Chern-Simons quantum mechanics and fractional angular momentum in atom system

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The model of a neutral planar atom which possesses a non-vanishing electric dipole momentum interacting with magnetic fields in a specific setting is studied. The energy spectra of this model and its reduced model, which is the limit of cooling down the atom to the negligible kinetic energy, are solved exactly. We show that the energy spectra of the reduced model can not be obtained directly from the full ones by taking the same limit. In order to match them, we must regularize the energy spectra of the full model when the limit of the negligible kinetic energy is taken. It is one of the characteristics of the Chern-Simons quantum mechanics. Besides it, the canonical angular momentum of the reduced model will take fractional values although the full model can only take integers. It means that it is possible to realize the Chern-Simons quantum mechanics and fractional angular momentum simultaneously by this model.

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I. INTRODUCTION

Chern-Simons quantum mechanics was firstly studied in [1]. It describes a charged planar particle confined by a quadratic scalar potential in the background of a constant external magnetic field. It is found that the reduced model, which is the zero-mass \(m \to 0\) limit of the full model, behaves non-trivially. Although the solutions to the classical equations of motion of the reduced model can be obtained directly from the full ones by taking the limit \(m \to 0\), the energy spectra of this reduced model can not be obtained straightforwardly from the full model’s in the same manner. Because the energy spectra of the full model will be divergent when the limit \(m \to 0\) is taken. In order to match them, one must regularize the spectra of the full theory artificially when this limit is taken. Besides it, the authors show that the eigenvalues of the canonical angular momentum take integer values in the full theory (in the unit of \(\hbar = 1\)). However, in the reduced model, it can only take positive half odd numbers.

The author of [2] studies an atom with a non-vanishing electric dipole momentum interacting with electromagnetic fields. It is found that the eigenvalues of the canonical angular momentum of the reduced model, which is obtained by cooling down the atom to the negligible kinetic energy, can only take positive half odd numbers. It is one of the characteristics of the Chern-Simons quantum mechanics. It means that one can use a neutral atom to realize Chern-Simons quantum mechanics which describes a charged particle before.

The fractional angular momentum is an interesting issue in physics [3]. It is well-known that in three-dimensional space, the eigenvalues of angular momentum should be quantized in the unit of \(\hbar/2\) due to the non-Abelian rotation group. However, this conclusion is incorrect in 2+1 dimensional space-time since the rotation group in two-dimensional space is an Abelian one which does not impose any constraints on the eigenvalues of the angular momentum. One of the ways to realize the fractional angular momentum in 2+1 dimensional space-time is to couple a charged particle with the Chern-Simons gauge field \([5, 7]\).

Ref. [8] provides an alternative approach to realize the fractional angular momentum by using a cold ion. The author considers an trapped planar ion coupling to two different types of magnetic potentials. One is the dynamical the other is the Aharonov-Bohm type which is generated by a long-thin flux-carried solenoid. It is showed that the eigenvalues of the canonical angular momentum take integers. However, the eigenvalues of the canonical angular momentum of the reduced model, which is obtained by cooling down the kinetic energy of the ion to its lowest level, take fractional values. The fractional part is proportional to the magnetic flux inside the solenoid.

In this paper, we propose a model which can realize the Chern-Simons quantum mechanics and the fractional angular momentum simultaneously. Different from previous studies [2, 8] which realize Chern-Simons quantum mechanics and fractional angular momentum by charged particles, we realize them simultaneously by a cold atom which possesses a non-vanishing electric dipole momentum. The organization of the paper is as follows: In the next section, we introduce the model and solve it exactly by an algebraic method. Then, in section III, we analyze the reduced model which is obtained by cooling down the atom to the limit of negligible small kinetic energy.
from the full model. We show that the energy spectra of the reduced model can not be obtained trivially from the full model by taking the same limit. In order to match them, we must regularize the energy spectra properly. It is one of the characteristics of the Chern-Simons quantum mechanics. Furthermore, we show that the eigenvalues of the canonical angular momentum of the reduced model take fractional values. Some remarks and further discussions will be given in the last section.

