Defect production due to quenching through a multicritical point

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Abstract. We study the generation of defects when a quantum spin system is quenched through a multicritical point by changing a parameter of the Hamiltonian as \( t/\tau \), where \( \tau \) is the characteristic timescale of quenching. We argue that when a quantum system is quenched across a multicritical point, the density of defects \( (n) \) in the final state is not necessarily given by the Kibble–Zurek scaling form \( n \sim 1/\tau^{d\nu/(2z+1)} \), where \( d \) is the spatial dimension, and \( \nu \) and \( z \) are respectively the correlation length and dynamical exponent associated with the quantum critical point. We propose a generalized scaling form of the defect density given by \( n \sim 1/\tau^{d/(2z_2)} \), where the exponent \( z_2 \) determines the behavior of the off-diagonal term of the \( 2 \times 2 \) Landau–Zener matrix at the multicritical point. This scaling is valid not only at a multicritical point but also at an ordinary critical point.

Keywords: integrable spin chains (vertex models), spin chains, ladders and planes (theory), quantum phase transitions (theory)
1. Introduction

The zero-temperature quantum phase transitions occurring in quantum many-body systems has been a challenging area of research for the past few years [1, 2]. The dynamics taking place in such systems on varying a parameter in the Hamiltonian in a definite fashion have come to the forefront only recently [3–7]. In this paper, we focus on the density of defects generated when a system, prepared in its ground state, is adiabatically quenched at a uniform rate [4–6], [8–21]. These works have their roots in the study of phase transitions in the early universe [22] which was extended to second-order phase transitions [23] and later to quantum spin chains [4]. The diverging relaxation time associated with a quantum critical point results in the failure of the system to follow its instantaneous ground state; this eventually leads to the generation of defects in the final state. When a parameter of the quantum Hamiltonian is varied as $t/\tau$, where $\tau$ is the characteristic timescale of the quenching, the Kibble–Zurek (KZ) argument [4, 5] predicts a density of defects in the final state that scales as $1/\tau^{d\nu/(z\nu+1)}$ in the limit $\tau \rightarrow \infty$. Here $\nu$ and $z$ denote the correlation length and dynamical exponents, respectively, characterizing the associated quantum phase transition of the $d$-dimensional quantum system. The KZ prediction has been verified for various exactly solvable spin models when quenched across a critical point [4, 6, 8, 9]. Various generalizations of the KZ scaling form have also been proposed for quenching through a gapless phase or along a gapless line [10, 17, 20]. Experimental verification of the dynamics of such systems can be realized by the trapped ultracold atoms in optical lattices; for a review see [24].

The generation of defects during the adiabatic quenching dynamics of a one-dimensional spin-$1/2$ $XY$ chain across a quantum critical point was studied in [6]. The Hamiltonian of the system is [25]

$$H = -\frac{1}{2} \sum_n (J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + h \sigma_n^z),$$

(1)

where the $\sigma$s are Pauli spin matrices satisfying the usual commutation relations. The strength of the transverse field is denoted by $h$, and $J_x$ and $J_y$ are the strengths of the interactions in the $x$ and $y$ directions, respectively. The phase diagram of the above model is shown in figure 1.

It was observed that when the transverse field $h$ is varied as $h = t/\tau$, the system crosses the Ising transition lines as shown in the figure, and the defect density scales
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Figure 1. The phase diagram of the anisotropic $XY$ model in a transverse field in the $h/(J_x + J_y) - \gamma$ plane, where $\gamma \equiv (J_x - J_y)/(J_x + J_y)$. The vertical bold lines given by $h/(J_x + J_y) = \pm 1$ denote the Ising transitions. The system is also gapless on the horizontal bold line $\gamma = 0$ for $|h| < J_x + J_y$. FM$_x$(FM$_y$) is a long-range ordered phase with ferromagnetic ordering in the $x(y)$ direction. The thick dashed line marks the boundary between the commensurate and incommensurate ferromagnetic phases. The thin dotted lines indicate the adiabatic and impulse regions when the field $h$ is quenched from $-\infty$ to $\infty$. The two points with coordinates $\gamma = 0$ and $h/(J_x + J_y) = \pm 1$ denoted by A and B are multicritical points.

