Einstein–Weyl structures and de Sitter supergravity

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Received 11 October 2011, in final form 9 March 2012
Published 2 May 2012
Online at stacks.iop.org/CQG/29/105006

Abstract

The geometric structure of the null solutions of de Sitter \(D = 5\) gauged supergravity coupled to vector multiplets is investigated. These solutions are Kundt metrics, constructed from a one-parameter family of three-dimensional Gauduchon–Tod base spaces. We give examples, including the near-horizon geometries previously found in Gutowski and Sabra (2011 J. High Energy Phys. JHEP05(2011)020). In addition, two special cases are considered in detail. In the first case, we consider solutions for which the Gauduchon–Tod base space is the Berger sphere. In the second case, we take the null 1-form Killing spinor bilinear to be recurrent, so that the holonomy of the Lévi-Cività connection is contained in \(\text{Sim}(3)\).

PACS number: 04.65.+e

1. Introduction

Gravitational models with vanishing or negative cosmological constant can be embedded in supergravity theories. In this sense, finding gravitational solutions in supergravity with zero or negative cosmological constant is simplified through looking for configurations admitting supersymmetry (see e.g. [2, 3], respectively). This is because the Killing spinor equations are first-order, and their integrability conditions ensure that many of the second-order field equations are satisfied.

Backgrounds of positive cosmological constant (de Sitter) are of interest due to processes like the higher dimensional analogue of the black hole anti-evaporation of [4],\(^4\) or in the study of the proposed dS/CFT correspondence [5], where bulk quantum gravity (or a supergravity approximation) on a \(d\)-dimensional de Sitter space would be dual to a Euclidean CFT on \(S^{d−1}\).

\(^4\) Where it was shown that under certain initial conditions (namely, when the perturbation away from the solution is non-vanishing, but its time derivative is), a four-dimensional quantum, asymptotically de Sitter, Schwarzschild solution of nearly maximal mass would grow, contrary to a classical thermodynamical intuition.

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Unfortunately, finding solutions with positive cosmological constant is not easy because gravity with a positive cosmological constant is generally not compatible with supersymmetry \[10, 11\]. However, one may consider a fake supergravity, obtained from the genuine one via analytic continuation. These kinds of constructions were originally proposed in \[12\] to study the stability of domain-wall solutions (see also \[13\]), and later used to suggest the correspondence between fake supersymmetric domain-wall solutions and pseudo-supersymmetric cosmological solutions \[14, 15\]. However, in this work we primarily consider fake supergravity as a solution-generating technique for obtaining cosmological solutions by solving the (pseudo)-Killing spinor equations arising from the analytically continued supersymmetry transformations of the fermionic fields in the original theory. In this regard, our solutions will generally not be supersymmetric, only achieving that in the limiting case of vanishing cosmological constant. The first analysis of pseudo-supersymmetric five-dimensional de Sitter solutions in this setting was performed in \[16, 17\]. The first systematic analysis, not depending on choosing a particular ansatz for the solution, was performed in \[18, 19\]. There, the ‘timelike’ solutions (those for which the 1-form Killing spinor bilinear is timelike) of the \(d = 5\) minimal theory were given in terms of a four-dimensional hyper-Kähler with torsion (HKT) base space. Similarly, \(d = 5\) fakeSUGRA coupled to Abelian vectors was studied in \[20\].

Following from this, in \[21\], solutions of the minimal five-dimensional de Sitter supergravity with a null Killing spinor 1-form bilinear were shown to be given by a family of backgrounds with Gauduchon–Tod (GT) base space. GT spaces \[23\] are special types of Einstein–Weyl 3-spaces, which were discovered in the context of hyper-Hermitian spaces admitting a tri-holomorphic Killing vector field. These spaces appear in the classification of solutions in four-dimensional de Sitter supergravity \[24, 25\], and as certain examples of solutions in five-dimensional de Sitter supergravity with hyper-Hermitian isometries \[20\], as well as in the classification of four-dimensional (Euclidean) gravitational instantons \[26\].

The solutions of \[21\], just like the solutions of ungauged \[2\] and gauged (anti-de Sitter) \[3\] theories, admit a geodesic, expansion-free, twist-free and shear-free null vector field \(N\). These geometries are known in four-dimensional general relativity as Kundt metrics \[27\]. In higher dimensions, Kundt geometries have been considered in \[28–30\]. In Minkowski or AdS theories, the null vector is always Killing, and for some special cases, it becomes covariantly constant, and the Kundt geometries are pp-waves. However, this turns out not to be the case in dS theory. In contrast the null vector of solutions of \[21\] is not Killing, but has an interesting property that under certain conditions, it becomes recurrent with respect to the Lévi-Cività connection.

Our work generalizes\(^6\) the analysis of \[21\] to de Sitter \(D = 5\) supergravity coupled to Abelian vector multiplets. To obtain this theory, we analytically continue the supersymmetry transformations of the gravitini as well as the gauginos. The vanishing of these transformations produces fake Killing spinor equations, and we consider pseudo-supersymmetric solutions admitting non-vanishing (fake) Killing spinors satisfying these equations.

We outline our work as follows. Section 2 gives a summary of the basic equations, conventions and the (fake) Killing spinor equations of de Sitter supergravity. We find that the (pseudo-supersymmetric) solutions generically preserve half of the pseudo-supersymmetry. As is the case with the classification of supersymmetric backgrounds of genuine supergravity

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\(^5\) A notable exception is \[9\], in which a four-dimensional, locally diffeomorphism-covariant, \(N^\prime = 1\) supersymmetric and chiral-invariant field theory for the \((2, 3/2)\) and \((1, 1/2)\) multiplets, where the spin-1 gauge field is coupled to the fermions, is presented.

\(^6\) One can see that the solutions of \[21\] are a limiting case of those presented here, where only the gravity supermultiplet was present.
theories, one obstruction to having maximally pseudo-supersymmetric solutions is given by
the gaugino equation, which is of an algebraic nature, and sets a condition on the pseudo-Killing spinor. It has been conjectured in [22] that the only theories with gaugino fields that admit non-trivial maximally supersymmetric backgrounds are chiral ones. We provide another instance for this phenomenon, as we find that the only maximally pseudo-supersymmetric background is five-dimensional de Sitter space, just like in the minimal theory of [21].

In section 3, we analyse the Killing spinor equations, focusing on the case where the 1-form spinor bilinear is null. The conditions obtained from the gravitino equation are derived from the analysis of the minimal pseudo-supersymmetric solutions in [21]. We also introduce local coordinates, and show that the solutions are given in terms of a one-parameter family of three-dimensional GT spaces. Imposing pseudo-supersymmetry, together with the Bianchi identity and the gauge field equations, is sufficient to ensure that all the remaining equations, with the exception of one component of the Einstein equations, hold automatically. In section 4, we give some simple examples of our solutions, including the near-horizon geometries found in [1], and an explicit model with one gauge multiplet, which under vanishing of the potential provides non-BPS solutions to a SUGRA theory. In section 5, we focus on solutions where the GT space is the Berger sphere [23, 31]. In section 6, we consider the conditions for which the null Killing spinor 1-form bilinear is recurrent, and investigate the properties of various scalar curvature invariants.

