A SIMPLE CONSTRUCTION OF THE FRACTIONAL BROWNIAN MOTION.

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ABSTRACT. In this work we introduce correlated random walks on $\mathbb{Z}$. When picking suitably at random the coefficient of correlation, and taking the average over a large number of walks, we obtain a discrete Gaussian process, whose scaling limit is the fractional Brownian motion. We have to use two radically different models for both cases $\frac{1}{2} \leq H < 1$ and $0 < H < \frac{1}{2}$. This result provides an algorithm for the simulation of the fractional Brownian motion, which appears to be quite efficient.

1. Introduction

The fractional Brownian motion appears to be a very natural object, for its three fundamental and characteristic features of being a continuous Gaussian process, self-similar and with stationary increments. By self-similar, we mean that there exists a real number $H \in [0, 1]$, such that the finite-dimensional distributions of $\{T^{-H}B_H(Tt), t \geq 0\}$ do not depend on $T$. The parameter $H$ is called the Hurst parameter or the index of self-similarity. Under these conditions, it is not hard to check that the covariance function must have the following form:

$$E[B_H(s)B_H(t)] = \frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H}).$$

A major consequence of this fact is that if we introduce the sequence of increments $G_H(j) = B_H(j) - B_H(j - 1), j = 1, 2, ...$, also called fractional Gaussian noise, we note that they are strongly correlated (for $H \neq \frac{1}{2}$). More precisely,

$$E[G_H(j)G_H(j + k)] \sim_{k \to \infty} H(2H - 1)k^{2H-2} \quad (1)$$

We denote the two radically different behaviours on both sides of $\frac{1}{2}$: for $H < 1/2$ the increments are all negatively correlated, which corresponds to a chaotic behavior, whereas for $H > 1/2$ the positive correlation between the increments corresponds to a more disciplined behaviour. We refer to Chapter 7 of Samorodnitsky and Taqqu \cite{1} for more information about fractional Brownian motion.

There is a well-known representation introduced by Mandelbrot and Van Ness \cite{8} of the fractional Brownian motion as an integral of a kernel function with respect to the usual Brownian motion. The method of approximation that uses the discretization of this integral leads to an algorithm obliging to store a lot of data in memory, and to deal with non smooth functions in the case $0 < H < \frac{1}{2}$. This inconvenience is nicely discussed by Carmona and Coutin in \cite{3}, and they propose a way to reduce it.

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The construction we propose is based on correlated random walks: it consists in discrete processes such that the law of each move is ruled by the value of the previous move. We refer to [1] and the references therein for more information about these processes. We note that the decay of correlation for such processes is exponential, but that a mixture of these processes, adapted to the value of the index of self-similarity, leads to walks whose correlations satisfy (1). The superposition of a large number of such walks yields a discrete Gaussian process whose correlations fulfill the conditions of Taqqu [13], so that its scaling limit is the fractional Brownian motion. We have to distinguish both cases \( \frac{1}{2} \leq H < 1 \) and \( 0 < H < \frac{1}{2} \). In the first case, the equivalence relation (1) is the only condition to check. The case \( 0 < H < \frac{1}{2} \) is more delicate: a compensation relation between the correlations has to be satisfied simultaneously. That leads us to introduce two different types of correlated random walks for both cases.

This construction reminds the constructions using renewal processes of Mandelbrot [8], Taqqu and Levy [12, 7]. It reminds even more the construction coming from traffic modeling by Taqqu, Willinger and Sherman [14]. To our knowledge, all these methods are restricted to the case \( \frac{1}{2} \leq H < 1 \). In that case, we shall discuss the differences between our construction and [14]. Finally, we want to mention that random media has already been used by Kesten and Spitzer [6] to get convergences towards certain self-similar processes with index of self-similarity bigger than \( \frac{1}{2} \).

In the last part, we discuss the speed of the convergence stated in the previous parts. We indicate how to deduce a quite fast algorithm that does not ask to keep a lot of datas in memory.

2. The case \( 1/2 \leq H < 1 \)

2.1. The correlated random walk. We introduce first our basic tool: the correlated random walk with persistence \( p \). It is a process evolving on \( \mathbb{Z} \) by jumps of \(+1\) or \(-1\), whose probability of making the same jump as the previous one is \( p \):

**Definition 1.** For any \( p \in [0,1] \), the correlated random walk \( X_p \) with persistence \( p \) is a \( \mathbb{Z} \)-valued discrete process, such that :

- \( X_0 = 0 \), \( P(X_0^p = -1) = 1/2 \), \( P(X_0^p = 1) = 1/2 \).

- \( \forall n \geq 1, \varepsilon^p_n := X_n^p - X_{n-1}^p \) equals \( 1 \) or \(-1\) a.s.

- \( \forall n \geq 1, P(\varepsilon^p_{n+1} = \varepsilon^p_n | \sigma(X_k^p, 0 \leq k \leq n)) = p \).

This process is not Markovian, but if we define a state as the position of the process on \( \mathbb{Z} \), coupled with the sign of its last jump, we have to deal with a Markov process on \( \mathbb{Z} \times \{-1,1\} \). In fact, this process consists in alternate falls and rises of i.i.d. geometric length with parameter \( p \).

We can compute the correlations between two steps distant from \( n \):

**Proposition 1.** \( \forall m \geq 1, n \geq 0, E[\varepsilon^p_m \varepsilon^p_{m+n}] = (2p - 1)^n \)

Proof: \( \forall n \geq 1, \) 
\[ E[\varepsilon^p_{n+1} \sigma(X_k, 0 \leq k \leq n)] = E[\varepsilon^p_{n+1} 1_{\varepsilon^p_n = 1} \sigma(X_k, 0 \leq k \leq n)] + E[\varepsilon^p_{n+1} 1_{\varepsilon^p_n = -1} \sigma(X_k, 0 \leq k \leq n)] = (2p - 1) \varepsilon^p_n - (2p - 1)(-\varepsilon^p_n) = (2p - 1)\varepsilon^p_n. \]

Consequently, conditioning by \( \sigma(X_k, 0 \leq k \leq m + n) \), we get 
\( \forall m \geq 1, n \geq 0, E[\varepsilon^p_m \varepsilon^p_{m+n}] = (2p - 1) E[\varepsilon^p_m \varepsilon^p_{m+n}] \).

