Quadratic and cubic spherically symmetric black holes in the modified teleparallel equivalent of general relativity: energy and thermodynamics

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Abstract
In Bahamonde et al (2019 arXiv:1907.10858 [gr-qc]), a spherically symmetric black hole (BH) was derived from the quadratic form of $f(T)$. Here we derive the associated energy, invariants of curvature, and torsion of this BH and demonstrate that the higher-order contribution of torsion renders the singularity weaker compared with the Schwarzschild BH of general relativity (GR). Moreover, we calculate the thermodynamic quantities and reveal the effect of the higher-order contribution on these quantities. Therefore, we derive a new spherically symmetric BH from the cubic form of $f(T) = T + \epsilon \left( \frac{1}{2} \alpha T^2 + \frac{1}{3} \beta T^3 \right)$, where $\epsilon \ll 1$, $\alpha$, and $\beta$ are constants. The new BH is characterized by the two constants $\alpha$ and $\beta$ in addition to $\epsilon$. At $\epsilon = 0$ we return to GR. We study the physics of these new BH solutions via the same procedure that was applied for the quadratic BH. Moreover, we demonstrate that the contribution of the higher-order torsion, $\frac{1}{3} \alpha T^2$ and $\frac{1}{3} \beta T^3$, may afford an interesting physics.

Keywords: $f(T)$ gravitational theories, asymptotes solutions, thermodynamics

1. Introduction
The black hole (BH) is the most interesting phenomena in the general relativity (GR) of Einstein and other modified gravitational theories [2–5]. This profound and sustained interest in the different approaches to BH physics can be investigated because of its relevance in astrophysics [6] and the numerous applications and methods that have initially evolved in the

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gravitational theories of different systems. Regarding the BH many topics including the horizons global structure, Hawking radiation and thermodynamic properties which are considered as the main goals for realizing the form of the space–time can be studied [7]. Although GR is a satisfactory gravitational theory, modified theories remain greatly desired. It has been proven that GR is a successful theory for isolated masses with length scales of the Solar System; however, this theory still faces disputes in the domains of cosmology and quantum scales.

There are many alternative gravitational theories to GR in the literature and the teleparallel equivalent of general relativity (TEGR), which was applied by Einstein in (1928) to unify gravity and electromagnetism [8] is among these theories. In this theory, the tetrad field is used as a dynamic field instead of the metric of GR. The affine connection is defined regarding the nonsymmetric Weitzenböck connection in the TEGR theory whereas the Live–Civita connection plays the affine connection in GR. The use of the Weitzenböck connection affords a space–time without curvature and the gravitational field is encoded in the torsion tensor, which is the difference between two Weitzenböck connections. The Lagrangian of the TEGR theory depends on the torsion scalar, $T$, which is mainly constructed from the torsion tensor. There are many physical models in the Solar System regarding the TEGR theory, as well as in cosmology [9–21].

Similar to $f(R)$ which is a generalization of GR in which the Palatini action that depends on the Ricci scalar, $R$, is replaced by an arbitrary analytic differentiable function [22–26], there is a generalization of the TEGR theory, which is called the $f(T)$ gravitational theory [22,27–30]. Although TEGR is formulated from a geometry (Weitzenböck geometry) that is different from GR (the Riemann geometry) both theories are equivalent from the viewpoint of field equations. Nevertheless, when we assume the generic forms of $f(R)$ and $f(T)$, the two theories become inequivalent [31–33]. The $f(T)$ theory is an interesting method of solving the challenges of dark energy and dark matters [34–41]. Moreover, in the frame of $f(T)$ there are many interesting BH and cosmological solutions that have interesting physics [42–52]. Here we considered the cubic form of the $f(T)$ gravitational theories in the framework of the spherically symmetric space–time to derive a new spherically symmetric BH. Additionally, we considered the BH that was derived in [1] to calculate its associated energy, invariants of curvature, and torsion, thereby demonstrating and show how the higher-order contribution of torsion.

This study is arranged as follows. In section 2, we briefly reviewed the TEGR and $f(T)$ formalisms. In section 3, we discussed the application of a tetrad field that possesses spherical symmetry in four-dimensions to the vacuum field equations of the $f(T)$ gravity after which the estimated solution of four-dimensions is derived for the quadratic form of $f(T)$ [1]. This BH asymptotically behaves as the flat space–time. The physical properties of this BH were studied by calculating its invariants in section 3.1. In sections 3.2 and 3.3 we calculated the energy content and derived the stability condition via the geodesic deviation respectively. In section 4, we presented the cubic form of the field equation, using $f(T) = T + \epsilon \left[ \alpha T^2 + \beta T^3 \right]$, and derived a new BH solution, up to $O(\epsilon)$, for these differential equations. In section 5 the physical properties of the BH which was derived in the cubic form were discussed and analyzed. In sections 6.1 and 6.2, we calculated the thermodynamic quantities like, Hawking temperature, entropy, heat capacity and Gibbs energy for the quadratic, and the cubic BH solutions, respectively. The final section was devoted to the discussion and conclusion.

2. $f(T)$ theory

Einstein used the TEGR theory, which is a gauge theory [8] to fulfill his dream of unifying gravity with electromagnetism. In this theory, $T$ was responsible for the gravitational field in
the same way $R$ was in the Riemann geometry. The affine connection of the TEGR theory (the Weitzenböck connection) is defined by the following connection \[ W^\lambda_{\mu\nu} := e_i^\lambda \frac{\partial}{\partial x^\mu} e_i^\nu, \] (1)

where $e_i^\mu$ is the dynamical tetrad in four-dimensions. The metric space–time is defined regarding the tetrad as follows:

\[ g_{\mu\nu} := \eta_{ij} e^i_\mu e^j_\nu, \] (2)

where $\eta_{ij} = (+, -, -, -)$ is a four-dimensional Minkowskian metric of the tangent space. Using the Weitzenböck connection, the torsion and contortion tensors are defined as follows:

\[ T^\alpha_{\mu\nu} := W^\alpha_{\nu\mu} - W^\alpha_{\mu\nu} = e_i^\alpha \left( \frac{\partial}{\partial x^\mu} e_i^\nu - \frac{\partial}{\partial x^\nu} e_i^\mu \right), \] (3)

\[ \gamma^\mu_{\nu\alpha} := -\frac{1}{2} \left( T^\mu_{\nu\alpha} - T^\nu_{\mu\alpha} - T^\alpha_{\mu\nu} \right). \] (4)

It is well known that the difference between Weitzenböck and Levi–Civita connections reproduces the contortion, as follows:

\[ \gamma^\mu_{\nu\alpha} = W^\mu_{\nu\alpha} - \Gamma^\mu_{\nu\alpha}. \] (5)

The superpotential tensor $\Sigma^\mu_{\nu\alpha}$ which is the antisymmetric tensor in the last two indices is defined as follows:

\[ \Sigma^\mu_{\nu\alpha} := \frac{1}{2} \left( \gamma^\mu_{\nu\alpha} + \delta^\mu_\nu T^\alpha_{\beta\mu} - \delta^\mu_\alpha T^\beta_{\nu\mu} \right). \] (6)

Using all the above data we can define $T$ of the TEGR theory according to equation (6):

\[ T := T^\alpha_{\mu\nu} \Sigma^\mu_{\nu\alpha}, \] (7)

