Nonexistence of nonnegative entire solutions of semilinear elliptic systems

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\begin{abstract}
We consider the second-order semilinear elliptic system \( \Delta u = p(x)v^\alpha, \Delta v = q(x)u^\beta \), where \( x \in \mathbb{R}^N, N \geq 3, \alpha \) and \( \beta \) are positive constants, \( p \) and \( q \) are nonnegative continuous functions. We prove that nontrival nonnegative entire solutions fail to exist if the functions \( p \) and \( q \) are of slow decay.
\end{abstract}

\section{1. Introduction}

We consider the second-order semilinear elliptic system of the following form

\begin{align}
\Delta u &= p(x)v^\alpha, \\
\Delta v &= q(x)u^\beta,
\end{align}

where \( x \in \mathbb{R}^N, N \geq 3, \alpha, \beta \) are positive numbers, \( p \) and \( q \) are nonnegative continuous functions defined on \( \mathbb{R}^N \). Our objective is to establish conditions for the nonexistence of nontrivial nonnegative entire solutions of (1). An entire solution of (1) is defined to be a vector-valued function \( (u, v) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N) \) which satisfies (1) at every point in \( \mathbb{R}^N \).

To formulate main result of this paper we introduce the functions \( \tilde{p}(r) \) and \( \tilde{q}(r) \) by

\[ \tilde{p}(r) = \begin{cases} 
\left( \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} p^{1/(1-\alpha)}(x) \, dS \right)^{1-\alpha}, & \alpha > 1, \\
\min_{|x|=r} p(x), & \alpha = 1,
\end{cases} \]

where \( x \in \mathbb{R}^N \), \( N \geq 3, \alpha, \beta \) are positive numbers, \( p \) and \( q \) are nonnegative continuous functions defined on \( \mathbb{R}^N \). Our objective is to establish conditions for the nonexistence of nontrivial nonnegative entire solutions of (1). An entire solution of (1) is defined to be a vector-valued function \( (u, v) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N) \) which satisfies (1) at every point in \( \mathbb{R}^N \).

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\min_{|x|=r} p(x), & \alpha = 1,
\end{cases} \]
\[ \tilde{q}(r) = \begin{cases} \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} q^{1/(1-\beta)}(x) \, dS \left(1-\beta\right), & \beta > 1, \\ \min_{|x|=r} q(x), & \beta = 1, \end{cases} \]

where \( r > 0, \omega_N \) is the surface area of the unit sphere in \( \mathbb{R}^N \). We set \( \tilde{p}(r) = 0 \) and \( \tilde{q}(r) = 0 \) if
\[
\int_{|x|=r} p^{1/(1-\alpha)}(x) \, dS = \infty \quad \text{and} \quad \int_{|x|=r} q^{1/(1-\beta)}(x) \, dS = \infty,
\]
respectively. We note that \( p(x) = \tilde{p}(|x|), \quad q(x) = \tilde{q}(|x|) \) when \( p \) and \( q \) are spherically symmetric functions (i.e. \( p(x) = p(|x|) \) and \( q(x) = q(|x|) \)). Suppose that \( \tilde{p} \) and \( \tilde{q} \) satisfy
\[
\tilde{p}(r) \geq \frac{L_1}{r^\lambda \ln^{\nu} r}, \quad \tilde{q}(r) \geq \frac{L_2}{r^\mu \ln^{\xi} r}, \quad r \geq r_0 > 1,
\]
where \( L_1 > 0, L_2 > 0 \) and \( \lambda, \nu, \mu, \xi \) are constants. Our main result is as follows.

