Non linear gluon evolution in path-integral form

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Abstract. We explore and clarify the connections between two different forms of the renormalisation group equations describing the quantum evolution of hadronic structure functions at small $x$. This connection is established via a Langevin formulation and associated path integral solutions that highlight the statistical nature of the quantum evolution, pictured here as a random walk in the space of Wilson lines. The results confirm known approximations, form the basis for numerical simulations and widen the scope for further analytical studies.

1 Introduction

A lot of effort is presently devoted to the study of hadronic systems with large parton densities, as revealed for instance in high energy scattering experiments. Of particular interest is the regime of parton saturation [1–9] expected to end the growth with increasing energy of the structure functions which is predicted by linear evolution equations [10,11] and is experimentally observed at HERA. These linear equations are in fact only justified in situations where the partons in the hadrons form a dilute system; they single out two-particle correlators (actually the single particle densities) as the only important contributions. In the high density regime however, all $n$-point functions have to be treated on the same footing, and the ensuing evolution is non-linear. Several formalisms have been developed recently to cope with this non linear regime [5, 6, 12–24]. (See also Refs. [25, 26] for recent reviews and more references.)

To be specific we shall focus on deep inelastic scattering where a virtual photon collides with a hadronic target. The crucial variable in the discussion is the large rapidity separation $\tau \equiv \ln(1/x) \sim \ln s$ between the projectile and the target; $s$ is the total energy squared and $x$ is the typical fraction of the hadron momentum carried by the partons seen by the photon. Large rapidity interval implies “small
and all formalisms exploit this kinematic situation to reduce the number, and select the type, of degrees of freedom involved in the event.

A key simplification comes from the recognition that the basic degrees of freedom, in an appropriate frame and for a suitable gauge, involve only the component $A^+(x^-, x)$ of the gauge field of the hadron, and this enters only through path ordered exponentials along $x^-$ (Wilson lines):

$$U^+(x) \equiv \text{P exp} \left( ig \int dx^- A^+_a (x^- x) t^a \right).$$  \hspace{1cm} (1.1)

(Throughout, we shall use light-cone coordinates, e.g., $k^\mu = (k^+, k^-, k)$, with $k^\pm \equiv (k^0 \pm k^3)/\sqrt{2}$, and $k$ is a vector in a plane perpendicular to the direction of propagation of the photon, to which we shall refer briefly as the “transverse plane”.) Thus the total cross section in deep inelastic $eA(\gamma^*A)$ scattering is of the schematic form ($r = x - y, b = x + y$) [4, 9, 27]:

$$\sigma_{\gamma^*A}(x, Q^2) = 2 \int dz \int d^2 r |\psi(z, r; Q^2)|^2 \int d^2 b \left( 1 - S_\tau(x, y) \right)$$  \hspace{1cm} (1.2)

which naturally separates into a photon wave function $\psi$, giving the probability that the virtual photon splits into a $q\bar{q}$ pair (a color “dipole”) with transverse size $r$, and a matrix element summarizing the QCD interaction of the dipole with the hadronic target:

$$S_\tau(x, y) \equiv \frac{1}{N_c} \langle \text{tr}(U^+(x)U(y)) \rangle_\tau.$$  \hspace{1cm} (1.3)

This form ensues if the energy is large enough for the scattering to be treated in the eikonal approximation with the $q\bar{q}$ pair not deflected from its light-like trajectory.

At this point, although one has achieved a reduction of degrees of freedom, the calculation of the average $\langle \ldots \rangle$ in Eq. (1.3) is still a very hard problem. Note that this average depends on the rapidity interval $\tau$, and rather than calculating directly the average, one can at least try to calculate its variation with $\tau$. This calculation has been approached by following two seemingly distinct routes. In the first one, one focuses on the averages of products of Wilson lines and obtains differential equations for their evolution with $\tau$ without having to specify explicitly the average on the target hadron [17, 18, 26]. Such equations form an infinite hierarchy coupling together correlators of Wilson lines of increasing order [17] (see also Refs. [18, 20, 24, 26]). Remarkably, this hierarchy is coded in a single functional evolution equation for an appropriate weight function [19] (see also Ref. [23]). The second approach builds on an explicit model for the hadron wavefunction which allows for a direct and explicit calculation of the average [5, 6, 8, 12, 13, 15, 16, 21, 22].

In both approaches, the evolution results from a renormalization group (RG) procedure in which quantum fluctuations of the gluonic fields are successively integrated out [15, 21, 22]. Reassuringly, as shown in Ref. [22], both approaches yield
identical results for the calculation of observables. Both lead to a description of quantum evolution of parton densities in terms of equations for probability distributions. A complete derivation and thorough proof of these correspondences, however, has not been available up to now; this is the main purpose of the current publication.

As we shall see, both pictures of the evolution can be visualized as a random walk in the space of Wilson lines. For each value of $\tau$, there exists some probability that a particular value of $U$ is realized, i.e., there is an associated probability distribution $Z_\tau[U]$ that allows the calculation of averages such as that in Eq. (1.3). This distribution satisfies a Fokker-Planck equation: this is the evolution equation of the first approach, which we shall refer too as the “$U$-representation”. If we go deeper in the details of the random walk, one finds that the value of $U$ at rapidity $\tau + d\tau$, call it $U_{\tau+d\tau}$, is related to $U_\tau$ by $U_{\tau+d\tau} = U_\tau \exp(-i\alpha_\tau d\tau)$, where $\alpha_\tau$ is a random variable — the component $A^+$ of a classical colour field — whose probability distribution $W_\tau[\alpha]$ is explicitly calculated in the second approach which provides also the corresponding evolution equation. We shall refer to this second approach as the “$\alpha$-representation”. The correspondence between the two approaches is easy to see from the point of view of the $\alpha$-representation. Indeed, in this representation, the matrix $U$ at rapidity $\tau$ is constructed step by step from some initial $U_0$ by (right) multiplication by a matrix of the form $\exp(-i\alpha_\eta)$ when $\alpha_\eta$ is a random variable. Since we know the probability distribution of all the random variables $\alpha_\eta$ where $\eta$ spans the rapidity interval — this is $W_\tau[\alpha]$ — one can easily calculate the probability $Z_\tau[U]$ that the random walk terminates at $U$.

Furthermore, one can relate the two evolution equations.

Further insight is gained by considering the random steps as a Langevin process, and by writing the average over the random variables as path integrals. Such path integrals provide a direct connection between the two approaches. This not only highlights the physical concepts driving the RG procedure with a clear reference to its statistical nature, but it also allows us to give simple formulae for the Langevin process, which may be convenient for numerical simulations. Investigations along these lines have actually already been carried out and will be available soon [28]. In Ref. [29], a different path integral has been proposed as a solution to the infinite hierarchy of coupled equations established in [17]. The precise relation between the results in [29] and the results that we shall obtain below in this paper is still to be explored.

In the next section we shall elaborate on the physics underlying the RG equations of Refs [19] and [22] and recall the explicit forms of the basic equations in the two representations introduced above. The formal relation between the two formalisms will be outlined. In particular the characteristics of the random process responsible for the evolution with rapidity will put forward. Then, in section 3 we show that much of the structure of the evolution equations can be identified in the simpler setting of the random walk in ordinary Euclidean space. The last section considers finally the relevant case of random walks on a group manifold. Conclusions are summarized at the end.
2 Nonlinear quantum evolution in two representations

As discussed in the previous section, the calculation of scattering observables at high energy reduces to the calculation of averages of products of Wilson lines. This can be implemented in several ways, and we shall review in this section two seemingly independent approaches which turn out to be equivalent. Our goal here is to recall the main physical ingredients. The rest of the paper will be devoted to a deeper understanding of the relation between the two approaches by analyzing their common mathematical grounds.

2.1 Wilson line operators and the $U$–representation

Consider the evolution of the cross-section (1.2) when the rapidity gap $\tau$ increases by an amount $d\tau$. We choose the frame so that $\tau_H \approx \tau \gg \tau_{dipole}$, where $\tau_H$ and $\tau_{dipole}$ are the rapidities of the hadron and the dipole, respectively. If initially $\alpha_s\tau_{dipole} \ll 1$, the dipole is just a quark-antiquark pair [26]. When $\tau_{dipole}$ is increased by an amount $d\tau$, so is the phase space for the emission of a gluon. If such a gluon is indeed emitted, then the interaction of the dipole with the target will involve the independent scattering of the quark, the antiquark, and the emitted gluon, off the colour field $A(x^-,x)$ of the target. This is the source for the change $dS_\tau \propto \alpha_s d\tau$ in the $S$–matrix element $S_\tau \equiv \frac{1}{N_c} \langle \text{tr}(U_x^\dagger U_y) \rangle_\tau$, for which the following evolution equation is obtained:

$$\partial_\tau \langle \text{tr}(U_x^\dagger U_y) \rangle_\tau = -\frac{\alpha_s}{2\pi^2} \int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2} \langle N_c \text{tr}(U_x^\dagger U_z) \text{tr}(U_z^\dagger U_y) \rangle_\tau,$$

where all the Wilson lines are in the fundamental representation, and $U_x \equiv U(x)$. Eq. (2.1) has been originally derived by Balitsky [17].