The result is then obtained by recurrence. \( \square \)
We now introduce an extra randomness in the persistence. We first denote by $P^p$ the law of $X^p$ for a given $p$. Now, considering a probability measure $\mu$ on $[0,1]$, we will call $P^\mu$, the annealed law of the correlated walk associated to $\mu$, i.e. the measure on $\mathbb{Z}^N$ defined by $P^\mu := \int_0^1 P^p d\mu(p)$.

Remark: unlike the situation in [13], the persistence does not depend on the level. Only one coin toss, according to $\mu$, decides for the whole environment.

Let $X^\mu$ be a process of law $P^\mu$. Let us now introduce the notation $\varepsilon^\mu_n := X^\mu_n - X^\mu_{n-1}$. From Proposition 1 we get the straightforward result:

**Proposition 2.** $\forall m \geq 1, n \geq 0, E[\varepsilon^\mu_n \varepsilon^\mu_{m+n}] = \int_0^1 (2p-1)^n d\mu(p)$.

### 2.2. Statement and proof of the result.

The goal now is to introduce a probability measure $\mu$ leading to the same equivalent as (1), mentioned in the introduction, so that by taking the average over a large number of trajectories, we approximate a discrete Gaussian process having the same properties as in [13], whose scaling limit is the fractional Brownian motion:

**Theorem 1.** Let $H \in ]1/2,1[$. Denote by $\mu^H$ the probability on $[1/2,1]$ with density $(1-H)2^{3-2H}(1-p)^{-2H}$.

Let $(X^\mu_i)_{i \geq 1}$ be a sequence of independent processes of law $P^\mu$,

$$\mathcal{L} \lim_{N \to \infty} \mathcal{L} \lim_{M \to \infty} c_H \frac{X^\mu_{[N]1} + \ldots + X^\mu_{[M]}}{N^{H} M^{1/2}} = B_H(t)$$

with $c_H = \sqrt{\frac{H(2H-1)}{4(3-2H)}}$.

$L$ means convergence in the sense of the finite-dimensional distributions, and $\mathcal{L}^D$ means convergence in the sense of the weak convergence in the Skorohod topology on $D[0,1]$, the space of cadlag functions on $[0,1]$.

Proof: The central limit theorem implies that $\mathcal{L} \lim_{M \to \infty} X^\mu_{[N]1} + \ldots + X^\mu_{[M]}$ is a discrete centered Gaussian process $(Y^H_k)_{k \geq 1}$, with stationary increments $G^H_k := Y^H_{k+1} - Y^H_k$ with $E[G^H_k] = 0$, $E[(G^H_k)^2] = 1$ and

$$\forall i,n \geq 0, \quad r(n) := E[G^H_i G^H_{i+n}] = (2 - 2H)2^{2-2H} \int_{1/2}^1 (2u-1)^n(1-u)^{-2H} du$$

$$r(n) = (2 - 2H) \int_0^1 v^n(1-v)^{-2H} dv$$

$$= (2 - 2H) \frac{\Gamma(n+1)\Gamma(2-2H)}{\Gamma(n+3-2H)} + O(\frac{1}{n})$$

$$\sim_{n \to \infty} \frac{\Gamma(3-2H)}{n^{2-2H}} \frac{1}{v^H} \frac{H(2H-1)}{n^{2-2H}}$$

So that,

$$E[c_H^2 (G^H_i + \ldots + G^H_j)^2] = c_H^2 \sum_{i=1}^N \sum_{j=1}^N r(|i-j|)$$

$$= c_H^2 (r(0) + \sum_{i=1}^{N-1} [r(0) + 2 \sum_{k=1}^i r(k)])$$

$$\sim_{n \to \infty} N^{2H}$$
(the last step consists simply in two successive comparisons between sums and integrals). A direct application of \( [13] \) (lemma 5.1) allows to conclude.

We can give also an analog statement for \( H = 1/2 \):

**Theorem 2.** Denote by \( \mu^\frac{1}{2} \) the uniform probability on \([\frac{1}{2},1] \).

Let \( (X^\mu_n)_{n \geq 1} \) be a sequence of independent processes of law \( P^\mu \),

\[
\mathcal{L} \xrightarrow{N \to \infty} \mathcal{L} \xrightarrow{M \to \infty} \frac{X^\mu_{\lfloor Nt \rfloor} + \ldots + X^\mu_{\lfloor Mt \rfloor}}{\sqrt{N \log N \sqrt{M}}} = B(t)
\]

where \( B \) is the classical Brownian motion, and \( c_\frac{1}{2} = \frac{1}{\sqrt{2}} \).

Proof: The scheme is the same as in Theorem 1. The difference here is that \( r(n) = 2 \int_0^1 (2u - 1)^n du = \frac{(1)}{n+1} \).

So that, \( r(0) + \sum_{i=1}^{N-1} [r(0) + 2 \sum_{k=1}^{i} r(k)] \sim 2N \log N \).

We conclude again, applying \([13]\) (lemma 5.1).

Remark: The order of the limits in both theorems is of big importance: the limit in the reverse order would bring 0, as far as for any fixed \( p \), a correlated random walk satisfies a central limit theorem with normalization \( \sqrt{\frac{N}{\log N}} \) (see \([4]\)).

We want now to compare previous theorems with the result of \([14]\): in \([14]\), the limit theorem consists also in taking the scaling limit of the average over a large number (tending to infinity) of i.i.d. copies of processes which are a succession of falls and rises. In the case of \([14]\), falls and rises are all independent with infinite variance.