**Theorem 1.1:** Let \( N \geq 3, \alpha \geq 1, \beta \geq 1, \alpha \beta > 1 \) and \( p \) and \( q \) satisfy (2). If at least one from the following conditions holds

(i) \( 2 - \mu + \beta(2 - \lambda) > 0 \);
(ii) \( 2 - \lambda + \alpha(2 - \mu) > 0 \);
(iii) \( 2 - \mu + \beta(2 - \lambda) = 0, \quad \lambda < 2, \quad 1 - \xi - \beta \nu > 0 \);
(iv) \( 2 - \lambda + \alpha(2 - \mu) = 0, \quad \mu < 2, \quad 1 - \nu - \alpha \xi > 0 \);
(v) \( \lambda = 2, \mu = 2, \quad 1 - \xi + \beta(1 - \nu) > 0 \);
(vi) \( \lambda = 2, \mu = 2, \quad 1 - \nu + \alpha(1 - \xi) > 0 \),

then there are not nontrivial nonnegative entire solutions of (1).

The problem of the existence and nonexistence of entire solutions of scalar elliptic equations has been investigated by many authors (see e.g. [1–11] and the references therein). Entire solutions for semilinear elliptic systems have been considered in many papers also (see, for example, previous works [12–25]). In particular, Theorem 1.1 was proved in [13,14] for \( \nu = \xi = 0 \). Lair in [21] established the existence of positive entire solutions to the elliptic system (1) with spherically symmetric coefficients \( p \) and \( q \). More precisely, the system (1) has positive entire solutions if \( \alpha \beta > 1 \) and \( p(|x|) \) and \( q(|x|) \) satisfy at least one of the following conditions

\[
\int_0^\infty tp(t) \left( t^{2-N} \int_0^t s^{N-3} \int_0^s \tau q(\tau) \, d\tau \, ds \right)^\alpha \, dt < \infty, \quad (3)
\]

\[
\int_0^\infty tq(t) \left( t^{2-N} \int_0^t s^{N-3} \int_0^s \tau p(\tau) \, d\tau \, ds \right)^\beta \, dt < \infty. \quad (4)
\]

This paper is organized as follows. Some auxiliary propositions are proved in Section 2. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we discuss the optimality of obtained results.
2. Auxiliary propositions

Let $P$ and $Q$ be nonnegative nonincreasing continuous functions and $h$ and $g$ be nonnegative continuous functions. We introduce the functions $y$ and $z$ in the following way

$$y(r) = \int_{R}^{r} (r-s)^m P(s) g^\alpha(s) \, ds, \quad z(r) = \int_{R}^{r} (r-s)^n Q(s) h^\beta(s) \, ds,$$

where $r > R > 0$, $m, n \in \mathbb{N}$.

Lemma 2.1: Let $m, n \in \mathbb{N}, b > 1, \alpha > 0, \beta > 0, \alpha \beta > 1$ and for $r \in [R, bR]$,

$$h(r) \geq \int_{R}^{r} (r-s)^m P(s) g^\alpha(s) \, ds, \quad g(r) \geq \int_{R}^{r} (r-s)^n Q(s) h^\beta(s) \, ds.$$

Then

$$y(\alpha \beta - 1) \int_{A}^{bR} \left( P(r) Q^\alpha(r) \right)^{\frac{1}{(n+1)\alpha + m + 1}} \, dr \leq C_y,$$

$$z(\alpha \beta - 1) \int_{A}^{bR} \left( Q(r) P^\beta(r) \right)^{\frac{1}{(m+1)\beta + n + 1}} \, dr \leq C_z,$$

where $A \in (R, bR), C_y$ and $C_z$ are positive constants which do not depend on $R$.

