A note concerning the Grundy and $b$-chromatic number of graphs

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Abstract

The Grundy number of a graph $G$ is the maximum number of colors used by the First-Fit coloring of $G$ and is denoted by $\Gamma(G)$. Similarly, the $b$-chromatic number $b(G)$ of $G$ expresses the worst case behavior of another well-known coloring procedure i.e. color-dominating coloring of $G$. We obtain some families of graphs $\mathcal{F}$ for which there exists a function $f(x)$ such that $\Gamma(G) \leq f(b(G))$, for each graph $G$ from the family. Call any such family $(\Gamma, b)$-bounded family. We conjecture that the family of $b$-monotone graphs is $(\Gamma, b)$-bounded and validate the conjecture for some families of graphs.

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1 Introduction

This note deals only with undirected graphs without any loops or multiple edges. By a Grundy coloring of a graph $G$ we mean any partition of $V(G)$ into independent subsets $C_1, \ldots, C_k$ such that for each $i, j \in \{1, \ldots, k\}$ with $i < j$, each vertex in $C_j$ has a neighbor in $C_i$. The maximum such value $k$ is called the Grundy number (also called First-Fit chromatic number) and denoted by $\Gamma(G)$ (also by $\chi_{FF}(G)$). It can be observed that $\Gamma(G)$ is equal to the maximum number of colors used by the First-Fit (greedy) coloring procedure in the graph $G$ [11]. The Grundy number and First-Fit coloring of graphs are important research areas in chromatic and algorithmic graph theory with full of papers e.g. [2, 3, 6, 7, 10, 11, 12].

By a color-dominating coloring of $G$ we mean any partition of $V(G)$ into independent subsets $C_1, \ldots, C_k$ such that for each $i$, the class $C_i$ contains a vertex say $v$ such

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that \( v \) has a neighbor in any other class \( C_j, j \neq i \). Denote by \( b(G) \) (also denoted by \( \varphi(G) \)) the maximum number of colors used in any color-dominating coloring of \( G \). It can be easily seen that \( b(G) \leq \Delta(G) + 1 \) and under some conditions the equality holds, e.g. \( d \)-regular graphs with at least \( 2d^3 \) vertices \[1\]. An algorithmic interpretation of \( b(G) \) is that it expresses the worst case behavior of the following coloring procedure. In any proper coloring \( C \) of a graph \( G \), a vertex \( v \) is said to be a color-dominating vertex if it has a neighbor with any other color except the color of \( v \). Let \( C \) be any arbitrary proper coloring of \( G \) and \( C_i \) be a color class in \( C \). If \( C_i \) does not contain any color-dominating vertex then each vertex of \( C_i \) can be removed from \( C_i \) and transferred to another suitable class. By this technique the class \( C_i \) is totally removed and number of colors is decreased by one. We repeat this method for all remaining color classes until we obtain a color-dominating coloring. Obviously, the final number of colors is at most \( b(G) \). The b-chromatic number of graphs introduced in \[4\] and widely studied in the literature \[8, 9\]. For a recent survey on b-chromatic number see \[5\]. A useful graph parameter relating to b-chromatic number of a graph \( G \) with non-increasing degree sequence \( d_1 \geq d_2 \geq \ldots \geq d_n \) is \( m(G) := \max\{i : d_i \geq i - 1\} \). It is known that \( b(G) \leq m(G) \) and for trees \( T \), \( b(T) \geq m(T) - 1 \[4\]. In this paper, a graph \( G \) is called b-monotone if for each induced subgraph \( H \) of \( G \) we have \( b(H) \leq b(G) \).

A first natural inquiry concerning the comparison of Grundy and b-chromatic numbers is to explore and generate families of graphs \( \{G_n\}_{n \geq 1} \) and \( \{H_n\}_{n \geq 1} \) such that \( b(G_n) - \Gamma(G_n) \to \infty \) and \( \Gamma(H_n) - b(H_n) \to \infty \). Based on the results of this paper, both of the above-mentioned situations may happen in the universe of graphs. But the first situation (i.e. families with bounded Grundy number and unbounded b-chromatic number) is more likely to happen because these families are more accessible.

