Takasaki’s rational fourth Painlevé-Calogero system and geometric regularisability of algebro-Painlevé equations

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Abstract
We propose a notion of regularisation which extends Okamoto’s construction of spaces of initial conditions for the Painlevé differential equations to the class of systems with globally finite branching about movable singularities in the sense of the algebro-Painlevé property. We illustrate this regularisation first in the case of a Hamiltonian system obtained by Takasaki as part of the Painlevé-Calogero correspondence, which is related by an algebraic transformation to the fourth Painlevé equation. Through a combination of compactification, blowups and removal of certain curves we obtain a space on which the system is everywhere either regular or regularisable by certain algebraic transformations. We provide an atlas for this space in which the system has a global Hamiltonian structure, with all Hamiltonian functions being polynomial in coordinates just as in the case of the Painlevé equations on Okamoto’s spaces. We also compare the surface associated with the Takasaki system with that of the fourth Painlevé equation, showing that they are related by a combination of blowdowns and a branched double cover. We provide more examples of regularisable systems.
algebro-Painlevé equations regularised in this way and also discuss applications of this generalised construction of the space of initial conditions to the identification and classification of algebro-Painlevé equations.

Keywords: Painlevé equations, space of initial conditions, rational surface, algebro-Painlevé property
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(Some figures may appear in colour only in the online journal)

1. Introduction

It is well-known that solutions of a linear ordinary differential equation (ODE) can have only fixed singularities in the complex plane, i.e. those whose locations are determined by the coefficients of the equation. For a nonlinear differential equation, solutions may have singularities whose locations depend on initial conditions and are, therefore, called movable. The Painlevé property for differential equations requires that all solutions are single-valued about all movable singularities, which is related to the ability to meaningfully define special functions in terms of the general solutions to such equations. This idea was pursued in the early 1900’s by P Painlevé and his student B Gambier, as well as R Fuchs, and they considered a large class of nonlinear second-order ODEs and classified those with this property into 50 classes. All equations could be solved in terms of known functions or in terms of solutions of what are now known as the six Painlevé equations, which we will denote by $P_J$, $J = I, \ldots, VI$. These equations have remarkable properties and have found applications in many areas of mathematics and mathematical physics (see, for example, [8] for further references).

In a seminal paper [26], Okamoto showed that each Painlevé equation is regular everywhere on a complex manifold providing an extended phase space for an equivalent first-order system, constructed through a combination of compactification, blowups and removal of certain curves. This associates a special kind of rational surface to the differential equation, the geometry of which explains many of its properties. Okamoto’s work was built on in a series of papers by Takano and collaborators [24, 25, 32] in which they proved that the manifolds for $P_{II}$–$P_{VI}$ essentially determine the equations (the case of $P_I$ was shown later independently by Chiba [5] and Iwasaki-Okada [18]), giving rise to the idea that geometry tells one everything about the Painlevé equations. This idea, with its origins in Okamoto’s work, has gone on to inform much of the recent study of Painlevé equations, especially their discrete versions [30], via the associated geometric objects (see the important survey [20] and references within).

The regularisation on Okamoto’s space relies heavily on the Painlevé property, but it is natural to ask whether equations with weaker restrictions on singularities can still be associated to rational surfaces. The main consideration to be made, in addressing such a question, is that of what the appropriate relaxation of the requirement of regularity is that would allow such an association of surface to equation, and how it should correspond to the nature of singularities of solutions.

In [21], Kecker and Filipuk considered a class of equations with the quasi-Painlevé property, i.e. that all movable solutions reachable by analytic continuation along finite length curves are at worst algebraic branch points, and established a kind of regularising transformation by which the construction of a space of initial conditions is possible. For a system of first-order differential equations

$$\frac{dq}{dt} = F_1(q, p, t), \quad \frac{dp}{dt} = F_2(q, p, t),$$

(1)
with the quasi-Painlevé property, they compactified the fibres of the phase space from \( \mathbb{C}^2 \) to \( \mathbb{CP}^2 \) and resolved all indeterminacies of the system through blowups. This yields local coordinates \((u, v)\) on the resulting space in which the system is of the form

\[
\frac{du}{dt} = \frac{G_1(u, v, t)}{v^k}, \quad \frac{dv}{dt} = \frac{G_2(u, v, t)}{v^k},
\]

where \(G_1, G_2\) are analytic in some neighbourhood of \(v = 0\), and \(G_1(u, 0, t), G_2(u, 0, t) \neq 0\), and \(k\) is a positive integer. Then by transferring the role of independent variable from \(t\) to \(v\) (a trick also employed in [6]) the system becomes

\[
\frac{du}{dv} = \frac{G_1(u, v, t)}{G_2(u, v, t)}, \quad \frac{dt}{dv} = \frac{v^k}{G_2(u, v, t)},
\]

and this defines regular initial value problems at \((u, v, t) = (h, 0, t_*)\). Inversion of the power series solutions \(t(v), u(v)\) of these regular initial value problems gives \(u(t), v(t)\) as Puiseux series, which under the birational transformation back to the original variables \(q, p\) correspond to \((k + 1)\)th root type branching about a movable singularity \(t_*\).

Our starting point in this paper is the question of what regularising transformations are possible when we impose a stronger requirement that solutions are not just locally but globally finitely branched in the sense of the algebro-Painlevé property, i.e. that all solutions are algebraic over the field of meromorphic functions on the universal cover of \(\mathbb{C}\setminus F\), where \(F\) is the finite set of fixed singularities of the equation.

We begin by studying a non-autonomous rational Hamiltonian system obtained by Takasaki [37] as part of his extension of the Painlevé-Calogero correspondence beyond the celebrated elliptic form of the sixth Painlevé equation due to Manin [23] (which built on the works of Fuchs [7] and Painlevé [29] in which similar forms had already appeared). Though Takasaki’s system does not possess the Painlevé property, we proceed along the same lines as Okamoto and construct a family of rational surfaces which provide a manifold on which the system is everywhere either regular or regularisable, and we show that this regularisation can be achieved through algebraic transformations rather than the interchanging of role of independent variable as in the quasi-Painlevé case outlined above.

Of course this algebraic regularisability is not unexpected given that the system is mapped by a rational (but not birational) transformation to the fourth Painlevé equation, and we describe this transformation on the level of the associated surfaces. We provide a global Hamiltonian structure of Takasaki’s system on this manifold and compare this to that for \(P_{IV}\) on Okamoto’s space due to Matano et al [24] (see also [32]). This allows us to establish the relationship between the curves removed from the surfaces constructed from the Takasaki system and the symplectic structure with respect to which its Hamiltonian structure is defined. In the Painlevé case, such curves form the irreducible components of an effective anticanonical divisor of the surface and have intersection configuration encoded in an affine Dynkin diagram, which forms the basis of Sakai’s classification of Painlevé equations and their discrete analogues in terms of the associated surfaces [30].

The study of the Takasaki system leads us to the main idea of the paper, namely that global regularisability on bundles of rational surfaces by similar algebraic transformations is the appropriate notion to allow for the construction of spaces of initial conditions for algebro-Painlevé equations. We introduce such a notion of regularisability and demonstrate it in a suite of examples, including one obtained by Halburd and Kecker in [10] by Nevanlinna-theoretic methods, and show this regularisability can also function as a kind of algebro-Painlevé test. For each of these examples we determine the intersection configuration of inaccessible divisors and discuss possible approaches to classification of algebro-Painlevé equations, with a view
to developing a classification scheme for the wider class of systems with the quasi-Painlevé property via the associated surfaces.

1.1. Okamoto’s space for the fourth Painlevé equation $P_{IV}$

For each of Painlevé equations Okamoto provided an equivalent first-order non-autonomous Hamiltonian system \[\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda} \left( \frac{d\lambda}{dt} \right)^2 + \frac{3}{2} \lambda^3 + 4t\lambda^2 + 2(t^2 - \alpha)\lambda + \frac{\beta}{\lambda},\] (4)

where $\alpha$ and $\beta$ are arbitrary complex parameters. The Okamoto Hamiltonian form of the equation (4) is given by

\[
\begin{align*}
\frac{df}{dt} &= \frac{\partial H_{Ok}}{\partial g} = 4fg - f^2 - 2tf - 2a_1, \\
\frac{dg}{dt} &= -\frac{\partial H_{Ok}}{\partial f} = 2fg - 2g^2 + 2tg + a_2, \\
H_{Ok}(f, g, t) &= 2fg^2 - (f^2 + 2tf + 2a_1)g - a_2f,
\end{align*}
\] (5)

where $a_1, a_2$ are complex parameters which we introduce here for convenience, corresponding to the root variables in the Sakai theory of rational surfaces associated with Painlevé equations [30]. The reason that the first order system (5) is regarded as a form of $P_{IV}$ is that by eliminating $g$ we obtain the equation (4) for the function $f(t)$ with parameters

\[\alpha = 1 - a_1 - 2a_2, \quad \beta = -2a_2^2.\] (6)

As a non-autonomous system of two first-order ODEs, the phase space for (5) is taken initially to be the product $\mathbb{C}^2_f \times \mathbb{C}_t$ (where subscripts indicate coordinates) as a trivial complex analytic fibre bundle over the independent variable space $\mathbb{C}_t$. To construct Okamoto’s space we first compactify the fibres of the phase space to $\mathbb{P}^1 \times \mathbb{P}^1$ then perform a sequence of eight blowups of the fibre over $t$ in order to resolve the indeterminacies of the system, the details of which are deferred to appendix A. We denote the points by $p_1, \ldots, p_8$, with the exceptional divisor arising from the blowup of $p_i$ being denoted by $E_i$. Some of the points are infinitely near, i.e. there are cases when $p_{i+1}$ lies on the exceptional divisor $E_i$. Through this procedure we obtain a rational surface $\mathcal{X}_{t,Ok}$, which we give a schematic depiction of on figure 1.

Next it is necessary to remove the support $D_{Ok}$ of the inaccessible divisors, on which the vector field defining the system is vertical with respect to the bundle structure over $\mathbb{C}_t$, which are the irreducible components of the unique effective anticanonical divisor of $\mathcal{X}_{t,Ok}$ and are indicated in blue on figure 1. After this we denote the resulting complex analytic fibre bundle by

\[
\begin{align*}
\pi : \mathcal{E}_{Ok} &\to \mathbb{C}_t, \\
\pi^{-1}(t) &= \mathcal{E}_{Ok} = \mathcal{X}_{t,Ok} - D_{Ok}.
\end{align*}
\] (7)
To work with the differential system on $E^{Ok}$ we use an atlas due to Matano et al [24], the construction of which we also recall in appendix A. This consists of four charts: one coming from the original $f, g$ coordinates, as well as charts to cover the parts of $E_2, E_4, E_8$ away from the anticanonical divisor, indicated in red on figure 1, on which the system is regular:

$$E^{Ok} = C^3_{f,g,t} \cup C^3_{x_1,y_1,t} \cup C^3_{x_2,y_2,t} \cup C^3_{x_3,y_3,t},$$

with gluing defined by

$$\begin{align*}
1/f &= x_1, & g &= x_1 (-a_2 + y_1 x_1), \\
1/f &= x_2(a_1 + x_2 y_2), & 1/g &= x_2, \\
1/f &= y_3, & 1/g &= \frac{y_3}{1 + y_3 (t + y_3 (a_1 + a_2 - 1 + x_3 y_3))},
\end{align*}$$

so the parts of $E_2, E_4,$ and $E_8$ away from the inaccessible divisors are covered by charts $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ respectively.

The fact that these charts are symplectic means system (5) extended to $E^{Ok}$ via this atlas possesses a global holomorphic Hamiltonian structure. As the system is non-autonomous and the gluing (9) includes transition functions which are $t$-dependent, such a structure is provided by a symplectic form $\omega^{Ok}$ on the fibre over $t$ along with a collection of Hamiltonian functions, one in each chart. In particular the atlas is required to be such that the symplectic form written in coordinates is independent of $t$. That is, each transition function

$$\varphi : C^3 \ni (x,y,t) \mapsto (X(x,y,t), Y(x,y,t), t) \in C^3,$$

with restriction to the fibre over $t$ denoted by

$$\varphi_t : C^2 \ni (x,y) \mapsto (X(x,y,t), Y(x,y,t)) \in C^2,$$

is required to satisfy

$$F(x,y)dx \wedge dy = \varphi_t^* G(X,Y)dx \wedge dy,$$

where $F(x,y)dx \wedge dy$ is the canonical symplectic form on $C^2$.
for some \( t \)-independent functions \( F, G \), where \( \text{d} \) denotes the exterior derivative on \( \mathbb{C}^2 \) and so \( t \) is treated as a constant. With such an atlas, the following standard fact ensures that a Hamiltonian structure for the differential equation in one chart extends to the whole space \( E^{Ok}_t \).

**Lemma 1.** With \( \phi \) as above satisfying the condition (10), then given \( H(x,y,t) \) there exists \( K(X,Y,t) \) (unique modulo functions of \( t \)) such that

\[
(F(x,y)\text{d}y \wedge \text{d}x - \text{d}H \wedge \text{d}t) = \phi^*(G(X,Y)\text{d}Y \wedge \text{d}X - \text{d}K \wedge \text{d}t),
\]

where \( \text{d} \) is the exterior derivative on \( \mathbb{C}^3 \) so \( t \) is treated as a variable. Further, the Hamiltonian system

\[
F(x,y)\frac{\text{d}x}{\text{d}t} = \frac{\partial H}{\partial y}, \quad F(x,y)\frac{\text{d}y}{\text{d}t} = -\frac{\partial H}{\partial x}
\]

is transformed under \( \phi \) to

\[
G(X,Y)\frac{\text{d}X}{\text{d}t} = \frac{\partial K}{\partial Y}, \quad G(X,Y)\frac{\text{d}Y}{\text{d}t} = -\frac{\partial K}{\partial X}.
\]

Note that the Hamiltonians will not generally coincide under the transition functions—there may be correction terms arising from the \( t \)-dependence of the gluing. In the present case we take the symplectic form \( \omega^{Ok}_t \) on the fibre \( E^{Ok}_t \) given in coordinates by

\[
\omega^{Ok}_t = \text{d}f \wedge \text{d}g = \text{d}x_1 \wedge \text{d}y_1 = \text{d}x_2 \wedge \text{d}y_2 = \text{d}x_3 \wedge \text{d}y_3,
\]

where the equalities are under the gluing (9) and similarly to above \( \text{d} \) denotes the exterior derivative on the fibre \( E^{Ok}_t \) so \( t \) is treated as a constant in the calculation. By lemma 1 the Hamiltonian function in the original \((f,g)\) coordinates provides a global Hamiltonian structure for the system on \( E^{Ok}_t \). In particular it turns out that the Hamiltonians in the other three charts are also polynomial in the coordinates:

\[
H^{Ok}_1 = H^{Ok}_1(f,g,t) = 2g^2 - (t^2 + 2tf + 2a_1)g - a_2f,
H^{Ok}_1 = H^{Ok}_1(x_1,y_1,t) = 2(a_2 - x_1y_1)(t + y_1(a_1 + a_2 - x_1y_1)) - x_1,
H^{Ok}_2 = H^{Ok}_2(x_2,y_2,t) = 2y_2 - (a_1 + x_2y_2)(2t + x_2(a_1 + a_2 + x_2y_2)),
H^{Ok}_3 = H^{Ok}_3(x_3,y_3,t) = 2(a_2 - 1)(a_1 + a_2 - 1)y_3 + x_3(1 + 2y_3(t + y_3(a_1 + 2a_2 - 2 + x_3y_3))).
\]

It follows that the system on \( E^{Ok}_t \) is then everywhere regular, and each fibre \( E^{Ok}_t \) is called a space of initial conditions. We refer to the total space \( E^{Ok}_t \) as the defining manifold, following [35], or as Okamoto’s space for \( P_{117} \). We use the term Okamoto surfaces to refer to the surfaces \( \Sigma^{Ok}_t \) before the removal of inaccessible divisors.

