Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics

Edmundo CAPELAS DE OLIVEIRA(1) Francesco MAINARDI(2) and Jayme VAZ Jr.(1)

(1) Department of Applied Mathematics, IMECC, University of Campinas, 13083-869 Campinas, SP, Brazil
E-mail: capelas@ime.unicamp.br, vaz@ime.unicamp.br
(2) Department of Physics, University of Bologna, Via Irnerio 46, I-40126 Bologna, Italy
Corresponding Author. E-mail: francesco.mainardi@unibo.it

Abstract

We revisit the Mittag-Leffler functions of a real variable $t$, with one, two and three order-parameters $\{\alpha, \beta, \gamma\}$, as far as their Laplace transform pairs and complete monotonicity properties are concerned. These functions, subjected to the requirement to be completely monotone for $t > 0$, are shown to be suitable models for non–Debye relaxation phenomena in dielectrics including as particular cases the classical models referred to as Cole–Cole, Davidson–Cole and Havriliak–Negami. We show 3D plots of the response functions and of the corresponding spectral distributions, keeping fixed one of the three order-parameters.

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Foreword

This E-print reproduces the revised version of the paper published in EPJ-ST, Vol. 193 (2011), pp. 161–171. The revision concerns the proper use of the terms relaxation function and response function in the literature on dielectrics. In the published paper, starting from Eq (1.1), the authors had referred the inverse Laplace transform of the complex susceptibility as the relaxation function. This is not correct because the inversion provides the so-called response function as pointed out to FM by Prof. Karina WERON (KW) to whom the authors are very grateful. As a matter of fact the relationship between the response function and the relaxation function can better be clarified by their probabilistic interpretation investigated in several papers by KW. As a consequence, interpreting the relaxation function as a survival probability $\Psi(t)$, the response function turns out to be the probability density function corresponding to the cumulative probability function $\Phi(t) = 1 - \Psi(t)$. Then, denoting by $\phi(t)$ the response function, we have

$$\phi(t) = -\frac{d}{dt}\Psi(t) = \frac{d}{dt}\Phi(t), \quad t \geq 0.$$ 

As KW has pointed out, both functions have very different properties and describe different physical magnitudes; only in the Debye (pure exponential) case the properties coincide. The relaxation function describes the decay of polarization whereas the response function its decay rate (the depolarization current). However, for physical realizability, both functions are required to be completely monotone with a proper spectral distribution so our analysis can be properly transferred from response functions to the corresponding relaxation functions, whereas the corresponding cumulative probability functions turn out to be Bernstein (or creep) functions, that is positive functions with a completely monotone derivative.

1 Introduction

It is well recognized that relaxation phenomena in dielectrics deviate more or less strongly from the classical Debye law for which the Laplace transform pair for complex susceptibility ($s = -i\omega$) and response function ($t \geq 0$) reads in an obvious notation,

$$\tilde{\xi}_D(s) = \frac{1}{1+s} \div \xi_D(t) = e^{-t}. \quad (1.1)$$
Here, for the sake of simplicity, we have assumed the frequency $\omega$ and the time $t$ normalized with respect to a characteristic frequency $\omega_D$ and a corresponding relaxation time $\tau_D = 1/\omega_D$.

In the literature a number of laws have been proposed to describe the non-Debye (or anomalous) relaxation phenomena in dielectrics, of which the most relevant ones are referred to Cole – Cole (C-C), Davidson – Cole (D-C) and Havriliak – Negami (H-N) laws, see e.g., the classical books by Jonscher [13, 14]. Several authors have investigated these laws from different points of view, including Karina Weron and her associates, see e.g., [15, 16, 29, 30], Hilfer [11, 12] and Hanyga and Seredyńska [7].

In particular, Hilfer surveyed the analytical expressions in the frequency and time domain for the main non-Debye relaxation processes and provided the response functions corresponding to the complex frequency-dependent Cole-Cole, Davidson-Cole and Havriliak-Negami susceptibilities in terms of Fox $H$–functions. This class of functions is quite general so it includes the Mittag-Leffler functions that we prefer to use to characterize the above laws in a more accessible way.