Proof: We prove only (6), the proof of (5) is similar. Note that

$$h(r) \geq y(r) \geq 0, \quad g(r) \geq z(r) \geq 0,$$

$$y^{(i)}(r) \geq 0, \quad z^{(i)}(r) = 0 \text{ for } i \in \{1, 2, \ldots, m\},$$

$$y^{(i)}(r) \geq 0, \quad z^{(i)}(r) = 0 \text{ for } i \in \{1, 2, \ldots, n\},$$

$$y^{(m+1)}(r) = m! P(r) g^\alpha(r) \geq 0, \quad z^{(n+1)}(r) = n! Q(r) h^\beta(r) \geq 0.$$

Thus, from (7), (10), we have

$$y^{(m+1)}(r) \geq m! P(r) z^\alpha(r),$$

$$z^{(n+1)}(r) \geq n! Q(r) y^\beta(r).$$

Now we multiply (11) by $z'(r)$ and integrate over $[R, r]$. Using the integration by parts, (8), (9) and the monotonicity of $P$, we obtain

$$y^{(m)}(r) z'(r) \geq \frac{m!}{\alpha + 1} P(r) (z(r))^\alpha + 1.$$ 

Repeating this process appropriately many times, we get

$$y(r)(z'(r))^{m+1} \geq \frac{m!}{(\alpha + 1)(\alpha + 2) \ldots (\alpha + m + 1)} P(r) (z(r))^\alpha m + 1.$$ (13)
A combination of (12) and (13) gives
\[ z^{(n+1)}(r)(z'(r))^{(m+1)\beta} \geq c_1 Q(r)P^\beta(r)(z(r))^{(\alpha+m+1)\beta}, \] (14)
where a positive constant \( c_1 \) does not depend on \( R \). From now on, without causing any confusion, we may use \( c_i, \ C_i, \ Cu, \ C_i, \ C_i \) or \( \hat{C}_i \) \((i = 0, 1, 2, \ldots)\) to denote various positive constants. Applying \( n \) times the operation of multiplying (14) by \( z'(r) \) and integration over \([R, r]\), we obtain
\[ (z'(r))^{(m+1)\beta+n+1} \geq c_2 Q(r)P^\beta(r)(z(r))^{(\alpha+m+1)\beta+n}. \] (15)
Without loss of generality, we can suppose that \( z(A) > 0 \). From (15), we find
\[ (z(r))^{-\frac{\alpha\beta-1}{(m+1)\beta+n+1}} (z'(r)) \geq (c_2 Q(r)P^\beta(r))^{\frac{1}{(m+1)\beta+n+1}}, \quad r \in [A, bR]. \]
Integrating this relation from \( A \) to \( bR \), then leads to the inequality
\[
C_z \left( z^{-\frac{\alpha\beta-1}{(m+1)\beta+n+1}} (A) - z^{-\frac{\alpha\beta-1}{(m+1)\beta+n+1}} (bR) \right) \geq \int_A^{bR} (Q(r)P^\beta(r))^{\frac{1}{(m+1)\beta+n+1}} dr,
\]
which completes the proof.

Let us introduce the functions \( u_k(r) \), \( v_k(r) \) by the following recurrence relationships
\[
\begin{align*}
u_0(r) &= 1, \quad v_0(r) = 1, \\
u_k(r) &= \frac{1}{N-2} \int_{\rho}^{r} \tilde{s}p(s) \left[ 1 - \left( \frac{s}{r} \right)^{N-2} \right] v_{k-1}(s) \, ds, \quad k \in \mathbb{N}, \\
v_k(r) &= \frac{1}{N-2} \int_{\rho}^{r} \tilde{s}q(s) \left[ 1 - \left( \frac{s}{r} \right)^{N-2} \right] u_{k-1}(s) \, ds, \quad k \in \mathbb{N}
\end{align*}
\]
and define the functions
\[
\begin{align*}
y_k(R) &= \int_R^{aR} (aR - s)^m s^{1-m} \tilde{p}(s) v_k^\alpha(s) \, ds, \\
z_k(R) &= \int_R^{aR} (aR - s)^n s^{1-n} \tilde{q}(s) u_k^\beta(s) \, ds,
\end{align*}
\]
where \( \rho > 0, R > 1, a > 1 \).

Now we prove an auxiliary statement which has independent interest.

