Strong-coupling Bose polarons in 1D: Condensate depletion and deformed Bogoliubov phonons

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We consider the interaction of a quantum impurity with a one-dimensional degenerate Bose gas forming a Bose-polaron. In three spatial dimensions the quasiparticle is typically well-described by the extended Fröhlich model, in full analogy with solid-state systems. This description, which assumes an undepleted condensate, fails however in 1D, where the backaction of the impurity on the condensate leads to a self-bound mean-field polaron for arbitrarily weak impurity-boson interactions.

We present a model that takes into account this backaction and describes the impurity-condensate interaction as coupling to phonon-like excitations of a deformed condensate. A comparison of polaron energies and masses to diffusion quantum Monte-Carlo simulations shows very good agreement already on the level of analytical mean-field solutions and is further improved when taking into account quantum fluctuations.

The polaron, introduced by Landau and Pekar [1, 2] to describe the interaction of an electron with lattice vibrations in a solid material, is a paradigmatic model of quasiparticle formation in condensed matter physics. A hallmark feature of the quasiparticle is mass enhancement: the bare electron becomes dressed by a cloud of lattice phonons which in turn affects its dynamical properties. The polaron concept has wide applications across condensed matter physics ranging from the description of charge transport in organic semiconductors to high-$T_c$ superconductivity [3, 4].

More recently, neutral atoms immersed in quantum gases have attracted much attention since they are experimentally accessible platforms allowing the study of polaron physics with high precision and in novel regimes. For example, the impurity-bath interaction can be tuned from weak to strong coupling employing Feshbach resonances [5]. In such systems, the impurity atom is immersed in a superfluid and a polaron is formed by its interaction with the collective excitations of the superfluid. The Fermi-polaron, i.e. an impurity interacting with a degenerate Fermi gas has been studied in a number of experiments in recent years [6–13]. In contrast, due to the large compressibility of a Bose gas, a large number of excitations can be generated, and interactions within the Bose gas can be important.

Theoretical works addressing the Bose polaron most often describe the interaction with the impurity as a coupling to Bogoliubov phonons of a uniform superfluid [15, 23]. The resulting (extended) Fröhlich Hamiltonian is formally identical to the one used in solid-state systems [24], amended with two-phonon scattering terms. However, as well known from the example of electrons in superfluid Helium, a strongly interacting impurity can also distort the superfluid itself [25]. This deformation creates a potential for the impurity which can lead to a self-bound state making Fröhlich model inadequate. In 3D, the normalized impurity-Bose interaction has to exceed a critical value for this, given by the inverse gas parameter [26, 27]. Since for typical condensates the gas parameter is very small, the extended Fröhlich model remains adequate even for strong impurity-Bose interactions and efficient approaches for its solution beyond the perturbative regime have been developed in the past, including variational [23] and renormalization group (RG) approaches [22, 28].

The situation is very different in 1D, which was experimentally realized in [14]. Here an arbitrarily weak deformation of the condensate leads to a self-localized impurity wave function. This restricts the accuracy of the Fröhlich model to the perturbative regime. In fact a comparison between exact diffusion Monte-Carlo (DMC) simulations of the full microscopic model with RG solutions of the extended Fröhlich model in [22] shows that this model is only accurate for weak interactions and breaks down completely for attractive interactions at intermediate interaction strengths.

In this Letter, we follow a different approach, and expand the Bose quantum field about the exact mean-field solution of the mobile impurity immersed in a superfluid. Such a treatment incorporates the backaction of the impurity already at the mean-field level [27], and quantum effects are taken into account by the coupling to phonon-like excitations of the deformed superfluid. Here the naive application of the standard expressions for the polaron mass, applicable for Fröhlich-type models, give non-sensical results and a generalization is needed. We derive analytical expressions for the polaron wave function in the mean-field approximation as well as for the polaronic mass and energy. We calculate quantum corrections by solving the Bogoliubov-de Gennes equations in a self-consistent approach. Our results are benchmarked against recent DMC results [22]. We find very good agreement in all regimes for repulsive interactions underpinning the hypothesis that expanding about the
non-uniform condensate is an excellent starting point. We also present results for attractive interactions. Here we find again very good agreement with DMC for the energy of the polaron but less good agreement for the polaron mass. We attribute this discrepancy to the existence of many-particle bound states in the attractive regime [22, 29].