On the other hand, for linear systems, the connection between weak dissipativity and positive definiteness of the response functions as well as between monotone energy decay and complete monotonicity of the response functions were discussed by Hanyga and Seredyńska [7], and in references therein, in terms of functions of Mittag-Leffler type. A subordination model of anomalous diffusion leading to the two-power-law relaxation responses has been proposed by Stanislavski et al. [30], where the authors have presented a novel two-power relaxation law and shown its relationship to the H-N law by using functions of Mittag-Leffler type.

Moreover, Novikov et al. [23] have discussed anomalous relaxation in dielectrics providing a differential equation with fractional derivatives to describe the relaxations of the C-C, C-D and H-N types in dielectrics. The corresponding solutions were presented in terms of Mittag-Leffler functions.

Fractional relaxation and governing equations for dielectrics characterized by the H-N response function has been pointed out by Sibatov and Uchaikin [28] who have presented a comparison of numerical results with experimental data.

The purpose of this paper is to present a general model for anomalous relaxation in dielectrics that includes as particular cases the classical C-C,
D-C and N-H laws. Our model is still based on the Mittag-Leffler functions but depending on three order-parameters that, in view of their complete monotonicity in the time domain, ensure the existence of a suitable spectrum of relaxation times required for the physical realizability. On the realizability requirements the reader can be addressed, in addition to [7], to the treatise by Zemanian [31].

In a subsequent paper we will investigate how to provide the differential equation of fractional order that is expected to govern our general model, so generalizing the approach and the results by Novikov et al.

The present paper is organized as follows.

In section 2 we recall the definitions of the most common functions of Mittag-Leffler type, namely those depending on one, two and three order-parameters \( \{\alpha, \beta, \gamma\} \). For these functions we exhibit the corresponding Laplace transform pairs and we discuss their properties of complete monotonicity. We show that the complete monotonicity is ensured if the independent variable is real and negative and the three order-parameters are subjected to the conditions \( 0 < \alpha, \beta, \gamma < 1 \) with \( \alpha \gamma \leq \beta \). For other properties of the Mittag-Leffler functions we refer the reader mainly to texts on Special Functions and Fractional Calculus, including i.e. [4, 10, 17, 18, 20, 24], just to cite some of the most recent ones. In fact, as pointed out by Gorenflo and Mainardi [5, 19], functions of Mittag-Leffler type enter as solutions of many problems dealt with fractional calculus so that they like to refer to the Mittag-Leffler function to as the Queen function of Fractional Calculus, in contrast with its role of a Cinderella function played in the past.

In section 3 we show that for special cases of the triplet \( \{\alpha, \beta, \gamma\} \) with \( \alpha \gamma = \beta \) our functions provide the response functions for the classical models of Cole–Cole, Davidson–Cole, and Havriliak–Negami. As a consequence, we expect that in the more general case \( \alpha \gamma \leq \beta \) the corresponding Mittag–Leffler functions, being completely monotone, can provide further models for processes of anomalous (non-Debye) relaxation in dielectrics. For some study-cases, taking fixed two of the three order-parameters, we provide 3D plots of the responses functions and of the corresponding spectral distributions, in order to better visualize the positivity and the variability of the considered functions.

Finally, section 4 is devoted to conclusions and final remarks together with a discussion on the direction of future work.
2 The Mittag-Leffler functions

2.1 Definitions

We start to recall the definition in the complex plane of the generalized Mittag-Leffler function introduced by Prabhakar [26], known as Prabhakar function or 3-parameter Mittag-Leffler function

\[
E_{\gamma}^{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n! \Gamma(\alpha n + \beta)} z^n, \quad \text{Re}\{\alpha\} > 0, \text{Re}\{\beta\} > 0, \text{Re}\{\gamma\} > 0, \quad (2.1)
\]

where

\[
(\gamma)_n = \gamma(\gamma + 1) \ldots (\gamma + n - 1) = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}.
\]

