COCYCLE DEFORMATIONS AND BRAUER GROUP ISOMORPHISMS

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Abstract. Let $H$ be a Hopf algebra over a commutative ring $k$ with unity and $\sigma : H \otimes H \to k$ be a cocycle on $H$. In this paper, we show that the Yetter-Drinfeld module category of the cocycle deformation Hopf algebra $H^\sigma$ is equivalent to the Yetter-Drinfeld module category of $H$. As a result of the equivalence, the “quantum Brauer” group $\text{BQ}(k, H)$ is isomorphic to $\text{BQ}(k, H^\sigma)$. Moreover, the group $\text{Gal}(H_R)$ constructed in [19] is studied under a cocycle deformation.

Introduction

Let $k$ be a commutative ring with unity and let $H$ be a Hopf algebra over $k$. In [6], Doi introduced a cocycle twisted Hopf algebra $H^\sigma$, called a cocycle deformation of $H$, for a 2-cocycle $\sigma : H \otimes H \to k$. It is shown in [7] that Drinfeld’s quantum double $D(H)$ is a cocycle deformation of the tensor product Hopf algebra $H^\ast \otimes H$. If $(H, R)$ is a coquasitriangular Hopf algebra, then $(H^\sigma, R^\sigma)$ is a coquasitriangular Hopf algebra as well (see [10, p61]), where $R^\sigma = (\sigma \tau) R \sigma^{-1}$ and $\tau$ is the flip map. It is well known that the braided monoidal category $\mathcal{M}_{H^\sigma}$ of right $H^\sigma$-comodule category with the braiding induced by $R^\sigma$ is equivalent to the braided monoidal category $\mathcal{M}_{H_R}$ of right $H^\sigma$-comodule category with the braiding induced by $R^\sigma$. Since the Brauer group $\text{Br}(C)$ is a group invariant of a braided monoidal category $C$ [17], we have a group isomorphism from the equivariant Brauer group $\text{BC}(k, H, R)$ to the equivariant Brauer group $\text{BC}(k, H^\sigma, R^\sigma)$ [4]. The significance of this isomorphism was demonstrated via Sweedler’s Hopf algebra $H_4$ which has a family of coquasitriangular structures $R_t$ indexed by the ground ring $k$. Since each $(H_4, R_t)$ is a cocycle deformation of $(H_4, R_0)$, we have group isomorphisms $\text{BC}(k, H_4, R_t) \cong \text{BC}(k, H_4, R_0)$, for all $t \in k$, where the later is known as the direct product group of the Brauer-Wall group $\text{BW}(k)$ and the additive group $k^+$ [18]. In this paper, we will show that the braided monoidal category equivalence between $\mathcal{M}_{H_R}$ and $\mathcal{M}_{H^\sigma_R}$ is carried out by an equivalence braided monoidal functor from the Yetter-Drinfeld module category $\mathcal{YD}^H$ to the Yetter-Drinfeld module category $\mathcal{YD}^{H^\sigma}$. As a consequence, we obtain a Brauer group isomorphism between $\text{BQ}(k, H)$ and $\text{BQ}(k, H^\sigma)$ for any cocycle deformation $H^\sigma$ of a Hopf algebra $H$ with a bijective antipode. When $(H, R)$ is a coquasitriangular Hopf algebra, then the group isomorphism restricts to the group isomorphism between the equivariant Brauer groups $\text{BC}(k, H, R)$ and $\text{BC}(k, H^\sigma, R^\sigma)$ obtained in [4]. The equivalence

1991 Mathematics Subject Classification. 16A16.

Key words and phrases. Yetter-Drinfeld module, Brauer group, Azumaya algebra.
of Yetter-Drinfeld module categories under cocycle deformation is of self-duality in the sense that the Yetter-Drinfeld module category $\mathcal{YD}^H$ is not only stable under cocycle deformation but also stable under dual-cocycle deformation. In other words, if $\theta \in H \otimes H$ is a dual-cocycle and $H_\theta$ is the dual-cocycle deformation of $H$, then the braided monoidal category $\mathcal{YD}^H_{\theta}$ is equivalent to the braided monoidal category $\mathcal{YD}^{H_\theta}$.

In the first section, we recall some basic definitions about the Yetter-Drinfeld module category of a Hopf algebra and the (equivariant) Brauer group of a (coquasitriangular) Hopf algebra. In Section 2, we show that for a Hopf algebra $H$ there exists a natural equivalence braided monoidal functor $\sigma$ from the Yetter-Drinfeld module category $\mathcal{YD}^H$ to the Yetter-Drinfeld module category $\mathcal{YD}^{H_\sigma}$ of the cocycle deformation $H_\sigma$ with respect to a cocycle $\sigma : H \otimes H \to k$ (see Theorem 2.3); and there exists a natural equivalence braided monoidal functor from $\mathcal{YD}^H$ to the Yetter-Drinfeld module category $\mathcal{YD}^{H_\theta}$ with respect to the dual cocycle $\theta \in H \otimes H$. When $(H, R)$ is a coquasitriangular Hopf algebra, then the equivalence functor $\sigma$ restricts to an equivalence functor from the full braided monoidal subcategory $\mathcal{M}^H$ of right $H$-comodules to the full braided monoidal subcategory $\mathcal{M}^{H_\sigma}$. It is well-known that the Brauer group $\mathrm{BQ}(k, H)$ is the Brauer group of the braided monoidal category $\mathcal{YD}^H$. Thus we obtain a group isomorphism from $\mathrm{BQ}(k, H)$ to $\mathrm{BQ}(k, H_\sigma)$. When $\sigma$ is a lazy cocycle, the Hopf algebra $H_\sigma$ is equal to $H$ and $\sigma$ induces an automorphism of the Brauer group $\mathrm{BQ}(k, H)$. We obtain that the second lazy cohomology group $H^2_\mathrm{L}(H)$ acts on $\mathrm{BQ}(k, H)$ by automorphisms (see Corollary 2.7).

Section 3 devotes to the study of the group $\mathrm{Gal}(\mathcal{H}_R)$ of bigalois objects constructed in [19] under cocycle deformation. If $(H, R)$ is a finite (faithfully projective) coquasitriangular Hopf algebra, we have a generalized cotensor product over the braided Hopf algebra $\mathcal{H}_R$ defined in the Yetter-Drinfeld module category $\mathcal{YD}^H$ (see [19] for more detail). The generalized cotensor product induces a group structure on the set $\mathrm{Gal}(\mathcal{H}_R)$ of quantum commutative $\mathcal{H}_R$-bigalois objects in the category $\mathcal{YD}^H$. The group $\mathrm{Gal}(\mathcal{H}_R)$ measures the equivariant Brauer group $\mathrm{BC}(k, H, R)$ via an exact group sequence:

$$1 \to \mathrm{Br}(k) \to \mathrm{BC}(k, H, R) \to \mathrm{Gal}(\mathcal{H}_R) \to 1.$$  

Now let $\sigma$ be a cocycle on $H$ and let $(H^\sigma, R^\sigma)$ be the cocycle deformation of $(H, R)$ with respect to $\sigma$. We show that the equivalence functor $\sigma$ defined in Section 2 commutes with the generalized cotensor product, and induces a group isomorphism from $\mathrm{Gal}(\mathcal{H}_R)$ to $\mathrm{Gal}(\mathcal{H}_R^\sigma)$. Moreover, the exact sequence (1) is stable under cocycle deformation. That is, we have the following commutative diagram of exact sequences:

$$1 \to \mathrm{Br}(k) \to \mathrm{BC}(k, H, R) \xrightarrow{\hat{\pi}} \mathrm{Gal}(\mathcal{H}_R) \xrightarrow{\pi} 1$$

$\sigma$

$$1 \to \mathrm{Br}(k) \to \mathrm{BC}(k, H^\sigma, R^\sigma) \xrightarrow{\hat{\pi}} \mathrm{Gal}(\mathcal{H}_R^\sigma) \xrightarrow{\pi} 1.$$
1. Preliminaries

1.1. Yetter-Drinfeld modules. Throughout, we work over a fixed commutative ring $k$ with unity. Unless otherwise stated, all algebras, Hopf algebras and modules are defined over $k$; all maps are $k$-linear; dim, $\otimes$ and Hom stand for dim$_k$, $\otimes_k$ and Hom$_k$, respectively. For the theory of Hopf algebras and quantum groups, we refer to [8, 10, 12, 14].

Let $H$ be a Hopf algebra with a bijective antipode $S$. A Yetter-Drinfeld $H$-module (simply, $YD$-$H$-module) $M$ is a crossed $H$-bimodule. That is, $M$ is at once a left $H$-module and a right $H$-comodule satisfying the following equivalent compatibility conditions:

\begin{align*}
\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)}m_{(1)} &= \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)}h_{(1)}, \\
\sum (h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} &= \sum h_{(2)} \cdot m_{(0)} \otimes h_{(3)}m_{(1)}S^{-1}(h_{(1)}),
\end{align*}

where $h \in H$, $m \in M$, and the sigma notations for a comodule and for a comultiplication can be found in the reference book [14]. The category $_H YD^H$ of Yetter-Drinfeld $H$-modules and their homomorphisms is a braided monoidal category with the braiding given by

$$\Phi : M \otimes N \rightarrow N \otimes M, \quad \Phi(m \otimes n) = \sum n_{(0)} \otimes n_{(1)} \cdot m$$

where $M, N \in _H YD^H$.

A Hopf algebra over $k$ is called a coquasitriangular (CQT) Hopf algebra if there is an invertible element $R \in (H \otimes H)^*$, the convolution algebra of $H \otimes H$, subject to the following conditions:

\begin{align*}
(CQT1). \quad R(h \otimes 1) &= R(1 \otimes h) = \varepsilon(h), \\
(CQT2). \quad R(g \otimes hl) &= \sum R(g_{(1)} \otimes l)R(g_{(2)} \otimes h), \\
(CQT3). \quad R(hl \otimes g) &= \sum R(h \otimes g_{(1)})R(l \otimes g_{(2)}), \\
(CQT4). \quad \sum R(g_{(1)} \otimes h_{(1)})g_{(2)}h_{(2)} &= \sum R(g_{(2)} \otimes h_{(2)})h_{(1)}g_{(1)},
\end{align*}

for all $g, h$ and $l \in H$. Now let $H$ be a CQT Hopf algebra with a CQT structure $R$. Note that (CQT4) is equivalent to one of the following:

\begin{align*}
(CQT'4). \quad \sum g_{(1)}R(g_{(2)} \otimes h) &= \sum R(g_{(1)} \otimes h_{(2)})S(h_{(1)})g_{(3)}h_{(3)}, g, h \in H, \\
(CQT'4*). \quad \sum h_{(1)}R(g \otimes h_{(2)}) &= \sum R(g_{(2)} \otimes h_{(2)})g_{(1)}h_{(1)}S^{-1}(g_{(3)}), g, h \in H.
\end{align*}

If $M$ is a right $H$-(or $H^{op}$)-comodule, the CQT structure $R$ induces a left $H$-module structure on $M$ as follows:

\begin{equation}
\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)}m_{(1)} = \sum m_{(0)}R(h \otimes m_{(1)})
\end{equation}

for $h \in H$ and $m \in M$, where we use the $\triangleright_1$ as we will have the second left $H$-action induced by the CQT structure $R$ in Section 2. The $H$-action $\triangleright_1$ together with the original $H$-coaction makes $M$ into a YD $H$-module. Denote by $\mathcal{M}_H^H$ the category of YD $H$-modules with the left $H$-module structure $\triangleright_1$ coming from the right $H$-comodule structure. It is obvious that $\mathcal{M}_H^H$ is a full braided monoidal subcategory of $_H YD^H$. 
Now let $H$ be a quasitriangular (QT) Hopf algebra, that is, $H$ is a Hopf algebra with an invertible element $R = \sum R^{(1)} \otimes R^{(2)}$ in $H \otimes H$ satisfying the following axioms ($r = R$):

1. $\sum \Delta(R^{(1)}) \otimes R^{(2)} = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)},$
2. $\sum \varepsilon(R^{(1)}) R^{(2)} = \sum R^{(1)} \varepsilon(R^{(2)}) = 1,$
3. $\sum R^{(1)} \otimes \Delta(R^{(2)}) = \sum R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)},$
4. $\Delta^{\text{op}}(h) R = R \Delta(h),$

where $\Delta^{\text{op}} = \tau \Delta$ is the comultiplication of the Hopf algebra $H^{\text{op}}$ and $\tau$ is the flip map.

If $A$ is a left $H$-module (algebra), then $A$ is simultaneously a YD $H$-module (algebra) with the right $H^{\text{op}}$-comodule structure given by

$$A \to A \otimes H^{\text{op}}, \quad a \mapsto (R^{(2)} \cdot a) \otimes R^{(1)}, \quad a \in A. \quad (5)$$

### 1.2. $H$-Azumaya algebras

A Yetter-Drinfeld $H$-module algebra (simply, YD $H$-module algebra) is a YD $H$-module algebra $A$ such that $A$ is a left $H$-module algebra and a right $H^{\text{op}}$-comodule algebra. For the details of $H$-(co)module algebras we refer to $[5, 10, 12, 14].$

Let $A$ and $B$ be two YD $H$-module algebras. We may define a *braided product*, still denoted $\#$, on the YD $H$-module $A \otimes B$:

$$(a \# b)(c \# d) = \sum a c_{(0)} \# (c_{(1)} \cdot b) d$$

for $a, c \in A$ and $b, d \in B$. The braided product $\#$ makes $A \# B$ a left $H$-module algebra and a right $H^{\text{op}}$-comodule algebra so that $A \# B$ is a YD $H$-module algebra.

Now let $A$ be a YD $H$-module algebra. The *$H$-opposite algebra* $\overline{A}$ of $A$ is the YD $H$-module algebra defined as follows: $\overline{A}$ equals $A$ as a YD $H$-module, but with multiplication given by the formula

$$\overline{a} \cdot \overline{b} = \sum b_{(0)} (b_{(1)} \cdot a)$$

for all $\overline{a}, \overline{b} \in \overline{A}$.

In $[2]$ we defined the Brauer group of a Hopf algebra $H$ by considering isomorphism classes of $H$-Azumaya algebras. A Yetter-Drinfeld $H$-module algebra $A$ is said to be $H$-Azumaya if it is finite (i.e., faithfully projective) as a $k$-module and if the following two Yetter-Drinfeld $H$-module algebra maps are isomorphisms:

$$F : A \# \overline{A} \to \text{End}(A), \quad F(a \# b)(x) = \sum a x_{(0)} (x_{(1)} \cdot b),$$
$$G : \overline{A} \# A \to \text{End}(A)^{\text{op}}, \quad G(\overline{a} \# b)(x) = \sum a_{(0)} (a_{(1)} \cdot x) b,$$

For a finite YD $H$-module $M$, the endomorphism algebra $\text{End}_k(M)$ is a Yetter-Drinfeld $H$-module algebra with the $H$-structures given by

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m),$$
$$\sum f_{(0)}(m) \otimes f_{(1)} = \sum f(m_{(0)}) m_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)}) \otimes (m_{(1)})$$

for $f \in \text{End}(M)$ and $m \in M$. The elementary $H$-Azumaya algebra $\text{End}(M)^{\text{op}}$ has the different $H$-structures from those of $\text{End}(M)$ (see $[2]$ for the details).
Two $H$-Azumaya algebras $A$ and $B$ are Brauer equivalent (denoted $A \sim B$) if there exist two finite YD $H$-modules $M$ and $N$ such that $A \# \text{End}(M) \cong B \# \text{End}(N)$. Note that $A \sim B$ if and only if $A$ is $H$-Morita equivalent to $B$ (see [2 Th.2.10]). The relation $\sim$ is an equivalence relation on the set $B(k, H)$ of isomorphism classes of $H$-Azumaya algebras and the quotient set of $B(k, H)$ modulo $\sim$ is a group, called the Brauer group of the Hopf algebra $H$, denoted $BQ(k, H)$.

If $(H, R)$ is a CQT Hopf algebra, then $BQ(k, H)$ contains a subgroup consisting of elements represented by $H$-Azumaya algebras whose $H$-module structures are induced by the CQT structure $R$ via [1]. Such an $H$-Azumaya algebra is called an $R$-Azumaya algebra. The subgroup, denoted $BC(k, H, R)$, is called the equivariant Brauer group of the CQT Hopf algebra $(H, R)$ [1].