The defining manifold \( E^{Ok}_t \) is foliated by solutions of the extended system, in a way that is closely related to the Painlevé property of the equation. For each Painlevé equation \( P_t \), Okamoto provided an equivalent Hamiltonian system (with respect to the canonical symplectic form), which is polynomial in the two dependent variables, and analytic on the set \( B_j \subset \mathbb{C} \) given by the complement of the finite set of fixed singularities of \( P_j \) in the complex plane. With phase space being the trivial complex analytic fibre bundle \( (\mathbb{C}^2 \times B_j, \pi, B_j) \), where \( \pi \) is projection onto the second factor \( B_j \), solutions of the Hamiltonian system give a nonsingular foliation of \( \mathbb{C}^2 \times B_j \) into disjoint complex one-dimensional leaves transverse to the fibres. However this foliation is not uniform, precisely because solutions may develop movable poles.

Along the same lines as outlined for \( P_{117} \) above, Okamoto [26] constructed for each \( P_j \) a complex analytic fibre bundle \( (E_j,\pi_j,B_j) \) that contains \( (\mathbb{C}^2 \times B_j,\pi,B_j) \) as a fibre subspace and
of which the extended Hamiltonian form induces a foliation into complex one-dimensional solution curves such that

- Every leaf is transverse to the fibres and intersects the subspace \( \mathbb{C}^2 \times B_J \).
- The foliation is uniform: for each point \( p_0 \in E_J, \pi_J(p_0) = t_0 \), any path \( \ell \) in \( B_J \) with starting point \( t_0 \) can be lifted to the leaf passing through \( p_0 \). In other words, any solution of the extended system of differential equations, with initial condition corresponding to \( p_0 \), can be holomorphically continued in \( E_J \) over \( \ell \).

The fact that Okamoto’s Hamiltonian form of the fourth Painlevé equation induces a uniform foliation comes from the Painlevé property, namely that every solution of the Hamiltonian system for an initial condition in the initial phase space \( \mathbb{C}^2 \times B_J \) can be meromorphically continued along any path in \( B_J \). However we emphasise that the regularisation on \( E_{Ok} \) alone is not sufficient to prove the Painlevé property and one requires, in addition to the regular initial value problems corresponding to singularities of solutions, certain auxiliary functions related to the global holomorphic Hamiltonian structure \[35\], which are similar to those appearing in other proofs of the Painlevé property \[11–14, 28, 33, 34\]. Nevertheless the existence of a defining manifold on which an ODE system becomes everywhere regular is certainly intimately related to the Painlevé property, and this paper forms one part of a broader project extending this relation between regularisability and special singularity structures to wider classes of equations.

1.2. Takasaki’s rational Painlevé-Calogero system related to P_{IV}

We consider the second-order rational Painlevé-Calogero system related to P_{IV} obtained by Takasaki \[37\], which in this paper we call the Takasaki system, given explicitly by

\[
\frac{dq}{dt} = \frac{\partial H_{Tak}}{\partial p} = p,
\]

\[
\frac{dp}{dt} = -\frac{\partial H_{Tak}}{\partial q} = \frac{8\beta}{q^2} + (t^2 - \alpha)q + \frac{t}{2}q^3 + \frac{3}{64}q^5,
\]

\[
H_{Tak}(q, p, t) = \frac{p^2}{2} + V(q, p, t),
\]

\[
V(q, p, t) = -\frac{1}{2}\left(\frac{q}{2}\right)^6 - 2t\left(\frac{q}{2}\right)^4 - 2(t^2 - \alpha)\left(\frac{q}{2}\right)^2 + \beta \left(\frac{q}{2}\right)^2,
\]

where \( \alpha, \beta \) are again arbitrary complex parameters.

The potential \( V \) is associated to the rank-one case of an extended Calogero system with rational potential \[15, 16\], has an isomonodromic Lax pair \[3\], and reduces in the \( \beta = 0 \) case to an equation isolated by Halburd and Kecker \[10\] through Nevanlinna-theoretic methods, as we will see in section 4.1. We also note that this system has been studied by various authors in connection with the fourth Painlevé equation both before \[1, 2, 17\] and after Takasaki’s paper \[38–40\].

Takasaki obtained the system above as part of the Painlevé-Calogero correspondence, which began with the discovery of a so-called elliptic form of the sixth Painlevé equation by Manin \[23\], building on the work of Fuchs \[7\] and Painlevé \[29\], which was recognised by Levin and Olshanetsky \[22\] as being related to a special case of one of Inozemtsev’s generalisations \[15, 16\] of the Calogero system \[4\]. Takasaki extended the correspondence from P_{VI} to the other Painlevé equations—for a more detailed account of the historical progression of ideas leading to this we recommend the excellent paper \[36\].
Eliminating $p$ from system (16) one obtains the Takasaki scalar equation given by
\[ \frac{d^2 q}{dt^2} = \frac{3q^5}{64} + \frac{tq^3}{2} + (t^2 - \alpha)q + \frac{8\beta}{q^3}, \tag{17} \]
which is related to the fourth Painlevé equation as follows. If $(q, p)$ solve the Takasaki system (or $q$ solves the scalar equation (17)), then we obtain a solution $\lambda$ of the fourth Painlevé equation (4) with the same parameters $\alpha, \beta$ given by
\[ \lambda = \left( \frac{d}{2} \right)^2. \tag{18} \]
This gives a rational (but not birational) transformation that maps a solution $(q, p)$ of the Takasaki system to a solution $(f, g)$ of the Okamoto Hamiltonian form of the fourth Painlevé equation, which is given explicitly by (31) and will be studied in section 3.

### 1.3. Outline of the paper

In section 2, we introduce the notion of quadratic regularisability of a system of two ODEs and present a defining manifold $E^\text{Tak}$ for the Takasaki system (16) on which it is regular or regularisable in the sense of this definition. We then present a symplectic structure for the fibre over $t$, with respect to which we derive a global Hamiltonian structure for the Takasaki system on $E^\text{Tak}$, in which the Hamiltonian in each chart is holomorphic. The explicit details of the construction of the manifold are deferred to appendix B.

In section 3 we show how the algebraic transformation from the Takasaki system to the Okamoto Hamiltonian form of $P_{IV}$ realises the space of initial conditions as a quotient. We also show how this map extended to the compact surfaces (without inaccessible divisors removed) is a combination of blowdowns and a double cover, and clarify the relation under this map between the divisors of the rational two-forms associated with the Takasaki and Okamoto Hamiltonian systems.

In section 4, we define a class of regularising transformations that generalise the quadratic regularisability observed in the case of the Takasaki system. We discuss the ways in which this provides a notion of defining manifold for equations with the algebro-Painlevé property, with reference to a number of examples, for which we construct the surfaces and determine the configuration of inaccessible divisors.

### 2. Geometric regularisation of the Takasaki system

The reason we choose the system (16) among Takasaki’s second-order Painlevé-Calogero systems as the object of study in this paper is that it is given in Hamiltonian form by a rational Hamiltonian function (others are elliptic, hyperbolic, or exponential). Thus we can still recast it as a rational vector field on an appropriate phase space and attempt to resolve indeterminacies through blowups. The observations we make in doing so, the details of which will follow, lead us to introduce the following notion.

**Definition 2.** Suppose a first-order system of ODEs is of the form
\[ \frac{du}{dt} = R_1(t, u, v^2), \quad \frac{dv}{dt} = \frac{1}{v} R_2(t, u, v^2), \tag{19} \]
Figure 2. Surface $\mathcal{X}^{Tak}$ for the Takasaki system, with inaccessible divisors in blue.

where $R_1(t,x,y)$ and $R_2(t,x,y)$ are rational functions of their arguments, both regular in $y$, with $R_2(t,x,y)$ nonzero at $y=0$. Then we say that such a system is *quadratically regularisable* in the variable $v$. Introducing $h=v^2$, the resulting system is regular in $h$ at $h=0$:

$$u' = R_1(t,u,h), \quad h' = 2R_2(t,u,h). \quad (20)$$

After resolving the indeterminacies of the Takasaki system, we find that in the coordinates for the last exceptional divisors arising in the sequence of blowups, the system is precisely of the form above. We also note that if after a sequence of blowups the system is quadratically regularisable in one chart $(u,v)$ for the last exceptional divisor arising in the sequence in the variable $v$, introduced in the canonical way according to the convention established in appendix A, then it is also quadratically regularisable in the other chart $(U,V)$, in the variable $V$, which follows from the relation

$$uv = V, \quad v = UV. \quad (21)$$

### 2.1. Defining manifold for the Takasaki system

Similarly to the case of the Okamoto Hamiltonian form of $P_{IV}$, we begin with the Takasaki system (16) as a rational system on the trivial bundle over $\mathbb{C}_t$ with fibre $\mathbb{C}^2_{q,p}$, which we compactify to $\mathbb{P}^1 \times \mathbb{P}^1$. We then perform a sequence of twenty blowups of the fibre over $t$, the details of which are provided in appendix B, denoting the centres of blowups by $q_i$ and exceptional divisors by $F_i$. Through this we obtain a rational surface $\mathcal{X}^{Tak}$ (we sometimes add a subscript $t$ to emphasise the fibre), which is schematically represented on figure 2. Note we assume here that $\beta \neq 0$, since in the $\beta = 0$ case some indeterminacies disappear and fewer blowups are required.

We next remove from the fibre certain divisors where the system is not regular or quadratically regularisable (indicated in blue on figure 2), which we refer to as inaccessible by analogy with the Okamoto case and whose support we denote $D^{Tak}$. After this we have the bundle $(E^{Tak}, \pi, \mathcal{C}_t)$, the fibre of which is $E_t^{Tak} = \mathcal{X}_t^{Tak} - D^{Tak}$. Through the blowups we obtain the following atlas for $E^{Tak}$, provided by the initial variables $(q,p)$ away from $q=0$, as well as charts to cover affine neighbourhoods of the exceptional divisors $F_i$ for $i = 3, 5, 14, 20$ away from $D^{Tak}$, as indicated in red on figure 2

$$E^{Tak} = (\mathbb{C}^3_{q,p,t} \setminus \{q=0\}) \cup \mathbb{C}_{q_3,p,t}^3 \cup \mathbb{C}_{q_3,p,t}^3 \cup \mathbb{C}_{q_4,p,t}^3 \cup \mathbb{C}_{q_{14},p,t}^3 \cup \mathbb{C}_{q_{20},p,t}^3, \quad (22)$$
with gluing defined according to
\[
q = v_5(4a_1 + u_3 v_3^2), \quad \frac{1}{p} = v_3,
\]
\[
q = v_5(-4a_1 + u_5 v_5^2), \quad \frac{1}{p} = v_5,
\]
\[
\frac{1}{q} = v_{14}, \quad \frac{1}{p} = v_{14}^3 \left( \frac{1}{8} + v_{14}^2 \left(t + v_{14}^2 \left(4(a_1 + 2a_2 - 2) + u_{14} v_{14}^2\right)\right) \right),
\]
\[
\frac{1}{q} = v_{20}, \quad \frac{1}{p} = v_{20}^3 \left(-\frac{1}{8} + v_{20}^2 \left(-t + v_{20}^2 \left(-4(a_1 + 2a_2) + u_{20} v_{20}^2\right)\right)\right).
\]
For the construction of this atlas see appendix B, and note that we have introduced the parameters \(a_1, a_2\) in the same way as in \(P_{IV}\), namely \(\alpha = 1 - a_1 - 2a_2, \beta = -2a_1^2\).

**Proposition 3.** The Takasaki system is everywhere either regular or quadratically regularisable on the bundle \(E^{Tak}\).

**Proof.** The main part of the result is the construction of the atlas in appendix B, after which it remains only to verify the assertion by direct calculation of the extension of the system (16) to \(E^{Tak}\) using the relations (23) as changes of variables. For example in the chart \((u_{14}, v_{14})\) we have
\[
\frac{du_{14}}{d\tau} = 64(a_2 - 1)(a_1 + a_2 - 1) + 2u_{14} + 16(a_1 + 2a_2 - 2)u_{14} v_{14}^4 + 3u_{14}^2 v_{14}^4,
\]
\[
\frac{dv_{14}}{d\tau} = -\frac{1}{v_{14}} + \frac{4}{v_{14}} + 32(a_1 + 2a_2 - 2)v_{14}^2 + 8u_{14} v_{14}^4,
\]
so the system is quadratically regularisable in \(v_{14}\) on the part of \(F_{14}\) visible in the \((u_{14}, v_{14})\) chart. The regularisability on the other exceptional divisors \(F_3, F_5, F_{20}\) is verified similarly, and the system in the original variables is regular away from \(q = 0\).

\[\square\]

2.2. Global Hamiltonian structure

The quadratic regularisability of the Takasaki system can also be seen in terms of a global Hamiltonian structure, in an analogous way to the global holomorphic Hamiltonian structure on Okamoto’s space for \(P_{IV}\) as outlined section 1.1. For this, we take the holomorphic symplectic form on the fibre \(E_1^{Tak}\) to be the extension of that with respect to which the Takasaki system (16) is defined, namely \(dq \wedge dp\). The appropriate atlas for \(E^{Tak}\) and collection of Hamiltonian functions are provided by the following.