Similar to the GR extension ($f(R)$) it was logical to modify the TEGR theory to include higher torsion orders, which enabled us to define the Lagrangian of $f(T)$, where $f$ is an analytical continues differentiable function of $T$:

\[ \mathcal{L} = \frac{1}{2\kappa} \int |e| f(T) \, d^4x, \] (8)

where $|e| = \sqrt{-g} = \det (e^\mu_\nu)$ is the determinate of the metric and $\kappa$ is a four-dimensional constant that is defined as $\kappa = \frac{8\pi G}{c^2}$, where $G$ is the Newtonian gravitational constant in four-dimensions and $c$ is the speed of light. The variation of equation (7) w.r.t. the tetrad field, $e^\mu_\nu$ affords the following field equations of $f(T)$ in the vacuum case as shown in [27]:

\[ S^\mu_\nu \partial_\mu T f_T \partial_\nu T f_T \partial_\nu T f_T \partial_\mu T f_T - f_T \partial_\mu T f_T - \frac{f}{4} \delta^\nu_\mu = \xi^\nu_\mu \equiv 0, \] (9)

where $f := f(T)$, $f_T := \frac{\partial f(T)}{\partial T}$ and $f_{TT} := \frac{\partial^2 f(T)}{\partial T^2}$. The application of the field equation (8) to a spherically symmetric tetrad field using the form of $f(T) = T + \epsilon \left[ \frac{1}{2} \alpha T^2 + \frac{1}{3} \beta T^3 \right]$ is shown in section 4.
3. Asymptotically stationary AdS black holes

In this section, the field equations of the higher order torsion theory, (equation (8)) were applied to the spherically symmetric space–time, thus affording the vielbein which is written in the spherical coordinate \((t, r, \theta, \phi)\) as shown in [1]:

\[
e^{\mu\nu} = \begin{pmatrix}
\sqrt{\tau} & 0 & 0 & 0 \\
0 & \sqrt{\tau} \cos(\theta) \sin(\theta) & r \cos(\theta) \cos(\theta) & -r \sin(\phi) \sin(\theta) \\
0 & 0 & \sqrt{\tau} \sin(\theta) & r \sin(\theta) \cos(\theta) \\
0 & 0 & 0 & \sqrt{\tau} \cos(\theta) & -r \sin(\phi) \sin(\theta)
\end{pmatrix}, \tag{9}
\]

where \(-\infty < t < \infty\), \(0 \leq r < \infty\), \(0 \leq \theta < \pi\), \(0 \leq \phi < 2\pi\), \(\mu\) and \(\nu\) are the two unknown functions of \(r\). Thus, the space–time, which can be generated by (9) is expressed as follows:

\[
dx^2 = \mu(r) \, dt^2 - \nu(r) \, dr^2 - r^2 \, d\Sigma^2,
\]

where \(d\Sigma^2 = d\theta^2 + \sin^2 \theta \, d\phi^2\) is the two dimensional sphere. Substituting equation (9) into equation (6), we evaluate \(T\) as follows:

\[
T = -2 \left( \sqrt{\nu(r)} - 1 \right) \left( r \mu'(r) - \mu(r) \sqrt{\nu(r)} + \mu(r) \right) \frac{1}{r^2 \mu(r) \nu(r)}.
\]

Applying equation (9) to the vacuum field equation (8) the following nonvanishing components could be obtained [1]:

\[
\xi^t_t = \frac{r \beta (\sqrt{\nu} - 1) \mu' + \mu(r \nu' + 2 \nu^{3/2} - 2 \nu)}{2 r^2 \mu^2} f_T + \frac{\left( \sqrt{\nu} - 1 \right)}{r \nu} T' f_{TT} + \frac{1}{4} f_T,
\]

\[
\xi^t_r = - \frac{r (\sqrt{\nu} - 2) \mu' + 2 \mu (\sqrt{\nu} - 1)}{2 r^2 \mu} f_T - \frac{1}{4} f_T
\]  \(\xi^\theta_\theta = \xi^\phi_\phi\)

\[
= \frac{-r^2 \nu \mu'^2 + r \nu (r \mu' - 4 \nu^{3/2} \mu' + 2 \nu (r \mu'' + 3 \mu')) + \mu^2 \left( -2 r \nu' - 8 \nu^{3/2} + 4 \nu^2 + 4 \nu \right)}{8 r^2 \mu^2 \nu^2} f_T
\]

\[+ \frac{r \mu' - 2 \mu (\sqrt{\nu} - 1)}{4 r \mu(r \nu(r))} T' f_{TT} - \frac{1}{4} f_T.
\]

Bahamonde et al [1] have solved the above system when \(f(T) = T + \frac{1}{4} \epsilon \alpha T^p\). In this section, we discussed the physics of the BH solution that was derived in [1] for the case \('p = 2'\):

\[
\mu(r) = \sigma + \epsilon \sigma_1, \quad \nu(r) = \sigma^{-1} + \epsilon \sigma_2,
\]

where

\[
\sigma = 1 - \frac{2m}{r}.
\]

1The abbreviations are represented as follows \(\mu(r) \equiv \mu, \nu(r) \equiv \nu, \mu' \equiv \frac{\partial \mu}{\partial r}\) and \(\nu' \equiv \frac{\partial \nu}{\partial r}\).
\[
\sigma_1 = \left[ -\frac{c_1}{r} + c_2 - \alpha \left( \frac{m^2 + 6mr + r^2}{mr^3} - \frac{16(1 - \frac{2m}{r})^{3/2}}{3m^2} \right) \\
+ \frac{(1 - \frac{2m}{r^3}) \ln \left( 1 - \frac{2m}{r} \right)}{2m^2} \left( 1 - \frac{2m}{r} \right) \right],
\]

\[
\sigma_2 = \left[ \frac{8c_1 - \frac{2c_3 c_1}{r}}{1 - \frac{2m}{r}} - \alpha \left( \frac{8(3m^2 - 7mr + 2r^2)}{3m^3 \left( 1 - \frac{2m}{r} \right)^{3/2}} \right) \\
+ \frac{25m - 23r}{r^3 \left( 1 - \frac{2m}{r} \right)^2} + \frac{\ln \left( 1 - \frac{2m}{r} \right)}{2mr \left( 1 - \frac{2m}{r} \right)^2} \right],
\]

\[
(16)
\]

with \(c_1\) and \(c_2\) are the constants of the integration. Equation (15) could be reduced to the Schwarzschild BH GR when \(\epsilon = 0\). In the following subsections we extracted the physics of equation (15) when \(\epsilon \neq 0\). From now on we will refer to solution (15) as ‘\(p = 2'\).