2. de Sitter \( N = 2 \) supergravity and Killing spinors

In this section, we briefly summarize the supergravity theory we shall consider, together with some conventions. The model which we will consider is \( \mathcal{N} = 2, D = 5 \) gauged supergravity coupled to \( n - 1 \) Abelian vector multiplets [35] whose bosonic action is given by

\[
S = \frac{1}{16\pi G} \int (-R + 2g^2 V) * 1 - Q_{IJ}(-dX^I \wedge *dX^J + F^I \wedge *F^J) - \frac{C_{IJK}}{6} F^I \wedge F^J \wedge A^K
\]  

(2.1)

where \( I, J \) and \( K \) take values 1, . . . , \( n \) and \( F^I = dA^I \) are the 2-forms representing gauge field strengths (one of the gauge fields corresponds to the graviphoton). The constants \( C_{IJK} \) are symmetric in \( \{I, J, K\} \); we will assume that \( Q_{IJ} \) (the gauge coupling matrix) is positive definite and invertible, with inverse \( Q^{IJ} \). As usual when considering all the vector fields on the same footing, one has to introduce a further constrain on the scalar fields \( X^I \),

\[
\frac{1}{6} C_{IJK} X^I X^J X^K = X_I X^I = 1. 
\]  

(2.2)

The fields \( X^I \) can thus be regarded as being functions of \( n - 1 \) unconstrained scalars \( \phi^r \). We list some useful relations associated with \( \mathcal{N} = 2, D = 5 \) gauged supergravity,

\[
Q_{IJ} = \frac{3}{2} X_I X_J - \frac{1}{2} C_{IJK} X^K, \\
Q_{IJ} x^I = \frac{1}{2} x^I, \\
Q_{IJ} dX^J = -\frac{3}{2} dX_I, \\
V = 9V_I V_J (X^I x^J - \frac{1}{2} Q_{IJ}),
\]  

(2.3)

where \( V_I \) are constants. The de Sitter supergravity theory is obtained by sending \( g^2 \) to \(-g^2\) in (2.1).

Pseudo-supersymmetric de Sitter solutions admit a Dirac spinor \( \eta \) satisfying a gravitino and gaugino Killing spinor equation. The gravitino Killing spinor equation is

\[
[\nabla_M + \frac{1}{8} g_{MN} H_N N \Gamma_M N \Gamma_N - \frac{1}{2} H_M \Gamma_N - g(\frac{3}{2} X \Gamma_M - \frac{3}{2} A_M)] \eta = 0,
\]  

(2.4)

This result was not presented at the time, and we include it here for completeness.
where we have defined
\[ V_I X^I = X, \quad V_I A_M = A_M, \quad X_I F^I_{MN} = H_{MN}. \] (2.5)

The gaugino Killing spinor equation is given by
\[ \left( \left( - F^I_{MN} + X^I H_{MN} \right) \Gamma^{MN} + 2 \nabla_M X^I T^M - 4 g V_J (X^I X^J - \frac{3}{2} \chi \chi^J) \right) \eta = 0. \] (2.6)

We shall adopt a mostly minus signature for the metric, which we write in a null frame as
\[ ds^2 = 2 e^{-\phi} e^{-\delta} \delta_{ij} e^i e^j. \] (2.7)
for \( i, j, k = 1, 2, 3. \)

We remark that all pseudo-supersymmetric solutions of de Sitter supergravity coupled to vector multiplets preserve at least half of the pseudo-supersymmetry. This follows from the fact that if \( \epsilon \) is a Killing spinor satisfying the gravitino and gaugino Killing spinor equations, then so is \( C^* \epsilon \), where \( C^* \) is the charge conjugation operator defined in [21], and furthermore \( \epsilon \) and \( C^* \epsilon \) are linearly independent over \( \mathbb{C} \). This also implies that any pseudo-supersymmetric solution preserving at least \( 3/4 \) of the pseudo-supersymmetry is automatically maximally pseudo-supersymmetric.

The maximally pseudo-supersymmetric solutions can be obtained by first demanding the vanishing of the gaugino KSE equation (2.6), which implies that the scalars \( X^I \) are constant, and that all the field strengths \( F^I \) are proportional to \( H \). On substituting these conditions into the gravitino KSE, equation (2.4), and relabelling the fields appropriately, one finds that (2.4) reduces to the gravitino KSE of the minimal de Sitter theory. One then has to examine the integrability condition obtained from this gravitino KSE, and demand that all the different terms (in equal powers of gammas) vanish. This gives additional conditions which determine the form of the solutions. Since the integrability condition for the KSE of the minimal gauged theory was given in [3], on relabelling of fields and replacing \( g \) with \( i g \), one finds that maximal pseudo-supersymmetry implies \( H = 0 \). The integrability condition then simplifies to
\[ R_{abcd} = -g^2 X^2 (g_{ac} g_{bd} - g_{ad} g_{bc}), \] (2.8)
which implies that the spacetime is locally dS5.

3. Analysis of the gravitino Killing spinor equation

We proceed with the analysis of the Killing spinor equations, concentrating on the case for which the 1-form spinor bilinear generated from the Killing spinor is null; our basis is chosen such that this bilinear 1-form is \( e^\psi \).

The analysis of the gravitino Killing spinor equation has already been completed in the case of the minimal de Sitter supergravity theory in [21], using spinorial geometry techniques introduced in [36], in which spinors are identified with differential forms, and are simplified using appropriately chosen gauge transformations. The conditions on the geometry and the fluxes obtained from (2.4) can be read off from the results of section 3 of [21] which are listed in equations (3.1)–(3.20). It should be noted that we do not incorporate any of the conditions obtained from the Bianchi identity in [21] here, because the 2-form flux \( H \) which appears in (2.4) is not the exterior derivative of \( A \), in contrast to the minimal theory. Also, in order to establish the correspondence between the gravitino equation analysed in [21] and (2.4), one makes the replacements
\[ F \rightarrow \frac{\sqrt{3}}{2} H, \quad \chi \rightarrow -2\sqrt{3} g X, \quad \chi A \rightarrow -3 g A, \] (3.1)
where the quantities on the lhs of these expressions are respectively the field strength, cosmological constant and gauge potential of the minimal theory using the conventions of [21].
Following the reasoning set out in [21], one can without loss of generality work in a gauge for which the conditions on the geometry are
\[ d\epsilon^- = 0 \]
\[ d\epsilon^+ = -3g\epsilon^+ \wedge A - \omega_- \epsilon^- \wedge \epsilon^i - \omega_{[i} \epsilon^{j]} \epsilon^i \wedge \epsilon^j \]
\[ d\epsilon^i = 2\omega_- \epsilon^i \wedge \epsilon^i + B \wedge \epsilon^i + 3gX \star_3 \epsilon^i \]
\[ \mathcal{L}_N \epsilon^i = 0, \quad (3.2) \]
where \( N \) is the vector field dual to \( e^- \), and \( \omega \) is the five-dimensional spin connection. \( B \) is a 1-form given in terms of the spin connection by
\[ B = -2\omega_+ \epsilon^i. \quad (3.3) \]
In addition, the 1-form \( A \) satisfies
\[ -3gA = -\omega_- e^- + B \quad (3.4) \]
and the 2-form flux \( H \) satisfies
\[ H = \star_3 B + gX e^+ \wedge e^- + \frac{1}{2} e^- \wedge \star_3 (\omega_- \epsilon^i \wedge \epsilon^j). \quad (3.5) \]
Here \( \star_3 \) denotes the Hodge dual taken on with respect to the one-parameter family of 3-manifolds \( E \) equipped with the metric
\[ ds_E^2 = \delta_{ij} \epsilon^i \wedge \epsilon^j \quad (3.6) \]
whose volume form \( \epsilon^3 \) satisfies
\[ \Gamma_{ijk} \eta = (\epsilon^3)_{ijk} \eta. \quad (3.7) \]
Furthermore, in this gauge, the spinor \( \eta \) can be taken to be constant, and satisfies
\[ \Gamma_+ \eta = 0 \quad (3.8) \]
or equivalently
\[ \Gamma_{+-} \eta = \eta. \quad (3.9) \]

