Stochastic transport equation with bounded and Dini continuous drift

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Abstract

The results established by Flandoli, Gubinelli and Priola (Invent. Math. 180 (2010) 1–53) for stochastic transport equation with bounded and Hölder continuous drift are generalized to bounded and Dini continuous drift. The uniqueness of $L^\infty$-solutions is established by the Itô–Tanaka trick partially solving the uniqueness problem, which is still open, for stochastic transport equation with only bounded measurable drift. Moreover the existence and uniqueness of stochastic diffeomorphisms flows for a stochastic differential equation with bounded and Dini continuous drift is obtained.

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1 Introduction

We are concerned with the following stochastic transport equation

\begin{equation}
\left\{ \begin{array}{l}
\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) + \sum_{i=1}^d \partial_{x_i} u(t,x) \circ \dot{B}_i(t) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \\
u(t,x)|_{t=0} = u_0(x), \quad x \in \mathbb{R}^d,
\end{array} \right.
\end{equation}

(1.1)
where \( \{B(t)\}_t = \{(B_1(t), B_2(t), \ldots, B_d(t))\}_t \) is a \( d \)-dimensional standard Brownian motion defined on a stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})\). The stochastic integration with notation \( \circ \) is interpreted in Stratonovich sense. Given \( T > 0 \), the drift coefficient \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) and the initial data \( u_0 : \mathbb{R}^d \to \mathbb{R} \) are measurable functions in \( L^1([0, T]; L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)) \) and \( L^1([0, T]; L^1_{\text{loc}}(\mathbb{R}^d)) \) respectively. We are interested in the existence and uniqueness of weak \( L^\infty \)-solutions for the stochastic equation (1.1).

There are many results on the weak \( L^\infty \)-solutions for the deterministic transport equation. The first remarkable result of the uniqueness solution in \( L^\infty([0, T] \times \mathbb{R}^d) \) was obtained by DiPerna and Lions [12] under the assumption \( b \in L^1([0, T]; W^{1,1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)) \) with suitable global conditions including \( L^\infty \)-bounds on spatial divergence. Then Ambrosio [1] weakened the condition \( W^{1,1}_{\text{loc}} \) to \( BV_{\text{loc}} \) and the uniqueness of weak \( L^\infty \)-solutions is obtained under the assumption that negative part of \( \text{div} b \) is in space \( L^1([0, T]; L^\infty(\mathbb{R}^d)) \). For general \( b \) just with \( BV_{\text{loc}} \) or Hölder regularity, the uniqueness of weak \( L^\infty \)-solutions for the deterministic equation fails and counterexamples have been constructed in many works [5, 7, 9, 11, 12]. Obviously, more restrictions need to be imposed on \( b \) to overcome the obstacle of nonuniqueness of solution in deterministic case.

However the appearance of the noise makes the solution unique under very general assumptions on the drift coefficient for ordinary differential equations [24, 32], so a natural idea is to investigate the effects of noise in transport equation. The first milestone result, founded by Flandoli, Gubinelli and Priola [18], showed the uniqueness of weak \( L^\infty \)-solutions just with assuming \( b \in L^\infty([0, T]; C^1_0(\mathbb{R}^d, \mathbb{R}^d)) \) and \( \text{div} b \in L^p([0, T] \times \mathbb{R}^d) \) for some \( \alpha > 0 \), \( p > 2 \). This is the first concrete example of a partial differential equation related to fluid dynamics that becomes well-posed with a suitable noise. A key step for this result is to perform differential computations on regularization of \( L^\infty \)-solutions by a commutator lemma. Unfortunately the strategy fails if \( b \) is not Sobolev differentiable. However, by observing the fact that stochastic differential equation (SDE)

\[
\begin{align*}
&dX(t) = b(t, X(t))dt + dB(t), \quad 0 < t \leq T, \\
&X(t)|_{t=0} = x \in \mathbb{R}^d,
\end{align*}
\]

defines a \( C^1 \) stochastic diffeomorphisms flow and, along the stochastic characteristic \( X(t) \), the integral \( \int_0^t \text{div} b(s, X(s, x))ds \) has a regularization, Flandoli, Gubinelli and Priola developed the commutator lemma to prove the uniqueness of solutions. On the other hand, for bounded measurable \( b \), Mohammed, Nilssen and Proske [28] also proved the existence, uniqueness and Sobolev differentiable stochastic flows for (1.2) by employing ideas from the Malliavin calculus coupled with new probabilistic estimates on the spatial weak derivatives of solutions of (1.2). Then, as an application, they obtained the existence and uniqueness of Sobolev differentiable weak solutions for (1.1) with every \( C^1(\mathbb{R}^d) \) initial data. Notice that in one result the stochastic flow \( \{X(t, x)\} \) is differentiable in \( x \) in the classical sense [18] and in the other result the stochastic flow \( \{X(t, x)\} \) is only Sobolev differentiable [28]. So the method developed by Mohammed, Nilssen and Proske [28] can not be adapted to establish the uniqueness of weak \( L^\infty \)-solutions to (1.1).

Recent result [3], by using a different philosophy, proved the uniqueness of weak \( L^\infty \)-solutions for (1.1) just with assuming the \( BV \) regularity for \( b \) but without the \( L^\infty \)-bounds on spatial divergence. There are also several other related works [4, 14, 16, 29, 35, 36]. However, for bounded
measurable $b$ and $\text{div} b \in L^p([0, T] \times \mathbb{R}^d)$ (for some $p \in [1, \infty)$), the uniqueness of weak $L^\infty$-solutions for (1.1) is still unknown.

This paper intend to give a partial answer for the above problem and novelties of the work are

- The uniqueness of weak $L^\infty$-solutions for the Cauchy problem (1.1) with bounded and Dini-continuous drift is established due to the existence of noise, while the corresponding deterministic equation has multiple solutions.

- The existence and uniqueness of stochastic diffeomorphisms flow for singular SDE (3.1) is established without Hölder continuity or Sobolev differentiability hypotheses on $b$.

- The maximum regularity for parabolic equations of second order with Hölder-Dini or strong Hölder or weak Hölder coefficients is established.

We follow the strategy of Flandoli, Gubinelli and Priola’s [18] to establish the existence of a stochastic $C^1$ diffeomorphisms flow for (3.1) by the Itô–Tanaka trick, then derive a commutator estimates to get the uniqueness for weak $L^\infty$-solution of (1.1). The main idea of Itô–Tanaka trick is to use a parabolic partial differential equation (PDE) to transform the original SDE (3.1) with irregular drift and regular diffusion to a new SDE (3.15) with regular drift and diffusion. Then by the equivalence between (3.1) and (3.15) we show the existence of the stochastic $C^1$ diffeomorphisms flow for SDE (3.1). There are also some recent works on the stochastic flows and SDEs [2, 15, 17, 19, 31, 33, 37].

In the following parts, we first derive the $W^{2, \infty}$ estimates for a class of second order parabolic PDEs with bounded and Dini continuous coefficients in section 2; then by using the $W^{2, \infty}$ estimates, the existence and uniqueness of stochastic flow of diffeomorphisms for SDE (3.1) is shown in section 3 by the Itô–Tanaka trick; last section is concerned with the existence and uniqueness of weak $L^\infty$-solutions to stochastic transport equation (1.1).

**Notations**

The letter $C$ denotes a positive constant, whose values may change in different places. For a parameter or a function $\kappa$, $C(\kappa)$ means the constant is only dependent on $\kappa$, and we also write it as $C$ if there is no confusion. $\mathbb{N}$ is the set of natural numbers. For every $R > 0$, $B_R := \{ x \in \mathbb{R}^d : |x| < R \}$. Almost surely is abbreviated to $a.s.$. Let $\Theta$ be a $\mathbb{R}^{d \times d}$-valued function $\Theta = (\Theta_{i,j}(x))_{d \times d}$ with norm $||\Theta(x)|| = \max_{1 \leq i \leq d, 1 \leq j \leq d} |\Theta_{i,j}(x)|$. For $\xi \in \mathbb{R}^d$, $|\xi| = (\sum_{i=1}^d \xi_i^2)^{1/2}$. $\mathbb{R}_+$ is the set of nonnegative real numbers and $\mathbb{R}_+ = \mathbb{R}_+ \cup \{+\infty\}$.

## 2 Parabolic PDEs with bounded and Dini coefficients

Let $T > 0$. Consider the following Cauchy problem

$$\begin{cases} \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + g(t, x) \cdot \nabla u(t, x) + f(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\ u(0, x) = 0, \quad x \in \mathbb{R}^d. \end{cases}$$

(2.1)
The function \( u(t, x) \) is called a strong solution of (2.1) if \( u \in L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^d)) \cap W^{1,\infty}([0, T]; L^\infty(\mathbb{R}^d)) \) such that for almost all \((t, x) \in [0, T] \times \mathbb{R}^d\), (2.1) holds. We have the following equivalent form for the strong solution.

**Lemma 2.1** Let \( f \in L^\infty([0, T]; C_0(\mathbb{R}^d)) \), \( g \in L^\infty([0, T]; C_0(\mathbb{R}^d; \mathbb{R}^d)) \) and \( u \in L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^d)) \cap W^{1,\infty}([0, T]; L^\infty(\mathbb{R}^d)) \), then \( u \) is a strong solution for (2.1) if and only if

\[
\begin{align*}
    u(t, x) &= \int_0^t K(t-s, \cdot) * (g(s, \cdot) \cdot \nabla u(s, \cdot))(x)ds \\
    &\quad + \int_0^t K(t-s, \cdot) * f(s, \cdot)(x)ds, \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^d, 
\end{align*}
\]

(2.2)

where \( K(t, x) = (2\pi t)^{-d/2} e^{-|x|^2/4t}, \ t > 0, \ x \in \mathbb{R}^d. \)

**Proof.** By the properties of the heat kernel \( K \), it is direct to verify that if \( u \) satisfies (2.2), for almost all \((t, x) \in [0, T] \times \mathbb{R}^d\), (2.1) holds. On the other hand, if \( u \) satisfies (2.1) then for every \( \psi \in C_0^\infty(\mathbb{R}^d) \) and every \( t \in [0, T] \)

\[
\begin{align*}
    \int_{\mathbb{R}^d} \int_0^t \partial_s u(s, x) \varphi(s, x)dsdx &= \frac{1}{2} \int_{\mathbb{R}^d} \int_0^t \Delta u(s, x) \varphi(s, x)dsdx + \int_{\mathbb{R}^d} \int_0^t g(s, x) \cdot \nabla u(s, x) \varphi(s, x)dsdx \\
    &\quad + \int_{\mathbb{R}^d} \int_0^t f(s, x) \varphi(s, x)dsdx,
\end{align*}
\]

with \( \varphi(s, x) = K(t-s, \cdot) * \psi(\cdot)(x). \)

Now integrating by parts yields

\[
\begin{align*}
    \int_{\mathbb{R}^d} u(t, x) \psi(x)dx &= \int_{\mathbb{R}^d} \int_0^t u(s, x)[\partial_s \varphi(s, x) + \frac{1}{2} \Delta \varphi(s, x)]dsdx \\
    &\quad + \int_{\mathbb{R}^d} \int_0^t g(s, x) \cdot \nabla u(s, x) \varphi(s, x)dsdx + \int_{\mathbb{R}^d} \int_0^t f(s, x) \varphi(s, x)dsdx \\
    &= \int_{\mathbb{R}^d} \int_0^t K(t-s, \cdot) * (g(s, \cdot) \cdot \nabla u(s, \cdot))(x)ds\psi(x)dx \\
    &\quad + \int_{\mathbb{R}^d} \int_0^t K(t-s, \cdot) * f(s, \cdot)(x)ds\psi(x)dx, \quad \text{for all } t \in [0, T],
\end{align*}
\]

then by the arbitrariness of \( \psi \) and continuity of \( u \) in \( x \), (2.2) holds. \( \square \)
Definition 2.1 An increasing continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a Dini function if

$$\int_{0^+} \frac{\phi(r)}{r} dr < +\infty. \tag{2.3}$$

A measurable function $h : \mathbb{R}^d \to \mathbb{R}$ is said to be Dini continuous if there is a Dini function $\phi$ such that

$$|h(x) - h(y)| \leq \phi(|x - y|). \tag{2.4}$$

We now state the main result of this section.

Theorem 2.1 Let $f \in L^\infty([0,T];C_b(\mathbb{R}^d))$ and $g \in L^\infty([0,T];C_b(\mathbb{R}^d;\mathbb{R}^d))$. Suppose that $r_0 \in (0,1)$ and there is a Dini function $\phi$ such that for every $x \in \mathbb{R}^d$

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq \phi(|x - y|), \quad \text{for all } y \in B_{r_0}(x), \quad t \in [0,T]. \tag{2.5}$$

(i) The Cauchy problem (2.1) has a unique strong solution $u$, and there is a constant $C(d,T)$ such that

$$\|u\|_{L^\infty([0,T];C_b^2(\mathbb{R}^d))} \leq C(d,T)(1 + \|f\|_{L^\infty([0,T];C_b(\mathbb{R}^d))} + \|g\|_{L^\infty([0,T];C_b(\mathbb{R}^d;\mathbb{R}^d))}). \tag{2.6}$$

Moreover, for every $1 \leq i, j \leq d$ and every $x, y \in \mathbb{R}^d$, there is another constant $C(d,T)$ such that

$$|\partial^2_{x_i,x_j} u(t,x) - \partial^2_{y_i,y_j} u(t,y)| \leq C(d,T) \left[ \int_{r \in |x-y|} \frac{\phi(r)}{r} \, dr + \phi(|x-y|) + |x-y| \int_{|x-y| < r \leq r_0} \frac{\phi(r)}{r^2} \, dr \right] 1_{|x-y| < r_0} + C(d,T)|x-y| \quad \text{for all } t \in [0,T]. \tag{2.7}$$

(ii) Let $g_n, n \in \mathbb{N}$, be a regularizing kernel, that is

$$g_n(x) = n^d g(nx) \quad \text{with} \quad 0 \leq g \in C_0^\infty(\mathbb{R}^d), \quad \text{supp}(g) \subset B_1, \quad \int_{\mathbb{R}^d} g(x) dx = 1. \tag{2.8}$$

Let $u^n$ be the unique strong solution of (2.1) with $f$ and $g$ replaced by $f^n(t,x) = f * g_n(t,x)$ and $g^n(t,x) = g * g_n(t,x)$ respectively. Then $u^n \in L^\infty([0,T];C_b^2(\mathbb{R}^d)) \cap W^{1,\infty}([0,T];C_b(\mathbb{R}^d))$ and satisfies (2.6)-(2.7) uniformly in $n$. Furthermore,

$$\lim_{n \to \infty} \|u^n - u\|_{L^\infty([0,T];C_b^2(\mathbb{R}^d))} = 0. \tag{2.9}$$

Proof. (i) We first prove the result for the case $g = 0$. By Lemma 2.1 we just need to show

$$u(t,x) = \int_0^t K(t-s,\cdot) * f(s,\cdot)(x) \, ds \tag{2.10}$$
is in $L^\infty([0,T]; C^2_b(\mathbb{R}^d))$ and (2.7) holds. In fact $u \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^d))$ is classical [27, Ch.4] by the explicit representation (2.10).

