Elementary excitations, Spectral weights and Experimental signatures in a Supersolid and a Fulde - Ferrell-Larkin-Ovchinnikov state

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(Dated: February 9, 2021)

We construct a Ginsburg Landau (GL) theory to study the phases of liquid, solid, superfluid, especially a possible supersolid and phase transitions among these phases in a unified framework. In this GL, we put the two competing orders between the solid component and the superfluid component on the same footing. We only introduce two parameters: \( v \) which is the repulsive interaction between the normal component and the local superfluid mode and \( g \) which is a periodically changing chemical potential for the local superfluid mode. The microscopic origins of \( v \) and \( g \) are given. By using this GL action, we study the superfluid to supersolid transition, normal solid to the supersolid transition and analyze the conditions for the existence of the supersolid. The non-classical rotational inertial (NCRI) in the SS state is calculated and found to be isotropic in bcc and fcc lattice, but weakly anisotropic in hcp lattice. We study the elementary low energy excitation inside a supersolid. We find that there are one upper branch and one lower branch longitudinal "supersolidon" modes inside the SS, while the transverse modes in the SS stay the same as those inside the NS. We also determine the corresponding spectral weights of the two branches. We work out the experimental signatures of these elementary excitations in Debye-Waller factor, density-density correlation, vortex loop interaction and specific heat. The estimated excess entropy due to vacancies seems consistent with data measured in the specific experiment in Helium 4. Detecting the two supersolidon modes by various equilibrium and thermodynamic experiments such as X-ray scattering, neutron scattering, acoustic wave attenuation and heat capacity can prove or disprove the existence of a supersolid in Helium 4. A toy model of supersolid wavefunction is analyzed. The difference and similarities with lattice supersolid are clearly demonstrated. Elementary excitations inside a Fulde - Ferrell-Larkin-Ovchinnikov (FFLO) state of superfluid are also discussed.

I. INTRODUCTION

A solid can not flow. It breaks a continuous translational symmetry into a discrete lattice translational symmetry. There are low energy lattice phonon excitations in the solid. While a superfluid can flow even through narrowest channels without any resistance. It breaks a global \( U(1) \) phase rotational symmetry and has the off-diagonal long range order (ODLRO). There are low energy superfluid phonon excitations in the superfluid. A supersolid is a state which breaks both the continuous translational symmetry and the global \( U(1) \) symmetry, therefore has both the crystalline order and the ODLRO. The possibility of a supersolid phase in \(^4\text{He} \) was theoretically speculated in 1970\textsuperscript{2,3}. Andreev and Lifshitz proposed the Bose-Einstein condensation of vacancies as the mechanism of the formation of supersolid\textsuperscript{4}. Chester wrote down a wavefunction which has both ODLRO and crystalline order and also speculated that a supersolid cannot exist without vacancies or interstitials\textsuperscript{5}. Leggett proposed that solid \(^4\text{He} \) might display Non-Classical Rotational Inertial (NCRI) which is a low temperature reduction in the rotational moment of inertia due to the superfluid component of solid \(^4\text{He} \)\textsuperscript{6}. Leggett also suggested that quantum tunneling of He atoms between neighboring sites in a crystal can also lead to a supersolid even in the absence of vacancies. Over the last 35 years, a number of experiments have been designed to search for the supersolid state without success. However, recently, by using torsional oscillator measurement, a PSU group lead by Chan observed a marked 1 \( \sim \) 2\% NCRI of solid \(^4\text{He} \) at \( \sim 0.2K \), both when embedded in Vycor glass and in bulk \(^4\text{He} \)\textsuperscript{7}. The authors suggested that the supersolid state of \(^4\text{He} \) maybe responsible for the NCRI. The PSU group also detected about \( 10^{-4} \) reduction in the rotational inertial in solid \(^4\text{He} \), but by performing blocked cell experiments, the authors concluded that the reduction is classical due to the motion of HD impurities clustering. So the PSU group did not find any NCRI in solid hydrogen\textsuperscript{7}.

The PSU experiments inspired extensive both theoretical\textsuperscript{12-15} and experimental\textsuperscript{16-17} interests in the very intriguing supersolid phase of \(^4\text{He} \). Now it was widely believed that disorders played key roles in the PSU experiments. There are two kinds of complementary theoretical approaches. The first is the microscopic numerical simulation\textsuperscript{12}. The second is the phenomenological approach\textsuperscript{12-14}. In this paper, we assume no disorders in our GL theory and study all the possible phases in different parameter regimes of the GL model. A solid has the order in the number density, while a superfluid has the order in the phase. The two phases are in totally different extremes of the state of matter. How to combine the two opposite extremes into a new state of matter, supersolid, is an important and interesting topic on its own. It is widely believed that the only chance to get a supersolid is that the solid is not perfect, namely, it is an incommensurate solid which has defects such as vacancies or interstitials, so the total number of bosons can fluctuate to give some rooms for the invasion of superfluid. Due to large zero point motions, there are in-
deed rapid exchanging between local \(^4\)He atoms in bulk \(^4\)He, but this local process will not lead to a global phase ordering. It was estimated from both X-ray scattering experiments\(^25\) and some microscopic calculations\(^12\) that the thermal excitation energies of a vacancy and interstitial are \(\epsilon_v \sim 10 K, \epsilon_i \sim 40 K\), so thermal fluctuations favor vacancies over interstitials. Because both have very high energy, so the thermal generated vacancies and interstitials are irrelevant around \(45\sim 200 mK\). However, it is still possible that there are quantum fluctuation generated vacancies and interstitials even at \(T=0\). It is still not known if quantum fluctuations favor vacancies over interstitials or not\(^12\).

In this paper, we will construct a phenomenological Ginsburg Landau (GL) theory to study all the possible phases and phase transitions in helium 4 system. We identify order parameters, symmetry and symmetry breaking patterns in all the phases. Particularly, we will address the following two questions: (1) What is the criterion for the existence of the SS state? (2) If the SS exists, what are the properties of the supersolid to be tested by possible new experiments. Let’s start by reviewing all the known phases in \(^4\)He. The density of a normal solid (NS) is defined as \(n(\vec{x}) = n_0 + \sum a \psi \delta_n \vec{G}_n\), where \(n_0 = n_{\text{NS}}\) and \(\vec{G}\) is any non-zero reciprocal lattice vector. In a normal liquid (NL), if the static liquid structure factor \(S(k)\) has its first maximum peak at \(k_n\), then near \(k_n\), \(S(k) \sim \frac{1}{r_n + c(G^2-k_n^2)^2}\). If the liquid-solid transition is weakly first order, it is known that the classical free energy to describe the NL-NS transition is\(^21\):

\[
\frac{\mathcal{F}}{\mathcal{F}_0} = \sum_{\vec{G}} \frac{1}{2} r_G^2 |\psi|_G^2 - w \sum_{\vec{G}_1, \vec{G}_2, \vec{G}_3} n_{\vec{G}_1} n_{\vec{G}_2} n_{\vec{G}_3} \delta_{\vec{G}_1 + \vec{G}_2 + \vec{G}_3, 0} + u_n \sum_{\vec{G}_1, \vec{G}_2, \vec{G}_3, \vec{G}_4} n_{\vec{G}_1} n_{\vec{G}_2} n_{\vec{G}_3} n_{\vec{G}_4} \delta_{\vec{G}_1 + \vec{G}_2 + \vec{G}_3 + \vec{G}_4, 0} + (1)
\]

where \(r_G = r_n + c(G^2-k_n^2)^2\) is the tuning parameter controlled by the pressure or temperature. Note that the average density \(n_0\) does not enter in the free energy. Because the instability happens at finite wavevector, Eqn.\(^1\) is an expansion in terms of small parameter \(n_0\) alone, it is not a gradient expansion! The GL parameters \(w, u_n\) may be determined by fitting the theoretical predictions with experimental data. It is easy to see that Eqn.\(^1\) is invariant under \(\vec{x} \rightarrow \vec{x} + \vec{a}\), \(n(\vec{x}) \rightarrow n(\vec{x} + \vec{a})\), \(n(\vec{G}) \rightarrow n(\vec{G}) e^{i\vec{G} \cdot \vec{a}}\) where \(\vec{a}\) is any vector. In the NL, \(\langle n(\vec{G}) \rangle \neq 0\), the translational symmetry is respected. In the NS, \(\langle n(\vec{G}) \rangle \neq 0\), the symmetry is broken down to translational invariance under only a lattice constant \(\vec{a} = R, \vec{G} \cdot R = 2\pi n, n(\vec{x}) \rightarrow n(\vec{x} + \vec{R})\). If \(\langle n(\vec{G}) \rangle \neq 0\). As shown in the Fig.1, the NL to NS transition only happens at finite temperature, so the classical theory is valid. Note that due to the lack of the \(Z_2\) symmetry of \(\delta n(\vec{x}) \rightarrow -\delta n(\vec{x})\), namely, \(n\vec{G} \rightarrow -n\vec{G}\), there is always a cubic \(w\) term which makes the the NL to NS a 1st order transition. The \(u_n\) term which is invariant under the \(Z_2\) symmetry is needed for the stability reason.

Of course, the Superfluid to Normal Liquid transition at finite temperature in the Fig.1 is the 3d XY transition described by\(^20\):

\[
\mathcal{F}_0 = K|\nabla \psi|^2 + \ell |\psi|^2 + u|\psi|^4 + \cdots
\]

where \(\psi\) is the complex order parameter and \(t\) is the tuning parameter controlled by the temperature or pressure. Eqn.\(^2\) is invariant under the global \(U(1)\) symmetry \(\psi \rightarrow \psi e^{i\theta}\). In the NL, \(\langle \psi \rangle \neq 0\), the symmetry is respected. In the SF, \(\langle \psi \rangle \neq 0\), the symmetry is broken. The GL parameters \(K, t, u\) may be determined by fitting the theoretical predictions with experimental data. In this paper, we always assume \(u > 0\).

The coupling between \(n(\vec{x})\) and \(\psi(\vec{x})\) consistent with all the symmetry can be written down as:

\[
f_{\text{int}} = g \delta n(\vec{x}) |\psi(\vec{x})|^2 + v(\delta n(\vec{x}))^2 |\psi(\vec{x})|^2 + \cdots
\]

where \(\delta n(\vec{x}) = n(\vec{x}) - n_0 = \sum a \psi \delta_n \vec{G}_n\). Note that the average density \(n_0\) does not enter in the interaction either. Eqn.\(^3\) is an expansion in terms of two small parameters \(\delta n(\vec{x})\) and \(\psi(\vec{x})\). The \(\psi\) term include the other terms in higher odd and even powers of \(\delta n(\vec{x})\) which are subleading to the \(g\) and \(v\) term. In an effective GL theory, \(n(\vec{x})\) and \(\psi(\vec{x})\) emerge as two independent order parameters.

Eqn.\(^4\) is invariant under both the translational symmetry \(\vec{x} \rightarrow \vec{x} + \vec{a}\), \(n(\vec{x}) \rightarrow n(\vec{x} + \vec{a})\), \(\psi(\vec{x}) \rightarrow \psi(\vec{x} + \vec{a})\) and the global \(U(1)\) symmetry \(\psi \rightarrow e^{i\theta}\). Note that it is important to keep both \(g\) and \(v\) term in the Eqn.\(^4\) because the \(g\) term changes the sign, while the \(v\) term is invariant under the Particle-Hole (PH) transformation \(\delta n(\vec{x}) \rightarrow -\delta n(\vec{x})\), so the sign of \(g\) makes a difference! Due to the two competing orders between the density fluctuation represented by \(\delta n(\vec{x})^2\) and the local superfluid mode represented by \(|\psi|^2\), we expect \(v\) to be always positive and is an increasing function of the pressure \(p\). The positive \(v\) term is also needed for the stability reason. On the other hand, we can view \(g\) as a periodic chemical potential with average zero acting on \(\psi\). It is easy to see the coupling is attractive \(g_l < 0\) for vacancies, but repulsive \(g_l > 0\) for interstitials. From Eqn.\(^4\) we can classify three kinds of solids: If \(g = 0\), the C-NS has the P-H symmetry, let’s call this kind of PH symmetric C-NS as NS-PH. In general, \(g \neq 0\), so there is no particle-hole symmetry in the C-NS, there are still two kinds: (1) vacancy like C-NS where the excitation energy of a vacancy is lower than that of an interstitial, named NS-v. (2) interstitial like C-NS where the excitation of an interstitial is lower than that of a vacancy, named NS-i. We expect that in contrast to \(v, g\) is an intrinsic parameter of solid helium 4 which depends on the mass of \(^4\)He atom and the potential between the \(^4\)He atoms, but not sensitive to the pressure \(p\). In SS-v and SS-i, \(\psi(\vec{x})\) stands for vacancies and interstitials respectively. The total density of the system is \(n_t(\vec{x}) = n(\vec{x}) \pm |\psi(\vec{x})|^2\) for SS-i and SS-v respectively where \(n(\vec{x})\) is the density
of number of solid sites and \(|\psi(x)|^2\) is the superfluid density. Both were treated as independent order parameters in the GL theory.

The GL equations are invariant under both the translational symmetry \(\mathbf{x} \rightarrow \mathbf{x} + \mathbf{a}, n(\mathbf{x}) \rightarrow n(\mathbf{x} + \mathbf{a}), n(\mathbf{G}) \rightarrow n(\mathbf{G}) e^{i\mathbf{G} \cdot \mathbf{a}}\), \(\psi(\mathbf{x}) \rightarrow \psi(\mathbf{x} + \mathbf{a})\) and the global \(U(1)\) symmetry \(\psi \rightarrow e^{i\theta}\psi\). In a NL, \(n_{\mathbf{G}} = 0, \langle \psi \rangle = 0\). In a SF, \(n_{\mathbf{G}} = 0, \langle \psi \rangle \neq 0\). In a NS, \(n_{\mathbf{G}} \neq 0, \langle \psi \rangle = 0\). While in a supersolid, \(n_{\mathbf{G}} \neq 0, \langle \psi \rangle \neq 0\). From the normal liquid (NL) side, one can approach both the solid and the superfluid. Inside the NL, \(t > 0\), \(\psi\) has a gap, so can be integrated out from Eqn.3 we recover the solid-liquid transition tuned by \(t_{\mathbf{G}}\) (Fig. 2). Inside the NL \(\langle n(\mathbf{x}) \rangle = n_0\), the density fluctuations of \(\delta n(x)\) is massive, so can be integrated out from Eqn.3 then we recover the NL to SF transition tuned by \(t\) in Eqn.2 (Fig. 1).

Quantum fluctuations can be incorporated by (1) \(n(\mathbf{x}) \rightarrow n(\mathbf{x}, \tau)\) and including \(\frac{i}{2} \rho_n (\partial_\tau n)\) in Eqn.1 (2) \(\psi(\mathbf{x}) \rightarrow \psi(\mathbf{x}, \tau)\) and including \(\psi^\dagger (\mathbf{x}, \tau) \partial_\tau \psi(\mathbf{x}, \tau)\) in Eqn.2. (3) Due to the lack of particle-hole symmetry in the normal solid, including additional terms like \(\delta n(\mathbf{x}, \tau) \psi^\dagger (\mathbf{x}, \tau) \partial_\tau \psi(\mathbf{x}, \tau)\) in Eqn.3. We will explicitly consider the quantum fluctuations in section III where we will discuss the quantum phase transition from the superfluid to the C-NS (either NS-v or NS-i). However, as shown in this paper, the quantum fluctuation terms are very important in the zero temperature quantum phase transition from SS-v (SS-i) to NS-v (NS-i) driven by the pressure (see Fig. 2) and to determine the low energy excitation spectra in a given phase.

The rest of the paper is organized as the following: In sec. II, for the first time, we construct a quantum Ginsburg-Landau action to study SF to the NS transition and also explicitly establish the Feynman relation from the QGL. In Sec.III, taking into account the couplings between the \(n\) and \(\psi\) sector encoded in Eqn.3, we study the superfluid to supersolid transition which is a simultaneous combination of the SFDW transition in \(\psi\) sector driven by the roton condensation at \(k_0 = k_r\) and the NS transition in the \(n\) sector driven by the divergence of the structure function at \(k_0 = k_r = k_n\) discussed in Sec.II. We also sketch the global phase diagrams to be confirmed and analyzed in the following sections. In Sec.IV, we approach the SS phase from the NS side and discuss the SFDW in SS-v and SS-i respectively. We analyze carefully the conditions for the existence of SS-v. We explicitly show that the SS-v is the possible ground state when the \(g_n\) is sufficiently negative (Fig. 3). The SS-i is less likely, but we still analyze SS-i in the same footing as SS-v. We also classify several common SS-i lattice structures. In Sec. V, by renormalization group analysis, we study the universality class of zero temperature quantum phase transition from normal solid (NS) to supersolid (SS) driven by the pressure. In Sec. VI, we calculate the NCRI’s in both the SF and SS states and find the NCRI in the \(hc\) SS lattice may be weakly anisotropic. In Sec.VII, we work out the non-topological elementary excitations and corresponding spectral weights inside the supersolids in both the isotropic solid case and the \(hc\) lattice structure case. Then in Sec.VIII, we study topological elementary excitations, namely, vortex loops and vortex lines in the SS by performing a duality transformation to the vortex loop representation. We also estimate the very large vortex core size and low critical velocity in the SS state. Then in the following sections, we study the experimental signatures of these low energy elementary excitations: In Sec. IX, we make key predictions on the elastic X-ray scattering amplitudes from all SS-v and SS-i structures classified in Sec. IV by calculating the Debye-Waller factor and the density-density correlation function. In Sec. X, we study the specific heat in the SS. In Sec. XI, inspired by the results achieved in Sec. VII, we discuss the elementary excitations inside a Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state of superconductors. Conclusions are summarized in the final section XII. In the appendix A, we discuss the properties of a tight-binding toy SS-v wavefunction. In the appendix B, we compare the properties of the SS with those of SS on extended boson Hubbard model in a lattice. Two short reports of these results appeared in 16,17.

