A NOTE ON SHAVGULIDZE’S PAPERS CONCERNING THE AMENABILITY PROBLEM FOR THOMPSON’S GROUP $F$

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The purpose of this note is to examine E. Shavgulidze’s publications [5], [6] and ArXiv postings [7], [8] in which he claims to have solved the amenability problem for Thompson’s group $F$. There is a consensus among those who have carefully studied these papers that not only do they contain errors, but that the errors are serious and do not seem to be repairable.

Unfortunately, both [5] and [6] have been published and are now being referred to as though they represent a correct solution to the amenability problem for $F$. This is further complicated by the fact that an inaccurate review of [5] has appeared in Math Reviews MR2486813. Also, reviews of [5] and [6] on Zentralblatt MATH treat these articles as correct. Finally, to my knowledge, Shavgulidze has made no public acknowledgment that there are problems with [5] and [6].

Before proceeding, I will briefly describe the papers and preprints of Shavgulidze under discussion. [5] is an extended abstract in which Shavgulidze announces that he has proved a general theorem which has, as a corollary, that Thompson’s group $F$ is amenable. Contrary to what is asserted in Math Reviews MR2486813, [5] does not contain proofs of the results it announces. [6] is a longer paper which contains the proof which is outlined in [5]. [7] is a preprint of [6] posted to the ArXiv. [8] is a preprint posted to the ArXiv which apparently attempts to repair the problem known to exist in [6], [7]. It should be noted however that [7] and [8] are separate postings to the ArXiv, not different versions of the same posting. Furthermore, [8] makes no reference to any of the other papers mentioned above and in particular does not indicate that there are errors in [6], [7].

The present note aims to both point to serious errors in these papers and to argue more generally why the approach taken by these papers seems unlikely to yield a solution to the amenability problem for $F$. Much of what is contained in this note was circulated to a limited number of people at Vanderbilt and SUNY Binghamton during Shavgulidze’s visit to the US in January 2010. It will be integrated...
into a broader survey article [4]. I would like to acknowledge the hard work of all those involved in the project of reading Shavgulidze’s postings to the ArXiv. I became actively involved in the reading project at a relatively late stage and the notes from Matt Brin’s seminar [1] and private communication with Matt Brin, Victor Guba, and Mark Sapir were very helpful. Much of what follows was precipitated by Matt Brin’s talk in the Topology and Geometric Group Theory Seminar at Cornell on 12/1/2009.

1. Specific problems with the proofs

There seems to be general agreement that the problems with the proof in [6] (and [7]) are limited to Lemma 2.4 (Lemma 5 of [7]) which asserts that a certain sequence of Borel measures $u_n$ ($n \in \mathbb{N}$) satisfies a condition which I will refer to as the mesh condition. There is agreement that the arguments of [7] show that the existence of a sequence of Borel measures $u_n$ ($n \in \mathbb{N}$) which satisfy the mesh condition and additionally an invariance condition (the conclusion of Lemma 6 of [7]) are sufficient to establish the amenability of $F$. It was observed by the original team of readers of [7] that the original proposed sequence of Borel measures does not in fact satisfy the mesh condition as was claimed in [7] (see pages 36-39 of [1]). During Shavgulidze’s visit to several US universities in January 2010, he proposed a revised sequence of measures $u_n$ ($n \in \mathbb{N}$), this time with finite support, claiming that they satisfied both the mesh and invariance conditions. The details of this construction were limited at the time and were supplied much later in [8].

Section 2 contains an elementary proof (unlike the one provided in [8]) that the amenability of $F$ follows from the existence measures such as those constructed in [8]. This argument makes all but pages 10 and 11 of [8] irrelevant and eliminates the need for the sophisticated analytical tools which Shavgulidze utilized in his proofs. A similar but more elaborate argument can be used to show that the existence of a sequence of Borel measures $u_n$ ($n \in \mathbb{N}$) satisfying the conclusions Lemmas 5 and 6 of [7] is equivalent to the amenability of $F$.

The paper [8] is full of typographical errors, minor mathematical errors, and in general is poorly written at a basic mechanical level. This in part makes it difficult to pinpoint the exact location of the real mistake. Still, a serious error is contained in Lemma 5. The proof concludes with an estimate concerning the sets $X^{0,l,n}$ but Lemma 5 concerns the sets $X^{m,l,n}$ for arbitrary $m$. No justification is given for this discrepancy. What seems to be implicit is that the functions
κₙ commute with the maps $f_1$ and $f_2$ (which are the generators of the group) and this is false. It is also clear that the construction on pages 10-11 would result in a Følner sequence which would violate the following theorems of [3]:

**Theorem 1.1.** [3] There is a constant $C$ such that if $A$ is a $C^{-n}$-Følner set in $F$ (with respect to the standard generating set) then $|A| \geq \exp_n(0)$, where $\exp_0(m) = m$ and $\exp_{n+1}(m) = 2^{\exp_n(m)}$.

**Theorem 1.2.** [3] If $\mu$ is a finitely additive probability measure on $T$ which is invariant with respect to the action of $F$, then for every sequence $I_i$ ($i \leq k$) of intervals satisfying $0 < \min I_i < \max I_i < \min I_{i+1} < \max I_{i+1} < 1$ for all $i < k$, it follows that $\mu$-a.e. $T$ satisfies that the cardinalities $|T \cap I_i|$ ($i \leq k$) form a strictly monotonic sequence.

