On Non-zero Degree Maps between Quasitoric 4-Manifolds

Đorđe Baralić
Mathematical Institute SASA
Belgrade, Serbia

Abstract
We study the map degrees between quasitoric 4-manifolds. Our results rely on Theorems proved by Duan and Wang in [13] and [14]. We determine the set $D(M, N)$ of all possible map degrees from $M$ to $N$ when $M$ and $N$ are certain quasitoric 4-manifolds. The obtained sets of integers are interesting, e.g. those representable as the sum of two squares $D(CP^2\#CP^2, CP^2)$ or the sum of three squares $D(CP^2\#CP^2\#CP^2, CP^2)$. Beside the general results about the map degrees between quasitoric 4-manifolds, the connections among Duan-Wang’s approach, the quadratic forms, the number theory and the lattices is established.

1 Introduction

The mapping degree is one of the earliest topological invariants and almost every textbook has section devoted to the definition and the calculations of this invariant. For given two orientable $n$-manifolds $M$ and $N$, every map $f : M \to N$ induces the homomorphism

$$f_* : H_*(M) \to H_*(N).$$

The degree of $f$ is defined as an integer such that

$$f_*([M]) = k[N],$$

where $[M] \in H_*(M)$ and $[N] \in H_*(N)$ are the fundamental class of $M$ and $N$ respectively. The one of fundamental question in manifold topology is:

Problem 1. For given two manifolds $M$ and $N$ what are the integers that could be realized as degree of some map

$$f : M \to N?$$

This is natural, but nontrivial and hard problem.

Definition 1.1. For given two closed orientable $n$-manifolds $M$ and $N$, $D(M, N)$ is the set of integers that could be realized as degree of a map from $M$ to $N$

$$D(M, N) = \{\text{deg} f \mid f : M \to N\}.$$
In dimension 2, the problem of determining $D(M, N)$ is completely solved. In dimension 3, the problem has been studied by several authors and for numerous classes of 3-manifolds is solved. The most important results about 3-manifolds could be found in the survey article of Wang. However, even in dimension 2 and 3 it is seen that $D(M, N)$ highly depends on the homotopy types of both $M$ and $N$.

From the standard topology course (and) we know several effective methods for calculating the mapping degree. Proposition 2.30 and Exercises 8. p. 258 in could be easily generalized and summarized in:

**Theorem 1.1.** For a map $f : M \rightarrow N$ between connected closed orientable $n$-manifolds and a point $y \in N$ such that $f^{-1}(y) = \{x_1, \ldots, x_k\}$ and there is ball $B \subset N$, $y \in B$ such that $f^{-1}(y)$ is the union of $k$ disjoint balls $B_1, \ldots, B_k$, $x_i \in B_i$ for every $i$, $1 \leq i \leq k$, the mapping degree $\deg f$ is the sum

$$\deg f = \sum_{i=1}^{k} \deg f | x_i$$

where $\deg f | x_i$ is the local map degree, i.e. the degree of map $f : \partial B_i \rightarrow \partial B$.

Theorem states that $\deg f$ evaluates the number of times the domain manifold $M$ "wraps around" the range manifold $N$ under the mapping $f$. This geometrical principle is the guiding idea in the most papers studying the mapping degrees. From Theorem it is easy to produce the map of any given degree into the sphere $S^n$. We take $k$ disjoint balls on $M$ and map their interiors by orientation preserving homeomorphism onto $S^n - \{\text{pt}\}$ and the rest of $M$ maps to the point $\{\text{pt}\}$, Figure 1. Thus, $D(M^n, S) = \mathbb{Z}$.

![Figure 1: The degree k map from $M^n$ to $S^n$.](image)

Every map $f : M \rightarrow N$ induces homomorphisms on homology $f_*$ and cohomology $f^*$. From the following commutative diagram (see p. 241)

$$
\begin{array}{ccc}
H_n(M; \mathbb{Z}) \times H^k(M; \mathbb{Z}) & \xrightarrow{\cap} & H_{n-k}(M; \mathbb{Z}) \\
\downarrow f_* & & \downarrow f_* \\
H_n(N; \mathbb{Z}) \times H^k(N; \mathbb{Z}) & \xrightarrow{\cap} & H_{n-k}(M; \mathbb{Z})
\end{array}
$$

we conclude that for non-zero degree map $f$, every $f^* : H^k(N; \mathbb{Z}) \rightarrow H^k(M; \mathbb{Z})$ is monomorphism. Thus there is no nonzero degree map $f : \Sigma^n \rightarrow M$ from homology sphere $\Sigma^n$ to the oriented closed manifold $M$ with nontrivial cohomology.
It is easy to produce the maps of zero degree, so \( 0 \in D(M, N) \). The identity map shows that \( 1 \in D(M, M) \). For the beginning, we reduce Problem 1 on:

**Problem 2.** For given two manifolds \( M \) and \( N \) is there any degree 1 map from \( M \) to \( N \)?

The following simple examples show that in general, the sets \( D(M, N) \) are different for distinct manifolds. Even for Problem 2 it is hard to say when the answer is positive or negative.

**Proposition 1.1.** For a simply connected closed orientable manifold \( M^{2n-1} \), the set \( D(M, \mathbb{R}P^{2n-1}) = 2\mathbb{Z} \).

**Proof:** The sphere \( S^n \) is the universal covering for \( \mathbb{R}P^n \) and let \( p \) be the covering map. Let \( f : M \to \mathbb{R}P^n \) be a map. We consider the following diagram

\[
\begin{array}{ccc}
S^n & \xrightarrow{\tilde{f}} & \mathbb{R}P^n \\
\downarrow & & \downarrow p \\
M & \xrightarrow{f} & \mathbb{R}P^n 
\end{array}
\]

Because \( \pi_1 M \) is trivial, there is lifting map \( \tilde{f} : M \to S^n \) of \( f \). From the functoriality of homology we have \( f_* = p_* \tilde{f}_* \) and thus \( \deg f = \deg p \cdot \deg \tilde{f} \). But \( \deg p = 2 \) when \( n \) is odd and so \( \deg \tilde{f} \) is even.

However, it is not hard to produce a degree \( 2k \) map. Take any degree \( k \) map from \( M \) to \( S^{2n-1} \) and compose it with \( p \). \( \square \)

**Proposition 1.2.** Let \( M \) be a simply connected closed orientable 3-manifold, and \( P \) the Poincaré homology sphere. Then \( D(M, P) = 120\mathbb{Z} \).

**Proof:** Here \( S^3 \) is the universal cover of \( P \) and degree of \( p \) is 120. The argument is the same as in the previous proof. \( \square \)

**Remark.** Proposition 1.2 is a special case of the result of Legrand-Matveev-Zieschang [22] about the calculation of the mapping degree from Seifert manifolds into the Poincaré homology sphere. However, this result together with [22] answered the question for Seifert manifolds.

The results about 3-manifolds usually suppose some additional geometrical or topological structure on manifolds. Until recent years there have been studied some special examples for manifolds in dimensions above 3, see [3], [12], and [18]. Haibao Duan and Shicheng Wang progress in the problem [13] and [14] is significant because it gives algebraic conditions for the existence of certain map degree between two given closed \((n-1)\)-connected \(2n\)-manifolds. Their algebraic conditions are obtained from topology of this wide class of manifolds. However, even in dimension 4 where the situation is the simplest, in general it is not easy to check these conditions.

