Topological superradiance in a degenerate Fermi gas

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(Dated: October 31, 2014)

We predict the existence of a topological superradiant state in a two-component degenerate Fermi gas in a cavity. The superradiant light generation in the transversely driven cavity mode induces a cavity-assisted spin-orbit coupling in the system and opens a bulk gap at half-filling. This mechanism can simultaneously drive a topological phase transition in the system, yielding a topological superradiant phase. We map out the steady-state phase diagram of the system in the presence of an effective Zeeman field, and identify a critical quadracritical point beyond which the topological and the conventional superradiant phase boundaries separate. We also propose to detect the topological phase transitions based on the unique signatures in the momentum-space density distribution.

PACS numbers: 67.85.Lm, 03.75.Ss, 05.30.Fk

Introduction.— For an ultracold atomic gas inside an optical cavity, the interplay between the atomic motion and the light field can often give rise to rich dynamical processes and novel many-body phases [1–6]. An important experimental achievement in these systems is the recent observation of the Dicke superradiance in a Bose-Einstein condensate (BEC) coupled to cavity light fields, where the backaction of the cavity photons induces an open-system version of the supersolid state in the BEC [1, 7–9]. For a degenerate spinless Fermi gas in a cavity, it has been shown theoretically that the atomic scattering of cavity photons in the presence of a Fermi surface can lead to the enhancement of superradiance due to the nesting effect [10–12]. Furthermore, at the superradiant (SR) phase transition, a bulk gap opens at the Fermi surface as a result of the atom-photon scattering [11].

Here, we show that for a spinful degenerate Fermi gas in a cavity, a superradiance-induced gap opening can lead to the stabilization of an exotic topological superradiant (TSR) state, where the SR light generation in the cavity is accompanied by the onset of topologically non-trivial properties of the Fermi gas [13, 14]. This novel steady-state topological phase is intimately connected to the cavity-assisted spin-orbit coupling (SOC) inherent in the system. While the synthetic SOC in ultracold atomic systems has been extensively studied following its experimental implementation [15–22], the dynamical SOC mediated by a cavity emerges as a promising setting with rich physical implications [23, 24].

As an illuminating example, we study the superradiance of a quasi-one-dimensional, noninteracting Fermi gas with cavity-assisted Raman processes. As illustrated in Fig. 1, the two relevant cavity modes are driven, respectively, by a longitudinal and a transverse pumping laser under the blue-detuning condition. Since the detuning between the two cavity modes are typically much larger than the energy scales of the relevant system dynamics [25], we may treat them independently. While the transversely driven cavity mode (mode A) and the transverse pumping laser couple the two hyperfine spin components of the Fermi gas in two separate Raman processes as the cavity-assisted SOC (Fig. 1 b), the longitudinally driven cavity mode (mode B) provides a stationary background optical lattice potential for the atoms, which is important for the observation of the TSR state under realistic parameters.

When the strength of the transverse pumping laser increases beyond a critical value, the system can undergo an SR transition, with macroscopic photon occupation in mode A and the emergence of a finite order parameter. Interestingly, the backaction of the SR transition on the Fermi gas depends on the initial state of the fermions. This is particularly important when the background lattice potential is half filled. At half filling, prior to the SR transition, the Fermi gas is either in a gapless metallic (M) state or a ferromagnetic insulator (I) state, depending on whether the effective Zeeman field \( m_z \) is smaller or greater than a critical value \( m_c \) (Fig. 1 c,d).

For \( m_z < m_c \), the SR transition induces a gap at finite momenta out of a gapless metallic Fermi gas (Fig. 1 c). This is because the spatially varying cavity field of mode A has twice the period of that of the background potential, so that the Raman processes effectively play the role of a cavity-assisted SOC which couples fermions with different hyperfine spins and with a momentum difference that spans half the Brillouine zone. Importantly, we show that the SR light generation in mode A and the bulk gap opening are accompanied by the change of topological
properties of the Fermi gas. The system is thus in an exotic TSR phase, whose topological nature can be confirmed by edge-state and winding-number calculations.

