Towards the Quantum Geometry of Saturated Quantum Uncertainty Relations: The Case of the \((Q, P)\) Heisenberg Observables

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It is a pleasure to dedicate this article to M. Norbert Hounkonnou on the occasion of his sixtieth birthday

Abstract

This contribution to the present Workshop Proceedings outlines a general programme for identifying geometric structures—out of which to possibly recover quantum dynamics as well—associated to the manifold in Hilbert space of the quantum states that saturate the Schrödinger-Robertson uncertainty relation associated to a specific set of quantum observables which characterise a given quantum system and its dynamics. The first step in such an exploration is addressed herein in the case of the observables \(Q\) and \(P\) of the Heisenberg algebra for a single degree of freedom system. The corresponding saturating states are the well known general squeezed states, whose properties are reviewed and discussed in detail together with some original results, in preparation of a study deferred to a separated analysis of their quantum geometry and of the corresponding path integral representation over such states.

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1 Introduction

Historically, Heisenberg’s uncertainty principle[1] has proved to be pivotal in the emergence of quantum mechanics as the conceptual paradigm for physics at the smallest distance scales. Still to this day the uncertainty principle remains a reliable guide in the exploration and the understanding of the physical consequences of the foundational principles of quantum dynamics.

In its original formulation, Heisenberg suggested that measurements of a quantum particle’s (configuration space) coordinate, \(q\), and (conjugate) momentum, \(p\), are intrinsically limited in their precision in a way such that

\[ \Delta q \Delta p \gtrsim \hbar, \quad \hbar = \frac{h}{2\pi} \approx 6.626 \times 10^{-34} \text{ J \cdot s}, \tag{1} \]

\( \hbar \) being the reduced Planck constant. Soon thereafter, Schrödinger[2] as well as Robertson[3] made this statement both more precise and more general for any given pair of self-adjoint, or at least hermitian quantum observables \(A\) and \(B\), in the form of the Schrödinger-Robertson uncertainty relation (SR-UR),

\[ (\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} (\langle -i [A, B] \rangle)^2 + \frac{1}{4} (\langle [A - \langle A \rangle, B - \langle B \rangle] \rangle)^2, \tag{2} \]

where as usual \((\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle\) and \((\Delta B)^2 = \langle (B - \langle B \rangle)^2 \rangle\), while \(\langle O \rangle\) denotes the normalised expectation value of any quantum operator \(O\) given an arbitrary (normalisable) quantum state (see Appendix A for notations and a derivation of the SR-UR). As a by-product one thus also obtains the less tight (but better known, and generalised) Heisenberg uncertainty relation (H-UR),

\[ (\Delta A) (\Delta B) \geq \frac{1}{2} |\langle -i [A, B] \rangle|. \tag{3} \]

In the case of the Heisenberg algebra, namely \([Q, P] = i\hbar \mathbb{I}\), indeed this becomes \(\Delta q \Delta p \geq \hbar/2\).

In the classical limit \(\hbar \to 0\), both terms of these inequalities vanish and the latter turn into strict equalities. The physical world however, is not classical since Planck’s constant albeit small as measured in our macroscopic units, definitely has a finite and non-vanishing value. Yet, in certain regimes of their Hilbert spaces dynamical quantum systems must display classical behaviour as we experience it through quantum observables some of which are of a macroscopic character. Indeed, any quantum system is specified through a set of quantum observables of which the algebra of commutation relations is represented by the Hilbert space which describes that quantum system and its quantum states. Given a particular choice of quantum observables and through measurements of the latter, experiments give access to the quantum states of such a system and enable their manipulation. If certain regimes of a quantum system display the hallmarks of a classical-like behaviour, certainly these regimes must correspond to quantum states which are as close as possible to being classical given a set ensemble of quantum observables characterising that system. In other words classical-like regimes of a quantum system which is characterised by a collection of quantum observables, need to correspond to quantum states which saturate as exact equalities the generalised Schrödinger-Robertson uncertainty relation related to that ensemble of quantum observables. Indeed saturated uncertainty relations leave the least room possible for a genuine quantum dynamical behaviour which would otherwise potentially lead to large differences in the values taken by the two terms involved in the inequalities expressing such uncertainty relations.

For reasons recalled in Appendix A, in the case of two observables the SR-UR is saturated by quantum states \(|\psi_0\rangle\) which are such that,

\[ [(A - \langle A \rangle) - \lambda_0 (B - \langle B \rangle)] |\psi_0\rangle = 0, \quad [A - \lambda_0 B] |\psi_0\rangle = [\langle A \rangle - \lambda_0 \langle B \rangle] |\psi_0\rangle, \tag{4} \]

Robertson extended this statement to an arbitrary number of observables in terms of the determinant of their covariance matrix of bi-correlations.
where the complex parameter $\lambda_0$ is given by the following combination of expectation values for the state $|\psi_0\rangle$,

$$
\lambda_0 = \frac{(B - \langle B\rangle)(A - \langle A\rangle)}{(\Delta B)^2} = \frac{(\Delta A)^2}{\langle (A - \langle A\rangle)(B - \langle B\rangle) \rangle}.
$$

(5)

Such saturating quantum states are parametrised by collections of continuous parameters, if only for the expectation values $\langle A \rangle$ and $\langle B \rangle$ as well as the ratio $\Delta A/\Delta B$, for instance. Indeed, especially when considered in the form of the second relation in (4), such states determine classes of quantum coherent-like states (see Refs. [4, 5] and references therein), which share many of the remarkable properties of the well known Schrödinger canonical coherent states for the Heisenberg algebra. In particular in order that their expectation values $\langle A \rangle$ and $\langle B \rangle$ retain finite non-vanishing classical values as $\hbar \to 0$, it is necessary that the saturating states $|\psi_0\rangle$ meeting the conditions (4) involve all possible linearly independent quantum states spanning the full Hilbert space of the system. Furthermore, usually the linear span of such coherent states encompasses the full Hilbert space, since they obey a specific overcompleteness relation or resolution of the unit operator, thereby providing a self-reproducing kernel representation of that Hilbert space.[4, 6].

In other words, given a set of quantum observables such saturating quantum states for the corresponding collection of uncertainty relations determine a specific differentiable submanifold of Hilbert space, out of which the full Hilbert space of the quantum system may a priori be reconstructed (provided a sufficient number of quantum observables is considered). In particular quantum amplitudes may then be given a functional path integral representation over that manifold of coherent states, which involves specific geometrical structures of that manifold[6, 7]. Indeed, very naturally that manifold comes equipped then not only with a (quantum) symplectic structure[3] but also with a (quantum) Riemannian metric structure[2, 7], both of these geometric structures being compatible with one another (and dependent, generally, on Planck’s constant).

A quantum geometric representation of the quantum system thus arises out of its Hilbert space given a choice of its quantum observables and through the associated uncertainty relation. It may even be that, for instance through the corresponding path integral, the quantum system itself may be reconstructed out of these geometric structures (provided the original choice of quantum observables be large enough).

Such an approach connects directly with, and expands on Klau der’s general programme of “Enhanced Quantisation” having been proposed for many years now (see Ref.[6] and references therein), as a path towards a geometrical formulation of genuine quantum dynamics which shares a number of similarities with other proposals for such geometrical formulations[8, 9]. For that same reason, the programme as briefly outlined above provides a possible avenue towards a further understanding of the underpinnings of the AdS/CFT correspondence and the holographic principle, for instance along lines similar to those having been explored already in Ref.[10].

While the general programme outlined above, based on saturated uncertainty relations and the geometry of the associated coherent-like quantum states, is offered here as a project of possible interest to Professor Norbert Hounkonnou in celebration as well of his sixtieth birthday and on the occasion of this COPROMAPH Workshop organised in his honour, the present paper only deals with the construction of the quantum states which saturate the Schrödinger-Robertson uncertainty relation in the case of the Heisenberg algebra for a single quantum degree of freedom, leaving for separate work a discussion of the ensuing geometric structures. Besides some results which presumably are original, most of those being presented herein certainly are available in the literature (see Refs.[11, 12, 13] and references therein) even though in a scattered form[10]. However this author did not find them discussed along the lines addressed here, nor could he find them all brought together in one single place, as made available in the present contribution with the purpose of providing a basis towards a pursuit of the projected programme aiming at a better understanding of the geometric structures inherent to quantum systems and their dynamics.

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2Because of the sesquilinear properties of the inner product defined over Hilbert space.

3Because of the hermitian and positive definite properties of the inner product defined over Hilbert space.

4For this reason no attempt is being made towards a complete list of references to the original literature which relates to many different fields of quantum physics.
Section 2 particularises the discussion to the Heisenberg algebra and identifies the saturating states for the SR-UR in the configuration space representation of that algebra. A construction in terms of Fock algebras and their canonical coherent states is then initiated in Section 3, beginning with a reference Fock algebra related to an intrinsic physical scale. Section 4 then presents the complete parametrised set of saturating quantum states, leading to the general class of the well known squeezed coherent states. Further specific results of interest for these states are then presented in Sections 5 and 6, to conclude with some additional comments in the Conclusions. Complementary material of a more pedagogical character as befits the Proceedings of the present COPROMAPH Workshop, is included in two Appendices.

2 The Uncertainty Relation for the Heisenberg Algebra

Given a single degree of freedom system whose configuration space has the topology of the real line, \( q \in \mathbb{R} \), let us consider the corresponding Heisenberg algebra with its conjugate quantum observables, \( Q \) and \( P \), such that

\[
[Q, P] = i\hbar \mathbb{I}, \quad Q^\dagger = Q, \quad P^\dagger = P.
\] (6)

The configuration and momentum space representations of this algebra are well known, based on the corresponding eigenstate bases, \( Q|q\rangle = q|q\rangle \) and \( P|p\rangle = p|p\rangle \), with \( q, p \in \mathbb{R} \). Our choices of normalisations and phase conventions for these bases states are such that

\[
\langle q|q'\rangle = \delta(q - q'), \quad \langle p|p'\rangle = \delta(p - p'), \quad \int_{-\infty}^{+\infty} dq \langle q|q\rangle = \mathbb{I} = \int_{-\infty}^{+\infty} dp \langle p|p\rangle.
\] (7)

\[
\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{qp}{\hbar}}, \quad \langle p|q\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{qp}{\hbar}}.
\] (8)

Consider an arbitrary (normalisable) quantum state \( |\psi_0\rangle \), which we assume also to have been normalised, \( \langle \psi_0|\psi_0\rangle = 1 \). In configuration space this state is represented by its wave function, \( \psi_0(q) = \langle q|\psi_0\rangle \in \mathbb{C} \). Let \( q_0 \) and \( p_0 \) be its real valued expectation values for the Heisenberg observables,

\[
q_0 = \langle \psi_0|Q|\psi_0\rangle, \quad p_0 = \langle \psi_0|P|\psi_0\rangle, \quad q_0, p_0 \in \mathbb{R},
\] (9)

and introduce the shifted or displaced operators

\[
\bar{Q} = Q - q_0, \quad \bar{P} = P - p_0,
\] (10)

which again define a Heisenberg algebra of hermitian (ideally self-adjoint) quantum observables, \( [\bar{Q}, \bar{P}] = i\hbar \mathbb{I}, \bar{Q}^\dagger = \bar{Q}, \bar{P}^\dagger = \bar{P} \). One also has \( (\Delta Q)^2 = \langle \psi_0|\bar{Q}^2|\psi_0\rangle \) and \( (\Delta P)^2 = \langle \psi_0|\bar{P}^2|\psi_0\rangle \).

The Schrödinger-Robertson uncertainty relation (SR-UR) then reads (see Appendix A),

\[
(\Delta Q)^2 (\Delta P)^2 \geq \frac{1}{4} \hbar^2 + \frac{1}{4} \langle \{\bar{Q}, \bar{P}\}\rangle^2,
\] (11)

\( \{\bar{Q}, \bar{P}\} \) being the anticommutator of \( \bar{Q} \) and \( \bar{P} \). As a corollary note that one then also has the looser Heisenberg uncertainty relation (H-UR),

\[
\Delta Q \Delta P \geq \frac{1}{2} \hbar.
\] (12)

However according to the general programme outlined in the Introduction, we are interested in identifying all quantum states that saturate the SR-UR, but not necessarily the H-UR. Quantum states that saturate the H-UR are certainly such that \( \langle \{\bar{Q}, \bar{P}\}\rangle = 0 \), namely they cannot possess any \( (\bar{Q}, \bar{P}) \) quantum correlation. The ensemble of states that saturate the SR-UR is thus certainly

\footnote{Hence the states \( |q\rangle \), say, are determined up to a \( q \)-independent overall global phase factor which remains unspecified, relative to which all other phase factors are then identified accordingly.}
larger than that which saturates the H-UR. What distinguishes these two sets of states will be made explicit later on.

For reasons recalled in Appendix A, those states which saturate the SR-UR are such that

$$[\tilde{Q} - \lambda_0 \tilde{P}] |\psi_0\rangle = 0, \quad [(Q - q_0) - \lambda_0 (P - p_0)] |\psi_0\rangle = 0, \quad (13)$$

where the complex parameter $\lambda_0$ takes the value,

$$\lambda_0 = \frac{1}{(\Delta P)^2} \left( \frac{1}{2} \langle \{ Q, P \} \rangle - \frac{1}{2} i \hbar \right) = (\Delta Q)^2 \frac{1}{2} \langle \{ Q, P \} \rangle + \frac{1}{2} i \hbar \cdot (14)$$

The defining equation (13) of saturating states for the $(Q, P)$ observables of the Heisenberg algebra is best solved by working in a wave function representation, say in configuration space.