The goal of this article is to improve their results in dimension 4 and apply them for the concrete class of 4-dimensional quasitoric manifolds. We get interesting results and we put the results of Duan and Wang in the interaction with recent development in toric topology and other ideas from topology and geometry. Quasitoric manifolds are topological generalizations of toric varieties and represent very important class of manifolds. They are studied intensively in past twenty years.

In Section 2 we give review on Duan and Wang work. We prove again Theorem 2 [13] with accent on some details omitted in their paper, because we give slightly improved results.
In Section 3 we briefly discuss topology and geometry of Hirzebruch surfaces and quasitoric manifolds. In dimension 4 we have complete classification of quasitoric manifolds. Cohomology and intersection form of these manifolds is known.

In Section 4 we study the mapping degrees between 4-dimensional quasitoric manifolds.

In Section 5 we study the mapping degrees between connected sums of \(\mathbb{C}P^2\) and discuss the connection of our problem and the problems about the lattice discriminants.

In Section 6 we formulate some general theorems about the mapping degrees between quasitoric 4-manifolds.

In Section 7 we briefly touch some possible extensions of our results for higher dimensions and their connections with other areas of mathematics.

2 Results of Duan and Wang

In the articles [13] and [14] are given theorems which significantly contribute to our knowledge about map degrees between closed orientable \(2n\)-manifolds. In this section, we prove Theorem 2 from [13], and extend Corollary 3 of Wang’s and Duan’s result.

Let \(M\) be a \(2n\)-dimensional closed, connected and orientable manifold, \(n > 1\) and let \(\bar{H}^n(M; \mathbb{Z})\) be the free part of \(H^n(M; \mathbb{Z})\). Then the cup product operator

\[\bar{H}^n(M; \mathbb{Z}) \otimes \bar{H}^n(M; \mathbb{Z}) \to H^{2n}(M; \mathbb{Z})\]

defines the intersection form \(X_M\) over \(\bar{H}^n(M; \mathbb{Z})\), which is bilinear and unimodular by Poincaré duality, see [16] Proposition 3.38. This form is \(n\)-symmetric in the sense that

\[X_M(x \otimes y) = (-1)^n X_M(y \otimes x)\).

Let \(\alpha = (\alpha_1, \ldots, \alpha_m)\) be a basis for \(\bar{H}^n(M; \mathbb{Z})\). Then \(X_M\) determines an \(m \times m\) matrix \(A = (a_{ij})\) where \(a_{ij}\) is given by

\[a_{ij} = \alpha_i \cup \alpha_j[M],\]

and \([M]\) is the fundamental class of \(H_{2n}(M)\).

Let \(f : M \to L\) be a map between two connected, closed and orientable \(2n\)-manifolds \(M\) and \(L\), and let \(f^*\) and \(f_*\) be the induced homomorphisms on the cohomology rings and homology rings. Let \(\alpha = (\alpha_1, \ldots, \alpha_m)\) and \(\beta = (\beta_1, \ldots, \beta_l)\) be basis for \(\bar{H}^n(M; \mathbb{Z})\) and \(\bar{H}^n(L; \mathbb{Z})\) respectively. The induced homomorphism \(f^*\) determines \(m \times l\) matrix \(P = (p_{ij})\) such that

\[f^*(\alpha_i) = \sum_{j=1}^l p_{ij} \beta_j,\]

for every \(i, 1 \leq i \leq m\).

**Theorem 2.1** (H. Duan, S. Wang). Suppose \(M\) and \(L\) are closed oriented \(2n\)-manifolds with intersection matrices \(A\) and \(B\) under some given basis \(\alpha\) for \(H^n(M; \mathbb{Z})\) and \(\beta\) for \(H^n(L; \mathbb{Z})\). If there is a map \(f : M \to L\) of degree \(k\) such that \(f^*(\beta) = \alpha P\), then

\[P^t A P = kB.\]

Moreover, if \(k = 1\), then \(X_L\) is isomorphic to a direct summand of \(X_M\).
**Proof:** For a map $f : M \to L$ holds $f_*([M]) = k[L]$. From the functoriality of the cup and the cap product functor we have

$$X_M(f^*(x) \otimes f^*(y)) = f^*(x) \cup f^*(y)[M] = f^*(x \cup y)[M] = (x \cup y)[M] = (x \cup y)k[L] = kX_L(x \otimes y)$$

for every $x, y \in \tilde{H}^n(L; \mathbb{Z})$. Thus the following diagram commutes

$$
\begin{array}{c}
\tilde{H}^n(L; \mathbb{Z}) \times \tilde{H}^n(L; \mathbb{Z}) \xrightarrow{X_L} \mathbb{Z} \\
\downarrow f^* \otimes f^* \\
\tilde{H}^n(M; \mathbb{Z}) \times \tilde{H}^n(M; \mathbb{Z}) \xrightarrow{X_M} \mathbb{Z},
\end{array}
$$

Consequently, for the basis $\alpha$ for $\tilde{H}^n(M; \mathbb{Z})$ and $\beta$ for $\tilde{H}^n(L; \mathbb{Z})$ this fact is written in form $P^tAP = kB$ where $f^*(\beta) = \alpha P$.

In particular, when $k = 1$ the restriction of $X_M$ on the subgroup $f^*(\tilde{H}^n(L; \mathbb{Z})) \subset \tilde{H}^n(M; \mathbb{Z})$ is isomorphic to $X_L$ and unimodular. By Orthogonal Decomposition Lemma [24], p. 5,

$$X_M = X_{f^*(\tilde{H}^n(L; \mathbb{Z}))} \oplus X_{H^\perp} = X_L \oplus X_{H^\perp},$$

where $H^\perp$ is the orthogonal complement of $f^*(\tilde{H}^n(L; \mathbb{Z}))$ and $X_{H^\perp}$ is the restriction of $X_M$ on $H^\perp$.

In the same paper Duan and Wang proved the following theorem that gives the complete criteria for the existence of degree $k$ map from a 4-manifold $M$ to a simply connected 4-manifold $L$.

**Theorem 2.2.** Suppose $M$ and $L$ are closed oriented 4-manifolds with intersection matrices $A$ and $B$ under given bases $\alpha$ for $\tilde{H}^2(M; \mathbb{Z})$ and $\beta$ for $\tilde{H}^2(L; \mathbb{Z})$. If $L$ is simply connected, then there is a map $f : M \to L$ of degree $k$ such that $f^*(\beta) = \alpha P$ if and only if

$$P^tAP = kB.$$

Moreover there is a map $f : M \to L$ of degree 1 if and only if $X_L$ isomorphic to a direct summand of $X_M$.

Duan and Wang proved in [13] Corollary 3 of Theorem 2.1. Implicitly, it is clear from their paper this corollary could be generalized.

**Corollary 2.1.** Suppose $M$ and $L$ are closed oriented $2n$-manifolds such that

$$\text{rank} \tilde{H}^n(M; \mathbb{Z}) = \text{rank} \tilde{H}^n(L; \mathbb{Z}) = 2r + 1.$$  

Then for any map $f : M \to L$, the absolute value of the degree of $f$ is a square of an integer.