While this equation for $S_\tau(x,y)$ involves only averages of products of Wilson lines, this is not a closed equation: the 2-point function is related to a 4-point function. However a closed equation is obtained when one allows $N_c$ to take arbitrary large values. Indeed, in this large $N_c$ limit, the product of traces in its r.h.s. factorizes,

$$\langle \text{tr}(U_x^\dagger U_z) \text{tr}(U_z^\dagger U_y) \rangle_\tau \longrightarrow \langle \text{tr}(U_x^\dagger U_z) \rangle_\tau \langle \text{tr}(U_z^\dagger U_y) \rangle_\tau \quad \text{for } N_c \rightarrow \infty,$$

so that eq. (2.1) reduces indeed to a closed equation for $S_\tau(x,y)$. In this form, the equation has been independently obtained by Kovchegov [18], within the Mueller’s dipole model [14]. It has attracted much interest recently [30–33].(See also Ref. [20] for still a different derivation, which is summing “fan” diagrams.)

In general, eq. (2.1) is just the first step in an infinite hierarchy of coupled evolution equations for the correlation functions of Wilson lines. In principle, these
equations can be all obtained in the formalism of Ref. [17], but, as recognized in Ref. [19], they are equivalent to a single evolution equation for an appropriate generating functional $Z_\tau[U]$ giving the probability that a certain field $U_\mathbf{x}$ is realized in a collision with rapidity gap $\tau$. In particular the averages of products of Wilson lines are obtained as:

$$\langle U_{x_1}^{(1)} \ldots U_{x_n}^{(1)} \rangle_\tau = \int [d\mu(U)] \ U_{x_1}^{(1)} \ldots U_{x_n}^{(1)} \ Z_\tau[U].$$  \hspace{1cm} (2.3)

Here, $U^{(1)}$ is a generic notation for either $U$ or $U^\dagger$ (in any representation of SU($N$)), and $d\mu(U)$ denotes the group invariant measure [36, 37] (see also Appendix). Eq. (2.3) is consistent with Balitsky’s equations if $Z_\tau[U]$ obeys the following equation:

$$\partial_\tau Z_\tau[U] = \frac{1}{2} \nabla^a x \chi^{ab}_{xy}[U] \nabla^b y Z_\tau[U],$$ \hspace{1cm} (2.4)

where we employ a summation and integration convention for repeated indices and transverse coordinates — in the reminder of the text we will only write integral signs where confusion might arise. $\nabla^a x$ is a Lie derivative which generates translations on the group manifold (see Appendix),

$$\chi^{ab}_{xy}[U] \equiv \frac{1}{\pi} \int \frac{d^2 z}{(2\pi)^2} \ K_{xyz} [(1 - U_x^\dagger U_z)(1 - U_z^\dagger U_y)]^{ab}$$ \hspace{1cm} (2.5)

with $U^{ab}$ in the adjoint representation, and

$$K_{xyz} \equiv \frac{(x-z) \cdot (y-z)}{(x-z)^2(y-z)^2}. \hspace{1cm} (2.6)$$

$\chi$ is of the form $e^\dagger e$ and hence positive definite. More precisely

$$\chi^{ab}_{xy}[U] = \int d^2 z \ e^{ac,l}(x, z) \ e^{bc,l}(y, z),$$ \hspace{1cm} (2.7)

which, according to our conventions, we could have written shortly as $\chi^{ab}_{xy}[U] = e^{ac,l}_{x,z} e^{bc,l}_{y,z}$. The “square root” factor in Eq. (2.7) is given by

$$e^{ab,l}(x, z) = \frac{1}{\sqrt{4\pi^3}} \frac{(x-z)^l}{(x-z)^2}(1 - U_x^\dagger U_z)^{ab}. \hspace{1cm} (2.8)$$

It is a matrix in color indices and a vector in transverse coordinates. With the help of

$$\sigma^a_x[U] \equiv \frac{1}{2} \nabla^b y \chi^{ab}_{xy}[U],$$ \hspace{1cm} (2.9)

Eq. (2.4) takes the familiar form of a Fokker-Planck equation

$$\partial_\tau Z_\tau[U] = \frac{1}{2} \nabla^a_x \nabla^b_y \ (\chi^{ab}_{xy}[U] \ Z_\tau[U]) - \nabla^a_x \ (\sigma^a_x[U] \ Z_\tau[U]),$$ \hspace{1cm} (2.10)
albeit a functional one, involving fields with values on a curved manifold. As for all Fokker-Planck equations, there is an underlying random process whose origin lies in the renormalisation group operation itself, and the elimination of degrees of freedom which accompanies a change in $\tau$. In fact, this aspect will be made the centerpiece of the formulation by recoding Eq. (2.4) as a random walk governed by a Langevin equation. However this short presentation does not make the elements of the random walks immediately visible; it does not either exhibit the important role of the background field of the target in determining the properties of the random walk. Those aspects are better seen with a picture of the target wave function which utilizes the field $\alpha$ rather than $U$ as the basic random variable, a description to which we will turn now.

2.2 The $\alpha$-representation

The model of hadron wavefunction that we shall use rests on the effective theory developed in Refs. [5, 6, 15, 22], which we briefly summarize (see Ref. [25] for details)

Because of the large rapidity gap between the dipole and the hadron, the latter appears to the dipole as a Lorentz contracted colour source for the small-x gluons to which it couples. We are then led to separate the hadron constituents into hard ones with large longitudinal momenta ($k^+ \gtrsim xP^+$), and soft ones with momenta $k^+ \lesssim xP^+$ (with $P^+$ = the total longitudinal momentum of the hadron). The hard partons include the valence quarks, as well as the partons created in the quantum evolution down to longitudinal momenta $xP^+$ and whose density increases with $1/x$. Together, these hard partons constitute a high density system of colour charges which, during the interaction with the dipole, can be regarded as frozen and constitute the source $\rho_a(x^-,x)$ of a classical field representing the soft gluons.

This classical field is assumed to be the solution of the Yang-Mills equations:

$$(D_\mu F^{\mu\nu})_a(x) = \delta^{\mu+}\rho_a(x^-,x),$$

where the source $\rho^a(x^-,x)$ is time-independent, i.e. independent of $x^+$. This assumption follows from the observation that the internal dynamics of the hadron is frozen during the relatively short interaction time $\Delta x^+ \approx 2xP^+/Q^2$ with the dipole. The classical field has just one independent component, which in the covariant gauge $\partial_\mu A^\mu = 0$ is the component $A^+_a(x^-,x) \equiv \alpha_a(x^-,x)$, with $-\partial_\perp^2\alpha_a(x^-,x) = \rho_a(x^-,x)$.

The source $\rho_a$, or equivalently the classical field $\alpha_a$, is treated as a random variable, with some gauge-invariant probability density $W_\tau[\alpha]$, which depends upon $\tau \equiv \ln(1/x)$ [6, 15]. In order to compute some observable $\mathcal{O}[\alpha]$ (to leading log accuracy), one first evaluates it with the classical solution, and then average the result with the weight function $W_\tau[\alpha]$:

$$\langle \mathcal{O} \rangle_\tau = \int [d\alpha] \mathcal{O}[\alpha] W_\tau[\alpha]. \quad (2.11)$$

This peculiar averaging, somewhat reminiscent of the mathematical description of amorphous materials like spin glasses, has been dubbed Colour Glass Con-
What is being done here is analogous to the Born-Oppenheimer approximation: the original quantum average is replaced by a classical average over the various possible configurations of the “frozen” degrees of freedom.

The probability distribution $W_\tau[\alpha]$ is not known directly, but its variation corresponding to freezing partons in the rapidity window $(\tau, \tau + d\tau)$ can be computed. These partons have longitudinal momenta $b\Lambda < k^+ < \Lambda$, where $\Lambda$ is the momentum scale corresponding to $\tau$ (i.e., $\tau = \ln(P^+/\Lambda)$) and $d\tau = \ln(1/b)$. The elimination of these degrees of freedom induces new correlations at the softer scale $b\Lambda$, which, to the accuracy of interest, can be accounted for by shifting the original field $\alpha^a(x^-, x)$ by a random quantity $\alpha^a_\tau(x)\,d\tau$ with only 1-point and 2-point correlation functions:

$$\langle \alpha^a_\tau(x) \rangle_\Lambda, \quad \langle \alpha^a_\tau(x)\alpha^b_\tau(y) \rangle_\Lambda,$$

(2.12)

where the subscript $\Lambda$ is here to remind us that the averages depend on the background field $\alpha(x^-, x)$ at the scale $\Lambda$.

