Quantum to classical transition and entanglement sudden death in Gaussian states under local heat bath dynamics

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Entanglement sudden death in spatially separated two-mode Gaussian states coupled to local thermal and squeezed thermal baths is studied by mapping the problem to that of the quantum-to-classical transition. Using Simon’s criterion concerning the characterisation of classicality in Gaussian states, the time to ESD is calculated by analysing the covariance matrices of the system. The results for the two-mode system at \(T = 0\) and \(T > 0\) for the two types of bath states are generalised to \(n\)-qubit system.

I. INTRODUCTION

Entanglement is one of the basic features that distinguish quantum systems from their classical counterparts and is the most useful resource in Quantum Information Theory (QIT)\(^\dagger\). It is indispensible for essential quantum information tasks like quantum computation and teleportation as well as super-dense coding and one-way communication, to name a few. However, more often than not, quantum systems do not operate while completely insulated from its environment. Unfortunately, quantum systems are inherently fragile and extremely sensitive to environmental interactions. It is a common belief that the evolution of a quantum system in contact with a dissipative environment results in an asymptotic transition to classicality \(2\,\dagger\) and thus an inevitable loss of entanglement, since classicality subsumes disentanglement. Recently however it has been shown \(3\,\dagger\) that there exists a certain class of two-qubit states which display a finite entanglement decay time. This phenomenon is aptly called Entanglement Sudden Death (ESD) and cannot be predicted from quantum decoherence which is an asymptotic phenomenon.

ESD has important and obvious implications for the success of quantum information tasks. Much research has been done on ESD in discrete quantum states. Qasimi \(\text{et al.}\)\(^\dagger\) have shown that ESD occurs for a class of two-qubit states called ‘\(X\)’ states at non-zero temperatures. The authors of the present paper have shown that there is ESD for all \(n\)-qubit states at finite temperature \(4\) and have also given a sufficiency condition for the existence of ESD for a system consisting of \(n\) number of \(d\)-dimensional subsystems as well. The study of the evolution of entanglement as well as that of its sudden death in continuous variable systems has also received a great attention, for example, the paper by Dodd \(\text{et al.}\)\(^\dagger\) deals with ESD from the point of view of a separable representation of the joint Wigner function of two-mode Gaussian states.

Their results do not have an explicit presence of temperature. Also, the time to ESD for the maximally entangled EPR state is shown to be a lot smaller than that of any other initially entangled two-mode Gaussian state. This is counter-intuitive as one would expect the ESD time for a maximally entangled state to be the largest. In \(1\), Diósi has given a bound on the time of ESD by using a theorem on ‘entanglement breaking quantum channels \(12\,\ddagger\,14\)’. In \(15\) Marek \(\text{et al.}\) have addressed a different problem. In this paper they obtained a class of states which is tolerant against the decoherence at zero temperature. People have studied the decoherence in infinite dimensional systems interacting with different kinds of bath and with different system-environment models \(16–19\).

Since classicality subsumes disentanglement, the ESD problem can be embedded into the corresponding quantum-to-classical transition problem. Thus, the time taken for the system to attain classicality will be an upper bound on the time at which entanglement dies. We use the criterion due to Simon \(20,\,21\) for checking the classicality of Gaussian states in terms of their covariance matrices \(V\), which enables us to characterise the quantum-to-classical transition. If \(I\) be the covariance matrix corresponding to the vacuum state, then Simon’s criterion states that Gaussian states attain classicality if and only if the difference \(V − I\) becomes positive semi-definite. Which is equivalent to Sudarshan-Glauber \(P\) function being positive. We use this condition to find out the transition time to classicality \(t_e\) and hence the time to ESD. Diósi \(\text{et al.}\)\(^\dagger\) discusses about the positivity of Wigner function and \(P\) function under Markovian evolution. We find that the behaviour of the Gaussian system, as regards to ESD, depends on the state and the temperature of the bath it is coupled to. We have shown that all Gaussian states show ESD at all temperatures \(T > 0\), whereas some do not at \(T = 0\) when coupled to a thermal bath. However, all states again show ESD at \(T = 0\) when the state of the bath is squeezed thermal.