Next we show $\partial^2_{x_i,x_j} u \in L^\infty([0,T]; C_b(\mathbb{R}^d))$ for every $1 \leq i, j \leq d$ and (2.7) holds. For this we first show that $\partial^2_{x_i,x_j} u \in L^\infty([0,T]; L^\infty(\mathbb{R}^d))$. Let $\theta \in (0,1/2)$. For $x \in \mathbb{R}^d$ and $t \in (0,T]$, we have

$$
\left| \partial^2_{x_i,x_j} u(t,x) \right| = \left| \int_0^t \int_{\mathbb{R}^d} \partial^2_{x_i,x_j} K(t-s,x-y) f(s,y) dy ds \right|
= \left| \int_0^t \int_{\mathbb{R}^d} \partial^2_{x_i,x_j} K(t-s,x-y) [f(s,y) - f(s,x)] dy ds \right|
= \left| \int_0^t \int_{|x-y|>(t-s)^\theta} \partial^2_{x_i,x_j} K(t-s,x-y) [f(s,y) - f(s,x)] dy ds \right|
+ \left| \int_0^t \int_{|x-y|\leq(t-s)^\theta} \partial^2_{x_i,x_j} K(t-s,x-y) [f(s,y) - f(s,x)] dy ds \right|
\leq C \|f\|_{L^\infty([0,T] \times \mathbb{R}^d)} \int_0^t \int_{|x-y|>(t-s)^\theta} \frac{1}{t-s} K(t-s,x-y) dy ds
+ \int_0^t \int_{|x-y|\leq(t-s)^\theta} \frac{1}{t-s} K(t-s,x-y) \phi(|x-y|) dy ds
+ C \int_0^t \int_{(t-s)^\theta \wedge r_0 \leq |x-y| \leq (t-s)^\theta} \frac{1}{t-s} K(t-s,x-y) |x-y| dy ds
\leq C \left[ \int_0^T \int_{|y|>s^\theta} \frac{1}{s} K(s,y) dy ds + \int_0^T \int_{|y|<s^\theta \wedge r_0} \frac{1}{s} K(s,y) \phi(|y|) dy ds \right]
+ C \int_{r_0^\theta \wedge r_0 \leq |y| \leq s^\theta} \int_{r_0^\theta}^T \frac{1}{s} K(s,y) |y| dy ds =: I_1 + I_2 + I_3, \tag{2.11}
$$

where the boundedness of $f$ is applied in the seventh line of (2.11), and in the last line $I_3 = 0$ for $T \leq r_0^\theta$. We first estimate $I_1$.

$$
I_1 = C \int_0^T \int_{s^\theta}^{R_2^2} s \frac{ds}{r} \int_{r>s^\theta} e^{-\frac{r^2}{2s}} r^{d-1} dr
= C \int_0^T \int_{s^\theta}^{\infty} s^{-1} ds \int_{s^\theta}^{\infty} e^{-\frac{r^2}{2s}} r^{d-1} dr.
$$
\[
= C \int_0^{\frac{1}{2}} s^{-1} ds \int_0^\infty e^{-\frac{s^2}{2} r^{d-1}} dr + C \int_0^{\frac{1}{2}} s^{-1} ds \int_0^\infty e^{-r_2 \frac{s^2}{2} r^{d-1}} dr
\]
\[
\leq C \int_0^{\frac{1}{2}} s^{-1} ds \int_0^\infty e^{-\frac{s^2}{2} r^{d-1}} dr + C \log(2T \lor 1) \int_0^{(T\lor^{\frac{1}{2}})^{\theta - \frac{1}{2}}} e^{-r_2 \frac{s^2}{2} r^{d-1}} dr
\]
\[
\leq C \left[ \log(2T \lor 1) + \int_0^{\frac{1}{2}} s^{-1} ds \int_0^\infty e^{-\frac{s^2}{2} r^{d-1}} dr \right]. \tag{2.12}
\]

Without loss of generality we assume that \(d\) is even. Otherwise, since \(\theta < 1/2\) and \(s \in (0, 1/2)\),
\[
\int_0^\infty e^{-\frac{s^2}{2} r^{d-1}} dr \leq \int_0^\infty e^{-\frac{s^2}{2} r} dr.
\]

Thus, there is a natural number \(m \geq 0\) such that \(d = 2m + 2\) and
\[
\int_0^\infty e^{-\frac{s^2}{2} r^{d-1}} dr = 2^m \int_{s^{2^m-1/2}}^\infty e^{-r_2 \frac{s^2}{2} r^{d-1}} dr =: 2^m J_m. \tag{2.13}
\]

Set \(s_0 = s^{2^m-1/2}\) and integrating by parts yields the following recurrence formula
\[
J_m = \int_{s^{2^m-1/2}}^\infty e^{-r_2 \frac{s^2}{2} r^{d-1}} dr = s_0^m e^{-s_0} + mJ_{m-1}.
\]

Then
\[
J_m \leq C(1 + s_0^m)e^{-s_0}. \tag{2.14}
\]

Now from (2.12)–(2.14)
\[
I_1 \leq C \left[ \log(2T \lor 1) + \int_0^{\frac{1}{2}} s^{-1}(1 + s^{(\theta - \frac{1}{2})(d-2)}) e^{-\frac{1}{2} s^{2\theta-1}} ds \right]
\]
\[
= C \left[ \log(2T \lor 1) + \int_0^{\infty} s^{-1}(1 + s^{(\frac{1}{2}-\theta)(d-2)}) e^{-\frac{1}{2} s^{1-2\theta}} ds \right] < +\infty. \tag{2.15}
\]

For \(I_2\), since \(\phi\) is a nonnegative increasing continuous function, we have
\[
I_2 \leq C \int_0^{T} \frac{\phi(s^\theta)}{s} ds \int K(s, y) dy = C \int_0^{T\theta} \frac{\phi(s)}{s} ds < +\infty. \tag{2.16}
\]
Last for $I_3$, we have

$$I_3 \leq C \int \int_0^T s^{-\frac{1}{2}} K(s, y) dy ds \leq C\sqrt{T} < +\infty.$$  \tag{2.17}$$

Now by (2.11) and (2.15)--(2.17), $u \in L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^d))$ and there is a constant $C > 0$, depending only on $d$ and $T$, such that

$$\|u\|_{L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^d))} \leq C(1 + \|f\|_{L^\infty([0, T]; C^2(\mathbb{R}^d))}).$$

So to prove $u \in L^\infty([0, T]; C^2_b(\mathbb{R}^d))$, we just need inequality (2.7). Notice that for $x, y \in \mathbb{R}^d$ and $|x - y| \geq r_0/2$, since $u \in L^\infty([0, T]; W^{2,\infty}(\mathbb{R}^d))$, there is constant $C(d, T)$ such that

$$|\partial^2_{x_i, x_j} u(t, x) - \partial^2_{y_i, y_j} u(t, y)| \leq \frac{C(d, T)r_0}{2} \leq C(d, T)|x - y|, \quad \text{for all } t \in [0, T].$$

We next show (2.7) for $0 < |x - y| < r_0/2$.

For every $1 \leq i, j \leq d$ and $0 < t \leq T$, every $x, y \in \mathbb{R}^d$ with $|x - y| \leq r_0/2$, from (2.10), we have

$$\partial^2_{x_i, x_j} u(t, x) - \partial^2_{y_i, y_j} u(t, y)$$

$$= \int_0^t ds \int_{|x - z| \leq 2|x - y|} \partial^2_{x_i, x_j} K(t - s, x - z)[f(s, z) - f(s, x)] dz$$

$$- \int_0^t ds \int_{|x - z| \leq 2|x - y|} \partial^2_{y_i, y_j} K(t - s, y - z)[f(s, z) - f(s, y)] dz$$

$$+ \int_0^t ds \int_{|x - z| > 2|x - y|} \partial^2_{y_i, y_j} K(t - s, y - z)[f(s, y) - f(s, x)] dz$$

$$+ \int_0^t ds \int_{|x - z| > 2|x - y|} [\partial^2_{x_i, x_j} K(t - s, x - z) - \partial^2_{y_i, y_j} K(t - s, y - z)][f(s, z) - f(s, x)] dz$$

$$= : J_1(t) + J_2(t) + J_3(t) + J_4(t).$$  \tag{2.18}$$

By the assumption (2.5)

$$|J_1(t)| \leq \int_0^t ds \int_{|x - z| \leq 2|x - y|} s^{\frac{d+2}{2}} e^{-\frac{|x - z|^2}{2s}} \phi(|x - z|) dz$$

$$= \int_{|z| \leq 2|x - y|} \phi(|z|) dz \int_0^t s^{\frac{d+2}{2}} e^{-\frac{|z|^2}{2s}} ds$$

$$\leq C \int_{|z| \leq 2|x - y|} \phi(|z|) \frac{dz}{|z|^d} = \int_0^{2|x - y|} \phi(r) \frac{dr}{r^d}$$

$$\leq C \int_{r \leq 2|x - y|} \phi(r) \frac{dr}{r^{d-1}}.  \tag{2.19}$$
Analogously, 

\[ |J_2(t)| \leq C \int_{r \leq 2|x-y|} \frac{\phi(r)}{r} dr. \]  \hspace{1cm} (2.20)

For \( J_3 \), by Gauss–Green’s formula

\[ |J_3(t)| = \left| \int_0^t ds \int_{|x-z|=2|x-y|} \partial_y K(t-s, y-z) n_i [f(s, y) - f(s, x)] dS \right| \]

\[ \leq C \int_0^t ds \int_{|x-z|=2|x-y|} |y - z| s^{-\frac{d+2}{2}} e^{-\frac{|y-z|^2}{2s}} \phi(|x - y|) dS \]

\[ \leq C \phi(|x - y|) \int_{|x-z|=2|x-y|} |y - z| dS \int_0^T s^{-\frac{d+2}{2}} e^{-\frac{|y-z|^2}{2s}} ds \]

\[ \leq C \phi(|x - y|) |x - y|^{d} \int_0^\infty s^{-\frac{d+2}{2}} e^{-\frac{|y-z|^2}{2s}} ds \]

\[ \leq C \phi(|x - y|) \int_0^\infty r^{\frac{d-2}{2}} e^{-r} dr \]

\[ \leq C \phi(|x - y|). \]  \hspace{1cm} (2.21)

For \( J_4(t) \), since \(|x - z| > 2|x - y|\), for every \( \xi \in [x, y] \) (the line with endpoints \( x \) and \( y \))

\[ \frac{1}{2} |x - z| \leq |\xi - z| \leq 2|x - z|. \]

Thanks to (2.5) and the mean value inequality, we acquire

\[ |J_4(t)| \leq C |x - y| \int_0^t ds \int_{|x-z|>2|x-y|} \left( \phi(|x - z|) 1_{|x-z|<r_0} + 1_{|x-z|\geq r_0} |f(s, x) - f(s, z)| \right) \times (t-s)^{-\frac{d+3}{2}} e^{-\frac{|x-z|^2}{8(t-s)^{d+1}}} dz. \]  \hspace{1cm} (2.22)

Observing that \( f \) is bounded, there is a constant \( C > 0 \) such that

\[ \sup_{s \in [0, T]} |f(s, x) - f(s, z)| \leq C, \text{ for all } |x - z| \geq r_0. \]  \hspace{1cm} (2.23)

Then by (2.22) and (2.23)

\[ |J_4(t)| \leq C |x - y| \int_{|x-z|>2|x-y|} \left( \phi(|x - z|) 1_{|x-z|<r_0} + 1_{|x-z|\geq r_0} \right) |x - z|^{-d-1} dz \int_0^\infty r^{\frac{d+1}{2}} e^{-r} dr \]

\[ \leq C |x - y| \int_{|x-z|>2|x-y|} \left( \phi(|x - z|) 1_{|x-z|<r_0} + 1_{|x-z|\geq r_0} \right) |x - z|^{-d-1} dz \]
\[
|\partial_{x_i,x_j} u(t,x) - \partial_{y_i,y_j} u(t,y)| 
\leq C|y-x| \int_{2|y-x|<r<r_0} \frac{\phi(r)}{r^2} dr + C|y-x| \int_{r>r_0} r^{-2} dr 
\leq C|y-x| \int_{2|y-x|<r<r_0} \frac{\phi(r)}{r^2} dr + C|y-x|. 
\]

(2.24)

Now combining (2.19), (2.20), (2.21) and (2.24), for all \(x, y \in \mathbb{R}^d\) and \(t \in [0,T]\)
\[
|\partial_{x_i,x_j} u(t,x) - \partial_{y_i,y_j} u(t,y)| 
\leq C(d,T) \left[ \int_{r\leq|x-y|} \frac{\phi(r)}{r^2} dr + \phi(|x-y|) + |x-y| \int_{2|x-y|<r\leq r_0} \frac{\phi(r)}{r^2} dr + |x-y| \right] 
\leq C(d,T) \left[ \int_{r\leq|x-y|} \frac{\phi(r)}{r^2} dr + \phi(|x-y|) + 2|x-y| \int_{2|x-y|<r\leq r_0} \frac{\phi(r)}{r^2} dr + |x-y| \right] 
\leq C \left[ \int_{r\leq|x-y|} \frac{\phi(r)}{r^2} dr + \phi(|x-y|) + |x-y| \int_{|x-y|<r\leq r_0} \frac{\phi(r)}{r^2} dr + |x-y| \right] 
\leq C \left[ \int_{r\leq|x-y|} \frac{\phi(r)}{r^2} dr + \phi(|x-y|) + |x-y| \int_{|x-y|<r\leq r_0} \frac{\phi(r)}{r^2} dr + |x-y| \right], 
\]

(2.25)

which implies (2.7).

Next we consider the case \(g \neq 0\). Notice that for \(g \in L^\infty([0,T];C_b(\mathbb{R}^d;\mathbb{R}^d))\) and satisfies (2.5), \(g \cdot \nabla u \in L^\infty([0,T];C_b(\mathbb{R}^d))\) and satisfies (2.5) with the righthand side replaced by \(C\phi\). Let \(\mathcal{H}\) be the set consisting of the functions in \(L^\infty([0,T];C_b(\mathbb{R}^d))\) with
\[
|\nabla v(t,x) - \nabla v(t,y)| \leq C\phi(|x-y|), \quad \text{for all } x \in \mathbb{R}^d, \ y \in B_{r_0}(x), \ t \in [0,T] 
\]
for \(v \in \mathcal{H}\) with some constant \(C > 0\). For \(v \in \mathcal{H}\) we define a mapping
\[
\mathcal{T} v(t,x) = \int_0^t K(t-s,\cdot) \ast (g(s,\cdot) \cdot \nabla v(s,\cdot))(x) ds + \int_0^t K(t-s,\cdot) \ast f(s,\cdot)(x) ds. 
\]
From (2.10), if \(f\) is in \(L^\infty([0,T];C_b(\mathbb{R}^d))\) and satisfies (2.5), for every \(x \in \mathbb{R}^d\) and \(y \in B_{r_0}(x)\)
\[
|\nabla u(t,x) - \nabla u(t,y)| = \left| \int_0^t \nabla K(t-s,\cdot) \ast f(s,\cdot)(x) ds - \int_0^t \nabla K(t-s,\cdot) \ast f(s,\cdot)(y) ds \right| 
\leq C \int_0^t \int_{\mathbb{R}^d} K(t-s,z) \left| f(s,x-z) - f(s,y-z) \right| dz ds 
\leq C\phi(|x-y|) \sqrt{t} \leq C\phi(|x-y|). 
\]

(2.26)
Then $T$ maps $H$ into $H$. Moreover for sufficient small $T = T_0$, a direct verification yields that the mapping $T$ is contractive. Then there is a unique $u \in H$ such that $u = T u$ and similar argument as the case $g = 0$ yields the existence and unique strong solutions of the Cauchy problem (2.1). Now by the classical extension technique we construct a strong solution on $[0, T]$ for any given $T > 0$ and get inequalities (2.6) and (2.7) on $[0, T]$.