At this place we need to digress on a subtle aspect of the problem which has to do with the longitudinal structure of the field. The random field $\alpha^a_\tau(x)$ is a two dimensional field which is in fact obtained by integrating over $x^-$ a fluctuation $\delta\alpha^a_\tau(x^-, x)$:

$$\alpha^a_\tau(x)\,d\tau \equiv \int dx^- \delta\alpha^a_\tau(x^-, x).$$

(2.13)

It is shown in [22] that the $x^-$ dependence is of the form:

$$\delta\alpha^a_\tau(x^-, x) \propto \theta(x^-) \frac{e^{-ib\Lambda x^-} - e^{-i\Lambda x^-}}{x^-},$$

(2.14)

so that the integral over $x^-$ in Eq. (2.13) produces a factor $\ln(1/b) = d\tau$. Eq. (2.14) shows that the shift in the classical field resulting from integrating out gluons in the chosen layer of $k^+$ has support at $1/\Lambda \lesssim x^- \lesssim 1/b\Lambda$, and therefore essentially no overlap with the original field $\alpha^a(x^-, x)$ at the scale $\Lambda$ (whose support is $x^- \lesssim 1/\Lambda$). This is an important feature which allows us to reconstruct step by step the Wilson lines involved in the calculation of high energy scattering matrix elements. Thus, for instance:

$$U^\dagger_{\tau + d\tau}(x) = e^{ig\int dx^- \delta\alpha^a_\tau(x^-, x)} U^\dagger_\tau(x) = e^{igd\tau\alpha^a_\tau(x)} U^\dagger_\tau(x),$$

(2.15)

since all the $x^-$ in $\delta\alpha^a_\tau(x^-, x)$ are larger than those contained in the Wilson line $U^\dagger_\tau(x)$. This being recognized, it is not essential to keep trace of the detailed $x^-$ dependance of the field exhibited in eq. (2.14), and we shall work with integrated quantities such as $\alpha_\tau(x)$: the ordering in $\tau$ takes care of the initial ordering in $x^-$. We are now in position to introduce the equation obeyed by the probability distribution $W_\tau[\alpha]$ [22]:

$$\partial_\tau W_\tau[\alpha] = \frac{1}{2} \frac{\delta^2}{\delta\alpha^a_\tau(x)\delta\alpha^b_\tau(y)} (\chi^a_{xy}[\alpha]W_\tau[\alpha]) - \frac{\delta}{\delta\alpha^a_\tau(x)} (\sigma^a_{x}[\alpha]W_\tau[\alpha]),$$

(2.16)
where $\chi^{ab}_{\alpha \beta}[\alpha]$ and $\sigma^a_{\alpha \beta}[\alpha]$ are given by Eqs. (2.5) and (2.9) respectively, in which $U$ and $U^\dagger$ represent Wilson lines in the adjoint representation, built from the classical field using Eq. (2.15) iteratively from $\tau_0$ to $\tau$. In Eq. (2.16), the functional derivatives are taken with respect to the random variable of the last rapidity bin, i.e. $\alpha_\tau$, which, according to the previous discussion, is the only one involved in going from $W_\tau[\alpha]$ to $W_{\tau + d\tau}[\alpha]$.

### 2.3 Path integral solutions and Langevin representations of the RG

As emerges from the discussion in the previous subsection, the two evolution equations in the $U$ and the $\alpha$ representations, summarize a random process by which the Wilson lines are built. This process is more directly visible in the $\alpha$ representation: there we see clearly that at each step in rapidity, the Wilson line is multiplied by a random matrix close to unity. The Wilson line at rapidity $\tau$ is given by

$$U^\dagger_\tau(x) = T \exp \left\{ i g \int_{\tau_0}^{\tau} d\eta \alpha_\eta(x) \right\} U^\dagger_0(x),$$

(2.17)

with, as already mentioned, the initial ordering in $x^-$ (cf. eq. (1.1)) replaced by an ordering in $\tau$. With this explicit representation at hand, one has already indications about the relation between the two approaches. Indeed, the Lie derivative which enters the equation for $Z[U]$, and which produces a “translation” of $U$, is equivalent when acting on $U$ such as (2.17) to:

$$\nabla^a U^\dagger = \frac{\delta U^\dagger}{\delta \alpha_a^\tau},$$

(2.18)

which is precisely the variation generated in the $\alpha$ representation. In fact one can write an explicit formula relating the two probability distributions $Z_\tau[U]$ and $W_\tau[\alpha]$:

$$Z_\tau[U] = \int [d\alpha] \delta_G(U^\dagger, T e^{i g \int_0^\tau d\eta \alpha_\eta}) W_\tau[\alpha].$$

(2.19)

This also immediately allows us to translate the equations for the two into each other, giving complete equivalence of the approaches.

Thus the basic problem of the evolution at small $x$ is that of a random walk in a space of two dimensional fields with values in $SU(N)$. The group nature of the variable introduces several complication in the random walk which will be dealt with in section 4. But the general structure of the problem can be uncovered without dealing with these complications. This will be done in the next section.
3 Preliminaries: Brownian motion in flat space

As recalled in the previous sections, the evolution with rapidity of scattering observables can be described by two distinct functional equations. Both can be interpreted as diffusion equations, but they have rather different structures and control the evolution of quite different object: a probability distribution on group valued fields in one case, a functional of gauge fields in the other. It is the purpose of this section to clarify the deep relation between these two formalisms by showing that they have counterparts in the description of ordinary Brownian motion.

We shall consider the Brownian motion of a particle in a $D$-dimensional Euclidean space. As we shall see, the probability to find the particle at point $x$ at time $\tau$, which we call $P(x, \tau)$, is related to $Z_\tau[U]$, with the coordinate $x$ corresponding to $U$. Similarly one may introduce a probability functional $W_\tau[v]$ over the random paths, where the paths are parametrized by the velocity $v$ of the particle at each time step. This functional $W_\tau[v]$ is the analog of $W_\tau[\alpha]$. It is related to $P(x, \tau)$ by an equation analogous to Eq. (2.19):

$$P(x, \tau) = \int [dv] \delta^{(D)}(x - x[v])W_\tau[v]$$  \hspace{1cm} (3.1)

with

$$x[v] = x_0 + \int_0^T d\tau' v(\tau')$$  \hspace{1cm} (3.2)

with $T$ an arbitrary time greater than $\tau$. Note that in the r.h.s. of Eq. (3.1) all the time-dependence is carried by the weight function $W_\tau[v]$. Of course, the paths which are physically relevant for computing $P(x, \tau)$ terminate at $\tau$: this information is encoded in the weight function $W_\tau[v]$.

The probability distribution $P(x, \tau)$ obeys a Fokker-Planck equation which, in analogy with Eq. (2.19) satisfied by $Z_\tau[U]$, we choose of the form

$$\dot{P}(x, \tau) = -HP(x, \tau)$$  \hspace{1cm} (3.3)

where the dot denotes the time derivative, $H$ is the following “hamiltonian”:

$$H \equiv -\partial^\alpha \left[ \partial^\beta \frac{1}{2} \chi^{\alpha\beta}(x) - \sigma^\alpha(x) \right].$$  \hspace{1cm} (3.4)

(with $\partial^\alpha = \partial/\partial x^\alpha$) and $\chi^{\alpha\beta}(x)$ is of the form:

$$\chi^{\alpha\beta}(x) \equiv e^{\alpha\gamma}(x)e^{\beta\gamma}(x).$$  \hspace{1cm} (3.5)

Thus $\chi^{\alpha\beta}(x)$ is symmetric and positive semi-definite, and therefore invertible. We shall also assume the following relation, analogous to Eq. (2.9):

$$\sigma^\alpha(x) = \frac{1}{2} \partial^\beta \chi^{\alpha\beta}(x)$$  \hspace{1cm} (3.6)
which, in particular, guarantees that $H$ in Eq. (3.4) is hermitian. The equation satisfied by $W_\tau$ will be found to be:

$$\partial_\tau W_\tau[v] = \frac{\delta}{\delta v_\tau^\alpha} \left[ \frac{1}{2} \frac{\delta}{\delta v_\tau^\beta} \left( \chi^{\alpha \beta}(x) W_\tau[v] \right) - \sigma^\alpha(x) W_\tau[v] \right]. \quad (3.7)$$

Note that the Fokker-Planck equation (3.3)–(3.4) for $P(x, \tau)$ follows indeed from Eq. (3.7) for $W_\tau[v]$ together with the relation (3.1), as can be checked by performing some integrations by parts with respect to $v$ in Eq. (3.1) and then using the identity:

$$\frac{\delta}{\delta v_\tau^\alpha} \delta(x - x[v]) = -\partial^\alpha \delta(x - x[v]). \quad (3.8)$$

Underlying all these equations is an elementary random process which controls the random walk. This elementary process is best seen at the level of a Langevin equation, from which in fact all equations can be derived. We shall make this Langevin equation the starting point of our discussion.