II. THE FULL MODEL

Our model is a planar neutral atom with a non-vanishing electric dipole momentum interacting with two magnetic fields. This atom is confined by a harmonic potential and the electric dipole momentum is perpendicular to the plane. The harmonic potential and the magnetic fields are arranged that the motion of the atom is rotationally symmetric. To be specific, the magnetic fields take the form

\[ B^{(1)} = \frac{\lambda_m}{2 \pi r} \mathbf{e}_r, \quad B^{(2)} = \frac{\rho_m r}{2} \mathbf{e}_r, \]

where \( \lambda_m \) is the magnetic charges per unit length on the long filament, \( \rho_m \) is the density of the magnetic monopoles and \( \mathbf{e}_r \) is the unit vector along the radial direction on the plane.

In 3-dimensional space, the Hamiltonian which governs the dynamics of an atom with a non-vanishing electric dipole momentum in the background of the magnetic field \( \mathbf{B} \) is

\[ H = \frac{1}{2m} (\mathbf{p} + \frac{d}{c^2} \mathbf{n} \times \mathbf{B})^2 - \frac{\hbar d}{2mc^2} \nabla \cdot \mathbf{B}, \quad (3) \]

in which \( \mathbf{p} = -i\hbar \nabla \) is the canonical momentum, \( d \) is the magnitude of the electric dipole momentum and \( \mathbf{n} \) is the unit vector along the electric dipole momentum. This Hamiltonian is the non-relativistic limit of a relativistic spin half neutral particle with a non-vanishing electric dipole momentum interacting with magnetic fields \( \mathbf{B} \). In our model we apply two magnetic fields \( B^{(1)} B^{(2)} \) simultaneously, i.e., \( \mathbf{B} = B^{(1)} + B^{(2)} \). In fact, only \( B^{(2)} \) contributes to the term \( \nabla \cdot \mathbf{B} \) since \( \nabla \cdot B^{(1)} = 0 \) in the area \( r \neq 0 \) where the atom moves. Therefore, the last term of the Hamiltonian \( (3) \) in our model becomes \( \nabla \cdot \mathbf{B} = \frac{\hbar d \rho_m}{(2mc^2)} \).

One can derive the He-Mckellar-Wilkens (HMW) effect \( B^{(1)} \) from the above Hamiltonian by only turning on the magnetic field \( B^{(1)} \). The HMW effect is firstly predicted by He and McKellar \( B^{(1)} \) and later independently by Wilkens \( B^{(1)} \). This effect states that a neutral particle with a non-vanishing electric dipole momentum will experience a topological phase if it moves around a line of magnetic monopole with its electric dipole momentum paralleling to the line. It shows that the effect of magnetic field \( B^{(1)} \) is purely geometrical. It does not influence the classical motion of the particle. The magnetic field \( B^{(1)} \) plays an analogous role as the magnetic potentials produced by the long-thin magnetic flux-carried solenoid in the Aharonov-Bohm (AB) effect \( B^{(1)} \). Therefore, the magnetic field \( B^{(1)} \) does not have dynamical contents. Its effect only arises in the quantum phase of the wave function.

The other aspect of the Hamiltonian \( (9) \) which has been studied is that the energy eigenvalue of the Hamiltonian \( (9) \) with only \( B^{(2)} \) turning on are analogous with Landau levels \( (13) \). Thus, it means that the Landau levels can also be realized by a neutral atom with a non-vanishing electric dipole momentum. It may afford a possible method to realize the quantum Hall effect \( (15) \) by neutral atoms.

Since the motion of the atom is on a plane which is perpendicular to the electric dipole momentum, we concentrate on this plane and write the Hamiltonian \( (3) \) on this plane as

\[ H = \frac{1}{2m} \left( p_i - \frac{d}{c^2} \epsilon_{ij} B_j \right)^2 - \frac{\hbar d \rho_m}{2mc^2}, \quad i, j = 1, 2, \quad (4) \]

where the summation convention is applied. Besides the magnetic fields \( B^{(1)} B^{(2)} \), we assume that the atom is trapped by a harmonic potential \( \frac{1}{2} K x_i^2 \). Thus, our model is described by the Hamiltonian

\[ H = \frac{1}{2m} \left( p_i - \frac{d}{c^2} \epsilon_{ij} B_j \right)^2 + \frac{1}{2} K x_i^2 - \frac{\hbar d \rho_m}{2mc^2}. \]

The Lagrangian corresponding to the above Hamiltonian is

\[ L = \frac{1}{2} m \dot{x}_i^2 + \frac{d}{c^2} \epsilon_{ij} \dot{x}_i B_j - \frac{1}{2} K x_i^2 + \frac{\hbar d \rho_m}{2mc^2}. \]

The canonical momenta with respect to variables \( x_i \) are defined by

\[ p_i = \frac{\partial L}{\partial \dot{x}_i} = m \dot{x}_i + \frac{d}{c^2} \epsilon_{ij} B_j. \]

The commutators among the canonical variables \( x_i, p_i \) are

\[ [x_i, x_j] = [p_i, p_j] = 0, \quad [x_i, p_j] = i \hbar \delta_{ij}. \]

which are the standard Heisenberg algebra.

