THEORETICAL AND NUMERICAL ANALYSIS OF A CLASS OF QUASILINEAR ELLIPTIC EQUATIONS

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Dedicated to Michel Pierre on his 70th birthday

Abstract. The purpose of this paper is to give a result of the existence of a non-negative weak solution of a quasilinear elliptic equation in the N-dimensional case, \( N \geq 1 \), and to present a novel numerical method to compute it. In this work, we assume that the nonlinearity concerning the derivatives of the solution are sub-quadratics. The numerical algorithm designed to compute an approximation of the non-negative weak solution of the considered equation has coupled the Newton method with domain decomposition and Yosida approximation of the nonlinearity. The domain decomposition is adapted to the nonlinearity at each step of the Newton method. Numerical examples are presented and commented on.

1. Introduction. Quasilinear elliptic problems present an important role in the modeling of a large range of real-world phenomena in the several fields such as chemotaxis, drug release, ecology, chemistry, cell processes, chemical engineering, physical systems, transport of contaminants in the environment, corrosion phenomena, industrial catalytic processes and the spread of diseases [31, 34].

Several authors have been interested in the mathematical analysis of the solutions for such problem and a lot of works has been published extensively in the literature to answer the several questions about the existence, the uniqueness, the regularity and the asymptotic behavior of the solutions of the considered problem. Various discussions in the earlier literature [1, 10, 11, 20, 28] are devoted to examining

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the classical solutions with Dirichlet or Neumann boundary conditions. During the past half-century, attention has been given to another definition of a solution called “weak solution”, see [2, 4, 6, 7, 5, 8, 12, 13, 14, 16, 32, 35], related to this notion different methods have been used, such as fixed-point theorem, topological degree, penalty method and technic of sub-solutions and super-solutions. Consequently, numerical works have been realized [3, 9, 17, 18, 19, 22, 23, 24, 27] to simulate the analytical solution which describes the behavior of the considered solution in the lower dimension to give an answer of the corresponding real phenomena.

The aim of this work is to examine the existence and to simulate the weak non-negative solution of a quasilinear elliptic problem modeled as follows:

\[
\begin{align*}
\alpha u(x) - \Delta u(x) + G(x, \nabla u(x)) &= F(x, u(x)) + f(x) \quad \text{in } \Omega, \\
u(x) &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\]

where \( \Omega \) is an open bounded set of \( \mathbb{R}^N \) with a smooth boundary \( \partial\Omega \), \( \alpha \) is a positive constant and the functions \( G, F \) are non-negative Caratheodory functions, the function \( f \) is non-negative belonging in \( L^1(\Omega) \).

The result of this paper can be applied to the following model problem:

\[
\begin{align*}
\alpha u(x) - \Delta u(x) + \|\nabla u(x)\|_0^q &= |u(x)|^p + f(x) \quad \text{in } \Omega, \\
u(x) &= 0 \quad \text{on } \partial\Omega,
\end{align*}
\]

where \( 1 \leq p, \) and \( 1 \leq q \leq 2 \).

When the data \( f \) is regular enough (belongs to \( L^\infty(\Omega) \)) and the nonlinearity has a critical growth with respect to the gradient \( (q \leq 2) \), Amann [10], Crandall [11], Choquet-Bruhat and Leray [20] have used the technic of sub-solutions and super-solutions to prove the existence of a classical solution of problem (1).

Lions [32] treated the problem (1) when the nonlinearity is convex and has sublinear growth with respect to the gradient \( (q = 1) \) and the data source belongs to \( L^2(\Omega) \), assumed the existence of a coupled of order function \( (u_1, u_2) \) and he proved the existence of a weak solution \( u \in H^1_0(\Omega) \) such that \( u_1 \leq u \leq u_2 \).

We are aiming to study the equation with general assumptions on the data \( f \) and the nonlinearity \( G \), the two essential ingredients to analyze this problem are the positivity of \( G(x, r) \) and that \( G(x, r) \) has a critical growth nonlinearity with respect to the gradient namely:

\[ G(x, r) \leq C_2(d_2(x) + \|r\|_0^2), \quad \text{where } d_2(x) \in L^1(\Omega) \quad \text{and} \quad C_2 > 0. \]

Basing on the technic of sub-solution and super-solution combined with the method of truncation, we give an existence result of a non-negative weak solution to the considered problem (1) and we introduce a numerical method to compute a numerical approximation of the weak solution in the bidimensional case. The general algorithm for numerical solution of these equations is one application of the Newton method to the discretized version of problem (1):

Find \( U \in \mathbb{R}^{N_h} \) such that \( KU = H(U), \)

where \( K \) is a sparse matrix, \( H : \mathbb{R}^{N_h} \rightarrow \mathbb{R}^{N_h} \) is a nonlinear operator, and \( N_h \) is the number of unknowns.

The Newton algorithm is given by:

\[
\begin{align*}
\text{choose } U^0 &\in \mathbb{R}^{N_h} \text{ in a neighborhood of the solution} \\
\text{and solve until convergence} \\
(K - H'(U^k) Id) (U^{k+1} - U^k) &= -K U^k + H(U^k),
\end{align*}
\]
where $H'(U^k)$ is the Jacobian matrix of the operator $H$ computed in $U^k$ and $Id$ is the identity matrix in $\mathbb{R}^{N_s} \times \mathbb{R}^{N_s}$. The convergence depends in particular on the choice of $U^0$, the existence and the uniqueness of solutions of the linear problem (4). In the case of problem (1) the matrix $K - H'(U^k)Id$ is often singular. Consider the following example:

$$
\begin{cases}
-u''(t) = \zeta u(t) + \beta \quad \text{in} \ (0,1), \\
u(0) = u(1) = 0,
\end{cases}
$$

where $\zeta$ and $\beta$ belong to $\mathbb{R}$. It is easy to verify that the problem (5) has an infinity of solutions when $\zeta = (2\pi p)^2$. For all $B \in \mathbb{R}$, $\exists u_B$ solution to (5) where:

$$
u_B(t) = \frac{\beta}{\zeta}(1 - \cos(p \pi t)) + B \sin(p \pi t).$$

If we consider a classical discretization of $u''$ using finite difference schema and choosing $\zeta$ an eigenvalues of the matrix $K$, the Newton schema is written as follows:

$$(K - \zeta Id)(U^{k+1} - U^k) = -K U^k + H(U^k).$$

Clearly the matrix $K - \zeta Id$ is singular and the equation (6) has not necessary a solution or an infinite number of solutions if $-K U^k + H(U^k) \in \text{Im}(K - \zeta Id)$.

To overcome this difficulty, we introduce a domain decomposition method to compute an approximation of $\delta U^k = U^{k+1} - U^k$ by the resolution of a sequence of problems of type (1) in $\Omega_i \subset \Omega$, such that $\Omega = \bigcup_{i=1,m} \Omega_i$, (see [32, 33]). The geometric domain decomposition is governed by the behavior of the nonlinearity. We obtain an adaptive domain decomposition method. In [17, 18, 19, 39], the authors have studied the application of domain decomposition methods to non symmetric and indefinite elliptic equations, but our approach of an adaptive domain decomposition w.r.t. the nonlinearity is new.

We have organized this paper as follows. Section 2 is devoted to present an existence result of a weak non-negative solution to (1). Section 3 describes the numerical method, in the first step we compute a super-solution using an adaptive domain decomposition method, for the second step we compute a sequence of solutions to an intermediate problem obtained by using the Yosida approximation of $G$, this sequence converges to the weak solution to (1). Finally, section 4 is devoted to report a set of numerical examples attesting the performances of the numerical algorithm.