In our setting, even if the lengths of falls and rises have finite variance under each \( P^p \) (they are geometric), the laws under \( P^\mu \) of the falls and the rises (which are all the same) have infinite variance: indeed, if we denote by \( L \) the length of a rise,

\[
P^\mu(L \geq n) = \int_0^1 P^p(L \geq n) d\mu^p(p) = \int_0^1 P^p(1-p)^{1-2p}dp \sim \Gamma(3-2H(n^2-2H)
\]

(for \( H = 1/2 \), we get \( P^\mu(L \geq n) \sim \frac{1}{n} \))

and we get the same kind of tail as in \([14]\).

The difference lies in the fact that the falls and the rises are not independent: indeed, if the first rise is short, it probably means that the environment \( p \) is small, so that the following fall will be probably short also. More precisely, if \( L_1 \) and \( L_2 \) denote the lengths of the first rise (resp. fall) and of its following fall (resp. rise),

\[
P^\mu(L_1 \geq n, L_2 \geq m) = E_{\mu^\mu}[P^p(L_1 \geq n, L_2 \geq m)] = E_{\mu^\mu}[p^{n+m}]
\]

\[
\sim \Gamma(3-2H(n+m)^2-2H)
\]

\( L_1 \) and \( L_2 \) are therefore not independent.

Note: Theorems 1 and 2 can be extended to any probability measure with moments equivalent to \( \frac{1}{n^2-2H} \), where \( L \) is a slowly varying function. The additional arguments can be found in \([13]\) and rely mainly on Karamata’s theorem in order to replace the naive comparison between sums and integrals we used at the end of
the proof. We chose $\mu^H$ in our statement for the simple formula it yields for $c_H$ and because it is the law of $1 - \frac{U^{2H}}{2}$ where $U$ is uniform on $[0,1]$, which makes it easy to simulate. At the end of the article, we will discuss the practical interest of other measures.

3. The case $0 < H < 1/2$

3.1. The alternating correlated random walk. We first remark that the correlated random walks of section 2, cannot provide negative correlations for the increments, at least for increments separated by an even time interval. The best we can hope is to get an alternate sign for the correlations. In order to get a process with always negative correlations (except for variances), we will consider the sequence of the sum of two consecutive increments. More precisely, if we consider, with the notations of section 2, the sequence $(\varepsilon_{2n+1}^p + \varepsilon_{2n+2}^p)_{n \geq 0}$, for any $p$ less than $1/2$, we get indeed a sequence of negatively correlated variables and it is also possible to exhibit a probability on $[0,1]$ such that the equivalence relation (1) mentioned in the introduction will be satisfied.

But, in the case $0 < H < 1/2$, this condition alone does not ensure a scaling limit for the scale $N^H$ (which is that time smaller than $\sqrt{N}$). It has to be allied to a compensation relation between all the correlations. We refer to [13] (section 5) for the statement of this condition, and we will explicit it further.

It is the reason why we have to introduce a different kind of walk, we will call alternating correlated random walk with persistence $p$. It is a process evolving on $\mathbb{Z}$ by jumps of +1 or -1, whose probability of making the same jump as the previous one is alternately $p$ and 0. In other words, one jump over two is the opposite of the previous one:

**Definition 2.** For any $p \in [0,1]$, the alternating correlated random walk $\tilde{X}_n^p$ with persistence $p$, is a $\mathbb{Z}$-valued discrete process, such that:

- $\tilde{X}_0^p = 0$, $P(\tilde{X}_1^p = -1) = 1/2$, $P(\tilde{X}_1^p = 1) = 1/2$.
- $\forall n \geq 1$, $\tilde{\varepsilon}_n^p := \tilde{X}_n^p - \tilde{X}_{n-1}^p$ equals 1 or -1 a.s.
- $\forall n \geq 1$, $P(\tilde{\varepsilon}_{2n}^p = \tilde{\varepsilon}_{2n-1}^p | \sigma(\tilde{X}_k^p, 0 \leq k \leq 2n-1)) = p$.
- $\forall n \geq 1$, $\tilde{\varepsilon}_{2n+1}^p = -\tilde{\varepsilon}_{2n}^p$.

As suggested in the introduction of this section, we will be actually interested by the process

$$Y_n^p := \frac{\tilde{X}_{2n}^p}{2\sqrt{p}}$$

(the importance of this normalization will appear later). The trajectories of this process take only two values, which are either $-1/\sqrt{p}$ and 0 or 0 and $1/\sqrt{p}$. The successive lengths of the time intervals during which the process stays on each value are independent and geometric with parameter $1 - p$.

We compute now the correlations of the increments of $Y_n^p$ i.e. of $\delta_n^p := Y_n^p - Y_{n-1}^p = \frac{1}{\sqrt{p}}(\varepsilon_{2n-1}^p + \varepsilon_{2n}^p)$ for $n \geq 1$.

**Proposition 3.** $\forall n \geq 1$, $E[(\delta_n^p)^2] = 1$

- $\forall n \geq 1$, $n \geq 1$, $E[\delta_n^p \delta_{m+n}] = p(1 - 2p)^{n-1}$

**Proof:** As in Proposition 1, everything is based on the following facts: $\forall n \geq 1$, ...
Proposition 4. We get the straightforward result:

\[ E[\xi_{2n}^p|\sigma(\hat{X}_k^p, 0 \leq k \leq 2n - 1)] = (2p - 1)\xi_{2n-1}^p, \]

\[ E[\xi_{2n+1}^p|\sigma(\hat{X}_k^p, 0 \leq k \leq 2n)] = -\xi_{2n}^p, \]