With the well-defined canonical variables \( x_i \) and \( p_i \), we can define the canonical angular momentum unambiguously as

\[ J = \epsilon_{ij} x_i p_j. \]

One can verify

\[ [J, H] = 0 \]

by a straightforward calculation. It indicates that the canonical angular momentum is a conservative quantity. The canonical angular momentum can also be written as
\( J = -i \partial / \partial \varphi \) with \( \varphi \) being the polar angular. Obviously, the eigenvalues of the canonical angular momentum can only be

\[ J_n = nh, \; n = 0, \pm 1, \pm 2, \ldots \]  

(11)

Based on the basic commutators \( [\varphi, p] = i\hbar \), we can solve the eigenvalues of the Hamiltonian \( (5) \) algebraically. In doing so, we introduce new variables \( \tilde{x}_i, \tilde{p}_i \) to replace \( x_i, p_i \) in Hamiltonian \( (5) \),

\[ \tilde{p}_i = p_i - \frac{d}{c^2} \epsilon_{ij} B_j, \quad \tilde{x}_i = x_i. \]  

(12)

The commutators among variables \( \tilde{x}_i, \tilde{p}_i \) can be derived from \( (8) \). They are

\[ [\tilde{x}_i, \tilde{x}_j] = 0, \quad [\tilde{x}_i, \tilde{p}_j] = i\hbar \delta_{ij}, \quad [\tilde{p}_i, \tilde{p}_j] = -\frac{i\hbar \rho_m}{c^2} \epsilon_{ij}. \]  

(13)

Replacing the variables \( x_i, p_i \) by \( \tilde{x}_i, \tilde{p}_i \) in Hamiltonian \( (5) \), we get

\[ H = \frac{\tilde{p}_i^2}{2m} + \frac{K}{2} \tilde{x}_i^2 + \frac{\hbar^2 \rho_m}{2mc^2}. \]  

(14)

Obviously, the commutators \( (13) \) are different from \( (8) \) because of the noncommutativity between \( \tilde{p}_i \). We map the variables \( (\tilde{x}_i, \tilde{p}_i) \) to a new set of variables \( (X_i, P_i) \) which satisfy the standard Heisenberg algebra. The map is

\[ X_i = \tilde{x}_i, \quad P_i = \tilde{p}_i + \frac{d\rho_m}{c^2} \epsilon_{ij} \tilde{x}_j. \]  

(15)

In terms of variables \( X_i, P_i \), we write the Hamiltonian \( (14) \) in the form

\[ H = \frac{P_i^2}{2m} + \frac{1}{2} m \Omega^2 X_i^2 + \omega \epsilon_{ij} X_i P_j - \frac{\hbar^2 \rho_m}{2mc^2}. \]  

(16)

where \( \omega^2 = \frac{\partial^2 \rho_m}{4m^2 c^4}, \quad \Omega^2 = \omega^2 + \frac{K}{m} \). Introducing the annihilation-creation operators

\[ a_i = \sqrt{\frac{m\Omega}{2\hbar}} (X_i + \frac{i}{m\Omega} P_i), \quad a_i^\dagger = \sqrt{\frac{m\Omega}{2\hbar}} (X_i - \frac{i}{m\Omega} P_i), \]  

(17)

and applying the Bogoliubov transformation

\[ A_1 = \sqrt{\frac{1}{2}} (a_1 + a_2), \quad A_1^\dagger = \sqrt{\frac{1}{2}} (-ia_1^\dagger + a_2^\dagger), \]
\[ A_2 = \sqrt{\frac{1}{2}} (a_1 + a_2), \quad A_2^\dagger = \sqrt{\frac{1}{2}} (-ia_2^\dagger + a_1^\dagger), \]  

(18)

in which \( A_i, A_i^\dagger \) satisfy \( [A_i, A_j^\dagger] = \delta_{ij} \), we write the Hamiltonian \( (15) \) in the form

\[ H = \hbar \omega_1 A_1^\dagger A_1 + \hbar \omega_2 A_2^\dagger A_2 + \hbar \Omega - \frac{\hbar^2 \rho_m}{2mc^2}. \]  

(19)

where \( \omega_1 = \Omega - \omega, \quad \omega_2 = \Omega + \omega \). Therefore, the Hamiltonian \( (15) \) is equivalent to the one which describes two uncoupled harmonic oscillators.