3.1. Singularities of ‘\(p = 2'\)

We calculate the invariant of equation (15), up to \(O(\epsilon)\), and obtained the following

\[
T^{\mu\nu\lambda}_\mu - \frac{16em}{r^2} + \frac{2m^2}{r^4} + \epsilon \left[ \frac{8c_1 + 12\alpha \ln(2)/m + 128\alpha/3m - 16mc_2}{r^3} \right. \\
+ \frac{2mc_1 + 3\alpha \ln(2) + 32\alpha/3 - 4m^2c_2}{r^4} \left. \right] + O \left( \frac{1}{r^6} \right),
\]

\[
T^{\nu}\mu = -\frac{16em}{r^2} - \frac{m^2}{r^4} + \frac{4m^3}{r^5} + \epsilon \left[ \frac{64\alpha/3m - 8mc_2 + 6\alpha \ln(2)/m + 4c_1}{r^3} \right. \\
+ \frac{2mc_2 - 16\alpha/3 - 3\alpha \ln(2)/2 - mc_1}{r^4} \left. \right] + O \left( \frac{1}{r^6} \right),
\]

\[
T(r) = \frac{8em}{r^2} + \frac{2m^2}{r^4} + \frac{4m^3}{r^5} + \epsilon \left[ \frac{3\alpha \ln(2) + 2mc_1 + 32\alpha/3 - 4m^2c_2}{r^2} \right. \\
+ \frac{9m \alpha \ln(2) + 6m c_2 + 32mc_1 - 12m c_2}{r^3} \left. \right] + O \left( \frac{1}{r^6} \right),
\]

\[
R^{\mu\nu\lambda\rho}_\mu = -\frac{48m^2}{r^6} + \epsilon \left[ \frac{96m^2 c_2 - 48mc_1 - 256\alpha - 72\alpha \ln(2)}{r^6} \right] + O \left( \frac{1}{r^{10}} \right),
\]

\[
R^{\mu\nu} = O \left( \epsilon^2 \right), \\
R = \epsilon \left[ \frac{16m^3 c_2 + 20m^2 c_1}{r^7} + \frac{20m^4 c}{r^8} \right] + O \left( \frac{1}{r^{10}} \right),
\]

\[
(17)
\]

where \(T^{\mu\nu\lambda}_\mu, T^{\nu}\mu, T, R^{\mu\nu\lambda\rho}_\mu, R^{\mu\nu}\) and \(R\), are the torsion tensor square, torsion vector square, torsion scalar, Kretschmann scalar, Ricci tensor square, and Ricci scalar. The above invariants indicated that there was a singularity (curvature singularity) at \(r = 0\). Close to \(r = 0\), the behaviors of \(T^{\mu\nu\lambda}_\mu, T^{\nu}\mu, T\) are given by \(T^{\mu\nu\lambda\rho}_\mu = T^{\nu}\mu, T = T \sim r^{-2}\).
in contrast with the solutions of the Einstein theory in GR and TEGR, which are given as
\[ T^{\alpha\beta\gamma} T_{\alpha\beta\gamma} = T' T = T \sim r^{-4}, \]
This demonstrates that the singularity of the higher-order torsion theory was much milder than the one obtained in GR and TEGR for the natural spherically symmetric case. This result suggests that these singularities are weak ones, according to Tipler and Krolak [54, 55], and the possibility of extending the geodesics beyond these regions. This would be discussed in subsequent studies.

3.2. Energy content of ‘p = 2’

Here, we calculated the energy content of equation (15). To do this, we applied the Hamiltonian density \( H \) which can be obtained from the Lagrangian equation (7) by rewriting it as follows:

\[ L = p \dot{q} - H. \]

The Hamiltonian of the \( f(T) \) gravitational theory is expressed as follows:

\[ H = e^{\alpha}_0 B^\alpha + \xi_j \Gamma^j^{ij} + \xi_i \Gamma^i, \quad \xi_{ij} = \frac{1}{2} e^\alpha_i e^\beta_j (T_{\alpha\beta\gamma} - T_{\beta\alpha\gamma}), \quad \xi_j = e^\alpha_0 e^\beta_j T_{\alpha\beta\gamma}, \quad (18) \]

where \( \xi_{ij} \) and \( \xi_j \) are the Lagrange multipliers. The components \( e^{\alpha}_0 \) do not exhibit time dependence thus this quantity could be considered as a Lagrange multiplier. The canonically conjugate momenta to \( e^\alpha_\alpha \) are denoted by \( \Pi^{\alpha\beta} \). In the configuration space one can obtain [56]:

\[ \Pi^{ac} = e^{\alpha}_a e^\beta_c \Pi^{\alpha\beta} = -4\pi e^{\alpha}_a e^\beta \Sigma^{\alpha\beta}_{\alpha\beta} f T = -4\pi \Sigma^{\alpha\beta}_{\alpha\beta} f T. \quad (19) \]

Therefore, writing \( H \) in terms of \( e^{\alpha}_a, \Pi^{\alpha\beta} \) and Lagrange multipliers is possible. To express a simple form of \( H \) some Lagrange multipliers and constraints were redefined [57] and equation (18) reads as follows:

\[ H = e^{\alpha}_0 C^\alpha + \frac{1}{2} \lambda_{ab} \Gamma^{ab}, \quad (20) \]

where \( e^{\alpha}_0 \) and \( \lambda_{ab} = -\lambda_{ba} \) are the Lagrange multipliers, and \( C^\alpha \) and \( \Gamma^{ab} \) are the first-class constraints. Solving the Hamilton field equations can aid the identifications of \( \lambda_{ij} = e_i^\alpha e_j^\beta \lambda_{\alpha\beta} = \frac{1}{2} e_i^\alpha e_j^\beta (T_{\alpha\beta\gamma} - T_{\beta\alpha\gamma}) \) and \( \lambda_{ij} = e_i^\alpha e_j^\beta \lambda_{\alpha\beta} = e_i^\alpha e_j^\beta T_{\alpha\beta\gamma}. \) The quantities \( \lambda_{ij} \) and \( \lambda_{ij} \) are the components of equation (21)

\[ \lambda_{\mu\nu} = e^\mu_\rho e^\nu_\sigma \lambda_{\rho\sigma}. \quad (21) \]

The constraint \( C^\alpha \) may be written in the following form:

\[ C^\alpha = -\partial_t \Pi^{\alpha\mu} f_T + f^\alpha, \quad (22) \]

where \( f^\alpha \) is a lengthy expression of the field quantities. Notably \( \partial_t \Pi^{\alpha\mu} \) is the only total divergence term of the momenta \( \Pi^{\alpha\mu} \), which emerged in the expression of \( C^\alpha \). The constraint \( C^\alpha = 0 \) inspired the definition of the gravitational energy–momentum \( P^{\alpha\mu} \) in four-dimensions in the integral form:

\[ P^{\alpha\mu} = -\int_V d^3 x \partial_t \Pi^{\alpha\mu}, \quad (23) \]

where \( V \) is the three-dimensional volume of the space [56].

\(^2\) Notably, the Latin indices were raised and lowered by the Minkowski metric.
Next the energy that was related to the BH was calculated using equation (15). Using equation (23), the necessary components for calculating the energy in the following form could be derived:

\[ \Sigma^{0(0)1} = \frac{r \sin(\theta) \sqrt{\nu}}{\nu}. \] (24)

Substituting equations (15) and (24) into equation (23), we obtained the energy content of the BH (15), up to \( O(\epsilon) \), in the following form:

\[ P^0 = E \approx m + \epsilon \left( \frac{44\alpha + 24mc_1 + 3\alpha \ln(2) - 4m^2c_2}{8m} \right) + O \left( \frac{1}{r} \right), \] (25)

which obtained \( E = m \) when \( \epsilon \rightarrow 0 \) which is ADM (Arnowitt, Deser Misner) mass [58]. Equation (25) is finite value and indicates that the energy depended on the coefficient of the higher-order torsion terms, \( \epsilon \), up to order \( O \left( \frac{1}{r} \right) \). Moreover, equation (23) indicates that the values of \( \epsilon, \alpha \) and \( c_1 \) must be positive and that of \( c_2 \) must be negative value or \( 8m^2 + \epsilon(44\alpha + 24mc_1 + 3\alpha \ln(2)) > 4m^2c_2 \).