**Theorem 4.** The Takasaki system on \(E^{Tak}\) has a global Hamiltonian structure with respect to the symplectic form on the fibre \(E_1^{Tak}\) extended from \(dq \wedge dp\) in the original variables, in which all Hamiltonian functions are polynomial in coordinates.

**Proof.** We take as an atlas for the bundle
\[
E^{Tak} = C^3_{31, 31, \tau} \cup C^3_{32, 31, \tau} \cup C^3_{31, 32, \tau} \cup C^3_{44, 34, \tau},
\]
with gluing defined by
\[
\begin{align*}
\frac{1}{x_1} &= -\frac{y_2^2}{8a_1 + x_2y_2^2}, \quad y_1 = y_2, \\
x_1 &= -\frac{y_3^2}{8a_2 + x_3y_3^2}, \quad \frac{1}{y_3} = y_3, \\
\frac{1}{x_1} &= \frac{4y_4^2}{1 + 8b\gamma^2 + 32(a_1 + a_2 - 1)y_4^2 - 4x_4y_4^6}, \quad \frac{1}{y_4} = y_4,
\end{align*}
\]

where notation has been recycled from the \( P_{IV} \) case and the charts \((x_i, y_i)\) are not to be confused with those from the symplectic atlas for \( E^{odd} \). The original variables \( q, p \) are related to these coordinates by

\[
q = y_1, \quad p = x_1y_1 - \frac{4a_1}{y_1} - ty_1 - \frac{y_1^3}{8},
\]

and the transition functions \((x_i, y_i) \mapsto (x_j, y_j)\) defined by (26) can be verified to be biholomorphisms on the overlaps of coordinate patches by direct calculation. In particular the parts of the exceptional divisors \( F_3, F_3, F_4, \) and \( F_20 \) away from \( D_{Tak} \) are given by \( y_2 = 0, y_1 = 0, y_4 = 0, \) and \( y_3 = 0 \) respectively. We take the symplectic form \( \omega_{Tak} = dq \wedge dp \) extended to the fibre \( E_{Tak} \), which is given in each chart by

\[
\omega_{Tak} = y_i dx_i \wedge dy_i,
\]

with respect to which the system is Hamiltonian:

\[
\frac{dx_i}{dt} = \frac{1}{y_i} \frac{\partial H_{Tak}^i}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{1}{y_i} \frac{\partial H_{Tak}^i}{\partial x_i},
\]

with Hamiltonians \( H_{Tak}^i \) being polynomial in \( x_i, y_i^2, t \), explicitly given (modulo functions of \( t \)) by

\[
\begin{align*}
H_{1}^{Tak}(x_1, y_1, t) &= 4a_1x_1 + \left(a_2 + tx_1 - \frac{x_1^2}{2}\right)y_1^2 + \frac{1}{8}x_1y_1^4, \\
H_{2}^{Tak}(x_2, y_2, t) &= -4a_1x_2 + \left(a_4 + a_2 + tx_2 - \frac{x_2^2}{2}\right)y_2^2 + \frac{1}{8}x_2y_2^4, \\
H_{3}^{Tak}(x_3, y_3, t) &= -\frac{1}{8}x_3 - (32a_2(a_1 + a_2 + tx_3)y_3^2 - 4(a_1 + 2a_2)x_3y_3^4 - \frac{1}{2}x_3^3y_3^6, \\
H_{4}^{Tak}(x_4, y_4, t) &= \frac{1}{8}x_4 - (32(a_1 + a_2 - 1)(a_2 - 1) - tx_4)y_4^2 + 4(a_1 + 2a_2 - 2)x_4y_4^4 - \frac{1}{2}x_4^2y_4^6.
\end{align*}
\]

**Remark 1.** In particular the fact that each Hamiltonian is polynomial in coordinates with only even powers of \( y_i \) ensures that the system in the chart \((x_i, y_i)\) is quadratically regularisable in the variable \( y_i \), and regular in \( x_i \) everywhere. This is analogous to how a polynomial Hamiltonian structure in an atlas providing canonical coordinates for the symplectic form guarantees regular initial value problems, as is the case for Okamoto’s spaces for the Painlevé equations.
3. The map from Takasaki to Okamoto surfaces

The following algebraic transformation was presented by Takasaki [37], and can be reproduced using that from the scalar equation (17) to P_{IV}. If \((q, p)\) solves the system (16) with \(\alpha = 1 - a_1 - 2a_2, \beta = -2a_1^2\), then the transformation

\[
\begin{align*}
    f(q, p) &= \left(\frac{q}{2}\right)^2, \\
    g(q, p) &= \frac{t}{2} + \frac{2a_1}{q^2} + \frac{p}{2q} + \frac{q}{16},
\end{align*}
\]

(31)
gives a solution \((f, g)\) to the Okamoto Hamiltonian form (5) of P_{IV}. Further, this transformation is, up to scaling, canonical:

\[
d_t q \wedge d_t p = 4d_t f \wedge d_t g,
\]

(32)
where the equality is under the above transformation and \(d_t\) is the exterior derivative on \(\mathbb{C}^2\) so \(t\) is treated as constant. The above transformation defines a rational, but not birational, map \(\varphi: \mathcal{X}_{\text{Tak}} \to \mathcal{X}_{\text{Ok}}\),

(33)
which we study in this section.

3.1. The rational map between open surfaces

We first consider the restriction of the map (33) to the fibres of the defining manifolds \(E_{\text{Tak}}, E_{\text{Ok}}\), so the surfaces with inaccessible divisors removed:

\[
\varphi: \mathcal{X}_{\text{Tak}} - D_{\text{Tak}} \to \mathcal{X}_{\text{Ok}} - D_{\text{Ok}}.
\]

(34)

Proposition 5. The rational map (34) is a morphism (i.e. has no indeterminacies) and has empty critical locus (i.e. does not blow down any curves and all fibres are finite). Further, its ramification locus consists of the parts away from \(D_{\text{Ok}}\) of the exceptional divisors \(E_2, E_4, E_8\) and the proper transform of \(\{f = 0\}\).

Proof. The proof is a direct calculation in coordinates, in which we use \(\bar{u}, \bar{v}\) to denote charts for the Okamoto surface \(\mathcal{X}_{\text{Ok}}\) as introduced in appendix A to distinguish them from \(u, v\) for the Takasaki surface \(\mathcal{X}_{\text{Tak}}\) as introduced in appendix B. It is straightforward to first verify that \(\varphi\) has no indeterminacies on \(\mathcal{X}_{\text{Tak}}\), after which the absence of any critical locus as well as the ramification are checked through the following calculations:

- The part of \(\mathcal{X}_{\text{Tak}} - D_{\text{Tak}}\) visible in the \((q, p)\)-chart (so on which in particular \(q \neq 0\)) is mapped to the \((f, g)\)-chart with \(f \neq 0\), and every point \((f, g)\) with \(f \neq 0\) has two distinct preimages in the \((q, p)\)-chart away from \(q = 0\).
- The part of the exceptional divisor \(F_5\) on \(\mathcal{X}_{\text{Tak}} - D_{\text{Tak}}\) is mapped to the part of the proper transform of \(\{f = 0\}\) on \(\mathcal{X}_{\text{Ok}}\) away from \(D_{\text{Ok}}\):

\[
(u_5, v_5)|_{v_5 = 0} \mapsto (f, g) = \left(0, \frac{u_5}{32a_1^2} + \frac{t}{2}\right),
\]

and every point on this proper transform has preimage under \(\varphi\) being a single point of \(F_5\).
Figure 3. Surface for the Takasaki Hamiltonian system after minimisation.

- The part of the exceptional divisor $F_3$ on $X_{Tak}$ away from $D_{Tak}$ is mapped to the part of $E_4$ on $X_{Ok}$ away from $D_{Ok}$:

$$\frac{u_3}{\sqrt{2}} v_3 = \frac{u_4}{\sqrt{2}} v_4 = 0$$

and every point on $E_4$ away from $D_{Ok}$ has preimage being a single point of $F_5$.

Similarly $F_{14}$ and $F_{20}$ are mapped to $E_8$ and $E_2$, respectively.

3.2. The rational map between compact surfaces

We now investigate the rational map $\varphi$ between the compact surfaces and the relation of the rational two-forms providing the symplectic structures for the Takasaki and Okamoto systems. In the case of $X_{Ok}$, as with surfaces associated with the other Painlevé equations, the configuration of the inaccessible divisors forming $D_{Ok}$ plays a defining role and they form the irreducible components of an effective anticanonical divisor of canonical type \[30\]. In particular they are all of self-intersection $-2$, with intersection configuration encoded in a Dynkin diagram of an affine root system, and give the pole divisor of the symplectic form used to define the Hamiltonian structure of $P_{IV}$. In this section we establish the role played by the components of $D_{Tak}$ in the symplectic structure with respect to which the Takasaki Hamiltonian system is defined.

In order to investigate the map $\varphi$ we first note that it is possible to perform a kind of minimisation of the Takasaki surface $X_{Tak}$ which does not affect the fibres $E_{i_{Tak}}$ of the defining manifold, since it is possible to contract inaccessible exceptional curves contained in $D_{Tak}$. This is done through a sequence of four blowdowns, the details of which are provided in appendix B. This gives a surface $X_{m_{Tak}}$, which we call the minimal surface for the Takasaki system. We give a schematic representation of this on figure 3, where the components of the image of $D_{Tak}$ under the blowdowns are labeled $C_i$, $i = 1, \ldots, 15$ as in appendix B and indicated in a combination of blue and magenta for reasons that will be explained below.

In particular the self-intersection numbers of these curves are given by

$$\langle C_4 \rangle^2 = -4, \quad \langle C_i \rangle^2 = -2 \quad \text{otherwise,}$$

(35)
Figure 4. Intersection configuration of inaccessible divisors for the Takasaki system.

and their intersection configuration is given on figure 4. It is natural to ask how this configuration of the curves \( C_i \) transforms into that of the inaccessible divisors on \( \mathcal{X}^{\text{Ok}} \), which is encoded in the Dynkin diagram of type \( E_6^{(1)} \). From now on we take \( \varphi \) as a map from the minimal surface \( \mathcal{X}^{\text{Tak}}_m \) to \( \mathcal{X}^{\text{Ok}} \). The fact that the transformation (31) is canonical in the sense of equation (32) means that the rational two-forms \( \omega^{\text{Tak}} \) and \( \omega^{\text{Ok}} \) extended from \( d\tau \land d\rho \) and \( 4d\tau \land d\sigma \) respectively are related by

\[
\varphi^* \omega^{\text{Ok}} = \omega^{\text{Tak}}. \tag{36}
\]

While the pole divisor of \( \omega^{\text{Ok}} \) provides the effective anticanonical divisor of \( \mathcal{X}^{\text{Ok}} \), it turns out that the anticanonical divisor class of \( \mathcal{X}^{\text{Tak}}_m \) is not effective.

**Proposition 6.** The divisor of \( \omega^{\text{Tak}} \) on \( \mathcal{X}^{\text{Tak}}_m \) is given by

\[- \mathrm{div} \omega^{\text{Tak}} = C_3 + 2C_4 + 5C_5 + 4C_6 + 3C_7 + 2C_8 + C_9 + 4C_{11} + 3C_{12} + 2C_{13} + C_{14} - F_3 - F_5 - F_{14} - F_{20}. \tag{37}\]

**Proof.** This is obtained by a standard computation rewriting \( \omega^{\text{Tak}} \) in charts to cover the exceptional divisors. For example the curve \( C_6 \) corresponds to \( F_9 \) and \( F_{10} \) and is given in the chart \((u_9, v_9)\) by \( v_9 = 0 \). The two-form in this chart is computed directly to be

\[
\omega^{\text{Tak}} = \frac{d_1 v_9 \land d_1 u_9}{v_9^2 (8 + u_9 v_9)^2}, \tag{38}\]

so we find the term \( 4C_6 \) in \(- \mathrm{div} \omega^{\text{Tak}} \). On the other hand the last exceptional divisors \( F_3, F_5, F_{14}, F_{20} \) appear with negative coefficients in \(- \mathrm{div} \omega^{\text{Tak}} \) because the symplectic form has zeroes along them, for example in the chart \((u_3, v_3)\) on \( F_3 \) we have

\[
\omega^{\text{Tak}} = v_3 d_1 v_3 \land d_1 u_3, \tag{39}\]

which is to be expected given the form of \( \omega^{\text{Tak}} \) in the atlas provided in theorem 4. \( \square \)

The unique effective anticanonical divisor of \( \mathcal{X}^{\text{Ok}} \) is given by

\[- \mathrm{div} \omega^{\text{Ok}} = D_0 + D_1 + 2D_2 + 3D_3 + 2D_4 + D_5 + 2D_6, \tag{40}\]

where the irreducible components \( D_j \) are the same as those giving \( D^{\text{Ok}} \) as in appendix A. Direct calculation in charts shows that \( \varphi \) has no indeterminacies on \( \mathcal{X}^{\text{Tak}}_m \), but that it blows down some of the curves \( C_i \), specifically those indicated in magenta on figure 3. We apply eight extra blowups to the images of these curves on \( \mathcal{X}^{\text{Ok}} \), to obtain what we call the extended Okamoto surface \( \tilde{\mathcal{X}}^{\text{Ok}} \), for which the projection is denoted

\[
\rho : \tilde{\mathcal{X}}^{\text{Ok}} \longrightarrow \mathcal{X}^{\text{Ok}}. \tag{41}\]
We use the notation $z_1, \ldots, z_8$ for the extra points on $X^{\text{Ok}}$ to be blown up, with the corresponding exceptional divisors being $L_1, \ldots, L_8$. The locations of the extra points are indicated on figure 5, and given explicitly in coordinates in appendix A.

