Abstract: The connection between Lefschetz formulae and zeta function is explained. As a particular example the theory of the generalized Selberg zeta function is presented. Applications are given to the theory of Anosov flows and prime geodesic theorems.

Key words: Lefschetz formula, Selberg zeta function, Anosov flows, prime geodesic theorem.

MSC: 11M36, 53D25, 11F72.

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Introduction

This is a survey on the connection between Lefschetz formulae and zeta functions in general and the work of the author on the generalized Selberg zeta function in particular.

A Lefschetz formula relates fixed points of an automorphism $f$ of some space to some global cohomology and the action of $f$ thereon. In the realm of geometrically defined zeta functions they serve to prove rationality by giving an interpretation of a zeta function as a “characteristic function” of the induced automorphism on the cohomology.

Such an interpretation is highly desirable for other types of zeta functions such as the Riemann zeta function. Since the Riemann zeta function is not rational the cohomology space in question should be infinite dimensional. In this paper we give some prototypes of zeta functions which can be interpreted in this way for some natural infinite dimensional cohomology groups.

Starting with discrete dynamical systems for which the classical Lefschetz formula ensures rationality of the zeta function, the case of suspensions, which is treated in some detail, serves as a guideline to find the suitable cohomology theory. It turns out that foliation cohomology, or rather reduced foliation cohomology does the trick.

The Selberg zeta function is an example of a zeta function which can be interpreted in this way. The corresponding Lefschetz formula can be generalized to a multi-dimensional Lefschetz formula for higher rank spaces. The latter can be applied to give meromorphic extension of generalized Selberg zeta functions or to prove higher rank prime geodesic theorems.
1 The classical Lefschetz formula

Let $M$ be a compact smooth manifold and let $f : M \to M$ be a diffeomorphism. We say that $f$ is regular if its graph $\Gamma(f)$ intersects the diagonal $\Delta \subset M \times M$ transversally only. This is equivalent to saying that for every fixed point $x$ of $f$ the differential $f_* : T_xM \to T_xM$ satisfies $\det(1 - f_* | T_xM) \neq 0$. In that case we define the index of the fixed point $x$ as

$$\text{ind}_f(x) \overset{\text{def}}{=} \text{sign} \left( \det(1 - f_* | T_xM) \right) \in \{ \pm 1 \}.$$ 

Note that the regularity implies that the fixed points of $f$ are isolated. Since $M$ is compact they are finite in number.

**Theorem 1.1 (Lefschetz trace formula)**

If $f$ is regular, then

$$\sum_{x = f(x)} \text{ind}_f(x) = \sum_{q=0}^{\dim M} (-1)^q \text{tr} \left( f^* | H^q(M) \right).$$

Here $H^q(M)$ denotes the cohomology of $M$ with, say, complex coefficients.

For a proof see [9].

Suppose that every iterate $f^n = f \circ \cdots \circ f$ of $f$ has only finitely many fixed points. In that case define the zeta function of $f$ as

$$Z_f(T) = \exp \left( \sum_{n=1}^{\infty} \frac{T^n}{n} \# \text{Fix}(f^n) \right),$$

where $\text{Fix}(f^n)$ is the set of fixed points of the map $f^n$. Since $\# \text{Fix}(f^n) = \sum_{d|n} \sum_{|o|=d} d$, where the outer sum runs over all positive divisors of $n$ and the inner sum runs over all $f$-orbits $o$ in $M$ of cardinality $d$. This implies that

$$Z_f(T) = \prod_o (1 - T^{|o|})^{-1},$$

where the product runs over all finite $f$-orbits in $M$. At first this is a formal power series in $T$. Under additional assumptions we can say more.
We say that $f$ is *strongly regular* if all iterates $f^n$, $n \in \mathbb{N}$ are regular and if $\text{ind}_{f^n}(x) = 1$ for every $x \in \text{Fix}(f^n)$. The latter condition follows for example if $M$ is a complex manifold and $f$ is holomorphic.

