A Simple 3D Isometric Embedding of the Flat Square Torus

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Abstract

Start with Gott (2019)'s envelope polyhedron (Squares—4 around a point): a unit cube missing its top and bottom faces. Stretch by a factor of 2 in the vertical direction so its sides become (2 × 1 unit) rectangles. This has 8 faces (4 exterior, 4 interior), 8 vertices, and 16 edges. F − E + V = 0, implying a (toroidal) genus = 1. It is isometric to a flat square torus. Like any polyhedron it has zero intrinsic Gaussian curvature on its faces and edges. Since 4 right angled rectangles meet at each vertex, there is no angle deficit and zero Gaussian curvature there as well. All meridian and latitudinal circumferences are equal (4 units long).

1 Introduction

Many video games including Pac-Man and Asteroids are played on a square TV screen with a toroidal geometry. Pac-Man, or a spaceship that disappears off the top of the screen immediately reappears at the bottom at same horizontal location. Likewise, if Pac-Man or a spaceship disappears off the right-hand edge of the screen it immediately reappears on the left-hand edge of the screen at the same vertical location. These games thus have a square Euclidean flat geometry but a toroidal topology. It takes seven colors in general to color nations on such a TV game map, rather than the four required on the plane. It is a map of a toroidal flatland universe of two dimensions, that obeys Euclidean geometry, has zero Gaussian curvature everywhere (every small circle around any point has a circumference of $2\pi r$, and the sum of angles in any triangle on the screen has a sum of angles of $180^\circ$). But the map has two boundary cuts, one across the top of the screen, one across the right of the screen, connected to the same cuts on the opposite sides. This game can be mapped onto a donut (or tire

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inner tube shape) in 3D space, but with distortion, since the tire has a curved
surface. It has positive Gaussian curvature on the outer circumference of the
tire, and negative Gaussian curvature on the interior of the tire, where the
geometry is saddle shaped. Can the game geometry be embedded in 3D space?
Videos illustrating this usually start with a square, flat sheet of paper. The
demonstrator then bends it into a cylinder and tapes it together. If the side
length of the square is $S$, the cylinder has a height of $S$ and a radius of $r$, where
$S = 2\pi r$. The demonstrator bends the paper; this does not change its intrinsic
Gaussian curvature of zero everywhere. A cylinder has extrinsic curvature, but
zero intrinsic curvature everywhere. One of the two boundary cuts has been
sown together and healed when the demonstrator tapes the two opposite edges
together. This leaves at the ends of the cylinder, two circular edges, which now
need to be taped together. Then the demonstrator tries fruitlessly to bend the
cylinder around to bring them together, and illustrates that it would tear and
crumple the paper to try to do so. One is trying to produce a tire inner tube
with curvature from a flat cylinder with no curvature. Doesn’t work we are told.

But if we lived in 4D Euclidean space with a metric $ds^2 = dx^2 + dy^2 + dz^2 +
dw^2$, we could do it by bending the cylinder around smoothly in the $w$ direction.
The result is a 2D flat torus living in 4D space (a Clifford torus):

$$r^2 = x^2 + y^2 = \frac{1}{2} = z^2 + w^2.$$  

This is a 2D surface defined by two equations, living in 4D Euclidean space.

Here $r$ is a constant $r = 1/\sqrt{2}$. Define $\theta$ and $\phi$ by $x = r \sin(\theta)$, $y = r \cos(\theta)$,
$z = r \sin(\phi)$, $w = r \cos(\phi)$ and we can show that the equations of the surface are
satisfied automatically since $\sin^2 + \cos^2 = 1$, and that using $(\theta, \phi)$ as coordinates
on the surface, the metric on the surface is

$$ds^2 = r^2 d\theta^2 + r^2 d\phi^2,$$
where $0 \leq \theta < 2\pi$, $0 \leq \phi < 2\pi$.

This is a flat metric with zero Gaussian curvature. It is a flat square with side
$S = 2\pi r = \sqrt{2}\pi$, our original TV game with no distortion, and no boundary
cuts. The vertical and horizontal coordinates are cyclic, and the boundary cuts
have been healed. This is an isometric embedding of the TV game geometry in
4D Euclidean space.

