Wasserstein Complexity of Quantum Circuits

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Abstract

Given a unitary transformation, what is the size of the smallest quantum circuit that implements it? This quantity, known as the quantum circuit complexity, is a fundamental property of quantum evolutions that has widespread applications in many fields, including quantum computation, quantum field theory, and black hole physics. In this letter, we obtain a new lower bound for the quantum circuit complexity in terms of a novel complexity measure that we propose for quantum circuits, which we call the quantum Wasserstein complexity. Our proposed measure is based on the quantum Wasserstein distance of order one (also called the quantum earth mover’s distance), a metric on the space of quantum states. We also prove several fundamental and important properties of our new complexity measure, which stand to be of independent interest. Finally, we show that our new measure also provides a lower bound for the experimental cost of implementing quantum circuits, which implies a quantum limit on converting quantum resources to computational resources. Our results provide novel applications of the quantum Wasserstein distance and pave the way for a deeper understanding of the resources needed to implement a quantum computation.

I. INTRODUCTION

How many elementary quantum gates does it take to synthesize a desired unitary transformation $U$? Computing this number—called the quantum circuit complexity of $U$—is one of the holy grails of quantum computation, especially noisy intermediate-scale quantum computation where quantum resources are scarce [1, 2]. By using fewer elementary gates, a quantum algorithm stands to take less time to produce its output; this in turn increases its prospect for achieving the highly sought goal of quantum-computational advantage [3, 4].

In the search for optimal quantum circuits that use as few elementary gates as possible, one has taken various approaches to quantum circuit synthesis, including: reinforcement learning [5], quantum Karnaugh maps [6], and ZX-calculus [7]. While these methods usually succeed at finding smaller circuits, they seldom achieve optimal circuit solutions [8]. Computing the quantum circuit complexity of unitaries is generally hard, and no efficient algorithms are known [9, 10].

Unexpected connections have been found between the notion of quantum circuit complexity and high energy physics. A solution to the wormhole-growth paradox has been proposed, which asks: in the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, which quantity in the CFT is dual to the wormhole volume [11]? Stanford and Susskind conjecture that the answer is the quantum circuit complexity of the boundary state [12]. In follow-up work by Brown and Susskind [13], they conjecture that the quantum circuit complexity grows linearly for an exponentially long time. This was subsequently formalized and proved by Haferkamp et al. [14]. (See also [15] for two short proofs of this result.)

Nielsen pioneered a geometric approach to find upper and lower bounds for the quantum circuit complexity [16]. In a seminal series of papers, Nielsen and collaborators developed various geometric notions related to quantum computation and proved that the quantum circuit complexity of a unitary, up to polynomial factors and technical caveats, is equal to the circuit cost, defined as the length of the shortest path between two points in some curved Riemannian geometry [16–19].

A recent fruitful approach to bound Nielsen’s circuit cost is to relate it to the quantum resources in a quantum circuit that refer to ingredients in the circuit responsible for a quantum speedup. A famous example of a quantum resource is nonstabilizerness, also known as magic. The Gottesman-Knill theorem proves that stabilizer circuits are efficiently classically simulable [20], but when supplemented with magic states, such circuits become hard to simulate [21–24]. Hence, magic could be thought of as a resource for a quantum speedup [25–33]. In recent works, the circuit cost was shown to be bounded below by the circuit’s entangling power [34], magic [35], and sensitivity [35].

In this letter, we study the circuit complexity of quantum circuits using the so-called Wasserstein distance. Classically, the Wasserstein distance is a metric on the space of probability distributions on a metric space that traces its origins to the works of Kantorovich [36], Vaserestein [37], and others. The classical Wasserstein distance has numerous applications, including optimal transport [36, 38, 39], image retrieval in computer vision [40], and Wasserstein generative adversarial networks (WGANs) in classical machine learning [41–43].

What is an appropriate quantum generalization of the classical Wasserstein distance that is useful for quantum computation? To answer this question, several quantum generalizations of the Wasserstein distance have been proposed. These include the quantum $W_2$ distances, pioneered by Carlen, Maas, Datta, Rouzé, Junge, and others, based on a Riemannian met-
ric on the manifold of quantum states [44–50] that generalize the classical Wasserstein distance of order 2. These W₂ distances have been shown to be related to the entropy and Fisher information, two important concepts in quantum information theory [48]. In addition, several candidates have been proposed that generalize the Wasserstein distance of order 1 [51–53]. In this letter, we focus on the quantum Wasserstein distance of order 1, proposed by De Palma, Marvian, Trevisan, and Lloyd [53], which can be regarded as a quantum version of the Hamming distance on n-qudit systems. This new version of the quantum Wasserstein distance has numerous applications in quantum information and quantum computation, such as quantum machine learning (where it is commonly referred to as the quantum earth mover’s distance) [54], quantum concentration inequalities [55], quantum differential privacy [56], characterizing limitations of variational quantum algorithms [57], etc. However, little is hitherto known about the connection between the quantum Wasserstein distance and the quantum circuit complexity.

In this work, we introduce a complexity measure for quantum circuits, called the quantum Wasserstein complexity, which we define as the maximal distance between the input and output states of the circuit, measured using the quantum Wasserstein distance of order 1. We show several useful properties of this complexity measure for quantum channels, such as subadditivity under concatenation, superadditivity under tensor product (or additivity for correlation enhanced complexity measures). Most importantly, we show a connection between the quantum Wasserstein complexity and the circuit complexity, where we show that the quantum Wasserstein complexity provides a lower bound for the circuit cost. Finally, we show that the quantum Wasserstein complexity also provides a lower bound on the experimental cost to implement quantum circuits.

