Curvature Estimates for the Ricci Flow I

Rugang Ye
Department of Mathematics
University of California, Santa Barbara

1 Introduction

In this paper we present several curvature estimates for solutions of the Ricci flow
\[ \frac{\partial g}{\partial t} = -2Ric. \] (1.1)

These estimates depend on the smallness of certain local $L^\frac{n}{2}$ integrals of the norm of the Riemann curvature tensor $|Rm|$, where $n$ denotes the dimension of the manifold. A key common property of these integrals is scaling invariance, thanks to the critical exponent $\frac{n}{2}$. Because of this property, they are very natural and contain particularly rich geometric information. (Note that in dimension 4, the norm square of the Riemann curvature tensor is closely related to the Gauss-Bonnet-Chern integrand.)

To formulate our results, we need some terminologies. Consider a Riemannian manifold $(M, g)$ ($g$ denotes the metric) possibly with boundary. For convenience, we define the distance between two points of $M$ to be $\infty$, if they belong to two different connected components. Consider a point $x \in M$. If $x$ is in the interior of $M$, we define the distance $d(x, \partial M) = d_g(x, \partial M)$ to be $\sup\{r > 0 : B(x, r) \text{ is compact and contained in the interior of } M\}$, where $B(x, r)$ denotes the closed geodesic ball of center $x$ and radius $r$. If $M$ has a boundary and $x \in \partial M$, then $d(x, \partial M)$ is the ordinary distance from $x$ to $\partial M$ and equals zero. (For example, $d(x, \partial M) = \infty$ if $M$ is closed.)

Notations Let $g = g(t)$ be a family of metrics on $M$. Then $d(x, y, t)$ denotes the distance between $x, y \in M$ with respect to the metric $g(t)$, and $B(x, r, t)$ denotes the closed geodesic ball of center $x \in M$ and radius $r$ with respect to the metric $g(t)$. The volume of $B(x, r, t)$ with respect to $g(t)$ will often be denoted by $V(x, r, t)$. We shall often use $dq$ to denote $dvol_{g(t)}$. These notations naturally extend when $g$ and (or) $t$ are replaced by other notations.
We set \( \alpha_n = \frac{1}{40(n-1)} \), \( \epsilon_0 = \frac{1}{42} \) and \( \epsilon_1 = \frac{\epsilon_0}{\sqrt{1+2\alpha_n \epsilon_0^2}} \). (These constants are not meant to be optimal. One can improve them by carefully examining the proofs.)

Our first result involves the concept of \( \kappa \)-noncollapsedness due to Perelman [P].

**Definition** Let \( g \) be a Riemannian metric on a manifold \( M \) of dimension \( n \). Let \( \kappa \) and \( \rho \) be positive numbers. We say that \( g \) is \( \kappa \)-noncollapsed on the scale \( \rho \), if \( g \) satisfies \( \text{vol}(B(x,r)) \geq \kappa r^n \) for all \( x \in M \) and \( r > 0 \) with the properties \( r < \rho \) and \( \sup\{|Rm|(x) : x \in B(x,r)\} \leq r^{-2} \). We say that a family of Riemannian metrics \( g = g(t) \) is \( \kappa \)-noncollapsed on the scale \( \rho \), if \( g(t) \) is \( \kappa \)-noncollapsed on the scale \( \rho \) for each \( t \) (in the given domain).

By [Theorem 4.1, P], a smooth solution \( g \) of the Ricci flow on \( M \times [0,T) \) for a closed manifold \( M \) and a finite \( T \) is \( \kappa \)-noncollapsed on the scale \( \sqrt{T} \), where \( \kappa \) depends on the initial metric and (an upper bound of) \( T \). For a variant of [Theorem 4.1, P] involving the scalar curvature, which implies [Theorem 4.1, P], see Theorem A.1 in Appendix A. By Theorem A.1, \( g \) is \( \kappa \)-noncollapsed on the scale \( \rho \) for an arbitrary positive number \( \rho \), where \( \kappa \) depends on the initial metric and (an upper bound of) \( T + \rho^2 \).

**Theorem A** For each positive number \( \kappa \) and each natural number \( n \geq 3 \) there are positive constants \( \delta_0 = \delta_0(\kappa, n) \) and \( C_0 = C_0(n, \kappa) \) depending only on \( \kappa \) and \( n \) with the following property. Let \( g = g(t) \) be a smooth solution of the Ricci flow on \( M \times [0,T) \) for a manifold \( M \) of dimension \( n \geq 3 \) and some (finite or infinite) \( T > 0 \), which is \( \kappa \)-noncollapsed on the scale \( \rho \) for some \( \kappa > 0 \) and \( \rho > 0 \). Consider \( x_0 \in M \) and \( 0 < r_0 \leq \rho \), which satisfy \( r_0 < d_{g(t)}(x_0, \partial M) \) for each \( t \in [0,T) \). Assume that

\[
\int_{B(x_0,r_0,t)} |Rm|(\cdot,t) dq(\cdot,t) \leq \delta_0
\]

for all \( t \in [0,T) \). Then we have

\[
|Rm|(x,t) \leq \alpha_n t^{-1} + (\epsilon_0 r_0)^{-2}
\]

whenever \( t \in (0,T) \) and \( d(x_0,x,t) < \epsilon_0 r_0 \), and

\[
|Rm|(x,t) \leq C_0 \max \left\{ \frac{1}{r_0^2}, \frac{1}{t} \right\} \sup_{0 \leq s \leq t} \left( \int_{B(x_0,\frac{1}{2} r(t),s)} |Rm|(\cdot,s) dq(\cdot,s) \right)^{\frac{2}{n}}
\]

whenever \( 0 < t < T \) and \( d(x_0,x,t) \leq \frac{1}{4} r(t) \), where \( r(t) = \epsilon_1 \min \{ r_0, \sqrt{t} \} \). (Obviously, it follows that if the assumptions hold on \([0,T]\), then the estimates (1.3) and (1.4) hold on \([0,T]\). This remark also applies to the results below.)
Note that the constant $\delta_0$ depends on $n$ decreasingly and depends on $\kappa$ increasingly, i.e. $\delta_0(n, \kappa)$ is a decreasing function of $n$ and an increasing function of $\kappa$. In contrast, the constant $C_0$ depends on $n$ increasingly and depends on $\kappa$ decreasingly. The dependences of the constants in Theorem B and Theorem C are of similar nature.

The following corollary is a consequence of Theorem A and Theorem A.1.

**Corollary A** Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ for a closed manifold $M$ of dimension $n \geq 3$ and some finite $T > 0$. Let $\rho > 0$ be a positive number. There is a positive constant $\delta_0 = \delta_0(n, T + \rho^2, g(0))$ depending on $n, T + \rho^2$ and $g(0)$ with the following property. Assume that (1.2) holds true for all $x_0 \in M$, all $t \in [0, T)$, and some $r_0$ satisfying $r_0 \leq \rho$ for all $t \in [0, T)$. Then (1.3) holds true for all $x \in M$ and $t \in [0, T)$. Consequently, $g(t)$ extends to a smooth solution of the Ricci flow over $[0, T')$ for some $T' > T$.

A corollary of Theorem A for the case $T = \infty$ analogous to Corollary B (under the assumption of $\kappa$-noncollapsedness) also holds true. We omit the statement. Our second result does not involve the condition of $\kappa$-noncollapsedness. Instead, smallness of $L^2$ integrals of $Rm$ over balls of varying center and radius measured against a volume ratio is assumed.

**Theorem B** For each natural number $n \geq 3$ there are positive constants $\delta_0 = \delta_0(n)$ and $C_0 = C_0(n)$ depending only on $n$ with the following property. Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ for a manifold $M$ of dimension $n \geq 3$ and some (finite or infinite) $T > 0$. Consider $x_0 \in M$ and $r_0 > 0$, which satisfy $r_0 \leq \text{diam}_{g(t)}(M, x_0)$ and $r_0 < d_{g(t)}(x_0, \partial M)$ for each $t \in [0, T)$. Assume that

\[
\int_{B(x, r, t)} |Rm|^{\frac{2}{n}}(\cdot, t) dq(\cdot, t) \leq \delta_0 \frac{V(x, r, t)}{r^n} \tag{1.5}
\]

whenever $t \in [0, T)$, $0 < r \leq \frac{r_0}{T}$ and $x \in B(x_0, \frac{r_0}{T}, t)$. Then we have

\[
|Rm|(x, t) \leq \alpha_n t^{-1} + (\epsilon_0 r_0)^{-2} \tag{1.6}
\]

whenever $t \in (0, T)$ and $d(x_0, x, t) < \epsilon_0 r_0$, and

\[
|Rm|(x, t) \leq C_0 \sup_{0 \leq s \leq t} \left( \frac{\int_{B(x, \frac{1}{2}r(t), s)} |Rm|^{\frac{2}{n}}(\cdot, s) dq(\cdot, s)}{V(x, \frac{1}{2}r(t), s)} \right)^{\frac{n}{2}} \tag{1.7}
\]

whenever $0 < t < T$ and $d(x_0, x, t) \leq \frac{1}{2} \epsilon_1 \min\{r_0, \sqrt{t}\}$, where $r(t) = \epsilon_1 \min\{r_0, \sqrt{t}\}$.