Denote the proper transform under $\rho$ of $D_i$ by $\tilde{D}_i \in \text{Div}(\tilde{X}^{\text{Ok}})$, so

$$
\begin{align*}
\tilde{D}_0 &= E_3 - E_4 - L_1 - L_2, \\
\tilde{D}_1 &= E_1 - E_2 - L_3 - L_4, \\
\tilde{D}_2 &= H_f - E_5 - E_6 - L_5, \\
\tilde{D}_3 &= E_5 - E_6 - L_5 - L_6, \\
\tilde{D}_4 &= E_6 - E_7 - L_6 - L_7, \\
\tilde{D}_5 &= E_7 - E_8 - L_7 - L_8, \\
\tilde{D}_6 &= H_g - E_5 - E_8.
\end{align*}
$$

(42)

The pole divisor of the rational two-form $\tilde{\omega}^{\text{Ok}} = \rho^*\omega^{\text{Ok}}$ is given in terms of these and the extra exceptional divisors $L_i$ by

$$
- \text{div}\tilde{\omega}^{\text{Ok}} = D_0 + \tilde{D}_1 + 2\tilde{D}_2 + 3\tilde{D}_3 + 2\tilde{D}_4 + \tilde{D}_5 + 2\tilde{D}_6 + 2L_4 + 4L_5 + 4L_6 + 2L_7.
$$

(43)

**Theorem 7.** After minimisation of the Takasaki surface, the map given by the transformation (31) decomposes as

$$
\rho \circ \tilde{\phi} : \chi^{\text{Tak}}_m \longrightarrow \chi^{\text{Ok}},
$$

where the factor

$$
\tilde{\phi} : \chi^{\text{Tak}}_m \longrightarrow \tilde{\chi}^{\text{Ok}},
$$

is a rational morphism and has no critical locus. Then $\tilde{\phi} : \chi^{\text{Tak}}_m \rightarrow \tilde{\chi}^{\text{Ok}}$ maps the curves $C_i$ according to

$$
\tilde{\phi} : \begin{cases} 
C_1 \mapsto L_2, & C_2 \mapsto L_1, & C_3 \mapsto \tilde{D}_0, & C_4 \mapsto \tilde{D}_0, & C_5 \mapsto \tilde{D}_3, \\
C_6 \mapsto L_0, & C_7 \mapsto \tilde{D}_4, & C_8 \mapsto L_7, & C_9 \mapsto \tilde{D}_5, & C_{10} \mapsto L_8, \\
C_{11} \mapsto L_5, & C_{12} \mapsto \tilde{D}_2, & C_{13} \mapsto L_4, & C_{14} \mapsto \tilde{D}_1, & C_{15} \mapsto L_3. 
\end{cases}
$$

(44)
and the last exceptional divisors $F_3, F_5, F_{14}, F_{20}$ by

$$\tilde{\varphi} : F_3 \mapsto H_f - E_3 - L_1, \quad F_5 \mapsto E_4 - L_2, \quad F_{14} \mapsto E_6 - L_8, \quad F_{20} \mapsto E_2 - L_3.$$  \hfill (45)

**Proof.** First, we verify that there are no indeterminacies of $\tilde{\varphi}$ on $X_{m}^{\text{Tak}}$ by direct calculation in charts. The fact that the critical locus is empty is verified by computing the map in charts for $X_{m}^{\text{Tak}}$ as well as the new coordinates $(r_5, s_5)$ and $(R_5, S_5)$ covering the exceptional divisors $L_4$ on $X_{m}^{\text{Ok}}$. For example the fact that $C_{11}$, corresponding to $F_{15} - F_{16}$, is mapped to $L_5$ can be verified by rewriting the transformation (31) in charts $(u_{15}, v_{15})$ and $(r_5, s_5)$, which gives

$$r_5 = \frac{(-64 v_{15} (t + 4 a_1 v_{15}^2) + u_{15} (1 + 8 v_{15}^2 + 32 a_1 v_{15}^4))^2}{64 (8 - u_{15} v_{15})^2},$$

$$s_5 = \frac{16 v_{15} (8 - u_{15} v_{15})}{64 v_{15} (t + 4 a_1 v_{15}^2) - u_{15} (1 + 8 v_{15}^2 + 32 a_1 v_{15}^4)},$$  \hfill (46)

into which substitution of the local equation $v_{15} = 0$ of $C_{11}$ leads to

$$(u_{15}, v_{15})|_{v_{15}=0} \mapsto (r_5, s_5) = \left(\frac{u_{15}}{4096}, 0\right),$$  \hfill (47)

which given that $s_5 = 0$ is a local equation of $L_5$ allows us to deduce that $C_{11}$ is mapped surjectively onto $L_5$. The rest of the calculations are similar, albeit with more complicated rational functions. 

**Proposition 8.** The action of $\tilde{\varphi}$ by pullback on components of the image $\tilde{\varphi}(D^{\text{Tak}})$ is given by:

$$\tilde{\varphi}^*: \begin{cases}  
  D_0 \mapsto 2C_3, & D_1 \mapsto 2C_{14}, & D_2 \mapsto 2C_{12}, & D_3 \mapsto 2C_5, \\
  D_4 \mapsto 2C_7, & D_5 \mapsto 2C_9, & D_6 \mapsto C_4, \\
  L_1 \mapsto C_2, & L_2 \mapsto C_1, & L_3 \mapsto C_{15}, & L_4 \mapsto C_{13}, \\
  L_5 \mapsto C_{11}, & L_6 \mapsto C_6, & L_7 \mapsto C_8, & L_8 \mapsto C_{10}.
\end{cases} \hfill (48)$$

**Proof.** This follows from direct calculation using local equations of the curves in charts, along the same lines as in the proof of theorem 7. For example, to see that $\tilde{\varphi}^*(D_0) = 2C_3$, we take the chart $(u_3, v_3)$ for $X_{m}^{\text{Ok}}$ as in appendix A, and relabel $u, v$ by $\tilde{u}, \tilde{v}$ to distinguish them from the charts for the minimised Takasaki surface $X_{m}^{\text{Tak}}$. In this chart $(\tilde{u}_3, \tilde{v}_3)$, the local equation of $D_0$ is $\tilde{v}_3 = 0$. On $X_{m}^{\text{Tak}}$ we take the chart $(u_1, v_1)$ as defined in appendix B, in which $C_3$ (the image of $F_1 - F_2 - F_3$ under the minimisation $\rho$) has local equation $v_1 = 0$. Rewriting the mapping $\tilde{\varphi}$ in these charts using their definitions in appendices A and B, we see that

$$\tilde{u}_3 = \frac{1}{64} \left(32 a_1 + u_1 (8 + 8 u_1 v_1^2 + u_1^3 v_1^4)\right),$$

$$\tilde{v}_3 = \frac{16 u_1^3 v_1^3}{32 a_1 + u_1 (8 + 8 u_1 v_1^2 + u_1^3 v_1^4)},$$  \hfill (49)

so the divisor $D_0$ with local equation $v_3 = 0$ is pulled back to twice the divisor defined by $v_1 = 0$, i.e. $2C_3$.

On the other hand, the cases in which divisors are pulled back without increase in multiplicity can be seen by similar calculations. For example to see that $\tilde{\varphi}^*(L_2) = C_1$, consider the chart $(r_2, s_2)$ for $X_{m}^{\text{Ok}}$ in which $L_2$ is given by $s_2 = 0$, and the chart $(u_2, v_2)$ for $X_{m}^{\text{Tak}}$, in which $C_1$ (which we recall is the image of $F_2 - F_3$ under $\rho$) is given by $v_2 = 0$. Calculating by direct substitution into the defining equation (31) of the mapping we obtain a relation of the form...
Figure 6. Behaviour of inaccessible divisors under the map $\varphi : \mathcal{X}_n^{\text{Ok}} \to \mathcal{X}_n^{\text{Ok}}$.

$s_2 = v_2 \left( u_2^2 + v_2 P(u_2, v_2) \right)$, where $P$ is polynomial in $u_2, v_2$, so in this case the divisor $L_2$ with local equation $s_2 = 0$ is pulled back to $C_1$, counted with multiplicity one as opposed to two. The rest of the calculations are similar.

So in particular we see that eight of the $-2$ curves $C_i$ are mapped to exceptional curves $L_j$ of the first kind, which can be understood through the formula

$$\varphi^* C \cdot \varphi^* C = (\deg \varphi) C \cdot C = 2 C \cdot C,$$

so in particular

$$\left(\varphi^* L_i\right)^2 = 2(L_i)^2 = -2.$$ 

These exceptional curves $L_j$ are then contracted by $\rho$, leading to the configuration of $-2$ curves providing $D_n^{\text{Ok}}$. We give a graphical depiction of the transformation of the configuration of curves $C_i$ to that of $D_j$ on figure 6, as well as the exceptional divisors $F_3, F_5, F_{14}, F_{20}$. 

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Of the curves $C_i$ on $X_n^{Tak}$, it is $C_3, C_4, C_5, C_7, C_9, C_{12}$, and $C_{14}$ (indicated in blue on figure 3) that are mapped to components of the anticanonical divisor of $X^{Ok}$, while the rest of the $C_i$ (indicated in magenta on figure 3) are mapped first to $-1$ curves under $\varphi$ then are contracted to points by $\rho$. The exceptional divisors $F_3, F_5, F_{14}, F_{20}$ which are not contained in $D^{Tak}$ first become $-2$ curves under $\varphi$ then become the divisors $E_4, H_f - E_3, E_5, E_2$ respectively on $X^{Ok}$.

4. Regularisation of algebro-Painlevé systems on rational surfaces

In this section we show that the mechanism by which the Takasaki system is associated to the surface $X_n^{Tak}$, namely quadratic regularisability in the sense of definition 2, can be generalised to give a similar mechanism to associate equations with globally finite branching of solutions about movable singularities to rational surfaces. We introduce a notion of regularisability by algebraic transformations and illustrate this phenomenon in a number of examples with the algebro-Painlevé property.

**Definition 9.** Consider an $n$th-order ODE

$$\frac{d^n y}{dt^n} = F \left( y, \frac{dy}{dt}, \ldots, \frac{d^{n-1} y}{dt^{n-1}}, t \right),$$

(52)

where $F$ is rational in $y$ and its derivatives, and locally analytic in $t$. Let the fixed singularities of the equation be the discrete set $\mathcal{F} \subset \mathbb{C}$, and let $B = \mathbb{C} \setminus \mathcal{F}$. The equation (52) is said to have the *algebro-Painlevé property* if all solutions are algebroid functions over $B$, by which we mean they are algebraic over the field of meromorphic functions on the universal cover of $B$.

The notion of the quadratic regularisability in definition 2 can be naturally extended as follows.

**Definition 10.** Suppose a first-order system of ODEs is of the form

$$u' = R_1(t, u, v^p), \quad v' = \frac{1}{v^{p-1}} R_2(t, u, v^p),$$

with $R_1(t, x, y), R_2(t, x, y)$ rational functions of their arguments, and $R_1(t, x, y), R_2(t, x, y)$ being regular in $y$ and nonzero at $y = 0$. Then we say that such a system is *nth-order monomially regularisable* in the variable $v$. Introducing $h = v^p$, the resulting system is regular in $h$ at $h = 0$:

$$u' = R_1(t, u, h), \quad h' = nR_2(t, u, h).$$

With this mechanism of association of equation to surface in hand, it is natural to ask whether intersection configurations of inaccessible divisors can be used to classify such equations, as is the case for the Painlevé equations. For a second-order differential equation related by an algebraic transformation to one of the Painlevé equations, one approach to the classification problem via intersection configurations of inaccessible divisors would be to describe the algebraic transformation in terms of a kind of graph surgery of the affine Dynkin diagram associated with the Painlevé equation. That is, similarly to how the intersection graph of inaccessible divisors of the Takasaki system in figure 4 transformed to the $E_6^{(1)}$ Dynkin diagram for $P_{IV}$, one could in principle define a class of graph surgeries arising from algebraic transformations on the level of surfaces and use these to classify algebro-Painlevé equations related by algebraic transformations to Painlevé equations. A similar approach could also be taken for equations related algebraically to autonomous systems with the Painlevé property such as those solved by elliptic functions. This would rely on knowledge of the possible intersection configurations of inaccessible divisors for the autonomous systems, which could be deduced from knowledge of the fibrations they preserve [9, 31]. It is *not a priori* clear whether
the class of second-order differential equations with the algebro-Painlevé property is exhausted by those related by algebraic transformations to second-order equations with the Painlevé property. The approach to the classification problem outlined above based on inaccessible divisors seems to be the natural one, but a thorough pursuit of such a programme is beyond the scope of the current paper. For the examples related by algebraic transformations to systems with the Painlevé property we present in this section, we give the intersection graphs of inaccessible divisors as well as how they degenerate to those of the underlying Painlevé system similarly to the Takasaki case as in figure 6. We will see various kinds of graph surgeries that can arise from the behaviour of inaccessible divisors under a rational map, including deletion of nodes as in the Takasaki case, as well as addition of nodes in a certain way as induced by the rational map having a point of indeterminacy somewhere on the inaccessible divisors.

4.1. Example 1

In [10], Halburd and Kecker considered a class of second-order equations, and isolated equations whose solutions are globally quadratic over the field of meromorphic functions. Under certain assumptions on the coefficients it is shown that if

\[ y'' = \frac{3}{4} y^5 + \sum_{k=0}^{4} a_k(t) y^k \]  

(53)

and \( y(t) + s_1(t) y(t) + s_2(t) = 0 \), then \( s_1 \) is proportional to \( a_4 \) (which can be set to zero without loss of generality) and \( s_2 \) reduces either to a solution of a Riccati equation or the equation

\[ w'' = \frac{w'^2}{2w} + \frac{3}{2} w^3 + 4(at + b)w^2 + 2((at + b)^2 - c)w, \]

(54)

which in the case \( a \neq 0 \) is equivalent to a special case of the fourth Painlevé equation and in the case \( a = 0 \) can be solved in terms of elliptic functions. We call the equation isolated in [10] from the class (53) the Halburd–Kecker equation, which is given explicitly by

\[ y'' = \frac{3}{4} y^5 - (2at + 2b)y^3 + ((at + b)^2 - 2c) y. \]

(55)

By taking \( s_2 = -y^2 \) one recovers the variant (54) of the fourth Painlevé equation above. Note that if one follows the approach of [10] but assuming \( s_1 \neq 0 \), then \( s_2 \) must satisfy an equation of second order and second degree. The question of how to write such an equation in the form of an equivalent system of two first order differential equations with rational right-hand sides is open, so this case falls beyond the scope of the current paper.