**Model and polaron mass.** Our starting point is a single impurity atom coupled to $N$ identical bosons in one dimension, described by the Hamiltonian

\[
\hat{H} = \int dx \hat{\phi}^\dagger(x) \left( -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \frac{g_{\text{BB}}}{2} \hat{\phi}(x) \hat{\phi}(x) - \mu \right) + g_{\text{IB}} \delta(x - \hat{X}) \hat{\phi}(x) + \hat{P}^2 \frac{2}{M},
\]

where $m$ (or $M$) denotes the mass of the bosons (impurity atom), $\hat{\phi}(x)$ is the Bose field operator, $g_{\text{BB}}$ ($g_{\text{IB}}$) are the boson-boson (boson-impurity) interaction strength, $\hat{X}$ ($\hat{P}$) denotes the position (momentum) operator of the impurity, and $\mu$ is the chemical potential. Throughout the paper, we set $\hbar = 1$ and employ periodic boundary conditions of length $L$. The relative interaction strength is denoted by $\eta = g_{\text{IB}}/g_{\text{BB}}$ and we introduce the healing length $\xi = 1/\sqrt{2m\mu}$ and the speed of sound $c = \sqrt{\mu/m}$.

Expanding the bosonic field operator in Eq. (1) around a homogenous condensate as $\hat{\phi}(x) = \sqrt{n_0} + \xi(x)$ with $n_0 = N/L$ leads to the extended Fröhlich Hamiltonian [23,24]. In this work, we choose a different starting point and consider the effects of the impurity already at the level of the condensate.

Before delving into the solutions of the mean-field equations, it is important to point out some fundamental differences between the ground state of the effective Fröhlich and the full Hamiltonian for finite momentum. For the Fröhlich model it is easy to show that for fixed total momentum, the ground state is indeed the polaronic solution and from there quantities like effective mass or energy can be calculated [22,33]. The situation is very different for (1). Indeed the ground state for finite momentum for this case is the uniformly boosted system. To see this we introduce the potential $\hat{\Omega} = \hat{H} - v \hat{P}_{\text{tot}}$, with total momentum $\hat{P}_{\text{tot}} = \hat{P}_{\text{B}} + \hat{P}$ where $\hat{P}_{\text{B}} = -i \int \hat{\phi}^\dagger(x) \partial_x \hat{\phi}(x) dx$. It is straightforward to see that finding the constrained ground state of (1) (with fixed total momentum) is equivalent to finding the unconstrained ground state of $\hat{\Omega}$ for a given $v$ which acts as a Lagrange multiplier. Introducing the unitary transformation $\hat{U}_{\text{cm}} = \exp \left( -i M_{\text{tot}} \frac{1}{2} \hat{X}_{\text{cm}} v \right)$, with $M_{\text{tot}} = Nm + M$ and $\hat{X}_{\text{cm}} = M_{\text{tot}}^{-1} \left( m \int dx x \hat{\phi}^\dagger(x) \hat{\phi}(x) + M \hat{X} \right)$ to boost into the center of mass frame one finds $\hat{\Omega} = \hat{U}_{\text{cm}}^{-1} \hat{H} \hat{U}_{\text{cm}} - \frac{1}{2} M_{\text{tot}} v^2$. With this expression, one can clearly relate eigenstates of $\hat{H}$ with those of $\hat{\Omega}$. In particular, the ground state for finite momentum (corresponding to finite $v$) is the boosted ground state and the effective mass of the polaron is independent of coupling strength and always equal to the total mass. Such a uniformly boosted system is precluded in the Fröhlich model since the mean-field configuration is chosen to carry no momentum.

