide rigorous guarantees of correctness. Inspired by the success of model checking in the context of hardware and software verification, the earliest formal verification methodologies in the context of deep neural networks focused on qualitative verification, i.e., whether a system satisfies a given specification. Prior work in this category has been following the model checking paradigm wherein a given DNN is encoded as a model using constraints grounded in a chosen theory. Then a satisfiability solver (often modulo the chosen theory) is invoked to check if there exists an execution of the system that violates the given specification [13,22,31,35].

The proposed techniques in this category appear to have three limitations. Firstly, they require white-box access to the models and specialized procedures to transform the DNNs to a specification, limiting their generality. Secondly, the performance of the underlying feasibility solver degrades severely with the usage of non-linear constraints, leading to analyses that do not scale to larger models. Thirdly, prior techniques are limited to deterministic neural networks, while extensive research effort has been invested in designing randomized DNNs, especially to enhance robustness [10,11,26].

Such qualitative verification considers only two scenarios: either a DNN satisfies the property, or it does not. However, neural networks are stochastically trained, and more importantly, they may run on inputs drawn from an unknown distribution at inference time. Properties of interest are thus often probabilistic and defined over an input distribution (e.g., differential privacy and fairness). Hence, qualitative verification is not suitable for such properties.

An alternative approach is to check how often a property is satisfied by a given DNN under a given input distribution. More specifically, one can assert that a DNN satisfies a property $\psi$ with a desirably high probability $1-\delta$. Unlike ad-hoc testing, quantitative verification [2] aims to provide soundness, i.e., when it confirms that $\psi$ is true with probability $p$, then the claim can be rigorously deduced from the laws of probability. For many practical applications, knowing that the chance of failure is controllably small suffices for deployment.
For instance, it has been suggested that road safety standards for self-driving cars specify sufficiently low failure rates of the perceptual sub-systems, against which implementations can be verified [21, 24, 45]. Further, we show the role of quantitative verification in the specific context of adversarial robustness (see Section 2.1).

In this paper, we present a new quantitative verification algorithm for DNNs called PROVERO, tackling the following problem: Given a logical property \( \psi \) specified over a space of inputs and outputs of a DNN and a numerical threshold \( \theta \), decide whether \( \psi \) is true for less than \( \theta \) fraction of the inputs. PROVERO is a procedure that achieves the above goal with proven soundness: When it halts with a ‘Yes’ or ‘No’ decision, it is correct with probability \( 1 - \delta \) and within approximation error \( \eta \) to the given \( \theta \). The verifier can control the desired parameters \( (\eta, \delta) \), making them arbitrarily close to zero. That is, the verifier can have controllably high certainty about the verifier’s output, and \( \theta \) can be arbitrarily precise (or close to the ground truth). The lower the choice of \( (\eta, \delta) \) used by the verifier, the higher is the running time.

PROVERO is based on sampling, and it makes only one assumption—the ability to take independent samples from the space over which \( \psi \) is defined\(^1\). This makes the verification procedure considerably general and stand-alone. The verifier only needs black-box access to the DNN, freeing it up from assuming anything about the internal implementation of the DNNs. The DNN can be deterministic or from a general family of non-deterministic DNNs. This allows checking probabilistic properties of deterministic DNNs and of randomization procedures defined over DNNs.

**Results.** We demonstrate that PROVERO has practical utility for certifying adversarial robustness. Suppose the prover claims that a DNN has at most \( 10^{-3} \) adversarial inputs in a certain space of inputs (e.g., in the \( L_p(\varepsilon, I) \) ball of a given image \( I )\). PROVERO provides a procedure that can verify the prover’s claim that the true ratio of adversarial examples is upper bounded by a quantity \( \theta \) (e.g., \( 10^{-3} \frac{\|I\|L_p}{\|I\|L_p(\varepsilon, \theta)} \)) with arbitrarily high precision, and with arbitrarily high certainty.

Based on this capability, we show a new metric of robustness called adversarial hardness, which can be computed using PROVERO with only black-box access. Adversarial hardness is an attack-agnostic metric for comparing the robustness of (say) two DNNs on a given input. We find that this metric is strongly correlated with the efficacy of today’s most prominent white-box attacks (c.f. PGD [29] and C&W [8]) on over 38 DNNs, including large ones such as ResNet and Inception. Both these white-box attacks very often do better when adversarial hardness metric is low and worse when it is high. This connection suggests that, when comparing DNN robustness, future evaluation of robustness could use adversarial hardness as a quantitative attack-agnostic metric, complementing the existing use of attacks.

Second, PROVERO scales to large DNNs. We compare PROVERO to ERAN [39], presently the most scalable state-of-the-art qualitative verification tool. We find that ERAN does not scale to large models (e.g., VGG16 on ImageNet), for which PROVERO quantitatively verifies in 2000 seconds. For smaller models where ERAN terminates, it often does not conclusively certify the absence of adversarial examples above small perturbation size (e.g., 0.03 on MNIST with \( L_\infty \) norm). PROVERO quantitatively certifies with high certainty for these models within a comparable running time. This explains how quantitative verification can complement qualitative verification by generating sound and high-confidence certificates, especially in cases where qualitative analysis fails to provide any conclusive results.

Lastly, PROVERO can quantitatively verify deterministic DNNs as well as randomized DNNs having only black-box access. As a case study, we demonstrate PROVERO on models that internally use randomization to improve as suggested in PixelDP [26]. PROVERO’s quantitative results are consistent with the qualitative robustness certificates that PixelDP itself generates. Further, we can calculate the adversarial hardness metric: the largest perturbation bound below which the model has very low adversarial example density (i.e. \( \theta \leq 10^{-2} \)) using PROVERO. We find that adversarial hardness in the PixelDP model is \( 5 \times \) larger than its qualitatively certified bounds.

**Contributions.** We claim the following contributions:

- We present a new quantitative verification algorithm for neural networks. The framework is fairly general: It assumes only black-box access to the model being verified, and assumes only the ability to sample from the input distribution over which the property is asserted.

- We implement our approach in a tool called PROVERO that embodies sound algorithms and scales quantitative verification for adversarial robustness to large real-world DNNs, both deterministic and randomized, within 1 hour.

- In the context of certifying adversarial robustness, our empirical evaluation presents a new attack-agnostic measure of robustness and shows that PROVERO can produce certificates with high confidence on instances where existing state-of-the-art qualitative verification does not provide conclusive results.

### 2 Application: Adversarial Robustness

For concreteness, we apply our approach to verifying the robustness of neural networks. In proving robustness, the analyst has to provide a space of inputs over which the robustness holds. Often, this space is defined by all inputs that are within a perturbation bound \( \varepsilon \) of a given input \( x \) in the \( L_p \) norm [16]. Different distance norms have been
used such as $L_0$, $L_1$, $L_2$ and $L_{\infty}$. The $L_p$ norm is defined as $||x - x'||_p = (|x'_1 - x_1|^p + |x'_2 - x_2|^p + \ldots + |x'_n - x_n|^p)^{1/p}$. A neural network $f$ is defined to be robust with respect to a given input $x$ if $\forall x'$ such that $||x - x'||_p < \epsilon$, we have $f(x) = f(x')$.

For a given neural network $f$ and input point $x$, there always exists some perturbation size beyond which there are one or more adversarial samples. We refer to this minimum perturbation with non-zero adversarial examples as $\epsilon_{\text{min}}$, which is the ground truth the security analyst wants to know. Attack procedures are best-effort methods which find upper bounds for $\epsilon_{\text{min}}$ but cannot provably show that these bounds are tight [1,7,48]. Verifications procedures aim to prove the absence of adversarial examples below a given bound, i.e., they can establish lower bounds for $\epsilon_{\text{min}}$. We call such verified lower bounds $\epsilon_{\text{ver} f}$. Most verifiers proposed to date for robustness checking are qualitative, i.e., given a perturbation size $\epsilon_{\text{ver} f}$, they output whether adversarial examples are absent within $\epsilon_{\text{ver} f}$. If the verification procedure is sound and outputs ‘Yes’, then it is guaranteed that there are no adversarial examples within $\epsilon_{\text{ver} f}$, i.e., the robustness property is satisfied. When the verifier says ‘No’, if the verifier is complete, then it is guaranteed that there are indeed adversarial examples within $\epsilon_{\text{ver} f}$. If the verifier is incomplete and prints ‘No’, the result is inconclusive.

Let us introduce a simple quantitative measure of robustness called the adversarial density. Adversarial density is the fraction of inputs around a given input $x$ which are adversarial examples. We explain why adversarial density is a practically useful quantity and much easier to compute for large DNNs than $\epsilon_{\text{min}}$. We can compute perturbation bounds below which the adversarial density is non-zero but negligibly small, and we empirically show these bounds are highly correlated with estimates of $\epsilon_{\text{min}}$ obtained by state-of-the-art attack methods.