2. The Yetter-Drinfeld module category of a cocycle deformation

Let $H$ be a Hopf algebra. Recall that a 2-cocycle on $H$ is a convolution invertible $k$-linear map $\sigma : H \otimes H \rightarrow k$ satisfying:

\[ \sum \sigma(g(1) \otimes h(1))\sigma(g(2)h(2) \otimes l) = \sum \sigma(h(1) \otimes l(1))\sigma(g \otimes h(2)l(2)), \]

or equivalently,

\[ \sum \sigma(g(1)h(1) \otimes l(1))\sigma^{-1}(g(2) \otimes h(2)l(2)) = \sum \sigma^{-1}(g \otimes h(1))\sigma(h(2) \otimes l), \]

and $\sigma(h \otimes 1) = \sigma(1 \otimes h) = \varepsilon(h)1$, for all $g, h, l \in H$. In general, the inverse $\sigma^{-1}$ of a 2-cocycle $\sigma$ on $H$ is not necessarily a 2-cocycle on $H$. But $\sigma^{-1}$ satisfies:

\[ \sum \sigma^{-1}(g(1)h(1) \otimes l)\sigma^{-1}(g(2) \otimes h(2)) = \sum \sigma^{-1}(g \otimes h(1)l(1))\sigma^{-1}(h(2) \otimes l(2)) \]

for all $g, h, l \in H$. We will use the equation (6) later in computations. Furthermore, a 2-cocycle $\sigma$ on $H$ satisfies the following identity by [3] Theorem 1.6(a5)]:

\[ \sum \sigma(h(1) \otimes S(h(2)))\sigma^{-1}(h(3) \otimes h(4)) = \varepsilon(h), \quad h \in H. \]

In [1], Y. Doi introduced a new Hopf algebra $H^\sigma$, called the $\sigma$-deformation of $H$ for a 2-cocycle $\sigma$ on $H$. The Hopf algebra $H^\sigma$ is equal to $H$ as a coalgebra. But its multiplication and its antipode are given by

\[ h \cdot \sigma h' = \sum \sigma(h(1) \otimes h'_1)h'_2(2)\sigma^{-1}(h(3) \otimes h'_3) \]

and

\[ S^\sigma(h) = \sum \sigma(h(1) \otimes S(h(2)))S(h(3))\sigma^{-1}(S(h(4)) \otimes h(5)), \]

respectively, where $h, h' \in H$ and $S$ is the antipode of $H$. If $S$ is bijective, then $S^\sigma$ is bijective as well and its inverse $(S^\sigma)^{-1}$ is given by

\[ (S^\sigma)^{-1}(h) = \sum \sigma^{-1}(h(5) \otimes S^{-1}(h(4)))S^{-1}(h(3))\sigma(S^{-1}(h(2)) \otimes h(1)) \]

for all $h \in H^\sigma$ (see [1, 10]).

From now on, $H$ will be a Hopf algebra with a bijective antipode $S$ and $\sigma$ will be a 2-cocycle on $H$ if it is not specified. We show in this section that the Yetter-Drinfeld module categories $H\mathcal{YD}_H$ and $H^\sigma\mathcal{YD}_{H^\sigma}$ are equivalent as braided monoidal categories. To this aim, we first construct a covariant functor from $H\mathcal{YD}_H$ to $H^\sigma\mathcal{YD}_{H^\sigma}$.
Lemma 2.1. Let $M$ be a YD $H$-module $M$. Then
(a) $M$ is a left $H^\sigma$-module with the $H^\sigma$-action given by
\begin{equation}
\begin{aligned}
h \mapsto m &= \sum (h_{(2)} \cdot m_{(0)})_{(0)} \sigma((h_{(2)} \cdot m_{(0)})(1) \otimes h_{(1)}) \sigma^{-1}(h_{(3)} \otimes m_{(1)}) \\
&= \sum h_{(3)} \cdot m_{(0)} \sigma(h_{(4)} m_{(1)} S^{-1}(h_{(2)}) \otimes h_{(1)}) \sigma^{-1}(h_{(5)} \otimes m_{(2)}),
\end{aligned}
\end{equation}
where $h \in H$ and $m \in M$.
(b) Denote by $\sigma(M)$ the left $H^\sigma$-module in (a). If $f : M \longrightarrow N$ is a YD $H$-module map, then $\sigma(f) = f : \sigma(M) \longrightarrow \sigma(N)$ is a left $H^\sigma$-module map.

Proof. (a) It is clear that $1 \mapsto m = m$ for all $m \in M$. Now let $h, l \in H^\sigma$ and $m \in M$. Then we have
\begin{align*}
l \mapsto (h \mapsto m) \\
&= \sum (l \mapsto (h_{(2)} \cdot m_{(0)})(0)) \sigma((h_{(2)} \cdot m_{(0)})(1) \otimes h_{(1)}) \sigma^{-1}(h_{(3)} \otimes m_{(1)}) \\
&= \sum l_{(2)} \cdot (h_{(2)} \cdot m_{(0)})(0) \sigma((l_{(2)} \cdot h_{(2)} \cdot m_{(0)})(0)(1) \otimes l_{(1)}) \\
&\hspace{1cm} \sigma^{-1}(l_{(3)} \otimes (h_{(2)} \cdot m_{(0)})(1)) \sigma((h_{(2)} \cdot m_{(0)})(2) \otimes h_{(1)}) \sigma^{-1}(h_{(3)} \otimes m_{(1)}) \\
&\hspace{1cm} + \sum l_{(2)} \cdot (h_{(3)} \cdot m_{(0)})(0) \sigma((l_{(2)} \cdot h_{(3)} \cdot m_{(0)})(0)(1) \otimes l_{(1)}) \\
&\hspace{1cm} \sigma(l_{(3)} \cdot m_{(0)})(1) \otimes h_{(1)}) \sigma^{-1}(l_{(4)} \otimes (h_{(3)} \cdot m_{(0)})(2) h_{(2)}) \\
&\hspace{1cm} \sigma^{-1}(l_{(5)} \otimes m_{(1)}) \\
&= \sum((l_{(3)} h_{(2)} \cdot m_{(0)})(0) \sigma((l_{(3)} h_{(2)} \cdot m_{(0)})(1) \otimes l_{(1)}) \\
&\hspace{1cm} \sigma((l_{(3)} h_{(2)} \cdot m_{(0)})(2) l_{(2)} \otimes h_{(1)}) \sigma^{-1}(l_{(4)} h_{(3)} \otimes m_{(1)}) \\
&\hspace{1cm} \sigma^{-1}(l_{(5)} \otimes h_{(4)}) \\
&= \sum((l_{(3)} h_{(2)} \cdot m_{(0)})(0) \sigma(l_{(1)} \otimes h_{(1)}) \sigma((l_{(3)} h_{(2)} \cdot m_{(0)})(1) \otimes l_{(2)} h_{(2)}) \\
&\hspace{1cm} \sigma^{-1}(l_{(4)} h_{(4)} \otimes m_{(1)}) \sigma^{-1}(l_{(5)} \otimes h_{(5)}) \\
&= \sum((l_{(3)} h_{(2)} \mapsto m) \sigma(l_{(1)} \otimes h_{(1)}) \sigma^{-1}(l_{(3)} \otimes h_{(3)}) \\
&= (l \mapsto h) \mapsto m.
\end{align*}

This shows that $M$ is a left $H^\sigma$-module.

(b) If $f : M \longrightarrow N$ is a YD $H$-module map, then $f$ is $H$-linear and $H$-colinear. By the definition of (11), we have for $h \in H^\sigma$ and $m \in M$,
\begin{align*}
f(h \mapsto m) &= \sum f((h_{(2)} \cdot m_{(0)})(0)) \sigma((h_{(2)} \cdot m_{(0)})(1) \otimes h_{(1)}) \sigma^{-1}(h_{(3)} \otimes m_{(1)}) \\
&= \sum h_{(3)} \cdot f(m_{(0)})(0) \sigma((h_{(2)} \cdot m_{(0)})(1) \otimes h_{(1)}) \sigma^{-1}(h_{(3)} \otimes f(m_{(1)})) \\
&= h \mapsto f(m).
\end{align*}

Following Lemma 2.1, we obtain a covariant functor $\sigma$ from the category $\mathcal{YD}^H$ to the category of left $H^\sigma$-modules. We show that the functor $\sigma$ is in fact a covariant functor into the category $\mathcal{YD}^{H^\sigma}$.

Lemma 2.2. Let $M$ be a YD $H$-module. Then $\sigma(M)$ is a YD $H^\sigma$-module with the inherited right $H$-comodule structure of $M$. 

Proof. Note that $H^\sigma = H$ as a coalgebra. Since $M$ is a right $H$-comodule, $\sigma(M) = M$ is also a right $H^\sigma$-comodule. By Lemma 2.1, $\sigma(M)$ is a left $H^\sigma$-module. It remains to show that $\sigma(M)$ satisfies the YD $H^\sigma$-module compatibility [2]. Now let $h \in H^\sigma$ and $m \in \sigma(M)$. We have

$$\sum (h(2) \rightarrow m(0)) \otimes (h(2) \rightarrow m(1)) \cdot \sigma h(1)$$

$$= \sum (h(3) \cdot m(0)) \otimes (h(3) \cdot m(0)) \cdot \sigma h(1) \cdot ((h(3) \cdot m(0)) \otimes h(1)) \cdot (h(3) \cdot m(0)) \cdot h(2) \cdot \sigma^{-1} (h(4) \otimes m(1))$$

$$= \sum h(2) \cdot m(0) \cdot \sigma ((h(2) \cdot m(0)) \otimes h(1)) \cdot (h(3) \cdot m(1) \cdot \sigma^{-1} (h(4) \otimes m(2)))$$

$$= \sum h(1) \cdot (\sigma((h(2) \cdot m(0)) \otimes h(1)) \cdot (h(3) \cdot m(1) \cdot \sigma^{-1} (h(4) \otimes m(3)))$$

$$= \sum (h(1) \rightarrow m(0)) \otimes h(2) \cdot \sigma m(1),$$

where the third equality follows from [2]. It follows that $\sigma(M)$ is a YD $H^\sigma$-module. \qed

Thus we have constructed a covariant functor $\sigma$ from the category $HYD^H$ to the category $HYD^{H^\sigma}$. We show that $\sigma$ is an equivalence monoidal functor and preserves the braidings. Consequently, the two braided monoidal categories $HYD^H$ and $HYD^{H^\sigma}$ are equivalent.

**Theorem 2.3.** Let $\sigma$ be a 2-cocycle on $H$. Then $\sigma : HYD^H \rightarrow HYD^{H^\sigma}$ is an equivalence braided monoidal functor.

Proof. By Lemma 2.1 and Lemma 2.2, $\sigma$ is a covariant functor. It remains to show that $\sigma$ is a braided monoidal functor and it has an inverse functor. For any YD $H$-modules $M$ and $N$, we define an isomorphism $\eta_{M,N}$:

$$\sigma(M) \otimes \sigma(N) \rightarrow \sigma(M \otimes N), m \otimes n \mapsto \sum m(0) \otimes n(0) \cdot \sigma^{-1} (n(1) \otimes m(1)).$$

It is easy to see that $(\sigma, \eta)$ is a monoidal functor from $HYD^H$ to $HYD^{H^\sigma}$. We show that $(\sigma, \eta)$ commutes with the braidings in the two categories. Recall that the braiding $\Phi$ of $HYD^H$ is defined by

$$\Phi_{M,N} : M \otimes N \rightarrow N \otimes M, \phi(m \otimes n) = \sum n(0) \otimes n(1) \cdot m$$

for $m \in M, n \in N$ and $M, N \in HYD^H$. We have to verify the following commutative diagram:

$$\sigma(M) \otimes \sigma(N) \xrightarrow{\Phi_{\sigma(M), \sigma(N)}} \sigma(N) \otimes \sigma(M)$$

$$\sigma(M \otimes N) \xrightarrow{\sigma(\Phi_{M,N})} \sigma(N \otimes M).$$
Let $m \in \mathcal{g}(M)$ and $n \in \mathcal{g}(N)$. Then we have

$$(\eta_{N,M}\Phi_{\mathcal{g}(M),\mathcal{g}(N)})(m \otimes n)$$

$$= \sum \eta_{N,M}(n(0) \otimes n(1) \mapsto m)$$

$$= \sum \eta_{N,M}(n(0) \otimes (n(2) \cdot m(0))_0)\sigma((n(2) \cdot m(0))_1 \otimes n(1))\sigma^{-1}(n(3) \otimes m(1))$$

$$= \sum n(0) \otimes (n(3) \cdot m(0))_0\sigma^{-1}((n(3) \cdot m(0))_1 \otimes n(1))\sigma((n(3) \cdot m(0))_2 \otimes n(2))$$

$$= \sum n(0) \otimes (n(1) \cdot m(0))\sigma^{-1}(n(2) \otimes m(1))$$

$$(\mathcal{g}(\Phi_{M,N})\eta_{M,N})(m \otimes n).$$

Next we show that the braided monoidal functor $\mathcal{g}$ has an inverse functor. Recall from [5, Lemma 1.2] that $\sigma^{-1}$ is a 2-cocycle on $H\sigma$. It is easy to see that $(H\sigma)^{\sigma^{-1}} = H$ as Hopf algebras. Thus $\sigma^{-1}$ induces a braided monoidal functor $(\mathcal{g}^{-1}, \xi)$ from $H^\sigma \mathcal{YD}H^\sigma$ to $\eta \mathcal{YD}H$ with $\mathcal{g}^{-1}(M) = M$ as right comodules, $\mathcal{g}^{-1}(f) = f$ and $\xi_{M,N} : \mathcal{g}^{-1}(M) \otimes \mathcal{g}^{-1}(N) \rightarrow \mathcal{g}^{-1}(M \otimes N)$, $m \otimes n \mapsto \sum m(0) \otimes n(0)\sigma(n(1) \otimes m(1))$

for all objects $M$, $N$ and all morphisms $f$ in the category $H^\sigma \mathcal{YD}H^\sigma$.

Now let $M$ be a YD $H$-module. Denote by $\rightarrow$ the left $(H\sigma)^{\sigma^{-1}} = H$-action on $\mathcal{g}^{-1}(\mathcal{g}(M))$. Then for any $m \in \mathcal{g}^{-1}(\mathcal{g}(M))$ and $h \in H$, we have

$$h \rightarrow m = \sum (h(2) \rightarrow m(0))_0\sigma^{-1}((h(2) \rightarrow m(0))_1 \otimes h(1))\sigma(h(3) \otimes m(1))$$

$$= \sum (h(3) \cdot m(0))_0\sigma((h(3) \cdot m(0))_2 \otimes h(2))\sigma^{-1}(h(4) \otimes m(1))$$

$$= \sum (h(3) \cdot m(0))_1 \otimes h(1))\sigma(h(5) \otimes m(2))$$

$$= \sigma^{-1}((h(3) \cdot m(0))_1 \otimes h(1))\sigma(h(5) \otimes m(2))$$

That is, $(\mathcal{g}^{-1}\mathcal{g})(M) = M$. On the other hand, it is not difficult to see that $\mathcal{g}^{-1}(\eta_{M,N}) \circ \mathcal{g}(\mathcal{g}^{-1}) \circ \mathcal{g}(\mathcal{g}^{-1})$ is the identity on $M \otimes N$, where we have $M \otimes N = (\mathcal{g}^{-1}\mathcal{g})(M) \otimes (\mathcal{g}^{-1}\mathcal{g})(N) = (\mathcal{g}^{-1}\mathcal{g})(M \otimes N)$ for any YD $H$-modules $M$ and $N$.

This shows that $(\mathcal{g}^{-1}, \xi) \circ (\mathcal{g}, \eta) = \text{id}$ on the braided monoidal category $H^\sigma \mathcal{YD}H^\sigma$. Similarly, we have $(\mathcal{g}, \eta) \circ (\mathcal{g}^{-1}, \xi) = \text{id}$ on the braided monoidal category $H^\sigma \mathcal{YD}H^\sigma$. Hence $\mathcal{g}$ is an equivalence braided monoidal functor. Consequently, $H^\sigma \mathcal{YD}H^\sigma$ and $H^\sigma \mathcal{YD}H^\sigma$ are equivalent braided monoidal categories.

Next we consider a CQT Hopf algebra $(H, R)$. Let $\sigma$ be a cocycle on $H$. By [7], we know that $H\sigma$ is also a CQT Hopf algebra with a CQT structure $R\sigma = (\sigma \tau) \ast (R \ast \sigma^{-1})$, that is,

$$R\sigma (g \otimes h) = \sum \sigma(h(1) \otimes g(1))R(g(2) \otimes h(2))\sigma^{-1}(g(3) \otimes h(3)).$$

for all $g, h \in H$. Now let $M \in \mathcal{M}_H$. In this case, the $H$-module structure is given by [8]. We show that the YD $H\sigma$-module $\mathcal{g}(M)$ is in $\mathcal{M}_H^\sigma$. Namely, the $H\sigma$-module structure of $\mathcal{g}(M)$ is induced by the CQT structure $R\sigma$ through [8]. Thus we get the following well-known result (see the dual version [8, Lemma XV 3.7]).
Corollary 2.4. Let $(H,R)$ be a CQT Hopf algebra and $\sigma$ be a cocycle on $H$. Then the equivalence braided monoidal functor $\sigma$ restricts to an equivalence braided monoidal functor from the category $\mathcal{M}_R^H$ to $\mathcal{M}_R^{H\sigma}$.

