ASYMPTOTICS OF MULTIVARIATE ORTHOGONAL POLYNOMIALS WITH HYPEROCTAHEDRAL SYMMETRY

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Abstract. We present a formula describing the asymptotics of a class of multivariate orthogonal polynomials with hyperoctahedral symmetry as the degree tends to infinity. The polynomials under consideration are characterized by a factorized weight function satisfying certain analyticity assumptions. As an application, the large-degree asymptotics of the Koornwinder-Macdonald $BC_N$-type multivariate Askey-Wilson polynomials is determined.

1. Introduction

It is known that the classical families of hypergeometric and basic hypergeometric orthogonal polynomials form a hierarchy, the Askey scheme, of which the most general member is given by the celebrated Askey-Wilson polynomials [AW, KS]. Other families of classical orthogonal polynomials, such as e.g. the Hermite, Laguerre, Jacobi, and Hahn polynomials, all turn out to be special (limiting) cases of these Askey-Wilson polynomials. Around a decade ago, Koornwinder introduced a multivariate generalization of the Askey-Wilson polynomials with hyperoctahedral symmetry [K], by building upon the pioneering works of Macdonald on families of orthogonal polynomials associated with root systems [M1, M2, M3]. As it turns out, these Koornwinder-Macdonald polynomials form again a master family in the sense that they contain all Macdonald families associated with the classical root systems as special cases [K, Di1], as well as certain multivariate versions of the Hermite, Laguerre, Jacobi, and Hahn polynomials (see e.g. the papers [BO, BF, Di3, Di4] and references therein). Over the past few years, the properties of the Koornwinder-Macdonald polynomials have been subject of investigation in a number of works, leading to a multivariate generalization of significant part of the theory surrounding the one-variable Askey-Wilson polynomials [Di2, Di4, O, Sa, DS, NK, St, Mi, C, Ra].

A fundamental problem in the theory of orthogonal polynomials is the question of their asymptotical behavior as the degree tends to infinity [Sz, D-Z, D]. For the Askey-Wilson polynomials, this asymptotics was determined by Ismail and Wilson [IW] (leading asymptotics) and by Ismail [I] (full asymptotic expansion). The main purpose of the present work is to lift this asymptotic analysis to the multivariate level. More specifically, our goal is to determine the large-degree asymptotics of the Koornwinder-Macdonald multivariate Askey-Wilson polynomials. Such large-degree asymptotics was determined recently for the Macdonald polynomials by Ruijsenaars [Ru] (for the type $A$ root systems) and by the present author [Di5] (for arbitrary reduced root systems). The current work should be seen as an extension of these results to the case of the Koornwinder-Macdonald polynomials, or from a

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more conceptual point of view, as an extension from reduced to nonreduced root systems. Following the ideas of Ruijsenaars, we in fact determine the asymptotics of orthogonal polynomials associated to a fairly large class of weight functions that factorize in terms of one-dimensional $c$-functions. However, whereas Ruijsenaars studies homogeneous symmetric polynomials, here in contrast we consider (Laurent) polynomials in $N$ variables invariant under the action of the hyperoctahedral group $\Sigma_N \ltimes \mathbb{Z}_2^N$ (thus passing from type $A$ to type $BC$ root systems). For a specific choice of the $c$-functions, we end up with the asymptotics of the Koornwinder-Macdonald polynomials.

It is important to emphasize that—at the multivariate level—the large-degree asymptotics considered here is not the only type of asymptotics of interest. Other types of asymptotical properties of multivariate orthogonal polynomials, involving their behavior as the number of variables tends to infinity, were for instance studied by Okounkov and Olshanski for the case of Jack’s hypergeometric degeneration ($q \to 1$) of the Macdonald polynomials associated with the type $A$ root systems [OO].

The paper is organized as follows. In Section 2 we first define our class of symmetric orthogonal polynomials. An asymptotic formula for these polynomials is presented in Section 3. In Section 4 we apply the asymptotic formula in question to determine the asymptotics of the Koornwinder-Macdonald polynomials. Finally, Sections 5 and 6 wrap up the paper via a series of results which—when linked together—combine into the proof of the fundamental asymptotic formula from Section 3.

2. Multivariate Orthogonal Polynomials

2.1. Symmetric Monomials. Let $W$ be the hyperoctahedral group given by the semidirect product of the permutation group $\Sigma_N$ and the $N$-fold product of the cyclic group $\mathbb{Z}_2$. The natural action of $w = (\sigma, \varepsilon) \in W$ on $\mathbb{R}^N$ is given by

$$x_w \equiv w(x) = (\varepsilon_1 x_{\sigma_1}, \ldots, \varepsilon_N x_{\sigma_N})$$

(with $\sigma \in \Sigma_N$ and $\varepsilon_j \in \{1, -1\}$ for $j = 1, \ldots, N$). Let $A^W$ be the algebra of $W$-invariant trigonometric polynomials on the torus

$$T_N = \mathbb{R}^N / (2\pi \mathbb{Z})^N.$$

The standard basis for $A^W$ is given by the symmetric monomials

$$m_\lambda(x) = \frac{1}{|W_\lambda|} \sum_{w \in W} e^{i\langle \lambda, x_w \rangle}, \quad \lambda \in \Lambda,$$

where $\langle \lambda, x \rangle = \sum_{j=1}^N \lambda_j x_j$,

$$\Lambda = \{ \lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0 \},$$

and $|W_\lambda|$ denotes the order of the stabilizer subgroup $W_\lambda = \{ w \in W \mid \lambda w = \lambda \}$.