\( \forall m \geq 1, \)

\[ E[\delta_{m}^p] = \frac{1}{4p}(E[(\xi_{2m-1}^p)^2] + E[(\xi_{2m}^p)^2] + 2E[\xi_{2m-1}^p\xi_{2m}^p]) = \frac{1}{4p}(2 + 2(2p - 1)) = 1 \]

\( \forall m \geq 1, n \geq 1, \) by successive conditionings,

\[ E[\delta_{m}^p\delta_{m+n}] = \frac{1}{4p}(1 - 2p)^{n-1}E[(\xi_{2m-1}^p + \xi_{2m}^p)(\xi_{2m+1}^p + \xi_{2m+2}^p)] = \frac{1}{4p}(1 - 2p)^{n-1}(-1 - 2(2p - 1) - (2p - 1)^2) = -p(1 - 2p)^{n-1} \]

We can already note the following (compensation) relation:

\[ E[(\delta_{m}^p)^2] + 2 \sum_{n \geq 1} E[\delta_{m}^p\delta_{m+n}] = 0 \quad (2) \]

Again we introduce an extra randomness in the persistence. We first denote by \( Q^p \) the law of \( Y^p \) for a given \( p \). Now, considering a probability measure \( \mu \) on \([0,1]\), we will call \( Q^\mu \), the annealed law of the correlated walk associated to \( \mu \), i.e. the measure on \( \mathbb{Z}^N \) defined by \( dQ^\mu := \int_0^1 Q^p d\mu(p) \).

Let \( Y^\mu \) be a process of law \( Q^\mu \). We introduce \( \delta_n^\mu := Y_n^\mu - Y_{n-1}^\mu \). From Proposition 3 we get the straightforward result:

**Proposition 4.** \( \forall m \geq 1, E[(\delta_{m}^\mu)^2] = 1. \)

\( \forall m \geq 1, n \geq 1, E[\delta_{m}^\mu \delta_{m+n}^\mu] = -\int_0^1 p(1 - 2p)^{n-1} d\mu(p). \)

Note: As the compensation relation (2) is satisfied "\( p \) by \( p \)" it remains true for the annealed correlations.

The goal now is to introduce a probability measure \( \mu \) leading to the same equivalent as (1), mentioned in the introduction.

### 3.2. Statement and proof of the result.

We proceed as in previous section:

**Theorem 3.** Let \( H \in [0,1/2] \).

Denote by \( \mu^H \) the probability on \([0,1/2]\) with density \((1 - 2H)^{2-2H}p^{-2H}\).

Let \((Y^\mu)^{i,j}_{i \geq 1}\) be a sequence of independent processes of law \( P^\mu_H \),

\[ \mathcal{L}^P \lim_{N \to \infty} \mathcal{L} \lim_{M \to \infty} c_H \frac{Y_{[NH]}^{H,1} + \ldots + Y_{[NH]}^{H,M}}{N^H \sqrt{M}} = B_H(t) \]

with \( c_H = \sqrt{\frac{2H}{1-(2-2H)}} \).

Proof: The central limit theorem implies that \( \mathcal{L} \lim_{M \to \infty} \frac{Y_{[NH]}^{H,1} + \ldots + Y_{[NH]}^{H,M}}{\sqrt{M}} \) is a discrete centered Gaussian process \((Z^H_k)_{k \geq 1}\), with stationary increments \( W^H_k := Z^H_{k+1} - Z^H_k \) with \( E[W^H_k] = 0 \), \( E[(W^H_k)^2] = 1 \) and

\( \forall i, n \geq 0, \quad r(n) := E[W^H_k W^H_{i+k}] = -(1 - 2H)2^{1-2H} \int_0^\frac{i}{2} (1 - 2u)^n u^{1-2H} du \)
\[ r(n) = -\frac{(1 - 2H)}{2} \int_0^1 (1 - v)^n v^{1 - 2H} dv \]
\[ = -\frac{(1 - 2H)}{2} \frac{\Gamma(n + 1) \Gamma(2 - 2H)}{\Gamma(n + 3 - 2H)} \]
\[ \sim_{n \to \infty} -\frac{(1 - 2H)}{2} \frac{\Gamma(n + 3 - 2H)}{\Gamma(n + 1) \Gamma(2 - 2H)} \frac{1}{n^{2 - 2H}} = \frac{1}{c_H^2} H(2H - 1) \]

So that,
\[ E[c_H^2(G_H^1 + \ldots + G_H^N)^2] = c_H^2 \sum_{i=1}^{N} \sum_{j=1}^{N} r(|i - j|) \]
\[ = c_H^2 (r(0) + \sum_{i=1}^{N-1} [r(0) + 2 \sum_{k=1}^{i} r(k)]) \]
\[ = c_H^2 (r(0) - 2 \sum_{i=1}^{N-1} \sum_{k=i+1}^{\infty} r(k)) \]
\[ \sim_{n \to \infty} N^{2H} \]

The last equality comes from the compensation relation (2), and the last step consists simply in two successive comparisons between sums and integrals. A direct application of [13] (lemma 5.1) allows to conclude.

4. Practical aspects

As mentioned above, taking the limit in reverse order, yields trivial processes. The question is: for a given number of steps \( N \), what kind of \( M \) are we supposed to take in order to approximate the right limit? We restrict our study to the law of \( B_H(1) \). We base our study on Berry-Esseen’s inequality, applied to the sequence of i.i.d. variables with law \( c_H X_N^{h_H} / N^H \) (resp. \( c_H Y_N^{h_H} / N^H \)) for \( H \geq 1/2 \) (resp. \( H < 1/2 \)).

What remains to compute, is an upper bound for the third moment of the absolute value of these variables.

We begin with the case \( H > 1/2 \).

Proposition 5. For \( H > 1/2 \), for \( N \) large enough,

\[ E[(c_H |X_N^H|)^3] \leq D_H N^{1-H}, \]

with \( D_H = \sqrt{\frac{6(2H-1)}{(H+1)(2H+1)}} \times c_H \).
Nota Bene: we express $D_M$ in terms of $c_M$, in order to have a formula that works for other measures, as it will be useful in the following.