We proceed as in the case of the Fröhlich model and eliminate the impurity position operator from (1) by a Lee-Low-Pines (LLP) type transformation $\hat{U}_{\text{LLP}} = \exp(-i\hat{X}\hat{P}_{\text{B}})$. Here, in contrast, the total momentum of the bosons $\hat{P}_{\text{B}}$ enters. $\hat{U}_{\text{LLP}}$ transforms to a co-moving frame, where the impurity is at the origin and its momentum is transformed to the conserved total momentum of the system which can be treated as a c-number $P$. At the same time, a new term that is quadratic in the boson field operators $(P - \hat{P}_{\text{B}})^2/2M$ emerges in the transformed Hamiltonian. In order to treat this it will prove helpful to introduce a Hubbard-Stratonovich field $\hat{u}$, which gives

\[
\hat{H}_{\text{LLP}}^{\text{B}} = \int dx \hat{\phi}^\dagger(x) \left( -\frac{1}{2m_r} \frac{\partial^2}{\partial x^2} + \frac{g_{\text{BB}}}{2} \hat{\phi}(x) \hat{\phi}(x) - \mu \right) + g_{\text{IB}} \delta(x - \hat{X}) \hat{\phi}(x) + \hat{P}^2 \frac{2}{M},
\]

where $\hat{u}$ satisfies $\hat{M} \hat{u} = P - \hat{P}_{\text{B}}$, and can thus be viewed as the impurity velocity. In the above equation, $m_r = (M + M)/Mm$ is the reduced mass and we defined a rescaled healing length $\xi = \sqrt{m/m_r} \xi$ and speed of sound $c = \sqrt{m/m_r} c$.

**Mean-field solutions.** We now expand $\hat{\phi}(x) = \phi(x) + \xi(x)$ and $\hat{u} = u + \delta \hat{u}$ where $\phi(x)$ and $u$ are chosen to
solve the mean-field equations of [2]

\[-\frac{1}{2m_e}\partial_x^2 + g_{BB}|\phi(x)|^2 - \mu + i\nu\partial_x\phi(x) = 0, \quad (3)\]

\[\partial_x\phi(x)|_{x=0} = 0 + 2m_e g_{BB}\phi(0), \quad (4)\]

subject to the boundary conditions \(\phi(\pm L/2) = 0\) and \(|\phi(\pm L/2)|^2 = n_0 + \mathcal{O}(1/L)\). Note that in order to remedy the problem of the uniformly boosted system being the ground state we require that the polaron is a local quantity. Thus the condensate must be stationary far away from the impurity up to \(1/L\) corrections. The correction to the density \(n_0\) of a homogeneous condensate can be understood by noting that the impurity locally repels (attracts) bosons which is compensated by the \(1/L\) corrections. Solutions of Eq. (3) exist in the literature where the phase is not periodic [32, 33]. These solutions correspond to unphysical sources at the boundary and lead to wrong predictions such as a negative polaron mass (see Supplemental Material). We instead find the mean-field solution \(\phi(x) = \sqrt{n(x)}e^{i\theta(x)}\)

\[n(x) = \frac{\mu}{g_{BB}}(1 - \beta \text{sech}^2(\sqrt{\beta/2}|x| + x_0/\xi) \quad (5)\]

with \(\beta = 1 - \frac{\eta^2}{x^2} + \mathcal{O}(1/L^2)\) and \(\mu = g_{BB}n_0 - (\partial_x^2 \theta_0) u + \mathcal{O}(1/L^2)\). If we consider the mean-field solution alone we fix \(n_0\) from \(n_0(1 + 2\sqrt{2}\beta\xi/L(1 - \tanh(\sqrt{\beta/2x_0/\xi}))) + \mathcal{O}(1/L^2)\). Upon considering quantum fluctuations later on, the mean-field density needs to be adjusted. For the phase we find \(\theta_0(x) = \theta_0(0) + (2f(0) - 2f(L/2))x/L\) with \(f(x) = \text{arctan}(\frac{\sqrt{4u^2\beta/\xi^2} - 2\beta - 1}{e^{\sqrt{2\beta(x + x_0)/\xi}} + 2\beta})\)

for \(x > 0\) and \(\theta_0(x) = 2f(0) - f(-x)\) for \(x < 0\). It is important to keep contributions up to first order of \(1/L\) when calculating the total boson momentum. Finally, we determine \(x_0\) through the jump condition [4] for the derivative. In the limit \(x = 0\) we find for \(g_{BB} > 0\): \(x_0 = \frac{\xi}{\sqrt{2}} \log(y)\), with \(y = \sqrt{1 + 8\eta^2 \xi^2/n_0^2 + 2\sqrt{2n_0\xi}\beta}\) and for \(g_{BB} < 0\), we have \(x_0 \to x_0^\ast = x_0 + i\pi/2\xi(2/\beta)^{1/2}\). In Fig. 1 mean-field predictions for condensate density and phase are shown for different interaction strength and a slowly moving impurity. From the analytical solution we can derive a parameter characterizing the relative condensate deformation

\[\eta/n_0\xi = \eta\sqrt{2}\gamma\]

with \(\gamma = \gamma_{me}/m\), where \(\gamma = 1/(2n_0^3\xi^3)\) is the so-called Tonks parameter of the 1D Bose gas [34, 35]. The deformation becomes sizeable if \(\eta/n_0\xi \sim 1\).