**Corollary B** Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ for a manifold $M$ of dimension $n \geq 3$ and some (finite or infinite) $T > 0$, such that
for all \( t \in [0, T) \). Assume that \((1.3)\) holds true for all \( x_0 \in M \), all \( t \in [0, T) \), and all \( 0 < r \leq r_0 \) for some positive number \( r_0 \) satisfying \( r_0 \leq \text{diam}_{g(t)}(M) \) for all \( t \in [0, T) \). Then \((1.6)\) and \((1.7)\) hold for all \( x \in M \) and \( t \in [0, T) \). Consequently, \( g(t) \) extends to a smooth solution of the Ricci flow over \([0, T']\) for some \( T' > T \) if \( T \) is finite. If \( T = \infty \), then \( g(t) \) subconverges smoothly as \( t \to T \).

Our third result does not involve the condition of \( \kappa \)-noncollapsedness, and employs only a fixed center and a fixed radius for \( L^2 \) integrals of the norm of the Riemann curvature tensor. But a lower bound for the Ricci curvature is assumed.

**Theorem C** For each natural number \( n \geq 3 \) there are positive constants \( \delta_0 = \delta_0(n) \) and \( C_0 = C_0(n) \) depending only on \( n \) with the following property. Let \( g = g(t) \) be a smooth solution of the Ricci flow on \( M \times [0, T) \) for a manifold of dimension \( n \geq 3 \) and some (finite or infinite) \( T > 0 \). Consider \( x_0 \in M \) and \( r > 0 \), which satisfy \( r_0 \leq \text{diam}_{g(t)}(M) \) and \( r_0 < d_{g(t)}(x_0, \partial M) \) for each \( t \in [0, T) \). Assume that

\[
\text{Ric}(x, t) \geq -\frac{n-1}{r^2} g(x, t)
\]  \( (1.8) \)

whenever \( t \in [0, T) \) and \( d(x_0, x, t) \leq r_0 \) \( (g(x, t) = g(t)(x) \) and \( \text{Ric}(x, t) \) is the Ricci tensor of \( g(t) \) at \( x \), and that

\[
\int_{B(x_0, r_0, t)} |\text{Rm}|^2(\cdot, t) dq(\cdot, t) \leq \delta_0 \frac{V(x_0, r_0, t)}{r_0^3}
\]  \( (1.9) \)

for all \( t \in [0, T] \). Then we have

\[
|Rm|(x, t) \leq \alpha_n t^{-1} + \left( \frac{1}{2} \epsilon_0 r_0 \right)^{-2}
\]  \( (1.10) \)

whenever \( t \in (0, T) \) and \( d(x_0, x, t) < \frac{1}{2} \epsilon_0 r_0 \), and

\[
|Rm|(x, t) \leq C_0 \sup_{0 \leq s \leq t} \left( \frac{\int_{B(x, \frac{1}{2} r(t), s)} |\text{Rm}|^2(\cdot, s) dq(\cdot, s)}{V(x, \frac{1}{2} r(t), s)} \right)^{\frac{n}{2}}
\]  \( (1.11) \)

whenever \( 0 < t < T \) and \( d(x_0, x, t) \leq \frac{1}{2} r(t) \), where \( r(t) = \epsilon_1 \min\{ \frac{1}{2} r_0, \sqrt{T} \} \). We also have

\[
|Rm|(x, t) \leq C_0 \left( \frac{r_0}{r(t)} \right)^2 \sup_{0 \leq s \leq t} \left( \frac{\int_{B(x_0, r_0, s)} |\text{Rm}|^2(\cdot, s) dq(\cdot, s)}{V(x_0, r_0, s)} \right)^{\frac{n}{2}}
\]  \( (1.12) \)

whenever \( 0 < t < T \) and \( d(x_0, x, t) \leq \frac{1}{2} r(t) \).
An elliptic analogue of (1.12) for Einstein metrics can be found in [An]. Obviously, Theorem C and its corresponding version for the modified Ricci flow (see discussions below) can also be applied to Einstein metrics.

**Corollary C** Let \( g = g(t) \) be a smooth solution of the Ricci flow on \( M \times [0, T) \) for a manifold \( M \) of dimension \( n \geq 3 \) and some (finite or infinite) \( T > 0 \), such that \( g(t) \) is complete for each \( t \in [0, T) \). Assume that (1.8) and (1.9) hold true for all \( x_0 \in M \) and some positive number \( r_0 \) satisfying \( r_0 \leq \text{rad}_{g(t)}(M) \) for all \( t \in [0, T) \). Then (1.10) and (1.11) holds for all \( x \in M \) and \( t \in [0, T) \). Consequently, \( g(t) \) extends to a smooth solution of the Ricci flow over \([0, T']\) for some \( T' > T \), if \( T \) is finite. If \( T = \infty \), then \( g(t) \) subconverges smoothly as \( t \to T \).

Note that the condition \( r_0 \leq \text{diam}_{g(t)}(M) \) appears in Theorem B and Theorem C, but not in Theorem A.

The above results extend to the modified Ricci flow

\[
\frac{\partial g}{\partial t} = -2\text{Ric} + \lambda(g, t)g
\]

with a scalar function \( \lambda(g, t) \) independent of \( x \in M \). The volume-normalized Ricci flow

\[
\frac{\partial g}{\partial t} = -2\text{Ric} + \frac{2}{n}\hat{R}g
\]

on a closed manifold, with \( \hat{R} \) denoting the average scalar curvature, is an example of the modified Ricci flow. We present two extensions of the curvature estimates to the modified Ricci flow.

**Extension I** Theorem A, Theorem B and Theorem C hold true for the modified Ricci flow (1.13), with the modification that the constants \( \delta_0 \) and \( C_0 \) in each theorem depend in addition on \( r_0^2|\min\{\inf_{[0,T]} \lambda, 0\}| \) which is assumed to be finite. (In other words, \( \delta_0 \) and \( C_0 \) depend in addition on a nonpositive lower bound of \( r_0^2 \lambda \).

**Extension II** Theorem A, Theorem B and Theorem C in the case \( T < \infty \) hold true for the modified Ricci flow (1.13), with the modification that the constants \( \delta_0 \) and \( C_0 \) in each theorem depend in addition on \( |\min\{\inf_{0 \leq t_1 < t_2 < T} \int_{t_1}^{t_2} \lambda, 0\}| \) which is assumed to be finite. (In other words, \( \delta_0 \) and \( C_0 \) depend in addition on a nonpositive lower bound of \( \int_{t_1}^{t_2} \lambda \).

In both extensions, the dependence of \( \delta_0 \) is decreasing, and the dependence of \( C_0 \) is increasing. Extension I can be proved by directly adapting the proofs of Theorem A, Theorem B and Theorem C. Extension II can be proved by converting the modified
Ricci flow into the Ricci flow, applying Theorem A, Theorem B and Theorem C, and then converting the obtained estimates back to the modified Ricci flow.

Similar results also hold for many other evolution equations. This will be presented elsewhere.

The results in this paper were obtained some time ago.

Analogous results involving other types of $L^p$ integrals of $|Rm|$, including the case $p < \frac{n}{2}$ and space-time integrals, will be presented in sequels of this paper. In particular, the case of space-time integrals is presented in [Ye3].

## 2 A Linear Parabolic Estimate

In this section we present a linear parabolic estimate based on Moser’s iteration, which will be needed for establishing our curvature estimates. First we fix some notations. Consider a Riemannian manifold $(M,g)$ of dimension $n$, possibly with boundary. Let $\Omega$ be a domain in $M$. The Sobolev constant $C_{S,g}(\Omega)$ is defined to be the smallest number $C_{S,g}(\Omega)$ such that

$$\|f\|_{n-1}^n \leq C_{S,g}(\Omega)\|\nabla f\|_1$$

for all Lipschitz functions $f$ on $\Omega$ with compact support contained in the interior of $\Omega$ (i.e. $\Omega - \partial M$), where $\| \cdot \|_p$ means the $L^p$-norm. More precisely,

$$C_{S,g}(\Omega) = \sup\{\|f\|_{n-1}^n : f \in C^1_c(\Omega), \|\nabla f\|_1 = 1\},$$

where $C^1_c(\Omega)$ is the space of $C^1$ functions on $\Omega$ with compact support contained in the interior of $\Omega$. As is well-known, $C_{S,g}(\Omega)$ equals the isoperimetric constant $C_{I,g}(\Omega)$, which is defined to be $\sup\{\frac{\text{vol}(\Omega')}{\text{vol}(\partial \Omega')} : \Omega' \subset \Omega$ is a $C^1$ domain in $\Omega$ with compact closure.$\}$

The $L^2$-Sobolev constant $C_{S,2,g}(\Omega)$ is defined to be the smallest number $C_{S,2,g}(\Omega)$ such that

$$\|f\|_{2n}^{2n} \leq C_{S,2,g}(\Omega)\|\nabla f\|_2.$$ 

There holds

$$C_{S,2,g}(\Omega) \leq \frac{2(n-1)}{n-2} C_{S,g}(\Omega).$$

Indeed, applying (2.1) to $|f|^{2(n-1)\frac{n-1}{n}}$ we deduce

$$\left( \int_\Omega |f|^{\frac{2n}{n-2}} dvol_g \right)^{\frac{n-1}{n}} \leq C_{S,g}(\Omega) \int |\nabla |f|^{\frac{2(n-1)}{n-2}} | dvol_g$$

$$\leq C_{S,g}(\Omega) \frac{2(n-1)}{n-2} \|\nabla f\|_2 \left( \int_\Omega |f|^{\frac{2n}{n-2}} dvol_g \right)^{\frac{1}{2}}.$$
Hence the claimed inequality follows.

