How far does the analogy between causal horizon-induced thermalization with the standard heat bath situation go?

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Abstract

After a short presentation of KMS states and modular theory as the unifying description of thermalizing systems we propose the absence of transverse vacuum fluctuations in the holographic projections as the principle reason for an area behavior (the transverse area) of localization entropy as opposed to the volume dependence of ordinary heat bath entropy. Thermalization through causal localization is not a property of QM but results from the omnipresent vacuum polarization in QFT and does not require a Gibbs type ensemble avaraging (coupling to a heat bath).

1 Posing the problem

Although thermal aspects which result from causal localization in local quantum physics permit a unified description with those generated by the standard coupling to a heat bath, there are some characteristic and important physical differences.

Whereas the temperature of local quantum subsystem which are causally protected from the rest of the world (e.g. a Hawking black hole or one of its recent non gravitational analogs) is determined by the geometry of that situation, the heat bath temperature can be continuously varied without making geometric changes (even if there should exist a maximal Hagedorn temperature as sometimes envisage in string theory). According to Bekenstein’s bold guess about an area behavior of a black hole entropy (related to Hawking quantum
aspects by the postulated classical fundamental Gibbs form of the 2nd fundamental thermodynamic law which made it more convincing), one should expect that for any successful attempt to define directly a quantum entropy associated with a causal horizon it should take the form of an area density as compared to the volume density in the standard heat bath case (with the area being the two-dimensional edge of a bifurcated horizon). The existence of the mentioned non gravitational analogs points towards a still poorly understood fundamental relation between geometry and thermal aspects of local quantum physics at a place where one would rather have expected the (elusive) quantum gravity or string theory.

In these lecture notes we will show that there is an important aspect of a properly defined algebraic holographic projection onto the horizon which sets the stage for an area density, namely the total absence of vacuum polarization on the horizon in directions transverse to the light ray direction \[5\]. We explicitly illustrate this phenomenon\(^1\) in the case of the Rindler-Unruh \[6\] wedge situation (for which the linear extension of the horizon is the lightfront) and argue that it any potential entropy-like measure for the impurity which results from restricting the vacuum to the horizon must necessarily follow an area law where the area is that of the edge of the wedge or its horizon. Without the normalizing use of a (still unknown) second fundamental law derived in the setting of horizon-caused quantum thermal behavior one can at best obtain a relative area density which determines the relation for different quantum matter content. Before this we will briefly sketch the unifying formalism for both kinds of thermal manifestation.

### 2 How modular theory of operator algebras unifies thermal aspects.

Let us briefly indicate how one gets from the standard description of heat bath coupled Gibbs ensembles to the more general unifying framework. The correlation functions of a QFT in a quantization box \(V\) (in order to obtain a discrete energy-momentum spectrum) coupled to a heat bath reservoir are computed with the well-known Gibbs formula

\[
\omega_\beta(A) := \frac{1}{Z_V} \text{tr} e^{-\beta H_V} A, \quad Z_V = \text{tr} e^{-\beta H_V}
\]

which assigns a (normalized) state\(^2\) on the algebra of bounded operators \(A \in \mathcal{A} = B(\mathcal{H})\). The Gibbs formula is meaningful as long as the partition function

\(^1\)According to the authors best knowledge this is the only case in which the quantum mechanical fluctuationless vacuum structure (and an ensuing transverse Galilei symmetry) appears in the midst of QFT without having done any nonrelativistic approximation.

\(^2\)The existence of inequivalent representations in the presence of infinitely many degrees of freedom and the structure of local algebras requires to make a distinction between a state (in the sense of an expectation value) and a state vector which implements this state (this is not necessary in \(B(\mathcal{H})\) algebras of QM which relates states one to one with unit rays).
exists (which requires a discrete Hamiltonian spectrum bounded below and with finite degeneracy, i.e. "boxed" systems). The difference to the vacuum situation becomes more visible on the level of the operator formalism which may be obtained from the state $\omega_\beta(\cdot)$ on $A$ (a set of correlation functions) by the canonical GNS (Gelfand, Neumark and Segal) construction [7]. Using the special property of density matrix states, one may implement the abstract GNS construction on a Hilbert space $H_{HS}$ whose vectors are Hilbert-Schmidt operators $\kappa$ i.e. $\text{tr}\kappa^*\kappa < \infty$

$$H_{HS} = \{\psi_\kappa \mid (\psi_\kappa, \psi_{\kappa'}) \equiv \text{tr}\kappa^*\kappa'\}$$

