Computations with Greater Quantum Depth Are Strictly More Powerful (Relative to an Oracle)

Matthew Coudron∗ Sanketh Menda†

September 24, 2019

Abstract

A conjecture of Jozsa [Joz06] states that any polynomial-time quantum computation can be simulated by polynomial-depth classical computation interleaved with logarithmic-depth quantum computation. This conjecture is a remarkable statement about the unresolved potential of combining classical and quantum computation. As a counterpoint we show that the Welded Tree Problem, which is an oracle problem that can be solved in quantum polynomial time as shown by Childs et al. [CCD+03], cannot be solved in the class which Jozsa describes. Therefore, even when interleaved with arbitrary polynomial-time classical computation, greater “Quantum Depth” leads to strictly greater computational ability in this relativized setting.

1 Introduction

1.1 Jozsa’s Conjecture

The inspiration behind our work came from the following fascinating conjecture of Jozsa.

“Conjecture: Any polynomial time quantum algorithm can be implemented with only $O(\log n)$ quantum layers interspersed with polynomial time classical computations.”

Richard Jozsa [Joz06, Section 8]

Intriguingly, this conjecture is known to hold for some of the most influential quantum algorithms. For example, Cleve and Watrous [CW00] showed that Shor’s algorithm for FACTORING can be implemented using log-depth polynomial-size quantum circuits with polynomial-time classical pre- and post-processing. Indeed, one might be able to use a similar methodology to parallelize many quantum algorithms that rely on the quantum Fourier transform.

Similarly, most oracle separations that show a quantum speedup seem to be consistent with Jozsa’s conjecture. The prototypical algorithms that achieve an exponential quantum speedup—SIMON’S PROBLEM [Sim97] and FORRELATION [Aar10, AA18]—both consist of constant-depth quantum circuits with oracle access, and polynomial-time classical pre- and post-processing. Thus, while they constitute oracle separations between P and BQP, they do not, on their own, suggest an oracle separation between BQP and the class that Jozsa considers. All of this could be taken as an indication that the class Jozsa describes is very powerful.

Note that in order to disprove Jozsa’s conjecture one must necessarily separate P from BQP as a prerequisite, and such a statement may be very difficult to prove as an unconditional mathematical fact. In this work we will prove a separation between BQP and the class that Jozsa describes, relative to an oracle. At the end, we also remark on the possibility of proving this separation based on a cryptographic assumption.

We recently learned of independent work by Nai-Hui Chia, Kai-Min Chung, and Ching-Yi Lai [CCL19], which obtains similar results showing that Jozsa’s conjecture fails relative to an oracle. We have coordinated with them to post our work on arXiv on the same day.

∗University of Waterloo, Canada. mcoudron@gmail.com
†University of Waterloo, Canada. sgmenda@gmail.com
1.2 Jozsa’s Class

In order to make our investigation of Jozsa’s conjecture more precise, in this subsection we define a hierarchy of complexity classes based on the conjecture. We begin by setting notation, followed by a definition of the hierarchy, and ending with a formal statement of Jozsa’s conjecture.

1.2.1 Quantum Circuits

For an introduction to uniform circuit families, quantum circuits, and the complexity classes not defined here see Watrous [Wat09].

The results in this paper are not sensitive to a choice of (reasonable, universal) gate set. Nevertheless, for concreteness, we assume that our classical circuits are composed of Toffoli gates, and that our quantum circuits are composed of Hadamard, Toffoli, and Phase gates. In addition, these circuits may contain query gates (discussed in Section 1.3.1), auxiliary qubit gates (which take no input and produce a qubit in the $|0\rangle$ state), and garbage gates (which take an input and produce no output.) For an introduction to quantum circuits, see Watrous [Wat11].

We assume, for simplicity and without loss of generality, that our quantum circuits have the following form: we receive an $n$-bit input, which is then padded with $p(n)$ qubits in the $|0\rangle$ state for some fixed polynomial $p$, we apply a unitary—the unitary purification of this circuit—to these $n + p(n)$ qubits, and measure the first qubit in the computational basis and consider that to be the output. In cases where we expect an $m$-bit output, we measure the first $m$ qubits.

**Definition 1.1.** Define a $(m,s)$-classical layer to be an $m$-input, $s$-output, depth-1 classical circuit.

**Definition 1.2.** Define a $(m,s)$-quantum layer to be an $m$-input, $s$-output, depth-1 quantum circuit.

**Definition 1.3.** We say that two consecutive circuits are compatible if the number of outputs of the first circuit is greater than or equal to the number of inputs of the second circuit. It is assumed that the extra outputs of the first circuit are traced out.

**Definition 1.4.** Define a $(m,s,d)$-classical tier to be an $m$-input, $s$-output, depth-$d$ classical circuit. In other words, a $(m,s,d)$-classical tier consists of $d$ compatible classical layers composed with each other.

**Definition 1.5.** Define an $(m,s,d)$-quantum tier to be an $m$-input $m$-output depth-$d$ quantum circuit followed by a measurement in the computational basis. In other words, a $(m,s,d)$-quantum tier consists of $d$ compatible quantum layers composed with each other, followed by a measurement in the computational basis.

**Definition 1.6.** Define a $(n,\ell,c,q)$-hybrid-quantum circuit to be a composition of $\ell$ circuits

$$C_1 \circ C_2 \circ \cdots \circ C_\ell$$

such that the following hold.

1. $C_1$ is an $(n,n_1,c)$-classical tier.
2. $C_\ell$ has at least one output.
3. For odd $i > 1$, $C_i$ is an $(n_{i-1},n_i,c)$-classical tier.
4. For even $i$, $C_i$ is an $(n_{i-1},n_i,q)$-quantum tier.

We define the output of this circuit to be the first bit of the output of $C_\ell$. 


1.2.2 A Hierarchy of Hybrid Quantum Circuits

Informally, $JC^i$ is the class of problems solvable by polynomial-size classical circuits with embedded $O(\log^i(n))$-depth quantum circuits, and $JC$ is the class problems solvable by polynomial-size classical circuits with embedded polylog$(n)$-depth quantum circuits. This notation is analogous to $NC^i$ and $NC$.

**Definition 1.7.** $JC^i$ is the class of promise problems solvable by a uniform family of $(n, \text{poly}(n), \text{poly}(n), O(\log^i(n)))$-hybrid-quantum circuits with probability of error bounded by $1/3$.

**Definition 1.8.** $JC$ is the union of $JC^i$ over all nonnegative $i$; in symbols,

$$JC = \bigcup_{i \geq 0} JC^i.$$  

1.2.3 Jozsa's Conjecture, More Formally

With these definitions in place, we can state Jozsa’s conjecture as follows.

**Conjecture 1.9 (Jozsa [Joz06]).** It holds that $JC^1 = BQP$.

We can also define a weaker version of this conjecture as follows.

**Conjecture 1.10.** It holds that $JC = BQP$.

1.3 Relativized Jozsa’s Conjecture

1.3.1 Oracles in the Quantum World

For an introduction to oracles in the quantum circuit model, see Section III.4 in [Wat09]. We recap some definitions for setting notation.

For us, an *oracle* $A$ is a collection $\{A_n : n \in \mathbb{N}\}$ of functions

$$A_n : \{0, 1\}^n \to \{0, 1\}^n,$$  

(3)

to which queries can be made at unit cost. We define

$$A(x) := A_{|x|}(x)$$  

(4)

where $|x|$ denotes the length of $x$. We use the term *black box* to refer to the restriction of an oracle to inputs of a fixed length.

We represent oracle queries by an infinite family

$$\{K_n : n \in \mathbb{N}\}$$  

(5)

of gates, one for each query length. Each gate $K_n$ is a unitary gate acting on $n + 1$ qubits, defined on the computational basis as

$$K_n |x\rangle |a\rangle \mapsto |x\rangle |a \oplus A(x)\rangle$$  

(6)

where $x \in \{0, 1\}^n$, $a \in \{1, 0\}^n$, and $A$ is the oracle under consideration.

**Multiple-bit queries versus single-bit queries.** As mentioned at the end of Section III.4 in [Wat09], one can use the Bernstein-Vazirani algorithm [BV97] to simulate multiple-bit queries with single-bit queries (after adapting the definition of $A_n$ appropriately.) Moreover, this can be performed without any non-constant depth-overhead, so our model (after slight modifications) is equivalent to the traditional single-bit query model.
1.3.2 Relativized Jozsa’s Class

A relativized circuit is one that may include query gates, and we say that a circuit queries a certain oracle \( A \) if the outputs to its queries are consistent with the oracle \( A \). A circuit may be consistent with many oracles; for example, a circuit that makes no queries is consistent with every oracle. Later on, we will show that if the queries of a circuit are consistent with a certain oracle, then it has a very low probability of success.

We define \( \text{JC}^A \) by replacing “quantum circuits” in the definition with “relativized quantum circuits querying an oracle \( A \).” Put differently, we modify the gate set for the quantum circuits to include query gates to \( A \).

1.3.3 Relativized Jozsa’s Conjecture

Relativized Jozsa’s Conjecture states that Jozsa’s Conjecture is true relative to all oracles.

**Conjecture 1.11** (Relativized Jozsa). For all oracles \( A \), it holds that \( (\text{JC}^{1})^A = \text{BQP}^A \).

As before, let us also define the weak version of this conjecture.

**Conjecture 1.12.** For all oracles \( A \), it holds that \( \text{JC}^A = \text{BQP}^A \).

1.4 The Welded Tree Problem

In this subsection, we review the Welded Tree Problem of Childs et al. [CCD+03]. We introduce the class of graphs we will consider, show how to turn them into black-boxes, and finally define the black-box problem we will consider.

1.4.1 Welded Trees

![Diagram of a welded tree](image)

**Figure 1:** An illustration of \( T_3 \).

**Definition 1.13.** A \( n \)-welded tree \( T_n \) is a combination of two balanced binary trees \( L \) and \( R \) of height \( n \), with the \( 2^n \) leaves of \( L \) identified with the \( 2^n \) leaves of \( R \) in a way such that \( R \) is a mirror image of \( L \). For an illustration see Figure 1. The leftmost vertex is termed \text{ENTRY} \) and the rightmost vertex is termed \text{EXIT}.

1.4.2 Random Welded Trees

**Definition 1.14.** A \( 1 \)-random \( n \)-welded tree \( T_n^1 \) is a combination of two balanced binary trees \( L \) and \( R \) of height \( n \) by connecting the leaves via a random cycle of edges which alternates between the leaves of \( L \) and the leaves of \( R \). For an illustration, see Figure 2.