For \( m_z > m_c \), the ferromagnetic bulk gap persists as the system crosses the SR transition (Fig 1d). The Fermi gas thus changes from a ferromagnetic I state to a topologically trivial SR state across the SR transition. However, we find that by further increasing the pumping strength, the bulk gap closes and opens up again at zero momentum (\( k = 0 \)), as the system crosses a topological phase transition to become TSR. By mapping out the steady-state phase diagram, we show that the TSR state here can be adiabatically connected to the TSR state at small Zeeman fields. We also identify a quadracritical point on the phase diagram where different phase boundaries merge. Finally, we propose to detect the TSR to SR topological phase transition by measuring the spin-related longitudinal (along \( \hat{x} \)) pumping by two linearly polarized lasers. (b) The transverse pumping laser and the corresponding cavity mode couple two different hyperfine states in two separate Raman processes. (c) Superradiance-induced gap opening under a small Zeeman field. (d) The persistence of the bulk gap across the SR transition at a large Zeeman field. In (c)(d), the solid (dashed) curves are the lowest band single-particle dispersion spectra in the first Brillouine zone before (after) the SR transition. The dispersion spectra of the spin-down species (solid green) are shifted by half the Brillouine zone relative to that of the spin-up species (solid red) due to the cavity-assisted SOC. The circles in (c) mark the positions of the gap opening.

FIG. 1: (Color online) (a) Illustration of the system setup. A quasi-one-dimensional Fermi gas is coupled to a two-mode optical cavity. The cavity is under both transverse (along \( -\hat{z} \)) and longitudinal (along \( \hat{z} \)) pumping by two linearly polarized lasers. The frequency \( \omega \), the periodic potential along \( \hat{x} \), and longitudinal (along \( \hat{z} \)) pumping by two linearly polarized lasers. (b) The transverse pumping laser and the corresponding cavity mode couple different hyperfine states in two separate Raman processes. (c) Superradiance-induced gap opening under a small Zeeman field. (d) The persistence of the bulk gap across the SR transition at a large Zeeman field. In (c)(d), the solid (dashed) curves are the lowest band single-particle dispersion spectra in the first Brillouine zone before (after) the SR transition. The dispersion spectra of the spin-down species (solid green) are shifted by half the Brillouine zone relative to that of the spin-up species (solid red) due to the cavity-assisted SOC. The circles in (c) mark the positions of the gap opening.

Adiabatically eliminating the excited states, we get the effective Rabi frequency of the cavity-assisted Raman processes: \( \eta = \Omega_A g_A / \Delta \) [26], where \( \Omega_A \) is the Rabi frequency of the transverse pumping laser, \( g_A \) is the single-photon Rabi frequency of the cavity mode \( A \), and the single-photon detuning \( \Delta \) is assumed to be the same for the Raman processes. Note that throughout the work, we will only consider the case where the background potential is half filled.

Adopting the mean-field approximation on the cavity fields, we focus on the steady-state solution of the driven-dissipative system, where the mean fields of the cavity modes are stationary. Integrating out the tightly confined transverse degrees of freedom, we have the effective one-dimensional Hamiltonian [27]:

\[
\hat{H} = \sum_{\sigma} \int d\mathbf{x} \left( \frac{p_{\sigma}^2}{2m} + V_0 + \xi_A |\alpha|^2 \cos^2(k_0 x) + \xi_{\sigma} m_z \right) \hat{\psi}_{\sigma} + \eta_A (\alpha^* + \alpha) \left[ \int d\mathbf{x} \hat{\psi}_{\sigma}^\dagger \cos(k_0 x) \hat{\psi}_{\sigma} + \text{H.C.} \right],
\]

where \( \hat{\psi}_{\sigma} \) (\( \sigma = \uparrow, \downarrow \)) are the effective one-dimensional fermionic field operators for different hyperfine states, \( \alpha \) is the cavity mean field for mode \( A \), and H.C. represents Hermitian conjugate. The atoms are subject to effective lattice potential (\( V_0 + \xi_A |\alpha|^2 \cos^2(k_0 x) \)), where \( \xi_A = g_A^2 / \Delta \), and \( V_0 \cos^2(k_0 x) \) is the lattice potential generated by mode \( B \). The effective pumping strength \( \eta_A = s \xi \), with the constant \( s \) accounting for the transverse integrals. Here, the cavity detuning \( \Delta_A = \omega_A - \omega_c \), \( k_0 \) is the wave vector of the cavity fields, \( \xi_{\sigma} = \pm \), and \( m_z \) describes the effective Zeeman field, tunable, for example, by applying a magnetic field.