(ii) Let $u^n$ be the unique strong solution of (2.1) with $f$ and $g$ replaced by $f^n$ and $g^n$ respectively. Since $f^n$ and $g^n$ are smooth in spatial variable, $u^n$ is smooth in spatial variable as well uniformly in time. Since $f$ and $g$ are bounded, $f^n$ and $g^n$ are bounded with

$$
\sup_n \|f^n\|_{L^\infty([0, T]; C_b(\mathbb{R}^d))} \leq \|f\|_{L^\infty([0, T]; C_b(\mathbb{R}^d))},
\sup_n \|g^n\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} \leq \|g\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))}.
$$

Due to (2.6)

$$
\sup_{n \geq 1} \|u^n\|_{L^\infty([0, T]; C_b^2(\mathbb{R}^d))} \leq C(d, T)(1 + \|f\|_{L^\infty([0, T]; C_b(\mathbb{R}^d))} + \|g\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))}).
$$

Moreover for every $x, y \in \mathbb{R}^d$ ($0 < |x - y| < r_0$) and every $t \in [0, T]$

$$
|f^n(t, x) - f^n(t, y)| = \left| \int_{\mathbb{R}^d} \varrho_n(z)[f(t, x - z) - f(t, y - z)]dz \right| 
\leq \phi(|x - y|) \int_{\mathbb{R}^d} \varrho_n(z)dz = \phi(|x - y|),
$$

and the above estimate is true for $g^n$ as well, so (2.7) holds uniformly in $n$.

By Lemma 2.1

$$
u_n(t, x) - u(t, x) = \int_0^t K(t - s, \cdot) * [g^n(s, \cdot) \cdot \nabla u^n(s, \cdot) - g(s, \cdot) \cdot \nabla u(s, \cdot)](x)ds
\quad + \int_0^t K(t - s, \cdot) * [f^n(s, \cdot) - f(s, \cdot)](x)ds.
$$

Then by a Gronwall type argument, we arrive at

$$
\|u^n - u\|_{L^\infty([0, T]; C_b^1(\mathbb{R}^d))} \leq C(\|f^n - f\|_{L^\infty([0, T]; C_b(\mathbb{R}^d))} + \|g^n - g\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))}),
$$

where $C > 0$ is independent of $n$.

On the other hand, for $f^n$ (and $g^n$) we have

$$
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |f^n(t, x) - f(t, x)| = \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \varrho_n(z)[f(t, x - z) - f(t, x)]dz \right| 
\leq C \int_{\mathbb{R}^d} \varrho_n(z)\phi(|z|)dz \to 0, \text{ as } n \to \infty.
$$
Combining (2.29) and (2.30)

\[ \lim_{n \to \infty} \| u^n - u \|_{L^\infty([0,T];C^1_0(\mathbb{R}^d))} = 0. \] (2.31)

It remains to show the convergence for \( \nabla^2 u^n \). For every \( 1 \leq i, j \leq d \), every \( t \in (0,T) \) and every \( x \in \mathbb{R}^d \)

\[
\left| \partial^2_{x_i x_j} u^n(t,x) - \partial^2_{x_i x_j} u(t,x) \right|
= \left| \int_0^t \int_{\mathbb{R}^d} \partial^2_{x_i x_j} K(t-s,x-y) \left( g^n(s,y) \cdot \nabla u^n(s,y) - g(s,y) \cdot \nabla u(s,y) \right) dyds \right|
+ \left| \int_0^t \int_{\mathbb{R}^d} \partial^2_{x_i x_j} K(t-s,x-y) \left( f^n(s,y) - f(s,y) \right) dyds \right|
\leq \left| \int_0^t \int_{\mathbb{R}^d} \partial^2_{x_i x_j} K(t-s,x-y) \left( g^n(s,y) \cdot \nabla u^n(s,y) - g(s,y) \cdot \nabla u(s,y) \right) dyds \right|
+ \left| \int_0^t \int_{\mathbb{R}^d} \partial^2_{x_i x_j} K(t-s,x-y) \left( f^n(s,y) - f(s,y) \right) dyds \right| =: H^n_1(t,x) + H^n_2(t,x). \quad (2.32)

First for \( H^n_2 \) we have

\[ H^n_2(t,x) = \left| \int_0^t \int_{|x-y|(t-s) > (t-s)^\theta} \partial^2_{x_i x_j} K(t-s,x-y) \left( f^n(s,y) - f^n(s,x) - f(s,y) + f(s,x) \right) dyds \right|
+ \left| \int_0^t \int_{|x-y|(t-s) < (t-s)^\theta} \partial^2_{x_i x_j} K(t-s,x-y) \left( f^n(s,y) - f^n(s,x) - f(s,y) + f(s,x) \right) dyds \right|
\leq C \| f^n - f \|_{L^\infty([0,T] \times \mathbb{R}^d)}
+ C \int_0^t \int_{|x-y|(t-s) > (t-s)^\theta \wedge r_0} \frac{1}{t-s} K(t-s,x-y) \left( f^n(s,y) - f^n(s,x) - f(s,y) + f(s,x) \right) dyds
+ C \int_0^t \int_{(t-s)^\theta \wedge r_0 \leq |x-y| \leq (t-s)^\theta} \frac{1}{t-s} K(t-s,x-y) \times \left( f^n(s,y) - f^n(s,x) - f(s,y) + f(s,x) \right) dyds, \quad (2.33)\]

for \( \theta \in (0,1/2) \). By (2.5) and (2.27),

\[ |f^n(s,y) - f^n(s,x) - f(s,y) + f(s,x)| \leq C \phi(|x-y|), \quad \text{for all } |x-y| < r_0. \]

From (2.16)

\[ \frac{1}{t-s} K(t-s,x-y) \phi(|x-y|) \in L^1(O), \quad O = \{(s,y) | s \in [0,t], |x-y| < (t-s)^\alpha \wedge r_0 \}. \]
By the assumption on \( f \) and the definition of \( f_n \), the integrand of the last term in (2.33) is integrable. Then by the dominated convergence theorem and (2.30), we have

\[
\lim_{n \to \infty} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} H^n_{2}(t, x) = 0.
\]

Noticing that \( g^n(s, y) \cdot \nabla u^n(s, y) \) and \( g(s, y) \cdot \nabla u(s, y) \) are of class \( L^\infty([0,T]; C_b(\mathbb{R}^d)) \), satisfy (2.5) and

\[
\lim_{n \to \infty} \| g^n \cdot \nabla u^n - g \cdot \nabla u \|_{L^\infty([0,T]; C_b(\mathbb{R}^d))} = 0.
\]

So the argument of convergence for \( H^n_1 \) is same as that for \( H^n_2 \), then we have

\[
\lim_{n \to \infty} \| \nabla^2 u^n - \nabla^2 u \|_{L^\infty([0,T]; \mathcal{C}_b(\mathbb{R}^d))} = 0,
\]

and combining (2.31) and (2.34), we show (2.9). \( \square \)

**Remark 2.1** There are some further properties of \( \nabla^2 u \) and \( \partial_t u \) by some estimates on the righthand side of (2.7). We consider a nonnegative Dini function \( \phi \). For every \( r_0 > 0 \), then

\[
\int_0^{r_0} \frac{\phi(r)}{r} \, dr < +\infty.
\]

On the other hand,

\[
\varepsilon \int_{\varepsilon < r \leq r_0} \frac{\phi(r)}{r^2} \, dr = \int_{0 < r \leq r_0} \frac{\phi(r)}{r^2} 1_{(\varepsilon, r_0)}(r) \varepsilon \, dr =: \int_{0 < r \leq r_0} h_\varepsilon(r) \, dr.
\]

Since \( h_\varepsilon(r) \leq \phi(r)/r \), by the dominated convergence theorem,

\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\varepsilon < r \leq r_0} \frac{\phi(r)}{r^2} \, dr = \int_{0 < r \leq r_0} \lim_{\varepsilon \to 0} h_\varepsilon(r) \, dr = 0.
\]

By (2.36), all the terms in the righthand side of (2.7) are meaningful. Moreover, by the above estimates, \( \nabla^2 u \) and \( \partial_t u \) are uniformly continuous in spatial variable uniformly in time.

**Remark 2.2** Theorem 2.1 generalizes the existing results for the Cauchy problem (2.1). By the classical parabolic theory ([27, Ch. 4], [23]), if \( f \in L^p([0,T] \times \mathbb{R}^d) \), \( g \in L^p([0,T] \times \mathbb{R}^d; \mathbb{R}^d) \) with \( p \in (1, +\infty) \), \( u \in L^p([0,T]; W^{2,p}(\mathbb{R}^d)) \) \( \cap W^{1,p}([0,T]; L^p(\mathbb{R}^d)) \). However, if \( f \in L^\infty([0,T] \times \mathbb{R}^d) \), \( g \in L^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d) \), in general \( u \notin L^\infty([0,T]; W^{2,\infty}(\mathbb{R}^d)) \) \( \cap W^{1,\infty}([0,T]; L^\infty(\mathbb{R}^d)) \). Nevertheless, recent result [35, Lemma 2.1] shows that if \( f \in L^p([0,T]; \mathcal{C}_b^2(\mathbb{R}^d)), g \in L^p([0,T]; \mathcal{C}_b^2(\mathbb{R}^d; \mathbb{R}^d)) \) with \( p \in (1, +\infty], \varsigma \in (0, 1) \), and \( \varsigma > 2/p \)

\[
u \in L^\infty([0,T]; \mathcal{C}_b^{2\varsigma-\frac{2}{p}}(\mathbb{R}^d)) \cap W^{1,\infty}([0,T]; \mathcal{C}_b^{\varsigma-\frac{2}{p}}(\mathbb{R}^d)).
\]

Here, by assuming the Dini continuity on \( f \) and \( g \), we establish the \( W^{2,\infty} \) estimates as well. Moreover, the second derivatives for spatial variable are also uniformly continuous.
Remark 2.3 (i) From (2.7), if there exists $\eta > 0$ such that for $r > 0$ small enough

$$\eta \int_0^r \frac{\phi(s)}{s} ds = \phi(r) \tag{2.37}$$

and

$$r \int_{r<s\leq r_0} \frac{\phi(s)}{s^2} ds \leq C\phi(r), \tag{2.38}$$

then the maximum Dini regularity for (2.1) holds. Now, we solve the integral equation (2.37) and get $\phi(r) = C_0 r^\eta$. Since the Dini function $\phi$ is given in (2.5), $\eta \in (0,1)$ is the best choice. Then,

$$r \int_{r<s\leq r_0} \frac{\phi(s)}{s^2} ds = \frac{C_0}{1-\eta} \left[ r^\eta - r r_0^\eta \right] \leq \frac{\phi(r)}{1-\eta},$$

for $r$ is sufficiently small. And now, we recover the classical Schauder theory for parabolic equations of second order.

(ii) If $\phi$ is only Dini continuous but not Hölder continuous, the functions given in the first and third terms on the right hand side of (2.7) will not preserve the same regularity as $\phi$, and thus the maximum regularity for (2.1) may be not true. For example, we choose $r_0 = 1/2$ and $\phi(r) = C|\log(r)|^{-\alpha}$ with some $\alpha > 1$, then there exist two positive real numbers $C_1(d,\alpha)$ and $C_2(d,\alpha)$ such that for $|x - y| > 0$ sufficiently small

$$\frac{C_1(d,\alpha)}{|\log(|x - y|)|^{\alpha-1}} \leq \int_{r>|x-y|} \frac{\phi(r)}{r} dr + |x - y| \int_{|x-y|<r\leq r_0} \frac{\phi(r)}{r^2} dr \leq \frac{C_2(d,\alpha)}{|\log(|x-y|)|^{\alpha-1}}.$$ 

However, $\phi(r) = o(|\log(r)|^{1-\alpha})$ as $r \to 0$. Therefore, if $\alpha \in (1,2)$, $\nabla^2 u(t,\cdot)$ may be only uniformly continuous but not Dini continuous.

To make the maximum regularity for (2.1) true, we need introduce the following notion.

Definition 2.2 Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone continuous function. Suppose $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $\phi(0) = 0$.

(i) We call $\phi$ a Hölder-Dini function with Hölder exponent $\vartheta \in (0,1)$ if $r^{-\vartheta}\phi(r) = \psi(r)$ is Dini but not Hölder continuous on $\mathbb{R}_+$;

(ii) We call $\phi$ a strong Hölder function with Hölder exponent $\vartheta \in (0,1)$ if
   (1) for every $\zeta \in (0,\vartheta)$, $r^{-\zeta}\phi(r)$ is Hölder continuous on $\mathbb{R}_+$;
   (2) $r^{-\vartheta}\phi(r) = \psi(r)$ is strictly monotone increasing and continuous but not Hölder continuous on $\mathbb{R}_+$;
   (3) $\psi(r) \downarrow 0$ as $r \downarrow 0$;

(iii) We call $\phi$ a weak Hölder function with Hölder exponent $\vartheta \in (0,1)$ if
   (1) for every $\zeta \in (0,\vartheta)$, $r^{-\zeta}\phi(r)$ is Hölder continuous on $\mathbb{R}_+$ and $r^{-\zeta}\phi(r) \to 0$ as $r \downarrow 0$;
   (2) $r^{-\vartheta}\phi(r) = \psi(r)$ is strictly monotone decreasing and continuous for $r \in (0,\infty)$;
A measurable function \( h: \mathbb{R}^d \to \mathbb{R} \) is said to be Hölder-Dini (strong Hölder or weak Hölder) continuous with Hölder exponent \( \vartheta \in (0, 1) \) if
\[
|h(x) - h(y)| \leq \phi(|x - y|),
\]
where \( \phi \) is a Hölder-Dini (strong Hölder or weak Hölder) function with Hölder exponent \( \vartheta \).

If \( f \) and \( g \) are Hölder-Dini or strong Hölder or weak Hölder continuous, then the maximum regularity for (2.1) is still true.

**Corollary 2.1** Let \( \phi \) be a Hölder-Dini or strong Hölder or weak Hölder function with Hölder exponent \( \vartheta \in (0, 1) \). Let \( f \in L^\infty([0, T]; C_b(\mathbb{R}^d)) \) and \( g \in L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d)) \) such that (2.5) holds. Then there exists a unique strong solution \( u \) to (2.1). Moreover, if \( \phi' \) is also continuous in \((0, \infty)\), then for every \( 1 \leq i, j \leq d \) and every \( x, y \in \mathbb{R}^d \), there is a constant \( C(d, T) \) such that
\[
|\partial^2_{x_i, x_j} u(t, x) - \partial^2_{y_i, y_j} u(t, y)| \leq C(d, T)\phi(|x - y|), \quad \text{for all } t \in [0, T].
\]

Before proving the above result, we need a useful lemma.

**Lemma 2.2 (L’Hospital’s rule [10], p. 346)** Suppose that we have one of the following cases,
\[
\lim_{r \to \lambda} \frac{f_1(r)}{f_2(r)} = \frac{0}{0} \quad \text{or} \quad \lim_{r \to \lambda} \frac{f_1(r)}{f_2(r)} = \frac{\pm \infty}{\pm \infty},
\]
where \( \lambda \) can be any real number, infinity or negative infinity. In these cases, we have
\[
\lim_{r \to \lambda} \frac{f_1(r)}{f_2(r)} = \lim_{r \to \lambda} \frac{f'_1(r)}{f'_2(r)}.
\]

**Proof of Corollary 2.1.** We only need to prove (2.40) for \(|x - y|\) sufficiently small. By (2.7) for every \( 1 \leq i, j \leq d \) and every \( x, y \in \mathbb{R}^d \) \(|x - y| < r_0\), there is a constant \( C(d, T) \) such that
\[
|\partial^2_{x_i, x_j} u(t, x) - \partial^2_{y_i, y_j} u(t, y)| \leq C \left[ \int_{r \leq |x - y|} \frac{\phi(r)}{r} dr + \phi(|x - y|) + |x - y| \int_{|x - y| < r \leq r_0} \frac{\phi(r)}{r^2} dr \right].
\]
From (2.41) if
\[
\limsup_{r \to 0} \frac{r}{\phi(r)} < +\infty \quad \text{and} \quad \limsup_{r \to 0} \frac{r_0}{\phi(r)} < +\infty,
\]
then for \(|x - y|\) small enough
\[
\max \{ |x - y| \int_{|x - y| < r \leq r_0} \frac{\phi(r)}{r^2} dr, \int_{|x - y| < r \leq r_0} \frac{\phi(r)}{r} dr \} \leq C \phi(|x - y|),
\]
thus (2.40) is proved.