### 2.1 Minimum Perturbation vs. Density

It is reasonable to ask why adversarial density is relevant at all for security analysis. After all, the adversary would exploit the weakest link, so the minimum perturbation size $\epsilon_{\text{min}}$ is perhaps the only quantity of interest. We present concrete instances where adversarial density is relevant.

First, we point to randomized smoothing as a defense technique, which has provable guarantees of adversarial robustness [10,11,26,53]. The defense uses a “smoothed” classifier $g$ that averages the behavior of a given neural net $f$ (called the base classifier) around a given input $x$. More specifically, given a base classifier $f$, the procedure samples perturbations of $x$ within $\epsilon$ from a specific noise distribution and outputs the most likely class $c$ such that $\text{argmax}_{c \in \mathcal{G}} \mathbb{P}_\epsilon[f(x + \epsilon) = c]$. Notice that this procedure computes the probability of $f$ returning class $c$—typically by counting how often $f$ predicts class $c$ over many samples—rather than considering the “worst-case” example around $x$. Said another way, these approaches estimate the adversarial density for each output class under some input distribution. Therefore, when selecting between two base classifiers during training, we should pick the one with the smallest adversarial density for the correct class, irrespective of their minimum adversarial perturbation size.

To illustrate this point, in Figure 1 we show two DNNs $f_1$ and $f_2$, as potential candidates for the base classifier in a randomized smoothing procedure. Notice that $f_1$ has a better (larger) minimum perturbation than $f_2$, but has a worse (larger) adversarial density because a majority of points within $\epsilon$ distance of $x$ are in the incorrect classification region. Therefore, the smoothened version of $f_2$ will classify $x$ correctly, while the smoothened $f_1$ will not. Picking the base classifier with the better adversarial density, rather than minimum perturbation, will lead to better accuracy in a smoothening defense.

![Figure 1: The decision boundaries of two binary classifiers $f_1$ (left) and $f_2$ (right) around an input $x$ are shown. The correct classification region for $x$ is shown in purple (solid), while the incorrect classification region is shown in red (hashed). The concentric circles show the equi-distant points from $x$ in $L_2$-norm drawn up to a bound $\epsilon$. The classifier $f_1$ has a better (larger) minimum perturbation than $f_2$, but has a worse (larger) adversarial density because a majority of points within $\epsilon$ distance of $x$ are in the incorrect classification region. Therefore, the smoothened version of $f_2$ will classify $x$ correctly, while the smoothened $f_1$ will not. Picking the base classifier with the better adversarial density, rather than minimum perturbation, will lead to better accuracy in a smoothening defense.](image-url)
the model internals and work only for deterministic neural networks. We show in this work that verifying adversarial density bounds is easy to compute even for large DNNs. We describe procedures that require only black-box access, the ability to sample from desired distributions and hence are attack-agnostic.

In particular, we show an empirical attack-agnostic metric for estimating robustness of a given DNN and input \( x \) called adversarial hardness. It is highest perturbation bound for which the adversarial density is below a suitably low \( \theta \). We can search empirically for the highest perturbation bound \( \epsilon_{\text{hard}} \), called the adversarial hardness, for which a sound quantitative certifier says ‘Yes’ when queried with \( (f, x, \epsilon_{\text{hard}}, \theta, \delta, \eta) \)—implying that \( f \) has suitably low density of adversarial examples for perturbation bounds below \( \epsilon_{\text{hard}} \).

Adversarial hardness is a measure of the difficulty of finding an adversarial example by uniform sampling. Surprisingly, we find that this measure strongly correlates with perturbation bounds produced by prominent white-box attacks (see Section 7). Given this strong correlation, we can effectively use adversarial hardness as a proxy for perturbation sizes obtained from specific attacks, when comparing the relative robustness of two DNNs.

We caution readers that adversarial hardness is a quantitative measure and technically different from \( \epsilon_{\text{min}} \), the distance to the nearest adversarial example around \( x \). But both these measures provide complementary information about the concentration of adversarial examples near a perturbation size.

## 3 Problem Definition

We are given a neural network and a space of its inputs over which we want to assert a desirable property \( \psi \) of the outputs of the network. Our framework allows one to check whether \( \psi \) is true for some specified ratio \( \theta \) of all possible values in the specified space of inputs. For instance, one can check whether most inputs, a sufficiently small number of inputs, or any other specified constant ratio of the inputs satisfies \( \psi \). The specified ratio parameter \( \theta \) is called a threshold.

Formally, let \( \psi(I, f, \Phi) \in \{0, 1\} \) be a property function over a neural network \( f \), a subset of all possible inputs, \( I \), to the neural network and user-specified parameters, \( \Phi \). We assume that we can efficiently draw samples from any probability distribution over \( I \) that the analyst desires. For a given distribution \( D \) over \( I \), let \( p_D = E_{x \sim D}[\psi(x, f, \Phi)] \). \( p_D \) can be viewed as the probability that \( \psi \) evaluates to 1 for \( x \) sampled from \( I \) according to \( D \). When clear from context, we omit \( D \) and simply use \( p \) to refer to \( p_D \).

Ideally, one would like to design an algorithm that returns ‘Yes’ if \( p \leq \theta \) and ‘No’ otherwise. Such exact quantification is intractable, so we are instead interested an algorithm \( A \) that returns ‘Yes’ if \( p \leq \theta \) and ‘No’ otherwise, with two controllable approximation parameters \((\eta, \delta)\). The procedure should be theoretically sound, ensuring that when it produces ‘Yes’ or ‘No’ results, it is correct with probability at least \( 1 - \delta \) within an additive bound on its error \( \eta \) from the specified threshold \( \theta \). Specifically, we say that algorithm \( A \) is sound if:

\[
Pr[A(\psi, \theta, \eta, \delta)] \text{ returns Yes } | p \leq \theta \geq 1 - \delta
\]

\[
Pr[A(\psi, \theta, \eta, \delta)] \text{ returns No } | p > \theta + \eta \geq 1 - \delta
\]

The analyst has arbitrary control over the confidence \( \delta \) about \( A \)’s output correctness and the precision \( \eta \) around the threshold. These values can be made arbitrarily small approaching zero, increasing the runtime of \( A \). The soundness guarantee is useful—it rigorously estimates how many inputs in \( I \) satisfy \( \psi \), serving as a quantitative metric of satisfiability.

The presented framework makes very few assumptions. It can work with any specification of \( I \), as long as there is an interface to be able to sample from it (as per any desired distribution) efficiently. The neural network \( f \) can be any deterministic function. In fact, it can be any “stateless” randomized procedure, i.e., the function evaluated on a particular input does not use outputs produced from prior inputs. This general class of neural networks includes, for instance, Bayesian neural networks [18] and many other ensembles of neural network architectures [3]. The framework permits specifying all properties of fairness, privacy, and robustness highlighted in recent prior work [2, 32], for a much broader class of DNNs.

Our goal is to present sound and scalable algorithms for quantitative verification, specifically targeting empirical performance for quantitatively certifying robustness of DNNs. The framework assumes black-box access to \( f \), which can be deterministic or non-deterministic. Our techniques can directly check qualitative certificates produced from randomized robustness-enhancing defenses, one example of which is the recent work called PixelDP [26] (see Section 7.3).

## 4 Approach Overview

### 4.1 Sampling

Given a property \( \psi \) over a samplable input space \( I \) and a neural network \( f \), our approach works by sampling \( N \) times independently from \( I \). We test \( f \) on each sample as input. Let \( X_i \) be a 0-1 random variable denoting the result of the trial with sample \( i \), where \( X_i = 1 \) if the \( \psi(x, f, \Phi) \) is true and \( X_i = 0 \) otherwise. Let \( X \) be the random variable denoting the number of trials in \( X_1, X_2, \ldots, X_N \) for which the property is true. Then, the standard Chernoff bounds (see [30]) given below form the main workhorse underlying our algorithms:

**Lemma 4.1** Given independent 0-1 random variables \( X_1, \ldots, X_N \), let \( X = \sum_{i=1}^{N} X_i \), \( \mu = \frac{E[X]}{N} \), and \( \hat{\mu} = \frac{X}{N} \). For \( 0 <
\[ \eta < 1: \]

\[ Pr[\hat{\theta} \geq \mu + \eta] \leq e^{\frac{-\eta^2}{2p}} \]

\[ Pr[\hat{\theta} \leq \mu - \eta] \leq e^{\frac{-\eta^2}{2p}} \]

Note that the probability \( p \) we are interested in bounding in our quantitative verification framework is exactly \( \mu \) in Lemma 4.1. More specifically, the probability depends on the neural network and distribution over the inputs, \( p = E_{x \sim D}[\psi(x,f,\Phi)] \), where \( D \) is a distribution over \( I \). Using a framework based on sampling and Chernoff bounds admits considerable generality. The only assumption in applying the Lemma 4.1 is that all samples are independent. If the neural network does not compute information during one trial (or execution under one sample) and use it in another trial, as is the case for all neural networks we study, trials will be independent. For any deterministic neural network, all samples are drawn independently and identically distributed in \( I \), so Chernoff bounds are applicable. For randomized DNNs, we can think of the \( i^{th} \) trial as evaluating a potentially different neural network (sampled from some distribution of functions) on the given sample. Here, the output random variables may not be identically distributed due to the randomization used by the neural network itself. However, Lemma 4.1 can still be used even for non-identically distributed trials but independent.