The cavity mean field \( \alpha \) for mode \( A \) under the stationary condition \( \partial \alpha / \partial t = 0 \) can be expressed as:

\[
\alpha = \frac{\eta_A \Theta}{\Delta_A + i \kappa - \xi_A \sum_{\sigma} \int d\mathbf{x} \langle \hat{\psi}_{\sigma}^\dagger \hat{\psi}_{\sigma} \rangle \cos^2(k_0 x)},
\]

where \( \kappa \) is the cavity decay rate, and we have defined the order parameter \( \Theta = \int d\mathbf{x} \theta(x) = \int d\mathbf{x} \left( \langle \hat{\psi}_{\uparrow}^\dagger \hat{\psi}_{\uparrow} \rangle + \text{H.C.} \right) \cos(k_0 x) \). We may then numerically diagonalize the effective Hamiltonian (1) while self-consistently imposing Eq. (2). For the numerical calculations, we will use the recoil energy \( E_r = \hbar k_0^2 / 2m \) as the unit of energy, where \( m \) is the mass of a fermion.

Topological superradiance.-- An outstanding feature of the system under Hamiltonian (1) is the existence of an SR transition when the effective transverse pumping strength \( \eta_A \) increases. Following Ref. [12], the critical pumping strength can be derived by evaluating the free energy using second-order perturbation. Integrating out the fermion fields, and requiring a vanishing coefficient for the second-order expansion in \( \Theta \) of the free energy,
the critical pumping strength is given by:

\[
\eta_c^A = \frac{1}{2} \sqrt{\frac{\Delta_A^2 + \kappa^2}{\Delta_A f}},
\]

(3)

where \(\Delta_A = \Delta_A - \xi_A \sum_{j,\sigma} \int dx |\varphi_{j,\sigma}|^2 \cos^2(k_0 x)n_F(\epsilon_j),\) \(f = \frac{1}{2} \sum_{j,j'} [M_{jj'}^2[n_F(\epsilon_{j'}) - n_F(\epsilon_j)]/(\epsilon_j - \epsilon_{j'})].\) Here, \(M_{jj'} = \sum_{\sigma \neq \sigma'} \int dx \varphi^*_{j,\sigma'} \varphi_{j',\sigma'} \cos(k_0 x)\), the Fermi-Dirac distribution \(n_F(x) = 1/(e^{(x-\mu)/k_B T} + 1)\) with chemical potential \(\mu\) and temperature \(T,\) and \(k_B\) is the Boltzmann constant. \(\{\varphi_{j,\uparrow}, \varphi_{j,\downarrow}\}\) is the eigen state of the Hamiltonian \(\rho_x^2/2m + V_0 \cos^2 k_0 x + m_z \sigma_z\) with eigen energy \(\epsilon_j\), and \(\sigma_z\) is the Pauli matrix.

Prior to the SR transition, \(\alpha = 0\) and the atoms experience the lattice potential \(V_0 \cos^2(k_0 x)\). After the SR transition, the macroscopic photon occupation of mode A not only modifies the background periodic potential, but also introduces a spatially varying cavity-assisted SOC term in the effective Hamiltonian (1), whose period is twice as that of the background potential. When the cavity field in mode A is small, we can neglect the small spin-dependent interband coupling. Taking the gauge transformation \(\hat{\psi}_\uparrow \rightarrow e^{i k_0 x} \hat{\psi}_\uparrow, \hat{\psi}_\downarrow \rightarrow \hat{\psi}_\downarrow\) in the tight-binding limit, we can map the Hamiltonian (1) onto the one underlying a chiral topological insulator whose ground state is topologically nontrivial below a critical Zeeman field at half filling [28, 29]. When \(\eta_A\) becomes large, the interband coupling is no longer negligible. However, we will show that the topological properties should persist even in the deep SR regime, leading to the TSR state.

To confirm the topological nature of the system, we diagonalize the effective Hamiltonian (1) at \(m_z = 0\) for a finite-size lattice under an open boundary condition, while solving the cavity field self-consistently following Eq. (2). The resulting cavity field and energy spectrum of the bulk states are shown in Fig. 2a and Fig. 2c, respectively. Apparently, a pair of edge states, with localized wave functions (Fig. 2d), emerge in the superradiance-induced bulk gap as soon as mode A becomes populated along with the appearance of a finite order parameter (Fig. 2b). The SR transition here is then topological as well, and the system is in the TSR phase beyond this TSR phase transition.