where $\pi(\cdot)$ denotes the representation of the algebra on $H_{HS}$. The HS Hilbert space is isomorphic to the tensor product of the original Hilbert space $H_{HS} \simeq H \otimes H$ since the linear combinations of "dyads" $|\psi\rangle \langle \varphi|$ from the tensor product upon closure in $H_{HS}$ generate the HS Hilbert space. This “doubling” entails that besides the left action [2] of the full algebra of bounded operators $B(H)$ on $H_{HS}$ there is a right action which in the HS description reads $\psi_\kappa \rightarrow \psi_{\kappa A}$ [7]. In order to distinguish between the left and right representation and to maintain the naturality of composition (representation) laws, one defines the right representation as a conjugate (antilinear) representation

$$\pi_l(A)\psi_\kappa = \pi(A)\psi_\kappa = \psi_{A\kappa}$$

$$\pi_r(A)\psi_\kappa = \psi_{\kappa A}^*$$

It is obvious that any right action commutes with any left action i.e. $\pi_r(A) \subseteq \pi(A)'$ (where the dash denotes the von Neumann commutant of $\pi(A)$ in $H_{HS}$) and in this particular case Haag, Hugenholtz and Winnink in their seminal paper had no problem to prove that in fact equality holds [8]. In $H_{HS} \simeq H \otimes H$ there are many more operators than in $\pi(A)$, e.g. the anti-unitary "flip" $J$

$$J\psi_\kappa := \psi_{\kappa s}, \quad J^2 = 1$$

$$J\pi(A)J = \pi_r(A)$$

which in the tensor product representation would simply interchange the bra and ket in a dyad.

Using now the fact that the Hilbert-Schmidt operator $\kappa_0 \equiv \rho^\top/2$ associated with the nondegenerate (no zero eigenvalue) Gibbs density matrix

$$\omega_\beta(A) = (\psi_{\kappa_0}, \pi(A)\psi_{\kappa_0}) = \text{tr}\kappa_0 A\kappa_0$$

is cyclic and separating with respect to the action $\pi(A)$ of the algebra (i.e. sufficiently entangled in $H \otimes H$ so that the application of this subalgebra permits to approximate any vector in $H \otimes H$ and that it is not possible to annihilate the entangled state with a nonzero operator from $\pi(A)$), one checks the validity of the relation (mainly an exercise in the correct application of definitions)

$$S\pi(A)\psi_{\kappa_0} = \pi(A)^*\psi_{\kappa_0}, \quad A \in A$$

where $S := J\pi(\rho^{1/2})\pi_r(\rho^{-1/2}) \subset S^2 \subset 1$
where the last relation is a notation for the fact that the unbounded operator $S$ is involutive on its domain. By an additional notational convention one gets this relation into the form where it may be viewed as a special illustration of a much more general operator algebra structure which is the famous Tomita-Takesaki modular theory of operator algebras [9].

**Theorem 1** (main theorem of the Tomita-Takesaki modular theory) Let $(A, \mathcal{H}, \Omega)$ denote a weakly closed (von Neumann) operator algebra $A$ acting on a Hilbert space $\mathcal{H}$, with $\Omega \in \mathcal{H}$ a vector on which $A$ acts in a cyclic and separating manner. Then there exists an antilinear closed involutive operator $S$ which has the dense subspace $A\Omega$ in its domain such that

\begin{equation}
SA\Omega = A^\ast \Omega, \quad A \in A
\end{equation}

\begin{equation}
S = J\Delta^\frac{1}{2}, \quad J\Delta = \Delta^{-1}J
\end{equation}

The polar decomposition of $S$ leads to an antiunitary $J$ and a positive $\Delta$ which in turn defines the unitary modular group $\Delta^it$. Their significance results from their adjoint action on the algebra

\begin{equation}
JAJ = A^\prime
\end{equation}

\begin{equation}
\sigma_t(A) \equiv \Delta^itA\Delta^{-it} \in A
\end{equation}

The modular automorphism $\sigma_t$ fulfills the following KMS property (with $\beta = -1$, see below)

\begin{equation}
\omega(\sigma_t(A)B) = \omega(B\sigma_{t-1}(A)B), \quad \omega(\cdot) \equiv (\Omega, \cdot) \quad \omega(\cdot)(\Omega)
\end{equation}

and depends only on the state $\omega$ (and not on its implementing vector $\Omega$).

