Theorem to generate Einstein–nonlinear Maxwell fields

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Abstract

We present a theorem in a \(d\)-dimensional static, spherically symmetric spacetime in generic Lovelock gravity coupled with a nonlinear electrodynamic source to generate solutions. The theorem states that irrespective of the order of the Lovelock gravity and nonlinear Maxwell (NLM) Lagrangian, for the pure electric field case the NLM equations are satisfied by virtue of the Einstein–Lovelock equations. Applications of the theorem, specifically to the study of black hole solutions in Chern–Simons theory is given. A radiating version of the theorem has been considered, which generalizes the Bonnor–Vaidya metric to the Lovelock gravity with a NLM field as a radiating source. We consider also the radiating power-Maxwell source (i.e. \((F_{\mu\nu} F^{\mu\nu})^q\), \(q = \text{finely tuned constant}\)) within the context of Lovelock gravity.

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1. Introduction

The string theory-motivated higher dimensional gravity, known as the Lovelock gravity, [1] attracted much interest in recent years [2]. This theory is known to admit the most general higher order invariants in such combinations that the field equations preserve their second-order form. These are important features concerning divergences at smaller scales and the ghost-free structure toward a quantum theory of gravity. The first few orders of the Lovelock Lagrangian are well known: zeroth/first order is just the cosmological constant \((\Lambda)/\text{Einstein–Hilbert (EH)}\) term. The second-order Lovelock term is also known as the Gauss–Bonnet (GB) term, which consists of quadratic invariants and is a highly non-trivial contribution, especially in higher dimensions [3]. Going to higher order corrections serves only to add contributions at higher levels of intricacy, which are crucial in a possible quantum gravity. In spite of all complications it is remarkable that in a spherically symmetric line element exact black hole solutions are found in higher order Lovelock gravity [4]. These metrics are sourced by
Maxwell and Yang–Mills (YM) fields. More recently, we obtained black holes in a power-YM source in Lovelock theory [5]. By the power-YM source it is meant that the source consists of a power of the YM invariant, i.e. $(F^a_{\mu\nu} F^{a\mu\nu})^q$. Here $F^a_{\mu\nu}$ refers to the YM field with a gauge group index $a$ and $q > 0$ is an arbitrary constant parameter. Although $q$ may stand arbitrary, energy and causality conditions restrict it to a certain set of admissible parameters [5].

By the same token, in this paper we consider Lovelock gravity of all higher orders coupled with a nonlinear Maxwell (NLM) source [6]. We prove a theorem, covering Lovelock terms to all orders—albeit with proper constant coefficients—which generates new metrics once the NLM energy momentum is known. The matter Lagrangian is given by $\mathcal{L}(\mathcal{F})$, in which $\mathcal{F} = F^a_{\mu\nu} F^{a\mu\nu}$ denotes the Maxwell invariant. The field equation satisfied by $F^a_{\mu\nu}$ will be referred to as the NLM equation. The theorem involves spherically symmetric metric ansatz together with the NLM Lagrangian, which leads to a general class of metrics. Static fields fall within the range of our theorem, which yields Born–Infeld (BI) electrodynamics as a particular example. A static, pure magnetic YM field constitutes another example of BI type.

For the first–second-order Lovelock terms the theorem addresses EH/GB gravities, resulting in metrics expressed in solutions of first–second-order algebraic equations. With equal ease, we extend this result to any higher order Lovelock gravity in terms of a higher order algebraic equation [7]. As an interesting example we obtain Chern–Simons–Born–Infeld (CSBI) black hole solutions in odd dimensions. We elaborate on $d = 5$, to show in particular that the thermodynamically well-behaving CS black hole preserves its good features when coupled with a BI source in the general relativity limit. We note that the combination of CS and BI types is in an artificial manner since CS/BI black holes arise naturally in odd/even dimensions. Such a combination works only when $^8F^a_{\mu\nu}$ (i.e. dual of $F^a_{\mu\nu}$) vanishes, which occurs in a restricted type of sources. BI black holes are known to arise naturally from Pfaffians only in even-dimensional spacetimes [5]. Depending on the topological parameter $\chi = 0, \pm 1$ and $\Lambda \geq 0$ we investigate the Hawking temperature of the five-dimensional CSBI black holes. Finally, we investigate the implications of our theorem in the Eddington–Bondi form of radiating metrics [8]. This is the time-dependent version of both the mass and charge so that the metrics become time dependent. That is, we extend the Bonnor–Vaidya (BV) [8] form of the radiating metric to the $d$-dimensional Lovelock gravity coupled with a NLM field. We consider various types of nonlinearities in an electromagnetic field as particular examples.

Organization of the paper is as follows. The theorem, its proof and applications are given in section 2. Generalization of the theorem to the radiating metrics in higher dimensions with NLM as source is studied in section 3. In section 4, we investigate the case of power-Maxwell nonlinearity. The paper is completed with the conclusion in section 5.

2. A theorem for solving Einstein–Lovelock–NLM equations

**Theorem.** Let the $d$-dimensional static spherically symmetric spacetime sourced by a nonlinear electromagnetic field be described by the action $(8\pi G = 1)$

$$S = \frac{1}{2} \int dx^d \sqrt{-g} \left\{ -\frac{(d-2)(d-1)}{3} \Lambda + \mathcal{L}_1 + \mathcal{L}_2 + \cdots + \mathcal{L}_{\frac{d}{2}} + \mathcal{L}(\mathcal{F}) \right\},$$

in which

$$\mathcal{L}_n = 2^{-n} a_{a_{1}...a_{n}} b_{b_{1}...b_{n}} R^{a_{1}b_{1}} \cdots R^{a_{n}b_{n}}, \quad n \geq 1.$$
is the \(n\)-th order Lovelock Lagrangian, \(\alpha_n\) is a real constant and the bracket \([\cdot]\) refers to the integer part. Herein \(\mathcal{F} = F_{\mu\nu} F^{\mu\nu}\) is the Maxwell invariant for a static field 2-form

\[
F = E(r) \, dr \wedge dr.
\]

