Recovery of Images with Missing Pixels using a Gradient Compressive Sensing Algorithm

Isidora Stanković, Student Member IET

Abstract: This paper investigates the possibility of reconstruction of images considering that they are sparse in the DCT transformation domain. Two approaches are considered. One when the image is pre-processed in the DCT domain, using 8x8 blocks. The image is made sparse by setting the smallest DCT coefficients to zero. In the other case the original image is considered without pre-processing, assuming the sparsity as intrinsic property of the analyzed image. A gradient based algorithm is used to recover a large number of missing pixels in the image. The case of a salt-and-paper noise affecting a large number of pixels is easily reduced to the case of missing pixels and considered within the same framework. The reconstruction of images affected with salt-and-paper impulsive is compared with the images filtered using a median filter. The same algorithm can be used considering transformation of the whole image. Reconstructions of black and white and colour images are considered.

Key words: image processing, compressive sensing, recovery, pepper and salt noise, median filter

1. Introduction

One of the basic signal processing and information theory results is the sampling or Shannon-Nyquist theorem. This theorem has provided a possibility to represent a continuous-time signal by a set of numbers representing the signal values at discrete time instants. The Nyquist theorem states that as long as the sampling frequency is at least twice as high as the maximum frequency present in the signal the signal can be fully recovered from the samples. If that requirement is met, the reconstruction will be alias-free. Shannon-Nyquist is a widely used technique in digital signal processing. The sampling theorem can be restated in the following way. If the Fourier transform of a signal is not of an infinite duration, but it is located within a frequency range defined by a maximal absolute negative value and maximal positive value, then in order to reconstruct the signal we do not need to know all signals values. It is sufficient to know just the signal values at some time instances defined by the sampling theorem.

Compressive Sensing (or Compressed Sampling) is a new, growing field in signal processing with an intensive development in the last ten years. In the centre of compressive sensing is the notion of a sparse signal. As it is known a signal can be transformed to different domains with a possibility to obtain the exact signal via the inverse transform. A signal is sparse in a transformation domain if the number of non-zero coefficients in that domain is much fewer than the number of signal samples. For a signal that has just a small number of non-zero coefficients in a transformation domain we can expect that a small number of non-zero coefficients can be recovered from a small set of signal samples. This is exactly the basic goal of the CS, to use a small number of randomly positioned samples to reconstruct all values of a sparse signal [1]-[6]. Since the introduction of CS, various techniques are developed to reconstruct a signal. They belong to two large groups. One of them is based on the signal analysis and reconstruction in the transformation domain where the signal is sparse and the other one is gradient based. The CS is already used in many signal processing areas. Image-related applications, such as medical imaging, signal compression, denoising, photography, holography, facial recognition, radar and array signal processing are some of the applications.

Real images usually have just a few DCT coefficients within each transformation block that should be considered as non-zero. Thus, they comply with the CS algorithm requirement that the image is a sparse signal in
the considered transformation domain [7]. An approach of
reconstructing when a large number of the missing pixels
is randomly positioned over the whole image will be
considered.

The paper is organized as follows. The problem
formulation is given in the section. A reconstruction
algorithm, with image recovery examples, is presented in
Section 3. The case of impulsive salt-and-paper noise is
redefined into the CS framework in Section 4. The results
are compared with median filtered results as well. The
conclusions are given in Section 5.

2. Basic Theory

The approach considered in this paper deals with
an image whose random set of pixels is available. The task
is to reconstruct the pixels that are not available. To make
the reconstruction possible, the picture sparsity in a
transformation (in this case DCT) domain should be
assumed. The DCT of an image is commonly calculated
using its into 8x8 blocks. Most of the images could be
considered as sparse in the DCT domain without any
processing. If we want to be sure that the original image,
which will be processed in our examples, is sparse we can
calculate the DCT of its 8x8 blocks and make some of the
coefficients, with lowest amplitudes, to be equal to zero.
Making the image sparse in the DCT domain in this way
will not make a big visual difference with respect to the
original image. Note that even if we do not use this step,
most of the images are sparse in the DCT domain, and the
presented methods can be used without pre-processing of
the original image. In Figure 1 the original image “Isi”
and a version with pre-processed DCT coefficients to
make it exactly sparse are presented. These images will be
used for most of our analysis and reconstruction.