It turns out to be an entire function. For \( \gamma = 1 \) we recover the 2-parameter Mittag-Leffler function (also known as Wiman or Agarwal generalized Mittag-Leffler function)

\[
E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (2.2)
\]

and for \( \gamma = \beta = 1 \) we recover the standard Mittag-Leffler function

\[
E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}. \quad (2.3)
\]

2.2 The Laplace transform pairs

Let us now consider the relevant formulas of Laplace transform pairs related to the above three functions, already known in the literature for \( \alpha, \beta > 0 \) and \( 0 < \gamma \leq 1 \), when the independent variable is real of type \( at \) where \( t > 0 \) is interpreted as time and \( a \) as a certain constant.

Let us start with the most general function. Substituting the series representation of the Prabhakar generalized Mittag-Leffler function in the Laplace transformation yields the identity

\[
\int_0^{\infty} e^{-st} t^\beta - 1 E_{\alpha,\beta}^{\gamma}(at^n) \, dt = s^{-\beta} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} \left( \frac{a}{s} \right)^n. \quad (2.4)
\]
On the other hand (binomial series)

\[
(1 + z)^{-\gamma} = \sum_{n=0}^{\infty} \frac{\Gamma(1 - \gamma)}{\Gamma(1 - \gamma - n)n!} z^n = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)n!} z^n .
\]  

(2.5)

Comparison of Eq. (2.4) and Eq. (2.5) yields the Laplace transform pair

\[
t^\beta - 1 E_{\alpha,\beta}^\gamma (at^\alpha) \div \frac{s^{-\beta}}{(1 - as^{-\alpha})^\gamma} = \frac{s^{\alpha\gamma - \beta}}{(s^\alpha - a)^\gamma}.
\]

(2.6)

Eq. (2.6) holds (by analytic continuation) for Re(\(\alpha\)) > 0, Re(\(\beta\)) > 0.

In particular we get the known Laplace transform pairs, see e.g. [18, 24],

\[
t^{\beta - 1} E_{\alpha,\beta}^\gamma (at^\alpha) \div \frac{s^{\alpha - \beta}}{s^\alpha - a} = \frac{s^{-\beta}}{1 - as^{-\alpha}},
\]

(2.7)

\[
E_{\alpha} (at^\alpha) \div \frac{s^{\alpha - 1}}{s^\alpha - a} = \frac{s^{-1}}{1 - as^{-\alpha}}.
\]

(2.8)

2.3 Complete monotonicity

We recall that a function \(f(t)\) with \(t \geq 0\) is completely monotone (CM) if it is positive and its derivatives are alternating in sign, namely

\[
(-1)^n f^{(n)}(t) > 0 , \ t \geq 0 .
\]

(2.9)

Thus, for the Bernstein theorem that states a necessary and sufficient condition for the CM, the function can be expressed as a real Laplace transform of non-negative (generalized) function, namely

\[
f(t) = \int_0^{\infty} e^{-rt} K(r) \, dr , \ K(r) \geq 0 , \ t \geq 0 .
\]

(2.10)

By the way, the determination of such non-negative function \(K(r)\) (the Laplace measure) is a standard method to prove the CM of a given function defined in the positive real axis \(\mathbb{R}^+\). In physical applications the function \(K(r)\) is usually referred to as the spectral distribution function, in that it is related to the fact that the process governed by \(f(t)\) can be expressed in terms of a continuous distribution of elementary (exponential) relaxation processes.
For the Mittag-Leffler functions in one and two-order parameters the conditions to be CM were proved respectively by Pollard [25] in 1948 and by Schneider [27] in 1996, see also Miller and Samko [21, 22] for further details. These conditions require the independent variable to be real and negative and the order-parameters such that $0 < \alpha \leq 1$ and $\alpha \leq \beta \leq 1$, respectively.

