GROTHENDIECK DUALITY MADE SIMPLE

AMNON NEEMAN

ABSTRACT. It has long been accepted that the foundations of Grothendieck duality are complicated. This has changed recently.

By “Grothendieck duality” we mean what, in the old literature, used to go by the name “coherent duality”. This isn’t to be confused with what is nowadays called “Verdier duality”, and used to pass as “ℓ-adic duality”. The footnote below comments on the historical inaccuracy of the modern terminology.

CONTENTS

1. Introduction 2
2. Background 3
  2.1. Reminder: formally inverting morphisms 3
  2.2. Reminder: derived categories and derived functors 3
  2.3. Conventions 7
3. Statements of the main results 7
  3.1. Generalities 7
  3.2. If \( f \) is smooth and proper 10
  3.3. Application: Serre duality 13
4. The proofs 14
  4.1. Simple proofs that have been around for decades 14
  4.2. The simple proofs discovered recently 20
5. Why did it take so long? 29
  5.1. The strangeness of the argument 29
  5.2. The historical block 31
  5.3. What finally woke us up 31
6. Future directions 32
  6.1. Assorted background material 32
  6.2. Foundational questions 37
  6.3. Computational problems 39
Appendix A. A computation of the base-change map \( u^* \to \mathbf{L}u^* \) when \( u : U \to C \) is an open immersion of curves 41

References 43

1 The prevailing current terminology—for duality in étale cohomology, that is “ℓ-adic duality”—is historically incorrect. The idea was originally due not to Verdier but to Grothendieck, see his work in SGA5 on what is nowadays called the formalism of the six operations. Since this survey is about coherent duality we elaborate no further.

2000 Mathematics Subject Classification. Primary 14F05, secondary 13D09, 18G10.

Key words and phrases. Derived categories, Grothendieck duality.

The research was partly supported by the Australian Research Council.
There are two classical paths to the foundations of Grothendieck duality: one due to Grothendieck and Hartshorne \[19\] and (much later) Conrad \[14\], and a second due to Deligne \[16\] and Verdier \[63\] and (much later) Lipman \[32\]. The consensus has been that both are unsatisfactory. If you listen to the detractors of the respective approaches: the first is a nightmare to set up, the second leads to a theory where you can’t compute anything. While exaggerated, each criticism used to have a kernel of truth to it. Lipman summed it up more circumspectly and diplomatically some years ago, by saying there is no royal road to the subject.

And Lipman is probably the person who worked hardest on simplifying the foundations.

In passing let us mention that, while the two foundational avenues to setting up the subject must obviously be related, the details of this link are far from clear—in fact they haven’t yet been fully worked out. And Lipman also happens to be the person who has tried hardest to understand this bridge.

Back to the history of the field: what happened is that in the mid 1970s the math community gradually started losing interest in the project, and by the mid 1980s the exodus was all but complete—most people had given up on improving the foundations and moved on. For three decades now there have been at most a dozen people actively working in the field—they break up into two groups: Lipman, his students and collaborators and Yekutieli, his students and collaborators. These two groups made regular, incremental progress, see for example Alonso, Jeremías and Lipman \[11, 2, 3, 4\], Lipman \[29, 30, 31\], Lipman, Nayak and Sastry \[33\], Lipman and Neeman \[34\], Nayak \[39, 40\], Porta, Shaul and Yekutieli \[49\], Shaul \[55, 56, 57\], Sastry \[50, 51\], Yekutieli \[60, 67, 68, 69, 65\] and Yekutieli and Zhang \[71, 72, 73, 74, 75, 76\].

And then, a couple of years ago, there was a seismic shift. In this article we attempt to describe the recent progress in a way accessible to the non-expert.

Let us stress that the term “non-expert” in the last paragraph is to be understood in the strong sense: the intended audience of this survey includes non-algebraic-geometers, and familiarity with derived categories isn’t assumed. The reason for the inclusive exposition is that the questions opened up by the recent progress might well interest people in diverse fields, the most obvious being Hochschild homology and cohomology. The Hochschild experts might wish to start with Computation \[4.2.11\] as well as Problems \[6.2.1, 6.2.2\] and \[6.3.3\]. Computation \[4.2.11\] spells out exactly where and how Hochschild homology and cohomology played a key role in the breakthrough we report. The three problems suggest obvious variants, explaining why each would be interesting to solve. To put it in a nutshell: up to the present time we—meaning the handful of algebraic geometers still working on the foundations of Grothendieck duality—have only been able to carry out the Hochschild homology computation of \[4.2.11\] which amounts to a very simple, special case of the general problem. Given how profoundly this baby computation has transformed our understanding of Grothendieck duality, we warmly invite mathematicians more adept and dextrous at using the Hochschild machinery to come to our aid.

**Acknowledgements.** The author would like to thank Leo Alonso, Asilata Bapat, Spencer Bloch, Jim Borger, Jesse Burke, Pierre Deligne, Anand Deopurkar, Luc Illusie, Ana Jeremías, Steve Lack, Joe Lipman, Bregje Pauwels, Pramath Sastry, Ross Street, Michel Van den Bergh, Amnon Yekutieli and an anonymous referee for questions, comments, corrections and improvements—based both on earlier versions of this manuscript, and on talks I’ve given presenting parts or all
2. Background

2.1. Reminder: formally inverting morphisms. Let \( \mathcal{C} \) be a category, and let \( S \subset \text{Mor}(\mathcal{C}) \) be some collection of morphisms. It is a theorem of Gabriel and Zisman [17] that one may form a functor \( F: \mathcal{C} \to S^{-1}\mathcal{C} \) so that

(i) The functor \( F \) takes every element of \( S \) to an isomorphism.
(ii) If \( H: \mathcal{C} \to \mathcal{B} \) is a functor, taking every element of \( S \) to an isomorphism, then there exists a unique functor \( G: S^{-1}\mathcal{C} \to \mathcal{B} \) rendering commutative the triangle

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & S^{-1}\mathcal{C} \\
\downarrow H & & \downarrow G \\
\mathcal{B} & & 
\end{array}
\]

We call this construction formally inverting the morphisms in \( S \).

Remark 2.1.1. On objects the functor \( F \) is the identity: the objects of \( S^{-1}\mathcal{C} \) are identical to those of \( \mathcal{C} \). But the morphisms in \( S^{-1}\mathcal{C} \) are complicated. Clearly any morphism of \( \mathcal{C} \) must have an image in \( S^{-1}\mathcal{C} \), but \( S^{-1}\mathcal{C} \) must also contain inverses of the images of morphisms in \( S \). And then we must be able to compose any finite string of these.

The morphisms of \( S^{-1}\mathcal{C} \) are in fact equivalence classes of such finite strings. The problem becomes to figure out when two such strings are equivalent, that is which strings must have the same composite in \( S^{-1}\mathcal{C} \). This is usually called the calculus of fractions of \( S^{-1}\mathcal{C} \). And without an understanding of this calculus of fractions the category \( S^{-1}\mathcal{C} \) is unwieldy.

The category \( S^{-1}\mathcal{C} \) can be dreadful in general, for example: it may happen that \( \mathcal{C} \) has small Hom-sets but \( S^{-1}\mathcal{C} \) doesn’t.

2.2. Reminder: derived categories and derived functors. Let \( \mathcal{A} \) be an abelian category. We will be looking at categories we will denote \( D_\mathcal{C}(\mathcal{A}) \). The category \( D_\mathcal{C}(\mathcal{A}) \) is as follows:

(i) The objects: an object in \( D_\mathcal{C}(\mathcal{A}) \) is a cochain complex of objects in \( \mathcal{A} \), that is a diagram in \( \mathcal{A} \)

\[
\cdots \to A^{-2} \to A^{-1} \to A^0 \to A^1 \to A^2 \to \cdots
\]

where the composite \( A^i \to A^{i+1} \to A^{i+2} \) vanishes for every \( i \in \mathbb{Z} \). The \( \mathcal{C} \) in \( D_\mathcal{C}(\mathcal{A}) \) stands for conditions: we may impose conditions on the objects, it may be that not every cochain complex belongs to \( D_\mathcal{C}(\mathcal{A}) \).

(ii) Morphisms: cochain maps are examples, that is commutative diagrams

\[
\cdots \to A^{-2} \to A^{-1} \to A^0 \to A^1 \to A^2 \to \cdots
\]

\[
\cdots \to B^{-2} \to B^{-1} \to B^0 \to B^1 \to B^2 \to \cdots
\]

where the rows are objects in \( D_\mathcal{C}(\mathcal{A}) \). But we also formally invert the cohomology isomorphisms.
Remark 2.2.1. In the special case of $D_C(A)$ the calculus of fractions is reasonably well understood, there’s a rich literature about it—but we will not explain this here. This means that, whenever we tell the reader about some computation of morphisms, we will be asking the beginner to accept it on faith. The expert will notice that all the computations we mention are easy.

Example 2.2.2. Let $R$ be a commutative ring, and let $A = R \text{-Mod}$ be the abelian category of $R$-modules. Then the category $D(R \text{-Mod})$ has for its objects all the cochain complexes of $R$-modules. The category $D_{K \text{-Flat}}(R \text{-Mod})$ has for its objects all the $K$-flat complexes, the category $D_{K \text{-Inj}}(R \text{-Mod})$ has for its objects all the $K$-injective complexes.

We remind the reader: a complex $F^*$ is $K$-flat if, for all acyclic complexes $A^*$, the complex $A^* \otimes_R F^*$ is acyclic. A complex $I^*$ is $K$-injective if, for all acyclic complexes $A^*$, the complex $\text{Hom}_R(A^*, I^*)$ is acyclic.

Next we recall functors. If $F : A \to B$ is an additive functor, we often want to extend $F$ to derived categories. But the simple-minded approach does not in general work, you cannot simply apply $F$ to the cochain complexes.

Example 2.2.3. Let $f : R \to S$ be a homomorphism of commutative rings, let $A = R \text{-Mod}$ and let $B = S \text{-Mod}$. Fix an object $A \in S \text{-Mod}$. We wish to consider the functor $A \otimes_R (-)$, that is the functor that takes the object $B \in R \text{-Mod}$ to the object $A \otimes_R B$ in $S \text{-Mod}$.

If we try to extend it to a functor $A \otimes_R (-) : D(R \text{-Mod}) \to D(S \text{-Mod})$ we run into the following problem: it is entirely possible to have a cochain map $B^* \to B^*$ of $R$-modules, inducing an isomorphism in cohomology, but where $A \otimes_R B^* \to A \otimes_R B^*$ is not a cohomology isomorphism. The special case where $B^* = 0$ is already problematic. A cochain map

$\cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots$

$\cdots \to B^{-2} \to B^{-1} \to B^0 \to B^1 \to B^2 \to \cdots$

is a cohomology isomorphism if $B^*$ is acyclic. But without some restrictions we would not expect $A \otimes_R B^*$ to be acyclic, meaning

$\cdots \to 0 \to 0 \to 0 \to 0 \to 0 \to \cdots$

$\cdots \to A \otimes_R B^{-2} \to A \otimes_R B^{-1} \to A \otimes_R B^0 \to A \otimes_R B^1 \to A \otimes_R B^2 \to \cdots$

will not be a cohomology isomorphism.

Construction 2.2.4. As above, suppose we are given an additive functor $F : A \to B$. The remedy is to pass to “derived functors”. The idea is as follows:

(i) Find a condition $C$ on the objects of $D_C(A)$, so that if a cochain map $A^* \to B^*$ between objects in $D_C(A)$ is a cohomology isomorphism then so is $FA^* \to FB^*$.

(ii) Prove that the natural functor $I : D_C(A) \to D(A)$ is an equivalence of categories.

Once we achieve (i) and (ii) above, we declare the derived functor of $F$ to be the composite

$D(A) \xrightarrow{I^{-1}} D_C(A) \xrightarrow{F} D(B)$
Example 2.2.5. Let us return to the situation of Example 2.2.3 we are given a ring homomorphism \( R \to S \) and an \( S \)-module \( A \). Then, while the functor \( F(-) = A \otimes_R (-) \) does not respect general cochain maps inducing cohomology isomorphisms, it does respect them when the cochain complexes are \( K \)-flat as in Example 2.2.2. It turns out that the natural functor \( I : D_{K-\text{Flat}}(R-\text{Mod}) \to D(R-\text{Mod}) \) is an equivalence of categories, and we define the functor \( A \otimes^L_R (-) : D(R-\text{Mod}) \to D(S-\text{Mod}) \) to be the composite

\[
\begin{align*}
D(R-\text{Mod}) & \xrightarrow{I^{-1}} D_{K-\text{Flat}}(R-\text{Mod}) \\
& \xrightarrow{A \otimes_R (-)} D(S-\text{Mod})
\end{align*}
\]

Now consider the functor \( \text{Hom}_R(A, -) : R-\text{Mod} \to S-\text{Mod} \). Once again this functor does not respect general cochain maps inducing cohomology isomorphisms. But it does respect them if the cochain complexes are \( K \)-injective as in Example 2.2.2 and the natural functor \( I : D_{K-\text{Inj}}(R-\text{Mod}) \to D(R-\text{Mod}) \) is an equivalence of categories. We define the functor \( R \text{Hom}_R(A, -) : D(R-\text{Mod}) \to D(S-\text{Mod}) \) to be the composite

\[
\begin{align*}
D(R-\text{Mod}) & \xrightarrow{I^{-1}} D_{K-\text{Inj}}(R-\text{Mod}) \\
& \xrightarrow{\text{Hom}_R(A, -)} D(S-\text{Mod})
\end{align*}
\]

Remark 2.2.6. If \( A \) is an abelian category, the category \( \mathcal{C}(A) \) has the same objects as \( D(A) \) but the only morphisms of \( \mathcal{C}(A) \) are the genuine cochain maps. Inverses of cohomology isomorphisms are not allowed.

Generalizing the discussion of Example 2.2.3 we will allow ourselves to derive additive functors \( \mathcal{C}(A) \to \mathcal{C}(B) \). For example: if \( A^* \) is an object of \( \mathcal{C}(S-\text{Mod}) \) there are standard functors

\[
A^* \otimes_R (-) : \mathcal{C}(R-\text{Mod}) \to \mathcal{C}(S-\text{Mod}) , \quad \text{Hom}_R(A^*, -) : \mathcal{C}(R-\text{Mod}) \to \mathcal{C}(S-\text{Mod}) .
\]

The derived functors \( A^* \otimes^L_R (-) \) and \( R \text{Hom}_R(A^*, -) \) are, respectively, the composites

\[
\begin{align*}
D(R-\text{Mod}) & \xrightarrow{I^{-1}} D_{K-\text{Flat}}(R-\text{Mod}) \\
& \xrightarrow{A^* \otimes_R (-)} D(S-\text{Mod})
\end{align*}
\]

\[
\begin{align*}
D(R-\text{Mod}) & \xrightarrow{I^{-1}} D_{K-\text{Inj}}(R-\text{Mod}) \\
& \xrightarrow{\text{Hom}_R(A^*, -)} D(S-\text{Mod})
\end{align*}
\]

Remark 2.2.7. As presented in Example 2.2.3 and Remark 2.2.6 the construction involves an arbitrary choice. More precisely: in Remark 2.2.6 the functor \( A^* \otimes^L_R (-) \) was defined by observing

(i) The natural functor \( I : D_{K-\text{Flat}}(R-\text{Mod}) \to D(R-\text{Mod}) \) is an equivalence of categories.

(ii) On the category \( D_{K-\text{Flat}}(R-\text{Mod}) \) the functor \( A^* \otimes_R (-) \) is well-defined in the obvious way, meaning that when we restrict to \( K \)-flat cochain complexes the classical functor \( A^* \otimes_R (-) \) respects cochain maps inducing cohomology isomorphisms.

This allowed us to form the functor \( A^* \otimes^L_R (-) \) as the composite

\[
\begin{align*}
D(R-\text{Mod}) & \xrightarrow{I^{-1}} D_{K-\text{Flat}}(R-\text{Mod}) \\
& \xrightarrow{A^* \otimes_R (-)} D(S-\text{Mod})
\end{align*}
\]

But the observant reader will note that

(iii) The natural functor \( I : D_{K-\text{Inj}}(R-\text{Mod}) \to D(R-\text{Mod}) \) is also an equivalence of categories.

(iv) On the category \( D_{K-\text{Inj}}(R-\text{Mod}) \) the functor \( A^* \otimes_R (-) \) is also well-defined in the obvious way.
Hence there is nothing to stop us from forming the composite
\[ \mathbf{D}(R\text{-Mod}) \xrightarrow{F^{-1}} \mathbf{D}_{K\text{-Inj}}(R\text{-Mod}) \xrightarrow{A^* \otimes_R (-)} \mathbf{D}(S\text{-Mod}) \]

And it turns out that the composite \( A^* \otimes_R (-) \), defined using (i) and (ii), does not in general agree with the composite defined using (iii) and (iv).

For a choice-free description one notes that, with the category \( \mathcal{C}(R\text{-Mod}) \) as in Remark 2.2.6 and with \( F: \mathcal{C}(R\text{-Mod}) \to D(R\text{-Mod}) \) the Gabriel-Zisman quotient map of Reminder 2.1, we have a triangle
\[
\begin{array}{ccc}
\mathcal{C}(R\text{-Mod}) & \xrightarrow{F} & \mathbf{D}(R\text{-Mod}) \\
\downarrow \gamma & & \downarrow A^* \otimes_R (-) \\
A^* \otimes_R (-) & \xrightarrow{\delta} & \mathbf{D}(S\text{-Mod})
\end{array}
\]

That is: there is a natural transformation from the composite functor \([A^* \otimes_R L (-)] \circ F\) to the functor \(A^* \otimes_R (-)\). And what turns out to be true is that the triangle above has the obvious universal property: for any triangle
\[
\begin{array}{ccc}
\mathcal{C}(R\text{-Mod}) & \xrightarrow{F} & \mathbf{D}(R\text{-Mod}) \\
\downarrow \gamma & & \downarrow G \\
A^* \otimes_R (-) & \xrightarrow{\delta} & \mathbf{D}(S\text{-Mod})
\end{array}
\]

of functors and natural transformation, there is a unique natural transformation \( \varphi: G \Rightarrow [A^* \otimes_R L (-)] \circ F \) such that \( \delta = \gamma \circ (\varphi F) \). In category theoretic language: the functor \( A^* \otimes_R L (-) \) is the right Kan extension of \( A^* \otimes_R (-) \) along \( F \).

This description as the right Kan extension is what earns the functor \( A^* \otimes_R L (-) \) the title of the left derived functor of the functor \( A^* \otimes_R (-) \), and is the reason for the \( L \) in the symbol. The functor \( R\text{Hom}(A^*, -) \), which we met in Example 2.2.5 and Remark 2.2.6 turns out to be the left Kan extension, meaning the triangle of functors and natural transformation is
\[
\begin{array}{ccc}
\mathcal{C}(R\text{-Mod}) & \xrightarrow{F} & \mathbf{D}(R\text{-Mod}) \\
\downarrow \gamma & & \downarrow R\text{Hom}(A^*, -) \\
\text{Hom}(A^*, -) & \xrightarrow{\delta} & \mathbf{D}(S\text{-Mod})
\end{array}
\]

and the universality is with respect to all such triangles. Being the left Kan extension earns the functor \( R\text{Hom}(A^*, -) \) the title of right derived functor of \( \text{Hom}(A^*, -) \), as well as the \( R \) in the symbol.
2.3. Conventions. Unless otherwise stated all rings are assumed commutative and noetherian, all schemes are assumed noetherian, and all morphisms of schemes are assumed of finite type. Since we will often deal with the derived category $\mathbf{D}(R\text{-Mod})$ we abbreviate it to $\mathbf{D}(R)$. If $X$ is a scheme we will use $\mathbf{D}_{\text{qc}}(X)$ as a shorthand for the category $\mathbf{D}_{\text{qc}}(\mathcal{O}_X\text{-Mod})$. That is: the objects are cochain complexes of sheaves of $\mathcal{O}_X$–modules, and the condition we impose is that the cohomology sheaves are quasicoherent.