With the analytical expressions for the condensate density and phase we can calculate the polaron energy \(E_p = E(g_{BB}) - E(g_{BB} = 0)\), and, using \(M/m^* = \lim_{\eta \to 0}(1 - \frac{\Delta}{\eta})\), see [25], the effective mass \(m^\ast\) of the polaron. This gives

\[E^{\pm}_{p} = g_{BB}n_0 \left(\frac{|y| \pm 1}{|y| \pm 1}\right)^2 + \frac{8}{3}n_0\xi \left(\frac{3|y| \pm 1}{(|y| \pm 1)^2}\right)\]

for the energies of the repulsive \((E_p^{\rho})\) and attractive \((E_p^{\sigma})\) polaron, and for the mass:

\[M/m^* = M(y^2 - 1)\frac{8n_0\xi m^\ast}{\sqrt{2} + M(y^2 - 1)}. \quad (8)\]

The closed analytical expressions Eqs. (8) and (8) are a central result of this work. It is interesting to note that for \(\eta \to \infty\), [7] approaches the energy of a dark soliton and the effective mass \(m^*\) goes to zero which is in contrast
to results from the extended Fröhlich Hamiltonian \cite{22}. At this point we note that on the attractive side the solution will collapse to a multi-particle bound state for $\eta \gg 1$, which can be easily seen by noting $E_p \to -\infty$ for $\eta \to -\infty$.

Quantum fluctuations. After expanding the fields in $H_{\text{LPP}}$, the quantum fluctuations we find up to second order in $\xi(x)$ and $\delta u$

$$
\hat{H}_{\text{LPP}}^{\xi} = \int dx \left[ \hat{\xi}(x) \left( -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + 2g_{\text{BB}}|\phi(x)|^2 - \mu + 
+ g_{\text{IB}}\delta(x) + iu\xi(x) \right) + \frac{g_{\text{BB}}}{2} \left( \phi(x)^2 \hat{\xi}(x)^2 + h.c. \right) \right] - i\delta u \int dx \left( \frac{\partial}{\partial x} \hat{\xi}(x) \phi(x) + \phi^*(x) \frac{\partial}{\partial x} \xi(x) \right) - \frac{1}{2} M\delta u^2,
$$

with $M\delta u = -i \int [\phi^*(x) \partial_x \xi(x) + \hat{\xi}(x) \partial_x \phi(x)] dx + O(\xi(x)^2)$, which can be diagonalized by a Bogoliubov rotation to a generalized basis of phonons on a deformed background. We note that for distances far away from the impurity i.e. $x \to \infty$ these phonons look like the ones of a homogeneous condensate. This allows us to extract the quantum depletion (see \cite{36} for a detailed discussion on how to regularize the arising UV divergences of the zero point energy). We find that the quantum corrected density far away from the impurity is given by $n_0 = n_{\text{MF}} - \frac{1}{2} g_{\text{BB}} n_{\text{MF}}^2$ and thus we have to add just the mean-field density accordingly. To diagonalize (9) we note that all terms involving $\delta u$ become non local and thus difficult to handle in general, except for the special case $p = 0$. This enables us to diagonalize (9) in its full form in this case and to calculate the polaron energy at rest. For a moving impurity we introduce an approximation treating $\delta u$ as a c-number and keep it as an undetermined variational parameter in the mean-field equations. After diagonalizing the remaining quadratic Hamiltonian (9) with $\delta u = 0$ the value of $u$ is determined self-consistently by enforcing the equations of motion for $u$ on the level of expectation values, i.e. $M u = P - \langle P_B \rangle$, where $\langle P_B \rangle = \int \phi^*(x) \xi \phi(x) dx - i \int \xi \partial_x \hat{\xi}(x) dx$, where the expectation value is taken with respect to the vacuum of the eigenmodes. For a more detailed description we refer to the Supplemental Material. Then it is straightforward to calculate the effective mass including the quantum corrections $M/m = M u/p$. As can be seen in Fig. 2(a), where the energies of the full and approximate solution of the BdG equations are shown, the approximate treatment of the Hubbard-Stratonovich field is very good.