II. MAIN RESULTS

Given the n-qudit system \( \mathcal{H} = (\mathbb{C}^d)^\otimes n \), let us define \( O(\mathcal{H}) \) to be the set of traceless, Hermitian operators on \( \mathcal{H} \), i.e.,

\[
O(\mathcal{H}) = \{ A \in \mathcal{L}(\mathcal{H}) : \text{Tr}[A] = 0, A^\dagger = A \},
\]

where \( \mathcal{L}(\mathcal{H}) \) denotes the set of linear operators on \( \mathcal{H} \). Also, let \( D(\mathcal{H}) \) denote the set of density operators on \( \mathcal{H} \). The quantum Wasserstein distance of order 1 [53] between two quantum states \( \rho, \sigma \in D(\mathcal{H}) \) is defined as:

\[
\|\rho - \sigma\|_{W_1} := \frac{1}{2} \min_{\{X_i\}} \left\{ \sum_i \|X_i\|_1 : \rho - \sigma = \sum_i X_i , X_i \in O(\mathcal{H}), \text{Tr}[X_i] = 0 \forall i \right\}, \tag{1}
\]

where \( \text{Tr}[\cdot] \) denotes the partial trace over the \( i \)-th subsystem, and \( \|X\|_1 = \text{Tr}[\sqrt{X^\dagger X}] \) denotes the trace norm of \( X \). This distance is also called the quantum W₁ distance or the quantum earth mover’s distance. Using the quantum Wasserstein distance lets one distinguish between two quantum states that differ locally; this is different from their global distinguishability. One example to show this distinction comes from considering the n-qubit states \( |0\rangle^\otimes n, |1\rangle^\otimes n^{-1} \) and \( |1\rangle^\otimes n \). The trace distances \( [58] \) between \( |0\rangle^\otimes n \) and \( |1\rangle^\otimes n^{-1} \) and between \( |0\rangle^\otimes n \) and \( |1\rangle^\otimes n \) are both equal to 1; hence, using the trace distance, we cannot distinguish which of \( |1\rangle^\otimes n^{-1} \) and \( |1\rangle^\otimes n \) is further away from \( |0\rangle^\otimes n \). However, the quantum W₁ distance between \( |0\rangle^\otimes n \) and \( |1\rangle^\otimes n^{-1} \) is equal to 1, while the quantum W₁ distance between \( |0\rangle^\otimes n \) and \( |1\rangle^\otimes n \) is equal to \( n \); hence in quantum W₁ distance, \( |1\rangle^\otimes n \) is further away from \( |0\rangle^\otimes n \) than \( |1\rangle^\otimes n^{-1} \).

Another example comes from considering the trace distance between \( |0\rangle^\otimes n \) and the Cat state \( |\text{Cat}_a\rangle = a|0\rangle^\otimes n + \sqrt{1 - |a|^2}|1\rangle^\otimes n \) with \( 0 < |a| < 1 \). The trace distance is \( \sqrt{1 - |a|^2} n \), which is very small when \( a \) is close to 1. However, the experimental resource, e.g., the number of gates, to transform \( |0\rangle^\otimes n \) to \( |\text{Cat}_a\rangle \) is proportional to the size of the system \( n \), independent of how close \( a \) is to 1. On the other hand, the quantum W₁ distance between \( |0\rangle^\otimes n \) and \( |\text{Cat}_a\rangle \) is \( \Omega((1 - |a|^2)n) \). This motivates us to consider the circuit complexity in terms of the quantum W₁ distance.

**Definition 1.** Given an n-qudit quantum channel \( \Lambda : D(\mathcal{H}) \to D(\mathcal{H}) \), the quantum Wasserstein complexity \( C_{W_1}(\Lambda) \) is the maximal distance between the input state and output state in quantum W₁ distance,

\[
C_{W_1}(\Lambda) := \max_{\rho \in D(\mathcal{H})} \|\rho - \Lambda(\rho)\|_{W_1}. \tag{2}
\]

By the convexity of the quantum W₁ distance, we need only to take the maximization over all pure states,

\[
C_{W_1}(\Lambda) = \max_{|\psi\rangle \in \mathcal{H}} \|\psi\rangle\langle\psi| - \Lambda(|\psi\rangle\langle\psi|)\|_{W_1}. \tag{3}
\]

To demonstrate applications of the quantum Wasserstein complexity, let us first study its basic properties.

**Proposition 2.** The quantum Wasserstein complexity \( C_{W_1}(\Lambda) \) satisfies the following properties:

1. **Faithfulness:** \( C_{W_1}(\Lambda) = 0 \) if and only if \( \Lambda \) is the identity map;

2. **Convexity:** \( C_{W_1}(\sum_i p_i \Lambda_i) \leq \sum_i p_i C_{W_1}(\Lambda_i) \), where \( p_i \geq 0 \) and \( \sum_i p_i = 1 \);

3. **Subadditivity under concatenation:** \( C_{W_1}(\Lambda_1 \otimes \Lambda_2) \leq C_{W_1}(\Lambda_1) + C_{W_1}(\Lambda_2) \);

4. **Cone additivity:** \( C_{W_1}(\Lambda_1 \otimes I) + C_{W_1}(\Lambda_2 \otimes I) \);

5. **Superadditivity under tensorization:** \( C_{W_1}(\Lambda_1 \otimes \Lambda_2) \geq C_{W_1}(\Lambda_1) + C_{W_1}(\Lambda_2) \);

6. **For a unitary channel \( U \), \( C_{W_1}(U) = C_{W_1}(U^\dagger) \);

7. **If the quantum channel \( \Lambda \) acts nontrivially on a k-qudit subsystem, then we have \( C_{W_1}(\Lambda) \leq k \).**
We prove Proposition 2 in Appendix B. Note that there exists a quantum channel $\Lambda$, such that the inequality (5) holds strictly. For example, for the single-qubit depolarizing channel $D_p(\cdot) = p\cdot I + (1-p)\text{Tr}[\cdot]/d$, it holds that $C_{W_1}(D_p) = (1-p)(1-1/d)$ and $C_{W_1}(D_p \otimes I) = (1-p)(1-1/d^2)$. Hence, the inequality in (5) holds strictly. See more examples in Table I.

The effects of the environment on quantum systems are usually taken into consideration, which means the ancilla qudits should be considered.

**Definition 3.** Given an $n$-qudit quantum channel $\Lambda : D(\mathcal{H}) \rightarrow D(\mathcal{H})$, the correlation-assisted quantum Wasserstein complexity $AC_{W_1}$ for an $n$-qudit quantum channel $\Lambda$ is defined as follows

$$AC_{W_1}(\Lambda) := \sup_{m \in \mathbb{Z}_{\geq 0}} C_{W_1}(\Lambda \otimes I_m),$$

where $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers, and $I_m$ denotes the $m$-qudit identity operator.

Note that $C_{W_1}(\Lambda \otimes I_m) \geq C_{W_1}(\Lambda \otimes I_{m-1}), \forall m \in \mathbb{Z}_{\geq 0}$ and $C_{W_1}(\Lambda) \leq n$. Hence, $AC_{W_1}$ is well-defined.