II. SUPERFLUID TO NORMAL SOLID TRANSITION

Let us start with the SF to the NS transition. In the superfluid state, if the multi-quasiparticle part can be neglected in the dynamic structure factor, the Feynman relation between the Landau quasi-particle dispersion relation in the \(\psi\) sector and the equal time structure factor in the \(n\) sector holds:

\[ \omega(q) = \frac{q^2}{2mS(q)} \] (4)

In the \(q \to 0\) limit, \(S(q) \sim q\), \(\omega(q) \sim q\) recovers the superfluid phonon spectrum near \(q = 0\) in Fig.1b. The first maximum peak in \(S(q)\) corresponds to the roton minimum in \(\omega(q)\) in the \(\psi_2\) sector (Fig.1b), namely, \(k_n = k_r\). As one increases the pressure \(p\), the interaction \(u\) in Eqn.4 also gets bigger and bigger, the first maximum peak of \(S(q)\) increases, the roton minimum \(\Delta_r\) gets smaller and smaller. Across the critical pressure \(p_c\) there are the two possibilities. (1) The resulting solid is a commensurate solid, then \(\langle \psi \rangle = 0\). In the NS, there is no remanent of the roton inside the SF, the supersolid phase does not exist as an equilibrium ground state. This is the SF to the C-NS transition. As to be shown in section IV, this happens when \(|g|\) is sufficiently small (Fig.1). (2) The resulting solid at high pressure is an incommensurate solid with zero point quantum fluctuations generated vacancies or interstitials whose condensation leads to the formation of the SS-v and SS-i respectively. In section IV, this happens when \(|g|\) is sufficiently large (Fig.2).

We will discuss case (1) in this section, then the most interesting case (2) in the next section.
The effective action inside the SF is:

\[ \mathcal{L}[\delta n, \theta] = i\delta n \partial_t \theta + \frac{1}{2} \rho_s (\nabla \theta)^2 + \frac{1}{2} \delta n V_n(\vec{q}) \delta n \]  

(5)

where \( \rho_s \) is the superfluid density and \( V_n(q) = a - bq^2 + cq^4 \) with \( a > 0, b > 0 \) is the density-density interaction between the \( ^4\text{He} \) atoms.

In the SF state, it is convenient to integrate out \( \delta n \) in favor of the phase field \( \theta \) to get the phase representation

\[ \mathcal{L}[\theta] = \frac{1}{2} V_n(q)(\partial_t \theta)^2 + \frac{1}{2} \rho_s (\nabla \theta)^2 \]  

(6)

where one can identify the compressibility \( \kappa^{-1} = \lim V_n(q) |_{q \to 0} = a \). The dispersion relation of the superfluid modes including higher orders of momentum can be extracted\(^{24}\).

\[ \omega^2 = [\rho_s V_n(\vec{q})] q^2 = 2 \rho_s q^2 (a - bq^2 + cq^4) \]  

(7)

where one can see the superfluid phonon velocity \( \nu^2 = \rho_s / \kappa \).

It is easy to see that the dispersion relation indeed has the form shown in Fig.1b with a roton minimum. Because the original instability comes from the density-density interaction \( V_n(q) \), it is convenient to integrate out the phase field in favor of the density fluctuation operator \( \delta n \). Neglecting the vortex excitations in \( \theta \) and integrating out the \( \theta \) in Eqn.5 leads to:

\[ \mathcal{L}[\delta n] = \frac{1}{2} \delta n(-\vec{q} \cdot -\omega_n) [\frac{\omega_n^2}{\rho_s q^2} + V_n(\vec{q})] \delta n(\vec{q}, \omega_n) \]  

(8)

where we can identify the dynamic pseudo-spin density-density correlation function \( S_n(\vec{q}, \omega_n) = \langle \delta n(-\vec{q} \cdot -\omega_n) \delta n(\vec{q}, \omega_n) \rangle = \frac{\omega_n^2}{ \omega_n^2 + \nu^2(q) q^2} \) where \( \nu^2(q) = \rho_s V_n(q) \) is the "momentum dependent" spin wave velocity. Obviously \( \nu^2(q = 0) = \nu^2 = \rho_s / \kappa \).

From the pole of the dynamic density-density correlation function, we can identify the speed of sound wave which is exactly the same as the spin wave velocity. This should be expected. From the analytical continuation \( \omega_n \to \omega + i\delta \) and taking the imaginary part, we can identify the dynamic structure factor:

\[ S_n^\omega(\vec{q}, \omega) = S_n(q) \delta(\omega - \nu(q) q) \]  

where \( S_n(q) = \rho_s q \pi / 2 \nu(q) \) is the equal time density correlation function shown in Fig.1b. As \( q \to 0, S_n(q) \to q \). The sum rules are:

\[ \int_0^\infty d\omega S_n^\omega(\vec{q}, \omega) = \frac{\rho_s q \pi}{2 \nu(q)} = S_n(q) \]

\[ \int_0^\infty d\omega S_n^w(\vec{q}, \omega) = \rho_s q^2 \pi / 2 \]

\[ \int_0^\infty d\omega S_n^\omega(\vec{q}, \omega) / \omega = \rho_s \pi / 2 \nu^2(q) \]  

(9)

where the first one just defines the equal time density correlation function, the second one gives the \( f \) sum rule which can be used to define the superfluid density \( \rho_s \), the third one can be used to define the compressibility \( \kappa: \int_0^\infty d\omega S_n^\omega(\vec{q}, \omega) / \omega |_{q \to 0} = \kappa \pi / 2 \).

The Feynman relation which relates the dispersion relation to the equal time structure factor is:

\[ \omega(q) = \frac{\int_0^\infty d\omega S_n^\omega(\vec{q}, \omega)}{\int_0^\infty d\omega S_n^2(\vec{q}, \omega)} = \frac{\rho_s \pi q^2}{2 S_n(q)} \]  

(10)

which takes exactly the same form as Eqn.4 if we identify \( \rho_s \sim 1/m \) with \( m \) the mass of \( ^4\text{He} \) mass. Therefore, we recovered the Feynman relation from our GL theory Eqn.5 which gives us the confidence that Eqn.5 is the correct starting action to study the the SF to SS transition. The density representation Eqn.8 is dual to the phase representation Eqn.6. However, the phase representation Eqn.6 contains explicitly the superfluid order parameter \( \psi \sim e^{i\theta} \) which can be used to characterize the superfluid order in the SF phase. While in Eqn.8 because the order parameter \( \psi \) is integrated out, the superfluid order is hidden, the signature of the superfluid phonon mode is encoded in the density sound mode, so it is not as effective as the phase representation in describing the SF state. However, as shown in the following, when describing the transition from the ES to the NS, the density representation Eqn.8 has a big advantage over the phase representation.

Because the instability is happening at \( q = q_0 \) instead of at \( q = 0 \), the vortex excitations in \( \theta \) remain uncritical through the SF to SS transition. Integrating them out will generate interactions among the density \( \delta n \):

\[ \mathcal{L}[\delta n] = \frac{1}{2} \delta n(-\vec{q} \cdot -\omega_n) [\frac{\omega_n^2}{\rho_s q^2} + V_n(\vec{q})] \delta n(\vec{q}, \omega_n) - w(\delta n)^3 + u(\delta n)^4 + \cdots \]  

(11)

where the momentum and frequency conservation in the quartic term is assumed. Note that the \( (\omega_n / q)^2 \) term in the first term stands for the quantum fluctuations of \( \delta n \) which is absent in the classical NL to NS transition Eqn.1. Because of the lack of \( Z_2 \) exchange symmetry, there is a cubic term in Eqn.11. Expanding \( V_n(q) \) near the roton minimum \( q_0 \) leads to the quantum Ginsburg-Landau action to describe the SF to the NS transition:

\[ \mathcal{L}[\delta n] = \frac{1}{2} \delta n [A_n \omega_n^2 + r + (q^2 - q_0^2)^2] \delta n - w(\delta n)^3 + u(\delta n)^4 + \cdots \]  

(12)

where \( r \sim p_{c1} - p \) and \( A_n \sim \frac{\rho_s q_0^2}{\mu m} \) which is non-critical across the transition. Just like Eqn.1 because the instability happens at the finite wavevector \( q = q_0 \), Eqn.12 is not a gradient expansion, but an expansion in terms of the small order parameter \( \delta n \). Again the average density \( n_0 \) does not appear in Eqn.12. The generic transition driven by the collapsing of roton minimum is from SF to NS instead of from the SF to the supersolid (SS). In the SF, \( r > 0, \langle \psi \rangle \neq 0, \langle \delta n \rangle = 0 \). In the NS, \( r < 0, \langle \psi \rangle = 0, \langle \delta n \rangle = \sum_i \langle \delta n_i \rangle e^{i\vec{G}_i \cdot \vec{r}} \) where \( \vec{G} \) are the
shortest reciprocal lattice vector of the resulting lattice. The corresponding phase diagram for the SF to the NS transition is shown in Fig.1.

III. SUPERFLUID TO SUPER SOLID TRANSITION

In the last section, we discuss the SF to the NS transition with the order parameter $\delta n$. In this section, we discuss the SF to the SS transition with both order parameters $\psi$ and $\delta n$. Then we have to also include the couplings between $\delta n$ and $\psi$ sector encoded in Eqn.3. Starting from the SF side with increasing pressure, we develop the theory based on the two facts (1) there is a roton minimum in the superfluid state. The instability to solid formation is driven by the gap diminishing at the roton minimum. As shown in this last section, the fact (2) is guaranteed by the Feynman relation Eqn.4 which relates the roton minimum to the peak of the structure factor. Across the phase boundary $p = p_{c2}$ in Fig.2, the resulting solid could be an in-commensurate solid (IC-NS) with vacancies or interstitials even at $T = 0$ whose condensation leads to the formation of the SS-v and SS-i respectively where $\langle \psi \rangle \neq 0$. There is still some remanent of the SF in the IC-NS, the supersolid phase does exist as an equilibrium ground state. In this section, we assume $|g|$ is sufficiently large in Fig.2 and study the SF to the SS transition across $p = p_{c2}$. Because $\psi$ is also critical through the transition, we can not simply integrate it out like in the last section. In fact, in the effective GL theory, we have to treat $n$ and $\psi$ on the same footing. From Eqn.1 and Eqn.2 we can see that $n$ and $\psi$ have very similar propagators, so the lattice formation in $n$ sector with $n(x) = n_0 + \sum G \delta(x-G)$ where $G = k_n$ and the superfluid density wave (SF DW) formation in $\psi$ sector with $\langle \psi(\vec{x}) \rangle = \psi_0 + \sum G \delta_G \langle \psi \rangle$ where $Q_m = k_r = k_n \sim 2\pi/\alpha \sim 2\pi/3.17A$ have to happen simultaneously. From Hansen-Verlet criterion, when $S(k_n)/n_0$ is sufficiently large, solidification in the $n$ sector occurs, so the roton minimum remains finite just before its condensation. The SF DW $\rho = |\psi|^{2}$ is simply locked to (or commensurate with) the underlying normal solid ($n$) lattice. In fact, this locking is dictated by the $\rho$ density- $\delta n$ density couplings in Eqn.3. If the coupling $g$ is attractive $g_v < 0$, the SF DW just coincides with the $n$ lattice. If it is repulsive $g_i > 0$, it then simply shifts the SF DW by suitable constants along the three unit vectors in the direct lattice. These constants will be determined in the next section for different $n$ lattices. Namely, the supersolid states consist of two inter-penetrating lattices formed by the $n$ lattice and the $\psi$ superfluid density wave. In fact, in a carefully prepared super-pressured sample, the roton minimum survives up to as high pressure as $p_r \sim 120$ bar. This fact suggests the roton minimum in the meta-stable state in a super-pressured sample is replaced by a SF DW which is commensurate with the $n$ lattice in the stable SS state. Strikingly, this pressure $p_r \sim 120$ bar is close to $p_{c2} \sim 170$ bar in Fig.2 where the NCRI was extrapolated to vanish in the PSU’s experiments. This consistency also lead some support to the above picture. Obviously, when the pressure is so high that $p > p_{c2}$ in Fig.2, C-NS is the only possible ground state, any remanent of the SF completely disappears. So the vertical axis in the Fig.2 shows the SS-SS-N series as the pressure $P$ increases at $T = 0$. Combining the roton condensation picture in the last section with the results reviewed in the introduction, we can sketch the following phase diagram Fig.2 of the complete QGL Eqs.1, 2, and 3.

As can be seen from the Fig.2, starting from the SF side, as the pressure is increased at a given temperature, there are two possible states (1) At very low $T < T_{SS}$, the Bose-Einstein condensation (BEC) of $\psi_2$ leads to the supersolid state where $\langle \psi_1 \rangle \neq 0$, $\langle \psi_2 \rangle \neq 0$, $\langle n_2 \rangle \neq 0$. The instability happens at a finite wavevector $k = k_r$ instead...
of at \( k = 0 \). (2) At higher temperature \( T_{SS} < T < T_{SF} \), there is a direct SF to NS transition. The SS state is certainly different from a conventional normal solid phase where \( \langle \psi_1 \rangle = \langle \psi_2 \rangle = 0, (n_G) \neq 0 \). In addition to the conventional translational and rotational orders characterized by \( n \), the SS also has the ODLRO characterized by \( \psi_1 \) and \( \psi_2 \). When decreasing the temperature at a given pressure, if \( p < p_c \), the NL becomes SF at \( T = T_{SF} \), the instability happens at \( k = 0 \). If \( p > p_c \), the NL becomes a NS first at \( T = T_m \), then there is a SFDW onset transition from the NS to a SS phase at \( T = T_{SS} \).

Because the SF to SS transition driven by the roton condensation can be either weakly or strong first order, in principle, Eqn.12 works precisely only in the SF side, it is not easy to study the precise nature of the SS state from the SF side. It turns out that it is more convenient to study the properties of the SS state from the NS side in the next section.

IV. THE NORMAL SOLID TO THE SUPERSOLID TRANSITION

Starting from the NS side, we assume that the local tunneling processes in the NS-PH only leads to local fluctuations of \( \psi \) with a gap \( \Delta(p) \) in Fig.3, so there is no long-range phase coherence and no supersolid in this case (\( T_{SS} = 0 \)). Taking the NS-PH state as the reference state, we will show that if the coupling \( g \) in Eqn.3 is sufficiently negative (or positive), the supersolids SS-v or SS-i can be realized by adding small number of vacancies to NS-v or interstitials to the NS-i (Fig.3). The attractive (repulsive) interaction \( g \) is shown to be crucial to raise the normal solid to the vacancy (interstitials) induced supersolid transition temperature \( T_{SS-v} \) \( (T_{SS-i}) \) above the zero temperature in the Fig.3. In fact, the temperature \( T_{SS} \) becomes an effective measure of the coupling strength \( g \). We find that SS-v is more likely than SS-i. However, in order to be complete, we study both cases on the same footing. It is also constructive to compare SS-v with SS-i even though the SS-i is unlikely to be relevant to the Helium 4 system. When the two kinds of SS show different properties, we treat them differently, when they share the same properties, we treat them just in the same notation SS. The SS phase naturally and consistently fits into the phase diagram.

In the NS, \( \psi \) stands for the vacancies or interstitials coming from the large zero point quantum fluctuations\( \Delta \). Inside the NS, the translational symmetry is already broken, we can simply set \( \hat{n}(\bar{x}) = n(\bar{x}) - n_0 = \sum_G n_G e^{iG \cdot \bar{x}} \) and put it into Eqn.4 to look at the effects of the coupling constants \( g \) and \( \nu \). Imagine that at a given pressure \( p > p_{c1} \sim 25 \text{ bar} \), if tuning \( g \rightarrow 0 \), but keeping \( \nu \) intact, then the normal solid becomes an asymptotically P-H symmetric normal solid (NS-PH) in Fig.3. Inside the NL, the mass of \( \psi \) at \( \bar{k} = 0 \) is \( t = T - T_{XY} \) which sets up the temperature scale of the problem. Inside the NS-PH, it is easy to see that the mass of \( \psi \) at \( \bar{k} = 0 \) is \( t_{NS-PH} = t + \nu \sum_G |n(G)|^2 \). If we take the temperature scale \( t = T - T_{XY} \) as the reference scale, then \( t_{NS-PH} = T + \Delta(p) \) where \( \Delta(p) = \nu \sum_G |n(G)|^2 - T_{XY} > 0 \) is the \( T = 0 \) gap for the local superfluid mode \( \psi \) at \( \bar{k} = 0 \) in the NS-PH. Because as the pressure \( p \) increases, the repulsive interaction \( \nu \) also increases, so it is reasonable to assume that \( \Delta(p) \) is a monotonic increasing function of \( p \). Because it is a first order transition across \( p_{c1} \), just like the roton gap \( \Delta_r > 0 \) remains finite just before the first order transition, \( \Delta(p) \) also remains finite just after the first order transition, namely, \( \Delta(p_{c1}^+) > 0 \) (Fig.3).