It is easy to see that \( C_{S,g}, C_{S,2,g} \) and \( C_{I,g} \) are nondecreasing, i.e. for example \( C_{S,2,g}(\Omega_1) \leq C_{S,2,g}(\Omega_2) \) if \( \Omega_1 \subset \Omega_2 \).

We set \( C_{S,g}(\Omega) = C_{S,g}(\Omega), \ C_{S,2,g}(\Omega) = C_{S,2,g}(\Omega) \) and \( C_{I,g}(\Omega) = C_{I,g}(\Omega) \).

The following result is taken from [Ye2]. We include the proof for the convenience of the reader, and for the reason of verifying the explicit dependence on the Sobolev constant, which is important for the curvature estimates in this paper.

**Theorem 2.1** Let \( M \) be a smooth manifold of dimension \( n \) and \( g = g(t) \) a smooth family of Riemannian metrics on \( M \) for \( t \in [0,T] \). Let \( f \) be a nonnegative Lipschitz continuous function on \( M \times [0,T] \) satisfying

\[
\frac{\partial f}{\partial t} \leq \Delta f + af
\]

on \( M \times [0,T] \) in the weak sense, where \( a \) is a nonnegative constant and \( \Delta = \Delta_{g(t)} \). Let \( x_0 \) be an interior point of \( M \). Then we have for each \( p_0 > 1 \) and \( 0 < R < d_{g(0)}(x_0, \partial M) \)

\[
|f(x,t)| \leq (1 + \frac{2}{n} p_0 C_{S,2}^{p_0} (ap_0 + \gamma + n/2 (1 + n/2)^2 \cdot \frac{1}{t} + (n + 2)^2 e^{-\lambda^* T} R^2)}^{\frac{n+2}{2p_0}} \cdot \left( \int_0^T \int_{B(x_0,R,0)} f^{p_0}(\cdot,t) dvol_{g(t)} dt \right)^{\frac{1}{p_0}},
\]

whenever \( 0 < t \leq T \) and \( d_{g(0)}(x_0, x) \leq \frac{R}{2} \), where \( \sigma_n \equiv \sum_0^\infty \frac{2^k}{(1+2^k)^k} \), \( \gamma \) denotes the maximum value of the trace of \( \frac{\partial g}{\partial t} \) on \( B(x_0,R,0) \times [0,T] \), \( \lambda_* \) denotes the minimum eigenvalue of \( \frac{\partial g}{\partial t} \) on \( B(x_0,R,0) \times [0,T] \), and \( C_{S,2} = \max_{0 \leq t \leq T} C_{S,g(t),2}(B(x_0,R,0)) \).

The same estimate holds if we replace \( B(x_0, R, 0) \) by \( B(x_0, R, T) \), and \(-\lambda_* \) by \( \lambda^* \), which denotes the maximum eigenvalue of \( \frac{\partial g}{\partial t} \) on \( B(x_0, R, T) \times [0,T] \).

**Proof.** We handle the case of \( B(x_0, R, 0) \), while the other case is similar. Let \( \eta \) be a non-negative Lipschitz function on \( M \) whose support is contained in \( B(x_0, R, 0) \). The partial differential inequality (2.3) implies for \( p \geq 2 \)

\[
\frac{1}{p} \frac{\partial}{\partial t} \int f^p \eta^2 dvol_{g(t)} \leq - \int \nabla (\eta^2 f^{p-1}) \cdot \nabla f dvol_{g(t)} + \int b f^p \eta^2 dvol_{g(t)} + \frac{1}{p} \int f^p \eta^2 \frac{\partial}{\partial t} dvol_{g(t)}.
\]

We’ll omit the notation \( dvol_{g(t)} \) below. We have

\[
- \int \nabla (\eta^2 f^{p-1}) \cdot \nabla f = - \frac{4(p-1)}{p^2} \int |\nabla (\eta f^{p/2})|^2 + \frac{4}{p^2} \int |\nabla \eta|^2 f^p
\]

\[
+ \frac{4(p-2)}{p^2} \int \nabla (\eta f^{p/2}) f^{p/2} \nabla \eta
\]

\[
\leq - \frac{2}{p} \int |\nabla (\eta f^{p/2})|^2 + \frac{2}{p} \int |\nabla \eta|^2 f^p,
\]

7
where $\nabla = \nabla_{g(t)}$. Therefore

$$\frac{\partial}{\partial t} \int f^p \eta^2 + 2 \int |\nabla (\eta f^{p/2})|^2 \leq 2 \int |\nabla \eta|^2 f^p + (pa + \frac{\gamma}{2}) \int f^p \eta^2. \quad (2.5)$$

Next we define for $0 < \tau < \tau' < T$

$$\psi(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq \tau, \\
\frac{(t - \tau)}{(\tau' - \tau)} & \text{if } \tau \leq t \leq \tau', \\
1 & \text{if } \tau' \leq t \leq T.
\end{cases}$$

Multiplying (2.5) by $\psi$, we obtain

$$\frac{\partial}{\partial t} \left( \psi \int f^p \eta^2 \right) + 2\psi \int |\nabla (\eta f^{p/2})|^2 \leq 2\psi \int |\nabla \eta|^2 f^p + \left( (pa + \frac{\gamma}{2}) \psi + \psi' \right) \int f^p \eta^2.$$

Integrating this with respect to $t$ we get

$$\int_t^{\tau'} f^p \eta^2 + 2 \int_\tau^{\tau'} \int |\nabla (\eta f^{p/2})|^2 \leq 2 \int_\tau^{\tau'} \int |\nabla \eta|^2 f^p + \left( (pa + \frac{\gamma}{2} + \frac{1}{\tau - \tau'}) \int \tau^{\tau'} \int f^p \eta^2 \right)$$

for $\tau' \leq t \leq T$. Applying this estimate and the Sobolev inequality we deduce

$$\int_\tau^{\tau'} \int f^p (1 + \frac{2}{n}) \eta^{\frac{n+2}{n}} \leq \left( \int \int f^p \eta^2 \right)^{2/n} \left( \int \int f^{p/2} \eta^{2/n} \right)$$

$$\leq C^2_{S,2} \left( \sup_{\tau' \leq t \leq T} \int f^p \eta^2 \right)^{2/n} \int_\tau^{\tau'} \int |\nabla (\eta f^{p/2})|^2$$

$$\leq C^2_{S,2} \left[ 2 \int_\tau^{\tau'} \int |\nabla \eta|^2 f^p + \left( (pa + \frac{\gamma}{2} + \frac{1}{\tau - \tau'}) \int \tau^{\tau'} \int f^p \eta^2 \right) \right]^{1+\frac{2}{n}}. \quad (2.6)$$

We put

$$H(p, \tau, R) = \int_\tau^T \int_{B(x_0, R, 0)} f^p$$

for $0 < \tau < T$ and $0 < R < d_g(x_0, \partial M)$. Given $0 < R' < R < d_g(x_0, \partial M)$, we define $\eta(x) = 1$ for $d(x_0, x, 0) \leq R'$, $\eta(x) = 1 - \frac{1}{R - R'} \left( d(x_0, x, 0) - R' \right)$ for $R' \leq d(x_0, x, 0) \leq R$, and $\eta(x) = 0$ for $d(x_0, x, 0) \geq R$. Noticing $|\nabla \eta| \leq \frac{1}{R - R'} e^{\frac{1}{2} \lambda t}$ we derive from (2.6)

$$H \left( p \left( 1 + \frac{2}{n} \right), \tau', R' \right) \leq C^2_{S} \left[ pa + \frac{\gamma}{2} + \frac{1}{\tau' - \tau} + \frac{2e^{-\lambda t}}{(R - R')^2} \right]^{1+\frac{2}{n}} H(p, \tau, R)^{1+\frac{2}{n}} \quad (2.7)$$
Now we fix $0 < R < d_{g(0)}(x_0, \partial M)$ and set $\mu = 1 + \frac{2}{n}$, $p_k = p_0 \mu^k$, $\tau_k = (1 - \frac{1}{\mu^2})t$ and $R_k = \frac{R}{2}(1 + \frac{1}{\mu^k})$ with $R = \frac{1}{2}dist_{g(0)}(x, \partial N)$. Then it follows from (2.7) that

$$H(p_k+1, \tau_k+1, R_{k+1})^{\frac{1}{p_k+1}} \leq C_S^{\frac{p_k}{2}} \left[ ap_k + \frac{\gamma}{2} + \frac{\mu^2}{\mu - 1} \cdot \frac{1}{t} + \frac{2e^{-\lambda T} \mu^2}{R^2(\mu - 1)^2} \right]^{\frac{1}{p_k}} \mu^{\frac{1}{p_k}} H(p_k, \tau_k, R_k)^{\frac{1}{p_k}} \leq C_S^{\frac{p_k}{2}} \left[ ap_0 + \frac{\gamma}{2} + \frac{\mu^2}{\mu - 1} \cdot \frac{1}{t} + \frac{2e^{-\lambda T} \mu^2}{R^2(\mu - 1)^2} \right]^{\frac{1}{p_k}} \mu^{\frac{2k}{p_k}} H(p_0, \tau_0, R_0)^{\frac{1}{p_0}}.$$

Hence

$$H(p_{m+1}, \tau_{m+1}, R_{m+1})^{\frac{1}{p_{m+1}}} \leq C_S^{\frac{p_m}{2}} \mu^{\frac{m}{p_m}} \sum_{k=0}^{m} \frac{1}{p_k} H(p_0, \tau_0, R_0)^{\frac{1}{p_0}}.$$

Letting $m \to \infty$ we arrive at (2.4). 

