An array of coupled nonlinear oscillators as a model for amplitude variations in intermodulation distortions from the auditory system

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Abstract The response of a nonlinear oscillator driven by two or more forces displays several frequency components corresponding to the linear combinations of the driving frequencies, known as intermodulation distortions (IMDs). The auditory system of all classes of vertebrates produces strong IMDs in the presence of sounds of two tones. This phenomenon is attributed to the nonlinearity of the biological acoustic sensors within the inner ear. In this work, we developed a numerical model to investigate the total IMDs generated by a chain of coupled nonlinear oscillators. Total nonlinear distortions exhibited variations in amplitudes with respect to the stimulus frequency. The variation profile was sensitive to the magnitude of the driving forces and the difference in the two driving frequencies. We found that the arrangement of the oscillators’ characteristic frequencies was necessary for the variations in total IMD level. Results from the numerical model agreed with measurements of IMDs from Chinese edible frog (Hoplobatrachus Rugulosus), whose inner ear is constituted by arrays of acoustic receptors with progressively increasing natural frequencies. In response to sounds of two tones, IMDs produced by the inner ear of Chinese edible frog exhibited variations in amplitudes with respect to the driving frequencies. The third-order distortion level displayed several peaks at high stimulating sound pressure level, and at large differences between the stimulus frequencies. Our findings suggest that an array of coupled nonlinear oscillator can serve as a simple model for the generation of nonlinear distortions by the inner ear.

1. Introduction
In response to driving forces of two or more frequencies, the displacement of a nonlinear oscillator displays several frequency components, each corresponding to a linear combination of those of the inputs. These side-band responses, termed intermodulation distortions (IMDs), are consequences of the nonlinear terms in the equation of motion governing the oscillator. This contrasts with a linear system possessing a saturation from which distortions are produced only at high input amplitudes.

The auditory system of vertebrates is highly nonlinear [1]. An inner ear stimulated by sounds of two frequencies (f₁ and f₂) generates strong third-order IMDs at 2f₁−f₂ and 2f₂−f₁. These acoustic distortions can be detected in the outer ear, known as Distortion Product Otoacoustic Emissions (DPOAEs). Measurements of DPOAEs from human and some lizard species reveal a quasi-periodic variation in the
distortion amplitude with respect to the stimulus frequency, a feature called DPOAE fine structure [2-3]. This suggests that the response of the inner ear cannot be accountable by a single nonlinear oscillator.

The auditory nonlinearity stems from hair cells, the biological acoustic sensors. Each hair cell consists of a cluster of villi, called hair bundle, which deflects upon mechanical stimulations by sound waves. The compliance of a hair bundle shows a nonlinear dependence on its displacement [4]. Hair bundles can produce strong nonlinear distortions in response to two-frequency signals, a phenomenon which could underlie DPOAEs. A model of a driven nonlinear oscillator poised near a supercritical Hopf bifurcation has been proposed to describe the generation of IMDs by a hair bundle [5].

The vertebrate inner ear consists of arrays of elastically coupled hair cells organized in an ascending or a descending order in their characteristic frequencies, known as tonotopic organization [6]. We developed a mathematical model of a chain of coupled nonlinear oscillators to investigate the production of nonlinear distortions. Depending on the parameters of the driving forces, total IMDs from a system of oscillators could display large variations in amplitude with respect to the driving frequency. Numerical results were then used to describe physiological recordings of DPOAEs from Chinese Edible frogs, a model animal which robustly produces nonlinear distortions.

2. Methods

2.1. Mathematical model of a chain of nonlinear oscillators

A chain of N noisy nonlinear oscillators, each subject to two driving forces is described in equation (1)

\[
\frac{dz_j}{dt} = (\mu + i\omega_{0,j})z_j - |z_j|^2 z_j + k(z_{j+1} - z_j) + k(z_{j-1} - z_j) + Fe^{2\pi f_1 t} + Fe^{2\pi f_2 t} + \eta(t),
\]

where \(Re(z_j)\) and \(\omega_{0,j}\) denote the displacement and the characteristic frequency of the \(j^{th}\) oscillator, respectively. All oscillators were poised at a supercritical Hopf bifurcation by setting the control parameter \(\mu = 0\). Elastic coupling strength between two adjacent oscillators is denoted by \(k\). To represent the free ends of an array of hair cells within the inner ear, the first and the last oscillators were...
connected to only one nearest neighbor. The noise term $\eta$ was Gaussian with zero mean, drawn from a normal distribution with a standard deviation of 100. Numerical simulations were performed by the 4th order Runge Kutta method at 0.1-ms time step. In all simulations, the number of oscillators was fixed at $N = 101$. The characteristic frequencies linearly increased from 200 Hz to 1,200 Hz to represent an auditory organ within the inner ear of American bullfrog (*Lithobates catesbeianus*). Numerical simulations and data analysis were performed in MATLAB.