Proof: We omit here the superscripts $\mu^H$ in the variables $X^\mu_{[t]}$'s and $\varepsilon^\mu_{[t]}$'s. $X_N = \sum_{k=1}^N \varepsilon_k$, with:
- $\varepsilon_k$ are Bernoulli(1/2),
- $\frac{c^2}{H} \text{Cov}(\varepsilon_k, \varepsilon_l) = r(|k-l|), $ with $r(n) \sim \frac{H(2H-1)}{n^{2H}}$.
We use Cauchy-Schwarz inequality to get:

$$E[(\frac{c_H|X_N|}{H})^3] \leq E[(\frac{c_H|X_N|}{H})^2] \frac{1}{2} E[(\frac{c_H|X_N|}{H})^4]^{1/2}$$

$$\sim \frac{c^2}{H} \sum_{1 \leq i \leq N} E[\varepsilon_i \varepsilon_i \varepsilon_i \varepsilon_i]^{1/2}$$

(using that the variance converges to 1 for large $N$).

Assume $i_4 \leq i_3 \leq i_2 \leq i_1$, we get, as in Proposition 1, by successive conditionings:

$$E[\varepsilon_{i_1} \varepsilon_{i_2} \varepsilon_{i_3} \varepsilon_{i_4}] = (2p-1)^{(i_1-i_2)+(i_3-i_4)}$$

So that,

$$E[\varepsilon_i \varepsilon_i \varepsilon_i \varepsilon_i] = r((i_1-i_2) + (i_3-i_4))$$

We see by using the equivalence between sums and integrals, as we did at the end of the proof of Theorem 1, that the sum is equivalent to:

$$4! \times \int_0^N \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} r((x_1-x_2) + (x_3-x_4)) dx_4 dx_3 dx_2 dx_1$$

$$\sim \frac{4!}{c^2} \times H(2H-1) \int_0^N \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} ((x_1-x_2) + (x_3-x_4))^{2H-2} dx_4 dx_3 dx_2 dx_1$$

$$= \frac{6}{c^2} \frac{(2H-1)N^{2H+2}}{(H+1)(2H+1)}$$

Now applying Berry-Esseen’s inequality, the error of the distribution function of the marginal at time 1, is dominated by $0.65 \times D_M \frac{N^{1-H}}{\sqrt{M}}$ (using the constant 0.65 of \[3\]), as far as the third moment is much bigger than the power $\frac{3}{2}$ of the variance).

We deduce that Theorem 1 remains true as soon as $M(N) \to \infty$ as $N \to \infty$ at a faster rate than $N^{2-2H}$.

**Corollary 1.** Let $M$ be a function on the integers such that $\frac{M(N)}{N^{2-2H}}$ tends to $\infty$,

$$\mathcal{L}^D \lim_{N \to \infty} c_H \frac{X^\mu_{[N]} + \ldots + X^\mu_{[M(N)]}}{\sqrt{M(N)}} = B_H(t)$$

Proof: Using the generalization of Berry Esseen’s inequality to multidimensional variables \[4\], we get the convergence for finite-dimensional marginals.

To get the weak convergence, we cannot use directly \[3\] as in previous section, and we prove the tightness of the family of processes, by checking Billingsley’s criteria \[2\] (Theorem 15.6):

Denote by $S_N(t) := \frac{X_{[2]}^{\mu,H} + \ldots + X_{[N]}^{\mu,H,M(N)}}{\sqrt{M(N)}}$. 
Let $1 \geq t_2 \geq t \geq t_1 \geq 0$, and $k \in \mathbb{N}$,

$$J_N(k, t_2, t, t_1) := E[|\frac{S_{N(t_2)} - S_{N(t)}}{N^{\frac{1}{2}}} - \frac{S_{N(t)} - S_{N(t_1)}}{N^{\frac{1}{2}}}|^k]$$

\[
\leq \frac{1}{N^{2H}} E[|S_{N(t_2) - t)|^2k]^\frac{1}{2} E[|S_{N(t - t_1)|^2k]^\frac{1}{2} \\
\forall k \in \mathbb{N}, \quad E[S_{N(t)|^{2k}}] = \frac{1}{M(N)^k} E[(X_{\frac{N}{|N|}}^{\mu, 1} + \ldots + X_{\frac{N}{|N|}}^{\mu, M(N)})^{2k}] \]

When we develop the polynomial with degree $2k$ inside the expectation, we notice that only the monomials of the type $(X^1)^{2\alpha_1} \ldots (X^M)^{2\alpha_M}$ have a non null contribution, as far as the $X^i$’s are centered and independent. This set has cardinal $O(M^k)$.

Now, we find, by the same means than in Proposition 1, that

$$\forall \alpha_1 \geq 1, \quad E[X_N^{2\alpha_1}] = O(N^{2\alpha_1 + (2H - 2)})$$

This leads to $E[S_N(t)^{2k}] = O((N^t)^{2k + (2H - 2)k}) = O((N^t)^{2Hk})$.

Hence, for some positive constant $C$,

\[
J_N(k, t_2, t, t_1) \leq C(t_2 - t)^{2HK} (t - t_1)^{2Hk} \leq C(t_2 - t_1)^{2Hk}.
\]

We choose $k > \frac{1}{2H}$ in order to satisfy Billingsley’s criteria. \[\square\]

We conjecture that this result remains true in the case of $[14]$ which proposes an answer to the question asked at the end of $[14]$.