2. An existence result. Throughout this paper we assume that:

$$f \in L^1(\Omega), f \geq 0,$$

$$F : \Omega \times \mathbb{R} \to [0, +\infty[ \text{ a Caratheodory function},$$

$$F(x,s) \in L^1(\Omega), F \text{ is nondecreasing with respect to } s,$$

$$G : \Omega \times \mathbb{R}^N \to [0, +\infty[ \text{ a Caratheodory function},$$

$$G(x,0) = \min \{G(x,r), r \in \mathbb{R}^N\} = 0 \text{ and } F(x,0) = 0,$$

$$G(x,r) \leq C_2 \left(\|r\|^2_{0} + d_2(x)\right), \text{ for all } r \in \mathbb{R}^N \text{ and } a.e. x \in \Omega,$$

with $d_2 \in L^1(\Omega)$ and $C_2 > 0$.

For $1 \leq s \leq \infty$ and $k$ a non-negative integer, the Sobolev spaces $W^{k,s}(\Omega)$ consist of the functions $f \in L^s(\Omega)$ such that for each index $|\eta| \leq k$, $\partial^\eta f$ exists in the weak sense and belongs to $L^s(\Omega)$. If $s = 2$ we write $W^{1,s}(\Omega) = H^1(\Omega)$. We denote $\|v\|_0$
the $L^2$ norm of $v$, $\|v\|_1$ the $H^1(\Omega)$ norm of $v$ and $|v|_s$ the classical semi-norm of $v$ in $W^{k,s}(\Omega)$.

The space $W_0^{1,s}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,s}(\Omega)$. If $s = 2$ we usually denote $W_0^{1,2}(\Omega)$ by $H^1(\Omega)$.

A function $f \in L_{loc}^s(\Omega)$ if $f_K$ belongs to $L^s(\Omega)$ for every compact $K \subset \Omega$.

The Sobolev space $W_{loc}^{1,s}(\Omega)$ is the set of functions $f \in L_{loc}^s(\Omega)$ such that $\nabla f \in L_{loc}^s(\Omega; \mathbb{R}^N)$.

First, we introduce the definition of a weak solution of the problem (1).

**Definition 2.1.** A function $u$ is a weak solution of the problem (1), if

$$
\begin{cases}
  u \in W_0^{1,1}(\Omega), & G(x, \nabla u) \text{ and } F(x, u) \in L^1(\Omega), \\
  \alpha u - \Delta u + G(x, \nabla u) = F(x, u) + f & \text{in } \mathcal{D}'(\Omega).
\end{cases}
$$

The main theorem of this section gives the existence in $W_0^{1,1}(\Omega)$ of a non-negative weak solution to the problem (1).

**Theorem 2.2.** Under hypotheses 7—12, and assuming that there exists $w$ such that

$$
\begin{cases}
  w \in W_0^{1,1}(\Omega), & F(x, w) \in L^1(\Omega), \\
  \alpha w - \Delta w = F(x, w) + f & \text{in } \mathcal{D}'(\Omega),
\end{cases}
$$

then problem (1) has a non-negative weak solution in $W_0^{1,1}(\Omega)$.

To prove the Theorem 2.2, we give first some preliminary results.

2.1. **An approximate equation.** In this section, we introduce a regular version of the problem (1), by taking $G_n(\cdot), F_n(\cdot)$ and $f_n$ regular approximations of $G(\cdot), F(\cdot)$ and $f$ which can be defined as follows:

$$
G_n(x, r) = \frac{G(x, r)}{1 + \frac{1}{n}G(x, r)}1_{[w \leq n]},
$$

$$
F_n(x, s) = \frac{F(x, s)}{1 + \frac{1}{n}F(x, s)}1_{[w \leq n]},
$$

$$
f_n = \min\{n, f\}1_{[w \leq n]}.
$$

We recall that the regular approximations $G_n, F_n$ and $f_n$ are bounded by $n$ and converge pointwise respectively to $G, F$ and $f$ as $n$ tends to $\infty$. Let us define the approximating problem:

$$
\begin{cases}
  \alpha u_n - \Delta u_n + G_n(x, \nabla u_n) = F_n(x, u_n) + f_n & \text{in } \mathcal{D}'(\Omega) \\
  u_n \in H_0^1(\Omega).
\end{cases}
$$

Since $0$ is a sub-solution of the problem (1), the function $w_n = \min\{n, w\}$ is a non-negative super-solution and the growth of $G_n$ is subquadratic, we can apply the classical works of [11], we conclude the existence of $u_n \in H_0^1(\Omega)$ a weak solution of (15), such that:

$$
0 \leq u_n \leq w_n \leq w.
$$

**Remark 1.** If we assume that the nonlinearity $G$ is convex with respect to the gradient, we regularize the problem (1) by taking $G_n(x, r)$ the Yosida approximation of $G(x, r)$, we have $G_n(x, r)$ is convex in $r$, converges to $G(x, r)$ as $n$ tends to $\infty$ and satisfies the following properties:

$$
G_n \leq G_{n+1} \leq G, \quad \|G_{n,r}(x, r)\|_{\infty} \leq n,
$$
where $G_{n,r}$ is a section of subdifferential of $G_n$ with respect to $r$. We define the approximate solution to the problem (1) as follows: we consider the sequence defined by $u_0 = w$ and for $n \geq 1$ we take $u_n$ solution of the following problem:

$$
\begin{align*}
\begin{cases}
    \alpha u_n - \Delta u_n + G_n(x, \nabla u_n) = F(x, u_{n-1}) + f & \text{in } D'(\Omega), \\
    u_n \in W^{1,1}_0(\Omega), G_n(x, \nabla u_n) \text{ and } F(x, u_{n-1}) \in L^1(\Omega).
\end{cases}
\end{align*}
$$

The rest of the proof follows the same technique introduced in [3].

2.2. A priori estimate. Let us consider, the function

$$
T_k(s) = \max \left\{ -k, \min(s, k) \right\}
$$

for $s \in \mathbb{R}$ and $k$ is a given real number.

We will show in the following lemma some necessary estimates which are useful in the proof of the Theorem 2.2.

**Lemma 2.3.** Let $u_n$ be a sequence defined as above. Then

(i) $\int_{\Omega} F_n(x, u_n) dx + \int_{\Omega} f_n(x) dx \leq C$,

(ii) $\int_{\Omega} G_n(x, \nabla u_n) dx \leq C$,

(iii) $\int_{\Omega} |\nabla T_k(u_n)|^2 \leq Ck$,

(iv) $\lim_{k \to \infty} \frac{1}{k} \int_{\Omega} |\nabla T_k(u_n)|^2 = 0$ uniformly on $n$,

where $C$ denotes a constant independent of $n$.

**Proof.** (i) By using (9) and (16), one has

$$
\int_{\Omega} F_n(x, u_n) dx + \int_{\Omega} f_n(x) dx \leq \int_{\Omega} F(x, w) dx + \int_{\Omega} f(x) dx \leq C.
$$

(ii) Integrating the equation (15) over $\Omega$, we have

$$
\int_{\Omega} \alpha u_n - \Delta u_n + \int_{\Omega} G_n(x, \nabla u_n) = \int_{\Omega} F_n(x, u_n) + \int_{\Omega} f_n.
$$

First, we recall that for every function $y \in W^{1,1}_0(\Omega)$ such that

$$
\begin{align*}
\begin{cases}
    -\Delta y = H, & \text{in } L^1(\Omega), \\
    y \geq 0,
\end{cases}
\end{align*}
$$

there exists a sequence $y_n \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which satisfies

$$
\begin{align*}
y_n \to y \text{ strongly in } W^{1,1}_0(\Omega), \\
\Delta y_n \to \Delta y \text{ strongly in } L^1(\Omega).
\end{align*}
$$