### 3.3. Analysis of the stability of the black hole with the geodesic deviation

The paths of a test particle in the gravitational field are described by the following equation:

\[ \frac{d^2 x^\alpha}{d \tau^2} + \left\{ \alpha \beta \rho \right\} \frac{dx^\beta}{d \tau} \frac{dx^\rho}{d \tau} = 0, \] (26)

which is known as the geodesic equations. In equation (26) \( \tau \) represents the affine connection parameter. The geodesic deviation possesses the form [59, 60]

\[ \frac{d^2 \xi^\alpha}{d \tau^2} + 2 \left\{ \frac{\sigma}{\mu \nu} \right\} \frac{dx^\mu}{d \tau} \frac{dx^\nu}{d \tau} + \left\{ \frac{\sigma}{\mu \nu} \right\} \frac{dx^\mu}{d \tau} \frac{dx^\nu}{d \tau} \xi^\alpha = 0, \] (27)

where \( \xi^\alpha \) is the four-vector deviation. Introducing (15) into (26) and (27), obtained the following:

\[ \frac{d^2 t}{d \tau^2} = 0, \quad \frac{1}{2} \mu'(r) \left( \frac{dr}{d \tau} \right)^2 - \nu \left( \frac{d\phi}{d \tau} \right)^2 = 0, \quad \frac{d^2 \theta}{d \tau^2} = 0, \quad \frac{d^2 \phi}{d \tau^2} = 0, \] (28)

and for the BH regarding the geodesic deviation (15) afforded the following:

\[ \frac{d^2 \xi^1}{d \tau^2} + \nu(r)\mu'(r) \frac{dr}{d \tau} \frac{d\xi^0}{d \tau} - 2r\nu'(r) \frac{d\xi^0}{d \tau} \frac{d\xi^3}{d \tau} \]
\[ + \left[ \frac{1}{2} \left( \mu'(r)\nu'(r) + \nu(r)\mu''(r) \right) \left( \frac{dr}{d \tau} \right)^2 - \left( \nu(r) + r\nu'(r) \right) \left( \frac{d\phi}{d \tau} \right)^2 \right] \xi^1 = 0, \]
\[ \times \frac{d^2 \xi^0}{d \tau^2} + \frac{\nu'(r) dr}{\nu(r) d \tau} \frac{d\xi^1}{d \tau} = 0, \quad \frac{d^2 \xi^2}{d \tau^2} + \left( \frac{d\phi}{d \tau} \right)^2 \xi^2 = 0, \quad \frac{d^2 \xi^3}{d \tau^2} + 2 \frac{d\phi}{d \tau} \frac{d\xi^1}{d \tau} = 0, \] (29)

\(^3\)The square parentheses in the quantities \( \Sigma^{0(0)1} \) refer to the tangent components, i.e. \( \Sigma^{0(0)1} = \beta^{10} = \xi^{01} \).
where $\mu(r)$ and $\nu(r)$ were defined from equation (15), $\nu'(r) = \frac{d\alpha(r)}{dr}$. Using the circular orbit

$$\theta = \frac{\pi}{2}, \quad \frac{d\theta}{d\tau} = 0, \quad \frac{dr}{d\tau} = 0,$$

we get

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{\mu'(r)}{r[2\mu(r) - r\mu'(r)]'}, \quad \left(\frac{dt}{d\tau}\right)^2 = \frac{2}{2\mu(r) - r\mu'(r)}.$$

Further, equation (29) can be rewritten as follows:

$$\frac{d^2\xi^1}{d\phi^2} + \mu(r)\mu'(r)\frac{d\xi^0}{d\phi} - 2r\mu(r)\frac{d^2\xi^0}{d\phi^2}$$

$$+ \left[\frac{1}{2} [\mu'^2(r) + \mu(r)\mu''(r)] \left(\frac{dr}{d\phi}\right)^2 - [\mu(r) + r\mu'(r)]\right] \xi^1 = 0,$$

$$\times \frac{d^2\xi^2}{d\phi^2} + \xi^2 = 0, \quad \frac{d^2\xi^0}{d\phi^2} + \mu'(r)\frac{dr}{d\phi} \frac{d\xi^1}{d\phi} = 0, \quad \frac{d^2\xi^3}{d\phi^2} + 2\frac{d\xi^1}{r d\phi} = 0.$$

The second equation of equation (32) corresponds to a simple harmonic motion, which indicates the stability on the plane $\theta = \pi/2$, assuming the remaining equations of (32) obtained solutions in the form of equation (33):

$$\xi^0 = \zeta_1 e^{i\alpha\phi}, \quad \xi^1 = \zeta_2 e^{i\beta\phi}, \quad \text{and} \quad \xi^3 = \zeta_3 e^{i\gamma\phi}.$$
where $\xi_1$, $\xi_2$ and $\xi_3$ are the constants and $\varphi$ is an unknown variable. Substituting equation (33) into (32), the stability condition for static spherically symmetric charged BH can be obtained in the following form:

$$\frac{3\mu\nu\sigma' - \sigma^2 \mu\nu' - 2\nu r^{3/2} \mu^{3/2} - r\mu\nu' + \mu\nu\mu''}{\mu\nu'} > 0.$$  \hspace{1cm} (34)

Equation (34) obtained the following solution:

$$\sigma^2 = \frac{3\mu\nu\sigma' - 2\nu r^{3/2} \mu^{3/2} - r\mu\nu' + \mu\nu\mu''}{\mu^2 \nu^2} > 0.$$  \hspace{1cm} (35)

Figure 1 which exhibits the regions where the BH solutions are stable and the regions where there are no possible stability is a plot of equation (35) for particular values of the model.

4. Cubic solution of equation (8)

In this section, we derived a novel spherically symmetric solution using the form $f(T) = T + \epsilon \left[ \frac{1}{\alpha} T^2 + \frac{1}{\beta} T^3 \right]$. To do this, we derived the cubic form of the field equation (8) as follows:

$$\xi_i = \frac{1}{3\rho \dot{\rho}^2} \left[ 3r^4 \dot{g}^2 \gamma_1 (1 - \gamma - r \dot{g}) + \epsilon \left\{ 12 \rho^2 \dot{g} \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) + 3r^2 \dot{g}^2 \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) \right] + 3r^2 \gamma_1 (1 - \gamma - r \dot{g}) + \epsilon \left\{ 6 \rho^2 \dot{g} \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) \right] + 6 \dot{g} \dot{\beta}^2 \right] \right].$$

$$\xi_i = \frac{1}{3\rho \dot{\rho}^2} \left[ 3r^4 \dot{g}^2 \gamma_1 (1 - \gamma - r \dot{g}) + \epsilon \left\{ 12 \rho^2 \dot{g} \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) \right] + 3r^2 \dot{g}^2 \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) \right] + 3r^2 \gamma_1 (1 - \gamma - r \dot{g}) + \epsilon \left\{ 6 \rho^2 \dot{g} \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) \right] + 6 \dot{g} \dot{\beta}^2 \right] \right].$$