We discuss an estimation-based strategy that applies Chernoff bounds in a straight-forward manner next. Such a solution has high sample complexity for quantitative verification of adversarial robustness. We then propose hypothesis-based solutions which are sound and have much better empirical sample complexity for real-world DNNs. Our proposed algorithms still rely only on Chernoff-style bounds, but are carefully designed to internally vary parameters (on which Chernoff bounds are invoked) to reduce the number of samples needed to dispatch the asserted property.

### 4.2 An Estimation-based Solution

One way to quantitatively verify a property through sampling is to directly apply Chernoff bounds to the empirical estimate of the mean \( \hat{\theta} \) in \( N \) trials. The solution is to take \( N > \frac{12ln(1/\delta)}{\eta^2} \) samples, and decide ‘Yes’ if \( \hat{\theta} \leq \theta + \frac{\eta}{2} \) and ‘No’ if \( \hat{\theta} > \theta + \frac{\eta}{2} \). This is a common estimation approach, for instance used in the prediction step in the certified defense mechanism PixelDP [26]. By Lemma 4.1, one can show that \( \hat{\theta} \) is within \( \pm \frac{\eta}{2} \) additive error of \( \theta \) with confidence higher than \( 1 - \delta \). Therefore, the procedure is sound since \( Pr[\theta \notin [\hat{\theta} - \frac{\eta}{2}, \hat{\theta} + \frac{\eta}{2}]] \leq \delta \) by Lemma 4.1, for all \( 0 \leq \delta \leq 1 \).

In this solution, the number of samples increase quadratically with decreasing \( \eta \). For example, if the \( \theta = 0.1, \eta = 10^{-3}, \delta = 0.01 \), directly applying Chernoff bounds will require over \( 55 \times 10^6 \) samples. Even for an optimized architecture such as BranchyNet [44] that reports 70.9 ms on average inference time per sample for a ResNet (152 layers) the estimation approach would take more than 8 days on GPU clusters (such as our evaluation platform). This is a prohibitive cost. For randomized DNNs, which internally compute expectations, the runtime of the estimation baseline approach can be even larger. For example, the randomization used in PixelDP can have \( 3 - 42 \times \) inference overhead compared to deterministic DNNs [26].

Our work provides new algorithms that utilize much fewer samples on average. In the example of BranchyNet mentioned above, if the true probability \( p \) is 0.3, our approach requires 4246 samples to return a ‘No’ answer with confidence greater than 0.99. The main issue with the estimation algorithm is that it does not utilize the knowledge of the given \( \theta \) in deciding the number of samples it needs.

### 4.3 Our Approach

The number of samples needed for Chernoff bounds depend on how far is the true probability \( p \) that we are trying to bound from the given threshold \( \theta \). Intuitively, if \( p \) and \( \theta \) are far apart in the interval \([0,1]\), then a small number of samples are sufficient to conclude with high certainty that \( p \leq \theta \) or \( p > \theta \) (for small \( \eta \)). The estimation approach takes the same number of samples irrespective of how far \( p \) and \( \theta \) are. Our algorithms terminate quickly by checking for “quick-to-test” hypothesis early, yielding the sample complexity comparable to the estimation only in the worst case.

We propose new algorithms, the key idea of which is to use cheaper (in sample complexity) hypothesis tests to decide
‘Yes’ or ‘No’ early. Given the threshold $\theta$ and the error $\eta$, the high-level idea is to propose alternative hypotheses on the left side of $\theta$ and on the right side of $\theta + \eta$. We choose the hypotheses and a sampling procedure such that if any of the hypotheses on the left side of $\theta$ are true, then we can return ‘Yes’. Similarly, if any of the hypotheses on the right side of $\theta + \eta$ are true, then we can return ‘No’. Thus, we can potentially return much faster when $p$ is further from $\theta$.

The overall meta-level structure of our algorithms is simple and follows Algorithm 1, called METAPROVERO. We pick certain number of samples, estimate the ratio $\hat{p}$ (by invoking TESTER in lines 3 and 6), and check if we can prove that conditions involving certain intermediate thresholds ($\theta_1$ and $\theta_2$) are satisfied with the desired input parameters ($\eta, \delta$) using Chernoff bounds. If the check succeeds, the algorithm can return ‘Yes’ or ‘No’; otherwise, the process repeats until a condition which guarantees soundness is met.

The internal thresholds are picked so as to soundly prove or refute that $p$ lies in certain ranges in $[0, 1]$. This is done by testing certain intervals $p$ is (or is not) in. For instance, line 3 tests a pair of hypotheses $p \leq \theta_1$ and $p > \theta_2$ simultaneously. Notice that for the intervals on the left side of $\theta$ ($\theta_1 < \theta_2 \leq \theta$, line 2 in Alg. 1) can result in proving that $p \leq \theta_1$ with desired confidence $\delta$ and error tolerance $\eta$. In this case, since $\theta_1 < \theta$, we will have proved the original hypothesis $p \leq \theta$ and the algorithm can return ‘Yes’ soundly. We call such intervals, which are to the left of $\theta$, as proving intervals. Conversely, refuting intervals are on the right side of $\theta + \eta$, such as the choice of $\theta_1$ and $\theta_2$ on line 5 in Alg. 1, which are larger than $\theta + \eta$. When we can prove that $p > \theta_2$ then we can soundly return ‘No’, because $\theta_2 > \theta + \eta$ then implies $p > \theta + \eta$.

The key building block of this algorithm is the TESTER sub-procedure (Algorithm 2), which employs samples to check hypotheses. Informally, the TESTER does the following: Given two intermediate thresholds, $\theta_1$ and $\theta_2$, if the true probability $p$ is either smaller than $\theta_1$ or greater than $\theta_2$, it returns ‘Yes’ or ‘No’ respectively with high confidence. If $p \in (\theta_1, \theta_2)$ then the tester does not have any guarantees. Notice that a single invocation of the TESTER checks two hypotheses simultaneously, using one set of samples. The number of samples needed are derived and proven sufficient in Section 5.2.

One can directly invoke TESTER with $\theta_1 = \theta$ and $\theta_2 = \theta + \eta$ but that might lead to a very large number of samples, $O(1/\eta^2)$. Thus, the key challenge is to judiciously call the TESTER on hypotheses with smaller sample complexity such that we can refute or prove faster in most cases. To this end, notice that METAPROVERO leaves out two algorithmic design choices: stopping conditions and the strategy for choosing the proving and refuting hypotheses (highlighted in Alg. 1). We propose and analyze an adaptive binary-search-style algorithm where we change our hypotheses based on the outcomes of our sampling tests (Section 5.1). We show that our proposed algorithm using this strategy is sound. When $p$ is extremely close to $\theta$, these algorithms are no worse than estimating the probability, requiring roughly the same number of samples. We defer the theoretical sample complexity analysis of our algorithm in Appendix A and focus on the empirical evaluation in the paper.

## 5 Algorithms

We provide an adaptive algorithm for quantitative certification that narrows the size of the intervals in the proof search, similar to a binary-search strategy (Section 5.1). This algorithm build on the base TESTER primitive which we explain in Section 5.2.

### 5.1 The BinPCertify Algorithm

We propose an algorithm BinPCertify (Algorithm 3) where instead of fixing the intervals beforehand we narrow our search by halving the intervals. The user-specified input parameters for BinPCertify are the threshold $\theta$, the error bound $\eta$ and the confidence parameter $\delta$. The interval creation strategy is off-loaded to the CREATEINTERVAL procedure outlined in Algorithm 4. The interval size $\alpha$ is initially set to the largest possible as TESTER would require less samples on wider intervals (Algorithm 4, lines 1-3). Then, the BinPCertify algorithm calls internally the procedure CREATEINTERVAL to create proving intervals (on the left side of $\theta$, Alg. 3, line 6) and refuting intervals (on the right side of $\theta + \eta$, Alg. 3, line 11). Note that for the refuting intervals, we keep the left-side fixed, $\theta_1 = \theta + \eta$ and for the proving intervals we keep the right-side fixed, i.e., $\theta_2 = \theta$. For each iteration of BinPCertify, the strategy we use is to halve the intervals by moving the outermost thresholds closer to $\theta$ (Alg. 4, lines 5-7). For these intermediate hypotheses, TESTER checks if it can prove or disprove the assert $p \leq \theta$ (lines 9 and 15). It continues to do so alternating the proving and refuting hypotheses until the size of both intervals becomes smaller than the error bound $\eta$ (line 16). If only on one side of the threshold CREATEINTERVAL returns intervals with size $\alpha > \eta$, BinPCertify checks those hypotheses. If the size of
the proving and refuting intervals returned by CREATE INTERVAL is \( \eta \), then the final check is directly invoked on \((\theta, \theta + \eta)\) and returned to line (18).