We do the same for the case $H = \frac{1}{2}$:

**Proposition 6.** For $N$ large enough,

$$E[(\frac{c_1}{\sqrt{N \log N}}\frac{X_N^{\frac{1}{2}}}{\sqrt{N \log N}})^3] \leq c_2 \times \frac{\sqrt{N}}{\log N}$$

Proof: As in previous proposition, $X_N = \sum_{k=1}^{N} \varepsilon_k$, with:

- $\varepsilon_k$ are Bernoulli(1/2),
- $\frac{c_1^2}{2} \text{Cov}(\varepsilon_k, \varepsilon_l) = r(|k - l|)$, with $r(n) \sim \frac{1}{n}\text{ as } n \to \infty$.

Similarly as in previous proposition, $E[(\frac{c_1}{\sqrt{N \log N}}\frac{X_N^{\frac{1}{2}}}{\sqrt{N \log N}})^4]^{\frac{1}{2}}$ is equivalent to:

$$\sim \frac{c_2^2}{N \log N} (\frac{4!}{c_2^2} \int_0^N \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} r((x_1 - x_2) + (x_3 - x_4)) dx_4 dx_3 dx_2 dx_1) \frac{1}{2}$$

But,

$$\int_0^N \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} \frac{1}{(x_1 - x_2) + (x_3 - x_4)} dx_4 dx_3 dx_2 dx_1 = \frac{N^3}{6}$$

Now applying Berry-Esseen’s inequality, the error of the distribution function of the marginal at time 1, is dominated by $1.3 \times \frac{\sqrt{N}}{\log N \sqrt{M}}$.

**Corollary 2.** Let $M$ be a function on the integers such that $\frac{M(N)}{\sqrt{N \log(N)}}$ tends to $\infty$,

$$\mathcal{L}^D \lim_{N \to \infty} c_1 \frac{X_{\frac{N}{|N|}}^{\frac{1}{2}} + \ldots + X_{\frac{N}{|N|}}^{\frac{1}{2}, M(N)}}{\sqrt{M(N) \times N \log(N)}} = B(t)$$

Finally, we treat the case $H < \frac{1}{2}$.
Proposition 7. For $H < 1/2$, for $N$ large enough,

$$E \left[ \left( \frac{CH[Y_N^H]}{N^H} \right)^3 \right] \leq D_H N^{\frac{1}{2} - H},$$

with $D_H = \sqrt{\frac{2H}{2H+1}} \times c_H$.

Proof: The proof is more delicate than in Proposition 5, because the computation of the fourth moment makes appear compensations (as in the variance computation), and we have to treat them first "p by p", before integrating against $\mu^H$.

Assume $i_4 \leq i_3 \leq i_2 \leq i_1$, we get, as in Proposition 1, by successive conditionings:

$$E[\delta^p(i_1-i_2)\delta^p(i_2-i_3)\delta^p(i_3-i_4)] = r_p(i_1-i_2)r_p(i_2-i_3)$$

where $r_p(0) = 1$ and $\forall n > 0$, $r_p(n) = -p(1-2p)^{n-1}$.

Now,

$$\sum_{1 \leq i_4 \leq N} \sum_{1 \leq i_4 \leq N} E[\delta^p(i_1-i_2)\delta^p(i_2-i_3)\delta^p(i_3-i_4)] = 12 \sum_{\max\{i_1,i_2\} \leq \min\{i_1,i_2\}} r_p(|i_1-i_2|)r_p(|i_3-i_4|) + O(N)$$

$$= 12 \sum_{\min\{i_1,i_2\} = 1} \left( \sum_{\max\{i_1,i_2\} = 1} r_p(|i_3-i_4|) \right) r_p(|i_1-i_2|) + O(N)$$

$$= 12 \sum_{\min\{i_1,i_2\} = 1} \left( \sum_{\max\{i_3,i_4\} = 1} (1-2p)^{\max\{i_3,i_4\}} \right) r_p(|i_1-i_2|) + O(N)$$

$$= 12 \sum_{\min\{i_1,i_2\} = 1} \left( \sum_{k=1}^{N-\min\{i_1,i_2\}} r_p(0) + 2 \sum_{l=1}^{N-\min\{i_1,i_2\}} r_p(l) \right) + O(N)$$

$$= 12 \sum_{\min\{i_1,i_2\} = 1} \left( \sum_{k=1}^{N-\min\{i_1,i_2\}} (1-2p)^k \right) + O(N)$$

The contribution of the exceptional situations $\max\{i_3,i_4\} = \min\{i_1,i_2\}$ is estimated by $O(N)$, because of the same compensations as the ones described in the above equalities.

We now use,

$$\int_0^1 (1-2p)^n d\mu^H(p) = -2 \sum_{k>n} r(k) \sim \frac{1}{c_H^2} \frac{2H}{n^{1-2H}}$$

We deduce,
\[
\frac{c_H^2}{N^{2H}} \left( \sum_{1 \leq i_4 \leq N} E[\delta_{i_4}] \right)^2 \sim \\
\frac{c_H^2}{N^{2H}} \left( \frac{2H}{c_H} \int_0^N \int_0^{x_2} \frac{dx_1}{(N - (x_2 - x_1))^{1-2H}} \right)^{\frac{1}{2}} \\
= c_H \times \sqrt{\frac{2H}{2H + 1}} N^{\frac{1}{2} - H}.
\]

Applying Berry-Esseen’s inequality, the error of the distribution function of the marginal at time 1, is dominated by \(0.65 \times D_H \frac{H^{1-H}}{\sqrt{M}}\).

**Corollary 3.** Let \(M\) be a function on the integers such that \(\frac{M(N)}{N^{1-H}}\) tends to \(\infty\),

\[
\mathcal{L}^D \lim_{N \to \infty} c_H \frac{X_{[N]}^H + \ldots + X_{[N]}^{H,M(N)}}{N^H \sqrt{M(N)}} = B_H(t)
\]

As a general conclusion, we can say that, using \(\mu_H\), the number of computations we have to make, in order to get a \(N\) steps trajectory, is a constant times \(N^{3-2H}\) for \(\frac{1}{2} < H\), and \(N^{2-2H}\) for \(0 < H < \frac{1}{2}\). In any case, it is a power of \(N\) between 1 and 2.