The above condition then reads,

$$\left[ (q - q_0) - \lambda_0 \left(-i \hbar \frac{d}{dq} - p_0 \right) \right] \psi_0(q) = 0. \quad (15)$$

Clearly its solution is

$$\psi_0(q) = N_0(q_0, p_0, \lambda_0) e^{\frac{i}{\hbar} q p_0} e^{-\frac{i}{2 \hbar} \lambda_0 (q-q_0)^2}, \quad \psi_0^*(q) = \bar{N}_0(q_0, p_0, \lambda_0) e^{-\frac{i}{\hbar} q p_0} e^{-\frac{i}{2 \hbar} \lambda_0 (q-q_0)^2}, \quad (16)$$

where $N_0(q_0, p_0, \lambda_0)$ is a complex valued normalisation factor still to be determined. Requiring the state $|\psi_0\rangle$ to be normalised to unity implies the following value for the norm of $N_0(q_0, p_0, \lambda_0),

$$|N_0(q_0, p_0, \lambda_0)| = \left(2\pi (\Delta Q)^2\right)^{-1/4}. \quad (17)$$

Its overall phase however, will be determined later on, once further phase conventions will have been specified. Note well that all quantum states saturating the SR-UR are of this simple form, specified in terms of four independent real parameters, namely $q_0, p_0, \Delta Q > 0$ (say) and $\langle \{ Q, P \} \rangle$ (in terms of which $\Delta P > 0$ is then also determined since $(\Delta Q)^2 (\Delta P)^2 = (\hbar^2 + \langle \tilde{Q}, \tilde{P} \rangle)^2 / 4$).

In the remainder of this paper, we endeavour to understand the structure of these saturating quantum states from the point of view of coherent states, as indeed the defining equation (13) invites us to do.

To conclude, let us also remark that for those saturating states such that in addition $\langle \{ Q, P \} \rangle = 0$, in this particular case which thus saturates the H-UR rather than the SR-UR we have the following results (with a choice of phase factor for the wave function which complies with the specifications to be addressed later on),

$$\langle \{ Q, P \} \rangle = 0 : \quad \lambda_0 = -\frac{i \hbar}{2 (\Delta P)^2} = -2 i \frac{(\Delta Q)^2}{\hbar}, \quad \frac{1}{\lambda_0} = -\frac{i \hbar}{2 (\Delta Q)^2}. \quad (18)$$

$$\psi_0(q) = \frac{1}{\left(2\pi (\Delta Q)^2\right)^{1/4}} e^{\frac{i}{\hbar} q p_0} e^{-\frac{i}{2 \hbar} \lambda_0 (q-q_0)^2}, \quad \Delta Q \Delta P = \frac{1}{2} \hbar. \quad (19)$$

Note well however, that even in this case the value of $\Delta P / \Delta Q = \hbar / (2(\Delta Q)^2)$ is still left as a free real and positive parameter. In the case of the ordinary Schrödinger canonical coherent states, which indeed saturate the H-UR, this latter ratio is implicitly set to a specific value in terms of physical parameters of the system under consideration.

### 3 A Reference Fock Algebra

A priori the quantum observables $Q$ and $P$ possess specific physical dimensions, of which the product has the physical dimension of $\hbar$. For the sake of the construction hereafter, let us denote by $\ell_0$ an intrinsic physical scale which has the same physical dimension as $Q$, so that the physical
dimension of $P$ is that of $\hbar/\ell_0$. For instance we may think of $Q$ as a configuration space coordinate measured in a unit of length, in which case $\ell_0$ has the dimension of length, hence the notation. However, note that the physical dimension of $\ell_0$ could be anything, as may be relevant given the physical system under consideration. Furthermore $\ell_0$ need not correspond to some fundamental physical scale or constant. The scale $\ell_0$ may well be expressed in terms of fundamental physical constants in combination with other physical parameters related to the system under consideration. In particular $\ell_0$ may involve Planck’s constant itself, $\hbar$, and thus change value in the classical limit $\hbar \to 0$ (as is the case for the ordinary harmonic oscillator of mass $m$ and angular frequency $\omega$, with then the natural choice $\ell_0 = \sqrt{\hbar/(m\omega)}$). The purpose of the intrinsic physical scale $\ell_0$ is to introduce a reference quantum Fock algebra, hence the corresponding reference canonical coherent states, in order to address the quantum content characterised by the defining equation (13) of the quantum states saturating the SR-UR of the Heisenberg algebra, which is indeed a condition characteristic of quantum coherent states.

Given the intrinsic physical scale $\ell_0$, let us thus introduce the following reference Fock operators,

$$a = \frac{1}{\sqrt{2}} \left( \frac{Q}{\ell_0} + i \frac{\ell_0}{\hbar} P \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{Q}{\ell_0} - i \frac{\ell_0}{\hbar} P \right),$$

with the inverse relations for the Heisenberg observables,

$$Q = \frac{1}{\sqrt{2}} \ell_0 \left( a + a^\dagger \right), \quad P = -\frac{i\hbar}{\ell_0^{3/2}} \left( a - a^\dagger \right),$$

which indeed generate the corresponding Fock and Heisenberg algebras, respectively,

$$[a, a^\dagger] = \mathbb{I}, \quad [Q, P] = i\hbar \mathbb{I}.$$  \hspace{1cm} (22)

The associated normalised reference Fock vacuum, $|\Omega_0\rangle$, such that

$$a|\Omega_0\rangle = 0, \quad \langle \Omega_0 | \Omega_0 \rangle = 1,$$  \hspace{1cm} (23)

is chosen with a phase relative to the overall phase implicitly chosen for the position eigenstates $|q\rangle$ such that

$$\langle q | \Omega_0 \rangle = (\pi \ell_0^2)^{-1/4} e^{-\frac{i}{2\ell_0^2} q^2}. \hspace{1cm} (24)$$

On account of the condition $a|\Omega_0\rangle = 0$ to be compared to the defining equation (13), it is clear that the reference Fock vacuum $|\Omega_0\rangle$ saturates not only the SR-UR but also the H-UR with vanishing expectation values for $q_0$, for $p_0$ and for the $(Q, P)$ correlator $\langle \{Q, P\} \rangle$, while the values for $\Delta Q$ and $\Delta P$ given by

$$\langle \Delta Q \rangle^2 = \frac{1}{2} \ell_0^2, \quad \langle \Delta P \rangle^2 = \frac{1}{2} \frac{\hbar^2}{\ell_0^2}, \quad \left( \frac{\Delta Q}{\ell_0} \right)^2 = \frac{1}{2}, \quad \left( \frac{\ell_0}{\hbar} \Delta P \right)^2 = \frac{1}{2}; \hspace{1cm} (25)$$

are such that,

$$(\Delta Q) (\Delta P) = \frac{1}{2} \hbar, \quad \left( \frac{\Delta Q}{\ell_0} \right) \left( \frac{\ell_0}{\hbar} \Delta P \right) = \frac{1}{2}, \quad \left( \frac{\Delta Q}{\ell_0} \right)^2 + \left( \frac{\ell_0}{\hbar} \Delta P \right)^2 = 1, \hspace{1cm} (26)$$

with in particular thus even the ratio $\Delta P/\Delta Q$ taking a predetermined value, $\Delta P/\Delta Q = \hbar/\ell_0^2$. As it turns out, all states saturating the SR-UR will be constructed out of this reference Fock vacuum (thereby also determining the overall phase of the wave function of these states, $\psi_0(q)$, left unspecified in (16) and (17) of Section 2).

In order to deal with the shifted or displaced observables $\bar{Q}$ and $\bar{P}$ which involve the expectation values $q_0$ and $p_0$, given the reference Fock algebra (20) let us introduce the following complex quantity,

$$u_0 = \frac{1}{\sqrt{2}} \left( \frac{q_0}{\ell_0} + i \frac{\ell_0}{\hbar} p_0 \right), \quad \bar{u}_0 = u_0^* = \frac{1}{\sqrt{2}} \left( \frac{q_0}{\ell_0} - i \frac{\ell_0}{\hbar} p_0 \right), \hspace{1cm} (27)$$
with the inverse relations,
\[ q_0 = \frac{1}{\sqrt{2}} \ell_0 (u_0 + \bar{u}_0), \quad p_0 = -\frac{i \hbar}{\ell_0 \sqrt{2}} (u_0 - \bar{u}_0). \] (28)

Correspondingly we have the Fock algebra of the associated shifted or displaced Fock generators,
\[ b(u_0) = a - u_0, \quad b^\dagger(u_0) = a^\dagger - \bar{u}_0, \quad \left[ b(u_0), b^\dagger(u_0) \right] = \mathbb{I}, \] (29)
which are such that,
\[ b(u_0) = \frac{1}{\sqrt{2}} \left( \frac{Q}{\ell_0} + \frac{\ell_0}{\hbar} \bar{P} \right), \quad b^\dagger(u_0) = \frac{1}{\sqrt{2}} \left( \frac{Q}{\ell_0} - \frac{\ell_0}{\hbar} \bar{P} \right), \] (30)
as well as,
\[ \bar{Q} = \frac{1}{\sqrt{2}} \ell_0 \left( b(u_0) + b^\dagger(u_0) \right), \quad \bar{P} = -\frac{i \hbar}{\ell_0 \sqrt{2}} \left( b(u_0) - b^\dagger(u_0) \right), \quad [\bar{Q}, \bar{P}] = i \hbar \mathbb{I}. \] (31)

The correspondence between the displaced Fock algebra and the reference one is best understood by considering the displacement operator\[4\] defined\[6\] in terms of the parameters \( u_0 \) or \((q_0, p_0)\),
\[ D(q_0, p_0) \equiv D(u_0) \equiv e^{u_0 a^\dagger - \bar{u}_0 a} = e^{-\frac{1}{2} |u_0|^2} e^{u_0 a^\dagger} e^{-\bar{u}_0 a}, \] (32)
\[ D(u_0) \equiv D(q_0, p_0) \equiv e^{-\frac{i}{\hbar} q_0 P + \frac{i}{\hbar} p_0 Q} = e^{\frac{\hbar}{2} q_0 p_0} e^{-\frac{i}{\hbar} q_0 P} e^{\frac{i}{\hbar} p_0 Q} = e^{-\frac{i}{\hbar} q_0 p_0} e^{\frac{i}{\hbar} q_0 P} e^{-\frac{i}{\hbar} p_0 Q}, \] (33)
which is a unitary operator defined over Hilbert space,
\[ D^\dagger(u_0) = D^{-1}(u_0) = D(-u_0). \] (34)

Indeed the following identities readily follow, which make explicit the displacement action of the displacement operator \( D(u_0) \) on the different quantities being involved,
\[ b(u_0) = D(u_0) a D^\dagger(u_0) = a - u_0, \quad b^\dagger(u_0) = D(u_0) a^\dagger D^\dagger(u_0) = a^\dagger - \bar{u}_0, \] (35)
\[ \bar{Q} = D(u_0) Q D^\dagger(u_0) = Q - q_0, \quad \bar{P} = D(u_0) P D^\dagger(u_0) = P - p_0, \] (36)
\[ D(u_0) |q\rangle = e^{\frac{\hbar}{2} q_0 p_0} e^{\frac{i}{\hbar} q_0 P} |q + q_0\rangle, \quad D(u_0) |p\rangle = e^{-\frac{\hbar}{2} q_0 p_0} e^{-\frac{i}{\hbar} p_0 Q} |p + p_0\rangle. \] (37)

Consequently the normalised Fock vacuum, \(|\Omega_0(u_0)\rangle\), of the displaced Fock algebra, such that \( b(u_0)|\Omega_0(u_0)\rangle = 0 \) and \( \langle \Omega_0(u_0) | \Omega_0(u_0) \rangle = 1 \), is obtained as being simply the displaced reference Fock vacuum since \( b(u_0)D(u_0) = D(u_0)a \),
\[ |\Omega_0(u_0)\rangle = D(u_0)|\Omega_0\rangle, \quad \langle \Omega_0(u_0) | \Omega_0(u_0) \rangle = 1, \] (38)
\[ b(u_0)|\Omega_0(u_0)\rangle = 0, \quad (a - u_0)|\Omega_0(u_0)\rangle = 0, \quad a|\Omega_0(u_0)\rangle = u_0|\Omega_0(u_0)\rangle. \] (39)

In other words, the Fock vacuum \(|\Omega_0(u_0)\rangle\) of the displaced Fock algebra is a canonical coherent state of the reference Fock vacuum \(|\Omega_0\rangle\). This also implies that all such states \(|\Omega_0(u_0)\rangle\) again saturate not only the SR-UR but also the H-UR with still a vanishing expectation value for the \((Q, P)\) correlator, \(\langle \{ Q, P \} \rangle\), but this time with non-vanishing expectation values for \(Q\) and \(P\) which are specified by the choice for \(u_0\),
\[ \langle \Omega_0(u_0) | Q \Omega_0(u_0) \rangle = q_0, \quad \langle \Omega_0(u_0) | P \Omega_0(u_0) \rangle = p_0, \quad \langle \Omega_0(u_0) | \{ \bar{Q}, \bar{P} \} \Omega_0(u_0) \rangle = 0, \] (40)
while the values for \(\Delta Q\) and \(\Delta P\) remain those of the reference Fock vacuum \(|\Omega_0\rangle\),
\[ \langle \Delta Q \rangle^2 = \frac{1}{2} \ell_0^2, \quad \langle \Delta P \rangle^2 = \frac{1}{2} \frac{\hbar^2}{\ell_0^2}, \quad \langle \Delta Q \rangle^2 \langle \Delta P \rangle^2 = \frac{1}{4} \hbar^2. \] (41)

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\[6\] All Baker-Campbell-Hausdorff (BCH) formulae necessary for this paper are discussed in Appendix B.
As is well known, the coherent states $|\Omega_0(u_0)\rangle$ possess some remarkable properties\cite{4,6}, of which two are worth to be emphasized in our discussion. Even though these states are not linearly independent among themselves as is made explicit by their non-vanishing overlap matrix elements, none of which is vanishing,

$$\langle \Omega_0(u_2)|\Omega_0(u_1)\rangle = e^{-\frac{1}{2}u_2|^2} e^{-\frac{1}{2}u_1|^2} = e^{-\frac{1}{2}(u_2 \bar{a}_1 - \bar{a}_2 u_1)} e^{-\frac{1}{2}|u_2 - u_1|^2}$$

$$= e^{\frac{1}{2\hbar}(q_2 q_1 - p_2 p_1)} e^{-\frac{1}{\hbar}(2\alpha^2)(p_2^2 - p_1^2)}, \quad (42)$$

their linear span over all possible values of the parameter $u_0 \in \mathbb{C}$ encompasses the complete Hilbert space of the system. As a matter of fact this latter result remains valid whatever the choice of normalised reference quantum state on which the displacement operator acts. Thus given an arbitrary state $|\chi_0\rangle$ normalised to unity, $\langle \chi_0|\chi_0\rangle = 1$, consider the states obtained from the action on it of $D(u_0)$ for all possible values of $u_0 \in \mathbb{C}$,

$$|u_0, \chi_0\rangle \equiv D(u_0) |\chi_0\rangle. \quad (43)$$

One then has the following overcompleteness relation in Hilbert space\cite{7},

$$\int_{\mathbb{C}} \frac{du_0 \, d\bar{u}_0}{\pi} |u_0, \chi_0\rangle \langle u_0, \chi_0| = \int_{\mathbb{R}^2} \frac{dq_0 dp_0}{2\pi \hbar} |u_0, \chi_0\rangle \langle u_0, \chi_0| = \mathbb{I}, \quad (44)$$

a result which may readily be established by computing the matrix elements of both terms of this equality in the $Q$ eigenstate basis, for instance. In particular, by choosing for $|\chi_0\rangle$ the reference Fock vacuum $|\Omega_0\rangle$, one obtains the overcompleteness relation for the displaced Fock vacua $|\Omega_0(u_0)\rangle$,

$$\int_{\mathbb{C}} \frac{du_0 \, d\bar{u}_0}{\pi} |\Omega_0(u_0)\rangle \langle \Omega_0(u_0)| = \int_{\mathbb{R}^2} \frac{dq_0 dp_0}{2\pi \hbar} |\Omega_0(u_0)\rangle \langle \Omega_0(u_0)| = \mathbb{I}. \quad (45)$$

This specific result will be shown to extend to all saturating states of the SR-UR.