2.2. Orthogonality. We will partially order the monomial basis $\{m_\lambda\}_{\lambda \in \Lambda}$ by means of the hyperoctahedral dominance order on $\mathbb{Z}^N$:

$$\lambda \geq \mu \iff \sum_{j=1}^\ell \lambda_j \geq \sum_{j=1}^\ell \mu_j \quad \text{for } \ell = 1, \ldots, N.$$
Let $\Delta(x)$ be an almost everywhere positive weight function on the torus $T_N$. We equip $A^W$ with an inner product structure associated to $\Delta$ via the definition

$$\langle f, g \rangle_\Delta = \frac{1}{(2\pi)^N} \int_{T_N} f(x)\overline{g(x)} \Delta(x) \, dx, \quad \forall f, g \in A^W$$

(where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$).

Application of the Gram-Schmidt process to the partially ordered monomial basis produces a basis $\{P_\lambda\}_{\lambda \in \Lambda}$ of $A^W$ of the form

$$P_\lambda(x) = \sum_{\rho \in \Lambda, \rho \preceq \lambda} a_{\lambda\rho} m_\rho(x), \quad \lambda \in \Lambda,$$

with coefficients $a_{\lambda\rho} \in \mathbb{C}$ such that

$$\langle P_\lambda, m_\rho \rangle_\Delta = 0 \quad \text{if} \quad \rho \prec \lambda \quad \text{and} \quad \langle P_\lambda, P_\lambda \rangle_\Delta = 1,$$

where we chose $a_{\lambda\lambda} > 0$ by convention. (In other words, the elements of this new basis are orthogonal when comparable in the partial order.)

### 2.3. Factorized Weight Functions.

From now on we will restrict our attention to a special class of $W$-invariant weight functions that factorize in terms of one-dimensional $c$-functions. Specifically, we consider weight functions of the form

$$\Delta(x) = \frac{1}{|W| C(x) C(-x)},$$

with $|W| = 2^N N!$ and

$$\begin{align*}
C(x) &= \prod_{1 \leq j < k \leq N} c_0(x_j + x_k) c_0(x_j - x_k) \prod_{1 \leq j \leq N} c_1(x_j).
\end{align*}$$

For technical reasons, it will be assumed that the $c$-functions $c_p(x)$, $p = 0, 1$ are of the form

$$c_0(x) = (1 - e^{-ix})^{-1} \hat{c}_0(e^{-ix}) \quad c_1(x) = (1 - e^{-2ix})^{-1} \hat{c}_1(e^{-ix}).$$

Here the reduced $c$-functions $\hat{c}_p(z)$ are taken to be: analytic on a closed disc $D_p = \{ z \in \mathbb{C} \mid |z| \leq g_p \}$ of radius $g_p > 1$, zero-free on an open environment of the origin containing the closed unit disc, real-valued for $z$ real, and normalized such that $\hat{c}_p(0) = 1$. It follows from these conditions that $\hat{c}_p(z)$ and $1/\hat{c}_p(z)$ have uniformly converging Taylor expansions on the closed unit disc of the form

$$\hat{c}_p(z) = 1 + \sum_{n=1}^\infty a_{n,p}^+ z^n \quad (p = 0, 1),$$

with $a_{n,0}^+ = O(e^{-\epsilon n})$ and $a_{n,1}^+ = O(e^{-\epsilon n/2})$ as $n \to \infty$ for

$$0 < \epsilon \leq \min(\log(g_0), 2 \log(g_1)).$$

Indeed, the asymptotic bound on the Taylor coefficients follows from the Cauchy formula $a_{n,p}^+ = \frac{1}{2\pi i} \oint_{|z|=g_p} \hat{c}_p(z) z^{-n-1} \, dz$ (whence $a_{n,p}^+ = O(e^{-n \log(g_p)})$ as $n$ tends to $\infty$).
3. Asymptotic Formulas

For $\lambda \in \Lambda$, we define

$$P^\infty_\lambda(x) = \sum_{w \in \cal W} C(x_w) e^{i(\lambda,x_w)}$$

(3.1)

and

$$m(\lambda) = \min_{j=1,\ldots,N} \lambda_j - \lambda_{j+1}$$

(3.2)

(with the convention that $\lambda_{N+1} = 0$). Let $\| \cdot \|_\Delta = \sqrt{\langle \cdot, \cdot \rangle_\Delta}$. The following theorem states that for $m(\lambda) \to \infty$ the strong $L^2$-asymptotics of the polynomials $P_\lambda(x)$ is given by the functions $P^\infty_\lambda(x)$, with an exponential error bound governed by the decay rate $\epsilon$ of the Taylor coefficients $\hat{c}_{n,p}$ of the reduced $c$-functions $\hat{c}_p(z)$.

**Theorem 3.1** (Asymptotic Formula A). One has that

$$\|P_\lambda - P^\infty_\lambda\|_\Delta = O(e^{-\epsilon m(\lambda)/2}) \quad \text{as} \quad m(\lambda) \to \infty.$$

If the polynomials $P_\lambda$ are moreover orthogonal when non-comparable in the partial order (i.e., if they form an orthonormal basis of $A^W$), then we have the following alternative error bound.