**Proposition 4.** The correlation-assisted Wasserstein complexity $AC_{W_1}(\Lambda)$ satisfies the following properties:

1. **Faithfulness**: $AC_{W_1}(\Lambda) = 0$ if and only if $\Lambda$ is the identity map;
2. **Convexity**: $AC_{W_1}(\sum_i p_i \Lambda_i) \leq \sum_i p_i AC_{W_1}(\Lambda_i)$, where $p_i \geq 0$ and $\sum_i p_i = 1$;
3. **Subadditivity under concatenation**: $AC_{W_1}(\Lambda_1 \circ \Lambda_2) \leq AC_{W_1}(\Lambda_1) + AC_{W_1}(\Lambda_2)$;
4. **Additivity under tensorization**: $AC_{W_1}(\Lambda_1 \otimes \Lambda_2) = AC_{W_1}(\Lambda_1) + AC_{W_1}(\Lambda_2)$.

The proof of Proposition 4 is provided in Appendix B. Note that, unlike $C_{W_1}$, $AC_{W_1}$ is additive under tensorization. Based on the definition, it is easy to see that $AC_{W_1}(\Lambda) \geq C_{W_1}(\Lambda)$. Also, there exists some quantum channel $\Lambda$, e.g., the 1-qudit depolarizing channel, such that $AC_{W_1}(\Lambda) > C_{W_1}(\Lambda)$. (See the examples in Table I).

After studying the basic properties of $C_{W_1}$ and $AC_{W_1}$, let us consider an application of the quantum Wasserstein complexity to the study of the circuit complexity of quantum circuits. The circuit complexity of a unitary operator $U$ is defined as the minimum number of basic gates needed to generate $U$ [59–61] (See Fig. 1). Here, we consider the circuit cost of quantum circuits, which was introduced by Nielsen et al. [16–18] as a geodesic distance from the identity operator to the target unitary with respect to a given metric, and was shown to provide a useful lower bound for the quantum circuit complexity.

Here, the circuit cost of a unitary $U \in SU(d^n)$ with respect to traceless Hermitian operators $h_1, \ldots, h_m$, supported on 2 qudits and normalized as $\|h_i\|_{\infty} = 1$, is defined to be

$$\text{Cost}(U) := \inf \int_0^1 \sum_{j=1}^m |r_j(s)|ds,$$  \hspace{1cm} (5)

where the infimum in (5) is taken over all continuous functions $r_j : [0, 1] \rightarrow \mathbb{R}$ that satisfy $H(s) = \sum_{j=1}^m r_j(s)h_j$ and $U = \mathcal{P} \exp \left( -i \int_0^1 H(s)ds \right)$, where $\mathcal{P}$ denotes the path-ordering operator. (See Fig. 2)

For an $n$-qudit Hamiltonian acting nontrivially on a $k$-qudit subsystem, there is a simple upper bound on the total change of the quantum Wasserstein complexity through unitary evolution.

**Lemma 5.** Given an $n$-qudit system with a $k$-qudit Hamiltonian $H$, the change of the quantum Wasserstein complexity for the unitary evolution $U_t = \exp(-iHt)$ with any time interval $\Delta t$ is bounded as follows

$$C_{W_1}(U_{t+\Delta t}) - C_{W_1}(U_t) \leq k.$$ \hspace{1cm} (6)

**Proof.** This comes from the fact that $C_{W_1}(U_{t+\Delta t}) = C_{W_1}(U_{\Delta t} \circ U_t) \leq C_{W_1}(U_{t}) + C_{W_1}(U_{\Delta t})$, and $C_{W_1}(U_{\Delta t}) \leq k$. \hfill $\Box$

To quantify the change of the quantum Wasserstein complexity in an infinitesimally small time interval, we need to
introduce the quantum Wasserstein rate in the unitary dynamics generated by a Hamiltonian \( H \), which will be useful for the connection between the quantum Wasserstein complexity and the circuit complexity. Given an \( n \)-qudit Hamiltonian \( H \) and an \( n \)-qudit pure state \( |\psi\rangle \), the quantum Wasserstein rate of the unitary \( U_i = \exp(-iHt) \) on the state \( |\psi\rangle \) is defined as

\[
R_{W_i}(\psi, H) = \lim_{\Delta t \to 0} \frac{\| |\psi\rangle - U_{\Delta t}|\psi\rangle\|}{\|U_{\Delta t}\|_{W_i}}. \tag{7}
\]

**Theorem 6** (Small incremental quantum Wasserstein complexity). Given an \( n \)-qudit system with the Hamiltonian \( H \) acting on a \( k \)-qudit subsystem, and an \( n \)-qudit state \( |\psi\rangle \), one has

\[
R_{W_i}(\psi, H) \leq 2\sqrt{2k} \|H\|_{\infty}, \tag{8}
\]

where \( \|\cdot\|_{\infty} \) denotes the operator norm.

We prove Theorem 6 in Appendix D. The small incremental quantum Wasserstein complexity is a key point to show the connection between circuit cost and the quantum Wasserstein complexity, for which we have the following relationship.

**Theorem 7.** Given a unitary \( U \), the circuit cost \( \text{Cost}(U) \) is lower bounded by the quantum Wasserstein complexity \( C_{W_i}(U) \) as

\[
\text{Cost}(U) \geq 4\sqrt{2}C_{W_i}(U). \tag{9}
\]

We prove Theorem 7 in Appendix D. This relation can be generalized to the correlation-assisted quantum Wasserstein complexity \( AC_{W_i} \) defined in (4).

**Corollary 8.** Under the assumptions of Theorem 7, we have

\[
\text{Cost}(U) \geq 4\sqrt{2}AC_{W_i}(U). \tag{10}
\]

Previously, the experimental cost of the quantum circuits was investigated using a weighted version of the Bures distance [62]. Here we find a new connection between the experimental cost \( R_U \) and the quantum Wasserstein complexity \( C_{W_i}(U) \) as follows:

**Theorem 9.** The experimental cost \( R(U) \) of an \( n \)-qubit quantum circuit \( U = \prod_i U_i \) is bounded from below by the quantum Wasserstein complexity \( C_{W_i}(U) \) as

\[
R(U) \geq \frac{1}{2}C_{W_i}(U). \tag{12}
\]

The proof of Theorem 9 is presented in Appendix D. A similar argument also works for \( AC_{W_i} \) defined in (4).

