ON AN $H^r(\text{curl}, \Omega)$ ESTIMATE FOR A MAXWELL-TYPE SYSTEM IN
CONVEX DOMAINS

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Abstract. In bounded convex domains, the regularity estimates of a vector field $u$ with its
$\text{div} u$, $\text{curl} u$ in $L^r$ space and the tangential components or the normal component of $u$ over
the boundary in $L^r$ space, are established for $1 < r < \infty$. As an application, we derive an
$H^r(\text{curl}, \Omega)$ estimate for solutions to a Maxwell-type system with an inhomogeneous boundary
condition in convex domains.

1. Introduction

This paper is concerned with the regularity of a vector field $u$ with its $\text{div} u, \text{curl} u \in L^r(\Omega)$ and the tangential components $\nu \times u$ or the normal component $\nu \cdot u$ on boundary
in $L^r(\partial\Omega)$, where $1 < r < \infty$, $\Omega$ is a bounded convex domain in $\mathbb{R}^3$ and $\nu(x)$ denotes the
unit outer normal vector at $x \in \partial\Omega$. Based on the established estimates, we then study
the well-posedness of the following Maxwell-type system

$$\text{curl}(A(x)\text{curl} u) + u = F + \text{curl} f \quad \text{in } \Omega, \quad \nu \times u = g \quad \text{on } \partial\Omega, \quad (1.1)$$

where the coefficient $A(x) = (a^{ij}(x))$ denotes a $3 \times 3$ matrix with real-valued, bounded,
measurable entries satisfying the uniform ellipticity condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^3 a^{ij} \xi_i \xi_j \leq \Lambda |\xi|^2$$

for all $\xi \in \mathbb{R}^3$ and for some positive constants $0 < \lambda < \Lambda < \infty$.

Before stating our main results we would like to mention that, the regularity estimates
of a vector field $u$ by means of $\text{div} u$ and $\text{curl} u$ are fundamental questions, and such
estimates are useful in the study of various partial differential systems including Navier-
Stokes equations in fluid mechanics, Maxwell’s equations in electromagnetism field, and
Ginzburg-Landau system for superconductivity. For smooth domains, the estimates on Sobolev spaces $W^{1,r}$ with $1 < r < \infty$ are well-known. We refer to [18, 26] for details.

In the case of non-smooth domains, Costabel in [6] considered the div-curl estimates
when $r = 2$ in Lipschitz domains and showed the $H^{1/2}(\Omega)$ regularity for vector fields. These results were generalized to $r \in (3/2 - \epsilon, 2 + \epsilon)$ with $\epsilon$ depending on the Lipschitz

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character of domains by D. Mitrea, M. Mitrea and J. Pipher (see [20]), and also the range for $r$ is sharp (see [11, 10]). It should also be noted in [15] that if the boundary $\partial \Omega \in C^1$, then one can obtain the corresponding estimates for $r \in (1, \infty)$. One may ask, under what additional conditions (weaker than $C^1$ regularity) for Lipschitz domains, the range for $r$ can be extended to the interval $(1, \infty)$?

Note that any convex domain is Lipschitz but may not be $C^1$, and also the convexity of the domain may improve the regularity, see for instance [3, 4, 12, 21]. Therefore, it is important to examine the estimates in convex domains. To state our results, we need to introduce the well-known Bessel potential spaces $L^r_\alpha(\mathbb{R}^3)$ and Besov spaces $B^{r,q}_\alpha(\Omega)$, see [15]. First, we define $L^r_\alpha(\mathbb{R}^3)$ by

$$L^r_\alpha(\mathbb{R}^3) = \{(I - \Delta)^{-\alpha/2}g : g \in L^r(\mathbb{R}^3)\}$$

with norm

$$\|f\|_{L^r_\alpha(\mathbb{R}^3)} = \|(I - \Delta)^{\alpha/2}f\|_{L^r(\mathbb{R}^3)},$$

where

$$(I - \Delta)^{\alpha/2} = \mathcal{F}^{-1} \left(1 + |\xi|^2\right)^{\alpha/2} \mathcal{F}$$

and $\mathcal{F}$ is the Fourier transform. Define $L^r_\alpha(\Omega)$ as the space of restrictions of functions in $L^r_\alpha(\mathbb{R}^3)$ to $\Omega$ with the usual quotient norm

$$\|f\|_{L^r_\alpha(\Omega)} = \inf \{\|h\|_{L^r_\alpha(\mathbb{R}^3)} : h = f \text{ in } \Omega\}.$$ 

Let $0 < \alpha < 1$, $1 \leq r \leq \infty$ and $1 \leq q < \infty$. We say that a function $f$ belongs to Besov space $B^{r,q}_\alpha(\mathbb{R}^3)$ if the norm

$$\|f\|_{B^{r,q}_\alpha(\mathbb{R}^3)} + \left(\int_{\mathbb{R}^3} \frac{\|f(x + t) - f(x)\|_{L^r}^q}{|t|^{3+\alpha q}} dt\right)^{1/q} < \infty.$$ 

Define the space $B^{r,q}_\alpha(\Omega)$ as the space of restrictions of functions in $B^{r,q}_\alpha(\mathbb{R}^3)$ to $\Omega$ with the usual quotient norm.

Suppose $1 < r < \infty$ and $\alpha > 0$. Then we have the following inclusion relations (see Theorem 5 in [23, Chapter V])

$$B^{r,2}_\alpha \subset L^r_\alpha \subset B^{r,r}_\alpha \quad \text{if } r \geq 2; \quad B^{r,r}_\alpha \subset L^r_\alpha \subset B^{r,2}_\alpha \quad \text{if } r \leq 2.$$ 

The first result now reads:

**Theorem 1.1.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^3$. Assume that $\text{div } u \in L^r(\Omega)$, $\text{curl } u \in L^r(\Omega)$ and $\nu \cdot u \in L^r(\partial \Omega)$ with $2 \leq r < \infty$. Then $u \in L^r_{1/r}(\Omega)$, and we have the estimate

$$\|u\|_{L^r_{1/r}(\Omega)} \leq C \left(\|\text{div } u\|_{L^r(\Omega)} + \|\text{curl } u\|_{L^r(\Omega)} + \|\nu \cdot u\|_{L^r(\partial \Omega)}\right),$$

where the constant $C$ depends on $r$ and the Lipschitz character of $\Omega$. 
To prove Theorem 1.1 we apply the Helmholtz-Weyl decomposition for vector fields in bounded domains (see [18, Theorem 2.1]):

\[ u = \nabla p + \text{curl} \ w_u. \]  

(1.3)

Our strategy is to get the estimates for the gradient part \( \nabla p \) and for the curl part \( \text{curl} \ w_u \) respectively. The gradient part \( \nabla p \) satisfies the Laplace equation with Neumann boundary condition, which can be established by the result of Geng and Shen in [12] for Laplace-Neumann problem. For the estimate of curl \( \w_u \), the vector \( \w_u \) satisfies a curl-curl system (see (2.1)). As the proof of Theorem 5.15(a) in [15] by Jerison and Kenig, it suffices to establish the \( L^\infty \) estimate for \( \text{curl} \ w_u \). To prove this, we shall use the technique developed by Cianchi and Maz’ya in [3, 4] in which the \( L^\infty \) gradient estimates of solutions to the divergence form elliptic systems with Uhlenbeck type structure were treated. At last, by the complex interpolation, we can obtain the \( L^{r_1/r}(\Omega) \) estimate for \( \text{curl} \ w_u \) if \( 2 \leq r < \infty \).

Remark 1.2. We need to mention that for Lipschitz domains, D. Mitrea, M. Mitrea and J. Pipher in [20] obtained the \( B^{p,2}_{1/r}(\Omega) \) estimates under the assumptions of Theorem 1.1 if \( r \in (3/2 - \varepsilon, 2] \) with \( \varepsilon \) depending on the Lipschitz character of domains. The \( B^{p,2}_{1/r}(\Omega) \) estimate for \( r \in (1, 3/2 - \varepsilon] \) is still open.

For the tangential component \( \nu \times u \) given, we have

**Theorem 1.3.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^3 \). Assume that \( u \in L^r(\Omega) \), \( \text{div} u \in L^r(\Omega) \), \( \text{curl} u \in L^r(\Omega) \) and \( \nu \times u \in L^r(\partial\Omega) \) with \( 1 \leq r < \infty \). Then

\[ \| \nu \times u \|_{L^r(\partial\Omega)} \leq C \left( \| \text{div} u \|_{L^r(\Omega)} + \| \text{curl} u \|_{L^r(\Omega)} + \| \nu \times u \|_{L^r(\partial\Omega)} \right), \]

(1.4)

where the constant \( C \) depends on \( r \) and the Lipschitz character of \( \Omega \). Also, we have \( u \in L^{r_1/r}(\Omega) \) if \( 2 \leq r < \infty \) and \( u \in B^{2,2}_{1/r}(\Omega) \) if \( 1 < r < 2 \).

