ASYMPTOTIC BEHAVIOR OF NON-EXPANDING PIECEWISE LINEAR MAPS IN THE PRESENCE OF RANDOM NOISE

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Abstract. We consider the perturbed dynamical system applied to non-expanding piecewise linear maps on [0,1] which describe simplified dynamics of a single neuron. It is known that the Markov operator generated by this perturbed system has asymptotic periodicity with period \( n \geq 1 \). In this paper, we give a sufficient condition for \( n > 1 \), asymptotic periodicity, and for \( n = 1 \), asymptotic stability. That is, we show that there exists a threshold of noises \( \theta^* \) such that the Markov operator generated by this perturbed system displays asymptotic periodicity (asymptotic stability) if a maximum value of noises is less (greater) than \( \theta^* \). This result indicates that an existence of phenomenon called mode-locking is mathematically clarified for this perturbed system.

1. Introduction. The non-expanding piecewise linear map, known as the Nagumo-Sato (NS) model [21], is described as

\[
S_{\alpha,\beta}(x) = \alpha x + \beta \pmod{1},
\]

where \( 0 < \alpha, \beta < 1 \). The NS model corresponds to a special case of Caianiello’s model [7], and it describes the simplified dynamics of a single neuron. It is known that the system (1) shows periodic behavior of the trajectory for almost every \((\alpha,\beta)\). The transformation has one discontinuous point when \( \alpha + \beta > 1 \), and this leads to a complicated structure for periodicity of the NS model. This structure is called a Farey structure defined in [22] and is presented graphically in Figure 1 which shows regions in which \( S_{\alpha,\beta} \) has a periodic point. An important feature of the Farey structure is that there exists a region in which \( S_{\alpha,\beta} \) has a periodic point with period \((m+n)\) between the region with period \( m \) and \( n \). Farey structure is known as Arnold tongues or phase locked regions which were originally studied in circle map models of cardiac arrhythmias [9, 12, 20]. These phase locked regions show the layered structure which is obtained by classifying the parameter space by the rotation number of the maps. In each phase locked region, the maps exhibits a phenomenon called mode-locking in which the rotation number remains constant over the whole region. Such regions have been also observed for other models (e.g. [5, 13, 24]). The NS model shows similar phase locking regions which have been observed in [8, 23], and analyzed in detail in [22]. Specifically, we found explicit parameter regions in which \( S_{\alpha,\beta} \) has rational rotation number \( l/n \) for all \( l/n \) in

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(0, 1). Using explicit formulas of these regions, in this paper, we will show analytically that the periodicity (i.e. rotation number) of $S_{\alpha, \beta}$ is not affected by small perturbations to iterations when parameters are chosen in each phase locked region. This shows stability of mode-locking under small perturbations by an additive noise (see Corollary 1).

In this paper, we consider a perturbed dynamical system in which noise is applied to the NS model $S_{\alpha, \beta}$, that is,

$$x_{t+1} = S_{\alpha, \beta}(x_t) + \xi_t \pmod{1} \text{ for } (\alpha, \beta) \in (0, 1)^2, \tag{2}$$

where $\{\xi_t\}$ are independent random variables each having same density $g$ satisfying $\text{supp}(g) = [0, \theta]$ with $\theta \in [0, 1]$ where $\text{supp}(g) = \{x \in [0, 1] \mid g(x) \neq 0\}$. We discuss two important asymptotic properties for the Markov operator [18, 19] corresponding to the model (2). It is well known that the Markov operator describes asymptotic behaviors of a trajectory. Especially, we focus on the properties of asymptotic periodicity and asymptotic stability which are introduced in §3. These asymptotic behaviors are observed and discussed in [10, 11, 18, 19]. Since the Markov operator generated by the system (2) is clearly constrictive (Definition 3.3), it should be either asymptotically periodic or asymptotically stable (Lemma 3.5). Our main result (Theorem 3.6) shows that, for almost all $(\alpha, \beta) \in (0, 1)^2$, there exists a threshold $\theta_*(\alpha, \beta)$ such that the Markov operator generated by the system (2) displays asymptotic periodicity if $\theta$ is less than $\theta_*(\alpha, \beta)$. On the other hand, if $\theta$ is greater than $\theta_*(\alpha, \beta)$, the Markov operator shows asymptotic stability. In [25], Provatas and Mackey have already showed the same result in the special case of rotation numbers $1/n$. Thus, our main theorem in this paper extends their result to all cases of rotation numbers $l/n$. Our method used for proving our main theorem follows the ideas in [25].

In [14], Inoue considered Frobenius- Perron operator for random maps and showed a sufficient condition for asymptotic periodicity and stability of the operator. This sufficient condition is satisfied automatically if the given transformations are expanding, but it is not clear for non-expanding maps. Therefore, our perturbed model is an examples that extends this result.

The organization of this paper is as follows. In §2, we describe a Farey structure and the concept of a rational characteristic sequence which is defined in [22]. We show some new properties of a rational characteristic sequence which are used for the proof of the main theorem. In §3, we state our main results. In §4, we show the numerical illustrations of our results which describe asymptotic behaviors of the Markov operator generated by the system (2).

2. Farey structure and rational characteristic sequence. In this section, we first recall the Farey structure of the NS model defined in [22]. This is a layered structure (see Fig.1) which shows the regions of parameter space in which $S_{\alpha, \beta}$ has a periodic point. Moreover, we recall a useful mathematical tool, a rational characteristic sequence, which was used to clarify the Farey structure in [22], and we will show further new properties of a rational characteristic sequence which play an important role in the proof of our main theorem.

2.1. Farey structure of the NS model. Let $Pr(n) := \{l < n \mid \text{GCD}(n, l) = 1\}$ for each $n \in \mathbb{N}$ and $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be two projections with $\pi_1(\alpha, \beta) = \alpha$ and $\pi_2(\alpha, \beta) = \beta$, and define a Farey structure as follows. Let $E$ be a bounded subset of $\mathbb{R}^2$ and let $\{D_{n,l}\}_{n \in \mathbb{N}, l \in Pr(n)}$ be a family of subsets of $E$ satisfying the following
properties. For each \( \alpha \in \pi_1(E) \), there exist real numbers \( B^U_{n,l}(\alpha) \) and \( B^L_{n,l}(\alpha) \) such that

\[
\pi_2(\pi_1^{-1}(\{\alpha\}) \cap D_{n,l}) = [B^L_{n,l}(\alpha), B^U_{n,l}(\alpha)].
\] (3)

We denote \( D_{n,l} \prec D_{n',l'} \) if \( B^U_{n,l}(\alpha) < B^L_{n',l'}(\alpha) \) holds for any \( \alpha \in \pi_1(E) \). We then consider a two parameter family of transformations of \([0,1) \), \( T_{\alpha,\beta} : [0,1) \rightarrow [0,1) \) \( (\alpha, \beta) \in E \). In this paper, if we write \((n,l)\), then \( n \) and \( l \) always satisfy \( n \in \mathbb{N} \geq 2 \) and \( l \in Pr(n) \).