**Proof:** Let $P$ be the matrix realized by $f$. By Theorem 2.1 we have

$$P^tAP = kB,$$

where $P$, $A$ and $B$ are square matrices of order $2r + 1$. By taking the determinant we have $|P|^2 |A| = k^{2r+1} |B|$. Since $A$ and $B$ are unimodular, then $|P|^2 = |k|^{2r+1}$. Thus, $|k|$ is a perfect square. \hfill\square

In papers [13] and [14], Duan and Wang developed technique for studying the non-zero degree maps between $(n-1)$-connected closed and oriented $2n$-manifolds. They demonstrated application on various concrete examples of manifolds. In Section 4, we continue in the same manner, trying to put the results in [13] Section 7 in more general context and classify the obtained results.
3 Quasitoric manifolds

Quasitoric manifolds from the general topological point of view are manifolds with a certain nice torus action. This notion appeared first in [11]. These objects are connected with toric varieties from algebraic geometry. In monograph [7] there is nice exposition about this topic. For purposes of our results we offer brief overview on them following this exposition.

3.1 Quasitoric manifolds and characteristic maps

Let \( (\mathbb{C}^*)^n \) denote the multiplicative group of complex numbers. The product \((\mathbb{C}^*)^n\) is algebraic torus. The torus \( T^n = \bigoplus_{i=1}^{n} S^1 \) is a subgroup of the algebraic torus \((\mathbb{C}^*)^n\) in the standard way:

\[
T^n = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1| = \cdots = |z_n| = 1 \}.
\]

The torus \( T^n \) acts on \( \mathbb{C}^n \) by standard diagonal multiplication and the quotient of this action is the positive cone \( \mathbb{R}_+^n \).

Let \( M^{2n} \) be a \( 2n \)-dimensional manifold with an action of the torus \( T^n \) - \( T^n \)-manifold.

**Definition 3.1.** A standard chart on \( M^{2n} \) is a triple \((U,f,\psi)\) where \( U \) is a \( T^n \)-stable open subset of \( M^{2n} \), \( \psi \) is an automorphism of \( T^n \), and \( f \) is a \( \psi \)-equivariant homeomorphism \( f : U \rightarrow W \) with some \((T^n\)-stable\) open subset \( W \subset \mathbb{C}^n \). i. e. \( f(t \cdot y) = \psi(t)f(y) \) for all \( t \in T^n \) and \( y \in U \). Say that a \( T^n \)-action on \( M^{2n} \) is locally standard if \( M^{2n} \) has a standard atlas, that is, every point of \( M^{2n} \) lies in a standard chart.

The orbit space for a locally standard action of \( T^n \) on \( M^{2n} \) is an \( n \)-dimensional manifold with corner and quasitoric manifolds correspond to the case when this orbit space is diffeomorphic, as manifold with corners to a simple polytope \( P^n \).

**Definition 3.2.** Given a combinatorial simple polytope \( P^n \), a \( T^n \)-manifold \( M^{2n} \) is called a quasitoric manifold over \( P^n \) if the following two conditions are satisfied:

1. the \( T^n \) action is locally standard;
2. there is a projection map \( \pi : M^{2n} \rightarrow P^n \) constant on \( T^n \)-orbits which maps every \( k \)-dimensional orbit to a point in the interior of a codimension-\( k \) face of \( P^n \), \( k = 0, \ldots, n \).

The \( T^n \)-action on a quasitoric manifold \( M^{2n} \) is free over the interior of the quotient polytope \( P^n \) and vertices of \( P^n \) correspond to the \( T^n \)-fixed points of \( M^{2n} \). Let \( F_1, \ldots, F_m \) be facets of \( P^n \). For every facet \( F_i \), the pre-image \( \pi^{-1}(\text{int} F_i) \) consists of codimension-one orbits with the same 1-dimensional isotropy subgroup, which we denote \( T(F_i) \), (see Figure 2). \( \pi^{-1}(\text{int} F_i) \) is an \( 2(n-i) \)-dimensional quasitoric submanifold over \( F_i \), with respect to the action of \( T^n/T(F_i) \) and we denote it \( M^{2(n-i)}_i \) and refer to it as the facial submanifold corresponding to \( F_i \). Its isotropy subgroup \( T(F_i) \) can be written as

\[
T(F_i) = \{ (z^{\lambda_1}, \ldots, z^{\lambda_m}) \in T^n \mid |z| = 1 \}.
\]
The vector $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{in})^t \in \mathbb{Z}^n$ is determined only up to sign and is called the \textit{facet vector} corresponding to $F_i$. The correspondence

$$l : F_i \mapsto T(F_i) \quad (1)$$

is called the \textit{characteristic map} of $M^{2n}$.

Let $G^{n-k}$ be a codimension-$k$ face written as an intersection of $k$ facets $G^{n-k} = F_{i1} \cap \cdots \cap F_{ik}$. Then the submanifolds $M_{i1}, \ldots, M_{ik}$ intersect transversally in a submanifold $M(G)^{2(n-k)}$, which we refer to as the \textit{facial submanifold} corresponding to $G$. The map $T(F_{i1}) \times \cdots \times T(F_{ik}) \to T^n$ is injective since $T(F_{i1}) \times \cdots \times T(F_{ik})$ is the $k$-dimensional isotropy subgroup of $M(G)^{2(n-k)}$. Thus, the vectors $\lambda_{i1}, \ldots, \lambda_{ik}$ form a part of an integral basis of $\mathbb{Z}^n$.

The correspondence

$$G^{n-k} \mapsto \text{isotropy subgroup of } M(G)^{2(n-k)}$$

extends the characteristic map (1) to a map from the face poset of $P^n$ to the poset of subtori of $T^n$.

**Definition 3.3.** Let $P^n$ be a combinatorial simple polytope and $l$ is a map from facets of $P^n$ to one-dimensional subgroups of $T^n$. Then $(P^n, l)$ is called a characteristic pair if $l(F_{i1}) \times \cdots \times l(F_{ik}) \to T^n$ is injective whenever $F_{i1} \cap \cdots \cap F_{ik} \neq \emptyset$.

The map $l$ directly extends to a map from the face poset of $P^n$ to the poset of subtori of $T^n$, so we have subgroup $l(G) \subset T^n$ for every face $G$ of $P^n$. As in the case of standard action of $T^n$ on $\mathbb{C}^n$, there is projection $T^n \times P^n \to M^{2n}$ whose fibre over $x \in M^{2n}$ is the isotropy subgroup of $x$. This argument we use for reconstructing the quasitoric manifold from any given characteristic pair $(P^n, l)$.

Given a point $q \in P^n$, we denote by $G(q)$ the minimal face containing $q$ in its relative relative. Define relation $\sim$ on $T^n \times P^n$ in following way $(t_1, p) \sim (t_2, q)$ if and only if $p = q$ and $t_1 t_2^{-1} \in l(G(q))$. Now set

$$M^{2n}(l) := (T^n \times P^n) / \sim.$$
The free action of $T^n$ on $T^n \times P^n$ obviously descends to an action on $(T^n \times P^n)/\sim$, with quotient $P^n$. The latter action is free over the interior of $P^n$ and its fixed points are vertices of $P^n$, (see Figure 2). Just as $P^n$ it is covered by the open sets $U_i$, based on the vertices and diffeomorphic to $\mathbb{R}^n_i$, so the space $(T^n \times P^n)/\sim$ is covered by open sets $(T^n \times U_i)/\sim$ homeomorphic to $(T^n \times \mathbb{R}^n_i)/\sim$, and therefore to $\mathbb{C}^n$. This implies that the $T^n$-action on $(T^n \times P^n)/\sim$ is locally standard, and therefore $(T^n \times P^n)/\sim$ is a quasitoric manifold.