Now let us check (2.42). Since $\phi$ is a Hölder-Dini or strong Hölder or weak Hölder function with Hölder exponent $\vartheta \in (0, 1)$, we can rewrite $\phi(r)$ by $r^\vartheta \psi(r)$ with $\psi$ a monotone continuous function. Applying Lemma 2.2, then

$$\lim_{r \to 0} \frac{\phi(r)}{r^\vartheta} = \lim_{r \to 0} \frac{r^\vartheta \psi(r)}{r^\vartheta} = \lim_{r \to 0} \left[ \frac{r^\vartheta}{\psi(r)} \right]^{-1}$$

and

$$\lim_{r \to 0} \frac{\int_0^r \phi(s) ds}{r^\vartheta} = \lim_{r \to 0} \frac{\int_0^r s^\vartheta \psi(s) ds}{r^\vartheta} = \lim_{r \to 0} \left[ \frac{r^\vartheta}{\psi(r)} \right]^{-1}. \quad (2.43)$$

If $\phi$ is a Hölder-Dini or strong Hölder function, then

$$\lim_{r \to 0} \log(\psi(r)) = -\infty, \quad (2.45)$$

and if $\phi$ is a weak Hölder function, then

$$\lim_{r \to 0} \log(\psi(r)) = +\infty. \quad (2.46)$$

From (2.45) and (2.46), by using Lemma 2.2 again, we gain

$$\lim_{r \to 0} \frac{r^\vartheta \psi(r)}{\psi(r)} = \begin{cases} \lim_{r \to 0} \frac{\log(\psi(r))}{\log(r)}, & \text{when } \phi \text{ is Hölder-Dini or strong Hölder;} \\ -\lim_{r \to 0} \frac{\log(\psi(r))}{\log(r)}, & \text{when } \phi \text{ is weak Hölder.} \end{cases} \quad (2.47)$$

When $\phi$ is a Hölder-Dini or strong Hölder function, then for every $\zeta > 0$, $\psi(r) \geq r^\zeta$ as $r \to 0$. Thus, there exists $\delta = \delta(\zeta) > 0$ such that

$$[\psi(r)]^{-1} \leq r^{-\zeta}, \quad \text{for all } r \in (0, \delta].$$

It follows that

$$\limsup_{r \to 0} \frac{\log(\psi(r))}{\log(r)} = \limsup_{r \to 0} \frac{\log([\psi(r)]^{-1})}{\log(r^{-1})} \leq \limsup_{r \to 0} \frac{\log(r^{-\zeta})}{\log(r^{-1})} = \zeta. \quad (2.48)$$

Since $\zeta > 0$ is arbitrary and $\log(\psi(r))/\log(r) \geq 0$, from (2.47) and (2.48), we conclude

$$\lim_{r \to 0} \frac{r^\vartheta \psi'(r)}{\psi(r)} = \lim_{r \to 0} \frac{\log(\psi(r))}{\log(r)} = 0. \quad (2.49)$$

When $\phi$ is a weak Hölder function, then for every $\zeta_1 > 0$, $r^{\zeta_1} \psi(r) \to 0$ as $r \to 0$. Thus, there exists $\delta_1 = \delta_1(\zeta_1) > 0$ such that

$$\psi(r) \leq r^{-\zeta_1}, \quad \text{for all } r \in (0, \delta_1],$$

which implies that

$$\limsup_{r \to 0} \frac{\log(\psi(r))}{\log(r^{-1})} \leq \limsup_{r \to 0} \frac{\log(r^{-\zeta_1})}{\log(r^{-1})} = \zeta_1. \quad (2.50)$$
Since $\zeta_1 > 0$ is arbitrary and $\log(\psi(r))/\log(r^{-1}) \geq 0$, from (2.47) and (2.50), we conclude
\[
\lim_{r \to 0} \frac{r \psi'(r)}{\psi(r)} = - \lim_{r \to 0} \frac{\log(\psi(r))}{\log(r^{-1})} = 0.
\]

Combining (2.49) and (2.51), then (2.42) holds. From this we complete the proof. □

Example 2.1 We choose $\phi(r) = Cr^{\vartheta} |\log(r)|^\alpha$ with $\vartheta \in (0, 1)$ and $\alpha \in \mathbb{R}$, then $\phi$ is a Hölder-Dini function if $\alpha < -1$, a strong Hölder function if $\alpha < 0$, Hölder continuous if $\alpha = 0$, a weak Hölder function if $\alpha > 0$. Using Lemma 2.2, we have
\[
\lim_{r \to 0} r \int_0^r \frac{\phi(s)}{s^2} ds \leq 1 + \frac{\vartheta}{\alpha} \phi(r),
\]
\[
r \int_0^r \phi(s) ds \leq 2 - \frac{\vartheta}{1 - \vartheta} \phi(r).
\]

Combining (2.7) and (2.52), the maximum regularity for (2.1) is preserved. This result as we know is new.

3 Stochastic flows for SDEs with bounded and Dini drift

Given real number $T > 0$, for $s \in [0, T]$ and $x \in \mathbb{R}^d$, consider the following SDE
\[
dX(s, t) = b(t, X(s, t)) dt + dB(t), \quad t \in (s, T], \quad X(s, t)|_{t=s} = x.
\]

We intend to show the existence of a stochastic flow for equation (3.1). First we give the following definition.

Definition 3.1 ([26], p. 114) A stochastic homeomorphisms flow of class $C^\beta$ with $\beta \in (0, 1)$ on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0 \leq t \leq T})$ associated to (3.1) is a map $(s, t, x, \omega) \mapsto X(s, t, x)(\omega)$, defined for $0 \leq s \leq t \leq T$, $x \in \mathbb{R}^d$, $\omega \in \Omega$ with values in $\mathbb{R}^d$, such that
(i) the process \( \{X(s, \cdot, x)\} = \{X(s, t, x), \ t \in [s, T]\} \) is a continuous \( \{\mathcal{F}_s\}_s \)-adapted solution of (3.1), for every \( s \in [0, T], \ x \in \mathbb{R}^d; \)

(ii) the functions \( X(s, t, x) \) and \( X^{-1}(s, t, x) \) are continuous in \( (s, t, x) \) and are of class \( C^3 \) in \( x \) uniformly in \( (s, t), \mathbb{P} \)-a.s., for all \( 0 \leq s \leq t \leq T; \)

(iii) \( X(s, t, x) = X(r, t, X(s, r, x)) \) for all \( 0 \leq s \leq r \leq t \leq T, \ x \in \mathbb{R}^d, \mathbb{P} \)-a.s., and \( X(s, s, x) = x. \)

Further if

(iv) for all \( 0 \leq s \leq t \leq T, \) the functions \( \nabla X(s, t, x) \) and \( \nabla X^{-1}(s, t, x) \) are continuous in \( (s, t, x), \mathbb{P} \)-a.s.,

it is called a stochastic diffeomorphisms flow.

Now we state the main result of the section.

**Theorem 3.1** Let \( b \in L^\infty([0, T]; C_0(\mathbb{R}^d; \mathbb{R}^d)) \) with integer \( d \geq 1. \) Suppose that \( r_0 \in (0, 1) \) and there is a Dini function \( \phi \) such that for every \( x \in \mathbb{R}^d \)

\[
|b(t, x) - b(t, y)| \leq \phi(|x - y|), \text{ for all } \ y \in B_{r_0}(x), \ t \in [0, T].
\]  (3.2)

In addition that for every \( p \geq 1, \) there is a small enough positive real number \( \delta = \delta(p) < r_0 \) such that \( F_\delta^p(\cdot) \) is increasing and concave on \([0, \delta] , \) where

\[
F_\delta(r) = \int_{s \leq r} \frac{\phi(s)}{s} ds + \phi(r) + r \int_{r \leq s \leq \delta} \frac{\phi(s)}{s^2} ds + r, \quad r \in [0, \delta].
\]  (3.3)

(i) For every \( s \in [0, T] \) and \( x \in \mathbb{R}^d, \) SDE (3.1) has a unique continuous adapted solution \( \{X(s, t, x)(\omega), \ t \in [s, T], \omega \in \Omega\}, \) which forms a stochastic diffeomorphisms flow. For every \( p \geq 1, \) there is a constant \( C(p, d, T) > 0 \) such that

\[
\sup_{0 \leq s \leq T} \mathbb{E} \sup_{t \leq T} |X(s, t, x)|^p + \sup_{x \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \mathbb{E} \sup_{t \leq T} \|\nabla X(s, t, x)\|^p \leq C(p, d, T).
\]  (3.4)

Moreover, for every \( p \geq 1 \) and every \( x, y \in \mathbb{R}^d, \)

\[
\sup_{0 \leq s \leq T} \mathbb{E} \sup_{s \leq t \leq T} |X(s, t, x) - X(s, t, y)|^p \leq C|x - y|^p
\]  (3.5)

and

\[
\sup_{0 \leq s \leq T} \mathbb{E} \sup_{s \leq t \leq T} \|\nabla X(s, t, x) - \nabla X(s, t, y)\|^p
\leq C \left[ \int_{|r - |x - y||} \frac{\phi(r)}{r} dr + \phi(|x - y|) + |x - y| \int_{|x - y| < r < r_0} \frac{\phi(r)}{r^2} dr \right]^p 1_{|x - y| < r_0} + C|x - y|^p. \quad (3.6)
\]

(ii) Let \( \varrho_n \) be given in (2.8) and \( b^n(t, x) = b \ast \varrho_n(t, x). \) Let \( X^n \) be the stochastic flows corresponding to the vector field \( b^n. \) Then for every \( p \geq 1, \)

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |X^n(s, t, x) - X(s, t, x)|^p \right] = 0
\]  (3.7)

and

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \mathbb{E} \left[ \sup_{s \leq t \leq T} \|\nabla X^n(s, t, x) - \nabla X(s, t, x)\|^p \right] = 0. \quad (3.8)
\]
Proof. (i) Let \( \lambda > 0 \) be a real number. Consider the following backward heat equation
\[
\begin{aligned}
\partial_t U(t, x) + \frac{1}{2} \Delta U(t, x) + b(t, x) \cdot \nabla U(t, x) &= \lambda U(t, x) - b(t, x), \quad (t, x) \in [0, T) \times \mathbb{R}^d, \\
U(T, x) &= 0, \quad x \in \mathbb{R}^d.
\end{aligned}
\]  
(3.9)

By Theorem 2.1, the above Cauchy problem has a unique solution \( U \in L^\infty([0, T]; C_b^2(\mathbb{R}^d; \mathbb{R}^d)) \cap W^{1, \infty}([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d)) \). Moreover, (2.6) is true and the constant \( C \) in (2.6) is independent of \( \lambda \). Further, for every \( x, y \in \mathbb{R}^d \) and \( t \in [0, T) \), there is a constant \( C(d,T) \) such that
\[
|\partial^2_{x_i, x_j} U(t, x) - \partial^2_{y_i, y_j} U(t, y)| \\
\leq C(d,T) \left[ \int_{|x-y|=r} \frac{\phi(r)}{r} dr + \phi(|x-y|) |x-y| + \int_{|x-y|<r_0} \phi(r) \frac{|x-y|}{r^2} dr \right] 1_{|x-y|<r_0} \\
+ C(d,T) |x-y|.
\]  
(3.10)

By Lemma 2.1, the unique strong solution \( U \) has the following representation
\[
U(t, x) = \int_0^{T-t} e^{-\lambda r} K(r, \cdot) * [b(t + r, \cdot) + b(t + r, \cdot) \cdot \nabla U(t + r, \cdot)](x) dr.
\]

For every \( 1 \leq i \leq d \), every \( x \in \mathbb{R}^d \) and \( t \in [0, T] \)
\[
|\partial_{x_i} U(t, x)| = \left| \int_0^{T-t} e^{-\lambda r} dr \int_{\mathbb{R}^d} \partial_{x_i} K(r, x - z) [b(t + \tau, z) + b(t + r, z) \cdot \nabla U(t + r, z)] dz \right|
\]
\[
\leq C \|b\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} (1 + \|b\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))}) \int_0^T r^{-\frac{1}{2}} e^{-\lambda r} dr
\]
\[
\leq C \|b\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} (1 + \|b\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))}) \lambda^{-\frac{1}{4}},
\]  
(3.11)
where the Hölder inequality is applied in the last inequality. Then letting \( \lambda \) tend to infinity in (3.11) yields
\[
\|\nabla U\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^{d \times d}))} \longrightarrow 0.
\]  
(3.12)

Therefore, there is a large real number \( \lambda_0 > 0 \) such that if \( \lambda \geq \lambda_0 \)
\[
\|\nabla U\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^{d \times d}))} \leq \frac{1}{2}.
\]  
(3.13)

Now for \( \lambda \geq \lambda_0 \), define \( \gamma(t, x) = x + U(t, x) \)
\[
\frac{1}{2} \leq \|\nabla \gamma\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^{d \times d}))} \leq \frac{3}{2}.
\]

Then \( \gamma(t) \) forms a nonsingular diffeomorphism of class \( C^2 \) uniformly in \( t \in [0, T] \) by the classical Hadamard theorem ([30, p.330]). Moreover, for every \( t \in [0, T] \), the inverse of \( \gamma(t) \) (denoted by \( \gamma^{-1}(t) \)) has bounded first and second spatial derivatives, uniformly in \( t \in [0, T] \), and
\[
\frac{2}{3} \leq \|\nabla \gamma^{-1}\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^{d \times d}))} \leq 2.
\]  
(3.14)
On the other hand, by the result of Kunita [25, Theorem 4.3, p.227], there is a unique stochastic homeomorphisms flow of class $C^\beta$ ($\beta \in (0, 1)$) defined by $Y(s, t)$. Moreover, for every $x, y \in \mathbb{R}^d$ and every $p \geq 1$, there is a constant $C(p, T) > 0$ such that

\[
\sup_{0 \leq s \leq T} \mathbb{E} \sup_{s \leq t \leq T} |Y(s, t, y)|^p \leq C(p, T) \tag{3.16}
\]

and

\[
\sup_{0 \leq s \leq T} \mathbb{E}\left[ \sup_{s \leq t \leq T} |Y(s, t, x) - Y(s, t, y)|^p \right] \leq C(p, T)|x - y|^p. \tag{3.17}
\]

