Berry-Esséen bound for drift estimation of fractional Ornstein Uhlenbeck process of second kind

Maoudo Faramba Balde, Rachid Belfadli, Khalifa Es-Sebaiy

Abstract

In the present paper we consider the Ornstein-Uhlenbeck process of the second kind defined as solution to the equation
\[ dX_t = -\alpha X_t dt + dY_t^{(1)}, \quad X_0 = 0, \]
where
\[ Y_t^{(1)} := \int_0^t e^{-s} dB_H^s \quad \text{with} \quad a_t = He^{\frac{t}{2}}, \quad B_H^s \]
is a fractional Brownian motion with Hurst parameter \( H \in (\frac{1}{2}, 1) \), whereas \( \alpha > 0 \) is unknown parameter to be estimated. We obtain the upper bound \( O(1/\sqrt{T}) \) in Kolmogorov distance for normal approximation of the least squares estimator of the drift parameter \( \alpha \) on the basis of the continuous observation \( \{X_t, t \in [0, T]\} \), as \( T \to \infty \). Our method is based on the work of [Kim and Park 2017], which is proved using a combination of Malliavin calculus and Stein’s method for normal approximation.

Keywords: Rate of convergence of CLT; Fractional Ornstein-Uhlenbeck processes; Least squares estimator; Malliavin calculus.

1 Introduction

Consider the fractional Ornstein-Uhlenbeck process (fOU) of the second kind, defined as the unique pathwise solution to

\[
\begin{cases}
    dX_t = -\alpha X_t dt + dY_t^{(1)}, & t \geq 0, \\
    X_0 = 0,
\end{cases}
\]

where
\[ Y_t^{(1)} := \int_0^t e^{-s} dB_H^s \quad \text{with} \quad a_t = He^{\frac{t}{2}}, \quad B_H^s \]
is a fractional Brownian motion with Hurst parameter \( H \in (\frac{1}{2}, 1) \), whereas \( \alpha > 0 \) is considered as unknown parameter.

Let \( \tilde{\alpha}_T \) be the least squares estimator (LSE) for the parameter \( \alpha \), proposed in the paper [Hu and Nualart 2010], which is defined by

\[ \tilde{\alpha}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \alpha - \frac{\int_0^T X_t dY_t^{(1)}}{\int_0^T X_t^2 dt}, \]

1 Cheikh Anta Diop University, Dakar, Senegal. Email: faramba88@gmail.com
2 M. F. Balde would like to acknowledge the NLAGA project of SIMONS foundation and the CEA-MITIC that partially supported this work.
3 Department of Sciences and Techniques, Cadi Ayyad University, Morocco. Email: belfadli@gmail.com
4 Department of Mathematics, Kuwait University, Kuwait. E-mail: khalifa.essebaiy@ku.edu.kw
where the integral with respect to $Y^{(1)}$ is interpreted in the Skorohod sense.

Azmoodeh and Morlanes (2013) proved that the LSE $\tilde{\alpha}_T$ is consistent and asymptotically normal for the whole range $H \in (\frac{1}{2}, 1)$, based on the continuous observation $\{X_t, 0 \leq t \leq T\}$ as $T \to \infty$.

However, the study of the asymptotic distribution of an estimator is not very useful in general for practical purposes unless the rate of convergence is known. To our knowledge, no result of the Berry-Esseen type is known for the distribution of the LSE $\tilde{\alpha}_t$ of the drift parameter $\alpha$ of the fOU of the second kind (1). The aim of the present work, in the case $H \in (\frac{1}{2}, 1)$, is to provide an upper bound of Kolmogorov distance for central limit theorem (CLT) of the LSE $\tilde{\alpha}_T$ in the following sense: There exists constant $0 < C < \infty$, depending only on $\alpha$ and $H$, such that for all sufficiently large positive $T$,

$$
\sup_{z \in \mathbb{R}} \left| P \left( \frac{\sqrt{T}}{\sigma_{\alpha,H}} (\alpha - \tilde{\alpha}_t) \leq z \right) - P(Z \leq z) \right| \leq \frac{C}{\sqrt{T}},
$$

where $Z$ denotes a standard normal random variable, and the positive constant $\sigma_{\alpha,H}$ is given by

$$
\sigma_{\alpha,H} := \frac{\alpha}{H(\alpha+1-H,2H-1)} \sqrt{2 \int_{(0,\infty)^3} F(y_1,y_2,y_3)dy_1dy_2dy_3 < \infty},
$$

where $\beta$ denotes the classical Beta function, $\sigma_{\alpha,H} < \infty$ (see Azmoodeh and Viitasaari (2015)), and the function $F$ is defined by

$$
F(y_1,y_2,y_3) := e^{-\alpha|y_1-y_3|}e^{-\alpha y_2}e^{(1-\frac{1}{H})(y_1+y_2+y_3)} \left| e^{-\frac{y_1}{\beta}} - e^{-\frac{y_2}{\beta}} \right|^{2H-2} \left| 1 - e^{-\frac{y_3}{\beta}} \right|^{2H-2}.
$$