As with \( n \)-welded trees, we term the leftmost vertex \text{ENTRY} \) the rightmost vertex \text{EXIT}. Notice that the \text{ENTRY} \) and \text{EXIT} vertices are distinguished as they are the only vertices with degree 2.
1.4.3 Graphs with Black-Box Access

In this paper welded tree graphs are objects which our algorithm will only have access to via a black-box which it can query about the neighbors of a given vertex. To stay consistent with [CCD+03] we are also going to assume that the graphs are edge-coloured. For this problem, we can pick an 9-edge coloring that does not make the problem easier (i.e., preserves the output probability in expectation over colourings).

Following Childs et al. [CCD+03], we pick a colouring as follows.

Arbitrarily label the vertices in odd columns with colors \{1, 2, 3\} and arbitrarily label the vertices in even columns with colors \{A, B, C\}. Then there is an induced edge coloring as follows: an edge joining an \(X\)-coloured vertex to a \(Y\)-coloured vertex has color \(XY\). For example, an edge joining a 1-coloured vertex and an \(A\)-coloured vertex has color 1\(A\).

**Definition 1.15.** A \((n, \Xi)\)-black-box graph \(G\) is a \(\Xi\)-edge coloured graph with \(O(n)\) vertices whose vertices are uniquely encoded by bit strings of length \(2n\). We say that a \(2n\)-bit string is valid with respect to \(G\) if it is the label of a vertex in \(G\).

Notice that a graph may have many different corresponding black-box graphs. Moreover, since the graph only has \(O(2^n)\) vertices, \(n + O(1)\) bits are enough to give every vertex a unique label. But we chose \(2n\)-bit labels so that there are exponentially more labels than there are vertices. Later on, this fact is used to argue that it is hard for an adversary to guess a valid label.

In this paper, we will only consider a restricted class of black-box graphs, ones corresponding to 1-random \(n\)-welded trees with some additional structure.

**Definition 1.16.** A 1-random \(n\)-welded black-box tree \(T\) is a \((n, 9)\)-black-box graph with the following additional structure.

1. \(T\) is a 1-random \(n\)-welded tree.
2. The entrance vertex has the label 0\(\cdots\)0.
3. The label 1\(\cdots\)1 is not used for a valid vertex. We will henceforth refer to this string by the name INVALID.

We have defined black-boxes but we haven’t seen how to query them. Well, the wait is over, the time is now.

**Definition 1.17.** A query \(K_T\) to a 1-random \(n\)-welded black-box tree \(T\) black-box tree is defined as

\[
K_T(x, c) := \begin{cases} 
  \text{c-neighbour of } x, & \text{if } x \text{ is a valid vertex with a } c\text{-neighbour} \\
  \text{INVALID,} & \text{otherwise}
\end{cases}
\]
where \( x \in \{0,1\}^{2n} \), \( c \)-neighbour of \( x \) (the vertex joined to \( x \) by an edge with color \( c \)) is a \( 2n \)-bit string, and \( \text{INVALID} := 1 \cdots 1 \). We define this as a unitary as

\[
K_T |x\rangle |c\rangle |0^{2n}\rangle \mapsto |x\rangle |c\rangle |y\rangle
\]

where \( x \in \{0,1\}^{2n} \), \( c \in \{1,\ldots,9\} \), and \( y \in \{0,1\}^{2n} \) is the label of the \( c \)-neighbour of \( x \).

**Definition 1.18** (querying a welded tree). Let \( T \) be a 1-random \( n \)-welded tree, and let \( C(T') \) be a relativized circuit that queries a 1-random \( n \)-welded black-box tree \( T' \) corresponding to \( T \). We define

\[
P[C(T)] := \mathbb{P}[C(T') \text{ returns the label of EXIT}],
\]

where the probability is over all 1-random \( n \)-welded black-box trees \( T' \) corresponding to \( T \). Put differently, the probability is over all \( 2^n \)-bit labellings of the graph \( T \). With this notation in place, we can say, a circuit \( C(T) \) queries a 1-random \( n \)-welded tree \( T \) and it is understood that we take the output probability over all 1-random \( n \)-welded black-box trees \( T' \) corresponding to \( T \).

1.4.4 The Welded Tree Problem

Given a family

\[
T := \{T_n : n \in \mathbb{N}\}
\]

of 1-random welded black-box trees, where \( T_n \) is a 1-random \( n \)-welded black-box tree, we define the welded tree problem relative to \( T \) as follows.

**WELDED TREE PROBLEM**

**Input:** \( 0^n \) for some \( n \in \mathbb{N} \).

**Output:** The label of the EXIT vertex in \( T_n \).

**Search versus Decision** Since all our results hold even if we repeat the algorithm \( O(n) \) times, the above mentioned search variant is equivalent to the following decision variant of this problem.

**DECISION WELDED TREE PROBLEM**

**Input:** \( 0^n \) for some \( n \in \mathbb{N} \) and \( i \in \{1,\ldots,n\} \).

**Output:** \( i \)th bit of the label of the EXIT vertex in \( T_n \).

So, in the remainder of the paper, we restrict our attention to the search variant.

**Query Length Equals Input Length** We assume that given an \( n \)-bit string as input, a quantum algorithm only queries \( T_n \) and not \( T_m \) for any \( m \neq n \), this is without loss of generality. The idea is to replace a circuit \( Q \) with an new circuit \( R \) in which all queries to \( T_m \) for \( m \neq n \) are hardcoded to \( \text{INVALID} \). From the description of our problem, it is immediate that the success probability of \( Q \) is no greater than \( R \).

1.4.5 Quantum Algorithm for the Welded Tree Problem

Childs et al. [CCD+03] gave an efficient quantum algorithm for the WELDED TREE PROBLEM using quantum walks.

**Theorem 1.19** (Childs et al. [CCD+03]). Given a family

\[
T := \{T_n : n \in \mathbb{N}\}
\]

of 1-random welded black-box trees, where \( T_n \) is a 1-random \( n \)-welded black-box tree. There is a quantum algorithm for the WELDED TREE PROBLEM \( (T) \) which takes \( \text{poly}(n) \) time and outputs the correct answer (the label of the EXIT vertex) with probability greater than \( 2/3 \). Succinctly, WELDED TREE PROBLEM \( (T) \in \text{BQP}^T \).
1.4.6 Classical Lower Bound for the Welded Tree Problem

Childs et al. [CCD+03] also gave the first classical lower bound for the WELDED TREE PROBLEM, which we use as a key tool in our proof. To be precise, we will use the following version of the lower bound, due Fenner and Zhang [FZ03], who improved on the analysis.

**Theorem 1.20 ([CCD+03, FZ03]).** Given a family

\[ T := \{ T_n : n \in \mathbb{N} \} \]  

of 1-random welded black-box trees, where \( T_n \) is a 1-random \( n \)-welded black-box tree. For sufficiently large \( n \), any classical algorithm for the WELDED TREE PROBLEM(\( T \)) that makes at most \( 2n^{2/3} \) queries outputs the correct answer (the label of the EXIT vertex) with probability at most \( O(n^{-2n/3}) \).

1.5 Some Definitions and Assisting Results

**Distances Between States**

**Definition 1.21.** Given two quantum states

\[ |\psi\rangle := \sum_x \alpha_x |x\rangle \quad \text{and} \quad |\varphi\rangle := \sum_x \beta_x |x\rangle, \]

define the Euclidean distance (or 2-norm distance) between them as

\[ \| |\psi\rangle - |\varphi\rangle \|_2 := \left( \sum_x |\alpha_x - \beta_x|^2 \right)^{1/2}, \]

and the 1-norm distance between them as

\[ \| |\psi\rangle - |\varphi\rangle \|_1 := \sum_x |\alpha_x - \beta_x|. \]

**Definition 1.22.** Given two probability distributions \( P \) and \( Q \), define the 1-norm distance between them as

\[ \| P - Q \|_1 := \sum_x |P(x) - Q(x)|. \]

**Lemma 1.23** ([BV97], Lemma 3.6). Let \( |\psi\rangle \) and \( |\varphi\rangle \) be quantum states such that

\[ \| |\psi\rangle - |\varphi\rangle \|_2 \leq \epsilon, \]

for some \( \epsilon \in [0, 1] \), and let \( P \) and \( Q \) be the probability distributions resulting from measuring these states (in the same basis). Then it holds that

\[ \| P - Q \|_1 \leq 8\epsilon. \]

**Intermediate Quantum States**

We are going to define a set of quantum states corresponding to the cross-section of quantum tiers querying a 1-random \( n \)-welded tree.

**Definition 1.24** (state at depth \( \ell \)). Let \( Q(T) \) be a \((m, d)\)-quantum tier, with input state \( |\psi_0\rangle \), and querying a 1-random \( n \)-welded tree \( T \). We define the state at depth \( \ell \), denoted by \( |\psi_\ell\rangle \), to be the state produced by the first \( \ell \) consecutive layers of \( Q(T) \) acting on \( |\psi_0\rangle \).
2 The Case of $n^{1/7}$ Tiers

In this section, we will give a query lower bound for $(n, \eta, 4^d, d)$-hybrid quantum circuits solving the welded tree problem. But our lower bound leads to a separation against BQP only when $d \in O(poly \log(n))$ and $\eta \leq n^{1/7}$. In other words, this only allows us to separate “few-tier JC” (JC where the hybrid quantum circuits are restricted to have at most $\leq n^{1/7}$ tiers) from BQP.

Our proof has two parts. First, we show that any $(n, \eta, 4^d, d)$-hybrid quantum circuit can be simulated by a classical algorithm that makes at most $4^\eta d$ oracle queries. Second, we combine this result with the classical lower bound for the welded tree problem (Theorem 1.20) to obtain the query lower bound.

The first part of the proof is formalized in the following Theorem which is the technical heart of our proof.

**Theorem 2.1.** Let $C(T)$ be a $(n, \eta, 4^d, d)$-hybrid-quantum circuit that queries a $1$-random $n$-welded tree $T$. Then there exists a classical algorithm $A(T)$ making $4^\eta d$ queries (and running in $\exp(n)$ time) such that the output probabilities of $C(T)$ and $A(T)$ differ in $1$-norm error by

$$
\|P_C - P_A\|_1 \leq g(n) \cdot 4^{\eta + 2} d + 2 n^2 - 2^{2n},
$$

where the output probabilities of $C(T)$ and $A(T)$ are defined over all possible labellings of the tree $T$.

Using Theorem 2.1 we can prove the following adaptive quantum query lower bound via an appeal to the classical query lower bound for the WELDED TREE PROBLEM (Theorem 1.20).