Discussion & Summary. Fig. 2 shows that already the mean-field solution improves the agreement with DMC simulations significantly for $g_{\text{IB}} > 0$ as compared to the Fröhlich model. Including quantum fluctuations leads to almost perfect agreement for the energy. We find however that the effective mass diverges for $g_{\text{IB}} \to \infty$, even after including quantum fluctuations, which is in contrast to the DMC results \cite{22}. This divergence is a characteristic of the 1D geometry and is, for example, also observed in the Tonks-limit \cite{37}. One would naively expect this to happen since for $\eta \gg 1$ the condensate is split into two halves by the impurity, preventing any transport of the condensate across it. The only possible contribution could come from tunneling which is highly suppressed for $\eta \gg 1$. The same reasoning explains why the quantum correction to the effective mass is most significant for intermediate couplings since here the classical current is reduced by the strong condensate deformation, but tunneling is still relevant. The question whether the effective mass actually saturates remains open and other approaches such as DMRG could shed more light on this.

In summary, we have developed a description of the Bose polaron in 1D that unperturbatively accounts for the backaction of the condensate. Since the density of phonons defined on such a deformed background remains small, their intrinsic interactions can be neglected to a good approximation. Our approach provides a quantitatively accurate and, to a large extent, analytical description of Bose polarons even for strong impurity-boson interactions. We expect that it will also allow a good description in 3D at and beyond the critical strength of the impurity-boson interaction for self-trapping.

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SUPPLEMENTAL MATERIAL

A. The mean-field solution

In this section, we discuss the mean-field equations used in the main text. The mean-field equations that need to be solved are

\[ \left( -\frac{1}{2m_{\text{red}}} \partial_x^2 + g_{\text{BB}} |\phi(x)|^2 - \mu + i\eta \partial_x \right) \phi(x) = 0 \]

\[ \partial_x \phi(x) \bigg|_{0^+} = 2m_{\text{red}} g_{\text{BB}} \phi(0) \]

\[ \phi(L/2) = \phi(-L/2) \]

\[ Mu = p - P_B. \]  

(9)

If we do not require periodic boundary conditions analytical solutions of the form \(|\phi(x)| e^{i\theta_1(x)}\) can be found in the literature [32, 33]. To make use of those solutions we make the following ansatz

\[ \phi(x) = \tilde{\phi}(x) e^{i\theta_1(x)}, \]  

(10)

where we introduced \(\theta_1(x)\) which will be of \(O(x/L)\) and is fixed later on to ensure periodicity of the phase for the mean-field solutions, giving the overall phase \(\theta(x) = \theta_0(x) + \theta_1(x)\). Upon inserting our ansatz into (9) we arrive at

\[ \left( -\frac{1}{2m_{\text{red}}} \partial_x^2 + g_{\text{BB}} |\tilde{\phi}(x)|^2 - \tilde{\mu} + i\eta \partial_x \right) \tilde{\phi}(x) = 0 \]

\[ \partial_x \tilde{\phi}(x) \bigg|_{0^+} = 2m_{\text{red}} g_{\text{BB}} \tilde{\phi}(0) \]

\[ e^{i\theta_1(L)} \tilde{\phi}(L/2) = \tilde{\phi}(-L/2) \]  

(11)

with the re-definitions \(\tilde{\mu} = \mu + (\partial_x \theta_1) u/M + O(1/L^2)\) and \(\tilde{\mu} = u - (\partial_x \theta_1)/m_1\). The solution for this problem is now given by [32, 33]

\[ |\tilde{\phi}(x)| = \sqrt{\mu/g_{\text{BB}}} \left( 1 - \beta \sech^2 \left( \sqrt{\beta/2} (|x| + x_0)/\bar{\xi} \right) \right)^{1/2}, \]

\[ \theta_0(x) = \begin{cases} f(x) & x > 0 \\ 2f(0) - f(-x) & x < 0, \end{cases} \]

\[ f(x) = \arctan \left( \frac{\sqrt{4u^2\beta/\bar{\xi}^2}}{\sqrt{2} - 2\beta + 1} \right). \]  