2. The Elementary proof

I will first review some notation. Let $D$ denote all finite subsets of $[0, 1]$ which contain $\{0, 1\}$. Let $\mathcal{T}$ denote the collection of all elements $T$ of $D$ such that, if $t_i$ ($i \leq k$) is the increasing enumeration of $T$, then for all $i < k$, there is a non-negative integers $p$ and $q$ such that $t_i = p/2^q$ and $t_{i+1} = (p + 1)/2^q$. If $(S, T)$ is a pair of elements of $\mathcal{T}$ of the same cardinality, then the increasing map from $S$ to $T$ extends linearly on the complement to a piecewise linear map from $[0, 1]$ to $[0, 1]$. The collection of all such maps under composition is one of the standard models of $F$ [2]. The standard generators for $F$ are the elements

$$x_0 = (\{0, \frac{1}{2}, \frac{3}{4}, 1\}, \{0, \frac{1}{4}, \frac{1}{2}\})$$

$$x_1 = (\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1\}, \{0, \frac{1}{4}, \frac{5}{8}, \frac{3}{4}\})$$

In what follows, generator will always refer $x_0$, $x_1$, $x_0^{-1}$, or $x_1^{-1}$. Set

$$I_n = \{1 - 2^{-i} : 0 \leq i \leq n + 1\} \cup \{1\}.$$ 

If $T$ is in $\mathcal{T}$ and $|I_n| = |T|$, then I will use $f_T$ to denote the element of $F$ represented by $(I_n, T)$.

While the amenability of the partial action of $F$ on $\mathcal{T}$ is equivalent to the amenability of $F$ (see [3] for a direct proof of this using similar notation to the present note), the action of $F$ on $D$ is easily seen to be amenable. In fact if $A \subseteq \mathbb{N}$ is an $\epsilon$-Følner set in $\mathbb{Z}$, then it is not hard to see that

$$\{\{0, 1 - 2^{n-2}, 1\} : n \in A\}$$

is a $2\epsilon$-Følner set with respect to the action of $F$ on $D$ (each generator has a translation error of $\epsilon$). It turns out, however, that if one requires
that the Følner sequence concentrates on sets of mesh at most $1/16$, then one again obtains a reformulation of the amenability of $F$.

**Proposition 2.1.** $F$ is amenable provided that, for every $\epsilon$ there is a finite $A \subseteq D$ such that

$$|A \triangle (A \cdot x_i)| < \epsilon |A|$$

for each $i < 2$ and for every $X$ in $A$, $||X|| \leq 1/16$.

**Definition 2.2.** If $X$ is in $D$, define $T(X)$ to be the maximum element of $T$ such that if $s < t$ are in $T$, then there is an $x$ in $X$ with $s \leq x < t$.

**Claim 1.** If $T$ is in $T$ and $g$ is represented by $(U, V)$ and $U$ is contained in $T$, then $g \cdot T$ is in $T$. Furthermore $g \circ f_T = f_{g \cdot T}$.

**Proof.** See [2].

**Claim 2.** If $X$ is in $D$ and $f$ is represented by $(U, V)$, $U$ is contained in $T(X)$, then $f \cdot T(X) = T(f \cdot X)$.

**Proof.** Let $X$, $f$ be as above. Since $f \cdot T(X)$ is in $T$, it is contained in $T(f \cdot X)$ (this follows from the fact that $f$ is increasing). This in turn implies $f^{-1} \cdot T(f \cdot X)$ is in $T$ and therefore is contained in $T(X)$. Hence $|T(X)| = |f \cdot T(X)| = |T(f \cdot X)|$ and therefore we must have $f \cdot T(X) = T(f \cdot X)$. 

**Claim 3.** If $||X|| \leq 1/16$, then $||T(X)|| \leq 1/8$. In particular $I_0$ and $I_1$ are contained in $T(X)$.

**Proof.** Suppose $X$ is as above and that $s < t$ are consecutive elements of $T(X)$. Since $T(X)$ is maximal, either there is no $x$ is $X$ such that $s \leq x < (s+t)/2$ or else there is no $x$ in $X$ such that $(s+t)/2 \leq x < t$. Hence there is an interval of length $(t-s)/2$ contained in $[0, 1]$ and disjoint from $X$. It follows that $(t-s)/2 \leq 1/16$ and therefore that $t-s \leq 1/8$.

Putting this together, we have that $i = 0, 1$ and $Z \subseteq D$ consists only of $Z$ such that $|Z| < 1/8$, then

$$\{f_{T(X)} : X \in (x_i \cdot Z)\} = \{f_{T(x_i, Z)} : Z \in Z\}$$

$$= \{f_{x_i \cdot T(Z)} : Z \in Z\} = \{x_i \circ f_{T(Z)} : Z \in Z\}$$

In particular, if additionally $Z$ is finite and

$$|(Z \cdot x_i) \triangle Z| < \epsilon |Z|$$

and $A = \{f_{T(Z)} : Z \in Z\}$, then $|(x_i \circ A) \triangle A| < \epsilon |A|$. This finishes the proof of Proposition 2.1.
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