**Theorem 1.2** (*Lefschetz determinant formula*)

*If $f$ is strongly regular, then*

$$Z_f(T) = \prod_{q=0}^{\dim M} \det (1 - Tf^* | H^q(M))^{(-1)^{q+1}}.$$  

*In particular, $Z_f(T)$ is a rational function in $T$.*

**Proof:** This is a simple consequence of Theorem 1.1. The assumptions imply that $\#\text{Fix}(f^n) = \sum_{x=f^n(x)} \text{ind}_{f^n}(x)$, so the Lefschetz formula gives

$$Z_f(T) = \exp \left( \sum_{n=1}^{\infty} \frac{T^n}{n} \sum_{q=0}^{\dim M} (-1)^q \text{tr} \left( f^* | H^q(M) \right) \right) = \prod_{q=0}^{\dim M} \exp \left( (-1)^q \text{tr} \sum_{n=1}^{\infty} \frac{(T f^*)^n}{n} | H^q(M) \right)$$

$$= \prod_{q=0}^{\dim M} \det (\exp(-\log(1 - Tf^*)) | H^q(M))^{(-1)^{q+1}}$$

$$= \prod_{q=0}^{\dim M} \det(1 - Tf^* | H^q(M))^{(-1)^{q+1}}.$$

□

Lefschetz formulae of the trace or the determinant type emerge in varying contexts throughout mathematics. As an example consider the Hasse-Weil zeta function $Z_V(T)$ of a smooth projective variety $V$ over a finite field $\mathbb{F}$. It is defined as

$$Z_V(T) = \exp \left( \sum_{n=1}^{\infty} \frac{T^n}{n} \#V(\mathbb{F}_n) \right),$$
where $\mathbb{F}_n$ is the extension of $\mathbb{F}$ of degree $n$, which is uniquely determined up to isomorphism. For this zeta function M.Artin, Verdier, and Grothendieck showed that

$$Z_V(T) = \prod_{j=0}^{2\dim M} \det(1 - T \text{Frob} | H^j_{l-ad}(V))^{(-1)^{j+1}},$$

where Frob is the Frobenius acting on the $l$-adic cohomology $H^j_{l-ad}(V)$ for a prime $l$ different from the characteristic of $\mathbb{F}$.

2 The classical Lefschetz formula for suspensions

Let $Y = (M \times \mathbb{R})/\mathbb{Z}$, where $\mathbb{Z}$ acts on $M \times \mathbb{R}$ by $k.(m, s) = (f^k(m), s - k)$. Then $Y$ is called the suspension of $M$. There is a natural flow $\phi_t$ on $Y$ given by

$$\phi_t[m, s] = [m, s + t].$$

This is the suspended flow of $f$. Since any closed orbit of $\phi_t$ must sit over a periodic point of $f$, the closed orbits of $\phi$ have integer lengths and for $n \in \mathbb{N}$ the closed orbits of length $n$ correspond to $f$-orbits of fixed points of $f^n$. For a closed orbit $c$ let $l(c)$ denote the length of the orbit. Note here that we use the differential geometric notion of closed orbits rather than the group theoretical one. To make this precise define a periodic point to be an element $(y, t) \in Y \times [90, \infty)$ such that $\phi_t(y) = y$. The period $t$ is not necessarily the prime period of $y$, ie, the smallest $t > 0$ with $\phi_t(y) = y$. A closed orbit is an equivalence class of periodic points where two periodic points $(y, t)$ and $(z, s)$ are called equivalent if $s = t$ and $\phi_r(y) = z$ for some $r \in \mathbb{R}$. A closed orbit $(y, t)$ is called primitive if $t$ is the prime period of $y$.

The projection to the second variable

$$\pi : Y \to \mathbb{R}/\mathbb{Z}$$

is a fibre bundle with fibre $M$. Let $T_M \subset TY$ be the sub-bundle of vectors tangent to fibres. These vectors are also called the vertical vectors. For $p \geq 0$ let $H^p(T_M)$ be the cohomology of the complex $\Gamma^\infty(\wedge^* T_M^*)$ with the natural
exterior differential. Then $H^p(T_M)$ is the space of sections of a vector bundle $E^p$ over $\mathbb{R}/\mathbb{Z}$ whose fibre is $H^p(M)$. This vector bundle can be described as

$$E^p = (H^p(M) \times \mathbb{R})/\mathbb{Z},$$

where $\mathbb{Z}$ acts by

$$k.(v, x) = (f^{*k}v, x - k).$$

A section $s$ of $E^p$ can be viewed as a map $s : \mathbb{R} \to H^p(M)$ satisfying $s(x + k) = f^{*k}s(x)$. The flow acts on $s$ by $\phi_t^*s(x) = s(x + t)$.