**Theorem 2.2.** Let $T$ be a $1$-random $n$-welded black-box tree, and let $C(T)$ be a $(n, \eta, 4^d, d)$-hybrid-quantum circuit that queries a $1$-random $n$-welded tree $T$, such that

$$
2\eta d < n/3.
$$

Then, for sufficiently large $n$, $C(T)$ finds the EXIT with probability at most

$$
O(n2^{-n/3}).
$$

**Proof.** Using Theorem 2.1 we can replace $C(T')$ with the corresponding classical algorithm $A(T')$ that makes $2^{2\eta d}$ queries with an exponential loss in acceptance probability.

$$
2^{2\eta d} < 2^{n/3}.
$$

From the classical lower bound (Theorem 1.20), it follows that, for sufficiently large $n$, the algorithm succeeds—that is, finds the EXIT—with probability at most

$$
O(n2^{-n/3}),
$$

as desired.

**Corollary 2.3.** Given a family $T := \{T_n : n \in \mathbb{N}\}$

of $1$-random welded black-box trees, where $T_n$ is a $1$-random $n$-welded black-box tree. For sufficiently large $n$, no JC algorithm with at most $n^{1/7}$ tiers succeeds in deciding the WELDED TREE PROBLEM($T$) (where the probability is taken over all labellings.)

The remainder of this section is devoted to a proof of Theorem 2.1.
2.1 The Simulation Algorithm

In this subsection, we will prove the following theorem:

**Theorem 2.4.** Consider a \((n, \eta, 4^d, d)\)-hybrid-quantum circuit \(C(T)\). There exists a relativized classical algorithm \(A(T)\) which simulates \(C(T)\) with 1-norm error

\[
g(n) \cdot 4^{(\eta+2)d+2} \cdot \frac{2^{n+2} - 2}{2^{2n}},
\]

(where \(g(n)\) is the width of \(C(T)\)) while making at most

\[
4^{d\eta}
\]

classical queries to \(T\).

**Definition 2.5.** Given a \((n, \eta, 4^d, d)\)-hybrid-quantum circuit \(C(T)\), for \(\zeta \leq \eta\), let \(C_{\zeta}(T)\) be the \(\zeta\)th tier of \(C(T)\) (whether that tier be quantum or classical); for \(\zeta \leq \eta\), let \(C^{\zeta}(T)\) be the hybrid-quantum circuit corresponding to the first \(\zeta\) tiers of \(C(T)\).

2.1.1 Outline of Proof of Theorem 2.4

Given an \((n, \eta, 4^d, d)\)-hybrid-quantum circuit \(C(T)\) we define a simulation algorithm \(A(T)\) below. First we will need to recall a few definitions.

Let \(Q(T)\) be a quantum tier in \(C(T)\). For a particular input bitstring \(x\) to \(Q(T)\), recall that \(|\psi_{\ell}\rangle\) denotes the state at depth \(\ell \leq d\) of \(Q(T)\) as in Definition 1.24. We wish to prove that, for every such input \(x\), the output distribution of \(A(T)\) is close to the output distribution of \(C(T)\) in trace distance. Therefore, we fix an arbitrary input \(x\) at this point and will suppress the appearance of \(x\) in our notation for the remainder of the proof.

While it may be impossible to compute a classical description of \(|\psi_{d}\rangle\) using only polynomially many classical queries to \(T\), the intuition behind our classical simulation \(A(T)\) of \(C(T)\) will instead be, at each depth \(\ell \leq d\), to maintain a classical description of a different quantum state \(|\phi_{\ell}\rangle\) which will be a close approximation of the state \(|\psi_{\ell}\rangle\) in trace distance. The state \(|\phi_{\ell}\rangle\) will be defined inductively by the algorithm \(A(T)\) beginning with the initial condition \(|\phi_{0}\rangle := |\psi_{0}\rangle := |x\rangle\) and proceeding with the simple update rule that \(A(T)\) faithfully classically simulates (in exponential time) everything that \(Q(T)\) does in layer \(\ell\), except for the points at which \(Q(T)\) queries the black-box \(T\) at an input bitstring which is not among the "previously known vertices" (defined later), in which case \(A(T)\) refrains from querying \(T\) and simply assumes (without justification) that the output of that query will be INVALID. As we will see below, this strategy allows \(A(T)\) to maintain a close approximation \(|\phi_{\ell}\rangle\) of \(|\psi_{\ell}\rangle\) while only making a polynomial number of classical queries to \(T\).
2.1.2 The Algorithm

**Algorithm 1:** $i$th-level ClassicalSimulationWrapper: $A^i$

/* Simulates $C^i(T)$ by composing the individual simulations of each of the first $i$ tiers of $C(T)$. */

**Input**: Relativized circuit $C(T)$ and blackbox $T$

**Output**: Simulated output of $C^i(T)$, in register $\text{OUT}$; set $V_{\text{known}}$ of currently known vertices, in register $V_{\text{KNOWN}}$

/* initialization */

1. $V_{\text{known}} \leftarrow \text{empty dictionary}$;
2. Query the $\text{ENTRANCE}$ vertex to get output $S$;
3. Set $V_{\text{known}}(\text{ENTRANCE}) \leftarrow S$;
4. $x \leftarrow 0^n$;
5. for each $j \in \{0, \ldots, i\}$ do
6.   if $C_j$ is a quantum tier then
7.     $x, V_{\text{known}} \leftarrow \text{QuantumTierSimulator}(C_j, x, V_{\text{known}}, T)$
8.   else
9.     $x, V_{\text{known}} \leftarrow \text{ClassicalTierSimulator}(C_j, x, V_{\text{known}}, T)$
10. return $x, V_{\text{known}}$

**Definition 2.6.** A dictionary data structure is a set of key-value pairs indexed by keys. In other words, a dictionary $D$ has the form

$$D = \{(x_1, y_1), (x_2, y_2), \ldots\}. \quad (28)$$

We could also look at the dictionary as a mapping

$$D(x_i) := y_i, \quad (29)$$

for each $i$.

**Definition 2.7 ($V_{\text{known}}$).** The dictionary $V_{\text{known}}$ has keys $x_i$, which are 2-tuples $(v, c) \in \{0,1\}^n \times \{1, \ldots, 9\}$. We store in $V_{\text{known}}(v, c)$ the vertex label of the $c$-neighbour of $v$. By default, the value in $V_{\text{known}}(v, c)$ is INVALID.

Sometimes, abusing notation, we set $V_{\text{known}}(v)$ to the output of querying a vertex $v$ (like in Line 3 of Algorithm 1), by this we mean that we query $(v, c)$ for each $c$ to get the label of the $c$-neighbour of $v$ (which can be INVALID) and then set that to be $V_{\text{known}}(v, c)$.

**Definition 2.8.** Let $A^i(T)|_{\text{OUT}}$ denote a modification of the algorithm $A^i(T)$ which only outputs the value in the OUT register. In other words, $A^i(T)|_{\text{OUT}}$ returns only $x$, rather than $(x, V_{\text{known}})$.

Before giving the tier simulation subroutines, we need the following definition.

**Definition 2.9.** Let $L$ be a classical or quantum layer in a relativized circuit $Q(T)$. We can divide $L$ into two disjoint layers, one called $L^T$ which applies all of the black-box query gates in $L$ in parallel, and one called $L^G$, which applies every other gate in $L$ in parallel. Moreover, one can split $L$ into $L^T$ and $L^G$ in linear time.
Algorithm 2: SimOracle

**Input**: Dictionary $V_{\text{known}}$ of known vertices, and blackbox $T$, quantum layer $L^T$ solely composed of query gates

**Output**: An array $S(z)$, and dictionary $V_{\text{known}}$ of currently known vertices

1. Set $z_{\text{temp}} \leftarrow z$;
2. Set $V_{\text{temp}} \leftarrow V_{\text{known}}$;
3. For each bitstring $z$ which has length equal to the input register of $L^T$ do
   4. For each query gate $K$ in $L^T$ do
      5. Let $z_K$ be the substring of $z$ which lies in the input register of $K$;
      6. Let $z_{K,x}, z_{K,c}, z_{K,y}$ be the three disjoint substrings of $z_K$ corresponding to the $x$-register, $c$-register, and $y$-register (respectively) of the input to gate $K$, as defined in Equation (8);
      7. If $V_{\text{known}}(z_{K,x}, z_{K,c})$ exists then
         8. Compute $z_{\text{out}} \leftarrow K(z)$ without any queries to $T$, by starting with $z_{\text{temp}}$, and replacing the substring $z_{K,y}$ in $z$ with the substring $V_{\text{known}}(z_{K,x}, z_{K,c})$;
         9. Set $z_{\text{temp}} \leftarrow z_{\text{out}}$;
      else if $z_{K,x} == V_{\text{known}}(\alpha, \beta)$ for some $\alpha, \beta$ then
         10. Then, use one classical query to $T$ to set $V_{\text{temp}}(z_{K,x}, z_{K,c}) \leftarrow T(z_{K,x}, z_{K,c})$;
          11. Compute $z_{\text{out}} \leftarrow K(z)$ by starting with $z_{\text{temp}}$, and replacing the substring $z_{K,y}$ in $z$ with the substring $V_{\text{known}}(z_{K,x}, z_{K,c})$;
          12. Set $z_{\text{temp}} \leftarrow z_{\text{out}}$;
      else
         13. Compute $z_{\text{out}} \leftarrow K(z)$ without any queries to $T$, by starting with $z_{\text{temp}}$, and replacing the substring $z_{K,y}$ in $z$ with the substring $\text{INVALID}$;
         14. Set $z_{\text{temp}} \leftarrow z_{\text{out}}$;
   15. Set $S(z) \leftarrow z_{\text{temp}}$;
   16. Set $V_{\text{known}} \leftarrow V_{\text{temp}}$;
4. Return $S, V_{\text{known}}$;

Algorithm 3: ClassicalTierSimulator

**Input**: Relativized circuit $C(T)$, input $x$, set $V_{\text{known}}$ of known vertices, and blackbox $T$

**Output**: Simulated output of $C(T)$ on input $x$, in register $\text{OUT}$, and set $V_{\text{known}}$ of currently known vertices, in register $V_{\text{KNOWN}}$

1. Initialize $w_0 \leftarrow x$;
2. Let $\eta$ be the number of layers in $C(T)$;
3. Let $L_i$ be the $i$th layer in $C(T)$;
4. For each $i \in \{1, \ldots, \eta\}$ do
   5. Factorize $L_i$ into a query layer $L_i^T$ and a non-query layer $L_i^G$;
   6. Compute $u_i \leftarrow L_i^G w_{i-1}$ without any queries;
   7. Compute $w_i, V_{\text{known}} \leftarrow \text{SimOracle}(u_i, V_{\text{known}}, T, L_i^T)$;
4. Return $w_\eta, V_{\text{known}}$
Lemma 2.10. Consider a \((n, \eta, 4^d, d)\)-hybrid-quantum circuit \(C(T)\). For all \(k \leq \eta\), it holds that

\[
\|A^k(T)|_{\text{out}} - C^k(T)\|_1 \leq g(n) \cdot 4^{(k+2)d+2} \cdot \frac{2^{n+2} - 2}{2^{2n}},
\]

(30)

where \(g(n)\) is the width of \(C^k(T)\).