**Theorem 2.2:** Let \( N \geq 3, \alpha \geq 1, \beta \geq 1, \alpha\beta > 1, r^{1-m} \tilde{p}(r) \) and \( r^{1-n} \tilde{q}(r) \) be nonincreasing functions on \((\rho, \infty)\) for some \( m, n \in \mathbb{N} \) and \( \rho > 0 \). If at least one from the following conditions holds
\[
\begin{align*}
\limsup_{R \to \infty} y_k^{\frac{\alpha\beta-1}{(m+1)\beta+n+1}} (R) &\int_{aR}^{bR} (s^{1-m} \tilde{p}(s)(s^{1-n} \tilde{q}(s))^\alpha)^{\frac{1}{(m+1)\beta+n+1}} \, ds = \infty \quad (19) \\
or \limsup_{R \to \infty} z_k^{\frac{\alpha\beta-1}{(m+1)\beta+n+1}} (R) &\int_{aR}^{bR} (s^{1-n} \tilde{q}(s)(s^{1-m} \tilde{p}(s))^\beta)^{\frac{1}{(m+1)\beta+n+1}} \, ds = \infty \quad (20)
\end{align*}
\]
for some \( k \in \mathbb{N} \) and \( a, b \) such that \( 1 < a < b \leq 2 \), then there are not nontrivial nonnegative entire solutions of (1).
Proof: Assume to the contrary that (1) has a nontrivial nonnegative entire solution \((u,v)\). Let \(\overline{u}(r), \overline{v}(r)\) denote the averages of \(u(x), v(x)\) over the sphere \(|x| = r\), respectively, that is,

\[
\overline{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} u(x) \, dS, \quad \overline{v}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} v(x) \, dS.
\]

Then we can proceed analogously as in \([26,27]\) to find that

\[
(r^{-N-1} u'(r))' \geq r^{-N-1} \overline{p}(r) \overline{v}^\alpha(r), \quad r > 0, \quad \overline{u}'(0) = 0, \quad \overline{u}'(0) = 0,
\]

(21)

\[
(r^{-N-1} v'(r))' \geq r^{-N-1} \overline{q}(r) \overline{u}^\beta(r), \quad r > 0, \quad \overline{v}'(0) = 0.
\]

(22)

Obviously, \(\overline{u}'(r) \geq 0, \overline{v}'(r) \geq 0, r \geq 0\) and \(\overline{u}(r) > 0, \overline{v}(r) > 0, r > \rho\) for some \(\rho > 0\). Integrating (21) and (22) twice over \([0, r]\), we have for \(r > \rho\)

\[
\overline{u}(r) \geq \overline{u}(0) + \frac{1}{N-2} \int_0^r s \overline{p}(s) \left[ 1 - \left( \frac{s}{r} \right)^{N-2} \right] \overline{v}^\alpha(s) \, ds
\]

(23)

and

\[
\overline{v}(r) \geq \overline{v}(0) + \frac{1}{N-2} \int_0^r s \overline{q}(s) \left[ 1 - \left( \frac{s}{r} \right)^{N-2} \right] \overline{u}^\beta(s) \, ds.
\]

(24)

Now the principle of mathematical induction is used to prove the estimates

\[
\overline{u}(r) \geq C_{2k} u_k(r), \quad \overline{v}(r) \geq C_{2k+1} v_k(r), \quad r > \rho,
\]

(25)

where \(u_k(r)\) and \(v_k(r)\) are defined in (16)–(18), \(k = 0, 1, 2, \ldots\). It is easy to see that (25) follows from (16) for \(k = 0\). Assume (25) is true for \(k = l - 1\). In view of (17), (18), (23), (24) and (25) with \(k = l - 1\), we have

\[
\overline{u}(r) \geq \frac{C_{2l-1}^\alpha}{N-2} \int_\rho^r s \overline{p}(s) \left[ 1 - \left( \frac{s}{r} \right)^{N-2} \right] v_{l-1}^\alpha(s) \, ds,
\]

