Even Subdivision-Factors of Cubic Graphs

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Abstract

We call a set $S$ of graphs an ”even subdivision-factor” of a cubic graph $G$ if $G$ contains a spanning subgraph $H$ such that every component of $H$ has an even number of vertices and is a subdivision of an element of $S$. We show that any set of 2-connected graphs which is an even subdivision-factor of every 3-connected cubic graph, satisfies certain properties. As a consequence, we disprove a conjecture which was stated in an attempt to solve the circuit double cover conjecture.

Keywords: circuit double cover, factor, frame, Petersen graph

1 Basic definitions and main results

For terminology not defined here we refer to [1]. There are several ways to describe that a spanning subgraph with certain properties exists in a cubic graph $G$.

A set $S$ of graphs is called a component-factor of $G$ if $G$ has a spanning subgraph $H$ such that every component of $H$ is an element of $S$, see [6]. Within the topic of circuit double covers the notion of a frame was introduced, see [3, 4, 7, 8]. Some slightly different definitions of a frame exist. Here, a frame of $G$ is a graph $F$ where every component of $F$ is either an even circuit or a 2-connected cubic graph such that the following holds: $G$ has a spanning subgraph $F'$ which is a subdivision of $F$ and every component of $F'$ has an even number of vertices. For our purpose it is useful to join these two concepts.

Definition 1.1 A set $S$ of graphs is called a subdivision-factor of a cubic graph $G$ if $G$ contains a spanning subgraph $H$ such that every component of $H$ is a subdivision of an element of $S$. If every component of $H$ has an even number of vertices then $S$ is called an even subdivision-factor of $G$.

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Example 1.2 Every 3-edge colorable cubic graph $G_3$ has a spanning subgraph consisting of even circuits, i.e. an even 2-factor. Hence, $\{C_2\}$ where $C_2$ denotes the circuit of length 2, is an even subdivision-factor of $G_3$. Reversely, if $\{C_2\}$ is an even subdivision-factor of a cubic graph $G$, it follows that $G$ is 3-edge colorable.

Thus an even subdivision-factor is a generalization of an even 2-factor. It was asked in a preprint of [4] whether $\{C_2\} \cup H$ where $H$ is a certain infinite family of hamiltonian cubic graphs, is an even subdivision-factor of every 3-connected cubic graph. In particular the following is conjectured in [4]. (A cubic graph $G$ which admits a 3-edge coloring such that each pair of color classes forms an hamiltonian circuit, is called a Kotzig graph, see [4, 7].)

Conjecture 1.3 Every 3-connected cubic graph has a spanning subgraph which is a subdivision of a Kotzig graph.

A positive answer to this conjecture would have solved the circuit double cover conjecture (CDCC), see [4]. For stating the main theorem which provides a negative answer to Conjecture 1.3 and the posed question above, we use two definitions.

Definition 1.4 Let $H_i, i \in \{1, 2\}$ be a subgraph of a graph $G$ or a subset of $V(G)$. Denote by $[H_1, H_2]$ the set of all paths with connect a vertex of $H_1$ with a vertex of $H_2$. Then, $d_G(H_1, H_2)$ or in short $d(H_1, H_2) := \min_{\alpha \in [H_1, H_2]} |E(\alpha)|$.

The parameter $l(G)$ below measures to which extend $G$ is not hamiltonian.

Definition 1.5 Let $G$ be a 2-connected graph. Denote by $U(G)$ the set of all circuits of $G$. Define

$$l(G) := \min_{C \in U(G)} \max_{v \in V(G)} d(C, v).$$

Let $S$ be a set of 2-connected graphs. Define $l_m(S) := \max_{G \in S} l(G)$ if this maximum exists; otherwise set $l_m(S) := \infty$.

Note that in the case of $G$ being hamiltonian, $l(G) = 0$. We state the main result.

Theorem 1.6 Let $S$ be a set of 2-connected graphs which is an even subdivision-factor of every 3-connected cubic graph, then $l_m(S) = \infty$. 

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Theorem 1.6 implies that there is no finite set of graphs which is an even subdivision factor of every 3-connected cubic graph. Note that Conjecture 1.3 remains open for cyclically 4-edge connected cubic graphs. A positive answer to this version would still solve the CDCC since a minimal counterexample to the CDCC is at least cyclically 4-edge connected. In order to prove Theorem 1.6, we prove Theorem 2.14 which concerns the iterated Petersen graph. From now on, we make preparations for the proof of Theorem 2.14.