**Theorem 3.2** (Asymptotic Formula B). If the basis $\{P_\lambda\}_{\lambda \in \Lambda}$ is orthogonal, then one has that

$$\|P_\lambda - P^\infty_\lambda\|_\Delta = O(\lambda_N^\epsilon e^{-\epsilon m(\lambda)}) \quad \text{as} \quad m(\lambda) \to \infty.$$

If the growth of $m(\lambda)$ and $\lambda_1$ is proportional, then the error bound of Theorem 3.2 is more efficient than that of Theorem 3.1. For instance, for $\lambda \in \Lambda$ fixed and strongly dominant (i.e. with $m(\lambda) > 0$), we get the following asymptotics along the discrete ray $\lambda N$.

**Corollary 3.3** (Ray Asymptotics). Let $\lambda \in \Lambda$ be fixed and strongly dominant. If the basis $\{P_\lambda\}_{\lambda \in \Lambda}$ is orthogonal, then one has that

$$\|P_\lambda - P^\infty_\lambda\|_\Delta = O(\epsilon^N e^{-\epsilon m(\lambda)}) \quad \text{as} \quad \epsilon \to \infty.$$

In case of polynomial reduced $c$-functions $\hat{c}_p(z)$, the asymptotic formula turns out to be exact for $m(\lambda)$ sufficiently large.

**Theorem 3.4** (Exact Asymptotics). If there exists a nonnegative integer $M$ such that $a_{n,0}^+ = 0$, $\forall n > M$ and $a_{n,1}^+ = 0$, $\forall n > 2M$, then one has that

$$P_\lambda(x) = \sum_{n=0}^{M-1} P^\infty_n(x)$$

for $m(\lambda) \geq M - 1$, where $N_\lambda^\infty \equiv \|P^\infty_\lambda\|_\Delta = 1$ if $m(\lambda) \geq M$.

For $M = 0$, the $c$-functions are of the form $c_0(x) = (1 - e^{-ix})^{-1}$ and $c_1(x) = (1 - e^{-2ix})^{-1}$, respectively, and the polynomials $P_\lambda(x)$ amount in this case to the characters of the symplectic Lie group $SP(2N; \mathbb{C})$ (with root system $C_N$). The formula of Theorem 3.4 boils then down to the Weyl character formula.

For $M = 1$, the $c$-functions are of the form $c_0(x) = (1 - te^{-ix})(1 - e^{-ix})^{-1} = (1 - t_0 e^{-ix})(1 - e^{-ix})^{-1} e^{-2ix})^{-1}$ (with $-1 < t, t_0, t_1 < 1$), respectively, and the polynomials $P_\lambda(x)$ amount in this case to Macdonald’s generalized Hall-Littlewood polynomials associated with the root system $BC_N$. The formula of Theorem 3.4 boils then down to the standard explicit representation for these polynomials (cf. Eq. (10.1) of [M1]).
When replacing the partial order $\geq$ in the Gram-Schmidt process of definition (2.6a), (2.6b) by the lexicographical ordering $\succeq$, one ends up with an orthonormal basis \{$P_{\lambda}\}_{\lambda \in \Lambda}$ of $AW$. It is clear from the proofs in Sections 5 and 3 that the asymptotics of the polynomials $P_{\lambda}(x)$, $\lambda \in \Lambda$ is again given by Theorem 3.1, by Theorem 3.2 (and thus Corollary 3.3), and by Theorem 3.4 (with the same asymptotic functions $P_{\lambda}^{\infty}(x)$ and the same error bound $\epsilon$). (The crux is that Proposition 5.1 and Lemma 6.2 below remain valid when replacing the partial order $\geq$ by the lexicographical linear refinement $\succeq$.) In particular, it follows from this observation that in the situation of Theorem 3.4, one has that $(P_{\lambda}, P_{\mu})_{\Delta} = \delta_{\lambda \mu}$ for $\lambda, \mu \in \Lambda$ such that $\min(m(\lambda), m(\mu)) \geq M - 1$ (even if $\lambda$ and $\mu$ are not comparable in the partial order $\geq$).

4. Specialization to Koornwinder-Macdonald Polynomials

By picking reduced c-functions $c_{p}(z)$, $p = 0, 1$, of the form
\begin{align}
  c_{0}(z) &= \frac{(tz; q)_{\infty}}{(qz; q)_{\infty}}, \\
  c_{1}(z) &= \frac{\prod_{r=0}^{3}(t_{r}; q)_{\infty}}{(qz^{2}; q)_{\infty}}, \\
\end{align}
where $(z; q)_{\infty} \equiv \prod_{n=0}^{\infty}(1 - zq^{n})$, and with parameters subject to the constraints
\begin{equation}
  0 < q < 1, \quad -1 < t, t_{r} < 1 \quad (r = 0, \ldots, 3),
\end{equation}
the weight function $\Delta(x)$ (2.4a), (2.4b) specializes to
\begin{equation}
  \Delta(x) = \frac{1}{2^{N}N!C(x)C(-x)},
\end{equation}
with
\begin{equation}
  C(x) = \prod_{1 \leq j < k \leq N} \frac{(te^{-i(x_{j}+x_{k})}, te^{-i(x_{j}-x_{k})}; q)_{\infty}}{(e^{-i(x_{j}+x_{k})}, e^{-i(x_{j}-x_{k})}; q)_{\infty}} \\
  \times \prod_{1 \leq j \leq N} \frac{\prod_{r=0}^{3}(t_{r}; q)_{\infty}}{(e^{-2ix_{j}}; q)_{\infty}}.
\end{equation}
(and $(z_{1}, z_{2}, \ldots, z_{k}; q)_{\infty} \equiv (z_{1}; q)_{\infty}(z_{2}; q)_{\infty} \cdots (z_{k}; q)_{\infty}$). It was shown by Koornwinder that the polynomials $P_{\lambda}(x)$, $\lambda \in \Lambda$ (2.6a), (2.6b), form an orthogonal system $K$ (i.e., the polynomials are also orthogonal when non-comparable in the partial order). The conditions on the parameters $q$, $t$ and $t_{0}, \ldots, t_{3}$ in Equation (4.1b) ensure that the reduced c-functions $c_{0}(z), c_{1}(z)$ in Equation (4.1a) satisfy the technical requirements stipulated in Section 2. In particular, for the bound on the decay rate of the Taylor coefficients $a_{n,p}^{+}$ we have $q_{0} = q^{-1}$ and $q_{1} = q^{-1/2}$, so we may choose (any) $\epsilon \in (0, \log(1/q))$.

