As a preliminary step, the radiation produced by a classical charged current coupled to a quantized $A_{\mu}$ is solved. To each order in $\alpha$, all infrared divergences cancel between the virtual $\gamma$'s and the real $\gamma$'s absorbed from the plasma or emitted into the plasma. When all orders of perturbation theory are summed, the finite answer predicts a suppression of radiation with $\omega < \alpha T$. The analysis of QED then consists of two steps. First, a general probability at $T \neq 0$ is organized so that all the virtual $e^\pm, \gamma$ are in the amplitudes and all the real $e^\pm, \gamma$ are in the phase space integrations. Next, the cancellations of IR divergences between virtual and real are demonstrated.

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1. INTRODUCTION

In studies of the quark-gluon plasma to be produced in ultrarelativistic heavy ion collisions and, more generally, in studies of field theory at finite temperature, a central concern is how the cancellation of infrared divergences comes about. Infrared divergences occur when an on-shell particle \((p^2 = m^2)\) emits a massless particle \((k^2 = 0)\). The resulting propagator is \([m^2 - (p - k)^2]^{-1} = [2p \cdot k]^{-1} = [2(E - |p| \cos \theta)|k]|^{-1}\) and so amplitudes for one-photon emission behave like \(\sim 1/k\) at low energy. At \(T = 0\) this behavior leads to logarithmic divergences, \(\int \frac{dk}{k}\), for each real and virtual photon. At \(T \neq 0\) it leads to linear divergences, \(\int \frac{dk n_B}{k}\), because of the Bose-Einstein function \(n_B = 1/[\exp(k/T) - 1]\). This paper discusses how all the infrared divergences cancel even when \(T \neq 0\).

2. IR CANCELLATION FOR A CLASSICAL CURRENT

2.1 Bremsstrahlung to Order \(\alpha\)

If a charged particle scatters while passing through a fixed-temperature plasma, it will radiate. If the radiated energy \(\omega\) is much smaller than the energy transfer in the collision, then the inelastic cross section factors:

\[
2\omega \frac{d\sigma}{d^3k \, dq^2} \approx 2\omega \frac{dP(q^2)}{d^3k} \frac{d\sigma}{dq^2}
\]

(1)

so that the collision cross section \(d\sigma/dq^2\) is independent of the photon energy-momentum \((\omega = |\vec{k}|)\). To first order in \(\alpha\) the probability of radiating is

\[
2\omega \frac{dP_1}{d^3k} = \sum_{\text{pol}} |\epsilon_\mu \cdot J^\mu|^2 \frac{1 + n_B(\omega)}{(2\pi)^3}
\]

(2)

When the radiated energy \(\omega\) is small, the current has a universal form regardless of the spin of the charged particle:

\[
J^\mu(k) = ie \left( \frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k} \right) e^{-k/2\Lambda}
\]

(3)

Here \(\Lambda\) is a momentum cutoff that is necessary later. The radiation is mostly parallel to \(\vec{p}\) or \(\vec{p}'\). When integrated over angles the result is

\[
\frac{dP_1}{d\omega} = \frac{A}{\omega} \left[ 1 + n_B(\omega) \right] e^{-\omega/2\Lambda}, \quad A(p \cdot p') = \frac{\alpha}{\pi} \left[ \frac{1}{v} \ln\left( \frac{1+v}{1-v} \right) - 2 \right]
\]

(4)

where \(v\) is defined by \(p \cdot p' = m^2 (1 - v^2)^{-1/2}\). At large momentum transfer \((Q \gg m)\), the behavior is \(\pi A \approx 4\alpha \ln(Q/m)\). Except for the statistical factor \(n_B\), (4) is a classical formula. When \(\omega \ll T\) this predicts \(dP_1/d\omega \approx AT/\omega^2\). This is totally unphysical because the energy radiated in low energy modes, below some \(E_{\text{max}}\), would be infinite: \(\int_{0}^{E_{\text{max}}} d\omega \omega dP_1/d\omega = \infty\).
2.2 Bremsstrahlung to All Orders in $\alpha$

To improve upon (4), couple the classical current (3) to the quantized radiation field $A_\mu$. The generating function for all multi-photon amplitudes can be obtained by a Gaussian functional integration. From this one obtains the exact multi-photon amplitudes $M$. The probability that any number $n$ of real photons in the plasma will radiate a net energy $\omega$ is

$$\frac{dP}{d\omega} = \sum_{n=1}^{\infty} \int d\Phi_1...d\Phi_n \delta(k_1^0 + ... k_n^0 - \omega) \frac{1}{n!} \sum_{\text{pol}} |M(k_1, ... k_n)|^2$$

(5)

$$d\Phi_i \equiv \frac{d^4k_i}{(2\pi)^3} \delta(k_i^2) \left[ \theta(k_i^0) + n_B(|k_i|) \right]$$

(6)

This weights photon emission ($k^0 > 0$) with the statistical factor $1 + n_B$; and photon absorption ($k^0 < 0$) with the factor $n_B$.