### 3.1 Langevin equation and path integral

Consider then a particle undergoing a random walk in a $D$-dimensional Euclidean space. Its coordinates $x^\alpha \,(\alpha, \beta = 1, \cdots, D)$ obey the following Langevin equation:

$$\dot{x}^\alpha(\tau) = \sigma^\alpha(x) + e^{\alpha \beta}(x) \nu^\beta(\tau), \quad (3.9)$$

where the dot denotes time derivative, and $\sigma^\alpha(x)$ and $e^{\alpha \beta}(x)$ are given regular functions of the coordinates characterizing the medium in which the particle propagates; finally, $\nu^\alpha(\tau)$ is a Gaussian white noise (see below). We shall also use the more compact notation:

$$\dot{x}(\tau) = \sigma(x) + e(x) \nu(\tau), \quad (3.10)$$

where $x$, $\sigma$ and $\nu$ are $D$-dimensional vectors and $e$ is a $D \times D$ matrix.

Strictly speaking, Eq. (3.9) is formal (the trajectory $x(\tau)$ of the random walk is not differentiable) and gets meaningful only with a discretization prescription. Here, we shall focus on the following discretization:

$$\frac{x_i - x_{i-1}}{\epsilon} = \sigma_{i-1} + e_{i-1} \nu_i, \quad (3.11)$$

and only briefly mention later other possible choices (see also Refs. [34, 35]). In Eq. (3.11), $\epsilon$ is the length of the time step, $x_i \equiv x(\tau_i)$ with $\tau_i = i\epsilon$, $i = 0, 1, \cdots, n$, and $x_0 \equiv x(\tau_0)$ is a given initial condition. We have also introduced the simplified notation $\sigma_i \equiv \sigma(x_i)$, $e_i \equiv e(x_i)$, to be used throughout.
The $\nu_i$'s are Gaussian random variables with zero expectation value and local 2-point correlations:

$$\langle \nu_i^\alpha \nu_j^\beta \rangle = \frac{1}{\epsilon} \delta^{\alpha \beta} \delta_{ij}. \quad (3.12)$$

In other words, the probability law for the noise variables $\nu_i$ is a normalized Gaussian distribution:

$$dP(\nu_i) \equiv \left( \frac{\epsilon}{2\pi} \right)^{D/2} e^{-\frac{\epsilon}{2} \nu_i \cdot \nu_i} d^D \nu_i, \quad (3.13)$$

the same for each time step, and for $n$ time steps it is the product of $n$ such distributions.

The particular discretization (3.11) of the Langevin equation is such that, for fixed $x_{i-1}$, $x_i$ is a random variable linearly related to $\nu_i$. This makes it easy to calculate the probability $P_\epsilon(x_i|x_{i-1})$ that the random walk brings the particle from position $x_{i-1}$ to $x_i$ in one time step, and this for any value of $\epsilon$. (We shall often refer to quantities such as $P_\epsilon(x_i|x_{i-1})$ as “elementary propagators”.) We have:

$$P_\epsilon(x_i|x_{i-1}) = \int dP(\nu_i) \delta^D(x_i - x_{i-1} - \epsilon(\sigma_{i-1} + e_{i-1}\nu_i)). \quad (3.14)$$

The integral over $\nu_i$ is easily performed, and yields:

$$P_\epsilon(x_i|x_{i-1}) = \frac{(\det \chi_{i-1})^{-1/2}}{(2\pi\epsilon)^{D/2}} e^{-\frac{1}{2\epsilon} (x_i - x_{i-1} - \epsilon \sigma_{i-1}) \chi_{i-1}^{-1} (x_i - x_{i-1} - \epsilon \sigma_{i-1})}, \quad (3.15)$$

where $\chi$ is obtained from $\epsilon$ via Eq. (3.5). Thus, for fixed $x_{i-1}$, the probability to find the particle at position $x_i$ is a gaussian centered around $x_i = x_{i-1} - \epsilon \sigma_{i-1}$, with a width $\epsilon \chi_{i-1}^{-1}$.

The coordinate $x_n$ of the particle after $n$ time step is also a random variable. The average of any function of $x_n$, say $f(x_n)$, can be obtained by first solving Eq. (3.11) for a given realization $\{\nu_1, \nu_2, \cdots, \nu_n\}$ of the random variables, and then averaging over all such realizations. Equivalently, one may first determine the probability density $P(x,\tau)$ to find the particle at point $x$ at time $\tau = n\epsilon$ knowing that it was at $x_0$ at time $\tau_0 = 0$. The average of $f(x)$ may then be obtained as

$$\langle f(x_n) \rangle = \langle f(x) \rangle_\tau = \int d^D x f(x) P(x,\tau) \quad (3.16)$$

The probability $P(x,\tau)$ is given by:

$$P(x,\tau) \equiv \int \prod_{i=1}^n dP(\nu_i) \delta^D(x - x_n[\nu]), \quad (3.17)$$
where \( x_n(\nu) \) is the solution of Eq. (3.11) for \( n \) time steps, and depends therefore on all the \( \nu_i \)'s with \( i \leq n \). Using repeatedly Eq. (3.14) one can easily show that the integral above can be rewritten as:

\[
P(x, \tau) = \int \prod_{i=1}^{n-1} d^Dx_i \ P_i(x|x_{n-1}) \prod_{i=1}^{n-1} P_i(x_i|x_{i-1}).
\]

(3.18)

By inserting in this expression the formula (3.15) for the elementary propagator, we finally get:

\[
P(x, \tau) = \int [dx] \delta(x - x_n) e^{-A[x]},
\]

(3.19)

with

\[
[dx] \equiv \prod_{i=1}^{n} \frac{d^Dx_i}{(2\pi\epsilon)^{D/2}} \prod_{i=1}^{n} (\det \chi_{i-1})^{-1/2},
\]

(3.20)

and

\[
A[x] \equiv \frac{\epsilon}{2} \sum_{i=1}^{n} \left( \frac{x_i - x_{i-1}}{\epsilon} - \sigma_{i-1} \right) \chi_{i-1}^{-1} \left( \frac{x_i - x_{i-1}}{\epsilon} - \sigma_{i-1} \right)
\]

(3.21)

The average \( \langle f(x) \rangle_{\tau} \) is then easily obtained:

\[
\langle f(x) \rangle_{\tau} = \int d^Dx \ f(x) \ P(x, \tau) = \int [dx] \ f(x_n) e^{-A[x]}
\]

(3.22)

An alternative expression for \( P(x, \tau) \) is obtained by integrating over auxiliary momentum variables which allows one to get rid of the determinant \( \det \chi_{i-1} \) in the measure and to reexpress \( A \) in terms of the matrix \( \chi_{i-1} \) rather than its inverse:

\[
P(x, \tau) = \int \prod_{j=1}^{n} \left[ d^Dx_j \frac{d^Dp_j}{(2\pi\epsilon)^D} \right] \delta(x - x_n) \exp \left\{ -\sum_{j=1}^{n} [ip_j \cdot (x_j - x_{j-1}) + \epsilon \mathcal{L}_j] \right\}
\]

(3.23)

with

\[
\mathcal{L}_j[x, p] \equiv \frac{1}{2} p_j \cdot \chi_{j-1} \cdot p_j - ip_j \cdot \sigma_{j-1}.
\]

(3.24)

The formulae (3.19) and (3.23) express the probability \( P(x, \tau) \) as a sum over paths specified by the set of coordinates \( \{x_0, x_1, \cdots x_{n-1}, x_n\} \), where \( x_0 \) is a given initial condition while all the other coordinates are random variables. A noteworthy feature of this expression is that, at step \( i \), the functions \( \chi(x) \) and \( \sigma(x) \) have to be evaluated at the point \( x_{i-1} \) attained in the preceding step. This property is of course intimately related to the special discretization of the Langevin equation, Eq. (3.11). One could naively expect that, in the limit \( \epsilon \to 0 \), \( x_{i-1} \) can be
replaced by $x_i$ in $\chi$ or $\sigma$. This is not so however, and it is important to keep
the arguments of $\chi$ and $\sigma$ as written in Eqs. (3.19) or (3.23). This is because
the typical paths contributing to the integral are such that $|x_i - x_{i-1}| \sim \sqrt{\epsilon}$.
Thus, the replacement of $x_{i-1}$ by $x_i$ within $\chi$ or $\sigma$ modifies the exponent in the
integrand of these formulae by terms of the same order of magnitude as the terms
that have been kept. Therefore, writing Eq. (3.23) in continuum notations, i.e., as

$$P(x, \tau) = \int_{x_0}^{x} \int \! [dx] \int \! [dp] \exp \left\{ \int_0^\tau \! d\eta \left[ -i p \cdot (\dot{x} - \sigma(x)) - \frac{1}{2} p \cdot \chi(x) \cdot p \right] \right\},$$

(3.25)
is ambiguous.

3.2 The Fokker-Planck equation

The path integral in Eq. (3.19) summarizes in a compact formula the averages
performed at each step of the Langevin process. In the limit $\epsilon \to 0$, it is possible
to obtain for $P(x, \tau)$ a differential equation which involves only the local prop-
erties of the medium in which the particle propagate, i.e. the functions $\chi$ and $\sigma$, and
with no explicit reference to the specific elementary random process. This
equation is a Fokker-Planck equation.