An image that is sparse in a transformation
domain can be reconstructed from a very reduced set of
available pixels. Assume that the image values are
available only for the pixels

\[ x(n,m), \text{ for } (n,m) = (n_1,m_1), (n_2,m_2), \ldots, (n_M,m_M) \]  

(1)

Based on the available pixels we form an image \( y(n,m) \)
that assumes the values of the original image at the
positions of available pixels. For the pixels that are not
available we will set their value to zero. This new image
is:

\[ y(n,m) = \begin{cases} x(n,m), & \text{if } (n,m) = (n_1,m_1), \ldots, (n_M,m_M) \\ 0, & \text{elsewhere} \end{cases} \]  

(2)

For missing pixels we can assume any value in the initial
step, since the algorithm should reconstruct the true image
values at these positions. For graphical representation of
missing pixels we will use the value 255 instead of 0. The
missing pixels are then represented by the white colour
(blank), as in Figure 2.

Figure 1: (a) Original image; (b) Image sparse in the DCT
domain

Image with 62% missing pixels

Figure 2: Image with a large number of missing pixels
3. Reconstruction Algorithm

The gradient algorithm is used on 8x8 blocks of the picture. The blocks with reconstructed pixels are then recombined into the final image. The two-dimensional gradient algorithm is similar to the one-dimensional one [8]-[10] except that we use the position of the value in x (rows) and y (columns) directions rather than a value in a sequence. The $L_1$-norm is used over two-dimensional space of image, i.e. as a sum the DCT values over the rows and columns.

Algorithm is implemented as follows [11]:

- For the available image pixels $x(n,m)$, for $n=n_1,n_2,...,n_M$ and $m=m_1,m_2,...,m_M$, a signal $y(n,m) = x(n,m)$ is formed, within one block, where $n$ represents the row and $m$ represents column of the image pixel. Other values are set to zero:

$$y(n,m) = \begin{cases} x(n,m), & \text{if } (n,m) = (n_1,m_1),..., (n_M,m_M) \\ 0, & \text{elsewhere} \end{cases}$$  

3)

- Then for each of $k \neq n_1,n_2,...,n_M$ and each of $l \neq m_1,m_2,...,m_M$ within the same block we form:

$$y_1^{(k,l)}(n,m) = y(n,m) + \Delta \delta(n-k,m-l)$$

$$y_2^{(k,l)}(n,m) = y(n,m) - \Delta \delta(n-k,m-l)$$  

4)

- The 2D-DCT is calculated as

$$Y_1^{(k,l)}(p,q) = \|DCT[y_1^{(k,l)}(n,m)]\|$$

$$Y_2^{(k,l)}(p,q) = \|DCT[y_2^{(k,l)}(n,m)]\|$$  

5)

Where $L_1$-norm of 2D-DCT is:

$$\|Y_1(k,l)\| = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} |Y_1^{(k,l)}(p,q)|$$

$$\|Y_2(k,l)\| = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} |Y_2^{(k,l)}(p,q)|$$  

6)

The gradient corresponding to the change of the $(k,l)$-th pixel $k \neq n_1,n_2,...,n_M$ and $l \neq m_1,m_2,...,m_M$ is:

$$Gr(k,l) = \frac{\|Y_1(k,l)\| - \|Y_2(k,l)\|}{2\Delta}$$  

7)

Each missing signal value $k \neq n_1,n_2,...,n_M$, $l \neq m_1,m_2,...,m_M$ is then changed in the direction of the gradient for a step $\mu$

$$y^{(i+1)}(k,l) = y^{(i)}(k,l) - \mu Gr(k,l)$$  

8)

The available signal values are not changed.

When the given parameters $\Delta, \mu$ do not improve the result, the step size is reduced. For example, as $\Delta \rightarrow \Delta/10$ and $\mu \rightarrow \mu/10$. The procedure is repeated until the desired reconstruction accuracy is achieved.