Each amplitude $M$ is infrared divergent from closed loops of virtual photons. Each integration $d\Phi$ over the real photons is infrared divergent. The virtual and real contributions exponentiate to give

$$\frac{dP}{d\omega} = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-i\omega z} \exp[\overline{R}(z)]$$

(7)

$$\overline{R}(z) = \int \frac{d^3k}{2k(2\pi)^3} J_\mu(k)J^\mu(k) \left( [1 + n_B]e^{ikz} + n_B e^{-ikz} - [1 + 2n_B] \right)$$

(8)

The term proportional to $1 + n_B$ represents stimulated emission; the term proportional to $n_B$ represents absorption; the term proportional to $1 + 2n_B$ represents the virtual photons (emitted and absorbed). At small $k$, $J_\mu \sim 1/k$ and $n_B \sim 1/k$. Nevertheless, all infrared divergences cancel in (8) and give a finite result. It is possible to compute $\overline{R}(z)$ and to compute the Fourier transform (7). The final result is

$$\frac{dP}{d\omega} = |\Gamma(A + i\omega/2\pi T)|^2 \frac{e^{\omega/2T} e^{-|\omega|/\Lambda}}{4\pi^2 T \Gamma(A)} \left( \frac{2\pi T}{\Lambda} \right)^A$$

(9)

where $A$ is the same function (4) as before. The most interesting feature of (9) is the appearance of two dimensionless scales: $A \ll 1$ and $\omega/T$. If $A\pi T \ll \omega$ then

$$\frac{dP}{d\omega} \approx \frac{A}{\omega} \left[ 1 + n_B(\omega) \right] e^{-\omega/\Lambda} \quad (A\pi T \ll \omega)$$

(10)

which agrees with the first-order result (4). However this does not apply at $\omega \ll T$. At small energy,

$$\frac{dP}{d\omega} \approx \frac{AT}{\omega^2 + (A\pi T)^2} \quad (\omega \ll T)$$

(11)
Naturally (10) and (11) agree in the region of overlap. Surprisingly, \( dP/d\omega \) is constant for \( \omega \ll A\pi T \) rather than increasing like \( 1/\omega^2 \). The interpretation of the small \( \omega \) suppression is that the quantity \( 2A\pi T = \Gamma_r \) is a damping rate produced by the radiation reaction as required by unitarity \([1]\).

3. REAL & VIRTUAL PARTICLES IN THERMAL FIELD THEORY

We now set aside the semiclassical approximation and turn to the full quantum field theory. Each species of particle has four types of propagator in thermal field theory, e.g. \( S_{ab} \) for electrons and \( D_{ab}^{\mu\nu} \) for photons with \( a, b = 1 \) or \( 2 \). It is possible to rewrite thermal probabilities as squares of amplitudes that contain only \( S_{11} \) and \( D_{11}^{\mu\nu} \), integrated over physical phase space.

For definiteness, consider the process \( e^- (p_1) + \text{plasma} \rightarrow \text{anything} \), with rate

\[
R(p_1) = \sum_{F,I} | < F|C|I > |^2 \frac{e^{-\beta E_i}}{Z} C \equiv [S, b^\dagger(p_1)]
\]  

(12)

Using completeness and thermofield dynamics \([2]\) one can write this as

\[
R(p_1) = < 0(\beta)|C^\dagger C|0(\beta) > = \sum_F | < F(\beta)|C|0(\beta) > |^2
\]  

(13)

where \( |F(\beta) > \) is a complete set of thermal modes. These correspond to real particles:

\[
R(p_1) = \sum_{\ell=2}^\infty \int d\Psi_2 ... d\Psi_\ell \sum_{n=0}^\infty \int d\Phi_1 ... d\Phi_n \frac{1}{n!} | M_{\ell,n}(p_1, ... p_\ell; k_1, ... k_n) |^2
\]  

(14)

where \( M_{\ell,n} \) is the amplitude for \( \ell \) real \( e \)'s or \( \bar{\tau} \)'s (including the initial one) and \( n \) real \( \gamma \)'s. By charge conservation, \( \ell \) is even. Here \( d\Psi \) is the fermion phase space similar to the photon phase space \( d\Phi \) in (6). The explicit amplitude has the form

\[
M_{\ell,n} = \sum_I | < I|[[a^\dagger, [a^\dagger, ... [b(p_2), [S, b^\dagger(p_1)]]]|I > \frac{e^{-\beta E_i}}{Z}
\]  

(15)

These amplitudes are related to Green functions that do not have the usual \( \exp(-\beta H)/Z \) but rather \( S \exp(-\beta H)/Z \). Consequently \( M_{\ell,n} \) has no “cut” propagators \( D_{21} = D_{12} \) or \( S_{21} = -S_{12} \). It is constructed entirely of \( D_{11}(k) \) and \( S_{11}(p) \). See also \([3,4]\).