3.2 Cohomology of quasitoric manifolds

The cohomology of quasitoric manifolds is determined by Davis and Januszkiewicz [11]. They constructed also CW structure on quasitoric manifolds with only even dimensional cells.

Let $F_1, \ldots, F_m$ be facets of a simple polytope $P^n$ and let $\mathbb{Z}[v_1, \ldots, v_m]$ be the polynomial algebra on $\mathbb{Z}$ with $m$ generators $v_1, \ldots, v_m$ one for each facet. The Stanley-Reisner ring of a simple polytope $P^n$ is the quotient ring $\mathbb{Z}(P^n) = \mathbb{Z}[v_1, \ldots, v_m]/I_P$ where $I_P$ is the ideal generated by all square-free monomials $v_1 v_2 \cdots v_i$ such that $F_{i_1} \cap \cdots \cap F_{i_s} = \emptyset$ in $P$, $i_1 < \cdots < i_s$.

Given a quasitoric manifold $M^{2n}$ with characteristic map $l : F_i \to T(F_i)$ and facet vectors $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{im})^t \in \mathbb{Z}^n$, $i = 1, \ldots, m$, define linear forms

$$\theta_i := \lambda_{i1} v_1 + \cdots + \lambda_{im} v_m \in \mathbb{Z}[v_1, \ldots, v_m], \quad 1 \leq i \leq n. \quad (2)$$

The images of these linear forms in the Stanley-Reisner ring $\mathbb{Z}(P^n)$ will be denoted by the same letters. Let $I_i$ denote the ideal in $\mathbb{Z}(P^n)$ generated by $\theta_1, \ldots, \theta_n$.

**Theorem 3.1** (Davis and Januszkiewicz). The cohomology ring of $M^{2n}$ is given by

$$H^*(M^{2n}; \mathbb{Z}) = \mathbb{Z}[v_1, \ldots, v_m]/(I_P + I_i) = \mathbb{Z}(P^n)/I_i,$$

where $v_i$ is the 2-dimensional cohomology class dual to the facial submanifold $M^{2(n-1)}_i$, $i = 1, \ldots, m$.

3.3 Hirzebruch surfaces and 4-dimensional quasitoric manifolds

Hirzebruch surfaces were introduced by Hirzebruch in [17] and they are algebraic surfaces over the complex numbers. As complex manifolds they are pairwise distinct while as smooth manifolds there are only two diffeomorphism types.

Given an integer $k$, the Hirzebruch surface is the complex manifold $\mathbb{C}P^k(\mathbb{C} + \mathbb{C})$, where $\mathbb{C}$ is the complex line bundle over $\mathbb{C}P^1$ with first Chern class $k$, and $\mathbb{C}P^k(\mathbb{C})$ denotes the projectivisation of a complex bundle. Each Hirzebruch surface is the total space of the bundle $H_k \to \mathbb{C}P^1$ with fibre $\mathbb{C}P^1$. For even $k$ the surface $H_k$ is diffeomorphic to $S^2 \times S^2$ and for odd $k$ to $\mathbb{C}P^2 \# \mathbb{C}P^2$, where $\mathbb{C}P^2$ denotes the space $\mathbb{C}P^2$ with reversed orientation. The Hirzebruch surface $H_k$ is quasitoric manifold which orbit space is a combinatorial square.

The topological classification problem (up to diffeomorphism) for quasitoric manifolds over a given simple polytope is intractable. Up to now, only some particular results are known. In [26] classification problem for quasitoric manifolds over polygons is completely solved.
Theorem 3.2. A quasitoric manifold of dimension 4 is diffeomorphic to connected sum of several copies of \( \mathbb{C}P^2 \), \( \overline{\mathbb{C}P}^2 \) and \( S^2 \times S^2 \).

Quasitoric manifolds are simply connected, so they are perfect test examples for the application of Theorem 2.2.

4 Mapping degrees between Quasitoric 4-manifolds

The mapping degree between some quasitoric 4-manifolds are studied in [13], where some simply examples illustrated application of Theorems 2.1 and 2.2. However, the results they got in simplest case are not trivial. In this section we focus our attention on more general cases of quasitoric 4-manifolds.

Quasitoric 4-manifolds are topologically classified by Theorem 3.2 and we could easily determine their intersection form. The matrices representing the intersection form for \( \mathbb{C}P^2 \), \( \overline{\mathbb{C}P}^2 \) and \( S^2 \times S^2 \) are

\[
\begin{bmatrix}
1 \\
-1
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

respectively. Thus, the intersection form for \( \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \) has matrix representation

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
\]

Theorem 3.2 states that a quasitoric 4-manifold is diffeomorphic to

\[
(\mathbb{C}P^2)^a \# (\overline{\mathbb{C}P}^2)^b \# (S^2 \times S^2)^c = \mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \cdots \# \overline{\mathbb{C}P}^2 \# S^2 \times S^2 \# \cdots \# S^2 \times S^2 .
\]

The intersection form of a given quasitoric manifold has representation in \((a+b+2c)\)-square matrix

\[
\begin{bmatrix}
I_{a \times a} & 0 & 0 \\
0 & -I_{b \times b} & 0 \\
0 & 0 & A_{c \times c}
\end{bmatrix},
\]

where \( A_{c \times c} \) is \(2c\)-square matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]

Let \( M = (\mathbb{C}P^2)^a \# (\overline{\mathbb{C}P}^2)^b \# (S^2 \times S^2)^c \) and \( N = (\mathbb{C}P^2)^d \# (\overline{\mathbb{C}P}^2)^e \# (S^2 \times S^2)^f \) be two quasitoric manifolds, and the matrices \( A \) and \( B \) their intersection forms respectively. From Theorem 2.2 follows that there is degree \( k \) map between \( M \) and \( N \).
if and only if there is \((a + b + 2c) \times (d + e + 2f)\) matrix \(P\) such that \(P^tAP = kB\).

Direct calculation gives that elements of \((d + e + 2f)\)-square matrix \(C = P^tAP\) are
\[
c_{ij} = \sum_{r=1}^{a} p_{ri}p_{rj} + \sum_{r=1}^{b} p_{a+r}i p_{a+r}j + \sum_{r=1}^{c} \left( p_{a+b+2r-1}i p_{a+b+2r}j + p_{a+b+2r}i p_{a+b+2r-1}j \right).
\]

Solving the equation \(P^tAP = kB\) is then equivalent to solving the certain system of Diophantine equations. There is no algorithm for solving a Diophantine equation, so there is not natural way to approach this problem. However, only \(d + e + f\) out of \(\frac{(d+e+f)(d+e+f+1)}{2}\) expressions is equal to \(\pm k\) and the others are 0.