Specialization of the results of Section 3 to the weight function $\Delta(x)$ (4.2a), (4.2b) immediately entails the main application of our asymptotic analysis.

Corollary 4.1 (Asymptotics of Koornwinder-Macdonald Polynomials). The asymptotics of the Koornwinder-Macdonald polynomials is governed by Theorem 3.1 Theorem 3.2 and Corollary 3.3 with asymptotic functions $P_{\lambda}^{\infty}(x)$ (3.1) characterized by the product c-function $C(x)$ (4.2b), and an error bound with a decay rate that is at least as fast as any $\epsilon$ taken from the interval $(0, \log(1/q))$. 
5. The Asymptotic Functions

This section exhibits some properties of the asymptotic functions \( P_\lambda^\infty(x) \) that are needed in the proof of the asymptotic formulas stated in Section 3.

**Proposition 5.1** (Partial Biorthogonality). Let \( \lambda, \mu \in \Lambda \) with \( \mu \leq \lambda \). Then

\[
\langle P_\lambda^\infty, m_\mu \rangle_\Delta = \begin{cases} 0 & \text{if } \mu < \lambda, \\ 1 & \text{if } \mu = \lambda. 
\end{cases}
\]

**Proof.** From the \( W \)-invariance of the weight function \( \Delta(x) = \frac{1}{e(x)c(-x)} \) it is clear that

\[
\langle P_\lambda^\infty, m_\mu \rangle_\Delta = \frac{1}{(2\pi)^N |W||W_\mu|} \int_{T_N} \frac{1}{C(x)C(-x)} \sum_{w_1 \in W} C(xw_1) e^{i(\lambda, xw_1)} \sum_{w_2 \in W} e^{-i(\mu, xw_2)} dx
\]

The integral on the last line picks up the constant term of the integrand (times \((2\pi)^N\)). It is immediate from our assumptions on the structure of the \( c \)-functions that \( 1/C(-x) \) has a (uniformly converging) Fourier expansion of the form \( 1 + \sum_{n \in \mathbb{Z}^N} a_n e^{i(n,x)} \), whence the constant term in question is equal to 1 if \( \lambda = \mu_w = \mu \) and equal to 0 otherwise. (Here we used the standard fact that \( \mu \geq \mu_w \) for all \( \mu \in \Lambda \) and \( w \in W \), cf. also Lemma 5.3 below.) \( \square \)

By factoring-off the denominators of the \( c \)-functions, one rewrites \( P_\lambda^\infty(x) \) as

\[
P_\lambda^\infty(x) = \delta^{-1}(x) \sum_{w \in W} \det(w) \hat{c}(x_w) e^{i(\lambda, x_w)} , \tag{5.1a}
\]

with

\[
\hat{c}(x) = \prod_{1 \leq j < k \leq N} \hat{c}_0(e^{-i(x_j + x_k)}) \hat{c}_0(e^{-i(x_j - x_k)}) \prod_{1 \leq j \leq N} \hat{c}_1(e^{-ix_j}) , \tag{5.1b}
\]

\[
\delta(x) = \prod_{1 \leq j < k \leq N} (e^{i(x_j + x_k)/2} - e^{-i(x_j + x_k)/2}) (e^{i(x_j - x_k)/2} - e^{-i(x_j - x_k)/2}) \times \prod_{1 \leq j \leq N} (e^{ix_j} - e^{-ix_j}) , \tag{5.1c}
\]

and

\[
\rho = \sum_{j=1}^N (N + 1 - j)e_j . \tag{5.1d}
\]

(Here \( e_j \) denotes the \( j^{\text{th}} \) unit vector in the standard basis of \( \mathbb{R}^N \).) We introduce the following polynomial truncation of the asymptotic function \( P_\lambda^\infty(x) \):

\[
P_\lambda^{(m)}(x) = \delta^{-1}(x) \sum_{w \in W} \det(w) \hat{c}^{(m)}(x_w) e^{i(\lambda, x_w)} , \tag{5.2a}
\]

with

\[
\hat{c}^{(m)}(x) = \prod_{1 \leq j < k \leq N} \hat{c}_0^{(m)}(e^{-i(x_j + x_k)}) \hat{c}_0^{(m)}(e^{-i(x_j - x_k)}) \prod_{1 \leq j \leq N} \hat{c}_1^{(m)}(e^{-ix_j}) . \tag{5.2b}
\]
The quotient $\Delta(x)$ consists of the first $m+1$ and $2m+1$ terms of the Taylor expansions of the reduced $c$-functions $c_0(z)$ and $c_1(z)$, respectively, i.e.

$$c_0^{(m)}(z) = 1 + \sum_{n=1}^{m} a_{n,0}^+ z^n, \quad c_1^{(m)}(z) = 1 + \sum_{n=1}^{2m} a_{n,1}^+ z^n \quad (5.2c)$$

(with the coefficients $a_{n,p}^+ (p = 0, 1)$ being defined by Eq. \(2.8\)).