To obtain this equation we note first that Eq. (3.18) allow us to write:

$$P(x, \tau) = \int d^D x_{n-1} P_t(x|x_{n-1}) P(x_{n-1}, \tau_{n-1}).$$

(3.26)

In order to calculate the integral over $x_{n-1}$, we use the fact that the propagator
$P_t(x|x_{n-1})$, considered as a function of $x_{n-1}$, is a gaussian which, when $\epsilon \to 0$,
is strongly peaked near $x$ (see Eq. (3.15)); it follows that, when $\epsilon$ is small,
one can get a good approximation of the integrand of Eq. (3.26) by expanding
$P(x_{n-1}, \tau_{n-1})$ in the vicinity of $x$. To do that systematically, it is in fact easier
to go back to Eq. (3.14), and expand the $\delta$-constraint, keeping all terms that will
contribute to $O(\epsilon)$ after averaging over the noise:

$$\delta^D \left( x - x_{n-1} - \epsilon (\sigma_{n-1} + e_{n-1}\nu_n) \right) = e^{-\epsilon(\sigma_{n-1} + e_{n-1}\nu_n) \cdot \partial} \delta^D (x - x_{n-1})$$

$$\approx \left( 1 - (\sigma_{n-1} + e_{n-1}\nu) \cdot \partial + \frac{\epsilon}{2} (\sigma_{n-1} + e_{n-1}\nu) \cdot \partial^2 \right) \delta^D (x - x_{n-1}),$$

(3.27)

where $\partial_\alpha = \partial/\partial x^\alpha$. Averaging over $\nu_n$ leaves us with the expression

$$P_\epsilon(x|x_{n-1}) = \left( 1 - \epsilon \sigma_{n-1}^\alpha \partial_\alpha + \frac{\epsilon}{2} \chi_{n-1}^{\alpha\beta} \partial_\alpha \partial_\beta \right) \delta^D (x - x_{n-1}) + O(\epsilon^{3/2}).$$

(3.28)

The first term obtained after plugging this formula into Eq. (3.26) is simply:

$$P(x, \tau_{n-1}) = P(x, \tau - \epsilon) \approx P(x, \tau) - \epsilon \dot{P}(x, \tau),$$

(3.29)
where \( \dot{P}(x, \tau) \equiv \partial_\tau P(x, \tau) \). In the remaining terms, we use identities like

\[
\sigma^\alpha (x_{n-1}) \partial_\alpha \delta^D (x - x_{n-1}) P(x_{n-1}, \tau_{n-1}) = \partial_\alpha (\delta^D (x - x_{n-1}) \sigma^\alpha (x)) P(x, \tau_{n-1}),
\]

(3.30)

to deduce that, in the limit \( \epsilon \to 0 \), \( P(x, \tau) \) obeys the following equation:

\[
\dot{P}(x, \tau) = \partial^\alpha \left[ \partial^\beta \left( \frac{1}{2} \chi^{\alpha\beta} (x) P(x, \tau) \right) - \sigma^\alpha (x) P(x, \tau) \right].
\]

(3.31)

### 3.3 Alternative discretizations

At this point, one could reverse the logics and consider the Fokker-Planck equation as our starting point. Then, path integral representations for the solution \( P(x, \tau) \) could be constructed in a standard way, for instance by exploiting the formal analogy between Eq. (3.31) and the Schrödinger equation in imaginary time. Because the coefficients \( \chi(x) \) and \( \sigma(x) \) in this equation are coordinate dependent, different orderings of the derivatives within the Hamiltonian (3.4) will lead to different path-integral representations (possibly involving derivatives of \( \chi \) and \( \sigma \)), which become however equivalent in the limit \( \epsilon \to 0 \).

To illustrate this, let us consider three alternative expressions of the Hamiltonian (3.4) which are obtained by successively commuting the derivatives with \( \chi \), and using Eq. (3.6) to simplify the results:

\[
H = - \left( \frac{1}{2} \partial^\alpha \partial^\beta \chi^{\alpha\beta} (x) - \partial^\alpha \sigma^\alpha (x) \right) \quad (3.32a)
\]

\[
= - \left( \frac{1}{2} \chi^{\alpha\beta} (x) \partial^\alpha \partial^\beta + \sigma^\alpha (x) \partial^\alpha \right) \quad (3.32b)
\]

\[
= - \frac{1}{8} \left\{ \partial^\alpha, \left\{ \partial^\beta, \chi^{\alpha\beta} (x) \right\} \right\} + \frac{1}{8} (\partial^\alpha \partial^\beta \chi^{\alpha\beta} (x)). \quad (3.32c)
\]

The form in Eq. (3.32a) is what naturally came out of our original Langevin discretization, the other two represent different but clearly equivalent orderings of momentum and coordinate operators; the term involving a double anticommutator in Eq. (3.32c) is in the so-called Weyl-ordered form. By using the last two expressions, Eqs. (3.32b) and (3.32c), within a canonical construction of the path-integral, one obtains expressions similar to Eq. (3.23), but with \( \sigma \) and \( \chi \) evaluated at \( x_i \) (for Eq. (3.32b)) and, respectively, at the midpoint \( \bar{x}_i \equiv (x_i + x_{i-1})/2 \) (for Eq. (3.32c)). The only difference with respect to Eq. (3.23) consists in the replacement of \( \mathcal{L}_i \) in that equation by the alternative expressions

\[
\mathcal{L}_i = \frac{1}{2} \chi^{\alpha\beta} (x_i) p_i^\alpha p_i^\beta + ip_i^\alpha \sigma^\alpha (x_i) \quad (3.33a)
\]

\[
= \chi^{\alpha\beta} (\bar{x}_i) p_i^\alpha p_i^\beta + \frac{1}{8} \partial^\alpha \partial^\beta \chi^{\alpha\beta} (\bar{x}_i). \quad (3.33b)
\]

It can be verified that the expressions of \( P(x, \tau) \) obtained from Eqs. (3.33) satisfy indeed the Fokker-Planck equation (3.31) in the continuum limit.
Taking this one step further one can translate these differently discretized path integrals into differently discretized Langevin systems. Let us illustrate this for the case of Eq. (3.32b) with (3.33a) as its discretization prescription for the path integral. One can verify that its elementary propagator can be written as the following integral over an auxiliary random noise

\[ P_i(x_i|x_{i-1}) = \int d\mathcal{P}(\nu_i) \delta^D(x_i - x_{i-1} - \epsilon(-\sigma_i + e_i\nu_i)). \]  (3.34)

The \( \delta \) function in this equation imposes the proper discretization of the corresponding Langevin equation. Note that this has \( \sigma \) and \( e \) evaluated at \( x_i \) instead of \( x_{i-1} \) and the sign of the \( \sigma \) term is changed compared to Eq. (3.11). For the Weyl ordered form, the \( \sigma \) term would be completely absent but additional terms would arise from eliminating the contribution \( e^{-\epsilon^2\delta^{\alpha\beta}\chi^{\alpha\beta}} \) from the measure in order to recover the structure of Eq. (3.34) necessary to read off the Langevin equation.

From here it is evident that continuous notations, which are not able to keep track of details of coordinate dependence, are ambiguous, both for path-integrals and Langevin descriptions. The naive continuum notation for the Langevin equation corresponding to Eq. (3.34) would leave us with the sign of the \( \sigma \) term reversed compared to that in Eq. (3.14), while “forgetting” the mid point prescription of the Weyl ordered version would lead to yet another form. The same holds for path-integral expressions involving the differently ordered \( \mathcal{L} \) given above.

The one common denominator of all these equivalent, but differently discretized, path-integrals and Langevin systems is the fact that they correspond to one common Fokker-Planck equation which is perfectly meaningful in continuum notation.

3.4 The velocity representation

Until now we have focused on the \( x_n \) as the important random variable. There are situations however where one needs to calculate observables which depend on the entire path, and not on the end point alone. Thus, we are led to consider distribution functions over the paths.

To proceed with the construction of such a distribution, it is useful to perform a change of variables, and to better specify the paths.