Another remarkable property of the states $|\Omega_0(u_0)\rangle$ which extends to all saturating states of the SR-UR is the following. Any finite order polynomial in the Heisenberg observables $Q$ and $P$ possesses a diagonal kernel integral representation in terms of the states $|\Omega_0(u_0)\rangle$, a result which extends the above overcompleteness relation valid specifically for the unit operator. Let us point out however, that this property applies specifically for the states $|\Omega_0(u_0)\rangle$ constructed out of the reference Fock vacuum $|\Omega_0\rangle$. Generically, it does not apply\cite{8} for other choices of reference state $|\chi_0\rangle$.

To establish such a result, first consider a general finite order polynomial in the operators $Q$ and $P$. Such a composite operator may always be brought into the form of a finite sum of normal ordered monomials relative to the reference Fock algebra $(a, a^\dagger)$. A generating function of such normal ordered monomials is provided by the operator $\exp(aa^\dagger) \exp(-\bar{a}a)$ with $\alpha$ and $(-\bar{\alpha} = \alpha^*)$ as independent generating parameters. Using the above overcompleteness relation and the fact that $a|\Omega_0(u_0)\rangle = u_0|\Omega_0(u_0)\rangle$, this generating operator may be given the following integral representation (see also (175) in Appendix B),

$$e^{\alpha a^\dagger} e^{-\bar{\alpha} a} = e^{\alpha^2} e^{-\bar{\alpha} a} e^{\alpha a^\dagger}$$

$$= \int_{\mathbb{C}} \frac{du_0 \, d\bar{u}_0}{\pi} |\Omega_0(u_0)\rangle \langle \Omega_0(u_0)| e^{\alpha a^\dagger}$$

$$= \int_{\mathbb{C}} \frac{du_0 \, d\bar{u}_0}{\pi} |\Omega_0(u_0)\rangle \left( e^{\alpha^2} e^{-\bar{\alpha} u_0} e^{\alpha \bar{u}_0} \right) \langle \Omega_0(u_0)|. \quad (46)$$

However the product of exponential factors appearing inside this integral is directly related to the diagonal matrix elements of the same operator for the coherent states $|\Omega_0(u_0)\rangle$,

$$\langle \Omega_0(u_0)| e^{\alpha a^\dagger} e^{-\bar{\alpha} a} |\Omega_0(u_0)\rangle = e^{\alpha \bar{u}_0} e^{-\bar{\alpha} u_0}, \quad e^{-\bar{\alpha} u_0} e^{\alpha \bar{u}_0} e^{-\bar{\alpha} u_0} = e^{\alpha \bar{u}_0} e^{-\bar{\alpha} u_0}, \quad (47)$$

\footnote{With $du_0 d\bar{u}_0 \equiv dRe u_0 \, dIm u_0$.}

\footnote{Unless of course, one considers a reference state which itself is again the Fock vacuum for some other Fock algebra constructed out of the Heisenberg algebra of the observables $Q$ and $P$, as is the case with the squeezed quantum states to be identified in Section 4.}
leading to the final diagonal kernel integral representation of the generating operator,
\[ e^{\alpha \hat{a}^\dagger} e^{-\alpha \hat{a}} = \int_{\mathcal{C}} \frac{du_0 du_0^*}{\pi} |\Omega_0(u_0)\rangle \left( e^{-\partial_{u_0} \partial_{u_0^*}} \langle \Omega_0(u_0)| e^{\alpha \hat{a}^\dagger} e^{-\alpha \hat{a}} |\Omega_0(u_0)\rangle \right) \langle \Omega_0(u_0)|. \] (48)

Therefore any composite operator, \( A \), which is a finite order polynomial in the observables \( Q \) and \( P \) possesses the following diagonal kernel integral representation over the states \( |\Omega_0(u_0)\rangle \),
\[ A = \int_{\mathcal{C}} \frac{du_0 du_0^*}{\pi} |\Omega_0(u_0)\rangle a(u_0, u_0^*) \langle \Omega_0(u_0)|, \] (49)
where the diagonal kernel \( a(u_0, u_0^*) \) is constructed as follows out of the diagonal matrix elements of \( A \) in the states \( |\Omega_0(u_0)\rangle \),
\[ a(u_0, u_0^*) = e^{-\partial_{u_0} \partial_{u_0^*}} A(u_0, u_0^*), \quad A(u_0, u_0^*) = \langle \Omega_0(u_0)|A|\Omega_0(u_0)\rangle. \] (50)

In terms of the parameters \((q_0, p_0)\), the same results are expressed as, with \(|\Omega_0(q_0, p_0)\rangle \equiv |\Omega_0(u_0)\rangle \),
\[ A = \int_{\mathbb{R}^2} \frac{dq_0 dp_0}{2\pi \hbar} |\Omega_0(q_0, p_0)\rangle a(q_0, p_0) \langle \Omega_0(q_0, p_0)|, \] (51)
where
\[ a(q_0, p_0) = \exp \left( -\frac{1}{2} \ell_0^2 q_0^2 - \frac{1}{2} \frac{\hbar^2}{\ell_0^2} p_0^2 \right) \langle \Omega_0(q_0, p_0)|A|\Omega_0(q_0, p_0)\rangle. \] (52)

4 Fock Algebras for the Saturating Quantum States

4.1 Reversible parametrisation packages

Let us now address the quantum state content, \( |\psi_0\rangle \), of the defining relation (L3) for the saturated SR-UR of the Heisenberg algebra of quantum observables \((Q, P)\), namely,
\[ [(Q - q_0) - \lambda_0 (P - p_0)] |\psi_0\rangle = 0, \] (53)
with
\[ \lambda_0 = \frac{1}{(\Delta P)^2} \left( \frac{1}{2} \langle \{Q, P\} \rangle - \frac{1}{2} i \hbar \right) = \frac{1}{2} \langle \{Q, P\} \rangle + \frac{\hbar}{2 i}. \] (54)

Besides the complex variable \( u_0 \) already representing the real expectation values \((q_0, p_0)\) given the physical scale \( \ell_0 \), let us now introduce the angular parameter \( \varphi \) related to the possible \((Q, P)\) correlation, such that \(-\pi/2 \leq \varphi \leq +\pi/2\) and defined by,
\[ \cos \varphi = \frac{1}{\sqrt{1 + \frac{\hbar}{\Delta P}(\{Q, P\})^2}}, \quad \sin \varphi = \frac{\hbar}{\sqrt{1 + \frac{\hbar}{\Delta P}(\{Q, P\})^2}}, \quad \tan \varphi = \frac{\hbar}{\Delta P}. \] (55)

Note that the saturated SR-UR is then expressed simply as,
\[ \Delta Q \Delta P = \frac{\hbar}{2} \frac{1}{\cos \varphi} = \frac{\hbar}{2} \sqrt{1 + \tan^2 \varphi} \geq \frac{\hbar}{2}, \quad \left( \frac{\Delta Q}{\ell_0} \right) \left( \frac{\Delta P}{\hbar} \right) = \frac{1}{2 \cos \varphi} = \frac{\hbar}{2} \sqrt{1 + \tan^2 \varphi} \geq \frac{\hbar}{2}, \] (56)
while the parameter \( \lambda_0 \) simplifies to,
\[ -\lambda_0 = \frac{i \hbar}{2 (\Delta P)^2} \frac{1}{\cos \varphi} e^{i \varphi} = i \frac{\Delta Q}{\Delta P} e^{i \varphi}, \] (57)
the latter expression thus also displaying explicitly the remaining fourth and last real (and positive) independent free parameter labelling the saturating states, namely the ratio \( \Delta Q/\Delta P \). In particular we then have for the operator which annihilates the saturating quantum states,
\[ \frac{1}{\Delta Q} [(Q - q_0) - \lambda_0 (P - p_0)] = \left( \frac{Q - q_0}{\Delta Q} + i e^{i \varphi} \frac{P - p_0}{\Delta P} \right) = \frac{1}{\sqrt{2}} \left( \frac{\ell_0}{\Delta Q} + \frac{i \hbar e^{i \varphi}}{\ell_0 \Delta P} \right) b(u_0) + \frac{1}{\sqrt{2}} \left( \frac{\ell_0}{\Delta Q} - \frac{\hbar e^{i \varphi}}{\ell_0 \Delta P} \right) b^\dagger(u_0). \] (58)
Consequently, when having in mind the reference Fock algebra \((a,a^\dagger)\) and in order to account for this last variable, \(\Delta Q/\Delta P\) or \(\Delta P/\Delta Q\), it proves useful to consider the following further definitions of properly normalised quantities, with \(\rho_\pm \geq 0\) and \(-\pi \leq \theta_\pm \leq +\pi\),

\[
\rho_\pm e^{i\theta_\pm} = \frac{1}{2\sqrt{2}\cos \varphi} \left( \frac{\ell_0}{\Delta Q} \pm \frac{\hbar e^{i\varphi}}{\ell_0 \Delta P} \right) = \frac{1}{\sqrt{2}} \left( \left( \frac{\ell_0}{\hbar} \Delta P \right) \pm \left( \frac{\Delta Q}{\ell_0} \right) e^{i\varphi} \right),
\]

so that,

\[
\cos \theta_\pm = \frac{1}{\rho_\pm \sqrt{2}} \left[ \left( \frac{\ell_0}{\hbar} \Delta P \right) \pm \cos \varphi \left( \frac{\Delta Q}{\ell_0} \right) \right], \quad \sin \theta_\pm = \frac{1}{\rho_\pm \sqrt{2}} \sin \varphi \left( \frac{\Delta Q}{\ell_0} \right),
\]

\[
\tan \theta_\pm = \frac{\pm \sin \varphi \left( \frac{\Delta Q}{\ell_0} \right)}{\cos \varphi \left( \frac{\Delta Q}{\ell_0} \right)} = \frac{\sin \varphi}{\cos \varphi \pm \frac{\ell_0}{\hbar} \frac{\Delta P}{\Delta Q}}.
\]

Since \(\rho_+^2 - \rho_-^2 = 1\), let us introduce finally a real parameter \(r\) such that \(0 \leq r < +\infty\), defined by,

\[
\cosh r = \rho_+ = \frac{1}{\sqrt{2}} \sqrt{\left( \frac{\ell_0}{\hbar} \Delta P \right)^2 + \left( \frac{\Delta Q}{\ell_0} \right)^2 + 1} \geq 1,
\]

\[
\sinh r = \rho_- = \frac{1}{\sqrt{2}} \sqrt{\left( \frac{\ell_0}{\hbar} \Delta P \right)^2 + \left( \frac{\Delta Q}{\ell_0} \right)^2 - 1} \geq 0.
\]

In terms of these quantities, the following notations prove to be useful later on as well,

\[
\zeta = e^{i\theta} \tanh r, \quad z = r e^{i\theta}, \quad e^{i\theta} = -e^{i(\theta_--\theta_+)} = e^{i(\theta_--\theta_+\pm \pi)}.
\]

Note that the complex variable \(z\) takes a priori all its values in the entire complex plane, while the complex variable \(\zeta\) takes all its values inside the unit disk in the complex plane.

Hence given the physical scale \(\ell_0\) and any quantum state saturating the SR-UR, the associated real quantities \(q_0, p_0, \langle \{Q, P\} \rangle\), \(\Delta Q > 0\) and \(\Delta P > 0\), of which four are independent because of the property \((\Delta Q)^2(\Delta P)^2 = \hbar^2 (1 + \langle \{Q, P\} \rangle^2 / \hbar^2) / 4\), determine in a unique manner through the above definitions the two independent complex quantities \(u_0\) and \(z\) in the complex plane. These two complex variables, \(u_0\) and \(z\), thus label all SR-UR saturating quantum states.