The algorithm BINPCERTIFY returns ‘Yes’ or ‘No’ with soundness guarantees as defined in Section 3. We give our main theorem here and defer the proof to Section 6.1:

**Theorem 5.1** For an unknown value \( p \in [0,1] \), a given threshold \( \theta \in [0,1], \eta \in (0,1), \delta \in (0,1] \), BINPCERTIFY returns a ‘Yes’ or ‘No’ with the following guarantees:

\[
Pr[\text{BINPCERTIFY returns Yes } | p \leq \theta] \geq 1 - \delta \\
Pr[\text{BINPCERTIFY returns No } | p > \theta + \eta] \geq 1 - \delta
\]

We say that the BINPCERTIFY fails or returns incorrect results if:

\[
(\text{BINPCERTIFY does not return ‘Yes’ } | p \leq \theta) \lor \\
(\text{BINPCERTIFY does not return ‘No’ } | p > \theta + \eta)
\]

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**Algorithm 3** BINPCERTIFY \((\theta, \eta, \delta)\)

1. \( \theta_1 = \theta_2 = 0 \)
2. \( \theta_1 = \theta_2 = 0 \)
3. \( n = 3 + \max(0, \log(\frac{\theta}{\eta})) + \max(0, \log(\frac{1-\theta-\eta}{\eta})) \)
4. \( \delta_{\text{min}} = \delta/n \)  \( \triangleright \) minimum confidence per call to TESTER
5. **while** True **do**
6. \( \theta_1, \theta_2 = \text{CREATE INTERVAL}(\theta, \theta_1, \theta_2, \eta, \text{True}) \)
7. **if** \( \theta_2 - \theta_1 > \eta \) **then**
8. interval = \((\theta_1, \theta_2)\)
9. ans = TESTER(interval, \( \delta_{\text{min}} \))
10. **if** ans == Yes **then** return Yes
11. \( \theta_1, \theta_2 = \text{CREATE INTERVAL}(\theta, \theta_1, \theta_2, \eta, \text{False}) \)
12. interval = \((\theta_1, \theta_2)\)
13. **if** \( \theta_2 - \theta_1 > \eta \) **then**
14. ans = TESTER(interval, \( \delta_{\text{min}} \))
15. **if** ans == No **then** return No
16. **if** \( \theta_2 - \theta_1 \leq \eta \) and \( \theta_2 - \theta_1 \leq \eta \) **then**
17. interval = \((\theta, \theta + \eta)\)
18. **return** TESTER(interval, \( \delta_{\text{min}} \))

**Algorithm 4** CREATE INTERVAL \((\theta, \theta_1, \theta_2, \eta, \text{left})\)

1. **if** \( \theta == 0 \) and \( \text{left} \) **then** return \((\theta, \theta + \eta)\)
2. **if** \( \theta_1 == 0 \) and \( \theta_2 == 0 \) and \( \text{left} \) **then** return \((0, \theta)\)
3. **if** \( \theta_1 == 0 \) and \( \theta_2 == 0 \) and \( \text{not left} \) **then** return \((\theta + \eta, 1)\)
4. \( \alpha = \theta_2 - \theta_1 \)
5. **if** \( \text{left} \) **then**
6. **return** \((\theta_2 - \max(\eta, \alpha/2), \theta_2)\)
7. **return** \((\theta_1, \theta_1 + \max(\eta, \alpha/2))\)

---

### 5.2 TESTER Primitive

The tester takes as input two thresholds \( \theta_1, \theta_2 \) such that \( \theta_1 < \theta_2 \) and confidence parameter \( \delta \) and returns ‘Yes’ when \( p \leq \theta_1 \) with confidence higher than \( 1 - \delta \) and ‘No’ when \( p > \theta_2 \) with confidence higher than \( 1 - \delta \). If \( p \in (\theta_1, \theta_2) \) the TESTER returns without guarantees.

The procedure for implementing the TESTER is simple. Following the procedure outlined in Algorithm 2, we take \( N = \frac{(\sqrt{\theta_1} + \sqrt{\theta_2})^2}{(\theta_2 - \theta_1)^2} \ln \frac{1}{\delta} \) number of independent samples and estimate the ratio of these 0-1 trials as \( \hat{p} \). The TESTER returns ‘Yes’ if \( \hat{p} < \theta_1 + \eta_1 \) and ‘No’ if \( \hat{p} > \theta_2 - \eta_2 \) where \( \eta_1 \) and \( \eta_2 \) are error parameters (lines 2 and 3). If \( \theta_1 + \eta_1 < \hat{p} < \theta_2 - \eta_2 \), our implementation returns ‘None’. The following lemma establishes the soundness of the tester, and follows directly from applying Chernoff bounds on \( \hat{p} \).

**Lemma 5.2** Given the thresholds \((\theta_1, \theta_2)\) and confidence parameter \( \delta \), TESTER has following soundness guarantees:

\[
Pr[\text{TESTER returns ‘Yes’ } | p \leq \theta_1] \geq 1 - \delta \\
Pr[\text{TESTER returns ‘No’ } | p > \theta_2] \geq 1 - \delta
\]

We say that the TESTER fails or returns incorrect results if:

\[
(\text{TESTER does not return ‘Yes’ } | p \leq \theta_1) \lor \\
(\text{TESTER does not return ‘No’ } | p > \theta_2)
\]

**Proof.** We show that the failure probability for the TESTER is bounded by \( \delta \). Given that TESTER sets \( \eta_1 = (\theta_2 - \theta_1) \frac{\sqrt{\theta_1}}{\sqrt{\theta_1} + \sqrt{\theta_2}} \) and \( N = \frac{(\sqrt{\theta_1} + \sqrt{\theta_2})^2}{(\theta_2 - \theta_1)^2} \ln \frac{1}{\delta} \), we can directly substitute in Lemma 4.1 to derive the failure probability that the tester does not return ‘Yes’ given \( p \leq \theta_1 \):

\[
Pr[\hat{p} \geq \theta_1 + \eta_1] \leq e^{-\frac{(\sqrt{\theta_1} + \sqrt{\theta_2})^2}{(\theta_2 - \theta_1)^2}(\frac{\theta_1 + \theta_2}{2})^{\frac{1}{2}} \ln \frac{1}{\delta}} = \delta
\]

Similarly, we bound the failure probability for TESTER returning ‘No’ given that \( p > \theta_2 \) for \( \eta_2 = \theta_2 - \theta_1 - \eta_1 = (\theta_2 - \theta_1) \frac{\sqrt{\theta_2}}{\sqrt{\theta_1} + \sqrt{\theta_2}} \) and \( N = \frac{(\sqrt{\theta_1} + \sqrt{\theta_2})^2}{(\theta_2 - \theta_1)^2} \ln \frac{1}{\delta} \), we can directly substitute in Lemma 4.1 and obtain \( Pr[\hat{p} \leq \theta_2 - \eta_2] \leq \delta \). We have shown that TESTER error probability is bounded by \( \delta \), which proves Lemma 5.2.

We present a detailed analysis of how we select the specific values of the internal parameters \((\eta_1, \eta_2, N)\) for TESTER in Section 6.2.

### 6 Soundness

In this section, we prove that our proposed algorithm satisfies soundness as defined in Section 3. The algorithm returns ‘Yes’ given that \( p \leq \theta \) or ‘No’ given that \( p > \theta + \eta \) with probability at least \( 1 - \delta \). We say that the algorithm fails if it returns ‘Yes’ when \( p > \theta + \eta \) or ‘No’ when \( p \leq \theta \).
BINPCERTIFY uses the TESTER primitive on certain intervals in sequence. Depending on strategy, the size of the intervals and the order of testing them differs. But, the algorithm terminates immediately if the TESTER returns ‘Yes’ on a proving interval or ‘No’ on a refuting one. The meta-algorithm captures this structure on line 2-5 and BINPCERTIFY algorithm instantiates this general structure. When none of these optimistic calls to TESTER are successful, the algorithm makes a call to the TESTER on the remaining interval in the worst case. Given the same basic structure of algorithms as per the meta-algorithm METAPROVERO, we now prove the following key theorem:

Lemma 6.1 Let E be the event that the algorithm \( A \in \{\text{BINPCERTIFY}\} \) fails, then \( Pr[E] \leq \delta \).

Proof. Let the algorithm under analysis be \( A \), which can either BINPCERTIFY or some other strategy that follows the METAPROVERO structure. Fix any input to \( A \), and consider the execution of \( A \) under those inputs. Without loss of generality, we can order the intervals tested by \( A \) in the sequence that \( A \) invokes the TESTER on them in that execution. Let the sequence of intervals be numbered from 1, ..., \( i \), for some value \( i \).