**Definition 2.1.** \( \{T_{\alpha,\beta}\}_{(\alpha,\beta) \in E} \) possesses a Farey structure in a parameter subspace \( E \subset \mathbb{R}^2 \) if there exists \( \{D_{n,l}\}_{(n,l)} \subset E \) satisfying the property (3) such that

(i) \( \text{Leb}(D_{n,l}) > 0 \) for all \((n,l)\),

(ii) for each \((n,l)\), \( T_{\alpha,\beta} \) with \( (\alpha, \beta) \in D_{n,l} \) has a periodic point with period \( n \),

(iii) \( D_{n+1,1} \prec D_{n,1} \) and \( D_{n,1} \prec D_{n+1,n} \) hold for every \( n \in \mathbb{N} \). If \( (n,l) \) and \( (n',l') \) satisfying \( nl - n'l = 1 \) and \( D_{n,l} \prec D_{n',l'} \), then \( D_{n,l} \prec D_{n+1,n' + l'} \prec D_{n',l'} \).

To state the next Proposition 1, which shows that \( \{S_{\alpha,\beta}\} \) has this Farey structure, we define two functions \( B^U_{n,l}(\alpha) \) and \( B^L_{n,l}(\alpha) \) and sets \( \{D_{n,l}\}_{(n,l)} \) as follows;

\[
B^U_{n,l}(\alpha) = (1 - \alpha) \left( \frac{1}{1 - \alpha^n} \sum_{m=1}^{n-1} k_m \alpha^m + 1 \right),
\] (4)

\[
B^L_{n,l}(\alpha) = (1 - \alpha) \left( \frac{1}{1 - \alpha^n} \sum_{m=1}^{n-1} k_m \alpha^m + 1 - \frac{\alpha^{n-1} - \alpha^n}{1 - \alpha^n} \right),
\] (5)

and

\[
D_{n,l} = \{ (\alpha, \beta) \in (0,1)^2 \mid B^L_{n,l}(\alpha) \leq \beta < B^U_{n,l}(\alpha) \},
\] (6)

where

\[
k_m := \left\lfloor \frac{(m+1)l}{n} \right\rfloor - \left\lfloor \frac{ml}{n} \right\rfloor \text{ for } m \in \mathbb{Z},
\] (7)
where \([x]\) is the integer part of \(x\). We call this sequence \(\{k_i\}_{i \in \mathbb{Z}}\) defined by (7) a **rational characteristic sequence**. We first introduce useful properties of the rational characteristic sequence. By using properties of the rational characteristic sequence, we will show an important inequality (Lemma 2.3) of the function \(F_{n,l}(i)\) which is used to prove Proposition 4. We first introduce useful properties of the rational characteristic sequence. See [22] for the details.

**Proposition 1.** ([22], Theorem 4.1) \(\{S_{\alpha,\beta}\}_{(\alpha,\beta) \in (0,1)^2}\) possesses the Farey structure in \((0,1)^2\) with \(\{D_{n,l}\}_{(n,l)}\) defined by (6).

To avoid messy calculations, for \(i = 2, \cdots, n\) we define

\[
F_{n,l}(i) = \frac{1}{1 - \alpha} \sum_{m=1}^{i-1} k_m \alpha^m, \quad \alpha \in (0, 1). \tag{8}
\]

**Remark 1.** The Proposition 1 show that the following relations hold for rational characteristic sequences \(\{k_m\}, \{k'_m\}\) and \(\{\hat{k}_m\}\) with respect to \((n,l)\), \((n',l')\) and \((n+n',l+l')\), respectively, with \(nl' - n'l = 1\) for \(\alpha \in (0, 1)\),

\[
B_{n',l'}(\alpha) > B_{n,l}(\alpha), \quad B_{n',l+1}(\alpha) > B_{n+n',l+l'}(\alpha), \quad B_{n+n',l+l'}(\alpha) > B_{U,l}(\alpha).
\]

By using explicit formulas (4), (5) and (8), these can be rewritten as follows respectively:

\[
F_{n',l'}(n') - F_{n,l}(n) > \frac{\alpha^{n'-1} - \alpha^n}{1 - \alpha^n} \quad \text{for} \quad \alpha \in (0, 1), \tag{9}
\]

\[
F_{n',l'}(n') - F_{n+n',l'+1}(n+n') > \frac{\alpha^{n'-1} - \alpha^n}{1 - \alpha^n} \quad \text{for} \quad \alpha \in (0, 1), \tag{10}
\]

\[
F_{n+n',l+l'}(n+n') - F_{n,l}(n) > \frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1 - \alpha^{n+n'}} \quad \text{for} \quad \alpha \in (0, 1). \tag{11}
\]

These inequalities will be used to prove the key lemma (Lemma 2.3). Note that these inequalities (9), (10) and (11) are the properties of a rational characteristic sequence, which is not necessary to the NS model.

**Remark 2.** The system \(S_{\alpha,\beta}\) has a periodic point with period \(n\) when \((\alpha, \beta) \in D_{n,l}\). The set of these periodic points with period \(n\) is given by

\[
\text{Per}_n(S_{\alpha,\beta}) = \left\{ \frac{\beta}{1 - \alpha} - A_i(\alpha) \bigg| i = 0, \cdots, n-1 \right\}, \tag{12}
\]

where, for \(i = 0, 1, \cdots, n-1,\)

\[
A_i(\alpha) = \frac{1}{1 - \alpha^n} \left( \sum_{m=0}^{i-1} k_m \alpha^{i-m-1} + \sum_{m=i}^{n-1} k_m \alpha^{n+i-m-1} \right). \tag{13}
\]

These periodic points in \(\text{Per}_n(S_{\alpha,\beta})\) will be useful for proving the main theorem (Theorem 3.6).

2.2. **Properties of rational characteristic sequence.** The rational characteristic sequence is known as mechanical words, rotation words or Christoffel words [3, 4] and good sequence in [22], and if \(l/n\) is replaced by an irrational number, then it is known as Sturmian words or characteristic sequence [2, 6]. One of our mathematical tools is the function \(F_{n,l}(i)\) (see Eq.(8)) defined by the rational characteristic sequence. By using properties of the rational characteristic sequence, we will show an important inequality (Lemma 2.3) of the function \(F_{n,l}(i)\) which is used to prove Proposition 4. We first introduce useful properties of the rational characteristic sequence. See [22] for the details.
Proposition 2. \cite[Proposition 2.2]{22} Let \( \{k_m\}_{m \in \mathbb{Z}} \) be a rational characteristic sequence with respect to \((n, l)\). We then have the following properties.

(i) \( k_{m+n} = k_m \) for \( m \in \mathbb{Z} \),
(ii) \( k_{n-1-m} = k_m \) for \( m \in \mathbb{Z}, m \notin n\mathbb{Z}, n\mathbb{Z} - 1 \),
(iii) \( k_{m-l} = k_m \) for \( m \in \mathbb{Z}, m \notin n\mathbb{Z}, n\mathbb{Z} - 1 \),

where \( l := \min\{t \in \mathbb{N} \mid tl = 1 \pmod{n}\} \). Note that \( k_0 = 0 \) and \( k_{n-1} = 1 \) always hold obviously.