We begin this article by writing down the master equation for a single harmonic oscillator (Sec. \(11\)) in contact with a thermal bath of infinitely many oscillators, taken at some finite non-zero temperature. We express

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the states of this system in the Sudarshan-Glauber $P$-representation and derive the corresponding quantum analog of the classical linear Fokker-Planck equation (Sec. III). This enables us to use the corresponding covariance matrices to characterize classicality of the single-mode system in the presence of thermal baths (Sec. IV). Once we have this mechanism in place, we use it to determine the action of the thermal bath (Sec. V) and squeezed thermal bath (Sec. VI) on a two-mode Gaussian state. Using Simon’s criterion, we calculate the time to classicality and hence are able to determine when a system will show ESD. Furthermore, using the two-mode analysis, we are able to predict the possibility of ESD in any $n$-mode Gaussian state since we make the important observation that Simon’s criterion $V - I \geq 0$ is independent of the number of modes of the system. Hence we are able to generalize our earlier statements to $n$-mode systems as well (Sec. VII). Section VIII provides the conclusion of the paper.

II. MASTER EQUATION FOR SINGLE MODE SYSTEM

The irreversible time evolution of the state $\rho_s(t)$ of a single harmonic oscillator coupled to a bath is described by the Lindblad master equation (ME) [23].

$$\frac{d}{dt} \rho_s(t) = -i\omega_0 [a^\dagger a, \rho_s(t)] + \gamma_0 (N + 1) \left\{ a \rho_s(t) a^\dagger - \frac{1}{2} a^\dagger a \rho_s(t) - \frac{1}{2} \rho_s(t) a^\dagger a \right\} + \gamma_0 (N) \left\{ a^\dagger \rho_s(t) a - \frac{1}{2} a a^\dagger \rho_s(t) - \frac{1}{2} \rho_s(t) a a^\dagger \right\}.$$  

(1)

The initial state of the entire system is of the form $\rho_s(0) \otimes \rho_b$, where $\rho_s(0)$ is the initial state of the system and the thermal state $\rho_b$ is the initial state of bath. Since the bath we are considering is very large, the state of the bath will not change during the evolution. The first term of the ME describes free evolution generated by the system Hamiltonian $H = \omega_0 a^\dagger a$ while the rest are interaction terms with the bath, where the dissipative coupling is provided through the damping rate $\gamma_0$. Here $N = (e^{\beta \omega_0} - 1)^{-1}$ is the mean number of quanta in a mode with frequency $\omega_0$. This ME can be used to describe, for example, the damping of an electromagnetic field inside a cavity where $a$ and $a^\dagger$ denote the creation and annihilation operators of the cavity mode. The mode outside the cavity plays the role of the environment with dissipative coupling rate $\gamma_0$.

III. COHERENT STATE REPRESENTATION

We can transform the master equation described by (1) into a continuous diffusion process by using the coherent state representation,

$$\rho_s(t) = \int d^2 \alpha P(\alpha, \alpha^*, t) |\alpha\rangle \langle \alpha|,$$

(2)

where $P(\alpha, \alpha^*, t)$ is the Sudarshan-Glauber $P$-function [24, 27] and the integration in (2) is over the entire complex plain. The quasiprobability distribution, called thus because $P(\alpha, \alpha^*, t)$ can take negative values for some $\alpha$, satisfies the normalization

$$\text{tr}_s \rho_s(t) = \int d^2 \alpha P(\alpha, \alpha^*, t) = 1.$$  

(3)

Substituting (2) into (1) and using the properties

$$a|\alpha\rangle \langle \alpha| = \alpha |\alpha\rangle \langle \alpha|,$$

$$a^\dagger |\alpha\rangle \langle \alpha| = \left( \frac{\partial}{\partial \alpha} + \alpha^* \right) |\alpha\rangle \langle \alpha|,$$

(4)

we get the following equation for the evolution of $P(\alpha, \alpha^*, t)$:

$$\frac{\partial}{\partial t} P(\alpha, \alpha^*, t) = - \left[ \left( -i\omega_0 - \frac{\gamma_0}{2} \right) \frac{\partial}{\partial \alpha} + \left( i\omega_0 - \frac{\gamma_0}{2} \right) \frac{\partial}{\partial \alpha^*} \right] P(\alpha, \alpha^*, t) + \gamma_0 N \frac{\partial^2}{\partial \alpha \partial \alpha^*} P(\alpha, \alpha^*, t).$$