Proof. Let $M \in \mathcal{M}_R^H$. To show $\sigma(M) \in \mathcal{M}_R^{H\sigma}$, we have to verify that the $H^\sigma$-action (11) on $\sigma(M)$ coincides with the $H^\sigma$-action, denoted $\cdot \sigma$, induced by $R^\sigma$ and given by (4). Indeed, let $m \in \sigma(M)$ and $h \in H^\sigma$. We have

$$h \cdot \sigma m = \sum m_{(0)} R^\sigma(h \otimes m_{(1)})$$

$$= \sum m_{(0)} \sigma(m_{(1)} \otimes h_{(1)}) R(h_{(2)} \otimes m_{(2)}) \sigma^{-1}(h_{(3)} \otimes m_{(3)})$$

$$= \sum m_{(0)} R(h_{(3)} \otimes m_{(1)}) \sigma(h_{(4)} m_{(2)} S^{-1}(h_{(2)}) \otimes h_{(1)}) \sigma^{-1}(h_{(5)} \otimes m_{(3)})$$

$$= \sum (h_{(3)} \cdot \sigma m_{(0)}) \sigma(h_{(4)} m_{(1)} S^{-1}(h_{(2)}) \otimes h_{(1)}) \sigma^{-1}(h_{(5)} \otimes m_{(2)})$$

$$= h \cdot m,$$

where we use (CQT4′) in the third equality. So $\sigma(M) \in \mathcal{M}_R^{H\sigma}$. \hfill $\Box$

Let $A$ be a YD $H$-module algebra, namely, an algebra in the category $H \mathcal{YD}^H$. Then $\sigma(A)$ is a YD $H^\sigma$-module algebra with the multiplication given by

$$m^\sigma : \sigma(A) \otimes \sigma(A) \xrightarrow{\eta_A,\sigma} \sigma(A \otimes A) \xrightarrow{\sigma(m)} \sigma(A),$$

where $m : A \otimes A \to A$ is the multiplication of $A$. Denote by $a \cdot b$ the product $m^\sigma(a \otimes b)$ for any $a,b \in \sigma(A)$. Then

$$a \cdot b = \sum a_{(0)} h_{(0)} \sigma^{-1}(h_{(1)} \otimes a_{(1)}), \quad a,b \in \sigma(A).$$

That is, $m^\sigma = m \ast (\sigma^{-1} \tau)$. Hence as an algebra $\sigma(A)$ is the same as $A_{\sigma^{-1}\tau}$ given in [10] (2.27), where $\tau$ is the usual flip map.

Recall from [17] that a braided monoidal functor $F$ from a braided monoidal category $C$ to a braided monoidal category $D$ sends an Azumaya algebra in $C$ to an Azumaya algebra in $D$, and consequently induces a group homomorphism from the Brauer group $\mathrm{Br}(C)$ to the Brauer group $\mathrm{Br}(D)$. If the functor $F$ is an equivalence functor, then the two Brauer groups are isomorphic. Thus the equivalence functor $\sigma$ from $H \mathcal{YD}^H$ to $H^\sigma \mathcal{YD}^{H^\sigma}$ sends an $H$-Azumaya algebra $A$ to the $H^\sigma$-Azumaya algebra $\sigma(A)$ with the product (12) and induces an group isomorphism from the Brauer group $\mathrm{BQ}(k,H)$ to the Brauer group $\mathrm{BQ}(k,H^\sigma)$.

Corollary 2.5. Let $H$ be a Hopf algebra and $\sigma$ be a cocycle on $H$. Then the Brauer groups $\mathrm{BQ}(k,H)$ and $\mathrm{BQ}(k,H^\sigma)$ are isomorphic. Furthermore, if $(H,R)$ is a CQT Hopf algebra, then the two equivariant Brauer groups $\mathrm{BC}(k,H,R)$ and $\mathrm{BC}(k,H^\sigma,R^\sigma)$ are isomorphic [4].

In general, if $\sigma$ is a 2-cocycle on $H$ and $\sigma_1$ is a 2-cocycle on $H^\sigma$, then $\sigma_1 \ast \sigma$ is a 2-cocycle on $H$ and $H^{\sigma_1 \ast \sigma} = (H^\sigma)^{\sigma_1}$ by [4] Lemma 1.4. Moreover, it is not difficult to check that $\sigma_1(\sigma(A)) = (\sigma_1 \ast \sigma)(A)$ as YD $H^{\sigma_1 \ast \sigma}$-modules (or module algebras) if $A$ is a YD $H$-module (or module algebra).

A 2-cocycle $\sigma$ is called lazy if for all $h,l \in H$

$$\sum \sigma(h_{(1)} \otimes l_{(1)}) h_{(2)} l_{(2)} = \sum h_{(1)} \cdot l_{(1)} \sigma(h_{(2)} \otimes l_{(2)}).$$
In other words, a lazy cocycle $\sigma$ commutes with the multiplication of $H$. It is clear that a 2-cocycle $\sigma$ is lazy if and only if $H^\sigma = H$ as Hopf algebras (Lemma 1.3), i.e., the multiplication `$\cdot$' of $H^\sigma$ is the same as the original multiplication of $H$. Thus, if $\sigma$ is a lazy 2-cocycle on $H$ then $\sigma^{-1}$ is also a 2-cocycle on $H$ by Lemma 1.2. The set of all lazy 2-cocycles forms a group denoted $Z^2_\text{lazy}(H)$. The following corollary tells us that the group $Z^2_\text{lazy}(H)$ acts on the Brauer group $\text{BQ}(k,H)$ by automorphisms.

**Corollary 2.6.** Let $\sigma$ be a 2-cocycle on $H$. If $\sigma$ is lazy, then $\sigma$ induces an automorphism of the Brauer group $\text{BQ}(k,H)$

$$\sigma : \text{BQ}(k,H) \to \text{BQ}(k,H), [A] \mapsto [\sigma(A)].$$

**Proof.** Follows Corollary 2.5. \qed

Note that in [13] P. Schauenburg introduced a cohomology group $H^2_\text{co}(H)$ which is the quotient group of $Z^2_\text{lazy}(H)$ modulo the subgroup $B^2_\text{co}(H)$ consisting of coboundary lazy cocycles. The cohomology group has been systematically studied by J. Bichion and G. Carnovale in [1], where the group is called the lazy cohomology group, denoted $H^2_\text{co}(H)$. Precisely, a lazy cocycle $\sigma$ is coboundary if there is an (invertible) lazy 1-cocycle $\mu : H \to k$ such that $\sigma(a,b) = \mu(a_{(1)})\mu(b_{(1)})\mu^{-1}(a_{(2)}b_{(2)})$ for all $a,b \in H$, where $\mu$ satisfies the relations: $\sum \mu(a_{(1)})a_{(2)} = \sum \mu(a_{(2)})a_{(1)}$ and $\mu(1) = 1$ for all $a \in H$. That is, $\mu$ is a normalized central element in $H^*$. We show that if $\sigma$ is a coboundary lazy cocycle, then the functor $\sigma$ is isomorphic to the identity functor of $\text{BQ}(k,H)$.

**Corollary 2.7.** Let $\sigma$ be a coboundary lazy 2-cocycle on $H$, then $\sigma$ is isomorphic to the identity functor. Consequently, the lazy cohomology group $H^2_\text{co}(H)$ acts on $\text{BQ}(k,H)$ by automorphisms.

**Proof.** Let $\sigma \in B^2_\text{co}(H)$. That is, $\sigma(g \otimes h) = \sum \mu(g_{(1)})\mu(h_{(1)})\mu^{-1}(g_{(2)}h_{(2)})$ for any $g,h \in H$, where $\mu \in H^*$ is an invertible element such that $\mu(1) = 1$ and $\sum \mu(h_{(1)}h_{(2)}) = \sum h_{(1)}\mu(h_{(2)})$ for any $h \in H$. It is clear that $\sigma^{-1}(g \otimes h) = \sum \mu(g_{(1)}h_{(1)})\mu^{-1}(g_{(2)})\mu^{-1}(h_{(2)})$ for any $g,h \in H$.

Now for any $M \in \text{BQ}(k,H)$, define

$$\zeta_M : M \to \sigma(M), \quad m \mapsto \sum m_{(0)}\mu(m_{(1)}).$$

Then $\zeta_M$ is a YD $H$-module isomorphism. In fact, since $\mu$ is invertible, $\zeta_M$ is a $k$-linear isomorphism. Now for any $m \in M$ and $h \in H$, we have

$$\rho(\zeta_M(m)) = \sum \rho(m_{(0)})\mu(m_{(1)}) = \sum m_{(0)} \otimes \mu(m_{(2)}) = \sum m_{(0)} \otimes \mu(m_{(1)})m_{(2)} = \sum \zeta_M(m_{(0)}) \otimes m_{(1)}.$$
and
\[
\zeta_M(m) = \sum (h \to m_{(0)}) \mu(m_{(1)})
\]
\[
= \sum (h_{(2)} \cdot m_{(0)})(0) \mu((h_{(2)} \cdot m_{(0)})(1) \otimes h_{(1)}) \sigma^{-1}(h_{(3)} \otimes m_{(1)}) \mu(m_{(2)})
\]
\[
= \sum (h_{(3)} \cdot m_{(0)})(0) \mu((h_{(3)} \cdot m_{(0)})(1)) \mu(h_{(1)}) \mu^{-1}((h_{(3)} \cdot m_{(0)})(2)) h_{(2)}
\]
\[
\mu(h_{(4)} m_{(1)}) m_{(2)}
\]
\[
= \sum (h_{(3)} \cdot m_{(0)})(0) \mu((h_{(3)} \cdot m_{(0)})(1)) \mu(h_{(1)}) \mu^{-1}((h_{(3)} \cdot m_{(0)})(2)) h_{(2)}
\]
\[
\mu(h_{(4)} m_{(1)}) m_{(2)}
\]
\[
= \sum (h_{(2)} \cdot m_{(0)})(0) \mu((h_{(2)} \cdot m_{(0)})(1)) \mu^{-1}((h_{(2)} \cdot m_{(0)})(2)) h_{(1)}
\]
\[
\mu(h_{(3)} m_{(1)})
\]
\[
= \sum (h_{(2)} \cdot m_{(0)})(0) \mu((h_{(2)} \cdot m_{(0)})(1)) h_{(1)} m_{(3)}
\]
\[
= \sum \zeta_M(h_{(2)} \cdot m_{(0)})(0) \mu^{-1}((h_{(2)} \cdot m_{(0)})(1)) h_{(1)} m_{(3)}
\]
\[
= \zeta_M(h \cdot m).
\]

This shows that \(\zeta_M\) is a YD \(H\)-module isomorphism from \(M\) to \(\Pi(M)\). It is easy to see that \(\zeta\) is a natural transformation from the identity functor \(\text{Id}\) to the functor \(\Pi\).

Furthermore, for any YD \(H\)-modules \(M\) and \(N\), we have the following commutative diagram:
\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{\zeta_M \otimes \zeta_N} & \Pi(M) \otimes \Pi(N) \\
(\eta_{M,N} \circ \zeta_M \otimes \zeta_N) \circ (\zeta_M \otimes N) & \xrightarrow{\eta_{M,N}} & \zeta_M \otimes \zeta_N \circ (\Pi(M) \otimes \Pi(N)).
\end{array}
\]

In fact, let \(m \in M\) and \(n \in N\). We have
\[
(\eta_{M,N} (\zeta_M \otimes \zeta_N))(m \otimes n) = \sum \eta_{M,N}(m_{(0)} \otimes n_{(0)}) \mu(m_{(1)}) \mu(n_{(1)})
\]
\[
= \sum \mu(m_{(0)} \otimes n_{(0)}) \sigma^{-1}(m_{(1)} \otimes m_{(1)}) \mu(m_{(2)}) \mu(n_{(2)})
\]
\[
= \sum \mu(m_{(0)} \otimes n_{(0)}) \mu(m_{(1)} m_{(1)})
\]
\[
= \sum (m \otimes n_{(0)}) \mu(m \otimes n_{(1)})
\]
\[
= \zeta_M(m \otimes n).
\]

It follows from the above commutative diagram that \(\zeta_A : A \rightarrow \Pi(A)\) is also an algebra homomorphism, and hence a YD \(H\)-module algebra isomorphism if \(A\) is a YD \(H\)-module algebra. Thus we have proved that \(B^2(H)\) acts trivially on the Brauer group \(BQ(k, H)\). Hence \(H^2(H)\) acts on \(BQ(k, H)\) by automorphisms. \(\square\)

For completeness, we now give the dual versions of Theorem 2.3 and Corollary 2.4. We will omit the detail of proofs and give the sketch of the construction. Let \(H\) be a Hopf algebra with a bijective antipode in the sequel. For an element \(r \in H \otimes H\), we let \(r_{12}\) and \(r_{23}\) stand for \(r \otimes 1\) and \(1 \otimes r\) in \(H \otimes H \otimes H\) respectively.

Recall from [10] that an invertible element \(\theta = \sum \theta^{(1)} \otimes \theta^{(2)} \in H \otimes H\) is called a dual 2-cocycle if it satisfies:
\[
\theta_{12}(\Delta \otimes \text{id})(\theta) = \theta_{23}((\text{id} \otimes \Delta)(\theta)).
\]
and \((\varepsilon \otimes \text{id})(\theta) = (\text{id} \otimes \varepsilon)(\theta) = 1\).

To a dual 2-cocycle \(\theta \in H \otimes H\) one may associate a new Hopf algebra \(H_\theta\). As an algebra \(H_\theta = H\) but with comultiplication given by
\[
\Delta_\theta(h) = \theta \Delta(h) \theta^{-1}, \quad h \in H_\theta.
\]
The antipode $S_\theta$ of $H_\theta$ is given by

$$S_\theta(h) = \sum \theta^{(1)} S(\theta^{(2)}) S(h) S((\theta^{-1})^{(1)}(\theta^{-1})^{(2)}), \quad h \in H_\theta,$$

where $\theta = \sum \theta^{(1)} \otimes \theta^{(2)}$ and $\theta^{-1} = \sum (\theta^{-1})^{(1)} \otimes (\theta^{-1})^{(2)}$ in $H \otimes H$. Note that $S_\theta$ is bijective since $S$ is bijective.

Let $M$ be a Yetter-Drinfeld $H$-module with the coaction $\rho(m) = \sum m_{(0)} \otimes m_{(1)}$, $m \in M$. Then one can define a new $H_\theta$-coaction $\rho_\theta : M \rightarrow M \otimes H_\theta$ by

$$\rho_\theta(m) = \sum \theta^{(1)} \cdot ((\theta^{-1})^{(2)} \cdot m_{(0)} \otimes \theta^{(2)}((\theta^{-1})^{(2)} \cdot m_{(1)})(\theta^{-1})^{(1)} = \sum \theta^{(1)}(\theta^{-1})^{(2)} \cdot m_{(0)} \otimes \theta^{(2)}(\theta^{-1})^{(2)} m_{(1)} S^{-1}((\theta^{-1})^{(2)})(\theta^{-1})^{(1)}.$$

Since $H_\theta = H$ as algebras, $M$ is a left $H_\theta$-module. It is straightforward to verify that the inherited $H_\theta$-module $M$ with the $H_\theta$-comodule structure $\rho_\theta$ given by (13) is a Yetter-Drinfeld $H_\theta$-module, denoted $\mathcal{YD}(M)$. Thus a dual cocycle $\theta$ induces a covariant functor $\mathcal{YD}$ from $\mathcal{H}_H^\text{YD}$ to $\mathcal{H}_H^\text{YD}$ sending a YD $H$-module $M$ to the YD $H_\theta$-module $\mathcal{YD}(M)$ and a YD $H$-module morphism $f$ to a YD $H_\theta$-module morphism $\mathcal{YD}(f) = f$.

For any two Yetter-Drinfeld $H$-modules $M$ and $N$, let

$$\phi_{M,N} : \mathcal{YD}(M) \otimes \mathcal{YD}(N) \rightarrow \mathcal{YD}(M \otimes N), \quad m \otimes n \rightarrow \theta^{-1}(m \otimes n) = \sum (\theta^{-1})^{(1)} \cdot m \otimes (\theta^{-1})^{(2)} \cdot n.$$

Then $\phi_{M,N}$ is a family of natural isomorphisms and $(\mathcal{YD}, \phi)$ gives a braided monoidal category equivalence from $\mathcal{H}_H^\text{YD}$ to $\mathcal{H}_H^\text{YD}$. We summarize our above discussion in the following dual version of Theorem 2.3.

**Theorem 2.8.** Let $\theta \in H \otimes H$ be a dual 2-cocycle. Then the functor $(\mathcal{YD}, \phi)$ is an equivalence braided monoidal functor from $\mathcal{H}_H^\text{YD}$ to $\mathcal{H}_H^\text{YD}$.

Now assume that $H$ is a QT Hopf algebra with a QT structure $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. Let

$$\mathcal{R}_\theta := \tau(\theta)\mathcal{R}\theta^{-1} = \sum \theta^{(2)} \mathcal{R}^{(1)}(\theta^{-1})^{(1)} \otimes \theta^{(1)} \mathcal{R}^{(2)}(\theta^{-1})^{(2)}.$$

Then $\mathcal{R}_\theta$ is a QT structure of $H_\theta$. In this case, restricting the functor $(\mathcal{YD}, \phi)$ on the subcategory $\mathcal{H}_H^\mathcal{R}$ gives a braided monoidal category equivalence from $\mathcal{H}_H^\mathcal{R}$ to $\mathcal{H}_H^\mathcal{R}_\theta$. Hence we have the following corollary dual to Corollary 2.4.