Proposition 3. \cite[Proposition 2.3]{22} Let \( \{k_m\}_{m \in \mathbb{Z}} \) be a rational characteristic sequence with respect to \((n, l)\) and \( \{k'_m\}_{m \in \mathbb{Z}} \) be another rational characteristic sequence with respect to \((n', l')\). If \( \frac{l}{n} < \frac{l'}{n'} \) and \( nl - n'l = 1 \), then the sequence \( \{k'_m\}_{m \in \mathbb{Z}} \) defined by

\[
\hat{k}_m := \begin{cases} 
  k_m & \text{for } m = 0, \ldots, n - 1 \\
  k'_{m-n} & \text{for } m = n, \ldots, n + n' - 1 
\end{cases}
\]

and \( \hat{m} := \hat{k}_m \) if \( \hat{m} = m + t(n + n') \) with \( m = 0, \ldots, n + n' - 1 \) and \( t \in \mathbb{Z}\setminus\{0\} \) is the rational characteristic sequence with respect to \((n + n', l + l')\).

We prepare useful equations for \( F_{n,l}(i) \) obtained from the above propositions.

Lemma 2.2.

\[
\begin{align*}
F_{n+n',l+l'}(i) &= \begin{cases} 
  F_{n,l}(i) & (i = 1, \ldots, n) \\
  \frac{1-a^n}{1-a^i} F_{n,l}(n) + \frac{a^n(1-a^{-n})}{1-a^i} F_{n',l'}(i-n) & (i = n+1, \ldots, n+n') 
\end{cases} 
\tag{14}
\end{align*}
\]

\[
\begin{align*}
F_{n+n',l+l'}(i) &= \begin{cases} 
  F_{n',l'}(i) & (i = 1, \ldots, n' - 1) \\
  F_{n',l'}(n') - \frac{a^{n'-1}}{1-a^n} & (i = n') \\
  \frac{1-a^{n'}}{1-a^n} F_{n',l'}(n') + \frac{a^n(1-a^{-n})}{1-a^i} F_{n,l}(i-n') - \frac{a^{n'-1}-a^{n'}}{1-a^i} & (i = n' + 1, \ldots, n' + n - 1) \\
  \frac{1-a^{n'}}{1-a^n} F_{n',l'}(n') + \frac{a^n(1-a^{-n})}{1-a^{n+n'}} F_{n,l}(n) - \frac{a^{n'-1}-a^{n'}}{1-a^{n+n'}} & (i = n' + n) 
\end{cases} 
\tag{15}
\end{align*}
\]

Proof. Equation (14) comes clearly from Proposition 3. Equation (15) is also obtained immediately by applying the property (ii) in Proposition 2 to \( \{k_i\} \).

The next lemma which is the most important for the proof of Proposition 4 is a new property of \( F_{n,l}(n) \).

Lemma 2.3. For any \((n, l)\), the inequality

\[
\frac{a^{n-1}-a^n}{1-a^n} < F_{n,l}(n) - F_{n,l}(i) < \frac{a^i}{1-a^i} \tag{16}
\]

holds for \( \alpha \in (0, 1) \) and \( i = 2, \ldots, n-1 \).

Proof. See appendix.
3. Random dynamical system. In this section, we first show a property of a preimage of zero by $S_{\alpha,\beta}$, and next prove the main theorem which states that a Markov operator generated by (2) has one of two different asymptotic properties depending on the maximum value $\theta$ of the noise $\xi$.

3.1. Preimage of zero for NS model. Lemma 3.6 in [17] show that if the set of preimages is finite, then the map has a periodic solution. The next proposition gives a new property of the preimage of zero for the NS model, which concludes that the set of preimages is finite for any parameter $(\alpha, \beta) \in D_{n,l}$. Remark that zero is a preimage of the discontinuity point of NS model. This result is used for the proof of Lemma 3.7.

Proposition 4. Assume that $(\alpha, \beta) \in D_{n,l}$, then

$$S_{\alpha,\beta}^{-i}(0) = \sum_{m=1}^{i} \frac{k_{n-i+m-1} - \beta}{\alpha^m} \in [0,1], \quad (i = 1, \cdots, n-1),$$

where $\{k_m\}$ is a rational characteristic sequence with respect to $(n,l)$. Moreover, for $i = n$, $S_{\alpha,\beta}^{-n}(0)$ is not in $[0,1]$.

Proof. Since the range of the map $S_{\alpha,\beta}$ is $[0, \alpha + \beta - 1] \cup [\beta, 1]$, it is obvious that, for any $x \in [0, 1]$, there exists $S_{\alpha,\beta}^{-i}(x)$ in $[0,1]$ when $S_{\alpha,\beta}^{-i}(x) \in [0, \alpha + \beta - 1] \cup [\beta, 1]$. Moreover, if there exists $S_{\alpha,\beta}^{-i}(0)$ in $[0, 1]$, this is unique. Next, for $t = 0, 1, 2, \cdots$, we write the map $S_{\alpha,\beta}$ as $x_{t+1} = S_{\alpha,\beta}(x_t) = \alpha x_t + \beta - \bar{k}_t$ for an initial point $x_0$, where $\bar{k}_t = 0$ if $x_t \in [0, \frac{1-\beta}{\alpha})$, and $\bar{k}_t = 1$ if $x_t \in \left[\frac{1-\beta}{\alpha}, 1\right)$. Then, $S_{\alpha,\beta}^{-1}(0) = \frac{\bar{k}_0 - \beta}{\alpha}$ and the following equation follows inductively,

$$S_{\alpha,\beta}^{-i}(0) = \sum_{m=1}^{i} \frac{\bar{k}_{n-i+m-1} - \beta}{\alpha^m} \quad \text{for} \quad i = 1, \cdots, n-1,$$

where $\{\bar{k}_m\}_{m=0}^{n-1} \in \{0,1\}^n$. Now we will show that there exists a sequence $\{\bar{k}_m\}_{m=0}^{n-1} \subset \{0,1\}$ such that above $S_{\alpha,\beta}^{-1}(0)$ is in $[0,1]$ for $i = 1, \cdots, n-1$. To show this, we choose a rational characteristic sequence $\{k_m\}_{m=0}^{n-1}$ with respect to $(n,l)$ as $\{\bar{k}_m\}_{m=0}^{n-1}$. More precisely, it is enough to put $\bar{k}_{i+m+1} = k_{n-i+m-1}$ for $i = 1, \cdots, n-1$ and $m = 1, \cdots, i$. Applying this, we will show

$$S_{\alpha,\beta}^{-i}(0) \in [\beta, 1) \text{ if } k_{n-i-1} = 0,$$

$$S_{\alpha,\beta}^{-i}(0) \in [0, \alpha + \beta - 1) \text{ if } k_{n-i-1} = 1.$$  

for $i = 2, \cdots, n-1$.