3. Statements of the main results

3.1. Generalities. We begin by setting up the framework.

Reminder 3.1.1. Suppose $f : X \rightarrow Y$ is a morphism of schemes. There are three induced functors on the derived categories $\mathbf{D}_{\text{qc}}(X)$

\[
\begin{array}{ccc}
\mathbf{D}_{\text{qc}}(X) & \xrightarrow{Lf^*} & \mathbf{D}_{\text{qc}}(Y) \\
\downarrow & & \downarrow \\
\mathbf{D}_{\text{qc}}(Y) & \xrightarrow{Rf_*} & \mathbf{D}_{\text{qc}}(X)
\end{array}
\]

where each functor is left adjoint to the one to its right; in category theoretic notation we write $Lf^* \dashv Rf_* \dashv f^\times$. We remind the reader what these functors do.

(i) The functor $Lf^*$ is the left-derived pullback functor. We compute it as in Construction 2.2.4: let $\mathbf{D}_{\text{qc},K\text{-Flat}}(Y)$ be the derived category of complexes of $\mathcal{O}_Y$–modules, which are $K$–flat and have quasicoherent cohomology, and let $I : \mathbf{D}_{\text{qc},K\text{-Flat}}(Y) \rightarrow \mathbf{D}_{\text{qc}}(Y)$ be the natural map. The functor $I$ happens to be an equivalence. To evaluate $Lf^*$ on an object $C \in \mathbf{D}_{\text{qc}}(Y)$ you first form $I^{-1}(C) \in \mathbf{D}_{\text{qc},K\text{-Flat}}(Y)$, then pull back to obtain the complex $f^{-1}I^{-1}(C)$ on $X$, and finally form on $X$ the tensor product $Lf^*C = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}I^{-1}(C)$.

(ii) The functor $Rf_*$ is the right-derived pushforward functor. Once again we compute it as in Construction 2.2.4: this time let $\mathbf{D}_{\text{qc},K\text{-Inj}}(X)$ be the derived category of complexes of $\mathcal{O}_X$–modules, which are $K$–injective and have quasicoherent cohomology, and let $I : \mathbf{D}_{\text{qc},K\text{-Inj}}(X) \rightarrow \mathbf{D}_{\text{qc}}(X)$ be the natural map. This functor $I$ also happens to be an equivalence. To evaluate $Rf_*$ on an object $D \in \mathbf{D}_{\text{qc}}(X)$ you first form $I^{-1}(D) \in \mathbf{D}_{\text{qc},K\text{-Inj}}(X)$, and then push forward to obtain the complex $Rf_*D = f_*I^{-1}(D)$ on $Y$.

(iii) The functor $f^\times$ is the mysterious one, Grothendieck duality is about understanding its properties. For a general $f$ it turns out that $f^\times$ need not be the derived functor of any functor on abelian categories, with the notation as in Remark 2.2.7. The reader might wish to look at Remark 6.1.1 and at Appendix A for further discussion of how unusual $f^\times$ can be.

Discussion 3.1.2. Since we’re after an understanding of the functor $f^\times$, we need to agree what the word “understanding” will mean. Recall that the adjunction $Rf_* \dashv f^\times$ gives a natural

1 Algebraic geometers might find the symbol $f^\times$ unfamiliar; the pre-2009 literature on Grothendieck duality talks almost exclusively about another functor $f'$! The functors $f^\times$ and $f'$ agree when $f$ is proper, but not in general. There is a discussion of $f'$ and its relation with $f^\times$ in Reminder 6.1.2 and a brief summary of the history in Remark 6.1.3. Until we reach that point, in this paper we will work exclusively with $f^\times$. 

isomorphism

\[
\text{Hom}(A, f^\times B) \xrightarrow{\varphi(A,B)} \text{Hom}(Rf_*, A, B)
\]

Putting \( A = f^\times B \) this specializes to a homomorphism

\[
\text{Hom}(f^\times B, f^\times B) \xrightarrow{\varphi(f^\times B,B)} \text{Hom}(Rf_*, f^\times B, B)
\]

which sends \( \text{id} : f^\times B \to f^\times B \) to the map \( \varepsilon : Rf_*, f^\times B \to B \), the counit of adjunction. Consider

\[
\text{Hom}(A, f^\times B) \xrightarrow{Rf_*} \text{Hom}(Rf_*, Rf_*, f^\times B) \xrightarrow{\text{Hom}(\varepsilon,-)} \text{Hom}(Rf_*, A, B)
\]

It is classical that naturality forces the composite to agree with \( \varphi(A,B) \). Summarizing:

**Conclusion 3.1.3.** If we could compute, for every \( B \in \mathcal{D}_{\text{qc}}(Y) \), the object \( f^\times B \) and the morphism \( \varepsilon : Rf_*, f^\times B \to B \), then we’d feel we understand the adjunction pretty well. After all the map \( \varphi(A,B) : \text{Hom}(A, f^\times B) \to \text{Hom}(Rf_*, A, B) \) would be explicit: given an element \( \alpha \in \text{Hom}(A, f^\times B) \), that is a morphism \( \alpha : A \to f^\times B \), then the map \( \varphi(A,B) \) would send \( \alpha \) to the composite

\[
Rf_* \xrightarrow{Rf_* \alpha} Rf_* f^\times B \xrightarrow{\varepsilon} B,
\]

which is an element \( \varepsilon \circ Rf_* \alpha \in \text{Hom}(Rf_*, A, B) \). OK: it wouldn’t be so clear how to go back, but classically people have been happy with understanding just this direction.

We will soon specialize to the case where \( f \) is smooth and proper, but the next result holds more generally and we state it in a strong form.

**Theorem 3.1.4.** Assume \( f : X \to Y \) is a finite-type morphism of noetherian schemes. If \( B \in \mathcal{D}_{\text{qc}}(Y) \) and \( C \in \mathcal{D}_{\text{qc}}(X) \) then there is a canonical natural isomorphism \( p_{B,C} : B \otimes^L Rf_* C \to Rf_* (Lf^* B \otimes^L C) \) and a canonical natural transformation \( \chi : Lf^* B \otimes^L f^\times \mathcal{O}_Y \to f^\times B \) such that the following pentagon commutes

\[
\begin{array}{ccc}
Rf_* [Lf^* B \otimes^L f^\times \mathcal{O}_Y] & \xrightarrow{p_{B,f^\times \mathcal{O}_Y}} & B \otimes^L Rf_* f^\times \mathcal{O}_Y \\
\text{Rf}_* \chi & & \text{id} \otimes \varepsilon \\
\text{Rf}_* f^\times B & \xrightarrow{\varepsilon} & B
\end{array}
\]

Furthermore: the map \( \chi \) is an isomorphism if and only if \( f \) is proper and of finite Tor-dimension.

The non-expert should view finite Tor-dimension as a technical condition that will be satisfied by all the \( f \’ s \) we will consider. We will discuss properness in Remark 3.2.2.

**Remark 3.1.5.** Suppose \( f \) is proper and of finite Tor-dimension. Then we have an isomorphism

\[
Lf^* B \otimes^L f^\times \mathcal{O}_Y \to f^\times B,
\]

and the commutative pentagon of Theorem 3.1.4 makes precise the compatibility of this isomorphism with the counit \( \varepsilon \) of the adjunction \( Rf_* \dashv f^\times \). Thus the import of Theorem 3.1.4 is that, as long as \( f \) is proper and of finite Tor-dimension, it suffices to compute \( f^\times \mathcal{O}_Y \) and the counit of adjunction \( \varepsilon : Rf_* f^\times \mathcal{O}_Y \to \mathcal{O}_Y \). We are reduced to studying a single object, namely \( \mathcal{O}_Y \in \mathcal{D}_{\text{qc}}(Y) \).
Reminder 3.1.6. The next reduction comes from the observation that the category \(D_{qc}(X)\) has many endofunctors. There are many diagrams

\[
\begin{array}{ccc}
\Gamma_W & \xrightarrow{c_W} & \Gamma_W \\
\downarrow & & \downarrow \\
D_{qc}(X) & \xrightarrow{id} & D_{qc}(X)
\end{array}
\]

That is: there are many choices of functors \(\Gamma_W : D_{qc}(X) \to D_{qc}(X)\), which come together with natural transformations \(c_W : \Gamma_W \to id\). The ones we have in mind are the Bousfield colocalizations. They come about as follows.

For every point \(p \in X\) let \(i_p : p \to X\) be the inclusion, which we view as a morphism of schemes \(i_p : \text{Spec}(k(p)) \to X\). Suppose we are given a set of points \(W \subset X\). The full subcategory \(D_{qc,W}(X) \subset D_{qc}(X)\) will be the subcategory of all objects supported on \(W\), we recall that this means

\[
D_{qc,W}(X) = \{ E \in D_{qc}(X) \mid \text{Let } p \not\in W\}.
\]

Let \(I_W : D_{qc,W}(X) \to D_{qc}(X)\) be the inclusion. A straightforward generalization of a theorem of Bousfield \cite{13} tells us that \(I_W\) has a right adjoint \(J_W : D_{qc}(X) \to D_{qc,W}(X)\), and the colocalizations we have in mind are the counits of adjunction \(c_W : I_W J_W \to id\).

We will give a concrete example later in this section.

Remark 3.1.7. Let the notation be as in Reminder 3.1.6. It is customary to choose the set of points \(W \subset X\) to be closed under specialization, meaning if \(p \in W\), and if \(q \in X\) belongs to the closure \(\overline{\{p\}}\) of \(p\), then \(q \in W\). The advantage is that for such \(W\) the Bousfield colocalization can be computed locally. More precisely: let \(u : U \to X\) be an open immersion, let \(D_{qc,U \cap W}(U) \subset D_{qc}(U)\) be the full subcategory of objects supported on \(U \cap W\); let \(J_{U \cap W} : D_{qc,U \cap W}(U) \to D_{qc}(U)\) be the inclusion and \(J_{U \cap W}\) its right adjoint, and let \(c_{U \cap W} : \Gamma_{U \cap W} \to \text{id}_{D_{qc,U \cap W}(U)}\) be the counit of the adjunction \(I_{U \cap W} \Rightarrow J_{U \cap W}\). Then the relation with the \(c_W : \Gamma_W \to \text{id}_{D_{qc}(X)}\) of Reminder 3.1.6 is simple: there is a canonical isomorphism \(Lu^* \Gamma_W \cong \Gamma_{U \cap W} \circ \text{id}\) making the triangle below commute.

If \(W\) isn’t specialization-closed this may fail, and the bottom line is that we feel infinitely more comfortable working with tools that lend themselves to local computations.

Given a \(c_W : \Gamma_W \to id\) as in Reminder 3.1.6 we can form the next gadget:

Definition 3.1.8. We define \(\rho_W\) to be the composite

\[
\begin{array}{ccc}
Rf_* \Gamma_W f^* \Omega_Y & \xrightarrow{Rf_* c_W f^*} & Rf_* f^* \Omega_Y \xrightarrow{\varepsilon} \Omega_Y
\end{array}
\]

Remark 3.1.9. Ideally, we would aim to choose \(c_W : \Gamma_W \to id\) in such a way that

(i) The composite \(\rho_W\) of Definition 3.1.8 is easy to compute.
(ii) From the computation of \(\rho_W\) we learn a lot about \(\varepsilon : Rf_* f^* \Omega_Y \to \Omega_Y\).
Of course we could make a dumb choice of \(c_W : \Gamma_W \to \text{id}\). For example: if we let \(c_W : \Gamma_W \to \text{id}\) be the identity map \(\text{id} \to \text{id}\), then \(\rho_W = \varepsilon\), we don’t lose any information in passing from \(\varepsilon\) to \(\rho_W\). but we also haven’t simplified the computation. Or if we choose \(\Gamma_W = 0\) then the computation of \(\rho_W\) becomes trivial, but worthless. The important thing is to choose \(c_W : \Gamma_W \to \text{id}\) wisely.

3.2. If \(f\) is smooth and proper. In the most classical case of the theory we have the following results:

**Theorem 3.2.1.** Assume \(f : X \to Y\) is smooth and proper, of relative dimension \(n\). Then there is a canonical isomorphism \(\theta : \Omega^n_f[n] \iso f^*\mathcal{O}_Y\).

**Remark 3.2.2.** We should explain the theorem, starting with the hypotheses: if the non-expert tried to guess what it means for \(f\) to be smooth and proper, chances are she was right about proper but wrong about smooth. Let us elaborate.

It is customary to consider the following two conditions, which a continuous map \(f : X \to Y\) of topological spaces can satisfy:

(i) \(f^{-1}(K)\) is compact whenever \(K \subset Y\) is compact.

(ii) The map \(f\) is universally closed. This means that, if \(f' : X' \to Y'\) is some pullback of \(f\), then \(f'(K) \subset Y'\) is closed whenever \(K \subset X'\) is closed.

In the category of locally compact Hausdorff spaces the two are equivalent, and a map satisfying these equivalent conditions is what’s normally called proper. As it happens the topological spaces that come up in algebraic geometry are rarely Hausdorff, and in the category of schemes (i) and (ii) aren’t equivalent. It turns out that the right way to define proper maps of schemes is to use (ii), this yields the theory one would intuitively expect.

But when it comes to smoothness algebraic geometers chose to be contrary. In differential geometry—and hence also in related topics like PDE—a smooth map of manifolds is defined to be a \(C^\infty\) map. With this definition algebraic geometers never consider any map that’s remotely non-smooth.

Even though the term “smooth map” was already in use in a well-established, clearly delineated context, algebraic geometers decided to steal the word and give it a different meaning. In this survey we follow the conventions of algebraic geometry: what we label a “smooth map” is what everyone else would dub a “submersion”. In algebraic geometry, a morphism \(f : X \to Y\) of manifolds is smooth if, at every point \(p \in X\), the derivative is a surjection \(t_p : T_p \to T_{f(p)}\). Here \(T_p\) is the tangent space at \(p\) and \(T_{f(p)}\) is the tangent space at \(f(p)\). The smooth map \(f\) has relative dimension \(n\) if the kernel of the linear map \(t_p : T_p \to T_{f(p)}\) is \(n\)-dimensional for every \(p \in X\).

Assume \(f\) is a smooth map of relative dimension \(n\) in the sense above, and let \(\Omega^*_X, \Omega^*_Y\) be the cotangent bundles of \(X, Y\) respectively. The pullback \(f^*\Omega^*_Y\) is naturally a subbundle of \(\Omega^*_X\), and the relative cotangent bundle is by definition the quotient \(\Omega^*_f = \Omega^*_X/f^*\Omega^*_Y\), often written \(\Omega^*_f\). It is a vector bundle of rank \(n\) over \(X\), whose top exterior power is what is usually written \(\Omega^{n}_f = \wedge^n\Omega^*_f\). The line bundle \(\Omega^n_f\) is called the relative canonical bundle of \(f\). Thus Theorem 3.2.1 is the assertion that \(f^*\mathcal{O}_Y\) is canonically isomorphic to the object \(\Omega^n_f[n] \in \text{D}_{\text{qc}}(X)\), which is the cochain complex with only one nonvanishing term, namely the relative canonical bundle \(\Omega^n_f\) in degree \(-n\).
Remark 3.2.3. With the notation as explained in Remark 3.2.2, Theorem 3.2.1 computes for us the object $f^*\mathcal{O}_Y$. By Remark 3.1.6, our mission would be accomplished if we could also compute the counit of adjunction $\varepsilon : Rf_*f^*\mathcal{O}_Y \rightarrow \mathcal{O}_Y$. In Remark 3.1.9 we noted that it might prove expedient to take advantage of some Bousfield colocalization $c_W : \Gamma_W \rightarrow \text{id}$. The traditional choice, which happens to be well-suited for the current computation, is to take $c_W : \Gamma_W \rightarrow \text{id}$ to be the Bousfield colocalization of Reminder 3.1.6, where the set of points $W \subset X$ is the union of the irreducible closed subsets $Z \subset X$ such that the composite map $Z \rightarrow X \rightarrow Y$ is generically finite.

It should be noted that our $\Gamma_W$ has been extensively studied and is very computable, the subject dealing with functors of this genre is called local cohomology. Most of what’s written about $\Gamma_W$ is in the commutative algebra literature.

Before the theorem it might help to illustrate the abstraction in a simple case.

Example 3.2.4. Suppose $Y = \text{Spec}(k)$ where $k$ is field. Then $X$ is smooth over the field $k$ and $n$-dimensional. Take a minimal injective resolution for $\Omega^n_Y[n] \cong f^*\mathcal{O}_Y$: that is form a cochain complex

$$0 \rightarrow I^{-n} \rightarrow I^{-n+1} \rightarrow \cdots \rightarrow I^{-1} \rightarrow I^0 \rightarrow 0$$

where $I^{-n}$ is the injective envelope of $\Omega^n_Y$, next $I^{-n+1}$ is the injective envelope of $I^{-n}/\Omega^n_Y$, and so on. Note that $X$ is regular and $n$-dimensional, hence the injective dimension of $\Omega^n_Y$ is $\leq n$. The minimal injective resolution must stop no later than $I^0$. The $W$ of Remark 3.2.3 is the set of all closed points in $X$. The corresponding Bousfield colocalization $c_W : \Gamma_W\Omega^n_Y[n] \rightarrow \Omega^n_Y[n]$ comes down to the cochain map

$$0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow I^0 \rightarrow 0$$

In the next theorem we will compute the composite $\rho_W$ of Definition 3.1.3 that is we propose to compute the map $Rf_*\Gamma_Wf^*\mathcal{O}_Y \xrightarrow{c_W} Rf_*f^*\mathcal{O}_Y \xrightarrow{\varepsilon} \mathcal{O}_Y$. This map identifies, via the isomorphism $\Omega^n_Y[n] \cong f^*\mathcal{O}_Y$ of Theorem 3.2.1 with a composite $Rf_*\Gamma_W\Omega^n_Y[n] \xrightarrow{Rf_*c_W} Rf_*\Omega^n_Y[n] \xrightarrow{\varepsilon} \mathcal{O}_Y$. Note that as $Y = \text{Spec}(k)$ we have $\mathcal{O}_Y = k$. Recalling Reminder 3.1.1(ii), we compute $Rf_*$ by applying $f_*$ to an injective resolution. That is, in our case $\rho_W$ comes down to a composite cochain map.

---

2The reader might recall that injective resolutions, even minimal ones, are only unique up to homotopy—hence it isn’t obvious that the zeroth sheaf $I^0$ is functorial in anything—and the formula that’s about to come, that is $I^0 = \Gamma_W\Omega^n_Y[n]$, seems unreasonable at first sight.

Given any two injective resolutions of $I^*$ and $J^*$ of $\Omega^n_Y[n]$ there are cochain maps $I^* \xrightarrow{\alpha} J^* \xrightarrow{\beta} I^*$, unique up to homotopy, so that $\alpha\beta$ and $\beta\alpha$ are homotopic to the identity. What is special here, because we are dealing with minimal injective resolutions of a line bundle $\Omega^n_Y$ on a regular scheme $X$, is that any homotopy must vanish. Hence the minimal injective resolution $I^*$ is unique up to canonical isomorphism, as is $I^0 = \Gamma_W\Omega^n_Y[n]$.

For the experts in commutative algebra: the way one proves the vanishing of any homotopy is by noting that the injective sheaf $I^{-j}$ is a direct sum of indecomposable injectives supported at points of dimension $j$, and there are no non-zero maps from an indecomposable injective supported at a point of dimension $j$ to an indecomposable injective supported at a point of dimension $j + 1$. 