**Winding number and steady-state phase diagram.** To characterize the stability of the TSR phase against Zeeman fields, we examine the steady-state phase diagram...
FIG. 4: (Color online) Steady-state phase diagram for an 80-site lattice with periodic boundary condition at a finite temperature $1/k_BT \sim 200E_r$. The solid curve is the phase boundary between the M and the TSR states; the thin dashed curve is the phase boundary between the M and the I states at $m_c = 0.132E_r$; the dash-dotted curve is the I to the trivial SR phase boundary; the dotted curve is the TSR to SR boundary. The phase boundaries merge at a quadracritical point (dot). The other parameters are the same as those used for Fig. 2. Inset: change of bulk gap before (solid) and after (dashed) crossing the phase boundaries at different locations.

on the $\eta_A-m_z$ plane. While the SR phase boundaries are calculated from Eq. (3), the topological phase boundaries can be determined by calculating the winding number, which serves as the bulk topological invariant.

Typically, the bulk topological invariant can be defined by a mapping from the first Brillouine zone to the spin space, and can be read out from the momentum-space spin texture $\langle \tilde{\sigma}_k \rangle = \langle \sigma_y \rangle k E_y + \langle \sigma_z \rangle k E_z$, where $\sigma_y$, $\sigma_z$ are the Pauli matrices. The expectation value is taken with respect to the lowest-band Bloch state in momentum space, which can be obtained by diagonalizing Hamiltonian (1) after taking the gauge transformation $\{ \hat{\psi}_\uparrow \rightarrow \hat{\psi}_\uparrow, \hat{\psi}_\downarrow \rightarrow -ie^{ik_0x}\hat{\psi}_\downarrow \}^{[27]}$. Here, the spin of a Bloch state lies in the $y-z$ plane, with the unit vector along the spin orientation being an element of a closed circle $S^1$. The winding number of the insulating state can be calculated by counting the number of times $S^1$ is covered by the mapping. The momentum-space spin textures of a finite-size system under different Zeeman fields are shown in Fig. 3. Starting from a TSR state, as the Zeeman field increases, the bulk gap typically closes and opens up again at $k = 0$ for a critical field value. We find that the spin texture at $k = 0$ undergoes an abrupt change (Fig. 3 a,b) as the critical point is crossed.

Particularly, running across the first Brillouine zone, the spin orientation covers a complete circle below the critical field (Fig. 3 a,c), yielding a winding number of 1; while the winding number is zero (Fig. 3 b,d) above the critical field. Apparently, as the Zeeman field increases, the TSR state can cross a topological phase boundary to become a trivial SR insulator.

With these understandings, we map out the steady-state phase diagram in Fig. 4. For $m_z < m_c$, the gapless Fermi gas at small $\eta_A$ can undergo a TSR phase transition when $\eta_A$ increases. While for $m_z > m_c$, the gapped Fermi gas first undergoes an SR transition to become a topologically trivial SR insulator, which can cross a topological phase boundary to become a TSR insulator as $\eta_A$ increases further. Notably, a quadracritical point appears on the phase diagram beyond which the topological and the SR phase boundaries separate. From the phase diagram, it is evident that the TSR state for $m_z < m_c$ and $m_z > m_c$ can be adiabatically connected.

Detection.— While the SR phase transition can be detected by monitoring the leaked photons on one end of the cavity, the experimental confirmation of the topological phase transition can be more challenging. However, we notice that the abrupt change in $\langle \sigma_z \rangle$ at the gap closing point in momentum space across the TSR to trivial SR phase transition is uniquely associated with change of winding number (see Fig. 3 a,b). As $\langle \sigma_z \rangle$ is related to the momentum-space density distribution of different spins in the lowest band, the abrupt change should manifest itself in the spin-selective momentum distribution. From Fig. 5 a, it is apparent that the momentum-space density distribution of the spin-up fermions at $k = 0$ also undergoes an abrupt change across the TSR to trivial SR phase boundary. This can in principle be detected using the time-of-flight imaging technique.