\section{Proof of Theorem A}

Proof of the estimate \((1.3)\)

By rescaling, we can assume $r_0 = 1$. Assume that the estimate \((1.3)\) does not hold. Then we can find for each $1 > \epsilon > 0$ a Ricci flow solution $g = g(t)$ on $M \times [0, T]$ for some $M$ and $T > 0$ with the properties as postulated in the statement of the theorem, such that $|Rm|(x, t) > \alpha_n t^{-1} + \epsilon^2$ for some $(x, t) \in M \times [0, T]$ satisfying $d(x_0, x, t) < \epsilon$.

We denote by $M_{\alpha_n}$ the set of pairs $(x, t)$ such that $|Rm|(x, t) > \alpha_n t^{-1}$. Consider an arbitrary positive number $A > 1$ such that $(2A + 1)\epsilon \leq \frac{1}{2}$. Following [Proof of Theorem 10.1, P], we choose $(\bar{x}, \bar{t}) \in M_{\alpha_n}$ with $0 < \bar{t} \leq \epsilon^2$, $d(x_0, \bar{x}, \bar{t}) < (2A + 1)\epsilon$, such that $|Rm|(\bar{x}, \bar{t}) > \alpha_n \bar{t}^{-1} + \epsilon^2$ and

$$|Rm|(x, t) \leq 4|Rm|(\bar{x}, \bar{t})$$

whenever

$$(x, t) \in M_{\alpha_n}, 0 < t \leq \bar{t}, d(x_0, x, t) \leq d(x_0, \bar{x}, \bar{t}) + A|Rm|(\bar{x}, \bar{t})^{-\frac{1}{2}}. \quad (3.2)$$

For the convenience of the reader, we reproduce here the argument in [Proof of Theorem 10.1, P] for the existence of $(\bar{x}, \bar{t})$. Let $(x_1, t_1)$ be an arbitrary point in $M \times [0, T)$ such that $0 < t_1 < \epsilon^2$, $d(x_0, x_1, t_1) < \epsilon$ and $|Rm|(x_1, t_1) > \alpha_n t_1^{-1} + \epsilon^2$. Now
if \((x_k, t_k)\) is already contructed, but does not have all the desired properties of \((\bar{x}, \bar{t})\), then we can find \((x_{k+1}, t_{k+1})\) satisfying

\[
(x_{k+1}, t_{k+1}) \in M_{\alpha_n}, \ 0 < t_{k+1} \leq t_k, d(x_0, x_{k+1}, t_{k+1}) \leq d(x_0, x_k, t_k) + A|Rm|(x_k, t_k)^{-\frac{1}{2}},
\]

such that \(|Rm|(x_{k+1}, t_{k+1}) > 4|Rm|(x_k, t_k)\). It follows that \(|Rm|(x_k, t_k) \geq 4^{k-1}|Rm|(x_1, t_1) \geq 4^{k-1} \epsilon^{-2}\). By (3.3), we then also have \(d(x_0, x_k, t_k) \leq (2A + 1) \epsilon\). Since \(g\) is smooth on \(M \times [0, T]\), the former estimate implies that the sequence \((x_k, t_k)\) must be finite. We can then take the last term in the sequence to be \((\bar{x}, \bar{t})\).

We set \(Q = |Rm|(\bar{x}, \bar{t})\). Note that \(Q > 1\) because \(\epsilon < 1\).

**Claim 1** If

\[
\bar{t} - \frac{1}{2} \alpha_n Q^{-1} \leq t \leq \bar{t}, d(\bar{x}, x, \bar{t}) \leq \frac{1}{10} AQ^{-\frac{1}{2}},
\]

then

\[
d(x_0, x, t) \leq d(x_0, \bar{x}, \bar{t}) + \frac{1}{2} AQ^{-\frac{1}{2}}.
\]

Note that (3.5) implies

\[
d(x_0, x, t) \leq (2A + 1) \epsilon + \frac{1}{2} AQ^{-\frac{1}{2}} \leq (\frac{5}{2} A + 1) \epsilon
\]

for \((x, t)\) satisfying (3.4).

**Proof of Claim 1**

Since \((\bar{x}, \bar{t}) \in M_{\alpha_n}\), we have \(Q \geq \alpha_n \bar{t}^{-1}\), so

\[
\bar{t} - \frac{1}{2} \alpha_n Q^{-1} \geq \frac{1}{2} \bar{t}.
\]

Consider \(x = \hat{x}\) and \(t = \hat{t}\) satisfying (3.4). By the triangular inequality, we have

\[
d(x_0, \hat{x}, \hat{t}) \leq d(x_0, \bar{x}, \bar{t}) + \frac{1}{10} AQ^{-\frac{1}{2}}.
\]

We estimate \(d(x_0, \hat{x}, \hat{t})\). For this purpose, consider the set \(I\) of \(t' \in [\hat{t}, \bar{t}]\) such that

\[
d(x_0, \hat{x}, t) \leq d(x_0, \bar{x}, \bar{t}) + \frac{1}{2} AQ^{-\frac{1}{2}}
\]
for all $t \in [\bar{t}, \bar{t}]$. Obviously, $I$ is open in $[\tilde{t}, \tilde{t}]$. Consider $t^* \in I$. For each $t \in [\bar{t}, \bar{t}]$, we apply [Lemma 8.3(b), P] to $x, \dot{x}$. We set $R = \frac{1}{2}AQ^{-\frac{1}{2}}$. For $x \in B(x_0, R, t)$ we have $|Rm|(x, t) \leq 4Q$ if $(x, t) \in M_{\alpha_n}$. If $(x, t) \notin M_{\alpha_n}$, we have by (3.7)

$$|Rm|(x, t) \leq \alpha_n t^{-1} \leq 2\alpha \tilde{t}^{-1} \leq 2Q.$$  \hfill (3.9)

For $x \in B(\dot{x}, R, t)$, we have $d(x_0, x, t) \leq d(x_0, \dot{x}, t) + d(\dot{x}, t, x, t) \leq d(x_0, \bar{x}, \bar{t}) + AQ^{-\frac{1}{2}}$. Hence $|Rm|(x, t) \leq 4Q$, if $(x, t) \in M_{\alpha_n}$. If $(x, t) \not\in M_{\alpha_n}$, we again obtain (3.9). By [Lemma 8.3(b), P], we have

$$\frac{d}{dt}d(x_0, \dot{x}, t) \geq -2(n - 1)\left(\frac{2}{3} \cdot 4Q \cdot \frac{1}{2}AQ^{-\frac{1}{2}} + 2A^{-1}Q^{\frac{3}{2}}\right) \geq -4(n - 1)(A + \frac{1}{A})Q^{\frac{3}{2}}.$$

Hence

$$d(x_0, \dot{x}, t^*) \leq d(x_0, \dot{x}, \bar{t}) + \frac{1}{2}\alpha_n Q^{-1} \cdot 4(n - 1)(A + \frac{1}{A})Q^{\frac{3}{2}} = d(x_0, \dot{x}, \bar{t}) + 2(n - 1)\alpha(1 + \frac{1}{A^2})AQ^{-\frac{1}{2}} \leq d(x_0, \dot{x}, \bar{t}) + \frac{1}{3}AQ^{-\frac{1}{2}}.$$

By the continuity of the distance function, the inequality (3.8) holds true in an open neighborhood of $t^*$ in $[\tilde{t}, \tilde{t}]$. It follows that $I$ is open in $[\tilde{t}, \tilde{t}]$. Hence we conclude that $I = [\tilde{t}, \tilde{t}]$. Consequently, we have $d(x_0, \dot{x}, \bar{t}) \leq d(x_0, \bar{x}, \bar{t}) + \frac{1}{2}AQ^{-\frac{1}{2}}$.