**Proposition 5.2** (Asymptotic Error Bound). One has that

$$P_\lambda^\infty(x) = P_\lambda^{(m)}(x) + \mathcal{E}_\lambda^{(m)}(x),$$

with

$$\|\mathcal{E}_\lambda^{(m)}\|_\Delta = O(e^{-\epsilon m}) \quad \text{as } m \to \infty$$

(uniformly in $\lambda$).

**Proof.** Let us write

$$\hat{C}(x) = \hat{c}^{(m)}(x) + \mathcal{R}^{(m)}(x).$$

The error between $P_\lambda^{(m)}(x)$ and $P_\lambda^\infty(x)$ is then given by (cf. Eqs. \(5.1\), \(5.2a\))

$$\mathcal{E}_\lambda^{(m)}(x) = \delta^{-1}(x) \sum_{w \in W} \det(w) \mathcal{R}^{(m)}(x_w) e^{i(\lambda + \rho \cdot x_w)}.$$  

The quotient $\Delta(x)/|\delta(x)|^2 = 1/(\hat{C}(x)\hat{C}(-x))$ is smooth on the torus $\mathbb{T}_N$ due to the absence of zeros in the $c$-functions. Hence, to prove the error bound on $\|\mathcal{E}_\lambda^{(m)}\|_\Delta$ it is enough to show that $\max_{x \in \mathbb{T}_N} (\mathcal{R}^{(m)}(x)) = O(e^{-\epsilon m})$. To this end we set $\hat{c}_p(z) = \hat{c}_p^{(m)}(z) + r_p^{(m)}(z)$, whence

$$\mathcal{R}^{(m)}(x) = \prod_{1 \leq j < k \leq N} \left( c_0^{(m)}(e^{-i(x_j + x_k)}) + r_0^{(m)}(e^{-i(x_j + x_k)}) \right)$$

$$\times \left( c_0^{(m)}(e^{-i(x_j - x_k)}) + r_0^{(m)}(e^{-i(x_j - x_k)}) \right)$$

$$\times \left( c_1^{(m)}(e^{-i(x_j - x_k)}) + r_1^{(m)}(e^{-i(x_j - x_k)}) \right)$$

$$- \prod_{1 \leq j < k \leq N} c_0^{(m)}(e^{-i(x_j + x_k)}) c_1^{(m)}(e^{-i(x_j - x_k)}) \prod_{1 \leq j \leq N} c_1^{(m)}(e^{-i x_j}).$$

The bound on $\max_{x \in \mathbb{T}_N} (\mathcal{R}^{(m)}(x))$ thus follows since $\max_{|z| = 1} (\hat{c}_p^{(m)}(z)) = O(1)$ and $\max_{|z| = 1} (r_p^{(m)}(z)) = O(e^{-\epsilon m})$, in view of the $O(e^{-\epsilon n})$ and $O(e^{-\epsilon n/2})$ decay rates of the expansion coefficients $a_{n,0}^+$ and $a_{n,1}^+$ for the reduced $c$-functions $\hat{c}_0(z)$ and $\hat{c}_1(z)$, respectively. \(\square\)

For $m$ sufficiently small, the polynomial truncation $P_\lambda^{(m)}(x)$ of the asymptotic function $P_\lambda^\infty(x)$ expands triangularly on the monomial basis. This observation hinges on the following lemma.

**Lemma 5.3.** Let $\lambda \in \Lambda$ and let

$$\mu = \lambda - \sum_{1 \leq j < k \leq N} \left( n_{j,k}^+ (e_j + e_k) + n_{j,k}^- (e_j - e_k) \right) + \sum_{1 \leq j \leq N} n_j e_j,$$

with $0 \leq n_{j,k}^+ n_{j,k}^- \leq m(\lambda)$ and $0 \leq n_j \leq 2m(\lambda)$. Then one has that

$$\mu_w \leq \lambda, \quad \forall w \in W.$$
Furthermore, if \( 0 \leq n^+_j, n^-_j < m(\lambda) \) and \( 0 \leq n_j < 2m(\lambda) \), then the equality \( \mu_w = \lambda \) is assumed if and only if \( w = Id \) (and all \( n^+_j, n^-_j, n_j \) vanish).

Proof. The components of \( \mu \) are given by

\[
\mu_j = \lambda_j - n_j - \sum_{1 \leq k < j} (n^+_k - n^-_k) - \sum_{j < k < N} (n^+_k + n^-_k),
\]

\( j = 1, \ldots, N \). It thus follows that

\[
\sum_{j=1}^{\ell} \varepsilon_j \mu_{\sigma_j} = \sum_{j \in J_+} (\lambda_{\sigma_j} - n_{\sigma_j} - \sum_{1 \leq k < \sigma_j} (n^+_k - n^-_k) - \sum_{\sigma_j < k < N} (n^+_k + n^-_k)) + \sum_{j \in J_-} (n_{\sigma_j} - \lambda_{\sigma_j} + \sum_{1 \leq k < \sigma_j} (n^+_k - n^-_k) + \sum_{\sigma_j < k < N} (n^+_k + n^-_k)) \leq \sum_{j \in J_+} (\lambda_{\sigma_j} + \sum_{1 \leq k < \sigma_j} n^-_k) + \sum_{j \in J_-} (n_{\sigma_j} - \lambda_{\sigma_j} + \sum_{1 \leq k < \sigma_j} n^+_k + \sum_{\sigma_j < k < N} n^+_k + \sum_{\sigma_j < k < N} n^-_k),
\]