[Nash (1954)] famously proved that there must exist a $C^1$ isometric embed-
ding of the flat square torus in 3D Euclidean space – but he provided no con-
structive prescription for doing so. (A $C^1$ isometric embedding has continuous
first derivatives, in other words it admits tangent planes that very smoothly,
being defined at every point. (A $C^1$ isometric map which is not $C^2$ has no
defined extrinsic curvature everywhere.) Recently, Borrelli, Jabrane, Lazarus,
and Thibert in their paper “Isometric Embedding of the Square Flat Torus in
Ambient Space”, [Borrelli et al. (2013)] have shown how to do it. It looks like a
torus, but with ripples on it. And there are ripples on those ripples tipped at
an angle, and tinier ripples on those ripples at another angle, ad infinitum. It
is sort of a smooth fractal. The amplitude of the ripples on the inner edge of
the torus is larger, making its circumference exactly as large as the smoother circumference on the outside. Beautiful pictures of it are shown in their paper. For his original work in this field, John Nash (who by the way often visited our Astrophysics Library in Princeton) received the Abel Prize in Mathematics. Many mathematicians regard this as a greater work on his part than his work on game theory for which he won the Nobel Prize.

While the embedding of Borelli, et al. is beautiful and important mathematically, it does not make a very practical object on which to play the game. It is so crinkled that it would be hard to even see the game portrayed on its corrugated surface. Yet it is isometric.

\section{Envelope Polyhedra}

We wish to propose here a simple $C^0$ isometric mapping of the flat square torus into 3D Euclidean space. Note that in Figure 1, Gott’s envelope polyhedron Squares—4 around a point has a toroidal topology. It is a cube with its top and bottom missing.

Figure 1: Some Finite Envelope Polyhedra from Gott (2019). Envelope polyhedra are regular polyhedra with regular polygons as faces, the arrangement of polygons around each vertex must be identical, but not all dihedral angles are equal, and some are $0^\circ$, allowing polygons to appear back to back. From left to right at top: Squares—4 around a point, Triangles—6 around a point, Squares—4 around a point (a unit cube missing its top and bottom—tipped on its side in this picture—a subject of this paper), and Triangles—8 around a point. At bottom, Squares—2 around a point (a dihedron—see Coxeter (1938)). To see this stereo view, touch your nose to the page and slowly withdraw it. You will see a blurry 3D image in the center which will come into focus.

The cube with its top and bottom faces missing has 4 exterior squares linked by the square holes at the top and bottom to 4 interior squares. It thus has 8 faces, 4 outside edges, 4 inside edges, and 8 edges at the top and bottom that link the exterior and interior faces. It has 8 vertices. Thus $F - E + V = 8 - 16 + 8 = 0 = 2(1 - g)$, giving this a genus of 1, which is a torus. Notice
that there is no Gaussian curvature on any of the faces, they are flat. There
is no Gaussian curvature on any of the edges; they are limits of flat surfaces of
partial cylinders whose radii shrink to zero. As in any polyhedron, the Gaussian
curvature is contained in delta-functions at the vertices. In this case since there
are 4 squares around every vertex, the angle deficit is zero at each vertex and
the integral of the Gaussian curvature at the vertex is zero. This is a geometric
surface which contains zero volume. The circumference around its equator on
the outside is 4 units (if the cube is of one-unit side length). This circumference
circles the 4 outside faces. It is obviously a geodesic. The inner circumference
circling the 4 inside faces is also 4 units. It is also a geodesic. What about
the upper edge: the four straight edges at the top that form the opening that
connects the outer and inner faces at the missing top face of the cube. These
seem to turn 90° four times as one circles the top of the cube. Yet this circuit
is also a geodesic. For any 2D surface imbedded in 3D Euclidean space, the
shortest curve connecting points $A$ and $B$ in the surface is a geodesic. Pick
any two points $A$ and $B$ in the four top edges. The shortest path from $A$ to $B$
staying in the surface will necessarily travel along those top edges. If the curve
dips down into any of the faces during part of its journey it will be longer.

