A NILPOTENCY CRITERION FOR FINITE GROUPS

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Abstract. Let $G$ be a finite group. In this short note, we give a criterion of nilpotency of $G$ based on the existence of elements of certain order in each section of $G$.

1. Introduction

The problem of detecting structural properties of finite groups by looking at element orders has been considered in many recent papers (see e.g. [1] and [3]-[6]). In the current note, we identify a new property detecting nilpotency of a finite group $G$ that uses the function

$$\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|$$

introduced and studied in [9]. The proof that we present is founded on the structure of minimal non-nilpotent groups (also called Schmidt groups) given by [8].

It is well-known that a finite nilpotent group $G$ contains elements of order $\exp(G)$. Moreover, all sections of $G$ have this property. Under the above notation, this can be written alternatively as

$$\varphi(S) \neq 0 \text{ for any section } S \text{ of } G.$$  \hspace{1cm} (1)

Our main theorem shows that the converse is also true, that is we have the following nilpotency criterion.

Theorem 1. Let $G$ be a finite group. Then $G$ is nilpotent if and only if $\varphi(S) \neq 0$ for any section $S$ of $G$.

Note that (1) implies

$$\varphi(S) \neq 0 \text{ for any subgroup } S \text{ of } G$$

and in particular

$$\varphi(G) \neq 0.$$  \hspace{1cm} (3)

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We observe that the condition (3) is not sufficient to guarantee the nilpotency of $G$, as shows the elementary example $G = \mathbb{Z}_6 \times S_3$; we can even construct a non-solvable group $G$ for which $\varphi(G) \neq 0$, namely $G = \mathbb{Z}_n \times H$, where $H$ is a simple group of exponent $n$. A similar thing can be said about the condition (2).

**Example.** Let $G$ be a nontrivial semidirect product of a normal subgroup isomorphic to $E(5^3) = \langle x, y \mid x^5 = y^5 = [x, y]^5 = 1, [x, y] \in Z(E(5^3)) \rangle$ by a subgroup $\langle a \rangle$ of order 3 such that $a$ commutes with $[x, y]$. Then $G$ is a non-CLT group of order 375, more precisely it does not have subgroups of order 75. We infer that its subgroups are: $G$, all subgroups contained in the unique Sylow 5-subgroup, all Sylow 3-subgroups, and all cyclic subgroups of order 15. Clearly, $G$ satisfies the condition (2), but it is not nilpotent.

Finally, we note that our criterion can be used to prove the non-nilpotency of a finite group by looking to its sections. In [9] we have determined several classes of groups $G$ satisfying $\varphi(G) = 0$, such as dihedral groups $D_{2n}$ with $n$ odd, non-abelian $P$-groups of order $p^{n-1}q$ ($p > 2, q$ primes, $q \mid p - 1$), symmetric groups $S_n$ with $n \geq 3$, and alternating groups $A_n$ with $n \geq 4$. These examples together with Theorem 1 lead to the following corollary.

**Corollary 2.** If a finite group $G$ contains a section isomorphic to one of the above groups, then it is not nilpotent.

2. **Proof of Theorem 1**

We will prove that a finite group all of whose sections $S$ satisfy $\varphi(S) \neq 0$ is nilpotent. Assume that $G$ is a counterexample of minimal order. Then $G$ is a Schmidt group since all its proper subgroups satisfy the hypothesis. By [3] (see also [2, 7]) it follows that $G$ is a solvable group of order $p^m q^n$ (where $p$ and $q$ are different primes) with a unique Sylow $p$-subgroup $P$ and a cyclic Sylow $q$-subgroup $Q$, and hence $G$ is a semidirect product of $P$ by $Q$. Moreover, we have:

- if $Q = \langle y \rangle$ then $y^q \in Z(G)$;
- $Z(G) = \Phi(G) = \Phi(P) \times \langle y^q \rangle$, $G' = P, P' = (G')' = \Phi(P)$;
- $|P/P'| = p^r$, where $r$ is the order of $p$ modulo $q$;
- if $P$ is abelian, then $P$ is an elementary abelian $p$-group of order $p^r$ and $P$ is a minimal normal subgroup of $G$;
- if $P$ is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$. 

We infer that $S = G/Z(G)$ is also a Schmidt group of order $p^r q$ which can be written as semidirect product of an elementary abelian $p$-group $P_1$ of order $p^r$ by a cyclic group $Q_1$ of order $q$ (note that $S_3$ and $A_4$ are examples of such groups). Clearly, we have
\[ \exp(S) = pq. \]
On the other hand, it is easy to see that
\[ L(S) = L(P_1) \cup \{Q_1^x \mid x \in S\} \cup \{S\}. \]
Thus, the section $S$ does not have cyclic subgroups of order $pq$ and consequently $\varphi(S) = 0$, a contradiction. This completes the proof.

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