**Corollary 10.** Under the same conditions as in Theorem 9, we have

\[
R(U) \geq \frac{1}{2}AC_{W_i}(U). \tag{13}
\]

**Example 1.** Consider the CNOT gate \( \exp(iH\pi/2) \), where \( H = \frac{1}{2}(I - Z_1 - X_2 + Z_1X_2) \). The experimental cost of CNOT is \( R(\text{CNOT}) = \pi \), and thus the quantum circuit \( U = \prod_i \text{CNOT}_{i,i+1} \), which can be used to generate the cat state \((|0\rangle^\otimes n + |1\rangle^\otimes n)/\sqrt{2} \), has the experimental cost \( R(U) = \pi n \). The quantum Wasserstein complexity is \( C_{W_i}(U) = \Theta(n) \). (See Table 1.) Thus, the quantum circuit \( U = \prod_i \text{CNOT}_{i,i+1} \) has an experimental cost which is equivalent to the quantum Wasserstein complexity, up to some constant factor. Also relation (12) holds.

### III. CONCLUSION

We introduce a new measure of quantum complexity, which we call quantum Wasserstein complexity. We provide a connection between this complexity and the circuit cost. This provides a useful lower bound for the circuit complexity of the shallow (or constant-depth) quantum circuits, in terms of the quantum Wasserstein complexity. We show that the quantum Wasserstein complexity provides a lower bound for the experimental cost of implementing quantum circuits.

Our results provide an application and operational interpretation of the quantum Wasserstein distance for shallow quantum circuits. This raises the interesting question: can one find a better bound for the circuit complexity of deep quantum circuits based on the quantum Wasserstein distance? To compute the quantum Wasserstein complexity, one needs to maximize

### TABLE I. We calculate (or estimate) the quantum Wasserstein complexity \( C_{W_i} \) and \( AC_{W_i} \) for several examples (See details in Appendix C). Here, \( f(n) = \Theta(n) \) means that there exist constants \( c_1, c_2 \) such that \( c_1n \leq f(n) \leq c_2n \) for any \( n \).

| Quantum Channel \( \Lambda \)                      | \( C_{W_i}(\Lambda) \)                     | \( AC_{W_i}(\Lambda) \)                     |
|---------------------------------------------|---------------------------------|---------------------------------|
| Single-qubit depolarizing channel \( D_p \) | \((1 - p)(1 - 1/d)\)            | \((1 - p)(1 - 1/d^2)\)               |
| Tensor product of depolarizing channels \( D_p^n \) | \(\geq n(1 - p)(1 - 1/d)\)     | \(\leq n(1 - p)(1 - 1/d^2)\)          |
| Tensor product of Hadamard gates \( H^{\otimes n} \) | \(n\)              | \(n\)                              |
| CNOT gates \( \prod_i \text{CNOT}_{i,i+1} \) | \(\Theta(n)\)                  | \(\Theta(n)\)                      |
over all pure states, making it difficult to compute. So one naturally asks: can one discover an efficient way to approximate the quantum Wasserstein complexity? It is also interesting to study quantum Wasserstein distances of order other than 1, and how they relate to circuit complexity in quantum computation.

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Appendix A: Basic properties of quantum Wasserstein norm of order 1

We state three basic properties of the quantum Wasserstein norm of order 1 [53], which is defined for all traceless, Hermitian operators $A$:

$$\|A\|_{W_1} := \frac{1}{2} \min_{\{x_i\}} \left\{ \sum_i \|x_i\|_1 : A = \sum_i x_i, \ x_i \in O(H), \ Tr[x_i] = 0 \ \forall i \right\}.$$  

Lemma 11 (De Palma et al. [53]). (Tensorization) Given an n-qudit traceless, Hermitian operator $A$, and a subset $S \subset [n]$ with complement $S^c$,

$$\|A\|_{W_1} \geq \|\text{Tr}_S[A]\|_{W_1} + \|\text{Tr}_{S^c}[A]\|_{W_1}.$$  \hfill (A1)

Lemma 12 (De Palma et al. [53]). Given an n-qudit traceless, Hermitian operator $A$ and a subset $S \subset [n]$ with $\text{Tr}_S[A] = 0$,

$$\|A\|_{W_1} \leq \frac{1}{2} \text{Tr}[A^2].$$  \hfill (A2)

Hence, for any n-qudit quantum channel $\Lambda$ acting nontrivially on k-qudit subsystems,

$$\|\rho - \Lambda(\rho)\|_{W_1} \leq k.$$  \hfill (A3)

Lemma 13 (De Palma et al. [53]). Given an n-qudit traceless, Hermitian operator $A$, we have

$$\frac{1}{2} \|A\|_1 \leq \|A\|_{W_1} \leq \frac{n}{2} \|A\|_1.$$  \hfill (A4)

Moreover, if there exists some $i$ such that $\text{Tr}_i[A] = 0$, then we have

$$\|A\|_{W_1} = \frac{1}{2} \|A\|_1.$$  \hfill (A5)

Appendix B: Basic properties of quantum Wasserstein complexity

Proposition 14 (Restatement of Proposition 2). The quantum Wasserstein complexity $C_{W_1}$ satisfies the following properties:

1. Faithfulness: $C_{W_1}(\Lambda) = 0$ if and only if $\Lambda$ is the identity map;
2. Convexity: $C_{W_1}(\sum_i p_i \Lambda_i) \leq \sum_i p_i C_{W_1}(\Lambda_i)$, where $p_i \geq 0$ and $\sum_i p_i = 1$;
3. Subadditivity under concatenation: $C_{W_1}(\Lambda_1 \circ \Lambda_2) \leq C_{W_1}(\Lambda_1) + C_{W_1}(\Lambda_2)$;
4. $C_{W_1}(\Lambda_1 \otimes I) + C_{W_1}(I \otimes \Lambda_2) \leq C_{W_1}(\Lambda_1 \otimes \Lambda_2)$;
5. Superadditivity under tensorization: $C_{W_1}(\Lambda_1 \otimes \Lambda_2) \geq C_{W_1}(\Lambda_1) + C_{W_1}(\Lambda_2)$;
6. For a unitary channel $U$, $C_{W_1}(U) = C_{W_1}(U^T)$;
7. If $\Lambda$ acts nontrivially on a k-qudit subsystem, then we have $C_{W_1}(\Lambda) \leq k$.