3.1. Analysis of the gaugino Killing spinor equation

The gaugino Killing spinor equation (2.6) can be rewritten as
\[ (2F^+_{+-} - F^+_{+i} \Gamma^i - F^+_j \epsilon^i \Gamma^j \epsilon^i + 2X^i H_{+-} + 2X^i H_{++} \Gamma^i + X^i H_{ij} \epsilon^i \wedge \epsilon^j + 2\nabla_+ X^i \Gamma^i + 4gX^i \wedge \epsilon^i + 6gQ^i J^i) \eta = 0, \quad (3.10) \]
where we have made use of the identity
\[ \Gamma^ij \eta = (\epsilon^3)^{ij} \Gamma^k \eta \quad (3.11) \]
and \( \Gamma^i = -\delta^i_0 \Gamma_0 \). On acting on (3.10) with the projectors \( \frac{1}{2}(1 \pm \Gamma_+) \), one obtains two equations of the form
\[ (\alpha + \beta \Gamma^i) \eta = 0 \quad (3.12) \]
for real \( \alpha, \beta \). As \( \eta \) is nonzero, the only solution of such an equation is \( \alpha = 0, \beta = 0 \), and on evaluating the resulting conditions on the fluxes and scalars obtained from these equations, one finds that
\[ \mathcal{L}_N X^i = 0 \quad (3.13) \]
\[ F_{i+}^I = 3g(XX^I - Q^I V_j) \]
\[ F_{ij}^I = X^I(\star B)_{ij} + \epsilon_{ij}^k \nabla_k X^I \]
\[ F_{i+}^I = 0, \]

where we adopt the convention that \((\epsilon^3)^{ijk} = (\epsilon^3)^{ijk}\), i.e. indices on the volume form are raised with the metric of signature \((+, +, +)\).

### 3.2. Introduction of local coordinates

As \(e^-\) is closed, one can introduce the local coordinates \(u, v, y^\alpha\) for \(\alpha = 1, 2, 3\) such that

\[ e^- = du, \quad N = \frac{\partial}{\partial v}, \quad e^\alpha = e^\alpha_\alpha dy^\alpha. \]  

We remark that possible \(du\) terms in \(e^\alpha\) can, without loss of generality, be removed by using a gauge transformation which leaves the spinor \(\eta\) invariant, as described in [21]. The three-dimensional dreibein \(e^\alpha_\alpha\) does not depend on \(v\), but in general depends on \(y^\alpha\) and \(u\), as do the scalars \(X^I\). We begin by determining the \(v\)-dependence of \(e^+\) and \(F^I\).

In order to determine the \(v\)-dependence of \(e^+\), note that (3.2) implies that

\[ \mathcal{L}_N e^+ = -3gA \]  

and (3.4) and (3.14) imply that

\[ \mathcal{L}_N A = 3g(X^2 - Q^I V_j) e^- \]  

and so on; combining these expressions, one obtains

\[ \mathcal{L}_N \mathcal{L}_N e^+ = -9g^2(X^2 - Q^I V_j) e^- \].

On integrating up (3.17), together with (3.4), implies that one can take, without loss of generality,

\[ A = 3g(X^2 - Q^I V_j) v du - \frac{1}{3g} B. \]  

Note also that if \(\tilde{\partial}\) denotes the exterior derivative restricted to hypersurfaces of constant \(u, v\), then (3.2) implies that

\[ \tilde{\partial} e^i = B \wedge e^i + 3g X \star_3 e^i. \]  

We remark that this structure is found in the study of a special class of Einstein–Weyl spaces [23] (see also [37]). This GT structure was also found in the analysis of the null class of pseudo-supersymmetric solutions in the minimal theory [21]8.

Furthermore, (3.13) and (3.2) imply that

\[ \mathcal{L}_N B = 0 \]  

and likewise (3.20) implies that

\[ \tilde{\partial} B + 3g \star_3 (XB + \tilde{\partial}X) = 0. \]  

It is then straightforward to integrate up (3.16) to find

\[ e^+ = dv - \frac{9}{2} g^2(X^2 - Q^I V_j) v^2 du + W du + v B + \phi e^i, \]  

where \(W\) is a \(v\)-independent function, and \(\phi = \phi^i e^i\) is a \(v\)-independent 1-form.

8 A brief introduction to GT spaces is given in the appendix of [21].
Next, we wish to determine the $v$-dependence of the field strengths $F^I$. First, note that

$$\mathcal{L}_N F^I = \mathcal{L}(i_N F^I) = \mathcal{L}(3g(XX^I - Q^{IJ} V_J)) \wedge du$$  \hspace{1cm} (3.24)

on making use of the Bianchi identity $dF^I = 0$. However, from (3.14) one also finds that

$$F^I = 3g(XX^I - Q^{IJ} V_J) e^+ \wedge e^- + \ast_3(X^I B + \tilde{d}X^I) + e^- \wedge S^I,$$  \hspace{1cm} (3.25)

where

$$S^I = S^I_\mu e^\mu.$$  \hspace{1cm} (3.26)

On taking the Lie derivative of this expression, and comparing with (3.24) one finds that

$$\mathcal{L}_N S^I = 3g(XX^I - Q^{IJ} V_J)(v B + du + \phi) \wedge du + \ast_3(XX^I B - \tilde{d}(XX^I - Q^{IJ} V_J) + T^I),$$  \hspace{1cm} (3.28)

where

$$T^I = T^I_\mu e^\mu.$$  \hspace{1cm} (3.29)

are $v$-independent 1-forms on $E$.