If \(\mathcal{L}(\mathcal{F})\) satisfies the NLM equation

\[
d(\ast F \, \mathcal{L}_\mathcal{F}) = 0
\]

with \(\mathcal{L}_\mathcal{F} = \frac{\partial}{\partial F}\) and \(\ast F\) the dual of \(F\), then the Einstein equations admit the solution

\[
d\mathcal{L}_\mathcal{F} = \ast F - 2(\mathcal{L} - 2\mathcal{L}_\mathcal{F}) \, dr.
\]

where \(\chi = 0, \pm 1\) and in which \(H(r)\) is the solution (or solutions) of the following algebraic equation of order \(\frac{d+2}{2}\):

\[
\sum_{k=1}^{\left[\frac{d+2}{2}\right]} \bar{a}_k H(r)^k = \frac{\Lambda}{3} + \frac{M}{r^{d-1}} - \frac{1}{(d - 2) r^{d-1}} \int r^{d-2}(\mathcal{L} - 2\mathcal{L}_\mathcal{F}) \, dr.
\]

Here \(m\) is an integration constant, \(\bar{a}_k = \prod_{i=3}^{2k} (d - i) \, \alpha_i\) and \(\bar{a}_1 = 1\).

**Proof.** Variation of the action with respect to the metric tensor \(g_{\mu\nu}\) yields the field equations in the form

\[
G_{\mu}^\nu = G_{\mu}^{rEH} + \sum_{k=2}^{\left[\frac{d+2}{2}\right]} \alpha_k G_{\mu}^{r(k)} + \frac{(d - 2)(d - 1)}{6} \Lambda \delta^\nu_\mu = T_{\mu}^\nu,
\]

where \(G_{\mu}^{rEH}\) is the Einstein tensor, while

\[
G_{\mu}^{r(k)} = \frac{1}{2(2^{k+1} r^{2k+1})} \delta_{a_1 b_1 \cdots a_k b_k} R_{a_1 b_1 \cdots}^{r k} R^{a_1 b_1 \cdots}.
\]

\(T_{\mu}^\nu\) is given by

\[
T_{\mu}^\nu = \frac{1}{2}(\mathcal{L} \delta_{\mu}^\nu - 4\mathcal{L}_\mathcal{F} F_{\mu\nu} F^{\mu\nu}),
\]

which clearly gives \(T_{\mu}^\nu = T_{\nu}^\mu = \frac{1}{2} \mathcal{L} - \mathcal{L}_\mathcal{F}\), stating that \(G_{\mu}^\nu = G_{\nu}^\mu\) and \(T_{\mu}^{\alpha} = \frac{1}{2} \mathcal{L}\). Now we introduce our metric as given by (5), where the choice of \(g_{\mu\nu} = - (g_{rr})^{-1}\) is a direct result of \(G_{\mu}^\nu = G_{\nu}^\mu\) up to a constant coefficient where we set it to be 1. By starting with the line element (5) one gets

\[
G_{\mu}^r = G_{rr} = \frac{(d - 2)}{2r^{d-2}} (r^{d-1} H(r))',
\]

and

\[
G_{\alpha}^{\beta} = \frac{1}{2r^{d-3}} (r^{d-1} H(r))'' - \frac{1}{2r^{d-3}} \sum_{k=2}^{\left[\frac{d+2}{2}\right]} \bar{a}_k (r^{d-1} H(r)^k)''\]

where \((\cdot)' = \frac{d}{dr} (\cdot)\). Equations (10) and (11) admit

\[
\left(\frac{r^{d-2}}{d-2} G_{\mu}^r \right)' = r^{d-2} G_{\alpha}^{\beta},
\]

and imposing Einstein equations yields

\[
\left(\frac{r^{d-2}}{d-2} \left( T_{\mu}^r - \frac{(d - 2)(d - 1)}{6} \Lambda \right) \right)' = r^{d-3} \left( T_{\alpha}^{\beta} - \frac{(d - 2)(d - 1)}{6} \Lambda \right).
\]
This is equivalent to
\[
\left( \frac{r^{d-2}}{d-2} \left( \frac{1}{2} \mathcal{L} - \mathcal{L}_5^5 \mathcal{F} \right) \right)' = r^{d-3} \left( \frac{1}{2} \mathcal{L} \right),
\] (14)
which integrates to
\[
r^{d-2} \mathcal{L}_5^5 \sqrt{|\mathcal{F}|} = \text{constant}.
\] (15)
This result is in line with the solution of the NLM equation (4) and the integration constant can be identified with the electric charge. Now recall from (7) that
\[
G_{t}^t + \frac{(d-2)(d-1)}{6} \Lambda = T_{t}^t,
\] (16)
or equivalently
\[
-\frac{(d-2)}{2r^{d-2}} \left( r^{d-1} H(r) \right)' - \frac{(d-2)}{2r^{d-2}} \sum_{k=2}^{\infty} \hat{a}_k (r^{d-1} H(r)^k)' = \frac{1}{2} \mathcal{L} - \mathcal{L}_5^5 \mathcal{F} - \frac{(d-2)(d-1)}{6} \Lambda,
\] (17)
which implies
\[
\sum_{k=1}^{\infty} \hat{a}_k H(r)^k = \frac{\Lambda}{3} + \frac{M}{r^{d-1}} - \frac{2}{(d-2)r^{d-1}} \int r^{d-2}(\mathcal{L} - 2 \mathcal{L}_5^5 \mathcal{F}) \, dr,
\] (18)
for the integration constant
\[
M = \frac{4m}{(d-2)}.
\]
This completes the proof of our theorem. □

**Remark 1.** In the theorem, we considered only a static electric field whose 2-form is 
\[ F = E(r) \, \text{d}t \wedge \text{d}r, \]
so that \[ \mathcal{F} = -2E^2 \]
and the Lagrangian \[ \mathcal{L}(\mathcal{F}), \]
this aided in the proof of the theorem. We must add that the theorem becomes of practical use whenever the expression on the right-hand side is integrable. Otherwise we should resort to the multipole expansions for the asymptotically flat spacetimes.