The concept of \((\chi_{FF}, \omega)\)-boundedness was introduced by Gyárfás and Lehel in \[3\]. Denote the size of a maximum clique in \( G \) by \( \omega(G) \). A family \( F \) is called \((\chi_{FF}, \omega)\)-bounded if there exists a function \( f(x) \) such that \( \Gamma(G) \leq f(\omega(G)) \) for each \( G \) from the family. Some \((\chi_{FF}, \omega)\)-bounded families were obtained in \[3, 6, 7, 12\]. In the next section we introduce \((\Gamma, b)\)-bounded families.

### 2 \((\Gamma, b)\)-bounded families of graphs

We say a family \( F \) is \((\Gamma, b)\)-bounded if there exists a function \( f(x) \) such that \( \Gamma(G) \leq f(b(G)) \), for each graph \( G \) from the family. Note that any \((\chi_{FF}, \omega)\)-bounded family is also \((\Gamma, b)\)-bounded. Also any family of graphs satisfying \( b(G) = \Delta(G) + 1 \) is \((\Gamma, b)\)-bounded. Some of such families were obtained in \[3\] and reported in \[5\]. With a similar manner we can define \((b, \Gamma)\)-bounded families. We can easily obtain a sequence of trees \( T_n \) such that \( \Gamma(T_n) \leq 3 \) for each \( n \), but \( b(T_n) \to \infty \). In fact, we
may consider \( T_n \) as a path with sufficiently large length and sufficiently many leaves attached to the vertices of the path. In this note we concentrate on \((\Gamma, b)\)-bounded families. The following proposition is useful.

**Proposition 1.** A family \( \mathcal{F} \) is \((\Gamma, b)\)-bounded if and only if for any sequence \( \{G_n\}_{n \geq 1} \) from \( \mathcal{F} \), \( \Gamma(G_n) \to \infty \) implies \( b(G_n) \to \infty \).

**Proof.** If \( \mathcal{F} \) is \((\Gamma, b)\)-bounded then the assertion trivially holds. To prove the other side, note that any infinite family of graphs is countable, so write \( \mathcal{F} = \{G_n\}_{n \geq 1} \). If necessary use a relabeling and assume that \( \{\Gamma(G_n)\}_{n \geq 1} \) is increasing. Assume that \( \Gamma(G_n) \to \infty \) (otherwise the assertion trivially holds). It implies \( b(G_n) \to \infty \). Hence, for each \( n \geq 1 \), there exists an integer \( N(n) \) such that \( b(G_i) \geq \Gamma(G_n) \) for each \( i \geq N(n) \). Now, define a function \( f \) by putting for each \( n \), \( f(b(G_n)) := b(G_{N(n)}) \). We have \( \Gamma(G_n) \leq f(b(G_n)) \) for each \( n \), as desired. \( \square \)

The following result shows that the family of tree graphs is \((\Gamma, b)\)-bounded.

**Proposition 2.** For any tree \( T \), \( \Gamma(T) \leq 2b(T) + 2 \).

**Proof.** Set \( |V(T)| = n \), \( \Gamma(T) = p \) and \( m(T) = m \). It is enough to show \( p \leq 2m \). Otherwise, \( p \geq 2m + 1 \). Let \( p - m = m + t + 1 \), for some \( t \geq 0 \). By the definition of \( m(T) \), for each \( k \geq m + 1 \), \( n - k + 1 \) vertices in \( T \) have degree at most \( k - 2 \). Take \( k = p - m - t \) and obtain that there exist \( n - p + m + t + 1 \) vertices of degree at most \( p - m - t - 2 \). From the other side, there exists a Grundy coloring of \( T \) using \( p \) colors. Then for each \( i \), at least \( i \) vertices have degree at least \( p - i \). Equivalently, at most \( n - i \) vertices of degree at most \( p - i - 1 \) exist in the graph. Combining these two bounds for \( i = m + t + 1 \), we obtain \( 2m + 2t + 2 \leq p \), a contradiction. \( \square \)

In the following, we denote the path on \( k \) vertices by \( P_k \). For any fixed graph \( H \), by \( 

\text{Forb}(H) \) we mean the family of all graphs \( G \) which does not contain \( H \) as induced subgraph. \( \text{Forb}(H_1, H_2) \) is defined similarly.