Let us also describe what is known about this estimation problem in the case of the Ornstein-Uhlenbeck process of the first kind, defined as solution to the equation

$$
dX_t = -\alpha X_t dt + dB_t^H, \quad X_0 = 0,
$$

where $\alpha$ is an unknown parameter, and $B_t^H$ is a fBm with Hurst parameter $H \in (0,1)$. The drift parameter estimation problem for (5) observed in continuous time and discrete time has been studied by using several approaches (see Kleptsyna and Le Breton (2002); Hu and Nualart (2010); Hu and Song (2013); Brouste and Iacus (2012); El Onsy et al. (2017); Douissi et al. (2019); Sottinen and Viitasaari (2018)). In a general case when the process (5) is driven by a Gaussian process, El Machkouri et al. (2016) studied the non-ergodic case corresponding to $\alpha < 0$. They provided sufficient conditions, based on the properties of the driving Gaussian process, to ensure that least squares estimators-type of $\alpha$ are strongly consistent and asymptotically Cauchy. On the other hand, using Malliavin calculus advances (see Nourdin and Peccati (2012)), Es-Sebaiy and Viens (2019) provided new techniques to statistical inference for stochastic differential equations related to stationary Gaussian processes, and they used their result to study drift parameter estimation problems for some
stochastic differential equations driven by fractional Brownian motion with fixed-time-step observations (in particular for the fOU X given in (5) with \( \alpha > 0 \)). Recently, a Berry-Esséen bound of the LSE of the drift parameter \( \alpha > 0 \) based on the continuous-time observation of the process (5) is provided in [Chen et al. (2019) and Chen and Li (2019)] for \( H \in [\frac{1}{2}, \frac{3}{4}] \) and \( H \in (0, \frac{1}{2}) \), respectively.

Our article is structured as follows. In section 2, we recall some basic elements of Malliavin calculus which are helpful for some of the arguments we use, and the result of Kim and Park (2017) used in this paper. In section 3, we provide a rate of convergence to normality of the LSE \( \tilde{\alpha}_t \) given in (2), for any \( \frac{1}{2} < H < 1 \).

2 Preliminaries

In this section, we briefly recall some basic elements of Gaussian analysis, and Malliavin calculus which are helpful for some of the arguments we use. For more details we refer to Nourdin and Peccati (2012) and Nualart (2006). We also give here the result of Kim and Park (2017) used in this paper.

Let \( B^H = \{ B_t^H, t \geq 0 \} \) be a fractional Brownian motion (fBm) with Hurst parameter \( H \in (0, 1) \) that is a centered Gaussian process, defined on a complete probability space \((\Omega, \mathcal{F}, P)\), with the covariance function

\[
R_H(t, s) = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).
\]

Let us now introduce the Gaussian process \( Y^{(1)}_t := \int_0^t e^{-s} dB_s^H, \ t \geq 0 \), with \( a_t = H e^{\frac{t}{H}} \).

Assume that \( \frac{1}{2} < H < 1 \). Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be a function of class \( C^1 \). Then, (see Bajja et al. (2017)),

\[
\int_s^t f(r) dY_r^{(1)} = \int_{a_s}^{a_t} f(a_u^{-1}) e^{-a_u^{-1}} dB_u,
\]

where \( a_u^{-1} = H \log(u/H) \). Moreover, for every \( f, g \) in \( C^1 \),

\[
E \left( \int_s^t f(r) dY_r^{(1)} \int_{a_s}^{a_u} g(r) dY_r^{(1)} \right) = H(2H-1) \int_{a_s}^{a_t} \int_{a_u}^{a_v} f(a_x^{-1}) g(a_y^{-1}) e^{-a_x^{-1} e^{-a_y^{-1}}} |x - y|^{2H-2} dxdy
\]

\[
= H^{2H+1}(2H-1) \int_{a_s}^{a_t} \int_{a_u}^{a_v} f(H \log(\frac{x}{H})) g(H \log(\frac{y}{H}))(xy)^{-H} |x - y|^{2H-2} dxdy \quad (6)
\]

\[
= \int_s^t \int_u^v f(w) g(z) r_H(w, z) dwdz, \quad (7)
\]

where \( r_H(w, z) \) is a symmetric kernel given by

\[
r_H(w, z) = H^{2H-1}(2H-1)(a_w a_z)^{-1-H} |a_w - a_z|^{2H-2} = H^{2H-1}(2H-1) \left( e^{w/H} e^{z/H} \right)^{-1-H} \left| e^{w/H} - e^{z/H} \right|^{2H-2}.
\]
In particular, we obtain the following covariance given in Kaarakka and Salminen (2011),

\[
\langle 1_{[s,t]}, 1_{[u,v]} \rangle_{\mathcal{H}} = E \left( (Y_t^{(1)} - Y_s^{(1)})(Y_v^{(1)} - Y_u^{(1)}) \right).
\]

Let \( \mathcal{E} \) denote the space of all real valued step functions on \( \mathbb{R} \). The Hilbert space \( \mathcal{H} \) is defined as the closure of \( \mathcal{E} \) endowed with the inner product

\[
E \left( (Y_t^{(1)} - Y_s^{(1)})(Y_v^{(1)} - Y_u^{(1)}) \right) = \int_s^t \int_u^v r_H(w, z)dwdz.
\]

The mapping \( 1_{[0,t]} \mapsto Y_t^{(1)} \) can be extended to a linear isometry between \( \mathcal{H} \) and the Gaussian space \( \mathcal{H}_1 \) spanned by \( Y^{(1)} \). We denote this isometry by \( \varphi : \mathcal{H} \mapsto Y^{(1)}(\varphi) \).