To obtain the estimate of \( \nu \cdot u \) on boundary, the method of the complex interpolation is no longer applied. Our strategy now is by introducing a divergence-free vector such that the boundary estimate can be reduced to the estimates of a double layer potential and the Laplace equation with Dirichlet boundary condition.

With Theorem 1.1 and Theorem 1.3 at our disposal, following the real variable method used in [11] by Geng in Lipschitz domains, we then study the \( H^r(\text{curl}; \Omega) \) well-posedness of the Maxwell-type system (1.1) in convex domains.

We mention that if the coefficient matrix \( A(x) \) is taken to be a constant, M. Mitrea, D. Mitrea and J. Pipher in [20] considered the \( L^p \) estimates of inhomogeneous boundary value problems for Maxwell equations in Lipschitz domains; while M. Mitrea in [19] showed the well-posedness in the Sobolev-Besov spaces \( H^{s,r}_0(\text{curl}; \Omega) \) with the smoothness index \( s \) and
the integrability index \( r \) belonging to \( R_\Omega \), where \( R_\Omega \) defined in [15] (also see [19] for details) is the optimal range of solvability of Poisson equation with inhomogeneous Dirichlet or Neumann boundary condition in Sobolev-Besov \( L^r_s(\Omega) \) spaces. For system (1.1) with the \( W^{s,\infty} \)-regular matrix \( A(x) \), Kar and Sini in [17] recently, by the perturbation argument, derived an \( H^{s,r}_0(\text{curl};\Omega) \) estimate if the indices \( (s, r) \) lie in a small region in the interior of \( R_\Omega \).

In contrast to the method used in [17], we will apply the real variable method which was used in [11] to treat the \( \text{div}(A(x)\nabla) \) operator, to the \( \text{curl}(A(x)\text{curl}) \) operator. As in [11], we also assume that the coefficient \( A(x) \) belongs to \( \text{VMO}(\Omega) \), that is
\[
\limsup_{r \to 0} \inf_{\rho \leq r} \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} \left| a^{ij}(x) - \frac{1}{|\Omega_\rho|} \int_{\Omega_\rho} a^{ij}(y) dy \right| dx = 0,
\]
where \( \Omega_\rho \) is the intersection \( \Omega \cap B_\rho \) with Lebesgue measure \( |\Omega_\rho| \), and \( B_\rho \) denotes the ball with radius \( \rho \) centered at the points of \( \Omega \). The following spaces \( H^r(\text{curl}, \Omega) \) for \( 1 < r < \infty \) are well known:
\[
H^r(\text{curl}, \Omega) = \{ u \in L^r(\Omega) : \text{curl} u \in L^r(\Omega) \}.
\]
For \( 1 < r < \infty \) and \( 0 < s < 1 \), we let \( B^{s,r}(\partial \Omega) \) denote the Besov space consisting of measurable functions on \( \partial \Omega \) such that
\[
\| f \|_{B^{s,r}(\partial \Omega)} := \| f \|_{L^r(\partial \Omega)} + \left( \int_{\partial \Omega} \int_{\partial \Omega} \frac{|f(P) - f(Q)|^r}{|P - Q|^{2+sr}} d\sigma(P) d\sigma(Q) \right)^{1/r} < \infty,
\]
and \( B^{-s,r/(r-1)}(\partial \Omega) \) is the dual of the Besov space \( B^{s,r}(\partial \Omega) \). Denote by \( \text{Div} \) the divergence operator on \( \partial \Omega \), the definition of which can be found in [20, p.143].

Now we state the \( H^r(\text{curl}, \Omega) \) estimate for system (1.1).

**Theorem 1.4.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^3 \). Assume that the coefficient matrix \( A(x) \) is symmetric, bounded measurable, uniformly elliptic and in \( \text{VMO}(\Omega) \). Let \( 1 < r < \infty \). Suppose that \( F \in L^r(\Omega) \), \( f \in L^r(\Omega) \), \( g \in L^r(\partial \Omega) \) with \( v \cdot g = 0 \) and \( \text{Div} g \in B^{-1/r,r}(\partial \Omega) \), then there exists a unique solution \( u \in H^r(\text{curl}, \Omega) \) of system (1.1), and the solution \( u \) satisfies the estimate
\[
\| u \|_{L^r(\Omega)} + \| \text{curl} u \|_{L^r(\Omega)} \leq C \left( \| F \|_{L^r(\Omega)} + \| f \|_{L^r(\Omega)} + \| \text{Div} g \|_{B^{-1/r,r}(\partial \Omega)} + \| g \|_{L^r(\partial \Omega)} \right),
\]
(1.5)
where the constant \( C \) depends on \( r \) and the Lipschitz character of \( \Omega \). Moreover, assume further that \( \text{div} F \in L^r(\Omega) \), then \( u \in L^r_{1/r}(\Omega) \) if \( 2 \leq r < \infty \) and \( u \in B^{1/r,2}_{1/r}(\Omega) \) if \( 1 < r < 2 \).

**Remark 1.5.** Using the proof of Theorem 1.4, if the domain \( \Omega \) is Lipschitz and \( r \in (3/2 - \epsilon, 3 + \epsilon) \) for some positive constant \( \epsilon \) depending on the Lipschitz character of \( \Omega \), we can also obtain the inequality (1.5). This can be viewed as an improvement of the \( s = 0 \) setting of Kar and Sini’s \( H^{s,r}_0(\text{curl};\Omega) \) estimate in [17], see Theorem 3.2 for details.
The organization of this paper is as follows. In Section 2, we first establish the $L^\infty$ estimates for vector fields with the normal component or the tangential components vanishing on the boundary. Then we will give the proofs of Theorem 1.1 and Theorem 1.3. In Section 3, applying Theorem 1.1 and Theorem 1.3, we prove Theorem 1.4. At last, we show the well-posedness of the Maxwell-type system in Lipschitz domains.

Throughout the paper, the bold typeface is used to indicate vector quantities; normal typeface will be used for vector components and for scalars.

2. Proofs of Theorem 1.1 and Theorem 1.3

Consider the system
\[
\text{curl curl } w = \text{curl } u, \quad \text{div } w = 0 \quad \text{in } \Omega, \quad \nu \times w = 0 \quad \text{on } \partial \Omega
\]  
and the system
\[
\text{curl curl } \hat{w} = \text{curl } u \quad \text{and} \quad \text{div } \hat{w} = 0 \quad \text{in } \Omega,
\]
\[
\nu \cdot \hat{w} = 0 \quad \text{and} \quad \nu \times \text{curl } \hat{w} = \nu \times u \quad \text{on } \partial \Omega.
\]

To define the respective weak solutions of systems (2.1) and (2.2), we introduce two spaces (15):
\[
X_r^\sigma \equiv \{u \in H^r(\text{curl}, \Omega) : \text{div } u = 0 \quad \text{in } \Omega, \quad \nu \cdot u = 0 \quad \text{on } \partial \Omega\},
\]
\[
V_r^\sigma \equiv \{u \in H^r(\text{curl}, \Omega) : \text{div } u = 0 \quad \text{in } \Omega, \quad \nu \times u = 0 \quad \text{on } \partial \Omega\},
\]
where $1 < r < \infty$.

**Definition 2.1.** We say $w$ is a weak solution to system (2.1) if $w \in V_r^\sigma$ and
\[
\int_\Omega \text{curl } w \cdot \text{curl } \Phi dx = \int_\Omega u \cdot \text{curl } \Phi dx
\]
for any $\Phi \in V_{r/(r-1)}^\sigma$.

We say $\hat{w}$ is a weak solution to system (2.2) if $\hat{w} \in X_r^\sigma$ and
\[
\int_\Omega \text{curl } \hat{w} \cdot \text{curl } \Phi dx = \int_\Omega u \cdot \text{curl } \Phi dx
\]
for any $\Phi \in X_{r/(r-1)}^\sigma$.