A method for proving CM suitable for our Mittag-Leffler functions (2.6) (2.7) (2.8) is that to arrive at the non-negative spectral distribution $K(r)$ by inverting the corresponding Laplace transform by using the Bromwich contour integral. Indeed this was the method followed by Gorenflo and Mainardi [5] to prove the CM of $E_{\alpha}(-t^{-\alpha})$ for $0 < \alpha < 1$ and to determine the corresponding spectral function. We recall the result

\[ E_{\alpha}(-t^{-\alpha}) = \int_{0}^{\infty} e^{-rt} K_{\alpha}(r) \, dr, \quad K_{\alpha}(r) = \frac{1}{\pi r} \frac{\sin(\alpha \pi)}{r^\alpha + 2 \cos(\alpha \pi) + r^\alpha} \geq 0. \]  

As a matter of fact the function $K_{\alpha}(r)$ is derived as an exercise in complex analysis by evaluating the contribution on the branch cut (the negative real axis) of the Bromwich integral and turns out to be provided by the so-called Titchmarsh formula

\[ K_{\alpha}(r) = -\frac{1}{\pi} \text{Im} \left\{ \left. \frac{s^{\alpha-1}}{s^{\alpha} + 1} \right|_{s = re^{i\pi}} \right\}. \]

(2.12)

We now prove that the function

\[ \xi_G(t) := t^\beta - 1 E_{\alpha,\beta}^\gamma(-t^\alpha), \]

(2.13)

with Laplace transform (as derived from (2.7) with $a = -1$)

\[ \tilde{\xi}_G(s) = \frac{s^{\alpha\gamma - \beta}}{(s^{\alpha} + 1)^\gamma}, \]

(2.14)

where the notation $\xi_G(t)$ been introduced for future convenience, is CM iff

\[ 0 < \alpha, \beta, \gamma \leq 1 \quad \text{with} \quad \alpha \gamma \leq \beta. \]

(2.15)

Furthermore, by using the Titchmarsh formula we will derive its spectral distribution function. For this purpose we take advantage of the requirements stated in the treatise by Gripenberg et al. [6], see Theorem 2.6, pp. 144-145, that provide necessary and sufficient conditions to ensure the CM of a
function $f(t)$ based on its Laplace transform $\tilde{f}(s)$. Hereafter, we recall this theorem by using our notation.

**Theorem** The Laplace transform $\tilde{f}(s)$ of a function $f(t)$ that is locally integrable on $\mathbb{R}^+$ and CM the following properties:

(i) $\tilde{f}(s)$ an analytical extension to the region $\mathbb{C} - \mathbb{R}^-$;
(ii) $\tilde{f}(x) = \tilde{f}^*(x)$ for $x \in (0, \infty)$;
(iii) $\lim_{x \to \infty} \tilde{f}(x) = 0$;
(iv) $\text{Im}\{\tilde{f}(s)\} < 0$ for $\text{Re}\{s\} > 0$;
(v) $\text{Im}\{s \tilde{f}(s)\} \geq 0$ for $\text{Im}\{s\} > 0$ and $\tilde{f}(x) \geq 0$ for $x \in (0, \infty)$.

Conversely, every function $\tilde{f}(s)$ that satisfies (i)–(iii) together with (iv) or (v), is the Laplace transform of a function $f(t)$, which is locally integrable on $\mathbb{R}^+$ and CM on $(0, \infty)$.

To prove the CM of $\xi_G(t)$ we must recognize that all the requirements (i)–(v) are satisfied for $\tilde{\xi}_G(s)$ under the conditions (2.15). We first recognize that for

$$0 < \alpha \leq 1, \ 0 < \beta \leq 1, \ 0 < \gamma \leq 1,$$

the requirements (i)–(ii) are satisfied. The requirement (iii) is satisfied for the additional condition $\alpha \gamma - \beta \leq 0$. Then it remains to prove that the requirement (v) is satisfied for the function

$$s \tilde{\xi}_G(s) = \frac{s^{\alpha \gamma - \beta + 1}}{(s^\alpha + 1)\gamma} = s^{1-\beta} \left( \frac{1}{1 + s^{-\alpha}} \right)^\gamma. \quad (2.16)$$