(12)

with \(\beta = 1 - v^2 + O(1/L^2)\) and \(\mu = g_{\text{BB}} n_0^{\text{MF}} - (\partial_x \theta_1(x)) u/M + O(1/L^2)\). The jump condition determines \(x_0\) through a polynomial of order three, but only one solution is stable. It is possible to extract quantities like the critical momentum from here, for a detailed discussion of this we refer to [32]. For finite momentum, the condition for \(x_0\) to be solved numerically but in the limit \(p \to 0\) we can find the analytical solutions stated in the paper. If we consider the mean-field solution alone and require the number of condensed particles \(N\) to stay constant on the mean-field level we fix \(n_0^{\text{MF}} = n_0 \left[ 1 + 2\sqrt{2}\bar{\xi}/L \left( 1 - \tanh(\sqrt{\beta/2x_0}/\bar{\xi}) \right) \right] + O(1/L^2). \) Lastly, we fix \(\theta_1(x)\) to ensure the periodicity of the phase by

\[ \theta_1(x) = 2[f(0) - f(L/2)] \frac{x}{L}. \]  

(13)

At this point we note that the \(1/L\) corrections are indeed important when calculating physical quantities. This can be seen by considering the Boson momentum \(P_B = \int n(x) \partial_x \theta(x) dx = \int n(x) \partial_x \theta_0(x) dx + n_0 [2(f(0) - f(L/2)). \) Form there we can derive the expressions for \(m^*\) and \(E_p\) given in the main text, which are both defined in the limit \(p \to 0\), which allows us to state them fully analytically.

B. Boundary conditions

In this section we will justify the importance of the periodic boundary conditions in correctly calculating the effective mass within mean-field theory. Even though this boundary condition leads to a constant density far away from the impurity, the phase is linearly changing at the order of \(1/L\). One might be tempted to use a solution where both density and phase are truly constant far away from the impurity. Which is supported by NSF PHY-1607611 and the Simons Foundation.
away from the impurity (up to exponentially small corrections). A solution with this different boundary condition would be given by (10) and (12), but with \( \theta_1(x) = 0 \). The effective mass can then be deduced from the wave function in the same way as was done in the main text and is plotted in Fig. 4. This effective mass decreases for increasing \( \eta \) and even becomes negative. This unphysical result qualitatively disagrees with DMC results, and also contains a phase jump at infinity which introduces a source term there. This implies that the system does not form a classical Hamiltonian system and, strictly speaking, functional derivatives cannot be taken. On a purely mean-field level this can be alleviated by modifying the functional derivatives by exactly this source term. This has been done in the context of solitons \[38, 39\]. Upon considering quantum fluctuations on top of the mean-field solution this will not be sufficient anymore and the phase has to be chosen to be periodic.

C. Quantum fluctuations

In the following, we give a short overview of the methods used to obtain the quantum corrections to the mean-field solutions. The major steps have been outlined in the main text, and thus we focus on the numerical details. An extensive overview of the techniques used here can be found in [40]. We note that this is equivalent to solving the resulting Bogoliubov-de Gennes equations. We start by discretizing \( \hat{H}_{\text{LLP}}^S \) from the main text after either making the approximation of treating \( u \) as a variational parameter or for \( p = 0 \) integrating out the \( \hat{u} \)-field. For all numerical results presented here, the discretization was done in real space and is therefore straightforward apart from the delta distribution, which was approximated by a Kronecker delta in the following way \( \delta(x) \rightarrow \delta_\delta,0/a \), where \( a \) is the discretization. This comes at the expense of not accounting correctly for the UV behavior. The deviation from the continuum UV behavior is due to discretizing the derivative operators. Nevertheless, for the observables we are interested in here the UV behavior is not essential, and we found fast convergence; thus, the diagonalization in real space is justified. For notational convenience, we omit the hats on all discretized operators.