Now, let us bound the probability of the event \( E_i \), which is when \( A \) tests intervals 1, ..., \( i \) and fails. To do so, we consider events associated with each invocation of TESTER \( j \in [1, i] \). Let \( R_j \) be the event that \( A \) returns immediately after invoking the TESTER on the \( j \)-th interval. Let \( C_j \) denote the event that TESTER returns a correct answer for the \( j \)-th interval. If \( E_i \) is true, then \( A \) terminates immediately after testing the \( i \)-th interval and fails. This event happens only if two conditions are met: First, \( A \) did not return immediately after testing intervals 1, ..., \( i-1 \); and second, \( A \) returns a wrong answer at \( i \)-th interval and does terminate. Therefore, we can conclude that the event \( E_i = C_i \cap R_i \cap \bar{R}_{i-1} \cap \cdots \cap \bar{R}_1 \). The probability of failure \( Pr[E_i] \) for each event \( E_i \) is upper bounded by \( Pr[C_i] \).

Lastly, let \( E \) be the total failure probability of \( A \). We can now use a union bound over possible failure events \( E_1, \ldots, E_n \), where \( n \) is the maximum number of intervals \( A \) can possibly test under any given input. Specifically:

\[
Pr[E] = Pr[E_1 \cup E_2 \ldots \cup E_n] \leq \sum_{i=1}^{n} Pr[E_i] \leq \sum_{i=1}^{n} Pr[C_i]
\]

By analyzing \( \sum_{i=1}^{n} Pr[C_i] \) in the context of each of our specific algorithm BINPCERTIFY, we show that the quantity is bounded by \( \delta \). Specifically, we invoke Lemma 6.2 presented in Section 6.1 to bound \( \sum_{i=1}^{n} Pr[C_i] \) below \( \delta \) for our adaptive algorithm.

6.1 Proof of Soundness: BINPCERTIFY

Lemma 6.1 directly implies Theorem 5.1. It remains to prove the required Lemma 6.2 that is invoked by Lemma 6.1 to establish the soundness of BINPCERTIFY.

Lemma 6.2 Under any given input \((\theta, \eta, \delta)\), let \( C_i \) be the event that \( i \)-th call made by BINPCERTIFY to the TESTER is correct and let \( n \) be the total number of calls to TESTER by BINPCERTIFY. Then, \( \sum_{i=1}^{n} Pr[C_i] \leq \delta \).

Proof. BINPCERTIFY halves the size of the intervals after each iteration until the intervals become smaller than the \( \eta \). On the left side of \( \theta \), it initially starts with an interval of size \( \theta \) (line 2 in Alg. 4). As a result we can upper bound the number of proving intervals by \( n_s = 1 + \log_{2} \frac{\theta}{\delta} \). Similarly, for the right side of \( \theta + \eta \), the number of refuting intervals is \( n_r \leq 1 + \log_{2} \frac{\theta + \eta}{n_s} \). Lastly, there is only 1 call to the TESTER on the interval \( \eta \) (line 19, Alg. 3). The total number of intervals tested in any one execution of BINPCERTIFY is at most \( n = 3 + \log_{2} \frac{\theta + \eta}{\delta} \). Each call to the TESTER is done with confidence parameter \( \delta_{min} = \frac{\delta}{\eta} \), therefore by Lemma 5.2, the failure probability of any call is at most \( \frac{\delta}{\eta} \). It follows:

\[
\sum_{i=1}^{n} Pr[C_i] \leq n \cdot \frac{\delta}{\eta} = \delta
\]

\[\square\]

6.2 TESTER Analysis

In this section, we show a detailed analysis of deriving the number of samples \( N = \frac{1}{(\theta_{\Delta} - \theta_{\Delta})^2} \ln 2 \) and \( \eta_1, \eta_2 \) parameters of the TESTER such that TESTER satisfies the soundness guarantees as outlined in Section 5.2. We denote the number of samples \( X \), for which \( \psi \) holds as \( X = \sum_{i=1}^{N} X_i \) and \( \hat{p} = \frac{1}{N} \sum_{i=1}^{N} X_i \) as the estimated probability after \( N \) samples. Using the estimated \( \hat{p} \), TESTER returns ‘Yes’ if \( \hat{p} \geq \theta_{\Delta} + \eta_1 \) and ‘No’ if \( \hat{p} \geq \theta_{\Delta} - \eta_2 \) with probability greater than \( 1 - \delta \). Otherwise, it returns ‘None’. In other words, given that \( p \leq \theta_{\Delta} \), we need to show that if TESTER takes \( n \) samples it fails the check on line 6 in Alg. 2 with a bounded failure probability of \( \delta \). Likewise, given that \( p > \theta_{\Delta} \) we need to show that the failure probability of the check on line 7 in Alg. 2 is bounded by \( \delta \). We want to choose values \( \eta_1, \eta_2 \in (0, 1) \) such that:

\[
Pr[\hat{p} \geq \theta_{\Delta} + \eta_1 \mid p \leq \theta_{\Delta}] \leq \delta \\
Pr[\hat{p} \geq \theta_{\Delta} - \eta_2 \mid p > \theta_{\Delta}] \leq \delta
\]

The key idea is to use one set of samples to check two hypotheses, \( p \leq \theta_{\Delta} \) and \( p > \theta_{\Delta} \), simultaneously. To do so, we find a point \( t \in (\theta_{\Delta}, \theta_{\Delta}) \) which we can serve as a decision boundary for \( \hat{p} \). Specifically, consider the value \( t \) such that:

\[
t = \theta_{\Delta} + \eta_1 = \theta_{\Delta} - \eta_2
\]

We choose \( t \) and the number of samples \( n \) in such a way that estimated \( \hat{p} \) is unlikely (with probability less than \( \delta \)) to be exceeding \( \theta_{\Delta} + \eta_1 \) if \( p \leq \theta_{\Delta} \) and to be less than \( \theta_{\Delta} - \eta_2 \)
if \( p > \theta_2 \). We illustrate this in Figure 3: it shows \((\eta_1, \eta_2)\) and the two probability distributions for \( \hat{p} \) given \( p = \theta_1 \) and \( p = \theta_2 \), respectively. The distributions for the case \( p < \theta_1 \) will be shifted further to the left, and the case \( p > \theta_2 \) will be shifted further to the right; so these are extremal distributions to consider. It can be seen that \( t \) is chosen such that \( \Pr[\hat{p} > t | p \leq \theta_1] \) as well as \( \Pr[\hat{p} < t | p > \theta_2] \) are bounded (shaded red) by probability \( \delta \). Next, we show how to find such \( t \) that requires smallest \( n \) when using Chernoff bounds.

Using the additive Chernoff bounds (Lemma 4.1), for a given \( \theta_1 \) and \( \theta_2 \), we can use the number of samples \( N \) is the maximum of the two quantities below:

\[
N \geq 3\theta_1 \frac{1}{\eta_1^2} \ln \frac{1}{\delta} \quad (2)
\]

\[
N \geq 2\theta_2 \frac{1}{\eta_2^2} \ln \frac{1}{\delta} \quad (3)
\]

Taking the maximum ensures that the probabilities of the both the hypotheses, \( p < \theta_1 \) and \( p > \theta_2 \), are being simultaneously upper bounded. From Eq. 1 we can express \( \eta_2 \) in terms of \( \theta_1, \theta_2, \eta_1 \) as \( \eta_2 = \theta_2 - \theta_1 - \eta_1 \). This allows us to express the required number of samples as functions of the input parameters \((\theta_1, \theta_2, \delta)\) and one variable \( \eta_1 \). More specifically, for Eq. 2 and Eq. 3, we obtain the functions \( g_1 \) and \( g_2 \):

\[
g_1(\theta_1, \eta_1, \delta) = 3\theta_1 \frac{1}{\eta_1^2} \ln \frac{1}{\delta}
\]

\[
g_2(\theta_1, \theta_2, \eta_1, \delta) = 2\theta_2 \left( \frac{1}{(\theta_2 - \theta_1 - \eta_1)^2} \ln \frac{1}{\delta} \right)
\]

Since the number of samples required must be \( \max(g_1, g_2) \), we find the \( \eta_1 \) at which \( \max(g_1, g_2) \) is the least. Since \( g_2 \) is a monotonically decreasing function in \( \eta_1 \) and \( g_1 \) is a monotonically increasing function in \( \eta_1 \), the minima of \( \max(g_1, g_2) \) is reached when the two bounds are equal. Thus the minimum \( N \) is found at the following value of \( \eta_1 \):

\[
\eta_1 = (\theta_2 - \theta_1) \frac{\sqrt{3}\theta_1}{\sqrt{3}\theta_1 + \sqrt{2}\theta_2}
\]

By fixing \( \eta_1 \), we have fixed our choice of \( t \) to \( \theta_1 + \eta_1 \). Substituting \( \eta_1 \) into Eq. 2 gives us the number of samples \( N = (\sqrt{3}\theta_1 + \sqrt{2}\theta_2)^2 \ln \frac{1}{\delta} \). The sufficiency of these parameter values is proven in Lemma 5.2.