Having found this expression for the flux, we continue by imposing two consistency conditions. First, we require that $V^I F^I = dA$, where $F^I$ is given by (3.28) and $A$ is obtained from (3.19). One finds the following condition on the 1-forms $T^I$:

$$V^I T^I = -\frac{1}{3g} \tilde{B} + 3g(XX^I - Q^{IJ} V_J) \phi,$$  \hspace{1cm} (3.30)

where $\tilde{B} = \mathcal{L}_\phi B$. We also impose $X^I F^I = H$, where $H$ is given in (3.5), and we note that

$$\omega_{-ij} = -\frac{1}{2}(v d\phi + \tilde{d}\phi - \phi \wedge B)_i + \frac{1}{2}(\tilde{e}^\mu)_j - \frac{1}{2}(\tilde{e}^\mu)_i.$$  \hspace{1cm} (3.31)

One finds an additional condition on $T^I$:

$$X^I T^I = -\frac{1}{3} \ast_3(\tilde{d}\phi - \phi \wedge B + \delta_i \tilde{e}^\mu \wedge e^\mu).$$  \hspace{1cm} (3.32)

Next, we impose the Bianchi identities $dF^I = 0$; one obtains the conditions

$$\tilde{d} T^I = \mathcal{L}_\phi \ast_3(X^I B + \tilde{d}X^I) + 3g \tilde{d}(XX^I - Q^{IJ} V_J) \wedge \phi + 3g(XX^I - Q^{IJ} V_J) \tilde{d}\phi$$  \hspace{1cm} (3.33)

and

$$\tilde{d} \ast_3(X^I B + \tilde{d}X^I) = 0.$$  \hspace{1cm} (3.34)

These conditions exhaust the content of the Killing spinor equations.

### 3.3. The gauge and Einstein equations

In addition to pseudo-supersymmetry, we require that the field equations be satisfied. We start by evaluating the gauge field equations

$$d \ast(Q_{12} F^I) + \frac{1}{2} C_{IJK} F^I \wedge F^K = 0,$$  \hspace{1cm} (3.35)

where the five-dimensional volume form $\epsilon^5$ is related to the three-dimensional volume form $\epsilon^3$ by

$$\epsilon^5 = e^+ \wedge e^- \wedge \epsilon^3.$$  \hspace{1cm} (3.36)
One obtains the following condition:
\[
\tilde{\alpha} \ast (Q_I \mathcal{T}^I) = -3g \nabla_{\alpha} \frac{Q_{IJ}}{2} \left( \left( \frac{1}{2} XX_I - V_I \right) e^3 + \frac{3}{2} \phi \wedge B \ast dX_I + \frac{1}{2} \tilde{\phi} \wedge (B X_I - \tilde{d}X_I) \\
+ 3g \phi \wedge \ast (\left( \frac{1}{2} XX_I - V_I \right) B - \frac{1}{2} X_I dX + \frac{1}{2} X \tilde{d}X_I + Q_{Ij} \tilde{d}Q^{IJ} V_I) \\
+ \frac{1}{2} C_{IJK} X^I \wedge \ast (X^K B + \tilde{d}X^K) \right) .
\]
(3.37)
If one considers the limit of this equation in the pure supergravity case (i.e. having only the gravity supermultiplet), it becomes satisfied automatically. This is the rather peculiar feature of the null case of minimal ISUGRA theories, which was noted before [21, 24].

Next, we consider the Einstein equations. It is straightforward to show that the integrability conditions of the Killing spinor equations imply that all components of the Einstein equations hold automatically, with the exception of the ‘=’ component. The Einstein equations are
\[
R_{\alpha \beta} + Q_{IJ} F_{\alpha \beta \gamma} F^{\gamma} - Q_{IJ} \nabla_{\alpha} X^I \nabla_{\beta} X^J + g_{\alpha \beta} \left( \frac{1}{2} \nabla I - X^I X^J V_I V_J \right) = 0
\]
(3.38)
and hence the ‘=’ component is
\[
R_{\alpha \beta} = Q_{IJ} F_{\alpha \beta \gamma} F^{\gamma} - Q_{IJ} \nabla_{\alpha} X^I \nabla_{\beta} X^J = 0.
\]
(3.39)
This equation imposes the additional condition
\[
\tilde{\nabla}^{\alpha} W + \nabla^{\alpha} (W B_{\gamma}) - \tilde{\nabla}^{\alpha} \phi_{\gamma} - 3 g \phi_{\gamma} V_I (T^I)_{\alpha} - 3 (\tilde{e}^I)_{\gamma} X_I (\ast (\ast e) T^I)_{j} + \frac{1}{2} C_{IJK} (X^I)_{I} (T^J)_{j} + X^J X^J = 0.
\]
(3.40)
We also require that the solution satisfy the scalar field equations. However, the integrability conditions of the Killing spinor equations, together with the gauge field equations, imply that the scalar field equations hold with no additional conditions.

### 3.4. Summary

In order to construct a supersymmetric solution in the null class, we introduce the local coordinates \( u, v, y^a \), together with a family of GT 3-manifolds GT, and write the metric as
\[
d\sigma^2 = 2 e^+ e^- - d\sigma_{GT}^2, \quad d\sigma_{GT}^2 = \delta_{ij} e^i e^j, \quad e^i = e_u^i dy^u,
\]
(3.41)
where the basis elements \( e^i \) do not depend on \( v \), but can depend on \( u \), and \( e^- = du \),
\[
e^+ = dv - \frac{1}{2} g^v v^2 (X^2 - Q^{ij} V_I V_J) du + W du + v B + \phi e^i,
\]
(3.42)
where \( W \) is a \( v \)-independent function, and \( B = B^i e_i, \phi = \phi v e^i \) are \( v \)-independent 1-forms. Metric (3.41) is actually a Kundt wave, of which we will say more in section 6. Because the base space is GT, the basis elements \( e^i \) satisfy
\[
\tilde{d} e^i = B \wedge e^i + 3g X \ast_3 e^i,
\]
(3.43)
where \( \tilde{d} \) is the exterior derivative restricted to hypersurfaces of constant \( v, u \) and \( \ast_3 \) is the Hodge dual on GT. The scalars satisfy
\[
\tilde{d} \ast_3 (X^I B + \tilde{d}X^I) = 0
\]
(3.44)
and
\[
\tilde{d} B + 3g \ast_3 (X B + \tilde{d}X) = 0.
\]
(3.45)
The field strengths are
\[
F^I = 3g (XX^I - Q^{ij} V_I) (dv + \phi) \wedge du + \ast_3 (X^I B + \tilde{d}X^I)
+ du \wedge (-3gv d(X^I - Q^{ij} V_I) + T^I),
\]
(3.46)
where
\[ T^I = T^I \, e^I \] (3.47)
are v-independent 1-forms on GT. The 1-forms \( T^I \) must further satisfy
\[ \tilde{d} \star (Q_I T^I) = -3g \mathcal{L}_{\frac{\partial}{\partial u}} \left( \left( \frac{3}{2} XX - V_I \right) \text{dvol}_{\text{GT}} \right) + \frac{3}{2} \phi \wedge B \wedge \tilde{d}X_I \\
+ \frac{3}{2} \tilde{d}\phi \wedge (B X_I - \tilde{d}X_I) + \frac{1}{2} C_{IJK} T^J \wedge \star_3 (X^K B + \tilde{d}X^K) \\
+ 3g \phi \wedge \star_3 \left( \left( \frac{3}{2} XX - V_I \right) B - \frac{3}{2} X_I \tilde{d}X + \frac{3}{2} X \tilde{d}X_I + Q_IJ \tilde{d}Q_JN \right), \] (3.48)
are v-independent 1-forms on GT. The 1-forms \( T^I \) must further satisfy
\[ \tilde{d} \star \left( Q_I T^I \right) = -\frac{1}{3g} B + 3g(X^2 - Q^J V_J) \phi, \] (3.49)
\[ X_I T^I = -\frac{1}{3} \star_3 (\tilde{d} \phi - \phi \wedge B + \delta_{ij} e^i \wedge e^j) \] (3.50)
and
\[ \tilde{d} T^I = \mathcal{L}_{\frac{\partial}{\partial u}} \star_3 (X^I B + \tilde{d}X^I) + 3g \tilde{d}(X^I - Q^J V_J) \phi. \] (3.51)
Finally, the function \( W \) is found by solving
\[ \nabla^2 W + \tilde{V}^I W_{Ii} - \tilde{V}^I \phi_i - 3g \phi_i V_I (T^I)_i - (\tilde{e}^i)_i - 3(\tilde{e}^i)_i X_I (\star_3 T^I)_i^j \\
+ \frac{1}{2} C_{IJK} X^K ((T^I)_i (T^J)_i^j + X^I X^J) = 0. \] (3.52)
We remark that (3.48), (3.51) and (3.52) always admit solutions; however, it is not a priori apparent that (3.49) and (3.50) can always be solved.