For a smooth and cylindrical random variable \( F = (Y^{(1)}(\varphi_1), \ldots, Y^{(1)}(\varphi_n)) \), with \( \varphi_i \in \mathcal{H} \), \( i = 1, \ldots, n \), and \( f \in C_b^{\infty}(\mathbb{R}^n) \) (\( f \) and all of its partial derivatives are bounded), we define its Malliavin derivative as the \( \mathcal{H} \)-valued random variable given by

\[
DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (Y^{(1)}(\varphi_1), \ldots, Y^{(1)}(\varphi_n)) \varphi_i.
\]

For every \( q \geq 1 \), \( \mathcal{H}_q \) denotes the \( q \)th Wiener chaos of \( Y^{(1)} \), defined as the closed linear subspace of \( L^2(\Omega) \) generated by the random variables \( \{ H_q(Y^{(1)}(h)) : h \in \mathcal{H}, \| h \|_{\mathcal{H}} = 1 \} \) where \( H_q \) is the \( q \)th Hermite polynomial. Wiener chaos of different orders are orthogonal in \( L^2(\Omega) \).

The mapping \( I_q(h^{\otimes q}) : = q! H_q(Y^{(1)}(h)) \) is a linear isometry between the symmetric tensor product \( \mathcal{H}^{\otimes q} \) (equipped with the modified norm \( \| \cdot \|_{\mathcal{H}^{\otimes q}} = \sqrt{q!} \| \cdot \|_{\mathcal{H}^{\otimes q}} \)) and \( \mathcal{H}_q \). For every \( f, g \in \mathcal{H}^{\otimes q} \) the following extended isometry property holds

\[
E(I_q(f)I_q(g)) = q!(f, g)_{\mathcal{H}^{\otimes q}}.
\]

What is typically referred to as the product formula on Wiener space is the version of the above formula before taking expectations (see Section 2.7.3 of Nourdin and Peccati (2012)).

In our work, beyond the zero-order term in that formula, which coincides with the expectation above, we will only need to know the full formula for \( q = 1 \), which is

\[
I_1(f)I_1(g) = \frac{1}{2}I_2(f \otimes g + g \otimes f) + \langle f, g \rangle_{\mathcal{H}}.
\]

(8)

Let \( \{ e_k, k \geq 1 \} \) be a complete orthonormal system in the Hilbert space \( \mathcal{H} \). Given \( f \in \mathcal{H}^{\otimes n}, g \in \mathcal{H}^{\otimes m} \), and \( p = 1, \ldots, n \wedge m \), the \( p \)-th contraction between \( f \) and \( g \) is the element of \( \mathcal{H}^{\otimes (m+n-2p)} \) defined by

\[
f \otimes_p g = \sum_{i_1, \ldots, i_p=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{\mathcal{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_p} \rangle_{\mathcal{H}^{\otimes p}}.
\]

Throughout the paper \( Z \) denotes a standard normal random variable, and \( C \) denotes a generic positive constant (perhaps depending on \( \alpha \) and \( H \), but not on anything else), which
Let us now state the result of Kim and Park (2017) we use in this paper. Recently, based on techniques relied on the combination of Malliavin calculus and Stein’s method (see, e.g., Nourdin and Peccati (2012)), Kim and Park (2017) provided an upper bound of the Kolmogrov distance for central limit theorem of sequences of the form $F_n/G_n$, where $F_n$ and $G_n$ are functionals of Gaussian fields, see Corollary 1 of Kim and Park (2017).

**Proposition 1 (Kim and Park (2017)).** Let $f_T, g_T \in \mathcal{S}_2$ for all $T \geq 0$, and let $b_T$ be a positive function of $T$ such that $I_2(f_T) + b_T > 0$ almost surely for all $T > 0$. Define for all sufficiently large positive $T$,

\[
\psi_1(T) := \frac{1}{b_T^2} \sqrt{(b_T^2 - 2\|f_T\|_{\mathcal{S}_2}^2)^2 + 8\|f_T \otimes_1 f_T\|_{\mathcal{S}_2}^2},
\]

\[
\psi_2(T) := \frac{2}{b_T^2} \sqrt{2\|f_T \otimes_1 g_T\|_{\mathcal{S}_2}^2 + 8\langle f_T, g_T \rangle_{\mathcal{S}_2}^2},
\]

\[
\psi_3(T) := \frac{2}{b_T^2} \sqrt{2\|g_T \|_{\mathcal{S}_2}^4 + 8\|g_T \otimes_1 g_T\|_{\mathcal{S}_2}^2}.
\]

Suppose that $\psi_i(T) \to 0$ for $i = 1, 2, 3$, as $T \to \infty$. Then there exists a positive constant $C$ such that for all sufficiently large positive $T$,

\[
\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left( \frac{I_2(f_T)}{I_2(g_T) + b_T} \leq z \right) - \mathbb{P} (Z \leq z) \right| \leq C \max_{i=1,2,3} \psi_i(T).
\]

### 3 Berry Esseen bound for CLT of LSE

Suppose that $\frac{1}{2} < H < 1$. Our main interest in this paper is to provide a Berry-Esséen bound for the LSE given in (2), of the drift parameter $\alpha > 0$ based on the continuous-time observation of the fOU of the second kind described by (1).