### 7 Evaluation

- **Scalability**: How large are the models PROVERO can work within a reasonable timeout? In comparable timeout, can qualitative analysis tools produce conclusive results for such models?
- **Utility in attack evaluations**: How does adversarial hard- ness computed with PROVERO compare with the efficacy of state-of-the-art attacks?
- **Applicability to randomized models**: Can PROVERO certify properties of randomized DNNs?
- **Performance**: How many samples are needed by our new algorithms compare to the estimation approach (Section 4.2)?

We implement our algorithms in a prototype tool called PROVERO and evaluate the robustness of 38 deterministic neural networks trained on 2 datasets: MNIST [25] and ImageNet [37]. For MNIST, we evaluate on 100 images from the model’s respective test set. In the case of ImageNet, we pick from the validation set as we require the correct label. Table 1 provides the size statistics of these models. In addition, we evaluate the randomized model publicly provided by PixelDP [26], which has a qualitative certificate of robustness as well.

**ERAN Benchmark (BM1)**. Our first benchmark consists of 33 moderate size neural networks trained on the MNIST dataset. These are selected to aid a comparison with a state-of-the-art qualitative verification framework called ERAN [39]. We selected all the models which ERAN reported on. These models range from 2-layer neural networks to 6-layer neural networks with up to about 90K hidden units. We use the images used to evaluate the ERAN tool.

| Dataset | Arch | Description | #Hidden Units |
|---------|------|-------------|---------------|
| MNIST(BM1) | PFNN | 6-layer feed-forward | 3010 |
| | convSmall | 2-layer convolutional | 3604 |
| | convMed | 3-layer convolutional | 4804 |
| | convBig | 6-layer convolutional | 34688 |
| | convSuper | 6-layer convolutional | 88500 |
| | skip | residual | 71030 |
| ImageNet (BM2) | VGG16 | 16-layer convolutional | 15,096,080 |
| | VGG19 | 19-layer convolutional | 16,380,056 |
| | ResNet50 | 50-layer residual | 36,968,684 |
| | Inception_v3 | 42-layer convolutional | 32,285,184 |
| | DenseNet121 | 121-layer convolutional | 49,775,084 |

\( 2 \)available to download from [https://github.com/eth-sri/eran](https://github.com/eth-sri/eran)
Larger Models (BM2). The second benchmark consists of 5 larger deep-learning models pretrained on ImageNet: VGG16, VGG19, ResNet50, InceptionV3 and DenseNet121. The pre-trained models were obtained via the Keras framework with Tensorflow backend. These have about 15 – 50M hidden units.

All experiments were run on GPU (Tesla V100-SXM2-16GB) with a timeout of 600 seconds per image for the MNIST, 2000 seconds for ImageNet models, and 3600 seconds for the randomized PixelDP model.

7.1 Utility in Attack Evaluation

PROVERO can be used to directly certify if the security analyst has a threshold they want to check, for example, obtained from an external specification. Another way to understand its utility is by relating the quantitative bounds obtained from PROVERO with those reported by specific attacks. When comparing the relative robustness of DNNs to adversarial attacks, a common evaluation methodology today is to find the minimum adversarial perturbation with which state-of-the-art attacks can produce at least one successful adversarial example. If the best known attacks perform worse on one DNN versus another, on a sufficiently many test inputs, then the that DNN is considered more robust.

PROVERO offers a new capability: we can measure the ratio of adversarial samples within a specified perturbation bound of a given test input \( x \) (see Section 2). Specifically, we can compute the adversarial density by uniformly sampling in the \( L_p \) ball of \( x \), and checking if the ratio of adversarial samples is very small (below \( \theta = 10^{-3} \)). By repeating this process for different perturbations bounds, we empirically determine the adversarial hardness (or \( \epsilon_{\text{hard}} \))—the largest perturbation bound below which PROVERO certifies the adversarial density to be that small (returns ‘Yes’) and above which PROVERO does not (returns ‘No’). We use density threshold \( \theta = 10^{-3} \), error tolerance \( \eta = 10^{-3} \), and confidence parameter \( \delta = 0.01 \).

PROVERO computes the adversarial hardness with blackbox access. As a comparison point, we evaluate against two white-box attacks — PGD [29] for \( L_{\infty} \) and C&W [8] for \( L_2 \) implemented in CleverHans [33] (v3.0.1) — 2 prominent attacks that are recommended for the \( L_p \) norms we consider [6]. White-box attacks enable the attacker complete access to internals; therefore, they are empirically more powerful than black-box attacks today. Both PGD and C&W are gradient-based adversarial attacks. For PGD, we perform 30 attacks on different values of \( \epsilon \) to identify the minimum value that an adversarial input can be identified. For C&W, we identify the best \( \epsilon \) it is able to identify for a given amount of resource (iterations).

Result 1. Our main empirical result in this experiment is that \( \epsilon_{\text{hard}} \) is strongly correlated with \( \epsilon_{\text{emp}} \). Figure 4 and Figure 5 show the correlation visually for two models: it shows that the perturbation bounds found by these two separate attacks are different, but both correlate with the adversarial hardness of the certification instance produced by PROVERO. The Pearson correlation for all models is reported in Table 2 for all cases where the compared white-box attacks are successful. The average Pearson correlation between the perturbation found by PGD, \( \epsilon_{\text{PGD}} \), and \( \epsilon_{\text{hard}} \) over all models is 0.858 and between the perturbation found by C&W, \( \epsilon_{\text{C&W}} \), and \( \epsilon_{\text{hard}} \) is 0.8438. We take 25 images per model to calculate the correlation. The significance level is high (p-value is below 0.01 for all cases).

Recall that PROVERO has the capability to directly certify if the security analyst is sound, so the estimate of adversarial hardness \( \epsilon_{\text{hard}} \) is close to the ground truth with high probability (99%). This metric is an attack-agnostic metric, computed by uniform sampling and without white-box ac-

| Models                     | \( \rho \) (PGD) | \( \rho \) (C&W) |
|----------------------------|-----------------|-----------------|
| convBigRELU__DiffAI        | 0.9617          | 0.7509          |
| convMedGRELU__PGDK_w_0.1  | 0.8143          | 0.6686          |
| convMedGRELU__PGDK_w_0.3  | 0.7699          | 0.6715          |
| convMedGRELU__Point        | 0.8461          | 0.982           |
| convMedSIGMOID__PGDK_w_0.1| 0.8533          | 0.8903          |
| convMedSIGMOID__PGDK_w_0.3| 0.9394          | 0.913           |
| convMedSIGMOID__Point      | 0.9424          | 0.9605          |
| convMedGTANH__PGDK_w_0.1  | 0.9521          | 0.9334          |
| convMedGTANH__PGDK_w_0.3  | 0.9567          | 0.8718          |
| convMedGTANH__Point        | 0.7592          | 0.9817          |
| convSmallRELU__DiffAI      | 0.9504          | 0.5127          |
| convSmallRELU__PGDK        | 0.7803          | 0.6411          |
| convSmallRELU__Point       | 0.893           | 0.9816          |
| convSuperRELU__DiffAI      | 0.687           | 0.3856          |
| DenseNet-res               | 0.7168          | 0.4879          |
| ffnRELU__PGDK_w_0.1 6_500  | 0.8932          | 0.9577          |
| ffnRELU__PGDK_w_0.3 6_500  | 0.7039          | 0.6496          |
| ffnRELU__Point 6_500       | 0.954           | 0.9788          |
| ffnSIGMOID__PGDK_w_0.1 6_500| 0.8706         | 0.8955          |
| ffnSIGMOID__PGDK_w_0.3 6_500| 0.9402         | 0.9201          |
| ffnSIGMOID__Point 6_500   | 0.8906          | 0.9489          |
| ffnTANH__PGDK_w_0.1 6_500 | 0.8156          | 0.9508          |
| ffnTANH__PGDK_w_0.3 6_500 | 0.9104          | 0.9485          |
| ffnTANH__Point 6_500       | 0.8998          | 0.8435          |
| mnist_conv_maxpool        | 0.9664          | 0.9699          |
| mnist_relu_3 100          | 0.9668          | 0.9448          |
| mnist_relu_3 50           | 0.9702          | 0.9298          |
| mnist_relu_4 1024         | 0.8945          | 0.9723          |
| mnist_relu_5 100          | 0.7472          | 0.9629          |
| mnist_relu_6 100          | 0.9845          | 0.9868          |
| mnist_relu_6 200          | 0.9699          | 0.9412          |
| mnist_relu_9 100          | 0.979           | 0.9725          |
| mnist_relu_9 200          | 0.8165          | 0.9652          |
| ResNet50                  | 0.7929          | 0.6932          |
| skip.DiffAI               | 0.7344          | 0.6298          |
| VGG16                     | 0.8064          | 0.8297          |
| VGG19                     | 0.7224          | 0.7335          |
| Inception-v3              | 0.5806          | 0.4866          |
cess to the model. The strong correlation shows that PGD and C&W attacks find smaller $\epsilon_{\text{min}}$ for easier certification instances, and larger $\epsilon_{\text{min}}$ for harder instances. This suggests that when comparing the robustness of two models, one can consider adversarial hardness as a useful attack-agnostic metric, complementing evaluation on specific attacks.