We use the regularity of $y_n$ to write

$$
\int_{\Omega} \Delta y_n = \int_{\partial \Omega} \frac{\partial y_n}{\partial \nu}.
$$
but \( y_n \geq 0 \) in \( \Omega \) and \( y_n = 0 \) on \( \partial \Omega \), then \( \frac{\partial y_n}{\partial \nu} \leq 0 \). By passing to the limit, we have \( \int_{\Omega} \Delta y \leq 0 \). Therefore

\[
\int_{\Omega} \Delta u_n \leq 0.
\]

Employing this fact, it follows from (17) that

\[
\int_{\Omega} G_n(x, \nabla u_n) \leq \int_{\Omega} F_n(x, u_n) + \int_{\Omega} f_n 
\leq \|F(w)\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}.
\]

(iii) Multiplying the equation (15) by \( T_k(u_n) \) and integrating over \( \Omega \), we obtain

\[
\int_{\Omega} \alpha u_n T_k(u_n) + \int_{\Omega} |\nabla T_k(u_n)|^2 + \int_{\Omega} G_n(x, \nabla u_n) T_k(u_n) 
= \int_{\Omega} F_n(x, u_n) T_k(u_n) + \int_{\Omega} f_n T_k(u_n) 
\leq k \left( \int_{\Omega} F(x, w) + \int_{\Omega} f \right) 
\leq Ck.
\]

Since \( u_n \) is non-negative, we have \( T_k(u_n) \geq 0 \), one gets

\[
\int_{\Omega} |\nabla T_k(u_n)|^2 \leq Ck.
\]

(iv) Concerning the equation satisfied by \( u_n \), we remark that

\[
\alpha u_n - \Delta u_n \leq F(x, w) + f \text{ in } D'(\Omega).
\]

For every \( 0 < M < k \), we multiply this inequality by \( T_k(u_n) \) and we integrate over \( \Omega \), one has

\[
\int_{\Omega} |\nabla T_k(u_n)|^2 \leq \int_{\Omega} T_k(u_n) \left( F(x, w) + f \right) + \int_{\Omega} T_k(u_n) \left( F(x, w) + f \right) 
\leq M \int_{\Omega} \left( F(x, w) + f \right) + k \int_{\Omega} \left( F(x, w) + f \right) 1_{\{u_n > M\}}.
\]

Hence,

\[
\frac{1}{k} \int_{\Omega} |\nabla T_k(u_n)|^2 \leq M \left( \|F(x, w)\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)} \right) + \int_{\Omega} \left( F(x, w) + f \right) 1_{\{u_n > M\}}.
\]

Fix \( \varepsilon > 0 \). Since \( u_n \) is bounded in \( L^1(\Omega) \), we have

\[
\|u_n \geq h\| = \int_{\{u_n \geq h\}} dx \leq \frac{1}{\varepsilon} \|u_n\|_{L^1(\Omega)} \leq \frac{C}{\varepsilon}.
\]
Therefore, there exists $h_\varepsilon$ independent of $n$ such that
\[
\int_\Omega \left( F(x, w) + f \right) 1_{[u_n > h_\varepsilon]} \leq \frac{\varepsilon}{2}.
\]
Taking $M = h_\varepsilon$ and letting $k$ tends to $\infty$, we obtain the desired result.

2.3. Proof of Theorem 2.2.

Proof. Employing (i) and (ii) of Lemma 2.3 and the compactness result of the following operator
\[
L^1(\Omega) \to W^{1,q}_0(\Omega)
\]
where $q \in [1, \frac{N}{N-1}]$ and $v$ is a solution to the following problem
\[
\left\{ \begin{array}{ll}
v \in W^{1,q}_0(\Omega) \\
\alpha v - \Delta v = \eta 
\end{array} \right. \quad \text{in} \; \mathcal{D}'(\Omega),
\]
we conclude the existence of a subsequence still denoted by $u_n$ for simplicity, such that
\[
u_n \to u \text{ strongly in } W^{1,q}_0(\Omega), \quad (u_n, \nabla u_n) \to (u, \nabla u) \text{ a.e in } \Omega.
\]
Moreover
\[
G_n(x, \nabla u_n) \to G(x, \nabla u) \text{ a.e in } \Omega,
\]
\[
F_n(x, u_n) \to F(x, u) \text{ a.e in } \Omega,
\]
the monotony of $F(x, \cdot)$ allows us to have
\[
|F_n(x, u_n)| \leq F(x, w) \text{ a.e in } \Omega.
\]
Using Lebesgue’s theorem, we obtain
\[
F_n(x, u_n) \to F(x, u) \text{ in } L^1(\Omega).
\]
Then there exists $\lambda$ a non-negative measure, see [38], such that :
\[
\alpha u_n - \Delta u_n + G_n(x, \nabla u_n) - F_n(x, u_n) - f_n
\]
\[
\to \alpha u - \Delta u + G(x, \nabla u) - F(x, u) - f + \lambda \in \mathcal{D}'(\Omega).
\]
On the other hand,
\[
\alpha u_n - \Delta u_n + G_n(x, \nabla u_n) = F_n(x, u_n) + f_n \to F(x, u) + f \text{ in } L^1(\Omega).
\]
Consequently
\[
\alpha u - \Delta u + G(x, \nabla u) \leq F(x, u) + f \text{ in } \mathcal{D}'(\Omega).
\] (18)
Therefore, to conclude the existence of a weak solution to the problem (1), we must establish the opposite inequality (18).

To this end, we consider the function $H \in C^1(\mathbb{R})$, such that $0 \leq H(s) \leq 1$ and
\[
H(s) = \begin{cases} 
0 & \text{if } |s| \geq 2 \\
1 & \text{if } |s| \leq 1
\end{cases}
\]
and the following test function
\[
v_n = \Psi \exp(-C_2 u_n)H(\frac{u_n}{k}),
\]
where \( C_2 \) is given by the assumption on \( G \)
\[
G(x, r) \leq C_2 \left( \|r\|_0^2 + d_2(x) \right)
\]
and \( \Psi \leq 0, \Psi \in H_0^1(\Omega) \cap L^\infty(\Omega) \). We multiply (15) by \( v_n \) and we integrate on \( \Omega \), we obtain
\[
\int_\Omega \nabla u_n \nabla \Psi \exp(-C_2 u_n) H \left( \frac{u_n}{k} \right) + \frac{1}{k} \int_\Omega \Psi |\nabla u_n|^2 \exp(-C_2 u_n) H' \left( \frac{u_n}{k} \right) \\
+ \int_\Omega (\alpha u_n + G_n(x, \nabla u_n) - C_2 |\nabla u_n|^2) \Psi \exp(-C_2 u_n) H \left( \frac{u_n}{k} \right) \\
= \int_\Omega F_n(x, u_n) \Psi \exp(-C_2 u_n) H \left( \frac{u_n}{k} \right) + \int_\Omega f_n \Psi \exp(-C_2 u_n) H \left( \frac{u_n}{k} \right).
\]
We note \( I_1, I_2, I_3, I_4 \) and \( I_5 \) the five terms in the last relation. By investigating separately each term, we get for the first one
\[
\lim_{n \to \infty} I_1 = \lim_{n \to \infty} \int_\Omega \nabla u_n \nabla \Psi \exp(-C_2 u_n) H \left( \frac{u_n}{k} \right) \\
= \lim_{n \to \infty} \int_\Omega \nabla T_k(u_n) \nabla \Psi \exp(-C_2 u_n) H \left( \frac{u_n}{k} \right).
\]
Using the fact that:
\[
\nabla \Psi \exp(-C_2 u_n) H \left( \frac{u_n}{k} \right) \text{ converges strongly to } \nabla \Psi \exp(-C_2 u) H \left( \frac{u}{k} \right) \text{ in } L^2(\Omega),
\]
and
\[
\nabla T_k(u_n) \text{ converges weakly to } \nabla T_k(u) \text{ in } L^2(\Omega),
\]
thanks to item (iii) of Lemma 2.3, we have
\[
\lim_{n \to \infty} I_1 = \int_\Omega \nabla u \nabla \Psi \exp(-C_2 u) H \left( \frac{u}{k} \right).
\]
For the second integral, since \( \exp(-C_2 u_n) \leq 1 \), we have
\[
\frac{1}{k} \int_\Omega \Psi |\nabla u_n|^2 \exp(-C_2 u_n) H' \left( \frac{u_n}{k} \right) \leq C \|\Psi\|_{L^\infty(\Omega)} \frac{1}{k} \int_\Omega |\nabla T_k(u_n)|^2.
\]
Then, by (iii) of Lemma 2.3, we have
\[
\lim_{n \to \infty} I_2 = 0 \text{ uniformly on } n.
\]
For \( I_3 \), using the assumption (11) and \( \Psi \leq 0 \), we have
\[
(\alpha u_n + G_n(x, \nabla u_n) - C_2 |\nabla u_n|^2 + d_2(x)) \Psi \exp(-C_2 u_n) H \left( \frac{u_n}{k} \right) \geq 0.
\]
Therefore, by applying Fatou’s lemma, we obtain
\[
\lim_{n \to \infty} I_3 \geq \int_\Omega (\alpha u + G(x, \nabla u) - C_2 |\nabla u|^2) \Psi \exp(-C_2 u) H \left( \frac{u}{k} \right).
\]
On the other hand, we have
\[
\lim_{n \to \infty} I_4 = \int_\Omega F(x, u) \Psi \exp(-C_2 u) H \left( \frac{u}{k} \right).
\]
and
\[ \lim_{n \to \infty} I_5 = \int_{\Omega} f \Psi \exp(-C_2 u) H\left(\frac{u}{k}\right). \]