Because (1) is linear, it is immediate to solve it explicitly; one then gets the following formula:

\[
X_t = e^{-\alpha t} \int_0^t e^{\alpha s} dY_s^{(1)}.
\]

From (2) we can write

\[
\alpha - \bar{\alpha}_T = \frac{\int_0^T X_t dY_t^{(1)}}{\int_0^T X_t^2 dt}.
\]

It follows from (3) that

\[
\frac{1}{\sqrt{T}} \int_0^T X_t dY_t^{(1)} = I_2^{Y^{(1)}}(h_T), \text{ with } h_T(s,t) := \frac{1}{2\sqrt{T}} e^{-\alpha|t-s|} 1_{[0,T]}(s,t).
\]
On the other hand, using the product formula (8),

\[
X_t^2 = \left( I_1 \left( e^{-\alpha(t-\cdot)}1_{[0,t]}(\cdot) \right) \right)^2
\]

\[
= I_2 \left( e^{-2\alpha t} e^{au} e^{av} 1_{[0,t]^2}(u,v) \right) + \left\| e^{-\alpha(t-\cdot)}1_{[0,t]}(\cdot) \right\|_{\mathcal{D}}^2.
\]

Let us introduce the positive constant

\[
\rho_{\alpha,H} := \frac{H^{2H}(2H-1)}{\alpha} \beta(H\alpha + 1 - H, 2H - 1).
\]

Thus

\[
\frac{1}{T \rho_{\alpha,H}} \int_0^T X_t^2 dt
\]

\[
= I_2 \left( \frac{1}{T \rho_{\alpha,H}} \int_0^T e^{-2\alpha t} e^{au} e^{av} 1_{[0,t]^2}(u,v) dt \right) + \frac{1}{T \rho_{\alpha,H}} \int_0^T e^{-2\alpha t} \left\| e^{au} 1_{[0,t]}(u) \right\|_{\mathcal{D}}^2 dt
\]

\[
=: I_2(g_T) + b_T,
\]

where

\[
b_T := \frac{1}{T \rho_{\alpha,H}} \int_0^T e^{-2\alpha t} \left\| e^{au} 1_{[0,t]}(u) \right\|_{\mathcal{D}}^2 dt,
\]

and

\[
g_T(u, v) := \frac{1}{2\alpha \rho_{\alpha,H}} e^{au} e^{av} e^{-2\alpha(u+v)} - e^{-2\alpha T} 1_{[0,T]^2}(u,v)
\]

\[
= \frac{1}{2\alpha \rho_{\alpha,H} T} \left( e^{-\alpha|u-v|} - e^{-2\alpha T} e^{au} e^{av} \right) 1_{[0,T]^2}(u,v)
\]

\[
= \frac{1}{\alpha \rho_{\alpha,H} \sqrt{T}} h_T(u, v) - l_T(u, v),
\]

with \( h_T \) is given by (11), and

\[
l_T(u, v) := \frac{1}{2\alpha \rho_{\alpha,H} T} e^{-2\alpha T} e^{au} e^{av} 1_{[0,T]^2}(u,v).
\]

Therefore, combining (10), (11) and (13), we get

\[
\frac{\sqrt{T}}{\sigma_{\alpha,H}} (\alpha - \tilde{\alpha}_T) = \frac{I_2(f_T)}{I_2(g_T) + b_T},
\]

where \( \sigma_{\alpha,H} \) is given by (3), and

\[
f_T := \frac{1}{\rho_{\alpha,H} \sigma_{\alpha,H}} h_T.
\]

In order to prove our main result we make use of the following technical lemmas.
Lemma 1. Let \( H \in (\frac{1}{2}, 1) \), and let \( b_T \) and \( f_T \) be the functions given by (14) and (17), respectively. Then, for all \( T > 0 \),

\[
|b_T - 1| \leq \frac{C}{T},
\]

(18)

\[
|1 - 2\|f_T\|_{\mathcal{S}}^2| \leq \frac{C}{T}.
\]

(19)

Consequently, for all \( T > 0 \),

\[
|b_T^2 - 2\|f_T\|_{\mathcal{S}}^2| \leq \frac{C}{T}.
\]

Proof. Using (6) and making the change of variables \( u = x/y \), we get

\[
\|e^{\alpha t}1_{[0,t]}(u)\|^2_{\mathcal{S}} = H(2H - 1) \int_{a_0}^{a_t} \int_{a_0}^{a_t} (x/H)^{\alpha - H} (y/H)^{\alpha - H} |x - y|^{2H - 2} dxdy
\]

\[
= 2H^{2H(1-\alpha)+1}(2H - 1) \int_{a_0}^{a_t} dy \int_{a_0}^{y} dx(xy)^{\alpha - H} |x - y|^{2H - 2}
\]

\[
= 2H^{2H(1-\alpha)+1}(2H - 1) \int_{a_0}^{a_t} dy y^{2H - 1} \int_{a_0/y}^{1} du u^{\alpha - H} |1 - u|^{2H - 2}
\]

\[
= 2H^{2H(1-\alpha)+1}(2H - 1) \int_{a_0}^{a_t} dy y^{2H - 1} \int_{0}^{a_0/y} du u^{\alpha - H} |1 - u|^{2H - 2}
\]

\[
= I_t - J_t,
\]