$$\xi_i = \frac{1}{3\rho \dot{\rho}^2} \left[ 3r^4 \dot{g}^2 \gamma_1 (1 - \gamma - r \dot{g}) + \epsilon \left\{ 12 \rho^2 \dot{g} \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) \right] + 3r^2 \dot{g}^2 \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) \right] + 3r^2 \gamma_1 (1 - \gamma - r \dot{g}) + \epsilon \left\{ 6 \rho^2 \dot{g} \left[ \dot{g}^{5/2} (16\beta - 12\sigma \dot{g} + 2r^2 \alpha) + 4 \dot{g}^{3/2} (r^2 \beta + 24 \dot{g}^2 + 6\beta - 3r^2 \alpha + \rho (r^4 \mu_1 - 3r^2 \alpha + 36\beta) + 6 \dot{g} \dot{\beta}^2 \right) \right] + 6 \dot{g} \dot{\beta}^2 \right] \right].$$

$$\xi_i = \frac{1}{12\rho^2 \dot{\rho}^2} \left[ 6r^3 \dot{g}^2 (2 \dot{g} + r \dot{g}'' - \epsilon \left\{ 6r^3 \dot{g} \left[ 4r^3 \dot{g} (r^2 \alpha + 8 \beta + 6r \beta \dot{g}) \right] \right. \right].}$$
From the second equation of equation (36) we obtained the following expression:

\[
\nu_1 = \frac{1}{3q^2/2\beta q^{2/2}} \left[ q^{1/2} \left( 3r^2 q^2 + 3q^2 (2q^2 + 24) + q (3q^2 r^2 - 12q^2 r^2 - 24) \right) \right. \\
- \left. 4(1 + q)(2q^2 + q(3q^2 + 28) + 2q) \right].
\]  

(37)

Further, using equation (38) in the third equation of (36) we obtained the following expression:

\[
\mu_1 = \frac{1}{1890q^{5/3}(1 - \frac{q}{r})} \left[ 10r^2 q^2 \left( 189r^3 c_4 q + 5859 \right) + 10q^{1/2} \right] \\
\times \left[ 1008 - 4032q + 112q^2 + 34608q^2 + 448q^4 \right] - 10r^2 q^2 \ln(q) \\
- 1323q^2 \ln(q^2) + 567q^3 \ln(q) - 189 \ln(q) + 567 \ln(2) - 1134 \ln(2)q \\
+ 567 \ln(2)q^2 - \beta q [105840 \ln(2) + 105840 \ln(q)] - 15120 \ln(q) \\
+ 150528q^{5/2} + 10c_4 r^4 q^2 \left[ 189 - 756q + 1134q^2 - 756q^2 \right] \\
+ 10c_4 r^3 q^3 [756q^3 - 189 + 756q - 1134q^2 - 189q^4] - 16052r^2 \alpha q^2 q \\
- 10 \alpha q^2 [3318q^4 + 2562 + 2226q - 504q^2 - 7224q^3] \\
+ \alpha q^2 [61425 - 88830q - 16065q^3 + 945q^4] + 3780q^4 \beta q^5 \\
+ 27936q^5 + 10r^2 q^2 \left( 4032q^2 - 8064q + 4032 \right). 
\]  

(38)
Substituting equation (39) in equation (38) we get the following:

\[

\nu_1 = \frac{1}{1890 r^4 \varrho^{5/2} (1 - \varrho)^2} \left[ 9450 c_4 \varrho^4 \varrho^{3/2} \{1 - \varrho^3\} + 10 c_3 r^2 \varrho^{3/2} \left[ 189 r \varrho^3 \right. \right.

\left. - 756 r \varrho^2 - 756 \varrho \right] + 10 \varrho^{3/2} [31752 \varrho \beta \ln(\varrho) - 4158 r^2 \alpha] \\

+ r^2 \alpha \varrho^{1/2} [5670 \varrho^3 \ln(2) - 3780 \varrho^2 \ln(\varrho) - 5670 \varrho^2 \ln(2) + 17955] \\

- 10 \beta \varrho^{1/2} [1512 \ln(\varrho) - 10584 \ln(2)] + 1890 \varrho^{3/2} [c_3 - rc_4] \\

+ 10 \alpha r^2 \varrho^{3/2} [2520 \varrho^3/2 + 567 \ln(2) - 189 \ln(\varrho)] - 10 \beta \left[ 1512 - 5586 \varrho^{1/2} \right. \\

- 7896 \varrho^{3/2} - 1890 \varrho^{13/2} - 12978 \varrho^5/2 + 37506 \varrho^9/2 - 7560 \varrho^{11/2} - 27552 \varrho^7/2 \right] \\

+ 29 340 \varrho^3/2 c_3 + 18 900 r^3 c_4 \varrho^{5/2} [\varrho - 1] + r^2 \alpha \varrho^{1/2} [18900 \varrho^2 + 5670 \varrho^4 \ln(\varrho) \\

+ 7560 \ln(\varrho) - 5670 \ln(2) + 945 \varrho^6 - 7560 \varrho^3 \ln(\varrho) - 7560 \varrho^{1/2} \\

+ \beta \varrho [108 272 \varrho^4 - 45 360 - 28 560 \varrho - 1023 792 \varrho^2 + 287 568 \varrho^3] \\

+ r^2 \alpha \varrho^2 \left[ 25 200 \varrho - 17 640 - 17 640 \varrho^3 - 7560 \varrho^4 - 74 655 \varrho^{5/2} + 83 160 \varrho^{3/2} \\

- 4725 \varrho^{1/2}\right] + 10 \beta \varrho^{3/2} [6048 \ln(\varrho) + 31 752 \ln(2) - 6776 \varrho^{9/2} - 168 \varrho^{11/2}] \\

+ 1890 r^4 c_4 \varrho^{11/2}] . \tag{40}
\]
From equations (39) and (40) we constructed the unknown functions $\mu$ and $\nu$ in the asymptotic form up to $O(\epsilon)$. Notably, the metric potential (37) using equations (39) and (40) was the solution to the field equation (8) up to $O(\epsilon)$ for the form $f(T) = T + \epsilon \left[ \frac{1}{2} \alpha T^2 + \frac{1}{4} \beta T^3 \right]$. In section 5 we extracted the physics of the solution of (37) using equations (39) and (40).

5. Main features of the cubic solution

Some features of the solution that was derived in the previous section were analyzed here.

The asymptote of the metric: by constructing the metric of solution (37) from equations (39) and (40) we could easily demonstrate that this solution asymptotically behaved as a flat space–time and that is an acceptable behavior.