As stated in the introduction, to prove Theorem 1.1 by applying the complex interpolation, the key step is to establish the $L^\infty$ estimate for the curl of solutions to the curl-type system (2.1). We need to mention that the proof of $L^\infty$ estimate is inspired by Cianchi and Maz’ya in [3, 4] where the divergence-type elliptic systems with Uhlenbeck type structure were treated.
We first establish an inequality for vector fields with the normal component or the tangential component vanishing in convex domains. A similar result can be found in [12, Lemma 2.2].

**Lemma 2.2.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^3 \) with smooth boundary. Let \( \mathbf{H} \in C^2(\Omega) \cap C^1(\Omega) \) satisfying \( \nu \cdot \mathbf{H} = 0 \) or \( \nu \times \mathbf{H} = 0 \) on \( \partial \Omega \). Then

\[
\int_{\{||\mathbf{H}|| = t\}} t |\nabla|\mathbf{H}|dS \leq \int_{\{||\mathbf{H}|| = t\}} t (|\text{curl}\mathbf{H}| + |\text{div}\mathbf{H}|) dS + \int_{\{||\mathbf{H}|| > t\}} (|\text{curl}\mathbf{H}|^2 + |\text{div}\mathbf{H}|^2) dx. \tag{2.3}
\]

**Proof.** We first note that

\[
\text{div}(\nabla||\mathbf{H}||) + \text{div}((\text{curl} \mathbf{H} \times \mathbf{H})) = \nabla \text{div} \mathbf{H} + |\nabla \mathbf{H}|^2 - |\text{curl}\mathbf{H}|^2.
\]

Then by Green’s formula, we have

\[
\int_{\{||\mathbf{H}|| > t\}} (\text{div}(\nabla||\mathbf{H}||) + \text{div}(\text{curl} \mathbf{H} \times \mathbf{H})) d\mathbf{x} = \int_{\{||\mathbf{H}|| = t\} \cap \partial \Omega} \sum_{i,j=1}^{3} \nu_i H_j \partial_j H_i dS + \int_{\{||\mathbf{H}|| = t\} \setminus \partial \Omega} (\text{curl} \mathbf{H} \times \mathbf{H} + \nabla||\mathbf{H}||) \cdot \nu(x) dS
\]

and

\[
\int_{\{||\mathbf{H}|| > t\}} (\nabla \text{div} \mathbf{H} + |\nabla \mathbf{H}|^2 - |\text{curl}\mathbf{H}|^2) d\mathbf{x} = \int_{\partial \{||\mathbf{H}|| > t\}} \nu \cdot \mathbf{H} \text{div} \mathbf{H} dS + \int_{\{||\mathbf{H}|| > t\}} (|\nabla \mathbf{H}|^2 - |\text{curl}\mathbf{H}|^2 - |\text{div}\mathbf{H}|^2) d\mathbf{x}.
\]

Therefore,

\[
\int_{\{||\mathbf{H}|| = t\} \cap \partial \Omega} \left( \sum_{i,j=1}^{3} \nu_i H_j \partial_j H_i - \nu \cdot \mathbf{H} \text{div} \mathbf{H} \right) dS = \int_{\{||\mathbf{H}|| = t\} \setminus \partial \Omega} (\mathbf{H} \text{div} \mathbf{H} - \text{curl} \mathbf{H} \times \mathbf{H} - \nabla||\mathbf{H}||) \cdot \nu(x) dS
\]

\[
+ \int_{\{||\mathbf{H}|| > t\}} (|\nabla \mathbf{H}|^2 - |\text{curl}\mathbf{H}|^2 - |\text{div}\mathbf{H}|^2) d\mathbf{x}.
\]

From [13, p.135-137] and by the condition \( \nu \cdot \mathbf{H} = 0 \) or \( \nu \times \mathbf{H} = 0 \) on \( \partial \Omega \), then it follows that

\[
\int_{\{||\mathbf{H}|| = t\} \cap \partial \Omega} \left( \sum_{i,j=1}^{3} \nu_i H_j \partial_j H_i - \nu \cdot \mathbf{H} \text{div} \mathbf{H} \right) dS \leq 0.
\]
This gives that

\[- \int_{\{|H| = t\}} \nabla|H||H| \cdot \nu(x) dS \leq \int_{\{|H| = t\}} t \left( |\text{curl } H| + |\text{div } H| \right) dS \]

\[+ \int_{\{|H| > t\}} \left( |\text{curl } H|^2 + |\text{div } H|^2 \right) dx. \tag{2.4} \]

Note that, for \(x \in \{|H| = t\} \cap \{|\nabla|H|| \neq 0\}\) we have

\[\nu(x) = -\frac{\nabla|H|}{|\nabla|H||}.\]

From Sard’s theorem, we know that

the image \(|H|(X)\) has Lebesgue measure 0, where \(X = \{|\nabla|H|| = 0\}\).

Then, the inequality (2.3) follows since (2.4). \(\square\)

To show the \(L^\infty\) estimate for \(\text{curl } w\) of system (2.1), it is necessary to introduce the well-known Lorentz spaces. Let \(f\) be a measurable function defined on \(\Omega\). We define the distribution function of \(f\) as

\[f_\ast(s) = \mu(\{|f| > s\}), \quad s > 0,\]

and the nonincreasing rearrangement of \(f\) as

\[f^\ast(t) = \inf\{s > 0, f_\ast(s) \leq t\}, \quad t > 0.\]

The Lorentz space is defined as

\[L^{m,q}(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable, } \|f\|_{L^{m,q}(\Omega)} < \infty\} \quad \text{with } 1 \leq m < \infty\]

equipped with the quasi-norm

\[\|f\|_{L^{m,q}(\Omega)} = \left( \int_0^\infty (t^{1/m} f^\ast(t))^q \frac{dt}{t} \right)^{1/q}, \quad 1 \leq q < \infty,\]

see for example [1, p.223-p.228] for a more precise definition. Furthermore, the property

that the Lebesgue space \(L^r(\Omega)\) is continuously imbedded into \(L^{m,q}(\Omega)\) if \(r > m\) will be used in the following proofs.

**Lemma 2.3.** Let \(\Omega\) be a bounded convex domain in \(\mathbb{R}^3\). Let \(u \in H^r(\text{curl}, \Omega)\) with \(r > 3\) and let \(w\) be the weak solution of system (2.1). Then we have

\[\|\text{curl } w\|_{L^\infty(\Omega)} \leq C \|\text{curl } u\|_{L^{3,1}(\Omega)}, \quad (2.5)\]

where the constant \(C\) depends on the Lipschitz character of the domain \(\Omega\).
Proof. We divide the proof into three steps.

Step 1. We prove (2.5) under the following assumptions:
(i) the vector \( u \in C^3(\Omega) \cap C^2(\bar{\Omega}) \);
(ii) the domain \( \Omega \) is smooth.

Let \( H = \text{curl} \, w \). From Lemma 2.2, we now have
\[
\int_{\{ |H| = t \}} t|\nabla|H|dS \leq \int_{\{ |H| = t \}} t|\text{curl} \, u|dS + \int_{\{ |H| > t \}} |\text{curl} \, u|^2dx. \tag{2.6}
\]
We need to mention that the inequality (2.6) is quite similar to the inequality (6.16) in [3]. Therefore, to obtain the estimate (2.5) under the assumptions (i) and (ii) the proof in [3] is applicable. For reader’s convenience, we give the outline of the proof in appendix.

Step 2. We remove the assumption (i). We take a sequence \( u_k \in C^3(\bar{\Omega}) \) such that \( u_k \) converges to \( u \) in \( H^r(\text{curl}, \Omega) \). Let \( w_k \) be the solution of system (2.1) with \( \text{curl} \, u \) replaced by \( \text{curl} \, u_k \). Then we have \( w_k \in C^3(\bar{\Omega}) \) and by (A.5) in appendix we have
\[
\| \text{curl} \, w_k \|_{L^\infty(\Omega)} \leq C(\Omega) \| \text{curl} \, u_k \|_{L^{3,1}(\Omega)}. \tag{2.7}
\]
From system (2.1), we know that \( w_k \in V^2_\sigma \) and \( \text{curl} \, w_k \in X^2_\sigma \). Note that the spaces \( X^2_\sigma \) and \( V^2_\sigma \) are both continuously imbedded into \( H^1(\Omega) \) in convex domains (see [2, Theorem 2.17]), then we can deduce that
\[
\| w_k \|_{H^1(\Omega)} + \| \text{curl} \, w_k \|_{H^1(\Omega)} \leq C(\Omega) \| \text{curl} \, u_k \|_{L^2(\Omega)}.
\]
Then there exists a vector \( w \in H^1(\Omega) \) such that \( w \) is the weak solution of system (2.1). Moreover, there exists a subsequence of \( \{ w_k \}_{k=1}^\infty \), still denoted by \( \{ w_k \}_{k=1}^\infty \), such that
\[
\text{curl} \, w_k \to \text{curl} \, w \quad \text{in} \ L^2(\Omega)
\]
and
\[
\text{curl} \, w_k \to \text{curl} \, w \quad \text{almost everywhere on} \ \Omega.
\]
From (2.7), the solution \( w \) satisfies the estimate (2.5).