![Figure 2. Effects of driving forces on the total IMD levels](image)

**Figure 2.** Effects of driving forces on the total IMD levels when (a) The driving force amplitude is fixed at $1.25 \times 10^5$ while $\Delta f$ increases and (b) when $\Delta f$ is fixed at 150 Hz while the driving amplitude. (c) The correlation bandwidth of the total IMD level at $2f_1-f_2$ decreases monotonically with $\Delta f$. (d) The correlation bandwidth of total IMD level at $2f_1-f_2$ decreases with the stimulus amplitude. In all panels, the coupling strength was fixed at $k = 1250$.

2.2. **Physiological recordings**

All experimental protocols were approved by the Animal Care and Use Committee of Chulalongkorn University. Chinese edible frogs (*Hoplobatrachus rugulosus*) were anesthetized with an intraperitoneal injection of sodium pentobarbital at 30 mg/kg body weight. Experiments were conducted in an anechoic chamber. A hollow plastic cylinder was sealed over the skin surrounding the frog’s tympanic membrane to serve as an artificial ear canal. The other end of the cylinder was attached to a sensitive microphone and two plastic tubes, each connected to a closed-field speaker. DPOAEs were elicited by sounds of two frequencies with equal intensity. All recordings were done at 50 kHz.

2.3. **Data analysis**

Finite-time Fourier transform with a Hanning window was calculated for each non-overlapping 0.1-s segment of each oscillator’s displacement or the pressure signal recorded from the microphone. For numerical data, the total IMDs was the sum of the complex Fourier component from each oscillator at the distortion frequencies. To extract the correlation bandwidth of the nonlinear distortion level, a linear fit was subtracted from the plot of IMD level as a function of driving frequency $f_1$. An autocorrelation function was calculated. The correlation bandwidth ($f_c$) was defined as the frequency $f_1$ at which the first zero-crossing was observed.

3. **Results**

We investigated the total nonlinear distortions from an array of nonlinear oscillators, all simultaneously driven by identical forces of two frequencies ($f_1$ and $f_2$), in which $f_1$ was varied between 0.2 kHz and 1.2 kHz at a constant value of $\Delta f = f_2-f_1$. Total IMDs were the magnitude of the sum of the complex Fourier components at the IMD frequencies from all oscillators. To facilitate comparisons to experimental data, we focused on the amplitude of the third order IMDs, i.e., distortions at frequencies $2f_1-f_2$ and $2f_2-f_1$. 

![Diagram](image)
The total IMDs produced by an array of uncoupled oscillators (k=0) exhibited a slight variation in amplitude as a function of the driving frequency (figure 1(a)). Coupling the oscillators enhanced the difference between the maximal and minimal IMD levels (figure 1(a)). Next, we extracted the magnitude and the phase of IMDs from individual oscillators from the complex Fourier component at the distortion frequencies. We found that the oscillators whose characteristic frequencies laid between $f_1-3\Delta f$ and $f_2$ displayed high level of $2f_1-f_2$ distortion (figures 1(b), 1(c)). IMD from each of these oscillators always showed a phase lead with respect to the distortion signal from its neighbor with lower characteristic frequency. Total IMDs arose from an interference of distortion signals generated by oscillators at different phases. A local minimum in the total IMD level corresponded to the greatest phase cancellations of these individual distortions (figure 1(c)). Similar characteristics were observed for IMDs at $2f_2-f_1$ (data not shown).

**Figure 3.** Physiological recordings of DPOAEs from Chinese Edible frogs. (a) DPOAEs were recorded by placing a plastic cylinder attached to a microphone around the eardrum. In response to sounds of two tones, the ear produces strong nonlinear distortions of several orders. (b) Upon an increase in $\Delta f$, the peak DPOAE level at $2f_1-f_2$ reduces as well as the occurrence of additional local maxima. The stimulus level was 80 dB. (c) A variation in the $2f_1-f_2$ distortion level is enhanced upon higher stimulus amplitude. $\Delta f$ was fixed at 100 Hz. (d) Correlation bandwidth of $2f_1-f_2$ DPOAE levels shows a decrease with respect to $\Delta f$. (e) Similarly, high-stimulus level decreases the correlation bandwidth of DPOAE amplitudes.