In the case that $k_{n-i-1} = 0$, we can rewrite (18) as follows by using the properties of rational characteristic sequences (Prop.2):

$$\frac{1-\alpha}{1-\alpha^i} \left( \sum_{m=1}^{i-1} k_m \alpha^m + 1 - \alpha^i \right) < \beta \leq \frac{1-\alpha}{1-\alpha^{i+1}} \left( \sum_{m=1}^{i-1} k_m \alpha^m + 1 \right).$$

Since $\beta$ satisfies $B_{n,l}^U(\alpha) \leq \beta < B_{n,l}^U(\alpha)$, it suffices to show that

$$\frac{1-\alpha}{1-\alpha^{i+1}} \left( \sum_{m=1}^{i-1} k_m \alpha^m + 1 \right) - B_{n,l}^U(\alpha) > 0,$$

$$B_{n,l}^L(\alpha) - \frac{1-\alpha}{1-\alpha^i} \left( \sum_{m=1}^{i-1} k_m \alpha^m + 1 - \alpha^i \right) > 0.$$
By using the explicit formulas (4) and (5), these inequalities can be rewritten as

\[
\frac{1}{1 - \alpha^{i+1}} \sum_{m=1}^{i} k_m \alpha^m - \frac{1}{1 - \alpha^n} \sum_{m=1}^{n-1} k_m \alpha^m + \frac{\alpha^{i+1}}{1 - \alpha^{i+1}} > 0, \tag{20}
\]

\[
\frac{1}{1 - \alpha^n} \sum_{m=1}^{n-1} k_m \alpha^m - \frac{1}{1 - \alpha^i} \sum_{m=1}^{i-1} k_m \alpha^m - \frac{\alpha^{n-1} - \alpha^n}{1 - \alpha^n} > 0, \tag{21}
\]

for any \( i = 2, \cdots, n - 1 \). These inequalities (20) and (21) are allowed by Lemma 2.3. On the other hand, in the case that \( k_{n-i-1} = 1 \), by similar arguments one can find that we should show next inequalities;

\[
\frac{1}{1 - \alpha^{i}} \sum_{m=1}^{\alpha} k_m \alpha^m - \frac{1}{1 - \alpha^n} \sum_{m=1}^{n-1} k_m \alpha^m + \frac{\alpha^{i}}{1 - \alpha^{i}} > 0, \tag{22}
\]

\[
\frac{1}{1 - \alpha^n} \sum_{m=1}^{n-1} k_m \alpha^m - \frac{1}{1 - \alpha^{i+1}} \sum_{m=1}^{i} k_m \alpha^m - \frac{\alpha^{n-1} - \alpha^n}{1 - \alpha^n} > 0, \tag{23}
\]

for any \( i = 2, \cdots, n - 2 \). These inequalities (22) and (23) are allowed by Lemma 2.3.

Finally, we can show that \( S_{\alpha, \beta}^{-1}(0) < 0 \) by using the inequality \( B_{\alpha, \beta}^L(\alpha) < \beta \). Thus, \( S_{\alpha, \beta}^{-1}(0) \) is in \([0, 1] \) uniquely for \( i = 1, \cdots, n - 1 \). These complete the proof. \( \square \)

3.2. Main theorems. Let \((X, \mathcal{A}, \mu)\) be a measure space, \( P : L^1 \to L^1 \) be a Markov operator \((Pf \geq 0, \|Pf\| = \|f\| \text{ if } f \geq 0, f \in L^1)\) and \( D = \{f \in L^1(X, \mathcal{A}, \mu) \mid f \geq 0, \|f\| = 1\} \). Here \( \|f\| \) is \( L^1 \)-norm of \( f \) relative to measure \( \mu \). An element in \( D \) is called a density. We follow [18] in our definitions and notation. Remark that for a finite state space Markov chain, a Markov operator corresponds to the transpose of the transition matrix.

**Definition 3.1.** \( \{P^t\} \) is said to be **asymptotically stable** if there exists a unique \( f_* \in D \) such that \( Pf_* = f_* \) and \( \lim_{t \to \infty} \|P^t f - f_*\| = 0 \) for every \( f \in D \).

**Definition 3.2.** \( \{P^t\} \) is said to be **asymptotically periodic** if there exists an integer \( r \), two sequences of nonnegative functions \( g_i \in D \) and \( h_i \in L^\infty, i = 1, \cdots, r \), and an operator \( Q : L^1 \to L^1 \) such that for every \( f \in L^1 \), \( Pf \) may be written in the form

\[
Pf(x) = \sum_{i=1}^{r} \lambda_i(f)g_i(x) + Qf(x), \tag{24}
\]

where

\[
\lambda_i(f) = \int_X f(x) h_i(x) \mu(dx). \tag{25}
\]

Moreover functions \( g_i \) and operator \( Q \) satisfy the following properties:

(i) \( g_i(x)g_j(x) = 0 \) for all \( i \neq j \);

(ii) For each integer \( i \) there exists a unique integer \( \rho(i) \) such that \( P g_i = g_{\rho(i)} \). Further \( \rho(i) \neq \rho(j) \) for \( i \neq j \) and thus operator \( P \) just serves to permute the functions \( g_i \).

(iii) \( \|P^t Qf\| \to 0 \) as \( t \to \infty \) for every \( f \in L^1 \).
Remark 3. Note that \( \{ P^t \} \) is asymptotically stable if and only if \( \{ P^t \} \) is asymptotically periodic with \( r = 1 \).

Definition 3.3. Let \( (X, \mathcal{A}, \mu) \) be a finite measure space, \( \mu(X) < \infty \). A Markov operator \( P \) is called constrictive if there exists a \( \delta > 0 \) and \( \kappa < 1 \) such that for every \( f \in D \) there is an integer \( t_0(f) \) for which
\[
\int_E P^t f(x) \mu(dx) \leq \kappa \quad \text{for all } t \geq t_0(f) \quad \text{and } E \text{ with } \mu(E) \leq \delta.
\] (26)

Proposition 5. ([19], Theorem 5.3.1) If \( P \) is a constrictive Markov operator, then \( \{ P^t \} \) is asymptotically periodic.

The next lemma is used for the proof of Theorem 3.6(ii).

Lemma 3.4. ([19], Theorem 5.6.1) Let \( P \) be a constrictive Markov operator. Assume there is a set \( A \subset X \) of nonzero measure, \( \mu(A) > 0 \), with the property that for every \( f \in D \) there is an integer \( t_0(f) \) such that \( P^t f(x) > 0 \) for almost all \( x \in A \) and all \( t > t_0(f) \). Then \( \{ P^t \} \) is asymptotically stable.

Next consider a finite measure space \( ([0, 1], \mathcal{B}([0, 1]), \mu) \) where \( \mu \) is the Lebesgue measure on \([0, 1]\) and a process on \([0, 1]\) defined by
\[
x_{t+1} = T(x_t) + \xi_t \pmod{1},
\] (27)
where \( T : [0, 1] \to [0, 1] \) is measurable and \( \xi_0, \xi_1, \cdots \) are independent random variables each having the same density \( g \). The Markov operator \( \mathcal{P} : L^1([0, 1]) \to L^1([0, 1]) \) of this system is defined by
\[
\mathcal{P} f(x) = \int_{[0, 1]} f(y) \sum_{i=0}^{1} g(x - T(y) + i) dy \quad \text{for } f \in L^1.
\] (28)
See [15] for the details of these arguments and the next theorem.