**Corollary 2.9.** [Lemma XV 3.7] Let $(H, \mathcal{R})$ be a QT Hopf algebra and let $\theta \in H \otimes H$ be a dual 2-cocycle. Then the equivalence braided monoidal functor $(\mathcal{YD}, \phi)$ restricts to an equivalence braided monoidal functor from $\mathcal{H}_H^\mathcal{R}$ to $\mathcal{H}_H^\mathcal{R}_\theta$.

Let $A$ be a Yetter-Drinfeld $H$-module algebra with multiplication $m : A \otimes A \rightarrow A$. It follows from Theorem 2.3 that $\mathcal{YD}(A)$ is a Yetter-Drinfeld $H_\theta$-module algebra with the multiplication given by

$$m_\theta : \mathcal{YD}(A) \otimes \mathcal{YD}(A) \xrightarrow{\delta} \mathcal{YD}(A \otimes A) \xrightarrow{\mathcal{YD}(m)} \mathcal{YD}(A).$$

Let $a \bullet b := m_\theta(a \otimes b)$ for any $a, b \in \mathcal{YD}(A)$. Then

$$a \bullet b = \sum ((\theta^{-1})^{(1)} \cdot a)((\theta^{-1})^{(2)} \cdot b), \quad a, b \in \mathcal{YD}(A).$$
If \( A \) is an \( H \)-Azumaya algebra, then \( \theta(A) \) is an \( H_\theta \)-Azumaya algebra. Thus, the functor \( \theta \) induces an group isomorphism from \( BQ(k, H) \) to \( BQ(k, H_\theta) \).

**Corollary 2.10.** Let \( H \) be a Hopf algebra and \( \theta \) be a dual 2-cocycle in \( H \). Then the Brauer group \( BQ(k, H) \) is isomorphic to the Brauer group \( BQ(k, H_\theta) \). Moreover, if \( (H, \mathcal{R}) \) is a QT Hopf algebra, then the two equivariant Brauer groups \( BM(k, H, \mathcal{R}) \) and \( BM(k, H_\theta, \mathcal{R}_0) \) are isomorphic.

A dual 2-cocycle \( \theta \in H \otimes H \) is said to be lazy if \( \theta \Delta(h) = \Delta(h)\theta \) for all \( h \in H \). This is equivalent to \( \Delta_\theta = \Delta \), i.e., \( H_\theta = H \) as Hopf algebras. Note that for a lazy dual 2-cocycle \( \theta \), its inverse \( \theta^{-1} \) is a dual 2-cocycle as well. Let \( Z_2(\theta) \) be the set of all lazy dual 2-cocycles in \( H \). Then \( Z_2(\theta) \) forms a group, which acts on the Brauer group \( BQ(k, H) \) by automorphisms like the group \( Z_2(\theta) \) does. Let \( B_2(\theta) \) be the subgroup of \( Z_2(\theta) \) consisting of coboundary lazy dual cocycle \( \theta \), i.e., \( \theta = (u \otimes u)\Delta(u^{-1}) \) for some central invertible element \( u \in H \) with \( \varepsilon(u) = 1 \).

**Corollary 2.11.** The cohomology group \( H^2_2(\theta) \) acts on the Brauer group \( BQ(k, H) \) by automorphisms.

**Example 2.12.** Let \( k \) be a field with \( ch(k) \neq 2 \). Let \( H_4 \) be Sweedler’s 4-dimensional Hopf algebra over \( k \) generated by two elements \( g \) and \( h \) satisfying the relations: \( g^2 = 1, \ v^2 = 0, \ gh + hg = 0 \). The comultiplication, the counit and the antipode of \( H_4 \) are given by

\[
\Delta(g) = g \otimes g, \quad \Delta(h) = 1 \otimes h + h \otimes g, \quad \varepsilon(g) = 1, \quad \varepsilon(h) = 0, \quad S(g) = g, \quad S(h) = gh.
\]

It is well-known that \( H_4 \) has a family of QT structures parameterized by \( t \in k \):

\[
\mathcal{R}_t = \frac{1}{2}((1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) + \frac{t}{2}(1 \otimes 1 + g \otimes g + 1 \otimes g - g \otimes 1))(h \otimes h).
\]

A straightforward computation shows that the group \( Z_2(\mathcal{R}_t) \) of the lazy dual 2-cocycles \( \theta_t \) is isomorphic to the additive group \( k^+ \), where \( \theta_t \) are given as follows:

\[
\theta_t = 1 \otimes 1 + \frac{t}{2}h \otimes gh, \quad t \in k.
\]

Note that \( H_4 \) is a self-dual Hopf algebra. It has a family of CQT structures \( R_t \) and the cohomology group \( H^2_2(\mathcal{R}_t) = Z_2^2(\mathcal{R}_t) \) is isomorphic to \( k^+ \) [Example 2.1], where \( R_t \) and lazy cocycles \( \sigma_t \) are given as follows:

| \( R_t \) | 1 | 1 | \( h \) | \( gh \) | \( \sigma_t \) | 1 | 1 | \( h \) | \( gh \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| \( g \) | 1 | -1 | 0 | 0 | \( g \) | 1 | 1 | 0 | 0 |
| \( h \) | 0 | 0 | \( t \) | -\( t \) | \( h \) | 0 | 0 | \( t/2 \) | -\( t/2 \) |
| \( gh \) | 0 | 0 | \( t \) | \( t \) | \( gh \) | 0 | 0 | \( t/2 \) | -\( t/2 \), |

where \( t \in k \). The action of the automorphism \( \sigma_{\mathcal{R}_t} \) on \( BQ(k, H_4) \) induced by a lazy cocycle \( \sigma_t \) is of particular interest, for \( \sigma_{\mathcal{R}_t} \) moves the equivariant Brauer group \( BC(k, H_4, R_t) \) to \( BC(k, H_4, R_{t-s}) \) and fixes the subgroup \( BW(k) \), the Brauer-Wall group of \( k \). By [12, 13] and Corollary 2.11, we have \( BC(k, H_4, R_t) \cong BW(k) \times k^+ \), where \( k^+ \cong k^+ \). It is not hard to show that \( k^+_s \cap k^+_t = 1 \) if \( s \neq t \) in \( k \). Thus, no distinct two elements \( \sigma_t \) and \( \sigma_{t'} \) act as the same automorphism of \( BQ(k, H_4) \). It follows that \( H^2_2(\mathcal{R}_t) \cong k^+ \) is a subgroup of the automorphism group of \( BQ(k, H_4) \).
Similarly, the group \( H_2^L(H_4) \cong k^+ \) is a subgroup of the automorphism group of \( BQ(k, H_4) \). We show that the intersection of the two subgroups \( H_2^L(H_4) \) and \( H_2^L(H_4) \) is trivial. One may take a while to compute that \( \sigma_s \) restricts to the identity automorphism of the subgroup \( \text{BM}(k, H_4, R) \) for any \( s \in k \). But \( \theta_{s'} \) moves \( \text{BM}(k, H_4, R) \) to the subgroup \( \text{BM}(k, H_4, R_{-s'}) \) for \( s' \in k \). Thus \( \theta_s \neq \sigma_s \) as automorphisms except for \( s = s' = 0 \).

3. The group of bigalois objects

Throughout this section \((H, R)\) will be a finite CQT Hopf algebra. In \cite{19}, the author constructed a group \( \text{Gal}(H_R) \) of \( H_R \)-bigalois objects, where \( H_R \) is a braided Hopf algebra in the category \( \mathcal{M}_R^H \), in order to compute the equivariant Brauer group \( \text{BC}(k, H, R) \). In this section, we study the group \( \text{Gal}(H_R) \) under a cocycle deformation. We will show that the group \( \text{Gal}(H_R) \) is stable under a cocycle deformation. That is, \( \text{Gal}(H_R) \) and \( \text{Gal}(H_{R''}) \) are isomorphic groups.

Recall that \( H_R \) is a braided Hopf algebra in the category \( \mathcal{M}_R^H \) defined by means of \( H \). As a coalgebra, \( H_R \) coincides with \( H \). The multiplication \( * \) and the antipode \( S_R \) are given by

\[
\begin{align*}
h \ast l &= \sum l_{(2)}h_{(2)}R(S^{-1}(l_{(3)})l_{(1)} \otimes h_{(1)}), \text{ and} \\
S_R(h) &= \sum S(h_{(2)})R(S^2(h_{(3)})S(h_{(1)}) \otimes h_{(4)})
\end{align*}
\]

respectively, where \( h, l \in H \). As an object in \( \mathcal{M}_R^H \), \( H_R \) has the adjoint coaction:

\[
\rho(h) = \sum h_{(2)} \otimes S(h_{(1)})h_{(3)}, \quad h \in H_R.
\]

For the detail of \( H_R \), the reader can refer to \cite{19} Theorem 4.1 or \cite{19} Lemma 2.1.

Let \( M \) be a YD \( H \)-module. We use the Sweedler notation \( \sum m_{(0)} \otimes m_{(1)} \) for the \( H \)-comodule structure of \( M \) as usual, and use the following summation notation for the dual \( H^* \)-comodule structure of the \( H \)-module structure of \( M \):

\[
(14) \quad M \to M \otimes H^*, \quad m \mapsto \sum m_{(0)} \otimes m_{(1)}.
\]

A natural left \( H_R \)-module structure stemming from the YD \( H \)-module structure of \( M \) is as follows:

\[
(15) \quad h \mapsto m = \sum S^{-1}(h_{(2)}) \triangleright_1 (h_{(1)} \cdot m) = \sum (h_{(2)} \cdot m_{(0)})R(S^{-1}(h_{(4)}) \otimes h_{(3)})m_{(1)}S^{-1}(h_{(1)}))
\]

for \( h \in H_R, \ m \in M, \) where \( \triangleright_1 \) stands for the action \( H \). Observe that the right \( H \)-comodule structure of \( M \) induces two left \( H \)-module structures. The first one is given by \( \triangleright_1 \), and the second one is given by

\[
(16) \quad h \triangleright_2 m = \sum m_{(0)}R(S(m_{(1)}) \otimes h) = \sum m_{(0)}R(m_{(1)} \otimes S^{-1}(h))
\]

for \( h \in H \) and \( m \in M \). Note that a right \( H \)-module \( M \) is simultaneously a YD \( H \)-module with the left \( H \)-action \( \triangleright_1 \). Now we define a right \( H_R \)-module structure on \( M \) as follows:

\[
(17) \quad m \leftarrow h = \sum S(h_{(1)}) \triangleright_2 (h_{(2)} \cdot m) = \sum (h_{(3)} \cdot m_{(0)})R(h_{(4)}m_{(1)}S^{-1}(h_{(2)}) \otimes h_{(1)})
\]
for \( m \in M \) and \( h \in \mathcal{H}_R \). It follows from [19] that \( M \) is an \( \mathcal{H}_R \)-bimodule with the actions (15) and (17), and consequently \( M \) is an \( \mathcal{H}_R^* \)-bicomodule. Denote the dual left and right \( \mathcal{H}_R^* \)-comodule structures by

- \( M \to \mathcal{H}_R^* \otimes M, \quad m \mapsto \sum m^{(0)} \otimes m^{(1)} \),
- \( M \to M \otimes \mathcal{H}_R^*, \quad m \mapsto \sum m^{(-1)} \otimes m^{(0)} \),

respectively. Then \( m \prec h = \sum \langle m^{(-1)}, h \rangle m^{(0)} \) and \( h \rhd m = \sum \langle m^{(1)}, h \rangle m^{(0)} \).

Similarly, we have a braided Hopf algebra \( \mathcal{H}_R^\sigma \) in the category \( \mathcal{M}_{R_{\sigma}} \), and one can define an \( \mathcal{H}_R^\sigma \)-bimodule structure on any \( \mathcal{YD}_{R^\sigma} \)-module \( M \). Consequently, \( M \) has an \( (\mathcal{H}_R^\sigma)^* \)-comodule structure similar to (15) and (17) respectively. Denote the left and right dual \( (\mathcal{H}_R^\sigma)^* \)-comodule structures by

- \( M \to (\mathcal{H}_R^\sigma)^* \otimes M, \quad m \mapsto \sum m^{<0>} \otimes m^{<1>}, \)
- \( M \to M \otimes (\mathcal{H}_R^\sigma)^*, \quad m \mapsto \sum m^{<1>} \otimes m^{<0>}, \)

respectively. If \( M \) is a \( \mathcal{YD}_{R^\sigma} \)-module, then the left \( \mathcal{H}_R^\sigma \)-module structure of \( M \) induces a dual right \( (\mathcal{H}_R^\sigma)^* \)-comodule structure similar to (14). Let us denote the right \( (\mathcal{H}_R^\sigma)^* \)-comodule structure by

\( M \to M \otimes (\mathcal{H}_R^\sigma)^*, \quad m \mapsto \sum m^{<0>} \otimes m^{<1>} \).

Now let \( \mathcal{O} M \) and \( \mathcal{O}_\sigma \) stand for the left and right \( \mathcal{H}_R^\sigma \)-coinvariant submodules respectively. That is,

\[
\mathcal{O} M = \{ m \in M | \sum m^{(-1)} \otimes m^{(0)} = \varepsilon \otimes m \} = \{ m \in M | m \prec h = \varepsilon(h)m, \forall h \in \mathcal{H}_R \} \]

and

\[
\mathcal{O}_\sigma = \{ m \in M | \sum m^{(0)} \otimes m^{(1)} = m \otimes \varepsilon \} = \{ m \in M | h \rhd m = \varepsilon(h)m, \forall h \in \mathcal{H}_R \}. \]

**Lemma 3.1.** [19] Lemma 2.5, p332] Let \( M \) be a \( \mathcal{YD}_{H^\sigma} \)-module. Then we have

(a) \( \mathcal{O}_\sigma = \{ m \in M | h \cdot m = h \rhd_1 m, \forall h \in H \} \),

(b) \( \mathcal{O} M = \{ m \in M | h \cdot m = h \rhd_2 m, \forall h \in H \} \).

The following lemma says that the left and right \( \mathcal{H}_R \)-invariant functors \( - \mathcal{O} \) and \( \mathcal{O}(-) \) are subfunctors from \( _{\mathcal{YD}^H} \) to \( ^{\mathcal{YD}^H} \).

**Lemma 3.2.** Let \( M \) be a \( \mathcal{YD}_{H^\sigma} \)-module. Then \( \mathcal{O}_\sigma \) and \( \mathcal{O} M \) are \( \mathcal{YD}_{H} \)-submodule of \( M \).
Proof. Let \( m \in M_o \). It follows from Lemma 3.1(a) that \( h \cdot m = h \triangleright_1 m = \sum m(0)_0 R(h \otimes m(1)) \) for any \( h \in H \). Hence we have

\[
\sum h \cdot m(0) \otimes m(1) = \sum h(1) \cdot m(0) \otimes S^{-1}(h(3))(h(2))m(1) \\
= \sum (h(2) \cdot m)(0) \otimes S^{-1}(h(3))(h(2) \cdot m)(1)h(1) \\
= \sum (h(2) \triangleright_1 m)(0) \otimes S^{-1}(h(3))(h(2) \triangleright_1 m)(1)h(1) \\
= \sum m(0) \otimes S^{-1}(h(3))m(1)h(1)R(h(2) \otimes m(2)) \\
= \sum m(0) \otimes R(h(1) \otimes m(1))S^{-1}(h(3))h(2)m(2) \\
= \sum m(0)R(h \otimes m(1)) \otimes m(2) \\
= \sum h \triangleright_1 m(0) \otimes m(1)
\]

for all \( h \in H \). It follows from Lemma 3.1(a) that \( \sum m(0) \otimes m(1) \in M_o \otimes H \). This shows that \( M_o \) is an \( H \)-submodule of \( M \). Then for any \( m \in M_o \) and \( h \in H \), we have \( h \cdot m = h \triangleright_1 m = \sum m(0)_0 R(h \otimes m(1)) \in M_o \). Hence \( M_o \) is an \( H \)-submodule of \( M \), and so \( M_o \) is a \( YD \) \( H \)-submodule of \( M \). Similarly, using Lemma 3.1(b), one can show that \( \sigma M \) is also a \( YD \) \( H \)-submodule of \( M \). \( \square \)

Next we show that the equivalence functor \( \sigma \) commutes with both \( \mathcal{H}_R \)-invariant functors \( \sigma(\cdot) \) and \( (-)_o \).