as $[6] 1/\sqrt{\tau}$. This is in agreement with the KZ prediction since the values of the critical exponents associated with the Ising transition are given by $z = \nu = 1$. On the other hand, if the interaction in the $x$ direction ($J_x$) is quenched in a similar fashion keeping $h$ and $J_y$ fixed [8], the defect density is again found to scale as $1/\sqrt{\tau}$ though the magnitude depends upon the values of $J_y$ and $h$. If $h < 2J_y$, the system crosses the anisotropic critical line ($J_x = J_y$) in addition to the Ising transition lines mentioned above, and hence the magnitude of the defects is increased. However, it was observed that if $J_x$ is quenched keeping $h = 2J_y$, the system crosses the multicritical point at $J_x = J_y$ and $h = 2J_y$, where the Ising and anisotropic transition lines meet. The density of defects in the final state generated in a passage through the above multicritical point shows a slower decay with $\tau$ given as $1/\tau^{1/6}$. Since the critical exponents associated with this multicritical point are given by $\nu = 1/2$ and $z = 2$, the above scaling relation does not follow from the KZ scaling relation $1/\tau^{d\nu/(2\nu+1)}$. It is this observation which motivated us to look for a generalized scaling relation valid even for a multicritical point. It should be noted here that this is the first attempt to provide a generalized scaling relation for defect density when the system

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is quenched linearly through a multicritical point, which has also been extended to the non-linear case in a recent work [29].

The paper is organized as follows. In section 2, we derive the general form for the scaling of defects and apply it in two models. Section 3 consists of concluding remarks.

2. General scaling

To propose a general scaling scheme valid even for a multicritical point using the Landau–Zener non-adiabatic transition probability [26, 27], let us consider a d-dimensional model Hamiltonian of the form

\[ H = \sum_{\vec{k}} \psi_{\dagger}(\vec{k})\left(\left(\lambda(t) + b(\vec{k})\right)\sigma^z + \Delta(\vec{k})\sigma^+ + \Delta^*(\vec{k})\sigma^-\right)\psi(\vec{k}), \tag{2} \]

where \( \sigma^+ = (\sigma^x + i\sigma^y) \), \( b(\vec{k}) \) and \( \Delta(\vec{k}) \) are model dependent functions, and \( \psi(\vec{k}) \) denotes the fermionic operators \( (\psi_1(\vec{k}), \psi_2(\vec{k})) \). The above Hamiltonian can represent, for example, a one-dimensional transverse Ising or XY spin chain [25], or an extended Kitaev model for \( d = 2 \) written in terms of Jordan–Wigner fermions [28]. We assume that the parameter \( \lambda(t) \) varies linearly as \( t/\tau \) and vanishes at the quantum critical point at \( t = 0 \), so that the system crosses a gapless point at \( t = 0 \) for the wavevector \( \vec{k} = \vec{k}_0 \). Without loss of generality, we set \( |\vec{k}_0| = 0 \). The parameters \( b(\vec{k}) \) and \( \Delta(\vec{k}) \) are assumed to vanish at the quantum critical point in a power-law fashion given by

\[ b(\vec{k}) \sim |\vec{k}|^{z_1} \quad \text{and} \quad \Delta(\vec{k}) \sim |\vec{k}|^{z_2}. \tag{3} \]

Many of the models described by equation (2) exhibit a quantum phase transition with the exponents associated with the quantum critical point being \( \nu = z = z_2 = 1 \). We shall however explore the more general case below.