Let $\varphi$ be a smooth function of compact support in $(0, \infty)$. Consider the operator $L^p_\varphi = \int_0^\infty \varphi(t) \phi_t^* dt$ on the space of $L^2$-sections of $E^p$.

**Theorem 2.1 (Lefschetz Formula for suspensions)**

$L^p_\varphi$ is of trace class and

$$\sum_{p=0}^{\dim M} (-1)^p \text{tr } L^p_\varphi = \sum_{c \text{ closed}} l(c_0) \text{ind}(c) \varphi(l(c)),$$

where the sum on the right hand side runs over all closed orbits of $\phi_t$ and $c_0$ is the primitive closed orbit underlying $c$.

Further, $\text{ind}(c) = \text{sign(det}(1 - \phi_{l(c)}^*| T_M))$.

The theorem can be reformulated as an identity of distributions on $(0, \infty)$ as follows.

**Corollary 2.2** As an identity of distributions on $(0, \infty)$ we have

$$\sum_{p=0}^{\dim M} (-1)^p \text{tr } (\phi_t^*| H^p(T_M)) = \sum_{c \text{ closed}} l(c_0) \text{ind}(c) \delta(t - l(c)),$$

where $\delta$ is the delta distribution.
Proof: (of the Theorem) Let \( s \) be a section of \( E^p \). Then

\[
L^p_\varphi s(x) = \int_{\mathbb{R}} \varphi(t) s(x + t) \, dt \\
= \int_{\mathbb{R}} \varphi(t - x) s(t) \, dt \\
= \sum_{k \in \mathbb{Z}} \int_0^1 \varphi(t - x + k) s(t + k) \, dt \\
= \sum_{k \in \mathbb{Z}} \int_0^1 \varphi(t - x + k) f^{*k}s(t) \, dt \\
= \int_{\mathbb{R}/\mathbb{Z}} K(x, t) s(t) \, dt,
\]

where \( K(x, t) \) is the kernel

\[
K(x, t) = \sum_{k \in \mathbb{Z}} \varphi(t - x + k) f^{*k}.
\]

This sum is locally finite and so \( K(x, t) \) is a smooth kernel. It follows that the operator \( L^p_\varphi \) is of trace class and that

\[
\text{tr } L^p_\varphi = \int_0^1 \text{tr } K(x, x) \, dx \\
= \sum_{k \in \mathbb{Z}} \varphi(k) \text{tr } f^{*k}.
\]

So that

\[
\sum_{p=0}^{\dim M} (-1)^p \text{tr } L^p_\varphi = \sum_{k \in \mathbb{Z}} \varphi(k) \sum_{p=0}^{\dim M} (-1)^p \text{tr } (f^{*k} | H^p(M)) \\
= \sum_{k \in \mathbb{N}} \varphi(k) \sum_{x = f^k(x)} \text{ind}_f(x).
\]

Since any closed orbit \( c \) of \( \phi_t \) gives \( l(c_0) \) points \( x \) with \( x = f^k(x) \) with \( k = l(c) \), the claim follows. \( \square \)
3 Foliation cohomology

The receptacle for the global side of the Lefschetz formula will be a foliation cohomology, a term to be defined in this section. See also [15]. A smooth foliation on a manifold $M$ is a smooth atlas consisting of coordinates $(x, y)$ with values in $\mathbb{R}^k \times \mathbb{R}^l$. A set of the form $\{y \equiv \text{const}\}$ is called a patch of the coordinate chart. The defining property of a foliation is that coordinate changes within the atlas map patches to patches. Thus a patch continues into a neighbouring coordinate set and thus extends to a $k$-dimensional immersed sub-manifold, called a leaf of the foliation. As an example consider $M = \mathbb{R}^2/\mathbb{Z}^2$. Fix $\alpha, \beta \in \mathbb{R} \times \mathbb{R}$ and consider the foliation on $M$ with leaves

$$L_{x, y} = (x, y) + \mathbb{R}(\alpha, \beta) \mod \mathbb{Z}^2,$$

then $L_{x, y}$ is the leaf through the point $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$.

If $\alpha/\beta$ is in $\mathbb{Q}$, then every leaf is compact. If $\alpha/\beta$ is not in $\mathbb{Q}$, then every leaf is non-compact and dense in $M$.