Taking this NS-PH as a reference state, we then gradually turn on \( g \) and see how the ground state evolves. In the presence of the periodic potential of \( n(x) \) lattice, \( \psi \) will form a Bloch wave. In principle, a full energy band calculation incorporating the interaction \( u \) is necessary to get the energy bands of \( \psi \). Fortunately, qualitatively important physical picture of GL Eqns.1, 2, 3 can be achieved without such a detailed energy band calculation. We expect that the self interaction \( u \) only leads to some renormalization of the parameters \( t \) and \( K \). Writing the \( g \) term in momentum space \( g \sum_G n(G)\psi^*(G)\psi(0) \) where

![Figure 2](image-url)
FIG. 3: (a) The gap $\Delta(p)$ of the local fluctuating superfluid mode $\psi$ in the P-H symmetric commensurate normal solid (NS-PH) which exists only at $g = 0$. We assume it is a monotonic increasing function of the pressure $p$. (b) The zero temperature phase diagram of $g_v$ versus the pressure $p$. The NS-PH only exists at $g_v = 0$. Any $g_v \neq 0$ will transfer the NS-PH into the NS-v. If the experimental is along the path I, then it is a 1st order SF to SS-v transition in Fig.1, if it is along the path II, it is a 1st order SF to SS-v transition in Fig.2.

$\vec{G} \neq 0$ and integrating out $\psi(\vec{G})$, we get a perturbative expansion on the eigen-energy $\epsilon_{\mu}(0)$ at the origin $\vec{K} = 0$, it is easy to see the $n$'s term is of the form $-(g^n)$. Up to the third order $g_{\mu}^3$, the series is:

$$
\epsilon_{\mu}(0) = t_{NS-PH} - g_{v0}^2 P \frac{n^2(\vec{G})}{K_{\mu} G_1^2} + g_{v0}^3 \sum_{\vec{G}_1, \vec{G}_2} \frac{n(\vec{G}_1)n(\vec{G}_2)n(-\vec{G}_1 - \vec{G}_2)}{K_{\mu} G_1^2 K_{\mu} G_2^2} + \cdots (13)
$$

where the term linear in $g$ vanishes, $\mu = v, i$ stands for vacancies and interstitials respectively and the shortest reciprocal lattice vector is $G = 2\pi/a$ with $a \sim 3.17A$ the lattice constant of the solid $^4He$, the $n(\vec{K})$ is $e^{-K^2 \alpha^2/4a^2}$ where $K$ is any reciprocal lattice vector can be taken as a Gaussian where $\alpha$ is the width of the exponential.

For vacancies, $g_v < 0$, without writing out the coefficients explicitly, the expansion is

$$
\epsilon_v(0) = t_{NS-PH} - g_v^2 - |g_v|^3 - |g_v|^4 - \cdots (14)
$$

so the coefficient has the same sign. Assuming the series converges, for any $g_v < 0$, we can write $\epsilon_v(0) = t - f_v(g_v)$ where the $f_v(g_v) \geq 0$, $f_v(0) = 0$ is a monotonic increasing function of $g_v$ and likely has no upper bound.

For interstitials, $g_i > 0$, without writing out the coefficients explicitly, the expansion is

$$
\epsilon_i(0) = t_{NS-PH} - g_i^2 + g_i^3 - g_i^4 + \cdots (15)
$$

so the coefficient has oscillating sign. Assuming the series converges, we can write $\epsilon_i(0) = t - f_i(g_i)$ which holds for any $g_i$. Because of the oscillating nature of the expansion coefficients, it is hard to judge the nature of the function of $f_i(g_i)$ except we know $f_i(0) = 0$. The different expansion series of $f_v(g_v)$ and $f_i(g_i)$ indicate that quantum fluctuations may favor vacancies over interstitials. However, for simplicity, we assume $f_v(g_v)$ is also a monotonic increasing function of $g_v$, so we can discuss vacancies and interstitials induced supersolids on the same footing. In the following, we discuss SS-v and SS-i respectively.

A. Vacancies induced supersolid: SS-v

The mass of $\psi_v$ was decreased to $t_{\psi_v} = T + \Delta(p) - f_v(g_v) = T - T_{SS-v}$ where $T_{SS-v}(p) = f_v(g_v) - \Delta(p)$ (Fig.3). Because $f_v(g_v)$ is a monotonic increasing function of $|g_v|$ and $f_v(0) = 0$, defining a critical value $f_v(g_{vc}) = \Delta(p_{c1})$, then when $|g_v| < |g_{vc}|$, $f_v(g_v) < \Delta(p_{c1})$, the $\psi_v$ mode remains massive, namely $\langle \psi_v \rangle = 0$. The C-NS remains to be the ground state even at $T = 0$. It is important to stress that even the solid is a C-NS, it still does not have the P-H symmetry. For $g_v < 0$, the C-NS is a vacancy-like C-NS (named as NS-v) where the vacancy excitation energy $\epsilon_v$ is lower than that of the interstitial $\epsilon_i$ (Fig.3b).

If $|g_v| > |g_{vc}|$, then $T_{SS-v}(p_{c1}) = f_v(g_v) - \Delta(p_{c1})$ is raised above the zero temperature, the SS-v state exists in the Fig.3b. $T_{SS-v}(p_{c1})$ is also proportional to the superfluid density measured in the experiments. The resulting solid is an in-commensurate solid with vacancies even at $T = 0$ whose condensation leads to $\langle \psi_v \rangle \neq 0$. Of course, the number of vacancies $n_v$ is quite small. The SS-v state has a lower energy than the NS-v state at sufficiently low temperature. As the pressure increases to $p_{c2}$, $T_{SS-v}(p_{c2}) = f_v(g_v) - \Delta(p_{c2}) = 0$ (Fig.3c). Then $f_v(g_v) = \Delta(p_{c2})$, so $T_{SS-v}(p) = \Delta(p_{c2}) - \Delta(p)$ which becomes an effective experimental measure of the energy gap $\Delta(p)$ in Fig.3a. Indeed, if reflecting the $\Delta(p)$ in Fig.3a with respect to the horizontal axis and then shift by $\Delta(p_{c2}) = f_v(g_v)$, then one recovers the $T_{SS-v}(p)$ in Fig.2. In the following, substituting the ansatz $\psi_v(\vec{x}) = ae^{i\varphi} \psi(\vec{x}) = e^{i\theta} e^{i\vec{Q}_m\cdot \vec{x}}$ we study the effects of $n$ lattice on $\psi_v = \psi_1 + \psi_2$. From Eqn[B] we can see $n(\vec{x})$ acts as a periodic potential on $\psi$. In order to get the lowest energy ground state, we must consider the following 4 conditions: (1) because any complex $\psi$ (up to a global phase) will lead to local supercurrents which is costly, so we can take $\psi$ to be real, so $\vec{Q}_m$ have to be paired as anti-nodal points. $\rho$ has to be even (2) as shown from the Feynman relation Eqn[H] $\vec{Q}_m, m = 1, \cdots, P$ are simply $P$ shortest reciprocal lattice vectors, then translational symmetry of the lattice dictates that $\epsilon(\vec{K} = 0) = \epsilon(\vec{K} = \vec{Q}_m)$, $\psi_1$ and $\psi_2$ have to condense at the same time (3) The point group symmetry of the lattice dictates $\Delta_m = \Delta$ and is real (4) for the vacancies case, the interaction is attractive $g_v < 0$, from Eqn[B] the SFDW-v simply sits on top of the n lattice, so the Superfluid Density wave $\rho = |\psi|^2$ simple sits on top of the n lattice as much as possible. This is reasonable, because vacancies are hopping on near the lattice.
Density wave \( \rho \) condition (4), for the SS-v, the interaction is attractive goes the same as those in the vacancies case except the arguments to determine the lattice structure of the SS-i. Similar thing can be defined for an interstitial-like C-NS (named as NS-i) when \( g_i < g_{ic} \).

The discussion is quite similar to the vacancy case except that (1) if \( g_i < g_{ic} \), the C-NS is an interstitial-like C-NS (named as NS-i) where the interstitial excitation energy \( \epsilon_i \) is lower than that of the vacancy \( \epsilon_v \) (2) if \( g_i > g_{ic} \), the resulting solid is an in-commensurate solid with interstitials even at \( T = 0 \) whose condensation leads to \( \langle \psi_i \rangle \neq 0 \), the SS-i state exists in the Fig.2. The arguments to determine the lattice structure of the SS-i goes the same as those in the vacancies case except the condition (4), for the SS-v, the interaction is attractive \( g_v < 0 \), the SFDW-v simply sits on top of the \( n \) lattice. However, for the interstitials case, the interaction is repulsive \( g_i > 0 \) favors \( \psi(x = 0) \sim 0 \), so the Superfluid Density wave \( \rho = |\psi|^2 \) can avoid the \( n \) lattice as much as possible. This is reasonable, due to the competing of the two orders, the superfluid component emerges from the places where the normal solid component is suppressed. It turns out that the the 4 conditions can fix the relative phase and magnitude of \( \psi_1 \) and \( \psi_2 \) to be \( \theta_2 = \theta_1 + \pi, \Delta = a/P \), namely:

\[
\psi_{ss-i} = \psi_0(1 - \frac{2}{P} \sum_{m=1}^{P/2} \cos \frac{\theta}{m} \cdot \bar{x})
\]

where \( \psi_0 = ae^{i\theta} \) depends on the temperature and pressure. Note that the crucial sign difference from the vacancies case which make an important difference from the X-ray scattering from SS-v and SS-i to be discussed in the next section. Again in contrast to a uniform superfluid, the magnitude of \( \psi \) is changing in space. This field satisfies the Bloch theorem with the crystal momentum \( \vec{k} = 0 \) and the Fourier components are \( \psi(\vec{K} = 0) = a, \psi(\vec{K} = \vec{Q}_m) = a/P \). They have the same sign and decay in magnitude. In principle, higher Fourier components may also exist, but they decay very rapidly, so can be neglected without affecting the physics qualitatively.

B. Interstitial induced supersolid: SS-i

The discussion is quite similar to the vacancy case except that (1) if \( g_i < g_{ic} \), the C-NS is an interstitial-like C-NS (named as NS-i) where the interstitial excitation energy \( \epsilon_i \) is lower than that of the vacancy \( \epsilon_v \) (2) if \( g_i > g_{ic} \), the resulting solid is an in-commensurate solid with interstitials even at \( T = 0 \) whose condensation leads to \( \langle \psi_i \rangle \neq 0 \), the SS-i state exists in the Fig.2. The arguments to determine the lattice structure of the SS-i goes the same as those in the vacancies case except the condition (4), for the SS-v, the interaction is attractive \( g_v < 0 \), the SFDW-v simply sits on top of the \( n \) lattice. However, for the interstitials case, the interaction is repulsive \( g_i > 0 \) favors \( \psi(x = 0) \sim 0 \), so the Superfluid Density wave \( \rho = |\psi|^2 \) can avoid the \( n \) lattice as much as possible. This is reasonable, due to the competing of the two orders, the superfluid component emerges from the places where the normal solid component is suppressed. It turns out that the the 4 conditions can fix the relative phase and magnitude of \( \psi_1 \) and \( \psi_2 \) to be \( \theta_2 = \theta_1 + \pi, \Delta = a/P \), namely:

\[
\psi_{ss-i} = \psi_0(1 - \frac{2}{P} \sum_{m=1}^{P/2} \cos \frac{\theta}{m} \cdot \bar{x})
\]

where \( \psi_0 = ae^{i\theta} \) depends on the temperature and pressure. Note that the crucial sign difference from the vacancies case which make an important difference from the X-ray scattering from SS-v and SS-i to be discussed in the next section. Again in contrast to a uniform superfluid, the magnitude of \( \psi \) is changing in space. This field satisfies the Bloch theorem with the crystal momentum \( \vec{k} = 0 \) and the Fourier components are \( \psi(\vec{K} = 0) = a, \psi(\vec{K} = \vec{Q}_m) = a/P \). They have the same sign and decay in magnitude. In principle, higher Fourier components may also exist, but they decay very rapidly, so can be neglected without affecting the physics qualitatively.
\[\psi(\vec{x}) = a[1 - \frac{1}{3}(\cos \vec{Q}_1 \cdot \vec{x} + \cos \vec{Q}_2 \cdot \vec{x} + \cos \vec{Q}_3 \cdot \vec{x})].\]

The maxima of the SFDW \(\psi_{\text{max}} = 3/2a\) appear in the middle of lattice points at \(\vec{a} = \pm \frac{1}{2}(\vec{a}_1 + \vec{a}_2).\) They form the dual lattice of the triangular lattice which is a honeycomb lattice. The honeycomb lattice is not a Bravais lattice which has two triangular sublattices \(A\) and \(B,\) but it can be considered as one triangular lattice \(A\) plus a basis. This can be understood intuitively: there are two equivalent ways to shift the \(n\) lattice, one way to get the sublattice \(A,\) the other to get the sublattice \(B.\) Putting \(A\) and \(B\) together forms the SFDW which takes the honeycomb lattice \((c2)\) \(\vec{Q}_i, i = 1, 2, 3, 4, 5, 6\) are the 6 shortest reciprocal lattice vectors generating a cubic lattice. The maxima of the SFDW \(\psi_{\text{max}} = 2a\) appear exactly in the middle of lattice points at the 8 points \(\vec{a} = \frac{1}{2}(\pm \vec{a}_1 \pm \vec{a}_2 \pm \vec{a}_3).\) So the SFDW forms the dual lattice of the cubic lattice which is also a cubic lattice.

(d) \(P = 8:\) \(\vec{Q}_i, i = 1, \ldots, 8\) form the 8 shortest reciprocal lattice vectors generating a \(bc\) reciprocal lattice which corresponds to a \(f\) \(c\) direct lattice. The field is \[\psi(\vec{x}) = a[1 - \frac{1}{3}(\cos \vec{Q}_1 \cdot \vec{x} + \cos \vec{Q}_2 \cdot \vec{x} + \cos \vec{Q}_3 \cdot \vec{x} + \cos \vec{Q}_4 \cdot \vec{x} + \cos \vec{Q}_5 \cdot \vec{x} + \cos \vec{Q}_6 \cdot \vec{x})].\]

The maxima of the SFDW \(\psi_{\text{max}} = 2a\) appear in all the edge centers such as \((1/2, 0, 0)\) etc. and the centers of any cube such as \((1/2, 1/2, 1/2)\) etc. It is easy to see these points can be obtained by simply shifting the \(n\) lattice by \((1/2, 1/2, 1/2),\) so the SFDW also forms a \(f\) \(c\) direct lattice.

(e) \(P = 12:\) \(\vec{Q}_i, i = 1, \ldots, 12\) form the 12 shortest reciprocal lattice vectors generating a \(f\) \(c\) reciprocal lattice which corresponds to a \(b\) \(c\) direct lattice. The field is \[\psi(\vec{x}) = a[1 - \frac{1}{3}(\cos \vec{Q}_1 \cdot \vec{x} + \cos \vec{Q}_2 \cdot \vec{x} + \cos \vec{Q}_3 \cdot \vec{x} + \cos \vec{Q}_4 \cdot \vec{x} + \cos \vec{Q}_5 \cdot \vec{x} + \cos \vec{Q}_6 \cdot \vec{x})].\]

The maxima of the SFDW \(\psi_{\text{max}} = 4/3a\) appear along any square surrounding the center of the cube such as \((1/2, 0, 0)\) or \((1/2, 0, \gamma)\) etc. In fact, one can achieve the SFDW lattice by shifting the \(c\) face centers, so all these points are on the edge centers and centers which are only discrete points on the square surrounding the center of the cube. We expect the continuous whole square is due to the artifact of the approximation \(\psi(\vec{x} = 0) \sim 0\) imposed. So the SFDW \(^4He\) in Vycor glass takes a \(b\) \(c\) lattice.