\[
\overline{v}(r) \geq \frac{C_{2l-2}^\beta}{N-2} \int_\rho^r s \overline{q}(s) \left[ 1 - \left( \frac{s}{r} \right)^{N-2} \right] u_{l-1}^\beta(s) \, ds,
\]

that is equivalent to (25) with \(k = l\), \(C_{2l} = C_{2l-1}^\alpha, C_{2l+1} = C_{2l-2}^\beta\).
Let \( R \leq s \leq r \leq bR \). Then \( s/r \geq 1/b \) and \((r-s)/s \leq b - 1 \leq 1\). Using mean value theorem, we obtain
\[
1 - \left( \frac{s}{r} \right)^{N-2} = \frac{r^{N-2} - s^{N-2}}{r^{N-2}} = \frac{(N-2)\xi^{N-3}(r-s)}{r^{N-2}} \geq \frac{N-2}{b^{N-2}} \frac{r-s}{s},
\]
where \( s \leq \xi \leq r \). Since \((r-s)/s \leq 1\), we conclude that
\[
1 - \left( \frac{s}{r} \right)^{N-2} \geq \frac{N-2}{b^{N-2}} \left( \frac{r-s}{s} \right)^m, \quad 1 - \left( \frac{s}{r} \right)^{N-2} \geq \frac{N-2}{b^{N-2}} \left( \frac{r-s}{s} \right)^n. \tag{26}
\]

Now (23), (24), (26) imply the inequalities
\[
\bar{u}(r) \geq c_3 \int_{R}^{r} (r-s)^m s^{1-m} \tilde{p}(s) \tilde{v}(s) \, ds,
\]
\[
\bar{v}(r) \geq c_3 \int_{R}^{r} (r-s)^n s^{1-n} \tilde{q}(s) \tilde{v}(s) \, ds,
\]
where \( c_3 = 1/b^{N-2}, \ R \geq \rho \). Applying Lemma 2.1 with \( h = \bar{u}, \ g = \bar{v}, \ P(s) = s^{1-m} \tilde{p}(s), \ Q(s) = s^{1-n} \tilde{q}(s), \ A = aR \), we have
\[
y^{\frac{a\beta-1}{(n+1)a+m+1}}(aR) \int_{aR}^{bR} (s^{1-m} \tilde{p}(s)(s^{1-n} \tilde{q}(s))^{\alpha})^{\frac{1}{(n+1)a+m+1}} \, ds \leq C_y, \tag{27}
\]
\[
z^{\frac{a\beta-1}{(m+1)\beta+n+1}}(aR) \int_{aR}^{bR} (s^{1-n} \tilde{q}(s)(s^{1-m} \tilde{p}(s))^{\beta})^{\frac{1}{(m+1)\beta+n+1}} \, ds \leq C_z. \tag{28}
\]

Combining (27), (28) with (25), one obtains
\[
(C_{2k+1}^\alpha y_k(R))^{\frac{a\beta-1}{(n+1)a+m+1}} \int_{aR}^{bR} (s^{1-m} \tilde{p}(s)(s^{1-n} \tilde{q}(s))^{\alpha})^{\frac{1}{(n+1)a+m+1}} \, ds \leq C_y,
\]
\[
(C_{2k+1}^{\beta} z_k(R))^{\frac{a\beta-1}{(m+1)\beta+n+1}} \int_{aR}^{bR} (s^{1-n} \tilde{q}(s)(s^{1-m} \tilde{p}(s))^{\beta})^{\frac{1}{(m+1)\beta+n+1}} \, ds \leq C_z,
\]
that contradicts (19), (20). \[\blacksquare\]

**Remark 1:** Let the conditions of Theorem 2.2 hold for the problem (1) with \( p(x) = p_0(x) \) and \( q(x) = q_0(x) \). Then from the proof of Theorem 2.2 we conclude that there are not nontrivial nonnegative entire solutions of (1) with any \( p(x) \) and \( q(x) \) satisfying the inequalities
\[
\tilde{p}(r) \geq \tilde{p}_0(r), \quad \tilde{q}(r) \geq \tilde{q}_0(r).
\]
3. The proof of Theorem 1.1