Proof. The proof proceeds by induction. We will begin with the base case \(k = 0\), and the statement for each larger \(k\) will be proven assuming the statement for \(k - 1\).

**Base Case:** Suppose \(k = 0\), then it is immediate that Equation (30) holds.

**Inductive Case:** For \(1 \leq k < \eta\), if we know that Equation (30) holds, then Lemma 2.11 implies that

\[
\|A^k(T)|_{\text{out}} - C^k(T)\|_1 \leq g(n) \cdot 4^{(k+2)d+2} \cdot \frac{2^{n+2} - 2}{2^{2n}} + g(n) \cdot 4^{(k+1)d+2} \cdot \frac{2^{n+2} - 2}{2^{2n}}
\]

(31)

\[
= (4^{(k+2)d+2} + 4^{(k+1)d+2})g(n) \cdot \frac{2^{n+2} - 2}{2^{2n}}
\]

(32)

\[
\leq (2 \cdot 4^{(k+2)d+2})g(n) \cdot \frac{2^{n+2} - 2}{2^{2n}}
\]

(33)

\[
\leq 4^{(k+1)+2d+2} \cdot g(n) \cdot \frac{2^{n+2} - 2}{2^{2n}},
\]

(34)

so Equation (30) also holds for \(k + 1\). This completes the inductive step.

Therefore, by induction, the lemma follows. \(\square\)

Lemma 2.11. Given a \((n, \eta, 4^d, d)\)-hybrid-quantum circuit \(C(T)\) with a hardcoded input (which is \(0^n\)). For all \(i < \eta\), if

\[
\|A^i(T)|_{\text{out}} - C^i(T)\|_1 \leq B
\]

(35)
for some bound $B$, then

$$\left\|A^{i+1}(T)_{\text{out}} - C^{i+1}(T)\right\|_1 \leq B + g(n) \cdot 4^{(i+1)d+2} \cdot \frac{2^{n+2} - 2}{2^{2n}},$$

(36)

where $g(n)$ is the width of $C_{i+1}$.

**Proof.** Define the random variable $x := A^i(T)_{\text{out}}$. By assumption, we have that

$$\left\|x - C^i(T)\right\|_1 = \left\|A^i(T)_{\text{out}} - C^i(T)\right\|_1 \leq B$$

(37)

Using the fact that, by definition, $C^{i+1}(T) = C_{i+1}(C^i(T))$ and applying the triangle inequality we get

$$I := \left\|A^{i+1}(T)_{\text{out}} - C^{i+1}(T)\right\|_1 = \left\|A^{i+1}(T)_{\text{out}} - C_{i+1}(C^i(T))\right\|_1 \leq \left\|A^{i+1}(T)_{\text{out}} - C_{i+1}(x)\right\|_1 + \left\|C_{i+1}(x) - C_{i+1}(C^i(T))\right\|_1.$$  

(38)

(39)

Since we can interpret $C_{i+1}$ as a quantum channel, and the trace norm is nonincreasing under quantum channels (see Theorem 9.2 in Nielsen and Chuang [NC10]) we get the bound

$$I \leq \left\|A^{i+1}(T)_{\text{out}} - C_{i+1}(x)\right\|_1 + \left\|x - C^i(T)\right\|_1.$$  

(40)

Plugging in Equation (37) we get

$$I \leq \left\|A^{i+1}(T)_{\text{out}} - C_{i+1}(x)\right\|_1 + B.$$  

(41)

We now have two cases.

**Case 1.** $C_{i+1}$ is a classical tier. Then $A^{i+1}(T) = \text{ClassicalTierSimulator}(C_{i+1}, A^i(T), T)$, we get

$$I \leq \left\|\text{ClassicalTierSimulator}(C_{i+1}, A^i(T), T)_{\text{out}} - C_{i+1}(x)\right\|_1 + B.$$  

(42)

Applying Lemma 2.12 (recall that, by assumption, the maximum depth of a classical circuit is $4^d$), we get

$$I \leq g(n) \cdot 4^d |V_{\text{known}}| \cdot \frac{2^{n+2} - 2}{2^{2n}} + B,$$

(43)

using Lemma 2.14 we obtain

$$I \leq g(n) \cdot 4^d \cdot 4^d \cdot \frac{2^{n+2} - 2}{2^{2n}} + B$$

(44)

$$\leq g(n) \cdot 4^{(i+1)d+2} \cdot \frac{2^{n+2} - 2}{2^{2n}} + B,$$

(45)

as desired.

**Case 2.** $C_{i+1}$ is a quantum tier. Then $A^{i+1}(T) = \text{QuantumTierSimulator}(C_{i+1}, A^i(T), T)$, we get

$$I \leq \left\|\text{QuantumTierSimulator}(C_{i+1}, A^i(T), T)_{\text{out}} - C_{i+1}(x)\right\|_1 + B.$$  

(46)

Applying Lemma 2.13, we get

$$I \leq 4g(n) \cdot 4^{d+1} |V_{\text{known}}| \cdot \frac{2^{n+2} - 2}{2^{2n}} + B,$$

(47)

$$\leq g(n) \cdot 4^{d+2} |V_{\text{known}}| \cdot \frac{2^{n+2} - 2}{2^{2n}} + B,$$

(48)
using Lemma 2.14 we obtain

\[ I \leq g(n) \cdot 4^{d+2} \cdot 4^d \cdot \frac{2^{n+2} - 2}{2^{2n}} + B \]

(49)

\[ \leq g(n) \cdot 4^{(i+1)d+2} \cdot \frac{2^{n+2} - 2}{2^{2n}} + B, \]

(50)
as desired.

\[ \square \]

**Lemma 2.12.** Let \((x, V\text{\_known})\) be the random variable produced by \(A^i(T)\). Let \(d\) is the depth of \(C_{i+1}\), \(g(n)\) the width of \(C_{i+1}\) and \(|V\text{\_known}|\) is the number of vertices in \(V\text{\_known}\). Then it holds that

\[ \|\text{ClassicalTierSimulator}(C_{i+1}, x, V\text{\_known}, T)\|_{\text{out}} - C_{i+1}(x)\|_1 \leq g(n) \cdot d|V\text{\_known}| \cdot \frac{2^{n+2} - 2}{2^{2n}}. \]

(51)

**Proof.** Notice that the output of \(C_{i+1}\) can be simulated by a classical algorithm that makes at most

\[ g(n) \cdot d|V\text{\_known}| \]

queries. Therefore, from the proof of Lemma 4 in [CCD⁺03] it follows that

\[ \|\text{ClassicalTierSimulator}(C_{i+1}, x, V\text{\_known}, T)\|_{\text{out}} - C_{i+1}(x)\|_1 \leq g(n) \cdot d|V\text{\_known}| \cdot \frac{2^{n+2} - 2}{2^{2n}}. \]

(53)

\[ \square \]

**Lemma 2.13.** Let \((x, V\text{\_known})\) be the random variable produced by \(A^\zeta(T)\), for some \(\zeta \in \{1, \ldots, \eta\}\). Say \(d\) is the depth of \(C_{\zeta+1}\), \(g(n)\) is the width of \(C_{\zeta+1}\), and \(|V\text{\_known}|\) is the number of vertices in \(V\text{\_known}\). Further, assume that

\[ g(n)4^d|V\text{\_known}| = \text{poly}(n). \]

(54)

Then the following statements hold.

1. It holds that

\[ \|\text{QuantumTierSimulator}(C_{\zeta+1}, x, V\text{\_known}, T)\|_{\text{out}} - C_{\zeta+1}(x)\|_1 \leq 4g(n) \cdot 4^{d+1}|V\text{\_known}| \cdot \frac{2^{n+2} - 2}{2^{2n}}. \]

(55)

2. \(\text{QuantumTierSimulator}(C_{\zeta+1}, x, V\text{\_known}, T)\) makes at most \(4^d \cdot |V\text{\_known}|\) queries to \(T\).

**Proof.** Let’s denote by \(\xi[\ell]\) the intermediate quantum state produced at the end of the first \(\ell\) layers of a circuit \(\xi\). For \(\ell \in \{1, \ldots, \eta\}\), let \(|\psi_\ell\rangle\) be the state defined in Line 10 of the QuantumTierSimulator subroutine. Let \(V^\ell\text{\_known}\) denote the known set of vertices at the point immediately before computing \(|\psi_\ell\rangle\), as defined in Line 9 of the QuantumTierSimulator subroutine. So, \(V^0\text{\_known}\) is identical to the set \(V\text{\_known}\) which is input to the QuantumTierSimulator subroutine.

First, we will show, by induction, that at every depth \(\ell \leq d\), it holds that

\[ |V^\ell\text{\_known}| \leq 4^\ell |V\text{\_known}| \]

(56)

and

\[ \|\psi_\ell - C_{i+1}[\ell]\|_1 \leq 4g(n) \cdot \left( \sum_{i=0}^{\ell} |V^i\text{\_known}| \right) \cdot \frac{2^{n+2} - 2}{2^{2n}}. \]

(57)

Combining (57) with (56), and using the sum of geometric series we get part 1 of the lemma.

**Base Case.** Suppose \(\ell = 0\), then it is immediate that Equation (57) holds because, by definition, \(|\psi_0\rangle = C_{i+1}[0] = |x\rangle\), and Equation (56) holds because, by definition, \(|V^0\text{\_known}| = 4^0 |V^0\text{\_known}|\).
Inductive Case. Suppose that, for a given $\ell \geq 0$, Equations (56) and (57) hold. We will now prove that it must also hold for $\ell + 1$.

We will first prove Equation 56 for $\ell + 1$. Recall that vertices in a random welded tree have degree at most 3. Therefore, since the $\text{SimOracle}(V^\ell_{\text{known}}, T, L^T)$ subroutine queries, at most, every vertex in $V^\ell_{\text{known}}$, we know that the new set $V^{\ell+1}_{\text{known}}$ of known vertices has at most $3|V^\ell_{\text{known}}|$ new vertices, plus the original $|V^\ell_{\text{known}}|$ vertices that were already contained in $V^\ell_{\text{known}}$ itself (since $V^\ell_{\text{known}} \subseteq V^{\ell+1}_{\text{known}}$ by definition). Thus it holds that

$$|V^{\ell+1}_{\text{known}}| \leq 4|V^\ell_{\text{known}}|, \quad (58)$$

and applying the induction hypothesis (Equation 56), we get

$$|V^{\ell+1}_{\text{known}}| \leq 4^{\ell+1}|V_{\text{known}}|. \quad (59)$$

This proves Equation 56 for $\ell + 1$.