This makes it obvious why the cochain map \( \varepsilon : Rf_*\Gamma_W \Omega^n_f [n] \to f_* I^0 \) of Remark 3.2.3 is a reasonable choice: it certainly doesn’t lose information, the map \( \varepsilon^0 : f_* I^0 \to k \) most definitely determines the cochain map \( \varepsilon : Rf_*\Gamma_W \Omega^n_f [n] \to f_* I^0 \).

Injective resolutions might be useful for proving a map is informative, but for computations one usually prefers other resolutions. Fortunately the functor \( \Gamma_W \) has other descriptions. For example there is a description in terms of local cohomology: it turns out that the above is isomorphic to

\[
Rf_*\Gamma_W \Omega^n_f [n] \cong \bigoplus_{p \in X} \mathbb{H}^{-n}(\Omega^n_p).
\]

That is, \( f_* I^0 = Rf_*\Gamma_W \Omega^n_f [n] \) is the direct sum, over all closed points \( p \in X \), of the \( n^{th} \) local cohomology of the sheaf \( \Omega^n_p \) at \( p \). The standard computation of local cohomology, via Čech complexes, tells us that \( f_* I^0 \) may be written as the quotient of \( J \), where \( J \) is the direct sum, over all closed points \( p \in X \), of the vector space of meromorphic differential forms at \( p \). A meromorphic form at \( p \) is the following list of data:

(i) The closed point \( p \in X \).

For parts (ii) and (iii) below we write \( O_{X,p} \) for the stalk at \( p \) of the structure sheaf \( O_X \).

(ii) A system of coordinates \( (x_1, \ldots, x_n) \) at \( p \), that is \( (x_1, \ldots, x_n) \) generate the maximal ideal of the local ring \( O_{X,p} \).

(iii) An expression

\[
\frac{f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{x_1^N x_2^N \cdots x_n^N}
\]

where \( f \in O_{X,p} \).

In Example 3.2.4 we simplified our life by assuming \( Y = \text{Spec}(k) \) with \( k \) a field. The general case is slightly more cumbersome to describe but similar. And now for the main result.

**Theorem 3.2.5.** Suppose \( f : X \to Y \) is a smooth and proper morphism of noetherian schemes, identify \( \Omega^n_f [n] \cong f^* \mathcal{O}_Y \) via the canonical isomorphism \( \theta \) of Theorem 3.2.1, and let \( c_W : \Gamma_W \to \text{id} \) be as chosen in Remark 3.2.3. Then the map \( \rho_W : Rf_*\Gamma_W \Omega^n_f [n] \to \mathcal{O}_Y \) of Definition 3.1.8 may be represented, in the notation of Example 3.2.4 (more accurately in the generalization of the notation to the case where \( Y \) is arbitrary), by the map \( J \to \mathcal{O}_Y \) taking a finite sum of meromorphic forms to the sum of their residues.

**Example 3.2.6.** Let us return to the situation of Example 3.2.4, where \( Y = \text{Spec}(k) \). But now assume further that \( k = \bar{k} \) is an algebraically closed field. Then the residue of a meromorphic
differential form is the obvious. In the expression
\[ f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \]
we may expand \( f \in \mathcal{O}_{X,p} \) into a Taylor series, which we view as an element of the completion \( \widehat{\mathcal{O}}_{X,p} \) of the ring \( \mathcal{O}_{X,p} \). This gives an expansion of the entire meromorphic form into a Laurent series
\[
\sum_{\text{all } k_i \leq N} a_{k_1, k_2, \ldots, k_n} \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}
\]
where the coefficients \( a_{k_1, k_2, \ldots, k_n} \) belong to the field \( k \). The map \( \rho_{W} \) takes the meromorphic form to \( a_{1,1, \ldots, 1} \), that is to the coefficient of \( dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \) in the Laurent series.

**Remark 3.2.7.** The discussion of Example 3.2.6 should explain why we made the simplifying assumption that \( Y = \text{Spec}(k) \) with \( k \) algebraically closed. We run into subtleties already when \( Y = \text{Spec}(k) \) but we drop the hypothesis that \( k = \overline{k} \). In this case the closed point \( p \) will not in general be \( k \)-rational, and we would not expect an element \( f \in \mathcal{O}_{X,p} \subset \widehat{\mathcal{O}}_{X,p} \) to have a Taylor expansion with coefficients in \( k \), in the generators \( \{x_1, x_2, \ldots, x_n\} \) of the maximal ideal. The definition of the residue of a meromorphic form becomes subtler.

### 3.3. Application: Serre duality.

Serre’s classical duality theorem is a special case; let us recall precisely how. But first a reminder: back in Remark 2.2.1 we disclosed that the non-expert will be asked to accept, on faith, all computations of Hom-sets in derived categories—for example the ones she’s about to witness.

Continue to assume that \( k \) is a field and \( f : X \rightarrow Y = \text{Spec}(k) \) is smooth and proper, of relative dimension \( n \). Let \( V \) be a vector bundle on \( X \). Then the adjunction \( Rf_{\ast} \dashv f^{\ast} \) tells us that, in the commutative square below, the map \( \varphi \) is an isomorphism
\[
\begin{array}{ccc}
\text{Hom}_{D_{qc}(X)}(\mathcal{V}[i], f^{\ast} \mathcal{O}_Y) & \xrightarrow{\varphi} & \text{Hom}_{D_{qc}(Y)}(Rf_{\ast} \mathcal{V}[i], \mathcal{O}_Y) \\
\downarrow & & \downarrow \\
\text{Hom}_{D_{qc}(X)}(\mathcal{V}[i], \Omega^n_X[n]) & \xrightarrow{} & \text{Hom}_{D_{qc}(Y)}(Rf_{\ast} \mathcal{V}[i], k)
\end{array}
\]
The bottom row simplifies to
\[
H^{n-i}(\mathcal{H}\text{om}(\mathcal{V}, \Omega^n_X)) \cong \text{Hom}(H^i(\mathcal{V}), k)
\]
meaning there is a nondegenerate pairing \( H^i(\mathcal{V}) \otimes H^{n-i}(\mathcal{H}\text{om}(\mathcal{V}, \Omega^n_X)) \rightarrow k \). The pairing is explicit. It takes a morphism \( \alpha : \mathcal{O}_X \rightarrow \mathcal{V}[i] \) and a morphism \( \beta : \mathcal{V} \rightarrow \Omega^n_X[n-i] \) to the image of \( 1 \in Rf_{\ast} \mathcal{O}_X \) under the composite
\[
\begin{array}{cccc}
Rf_{\ast} \mathcal{O}_X & \xrightarrow{Rf_{\ast} \alpha} & Rf_{\ast} \mathcal{V}[i] \xrightarrow{Rf_{\ast} \beta[i]} & Rf_{\ast} \Omega^n_X[n] \\
\downarrow & & & \downarrow \\
& & & k
\end{array}
\]
After all: Discussion 3.1.2 and Conclusion 3.1.3 combine to tell us that the map $\beta[i] : \mathcal{V}[i] \to \Omega^n_X[n] \cong f \times \mathcal{O}_Y$ goes under the bijection $\varphi$ to the composite

$$RF_*\mathcal{V}[i] \xrightarrow{RF_*\beta[i]} RF_*\Omega^n_X[n] \xrightarrow{\varepsilon} k$$

and all we do is evaluate this composite at the element $\alpha : \mathcal{O}_X \to \mathcal{V}[i]$ of the vector space $H^i(\mathcal{V}) = H^i(\mathcal{Hom}(\mathcal{O}_X, \mathcal{V})) = \text{Hom}(\mathcal{O}_X, \mathcal{V}[i])$.

**Remark 3.3.1.** We can view Grothendieck duality as being Serre duality on steroids. Being macho, Grothendieck duality doesn’t restrict the scheme $Y$ to be the one-point space, doesn’t assume the map $f$ to be smooth and proper, and doesn’t confine itself to only dealing with vector bundles.

### 4. The proofs

Modulo technicalities, the theorems of Section 3 are all contained in Hartshorne [19]. More precisely: the theorems in Hartshorne [19] aren’t quite as clean or general, Section 3 is comprised of several technical improvements on those original assertions. However: with one inessential exception—which we will mention at the very end of Sketch 4.1.1—all the improvements had been obtained by the mid-1990s. In other words: none of the results in Section 3 is younger than two decades.

What is new is that we can now prove *every one of these statements* simply and directly, sidestepping the customary circuitous routes and long detours, and bypassing all of the traditional stopovers on distant planets. We will next discuss where the reader can find these simple, formal proofs. This naturally divides into two parts.

#### 4.1. Simple proofs that have been around for decades

Let us begin with Reminder 3.1.1: we gave an explicit construction of the functors $Lf^*$ and $RF_*$, and asserted the existence of a functor $f^\times$ right adjoint to $RF_*$. The first short and formal proof of the existence of $f^\times$ may be found in Deligne’s appendix [16] to Hartshorne’s book [19]. In that proof the schemes are assumed noetherian and the derived categories are of bounded below complexes. The reader may find more general theorems in Balmer, Dell’Ambrogio and Sanders [9] and in [45, 42], with the strongest theorem to date covering the case where $f$ is any concentrated morphism of quasicompact, quasiseparated algebraic stacks. The modern proofs work by showing that $RF_*$ respects coproducts and applying Brown representability.

Now we turn to Theorem 3.1.4 and for the reader’s convenience we recall the statement

**Theorem 3.1.4 [reminder].** Assume $f : X \to Y$ is a finite-type morphism of noetherian schemes. Then there is a canonical natural isomorphism $p_{A,B'} : A \otimes^L RF_*B' \to RF_*(Lf^*A \otimes^L B')$ and a canonical natural transformation $\chi : LF^*A \otimes^L f^\times \mathcal{O}_Y \to f^\times A$ such that the following pentagon commutes

$$
\begin{array}{ccc}
RF_*\left[LF^*A \otimes^L f^\times \mathcal{O}_Y\right] & \xrightarrow{p_{A,f^\times \mathcal{O}_Y}^{-1}} & A \otimes^L RF_*f^\times \mathcal{O}_Y \\
& \xrightarrow{\text{id} \otimes \varepsilon} & A \otimes^L \mathcal{O}_Y \\
& \xrightarrow{RF_*\chi} & RF_*f^\times A \\
& \varepsilon & \xrightarrow{\varepsilon} & A
\end{array}
$$
Furthermore: the map $\chi$ is an isomorphism if and only if $f$ is proper and of finite Tor-dimension.

The modern proof may be found in [13], and the reader might also wish to look at [9] for a generalization of Theorem 3.1.3 proved by the same techniques. To emphasize the formal nature of the argument we give an outline.

**Sketch 4.1.1.** The functor $L_f^*$ is strong monoidal—it respects the tensor product, there is a natural isomorphism $Lf^*(A \otimes^L B) \longrightarrow (L_f^*A) \otimes^L (L_f^*B)$. If we put $B = Rf_*B'$ this gives the first map in the composite

$$Lf^*(A \otimes^L Rf_*B') \longrightarrow (Lf^*A) \otimes^L (Lf^*Rf_*B') \longrightarrow \text{id} \otimes \varepsilon' \longrightarrow (Lf^*A) \otimes^L B'',$$

where $\varepsilon': Lf^*Rf_* \longrightarrow \text{id}$ is the counit of the adjunction $Lf^* \dashv Rf_*$. Adjunction, applied to the highlighted composite above, gives a corresponding map $p_{A,B'}: A \otimes^L Rf_*B' \longrightarrow Rf_*[(Lf^*A) \otimes^L B']$. The map $p$ is an isomorphism, the so-called classical “projection formula”. But since we are into seeing what part of the theory is formal, let us indicate the modern proof that $p$ is an isomorphism.

If $A$ is a perfect complex then it is “strongly dualizing”[3]—in particular there exists a dual complex $A^\vee$ and a canonical isomorphism $A \otimes^L (-) \cong \mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}(A^\vee, -)$. The following string of isomorphisms

$$\text{Hom}(C, A \otimes^L Rf_*B') \cong \text{Hom}(C, \mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}(A^\vee, Rf_*B'))$$

$$\cong \text{Hom}(C \otimes^L A^\vee, Rf_*B')$$

$$\cong \text{Hom}((Lf^*C) \otimes^L (Lf^*A^\vee), B')$$

$$\cong \text{Hom}((Lf^*C) \otimes^L (Lf^*A), B')$$

$$\cong \text{Hom}([Lf^*C), \mathcal{R}\mathcal{H}\mathcal{O}\mathcal{M}((Lf^*A)^\vee, B')]$$

$$\cong \text{Hom}([C, Rf_*((Lf^*A) \otimes^L B')]$$

holds for every $C$ and is a formal consequence of the definition of strongly dualizable objects—in particular the fifth isomorphism is formal, any strict monoidal functor must take strongly dualizable objects to strongly dualizable objects, and must respect duals. Yoneda gives that the isomorphism of Hom-sets must come from an isomorphism $A \otimes^L Rf_*B' \cong Rf_*((Lf^*A) \otimes^L B')$, and it is an exercise in the definitions to check that this isomorphism is induced by the map $p_{A,B'}$. In other words: the map $p_{A,B'}$ induces an isomorphism as long as $A$ is strongly dualizing—in the category $D_{\text{qc}}(Y)$ this means as long as $A$ is a perfect complex.

Thus the subcategory of all $A$’s for which the map $p_{A,B'}$ induces an isomorphism, for every $B'$, contains the perfect complexes and is closed under suspensions, triangles and coproducts. The closure under triangles and coproducts is because the functors $Lf^*$ and $Rf_*$ both respect triangles and coproducts. But then [15, Lemma 3.2] tells us that every object $A \in D_{\text{qc}}(Y)$ belongs—the map $p_{A,B'}$ is an isomorphism for all pairs $A, B'$.

---

3Recall that an object $A$ in a monoidal category is strongly dualizing if there exists a dual object $A^\vee$ and maps $A^\vee \otimes A \longrightarrow 1$ and $1 \longrightarrow A \otimes A^\vee$ so that the composites $A \longrightarrow A \otimes A^\vee \otimes A \longrightarrow A$ and $A^\vee \longrightarrow A^\vee \otimes A \longrightarrow A^\vee$ are both the identity. For the monoidal category $D_{\text{qc}}(X)$ the strongly dualizing objects are the perfect complexes.
Once we know that the map \( p \) is an isomorphism we can repeat the idea. Put \( B' = f^*B'' \), and then \( \rho_{A,B''}^{-1} \) gives the first map in the composite

\[
\begin{array}{ccc}
\mathbf{R}f_* \left[ (\mathbf{L}f^*A) \otimes^L (f^*B'') \right] & \xrightarrow{\rho_{A,f^*B''}^{-1}} & A \otimes^L \mathbf{R}f_* f^*B'' \\
\xrightarrow{id \otimes \varepsilon} & & \xrightarrow{} A \otimes^L B''
\end{array}
\]

In this composite \( \varepsilon : \mathbf{R}f_* f^*B'' \rightarrow B'' \) is the counit of the adjunction \( \mathbf{R}f_* \dashv f^* \). Adjunction tells us that the composite corresponds to a map \( \chi(A,B'') : (\mathbf{L}f^*A) \otimes^L (f^*B'') \rightarrow f^*(A \otimes^L B'') \). The map \( \chi = \chi_A \) of Theorem 3.1.4 is just the special case where \( B'' = \mathcal{O}_Y \). The commutativity of the pentagon of Theorem 3.1.4 simply spells out what it means for the map \( \chi_A \) to correspond, under the adjunction \( \mathbf{R}f_* \dashv f^* \), to the composite above—in other words the pentagon commutes by definition.

It remains to discuss when the map \( \chi \) is an isomorphism.

One can easily write down a string of isomorphisms, much like those we used above to study \( \rho_{A,B''} \), which combine to show that the map \( \chi_A \) is an isomorphism as long as \( A \) is a perfect complex; see [45, top of page 228]. Consider the category of all \( A \)'s so that the map \( \chi_A \) is an isomorphism—it contains the perfect complexes, is closed under suspensions and triangles, and also closed under coproducts if \( f^* \) respects coproducts. Thus, as long as \( f^* \) respects coproducts, the map \( \chi_A \) is an isomorphism for every \( A \). In fact the condition that \( f^* \) respects coproducts is necessary and sufficient for the map \( \chi \) to be an isomorphism.

Now [45] Lemma 5.1 comes to our aid: given a pair of adjoint triangulated functors \( F \dashv G \), between compactly generated triangulated categories, the right adjoint \( G \) respects coproducts if and only if the left adjoint \( F \) respects compact objects.\(^4\) Specializing to the pair of adjoint functors \( \mathbf{R}f_* \dashv f^* \), this tidbit of formal nonsense says that the functor \( f^* \) respects coproducts if and only if \( \mathbf{R}f_* \) takes perfect complexes to perfect complexes. Thus we have transformed a question about the right adjoint \( f^* \), which is mysterious, into a problem about its left adjoint \( \mathbf{R}f_* \), which is explicit and computable. It is an old theorem of Illusie [26, Exposé III, Corollaire 4.3.1(a)] that if \( f \) is proper and of finite Tor-dimension then \( \mathbf{R}f_* \) respects perfect complexes. The converse [which we don’t use in this article], that is the theorem that if \( \mathbf{R}f_* \) respects perfect complexes then \( f \) must be proper and of finite Tor-dimension, is much more recent. It may be found in [34].

This concludes our discussion of Theorem 3.1.4. Before proceeding to the remaining two theorems we include a brief interlude, recalling a couple of base-change maps.

**Reminder 4.1.2.** Let us begin at the formal level. Suppose we are given four categories \( W, \mathcal{X}, \mathcal{Y} \) and \( \mathcal{Z} \), as well as pairs of adjoint functors

\[
\begin{array}{ccc}
\gamma : \mathcal{Y} & \xrightarrow{\Gamma} & W \\
\mathcal{X} & \xrightarrow{\beta} & \mathcal{Z}
\end{array}
\]

Recall: this is shorthand for the assertion that \( \Gamma : \mathcal{W} \rightarrow \mathcal{Y} \) is right adjoint to \( \gamma : \mathcal{Y} \rightarrow \mathcal{W} \), and \( \beta : \mathcal{Z} \rightarrow \mathcal{X} \) is right adjoint to \( \beta : \mathcal{X} \rightarrow \mathcal{Z} \). Assume further that we are given a pair of functors

\[
\begin{array}{ccc}
\alpha : \mathcal{X} & \xrightarrow{} & W \\
\mathcal{Z} & \xrightarrow{} & \mathcal{Y}
\end{array}
\]

\(^4\)Let \( \mathcal{T} \) be a triangulated category with coproducts. An object \( C \in \mathcal{T} \) is *compact* if \( \operatorname{Hom}(C,-) \) respects coproducts—for this survey what’s relevant is that the compact objects in \( \mathbf{D}_{qc}(\mathcal{X}) \) are the perfect complexes.
Then there is a canonical bijection between natural transformations

\[
\begin{array}{ccc}
W & \xrightarrow{\alpha} & X \\
\downarrow^{\gamma} & & \downarrow^{\delta} \\
\gamma & \xleftarrow{\rho} & \beta \\
\downarrow & & \downarrow \\
Y & \xleftarrow{\gamma} & Z \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
W & \xrightarrow{\alpha} & X \\
\downarrow^{\Gamma} & & \downarrow^{\delta} \\
\Gamma & \xleftarrow{\sigma} & B \\
\downarrow & & \downarrow \\
Y & \xleftarrow{\gamma} & Z \\
\end{array}
\]

The formula is explicit: if \(\eta(\gamma \dashv \Gamma)\) and \(\varepsilon(\gamma \dashv \Gamma)\) are, respectively, the unit and counit of the adjunction \(\gamma \dashv \Gamma\), while \(\eta(\beta \dashv B)\) and \(\varepsilon(\beta \dashv B)\) are, respectively, the unit and counit of the adjunction \(\beta \dashv B\), then the formulas are that the bijection takes \(\rho : \gamma \delta \rightarrow \alpha \beta\) and \(\sigma : \delta B \rightarrow \Gamma\alpha\), respectively, to

\[
\begin{array}{ccc}
\delta B & \xrightarrow{\eta(\gamma \dashv \Gamma)} & \Gamma \gamma \delta B \\
\downarrow^{\gamma \delta} & & \downarrow \\
\gamma \delta B \beta & \xrightarrow{\sigma} & \gamma \Gamma \alpha \beta \\
\downarrow & & \downarrow \\
\delta B \eta(\beta \dashv B) & \xrightarrow{\varepsilon(\beta \dashv B)} & \Gamma \alpha \beta \\
\end{array}
\]

The standard terminology in category theory is that \(\rho\) and \(\sigma\) are mates of each other. More explicitly: \(\rho\) is the left mate of \(\sigma\), and \(\sigma\) is the right mate of \(\rho\).