We prove the following lemma:

**Lemma 4.1.** Let \(B_{n \times k}\) and \(C_{k \times n}\) be arbitrary matrices such that \(k < n\). Then
\[
\det |B \cdot C| = 0.
\]

**Proof:** Denote \(A = B \cdot C\). Then \(a_{ij} = \sum_{r=1}^{k} b_{ir}c_{rk}\). Let \(B'\) and \(C'\) be \(n\) square matrices such that
\[
B' = \begin{bmatrix} B & 0 \end{bmatrix}, \quad C' = \begin{bmatrix} C & 0 \end{bmatrix}.
\]

It is obviously that
\[
B' \cdot C' = B \cdot C = A,
\]
and consequently
\[
0 = \det B' \cdot \det C' = \det A.
\]

\(\square\)

**Theorem 4.1.** Let \(M\) and \(N\) be closed, connected and oriented \(2n\)-manifolds such that \(1 \leq \text{rank} H^n(M;\mathbb{Z}) < \text{rank} H^n(N;\mathbb{Z})\). Then there is no non-zero degree map \(f : M \to N\).

**Proof:** This is corollary of Lemma 4.1 applied for matrices \(P^t\) and \(A \cdot P\). According to Lemma
\[
0 = k^{\text{rank} H^n(N;\mathbb{Z})} \det B
\]
and because of unimodularity of the intersection form \(k = 0\).

\(\square\)

**Theorem 4.2.** If there is a degree \(k\) map \(f : M \to N\) between two \(4\)-manifolds, then there is a degree \(k\) map from \(M \# CP^2\) to \(N\).

**Proof:** Let \(A\) be the intersection matrix for \(M\), \(B\) for \(N\) and \(P\) matrix such that
\[
P^tAP = kB.
\]
Then \(A' = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix}\) is the intersection matrix for \(M \# CP^2\). The matrix \(P' = \begin{bmatrix} 0 \\ P \end{bmatrix}\) clearly satisfies
\[
(P')^tA'P' = kB.
\]

\(\square\)

In the same fashion we can prove the following theorem:

**Theorem 4.3.** If there is a degree \(k\) map \(f : M \to N\) between two \(4\)-manifolds, then there is a degree \(k\) map from \(M \# (S^2 \times S^2)\) to \(N\).
Corollary 4.1. If there is a degree $k$ map $f : M \to N$ between two 4-manifolds, then there is a degree $k$ map from $M\#Q$ to $N$ for every 4-manifold $Q$.

Theorem 4.4. If there are degree $k$ maps $f : M \to N$ and $g : M' \to N'$ between 4-manifolds, then there is a degree $k$ map from $M\#M'$ to $N\#N'$.

Proof: Let $A$ and $A'$ be the intersection matrices for $M$ and $M'$, $B$ and $B'$ for $N$ and $N'$, $P$ and $P'$ matrices such that
\[ P^t A P = kB, \]
\[ P'^t A' P' = kB'. \]
We easily check that
\[
\begin{bmatrix} P^t & 0 \\ 0 & P'^t \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} \cdot \begin{bmatrix} P & 0 \\ 0 & P' \end{bmatrix} = k \begin{bmatrix} B & 0 \\ 0 & B' \end{bmatrix}. \]

\(\square\)

4.1 Maps to $\mathbb{C}P^2$

We study maps from quasitoric manifolds to $\mathbb{C}P^2$ (and $\overline{\mathbb{C}P^2}$). Let $M$ be a quasitoric manifold diffeomorphic to $(\mathbb{C}P^2)^a \# (\overline{\mathbb{C}P^2})^b \# (S^2 \times S^2)^c$.

Theorem 2.2 reduces problem on the existence of nontrivial solution of Diophantine equation
\[
\sum_{i=1}^a p_{1i}^2 - \sum_{i=1}^b p_{a+i1}^2 - \sum_{i=1}^c p_{a+b+2i-1} p_{a+b+2i} = k. \tag{6}
\]

Theorem 4.5.
- If $a \geq 1$ (or $b \geq 1$) and $c \geq 1$ then the equation \((6)\) have solution for every $k \in \mathbb{Z}$.
- If $a \geq 4$ and $b = c = 0$ then the equation \((6)\) have solution for every nonnegative integer $k$ and no solution for negative $k$.
- If $b \geq 4$ and $a = c = 0$ then the equation \((6)\) have solution for every integer $k \leq 0$ and no solution for positive $k$.
- If $a = 3$ and $b = c = 0$ then the equation \((6)\) have solution for every nonnegative integer $k \neq 4^p(8q + 7)$ and no solution for positive integers $k = 4^p(8q + 7)$ and negative integers.
- If $b = 3$ and $a = c = 0$ then the equation \((6)\) have solution for every integer $k \neq -4^p(8q + 7)$ and no solution for negative integers $k = -4^p(8q + 7)$ and positive integers.
- If $a = 2$ and $b = c = 0$ then the equation \((6)\) have solution for every nonnegative integer $k$ such that every prime number $4p - 1$ that divides $k$ occurs even time in the prime factorization of $k$ and no solutions in other cases.
- If $b = 2$ and $a = c = 0$ then the equation \((6)\) have solution for every integer $k \leq 0$ such that every prime number $4p - 1$ that divides $|k|$ occurs even time in the prime factorization of $|k|$ no solutions in other cases.
• If \(a = b = 1\) and \(c = 0\) then the equation \(a\) have solution for every integer 
\(k \neq 4p + 2\) and no solution for \(k = 4p + 2\)

• If \(a = 1\) and \(b = c = 0\) then\(\) have solution for every integer that is square of 
an integer and no solution in other cases

• If \(b = 1\) and \(a = c = 0\) then \(\) have solution for every integer that is square of 
an integer multiplied by \(-1\) and no solution in other cases

• If \(a = b = 0\) and \(c \geq 1\) then the equation \(\) have solution for every even integer 
\(k\)

Proof: We observe that every integer could be written as \(u^2 + 2vw\) where \(u, v\) and 
\(w\) are integers. This guarantees existence of \(k\) degree map \(f : \mathbb{C}P^2(S^3 \times S^3) \rightarrow \mathbb{C}P^2\). 

Theorems 4.2 and 4.3 extend the result for cases \(a \geq 1\) and \(c \geq 1\). Since every 
nonnegative integer has decomposition into the sum of four squares of integers, there 
is \(k \geq 0\) degree map when \(a \geq 4\) and \(b = c = 0\).

The case \(a = 3\) and \(b = c = 0\) is interesting, it follows from the nontrivial result of 
Legendre (1798) and Gauss (1801) that nonnegative integer has presentation as the 
sum of three squares of integers iff it is not of the type \(4p(8q + 7)\).

If \(a = 2\) and \(b = c = 0\) then for any map \(f : \mathbb{C}P^2\mathbb{C}P^2 \rightarrow \mathbb{C}P^2\), \(\deg f\) must be the 
sum of two squares. It is well known fact that nonnegative integer \(k\) could be written 
as the sum of two squares iff every prime number \(4p - 1\) that divides \(k\) occurs 
even time in the prime factorization of \(k\). For \(k = u^2 + v^2\) we take the matrix

\[
P = \begin{bmatrix} u & v \\ v & -u \end{bmatrix}
\]

and get the map of degree \(k\).