The relation to the HHW work is most directly established via the validity of the KMS property which replaces the Gibbs formula in the thermodynamic limit

\begin{equation}
\omega_\beta(\alpha_t(A)B) = \omega_\beta(B\alpha_{t+i\beta}(A)) \quad \exists F_{A,B}(z), F_{A,B}(t) = \omega_\beta(B\alpha_t(A)), F_{A,B}(t+i\beta) = \omega_\beta(\alpha_t(A)B)
\end{equation}

where the second line expresses the analytic content of the KMS condition in more careful terms: there exist an analytic functions $F_{A,B}(z)$ which interpolates between the thermal expectation values of operator products taken in different orders; this function is analytic in the strip $0 < \text{Im} z < \beta$ and has continuous boundary values on both margins which relate to the two different orders.

The nontriviality of the T-T proof relates to the fact that the J-transformation property into the commutant and the automorphic action of $\sigma_t$ turns out to be

3Since $\pi_r(\rho^{-\frac{1}{2}})$ is unbounded (even for Hamiltonians with one-sided unbounded spectrum).
much harder. Specializing again to the Gibbs setting, its HHW tensor product structure re-appears in the modular setting in the following way

\[ H_{\text{mod}} \equiv \pi(H) - \pi_v(H) \]

\[ \Delta^{it} \equiv e^{-i\beta t H_{\text{mod}}} \quad S = J \Delta^{\frac{t}{2}} \]

\[ H_{\text{mod}} \psi_{\kappa_0} = 0, \quad \Delta^{it} \psi_{\kappa_0} = \psi_{\kappa_0} \]

\[ \Delta^{-it} \pi(A) \Delta^{it} = \pi(\alpha^{it}(A)) \quad \alpha^{it}(A) = Ade^{iHt}(A) \equiv e^{iHt} Ae^{-iHt} \]

Whereas in vacuum QFT the energy operator \( H \) (obtained by integrating the energy density) is the generator of the translation, in the heat bath situation it is the doubled Hamiltonian \( H_{\text{mod}} \) which leaves the thermal reference state invariant, generates the translation symmetry and has finite fluctuations in the thermodynamic limit. The doubling of Fock space through tensoring can be used to arrange the computational scheme in such a way that the recipe parallels the Feynman rules for the zero temperature case. In this form it gained widespread popularity under the name “Thermo Field Theory”\(^4\) (at this School it was used in M. C. Abdalla’s talk). Besides of being more general, the KMS setting is more faithful to the main aim of theoretical physics which is the de-mystification of nature. According to the above remarks the tensor structure of the Thermo Field Theory is lost in the thermodynamic limit in which case one can simply use the modular \( J \)-operation to define the “Tilde” fields of TFT \cite{14}.

The KMS framework is also very successful in showing the equivalence between the Matsubara imaginary time (discrete energy) and real time (e.g. TFT) formulations. The mathematical proof is bases on the use of very nontrivial Carlssonian type of theorems \cite{15}. It also leads to an extension of the KMS analyticity region (the relativistic KMS \cite{11}) and to an understanding of the perseverance of dissipative effects in the timelike asymptotic behavior \cite{12} which is important for the avoidance of perturbative infrared divergencies.

In the next section it will be shown that the modular framework is capable of incorporating both the heat bath- and the localization- caused thermal properties.

\section{Thermal aspects caused by vacuum polarization on horizons}

The Reeh-Schlieder theorem \cite{7} of QFT (the localization-generated “operator-state” relation in the more folkloristic terminology often used in conformal QFT) insures that these properties are always fulfilled as long as the localization region has a nontrivial causal disjoint. Although the Tomita-Takesaki theorem asserts that the modular KMS automorphism always exists in these cases, the

\footnote{The authors who introduced it \cite{13} apparently were not aware of the close connection to the older Haag-Hugenholtz-Winnink KMS-based formulation and this had the effect that the majority of thermal practitioners up to this date remained unaware of the connection with the more fundamental modular theory (see however \cite{14})}
physical interpretation up to now has been restricted to cases of geometric (diffeomorphism, non-fuzzy) action of the modular group\(^5\) \(\sigma_t\) which in the context of Minkowski space leaves only the Lorentz boosts of wedge region (whereas in CST there are many models with horizon-preserving Killing symmetries). The best known illustration without curvature is Unruh’s Gedankenexperiment in which the observables localized in a Rindler wedge region bounded by a causal horizon are realized by a family of uniformly accelerated observers whose Hamiltonian is proportional to the Lorentz boost. The relation with the Tomita-Takesaki modular theory was first noticed by Sewell who observed on the basis of prior work on the modular theory of wedge algebras by Bisognano and Wichmann that Unruh’s Gedankenexperiment can be viewed as a physical realization of the B-W mathematical results (for a simple but enlightening presentation see [16]).