**Claim 2** If $(x, t)$ satisfies (3.4), then the estimate (3.1) holds.

Indeed, consider $(x, t)$ satisfying (3.4). If $(x, t) \in M_{\alpha_n}$, then (3.5) implies that the estimate (3.1) holds. If $(x, t) \not\in M_{\alpha_n}$, then we have $|Rm|(x, t) \leq 2Q$ as in (3.9). So (3.1) also holds.

Now we take $\epsilon = \frac{1}{42}$ and $A = 10$. Then $\frac{1}{10}A < 1$ and $(\frac{3}{2}A + 1)\epsilon = 1$. So (3.6) implies

$$B(\bar{x}, Q^{-\frac{1}{2}}, \tilde{t}) \subset B(x_0, 1, t)$$  \hfill (3.10)

for $t \in [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$, and hence

$$\int_{B(\bar{x}, Q^{-\frac{1}{2}}, \tilde{t})} |Rm|^{\frac{2}{3}}(\cdot, t) dq(\cdot, t) \leq \delta_0$$  \hfill (3.11)

for $t \in [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$. Moreover, Claim 2 implies that the estimate (3.1) holds on $B(\bar{x}, Q^{-\frac{1}{2}}, \tilde{t}) \times [\bar{t} - \frac{1}{2}\alpha_n Q^{-1}, \bar{t}]$. We shift $\bar{t}$ to the time origin and rescale $g$ by the factor
Q to obtain a Ricci flow solution $\bar{g}(t) = Qg(\bar{t} + Q^{-1}t)$ on $M \times [-\frac{1}{2}\alpha_n, 0]$. Now we’ll deal with $\bar{g}$ and all the quantities will be associated with $\bar{g}$. We have for $\bar{g}$

$$|Rm|(\bar{x}, 0) = 1,$$

and

$$|Rm|(x, t) \leq 4$$

whenever

$$-\frac{1}{2}\alpha_n \leq t \leq 0, d(\bar{x}, x, 0) \leq 1.$$ 

Moreover there holds

$$\int_{B(\bar{x}, 1, 0)} |Rm|^\frac{n}{2}(\cdot, t)dq(\cdot, t) \leq \delta_0$$

for $t \in [-\frac{1}{2}\alpha_n, 0]$. (Note that the geodesic balls are associated with $\bar{g}$. ) By the $\kappa$-noncollapsedness assumption, $\bar{g}$ is $\kappa$-noncollapsed on the scale $Q^\frac{n}{2}\rho \geq \rho \geq r_0 = 1$. Finally, one readily verifies that

$$d_{\bar{g}(0)}(\bar{x}, \partial M) > \frac{1}{2}.$$ 

Applying the $\kappa$-noncollapsedness property of $\bar{g}$ and (3.13) we deduce

$$V(\bar{x}, r, 0) \geq \kappa r^n$$

for all $0 < r \leq \frac{1}{2}$. By (3.13), (3.16), (3.15) and (2.2) we can apply Theorem B.1 in Appendix B to infer

$$C_{S, 2, \bar{g}(0)}(B(\bar{x}, \rho(n, \kappa), 0)) \leq C_1(n) \equiv \frac{2(n - 1)}{n - 2} \frac{\omega_{n-1}}{2^{\frac{\alpha-1}{\alpha}}\omega_{n-1}}$$

for a positive constant $\rho(n, \kappa)$ depending only on $n$ and $\kappa$. By the curvature bound (3.13) and the argument in [Ye1] for the evolution of the Sobolev constant, we then infer

$$C_{S, 2, \bar{g}(t)}(B(\bar{x}, \rho(n, \kappa), 0)) \leq C_2(n)$$

for $t \in [-\frac{1}{2}\alpha_n, 0]$, where $C_2(n)$ is a positive constant depending only on $n$.

On the other hand, it is easy to see that $\rho(n, \kappa) \leq \frac{1}{16}$. Hence the curvature bound (3.13) and the Ricci flow equation imply that $B(\bar{x}, \rho(n, \kappa), 0) \subset B(\bar{x}, 1, t)$ for $t \in [\bar{\alpha}_n, 0]$, where $\bar{\alpha}_n \leq \frac{1}{2}\alpha_n$ is a positive constant depending only on $n$. It follows that

$$\int_{B(\bar{x}, \rho(n, \kappa), 0)} |Rm|^\frac{n}{2}(\cdot, t)dq(\cdot, t) \leq \delta_0$$

(3.19)
for $t \in [-\bar{\alpha}_n, 0]$.

Now we appeal to the evolution equation of $Rm$ associated with the Ricci flow

$$\frac{\partial Rm}{\partial t} = \Delta Rm + B(Rm, Rm),$$

(3.20)

where $B$ is a certain quadratic form. It implies

$$\frac{\partial}{\partial t}|Rm| \leq \Delta|Rm| + c(n)|Rm|^2$$

(3.21)

for a positive constant $c(n)$ depending only on $n$. On account of (3.13), (3.18) and (3.19) we can apply Theorem 2.1 to (3.21) with $p_0 = \frac{n}{2}$ to deduce

$$|Rm|(\bar{x}, 0) \leq \left(1 + \frac{2}{n}\right)^{\frac{2\sigma_\alpha}{n}}C_2(n)^2C_3(n)\left(\int_{0}^{\bar{\alpha}_n} \int_{B(x_0, \rho(n, \kappa), 0)} |Rm|_{B(x_0, \rho(n, \kappa), 0)} dq(q, t)\right)^{\frac{2}{n}}$$

$$\leq \left(1 + \frac{2}{n}\right)^{\frac{2\sigma_\alpha}{n}}C_2(n)^2C_3(n, \kappa)(\bar{\alpha}_n \delta_0)^{\frac{2}{n}},$$

(3.22)

where

$$C_3(n, \kappa) = \left(2c(n)n + 4n(n - 1) + \frac{n}{2}(1 + \frac{n}{2})^2 \cdot \frac{1}{\bar{\alpha}_n} + \frac{(n + 2)^2}{4\rho(n, \kappa)^2} e^{8(n-1)\bar{\alpha}_n}\right)^{\frac{\alpha + \kappa}{n}}.$$  

We deduce $|Rm|(\bar{x}, 0) \leq \frac{1}{2}$, provided that we define

$$\delta_0 = \frac{1}{2^{\frac{2}{n}}}(1 + \frac{2}{n})^{-\frac{\sigma_\alpha}{n}}C_2(n)^{-\frac{1}{n}}C_3(n, \kappa)^{-\frac{1}{n}} \bar{\alpha}_n^{-1}.$$  

But this contradicts (3.12). Hence the estimate (1.3) has been established.

**Proof of the estimate (1.4)**

Consider a fixed $t_0 \in (0, T)$. If the ratio $\frac{t_0}{t_0} \geq 1$, we rescale $g$ by $t_0^{-1}$. If $\frac{t_0}{t_0} \leq 1$, we rescale $g$ by $\rho_1^{-2}$. We handle the former case, while the latter is similar. For the rescaled flow $t_0^{-1} g(t_0 t)$ on $[0, t_0^{-1} T)$ we have by (1.3)

$$|Rm|(x, t) \leq 2\alpha_n + \epsilon_0^2$$

(3.23)

whenever $\frac{1}{2} \leq t < t_0^{-1} T$ and $d(x, x, t) < t_0^{-\frac{1}{2}} \rho_0 \epsilon_0$. We rescale the flow further by the factor $\lambda_n \equiv 2\alpha_n + \epsilon_0^2 > 42$ to obtain $\bar{g}(t) = \lambda_n t_0^{-1} g(\lambda_n^{-1} t_0 t)$ on $[0, \lambda_n t_0^{-1} T)$. Note that the time $t_0$ is transformed to the time $\lambda_n$. We deal with $\bar{g}$ and all quantities will be associated with $\bar{g}$. We have for $\bar{g}$

$$|Rm|(x, t) \leq 1$$

(3.24)
whenever \( \frac{1}{2} \leq t < \lambda_n t_0^{-1} \) and \( d(x_0, x, t) < \sqrt{\lambda_n t_0^{-2}} r_0 \varepsilon_0 \). Note \( \sqrt{\lambda_n t_0^{-2}} r_0 \varepsilon_0 > 1 \) and \( \sqrt{\lambda_n t_0^{-2}} \rho \geq \sqrt{\lambda_n t_0^{-2}} r_0 \geq \sqrt{\lambda_n} > 6 \). It follows in particular that \( \tilde{g} \) is \( \kappa \)-noncollapsed on the scale 6. We also have \( d_{\tilde{g}(t)}(x_0, \partial M) \geq \lambda_n \) for all \( t \in [0, \lambda_n t_0^{-1} T) \). We deduce

\[
V(x, r, \lambda_n) \geq \kappa r^n \tag{3.25}
\]

whenever \( d(x_0, x, \lambda_n) \leq \frac{1}{2} \) and \( 0 < r \leq \frac{1}{2} \). Now we can argue as in the above proof of the estimate (1.3) to infer \( B(x, \frac{1}{4}, \lambda_n) \subset B(x, \frac{1}{2}, t) \) and

\[
C_{S,2, \tilde{g}(t)}(B(x, \rho(n, \kappa), \lambda_n)) \leq C_2(n) \tag{3.26}
\]