In the case \(a = b = 1\) and \(c = 0\), the map degree must be the difference of two 
squares and this is possible iff the integer is not equal \(2\) modulo \(4\). Again, it is easy 
to check

\[
P = \begin{bmatrix} u & v \\ v & -u \end{bmatrix}
\]

gives map of degree \(u^2 - v^2\).

In the same manner could be proved other cases so we omit the rest of proof. \(\square\)

4.2 Maps to \(S^2 \times S^2\)

Maps from \(\mathbb{C}P^2\mathbb{C}P^2\) and \(\mathbb{C}P^2\mathbb{C}P^2\) to \(S^2 \times S^2\) are studied in [13]. We are interested 
in the mapping degrees from an arbitrary quasitoric manifold to \(S^2 \times S^2\).

Proposition 4.1. There is no non-zero degree map from \((\mathbb{C}P^2)^\# k\) \((\mathbb{C}P^2)^\# k\) to 
\(S^2 \times S^2\).

Proposition 4.2. For every integer \(k\) there is a degree \(k\) map from \((S^2 \times S^2)^\# k\) to 
\(S^2 \times S^2\).

Proposition 4.3. For every integer \(k\) there is a degree \(k\) map from \((\mathbb{C}P^2)^\# 2\mathbb{C}P^2\) \n\((\mathbb{C}P^2)^\# (\mathbb{C}P^2)^\# 2\) to \(S^2 \times S^2\).

Theorem 4.6. Let \(M\) be a quasitoric manifold such that \(c \geq 1\), then there is a degree 
k map from \(M\) to \(S^2 \times S^2\) for every integer \(k\).
Proof: We easily check that matrix $P$ such that

$$P = \begin{bmatrix} 0 & 0 & 1 \\ k & 0 & 0 \end{bmatrix},$$

satisfy Theorem 2.2. \hfill $\square$

**Theorem 4.7.** Let $M$ be a quasitoric manifold such that $a \geq 2$ and $b \geq 1$ (or $a \geq 1$ and $b \geq 2$) then there is a degree $k$ map from $M$ to $S^2 \times S^2$ for every integer $k$.

**Proof:** We easily check that matrix $P$ such that

$$P = \begin{bmatrix} 0_{(a-2)\times2} & 0 \\ k & 0 \\ 0 & 1 \\ -k & 1 \\ 0_{(b+2c-1)\times2} \end{bmatrix},$$

satisfy Theorem 2.2. \hfill $\square$

### 4.3 Maps to $\mathbb{CP}^2 \sharp \mathbb{CP}^2$

According to [13] there is no nonzero degree map from $\mathbb{CP}^2 \sharp \mathbb{CP}^2$ and $S^2 \times S^2$ to $\mathbb{CP}^2 \sharp \mathbb{CP}^2$. For other quasitoric manifolds the sets $D(M, \mathbb{CP}^2 \sharp \mathbb{CP}^2)$ are richer and interesting.

**Proposition 4.4.** There is a degree $k$ map $f : (\mathbb{CP}^2)^{12} \to (\mathbb{CP}^2)^{12}$ if and only if $k \geq 0$ and every prime number $4p - 1$ that divides $k$ occurs even time in the prime factorization of $k$.

**Proposition 4.5.** There is a degree $k$ map $f : (S^2 \times S^2)^{2n} \to (\mathbb{CP}^2)^{2n}$ $n \geq 2$ if and only if $k$ is even number.

**Proposition 4.6.** For every integer $k$ there is a degree $k$ map $f : \mathbb{CP}^2 \sharp \mathbb{CP}^2$(S^2 \times S^2) \to (\mathbb{CP}^2)^{12}$.

**Theorem 4.8.** There is a degree $k$ map $f : (\mathbb{CP}^2)^{12}(S^2 \times S^2) \to \mathbb{CP}^2 \sharp \mathbb{CP}^2$ if and only if $k$ is the square of an integer or the twice square of an integer.

**Theorem 4.9.** There is $k$ degree map $f : (\mathbb{CP}^2)^{12}(S^2 \times S^2) \to \mathbb{CP}^2 \sharp \mathbb{CP}^2$ if and only if $k \geq 0$ and every prime number $4p - 1$ that divides $k$ occurs even time in the prime factorization of $k$.

**Proof:** Let

$$P = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}.$$  

We are solving the system

$$a^2 + c^2 + 2ge = b^2 + d^2 + 2fh = k \tag{7}$$

$$ab + cd + ef + gh = 0. \tag{8}$$

13
It is clear that when \( k = m^2 + n^2 \) we have solution \( a = d = m, \ b = n, \ c = -n \) and \( q = e = h = k = 0 \).

We are going to prove that \( k \) has the form \( m^2 + n^2 \).

\[
(k - (a^2 + c^2))(k - (b^2 + d^2)) = 4gefh = 2(ab + cd)^2 - 2(ge)^2 - (fh)^2
\]

\[
k^2 + (bc - ad)^2 + (gh - ef)^2 = k(a^2 + b^2 + c^2 + d^2).
\]

It is clear that \( k \geq 0 \). Let \( k \) be minimal integer such there is prime number \( q = 4p - 1 \) such that \( (q^{2p+1}) \parallel k \) and there exist the solution of system. Then \( q^{2p+1} | (bc - ad)^2 + (gh - ef)^2 \) and consequently \( q^{2p+1} | bc - ad \) and \( q^{2p+1} | gh - ef \). Thus, \( q^{2p+2} \) divides the left side, and we have that \( q | a^2 + b^2 + c^2 + d^2 \). Combining this with \( q | ad - bc \) we get \( q(a - d)^2 + (b + c)^2 \) and \( q(a + d)^2 - (b - c)^2 \). Thus, \( a \equiv b \equiv c \equiv d \equiv 0 \mod q \) and \( q | ge, q | gh \). So there exist the solution for \( \frac{k}{q^2} \), contradicting the minimality of \( k \). This proves \( k \) could be written as the sum of two squares of integers. □

Using the same approach we obtain the following theorems. We omit the proofs because they are more or less analogous to the proofs of the previous statements.

**Theorem 4.10.** There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{13} \to \mathbb{C}P^2 \mathbb{C}P^2 \) if and only if \( k \geq 0 \) and every prime number \( 4p - 1 \) that divides \( k \) occurs even time in the prime factorization of \( k \).

**Theorem 4.11.** There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{2n} \to \mathbb{C}P^2 \mathbb{C}P^2 \ n \geq 4 \), if and only if \( k \) is nonnegative integer.

**Theorem 4.12.** There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{13} \times (S^2 \times S^2) \to \mathbb{C}P^2 \mathbb{C}P^2 \) for every integer \( k \).

**Theorem 4.13.** There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{12} \mathbb{C}P^2 \to \mathbb{C}P^2 \mathbb{C}P^2 \) if and only if \( k \geq 0 \) and every prime number \( 4p - 1 \) that divides \( k \) occurs even time in the prime factorization of \( k \).

**Theorem 4.14.** There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{13} \mathbb{C}P^2 \to \mathbb{C}P^2 \mathbb{C}P^2 \) if and only if \( k \geq 0 \) and every prime number \( 4p - 1 \) that divides \( k \) occurs even time in the prime factorization of \( k \).

**Theorem 4.15.** There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{2n} \mathbb{C}P^2 \to \mathbb{C}P^2 \mathbb{C}P^2 \ n \geq 4 \) if and only if \( k \geq 0 \).