Putting $s = re^{i\phi}$ with $0 < \phi < \pi$, upper half $\mathbb{C}^+$ ($\text{Im}\{s\} > 0$) we can get

$$s \tilde{\xi}_G(s) = (r e^{i\phi})^{1-\beta} \frac{(1 + r^{-\alpha} e^{i\phi \alpha})^\gamma}{|1 + s^{-\alpha}|^{2\gamma}}. \quad (2.17)$$

Taking its imaginary part we can write

$$\text{Im}\{s \tilde{\xi}_G(s)\} = \frac{1}{\Omega} \text{Im} \{(r e^{i\phi})^{1-\beta}(1 + r^{-\alpha} e^{i\phi \alpha})^\gamma\}, \quad \text{with } \Omega = |1 + s^{-\alpha}|^{2\gamma} > 0. \quad (2.18)$$

Thus, we must prove that this imaginary part is always non-negative in the upper half-plane $\mathbb{C}^+$. Setting

$$z_1 := r^{1-\beta} e^{i\phi (1-\beta)}, \ z_2 := r^{-\alpha} e^{i\alpha \phi},$$
we recognize that $z_1$ lies in $\mathbb{C}^+$ being $0 < \beta \leq 1$, that $z_2$ and consequently $1 + z_2$ lies in $\mathbb{C}^+$ being $0 < \alpha \leq 1$ and the same for $(1 + z_2)^\gamma$ being $0 < \gamma \leq 1$. We therefore conclude that under these requirement on $\{\alpha, \beta, \gamma\}$ we get

$$\text{Im}\{s \tilde{\xi}_G(s)\} \geq 0. \quad (2.19)$$

As a matter of fact we have proved a noteworthy result stated without explicit prove by Mainardi [18] in his recent book. For this purpose we have adopted the same method followed by Hanyga and Seredyńska [7] who have limited themselves to prove, starting form the Laplace transform pair (2.6) with $\beta = 1$, and basing on [6],

$$E_{\alpha,1}^\gamma(-t^\alpha) \text{ CM if } 0 < \alpha \leq 1, \ 0 < \gamma \leq 1. \quad (2.20)$$

So we have extended their result for $0 < \beta < 1$.

### 2.4 The spectral distribution function

The Bernstein theorem ensures that our response function $\xi_G(t)$, being CM, can be expressed with a *spectral distribution function* $K_{\alpha,\beta}^\gamma(r) \geq 0$ such that

$$\xi_G(t) = \int_0^\infty e^{-rt} K_{\alpha,\beta}^\gamma(r) \, dr. \quad (2.21)$$

In view of Titchmarsh formula (2.12) applied to the Laplace transform (2.14) we get

$$K_{\alpha,\beta}^\gamma(r) := \frac{e^{-\beta}}{\pi} \text{Im} \left\{ e^{i\pi \beta} \left( \frac{r^\alpha + e^{-i\pi \alpha}}{r^\alpha + 2 \cos \pi \alpha + r^{-\alpha}} \right)^\gamma \right\}. \quad (2.22)$$

We note that for $\beta = \gamma = 1$ we obtain from (2.22) the spectral distribution function of the classical Mittag-Leffler function outlined in (2.11).
3 Mathematical models for dielectric relaxation

We intend to discuss how the Laplace transform pair outlined in Eqs. (2.13)-(2.14) coupled with the condition (2.15) on the 3-order parameters can be considered as the pair $\xi_G(t) \ (t \geq 0)$ and $\tilde{\xi}_G(s) \ (s = i\omega)$ for a possible mathematical model of the response function and the complex susceptibility in the framework of a general relaxation theory of dielectrics. We first show how the three classical models referred to Cole–Cole (C-C), Davidson–Cole (D-C) and Havriliak–Negami (H-N) for $\alpha\gamma = \beta$ are contained in our general model when $\alpha\gamma = \beta$ according to the scheme

$$
\alpha\gamma = \beta \text{ with } \begin{cases} 0 < \alpha < 1, \beta = \alpha, \gamma = 1 & \text{C-C } \{\alpha\}, \\ \alpha = 1, \beta = \gamma, 0 < \gamma < 1 & \text{D-C } \{\gamma\}, \\ 0 < \alpha < 1, 0 < \gamma < 1 & \text{H-N } \{\alpha, \gamma\}. \end{cases} \quad (3.1)
$$