(5)

This is structurally similar to the classical linear Fokker-Planck equation [20] and can be solved using the Gaussian ansatz

$$P(\alpha, \alpha^*, t) = \frac{1}{\pi \sigma^2(t)} \exp \left[ - \frac{|\alpha - \beta(t)|^2}{\sigma^2(t)} \right],$$

(6)

given the initial condition $P(\alpha, \alpha^*, 0) = \delta^2(\alpha - \alpha_0)$, where $|\alpha_0\rangle$ is the initial coherent state. Here the mean amplitude is given by $\beta(t) = \int d^2 \alpha \alpha P(\alpha, \alpha^*, t) = \alpha_0 e^{(-i\omega_0 - \gamma_0/2)t}$ and the variance is $\sigma^2(t) = N(1 - e^{-\gamma_0 t})$. Since we are interested in dissipation, we choose to ignore the $\omega_0$ term which only contributes to free evolution. The linearity of the evolution map ensures that any density matrix $\rho(0) = \int P(\lambda, \lambda^*, 0)|\lambda\rangle \langle \lambda| d^2 \lambda$ will evolve to $\rho(t) = \int P(\alpha, \alpha^*, t)|\alpha\rangle \langle \alpha| d^2 \alpha$, where $P(\alpha, \alpha^*, t) = \int P(\lambda, \lambda^*, 0) \exp \left( -|\alpha - \beta(t)|^2 \right) d^2 \beta$ with $\beta = \lambda e^{-\gamma_0 t/2}$. Therefore, thermal evolution of the $P$-distribution mani-
fests as a convolution of the $P$ and the thermal distribution $P_{th} = \exp(-|\alpha|^2/\sigma^2(t))$.

**IV. COVARIANCE MATRIX**

Our aim is to express the evolution of the state in terms of the covariance matrix, and to that end we write down the symmetric characteristic function $\chi(\alpha, t)$

$$\chi(\alpha, t) = e^{-|\alpha|^2/2} F[P(\alpha, \alpha^*, t)] = \exp\left(-\frac{X^T V X}{4}\right),$$

where $F$ denotes the Fourier transform and $V$ is the covariance matrix. The vector $X^T = (q, p)$ where $q = \frac{1}{\sqrt{2}}(\alpha + \alpha^*)$ and $p = \frac{1}{\sqrt{2}}(\alpha^* - \alpha)$ are the position and momentum variables. The evolution of $\chi(\alpha, t)$ follows from that of $P(\alpha, \alpha^*, t)$ (equation 6). We have, from (7)

$$\chi(\alpha, t)e^{i\omega_0 t} = \chi(\beta e^{-\gamma_0 t}, 0)e^{-|\alpha|^2/2 + \gamma_0 t}e^{-i\omega_0 t} = \chi(\alpha, t).$$

The covariance matrix $V$ can be written as:

$$V_{ij} = \int (XX^T)_{ij} F^{-1} \chi(\alpha, t)d^2q'd^2p,$$

and hence

$$V(t) = e^{-\gamma_0 t}V(0) + \left(\frac{N}{2} + 1\right)(1 - e^{-\gamma_0 t})I.$$

A Gaussian channel is a map that takes Gaussian states to Gaussian states, an example of which is the evolution, given by Eq. (1) for a harmonic oscillator coupled with a thermal bath. The evolution of the covariance matrix $V$ of the system, under the action of a general Gaussian channel, can be characterized by two matrices $A$ and $B$:

$$V_f = AVA^T + B,$$

where $B$ is a positive operator [29]. For a thermal bath, we have from (10), $A = e^{-\gamma_0 t/2}I$ and $B = \left(\frac{N}{2} + 1\right)(1 - e^{-\gamma_0 t})I$. We can hence characterize the action of a thermal bath completely using these two matrices. The characterization, given by Eq. (10), guarantees that $V(t)$ of Eq. (10) is a bona fide covariance matrix for all finite time $t$.