**Proof:** Theorem 1.1 is proved in [13,14] under the conditions (i) and (ii). Let us consider the problem (1) with \( p(x) \) and \( q(x) \) such that

\[
\tilde{p}(r) = \frac{L_1}{r^\lambda \ln^\nu r}, \quad \tilde{q}(r) = \frac{L_2}{r^\mu \ln^\xi r}, \quad r \geq r_0 > 1, \tag{29}
\]

Let (iii) hold, that is,

\[
2 - \mu + \beta(2 - \lambda) = 0, \quad \lambda < 2, \quad 1 - \xi - \beta \nu > 0. \tag{30}
\]

Using the principle of mathematical induction, we prove the estimates

\[
v_{2k}(r) \geq \tilde{C}_{2k}(\ln r) \frac{(a^k b^{k-1})(1-\beta \nu - \xi)}{a^{\beta-1}}, \tag{31}
\]

\[
u_{2k+1}(r) \geq \tilde{C}_{2k+1} r^{2-\lambda} (\ln r) \frac{(a^k b^{k-1})(1-\beta \nu - \xi) \alpha}{a^{\beta-1}} - v, \tag{32}
\]

where \( u_k(r) \) and \( v_k(r) \) are defined in (16) – (18), \( k = 0, 1, 2, \ldots, r > r_k \) for some \( r_k > r_* = \max(\rho, r_0) \). First, we prove (31), (32) for \( k = 0 \). Then (31) follows from (16). Easy to see that

\[
1 - \left( \frac{s}{r} \right)^{N-2} \geq \frac{r - s}{r} \quad \text{for } s \in (0, r). \tag{33}
\]

From (16), (17), (29), (33), we conclude

\[
u_1(r) \geq \frac{L_1}{r} \int_{r_*}^r (r - s)s^{1-\lambda} \ln^{-\nu} s \, ds.
\]

Using (30) and the integration by parts, we obtain

\[
u_1(r) \geq \tilde{C}_1 r^{2-\lambda} \ln^{-\nu} r, \quad r > r_1
\]

for a suitable choice \( \tilde{C}_1 \) and \( r_1 \). Assume (31) and (32) are true for \( k = l - 1 \). In view of (18), (29), (30), (33) and (32) with \( k = l - 1, \) we have

\[
v_{2l} \geq L_2 \tilde{C}_{2l-1} \int_{r_*}^r s^{1-\mu + (2-\lambda)\beta} (\ln s)^{-\xi - \beta \nu + \frac{(a^{l-1} b^{l-1})(1-\xi - \beta \nu) \alpha \beta}{a^{\beta-1}}} \left( 1 - \left( \frac{s}{r} \right)^{N-2} \right) \, ds
\]

\[
\geq \frac{L_2 \tilde{C}_{2l-1}}{r} \int_{r_*}^r \frac{r - s}{s} (\ln s)^{-\xi - \beta \nu + \frac{(a^{l-1} b^{l-1})(1-\xi - \beta \nu) \alpha \beta}{a^{\beta-1}}} \, ds.
\]

Integrating by parts on the right side of last inequality, we deduce (31) with \( k = l \). It follows from (17), (29), (30), (33) and (31) with \( k = l - 1, \) that

\[
u_{2l+1}(r) \geq \frac{L_1}{r} \int_{r_*}^r (r - s)s^{1-\lambda} (\ln s)^{-\nu + \frac{(a^{l+1} b^{l})(1-\xi - \beta \nu) \alpha}{a^{\beta-1}}} \, ds.
\]