On the other hand, by $X(s, t) = \gamma^{-1}(t, Y(s, t))$, (3.1) also defines a unique stochastic homeomorphisms flow of class $C^\beta$ ($\beta \in (0, 1)$). Moreover by (3.16) and (3.17) for every $p \geq 1$ and every $x, y \in \mathbb{R}^d$

\[
\sup_{0 \leq s \leq T} \mathbb{E} \sup_{s \leq t \leq T} |X(s, t, x)|^p \leq 2^{p-1} \mathbb{E} \sup_{0 \leq s \leq T} \sup_{s \leq t \leq T} \left[ |Y(s, t, \gamma(s, x))|^p + |\gamma^{-1}(t, Y(s, t, \gamma(s, x))) - \gamma^{-1}(t, \gamma(s, x))|^p \right] \leq C(p, T) + 2^{p-1} \|\nabla \gamma^{-1}\|_{L^\infty([0, T]; C_b(\mathbb{R}^d, \mathbb{R}^d \times d))} \sup_{0 \leq s \leq T} \mathbb{E} \sup_{s \leq t \leq T} |Y(s, t, \gamma(s, x)) - \gamma(s, x)|^p \leq C(p, d, T) \tag{3.18}
\]

and

\[
\sup_{0 \leq s \leq T} \mathbb{E}\left[ \sup_{s \leq t \leq T} |X(s, t, x) - X(s, t, y)|^p \right] = \sup_{0 \leq s \leq T} \mathbb{E}\left[ \sup_{s \leq t \leq T} |\gamma^{-1}(t, Y(s, t, \gamma(s, x))) - \gamma^{-1}(t, Y(s, t, \gamma(s, y)))|^p \right] \leq C\|\nabla \gamma^{-1}\|_{L^\infty([0, T]; C_b(\mathbb{R}^d, \mathbb{R}^d \times d))} \sup_{0 \leq s \leq T} \mathbb{E}\left[ \sup_{s \leq t \leq T} |Y(s, t, \gamma(s, x)) - Y(s, t, \gamma(s, y))|^p \right] \leq C\|\nabla \gamma^{-1}\|_{L^\infty([0, T]; C_b(\mathbb{R}^d, \mathbb{R}^d \times d))} \|Y\|_{L^\infty([0, T]; C_b(\mathbb{R}^d, \mathbb{R}^d \times d))} \|x - y\|^p \leq C(p, d, T)|x - y|^p. \tag{3.19}
\]

Now we just need to prove the continuity of $\nabla_x X(s, t, x)$ and the inequality (3.6). For this we consider $\nabla_y Y(s, t, y)$. First the stochastic flow $\{X(s, t)(\cdot)\}$ is weak differentiable, that is, $X(s, t)(\cdot) \in
L^2(\Omega;W_{loc}^{1,p}(\mathbb{R}^d;\mathbb{R}^d)) ([28, Theorem 3]), so does the stochastic flow \{Y(s,t)(\cdot)\}. Then differentiate \(Y(s,t)\) with respect to the initial data and denoting the derivative by \(\xi(s,t,y)\), we have

\[
d\xi(s,t,y) = \lambda \nabla U(t,\gamma^{-1}(t,Y(s,t,y)))\nabla \gamma^{-1}(t,Y(s,t,y))\xi(s,t,y)dt + \nabla^2 U(t,\gamma^{-1}(t,Y(s,t,y)))\nabla \gamma^{-1}(t,Y(s,t,y))\xi(s,t,y)dB(t)
\]

\[
= : A_1(t,Y(s,t,y))\xi(s,t,y)dt + A_2(t,Y(s,t,y))\xi(s,t,y)dB(t),
\]

(3.20)

with \(\xi(s,t,y)|_{t=s} = I\).

Notice that the equation (3.20) is a linear SDE with bounded coefficients \(A_1, A_2\) depending on the process \(Y(s,t,y)\), by the Cauchy–Lipschitz theorem, there is a unique strong solution for equation (3.20). Moreover for \(d = 1\), the unique strong solution is represented by

\[
\xi(s,t,y) = \exp \left( \int_s^t [A_1(r,Y(s,r,y)) - \frac{1}{2} A_2^2(r,Y(s,r,y))]dr + \int_s^t A_2(r,Y(s,r,y))dB(r) \right). \tag{3.21}
\]

By (3.10) and Remark 2.1, \(\nabla^2 U(t,x)\) is uniformly continuous in \(x\), then \(A_1(s,r,Y(s,\cdot,\cdot))\) and \(A_2(s,r,Y(s,\cdot,\cdot))\) are continuous in \([s,T] \times \mathbb{R}^d\) almost surely. Then for \(d = 1\), the process \(\xi(s,t,y)\) is continuous in \((s,t,y)\) almost surely, and for every \(p \geq 1\) there is a constant \(C(p,T) > 0\) such that

\[
\sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \sup_{s \leq t \leq T} \mathbb{E} \|\xi(s,t,y)\|^p \leq C(p,T). \tag{3.22}
\]

For general \(d > 1\), the unique strong solution also has an obvious representation which is analogue of (3.21), then using the same argument, the continuity of \(\xi(s,t,y)\) in \((s,t,y)\) and the inequality (3.22) also hold. Then using the relationship between \(X\) and \(Y\), and (3.18), (3.4) holds true.

For every \(x,y \in \mathbb{R}^d\), write \(Y(s,t,x), Y(s,t,y), \xi(s,t,x), Y(s,t,x) - Y(s,t,y), \xi(s,t,x) - \xi(s,t,y), U(t,\gamma^{-1}(t,Y(s,t,x)))\) and \(U(t,\gamma^{-1}(t,Y(s,t,y)))\) by \(Y(x), Y(y), \xi(x), \xi(y), Y(x,y), \xi(x,y), U(\gamma^{-1}(Y(x)))\) and \(U(\gamma^{-1}(Y(y)))\), respectively. Then for every \(q \geq 2\), by using Itô’s formula

\[
d\|\xi(x,y)\|^q = q\lambda\|\xi(x,y)\|^{q-2}\langle \xi(x,y), \nabla U(\gamma^{-1}(Y(x)))\nabla \gamma^{-1}(Y(x))\xi(x) - \nabla U(\gamma^{-1}(Y(y)))\nabla \gamma^{-1}(Y(y))\xi(y) \rangle dt
\]

\[
+ \frac{1}{2} q(q-1)\|\xi(x,y)\|^{q-2} tr([\nabla^2 U(\gamma^{-1}(Y(x)))\nabla \gamma^{-1}(Y(x))\xi(x) - \nabla^2 U(\gamma^{-1}(Y(y)))\nabla \gamma^{-1}(Y(y))])d\xi(x)y
\]

\[
\times \xi(y)]\nabla^2 U(\gamma^{-1}(Y(x)))\nabla \gamma^{-1}(Y(x))\xi(x) - \nabla^2 U(\gamma^{-1}(Y(y)))\nabla \gamma^{-1}(Y(y))\xi(y)]^2 dt
\]

\[
+ q\|\xi(x,y)\|^{q-2}\langle \xi(x,y), [\nabla^2 U(\gamma^{-1}(Y(x)))\nabla \gamma^{-1}(Y(x))\xi(x)
\]

\[
- \nabla^2 U(\gamma^{-1}(Y(y)))\nabla \gamma^{-1}(Y(y))\xi(y)]dB(t) \rangle
\]

\[
\leq C(q) \left[ \|\xi(x,y)\|^{q-1}\|\nabla U(\gamma^{-1}(Y(x)))\nabla \gamma^{-1}(Y(x))\xi(x) - \nabla U(\gamma^{-1}(Y(y)))\nabla \gamma^{-1}(Y(y))\xi(y)\| + \|\xi(x,y)\|^{q-2}\|\nabla^2 U(\gamma^{-1}(Y(x)))\nabla \gamma^{-1}(Y(x))\xi(x) - \nabla^2 U(\gamma^{-1}(Y(y)))\nabla \gamma^{-1}(Y(y))\xi(y)\|^2 \right] dt
\]

\[
+ q\|\xi(x,y)\|^{q-2}\langle \xi(x,y), [\nabla^2 U(\gamma^{-1}(Y(x)))\nabla \gamma^{-1}(Y(x))\xi(x)
\]

\[
- \nabla^2 U(\gamma^{-1}(Y(y)))\nabla \gamma^{-1}(Y(y))\xi(y)]dB(t) \rangle. \tag{3.23}
\]
Notice that

\[ \| \nabla U(\gamma^{-1}(Y(x))) \nabla \gamma^{-1}(Y(x)) \xi(x) - \nabla U(\gamma^{-1}(Y(y))) \nabla \gamma^{-1}(Y(y)) \xi(y) \| \]
\leq \| \nabla U(\gamma^{-1}(Y(x))) \nabla \gamma^{-1}(Y(x)) \xi(x) - \nabla U(\gamma^{-1}(Y(y))) \nabla \gamma^{-1}(Y(y)) \xi(x) \|
+ \| \nabla U(\gamma^{-1}(Y(y))) \nabla \gamma^{-1}(Y(x)) \xi(x) - \nabla U(\gamma^{-1}(Y(y))) \nabla \gamma^{-1}(Y(y)) \xi(x) \|
+ \| \nabla U(\gamma^{-1}(Y(y))) \nabla \gamma^{-1}(Y(y)) \xi(x) - \nabla U(\gamma^{-1}(Y(y))) \nabla \gamma^{-1}(Y(y)) \xi(y) \|
\leq \| \nabla^2 U \|_{L^\infty([0,T];C_b(R^d;R^d \otimes \mathbb{R}^d))} \| \nabla \gamma^{-1} \|_{L^\infty([0,T];C_b(R^d;R^d \otimes \mathbb{R}^d))} \| \xi(x) \|
+ \| \nabla^2 U \|_{L^\infty([0,T];C_b(R^d;R^d \otimes \mathbb{R}^d))} \| \nabla \gamma^{-1} \|_{L^\infty([0,T];C_b(R^d;R^d \otimes \mathbb{R}^d))} \| \xi(x) \|
+ \| \nabla^2 U \|_{L^\infty([0,T];C_b(R^d;R^d \otimes \mathbb{R}^d))} \| \nabla \gamma^{-1} \|_{L^\infty([0,T];C_b(R^d;R^d \otimes \mathbb{R}^d))} \| \xi(x) \|
\leq C \left[ \| \nabla^2 U(\gamma^{-1}(Y(x)) \right.
\left. - \nabla^2 U(\gamma^{-1}(Y(y))) \| \xi(x) \| + \| Y(x, y) \| \xi(x) \| + \| \xi(x) \| \right], \quad (3.25)

then for \( t \in [s, T] \)

\[ \mathbb{E} \| \xi(x, y) \|^q(t) \]
\leq C(q, d) \mathbb{E} \int_s^t \| \xi(x, y) \|^q(r) \, dr + C(q, d) \mathbb{E} \int_s^t \| \xi(x, y) \|^q - 1(r) |Y(x, y)|_r \| \xi(x) \| \| \xi(x) \| \, dr
+ C(q, d) \mathbb{E} \int_s^t \| \xi(x, y) \|^q - 2(r) \| Y(x, y) \|^2 \, \| \xi(x) \| \| \xi(x) \| \, dr
+ C(q, d) \mathbb{E} \int_s^t \| \xi(x, y) \|^q - 2(r) \| \nabla^2 U(\gamma^{-1}(Y(x))) - \nabla^2 U(\gamma^{-1}(Y(y))) \| \| \xi(x) \| \| \xi(x) \|^2 \, dr
\leq C(q, d) \mathbb{E} \int_s^t \| \xi(x, y) \|^q(r) \, dr + C(q, d) \mathbb{E} \int_s^t \| Y(x, y) \|^q(r) \| \xi(x) \|^q(r) \, dr
+ C(q, d) \mathbb{E} \int_s^t \| \nabla^2 U(\gamma^{-1}(Y(x)) \right.
\left. - \nabla^2 U(\gamma^{-1}(Y(y))) \| q \| \xi(x) \| \| \xi(x) \|^q \, dr
\]
Moreover by (3.16) and (3.22)

\[
E\|\xi(x,y)\|^q(t) 
\leq C(q,d,T) \left[ |x-y|^q + \left( E \int_s^t \|\nabla^2 U(\gamma^{-1}(X(x))) - \nabla^2 U(\gamma^{-1}(Y(y)))\|^{2q} dr \right)^{\frac{1}{2}} \right].
\] (3.27)

Further for every \( p \geq 2 \), by the Burkholder–Davis–Gundy inequality

\[
\sup_{0 \leq s \leq T} E \sup_{s \leq t \leq T} \|\xi(x,y)\|^p 
\leq C(p,d,T) |x-y|^p + C(p,d,T) E \left[ \int_0^T \|\nabla^2 U(\gamma^{-1}(X(x))) - \nabla^2 U(\gamma^{-1}(Y(y)))\|^{4p} dr \right]^\frac{1}{4}.
\] (3.28)

For given \( p \geq 2 \), by the assumption in Theorem 3.1, there is a small enough real number \( \delta = \delta(p) \) such that \( F_\delta^{4p}(\cdot) \) (see (3.3)) is increasing and concave. On the other hand, since \( U \in L^\infty([0,T];C^2_{\delta}(\mathbb{R}^d;\mathbb{R}^d)) \) and (3.10) holds, for every \( x,y \in \mathbb{R}^d \), and the given real number \( \delta(p) > 0 \), there is a constant \( C(\delta) > 0 \) such that

\[
\sup_{0 \leq t \leq T} \|\nabla^2 U(t,x) - \nabla^2 U(t,y)\| \leq C(\delta) F_\delta(|x-y|) 1_{|x-y| < \delta} + C(\delta)|x-y| 1_{|x-y| \geq \delta}.
\]

Then by (3.14) and the fact that \( F_\delta^{4p}(\cdot) \) is increasing

\[
\sup_{0 \leq s \leq T} E \sup_{s \leq t \leq T} \|\xi(x,y)\|^p 
\leq C|x-y|^p + C E \left[ \int_0^T F_\delta^{4p}(|\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))|) 1_{|\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))| < \delta} dt \right]^\frac{1}{4} + C E \left[ \int_0^T |\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))|^{4p} 1_{|\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))| \geq \delta} dt \right]^\frac{1}{4}
\leq C|x-y|^p + C \left[ \int_0^T E F_\delta^{4p}(|\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))| 1_{|\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))| < \delta}) dt \right]^\frac{1}{4} + C \left[ \int_0^T E |\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))|^{4p} dt \right]^\frac{1}{4}.
\]
\[ \begin{align*}
& \leq C|x - y|^p + C \left[ \int_0^T F_3^{2p}(E[|\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))|1_{|\gamma^{-1}(Y(x)) - \gamma^{-1}(Y(y))| < \delta}] \right]^{\frac{1}{2}} dt \\
& \leq C|x - y|^p + C F_*^{\delta}(2C_1|x - y|),
\end{align*} \]

(3.29)

where \( C_1 = C(p,T) \lor 1 \) and \( C(p,T) \) is given in (3.17).