Then, we obtain
\[
\begin{align*}
\int_{\Omega} \nabla u \nabla \Psi \exp(-C_2 u) H\left(\frac{u}{k}\right) \\
+ \int_{\Omega} (\alpha u + G(x, \nabla u) - C_2 |\nabla u|^2) \Psi \exp(-C_2 u) H\left(\frac{u}{k}\right) + \beta\left(\frac{1}{k}\right) \\
\leq \int_{\Omega} (F(x, u) + f) \Psi \exp(-C_2 u) H\left(\frac{u}{k}\right),
\end{align*}
\]

where \( \beta(\varepsilon) \) denotes a quantity that tends to 0 when \( \varepsilon \) tends to 0.

Now, we choose \( \Psi = -\phi \exp(C_2 u) H\left(\frac{u}{k}\right) \), where \( \phi \geq 0, \phi \in D(\Omega) \), we have
\[
\begin{align*}
- \int_{\Omega} \nabla u \nabla \phi H^2\left(\frac{u}{k}\right) - \int_{\Omega} C_2 |\nabla u|^2 \phi H^2\left(\frac{u}{k}\right) - \frac{1}{k} \int_{\Omega} |\nabla u|^2 \phi H\left(\frac{u}{k}\right) H'\left(\frac{u}{k}\right) \\
- \int_{\Omega} \phi(\alpha u + G(x, \nabla u) - C_2 |\nabla u|^2) H^2\left(\frac{u}{k}\right) + \beta\left(\frac{1}{k}\right) \\
\leq - \int_{\Omega} (F(x, u) + f) \phi H^2\left(\frac{u}{k}\right).
\end{align*}
\]

By developing calculations and remarking that the third term is equivalent to \( \beta\left(\frac{1}{k}\right) \), we can write:
\[
\begin{align*}
- \int_{\Omega} \nabla u \nabla \phi H^2\left(\frac{u}{k}\right) - \int_{\Omega} \phi(\alpha u + G(x, \nabla u)) H^2\left(\frac{u}{k}\right) + \beta\left(\frac{1}{k}\right) \\
\leq - \int_{\Omega} (F(x, u) + f) \phi H^2\left(\frac{u}{k}\right).
\end{align*}
\]

We finally pass to the limit as \( k \) tends to \( \infty \) and we use the fact that \( \lim_{k \to \infty} H\left(\frac{u}{k}\right) = 1 \) to conclude, for every \( \phi \geq 0, \phi \in D(\Omega) \) we have
\[
\int_{\Omega} \nabla u \nabla \phi + \int_{\Omega} \phi(\alpha u + G(x, \nabla u)) \geq \int_{\Omega} (F(x, u) + f) \phi.
\]

This allows us to get the existence of a weak non-negative solution of the problem (1).

3. **Numerical method.** In this section we present the numerical method to compute an approximation of the non-negative solution of the problem (1) in \( H^1_0(\Omega) \) in the context of the Remark 1 and in the two-dimensional case. Formally, the algorithm can be formulated in the following way:

1) find \( w \in H^1_0(\Omega) \) such that:
\[
\alpha w(x) - \Delta w(x) \geq F(x, w) + f \text{ in } \Omega.
\]
2) given \( u_0 = w \) we compute a sequence, \( \{ u_n \}_{n} \), where \( u_{n+1} \) is the solution in \( H^1_0(\Omega) \) of the nonlinear equation:

\[
\alpha u_{n+1}(x) - \Delta u_{n+1}(x) + G_{n+1}(x, \nabla u_{n+1}) = F(x, u_n) + f \quad \text{in} \quad \Omega. \quad (20)
\]

Both problems (19) and (20) are nonlinear, and if the problem (19) has a solution, then thanks to the Theorem 2.2, the problem (20) has also a solution.

3.1. Discretization. Let \( \Omega \) be a polygon in \( \mathbb{R}^2 \). We denote \( T_h \) a geometrically conformal triangulation of \( \Omega \). That means a set of triangles \( (T_i)_{i=1}^{N_h} \) such that:

- \( \bar{\Omega} = \bigcup_{i=1}^{N_h} T_i. \)
- If \( i \neq j \) then \( T_i \cap T_j = \emptyset \), or a common vertex or a common edge.

Let denote \( h = \max_{T \in T_h} h_T \) where \( h_T \) is the external diameter of the triangle \( T \). Let \( d_T \) denotes the internal diameter of the triangle \( T \). We assume that there exist two numbers \( \sigma_1 > 0 \) and \( \sigma_2 > 0 \) such that \( \sigma_1 h \leq d_T \leq \sigma_2 h \).

We have associated to the triangulation \( T_h \) an approximation space of continuous and piece-wise linear functions:

\[
V_h = \{ v_h \in C^0(\Omega) / v_h \big|_{T} \in \mathbb{P}_1(T) \text{ for every } T \in T_h \},
\]

where \( \mathbb{P}_1(T) \) is the set of linear functions defined in \( T \).

Then we look for a numerical approximation of the non-negative solution of the problem (1) in \( X_h = V_h \cap H^1_0(\Omega) \).

3.2. Numerical approximation of the super-solution. The numerical weak formulation associated to problem (19) is the following:

\[
\left\{ \begin{array}{l}
\text{find } w_h \in X_h \text{ such that:} \\
\quad a_1(w_h, v_h) = L_1(v_h) \quad \forall \, v_h \in X_h,
\end{array} \right. \quad (21)
\]

where

\[
a_1(w_h, v_h) = \int_{\Omega} \nabla w_h \nabla v_h \, dx + \int_{\Omega} \alpha w_h v_h \, dx - \int_{\Omega} F(x, w_h) v_h \, dx,
\]

and

\[
L_1(v_h) = \int_{\Omega} f v_h \, dx.
\]

The problem (21) is nonlinear, then we consider Newton’s algorithm to compute an approximation of the solution.