5.1. Singularities of the cubic solution

We calculated the invariant of solution (37) from equations (39) and (40) and obtained the following:

$$ T^{\mu \nu} T_{\mu \nu} = 16\epsilon M^2 \ln^2 \left( \frac{r}{\rho} \right) + 45360 \epsilon^4 M^4 \ln^4 \left( \frac{r}{\rho} \right) + \left[ \frac{1}{2} \alpha M^2 (15120 c_3 - 30240 M c_4) + \alpha M^2 (80640 + 22680 \ln(2) + 15120 c_3 + 3780 \ln(2)) + \beta (105840 \ln(2) + 32768) \right] 7560 M^4 r^2 + O \left( \frac{1}{r^2} \right), $$

$$ T^{\mu \nu} T_{\nu \mu} = -16\epsilon M^2 \ln^2 \left( \frac{r}{\rho} \right) - 5040 \epsilon^4 M^4 \ln^4 \left( \frac{r}{\rho} \right) + \left[ \frac{1}{2} \alpha M^2 (15120 c_3 - 30240 M c_4) + \alpha M^2 (80640 + 22680 \ln(2) + 15120 c_3 + 3780 \ln(2)) + \beta (105840 \ln(2) + 32768) \right] 5040 M^4 r^2 + O \left( \frac{1}{r^2} \right), $$

$$ T_{(r)} = 8\epsilon M^3 \ln^3 \left( \frac{r}{\rho} \right) - 15120 \epsilon^3 M^3 \ln^3 \left( \frac{r}{\rho} \right) + \left[ \frac{1}{2} \alpha M^2 (15120 c_3 - 30240 M c_4) + \alpha M^2 (80640 + 22680 \ln(2) + 15120 c_3 + 3780 \ln(2)) + \beta (105840 \ln(2) + 32768) \right] 7560 M^4 r^2 + O \left( \frac{1}{r^2} \right), $$

$$ R^\mu_{\nu \rho \sigma} R_{\mu \nu \rho \sigma} = \frac{48 \epsilon M^2}{r^2} + 8\epsilon \left[ \frac{M^2 (3835 \ln(2) + 10080) + \beta (13230 \ln(2) + 4906) + M^2 (1890 c_3 - 3780 M c_4)}{945 M^4} \right] + O \left( \frac{1}{r^2} \right), $$

$$ R^\nu_{\mu \nu \rho} R_{\mu \nu \rho} = O \left( \frac{1}{r^2} \right), R = \epsilon \left[ \frac{M^2 (3835 \ln(2) + 10080) + \beta (13230 \ln(2) + 4906) + M^2 (1890 c_3 - 3780 M c_4)}{945 M^4} \right] + O \left( \frac{1}{r^2} \right). $$

(41)

Due to the non-linearity of the field equation (8), equation (41) do not coincides to equation (17) when the parameter $\beta = 0$. This is similar to the BH presented in [61] in which the authors presented a BH solution for $f(T) = T + \frac{\alpha}{2} T^2 + \frac{\beta}{4} T^3$ and when $\beta = 0$ this BH does not reduce to the BH solution presented in [50] for $f(T) = T + \frac{\alpha}{2} T^2$.

Same discussion carried out for the quadratic solution given in Subsection IIIA can be applied to the invariants of the cubic case given by equation (41).

Using the same procedure of the quadratic form done for the calculation of the energy we calculated the energy of solution (37) from equations (39) and (40) and obtained the following:

$$ E = M + \epsilon \left[ 11340 M^4 c_4 - 3780 M^3 c_3 - 8505 M^2 \ln(2) + \beta (16384 - 39690 \ln(2)) \right] 7560 M^3. $$

(42)

Equation (42) reveals that $7560 M^3 + \epsilon [11340 M^4 c_4 - 3780 M^3 c_3 - 8505 M^2 \ln(2) + \beta (16384 - 39690 \ln(2))] > 0$. 

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Further, we calculated the geodesic deviation of solution (37) from equations (39) and (40) and derived the condition of stability. This condition was quite lengthy and was not presented here. However, figure 2 shows its behavior for particular values of the model. This figure shows the regions where the BH solution are stable and those where there is no possible stability.

6. Thermodynamics of ‘p = 2’ and cubic black holes

In this section, we investigated the thermodynamics behavior of the BH solutions (15) and (37), from equations (39) and (40), that are related to the quadratic and cubic forms of the field equation (8). To do this, we gave the basic definitions of the thermodynamical quantities.

6.1. Thermodynamics of ‘p = 2’

The metric potential of the temporal component of equation (15) takes the following form:

\[
\mu(r) = 1 - \frac{2m}{r} + \epsilon \left[ - \frac{c_1}{r} + c_2 - \alpha \left( \frac{m^2 + 6mr + r^2}{mr^3} - \frac{16(1 - \frac{2m}{r})^{1/2}}{3m^2} \right) \right. \\
+ \left. \frac{(1 - \frac{3m}{r})}{2m^2} \ln \left( 1 - \frac{2m}{r} \right) \right]
\]

\[
\simeq 1 + \epsilon \left[ c_2 + \frac{\alpha \left( 64 - 9 \ln(2) \right)}{12m^2} \right] - \frac{1}{r} \left[ 2m + \epsilon \left( c_1 + \frac{16\alpha}{m} \right) \right] - O \left( \frac{1}{r^5} \right). \quad (43)
\]

Equation (54) was drawn in figure 3(a), the plot indicates the one horizon of the BH which can be obtained in a precise form from the solution of equation \( \mu(r) = 0 \). This horizon is known as the event horizon \( r_h \). One can calculate the total mass contained \( r_h \). This can be done by setting \( \mu(r_h) = 0 \), and then we obtain the horizon mass–radius relation in the following form

\[
m_h = \frac{3r_h^2}{6r_h} + \epsilon \left( 3r_h^2c_2 - 32\alpha - 3r_h c_1 - 9\alpha \ln(2) \right). \quad (44)
\]

Equation (44) is plotted in figure 3(b), whereas \( m_h \) has positive and negative values, the BH has one horizon when \( m_h = m_{\text{min}} \). This result is consistent with figure 3(a).

The Hawking temperature is usually defined as \([62–65]\)

\[
T_h = \frac{\mu'(r_h)}{4\pi}, \quad (45)
\]

where the event horizon \( r = r_h \) is the positive solution of the equation \( \mu(r_h) = 0 \) which satisfies \( \mu'(r_h) \neq 0 \). In the framework of \( f(T) \) gravity, the entropy is given by \([66, 67]\)

\[
S(r_h) = \frac{1}{4} Af_T(r_h), \quad (46)
\]

where \( A \) represents the area.

The constraint \( \mu(r_h) = 0 \) yields

\[
r_{h, \mu=2}'' \simeq \frac{12m^2 + \epsilon \left( 6mc_1 + 32\alpha - 12c_2m^2 \right) + 9\alpha \ln(2) \}}{6m}. \quad (47)
\]

When \( \epsilon = 0 \) we get the GR limit.
Figure 3. Schematic plots of the thermodynamical quantities of the BH solution (15) for positive value of $\epsilon$: (a) typical behavior of the metric function $\mu(r)$ given by (15); (b) the horizon mass–radius relation (44); (c) typical behavior of the horizon entropy, which shows that $S_h$ increases quadratically as $r_h$ increases and we have always a positive entropy; (d) typical behavior of the horizon entropy, (49), and (e) the heat capacity, (51), which show that both vanish at $r_h$ and we have always positive temperature and negative heat capacity which shows that the BH is unstable; (f) typical behavior of the horizon Gibbs free energy which shows that $G_h$ have negative value as $r < r_h$ and positive value as $r > r_h$ for positive $\epsilon$.

From equation (46), the entropy of solution (15) takes the form

$$S_{h'}^{\gamma=2'} = \frac{\pi r_h^5 + 2\epsilon \alpha m^2 (4m + 3r_h)}{r_h^3},$$

(48)

which shows that when $\epsilon = 0$ we get the GR entropy. Equation (48) shows that the parameters $\epsilon$ and $\alpha$ should be either positive or negative to get positive entropy otherwise the entropy will have a negative quantity. The behavior of equation (48) is shown in figure 3(c) for positive values of $\epsilon$ and $\alpha$ which shows positive value of entropy.