Lemma 3.5. ([15], Theorem 2.8) The Markov operator \( \mathcal{P} : L^1([0, 1]) \to L^1([0, 1]) \) defined by (28) is constrictive, and this means that, the sequence \( \{ \mathcal{P}^t \} \) is asymptotically periodic.

Now we come to consider our random dynamical system (2). From Lemma 3.5, we know that the Markov operator \( \mathcal{P} : L^1([0, 1]) \to L^1([0, 1]) \) defined by (28) generated by (2) is asymptotically periodic. Therefore, the next main theorem gives a sufficient condition for \( r > 1 \) (asymptotic periodicity) and for \( r = 1 \) (asymptotic stability).

Theorem 3.6. Let \( \mathcal{P} \) be the Markov operator corresponding to system (2). Fix \( n \in \mathbb{N} \) and \( l \in Pr(n) \). Assume that \( (\alpha, \beta) \in D_{n,l} \). Then there exists \( \theta_*(\alpha, \beta) = \theta_*(\alpha, \beta, n, l) \in [0, 1] \) such that
(i) if \( \theta \leq \theta_*(\alpha, \beta) \), then \( r = n > 1 \), and \( \{ \mathcal{P}^t \} \) is asymptotically periodic with period \( n \),
(ii) if \( \theta > \theta_*(\alpha, \beta) \), then \( r = 1 \), and \( \{ \mathcal{P}^t \} \) is asymptotically stable,
where \( r \) is the number defined in Definition 3.2, and
\[
\theta_*(\alpha, \beta) = \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n} - \beta + B_{n,l}^L(\alpha).
\] (29)

Remark 4. Note that the inequality \( \theta \leq \theta_*(\alpha, \beta) \) means \( (\alpha, \beta + \xi) \in D_{n,l} \) with arbitrary \( \xi \in [0, \theta] \).
We first show the next result which is a key lemma for the proof of Theorem 3.6(i).

**Lemma 3.7.** Assume that \((\alpha, \beta), (\alpha, \beta + \theta) \in D_{n,l}\) for \(n \in \mathbb{N}\) and \(l \in Pr(n)\). For \(i = 0, \ldots, n-1\), let \(G_i\) be an interval defined by

\[
G_i = \left[\frac{\beta}{1-\alpha} - A_i(\alpha), \frac{\beta + \theta}{1-\alpha} - A_i(\alpha)\right],
\]

where \(A_i(\alpha)\) is defined by (13). Then, for \(i = 0, 1, \ldots, n-2\),

\[
x_{t+1} \in G_{i+1} \quad \text{if} \quad x_t \in G_i \quad \text{and} \quad x_{t+1} \in G_0 \quad \text{if} \quad x_t \in G_{n-1},
\]

where \(x_{t+1}\) is determined by the system (2). Moreover, there exists a number \(N \in \mathbb{N}\) such that \(x_{t+1} \in \bigcup_{i=0}^{n-1} G_i\) for \(t > N\) and a.e.\(x_0 \in [0, 1]\) \(\setminus \bigcup_{i=0}^{n-1} G_i\).

**Proof.** First we show the statement (31). Let \(\{k_i\}_{i \in \mathbb{Z}}\) be a rational characteristic sequence with respect to \((n, l)\). From the definition of \(A_i(\alpha)\), the following relations hold:

\[
A_{i+1} = \alpha A_i(\alpha) + k_i \quad \text{for} \quad i = 0, \ldots, n-2 \quad \text{and} \quad A_0 = \alpha A_{n-1}(\alpha) + k_{n-1}.
\]

Since \(x_{t+1} = S_{\alpha,\beta}(x_t) + \xi_t \pmod{1} = \alpha x_t + \beta - k_t + \xi_t\) for all \(x_t \in G_i\) and any noise \(\xi_t \in [0, \theta]\), by using equation (32), we have

\[
x_{t+1} \in \left[\alpha \left(\frac{\beta}{1-\alpha} - A_i(\alpha)\right) + \beta - k_i + \xi_t, \alpha \left(\frac{\beta + \theta}{1-\alpha} - A_i(\alpha)\right) + \beta - k_i + \xi_t\right]
\]

\[
= \left[\frac{\beta}{1-\alpha} - A_{i+1}(\alpha) + \xi_t, \frac{\beta + \theta}{1-\alpha} - A_{i+1}(\alpha) - (\theta - \xi_t)\right]
\]

\[
\subset \left[\frac{\beta}{1-\alpha} - A_{i+1}(\alpha), \frac{\beta + \theta}{1-\alpha} - A_{i+1}(\alpha)\right] = G_{i+1}
\]

Therefore, \(x_{t+1} \in G_{i+1}\) holds for \(x_t \in G_i\). In the same fashion, we can show that \(x_{t+1} \in G_0\) holds for all \(x_t \in G_{n-1}\) and any noise \(\xi_t \in [0, \theta]\).

Next we will show that, for a.e.\(x_0 \in [0, 1]\) \(\setminus \bigcup_{i=0}^{n-1} G_i\), there exists a number \(N \in \mathbb{N}\) such that \(x_{t+1} \in \bigcup_{i=0}^{n-1} G_i\) for \(t > N\). Let the set \(M\) be the interval \([S_{\alpha,\beta+\theta}(0), S_{\alpha,\beta}^{-1}(0)]\). Since the \(\xi_t\) have density \(g\) with \(\text{supp}\{g\} = [0, \theta]\), by using Proposition 4, we have

\[
\bigcup_{(\xi_0, \ldots, \xi_{n-1}) \in [0, \theta]^{n-1}} S_{\alpha,\beta+\xi_{n-1}}^{-1} \cdots S_{\alpha,\beta+\xi_1}^{-1} (M) \subset [S_{\alpha,\beta+\theta}(0), S_{\alpha,\beta}^{-1}(0)] \subset [0, 1],
\]

for \(i = 2, \ldots, n-1\). Moreover, we have the following relation from Proposition 4,

\[
\bigcup_{(\xi_0, \ldots, \xi_{n-1}) \in [0, \theta]^{n-1}} S_{\alpha,\beta+\xi_{n-1}}^{-1} \cdots S_{\alpha,\beta+\xi_0}^{-1} (M) \not\subset [0, 1].
\]

This means that there do not exist any initial points which belong to \(M\) after the \(n\)-th iteration by the system (2). For any interval \(I\), if \(I \cap M \neq \emptyset\), then \(\text{supp}\{P_I\}\) may be divided into two intervals since \(S_{\alpha,\beta}\) has a discontinuity point in \(M\). However, (34) and (35) shows that such divisions occur less than \((n-1)\) times. If some interval \(I\) develops by this system without divisions, the Lebesgue measure of iterated sets goes to \(\frac{\theta}{1-\alpha}\) after many iterations. From these facts, consider the iteration of an entire space \([0, 1]\), one have

\[
\lim_{t \to \infty} \left|\text{supp}\{P^t_{I[0,1]}\}\right| \leq \frac{n \theta}{1-\alpha}.
\]
On the other hand, since the set $\cup G_i$ is invariant for the iteration, we have that
\[
\left| \text{supp}\{P^t 1_{[0,1]}\} \right| \geq \sum_{i=0}^{n-1} |G_i| = \frac{n\theta}{1-\alpha} \quad \text{for} \quad t \geq 1.
\] (37)
Therefore, we have
\[
\lim_{t \to \infty} \left| \text{supp}\{P^t 1_{[0,1]}\} \right| = \frac{n\theta}{1-\alpha}.
\] (38)
Finally, since $|\cup_i G_i| = \sum_i |G_i| = \frac{n\theta}{1-\alpha}$, we can find that there exists a number $N \in \mathbb{N}$ such that $x_{t+1} \in \bigcup_{i=0}^{n-1} G_i$ for $t > N$ and a.e. $x_0 \in [0,1] \setminus \bigcup_{i=0}^{n-1} G_i$. This completes the proof. \)

Let $c$ be the discontinuity point of NS model, i.e. $c = \frac{1-\beta}{\alpha}$. Then we have the following corollary of Lemma 3.7.