2 The iterated Petersen graph

We denote by $P_{10}$ the Petersen graph and we set $P := P_{10} - z$, $z \in V(P_{10})$. The iterated Petersen graph which is defined next has already been introduced in [2].

Definition 2.1 Let $G$ be a graph with $d(v) \in \{2, 3\}$, $\forall v \in V(G)$. A $P$-inflation at $v_0 \in V(G)$ is defined as the following operation: add $P$ to $G - v_0$ and connect each former neighbor of $v_0$ to one distinct 2-valent vertex of $P$. $G^0, G^1, G^2, ..., G^k$ with $k \in \mathbb{N}$ and $G^0 := G$, is the sequence of graphs where $G^i$, $i \in \{1, 2, ..., k\}$ results from $G^{i-1}$ by applying the $P$-inflation at every vertex in $G^{i-1}$. We call $P^k$ for $k \geq 1$ an iterated Petersen graph.

Obviously, $G^k$ is cubic if $G$ is cubic. If $G$ is not cubic, then $G$ and $G^k$ have the same number of vertices of degree 2. See Figure 1 for an illustration of Def. 2.1. Note that if we remove in the illustration of $G^i$ the dangling edges, we obtain $P^i$, $i = 1, 2$.

Definition 2.2 Let $W_k$, $k \in \mathbb{N}$ denote the set of the three 2-valent vertices of $P^k$ and set $d_k := \max \{ d(W_k, v) \mid v \in V(P^k) \}$. If a graph $X$, say, is isomorphic to $P^k$, then $W_k(X)$ denotes the set of the three 2-valent vertices.

Proposition 2.3 Let $k \in \mathbb{N}$, then $d_k = 2^{2k+1} - 1$.

Proof: The statement obviously holds for $k = 0$. Consider $P^k$ for $k > 0$ and set $j_k := \min \{ |V(\alpha)| \mid \alpha \in [w_1, w_2] \}$ with $\{w_1, w_2\} \subseteq W_k$ and $w_1 \neq w_2$. Let $k \geq 1$, then $P^k$ contains 9 disjoint copies of $P^{k-1}$. $P$ results from $P^k$ by contracting each of them to a distinct vertex. Hence, every copy $P'$ of $P^{k-1}$ in $P^k$ corresponds to a vertex in $P$. We say a path $\alpha$ traverses $P' \subseteq P^k$ if $\alpha$ contains a subpath $\alpha' \subseteq P'$ which connects two distinct vertices of $W_{k-1}(P')$. Every shortest path in $P^k$ which connects $w_1$ with $w_2$, traverses exactly 4
Figure 1: A vertex in a cubic graph $G$ and the corresponding copies of $P^{k-1}$ in $G^i$, $i = 1, 2$.

copies of $P^{k-1}$ and thus $j_k = 4 j_{k-1}$. Since $j_0 = 4$, we obtain

$$j_k = 4^{k+1}.$$  \hfill (1)

Let $k \in \mathbb{N}$. Set $b_k := \max_{v \in V(P^k)} d(w_1, v)$ and $B_k := \{ v \in V(P^k) \mid d(v, w_1) = b_k \}$.

We claim that

$$B_k = W_k - \{ w_1 \}.$$  \hfill (2)

We proceed by induction on $k$. For $k = 0$, the statement holds. Let $P' \subseteq P^k$ be a copy of $P^{k-1}$ with $v_0 \in B_k \cap V(P')$. Then obviously $P'$ corresponds to a 2-valent vertex of $P$. Let $q_1, q_2$ denote the two distinct vertices of $P'$ which form together a vertex cut of $P^{k-1}$ and which are both contained in $W_{k-1}(P')$.

Then, $d(w_1, q_1) = d(w_1, q_2)$. The induction assumption for $k - 1$ on $P'$ implies that $v_0 \in W_{k-1}(P')$. Since $v_0 \notin \{ q_1, q_2 \}$, $v_0 \in W_k - \{ w_1 \}$. Hence the claim is proven.