**Remark 2.** In the case of a pure magnetic field the theorem is applicable only if \[ \mathcal{F} = \mathcal{F}_{\mu \nu} \]
is only a function of \( r \). This implies
\[
\left( \frac{r^{d-2}}{d-2} T_{t}^t \right)' = r^{d-3} T_{t}^t \hat{\theta}_i,
\] (19)
which leads to the same metric function in the form
\[
\sum_{k=1}^{\infty} \hat{a}_k H(r)^k = \frac{\Lambda}{3} + \frac{M}{r^{d-1}} - \frac{2}{(d-2)r^{d-1}} \int r^{d-2}T_{t}^t \, dr.
\] (20)

**Remark 3.** In the case of a general energy–momentum tensor
\[
T_{\mu}^\nu = \text{diag}(T_{t}^t, T_{r}^r, T_{\theta_1}^\theta_1, \ldots),
\] (21)
in which
\[
T_{t}^t = T_{r}^r, \quad T_{\theta_1}^{\theta_1} = T_{\theta_2}^{\theta_2} = \ldots
\] (22)
and
\[
\left( \frac{r^{d-2}}{d-2} T_{t}^t \right)' = r^{d-3} T_{t}^t \hat{\theta}_i,
\] (23)
again, a solution in the form of (5) satisfies the Einstein equations with \( H(r) \) given by (20).
Remark 4. The case of $\mathcal{L} - 2\mathcal{L}_F = 0$ must be excluded, since it implies a Lagrangian of the form $\mathcal{L} = \sqrt{F}$, which fails to satisfy the energy and causality conditions [5]. This form of the Lagrangian lacks also the linear Maxwell limit.

Example 1. As an application we consider the case of a pure electric Einstein–Born–Infeld (EBI) black hole solution. The pure electric BI Lagrangian can be written as [6]

$$\mathcal{L}(F) = 4\beta^2 \left(1 - \sqrt{1 + \frac{F}{2\beta^2}}\right),$$

(24)

where $F$ is the electric field invariant given by $F = 2F_{tr}F^{tr} = -2E(r)^2$. Note that for $\beta \to \infty$ we recover the standard, linear Maxwell Lagrangian. Consequently

$$\mathcal{L}_F = -\frac{1}{\sqrt{1 + \frac{F}{2\beta^2}}},$$

(25)

and therefore upon solving the NLM equation one finds

$$E = \frac{q\beta}{\sqrt{q^2 + \beta^2 r^{2(d-2)}}}.$$  

(26)

Finally we obtain

$$H(r) = \frac{\Lambda}{3} + \frac{M}{r^{d-1}} - \frac{2}{(d-2)r^{d-1}} \int r^{d-2} T_{t'} \, dr$$

$$= \frac{\Lambda}{3} + \frac{4m}{(d-2)r^{d-1}} - \frac{4\beta^2}{(d-1)(d-2)} \left(1 - \sqrt{1 + \frac{q^2}{\beta^2 r^{2(d-2)}}}\right)$$

$$- \frac{4}{(d-1)(d-3)} \frac{q^2}{r^{2(d-2)}} \, _2F_1\left(\frac{1}{2}, \frac{d-3}{2(d-2)}, \frac{d-7}{2(d-2)}, \frac{q^2}{\beta^2 r^{2(d-2)}}\right),$$

(27)

in which $_2F_1$ stands for the hypergeometric function.

Example 2. Another example for the case of a non-electric field is given by the Einstein–Yang–Mills (EYM) nonlinear electrodynamics black hole solution. In fact in [9] we find that

$$T_{t'} = T_{r'} = 2\beta^2 \left(1 - \sqrt{1 + \frac{(d-2)(d-3)Q^2}{2\beta^2 r^4}}\right),$$

(28)

and

$$T_{\theta i} = 2\beta^2 \left(1 - \sqrt{1 + \frac{(d-2)(d-3)Q^2}{2\beta^2 r^4}}\right) + \frac{2(d-3)Q^2}{r^4 \sqrt{1 + \frac{(d-2)(d-3)Q^2}{2\beta^2 r^4}}},$$

(29)

which clearly satisfies the conditions of the theorem and the Einstein equations admit a black hole solution with

$$H(r) = \frac{\Lambda}{3} + \frac{4m}{(d-2)r^{d-1}} - \frac{4\beta^2}{(d-1)(d-2)}$$

$$+ \frac{4\beta^2}{(d-2)r^{d-1}} \int dr \, r^{d-4} \sqrt{r^4 + \frac{(d-2)(d-3)Q^2}{2\beta^2}},$$

(30)

in line with the solution given in [9].
Example 3. Our next example will be the general form of the energy–momentum tensor given by Salgado [10], which states that
\[ T_{\mu(\text{Diag})}^\nu = \frac{C}{r^{n(1-k)}} [1, 1, k, \ldots, k], \quad (C, k : \text{constants}), \]
admits solutions for Einstein equations. Now we show that this is a natural result for the case of remark 3. In remark 3 let us consider
\[ T_{t}^{t} = T_{r}^{r}, \quad T_{\theta i}^{\theta i} = kT_{t}^{t}; \quad (31) \]
then, from equation (23) we obtain
\[ \left( \frac{r^{d-2}}{d-2} T_{t}^{' t} \right)^{'} = kr^{d-3}T_{t}^{' t}, \quad (32) \]
or, in a straightforward calculation, one finds
\[ T_{t}^{t} = C_{r n(1-k)}, \quad (33) \]
where \( C \) is an integration constant. This verifies that the theorem proved by Salgado [10] turns out to be a particular case of our more general theorem.