**Lemma 3.3.** Let \( M \) be a \( YD \) \( H \)-module. Then \( \sigma(\mathcal{H}_R) = \sigma(A) \sigma(M) = \sigma(M) \) as \( k \)-modules, and hence \( \sigma(M_o) = \sigma(\mathcal{H}_R) \sigma(M) = \sigma(\mathcal{H}_R) \sigma(M) = \sigma(M_o) \) as \( YD \) \( H^o \)-modules. In particular, if \( A \) is a \( YD \) \( H \)-module algebra, then \( \sigma(A_o) = \sigma(\mathcal{H}_R) \sigma(A) = \sigma(\mathcal{H}_R) \sigma(A) \) as \( YD \) \( H^o \)-module algebras.

**Proof.** By Corollary 2.7 and its proof, one can see \( M_o \subseteq \sigma(M) \). Since \( \sigma(M) \) is a \( YD \) \( H^o \)-module and \( \sigma^{-1} \) is a 2-cocycle on \( H^o \), we also have \( \sigma(M)_o \subseteq \sigma^{-1}(\sigma(M)) \) is a \( YD \) \( H^o \)-module. Hence \( \sigma(M)_o = M_o \).

Let \( m \in \sigma M \) and \( h \in H^o \). Then it follows from Lemma 3.1 and Lemma 3.2 that

\[
\sum h \triangleright_2 m = \sum m(0) R^o(m(1) \otimes (S^{o})^{-1}(h)) \\
= \sum m(0) R^o(m(1) \otimes S^{-1}(h(3))h(4)) \sigma(S^{-1}(h(2)) \otimes h(1)) \\
\sigma^{-1}(h(4))S^{-1}(h(3)) \sigma(S^{-1}(h(2)) \otimes h(1)) \sigma^{-1}(h(7))S^{-1}(h(6)) \\
= \sum m(0)R(m(1) \otimes S^{-1}(h(4))h(5) \otimes m(1)_0)R(m(2) \otimes S^{-1}(h(4))) \\
\sigma^{-1}(m(3))S^{-1}(h(3)) \sigma(S^{-1}(h(2)) \otimes h(1)) \sigma^{-1}(h(7))S^{-1}(h(6)) \\
= \sum m(0)R(m(1) \otimes S^{-1}(h(4))) \sigma(S^{-1}(h(8)) \otimes h(7))S^{-1}(h(5)) \\
\sigma^{-1}(m(3))S^{-1}(h(4)) \otimes h(1)) \sigma^{-1}(m(4)) \otimes S^{-1}(h(3))h(2) \\
\sigma^{-1}(h(10)) \otimes S^{-1}(h(9)) \\
= \sum h(4) \triangleright_2 m(0) \sigma(S^{-1}(h(8)) \otimes h(5)m(1)_0S^{-1}(h(3))) \\
\sigma(S^{-1}(h(7))h(6)m(2)S^{-1}(h(2)) \otimes h(1)) \sigma^{-1}(h(10)) \otimes S^{-1}(h(9)) \\
= \sum m(0) \sigma(h(4)m(1)_0S^{-1}(h(2)) \otimes h(1)) \\
\sigma(S^{-1}(h(6)) \otimes h(5)m(2)) \sigma^{-1}(h(8)) \otimes S^{-1}(h(7))h(6)m(3) \\
= \sum h(3) \cdot m(0) \sigma(h(4)m(1)_0S^{-1}(h(2)) \otimes h(1)) \sigma(h(5)) \otimes m(2) \\
= \sum h \cdot m(0)
\]
where we use (CQT4') and (17) to obtain the fourth equality, use (16) to obtain the sixth equality and use (14) to get the seventh equality. Again, from Lemma 3.1(b) one gets \( m \in \sigma(M) \), and so \( \sigma M \subseteq \sigma(M) \). Now replacing \( H, \sigma \) and \( M \) with \( H^\sigma, \sigma^{-1} \) and \( \sigma(M) \) respectively, one also gets \( \sigma(M) \subseteq \sigma^{-1}(\sigma(M)) = \sigma M \). Hence we have \( \sigma(M) = \sigma M \).

If \( A \) is a YD \( H \)-module algebra, then \( A_c \) and \( \sigma A \) are subalgebras of \( A \). Thus the last statement follows immediately. \( \square \)

Given two YD \( H \)-modules \( M \) and \( N \), a generalized cotensor product \( M \wedge N \) was introduced in [19]. That is,
\[
M \wedge N = \left\{ \sum_{i} m_i \otimes n_i \in M \otimes N \mid \sum_{i} (m_i \triangleleft h) \otimes n_i = \sum_{i} m_i \otimes (h \triangleright n_i), \forall h \in \mathcal{H}_R \right\}.
\]

Observe that \( M \wedge N \) is still an \( \mathcal{H}_R \)-bimodule with the left and right \( \mathcal{H}_R \)-module structures stemming from the left \( \mathcal{H}_R \)-module structure of \( M \) and the right \( \mathcal{H}_R \)-module structure of \( N \). In fact, this \( \mathcal{H}_R \)-bimodule structure of \( M \wedge N \) comes from a YD \( H \)-module structure on \( M \wedge N \) by (15) and (17). The YD \( H \)-module structure of \( M \wedge N \) can be described as follows:

The left \( H \)-action on \( M \wedge N \) is given by

\[
(18) \quad h \cdot \sum_{i} (m_i \otimes n_i) = \sum_{i} h_{(1)} \cdot m_i \otimes h_{(2)} \triangleright 1 n_i = \sum_{i} h_{(1)} \triangleright 2 m_i \otimes h_{(2)} \cdot n_i,
\]

where \( \sum_{i} m_i \otimes n_i \in M \wedge N \) and \( h \in H \). The right \( H \)-comodule structure of \( M \wedge N \) inherits from \( M \otimes N \). That is,

\[
(19) \quad M \wedge N \to (M \wedge N) \otimes H, \quad \sum_{i} m_i \otimes n_i \mapsto \sum_{i} (m_{i(0)} \otimes n_{i(0)}) \otimes n_{i(1)} m_{i(1)}.
\]

Let \( \sum_{i} m_i \otimes n_i \in M \otimes N \). Then by [19] Lemma 2.9 we know that \( \sum_{i} m_i \otimes n_i \in M \wedge N \) if and only if

\[
(20) \quad \sum_{i} h_{(1)} \cdot m_i \otimes h_{(2)} \triangleright 1 n_i = \sum_{i} h_{(1)} \triangleright 2 m_i \otimes h_{(2)} \cdot n_i, \ \forall h \in H.
\]

Now we show that the equivalence functor \( \sigma \) preserves the generalized cotensor product \( \wedge \).

**Lemma 3.4.** Let \( M \) and \( N \) be YD \( H \)-modules. Then \( \sigma(M) \wedge \sigma(N) \cong \sigma(M \wedge N) \) as YD \( H^\sigma \)-modules.

**Proof.** By Theorem 2.3 and its proof, we have a YD \( H^\sigma \)-module isomorphism

\[
\eta_{M,N}^{-1} : \sigma(M \otimes N) \to \sigma(M) \otimes \sigma(N), \quad x \otimes y \mapsto \sum x_{(0)} \otimes y_{(0)} \sigma(y_{(1)}) \otimes x_{(1)}.
\]

We show that \( \eta_{M,N}^{-1} \) restricts to an isomorphism from \( \sigma(M \wedge N) \) to \( \sigma(M) \wedge \sigma(N) \). In order to simplify the computation, we write \( x \otimes y \) for an element \( \sum x_i \otimes y_i \) in \( M \wedge N \). Let \( x \otimes y \in M \wedge N \). We show that the element \( \eta_{M,N}^{-1}(x \otimes y) = \sum x_{(0)} \otimes y_{(0)} \sigma(y_{(1)}) \otimes x_{(1)} \) sits in \( \sigma(M) \wedge \sigma(N) \). By the proof of [19] Proposition 2.10, \( M \wedge N \) is an \( H \)-subcomodule of \( M \otimes N \). Hence \( \sum (x_{(0)} \otimes y_{(0)}) \otimes x_{(1)} \in (M \wedge N) \otimes H \).

Following 2.8 we have the identity for all \( h \in H \):

\[
(21) \quad \sum h_{(1)} \cdot x_{(0)} \otimes h_{(2)} \triangleright 1 y_{(0)} \otimes y_{(1)} x_{(1)} = \sum h_{(1)} \triangleright 2 x_{(0)} \otimes h_{(2)} \cdot y_{(0)} \otimes y_{(1)} x_{(1)}.
\]
Now for any $h \in H^\sigma$, we verify the equation \((20)\) for the element $\eta_{\delta,\zeta}^{-1}(x \otimes y)$:

\[
\sum h(1) \rightarrow x(0) \otimes h(2) \rightarrow y(0) \sigma(y(1) \otimes x(1)) \\
\sum (h(2) \cdot x(0))(0) \otimes y(0) \sigma((h(2) \cdot x(0))(1) \otimes h(1)) \sigma^{-1}(h(3) \otimes x(1)) \\
\sigma(y(1) \otimes h(4)) R(h(5) \otimes y(2)) \sigma^{-1}(h(6) \otimes y(3)) \sigma(y(4) \otimes x(2)) \\
\sum (h(2) \cdot x(0))(0) \otimes y(0) \sigma((h(2) \cdot x(0))(1) \otimes h(1)) \sigma^{-1}(h(3) \otimes x(1)) \\
\sigma(y(1) \otimes h(4)) \sigma(y(2) h(5) \otimes x(2)) R(h(6) \otimes y(3)) \sigma^{-1}(h(7) \otimes y(4) x(3)) \\
\sum (h(2) \cdot x(0))(0) \otimes y(0) \sigma((h(2) \cdot x(0))(1) \otimes h(1)) \sigma^{-1}(h(3) \otimes x(1)) \\
\sigma(y(1) \otimes h(4)) \sigma(y(2) h(5) \otimes x(2)) R(h(6) \otimes y(3)) \sigma^{-1}(h(7) \otimes y(4) x(3)) \\
\sum (h(2) \cdot x(0))(0) \otimes y(0) \sigma((h(2) \cdot x(0))(1) \otimes h(1)) R(h(5) \otimes y(1)) \sigma^{-1}(h(6) \otimes y(2) x(2)) \\
\sigma(h(6) y(1) S^{-1}(h(4)) \otimes h(3) x(1)) \sigma^{-1}(h(7) \otimes y(2) x(2)) \\
\sum (h(3) \cdot x(0))(0) \otimes (h(4) \cdot y(0))(0) \sigma((h(3) \cdot x(0))(1) \otimes h(1)) \\
\sigma((h(4) \cdot y(0))(1) \otimes (h(3) \cdot x(0))(2) h(2)) \sigma^{-1}(h(5) \otimes y(1) x(1)) \\
\sum (h(3) \cdot x(0))(0) \otimes (h(4) \cdot y(0))(0) \sigma((h(3) \cdot x(0))(1) \otimes h(1)) \\
\sigma((h(4) \cdot y(0))(1) \otimes (h(3) \cdot x(0))(2) h(2)) \sigma^{-1}(h(5) \otimes y(1) x(1)).
\]

On the other hand, we have:

\[
\sum h(1) \rightarrow x(0) \otimes h(2) \rightarrow y(0) \sigma(y(1) \otimes x(1)) \\
\sum x(0) \otimes (h(3) \cdot y(0))(0) R\sigma(x(1) \otimes (S^\sigma)^{-1}(h(1))) \\
\sigma((h(3) \cdot y(0))(1) \otimes h(2)) \sigma^{-1}(h(4) \otimes y(1)) \sigma(y(2) \otimes x(2)) \\
\sum x(0) \otimes (h(9) \cdot y(0))(0) \sigma((S^\sigma)^{-1}(h(2)) \otimes h(1)) \sigma((S^\sigma)^{-1}(h(5)) \otimes x(1)) \\
R(x(2) \otimes (S^\sigma)^{-1}(h(4)) \otimes (S^\sigma)^{-1}(h(3)) \otimes x(1)) \sigma^{-1}(h(7) \otimes (S^\sigma)^{-1}(h(6))) \\
\sigma((h(9) \cdot y(0))(1) \otimes h(8)) \sigma((h(10) y(1)) \otimes x(4)) \sigma^{-1}(h(11) \otimes y(2) x(5)) \\
\sum x(0) \otimes (h(12) \cdot y(0))(0) \sigma((x(3) S^{-1}(h(4)) \otimes h(1)) \sigma^{-1}(x(4) \otimes S^{-1}(h(3)) h(2)) \\
R(x(1) \otimes S^{-1}(h(6)) \otimes S^{-1}(h(8)) \otimes h(7) x(2) S^{-1}(h(5))) \sigma^{-1}(h(10) \otimes S^{-1}(h(9))) \\
\sigma(y(12) \cdot y(0))(1) \otimes y(11)) \sigma(h(13) y(1) \otimes x(5)) \sigma^{-1}(h(14) \otimes y(2) x(6)) \\
\sum (h(4) \cdot 2 x(0))(0) \otimes (h(11) \cdot y(0))(0) \sigma((x(2) S^{-1}(h(2)) \otimes h(1)) \\
\sigma(S^{-1}(h(6)) \otimes h(5) x(1) S^{-1}(h(3)) \sigma^{-1}(h(8) \otimes S^{-1}(h(7)))) \\
\sum (h(11) \cdot y(0))(1) \otimes h(9)) \sigma(h(11) \cdot y(0))(2) h(10) \otimes x(3) \sigma^{-1}(h(12) \otimes y(1) x(4)) \\
\sum (h(3) \cdot 2 x(0))(0) \otimes (h(9) \cdot y(0))(0) \sigma((x(1) S^{-1}(h(2)) \otimes h(1)) \\
\sigma^{-1}(h(6) \otimes S^{-1}(h(5)) \otimes (h(3) \cdot 2 x(0))(1)) \sigma((h(7) \otimes x(2)) \sigma(h(9) \cdot y(0))(1) \otimes h(8) x(3)) \sigma^{-1}(h(10) \otimes y(1) x(4)) \\
\sum (h(3) \cdot 2 x(0))(0) \otimes (h(10) \cdot y(0))(0) \sigma((x(1) S^{-1}(h(2)) \otimes h(1)) \\
\sigma(h(6) S^{-1}(h(5)) \otimes (h(3) \cdot 2 x(0))(1)) \sigma^{-1}(h(7) \otimes S^{-1}(h(4)) h(3) \cdot 2 x(0))(2)) \\
\sigma(h(8) \otimes x(2)) \sigma((h(10) \cdot y(0))(1) \otimes h(9) x(3)) \sigma^{-1}(h(11) \otimes y(1) x(4)).
\]
\[ \sum (h_{(5)} \triangleright x_{(0)})_0 \otimes (h_{(10)} \cdot y_{(0)})_0 \sigma(x_{(1)}S^{-1}(h_{(2)}) \otimes h_{(1)}) \\
= \frac{1}{2} \sum (h_{(5)} \triangleright x_{(0)})_0 \otimes (h_{(8)} \cdot y_{(0)})_0 \sigma((h_{(10)} \cdot y_{(0)})_0 \otimes h_{(9)}x_{(3)}) \sigma^{-1}(h_{(11)} \otimes y_{(1)}x_{(4)}) \\
\] 

Finally, we verify that the restriction of \( H\sigma \) on \( M, N \) is obviously an \( H^\sigma \)-comodule map and consequently a YD \( H^\sigma \)-module isomorphism. Let \( x \otimes y \in g(M \otimes N) \) and \( h \in H^\sigma \). Then by Lemma 2.4 and 18 we have

\[ h \cdot \eta_{M,N}^{-1}(x \otimes y) = \sum h_{(1)} \rightharpoonup x_{(0)} \otimes h_{(2)} \triangleright y_{(0)} \sigma(y_{(1)} \otimes x_{(1)}) = \sum h_{(1)} \rightharpoonup x_{(0)} \otimes h_{(2)} \rightharpoonup y_{(0)} \sigma(y_{(1)} \otimes x_{(1)}) \]
algebras in $\mathcal{M}$

Proposition 2.11 that
the equivalence functor $\sigma$

Let

$\eta$

$\mathcal{M}$

On the other hand, we have

$\sigma^{-1}(h_2 \cdot x_2) = \sigma^{-1}(h_7 \cdot y_3 x_4)$

$\sum(h_3 \cdot x_1) \otimes y_0 \sigma((h_3 \cdot x_1) \otimes h_1)$

Therefore, we have

$\eta_{M,N}^{-1}(h \cdot (x \otimes y)) = h \cdot \eta_{M,N}^{-1}(x \otimes y)$. That is, $\eta_{M,N}^{-1}$ is an $H^\sigma$-module homomorphism, and whence $\mathfrak{g}(M) \wedge \mathfrak{g}(N) \cong \mathfrak{g}(M \wedge N)$ as YD $H^\sigma$-modules. 

Now let $A$ and $B$ be YD $H$-module algebras. Then $A$ and $B$ can be regarded as algebras in $\mathcal{M}_R^H$ by forgetting the $H$-module structures of $A$ and $B$. Then we can consider the braided product of $A$ and $B$ in $\mathcal{M}_R^H$. Denote by $A \#_R B$ the braided product to differ from the braided product $A \# B$ in $\mathcal{M}_R^H$. It follows from [19, Proposition 2.11] that $A \wedge B$ is a subalgebra of $A \#_R B$ and the algebra $A \wedge B$ is a YD $H$-module algebra with the $H$-structures given by [18] and [19]. We show that the equivalence functor $\mathfrak{g}$ preserves the generalized cotensor product of algebras.