(20)

where

\[
I_t = 2H^{2H(1-\alpha)+1}(2H - 1) \beta(H\alpha + 1 - H, 2H - 1) \int_{a_0}^{a_t} y^{2H - 1} dy
\]

\[
= \frac{H^{2H}(2H - 1)}{\alpha} \beta(H\alpha + 1 - H, 2H - 1)(e^{2\alpha t} - 1).
\]

Moreover,

\[
\frac{1}{T} \int_0^T e^{-2\alpha t} I_t dt = \frac{H^{2H}(2H - 1)}{\alpha} \beta(H\alpha + 1 - H, 2H - 1) \left( 1 + \frac{e^{-2\alpha t} - 1}{2\alpha T} \right).
\]

Thus

\[
\left| \frac{1}{T \rho_{\alpha,H}} \int_0^T e^{-2\alpha t} I_t dt - 1 \right| = \frac{1 - e^{-2\alpha t}}{2\alpha T} \leq \frac{1}{2\alpha T}.
\]

(21)
On the other hand,

\[ |J_t| \leq 2H^{2H(1-\alpha)+1}(2H-1) \int_{a_0}^{a_1} dy y^{2H\alpha-1}(a_0/y)^{H\alpha} \int_0^{a_0/y} du u^{-H}(1-u)^{2H-2} \]
\[ \leq 2H^{2H-\alpha H+1}(2H-1) \int_{a_0}^{a_1} dy y^{H\alpha-1} \int_0^1 du u^{-H}(1-u)^{2H-2} \]
\[ = 2H^{2H-\alpha H+1}(2H-1) \beta(1-H, 2H-1) \frac{e^{at} - 1}{H\alpha} \]
\[ \leq Ce^{at}. \]

Hence,

\[ \frac{1}{T} \int_0^T e^{-2\alpha t}|J_t|dt \leq \frac{C}{T} \int_0^T e^{-\alpha t}dt \leq \frac{C}{T}. \tag{22} \]

Therefore, combining (20), (21) and (22), we deduce (18).

Now let us prove (19). First we decompose the integral \( \int_{[0,T]_i} \) into

\[ \int_{[0,T]_i} = \int_{\bigcup_{i=1}^5 A_i,T} = \sum_{i=1}^5 \int_{A_i,T}, \tag{23} \]

where

\[ A_{1,T} = \bigcup_{i=1}^{8} D_{i,T}, \quad A_{2,T} = \bigcup_{i=9}^{12} D_{i,T}, \quad A_{3,T} = \bigcup_{i=13}^{16} D_{i,T}, \quad A_{4,T} = \bigcup_{i=17}^{20} D_{i,T}, \quad A_{5,T} = \bigcup_{i=21}^{24} D_{i,T}, \]

with

\[ D_{1,T} := \{ 0 < x_1 < x_2 < x_3 < x_4 < T \}, \quad D_{2,T} := \{ 0 < x_1 < x_2 < x_4 < x_3 < T \}, \]
\[ D_{3,T} := \{ 0 < x_2 < x_1 < x_3 < x_4 < T \}, \quad D_{4,T} := \{ 0 < x_2 < x_1 < x_4 < x_3 < T \}, \]
\[ D_{5,T} := \{ 0 < x_3 < x_4 < x_1 < x_2 < T \}, \quad D_{6,T} := \{ 0 < x_3 < x_4 < x_2 < x_1 < T \}, \]
\[ D_{7,T} := \{ 0 < x_4 < x_3 < x_1 < x_2 < T \}, \quad D_{8,T} := \{ 0 < x_4 < x_3 < x_2 < x_1 < T \}, \]
\[ D_{9,T} := \{ 0 < x_1 < x_3 < x_2 < x_4 < T \}, \quad D_{10,T} := \{ 0 < x_3 < x_1 < x_4 < x_2 < T \}, \]
\[ D_{11,T} := \{ 0 < x_2 < x_4 < x_1 < x_3 < T \}, \quad D_{12,T} := \{ 0 < x_4 < x_2 < x_3 < x_1 < T \}, \]
\[ D_{13,T} := \{ 0 < x_1 < x_3 < x_4 < x_2 < T \}, \quad D_{14,T} := \{ 0 < x_3 < x_1 < x_2 < x_4 < T \}, \]
\[ D_{15,T} := \{ 0 < x_2 < x_4 < x_3 < x_1 < T \}, \quad D_{16,T} := \{ 0 < x_4 < x_2 < x_1 < x_3 < T \}, \]
\[ D_{17,T} := \{ 0 < x_1 < x_4 < x_2 < x_3 < T \}, \quad D_{18,T} := \{ 0 < x_4 < x_1 < x_3 < x_2 < T \}, \]
\[ D_{19,T} := \{ 0 < x_2 < x_3 < x_1 < x_4 < T \}, \quad D_{20,T} := \{ 0 < x_3 < x_2 < x_4 < x_1 < T \}, \]
\[ D_{21,T} := \{0 < x_1 < x_4 < x_3 < x_2 < T\}, \quad D_{22,T} := \{0 < x_4 < x_1 < x_2 < x_3 < T\}, \]
\[ D_{23,T} := \{0 < x_3 < x_2 < x_1 < x_4 < T\}, \quad D_{24,T} := \{0 < x_2 < x_3 < x_4 < x_1 < T\}. \]