Let $\mathcal{F}$ be a foliation on the manifold $M$ and let $T_\mathcal{F} \subset TM$ be the sub-bundle of all vectors tangent to leaves. Since a sub-manifold is uniquely determined by its tangent bundle, a foliation $\mathcal{F}$ is uniquely determined by its tangent bundle $T_\mathcal{F}$. Not every sub-bundle $T$ of $TM$ is tangent to a foliation. A sub-bundle that is tangent to a foliation is called integrable. There are other characterizations of integrability. For instance, a bundle $T \subset TM$ is integrable if and only if for any two vector fields $X, Y$ with $X, Y \in \Gamma^\infty(T)$ it follows that $[X, Y] \in \Gamma^\infty(T)$. Let

$$\Omega^p_\mathcal{F} \overset{\text{def}}{=} \Gamma^\infty(\wedge^p T^*_\mathcal{F}).$$

Using foliation coordinates one shows that the exterior differential of the de Rham complex of $M$ induces a differential $d : \Omega^p_\mathcal{F} \to \Omega^{p+1}_\mathcal{F}$, with $d^2 = 0$. The foliation cohomology is

$$H^\bullet(\mathcal{F}) \overset{\text{def}}{=} \ker(d)/\text{im}(d).$$

For any $p$ the space $\Omega^p_\mathcal{F}$ is a Fréchet space, but the differential $d$ does not in general have closed image, which implies that the quotient topology may be
non-Hausdorff. Thus it seems natural to define the reduced foliation cohomology as the corresponding Hausdorff quotient, i.e.,

\[ \overline{H}^\bullet(F) \overset{\text{def}}{=} \ker(d)/\operatorname{im}(d). \]

Let \( E \) be a vector bundle which has a flat connection along the leaves. Then we have a differential on the \( E \)-valued differential forms and we can form the corresponding reduced cohomology which we write as \( \overline{H}^\bullet(F \otimes E) \).

## 4 Anosov flows

A smooth flow \( \phi_t \) on a compact manifold \( M \) is called an Anosov flow \[^1\text{9}\], if the tangent bundle \( TM \) of \( M \) splits as

\[ TM = T_0 \oplus T_s \oplus T_u, \]

where \( T_0 \), the neutral bundle, is of rank one and generated by the flow \( \phi_t \). Note that this implies that the flow has no fixed points. Next, \( T_s \), the stable bundle consists of all \( v \in TM \) such that \( \|\phi_{t^*}v\| \to \infty \) as \( t \to +\infty \) for any Riemannian metric on \( M \). Since \( M \) is compact this property does not depend on the choice of the metric. Finally, \( T_u \), the unstable bundle comprises all vectors \( v \in TM \) such that \( \|\phi_{t^*}v\| \to \infty \) as \( t \to -\infty \).

It turns out that the bundles \( T_u, T_s \) are integrable so there are corresponding foliations, the unstable and the stable foliation. For instance, two points \( m, n \in M \) lie in the same leaf of the stable foliation if and only if \( d(\phi_t m, \phi_t n) \) tends to zero as \( t \to +\infty \), where \( d \) is the distance function of any Riemannian metric on \( M \).

Suppose that \( \dim M > 1 \). Since the manifold is compact, both \( T_u \) and \( T_s \) have to be nonzero, so the smallest dimension in which a nontrivial Anosov flow can exist is three. An example is given as follows. Let \( \Gamma \subset G = \text{SL}_2(\mathbb{R})/\pm 1 \) be a discrete cocompact subgroup and set \( M = \Gamma \backslash G \). Let \( H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

Then for every \( t \in \mathbb{R} \) the matrix \( \exp(tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \) can be considered an element of \( G \). On \( M \) we get the flow

\[ \phi_t(\Gamma g) = \Gamma g \exp(tH). \]
Then $\phi$ is Anosov. The stable leaf through $\Gamma g$ is given by $l_g = \Gamma g N$, where

$$N = \left\{ \pm \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \middle| x \in \mathbb{R} \right\}.$$

Another way to get Anosov flows is to suspend a Anosov diffeomorphism. A diffeomorphism $f : M \to M$ is called Anosov if the tangent bundle $TM$ decomposes into a sum $T_s \oplus T_u$ of a stable and an unstable bundle which are defined as in the flow case. The suspension of an Anosov diffeomorphism is an Anosov flow.