Proof. Property (1) follows directly from the faithfulness of the quantum $W_1$ norm, properties (2)–(4) follow directly from the triangle inequality of the quantum $W_1$ norm, and property (5) follows from the tensorization of $W_1$ norm (see Lemma 11) as follows

$$\|\rho - \Lambda_1 \otimes \Lambda_2(\rho)\|_{W_1} \geq \|\rho_1 - \Lambda_1(\rho_1)\|_{W_1} + \|\rho_2 - \Lambda_2(\rho_2)\|_{W_1},$$

where $\rho_i$ (for $i = 1, 2$) denotes the corresponding reduced state. Property (6) follows directly from the definition of the quantum Wasserstein complexity. Finally, Property (7) follows from Lemma 12. \hfill $\Box$

Proposition 15 (Restatement of Proposition 4). The correlation-assisted Wasserstein complexity $AC_{W_1}(\Lambda)$ satisfies the following properties:

1. Faithfulness: $AC_{W_1}(\Lambda) = 0$ if and only if $\Lambda$ is the identity map:
(2) Convexity: \( AC_W(\sum p_i A_i) \leq \sum p_i AC_W(A_i) \), where \( p_i \geq 0 \) and \( \sum p_i = 1 \);

(3) Subadditivity under concatenation: \( AC_W(A_1 \circ A_2) \leq AC_W(A_1) + AC_W(A_2) \);

(4) Additivity under tensorization: \( AC_W(A_1 \otimes A_2) = AC_W(A_1) + AC_W(A_2) \).

Proof. The property (1) follows from the faithfulness of the quantum \( W \) distance. Properties (2) and (3) follow from the triangle inequality of the quantum \( W \) distance. Hence, it suffices for us to prove the property (4). Since

\[
C_W(A_1 \otimes A_2) \leq C_W(A_1 \otimes I) + C_W(A_2 \otimes I),
\]

by the definition of \( EW_1 \), we have

\[
\text{AC}_W(A_1 \otimes A_2) \leq \text{AC}_W(A_1) + \text{AC}_W(A_2).
\]

Besides, since

\[
C_W(A_1 \otimes A_2) \geq C_W(A_1) + C_W(A_2),
\]

we have

\[
\text{AC}_W(A_1 \otimes A_2) \geq \text{AC}_W(A_1) + \text{AC}_W(A_2).
\]

\[\square\]

Appendix C: Estimation of \( C_W \) and \( AC_W \): interesting examples

Let us start with the simplest case, where \( U \) is a single-qudit unitary.

Claim 16. For a 1-qudit unitary \( U \), we have

\[
C_W(U) = AC_W(U).
\]

Proof. For any 1-qudit unitary and pure state \( |\psi\rangle \), we have

\[
\| |\psi\rangle\langle\psi| - U|\psi\rangle\langle\psi|U^\dagger\|_W = \frac{1}{2} \| |\psi\rangle\langle\psi| - U|\psi\rangle\langle\psi|U^\dagger\|_1 = \sqrt{1 - |\langle\psi|U|\psi\rangle|^2}.
\]

Let \( U = \sum_j \exp(i\theta_j)|a_j\rangle\langle a_j| \) be the eigenvalue decomposition of \( U \). Then, the state \( |\psi\rangle \) can be written as \( |\psi\rangle = \sum_j c_j|a_j\rangle \), from which it follows that \( |\langle\psi|U|\psi\rangle|^2 = |\sum_j c_j^2 \exp(i\theta_j)|^2 = |\sum_j d_j \exp(i\theta_j)|^2 \), where \( d_j = |c_j|^2 \). Hence

\[
C_W(U) = \max_{\psi} \| |\psi\rangle\langle\psi| - U|\psi\rangle\langle\psi|U^\dagger\|_W = \max_{\psi} \sqrt{1 - |\langle\psi|U|\psi\rangle|^2} = \sqrt{1 - \min_{d_j} \sum_j d_j \exp(i\theta_j)^2}.
\]

Let us consider \( U \otimes I_m \) for any \( m \geq 1 \). For any \((m+1)\)-qudit state \( |\psi\rangle = \sum_j \sqrt{\lambda_j} |j\rangle \), with \( |j\rangle \) being a basis on \( m \)-qudit systems and \( \sum_j \lambda_j = 1 \), we have

\[
\langle\psi|U \otimes I_m|\psi\rangle = \sum_j \lambda_j \langle j|U|j\rangle,
\]

where for each \( j \), \( |j\rangle = \sum_k \sqrt{\lambda_k e^{i\theta_k}} |a_k\rangle \), and

\[
\langle j|U|j\rangle = \sum_k d_{jk} \exp(i\theta_k).
\]

Hence

\[
\langle\psi|U \otimes I_m|\psi\rangle = \sum_{j,k} \lambda_j d_{jk} \exp(i\theta_k) = \sum_k f_k \exp(i\theta_k),
\]

where \( f_k = \sum_j \lambda_j d_{jk} \), and \( \sum_k f_k = 1 \). Hence, we have the following expression for \( AC_W(U) \):

\[
AC_W(U) = \sup_m \max\| |\psi\rangle\langle\psi| - U \otimes I_m|\psi\rangle\langle\psi|U^\dagger \otimes I_m\|_W = \sqrt{1 - \min f_k \sum f_k \exp(i\theta_k)^2}.
\]
which implies that $AC_{W_1}(U) = C_{W_1}(U)$.

**Example 1: Depolarizing channel**

Let us consider the single-qudit depolarizing channel, for which we have the following statement.

**Claim 17.** For the 1-qudit depolarizing channel $D_p(\cdot) = p(\cdot) + (1 - p) \text{Tr} [\cdot] I / d$, we have

$$C_{W_1}(D_p) = (1 - p)(1 - 1/d),$$

and

$$AC_{W_1}(D_p) = (1 - p)(1 - 1/d^2).$$

**Proof.** For any 1-qudit state, we have

$$\|\langle \psi | - D_p(\langle \psi | \psi \rangle) \|_{W_1} = \frac{1}{2} \|\langle \psi | - D_p(\langle \psi | \psi \rangle) \|_1 = (1 - p)(1 - 1/d).$$

Hence, we have

$$C_{W_1}(D_p) = (1 - p)(1 - 1/d).$$

Now, let us take a 2-qudit state $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_j |j\rangle |j\rangle$ and the quantum channel $D_p \otimes I$. Then,

$$\|\langle \psi | - D_p \otimes I(\langle \psi | \psi \rangle) \|_{W_1} = \frac{1}{2} \|\langle \psi | - D_p \otimes I(\langle \psi | \psi \rangle) \|_1 = (1 - p)(1 - 1/d^2).$$