It can be easily seen that the Halburd–Kecker equation is equivalent to the \( \beta = 0 \) case of the scalar Takasaki equation:

**Proposition 11.** The equation (55) for \( y(\bar{t}) \) with parameters \( a, b, c \) is equivalent to equation (17) with \( \beta = 0 \) for \( q(t) \) via the change of variables

\[ y(\bar{t}) = Aq(t), \quad Bt = \bar{a}t + b, \quad \text{where} \quad A^2 = \frac{-B}{4}, \quad B^2 = a, \quad \alpha = \frac{2c}{a}. \]

(56)

By putting \( q = y \), \( p = y' \), we may rewrite the Halburd–Kecker equation as an equivalent first-order system

\[
\frac{dq}{dt} = p, \\
\frac{dp}{dt} = \frac{3}{4} q^2 - 2(at + b)q^3 + ((at + b)^2 - 2c) q.
\]

(57)
Considering this on the trivial bundle over $\mathbb{C}$ with fibre $\mathbb{P}^1 \times \mathbb{P}^1$ in the same way as for the Takasaki system in appendix B, we find initially only one point $(Q, P) = (1/q, 1/p) = (0, 0)$ that requires blowing up, so the five blowups over $(q, P) = (0, 0)$ are not required here. To resolve the indeterminacy of the vector field require essentially the same sequence of blowups over this point as in the Takasaki case, i.e. three blowups of points on the proper transform of $P = 0$, followed by two cascades of six points each. The configuration of points is the same as in appendix B but without the blowups of $q_1, \ldots, q_5$, so there are 15 blowups performed in total but we maintain the enumeration of the remaining points from the Takasaki case, so there are initially three repeated blowups before the cascade splits:

$$q_6 : (Q, P) = (0, 0) \leftrightarrow q_7 \leftrightarrow q_8 \left\{ \begin{array}{c} q_9 \leftrightarrow q_{10} \leftrightarrow \cdots \leftrightarrow q_{14}, \\ q_{15} \leftrightarrow q_{16} \leftrightarrow \cdots \leftrightarrow q_{20}. \end{array} \right.$$  (58)

Through these blowups we obtain a surface $\mathcal{X}_{HK}$, with the precise locations of points in coordinates of the points being the only difference from the Takasaki case. We present only the charts for the pair of final exceptional divisors, which together with the original variables $(q, p)$ provide an atlas for a manifold on which the Halburd–Kecker system (57) is regularisable.

**Proposition 12.** The Halburd–Kecker system (57) is everywhere either regular or quadratically regularisable on the bundle

$$E_{HK} = \mathbb{C}^3_{q,p,t} \cup \mathbb{C}^3_{u_{14},v_{14},t} \cup \mathbb{C}^3_{u_{20},v_{20},t},$$  (59)

with gluing defined according to

$$\frac{1}{q} = v_{14}, \quad \frac{1}{p} = v_{14}^3 \left( 2 + v_{14}^3 \left( 4 (at + b) + v_{14}^2 \left( 8 (b^2 + c) + 4a (2at^2 + 4bt - 1) + u_{14}v_{14}^2 \right) \right) \right),$$

$$\frac{1}{q} = v_{20}, \quad \frac{1}{p} = v_{20}^3 \left( -2 + v_{20}^3 \left( -4 (at + b) + v_{20}^2 \left( -8 (b^2 + c) - 4a (2at^2 + 4bt - 1) + u_{20}v_{20}^2 \right) \right) \right).$$  (60)

In particular the system rewritten in the variables $(u_{14}, v_{14})$, respectively $(u_{20}, v_{20})$, satisfies definition 10 with $n = 2$.

**4.2. Example 2**

Let us consider the following equation

$$3q^2 q^{'''} + \frac{3}{2} q q^{''} - 2Bq^6 + Btq^3 + \frac{A}{2q^3} = 0,$$  (61)

where $A$ and $B$ are arbitrary constants which we assume to be nonzero. The local expansion at $t_0$ is $q \sim (t - t_0)^{-2/3}$. By a cubic change of variables $w = q^3$ it is reduced to the Painlevé XXXIV equation

$$w^{''} = \frac{w^2}{2w} + Bw(2w - t) - \frac{A}{2w^2},$$

which possesses the Painlevé property, so the algebraic transformation ensures that equation (61) has the algebro-Painlevé property. By introducing $q' = p$ we can easily re-write...
equation (61) in the form of a system of two first order differential equations for \( q \) and \( p \) with rational right-hand sides. Extending this to \( \mathbb{P}^1 \times \mathbb{P}^1 \) in the usual way, we require 24 blowups in total. The points are grouped into a cascade of eight points over \((q,p) = (0, \infty)\) which splits in two after the second blowup according to

\[
p_1 : (q,p) = (0, \infty) \leftrightarrow p_2 \leftrightarrow \begin{cases} p_3 \leftrightarrow p_4 \leftrightarrow p_5, \\ p_6 \leftrightarrow p_7 \leftrightarrow p_8, \end{cases}
\]  

and one long cascade \( p_9 \leftrightarrow \cdots \leftrightarrow p_{24} \) of 16 points over \( (q,p) = (\infty, \infty) \). After this we see that the resulting three systems in the final charts are all cubically regularisable, satisfying definition 10 with \( n = 3 \).

**Proposition 13.** The system equivalent to equation (61) via \( q' = p \) is everywhere either regular or cubically regularisable on the bundle

\[
(\mathbb{C}_{q,p,t}^3 \setminus \{q = 0\}) \cup \mathbb{C}_{u_5,v_5,t}^3 \cup \mathbb{C}_{u_6,v_6,t}^3 \cup \mathbb{C}_{u_{24},v_{24},t}^3,
\]

with gluing defined according to

\[
\begin{align*}
q &= v_5, & \frac{1}{p} &= v_5^2 \left( \frac{3}{\sqrt{A}} + u_5 v_5^3 \right), \\
q &= v_8, & \frac{1}{p} &= v_8^2 \left( -\frac{3}{\sqrt{A}} + u_8 v_8^3 \right), \\
\frac{1}{q} &= \frac{v_{24}^2}{371} \left( 3366B - 432B^4 v_{24}^6 + 64 B^5 v_{24}^8 + 177147 u_{24} v_{24}^{12} \right), \\
\frac{1}{p} &= \frac{v_{24}^2}{372} \left( 3366B - 432B^4 v_{24}^6 + 64 B^5 v_{24}^8 + 177147 u_{24} v_{24}^{12} \right)^2.
\end{align*}
\]  

The Painlevé XXXIV equation \( P_{XXXIV} \) is equivalent to Okamoto’s Hamiltonian form of \( P_{II} \), and has space of initial conditions with surface type \( E_7^{(1)} \). While we omit details, the algebraic transformation from the equation (61) to \( P_{XXXIV} \) gives a rational map between surfaces which can be studied along the same lines as that from \( X_{Tak} \) to \( X_{Ok} \) in section 3. Calculating the behaviour of inaccessible divisors under the rational map, we see that there is an indeterminacy at the point of intersection of two inaccessible divisors on the surface constructed from the equation (61), which is blown up by the map. On the level of the intersection graph of inaccessible divisors this corresponds to a node being added. Then similarly to the Takasaki case in section 3 we see that most of the inaccessible divisors are contracted, with the \( E_7^{(1)} \) diagram appearing as a subdiagram corresponding to inaccessible divisors along which the map is ramified. We give a schematic example of this in figure 7, where the uppermost diagram is the intersection configuration of inaccessible divisors for the equation (61), the first arrow indicates the addition of the node from the blowup of the point at the intersection of inaccessible divisors as described above corresponding to the edge labelled by +, and the inaccessible divisors contracted by the map are indicated in magenta while those that become the nodes of the \( E_7^{(1)} \) diagram are in blue.
4.3. Example 3

Though the examples so far have been related by \( t \)-independent algebraic transformations to second-order equations with the Painlevé property, we now present an example related by a \( t \)-dependent transformation to the fourth Painlevé equation. The equation is

\[
(q^3 - t^3)q'' = \left( \frac{2t^3}{q} - \frac{1}{2}q^7 \right) (q')^2 - 3t^2 q' + \frac{1}{2} q^{10} - \left( 2t^3 - \frac{4}{3} t \right) q^7 \\
+ \left( 3t^6 - 4t^4 + \frac{2}{3} t^2 - \frac{2}{3} \right) q^4 - t \left( 2t^6 - 4t^4 + \frac{4}{3} t^2 - \frac{4}{3} t^2 - 2 \right) q \\
+ \left( \frac{1}{2} t^{12} - \frac{4}{3} t^{10} + \frac{2}{3} t^8 - \frac{2}{3} t^6 - \frac{1}{2} t^4 + \frac{\beta}{3} \right) \frac{1}{q^2},
\]

whose local expansion at \( t_0 \) is \( q \sim (t - t_0)^{-1/3} \). This was obtained by requiring that a cubic change of variables

\[
\lambda = q^3 - t^3
\]

transforms (65) to the fourth Painlevé equation in the form (4). We rewrite this as the first-order system

\[
q' = -\frac{q^4}{3} + \frac{2t(t^3 - 1)q}{3} + \frac{3t^2 + 2t^4 - \beta - 2a_1}{3q^2} + p \left( \frac{4q^3}{3} - \frac{4q^3}{3q^2} \right), \\
p' = -2p^2 + 2p(q^3 - t^3 + t) + a_2
\]
Proposition 14. The system (67) is everywhere either regular or cubically regularisable on the bundle

$$C_{q,p,t} \cup C_{u_8,v_8,t} \cup C_{u_{10},v_{10},t} \cup C_{u_{12},v_{12},t} \cup C_{u_{24},v_{24},t}$$

with gluing defined as follows, in which $\zeta$ is a primitive cube root of unity:

$$\frac{1}{q} = v_6, \quad p = v_6^3 (-a_2 + u_6 v_6^3),$$

$$q = t + v_8, \quad \frac{1}{p} = v_8 \left( \frac{3t^2}{a_1} + u_8 v_8 \right),$$

$$q = \zeta t + v_{10}, \quad \frac{1}{p} = v_{10} \left( \frac{3\zeta^{-1} t^2}{a_1} + u_{10} v_{10} \right),$$

$$q = \zeta^{-1} t + v_{12}, \quad \frac{1}{p} = v_{12} \left( \frac{3\zeta^2 t^2}{a_1} + u_{12} v_{12} \right),$$

$$\frac{1}{q} = v_{24}, \quad \frac{1}{p} = v_{24}^3 \left( 2 + 2t(r^2 - 2)v_{24}^3 + 2 (r^2 (r^2 - 2)^2 + 2a_0) v_{24}^6 + u_{24} v_{24}^9 \right).$$

In this case, the system is genuinely regular on the parts of the final exceptional divisors in each of the three charts $(u_8,v_8)$, $(u_{10},v_{10})$, and $(u_{12},v_{12})$, while in the other charts it is cubically regularisable in $v_6$, respectively $v_{24}$.

We give a schematic description of the behaviour of inaccessible divisors under the algebraic map as a degeneration of the intersection diagram in figure 8 similarly to the previous example. In this case the degeneration involves both deletion of nodes corresponding to contraction of curves as well as merging of nodes as indicated by dashed arrows. This comes from the fact that there are inaccessible divisors which are not contracted and are away from the ramification locus of the map and form a triple cover of one of the inaccessible divisors of $P_{IV}$.
4.4. Example 4

We now consider an example which is not transformable to any of the differential Painlevé equations but still possesses the algebro-Painlevé property. The equation is given by

\[ q'' = \frac{2q^2}{3} - 2tq^4 + 2 \left( t^2 + \frac{A}{3} \right) q - \frac{2(t^3 + At + B)}{3q^2} - \frac{2(q')^2}{q}, \]  

and is equivalent to the first-order system

\[ q' = -q^2 + \frac{2qt}{3} + \frac{1 - A - t^2}{3q^2} + \frac{2p}{3q^2}, \]
\[ p' = 2pq^3 - 2tp - B, \]

which was obtained by letting \( f = q^3 - t, \ g = p \) in

\[ f' = 2g - f^2 - A, \quad g' = 2fg - B, \]

where \( A \) and \( B \) are arbitrary constants. The system (72) is an autonomous limit of the Okamoto Hamiltonian form of the second Painlevé equation, and in particular is Liouville integrable and may be solved in terms of elliptic functions.

For system (71) we again require 24 blowups, consisting of one cascade of six points \( p_1 : (q,p) = (\infty,0) \leftarrow p_2 \leftarrow \cdots \leftarrow p_6 \), as well as another cascade of 18 points \( p_7 : (q,p) = (\infty,\infty) \leftarrow p_8 \leftarrow \cdots \leftarrow p_{24} \).

**Proposition 15.** The system (71) is everywhere either regular or cubically regularisable on the bundle

\[ C_{q,p,t} \cup C_{v_6, v_{24}, t} \cup C_{v_{24}, v_{24}, t}, \]

with gluing defined by

\[ \frac{1}{q} = v_6, \quad p = v_6^3 \left( B + u_6 v_6^3 \right), \]
\[ \frac{1}{q} = v_{24}, \quad \frac{1}{p} = v_{24}^6 + 2v_{24}^9 + v_{24}^{12} \left( 3r^2 - A + (4r^3 - 4Ar + B)v_{24}^3 + u_{24}v_{24}^6 \right). \]

We give the degeneration of graphs corresponding to the behaviour of inaccessible divisors under the algebraic transformation from system (71) to system (72) in figure 9. Note that the space of initial conditions for the autonomous system (72) is a rational surface admitting an elliptic fibration, and the inaccessible divisors form a singular fibre of type III* in the Kodaira classification with intersection configuration given by the same \( E_7^{(1)} \) Dynkin diagram as \( P_II \).
4.5. Algebraic regularisability as an algebro-Painlevé test

We now show that the requirement of algebraic regularisability in the sense of definition 10 of a system of rational first-order differential equations after resolution of indeterminacies can serve as a kind of counterpart to the Painlevé test to isolate equations with the algebro-Painlevé property. We first show that requiring quadratic regularisability after the same sequence of blowups recovers the Takasaki system, in the same spirit as the uniqueness results for global Hamiltonian structures of differential equations on Okamoto’s spaces [5, 18, 24, 25, 32] which show in essence that the only globally regular Hamiltonian system that can exist on $E_J$ is the extension of the Okamoto Hamiltonian form of $P_J$.

**Theorem 16.** If the system

$$q' = p, \quad p' = \frac{3q^5}{64} + \sum_{k=-3}^{4} a_k(t)q^k,$$  \hspace{1cm} (75)

is quadratically regularisable on a bundle of rational surfaces obtained by blowing up points in the same configuration as $X^{\text{Tak}}$, then it must coincide with the Takasaki system up to affine changes of independent variable.