**Corollary 1.** Assume that $(\alpha, \beta) \in D_{n,l}$ and $\theta \leq \theta_*(\alpha, \beta)$. Then the rotation number of the perturbed NS model (2) is given by
\[
\rho = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} 1_{[c,1)}(x_i) = \frac{l}{n}
\] (39)
for $\mu$-a.e. $x_0$ and almost every realization of the system.

**Proof.** Let $\{k_i\}_{i \in \mathbb{Z}}$ be a rational characteristic sequence with respect to $(n,l)$. When $\theta \leq \theta_*(\alpha, \beta)$, there exists $t$ such that $x_i \in G_0$ and $x_{i+t}$ moves in $\{G_i\}_{i=0}^{n-1}$ periodically. Since $k_i = 1$ if and only if $x_i \in [c,1)$, and $\sum_{i=0}^{n-1} k_i = l$, one can immediately calculate $\rho(x_0) = \frac{l}{n}$ for any initial point $x_0$. \)

**Proof of Theorem 3.6(i).** From Lemma 3.5, we have shown that $\{P^t\}$ is asymptotically periodic that $r = n$ and that the period equals $n$. This follows by using Lemma 3.7, since the permutation in the definition of asymptotic periodicity for $\{P^t\}$, (ii) in Definition 3.2, becomes
\[
\rho = \begin{pmatrix} 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & 0 \end{pmatrix}.
\] (40)
The proof of part (i) of the theorem is completed. \)

**Proof of Theorem 3.6(ii).** We use the idea based on the method in [25].

First consider the case $\beta = P_l^{n,1}(\alpha)$ and $\theta > \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n}$ for every $\alpha \in (0,1)$. In this case, the attracting region of phase space is given by
\[
G := \bigcup_{i=0}^{n-1} G_i \text{ (mod 1)},
\] (41)
where $G_i$ is defined by equation (30). Note that $\{G_i\}$ are not always disjoint for $\theta > \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n}$. Suppose $f_0$ is supported on $G$. If not, then $P^t f_0$ will be supported on $G$ for sufficient large $t$. Next take an arbitrary interval $(a_0,b_0) \in G$ and show that the restriction of $f_0$ to $(a_0,b_0)$, written as $f'_0 = f_0 |_{(a_0,b_0)}$, will eventually spread out to fill the entire attracting region $G$. Now assume $(a_0, b_0) \in G_i (\text{mod 1})$. Then there is the part of $P^n f_0$ which returns to the set $G_i (\text{mod 1})$ after the $n$-th iteration of $f'_0$. Then iterate this remainder $n$ times more, again retaining only that part in
the set \( G_i(\text{mod } 1) \). By continuing this procedure and considering the times \( t = nk \), \( k = 1, 2, 3, \cdots \), some algebra yields

\[
\text{supp}\{P^{nk} f'_0\} = \left[\max\{\min G_i, q_1\}, \min\{\max G_i, q_2\}\right]
\]

where

\[
\begin{align*}
\min G_i &= \frac{\beta}{1 - \alpha} - A_i(\alpha), \quad \max G_i = \frac{\beta + \theta}{1 - \alpha} - A_i(\alpha), \\
q_1 &= \alpha^{nk} a_0 + \beta \sum_{j=0}^{nk-1} \alpha^j - \sum_{j=0}^{nk-1} k_{i+j}, \\
q_2 &= \alpha^{nk} b_0 + (\beta + \theta) \sum_{j=0}^{nk-1} \alpha^j - \sum_{j=0}^{nk-1} k_{i+j}.
\end{align*}
\]

Taking the limit \( k \to \infty \), we have

\[
\begin{align*}
\lim_{k \to \infty} q_1 &= \frac{\beta}{1 - \alpha} - \left(\sum_{j=0}^{n-1} \alpha^{n-1-j} k_{i+j}\right) \sum_{k'=0}^{\infty} \alpha^{nk'} \\
&= \frac{\beta}{1 - \alpha} - \frac{1}{1 - \alpha} \sum_{j=0}^{n-1} \alpha^{n-1-j} k_{i+j} \\
&= \frac{\beta}{1 - \alpha} - A_i(\alpha),
\end{align*}
\]

and

\[
\lim_{k \to \infty} q_2 = \frac{\beta + \theta}{1 - \alpha} - A_i(\alpha).
\]

Therefore (42) becomes

\[
\lim_{k \to \infty} \text{supp}\{P^{nk} f'_0\} = G_i.
\]

Next we will show that there exists \( \epsilon > 0 \) such that

\[
\text{supp}\{P_1 G_{\phi(n-1)^{-1}}\} = G_{\phi(n-1)} \cup [0, \epsilon],
\]

where \( \phi(i) = i\hat{t}(\text{mod } n) \). The integer \( \hat{t} \) is defined in Proposition 2. Note that \( G_{\phi(n-1)} \) is the closest interval to 1 in all \( \{G_i\} \). To obtain equation (44), we need to demonstrate that

\[
\frac{\beta + \theta}{1 - \alpha} - A_{\phi(n-1)} > 1.
\]