The constant in factor, is of big importance for practical simulation. We took the best constant in the Berry-Esseen’s inequality we could find in the literature, even if it is bigger than the best constant possible conjectured by Esseen, i.e. \(\frac{3+\sqrt{10}}{6\sqrt{2\pi}} \approx 0.41\) that would gain in time a squared factor equal to 2.5 (see [15] for a nice discussion on this subject).

But more important is to note that the constant can be considerably ameliorated by using other measures than \(\mu_H\), providing smaller \(c_H\)’s. It is the case for the real-indexed sequence of probabilities \((\mu_{H,k})_{k>0}\), defined below:

- \(\forall H \in [\frac{1}{2}, 1]\),

\[
d\mu_{H,k}(p) := 2^{k+1-2H} \frac{\Gamma(k + 2 - 2H)}{\Gamma(k) \Gamma(2 - 2H)} (p - \frac{1}{2})^{k-1} (1 - p)^{1-2H} \mathcal{H}_{\frac{1}{2}, 1} \{p\} dp
\]

which is just the law of \(\frac{1+\beta(k, 2-2H)}{2}\) and coincides with \(\mu_H\) for \(k = 1\) (where \(\beta(a, b)\) denotes the Beta variable with parameters \(a\) and \(b\)).

- \(\forall H \in [0, \frac{1}{2}]\),

\[
d\mu_{H,k}(p) := 2^{k-2H} \frac{\Gamma(k + 1 - 2H)}{\Gamma(k) \Gamma(1 - 2H)} (\frac{1}{2} - p)^{k-1} p^{2H - 1} \mathcal{H}_{\frac{k}{2}, 1} \{p\} dp
\]

which is just the law of \(\frac{\beta(1-2H, k)}{2}\) and coincides with \(\mu_H\) for \(k = 1\).

- For \(H > \frac{1}{2}\), we obtain \(c_{H,k} = \sqrt{\frac{H(1-2H)}{\Gamma(k + 2 - 2H)}} \sim \sqrt{\frac{H(1-2H)}{k}}\).

So that, using \(\mu_{H,k}\) for large \(k\), yields an error equivalent to:

\[
0.65 \sqrt{\frac{6H(2H-1)^2}{(H+1)(2H+1)}} \times \left( \frac{N}{k} \right)^{1-H} \frac{1}{\sqrt{M}}
\]

- For \(H = \frac{1}{2}\), we obtain \(c_{\frac{1}{2}, k} = \frac{1}{\sqrt{2k}}\).
The error is equivalent to:

\[
\frac{0.65}{\log N} \times \left( \frac{N}{k} \right)^{\frac{1}{2}} \frac{1}{\sqrt{M}}
\]

- For \( H < \frac{1}{2} \), we obtain \( c_{H,k} = \sqrt{\frac{2H(k)}{\log(k+1-2H)}} \sim \frac{\sqrt{2H}}{k^{\frac{1}{2}-H}} \).

The error is equivalent to:

\[
0.65 \sqrt{\frac{4H^2}{2H+1}} \times \left( \frac{N}{k} \right)^{\frac{1}{2}+H} \frac{1}{\sqrt{M}}
\]

Using \( \mu_{H,k} \) instead of \( \mu_H \), allows to gain for \( M \) a factor \( k^{2-2H} \) (resp. \( k^{1-2H} \)) for \( H > \frac{1}{2} \) (resp. for \( H < \frac{1}{2} \)). Loosely speaking, it erases the noise generated by the parameters between \( \frac{1}{2} \) and a fixed constant smaller than 1. The trouble making \( k \) increase, is that it damages the value of the covariance of \( X_N \) (resp. \( Y_N \)), but we can allow any \( k = O(1) \). So that we obtain an algorithm with any \( M(N) = 1/o(1) \) number of trajectories. As a result, our algorithm requires a number of computations of the order \( N/o(1) \), for any \( o(1) \). Moreover, we have only \( M(N) \) real datas to keep in memory along the whole procedure, corresponding to the coin-tossed parameters of the walks, and \( M(N) \) integers (-1 or 1), giving the last moves of the walks.

We want to give now a second family of measures \( (\mu'_{H,k})_{k>0} \):

- For \( H > \frac{1}{2} \), \( \mu'_{H,k} \) is the law of \( 1 - \frac{\left(1-U_{\frac{k}{2}}\right)^{\frac{1}{2}-H}}{\sqrt{k}} \).

An easy computation gives \( c'_{H,k} = \frac{\sqrt{k}}{\sqrt{e}} \) (error: 0.65 \( \times D_H N^{1-H} \sqrt{e/k} M \)).

- For \( H < \frac{1}{2} \), \( \mu'_{H,k} \) is the law of \( \frac{\left(1-U_{\frac{k}{2}}\right)^{\frac{1}{2}-H}}{\sqrt{k}} \).

Again, an easy computation gives \( c'_{H,k} = \frac{\sqrt{k}}{\sqrt{e}} \) (error: 0.65 \( \times D_H N^{1-H} \sqrt{e/k} M \)).

The advantage of this family is obviously the easy simulation it provides. The error is estimated by a term containing \( \frac{1}{\sqrt{k}} \), that seems to be better than the last one, but the damages on the variances grow faster than for \( \mu_{H,k} \). Actually the scale \( \sqrt{k} \) corresponds, in the previous family, to the scale \( k^{1-H} \) (resp. \( k^{\frac{1}{2}-H} \)) for \( H > \frac{1}{2} \) (resp. \( H < \frac{1}{2} \)). The drawback of this family, is that the theoretical computations of the variance are not very explicit.