**Proposition 3.5.** Let $A$ and $B$ be YD $H$-module algebras. Then $\mathfrak{g}(A) \wedge \mathfrak{g}(B)$ and $\mathfrak{g}(A \wedge B)$ are isomorphic as YD $H^\sigma$-module algebras.
Proof. By Lemma 3.3 and its proof, it is enough to show that the map

$$\eta_{M,N}^{-1}: \mathfrak{g}(A \otimes B) \to \mathfrak{g}(A) \otimes \mathfrak{g}(B), a \otimes b \mapsto \sum a^{(0)} \otimes b^{(0)} \sigma(b^{(1)} \otimes a^{(1)})$$

is an algebra homomorphism from $\mathfrak{g}(A \#_R B)$ to $\mathfrak{g}(A) \#_R \mathfrak{g}(B)$. Obviously, $\eta_{M,N}^{-1}(1 \# 1) = 1 \# 1$. Now let $a, a' \in A$ and $b, b' \in B$. Denote by $\cdot$ the product in $\mathfrak{g}(A) \#_R \mathfrak{g}(B)$.

Then we have

$$\eta_{M,N}^{-1}(a \# b) \cdot \eta_{M,N}^{-1}(a' \# b') = \sum (a^{(0)} \# b^{(0)}) \cdot (a'^{(0)} \# b'^{(0)}) \sigma(b^{(1)} \otimes a^{(1)}) \sigma(b'^{(1)} \otimes a'^{(1)})$$

$$= \sum (a^{(0)} \cdot a'^{(0)} \# b^{(0)} \cdot b'^{(0)}) R(a^{(0)} \otimes b^{(0)}) \sigma(b^{(1)} \otimes a^{(1)}) \sigma(b'^{(1)} \otimes a'^{(1)})$$

$$= \sum (a^{(0)} a'^{(0)} \# b^{(0)} b'^{(0)}) \sigma^{-1}(a^{(1)} \otimes a^{(1)}) \sigma^{-1}(b^{(1)} \otimes b^{(1)}) R(a'^{(3)} \otimes b^{(3)}) \sigma(b^{(5)} \otimes a^{(2)}) \sigma(b'^{(5)} \otimes a'^{(2)})$$

$$= \sum (a^{(0)} a'^{(0)} \# b^{(0)} b'^{(0)}) \sigma^{-1}(a^{(1)} \otimes a^{(1)}) \sigma(b^{(1)} \otimes a^{(1)}) \sigma(b'^{(1)} \otimes a'^{2} a'^{(3)} \sigma^{-1}(a'^{(6)} \otimes b^{(5)} a^{(2)})$$

On the other hand, we have

$$\eta_{M,N}^{-1}(a \# b) \cdot \eta_{M,N}^{-1}(a' \# b') = \sum (a^{(0)} \# b^{(0)}) (a'^{(0)} \# b'^{(0)}) \sigma^{-1}(b^{(1)} a^{(1)} \otimes b^{(1)} a^{(1)})$$

$$= \sum \eta_{M,N}^{-1}(a^{(0)} a'^{(0)} \# b^{(0)} b'^{(0)}) R(a'^{(3)} \otimes b^{(3)}) \sigma^{-1}(b'^{(5)} a^{(2)} \otimes b^{(5)} a^{(2)})$$

$$= \sum (a^{(0)} a'^{(0)} \# b^{(0)} b'^{(0)}) \sigma(b^{(1)} b^{(1)} \otimes a^{(1)} a^{(1)}) R(a'^{(2)} \otimes b^{(2)}) \sigma^{-1}(b'^{(2)} a'^{(3)} \otimes b^{(3)} a^{(2)}).$$
Hence \( \eta_{M,N}^{-1}(a\#b) \cdot \eta_{M,N}^{-1}(a'\#b') = \eta_{M,N}^{-1}((a\#b)\bullet (a'\#b')) \). Thus, \( \eta_{M,N}^{-1} \) is an algebra homomorphism from \( \underline{\sigma}(A\#_R B) \) to \( \underline{\sigma}(A)\#_R \underline{\sigma}(B) \). \( \square \)

Let \( A \) be a YD \( H \)-module algebra. Recall from [19, Definition 3.1] that the extension \( A/A_0 \) is said to be a right \( \mathcal{H}^*_R \)-Galois extension if the \( k \)-linear map

\[
\beta^*: A \otimes_{A_0} A \to A \otimes \mathcal{H}^*_R, \quad \beta^*(a \otimes b) = \sum a^{(0)} b \otimes a^{(1)}
\]

is an isomorphism. Similarly, the extension \( A/A_0 \) is said to be left Galois if the \( k \)-linear map

\[
\beta^!: A \otimes_{A_0} A \to \mathcal{H}^*_R \otimes A, \quad \beta^!(a \otimes b) = \sum b^{(-1)} \otimes ab^{(0)}
\]

is an isomorphism. If, in addition, the subalgebra \( A \) (or \( A_0 \)) is trivial and \( A \) is faithfully flat over \( k \), then \( A \) is called a left (or right) \( \mathcal{H}^*_R \)-Galois object. Denote by \( \mathcal{E}(\mathcal{H}_R) \) the category of YD \( H \)-module algebras which are \( \mathcal{H}^*_R \)-bigalois objects. The morphisms in \( \mathcal{E}(\mathcal{H}_R) \) are YD \( H \)-module algebra homomorphisms (or equivalently, isomorphisms). If \( A \) and \( B \) are two objects of \( \mathcal{E}(\mathcal{H}_R) \), then \( A \wedge B \) is an object of \( \mathcal{E}(\mathcal{H}_R) \) by [19, Proposition 3.2].

Let \( H^* \) be the dual Hopf algebra of \( H \). Then \( H^* \) is a YD \( H \)-module algebra with the \( H \)-structures as follows: For \( h^*, p \in H^* \) and \( h \in H \), define

\[
\begin{align*}
\cdot & \quad h \cdot p = \sum p_{(1)}(p_{(2)}, h), & H\text{-action}, \\
\cdot^* & \quad h^* \cdot p = \sum h^*_{(2)} p S^{-1}(h^*_{(1)}), & H^*\text{-coaction}
\end{align*}
\]

where we use \( S \) for the antipodes of both \( H \) and \( H^* \) in order to simplify the notations and we will do the same in the sequel. By [19, Lemma 3.3], \( H^* \) is an object in \( \mathcal{E}(\mathcal{H}_R) \). Denote by \( I \) the object \( H^* \) described above. It follows from [19, Proposition 3.4] that \( \mathcal{E}(\mathcal{H}_R) \) is a monoidal category with the product \( \wedge \) and the unit \( I \).

Similarly, we have the monoidal category \( \mathcal{E}(\mathcal{H}^*_R) \) for the CQT Hopf algebra \( (H^*, R^*) \). Our main task in the rest of this section is to show that the equivalence functor \( \underline{\sigma} \) restricts to an equivalence monoidal functor from the monoidal category \( \mathcal{E}(\mathcal{H}) \) to \( \mathcal{E}(\mathcal{H}^*_R) \). To this end, we first look at the unit \( I^* \) of \( \mathcal{E}(\mathcal{H}^*_R) \) and show that \( \underline{\sigma} \) sends the unit \( I \) of \( \mathcal{E}(\mathcal{H}) \) to \( I^* \).

Let \( (H^*)^* \) be the dual Hopf algebra of \( H^* \). Then \( I^* = (H^*)^* \) as an algebra. The \( H^* \)-structures of \( I^* \) is given by

\[
\begin{align*}
\cdot & \quad h \cdot^* p = \sum h_{(1)} \otimes h_{(2)}(S^{-1}(h_{(1)})), & H^*\text{-action}, \\
\cdot & \quad h^* \cdot^* p = \sum h^*_{(2)} p S^{-1}(h^*_{(1)}), & H^*\text{-coaction}
\end{align*}
\]

where \( h^*, p \in (H^*)^* \) and \( h \in H^* \), and we use the sigma notation \( \sum h_{(1)}^* \otimes h_{(2)}^* \) for the comultiplication of an element \( h^* \in (H^*)^* \) to differ from the comultiplication \( \sum h^*_{(1)} \otimes h^*_{(2)} \) of \( h^* \in H^* \). Identify \( (H \otimes H)^* \) with \( H^* \otimes H^* \), we may assume that \( \sigma, \sigma^{-1} \in H^* \otimes H^* \) and may write \( \sigma = \sum \sigma^{(1)} \otimes \sigma^{(2)} \) and \( \sigma^{-1} = \sum (\sigma^{-1})^{(1)} \otimes (\sigma^{-1})^{(2)} \) in \( H^* \otimes H^* \). Note that \( (H^*)^* = H^* \) as algebras. Then for any \( h^* \in H^* = (H^*)^* \) we have

\[
\sum h_{(1)}^* \otimes h_{(2)}^* = \sum \sigma(h_{(1)}^* \otimes h_{(2)}^*) \sigma^{-1} = \sum \sigma^{(1)} h_{(1)}^* (\sigma^{-1})^{(1)} \otimes \sigma^{(2)} h_{(2)}^* (\sigma^{-1})^{(2)}.
\]
Let $\chi$ be a $k$-linear map defined by
\begin{equation}
(24) \quad \chi : H \to H, \quad \chi(h) = \sum \sigma^{-1}(h_{(4)} \otimes S^{-1}(h_{(3)})h_{(1)})h_{(2)}, \ h \in H.
\end{equation}

Consequently, we have a $k$-linear map dual to $\chi$
\begin{equation}
(25) \quad \chi^* : H^* \to H^*, \quad \langle \chi^*(h^*), h \rangle = \langle h^*, \chi(h) \rangle, \ h^* \in H^*, h \in H.
\end{equation}

**Lemma 3.6.** $\chi$ is a $k$-linear isomorphism from $H$ to $H$ with the inverse given by
\[\chi^{-1}(h) = \sum \sigma^{-1}(S^{-1}(h_{(5)}) \otimes h_{(1)})\sigma(S^{-1}(h_{(4)}) \otimes h_{(3)})h_{(2)}, \ h \in H.\]

Consequently, $\chi^*$ is a $k$-linear isomorphism as well.

**Proof.** Let $\lambda$ be the $k$-linear map given by $\lambda(h) = \sum \sigma^{-1}(S^{-1}(h_{(5)}) \otimes h_{(1)})\sigma(S^{-1}(h_{(4)}) \otimes h_{(3)})h_{(2)}, \ h \in H$. Then for any $h \in H$ we have
\[
(\lambda \chi)(h) = \sum \sigma^{-1}(h_{(4)} \otimes S^{-1}(h_{(3)})h_{(1)})\lambda(h_{(2)})
= \sum \sigma^{-1}(h_{(8)} \otimes S^{-1}(h_{(7)})h_{(1)})\sigma^{-1}(S^{-1}(h_{(6)}) \otimes h_{(2)})
\sigma(S^{-1}(h_{(5)}) \otimes h_{(4)})h_{(3)}
\begin{align*}
&\sum \sigma^{-1}(h_{(7)}S^{-1}(h_{(6)}) \otimes h_{(1)})\sigma^{-1}(h_{(8)} \otimes S^{-1}(h_{(5)}))
\sigma(S^{-1}(h_{(4)}) \otimes h_{(3)})h_{(2)}
\sum \sigma(h_{(6)}S^{-1}(h_{(5)}) \otimes h_{(2)})\sigma^{-1}(h_{(7)} \otimes S^{-1}(h_{(4)})h_{(3)})h_{(1)}
= \ h.
\end{align*}
\]

Similarly, one can check that $(\chi \lambda)(h) = h$ for any $h \in H$. It follows that $\chi$ is an isomorphism with $\chi^{-1} = \lambda$, and so $\chi^*$ is also an isomorphism. \hfill \Box

We are ready to show that $\mathfrak{a}(I) \cong I^\sigma$ in $E(H_{R^\sigma}^\sigma)$.

**Lemma 3.7.** $\mathfrak{a}(I)$ and $I^\sigma$ are isomorphic YD $H^\sigma$-module algebras.

**Proof.** By Lemma 3.6, $\chi^* : \mathfrak{a}(H^*) \to (H^*)^*$ is a $k$-linear isomorphism. Hence it is enough to show that $\chi^*$ is a YD $H^\sigma$-module algebra homomorphism from $\mathfrak{a}(I)$ to $I^\sigma$. Let $h^* \in (H^*)^*, \ p \in \mathfrak{a}(H^*)$ and $h \in H^\sigma$. Then we have
\[
\langle \chi^*(h^* \cdot p), h \rangle = \langle h^* \cdot p, \chi(h) \rangle
= \sum \sigma^{-1}(h_{(4)} \otimes S^{-1}(h_{(3)})h_{(1)})h_{(2)}pS^{-1}(h_{(1)}^*), h_{(2)}
\begin{align*}
&\sum \sigma^{-1}(h_{(6)} \otimes S^{-1}(h_{(5)})h_{(1)})h_{(2)}\langle p, h_{(3)} \rangle(S^{-1}(h_{(1)}^*)), h_{(4)}
= \sum \sigma^{-1}(h_{(6)} \otimes S^{-1}(h_{(5)})h_{(1)})\langle h^*, S^{-1}(h_{(4)})h_{(2)} \rangle(p, h_{(3)})
\end{align*}
\]
and

\[
\langle h^* \cdot \chi^* (p), h \rangle = \sum (h^*_{<2>} \chi^* (p) (S^*)^{-1} (h^*_{<1>}), h) \\
= \sum (h^*_{<2>}, h_{(1)}) \langle \chi^* (p), h_{(2)}, (S^*)^{-1} (h^*_{<1>}), h_{(3)} \rangle \\
= \sum (p, \chi (h_{(2)})) \langle h^*, (S^*)^{-1} (h_{(3)}), \cdot \cdot \cdot, h \rangle \\
= \sum \sigma^{-1} (h_{(5)} \otimes S^{-1} (h_{(4)}), h_{(2)}) (p, h_{(3)} \otimes \sigma (S^{-1} (h_{(7)}) \otimes h_{(6)})) \\
\sigma^{-1} (h_{(10)} \otimes S^{-1} (h_{(9)}), h_{(8)}), h_{(5)}) (p, h_{(3)} \otimes \sigma (S^{-1} (h_{(9)}), h_{(8)})) \\
\sigma^{-1} (h_{14} \otimes S^{-1} (h_{13}), h_{(11)}), h_{(1)}) (h^*, S^{-1} (h_{11}) \otimes h_{(3)}) \\
\sum \sigma^{-1} (S^{-1} (h_{11}), h_{(6)} \otimes S^{-1} (h_{5}), h_{(3)}) (p, h_{(3)}) \\
\sum \sigma^{-1} (S^{-1} (h_{11}), h_{(6)} \otimes S^{-1} (h_{10}), h_{(7)}) \\
\sigma^{-1} (S^{-1} (h_{14} \otimes S^{-1} (h_{13}), h_{(11)}), h_{(1)}) (h^*, S^{-1} (h_{12}) \otimes h_{(3)}) \\
\sum \sigma^{-1} (h_{6} \otimes S^{-1} (h_{5}), h_{(1)}), h_{(1)}) (h^*, S^{-1} (h_{4}) \otimes h_{(2)}) (p, h_{(3)}).
\]

So \( \chi^* (h^* \cdot p) = h^* \cdot \chi^* (p) \) and \( \chi^* \) is an \( H^\sigma \)-comodule homomorphism from \( \sigma(I) \) to \( I^\sigma \).

Now we choose a pair of dual bases \( \{ h_{1}, h_{2}, \cdots, h_{n} \} \) in \( H \) and \( \{ h^*_{1}, h^*_{2}, \cdots, h^*_{n} \} \) in \( H^* \) as \( H \) is finitely generated projective. Then the comodule structures of \( I \) and \( I^\sigma \) are given by

\[
I \rightarrow I \otimes H, \quad p \rightarrow \sum_{i=1}^{n} h^*_{i} \cdot p \otimes h_{i} = \sum_{i} p_{(0)} \otimes p_{(1)}
\]

and

\[
I^\sigma \rightarrow I^\sigma \otimes H^\sigma, \quad p \rightarrow \sum_{i=1}^{n} h^*_{i} \cdot \sigma p \otimes h_{i},
\]
Hence, it follows that

\[ p \mapsto \chi \]

Finally, we verify that \( \chi \) is an algebra map. Let \( p, q \in g(I) \) and \( h \in H^\sigma \). Then

\[
\langle \chi^*(p \cdot q), h \rangle = \sum (p_0, q_0) \cdot \chi(h) \sigma^{-1}(q_1 \otimes p_1)
\]

\[
= \sum (h^*_1 \cdot p, h^*_2 \cdot q, h_2) \sigma^{-1}(h_4 \otimes S^{-1}(h_3)h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (h^*_2 \cdot h^*_1 \cdot h_4, h_2) \sigma^{-1}(h_4 \otimes S^{-1}(h_3)h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (h^*_2 \cdot h^*_1 \cdot h_4, h_2) \sigma^{-1}(h_4 \otimes S^{-1}(h_3)h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (p, h_2) \sigma^{-1}(h_9) \sigma^{-1}(h_9, S^{-1}(h_{10})h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (p, h_2) \sigma^{-1}(h_9) \sigma^{-1}(h_9, S^{-1}(h_{10})h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (p, h_2) \sigma^{-1}(h_9) \sigma^{-1}(h_9, S^{-1}(h_{10})h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \langle \chi^*(p), h \rangle \cdot \chi^*(q), h \rangle.
\]

It follows that \( \chi^* \) is an \( H^\sigma \)-module homomorphism from \( g(I) \) to \( I^\sigma \).