Unfortunately, a spherical \(k = Q\) surface can not lead to lattices with different lengths of primitive reciprocal lattice vectors such as a \(h\) \(c\) \(p\) lattice. This is similar to the classical liquid-solid transition described by Eqn.\ref{eqn:liquid-solid} where a single maximum peak in the static structure factor can not lead to a \(h\) \(c\) \(p\) lattice. Another difficulty with the \(h\) \(c\) \(p\) lattice is that the \(h\) \(c\) \(p\) lattice is not a Bravais lattice, it consist of two inter-penetrating simple hexagonal lattices shifted by \(\vec{a} = \frac{1}{2}(\vec{a}_1 + \vec{a}_2 + \vec{a}_3).\) Here we can simply take the experimental fact that \(n\) forms a \(h\) \(c\) \(p\) lattice without knowing how to produce such a lattice from a GL theory Eqn.\ref{eqn:GL}. Despite the technical difficulty, because for an idea \(h\) \(c\) \(p\) lattice \(c/a = \sqrt{8}/3,\) an \(h\) \(c\) \(p\) lattice has 12 nearest neighbors, so its local environment may resemble that of an \(f\) \(c\) \(c\) lattice. We expect the physics (except the anisotropy of NCRI in the \(h\) \(c\) \(p\) lattice to be discussed in the next two sections) is qualitatively the same as that in \(f\) \(c\) \(c\) \(c\) direct lattice. From the insights achieved from the other lattices, one can achieve the SFDW lattice by shifting the \(h\) \(c\) \(p\) lattice of \(n\) by \(\vec{a} = \frac{1}{2}(\vec{a}_1 + \vec{a}_2) + \vec{a}_3.\)

C. Discussions on both SS-v and SS-i

In fact, for both SS-v and SS-i, we can write \(n\) \(\psi\) sector in a more symmetric way: \[n(\vec{x}) = n_0 + \sum' n \gamma e^{i \vec{G} \cdot \vec{x}}, \psi(\vec{x}) = \psi_0 + \sum' \psi \gamma e^{i \vec{G} \cdot \vec{x}}.\] It is easy to see that in the SS, the translational symmetry \(\vec{x} \rightarrow \vec{x} + \vec{a}, n \gamma \rightarrow n \gamma e^{i \vec{G} \cdot \vec{a}}, \psi \gamma \rightarrow \psi \gamma e^{i \vec{G} \cdot \vec{a}}\) is broken down to \(\vec{a} = \vec{R}, n \gamma \rightarrow e^{i \vec{R} \cdot \vec{G}} n \gamma, \psi \gamma \rightarrow e^{i \vec{R} \cdot \vec{G}} \psi \gamma = \psi \gamma.\) In the Fig.2, in the \(NL\) side, as the temperature is lowered, the symmetry breaking happens in \(\psi_1\) at \(k = 0,\) the NL gets into the SF. However, as shown in this section, in the \(NS\) side, the symmetry breaking happens in both the \(\psi_1\) at \(k = 0\) and \(\psi_2\) sector at \(P\) discrete points simultaneously, the NL gets to the SS state at a much lower critical temperature \(T_{SS}\). The results achieved from the \(NS\) side in this section indeed confirm Fig.2 achieved from the roton condensation picture in the SF phase in the last section.

V. THE ZERO TEMPERATURE TRANSITION FROM NS TO SS DRIVEN BY THE PRESSURE

The analysis so far focused on finite temperature and mean field level. Some interesting physics near the \textit{finite temperature} NS to SS transition in Fig.1 was explored \textit{in bold} by considering the coupling of elastic degree of freedoms to the SF mode. For example, the sound velocity will acquire a dip similar to the specific heat cusp in the \(\lambda\) transition in superfluid helium. In this and following sections, I will push the Ginsburg-Landau (GL) theory in the previous sections to zero temperature and to include all the possible low energy fluctuations above the mean filed solutions achieved in the previous sections. Particularly, I will work out the elementary low energy excitations including vortex loop excitations in a SS and study how they defer from the low energy excitations in solids and superfluids. In principle, if these elementary low energy excitations can be detected by X-ray scattering, neutron scattering, acoustic wave attenuation and heat capacity experiments in solid helium 4 can prove or disprove the existence of the supersolid in helium 4. In practice, the detection may still be complicated by sample quality. No matter if a supersolid indeed exists in helium 4, these results should be interesting in its own and may have application in other systems.

So far, we only look at the mean field solutions corresponding to vacancies and interstitials. Here we discuss excitations above the mean field solutions. It turns out that the excitations in both cases are the same, so we
discuss both cases at same time. In the SS-v and SS-i, the wavefunctions can be written as

\[ \psi_{ss} = \psi_0 (1 \pm \frac{1}{2} \sum_{m=1}^{P/2} \cos \theta_m \cdot \hat{x}) \]

where ± sign corresponds to SS-v and SS-i respectively. Obviously, there are also topological defects in the phase winding of \( \theta \) which are vortices. At \( T < T_{SS} \), the vortices can only appear in tightly bound pairs. However, as \( T \rightarrow T_{SS} \), the vortices start to become liberated, this process renders the total NCRI to vanish above \( T > T_{SS} \).

In addition to the superfluid \( \theta \) mode in SS states, there are also lattice phonon modes \( \bar{u} \) in both \( n \) sector and \( \psi_2 \) sector. However, it is easy to see that the coupling Eqn.3 is invariant under \( \hat{x} \rightarrow \hat{x} + \bar{u} \), \( n(G) \rightarrow n(G) e^{i \hat{G} \cdot \bar{u}} \), \( \psi(G) \rightarrow \psi(G) e^{i \hat{G} \cdot \bar{u}} \), so the lattice phonon modes in \( \psi \) are locked to those in the conventional \( n \) lattice. This is expected because there is only one kind of translational symmetry breaking, therefore only one kind of lattice phonons. Inside the NS side, the translational symmetry is already broken, so we can parameterize the density deviation order parameter \( \delta n(\hat{x}, \tau) = n(\hat{x}, \tau) - n_0 \) and the SF complex order parameter \( \psi(\hat{x}, \tau) \) as:

\[ \delta n(\hat{x}, \tau) = \sum_{G} n(G) e^{i \hat{G} \cdot (\hat{x} + \bar{u}(\hat{x}, \tau))} \]

\[ \psi(\hat{x}, \tau) = \psi_0(\hat{x}, \tau) [1 \pm \frac{1}{P} \sum_{G} e^{i \hat{G} \cdot (\hat{x} + \bar{u}(\hat{x}, \tau))}] \]  

where the \( \psi_0(\hat{x}, \tau) = |\psi_0(\hat{x}, \tau)| e^{i \theta(\hat{x}, \tau)} \) is the SF order parameter, \( \bar{u}(\hat{x}, \tau) \) are the 3 lattice phonon modes, the \( \pm \) means vacancy or interstitials induced supersolids respectively, \( n_\psi = n_{\bar{G}} \) the \( - \) means sum over the shortest non-zero reciprocal lattice vector \( \bar{G} \) and \( P \) is the number of them. From Eqn.19 we can identify the SS density order parameter \( \rho_\bar{G}(\hat{x}, \tau) = e^{i \hat{G} \cdot \bar{u}(\hat{x}, \tau)} \). The effective action to describe the NS to SF transition at \( T = 0 \) consistent with all the lattice symmetries and the global \( U(1) \) symmetry is:

\[ \mathcal{L} = \psi_0^\dagger \partial_\tau \psi_0 + c_{\alpha \beta} \partial_\alpha \psi_0^\dagger \partial_\beta \psi_0 + r |\psi_0|^2 + g |\psi_0|^4 + \frac{1}{2} \rho_\alpha (\partial_\tau u_\alpha)^2 + A_{\alpha \beta \gamma} u_\alpha \partial_\tau u_\beta \partial_\tau u_\gamma + \alpha_{\alpha \beta} u_\alpha \partial_\tau \psi_0^\dagger \partial_\tau \psi_0 + \alpha_{\alpha \beta} \partial_\tau \psi_0^\dagger \partial_\tau \psi_0 + \alpha_{\alpha \beta} \partial_\tau |\psi_0|^2 + \cdots \]  

where \( r = p - pc_2 \approx 170 \text{ bar} \) (Fig.5), \( \rho_\alpha \) is the normal density, \( u_\alpha = \frac{1}{2} (\partial_\alpha x + \partial_\alpha u) \) is the strain tensor, \( \lambda_{\alpha \beta \gamma} \) are the bare stress tensor dictated by the symmetry of the lattice, it has 5 (2) independent elastic constants for a hcp (isotropic) lattice. For a uniaxial lattice such as bcp lattice, all the coefficients \( c_{\alpha \beta}, \alpha_{\alpha \beta}, \alpha_{\alpha \beta} \) take the same form \( c_{\alpha \beta} = c_n a_n b_n + c_L (\delta_{\alpha \beta} - n_n b_n) \) where \( \bar{u} \) is a unit vector along the uni-axis. In the NS state \( r > 0, \psi_0(\hat{x}, \tau) = 0 \), the 3 lattice phonon modes \( \bar{u}(\hat{x}, \tau) \) become the 3 ordinary ones. While inside the SS state \( r < 0, \psi_0(\hat{x}, \tau) \neq 0 \). From the parameterizations of \( \psi(\hat{x}, \tau) \), we can see if the prefactor \( \langle \psi_0(\hat{x}, \tau) \rangle \neq 0 \), then \( \psi(\hat{x}, \tau) \) condenses at both \( \bar{G} = 0 \) and any other non-zero reciprocal lattice vectors \( \bar{G} \) to form the superfluid density wave (SDFD \( \rho_\theta^2 = |\psi(\hat{x}, \tau)|^2 \)) inside the SS. The \( a_{\alpha \beta} \) and \( a_{\alpha \beta} \) couplings come from the original couplings \( \delta n(\hat{x}, \tau) \psi_0^\dagger \partial_\tau \psi_0 \) and \( \delta n(\hat{x}, \tau) |\psi_0|^2 \) respectively in the GL in Sect.I. If setting all the couplings between \( \psi_\alpha \) and \( u_\alpha \) vanish, the \( \psi_0 \) sector describes the SF to Mott insulator transition in a rigid underlying lattice. So we will study how the NS to SS transition at \( T = 0, p = pc_2 \) in a rigid lattice is affected by its coupling to a quantum fluctuating lattice. Under the Renormalization group (RG) transformation, \( \tau' = \tau/\tau^2, x' = x/b \), and \( \psi' = \psi/\sqrt{b} \). When we choose \( z = 2, Z = b^{-d/2}, g' = g b^{2-d} \). It is well known that the SF to Mott insulator transition in a rigid underlying lattice has the mean field exponents with \( z = 2, \nu = 1/2, \eta = 0 \) at \( d \geq 2d \). We also choose \( u_\alpha = u_\alpha/Z \), then \( \rho_\alpha = b^{-d} \rho_n \), so the lattice phonon kinetic energy term is irrelevant near the QCP. It is easy to see \( a_0 = b^{-d/2}a_0 \), so \( a_0 \) is always irrelevant. \( a_1 = b^{-d/2}a_1 \), so both \( g \) and \( a_1 \)'s upper critical dimension is \( a_n = 2 \), so, in principle, a \( e = 2-d \) expansion is possible for both \( g \) and \( a_1 \). However, both are irrelevant at \( d = 3 \). We conclude that NS to SS transition at \( T = 0 \) remains the same as that in a rigid lattice described by

\[ \mathcal{L}_{\text{LT}=0} = \psi_0^\dagger \partial_\tau \psi_0 + c_{\alpha \beta} \partial_\alpha \psi_0^\dagger \partial_\beta \psi_0 + r |\psi_0|^2 + g |\psi_0|^4 + \cdots \]  

Namely, it is a transition with mean field exponents \( z = 2, \nu = 1/2, \eta = 0 \).

If neglecting the \( \tau \) dependence by setting \( u_\alpha(x, \tau) = u_\alpha(x) \), \( \psi_0(\hat{x}, \tau) = \psi_0(\hat{x}) \), then Eqn.20 reduces to the classical action studied in. The classical case, \( x' = x/b, \psi' = \psi/\sqrt{b}, u_\alpha' = u_\alpha/Z \), if we choose \( Z = b^{-d/2}/2 \), then \( g' = gb^{2-d}/a_1 \), so both \( g \) and \( a_1 \)'s upper
critical dimension is $d_c = 4$. So in principle, a $\epsilon = 4 - d$ expansion is possible for both $g$ and $a_1$, but the putting $\epsilon = 1$ for $d = 3$. In Ref. it was shown that due to the specific heat exponent of the 3d $XY$ model $\alpha = -0.012 < 0$, the $a_1$ coupling is irrelevant, so the NS to SS transition remains to be a classical 3d $XY$ transition at finite temperature given by:

$$\mathcal{L}_T = K_{NS} |\nabla \psi_0|^2 + t_{NS} |\psi_0|^2 + u_{NS} |\psi_0|^4 + \cdots$$

where $t_{NS} = T - T_{SS}$ and $T_{SS-v}(p) = f_v(g_v) - \Delta(p) = \Delta(p_{c2}) - \Delta(p)$ and $T_{SS-i}(p) = f_i(g_i) - \Delta(p) = \Delta(p_{c2}) - \Delta(p)$. Obviously, it is still a 3d $XY$ transition. It was shown in Ref. that the coupling to the phonon mode $\bar{u}$ will not change the universality class at finite temperature. As the pressure increased to $p_{c2}$, $T_{SS-v}$ or $T_{SS-i}$ are suppressed to zero, the system becomes a C-NS where $\langle \psi \rangle = 0$. It is important to stress that even at $p > p_{c2}$, the solid is a C-NS, it still does not have the P-H symmetry because $g \neq 0$. For $g_v < 0$, the C-NS is vacancy like (NS-v) where the vacancy excitation energy is lower than that of the interstitial. For $g_i > 0$, the C-NS is interstitial like (NS-i) where the interstitial excitation energy is lower than that of the vacancy.

**VI. THE NCRI OF SF AND SS STATES AT $T = 0$ AND $T > 0$.**

In order to calculate the superfluid density $\rho_s$ explicitly, we need to look at how the system’s free energy responds to a fictitious gauge potential $\tilde{A}$. We find that when $\Delta \gg \Delta_c$, $\rho_s(T = 0) = \rho = \int d^d x |\psi(x, \tau)|^2 = \int d^d x (|\psi_1(x, \tau)|^2 + |\psi_2(x, \tau)|^2)$ where the crossing terms between $\psi_1$ and $\psi_2$ drop out due to the momenta conservation.

In the SF state, at low $T$, the quantum fluctuations induced by the pressure are important. Let’s first look at the quantum phase fluctuations. The phase fluctuation action is given by $\mathcal{L}_p = \frac{1}{2\hbar} \sum_{\omega_n} \int \frac{d^d k}{(2\pi)^d} (\omega_n^2 + k^2) |\theta(\tilde{k}, \omega_n)|^2$ where $g = \frac{1}{p_s}$ controls the strength of quantum phase fluctuations and the superfluid phonon velocity has been set equal to 1 for simplicity. It is easy to see that at $T = 0$, $|\theta(x, \tau)|_{T=0}$ is infra-red (IR) finite, so the quantum phase fluctuations alone will not lead to any instability. However, it will lead to superfluid density depletion even before reaching the phase boundary of SF to SS transition in Fig.1 and Fig.2, although the depletion may be quite small. This fact explains why $T_{SF}(p)$ bends to the left slightly as the pressure $p$ increases. At finite $T$, the thermal fluctuations $\langle \theta^2(x, \tau) \rangle_T = \langle \theta^2(x, \tau) \rangle_{T=0} - T^{d-1} \log(T)$ lead to $\rho_s(T) = \rho_s(T = 0) - cT^2$ at $d = 3$. It is well known the superfluid density $\rho(T) \sim \rho(T = 0) - aT^2$, while the Bose condensation density $n_b(T) \sim n_b(T = 0) - bT^2$. So strictly speaking, $\psi$ sector can only describe the Bose condensation. This is expected, because the $\sigma$ sector in the SF phase also contributes to the superfluid density, but not to the BEC.

Then let’s look at the roton fluctuations. Setting the roton gap $\Delta_2 = \Delta^2$, at $T = 0$, the quantum roton fluctuations $\langle \phi^2(x, \tau) \rangle_{T=0} \sim \frac{\log(\Lambda)}{\Delta} \log(\frac{\Lambda}{\Delta})$ when $\Lambda \to 0$ which signifies the instability to the lattice formation. Due to this IR divergence, the 1st order SF to SS transition may happen well before $\Delta$ becomes zero, namely, at $\Delta = \Delta_c > 0$. This is consistent with the picture described in Sec.III. At finite $T$, the thermal roton fluctuations $\langle \phi^2(x, \tau) \rangle - \langle \phi^2(x, \tau) \rangle_{T=0} \sim (\log(\frac{\Lambda}{\Delta}) \sim \frac{1}{\Delta} T^2$ when $T \ll \Delta \ll \Lambda$.

In the SS-v and SS-i states, the $n(x)$ forms a lattice, at the same time, the unstable roton part is replaced by a stable SFDW formation commensurate with the underlying $n$ lattice. Obviously, the $n(x)$ normal lattice takes away the vast majority of density from the superfluid density even at $T = 0$. It can be shown that superfluid density from the $\psi_1$ sector is isotropic $\rho_1 \sim K a^2$, while the superfluid density from $\psi_2$ sector turns out to be anisotropic $\rho_{2,ij} \sim \sum_{m=1}^\infty |\Delta_m|^2 Q_{m1} Q_{m2}/Q^2$ where $\Delta_m = \pm a/P$ for SS-v or SS-i. Therefore, the total superfluid density in SS phase is:

$$\rho_{ij}^{SS} \sim \rho_1 \delta_{ij} + \rho_{2,ij} \sim a^2 (K \delta_{ij} + \frac{1}{P^2} \sum_{m=1}^P Q_{m1} Q_{m2}/Q^2)$$

Taking $P = 6$, the anisotropy is quite small. Solid $^4$He in a bulk takes a $hcp$ lattice with $c/a \sim 1.63$ which is quite close to the ideal value $c/a = \sqrt{8/3}$. The three primitive reciprocal lattice vectors are $G_1 = G_2 = \frac{\pi}{\sqrt{3}a}, G_3 = \frac{\pi}{c}$.