After discretization the Hamiltonian can be written as

\[
\hat{H}_{\text{LLP}}^S = \sum_{ij} \left[ A_{ij} \phi_i^\dagger \phi_j + \frac{1}{2} (B_{ij} \phi_i^\dagger \phi_j + B_{ij}^* \phi_i \phi_j) \right]
\]

\[
= \frac{1}{2} \Phi^\dagger M \Phi - \frac{1}{2} \text{Tr}(A),
\]

where \( \Phi^\dagger = [\phi_1^\dagger, \phi_2^\dagger, \ldots, \phi_n^\dagger, \phi_1, \phi_2, \ldots, \phi_n] \) is the discrete version of \( \hat{\xi}(x) \) and \( M \) is the semi positive definite matrix

\[
M = \begin{bmatrix} A & B \\ B^* & A^* \end{bmatrix}.
\]

At this point we already note that the trace term is of fundamental importance in 1D since it renders results like the zero point energy finite without performing additional regularization. Following the steps outlined in [40] we now diagonalize

\[
\nu M = \begin{bmatrix} A & B \\ -B^* & -A^* \end{bmatrix},
\]

and thus find \( T \) such that \( T^\dagger M T = \text{diag}(\omega_1, \omega_2, \ldots, \omega_n, \omega_1, \omega_2, \ldots, \omega_n) \), while guaranteeing \( T^\dagger \eta T = \nu \), which allows us to introduce new bosonic operators \( \Psi^\dagger = [b_1^\dagger, b_2^\dagger, \ldots, b_n^\dagger, b_1, b_2, \ldots, b_n] \) through

\[
\Phi = T \Psi,
\]

for which the Hamiltonian takes diagonal form. The new operators \( b_i \) can be interpreted as quasiparticle-like bosonic excitations with eigenenergy \( \omega_i \). For a stable polaron the energy of those excitations is minimized, i.e. the system is in its vacuum state \( |0\rangle \) with respect to the \( b_i \). From here it is then easy to verify that the quantum corrections to the expectation value of an observable of the form \( O_Q = \sum_{ij} O_{ij} \phi_i^\dagger \phi_j \) is

\[
\langle O_Q \rangle = \langle 0 | \Psi^\dagger T^\dagger \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} T \Psi | 0 \rangle
\]

\[
= \langle 0 | \Psi^\dagger \begin{bmatrix} C & D \\ E & F \end{bmatrix} \Psi | 0 \rangle
\]

\[
= \text{Tr}(F).
\]

To conclude this section, we will comment on the IR (infrared) divergences, that are characteristic in 1D systems and how they are dealt with here. First, we note that quantities like the two-point function

\[
\langle \phi_i^\dagger \phi_i \rangle = \langle 0 | (\Psi^\dagger T^\dagger)_(i)(T \Psi)_i | 0 \rangle \sim L,
\]

are indeed IR divergent in our treatment. For the global quantities and \( p = 0 \), this can be dealt with as outlined in the main text by considering the zero point energy

\[
E = \frac{1}{2} \sum_i \omega_i - \text{Tr}(A),
\]