Conversely, given the two complex variables \(u_0\) and \(z\) taking any values in the complex plane, in terms of the physical scale \(\ell_0\) there corresponds to these a SR-UR saturating quantum state, \(|\psi_0\rangle\), whose relevant expectation values are constructed as follows. On the one hand for the Heisenberg observables \(Q\) and \(P\), their expectation values are

\[
q_0 = \frac{1}{\sqrt{2}} \ell_0 (u_0 + \bar{u}_0), \quad p_0 = -\frac{i\hbar}{\ell_0 \sqrt{2}} (u_0 - \bar{u}_0),
\]

while on the other hand their uncertainties are such that,

\[
\left( \frac{\Delta Q}{\ell_0} \right)^2 + \left( \frac{\ell_0}{\hbar} \Delta P \right)^2 = \cosh 2r \geq 1, \quad \left( \frac{\Delta Q}{\ell_0} \right)^2 - \left( \frac{\ell_0}{\hbar} \Delta P \right)^2 = \cos \theta \sinh 2r,
\]

namely

\[
\left( \frac{\Delta Q}{\ell_0} \right)^2 = \frac{1}{2} \left( \cosh 2r + \cos \theta \sinh 2r \right), \quad \left( \frac{\ell_0}{\hbar} \Delta P \right)^2 = \frac{1}{2} \left( \cosh 2r - \cos \theta \sinh 2r \right),
\]

with thus the saturated SR-UR expressed as

\[
(\Delta Q)^2 (\Delta P)^2 = \frac{1}{4} \hbar^2 \left( 1 + \sin^2 \theta \sinh^2 2r \right).
\]
Furthermore the \((Q, P)\) correlation of these states \(|\psi_0\rangle\) is then determined as,

\[
\frac{1}{\hbar}\langle \{\bar{Q}, \bar{P}\} \rangle = \tan \varphi = \sin \theta \sinh(2r),
\]

with

\[
\cos \varphi = \frac{1}{\sqrt{1 + \sin^2 \theta \sinh^2(2r)}}, \quad \sin \varphi = \frac{\sin \theta \sinh(2r)}{\sqrt{1 + \sin^2 \theta \sinh^2(2r)}}.
\]

The particular case of \((Q, P)\) uncorrelated saturating states is worth a separate discussion. This situation, characterised by the vanishing correlation \(\langle \{\bar{Q}, \bar{P}\} \rangle = 0\), corresponds to the phase value \(\varphi = 0\). One then finds

\[
\cos \theta_\pm = \text{sgn}\left(\frac{\ell_0}{\hbar} \Delta P \pm \frac{\Delta Q}{\ell_0}\right), \quad \sin \theta_\pm = 0.
\]

Consequently in such a case \(\theta_+ = 0\), while the value for \(\theta_-\) is determined as follows,

\[
\text{if} \quad \frac{\ell_0}{\hbar} \Delta P - \frac{\Delta Q}{\ell_0} > 0: \quad \theta_- = 0; \quad \text{if} \quad \frac{\ell_0}{\hbar} \Delta P - \frac{\Delta Q}{\ell_0} < 0: \quad \theta_- = \pm \pi,
\]

leading to the value for \(\theta \equiv \theta_- - \theta_+ (\text{mod } 2\pi)\) given as,

\[
\text{if} \quad \frac{\ell_0}{\hbar} \Delta P - \frac{\Delta Q}{\ell_0} > 0: \quad \theta = \pm \pi (\text{mod } 2\pi); \quad \text{if} \quad \frac{\ell_0}{\hbar} \Delta P - \frac{\Delta Q}{\ell_0} < 0: \quad \theta = 0 (\text{mod } 2\pi).
\]

Nonetheless the value for \(r\) remains arbitrary,

\[
\cosh r = \frac{1}{\sqrt{2}} \left(\frac{\ell_0}{\hbar} \Delta P + \frac{\Delta Q}{\ell_0}\right), \quad \sinh r = \frac{1}{\sqrt{2}} \left|\frac{\ell_0}{\hbar} \Delta P - \frac{\Delta Q}{\ell_0}\right|.
\]

On the other hand, in terms of the \((u_0, z)\) parametrisation \((Q, P)\) uncorrelated saturating states correspond to either one of the two values \(\theta = 0, \pm \pi (\text{mod } 2\pi)\), thus leading to the quantities,

\[
\text{if} \quad \theta = 0: \quad \frac{\Delta Q}{\ell_0} = \frac{1}{\sqrt{2}} e^r, \quad \frac{\ell_0}{\hbar} \Delta P = \frac{1}{\sqrt{2}} e^{-r}, \quad \frac{\ell_0}{\hbar} \Delta P - \frac{\Delta Q}{\ell_0} = -\sqrt{2} \sinh r < 0;
\]

\[
\text{if} \quad \theta = \pm \pi: \quad \frac{\Delta Q}{\ell_0} = \frac{1}{\sqrt{2}} e^{-r}, \quad \frac{\ell_0}{\hbar} \Delta P = \frac{1}{\sqrt{2}} e^r, \quad \frac{\ell_0}{\hbar} \Delta P - \frac{\Delta Q}{\ell_0} = \sqrt{2} \sinh r > 0.
\]

Of course, these results are consistent with those derived above.

Given the latter expressions for \(\Delta Q/\ell_0\) and \(\ell_0 \Delta P/\hbar\), it is clear why the parameter \(r \geq 0\) is known as the squeezing parameter, while all such \((Q, P)\) uncorrelated states then all saturate the H-UR rather than the SR-UR whatever the value for \(r\). In addition, as the value for the correlation parameter \(\theta\) or \(\varphi\) varies away from \(\theta = 0, \pm \pi (\text{mod } 2\pi)\) or \(\varphi = 0 (\pm \pi/2 \leq \varphi \leq \pi/2)\), respectively, both these quantities remain limited within a finite interval whose width is set by the squeezing parameter \(r\),

\[
\frac{1}{\sqrt{2}} e^{-r} \leq \frac{\Delta Q}{\ell_0}, \quad \frac{\ell_0}{\hbar} \Delta P \leq \frac{1}{\sqrt{2}} e^r.
\]

In particular when \(r = 0\), corresponding to \(z = 0\) and thus to an irrelevant value for \(\theta\), one has the specific situation that \(\Delta Q/\ell_0 = 1/\sqrt{2} = \ell_0 \Delta P/\hbar\) in addition to the fact that \(\langle \{\bar{Q}, \bar{P}\} \rangle = 0\), namely the fact that \(\varphi = 0\), thereby leaving as only remaining free parameter the complex variable \(u_0\) for these states which saturate the H-UR rather than the SR-UR. This situation corresponds exactly to the displaced Fock vacua and coherent states \(|\Omega_0(u_0)\rangle = D(u_0)|\Omega_0\rangle\) constructed in Sect.3 out of the reference Fock vacuum \(|\Omega_0\rangle\).

Consequently in this paper those quantum states that saturate the SR-UR are referred to generally as “squeezed states” (or squeezed coherent states, since they turn out to correspond to coherent states as well, as discussed hereafter). Note that if the parameter \(z\) is purely real, whether positive or negative (thus corresponding to \(\theta = 0\) or \(\theta = \pm \pi (\text{mod } 2\pi)\), respectively), such
squeezed states have no \((Q, P)\) correlation. While if \(z\) is strictly complex with \(\theta \neq 0, \pm \pi \) \((\text{mod} \ 2\pi)\) those squeezed states have a non-vanishing \((Q, P)\) correlation. If the distinction needs to be emphasized, in this paper these situations will be referred to as “uncorrelated” and “correlated” squeezed states, respectively. Note that uncorrelated squeezed states saturate the SR-UR, while correlated squeezed states saturate the SR-UR but not the H-UR. In the literature some authors reserve the term “squeezed states” specifically to uncorrelated squeezed states thus with \(z\) strictly real and which saturate the H-UR, while to emphasize the distinction correlated squeezed states are then referred to as “intelligent states” which saturate the SR-UR \(^{11} \) \(^{12} \) \(^{13} \). However given the considerations of this paper based on the SR-UR leading to this general class of squeezed states, whether \((Q, P)\) correlated or not it seems preferable to refer to all of these as squeezed states. Furthermore when time evolution of such states is considered \(^{10} \) the value for \(\theta\) certainly evolves in time, thereby generating correlated squeezed states out of what could have been initially uncorrelated ones.

Another reason why it is legitimate to consider on a same footing correlated and uncorrelated squeezed states is the following fact. The general Schrödinger-Robertson uncertainty defined as

\[
\langle A^2 \rangle \langle B^2 \rangle - \langle AB \rangle \langle BA \rangle \geq 0,
\]

\[
(\Delta A)^2 (\Delta B)^2 - \left( \frac{1}{2} \langle \{A, B\} \rangle \right)^2 \geq \left( \frac{1}{2} \langle (-i) [A, B] \rangle \right)^2,
\]

which in the case of the Heisenberg observables reads,

\[
\langle \hat{Q}^2 \rangle \langle \hat{P}^2 \rangle - \langle \hat{Q} \hat{P} \rangle \langle \hat{P} \hat{Q} \rangle \geq 0,
\]

\[
(\Delta Q)^2 (\Delta P)^2 - \hbar^2 \left( \frac{1}{2\hbar} \langle [\hat{Q}, \hat{P}] \rangle \right)^2 \geq \frac{1}{4} \hbar^2.
\]

General squeezed states thus minimize the l.h.s. of these inequalities, whether correlated or uncorrelated, namely whether the parameter \(z\) is strictly complex or strictly real, respectively.

4.2 Correlated squeezed Fock algebras and their vacua

Coming back now to the operator \((5\mathbf{S})\) which annihilates the saturating states, note that it may be expressed in the form

\[
2 \cos \varphi e^{i\theta} \left( \cosh r b(u_0) - e^{i\theta} \sinh r b^\dagger(u_0) \right) = 2 \cos \varphi e^{i\theta} \cosh r \left( b(u_0) - \zeta b^\dagger(u_0) \right).
\]

Consequently let us now introduce the correlated displaced squeezed Fock algebra generators defined as \(^{11} \)

\[
b(z, u_0) = \cosh r b(u_0) - e^{i\theta} \sinh r b^\dagger(u_0) = \cosh r (a - u_0) - e^{i\theta} \sinh r \left( a^\dagger - \bar{u}_0 \right),
\]

\[
b^\dagger(z, u_0) = -e^{-i\theta} \sinh r b(u_0) + \cosh r b^\dagger(u_0) = -e^{-i\theta} \sinh r (a - u_0) + \cosh r \left( a^\dagger - \bar{u}_0 \right),
\]

which are such that

\[
[b(z, u_0), b^\dagger(z, u_0)] = \mathbb{I}.
\]

while a specific choice of overall phase factor has been effected for \(b(z, u_0)\) and \(b^\dagger(z, u_0)\), consistent with the fact that \(b(0, u_0) = b(u_0)\) and \(b^\dagger(0, u_0) = b^\dagger(u_0)\).

Obviously the SR-UR saturating or squeezed quantum states are the normalised Fock vacua of these displaced squeezed Fock algebras \((b(z, u_0), b^\dagger(z, u_0))\). Let us denote these Fock states as

\(^{10} \) In the simple situation of the harmonic oscillator of mass \(m\) and angular frequency \(\omega\), and by choosing then \(\ell_0 = \sqrt{\hbar/(m\omega)}\), all these general squeezed states evolve coherently into one another with parameters \(u_0\) and \(z\) whose time dependence is given by \(u_0(t) = u_0 e^{i\omega t}\) and \(z(t) = z e^{2i\omega t}\).

\(^{11} \) Note the slight abuse of notation which is without consequence, which consists in denoting as a dependency on \(z\) a dependence of \((b(z, u_0), b^\dagger(z, u_0))\) which is in fact separate in \(r\) and in \(e^{i\theta}\) while \(z = re^{i\theta}\).
and thus also Fock vacua

\[ |\Omega_z(u_0)\rangle = D(u_0) |\Omega_z\rangle, \quad b(z, u_0) |\Omega_z(u_0)\rangle = D(u_0) a(z) |\Omega_z\rangle = 0. \tag{86} \]

Note that from the last of these two identities it follows that general squeezed states \(|\Omega_z(u_0)\rangle\) with \(u_0 \neq 0\) are also coherent states of the squeezed \((a(z), a^\dagger(z))\) Fock algebras. Indeed by introducing the quantities

\[
u_0(z) \equiv \cosh r u_0 - e^{i\theta} \sinh r \bar{u}_0 = \cosh r (u_0 - \zeta \bar{u}_0), \quad \nu_0(0) = u_0, \\
\bar{u}_0(z) \equiv -e^{-i\theta} \sinh r u_0 + \cosh r \bar{u}_0 = \cosh r (\bar{u}_0 - \bar{\zeta} u_0), \quad \bar{u}_0(0) = \bar{u}_0, \tag{87} \]

one has,

\[ a(z) |\Omega_z(u_0)\rangle = u_0(z) |\Omega_z(u_0)\rangle, \tag{88} \]

as follows also from the identity,

\[ b(z, u_0) = a(z) - u_0(z), \tag{89} \]

which shows that the \((b(z, u_0), b^\dagger(z, u_0))\) Fock algebras are shifted versions of the \((a(z), a^\dagger(z))\) Fock algebras. As a matter of fact it may readily be checked that one has, independently from the value for \(z\),

\[ u_0(z)a^\dagger(z) - \bar{u}_0(z)a(z) = u_0 a^\dagger - \bar{u}_0 a, \tag{90} \]

so that,

\[ D(u_0) = e^{-\frac{i}{\hbar} \nu_0 P + \frac{1}{\hbar} \bar{\nu}_0 Q} = e^{u_0 a^\dagger - \bar{u}_0 a} a^\dagger(z) - \bar{u}_0(z) a(z), \tag{91} \]

a property which thus explains the above results.