Given a starting point \( w_h^0 \in X_h \), the iterative Newton method computes a sequence \( w_h^k \) where \( w_h^0 \in X_h \) and \( w_h^{k+1} = w_h^k + \delta \). The update \( \delta \in X_h \) is the solution of the following equation:

\[
a_2(\delta, v_h) = \int_{\Omega} \nabla \delta \nabla v_h \, dx + \int_{\Omega} \alpha \delta v_h \, dx - \int_{\Omega} \frac{\partial F}{\partial r}(x, w_h^k) \delta v_h \, dx = L_2(v_h), \quad (22)
\]

with

\[
L_2(v_h) = - \int_{\Omega} \nabla w_h^k \nabla v_h \, dx - \int_{\Omega} \alpha w_h^k v_h \, dx + \int_{\Omega} F(x, w_h^k) v_h \, dx + \int_{\Omega} f v_h \, dx,
\]

for all \( v_h \in X_h \).

In problem (22), we denote \( \frac{\partial F}{\partial r}(x, r) \) the subdifferential of \( F(x, r) \) w.r.t. \( r \).

To study the well-posedness of the problem (22), we introduce the following notation:
We assume that $c(x)$ is bounded and $g^k \in L^2(\Omega)$. We denote $c_\infty = \|c\|_{L^\infty(\Omega)}$.

Then, at each $k$ iteration, problem (22) can be reformulated in the following way:

$$\begin{aligned}
\begin{cases}
\text{find } v \in H^1_0(\Omega) \text{ such that:} \\
a_3(v, w) = L_3(w) \quad \forall \ w \in H^1_0(\Omega),
\end{cases}
\end{aligned}$$

where

$$a_3(v, w) = \int_\Omega \nabla v \nabla w \, dx + \int_\Omega (\alpha + c(x))v(x)w(x)dx,$$

$$L_3(w) = \int_\Omega g(x)w(x)dx.$$  

For all $v, w \in H^1_0(\Omega)$ we have:

$$|a_3(v, w)| \leq |v|_1|w|_1 + (\alpha + c_\infty)\|v\|_0\|w\|_0.$$  

Using Poincaré’s inequality, we have

$$\|v\|_0 \leq C(\Omega)\|v\|_1,$$

where $C(\Omega) = \frac{\text{meas}(\Omega)}{c_0}$, and $c_0 = \pi$.

Consequently, the bilinear form $a_3(\cdot, \cdot)$ is continuous in $H^1_0(\Omega) \times H^1_0(\Omega)$ and we have:

$$|a_3(u, v)| \leq (1 + \frac{(c_\infty + \alpha)}{c_0^2}\text{meas}(\Omega)^2)\|u\|_1\|v\|_1.$$  

The linear form $L_3$ is continuous in $H^1_0(\Omega)$ and

$$|L(w)| \leq \|g\|_0\|w\|_0 \leq C(\Omega)\|g\|_0\|w\|_1.$$  

Moreover the bilinear form $a_3(\cdot, \cdot)$ verifies:

$$a_3(w, w) = \int_\Omega \|\nabla w\|^2 \, dx + \int_\Omega (\alpha + c(x))w^2 \, dx \geq \left(1 + \frac{(\alpha - c_\infty)}{c_0^2}\text{meas}(\Omega)^2\right)\tilde{C}\|w\|_1^2.$$  

If $\left(1 + \frac{(\alpha - c_\infty)}{c_0^2}\text{meas}(\Omega)^2\right)$ is non-negative, the bilinear form $a_3(\cdot, \cdot)$ is elliptic in $H^1_0(\Omega)$ and consequently problem (23) has a unique solution in $H^1_0(\Omega)$.

But the sign of $\alpha + c(x)$, when $x \in \Omega$, is not always non-negative, the bilinear form $a_3(\cdot, \cdot)$ may not be coercive.

In fact the bilinear form (24) is coercive if $(\alpha + c(x)) \geq 0, \ x \in \Omega$ or if:

$$\text{meas}(\Omega) < \frac{c_0}{\sqrt{c_\infty - \alpha}} \quad \text{when } c_\infty > \alpha.$$  

Moreover, there exists a set of eigenvalues $\lambda_l$ of the second order derivation operator such that there is a unique solution for the variational problem provided that $\lambda_l \neq c_\infty - \alpha$, see Agmon [1]. In the other case may be an infinite number of solutions. Thus the well posedness of the Newton step, that means the existence of a unique solution to (23), depends on the behavior of $c_\infty = \|\frac{\partial F(x, w^k(x))}{\partial r}\|_\infty$. So, to compute a numerical solution of (23) using classical finite element when the domain $\Omega$ does not verify condition (25), we introduce a domain decomposition method.
Remark 2. If \( a_3(\cdot, \cdot) \) is elliptic, \( \delta \in H^2(\Omega) \cap H^1_0(\Omega) \) and \( \delta_h \in X_h \) we have, see [15, 21]:

\[
||\delta - \delta_h||_0 + h||\delta - \delta_h||_1 \leq \beta h^2|\delta|_2.
\]

3.3. Domain decomposition method. In the case where \( \Omega \) does not verify condition (25), we introduce the Schwarz domain decomposition method, see [37], to solve (23) at each iteration of the Newton method.

First, we consider a geometrical conform domain decomposition of \( \Omega \) in \( m \) subdomains \( \Omega_i \) such that:

- \( \Omega_i = \bigcup_{T \in T_{h,i}} T \) with \( T_{h,i} \subset T_h \),
- \( \Omega = \bigcup_{i=1}^m \Omega_i \),
- \( \Omega_i \cap \Omega_j = \Omega_{i,j} \), \( i = 1, \ldots, m \), when \( \Omega_i \cap \Omega_j \neq \emptyset \),
- \( \text{meas}(\Omega_i) \leq \frac{\pi}{\sqrt{c_\infty} - \alpha} \) if \( c_\infty > \alpha \), in other case domain decomposition is not necessary,
- let denote \( \partial \Omega_i \cap \Omega_j = \Gamma_{i,j} \), \( i, j = 1, \ldots, m \), if \( \Omega_i \cap \Omega_j \neq \emptyset \) for \( j=1, \ldots, m \).

The local approximation spaces associated to the geometrical domain decomposition and the Schwarz’s method are the following:

- \( V^i_{h,0} = \{ p \in C^0(\Omega_i); p|_T \in P_1(T), T \in T_{h,i} \} \),
- \( X^i_{h,0} = V^i_{h,0} \cap H^1_0(\Omega_i) \).

The local bilinear form at the \( l \)-iteration of the Schwarz’s method is given by:

\[
a^i_l(w_h, v_h) = \int_{\Omega_i} \nabla w_h \nabla v_h \, dx + \int_{\Omega_i} \alpha w_h v_h \, dx - \int_{\Omega_i} \frac{\partial F}{\partial r}(x, w_h^k)w_h v_h \, dx,
\]

and

\[
L^i_l(v_h) = -\int_{\Omega_i} \nabla w_h^k \nabla v_h \, dx - \int_{\Omega_i} \alpha w_h^k v_h \, dx + \int_{\Omega_i} F(x, w_h^k) v_h \, dx + \int_{\Omega_i} f v_h \, dx.
\]

If \( \Omega_i \) verifies condition (25), given \( w^k \) the bilinear form \( a^i_l \) is elliptic in \( X^i_{h,0} \) and the linear form \( L^i_l \) is continuous for all \( i = 1, \ldots, m \). Then the domain decomposition algorithm to compute \( \delta_h \) can be formulated in the following way: given \( \delta^0_h \in X_h \), for \( l \geq 0 \), we compute \( \delta^{l+1}_h \) in \( X_h \) such that:

\[
\delta^{l+1}_h = \delta^l_h + \sum_{i=1}^m \delta^l_{h,i},
\]

where \( \delta^l_{h,i} \) denotes the extension of \( \delta^l_{h,i} \) by 0 in \( \Omega \setminus \Omega_i \). The function \( \delta^l_{h,i} \) is the solution of the following problem:

\[
\begin{cases}
\text{find } \delta^l_{h,i} \in X^i_{h,0} \text{ such that:} \\
\quad a^i_3(\delta^l_{h,i}, v_h) = L^i_l(v_h) - a^i_3(\delta^l_h, v_h) \quad \forall \, v_h \in X^i_{h,0}.
\end{cases}
\]

If \( a^i_3(\cdot, \cdot) \) is elliptic in \( X^i_{h,0} \), problem (26) has an unique solution. In [33], the convergence of Schwarz’s domain decomposition method is proved. Consequently each Newton step is well posed.
3.4. **Convergence of Newton’s method applied to problem (21).** Let \( N \) be the number of interior nodes of the triangulation. Let \( \{ \phi_i \}_{1 \leq i \leq N} \) be a classical Lagrangian bases of \( X_h \). Then,

\[
w_h(x) = \sum_{i=1}^{N} w_{h,i} \phi_i(x),
\]

where \( w_{h,i} = w_h(s_i) \), and \( s_i \) is an internal node of the triangulation \( T_h \). We denote \( w_h \) the vector \( \{ w_{h,i} \}_{i=1}^{N} \).