Let $k \geq 1$ and let now $P' \subseteq P^k$ be a copy of $P^{k-1}$ with $x \in V(P')$ and $d(x, W_k) = d_k$, see Def. 2.2. Obviously, $P'$ corresponds to a vertex of degree $3$ in $P$. Let $\alpha_x \subseteq P^k$ connect $x$ with a vertex of $W_k$ and satisfy $|E(\alpha_x)| = d_k$. Hence $\alpha_x$ is a shortest path and traverses exactly one copy of $P^{k-1}$ which corresponds to a 2-valent vertex of $P$. By applying (2) on $P'$ we conclude that $x \in W_{k-1}(P')$. Thus, $|E(\alpha_x)| = 2 j_{k-1} - 1$ which finishes the proof.

**Corollary 2.4** $l(P_{10}) = 1$ and $l(P_{10}^k) = 2^{2k-1}$, $\forall k \geq 1$.  

Proof: Since $P_{10}$ has no hamiltonian circuit but $P_{10} - v_0$ is hamiltonian for every $v_0 \in V(P_{10})$, $l(P_{10}) = 1$. Let $k \geq 1$, then $P^k_{10}$ contains ten disjoint copies of $P^{k-1}$ which we denote by $X_i$, $i = 1, 2, \ldots, 10$. Every circuit in $P^k_{10}$ is vertex-disjoint with at least one $X_i$ since otherwise it would imply that $P_{10}$ is hamiltonian. Hence, $l(P^k_{10}) \geq d_{k-1} + 1$. It is not difficult to see that $P^k_{10}$ contains a circuit $C$ which passes through $X_i$ for $i = 1, 2, \ldots, 9$ and satisfies $W_{k-1}(X_i) \subseteq V(C)$. By the properties of $C$ and since $\bigcup_{i=1}^{10} V(X_i) = V(P^k_{10})$, it follows that $d(C, v) \leq d_{k-1} + 1$, $\forall v \in V(P^k_{10})$. Hence, $l(P^k_{10}) = d_{k-1} + 1$ and by applying Prop. 2.3 the proof is finished.

2.1 $f$-matchings and P-inflations

Definition 2.5 A matching $M$ of a cubic graph $G$ is called an $f$-matching if every component of $G - M$ is 2-connected and has an even number of vertices.

Lemma 2.6 Suppose a cubic graph $G$ has a minimal 3-edge cut $E_0$. Then for every $f$-matching $M$ of $G$, $|M \cap E_0| \in \{0, 1\}$.

Proof: Suppose $|M \cap E_0| = 3$. Since $E_0$ is a minimal edge-cut, $G - E_0$ consists of two components which have both an odd number of vertices. Let $L$ be one of them. Then $L - M$ and thus $G - M$ contains at least one component which has an odd number of vertices, in contradiction to Def. 2.5.

Suppose $|M \cap E_0| = 2$. Then the one edge of $E_0$ which is not contained in $M$ is a bridge in $G - M$ which contradicts Def. 2.5. Hence the proof is finished.

Lemma 2.7 Let $E_0 := \{e_1, e_2, e_3\}$ be a minimal 3-edge cut in a 2-connected cubic graph $G$ such that $P$ is one component of $G - E_0$. Then for every $f$-matching $M$ of $G$ the following is true.

(1) Consider $P \subseteq G$ as a graph and $M$ restricted to $P$. Then $P - M$ is connected.

(2) $G - M$ contains a 3-valent vertex within $V(P)$, i.e. at least one vertex of $P \subseteq G$ is not matched by $M$.

Proof: Let $W_0 := \{w_1, w_2, w_3\}$ denote the set of the 2-valent vertices of $P$ and let $e_i \in E_0$ be incident with $w_i$, $i = 1, 2, 3$. By Lemma 2.6 $|M \cap E_0| \in \{0, 1\}$.

Proof of the first statement:

Case 1. $|M \cap E_0| = 0$.

All $w_i$'s are contained in the same component $L$, say, of $P - M$ since otherwise
one component of $G - M$ would have $e_i$, for some $i \in \{1, 2, 3\}$ as a bridge in contradiction to Def. 2.5. Suppose by contradiction that $P - M$ has another component $L'$. Since $V(L') \cap W_0 = \emptyset$, $L'$ is not only a component of $P - M$ but also of $G - M$. By Def. 2.5 $L' \subseteq P$ is 2-connected and thus contains a circuit. There is exactly one circuit $C'$ in $P$ which contains no vertex of $W_0$, see Figure 1. Then $e_i, i = 1, 2, 3$ is a bridge in $G - M$ contradicting Def. 2.5. Hence $P - M$ is connected.