Applying the integration by parts, we prove (32) with \( k = l \).
Now we check (20). To do it we estimate the multipliers on left side of (20). For the convenience, we denote
\[ \sigma_k = -\xi - \beta v + \frac{(\alpha^k \beta^k - 1)(1 - \xi - \beta v)\alpha \beta}{\alpha \beta - 1}. \]
It is easy to see that \( \sigma_k > 0 \) for large values of \( k \). Applying (2), (29), (30), (32) and mean value theorem, we get
\[
z_{2k+1}(R) = \int_{aR}^{aR} (aR - s)^n (s) u_{2k+1}^\beta (s) \, ds
\]
\[
\geq L_2 \bar{c}^\beta_{2k+1} \int_{aR}^{aR} (aR - s)^n \left( s^{-\lambda} - \mu + (2 - \lambda) \beta \right) \ln \sigma_k \, ds
\]
\[
\geq L_2 \bar{c}^\beta_{2k+1} \int_{aR}^{aR} (aR - s)^n \left( aR \right)^{\frac{n}{2}} s^{-1} \ln \sigma_k \, ds
\]
\[
\geq \left( \frac{a - 1}{a + 1} \right)^n \bar{L}_{2k+1} \ln \sigma_k R
\]
for large values of \( R \) and \( k \). Using (29), (30) and mean value theorem, we find
\[
\int_{aR}^{bR} (s^{-\lambda} - \mu + (1 - m) \beta - \lambda) \ln \tau_1 (aR - s) \, ds
\]
\[
= L_2 L_1^\beta \int_{aR}^{bR} (s^{-\lambda} - \mu + (1 - m) \beta - \lambda) \ln \tau_1 (aR - s) \, ds
\]
\[
= L_2 L_1^\beta \int_{aR}^{bR} s^{-1} (\ln s)^{\frac{\xi + \beta v}{\beta + n + 1}} \, ds
\]
\[
= L_2 L_1^\beta \frac{\beta + n + 1}{\beta + n + 1} \frac{\beta + n + 1 - \xi - \beta v}{\gamma}
\]
\[
\times \left[ \left( \ln (bR) \right) \frac{(\beta + n + 1 - \xi - \beta v)}{(\beta + n + 1 + \beta + n + 1)} - \left( \ln (aR) \right) \frac{(\beta + n + 1 - \xi - \beta v)}{(\beta + n + 1 + \beta + n + 1)} \right]
\]
\[
= L_2 L_1^\beta \frac{b - a}{\gamma} (\ln (\gamma R))^{-\frac{\xi + \beta v}{\beta + n + 1}}.
\]
where \( R > r_0, \gamma \in (a, b) \). Now from (34), (35), we conclude
\[
\limsup_{R \to \infty} z_{2k+1}(R) \geq \hat{C}_k \lim_{R \to \infty} \left( \ln R \right) \left( \frac{\alpha^k \beta^k}{\beta + n + 1} \right)^{\frac{1}{\beta + n + 1}} = \infty
\]
for large values of \( k \). Since \( \lambda < 2 \) and \( \mu > 2 \) the functions \( r^{-\lambda} \bar{q}(r) \) and \( r^{-m} \bar{p}(r) \) are non-increasing for any \( n \in \mathbb{N}, m > 1 - \lambda \) and large \( r \). Thus, by Theorem 2.2 and Remark 1 there are not nontrivial nonnegative entire solutions of (1).
The case (iv) is treated in a similar way.

We note that (v) follows from (vi) if \( \nu < 1 \) and \( \xi \geq 1 \), (vi) follows from (v) if \( \nu \geq 1 \) and \( \xi < 1 \), and (v) and (vi) are equivalent if \( \nu < 1 \) and \( \xi < 1 \). So, we can prove the theorem for (v) under the condition \( \nu < 1 \) and for (vi) under the condition \( \xi < 1 \).

Let (v) and (vi) hold, that is,

\[
\lambda = 2, \quad \mu = 2, \quad 1 - \xi + \beta(1 - \nu) > 0, \quad \nu < 1.
\]  

As we noted the conditions \( \nu < 1, m \) and \( \nu < \xi \) are nonincreasing functions for any \( m, n \in \mathbb{N} \) and large \( r \). Applying Theorem 2.2 and Remark 1 again, we prove that there are not nontrivial nonnegative entire solutions of (1).