Now by \( X(s,t,x) = \gamma^{-1}(t,Y(s,t,\gamma(s,x))) \), for every \( p \geq 2 \) and every \( x, y \in \mathbb{R}^d \)

\[ \begin{align*}
& \|\nabla X(s,t,x) - \nabla X(s,t,y)\| \\
& = \|\nabla \gamma^{-1}(t,Y(s,t,\gamma(s,x)))\nabla Y(s,t,\gamma(s,x))\nabla \gamma(s,x) \\
& \quad - \nabla \gamma^{-1}(t,Y(s,t,\gamma(s,y)))\nabla Y(s,t,\gamma(s,y))\nabla \gamma(s,y)\| \\
& = \|\nabla \gamma^{-1}(t,Y(s,t,\gamma(s,x))) - \nabla \gamma^{-1}(t,Y(s,t,\gamma(s,y)))\|\nabla Y(s,t,\gamma(s,x))\nabla \gamma(s,x) \\
& \quad + \|\nabla \gamma^{-1}(t,Y(s,t,\gamma(s,y)))\|\nabla Y(s,t,\gamma(s,y))\|\nabla \gamma(s,y)\| \\
& \leq \|\nabla \gamma^{-1}\|_{L^\infty([0,T];C^\beta(\mathbb{R}^d;\mathbb{R}^{d \times d}))}^2 \|Y(s,t,\gamma(s,x)) - Y(s,t,\gamma(s,y))\|\|\nabla Y(s,t,\gamma(s,x))\| \\
& \quad + \|\nabla \gamma^{-1}\|_{L^\infty([0,T];C^\beta(\mathbb{R}^d;\mathbb{R}^{d \times d}))}^2 \|\nabla Y(s,t,\gamma(s,y))\|\|\nabla \gamma(s,y)\| \\
& \leq C \left[ |Y(s,t,\gamma(s,x)) - Y(s,t,\gamma(s,y))| + \|\nabla Y(s,t,\gamma(s,x))\| + \|\nabla Y(s,t,\gamma(s,y))\| \right]^{\frac{1}{2}}.
\end{align*} \]

Thanks to the Hölder inequality, (3.22) and (3.29), for every \( x, y \in \mathbb{R}^d \), we have that

\[ \begin{align*}
& \sup_{0 \leq s \leq T} E\left[ \sup_{s \leq t \leq T} \|\nabla X(s,t,x) - \nabla X(s,t,y)\|^p \right] \\
& \leq C \sup_{0 \leq s \leq T} E\left[ \sup_{s \leq t \leq T} \|\nabla Y(s,t,\gamma(s,x)) - Y(s,t,\gamma(s,y))\|^p \right]^{\frac{1}{2}} \left( E\|\nabla Y(s,t,\gamma(s,x))\|^2 \right)^{\frac{1}{2}} \\
& \quad + \sup_{0 \leq s \leq T} E\left[ \sup_{s \leq t \leq T} \|\nabla Y(s,t,\gamma(s,x)) - \nabla Y(s,t,\gamma(s,y))\|^p \right] + C|x - y|^p \\
& \leq C|x - y|^p + CF_3^{\phi}(2C_1|\gamma(s,x) - \gamma(s,y)|) \\
& \leq C|x - y|^p + CF_3^{\phi}(3C_1|x - y|),
\end{align*} \]

(3.30)

where in the last inequality we have used \( \|\nabla \gamma\|_{L^\infty([0,T];C^\beta(\mathbb{R}^d;\mathbb{R}^{d \times d}))} \leq 3/2 \). Then for every \( x, y \in \mathbb{R}^d \) such that \( 0 < |x - y| < \delta/(3C_1) \lt r_0/(3C_1) \), by similar calculations for (2.25)

\[ \begin{align*}
& \sup_{0 \leq s \leq T} E\left[ \sup_{s \leq t \leq T} \|\nabla X(s,t,x) - \nabla X(s,t,y)\|^p \right] \\
& \leq C \left[ \int_{r \leq 3C_1|x - y|} \frac{\phi(r)}{r^2} dr + \phi(|x - y|) + |x - y| \int_{3C_1|x - y| < r \leq \delta} \frac{\phi(r)}{r^2} dr + |x - y| \right]^p + C|x - y|^p \\
& \leq C \left[ \int_{r \leq |x - y|} \frac{\phi(r)}{r^2} dr + \phi(|x - y|) + |x - y| \int_{|x - y| < r \leq r_0} \frac{\phi(r)}{r^2} dr \right]^p + C|x - y|^p.
\end{align*} \]

(3.31)
On account of (3.31) and (3.4), by using the Hölder inequality, we gain (3.6).

(ii) Let $U^n$ be the unique solution of the parabolic problem (3.9) associated to $b^n$, i.e. $b$ is replaced by $b^n$ in equation (3.9). By Theorem 2.1 (ii), $U^n \in L^\infty([0,T];C^2_b(\mathbb{R}^d;\mathbb{R}^d)) \cap W^{1,\infty}([0,T];C_b(\mathbb{R}^d;\mathbb{R}^d))$ and

$$\lim_{n \to \infty} \|U^n - U\|_{L^\infty([0,T];C^2_b(\mathbb{R}^d;\mathbb{R}^d))} = 0. \tag{3.32}$$

We set $\gamma_n(t,x) := x + U^n(t,x)$, then if $\lambda \geq \lambda_0$, $\{\gamma_n(t,x) := x + U^n(t,x)\}_n$ form nonsingular diffeomorphisms of class $C^2$ uniformly in $t \in [0,T]$ and $n$. Moreover,

$$\lim_{n \to \infty} \|\gamma_n - \gamma\|_{L^\infty([0,T];C^2_b(\mathbb{R}^d;\mathbb{R}^d))} = \lim_{n \to \infty} \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T];C^2_b(\mathbb{R}^d;\mathbb{R}^d))} = 0. \tag{3.33}$$

To prove (3.7) and (3.8), it is sufficient to show for every $p \geq 2$

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \mathbb{E}[\sup_{s \leq t \leq T} |Y^n(s,t,y) - Y(s,t,y)|^p] = 0 \tag{3.34}$$

and

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \mathbb{E}[\sup_{s \leq t \leq T} \|\nabla Y^n(s,t,y) - \nabla Y(s,t,y)\|^p] = 0, \tag{3.35}$$

where $Y^n$ satisfies

$$dY^n(s,t) = \lambda U^n(t,\gamma_n^{-1}(t,Y^n(s,t)))dt + [I + \nabla U^n(t,\gamma_n^{-1}(t,Y^n(s,t)))]dB(t), \quad t \in (s,T], \quad Y^n(s,t)|_{t=s} = y. \tag{3.36}$$

For simplicity, we write $Y(s,t)$, $Y^n(s,t)$, $U(t,\gamma^{-1}(t,Y^n(s,t)))$ and $U^n(t,\gamma_n^{-1}(t,Y^n(s,t)))$ by $Y$, $Y^n$, $U(\gamma^{-1}(Y))$ and $U^n(\gamma_n^{-1}(Y^n))$, respectively. For every $q \geq 2$, by using the Itô formula to $|Y^n - Y|^q$

$$d|Y^n - Y|^q = q|Y^n - Y|^{q-2}(Y^n - Y,U^n(\gamma_n^{-1}(Y^n)) - U(\gamma^{-1}(Y)))dt$$

$$+ \frac{1}{2} q(q-1)|Y^n - Y|^{q-2} tr(\nabla U^n(\gamma_n^{-1}(Y^n)) - \nabla U(\gamma^{-1}(Y)))$$

$$\times [U^n(\gamma_n^{-1}(Y^n)) - U(\gamma^{-1}(Y))]^T dt$$

$$+ q|Y^n - Y|^{q-2} [\nabla U^n(\gamma_n^{-1}(Y^n)) - \nabla U(\gamma^{-1}(Y))]dB(t)$$

$$\leq C(q) \left[ |Y^n - Y|^{q-1} ||U^n(\gamma_n^{-1}(Y^n)) - U(\gamma^{-1}(Y))||$$

$$+ |Y^n - Y|^{q-2} ||\nabla U^n(\gamma_n^{-1}(Y^n)) - \nabla U(\gamma^{-1}(Y))||^2 \right] dt$$

$$+ q|Y^n - Y|^{q-2} |\nabla U^n(\gamma_n^{-1}(Y^n)) - \nabla U(\gamma^{-1}(Y))|^2 dB(t). \tag{3.37}$$

Then for every $t \in [s,T]$

$$\mathbb{E}|Y^n - Y|^q(t)$$

$$\leq C(q) \mathbb{E} \int_s^t |Y^n - Y|^{q-1}(r)||U^n(\gamma_n^{-1}(Y^n)) - U(\gamma^{-1}(Y))||^2(r)dr$$

$$+ C(q) \mathbb{E} \int_s^t |Y^n - Y|^{q-2}(r)||\nabla U^n(\gamma_n^{-1}(Y^n)) - \nabla U(\gamma^{-1}(Y))||^2(r)dr. \tag{3.38}$$
First we have
\[ ||U^n(\gamma_n^{-1}(Y^n)) - U(\gamma^{-1}(Y))|| \]
\[ = ||U^n(\gamma_n^{-1}(Y^n)) - U(\gamma^{-1}(Y^n)) + U(\gamma^{-1}(Y^n)) - U(\gamma^{-1}(Y^n)) + U(\gamma^{-1}(Y^n)) - U(\gamma^{-1}(Y))|| \]
\[ \leq ||U^n - U||_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} + \|\nabla U\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} \]
\[ + \|\nabla U\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} \|\nabla \gamma^{-1}\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} ||Y^n - Y|| \]
\[ \leq C \left[ ||U^n - U||_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} + \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} + ||Y^n - Y|| \right] \] (3.39)
and similarly
\[ \|\nabla U^n(\gamma_n^{-1}(Y^n)) - \nabla U(\gamma^{-1}(Y))\|^2 \]
\[ \leq C \left[ \|\nabla U^n - \nabla U\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} \right. \]
\[ + \|\nabla U\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} \|\nabla \gamma^{-1}\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} \] \[ \left. \|\nabla \gamma^{-1}\|_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} ||Y^n - Y|| \right]^2 \]
\[ \leq C \left[ \|\nabla U^n - \nabla U\|^2_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} + \|\gamma_n^{-1} - \gamma^{-1}\|^2_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} + ||Y^n - Y||^2 \right] . \] (3.40)

Since \( Y_n \) and \( Y \) satisfy (3.36) and (3.20), respectively, and the coefficients are globally Lipschitz continuous, for every \( q \geq 2 \), there is a positive constant \( C(q) \) such that
\[ \sup_{n} \sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \sup_{s \leq t \leq T} \mathbb{E}[|Y^n(s, t, y) - Y(s, t, y)|^q] \leq C(q) . \] (3.41)

Then by (3.38)–(3.41) and a Grönnwall type argument, we obtain
\[ \sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \sup_{s \leq t \leq T} \mathbb{E}[|Y^n(s, t, y) - Y(s, t, y)|^q] \]
\[ \leq C(q, T) \left[ ||U^n - U||^q_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} + \|\gamma_n^{-1} - \gamma^{-1}\|^q_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} \right] . \] (3.42)

Now for every \( p \geq 2 \), by (3.37) and the Burkholder–Davis–Gundy inequality
\[ \mathbb{E} \sup_{s \leq t \leq T} |Y^n(s, t, y) - Y(s, t, y)|^p \]
\[ \leq C(p) \mathbb{E} \int_{s}^{T} |Y^n - Y|^{p-1}(r)||U^n(\gamma_n^{-1}(Y^n)) - U(\gamma^{-1}(Y))||(r)dr \]
\[ + C(p) \mathbb{E} \int_{s}^{T} |Y^n - Y|^{p-2}(r)||\nabla U^n(\gamma_n^{-1}(Y^n)) - \nabla U(\gamma^{-1}(Y))||^2(r)dr \]
\[ + C(p) \mathbb{E} \left[ \int_{s}^{T} |Y^n - Y|^{2p-2}(r)||\nabla U^n(\gamma_n^{-1}(Y^n)) - \nabla U(\gamma^{-1}(Y))||^2(r)dr \right]^{\frac{1}{2}} . \] (3.43)

Then thanks to (3.39), (3.40), (3.42) and the Grönnwall inequality,
\[ \sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \sup_{s \leq t \leq T} \mathbb{E}[|Y^n(s, t, y) - Y(s, t, y)|^p] \]
\[ \leq C(p, T) \left[ ||U^n - U||^p_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} + \|\gamma_n^{-1} - \gamma^{-1}\|^p_{L^\infty([0,T],C_b(\mathbb{R}^d \times \mathbb{R}^d))} \right] . \] (3.44)
which implies (3.34) by (3.32) and (3.33).

Differentiate $Y^n(s, t, y)$ with respect to initial data (denoted by $\xi^n(s, t, y)$ or $\xi^n$), then $\xi^n$ satisfies equation (3.20) with $b$ replaced by $b^n$, and by the boundedness of the coefficients and initial value (I), for every $q \geq 2$, there is a positive real number $C(q)$ such that

$$\sup_{n} \sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \sup_{s \leq t \leq T} \mathbb{E}[\|\xi^n(s, t, y)\|^q] \leq C(q).$$

(3.45)

Moreover observing that

$$\| \nabla U^n(\gamma^{-1}_n(Y^n)) \nabla \gamma^{-1}_n(Y^n) \|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d))} \| \nabla \gamma^{-1}_n \|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d))} \| \xi^n \|$$

$$\leq \| \nabla U^n - \nabla U \|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d))} \| \nabla \gamma^{-1}_n \|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d))} \| \xi^n \|$$

and by the boundedness of $\| \nabla^2 U \|

$$\| \nabla^2 U^n(\gamma^{-1}_n(Y^n)) \nabla \gamma^{-1}_n(Y^n) \|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d))} \| \nabla \gamma^{-1}_n \|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d))} \| \xi^n \|$$

$$\leq C \left[ \| \nabla^2 U^n - \nabla^2 U \|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d))} + \| \nabla \gamma^{-1}_n - \nabla \gamma^{-1} \|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d))} + \| Y^n - Y \| \right] \| \xi^n \|$$

(3.46)
then by similar calculations for (3.37)–(3.44), we have
\[
\sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} \sup_{s \leq t \leq T} \mathbb{E} \sup_{p} \|\xi^n(s, t, y) - \xi(s, t, y)\|^p \\
\leq C(p, T) \left[ \|U^n - U\|_{L^\infty([0,T]; C_b^1(\mathbb{R}^d; \mathbb{R}^d))} + \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b^1(\mathbb{R}^d; \mathbb{R}^d))} \right] \\
+ C(p, T) \sup_{y \in \mathbb{R}^d} \mathbb{E} \left[ \int_{s}^{T} \|\nabla^2 U(\gamma_n^{-1}(Y^n)) - \nabla^2 U(\gamma^{-1}(Y))\|^{2p} \|\xi^n\|^{2p} dt \right]^{\frac{1}{p}} \\
\leq C(p, T) \left[ \|U^n - U\|_{L^\infty([0,T]; C_b^1(\mathbb{R}^d; \mathbb{R}^d))} + \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b^1(\mathbb{R}^d; \mathbb{R}^d))} \right] \\
+ C(p, T) \sup_{y \in \mathbb{R}^d} \mathbb{E} \left[ \int_{0}^{T} \|\nabla^2 U(\gamma_n^{-1}(Y^n)) - \nabla^2 U(\gamma^{-1}(Y))\|^{4p} dt \right]^{\frac{1}{4}}. \tag{3.48}
\]