whenever \( \lambda_n - \alpha_n \leq t \leq \lambda_n \) and \( d(x_0, x, \lambda_n) \leq \frac{1}{2} \), where \( \alpha_n > 0 \) and \( C_2(n) > 0 \) depend only on \( n \), and \( 0 < \rho(n, \kappa) \leq \frac{1}{16} \) depends only on \( n \) and \( \kappa \). (We retain the notations \( \alpha_n, \rho(n, \kappa) \) and \( C_2(n) \) although they may be different from their values before.) Now we can apply Theorem 2.1 on \( B(x, \rho(n, \kappa), \lambda_n) \times [\lambda_n - \alpha_n, \lambda_n] \) for each \( x \in B(x_0, \frac{1}{2}, \lambda_n) \) to derive (for \( \tilde{g} \))

\[
|Rm|(x, \lambda_n) \leq \tilde{C}_0(n, \kappa) \left( \int_{\lambda_n - \alpha_n}^{\lambda_n} \int_{B(x, \rho(n, \kappa), \lambda_n)} |Rm|^{\frac{n}{2}}(\cdot, t) dq(\cdot, t) dt \right)^{\frac{2}{n}} 
\]

\[
\leq C_0(n, \kappa) \sup_{\lambda_n - \alpha_n \leq t \leq \lambda_n} \left( \int_{B(x, \frac{1}{4}, t)} |Rm|^{\frac{n}{2}}(\cdot, t) dq(\cdot, t) \right)^{\frac{2}{n}} \tag{3.27}
\]

for all \( x \in B(x_0, \frac{1}{2}, \lambda_n) \), where \( \tilde{C}_0(n, \kappa) > 0 \) depends only on \( n \) and \( \kappa \) and \( C_0(n, \kappa) = \alpha_n \tilde{C}_0(n, \kappa) \). Scaling back to \( g \) we then arrive at the desired estimate (1.4) (with \( t_0 \) in place of \( t \)).

\[\blacksquare\]

4 Proof of Theorem B

Proof of the estimate (1.6)

Assume that the estimate (1.6) fails to hold. Then we carry out the same construction as in the proof of Theorem A. Again we assume \( r_0 = 1 \) and choose \( \varepsilon = \frac{1}{12} \) and \( A = 10 \). We deal with the rescaled flow \( \tilde{g} \) and all quantities will be associated with \( \tilde{g} \). By (1.5) we have for \( \tilde{g} \), in place of (3.14)

\[
\int_{B(\bar{x}, rt)} |Rm|^{\frac{n}{2}}(\cdot, t) dq(\cdot, t) \leq r^{-n} \delta_0 V(\bar{x}, r, t) \tag{4.1}
\]

for all \( 0 < r \leq \frac{1}{2} \) and \( t \in (-\frac{1}{2} \alpha_n, 0] \). As before, we also have for \( \tilde{g} \)

\[
|Rm|(\bar{x}, 0) = 1 \tag{4.2}
\]
\[ |Rm|(x, t) \leq 4 \quad (4.3) \]

whenever
\[ -\frac{1}{2}\alpha_n \leq t \leq 0, \quad d(\bar{x}, x, 0) \leq 1. \quad (4.4) \]

Moreover, we have
\[ d_{\bar{g}(0)}(\bar{x}, \partial M) > \frac{1}{2} \quad (4.5) \]

and
\[ \text{diam}_{\bar{g}(t)}(M) \geq 1 \quad (4.6) \]

for all \( t \in [-\frac{1}{2}\alpha_n, 0] \). By (4.3), (4.5) and the Ricci flow equation we have
\[ d_{\bar{g}(t)}(\bar{x}, \partial M) \geq \frac{1}{3} \quad (4.7) \]

for all \( t \in [-\alpha^*_n, 0] \), where \( \alpha^*_n \leq \frac{1}{2}\alpha_n \) depends only on \( n \).

By (4.3), (4.6) and (4.7) we can apply Theorem B.2 in Appendix B to deduce
\[ C_{S,2,\bar{g}(t)}(B(\bar{x}, \frac{1}{12}, t)) \leq \frac{C_5(n)}{V(\bar{x}, \frac{1}{12}, t)^\frac{n}{4}} \quad (4.8) \]

for \( t \in [-\alpha^*_n, 0] \), with a positive constant \( C_5(n) \) depending only on \( n \). On the other hand, (4.3) implies that
\[ B(\bar{x}, \frac{1}{14}, t_1) \subset B(\bar{x}, \frac{1}{13}, t_2) \subset B(\bar{x}, \frac{1}{12}, t_3) \quad (4.9) \]

for all \( t_1, t_2 \) and \( t_3 \in [-\alpha_n, 0] \), with a positive constant \( \alpha_n \leq \min\{\alpha^*_n, \frac{1}{2}\alpha_n\} \) depending only on \( n \). Consequently, we have
\[ \int_{B(\bar{x}, \frac{1}{14}, 0)} |Rm|(\cdot, t) dq(\cdot, t) \leq (13)^n \delta_0 vol_{\bar{g}(t)}(B(\bar{x}, \frac{1}{13}, t)) \quad (4.10) \]

and
\[ C_{S,2,\bar{g}(t)}(B(\bar{x}, \frac{1}{14}, 0)) \leq \frac{C_5(n)}{V(\bar{x}, \frac{1}{12}, t)^\frac{n}{4}} \quad (4.11) \]

for all \( t \in [-\alpha_n, 0] \). Moreover, (4.9) combined with (4.3) leads via the Ricci flow equation to
\[ \min_{-\alpha_n \leq t \leq 0} V(\bar{x}, \frac{1}{12}, t) \geq e^{-4n(n-1)\alpha_n} \max_{-\alpha_n \leq t \leq 0} V(\bar{x}, \frac{1}{13}, t) \quad (4.12) \]
for each \( t \in [-\bar{\alpha}_n, 0] \). Now we apply Theorem 2.1 to deduce

\[
|Rm|(\bar{x}, 0) \leq (1 + \frac{2}{n})^{\frac{2\sigma_n}{n}} \frac{C_5(n)^2C_6(n)}{\min_{-\bar{\alpha}_n \leq t \leq 0} V(\bar{x}, \frac{1}{12}, t)^{\frac{2}{n}}} \left( \int_{B(\bar{x}, \frac{1}{11}, 0)} |Rm| \cdot (\cdot, t) dq(\cdot, t) \right)^{\frac{1}{2n}}
\]

\[
\leq 169(1 + \frac{2}{n})^{\frac{2\sigma_n}{n}} C_5(n)^2C_6(n)(\bar{\alpha}_n\delta_0)^{\frac{2}{n}} \max_{-\bar{\alpha}_n \leq t \leq 0} V(\bar{x}, \frac{1}{13}, t)^{\frac{2}{n}}
\]

\[
\leq 169(1 + \frac{2}{n})^{\frac{2\sigma_n}{n}} C_5(n)^2C_6(n)(\bar{\alpha}_n\delta_0)^{\frac{2}{n}} e^{8(n-1)\bar{\alpha}_n}, \quad (4.13)
\]

with a positive constant \( C_6(n) \) depending only on \( n \). Choosing

\[
\delta_n = \frac{1}{2^{7}(13)^n}(1 + \frac{2}{n})^{-\sigma_n} C_5(n)^{-n}C_6(n)^{-\frac{2}{n}} \bar{\alpha}_n^{-1} e^{-8(n-1)\bar{\alpha}_n}
\]

we then obtain \( |Rm|(\bar{x}, 0) \leq \frac{1}{2} \), contradicting (4.2).

**Proof of the estimate (1.7)**

Consider a fixed \( t_0 \in (0, T) \). As in the corresponding part of the proof of Theorem A, we present the case \( \frac{r^2}{t_0} \geq 1 \), and rescale \( g \) by \( t_0^{-1} \). Again we have for the rescaled flow \( t_0^{-1}g \) on \([0, t_0^{-1}T]\)

\[
|Rm|(x, t) \leq 2\alpha_n + \epsilon_0^{-2} \quad (4.14)
\]

whenever \( \frac{1}{2} \leq t < t_0^{-1}T \) and \( d(x_0, x, t) < t_0^{-\frac{3}{2}}r_0\epsilon_0 \). As before, we rescale the flow further by the factor \( \lambda_n \equiv 2\alpha_n + \epsilon_0^{-2} > 42 \) to obtain \( \bar{g}(t) = \lambda_nt_0^{-1}g(\lambda_n^{-1}t_0t) \) on \([0, \lambda_nt_0^{-1}T]\). Again, the time \( t_0 \) is transformed to the time \( \lambda_n \). As before we deal with \( \bar{g} \) and all quantotoes are associated with \( \bar{g} \). We have \( \sqrt{\lambda_n}t_0^{-\frac{3}{2}}r_0\epsilon_0 > 1 \) and the curvature estimate for \( \bar{g} \)

\[
|Rm|(x, t) \leq 1 \quad (4.15)
\]

whenever \( \frac{\lambda_n}{2} \leq t < \lambda_nt_0^{-1}T \) and \( d(x_0, x, t) \leq \sqrt{\lambda_n}t_0^{-\frac{3}{2}}r_0\epsilon_0 \). We also have \( d_{\bar{g}(t)}(x_0, \partial M) \geq \lambda_n \) and \( diam_{\bar{g}(t)}(M) \geq \lambda_n \) for all \( t \in [0, \lambda_nt_0^{-1}T] \).