Final remarks.— Experimentally, quasi-one-dimensional Fermi gases can be prepared by imposing a two-dimensional lattice potential with tight transverse confinement, leading to an array of quasi-
one-dimensional atomic gases with large number of atoms. The increase of atom number implies that the TSR phase and the corresponding phase transition can be observed with more favorable parameters than what we consider here. Finally, we note that our scheme can be straightforwardly extended to quasi-two-dimensional Fermi gases in a cavity, where a TSR phase should also exist, as the tight-binding Hamiltonian just beyond the SR transition therein can be mapped onto that of a quantum anomalous Hall system [29].

Acknowledgments.— We thank Yu Chen, Dong-Sheng Ding, Su Wang, Zhen-Biao Yang, and Hui Zhai for helpful discussions. This work is supported by NFRP (2011CB921200, 2011CBA00200), NKBRP (2013CB922000), NNSF (60921091), NSFC (11105134, 11274009, 11374283), the Fundamental Research Funds for the Central Universities (WK2470000006), and the Research Funds of Renmin University of China (10XNL016).

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Supplementary material

A. Effective Hamiltonian

In this section, we provide more details on the derivation of the effective Hamiltonian in the main text (Eq. (1) therein). The two relevant cavity modes, mode A and mode B, have a frequency difference of hundreds of MHz [1]. Mode B is then effectively decoupled from the relevant dynamics of the system due to the large detuning. The effect of mode B is a stationary background lattice potential \( V_0 \cos^2(k_0 x) \), where \( k_0 \) is the wave vector of mode B, which only differs from that of mode A by a negligibly small amount. In the following, we will use \( k_0 \) for the wave vector of mode A as well.

For a quasi-one-dimensional cavity Fermi gas illustrated in Fig. 1 in the main text, the internal dynamics of a single particle under the cavity mode A and the transverse pumping field can be described by the Hamiltonian

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I,
\]

where

\[
\mathcal{H}_0 = \hbar \omega_c \hat{a}^\dagger \hat{a} + \sum_{j=1,2} \hbar \omega_j |j \rangle \langle j| + \sum_{\sigma=\uparrow, \downarrow} \hbar \omega_\sigma |\sigma \rangle \langle \sigma|,
\]

\[
\mathcal{H}_I = -\left( \Omega_{1\uparrow} \sigma_{1\uparrow}^+ + \Omega_{2\downarrow} \sigma_{2\downarrow}^+ \right) e^{-i(\omega_A t + k_0 z)} - \left( g_{1\downarrow} \sigma_{1\downarrow}^+ \hat{a} + g_{2\uparrow} \sigma_{2\uparrow}^+ \hat{a} \right) \cos \left( k_0 x \right) + c.c.
\]

Here \( \omega_j \) (\( \omega_\sigma \)) denotes the eigen frequency of the exited states (the ground states) respectively. The ground states are typically split by an effective Zeeman field, i.e., \( \omega_\uparrow = -\omega_\downarrow = m_z \). We assume the Zeeman field to be along the \( \hat{z} \) axis, which defines the quantization axis. The resonant frequency of mode A is denoted as \( \omega_c \). The linearly polarized transverse pumping field propagates in the \(-\hat{z}\) direction and has a frequency of \( \omega_A \). The raising operator \( \sigma_{j\sigma}^+ = |j \rangle \langle \sigma| \).

The two Raman channels are respectively driven by the two circularly polarized components of the transverse pumping field, and the effective Rabi frequency of the Raman processes is denoted as \( \Omega_{j\sigma} \). \( g_{j\sigma} \) denotes the single-photon Rabi frequency of the cavity-atom coupling. For simplicity, we assume \( g_{1\downarrow} = g_{2\uparrow} = g_A \) and \( \Omega_{1\uparrow} = \Omega_{2\downarrow} = \Omega_A \), which can always be satisfied by adjusting the system parameters for the commonly used fermion species such as \(^6\text{Li} \) and \(^{40}\text{K} \).