If (vi) and \( \xi < 1 \) hold then the theorem is proved in a very similar manner. \( \blacksquare \)

**Remark 2:** We need the conditions \( \alpha \geq 1, \beta \geq 1 \) to pass to the inequalities (21), (22) for the averages of \( u(x), v(x) \) in the proof of Theorem 2.2. We guess that Theorem 1.1 holds without these assumptions.

**Remark 3:** Let \( p \) and \( q \) have spherical symmetry and \( (u, v) \) be a nonnegative spherically symmetric entire solution of (1). If \( \alpha > 0, \beta > 0 \) then \( (u, v) \) satisfies the following problem

\[
(r^{N-1}u'(r))' = r^{N-1}p(r)v^\alpha(r), \quad r > 0, \quad u'(0) = 0,
\]

\[
(r^{N-1}v'(r))' = r^{N-1}q(r)u^\beta(r), \quad r > 0, \quad v'(0) = 0.
\]

As we noted the conditions \( \alpha \geq 1, \beta \geq 1 \) of Theorem 1.1 and Theorem 2.2 are used for (21) and (22) only. Hence we can state in Theorem 1.1 and Theorem 2.2 the nonexistence of nontrivial nonnegative spherically symmetric entire solutions of (1) without the assumptions \( \alpha \geq 1, \beta \geq 1 \).

### 4. The optimality of Theorem 1.1

In this section, we show the optimality of Theorem 1.1. We assume that the system (1) has spherically symmetric coefficients \( p(x) = p(|x|) \), \( q(x) = q(|x|) \) which satisfy the inequalities

\[
p(r) \leq \frac{L_3}{r^\lambda \ln^\nu r}, \quad q(r) \leq \frac{L_4}{r^\mu \ln^\xi r}, \quad r \geq r_1 > 1,
\]  

where \( L_3 > 0, L_4 > 0 \).
The following statement is proved by a direct verification of the conditions (3) and (4).

**Theorem 4.1:** Let \( N \geq 3, \alpha \beta > 1 \) and \( p \) and \( q \) satisfy (38). If one from the following conditions holds

(i) \( 2 - \mu + \beta (2 - \lambda) < 0, \, 2 - \lambda + \alpha (2 - \mu) < 0; \)
(ii) \( 2 - \mu + \beta (2 - \lambda) = 0, \, \lambda < 2, \, 1 - \xi - \beta \nu < 0; \)
(iii) \( 2 - \lambda + \alpha (2 - \mu) = 0, \, \mu < 2, \, 1 - \nu - \alpha \xi < 0; \)
(iv) \( \lambda = 2, \, \mu = 2, \, 1 - \xi + \beta (1 - \nu) < 0, \, 1 - \nu + \alpha (1 - \xi) < 0, \)

then (1) has positive entire solutions.

**Figure 1.** \( \nu \) and \( \xi \) are any.

**Figure 2.** \( 2 - \mu + \beta (2 - \lambda) = 0 \) and \( \lambda < 2. \)

**Figure 3.** \( 2 - \lambda + \alpha (2 - \mu) = 0 \) and \( \mu < 2. \)
Figure 4. \( \lambda = 2 \) and \( \mu = 2 \).

Let the conditions of Theorem 1.1 and Theorem 4.1 hold with \( L_3 \geq L_1, L_4 \geq L_2 \). Figures 1–4 show values of the parameters \( \lambda, \mu, \nu \) and \( \xi \) in (2) and (38) providing the nonexistence and existence of nontrivial nonnegative entire solutions of (1).

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

A. Gladkov is supported by the ‘RUDN University Program 5-100’ and the State program of fundamental research of Belarus [grant number 1.2.03.1].

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