The Hamiltonian \( (19) \) is already in the diagonal form. The eigenvalues can be read directly. They are

\[ E_{n_1, n_2} = (n_1 + \frac{1}{2}) \hbar \omega_1 + (n_2 + \frac{1}{2}) \hbar \omega_2 - \frac{\hbar^2 \rho_m}{2mc^2} \]  

(20)

where \( n_1, n_2 = 0, 1, 2, \ldots \).

We can express the canonical angular momentum \( (9) \) in terms of annihilation-creation operators \( A_i, A_i^\dagger \). It is

\[ J = \hbar (A_1^\dagger A_1 - A_2^\dagger A_2). \]  

(21)

The conversation and the eigenvalues of the canonical angular momentum \( (10, 11) \) will be more transparent if we write it in terms of \( A_i, A_i^\dagger \).

### III. THE REDUCED MODEL AND THE FRACTIONAL ANGULAR MOMENTUM

In this section, we study the reduced model which is obtained by cooling down the atom to the negligible kinetic energy \( (18) \). Mathematically, it is equivalent to take the limit \( m \to 0 \) in the Hamiltonian \( (5) \) or Lagrangian \( (6) \).

In order to keep the Hamiltonian \( (5) \) finite, we have no other choices but to set

\[ p_i - \frac{d}{c^2} \epsilon_{ij} B_j = 0. \]  

(22)

These equations in fact, lead to the relations among the variables \( x_i \) and \( p_i \). Thus, they are the constraints \( (19) \) and labeled as

\[ \phi_i^{(0)} = p_i - \frac{d}{c^2} \epsilon_{ij} B_j \approx 0, \]  

(23)

in which ‘\( \approx \’) is weak equivalent. It means that \( (23) \) are only valid on the constraints surface. These constraints can also be obtained from the Lagrange description. After taking the limit of \( m \to 0 \), the Lagrangian \( (6) \) becomes

\[ L_r = \frac{d}{c^2} \epsilon_{ij} \dot{X}_i B_j - \frac{1}{2} K \dot{x}_i^2 + \frac{\hbar^2 \rho_m}{2mc^2}. \]  

(24)

Introducing the canonical momenta with respect to \( x_i \), we get

\[ p_i = \frac{\partial L}{\partial \dot{X}_i} = \frac{d}{c^2} \epsilon_{ij} B_j. \]  

(25)

Evidently, the introduction of canonical momenta leads to the primary constraints. They are nothing but \( (23) \).

The Poisson brackets among constraints \( \phi_i^{(0)} \) are

\[ \{ \phi_i^{(0)}, \phi_j^{(0)} \} = -\frac{d \rho_m}{c^2} \epsilon_{ij}. \]  

(26)
Since \( \{ \phi_i^{(0)}, \phi_j^{(0)}\} \neq 0 \), they belong to the second class and there are no secondary constraints. Because of the second class nature, the constraints can be regarded as a ‘strong’ equivalent.

The Hamiltonian of the reduced model is the zero-mass limit of \((3)\). Equivalently, it can also be read from the Lagrangian \((2)\) directly since it is in the first-order form \((20)\) as

\[
H_r = \frac{1}{2} K x_i^2 - \frac{\hbar d \rho_m}{2mc^2}. \tag{27}
\]

The eigenvalues of this reduced Hamiltonian can be obtained once the commutation relations between \(x_i\) are determined. The classical version of the commutators of \(x_i, \text{i.e.} \) the Dirac brackets among variables \(x_i\) are defined by \((19)\)

\[
\{x_i, x_j\}_D = \{x_i, x_j\} - \{x_i, \phi_m^{(0)}\} \{\phi_m^{(0)}, \phi_n^{(0)}\}^{-1} \{\phi_n^{(0)}, x_j\}. \tag{28}
\]

After some algebraic calculations, we arrive at

\[
[x_i, x_j] = i\hbar \{x_i, x_j\}_D = \frac{i\hbar c^2}{d \rho_m} \epsilon_{ij}. \tag{29}
\]