Further we apply same calculations for (3.29) to \(\|\nabla^2 U(\gamma_n^{-1}(Y^n)) - \nabla^2 U(\gamma^{-1}(Y))\|\) to get
\[
\sup_{y \in \mathbb{R}^d} \mathbb{E} \left[ \int_{0}^{T} \|\nabla^2 U(\gamma_n^{-1}(Y^n(t, y))) - \nabla^2 U(\gamma^{-1}(Y(t, y)))\|^{4p} dt \right]^{\frac{1}{4}} \\
\leq C(p, T, \delta) \sup_{y \in \mathbb{R}^d} \mathbb{E} \left[ \int_{0}^{T} \mathbb{E} \left[ \sup_{0 \leq s \leq \tau} |\gamma_n^{-1}(Y(t, y)) - \gamma^{-1}(Y(t, y))| \right]^{4p} dt \right]^{\frac{1}{4}} \\
+ C(p, T, \delta) \sup_{y \in \mathbb{R}^d} \mathbb{E} \left[ \int_{0}^{T} \mathbb{E} \left[ \sup_{0 \leq s \leq \tau} |\gamma_n^{-1}(Y(t, y)) - \gamma^{-1}(Y(t, y))| \right]^{4p} dt \right]^{\frac{1}{4}} \\
\leq C \left[ \int_{0}^{T} \mathbb{E} \left[ \sup_{0 \leq s \leq \tau} |\gamma_n^{-1}(Y(t, y)) - \gamma^{-1}(Y(t, y))| \right]^{4p} dt \right]^{\frac{1}{4}} + C \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} \\
\leq C \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} + CF_\delta^p \left( \mathbb{E} \sup_{y \in \mathbb{R}^d} \sup_{0 \leq s \leq T} |Y^n(s, r, y) - Y(s, r, y)| + \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} \right) \\
\leq C \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} + CF_\delta^p \left( \mathbb{E} \|U^n - U\|_{L^\infty([0,T]; C_b^1(\mathbb{R}^d; \mathbb{R}^d))} + \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} \right), \tag{3.49}
\]
where in the last inequality we use (3.44), and in the third inequality we use
\[
|\gamma_n^{-1}(Y^n) - \gamma^{-1}(Y)| \leq \|\nabla\gamma_n^{-1}\|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} |Y^n - Y| + \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} \\
\leq C |Y^n - Y| + \|\gamma_n^{-1} - \gamma^{-1}\|_{L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d))}.
\]
Now from (3.48) and (3.49), (3.35) holds by (3.32) and (3.33). \(\square\)

**Remark 3.1** Let \(b^n\) be given in Theorem 3.1 (ii). By Liouville’s theorem, since \(b^n\) is smooth, we have the following Euler identity
\[
\det(\nabla_x X^n(s, t, x)) = \exp \left( \int_{s}^{t} \text{div} b^n(r, X^n(s, r, x)) dr \right), \quad 0 \leq s \leq t \leq T,
\]
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where \( \det(\cdot) \) denotes the determinant of a matrix. In view of (3.7) and (3.8), if \( b \in L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d)) \) such that (3.2) holds, and \( \text{div} b \in L^1([0,T]; L^1_{\text{loc}}(\mathbb{R}^d)) \), up to choosing a subsequence, one derives

\[
\det(\nabla_x X(s,t,x)) = \exp \left( \int_s^t \text{div}(r, X(s,r,x)) dr \right), \quad 0 \leq s \leq t \leq T. \tag{3.50}
\]

Since the inverse of \( X(X^n): X^{-1}(X_n^{-1}) \) satisfies an equation which has the same form as the original one except the drift has opposite sign, by Theorem 3.1 we have

**Corollary 3.1** Let \( b, b^n, X(s,t,x) \) and \( X^n(s,t,x) \) be stated in Theorem 3.1. For every \( p \geq 1 \), there is a constant \( C(p,d,T) > 0 \) such that

\[
\sup_{0 \leq s \leq T} \mathbb{E} \sup_{s \leq t \leq T} |X^{-1}(s,t,x)|^p + \sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{0 \leq s \leq T} \|\nabla X^{-1}(s,t,x)\|^p \leq C(p,d,T) \tag{3.51}
\]

and

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{0 \leq s \leq T} \|X^{-1}_n(s,t,x) - X^{-1}(s,t,x)\|^p = \lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{0 \leq s \leq T} \|\nabla X^{-1}_n(s,t,x) - \nabla X^{-1}(s,t,x)\|^p = 0. \tag{3.52}
\]

Moreover, for every \( x,y \in \mathbb{R}^d \), there is another constant \( C(p,d,T) > 0 \) such that

\[
\sup_{0 \leq s \leq T} \mathbb{E} \left[ \sup_{s \leq t \leq T} |X^{-1}(s,t,x) - X^{-1}(s,t,y)|^p \right] \leq C(p,d,T) |x - y|^p \tag{3.53}
\]

and

\[
\sup_{0 \leq s \leq T} \mathbb{E} \left[ \sup_{s \leq t \leq T} \|\nabla X^{-1}(s,t,x) - \nabla X^{-1}(s,t,y)\|^p \right] \leq C \left[ \int_{r < |x-y|} \frac{\phi(r)}{r} dr + \phi(|x-y|) + |x-y| \int_{|x-y| < r \leq r_0} \frac{\phi(r)}{r^2} dr \right]^p 1_{|x-y| < r_0} + C |x-y|^p. \tag{3.54}
\]

**Remark 3.2** (i) Let \( r_0 = 1/2 \) and \( \phi(r) = C |\log(r)|^{-\alpha} \) for \( r \in (0,r_0) \) with some \( \alpha > 1 \). From Remark 2.3 (ii)

\[
\int_{r < |x-y|} \frac{\phi(r)}{r} dr + \phi(|x-y|) + |x-y| \int_{|x-y| < r \leq r_0} \frac{\phi(r)}{r^2} dr + |x-y| \leq \frac{C(\alpha)}{|\log(|x-y|)|^{\alpha-1}}. \]

Let \( F_\delta \) be defined by (3.3). We get

\[
F_\delta(r) \leq \frac{C}{|\log(r)|^{\alpha-1}} =: \tilde{F}_\delta(r), \quad r \in [0,\delta], \quad \delta < \frac{1}{2}. \tag{3.55}
\]

Notice that given \( p \geq 1 \), if \( r \in (0,\delta) \) and \( \delta < \exp(p-p\alpha-1) \),

\[
\frac{d}{dr} \frac{1}{|\log(r)|^{p(\alpha-1)}} = \frac{p(\alpha-1)}{r|\log(r)|^{p(\alpha-1)+1}} \geq 0 \tag{3.56}
\]

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and
\[
\frac{d}{dt^2} \frac{1}{\log(r)|p^{(\alpha-1)}} = \frac{p(\alpha-1)}{|\log(r)|^{p(\alpha-1)+1}r^2} \left[ \frac{p\alpha - p + 1}{|\log(r)|} - 1 \right] \leq 0, \tag{3.57}
\]
then the function \( F^{p}_{x}(r) = C(p\log(r))^{-p(\alpha-1)} \) is increasing and concave (we define \(|\log(0)|^{-1} = 0\) on \([0, \delta]\). Thus the assumptions in Theorem 3.1 hold. Theorem 3.1 is applicable mutatis mutandis.

From Remark 3.2, we draw the following result.

**Corollary 3.2** Let \( b \in L^{\infty}([0, T]; C_{0}(\mathbb{R}^{d}; \mathbb{R}^{d})) \). Suppose that there are two real numbers \( C > 0 \) and \( \alpha > 1 \) such that for every \( x \in \mathbb{R}^{d} \)
\[
|b(t, x) - b(t, y)| \leq \frac{C}{|\log(|x - y|)|^{\alpha}}, \text{ for all } y \in B_{\frac{1}{2}}(x), t \in [0, T]. \tag{3.58}
\]

(i) SDE (3.1) has a unique continuous adapted solution \( \{X(s, t, x), t \in [s, T], \omega \in \Omega\} \), which forms a stochastic diffeomorphisms flow.

(ii) Estimates (3.4)–(3.5) hold for \( X \) and denote its inverse by \( X^{-1} \), then (3.51) and (3.53) hold as well. Moreover, for every \( x, y \in \mathbb{R}^{d} \)
\[
\sup_{0 \leq s \leq T} \mathbb{E} \sup_{s \leq t \leq T} \left[ \|\nabla X(s, t, x) - \nabla X(s, t, y)\| + \|\nabla X^{-1}(s, t, x) - \nabla X^{-1}(s, t, y)\| \right]^{p} \leq C(p, T) \left[ \frac{1_{|x - y| < \frac{1}{2}}}{|\log(|x - y|)|^{p(\alpha - 1)}} + |x - y|^{p(\alpha - 1)} 1_{|x - y| \geq \frac{1}{2}} \right]. \tag{3.59}
\]

4 Stochastic transport equations

**Definition 4.1** Let \( b \in L^{1}([0, T]; L^{1}_{\text{loc}}(\mathbb{R}^{d}; \mathbb{R}^{d})) \) such that \( \text{div} b \in L^{1}([0, T]; L^{1}_{\text{loc}}(\mathbb{R}^{d})) \), and let \( u_{0} \in L^{\infty}(\mathbb{R}^{d}) \). A stochastic field \( u \) is called a weak \( L^{\infty} \)-solution of (1.1) if \( u \in L^{\infty}(\Omega \times [0, T]; L^{\infty}(\mathbb{R}^{d})) \) and for every \( \varphi \in C_{0}^{\infty}(\mathbb{R}^{d}) \), \( \int_{\mathbb{R}^{d}} \varphi(x)u(t, x)dx \) has a continuous modification which is an \( \mathcal{F}_{t} \)-semimartingale and for every \( t \in [0, T] \)
\[
\int_{\mathbb{R}^{d}} \varphi(x)u(t, x)dx = \int_{\mathbb{R}^{d}} \varphi(x)u_{0}(x)dx + \int_{0}^{t} \int_{\mathbb{R}^{d}} \text{div}(b(s, x)\varphi(x))u(s, x)dxds + \sum_{i=1}^{d} \int_{0}^{t} \partial_{x_{i}}\varphi(x)u(s, x)dx, \quad \mathbb{P}-a.s. \tag{4.1}
\]

Then we state our main result.

**Theorem 4.1 (Existence and uniqueness)** Let \( d \geq 1 \). Suppose \( b \in L^{\infty}([0, T]; C_{0}(\mathbb{R}^{d}; \mathbb{R}^{d})) \) such that (3.2) and (3.3) hold. Further suppose that \( \text{div} b \in L^{1}([0, T]; L^{1}_{\text{loc}}(\mathbb{R})) \) for \( d = 1 \) or there exists \( q \in (2, +\infty) \) such that
\[
\text{div} b \in L^{q}([0, T] \times \mathbb{R}^{d}), \quad d \geq 1. \tag{4.2}
\]
Then there exists a unique weak \( L^\infty \)-solution to the Cauchy problem (1.1). Moreover, the unique weak solution can be represented by \( u(t, x) = u_0(X^{-1}(t, x)) \), with \( X(t, x) \) being the unique strong solution of the associated stochastic differential equation (3.1) with \( s = 0 \).

**Proof.** First, we prove that \( u(t, x) = u_0(X^{-1}(t, x)) \) is a weak \( L^\infty \)-solution of (1.1). Let \( b^n \) and \( X^n \) be given in Theorem 3.1, and let \( X_n^{-1} \) be the inverse of \( X^n \). Since \( b^n \) is smooth, \( u^n = u_0(X_n^{-1}) \) is the unique weak \( L^\infty \)-solution of the following Cauchy problem ([18, Theorems 16 and 20])

\[
\begin{aligned}
\partial_t u^n(t, x) + b^n(t, x) \cdot \nabla u^n(t, x) + \sum_{i=1}^d \partial_{x_i} u^n(t, x) \circ \dot{B}_i(t) &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
|u^n(t, x)|_{t=0} &= u_0(x), \quad x \in \mathbb{R}^d.
\end{aligned}
\]

(4.3)

Then for every \( \varphi \in C_0^\infty(\mathbb{R}^d) \) and every \( t \in [0, T] \)

\[
\int_{\mathbb{R}^d} \varphi(x) u_0(X_n^{-1}(t, x)) dx = \int_{\mathbb{R}^d} \varphi(x) u_0(x) dx + \int_0^t \int_{\mathbb{R}^d} \text{div}(b^n(s, x) \varphi(x)) u_0(X_n^{-1}(s, x)) dx ds
\]

\[
+ \sum_{i=1}^d \int_0^t \partial_i B_i(s) \int_{\mathbb{R}^d} \partial_{x_i} \varphi(x) u_0(X_n^{-1}(s, x)) dx ds
\]

\[
= \int_{\mathbb{R}^d} \varphi(x) u_0(x) dx + \int_0^t \int_{\mathbb{R}^d} \text{div}(b^n(s, x) \varphi(x)) u_0(X_n^{-1}(s, x)) dx ds
\]

\[
+ \sum_{i=1}^d \int_0^t dB_i(s) \int_{\mathbb{R}^d} \partial_{x_i} \varphi(x) u_0(X_n^{-1}(s, x)) dx ds
\]

\[
+ \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^d} \Delta \varphi(x) u_0(X_n^{-1}(s, x)) dx \quad \mathbb{P} - \text{a.s.}
\]

(4.4)

where in the last inequality we have used the relationship between the Stratonovich integral and the Itô integral.

By (2.29) and (4.2)

\[
\lim_{n \to \infty} \|b^n - b\|_{L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d))} = 0
\]

(4.5)

and

\[
\lim_{n \to \infty} \|\text{div}(b^n \varphi) - \text{div}(b \varphi)\|_{L^1([0, T]; L^1(\mathbb{R}^d))} = 0.
\]

(4.6)

Now thanks to Corollary 3.1, by taking \( n \) to infinity in (4.4), (4.1) holds for \( u(t, x) = u_0(X^{-1}(t, x)) \).

It remains to check the uniqueness and observing that the equation is linear, it suffices to prove that \( u \equiv 0 \) a.s. if the initial data vanishes. We only check the uniqueness for \( d > 1 \), the case \( d = 1 \) being similar and easier. Let \( g_n \) be given by (2.8) and set \( u_n = u \ast g_n \), then

\[
\partial_t u_n(t, x) + b(t, x) \cdot \nabla u_n(t, x) + \sum_{i=1}^d \partial_{x_i} u_n(t, x) \circ \dot{B}_i(t) = e_n(t, x),
\]

(4.7)
with

\[ e_n(t, x) = b(t, x) \cdot \nabla u_n(t, x) - (b \cdot \nabla) u_n(t, x). \]  

(4.8)

For every \( t > 0 \), the Itô’s formula yields that

\[ u_n(t, X(t, x)) = \int_0^t e_n(s, X(s, x)) ds, \]

which implies for every \( \varphi \in C^\infty_0(\mathbb{R}^d) \) and almost all \( \omega \in \Omega \)

\[
\int_{\mathbb{R}^d} u_n(t, X(t, x)) \varphi(x) dx
\]

\[
= \int_0^t \int_{\mathbb{R}^d} e_n(s, X(s, x)) \varphi(x) dx ds
\]

\[
= \int_0^t \int_{\mathbb{R}^d} e_n(s, x) \varphi(X^{-1}(s, x)) \det(\nabla_x X^{-1}(s, x)) dx ds
\]

\[
= - \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ \text{div}b(s, x) - \text{div}b(s, y) \right] \varphi(X^{-1}(s, x)) \det(\nabla_x X^{-1}(s, x)) u(s, y) g_n(x - y) dy dx ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [b(s, x) - b(s, y)] \cdot \nabla_x (\varphi(X^{-1}(s, x))) \det(\nabla_x X^{-1}(s, x)) u(s, y) g_n(x - y) dy dx ds
\]

\[
= : I_1^n(t) + I_2^n(t). \]  

(4.9)

In view of Theorem 3.1, \( \varphi(X^{-1}(s, x)) \det(\nabla_x X^{-1}(s, x)) \) is continuous in \((s, x)\). We claim that for almost all \( \omega \in \Omega \), \( \varphi(X^{-1}(s, \cdot)) \) has a compact support in \( x \) uniformly in \( s \). In fact, without loss of generality we assume there is a real number \( R > 0 \) such that the support of \( \varphi \) is in \( B_R \), if the assertion is false, there is a sequence \( \{(t_k, x_k)\}_{k \geq 1} \subset [0, t] \times B_R \) such that \( \lim_{k \to \infty} |X(\omega, t_k, x_k)| = +\infty \). But on the other hand, by Theorem 3.1, for every \( k \geq 1 \)

\[
|X(\omega, t_k, x_k)| \leq |X(\omega, t_k, x_k) - X(\omega, t_k, x_1)| + |X(\omega, t_k, x_1)|
\]

\[
\leq C|x_k - x_1| + |X(\omega, t_k, x_1)|
\]

\[
\leq 2CR + \sup_{0 \leq s \leq T} |X(\omega, s, x_1)| \leq C.
\]

Therefore, by (4.2), taking \( n \) to infinity yields \( I_1^n(t) \to 0 \), \( \mathbb{P} \)-a.s.