Case 2. $|M \cap E_0| = 1$. Let w.l.o.g. $M \cap E_0 = \{e_3\}$. Then $w_1$ and $w_2$ are contained in the same component $L$, say, of $P - M$ otherwise $e_i, i \in \{1, 2\}$ is a bridge of $G - M$. Suppose by contradiction that $P - M$ has another component $L'$. Since $e_3$ is matched and $w_i \in V(L), i = 1, 2, L'$ is not only a component of $P - M$ but also of $G - M$. By Def. 2.5 $L'$ is 2-connected and thus contains a circuit $C'$. Since $L$ is a component, $L$ contains a path $\beta$ (which is vertex-disjoint with $C'$) connecting $w_1$ with $w_2$. $P_{10}$ is obtained from $P$ and $E_0$ by identifying the three endvertices of $e_i, i = 1, 2, 3$ which are not in $P$. Then $\beta$ and $C'$ correspond to two disjoint circuits in $P_{10}$ which form a 2-factor of $P_{10}$. Hence $C' = L'$, and $L'$ is a circuit of length 5 which contradicts Def. 2.5.

Proof of the second statement:

Suppose by contradiction that every vertex of $P$ is matched by $M$. Since $|V(P)|$ is odd and by Lemma 2.6 $|E_0 \cap M| = 1$. Such matching $M$ covering $V(P)$ corresponds to a perfect matching of $P_{10}$. Hence, $P - M$ consists of a path and a circuit $C$ of length 5. Then $C$ is also a component of $G - M$ which contradicts Def. 2.5.

**Lemma 2.8** Let $G$, $E_0$ and $P$ be as in the previous lemma. Let $\alpha$ be a path in $G$ which passes through $P$, i.e. $\alpha$ has no endvertex in $P$ and $|E(\alpha) \cap E_0| = 2$. Then for every $f$-matching $M$ with $E(\alpha) \cap M = \emptyset$ the following is true: $G - M$ contains a 3-valent vertex within $V(\alpha) \cap V(P)$, i.e. at least one vertex of $V(\alpha) \cap V(P)$ is not matched by $M$.

Proof: Suppose by contradiction that every vertex of $V(\alpha) \cap V(P)$ is matched by $M$. Then $\alpha \cap P$ is a component of $P - M$ and thus by Lemma 2.7 (1) the only component of $P - M$. Since $\alpha \cap P$ contains no 3-valent vertex we obtain a contradiction to Lemma 2.7 (2) which finishes the proof.

**Proposition 2.9** Let $G$ be a 2-connected cubic graph and $v_0 \in V(G)$. Denote by $G'$ the cubic graph which is obtained from $G$ by applying the $P$-inflation at $v_0$. Then $G' - M'$ is 2-connected for every $f$-matching $M'$ of $G'$ if and only if $G - M$ is 2-connected for every $f$-matching $M$ of $G$.  

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Proof: Denote by $P'$ the subgraph of $G'$ which is isomorphic to $P$ and corresponds to $v_0 \in V(G)$.

Suppose by contradiction that $M'$ is an $f$-matching of $G'$ such that $G' - M'$ is not 2-connected whereas $G - M$ is 2-connected for every $f$-matching $M$ of $G$. Set $M'_1 := \{ e \in M' \mid e \notin E(P') \}$. Denote by $M_1$ the subset of $E(G)$ which corresponds to $M'_1$. Then,

$$ (G' - M')/V(P') = G - M_1 \quad (3) $$

We show that $M_1$ is an $f$-matching. Lemma 2.6 implies that $v_0 \in V(G)$ is covered by at most one edge of $M_1$. Hence, $M_1$ is a matching of $G$. Since $P' - M'$ is connected by Lemma 2.7 (1), equation (3) implies that $G - M_1$ has the same number of components as $G' - M'$. Contracting an edge or shrinking a subset of vertices in a bridgeless graph does not create a bridge. Therefore and since $G' - M'$ is bridgeless by Def. 2.5, equation (3) implies that $G - M_1$ is bridgeless. Every component of $G - M_1$ has a corresponding isomorphic component in $G' - M'$ (and thus an even number of vertices) with the one exception of the component $L_0$, say, which contains $v_0$. $P' - M'$ is connected by Lemma 2.7 (1). Denote by $L'_0$ the component of $G' - M'$ with $(P' - M') \subseteq L'_0$. $V(L'_0)$ differs from $V(L_0)$ by containing the vertices of $V(P' - M')$ instead of $v_0$. Since $|V(L'_0)|$ is even by Def. 2.5 and both $|V(P' - M')|$ and $|\{v_0\}|$ are odd, $|V(L_0)|$ is even. Hence $M_1$ is an $f$-matching of $G$. Since $G - M_1$ is not 2-connected we obtain a contradiction to the assumption in the beginning.