Let denote \( K \in \mathbb{R}^{N \times N} \) the following matrix:

\[
K_{i,j} = \int_{\Omega} (\nabla \phi_i(x) \nabla \phi_j(x) + \alpha \phi_i(x) \phi_j(x)) \, dx.
\]

Let denote \( H(w_h) \) the following vector function:

\[
H(w_h)_i = \int_{\Omega} F(x,w_h(x)) \phi_i(x) \, dx
\]

and we denote \( f \) the vector:

\[
f_i = \int_{\Omega} f(x) \phi_i(x) \, dx.
\]

Let denote \( A : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h} \) the nonlinear application:

\[
A(w_h) = K w_h - H(w_h).
\]

So the finite dimensional equivalent formulation of the nonlinear problem (21) is the following:

\[
A(w_h) = f.
\]

Let denote \( M(w_h^k) \in \mathbb{R}^{N \times N} \) the following matrix:

\[
M_{i,j} = \int_{\Omega} \left( \frac{\partial F}{\partial r}(x,w_h^k(x)) \right) \phi_i(x) \phi_j(x) \, dx.
\]

Now, we denote \( J(w_h) : \mathbb{R}^{N_h} \to \mathbb{R}^{N_h} \) the linear application:

\[
J(w_h)\delta_h = K \delta_h - M(w_h^k)\delta_h.
\]

Then the Newton iteration applied to problem (21) can be reformulated in the finite dimensional frame in the following equivalent way: given \( w_h^0 = 0 \) at each iteration we compute \( w_h^{k+1} = w_h^k + \delta_h \in \mathbb{R}^N \) where \( \delta_h \in \mathbb{R}^N \) is the solution of the linear system:

\[
J(w_h^k)\delta_h = -A(w_h^k) - f.
\]

If we assume that the domain decomposition condition proposed in the subsection 3.3 verifies constraint (25) in the measure of sub-domains, the matrix \( J(w_h^k) \) is positive definite.

Then there exists \( \gamma \) such that \( (J(w_h^k)\delta_h, \delta_h) \geq \gamma \| \delta_h \|_0^2 \) \( \forall k \) and \( \delta_h \in \mathbb{R}^{N_h} \). Consequently \( \| J^{-1}(w_h^k) \|_0 < \frac{1}{\gamma} \). Then, according to the Kantorovich’s theorem, see [22, 30], the sequence \( \{ w_h^k \}_k \) converges to \( w_h \) and the associated \( w_h \) is a solution of the problem (21) in \( X_h \). But, the rate of convergence is correlated to the regularity of the nonlinear operator \( A \).
Remark 3. In the case of our model problem:
\[
\begin{cases}
\alpha w(x) - \Delta w(x) = |w(x)|^p + f(x) & \forall x \in \Omega \\
w(x) = 0 & \forall x \in \partial \Omega,
\end{cases}
\]
where \( p \geq 1, f \geq 0 \) belongs to \( L^1(\Omega) \), we have:
- \( w_h \geq 0 \) in \( \Omega \).
- If \( p \geq 2 \), the derivative of \( A \), \( J(\cdot) \) is Lipschitz, then, applying the Kantorovich’s theorem, the Newton’s method converges quadratically.
- If \( 1 \leq p < 2 \), \( J(\cdot) \) is \((p - 1)\)-Hölder-continuous. According to the results announced in \([25, 26, 29]\), the Newton’s method converges at order \( p \).

3.5. Numerical approximation of the solution of problem (20). If \( u_{0,h} = w_h \), in the next step of the algorithm we compute a sequence \( \{u_{n,h}\}_n \subset X_h \) where \( u_{n,h} \) a numerical approximation of the solution of the following problem:
\[
\begin{cases}
\alpha u_n - \Delta u_n + G_n(x, \nabla u_n) = F(x, u_{n-1}) + f & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( G_n \) is the Yosida approximation of \( G \), that means \( G_n, G \) is convex, \( G_n < G \) and \( G_n \to G \) when \( n \to \infty \).

Let us denote by \( \bar{\pi}(\cdot, \cdot) \) the bilinear form associate to problem (27). The bilinear form \( \bar{\pi}(\cdot, \cdot) \) is defined by:
\[
\bar{\pi}(u_{n,h}, v_h) = \int_{\Omega} \alpha u_{n,h} v_h dx + \int_{\Omega} \nabla u_{n,h} \nabla v_h dx + \int_{\Omega} G(x, u_{n,h}) v_h dx
\]
and
\[
\mathcal{L}(v_h) = \int_{\Omega} (F(x, u_{n-1,h}) + f(x)) v_h dx.
\]

The numerical approximation \( u_{n,h} \) of \( u_n \) is the solution of the following problem:
\[
\begin{cases}
\text{find } u_{n,h} \in \bar{X}_h \text{ such :} \\
\bar{\pi}(u_{n,h}, v_h) = \mathcal{L}(v_h) \forall v_h \in \bar{X}_h.
\end{cases}
\]

(28)

The numerical solution of problem (28) will be similar at those implemented to compute the super-solution. Applying the Newton method, we compute a sequence \( \{u_{k+1, h}\}_k \subset X_h \) such that, given \( u_{0, n+1, h} = u_{n, h} \), the new iteration \( u_{k+1, h} = u_{n+1, h} + \theta_h \), where the update \( \theta_h \) is the solution in \( X_h \) of the following equation:
\[
\tilde{a}(\theta_h, v_h) = \tilde{L}(v_h) \forall v_h \in X_h,
\]
where
\[
\tilde{a}(\theta_h, v_h) = \int_{\Omega} \alpha \theta_h v_h dx + \int_{\Omega} \nabla \theta_h \nabla v_h dx + \int_{\Omega} d(x) \nabla \theta_h v_h dx
\]
and
\[
\tilde{L}(v_h) = \int_{\Omega} g v_h dx.
\]

We denote \( d(x) \) the subdifferential of \( G_n \) evaluated in \( \nabla u_{n,h}^k \), that means:
\[
d(x) = \frac{\partial G_n}{\partial r}(x, \nabla u_{n,h}^k(x))
\]
and we denote
\[
g(x) = -G_n(x, \nabla u_{n,h}^k) + F(x, u_{n-1}) + f(x) + \Delta u_{n,h}^k(x) - \alpha u_{n,h}^k(x).
\]

The bilinear form \( \tilde{a}(\cdot, \cdot) \) is continuous in \( X_h \times X_h \) and the linear form \( \tilde{L} \) is continuous in \( X_h \).
Let’s now study the ellipticity of the form \( \tilde{a} \), for every \( \theta_h \in X_h \), we have:

\[
\tilde{a}(\theta_h, \theta_h) = \int_\Omega \alpha \theta_h^2 + \int_\Omega |\nabla \theta_h|^2 + \int_\Omega d(x)(\nabla \theta_h)\theta_{\gamma h}
\]

\[
= \int_\Omega \alpha \theta_h^2 + \int_\Omega |\nabla \theta_h|^2 + \frac{1}{2} \int_\Omega d(x)\nabla (\theta_h^2).
\]