We will now prove Equation (57) for $\ell + 1$. As defined in Line 7 of the QuantumTierSimulator subroutine, we have that $|\phi_{\ell+1}| = L^G_{\ell+1} |\psi_\ell\rangle$. Recall that we can compute a classical description of $|\phi_{\ell+1}\rangle$ in exponential time, and without using any queries to $T$. This is because we have a classical description of $|\psi_\ell\rangle$ from the previous iteration of the QuantumTierSimulator subroutine, and the layer $L^G_{\ell+1}$ only applies standard quantum gates and no black-box queries to $T$.

The more complicated step in the QuantumTierSimulator subroutine is to compute $|\psi_{\ell+1}\rangle$, which, according to Lines 9–10, involves executing the command

$$S, V^{\ell+1}_{\text{known}} \leftarrow \text{SimOracle}(V^\ell_{\text{known}}, T, L^T_{\ell+1}), \quad (60)$$

and then setting $|\psi_{\ell+1}\rangle \leftarrow \sum_z c_z |S(z)\rangle$. Define the set

$$\text{Outliers} := \{ z : |S(z)\rangle \neq L^T_{\ell+1} |z\rangle \}. \quad (61)$$

and notice that if $|S(z)\rangle \neq L^T_{\ell+1} |z\rangle$ then they are unequal classical basis states and, therefore, perpendicular. Let us decompose $|\phi_{\ell+1}\rangle = \sum_z c_z |z\rangle$. Making use of (61) we can restate the fidelity between the simulated state and the true state as

$$\text{F}(|\psi_{\ell+1}\rangle, L^T_{\ell+1} |\phi_{\ell+1}\rangle) = \langle \psi_{\ell+1} | L^T_{\ell+1} |\phi_{\ell+1}\rangle$$

$$= \sum_z |c_z|^2 \langle S(z) | L^T_{\ell+1} |z\rangle$$

$$= \sum_{z \notin \text{Outliers}} |c_z|^2$$

$$= 1 - \sum_{z \in \text{Outliers}} |c_z|^2 \quad (65)$$

Thus, by applying the Fuchs-van de Graaf inequalities [FvdG99] we get

$$\left\| |\psi_{\ell+1}\rangle - L^T_{\ell+1} |\phi_{\ell+1}\rangle \right\|_1 \leq 2 \sqrt{\text{F}(|\psi_{\ell+1}\rangle, L^T_{\ell+1} |\phi_{\ell+1}\rangle)}$$

$$\leq 2 \sqrt{\sum_{z \notin \text{Outliers}} |c_z|^2}. \quad (66)$$

We will now use Equation (67) to argue that $|\psi_{\ell+1}\rangle$, and $L^T_{\ell+1} |\phi_{\ell+1}\rangle$ must be very close to the same state.

Consider, as a thought experiment, a classical algorithm $\mathcal{B}$ which begins with the classical description of $|\phi_{\ell+1}\rangle = \sum_z c_z |z\rangle$ and attempts to guess a valid vertex which is not contained in $V_{\text{known}}$ by first sampling a random $z$ with probability $|c_z|^2$, and then randomly choosing a query gate $K$ in $L^T_{\ell+1}$ and returning the substring of $z$ which lies in the input register of $K$.

By the definition of the subroutine SimOracle, the set Outliers contains those $z$ which, in the input register of at least one query gate $K$ in $L^T_{\ell+1}$, have a substring which is the label of a valid vertex outside of
Thus, if \( B \) successfully guesses a \( z \in \text{Outliers} \), which happens with probability \( \sum_{z \in \text{Outliers}} |c_z|^2 \), and further happens to guess the correct \( K \) in \( L^\ell_{\ell+1} \), which happens with probability at least \( 1/g(n) \) (recall that \( g(n) \) is the entire width of \( C^{\ell+1} \)) then \( B \) has successfully guessed the substring of \( z \) in the input of register to \( K \), which is a valid vertex that is not contained in \( V^\ell_{\text{known}} \). Therefore, the success probability of \( B \) is at least

\[
\frac{1}{g(n)} \cdot \sum_{z \in \text{Outliers}} |c_z|^2.
\]  

However, the algorithm \( B \) only uses at most \( |V^\ell_{\text{known}}| \) classical queries to \( T \) because it only uses a classical description of \( |\Phi^\ell+1 \rangle \) (which can be computed with \( |V^\ell_{\text{known}}| \) queries by definition). Thus, it follows from the proof of Lemma 4 of \([\text{CCD}^+03] \) that

\[
\frac{1}{g(n)} \sum_{z \in \text{Outliers}} |c_z|^2 \leq \Pr[\text{B succeeds}] \leq |V^\ell_{\text{known}}| \cdot \frac{2^{n+2} - 2}{2^{2n}} \]  

Combining Equation (69) with Equation (67) gives

\[
\left\| |\psi_{\ell+1}\rangle - L^T_{\ell+1}L^G_{\ell+1} |\psi_{\ell}\rangle \right\|_1 = \left\| |\psi_{\ell+1}\rangle - L^T_{\ell+1} |\psi_{\ell+1}\rangle \right\|_1 \leq 4\sqrt{g(n)} |V^\ell_{\text{known}}| \cdot \frac{2^{n+2} - 2}{2^{2n}} \]

But we assumed that \( g(n)4^4|V_{\text{known}}| = \text{poly}(n) \) (Equation (54)) and we proved that \( |V^\ell_{\text{known}}| \leq 4^{\ell+1}|V_{\text{known}}| \) (Equation (59)) (Recall that by assumption \( \ell + 1 \leq d \).) And, by definition, \( \frac{2^{n+2} - 2}{2^{2n}} = 1/\exp(n) \) so it holds that \( g(n)|V^\ell_{\text{known}}| \cdot \frac{2^{n+2} - 2}{2^{2n}} \leq 1 \) for sufficiently large \( n \). Therefore,

\[
\left\| |\psi_{\ell+1}\rangle - L^T_{\ell+1}L^G_{\ell+1} |\psi_{\ell}\rangle \right\|_1 \leq 4g(n) |V^\ell_{\text{known}}| \cdot \frac{2^{n+2} - 2}{2^{2n}}. \]

Combining Equation (72) with the inductive assumption (Equation (57)) and using the fact that the trace norm is nonincreasing under quantum channels we get

\[
\left\| |\psi_{\ell+1}\rangle - C_{\ell+1}|\ell+1| \right\|_1 \leq \left\| |\psi_{\ell+1}\rangle - L^T_{\ell+1}L^G_{\ell+1} |\psi_{\ell}\rangle \right\|_1 + \left\| L^T_{\ell+1}L^G_{\ell+1} |\psi_{\ell}\rangle - L^T_{\ell+1}L^G_{\ell+1} C_{\ell+1}|\ell| \right\|_1 \]

\[
\leq \left\| |\psi_{\ell+1}\rangle - L^T_{\ell+1}L^G_{\ell+1} |\psi_{\ell}\rangle \right\|_1 + \left\| |\psi_{\ell}\rangle - C_{\ell+1}|\ell| \right\|_1 \]

\[
\leq 4g(n) |V^\ell_{\text{known}}| \cdot \frac{2^{n+2} - 2}{2^{2n}} + 4g(n) \cdot \left( \sum_{i=0}^{\ell} |V^i_{\text{known}}| \right) \frac{2^{n+2} - 2}{2^{2n}} \]

\[
\leq 4g(n) \cdot \left( \sum_{i=0}^{\ell+1} |V^i_{\text{known}}| \right) \frac{2^{n+2} - 2}{2^{2n}} \]

This proves that Equation (57) holds for \( \ell + 1 \). This completes part 1 of the lemma.

To prove part 2 of the lemma note that all of the queries to \( T \) made by \text{QuantumTierSimulator}(C_{\zeta+1}, x, V_{\text{known}}, T) \) are contained in \( V^d_{\text{known}} \). Since we have just proved Equation (56) for all \( \ell \leq d \), it follows that the number of queries made by \text{QuantumTierSimulator}(C_{\zeta+1}, x, V_{\text{known}}, T) \) is at most \( |V^d_{\text{known}}| \leq 4^d|V_{\text{known}}| \).

\[
\square
\]

**Lemma 2.14.** Let \((x, V_{\text{known}})\) be the random variable produced by \( A^\zeta \), for some \( \zeta \in \{1, \ldots, \eta\} \). Say \( d \) is the depth of \( C_{\zeta+1} \), \( g(n) \) is the width of \( C_{\zeta+1} \), and \( |V_{\text{known}}| \) is the number of vertices in \( V_{\text{known}} \). Then it holds that

\[
|V_{\text{known}}| \leq 4^{d\zeta}. \]  

16
Proof. We will prove this by induction on \( \zeta \in \{1, \ldots, \eta\} \). The base case is immediate. Induction step follows from Equation (56) which we proved in the preceding lemma.

Lemma 2.15. \( A^\zeta(T) \), for some \( \zeta \in \{1, \ldots, \eta\} \), makes at most

\[ 4^{d\zeta} \]

queries, where \( d \) is the maximum depth of the quantum tiers and \( 4^d \) is the maximum depth of the classical tiers.

Proof. Immediate from Lemma 2.14. \( \square \)

3 The Information Bottleneck, and the Case of poly(\( n \)) Tiers.

The reason that Algorithm 1 in Section 2 is sufficient to prove Theorem 2.1 with \( n^{1/7} \) quantum tiers (interleaved with the same number of classical tiers of arbitrary polynomial depth), is because the set \( V_{\text{known}} \) grows only by a multiple of \( n \) for each quantum tier encountered. This means that, when simulating a circuit with only \( n^{1/7} \) quantum tiers, the set \( V_{\text{known}} \) only has subexponential size by the end of Algorithm 1 and the Algorithm is, therefore, still subject to the classical lower bound Theorem 1.20. However, this, while encouraging, is not sufficient to prove that the Welded Tree Problem is not solvable in JC, because JC includes circuits in which the number of tiers is some arbitrary polynomial \( \text{poly}(n) \) in the input size \( n \).