Then, we consider some study-cases when the inequality $\alpha\gamma < \beta$ holds. For this purpose we exhibit 3D plots for the response function $\xi_G(t)$, see (2.13), and the corresponding spectral distribution $K_{\alpha,\beta}^\gamma(r)$, see (2.21) keeping fixed two of the three-order parameters.

3.1 The classical dielectric functions

The Cole–Cole relaxation model The C-C relaxation model is a non-Debye relaxation model depending on one parameter, say $\alpha \ (0 < \alpha < 1)$, see [1, 2], that for $\alpha = 1$ reduces to the standard Debye model.

$$
\tilde{\xi}_{\mathrm{C-C}}(s) = \frac{1}{1 + s\alpha} \div \xi_{\mathrm{C-C}}(t) = t^{\alpha - 1} E_1^{\alpha}(-t^\alpha) = -\frac{d}{dt} E_\alpha(-t^\alpha), \ 0 < \alpha < 1. \quad (3.2)
$$

The Davidson–Cole relaxation model The D-C relaxation model is a non-Debye relaxation model depending on one parameter, say $\gamma \ (0 < \gamma < 1)$, see [3], that for $\gamma = 1$ reduces to the standard Debye model. The corresponding complex susceptibility $(s = -i\omega)$ and response function read

$$
\tilde{\xi}_{\mathrm{D-C}}(s) = \frac{1}{(1 + s)^\gamma} \div \xi_{\mathrm{D-C}}(t) = t^{\gamma - 1} E_{1,\gamma}^{\gamma}(-t) = \frac{t^{\gamma - 1}}{\Gamma(\gamma)} e^{-t}, \ 0 < \gamma < 1. \quad (3.3)
$$
The Havriliak–Negami relaxation model The H-N relaxation model is a non-Debye relaxation model depending on two parameters, say $\alpha$ ($0 < \alpha < 1$) and $\gamma$ ($0 < \gamma < 1$), see [8, 9], that for $\alpha = \gamma = 1$ reduces to the standard Debye model. The corresponding complex susceptibility and response function read

$$\tilde{\xi}_{\text{H-N}}(s) = \frac{1}{(1 + s^\alpha)\gamma} \div \tilde{\xi}_{\text{H-N}}(t) = t^{\alpha\gamma - 1} E^{\gamma}_{\alpha,\alpha\gamma}(-t^\alpha), \quad 0 < \alpha, \gamma < 1. \quad (3.4)$$

We recognize that the H-N relaxation model for $\gamma = 1$ reduces to the C-C model, while for $\alpha = 1$ to the D-C model. We also note that whereas for the C-C and H-N models the corresponding response functions decay like a certain negative power of time, the D-C response function exhibits an exponential decay.

3.2 Survey of the general response functions with their spectral distributions

In order to visualize the effects of varying the three order-parameters in our general model we survey some particular cases by exhibiting separately the response functions $\xi_{G}(t)$ for $0 < t < 10$ given by Eq. (2.21) and the corresponding spectral distributions for $0 < r < 10$ given by Eq. (2.22). For this purpose we provide 3D plots by keeping fixed two of the three order-parameters $\{\alpha, \beta, \gamma\}$, all of them less than unity and subjected to the condition $\alpha \gamma < \beta$.

In Figs. 1. and 2., for fixed $\alpha = 1/2$ we compare versus $\gamma \in (0, 1)$ the plots of $\xi_{G}(t)$ and of the corresponding spectral distribution for $\beta = \gamma/2, 2\gamma/3$.