Assuming \( \text{div}(d(x)) \in L^\infty(\Omega) \) we obtain, see [36]:

\[
\tilde{a}(\theta_h, \theta_h) = \int_\Omega \alpha \theta_h^2 + \int_\Omega |\nabla \theta_h|^2 - \frac{1}{2} \int_\Omega \text{div}(d(x)) \theta_h^2,
\]

\[
= \int_\Omega |\nabla \theta_h|^2 + \int_\Omega \left( \alpha - \frac{1}{2} \text{div}(d(x)) \right) \theta_h^2.
\]

Let us denote \( d_\infty \) the supremum norm of \( \text{div} \,(d(x)) \), then we have:

\[
\tilde{a}(\theta_h, \theta_h) \geq \int_\Omega |\nabla \theta_h|^2 + (\alpha - d_\infty) \int_\Omega \theta_h^2.
\]

Thanks to Poincaré’s inequality we obtain:

\[
\tilde{a}(\theta_h, \theta_h) \geq \int_\Omega |\nabla \theta_h|^2 + (\alpha - d_\infty) C(\Omega)^2 \int_\Omega |\nabla \theta_h|^2
\]

\[
\geq (1 + (\alpha - d_\infty) C(\Omega)^2) \|\nabla \theta_h\|_0^2
\]

\[
\geq (1 + (\alpha - d_\infty) C(\Omega)^2) C_1 \|\theta_h\|_1^2.
\]

Consequently, \( \tilde{a} \) is elliptic when \( \alpha - d_\infty > 0 \) or if the domain \( \Omega \) verifies the following inequality:

\[
\text{meas}(\Omega) < \frac{\pi}{\sqrt{d_\infty - \alpha}}, \text{ when } \alpha - d_\infty < 0.
\]

Then if \( \tilde{a}(\theta_h, \theta_h) \) is elliptic, the Newton update step is well posed. In the other case we can apply the same domain decomposition method used to compute the super-solution.

The convergence of Newton’s method can be obtained using the same arguments applied to the case of the super-solution, adding the convexity of \( G_\alpha \). The rate of convergence in the case of the model problem \( (2) \) will be \( q \) according to the results in [29].

4. Numerical simulations. In this section, we propose some numerical tests of the algorithm presented in the previous section.

The first one concerns an elementary case where we know the solution, in order to test the precision of the algorithm.

\[
\left\{ \begin{array}{ll}
\alpha u(x, y) - \Delta u(x, y) + \|\nabla u(x, y)\|^2 = \mu |u(x, y)|^3 + f(x, y) & \forall (x, y) \in \Omega \\
u(x, y) = 0 & \forall (x, y) \in \partial \Omega,
\end{array} \right.
\]

where \( u(x, y) = 10x(1-x)y(1-y), \alpha = 1, \mu = 10 \) and \( \Omega = (0,1) \times (0,1) \). In Figure 1, we plot the super-solution obtained after 5 iterations of Newton’s algorithm, when the \( L^2 \)-norm of the update is of order \( 10^{-5} \). The space discretization step is \( dx = dy = 0.02 \) and the domain was decomposed in 4 sub-domains.

In a second step, given \( u_{\alpha,h} = w_h \), we compute \( u_{n,h} \in X_h \) for \( n = 1, 2, \ldots, 10 \). In Figure 2, we plot the numerical approximation of the solution of the problem \( (30) \) obtained after 10 iterations of the Yosida approximation.
Figure 1. The computed super-solution $w_h$

Figure 2. The numerical approximation of the solution of (30)

| n   | 1    | 2    | 3    | ... | 8    | 9    | 10  |
|-----|------|------|------|-----|------|------|-----|
| Error | 0.6  | 0.281| 0.1033| ... | 1.2 $10^{-3}$ | 3.05 $10^{-3}$ | 8.21 $10^{-6}$ |
| #Newton iteration | 5    | 7    | 9    | ... | 7    | 6    | 6   |

Table 1. The $L^2$-norm of error between $u$ and $u_{n,h}$

In Table 1, we summarized the evolution of the error between the exact solution $u$ and the numerical solution $u_{n,h}$ w.r.t. $n$. Table 1 shows the number of Newton’s iteration needed to achieve the convergence. In Figure 3, we plot the evolution of the error w.r.t. $n$. 
In this example we have verified that the algorithm converges in few number of iterations and the error is the order less than $dx^2$, where $dx$ is the mesh step size.

Let consider now a new example where the source function $f$ is an addition of Dirac functions.

\[
\begin{array}{l}
\alpha u(\boldsymbol{\tau}) - \Delta u(\boldsymbol{\tau}) + |\nabla u(\boldsymbol{\tau})|^2 = \mu(\boldsymbol{\tau}) |u(\boldsymbol{\tau})|^2 + \sum_{j=1}^{3} K_j \delta_{\boldsymbol{\tau}_j} \forall \boldsymbol{\tau} \in \Omega \\
u(\boldsymbol{\tau}) = 0 \forall \boldsymbol{\tau} \in \partial \Omega.
\end{array}
\] (31)

where $\boldsymbol{\tau} = (x, y)$, $\Omega = (0, 1) \times (0, 1)$, $\mu(\boldsymbol{\tau}) = 6$, $\alpha = 1$, $K_1 = 5 = K_2$, $K_3 = 10$, $\boldsymbol{\tau}_1 = (0.1, 0.2)$, $\boldsymbol{\tau}_2 = (0.5, 0.6)$ and $\boldsymbol{\tau}_3 = (0.9, 0.9)$.

In Figure 4, we have plotted the computed super-solution obtained after five iterations.

**Figure 3.** Evolution of the error w.r.t. $n$

**Figure 4.** The numerical approximation of the super-solution of (31)
Newton iterations when the $L^2$-norm of the update is of order $10^{-4}$.

The space step size is $dx = dy = 0.02$ and the number of sub-domains goes from
9 to 16. At the second iteration of Newton’s method, we have 9 sub-domains and
from the third iteration the number of sub-domains needed is 16. Figure 5 shows

![Figure 5. The final sub-domains decomposition](image)

the last configuration of the sub-domain decomposition. In Table 2, we show the

| Newton iteration | 1 | 2 | 3 | 4 |
|------------------|---|---|---|---|
| Norm of the Newton update | 4.21 | 0.17 | 0.026 | $6.4 \times 10^{-4}$ |
| # Schwarz iteration | - | 20 | 14 | 6 |
| Norm of the Schwarz update | - | $8.6 \times 10^{-4}$ | $9.1 \times 10^{-4}$ | $8.8 \times 10^{-4}$ |
| # sub-domains | 1 | 9 | 16 | 16 |

Table 2. The behavior of the algorithm computing the super-solution

evolution of the algorithm computing the super-solution. It needs four iterations
of Newton’s method to obtain an infinite norm of the update of order $10^{-4}$. At
each Newton’s algorithm step, we give the number of iterations of the domain
decomposition algorithm to obtain the convergence, when the relative norm of the
update is of order $10^{-4}$. Once the approximation of the super-solution computed,
we solve problem (20) for $n = 1, 2, ..., 20$. In Figure 6, we have plotted the solution
obtained at the iteration $n = 20$.

| n    | 1   | 2   | ... | 6   | 7   | ... | 19  | 20  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|
| Norm of the update | 0.1009 | 0.1037 | ... | 0.0232 | 0.018 | ... | 0.0065 | 0.0062 |
| # Newton iteration | 7   | 8   | ... | 8   | 9   | ... | 9   | 8   |

Table 3. The behavior of the algorithm computing the solution
of the problem (20)
In Table 3, we summarized the number of the Newton’s iterations needed at each Yosida’s algorithm step to achieve the convergence, when the norm of the Newton update is less than $10^{-7}$. We report also the $L^2$-relative norm of the update between the step $n$ and $n+1$ when $G_n$ changes to $G_{n+1}$. The algorithm stops when the correction between two Yosida’s algorithm steps is less than $10^{-3}$.