This raises the question whether the Bekenstein area behavior could also be seen as a (classical) manifestation of the same vastly general local quantum physics mechanism which is responsible for the appearance of a temperature via vacuum polarization from quantum localization. Trying to answer this question with standard box quantization methods and ad hoc cut-offs (in order to obtain an entropy via degree of freedom counting) proved to be inconclusive [17]. According to the above ideas the relevant question should be whether by physically motivated ideas (i.e. by remaining within the given local theory and thus avoiding ad hoc locality-violating cut-offs) one can associate a localization entropy with the Lorentz boost in its role as the modular group of the wedge algebra. If one would be able to show that in this Unruh test case there exists an area density of localization entropy which is the counterpart of the well established horizon-affiliated KMS localization temperature, then the present thesis that the existing successful framework of QFT, if extended by some new concepts derived from the old principles, would gain strength and the many attempts to invoke speculative physics and the blue yonder (borrowing a phrase from Feynman) may use their strong spell which they exerts especially on young members of the physics community. If this (despite the encouraging signs below) should turn out to be disproved by future more detailed computations, one at least would have a theoretically more solid point of departure and a better guide for speculative endeavors.

There is obviously no chance in QFT to directly assign an entropy to the modular operator of the Unruh wedge situation (which is the Lorentz boost). Although the lightfront holography of the Bisognano-Wichmann/Unruh-Rindler wedge algebra is easily seen to map the boost into the generator of scale transformations, one is still stuck with a non trace class operator. The restriction of the global vacuum to the horizon algebra assures the thermalization in the sense of a geometrically determined (Hawking) temperature, but it does not

\(^5\)The vacuum modular group of a double cone algebra is an example of what is meant by fuzzy action. It is believed that in the standard formulation (where such algebra is generated by smeared fields with double cone supported test functions) the fuzzy modular actions are support-preserving maps of test function spaces (probably related to pseudo-differential operators) which are asymptotically geometric at the causal horizon [19].
help in getting closer to trace class properties, although (as a result of the loss of transverse vacuum polarization, see below) it reduces the problem of understanding of an area density to that of entropy of the vacuum restricted to a halfline algebra (=halfcircle in the compact description of chiral theories). The essential step for getting a density matrix from the local restriction of the vacuum is to allow the halfline localization to be “fuzzy” by an “ε-roughening” of the boundary endpoints. This split process of leaving a distance ε between the halfcircle localization of the chiral algebra and that of its commutant (the opposite halfcircle) is the opposite analog of the thermodynamic limit namely instead of starting from Gibbs states in order to approach the KMS thermodynamic limit state one wants to search for density matrix states associated with fuzzy boundaries which in the limit ε → 0 lead back to the KMS state. According to our previous considerations this split-restricted vacuum state is a vector in the two-fold tensor product of a (ground state) Fock space with itself. As in the Thermo Field formalism this vector is highly entangled and becomes impure upon restriction to the “physical” tensor factor. In the limit the thermalized physical vacuum becomes orthogonal to the split tensorproduct vacuum, in fact both vacua are cyclic and separating reference states which belong to inequivalent representation of a suitably defined C* tensorproduct algebra. The simple structure of local chiral algebras (generalized W-algebras) permits to argue that the speed of vanishing of overlaps is dominated by powers of ε (whereas area densities of partition functions diverge according to inverse ε-powers which suggests that the divergence of the split entropy should go universally like -ln ε. Formulating this expectation as an universality conjecture one obtains the statement that the holographic universality classes (different ambient matter content may be holographically mapped into the same chiral theory) lead to (class-dependent) numerical coefficients multiplying -ln ε so that the split property can only determine finite ratios of area densities. Thus the area density resulting from the split property can only be a relative entropy density; the holographic formalism together with the split property can never produce a normalized entropy density. In fact in view of all the black hole analogs one does not even want a normalized entropy on this level of discussion because besides the principles of local quantum physics (causal propagation in a local quantum context) we have not used properties which would distinguish between the different analogs and the Hawking gravitational black hole which would set the different scales in a Bekenstein entropy argument for those analogs; one expects the surface “gravity” (the Unruh acceleration) related to the numerical factors between the geometrically determined modular automorphism and the “Hamiltonian” to set this scale. The absolute normalization can only come (as in the Bekenstein-Hawking case) through the

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In TFT the physical algebras are generated from the original observables and the commutant (the “tilde” operators) do not occupy any “geometric territory”, whereas in the horizon-caused thermalization the commutant is localized “behind the horizon”. There is however the curious observation that for systems with no transverse direction in their holographic projections (i.e. 2-dimensional QFT), the distinction between the heat bath “shadow world” and the real world “behind the horizon” becomes blurred. 