Noticing that

\[
\nabla_x (\varphi(X^{-1}(s, x))) \det(\nabla_x X^{-1}(s, x))
\]

\[
= \nabla_{X^{-1}} \varphi(X^{-1}(s, x)) \nabla_x X^{-1}(s, x) \det(\nabla_x X^{-1}(s, x)) + \varphi(X^{-1}(s, x)) \nabla_x (\det(\nabla_x X^{-1}(s, x))),
\]

if for almost all \( \omega \in \Omega \), \( \nabla_x (\det(\nabla_x X^{-1}(\cdot, \cdot))) \in L_1([0, T]; L_{loc}^1(\mathbb{R}^d)) \), by the dominated convergence theorem, \( I_2^n(t) \to 0 \), \( \mathbb{P} \)-a.s. Now let us check \( \nabla_x (\det(\nabla_x X^{-1}(\cdot, \cdot))) \in L_1([0, T]; L_{loc}^1(\mathbb{R}^d)) \) and it is
equivalent to show that $\nabla_x \det(\nabla_x X, \cdot) \in L^1([0, T]; L^1_{loc}(\mathbb{R}^d))$. From (3.50)

$$
\nabla_x \det(\nabla_x X(t, x)) = \det(\nabla_x X(t, x)) \nabla_x \int_0^t \text{div} b(s, X(s, x)) ds.
$$

By Theorem 3.1, $\det(\nabla_x X(s, x))$ is continuous in $(s, x)$, we need to show

$$
\nabla_x \int_0^t \text{div} b(s, X(s, x)) ds \in L^1([0, T]; L^1_{loc}(\mathbb{R}^d)), \quad \mathbb{P} - \text{a.s.}.
$$

To do this we consider the following backward parabolic equation

$$
\begin{cases}
\partial_t V(t, x) + \frac{1}{2} \Delta V(t, x) + b(t, x) \cdot \nabla V(t, x) = \text{div} b(t, x), & (t, x) \in (0, T) \times \mathbb{R}^d, \\
V(T, x) = 0, & x \in \mathbb{R}^d.
\end{cases}
$$

(4.11)

Noticing $\text{div} b \in L^q([0, T] \times \mathbb{R}^d)$, there is a unique $V \in L^q([0, T]; W^{2, q}(\mathbb{R}^d)) \cap W^{1, q}([0, T]; L^q(\mathbb{R}^d))$ solving (4.11), and there is a constant $C(q, T) > 0$ such that

$$
\|V\|_{L^q([0, T]; W^{2, q}(\mathbb{R}^d))} + \|\partial_t V\|_{L^q([0, T] \times \mathbb{R}^d)} \leq C(q, T) \|\text{div} b\|_{L^q([0, T] \times \mathbb{R}^d)}.
$$

(4.12)

Moreover, by Sobolev’s imbedding theorem,

$$
\sup_{0 \leq t \leq T} \|V\|_{W^{1, q}(\mathbb{R}^d)} \leq C(q, T) \|\text{div} b\|_{L^q([0, T] \times \mathbb{R}^d)}.
$$

(4.13)

Next we assume that for every $t \in [0, T]$, $V(t)$ is smooth, otherwise one can follow an approximation argument, then the Itô’s formula yields

$$
V(t, X(t, x)) - V(0, x) - \int_0^t \nabla V(s, X(s, x)) dB(s) = \int_0^t \text{div} b(s, X(s, x)) ds,
$$

which implies that

$$
\nabla_x \int_0^t \text{div} b(s, X(s, x)) ds = \nabla_X V(t, X(t, x)) \nabla_x X(t, x) - \nabla_x V(0, x)
$$

- $\int_0^t \nabla^2_X V(s, X(s, x)) \nabla_x X(s, x) dB(s).
$$

(4.14)

By Theorem 3.1, (4.12) and (4.13), for almost all $\omega \in \Omega$, the first two terms in the righthand hand side in (4.14) are in $L^1([0, T]; L^1_{loc}(\mathbb{R}^d))$. For every $R > 0$

$$
\mathbb{E} \left\| \int_0^T \int_{|x| \leq R} \left( \int_0^t \nabla^2_X V(s, X(s, x)) \nabla_x X(s, x) dB(s) \right)^2 dx dt \right\|^2
$$

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\[ = \int_0^T \int_{|x| \leq R} \mathbb{E} \int_0^t |\nabla_X^2 V(s, X(s, x))\nabla_X X(s, x)|^2 \text{d}sd\text{d}t \]

\[ \leq C \int_0^T \int_{|x| \leq R} \left[ \mathbb{E}\|\nabla_X^2 V(s, X(s, x))\|^q \right]^{\frac{2}{q}} \left[ \mathbb{E}\|\nabla_X X(s, x)\|^\frac{2p}{p-q} \right]^{\frac{p-q}{q}} \text{d}sx \]

\[ \leq C \sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{0 \leq s \leq T} |\nabla_X X(s, x)|^q \left[ \int_0^T \int_{|x| \leq R} |\nabla_X^2 V(s, X(s, x))|^q \text{d}sx \right]^{\frac{2}{q}} \]

\[ \leq C \left[ \sup_{x \in \mathbb{R}^d} \mathbb{E} \sup_{0 \leq s \leq T} |\nabla_X X^{-1}(s, x)|^q \right]^{\frac{2}{q}} \left[ \int_0^T \int_{|x| \leq R} |\nabla_X^2 V(s, x)|^q \text{d}dx \right]^{\frac{2}{q}} < +\infty, \quad (4.15) \]

where in the fifth line we have used (3.4), and in the last inequality we have used (3.51) and (4.12).

Then from (4.9)–(4.15), by taking \( n \) to infinity, we have \( \int_{\mathbb{R}^d} u(t, X(t, x))\varphi(x)\text{d}x = 0 \), that is \( u(t, X(t, x)) = 0 \) for almost everywhere \( x \in \mathbb{R}^d \) and almost all \( \omega \in \Omega \). Because \( X(t, x) \) is a stochastic diffeomorphisms flow associated with (3.1) with \( s = 0 \), we have \( u(t, x) = 0 \) for almost everywhere \( x \in \mathbb{R}^d \) and almost all \( \omega \in \Omega \). The proof is complete. \( \Box \)

**Remark 4.1** Without the stochastic perturbation, even if the drift is bounded and Hölder continuous, the deterministic equation possesses multiple \( L^\infty \)-solutions ([18, Section 6.1]). So the noise has a regularization effect.

**Remark 4.2** (i) SDE (3.1) with \( s = 0 \) has a unique continuous adapted solution \( \{X(t, x), t \in [0, T], \omega \in \Omega\} \), which forms a stochastic flow of diffeomorphisms, and by Theorem 4.1 (1.1) has a unique weak \( L^\infty \)-solution which can be written by \( u_0 X^{-1}(t, x) \). Thus, if \( u_0 \in C_b(\mathbb{R}^d) \), for almost all \( \omega \in \Omega \), and every \( t \in [0, T] \), \( u(t, \cdot) \) is bounded and continuous. Moreover, if \( u_0 \in W^{1,p}(\mathbb{R}^d) \) with \( p \in [1, +\infty) \), we have the following chain rule

\[ \nabla_x (u_0(X^{-1}(t, x))) = \nabla_x u_0(X^{-1}(t, x)) \nabla_x X^{-1}(t, x), \]

so \( u(t, \cdot) \in W^{1,p}_{\text{loc}}(\mathbb{R}^d) \) almost surely, for every \( t \in [0, T] \). However, for the deterministic equation, even if the uniqueness is established, the persistence of the above properties (continuity and Sobolev differentiability) for solutions are missing [8].

(ii) We can further establish the existence and uniqueness of \( W^{1,p} \)-solutions with \( p \in [1, +\infty) \) as well if \( b \in L^\infty([0, T]; C_b(\mathbb{R}^d; \mathbb{R}^d)) \) such that (3.2) holds. More precisely there is a unique \( u \in L^\infty(\Omega \times [0, T]; L^p(\mathbb{R}^d)) \) such that

(1) for every \( \varphi \in C_0^\infty(\mathbb{R}^d) \), \( \int_{\mathbb{R}^d} \varphi(x) u(t, x) \text{d}x \) has a continuous modification which is an \( \mathcal{F}_t \)-semimartingale and for every \( t \in [0, T] \), (4.1) holds;

(2) for almost all \( \omega \in \Omega \), \( \nabla u \in L^\infty([0, T]; L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)) \).
The proof is similar to that of Theorem 4.1 ([35, Theorem 1.1], [18, Theorem 25]). However, if \( d \geq 2 \), the deterministic equation does not exist such strong solutions ([35, Theorem 1.2]), which means that noise prevents the singularity for solutions.

(iii) There are also some results on \( \cap_{p \geq 1} W_{loc}^{1,p} \) solutions ([16]) and \( L^p \) solutions ([6]).

5 Conclusions

Recently, there have been a broad research on the uniqueness of \( L^\infty \) solutions for the stochastic transport equation

\[
\begin{aligned}
\partial_t u(t, x) + b(t, x) \cdot \nabla u(t, x) + \sum_{i=1}^d \partial_{x_i} u(t, x) \circ \dot{B}_i(t) &= 0, & (t, x) &\in (0, T) \times \mathbb{R}^d, \\
|u(t, x)|_{t=0} = u_0(x), & x &\in \mathbb{R}^d,
\end{aligned}
\]

(5.1)

with non-Lipschitz drift. Most of these works are concentrated on the drift which is Hölder continuous in spatial variable uniformly in time. The question for the uniqueness when \( b \) is only bounded is still open. In this study, we established the existence and uniqueness of \( L^\infty \) solutions only assuming \( b \) is bounded and Dini continuous in spatial variable uniformly in time. Compared with the existing research, the result is new.

We solve the Cauchy problem (5.1) by the method of stochastic characteristics. Therefore, we should prove the existence of stochastic diffeomorphisms flow for the following stochastic differential equation

\[
\begin{aligned}
dX(t) = b(t, X(t)) dt + dB(t), & t &\in (0, T], \\
X(t)|_{t=0} = x.
\end{aligned}
\]

(5.2)

To reach the goal, we use the Itô-Tanaka trick to transform the SDE (5.2) with bounded and Dini continuous drift to an equivalent new SDE with Lipschitz coefficients via a non-singular diffeomorphism \( \Phi(t, x) = x + U(t, x) \), where \( U(T - t, x) =: V(t, x) \) satisfies a vector-valued parabolic partial differential equation of second order which has the form

\[
\begin{aligned}
\partial_t V(t, x) &= \frac{1}{2} \Delta V(t, x) + b(t, x) \cdot \nabla V(t, x) + b(t, x) - \lambda V(t, x), & (t, x) &\in (0, T) \times \mathbb{R}^d, \\
V(0, x) &= 0, & x &\in \mathbb{R}^d.
\end{aligned}
\]

(5.3)

There are two things we need to do. The first one is choosing a proper function space on which the Itô formula is applicable, and the second one is the boundedness estimate for the gradient of \( V \). We accomplish these issues by fetching \( L^\infty(0, T; \mathcal{C}_b^2(\mathbb{R}^d)) \cap W^{1,2}(0, T; \mathcal{C}_b(\mathbb{R}^d)) \) as the workspace. When \( b \in L^\infty(0, T; \mathcal{C}_b^2(\mathbb{R}^d; \mathbb{R}^d)) \), these estimates for solutions have been established by Flandoli, Gubinelli and Priola [18]. Noticing that, here we only assume that \( b \) is Dini continuous in \( x \), so we should extend the Schauder theory for (5.3) to \( W^{2,\infty} \) theory. We accomplish these estimates in Section 2, and then establish the stochastic diffeomorphisms flow for (5.2) in Section 3. These results are new as well.

We remark that the method used to establish \( W^{2,\infty} \) estimates for (5.3) can be applied to found the \( W^{1,\infty} \) estimates for solutions of second order parabolic equations driven by Browian motion or general Lévy noise. The \( W^{1,p} \) (\( p \in [2, \infty) \)) theory, stochastic BMO estimates and Schauder theory
have been founded by many researchers, see [22, 21, 13, 20, 34]. There are few works to deal the $W^{1,\infty}$ estimates. Therefore, the study of the $W^{1,\infty}$ property of solutions to stochastic parabolic equations is of very high importance. For simplicity, and without loss of generality, here we only give a brief calculation for the $W^{1,\infty}$ estimate for heat equation driven by Brownian noise:

\[
\begin{cases}
  du(t, x) = \frac{1}{2} \Delta u(t, x) dt + f(t, x) dB(t), & (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(0, x) = 0, & x \in \mathbb{R}^d,
\end{cases}
\]  

(5.4)

where $f$ is bounded and Dini continuous in $x$ uniformly in $t$. From (5.4), then

\[
u(t, x) = \int_0^t K(t - s, \cdot) \ast f(s, \cdot)(x) dB(s), \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\]

(5.5)

where $K(t, x) = (2\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}$, $t > 0$, $x \in \mathbb{R}^d$. For $1 < i < d$, we first differentiate $u$ in $x_i$, and then use the Itô isometry, to get

\[
\mathbb{E} |\partial_{x_i} u(t, x)|^2
\]

\[
= \int_0^t \int_{\mathbb{R}^d} |\partial_{x_i} K(t - s, x - y) f(s, y) dy|^2 ds
\]

\[
= \int_0^t \int_{|x - y| > (t-s)\theta} \partial_{x_i} K(t - s, x - y) [f(s, y) - f(s, x)] dy ds
\]

\[
+ \int_0^t \int_{|x - y| \leq (t-s)\theta} \partial_{x_i} K(t - s, x - y) [f(s, y) - f(s, x)] dy ds
\]

\[
\leq 2 \int_0^t \int_{|x - y| > (t-s)\theta} \partial_{x_i} K(t - s, x - y) [f(s, y) - f(s, x)] dy ds
\]

\[
+ 2 \int_0^t \int_{|x - y| \leq (t-s)\theta} \partial_{x_i} K(t - s, x - y) [f(s, y) - f(s, x)] dy ds,
\]

where $\theta \in (0, 1/2)$. Similar calculations from (2.12) to (2.17) used here again, we conclude that

\[
\mathbb{E} |\partial_{x_i} u(t, x)|^2 < \infty.
\]

Moreover, we also get an analogue of (2.7) for $\partial_{x_i} u$, i.e. for every $x, y \in \mathbb{R}$ and every $t \in [0, T]$

\[
\left[\mathbb{E} |\partial_{x_i} u(t, x) - \partial_{x_i} u(t, y)|^2\right]^{\frac{1}{2}}
\]

\[
\leq C \left[ \int_{\frac{1}{2}|x - y|}^{|x - y|} \phi(r) \frac{dr}{r} + \phi(|x - y|) + |x - y| \int_{|x - y| < r \leq r_0} \phi(r) \frac{dr}{r^2} \right] 1_{|x - y| < r_0} + C|x - y|.
\]

Moreover, if $f$ is Hölder-Dini or strong Hölder or weak Hölder continuous with the Hölder-Dini or strong Hölder or weak Hölder function $\phi$, then

\[
\left[\mathbb{E} |\partial_{x_i} u(t, x) - \partial_{x_i} u(t, y)|^2\right]^{\frac{1}{2}} \leq C \phi(|x - y|).
\]

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