![Figure 4: Correlation graph between $L_\infty$ bounds provided by PROVERO and PGD for a fully connected feedforward with sigmoid (FFNN) on MNIST.](image)

![Figure 5: Correlation graph between $L_2$ bounds provided by PROVERO and C&W for a fully connected feedforward (FFNN) with sigmoid on MNIST.](image)

## 7.2 Scalability

We test PROVERO on 38 models, which range from $3K - 50M$ hidden units. We select 100 input images for each model and retain all those inputs for which the model correctly classifies. We tested 11 perturbation size ($L_\infty$) from 0.01 to 0.25 for BM1 and 4 perturbation size ($L_2$) from 2/255 to 14/255 for BM2. This results in a total of 36971 test images for 38 models. We run each test image with PROVERO with the following parameters: $(\theta = 0.0001, \eta = 0.001, \delta = 0.01, \epsilon = 0.01)$ for BM1 $(\theta = 0.001, \eta = 0.001, \delta = 0.01, \epsilon = 0.01)$ for BM2. We find that PROVERO scales producing answers within the timeout of 600 seconds for BM1 and 2000 seconds for BM2 per test image. Less than 2% input cases for BM2 and less than 5% for BM1 return ‘None’, i.e., PROVERO cannot certify conclusively that there are less or more than than the queried thresholds. For all other cases, PROVERO provides a ‘Yes’ or ‘No’ results.

As a comparison point, we tried to compare with prior work on quantitative verification [2], prototyped in a tool called NPAQ. While NPAQ provides direct estimates rather than the ‘Yes’ or ‘No’ answers PROVERO produces, we found that NPAQ supports only a sub-class of neural networks (BNNs) and of much smaller size. Hence, it cannot support or scale for any of the models in our benchmarks, BM1 and BM2.

Secondly, we tested ERAN which is considered as the most scalable qualitative verification tool. We failed to run the large BM2 models. ERAN initially was not able to parse these models. After direct correspondence with the authors of ERAN, the authors added support for requisite input model formats. After applying these changes, we confirmed that the current implementation of ERAN times out on all the BM2 models. PROVERO finishes on these within a timeout of 2000 seconds.

We successfully run ERAN on BM1, which are smaller benchmarks that ERAN reported on. There are total of 4 analyses in ERAN. We evaluate on the DeepZono and DeepPoly domains but for the DeepPoly, on our evaluation platform, ERAN runs out of memory and could not analyze the neural networks in BM1. The remaining analyses, RefineZono and RefinePoly are known to achieve or improve the precision of the DeepZono or DeepPoly domain at the cost of larger running time by calling a mixed-integer programming solver [41]. Hence, we compare with the most scalable of these 4 analyses, namely the DeepZono domain.

**Result 2.** Figure 6 plots the precision of ERAN against PROVERO for all 11 perturbation sizes. We find that for a perturbation size of more than 0.05, ERAN’s results are inconclusive, i.e., the analysis reports neither ‘Yes’ nor ‘No’ for more than 50% of the inputs, likely due to imprecision in over-approximations of the analysis. Figure 6 shows that the verified models reduces for higher perturbations $\epsilon$. This is consistent with the findings in the ERAN paper: ERAN either takes longer or is more imprecise for non-robust models and higher $\epsilon$ values [41]. In all cases where ERAN is inconclusive, PROVERO successfully finishes within the 600 second timeout for all 35431 test images and values of $\epsilon$. In above 95% of these cases, PROVERO produces high-confidence ‘Yes’ or ‘No’ results.

As a sanity check, on cases where ERAN conclusively out-
puts a ‘Yes’, PROVERO also reports ‘Yes’. With comparable running time, PROVERO is able to obtain quantitative bounds for all perturbation sizes. From these experimental results, we conclude PROVERO is a complementary analysis tool that can be used to quantitatively certify larger models and for larger perturbation sizes, for which our evaluated qualitative verification framework (ERAN) is inconclusive. To the best of our knowledge, this is the first work to give any kind of sound quantitative guarantees for such large models.

### 7.3 Applicability to Randomized DNNs

So far in our evaluation we have focused on deterministic DNNs, however, PROVERO can certify the robustness of randomized DNNs. To demonstrate this generality, we take a model obtained by a training procedure called PixelDP that adds differentially private noise to make the neural network more robust. The inference phase of a PixelDP network uses randomization: instead of picking the label with the maximum probability, it samples from the noise layer and calculates an expectation of the output value. PixelDP also produces a certified perturbation bound for which it guarantees the model to be robust for a given input image. Note that qualitative verification tools such as ERAN require white-box access and work with deterministic models, so they would not be able to verify the robustness of randomized PixelDP DNNs.

We contacted the authors to obtain the models used in PixelDP [26]. The authors pointed us to the PixelDP ResNet28-10 model trained on CIFAR10 as the main representative of the technique. We randomly select 25 images in the CIFAR10 dataset and for each image we obtain the certified perturbation bound $\epsilon_{\text{PixelDP}}$ produced by PixelDP itself. We configure PixelDP to internally estimate $\epsilon_{\text{PixelDP}}$ using 25 samples from the noise layer as recommended in their paper. This bound, $\epsilon_{\text{PixelDP}}$, is the maximum bound for which PixelDP claims there are no adversarial samples within the $L_2$ ball.

We use PROVERO to check the certificate $\epsilon_{\text{PixelDP}}$ produced by PixelDP, using the following parameters $\theta = 0.01, \eta = 0.01, \delta = 0.01$. PROVERO reports ‘Yes’, implying that the model has adversarial density under these bounds. Under the same threshold $\theta = 10^{-2}, \eta = 10^{-2}$ we tested for larger perturbation sizes: from $\epsilon = 0.1$ to 0.25.

**Result 3.** Our findings in this experiment are that PROVERO can certify low adversarial density for perturbation bounds much larger than the qualitative certificates produced by PixelDP. In particular, PROVERO certifies that for $\epsilon = 5 \times \epsilon_{\text{PixelDP}}$, the PixelDP model has less than $10^{-2}$ adversarial examples with confidence at least 99%. PROVERO offers complimentary quantitative estimates of robustness for PixelDP.

### 7.4 Performance

Our estimation solution outlined in Section 4.2 applies Chernoff bounds directly. For a given precision parameter $\eta$, it requires a large number of samples, within a factor of $1/\eta^2$. While we do not escape from this worst case, we show that our proposed algorithms improve over the estimation baseline empirically. To this end, we record the number of samples taken for each test image and compare it to the number of samples as computed for the estimation approach.

**Result 4.** We find that PROVERO requires $27 \times$ less samples than the estimation approach for values of $\eta = 0.01, \theta = 0.01$ and $687 \times$ less samples for values of $\eta = 0.001, \theta = 0.0001$ for BM1. For larger models in BM2 and values of $\eta = 0.001, \theta = 0.001$, PROVERO requires $74 \times$ less samples than the estimation solution and $\eta = 0.01, \theta = 0.01$.

In our implementation, we use a batch-mode to do the inference for models in BM1 and BM2, speeding up the running time of the sampling process by a factor of $3 \times$. This leads to average times of 29.78 seconds per image for BM1 for a batch size of 128. For the models in BM2 we used different batch sizes so we report the average time per sample as 0.003 seconds which can be used to derive the running time based. For ($\theta = 0.0001, \eta = 0.001$) and all values of $\epsilon$, the average number of samples over all images is 80,424 (median value is 83,121, standard deviation is 14,303). For VGG16 the average number of samples is around 657,885, for VGG19 it is around 644,737 and for InceptionV3 664,397 samples, respectively, for $\theta = 0.001, \eta = 0.001$. For both $\theta = 0.01, \eta = 0.01$ and $\theta = 0.001, \eta = 0.001$, for VGG16 the average number of samples is 737,297, for VGG19 it is 722,979 and for InceptionV3 744,792 average number of samples, resulting in about $74 \times$ less samples than the estimation approach.

For the PixelDP model, PROVERO takes 20,753 samples. Since PixelDP models internally take 25 samples from the
noise layer, the time taken for inference on one given input itself is higher. PROVERO reports an average running time per test image of roughly 1,500 seconds.

8 Related Work

Qualitative Neural Network Verification. The formal verification of neural networks asks whether the neural network satisfies a certain property, i.e., a neural network is robust if all perturbations within a bound are correctly labeled by the neural network. The approaches following classical verification methodology, also sometimes called qualitative verification, focus on the design of techniques that return yes if the neural network is robust or produce a counterexample that violates the robustness property. In the context of deep neural networks, the earliest attempt to verification is due to Pulina and Tacchela [35], which employs an abstraction-refinement technique to verify neural network safety. The recent verification methodologies have sought to exploit the advances in combinatorial solving, thereby consisting of satisfiability modulo theories (SMT) solvers-based approaches [13, 19, 22, 23, 31] or integer linear programming approaches [9, 27, 47]. While significant progress has been made recently to extend the verification to convolutional layers with piecewise-linear activation function, the combinatorial solving-based approaches typically scale only to a few layers.