As the reader may have guessed, this terminology was invented by Australian category theorists.

**Construction 4.1.3.** Any commutative square of schemes

\[
\begin{array}{ccc}
W & \xrightarrow{u} & X \\
\downarrow^{f} & & \downarrow^{g} \\
Y & \xleftarrow{v} & Z \\
\end{array}
\]

gives rise to squares

\[
\begin{array}{ccc}
\mathsf{D}_{\text{qc}}(W) & \xrightarrow{\mathsf{L}u^*} & \mathsf{D}_{\text{qc}}(X) \\
\downarrow^{\mathsf{L}f^*} & & \downarrow^{\mathsf{L}g^*} \\
\mathsf{D}_{\text{qc}}(Y) & \xleftarrow{\mathsf{L}v^*} & \mathsf{D}_{\text{qc}}(Z) \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathsf{D}_{\text{qc}}(W) & \xleftarrow{\mathsf{L}u^*} & \mathsf{D}_{\text{qc}}(X) \\
\downarrow^{\mathsf{R}f_*} & & \downarrow^{\mathsf{R}g_*} \\
\mathsf{D}_{\text{qc}}(Y) & \xrightarrow{\mathsf{L}v^*} & \mathsf{D}_{\text{qc}}(Z) \\
\end{array}
\]

where \(\tau\) is the canonical isomorphism \(\tau : \mathsf{L}f^* \mathsf{L}v^* \rightarrow \mathsf{L}u^* \mathsf{L}g^*\). Reminder [4.1.2] yields a right mate for \(\tau\), producing a natural transformation we will call \(\beta : \mathsf{L}v^* \mathsf{R}g_* \rightarrow \mathsf{R}f_* \mathsf{L}u^*\). The classical flat base-change theorem tells us

**4.1.3.1.** The map \(\beta\) is an isomorphism if the square is cartesian—meaning a pullback square in the category of schemes—and if \(v\) is flat.

**Construction 4.1.4.** As in Construction [4.1.3] assume given a commutative square of schemes

\[
\begin{array}{ccc}
W & \xrightarrow{u} & X \\
\downarrow^{f} & & \downarrow^{g} \\
Y & \xleftarrow{v} & Z \\
\end{array}
\]
If the base-change map \( Rf_!Lu^* \leftarrow L\nu^*Rg_* \) is an isomorphism, for example if we are in the situation of 4.1.3.1, we may apply Reminder 4.1.2 to the squares

\[
\begin{array}{ccc}
D_{qc}(W) & \leftarrow & L\nu^* D_{qc}(X) \\
Rf_* & \uparrow & Rg_* \\
D_{qc}(Y) & \leftarrow & L\nu^* D_{qc}(Z)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
D_{qc}(W) & \leftarrow & L\nu^* D_{qc}(X) \\
f^\times & \uparrow & g^\times \\
D_{qc}(Y) & \leftarrow & L\nu^* D_{qc}(Z)
\end{array}
\]

and produce a right mate for \( \beta^{-1} \). We will write this base-change map as \( \Phi = \Phi(\Diamond) : Lu^*g^\times \rightarrow f^\times L\nu^* \). And the relevant theorem tells us

**4.1.4.1.** The map \( \Phi(\Diamond) \) is an isomorphism if, in addition to the hypotheses of 4.1.3.1 already made to define \( \Phi(\Diamond) \), we assume that \( g \) is proper and one of the following holds:

(i) \( f \) is of finite Tor-dimension.

(ii) We restrict the functors to \( D^{+}_{qc}(Z) \subset D_{qc}(Z) \); that is we only evaluate the map on bounded-below complexes.

**Remark 4.1.5.** The assertion 4.1.3.1 goes all the way back to Grothendieck in the 1950s; it’s an easy consequence of the fact that one can compute \( Rf_* \) using Čech complexes.

There is a (complicated) proof of 4.1.4.1(ii) in Hartshorne [19], towards the end of the book. The first short and formal proof of 4.1.4.1(ii) is due to Verdier [63]. Over the years there have been technical improvements, and the strong version stated in 4.1.4.1(i) is recent. The reader is referred to [42, Lemma 5.20] for the proof.

For the applications in this article we do not need such refined forms of 4.1.4.1. Verdier’s old theorem suffices.

It’s time to move on to the remaining business: the proofs of Theorems 3.2.1 and 3.2.5. Both theorems are assertions involving a certain natural map \( \theta \); the first step is to construct this \( \theta \). We’re about to do this to show that it can be done formally, using nothing more than the base-change map of Construction 4.1.4, the counit of some adjunction, and the Hochschild-Kostant-Rosenberg Theorem. The reader willing to skip the construction should proceed directly to Remark 4.1.9.

**Construction 4.1.6.** Suppose \( f : X \rightarrow Y \) is flat and proper. Consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\delta} & X \times_Y X & \xrightarrow{\pi_1} & X \\
\downarrow{\pi_2} & & \phi & & \downarrow{f} \\
X & & & f & \rightarrow & Y
\end{array}
\]

where \( \delta : X \rightarrow X \times_Y X \) is the diagonal map. Since \( f \) is both flat and proper the hypotheses of 4.1.4.1 are satisfied, and the map \( \Phi(\Diamond) \) is an isomorphism. We have the following composites,
with the first two defining the maps $\alpha$ and $\gamma$ that go into producing the third

\[ f^\times \xrightarrow{\sim} L^\delta R^3 f^\times \xrightarrow{L^\delta \Phi(\Diamond)} L^\delta R^2 L f^* \]

\[ R^\delta \xrightarrow{\sim} R^\delta \alpha \pi_2^X \xrightarrow{\beta \pi_2^X} \pi_2^X \]

\[ L^\delta R^\delta L f^* \xrightarrow{L^\delta \gamma L f^*} L^\delta R^2 L f^* \xrightarrow{\alpha^{-1}} f^\times \]

The natural isomorphisms in the first two composites are because $\pi_1 \delta = \pi_2 \delta = \text{id}$, hence $L^\delta R^1 \cong L(\text{id})^* = \text{id}^X \cong \delta^X \pi_2^X$. Since the map $\Phi(\Diamond)$ is an isomorphism the first row composes to an isomorphism, allowing us to form the third row. Evaluating the third row above at the object $O_Y \in D_{\text{qc}}(Y)$ we obtain the second and third maps in the composite below defining $\zeta$

\[ L^\delta R^\delta O_X \xrightarrow{L^\delta R^\delta L f^* O_Y} L^\delta R^2 L f^* O_Y \xrightarrow{\alpha^{-1}} f^\times O_Y \]

The object $L^\delta R^\delta O_X$ is something we know and love—it is the derived category version of the tensor product of $O_X$ with itself over $O_{X \times Y \times X}$. Formal nonsense tells us that $L^\delta R^\delta O_X$ is a commutative monoid in the monoidal category $D_{\text{qc}}(X)$, hence its sheaf cohomology $\mathcal{H}^*(L^\delta R^\delta O_X)$ is a graded commutative ring. There is an obvious ring homomorphism

\[ \wedge^* \mathcal{H}^{-1}[L^\delta R^\delta O_X] \xrightarrow{\wedge^* \mathcal{H}^1} \mathcal{H}^*(L^\delta R^\delta O_X) \]

from the exterior algebra on $\mathcal{H}^{-1}[L^\delta R^\delta O_X] = \mathcal{H}_f^1$ to the ring $\mathcal{H}^*(L^\delta R^\delta O_X)$. So far we have only assumed $f$ flat and proper.

Now assume $f$ is smooth and proper, of relative dimension $n$. The Hochschild-Kostant-Rosenberg Theorem [21] (see also Lipman [30, Proposition 4.6.3]) tells us that the homomorphism of graded rings of the paragraph above is an isomorphism. In particular we deduce

(i) The cohomology sheaves $\mathcal{H}^i(L^\delta R^\delta O_X)$ vanish for $i < -n$.

(ii) We have constructed a natural map

\[ \Omega_f^n \xrightarrow{\wedge^n \mathcal{H}^{-1}[L^\delta R^\delta O_X]} \mathcal{H}^{-n}[L^\delta R^\delta O_X] \]

Formal nonsense, about $t$-structures in triangulated categories, tells us that the map of cohomology sheaves $\Omega_f^n \rightarrow \mathcal{H}^{-n}[L^\delta R^\delta O_X]$ can be realized as $\mathcal{H}^{-n}$ of a unique morphism $\psi : \Omega_f^n[n] \rightarrow L^\delta R^\delta O_X$ in the derived category $D_{\text{qc}}(X)$. And the map $\theta$ of Theorem 3.2.1 is defined to be the composite

\[ \Omega_f^n[n] \xrightarrow{\psi} L^\delta R^\delta O_X \xrightarrow{\zeta} f^\times O_Y \]

The precise version of Theorem 3.2.1 now says:

**Theorem 4.1.7.** If $f$ is smooth and proper then the map $\theta$ of Construction 4.1.6 is an isomorphism.
Remark 4.1.8. In passing we mentioned that the Verdier version of \[4.1.4.1\] is sufficient for this paper—the reason is that in the proof we will only evaluate the base-change maps \(\Phi(\Diamond)\) on the objects like \(\mathcal{O}_Y\) or \(\pi^*\mathcal{O}_X\), which are bounded below. These are the only objects that come up in the definition of the map \(\theta\) of Construction \[4.1.6\].

Lipman lectured about the approach to the map \(\theta\), presented in Construction \[4.1.6\] already in the 1980s. But it only appeared in print relatively recently, it may be found in Alonso, Jeremías and Lipman \[3\] Example 2.4 and Proposition 2.4.2]. The map \(\theta\) has older avatars, for example in Verdier \[63]\—although the fact that Lipman’s and Verdier’s maps coincide was proved only recently, see \[35]\.

Remark 4.1.9. So far we have, up to one application of the Hochschild-Kostant-Rosenberg theorem, set up all the players entirely in formal nonsense fashion. And the historical asides along the way tell the reader that this much was known by the mid-1990s. Until now we haven’t met anything younger than two decades.

It remains to discuss the proofs of Theorems \[3.2.1\] and \[3.2.5\] and this is where there has been major progress in the last few years. We open a new section for this.

4.2. The simple proofs discovered recently. Now that we have defined the maps occurring in Theorems \[3.2.1\] and \[3.2.5\] it remains to prove the claims of the theorems—we need to show that the maps do as the theorems assert they do. This should be a local problem, but for the longest time no one understood how to do this local computation simply and elegantly. Recall the first paragraph of the Introduction: there are two paths to the foundations of the subject, and in this article we’ve been following the one pioneered by Deligne and Verdier. The objection to this approach has long been that it leads to a theory where you can’t compute anything. We’ve now reached the stage where a computation is in order—it should hardly come as a surprise that, until very recently, this was the point where the trail we have been on seemed to peter out, with the general direction appearing blocked and impenetrable.

We should probably explain, and for the purpose of clarity let us narrow our attention to just Theorem \[3.2.1\]. Theorem \[3.2.5\] is similar but a touch more technical.

Remark 4.2.1. Since we will say almost nothing about the proofs of Theorem \[3.2.5\] old or new, we should in passing acknowledge its long history and give references. The theorem was first sketched in Hartshorne \[19\] pp. 398–400]. The case of varieties over a perfect field was completely worked out in Lipman \[29\]. Lipman’s main results were generalized in Hübl and Sastry \[25\]; see their Residue Theorem (iii) on p. 752 and its generalization (iii) on p. 785.

The results cited in the paragraph above all came with complicated proofs. The existence of a simple proof is a recent surprise, none of the experts—the dozen of us—expected such a thing, it is part of the exciting developments of the last few years. The reader can find the simple proof in \[47\] Section 2], and we will say a tiny bit more about this proof in \[5.1\].

Strategy 4.2.2. Back to Theorem \[3.2.1\] in Construction \[4.1.6\] we defined a map \(\theta : \Omega^n_f[n] \to f^*\mathcal{O}_Y\), and Theorem \[3.2.1\] asserts that \(\theta\) is an isomorphism. Surely we should be able to prove this locally—a morphism in \(\mathsf{D}_{qc}(X)\) is an isomorphism if and only if it restricts to an isomorphism on an open affine cover. Let us follow our noses and proceed the way such arguments usually work, to see where the obstacle lies, and to pinpoint the insight which uncovered an unconvoluted path circumventing this hurdle.
Lemma 4.2.3. Suppose we are given a commutative square of schemes
\[
\begin{array}{ccc}
U & \overset{u}{\longrightarrow} & X \\
g & \downarrow & \leftarrow & f \\
V & \overset{v}{\longrightarrow} & Y
\end{array}
\]
where the maps \( u \) and \( v \) are open immersions. Then there is a canonical natural isomorphism \( Lu^*\Omega^n_f[n] \cong \Omega^n_g[n] \).

Proof. Obvious. \(\square\)

Reminder 4.2.4. We are given a morphism \( f : X \to Y \), which we assume smooth and proper. We wish to show that the map
\[
\Omega^n_f[n] \overset{\theta}{\longrightarrow} f^*\mathcal{O}_Y
\]
of Construction 4.1.6 is an isomorphism, and the plan is to do this by studying it locally.

Reduction 4.2.5. Let us begin by showing that the problem is local in \( Y \). Suppose that \( v : V \to Y \) is an open immersion with \( V \) affine. Next we

(i) Form the pullback square of schemes
\[
\begin{array}{ccc}
U & \overset{u}{\longrightarrow} & X \\
g & \downarrow & \leftarrow & f \\
V & \overset{v}{\longrightarrow} & Y
\end{array}
\]
It clearly suffices to show that, for every \( v : V \to Y \) as above, the map \( Lu^*\theta \) is an isomorphism. Our reduction is about simplifying \( Lu^*\theta \).

Let us pass from the square (\( \heartsuit \)) to the square of derived categories
\[
\begin{array}{ccc}
\mathbf{D}_{\text{qc}}(U) & \overset{Lu^*}{\longrightarrow} & \mathbf{D}_{\text{qc}}(X) \\
g^* & \uparrow & \leftarrow & f^* \\
\mathbf{D}_{\text{qc}}(V) & \overset{Lv^*}{\longrightarrow} & \mathbf{D}_{\text{qc}}(Y)
\end{array}
\]
The base-change map \( \Phi(\heartsuit) : Lu^*f^* \to g^*Lv^* \) is an isomorphism by 4.1.11. Applying to the object \( \mathcal{O}_Y \in \mathbf{D}_{\text{qc}}(Y) \), and recalling that \( Lv^*\mathcal{O}_Y = \mathcal{O}_V \), we deduce

(ii) We have produced an isomorphism
\[
Lu^*f^*\mathcal{O}_Y \overset{\Phi(\heartsuit)}{\longrightarrow} g^*\mathcal{O}_V
\]
Next we apply Lemma 4.2.3 to the square (\( \heartsuit \)) in (i), obtaining

(iii) We have an isomorphism
\[
Lu^*\Omega^n_f[n] \overset{\Phi(\heartsuit)}{\longrightarrow} \Omega^n_g[n]
\]
We are trying to show that the map \( \Omega^n_f \rightarrow f^* \mathcal{O}_Y \) is an isomorphism, and we have already agreed in (i) that it suffices to check that \( L^* \theta \) is an isomorphism. By (ii) and (iii) above \( L^* \theta \) rewrites as some map \( \Omega^n_g \rightarrow g^* \mathcal{O}_V \) and it is an exercise in the definitions to check that this map is nothing other than the \( \theta \) corresponding to \( g : U \rightarrow V \). In other words we are reduced to proving Theorem 3.2.1 in the case where \( Y \) is affine.

**Reduction 4.2.6.** Reduction 4.2.5 tells us that it suffices to prove Theorem 3.2.1 for smooth, proper morphisms \( f : X \rightarrow Y \) with \( Y \) affine. If we follow the usual yoga, the next step should be to reduce to the case where \( X \) is also affine. We want to prove that the map \( \theta \) of Construction 4.1.6 is an isomorphism, and it certainly suffices to show that \( L^* \theta \) is an isomorphism for every open immersion \( u : U \rightarrow X \) with \( U \) affine. Choose therefore an open immersion \( u : U \rightarrow X \), and we would like to express \( L^* \theta \) in some form that renders it easily computable. Now \( \theta \) is a map \( \Omega^n_f \rightarrow f^* \mathcal{O}_Y \) and we are embarking on a study of \( L^* \theta \), which is a morphism

\[
L^* \Omega^n_f \rightarrow L^* f^* \mathcal{O}_Y
\]

Let us begin by simplifying \( L^* \Omega^n_f \). To this end we study the commutative square of schemes

\[
\begin{array}{ccc}
U & \xrightarrow{u} & X \\
\downarrow{f_*} & & \downarrow{f} \\
Y & \xrightarrow{id} & Y
\end{array}
\]

Applying Lemma 4.2.3 produces an isomorphism

\[
L^* \Omega^n_f \rightarrow \Omega^n_{f_*}[n]
\]

In the usual jargon, our reduction so far tells us that the “object \( \Omega^n_f \) is local in \( X \).”

**Caution 4.2.7.** The simple-minded way to proceed doesn’t work, unfortunately the object \( f^* \mathcal{O}_Y \) isn’t local in \( X \). If we look at the composite \( U \xrightarrow{u} X \xrightarrow{f} Y \), then \( L^* f^* \mathcal{O}_Y \) is not in general isomorphic to \( (fu)^* \mathcal{O}_Y \). See Sketch 5.1.1 for a counterexample.

**Reduction 4.2.8.** Caution 4.2.7 tells us what doesn’t work when we try to simplify \( L^* f^* \), and it’s now time to present the way around the difficulty. To this end consider the following
from which we obtain a diagram of derived categories

\[
\begin{array}{c}
\text{D}_{\text{qc}}(U) & \xrightarrow{\mathcal{L}u^*} & \text{D}_{\text{qc}}(X) \\
\text{D}_{\text{qc}}(U \times_Y U) & \xrightarrow{\mathcal{L}(u \times \text{id})^*} & \text{D}_{\text{qc}}(U \times_Y U) \\
\text{D}_{\text{qc}}(U \times_Y U) & \xrightarrow{\mathcal{L}(u \times \text{id})^*} & \text{D}_{\text{qc}}(X \times_Y U) \\
\mathcal{L}(u \times \text{id})^* & \xrightarrow{\pi_2^*} & \text{D}_{\text{qc}}(X) \\
\text{D}_{\text{qc}}(U) & \xrightarrow{\mathcal{L}f^*} & \text{D}_{\text{qc}}(Y) \\
\end{array}
\]

We would like to simplify the expression \(\mathcal{L}u^* f^\times\), and in the diagram above we have a subdiagram which obviously commutes up to canonical natural isomorphism

The squares (\(\diamondsuit\)) and (\(\heartsuit\)) in the commutative diagram of schemes satisfy the hypotheses of \(\text{[4.1.4.1]}\) and, up to the natural isomorphisms induced by \(\Phi(\diamondsuit)\) and \(\Phi(\heartsuit)\) composed with the canonical
natural isomorphism of the diagram above, the following subdiagram must also commute

\[
\begin{array}{cccc}
D_{qc}(U) & \xrightarrow{L\delta^*} & D_{qc}(X) & \\
\downarrow{L\delta^*} & & & \downarrow{L(id)^* = id} \\
D_{qc}(U \times_Y U) & \xrightarrow{L(u \times id)^*} & D_{qc}(X \times_Y U) & \\
\downarrow{id = id^*} & & & \downarrow{f^*} \\
D_{qc}(U \times_Y U) & \xrightarrow{\pi_2^*} & D_{qc}(X \times_Y U) & \xleftarrow{L(u \times id)} \\
\end{array}
\]

Up until now everything is entirely classical.