The Schrödinger equation describing the time evolution of the system when \( \lambda \) is quenched is given by \( i\partial\psi/\partial t = H\psi \) (where we set Planck’s constant \( \hbar = 1 \)). Using the Hamiltonian in equation (2), we can write

\[ \begin{align*}
   i\frac{\partial \psi_1(\vec{k})}{\partial t} &= \left(\frac{t}{\tau} + b(\vec{k})\right) \psi_1(\vec{k}) + \Delta(\vec{k})\psi_2(\vec{k}), \\
   i\frac{\partial \psi_2(\vec{k})}{\partial t} &= -\left(\frac{t}{\tau} + b(\vec{k})\right) \psi_2(\vec{k}) + \Delta^*(\vec{k})\psi_1(\vec{k}). \tag{4} \end{align*} \]

One can now remove \( b(\vec{k}) \) from the above equations by redefining \( t/\tau + b(\vec{k}) \rightarrow t \); thus the exponent \( z_1 \) defined in equation (3) does not play any role in the following calculations. Defining a new set of variables \( \psi_1(\vec{k}) = \tilde{\psi}_1(\vec{k}) \exp(\int t' dt'/\tau) \) and \( \psi_2(\vec{k}) = \tilde{\psi}_2(\vec{k}) \exp(-\int t' dt'/\tau) \), we arrive at a time evolution equation for \( \psi_1(\vec{k}) \) given by

\[ \left(\frac{d^2}{dt^2} - 2it\frac{d}{dt} + |\Delta(\vec{k})|^2\right) \psi_1(\vec{k}) = 0. \tag{5} \]

Further rescaling \( t \rightarrow t\tau^{1/2} \) leads to

\[ \left(\frac{d^2}{dt^2} - 2it\frac{d}{dt} + |\Delta(\vec{k})|^2\tau\right) \psi_1(\vec{k}) = 0. \tag{6} \]
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If the system is prepared in its ground state at the beginning of the quenching, i.e., \( \psi_1(\vec{k}) = 1 \) at \( t = -\infty \), the above equation suggests that the probability of the non-adiabatic transition, \( p_k = \lim_{t \to +\infty} |\psi_1(\vec{k})|^2 \), must have a functional dependence on \( |\Delta(\vec{k})|^2 \tau \) of the form
\[
p_k = f(|\Delta(\vec{k})|^2 \tau).
\]

The analytical form of the function \( f \) is given by the general Landau–Zener formula [26, 27]. The defect density in the final state is therefore given by
\[
n = \int \frac{d^dk}{(2\pi)^d} f(|\Delta(\vec{k})|^2 \tau) = \int \frac{d^dk}{(2\pi)^d} f(|\vec{k}|^{2z_2} \tau).
\]

The scaling \( k \to k^{2z_2} \tau \) finally leads to a scaling of the defect density given by
\[
n \sim 1/\tau^{d/(2z_2)}.
\]

We shall recall the example of the quenching dynamics of the transverse XY spin chain when the field or the interaction is quenched [6, 8]. When the system is quenched across the Ising or anisotropic critical line by linearly changing \( h \) or \( J_x \) as \( t/\tau \), \( \Delta(\vec{k}) \) vanishes at the critical point as \( \Delta(\vec{k}) \sim |\vec{k}| \) yielding \( z_2 = z = 1 \); hence the generalized scaling form given in equation (9) matches with the Kibble–Zurek prediction with \( \nu = z = 1 \). On the other hand, when the system is swept across the multicritical point \( (J_x = J_y, h = 2J_y) \) by quenching the interaction \( J_x = t/\tau \) with \( h = 2J_y \), the equivalent \( 2 \times 2 \) Hamiltonian matrix of the Jordan–Wigner fermions in an appropriate basis can be written as [8]
\[
\begin{bmatrix}
J_x + J_y(\cos 2k + 2 \cos k) & J_y(\sin 2k + 2 \sin k) \\
J_y(\sin 2k + 2 \sin k) & -J_x - J_y(\cos 2k + 2 \cos k)
\end{bmatrix}.
\]