Inverting the Bogoliubov transformations \([84]\), one finds,

\[ a = \cosh r a(z) + e^{i\theta} \sinh r a^\dagger(z), \quad a^\dagger = e^{-i\theta} \sinh r a(z) + \cosh r a^\dagger(z), \tag{92} \]

hence likewise for the variables \(u_0\) and \(u_0(z)\),

\[ u_0 = \cosh r u_0(z) + e^{i\theta} \sinh r \bar{u}_0(z), \quad \bar{u}_0 = e^{-i\theta} \sinh r u_0(z) + \cosh r \bar{u}_0(z). \tag{93} \]

In terms of the Heisenberg observables, these definitions translate into,

\[
\begin{align*}
  a(z) & = \frac{1}{\sqrt{2}} \left( \cosh r - e^{i\theta} \sinh r \right) \frac{Q}{\ell_0} + \frac{i}{\sqrt{2}} \left( \cosh r + e^{i\theta} \sinh r \right) \frac{\ell_0}{\hbar} P, \\
  a^\dagger(z) & = \frac{1}{\sqrt{2}} \left( \cosh r - e^{-i\theta} \sinh r \right) \frac{Q}{\ell_0} - \frac{i}{\sqrt{2}} \left( \cosh r + e^{-i\theta} \sinh r \right) \frac{\ell_0}{\hbar} P, \tag{94}
\end{align*}
\]

\[^{12}\text{In the same way that } b(u_0)|\Omega_0(u_0)\rangle = 0, a_i|\Omega_0(u_0)\rangle = u_0|\Omega_0(u_0)\rangle \text{ and } b(u_0) = a - u_0, \text{ corresponding to the case with } z = 0.\]
with the inverse relations,
\[
\frac{Q}{\ell_0} = \frac{1}{\sqrt{2}} \left( \cosh r + e^{-i\theta} \sinh r \right) a(z) + \frac{1}{\sqrt{2}} \left( \cosh r + e^{i\theta} \sinh r \right) a^\dagger(z),
\]
\[
\frac{\ell_0 P}{\hbar} = -\frac{i}{\sqrt{2}} \left( \cosh r - e^{-i\theta} \sinh r \right) a(z) + \frac{i}{\sqrt{2}} \left( \cosh r - e^{i\theta} \sinh r \right) a^\dagger(z),
\]
so that for the corresponding parameters \(u_0(z), \bar{u}_0(z), q_0\) and \(p_0\),
\[
u_0(z) = \frac{1}{\sqrt{2}} \left( \cosh r - e^{i\theta} \sinh r \right) \frac{q_0}{\ell_0} + \frac{i}{\sqrt{2}} \left( \cosh r + e^{i\theta} \sinh r \right) \frac{\ell_0}{\hbar} p_0,
\]
\[
u_\bar{0}(z) = \frac{1}{\sqrt{2}} \left( \cosh r - e^{-i\theta} \sinh r \right) \frac{q_0}{\ell_0} - \frac{i}{\sqrt{2}} \left( \cosh r + e^{-i\theta} \sinh r \right) \frac{\ell_0}{\hbar} p_0,
\]
while
\[
u_0(z) = \frac{1}{\sqrt{2}} \left( \cosh r + e^{-i\theta} \sinh r \right) \frac{q_0}{\ell_0} - \frac{i}{\sqrt{2}} \left( \cosh r + e^{i\theta} \sinh r \right) \frac{\ell_0}{\hbar} p_0,
\]
so that for the corresponding parameters \(u_0(z), \bar{u}_0(z), q_0\) and \(p_0\),
\[
u_0(z) = \frac{1}{\sqrt{2}} \left( \cosh r - e^{i\theta} \sinh r \right) \frac{q_0}{\ell_0} + \frac{i}{\sqrt{2}} \left( \cosh r + e^{i\theta} \sinh r \right) \frac{\ell_0}{\hbar} p_0,
\]
\[
u_\bar{0}(z) = \frac{1}{\sqrt{2}} \left( \cosh r - e^{-i\theta} \sinh r \right) \frac{q_0}{\ell_0} - \frac{i}{\sqrt{2}} \left( \cosh r + e^{-i\theta} \sinh r \right) \frac{\ell_0}{\hbar} p_0.
\]

4.3 Squeezed Fock vacua and SR-UR saturating quantum states

Having understood that the SR-UR saturating states are the displaced coherent states of the squeezed Fock vacua \(|\Omega_z\rangle\), namely \(|\Omega_z(u_0)\rangle = D(u_0)|\Omega_z\rangle\), let us finally turn to the construction of the latter which are characterised by the condition that \(a(z)|\Omega_z\rangle = 0\) with
\[
\alpha(z) = \cosh r \alpha - e^{i\theta} \sinh r \alpha^\dagger.
\]

Given that the corresponding Bogoliubov transformation, linear in both generators of the reference Fock algebra \((a, a^\dagger)\), is unitary, necessarily it corresponds to a unitary operator acting on Hilbert space of the following form, defined up to an arbitrary global phase factor set here to a trivial value,
\[
S(\alpha) = \exp \left( \frac{1}{2} \alpha a^\dagger a - \frac{1}{2} \bar{\alpha} a^2 \right), \quad \alpha \in \mathbb{C},
\]
\[
S(\alpha) a S(\alpha)^\dagger = \cosh \rho a - e^{i\phi} \sinh \rho a^\dagger, \quad S(\alpha) a^\dagger S(\alpha)^\dagger = -e^{-i\phi} \sinh \rho a + \cosh \rho a^\dagger,
\]
the parameter \(\alpha\) being represented as \(\alpha = \rho e^{i\phi}\) with \(\rho \geq 0\).

Consequently by choosing \(\alpha = z\), one finds for the squeezed Fock algebras \((a(z), a^\dagger(z))\),
\[
a(z) = S(z) a S(z)^\dagger, \quad a^\dagger(z) = S(z) a^\dagger S(z)^\dagger, \quad \left[a(z), a^\dagger(z)\right] = I,
\]
while their normalised squeezed Fock vacua \(|\Omega_z\rangle\) are constructed as follows out of the reference Fock vacuum \(|\Omega_0\rangle\), since \(a(z)|\Omega_0\rangle = S(z)|\Omega_0\rangle = 0\),
\[
\langle \Omega_z|\Omega_z\rangle = 1.
\]

Given the displacement operator \(D(u_0)\), let us also introduce the operators
\[
S(z, u_0) \equiv \exp \left( \frac{1}{2} z(a^\dagger - \bar{u}_0)^2 - \frac{1}{2} \bar{z}(a - u_0)^2 \right),
\]
which obey the following properties\textsuperscript{13},

\[ D(u_0) S(z) = S(z, u_0) D(u_0), \quad S(z, u_0) = D(u_0) S(z) D^\dagger(u_0). \] (105)

Hence finally all normalised quantum states that saturate the Schrödinger-Robertson uncertainty relation for the Heisenberg observables \((Q, P)\) are given by the algebraic representation

\[ |\psi_0(z, u_0)\rangle \equiv |\Omega_z(u_0)\rangle = e^{u_0 a^{\dagger} - \bar{u}_0 a} e^{\frac{1}{2}z a^{\dagger}2 - \frac{1}{2}z^2 a^2} |\Omega_0\rangle = D(u_0) S(z) |\Omega_0\rangle = S(z, u_0) D(u_0) |\Omega_0\rangle = S(z) D(u_0(z)) |\Omega_0\rangle, \] (106)

\(|\Omega_0\rangle\) being the normalised Fock vacuum of the reference Fock algebra \((a, a^\dagger)\). Note that this construction also fixes the absolute phase factor for all these saturating states, relative to the choice of phase made for the state \(|\Omega_0\rangle\). The overall phase factor for the wave function of the saturating states, \(\psi_0(q; z, u_0) \equiv \langle q | \psi_0(z, u_0) \rangle\) in Eq.(106), will be determined accordingly in Sect.\textsuperscript{5}

5 Overcompleteness and Kernel Representation

In Sect.\textsuperscript{3} two remarkable properties of the canonical coherent states, \(|\Omega_0(u_0)\rangle\), were emphasized. Let us now consider how these properties extend to the general squeezed coherent states \(|\Omega_z(u_0)\rangle\), beginning with the overcompleteness property.

As established in Eq.(\textsuperscript{14}), given any normalised reference state \(|\chi_0\rangle\), one has the following representation of the unit operator on the considered Hilbert space,

\[ \int_C \frac{du_0}{\pi} D(u_0) |\chi_0\rangle \langle \chi_0| D^\dagger(u_0) = \int_{\mathbb{R}^2} \frac{dq_0 dp_0}{2\pi \hbar} D(q_0,p_0) |\chi_0\rangle \langle \chi_0| D^\dagger(q_0,p_0) = \mathbb{I}. \] (107)

Hence by choosing \(|\chi_0\rangle = |\Omega_z\rangle\) so that \(D(u_0)|\chi_0\rangle = |\Omega_z(u_0)\rangle\), given any fixed value for \(z \in \mathbb{C}\) one has the overcompleteness property for the SR-UR saturating states,

\[ \int_C \frac{du_0}{\pi} |\Omega_z(u_0)\rangle \langle \Omega_z(u_0)| = \int_{\mathbb{R}^2} \frac{dq_0 dp_0}{2\pi \hbar} |\Omega_z(u_0)\rangle \langle \Omega_z(u_0)| = \mathbb{I}, \] (108)

which thus generalises the overcompleteness relation in Eq.(\textsuperscript{15}) (which corresponds to the case \(z = 0\)). Note well however, that this identity involves an integral over the entire complex plane only for the complex variable \(u_0\), independently of the value for \(z\) which is fixed but arbitrary. Since the states \(|\Omega_z\rangle = S(z)|\Omega_0\rangle\) involve those Fock states built from the reference Fock algebra \((a, a^\dagger)\) which include only an even number of the corresponding \(a^\dagger\) Fock quanta and thereby span only half the Hilbert space under consideration, a similar identity involving rather such an integral only over the complex plane of \(z\) values but with a fixed value now for \(u_0\), cannot apply. However, given any arbitrary normalisable and normalised integration measure \(\mu(z, \bar{z})\) over the complex plane for all values of \(z\), provides still for a generalised form of overcompleteness relation involving then all the saturating states,

\[ \int_{\mathbb{C}^2} \frac{du_0}{\pi} \frac{dz}{\pi} \frac{d\bar{z}}{\pi} \mu(z, \bar{z}) |\Omega_z(u_0)\rangle \langle \Omega_z(u_0)| = \mathbb{I}, \quad \int_C \frac{dz}{\pi} \frac{d\bar{z}}{\pi} \mu(z, \bar{z}) = 1. \] (109)

Let us now consider the possibility of a diagonal kernel integral representation of operators. Given a fixed but arbitrary value for \(z\), any finite order polynomial in the observables \(Q\) and \(P\) may be written as a linear combination of monomials which are expressed in normal ordered form with respect to the Fock algebra \((a(z), a^\dagger(z))\). Let us thus consider again the generating function of such normal ordered monomials, namely the operator \(\exp(\alpha a^\dagger(z))\exp(-\bar{\alpha}a(z))\) with

\textsuperscript{13}Note that because of (108), one also has the identity \(D(u_0) S(z) = S(z) D(u_0(z))\) with \(u_0(z) = \cosh r(u_0 - \zeta u_0)\). The author thanks Victor Massart for a remark on this point.
generating parameters $\alpha \in \mathbb{C}$ and $(-\bar{\alpha} = -\alpha^*)$. Following the same line of analysis as in Sect. 3, one has,

$$e^{\alpha a^\dagger} e^{-\bar{\alpha} a} = e^{\alpha a^\dagger} e^{-\bar{\alpha} a} e^{\alpha a^\dagger}$$

$$= \int_C \frac{du_0 \, d\bar{u}_0}{\pi} e^{\alpha a^\dagger} e^{-\bar{\alpha} a} \, \langle \Omega_z(u_0) \rangle \langle \Omega_z(u_0) | e^{\alpha a^\dagger}$$

$$= \int_C \frac{du_0 \, d\bar{u}_0}{\pi} \langle \Omega_z(u_0) \rangle e^{\alpha a^\dagger} e^{-\bar{\alpha} u_0(z)} e^{\alpha a_0(z)} \langle \Omega_z(u_0) | \quad (110)$$

$$= \int_C \frac{du_0 \, d\bar{u}_0}{\pi} \langle \Omega_z(u_0) \rangle \left[ e^{-\partial_{u_0(z)} \partial_{\bar{u}_0(z)}} \langle \Omega_z(u_0) | e^{\alpha a^\dagger} e^{-\bar{\alpha} a} | \Omega_z(u_0) \rangle \right] \langle \Omega_z(u_0) |.$$}

Consequently, any finite order polynomial in the Heisenberg observables $Q$ and $P$ may be given the following diagonal kernel integral representation, whatever the fixed but arbitrary value for the complex squeezing parameter $z$,

$$A = \int_C \frac{du_0 \, d\bar{u}_0}{\pi} \langle \Omega_z(u_0) \rangle a(z, \bar{z}; u_0, \bar{u}_0) \langle \Omega_z(u_0) |,$$}

where the diagonal kernel is defined as,

$$a(z, \bar{z}; u_0, \bar{u}_0) = e^{-\partial_{u_0(z)} \partial_{\bar{u}_0(z)}} \langle \Omega_z(u_0) | A | \Omega_z(u_0) \rangle.$$}

More generally given the normalised integration measure $\mu(z, \bar{z})$, one may extend this representation to,

$$A = \int_{C^2} \frac{du_0 \, d\bar{u}_0}{\pi} \frac{dz \, d\bar{z}}{\pi} \mu(z, \bar{z}) \langle \Omega_z(u_0) \rangle a(z, \bar{z}; u_0, \bar{u}_0) \langle \Omega_z(u_0) |,$$}

In the above representations the second order differential operator $\partial_{u_0(z)} \partial_{\bar{u}_0(z)}$ may also be expressed as,

$$\partial_{u_0(z)} \partial_{\bar{u}_0(z)} = \frac{1}{2} e^{i\theta} \sinh 2r \partial^2_{u_0} + \frac{1}{2} e^{-i\theta} \sinh 2r \partial^2_{\bar{u}_0} + \cosh 2r \partial_{u_0} \partial_{\bar{u}_0}$$

$$= (\cosh 2r + \cos \theta \sinh 2r) \frac{1}{2} e^{i\theta} \partial^2_{u_0} + (\cosh 2r - \cos \theta \sinh 2r) \frac{1}{2} e^{-i\theta} \partial^2_{\bar{u}_0} + \hbar \sin \theta \partial_{u_0} \partial_{\bar{u}_0},$$}

while it is worth noting that

$$du_0 \, d\bar{u}_0 = du_0(z) \, d\bar{u}_0(z).$$}

Indeed, since (see (110))

$$|\Omega_z(u_0) \rangle = D(u_0) |\Omega_z \rangle = e^{u_0(z) a^\dagger(z) - \bar{u}_0(z) a(z)} |\Omega_z \rangle = e^{-\frac{1}{2} |u_0(z)|^2} e^{u_0(z) a^\dagger(z)} |\Omega_z \rangle,$$}

the matrix element $\langle \Omega_z(u_0) | A | \Omega_z(u_0) \rangle$ is first a function of $u_0(z)$ and $\bar{u}_0(z)$ rather than directly a function of $u_0$ and $\bar{u}_0$ independently of the value for $z$.