Introducing the time-dependent rotation, \( U(t) = \exp \left[ i \left( \sum_{j=1,2} |j \rangle \langle j| + \hat{a} \hat{a}^\dagger \right) \omega_A t \right] \), and adiabatically eliminating the two exited states, we have

\[
\mathcal{H}_{eff} = -\left[ \Delta_A - \xi_A \cos^2(k_0 x) \right] \hat{a} \hat{a}^\dagger + \hbar m_z \sigma_z + \eta \left( \hat{a} \right) \left( \langle \downarrow | e^{i k_0 z} + c.c. \rangle \right) \cos \left( k_0 z \right),
\]

where \( \Delta_A = \omega_c - \omega_A \) is the cavity detuning, \( \xi_A = g_A^2 / \Delta, \) and \( \eta = \Omega_A g_A / \Delta \) is the effective pumping strength with the single-photon detuning \( \Delta = \omega_1 - \omega_2 - \omega_A \).

Including the kinetic energy term, the radial trapping potential \( U(r) \) and the background lattice potential \( V_0 \cos^2(k_0 x) \), we can write the effective Hamiltonian in second quantization,

\[
\hat{H} = \sum_{\sigma} \int dr \hat{\Psi}_\sigma^\dagger \left[ \frac{p_r^2}{2m} + U(r) + \left( V_0 + \xi_A \hat{a} \hat{a}^\dagger \right) \cos^2(k_0 x) + \xi_m m_z \right] \hat{\Psi}_\sigma - \Delta_A \hat{a} \hat{a}^\dagger
\]

\[
+ \eta \left[ \int dr \hat{\Psi}_\downarrow^\dagger \left( \hat{a} e^{i k_0 z} + \hat{a}^\dagger e^{-i k_0 z} \right) \cos \left( k_0 x \right) \hat{\Psi}_\downarrow + c.c. \right].
\]

For a quasi-one-dimensional gas with tight radial confinement in the \( y-z \) plane, we assume only the ground state of the radial degree of freedom is occupied. The field operator can then be written as \( \hat{\Psi}_\sigma (r) = \sqrt{\frac{2}{\pi \rho^2}} \hat{\Psi}_\sigma (x) \exp \left( -\frac{x^2 + z^2}{\rho^2} \right) \), where \( \rho \) is the characteristic width of the radial harmonic confinement. Integrating the radial degrees of freedom, the effective one-dimensional Hamiltonian for the Fermi gas is

\[
\hat{H} = \sum_{\sigma} \int dx \hat{\psi}_\sigma^\dagger \left[ \frac{p_x^2}{2m} + \left( V_0 + \xi_A \hat{a} \hat{a}^\dagger \right) \cos^2(k_0 x) + \xi_m m_z \right] \hat{\psi}_\sigma
\]

\[
+ \eta_A \left( \hat{a} \hat{a}^\dagger \right) \left[ \int dx \hat{\psi}_\downarrow^\dagger \cos \left( k_0 x \right) \hat{\psi}_\downarrow + c.c. \right].
\]
where \( \eta_A = s \eta \) with a ratio \( s = e^{-k_0^2 \rho^2/\delta} \), which reduces to 1 when the system is exactly one dimensional.

We then write down the equations of motion for \( \dot{\hat{a}} \)

\[
i \dot{\hat{a}} = \hat{a} \left[ \xi_A \sum_{\sigma} \int dx \hat{\psi}^\dagger \cos(k_0 x) \hat{\psi}_\sigma - \Delta_A - i \kappa \right] + \eta_A \left[ \int dx \hat{\psi}^\dagger \cos(k_0 x) \hat{\psi}_+ + h.c. \right].
\]

Taking the mean field approximation \( \langle \dot{\hat{a}} \rangle = \alpha \) and imposing the stationary condition \( \partial \alpha / \partial t = 0 \), we have the steady-state cavity field

\[
\alpha \approx \frac{\eta_A \int dx \cos(k_0 x) \left[ \hat{\psi}^\dagger \hat{\psi}_\gamma + h.c. \right]}{\Delta_A + i \kappa - \xi_A \sum_{\sigma} \int dx \langle \hat{\psi}^\dagger_{\sigma} \hat{\psi}_{\sigma} \rangle \cos^2(k_0 x)},
\]

where \( \kappa \) is the cavity decay. Replacing the cavity field operators with their mean fields in Eq. (S8), we arrive at the effective Hamiltonian (1) in the main text.