(5.3)

where \( J_+ = \{ 1 \leq j \leq \ell \mid \varepsilon_j = +1 \} \) and \( J_- = \{ 1 \leq j \leq \ell \mid \varepsilon_j = -1 \} \). The proof of the first part of the lemma now hinges on successive application of the following three elementary ‘transportation’ inequalities

\[
\lambda_j + n \leq \lambda_{j-1} \quad (A), \quad n - \lambda_j \leq -\lambda_{j+1} \quad (B), \quad m - \lambda_N \leq \lambda_N \quad (C),
\]

for \( 0 \leq n \leq m(\lambda) \) and \( 0 \leq m \leq 2m(\lambda) \). Indeed, ordering of the components \( \lambda_{\sigma_j}, j \in J_+ \) from small to large and iterated application of inequality (A) (so as to ‘transport’ to \( \lambda_1, \lambda_2, \ldots, \lambda_{|J_+|} \), respectively) readily entails that

\[
\sum_{j \in J_+} (\lambda_{\sigma_j} + \sum_{1 \leq k < \sigma_j} n^-_k) \leq \lambda_1 + \lambda_2 + \cdots + \lambda_{|J_+|}
\]

(5.4)

(where \( |J_+| \) denotes the number of elements of \( J_+ \)). In a similar way we obtain that

\[
\sum_{j \in J_-} (n_{\sigma_j} - \lambda_{\sigma_j} + \sum_{1 \leq k < \sigma_j} n^+_k + \sum_{\sigma_j < k < N} n^+_k) \leq \sum_{j \in J_-} \left( n_{\sigma_j} + \sum_{1 \leq k < \sigma_j} n^+_k + \sum_{\sigma_j < k < N} n^+_k \right) - (\lambda_{N-|J_-|+1} + \lambda_{N-|J_-|+2} + \cdots + \lambda_N)
\]

\[
(5.5)
\]

Here the inequality (i) is inferred by ordering \(-\lambda_{\sigma_j}, j \in J_- \) from large to small, followed by iterated application of inequality (B) (‘transporting’ to \(-\lambda_{N+1-j}, j = 1, \ldots, |J_-| \), respectively); the inequality (ii) then follows by iterated application of, respectively, inequality (B) (‘transporting’ from \(-\lambda_{N+1-j} \) to \(-\lambda_N \)), inequality (C) (‘flipping’ the sign from \(-\lambda_N \) to \(+\lambda_N \)), and inequality (A) (‘transporting’ back
to $\lambda_{|J_+|+j}$ (with $j = 1, \ldots, |J_-|$). Combination of Eqs. (6.3), (6.4) and (6.5) now gives that $\sum_{j=1}^\ell \varepsilon_j \mu_{\sigma_j} \leq \sum_{j=1}^\ell \lambda_j$ for $\ell = 1, \ldots, N$, which proves the first part of the lemma. To prove the second part, one observes that the elementary inequalities (A), (B), and (C), become strict for $0 \leq n < m(\lambda)$ and $0 \leq m < 2m(\lambda)$. Hence, for $0 \leq n_j^+, n_j^- < m(\lambda)$ and $0 \leq n_j < 2m(\lambda)$ the inequality in Eq. (5.6) becomes strict unless $\sigma(J_+) = \{1, 2, \ldots, |J_+|\}$, and the inequality in Eq. (5.5) becomes strict unless $J_- = \emptyset$ (so $J_+ = \{1, 2, \ldots, \ell\}$). Thus, the upshot is that now $\sum_{j=1}^\ell \varepsilon_j \mu_{\sigma_j} = \sum_{j=1}^\ell \lambda_j$ for $\ell = 1, \ldots, N$ if and only if $\{\sigma_1, \ldots, \sigma_\ell\} = \{1, \ldots, \ell\}$ for $\ell = 1, \ldots, N$ and $\varepsilon_j = +1$ for $j = 1, \ldots, N$, i.e., if and only if $w = \text{Id}$ (whence all $n_j^+, n_j^-$ vanish).

**Proposition 5.4** (Triangularity). For $\lambda \in \Lambda$ and $m \leq m(\lambda) + 1$, one has that

$$P^{(m)}_{\lambda}(x) = \sum_{\mu \in \Lambda, \mu \leq \lambda} a^{(m)}_{\lambda\mu} m_\mu(x),$$

with $a^{(m)}_{\lambda\mu} \in \mathbb{C}$. Furthermore, for $m \leq m(\lambda)$ the polynomial $P^{(m)}_{\lambda}(x)$ is monic (i.e. $a^{(m)}_{\lambda\lambda} = 1$).