Thus, up to a constant, the reduced Hamiltonian \((27)\) is equivalent to a one-dimensional harmonics oscillator. Its eigenvalues can be read directly from the Hamiltonian \((27)\) and commutation relation \((29)\) as

\[
E_n = \frac{\hbar c^2}{d \rho_m} (n + \frac{1}{2}) - \frac{\hbar d \rho_m}{2mc^2}. \tag{30}
\]

It is natural to wonder whether the energy spectra of the reduced model \((30)\) can be obtained directly from \((20)\) by taking the limit of \(m \to 0\). A straightforward calculation shows that

\[
\begin{align*}
\lim_{m \to 0} \omega_1 & \to \frac{K c^2}{d \rho_m}, \\
\lim_{m \to 0} \omega_2 & \to \infty. \tag{31}
\end{align*}
\]

Obviously, one of the frequencies, namely, \(\omega_2\), will tend to infinity when the atom is cooled down to the negligible kinetic energy. Therefore, the energy spectra of the full model will be divergent when the limit \(m \to 0\) is taken. The similar problem is also appeared in the Chern-Simons quantum mechanics \((1)\). In order to get a physical result, we must get rid of the infinity artificially. We set

\[
\omega_2 = 0 \tag{32}
\]

when the limit \(m \to 0\) is taken.

The reason we set \(\omega_2 = 0\) is that in this limit, the energy gap of one of the harmonic oscillators \((20)\) tends to infinity. Thus, the energy gaps of this harmonic oscillator is too large to be excited. That is to say, this harmonics oscillator is completely frozen in this limit. Thus, we can get rid of them safely. The remaining part of the energy spectra \((20)\) matches the reduced spectra \((30)\) exactly.

Now, we focus on the rotation property of the reduced model. The angular momentum of the reduced model is also defined as \(J_r = \epsilon_{ij} x_i p_j\), in which the canonical momenta \(p_i\) are given by \((24)\). Substituting \((24)\) into \(J_r = \epsilon_{ij} x_i p_j\), we get

\[
J_r = \epsilon_{ij} x_i p_j = -\frac{d}{2c^2} (\rho_m x_i^2 + \frac{\lambda_m}{\pi}). \tag{33}
\]

It is analogous to a one-dimensional harmonic oscillator. With the help of commutators \((20)\), we can get the eigenvalues of \(J_r\) easily as

\[
J_{rn} = -(n + \frac{1}{2}) \hbar - \frac{d \lambda_m}{2\pi c^2}, \quad n = 0, 1, 2, \cdots. \tag{34}
\]

Interestingly, the eigenvalues of the reduced canonical angular momentum can only take negative half odd numbers if the magnetic field \(B^{(1)}\) is not considered.

The result \((34)\) shows that the eigenvalue of the canonical angular momentum can take fractional values. Apart from a minus sign, the fractional part is proportional to \(\lambda_m\), which is the magnetic charges per unit along the filament.

\section*{IV. REMARKS AND FURTHER DISCUSSIONS}

In this paper, we propose a model to realize the Chern-Simons quantum mechanics and fractional angular momentum simultaneously. The model contains a trapped atom which possesses a non-vanishing electric dipole momentum and two magnetic fields. The results of the present paper can also be understood from the electromagnetic duality point of view.

Since the discovery of AB effect \((11)\), Aharonov and Casher predicted that a neutral particle with a non-vanishing magnetic dipole momentum will acquire a topological phase if it moves around a uniformly charged infinitely long filament with its direction parallel to the filament. It is the Aharonov-Casher (AC) effect \((22–26)\). It is generally accepted that the AB effect is dual to the AC effect in the sense that the solenoid in the AB effect can be thought of a linear array of magnetic dipoles and exchanges the magnetic dipoles with the electric charges \((22, 27)\). Therefore, for the AC effect one has a line of electric charges and a particle with a magnetic dipole momentum moving around this line.

The Hamiltonian which describes the interaction between a neutral atom with a non-vanishing magnetic dipole momentum and electric fields is

\[
H = \frac{1}{2m} (p - \frac{\mu}{c^2} \mathbf{n} \times \mathbf{E})^2 + \frac{\mu}{2mc^2} \nabla \cdot \mathbf{E}. \tag{35}
\]

where \(\mu\) is the magnitude of the magnetic dipole momentum. It is the non-relativistic limit of a relativistic spin half neutral particle with a non-vanishing magnetic dipole momentum interacting with electric fields \((22)\).
In [21], the authors propose to realize the fractional angular momentum from the above Hamiltonian by applying two electric fields \( E = E^{(1)} + E^{(2)} \) and confining it by a harmonic trap potential. The electric fields are