Step 3. We remove the assumption (ii). We look for a sequence \( \{ \Omega_m \}_{m\in\mathbb{N}} \) of bounded domains \( \Omega_m \subset \Omega \) such that \( \Omega_m \in C^\infty, \Omega_m \to \Omega \) as \( m \to \infty \) with respect to the Necas-Verchota’s approximation, see [22, 25]. Let \( w_m \) be the solution of system (2.1) with the domain \( \Omega \) replaced by \( \Omega_m \). Then by (A.5) in appendix we have
\[
\| \text{curl} \, w_m \|_{L^\infty(\Omega_m)} \leq C \| \text{curl} \, u \|_{L^{3,1}(\Omega)}. \tag{2.8}
\]
where the constant \( C \) depends on the Lipschitz character of \( \Omega_m \), and hence depends on the Lipschitz character of \( \Omega \).

From system (2.1), we can also conclude that \( w_m \in V^2_\sigma \) and \( \text{curl} \, w_m \in X^2_\sigma \). Then by Theorem 2.17 in [2] again, we have
\[
\| w_m \|_{H^1(\Omega_m)} + \| \text{curl} \, w_m \|_{H^1(\Omega_m)} \leq C(\Omega) \| \text{curl} \, u \|_{L^2(\Omega)}.
\]

Let \( \tilde{w}_m \) be the extension of \( w_m \) such that \( \tilde{w}_m \) is 0 outside of \( \Omega_m \). Then we obtain that \( \tilde{w}_m \) converges to \( w \) weakly in \( L^2(\Omega) \) and \( \text{curl} \, \tilde{w}_m \) converges to \( \text{curl} \, w \) weakly in \( L^2(\Omega) \), where \( w \in H^1(\Omega) \) is the weak solution of system (2.1). From (2.9), for any compact subset \( K \) of \( \Omega \) we have

\[
\text{curl} \, w_m \to \text{curl} \, w \quad \text{almost everywhere on any compact set } K.
\]

By (2.8), the solution \( w \) satisfies the estimate (2.5). We finish our proof. \( \Box \)

By Theorem 2.2 and Theorem 2.5 in [3], then from Lemma 2.3 and the Helmholtz-Weyl decomposition (1.3), we immediately get

**Corollary 2.4.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^3 \). Let \( u \in L^{3,1}(\Omega) \), \( \text{div} \, u \in L^{3,1}(\Omega) \) and \( \text{curl} \, u \in L^{3,1}(\Omega) \). Assume further that \( \nu \times u = 0 \) or \( \nu \cdot u = 0 \) on \( \partial \Omega \), then \( u \in L^\infty(\Omega) \) and we have the estimate

\[
\| u \|_{L^\infty(\Omega)} \leq C \left( \| \text{div} \, u \|_{L^{3,1}(\Omega)} + \| \text{curl} \, u \|_{L^{3,1}(\Omega)} \right),
\]

where the constant \( C \) depends only on the Lipschitz character of \( \Omega \).

Next, we prove the \( L^r_{1/r}(\Omega) \) estimate for \( \text{curl} \, w \) of system (2.1).

**Lemma 2.5.** Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^3 \). Let \( u \in H^r(\text{curl}, \Omega) \) with \( r > 2 \) and let \( w \) be the weak solution of system (2.1). Then we have

\[
\| \text{curl} \, w \|_{L^r_{1/r}(\Omega)} \leq C \| \text{curl} \, u \|_{L^r(\Omega)},
\]

where the constant \( C \) depends on \( r \) and the Lipschitz character of the domain \( \Omega \).

**Proof.** The proof is similar to that of Theorem 5.15(a) in [15]. Let \( \mathcal{E} \) be Stein’s extension operator mapping from functions on \( \Omega \) to functions on \( \mathbb{R}^3 \) (see [23]). Denote by \( \Lambda^z \) the fractional integral operator

\[
\Lambda^z f = \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{z/2} \mathcal{F} f(\xi) \right).
\]

Then we define the mapping

\[
\mathcal{M}_z: \quad \text{curl} \, u \mapsto \Lambda^z \mathcal{E} \text{curl} \, w.
\]

From Lemma 2.3 for \( \text{Re} z = 0 \) the mapping \( \mathcal{M} \) maps \( L^{3,1}(\Omega) \) (and hence \( L^\infty \)) to \( BMO(\mathbb{R}^3) \). For \( \text{Re} z = 1 \), it maps \( L^2(\Omega) \) to \( L^2(\mathbb{R}^3) \). Therefore, by the complex interpolation, when \( z = 2/r \) it maps \( L^r(\Omega) \) to \( L^r(\mathbb{R}^3) \), which proves that if \( \text{curl} \, u \in L^r(\Omega) \), then \( \text{curl} \, w \in L^r_{1/r}(\Omega) \). This shows that (2.10) holds. \( \Box \)

We now begin to prove our main theorems.
Proof of Theorem 1.1. Consider the following Laplace equation with Neumann boundary condition
\[ \Delta p = \text{div} \mathbf{u} \quad \text{in } \Omega, \quad \frac{\partial p}{\partial \nu} = \nu \cdot \mathbf{u} \quad \text{on } \partial \Omega. \] (2.11)
Let
\[ \tilde{p} = -\frac{1}{4\pi} \int_{\Omega} \frac{1}{|x-y|} \text{div} \mathbf{u}(y) dy. \] (2.12)
Then the function \( p - \tilde{p} \) satisfies
\[ \Delta (p - \tilde{p}) = 0 \quad \text{in } \Omega, \quad \frac{\partial (p - \tilde{p})}{\partial \nu} = \nu \cdot \mathbf{u} - \frac{\partial \tilde{p}}{\partial \nu} \quad \text{on } \partial \Omega. \]
The solvability of the solution \( p - \tilde{p} \) to the above equation can be found in [12, Theorem 1.1], which implies the solvability of problem (2.11). Moreover, Theorem 1.1 in [12] gives the estimate
\[ \| \nabla (p - \tilde{p}) \|_{L^r(\Omega)} \leq C(r, \Omega) \left( \| \nu \cdot \mathbf{u} \|_{L^r(\partial \Omega)} + \| \frac{\partial \tilde{p}}{\partial \nu} \|_{L^r(\partial \Omega)} \right) \]
where we have used the trace theorem and the Calderon-Zygmund inequality in the last inequality. Applying the Calderon-Zygmund inequality again for \( \tilde{p} \), we have
\[ \| \nabla \tilde{p} \|_{L^r(\Omega)} \leq C(r, \Omega) \left( \| \nu \cdot \mathbf{u} \|_{L^r(\partial \Omega)} + \| \text{div} \mathbf{u} \|_{L^r(\Omega)} \right) \quad \text{for } 2 < r < \infty. \] (2.13)
Now we let
\[ \tilde{\mathbf{u}} = \nabla p + \text{curl } \mathbf{w}, \]
where \( p \) is defined in (2.11) and \( \mathbf{w} \) is the weak solution of system (2.1). Then we have
\[ \text{div}(\tilde{\mathbf{u}} - \mathbf{u}) = 0, \quad \text{curl}(\tilde{\mathbf{u}} - \mathbf{u}) = 0 \quad \text{in } \Omega, \quad \nu \cdot (\tilde{\mathbf{u}} - \mathbf{u}) = 0 \quad \text{on } \partial \Omega, \]
which shows that \( \tilde{\mathbf{u}} = \mathbf{u} \) in \( \Omega \). Therefore, the inequality (1.2) holds true since (2.13) and (2.10). We finish our proof. \( \square \)