The corresponding Schrödinger equations are
\[
\begin{align*}
\frac{i}{\tau} \frac{\partial \tilde{\psi}_1(\vec{k})}{\partial t} &= \left( \frac{t}{\tau} + J_y(\cos 2k + 2 \cos k) \right) \tilde{\psi}_1(\vec{k}) + J_y(\sin 2k + 2 \sin k)\tilde{\psi}_2(\vec{k}), \\
\frac{i}{\tau} \frac{\partial \tilde{\psi}_2(\vec{k})}{\partial t} &= J_y(\sin 2k + 2 \sin k)\tilde{\psi}_1(\vec{k}) - \left( \frac{t}{\tau} + J_y(\cos 2k + 2 \cos k) \right) \tilde{\psi}_2(\vec{k}).
\end{align*}
\]

At the quantum critical point \( J_x = J_y \), the diagonal term \( b(k) = J_y(\cos 2k + 2 \cos k) \) goes as \( -J_y - J_y|\pi - k|^2 \near k = \pi \). Hence the dynamical exponent is given by \( z = z_1 = 2 \) at this multicritical point. Note that in this example, the critical point is not crossed at \( t = 0 \); however, one can shift the time so that \( b'(k) \sim |\pi - k|^{z_1} \), which would ensure that the quantum critical point is crossed at \( t = 0 \).

On the other hand, the off-diagonal term \( \Delta(k) = J_y(\sin 2k + 2 \sin k) = |\pi - k|^3 \) leads to the density of defect scaling as \( 1/\tau^{1/6} \); this is in agreement with the generalized scaling relation proposed in equation (9) with \( z_2 = 3 \). Figure 2 shows the numerical integration of equation (10) which confirms the defect scaling exponent of \(-1/6\).

Finally, let us comment on the dynamics of an exactly solvable transverse Ising model with an additional three-spin interaction which is also quenched through a multicritical point [9] by varying the transverse field as \( h = t/\tau \). It has been observed that the defect density \( n \) scales as \( 1/\tau^{1/6} \) which again does not support the KZ scaling form. The three-
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Figure 2. $n$ versus $\tau$ obtained by numerically solving equation (10) at the multicritical point with $h = 10$ and $J_y = 5$. The line has a slope of $-0.16$.

The spin interacting Hamiltonian is given by [30]

$$H = -\frac{1}{2} \sum_i \sigma_i^z [h + J_3 \sigma_{i-1}^x \sigma_{i+1}^x] + \frac{J_x}{2} \sum_i \sigma_i^x \sigma_{i+1}^x.$$  \hspace{1cm} (11)

We shall henceforth set $J_x = 1$. The equivalent $2 \times 2$ Hamiltonian matrix of the Jordan-Wigner fermions in the momentum representation takes the form

$$\begin{bmatrix} h(t) + \cos k - J_3 \cos 2k & i(\sin k - J_3 \sin 2k) \\ -i(\sin k - J_3 \sin 2k) & -(h(t) + \cos k - J_3 \cos 2k) \end{bmatrix}.$$  

It may be noted that by virtue of a duality transformation, this model can be mapped to a transverse XY model with competing interactions for the $x$ and $y$ components of the spin [30]. The multicritical point in the phase diagram of this model is at $h = -1$ and $J_3 = 1/2$. We observe that the off-diagonal term $\sin k - J_3 \sin 2k$ scales as $|\vec{k}|^3$ at the multicritical point; therefore the defect density scales as $1/\tau^{1/6}$ as expected from the general scaling relation proposed here. The importance of the multicritical point has also been observed in non-linear quenching of different models [29].

3. Conclusions

We have shown that the density of defects $n$ produced when a system is quenched through a multicritical point does not follow the KZ scaling relation $1/\tau^{d\nu/(z\nu+1)}$. We then proved a new scaling form which is not only valid at an ordinary quantum critical point but is also valid at a multicritical point. We argue that for a system which is swept across a multicritical point in a phase diagram, it is the exponent $z_2$ defined above which appears in the scaling of the defect density given in equation (9). However, for a passage through an ordinary critical point in many models, $z_2 = z = 1$, and equation (9) reproduces the conventional KZ scaling form with $\nu = z = 1$.

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