This implies that

$$AC_{W_1}(D_p) \geq (1 - p)(1 - 1/d^2).$$

Besides, let us prove that for any integer $m \geq 0$,

$$C_{W_1}(D_p \otimes I_m) \leq (1 - p)(1 - 1/d^2).$$

For any $(m + 1)$-qudit state $|\psi\rangle$, we have

$$\|\langle \psi | - D_p \otimes I_m(\langle \psi | \psi \rangle) \|_{W_1} = (1 - p) \|\langle \psi | - \frac{1}{d} \otimes \text{Tr}_1 [\langle \psi | \psi \rangle] \|_{W_1} \leq (1 - p) \|\langle \psi | - \frac{1}{d^2} \sum_{s,t \in Z_d} X^s Z^t |\psi\rangle \langle Z^{-t} X^{-s} |\psi\rangle \|_{W_1} \leq (1 - p) \|\langle \psi | - X^s Z^t |\psi\rangle \langle Z^{-t} X^{-s} |\psi\rangle \|_1,$$

where the third line comes from the fact that

$$\frac{1}{d^2} \sum_{(s,t) \in Z_d \times Z_d} X^{s} Z^{t} Z^{-t} X^{-s} = \text{Tr} [I / d],$$

the fifth line comes from the convexity of the quantum $W_1$ distance, and the last line comes from the fact that $\text{Tr}_1 [\langle \psi | - X^s Z^t |\psi\rangle \langle Z^{-t} X^{-s} |\psi\rangle] = 0$, and thus

$$\|\langle \psi | - X^s Z^t |\psi\rangle \langle Z^{-t} X^{-s} |\psi\rangle \|_{W_1} = \frac{1}{2} \|\langle \psi | - X^s Z^t |\psi\rangle \langle Z^{-t} X^{-s} |\psi\rangle \|_1.$$
Corollary 18. For the n-fold tensor product of the depolarizing channel $D^\otimes_n^p$, we have

$$n(1 - p)(1 - 1/d) \leq C_{W_1}(D^\otimes_n^p) \leq n(1 - p)(1 - 1/d^2)$$  \hspace{1cm} (C5)

and

$$AC_{W_1}(D^\otimes_n^p) = n(1 - p)(1 - 1/d^2).$$  \hspace{1cm} (C6)

Example 2: n-fold tensor product of Hadamard gates

Let us consider the n-fold tensor product of Hadamard gates, i.e., $H^\otimes^n$. The quantum Wasserstein complexity of $H^\otimes^n$ is

$$C_{W_1}(H^\otimes^n) = n.$$  \hspace{1cm} (C7)

Proof of (C7). The inequality $C_{W_1}(H^\otimes^n) \leq n$ holds because $\|\rho - \sigma\|_{W_1} \leq \frac{n}{2} \|\rho - \sigma\|_1 \leq n$, which follows from Lemma 12. To prove that $C_{W_1}(H^\otimes^n) \geq n$, we use the fact that for the Hadamard gate $H$, there exists some pure state $|\phi\rangle$ such that $H|\phi\rangle \perp |\phi\rangle$. Hence,

$$C_{W_1}(H^\otimes^n) \geq \|\langle H|\phi\rangle\langle H^\otimes^n|\phi\rangle - |\phi\rangle\langle\phi\rangle\|_{W_1}$$

$$= \sum_i \|\langle H|\phi\rangle\langle H^\otimes^n|\phi\rangle - |\phi\rangle\langle\phi\rangle\|_{W_1}$$

$$= \frac{1}{2} \sum_i \|\langle H|\phi\rangle\langle H^\otimes^n|\phi\rangle - |\phi\rangle\langle\phi\rangle\|_1$$

$$= n,$$

where the first line follows from the definition of $C_{W_1}$; the second line follows from the tensorization of the quantum $W_1$ distance, i.e., $\|\rho_1 \otimes \rho_2 - \sigma_1 \otimes \sigma_2\|_{W_1} = \|\rho_1 - \sigma_1\|_{W_1} + \|\rho_2 - \sigma_2\|_{W_1}$ [53]; the third line follows from the fact that $\|\rho - \sigma\|_{W_1} = \frac{1}{2} \|\rho - \sigma\|_1$ for 1-qudit states $\rho, \sigma$ [53]; and the last line follows from the fact that $\|\rho_1 - \sigma_1\|_1 = 1$ as $H|\phi\rangle \perp |\phi\rangle$.

Besides, as $C_{W_1}(H^\otimes^n \otimes I_n) \leq n$, we also have

$$AC_{W_1}(H^\otimes^n) = n.$$  \hspace{1cm} (C8)

Example 3: CNOT gates

Let us consider the n-qubit quantum circuit $\prod_i \text{CNOT}_{i,i+1}$ (See Fig. 3). Then we have

$$C_{W_1}\left(\prod_i \text{CNOT}_{i,i+1}\right) = \Theta(n),$$  \hspace{1cm} (C9)

where $f(n) = \Theta(n)$ means that there exist constants $c_1, c_2$ such that $c_1 n \leq f(n) \leq c_2 n$ for any $n$. (1) The upper bound $C_{W_1}\left(\prod_i \text{CNOT}_{i,i+1}\right) \leq n$ holds for the same reason as the one in the above example. (2) Next, we show that $C_{W_1}\left(\prod_i \text{CNOT}_{i,i+1}\right) \geq n/2$. To this end, let us take the input state to be $|+\rangle \otimes |0\rangle^{n-1}$. Then, the output state is $\prod_i \text{CNOT}_{i,i+1} |+\rangle \otimes |0\rangle^{n-1} = \frac{1}{\sqrt{2}}(|0\rangle^n + |1\rangle^n)$, which we denote as $|\text{Cat}_{1/2}\rangle$. Hence

$$C_{W_1}\left(\prod_i \text{CNOT}_{i,i+1}\right) \geq \| |+\rangle \langle +| \otimes |0\rangle^{n-1} - |\text{Cat}_{1/2}\rangle \langle \text{Cat}_{1/2}|\|_{W_1}$$

$$\geq \frac{1}{2} \sum_i \|\rho_i - \sigma_i\|_1 = n/2,$$

where $\rho_i$ (or $\sigma_i$) is the reduced state of $|+\rangle \langle +| \otimes |0\rangle^{n-1}$ (or $|\text{Cat}_{1/2}\rangle \langle \text{Cat}_{1/2}|$) on the $i$-th position, and the second inequality comes from the tensorization of the quantum $W_1$ distance.