Of course there is nothing to stop us from looking at the base-change map of the square (♣) in the large diagram of schemes at the beginning of this Reduction. The square is cartesian, the horizontal map \((u \times id)\) is flat, and Construction 4.1.4 provides us with a morphism \(\Phi(♣) : L(id)^*(u \times id)^* \to id^*L(u \times id)^*\), or more simply \(\Phi(♣) : (u \times id)^* \to L(u \times id)^*\). It might seem idiotic\(^5\) to study \(\Phi(♣)\), after all the hypotheses of 4.1.4.1 don’t hold, the vertical map on the right decidedly isn’t proper. And just in case the reader was wondering: it’s not just that the hypotheses of 4.1.4.1 aren’t satisfied—neither is the conclusion, the map \(\Phi(♣)\) is known not to be an isomorphism in general.

The recent insight says

4.2.8.1. Consider the following extract from our large diagram of derived categories

\[
\begin{array}{cccc}
D_{qc}(U) & \xrightarrow{L\delta^*} & D_{qc}(X) & \\
\downarrow{L\delta^*} & & & \downarrow{Lf^*} \\
D_{qc}(U \times_Y U) & \xrightarrow{L(id)^*} & D_{qc}(U \times_Y U) & \xleftarrow{(u \times id)^*} \\
\downarrow{id^*} & & & \downarrow{\pi_2^*} \\
D_{qc}(U \times_Y U) & \xrightarrow{L(u \times id)^*} & D_{qc}(X \times_Y U) & \\
\end{array}
\]

Then the composites from bottom right to top left agree, more precisely the map \(L\delta^*\Phi(♣)\) gives an isomorphism \(L\delta^*(u \times id)^* \to L\delta^*L(u \times id)^*\).

\(^5\)In September 2016 the author presented the results (in a more technical incarnation) at a seminar at the University of Utah. The audience consisted of algebraic geometers and commutative algebraists. And this expert audience burst out laughing when told that the study of the base-change map \(\Phi(♣)\) is what underpins the recent progress—to the experts this was hilarious.
With the aid of the isomorphism of 4.2.8.1 we deduce that, up to all the isomorphisms above, the diagram

also commutes. This simplifies to

and the punchline is that we have found an isomorphism of the functor $L_{u^*}f^*$ with the composite $L\delta^*\pi_2^*L(fu)^*$, and in that composite all the schemes are affine.

**Remark 4.2.9.** Reductions 4.2.6 and 4.2.8 transform the problem into one in which all the schemes are affine. It remains to show that

(i) The affine problem is tractable, in other words we can do the computation needed in the affine case. The reader can find in [27, Theorem 4.2.4] a working out of what needs to be computed, and in [17, Section 1] the computation is actually carried out. Below we present a rough outline, in Reminder 4.2.10 and Computation 4.2.11.

(ii) We should also say something about the proof of 4.2.8.1 after all this is the crux of what’s new to the approach of the current document. The expert is referred to [27, (2.3.5.1)] for a general result, which implies 4.2.8.1 as a special case. In the interest of making the subject accessible to the non-expert we give a fairly complete, self-contained treatment in Sketch 4.2.12 below.

Let us first recall

**Reminder 4.2.10.** If $R$ is a ring let $D(R)$ denote the (unbounded) derived category of cochain complexes of $R$–modules. From [12, Theorem 5.1] we learn that, for any ring $R$, the canonical functor $D(R) \rightarrow D_{qc}[\text{Spec}(R)]$ is an equivalence. Given a ring homomorphism $f : R \rightarrow S$ we have an induced map of schemes, and by abuse of notation we will write it $f : \text{Spec}(S) \rightarrow$
Spec\( (R) \). The diagram

\[
\begin{array}{ccc}
D_{qc}[\text{Spec}(S)] & \xrightarrow{Rf_*} & D_{qc}[\text{Spec}(R)] \\
Lf^* & & \downarrow \\
D(S) & \xrightarrow{f^*} & D(R)
\end{array}
\]

of Reminder 3.1.1 identifies with

\[
\begin{array}{ccc}
D(S) & \xrightarrow{Lf^*} & D(R) \\
\downarrow & & \downarrow \\
f^* & = & f^*
\end{array}
\]

where \( f^* : D(S) \rightarrow D(R) \) is the forgetful functor—it takes an object of \( D(S) \), that is a cochain complex of \( S \)–modules, to itself viewed as a complex of \( R \)–modules. As \( f^* \) is exact there is no need to derive it, we have \( Rf_* = f_* \). With the notation as in Example 2.2.5 the functor \( Lf^* \), being the left adjoint of \( f^* \), is given by the formula \( Lf^*(-) = S \otimes \mathcal{O}_R(-) \), while \( f^* \), being the right adjoint of \( f_* \), is the functor \( f^*(-) = R\text{Hom}_R(S,-) \). And the units and counits of the adjunctions are all explicit.

**Computation 4.2.11.** In view of Reminder 4.2.10 achieving Remark 4.2.9(i) has to be straightforward—it’s just a matter of untangling the definitions and then doing a computation. We are given affine schemes \( U \) and \( Y \), hence we may write \( Y = \text{Spec}(R) \) and \( U = \text{Spec}(S) \). The morphism of schemes \( fu : U \rightarrow Y \) corresponds to a ring homomorphism \( \sigma : R \rightarrow S \), and we have an isomorphism \( U \times_Y U = \text{Spec}(S^\sigma) \) with \( S^\sigma = S \otimes_R S \). Reductions 4.2.6 and 4.2.8 produce isomorphisms in the category \( D_{qc}(U) \)

\[
\begin{align*}
L_u^*\Omega^n_{fu}[n] & \cong \Omega^n_{fu}[n] \\
L_u^*f^*\mathcal{O}_Y & \cong \mathcal{L}\delta^*\pi_2^*\mathcal{L}(fu)^*\mathcal{O}_Y
\end{align*}
\]

Using the descriptions on the right-hand-side, one checks that the equivalence \( D_{qc}(U) \cong D(S) \) takes these objects, respectively, to \( \text{Tor}_n^*(S,S)[n] \) and \( S \otimes_R S^{\delta_0} \text{RHom}_R(S,S) \). And the map \( L_u^*\theta \) is nothing other than the composite

\[
\begin{array}{ccc}
\text{Tor}_n^*(S,S)[n] & \xrightarrow{id \otimes_R I} & S \otimes_R S^{\delta_0} \\
\xrightarrow{id \otimes_R I} & S \otimes_R S^{\delta_0} \text{RHom}_R(S,S)
\end{array}
\]

where \( I : S \rightarrow \text{RHom}_R(S,S) \) is the obvious inclusion. It remains to show that, for \( S \) smooth over \( R \) of relative dimension \( n \), the composite above is an isomorphism. The reader can find the computation in [47, Section 1].

**Sketch 4.2.12.** It remains to deliver on the promise of Remark 4.2.9(ii), we should say something about the proof of 4.2.8.1. The argument below is reasonably detailed—it may safely be skipped, the reader should feel free to proceed directly to Section 5.
We remind the reader: we consider the diagram

\[
\begin{array}{c}
\text{D}_{\text{qc}}(U) \\
\downarrow \text{id}^* \\
\text{D}_{\text{qc}}(U \times_Y U) \\
\downarrow L(u \times \text{id})^* \\
\text{D}_{\text{qc}}(X \times_Y U) \\
\end{array}
\]

and the assertion is that the functor \(L\delta^*\) takes the base-change map \(\Phi(\bullet): (u \times \text{id})^* \longrightarrow L(u \times \text{id})^*\) to an isomorphism. To simplify the notation we will write \(v\) for the map \((u \times \text{id}): U \times_Y U \longrightarrow X \times_Y U\).

Now the diagonal map \(\delta: U \longrightarrow U \times_Y U\) is a closed immersion, hence the map \(R\delta_* = \delta_*\) is conservative—it suffices to prove that \(R\delta_*L\delta^*\) takes \(\Phi(\bullet)\) to an isomorphism. But we have isomorphisms

\[\text{R}\delta_*L\delta^*(-) \cong \text{R}\delta_*[L\delta^*(-) \otimes^L \mathcal{O}_U] \cong (-) \otimes^L \text{R}\delta_*\mathcal{O}_U\]

where the second is the isomorphism \(p^{-1}\) of the projection formula, see Sketch 4.1.1. Consider therefore the two full subcategories of \(\text{D}_{\text{qc}}(U \times_Y U)\) given by

\[\mathcal{D} = \left\{ s \in \text{D}_{\text{qc}}(U \times_Y U) \middle| \begin{array}{l}
L\iota^* s = 0 \text{ for all } i: \text{Spec}(K) \longrightarrow U \times_Y U - \Delta \\
\text{where } K \text{ is a field and where } \Delta \subset U \times_Y U \text{ is the diagonal} \\
\text{the functor } (-) \otimes^L s \text{ takes } \Phi(\bullet) \text{ to an isomorphism}
\end{array} \right\}
\]

The object \(R\delta_*\mathcal{O}_U\) clearly belongs to \(\text{D}_{\text{qc}}(U \times_Y U)\), and we wish to show that it belongs to \(\mathcal{D}\). It certainly suffices to prove that \(\text{D}_{\text{qc}}(U \times_Y U)\) is contained in \(\mathcal{D}\).

But both \(\mathcal{D}\) and \(\text{D}_{\text{qc}}(U \times_Y U)\) are localizing tensor ideals, and [14] Corollary 3.4 tells us that, as a localizing tensor ideal, \(\text{D}_{\text{qc}}(U \times_Y U)\) is generated by the perfect complexes inside \([\mathcal{D}]\). It suffices to show that the perfect complexes in \(\text{D}_{\text{qc}}(U \times_Y U)\) all belong to \(\mathcal{D}\). In other words: it suffices to prove that, for every perfect complex \(P\) supported on the diagonal, the functor \((-) \otimes^L P\) takes \(\Phi(\bullet)\) to an isomorphism.

The morphism \(v: U \times_Y U \longrightarrow X \times_Y U\) is an open immersion, hence the counit of adjunction \(\varepsilon: L v^* Rv_* \longrightarrow \text{id}\) is an isomorphism. Consequently \(Rv_*\) is fully faithful, and we have an isomorphism \(L v^* Rv_* P \cong P\). It therefore suffices to show that, for every perfect complex \(P\) in \(\text{D}_{\text{qc}}(U \times_Y U)\) supported on the diagonal, the functor

\[Rv_*[(-) \otimes^L P] \cong Rv_*[(-) \otimes^L L v^* Rv_* P] \cong Rv_*(-) \otimes^L Rv_* P\]

takes \(\Phi(\bullet)\) to an isomorphism, or to put it differently the functor \((-) \otimes^L Rv_* P\) takes \(Rv_*\Phi(\bullet)\) to an isomorphism. Now let \(P \in \text{D}_{\text{qc}}(U \times_Y U)\) be a perfect complex supported on the diagonal

\[\text{Note that } U \times_Y U \text{ is affine, so for us the old version in [14] suffices. We should mention that [14] builds on earlier papers by Hopkins [22] and Thomason and Trobaugh [60]. The reader can find later improvements in: the union of Thomason [59, Lemma 3.4] and Alonso, Jeremías and Souto [3 Corollary 4.11 and Theorem 4.12] generalize the result to all noetherian schemes, while Balmer and Favi [10] give a formal generalization to the world of tensor triangulated categories.}\]
and let $\Gamma \subset X \times Y$ be the graph of the map $u : U \to X$. The following is a cartesian square of open immersions

$$
\begin{array}{ccc}
U \times_Y U - \Delta & \xrightarrow{\alpha} & U \times_Y U \\
\beta \downarrow & & \downarrow \nu \\
X \times_Y U - \Gamma & \xrightarrow{\gamma} & X \times_Y U
\end{array}
$$

By Lemma 4.1.3.1 we have the first isomorphism below

$$L\gamma^*R\nu_*P \cong R\beta_*L\alpha^*P \cong 0,$$

where the second isomorphism is because $P$ is supported on $\Delta$ and hence $L\alpha^*P = 0$. Thus both $L\gamma^*R\nu_*P = 0$ and $L\nu^*R\nu_*P \cong P$ are perfect complexes, and as $U \times_Y U$ and $X \times_Y U - \Gamma$ form an open cover for $X \times_Y U$ we deduce

(i) If $P \in D_{qc}(U \times_Y U)$ is a perfect complex supported on the diagonal then $R\nu_*P$ is a perfect complex on $X \times_Y U$, supported on the graph $\Gamma \subset X \times_Y U$ of the map $u : U \to X$.

The reader can amuse herself by proving

(ii) The map $\Phi(\alpha) : v^* \to L\nu^*$ is taken by $R\nu_*$ to the composite

$$R\nu_*v^* \xrightarrow{\varepsilon} \mathrm{id} \xrightarrow{\eta} R\nu_*L\nu^*$$

where $\varepsilon : R\nu_*v^* \to \mathrm{id}$ is the counit of the adjunction $R\nu_* \dashv v^*$, while $\eta : \mathrm{id} \to R\nu_*L\nu^*$ is the unit of the adjunction $Lv^* \dashv R\nu_*$.

Let $Q \in D_{qc}(X \times_Y U)$ be a perfect complex supported on $\Gamma$. Then its dual $Q^!$ is also a perfect complex supported on $\Gamma$, and we have $(-) \otimes LQ \cong \mathbb{R}\text{Hom}(Q^!, -)$. From (i) and (ii) above it suffices to prove

(iii) For all perfect complexes $Q \in D_{qc}(X \times_Y U)$ supported on $\Gamma$, the functor $(-) \otimes LQ$ takes the map $\eta$ of (ii) to an isomorphism.

(iv) For all perfect complexes $Q \in D_{qc}(X \times_Y U)$ supported on $\Gamma$, the functor $\mathbb{R}\text{Hom}(Q, -)$ takes the map $\varepsilon$ of (ii) to an isomorphism.

To establish (iv) it suffices, by Lemma 3.2, to show that for all pairs of perfect complexes $C, Q \in D_{qc}(X \times_Y U)$, with $Q$ supported on $\Gamma$, the functor $\text{Hom}(C, -)$ takes $\mathbb{R}\text{Hom}(Q, \varepsilon)$ to an isomorphism. Now observe the isomorphism of functors

$$\text{Hom}(C, \mathbb{R}\text{Hom}(Q, -)) \cong \text{Hom}(C \otimes LQ, -)$$

As $C$ and $Q$ are both perfect and $Q$ is supported on $\Gamma$, the complex $C \otimes LQ$ is perfect and is supported on $\Gamma$. Hence (iv) would follow from

(v) For all perfect complexes $Q \in D_{qc}(X \times_Y U)$ supported on $\Gamma$, the functor $\text{Hom}(Q, -)$ takes the map $\varepsilon$ of (ii) to an isomorphism.

Let $Q \in D_{qc}(X \times_Y U)$ be a perfect complex supported on $\Gamma$. Then $L\nu^*Q$ is a perfect complex supported on $\Delta$. The map $\eta : Q \to R\nu_*L\nu^*Q$ is an isomorphism on the open set $U \times_Y U \subset X \times_Y U$, and is an isomorphism on $(X \times_Y U) - \Gamma$ because both $Q$ and $R\nu_*L\nu^*Q$ vanish outside $\Gamma$. Since the open sets $U \times_Y U$ and $(X \times_Y U) - \Gamma$ cover $X \times_Y U$ it follows that $\eta : Q \to R\nu_*L\nu^*Q$ is an isomorphism. Putting $A = L\nu^*Q$ we deduce that (iii) and (v) would follow from

(vi) For all $A \in D_{qc}(U \times_Y U)$, the functor $(-) \otimes L \to U_*A$ takes $\eta$ to an isomorphism.

(vii) For all $A \in D_{qc}(U \times_Y U)$, the functor $\text{Hom}(R\nu_*A, -)$ takes $\varepsilon$ to an isomorphism.
To see (vi) observe the isomorphism of the projection formula

\[ (\cdot) \otimes^L Rv_* A \cong Rv_* [Lv^*(\cdot) \otimes^L A] \]

Since the functor \( Lv^* \) takes \( \eta \) to an isomorphism so does the right-hand-side above, and hence also the left-hand-side.

To see (vii) observe the commutative square

\[
\begin{array}{ccc}
\text{Hom}(Rv_*Lv^*(-), ?) & \sim & \text{Hom}(-, Rv_*v^x(?)) \\
\downarrow \text{Hom}(\eta, ?) & & \downarrow \text{Hom}(\eta, ?) \\
\text{Hom}(-, ?) & \sim & \text{Hom}(-, ?)
\end{array}
\]

We wish to show that the vertical map on the right is an isomorphism when evaluated at \((-) = Rv_* A\), and the vertical map on the left makes it clear, after all \( \eta Rv_* : Rv_* \to Rv_*Lv^*Rv_* \) is an isomorphism.

5. Why did it take so long?

In some sense the ingredients of the proof were available already in the 1960s, but back then no one thought of applying the tools of homotopy theory—for example Brown representability—to problems in algebraic geometry. The methods employed in the classical proofs are fundamentally unsuited for the approach presented here. To mention just one facet: in the argument given here we relied heavily on the full power of the derived tensor product. The pre-1990 literature on Grothendieck duality all worked in the bounded-below derived category, where the derived tensor product exists only under strong restrictions and is next to useless.

That said, with the exception of 4.2.8.1 the ingredients of the argument were all available by the mid-1990s. And in Sketch 4.2.12 the reader learned that the proof of 4.2.8.1 could also have been given two decades ago. So why did we fail to see this?

5.1. The strangeness of the argument. The reader should appreciate the bizarreness of looking at the base-change map in 4.2.5.1. It might help to elaborate a little by sketching a related and slightly easier computation.

Sketch 5.1.1. Let \( k \) be a field, let \( f : X \to Y \) be the projection \( f : \mathbb{P}^n_k \to \text{Spec}(k) \), and let \( u : U \to X \) be the open immersion \( \mathbb{A}^n_k \to \mathbb{P}^n_k \). In Caution 4.2.7 we warned the reader that the functor \( u^*f^\times \) is very different from \( Lu^*f^\times \). Let us work out just how different by evaluating on \( \mathcal{O}_Y \). Theorem 3.2.1 gives an isomorphism \( \theta : \Omega^2_f[n] \to f^\times \mathcal{O}_Y \), which means that \( Lu^*f^\times \mathcal{O}_Y \cong Lu^*\Omega^2_f[n] \cong \Omega^2_{\mathbb{A}^n_k}[n] \cong \mathcal{O}_U[n] \), after all the canonical bundle of \( U = \mathbb{A}^n_k \) is trivial. Under the equivalence \( D(S) \cong D_{\text{qc}}(U) \) of Reminder 4.2.10 and Computation 4.2.11 the object \( Lu^*f^\times \mathcal{O}_Y \in D_{\text{qc}}(U) \) identifies with \( S[n] \in D(S) \), that is the cochain complex obtained by placing the \( S \)-module \( S \) in dimension \(-n\).