**Theorem 4.16.** There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{12} \mathbb{C}P^2 \to \mathbb{C}P^2 \mathbb{C}P^2 \) for every integer \( k \).

**Corollary 4.2.** If \( M \) is a quasitoric 4-manifold such that \( \text{rank} \mathcal{H}^2(M; \mathbb{Z}) \geq 5 \) and \( b + 2c \geq 2 \) then for every integer \( k \) there is a degree \( k \) map \( f : M \to \mathbb{C}P^2 \mathbb{C}P^2 \).

### 4.4 Maps to \( \mathbb{C}P^2 \mathbb{C}P^2 \)

**Proposition 4.7.** There is no non-zero degree map \( f : \mathbb{C}P^{2n} \to \mathbb{C}P^2 \mathbb{C}P^2 \).

**Proposition 4.8.** There is a degree \( k \) map \( f : \mathbb{C}P^2 \mathbb{C}P^2 \to \mathbb{C}P^2 \mathbb{C}P^2 \) iff \( k \neq 4t + 2 \).
Theorem 4.17. For every integer \( k \) there is a degree \( k \) map \( f : (\mathbb{C}P^2)^{\sharp 2} \to \mathbb{C}P^{2\sharp 2}\).

Proof: Let

\[ P = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}. \]

We are looking for the solutions of

\[ a^2 + c^2 - e^2 = f^2 - b^2 - d^2 = k \]  \hspace{1cm} (9)
\[ ab + cd - ef = 0. \]  \hspace{1cm} (10)

For \( k \neq 4t + 2 \) it is known that there are integers \( m \) and \( n \) such that \( k = m^2 - n^2 \). In this case, \( a = b = 0, c = f = m \) and \( d = e = n \) finishes job. For \( k = 4t + 2 \), we could take \( a = 1, b = 2, c = 2t + 1, d = 2t + 2, e = 2t \) and \( f = 2t + 3 \). \( \square \)

Theorem 4.18. For every integer \( k \) there is a degree \( k \) map \( f : \mathbb{C}P^2\sharp(S^2 \times S^2) \to \mathbb{C}P^{2\sharp 2}\).

Proof: Let

\[ P = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}. \]

We are looking for the solutions of the system

\[ a^2 + 2ce = b^2 + 2df = k \]  \hspace{1cm} (11)
\[ ab + cd - ef = 0. \]  \hspace{1cm} (12)

For \( k = 2t \) we could take \( a = b = 0, c = d = t, e = 1 \) and \( f = -1 \). For \( k = 2t + 1 \), we could take \( a = b = d = e = 2t + 1, e = -t \) and \( f = -t - 1 \). \( \square \)

Corollary 4.3. For every quasitoric 4-manifold \( M \) such that \( b \geq 1 \) or \( c \geq 1 \) and every integer \( k \) there is a degree \( k \) map \( f : M \to \mathbb{C}P^{2\sharp 2}\).

5 Orthogonal lattices and maps between connected sums of \( \mathbb{C}P^2 \)

In this section we are focused on the maps between connected sums of \( \mathbb{C}P^2 \). Our main interest is in the mapping degrees

\[ f : (\mathbb{C}P^2)^{\sharp n} \to (\mathbb{C}P^2)^{\sharp n}. \]

Proposition 5.1. There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{\sharp 2n-1} \to (\mathbb{C}P^2)^{\sharp 2n-1}, n \geq 1 \) if and only if \( k \) is the square of an integer.

Theorem 5.1. There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{\sharp 4} \to (\mathbb{C}P^2)^{\sharp 4} \) if and only if \( k \) is nonnegative integer.

Proof: We use the fact that every nonnegative integer could be written as the sum of four squares

\[ k = a^2 + b^2 + c^2 + d^2. \]
Then the matrix
\[
P = \begin{bmatrix}
  a & b & c & d \\
  b & -a & -d & c \\
  c & d & -a & -b \\
  d & -c & b & -a
\end{bmatrix}
\]
guarantees the existence of a degree \( k \) map. □

Theorem 5.1 together with Theorem 4.4 implies that:

**Corollary 5.1.** There is a degree \( k \) map \( f : (\mathbb{C}P^2)^{4n} \to (\mathbb{C}P^2)^{4n} \), \( n \geq 1 \) if and only if \( k \) is nonnegative integer.

The remaining case to determine the mapping degrees \( f : (\mathbb{C}P^2)^{2^{4n}+2} \to (\mathbb{C}P^2)^{2^{4n}+2}, \) \( n \geq 1 \) is still open. Proposition 4.4 implies that the set of all integers that could be written as the sum of two squares belongs to \( D((\mathbb{C}P^2)^{2^{4n}+2}, (\mathbb{C}P^2)^{2^{4n}+2}) \). We could not give the answer even in the case \( f : (\mathbb{C}P^2)^{2^6} \to (\mathbb{C}P^2)^{2^6} \), but we checked directly that there is no degree 3, 7, 11, 15, 19 and various other cases that afford conjecture that \( D((\mathbb{C}P^2)^{2^6}, (\mathbb{C}P^2)^{2^6}) \) is the set of integers that could be written as the sum of squares. Generally, we suppose:

**Conjecture 1.** The set \( D((\mathbb{C}P^2)^{2^{4n}+2}, (\mathbb{C}P^2)^{2^{4n}+2}) \) is the set of nonegative integers such that every prime number \( 4p - 1 \) that divides \( k \) occurs even time in the prime factorization of \( k \).

Conjecture 1 could be reformulated in the following way:

There is an integer matrix \( P = [p_{ij}] \) \( 1 \leq i, j \leq 4n + 2 \) such that
\[
\sum_{j} p_{ij}^2 = k
\]
for every \( i = 1, \ldots, 4n + 2 \) and
\[
\sum_{t} p_{it}p_{jt} = 0
\]
for every \( i \neq j \) iff \( k \) could be written as the sum of two squares?

We could think about the columns of \( P \) as the vectors in \( \mathbb{R}^{4n+2} \). Let observe that matrix \( P \) would satisfy the equality case in the famous Hadamard’s Inequality (see [8], p. 108). This means that if we look at the columns of \( P \) as generators of the lattice (which is the sublattice of \( \mathbb{Z}^{4n+2} \)) the integer \( k \) is its discriminant. Our question is what are the values of discriminants of orthogonal integer lattices in \( \mathbb{R}^{4n+2} \) with equal lengths of generators. The matrices that satisfy equality case of Hadamard’s Inequality are frequently seen in mathematics. Those with entries \(-1\) and \(1\) are called Hadamard’s matrices (see [1]). There are no \((4n + 2) \times (4n + 2)\) Hadamard’s matrices by the result of Paley from 1933. We think that Conjecture 1 is highly connected with studying the orthogonal lattices and their discriminants.

6 Some observations about maps between quasitoric 4-manifolds

In the previous sections, we saw several examples of the sets \( D(M, N) \) when \( M \) and \( N \) are quasitoric 4-manifolds. We are not able to determine this set in general for
quasitoric 4-manifolds but due to Theorem 4.4 and Corollary 4.1 and special cases from Section 4 we could determine it for various manifolds, and in most of cases say something about them.