Variations in the total IMD levels strongly depended on the organization of the characteristic frequencies. Total distortions from a chain of oscillators with randomly organized characteristic frequencies showed a fluctuation around a constant level (figure 1(a)). Analysis of IMDs from individual oscillators revealed that high-level distortions were produced by many oscillators. However, there was no correlations between the oscillator’s characteristic frequency and IMD level or IMD phase.

The variation profile of the total IMD levels largely depended on the driving forces. When the driving frequency $f_1$ was varied with $\Delta f$ fixed at a high value (≥200 Hz), the elicited total IMD exhibited a more strongly attenuated local minimum. An additional peak in the total IMD level became more distinctive upon larger $\Delta f$ (figure 2(a)). In association with this, the correlation bandwidth of the IMD level as a function of $f_1$ showed a progressive decrease (figure 2(c)). Next, with a fixed $\Delta f = 150$ Hz, a greater degree of variations in the total IMD level was observed at higher driving forces (figure 2(b)). The correlation bandwidth correspondingly reduced (figure 2(d)). Upon a further increase in the input amplitude, the total third-order IMD level displayed less variation with the driving frequency, resulting in an increase in the correlation bandwidth.

We compared results from our numerical simulations to physiological measurements of DPOAEs from Chinese Edible frogs. Nonlinear distortions were detected upon stimulation by sounds of two tones (figure 3(a)). We observed a strong variation in DPOAE level with respect to the stimulus frequency. At a constant input sound pressure level, an increase in $\Delta f$ revealed a fine structure in the DPOAE level: an additional local peak in the distortion level at $2f_1-f_2$ was observed (figure 3(b)). Similarly, higher
stimulus level elicited multiple peaks in the distortion level (figure 3(c)). In both cases, the correlation length of the distortion spectra decreased with both driving force parameters (figures 3(d), 3(e)).

4. Discussions
The variation in the total IMD level illustrated in this work resembles DPOAE fine structure observed in several vertebrate species. The amplitude of the third-order nonlinear distortions emitted by the inner ear exhibited a quasi-periodic variation with respect to the stimulus frequency. This feature has been attributed to an interference of distortion signals produced by two groups of hair cells, each containing a small number of cells that generates distortions of nearly identical phase [2]. Our model, on the other hand, demonstrates that qualitatively similar effects can be accountable by a large number of coupled oscillators that produces IMDs of distinct phases.

We observed a fine structure in DPOAEs recorded from the ear of Chinese edible frog. Our model could predict the dependence of the distortion level variation on the input sound amplitude and the difference between the two driving frequencies. At low-level stimulus, total IMDs are contributed by a small number of oscillators whose characteristic frequencies are sufficiently close to those of the stimuli. The response of the system thus resembles that of a single nonlinear oscillator. Stronger inputs elicit distortions from many oscillators over a broad range of characteristic frequencies, resulting in a large variation in phases of individual signals which could lead to a phase cancellation. Similar mechanism could underly several peaks in DPOAE observed at greater difference in the driving frequencies.

Note that some contradictions between the IMD variation profiles obtain from the model and the physiological recordings are observed, such as a prominent peak in DPOAE level near 0.5 kHz (figure 3(b), 3(c)). This is potentially due to our simplified model which does not incorporate the variations in the coupling strength and fluctuations in the difference between the characteristic frequencies of two adjacent oscillators. In addition, the bandpass filtering of the middle ear is not considered in the model.

5. Conclusions
We propose a model of a chain of elastically coupled nonlinear oscillators to describe the generation of third-order acoustic distortions from the inner ear. We found that a system of oscillators can produce total IMDs that displays strong amplitude variations whose profile is sensitive to the driving force parameters. Similar characteristics are observed in the level of the third-order DPOAEs produced by the frog’s inner ear. An increase in the stimulus amplitude and the difference in the stimulus frequencies enhanced the variations in the distortion levels. This suggests that the generation of nonlinear distortions by the inner ear could be partially accountable by an array of nonlinear oscillators.

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