We can estimate the anisotropy of the NCRI in the $hcp$ lattice. If the rotation axis is along the $c$ axis, the NCRI is $\rho_{11} \sim K a^2 + v_r \sum_{m=1}^6 |\Delta_m|^2 Q_{m1} Q_{m2}/Q^2$ If the rotation axis is along the $a$ or $b$ axis, the NCRI is $\rho_{33} \sim K a^2 + v_r \sum_{m=1}^6 |\Delta_m|^2 Q_{m3} Q_{m2}/Q^2$. The anisotropy mainly comes from the $\psi_2$ sector. Setting $s = G_1/G_3$, for the ideal value $s = \frac{2\sqrt{2}}{3} > 1$, so $\rho_{11} > \rho_{33}$. Namely, the NCRI response is larger when one is rotating the sample around the $c$ axis than that when one is rotating the sample around the $a$ or $b$ axis. However, as the pressure is increased, $s$ decreases, the anisotropy of the NCRI also decreases. In the PSU experiments, the samples are poly-crystal, the relative orientation of the rotation axis to the $c$ axis is not known, so it’s hard to test this prediction with poly-crystals.

**VII. NON-TOPOLOGICAL ELEMENTARY EXCITATIONS AND SPECTRAL WEIGHTS IN THE SS**

Classical non-equilibrium hydrodynamics in SS was investigated for a long time. These hydrodynamics will break down at very low temperature where quantum fluc-
tions dominate. However, the quantum nature of the excitations in the SS has not been studied yet. Here, we will study the quantum characteristics of low energy excitations in the SS. For example, how the phonon spectra in the SS differ from that in a NS and how the SF mode in the SS differs from that in a SF. Inside the SS, \( \langle \psi_0(\vec{x}, \tau) \rangle = a \), we can write \( \psi_0(\vec{x}, \tau) = \sqrt{a + \delta \rho e^{ib(\vec{x}, \tau)}} \) and plug it into the Eqn [20]

\[
\mathcal{L} = i \delta \rho \partial_\tau \theta + \rho_{\alpha\beta}^s \partial_\alpha \theta \partial_\beta \theta + \frac{1}{2} \delta \rho S^{-1} \delta \rho
+ \frac{1}{2} \left[ \rho_\alpha \left( \partial_\tau u_\alpha \right)^2 + \lambda_{\alpha\beta\gamma} u_{\alpha\beta} u_{\gamma\delta} \right]
+ a\delta_{\alpha\beta} i \partial_\tau \theta + a^1_{\alpha\beta} u_{\alpha\beta} \delta \rho + \cdots \tag{24}
\]

where we already dropped \( i \delta_\tau \theta \) term which is irrelevant inside the SS phase (although it is very important in describing the \( T = 0 \) NS to SS transition discussed in the last section), \( S_0 \) is the bare density-density correlation function. Integrating out the massive magnitude \( \delta \rho \) fluctuations, we get:

\[
\mathcal{L} = - \frac{S_0}{2} (i \delta_\tau \theta + a^1_{\alpha\beta} u_{\alpha\beta})^2 + \rho^s_{\alpha\beta} \partial_\alpha \theta \partial_\beta \theta
+ \frac{1}{2} \left[ \rho_\alpha \left( \partial_\tau u_\alpha \right)^2 + \lambda_{\alpha\beta\gamma} u_{\alpha\beta} u_{\gamma\delta} \right]
+ a\delta_{\alpha\beta} i \partial_\tau \theta + \cdots \tag{25}
\]

Expanding the square, we get the effective action describing the low energy modes inside the SS phase:

\[
\mathcal{L} = \frac{1}{2} \left\{ \kappa \left( \partial_\tau \theta \right)^2 + \rho^s_{\alpha\beta} \partial_\alpha \theta \partial_\beta \theta \right\}
+ \frac{1}{2} \left[ \rho_\alpha \left( \partial_\tau u_\alpha \right)^2 + \lambda_{\alpha\beta\gamma} u_{\alpha\beta} u_{\gamma\delta} \right] + a\delta_{\alpha\beta} u_{\alpha\beta} \partial_\tau \theta \tag{26}
\]

where \( \kappa \) is the SF compressibility defined in Eqn [6] and \( \rho^s_{\alpha\beta} \) is the SF stiffness which has the same symmetry as \( a_0 \alpha, \alpha_{\beta} = a^0_\alpha + S_0 a^1_\alpha \) where \( S_0 \) is the bare SF density correlation function. Again, the \( \rho_\alpha \) is the normal density, the \( \lambda_{\alpha\beta\gamma} \) is the stress tensor. Obviously, the last term is the crucial Berry phase coupling term which couples the lattice phonon modes to the SF mode. The factor of \( i \) is important in this coupling. By integration by parts, this term can also be written as \( a_{\alpha\beta} \left( \partial_\tau u_\alpha \partial_\beta \theta + \partial_\alpha u_\beta \partial_\tau \theta \right) \) which has the clear physical meaning of the coupling between the SF velocity \( \partial_\tau \theta \) and the velocity of the lattice vibration \( \partial_\alpha u_\beta \). It is this term which makes the low energy modes in the SS to have its own characteristics which could be detected by experiments. The invariance under the Galilean transformation dictates that

\[
a_{\alpha\beta} = \rho_\alpha \delta_{\alpha\beta} - \rho^s_{\alpha\beta}.
\]

In this section, we neglect the topological vortex loop excitations in Eqn [24]. In the next section, we will discuss these vortex loop excitations in detail. In the following, we discuss two extreme cases: isotropic solid and hcp lattice separately. Usual samples are between the two extremes.

A. Isotropic solid

A truly isotropic solid can only be realized in a highly poly-crystalline sample. Usual samples are not completely isotropic. However, we expect the simple physics brought about in an isotropic solid may also apply qualitatively to other samples which is very poly-crystalline.

For an isotropic solid, \( \lambda_{\alpha\beta\gamma} = \delta \lambda_{\alpha\beta} \delta_{\gamma\delta} + \mu (\delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) \) where \( \lambda \) and \( \mu \) are Lamé coefficients, \( \rho_{\alpha\beta}^s = \rho^s \delta_{\alpha\beta}, a_{\alpha\beta} = \delta \alpha_{\beta} \) where \( a = \rho_n - \rho_s \). In \( (\vec{q}, \omega_n) \) space, Eqn [24] becomes:

\[
\mathcal{L}_{is} = \frac{1}{2} \left[ \rho_n \omega_n^2 + (\lambda + 2\mu) q^2 \right] \left[ u_i (\vec{q}, \omega_n) \right]^2
+ \frac{1}{2} \left[ \kappa \omega_n^2 + \rho_s q^2 \right] \left[ \theta (\vec{q}, \omega_n) \right]^2
+ a q \omega_n u_i (\vec{q}, \omega_n)
+ \frac{1}{2} \left[ \rho_n \omega_n^2 + \mu q^2 \right] \left[ u_i (\vec{q}, \omega_n) \right]^2 \tag{27}
\]

where \( u_i (\vec{q}, \omega_n) = i q u_i (\vec{q}, \omega_n) / q \) is the longitudinal component, \( u_i (\vec{q}, \omega_n) = i e_i q u_i (\vec{q}, \omega_n) / q \) are transverse components of the displacement field. Note that Eqn [27] shows that only longitudinal component couples to the superfluid \( \theta \) mode, while the two transverse components are unaffected by the superfluid mode. This is expected, because the superfluid mode is a longitudinal density mode itself which does not couple to the transverse modes.

From Eqn [27] we can identify the longitudinal-longitudinal phonon correlation function:

\[
\langle u_i u_i \rangle = \frac{\kappa \omega_n^2 + \rho_s q^2}{(\kappa \omega_n^2 + \rho_s q^2)(\rho_n \omega_n^2 + (\lambda + 2\mu) q^2) + a^2 q^2 \omega_n^2} \tag{28}
\]

The \( \langle \theta \theta \rangle \) and \( \langle u_i \theta \rangle \) correlation functions can be similarly written down. By doing the analytical continuation \( \omega_n \rightarrow \omega + i \delta \), we can identify the two poles of all the correlation functions at \( q^2 = v^2 + v_s^2 \) where the two velocities \( v_\pm \) is given by \( v^2_{\pm} = \kappa (\lambda + 2\mu) + \rho_s \rho_n + a^2 \pm \sqrt{\kappa (\lambda + 2\mu) + \rho_s \rho_n + a^2 - 4\kappa \rho_\alpha \rho_n (\lambda + 2\mu) / 2\kappa \rho_\alpha} \). When setting \( a = 0 \), then \( v^2_s \) reduces to the longitudinal phonon velocity \( v^2_p = (\lambda + 2\mu) / \rho_n \) and the superfluid velocity \( v^2_s = \rho_s / \kappa \) respectively. Of course, the transverse phonon velocity \( v^2_p = \mu / \rho_n \) is untouched. For notation simplicity, in the following, we just use \( v_p \) for \( v^2_p \). Inside the SS, due to the very small superfluid density \( \rho_s \), it is expected that \( v_p > v_s \). In fact, in isotropic solid \( He^4 \), it was measured that \( v_p \sim 450 \) to \( 500m/s, v_s \sim 230 \) to \( 320m/s \) and \( v_s \sim 366m/s \) near the melting curve. It is easy to show that \( v_s > v_p > v_s > v \) and \( v^2_s + v^2_p > v^2_s + v^2_\pm \), but \( v_s + v \rightarrow v_p + v_s \), so \( v_s + v \rightarrow v_p + v_s \) (see Fig.1). Note that because the Galilean invariance dictates \( \omega = \rho_n - \rho_s \), for \( \rho_n \ll \rho_s \), one can see \( \rho_s \rho_n + a^2 \gg \rho_s \rho_n \), so \( v_s \) ( \( v_p \) ) are considerably above ( below ) \( v_p \) ( \( v_s \)), so the two supersolidons, especially the softening of the lower branch, may be easily distinguished by possible neutron scattering experiments.
The upper dashed line) and the superfluid mode \( \omega_s = v_s q \) (the lower dashed line) leads to the two new "supersolidon" modes \( \omega_{\pm} = v_{\pm} q \) (solid lines) in the SS. Their corresponding spectral weights are listed in Eqs. (20) and (30). These two new supersolid modes and the spectral weights should be detected by inelastic neutron scatterings. (b) The excitations inside the lattice phonons and superfluid phonons, the first and second term give give the excitation energies and the two corresponding spectral weights.

Very similarly, we can find:

\[
\text{Im}(\langle u_l u_l \rangle)_{\omega \rightarrow \omega + i \delta} = \frac{v_p^2 - v_q^2}{v_+^2 - v_-^2} \frac{\pi}{2\rho_n v_+ q} [\delta(\omega - v_+ q) - \delta(\omega + v_+ q)]
\]

\[
- \frac{v_p^2 - v_q^2}{v_+^2 - v_-^2} \frac{\pi}{2\rho_n v_- q} [\delta(\omega - v_- q) - \delta(\omega + v_- q)]
\]

It is easy to see that the second term can be achieved from the first term just by \( v_+ \leftrightarrow v_- \). Setting \( a = 0 \), then \( v_+ = v_p, v_- = v_s \), the second term just vanishes, the first term recovers the excitation spectrum of the lattice phonons. When \( a \neq 0 \), then Eqn. (29) becomes a mixing of the lattice phonons and superfluid phonons, the first and second term give the excitation energies and the two corresponding spectral weights.

Very similarly, we can find:

\[
\text{Im}(\Theta \theta)_{\omega \rightarrow \omega + i \delta} = \frac{v_p^2 - v_q^2}{v_+^2 - v_-^2} \frac{\pi}{2\rho_n v_+ q} [\delta(\omega - v_+ q) - \delta(\omega + v_+ q)]
\]

\[
- \frac{v_p^2 - v_q^2}{v_+^2 - v_-^2} \frac{\pi}{2\rho_n v_- q} [\delta(\omega - v_- q) - \delta(\omega + v_- q)]
\]

It is easy to see that the second term can be achieved from the first term just by \( v_+ \leftrightarrow v_- \). Setting \( a = 0 \), then \( v_+ = v_p, v_- = v_s \), the first term just vanishes, the second term recovers the excitation spectrum of the superfluid phonons. When \( a \neq 0 \), then Eqn. (30) becomes a mixing of the lattice phonons and superfluid phonons, the first and second term give the excitation energies and the two corresponding spectral weights.

The two longitudinal modes in the SS can be understood from an intuitive picture: inside the NS, it was argued in\(^{40}\) that there must be a diffusion mode of vacancies in the NS. Inside the SS, the vacancies condense and lead to the extra superfluid mode. So the diffusion mode in the NS is replaced by the SF mode in the SS. The analysis in this subsection is similar to the atom-photon polaritons inside a cavity\(^{27,28}\) or exciton-photon polaritons of excitons in quasi-two dimensional quantum wells inside a planar cavity\(^{29}\).

B. \underline{hcp crystal}

Usual single hcp crystal samples may also contain dislocations, grain boundaries. Here we ignore these line and plane defects and assume that there are only vacancies whose condensation leads to the superfluid density wave inside the supersolid discussed in Sect.IV.

For a uni-axial crystal such as an hcp lattice, the action is:

\[ \mathcal{L}_{\text{hcp}} = \frac{1}{2} \rho_n (\partial_r u_n)^2 + K_{11} (u_{xx}^2 + u_{yy}^2) + 2K_{12} u_{xx} u_{yy} + 2K_{33} u_{xx}^2 + 2K_{13} (u_{xx} u_{yy}) u_{zz} + 2(K_{11} - K_{12}) u_{xy}^2 + K_{44} (u_{yz}^2 + u_{xz}^2) \]

\[ + \frac{1}{2} [\kappa (\partial_r \theta)^2 + \rho_s^2 (\partial_r \theta)^2 + \rho_s^3 ((\partial_r \theta)^2 + (\partial_y \theta)^2)] + \frac{1}{2} [\alpha \partial_z u_z + a (\partial_z u_x + \partial_y u_y)] i\partial_z \theta \quad (31) \]

If \( \dot{q} \) is along \( \hat{z} \) direction, namely \( q_z \neq 0, q_x = q_y = 0 \), then Eqn. (31) simplifies to:

\[ \mathcal{L}_{\text{hpc}}^{\hat{z}} = \frac{1}{2} \rho_n (\omega_n^2 + K_{33} q_z^2) u_z (q_z, \omega_n)^2 + \frac{1}{2} [\omega_n^2 + \rho_s^2 q_z^2] \theta (q_z, \omega_n)^2 - i a q_z \omega_n u_z (-q_z - \omega_n) \theta (q_z, \omega_n) + \frac{1}{2} [\rho_n \omega_n^2 + K_{44} q_z^2/4] u_z (q_z, \omega_n)^2 \quad (32) \]

where \( |u_z (q_z, \omega_n)|^2 = |u_x (q_z, \omega_n)|^2 + |u_y (q_z, \omega_n)|^2 \) stand for the two transverse modes with the velocity \( v_r^2 = K_{44}/4\rho_n \). The superfluid mode only couples to the longitudinal \( u_z \) mode, while the two transverse modes \( u_x, u_y \) are decoupled. Eqn. (32) is identical to Eqn. (27) after the replacement \( u_z \rightarrow u_1, K_{33} \rightarrow \lambda + 2\mu, a \rightarrow a \). It was found that \( v_{pz} \sim 540m/s, v_r \sim 250m/s \) when \( \dot{q} \) is along the \( \hat{z} \) direction. Fig. 2 follows after these replacements.

Similarly, we can work out the action in the \( xy \) plane where \( q_r = 0, q_z \neq 0, q_y = 0 \). Then \( u_z \) mode is decoupled, only \( u_x, u_y \) modes are coupled to the superfluid mode:

\[ \mathcal{L}_{\text{hcp}}^{xy} = \frac{1}{2} [\rho_n (\partial_r u_n)^2 + K_{11} (u_{xx}^2 + u_{yy}^2) + 2K_{12} u_{xx} u_{yy} + 2K_{33} u_{xx}^2 + 2K_{13} (u_{xx} u_{yy}) u_{zz} + 2(K_{11} - K_{12}) u_{xy}^2 + K_{44} (u_{yz}^2 + u_{xz}^2) \]

\[ + \frac{1}{2} [\kappa (\partial_r \theta)^2 + \rho_s^2 (\partial_r \theta)^2 + \rho_s^3 ((\partial_r \theta)^2 + (\partial_y \theta)^2)] + \frac{1}{2} [\alpha \partial_z u_z + a (\partial_z u_x + \partial_y u_y)] i\partial_z \theta \quad (33) \]
where $\alpha, \beta = x, y$. By comparing Eqn.33 with Eqn.27, we can see that $K_{11} \rightarrow \lambda + 2\mu, K_{12} \rightarrow \lambda$, so all the discussions in the isotropic case can be used here after the replacements. Fig.2 follows after these replacements. Namely, only the longitudinal component in the $xy$ plane is coupled to the $\theta$ mode, while the transverse mode in the $xy$ plane with velocity $v_{txy} = (K_{11} - K_{12})/2\rho_0$ is decoupled. Obviously the transverse mode along $\hat{z}$ direction $u_z$ mode with the velocity $v_{tx}^2 = K_{44}/4\rho_0$ is also decoupled. Note that the two transverse modes have different velocities. It was found that $v_{tx} \sim 455m/s, v_{tz} \sim 255m/s, v_{txy} \sim 225m/s$ when $\vec{q}$ is along the $xy$ plane.