**Proof.** It is immediate from the definition that the truncated asymptotic function $P^{(m)}_{\lambda}(x)$ can be written as a finite linear combination of symmetric polynomials of the form

$$\delta^{-1}(x) \sum_{w \in W} \det(w)e^{i(\lambda + \rho, x_w)},$$

where

$$n = \sum_{1 \leq j < k \leq N} \left( n_{jk}^+(e_j + e_k) + n_{jk}^-(e_j - e_k) \right) + \sum_{1 \leq j \leq N} n_j e_j, \quad (5.6b)$$

with $0 \leq n_{j^+}, n_{j^-} \leq m$ and $0 \leq n_j < 2m$. The polynomial in Eq. (5.6b) vanishes if $\lambda + \rho - n$ is singular with respect to the action of the Weyl group (i.e. if it has a nontrivial stabilizer) and otherwise it is equal, possibly up to a sign, to the Weyl character

$$\chi_\mu = \delta^{-1}(x) \sum_{w \in W} \det(w)e^{i(\mu + \rho, x_w)}, \quad (5.7)$$

where $\mu$ is the unique dominant weight in the translated Weyl orbit $W(\lambda + \rho - n) - \rho$. It follows from the first part of Lemma 5.3 and the assumption $m \leq m(\lambda) + 1 = m(\lambda + \rho)$, that $\mu \leq \lambda$. It moreover follows from the second part of Lemma 5.3 that for $m \leq m(\lambda)$, one has that $\mu = \lambda$ if and only if all $n_{j^+}, n_{jk}^-$ and $n_j$ are zero.

The proposition now follows from the well-known fact that the Weyl characters are monic $W$-invariant polynomials that expand triangularly on the basis of monomial symmetric functions.

We conclude this section with estimates for the norm of the asymptotic function $P^\infty_{\lambda}(x)$ and the for the leading coefficient in the monomial expansion of the normalized polynomial $P^\infty_{\lambda}(x)$.

**Proposition 5.5** (Norm Estimate). One has that

$$\|P^\infty_{\lambda}\|_\Delta = 1 + O(e^{-c m(\lambda)}) \quad \text{as } m(\lambda) \to \infty.$$
Proof. It is clear that
\[ \|P_\lambda^\infty\|_\Delta^2 = \langle P_\lambda^\infty, P_\lambda^\infty \rangle_\Delta = \langle P_\lambda^\infty, P_\lambda^{(m(\lambda))} \rangle_\Delta + \langle P_\lambda^\infty, \epsilon_\lambda^{(m(\lambda))} \rangle_\Delta. \]

The proposition now follows from the observation that \( \langle P_\lambda^\infty, P_\lambda^{(m(\lambda))} \rangle_\Delta = 1 \) by Propositions 5.1 and 5.4 combined with the error estimate \( \epsilon_\lambda^{(m(\lambda))} \) by Proposition 5.2 (using also that \( \max_{x} \) and that \( \Delta(x) \))

\[ (\text{Leading Coefficient}) \]

A sequence of elementary manipulations entails that
\[ (i) \quad a_{\lambda\lambda} = \frac{\langle P_\lambda, P_\lambda^{\infty} \rangle_\Delta}{\langle P_\lambda, P_\lambda^{(m(\lambda))} + \epsilon_\lambda^{(m(\lambda))} \rangle_\Delta} \]
\[ (ii) \quad a_{\lambda\lambda} = \frac{\langle P_\lambda, a_{\lambda\lambda}^{-1} P_\lambda + \epsilon_\lambda^{(m(\lambda))} \rangle_\Delta}{\langle P_\lambda, a_{\lambda\lambda}^{-1} P_\lambda + \epsilon_\lambda^{(m(\lambda))} \rangle_\Delta} \]
\[ (iii) \quad a_{\lambda\lambda} = a_{\lambda\lambda}^{-1} \quad \text{as} \quad m(\lambda) \to \infty. \]

Let \( N_\lambda = a_{\lambda\lambda}^{-1} \), where \( a_{\lambda\lambda} > 0 \) represents the leading coefficient of the polynomial \( P_\lambda(x) \) in the monomial basis.

**Proposition 5.6 (Leading Coefficient).** One has that
\[ N_\lambda = 1 + O(e^{-\epsilon m(\lambda)}) \quad \text{as} \quad m(\lambda) \to \infty. \]

Proof. A sequence of elementary manipulations entails that
\[ a_{\lambda\lambda} = \frac{\langle P_\lambda, P_\lambda^{\infty} \rangle_\Delta}{\langle P_\lambda, P_\lambda^{(m(\lambda))} \rangle_\Delta} \]

\[ \text{whence} \quad N_\lambda^{-1} = a_{\lambda\lambda} = 1 + O(e^{-\epsilon m(\lambda)}). \]

6. **Proofs of the Main Theorems**

By combining the properties in Section 6, we arrive at the proofs of the theorems stated in Section 5.

6.1. **Proof of Theorem 5.1** Straightforward manipulations reveal that
\[ \|P_\lambda - P_\lambda^{\infty}\|_\Delta^2 = \langle P_\lambda, P_\lambda \rangle_\Delta - \langle P_\lambda, P_\lambda^{\infty} \rangle_\Delta - \langle P_\lambda^{\infty}, P_\lambda \rangle_\Delta + \langle P_\lambda^{\infty}, P_\lambda^{\infty} \rangle_\Delta \]

\[ (i) \quad 1 - 2N_\lambda^{-1} + \|P_\lambda^{\infty}\|_\Delta^2 \]

\[ (ii) \quad O(e^{-\epsilon m(\lambda)}), \]