Hence, we also have

$$AC_{W_1}\left(\prod_i \text{CNOT}_{i,i+1}\right) = \Theta(n).$$  \hspace{1cm} (C10)
Appendix D: Application in quantum circuit complexity and experimental cost

Lemma 19. Given an $n$-qudit system with a Hamiltonian $H$ acting on a $k$-qudit subsystem, and an $n$-qudit state $\psi$, for the unitary $U_\Delta = \exp(-iH\Delta)$, one has

$$\left\| \langle \psi | - U_\Delta | \psi \rangle \psi U_\Delta^\dagger \right\|_{W_1} \leq 2\sqrt{2} \frac{d^2 - 1}{d^2} \| H \|_\infty |\Delta| e^{\| H \|_\infty |\Delta|},$$

which implies that

$$R_{W_1}(\psi, H) \leq 2\sqrt{2} \| H \|_\infty.$$  \hfill (D1)

Proof. Since $H$ acts on a $k$-qudit subsystem, there exists a subsystem $S$ for which $\text{Tr}_S \left[ |\psi\rangle\langle \psi | - U_\Delta |\psi \rangle \psi U_\Delta^\dagger \right] = 0$. Hence, by Lemma 12, we have

$$\left\| |\psi\rangle\langle \psi | - U_\Delta |\psi \rangle \psi U_\Delta^\dagger \right\|_{W_1} \leq d^2 - 1 \left\| |\psi\rangle\langle \psi | - U_\Delta |\psi \rangle \psi U_\Delta^\dagger \right\|_1 = 2d^2 - 1 \sqrt{1 - |\langle \psi \rangle U_\Delta \rangle |^2},$$

Let us define $|\psi_\Delta\rangle = U_\Delta |\psi \rangle$. Then, the Taylor expansion of $|\psi_\Delta\rangle\langle \psi_\Delta |$ is

$$|\psi_\Delta\rangle\langle \psi_\Delta | = |\psi\rangle\langle \psi | + i\Delta t [H, |\psi\rangle\langle \psi |] + \frac{(i\Delta t)^2}{2!} [H, [H, |\psi\rangle\langle \psi |]] + \cdots$$

where $[A, B] = AB - BA$ denotes the commutator between $A$ and $B$. Then

$$1 - |\langle \psi | U_\Delta \rangle |^2 = 1 - \left[ \text{Tr} \left( |\psi\rangle\langle \psi |^2 \right) + i\Delta t \text{Tr} \left( |\psi\rangle\langle \psi | [H, |\psi\rangle\langle \psi |] \right) + \frac{(i\Delta t)^2}{2!} \text{Tr} \left( |\psi\rangle\langle \psi | [H, [H, |\psi\rangle\langle \psi |]] \right) + \cdots \right] = -\frac{(i\Delta t)^2}{2!} \text{Tr} \left( |\psi\rangle\langle \psi | [H, [H, |\psi\rangle\langle \psi |]] \right) + \cdots$$

where $\text{Tr} \left( |\psi\rangle\langle \psi | [H, |\psi\rangle\langle \psi |] \right) = 0$. Let us define $\text{ad}_H$ as $\text{ad}_H(A) := [H, A]$. Then, the above formula can be rewritten as

$$1 - |\langle \psi | U_\Delta \rangle |^2 = -\sum_{k \geq 2} \frac{(i\Delta t)^k}{k!} \text{Tr} \left( |\psi\rangle\langle \psi | \text{ad}_H^k(|\psi\rangle\langle \psi |) \right).$$

For each term $\text{Tr} \left( |\psi\rangle\langle \psi | \text{ad}_H^k(|\psi\rangle\langle \psi |) \right)$, by Hölder’s inequality, we have

$$\left| \text{Tr} \left( |\psi\rangle\langle \psi | \text{ad}_H^k(|\psi\rangle\langle \psi |) \right) \right| \leq \left\| \text{ad}_H^k(|\psi\rangle\langle \psi |) \right\|_\infty \leq \| H \|_\infty^k 2^k,$$

which implies that

$$|1 - |\langle \psi | U_\Delta \rangle |^2| \leq 2 \| H \|_\infty^2 |\Delta| \exp(2 \| H \|_\infty |\Delta|).$$

Hence, we have

$$\left\| |\psi\rangle\langle \psi | - U_\Delta |\psi \rangle \psi U_\Delta^\dagger \right\|_{W_1} \leq 2\sqrt{2} d^2 - 1 \| H \|_\infty |\Delta| e^{\| H \|_\infty |\Delta|}. \hfill (D3)$$

FIG. 3. A circuit diagram for the circuit of cascading CNOT gates $\prod_i \text{CNOT}_{i,i+1}$. 

\hfill \square
Theorem 20. Given a unitary $U$, the circuit cost $\text{Cost}(U)$ is lower bounded in terms of $C_{W_1}(U)$ as follows

$$\text{Cost}(U) \geq 4\sqrt{2}C_{W_1}(U).$$

(D5)

Proof. First, let us take a Trotter decomposition of $U$ such that for arbitrarily small $\varepsilon > 0$,

$$\|U - V_N\|_\infty \leq \varepsilon,$$

where $V_N$ is defined as follows

$$V_N := \prod_{i=1}^{N} W_i,$$

$$W_i := \exp \left(-\frac{i}{N} \sum_{j=1}^{m} r_j \left(\frac{1}{N}\right) h_j\right).$$

and

$$W_i = \lim_{l \to \infty} W_i^{(l)},$$

$$W_i^{(l)} := (W_{i,1}^{1/l} \cdots W_{i,m}^{1/l})^l,$$

$$W_{i,j} := \exp \left(-\frac{i}{N} r_j \left(\frac{1}{N}\right) h_j\right).$$

where $\|h_j\|_\infty \leq 1$ for any $j$. Let us define $|\psi_1\rangle = W_i |\psi_{i-1}\rangle$ with $|\psi_0\rangle = |\psi\rangle$, then by the triangle inequality of the quantum $W_1$ distance, we have

$$\|\langle \psi | V_N | \psi \rangle V_N^2\|_{W_i} \leq \sum_{i=1}^{N} \|\langle \psi_{i-1} | \psi_{i-1} \rangle - |\psi_i\rangle \langle \psi_i |\|_{W_i}.$$  

(D6)