Along any general direction $\vec{q}$, strictly speaking, one can not even define longitudinal and transverse modes, so the general action Eqn.31 should be used. Despite the much involved $4 \times 4$ matrix diagonalization in $u_x, u_y, u_z, \theta$, we expect the qualitative physics is still described by Fig.2.

In principle, inelastic neutron scattering experiments or acoustic attenuation experiments can be used to detect the predicted the low energy excitation spectra in the SS shown in Fig.2.

**VIII. TOPOLOGICAL ELEMENTARY EXCITATIONS IN SS: VORTEX LOOPS AND VORTEX**

In the last section, we studied the low energy excitations shown in the Fig.1 by neglecting the topological vortex loop. Here, we will study how the vortex loop interaction in SS differ from that in the SF. For simplicity, in the following, we only focus on the isotropic case. The formulations can be generalized to the hcp case straightforwardly. We can perform a duality transformation on Eqn.24 to the vortex loop representation:

$$\mathcal{L}_v = \frac{1}{2K_\mu} (\epsilon_{\mu\nu\lambda\sigma} \partial_\nu a_{\lambda\sigma} - a_{\mu} \partial_\mu a_{\nu})^2 + i2\pi a_{\mu\nu} j^\nu_{\mu}$$  (34)

where $\mu, \nu, \lambda, \sigma$ stand for space and time, but $\alpha, \beta$ stand for the space components only, $K_\theta = K_\nu = \rho_s$ and $a_{\mu\nu} = -a_{\nu\mu}$ is an anti-symmetric tensor gauge field and $j^\nu_{\mu} = \frac{i}{\hbar} \epsilon_{\mu\nu\lambda\sigma} \partial_\lambda \partial_\sigma \theta$ is the anti-symmetric tensor vortex loop current due to the topological phase winding in $\theta$.

Eqn.34 has the gauge invariance $a_{\mu\nu} \rightarrow a_{\mu\nu} + \partial_\nu \chi_{\mu} - \partial_\mu \chi_{\nu}$, where $\chi_{\mu}$ is any 4-component field. It is the most convenient to choose the Coulomb gauge $\partial_\mu a_{\nu} = 0$ to get rid of the longitudinal component, then the transverse component is $a_\nu = i\epsilon_{\alpha\beta\gamma} q_\alpha a_{\beta\gamma}/q$. It can be shown that $|a_\nu|^2 = 2|a_{\alpha}\beta|^2$. Then Eqn.34 is simplified to:

$$\mathcal{L}_v = \frac{1}{2} |\rho_n \omega_n^2 + (\lambda + 2\mu + a^2/\kappa) q^2| |u_1(q, \omega_n)|^2 + \frac{1}{2} (q^2/\kappa + \omega_n^2/\rho_s) |a_\nu|^2 + \frac{2}{\rho_s} q^2 |a_{\alpha}\beta|^2
- a q^2/\kappa u_{\nu}(-q, -\omega_n) a_{\nu}(-q, \omega_n)
+ i2\pi j^{\nu}_{\alpha\beta} a_{\alpha}\beta + i2\pi j^{\nu}_{\gamma\delta}(q, \omega_n)$$  (35)

where the transverse phonon mode $u_1$ was omitted, because it stays the same as in the NS shown in Eqn.27.

It is easy to see that only $a_1$ has the dynamics, while $a_{\alpha}\beta$ is static. This is expected, because although $a_{\mu\nu}$ has 6 non-vanishing components, only the transverse component $a_1$ has the dynamics which leads to the original gapless superfluid mode $\omega^2 = v_1^2 q^2$. Eqn.35 shows that the coupling is between the longitudinal phonon mode $u_1$ and the transverse gauge mode $a_1$. The vortex loop density is $j_{\omega} = \frac{i}{\hbar} \epsilon_{\alpha\beta\gamma\delta} \partial_\beta \partial_\gamma \omega$, and the vortex current is $j^{\nu}_{\alpha\beta} = \frac{i}{\hbar} \epsilon_{\alpha\beta\gamma\delta} \partial_\beta \partial_\gamma \nu$. Integrating out the $a_{\alpha}\beta$, we get the vortex loop density-density interaction:

$$\pi \rho_1 \int_0^\beta d\tau \int d^3x d^3y j_{\omega}(\vec{x}, \tau) \frac{1}{|x - y|} j^\omega_{\alpha}(\vec{y}, \tau)$$  (36)

Namely, the vortex loop density-density interaction in SS stays as $1/r$ which is the same as that in NS! Therefore, a single vortex loop energy and the critical transition temperature $T_{3d}$ in Fig.1 is solely determined by the superfluid density $\rho_s$ independent of any other parameters in Eqn.24 except that the vortex core of the vortex loop is much larger than that in a superfluid.

In a cylindrical geometry used in the torsional oscillator experiment, the vortex loops will become straight vortex line along the rotational axis. In the SF phase, a single vortex costs a lot of energy $E_{v}^{SF} = \frac{\rho_s^2 \hbar^2}{4 \pi m^2} \ln \frac{R}{\xi_S}$ where $m$ is the mass of He atom, $R$ is the system size and $\xi_S \sim a$ is the core size of the vortex. This energy determines the critical velocity in SF $v_c^{SF} > 30cm/s$. Because the long distance behavior of SS is more or less the same as SF, we can estimate its single vortex energy $E_{v}^{SS} = \frac{\rho_s^2 \hbar^2}{4 \pi m^2} \ln \frac{R}{\xi_S}$ where $\rho_s^{SS}$ is the global superfluid density inside the SS. We expect the core size of a super-solid vortex $\xi_{SS} \sim 1/\Lambda \gg 1/k_r \sim a \sim \xi_{SS}$. So inside the SS vortex core, we should also see the lattice structure of $T_{3d}^{2}$. This is similar to the phenomenon that DW ordered states were detected in the vortex core of high temperature superconductors. In fact, because $\psi(x)$ stands for vacancies or interstitials, we expect that $\xi_{SS}$ should be of the order of the average spacing between the interstitials or vacancies in the SS. It is interesting to see if neutron or light scattering experiments can test this prediction. Compared to $E_{v}^{SF}$, there are two reductions, one is the superfluid density, another is the increase of the vortex core size $\xi_{SS} \gg \xi_{SF}$. These two factors contribute to the very low critical velocity $v_{c}^{SS} \sim 30\mu m/s$. Of course, the reduction from the increase of the vortex core is negligible because of the logarithmic dependence.

Integrating out the $a_{\alpha}\beta$, we get the vortex loop current-current interaction:

$$2\pi^2 j^{\nu}_{\alpha\beta}(\vec{q}, -\omega_n) D_{\alpha\beta\gamma\delta}(\vec{q}, \omega_n) j_{\omega}^\gamma(\vec{q}, \omega_n)$$  (37)

where $D_{\alpha\beta\gamma\delta}(\vec{q}, \omega_n) = (\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} - \frac{q_n q_\delta}{q^2} \delta_{\alpha\gamma} - \frac{q_n q_\delta}{q^2} \delta_{\alpha\beta} - \frac{q_n q_\delta}{q^2} \delta_{\gamma\delta} + \frac{q_n q_\delta}{q^2} \delta_{\alpha\beta}) D_1(q, \omega_n)$ where $D_1(q, \omega_n)$ is the $a_{\alpha\beta}$ propagator. Defining $\Delta D_i(q, \omega_n) = D_{iSS}(q, \omega_n) -$
\[ D_t^{SF} (\hat{q}, \omega_n) \text{ as the difference between the } a_t \text{ propagator in the SS and the SF, then from Eqn[35] we can get:} \]
\[
\Delta D_t = \frac{a_t^2 \rho_s^2 q^4}{\kappa \rho_n (\omega_n^2 + v_+^2 q^2)(\omega_n^2 + v_-^2 q^2)} \tag{38}
\]
\[
\Delta D_t (\vec{x} - \vec{x}', \tau = 0) = \frac{a_t^2 \rho_s^2}{4 \pi^2 \kappa \rho_n^2} \frac{v_+ + v_- + v_s}{(v_s + v_+)(v_s + v_-) v_v v_s (\vec{x} - \vec{x}')^2} \tag{39}
\]

Namely, the vortex current-current interaction in SS is stronger than that in the SF with the same parameters \(\kappa, \rho_s\)!

**IX. X-RAY SCATTERING FROM THE SS**

Let’s look at the prediction of our theory on X-ray scattering from the SS. For a lattice with \(j = 1, \ldots, n\) basis located at \(d_j\), the geometrical structure factor at the reciprocal lattice vector \(\vec{K}\) is \(S(\vec{K}) = \sum_{j=1}^{n} f_j (\vec{K}) e^{i \vec{K} \cdot \vec{d}_j}\) where \(f_j\) is the atomic structure factor of the basis at \(d_j\). The X-ray scattering amplitude \(I(\vec{K}) \sim |S(\vec{K})|^2\). Again, we discuss SS-v and SS-i respectively.

### A. X-ray scattering from the SS-v

In this case, because the superfluid density wave simply sits on the \(n\) lattice, so the X-ray scattering is very similar to that from NS at mean field level. However, as shown in C, quantum and thermal fluctuations will still make the X-ray scattering from the SS-v different from that from the NS.

### B. X-ray scattering from the SS-i

In this case, as shown in section VI, the superfluid density wave is shifted from \(n\) lattice, so the X-ray scattering is different from that from NS even at mean field level. For simplicity, we first take the \(sc\) lattice to explain the main points, then list the X-ray scattering from all the other lattices classified in section VI.

(a) Simple Cubic lattice. For the SS in the \(sc\) lattice, as shown in (c2) of the last section, the local superfluid density attains its maximum at the dual lattice points of the \(sc\) lattice. Then \(d_1^1 = 0, d_1^2 = \frac{2}{3}(\hat{i} + \hat{j} + \hat{k}), d_2^1 = \frac{2}{3}(\hat{i} + \hat{j} + \hat{k}), d_1^1 = \frac{2}{3}(\hat{i} + \hat{j} + \hat{k})\), then taking the ratio of the geometric structure factor of SS over that of the NS \(S_{SS}(\vec{K})/S_{NS} = 1 + f(-1)^{n_1+n_2+n_3}\) where \(f \sim \rho_s^{max} \sim a^2\). It is \(1 + f\) for even \(\vec{K}\) and \(1 - f\) for odd \(\vec{K}\).

For simplicity, we just give the expression for the equal time

(b) Triangular lattice. \(d_1 = 0, d_2 = \frac{1}{2}(\hat{a}_1 + \hat{a}_2), \vec{K} = n_1 \hat{a}_1 + n_2 \hat{b}_2\), then taking the ratio of the geometric structure factor of SS over that of the SS \(S_{SS}(\vec{K})/S_{NS} = 1 + 2 f \cos \frac{2\pi}{3}(n_1 + n_2)\). This result could be relevant to possible 2d excitonic superfluid in electron-hole bilayer system to be briefly mentioned in section X.

(c) \(hcp\) lattice. \(hcp\) is not a Bravais lattice. In NS, \(d_1 = 0, d_2 = \frac{3}{4}(\hat{a}_1 + \hat{a}_2) + \hat{a}_3/2, \vec{K} = n_1 \hat{a}_1 + n_2 \hat{b}_2 + n_3 \hat{b}_3\), then \(S_{NS}(\vec{K}) = 1 + e^{i2\pi(\frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c})}\). In the SS, there are \(2\) more additional basis at \(\vec{l}_1 = \frac{3}{4}(\hat{a}_1 + \hat{a}_2) + \hat{a}_3/4, \vec{l}_2 = 3\hat{a}_3/4\), then \(S_{SS}(\vec{K}) = S_{NS}(\vec{K}) + f e^{i2\pi(\frac{n_1}{a} + \frac{n_2}{b} + \frac{n_3}{c})} + f e^{-i\frac{2\pi}{3} n_3} S_{NS}(\vec{K})\).

(d) \(bcc\) lattice. We think \(bcc\) lattice as a \(sc\) lattice plus a basis, \(d_1 = 0, d_2 = \frac{3}{4}(\hat{i} + \hat{j} + \hat{k}), \vec{K} = \frac{2}{3}(n_1 \hat{i} + n_2 \hat{j} + n_3 \hat{k}),\) then \(S_{NS}(\vec{K}) = 1 + (-1)^{n_1+n_2+n_3}\) which is \(2\) for even \(\vec{K}\) and \(0\) for odd \(\vec{K}\). In the SS, there are \(3\) more additional basis at \(\vec{l}_1 = \frac{3}{4}(\hat{i} + \hat{j}), \vec{l}_2 = \frac{3}{4}(\hat{i} + \hat{k}), \vec{l}_3 = \frac{3}{4}(\hat{j} + \hat{k}),\) then \(S_{SS}(\vec{K}) = S_{NS}(\vec{K}) + f(-1)^{n_1+n_2+(-1)^{n_1+n_3}}\).

(e) \(fcc\) lattice. We think \(fcc\) lattice as a \(sc\) lattice plus \(4\) basis located at \(d_1 = 0, d_2 = \frac{3}{4}(\hat{i} + \hat{j}), d_3 = \frac{3}{4}(\hat{i} + \hat{k}), d_4 = \frac{3}{4}(\hat{j} + \hat{k}), \vec{K} = \frac{2}{3}(n_1 \hat{i} + n_2 \hat{j} + n_3 \hat{k}),\) then \(S_{NS}(\vec{K}) = 1 + f(-1)^{n_1+n_2+(-1)^{n_1+n_3+(-1)^{n_2+n_3}}}.\) In the SS, there is one more additional basis located at \(\vec{l}_4 = \frac{3}{4}(\hat{i} + \hat{j} + \hat{k}),\) then \(S_{SS}(\vec{K}) = S_{NS}(\vec{K}) + f(-1)^{n_1+n_2+n_3}.\) It is easy to see that in \(bcc\) and \(fcc\) lattices, we need simply exchange \(d\) vectors for the NS and the \(l\) vectors for SFDW.

We conclude that the elastic X-ray scattering intensity from the SS-i has an additional modulation over that of the NS. The modulation amplitude is proportional to the maxima of the superfluid density \(\rho_{s}^{max} \sim a^2\) which is the same as the NCRI observed in the PSU’s torsional oscillator experiments.

### C. Debye-Waller factor in the X-ray scattering from the SS-v and SS-i

It is known that due to zero-point quantum motion in any NS at very low temperature, the X-ray scattering amplitude \(I(\vec{G})\) will be diminished by a Debye-Waller
\[ (DW) \text{ factor } \sim e^{-\frac{4}{3}G^2(u_0^2)} \] where \( u_0 \) is the lattice phonon modes in Eqn\[24\]. In Eqn\[24\] if the coupling between the \( \vec{u} \) and \( \theta \) were absent, then the DW factor in the SS would be the same as that in the NS. By taking the ratio \( I_{SS}(\vec{G})/I_{NS}(\vec{G}) \) at a given reciprocal lattice vector \( \vec{G} \), then this DW factor drops out. However, due to this coupling, the \( \langle u_0^2 \rangle \) in SS is different than that in NS, so the DW factor will not drop out in the ratio. In this subsection, we will calculate this ratio and see how to take care of this factor when comparing with the X-ray scattering data.

As identified below Eqn\[19\] the density order parameter at the reciprocal lattice vector \( \vec{G} \) is \( \rho_G(\vec{x},\tau) = e^{i\vec{G} \cdot \vec{w}(\vec{x},\tau)} \), then \( \langle \rho\vec{G}(\vec{x},\tau) \rangle = e^{-\frac{4}{3}G_iG_j(u_iu_j)} \). The Debeye-Waller factor:

\[ I(\vec{G}) = \langle |\rho\vec{G}(\vec{x},\tau)|^2 \rangle = e^{-\frac{4}{3}G_iG_j(u_iu_j)} \] (40)

\[ (\Delta u^2)_i = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\beta} \sum_{\omega_n} \left[ (\kappa\omega_n^2 + \rho_nq^2)(\rho_n\omega_n^2 + (\lambda + 2\mu)q^2) + a^2q^2\omega_n^2 \right] \rho_n\omega_n^2 + (\lambda + 2\mu)q^2 \] (43)

Obviously, \( (\Delta u^2)_i < 0 \), namely, the longitudinal vibration amplitude in SS is smaller that that in NS. The \( \alpha(\vec{G})(T = 0) = e^{-\frac{4}{3}G^2(\Delta a^2)} \) > 1. This is expected, because the SS state is the ground state at \( T < T_{SS} \), so the longitudinal vibration amplitude should be reduced compared to the corresponding NS with the same parameters \( \rho_n, \lambda, \mu \).