Consequently, techniques to address scalability often sacrifice completeness: abstract interpretation-based techniques [14, 39-41] are among the most scalable verification procedures that over-approximate the activation functions with abstract domains. If the procedure returns Yes, then the neural network is robust with respect to the particular perturbation bound; otherwise, the network can be either robust or have adversarial examples, i.e., the approach has false positives. Similarly, optimization-based approaches [12, 36, 53] produce a certificate of robustness for a given input: a lower bound on the minimum perturbation bound $\varepsilon_{\text{min}}$ such that there are no adversarial examples within that a $\varepsilon_{\text{min}}$ ball. These techniques are, however, incomplete and catered to only ReLU activations [53] and fully-connected layers [56].

A promising line of incomplete techniques has been proposed employing two complementary techniques: Lipschitz computation and linear approximations. Hein and Andriushchenko [17] propose an analytical analysis based on the Lipschitz constant, but the approach assumes that a differentiable activation function (thus excluding ReLU activations) and can handle only two layers. Boopathy et al [5], Weng et al. [51], Zhang et al [56] have further improved the scalability of techniques based on Lipschitz constants and linear approximations, achieving scalability up to 80,000 hidden units. While the lower and upper bounds are sound, their tightness is not guaranteed. In another line of work, Weng et al. [52] employ extreme value theory to estimate a lower bound on $\varepsilon_{\text{min}}$ albeit without theoretical guarantees of the soundness of the obtained bounds.

As discussed earlier, our work differs fundamentally from classical verification approaches in our focus on the development of a quantitative verification framework with rigorous guarantees on computed estimates.

Quantitative Verification. While qualitative verification provides a binary answer to the question of whether there exists an input that violates a property, in quantitative approaches, the goal is to estimate what proportion of inputs violate the property. Webb et al. [50] proposed a Monte Carlo-based approach to estimating the ratio of adversarial examples. In particular, the authors propose to use a technique that is catered for rare event estimation. In this work, we take a property testing approach wherein we ask if the proportion of inputs that violate a property is less than a given threshold (in which case we say yes) or it is $\eta$-far from the threshold (in which case we say no). While both approaches are black-box and rely on sampling, PROVERO returns a yes or no answer with user-specified high confidence, whereas the statistical approach proposed by Webb et al. does not provide such guarantees.

The approach of Baluta et al. [2] is similar to our work, in spirit. Baluta et al. propose a white-box approach, called NPAQ, by encoding the neural network into a logical formula and employ model counting to estimate the proportion of adversarial inputs. The lack of efficient model counters restricts the authors to instantiating the approach to a particular type of neural networks, namely binarized neural networks, thus limiting the applicability to other types of neural networks. While NPAQ does provide PAC-style guarantees, its scalability is limited to binarized neural networks of around 50,000 parameters. Narodytska et al. [32] propose a similar approach for assessing machine learning explanations. In contrast, PROVERO scales to significantly larger DNNs than any of the previous work while preserving the formal guarantees of the white-box approach, i.e., PROVERO returns an answer with user-specified error and confidence parameters.

9 Conclusion

We have presented a quantitative verification algorithm for neural networks which has soundness guarantees and empirical scalability. The framework makes minimal assumptions. We have shown its practical utility in certifying adversarial robustness. Specifically, we present a new attack-agnostic metric of hardness of finding adversarial examples which can be computed using our approach and we find that correlates well with the efficacy of 2 effective attacks today. PROVERO provides complementary utility compared to qualitative verification tools by allowing analysts to certify DNN robustness with high confidence efficiently.
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We now provide an upper bound on the number of samples required by $\text{BINPCERTIFY}$. Throughout the analysis, for an interval of size $\alpha$, the number of samples needed by $i-$th call to the $\text{TESTER}$ for the interval of size $\alpha$ is $s_i(\alpha, \delta) \leq \frac{(\sqrt{3} + 2)^2}{\alpha^2} \ln \frac{1}{\delta}$. The $\text{BINPCERTIFY}$ algorithm halves the interval size on each iteration (call to $\text{CREATEINTERVAL}$, lines 6 and 11 in Alg. 3) until the interval size for both the proving and refuting intervals become smaller than $\eta$. When that happens, $\text{BINPCERTIFY}$ calls the $\text{TESTER}$ on the interval $(\theta, \theta + \eta)$ and returns directly from there, lines 16–18, Alg. 3. The $\text{CREATEINTERVAL}$ method starts by creating intervals of size $\theta$ for the proving intervals and $1 - \theta - \eta$ for the refuting intervals.

**Theorem A.1** The number of samples required by $\text{BINPCERTIFY}$ is upper bounded by $O(k_1 + k_2 + k_3)$, where

1. $k_1 = \frac{4}{3}(\frac{1}{\bar{\alpha}^2} - \frac{1}{\eta^2})(\sqrt{3} + \sqrt{2})^2 \ln^2 \frac{1}{\delta}$
2. $k_2 = \frac{4}{3}(\frac{1}{\bar{\alpha}^2} - \frac{1}{4(1 - \theta - \eta)^2})(\sqrt{3} + \sqrt{2})^2 \ln^2 \frac{1}{\delta}$
3. $k_3 = \frac{(\sqrt{3} + \sqrt{2})^2}{\eta^2} \ln^2 \frac{1}{\delta}$

and $n = 3 + \max(0, \log \frac{\delta}{\eta}) + \max(0, \log \frac{1 - \theta - \eta}{\eta})$.

**Proof.** We shall use the notations described at the beginning of Section A. Let $Z_i$ denote the number of samples required by the $\text{BINPCERTIFY}$ during the $i$-th iteration of the loop. Note that $\text{BINPCERTIFY}$ first invokes $\text{TESTER}$ for the interval $(0, \theta)$, therefore, we have

$$Z_1 \leq s_1(\theta, \delta_{\min}) + \Pr[R_i]s_2(\frac{\theta}{2}, \delta_{\min})$$

For all the rest of the iterations until the final iteration, i.e., when the condition in line 16 is satisfied, we have the following bounds on $Z_i$.

$$Z_i \leq s_{2i-1}(\frac{\theta}{2^{i-1}}, \delta_{\min}) + \Pr[R_{2i-1}]$$

Let $Z_f$ denote the number of samples by $\text{TESTER}$ in line 18. We have $Z_f \leq s_n(\frac{\theta}{2^n}, \delta_{\min})$. Finally, observe that

$$Z = Z_1 + \Pr[R_2]Z_2 + \Pr[R_{2(i-1)}]Z_3 + \cdots + \Pr[R_{2n-1}]Z_f$$

Also, the number of proving intervals before the last call is $n_1 \leq 1 + \log \frac{\theta}{\eta}$ and, respectively, $n_r \leq 1 + \log \frac{1 - \theta - \eta}{\eta}$ refuting intervals. Therefore,

$$Z \leq s_1(\theta, \delta_{\min}) + \sum_{i=2}^{n_1} s_{2i-1}(\frac{\theta}{2^{i-1}}, \delta_{\min}) \prod_{j=1}^{2i-2} \Pr[R_j] + \sum_{i=2}^{n_r} s_{2i}(1 - \frac{\theta - \eta}{2^{i-1}}, \delta_{\min}) \prod_{j=1}^{2i-1} \Pr[R_j] + s_n(\frac{\theta}{2^n}, \delta_{\min}) \prod_{j=1}^{n-1} \Pr[R_j],$$

where $n_1 \leq 1 + \log \frac{\theta}{\eta}$, $n_r \leq 1 + \log \frac{1 - \theta - \eta}{\eta}$.

The following bounds can be obtained by simple algebraic computations:

1. $\sum_{i=2}^{n_1} s_{2i-1}(\frac{\theta}{2^{i-1}}, \delta_{\min}) \leq \frac{4}{3}(\frac{1}{\bar{\alpha}^2} - \frac{1}{\eta^2})(\sqrt{3} + \sqrt{2})^2 \ln^2 \frac{1}{\delta_{\min}}$
2. $\sum_{i=2}^{n_r} s_{2i}(\frac{1 - \theta - \eta}{2^{i-1}}, \delta_{\min}) \leq \frac{4}{3}(\frac{1}{\bar{\alpha}^2} - \frac{1}{4(1 - \theta - \eta)^2})(\sqrt{3} + \sqrt{2})^2 \ln^2 \frac{1}{\delta_{\min}}$
3. \( s_n(\frac{\theta}{\eta}, \delta_{\min}) = \frac{(\sqrt{3\theta} + \sqrt{2(\theta + \eta)})^2}{\eta^2} \ln \frac{1}{\delta_{\min}}. \)

The statement of the theorem follows directly from the above bounds and by noting \( Pr[\bar{R}_i] \leq 1 \) and \( \delta_{\min} = \delta/n \quad \square \)

The above analysis is conservative and an interesting direction of future work would be to analyze \( Pr[\bar{R}_i] \) under given distribution of \( \rho \).