Examples of Anosov diffeomorphisms are constructed as follows. Let $G$ be a semi-simple Lie group and $\Gamma \subset G$ a discrete cocompact subgroup. The quotient $M = \Gamma \backslash G$ is called a nilmanifold. Let $f : G \to G$ be an automorphism with $f(\Gamma) = \Gamma$. Then $f$ induces a diffeomorphism on $M$ denoted by the same letter. Suppose that the differential $f_* : T_e G \to T_e G$ at the unit element $e$ of $G$ is hyperbolic in the sense that for every eigenvalue $\lambda \in \mathbb{C}$ of $f_*$ we have $|\lambda| \neq 1$. Then the induced diffeomorphism is Anosov. Such a diffeomorphism is called an algebraic Anosov diffeomorphism. There is a conjecture [14] that states that up to finite covering every Anosov diffeomorphism should be topologically conjugate to an algebraic one. This means that up to finite covering for every Anosov diffeomorphism $f$ on a smooth manifold $M$ there should be an algebraic Anosov diffeomorphism $g$ on some nil-manifold $\Gamma \backslash G$ and a homeomorphism $\varphi : M \to \Gamma \backslash G$ such that $f = \varphi^{-1} g \varphi$.

The following conjecture was, in a slightly different setting and formulation, first given by V. Guillemin [10] and later by S. Patterson [16]. For a flow, a closed orbit is considered to come with a multiplicity, so if you have a closed orbit $c$, then you can go through it twice and get a different closed orbit $c^2$. So for every closed orbit $c$ there is an underlying primitive closed orbit $c_0$ such that $c$ is a power of $c_0$ but $c_0$ is not the power of a shorter orbit.

**Conjecture 4.1** Let $\phi_t$ be a Anosov flow with stable foliation $F_s$. Then, as an identity of distributions on $(0, \infty)$ we have

$$\sum_{p=0}^{\text{rank} F_s} (-1)^p \text{tr} (\phi_t^* | \bar{H}^p(F_s)) = \sum_{c \text{ closed}} l(c_0) \frac{\delta(t - l(c))}{\det(1 - \phi_t^* | T_{s,x})},$$

where on the right hand side $x$ is any point on the orbit $c$. 
In the case of a flow which is suspended from an algebraic diffeomorphism, C. Deninger and the author have proved this conjecture [8].

**Theorem 4.2 (C. Deninger-AD)**

The Guillemin-Patterson conjecture is true for flows suspended from algebraic Anosov diffeomorphisms. More specifically, if \( f : M \to M \) is an algebraic Anosov diffeomorphism with stable bundle \( \mathcal{F}_s \), then the reduced cohomology \( \tilde{H}^\bullet(\mathcal{F}_s) \) is finite dimensional and

\[
\sum_{p=0}^{\text{rank} \mathcal{F}_s} (-1)^p \text{tr} (f^p \mid \tilde{H}^p(\mathcal{F}_s)) = \sum_{x=f(x)} \frac{\det(1-f_s \mid T_{s,x})}{\det(1-f_s \mid T_x M)}.
\]

For the proof one shows that the foliation cohomology is canonically isomorphic to Lie algebra cohomology with trivial coefficients. This is shown inductively using the Hochschild-Serre spectral sequence iteratedly.

## 5 The Selberg zeta function

The Selberg zeta function for a compact Riemannian surface \( Y \) of genus \( g \geq 2 \) is defined by

\[
Z_Y(s) \overset{\text{def}}{=} \prod_{c_0} \prod_{N \geq 0} (1 - e^{-(s+N)l(c_0)})
\]

where the first product runs over all primitive closed geodesics in \( Y \) which is equipped with the hyperbolic metric. Selberg showed in [18] that \( Z_Y \) extends to an entire function which satisfies a generalized Riemann hypothesis insofar as all zeros are in \( \mathbb{R} \cup (\frac{1}{2}i\mathbb{R}) \).

In [2] P. Cartier and A. Voros gave the following determinant expression.

**Theorem 5.1 (Cartier-Voros)**

We have

\[
Z_Y(\frac{1}{2} + s) = \left( e^{s^2} \det \left( (\Delta_d + \frac{1}{4})^\frac{1}{2} + s \right) \right)^{2g-2} \det \left( (\Delta - \frac{1}{4}) + s^2 \right).
\]

Here \( \Delta \) is the Laplace operator on functions of \( Y \) and \( \Delta_d \) is the Laplace operator on the sphere \( S^2 \).
This theorem can be proved by means of the trace formula inserting test functions of the form $f(\Delta)$ where $\Delta$ is the Laplace operator and $f$ a sufficiently nice function on the spectrum of $\Delta$. So either powers of the resolvent kernel, or heat or wave kernels will do. These methods can be generalized to locally symmetric spaces of rank one [1, 3], but they will not neatly generalize to higher rank because any functional calculus of the given sort can not distinguish contributions of split tori of the same dimension which are not conjugate. Since the zeta functions which are attached to such tori show different analytical behaviour, a separation indeed is necessary.