4. CANCELLATION OF INFRARED DIVERGENCES

To analyze the rate (14) one can repeat the analysis of Yennie, Frautschi, and Suura \([5]\) with slight modifications. Virtual photons inside the amplitude \( M_{\ell,n} \) can cause an IR divergence when they are on-shell and attached to “external” fermions \( p_1, ... p_\ell \). Each end of a photon line attached to an “external” fermion gives a multiplicative factor \( e2p^\mu /2p \cdot k \) plus non-IR terms. At each order of \( \alpha \), the amplitude has a maximum IR
divergence $\alpha^n$ plus non-leading divergences $\alpha^{n-1}, \alpha^{n-2}, \ldots$. When summed to all orders, the leading and non-leading divergences all exponentiate:

$$M_{\ell,n} = \exp(V) M_{\ell,n}^{\text{fin}} \quad V = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sum_{a,b=1}^{\ell} \frac{e_a e_b p_a \cdot p_b}{(p_a \cdot k)(p_b \cdot k)} e^{-k/\Lambda} \quad (16)$$

$V$ by itself is IR divergent. For real photons, infrared divergences occur in the integrations $d\Phi$ in (14). For fixed fermion momenta, define

$$R_{\ell}(p_1, \ldots p_\ell) = \sum_{n=0}^{\infty} \int d\Phi_1 \ldots d\Phi_n \frac{1}{n!} |M_{\ell,n}|^2 \quad (17)$$

Since $d\Phi \sim kdk[\theta(k^0) + n_B]$, divergences arise when $M_{\ell,n} \sim 1/k$. When $n$ real photon lines are attached to the “external” fermions in all possible ways, they each give multiplicative factors $e^{2p^\mu/2p \cdot k}$ plus non-IR terms. The contribution to (17) of $n$ real photons has an IR divergence $\alpha^n$ plus non-leading divergences $\alpha^{n-1}, \alpha^{n-2}, \ldots 1$. When summed over $n$, the leading and non-leading divergences all exponentiate to give:

$$R_{\ell}(p_1, \ldots p_\ell) = \int \frac{dP}{d\omega} \int d\Phi_1 \ldots d\Phi_m \beta_{\ell,m}(k_1, \ldots k_m)/m! \quad (18)$$

All IR divergences are contained in the quantity

$$\frac{dP}{d\omega} = \int_\infty^\infty \frac{dz}{2\pi} e^{-iz\omega} \exp[R(z)] \quad (19)$$

$$R(z) = \int \frac{d^3k}{2k(2\pi)^3} \sum_{a,b=1}^{\ell} \frac{e_a e_b p_a \cdot p_b}{(p_a \cdot k)(p_b \cdot k)} ([1 + n_B]e^{ikz} + n_B e^{-ikz} - [1 + 2n_B]) e^{-k/\Lambda} \quad (20)$$

This integration is IR finite because of the delicate cancellation between the virtual photons and the real photons (emitted + absorbed). The calculation is the same as in Sec. 2 with the result

$$\frac{dP}{d\omega} = |\Gamma(\frac{A}{2} + i \frac{\omega}{2\pi T})|^2 \frac{e^{\omega/2T} e^{-|\omega|/\Lambda}}{4\pi^2 f T \Gamma(f)} \left( \frac{2\pi T}{\Lambda} \right)^A \quad (21)$$

but now $A$ depends on all the “external” fermion momenta:

$$A = -\sum_{a,b=1}^{\ell} \frac{e_a e_b}{8\pi^2 v_{ab}} \ln \left( \frac{1 + v_{ab}}{1 - v_{ab}} \right) \geq 0 \quad (22)$$

and $v_{ab}$, defined by $p_a \cdot p_b/m^2 = (1 - v_{ab}^2)^{-1/2}$, is relative velocity of charge $a$ in the rest frame of $b$. 5
5. CONCLUSIONS

For fixed momenta of the “external” fermions, the rate (18) is IR finite because (a) the function $dP/d\omega$ is finite and (b) the integration $\int d\omega dP/d\omega$ is finite due to the behavior $dP/d\omega \to \text{constant}$ as $\omega \to 0$.

The full rate requires integration over the thermalized fermions:

$$R(p_1) = \sum_{\ell=2}^{\infty} \int \cdots \int d\Psi_2 \cdots d\Psi_\ell \ R_\ell(p_1, \ldots p_\ell) \quad (23)$$

The fermion integrations do not affect the infrared finiteness of $R(p_1)$. However, there will still be Coulomb divergences that arise when any one of the momentum transfers vanishes: $(p_a - p_b)^2 \to 0$. It is known that the usual $T = 0$ Coulomb divergence $\int \sin \theta d\theta / \theta^4$ is reduced to logarithmic, $\int \sin \theta d\theta / \theta^2$ at $T \neq 0$ due to Braaten-Pisarski resummation [6].

For definiteness, the process $e^- + \text{plasma} \to \text{anything}$ was treated specifically. For a general process $\{A\} + \text{plasma} \to \{B\} + \text{anything}$, where $\{A\}$ and $\{B\}$ are any sets of $e^\pm, \gamma$, the same arguments apply and show that all infrared divergences cancel. However, the logarithmic Coulomb divergence remains.

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