whence \( \|P_\lambda - P_\lambda^{\infty}\|_\Delta = O(e^{-\epsilon m(\lambda)/2}). \) Step (i) hinges on Eq. 2.6a and Proposition 5.3 which implies that \( \langle P_\lambda, P_\lambda^{\infty} \rangle_\Delta = \langle P_\lambda^{\infty}, P_\lambda \rangle_\Delta = a_{\lambda\lambda} = N_\lambda^{-1}. \) Step (ii) follows from the estimates in Proposition 5.3 and Proposition 5.6.
6.2. Proof of Theorem 3.2. From Proposition 5.1 and Eqs. (6.1a), (6.1b) it is immediate that for \( m \leq m(\lambda) \)

\[
P_{\lambda}^{(m)}(x) = N_{\lambda} P_{\lambda}(x) + \sum_{\mu \in \Lambda, \mu < \lambda} b_{\lambda \mu}^{(m)} P_{\mu}(x),
\]

where \( N_{\lambda} = a_{\lambda \lambda}^{-1} \) and \( b_{\lambda \mu}^{(m)} \in \mathbb{C} \).

**Lemma 6.1.** If the basis \( \{ P_{\lambda} \}_{\lambda \in \Lambda} \) is orthonormal, then

\[
|b_{\lambda \mu}^{(m)}| \leq ||\mathcal{E}_{\lambda}^{(m)}||_{\Delta}
\]

for \( \lambda, \mu \in \Lambda \) with \( \mu < \lambda \) and \( m \leq m(\lambda) \).

**Proof.** Respectively applying Eq. (6.1), Proposition 5.2, and Proposition 5.1 readily entails that

\[
b_{\lambda \mu}^{(m)} = \langle P_{\lambda}^{(m)}, P_{\mu} \rangle_{\Delta} = \langle P_{\lambda}^{\infty} - \mathcal{E}_{\lambda}^{(m)}, P_{\mu} \rangle_{\Delta} = -\langle \mathcal{E}_{\lambda}^{(m)}, P_{\mu} \rangle_{\Delta}.
\]

Hence \( |b_{\lambda \mu}^{(m)}| \leq ||\mathcal{E}_{\lambda}^{(m)}||_{\Delta} \) by the Cauchy-Schwarz inequality. \( \square \)

**Lemma 6.2.** For \( \lambda \in \Lambda \) let \( A_{\lambda}^{W} = \text{Span}\{ m_{\mu} \}_{\mu \in \Lambda, \mu \leq \lambda} \). Then

\[
\dim(A_{\lambda}^{W}) \leq (1 + \lambda_{1})^{N}.
\]

**Proof.** Immediate from the observation that for \( \mu \in \Lambda \) the inequality \( \mu \leq \lambda \) implies that \( 0 \leq \mu_{j} \leq \lambda_{1} \) for \( j = 1, \ldots, N \). \( \square \)

The error bound of Theorem 3.2 now follows from the estimates

\[
\| P_{\lambda} - P_{\lambda}^{\infty} \|_{\Delta} \leq (i) \| P_{\lambda} - P_{\lambda}^{(m)} \|_{\Delta} + ||\mathcal{E}_{\lambda}^{(m)}||_{\Delta}
\]

\[
\leq (ii) |N_{\lambda} - 1| + \dim(A_{\lambda}^{W}) ||\mathcal{E}_{\lambda}^{(m)}||_{\Delta},
\]

whence \( \| P_{\lambda} - P_{\lambda}^{\infty} \|_{\Delta} = O(\lambda_{1}^{N} e^{-e^{m(\lambda)}}) \) by Proposition 5.2, Proposition 5.6, and Lemma 6.2. Here step (i) hinges on Proposition 5.2 and in step (ii) we employed Eq. (6.1) combined with Lemma 6.1.

6.3. Proof of Theorem 3.3. If the reduced \( \epsilon \)-functions \( \hat{c}_{0}(z) \) and \( \hat{c}_{1}(z) \) are polynomial in \( z \) of degree at most \( M \) and \( 2M \), respectively, then \( P_{\lambda}^{\infty}(x) = P_{\lambda}^{(m)}(x) \) (i.e. \( \mathcal{E}_{\lambda}^{(m)} = 0 \)) for \( m \geq M \). Hence, by Proposition 5.2, one has in this case that

\[
P_{\lambda}^{\infty}(x) = \sum_{\mu \in \Lambda, \mu \leq \lambda} a_{\lambda \mu}^{\infty} m_{\mu}(x) \quad (a_{\lambda \mu}^{\infty} \in \mathbb{C}),
\]

provided \( m(\lambda) \geq M - 1 \). Invoking of Proposition 5.1 and comparing with the defining relations for \( P_{\lambda}(x) \) in Eqs. (2.6a), (2.6b), leads to the conclusion that the asymptotic functions in question coincide with the latter polynomials up to normalization

\[
P_{\lambda}(x) = \frac{P_{\lambda}^{\infty}(x)}{\| P_{\lambda}^{\infty} \|_{\Delta}} \quad \text{for } m(\lambda) \geq M - 1.
\]

The (square of the) normalization factor reads

\[
\| P_{\lambda}^{\infty} \|_{\Delta}^{\text{Eq. (6.2)}} = \langle P_{\lambda}^{\infty}, \sum_{\mu \in \Lambda, \mu \leq \lambda} a_{\lambda \mu}^{\infty} m_{\mu} \rangle_{\Delta} \stackrel{\text{Prop. 5.1}}{=} a_{\lambda \lambda}^{\infty},
\]

which, by Proposition 5.4, is equal to 1 when \( m(\lambda) \geq M \).
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