Let now $Y$ denote a compact Riemannian manifold of odd dimension. Let $SY$ denote the sphere bundle of $Y$, i.e.,

$$SY \overset{\text{def}}{=} \{ v \in TY \mid \|v\| = 1 \}.$$ 

On $SY$ there is a natural flow, the geodesic flow of $Y$. It is defined as follows. Let $t > 0$. A point $p$ of $SY$ comprises a point in $Y$ plus a direction. If you walk along the unique geodesic in that direction for the time $t$, you get a new point and a new direction, the one you came along in. Thus you get a new point $\phi_t p$ in $SY$. It is clear that closed orbits of the geodesic flow correspond to closed geodesics in $Y$. It turns out that $\phi_t$ is Anosov. Let $F_s$ denote its stable foliation. Define the Selberg zeta function in this setting as

$$Z_Y(s) \overset{\text{def}}{=} \prod_{c_0} \prod_{N \geq 0} \det(1 - e^{-s(l(c_0))}\phi_{l(c_0)}\mid S^N(T_{s,x})), $$

where $x$ is any point on the primitive closed orbit $c_0$ of the geodesic flow $\phi_t$, and $S^N$ denotes the $N$th symmetric power.

**Theorem 5.2** (Lefschetz determinant formula, AD)

$$Z_Y(s) = \prod_{q \geq 0} \det(F + s \mid \tilde{H}^q(F_s))(-1)^{q+1}. $$

This theorem can be proved as follows. First one uses the Lefschetz trace formula for rank one spaces [13] to see that the divisors on both sides agree. The actual existence of the determinants follows from the work of G. Illies [12]. This proves the identity up to a factor of the form $e^{P(s)}$, where $P$ is a polynomial. Finally one uses the asymptotic of the regularized determinants [4] to conclude the proof of the theorem.
6 The Lefschetz formula for higher rank

Let $G$ be a connected semisimple Lie group with finite center. Let $X = G/K$ be the attached globally symmetric space, where $K \subset G$ is a maximal compact subgroup. Let $Y = \Gamma \backslash X = \Gamma \backslash G/K$, where $\Gamma$ is a discrete, cocompact, torsion-free subgroup of $G$. Then $Y$ is a locally symmetric space. Let $SX$ and $SY$ denote the sphere bundles, then $SY = \Gamma \backslash SX$. We want to understand the $G$-orbit structure of $SX$. For this recall that $G$ acts transitively on $X = G/K$. So we get

$$G \backslash SX = K \backslash S e K X = W \backslash S(A),$$

where $A$ is a maximal split torus in $G$, so $A \cong \mathbb{R}^r$ as a Lie group, and $S(A)$ is the sphere of norm one elements in $A$. Finally, $W$ denotes the Weyl group $W = N(A)/Z(A)$, the quotient of the normalizer of $A$ and the centralizer of $A$. Then $W$ is a finite reflection group acting on $A$.

The set $G \backslash SX$ can be identified with the set of norm-1 elements of a closed positive Weyl chamber, and so, $G \backslash SX$ has the orbifold structure of a polytope and the latter can be viewed as a subset of $A$ in a natural way. Let $f$ be a facet of this polytope and let $A_f \subset A$ be the subgroup generated by $f$. We say that $f$ is cuspidal if $A_f$ is the split part of a Cartan subgroup of $G$. As an example consider $G = SL_3(\mathbb{R})$. Then $G \backslash SX$ has dimension one, so is a closed interval and has three facets, the open one and the two endpoints. In this case each facet is cuspidal. For $G = SL_4(\mathbb{R})$ the polytope is the two dimensional simplex, so it has 7 facets. With the exception of one of the vertices each facet is cuspidal. Generally, the open facet always is cuspidal and the bigger the dimension of a facet, the more likely it will be cuspidal.