Therefore, using (7), (23), and setting
\[
m_H(x_1, x_2, x_3, x_4) := e^{-\alpha|x_1-x_3|}e^{-\alpha|x_2-x_4|}r_H(x_1, x_2)r_H(x_3, x_4),
\]
we have
\[
\|h_T\|_{H^2}^2 = \frac{1}{4T} \int_{[0,T]^4} m_H(x_1, x_2, x_3, x_4)dx_1 \ldots dx_4
\]
\[
= \frac{1}{4T} \left( \int_{A_1,T} + \int_{A_2,T} + \int_{A_3,T} + \int_{A_4,T} + \int_{A_5,T} \right) m_H(x_1, x_2, x_3, x_4)dx_1 \ldots dx_4
\]
\[
= \frac{1}{4T} \left( 8 \int_{D_{1,7,T}} + 4 \int_{D_{9,7,T}} + 4 \int_{D_{13,7,T}} + 4 \int_{D_{17,7,T}} + 4 \int_{D_{21,7,T}} \right) m_H(x_1, x_2, x_3, x_4)dx_1 \ldots dx_4
\]
\[
=: 2I_{1,T} + I_{2,T} + I_{3,T} + I_{4,T} + I_{4,T}, \quad (24)
\]
where we used the fact that
\[
\int_{D_{1,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4 = \ldots = \int_{D_{8,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4,
\]
\[
\int_{D_{9,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4 = \ldots = \int_{D_{12,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4,
\]
\[
\int_{D_{13,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4 = \ldots = \int_{D_{16,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4,
\]
\[
\int_{D_{17,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4 = \ldots = \int_{D_{20,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4,
\]
\[
\int_{D_{21,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4 = \ldots = \int_{D_{24,T}} m_H(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4.
\]
Let us now estimate \( I_{1,T} \). Making the change of variables \( y_3 = x_4 - x_1, \ y_2 = x_4 - x_2, \)
\[ y_1 = x_4 - x_3, \text{ and } y_4 = x_4, \text{ we obtain} \]
\[
\frac{1}{H^{4H-2}(2H-1)^2} I_{1,T}
\]
\[
= \frac{1}{T} \int_0^T dx_4 \int_{0<x_1<x_2<x_3<x_4} dx_1 dx_2 dx_3 e^{-\alpha|x_1-x_3|} e^{-\alpha|x_2-x_4|} e^{(1/H-1)(x_1+x_2+x_3+x_4)}
\]
\[
\times e^{x_1/H} - e^{x_2/H} \left| e^{x_3/H} - e^{x_4/H} \right|^{2H-2}
\]
\[
= \frac{1}{T} \int_0^T dy_1 \int_{0<y_1<y_2<y_3} dy_1 dy_2 dy_3 F(y_1, y_2, y_3)
\]
\[
= \frac{1}{T} \int_0^T dy_1 \int_{y_1<y_2<y_3<\infty} dy_1 dy_2 dy_3 - \int_0^T dy_1 \int_{y_1}^{\infty} dy_3 \int_0^{y_3} dy_2 \int_0^{y_2} dy_1 \ F(y_1, y_2, y_3)
\]
\[
= \left[ \int_{y_1<y_2<y_3<\infty} dy_1 dy_2 dy_3 - \frac{1}{T} \int_0^T dy_1 \int_{y_1}^{\infty} dy_3 \int_0^{y_3} dy_2 \int_0^{y_2} dy_1 \right] F(y_1, y_2, y_3), \tag{25}
\]

where the function \( F \) is given by \[ \tag{1} \] Moreover,
\[
\frac{1}{T} \int_0^T dy_1 \int_{y_1}^{\infty} dy_3 \int_0^{y_3} dy_2 \int_0^{y_2} dy_1 F(y_1, y_2, y_3)
\]
\[
\leq \frac{1}{T} \int_0^T dy_1 \int_{y_1}^{\infty} dy_3 \int_0^{y_3} dy_2 \int_0^{y_2} dy_1 e^{-\alpha y_3} e^{(1-\frac{1}{H})(y_1+y_2+y_3)} \left| e^{-\frac{y_2}{\beta}} - e^{-\frac{y_3}{\beta}} \right|^{2H-2} \left| 1 - e^{-\frac{y_2}{\beta}} \right|^{2H-2}
\]
\[
\leq \frac{H\beta(1-H,2H-1)}{T} \int_0^T dy_1 \int_{y_1}^{\infty} dy_3 \int_0^{y_3} dy_2 e^{-\alpha y_3} e^{(1-\frac{1}{H})(y_1+y_2+y_3)} \left| e^{-\frac{y_2}{\beta}} - e^{-\frac{y_3}{\beta}} \right|^{2H-2}
\]
\[
= \frac{H\beta(1-H,2H-1)}{T} \int_0^T dy_1 \int_{y_1}^{\infty} dy_3 \int_0^{y_3} dy_2 e^{-\alpha y_3} e^{(1-\frac{1}{H})(y_1+y_2)} \left| 1 - e^{-(y_3-y_2)/H} \right|^{2H-2}
\]
\[
= \frac{H\beta(1-H,2H-1)}{T} \int_0^T dy_1 \int_{y_1}^{\infty} dy_3 \int_0^{y_3} dy_2 e^{-\alpha y_3} e^{(1-\frac{1}{H})x_2} \left| 1 - e^{-\frac{x_2}{\beta}} \right|^{2H-2}
\]
\[
\leq \frac{(H\beta(1-H,2H-1))^2}{T} \int_0^T dy_1 \int_{y_1}^{\infty} dy_3 e^{-\alpha y_3}
\]
\[
\leq \frac{(H\beta(1-H,2H-1))^2}{\alpha^2 T}. \tag{26}
\]