For each $\|\langle \psi_{i-1} | \psi_{i-1} \rangle - |\psi_i\rangle \langle \psi_i |\|_{W_i}$, we have

$$\|\langle \psi_{i-1} | \psi_{i-1} \rangle - |\psi_i\rangle \langle \psi_i |\|_{W_i} = \|\langle \psi_{i-1} | \psi_{i-1} \rangle - W_i |\psi_{i-1}\rangle \langle \psi_{i-1} | W_i^\dagger \|_{W_i}$$

$$\leq \lim_{l \to \infty} \sum_{j} \frac{4\sqrt{2}}{N} |r_j \left(\frac{1}{N}\right)| \exp \left(\frac{1}{N} r_j \left(\frac{1}{N}\right)\right)$$

$$= \frac{4\sqrt{2}}{N} \sum_{j} |r_j \left(\frac{1}{N}\right)|,$$

where the last inequality comes from the triangle inequality of quantum $W_1$ distance and Lemma 19 by taking $\Delta t = \frac{1}{l}$. Therefore,

$$\|\langle \psi | V_N | \psi \rangle V_N^2\|_{W_i} \leq \frac{4\sqrt{2}}{N} \sum_{i=1}^{N} \sum_{j=1}^{m} |r_j \left(\frac{1}{N}\right)|.$$  

(D7)

Since the circuit cost can be expressed as

$$\text{Cost}(U) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{m} |r_j \left(\frac{1}{N}\right)|,$$

we have

$$\text{Cost}(U) \geq 4\sqrt{2}C_{W_1}(U).$$

(D8)

Note that Lemma 19 also holds for $(n+m)$-qudit systems, and hence the statement also holds for $AC_{W_1}$. Therefore, the following corollary follows immediately.

Corollary 21. Given a unitary $U$, the circuit cost $\text{Cost}(U)$ is lower bounded in terms of $AC_{W_1}(U)$ as follows:

$$\text{Cost}(U) \geq 4\sqrt{2}AC_{W_1}(U).$$

(D9)
Theorem 22. The experimental cost $\mathcal{R}_U$ of an $n$-qubit quantum circuit $U = \prod U_i$ is lower bounded in terms of $C_{W_1}(U)$ as follows:

$$\mathcal{R}(U) \geq \frac{1}{2} C_{W_1}(U).$$  \hfill (D10)

Proof. For any pure $n$-qubit state $|\psi\rangle$, let us define

$$|\psi_{i+1}\rangle := U_i |\psi_i\rangle, \quad \text{with} \quad |\psi_i\rangle := |\psi\rangle.$$  \hfill (D11)

And we also define

$$|\psi_{i,t}\rangle := e^{iH_{t-i}} |\psi_i\rangle,$$

as the intermediate state at time $t_i \in [0, T_i]$ while implementing $U_i$ with $|\psi_{i,0}\rangle = |\psi_i\rangle$, $|\psi_{i,T_i}\rangle = |\psi_{i+1}\rangle$.

Since $U_i$ acts on only $k_i$ qubits, by Lemma 12, we have

$$\|\psi_{i+1}\rangle\langle\psi_{i+1}| - |\psi_i\rangle\langle\psi_i|\|_{W_1} \leq k_i \|\psi_{i+1}\rangle\langle\psi_{i+1}| - |\psi_i\rangle\langle\psi_i|\|_1.$$  \hfill (D13)

Besides, it was shown in [70] that the quantum speed limit for the trace norm is bounded by the trace norm of the derivative of the state, that is,

$$\|\psi_{i+1}\rangle\langle\psi_{i+1}| - |\psi_i\rangle\langle\psi_i|\|_1 \leq \int_0^{T_i} \|d|\psi_{i,t}\rangle\langle\psi_{i,t}|/dt\|_1 dt,$$

where $d|\psi_{i,t}\rangle\langle\psi_{i,t}|/dt = de^{iH_{t-i}} |\psi_i\rangle\langle\psi_i|e^{-iH_{t-i}}/dt = i[H_{t-i}, |\psi_{i,t}\rangle\langle\psi_{i,t}|]$. The trace norm of the derivative of the state under unitary evolution $e^{iH_{t}}$ is bounded above by the square root of the variance of the Hamiltonian $H_{t}$, i.e.,

$$\|d|\psi_{i,t}\rangle\langle\psi_{i,t}|/dt\|_1 \leq 2 \sqrt{\text{Var}_{\psi_{i,t}}[H_{t}]}.$$  \hfill (D15)

where

$$\text{Var}_{\psi_{i,t}}[H_{t}] = \text{Tr} \left[H_{t}^2 |\psi_{i,t}\rangle\langle\psi_{i,t}| \right] - \text{Tr} \left[H_{t} |\psi_{i,t}\rangle\langle\psi_{i,t}| \right]^2.$$  \hfill (D16)

Note that $\text{Var}_{\psi_{i}}[H_{t}] \leq E_t^2$ with $E_t = (h_{2i} - h_1)/2$, and so we have

$$\|\psi_{i+1}\rangle\langle\psi_{i+1}| - |\psi_i\rangle\langle\psi_i|\|_1 \leq 2E_i T_i.$$  \hfill (D17)

Hence,

$$\|\psi_{i+1}\rangle\langle\psi_{i+1}| - |\psi_i\rangle\langle\psi_i|\|_{W_1} \leq 2k_i E_i T_i.$$  \hfill (D18)

Therefore, for the quantum circuit $U = \prod U_i$ with any input state $|\psi\rangle$,

$$\|\psi\langle\psi - U|\psi\rangle\langle\psi|^U\|_{W_1} \leq \sum_i \|\psi_{i+1}\rangle\langle\psi_{i+1}| - |\psi_i\rangle\langle\psi_i|\|_{W_1} \leq \sum_i k_i \|\psi_{i+1}\rangle\langle\psi_{i+1}| - |\psi_i\rangle\langle\psi_i|\|_1 \leq 2 \sum_i k_i E_i T_i = 2\mathcal{R}(U),$$

where the first inequality comes from the triangle inequality, the second inequality comes from (D13), and the third inequality comes from (D17). \hfill \square

The proof given for the above theorem also works for the case where we have $m$ qubits as ancillas. Hence, the following corollary follows immediately.

Corollary 23. The experimental cost of an $n$-qubit quantum circuit $\mathcal{R}(U)$ is lower bounded in terms of $AC_{W_1}(U)$ as follows:

$$\mathcal{R}(U) \geq \frac{1}{2} AC_{W_1}(U).$$  \hfill (D19)