Fix a cuspidal facet $f$. Let $C_f$ denote the set of all closed geodesics in $Y$ that lift into $f$. Every $c \in C_f$ gives a point $a_c$ in the positive Weyl chamber $A_f^+$ by taking the corresponding point in $S(A_f^+)$ and multiplying it with the length of $c$.

The pullback $Gf$ of the facet $f$ is, as a $G$-set, isomorphic to $(G/K_f) \times e$, where $e$ is a cell on which $G$ acts trivially and $K_f = Z(A) \cap K$. So $A_f$ acts on $G/K_f$ and on $\Gamma \backslash G/K_f$ by right multiplication. This action is Anosov in the sense that the tangent bundle of $G/K_f$ or $\Gamma \backslash G/K_f$ decomposes as $T_0 \oplus T_n \oplus T_s \oplus T_u$, where $T_n$ is an additional neutral bundle on which $A_f$ preserves norms, $T_s$ is the stable bundle, which comprises all vectors that tend to zero under the
positive Weyl cone. The unstable bundle $T_u$ finally consists of all vectors which tend to zero under the negative Weyl cone $A^-_f = \{ a^{-1} \mid a \in A^+_f \}$.

**Theorem 6.1 (Lefschetz trace formula, AD)**

Let $\mathcal{F}_s$ denote the stable foliation. As a distribution on $A^+_f$ we have that

$$
\sum_{p,q \geq 0} (-1)^{p+q+\text{rank}\mathcal{F}_s} \text{tr}\left( a \mid \bar{H}^q(\mathcal{F}_s \otimes \wedge^p T_n) \right)
$$

equals

$$
\sum_{c \text{ closed/homotopy}} \frac{\lambda_c \chi(A_f \setminus X_c) \delta(a - a_c)}{\det(1 - \phi(t) \mid T_{s,x})},
$$

where $\lambda_c$ is the volume of the unique compact $A_f$-orbit that contains $c$. Further $X_c$ is the union of all closed geodesics homotopic to $c$ and $\chi(A_f \setminus X_c)$ is the Euler-characteristic of the quotient of $X_c$ modulo the $A_f$-action.

The proof [6] requires a geometric construction of a test function to be plugged into the trace formula. This test function is built in a way as to have non-trivial orbital integrals only on conjugates of a prescribed Cartan subgroup $H$. It is defined by $f(ghg^{-1}) = \eta(g)\varphi(h)$, where $\eta$ is a suitable function on the homogeneous space $G/H$. The spectral interpretation in terms of foliation cohomology is a consequence of the Osborne conjecture [11].

7 The higher rank Selberg zeta function

Suppose there is a cuspidal facet $f$ of dimension zero, i.e., $f$ is a vertex of the polysimplex $G \setminus SX$. Then $A_f$ is one dimensional. Let $F$ be the positive infinitesimal generator of norm 1. Then $F$ can also be viewed as the infinitesimal generator of the geodesic flow, i.e., $F = \frac{d}{dt}|_{t=0}\phi_t$. Define the generalized Selberg zeta function (AD) as

$$
Z_f(s) \overset{\text{def}}{=} \prod_{c \in C_f} \prod_{N \geq 0} \text{det}\left( 1 - e^{-s(c)\phi(t)_{(c),x}} \mid S^N(T_s) \right)^{\chi(A_f \setminus X_c)}. 
$$
Theorem 7.1 (AD) The double product defining $Z_f(s)$ converges if $\text{Re}(s) \gg 0$. The function $Z_f(s)$ extends to a meromorphic function on the plane. It has finitely many poles which are located at real numbers and under a suitable normalization of the metric all poles and zeros lie in $\mathbb{R} \cup (\frac{1}{2} + i\mathbb{R})$. The vanishing order at $s = \lambda$ equals

$$(-1)^{\text{rank} F_s} \sum_{p,q \geq 0} (-1)^{p+q} \dim H^q(F_s \otimes \wedge^p T_n)_{\lambda-\text{eigenspace of } F}$$

The proof uses the Lefschetz formula, resp. an extension to tempered distributions of the latter. One plugs in a test function that has the property that the geometric side of the Lefschetz formula equals a high logarithmic derivative of the Selberg zeta function.

8 The higher rank prime geodesic theorem

We will first state the classical prime geodesic theorem. Let $Y$ denote a compact Riemannian surface of genus $\geq 2$ equipped with the hyperbolic metric.