Combining \[ \tag{25} \] and \[ \tag{26} \] we deduce
\[
\left| I_{1,T} - H^{4H-2}(2H-1)^2 \int_{y_1<y_2<y_3<\infty} dy_1 dy_2 dy_3 F(y_1, y_2, y_3) \right| \leq \frac{C}{T}. \tag{27}
\]

Moreover,
\[
\left| I_{1,T} - H^{4H-2}(2H-1)^2 \int_{y_1<y_2<y_3<\infty} F(y_1, y_2, y_3) dy_1 dy_2 dy_3 \right| \leq \frac{C}{T}, \tag{28}
\]
since
\[
\int_{0<y_1<y_2<y_3<\infty} F(y_1, y_2, y_3) dy_1 dy_3 dy_2 = \int_{0<y_1<y_2<y_3<\infty} F(y_1, y_2, y_3) dy_1 dy_2 dy_3.
\]

Using similar arguments as above, we can conclude
\[
|I_{2,T} - H^{4H-2}(2H-1)^2 \int_{0<y_2<y_1<y_3<\infty} F(y_1, y_2, y_3) dy_2 dy_3 dy_1| \leq \frac{C}{T},
\]
(29)
\[
|I_{3,T} - H^{4H-2}(2H-1)^2 \int_{0<y_2<y_1<y_3<\infty} F(y_1, y_2, y_3) dy_2 dy_3 dy_1| \leq \frac{C}{T},
\]
(30)
\[
|I_{4,T} - H^{4H-2}(2H-1)^2 \int_{0<y_3<y_1<y_2<\infty} F(y_1, y_2, y_3) dy_3 dy_1 dy_2| \leq \frac{C}{T},
\]
(31)
\[
|I_{5,T} - H^{4H-2}(2H-1)^2 \int_{0<y_3<y_1<y_2<\infty} F(y_1, y_2, y_3) dy_3 dy_2 dy_1| \leq \frac{C}{T}.
\]
(32)

Combining (24), (27)–(32) and the fact that
\[
(0, \infty)^3 = \{0 < y_1 < y_2 < y_3\} \cup \{0 < y_1 < y_3 < y_2\} \cup \{0 < y_2 < y_1 < y_3\} \\
\cup \{0 < y_2 < y_3 < y_1\} \cup \{0 < y_3 < y_1 < y_2\} \cup \{0 < y_3 < y_2 < y_1\},
\]
we deduce that
\[
\left| \|h_T\|_{\mathcal{H}/2}^2 - H^{4H-2}(2H-1)^2 \int_{(0, \infty)^3} F(y_1, y_2, y_3) dy_1 dy_2 dy_3 \right| \leq \frac{C}{T},
\]
(33)
which proves (19).

\[\square\]

**Lemma 2.** Suppose \(H \in (\frac{1}{2}, 1)\). Let \(g_T\) and \(f_T\) be the functions given by (15) and (17), respectively. Then, for all \(T > 0\),
\[
\|f_T \otimes_1 f_T\|_{\mathcal{H}/2} \leq \frac{C}{\sqrt{T}},
\]
(34)
\[
\|g_T\|_{\mathcal{H}/2} \leq \frac{C}{\sqrt{T}},
\]
(35)
\[
\|g_T \otimes_1 g_T\|_{\mathcal{H}/2} \leq \frac{C}{T^{3/2}},
\]
(36)
\[
\|f_T \otimes_1 g_T\|_{\mathcal{H}/2} \leq \frac{C}{T},
\]
(37)
\[
|\langle f_T, g_T \rangle_{\mathcal{H}/2}| \leq \frac{C}{\sqrt{T}}.
\]
(38)
Proof. Setting $F_T := I_2(f_T)$, it follows from Lemma 5.2.4 of Nourdin and Peccati (2012) that

$$8\|f_T \otimes_1 f_T\|_{\mathcal{H}^2} = Var\left(\frac{1}{2}\|DF_T\|_{\mathcal{H}^2}^2\right).$$

Further, using Lemma 5.1 of Azmoodeh and Viitasaari (2015), we have

$$Var\left(\frac{1}{2}\|DF_T\|_{\mathcal{H}^2}^2\right) \leq C T.$$

Thus the inequality (34) is obtained.

Since for every $(u, v) \in [0, T]^2$, $g_T(u, v) \geq 0$, $h_T(u, v) \geq 0$ and $l_T(u, v) \geq 0$, then, using (15), we get

$$\|g_T\|_{\mathcal{H}^2} \leq \frac{1}{\alpha \rho_{\alpha, H} \sqrt{T}} \|h_T\|_{\mathcal{H}^2}.$$

Combining this with (33), we obtain (35). Similarly, using (15), (17) and (34), we have

$$\|g_T \otimes_1 g_T\|_{\mathcal{H}^2}^2 \leq C T \|h_T \otimes_1 h_T\|_{\mathcal{H}^2}^2 \leq C T \|f_T \otimes_1 f_T\|_{\mathcal{H}^2}^2 \leq C T^3,$$

which implies (36).