Use the assumptions \( \beta = B_{n,\hat{t}}^L(\alpha) \) and \( \theta > \frac{\alpha^{n-1}(1-\alpha)^2}{1-\alpha^n} \), we have

\[
\frac{\beta + \theta}{1 - \alpha} - A_{\phi(n-1)} > 1 + \frac{1}{1 - \alpha^n} \left(\sum_{m=0}^{n-1} k_m \alpha^m \right.
\]

\[
- \left. \phi(n-1)^{-1} \sum_{m=0}^{\phi(n-1)-1} k_m \alpha^{\phi(n-1)-m-1} - \sum_{m=\phi(n-1)}^{n-1} k_m \alpha^{n+\phi(n-1)-m-1}\right).
\]
By using the properties of a rational characteristic sequence (Prop.2), we can see
\[ \sum_{m=0}^{n-1} k_m \alpha^m - \sum_{m=0}^{\phi(n-1)-1} k_m \alpha^{\phi(n-1)-m-1} - \sum_{m=\phi(n-1)}^{n-1} k_m \alpha^{n+\phi(n-1)-m-1} = 0. \] (46)
Thus equation (45) holds. From equation (43), there exists a large number \( N_0 \)
such that \( \text{supp}\{P_{N_0}^t f_0\} = G_1 \). In particular, we can choose the number \( N_1 \geq N_0 \)
such that \( \text{supp}\{P_{N_1}^t f_0\} = G_{\phi(n-2)} \). From (44), \( \text{supp}\{P_{N_1+1}^t f_0\} = G_{\phi(n-1)} \cup [0, \epsilon] \).
Allowing \([0, \epsilon] \in G_0\) to play the role of \([a_0, b_0]\) above, eventually \([0, \epsilon]\) will spread over the entire interval \( G_0 \). Thus, there exists a large number \( N_2 \) such that
\[ \text{supp}\{P_{N_2}P_{N_1+1}^t f_0\} = G_{\phi(n-1)} \cup G_0. \] (47)
Continuing this argument, \((a_0, b_0)\) will spread over the entire \( G \), i.e.,
\[ \lim_{t \to \infty} \text{supp}\{P_t f_0\} = G. \] (48)
Since \((a_0, b_0)\) was an arbitrary component of a density on \([0, 1]\), any component of
an initial density will spread to cover the whole attracting part \( G \), i.e.,
\[ \lim_{t \to \infty} \text{supp}\{P_t f_0\} = G \quad \text{for} \quad f_0 \in D. \] (49)
Lemma 3.4 shows that if the condition (49) holds for the operator \( P \), then the
iterates \( \{P_t f_0\} \) will be asymptotically stable for all \( f_0 \in D \).

Next, we consider the case \( \beta > B_{n,l}^L(\alpha) \) and \( \theta + \beta - B_{n,l}^L(\alpha) > \frac{\alpha^{-1}(1-\alpha)^2}{1-\alpha} \). In this case, since there exists \((n', l')\) such that \((\alpha, \beta + \theta) \in D_{n', l'}\), replacing \( n \) and \( l \) by \( n + n' \) and \( l + l' \) respectively in the argument of the previous case \( \beta = B_{n,l}^L(\alpha) \) is enough to prove this case \( \beta > B_{n,l}^L(\alpha) \). And we can show that any component of
an initial density spreads to cover the whole attracting part of the phase space. This completes the proof.

In addition to Theorem 3.6, the argument used in the proof of Theorem 3.6
(ii) plays an important role to obtain the next result which shows an asymptotic
behavior for the parameter satisfying \( \beta = B_{n,l}^U(\alpha) \), which implies that \( S_{\alpha, \beta} \) may
not have periodic point. The case \( \alpha = 1/2, \beta = B_{n,l}^L(1/2) = 17/30, \theta = 1/15 \) of
behavior was observed in [19, 22], numerically. And recently, Kaijsrer [16] showed
that it displays asymptotic stability in this special case. The next theorem shows
the asymptotic stability for all parameter satisfying \( \beta = B_{n,l}^U(\alpha) \).

**Theorem 3.8.** Let \( \overline{P} \) be the Markov operator corresponding to system (2) and give
a parameter \( (\alpha, \beta) \in [0, 1]^2 \) satisfying \( \beta = B_{n,l}^U(\alpha) \). Then, \( \{P_t\} \) is asymptotically
stable for any \( \theta > 0 \).

**Proof.** If we take \( \beta = B_{n,l}^U(\alpha) \), then, for any \( \theta > 0 \), there exists \((n', l')\) such that
\((\alpha, \beta + \theta) \in D_{n', l'}\). Replacing \( n \) and \( l \) by \( n + n' \) and \( l + l' \) respectively in the argument
of the previous case \( \beta = B_{n,l}^L(\alpha) \) in Theorem 3.6 (ii) is enough to prove this case
\( \beta = B_{n,l}^U(\alpha) \) and \( \theta > 0 \). Then we can derive the equation (49) for any initial density
\( f_0 \in D \), and show that any component of an initial density spreads to cover the
whole attracting part of the phase space. \( \Box \)
4. Numerical results. In this section, we show the numerical illustrations of our results, which describe the asymptotic behavior of the Markov operator corresponding to system (2). In Figure 2, illustrated histograms approximately describe evolutions of density by Markov operator \( \overline{P} \) of system (2), \( \{ \overline{P}f_0 \} \), with \( \alpha = 1/2, \beta = B_{3,1}^U(1/2) = 4/7, \) and \( \theta = \theta_*(1/2, 4/7) = 1/14 \) for an initial density \( f_0 = 1_{[0,1]} \). In this case, \( \overline{P} \) generated by this system has the asymptotic periodicity with period 3 from Theorem 3.9(i). Indeed, Figure 2 indicates that the sequence \( \{ \overline{P} f_0 \} \) has period 3, and these densities are repeated even after \( t = 100,000 \).

![Figure 2](image-url)

**Figure 2.** Asymptotic periodicity illustrated. Here we show histograms obtain after iterating 5,000,000 initial points uniformly distributed on \([0,1]\) with \( \alpha = 1/2, \beta = 4/7, \) and \( \theta = 1/14 \) in Equation (2) for (a) \( t = 200 \); (b) \( t = 201 \); (c) \( t = 202 \); and (d) \( t = 203 \). A correspondence of the histograms for \( t = 200 \) and \( t = 203 \) indicates that the sequence of densities has period 3.

On the other hand, in Figure 3, we pick \( \alpha = 1/2, \beta = B_{3,1}^U(1/2) = 4/7 \) and \( \theta = \theta_*(1/2, 4/7) + 0.02 = 1/14 + 0.02 \) for system (2). In this case, \( \overline{P} \) has the asymptotic stability from Theorem 3.9(ii). Indeed, after \( t \)-th iterations \( (t \geq 100) \), the sequence \( \{ \overline{P}^t f_0 \} \) goes to the density of Figure 3 for the initial density \( f_0 = 1_{[0,1]} \).

![Figure 3](image-url)

**Figure 3.** Asymptotic stability illustrated. Here we show histograms obtain after 200 iterating 5,000,000 initial points uniformly distributed on \([0,1]\) with \( \alpha = 1/2, \beta = 4/7, \) and \( \theta = 1/14 + 0.02 \) in Equation (2).

Finally, in Figure 4, we show the case \( \alpha = 1/2, \beta = B_{3,1}^U(1/2) = 17/30 \) and \( \theta = 1/15 \) for system (2). In this case, \( \overline{P} \) has the asymptotic stability from Theorem 3.13. Indeed, after 100,000 iterations, the sequence \( \{ \overline{P}^t f_0 \} \) goes to the density of Figure 4(d) for the initial density \( f_0 = 1_{[0,1]} \).
Appendix A. Special type of induction based on Farey series. We gave the proof of Lemma 2.3 by using a special type of induction as follows;

Step(1): The inequality holds for \((n, 1)\) and \((n, n−1)\) for \(n \in \mathbb{N}_{\geq 2}\).

Step(2): Assume that the inequality holds for \((n, l)\) and \((n′, l′)\) with \(nl'−n'l = 1\). Then, the inequality holds for \((n + n', l + l')\).