**Proof.** We start by resolving indeterminacies of system (75) maintaining genericity assumptions on the unknown coefficients $a_k(t)$. We only assume that $a_{-3} \neq 0$. We first find a sequence of five points of indeterminacy to blow up (with the same configuration as in the Takasaki case as indicated on the left-hand side of figure 2), the locations of which are as follows (below $i^2 = -1$):

\[
q_1 : (q, p) = (0, 0) \quad \leftarrow \quad \begin{cases} 
q_2 : (u_1, v_1) = (-i\sqrt{a_{-3}(t)}, 0) \\
q_3 : (u_2, v_2) = (-a_{-2}(t), 0) \\
q_4 : (u_1, v_1) = (i\sqrt{a_{-3}(t)}, 0) \\
q_5 : (u_4, v_4) = (-a_{-2}(t), 0)
\end{cases}
\]

After blowing up $q_1$ and $q_2$, there are no further points of indeterminacy over $q_1$. We next find the following sequence of points over $(Q, P) = (0, 0)$, which splits after the blowup of $q_8$ just as in the Takasaki case:

\[
q_6 : (Q, P) = (0, 0) \quad \leftarrow \quad q_7 : (U_6, V_6) = (0, 0) \quad \leftarrow \quad q_8 : (u_7, v_7) = (0, 0) \quad \leftarrow \quad \begin{cases} 
q_9 \\
q_{15}
\end{cases}
\]

The cascade over the point $p_9$ is given by

\[
p_9 : (u_8, v_8) = (8, 0) \\
\uparrow \\
p_{10} : (u_9, v_9) = (F_1(a_4(t)), 0) \\
\uparrow \\
p_{11} : (u_{10}, v_{10}) = (F_2(a_3(t), a_4(t)), 0) \\
\uparrow \\
p_{12} : (u_{11}, v_{11}) = (F_3(a_2(t), a_3(t), a_4(t), a'_4(t)), 0) \\
\uparrow \\
p_{13} : (u_{12}, v_{12}) = (F_4(a_1(t), a_2(t), a_3(t), a'_4(t), a_4(t), a'_4(t)), 0) \\
\uparrow \\
p_{14} : (u_{13}, v_{13}) = (F_5(a_0(t), a_1(t), a_2(t), a'_4(t), a_3(t), a'_4(t), a_4(t), a'_4(t), a'_4(t)), 0)
\]

\[
5685
\]
where \( F_1, \ldots, F_3 \) are known polynomial functions of their arguments which we omit for conciseness. The cascade over the point \( p_{15} \) is similar, given by

\[
\begin{align*}
p_{15} &: (u_8, v_8) = (8, 0) \\
\uparrow \\
p_{16} &: (u_{15}, v_{15}) = (G_1(a_4(t)), 0) \\
\uparrow \\
p_{17} &: (u_{16}, v_{16}) = (G_2(a_5(t), a_4(t)), 0) \\
\uparrow \\
p_{18} &: (u_{17}, v_{17}) = (G_3(a_2(t), a_3(t), a_4(t), a'_3(t)), 0) \\
\uparrow \\
p_{19} &: (u_{18}, v_{18}) = (G_4(a_1(t), a_2(t), a_3(t), a'_3(t), a_4(t), a'_4(t)), 0) \\
\uparrow \\
p_{20} &: (u_{19}, v_{19}) = (G_5(a_0(t), a_1(t), a_2(t), a'_2(t), a_3(t), a'_3(t), a_4(t), a'_4(t), a'_5(t)), 0),
\end{align*}
\]

where \( G_1, \ldots, G_5 \) are known polynomials in their arguments. After the blowups of \( q_1, \ldots, q_{20} \), there are no indeterminacies remaining, so we see that without additional assumptions on the coefficients we have the same point configuration as \( \chi^{Tak} \).

Next we need to impose the condition of quadratic regularisability on the systems in coordinate charts covering the exceptional divisors from the blowups of points \( q_3, q_5, q_{14} \) and \( q_{20} \). In general, whenever we impose any conditions on the coefficients \( a_i(t) \) we need to check that the points of indeterminacy do not disappear (and also new points do not appear) and our calculations are still valid (the locations of points might change but the configuration stays the same). We do so every time by running the calculations from the beginning. The system in coordinates \( u_3 \) and \( v_3 \) after the blowup of the point \( q_3 \) is of the form

\[
\begin{align*}
u_3' &= \frac{P^{(1)}_3(t, u_3, v_3)}{v_3^2 Q^{(1)}_3(t, u_3, v_3)}, \\
v_3' &= \frac{P^{(2)}_3(t, u_3, v_3)}{v_3^2 Q^{(2)}_3(t, u_3, v_3)},
\end{align*}
\]

(76)

with \( P^{(j)}_3(t, u_3, 0) \) and \( Q^{(j)}_3(t, u_3, 0) \) \((j = 1, 2)\) nonzero and independent of \( u_3 \). To force the cancellation of a factor of \( v_3 \) in the rational function giving \( u_3' \), we must set \( P^{(1)}_3(t, u_3, 0) = 0 \), leading to

\[
a_3(t)^{3/2}(2\sqrt{a_{-3}(t)a_{-1}(t)} + ia'_{-3}(t)) = 0.
\]

(77)

After imposing this condition we confirm that the system above becomes

\[
\begin{align*}
u_3' &= \frac{\tilde{P}^{(1)}_3(t, u_3, v_3)}{v_3^2 \tilde{Q}^{(1)}_3(t, u_3, v_3)}, \\
v_3' &= \frac{\tilde{P}^{(2)}_3(t, u_3, v_3)}{v_3^2 \tilde{Q}^{(2)}_3(t, u_3, v_3)},
\end{align*}
\]

(78)

so one power of \( v_3 \) in the denominator in the first equation has cancelled as required.

Now, to make \( u_3' \) regular and nonzero in \( v_3 \) at \( 0 \), we must set \( \tilde{P}^{(1)}_3(t, u_3, 0) = 0 \). Collecting the coefficients of \( u_3 \) in this expression we find that \( a_{-2}(t) = 0 \) and \( a_0(t) = 0 \) (recall that \( a_{-3}(t) \neq 0 \)). Another condition that we need to impose is that \( u_3' \) is even in \( v_3 \), which leads to \( a_2(t) = a_4(t) = 0 \). Combining this with the previously determined coefficients \( a_0(t), a_{-2}(t), \) which are equal to zero, and \( a'_{-3}(t) \) from (77), we find that the system in \( u_3 \) and \( v_3 \) coordinates is quadratically regularisable, that is, we also have that expression \( v_3 v_3' \) is regular and nonzero at \( v_3 = 0 \) and even in \( v_3 \).
We could also have begun with the system in coordinates \(u_5\) and \(v_5\) after the last blowup of the point \(p_5\), which with generic coefficient functions \(a_k(t)\) is of the form

\[
\begin{align*}
    u'_5 &= \frac{P_5^{(1)}(t, u_5, v_5)}{v_5^2 Q_5^{(1)}(t, u_5, v_5)}, \\
    v'_5 &= \frac{P_5^{(2)}(t, u_5, v_5)}{v_5 Q_5^{(2)}(t, u_5, v_5)}
\end{align*}
\]

(79)

with \(P_5^{(j)}(t, u_5, 0)\) and \(Q_5^{(j)}(t, u_5, 0)\) \((j = 1, 2)\) nonzero and independent of \(u_5\). As above, quadratic-regularisability requires us to set the expression \(P_5^{(1)}(t, u_5, 0) = 0\), which leads to

\[
a_3(t)^{3/2}(2\sqrt{a_{-3}(t)}a_{-1}(t) - ia'_{-3}(t)) = 0.
\]

(80)

After imposing this condition the system above would simplify to

\[
\begin{align*}
    u'_5 &= \frac{\tilde{P}_5^{(1)}(t, u_5, v_5)}{v_5 \tilde{Q}_5^{(1)}(t, u_5, v_5)}, \\
    v'_5 &= \frac{\tilde{P}_5^{(2)}(t, u_5, v_5)}{v_5 \tilde{Q}_5^{(2)}(t, u_5, v_5)},
\end{align*}
\]

(81)

so again one power of \(v_5\) in the denominator in the first equation has disappeared. Now, to make \(u'_5\) regular and nonzero in \(v_5\) at 0, we must set \(\tilde{P}_5^{(1)}(t, u_5, 0) = 0\). Collecting the coefficients of \(u_5\) in this expression we find that \(a_{-2}(t) = 0\), \(a_0(t) = 0\). Thus, we obtain the same conditions as above for system in coordinates \(u_5, v_5\).

Next we impose condition that \(u'_5\) is even in \(v_5\). This leads again to \(a_2(t) = a_4(t) = 0\). Under this assumptions together with previously determined coefficients \(a_0(t), a_{-2}(t)\), which are all equal to zero, and \(a_{-3}(t)\) from (80), we find that the system in \(u_5\) and \(v_5\) coordinates is quadratically regularisable, that is, we also have that the expression \(v_5 v'_5\) is regular and nonzero at \(v_5 = 0\) and even in \(v_5\).

Adding and subtracting (77) and (80) yields \(a'_{-3}(t) = 0\) and \(a_{-1}(t) = 0\), so for quadratic regularisability in the charts \((u_3, v_3)\) and \((u_5, v_5)\) we must have

\[
a_{-2}(t) = a_{-1}(t) = a_0(t) = a_2(t) = a_4(t) = a'_{-3}(t) = 0.
\]

(82)

Without any specialisation of coefficient functions, the system in the coordinates \(u_{14}\) and \(v_{14}\) after the last blowup of the point \(p_{14}\) is of the form

\[
\begin{align*}
    u'_{14} &= \frac{P_{14}^{(1)}(t, u_{14}, v_{14})}{v_{14}^2 Q_{14}^{(1)}(t, u_{14}, v_{14})}, \\
    v'_{14} &= \frac{P_{14}^{(2)}(t, u_{14}, v_{14})}{v_{14} Q_{14}^{(2)}(t, u_{14}, v_{14})}
\end{align*}
\]

(83)

with \(P_{14}^{(j)}(t, u_{14}, 0)\) and \(Q_{14}^{(j)}(t, u_{14}, 0)\) \((j = 1, 2)\) nonzero and independent on \(u_{14}\). Moreover, if we set expression \(P_{14}^{(1)}(t, u_{14}, 0) = 0\), which gives some (cumbersome) relation between coefficients \(a_k(t)\) and their derivatives, we see that the system above would simplify to

\[
\begin{align*}
    u'_{14} &= \frac{\tilde{P}_{14}^{(1)}(t, u_{14}, v_{14})}{v_{14} \tilde{Q}_{14}^{(1)}(t, u_{14}, v_{14})}, \\
    v'_{14} &= \frac{\tilde{P}_{14}^{(2)}(t, u_{14}, v_{14})}{v_{14} \tilde{Q}_{14}^{(2)}(t, u_{14}, v_{14})}.
\end{align*}
\]

(84)

Now, it is convenient to recalculate everything with the coefficient functions specialised according to (82), so that the condition \(P_{14}^{(1)}(t, u_{14}, 0) = 0\) becomes

\[
a_{14}'(t) = \frac{1}{2}(a_{14}'(t) - 8a_3(t)a'_{-3}(t)),
\]

(85)

imposing which we find that the system becomes quadratically regularisable in the chart \((u_{14}, v_{14})\).
Similarly, with generic coefficient functions the system in the original coordinates \( u_{20} \) and \( v_{20} \) after the last blowup of the point \( p_{20} \) is of the form

\[
\begin{align*}
u'_{20} &= \frac{P^{(1)}_{20}(t, u_{20}, v_{20})}{v_{20}^2Q^{(1)}_{20}(t, u_{20}, v_{20})}, \\
u'_{20} &= \frac{P^{(2)}_{20}(t, u_{20}, v_{20})}{v_{20}^2Q^{(2)}_{20}(t, u_{20}, v_{20})},
\end{align*}
\]

with \( P^{(j)}_{20}(t, u_{20}, 0) \) and \( Q^{(j)}_{20}(t, u_{20}, 0) \) \((j = 1, 2)\) nonzero and independent of \( u_{20} \). Moreover, if we set expression \( P^{(1)}_{20}(t, u_{20}, 0) = 0 \), which gives some (cumbersome) relation between coefficients \( a_k(t) \) and their derivatives, we see that the system above would also simplify to

\[
\begin{align*}
u'_{20} &= \frac{\tilde{P}^{(1)}_{20}(t, u_{20}, v_{20})}{v_{20}^2\tilde{Q}^{(1)}_{20}(t, u_{20}, v_{20})}, \\
u'_{20} &= \frac{\tilde{P}^{(2)}_{20}(t, u_{20}, v_{20})}{v_{20}^2\tilde{Q}^{(2)}_{20}(t, u_{20}, v_{20})}.
\end{align*}
\]

Again recalculating the system in this chart with the assumptions (82), we see that the condition \( P^{(1)}_{20}(t, u_{20}, 0) = 0 \) becomes

\[
a_3''(t) = \frac{1}{2}(8a_3(t)a_3'(t) - a_3'(t)),
\]

imposing which causes the system to become quadratically regularisable in the chart \((t_{20}, v_{20})\).

The final step is to solve the conditions (82), (85) and (88) and show that this must recover the Takasaki system without loss of generality. We see that \( a_{-3}(t) = c \), where \( c \) is a constant. Next, from (85) and (88) we find \( a_3''(t) = 0 \) and \( a_3'(t) = 8a_3(t)a_3'(t) \), from which we can deduce that \( a_3(t) \) is linear in \( t \) and \( a_1(t) \) is quadratic in \( t \). If we let

\[
a_3(t) = c_1 + c_2t,
\]

then

\[
a_1(t) = c_3 + 8tc_1c_2 + 4t^2c_2^2.
\]

We also have that conditions (85) and (88) are satisfied. By changes of independent variable \( t \mapsto at + b \) we can without loss of generality set particular values of the constants \( c_i \), \( i = 1, 2, 3 \), as \( c_1 = 0, c_2 = 1/2, c_3 = -1 + a_1 + 2a_2 \) and \( c = -16a_1^2 \), which gives \( a_{-3}(t) = 8\beta, a_3(t) = t/2, a_1(t) = t^2 - \alpha \), so we completely recover the Takasaki system. This proves the theorem.

The following is proved along exactly the same lines as the above proposition. Note that in comparison with [10] we do not make any Nevanlinna type estimates and the whole procedure is computational.

**Proposition 17.** If the equation

\[
y'' = \frac{3}{4}y^5 + \sum_{k=0}^{3} a_k(t)y^k
\]

is quadratically regularisable on a bundle of rational surfaces obtained by blowing up points in the same configuration as \( \mathcal{X}^{HK} \), then it must coincide with the Halburd–Kecker equation up to affine changes of independent variable.

**Remark 2.** If we assume at the beginning that

\[
y'' = \frac{3}{4}y^5 + \sum_{k=0}^{4} a_k(t)y^k
\]
and quadratic regularisability, the computations and conditions are a bit more involved and include \( a_4 \) but it is necessary to take \( a_4 = 0 \) for quadratic regularisability.