**V. ESD OF TWO-MODE GAUSSIAN STATE**

Consider a two-mode system coupled to two identical local thermal baths. Let us assume that the initial two-mode $(4 \times 4)$ covariance matrix $V_0$ represents an entangled state. Its subsequent evolution is given by

$$V(t) = (A \oplus A) V(0)(A \oplus A)^T + (B \oplus B) = e^{-\gamma_0 t}V(0) + \left(\frac{N}{2} + 1\right)(1 - e^{-\gamma_0 t})I.$$

where $A = e^{-\gamma_0 t/2}I$ and $B = \left(\frac{N}{2} + 1\right)(1 - e^{-\gamma_0 t})I$. From a quantum optics point of view, we know [20], in view of the $P$-representation, that the state $\rho = \int d\alpha d\beta P(\alpha, \beta)|\alpha\rangle \otimes |\beta\rangle \langle \beta|\langle \alpha|$ is classical only when $P(\alpha, \beta)$ is non-negative for all $\alpha$ and $\beta$. This interpretation of classicality can be translated into the language of the covariance matrix $V(t)$. Thus, a two-mode Gaussian state will be classical at some time $t$ if and only if $V(t) \geq I$ [20]. Clearly, being entangled, the initial covariance matrix satisfies: $V(0) < I$. However, since the evolution (12) is dissipative, the system will attain classicality after a time $t_c$ so that $V(t_c) \geq I$. This condition is equivalent to $n_{\min}(t_c) \geq 1$, where $n_{\min}(t_c)$ is the smallest eigenvalue of $V(t_c)$. Using (12), it can be written as the evolution of $n_{\min}(0)$ which is the smallest eigenvalue of $V_0$. Since $V(0)$ is not classical, $n_{\min}(0) < 1$.

$$n_{\min}(t_c) = e^{-\gamma_0 t_c}n_{\min}(0) + \left(\frac{N}{2} + 1\right)(1 - e^{-\gamma_0 t_c}).$$

Here we find that $n_{\min}(t_c) \geq 1$ always for $t_c \geq \frac{1}{\gamma_0} \ln \left(\frac{N}{n_{\min}(0) + 1}\right)$. An appropriate choice of $V_0$ allows us to make $n_{\min}(0)$ arbitrarily small (which is the case in EPR states, which are maximally entangled initially) and thus get an upper bound on the transition time $t_c$, given by $t_{\max} = \frac{1}{\gamma_0} \ln \left(\frac{N}{n_{\min}(0) + 1}\right)$. For $T > 0$, we have $N > 0$ and hence $t_{\max}$ is finite and non-zero. This proves that there is always ESD at non-zero temperatures. However, when $T = 0$, i.e. $N = 0$, we have $t_{\max} \rightarrow \infty$ and hence no quantum-to-classical transition is seen at finite times. This does not, however, rule out ESD since non-classicality does not necessarily imply that there is entanglement.

For $T = 0$, let us consider a particular form of the initial covariance matrix representing a symmetric two-mode Gaussian state.

$$V(0) = \begin{pmatrix} n & 0 & k_x & 0 \\ 0 & n & 0 & -k_y \\ k_x & 0 & n & 0 \\ 0 & -k_y & 0 & n \end{pmatrix}.$$
at non-zero temperature. For \( T = 0 \), there exist states which do not show ESD \([9]\).

VI. SQUEEZED THERMAL BATH

If the initial state of the bath is a squeezed thermal state, then the master equation \([11]\) is replaced by \([31]\):

\[
\frac{d}{dt}\rho_s(t) = -i\omega_0[a^\dagger a, \rho_s(t)] + \gamma_0(N + 1)\left\{a\rho_s(t)a^\dagger - \frac{1}{2}a^\dagger a\rho_s(t) - \frac{1}{2}\rho_s(t)a^\dagger a\right\} + \gamma_0N\left\{a^\dagger \rho_s(t)a - \frac{1}{2}aa^\dagger \rho_s(t) - \frac{1}{2}\rho_s(t)aa^\dagger\right\} \nonumber
\]

\[
- \frac{\gamma_0}{2}M^s\{2a\rho_s - a^2\rho - \rho a^2\} \nonumber
\]

\[
- \frac{\gamma_0}{2}M\{2\rho a^\dagger - (a^\dagger)^2\rho - \rho (a^\dagger)^2\},
\]

where \( M = -\frac{1}{2}\sinh(2r)e^{\phi}(2N_{th} + 1) \) and \( 2N + 1 = \cosh(2r)(2N_{th} + 1) \). The quantity \( N_{th} \) is the average number of photons in the thermal state and \( r \) and \( \phi \) are the squeezing parameters. Repeating the same procedure as earlier, we finally write down the evolution in terms of covariance matrices:

\[
V(t) = e^{-\gamma_0 t}V(0) + (1 - e^{-\gamma_0 t})V_\infty,
\]

where \( V_\infty \) is given by

\[
V_\infty = \left( \begin{array}{cc} \frac{N}{2} + 1 + \text{Re}\{M\} & \text{Im}\{M\} \\ \text{Im}\{M\} & \frac{N}{2} + 1 - \text{Re}\{M\} \end{array} \right).
\]