The Hawking temperatures of solution (15) takes the form,

$$T_{h'}^{\gamma=2'} \simeq \frac{3r_h^2 + \epsilon [\alpha (64 - 9 \ln(2)) + 3c_2 r_h^3]}{12\pi r^3}.$$

(49)

Equation (49) shows that when $\epsilon = 0$ we get the Hawking temperature of Schwarzschild BH. We depicted the Hawking’s temperature in figure 3(d) for positive values of $\epsilon$ and $\alpha$. Figure 3(d) proves that we do have a positive temperature for the BH (15) and the temperature may take a negative value when either $\epsilon$ or $\alpha$ become a negative.
The stability of the BH solution is an important topic that can be studied on the dynamical and the perturbative levels [60, 68, 69]. To investigate the thermodynamical stability of BH solution one derives the formula of the heat capacity $C(r_h)$ at the event horizon. The event horizon heat capacity is given by the following from [70–72]:

$$ C_h \equiv C(r_h) = \frac{\partial m_h}{\partial T} = \frac{\partial m_h}{\partial r_h} \left( \frac{\partial T}{\partial r_h} \right)^{-1}. $$  \hspace{1cm} (50)

The BH will be thermodynamically stable, if its heat capacity $C_h$ is positive, and will be unstable if $C_h$ is negative. Using (44) and (49) into (50), we obtain the heat capacity as

$$ C_{h}^{\prime} = \frac{2\pi r_h^3}{3} \left( \frac{3r_h^2 + \epsilon \left( 3r_h^2 c_2 + 9\alpha \ln(2) + 32\alpha \right)}{64 \alpha^2} \right). $$  \hspace{1cm} (51)

Equation (51) shows that $C_h$ does not locally diverge and the BH has no phase transition of second-order. The heat capacity is depicted in figure 3(e) which shows that $C_h < 0$ where $r_h < r_d$ and the BH is thermodynamically unstable. The main reason that makes the heat capacity negative is the derivative of Hawking temperature and this is consistent with the nature of Schwarzschild BH which can be discovered when $\epsilon = 0$. In the non-vanishing of $\epsilon$ we can create a positive heat capacity but the price of this is to accept the Hawking temperature to has a negative value.

The Gibbs free energy is given by [67, 73]

$$ G(r_h) = m(r_h) - T(r_h) S(r_h). $$  \hspace{1cm} (52)

The quantities $m(r_h)$, $T(r_h)$ and $S(r_h)$ are the mass, temperature and entropy at the event horizon, respectively. From equations (44), (48) and (49) in (52), we obtain

$$ G_{\text{equation(15)}} = \frac{3r_h^5 + \epsilon \left( 3r_h^2 c_2 - 9\alpha r_h^2 \ln(2) - 128\alpha r_h + 64m^2 \right) + m^2 \left( 3\alpha r_h + 4\alpha m^2 \right)}{12r_h^4}. $$  \hspace{1cm} (53)

We depict the Gibbs energy of the BH (15) in figure 3(f), which indicates that the Gibbs energy has negative values at large $r < r_h$ and positive value when $r > r_h$. We note that for $\epsilon = 0$, the Schwarzschild BH is recovered which is shown in figure 3(f) by the blue dot curve. Interestingly, for the negative value of $\epsilon$, Gibbs energy is always positive, and as we discuss before that the price of this is the negative Hawking temperature. We depict the case of a negative value of $\epsilon$ in figure 4.

6.2. Thermodynamics of the cubic solution

The metric potential of the temporal component of solution (37), using equations (39) and (40) takes the form

$$ \mu(r) = 1 - \frac{2m}{r} + \epsilon \left[ c_1 + \frac{\alpha}{m} \right] \left( \frac{m^2 + 6mr + r^2}{m^2} - \frac{16(1 - \frac{2m}{r})^{3/2}}{3m^2} \right) $$
$$ + \left( \frac{1 - \frac{2m}{r}}{2m^2} \right) \ln \left( 1 - \frac{2m}{r} \right) \right] $$
$$ \simeq 1 + \epsilon \left[ c_2 + \frac{\alpha}{12m^2} \left( 64 - 9 \ln(2) \right) \right] - \frac{1}{r} \left[ 2m + \epsilon \left( c_1 + \frac{16\alpha}{m} \right) \right] - O \left( \frac{1}{r^2} \right). $$  \hspace{1cm} (54)
Equation (54) is drawn in figure 5(a), the plot shows that the BH could have one horizon at the root of \( \mu(r) = 0 \), this is the event horizon \( r_h \). The horizon mass–radius relation of solution (37), using equations (39) and (40) has the form

\[
m_h = \frac{135n_b^4 + \epsilon [135n_b^4c_2 - 405\alpha r_h^2 - 135n_b^3c_1 - 405n\alpha \ln(2)r_h^2 - 16348\beta]}{720r_h^3}.
\] (55)

We plot the above relation in figure 5(b), whereas \( m_h \) has positive and negative values, the BH has one horizon when \( m_h = m_{\text{min}} \). This result is in agreement with figure 5(a).

The constraint \( \mu(r_h) = 0 \) yields

\[
r_{\text{cubic}} \approx \frac{3780m^4 + \epsilon [4096\beta + 10080\alpha m^2 + 1890c_1m^3 + 13230\alpha \ln(2) + 2835\alpha m^2 \ln(2) - 3780c_2m^4]}{1890m^3}.
\] (56)

When \( \epsilon = 0 \) equation (56) gives the GR limit.
Figure 5. Schematic plots of thermodynamical quantities of the BH solution (37), using equations (39) and (40) for positive value of $\epsilon$: (a) typical behavior of the metric function $\mu(r)$ given by (54); (b) the horizon mass–radius relation (44); (c) typical behavior of the horizon entropy, which shows that $S_h$ increases quadratically as $r_h$ increases and we have always a positive entropy; (d) typical behavior of the horizon temperature, and (e) the heat capacity, (51), which show that both vanish at $r_h$, and we have always positive temperature and negative heat capacity; (f) typical behavior of the horizon Gibbs free energy which shows that $G_h$ could have negative value at $r < r_h$ and positive value at $r > r_h$ for negative value of $\epsilon$.

From equation (46), the entropy of solution (37) takes the form

\[
S_{\text{cubic}} = \frac{\pi}{r_h^2(r_h^2 - 4m^2)} \left[ r_h^4(r_h^2 - 4m^2)^2 + 4\epsilon \left[ r_h^6\alpha + 16\beta m^4 + 8\alpha r_h^2 m^4 - 6\alpha r_h^4 m^2 + 8\beta m^3 \right] r_h^2 - m[16\beta m^2 - \alpha r_h^4] + 4\beta m^3 \right] \right]^{1/2} + \frac{1}{2m_h} + 4\beta r_h^2 \left[ 2r_h^2 - 7m^2 \right],
\]

(57)

which shows that when $\epsilon = 0$ we recover the entropy of GR. Equation (57) shows that the parameters $\epsilon$, $\alpha$ and $\beta$ should be either positive or negative to get positive entropy otherwise the entropy will have a negative quantity. The behavior of equation (57) is shown in figure 5(c) for positive values of $\epsilon$ and $\alpha$ which shows positive value of entropy.
Figure 6. Schematic plots of thermodynamical quantities of the BH solution (37) for negative value of $\epsilon$: (a) typical behavior of the metric function $\mu(r)$ given by (37); (b) the horizon mass–radius relation (44); (c) typical behavior of the horizon entropy, which shows that $S_h$ increases quadratically as $r_h$ increases and then decrease; (d) typical behavior of the heat capacity, (59), which shows that it vanish at $r_h$, and as $r < r_h$ we have a positive heat capacity and as $r > r_h$ we have a negative value of the BH; (e) typical behavior of the horizon Gibbs free energy which shows that $G_h$ have always positive value for negative value of $\epsilon$.