To address this more general setting we now employ a new idea which we will refer to informally as the “Information Bottleneck”. The intuition is that, while a circuit in JC may have \( \text{poly}(n) \) tiers, its width is also bounded by some polynomial \( g(n) \). Therefore, while Algorithm 1 tracks a set \( V_{\text{known}} \) of known vertices that grows exponentially large as it increases through \( \text{poly}(n) \) tiers, it seems intuitive that after the end of each quantum tier, only \( g(n) \) (or \( \text{poly}(n, g(n)) \), say) of those known vertices should “actually matter” to the JC circuit being simulated. This is because the width of the circuit bounds the amount of classical information that can be passed from one tier of the circuit to the next. Note that the information passed between tiers is necessarily a classical bit string by definition. In this section we will make this intuition more formal and use it to prove an oracle separation between JC and BQP.

We begin by noting that, in the setting of \( \text{poly}(n) \) tiers, we can assume WLOG that every tier is a quantum tier because a classical tier of polynomial depth can always be implemented as the composition of polynomially many quantum tiers of logarithmic depth. For simplicity of notation we will take this interpretation WLOG for the remainder of this section. So, we will use the following modification of Definition 1.6.

Definition 3.1. Define a \((n, \ell, q, g)\)-hybrid-quantum circuit to be a composition of \( \ell \) circuits

\[ C_1 \circ C_2 \circ \cdots \circ C_\ell \]

such that the following hold.

1. \( C_1 \) is an \((n, n_1, q)\)-quantum tier.
2. \( C_\ell \) has at least one output.
3. For all \( i \), \( C_i \) is an \((n_{i-1}, n_i, q)\)-quantum tier.
4. Every tier \( C_i \) has width at most \( g \).

The first step in proving the oracle separation is to augment our previous algorithm for classically simulating a relativized JC circuit by adding a subroutine called Bottleneck which limits the growth of the set \( V_{\text{known}} \) to be polynomial in the number of tiers rather than exponential. The challenge is to do this while also preserving the properties of \( V_{\text{known}} \) required for the rest of the algorithm to work.
3.1 The New Algorithm

Given a \((n, \ell(n), q(n), g(n))\)-hybrid-quantum circuit \(C(\cdot)\) where \(\ell(n) = \text{poly}(n)\), \(q(n) = \text{polylog}(n)\), and \(g(n) = \text{poly}(n)\), the outer loop for the algorithm for simulating \(C(T)\) is as follows:

**Algorithm 5:** \(i\)-th-level ClassicalSimulationWrapper2 with bottleneck: \(M^i\)

```plaintext
/* Simulates \(C^i(T)\) by composing the individual simulations of each of the first \(i\) tiers of \(C(T)\). */

**Input:** Relativized circuit \(C(T)\) and blackbox \(T\)

**Output:** Simulated output of \(C^i(T)\), in register \(\text{OUT}\); set \(V_{\text{known}}\) of currently known vertices, in register \(V_{\text{KNOWN}}\)

1. if \(i == 0\) then
2. \(V_{\text{known}} \leftarrow \text{empty dictionary};\)
3. \(V_{\text{hist}}^{\text{known}} \leftarrow \text{empty dictionary} \)
4. Query the \(\text{ENTRANCE}\) vertex to get output \(S;\)
5. Set \(V_{\text{known}}(\text{ENTRANCE}) \leftarrow S;\)
6. \(x \leftarrow 0^n;\)
7. if \(i > 0\) then
8. \(y, V_{\text{known}}^{\text{init}}, V_{\text{hist}}^{\text{known}} \leftarrow M^{i-1}(C(T), T);\)
9. \(x, V_{\text{known}}^{\text{final}}, V_{\text{hist}}^{\text{known}} \leftarrow \text{BottleneckQuantumTierSimulator}(C_i, i - 1, y, V_{\text{known}}^{\text{init}}, V_{\text{hist}}^{\text{known}}, T);\)
10. return \(x, V_{\text{known}}, V_{\text{hist}}^{\text{known}}\)
```

For \(a \in \{1, 2, 3\}\) we will let \(M^i(C(T), T)[a]\) denote the \(a\)th element of the tuple \(x, V_{\text{known}}, V_{\text{hist}}^{\text{known}}\) output by \(M^i(C(T), T)\).

The new subroutine BottleneckQuantumTierSimulator is defined below. It works in a very similar manner to the earlier defined QuantumTierSimulator except that it uses another new subroutine Bottleneck between each simulated quantum layer in order to prevent the set of explored vertices from growing too large. One similarity between the two simulators is that the only randomness used in either of them (or in all of \(M^i\)) is when they sample from the computational basis elements \(|z\rangle\) of a quantum state \(|\psi\rangle = \sum_z c_z |z\rangle\) according to the probability distribution given by \(|c_z|^2\). Since the number of bits in \(|z\rangle\) is at most \(g(n)\), this process can be done with \(10g(n)\) random bits. Since this sampling process is only done once per quantum layer, of which there are only \(q(n)\) per quantum tier, it follows that only \(10\ell(n)q(n)g(n)\) random bits are required for the entire algorithm \(M^{i(n)}\). For concreteness in the rest of the argument we name this seed randomness \(r\) (\(|r| \leq 10\ell(n)q(n)g(n)\)), and we consider \(M^i\) to be a deterministic algorithm which is a function of \(r\). When necessary we will use the notation \(M^i_r\) to highlight this, although we will omit the \(r\) when it is not relevant.
Algorithm 6: Bottleneck-QuantumTierSimulator

**Input**: Relativized tier \(Q(T)\), number \(j\) of the current tier, input \(x\), dictionary \(V_{\text{known}}\) of currently known vertices, dictionary \(V_{\text{hist}}\) of all vertices ever encountered, and blackbox \(T\)

**Output**: Simulated output of \(Q(T)\) on input \(x\), in register \(\text{OUT}\), and dictionary \(V_{\text{known}}\) of currently known vertices, in register \(V_{\text{KNOWN}}\)

1. Initialize \(|\psi_0\rangle \leftarrow |x\rangle\);
2. Initialize \(V_{\text{known}}^0 \leftarrow \text{Bottleneck}(j-1, x, V_{\text{hist}}^\emptyset)\);
   /* Here \(\emptyset\) represents the empty dictionary */
3. Let \(d\) be the number of layers in \(Q(T)\);
4. Let \(L_\ell\) be the \(\ell\)th layer in \(Q(T)\);
5. for each \(\ell \in \{1, \ldots, d\}\) do
6.   factorize \(L_\ell\) into a query layer \(L_\ell^T\) and a non-query layer \(L_\ell^C\);
7.   Compute \(|\phi_\ell\rangle \leftarrow L_\ell^C |\psi_{\ell-1}\rangle\) (in exponential time) without any queries;
8.   Expand \(|\phi_\ell\rangle\) in the classical basis as \(|\phi_\ell\rangle = \sum_z c_z |z\rangle\);
9.   Compute \(S, V_{\text{temp}}^\ell \leftarrow \text{SimOracle}(V_{\text{known}}^{\ell-1}, T, L_\ell^T)\);
10. Define \(|\psi_\ell\rangle \leftarrow \sum_z c_z |S(z)\rangle\);
    /* Note that \(|S(z)\rangle\) is a bitstring for all \(z\), by definition of SimOracle. */
11. Update \(V_{\text{hist}}^\ell \leftarrow \text{Merge}(V_{\text{hist}}^{\ell-1}, V_{\text{known}}^\ell, V_{\text{temp}}^\ell)\);
12. Set \(V_{\text{known}} \leftarrow V_{\text{known}}^\ell\);
13. Update \(V_{\text{hist}}^\ell \leftarrow \text{Merge}(V_{\text{hist}}^\ell, V_{\text{known}}^\ell)\);
14. Let \(x\) be the output of a classical basis measurement on \(|\psi_\eta\rangle\);
15. return \(x, V_{\text{known}}, V_{\text{hist}}\)

**Definition 3.2.** For any dictionary \(V\), and bitstring \(s \in \{0, 1\}^{g(n)}\), let
   \[
   \mathcal{T}_V^{s,r} := \{\text{blackboxes } P \text{ such that } s = M_s^r(C(P), P)[1] \text{ and } P \text{ is consistent with } V\}.
   \] (80)

When we say \(P\) is consistent with \(V\), we mean that the black-box \(P\) and the dictionary \(V\) agree on all the labels, colors, and adjacencies specified by \(V\). Note that \(\mathcal{T}_V^{s,r}\) is a well defined set because \(M_s^r(C(P), P)[1]\) is a deterministic function of \(P\) (for the fixed random seed \(r\)).

Let,
   \[
   \mathcal{T}_V := \{\text{blackboxes } P \text{ such that } P \text{ is consistent with } V\}.
   \] (81)

Let \(\mathcal{T} = \cup_s \mathcal{T}_V^{s,r}\) denote the set of all blackboxes \(P\) for the Welded Tree \(T\). (Note that every blackbox \(P\) is consistent with the empty dictionary \(V = \emptyset\).)

**Definition 3.3.** For a set of blackboxes \(\mathcal{S}\) we let
   \[
   \mathbb{P}_{P \in \mathcal{S}}[b \text{ is a valid label in } P]
   \] denote the probability that \(b\) is a valid label in \(P\) when \(P\) is selected uniformly at random from the set \(\mathcal{S}\).
Algorithm 7: Bottleneck

Input: Index $i$ of current quantum tier, bit string $x$, dictionaries $V^\text{current}_{\text{known}} \subseteq V^\text{hist}_{\text{known}}$ of initially known and finally known vertices (respectively)

Output: Dictionary $V_{\text{known}}$ of "effectively known" vertices, satisfying $V^\text{current}_{\text{known}} \subseteq V_{\text{known}} \subseteq V^\text{hist}_{\text{known}}$