Now let’s compute \( u^*f^\times \mathcal{O}_Y \cong (fu)^\times \mathcal{O}_Y \). The morphism \( fu : U \to Y \) is a map of affine schemes, and in Reminder 4.2.10 we noted that the computation of \( (fu)^\times \) can be carried over to \( (fu)^\times : D(R) \to D(S) \) and is given explicitly by the formula \( (fu)^\times(-) = R\text{Hom}_R(S, -) \). In our case \( R \) is the field \( k \), \( S = k[x_1, \ldots, x_n] \) is the polynomial ring, and \( (fu)^\times \mathcal{O}_Y \) computes to be \( R\text{Hom}_k(S, k) = \text{Hom}_k(S, k) \), which is a gigantic injective \( S \)-module placed in degree zero. The
reader can check [46] to see just how gargantuan this injective module is. It turns out to depend on the cardinality of \( k \).

Now consider the cartesian square

\[
\begin{array}{ccc}
\mathbb{A}^n_k & \xrightarrow{id} & \mathbb{A}^n_k \\
\uparrow{id} & & \uparrow{u} \\
\mathbb{A}^n_k & \xrightarrow{u} & \mathbb{P}^n_k
\end{array}
\]

The bottom horizontal map is flat, so there is a base-change map \( \Phi : L(id)^*u^* \to id^*Lu^* \); more simply we can write it as \( \Phi : u^* \to Lu^* \). Since \( \Phi \) is a natural transformation between two functors \( D_{\mathcal{Q}C}(\mathbb{A}^n_k) \to D_{\mathcal{Q}C}(\mathbb{P}^n_k) \) we may evaluate it at the object \( f^*\mathcal{O}_{\text{Spec}(k)} \in D_{\mathcal{Q}C}(\mathbb{P}^n_k) \), producing a morphism \( \psi = \Phi |_{f^*\mathcal{O}_{\text{Spec}(k)}} : u^*f^*\mathcal{O}_{\text{Spec}(k)} \to Lu^*f^*\mathcal{O}_{\text{Spec}(k)} \) in the category \( D_{\mathcal{Q}C}(\mathbb{A}^n_k) \). The equivalence \( D_{\mathcal{Q}C}(\mathbb{A}^n_k) \cong \mathcal{D}(S) \) must take it to a morphism

\[
\psi : \text{Hom}_k(S,k) \to S[n]
\]

Doesn’t it seem absurd to study this map?

But this is exactly what we do in the simple, recent proof of Theorem 3.2.5, which we haven’t discussed in this article. The next paragraph gives a quick sketch—the non-experts may wish to skip ahead to \( \S6.2 \).

Let \( W \subset X \) be as in the statement of Theorem 3.2.5—that is \( W \) is the set of closed points in \( X = \mathbb{P}^n \), and hence \( U \cap W \subset U \) the set of closed points in \( U = \mathbb{A}^n_k \). The recent proof of Theorem 3.2.5 hinges on the observation that the functor \( \Gamma_{U\cap W} \) takes the map \( \psi \) above to an isomorphism. In Example 3.2.4 we told the reader how to compute \( \Gamma_{U\cap W}Lu^*f^*\mathcal{O}_Y \), and the analogous recipe gives that \( \Gamma_{U\cap W}Lu^*f^*\mathcal{O}_Y \cong Lu^*\Gamma_{U\cap W}f^*\mathcal{O}_Y \) can be computed by forming a minimal injective resolution for \( S[n] \), that is the complex

\[
\begin{array}{cccccccc}
0 & \to & I^{-n} & \to & I^{-n+1} & \to & \cdots & \to & I^{-1} & \to & I^0 & \to & 0
\end{array}
\]

and putting \( I^0 = \Gamma_{U\cap W}Lu^*f^*\mathcal{O}_Y \). The morphism \( \psi \) in the category \( \mathcal{D}(S) \) may be represented by a cochain map of injective resolutions

\[
\begin{array}{cccccccc}
0 & \to & 0 & \to & 0 & \to & \cdots & \to & 0 & \to & \text{Hom}_k(S,k) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \uparrow{\psi} & & \downarrow & & \downarrow
\end{array}
\]

and the functor \( \Gamma_{U\cap W} \) takes this to a map \( \Gamma_{U\cap W}(\psi) : \Gamma_{U\cap W}\text{Hom}_k(S,k) \to I^0 \). We haven’t yet told the reader how to compute \( \Gamma_{U\cap W}\text{Hom}_k(S,k) \); the formula is that \( \Gamma_{U\cap W}\text{Hom}_k(S,k) \subset \text{Hom}_k(S,k) \) is the submodule of all elements supported at the closed points. That is: an element \( e \in \text{Hom}_k(S,k) \) belongs to the submodule \( \Gamma_{U\cap W}\text{Hom}_k(S,k) \) if there exist a finite number of maximal ideals \( m_1, m_2, \ldots, m_r \subset S \) and an integer \( N > 0 \) so that the ideal \( m_1^n m_2^n \cdots m_r^n \) annihilates \( e \). So the assertion is that the obvious composite

\[
\begin{array}{ccc}
\Gamma_{U\cap W}\text{Hom}_k(S,k) & \to & \text{Hom}_k(S,k) \\
\downarrow & & \downarrow{\psi} \\
I^0 & \to & I^0
\end{array}
\]

\[\text{The isomorphism comes from Remark 3.2.4}\]
5.2. The historical block. The fact that the key new idea is so outlandish is only part of our excuse for taking so long. There were also the historical circumstances.

The foundations of Grothendieck duality was a lively, active field from about the mid-1960s until well into the 1970s. And then the interest gradually faded. The mathematical community accepted that the foundations were complicated, and stopped thinking about it. As an editor of one journal put it, when rejecting a recent paper of mine on the subject: “...your paper will be read by only the hardcore people...people have moved on...”. It’s pretty accurate to say that, in the last three decades, there have been two groups of people who have studied the foundations of Grothendieck duality, in the old-fashioned setting of classical, ordinary schemes: Lipman and his students and collaborators on one side, and Yekutieli and his students and collaborators on the other. It’s not the whole story, one can point to exceptions such as Conrad [14], but it is a good approximation of the truth. In this I must count as one of Lipman’s collaborators—every few years we run into each other, and he persuades me to return to the problem.

Lipman's approach has long been orthogonal to Yekutieli’s: Yekutieli accepted the Grothendieck formalism, while Lipman has largely followed the approach of Deligne and Verdier. In this survey we’ve said next to nothing about the Grothendieck angle, making any comparison difficult—one could fairly say it takes an expert to recognize that the two subjects are truly one and the same. In fact: checking the details, that is verifying that the maps defined in the two theories agree, is often nontrivial. Just ask Lipman—he’s probably the person who has tried hardest.

To put it in a nutshell: the story isn’t that hundreds of people were feverishly working away at the problem, and this multitude overlooked the obvious for twenty-some years. What actually transpired is that at most a dozen of us were still studying foundational questions, half of us were exploring what turned out to be the wrong continent, and the solution, when it ultimately came, scored pretty high on the weirdness scale.

5.3. What finally woke us up. Back to the recent progress: the stimulus which, at long last, nudged us into probing in the right direction came from the work of Avramov and Iyengar [7], and later Avramov, Iyengar, Lipman and Nayak [8]. They discovered puzzling formulas, and since then there has been a proliferation—the reader can find increasingly general formulas in [27, 43]. Let us present one example. Suppose we are given composable morphisms of schemes $U \xrightarrow{u} X \xrightarrow{f} Y$, and assume that $Y = \text{Spec}(R)$ and $U = \text{Spec}(S)$ are affine. Suppose the map $u$ is an open immersion, the map $f$ is proper and the composite $fu : U \longrightarrow Y$ is flat\footnote{The flatness of $fu$ isn’t crucial, it suffices for $fu$ to be of finite Tor-dimension—the formula still holds, it is a special case of Avramov, Iyengar, Lipman and Nayak [8] (4.6.1)]. But in the finite-Tor-dimension generality we don’t have a proof that’s elementary. We will return to this in Problem 6.2.3.}. The formula we happened to choose says that, for any $N \in D(R)$,

$$L_u^*f^*N \cong S \otimes_S^L \mathcal{R}\hom_R(S,S \otimes_R^L N),$$

where $S^e = S \otimes_R S$. Where on earth did this come from?

The original presentation, in the articles [7, 8], touted the left-hand-side as a great simplification of the right-hand-side. Even the name given to the formulas—the “Reduction Formulas”—reflects this perspective. The formulas were first proved using the full strength of the existing
theory of Grothendieck duality. It was not until [27] that we came up with an elementary proof of the formulas\(^3\), and not until [47] that the right-hand-side became a computational tool for working out what's on the left.

What can I say: we were slow to see the light.

6. Future directions

For the last three decades Grothendieck duality has been a small, niche subject, with only a handful of dedicated practitioners. Granted that, the reader might well wonder what this section could possibly be about. What conceivable future can there be in a moribund, small field, long abandoned by the hordes?

There are two customary causes for the waning of a subject: it may die because the important questions have all been satisfactorily answered, or else because it hits a brick wall, and no one has any idea how to advance. In both cases a rebirth is possible, but it takes something of an earthquake. There needs to be a startling new development, a major new insight, or vital new questions that come up.

Grothendieck duality is an example of a subject that died because people got stuck—there were plenty of simple, natural questions left, but no really good ideas on how to tackle them. When we view the recent progress against this background, it can only be a matter of time before the field picks up again—if nothing else, the developments offer a radically new slant on the what’s known. In this section we will sketch some of the obvious problems that now seem within reach. But before the open questions we need to brush up on facts that are known but haven’t been covered yet—until now we have made a conscious effort to be minimal in our use of category theory, if a category or a functor was dispensable we omitted it.

6.1. Assorted background material. We have told the reader how to compute the functor \(f^\times\) when \(f\) is smooth and proper. There is another classical situation in which the computation of \(f^\times\) is understood, let us recall.

Remark 6.1.1. The category \(D_{qc}(X)\) is monoidal, it has a tensor product. This tensor product has a right adjoint: there is a functor \(\mathcal{R}\text{Hom}_{D_{qc}(X)}(-,?)\) and an isomorphism, natural in everything in sight,

\[
\text{Hom}[A \otimes L B, C] \cong \text{Hom}[A, \mathcal{R}\text{Hom}_{D_{qc}(X)}(B,C)]
\]

One way to construct it is to fix \(B\) and note that, since the functor \((-) \otimes L B\) respects coproducts, Brown representability gives a right adjoint \(\mathcal{R}\text{Hom}_{D_{qc}(X)}(B,-)\). Now let \(f : X \rightarrow Y\) be a morphism of schemes and suppose \(A, C\) are objects of \(D_{qc}(Y)\). We remind the reader of the

\(^3\)As exposed in [27] the proof doesn’t seem elementary—the article [27] was written for an expert audience. But the formula is a minor variant of Reduction 4.2.8 and, as presented in this document, the proof is manifestly elementary. And the truth is that, modulo peeling away the generality in [27] and dusting off superfluous fluff, the proof here is identical to the proof there.
following string of isomorphisms
\[
\text{Hom}[A, Rf_\ast f^\times C] \cong \text{Hom}[Lf^\ast A, f^\times C] \\
\cong \text{Hom}[Lf^\ast A \otimes^L O_X, f^\times C] \\
\cong \text{Hom}[Rf_\ast (Lf^\ast A \otimes^L O_X), C] \\
\cong \text{Hom}[A \otimes^L Rf_\ast O_X, C] \\
\cong \text{Hom}[A, \mathcal{R}\text{Hom}_{\mathcal{D}_{qc}(Y)}(Rf_\ast O_X, C)]
\]
where the second isomorphism is because $O_X$ is the unit of the tensor, the fourth comes from the projection formula, and the others are all by adjunction. Yoneda tells us that we have produced an isomorphism, natural in $C$,
\[
Rf_\ast f^\times C \cong \mathcal{R}\text{Hom}_{\mathcal{D}_{qc}(Y)}(Rf_\ast O_X, C)
\]
When $f$ is an affine morphism this looks great—after all for an affine morphism $f$ the functor $Rf_\ast$ is informative, especially when we view it as a functor from $\mathcal{D}_{qc}(X)$ to $\mathcal{D}_{qc}(f_\ast O_X\text{-Mod})$.

The problem is that, in general, the expression $\mathcal{R}\text{Hom}_{\mathcal{D}(Y)}(Rf_\ast O_X, C)$ isn’t overly computable. What is frequently far more amenable to calculation is $\mathcal{R}\text{Hom}_{\mathcal{D}(X)}(Rf_\ast O_X, C)$, which is related. That is: we compute the internal Hom not in the category $\mathcal{D}_{qc}(Y)$ but in the larger category $\mathcal{D}(Y)$, whose objects are all cochain complexes of sheaves of $O_Y$–modules. This functor has the property that the cohomology sheaves $\mathcal{F}$ $[\mathcal{R}\text{Hom}_{\mathcal{D}(Y)}(Rf_\ast O_X, C)]$ are what one would hope for. This means: for every open immersion $u: U \hookrightarrow X$ consider the $\Gamma(U, O_Y)$–module $\text{Hom}_{\mathcal{D}_{qc}(U)}(Lu^\ast Rf_\ast O_X, Lu^\ast C[i])$. This comes with obvious restriction maps, elevating the construction to a presheaf of $O_Y$–modules on $Y$. And the sheaf $\mathcal{F} [\mathcal{R}\text{Hom}_{\mathcal{D}(Y)}(Rf_\ast O_X, C)]$ turns out to be the sheafification of this presheaf. Unfortunately in order to pass, from the computable $\mathcal{R}\text{Hom}_{\mathcal{D}(Y)}(Rf_\ast O_X, C)$ to the desired $\mathcal{R}\text{Hom}_{\mathcal{D}_{qc}(Y)}(Rf_\ast O_X, C) \cong Rf_\ast f^\times C$, one has to “quasicoherate”—not the world’s most transparent process.\(^{10}\)

There are cases where the computable Hom already has quasicoherent cohomology, in which case the two functors agree and the quasicoherator does nothing. Since we’re making the assumption that $f$ is an affine map (this is the case in which the computation of $Rf_\ast f^\times C$ will carry useful information about $f^\times C$), what is relevant for us is that $Rf_\ast f^\times C \cong \mathcal{R}\text{Hom}_{\mathcal{D}_{qc}(Y)}(Rf_\ast O_X, C)$ is given by the more computable expression if one of the conditions below holds:

(i) $f$ is finite and $C$ is bounded below.
(ii) $f$ is finite and of finite Tor-dimension, and $C$ is arbitrary.

In passing we note that, historically, the most widely used special case of the above has been where $f$ is a closed immersion—possibly of finite Tor-dimension.

There isn’t a whole lot more concrete computational knowledge about $f^\times$: in the body of the article we told the reader what is known when $f$ is smooth and proper, and Remark 6.1.1 tells us some more when $f$ is a finite map, possibly of finite Tor-dimension. Nevertheless one can use these (admittedly limited) pieces of information to deduce useful facts. But for this it helps to know another functor, a close cousin of $f^\times$. Before we introduce it, a reminder might help.

In Reduction 4.2.8 we met the following situation: we were given composable morphisms of schemes $U \xrightarrow{u} X \xrightarrow{f} Y$, with $u$ an open immersion and $f$ proper. The Reduction was all about

---

\(^{10}\)In Appendix A we compute a simple example to illustrate the point.
computing the functor $L u^* f^\times$, and until now we haven’t mentioned that $L u^* f^\times$ depends only on the composite map $U \xrightarrow{f u} Y$. That is:

**Reminder 6.1.2.** If $g : U \rightarrow Y$ is any separated morphism of finite type, Nagata’s theorem \[38\] allows us to choose a factorization of $g$ as $U \xrightarrow{u} X \xrightarrow{f} Y$, with $u$ an open immersion and $f$ proper. We then define $g^! = L u^* f^\times$, and it is a theorem that $g^!$ is independent of the factorization up to canonical isomorphism.\[14\] Classically the bulk of Grothendieck duality has been devoted to the study of the functors $g^!$. See Remark 6.1.4 below.

Having introduced $g^!$, we are now in a position to restate the recent progress in terms of it. Suppose we are given, as in the last paragraph, a map of schemes $g : U \rightarrow Y$ which factorizes as $U \xrightarrow{u} X \xrightarrow{f} Y$, with $u$ an open immersion and $f$ proper. The square

\[
\begin{array}{ccc}
U & \xrightarrow{id} & U \\
id & & \downarrow \phi \\
U & \xrightarrow{u} & X
\end{array}
\]

is cartesian, hence it has a base-change map $\phi : u^\times \rightarrow L u^*$ as in Construction 4.1.4. The recent discoveries can be summarized as saying

(i) The induced map $\Phi f^\times : u^\times f^\times \rightarrow L u^* f^\times$ is independent of the factorization, and gives an unambiguous map $g^\times \rightarrow g^!$. The construction taking $g$ to $g^!$ yields a 2-functor we will denote $(-)^!$—in other words there is a compatibility with composition. And the map $g^\times \rightarrow g^!$ is compatible too, it is a morphism of 2-functors $\psi : (-)^\times \rightarrow (-)^!$ with many reasonable naturality properties. The reader can find this worked out in \[27\], and in greater generality in \[42\].

(ii) There are interesting situations in which some natural functor $\Gamma$ takes $\psi(g) : g^\times \rightarrow g^!$ to an isomorphism. We have encountered two examples, namely \[4.2.8.1\] and Sketch 5.1.1 For the general theory see \[27\] \[42\].

**Application 6.1.3.** Suppose $f : X \rightarrow Y$ is proper. Using the 2-functor $(-)^!$ one can prove the following

(i) If $G \in D^+_\text{coh}(Y)$ then $f^X G \in D^+_\text{coh}(X)$. Here $D^+_\text{coh} \subset D_{\text{qc}}$ is the full subcategory of all objects with coherent cohomology sheaves, which vanish in sufficiently negative degrees.

Now suppose $f$ is not only proper, but also of finite Tor-dimension. Then

(ii) If $G \in D^b_{\text{coh}}(Y)$ then $f^X G \in D^b_{\text{coh}}(X)$. Here $D^b_{\text{coh}} \subset D_{\text{qc}}$ is the full subcategory of complexes with bounded, coherent cohomology sheaves.

(iii) $G \in D_{\text{coh}}(Y)$ then $f^X G \in D_{\text{coh}}(X)$. Here $D_{\text{coh}} \subset D_{\text{qc}}$ is the full subcategory of complexes with coherent cohomology sheaves, vanishing in sufficiently positive degrees.

\[\text{Footnote: If one follows the Grothendieck-Hartshorne path to the subject, which we will discuss a little more in Remark 6.1.4 then the way to see the isomorphism $g^! \cong L u^* f^\times$ is that the theory sets up a functor $g^!$, one shows that for open immersions there is a natural isomorphism $u^! \cong L u^*$, for proper maps $f$ there is a natural isomorphism $f^! \cong f^\times$, and for composable maps there is a natural isomorphism $g^! = (fu)^! \cong u^! f^!$. But both Deligne \[16\] and Verdier \[63\] sketched an argument hinting how to prove directly the independence of factorization of $L u^* f^\times$, using only \[4.1.4\] applied to suitable cartesian squares. There is some more detail in Lipman’s book \[32\], and for a fuller argument, which works for algebraic stacks and hence must handle the 2-category technicalities more carefully, the reader is referred to \[42\].}\]
Thus if \( f \) is proper the adjoint pair \( Rf_* : D_{qc}(X) \rightleftarrows D_{qc}(Y) : f^\times \) restricts to an adjunction
\[
Rf_* : D_{coh}^+(X) \rightleftarrows D_{coh}^+(Y) : f^\times.
\]
When \( f \) is not only proper but also of finite Tor-dimension, we also have two more adjoint pairs, namely \( Rf_* : D_{coh}^-(X) \rightleftarrows D_{coh}^-(Y) : f^\times \) and
\[
Rf_* : D_{coh}^{b}(X) \rightleftarrows D_{coh}^{b}(Y) : f^\times.
\]

And the proofs of (i), (ii) and (iii) go as follows: the assertions are local in \( X \), hence it suffices to show that, for \( u : U \rightarrow X \) an open immersion from a (sufficiently small) open affine subset, the map \( Lu^* f^\times = (fu)^! \) satisfies the properties. But \((-)^!\) is a 2-functor. We are allowed to factor \( fu \) in some other way, for example as a composite \( fu = ghk \), and if we choose our factorization wisely then \( (fu)^! = k^! h^! g^! \) might be more computable. For example \( g, h, k \) might fall into the classes where we understand \((-)^!\): open immersions, maps that are smooth and proper, and finite morphisms (possibly of finite Tor-dimension). For details the reader is referred to [43, Lemma 3.12 and its proof]

Thus the abstract nonsense approach does recover “coherent duality” as it was traditionally understood—meaning about complexes with coherent cohomology. The reader might also wonder about the relation of (for example) the categories \( D^{b}_{coh}(X) \) and \( D^{b}(coh/X) \). The relation is well-understood by now, but falls outside the scope of Grothendieck duality—it isn’t about the functors \( f^\times \) or \( f^! \), it’s a formal question about the interplay among the myriad derived categories one can associate to a single scheme \( X \). We leave this out of the survey—we’ve barely started Section 5, and we’ve already been bombarded with a hail of new categories and functors.