We are now in the position to show Theorem 1.3. In the proof, we shall use the symbol \((h)^*\) to denote the nontangential maximal function of \( h \) in \( \Omega \), defined as
\[ (h)^*(x) = \sup \{ |u(y)|, y \in \Omega, |x-y| < 2 \text{dist}(y, \partial \Omega) \}, \quad x \in \partial \Omega; \]
we also introduce the tangential derivative of a function \( \psi \) defined on \( \partial \Omega \) by \( \nabla_{\text{tan}} \psi \), we refer to [21, p.2518] for its definition, in particular, if \( \psi \) is a Lipschitz function then \( \nabla_{\text{tan}} \psi = \nu \times \nabla \psi \) almost everywhere on \( \partial \Omega \).
Proof of Theorem 1.3 Let \( \hat{p} \) be the weak solution of Laplace equation
\[
\Delta \hat{p} = \text{div} \ u \quad \text{in } \Omega, \quad \hat{p} = 0 \quad \text{on } \partial \Omega,
\]
and let \( \tilde{p} \) be defined as (2.12). The function \( \hat{p} - \tilde{p} \) satisfies
\[
\Delta (\hat{p} - \tilde{p}) = 0 \quad \text{in } \Omega, \quad \hat{p} - \tilde{p} = -\tilde{p} \quad \text{on } \partial \Omega.
\]
For \( 1 < r < \infty \), we have
\[
\| (\nabla (\hat{p} - \tilde{p}))^* \|_{L^r(\partial \Omega)} \leq C(r, \Omega) \left( \| \tilde{p} \|_{L^r(\partial \Omega)} + \| \nabla \tan \tilde{p} \|_{L^r(\partial \Omega)} \right) \leq C(r, \Omega) \| \tilde{p} \|_{W^{2,r}(\Omega)},
\]
where the first inequality follows from Theorem 3.11 in [21], and the last inequality holds true since the trace theorem. Then we have, by the Calderon-Zygmund inequality for \( \tilde{p} \),
\[
\| \nabla \hat{p} \|_{L^r(\partial \Omega)} \leq C \| \text{div} \ u \|_{L^r(\Omega)} \quad \text{for } 1 < r < \infty,
\]
where the constant \( C \) depends on \( r \) and the Lipschitz character of \( \Omega \).

Let \( \hat{w} \) be the weak solution of system (2.2). Then we introduce
\[
\hat{v}(x) = \phi(x) - \zeta(x)
\]
with
\[
\phi(x) = \frac{1}{4\pi} \int_{\Omega} \frac{\text{curl} \ u(y)dy}{|x - y|}, \quad \zeta(x) = \frac{1}{4\pi} \int_{\partial \Omega} \frac{\nu \times u(y)dS_y}{|x - y|}.
\]
Using Green's formula, we have
\[
\int_{\partial \Omega} \frac{1}{|x - y|} \nu \times u(y)dS_y = \int_{\Omega} \frac{1}{|x - y|} \text{curl} \ u(y)dy + \int_{\Omega} \nabla_y \left( \frac{1}{|x - y|} \right) \times u(y)dy
\]
\[
= \int_{\Omega} \frac{1}{|x - y|} \text{curl} \ u(y)dy - \int_{\Omega} \nabla_x \left( \frac{1}{|x - y|} \right) \times u(y)dy.
\]
The last integral of the above equality is divergence-free, and hence we have \( \text{div} \ \hat{v} = 0 \) in \( \Omega \). By noting that \( \Delta \zeta = 0 \) in \( \Omega \), we then obtain
\[
\text{curl} \ \text{curl} \ \hat{v} = \text{curl} \ u \quad \text{in } \Omega.
\]
In the following, we establish the estimate of \( \text{curl} \ \hat{v} \). From the trace theorem and the Calderon-Zygmund inequality, it follows that
\[
\| \text{curl} \phi \|_{L^r(\partial \Omega)} \leq C(r, \Omega) \| \phi \|_{W^{2,r}(\Omega)} \leq C(r, \Omega) \| \text{curl} \ u \|_{L^r(\Omega)}.
\]
Therefore, it suffices to establish the estimate of \( \text{curl} \ \zeta \). Applying Theorem 1.1 in [12] again (since \( \Delta \zeta = 0 \) in \( \Omega \)), we have
\[
\| (\text{curl} \ \zeta)^* \|_{L^r(\partial \Omega)} \leq C(r, \Omega) \left\| \frac{\partial \zeta}{\partial \nu} \right\|_{L^r(\partial \Omega)} \quad \text{for } 1 < r < \infty.
\]
By the equality (see e.g. [9, 10])
\[
\frac{\partial \zeta}{\partial \nu} = 2\pi \nu \times u + \int_{\partial \Omega} \frac{1}{|x - y|} \nu \times u(y) dS_y,
\]
then noting that we have, from [9, Theorem 1.0] and [5],
\[
\left\| \int_{\partial \Omega} \frac{1}{|x - y|} \nu \times u(y) dS_y \right\|_{L^r(\partial \Omega)} \leq C(r, \Omega) \|\nu \times u\|_{L^r(\partial \Omega)},
\]
we immediately obtain the estimate
\[
\|\text{curl } \zeta^*\|_{L^r(\partial \Omega)} \leq C(r, \Omega) \|\nu \times u\|_{L^r(\partial \Omega)}.
\]

Combining with the estimate of curl $\phi$, we now get
\[
\|\text{curl } \mathbf{v}\|_{L^r(\partial \Omega)} \leq C \left( \|\text{curl } u\|_{L^r(\Omega)} + \|\nu \times u\|_{L^r(\partial \Omega)} \right) \quad \text{for } 1 < r < \infty,
\]
where the constant $C$ depends on $r$ and the Lipschitz character of $\Omega$.

Let $\mathbf{h} = \mathbf{w} - \mathbf{v}$. Then we have
\[
\text{curl } \text{curl } \mathbf{h} = 0 \quad \text{and} \quad \text{div } \mathbf{h} = 0 \quad \text{in } \Omega,
\]
\[
\nu \times \text{curl } \mathbf{h} = \nu \times (u - \text{curl } \mathbf{v}) \quad \text{on } \partial \Omega.
\]

From the first equation, there exists a function $\hat{\phi}$ with $\int_{\partial \Omega} \hat{\phi} dx = 0$ such that $\text{curl } \mathbf{h} = \nabla \hat{\phi}$ in $\Omega$. Then from the boundary condition, $\hat{\phi}$ satisfies
\[
\Delta \hat{\phi} = 0 \quad \text{in } \Omega; \quad \nabla \text{tan } \hat{\phi} = \nu \times (u - \text{curl } \mathbf{v}) \quad \text{on } \partial \Omega.
\]

From Theorem 3.11 in [21] we have, for $1 < r < \infty$,
\[
\|\nabla \hat{\phi}\|_{L^r(\partial \Omega)} \leq C(r, \Omega) \left( \|\nu \times (u - \text{curl } \mathbf{v})\|_{L^r(\partial \Omega)} \right).
\]

From (2.15) and the above inequality, it follows that
\[
\|\text{curl } \mathbf{h}\|_{L^r(\partial \Omega)} \leq C(r, \Omega) \left( \|\text{curl } u\|_{L^r(\Omega)} + \|\nu \times u\|_{L^r(\partial \Omega)} \right) \quad \text{for } 1 < r < \infty.
\]

Therefore, by (2.15) again we have
\[
\|\text{curl } \mathbf{w}\|_{L^r(\partial \Omega)} \leq C \left( \|\text{curl } u\|_{L^r(\Omega)} + \|\nu \times u\|_{L^r(\partial \Omega)} \right) \quad \text{for } 1 < r < \infty,
\]
where the constant $C$ depends on $r$ and the Lipschitz character of $\Omega$.

If we let
\[
\hat{u} = \nabla \hat{\phi} + \text{curl } \mathbf{w},
\]
then we have
\[
\text{div}(\hat{u} - u) = 0, \text{curl}(\hat{u} - u) = 0 \quad \text{in } \Omega, \quad \nu \times (\hat{u} - u) = 0 \quad \text{on } \partial \Omega.
\]

This gives $\hat{u} = u$ in $\Omega$. Therefore, the inequality (1.4) holds true since (2.14) and (2.16).

Using Corollary 10.3(c) in [20], we finish our proof. \qed
3. Proof of Theorem 1.4

We first prove a weak reverse Hölder inequality near the boundary for a curl-type system with the coefficient matrix symmetric and uniformly elliptic.