After evaluating the frequency summation in Eqn\[13\] we get:

\[ (\Delta u^2)_i(T) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\rho_n} \coth \beta v_+q/2 \frac{v_2^2}{2v_+q} - \coth \beta v_-q/2 \frac{v_2^2}{2v_-q} - \frac{v_2^2}{v_+^2 - v_-^2}(\coth \beta v_+q/2 - \coth \beta v_-q/2) \] (44)

At \( T = 0 \), the above equation simplifies to:

\[ (\Delta u^2)_i(T = 0) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\rho_n} \frac{v_2^2}{2v_+q} - \frac{v_2^2}{2v_-q} \frac{1}{v_+ + v_-} \] (45)

\[ = -\frac{v_2^2}{(v_+ + v_-)v_+} \frac{\Lambda^2}{8\pi^2\rho_n} = -\frac{\Lambda^2}{\kappa\rho_n (v_+ + v_-)v_+ (v_+ + v_-)v_p} \frac{8\pi^2\rho_n}{\rho_n (v_+ + v_-)v_+ (v_+ + v_-)v_p} \] (45)

where \( \Lambda \sim 1/a \) is the ultra-violet cutoff and we have used the fact \( v_+ + v_- > v_p + v_+ \).

where the phonon-phonon correlation function is:

\[ \langle u_iu_j \rangle = \langle u_iu_j \rangle \hat{q}_i \hat{q}_j + \langle u_iu_j \rangle (\delta_{ij} - \hat{q}_i \hat{q}_j) \] (41)

where \( \hat{q}_i \hat{q}_j = \frac{q_i q_j}{q^2} \).

Then substituting Eqn\[11\] into Eqn\[40\] leads to:

\[ \alpha(\vec{G}) = I_{SS}(\vec{G})/I_{NS}(\vec{G}) = e^{-\frac{4}{3}G^2(u_0^2)(\vec{x},\tau)_{NS} - (u_0^2)(\vec{x},\tau)_{SS}} \] (42)

where the transverse mode drops out, because it stays in the same in the SS and in the NS.

Defining \( \langle \Delta u^2 \rangle_i(q,q') = \langle \rho\vec{G}(\vec{x},\tau) \rangle_{SS} - \langle \rho\vec{G}(\vec{x},\tau) \rangle_{NS} \) and \( (\Delta u^2)_i = \langle \Delta u^2 \rangle_i(q) \) it is easy to see:

\[ \langle \Delta u^2 \rangle_i(q) = \int \frac{d^3q}{(2\pi)^3} \sum_{\omega_n} (\Delta u^2)^2_i(q,\omega_n) = \int \frac{d^3q}{(2\pi)^3} (\Delta u^2)(q) \] (43)

By subtracting Eqn\[15\] from Eqn\[14\] we get:

\[ (\Delta u^2)_i(T) - (\Delta u^2)_i(T = 0) = \frac{(v_+ - v_\mu)(v_+ + v_\mu) - (v_+ - v_-)(v_+ + v_-)}{v_+ + v_- v_\mu v_\mu} (k_BT)^2 > 0 \] (46)

Namely, the difference in the ratio will decrease as \( T^2 \) as the temperature increases. Of course, when \( T \) approaches \( T_{SS} \) from below, the difference vanishes, the \( \alpha(\vec{G}) \) will approach 1 from above, the SS turns into a NS.

D. Density-density correlations

The density-density correlation function in the SS is:

\[ \langle \rho\vec{G}(\vec{x},\tau) \rangle_{SS} = e^{-\frac{4}{3}G_iG_j(u_i(\vec{x},\tau) - u_i(\vec{x}',\tau)) (u_j(\vec{x},\tau) - u_j(\vec{x}',\tau))} \] (47)

where \( t \) is the real time.

For simplicity, we only evaluate the equal-time correlation \( \langle \rho\vec{G}(\vec{x},\tau) \rho\vec{G}(\vec{x}',\tau) \rangle = \langle \rho\vec{G}(\vec{x},\tau) \rangle \langle \rho\vec{G}(\vec{x}',\tau) \rangle \) where \( \tau \) is the imaginary time. It is instructive to compare the density order in SS with that in a NS by looking at the ratio of the density correlation function in the SS over the NS:

\[ \alpha_r(\vec{x} - \vec{x}') = \langle \rho\vec{G}(\vec{x},\tau) \rangle_{SS}/\langle \rho\vec{G}(\vec{x}',\tau) \rangle_{NS} = e^{-\frac{4}{3}G^2\Delta D_r(\vec{x} - \vec{x}')} \] (48)

It is easy to find that

\[ \Delta D_r(\vec{x} - \vec{x}') = \int \frac{d^3q}{(2\pi)^3} \left( 2e^{-i\vec{q}\cdot(\vec{x} - \vec{x}')} - e^{-i\vec{q}\cdot(\vec{x} - \vec{x}')} \right) (\Delta u^2)(q) \] (49)
where \( (\Delta u^2)_{(q)} \) is defined above Eqn.43 and is the integrand in Eqn.41.

At \( T = 0 \), the above equation can be simplified to

\[
\Delta D_p(\vec{x} - \vec{x}'') = \frac{(v_+ + v_- - v_p - v_s)}{(v_+ + v_-)v_p} \frac{1}{2\pi^2 \rho_n (\vec{x} - \vec{x}'')^2}
\]

(50)

So we conclude that \( \alpha_p(\vec{x} - \vec{x}'') < 1 \), namely, the density order in SS is \textit{weaker} than the NS with the corresponding parameters \( \rho_n, \lambda, \mu \). This is expected because the density order in the SS is weakened by the presence of moving vacancies.

Unfortunately, so far, the X-ray scattering data is limited to high temperature \( T > 0.8K > T_{ss} \). X-ray scattering experiments on lower temperature \( T < T_{ss} \) are being performed to test these predictions.

X. SPECIFIC HEAT IN THE SS

It is well known that at low \( T \), the specific heat in the NS is \( C^{NS} = C^{NS}_s + C^{NS}_v \) where \( C^{NS}_s = \frac{2\pi^2}{15} k_B \left( \frac{k_B T}{\hbar v_+} \right)^3 \) is from the longitudinal phonon mode and \( C^{NS}_v = 2 \times \frac{2\pi^2}{15} k_B \left( \frac{k_B T}{\hbar v_-} \right)^3 \) is from the two transverse phonon modes, while \( C_{van} \) is from the vacancy contribution. \( C_{van} \) calculated in \ref{43} by assuming 3 different kinds of models for the vacancies. So far, there is no consistency between the calculated \( C_{van} \) and the experimentally measured one\ref{15,39}. The specific heat in the SF \( C^{SF}_s = \frac{2\pi^2}{15} k_B \left( \frac{k_B T}{\hbar v_-} \right)^3 \) is due to the SF mode \( \theta \). In this subsection, we focus on the specific heat inside the SS. From Eqn.27 we can find the specific heat in the SS:

\[
C^S_s = \frac{2\pi^2}{15} k_B \left( \frac{k_B T}{\hbar v_+} \right)^3 + \frac{2\pi^2}{15} k_B \left( \frac{k_B T}{\hbar v_-} \right)^3 + C^{tp} \tag{51}
\]

where \( C^{tp} \) stands for the contributions from the phonon modes which are the same as those in the NS (Fig.7).

It was argued in\ref{16}, the critical regime of finite temperature NS to SS transition in Fig.1 is much narrower than the that of SF to the NL transition, so there should be an jump in the specific heat at \( T = T_{SS} \) (Fig.7). Equ.51 shows that at \( T < T_{SS} \), the specific heat still takes \( T^3 \) behavior and is dominated by the \( \nu \) mode in Fig.6.

From Eqn.51 it is easy to evaluate the entropy inside the SS:

\[
S_{SS}(T_{SS}) = \frac{2\pi^2}{45} k_B \left( \frac{k_B T_{SS}}{\hbar} \right)^3 \left( \frac{1}{v_+} + \frac{1}{v_-} \right) + S^{tp} \tag{52}
\]

where \( S^{tp} \) is the entropy due to the 2 transverse modes.

The \textit{excess} entropy due to the vacancy condensation is:

\[
\Delta S = \int_0^{T_{SS}} dT C_{van} / T = \frac{2\pi^2}{45} k_B \left( \frac{k_B T_{SS}}{\hbar} \right)^3 \left( \frac{1}{v_+} + \frac{1}{v_-} \frac{1}{v_p} \right) \tag{53}
\]

FIG. 7: The specific heat of a supersolid

Obviously, \( \Delta S > 0 \) is due to the lower branch \( v_- < v_p \). At \( a = 0 \), the above equation reduces to \( \Delta S = \frac{2\pi^2}{45} k_B (k_B T_{SS})^3 \) which is simply the vacancy condensation into SF. Using the molar volume \( v_0 \sim 20cm^3/mole \) of solid \( ^4He \) and \( T_{ss} \sim 100mK \), we can estimate the \( \Delta S \) per mole is \( \sim 10^{-3} R \) where \( R \) is th gas constant. This estimate is 3 orders magnitude smaller than that in\ref{58} where the SS state was taken simply as the boson condensation of non-interacting vacancies. Our estimate is indeed consistent with recent experiment on specific heat\ref{44}. The linear \( \sim T \) behaviour found in heat capacity experiment can only be due to disorder or \(^3He \) impurities.

XI. ELEMENTARY EXCITATIONS IN FULDE - FERRELL-LARKIN-OVCHINNIKOV (FFLO) STATE OF SUPERCONDUCTORS

The QGL theories constructed for the supersolid in the previous sections may also be used to study the elementary excitations in the inhomogeneous Fulde - Ferrell-Larkin-Ovchinnikov (FFLO) state of superconductors\ref{57-60}. As argued in the following, the FFLO state can be taken as a fermionic supersolid. When the number of spin up electron is equal to the number of down spin electron \( n_\uparrow = n_\downarrow \), the pairing is at \( k = 0 \) only. If there is a mismatch \( n_\uparrow - n_\downarrow \), then pairing will shift to a non-zero momentum \( q_0 = k_{F \uparrow} - k_{F \downarrow} \). By using the GL theory near the normal to the FFLO state (Fig.8), at a mean field level, the authors in\ref{61} found the most favorable lattice structures of the FFLO state is the stripe state (LO state) in large number of parameter regimes. The FFLO state maybe considered as a weak coupling (or fermionic) analog of the (bosonic) supersolid. Indeed, the Fig.8 is similar to the Fig.2 after identifying the chemical potential difference \( \delta \mu \) with the pressure \( p \). It is important to point out a few important differences between the GL for the bosons developed in this paper and the GL for the FFLO state used in\ref{61}: (1) In the former, due to the lack of the \( Z_2 \) symmetry, there is a cubic term in Eqn.1. But in the latter, due to the presence of the \( Z_2 \) symmetry, there is no such a cubic term. (2) In the latter, the transition from the normal
state to the FFLO state is a classical Lifshitz type transition, so there is no zero momentum pairing as shown in Eqn.51 and in Eqn.54 in contrast to the second equation in Eqn.54 where there is always a zero momentum BEC condensation. (3) In the latter, there are large number of unpaired fermions which are the normal component of the system. Their distributions also take the same lattice structure as the FFLO state. They play a similar role as the normal lattice component in the first equation of 19, but they form extended Bloch waves in the underlying FFLO lattice and are integrated out in the GL theory in 49.

So far all the analysis in a FFLO state are only at a mean field level. In this appendix, we study the elementary excitations above the mean field solutions. Just from symmetry breaking point of views, the FFLO state breaks both the translational symmetry and the $U(1)$ symmetry, so it also has both the diagonal order and off-diagonal order, so the low energy modes should contain both the lattice phonon mode and the superfluid phonon mode. The pairing order parameter can be written as:

$$\psi_{\text{FFLO}}(\vec{x}, \tau) = \psi_0 \sum_\mathbf{G} e^{i\mathbf{G} \cdot \vec{x} + \Phi(\vec{x}, \tau)}$$

where $G = q_0$, the $\psi_0(\vec{x}, \tau) = \Delta e^{i\theta(\vec{x}, \tau)}$ is the pairing order parameter, $\Phi(\vec{x}, \tau)$ are the lattice phonon modes. Note the absence of the zero momentum pairing. For a charged system such as an electron system, due to the Higgs mechanism, the Goldstone mode $\theta(\vec{x}, \tau)$ will be just eaten by the gauge field. However, for a neutral system such as the pairing between two species of fermions with unequal populations in cold atom systems across a Feshbach resonance, the Goldstone mode $\theta(\vec{x}, \tau)$ stays. The coupling between the two phonon modes in Eqn.54 are also described by a equation similar to Eqn.27. The only difference is that the lattice structure is a LO state instead of an isotropic solid. After taking this difference into account, the elementary excitations inside the FF state are similar to those in the Fig.6a and the corresponding spectral weights can be worked out similarly. The experimental signatures of the elementary excitations can also be worked out similarly.

There exist other kinds of fermionic supersolids in addition to the FFLO state. In some fermionic systems, it is easy to see the coexistence of CDW and Superconductivity (SC), so "fermionic supersolid" phases are common. For instance, a quasi-two-dimensional system NbSe$_2$ has a transition to an incommensurate CDW phase at some high temperature $T_{C_{\text{CDW}}}$ and then a transition to a phase with coexisting CDW and SC order at a lower temperature $T_{SC}$. The CDW is a pairing in particle-hole channel at $2k_F$, its order parameter is $\psi_{\text{CDW}} = \langle c_\sigma^\dagger(\vec{k})c_\sigma(\vec{k} + \vec{Q})\rangle$ where $\vec{Q}$ is the ordering wave vector of the CDW. The SC is a pairing in particle-particle channel also across the Fermi surface. Its order parameter is $\psi_{\text{SC}} = \langle c_\uparrow^\dagger(\vec{k})c_\downarrow^\dagger(-\vec{k})\rangle$. Both order parameters are composite order parameters. Different parts of Fermi surface can do the two jobs separately (see, for example, 49). In contrast to the bosonic SS where there is a density operator $n$ and a complex order parameter $\psi$, both order parameters $\psi_{\text{CDW}}$ and $\psi_{\text{SC}}$ are complex order parameters. We can see that although the formation of a supersolid in a bosonic system has completely different mechanism than in fermionic case, from symmetric point of view, the QGL theories constructed developed in previous sections could also be used to describe the interplay between the two complex order parameters, the properties of the "fermionic supersolid" (FSS) and the transition from the FSS to the CDW. However, the fermionic excitations near the nodes maybe also be important and need to be taken into account.

\section{Conclusions}

In this paper, assuming there are quantum fluctuations generated vacancies or interstitials at zero temperature, we constructed a two component QGL theory to map out a possible global phase diagram, analyze carefully the conditions for the existence of the supersolid and study all the phases and phase transitions in a unified view. The only new parameters introduced in the GL theory in this paper is the coupling $g$ and $v$ between the $n$ sector (or normal solid part) and the $\psi$ sector (or the superfluid part) in Eqn.53. We investigated the SS state from both the SF and the NS side and found completely consistent description of the properties of the SS state. Starting from the SF side with increasing the pressure, we developed the theory basing on the two facts (1) there is a roton minimum in the superfluid state (2) the instability to solid formation is driven by the gap diminishing at the roton minimum. By increasing the pressure from the superfluid side, there are two possible scenarios (1) the SF to the C-NS transition in Fig.1 was described by the QGL action Eqn.12 first derived in this paper (2) the SF to the SS transition in Fig.2 is a simultaneous combination of the SFDW transition in the $\psi$ sector driven by the roton condensation at $k_0 = k_r$ and the NS transition in the $n$ sector driven by the divergence in the structure
function \( k_0 = k_n = k_r \). The superfluid becomes a SS at lower temperature and a NS at higher temperature (Fig.2). Then we also approached the SS state from the NS side. Depending on the sign and strength of the coupling \( g \) between the solid and superfluid, we also found two possible scenarios: (1) If \( |g| \) is sufficiently small (Fig.4), then the resulting solid at \( T = 0, p_{c1} < p < p_{c2} \) is a commensurate normal solid (C-NS). The SS state does not exist as a ground state. However, it may still exist as a metastable state. The QGL action to describe this SF to NS transition in Fig.1 was developed in Sec.II. (2) If \( |g| \) is sufficiently large (Fig.4), the resulting solid at \( T = 0, p_{c1} < p < p_{c2} \) is an incommensurate solid with zero point quantum fluctuations generated vacancies if it is negative and interstitials if it is positive (Fig.4). The condensation of the vacancies or interstitials lead to the formation of the SS-v and SS-i respectively. The SS state has lower energy than the NS state at \( T = 0 \). The \( T_{SS-v} \) (\( T_{SS-i} \)) is an effective measure of the strength of the interaction \( g \) in the SS-v (SS-i) supersolid. There is no particle-hole symmetry relating \( T_{SS-v} \) to \( T_{SS-i} \). Many physical consequences came out of this single parameter \( g \). Our results on supersolid should be independent of many microscopic details and universal.

Just like the SF is a uniform two-component phase consisting of superfluid and normal component at any finite temperature, the SS state is a uniform two-component phase consisting of a superfluid density wave (SFDW) and a normal solid component even at zero temperature. The SFDW in the SS-v coincides with the underlying normal solid. While the SFDW in the SS-i state is just a dual lattice to the underlying normal solid. This important fact leads to one of the key predictions in this paper: the X-ray scattering intensity from the SS-v is similar to that of NS at mean field level, while the X-ray scattering intensity from the SS-i ought to have an additional modulation over that of the NS. The modulation amplitude is proportional to the Non-Classical Rotational-Inertial (NCRI) observed in the torsional oscillator experiments. However, the X-ray scatterings from SS-v and SS-i will be modified by the Debye-Waller factor calculated in sec. IV. The NS-v (NS-i) to SS-v (SS-i) transition is described by a 3d XY model with much narrower critical regime (Fig.4).