**Remark 6.1.4.** Now that we have introduced \((-)^!\) we can recall a historical point. The Grothendieck approach is entirely in terms of the 2-functor \((-)^!\), the 2-functor \((-)^\times\) never comes up. That is: starting with a separated, finite-type morphism of noetherian schemes \( f : X \rightarrow Y \), Grothendieck goes through an intricate, arduous procedure to arrive at a functor which turns out to agree with the \( f^! \) of Reminder 6.1.2. In the Grothendieck setting it’s a major theorem that, for proper \( f \), the functor \( f^! \) is right adjoint to \( Rf_* \), that is \( f^! \) satisfies the defining property of \( f^\times \). Of course if you take Remark 6.1.2 as the definition of \( f^! \) then this theorem becomes trivial: for a proper \( f \) we choose the Nagata factorization \( X \xrightarrow{\text{id}} X \xrightarrow{f} Y \), and by definition \( f^! = \text{id}^! f^\times = f^\times \).

We have already mentioned that, for general \( f \), the existence of the right adjoint \( f^\times \) for the functor \( Rf_* \) was first proved in Deligne [16]. The article [16] notes that \( f^\times \) agrees with \( f^! \) for proper \( f \), and for non-proper \( f \) Deligne dismisses \( f^\times \) as too undeserving even to be graced with a name, a functor that doesn’t lend itself to calculation. For non-proper \( f \) the functor \( f^\times \) was deemed worthless and consigned to the trash heap of obscurity, until the 1990s it remained nameless.

Until the 1980s, the Deligne-Verdier approach to Grothendieck duality existed only as an offhand aside in Verdier [63], a remark saying such a theory should be possible. No one took the trouble to check the details. Lipman plunged into this project sometime in the late 1980s—it took him the best part of two decades, and the outcome was the book [52]. In his development of the subject Lipman chose not to ignore the right adjoint of \( Rf_* \) when \( f \) isn’t proper. He brushed
off Deligne’s disparaging appraisal of this functor—he christened it $f^\times$, and then went on to study its properties.

As it turns out Lipman was right in resurrecting the maligned $f^\times$ from oblivion: the key to the new progress is to study $f^\times$ for non-proper maps. In particular the computability of $f^\times$, when $f$ is a morphism of affine schemes, plays a key role—see Remark 6.2.10.

**Remark 6.1.5.** Although none of the results presented in this survey is new, the published expositions of the recent work are all addressed to the expert—they are all couched in terms of $g^!$, its relation with $g^\times$, and how to use the relation for computations. Thus, even though the results give new and much simplified proofs of concrete, old theorems, you have to be an expert to discern this from the available manuscripts. This was inevitable: after all when we write research papers we’d like to have them published. When the subject happens to be out of fashion this adds an extra hurdle—the editor will naturally worry that the paper is likely to have only a miniscule readership and a meager impact (as measured by citations). But, even with the most sympathetic of editors, the paper will go to referees who will undoubtedly be experts. Thus the authors will invariably try to write to impress the experts. The focus will be on new results, not on simple proofs of old theorems—and even when such simplifications are present the fact may well be hidden, buried deep under a mountain of technicalities.

That said, the current survey is an attempt to open up the field to non-specialists. If the subject is to have a revival then it’s imperative for the main points to be widely accessible. To keep this document as readable as possible we have, up to Section 6 avoided all but the most indispensable machinery. We are about to delve into areas where progress now seems within reach, and this is forcing us to first recall a little more background. We have to catch up a little, before the open questions can be stated clearly.

We have explained the 2-functor $(-)^!$. The next step is

**Reminder 6.1.6.** We begin with a couple of old definitions

(i) Let $X$ be a noetherian scheme. An object $D \in D^b_{coh}(X)$ is a **dualizing complex** if $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(-, D)$ takes $D_{coh}(X)$ to itself and furthermore the natural map $0_X \rightarrow \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(D, D)$ is an isomorphism. Equivalently: the functor $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(-, D)$ induces an equivalence $D_{coh}(X)^{op} \rightarrow D_{coh}(X)$.

(ii) Let $f : X \rightarrow Y$ be a flat, finite-type morphism of noetherian schemes, and consider the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\delta} & X \times_Y X \\
\downarrow{\pi_2} & & \downarrow{\pi_1} \\
X & \xrightarrow{f} & Y
\end{array}
$$

A dualizing complex $D \in D_{coh}(X)$ is **$Y$–rigid** if it comes together with an isomorphism $D \rightarrow \delta^\times [L\pi_1^* D \otimes^L L\pi_2^* D]$.

An old theorem tells us

---

12 The drawback of Lipman’s notation is that, when handwritten, the symbol $f^\times$ is barely distinguishable from $f^*$. This renders it a calligraphic challenge to give blackboard talks in the subject.

13 In Remark 6.1.3 we noted that there are two classical functors $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}$, namely the one with values in $D_{qc}(X)$ and the one with values in $D(X)$. Since we are only considering $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(C, D)$ with $C \in D^b_{coh}(X)$ and $D \in D^b_{coh}(X) \subset D^\times_{qc}(X)$, we are in a situation where the two are classically known to agree.
(iii) Assume $X \xrightarrow{f} Y \xrightarrow{g} Z$ are finite-type, flat, separated morphisms of noetherian schemes. Then $f^! : \mathcal{D}_{\text{qc}}(Y) \to \mathcal{D}_{\text{qc}}(X)$ takes $Z$-rigid dualizing complexes to $Z$-rigid dualizing complexes.

For the sake of historical accuracy: the fact that $f^!$ takes dualizing complexes to dualizing complexes goes all the way back to the dawn of the theory. The concept of rigid dualizing complexes started with Van den Bergh [62]. The formulation given here follows Lipman’s reworking of Van den Bergh’s result. And Yekutieli and Zhang [73, 74, 75, 76] pursued this in depth, it was their contribution to simplifying and extending the Grothendieck approach to the subject.

In other words: in an alternative universe, a survey of the field would begin with rigid dualizing complexes and build up from there. It just so happens that the Deligne-Verdier angle on the subject was the first to achieve the status of satisfactorily cleaning up the foundations.

Reminder 6.1.7. We should also remind the reader that there is an interesting noncommutative version. Since I’m not quite sure what a general noncommutative scheme should be, I will confine the discussion to affine noncommutative schemes.

Let $R$ be a noetherian, commutative ring and let $S$ be a flat, finitely generated, associative $R$-algebra which is right and left noetherian (but not necessarily commutative). Set $S^\ast = S \otimes_R S^{\text{op}}$. Following Yekutieli [64], a dualizing complex is an object $D \in \mathcal{D}(S^\ast)$ such that the functor $\underline{\text{RHom}}(-, D)$ yields an equivalence of categories $\mathcal{D}^b(S-\text{mod})^{\text{op}} \to \mathcal{D}^b(S^{\text{op}}-\text{mod})$. The paper [66] goes on to study the graded situation and produce some examples of dualizing complexes. If $S$ is commutative then dualizing complexes in $\mathcal{D}(S^\ast)$, in Yekutieli’s sense, can be shown to agree with dualizing complexes in $\mathcal{D}^b_{\text{coh}}(\text{Spec}(S))$, as recalled in Reminder 6.1.6(i).

Following Van den Bergh [62], the dualizing complex $D \in \mathcal{D}(S^\ast)$ is $R$–rigid if it comes together with an isomorphism

$$D \cong \underline{\text{RHom}}_{S^\ast}(S, D \otimes_L^R D).$$

If $R$ is a field, and $S$ has a filtration whose associated graded ring is commutative and finitely generated as an $R$–algebra, then [62] cleverly shows that a rigid dualizing complex exists. When $S$ is commutative then an $R$–rigid dualizing complexes in $\mathcal{D}(S^\ast)$, in Van den Bergh’s sense, can be shown to agree with the $\text{Spec}(R)$–rigid dualizing complex in $\mathcal{D}^b_{\text{coh}}(\text{Spec}(S))$, as recalled in Reminder 6.1.6(ii).

There has been literature pursuing this further, for a couple of early papers the reader is referred to Yekutieli and Zhang [71, 72]. But it’s now high time to move on to the open problems.

6.2. Foundational questions. And now we are ready to state the first open question. This one is based not on the very recent work, it derives from the formulas of Avramov and Iyengar [7, 8] that finally opened our eyes. Fittingly the first open question is about noncommutative algebraic geometry, which is unquestionably a hot field nowadays.

Problem 6.2.1. Let $g : k \to R$ be any finite-type, flat homomorphism of noetherian, commutative rings. Let the relation between the rings $R$ and $S$ be as in Reminder 6.1.7, meaning $S$ is an $R$–algebra satisfying all the hypotheses of Reminder 6.1.7. Let $N \in \mathcal{D}(R-\text{mod}) \cong \mathcal{D}^b_{\text{coh}}(\text{Spec}(R))$ be a $k$–rigid dualizing complex.

Question: is $S \otimes_S^L \text{RHom}_R(S, S \otimes_R^L N)$ a $k$–rigid dualizing complex in $\mathcal{D}(S^\ast-\text{mod})$?

We should remark that, if $S$ is commutative, this follows from the isomorphism $f^! N \cong S \otimes_S^L \text{RHom}_R(S, S \otimes_R^L N)$ of the Reduction Formula, which we met in [5.3] coupled with Reminder 6.1.6(iii).
But we don’t yet understand the theory well enough to have a simple, direct proof of Remind
ner 6.1.6(iii) in the affine (commutative) case, hence have no idea if the result can be extended
to the noncommutative context.

We should note that the case $k = R$ is already interesting. In fact let us confine ourselves
to the case where the ring $k = R$ is Gorenstein; in this case it is known that $R \in D(R\text{-mod})$ is an
$R$–rigid dualizing complex, and the question specializes to: is $S \otimes^L_R \mathcal{RHom}_{k}(S,S)$ an $R$–rigid
dualizing complex in $D(S^e\text{-mod})$? The reader should note that, in the noncommutative setting,
$R$–rigid dualizing complexes are known to exist only when $R$ is a field, in particular the existence
results to date all assume equal characteristic.

Problem 6.2.2. The work of Avramov, Iyengar and Lipman [6, Section 3] suggests an alternative
definition for rigid dualizing complexes. With $k \to R \to S$ as in Problem 6.2.1 we can declare
that a dualizing complex $D \in D(S^e)$ is AIL–$k$–rigid if it comes with an isomorphism
$$D \cong \mathcal{RHom}_{S}[\mathcal{RHom}_{S}(D, S \otimes^L_R \mathcal{RHom}_{k}(S,S)), D]$$
From the Reduction Formula of Avramov, Iyengar, Lipman and Nayak [8, (4.1.1)] we have that,
as long as $k$ is regular and finite dimensional and $S$ is commutative, the two notions of rigidity
agree.

Question 1: Do the two notions coincide when $S$ isn’t commutative?

Question 2, assuming the notions are different: Suppose $N \in D(R)$ is an AIL–$k$–rigid dualizing
complex. Is $S \otimes^L_S \mathcal{RHom}_{R}(S, S \otimes^L_R N)$ also an AIL–$k$–rigid dualizing complex in $D(S^e\text{-mod})$?

The third foundational problem is about a derived stack version of the theory—as it happens
derived stacks are also much in vogue nowadays.

Problem 6.2.3. In many of the theorems we had to assume flatness, or at the very minimum
finite Tor-dimension. The modern way to get around this is to work in the setting of derived
algebraic geometry.

Question: is there an incarnation of the theory in derived algebraic geometry?

The Yekutieli school was the first to successfully employ DG methods in Grothendieck duality:
see Yekutieli and Zhang [70, 75, 76], Yekutieli [68, survey], and more recently Shaul [57]. The
Lipman school, inspired by the successes of the Yekutieli school, followed suit: the affine case
of the Reduction Formulas, of Avramov and Iyengar, does extend from the flat case presented
in §5.3 to the case where the map $R \to S$ is of finite Tor-dimension. The proof given in Avramov,
Iyengar, Lipman and Nayak [8, Section 4] goes by way of differential graded algebras. See also [6]
for further instances, of the Lipman school exploiting the DG methods introduced by Yekutieli
and Zhang.

For some planned future projects see Yekutieli [64]. Lipman is also interested in pursuing
further the methods of derived algebraic geometry—he has been working his way through Lurie’s
book [37]—but I’m not aware of any manuscripts yet. In any case: at this point the subject is in
its infancy, all are welcome to join in.

Remark 6.2.4. The work in Hafiz Khusyairi 2017 PhD thesis might be relevant to Prob-
lem 6.2.3—the thesis is entirely in the setting of old-fashioned, ordinary, commutative schemes,
but uses the formulas of Avramov and Iyengar as the starting point for setting up the theory. It
then proceeds to develop the usual functoriality properties from there. Since the formulas have
a DG analog Khusyairi’s work might generalize.
6.3. Computational problems. In the previous section we sketched three foundational problems, about extending the theory to noncommutative and to derived algebraic geometry—both of which are “in” fields nowadays.

Let us now return to more classical problems. In the old, traditional world of ordinary, commutative algebraic geometry the foundations of Grothendieck duality have reached a reasonably satisfactory state. It is feasible to introduce the players and describe the relations among them in what could plausibly be called a short space, and it is possible to do so in such a way that the traditional computations become transparent and brief.

But the problem is that the traditional computations are limited. Let us assume \( f : X \to Y \) proper and of finite Tor-dimension, in which case Remark 3.1.6 allows us to reduce the problem to computing \( f^* \mathcal{O}_Y \) and the map \( \varepsilon : Rf_* f^* \mathcal{O}_Y \to \mathcal{O}_Y \). The classical literature gives us a good understanding in the case where \( f \) is smooth, and an understanding of some sort in situations that are easily reduced to the smooth case, for example when \( f \) is Cohen-Macaulay. Beyond that, what’s known is not all that useful.

As it happens Nayak and Sastry are in the process of writing up a comprehensive account of what is known about the computations. Before long there will be a manuscript containing everything that has been figured out so far—it will be valuable to have it all assembled in one place and the connections worked out. Once the document is ready the interested reader will be able to see, in print and in detail, just how paltry our understanding really is.

Remark 6.3.1. The last paragraphs should not be interpreted as belittling the traditional case of Grothendieck duality, the special case where \( f : X \to Y \) is smooth and proper—this classical situation is already fascinating and has spawned a rich literature spanning many decades. Specializing further, assume that \( Y = \text{Spec}(k) \) is a point and \( f : X \to Y \) is smooth, proper and of relative dimension 1, and we find ourselves in ancient territory—we’re in the well-understood case of duality for curves. The reader can find an excellent exposition of the old approaches in Serre [54, pp. 25-34 and pp. 76-81], and [as far as the author knows] the cleverest, most recent idea is already five decades old, see Tate [58]. The case where \( Y = \text{Spec}(k) \) is still a point, \( f : X \to Y \) is still smooth and proper, but the relative dimension is arbitrary is classical Serre duality, the reader is referred to Serre [53], and also to the sketch presented in Section 3.3. In Be˘ılinson [11] we learn how to generalize Tate’s clever trick to higher dimension.

If \( k = \mathbb{C} \) then \( X \) is a smooth, compact Kähler manifold, and the Hodge decomposition theorem identifies \( H^n(\Omega^n_f) = Rf_* \Omega^n_f[n] \) with \( H^{n,n}(X) = H^{2n}(X, \mathbb{C}) \cong \mathbb{C} \), where the last isomorphism is by Poincaré duality. We would therefore expect to be able to understand the residue map from this perspective too. The reader can find this explored in Harvey [20], Tong [61], and more recently in Sastry and Tong [52].

Now let us return to the generality of the relative case: that is \( f : X \to Y \) is assumed smooth and proper but \( Y \) is an arbitrary noetherian scheme. We know, from the results surveyed in this article, that \( f^* \mathcal{O}_Y \) is canonically isomorphic to \( \Omega^n_f[n] \), and that the counit of adjunction \( \varepsilon : Rf_* f^* \mathcal{O}_Y \to \mathcal{O}_Y \) is determined by the map taking a relative meromorphic \( n \)-form to its residue. In this article we presented a very recent approach to these theorems, we should say something about the older methods—after all understanding the relationship of the old tack with the new might well prove fruitful and illuminating.

The first issue is that all the data must be compatible with composition. That is: if \( X \to Y \to Z \) are composable morphisms of schemes, both of which are smooth and proper, then the
The composite is smooth and proper and we have a string of canonical isomorphisms
\[ \Omega_{g_f}^{m+n}[m+n] \cong (gf)^* \mathcal{O}_Z \cong f^* g^* \mathcal{O}_Z \cong Lf^* g^* \mathcal{O}_Z \otimes^L f^* \mathcal{O}_Y \cong Lf^* \Omega_{g}^m[m] \otimes^L \Omega_Y^n[n] \]

The reader might wonder whether the composite is the obvious isomorphism—not surprisingly the answer turns out to be Yes, see Lipman and Sastry [36]. Furthermore the counits of adjunction must be compatible. We have a counit of adjunction
\[ \varepsilon : R(gf)_* (gf)^* \mathcal{O}_Z \rightarrow \mathcal{O}_Z, \]

Rewriting this in terms the string of canonical isomorphisms above yields a diagram which must commute
\[
\begin{array}{ccc}
Rg_* (\Omega^m_g[m] \otimes^L Rf_* \Omega^n_f[n]) & \xrightarrow{\varepsilon(f)} & R(gf)_* \Omega_{g_f}^{m+n}[m+n] \\
\varepsilon(f) & & \varepsilon(g) \\
Rg_* (\Omega^m_g[m] \otimes^L \mathcal{O}_Y) & \cong & Rg_* \Omega^m_g[m] \mathcal{O}_Z \end{array}
\]

and the commutativity can be interpreted as a compatibility condition on the residue maps.

In the previous paragraph we learned that the counit of adjunction \( \varepsilon : Rf_* \Omega^n_f[n] \rightarrow \mathcal{O}_Y \), and hence the closely related residue map \( \rho : Rf_* \Gamma_W \Omega^n_f[n] \rightarrow \mathcal{O}_Y \), must be compatible with composition. It’s even easier to see that \( \rho \) must be compatible with flat base change. For the purpose of computations, the compatibility with flat base change allows us to assume that \( Y = \text{Spec}(R) \) is affine—and if it helps we may even assume that \( R \) is a (strictly) henselian or even a complete local ring. Grothendieck’s GFGA result [18, Théorème 5.1.4] allows us to replace \( X \) by its formal completion, and for some time now the experts have been pursuing the idea that doing so might lead to a better understanding of \( \rho \). For a more extensive treatment the reader is referred to the forthcoming article by Nayak and Sastry; but see also Alonso, Jeremías and Lipman [1, 2], Lipman Nayak and Sastry [33], Nayak [39], Nayak and Sastry [41] and Sastry [51].