4. Some simple examples

4.1. Near-horizon geometries

The near-horizon geometries found in [1] are all examples of pseudo-supersymmetric solutions in the null class. We take \( \frac{\partial}{\partial u} \) as a symmetry of the full solution, and set \( X_I \) to be constant, and \( W = 0, \phi = 0, T^I = 0 \). Then the remaining conditions on the geometry simplify considerably, and one finds
\[ e^- = du \]
\[ e^+ = dv - \frac{1}{2} g^2 (X^2 - Q^J V_J) v^2 \, du + vB \]
\[ e^i = e^i_\nu \, dy^\nu, \] (4.1)
where
\[ \tilde{d} e^i = B \wedge e^i + 3g X \star_3 e^i \] (4.2)
and the gauge field strengths are
\[ F^I = 3g(X^I - Q^J V_J) \, du \wedge dv + X^I \star_3 B, \] (4.3)
with the 1-form \( B \) satisfied,
\[ \tilde{d} B + 3g X \star_3 B = 0. \] (4.4)
This solution can be interpreted as the (pseudo-supersymmetric) near-horizon geometry of a (possibly non-pseudo-supersymmetric) black hole. The case for which \( V_I X^I = 0, B \neq 0 \) is of particular interest, as the spacetime geometry is \( M_3 \times S^2 \), where \( M_3 \) is a \( U(1) \) vibration over \( \text{AdS}_2 \) related to the near-horizon extremal Kerr solution, and the spatial cross sections of the event horizon are \( S^1 \times S^2 \).

9
4.2. A small model of real special geometry

A particularly simple class of solutions can be constructed with only one vector multiplet. Since the five-dimensional gravity multiplet does not contain scalars, there is only one physical scalar field $\varphi$, and we choose the only non-vanishing value for the symmetric constant $C_{IK}$ to be given by $C_{122} = 1$. With this choice, 

$$X^I = \left( \frac{\varphi^{-2}}{\sqrt{2}\varphi} \right), \quad X_I = \frac{1}{3} \left( \frac{\varphi^2}{\sqrt{2}\varphi} \right);$$

$$Q_{IJ} = \frac{1}{2} \text{diag} \left( \varphi^4, \varphi^{-2} \right), \quad Q_{IJ} = \frac{2}{\sqrt{3}} \text{diag} \left( \varphi^{-4}, \varphi^2 \right).$$

The equations resulting from the classification of the theory, summarized in subsection 3.4, must also be satisfied; however, we shall not present this analysis here.

This model is interesting as it provides a simple setting for non-supersymmetric solutions to a supersymmetric theory. The potential is given by

$$V = \frac{9}{\sqrt{3}} V_2 (V_2^2 \varphi^2 + 2\sqrt{2} V_1 \varphi^{-1}).$$

One can immediately see that if $V_2 = 0$, the theory is supersymmetric (i.e. it was the definite positiveness of $V$ that granted us a de Sitter-like $fSUGRA$ structure). This solution, however, is non-BPS, since the presence of $V_1$ means that its Killing spinor equation is not that of standard five-dimensional SUGRA. This kind of behaviour was already present in the classification of four-dimensional $fSUGRA$ (see section 4 of [25]) and one can also here make use of the oxidation/dimensional-reduction relations between supergravity theories (see e.g. [33] or section 5.3 in [34]) to obtain solutions to minimal $N = (2, 0) d = 6$ supergravity.

To achieve this, we make use of the results developed in [33], which provide the five-dimensional action (obtained through a KK compactification over an $S^1$) and compare it to the action for our model. The fields in these actions, however, do not promptly correspond, and they have to be appropriately identified. The dimensionally reduced action is given by

$$S = \int d^5x \sqrt{|g|} k \left( -R - \frac{1}{4} k^2 F^2(A) - \frac{1}{4} k^{-2} F^2(B) + \frac{\epsilon^\mu\nu\rho\sigma\tau}{8\sqrt{|g|}} k^{-1} F(A)_\mu F(B)_\rho F(B)_\sigma F(B)_\tau \right).$$

The kinetic term for the graviton in this action does not have a canonical form, so we proceed to rescale the metric by a scalar $k$, and hence unveil the kinetic term for the scalars hidden in the Einstein–Hilbert term

$$g_{\mu\nu} \rightarrow k^2 g_{\mu\nu}.$$ 

The action hence becomes

$$S = \int d^5x \sqrt{|g|} \left( -R + \frac{4}{3} k^{-2} (\partial k)^2 - \frac{1}{4} k^4 F^2(A) - \frac{1}{4} k^{-4} F^2(B) + \frac{\epsilon}{8\sqrt{|g|}} F(A) F(B) B \right).$$

We then compare this action with that of our model

$$S = \int dvol \left( -R + 3 \varphi^{-2} (\partial \varphi)^2 - \frac{1}{4} \varphi^4 (F^1)^2 - \frac{1}{4} \varphi^{-2} (F^2)^2 \right)$$

$$- \frac{\epsilon}{24\sqrt{|g|}} F^2 F^2 A^1 - \frac{\epsilon}{12\sqrt{|g|}} F^1 F^2 A^2,$$ 

9 We have adapted the conventions of [33] to those of here. In particular, the Riemann tensor has the opposite defining sign.
where the topological term is integrated by parts to identify the gauge fields. Upon inspection, 
\[ k = \alpha \phi^2, \]
where \( \alpha \) is just a real constant of integration, and
\[\begin{align*}
A &= -a^{-\frac{1}{2}}A^1 \\
B &= \pm a^{-\frac{1}{2}}A^2.
\end{align*}\]

(4.11)

To identify the remaining elements, we consider the supersymmetric variation of the gaugino, i.e. equation (2.6), and that obtained from the dimensional reduction of the gravitino KSE of the six-dimensional theory (see e.g. equation (1.15) in [33]). We obtain that
\[ A_\mu = -A^1_\mu, \quad B_\mu = -A^2_\mu \]
and
\[ k = -\phi^2. \]

This gives the identification of fields, and one can use the equations of the reduction on a circle to obtain the six-dimensional ones
\[ g^{(6)}_{\mu\nu} = g^{(5)}_{\mu\nu} - \phi^3 A^1_{\mu} A^1_{\nu}, \quad g^{(6)}_{\mu\nu} = -\phi^3, \quad H^{(6)}_{\alpha\beta\gamma} = \phi^{-\frac{1}{2}} F^2_{\alpha\beta\gamma}. \]

(4.12)

The geometries obtained in this manner, lifted from solutions to the model of (4.5) that fulfil the constraining equations of subsection 3.4, are solutions to minimal \( N = (2, 0) \) \( d = 6 \) SUGRA with (the bosonic part of the) action given by
\[ \int d^6x \sqrt{|g|} \left( R + \frac{1}{24}(H^-)^2 \right), \]

(4.13)

where as usual one considers the anti-self-duality of \( H^- \) as an additional constraint on the theory, rather than on the actual action.