**Proposition 3.** \( \text{Forb}(P_k) \) is \((\Gamma, b)\)-bounded if and only if \( k \leq 5 \).

**Proof.** Define a bipartite graph \( B_t \), \( t \geq 2 \) as follows. Take a complete bipartite graph \( K_{t,t} \) and remove the edges of a matching of size \( t - 1 \) from the graph and call it \( B_t \). It’s easily seen that \( \Gamma(B_t) = t + 1 \). It can also be shown that \( b(B_t) = 2 \). Note that \( B_t \) contains \( P_5 \) as induced subgraph but not \( P_k \) for each \( k \geq 6 \) and hence is \( P_k \)-free for each \( k \geq 6 \). Therefore, the family of \( P_k \)-free graphs is not \((\Gamma, b)\)-bounded for \( k \geq 6 \).

Assume now that \( G \) is any \( P_5 \)-free graph. A result of Kierstead et al. \( \square \) asserts that the family of \( P_5 \)-free graphs is \((\chi_F, \omega)\)-bounded. It follows that the very family is \((\Gamma, b)\)-bounded. \( \square \)
We say a graph $G$ is $b$-monotone if for each induced subgraph $H$ of $G$ we have $b(H) \leq b(G)$. The family of non $b$-monotone graphs is not $(\Gamma, b)$-bounded. For this purpose it’s enough to consider the graphs $B_t$, $t \geq 2$, introduced in the proof of Proposition 3. Recall that $\Gamma(B_t) = t + 1$ but $b(B_t) = 2$, for each $t \geq 2$. Also $B_t$ is not $b$-monotone for each $t \geq 4$, because by removing the two vertices of degree $t$ in $B_t$ we obtain a subgraph with $b$-chromatic number $t - 1$. We make the following conjecture.

**Conjecture.** There exists a function $f(x)$ such that if $G$ is any $b$-monotone graph then $\Gamma(G) \leq f(b(G))$.

The next proposition proves that the conjecture is valid for all $K_{t,t}$-free graphs, for any fixed integer $t$. We need to define a tree $R_k$ of radius two. Take a vertex $v$ of degree $k - 1$ as the root of $R_k$. Let $v_1, \ldots, v_{k-1}$ be the children of $v$. For each $i$, attach $k - 2$ vertices of degree one to $v_i$. These vertices are all distinct so that $R_k$ contains $(k - 1)(k - 2)$ leaves. It is easily seen that $R_k$ admits a $b$-coloring using $k$ colors, where $v_i$ is color-dominating vertex of color $i$.

**Proposition 4.** Let $t \geq 2$ be any fixed integer and $\{G_n\}_{n \geq 1}$ be any sequence of $K_{t,t}$-free $b$-monotone graphs. Then $\Gamma(G_n) \to \infty$ implies $b(G_n) \to \infty$.

**Proof.** Assume on the contrary that $\{G_n\}_{n \geq 1}$ is a sequence of $K_{t,t}$-free $b$-monotone graphs with $\Gamma(G_n) \to \infty$ but for some integer $p$ and any $n$, $b(G_n) \leq p$. Since $G_n$ is $b$-monotone then $R_{p+1}$ is not an induced subgraph of $G_n$. Hence $G_n$ is $(K_{t,t}, R_{p+1})$-free for each $n$. A result of Kierstead and Penrice [6] asserts that $\{G_n\}_{n \geq 1}$ is $(\chi_{FF}, \omega)$-bounded. Hence for some function $f(x)$, $\Gamma(G_n) \leq f(\omega(G_n)) \leq f(b(G_n)) \leq f(p)$, a contradiction. □

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