Take this cube and lengthen it in the vertical direction by a factor of 2. Now
there is a missing square at the top and bottom, and there are 4 rectangular
sides made up of 2:1 rectangles. There are 4 exterior rectangles of length 2
units and width 1 unit, and 4 interior rectangles of length 2 units and width 1
unit. The equatorial circumference of any horizontal geodesic traversing the 4
exterior rectangular faces is 4 units as before. But now consider the longitudinal
circumference of the torus. Starting at a bottom edge, one first traverses an
outside rectangular face traveling 2 units, one then travels over the top edge and
traverses the inside rectangular face traveling another 2 units and reaches the
bottom edge where one started. These are now the vertical lines in the original
game, 4 units long, while the equatorial geodesics, also 4 units in circumference,
are the horizontal lines in the original game. This is a flat square torus.

3 Origami

Go back to the original demonstration. Bend the original square piece of pa-
per containing an Asteroids game into a cylinder with the game on the out-
side. Tape it together to form a complete cylinder as before: the top of the
original square piece of paper is now taped to the bottom. Now instead of giving
up, we are solving the problem with origami.

Origami has an interesting history in mathematics. In 1980, Hisashi Abe
solved the angle trisection problem by using origami: the angle trisection takes
7 steps, see for example [Alperin (2005)] and [Fuchs (2011)] for discussion, proofs
and the axiomatic basis for this. It didn’t occur to the ancient Greeks to
think of a plane as a piece of paper that could be folded. Likewise, the $\mathbb{C}^1$
embeddings of [Nash (1954)] and [Borrelli et al. (2013)] by definition do not allow
folds, although it is only the intrinsic geometry of the embedded surface that
we are really interested in.

Now take the horizontal cylinder you have just formed lying on a table, where the taping line is flush with the table, and squash the cylinder flat: crease it making two folds, opposite each other on the cylinder and equidistant from the taping line. Let the square have side length of 4 units. One now has a flattened, open-ended envelope (4 units long and 2 units wide) with the game printed on the top and bottom outside sides of this envelope. Flip it over so the taped edges are on the top side. Now pick up the two open ended ends (which are 2 units wide) and 4 units apart and fold them inward: fold it in quarters so that it makes 4 rectangles, each 2 units by 1 unit. Mate the two open-ended slits and tape them together to form a tall, square, open-ended box with 4 rectangular sides, each 2 units tall and 1 unit wide. This is the elongated box-like structure we have just discussed above. See Figures 2 and 3 below. We have made the open-ended boxlike surface out of our original flat square and we have taped the opposite sides of the square together as required to make the square torus. The spaceship shoots an asteroid with a ray gun.

A literature search revealed that people have come close, but failed, to find this solution earlier. The Zalgaller (2000) paper, “Some Bendings of a Long Cylinder” came up with some polyhedron solutions for a long rectangular torus. These enclose volume. Basically, he makes a long triangular prism by origami first, makes origami fold crimps in it and ends up with an n-gonal toroidal polyhedron, where the sum of face angles around each vertex is 360°. As part of his introduction, he gives a construction where he takes a long rectangle and folds in in half and tapes it together to make a long, ironed sleeve. Then he takes the open cuff ends and creates two folds in the sleeve so that he can bring the two cuff ends back together at the center and tape them together in the center. Now he has a four-folded surface ironed flat in the plane. This is really a long rectangular flat torus. It has a disadvantage that half of its map of the video game is buried on inside faces of the ironed flat surface of the plane so that you can’t see them. He calls this a direct flat torus. He seems to not realize that this construction could be used to make a flat square torus embedded in a 2D Euclidean space as well, instead of the long one he is making.

There is a wonderful YouTube video uploaded on June 8, 2015 by Segerman (2015) titled “Hinged Flat Torus” showing him folding a flat polyhedron to form a torus. It is a bending of a long cylinder as well. It has 10 rectangles around its outer equatorial surface forming a decagonal prism with open top and bottom. Then a zig-zag configuration of 20 long triangles that connect top vertices with rotated vertices on the decagonal edge below. This encloses a volume. This is called a piecewise linear flat torus. Unfolded, it shows a row of 10 rectangles along the bottom and 20 long, tipped triangles making a parallelogram on the top. This figure is flat and would tile the plane. There is no Gaussian curvature at any vertex; the sum of angles around each vertex is 360° with three triangles and two rectangles around each point. This is longer than it is wide and so does not solve the flat torus embedding. Guy Valette saw a torus like this at Oberwolfach over 30 years ago. Thus, we don’t know who invented it.