5. Solutions with compact Einstein–Weyl structure

In this section, we concentrate on the case for which the GT space is compact and without boundary for all \( u, \) and \( X^I, T^I, \phi, W \) are smooth. The near-horizon geometries of the previous section are special examples of such solutions.

To proceed, consider (3.44) and contract with \( X^I. \) One finds that
\[ \tilde{\nabla}^I B_i + \frac{i}{2} Q_{IJ} \tilde{\nabla}^J X^I \tilde{\nabla} X^J = 0. \]

(5.1)

On integrating over GT, one finds
\[ \int_{GT} Q_{IJ} \tilde{\nabla}^J X^I \tilde{\nabla} X^J = 0 \]

(5.2)

and assuming that \( Q_{IJ} \) is positive definite, this implies that \( X^I = X^I(u), X = X(u). \)

We shall consider (3.43) with \( X \neq 0 \) and take GT to be the Berger sphere\(^\text{11}\) [23, 31]; one can write
\[ ds^2_{GT} = \frac{\cos^2 \mu}{g^2 X^2} \left( \cos^2 \mu (\sigma^2_L)^2 + (\sigma^2_R)^2 \right), \]
\[ B = \sin \mu \cos \mu \sigma^2_L, \]

(5.3)

where \( \mu = \mu(u), \) and \( \sigma^2_L \) are the left-invariant 1-forms on \( SU(2) \) satisfying
\[ \tilde{\alpha} \sigma^2_R = -\frac{1}{2} e^{i\theta} \sigma^2_L \wedge \sigma^2_L. \]

10 In flat indices, the six-dimensional space is labelled by \( a = \{0, 1, 2, 3, 4\} \) and \( z, \) where \( a \) span the five-dimensional space.

11 If \( X = 0, \) then GT is either \( S^4 \times S^1 \) or \( T^3, \) according to \( B \neq 0 \) or \( B = 0, \) respectively. We do not consider these cases here.
Next, note that (3.51) and (3.49) can be solved, making use of (3.45), to give

\[ T^I = -\mathcal{L}_\phi \left( \frac{X^I}{3gX} \right) + 3g(XX^I - Q^I V_j) \phi + \Theta^I, \]  

(5.4)

where \( \Theta^I \) are 1-forms on \( GT \) satisfying

\[ V_I \Theta^I = 0 \]  

(5.5)

and

\[ \mathfrak{d} \Theta^I = 0. \]  

(5.6)

Next, simplify (3.48) using (3.50) and (3.51), using the identity

\[ \mathfrak{d}((\delta_i \dot{e}^I \wedge e^I) = 2B \wedge (\delta_i \dot{e}^I \wedge e^I) + 9gX d\text{vol}_{GT} + 3gX L_{\Xi} d\text{vol}_{GT}. \]  

(5.7)

After some manipulation, one finds that (3.48) can be rewritten as

\[ \mathfrak{d} \left( \frac{3}{4} X_I \delta_i \dot{e}^I \wedge e^I - \frac{1}{3gX} Q_{ij} T^I \wedge B - \frac{1}{X} V_I \phi \wedge B \right) + \frac{9g}{2} g \left( \frac{1}{2} X_I - XX_I \right) d\text{vol}_{GT} + \left( \frac{3gV_I - \frac{9gX_I}{4} \right) L_{\Xi} d\text{vol}_{GT} \]

\[ + \frac{1}{2gX} (\dot{X}_I \mathbb{B} \wedge B + X_I (L_{\Xi}, \mathbb{B}) \wedge B) = 0. \]  

(5.8)

For the Berger sphere, the second and third lines of this expression can be written in the form \( Q^I(u) d\text{vol}_{GT} \), and hence on integrating over \( GT \), one obtains two separate conditions:

\[ \mathfrak{d} \left( \frac{3}{4} X_I \delta_i \dot{e}^I \wedge e^I - \frac{1}{3gX} Q_{ij} T^I \wedge B - \frac{1}{X} V_I \phi \wedge B \right) = 0 \]  

(5.9)

and

\[ \frac{9g}{2} g \left( \frac{1}{2} X_I - XX_I \right) d\text{vol}_{GT} + \left( \frac{3gV_I - \frac{9gX_I}{4} \right) L_{\Xi} d\text{vol}_{GT} \]

\[ + \frac{1}{2gX} (\dot{X}_I \mathbb{B} \wedge B + X_I (L_{\Xi}, \mathbb{B}) \wedge B) = 0. \]  

(5.10)

On using (5.4), (5.9) and (3.50) can be rewritten as

\[ \mathfrak{d} \left( \mathbb{B} \Theta^I + \frac{1}{3gX} \Theta^I \wedge B + 3g(XX^I - Q^I V_j) \mathbb{B} \phi + \frac{1}{2} X_I \delta_i \dot{e}^I \wedge e^I \right) = 0 \]  

(5.11)

and

\[ - L_{\Xi} \left( \frac{1}{3gX} \mathbb{B} \right) + gX \phi + X_I \Theta^I = - \frac{1}{3} \mathbb{B} \left( \mathfrak{d} \phi - \phi \wedge B + \delta_i \dot{e}^I \wedge e^I \right). \]  

(5.12)

On contracting (5.11) with \( X_I \) and using (5.12), one finds

\[ L_{\Xi} d\text{vol}_{GT} = - \frac{X}{X} d\text{vol}_{GT} \]  

(5.13)

which for the Berger sphere implies that the squashing of the \( S^3 \) is \( u \)-independent, i.e. \( \mu \) is constant in (5.3), and the \( u \) dependence of the metric on \( GT \) is in the overall conformal factor of \( X^{-2} \). It follows that

\[ L_{\Xi} \mathbb{B} = - \frac{X}{X} \mathbb{B} \]  

(5.14)

and hence (5.10) can be simplified to give

\[ XX_I - \frac{1}{2} XX_I - \frac{X}{X} V_I + \frac{1}{18g^2 X^2} (-XX_I - XX_I) B^2 = 0. \]  