Lemma 3.1. Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^3 \), and let the matrix \( A(x) \) be symmetric, bounded measurable, uniformly elliptic and in \( \text{VMO}(\Omega) \). Let \( Q \in \partial \Omega \) and \( 0 < s < s_0 \) for some \( s_0 \). Suppose that \( H \) satisfies

\[
\text{curl}\left(A(x) \text{curl } H \right) = \text{curl}\left((1 - \varphi)\Psi\right) \quad \text{and} \quad \text{div } H = 0 \quad \text{in } \Omega
\]

with the boundary condition \( \nu \times H = 0 \) on \( \partial \Omega \), where \( \Psi \in L^r(\Omega) \) and \( \varphi \in C^\infty(\mathbb{R}^3) \) is a cut-off function such that \( \varphi = 1 \) on \( B(Q,4s) \) and \( \varphi = 0 \) outside of \( B(Q,8s) \). Then for any \( r > 2 \) we have

\[
\left\{ \frac{1}{s^3} \int_{\Omega \cap B(Q,s)} \left| \text{curl } H \right|^r dx \right\}^{1/r} \leq C \left\{ \frac{1}{s^3} \int_{\Omega \cap B(Q,2s)} \left| \text{curl } H \right|^2 dx \right\}^{1/2}, \quad (3.1)
\]

where the constant \( C \) depends on \( r, s_0 \) and the Lipschitz character of \( \Omega \).

Proof. From the assumptions, we have \( \text{curl}\left(A(x) \text{curl } H \right) = 0 \) in \( B(Q,4s) \). Thus there exists a function \( \phi \) defined on \( B(Q,4s) \) such that

\[
A(x) \text{curl } H = \nabla \phi. \quad (3.2)
\]

Then \( \phi \) satisfies

\[
\text{div}\left(A^{-1}(x)\nabla \phi \right) = 0 \quad \text{in } B(Q,4r) \cap \Omega, \quad \nu \cdot (A^{-1}(x)\nabla \phi) = 0 \quad \text{on } B(Q,4r) \cap \partial \Omega.
\]

Based on Theorem 2.1 in [12], then from Lemma 4.1, Lemma 4.2 and Theorem 2.1 in [11], it follows that

\[
\left\{ \frac{1}{s^3} \int_{\Omega \cap B(Q,s)} |\nabla \phi|^r dx \right\}^{1/r} \leq C \left\{ \frac{1}{s^3} \int_{\Omega \cap B(Q,2s)} |\nabla \phi|^2 dx \right\}^{1/2}.
\]

By applying the inequality

\[
\frac{1}{A^r}|\nabla \phi|^r \leq |A^{-1}(x)\nabla \phi|^r \leq \frac{1}{A^r}|\nabla \phi|^r,
\]

and then using (3.2), we immediately get (3.1). \( \square \)

We now give the proof of Theorem 1.4.

Proof of Theorem 1.4. We decompose

\[
\mathbf{u} = \mathbf{u}_1 + \nabla \mathbf{u}_2 + \mathbf{u}_3,
\]

where \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \) are to be determined.
Step 1. Construct $u_1$. Consider the following Neumann problem
\[
\text{div}(A^{-1}(x) \nabla \phi) = 0 \quad \text{in } \Omega, \quad \nu \cdot (A^{-1}(x) \nabla \phi) = \text{Div } g \quad \text{on } \partial \Omega.
\]
This problem studied by Geng in [11] is solvable in Lipschitz domains if \(\text{Div } g \in B^{-1/r,r}(\partial \Omega)\) with \(3/2 - \epsilon < r < 3 + \epsilon\), see [11, Lemma 5.2]. To prove this, it suffices to establish a weak reverse Hölder inequality
\[
\left( \frac{1}{s^3} \int_{B(x_0,s) \cap \Omega} |\nabla v|^r dx \right)^{\frac{1}{r}} \leq C_0 \left( \frac{1}{s^3} \int_{B(x_0,2s) \cap \Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}}
\]
for any \(0 < s < s_0\) (\(s_0\) depends on the domain) and any \(v \in W^{1,2}(B(x_0,2s) \cap \Omega)\) satisfying the above Neumann problem in \(B(x_0,2s) \cap \Omega\) with the boundary condition \(\text{Div } g = 0\) on \(B(x_0,2s) \cap \partial \Omega\), see Theorem 1.1 and Lemma 5.1 in [11]. For Lipschitz domains, the weak reverse Hölder inequality only holds for \(2 < r \leq 3 + \epsilon\) (see [11, Lemma 4.1]). However, for any convex domains, the range of the index \(r\) can be extended to \(2 < r < \infty\), which may be proved by applying Theorem 2.1 in [12] to Lemma 4.1, Lemma 4.2 and Theorem 2.1 in [11]. Based on this, the conclusion of Lemma 5.2 in [11] can be obtained for any \(1 < r < \infty\) if the domain \(\Omega\) is convex. That is, the above Neumann problem is solvable for any \(1 < r < \infty\), and we can deduce the estimate
\[
\|\nabla \phi\|_{L^r(\Omega)} \leq C\|\text{Div } g\|_{B^{-1/r,r}(\partial \Omega)},
\]
where the constant \(C\) depends on \(r\) and the Lipschitz character of \(\Omega\).

We now solve the following div-curl system
\[
\text{curl } u_1 = A^{-1}(x) \nabla \phi, \quad \text{div } u_1 = 0 \quad \text{in } \Omega, \quad \nu \times u_1 = g \quad \text{on } \partial \Omega.
\]
By the proof of Theorem 10.1 in [20], we can conclude that there exists a unique solution in \(L^r(\Omega) \cap L^2(\Omega)\) space to this system. Applying Theorem 1.3 we have \(\nu \cdot u \in L^r(\partial \Omega)\) and the estimate (1.4) holds. From the integral representation formula for vector fields (see [20, Theorem 3.2]) and recalling that \(A(x)\) is positive, then we obtain the estimate
\[
\|u_1\|_{L^r(\Omega)} \leq C(r,\Omega) \left( \|\nabla \phi\|_{L^r(\Omega)} + \|g\|_{L^r(\partial \Omega)} \right),
\]
see the estimate of \(\zeta(x)\) in the proof of Theorem 1.3 or we may use Corollary 10.3(c) in [20]. Combining with the estimate for \(\nabla \phi\) and by the first equation in the div-curl system, we immediately get
\[
\|u_1\|_{L^r(\Omega)} + \|\text{curl } u_1\|_{L^r(\Omega)} \leq C \left( \|\text{Div } g\|_{B^{-1/r,r}(\partial \Omega)} + \|g\|_{L^r(\partial \Omega)} \right),
\]
where the constant \(C\) depends on \(r\) and the Lipschitz character of \(\Omega\).

Step 2. Construct $u_2$. By Theorem 1.3 in [12], we take the Helmholtz decomposition to $F$ and to $u_1$ :
\[
F = \nabla p_F + \text{curl } w_F, \quad u_1 = \nabla p_{u_1} + \text{curl } w_{u_1}.
\]
Let $u_2 \in W^{1,r}_0(\Omega)$ be the weak solution of the form

$$\int_\Omega \nabla u_2 \cdot \nabla \psi = \int_\Omega (\nabla p F - \nabla p u_1) \cdot \nabla \psi$$

for any $\psi \in W^{1,r/(r-1)}_0(\Omega)$.

Then there exists a constant $C$ depending on $r$ and the Lipschitz character of $\Omega$ such that (see e.g. [16])

$$\| \nabla u_2 \|_{L^r(\Omega)} \leq C \left( \| F \|_{L^r(\Omega)} + \| u_1 \|_{L^r(\Omega)} \right),$$

where the last inequality follows from Theorem 1.3 in [12].

Step 3. Construct $u_3$. Consider the system

$$\text{curl } (A(x) \text{curl } u_3) + u_3 = F - u_1 - \nabla u_2 + \text{curl } f \quad \text{in } \Omega,$$

$$\nu \times u_3 = 0 \quad \text{on } \partial \Omega.$$  \hspace{1cm} (3.3)

Now we have $\text{div} (F - u_1 - \nabla u_2) = 0$ in $\Omega$. By Poincaré’s lemma (see [8, p.214]), there exists a vector $\omega \in L^r(\Omega)$ such that $\text{curl } \omega = F - u_1 - \nabla u_2$ and $\omega$ satisfies the estimate

$$\| \omega \|_{L^r(\Omega)} \leq C(r, \Omega) \left( \| F \|_{L^r(\Omega)} + \| u_1 \|_{L^r(\Omega)} + \| \nabla u_2 \|_{L^r(\Omega)} \right).$$  \hspace{1cm} (3.4)

To obtain the existence of $u_3$, we first assume $r \geq 2$. From the Lax-Milgram Lemma, it follows that $u_3 \in H^1(\Omega)$. For $1 < r < 2$, it is necessary to establish the a priori estimate for $u_3$, then take the usual approximation argument to obtain the existence.