Finally, we verify that \( \chi^* \) is an algebra map. Let \( p, q \in g(I) \) and \( h \in H^\sigma \). Then

\[
\langle \chi^*(p \cdot q), h \rangle = \sum (p_0, q_0, \chi(h)) \sigma^{-1}(q_1 \otimes p_1)
\]

\[
= \sum (h^*_1 \cdot p, h^*_2 \cdot q, h_2) \sigma^{-1}(h_4 \otimes S^{-1}(h_3)h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (h^*_2 \cdot h^*_1 \cdot h_4, h_2) \sigma^{-1}(h_4 \otimes S^{-1}(h_3)h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (h^*_2 \cdot h^*_1 \cdot h_4, h_2) \sigma^{-1}(h_4 \otimes S^{-1}(h_3)h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (p, h_2) \sigma^{-1}(h_9) \sigma^{-1}(h_9, S^{-1}(h_{10})h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (p, h_2) \sigma^{-1}(h_9) \sigma^{-1}(h_9, S^{-1}(h_{10})h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \sum (p, h_2) \sigma^{-1}(h_9) \sigma^{-1}(h_9, S^{-1}(h_{10})h_{11}) \sigma^{-1}(h_j \otimes h_i)
\]

\[
= \langle \chi^*(p), h \rangle \cdot \chi^*(q), h \rangle.
\]

Hence \( \chi^*(p \cdot q) = \chi^*(p) \chi^*(q) \) for all \( p, q \in g(I) \). It is clear that \( \chi^*(\varepsilon) = \varepsilon \). Therefore, \( \chi^* \) is an algebra map from \( g(I) \) to \( I^\sigma \).
Let $\omega : H \otimes H \to k$ be a $k$-linear map. Then $\omega$ induces two $k$-linear maps from $H$ to $H^*$. They are,
\[\omega_l : H \to H^*, \quad \langle \omega_l(h), x \rangle = \omega(h \otimes x), \quad h, x \in H\]
and
\[\omega_r : H \to H^*, \quad \langle \omega_r(h), x \rangle = \omega(x \otimes h), \quad h, x \in H.\]
For the 2-cocycle $\sigma$ and its inverse $\sigma^{-1}$, we have four $k$-linear maps $\sigma_l, \sigma_r, (\sigma^{-1})_l$ and $(\sigma^{-1})_r$. Note that $(\sigma^{-1})_l$ (or $(\sigma^{-1})_r$) coincides with the convolution inverse $\sigma_l^{-1}$ (or $\sigma_r^{-1}$) of $\sigma_l$ (or $\sigma_r$) in $\text{Hom}(H, H^*)$.

**Lemma 3.8.** Let $A$ be a YD $H$-module algebra. Then the $k$-linear map
\[\phi : A \otimes H^* \to A \otimes H^*, \quad a \otimes h^* \mapsto \sum a(0) \otimes \sigma_l(a(1))(2) h^* S^{-1}(\sigma_l(a(1))(1))\]
is an isomorphism with the inverse given by
\[\phi^{-1} : A \otimes H^* \to A \otimes H^*, \quad a \otimes h^* \mapsto \sum a(0) \otimes \sigma_l^{-1}(a(1))(2) h^* S^{-1}(\sigma_l^{-1}(a(1))(1)),\]
where $H^*$ is the dual Hopf algebra of $H$.

**Proof.** Identify $A \otimes H^*$ with $\text{Hom}(H, A)$. For $a \in A, h^* \in H^*$ and $h \in H$, we then have
\[(\phi^{-1} \phi)(a \otimes h^*) = \sum a(0) \otimes \sigma_l^{-1}(a(1))(2) \sigma_l(a(2))(2) h^* S^{-1}(\sigma_l(a(2))(1)) S^{-1}(\sigma_l^{-1}(a(1))(1)).\]
It leads to the following equations:
\[[(\phi^{-1} \phi)(a \otimes h^*)](h) = \sum a(0) \langle \sigma_l^{-1}(a(1))(2), h(1) \rangle \langle \sigma_l(a(2))(2), h(2) \rangle \langle h^*, h(3) \rangle \langle S^{-1}(\sigma_l(a(2))(1)), h(4) \rangle \langle S^{-1}(\sigma_l^{-1}(a(1))(1)), h(5) \rangle = \sum a(0) \langle \sigma_l^{-1}(a(1)), h(5) \rangle \langle h(1) \rangle \langle \sigma_l(a(2)), h(4) \rangle \langle h(2) \rangle \langle h^*, h(3) \rangle \langle \sigma_l^{-1}(a(1) \otimes S^{-1}(h(5))), h(1) \rangle = \sum a(0) \langle \sigma_l^{-1}(a(1) \otimes S^{-1}(h(5))), h(1) \rangle \langle \sigma_l(a(2)), h(4) \rangle \langle h(2) \rangle \langle h^*, h(3) \rangle = a \langle h^*, h \rangle = (a \otimes h^*)(h).\]
It follows that $\phi^{-1} \phi = \text{id}$. Similarly, one can show that $\phi \phi^{-1} = \text{id}$. \hfill $\Box$

**Lemma 3.9.** Let $A$ be a YD $H$-module algebra. Then the $k$-linear map
\[\psi : H^* \otimes A \to H^* \otimes A, \quad h^* \otimes a \mapsto \sum \sigma_r(a(1))(2) h^* S^{-1}(\sigma_r(a(1))(1)) \otimes a(0)\]
is an isomorphism with the inverse given by
\[\psi^{-1} : H^* \otimes A \to H^* \otimes A, \quad h^* \otimes a \mapsto \sum \sigma^{-1}_r(a(1))(2) h^* S^{-1}(\sigma^{-1}_r(a(1))(1)) \otimes a(0).\]

**Proof.** Similar to the proof of Lemma 3.8. \hfill $\Box$

Now we are ready to show that $\sigma$ sends any object of $\mathcal{E}(H_R)$ into $\mathcal{E}(\mathcal{H}_R^{\sigma})$.

**Proposition 3.10.** Let $A$ be a YD $H$-module algebra. Then
(a) $\mathcal{E}(A)/\mathcal{E}(A)_\sigma$ is a right $(\mathcal{H}_R^{\sigma})^*$-Galois extension if and only if $A/A_\sigma$ is a right $\mathcal{H}_R$-Galois extension.
(b) $\mathcal{E}(A)/\mathcal{E}(A)_\sigma$ is a left $(\mathcal{H}_R^{\sigma})^*$-Galois extension if and only if $A/\sigma A$ is a left $\mathcal{H}_R$-Galois extension.
Proof. By [19, Eq.(12)], the Galois map \( \beta^r : A \otimes_{A_s} A \to A \otimes \mathcal{H}^*_R \) is given by

\[
\beta^r(a \otimes b) = \sum a_{[0]}b \otimes a_{[1]}S^{-1}(R_r(a_{[0]})), \quad a, b \in A.
\]

Similarly, the Galois map \( \beta^r_\sigma : g(A) \otimes g(A) \to g(A) \otimes (\mathcal{H}^*_R)^* \) is given by

\[
\beta^r_\sigma(a \otimes b) = \sum a_{<0>}(0) \cdot b \otimes a_{<1>} (S^\sigma)^{-1}(R^r_\sigma(a_{<0>}(1))) = \sum a_{<0>}(0)b_{<0>}(0) \otimes a_{<1>} (S^\sigma)^{-1}(R^r_\sigma(a_{<0>}(2))).
\]

By Lemma 3.2 and Lemma 3.3, it is straightforward to check that the \( k \)-linear isomorphism \( A \otimes A \to \sigma(A) \otimes \sigma(A) \), \( a \otimes b \mapsto \sum a_{[0]} \otimes b_{[0]} \sigma(b_{[1]} \otimes a_{[1]}) \) induces an isomorphism

\[
f : A \otimes_{A_s} A \to g(A) \otimes g(A), \quad a \otimes b \mapsto \sum a_{[0]} \otimes b_{[0]} \sigma(b_{[1]} \otimes a_{[1]}).
\]

As \( k \)-modules, we may regard \( g(A) \otimes (\mathcal{H}^*_R)^* = A \otimes \mathcal{H}^*_R = A \otimes H^* \). We claim that the following diagram is commutative:

\[
\begin{array}{ccc}
A \otimes_{A_s} A & \xrightarrow{\beta^r} & A \otimes \mathcal{H}^*_R \\
f \downarrow & & \downarrow (id \otimes \chi^*) \phi \\
g(A) \otimes g(A) & \xrightarrow{\beta^r_\sigma} & g(A) \otimes (\mathcal{H}^*_R)^*,
\end{array}
\]

where \( \chi^* \) and \( \phi \) are isomorphisms given in [25] and Lemma 3.8, respectively. In fact, let \( a, b \in A \). Then we have

\[
(\beta^r_\sigma f)(a \otimes b) = \sum a_{<0>}(0)b_{<0>}(0) \otimes a_{<1>}(0)(S^\sigma)^{-1}(R^r_\sigma(a_{<0>}(2)))
\]

and

\[
((id \otimes \chi^*) \phi \beta^r)(a \otimes b) = \sum a_{[0]}b_{[0]} \otimes \chi^*(\sigma_1(b_{[1]}a_{[0]}(1))(2)a_{[1]}(1))S^{-1}(R_r(a_{[0]}(2)))
\]

\[
S^{-1}(\sigma_1(b_{[1]}a_{[0]}(1))(1)).
\]
Identify $A \otimes H^*$ with Hom($H, A$), then we have for any $h \in H$,

$$
[(\beta^*_f)(a \otimes b)][h]
= \sum a_{(0)} \otimes b_{(0)} \langle a_{(0)} \otimes h_{(1)}, R_{\sigma}(a_{(0)} \otimes (2)), (S^\sigma)^{-1}(h_{(2)}) \rangle
\sigma^{-1}(b_{(1)} \otimes a_{(0)} \otimes \sigma^2, (S^\sigma)^{-1}(h_{(2)}) \rangle
\sigma^{-1}(h_{(3)} \otimes b_{(0)} \otimes a_{(0)} \otimes S^{-1}(h_{(4)}), h_{(5)} \otimes h_{(9)} \otimes h_{(10)} \rangle
R(S^{-1}(h_{(6)}) \otimes h_{(7)}, h_{(8)} \otimes h_{(9)} \otimes h_{(10)} \rangle
\sigma^{-1}(h_{(9)} \otimes h_{(10)}, S^{-1}(h_{(11)}), h_{(12)} \rangle
\sigma^{-1}(b_{(0)}, (S^\sigma)^{-1}(h_{(2)}), (S^\sigma)^{-1}(h_{(3)}), h_{(4)} \otimes h_{(5)} \rangle
\sigma^{-1}(h_{(6)} \otimes h_{(7)}, h_{(8)} \otimes h_{(9)} \rangle
\sigma^{-1}(b_{(3)} \otimes a_{(3)} \rangle
\sum a_{(0)} \otimes b_{(0)} \sigma^{-1}(b_{(1)} \otimes (h_{(3)} \otimes a_{(3)}), h_{(4)} \otimes (h_{(5)} \otimes a_{(5)}))
\sum b_{(2)} \otimes h_{(2)} \otimes h_{(3)} \otimes a_{(6)} \otimes h_{(11)} \otimes S^{-1}(h_{(12)}) \rangle
\sum b_{(3)} \otimes h_{(3)} \otimes a_{(5)} \rangle
\sum b_{(4)} \otimes h_{(4)} \otimes h_{(5)} \rangle
\sum b_{(5)} \otimes h_{(5)} \rangle
$$
and

\[
[(\id \otimes \chi^*) \phi \beta_r^r](a \otimes b)(h) \\
= \sum a_{[0]} b_{[0]} \sigma_{(b_{[1]} a_{[0]}(1))}(h_7) S^{-1}(h_6) h_{[1]}(\sigma(b_{[1]} a_{[0]}(1)), h_{[2]}) \\
= \sum a_{[0]} b_{[0]} \sigma^{-1}(h_7) S^{-1}(h_6) h_{[1]}(\sigma(b_{[1]} a_{[0]}(1)), h_{[2]}) \\
= \sum a_{[0]} b_{[0]} \sigma^{-1}(h_7) S^{-1}(h_6) h_{[1]}(\sigma(b_{[1]} a_{[0]}(1)) \otimes S^{-1}(h_5) h_{[2]}) \\
= \sum (h_{[3]} \cdot a_{[1]} b_{[0]} \sigma^{-1}(h_7) h_{[1]}(\sigma(b_{[1]} a_{[0]}(1)) \otimes S^{-1}(h_5) h_{[2]}) R(S^{-1}(h_{[4]}) \otimes (h_{[3]} \cdot a_{[2]})).
\]

Since \( \chi^* \) and \( \phi \) are isomorphisms, it follows from the foregoing commutative diagram that \( \beta_r^r \) is an isomorphism if and only if \( \beta^r \) is an isomorphism. This completes the proof of Part (a).

Part (b) follows from a similar argument by using the isomorphisms \( \psi \) in Lemma 3.8 and \( \chi^* \).

Combining Lemma 3.3, Proposition 3.5, Lemma 3.7 and Proposition 3.10, we now arrive at the monoidal equivalence between \( \mathcal{E}(\mathcal{H})_R \) and \( \mathcal{E}(\mathcal{H}_R) \).

**Theorem 3.11.** \( \mathcal{E}(\mathcal{H}_R) \) and \( \mathcal{E}(\mathcal{H}_R) \) are equivalent monoidal categories.

Let \( \mathcal{E}(\mathcal{H}_R) \) be the set of the isomorphism classes of objects in \( \mathcal{E}(\mathcal{H}_R) \). Then \( \mathcal{E}(\mathcal{H}_R) \) is a semigroup with the product induced by the generalized cotensor product \( \wedge \).

Denote by \( \text{Gal}(\mathcal{H}_R) \) the subset of \( E(\mathcal{H}_R) \) consisting of the isomorphism classes of objects in \( \mathcal{E}(\mathcal{H}_R) \) that are quantum commutative in \( H \mathcal{YD}^H \), where a \( \mathcal{YD}^H \)-module algebra \( A \) is quantum commutative in case it satisfies

\[
ab = \sum b_{[0]} (b_{[1]} \cdot a), \quad \text{for all } a, b \in A.
\]

The subset \( \text{Gal}(\mathcal{H}_R) \) is a group (see [19, Theorem 3.9]). Since the quantum commutativity \( 26 \) is defined by the braiding of the category \( H \mathcal{YD}^H \) and \( \varpi \) is a braided monoidal functor, \( \varpi(A) \) is quantum commutative in \( H \mathcal{YD}^H \) if \( A \) is quantum commutative in \( H \mathcal{YD}^H \). Thus we obtain the following main result of this section.

**Theorem 3.12.** The functor \( \varpi \) induces a group isomorphism from \( \text{Gal}(\mathcal{H}_R) \) to \( \text{Gal}(\mathcal{H}_R) \) sending an element \( [A] \) to the element \( [\varpi(A)] \).

If \( \sigma \) is a lazy 2-cocycle on \( H \), then \( \sigma \) induces a group isomorphism from \( \text{Gal}(\mathcal{H}_R) \) to \( \text{Gal}(\mathcal{H}_R) \). For example, every CQT structure \( R_t \) of \( H_4 \) in Example 2.12 is equal to \( R_0^2 \) for some lazy cocycle \( \sigma \). By Theorem 3.12, we have \( \text{Gal}(\mathcal{H}_R) \cong \text{Gal}(\mathcal{H}_R) \).

Similar to Corollary 2.7, we have that \( H_2^L(H) \) acts on \( \text{Gal}(\mathcal{H}_R) \) by automorphisms.

To end this section, we show that the exact sequence \( 11 \) is stable under the equivalence functor \( \varpi \). For simplifying notations we will write in the sequel \( M_0 \) for the coinvariant subset \( M \mathcal{C}oH \) of right \( H \)-comodule \( M \):

\[
\left\{ m \in M \mid \rho(m) = \sum m_{[0]} \otimes m_{[1]} = m \otimes 1 \right\}.
\]
Let $A$ be a right $H^{op}$-comodule algebra. Recall that $A/A_0$ is called an $H^{op}$-Galois extension if the Galois map is bijective:

$$\beta : A \otimes A_0 \to A \otimes H^{op}, \quad a \otimes b \mapsto \sum ab_{(0)} \otimes b_{(1)}.$$

If, in addition, $A$ is $R$-Azumaya, then we say that $A$ is a Galois $R$-Azumaya algebra. It is known that any element of $\text{BC}(k, H, R)$ can be represented by a Galois $R$-Azumaya algebra [19, Corollary 4.2].