The self-consistent calculation of cavity field can be divided into following several steps: 1. Solve Hamiltonian (Eq. (S6)) by substituting an input cavity field \( \alpha_0 \) for the cavity operator \( \hat{a} \); 2. Solve for the chemical potential from the number equation, \( N = \sum_{\sigma} \int dx \langle \hat{\psi}^\dagger_{\sigma} \hat{\psi}_{\sigma} \rangle \); 3. Calculate the cavity field \( \alpha \) with Eq. (S8); 4. Stop the calculation when \( |\alpha - \alpha_0| < \epsilon \), where \( \epsilon \) is a given precision. Otherwise, update \( \alpha_0 \) with \( \alpha \) and repeat 1-4.

### B. Superradiant Critical Point

The critical point of the superradiant phase transition can be determined from the second-order perturbation theory. Integrating out the fermion fields and replacing the cavity field with its steady-state value Eq. (S8), we derive the free energy to the second order of the order parameter \( \Theta = \int dx \cos(k_0 x) \left[ \langle \hat{\psi}^\dagger \hat{\psi}_\gamma + h.c. \right] \)

\[
F_\alpha = - \left[ \frac{\Delta_A}{\Delta_A^2 + \kappa^2} + \chi_\eta \frac{4\Delta_A^2}{(\Delta_A^2 + \kappa^2)^2} \right] (\eta_A \Theta)^2.
\]

Here the effective cavity detuning \( \Delta_A \) and the susceptibility \( \chi_\eta \) are given by

\[
\Delta_A = \Delta_A - V_0 \sum_j V_{jj'} \eta_{F}(\epsilon_j),
\]

\[
\chi_\eta = \eta_A f = - \frac{\eta_A^2}{2} \sum_{j,j'} M_{jj'} \left[ n_{F}(\epsilon_j) - n_{F}(\epsilon_{j'}) \right]/(\epsilon_j - \epsilon_{j'}),
\]

where \( \varphi_j(x) = \{ \varphi_{j\uparrow}, \varphi_{j\downarrow} \}^T \) is the eigen function of the Hamiltonian \( \hat{H}_0 = p^2/2m + V_0 \cos^2(k_0 x) + m \sigma_z \) with an eigen energy \( \epsilon_j \), the Fermi-Dirac distribution \( n_{F}(\epsilon_j) = 1/[e^{(\epsilon_j-\mu)/k_BT} + 1] \) with chemical potential \( \mu \) and temperature \( T \).

The matrix elements \( V_{jj'} \) and \( M_{jj'} \) are given as

\[
V_{jj'} = \sum_{\sigma} \int dx \varphi_{j\sigma}^* (x) \cos^2(k_0 x) \varphi_{j'\sigma}(x),
\]

\[
M_{jj'} = \sum_{\sigma \neq \sigma'} \int dx \varphi_{j\sigma}^* \cos(k_0 x) \varphi_{j'\sigma'}. \]

The critical pumping strength for the superradiant phase transition can be derived by requiring a vanishing second-order expansion coefficient of the free energy

\[
\eta_A^* = \frac{1}{2} \sqrt{\frac{\Delta_A^2 + \kappa^2}{-\Delta_A f}}.
\]

where \( f \) is defined through Eq. (S11).
Beyond the superradiant critical point, both the cavity field \( \alpha \) and the cavity-assisted Raman term \( \eta_A (\alpha + \alpha^*) \) become finite. From the effective one-dimensional Hamiltonian Eq. 1, the cavity-assisted Raman term induces a spin flip with spatial-dependent rate. When the system just crosses the critical point, the Raman term is much smaller than the initial lattice depth \( V_0 \) and the tight-binding approximation is applicable at low temperatures. Furthermore, the interband spin-dependent coupling, such as the on-site spin flip etc., is also small. Dropping these interband coupling terms, we can write the single-band tight-binding Hamiltonian corresponding to Eq. 1 in the main text as

\[
\hat{H}_{TI} = - \sum_{\langle i,j \rangle, \sigma} t_{ij} \hat{\psi}_{i\sigma}^\dagger \hat{\psi}_{j\sigma} + \sum_{\langle i,j \rangle} \left( \xi_{s0} \hat{\psi}_{i\uparrow}^\dagger \hat{\psi}_{j\uparrow} + h.c. \right) + m_z \sum_{i, \sigma} \xi_{s} \hat{\psi}_{i\sigma}^\dagger \hat{\psi}_{i\sigma}, \tag{S14}
\]