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validity of a second fundamental thermodynamic law in which the entropy is related to other quantities. Bekenstein takes the classical Gibbs form of this law, but the problem in the present setting would be to find out if and how such a law can be derived outside the classical heat bath setting. Here we are entering an unknown area of QFT in which however the questions seem to be well-posed.

From a pragmatic viewpoint the different steps in the argument all serve to extract a well-defined additive (under correlation-free subdivisions) measure of impurity for the horizon-restricted vacuum (alias wedge-restricted vacuum). This is quite different from [17]; the box of those authors should be causally completed to a double cone, but even then a treatment paralleling the present would be much more difficult as a result of the nongeometric nature of the associated modular group [5] [6].

The remainder of this section will be used to present the argument about the absence of transverse vacuum correlations in the holographic lightfront projection. For brevity (and pedagogical reasons) we limit the presentation to the holographic lightfront projection of scalar free fields. In that case one finds that in terms of Weyl generators the result looks as follows

\[ W(f) := e^{iA(f)}, \quad A(f) = \int A(x)f(x)d^4x \]  

(12)

\[ W(g, f_\perp) \mapsto W_{LF}(g, f_\perp) = e^{iA_{LF}(g, f_\perp)} \]

with

\[ A_{LF}(g, f_\perp) = \int a^\ast(p_-, p_\perp)g(p_-)f_\perp(x_\perp)\frac{dp_-}{2|p_-|}d^2p_\perp + h.c. \]  

(13)

\[ \langle W(g, f_\perp)W(g', f'_\perp) \rangle = \langle W(g, f_\perp) \rangle \langle W(g', f'_\perp) \rangle \quad \text{if} \quad \text{supp} f \cap \text{supp} f' = \emptyset \]

The second line formulates lightfront restriction on the dense set of wedge supported test functions which factorize into a longitudinal and a transverse part [5] [6]. The third line is the statement that holographically projected Weyl generators (and therefore also the algebras they generate) have no transverse fluctuations; the holographic projection compresses all vacuum fluctuations into the lightlike direction. This reduces the problem of horizon-associated entropy to the problem of looking for an area (the area of the edge of the wedge which limits the upper horizon) density of entropy as mentioned before. and should be interpreted as the localization entropy of a halfline in a chiral theory. It turns out that the lightfront holography leads to a QFT with a seven parametric symmetry subgroup of the Poincaré group which contains in particular a transverse Galilei group which results from the holographic projection of the “translations” contained in Wigner’s 3-parametric “little group” of the lightray in the lightfront. This is also true in the general non-free situation.

We will skip the generalization of the proof to interacting theories since an explanation of the methods (involving modular inclusions and intersections [5] [6]) goes beyond the scope of this talk.
4 Concluding remarks

The main aim of these notes was two-fold, on the one hand we have recalled that there exists a unified formalism for heat bath and localization caused thermalization and on the other hand we emphasized that the most startling difference between the two cases shows up in the total absence of vacuum transverse polarization which is the prerequisite for area proportionality of entropy. These considerations did not yet solve the existence of a horizon-associated (relative) area density of entropy. In order to arrive at a Bekenstein like formula one still has to prove a universality conjecture and (for its normalization) and derive a second thermodynamic law in which this quantum localization entropy enters. However the remarkable area dependence of any would-be entropy (versus the standard volume dependence of heat bath entropy) is already secured on the present level of understanding. The area density of entropy (assuming that the conjectures can be established) inherits the universality of the holographic projection. However (in line with the experience that classical laws suffer modifications which depend on the kind of quantum matter) in the presence of quantum matter one perhaps should not expect total universality of the area density as in Bekenstein’s formula.

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7The holographic universality classes are in some sense bigger than the short distance classes since the resulting chiral algebras are one-dimensional and local i.e. there can be no anomalous dimensions (non halfinteger critical indices).
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