Corollary 2.10 For every $f$-matching $M$ of $P_{10}^k$, $k \in \mathbb{N}$, $P_{10}^k - M$ is homeomorphic to a 2-connected cubic graph.

Proof: $P_{10} - M$ is not a circuit since it would imply that $P_{10}$ is hamiltonian. Therefore and since every bridgeless disconnected subgraph of $P_{10}$ consists of two circuits of length 5, $P_{10} - M$ is homeomorphic to a 2-connected cubic graph. Since $P_{10}^k$ is not hamiltonian and results from $P_{10}$ by $P$-inflations and since Proposition 2.9 can be applied after each $P$-inflation, the corollary follows.
2.2 Frames

Lemma 2.11 Let \( k \in \mathbb{N} \), then \( P_{10}^k \) is a frame of \( P_{10}^{k+1} \).

Proof: Let \( M \) be a matching of \( P_{10}^{k+1} \) such that every copy of \( P \) in \( P_{10}^{k+1} \) is matched as in Figure 2. \( M \) is illustrated by dashed lines. Then \( M \) is a \( f \)-matching of \( P_{10}^{k+1} \) and the cubic graph homeomorphic to \( P_{10}^{k+1} - M \) is \( P_{10}^k \). Hence \( P_{10}^k \) is a frame of \( P_{10}^{k+1} \).

Definition 2.12 Let \( \alpha \) be a path in a graph \( G \), then \( p(\alpha) \) denotes the number of distinct copies of \( P \) with which \( \alpha \) has a non-empty vertex-intersection. For \( H_i \subseteq G \), \( i = 1, 2 \), we define \( p[H_1, H_2] := \min \{ p(\alpha) \mid \alpha \in [H_1, H_2] \} \) and we set \( p_k := \max \{ p[v, W_k] \mid v \in V(P^k) \} \), \( k \in \mathbb{N} \).

Lemma 2.13 Let \( k \in \mathbb{N} \), then \( p_{k+1} = 2^{2k+1} \) and \( p_0 = 1 \).

Proof: Clearly, \( p_0 = 1 \). Let \( P(x) \) and \( P(y) \) denote two distinct copies of \( P \) in \( P_{10}^k \), \( k \in \mathbb{N} \) with \( x \in V(P(x)) \) and \( y \in V(P(y)) \). Let \( x' \) (\( y' \)) be the vertex in \( P_{10}^k \) which corresponds to \( P(x) \) (\( P(y) \)) by regarding \( P_{10}^k \) as the graph which is obtained from \( P_{10}^{k+1} \) by contracting every copy of \( P \). Then for every path \( \alpha \in [x, y] \) and its corresponding path \( \alpha' \in [x', y'] \), \( p(\alpha) = |V(\alpha')| \). Hence, \( p[x, y] \geq d(x', y') + 1 \). Since for every given path \( \beta' \in [x', y'] \), there is a path \( \beta \in [x, y] \) with \( p(\beta) = |V(\beta')| \), \( p[x, y] = d(x', y') + 1 \). Therefore, \( p_{k+1} = d_k + 1 \) (Def. 2.2) and by applying Prop. 2.3 the proof is finished.

Theorem 2.14 Let \( \mathcal{F}(k) \) be the set of frames of \( P_{10}^k \), \( k \in \mathbb{N} \), then

1. every frame \( G \) of \( P_{10}^k \) is cubic and 2-connected, and
2. \( \min_{G \in \mathcal{F}(k)} l(G) = \begin{cases} k & \text{for } k \in \{0, 1\} \\ 2^{2k-3} & \text{for } k \geq 2 \end{cases} \).

Proof: Corollary 2.10 implies that every element of \( \mathcal{F}(k) \) is cubic and 2-connected. For \( k = 0 \), the equality above holds since \( K_{3,3} \) is a frame of \( P_{10} \) and \( l(K_{3,3}) = 0 \).