In Figs. 3. and 4., for fixed $\gamma = 1/2$ we compare versus $\alpha \in (0, 1)$ the plots of $\xi_{G}(t)$ and of the corresponding spectral distribution for $\beta = \alpha/2, 2\alpha/3$.

In Figs. 5. and 6., for fixed $\alpha = 1/2, \gamma = 1$ and $\alpha = 2/3, \gamma = 1/2$ we compare versus $\beta \in (1/3, 1)$ the plots of $\xi_{G}(t)$ and of the corresponding spectral distribution.
Figure 1: Plot of the response Function $\xi_G(t)$ versus $\gamma \in (0, 1)$ and $t \in (0, 10)$ for \(\{\alpha = 1/2, \beta = \gamma/2, \gamma\}\) (dark grey/blue surface) and \(\{\alpha = 1/2, \beta = 2\gamma/3, \gamma\}\) (light grey/yellow surface).

Figure 2: Plot of the Spectral Distribution Function $K_{1/2, \gamma/2}^\gamma(r)$ (dark grey/blue surface) and $K_{1/2, 2\gamma/3}^\gamma(r)$ (light grey/yellow surface) for $0 < \gamma < 1$ and $0 \leq r \leq 10$. 
Figure 3: Plot of the response Function $\xi_G(t)$ versus $\alpha \in (0, 1)$ and $t \in (0, 10)$ for $\{\alpha, \beta = \alpha / 2, \gamma = 1/2\}$ (dark grey/blue surface) and $\{\alpha, \beta = 2\alpha / 3, \gamma = 1/2\}$ (light grey/yellow surface).

Figure 4: Plot of the Spectral Distribution Function $K^{1/2}_{\alpha, \alpha/2}(r)$ (dark grey/blue surface) and $K^{1/2}_{\alpha, 2\alpha/3}(r)$ (light grey/yellow surface) for $0 < \alpha < 1$ and $0 \leq r \leq 10$. 
Figure 5: Plot of the response Function $\xi_G(t)$ versus $\beta \in (1/3, 1)$ and $t \in (0, 10)$ for $\{\alpha = 1/2, \beta, \gamma = 1\}$ (dark grey/blue surface) and $\{\alpha = 2/3, \beta = 2\alpha/3, \gamma = 1/2\}$ (light grey/yellow surface).

Figure 6: Plot of the Spectral Distribution Function $K_{1/2,\beta}(r)$ (dark grey/blue surface) and $K_{2/3,\beta}(r)$ (light grey/yellow surface) for $1/3 < \beta < 1$ and $0 \leq r \leq 10$. Note that $K_{2/3,\beta}(r) < 0$ when the condition $\beta \geq \alpha\gamma$ is not satisfied.
4 Conclusions

In this paper we have presented a quite general mathematical model for dielectrics that exhibit deviations from the standard relaxation Debye law. This model is based on a time response function expressed in terms of a Mittag-Leffler function with three order-parameters. A restriction on these parameters is required to ensure its complete monotonicity for $t > 0$, so that the resulting relaxation process can be seen as a continuous distribution of elementary exponential processes by means of a corresponding (non-negative) spectral distribution function.

Our approach allows one to derive from a unique mathematical framework the classical Cole-Cole, Davidson–Cole and Havriliak–Negami models that are usually adopted in the literature. However, other laws can be derived from our model that could better fit some experimental data.

For some study–cases we have exhibited plots of the response functions (versus $t$) along with their corresponding spectral distributions (versus $r$) keeping fixed two of the three order-parameters $\{\alpha, \beta, \gamma\}$ of our Mittag-Leffler function. We can note that when the condition $0 < \alpha \gamma \leq \beta \leq 1$ is not satisfied the spectral distribution functions turn out to be negative.

In a future paper we are interested to provide the fractional differential equation governing our general response function extending the results obtained by Novikov et al. [23] and by Stanislavski et al. [30] for the Havriliak–Negami model.

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