Example 4. One may note that a proper choice of \( \tilde{\alpha}_k \) leads to a Chern–Simons (CS) [11] gravity in odd dimensions. To do so we set
\[ \tilde{\alpha}_k = \tilde{\alpha}_1 \quad \text{for} \quad k \geq 2 \quad \text{and} \quad \frac{\Lambda}{3} = \tilde{\alpha}_0 / \tilde{\alpha}_1, \quad (34) \]
and we rewrite (18) as
\[ \sum_{k=0}^{\left[ \frac{d-1}{2} \right]} \tilde{\alpha}_k H(r)^{k} = \tilde{\alpha}_1 \left( \frac{M}{r^{d-1}} - \frac{1}{(d-2)r^{d-1}} \int r^{d-2}(L - 2L_{x,F}) \, dr \right). \quad (35) \]
Now we choose
\[ \tilde{\alpha}_k = \left( \frac{\left( \frac{d-1}{2} \right)}{k} \right) \ell^{2k-d}, \quad (36) \]
where
\[ -\frac{\Lambda}{3} = \frac{\tilde{\alpha}_0}{\tilde{\alpha}_1} = \frac{\ell^{-2}}{\left[ \frac{d-1}{2} \right]}, \quad (37) \]
to get from the binomial expansion
\[ (1 + \ell H(r))^{\left[ \frac{d-1}{2} \right]} = \ell^{d-2} \tilde{\alpha}_1 \left( \frac{M}{r^{d-1}} - \frac{1}{(d-2)r^{d-1}} \int r^{d-2}(L - 2L_{x,F}) \, dr \right). \quad (38) \]
This implies that
\[ H(r) = -\frac{1}{\ell^2} + \frac{\sigma}{\ell^2} \left[ \ell^{d-2} \tilde{\alpha}_1 \left( \frac{M}{r^{d-1}} - \frac{1}{(d-2)r^{d-1}} \int r^{d-2}(L - 2L_{x,F}) \, dr \right) \right]^{\frac{1}{d-2}}, \quad (39) \]
where \( \sigma = +1 \) if \( \left[ \frac{d-1}{2} \right] \) is an odd integer and \( \sigma = \pm 1 \) if \( \left[ \frac{d-1}{2} \right] \) is an even integer.

The latter equation for \( d \rightarrow \text{odd}, \chi = 1 \) admits \( (\ell^2 > 0) \)
\[ f(r) = 1 + \frac{\ell^2}{\ell^2} - \sigma \left[ \ell \tilde{\alpha}_1 \left( M - \frac{2B(r)}{(d-2)} \right) \right]^{\frac{d-1}{2}}, \quad (40) \]
where \( f(r) = (\chi - r^2 H(r)) \) is the metric function and \( B(r) = \int z^{d-2} T_i(z) \, dz \). This metric function by using (39) and (40) becomes
\[
f(r) = 1 + \frac{r^2}{\ell^2} - \sigma \left[ m + 1 - \frac{(d - 1) B(r)}{(d - 2) \ell^{(d-3)}} \right]^{\frac{1}{d-1}} ,
\]
in which the new integration constant \( m \) is related to \( M \) \[11\]. In what follows we investigate some thermodynamic properties of this solution.

1. With \( \sigma = +1 \) or \( \frac{d-1}{2} \) is an odd integer \( (d = 7, 11, 15, \ldots) \). In this case one finds Hawking’s temperature as
\[
T_H = \frac{1}{4\pi} \int f'(r_+) = \frac{1}{2\pi} \left( \frac{r_+}{\ell^2} + \frac{r_+^2 \ell^2}{(d-2)(\ell^2 + r_+^2)^{\frac{d-1}{2}}} \right).
\]
The specific heat capacity of the black hole for constant charge is defined by
\[
C_q = T_H \left( \frac{\partial S}{\partial T_H} \right)_q , \quad S = \frac{(d-1) \ell^{\frac{d+1}{2}}}{4 (\ell^2)} r_+^{d-2} ,
\]
and is obtained as
\[
C_q = \frac{(d-1)(d-2) \pi^{\frac{d+1}{2}} r_+^{d-2} \Psi}{\Gamma \left( \frac{d+1}{2} \right)} \]
where
\[
\Psi = (d - 2) + \frac{\ell^2 r_+^2}{(\ell^2 + r_+^2)^{\frac{d+1}{2}}} T_i' (r_+),
\]
(45)
\[
\Psi = (d - 2) + \frac{\ell^2 r_+^2}{(\ell^2 + r_+^2)^{\frac{d+1}{2}}} \left[ r_+ \left( \ell^2 + r_+^2 \right) \frac{\partial}{\partial r} T_i' (r_+) + T_i' (r_+) \left( 3 \ell^2 - r_+^2 (d - 6) \right) \right].
\]
(46)

2. With \( \sigma = \pm 1 \) or \( \frac{d-1}{2} \) is an even integer \( (d = 5, 9, 13, \ldots) \). It is clear that in this case \( \sigma = -1 \) does not claim any horizon and therefore it is out of our interest but for the \( \sigma = 1 \) branch the Hawking temperature and the specific heat capacity are given as (42) and (44) but there exists an additional constraint on the free parameter in order to have a real metric function.