Let us then consider as new random variables the velocities:

\[ v_i \equiv \frac{x_i - x_{i-1}}{\epsilon} = \sigma_{i-1} + e_{i-1}\nu_i, \]  (3.35)

for \( i = 1, 2, \ldots, N \), where \( T = N\epsilon \) is a fixed time much larger than all times of interest. The coordinates of a path are easily expressed in terms of these:

\[ x_{i-1} = x_0 + \epsilon \sum_{j=1}^{i-1} v_j \equiv x_{i-1}[v]. \]  (3.36)
Note that this change of variables is not a linear transformation since, as is evident from Eq. (3.36), the coefficients $\sigma_{i-1} = \sigma(x_{i-1})$ and $e_{i-1} = e(x_{i-1})$ in Eq. (3.35) for $v_i$ involve the velocities $v_j$ at all the steps $j = 1, 2, \ldots, i - 1$. However, $\sigma_{i-1}$ and $e_{i-1}$ do not depend upon $v_i$ itself, so that the corresponding Jacobien is rather simple, being the determinant of a triangular matrix. Accordingly, the probability distribution for the $v_i$'s is immediately obtained from that of the $x_i$'s in Eq. (3.19):

$$dP[v] = [dv] e^{-A[v]}, \quad (3.37)$$

with

$$[dv] \equiv \prod_{i=1}^{N} \left( \frac{\epsilon}{2\pi} \right)^{D/2} d^D v_i \left( \det \chi_{i-1} \right)^{-1/2}, \quad (3.38)$$

and

$$A[v] \equiv \frac{\epsilon}{2} \sum_{i=1}^{N} \left( v_i - \sigma_{i-1} \right) \chi_{i-1}^{-1} \left( v_i - \sigma_{i-1} \right). \quad (3.39)$$

We can write $dP[v]$ in the convenient factorized form:

$$dP[v] \equiv \prod_{i=1}^{N} d^D v_i P_i[v]. \quad (3.40)$$

Although this factorization property is a major advantage of the velocity representation, it should be kept in mind that the notation is deceiving since the different factors do not correspond to independent probabilities: the probability density $P_i[v]$ for the velocity at step $i$ depends also upon the velocities $v_1, v_2, \ldots, v_{i-1}$. Giving this, a convenient way to check the normalization condition $\int dP[v] = 1$ (or to do any other similar calculation) is to perform the integrations over the $v_i$'s in decreasing order of $i$, from $i = N$ down to $i = 1$ (notice that Eq. (3.37) is truly a Gaussian in $v_N$).

Let us turn now to the specification of the paths. The paths which contribute to $P(x, \tau)$ have a number $n = \tau/\epsilon$ of time steps, and the probability $w_n(v_1, \ldots, v_n)$ that a given such path is realized in the random walk is given by Eq. (3.40) in which $N$ is replaced by $n$, that is

$$w_n(v_1, \ldots, v_n) = \prod_{i=1}^{n} d^D v_i P_i[v]. \quad (3.41)$$

Now we want to be able to compare such distributions for different numbers of time steps, say $n$ and $n + 1$, and eventually obtain a differential equation in the limit of infinitesimal time steps. Clearly one cannot directly compare $w_n$ and $w_{n+1}$ since these have different numbers of variables. To allow such a comparison,
one possibility is to consider \( w_n \) as a function of \( n + 1 \) variables in which the last one, \( v_{n+1} \), is set equal to zero. A normalized distribution which achieves this goal is \( w_n(v_1, \cdots, v_n)\delta(v_{n+1}) \). More generally, we shall consider paths with a fixed number \( N \) of time steps such that the velocities associated with the last \( N - n \) steps vanish, while the velocities of the first \( n \) steps are random variables chosen with the distribution given above. That is, we define the following probability distribution:

\[
W_n[v] \equiv \prod_{i=1}^{n} P_i[v] \prod_{j=n+1}^{N} \delta^D(v_j). \tag{3.42}
\]

This distribution depends on \( n \), that is on time, and satisfies an evolution equation that we now determine. To this aim, we use Eq. (3.42) to write:

\[
W_n[v] - W_{n-1}[v] = \left( \prod_{i=1}^{n-1} P_i[v] \right) \left( \prod_{j=n+1}^{N} \delta^D(v_j) \right) \delta^D(v_n). \tag{3.43}
\]

Then we note that \( P_n[v] \), considered as a function of \( \epsilon v_n \), is a gaussian peaked at \( v_n = \sigma_{n-1} \), and proceed as for Eq. (3.28) to write:

\[
P_n[v] \approx \left( 1 - \sigma_{n-1}^2 \frac{\partial}{\partial v_n^\alpha} + \frac{1}{2\epsilon} \chi_{n-1}^{\alpha\beta} \frac{\partial^2}{\partial v_n^\alpha \partial v_n^\beta} \right) \delta^D(v_n). \tag{3.44}
\]

In writing such an expansion, we anticipated on the fact that the relevant variable is actually \( \epsilon v \) rather than \( v \); it is in the variable \( \epsilon v_n \) that the width of the gaussian is of order \( \epsilon \); furthermore, as we shall see shortly, \( \epsilon v_n \) is the variable which is needed to turn \( W_n[v] \) into a functional of the continuous function \( v(t) \).

Using the expansion (3.44) in Eq. (3.43) one easily obtains:

\[
W_n[v] - W_{n-1}[v] = \left( \frac{1}{2\epsilon} \frac{\partial^2}{\partial v_n^\alpha \partial v_n^\beta} \chi_{n-1}^{\alpha\beta} - \frac{\partial}{\partial v_n^\alpha} \epsilon \sigma_{n-1}^{\alpha} \right) W_{n-1}[v]. \tag{3.45}
\]

Recall that \( \chi_{n-1} \) and \( \sigma_{n-1} \) involve only the velocities \( v_i \) with \( i \leq n - 1 \), cf. Eq. (3.36). However, since the function \( W_{n-1}[v] \) on the r.h.s. is proportional to \( \prod_{i \geq n} \delta^D(v_i) \) one may replace \( x_{n-1} \) by \( x[v] \) in \( \chi_{n-1} \) and \( \sigma_{n-1} \), with

\[
x[v] \equiv x_0 + \epsilon \sum_{i=1}^{N} v_i, \tag{3.46}
\]

and consequently \( \chi_{n-1} \) by \( \chi(x) \) and \( \sigma_{n-1} \) by \( \sigma(x) \). The continuum limit \( \epsilon \to 0 \), \( N \to \infty \), and \( n \to \infty \) with \( n\epsilon = \tau \) and \( N\epsilon = T \) fixed, can then be taken. In doing so, we use the formula

\[
\frac{1}{\epsilon} \frac{\partial}{\partial v_n} \to \frac{\delta}{\delta v(\tau)}, \tag{3.47}
\]
which transforms the partial derivative w.r.t. \( v_n \) into a functional derivatives w.r.t. \( v(\tau) \). We thus obtain the following functional differential evolution equation for \( W_\tau[v] \):

\[
\frac{\partial}{\partial \tau} W_\tau[v] = \frac{\delta}{\delta v_\alpha^{\tau}} \left[ \frac{1}{2} \frac{\delta}{\delta v_\beta^{\tau}} \left( \chi^{\alpha\beta}(x) W_\tau[v] \right) - \sigma^{\alpha}(x) W_\tau[v] \right],
\]

where \( v_\tau \equiv v(\tau) \) and \( x \equiv x[v] \) given by Eq. (3.2).

Once the probability distribution over the paths \( W_\tau[v] \) is known, it is of course easy to calculate probability \( P(x, \tau) \) according to Eq. (3.1). The only quantity which depends on \( \tau \) on the r.h.s. of this equation is \( W_\tau \). Taking the time derivative and using the equation (3.48), one easily reconstructs the Fokker-Planck equation (3.31) satisfied by \( P(x, \tau) \).

One may also rewrite the probability distribution (3.42) in the alternative discretization in which \( \sigma \) and \( \chi \) are evaluated at \( x_i \), as in Eq. (3.33a). As in Sect. 3.3, this requires using the relation (3.6) between \( \chi \) and \( \sigma \). One thus obtains:

\[
W_n[v] = \int \prod_{i=1}^N \left( \frac{\epsilon}{2\pi} \right)^D d^D p_i \exp \left\{ -i \epsilon \sum_{i=1}^N p_i \cdot v_i \right\} \\
\times \exp \left\{ -\epsilon \sum_{i=1}^n \left( p_i \cdot \chi(x_i) \cdot p_i + 2i p_i \cdot \sigma(x_i) \right) \right\},
\]

which can be recast in continuum notations as follows:

\[
W_\tau[v] = \int [d\eta] \exp \left\{ -i \int_0^\infty d\eta \cdot p_\eta \cdot v_\eta \right\} \\
\times \exp \left\{ -\frac{1}{2} \int_0^\tau d\eta \left( p_\eta \cdot \chi(x_\eta) \cdot p_\eta + 2i p_\eta \cdot \sigma(x_\eta) \right) \right\}
\]

with \( x_\eta = x_\eta[v] \) given by the continuous form of Eq. (3.36):

\[
x_\eta[v] = x_0 + \int_0^\eta d\tau' v(\tau')
\]

Although, for the reasons discussed in relation with Eqs. (3.25) and (3.34) above, such continuous notations can be ambiguous, they can be convenient. For instance, when used with care, they allow simple formal operations, like functional differentiations. In particular, it is easy to verify in this way that Eq. (3.50) satisfies equation (3.48). To this aim, it is convenient to use the alternative form of Eq. (3.48) which is obtained after commuting the two functional derivatives in its r.h.s. through \( \chi \) (i.e., the analog of Eq. (3.32b)). In this check, the following identity is also useful:

\[
\frac{\delta}{\delta v_\alpha^{\tau_1}} \int_0^\tau d\eta F(x_\eta) = \theta(\tau - \tau_1) \int_{\tau_1}^\tau d\eta \partial^\alpha F(x_\eta),
\]

which vanishes for \( \tau_1 \geq \tau \).
4 Brownian motion in the space of Wilson lines

After all these preparations, we finally come to the study of the Brownian motion in the space of Wilson lines, which is our main interest here. The forthcoming discussion will in fact follow closely that in Sect. 3: as we shall see, the evolution equations (2.4) for $Z_{\tau}[U]$ and (2.16) for $W_{\tau}[\alpha]$, are the close generalizations of Eqs. (3.31) and (3.48), respectively.