**Remark 6.3.2.** In the special case of smooth and proper morphisms \( f : X \rightarrow Y \), Remark 6.3.1 surveyed some of the work done in the quest for a better understanding of Grothendieck duality. The opening paragraphs of Section 6.3 dismissed what’s known about more general \( f \) as being of limited computational value.

There is some literature: the reader might wish to look at Huang [23, 24], Kersken [28], Paršin [48] and Yekutieli [67] (see also the appendix by Sastry).

**Problem 6.3.3.** Now put the recent results at center stage—they should allow us to go further with the computations. At least when \( f \) is flat, we have a simple and explicit formula for \( f^! \). The generalization of Reduction 4.2.8 gives an isomorphism \( f^! = L\delta^* \pi^* Lf^* \), where \( \delta : X \rightarrow X \times_Y X \) is the diagonal map and \( \pi : X \times_Y X \rightarrow X \) is the (second) projection. Any colocalization \( c : \Gamma \rightarrow \text{id} \), where \( \Gamma \) takes the map \( \psi : f^* \rightarrow f^! \) to an isomorphism, will permit us to form the composite
\[
\begin{array}{cccc}
Rf_* \Gamma f^! & \xrightarrow{Rf_* (\Gamma \psi)^{-1}} & Rf_* \Gamma f^x & \xrightarrow{Rf_* c f^x} & Rf_* f^x & \xrightarrow{\varepsilon} & \text{id}
\end{array}
\]
which should be computable, at least in the special case where \( X \) and \( Y \) are affine. For suitable choices, of the colocalization \( c : \Gamma \to \text{id} \), the composite should deliver useful information about \( \varepsilon \)—and those of us competent to carry out the computations should be able to learn much more about the map \( \varepsilon : Rf_* f^* \to \text{id} \).

The computations will involve Hochschild homology and cohomology—terms like \( S \otimes S, \text{RHom}_R(S, S \otimes_R N) \) are bound to appear. Fortunately the world is full of experts in Hochschild homology and cohomology, and once they take an interest they will undoubtedly be able to move these computations much further than the handful of us, the few people who have been working on Grothendieck duality. Let’s face it: in our tiny group none is adept at handling the Hochschild machinery. The Hochschild experts should feel invited to move right in.

**Appendix A. A computation of the base-change map** \( u^* \to Lu^* \) **when** \( u : U \to C \) **is an open immersion of curves**

Let \( k \) be a field, let \( C \) be a complete algebraic curve smooth over \( k \), and let \( p \in C \) be a \( k \)-rational point. Put \( U = C - \{ p \} \) and let \( u : U \to C \) be the open immersion. The square

\[
\begin{array}{ccc}
U & \xrightarrow{u} & U \\
\downarrow & & \downarrow \\
U & \xrightarrow{u} & C
\end{array}
\]

is cartesian and the horizontal maps are flat, and Construction 4.1.4 yields a base-change map \( \Phi : u^* \to Lu^* \). Let \( \mathcal{L} \) be a line bundle on \( C \); we propose to compute the map \( \Phi(\mathcal{L}) : u^* \mathcal{L} \to Lu^* \mathcal{L} \). In Remark 6.1.1 we met the isomorphism

\[
\text{R}u_* u^* \mathcal{L} \cong \text{RHom}_{\text{D}_{\text{qc}}(C)}(\text{R}u_* \mathcal{O}_U, \mathcal{L})
\]

One may check that the counit of adjunction \( \varepsilon : \text{R}u_* u^* \mathcal{L} \to \mathcal{L} \) is the map obtained by applying the functor \( \text{RHom}_{\text{D}_{\text{qc}}(C)}(\cdot, \mathcal{L}) \) to the morphism \( \mathcal{O}_C \to \text{R}u_* \mathcal{O}_U \). And the map \( u^* \to Lu^* \) is just the composite \( u^* \to \varepsilon^{-1}u^* \to Lu^* \text{R}u_* u^* \to Lu^* \), where \( \varepsilon : Lu^* \text{R}u_* u^* \to \text{id} \) is the (invertible) counit of adjunction.

So much for abstract nonsense. Concretely we are reduced to computing what the functor \( \text{RHom}_{\text{D}_{\text{qc}}(C)}(\cdot, \mathcal{L}) \) does to the morphism \( \mathcal{O}_C \to \text{R}u_* \mathcal{O}_U \), after which we will apply \( Lu^* \). Now the map \( \mathcal{O}_C \to \text{R}u_* \mathcal{O}_U = u_* \mathcal{O}_U \) is the direct limit, as \( n \to \infty \), of the maps \( \mathcal{O}_C \to \mathcal{O}_C(np) \). This means that, in the derived category \( \text{D}_{\text{qc}}(C) \), we need to compute the homotopy inverse limit of the sequence \( \mathcal{L}(-np) \to \mathcal{L} \). This is what we will now do.

Let \( R = \mathcal{O}_{C,p} \), that is the stalk at \( p \) of the structure sheaf \( \mathcal{O}_C \). Let \( m \subset R \) be the maximal ideal. For each \( n \) we have a triangle

\[
\mathcal{L}(-np) \to \mathcal{L} \to \mathcal{L} \otimes R/m^n
\]

and taking homotopy inverse limits over \( n \) will yield a triangle. In general I find homotopy inverse limits difficult, but in the case of the inverse system \( \mathcal{L} \otimes R/m^n \) it isn’t so bad.

Let us first take the homotopy inverse limit in the category \( \text{D}(C) \), where we allow all complexes of sheaves of \( \mathcal{O}_C \)-modules—not only ones with quasicoherent cohomology, see Remark 6.1.1 for a discussion. Let \( i : p \to C \) be the inclusion of \( p \); we turn it into a map of ringed spaces by giving \( p \)
the structure sheaf $R$. The functor $i_*$ (extension by zero) is exact, and has an exact left adjoint—the functor taking a sheaf to its stalk at $p$. Hence the induced functor $i_*: \mathbf{D}(p) \to \mathbf{D}(C)$ respects products and therefore homotopy inverse limits. Thus the homotopy inverse limit of the system $\mathcal{L} \otimes R/m^n$ can be computed in $\mathbf{D}(p)$, and it comes down to the sheaf $\mathcal{L} \otimes i_*\hat{R}$, the extension by zero of the completion of the stalk at $p$ of $\mathcal{L}$.

This sheaf is manifestly not quasicoherent—to compute the homotopy inverse limit in the category $\mathbf{D}_{qc}(C)$ we need to derived quasicoherate. That is: we replace by an injective resolution and then quasicoherate. The injective resolution is easy enough: if $K$ is the quotient field of $R$ and $\hat{K}$ its $m$-adic completion (i.e. the quotient field of $\hat{R}$), then an injective resolution of $\hat{R}$ as an $R$–module is given by $\hat{K} \to \hat{K}/\hat{R}$, and $i_*\hat{K} \to i_*[\hat{K}/\hat{R}]$ is an injective resolution of $i_*\hat{R}$ in the category of sheaves of $\mathcal{O}_C$–modules. The module $i_*[\hat{K}/\hat{R}]$ is quasicoherent as it stands, and the quasicoherator takes $i_*\hat{K}$ to the constant sheaf $\hat{K}$. The morphism $\mathcal{L} \to \text{Holim} (\mathcal{L} \otimes R/m^n)$ becomes identified with the cochain map

$\begin{array}{cccccccc}
0 & \to & \mathcal{L} & \to & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{L} \otimes \hat{K} & \to & \mathcal{L} \otimes i_*[\hat{K}/\hat{R}] & \to & 0 \\
\end{array}$

Applying the functor $L u^* = u^*$, that it restricting to $U \subset C$, kills the sheaf $i_*[\hat{K}/\hat{R}]$. We deduce the map of cochain complexes

$\begin{array}{cccccccc}
0 & \to & u^*\mathcal{L} & \to & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \to & u^*\mathcal{L} \otimes \hat{K} & \to & 0 & \to & 0 \\
\end{array}$

And the map $u^x\mathcal{L} \to Lu^*\mathcal{L}$ is obtained by completing the triangle, it is the cochain map

$\begin{array}{cccccccc}
0 & \to & u^x\mathcal{L} & \to & u^*\mathcal{L} \otimes \hat{K} & \to & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \to & u^x\mathcal{L} & \to & 0 & \to & 0 \\
\end{array}$

To sum it up: the curve $U$ is affine, let’s say $U = \text{Spec}(S)$. The line bundle $u^x\mathcal{L} \in \mathbf{D}_{qc}(U)$ corresponds, under the equivalence $\mathbf{D}_{qc}(U) \cong \mathbf{D}(S)$, to the rank–1 projective $S$–module $L = \Gamma(U, \mathcal{L})$. Consider the short exact sequence of $S$–modules

$\begin{array}{cccccccc}
0 & \to & L & \to & L \otimes \hat{K} & \to & L \otimes \frac{\hat{K}}{L} & \to & 0 \\
\end{array}$

The morphism $u^x\mathcal{L} \to Lu^*\mathcal{L}$ in the derived category $\mathbf{D}_{qc}(U)$ corresponds, under the equivalence $\mathbf{D}_{qc}(U) \cong \mathbf{D}(S)$, to the map $L \otimes \frac{\hat{K}}{L}[-1] \to L$ that is represented by the short exact sequence.

Now there is an isomorphism of $S$–modules $\text{Hom}_k(S, k) \cong \frac{L \otimes \hat{K}}{L}$. The bad way to see this is as follows: both are injective $S$–modules, and the indecomposable injectives have the same multiplicity on both sides, see [46]. But in the case where $\mathcal{L}$ is the canonical bundle $\Omega^1_C$ we know there is a canonical isomorphism. If $f : C \to \text{Spec}(k)$ is the projection to a point, then
\[ f^*k \cong \Omega^1_U[1] \] canonically, hence

\[ \text{Hom}_k(S,k) \cong (fu)^*k \cong u^*f^*k \cong u^*\Omega^1_C \cong \frac{\Omega^1_U \otimes \hat{K}}{\Omega^1_U} \]

where the isomorphisms are all canonical.

References

[1] Leovigildo Alonso Tarrio, Ana Jeremías López, and Joseph Lipman, \textit{Local homology and cohomology on schemes}, Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 1, 1–39.
[2] , \textit{Studies in duality on Noetherian formal schemes and non-Noetherian ordinary schemes}, Contemporary Mathematics, vol. 244, American Mathematical Society, Providence, RI, 1999.
[3] , \textit{Bivariance, Grothendieck duality and Hochschild homology I: Construction of a bivariant theory}, Asian J. Math. 15 (2011), no. 3, 451–497.
[4] , \textit{Bivariance, Grothendieck duality and Hochschild homology, II: The fundamental class of a flat scheme-map}, Adv. Math. 257 (2014), 365–461.
[5] Leovigildo Alonso Tarrio, Ana Jeremías López, and María José Souto Salorio, \textit{Bousfield localization on formal schemes}, J. Algebra 278 (2004), no. 2, 585–610.
[6] Luchezar Avramov, Srikanth B. Iyengar, and Joseph Lipman, \textit{Reflexivity and rigidity for complexes, II: Schemes}, Algebra Number Theory 5 (2011), no. 3, 379–429.
[7] Luchezar L. Avramov and Srikanth B. Iyengar, \textit{Gorenstein algebras and Hochschild cohomology}, Michigan Math. J. 57 (2008), 17–35, Special volume in honor of Melvin Hochster.
[8] Luchezar L. Avramov, Srikanth B. Iyengar, Joseph Lipman, and Suresh Nayak, \textit{Reduction of derived Hochschild functors over commutative algebras and schemes}, Adv. Math. 223 (2010), no. 2, 735–772.
[9] Paul Balmer, Ivo Dell’Ambrogio, and Beren Sanders, \textit{Grothendieck-Neeman duality and the Wirthmüller isomorphism}, Compos. Math. 152 (2016), no. 8, 1740–1776.
[10] Paul Balmer and Giordano Favi, \textit{Generalized tensor idempotents and the telescope conjecture}, Proc. Lond. Math. Soc. (3) 102 (2011), no. 6, 1161–1185.
[11] Alexandre Beilinson, \textit{Residues and adèles}, Funktsional. Anal. i Prilozhen. 14 (1980), no. 1, 44–45.
[12] Marcel Bökstedt and Amnon Neeman, \textit{Homotopy limits in triangulated categories}, Compositio Math. 86 (1993), 209–234.
[13] A.K. Bousfield, \textit{The localization of spectra with respect to homology}, Topology 18 (1979), 257–281.
[14] Brian Conrad, \textit{Grothendieck duality and base change}, Lecture Notes in Mathematics, vol. 1750, Springer-Verlag, Berlin, 2000.
[15] , \textit{Deligne’s notes on Nagata compactifications}, J. Ramanujan Math. Soc. 22 (2007), no. 3, 205–257.
[16] Pierre Deligne, \textit{Cohomology à support propre en construction du foncteur f}, Residues and Duality, Lecture Notes in Mathematics, vol. 20, Springer–Verlag, 1966, pp. 404–421.
[17] Peter Gabriel and Michel Zisman, \textit{Calculus of fractions and homotopy theory}, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967.
[18] Alexandre Grothendieck, \textit{Éléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents I}, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 425–511.
[19] Robin Hartshorne, \textit{Residues and duality}, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966.
[20] F. Reese Harvey, \textit{Integral formulae connected by Dolbeault’s isomorphism}, Rice Univ. Studies 56 (1970), no. 2, 77–97 (1971).
[21] Gerhard Hochschild, Bertram Kostant, and Alex Rosenberg, \textit{Differential forms on regular affine algebras}, Trans. Amer. Math. Soc. 102 (1962), 383–408.
[22] Michael J. Hopkins, \textit{Global methods in homotopy theory}, Homotopy Theory—Proceedings of the Durham Symposium 1985, London Math. Soc. Lecture Notes Series, vol. 117, Cambridge University Press, 1987, pp. 73–96.
[23] I-Chiau Huang, \textit{An explicit construction of residual complexes}, J. Algebra 225 (2000), no. 2, 698–739.
[24] , \textit{The residue theorem via an explicit construction of traces}, J. Algebra 245 (2001), no. 1, 310–354.
[25] Reinhold Hüb and Pramathanath Sastry, *Regular differential forms and relative duality*, Amer. J. Math. 115 (1993), no. 4, 749–787.

[26] Luc Illusie, *Conditions de finitude*, Théorie des intersections et théorème de Riemann-Roch, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6, Exposé III), pp. 222–273. Lecture Notes in Mathematics, Vol. 225.

[27] Srikanth B. Iyengar, Joseph Lipman, and Amnon Neeman, *Relation between two twisted inverse image pseudofunctors in duality theory*, Compos. Math. 151 (2015), no. 4, 735–764.

[28] Masumi Kersken, *Cousinkomplex und Nennersysteme*, Math. Z. 182 (1983), no. 3, 389–402.

[29] Joseph Lipman, *Dualizing sheaves, differentials and residues on algebraic varieties*, Astérisque (1984), no. 117, ii+138.

[30] Joseph Lipman, *Residues and traces of differential forms via Hochschild homology*, Contemporary Mathematics, vol. 61, American Mathematical Society, Providence, RI, 1987.

[31] Joseph Lipman, *Lectures on local cohomology and duality*, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Mathematics, vol. 1727, Springer, Berlin, 1999, pp. 39–89.

[32] Joseph Lipman and Pramathanath Sastry, *Pseudofunctorial behavior of Cousin complexes on formal schemes*, Variance and duality for Cousin complexes on formal schemes, Contemp. Math., vol. 375, Amer. Math. Soc., Providence, RI, 2005, pp. 137–192.

[33] Joseph Lipman and Suresh Nayak, *Transitivity in duality for formal schemes 1*, (2017), preprint.

[34] Joseph Lipman and Amnon Neeman, *Quasi-perfect scheme maps and boundedness of the twisted inverse image functor*, Illinois J. Math. 51 (2007), 209–236.

[35] Joseph Lipman and Suresh Nayak, *Compactification for essentially finite-type maps*, Adv. Math. 222 (2009), no. 2, 527–546.

[36] Suresh Nayak and Pramathanath Sastry, *The relation between Grothendieck duality and Hochschild homology*, To appear in the proceedings of the TIFR international colloquium, 2016.
[53] Jean-Pierre Serre, *Un théorème de dualité*, Comment. Math. Helv. 29 (1955), 9–26.
[54] _______, *Groupes algébriques et corps de classes*, Publications de l’institut de mathématique de l’université de Nancago, VII. Hermann, Paris, 1959.
[55] Liran Shaul, *Reduction of Hochschild cohomology over algebras finite over their center*, J. Pure Appl. Algebra 219 (2015), no. 10, 4368–4377.
[56] _______, *Relations between derived Hochschild functors via twisting*, Comm. Algebra 44 (2016), no. 7, 2898–2907.
[57] _______, *The twisted inverse image pseudofunctor over commutative DG rings and perfect base change*, Adv. Math. 320 (2017), 279–328.
[58] John Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. (4) 1 (1968), 149–159.
[59] Robert W. Thomason, *The classification of triangulated subcategories*, Compositio Math. 105 (1997), 1–27.
[60] Robert W. Thomason and Thomas F. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift (a collection of papers to honor Grothendieck’s 60’th birthday), vol. 3, Birkhäuser, 1990, pp. 247–435.
[61] Yue Lin L. Tong, *Integral representation formulae and Grothendieck residue symbol*, Amer. J. Math. 95 (1973), 904–917.
[62] Michel Van den Bergh, *Existence theorems for dualizing complexes over non-commutative graded and filtered rings*, J. Algebra 195 (1997), no. 2, 662–679.
[63] Jean-Louis Verdier, *Base change for twisted inverse images of coherent sheaves*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 393–408.
[64] Amnon Yekutieli, *The derived category of sheaves of commutative dg rings (preview)*, arXiv:1312.6411.
[65] _______, *Duality and tilting for commutative dg rings*, arXiv:1312.6411.
[66] _______, *Dualizing complexes over noncommutative graded algebras*, J. Algebra 153 (1992), no. 1, 41–84.
[67] _______, *An explicit construction of the Grothendieck residue complex*, Astérisque (1992), no. 208, 127, With an appendix by Pramathanath Sastry.
[68] _______, *Rigid dualizing complexes via differential graded algebras (survey)*, Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 452–463.
[69] _______, *The squaring operation for commutative DG rings*, J. Algebra 449 (2016), 50–107.
[70] Amnon Yekutieli and James J. Zhang, *Rigid dualizing complexes on schemes*, arXiv:math.AG/0405570.
[71] _______, *Serre duality for noncommutative projective schemes*, Proc. Amer. Math. Soc. 125 (1997), no. 3, 697–707.
[72] _______, *Rings with Auslander dualizing complexes*, J. Algebra 213 (1999), no. 1, 1–51.
[73] _______, *Dualizing complexes and perverse modules over differential algebras*, Compos. Math. 141 (2005), no. 3, 620–654.
[74] _______, *Dualizing complexes and perverse sheaves on noncommutative ringed schemes*, Selecta Math. (N.S.) 12 (2006), no. 1, 137–177.
[75] _______, *Rigid complexes via DG algebras*, Trans. Amer. Math. Soc. 360 (2008), no. 6, 3211–3248.
[76] _______, *Rigid dualizing complexes over commutative rings*, Algebr. Represent. Theory 12 (2009), no. 1, 19–52.

Centre for Mathematics and its Applications, Mathematical Sciences Institute, Building 145, The Australian National University, Canberra, ACT 2601, AUSTRALIA

Email address: Amnon.Neeman@anu.edu.au