To complete this section we give an example for the CSBI black hole \( (\chi = 1, \sigma = 1) \) in five dimensions. To do so we recall that in five dimensions
\[
B(r) = \int z^3 T_i(z) \, dz = \frac{\beta^2 r^4}{2} \left( 1 - \sqrt{1 + \frac{q^2}{\beta^2 r^6}} + \frac{3q^2}{4r^2} F_1 \left( 1.1.4 - \frac{q^2}{\beta^2 r^6} \right) \right)
\]
and therefore
\[
f(r) = 1 + \frac{r^2}{\ell^2} = \left\{ m + 1 - \left[ \frac{2\beta^2 r^4}{3\ell^2} \left( 1 - \sqrt{1 + \frac{q^2}{\beta^2 r^6}} \right) + \frac{q^2}{\ell^2 r^2} F_1 \left( 1.1.4 - \frac{q^2}{\beta^2 r^6} \right) \right] \right\}^{\frac{1}{2}}.
\]
(48)

In figure 1 we plot \( f(r) \) for the specific choices of mass, charge and the cosmological constant. The corresponding temperature and specific heat capacity plots are given in figures 2 and 3.
Figure 1. The plot of $f(r)$ (equation (48)), for the specific parameters $m = \ell = q = 1$, as $\beta$ ranges from 0 to $\infty$. It is seen that black hole solutions are available only for $\beta \leq \beta_{\text{critical}} = 0.2276$.

Figure 2. Hawking temperature $T_H$ versus event horizon radius $r_+$, for the specific parameters $\ell = q = 1$ and for $\beta \in (0, \beta_{\text{critical}}]$. Taking the absolute value of $T_H$ automatically deletes the negative temperatures as non-physical.

Example 5. (Clouds of strings as source): we consider another application of the theorem that incorporates the energy–momentum tensor representing clouds of string-type matter fields [12]. More recently [13], clouds of string-type energy–momentum are considered in the
Einstein–Gauss–Bonnet (EGB) gravity. In remarks 2 and 3, we considered the case for a general energy–momentum tensor and its corresponding solution. Clouds of string-type matter fields obey the condition imposed on the energy–momentum tensor stated in remark 3, such that

\[ T_{tt} = T_{rr} = a r^d - 2, \quad a = \text{real constant}, \] (49)

which leads to

\[ T_{\theta \theta} = 0. \] (50)

Therefore the general solution for the metric function is obtained from the algebraic equation

\[ \sum_{k=1}^{\frac{d-2}{2}} \tilde{\alpha}_k H(r)^k = \frac{\Lambda}{3} + \frac{M}{r^{d-1}} - \frac{2a}{(d-2)r^{d-2}}. \] (51)

The solution for \( H(r) \) generalizes the solution obtained in [13] to higher order Lovelock gravity. And hence, our general solution includes the solution obtained in [13] if we restrict the spacetime dimension to \( d = 5 \). For this particular case the solution is obtained from

\[ H(r) + \tilde{\alpha}_2 H(r)^2 = \frac{\Lambda}{3} + \frac{M}{r^3} - \frac{2a}{3r^3}, \] (52)

or equivalently

\[ f(r) = \chi + \frac{r^2}{4\tilde{\alpha}_2} \left( 1 \pm \sqrt{1 + 8\tilde{\alpha}_2 \left( \frac{\Lambda}{3} + \frac{M}{r^4} - \frac{2a}{3r^3} \right)} \right). \] (53)

This solution is nothing but the black hole in EGB gravity in the presence of string cloud-type matter fields [12]. In this theory one can easily check that the corresponding energy–momentum tensor in \( d \)-dimensions is given by

\[ T_{\mu \nu} = \text{diag} \left( \frac{a}{r^{d-2}}, \frac{a}{r^{d-2}}, 0, 0, \ldots \right). \] (54)
At this stage we wish to go a step further to relate our solution to the CS solution in odd $d$-dimensions as ($\chi = 1$)

$$f(r) = 1 + \frac{r^2}{\ell^2} - \sigma \left[ m + 1 - \frac{(d - 1)a r}{(d - 2)\ell^{(d-3)}} \right]^{\frac{1}{2}}$$

which in five dimensions becomes

$$f(r) = 1 + \frac{r^2}{\ell^2} \pm \sqrt{m + 1 - \frac{4a r}{3\ell^2}}.$$  \hspace{1cm} (56)

The thermodynamic properties of this solution are also investigated. The Hawking temperature ($T_H$) and the heat capacity $C_a$ at a constant string parameter $a$ calculated at the location of the event horizon ($r_h$) give by

$$T_H = \frac{\ell^2 (3r_h + a) + 3r_h^3}{3\pi l^2 (l^2 + l^2)}$$

$$C_a = \frac{3\pi r_h^2 (l^2 + r_h) [l^2 (3r_h + a) + 3r_h^3]}{2 [l^2 (6r_h^2 + 3l^2 - 2ar_h) + 3r_h^3]}.$$  \hspace{1cm} (58)

We also analyzed the thermodynamic stability, which is indicated by the positive heat capacity $C_a$. If the heat capacity has unbounded discontinuity at particular points of $r_h$, this implies possible phase change from the stable to unstable black hole solution. As illustrated in figure 4 the transitions from the stable to unstable black hole solution are not continuous and therefore a possible Hawking–Page-type phase transition occurs [15]. The occurrence of the phase transition crucially depends on the ratio of $a/l$. In figure 4, the values of this ratio that creates phase transitions are depicted.