We could think about problem in this manner. We decompose $M$ and $N$ as the connected sums of $\mathbb{C}P^2$, $\mathbb{C}P^2$, and $S^2 \times S^2$. From the system of Diophantine equations we could notice some general restriction on degree $k$ map, for example that should be positive or negative or even or the sum of certain number of squares or no restriction. Then we are working backward. We studied maps to $\mathbb{C}P^2$, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ etc. We could combine this results by repeatedly application of Theorem 4.4 and hope to produce a degree $k$ map $f : M \to N$. This is not always possible to do, but this is the algorithm for generating new examples.

**Example 6.1.** Let $l$, $m$ and $n$ be positive integers such that $l \geq m + n$, then there is a degree $k$ map

$$f : (S^2 \times S^2)^l \to (\mathbb{C}P^2)^m \# (\mathbb{C}P^2)^n \# (S^2 \times S^2)^m$$

if and only if $k$ is even number.

**Example 6.2.** Let $m$ and $n$ be positive integers such that $m \geq n$, then there is a degree $k$ map

$$f : (\mathbb{C}P^2)^{2m+1} \to (\mathbb{C}P^2)^m (S^2 \times S^2)^n$$

iff $k$ is the square of an integer.

**Example 6.3.** Let $l$, $m$, $n$ and $p$ be positive integers such that $p \geq n$ and $l \geq m$, then for every integer $k$ there is a degree $k$ map

$$f : (\mathbb{C}P^2)^p \# (\mathbb{C}P^2)^n (S^2 \times S^2)^l \to (\mathbb{C}P^2)^m \# (\mathbb{C}P^2)^n \# (S^2 \times S^2)^m$$

Having this idea in mind we get the following theorem:

**Theorem 6.1.** Let $M$ be a given quasitoric 4-manifold. Then there exist integers $a_0$, $b_0$ and $c_0$ such that for every integers $a$, $b$ and $c$ such that $a \geq a_0$, $b \geq b_0$ and $c \geq c_0$, and

$$D((\mathbb{C}P^2)^a \# (\mathbb{C}P^2)^b \# (S^2 \times S^2)^c, M) = \mathbb{Z}.$$ 

This theorem states that there are infinitely many manifolds that could be mapped to $M$ with any degree.

## 7 Concluding remarks

In the previous sections we made the numerous calculations. We used the number theory for determination of the mapping degree between special class of manifolds. According to the result of Duan and Wang all these observations are relevant at least as the necessary condition in more general case of $(n - 1)$-connected $2n$ manifolds. One direction for further research is to connect our result with higher dimensional manifolds that have the same intersection forms. Duan and Wang also gave sufficient condition that one need to check for the existence of degree $k$ map. Our results showed that even in the simplest case of 4-manifolds, it is hopeless to directly check conditions.
If we stay on 4-manifolds, the famous result [19] of Freedman gives classifications of 4-manifolds in the terms of their intersection forms. We have already noticed the connection of the problem with the problems about lattices and quadratic forms. So it is naturally to study problem from this point of view. We expect that progress in any of this area may lead to progress in our problem and vice versa.

The partially results we got in the case of quasitoric 4-manifolds give hope to fully answer the question for this class of manifolds. We think that other more sophisticated technics of the number theory could at least enlarge our knowledge about the mapping degrees.

Acknowledgements

The author is grateful to Rade Živaljević and Vladimir Grujić for fruitful discussion and comments.

References

[1] S. S. Agaian, *Hadamard Matrices and Their Applications*, Springer-Verlag (1985)
[2] Torsten Asselmeyer-Maluga and Carl H. Brans, *EXOTIC SMOOTHNESS AND PHYSICS - Differential Topology and Spacetime Models*, World Scientific (2007)
[3] H. J. Baues, *The degree of maps between certain 6-manifolds*, Compositio Math. 110 (1998) 51-64.
[4] Glen Bredon, *Topology and Geometry*, Springer-Verlag, 1993.
[5] I. Berstein and A. L. Edmonds, *On the construction of branched coverings of low-dimensional manifolds*, Transactions of American Mathematical Society 247 (1979), 87-124.
[6] V. Buchstaber, T. Panov and N. Ray, *Spaces of polytopes and cobordisms of quasitoric manifolds*, Moscow Math. Journal 7 (2007), no. 2, 219-242.
[7] V. Buchstaber and T. Panov, *Torus Actions and their applications in topology and combinatorics*, AMS University Lecture Series, volume 24, (2002).
[8] P. S. Bullen, *A Dictionary of Inequalities*, Chapman & Hall (1998)
[9] David M. Burton, *ELEMENTARY NUMBER THEORY*, McGraw-Hill Higher Education (2007)
[10] John H. Conway, *The Sensual Quadratic Form*, The Mathematical Association of America (1997)
[11] M. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991), no. 2, 417451.
[12] Haibao Duan, *Self-maps on the Grassmannian of complex structure*, Compositio Math. 132 (2002), 159-175.
[13] H. Duan and S. Wang, *Non-zero Degree Maps between 2n-manifolds*, [arXiv:math/0402119v1 [math.GT]]
[14] H. Duan and S. Wang, *The degrees of maps between manifolds*, Mathematische Zeitschrift

[15] Allan L. Edmonds, *Deformation of Maps to Branched Coverings in Dimension Two*, The Annals of Mathematics, Second Series, Vol. 110, No. 1 (1979), 113-125.

[16] Allen Hatcher, *Algebraic Topology*, Cambridge University Press (2001)

[17] F. Hirzebruch, *Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten*, Math. Ann. 124 (1951), 77-86.

[18] S. Feder and S. Gitler, *Mappings of quaternionic projective space*, Bol. Soc. Mat. Mexicana 18 (1973), 33-37.

[19] M. H. Freedman, *The topology of 4-manifolds*, J. Diff. Geom. 17 (1982), 337-453.

[20] H. Kneser, *Glätung von Flächenabbildungen*, Math. Ann. (1928), 609-617.

[21] C. Hayat-Legrand, S. Wang and H. Zieschang, *Minimal Seifert manifolds*, Math. Ann. 308 (1997), 673-700.

[22] C. Hayat-Legrand, S. Wang and H. Zieschang, *Computer calculation of the degree of maps into the Poincaré homology sphere*, Experimental Mathematics 10 (2001), 497-508.

[23] M. Kreck and D. Crowlet, *Hirzebruch surfaces*, Bulletin of the Manifold Atlas, 2011, 19-22.

[24] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Berlin Heidelberg New York

[25] Melvyn B. Nathanson, *Elementary Methods in Number Theory*, Springer (1991)

[26] P. Orlik and F. Raymond, *Actions of the torus on 4-manifolds*, Trans. Amer. Math. Soc. 152 (1970), 531-559.

[27] P. Sankaran and S. Sarkar, *Degrees of maps between Grassmann manifolds*, to appear in Osaka J. Math.

[28] James J. Tattersall *Elementary number theory in nine chapters*, Cambridge University Press (1999)

[29] Schicheng Wang, *Non-zero Degree Maps between 3-manifolds*, arXiv:math/0304301v1 [math.GT]

[30] Günter M. Ziegler, *Lectures on Polytopes*, Springer (1995)

DORDE BARALIĆ, Mathematical Institute SASA, Kneza Mihaila 36, p.p. 367, 11001 Belgrade, Serbia
E-mail address: djbaralic@mi.sanu.ac.rs