(5.15)
On contracting this expression with $X^I$ and $Q^J V_J$, one finds

$$X B^2 = 0, \quad (X^2 - Q^J V_J) X = 0. \quad (5.16)$$

Suppose first that $X^2 - Q^J V_J \neq 0$. Then $X = 0$ and (5.15) further implies that

$$\tilde{X}_I = 0 \quad (5.17)$$

also. Furthermore, contracting (5.11) with $V_J$ gives

$$\tilde{d} \star_3 \phi = 0 \quad (5.18)$$

so (5.11) is

$$\tilde{d} \star_3 \Theta^I + \Theta^I \wedge \star_3 B = 0. \quad (5.19)$$

As $\tilde{d} \Theta^I = 0$ and the Berger sphere is simply connected, this implies that the $\Theta^I$ are exact

$$\Theta^I = \tilde{d} H^I, \quad (5.20)$$

where

$$\tilde{\nabla}^2 H^I + B^i \tilde{\nabla}_i H^I = 0. \quad (5.21)$$

It follows that $\tilde{d} H^I = 0$, and so $\partial_\mathcal{H} = 0$. Also, (3.52) implies that $W = W(u)$. It remains to consider the condition

$$gX \phi = -\frac{1}{\tilde{\nabla}^2} \star_3 (\tilde{d} \phi - \phi \wedge \mathcal{B}). \quad (5.22)$$

To proceed, note that the Ricci tensor of GT is

$$\tilde{R}_{ij} = (B^i B_i + \frac{5}{2} s^2 X^2) \delta_{ij} - B_i B_j \quad (5.23)$$

as a consequence of (3.43). After some manipulation, it can also be shown that (5.22) implies

$$\tilde{\nabla}_j \phi^2 + B^i \tilde{\nabla}_i \phi^2 = 2 \tilde{\nabla}_j (\phi) \tilde{\nabla}_j (\phi) + 3 (B^2 \phi^2 - (\mathcal{B} \phi)^2). \quad (5.24)$$

As the rhs of this expression is a sum of two non-negative terms, it follows from the maximum principle that $\phi^2$ is constant, and

$$\tilde{\nabla}_j (\phi) = 0, \quad B^2 \phi^2 - (\mathcal{B} \phi)^2 = 0. \quad (5.25)$$

These conditions also imply that one can take, without loss of generality,

$$\phi = k \sigma^I \quad (5.26)$$

for constant $k$, irrespective of whether $B$ vanishes or not. Also note that by making a coordinate transformation of the form

$$\tilde{u} = f(u), \quad v = h(u) \tilde{u} + g(u), \quad \psi = \tilde{\psi} + \ell(u), \quad (5.27)$$

where we take the vector field dual to $\sigma^I$ to be $\frac{\tilde{d}}{\tilde{\nabla}^2}$, one can choose the functions $f, h, g, \ell$ appropriately such that the form of the metric and gauge field strengths is preserved, and in the new coordinates, $W = 0$.

To summarize: if $X^2 - Q^J V_J \neq 0$, then the Berger sphere squashing parameter $\mu$ and $X^I$ are constant, and

$$ds^2 = 2 du \left( dv - \frac{9}{2} s^2 v^2 (X^2 - Q^J V_J) du + (v \sin \mu \cos \mu + k) \sigma_L \right) - \frac{\cos^2 \mu}{9 s^2 X^2} \left( \cos^2 \mu (\sigma_L^I)^2 + (\sigma_L^I)^2 + (\sigma_L^I)^2 \right) \quad (5.28)$$

$$F^I = 3g(XX^I - Q^J V_J) \, dv \wedge du + \frac{X^I}{3gX} \sin \mu \cos \mu \mu \sigma_L \wedge \sigma_L. \quad (5.28)$$

where $k$ is constant.
In the special case, $X^2 - Q^I V_I V_J = 0$; then there are two possibilities. If $B \neq 0$, then (5.15) implies that the $X^I$ are again constant, whereas if $B = 0$, then (5.15) implies that

$$X_I = \frac{1}{3} X^{-1} V_I + X^2 Z_I$$  \hspace{1cm} (5.29)$$

for constant $Z_I$. Also (5.11) and (5.12) imply

$$H^I = \frac{3}{X} (X X^I - Q^I V_J) X^L H^L = L^I$$  \hspace{1cm} (5.30)$$

and

$$\phi = -\frac{1}{2X} \tilde{d}(X_I H^I) + X_I H^L B + k(u) \sigma^3,$$  \hspace{1cm} (5.31)$$

where $H^I$ are functions, $\sigma^I = \tilde{d} H^I$ and $L^I = L^I(u)$ satisfy $X_I L^I = 0$.

6. Solutions with a recurrent vector field

The geometry which we have found (3.41) is a five-dimensional Kundt wave [27]. This is a kind of metric in the Walker form [38] that admits a null vector generating a geodesic null congruence that is hypersurface orthogonal, non-expanding and shear-free. It is a special case of the higher dimensional Kundt metrics studied in [29, 28].

To see this, consider the null vector field $N = \partial/\partial v$. The congruence of integral curves affinely parametrized by $v$ fulfills $\nabla_N N = 0$ (geodesic). $N$ is hypersurface orthogonal, i.e. $e^v \wedge de^v = 0$, which means that the congruence is twist-free. It is also non-expanding $\nabla_{(\mu} N_{\nu)} = 0$ and shear-free $\nabla_{(\mu} N_{\nu)} \nabla^{(\mu} N^{\nu)} = 0$. Hence, it is a Kundt metric.

As in the case of minimal de Sitter supergravity, the null vector field $N$ is not Killing. We shall consider the necessary and sufficient conditions for $N$ to be recurrent, which places additional restrictions on the holonomy of the Lévi-Civita connection. Recurrency with respect to this connection is defined as

$$\nabla_{(\mu} N_{\nu)} = C_{\mu} N^\nu,$$  \hspace{1cm} (6.1)$$

where $C$ is the recurrent 1-form [39]. $D$-dimensional geometries that allow recurrent vector fields have their holonomy group contained in the similitude group $Sim(D - 2)$. This is a $(D^2 - 3D + 4)/2$-dimensional subgroup of the Lorentz group $SO(D - 1, 1)$, and it is isomorphic to the Euclidean group $E(D - 2)$ augmented by homotheties. It is also the maximal proper subgroup of the Lorentz group, and hence connections admitting $Sim(D - 2)$ have a minimal (non-trivial) holonomy reduction.

Theories with the similitude group have received some attention in the past few years, as they have been shown to hold interesting physical features. They are linked to theories with vanishing quantum corrections [40] and to the recently proposed theories of Very Special Relativity [41] and General Very Special Relativity [42].