In a similar way, the definition of \( n \)-th order monomial regularisability can also be useful to eliminate whole classes of equations that have no algebroid solutions of certain types. Let us consider, for instance, equation

\[
y'' = y^7 + \sum_{k=0}^{5} a_k(t)y^k
\]  

(91)

or equation

\[
y'' = y^4 + \sum_{k=0}^{2} a_k(t)y^k
\]  

(92)

and assume that the equivalent systems are cubically regularisable (that is, the function \( w = y^3 \) will be meromorphic). After resolving all indeterminacies through blowups, we verify that we cannot cubically regularise the systems for any choice of \( a_k \)'s. We remark that this can also be seen from local expansions of the solutions of the equations.

5. Discussion

To summarise, the guiding idea of this paper is the interpretation of Okamoto’s spaces of initial conditions for the Painlevé equations as a mechanism by which equations with a special singularity structure can be associated to rational surfaces via an appropriate notion of regularisation. The main goal of this paper is to find the appropriate extension of this construction to the wider class of equations with the algebro-Painlevé property through a more general notion of regularisation. We proposed algebraic regularisability to play this role, which is stronger than the regularising transformations for quasi-Painlevé equations appearing in [21]. We must remark that the examples from the second-order non-autonomous algebro-Painlevé class we consider are related to Painlevé equations or their solvable autonomous limits via algebraic transformations, so admittedly the geometric picture is unlikely to give new insights in terms of properties of the equations, since everything must map down to the Painlevé case where the geometric picture is well-understood.

The question of whether there exist algebro-Painlevé equations of second-order which are not transformable algebraically to second-order differential equations with the Painlevé property is not \( a \) priori clear, but our framework for constructing spaces of initial conditions could be useful to isolate such examples, if they exist. To this end, it will be interesting to find an algorithmic procedure to find classes of equations with globally finite branching, that is, equations satisfying both a second-order differential equation \( y'' = F(t,y,y') \) and an algebraic relation with meromorphic coefficients \( y^n + s_1(t)y^{n-1} + \ldots + s_{n-1}(t)y + s_n(t) = 0 \). One approach is to find a differential equation for \( s_n \) (of second order and of degree higher than one) and find conditions on other coefficients of both the algebraic and differential equations necessary for \( s_n \) to be a meromorphic function. The difficulty here lies in finding an equivalent system of first order differential equations for which the method of blowing up singularities works well.

The construction presented here also has potential applications in proving the algebro-Painlevé property for systems for which have this property but whose relation to a Painlevé equation is not known. In the case of the differential Painlevé equations, the regularisation on the defining manifold is not sufficient to prove the Painlevé property—having regular initial
value problems everywhere is one part, but afterwards certain auxiliary functions are required to complete the proofs. The geometry of the defining manifold can to an extent give clues as to the construction of appropriate auxiliary functions as in [35], so it is natural to ask if methods of proofs of the algebro-Painlevé property could be constructed purely from manifolds on which a system is algebraically regularisable.

We believe that the formulation we have presented also has value as a possible approach to classification of algebro- and quasi-Painlevé equations via associated surfaces. In the Painlevé case the classification is based on intersection configurations of inaccessible divisors, along which the symplectic form for the Hamiltonian structure of the system has poles. In the algebro-Painlevé case, algebraic regularisability can also be seen in terms of properties of Hamiltonian structures of the systems on their defining manifolds, but their relation to the symplectic structure is different. As we have shown for the examples in section 4 related to Painlevé equations or their autonomous limits, the transformation can be seen as a diagram surgery of the intersection graph of inaccessible divisors, which with some work to could be turned into a classification scheme for algebro-Painlevé equations. This would require a formulation of a class of diagram surgeries which can arise from rational maps to surfaces associated with equations with the Painlevé property, and could potentially yield clues as to how a classification scheme could be developed for the wider class of quasi-Painlevé equations. A Hamiltonian formulation of the regularisability of quasi-Painlevé equations involved in the construction of their spaces of initial conditions will be presented in a subsequent manuscript as the next step towards a potential solution of this classification problem for quasi-Painlevé equations.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Surfaces associated with the Okamoto system

A.1. Blowups

We begin with \((f, g)\) as coordinates on \(\mathbb{C}^2\), which we compactify to \(\mathbb{P}^1 \times \mathbb{P}^1\) by introducing \(F = 1/f, G = 1/g\), so \(\mathbb{P}^1 \times \mathbb{P}^1\) is covered by the four affine charts \((f, g), (F, g), (f, G), (F, G)\). In performing the sequence of eight blowups to obtain \(X^{\text{Ok}}\) we use the following convention for introducing charts. After blowing up a point \(p_i\) given in some affine chart \((x, y)\) by
\((x, y) = (x^*, y^*)\), the exceptional divisor \(E_i \cong \mathbb{P}^1\) replacing \(p_i\) can be covered by two local affine coordinate charts \((u_i, v_i)\) and \((U_i, V_i)\) given by

\[
\begin{align*}
x - x^* &= u_i v_i, & y - y^* &= v_i, \\
x - x^* &= V_i, & y - y^* &= U_i V_i.
\end{align*}
\tag{93}
\]

In particular the exceptional divisor \(E_i\) has in these charts local equations \(v_i = 0\), respectively \(V_i = 0\). Below we present the locations in coordinates of the points to be blown up to obtain \(\mathcal{X}^{\text{Ok}}\) as on figure 1, with arrows \(p_{i-1} \leftarrow p_i\) indicating that \(p_i\) lies on the exceptional divisor arising from the blowup of \(p_{i-1}\). We use the convention (93) for charts, so e.g. \(p_1 : (F, g) = (0, 0) \leftarrow p_2 : (U_1, V_1) = (-a_2, 0)\) indicates that the coordinates \((U_1, V_1)\) in which we find \(p_2\) are defined by \(F = V_1, \ g = U_1 V_1\),

\[
\begin{align*}
p_1 : (F, g) &= (0, 0) & p_2 : (U_1, V_1) &= (-a_2, 0) \\
p_3 : (f, G) &= (0, 0) & p_4 : (u_3, v_3) &= (a_1, 0) \\
p_5 : (F, G) &= (0, 0) & p_6 : (U_5, V_5) &= (2, 0) \\
\uparrow & & p_7 : (u_6, v_6) &= (-4t, 0) \\
\uparrow & & p_8 : (u_7, v_7) &= (4(1 - a_1 - a_2 + 2t^2), 0)
\end{align*}
\]

The inaccessible divisors whose support \(D^{\text{Ok}}\) is removed from \(\mathcal{X}^{\text{Ok}}\) as part of the construction of Okamoto’s space are given by

\[
\begin{align*}
D_0 &= E_3 - E_4, & D_4 &= E_6 - E_7, \\
D_1 &= E_1 - E_2, & D_5 &= E_2 - E_8, \\
D_2 &= H_f - E_1 - E_5, & D_6 &= H_g - E_3 - E_5, \\
D_3 &= E_5 - E_6, &
\end{align*}
\tag{94}
\]

where \(H_f, H_g\) denote the total transforms of lines of constant \(f, g\), respectively on \(\mathbb{P}^1 \times \mathbb{P}^1\), so for example \(D_2 = H_f - E_1 - E_5\) is the proper transform of the line \(F = 0\). In particular

\[
D^{\text{Ok}} = \bigcup_{i=0}^6 D_i.
\tag{95}
\]

Note that we have used a slightly different enumeration to that in [20], which we have chosen to make the comparison of intersection graphs of inaccessible divisors on \(\mathcal{X}^{\text{Tak}}\), \(\mathcal{X}^{\text{Ok}}\) neater.

### A.2. Symplectic atlas

While the charts introduced in the canonical way in the previous section are sufficient to provide an atlas for \(E^{\text{Ok}}\) to which the system (5) extends to be everywhere regular, to obtain the atlas due to Matano et al [24] (see also [32]) in which all Hamiltonians are polynomial, one additional change of coordinates is required.

The coordinates introduced after the blowups of \(p_1, \ldots, p_4\) remain unchanged from above, but after the blowup of \(p_5\) one must make the local change of coordinates

\[
U_5 \mapsto \tilde{U}_5 = \frac{1}{U_5}, \quad V_5 \mapsto \tilde{V}_5 = V_5.
\tag{96}
\]
After this, we obtain the charts from the atlas (them from those above. These are given by the following:

\[ p_5 : (F, G) = (0, 0) \quad \leftarrow \quad p_6 : (\bar{U}_5, \bar{V}_5) = (1/2, 0) \]

\[ \uparrow \]

\[ p_7 : (\bar{u}_6, \bar{v}_6) = (1, 0) \]

\[ \uparrow \]

\[ p_8 : (\bar{u}_7, \bar{v}_7) = (a_1 + a_2 - 1, 0) \]

After this, we obtain the charts from the atlas (8) by letting

\[ (x_1, y_1) = (u_2, v_2), \quad (x_2, y_2) = (v_4, u_4), \quad (x_3, y_3) = (\bar{u}_8, \bar{v}_8), \]

which are related to the coordinates \((f, g)\) by the gluing (9). The fact that these provide canonical coordinates for \(\omega^{Ok}\) as in (14) is verified by direct calculation.

### A.3. Extended Okamoto surfaces

The locations of the extra points \(z_1, \ldots, z_8\) on \(\mathcal{X}^{Ok}\) to be blown up to obtain \(\tilde{\mathcal{X}}^{Ok}\) as indicated on figure 5 are given by

\[
\begin{align*}
  z_1 : (u_3, v_3) &= (0, 0), & z_1 &\in D_0 \cap (H_j - E_j), \\
  z_2 : (U_4, V_4) &= (0, 0), & z_2 &\in D_0 \cap E_4, \\
  z_3 : (U_2, V_2) &= (0, 0), & z_3 &\in D_1 \cap D_2, \\
  z_4 : (u_1, v_1) &= (0, 0), & z_4 &\in D_1 \cap E_2, \\
  z_5 : (u_5, v_5) &= (0, 0), & z_5 &\in D_3 \cap D_2, \\
  z_6 : (U_6, V_6) &= (0, 0), & z_6 &\in D_3 \cap D_4, \\
  z_7 : (U_7, V_7) &= (0, 0), & z_7 &\in D_4 \cap D_5, \\
  z_8 : (U_8, V_8) &= (0, 0), & z_8 &\in D_5 \cap E_5,
\end{align*}
\]

(98)

where the coordinates \((u_i, v_i)\), \((U_i, V_i)\) are those introduced using the point locations in appendix A.1, so without the coordinate change on \(E_5\) which was used to construct the symplectic atlas.

For computations on \(\tilde{\mathcal{X}}^{Ok}\) we use charts \((r_i, s_i)\) and \((R_i, S_i)\) to cover the exceptional divisor \(L_i = \rho^{-1}(z_i)\), defined according to the same convention as \((u_i, v_i)\) and \((U_i, V_i)\), namely

\[
\begin{align*}
  u_3 &= r_3 s_1, & v_3 &= s_1, & u_3 &= S_1, & v_3 &= R_3 S_1, \\
  U_4 &= r_2 s_2, & V_4 &= s_2, & U_4 &= S_2, & V_4 &= R_2 S_2, \\
  U_2 &= r_3 s_3, & V_2 &= s_3, & U_2 &= S_3, & V_2 &= R_3 S_3, \\
  u_1 &= r_4 s_4, & v_1 &= s_4, & u_1 &= S_4, & v_1 &= R_4 S_4, \\
  u_5 &= r_5 s_5, & v_5 &= s_5, & u_5 &= S_5, & v_5 &= R_5 S_5, \\
  U_6 &= r_6 s_6, & V_6 &= s_6, & U_6 &= S_6, & V_6 &= R_6 S_6, \\
  U_7 &= r_7 s_7, & V_7 &= s_7, & U_7 &= S_7, & V_7 &= R_7 S_7, \\
  U_8 &= r_8 s_8, & V_8 &= s_8, & U_8 &= S_8, & V_8 &= R_8 S_8,
\end{align*}
\]

(99)

so that \(L_i\) has local equation \(s_i = 0\), respectively \(S_i = 0\).
Appendix B. Surfaces associated with the Takasaki system

B.1. Blowups

Just as in the Okamoto case we begin with \((q, p)\) as coordinates on \(\mathbb{C}^2\), which we compactify to \(\mathbb{P}^1 \times \mathbb{P}^1\) by introducing \(Q = 1/q, P = 1/p\), so \(\mathbb{P}^1 \times \mathbb{P}^1\) is covered by the four affine charts \((q, p), (Q, p), (q, P), (Q, P)\). We now present the locations of points \(q_1, \ldots, q_{20}\) as depicted on figure 2, using precisely the same convention as in appendix A for introduction of charts after a blowup, recycling the notation \((u_i, v_i), (U_i, V_i)\). We first blowup up \(q_1\), then find two distinct point \(q_2\) and \(q_4\) on the exceptional divisor \(F_1\). Two blowups are then performed over each of these, the centres of which are given in coordinates as follows.

\[ q_1 : (Q, P) = (0, 0) \quad \left\{ \begin{array}{l}
q_2 : (u_1, v_1) = (4a_1, 0) \\
q_4 : (u_1, v_1) = (-4a_1, 0)
\end{array} \right. \quad q_3 : (u_2, v_2) = (0, 0) \quad q_5 : (u_4, v_4) = (0, 0) \]

After four repeated blowups over \(q_6\) we again find two distinct point on the exceptional divisor \(F_8\):

\[ q_6 : (Q, P) = (0, 0), \quad q_7 : (U_6, V_6) = (0, 0), \quad q_8 : (u_7, v_7) = (0, 0) \quad \left\{ \begin{array}{l}
q_9 : (u_9, v_9) = (8, 0) \\
q_{15}
\end{array} \right. \]

We perform six blowups over each of these, of points given in coordinates over \(q_9\) as given by the following:

\[ q_9 : (u_9, v_9) = (8, 0) \quad \left\{ \begin{array}{l}
q_{10} : (u_9, v_9) = (0, 0) \\
q_{11} : (u_{10}, v_{10}) = (-64r, 0) \\
q_{12} : (u_{11}, v_{11}) = (0, 0) \\
q_{13} : (u_{12}, v_{12}) = (256(2 + 2t^2 - a_1 - 2a_2), 0) \\
q_{14} : (u_{13}, v_{13}) = (0, 0).
\end{array} \right. \]

Similarly over \(q_{15}\) we perform six blowups:

\[ q_{15} : (u_{18}, v_{18}) = (-8, 0) \quad \left\{ \begin{array}{l}
q_{16} : (u_{15}, v_{15}) = (0, 0) \\
q_{17} : (u_{16}, v_{16}) = (64r, 0) \\
q_{18} : (u_{17}, v_{17}) = (0, 0) \\
q_{19} : (u_{18}, v_{18}) = (-256(2t^2 - a_1 - 2a_2), 0) \\
q_{20} : (u_{19}, v_{19}) = (0, 0).
\end{array} \right. \]

We note that in the above point locations, there are several instances of points lying on the proper transforms of exceptional divisors from prior blowups. These are as follows, where we use \(B_{q_1, \ldots, q_{20}}\) to denote the projection from the blowups of \(q_1, \ldots, q_j\).
• $q_7$ lies on the proper transform under $\text{Bl}_{q_6}$ of the line $\{P = 0\}$.
• $q_8$ lies on the proper transform under $\text{Bl}_{q_8}$ of the line $\{P = 0\}$.