Set \( Q := P_{10}^k \) with \( k \geq 1 \). Let \( M \) be an \( f \)-matching of \( Q \). Denote the 2-connected cubic graph which is homeomorphic to \( Q - M \) by \( \overline{Q}(k) \). Suppose that \( M \) is chosen in such a way that \( l(\overline{Q}(k)) \) is minimal.

A subgraph of \( \overline{Q}(k) \) is denoted by \( \overline{H} \), say, and the corresponding subgraph in \( Q - M \) and \( Q \) by \( H \).
Let $\overline{C}$ be a circuit of $\overline{Q}(k)$ such that $\max_{v \in \overline{Q}(k)} d_{\overline{Q}(k)}(\overline{C}, v) = l(\overline{Q}(k))$. $Q$ contains ten disjoint induced subgraphs isomorphic to $P^{k-1}$. If we contract each of them to a distinct vertex, we obtain $P_{10}$. Hence $C$ does not pass through each of them since otherwise it would imply that $P_{10}$ is hamiltonian. Let us denote one copy of $P^{k-1}$ in $Q$ which is vertex-disjoint with $C$, by $X$.

Figure 2: A matching of a copy of $P$ in $P^{k+1}$.

Let $\{v_1, v_2\} \subseteq V(X)$, then Def. 2.12 implies, if $v_1$ and $v_2$ are contained in the same copy of $P$, that $p[v_1, W_{k-1}(X)] = p[v_2, W_{k-1}(X)]$. Therefore and by Lemma 2.7 (2) there is a vertex $x \in V(X)$ which is not matched by $M$ and which satisfies, $p[x, W_{k-1}(X)] = p_k - 1$, see Def. 2.12. Denote also by $x$ the corresponding vertex in $Q(k)$.

Let $\alpha_x \subseteq Q(k)$ be a path of length $d(x, \overline{C})$ which connects $x$ with $\overline{C}$. By the definition of $x$, $p(\alpha_x) \geq p_k - 1$. Since $V(C) \cap V(X) = \emptyset$, $\alpha_x$ passes through at least $p_k - 1 - 1$ distinct copies of $P$. For every such copy of $P$, $\pi_x$ contains by Lemma 2.8 at least one vertex. Since $\alpha_x$ starts and ends in a vertex of degree 3 which is not contained in any of these copies of $P$, $|V(\alpha_x)| \geq p_k - 1 + 1$. Thus and by definition of $\overline{C}$ and $\pi_x$,

$$l(\overline{Q}(k)) \geq d(x, \overline{C}) \geq p_k - 1 \quad (4)$$

Consider $k = 1$. By inequality (4), $l(\overline{Q}(1)) \geq p_0$. Since $p_0 = 1$ (Lemma 2.13) and since $P_{10}$ is a frame of $Q$ (Lemma 2.11) with $l(P_{10}) = 1$ (Corollary 2.4), $l(\overline{Q}(1)) = 1$.

Consider $k > 1$. By inequality (4) and by Lemma 2.13 $l(\overline{Q}(k)) \geq 2^{2k-3}$. Since by Lemma 2.11 $P^{k-1}_{10}$ is a frame of $Q$ and since by Corollary 2.4 $l(P^{k-1}_{10}) = 2^{2k-3}$, $l(\overline{Q}(k)) = 2^{2k-3}$ which finishes the proof.

**Corollary 2.15** Every $P^{k}_{10}$, $k \geq 1$ is a counterexample to Conjecture 1.3.

**Corollary 2.16** For every set $S_0$ of 2-connected graphs with $l_m(S_0) \neq \infty$, there is an infinite set $\mathcal{S}$ of 3-connected cubic graphs with the following property: for every $G \in \mathcal{S}$, $S_0$ is not an even subdivision-factor of $G$.  

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Proof: Replace every element in $S_0$ which contains a 2-valent and a 3-valent vertex by its homeomorphic cubic graph. Denote this set by $T_0$. We observe that if $S_0$ is an even subdivision-factor of a cubic graph $H$, say, then $T_0$ is also an even subdivision-factor of $H$. Moreover, $l_m(T_0) \leq l_m(S_0)$. Set $\mathcal{S} := \{ P_k^{10} | 2^{2k-3} > l_m(T_0), k \geq 2 \}$. Theorem 2.14 implies that for every $G \in \mathcal{S}$, $T_0$ is not an even subdivision-factor of $G$. By the above observation, the same holds for $S_0$ which finishes the proof.

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