1. if $|\mathcal{T}^i_{\text{Vcurrent},x,r}| < 2^{-n(g(n)+|r|)}|\mathcal{T}^i_{\text{Vknown}}|$ then
2. ABORT and guess a random label for the EXIT vertex of the entire welded tree problem on $T$.
3. Initialize $V_{\text{known}} = \text{V}_{\text{current}}$.
4. Our goal is to build $V_{\text{known}}$ into a set satisfying $V_{\text{known}} \subseteq V_{\text{known}} \subseteq V^\text{hist}_{\text{known}}$ and:
   \[ \forall b \in \{0,1\}^{2n} \text{ such that } b \text{ does not appear in } V_{\text{known}}: \]
   \[ \mathbb{P}_{P \in \mathcal{T}_{V_{\text{known}}}^i} \left[ b \text{ is a valid label in } P \right] \leq 2^{-n/100} \] (82)
   \[ \text{/* Note that one can compute the set } \mathcal{T}_{V_{\text{known}}}^i \text{ in doubly exponential time without using any queries to } T \text{ (as we can use } V_{\text{known}} \text{ to lookup the values of the queries.) We can then compute the LHS of Equation (83) using the same resources. With this ability we can use the following greedy algorithm to add to } V_{\text{known}}: */\]
5. while Equations (82), (83) are not true do
6. Compute an arbitrary $b'$ violating Equations (82), (83) (This can be done in doubly exponential time);
7. if $b'$ does not appear in $V^\text{hist}_{\text{known}}$ then
8. ABORT and guess a random label for the EXIT vertex of the entire welded tree problem on $T$
9. If $b'$ does appear in $V^\text{hist}_{\text{known}}$, then add $b'$ and its children, and edge colors in $V^\text{hist}_{\text{known}}$ to the dictionary $V_{\text{known}}$, and continue.
10. Let $V^\text{complete}_{\text{known}} \subseteq V^\text{hist}_{\text{known}}$ be the minimum size subtree (rooted at ENTRANCE) of $V^\text{hist}_{\text{known}}$ which contains $V_{\text{known}}$:
   \[ /* \text{Since } V_{\text{known}} \subseteq V^\text{hist}_{\text{known}} \text{ and } V^\text{hist}_{\text{known}} \text{ is a tree rooted at ENTRANCE we can compute } V^\text{complete}_{\text{known}} \text{ without any queries to } T, \text{ only look-ups to } V^\text{hist}_{\text{known}}. \text{ We will see in the analysis that this does not adversely increase the size of } V_{\text{known}}. */\]
11. $V_{\text{known}} \leftarrow V^\text{complete}_{\text{known}}$;
12. return $V_{\text{known}}$

3.2 Analysis of the Algorithm $\mathcal{M}^i$

Lemma 3.4. In the classical simulation algorithm $\mathcal{M}^k$ which simulates the first $k$ tiers of $C(T)$, the set of all encountered vertices after $k$ tiers $V^\text{hist}_{\text{known}} = \mathcal{M}^k(T)[3]$ has size $|V^\text{hist}_{\text{known}}| \leq kq(n)2^{q(n)}2n(g(n)+|r|)$

Proof. Lemma 3.7 shows that, within each call of BottleneckQuantumTierSimulator, $|V^i_{\text{known}}| \leq 2n(g(n)+|r|)$. Since SimOracle at most doubles the size of the set $V_{\text{known}}$ that it acts on, and since, by Lemma 3.7, Bottleneck adds at most $2n(g(n)+|r|)$ to the size of the set $V_{\text{known}}$ that it acts on, we have that, within BottleneckQuantumTierSimulator, $|V^i_{\text{known}}| \leq 2 |V^{i-1}_{\text{known}}| + 2n(g(n)+|r|)$. Following this recursion through $q(n)$ quantum layers yields $|V^{(n)}_{\text{known}}| \leq q(n)2^{q(n)}2n(g(n)+|r|)$. At the end of BottleneckQuantumTierSimulator, $V^{(n)}_{\text{known}}$ is merged with $V^\text{hist}_{\text{known}}$, thus increasing $V^\text{hist}_{\text{known}}$ in size by at most $q(n)2^{q(n)}2n(g(n)+|r|)$. The subroutine BottleneckQuantumTierSimulator is run $k$ times in $\mathcal{M}^k$, and so, by the end of $\mathcal{M}^k$, we have that $|V^\text{hist}_{\text{known}}| \leq kq(n)2^{q(n)}2n(g(n)+|r|)$.
Theorem 3.5. Let \( \mathcal{T} \) be the set of all Welded Tree black-boxes. Given a \((n, \ell(n), q(n), g(n))\)-hybrid-quantum circuit \( C(T) \), which solves the Welded Tree problem with probability \( p \) we have that

\[
\Pr_{T \in \mathcal{T}}[C(K^{\ell(n)}(C(T), T)) \text{ solves the Welded Tree problem}] \geq p - 2^{-n/400}
\]

Proof. Recall the analysis of algorithm \( A_i \). That same analysis could be used to analyze \( M_i \) except for one point, at Equation (69), where we use that the set \( V_{\ell+1}^{\text{known}} \) has the property that the probability of guessing a valid label outside of \( V_{\ell+1}^{\text{known}} \) is at most \( \frac{2^{n+i+2}}{2^n} \). In the analysis of \( M_i \), within the subroutine BottleneckQuantumTierSimulator the set \( V_{\ell+1}^{\text{known}} \) is defined differently than in \( A_i \) (because of the subroutine Bottleneck), and so Equation (69) no longer holds. However, it follows from Lemma 3.7 that, so long as Bottleneck does not ABORT, the probability of guessing a valid label outside of \( V_{\ell+1}^{\text{known}} \) is at most \( 2^{-n/100} \). Therefore, in the case that Bottleneck never calls ABORT in the entire course of \( M_i \), we may use the same analysis for \( M_i \) as we did for \( A_i \), except replacing the \( \frac{2^{n+i+2}}{2^n} \) in the analysis of \( A_i \) with a \( 2^{-n/100} \).

It follows that an error in the simulation of \( 2^{-n/200} \) (in the trace norm) is incurred for every quantum layer. Since there are \( kq(n) \) quantum layers in \( C^k(T) \), this analysis gives

\[
\|M^k(T)[1] - C^k(T)\|_1 \leq kq(n)2^{-n/200} + \Pr[M^k \text{ calls ABORT}], \quad (84)
\]

We know from Lemma 3.8 that Bottleneck has less than \( 2^{-n/8} \) probability of \text{ABORT} every time that it is called in \( M^k \). Since Bottleneck is only called once per layer, and \( kq(n) \) quantum layers in \( C^k(T) \) it follows by union bound that

\[
\Pr[M^k \text{ calls ABORT}] \leq kq(n)2^{-n/8}
\]

Thus,

\[
\|M^k(T)[1] - C^k(T)\|_1 \leq kq(n)2^{-n/200} + \Pr[M^k \text{ calls ABORT}] \leq kq(n)2^{-n/200} + kq(n)2^{-n/8} \quad (85)
\]

\[
\leq 2kq(n)2^{-n/200} \leq \ell(n)q(n)2^{-n/200}, \quad (86)
\]

The desired result follows for sufficiently large \( n \). \( \square \)

Theorem 3.6. No algorithm in JC can solve the Welded Tree Oracle problem with probability higher than \( 2^{-\Omega(n)} \).

Proof. Note that, by Lemma 3.4 we know that the size of \( V_{\text{known}}^{\text{hist}} \) at the end of \( M^k \) is at most \( kq(n)2^{\ell(n)}2n(g(n) + |r|) \). This quantity is pseudopolynomial because \( q(n) = \text{polylog}(n) \), and \( |r| = \text{poly}(n) \) (recall that \( |r| \leq 10\ell(n)q(n)g(n) \)).

With this observation, the desired result follows for sufficiently large \( n \) by combining Theorem 3.5 with the classical lower bound Theorem 1.20. \( \square \)

Lemma 3.7. In the context of algorithm \( M^k \), when the subroutine Bottleneck\((i, x, V_{\text{known}}^{\text{current}}, V_{\text{known}}^{\text{hist}}) \) does not \text{ABORT}, the set \( V_{\text{known}} := \text{Bottleneck}(i, x, V_{\text{knowninit}}, V_{\text{knownfinal}}) \) has size at most \( |V_{\text{known}}^{\text{current}}| + 2n(g(n) + |r|) \), and has the property that,

\[
\forall b \in \{0, 1\}^{2^n} \text{ such that } b \text{ does not appear in } V_{\text{known}} : \quad (87)
\]

\[
\Pr_{P \in \mathcal{T}_{V_{\text{known}}^{\text{hist}}}}[b \text{ is a valid vertex in } P] \leq 2^{-n/100} \quad (88)
\]
Proof. The “While” loop within Bottleneck$(i, x, V_{\text{current}}, V_{\text{hist}}, V_{\text{known}})$ is defined to continue iterating until Equations (87), (88) are true. So, under our assumption that the “While” loop returns a valid $V_{\text{known}}$ without ever calling the ABORT command, we know that Equations (87) (88) are satisfied. Since $V_{\text{known}}$ grows in size by at most 1 label for each iteration of the “While” loop, it suffices to prove that, so long as no ABORT occurs, the “While” loop finishes and returns a valid answer $V_{\text{known}}$ after at most $2(g(n) + |r|)$ iterations. To prove this let $V_{\text{known}}^j$ denote the set $V_{\text{known}}$ as it is defined within the $j$th iteration of the “While” loop. We will show that the number of iterations $j$ cannot exceed $2(g(n) + |r|)$ by showing both a necessary upper bound and a necessary lower bound on the quantity $|T_{\text{current}}^j|_{V_{\text{known}}^j,x,r}$, which contradict each other when $j$ exceeds $2(g(n) + |r|)$.

The Lower Bound. By repeated application of Lemma A.4 we get
\[
|T_{\text{current}}^j|_{V_{\text{known}}^j,x,r} \geq 2^{-\frac{\mu}{\pi}} |T_{\text{current}}^j|_{V_{\text{known}}^j,x,r}, \tag{89}
\]
Since we are assuming that Bottleneck$(i, x, V_{\text{current}}, V_{\text{hist}}, V_{\text{known}})$ did not ABORT, we know from line 1 and 2 of Bottleneck that it must be the case that $|T_{\text{current}}^j|_{V_{\text{known}}^j,x,r} \geq 2^{-n(g(n) + |r|)} |T_{\text{current}}^j|_{V_{\text{known}}^j}$. Thus,
\[
|T_{\text{current}}^j|_{V_{\text{known}}^j,x,r} \geq 2^{-\frac{\mu}{\pi} - n(g(n) + |r|)} |T_{\text{current}}^j|_{V_{\text{known}}^j}. \tag{90}
\]

The Upper Bound. By repeated application of Lemma A.2 we get
\[
|T_{\text{current}}^j|_{V_{\text{known}}^j,x,r} \leq \left(\frac{1}{2^{2n} - |V_{\text{known}}|}\right)^j |T_{\text{current}}^j|_{V_{\text{known}}^j}, \tag{91}
\]
Since $|V_{\text{current}}| \leq 2^n$, we have
\[
|T_{\text{current}}^j|_{V_{\text{known}}^j,x,r} \leq \left(\frac{1}{2^{2n} - 2^n}\right)^j |T_{\text{current}}^j|_{V_{\text{known}}^j} \leq \left(\frac{1}{2^n (2^n - 1)}\right)^j |T_{\text{current}}^j|_{V_{\text{known}}^j} \leq \left(\frac{1}{2^n}\right)^j |T_{\text{current}}^j|_{V_{\text{known}}^j} = (2^{-n})^j |T_{\text{current}}^j|_{V_{\text{known}}^j} = 2^{-nj} |T_{\text{current}}^j|_{V_{\text{known}}^j}. \tag{96}
\]

But notice that (96) contradicts (90) when $j \geq 2(g(n) + |r|)$ (say). Therefore, we have shown that the “While” loop cannot run for more than $k \leq 2(g(n) + |r|)$ iterations, and the set $V_{\text{known}}^k$ at the end of the “While” loop satisfies $|V_{\text{known}}^k| \leq |V_{\text{current}}| + 2(g(n) + |r|)$.