The Wilson line $U(x)$ is a field, with the coordinate $x$ leaving in the transverse plane. In what follows, we shall often omit these transverse coordinates in intermediate formulae, in order to simplify the notation and focus on the true difficulties. These coordinates will be reinserted in the final formulae, or whenever ambiguities may arise from omitting them. Similarly, we shall alternate between discrete and continuous time: most of the formulae are established for discrete times, and while many of them have a straightforward continuous extension, one should keep in mind that the use of continuous notations may hide ambiguities, as emphasized in the previous section.

4.1 Langevin equation and the $\alpha$-representation

The physical random variable in the evolution is the elementary contribution $\alpha_{\tau}(x)$ to the classical field of the target arising from integrating out gauge field fluctuations in the the rapidity strip $[\tau, \tau + d\tau]$. Such a contribution changes the Wilson line according to ($\tau = n\epsilon, \alpha_{\tau}d\tau = \alpha_{n}\epsilon$):

$$U_i = U_{i-1}e^{-i\alpha_{\epsilon}^{\alpha}} , \quad U_i^{\dagger} = e^{i\alpha_{\epsilon}^{\alpha}}U_{i-1}^{\dagger}. \quad (4.1)$$

A path in the space of $U$ fields is thus defined as the set of values $\{U_n^{\dagger}, U_{n-1}^{\dagger}, \cdots, U_0^{\dagger}\}$, with:

$$U_n^{\dagger} = e^{i\alpha_n}U_{n-1}^{\dagger} = \cdots = e^{i\alpha_n}e^{i\alpha_{n-1}} \cdots e^{i\alpha_1}U_0^{\dagger}, \quad (4.2)$$

In continuous notations, and with spatial coordinates included, a path is given by Eq. (2.17) in Sect. 2.3. Note that in the previous formulae, and in most formulae to come in this section, we set $g = 1$ for simplicity. To reintroduce the coupling constant $g$ in the final results, it is enough to reinsert it in the exponent of the Wilson lines.

The random variables $\alpha_{\epsilon}^{\alpha}(x)$ at step $i$ are selected according to a Gaussian probability distribution with the following characteristics:

$$\langle \alpha_{i}^{\alpha}(x) \rangle = \sigma_{i-1}^{\alpha}(x), \quad \langle \alpha_{i}^{\alpha}(x) \alpha_{i}^{\beta}(y) \rangle = \frac{1}{\epsilon} \chi_{i-1}^{\alpha\beta}(x, y), \quad (4.3)$$

These quantities are calculated in terms of the Wilson lines obtained at the step $i - 1$, according to Eqs. (2.5) and (2.9) in Sect. 2.
Equivalently, the variables \( \alpha_i^a(x) \) may be obtained by solving the Langevin equation (with \( \epsilon_{ab,l}^{i-1} \) given by Eq. (2.8) with \( U \equiv U_{i-1} \)):

\[
\alpha_i^a(x) = \sigma_i^a(x) + \int_x c_{i-1}^{ab,l}(x,z) \nu_i^{b,l}(z),
\]

(4.4)

where \( \nu_i^a(x) \) is an auxiliary Gaussian white noise:

\[
\langle \nu_i^{a,l}(x) \rangle = 0, \quad \langle \nu_i^{a,l}(x) \nu_j^{b,k}(y) \rangle = \frac{1}{\epsilon} \delta_{ij} \delta^{ab} \delta^{lk} \delta^{(2)}(x-y),
\]

(4.5)

By averaging over the noise the solution of Eq. (4.4) one indeed recovers the expectation value and correlator of the random variables \( \alpha_i^a(x) \) as given by Eq. (4.3).

The Langevin equation (4.4) plays the same role for the random walk on the group manifold as Eq. (3.35) for the random walk in a flat space, with the gauge field \( \alpha_i^a(x) \) playing here the role of the velocity \( v_i \). But in contrast to what happens in the simple example of Sec. 3.4, the relation between the “coordinate” \( U_i \) and the “velocity” \( \alpha_i \) is here non trivial, because of the non-trivial geometry of the group manifold. However, as long as we characterize the random walk by small displacements in the tangent space, the special geometry of the group manifold plays essentially no role (it only enters through the dependence of \( \sigma \) and \( \chi \) on the Wilson lines). The mathematical analogy with ordinary Brownian motion in the velocity representation is then complete, and the formulae obtained in Sect. 3.4 have an immediate translation to the present problem.

Thus the weight function \( W_n[\alpha] \) is given by the analog of Eqs. (3.42) and (3.40), namely:

\[
W_n[\alpha] = \prod_{i=1}^n P_i[\alpha] \prod_{j=n+1}^N \delta[\alpha_j],
\]

(4.6)

where

\[
P_i[\alpha] \equiv \left( \frac{\epsilon}{2\pi} \right)^{D/2} (\det \chi_{i-1})^{-1/2} e^{-A_i[\alpha]},
\]

(4.7)

\[
A_i[\alpha] = \int_{x,y} \left( \alpha_i - \sigma_{i-1} \right)_x^a \chi_{i-1}^{ab} \left( \alpha_i - \sigma_{i-1} \right)_y^b,
\]

(4.8)

and

\[
\delta[\alpha_i] \equiv \prod_a \prod_x \delta(\alpha_i^a(x)).
\]

(4.9)

We assume here a suitable discretization of the transverse plane with lattice points \( x \). In Eq. (4.7), \( D \) is the total number of degrees of freedom, \( D = (N_c^2 - 1)N_x \) (i.e., the number of colors times the number of points in the transverse lattice). Recall that \( \sigma_{i-1} \), \( \chi_{i-1} \) are functionals of \( U_{i-1} \) so that \( P_i[\alpha] \) depend on all the
variables $\alpha_1, \alpha_2, \cdots, \alpha_{i-1}$ (c.f. Eq. (4.2)). It is readily verified, by proceeding as indicated after Eq. (3.40), that the weight function (4.6) is properly normalized:

$$\int [d\alpha] \, W_n[\alpha] = 1, \quad [d\alpha] \equiv \prod_i \prod_a \prod x \, d\alpha_i^a(x).$$

Since the inverse matrix $\chi$ is not immediately available, it is convenient to introduce an extra integration over auxiliary momenta $\pi_i^a(x)$ and transform Eq. (4.6) into

$$W_n[\alpha] = \int [d\pi] \exp \left\{ -i \sum_{i=1}^N \int_x \pi_i^a \alpha_i^a \right\} \times \exp \left\{ -\epsilon \sum_{i=1}^n \left( \frac{1}{2} \int_{x,y} \pi_i^a \chi_{x,y}^{ab} \pi_i^b - i \int_x \pi_i^a \sigma_i^a \right) \right\}.$$  (4.11)

where $[d\pi] = \prod_{i=1}^N \prod_a \prod_x (\epsilon/2\pi) \, d\pi_i^a(x)$. It can then be easily verified by direct differentiation that $W_\tau[\alpha]$ is solution of the following functional evolution equation which is the immediate translation of Eq. (3.48):

$$\partial_\tau W_\tau[\alpha] = \frac{\delta}{\delta \alpha_i^a(x)} \left[ \frac{1}{2} \frac{\delta}{\delta \alpha_j^b(y)} \left( \chi_{x,y}^{ab}[\alpha] W_\tau[\alpha] \right) - \sigma_i^a[\alpha] W_\tau[\alpha] \right].$$  (4.12)

As expected, this is the same as the renormalization group equation in the $\alpha$-representation, Eq. (2.16).

### 4.2 The Fokker-Planck equation in the space of $U$-fields

Let us now define $Z_n[U]$ as the probability density for the random variable to take the value $U$ at time $\tau = n\epsilon$, knowing that it is $U_0$ at time 0. This is the analog of the probability density $P(x,t)$ introduced in Sect. 3. The normalization and initial conditions are written as

$$\int d\mu(U) \, Z_\tau[U] = 1 \quad Z_0[U] = \delta_G(U,U_0),$$  (4.13)

where $d\mu(U)$ is the Haar measure and $\delta_G(U,V)$ is the group-invariant $\delta$-function (see Appendix).