### 6 Correlated Squeezed State Wavefunctions

#### 6.1 Squeezed state configuration space wave functions

Having fully identified, in the form recalled hereafter, the quantum states that saturate the Schrödinger-Robertson uncertainty relation for the Heisenberg algebra of the observables $Q$ and $P$, inclusive of their phase since that of the reference Fock vacuum has been specified,

$$|\psi_0(z, u_0) \rangle = |\Omega_z(u_0) \rangle = D(u_0) S(z) |\Omega_0 \rangle,$$}

we may reconsider the construction of the wave function representation of these states, say in configuration space.
In terms now of the notations and parametrisations introduced throughout the discussion, the expression for the wave functions of these states as determined in (16) and (17) read\footnote{Since one has the relations
\[ \frac{1}{2} \Delta \theta \frac{e^{-i \phi}}{\Delta z} = - \frac{1}{2} \Delta z \frac{e^{-i \phi}}{\Delta \theta} \]
}

\[
\psi_0(q; z, u_0) \equiv \langle q| \Omega_z(u_0) \rangle = \left( \pi \ell_0^2 \right)^{-1/4} \left( \cosh 2r + \cos \theta \sinh 2r \right)^{-1/4} \times \\
\times e^{i \varphi(z,u_0)} e^{\frac{i}{2} q p_{0} \ell_0} \exp \left( - \frac{1}{2} \frac{1}{\cosh 2r + \cos \theta \sinh 2r} \left( \frac{q - q_0}{\ell_0} \right)^2 \right),
\]

(118)

where \( \varphi(z,u_0) \) is the phase factor still to be determined. Thus in particular, when \( u_0 = 0 \),

\[
\langle q| \Omega_z(0) \rangle = \langle q| \Omega_z \rangle = \left( \pi \ell_0^2 \right)^{-1/4} \left( \cosh 2r + \cos \theta \sinh 2r \right)^{-1/4} \times \\
\times e^{i \varphi(z,0)} \exp \left( - \frac{1}{2} \frac{1}{\cosh 2r + \cos \theta \sinh 2r} \left( \frac{q}{\ell_0} \right)^2 \right).
\]

(119)

However since the displacement operator’s action on \( Q \) eigenstates is such that

\[
D(u_0)|q\rangle = e^{\frac{i}{2} q_0 p_{0} \ell_0} e^{\frac{i}{2} q p_{0}} |q + q_0\rangle, \quad \langle q| D(u_0) = \langle q| D^\dagger(-u_0) = \langle q - q_0| e^{-\frac{i}{2} q_0 p_{0} \ell_0} e^{\frac{i}{2} q p_{0}},
\]

(120)

one has,

\[
\langle q| \Omega_z(u_0) \rangle = \langle q| D(u_0) S(z)| \Omega_0 \rangle = e^{-\frac{i}{2} q_0 p_{0} \ell_0} e^{\frac{i}{2} q p_{0}} \langle q - q_0| \Omega_z(0) \rangle,
\]

(121)

which, given the above two expressions for \( \langle q| \Omega_z(u_0) \rangle \) and \( \langle q| \Omega_z(0) \rangle \), thus implies that

\[
e^{i \varphi(z,u_0)} = e^{-\frac{i}{2} q_0 p_{0} \ell_0} e^{i \varphi(z,0)}.
\]

(122)

The final determination of the phase factor \( \varphi(z,0) \) is based now on the following relation between specific Fock state overlaps,

\[
\langle \Omega_0| \Omega_z \rangle = \int_{-\infty}^{+\infty} dq \langle \Omega_0|q\rangle \langle q| \Omega_z \rangle.
\]

(123)

The function \( \langle q| \Omega_z \rangle \) is specified in (119) in terms of \( e^{i \varphi(z,0)} \), while given the choice of phase for the reference Fock vacuum \( |\Omega_0\rangle \) its own wave function was determined earlier on to be simply,

\[
\langle q| \Omega_0 \rangle = \left( \pi \ell_0^2 \right)^{-1/4} e^{-\frac{1}{2} q^2 \ell_0^2}.
\]

(124)

On the other hand since the l.h.s. of the overlap (123) corresponds to \( \langle \Omega_0|S(z)| \Omega_0 \rangle \), clearly this latter quantity does not involve any phase factor left unspecified. Consequently the Gaussian integration in (123) determines the overall phase factor \( \varphi(z,u_0) \) of the wave functions (118).

The evaluation of \( \langle \Omega_0| \Omega_z \rangle \) is readily achieved by using the BCH formula (203) of Appendix B for the squeezing operator \( S(z) \),

\[
S(z) = e^{\frac{i}{2} \zeta a^\dagger a} e^{\ln(1-|\zeta|^2) \frac{i}{2} (a^\dagger a + \frac{1}{2})} e^{-\frac{i}{2} \zeta a^\dagger a}, \quad \zeta = e^{i \theta} \tanh r, \quad z = r e^{i \theta}.
\]

(125)

Hence,

\[
\langle \Omega_0| \Omega_z \rangle = \langle \Omega_0|S(z)| \Omega_0 \rangle = \left( 1 - \tanh^2 r \right)^{1/4} = \left( \cosh r \right)^{-1/2}.
\]

(126)

When combined with the normalisation factor in (119), the Gaussian integration in (123) leads to a factor which may be brought into the form of this last factor \( \left( \cosh r \right)^{-1/2} \) being multiplied by a specific phase factor. In order to express the thereby determined phase factor \( \varphi(z,0) \), let us introduce two last angular parameters \( \theta_{\pm}(z) \) defined by,

\[
\cos \theta_{\pm}(z) = \frac{\cosh r \pm \cos \theta \sinh r}{\sqrt{\cosh 2r \pm \cos \theta \sinh 2r}}, \quad \sin \theta_{\pm}(z) = \frac{\pm \sin \theta \sinh r}{\sqrt{\cosh 2r \pm \cos \theta \sinh 2r}},
\]

(127)
$$\tan \theta_{\pm}(z) = \frac{\pm \sin \theta \sinh r}{\cosh r \pm \cos \theta \sinh r}, \quad (128)$$

and such that,

$$\cos(\bar{\theta}_+(z) - \bar{\theta}_-(z)) = \frac{1}{\sqrt{\cosh^2 2r - \cos^2 \theta \sinh^2 2r}} \sin \theta \sinh 2r$$
$$\sin(\bar{\theta}_+(z) - \bar{\theta}_-(z)) = \frac{\sin \theta \sinh 2r}{\sqrt{\cosh^2 2r - \cos^2 \theta \sinh^2 2r}} \quad (129)$$

On completing the Gaussian integration in (123) (which requires some little work for simplifying some intermediate expressions), one then finally determines that

$$e^{i\varphi(z,0)} = e^{-\frac{1}{2} \bar{\theta}_+(z)}. \quad (130)$$

In conclusion the complete expression for the configuration space wave function representations of all the states that saturate the Schrödinger-Robertson uncertainty relation for the Heisenberg observables $Q$ and $P$ is given as,

$$\psi_0(q; z, u_0) \equiv \langle q|\Omega_z(u_0) \rangle = (\pi \ell_0^2)^{-1/4} \left(\cosh 2r + \cos \theta \sinh 2r\right)^{-1/4} e^{-\frac{1}{2} \bar{\theta}_+(z)} \times$$

$$\times e^{-\frac{1}{4} \frac{q}{\ell_0} \eta_{q_0}^0} e^{\frac{i}{2} \ell_0 q_0} \exp \left(-\frac{1}{2} \frac{1 - i \sin \theta \sinh r}{\cos 2r + \cos \theta \sinh r} \left(\frac{q - q_0}{\ell_0}\right)^2\right). \quad (131)$$

Note that because of the following identities,

$$\cosh r \pm e^{i\theta} \sinh r = \sqrt{\cosh 2r \pm \cos \theta \sinh 2r} e^{i\bar{\theta}_\pm},$$
$$\cosh r \pm e^{-i\theta} \sinh r = \sqrt{\cosh 2r \pm \cos \theta \sinh 2r} e^{-i\bar{\theta}_\pm}, \quad (132)$$

the same wave functions have also the equivalent representations,

$$\psi_0(q; z, u_0) \equiv \langle q|\Omega_z(u_0) \rangle = (\pi \ell_0^2)^{-1/4} \left(\cosh 2r + \cos \theta \sinh 2r\right)^{-1/4} e^{-\frac{1}{2} \bar{\theta}_+(z)} \times$$

$$\times e^{-\frac{1}{4} \frac{q}{\ell_0} \eta_{q_0}^0} e^{\frac{i}{2} \ell_0 q_0} \exp \left(-\frac{1}{2} \frac{1 - i \sin \theta \sinh r}{\cos 2r + \cos \theta \sinh r} \left(\frac{q - q_0}{\ell_0}\right)^2\right), \quad (134)$$

and

$$\psi_0(q; z, u_0) \equiv \langle q|\Omega_z(u_0) \rangle = (\pi \ell_0^2)^{-1/4} \left(\cosh 2r + \cos \theta \sinh 2r\right)^{-1/4} e^{-\frac{1}{2} \bar{\theta}_+(z)} \times$$

$$\times e^{-\frac{1}{4} \frac{q}{\ell_0} \eta_{q_0}^0} e^{\frac{i}{2} \ell_0 q_0} \exp \left(-\frac{1}{2} \sqrt{\frac{\cosh 2r - \cos \theta \sinh 2r}{\cosh 2r + \cos \theta \sinh 2r}} e^{i(\bar{\theta}_-(z) - \bar{\theta}_+(z))} \left(\frac{q - q_0}{\ell_0}\right)^2\right). \quad (135)$$

Furthermore note that,

$$(\cosh 2r \pm \cos \theta \sinh 2r)^{-1/4} e^{-\frac{1}{2} \bar{\theta}_\pm(z)} = \left(\cosh r \pm e^{i\theta} \sinh r\right)^{-1/2}, \quad (136)$$

a relation which invites us to consider finally the result in the following form,

$$\psi_0(q; z, u_0) \equiv \langle q|\Omega_z(u_0) \rangle = (\pi \ell_0^2)^{-1/4} \left(\cosh r + \cos \theta \sinh r\right)^{-1/2} \times$$

$$\times e^{-\frac{1}{4} \frac{q}{\ell_0} \eta_{q_0}^0} e^{\frac{i}{2} \ell_0 q_0} \exp \left(-\frac{1}{2} \cosh r - \cos \theta \sinh r \left(\frac{q - q_0}{\ell_0}\right)^2\right). \quad (137)$$
6.2 The fundamental overlap $\langle \Omega_{z_2}(u_2)|\Omega_{z_1}(u_1) \rangle$ of squeezed states

As a last quantity to be determined in this paper, let us consider the overlap of two arbitrary general squeezed coherent states, associated to the pairs of variables $(z_2, u_2)$ and $(z_1, u_1)$,

$$\langle \Omega_{z_2}(u_2)|\Omega_{z_1}(u_1) \rangle = \int_{-\infty}^{+\infty} dq \langle \Omega_{z_2}(u_2)|q \rangle \langle q|\Omega_{z_1}(u_1) \rangle. \quad (138)$$

Even though a little tedious, the evaluation of the Gaussian integral in (138) is straightforward enough. It leads to the following equivalent expressions, by relying on a number of the identities pointed out above. In one form one finds,

$$\langle \Omega_{z_2}(u_2)|\Omega_{z_1}(u_1) \rangle = \left( \cosh r_2 \cdot \cosh r_1 \right)^{-1/2} \left( 1 - \bar{\zeta}_2 \bar{\zeta}_1 \right)^{-1/2} e^{\frac{i}{\hbar} (q_2 p_2 - q_1 p_1)} e^{-\frac{1}{\hbar} G_{(2)}(2,1)}, \quad (140)$$

where the Gaussian quadratic form $G_{(2)}(2,1)$ is given as

$$G_{(2)}(2,1) = \frac{1}{1 - \zeta_2 \zeta_1} \left[ (1 - \bar{\zeta}_2) (1 - \zeta_1) \left( \frac{q_2 - q_1}{\ell_0} \right)^2 - 2i (\bar{\zeta}_2 - \zeta_1) \left( \frac{q_2 - q_1}{\ell_0} \right) (\ell_0 \hbar (p_2 - p_1)) + (1 + \bar{\zeta}_2) (1 + \zeta_1) \left( \frac{\ell_0 \hbar (p_2 - p_1)}{2} \right)^2 \right]. \quad (141)$$

In terms of the variables $(u_2, u_1)$ and $(\zeta_2, \zeta_1)$, the same expression writes as,

$$\langle \Omega_{z_2}(u_2)|\Omega_{z_1}(u_1) \rangle = \left( \cosh r_2 \cdot \cosh r_1 \right)^{-1/2} \left( 1 - \bar{\zeta}_2 \bar{\zeta}_1 \right)^{-1/2} e^{-\frac{1}{\hbar} (u_2 \bar{u}_1 - \bar{u}_2 u_1)} e^{-\frac{1}{\hbar} G_{(2)}(2,1)}, \quad (142)$$

with this time, in a further streamlined form for the Gaussian quadratic factor,

$$\frac{1}{2} G_{(2)}(2,1) = \frac{1}{1 - \zeta_2 \zeta_1} \left( (1 + \bar{\zeta}_2 \bar{\zeta}_1)(u_2 - u_1)^2 - \bar{\zeta}_2 (u_2 - u_1)^2 - \zeta_1 (\bar{u}_2 - \bar{u}_1)^2 \right) \quad (143)$$