The final step of the Bottleneck subroutine connects every new vertex in $V_{\text{known}}^k$ to the ENTRANCE node to create $V_{\text{known}}^{\text{complete}}$. This requires an overhead of at most $n$, and so $|V_{\text{known}}^{\text{complete}}| \leq |V_{\text{current}}|_{V_{\text{known}}^j} + 2n(g(n) + |r|)$.

\begin{lemma}
In the context of algorithm $M'$, the subroutine Bottleneck$(i, x, V_{\text{current}}, V_{\text{hist}}, V_{\text{known}})$ does not ABORT with probability higher than $2^{-n/8}$.
\end{lemma}
Proof. We bound the probability that Bottleneck \((i, x, V_{\text{knowninit}}, V_{\text{knownfinal}})\) calls \texttt{ABORT} by considering the two separate \texttt{ABORT} cases. The probability of an \texttt{ABORT} in line 1 and 2 of Bottleneck is less than

\[
\mathbb{P}_{x,r|V_{\text{current}}} \left[ \left| \mathcal{T}_{V_{\text{current}}}^{x \rightarrow r} \right| < 2^{-n(g(n) + |r|)} \right] = \sum_{x,r: \left| \mathcal{T}_{V_{\text{current}}}^{x \rightarrow r} \right| < 2^{-n(g(n) + |r|)}} \mathbb{P}_{x,r|V_{\text{current}}} \leq \sum_{x,r: \left| \mathcal{T}_{V_{\text{current}}}^{x \rightarrow r} \right| < 2^{-n(g(n) + |r|)}} 2^{-n(g(n) + |r|)}
\]

\[
\leq \sum_{x,r} 2^{-n(g(n) + |r|)} \leq 2|x| + |r| 2^{-n(g(n) + |r|)} \leq 2g(n) + |r| 2^{-n(g(n) + |r|)} \leq 2^{-(n-1)} (g(n) + |r|) \leq 2^{-(n-1)}
\]

Now let’s analyze the probability of an \texttt{ABORT} in lines 8 and 9. In this case we must have a label \(b'\) which is does not appear in \(V_{\text{known}} \subseteq V_{\text{hist}}^{\text{known}}\) or \(V_{\text{hist}}^{\text{known}'}\) and yet

\[
\mathbb{P}_{p \in \mathcal{T}_{V_{\text{known}}}^{x \rightarrow r}} \left[ b' \text{ is a valid label in } P \right] \geq 2^{-n/100}.
\]

Since the entire algorithm up to this point has only made \(|V_{\text{hist}}^{\text{known}}|\) classical queries, we know from Lemma 4 of [CCD+03], that with our fixed \(r\) but over the randomness in the black-box, the probability of guessing a valid label outside of \(V_{\text{hist}}^{\text{known}}\) is at most

\[
\left| V_{\text{hist}}^{\text{known}} \right| \leq \frac{2^{n+2} - 2}{2^{n}} \leq 2^{-n/2}
\]

Here the inequality follows for sufficiently large \(n\) because, by Lemma 3.4 we have that \(|V_{\text{hist}}^{\text{known}}| \leq kq(n) 2^{n} 2n(g(n) + |r|)\) which is less than \(2^{-n/2}\) for sufficiently large \(n\) because \(q(n) = \text{polylog}(n)\), and \(|r| = \text{poly}(n)\) (recall that \(|r| \leq 10\epsilon(n) q(n) g(n)\)). However, since (for our fixed \(r\)) \(V_{\text{known}}^{\text{known}}\) and \(x\) are deterministic functions of \(V_{\text{hist}}^{\text{known}'}\), it follows that the probability of arriving in the above situation can be at most \(\frac{2^{-n/2}}{2^{-n/100}} \leq 2^{-n/4}\). Any higher probability of this event would yield an algorithm for guessing a valid label outside of \(V_{\text{hist}}^{\text{known}}\) that contradicts Lemma 4 of [CCD+03].

Note that lines 8 and 9 are run at most \(2(g(n) + |r|)\) times. Thus, union bounding over the two different \texttt{ABORT} cases gives an upper bound on the total abort probability of \(2(g(n) + |r|) 2^{-n/4} + 2^{-(n-1)} \leq 2^{-n/8}\) where the inequality holds for sufficiently large \(n\) because \(g(n)\) and \(|r|\) scale polynomially.

\section{Open Problems}

1. Is it possible to quantum-VBB-obfuscate pseudorandom blackboxes \(T\) for the Welded Tree Problem, thereby allowing us to instantiate our separation between JC and BQP based on a cryptographic assumption?

2. Can we instantiate our separation assuming just post-quantum classical indistinguishability obfuscation for Welded Tree blackboxes \(T\)?

\section*{Acknowledgements}

We thank Richard Cleve, Aram Harrow, John Watrous, and Umesh Vazirani for helpful comments and discussions.
References

[AA18] Scott Aaronson and Andris Ambainis. Forrelation: A problem that optimally separates quantum from classical computing. SIAM J. Comput., 47(3):982–1038, 2018. doi:10.1137/15M1050902.

[Aar10] Scott Aaronson. BQP and the polynomial hierarchy. In Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, Cambridge, Massachusetts, USA, 5-8 June 2010, pages 141–150, 2010. doi:10.1145/1806689.1806711.

[BV97] Ethan Bernstein and Umesh V. Vazirani. Quantum complexity theory. SIAM J. Comput., 26(5):1411–1473, 1997. doi:10.1137/S0097539796300921.

[CCD+03] Andrew M. Childs, Richard Cleve, Enrico Deotto, Edward Farhi, Sam Gutmann, and Daniel A. Spielman. Exponential algorithmic speedup by a quantum walk. In Proceedings of the 35th Annual ACM Symposium on Theory of Computing, June 9-11, 2003, San Diego, CA, USA, pages 59–68, 2003. doi:10.1145/780542.780552.

[CCL19] Nai-Hui Chia, Kai-Min Chung, and Ching-Yi Lai. On the need for large quantum depth, 2019. (in preparation).

[CW00] Richard Cleve and John Watrous. Fast parallel circuits for the quantum fourier transform. In 41st Annual Symposium on Foundations of Computer Science, FOCS 2000, 12-14 November 2000, Redondo Beach, California, USA, pages 526–536, 2000. doi:10.1109/SFCS.2000.892140.

[FvdG99] Christopher A. Fuchs and Jeroen van de Graaf. Cryptographic distinguishability measures for quantum-mechanical states. IEEE Trans. Information Theory, 45(4):1216–1227, 1999. doi:10.1109/18.761271.

[FZ03] Stephen A. Fenner and Yong Zhang. A note on the classical lower bound for a quantum walk algorithm, 2003. arXiv:arXiv:quant-ph/0312230.

[Joz06] Richard Jozsa. An introduction to measurement based quantum computation. NATO Science Series, III: Computer and Systems Sciences. Quantum Information Processing-From Theory to Experiment, 199:137–158, 2006. arXiv:quant-ph/0508124.

[NC10] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press, 2010. doi:10.1017/CBO9780511976667.

[Sim97] Daniel R. Simon. On the power of quantum computation. SIAM J. Comput., 26(5):1474–1483, 1997. doi:10.1137/S0097539796298637.

[Wat09] John Watrous. Quantum computational complexity. In Encyclopedia of Complexity and Systems Science, pages 7174–7201. Springer, 2009. doi:10.1007/978-0-387-30440-3_428.

[Wat11] John Watrous. Guest column: an introduction to quantum information and quantum circuits 1. SIGACT News, 42(2):52–67, 2011. doi:10.1145/1998037.1998053.
Appendix

For some fixed $i > 0$, let $y, x, V_{\text{known}}, V_{\text{knowninit}},$ and $V_{\text{knownfinal}}$ be as in Algorithm 7.

Lemma A.1. For any dictionary $V$, it holds that

$$|T_V| = \frac{(2^{2n} - |V|)!}{(2^n - |V|)!}. \quad (101)$$

Proof. Recall that there are $2^{2n} - |V|$ unused labels and $2^n - |V|$ vertices that need a label. In other words, $|T_V|$ is equal to the number of $(2^n - |V|)$-permutations of $2^{2n} - |V|$, which is

$$\frac{(2^{2n} - |V|)!}{(2^n - |V|)!}. \quad (102)$$

\hfill \square

Lemma A.2. When we add a label $b$ to $V_{\text{known}}$ to get $V_{\text{known}}^\text{new}$, we get

$$|T_{V_{\text{known}}}^{i,b,x,r}V_{\text{known}}^\text{new}| \leq \frac{1}{2^{2n} - |V_{\text{known}}|} |T_{V_{\text{known}}}^{|V_{\text{known}}^\text{new}|}. \quad (103)$$

Proof. It follows from Lemma A.1 that

$$|T_{V_{\text{known}}}^{|V_{\text{known}}^\text{new}| \leq \frac{1}{2^{2n} - |V_{\text{known}}|} |T_{V_{\text{known}}}^{|V_{\text{known}}|}$$

The proof follows by using Lemmas A.3. \hfill \square

The following two lemmas follow from the definition of $T_{V,x,r}$ and $T_V$.

Lemma A.3.

$$|T_V^{x,r}| \leq |T_V| \quad (104)$$

Lemma A.4. When we add a label $b$ to $V_{\text{known}}$ to get the new dictionary which we will call $V_{\text{known}}^\text{new}$, we get

$$|T_{V_{\text{known}}}^{i,b,x,r}V_{\text{known}}^\text{new}| \geq \mathbb{P}_{P \in T_{V_{\text{known}}}^{i,b,x,r}, P \text{ is consistent with } b} |T_{V_{\text{known}}}^{i,b,x,r}|. \quad (105)$$

Now we want to bound the number of valid vertices $v \notin V_{\text{known}}$ we can guess given $(x, V_{\text{known}})$. We keep adding vertices $v \notin V_{\text{known}}$, one at a time, to $V_{\text{known}}$ to obtain $V_{\text{known}}^\text{new}$. First, lets make the following observation, which follows from Lemma 2.13.