The Hawking temperatures of solution (37) takes the form,

$$T_{hcubic} \approx \frac{3r_h^2 + \epsilon[\alpha(64 - 9 \ln(2)) + 3c_2r_h^2]}{12\pi r_h^3},$$

which is identical with the quadratic form up to $\epsilon$ but with different value of $\alpha$ to unify the values through all the calculations of cubic case. Therefore, same discussions carried out for the case of quadratic can be applied here.

Using (55) and (58) into (50), we obtain the heat capacity as

$$C_{hcubic} = \frac{2\pi 45r_h^4 + \epsilon[45r_h^4c_2 + 135r_h^2\alpha \ln(2) + 480r_h^2\alpha + 16348\beta]}{45 \epsilon[9\alpha \ln(2) - r_h^2c_2 - 64\alpha] - r_h^2}.$$ (59)

Equation (59) shows that $C_h$ does not locally diverge and the BH has no phase transition of second-order. The heat capacity is depicted in figure 5(e) which shows that $C_h < 0$ where $r_h < r_{dg}$ and the BH is thermodynamically unstable. The main reason that makes the heat capacity negative is the derivative of Hawking temperature and this is consistent with the Schwarzschild BH which can be discovered when $\epsilon = 0$. In the non-vanishing of $\epsilon$ we can
Table 1. The main results thermodynamic according to the values of $\epsilon$, i.e. positive or negative values.

| Quadratic | Cubic |
|-----------|-------|
| $\epsilon$ | Entropy | Temperature | Heat capacity | Gibb’s free energy |
| Zero      | Positive | Positive | Negative | Positive |
| $>0$ Zero | Positive | Positive | Negative | Positive(conditional) |
| $<0$ Zero | Positive(conditional) | Positive(conditional) | Positive(conditional) | Positive |

| Quadratic |
|-----------|
| Create a positive heat capacity but the price of this is to accept the Hawking temperature to have a negative value. |

From equations (55), (57) and (58) in (52), we obtain

$$G_{\text{cubic}} = \frac{\pi}{540r_h^3(r_h^2 - 4m^2)^2} \left[ 135r_h^4(r_h^2 - 4m^2)^2 - \epsilon \left\{ 4 \left( 1575r_h^6\alpha \right. \right. \\
+ 9272r_h^4 \beta + 133 232 \beta m^4) - 135r_h^7 \left[ r_h c_2 - 2c_1 \right] \\
- 16r_h m^2 \beta \left[ 17 329r_h - 270m \right] + 1080c_2 m^2 r_h^4 \left[ r_h^2 - 2m^2 \right] \\
- 2160m^2 r_h^3 c_1 [r_h^2 - 2m^2] + 405r_h^6 \alpha \ln(2) \\
+ \left\{ 2160r_h m^2 [4\beta + \alpha r_h^2] + 540r_h^2 \alpha m [r_h^2 - 2m^2] \right\} \sqrt{r_h^2 + 2mr_h} \\
- 4320\beta m [2m^2 - 4r_h^2] - 540r_h^3 [8\beta + r_h^2 \alpha] \right\} \frac{\sqrt{r_h^2 + 2mr_h}}{\ln(2)} \\
+ 10r_h^2 m^2 \alpha \left[ 4932r_h^2 - 9648m^2 - 648m^2 \ln(2) \\
+ 324r_h^2 \ln(2) \right] \right] . \quad (60)$$

We depict Gibbs energy of the BH (37) in figure 5(f), which indicates that the Gibbs energy has negative values at large $r < r_h$ and positive value when $r > r_h$. We note that for $\epsilon = 0$, the Schwarzschild BH is recovered which is shown in figure 5(f) by the blue dot curve. Interestingly, for negative value of $\epsilon$, Gibbs energy is always positive and as we discuss before that the price of this is the negative Hawking temperature. We depict the case of negative value of $\epsilon$ in figure 6.

We summarize the results of thermodynamics of the quadratic and cubic cases in table 1. As this table shows that when the parameter $\epsilon$ takes positive value we have good models.

7. Discussion and conclusions

In this study, we further studied $f(T)$ in the framework of the spherically symmetric space–time. To do this, we used a physical tetrad field space–time possessing two unknown functions. This study aims to determine the physics of the BH that was derived in [1] for the quadratic form of $f(T)$. We calculated the invariants of this BH and demonstrated that the behavior of the tensors $T^{\alpha\beta\gamma} T_{\alpha\beta\gamma} = T^\alpha T_\alpha = T \sim r^{-2}$ are similar to the results obtained before in the frame of $f(T)$ [50, 61], and dissimilar to the solutions of GR and TEGR, which behaved as $T^{\alpha\beta\gamma} T_{\alpha\beta\gamma} = T^\alpha T_\alpha = T \sim r^{-4}$. This clearly indicated that the contribution of the
higher-order torsion made the singularity of $T^{\alpha\beta\gamma} T_{\alpha\beta\gamma}$, $T^\alpha T^\alpha$, and $T$ milder. Additionally, we calculated the energy content and revealed the contribution of the higher-order torsion. Finally, we calculated the stability of this BH via the geodesic deviation and revealed the regions where the BH exhibited stability, as shown in figure 1.

In the second part, we presented the cubic form of the field equations using $f(T) = T + \epsilon \left( \frac{1}{2} \alpha T^2 + \frac{1}{3} \beta T^3 \right)$ and derived the asymptotic form of these field equations up to $O(\epsilon)$. The derived BH solution was characterized by the mass, the two constants of integrations and the three constants that characterized the form $f(T)$. The asymptotic form of this BH behaved like a flat space–time. We calculated the invariants of this BH and its energy thereby revealed the contribution of the higher-order torsion to the energy. Furthermore, we calculated the geodesic deviation and revealed where the regions of stability in the BH. To study the BH solutions of this study in-depth we calculated the thermodynamic quantities including the Hawking temperature, entropy, heat capacity, and Gibbs energy for the quadratic and cubic forms of the BH solutions. For the quadratic form, we used the same values of the constants that are used for the stability study. We studied all the thermodynamical quantities with two values of the parameters $\epsilon$ the positive value and negative values as shown in figures 3 and 4. The most beneficial results that were discussed were those of the entropy, Hawking temperature, heat capacity, and Gibbs energy. At $\epsilon > 0$ positive values were always obtained for the entropy and temperature while a negative value was obtained for the heat capacity. Regarding the Gibbs energy, we obtained a negative value at $r_{dg} < r_h$ and a positive one at $r_{dg} > r_h$. At $\epsilon < 0$ those quantities were negative regarding the entropy at $r_{dg} < r_h$ and positive at $r_{dg} > r_h$. Furthermore, regarding the temperature and heat capacity the values were negative at $r_{dg} < r_h$ and positive at $r_{dg} > r_h$. Regarding the Gibbs energy it was always positive and we concluded that $\epsilon$ of the quadratic form might be positive or negative values and both of them afforded the physical thermodynamic quantities. The same discussion is applicable for the cubic BH.

Data availability statement

No new data were created or analyzed in this study.

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