To describe the inaccessible divisors on the resulting surface $\chi^\text{Tak}$ we let $H_q$, $H_p$ be divisors giving the total transforms of lines of constant $q$, $p$ respectively. The inaccessible divisors are

$$
H_q - F_1, \quad F_1 - F_2 - F_4, \quad F_2 - F_3, \quad F_4 - F_5,
$$

$$
H_p - F_1 - F_6 - F_7 - F_8, \quad F_8 - F_9 - F_{15}, \quad F_7 - F_8, \quad F_6 - F_7, \quad H_q - F_6,
$$

$$
F_9 - F_{10}, \quad F_{10} - F_{11}, \quad F_{11} - F_{12}, \quad F_{12} - F_{13}, \quad F_{13} - F_{14},
$$

$$
F_{15} - F_{16}, \quad F_{16} - F_{17}, \quad F_{17} - F_{18}, \quad F_{18} - F_{19}, \quad F_{19} - F_{20},
$$

(100)

for which we do not introduce labels since they will change under the minimisation in what follows.

### B.2. Minimisation

We note that among the components of $D^\text{Tak}$ as listed in (100), there are two exceptional curves of the first kind, i.e. those of self-intersection $-1$. These are $H_q - F_1$, which is the proper transform of the line $\{q = 0\}$, and $H_p - F_6$, which is the proper transform of the line $\{Q = 0\}$.

Under the birational morphism contracting both of these, we note that another component $F_6 - F_7$ is sent to an exceptional curve of the first kind which we can then contract. Blowing down this way, we again see another component becoming of self-intersection $-1$, namely $F_7 - F_8$.

Under the projection

$$
\sigma : \chi^\text{Tak} \to \chi^\text{m},
$$

(101)

from this sequence of four blowdowns, we find no more components of $\sigma(D^\text{Tak})$ which are contractable through blowdowns. This leads to the curves $C_i$, $i = 1, \ldots, 15$, as shown on figure 3, given explicitly as divisors on $\chi^\text{Tak}_m$ by

$$
C_1 = F_2 - F_3, \quad C_6 = F_9 - F_{10}, \quad C_{11} = F_{15} - F_{16},
$$

$$
C_2 = F_4 - F_5, \quad C_7 = F_{10} - F_{11}, \quad C_{12} = F_{16} - F_{17},
$$

$$
C_3 = \sigma(F_1 - F_2 - F_4), \quad C_8 = F_{11} - F_{12}, \quad C_{13} = F_{17} - F_{18},
$$

$$
C_4 = H_p - F_1 - F_6 - F_7 - F_8, \quad C_9 = F_{12} - F_{13}, \quad C_{14} = F_{18} - F_{19},
$$

$$
C_5 = \sigma(F_8 - F_9 - F_{14}), \quad C_{10} = F_{13} - F_{14}, \quad C_{15} = F_{19} - F_{20},
$$

(102)

where we write the image $\sigma(D)$ simply as $D$ if the divisor $D$ is disjoint from the curves that are blown down by $\sigma$, since in this case $D$ and $\sigma(D)$ are isomorphic.

### B.3. Symmetries

In [8] the following Bäcklund transformation for the fourth Painlevé equation is given. Let $\lambda = \lambda(t)$ be the solution of the fourth Painlevé equation with parameters $\alpha$ and $\beta$, then a new solution $\lambda_1 = \lambda_1(t)$ with parameters $\alpha_1$ and $\beta_1$ is given by

$$
\lambda_1 = \frac{\lambda' - \mu_1 \lambda^2 - 2 \mu_1 t \lambda - \mu_2 b}{2 \mu_1 \lambda}
$$

(103)

with

$$
\alpha_1 = \frac{1}{4}(2 \mu_1 - 2 \alpha + 3 \mu_1 \mu_2 b), \quad \beta_1 = -\frac{1}{2}(1 + \mu_1 \alpha + \mu_2 b/2)^2,
$$

(104)
Table B1. Algebraic Bäcklund transformation symmetries of the Takasaki system

| F(q, p, ˜q, ˜p) = 0 | G(x₁, y₁, ˜x₁, ˜y₁) = 0 | ˜a₁ | ˜a₂ |
|---------------------|----------------------|------|------|
| ˜q² = q² (1 - 8α₁q^{a₁}) | ˜x₁ = x₁ - \frac{8α₁}{2q(1-4q^{a₁})} | 1 - a₂ | 1 - a₁ |
| ˜q² = q² (1 + \frac{64α₂}{q^{a₂} - 64q^{a₂}}) | ˜x₁ = x₁ | a₁ + a₂ | a₁ - a₂ |
| ˜q² = \frac{q²}{q^{a₂} - 64q^{a₂} - 32α₁} | ˜x₁ = x₁ | 1 - a₁ - a₂ |

where \( \mu_1^2 = \mu_2^2 = 1 \) and \( b^2 = -2\beta \). Using the change of variables we can easily verify the corresponding Bäcklund transformation for the Takasaki equation (17). Let \( q = q(t) \) be the solution with parameters \( \alpha \) and \( \beta \) then the new solution \( q_1 = q_1(t) \) with parameters \( \alpha_1 \) and \( \beta_1 \) as above is given by

\[
q_1 = \frac{\mu_3}{q} \sqrt{\frac{8qq'q^{a₁}q^{a₂} - 8μ_1q^{a₁}q^{a₂} - 16μ_2b}{2μ₁}}
\]

(105)

with additionally \( μ_3^2 = 1 \). Note that according to [8, theorem 25.3] we do not have a composition of Bäcklund transformations that would give \( β = 0 \), so this formula does not provide a symmetry in the \( β = 0 \) case.

Alternatively we may consider the generators of the whole extended affine Weyl group \( \tilde{W}(A_2^{(1)}) \) of symmetries of \( P_{IV} \) using their form as birational transformations of the variables \( (f, g) \) from system (5). Lifting these under the transformation (31), we find algebraic symmetries of the Takasaki system (16), in terms of both the original \( (q, p) \) variables and \( (x₁, y₁) \) from the atlas (23). Note that for conciseness we consider only the symmetries of (5) which fix the independent variable \( t \), so elements of \( \text{Aut}(A_2^{(1)}) \) of order two are excluded, though these can be lifted in a similar way. The symmetries of the Takasaki system take the form of a vector of algebraic relations

\[
F(q, p, ˜q, ˜p) = 0 \quad \text{(resp. } G(x₁, y₁, ˜x₁, ˜y₁) = 0 \text{)},
\]
such that if \( (q(t), p(t)) \) (resp. \( (x₁(t), y₁(t)) \)) solve the Takasaki system with parameters \( a₁ \), then \( (˜q(t), ˜p(t)) \) (resp. \( (˜x₁(t), ˜y₁(t)) \)) solve the system with parameters \( ˜a₁ \). We give the transformations of variables and parameters in table B1 corresponding to the generators \( s₀, s₁, s₂, \) and \( ρ = \pi₂π₁ \) of the symmetry group written in [20, section 8.4.20], where \( a₀ = 1 - a₁ - a₂ \) is introduced for convenience.

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References

[1] Bassom A P, Clarkson P A, Hicks A C and McLeod J B 1992 Integral equations and exact solutions for the fourth Painlevé equation Proc. R. Soc. A 437 1–24
[2] Clarkson P A and McLeod J B 1992 Integral equations and connection formulae for the Painlevé equations Painlevé Transcendents, Their Asymptotics and Physical Applications ed P Winternitz and D Levi (Springer) pp 1–31 (available at: https://link.springer.com/chapter/10.1007/978-1-4899-1158-2_1)
[3] Bertola M, Cafasso M and Rubtsov V 2018 Noncommutative Painlevé equations and systems of Calogero type Commun. Math. Phys. 363 503–30
[4] Calogero F 1971 Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials J. Math. Phys. 12 419–36
[5] Chiba H 2016 The first, second and fourth Painlevé equations on weighted projective spaces J. Differ. Equ. 260 1263–313
[6] Filipuk G and Halburd R 2009 Movable algebraic singularities of second-order ordinary differential equations J. Math. Phys. 50 023509
[7] Fuchs R 1905 Sur quelques équations différentielles linéaires du second ordre C. R. Acad. Sci., Paris 141 555–88
[8] Gromak V I, Laine I and Shimomura S 2002 Painlevé Differential Equations in the Complex Plane (De Gruyter Studies in Mathematics vol 28) (Walter de Gruyter & Co)
[9] Guillot A 2019 Meromorphic vector fields with single-valued solutions on complex surfaces Adv. Math. 354 106742
[10] Halburd R and Kecker T 2014 Local and global finite branching of solutions of ordinary differential equations Proc. Workshop on Complex Analysis and its Applications to Differential and Functional Equations Publ. Univ. East. Finl. Rep. Stud. For. Nat. Sci., 14, Univ. East. Finl., Fac. Sci. For. Joensuu pp 57–58
[11] Hinkkanen A and Laine I 1999 Solutions of the first and second Painlevé equations are meromorphic J. Anal. Math. 79 345–77
[12] Hinkkanen A and Laine I 2001 Solutions of a modified third Painlevé equation are meromorphic J. Anal. Math. 85 323–37
[13] Hinkkanen A and Laine I 2001 Solutions of a modified fifth Painlevé equation are meromorphic Papers on analysis 83 133–46 Rep. Univ. Jyväskylä Dep. Math. Stat. Univ. Jyväskylä
[14] Hinkkanen A and Laine I 2004 The meromorphic nature of the sixth Painlevé transcendents Papers on analysis 83 319–42
[15] Inozemtsev V I and Meshcheryakov D V 1985 Extension of the class of integrable dynamical systems connected with semisimple Lie algebras Lett. Math. Phys. 9 13–18
[16] Inozemtsev V I 1989 Lax representation with spectral parameter on a torus for integrable particle systems Lett. Math. Phys. 17 11–17
[17] Its A R and Kapaev A A 1998 Connection formulae for the fourth Painlevé transcendent; Clarkson-McLeod solution J. Phys. A 31 4073–113
[18] Iwasaki K and Okada S 2016 On an orbifold Hamiltonian structure for the first Painlevé equation J. Math. Soc. Japan 68 961–74
[19] Joshi N and Radnović M 2016 Asymptotic behaviour of the fourth Painlevé transcendents in the space of initial values Constr. Approx. 44 195–231
[20] Kajiwara K, Noumi M and Yamada Y 2017 Geometric aspects of Painlevé equations J. Phys. A 50 073001
[21] Uludağ A and Filipuk G 2022 Regularising transformations for complex differential equations with movable algebraic singularities Math. Phys. Anal. Geom. 25 43
[22] Levin A M and Olshanetsky M A 2000 Painlevé-Calogero Correspondence Calogero-Moser-Sutherland Models (CRM Ser. Math. Phys.) (Springer) pp 313–32
[23] Manin Y I 1998 6th Painlevé Equation, Universal Elliptic Curve and Mirror of P2 vol 2 (American Mathematical Society) Adv. Math. Sci., p 39
[24] Matano T, Matumiya A and Takano K 1999 On some Hamiltonian structures of Painlevé systems. II J. Math. Soc. Japan 51 843–66
[25] Matumiya A 1997 On some Hamiltonian structures of Painlevé systems III Kumanoto J. Math. 10 45–73
[26] Okamoto K 1979 Sur les feuilletages associés aux équations du second ordre à points critiques fixés de P. Painlevé (French) [On foliations associated with second-order Painlevé equations with fixed critical points] Japan. J. Math. (N.S.) 5 1–79
[27] Okamoto K 1980 Polynomial Hamiltonians associated with Painlevé equations, I Proc. Japan Acad. A 56 264–8
[28] Okamoto K and Takano K 2001 The proof of the Painlevé property by Masao Hukuhara Funkcial. Ekvac. 44 201–17
[29] Painlevé P 1906 Sur les équations différentielles du second ordre a points critique fixés C.R. Acad. Sci., Paris 143 1111–7
[30] Sakai H 2001 Rational surfaces associated with affine root systems and geometry of the Painlevé equations Commun. Math. Phys. 220 165–229
[31] Sakai H 2013 Ordinary Differential Equations on Rational Elliptic Surfaces Symmetries, Integrable Systems and Representations (Springer Proc. in Mathematics & Statistics vol 40) pp 515–41
[32] Shioda T and Takano K 1997 On some Hamiltonian structures of Painlevé systems. I Funkcial. Ekvac. 40 271–91
[33] Shimomura S 2003 Proofs of the Painlevé property for all Painlevé equations Japan. J. Math. (N.S.) 29 159–80
[34] Steinmetz N 2000 On Painlevé’s equations I, II and IV J. Anal. Math. 82 363–77
[35] Takasaki K 1998 Defining Manifolds for Painlevé Equations Toward the Exact WKB Analysis of Differential Equations, Linear or non-Linear vol 204 (Kyoto University Press) pp 261–9
[36] Takasaki K 2000 Painlevé–Calogero correspondence analysis of Painlevé equations Sūrikaisekikenkyūsho Kōkyūroku (Kyoto University RIMS) pp 71–88 (available at: https://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/40970/1/1203_08.pdf)
[37] Takasaki K 2001 Painlevé–Calogero correspondence revisited J. Math. Phys. 42 1443–73
[38] Wong R and Zhang H Y 2009 On the connection formulas of the fourth Painlevé transcendent Anal. Appl. 4 419–48
[39] Xia J, Xu S-X and Zhao Y-Q 2022 Singular asymptotics for the Clarkson–McLeod solutions of the fourth Painlevé equation Physica D 434 133254
[40] Xia J, Xu S-X and Zhao Y-Q 2023 Clarkson–McLeod solutions of the fourth Painlevé equation and the parabolic cylinder-kernel determinant J. Differ. Equ. 352 249–307