That the dependence of this result on the different variables parametrising the SR-UR states $|\Omega_{z_2}(u_2)\rangle$ and $|\Omega_{z_1}(u_1)\rangle$ comes out as established above may be understood from the following two identities (for the first, see [125]),

$$S(z)|\Omega_0\rangle = \left( \cosh r \right)^{-1/2} e^{\frac{1}{\hbar} \zeta \zeta^*} |\Omega_0\rangle, \quad (144)$$

$$D^\dagger(u_2)D(u_1) = D(-u_2)D(u_1) = e^{\frac{i}{\hbar} (q_2 p_1 - q_1 p_2)} D(u_1 - u_2) = e^{-\frac{1}{\hbar} (u_2 \bar{u}_1 - \bar{u}_2 u_1)} D(u_1 - u_2), \quad (145)$$

which imply the relation,

$$\langle \Omega_{z_2}(u_2)|\Omega_{z_1}(u_1) \rangle = \left( \cosh r_2 \cdot \cosh r_1 \right)^{-1/2} e^{\frac{i}{\hbar} (q_2 p_1 - q_1 p_2)\langle \Omega_0|e^{\frac{1}{\hbar} \zeta \zeta^*} D(u_1 - u_2) e^{\frac{1}{\hbar} \zeta \zeta^*}|\Omega_0\rangle}. \quad (146)$$

Hence the above evaluations have also established the corresponding general matrix element,

$$\langle \Omega_0|e^{\frac{1}{\hbar} \zeta \zeta^*} D(u) e^{\frac{1}{\hbar} \zeta \zeta^*}|\Omega_0\rangle = \left( 1 - \bar{\zeta}_2 \bar{\zeta}_1 \right)^{-1/2} \exp \left\{ -\frac{1}{2} \frac{1}{1 - \bar{\zeta}_2 \bar{\zeta}_1} \left( u - \bar{\zeta}_2 \bar{u} \right)^* \left( u - \zeta_1 \bar{u} \right) \right\}. \quad (147)$$
7 Conclusions

In this contribution to the present Workshop Proceedings we have explored the first step in the
general programme outlined in the Introduction, in the case of the quantum observables $Q$ and $P$
defining the Heisenberg algebra. Namely, generally given a quantum system characterised by a set
of quantum observables, one considers the set of quantum states that saturate the Schrödinger-
Robertson uncertainty relation corresponding to this set of observables. Such states are the closest
possible to displaying a classical behaviour of the quantum system while, being determined by
a condition characteristic of coherent-like quantum states, they are parametrised by a collection
of continuous variables and therefore define a specific submanifold within the Hilbert space of
the quantum system. Correspondingly there arise specific geometric structures associated to this
manifold, compatible with one another, namely both a quantum symplectic structure as well as a
quantum Riemannian metric. It may even be possible to reconstruct the quantum dynamics of the
system from that geometric data as well as a choice of Hamiltonian operator represented through
its diagonal matrix elements for the saturating states, thereby offering a geometric formulation
of quantum systems and their dynamics through a path integral representation.

This general programme is initiated herein, in the case of observables of the Heisenberg
algebra for a single degree of freedom quantum system as an illustration. Correspondingly the
saturating quantum states are the so-called and well known general squeezed states, for which
many properties and results were reviewed and presented together with quite many details and
some original results, with the hope that some readers could become interested in taking part in
such an exploration in the case of other possible choices of quantum observables and the ensuing
saturating quantum states. For instance, the affine quantum algebra of scale transformations,
$[Q,D] = i\hbar Q$ (say with $D = (QP + PQ)/2$), does also play an important role in quite many
quantum systems [1-4, 7, 14], with its own coherent states. To this author’s best knowledge, the
analogue states for the affine algebra of the squeezed states for the Heisenberg algebra remain
to be fully understood. Other situations may be thought of as well, such as for instance the
operators and uncertainty relation related to the factorisation of a quantum Hamiltonian along
the lines and methods of supersymmetric quantum mechanics, $H = A \dagger A + E_0$.

Essential to such a programme is the evaluation of the overlap of the saturating quantum
states given a set of quantum observables. In particular this quantity encodes the data necessary
in identifying the inherent geometric structures, as well as in the construction of the quantum path
integral of the system over the manifold in Hilbert space associated to the saturating quantum
states. Usually overcompleteness relations ensue, implying that the overlap of saturating states
determines a reproducing kernel representation of the Hilbert space.

In this contribution the discussion concludes with the evaluation of this reproducing kernel
for the general squeezed states of the Heisenberg algebra which saturate the Schrödinger-
Robertson uncertainty relation. We defer to a separate publication an analysis of the correspond-
ing symplectic and Riemannian geometric structures, as well as of the path integral representation
of the quantum system over the associated manifold of squeezed states, $|\Omega_z(u)\rangle = D(u)S(z)|\Omega_0\rangle$.
All of these considerations are to follow from the quantities $\langle \Omega_{z_2}(u_2)|\Omega_{z_1}(u_1)\rangle$.

Thus in particular the overlap $\langle \Omega_{z_2}(u_2)|\Omega_{z_1}(u_1)\rangle$ determines a reproducing kernel
representation of the Hilbert space of the Heisenberg algebra of observables $Q$ and $P$. Indeed, given the
generalised overcompleteness relation (109), obviously one has the property,

$$\langle \Omega_{z_2}(u_2)|\Omega_{z_1}(u_1)\rangle = \int_{\mathbb{C}^2} \frac{du_3}{\pi} \frac{dz_3}{\pi} \mu(z_3, \bar{z}_3) \langle \Omega_{z_2}(u_2)|\Omega_{z_3}(u_3)\rangle \langle \Omega_{z_3}(u_3)|\Omega_{z_1}(u_1)\rangle.$$  (148)

We plan to report elsewhere on such applications and further developments of the results of the
present contribution, as well as on the general programme outlined in the Introduction. This
programme is offered here as a token of genuine and sincere appreciation for Professor Norbert
Hounkonnou’s constant interest and many scientific contributions of note, and certainly for his
unswerving efforts as well towards the development of mathematical physics in Benin, in Western
Africa and on the African continent, to the benefit of the younger and future generations, and
this on the occasion of this special COPROMAPH Workshop celebrating his sixtieth birthday.
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Appendices

Appendix A: Cauchy-Schwarz inequality and quantum uncertainty relations

In the first part of this Appendix, for the purpose of the present paper it proves useful to reconsider specific arguments leading to the Cauchy-Schwarz inequality. In the second part this inequality is applied to establish the Schrödinger-Robertson uncertainty relation (SR-UR) given any two quantum observables.

Let |ψ₁⟩ and |ψ₂⟩ be any two (normalisable and non-vanishing) quantum states, and consider their arbitrary complex linear combination, say in the form,

\[ |ψ⟩ = |ψ₁⟩ + iλe^{iφ}|ψ₂⟩, \quad φ, λ ∈ \mathbb{R}, \]  

(149)

with φ a phase factor and λ a real parameter. Since the sesquilinear and hermitian inner product of Hilbert space is positive definite, the norm of the state |ψ⟩ is positive definite whatever the values for these two parameters,

\[ P(λ) \equiv ⟨ψ|ψ⟩ = λ²⟨ψ₂|ψ₂⟩ + iλ(ie^{iφ}⟨ψ₁|ψ₂⟩ − e^{−iφ}⟨ψ₂|ψ₁⟩) + ⟨ψ₁|ψ₁⟩ ≥ 0. \]

(150)

Note that the l.h.s. of this inequality is a real quadratic polynomial in λ ∈ \mathbb{R} with real coefficients, P(λ), of which the coefficient in λ² is strictly positive. Hence the parabolic graph of this function of λ lies entirely in the upper half plane and this polynomial has no real roots in λ, unless the state |ψ⟩ itself vanishes identically in which case the two roots are degenerate and real for just a unique and specific set of values for the parameters φ and λ such that the parabola P(λ) has its minimum just touching the horizontal coordinate axis in λ. Consequently the discriminant of this real quadratic form in λ is negative, namely

\[ ⟨ψ₁|ψ₁⟩⟨ψ₂|ψ₂⟩ ≥ −\frac{1}{4}(e^{iφ}⟨ψ₁|ψ₂⟩ − e^{−iφ}⟨ψ₂|ψ₁⟩)^² ≥ 0. \]

(151)

This inequality is the tightest when the quantity on the r.h.s. of this relation, which is still a function of the parameter φ, reaches its maximal value. As readily established this maximum is obtained for a phase factor φ = φ₀ such that

\[ e^{2iφ₀} = −\frac{⟨ψ₂|ψ₁⟩}{⟨ψ₁|ψ₂⟩}, \]

(152)

namely,

\[ e^{iφ₀}⟨ψ₁|ψ₂⟩ = −e^{−iφ₀}⟨ψ₂|ψ₁⟩, \quad ie^{iφ₀}⟨ψ₁|ψ₂⟩ = −ie^{−iφ₀}⟨ψ₂|ψ₁⟩ = (ie^{iφ₀}⟨ψ₁|ψ₂⟩)^* \]

(153)

Given this choice, the tightest discriminant inequality in (151) reduces to the well-known Cauchy-Schwarz inequality

\[ ⟨ψ₁|ψ₁⟩⟨ψ₂|ψ₂⟩ ≥ |⟨ψ₁|ψ₂⟩|^². \]

(154)

Having set the phase factor as φ = φ₀ the polynomial P(λ) may be organised in the following form,

\[ P(λ) = ⟨ψ₂|ψ₂⟩ \left[ (λ + ie^{iφ₀}⟨ψ₁|ψ₂⟩)^² + \frac{⟨ψ₁|ψ₁⟩⟨ψ₂|ψ₂⟩ − |⟨ψ₁|ψ₂⟩|^²}{⟨ψ₂|ψ₂⟩^²} \right] ≥ 0. \]

(155)
Hence making now the additional choice \( \lambda = \lambda_{CS} \) such that,

\[
\lambda_{CS} = -ie^{i\phi_0} \frac{\langle \psi_1 | \psi_2 \rangle}{\langle \psi_2 | \psi_2 \rangle} = ie^{-i\phi_0} \frac{\langle \psi_2 | \psi_1 \rangle}{\langle \psi_2 | \psi_2 \rangle}, \quad ie^{i\phi_0} \lambda_{CS} = -\frac{\langle \psi_2 | \psi_1 \rangle}{\langle \psi_2 | \psi_2 \rangle}, \tag{156}
\]

\( \lambda_{CS} \) being indeed a real quantity on account of the properties in (153), one has,

\[
\langle \psi | \psi \rangle = P(\lambda_{CS}) = \frac{\langle \psi_2 | \psi_2 \rangle}{\langle \psi_2 | \psi_2 \rangle} \left( \frac{\langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle - |\langle \psi_1 | \psi_2 \rangle|^2}{\langle \psi_2 | \psi_2 \rangle^2} \right) \geq 0. \tag{157}
\]

Consequently, besides the Cauchy-Schwarz inequality (154), one also concludes that this inequality is saturated into a strict equality provided the states |\( \psi_1 \rangle \) and |\( \psi_2 \rangle \) are such that |\( \psi \rangle = 0 \) for these choices of parameters \( \varphi = \varphi_0 \) and \( \lambda = \lambda_{CS} \), namely

\[
|\psi_1 \rangle - \frac{\langle \psi_2 | \psi_1 \rangle}{\langle \psi_2 | \psi_2 \rangle} |\psi_2 \rangle = 0 \iff \langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle = |\langle \psi_1 | \psi_2 \rangle|^2. \tag{158}
\]

Let now \( A \) and \( B \) be two arbitrary quantum observables, namely hermitian (and ideally, self-adjoint) operators acting on Hilbert space, \( A^\dagger = A \) and \( B^\dagger = B \), and consider an arbitrary (normalisable and non-vanishing) quantum state |\( \psi_0 \rangle \). Whatever choice of quantum operator \( O \), its expectation value for that state |\( \psi_0 \rangle \) is denoted as

\[
\langle O \rangle = \frac{\langle \psi_0 | O | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}. \tag{159}
\]

In particular for the observables \( A \) and \( B \) we have their real valued expectation values

\[
a_0 = \langle A \rangle, \quad b_0 = \langle B \rangle, \quad a_0, b_0 \in \mathbb{R}, \tag{160}
\]

which are used to shift these observables as follows,

\[
\bar{A} = A - a_0\mathbb{1}, \quad \bar{B} = B - b_0\mathbb{1}, \quad \bar{A}^\dagger = \bar{A}, \quad \bar{B}^\dagger = \bar{B}, \tag{161}
\]

such that \( [\bar{A}, \bar{B}] = [A, B] \). Consequently we have

\[
(\Delta A)^2 = \langle \bar{A}^2 \rangle, \quad (\Delta B)^2 = \langle \bar{B}^2 \rangle. \tag{162}
\]

In order to establish the SR-UR from the Cauchy-Schwarz inequality, let us consider the following two quantum states,

\[
|\psi_1 \rangle = \frac{1}{\sqrt{\langle \psi_0 | \psi_0 \rangle}} \bar{A} |\psi_0 \rangle, \quad |\psi_2 \rangle = \frac{1}{\sqrt{\langle \psi_0 | \psi_0 \rangle}} \bar{B} |\psi_0 \rangle, \tag{163}
\]

which are such that

\[
\langle \psi_1 | \psi_1 \rangle = (\Delta A)^2, \quad \langle \psi_2 | \psi_2 \rangle = (\Delta B)^2, \quad \langle \psi_1 | \psi_2 \rangle = \langle \bar{A} \bar{B} \rangle, \quad \langle \psi_2 | \psi_1 \rangle = \langle \bar{B} \bar{A} \rangle = \langle \bar{A} \bar{B} \rangle^*. \tag{164}
\]

Consequently the Cauchy-Schwarz inequality (154) reads,

\[
(\Delta A)^2 (\Delta B)^2 \geq |\langle \bar{A} \bar{B} \rangle|^2, \quad (\Delta A)^2 (\Delta B)^2 \geq (\Delta \bar{A})^2 (\Delta \bar{B})^2. \tag{165}
\]

Alternatively by expressing the quantity \( \langle \bar{A} \bar{B} \rangle \) in terms of the commutator and anti-commutator of the operators \( \bar{A} \) and \( \bar{B} \) which then separate its real and imaginary parts\textsuperscript{15}, namely

\[
\langle \bar{A} \bar{B} \rangle = \frac{1}{2} \{ \bar{A}, \bar{B} \} + \frac{1}{2} \{ [\bar{A}, \bar{B}] \} = \frac{1}{2} i \langle (\bar{A}) [\bar{B}] \rangle + \frac{1}{2} \langle \{ \bar{A}, \bar{B} \} \rangle, \tag{166}
\]