which is UV and IR finite and then taking adequate derivatives (i.e. with respect to the chemical potential for the depletion). When considering \( \hat{H}_{\text{LLP}}^S \) for \( p \neq 0 \) without any approximations, the phonon momentum seems to be IR divergent and also for the polaron energy we found a system size dependence. Lastly, we remark that in the approximate treatment, i.e. when viewing \( u \) as a variational parameter, the phonon momentum remains IR and UV finite. Therefore, we conclude that all results presented in the main text are cut-off independent, and no divergences occur.
[1] L. Landau, Phys. Z. Sowjetunion 3, 644 (1933).
[2] L. D. L. S. I. Pekar, Zh. Eksp. Teor. Fiz. (1946).
[3] N. Mott, Phys. C Supercond. 205, 191 (1993).
[4] A. S. Alexandrov and J. T. Devreese, Advances in Polaron Physics Vol. 159 (Springer-Verlag, Berlin, 2010).
[5] H. Bässler and A. Köhler, Top. Curr. Chem. 312, 1 (2012).
[6] A. Schirotzek, C. H. Wu, A. Sommer, and M. W. Zwierlein, Phys. Rev. Lett. 109 (2012).
[7] Y. Zhang, W. Ong, I. Arakelyan, and J. E. Thomas, Phys. Rev. Lett. 108 (2012).
[8] C. Kohstall, M. Zaccanti, M. Jag, A. Trenkwalder, P. Massignan, G. M. Bruun, F. Schreck, and R. Grimm, Nature 485, 615 (2012).
[9] M. Koschorreck, D. Pertot, E. Vogt, H. Fröhlich, M. Feld, and M. Köhl, Nature 485, 619 (2012).
[10] F. Scazza, G. Valtolina, P. Massignan, A. Recati, A. Amico, A. Burchianti, C. Fort, M. Inguscio, M. Zaccanti, and G. Roati, Phys. Rev. Lett. 118 (2017).
[11] M. Cetina, M. Jag, R. S. Lous, J. T. Walraven, R. Grimm, R. S. Christensen, and G. M. Bruun, Phys. Rev. Lett. 115 (2015).
[12] M. Cetina, M. Jag, R. S. Lous, I. Fritsche, J. T. Walraven, R. Grimm, J. Levinsen, M. M. Parish, R. Schmidt, M. Knap, and E. Demler, Science 354, 96 (2016).
[13] M. Parish and J. Levinsen, Phys. Rev. B 94 (2016).
[14] J. Catani, G. Lamporesi, D. Naik, M. Gring, M. Inguscio, F. Minardi, A. Kantian, and T. Giamarchi, Phys. Rev. A 85, 023623 (2012).
[15] N. B. Jørgensen, L. Wacker, K. T. Skalmstang, M. M. Parish, J. Levinsen, R. S. Christensen, G. M. Bruun, and J. J. Arlt, Phys. Rev. Lett. 117, 055302 (2016).
[16] M. G. Hu, M. J. Van De Graaff, D. Kedar, J. P. Corson, E. A. Cornell, and D. S. Jin, Phys. Rev. Lett. 117, 055301 (2016).
[17] Z. Z. Yan, Y. Ni, C. Robens, and M. W. Zwierlein, arXiv:1904.02685.
[18] S. P. Rath and R. Schmidt, Phys. Rev. A 88, 053632 (2013).
[19] G. E. Astrakharchik and L. P. Pitaevskii, Phys. Rev. A 70, 013608 (2004).
[20] J. Tempere, W. Casteels, M. K. Oberthaler, S. Knoop, E. Timmermans, and J. T. Devreese, Phys. Rev. B 80, 184504 (2009).
[21] W. Casteels, T. Van Cauteren, J. Tempere, and J. T. Devreese, Laser Phys. 21, 1480 (2011).
[22] F. Grusdt, G. E. Astrakharchik, and E. Demler, New J. Phys. 19, 103035 (2017).
[23] Y. E. Shchadilova, R. Schmidt, F. Grusdt, and E. Demler, Phys. Rev. Lett. 117 (2016).
[24] H. Fröhlich, Adv. Phys. 3, 325 (1954).
[25] J. P. Hernandez, Rev. Mod. Phys. 63, 675 (1991).
[26] F. M. Cucchietti and E. Timmermans, Phys. Rev. Lett. 96 (2006).
[27] A. A. Blinova, M. G. Boshier, and E. Timmermans, Phys. Rev. A 88, 75 (2013).
[28] F. Grusdt and E. Demler, arXiv:1510.04934.
[29] L. A. Ardila and S. Giorgini, Phys. Rev. A 92, 33612 (2015).
[30] T. D. Lee, F. E. Low, and D. Pines, Phys. Rev. 90, 297 (1953).
[31] G. D. Mahan, Many-Particle Physics (Berlin: Springer, 2000).
[32] V. Hakim, Phys. Rev. E 55, 2835 (1997).
[33] T. Tsuzuki, J. Low Temp. Phys. 4, 441 (1971).
[34] M. Girardeau, J. Math. Phys. 1, 516 (1960).
[35] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[36] L. Salasnich and F. Teglio, Phys. Rep. 640, 1 (2016).
[37] L. Parisi and S. Giorgini, Phys. Rev. A 95, 1 (2017).
[38] M. Ivan and V. S. Gerdjikov, Phys. Rev. A 47 (1993).
[39] L. V. Barashenkov and E. Y. Panova, Phys. D 69, 114 (1993).
[40] M. Xiao, arXiv:0908.0787.