5. **Conclusion.** In this work, a theoretical result of the existence of a non-negative solution in $W^{1,1}_0(\Omega)$ of the quasilinear problem (1) was presented when we assume a quadratic growth of the nonlinearity on the gradient of the solution, for a regular or a non-regular source function, in particular in the case of a sum of point measurements.

In the third section of this paper, based on the theoretical methodology, we have developed a numerical method allowing to calculate an approximation of the non-negative solution of problem (1) using the Newton method and the additive Schwarz domain decomposition techniques. The domain decomposition is adapted at each Newton method step. Then, starting with the computed non-negative super-solution, we have built a sequence of solutions of the problems implementing the Yosida’s approximation of $G$ that converges towards the solution of our problem.

At the end of this paper, in Section 4, we have given some numerical examples attesting the performance of the numerical method presented.

The numerical method developed to compute the super-solution of (1) in $N \geq 1$ dimensions can be used to solve parabolic quasilinear problems. At each iteration in time, considering an adaptive step of time and we can be able to solve the resulting, quasilinear and static problem using the algorithm presented in Section 3.

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REFERENCES

[1] S. Agmon, Lectures on Elliptic Boundary Value Problems, Van Nostrand Mathematical Studies, D. Van Nostrand Co., Inc., Princeton, 1965.

[2] N. Alaa, Etude d’Équations Elliptiques Non-Linéaires à Dépendance Convexe en le Gradient et à Données Mesures, Ph.D thesis, Université de Nancy I, 1989.

[3] N. Alaa, A. Cheggour and J. R. Roche, Mathematical and numerical analysis of a class of nonlinear elliptic equations in the two dimensional case, Numerical Mathematics and Advanced Applications, Springer, Berlin, 2006, 926–934.

[4] N. Alaa, F. Maach and I. Mounir, Existence for some quasilinear elliptic systems with critical growth nonlinearity and $L^3$ data, J. Appl. Anal., 11 (2005), 81–94.

[5] N. Alaa and I. Mounir, Global existence for reaction-diffusion systems with mass control and critical growth with respect to the gradient, J. Math. Anal. Appl., 253 (2001), 532–557.

[6] N. Alaa and M. Iguernane, Weak periodic solutions of some quasilinear parabolic equations with data measures, JIPAM. J. Inequal. Pure Appl. Math., 3 (2002), 14pp.

[7] N. Alaa, Solutions faibles d’équations paraboliques quasi-linéaires avec données initiales mesures, Ann. Math. Blaise Pascal, 3 (1996), 1–15.

[8] N. Alaa and M. Pierre, Weak solution of some quasilinear elliptic equations with data measures, SIAM J. Math. Anal., 24 (1993), 23–35.

[9] N. Alaa and J. R. Roche, Theoretical and numerical analysis of a class of nonlinear elliptic equations, Mediterr. J. Math, 2 (2005), 327–344.

[10] H. Amann, Fixed point equations and nonlinear eigenvalue problems in order Banach spaces, SIAM Rev., 18 (1976), 620–709.

[11] H. Amann and M. G. Crandall, On some existence theorems for semilinear elliptic equations, Indiana Univ. Math. J., 27 (1978), 779–790.

[12] P. Baras and M. Pierre, Critères d’existence de solutions positives pour des équations semi-linéaires non monotones, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2 (1985), 185–212.

[13] A. Bensoussan, L. Boccardo and F. Murat, On a nonlinear partial differential equation having natural growth terms and unbounded solution, Ann. Inst. Henri Poincaré Anal. Non Linéaire, 5 (1988), 347–364.

[14] L. Boccardo, F. Murat and J. P. Puel, Existence results for some quasilinear parabolic equations, Nonlinear Anal., 13 (1989), 373–392.

[15] S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods, Texts in Applied Mathematics, 15, Springer-Verlag, New York, 1994.

[16] H. Brezis and W. Strauss, Semilinear second-order elliptic equations in $L^1$, J. Math. Soc. Japan, 25 (1973), 565–590.

[17] X. C. Cai and O. B. Widlund, Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems, SIAM J. Numer. Anal., 30 (1993), 936–952.

[18] X. C. Cai and O. B. Widlund, Domain decomposition algorithms for indefinite elliptic problems, SIAM J. Sci. Statist. Comput., 13 (1992), 243–258.

[19] X. C. Cai, An optimal two-level overlapping domain decomposition method for elliptic problems in two and three dimensions, SIAM J. Sci. Comput., 14 (1993), 239–247.

[20] Y. Choquet-Bruhat and J. Leray, Sur le problème de Dirichlet, quasilineaire, d’ordre 2, C. R. Acad. Sci. Paris, Sér. A-B, 274 (1972), A81–Å85.

[21] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, Studies in Mathematics and its Applications, 4, North-Holland Publishing Co., Amsterdam-New-York-Oxford, 1978.

[22] P. Deuflhard, Newton Methods for Nonlinear Problems. Affine Invariance and Adaptive Algorithms, Springer Series in Computational Mathematics, 35, Springer-Verlag, Berlin, 2004.

[23] M. Dryja, B. F. Smith and O. B. Widlund, Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions, SIAM J. Numer. Anal., 31 (1994), 1662–1694.

[24] M. Dryja and O. B. Widlund, Domain decomposition algorithms with small overlap. Iterative methods in numerical linear algebra, SIAM J. Sci. Comput., 15 (1994), 604–620.

[25] J. A. Ezquerro and M. A. Hernandez, On an application of Newton’s method to nonlinear operators with $w$-conditioned second derivative, BIT, 42 (2002), 519–530.

[26] J. A. Ezquerro and M. A. Hernandez, On the $R$-order of convergence of Newton’s method under mild differentiability conditions, J. Comput. Appl. Math., 197 (2006), 53–61.

[27] M. Gander, A waveform relaxation with overlapping splitting for reaction diffusion equations, Numer. Linear Algebra Appl., 6 (1999), 125–145.
[28] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, 224, Springer-Verlag, Berlin, 2001.

[29] M. A. Hernandez, The Newton method for operators with Hölder continuous first derivative, *J. Optim. Theory Appl.*, **109** (2001), 631–648.

[30] C. T. Kelly, *Iterative Methods for Linear and Nonlinear Equations*, Frontiers in Applied Mathematics, 16, SIAM, Philadelphia, PA, 1995.

[31] S. A. Levin, Models in ecotoxicology: Methodological aspects, in *Applied Mathematical Ecology*, Biomathematics, 18, Springer, Berlin, 1989, 315–321.

[32] P. L. Lions, Résolution de problèmes elliptiques quasilinéaires, *Arch. Rational Mech. Anal.*, **74** (1980), 335–353.

[33] P. L. Lions, On the Schwarz alternating method. I, in *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia, PA, 1988, 1–42.

[34] J. D. Murray, *Mathematical Biology*, Biomathematics, 19, Springer-Verlag, Berlin, 1993.

[35] A. Porretta, Existence for elliptic equations in $L^1$ having lower order terms with natural growth, *Portug. Math.*, **57** (2000), 179–190.

[36] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*, Springer Series in Computational Mathematics, 23, Springer-Verlag, Berlin, 1994.

[37] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1999.

[38] J. T. Schwartz, *Nonlinear Functional Analysis*, Mathematics and its Applications, Gordon and Breach Science Publishers, New York-London-Paris, 1969.

[39] B. F. Smith, P. E. Bjorstad and W. D. Gropp, *Domain Decomposition. Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, Cambridge, 1996.

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