Now we can apply the arguments in the above proof of (1.6). Since the condition \( d_{\bar{g}(t)}(x_0, \partial M) \geq \frac{1}{3} \) is now replaced by \( d_{\bar{g}(t)}(x_0, \partial M) \geq \lambda_n > 1 \) and we have \( \sqrt{\lambda_n}t_0^{-\frac{3}{2}}r_0\epsilon_0 > 1 \) (for the purpose of applying (4.15)), we can replace the radii \( \frac{1}{14}, \frac{1}{13} \)
and $\frac{1}{12}$ by $\frac{1}{10}, \frac{1}{5}$ and $\frac{1}{8}$. We deduce
\[
|Rm|(x, \lambda_n) \leq \frac{C_7(n)}{\min_{\lambda_n - \alpha_n \leq t \leq \lambda_n} V(x, \frac{1}{8}, t)^{\frac{n}{4}}} \left( \int_{\lambda_n}^{\lambda_n} \left( \int_{B(x, \frac{1}{10}, \lambda_n)} |Rm|^{\frac{n}{2}}(\cdot, t) dq(\cdot, t) \right)^{\frac{2}{n}} \right)
\]
\[
\leq C_0(n) \sup_{\lambda_n - \alpha_n \leq t \leq \lambda_n} \left( \frac{\int_{B(x, \frac{1}{8}, t)} |Rm|^{\frac{n}{2}}(\cdot, t) dq(\cdot, t)}{V(x, \frac{1}{8}, t)} \right)^{\frac{2}{n}},
\]
with positive constant constants $C_7(n), \alpha_n$ and $C_0(n)$ depending only on $n$, whenever $d(x_0, x, \lambda_n) \leq \frac{1}{2}$. Scaling back to $g$ we then arrive at the desired estimate (4.17) (with $t_0$ in place of $t$).

## 5 Proof of Theorem C

### Proof of Theorem C

We establish the condition (1.5). Then the theorem follows from Theorem B. By rescaling we can assume $r_0 = 1$. Then (1.8) becomes
\[
Ric \geq -(n - 1)g.
\]
By (1.9), we have now
\[
\int_{B(x, 1, t)} |Rm|^{\frac{n}{2}}(\cdot, t) dq(\cdot, t) \leq \delta_0 V(x, 1, t)
\]
for all $t \in [0, T)$. By Bishop-Gromov relative volume comparison, we have
\[
V(x, R, t) \leq \frac{V_{-1}(R)}{V_{-1}(r)} V(x, r, t) \leq C(n) \frac{V(x, r, t)}{r^n},
\]
with a positive constant $C(n)$ depending only on $n$, provided that $t \in [0, T], d(x_0, x, t) < 1$, and $0 < r < R \leq 1 - d(x_0, x, t)$. Here $v_{-1}(r)$ denotes the volume of a geodesic ball of radius $r$ in $\mathbb{H}^n$, the $n$-dimensional hyperbolic space (of sectional curvature $-1$). If $t \in [0, T)$ and $d(x_0, x, t) \leq \frac{1}{4}$, we then have $B(x_0, \frac{1}{4}, t) \subset B(x, \frac{1}{2}, t) \subset B(x_0, 1, t)$. Consequently,
\[
V(x, r, t) \geq C(n)^{-1} r^n V(x, \frac{1}{2}, t) \geq C(n)^{-1} r^n V(x_0, \frac{1}{4}, t) \geq 4^{-n} C(n)^{-2} r^n V(x_0, 1, t)
\]
\[(5.4)\]
for $0 < r \leq \frac{1}{2}$. This leads to
\[
V(x, r, t) \geq 4^{-n}C(n)^{-2} \frac{r^n}{r_0^n} V(x_0, r_0, t)
\]  
(5.5)
as long as $t \in [0, T)$, $d(x_0, x, t) \leq \frac{1}{4}r_0$ and $0 < r \leq \frac{1}{2}r_0$. Hence we infer
\[
\int_{B(x,r,t)} |Rm|^\frac{2}{n}(\cdot,t) dq(\cdot,t) \leq \int_{B(x_0,r_0,t)} |Rm|^\frac{2}{n}(\cdot,t) dq(\cdot,t) \leq \delta_0 \frac{V(x_0,r_0,t)}{r_0^n} \leq 4^nC(n)^2\delta_0 \frac{V(x,r,t)}{r^n}
\]  
(5.6)
whenever $t \in [0, T)$, $d(x_0, x, t) \leq \frac{1}{4}r_0$ and $0 < r \leq \frac{1}{2}r_0$. Choosing $\delta_0$ to be the $\delta_0$ in Theorem B multiplied by $4^{-n}C(n)^{-2}$ and replacing $r_0$ by $\frac{r_0}{2}$ we then have all the conditions of Theorem B. The desired estimate follows.

\section*{Appendices}

\section{A $\kappa$-Noncollapsingness}

In this appendix we present a stronger version of [Theorem 4.1, P], namely Theorem A.1 regarding the $\kappa$-noncollapsed property of the Ricci flow at finite times. We observed this version in the beginning of 2003. Later, we learnt that Perelman also made the same observation, see [KL].

\textbf{Definition} Let $g$ be a Riemannian metric on a manifold $M$ of dimension $n$. Let $\kappa$ and $\rho$ be positive numbers. We say that $g$ is $\kappa$-noncollapsed on the scale $\rho$ relative to the positive part of the scalar curvature (or relative to upper bounds of the scalar curvature), if $g$ satisfies $vol(B(x,r)) \geq \kappa r^n$ for all $x \in M$ and $r > 0$ satisfying $r < \rho$ and $\sup\{R(x) : x \in B(x,r)\} \leq r^{-2}$. (Note that $\sup\{R(x) : x \in B(x,r)\} \leq r^{-2}$ is equivalent to $\sup\{R^+(x) : x \in B(x,r)\} \leq r^{-2}$. Hence the terminology “the positive part of the scalar curvature”.) We say that a family of Riemannian metrics $g = g(t)$ is $\kappa$-noncollapsed on the scale $\rho$ relative to the positive part of the scalar curvature, if $g(t)$ is $\kappa$-noncollapsed on the scale $\rho$ relative to the positive part of the scalar curvature for each $t$ (in the given domain).

Obviously, if $g$ is $\kappa$-noncollapsed on the scale $\rho$ relative to the positive part of the scalar curvature, then it is $\tilde{\kappa}$-noncollapsed on the scale $\rho$, where $\tilde{\kappa} = c(n)\kappa$ for a positive constant $c(n)$ depending only on the dimension $n$. 

18
Theorem A.1 Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ for a closed manifold $M$ of dimension $n \geq 2$ and some finite $T > 0$. Let $\rho > 0$ be an arbitrary positive number. Then $g$ is $\kappa$-noncollapsed on the scale $\rho$ relative to the positive part of the scalar curvature for $t \in [0, T)$, where $\kappa = \kappa(T + \rho^2, g(0))$ depends on (an upper bound of) $T + \rho^2$ and the initial metric $g(0)$.

Proof. Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ for a closed manifold $M$ of dimension $n \geq 2$ and some finite $T > 0$. Assume that there is no $\kappa > 0$ such that $g$ is $\kappa$-noncollapsed on the scale $\rho$ relative to the positive part of the scalar curvature for $t \in [0, T)$. Then there is a sequence of times $t_k \in [0, T)$ with $t_k \to T$, a sequence of points $x_k \in M$, and a sequence of positive numbers $r_k < \rho$ such that for all $k$

$$V(x_k, r_k, t_k) \leq \frac{1}{2k} \omega_n r_k^n$$

and

$$R(\cdot, t_k) \leq r_k^{-2}$$

on $B(x_k, r_k, t_k)$. Here $\omega_n$ denotes the volume of the $n$-dimensional euclidean ball of radius 1.