\textsuperscript{15}Note that \((\bar{A})[\bar{B}]\) and \(\{ \bar{A}, \bar{B} \}\) are hermitian (or even self-adjoint if \( A \) and \( B \) are self-adjoint) operators whose expectations values are thus real.
\[
\langle B \bar{A} \rangle = \langle AB \rangle^* = -\frac{1}{2i} \langle (-i) [\bar{A}, B] \rangle + \frac{1}{2} \langle \{ \bar{A}, B \} \rangle,
\]
(167)
one obtains the inequality in the Schrödinger-Robertson form,
\[
(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} ((-i) [A, B])^2 + \frac{1}{4}(\{ \bar{A}, B \})^2.
\]
(168)
As a by-product one also derives the looser generalised Heisenberg uncertainty relation,
\[
(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} ((-i) [A, B])^2, \quad (\Delta A) (\Delta B) \geq \frac{1}{2} ((-i) [A, B])).
\]
(169)
Furthermore given (158) the SR-UR (168) is saturated, namely \((\Delta A)^2 (\Delta B)^2 = \langle B \bar{A} \rangle \langle B \bar{A} \rangle\), provided the state \(|\psi_0\rangle\) is such that,
\[
\left( \bar{A} - \frac{\langle B \bar{A} \rangle}{(\Delta B)^2} \right) |\psi_0\rangle = 0, \quad (\bar{A} - \lambda_0 \bar{B}) |\psi_0\rangle = 0, \quad (A - \lambda_0 B) |\psi_0\rangle = (\langle A \rangle - \lambda_0 \langle B \rangle) |\psi_0\rangle,
\]
(170)
where the complex parameter \(\lambda_0\) is given by \(\lambda_0 = (\bar{B} \bar{A}) / (\Delta B)^2 = (\Delta A)^2 / \langle B \bar{A} \rangle\), namely
\[
\lambda_0 = \frac{1}{(\Delta B)^2} \left( \frac{1}{2} \langle \{ \bar{A}, B \} \rangle - \frac{1}{2i} \langle (-i) [A, B] \rangle \right) = (\Delta A)^2 \left( \frac{1}{2} \langle \{ A, B \} \rangle + \frac{1}{2i} \langle (-i) [A, B] \rangle \right).
\]
(171)
Appendix B: Baker-Campbell-Hausdorff formulae

This Appendix is structured in three parts. The first recalls a basic Baker-Campbell-Hausdorff (BCH) formula. The second part discusses recent results established in Ref. [15] based on a construction of the most general BCH formula which is also outlined. Finally the third part applies these results to a SU(1,1) algebra directly related to the general squeezed coherent states arising as the states saturating the Schrödinger-Robertson uncertainty relation.

Given any two operators, \(A\) and \(B\), the following basic BCH is well known,\(^{[16]}\)
\[
e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \cdots = e^{ad \cdot A} B,
\]
(172)
where the (Lie algebra) adjoint action of the operator \(A\) on an operator \(X\) is defined by
\[
ad A \cdot X \equiv \{ A, X \}.
\]
(173)
This identify is to be used throughout hereafter. Note that it also implies
\[
e^A e^B e^{-A} = e^{e^{ad A} B}, \quad e^A e^B = e^{e^{ad A} B} e^A.
\]
(174)
Hence in particular when \([A, B]\) commutes with both \(A\) and \(B\), we have simply
\[
e^A e^B = e^{[A, B]} e^B e^A.
\]
(175)
In order to establish the general BCH formula, first let us consider some operator \(A(\lambda)\) function of a parameter \(\lambda\). Then the following identities apply,\(^{[17]}\)
\[
e^{-A(\lambda)} \frac{d}{d\lambda} e^{A(\lambda)} = \int_0^1 dt e^{-t A(\lambda)} \frac{dA(\lambda)}{d\lambda} e^{t A(\lambda)} = \int_0^1 dt e^{-t ad A(\lambda)} \frac{dA(\lambda)}{d\lambda} = \Phi(-ad A(\lambda)) \frac{dA(\lambda)}{d\lambda},
\]
(176)
\(^{[16]}\)It suffices to consider the generating operator in \(\lambda\), \(e^{\lambda A} B e^{-\lambda A}\), expanded in series in \(\lambda\).
\(^{[17]}\)As a reminder we have \(\int_0^1 dt (1-t)^n t^m = n! m!/(n+m+1)!\) as well as \(\int_0^1 dt t^n = 1/(n+1)\), hence in particular \(\frac{d}{d\lambda} e^{A(\lambda)} = \int_0^1 dt e^{(1-t) A(\lambda)} \frac{dA(\lambda)}{d\lambda} e^{t A(\lambda)}\).
where the function $\Phi(x)$ is given as
\[
\Phi(x) = \int_0^1 dt x^t = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n = \frac{e^x - 1}{x}.
\] (177)

This function being such that
\[
\Phi(-\ln x) = \frac{x - 1}{x \ln x} = \frac{1}{\Psi(x)},
\] it proves useful to also introduce the function $\Psi(x)$ defined by\(^{18}\)
\[
\Psi(x) = \frac{x \ln x}{x - 1} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^n, \quad \Phi(-\ln x) \Psi(x) = 1.
\] (179)

Given two operators $A$ and $B$, the general BCH formula provides an expression for the operator $C$ defined by
\[
C = \ln(e^A e^B), \quad e^C = e^A e^B.
\] (180)

In order to establish this expression, let us introduce the generating operator $C(\lambda)$ such that
\[
e^{C(\lambda)} = e^A e^{\lambda B}, \quad C(\lambda) = \ln(e^A e^{\lambda B}), \quad C(0) = A.
\] (181)

The adjoint action of the operator $C(\lambda)$ on any operator $D$ is given as,
\[
e^{C(\lambda)} D e^{-C(\lambda)} = e^A e^{\lambda B} D e^{-\lambda B} e^{-A}, \quad \text{namely,} \quad e^{ad C(\lambda)} D = e^{ad A} e^{\lambda ad B} D,
\] (182)
which implies,
\[
e^{ad C(\lambda)} = e^{ad A} e^{\lambda ad B}, \quad ad C(\lambda) = \ln(e^{ad A} e^{\lambda ad B}).
\] (183)

On the other hand, since
\[
e^{-C(\lambda)} \frac{d}{d\lambda} e^{C(\lambda)} = e^{-\lambda B} e^{-A} \frac{d}{d\lambda} e^{\lambda B}, \quad \text{namely,} \quad \Phi(-ad C(\lambda)) \frac{dC(\lambda)}{d\lambda} = B,
\] (184)

necessarily
\[
\frac{dC(\lambda)}{d\lambda} = \Psi(e^{ad A} e^{\lambda ad B}) B.
\] (185)

Given the integration condition $C(0) = A$, finally the following general BCH formulae applies for the operator $C$,
\[
C = \ln(e^A e^B) = A + \int_0^1 d\lambda \Psi(e^{ad A} e^{\lambda ad B}) B
\] 
\[
= A + B - \int_0^1 d\lambda \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\| - e^{ad A} e^{\lambda ad B}\right)_n B
\] 
\[
= A + B + \int_0^1 d\lambda \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\| - e^{ad A} e^{\lambda ad B}\right)^{n-1} \left(\frac{e^{ad A} - \|}{ad A}\right) [A, B].
\] (186)

In particular when $[A, B]$ commutes with both $A$ and $B$, one has the well known BCH formula,
\[
C = \ln(e^A e^B) = A + B + \frac{1}{2}[A, B], \quad e^A e^B = e^{A+B+\frac{1}{2}[A,B]} = e^{\frac{1}{2}[A,B]} e^{A+B},
\] (187)

which implies again the result in (175).

It is the last form for the BCH formula in (186), which is the starting point of the recent analysis of Ref.[15] which manages to sum up the BCH formula in closed form in the case of operators $A$ and $B$ whose commutator is of the form,
\[
[A, B] = uA + vB + c\mathbb{1},
\] (188)

\(^{18}\)Note that $\Psi(x^{-1}) = 1/\Phi(x) = x/(e^x - 1) = \sum_{n=0}^{\infty} B_n x^n/n!$ is a generating function for Bernoulli numbers, $B_n$. The author thanks Christian Hagendorf for pointing this out to him.
where \( u, v \) and \( c \) are constant parameters. Indeed in such a situation one has,

\[
\begin{align*}
\text{ad} A \cdot [A, B] &= v [A, B], \\
\text{ad} B \cdot [A, B] &= -u [A, B], \\
e^{\text{ad} A} [A, B] &= e^v [A, B], \\
e^{\lambda \text{ad} B} [A, B] &= e^{-\lambda u} [A, B],
\end{align*}
\]

which implies that

\[
\ln(e^A e^B) = A + B + f(u,v) [A,B],
\]

where \( f(u,v) \) is a simple function determined from (186) in the form

\[
f(u,v) = \frac{e^v - 1}{v} \int_0^1 d\lambda \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left( 1 - e^{v e^{-\lambda v}} \right)^{n-1}.
\]

A direct evaluation finds for this function, which proves to be symmetric,

\[
f(u,v) = \frac{ue^u(e^v - 1) - ve^v(e^u - 1)}{uv(e^u - e^v)} = \frac{u(1 - e^{-v}) - v(1 - e^{-u})}{uv(e^{-v} - e^{-u})} = f(v,u),
\]

with the distinguished value \( f(0,0) = 1/2 \) (in agreement with (187)).

Finally given the reference Fock algebra of operators \( a \) and \( a^\dagger \) introduced in Section 3 consider the operators

\[
K_0 = \frac{1}{2} \left( a^\dagger a + \frac{1}{2} \right), \quad K_+ = \frac{1}{2} a^\dagger a^2, \quad K_- = \frac{1}{2} a^2,
\]

which generate a SU(1,1) algebra of transformations acting on the Hilbert space representing the Heisenberg algebra of observables \( Q \) and \( P \),

\[
[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0.
\]

Independently of the representation which realises this SU(1,1) algebra, let us apply the result (190) to specific combinations of operators \( K_0 \) and \( K_\pm \) obeying this algebraic structure.

To begin with consider the following operator

\[
e^{\alpha K_+} e^{\gamma K_0} e^{-\bar{\alpha} K_-},
\]

where \( \alpha \) is an arbitrary complex parameter such that \( |\alpha| < 1 \) and \( \gamma = \ln(1 - |\alpha|^2) \). In order to apply (190), let us rewrite this operator in the form (195),

\[
e^{\alpha K_+} e^{\gamma K_0} e^{-\bar{\alpha} K_-} = e^{\alpha K_+} e^{\gamma_s K_0} e^{\gamma_{-s} K_0} e^{-\bar{\alpha} K_-},
\]

where

\[
\gamma_s = \ln(1 + s|\alpha|), \quad \gamma_+ + \gamma_- = \gamma = \ln(1 - |\alpha|^2), \quad s = \pm 1.
\]

Since we have

\[
[\alpha K_+, \gamma_s K_0] = -\gamma_s (\alpha K_+), \quad [\gamma_{-s} K_0, -\bar{\alpha} K_-] = -\gamma_{-s} (-\bar{\alpha} K_-),
\]

the BCH formula (190) then applies separately to the first two, and the last two factors in (196). With the evaluation of the corresponding values for the function \( f(u,v) \), one then finds,

\[
\ln(e^{\alpha K_+} e^{\gamma_s K_0}) = \frac{s}{|\alpha|} \gamma_s \alpha K_+ + \gamma_s K_0 \equiv \tilde{A},
\]

\[
\ln(e^{\gamma_{-s} K_0} e^{-\bar{\alpha} K_-}) = \frac{s}{|\alpha|} \gamma_{-s} \bar{\alpha} K_- + \gamma_{-s} K_0 \equiv \tilde{B},
\]

namely so far,

\[
e^{\alpha K_+} e^{\gamma K_0} e^{-\bar{\alpha} K_-} = e^{\tilde{A}} e^{\tilde{B}}.
\]

\[\text{Note that for all practical purposes the results of Ref.15 remain valid as stated if the term ci stands for an operator which commutes with both A and B.}\]
However the values for $\gamma_s$ are chosen not only such that $\gamma_s + \gamma_{-s} = \gamma$ but also such that the BCH formula (190) may be applied once again to the latter product, which requires that the commutator of $\tilde{A}$ and $\tilde{B}$ be again a linear combination of these same two operators,

$$ [\tilde{A}, \tilde{B}] = -\gamma_{-s} \tilde{A} - \gamma_s \tilde{B}. $$

The final evaluation of the BCH formula for (195) then reduces to the determination of the value $f(-\gamma_{-s}, -\gamma_s)$. In order to present this BCH formula, since $|\alpha| < 1$ let us introduce the following parameters, with $0 \leq r < \infty$ and $-\pi \leq \theta < +\pi$,

$$ |\alpha| = \tanh r, \quad \alpha = e^{i\theta} \tanh r, \quad z = e^{i\theta} r. $$

One then has finally,

$$ e^\alpha K_+ e^{\ln(1-\tanh^2 r)} K_0 e^{-\bar{\alpha} K_-} = e^{z} K_+ - \bar{z} K_- . $$

A similar procedure may be applied to the operator

$$ e^{-\bar{\alpha} K_-} e^{-\gamma K_0} e^\alpha K_+ . $$

However given the following inner automorphism of the SU(1,1) algebra,

$$ K_0 \longleftrightarrow -K_0, \quad K_+ \longleftrightarrow -K_-, \quad K_- \longleftrightarrow -K_+, $$

from (203) one readily has (with then also $\alpha \leftrightarrow \bar{\alpha}$ and $z \leftrightarrow \bar{z}$),

$$ e^{-\bar{\alpha} K_-} e^{-\ln(1-\tanh^2 r)} K_0 e^{\alpha K_+} = e^{z} K_+ - \bar{z} K_- = S(z) = e^{\alpha K_+} e^{\ln(1-\tanh^2 r)} K_0 e^{-\bar{\alpha} K_-}. \quad (206) $$

The results (203) and (206), which thus apply to the squeezing operator $S(z)$ introduced in Section 4.3, are stated in Ref.[17] by establishing them in the defining representation of the SU(1,1) algebra. Here they are derived solely from the structure of the SU(1,1) algebra and independently of the representation of that algebra, by using the conclusions of Ref.[15].

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