For our solutions, note that

$$\nabla_{(\mu} N_{\nu)} = \frac{1}{2} B_{\mu \nu}$$  \hspace{1cm} (6.2)$$

and so a necessary condition for the $N$ to be recurrent is $B = 0$. In fact, it is also sufficient, and one finds that if $B = 0$, then

$$\nabla_{(\mu} N_{\nu)} = -9g^2 (X^2 - Q^I V_I V_J)_{(\mu} N_{\nu)}.$$  \hspace{1cm} (6.3)$$

In the remainder of this section, we take $B = 0$, and investigate the resulting conditions imposed on the geometry. To proceed, first consider (3.43). If $X \neq 0$, then the GT space is $S^4$, whereas if $X = 0$, it is flat. Next consider (3.44); on contracting with $X_J$, this condition is equivalent to

$$Q_{IJ} \tilde{\nabla}^I \tilde{\nabla}^J = 0$$  \hspace{1cm} (6.4)$$
and assuming that $Q_{IJ}$ is positive definite, this implies that $X^I = X^I(u)$. Then (3.49) and (3.51) further imply that
\[ T^I = 3g(XX^I - Q^{IJ}V_J)\phi + K^I \]  
(6.5)

where $K^I$ are 1-forms on $\text{GT}$ satisfying
\[ \tilde{d}K^I = 0, \quad V_I K^I = 0 \]  
(6.6)

and (3.50) simplifies to
\[ gX\phi + X_I K^I = -\frac{1}{4} \star_3 \tilde{d}\phi. \]  
(6.7)

The conditions obtained from (3.48) are
\[ \dot{X} = 0 \]  
(6.8)

and
\[ \tilde{d} \star_3 K^I = -3g(XX^I - Q^{IJ}V_J)\tilde{d} \star_3 \phi + 3gXX^I d\text{vol}_{\text{GT}}. \]  
(6.9)

In particular, note that if $X^2 - Q^{IJ}V_I V_J \neq 0$, then on contracting this condition with $V_I$, one finds that
\[ \tilde{d} \star_3 \phi = 0, \quad \tilde{d} \star_3 K^I = 3gXX^I d\text{vol}_{\text{GT}}. \]  
(6.10)

The function $W$ must satisfy
\[ \tilde{\nabla}^2 W = \tilde{\nabla}^2 \phi_I + 9g^2(X^2 - Q^{IJ}V_J) \phi \cdot \phi - \frac{1}{2} C_{IKL}X^K((T^I)_I)(T^J)_J + \dot{X}^2 \dot{X}^I \]  
(6.11)

as a consequence of (3.52).

Given $K^I, \phi, X^I, W$ satisfying these conditions, the metric and field strengths are
\[ ds^2 = 2 du (dv - \frac{1}{2} g^{IJ} \phi^I v_J) du + W du + \phi^I du - d\text{vol}_{\text{GT}} \]  
(6.12)

As a simple example, take $X \neq 0, X^I$ constant with $K^I = 0$. Then the GT space is $S^3$. All of the conditions are satisfied if one takes $\phi = \xi_i \sigma_i^J$, where $\xi_i = \xi_i(u)$, and with this choice of $\phi$, (6.11) implies that $W$ is a $(u)$-dependent harmonic function on $S^3$. This solution describes gravitational waves propagating through a generalized squashed Nariai universe. Generically, these waves will be plane-fronted waves, as $N$ is not a Killing vector. However, if $X^2 - Q^{IJ}V_I V_J = 0$, they are pp-waves.

Alternatively, taking again $X^I$ constant with $X \neq 0$, one can instead set $\phi = 0$ and
\[ K^I = K^I(u) \sigma^I_u. \]  
(6.13)

Then the conditions which must be satisfied are
\[ V_I K^I = X_I K^I = 0, \quad \tilde{\nabla}^2 W = Q_{IJ}(K^I)_J. \]  
(6.14)

In this case, if $K^I \neq 0$, one must have non-vanishing $W$.

It was shown that for recurrent solutions in the minimal theory, all scalar curvature invariants constructed purely algebraically from the Riemann tensor are constant [21]. By computing the Ricci scalar for the recurrent solutions constructed here, it can be seen that a necessary and sufficient condition for the Ricci scalar to be constant is that $Q^{IJ}V_I V_J$ is constant. This condition is also sufficient to ensure that all the other algebraic scalar curvature invariants are also constant. To see this, define
\[ \psi_{\mu\nuk} = \frac{2}{3} \nabla_{\mu} \nabla_{[\nu} \phi_{k]} + \frac{1}{3} \nabla_{\nu} \nabla_{[\mu} \phi_{k]} - \frac{1}{3} \nabla_k \nabla_{[\nu} \phi_{\mu]} \]  
(6.15)

\[ \theta_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} W + 9g^2(X^2 - Q^{IJ}V_J) v_{[\mu} \phi_{\nu]} + \frac{1}{4} (d\phi)_{[\mu} (d\phi)_{\nu]}, \]
and note that
\[ N^\mu \psi_{\mu\nu\lambda} = 0, \quad N^\mu \theta_{\mu\nu} = 0. \tag{6.16} \]

The Riemann tensor satisfies
\[ R_{\mu\nu\lambda\tau} = (R^0)_{\mu\nu\lambda\tau} + 4 N_{[\mu\nu]} [\lambda, \tau] + N_{\mu} \psi_{\nu\lambda\tau} - N_{\nu} \psi_{\mu\lambda\tau} + N_{\lambda} \psi_{\nu\tau\mu} - N_{\tau} \psi_{\lambda\mu\nu}, \tag{6.17} \]
where \( R^0 \) is the Riemann tensor of the metric \( g^0 \) obtained from the metric in (6.12) setting \( \omega = 0, \phi = 0 \), where \( (R^0)_{\mu\nu\lambda\tau} = g^0_{\mu\kappa} (R^0)_{\kappa\nu\lambda\tau} \). The metric \( g^0 \) is the metric on \( \text{AdS}_2 \times \text{GT}, \mathbb{R}^{1,1} \times \text{GT} \) and \( dS_2 \times \text{GT} \) if \( X^2 - Q^{IJ} V^IV^J \) is negative, zero or positive. It then follows, from exactly the same reasoning as set out for the minimal case in [21], that all algebraic scalar curvature invariants constructed from the metric \( g \) and Riemann tensor \( R \) are identical to the same invariants constructed from \( g^0, R^0 \) and hence are constant. The status of scalar curvature invariants constructed from covariant derivatives of the Riemann tensor remains to be determined.

For \( B \neq 0 \), one can also provide the construction with a \( \text{Sim} \)-holonomy structure. This is relevant for the embedding of the Berger sphere considered in section 5. By considering the gravitino Killing spinor equation, one obtains
\[ \nabla_{\mu} N_{\nu} = -3 g A_{\mu} N_{\nu} + \frac{1}{2} S_{\mu\nu} \psi = -3 g A_{\mu} N_{\nu}, \tag{6.18} \]
Next, define a new covariant derivative \( D \) by
\[ D_{\mu} N_{\nu} = \nabla_{\mu} N_{\nu} - S_{\mu\nu} \psi = -3 g A_{\mu} N_{\nu}, \quad \text{where} \quad S = \frac{1}{2} \psi [XI F^I]. \tag{6.19} \]
can be interpreted as a totally antisymmetric torsion 3-form; hence, the new connection is metric-compatible. It is clear that \( N \) is recurrent with respect to the connection \( D \), and consequently its holonomy is a subgroup of \( \text{Sim}(3) \) [39].

Acknowledgments

The work of WS was supported in part by the National Science Foundation under grant number PHY-0903134. JG was supported by the EPSRC grant EP/F069774/1. AP was supported by a CSIC scholarship JAEPre-07-00176. AP would like to thank Patrick Meessen and Tomás Ortín for very useful discussions and technical guiding, as well as the Department of Fundamental Physics at the Chalmers Tegniska Högskola, and in particular Ulf Gran, for a research stay during the spring/summer of 2010.

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