Had they allowed polygons to appear back to back with dihedral angles of 0°
Figure 2: The square flat torus showing an *Asteroids* game, top and bottom edges are identified as are left and right edges.
Figure 3: Folding the square flat torus into a simple boxlike surface isometrically embedded in 3D space that is $C^0$. 
as in Gott’s envelope polyhedra they could have used the 10 rectangles alone to have formed an open-ended decagonal prism with rectangles meeting 4 around a point (at the top and bottom decagonal edges). Make the rectangles tall enough (5:1) and one would have an envelope polyhedron that was an embedding of the square flat torus.

We propose another origami construction which embeds the flat torus in 3D space. Start with a flat square 4 units on a side. Fold it in half, and tape the two sides together. Now bend this 4-unit long, 2-unit wide rectangular envelope of zero thickness into a cylinder by taping the two open ends together. One now has an open-ended cylinder 2 units tall with a circumference of $2\pi r = 4$ units. The equatorial circumference is 4 units. The longitudinal circumference is also 4 units, since starting at the bottom circular edge traveling upward on the outside of the cylinder 2 units, crossing over the upper edge and traveling downward 2 units one reaches the bottom edge where one started. All the longitudinal and equatorial circumferences are geodesics. The circular edges at the top and bottom are also geodesics, being straight lines in the original flat square. Take the original square, tile the plane with images of it. Pick any two points $A$ and $B$ in the original square and connect them with a straight line connecting their closest pair of images in the tiling and this will be the geodesic connecting them on the surface. Such a geodesic encountering the top or bottom edge will make an equal angle with the edge on the other side. Unfolded, the line will be straight. This is a surface that is the limit of an $N$-gonal prism without its top and bottom $N$-gons as $N$ goes to infinity. The prisms have $N$ external rectangular faces, $N$ internal rectangular faces, $N$ external edges, $N$ internal edges, $2N$ top and bottom edges linking the internal and external edges, and $2N$ vertices. Thus, they have $F - E + V = 2N - 4N + 2N = 0 = 2(1 - g)$, so $g = 1$ and each is a torus.

An interesting side note: starting with the same flat square torus 4 units by 4 units, one can make this out of 16 squares. These meet 4 around a point, like in a checkerboard. This can be embedded in 3D space as we have described above: 8 squares on the inside of a 2x1x1 hollow box, and 8 squares on the outside. This is a cube stretched vertically by a factor of 2 missing its top and bottom. Dihedral angles at the top and bottom edges are $0^\circ$. But it is also possible to embed this in 4D Euclidean space as 16 square faces which are a subset of the square faces of a tesseract (or hyper-cube). The tesseract has a surface composed of 8 cubic cells. Unfold the tesseract in a cross like shape as shown in the Salvador Dali 1954 painting Corpus Hypercubus. Four cubes form the vertical post of this cross. Consider the four sides of each of these four cubes, they form a tube with cylindrical topology whose surface consists of 16 squares. When this polyhedral net is folded back up into the tesseract, the top of the cross will meet the bottom, to make this a torus with 16 square faces. This is a two-sided surface. It divides the tesseract surface into two equivalent parts, like the Clifford Torus divides the 3-sphere into two equivalent parts. On this torus, the square faces meet 4 around a point.
References

Alperin R.C. Trisections and Totally Real Origami // The American Mathematical Monthly. 2005. 112, 3. 200–211.

Borrelli V., Jabrane S., Lazarus F., Thibert B. Isometric Embedding of the Square Flat Torus in Ambient Space // Ensaios Mathematicos. 2013. 24. 1–91.

Coxeter H.S.M. Regular Skew Polyhedra in Three and Four Dimensions, and Their Topological Analogues // Proceedings of the London Mathematical Society. 2. 1938. 43, 216. 33.

Fuchs C. Angle Trisection with Origami and Related Topics // Elemente der Mathematik. 2011. 66. 121–131.

Gott J.R. Envelope Polyhedra. 2019. arXiv:1908.05395v1.

Nash J.F. C1-isometric embeddings // Annals of Mathematics. 1954. 60, 3. 383–396.

Segerman H. Hinged Flat Torus. 2015. YouTube video.

Zalgaller V.A. Some Bendings of a Long Cylinder // Journal of Mathematical Sciences. 2000. 100, 3. 2228–2238.