We now give the estimate for $u_3$. Note that $u_3 \in L^6(\Omega)$ by the imbedding theorem. By Poincaré’s lemma again, there exists a vector $\psi \in L^6(\Omega)$ such that $\text{curl } \psi = F - u_1 - \nabla u_2$. Actually, by Theorem 1.3 in [12] we can further let $\psi$ satisfy $\text{div} \psi = 0$ in $\Omega$ and $\nu \cdot \psi = 0$ on $\partial \Omega$.

From Corollary 2.3, we have the estimate

$$\| \psi \|_{L^\infty(\Omega)} \leq C \| u_3 \|_{L^{3,1}(\Omega)}.$$  \hspace{1cm} (3.5)

Since $H^1(\Omega)$ is continuously imbedded into the Lorentz space $L^{3,1}(\Omega)$ and by $H^1$ estimate for $u_3$, we can obtain

$$\| \psi \|_{L^r(\Omega)} \leq C \| \psi \|_{L^\infty(\Omega)} \leq C \| u_3 \|_{H^1(\Omega)} \leq C \left( \| \omega \|_{L^2(\Omega)} + \| f \|_{L^2(\Omega)} \right),$$

where the constants $C$ depend on $r$ and the Lipschitz character of $\Omega$.

Let $\Psi = \omega + f - \psi$. Then $u_3$ satisfies the system

$$\text{curl } (A(x) \text{curl } u_3) = \text{curl } \Psi, \quad \text{div } u_3 = 0 \quad \text{in } \Omega, \quad \nu \times u_3 = 0 \quad \text{on } \partial \Omega.$$  \hspace{1cm} (3.3)

Based on the weak reverse Hölder inequality (Lemma 3.1), the proof of Theorem 1.1 in [11] with the $\nabla$ operator replaced by the curl operator is also applicable. Thus, we can deduce that

$$\| \text{curl } u_3 \|_{L^r(\Omega)} \leq C(r, \Omega) \| \Psi \|_{L^r(\Omega)}.$$
Therefore, by Theorem 1.3 (as the estimate of \( u_1 \)) we have that
\[
\|u_3\|_{L^r(\Omega)} + \|\text{curl} u_3\|_{L^r(\Omega)} \leq C(r, \Omega) \|\Psi\|_{L^r(\Omega)}
\]
Since \( \Psi = \omega + f - \psi \) and the estimate (3.3) on \( \psi \), we then get
\[
\|u_3\|_{L^r(\Omega)} + \|\text{curl} u_3\|_{L^r(\Omega)} \leq C(r, \Omega) \left( \|\omega\|_{L^r(\Omega)} + \|f\|_{L^r(\Omega)} \right).
\]
From (3.4), we now have
\[
\|u_3\|_{L^r(\Omega)} + \|\text{curl} u_3\|_{L^r(\Omega)} \leq C(r, \Omega) \left( \|F\|_{L^r(\Omega)} + \|\text{curl} u_2\|_{L^r(\Omega)} + \|\|\|_{L^r(\Omega)} \right),
\]
Plugging the estimates of \( u_1 \) (step 1) and of \( \text{curl} u_2 \) (step 2) back to the above inequality, then noting that \( u = u_1 + \text{curl} u_2 + u_3 \), we finally obtain that, for \( 2 \leq r < \infty \),
\[
\|u\|_{L^r(\Omega)} + \|\text{curl} u\|_{L^r(\Omega)} \leq C \left( \|F\|_{L^r(\Omega)} + \|f\|_{L^r(\Omega)} + \|\text{Div} g\|_{B^{-1/r, r}(\partial\Omega)} + \|g\|_{L^r(\partial\Omega)} \right),
\]
where the constant \( C \) depends only on \( r \) and the Lipschitz character of \( \Omega \).

To obtain the a priori estimate for \( u_3 \) if \( 1 < r < 2 \), we take the duality argument. For any given vector \( G \in L^{r/(r-1)}(\Omega) \), we solve the following system
\[
\text{curl}(A(x) \text{curl} v) + v = G \quad \text{in} \ \Omega, \quad \nu \times v = 0 \quad \text{on} \ \partial\Omega.
\]
From (3.7), we have the estimate for \( v \):
\[
\|v\|_{L^{r/(r-1)}(\Omega)} + \|\text{curl} v\|_{L^{r/(r-1)}(\Omega)} \leq C(r, \Omega) \|G\|_{L^{r/(r-1)}(\Omega)}.
\]
Let \( \langle \cdot, \cdot \rangle \) denote the duality pairing between \( L^r(\Omega) \) and \( L^{r/(r-1)}(\Omega) \). Since \( A(x) = A^T(x) \), we have
\[
\langle u_3, G \rangle = \langle u_3, \text{curl}(A(x) \text{curl} v) + v \rangle = \langle \text{curl}(A(x) \text{curl} u_3) + u_3, v \rangle.
\]
From (3.3), it follows that
\[
\langle u_3, G \rangle = \langle \text{curl}(\omega + f), v \rangle = \langle \omega + f, \text{curl} v \rangle
\]
Combining with (3.8), we have
\[
\|u_3\|_{L^r(\Omega)} \leq C \left( \|\omega\|_{L^r(\Omega)} + \|f\|_{L^r(\Omega)} \right).
\]
To obtain the estimate for \( \text{curl} u_3 \), we solve the following system
\[
\text{curl}(A(x) \text{curl} m) + m = \text{curl} h \quad \text{in} \ \Omega, \quad \nu \times m = 0 \quad \text{on} \ \partial\Omega
\]
for any given vector \( h \in L^{r/(r-1)}(\Omega) \). From (3.7), we have the estimate for \( m \):
\[
\|m\|_{L^{r/(r-1)}(\Omega)} + \|\text{curl} m\|_{L^{r/(r-1)}(\Omega)} \leq C(r, \Omega) \|h\|_{L^{r/(r-1)}(\Omega)}.
\]

Then
\[ \langle \text{curl} \, u_3, h \rangle = \langle u_3, \text{curl} \, h \rangle = \langle \omega + f, \text{curl} \, m \rangle. \]

This shows the estimate, by (3.9),
\[ \| \text{curl} \, u_3 \|_{L^r(\Omega)} \leq C(r, \Omega) \left( \| \omega \|_{L^r(\Omega)} + \| f \|_{L^r(\Omega)} \right). \]

Therefore, for any \( 1 < r < \infty \), we always have the estimate (3.6).

From step 1-step 3, we now have the inequality (1.5). The uniqueness is obvious since
\[ \text{div} \, F \in L^r(\Omega), \text{ then } \text{div} \, u \in L^r(\Omega). \]
It follows from Theorem 1.3 that we have \( u \in L^r_{1/r}(\Omega) \) if \( 2 \leq r < \infty \) and \( u \in B^{r,2}_{1/r}(\Omega) \) if \( 1 < r < 2 \). We end our proof.

Finally, we consider the Maxwell-type system (1.1) in Lipschitz domains. For simplicity, we let \( g = 0 \). This system was studied in the space \( H_0^{s,r}(\text{curl} \, ; \Omega) \) by Kar and Sini, see [17].

When \( s = 0 \), they gave a condition that characterizes the range of \( r \) such that the problem is well-posed, see Remark 2.2 in [17]. However, we may notice that by this condition it is not easy to check how large the range for \( r \) is.

Based on Lemma 3.3 below and the proof of Theorem 1.4, we say, to show the well-posedness of this problem, the condition given by Kar and Sini is not needed if the coefficient matrix \( A(x) \) is symmetric, bounded measurable, uniformly elliptic and in \( \text{VMO}(\Omega) \).

Denote by
\[ I = \left\{ r \mid \frac{2}{3} \left( 1 - \frac{1}{p_\Omega} \right) < \frac{1}{r} < \frac{1}{3} \left( \frac{2}{p_\Omega} + 1 \right) \right\}, \]
where \( p_\Omega \) is determined by the Lipschitz character of the domain \( \Omega \), see [19].

**Theorem 3.2.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^3 \). Assume that the coefficient matrix \( A(x) \) is symmetric, bounded measurable, uniformly elliptic and in \( \text{VMO}(\Omega) \). Suppose that \( F \in L^r(\Omega) \) and \( f \in L^r(\Omega) \) with \( r \in I \), then there exists a unique solution \( u \in H^r(\text{curl}, \Omega) \) of system (1.1) with \( g = 0 \), and the solution \( u \) satisfies the estimate
\[ \| u \|_{L^r(\Omega)} + \| \text{curl} \, u \|_{L^r(\Omega)} \leq C \left( \| F \|_{L^r(\Omega)} + \| f \|_{L^r(\Omega)} \right), \]
where the constant \( C \) depends on \( r \) and the Lipschitz character of \( \Omega \).