We illustrate our results by three graphs corresponding to three different parameters of \( H \), with \( N = 1000 \), and a theoretical error smaller than 10% (we indicate the time it takes for Matlab to draw a graph):

- For \( H = 0.25 \), we take \( M = 200 \) and \( k = 1 \) (15 seconds) (Fig. 1).
- For \( H = 0.5 \), we can see that the convergence in Theorem 2 is the slowest one, and we will use the simple random walk to simulate it. (Fig. 2)
- For \( H = 0.75 \), we take \( M = 400 \) and \( k = 0.5 \) (25 seconds) (Fig. 3).

Remark: The interest of taking large \( k \) appears when \( N \) is very large, especially if we want to preserve a small error on the whole trajectory. In the case \( H > \frac{1}{2} \), we were even obliged to take \( k \) smaller than one.

We remark the different behaviours of the trajectories: let us recall that the Hausdorff dimension of the trajectories are a.s. equal to \( 2-H \), and we notice that the variances of the process between 0 and 1 become larger when \( H \) decreases.

As we noticed just above, we may be limited by the fidelity of the covariance of our process. In this spirit, it is quite interesting to note that the autocovariance
function of the Gaussian noise $G_H(j)$ of the introduction is, up to a shift, the sequence of moments of a probability measure on $[0, 1]$. I first remarked it by checking the conditions of Haussdorff theorem (p. 226), but Marc Yor gave me kindly, the method to get the explicit density of this measure, and I present it here. This brings a third (the last !) family of probability measures:

**Proposition 8.** Let $H \in [\frac{1}{2}, 1]$. Consider the family of probability measures $(\nu_{H,k})_{k>0}$ on $[\frac{1}{2}, 1]$, with density $C(H,k) \times (1-p)^2(2p-1)k^{-1}(\ln(\frac{1}{2p-1}))^{-1-2H}$, and

$$C(H,k) := \frac{16H(2H-1)}{(2-2H)} \times ((k+2)^{2H} - 2(k+1)^2H + k^{2H})^{-1}$$

$$\forall n \geq 0, \int_0^1 (2p-1)^n \, d\nu_{H,k}(p) = \frac{(n+k+2)^{2H} - (n+k+1)^{2H} + (n+k)^{2H}}{(k+2)^{2H} - 2(k+1)^2H + k^{2H}}$$

Proof: $\forall n \geq 1,$

$$\frac{1}{2}((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}) = H \int_0^1 (n+t)^{2H-1} - (n+t-1)^{2H-1} \, dt$$

$$= H (2H-1) \int_0^1 \int_0^1 (n+t+s-1)^{2H-2} \, ds \, dt$$

$$= \frac{H (2H-1)}{\Gamma(2-2H)} \int_0^1 \int_0^1 \int_0^{+\infty} e^{-(n+t+s-1)u} u^{1-2H} \, dudsdt$$

$$= \frac{H (2H-1)}{\Gamma(2-2H)} \int_0^{+\infty} e^{-nu} (1+u)^2 e^{u} u^{-1-2H} \, du$$

$$= \frac{H (2H-1)}{\Gamma(2-2H)} \int_0^1 x^n (1-x)^2 (\ln\frac{1}{x})^{-1-2H} \, dx$$

The end of the proof is straightforward, by change of variable. \hfill \square

Note: The normalization corresponding to $\nu_{H,k}$ is:

$$c_{H,k} = \frac{(k+2)^{2H} - (k+1)^{2H} + k^{2H}}{(k+2)^{2H} - 2(k+1)^2H + k^{2H}} \times \frac{\sqrt{H(2H-1)} e^{-nu}}{\Gamma(2-2H)} \sim c_{H,k}$$

The same can be done for $H \in [0, \frac{1}{2}]$:

**Proposition 9.** Let $H \in [0, \frac{1}{2}]$. Consider the family of probability measures $(\nu_{H,k})_{k>0}$ on $[0, \frac{1}{2}]$, with density $C(H,k) \times p(2p-1)^{-1}(\ln(\frac{1}{1-2p}))^{-1-2H}$, and

$$C(H,k) := \frac{8H}{\Gamma(1-2H)} \times ((k+1)^{2H} - k^{2H})^{-1}$$

$$\forall n \geq 1, \int_0^1 p(2p-1)^{n-1} \, d\nu_{H,k}(p) = \frac{(n+k+1)^{2H} - 2(n+k)^{2H} + (n+k-1)^{2H}}{2((k+1)^{2H} - k^{2H})}$$

Proof: $\forall n \geq 1,$

$$\frac{1}{2}((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}) = H \int_0^1 (n+t)^{2H-1} - (n+t-1)^{2H-1} \, dt$$

$$= \frac{H}{\Gamma(1-2H)} \int_0^1 \int_0^{+\infty} (e^{-(n+t)u} - e^{-(n+t-1)u}) u^{-2H} \, dudsdt$$

$$= -\frac{H}{\Gamma(1-2H)} \int_0^{+\infty} e^{-nu} (e^{u} - 1)^2 u^{-1-2H} \, du$$

$$= -\frac{H}{\Gamma(1-2H)} \int_0^1 x^n (\frac{1-x}{x})^2 (\ln\frac{1}{x})^{-1-2H} \, dx$$
We find the normalization, using
\[ \sum_{n \geq 1} (n + k + 1)^{2H} - 2(n + k)^{2H} + (n + k - 1)^{2H} = k^{2H} - (k + 1)^{2H}, \]
and the relation \( 1 = 2 \sum_{n \geq 1} p(1 - 2p)^{n-1} \).

\( \Box \)

Note: The normalization corresponding to \( \nu_{H,k} \) is:
\[ c'_{H,k} = (\frac{(k + 1)^{2H} - k^{2H}}{k^{2H}})^{1/2} \sim c_{H,k}. \]

These measures \( \nu_{H,k} \) have an interest, as far as \( X^\nu_N \) (resp. \( Y^\nu_N \)) have explicit covariances matching quite well with the covariances of the fractional Brownian motion. On the other hand, they don’t seem very easy to simulate.
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