It is well known that

$$\|f_T \otimes_1 g_T\|_{\mathcal{H}^2}^2 = \langle f_T \otimes_1 f_T, g_T \otimes_1 g_T \rangle_{\mathcal{H}^2},$$

due to a straightforward application of the definition of contractions and Fubini theorem. Thus, from (34) and (36), we obtain

$$\|f_T \otimes_1 g_T\|_{\mathcal{H}^2}^2 \leq \|f_T \otimes_1 f_T\|_{\mathcal{H}^2} \|g_T \otimes_1 g_T\|_{\mathcal{H}^2} \leq C T^2,$$

which leads to (37).

Finally, the inequality (38) is a direct consequence of (19) and (35). The proof of the lemma is thus complete.

Our main result is the following theorem. It is a consequence of Proposition 1, Lemma 1 and Lemma 2.

**Theorem 1.** Suppose $H \in (\frac{1}{2}, 1)$. Then, there exists constant $0 < C < \infty$, depending only on $\alpha$ and $H$, such that for all sufficiently large positive $T$,

$$\sup_{z \in \mathbb{R}} \left| P\left(\frac{\sqrt{T}}{\sigma_{\alpha, H}} (\alpha - \tilde{\alpha}_t) \leq z\right) - P(Z \leq z)\right| \leq \frac{C}{\sqrt{T}},$$

where $\sigma_{\alpha, H}$ is given by (3).

12
References

Azmoodeh, E., Morlanes, G. I. (2013). Drift parameter estimation for fractional Ornstein-Uhlenbeck process of the second kind. Statistics. 49(1), 1-18.

Azmoodeh, E., Viitasaari, L. (2015). Parameter estimation based on discrete observations of fractional Ornstein-Uhlenbeck process of the second kind. Statist. Infer. Stoch. Proc. 18(3), 205-227.

Bajja, S., Es-Sebaiy, K., Viitasaari, L. (2017). Least squares estimator of fractional Ornstein-Uhlenbeck processes with periodic mean. J. Korean Statist. Soc. 46(4), 608-622.

Brouste, A., Iacus, S. M. (2012). Parameter estimation for the discretely observed fractional Ornstein-Uhlenbeck process and the Yuima R package. Comput. Stat. 28(4), 1529-1547.

Chen, Y., Kuang, N., Li, Y. (2019). Berry-Eséeen bound for the parameter estimation of fractional Ornstein-Uhlenbeck processes. Stochastics and Dynamics 20(1): 2050023.

Chen, Y., Li, Y. (2019). Berry-Eséeen bound for the parameter estimation of fractional Ornstein-Uhlenbeck processes with the hurst parameter $H \in (0, \frac{1}{2})$. Communications in Statistics-Theory and Methods, 1-18. DOI: 10.1080/03610926.2019.1678641.

Douissi, S., Es-Sebaiy, K., Viens, F. (2019). Berry-Eséeen bounds for parameter estimation of general Gaussian processes. ALEA, Lat. Am. J. Probab. Math. Stat., 16, 633-664.

El Machkouri, M., Es-Sebaiy, K., Ouknine, Y. (2016). Least squares estimator for non-ergodic Ornstein-Uhlenbeck processes driven by Gaussian processes. Journal of the Korean Statistical Society 45(3), 329-341.

El Onsy, B., Es-Sebaiy, K., Viens, F. (2017). Parameter estimation for a partially observed Ornstein-Uhlenbeck process with long-memory noise. Stochastics, 89(2), 431-468.

Es-Sebaiy, K., Viens, F. (2019). Optimal rates for parameter estimation of stationary Gaussian processes. Stochastic Processes and their Applications, 129(9), 3018-3054.

Hu, Y., Nualart, D. (2010). Parameter estimation for fractional Ornstein Uhlenbeck processes. Statistics and Probability Letters, 80(11-12), 1030-1038.

Hu, Y., Song, J. (2013). Parameter estimation for fractional Ornstein-Uhlenbeck processes with discrete observations. F. Viens et al (eds), Malliavin Calculus and Stochastic Analysis: A Festschrift in Honor of David Nualart, 427-442, Springer.

Kaarakka, T., Sahlminen, P. (2011). On Fractional Ornstein-Uhlenbeck process. Communications on Stochastic Analysis, 5, 121-133.

Kim, Y. T., Park, H. S. (2017). Optimal Berry-Esseen bound for statistical estimations and its application to SPDE. Journal of Multivariate Analysis, 155, 284-304.
Kleptsyna, M., Le Breton, A. (2002). Statistical analysis of the fractional Ornstein-Uhlenbeck type process. Statistical Inference for Stochastic Processes, 5, 229-241.

Nourdin, I., Peccati, G. (2012). Normal approximations with Malliavin calculus: from Stein’s method to universality. Cambridge Tracts in Mathematics 192. Cambridge University Press, Cambridge.

Nualart, D. (2006). The Malliavin calculus and related topics. Springer-Verlag, Berlin.

Sottinen, T., Viitasaari, L. (2018). Parameter estimation for the Langevin equation with stationary-increment Gaussian noise. Statistical Inference for Stochastic Processes, 21(3), 569-601.