We also studied the zero temperature quantum phase transition from the SS to the NS driven by the pressure near the upper critical pressure \( p = p_{c2} \) in Fig.5. We found that the coupling to the quantum fluctuation of the underlying lattice is irrelevant, so the transition stays the same universality class as the superfluid to Mott insulator transition in a 3 dimensional rigid optical lattice. The finite temperature transition from the SS to the NS in Fig.1 was studied previously in \(^{14}\) and \(^{23}\) in different contexts. It was found that the coupling to classical elastic degree of freedoms will not change the universality class of the 3D XY transition. However, we found that the coupling to quantum lattice phonons is very important inside the SS and leads to two “supersolidon” modes \( \omega_{\pm} = v_\pm q \) (one upper branch and one lower branch) shown in Fig.2. Their corresponding spectral weights are also worked out. The transverse modes in the SS stays the same as those in the NS. Detecting the two supersolidon modes with the corresponding spectral weights, especially, the lower branch \( \omega_- \) mode by neutron scattering\(^{17}\) or acoustic wave attention experiments\(^{38}\) is a smoking gun experiment to prove or disprove the existence of the SS in helium \(^4\)He. The \( \omega_- \) is estimated to considerably lower than the sound speed in the superfluid. Then we calculated the experimental signature of the two supersolidon modes. We found that the longitudinal vibration in the SS is smaller than that in the NS (with the same corresponding solid parameters), so the Debye-Waller factor at a given reciprocal lattice vector is larger than that in the NS. The density-density correlation function in the SS is weaker than that in the NS. By going the to the dual vortex loop representation, we found the vortex loop density-density interaction in SS stays the same as that in the SF (with the same corresponding superfluid parameters), so the vortex loop energy and the corresponding SS to NS transition temperature is solely determined by the superfluid density and independent of any other parameters. The vortex current-current interaction is stronger than that in the SF. The specific heat in the SS is still given by the sum from the transverse phonons and the two supersolidon modes and still shows \( \sim T^3 \) behaviors. The supersolidon part is dominated by the lower branch. The NCRI is only weakly anisotropic in the SS phase for \( ^{4}\)He lattice. In principle, all these predictions can be tested by experimental techniques such as X-ray scattering, neutron scattering, acoustic wave attentuations\(^{38,39}\) and heat capacity.

It may be instructive to make some analogy of Fig.1 and Fig.2 at \( T < T_{SF} \) to Type-I and type-II superconductors with the pressure \( p \) playing the role of the magnetic field \( H \): Fig.1 is similar to Type-I superconductor with SF identified as the Messiner state, the NS as the normal state, the critical pressure \( p_c \) identified as the critical magnetic field \( H_c \). Fig.2 is similar to Type II superconductor with SF identified as the Messiner state, the SS as the mixed vortex lattice state which also breaks both translational order and the global \( U(1) \) symmetry, the NS as the normal state, the lower and upper critical pressures \( p_{c1} \) and \( p_{c2} \) identified as the lower and upper critical magnetic fields \( H_{c1} \) and \( H_{c2} \). In superconductors, it is the \( \kappa = \lambda/\xi \) to determine Type I and Type II and if the vortex lattice is a stable intermediate state or not as the magnetic field is increased. In Helium 4, it is the sign and strength of the coupling constant \( g \) in Eqn.\(^3\) to determine the Fig.1 and Fig.2 and if the SS is a stable intermediate state or not as the pressure is increased. So the pressure \( p \) and the coupling \( g \) in the formation of SS-v play the role of the magnetic field \( H \) in the formation of the mixed state of superconductors. Note that in superconductors, \( H \) and \( \kappa \) are two independent parameters, in \(^4\)He, \( p \) and \( g \) are also two independent
parameters.

The GL theory developed in this paper put the competing orders of superfluid and solid in the unified framework. We suggest that even supersolid may not be realized in $^4$He system, it has its own intrinsic, scientific interests and may be realized in other continuous bosonic and fermionic systems. For example, in symmetric electron-hole bilayer systems, it was shown in Ref. 36–38, it is quite possible that there may a narrow window of ESS where both order parameters are non-vanishing $\langle \psi \rangle \neq 0, \langle n_G \rangle \neq 0$ intervening between the ESF and two weakly coupled Winger crystal. A similar GL theory can be used to study the elementary excitations inside a FFLO state and some fermionic systems with coexistence of CDW and superconductivity.

Acknowledgement

I thank P. W. Anderson, M. Chan, T. Clark, Milton Cole, B. Halperin, Jason Ho, T. Leggett, T. Lubensky, Mike Ma, G. D. Mahan, S. Sachdev and F. C. Zhang for helpful discussions, A. T. Dorsey for pointing out Ref. 36 to me. I also thank the hospitality of Y. Chen, Z. Wang and F. C. Zhang during my visit at Hong Kong University, Yu Lu during my visit at Institute for Theoretical Physics in Beijing, China. The research at KITP was supported by the NSF grant No. PHY99-07949. The research at KITP-C is supported by the Project of Knowledge Innovation Program (PKIP) of Chinese Academy of Sciences.

Appendix A: Discussions on a tight-binding toy

Supersolid ground state wavefunction

The Ginzburg-Landau theory constructed in the main text is based on order parameters and symmetries. It should hold irrespective many microscopic details such as what is the mechanism responsible for the formation of the supersolid. Despite there are many microscopic calculations for $^4$He, constructing a microscopic theory for supersolid is very difficult. In this appendix, I will discuss a well known toy SS wavefunction and clarify a few concepts related to global phase-number uncertainty relation and the role of vacancies or interstitials in the formation of SS. We also clarify the physical meaning of the order parameters $n$ in Eqn. 1 and $\psi$ in Eqn. 2. However, because the toy wavefunction may miss some important physics in bulk $^4$He systems. For example, due to the very peculiar potential well in the solid $^4$He which has a local shallow maximum at the lattice site, the tight binding model maybe crude, so the discussion is intuitive and instructive. Eqn. [A1] is also a ground state wavefunction, so does not include the lattice phonon interaction.

The toy wavefunction of a supersolid takes the BCS like form

$$|SS\rangle = \prod_{i=1}^{N}(u + v|b_i^\dagger\rangle)|0\rangle \tag{A1}$$

where $N$ is the number of sites, $u \neq 0$ and $|u|^2 + |v|^2 = 1$. If setting $u = 0$, the state reduces to a commensurate solid without any vacancies $|CS\rangle = \prod_{i=1}^{N}b_i^\dagger|0\rangle$. The commensurate solid (CS) is an exact eigenstate of the boson number operator $N_b = \sum_{i=1}^{N}n_i$ with the eigenvalue $N_b = N$, so has no chance to become phase ordered. Adding a superfluid component to the CS leads to the SS in Eqn. [A1]. If setting $v = 0$, the state reduces to the vacuum state $|0\rangle$.

If setting $u = \cos \frac{\theta}{2}, v = \sin \frac{\theta}{2}e^{i\phi}$, then Eqn. [A1] becomes:

$$|SS, \phi\rangle = \prod_{i=1}^{N}(\cos \frac{\theta}{2} + \sin \frac{\theta}{2}e^{i\phi}|b_i^\dagger\rangle)|0\rangle \tag{A2}$$

where $\theta \neq \pi$.

By construction, the state has the translational order with the average boson density $\langle n_i \rangle = |v|^2 = \sin^2 \theta/2$, so the average vacancy density is $|u|^2 = \cos^2 \theta/2$. The total number of bosons $N_b = N \sin^2 \theta/2$, the number of vacancies $N_v = N \cos^2 \theta/2$. It is easy to see that $|SS\rangle$ also has the ODLO with $|b_i\rangle = u^*|v\rangle = \frac{1}{2} \sin \theta e^{i\phi}$, so $|SS\rangle$ is indeed a supersolid state. The angle $\theta$ controls the magnitude, while the phase $\phi$ controls the phase of the condensation. Defining $b_{k=1} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N}i_k b_i$ which satisfy the boson commutation relation $[b_i^\dagger, b_{j}] = 1$, the boson operator at zero momentum is $n_0 = b_{k=1}^\dagger b_{k=1} = 0$, the total number of bosons at the zero momentum state is $N_0 = \langle SS|n_0|SS\rangle = \frac{N}{2} \sin^2 \theta = N_b \cos^2 \theta/2 < N_b = N \sin^2 \theta/2 < N_v = N \cos^2 \theta/2$. So the condensate is less than both the boson density and the vacancy density. At integer filling $n = 1$, the non-interacting BEC state $|SF\rangle = \frac{1}{\sqrt{N}}\prod_{i=1}^{N}|b_i^\dagger=0\rangle|0\rangle$, then $N_0 = \langle SF|n_0|SF\rangle = N_b = N$. Obviously, this non-interacting BEC state is not included in the family in the Eqn. [A1].

A supersolid state $|SS, N_b\rangle$ with $N_b$ bosons is given by:

$$|SS, N_b\rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-iN_0 \phi} |SS, \phi\rangle \tag{A3}$$

where the total boson number $N_b$ and the global phase $\phi$ are two Hermitian conjugate variables satisfying the commutation relation: $[N_b, \phi] = i\hbar$. It leads to the uncertainty relation $\Delta N_b \Delta \phi > 1$.

It is easy to see $\langle N_b \rangle = \langle |SS\rangle |N| |SS\rangle = N \sin^2 \theta/2, \Delta N_b = \sqrt{\langle N_b^2 \rangle - \langle N_b \rangle^2} = \sqrt{N} |v| = \sqrt{N} \frac{1}{2} \sin \theta, \Delta N_b/\langle N_b \rangle = \sqrt{\frac{1}{N}} |v|/\sin \theta$. If $\theta \neq \pi$, the absolute boson number fluctuation $\Delta N_b \sim \sqrt{N}$ is quite large, so $\Delta \theta$ could be quite small, so one can get a phase ordered state. On the other hand, the relative boson number fluctuation $\Delta N_b/\langle N_b \rangle \sim \sqrt{\frac{1}{N}}$ is quite small, so one can still measure the average boson number accurately. The first quantization form of the Eqn. [A3] can be derived by the same method used in Ref. 19.
Because in the SS state, there is a global phase ordering in $\phi$, so its conjugate variable is the total number of particles $N_b$ as shown in Eqn.1. The local tunneling or exchanging processes stressed in [2] may not cause the total number fluctuations, therefore may not cause the global phase ordering leading to the supersolid phase. Eqn.1 implies that the vacancies in an incommensurate solid could lead to the formation of a supersolid. If this indeed happens, in the GL theory constructed for SS-V in the main text, the lattice sites are represented by $n(x)$, while the vacancies are represented by $\psi$. Of course, the trial ground wavefunction $A_1$ does not include fluctuations, so no information on the lattice phonons and superfluid phonons, therefore no supersolidons in the Fig.6a. A toy wavefunction for SS-i was not written down so far, because the interstitials are moving between lattice sites, so there is no tight binding limit. The interstitial case is described by the extended Bloch state in Section IV-B. In the SS-i, the bosons are represented by $n(x)$, while the interstitials are represented by $\psi$.

### Appendix B: Comparisons with Supersolids on lattices

The extended boson Hubbard model (EBHM) with various kinds of interactions, on all kinds of lattices and at different filling factors is described by the following Hamiltonian:

$$H = -t \sum_{\langle ij \rangle} (b_i^\dagger b_j + h.c.) - \mu \sum_i n_i + \frac{U}{2} \sum_i n_i(n_i - 1) + V_1 \sum_{\langle ij \rangle} n_i n_j + V_2 \sum_{\langle\langle ij \rangle\rangle} n_i n_k + \cdots \quad (B1)$$

where $n_i = b_i^\dagger b_i$ is the boson density, $t$ is the nearest neighbor hopping amplitude. $U, V_1, V_2$ are onsite, nearest neighbor (nn) and next nearest neighbor (nnn) interactions respectively, the ... may include further neighbor interactions and possible ring-exchange interactions. A supersolid in Eqn.1 is defined as the simultaneous orderings of ferromagnet in the XY component (namely, $\langle b_i \rangle \neq 0$) and CDW in the $Z$ component. In [2], we studied all the possible phases and phase transitions in the EBHM in bipartite lattices such as a honeycomb and square lattice near half filling. We show that there are two consecutive transitions at zero temperature driven by the chemical potential: in the Ising limit, a Commensurate-Charge Density Wave (CDW) at half filling to a narrow window of CDW supersolid, then to an Incommensurate-CDW; in the easy-plane limit, a Commensurate-Valence Bond Solid (VBS) at half filling to a narrow window of VBS supersolid, then to an Incommensurate-VBS. The first transition is second order in the same universality class as the Mott to insulator transition, therefore has the exact critical exponents $z = 2, \nu = 1/2, \eta = 0$ with logarithmic corrections, while the second one is first order. Liu and Fisher also studied the C-CDW to the CDW-SS transition and concluded that $z = 1$ in contrast to $z = 2$ achieved in [50]. We found that the phase diagram in the Ising limit is similar to the reentrant "superfluid" in a narrow region of coverages in the second layer of $^4$He adsorbed on graphite detected by Crowell and Reppy’s torsional oscillator experiment in 1993 [52]. Indeed, the data in the torsional oscillator experiment in [52] do not show the characteristic form for a 2d $^4$He superfluid film, instead it resembles that in [7] characteristic of a supersolid in terms of the gradual onset temperature of the NCRI, the unusual temperature dependence of $T_{SS}$ on the coverage. Of course, both experiments may due to phase separations instead of the SS phase. Very recently, the author studied various kinds of supersolids in frustrated lattices such as triangular and Kagome lattices [51].

By comparing Eqs.11 and 21 in the continuum with the Eqs in [11] on the lattice, one can see the crucial difference between a lattice system with a continuous system: Eqn.11 stands for a spontaneously formed lattice with phonon excitations. While Eqn.21 is a fixed external lattice with no such phonon excitations. This crucial difference is responsible for the difference in the elementary excitation spectra in the two kinds of supersolid shown in Fig.6. So the reentrant lattice SS discussed in [33,50,51] is different from the bulk $^4$He SS state discussed in this paper, although both kinds of supersolids share some interesting common properties. In both cases, the SF to SS transition is driven by the closing of the gap of the roton minimum, so the transition could be first or second order at mean field level. However, due to the fluctuation of the underlying superfluid order, the transition turns out to be the first order as confirmed by the QMC simulations. On the lattice, the manifold of the roton minimum consists of discrete points due to the lattice symmetry. However, in $^4$He, as shown in sections III and IV, the manifold of the roton minimum is a continuous surface, so the transition must be first order. In the former, there is a periodic substrate or spacer potential which breaks translational symmetries at the very beginning. The filling factor is controlled by an external potential. At integer filling factors or half filling factor, there are particle-hole symmetry (PH) for excitations which ensures the number of particles is equal to that of vacancies and adding interstitials is equivalent to adding vacancies respectively [55,56]. While in the latter, the lattice results from a spontaneous translational symmetry breaking driven by the pressure as shown in the Fig.2, if there are vacancies or interstitials in the ground state has to be self-determined by ground state energy minimization. There is usually no particle-hole symmetry for excitations. Due to this absence of symmetry, the number of vacancies is usually not equal to that of interstitials. So the theory developed in this paper on bulk $^4$He is different from the lattice theory developed in [53,50]. As shown in the appendix A and in [50,51], one common fact of both supersolids is that both are due to
vacancies or interstitials. Lattice supersolids can also be described by doping the adjacent CDW either by vacancies or interstitials, so are classified as two types $SS-v$ and $SS-i$. In the hard-core limit, $SS-v$ and $SS-i$ are simply related by P-H transformation. However, as shown in section IV, in Helium-4 supersolid, there is no particle-hole symmetry relating $T_{SS-v}$ to $T_{SS-i}$ (Fig. 3).

So the coupling constant $y$ in Helium 4 plays a similar role as the chemical potential $\mu$ in the lattice models, the gap $\Delta(p)$ in the NS-PH which tunes the distance from the NS-PH to the SF plays a similar role as the gap in the CDW which tunes the distance from the CDW to the superfluid. It was shown in, the lattice supersolids existing at commensurate 1/2 filling factors in frustrated lattices such as triangular lattice is just the coexistence of SS-v and SS-i. Combined with the results in, we conclude that $^4$He supersolid can exist both in bulk and on substrate, while although $^3$He supersolid may not exist in the bulk, but it may exist on wisely chosen substrates. Lattice supersolid could also be realized in optical lattices in ultra-cold atomic experiment. However, in both continuum and on lattices, SS states could be unstable against phase separations. For example, the vacancies or interstitials in the in-commensurate solid can simply move to the boundary of the sample instead of boson condensation, namely, it will turn into a commensurate solid. This case is included in the C-NS case in the paper anyway. Due to its negative compressibility, the instability of lattice SS against phase separation was demonstrated in some lattice models in.

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