\[
E^{(1)} = \frac{\lambda_e}{2\pi\epsilon_0}\varepsilon_r, \tag{36}
\]

\[
E^{(2)} = \frac{\rho_e}{2\epsilon_0}\varepsilon_r, \tag{37}
\]

where \( \lambda_e \) and \( \rho_e \) are the charges per unit length on the long filament and the charge density respectively, \( \epsilon_0 \) is the dielectric constant. After taking the limit of cooling down the kinetic energy of the atom to its lowest level, one gets the canonical angular momentum. It is [21]

\[
J = \frac{\mu}{2c^2\epsilon_0}(\rho_e x_i^2 + \frac{\lambda_e}{\pi}) \tag{38}
\]

where \( x_i \) are on the plane which is perpendicular to the magnetic dipole momentum. They satisfy the commutation relations

\[
[x_i, x_j] = -i\hbar\epsilon_{0i}c^2 \frac{\epsilon_0}{\mu\rho_e}, \tag{39}
\]

Obviously, the eigenvalues of the canonical angular momentum can be read directly from (38) and commutation relation (39) as [21]

\[
J_n = (n + \frac{1}{2})\hbar + \frac{\mu\lambda_e}{2c^2\epsilon_0}. \tag{40}
\]

However, it is argued that it is the HMW effect rather than the AB effect which is dual to the AC effect [4]. Because there is a natural correspondence between the AC and HMW effects: the magnetic dipole momentum in the AC effect corresponds to the electric dipole momentum in the HMW effect, the electric charges in the AC effect corresponds to the magnetic charges in the HMW effect.

The duality between the AC effect and the HMW effect can be expressed exactly as

\[
\begin{align*}
\begin{bmatrix} E \\ d \end{bmatrix} & \rightarrow \frac{1}{\sqrt{\epsilon_0\mu_0}} \begin{bmatrix} B \\ \mu \end{bmatrix}, \\
\begin{bmatrix} \lambda_e \\ \rho_e \end{bmatrix} & \rightarrow \sqrt{\frac{\epsilon_0}{\mu_0}} \begin{bmatrix} \lambda_m \\ \rho_m \end{bmatrix}
\end{align*} \tag{41}
\]

and

\[
\begin{align*}
\begin{bmatrix} B \\ \mu \end{bmatrix} & \rightarrow -\frac{1}{\sqrt{\epsilon_0\mu_0}} \begin{bmatrix} E \\ d \end{bmatrix}, \\
\begin{bmatrix} \lambda_m \\ \rho_m \end{bmatrix} & \rightarrow -\sqrt{\frac{\epsilon_0}{\mu_0}} \begin{bmatrix} \lambda_e \\ \rho_e \end{bmatrix}
\end{align*} \tag{42}
\]

where the minus arises from the asymmetric nature of the electromagnetic duality.

It is easy to check that the AC effect is dual to the HMW effect through (41) and the HMW effect is dual to the AC effect through (42). Furthermore, the Maxwell equations are also invariant under the dualities (41), (42) and \( J_e \rightarrow \sqrt{\frac{\epsilon_0}{\mu_0}}J_m \), \( J_m \rightarrow -\sqrt{\frac{\mu_0}{\epsilon_0}}J_e \) with \( J_e (J_m) \) being the electric (magnetic) current density provided the magnetic monopoles are presented [28]. Interestingly, the authors of [29] predict that there is a new topological effect which is dual to the AB effect according to the duality relations (41), (42).

These duality relations are also applied in [14, 29]. In [14], the authors find that the energy spectra of an electric dipole momentum in the background of the magnetic field [2] are dual to the spectra of a magnetic dipole momentum interacting with the electric field [37] through (41), (42) [30].

The dualities (41), (42) not only hold in the Hamiltonians [8] and [35], but also in the canonical angular momentum [33] and [35] as well as the commutation relations between \( x_1 \) in the reduced models [29] and [39]. Therefore, the studies of the present paper can be regarded as the electromagnetic duality of the [21] from the point of view (41), (42). It may be curious that the eigenvalues of the angular momentum (41) take negative values. The minus sign in the eigenvalues of the angular momentum (41) originates from [33], which is the direct result of the electromagnetic dualities (41), (42). It can also be understood classically since the atom can only stay in the stable equilibrium provided the direction of the classical angular momentum is inverse to the electric dipole momentum.

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