3. A generalization to the radiating metrics

Following BV [8], we consider a nonlinear electrodynamic Lagrangian $\mathcal{L}(\mathcal{F})$, a null current $\mathbf{J}$ and a coupling term $A^\mu J^\mu$ added to the original Lagrangian such that

$$d(\star F \mathcal{L}_\phi) = \star J.$$  \hspace{1cm} (59)

Now, we consider the $d$-dimensional version of the BV metric

$$ds^2 = -(\chi - r^2 H(r, u)) du^2 + 2e^r dr du + r^2 d\Omega^2_{d-2},$$

with the outgoing null coordinate $u$ and the field 2-form

$$\mathbf{F} = E(u, r) dr \wedge du.$$  \hspace{1cm} (61)

This gives

$$\star \mathbf{F} = E(u, r) \sqrt{-g} d\theta_1 \wedge d\theta_2 \cdots \wedge d\theta_{d-2},$$

in which $g = \det(g_{\mu\nu})$ and the Einstein equation is given by

$$G^\nu_{\mu} = T^\nu_{\mu}(\text{em}) + T^\nu_{\mu}(\text{fluid}).$$  \hspace{1cm} (63)

Here

$$G^\nu_{\mu} = G^\nu_{\mu}(\text{EH}) + \sum_{k=2}^{d-1} \alpha_k G^\nu_{\mu}(\text{b}),$$  \hspace{1cm} (64)

$$T^\nu_{\mu}(\text{em}) = \frac{1}{2} \left( \partial^\nu F_{\mu\lambda} - 4 \delta_{\nu}^\lambda F_{\mu\lambda} \right).$$
Figure 4. Heat capacity $C_a$ versus the horizon radius $r_h$ for specific parameters $a = 5$ and $\ell = 1$. The singularity in $C_a$ and therefore occurrence of Hawking–Page phase transition is clearly seen. The shaded region in the inscribed figure depicts $a\ell$ versus $r_h$ for which such a transition occurs.

and

$$T_{\mu}^{\nu(\text{dual})} = -V_{\nu} V_{\mu},$$  \hspace{2cm} (65)$$

for a null vector $V_{\mu}$. We now start with the NLM equation (59), which leads to

$$d(*F) = d(E(u, r) L e^{-\frac{\ell}{2}} dr \wedge d \theta_1 \wedge d \theta_2 \cdots \wedge d \theta_{d-2})$$

$$= [(E(u, r) L e^{-\frac{\ell}{2}}) r dr + (E(u, r) L e^{-\frac{\ell}{2}}) u du] \wedge d \theta_1 \wedge d \theta_2 \cdots \wedge d \theta_{d-2} = *J.$$  \hspace{2cm} (66)$$

By using the relation between the 1-form current $J$ and its dual, i.e. $J = (-1)^{d-d} * J$, one finds

$$J = (-1)^{d-d} [(E(u, r) L e^{-\frac{\ell}{2}}) r dr \wedge d \theta_1 \wedge d \theta_2 \cdots \wedge d \theta_{d-2})$$

$$+ (E(u, r) L e^{-\frac{\ell}{2}}) u du] * (dr \wedge d \theta_1 \wedge d \theta_2 \cdots \wedge d \theta_{d-2}).$$  \hspace{2cm} (67)$$

From the metric we find

$$* (dr \wedge d \theta_1 \wedge d \theta_2 \cdots \wedge d \theta_{d-2}) = \frac{(-1)^{d-1}}{\sqrt{-g}} (dr - \epsilon f du),$$  \hspace{2cm} (68)$$

$$* (du \wedge d \theta_1 \wedge d \theta_2 \cdots \wedge d \theta_{d-2}) = \frac{(-1)^{d-1}}{\sqrt{-g}} du$$  \hspace{2cm} (69)$$

and therefore

$$J = (E(u, r) L e^{-\frac{\ell}{2}}) r \frac{1}{r^{d-2}} (dr - \epsilon f du) - (E(u, r) L e^{-\frac{\ell}{2}}) u du.$$  \hspace{2cm} (70)$$
This current is going to be null, i.e. \( J_\mu = (J_u, 0, \ldots, 0) = J_u \delta_\mu^u \), which means that \( (E(u, r) \mathcal{L}_F r^{d-2})_u = 0 \) integrates to
\[
E(u, r) \mathcal{L}_F r^{d-2} = Q(u)
\]
(71)
or
\[
E(u, r) \mathcal{L}_F = \frac{Q(u)}{r^{d-2}},
\]
(72)
for a \( u \)-dependent charge \( Q(u) \). After considering these results we find
\[
F = \frac{Q(u)}{r^{d-2}} dr \wedge du,
\]
(73)
\[
J = -(-1)^{d-1} (E(u, r) \mathcal{L}_F)_u du = (-1)^d \frac{Q(u)}{r^{d-2}} du,
\]
(74)
where \( \dot{Q}(u) = \frac{dQ(u)}{du} \). The explicit form of the Maxwell field can be expressed as
\[
F = F_{\mu \nu} F^{\mu \nu} = 2 F_{ru} F^{ru} = 2 F_{ru} (g^{\alpha \sigma} g^{\mu \nu} F_{\alpha \sigma}),
\]
(75)
while our metric tensors \( g_{\mu \nu} \) and \( g^{\mu \nu} \) are
\[
g_{\mu \nu} = \begin{pmatrix} -f & \epsilon & 0 & 0 & \ldots \\ \epsilon & 0 & 0 & \ldots \\ 0 & 0 & r^2 & 0 & \ldots \\ 0 & 0 & 0 & r^2 \sin^2 \theta & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},
g^{\mu \nu} = \begin{pmatrix} 0 & \frac{1}{r} & 0 & 0 & \ldots \\ \frac{1}{r} & f & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \frac{1}{r \sin \theta} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}
\]
(76)
giving
\[
F = -2 (F_{ru})^2.
\]
(77)
The energy–momentum tensor components are given by
\[
T^{(em)}_{\mu} = \frac{1}{2} (\mathcal{L} - 4 \mathcal{L}_F F_{\mu \nu} F^{\mu \nu}) = \frac{1}{2} (\mathcal{L} - 2 \mathcal{L}_F F),
\]
(78)
\[
T^r_{(em)} = \frac{1}{2} (\mathcal{L} - 4 \mathcal{L}_F F_{ ru} F^{ ru}) = \frac{1}{2} (\mathcal{L} - 2 \mathcal{L}_F F) = T^{(em)}_u,
\]
(79)
\[
T^0_{(em)} = \frac{1}{2} \mathcal{L},
\]
(80)
\[
T^r_{(em)} = T^u_{(em)} = 0.
\]
(81)
The null-fluid current vector is \( V_\mu = (V_u, 0, \ldots, 0) = V_u \delta_\mu^u \) and therefore
\[
V^\mu = \epsilon \delta^\mu_u V_u,
\]
(82)
implying that
\[
g_{\mu \nu} V^\mu V^\nu = g_{rr} (V^r)^2,
\]
(83)
which obviously vanishes and
\[
T^{(fluid)}_{\mu} = -V^\mu V_\nu = -V^r V_u \delta^\nu_u \delta^\mu_r = -\epsilon (V_u)^2 \delta^\nu_u \delta^\mu_r.
\]
(84)
Finally, we give the explicit form of the energy–momentum tensor as
\[
T^r_\mu = \begin{pmatrix} \frac{1}{2} (\mathcal{L} - 2 \mathcal{L}_F F) & 0 & 0 & 0 & \ldots \\ -\epsilon (V_u)^2 & \frac{1}{2} (\mathcal{L} - 2 \mathcal{L}_F F) & 0 & 0 & \ldots \\ 0 & 0 & \frac{1}{2} \mathcal{L} & 0 & \ldots \\ 0 & 0 & 0 & \frac{1}{2} \mathcal{L} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.
\]
(85)
Similarly the Einstein tensor can be expressed by