Theorem 8.1 (Prime Geodesic theorem)
For $x > 0$ let

$$\pi(x) \overset{\text{def}}{=} \# \{ c_0 \text{ prime } | e^{l(c_0)} \leq x \}.$$ 

Then under a suitable scaling of the metric,

$$\pi(x) \sim \frac{x}{\log x},$$

as $x \to \infty$.

To motivate the higher rank case we will rewrite this theorem in the Chebyshev form. Define the Chebyshev function by

$$\psi(x) \overset{\text{def}}{=} \sum_{c \leq x \text{ prime} | e^{l(c_0)} \leq x} l(c_0).$$
Here the sum runs over all closed geodesics with the given length restriction and $c_0$ is the prime geodesic underlying $c$. The prime geodesic theorem is equivalent to saying that, as $x \to \infty$,

$$\psi(x) \sim x.$$ 

Now let $X = G/K$ be a globally symmetric space as before and let $Y = \Gamma \backslash X$ be a compact quotient. Let $f$ be the open facet of $G \backslash SX$. The closed geodesics $c \in C_f$ are also called the regular geodesics. Every $c \in C_f$ gives a point $a_c \in A^+$. On $A^+$ there are canonical coordinates stemming from primitive roots. To $c \in C_f$ we can thus attach coordinates $c_1, \ldots, c_r > 0$.

**Theorem 8.2 (Higher Rank Prime Geodesic Theorem, AD)**

For $x_1, \ldots, x_r > 0$ let

$$\psi(x_1, \ldots, x_r) \overset{\text{def}}{=} \sum_{c: \varepsilon^j \leq x_j \text{ for } j = 1, \ldots, r} \lambda_c,$$

where $\lambda_c$ is the volume of the unique maximal flat containing $C$. Then, as $x_1, \ldots, x_r \to \infty$,

$$\psi(x_1, \ldots, x_r) \sim x_1 \cdots x_r.$$ 

To prove this theorem [7] one uses a generalization of the logarithmic derivative of the Selberg zeta function in several variables. as well as methods from analytical number theory (Tauberian Theorems) extended to several variables.

We will close this section with a number theoretical application of the higher rank prime geodesic theorem. This requires the prime geodesic theorem for locally symmetric manifolds $Y = \Gamma \backslash X = \Gamma \backslash G/K$ which are not compact but of finite volume. In that case one needs to employ the Arthur-Selberg trace formula which is quite harder to handle than the classic Selberg trace formula. So it is not surprising that results in this setting are a lot more fragmentary at the moment.

In [17] P. Sarnak proved the prime geodesic theorem for the arithmetic group $\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})/\text{SO}(2)$ and inferred the following theorem.
Theorem 8.3 (Sarnak 83)

Let
\[ \pi_{2,0}(x) \overset{\text{def}}{=} \sum_{\mathcal{O} : e^{R(\mathcal{O})} \leq x} h(\mathcal{O}), \]
where the sum ranges over all orders \( \mathcal{O} \) in real quadratic number fields, \( R(\mathcal{O}) \) denotes the regulator of the order \( \mathcal{O} \), and \( h(\mathcal{O}) \) the class number of \( \mathcal{O} \). Then, as \( x \to \infty \),
\[ \pi_{2,0}(x) \sim \frac{x}{\log x}. \]

Together with W. Hoffmann the author was recently able to prove a similar result for \( SL_3(\mathbb{Z}) \).

Theorem 8.4 (W Hoffmann-AD, 02) Let
\[ \pi_{1,1}(x) \overset{\text{def}}{=} \sum_{\mathcal{O} : e^{R(\mathcal{O})} \leq x} h(\mathcal{O}), \]
where the sum runs over all orders \( \mathcal{O} \) in number fields \( F \) which have one real and two complex embeddings. Then, as \( x \to \infty \),
\[ \pi_{1,1}(x) \sim \frac{x}{\log x}. \]

For the proof one employs the Arthur-Selberg trace formula. First one plugs in functions that vanish on parabolically degenerate elements to obtain a simplified trace formula whose geometric side comprises orbital integrals only. Next one constructs test functions that are products of virtual characters and twisted heat kernels to single out the relevant classes. The Mellin transform of the resulting contribution is an analytic function bearing similarity to the logarithmic derivative of the Selberg zeta function. The continuous spectral contributions cannot be computed but for the purpose of the theorem it suffices to give a good estimate.
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