Now let $A$ be a right $H^{op}$-comodule algebra such that $A/A_0$ is Galois. Denote by $\pi(A)$ the centralizer subalgebra $C_A(A_0)$ of $A_0$ in $A$. It is clear that $\pi(A)$ is an $H^{op}$-subcomodule subalgebra of $A$. The Miyashita-Ulbrich-Van Oystaeyen action [11, 15, 16] of $H$ on $\pi(A)$ is given by

$$(27) \quad h \cdot a = \sum X_i(h)aY_i(h), \quad a \in \pi(A), h \in H,$$

where $\sum X_i(h) \otimes Y_i(h) = \beta^{-1}(1 \otimes h)$ for $h \in H$. It is well known that the right $H^{op}$-comodule algebra $\pi(A)$ together with the action (27) is a YD $H$-module algebra (e.g., see [2, 15]). Moreover, $\pi(A)$ is quantum commutative in $H^{\text{YD}}$. If, in addition, $A$ is $R$-Azumaya, $\pi(A)$ is an object in $\text{Gal}(H_{R^{\sigma}})$ (see [19, Proposition 4.6]). Thus, $\pi$ induces a group homomorphism

$$\tilde{\pi} : \text{BC}(k, H, R) \to \text{Gal}(H_{R^{\sigma}}), \quad [A] \mapsto [\pi(A)],$$

where $[A]$ is taken in $\text{BC}(k, H, R)$ such that $A$ is a Galois $R$-Azumaya algebra. The group homomorphism $\tilde{\pi}$ fits in the exact sequence (1) of groups:

$$1 \to \text{Br}(k) \to \text{BC}(k, H, R) \xrightarrow{\tilde{\pi}} \text{Gal}(H_{R^{\sigma}}).$$

Similarly, we have a group homomorphism $\tilde{\pi} : \text{BC}(k, H^{\sigma}, R^{\sigma}) \to \text{Gal}(H^{\sigma}_{R^{\sigma}})$ and such an exact group sequence (1) for $\text{BC}(k, H^{\sigma}, R^{\sigma})$ and $\text{Gal}(H^{\sigma}_{R^{\sigma}})$. We will prove that the exact sequence (1) is stable under a cocycle deformation. We will need the following isomorphism $\xi$ later on.

**Lemma 3.13.** Let $A$ be a right $H^{op}$-comodule algebra. Then the $k$-linear map

$$\xi : A \otimes H \to A \otimes H, \quad a \otimes h \mapsto \sum a_{(0)} \otimes h_{(4)} \sigma(S^{-1}(h_{(2)}) \otimes h_{(1)}) \sigma^{-1}(S^{-1}(h_{(3)}) \otimes a_{(1)})$$

is an isomorphism with the inverse given by

$$\xi^{-1}(a \otimes h) = \sum a_{(0)} \otimes h_{(3)} \sigma^{-1}(h_{(2)} \otimes S^{-1}(h_{(1)}) a_{(1)}), \quad a \in A, h \in H.$$

**Proof.** Let $\varphi : A \otimes H \to A \otimes H$ be a $k$-linear map given by

$$\varphi(a \otimes h) = \sum a_{(0)} \otimes h_{(3)} \sigma^{-1}(h_{(2)} \otimes S^{-1}(h_{(1)}) a_{(1)}).$$
Then for any \( a \otimes h \in A \otimes H \), we have

\[
(\varphi \xi)(a \otimes h) = \sum \varphi(a_0) \otimes h_4)\sigma(S^{-1}(h_2) \otimes h_{11})\sigma^{-1}(S^{-1}(h_{13}) \otimes a_{11})
\]
\[
= \sum (a_0 \otimes h_6)\sigma(S^{-1}(h_4) \otimes h_{11})\sigma^{-1}(S^{-1}(h_{13}) \otimes a_{12})
\]
\[
= \sum (a_0 \otimes h_7)\sigma(S^{-1}(h_4) \otimes h_{11})\sigma^{-1}(S^{-1}(h_{13}) \otimes a_{12})
\]
\[
= \sum (a \otimes h_5)\sigma(S^{-1}(h_4) \otimes h_{11})\sigma^{-1}(S^{-1}(h_{13}) \otimes a_{12})
\]
\[
= a \otimes h
\]

and

\[
(\xi \varphi)(a \otimes h) = \sum \xi(a_0) \otimes h_3)\sigma^{-1}(h_2 \otimes S^{-1}(h_{11})a_{11})
\]
\[
= \sum (a_0 \otimes h_6)\sigma(S^{-1}(h_4) \otimes h_{13})\sigma^{-1}(S^{-1}(h_{15}) \otimes a_{11})
\]
\[
= \sum (a_0 \otimes h_7)\sigma(S^{-1}(h_4) \otimes h_{11})\sigma^{-1}(S^{-1}(h_{13}) \otimes a_{12})
\]
\[
= \sum (a \otimes h_5)\sigma(S^{-1}(h_4) \otimes h_{11})\sigma^{-1}(S^{-1}(h_{13}) \otimes a_{12})
\]
\[
= a \otimes h
\]

This shows that \( \xi \) is an isomorphism and \( \xi^{-1} = \varphi \). \( \square \)

Let \( A \) be a right \( H^{\text{op}} \)-comodule algebra. Recall that the product of \( \varrho(A) \) is given by \( a \bullet b = \sum a_{(0)}b_{(0)}\sigma^{-1}(a_{11} \otimes b_{11}) \), for all \( a, b \in A \). Thus, if \( a \in A_0 \), then \( a \bullet b = ab \) and \( b \bullet a = ba \) for all \( b \in A \). Moreover, \( \varrho(A)_0 = A_0 \) as algebras because \( H^* = H \) as coalgebras and \( \varrho(A) = A \) as right \( H \)-comodules.

**Lemma 3.14.** Let \( A \) be a right \( H^{\text{op}} \)-comodule algebra. Then \( \varrho(A)/\varrho(A)_0 \) is an \( (H^*)^{\text{op}} \)-Galois extension if and only if \( A/A_0 \) is an \( H^* \)-Galois extension.

**Proof.** Note that \( \varrho(A) \otimes_{\varrho(A)_0} \varrho(A) = A \otimes_{A_0} A \) as \( k \)-modules. We claim that the following diagram commutes:

\[
\begin{array}{ccc}
\varrho(A) \otimes_{\varrho(A)_0} \varrho(A) & \xrightarrow{\beta^*} & \varrho(A) \otimes H^* \\
\downarrow & & \downarrow \xi \\
A \otimes_{A_0} A & \xrightarrow{\beta} & A \otimes H,
\end{array}
\]
where \( \beta^\sigma \) is the Galois map for the right \((H^\sigma)^{op}\)-comodule algebra \( \mathfrak{g}(A) \) and \( \xi \) is the isomorphism given in Lemma 3.13. Indeed, given \( a, b \in \mathfrak{g}(A) \), we have

\[
(\xi \beta^\sigma)(a \otimes b) = \xi(\sum a \cdot b(0) \otimes b(1))
\]

\[
= \sum a(a(0)b(0) \otimes b(2))\sigma^{-1}(b(1) \otimes a(1))
\]

\[
= \sum(a_0b_0 \otimes b_6)\sigma(S^{-1}(b_4) \otimes b_3)
\]

\[
= \sum S^{-1}(b_5) \otimes (b_1a_1)\sigma^{-1}(b_2) \otimes a(1)
\]

\[
= \beta(a \otimes b).
\]

It follows that \( \beta^\sigma \) is isomorphic if and only if \( \beta \) is isomorphic. That is, \( \mathfrak{g}(A)/\mathfrak{g}(A)_0 \) is \((H^\sigma)^{op}\)-Galois if and only if \( A/A_0 \) is \( H^{op}\)-Galois.

**Theorem 3.15.** The following diagram commutes with exact rows:

\[
\begin{array}{ccc}
1 & \longrightarrow & \text{Br}(k) \\
& & \longrightarrow \\
& & \text{BC}(k, H, R) \xrightarrow{\pi} \text{Gal}(H_R) \\
& & \downarrow \pi \\
& & \text{BC}(k, H^\sigma, R^\sigma) \xrightarrow{\pi} \text{Gal}(H^\sigma_R).
\end{array}
\]

**Proof.** It is enough to verify that \( \pi \sim \pi \sim \pi \) as group homomorphisms from \( \text{Br}(k, H, R) \) to \( \text{Gal}(H^\sigma_R) \). Let \( A \) be a right \( H^{op}\)-comodule algebra. Suppose that \( A \) is a Galois \( R\)-Azumaya algebra. By Corollary 2.24 and Lemma 3.13 the right \((H^\sigma)^{op}\)-comodule algebra \( \mathfrak{g}(A) \) is a Galois \( R^\sigma\)-Azumaya algebra. Since \( \mathfrak{g}(A)_0 = A_0 \) and \( a \otimes b = ab \) for all \( a \in A_0 \) and \( b \in \mathfrak{g}(A) \), we have that \( C_{\mathfrak{g}(A)}(\mathfrak{g}(A)_0) = C_A(A_0) \). This means that \( \pi(\mathfrak{g}(A)) = \pi(A) \) as right \( H\)-comodules since \( H^\sigma = H \) as coalgebras. It follows that \( \pi(\mathfrak{g}(A)) = \mathfrak{g}(\pi(A)) \) as right \((H^\sigma)^{op}\)-comodule algebras. It remains to verify that the \( H^\sigma\)-actions on \( \pi(\mathfrak{g}(A)) \) and \( \mathfrak{g}(\pi(A)) \) coincide.

Let \( \sum X_i(h) \otimes Y_i(h) = \beta^{-1}(1 \otimes h) \) in \( A \otimes A, A \) for any \( h \in H \). Then the \( H \)-action on \( \pi(A) \) is given by \( \beta^\sigma \). By Lemma 3.14 the proof of Lemma 3.14 we know that

\[
(\beta^\sigma)^{-1}(1 \otimes h) \otimes Y_i(h) = \sum \sigma(S^{-1}(h(2)) \otimes h(1))X_i(h(3)) \otimes Y_i(h(3)),
\]

for all \( h \in H^\sigma \). Hence the \( H^\sigma\)-action on \( \pi(\mathfrak{g}(A)) \) is given by

\[
h \cdot a = \sum \sigma(S^{-1}(h(2)) \otimes h(1))X_i(h(3)) \cdot a \cdot Y_i(h(3)), \quad h \in H^\sigma, a \in \pi(\mathfrak{g}(A)).
\]

If we define two right \( H \)-comodule structures on \( A \otimes A_0 \) \( A \) and \( A \otimes H^{op} \) by \( a \otimes b \mapsto \sum a \otimes b(0) \otimes b(1) \) and \( a \otimes h \mapsto \sum a \otimes h(1) \otimes h(2) \) respectively, then it is easy to see that the Galois map \( \beta \) is an \( H \)-comodule isomorphism with respect to the comodule structures. It follows that

\[
\sum X_i(h) \otimes Y_i(h(0)) \otimes Y_i(h(1)) = \sum X_i(h(1)) \otimes Y_i(h(1)) \otimes h(2)
\]

for all \( h \in H \). On the other hand, if we define two right \( H \)-comodule structures on \( A \otimes A_0 \) \( A \) and \( A \otimes H^{op} \) by \( a \otimes b \mapsto \sum a_0 \otimes b \otimes a(1) \) and \( a \otimes h \mapsto \sum a_0 \otimes h(2) \otimes h(2) \) respectively, then it follows that

\[
\sum X_i(h) \otimes Y_i(h(0)) \otimes Y_i(h(1)) = \sum X_i(h(1)) \otimes Y_i(h(1)) \otimes h(2)
\]

for all \( h \in H \).
respectively, then $\beta$ is also an $H$-comodule isomorphism with respect to the foregoing defined comodule structures. It follows that

$$
\sum X_i(h(0)) \otimes Y_i(h) \otimes X_i(h(1)) = \sum X_i(h(2)) \otimes Y_i(h(2)) \otimes S^{-1}(h(1))
$$

for all $h \in H$. Now we have

$$
h \cdot \sigma a = \sum \sigma(S^{-1}(h(2)) \otimes h(1)) X_i(h(3)) \bullet a \bullet Y_i(h(3))
$$

$$
= \sum \sigma(S^{-1}(h(2)) \otimes h(1)) X_i(h(3)) \bullet (a_0 Y_i(h(3)) \otimes a_1) \sigma^{-1}(Y_i(h(3)) \otimes a_1)
$$

$$
= \sum \sigma(S^{-1}(h(2)) \otimes h(1)) X_i(h(3)) \bullet (a_0 Y_i(h(3))) \sigma^{-1}(h(4) \otimes a_1)
$$

$$
= \sum \sigma(S^{-1}(h(2)) \otimes h(1)) X_i(h(3)) \bullet (a_0 Y_i(h(3))) \sigma^{-1}(h(4) \otimes a_1)
$$

$$
\sum X_i(h(5)) a_0 Y_i(h(5)) \sigma^{-1}(h(7) \otimes a_1) \otimes S^{-1}(h(3)) h(2))
$$

$$
\sigma(h(6)) a_1 S^{-1}(h(4)) \otimes h(1) \sigma^{-1}(h(8) \otimes a_1)
$$

$$
= \sum X_i(h(3)) a_0 Y_i(h(3)) \sigma(h(4) a_1 S^{-1}(h(2)) \otimes h(1)) \sigma^{-1}(h(5) \otimes a_2)
$$

$$
\sum h(3) \cdot a_0 \sigma(h(4) a_1 S^{-1}(h(2)) \otimes h(1)) \sigma^{-1}(h(5) \otimes a_2)
$$

Thus we have proved that $h \cdot \sigma a = h \rightarrow a$ for all $a \in \pi(\sigma(A)) = \sigma(\pi(A))$ and $h \in H^\sigma$. Whence, $\pi(\sigma(A)) = \sigma(\pi(A))$ as YD $H^\sigma$-module algebras.

ACKNOWLEDGMENTS

The first author would like to thank the School of Mathematics, Statistics and Computer Science, Victoria University of Wellington for their hospitality during his visit in 2004. He is grateful to the URF of VUW for the financial support. He is also supported by NSF of China (No. 10471121).

The second author is supported by the Marsden Fund.

REFERENCES

[1] J. Bichon and G. Carnovale, Lazy cohomology: an analogue of the Schur multiplier for Hopf algebras, preprint.
[2] S. Caenepeel, F. Van Oystaeyen and Y. H. Zhang, Quantum Yang-Baxter module algebras, K-Theory 8 (1993), 231-255.
[3] S. Caenepeel, F. Van Oystaeyen and Y. H. Zhang, The Brauer group of Yetter-Drinfeld module algebras, Trans. Amer. Math. Soc. 349(9) (1997), 3737-3771.
[4] G. Carnovale, Some isomorphisms for the Brauer groups of a Hopf algebra, preprint.
[5] H. X. Chen, Skew pairing, cocycle deformations and double crossproducts, Acta Math. Sinica, English Ser. 15(2) (1999), 225-234.
[6] Y. Doi, Braided bialgebras and quadratic bialgebras, Comm. Algebra 21 (1993), 1731-1749.
[7] Y. Doi and M. Takeuchi, Multiplication alteration by 2-cocycles-The quantum version, Comm. Algebra 22 (1994), 5715-5732.
[8] C. Kassel, Quantum groups, Springer-Verlag, New York, 1995.
[9] S. Majid, Braided groups, J. Pure and Appl. Algebra 86 (1993), 187-221.
[10] S. Majid, Foundations of quantum group theory, Cambridge Univ. Press, Cambridge, 1995.
[11] Y. Miyashita, An exact sequence associated with a generalized crossed product, Nagoya Math. J. 49 (1973), 21-51.
[12] S. Montgomery, Hopf Algebras and their actions on rings, CBMS Series in Math., Vol.82, Am. Math. Soc., Providence, 1993.
[13] P. Schauenburg, Hopf bimodules, coquasialgebras, and an exact sequence of Kac, Adv. in Math. 165(2002), 194-263.
[14] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[15] K.-H. Ulbrich, Galoiserweiterungen von nicht-kommutativen Ringen, Comm. Algebra 10 (1982), 655-672.
[16] F. Van Oystaeyen, Pseudo-places algebras and the symmetric part of the Brauer group, Ph.D dissertation, March 1972, Vrije Universiteit Amsterdam.
[17] F. Van Oystaeyen and Y. H. Zhang, The Brauer group of braided monoidal category, J.Alg.202(1998), 96-128.
[18] F. Van. Oystaeyen and Y. H. Zhang, The Brauer group of the Sweedler’s Hopf algebra $H_4$, Proc. Amer. Math. Soc. 129 (2001), 371-380.
[19] Yinhuo Zhang, An exact sequence for the Brauer group of a finite quantum group, J. Alg. 272(2004), 321-378.

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