\[
G^{\nu}_{\mu} = \begin{pmatrix}
-\frac{d-2}{2r^{d-2}} \left( r^{d-1} \sum_{i=1}^{\frac{d+1}{2}} \tilde{\alpha}_i H' \right)' & 0 & 0 & \cdots \\
\frac{d-2}{2r^{d-2}} \frac{d}{d-2} \sum_{i=1}^{\frac{d+1}{2}} \tilde{\alpha}_i H^i & -\frac{d-2}{2r^{d-2}} \left( r^{d-1} \sum_{i=1}^{\frac{d+1}{2}} \tilde{\alpha}_i H' \right)' & 0 & \cdots \\
0 & 0 & -\frac{1}{2r^{d-2}} \left( r^{d-1} \sum_{i=1}^{\frac{d+1}{2}} \tilde{\alpha}_i H' \right)'' & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]  

(86)

where

\[
f(u, r) = 1 - r^2 H(u, r).
\]  

(87)

After all these arrangements we are now ready to solve the Einstein equations. We start with the \(uu\) component

\[
-\frac{d-2}{2r^{d-2}} \left( r^{d-1} \sum_{i=1}^{\frac{d+1}{2}} \tilde{\alpha}_i H' \right)' = - (\mathcal{L} - 2\mathcal{L}_F \mathcal{F}) \to \sum_{i=1}^{\frac{d+1}{2}} \tilde{\alpha}_i H^i
\]

\[
= \frac{M(u)}{r^{d-1}} - \frac{2}{(d-2)r^{d-1}} \int r^{d-2} T^{\nu}_{\nu} \, dr.
\]  

(88)

One can easily check that the \(rr\) component gives the same result while the angular components give

\[
-\frac{1}{2r^{d-3}} \left( r^{d-1} \sum_{i=1}^{\frac{d+1}{2}} \tilde{\alpha}_i H' \right)'' = \frac{1}{2} \mathcal{L} \to \left( \frac{d-2}{d-2} \left( \mathcal{L} - 2\mathcal{L}_F \mathcal{F} \right) \right)' = r^{d-3} \mathcal{L},
\]  

(89)

which is nothing but the NLM equation (14), already satisfied. In the last step we work on \(G^{\nu}_{\mu} = T^{\nu}_{\mu}\) giving

\[
\frac{d-2}{2r} \frac{\partial}{\partial u} \sum_{i=1}^{\frac{d+1}{2}} \tilde{\alpha}_i H^i = -\epsilon (V_u)^2
\]  

(90)

and therefore

\[
\frac{d-2}{2r} \left( \frac{M(u)}{r^{d-1}} - \frac{2}{(d-2)r^{d-1}} \int r^{d-2} T^{\nu}_{\nu} \, dr \right) = -\epsilon (V_u)^2,
\]  

(91)

which finally determines the null-fluid current vector by

\[
V_u^2 = -\epsilon \left( \frac{M(u)}{r^{d-1}} - \frac{1}{(d-2)r^{d-1}} \int dr r^{d-2} (\mathcal{L} - 2\mathcal{L}_F \mathcal{F}) \right).
\]  

(92)

4. Radiating power-Maxwell source in Lovelock gravity

In this section our choice for the NLM field consists of a particular kind, namely

\[
\mathcal{L}(\mathcal{F}) = -\mathcal{F}^q
\]  

(93)

in which \(q\) stands for a constant parameter and \(\mathcal{F} = F_{\mu\nu} F^{\mu\nu}\), as before. Let us note that this particular version of nonlinearity attracted considerable interest in recent years [14]. Although
the parameter \( q \) (i.e. the power) may be assumed arbitrary, imposition of the energy conditions with other requirements restricts \( q \) to a limited set of values. The rest of the gravitational action will be chosen as in the previous sections. Similarly, our choice of the line element follows that of (60). The electromagnetic field 2-form turns out to be

\[
F = \frac{Q(u)}{r^{d-2}} \, d\tau \wedge du, \tag{94}
\]

and the energy–momentum component \( T_{r'} \)

\[
T_{r'} = \frac{1}{2} (\mathcal{L} - 2\mathcal{L}_F) = (q - \frac{1}{2}) \, \mathcal{F}' , \quad (q \neq \frac{1}{2}). \tag{95}
\]

so that

\[
\mathcal{F}' = (-1)^q \left( \frac{2Q^2(u)}{q^2 r^{2d-2}} \right)^{\frac{q}{q-1}}. \tag{96}
\]