By the definition of the Farey series, it is obvious that the above induction shows the inequality holds for all \((n, l)\). The details of the Farey series is to be found in [1].

Definition A.1. The set of Farey series of order \(n\), denoted by \(F_n\), is the set of reduced fractions in the closed interval \([0, 1]\) with denominators \(\leq n\), listed in increasing order of magnitude.

The following theorem is well known.

Theorem A.2. ([1], Theorem 5.5) The set \(F_{n+1}\) includes \(F_n\). Each fraction in \(F_{n+1}\) which is not in \(F_n\) is the mediant of a pair of consecutive fractions in \(F_n\). Moreover, if \(\frac{a}{b} < \frac{c}{d}\) are consecutive in any \(F_n\), then they satisfy the unimodular relation \(bc−ad = 1\).

Proof of Lemma 2.3. (I) First we consider the following inequality;

\[
F_{n,i}(n) - F_{n,i}(i) > \frac{\alpha^{n−1}n − \alpha^n}{1−\alpha^n}. \tag{50}
\]

We used a special type of induction for \((n, l)\), which is associated to the Farey series.

(I-1) We first show the inequality (50) in the case \((n, 1)\) and \((n, n−1)\) for any \(n \geq 2\).

(I-1-1) When \(\{k_m\}_{m \in \mathbb{Z}}\) is a rational characteristic sequence with respect to \((n, 1)\), one can see that \(k_m = 0\) for \(m = 0, \ldots, n−2\) and \(k_{n−1} = 1\), and obtain inequality (50) immediately for any \(i = 2, \ldots, n−1\).

(I-1-2) When \(\{k_m\}_{m \in \mathbb{Z}}\) is a rational characteristic sequence with respect to \((n, n−1)\), one can see that \(k_m = 1\) for \(m = 1, \ldots, n−1\). Thus the inequality (50) can be also shown by elementary calculations.

(I-2) Next, assume that the inequality (50) holds for \((n, l)\) and \((n′, l')\) with \(nl'−n'l =
1. That is, next two inequalities hold;
\[ F_{n,l}(n) - F_{n,l}(i) > \frac{\alpha^{n+1} - \alpha^n}{1 - \alpha^n} \quad \text{for} \quad i = 2, \cdots, n-1, \quad (51) \]
\[ F_{n',l'}(n') - F_{n',l'}(i) > \frac{\alpha^{n'+1} - \alpha^{n'}}{1 - \alpha^{n'}} \quad \text{for} \quad i = 2, \cdots, n'-1, \quad (52) \]

where \( \{k_m\}_{m \in \mathbb{Z}} \) and \( \{k'_m\}_{m \in \mathbb{Z}} \) are rational characteristic sequences with respect to \((n,l)\) and \((n',l')\) respectively. Then, we will show that the inequality (50) holds for \((n + n',l + l')\). That is, we will show that the next value is positive for any \( i = 2, \cdots, n + n' - 1 \) and \( \alpha \in (0,1) \);
\[ F_{n+n',l+l'}(n + n') - F_{n+n',l+l'}(i) = \frac{\alpha^{n+n'+1} - \alpha^{n+n'}}{1 - \alpha^{n+n'}}. \quad (53) \]

where the sequence \( \{k'_m\}_{m \in \mathbb{Z}} \) is the rational characteristic sequence with respect to \((n + n',l + l')\). From the Proposition 3, \( \{k_m\}_{m \in \mathbb{Z}} \) can be made from \( \{k_m\}_{m \in \mathbb{Z}} \) and \( \{k'_m\}_{m \in \mathbb{Z}} \).

**I-2-1** For \( i = 2, \cdots, n-1 \), by using (11) and (51),
\[ \operatorname{Eq.(53)} = F_{n+n',l+l'}(n + n') - F_{n,l}(i) - \frac{\alpha^{n+n'+1} - \alpha^{n+n'}}{1 - \alpha^{n+n'}} > 0. \]

**I-2-2** For \( i = n, n + 1 \), it can be obtained from (11) and (14).

**I-2-3** For \( i = n + 2, \cdots, n + n' - 1 \), by using (14),
\[ \operatorname{Eq.(53)} = \frac{1 - \alpha^n}{1 - \alpha^{n+n'}} F_{n,l}(n) + \frac{\alpha^n(1 - \alpha^{n'})}{1 - \alpha^{n+n'}} F_{n',l'}(n') - \frac{1 - \alpha^n}{1 - \alpha^i} F_{n,l}(n) + \frac{\alpha^n(1 - \alpha^{i-n})}{1 - \alpha^i} F_{n',l'}(i - n) - \frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1 - \alpha^{n+n'}}. \quad (54) \]

Applying the assumption (52) to the fourth term of (54), we can calculate as follows;
\[ \operatorname{Eq.(53)} > \left( \frac{1 - \alpha^n}{1 - \alpha^{n+n'}} \right) \left( \frac{1 - \alpha^{i-n}}{1 - \alpha^i} \right) \left( \frac{\alpha^{n+n'-1} - \alpha^{n+n'}}{1 - \alpha^n} \right). \]

Then it was proven that equation (53) is positive for \( i = n + 2, \cdots, n + n' - 1 \) from the inequality (9). From these arguments (I-1) and (I-2), for any \((n,l)\), the inequality (50) is proved.

**II** Next, we consider another inequality;
\[ F_{n,l}(n) - F_{n,l}(i) < \frac{\alpha^i}{1 - \alpha^i}. \quad (55) \]

This proof is similar to the previous inequality (50). The key idea is to switch the roles of \( n \) and \( n' \).

**II-1** In the case \((n,1)\) and \((a, n-1)\) for any integer \( n \geq 2 \), the inequality (55) can be clarified by elementary calculations because of same reasons with (I-1).

**II-2** Next, assume that the inequality (55) holds for \((n,l)\) and \((n',l')\) with \( nl' = nl = 1 \). That is, next two inequalities hold;
\[ F_{n,l}(n) - F_{n,l}(i) < \frac{\alpha^i}{1 - \alpha^i} \quad \text{for} \quad i = 2, \cdots, n-1, \quad (56) \]
\[ F_{n',l'}(n') - F_{n',l'}(i) < \frac{\alpha^i}{1 - \alpha^i} \quad \text{for} \quad i = 2, \cdots, n'-1, \quad (57) \]
where \( \{k_m\}_{m \in \mathbb{Z}} \) and \( \{k'_m\}_{m \in \mathbb{Z}} \) are rational characteristic sequences with respect to \((n, l)\) and \((n', l')\) respectively. Then, we will show that the inequality (55) holds for \((n + n', l + l')\). That is, we will show that the next value is positive for any \(i = 2, \cdots, n + n' - 1\) and \(\alpha \in (0, 1)\):

\[
\frac{\alpha^i}{1 - \alpha^i} - F_{n+n',l+l'}(n + n') + F_{n+n',l+l'}(i),
\]

(58)

where the sequence \(\{k_m\}_{m \in \mathbb{Z}}\) is the rational characteristic sequence with respect to \((n + n', l + l')\).

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