where \( \xi_{\uparrow,\downarrow} = \pm 1 \), and

\[
ts_a = \int dx \phi_0^\dagger (x) \left[ \frac{\beta^2_a}{2m} + \left( V_0 + \xi_A |\alpha|^2 \right) \cos^2 (k_0 x) \right] \phi_1 (x), \tag{S15}\]

\[
t_{s0}^{\pm 1} = \pm (-1)^j \eta_A (\alpha^* + \alpha) \int dx \phi_0^\dagger (x) \cos (k_0 x) \phi_1 (x).
\]

Here, \( \hat{\psi}_{j\sigma} \) is the atomic annihilation operator of spin \( \sigma \) on site \( j \). \( \phi_j (x) \) is the lowest-band Wannier function on the \( j \)-th site. Similar to Ref. [2], using the local unitary transformation \( \hat{\psi}_{j\uparrow} \rightarrow (-1)^j \hat{\psi}_{j\downarrow} \) and performing the Fourier transform, we can write the effective Hamiltonian in momentum space,

\[
\hat{H}_{TI} = \sum_{i,k \in \mathcal{BZ}} \left( \hat{\psi}_{k\uparrow}^\dagger \hat{\psi}_{k\downarrow}^\dagger \right) \left[ h_y (k) \sigma_y + h_z (k) \sigma_z \right] \left( \hat{\psi}_{k\uparrow} \hat{\psi}_{k\downarrow} \right), \tag{S16}\]

where the summation is over the first Brillouin zone, \( h_y (k) = 2t_{so} \sin (ka) \) and \( h_z (k) = m_z - 2t_s \cos (ka) \), with the lattice constant \( a = \pi/k_0 \). The topological nature of the ground state can be captured by the so-called winding number, which characterizes the rotation of the spin components of the Hamiltonian in the first Brillouin zone. Mathematically, the winding number is defined as \[3\]

\[
\mathcal{W} = \sum_{\nu, \nu'} \oint \frac{dk}{4\pi} \epsilon_{\nu, \nu'} \hat{h}_{\nu'}^\dagger (k) \partial_k \hat{h}_\nu (k), \tag{S17}\]

where \( \epsilon_{y,z} = -\epsilon_{z,y} = 1 \). The winding number is 1 for a topologically nontrivial state with \( m_z < 2t_s \), and 0 for a topologically trivial state with \( m_z > 2t_s \). It is worth noting that a homeomorphic transformation cannot change the topology of a system. Therefore this winding number also characterizes the topology of the original Hamiltonian in Eq. (S6).

When the cavity-assisted Raman term becomes large, the interband couplings can no longer be neglected. For example, the on-site spin flip becomes appreciable, as is reflected in the emergence of a finite and large \( \theta (x) \) (see Fig. 2b in the main text). Therefore, when the system is deep in the superradiant region, we need to numerically evaluate the winding number to determine the topological phase boundary. For that purpose, we introduce the local gauge transformation \( \hat{\psi}_{\downarrow} \rightarrow -ie^{i\kappa x} \hat{\psi}_{\downarrow} \), to map the Hamiltonian in Eq. (S6) into its homeomorphic form,

\[
\hat{H} = \int dx \left( \hat{\psi}_{\uparrow}^\dagger \hat{\psi}_{\uparrow} + \hat{\psi}_{\downarrow}^\dagger \hat{\psi}_{\downarrow} \right) \left( \frac{\beta^2_a}{2m} + V (x) + m_z \right) - im (x) e^{ik_0 x} + M (x) \left( \frac{(\nu_a + ka)^2}{2m} + V (x) - m_z \right) \left( \hat{\psi}_{\uparrow}^\dagger \hat{\psi}_{\downarrow} \right), \tag{S18}\]

where \( (V_0 + \xi_A |\alpha|^2 \cos (k_0 x) \right) \) and \( M (x) = \eta_A (\alpha^* + \alpha) \cos (k_0 x) \). We map out the topological phase boundaries on the steady-state phase diagram (see Fig. 4 in the main text) by calculating the winding number of the ground state of Eq. (S18). The topological phase transition in the phase diagram is also confirmed with the edge-state calculations.
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