Following the procedure of the previous section further, which led us to condition (88), we now obtain, in analogy,

\[
\sum_{i=1}^{\lfloor \frac{d-1}{2} \rfloor} \tilde{\alpha}_i H^i = \frac{M(u)}{r^{d-1}} - h_d(u) r^{\frac{2q(d+1)}{d-1}}, \tag{97}
\]

in which

\[
h_d(u) = (-1)^q \left( \frac{2q - 1}{d - 2} \right) \left( \frac{2Q^2(u)}{q^2} \right)^{\frac{q}{q-1}}. \tag{98}
\]

It is observed that an arbitrary \( q \) does not guarantee the reality of the metric function, which forces us to choose \( q \) appropriately.

The particular dimensionality \( d = 5 \) brings in significant simplicity to the foregoing expressions such as

\[
\tilde{\alpha}_1 H + \tilde{\alpha}_2 H^2 = \frac{M(u)}{r^4} - h_5(u) r^{\frac{2q(4q+1)}{3q-1}}, \tag{99}
\]

where \( \tilde{\alpha}_1 = 1 \) and \( \tilde{\alpha}_2 = 2\alpha \). The quadratic equation for \( H \) can easily be solved and the corresponding metric function \( f(r, u) \) is determined by

\[
f(r, u) = \chi + \frac{r^2}{4\alpha} \left( 1 \pm \sqrt{1 + 8\alpha \left( \frac{M(u)}{r^4} - h_5(u) r^{\frac{2q+1}{3q-1}} \right)} \right). \tag{100}
\]

In analogy to equation (82), the null-current component for the present case turns out to be

\[
V^2_u = -\epsilon \frac{3}{2} \left( \frac{M(u)}{r^4} - h_5(u) r^{\frac{2q+1}{3q-1}} \right)^2 \tag{101}
\]

in which, as usual, a ‘dot’ represents \( \frac{d}{du} \).

To be able to proceed further with the thermodynamical properties in the present choice of \( d = 5 \) (and \( \chi = +1 \)), we must determine the apparent horizon through \( f(r, u) = 0 \). This leads us to the algebraic relation

\[
2\alpha - M(u) + r_{\text{H}}^2 + h_5(u) r_{\text{H}}^k = 0 \tag{102}
\]

in which \( r_{\text{H}} \) denotes apparent horizon (if any) and we have abbreviated

\[
k = 2 \left( \frac{5q - 6}{2q - 1} \right). \tag{103}
\]
Table 1. The relation between the parameters $k$ and $q$ according to equation (103).

| $k$ | 0 | 1 | 2 | 3 | 4 | 6 | $\cdots$ |
|-----|---|---|---|---|---|---|---------|
| $q$ | $\frac{6}{5}$ | $\frac{11}{3}$ | $\frac{5}{5}$ | $\frac{9}{2}$ | 4 | $-3$ | $\cdots$ |

Table 1 shows the relation between $q$ and $k$ for certain leading numbers of interest. With reference to table 1, we can find a finely tuned set of $q$ powers so that $h_5(u)$ will be real. Clearly, any odd/even integer $q$ will do the job, whereas non-integer $q$'s will not serve the purpose. For instance, $q = 4$ is a good choice. Accordingly, we find

$$h_5(u) = \frac{7}{3} \left( \frac{Q^2(u)}{8} \right)^{4/7} > 0,$$

and

$$r_5(u) = \frac{1}{2} \left( \sqrt{1 + 4h_5(u)(M(u) - 2\alpha)} - 1 \right) \quad M(u) > 2\alpha. \quad (105)$$

The corresponding metric function takes the form

$$f(r, u) = 1 + \frac{r^2}{4\alpha} \left( 1 \pm \sqrt{1 + 8\alpha \frac{M(u)}{r^4} - h_5(u)} \right). \quad (106)$$

Different choices of $q$ values from table 1 can be treated in a similar manner to obtain the corresponding $f(r, u)$ function, which we shall not discuss any further in this paper.

5. Conclusion

Methods to generate new solutions for various energy momenta are available in the literature. In this regard, recently we have proved a theorem that generalizes Salgado’s theorem [7]. The whole issue in such a problem is the physical significance of the energy momentum under consideration. Scalar field source, for instance, is known to work only in $d = 4$. Linear Maxwell electrodynamics has already been incorporated in higher dimensional Lovelock gravity [2, 4]. Our example 5, which is a particular version of the theorem, concerns a cloud of strings without the Maxwell field. We explore thermodynamic properties of such a cloud for a particular solution in $d = 5$ spacetime dimension, representing CS black holes, which undergoes a Hawking–Page phase transition (figure 4). In this paper we extended the problem to cover a NLM source. Among this class of theories BI is the most familiar one, which arises as a particular example of the theorem. The CSBI metrics that we found by using our theorem turn out to be thermodynamically stable. Next, we address the dynamical problem that covers time-dependent metrics and NLM fields. Stated otherwise, we have generalized the well-known dynamical BV metric to the general Lovelock theory with NLM sources. A radiating null current source naturally accompanies the radiating energy–momentum of such metrics, which lose mass and charge. It is needless to remark, finally, that the opposite problem of the 'shining star', namely the collapse of time-dependent energy–momentum and null radiation current, is also solved in the same theory.

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