**Proof.** The proof is quite similar to that of Theorem 1.4, we here omit it.

**Lemma 3.3.** Let \( \Omega \) be a Lipschitz domain and let \( r \in I \). For any \( \psi \in L^3(\Omega) \) with \( \text{div} \, \psi = 0 \) in \( \Omega \), there exists a vector \( \omega \in L^r(\Omega) \) with \( \text{div} \, \omega = 0 \) in \( \Omega \) and \( \nu \cdot \omega = 0 \) on \( \partial \Omega \) such that \( \psi = \text{curl} \, \omega \), and we have the estimate
\[ \| \omega \|_{L^r(\Omega)} \leq C \| \psi \|_{L^3(\Omega)}, \]
where the constant \( C \) depends on \( r \) and the Lipschitz character of \( \Omega \).
Proof. It suffices to show that the inequality (3.10) holds for \( r > 3 \). The method of our proof goes back to [6]. As in [6], take \( R \) sufficiently large such that \( \Omega \subset B_R \). Let \( \chi \) be the solution of the equation

\[
\Delta \chi = 0 \quad \text{in} \quad B_R \setminus \Omega; \quad \frac{\partial \chi}{\partial \nu} = \nu \cdot \psi \quad \text{on} \quad \partial \Omega; \quad \frac{\partial \chi}{\partial \nu} = 0 \quad \text{on} \quad \partial B_R.
\]

It follows that

\[
\| \nabla \chi \|_{L^3(B_R \setminus \Omega)} \leq C(\Omega) \| \nu \cdot \psi \|_{B^{-1/3,3}(\partial \Omega)} \leq C(\Omega) \| \psi \|_{L^3(\Omega)}.
\]

Let

\[
f = \psi \quad \text{in} \quad \Omega; \quad f = \nabla \chi \quad \text{in} \quad B_R \setminus \Omega; \quad f = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus B_R.
\]

Then we have

\[
\text{div } f = 0 \quad \text{in the sense of distribution in } \mathbb{R}^3.
\]

Denote by

\[
v = \text{curl } \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y) dy.
\]

By the Calderon-Zygmund inequality we have

\[
\| v \|_{L^r(\Omega)} \leq C \| v \|_{W^{1,3}(B_R)} \leq C \| f \|_{L^3(\mathbb{R}^3)} \leq C \left( \| \psi \|_{L^3(\Omega)} + \| \nabla \chi \|_{L^3(B_R \setminus \Omega)} \right) \leq C \| \psi \|_{L^3(\Omega)},
\]

where the constants \( C \) depend only on the Lipschitz character of \( \Omega \).

Introduce \( h = \omega - v \). Then

\[
\text{curl } h = 0, \quad \text{div } h = 0 \quad \text{in} \quad \Omega; \quad \nu \cdot h = -\nu \cdot v \quad \text{on} \quad \partial \Omega.
\]

Thus there exists a function \( \varphi \) such that \( h = \nabla \varphi \) in \( \Omega \). This gives that

\[
\Delta \varphi = 0 \quad \text{in} \quad \Omega; \quad \frac{\partial \varphi}{\partial \nu} = -\nu \cdot v \quad \text{on} \quad \partial \Omega.
\]

Therefore,

\[
\| h \|_{L^r(\Omega)} = \| \nabla \varphi \|_{L^r(\Omega)} \leq C(r, \Omega) \| \nu \cdot v \|_{B^{-1/r,r}(\partial \Omega)} \leq C(r, \Omega) \| v \|_{L^r(\Omega)}.
\]

Combining with (3.11), we obtain the inequality (3.10). We end our proof.

\[\square\]

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Appendix A. Proof of inquality \((2.5)\)

In this section we give the proof of \((2.5)\) in Lemma 2.3 if \(u \in C^3(\Omega) \cap C^2(\bar{\Omega})\) and the domain \(\Omega\) is smooth. The proof due to Cianchi and Maz’ya (see [3]).

**Proof.** Introduce the distribution function of \(v\) (see [3]):

\[
\mu_v(t) = \mu(\{|v| > t\}), \quad t > 0,
\]

and the nonincreasing rearrangement of \(v\):

\[
v^*(s) = \sup\{t > 0, \mu_v(t) > s\}, \quad t > 0.
\]

(A.1)

By the isoperimetric inequality and the coarea formula (see [3, Lemma 5.2]), for \(t \geq |H|^*(|\Omega|/2)\) we can obtain that

\[
\left(\int_{\{|H| = t\}} |\nabla|H||dS\right)^{-1} \leq C \left(-\mu'_H(t)\right) \mu_H^{-\frac{4}{3}}(t),
\]

where the constant \(C\) depends on the Lipschitz character of \(\Omega\). Note that (see the inequality (6.38) in [3])

\[
\int_{\{|H| = t\}} |\text{curl}\,u|dS \leq \left(-\frac{d}{dt} \int_{\{|H| > t\}} |\text{curl}\,u|^2dx\right)^{1/2} \left(\int_{\{|H| = t\}} |\nabla|H||dS\right)^{1/2}.
\]

Denote by \(t_0 := |H|^*(|\Omega|/2)\). Then for any \(T\) satisfying \(t_0 < T < \|H\|_{L^\infty(\Omega)}\) we have

\[
\int_{t_0}^{t_0} \left(\mu'_H(t) \frac{d}{dt} \int_{\{|H| > t\}} |\text{curl}\,u|^2dx\right)^{1/2} \left(\int_{\{|H| = t\}} |\nabla|H||dS\right)^{-1/2} dt \\
\leq C(\Omega) \int_{t_0}^{T} \left(\mu'_H(t) \frac{d}{dt} \int_{\{|H| > t\}} |\text{curl}\,u|^2dx\right)^{1/2} \mu_H^{-\frac{2}{3}}(t) dt.
\]

Since

\[
\int_{t_0}^{T} \left(\mu'_H(t) \frac{d}{dt} \int_{\{|H| > t\}} |\text{curl}\,u|^2dx\right)^{1/2} \mu_H^{-\frac{2}{3}}(t) dt \\
\leq \int_{t_0}^{\|\Omega\|} \left(\frac{d}{ds} \int_{\{|H| > |H|^*(s)\}} |\text{curl}\,u|^2dx\right)^{1/2} s^{-\frac{2}{3}} ds
\]

and from [4, Proposition 3.4, Lemma 3.5] (also see [3]), we have

\[
\int_{t_0}^{\|\Omega\|} \left(\frac{d}{ds} \int_{\{|H| > |H|^*(s)\}} |\text{curl}\,u|^2dx\right)^{1/2} s^{-\frac{2}{3}} ds \leq C(\Omega)\|\text{curl}\,u\|_{L^{3,1}(\Omega)}.
\]

Then we obtain that

\[
\int_{t_0}^{T} \left(\int_{\{|H| = t\}} |\nabla|H||dS\right)^{-1} \int_{\{|H| = t\}} |\text{curl}\,u|dS dt \leq C(\Omega)\|\text{curl}\,u\|_{L^{3,1}(\Omega)}.
\]

(A.3)
Similarly, we have (Lemma 3.6)
\[
\int_{t_0}^{T} \left( \int_{\{H=r\}} |\nabla|H||dS \right)^{-1} \int_{\{H>\rho\}} |\text{curl}\ u|^2 dx dt \leq C(\Omega) \|\text{curl}\ u\|_{L^{3,1}(\Omega)}^2 \tag{A.4},
\]
where the constants \(C\) in (A.3) and (A.4) depend on the Lipschitz character of \(\Omega\).

Therefore, from (2.3), (A.3) and (A.4) we have
\[
T^2 - t_0^2 \leq CT \|\text{curl}\ u\|_{L^{3,1}(\Omega)} + C \|\text{curl}\ u\|_{L^{3,1}(\Omega)}^2.
\]

Note that
\[
t_0 := |H|^*(|\Omega|/2) \leq \frac{2}{|\Omega|} \int_{\Omega} |H| dx \leq C(\Omega) \|\text{curl}\ u\|_{L^{3,1}(\Omega)}.
\]

Then we have
\[
T \leq C(\Omega) \|\text{curl}\ u\|_{L^{3,1}(\Omega)}.
\]

Now letting \(T \to \|H\|_{L^\infty(\Omega)}\), we obtain that
\[
\|\text{curl}\ w\|_{L^\infty(\Omega)} \leq C(\Omega) \|\text{curl}\ u\|_{L^{3,1}(\Omega)}. \tag{A.5}
\]

We end our proof. \(\square\)

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