For a fixed $k$ we set $\rho_{k,j} = r_k^{2j}$. Choose the largest $j$ such that $V(x_k, \rho_{j}, t_k) \leq \frac{1}{2k} \omega_n \rho_j^n$. Since $\frac{V(x_k, \rho_{j}, t_k)}{\omega_n \rho_j^n} \to 1$ as $\rho \to 0$, such a $j$ exists. Then we have $V(x_k, \rho_{k,j+1}, t_k) > \frac{1}{2k} \omega_n \rho_{k,j+1}^n$. We replace the value of $r_k$ by $\rho_{k,j}$ with this largest $j$. Then we have in addition to (1.1) and (1.2)

$$V(x_k, \frac{r_k}{2}, t_k) > \frac{1}{2k} \omega_n (\frac{r_k}{2})^n. \quad (1.3)$$

Next we consider the entropy functional of Perelman [P]

$$W(g, f, \tau) = \int_M \left[ \tau (|\nabla f|^2 + R) + f - n \right] (4\pi \tau)^{-\frac{n}{2}} e^{-f} dvol \quad (1.4)$$

for smooth metrics $g$ and Lipschitz functions $f$ on $M$, and $\tau > 0$, under the side condition

$$(4\pi \tau)^{-\frac{n}{2}} \int_M e^{-f} dvol = 1, \quad (1.5)$$

where all geometric quantities are associated with $g$. We construct a sequence of Lipschitz functions $f_k$ on $M$ satisfying the side condition (1.5) for $g = g(t_k)$ such that $W(g(t_k), f_k, r_k^2) \to -\infty$. For $\delta > 0$ we set $\psi_\delta(t) = 1$ for $0 \leq t \leq \frac{1}{2}$, $\psi_\delta(t) = \delta$ for
\( t \geq 1 \), and \( \psi_\delta(t) = \frac{2(1+\delta^2-t)}{1+2\delta} \) for \( \frac{1}{2} \leq t \leq 1 \). Let \( \Lambda > 0 \). Following [P] we then set \( f_k = -2 \log(\Lambda \psi_\delta(r_k)) \), where \( r(x) = d(x_k, x, t_k) \). Obviously,

\[
\int_{B(x_k, r_k, t_k)} f_k e^{-f_k} (4\pi r_k^2)^{-\frac{n}{2}} dvol_{g(t_k)} = -2 (4\pi)^{-\frac{n}{2}} V(x_k, r_k, t_k) \Lambda^2 \log \Lambda. \tag{1.6}
\]

By the conditions (1.1), (1.3) and (1.2), this integral is dominating in \( W(g(t_k), f_k, r_k^2) \), provided that \( \delta \) is small and \( \Lambda \) is large. It is easy to choose \( \Lambda = \Lambda_k \approx \omega_n^{-1} (4\pi)^{\frac{n}{2}} k^{2^{n+1}} \) and \( \delta = \delta_k \) with \( \delta_k^2 \log \delta_k \approx -\frac{1}{2} r_k^n \log_{g(t_k)}(M)^{-1} \) such that

\[
W(g(t_k), f_k, r_k^2) < -\log \Lambda_k
\]

and

\[
(4\pi)^{-\frac{n}{2}} \int_M e^{-f_k} dvol_{g(t_k)} = 1. \tag{1.8}
\]

For a fixed \( k \) let \( \tau(t) = t_k - t + r_k^2 \) and \( f \) be the solution of the equation

\[
\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2} \tau
\]

on \([0, t_k]\) associated with \( g = g(t) \), with the initial value \( f(\cdot, t_k) = f_k \). By the monotonicity of the entropy functional [P], we have for \( \bar{f}_k = f(\cdot, 0) \)

\[
W(g(0), \bar{f}_k, t_k + r_k^2) \leq W(g(t_k), f_k, r_k^2) < -\log \Lambda_k. \tag{1.10}
\]

Moreover, we also have

\[
(4\pi(t_k + r_k^2))^{-\frac{n}{2}} \int_M e^{-f_k} dvol_{g(0)} = 1. \tag{1.11}
\]

However, \( t_k + r_k^2 \leq T + \rho^2 \). Hence the logarithmic Sobolev inequality implies that \( W(g(0), \bar{f}_k, t_k + r_k^2) \) is bounded from below by a finite constant independent of \( k \) (see [R] and [Y4]). This contradicts (1.10). \( \blacksquare \)

## B Estimates of the Sobolev Constant

In this appendix we present two estimates for the Sobolev constant.

**Lemma B.1** Let \((M, g)\) be a Riemannian manifold of dimension \( n \). Assume that the sectional curvatures \( K_g \) of \( g \) satisfies \( \kappa_1 \leq K_g \leq \kappa_2 \) on a geodesic ball \( B(p, r_0) \) in \((M, g)\), such that \( r_0 \leq d(p, \partial M) \). Set \( r_1 = \frac{1}{4} \min\{r_0, \frac{\pi}{4\sqrt{\kappa_2}}\} \). Then the injectivity radius \( i(q) \) at any \( q \in B(p, r_1) \) satisfies

\[
i(q) \geq r_2, \tag{B.1}
\]
where
\[ r_2 = \frac{r_1}{2} \left( 1 + \frac{V_{\kappa_1} (2r_1)^2}{\text{vol}_g (B(p, r_1)) V_{\kappa_1} (r_1)} \right)^{-1}, \tag{B.2} \]
\[ r_1 = \frac{1}{4} \min \{ r_0, \frac{\pi}{\sqrt{\kappa_2}} \}, \]
and for any \( r > 0 \), \( V_{\kappa_1} (r) \) denotes the volume of a geodesic ball of radius \( r \) in the \( n \)-dimensional model space (a simply connected complete Riemannian manifold) of sectional curvature \( \kappa_1 \).

Proof. The estimate (B.1) follows from the proof of a similar estimate [(4.23), CGT] in [CGT]. In [CGT], the estimate [(4.23), CGT] is proved for complete manifolds. Under the above assumption about the radius \( r_0 \), one can easily check that the involved geodesics in the relevant arguments in [CGT] all stay inside \( B(p, r_0) \), and hence those arguments all go through.

**Theorem B.2** Let \((M, g)\) be a Riemannian manifold of dimension \( n \). Assume that the sectional curvature \( K_g \) of \( g \) satisfies \( \kappa_1 \leq K_g \leq \kappa_2 \) on a geodesic ball \( B(p, r_0) \) in \((M, g)\), such that \( r_0 \leq d(p, \partial M) \). Let \( r_1 \) and \( r_2 \) be defined as in Lemma A.1. Then we have
\[ C_{S, g} (B(x, r)) \leq \frac{\omega_{n-1}}{2 \frac{n-1}{n} \omega_{n-1}} \] for all \( 0 < r \leq \frac{r_2}{2} \), where \( \omega_n \) denotes the volume of the unit sphere in the \((n + 1)\)-dimensional Euclidean space.

Proof. This follows from Lemma B.1 and Croke’s isoperimetric inequality, see [C] or [Proposition V.2.3(a), Ch].

**Theorem B.3** Let \((M, g)\) be a Riemannian manifold of dimension \( n \). Assume that the Ricci curvature \( \text{Ric}_g \) of \( g \) satisfies \( \text{Ric}_g \geq -\kappa (n-1) \) for a nonnegative constant \( \kappa \) on a geodesic ball \( B(p, r_0) \) in \((M, g)\), such that \( r_0 \leq \min \{ d(p, \partial M), \text{diam}(M) \} \). Then we have
\[ C_{S, g} (B(p, r)) \leq C(n, \kappa) \left( \frac{V_{-\kappa} (r)}{\text{vol}_g (B(p, r))} \right)^{\frac{1}{n}} \] for all \( 0 < r \leq \min \{ \frac{r_0}{4}, 1 \} \), where \( C(n, \kappa) \) is a positive constant depending only on \( n \) and \( \kappa \).

Proof. The estimate (B.4) was established in [An] for complete Riemannian manifolds. Its proof in [An], which is of local nature, can be carried over to the present situation, because the involved geodesics all stay inside of \( B(p, r_0) \). Note that the proof in [An] is based on some inequalities established in [G]. Those inequalities also carry over to the present situation for the same reason.

**Remark** Obviously, Theorem B.3 leads to a different estimate of the Sobolev constant under the conditions of Theorem B.2. This estimate can replace Theorem B.2 in the proof of Theorem A.

21
References

[An] M. Anderson, The L2-structure of moduli spaces of Einstein metrics on 4-manifolds, Geometric and Functional Analysis 2 (1992), 29-89.

[C] C. Croke, Some isoperimetric inequalities and eigenvalue estimates, Ann. Sci. École Norm. Sup. 13 (1980), 419-435.

[CGY] J. Cheeger, M. Gromov & M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Diff. Geom. 17 (1982), 15-53.

[Ch] I. Chavel, Isoperimetric Inequalities, Cambridge University Press 2001.

[G] M. Gromov, Paul Levy’s isoperimetric inequality, Chapter C of “Metric Structures For Riemannian and Non-Riemannian Spaces” by M. Gromov, Birkhäuser, 1998.

[H] R. Hamilton, The formation of singularities in the Ricci flow, Surveys in Differential Geometry, Vol.II, Internat. Press, Cambridge, 1995, 7-136.

[P] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, http://arXiv.org/abs/math.DG/0211159.

[R] O. S. Rothaus, Logarithmic Sobolev inequalities and the spectrum of Schrödinger operators, J. Funct. Anal. 42 (1981), 110-120.

[Sh] W.-X. Shi, Deforming the metric on complete Riemannian manifolds, J. Diff. Geom. 30 (1989), 223-301.

[Ye1] R. Ye, Ricci flow, Einstein metrics and space forms, Tran. Am. Math. Soc. 338 (1993), 871-895.

[Ye2] R. Ye, Ricci flow and manifolds of negatively pinched curvature, preprint, 1990.

[Ye3] R. Ye, Curvature estimates for the Ricci flow II, preprint 2005, available at http://xxx.lanl.gov.

[Ye4] R. Ye, Notes on the logarithmic Sobolev inequality and its application to the Ricci flow.