In this case, for a squeezed thermal bath, \( A = e^{-\gamma_0 t}I \) and \( B = (1 - e^{-\gamma_0 t})V_\infty \). If there is no squeezing (i.e. if we set \( M = 0 \) and \( N = N_{th} \)), we get \( V_\infty = (N/2 + 1)I \) and thus recover the unsqueezed result namely, Eq. \((12)\).

Thus the evolution of the two-mode Gaussian state can be written as

\[
V(t) = e^{-\gamma_0 t}V(0) + (1 - e^{-\gamma_0 t})(V_\infty \oplus V_\infty).
\]

If the smallest eigenvalue of \( V(0) \) is \( n(0) \) then the classicality condition \( V(t) \geq I \) can be written as \([32]\):

\[
n(t) \geq e^{-\gamma_0 t}n(0) + (1 - e^{-\gamma_0 t})\left(\frac{N}{2} + 1 - |M|\right) \geq 1,
\]

\[
i.e., t \geq -\frac{1}{\gamma_0}\ln\left(\frac{N - 2|M|}{N + 2|\sqrt{2}| - 2n(0)}\right),
\]

where in Eq. \((19)\), \( n(t) \) is the smallest eigenvalue of \( V(t) \) and \( \left(\frac{N}{2} + 1 - |M|\right) \) is the smallest eigenvalue of \( V_\infty \). Since \( n(0) < 1 \), \( t_c \) is finite and positive. Therefore, the transition to classicality and thus ESD is ensured for the squeezed thermal bath for non-zero temperatures. However, unlike the thermal bath case, we see here that \( N \) does not become zero with temperature \( T \) and hence a quantum-to-classical transition always happens at zero temperature for the squeezed thermal bath which ensures ESD.

VII. ESD FOR \( n \)-MODE GAUSSIAN STATES

In this section we generalize the result mentioned previously for \( n \)-mode Gaussian states in the presence of local thermal and squeezed thermal baths. Consider a covariance matrix \( V_n(0) \) corresponding to an \( n \)-mode Gaussian state. Let the matrices \( A \) and \( B \) characterise the single mode Gaussian channel for a given bath. The evolution of the covariance matrix can be written as [as a generalization of Eq. \((12)\)],

\[
V_n(t) = (A \oplus A \oplus \cdots \oplus A)V_n(0)(A \oplus A \oplus \cdots \oplus A)^T + (B \oplus B \oplus \cdots \oplus B),
\]

If \( n(0) \) is the smallest eigenvalue of \( V_n(0) \), the classicality condition \( V_n(t) \geq I \) gives rise to an equation which is same as Eq. \((13)\) (Eq. \((19)\) representing \( n_{min}(t)(n(t)) \) as the smallest eigenvalue of \( V_n(t) \) in the case of the thermal (squeezed thermal) bath. Therefore, all \( n \)-mode Gaussian states show quantum-to-classical transition at finite temperature.

VIII. CONCLUSION

In this article we have analysed the phenomenon of entanglement sudden death for the case of Gaussian states coupled to local identical thermal and squeezed thermal baths. We have mapped this problem of non-asymptotic decay of entanglement to the quantum-to-classical transition phenomenon by noting that classicality subsumes separability. Using the powerful criterion given by Simon, we have characterized the classicality of the Gaussian states and calculated the time taken to attain classicality. We have shown whether ESD is possible at zero temperature as well as at non-zero temperatures for the two different types of baths considered. We have been able to generalize our two-mode result to a general \( n \)-mode system by exploiting the mode-independence of Simon’s criterion. Finally, we have noted the similarity that the ESD results for Gaussian systems have with discrete system results \([8]\). We believe that the same might hold for non-Gaussian states as well, work on which is currently in progress.

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