Applying the algorithm on the sparse image, the reconstructed image is obtained, Figure 3.

4. Filtering Impulsive Noise

In some applications the signal is disturbed by a strong impulsive noise in images known as “salt and pepper”. The image affected by a large number of impulsive noise pixels has many black and white pixels, manifesting themselves as salt and pepper noise. A classical approach to reconstruct this picture is in using a Median filter.

To illustrate how Median filter works, image “Isi” is used shown in Figure 6 (a). The image affected with 100 salt and pepper pixels (out of 240x240= 57600 pixels), shown in Figure 6 (b), is considered.
Doing the same procedure as for 75% missing samples, the reconstructed image is shown in Figure 5.

Using both 3x3 and 5x5 Median filter, the picture is filtered and shown in Figure 7.

Figure 4: Image with 75% missing samples

Figure 5: Reconstructed image from the image with 75% missing samples

Figure 6: (a) Original image; (b) Image affected with a salt-and-paper noise in small number of pixels

Both (a) and (b) removed the salt and pepper noise but the image became more blurry in both cases. If the picture is highly affected with noise, as shown in Figure 8 the Median filter will not give as good results as before. The reconstruction of the image using Median filter is shown in Figure 9

Figure 7: Picture filtered using both 3x3 and 5x5 Median filter

Figure 8: Image with high amount of noise

Figure 9: Reconstructed image using Median filter
We can conclude that both 3x3 and 5x5 Median filters not only let some noise to pass, but also blurred the picture, degrading quality. Next we will show that these pictures can be reconstructed using the CS algorithms introduced before.

In Compressive Sensing, we define the noisy pixels of the image as missing and do the reconstruction with the algorithm defined earlier [12]-[15]. Let us consider the same noisy image as in Figure 8. We know that the salt and pepper noise is either black or white, so in pixel values it is either 0 or 255 respectively. The assumption made is then:
if \( y(n,m) = 255 \), signal sample discarded,
initial value set to neutral value \( y(n) = 128 \)

\[
\text{(9)}
\]

if \( y(n,m) = 0 \), signal sample discarded,
initial value set to \( y(n) = 128 \)

In this case, for missing samples the value 255 is used instead of 0. It represents the white colour (blank).

The image with impulsive noise and the image with discarded samples are shown in Figure 10.

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After going through the algorithm from equations (3)-(8) the new, reconstructed image is shown in Figure 11.

Figure 11: Reconstructed image using a CS gradient algorithm

The algorithm is also tested on colour images. For a colour images, each primary colour (red, green and blue) is reconstructed separately. The presented reconstruction method is used without pre-processing of the original image to make it sparse in an artificial manner by setting the DCT coefficients to zero. The number of corrupted pixels is 50% of the total number of pixels. The reconstruction of colour image "Autumn" is shown in Figure 12.

5. Conclusions

In the image processing the common DCT blocks of 8x8 are considered. As images, in general, may not be sparse, two approaches are used. One is to make the analysed image to be sparse in the 2D-DCT domain, by considering only a few of its largest values as non-zero. The processing and reconstruction is then based on this picture. Since common images contain just a few significant 2D-DCT coefficients in the 8x8 blocks this step can be also avoided and the original image, without pre-processing can considered as sparse. This is done in the second approach presented in this paper. The reconstruction was done with compressive sensing gradient algorithm. Both approaches produced similar results for the analysed image. The case of missing samples can be considered due to reduced number of available samples or due to our desire to use a small number of signal samples in the process of information storage or transmission. One more case that reduces to the compressive sensing formulation is the presence of strong, salt-and-paper, noise in the images. After the noisy pixels are detected they can be discarded and considered as missing. The compressive algorithm is then used to recover the image original pixels. It has been shown that the recovery can be succesful with up to 75% missing
pixels. The algorithm has been applied on both black and white and colour images. In both cases the images are recovered with a high accuracy.

Figure 12: (a) Image with missing/corrupted samples; (b) Reconstruction after 3 iterations; (c) Reconstructed image

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