According to the discussion in Sect. 2.3, the probability $Z_n[U]$ can be calculated from the weight function $W_n[\alpha]$, via Eq. (2.19) (for more transparency, we suppress space coordinates):

$$Z_n[U] = \int [d\alpha] \, W_n[\alpha] \delta_G(U^\dagger, U_n^\dagger[\alpha]),$$  (4.14)

where $U_n^\dagger$ is given in terms of the $\alpha_i$’s by Eq. (4.2). (In fact, one could as well replace $U_n^\dagger$ by $U_N^\dagger$ in the r.h.s. of this equation since $W_n[\alpha]$ ensures that $\alpha_j = 0$ for any $j > n$). In continuous notations, Eq. (4.14) becomes Eq. (2.19).
By exploiting the factorized structure of \(W_n[U]\), Eq. (4.6), one can verify that \(Z_n[U]\) satisfies a recurrence formula similar to Eq. (3.26), that is,

\[
Z_n[U] = \int d\mu(U_{n-1}) \, Z_i[U|U_{n-1}] \, Z_{n-1}[U_{n-1}],
\]

(4.15)

and the elementary propagator \(Z_e[U|U_{n-1}]\) is given by an equation analogous to Eq. (4.14):

\[
Z_e[U_i|U_{i-1}] = \int [d\alpha_i] \, P_i[\alpha] \, \delta_G(U_i^\uparrow, e^{i\epsilon\alpha_i} U_i^{\uparrow 1}_{i-1}).
\]

(4.16)

Note that for fixed \(U_{i-1}\), the probability \(P_i[\alpha]\) is in fact a function of \(\alpha_i\) given by Eqs. (4.7) and (4.8). We expect \(Z_e[U_i|U_{i-1}]\) to differ from unity by terms which vanishes as \(\epsilon \to 0\). In fact the deviation is of order \(\epsilon\), as we now show. To proceed, we note that the weight function \(P_i[\alpha]\) is a Gaussian of width \(\epsilon\) in the variable \(\epsilon\alpha\), and we perform an expansion of the group \(\delta\)-function in Eq. (4.16) up to quadratic order in \(\epsilon\alpha_i \sim \sqrt{\epsilon}\). To perform this expansion, we apply the identity (A.7) derived in Appendix and obtain, in direct generalization of Eq. (3.27),

\[
\delta_G(U_1^\uparrow, e^{i\epsilon\alpha_i} U_1^{\uparrow 1}_{i-1}) = \delta_G(e^{-i\epsilon\alpha_i} U_1^\uparrow, U_1^{\uparrow 1}_{i-1}) = e^{-i\epsilon\alpha_i \nabla a} \delta_G(U_1^\uparrow, U_1^{\uparrow 1}_{i-1})
\]

(4.17)

where the Lie derivatives act on \(U_i\) (not on \(U_{i-1}\)). By inserting this expression in Eq. (4.16) and performing the integral over \(\alpha_i\), we finally arrive at

\[
Z_e[U|U_{n-1}] = \left\{ 1 + \epsilon \left( -\sigma_{n-1}^a \nabla^a + \frac{1}{2} \chi_{n-1}^{ab} \nabla^a \nabla^b \right) \right\} \delta_G(U_1^\uparrow, U_1^{\uparrow 1}_{n-1}) + \mathcal{O}(\epsilon^{3/2}).
\]

(4.18)

The steps needed to obtain the equation satisfied by \(Z_\tau[U]\) in the continuum limit are now identical to those leading to the Fokker-Planck equation (3.31). Reinstating coordinate dependence, one gets

\[
\partial_\tau Z_\tau[U] = \frac{1}{2} \nabla_x^a \nabla_y^b \chi_{xy}^{ab}[U] Z_\tau[U] - \nabla_x^a \sigma_x^a[U] Z_\tau[U],
\]

(4.19)

which is Eq. (2.4).

On a more formal level, this equation could have been simply obtained by taking the time derivative of Eq. (2.19) relating \(Z_\tau[U]\) and \(W_\tau[\alpha]\). Noting that only \(W_\tau\) depends on time in the r.h.s. of this equation, and using Eq. (4.12) for \(\partial_\tau W_\tau\), one obtains:

\[
\partial_\tau Z_\tau[U] = \int [d\alpha] \, \partial_\tau W_\tau \, \delta_G(U_\uparrow, T e^{i \int_0^\tau d\eta \alpha_\eta} U_0^\uparrow)
\]

\[
= \int [d\alpha] \, W_\tau[\alpha] \left\{ \frac{1}{2} \chi_{xy}^{ab}[\alpha] \frac{\delta^2}{\delta \alpha_x^a(x) \delta \alpha_y^b(y)} + \sigma_x^a[\alpha] \frac{\delta}{\delta \alpha_x^a(x)} \right\} \delta_G(U_\uparrow, T e^{i \int_0^\tau d\eta \alpha_\eta} U_0^\uparrow)
\]

(4.20)
In the second line, we have performed integrations by parts w.r.t. $\alpha$, and then restricted the integral in the exponent of the Wilson line to $\eta \leq \tau$, taking advantage that the weight function $W_\tau[\alpha]$, imposes $\alpha^a_0 = 0$ for any $\eta > \tau$ (c.f. Eq. (4.6)). We then observe that the functional derivatives acting on this particular Wilson line have the same effect as the Lie derivative $\nabla^a$. Setting $V^\dagger \equiv T \exp(i \int_0^\tau d\eta \alpha^a_\eta)U_0^\dagger$, we have indeed (see Eq. (A.6)): 

$$\frac{\delta}{\delta \alpha^a_\tau(y)} V^\dagger(x) = it^a V^\dagger(y) \delta^{(2)}(x - y) = \nabla^a_y V^\dagger(x). \quad (4.21)$$

This allows us to trade the functional derivatives in Eq. (4.20) for Lie derivatives, e.g.,

$$\frac{\delta}{\delta \alpha^a_\tau} \delta_G(U^\dagger, T e^{i \int_0^\tau d\eta \alpha^a_\eta} U_0^\dagger) = \nabla^a_v \delta_G(U^\dagger, V^\dagger) = -\nabla^a_U \left( U^\dagger, T e^{i \int_0^\tau d\eta \alpha^a_\eta} U_0^\dagger \right), \quad (4.22)$$

and thus to immediately recover Eq. (4.19).

### 4.3 Path integrals in the space of $U$-fields

We now return to Eq. (4.14) and use the known expression (4.11) for $W_n[\alpha]$ to derive a path-integral representation for $Z_n[U]$ directly in terms of the $U_i$'s, rather than the $\alpha_i$'s.

We proceed as usual, divide the rapidity interval $\tau$ into $n$ small elements of length $\epsilon$, and use repeatedly Eq. (4.15) to deduce that

$$Z_\tau[U] = \int \prod_{i=1}^{n-1} d\mu(U_i) Z_v[U|U_{n-1}] Z_v[U_{n-1}|U_{n-2}] \cdots Z_v[U_1|U_0]. \quad (4.23)$$

Next, we define

$$U_i^\dagger U_{i-1} \equiv e^{i \omega_i^a t^a}, \quad i \epsilon \omega_i^a = 2 \text{tr} t^a \ln(U_i^\dagger U_{i-1}). \quad (4.24)$$

The $\omega_i$'s are uniquely determined as long as the matrix $U_i^\dagger U_{i-1}$ is sufficiently close to the unit matrix, which will be the case.

Consider then the elementary propagator $Z_v[U_i|U_{i-1}]$ as given by Eq. (4.16). Using the parametrization (4.24), one can express the group $\delta$-function in Eq. (4.16) as an ordinary $\delta$-function for the random variable $\alpha^a_i$ over which we need to integrate (see Eq. (A.19) in Appendix):

$$\mu(\epsilon \omega_i) \delta_G(e^{i \epsilon \omega_i^a t^a}, e^{i \epsilon \alpha_i^a t^a}) = \delta(\epsilon \omega_i - \epsilon \alpha_i). \quad (4.25)$$

It is then straightforward to perform the integral over $\alpha_i^a$ in Eq. (4.16), and obtain:

$$Z_v[U_i|U_{i-1}] = \frac{(\det \chi_{i-1})^{-1/2}}{(2\pi \epsilon)^{D/2}} \prod_x \mu(\epsilon \omega_i(x))^{-1} \times \exp \left\{ -\frac{\epsilon}{2} \int_{x,y} (\omega_i - \sigma_i - 1)_x [\chi_{i-1}]^{ab} (\omega_i - \sigma_i - 1)_y \right\}. \quad (4.26)$$
where it is understood that $\omega^a_i(x)$ is a function of $U_i(x)$ and $U_{i-1}(x)$ obtained by inverting the relations (4.24). Eq. (4.26) is the analog of Eq. (3.15), but it is actually more complicated than the latter because $\omega_i$ is non-linearly related to the neighboring group elements $U_i$ and $U_{i-1}$ (cf. Eq. (4.24)), as required by the geometry of the group manifold.

Eq. (4.26) implies that, typically, $\varepsilon \omega^a_i \sim \sqrt{\varepsilon}$, which is consistent with Eq. (4.25) and the previous estimates for $\alpha_i$ (cf. Eq. (4.3)). This shows that, in order to control Eq. (4.26) to the accuracy of interest (i.e., up to and including terms of order $\varepsilon$), it is enough to evaluate the measure factor $\mu(\varepsilon \omega_i)$ to quadratic order in $\varepsilon \omega$. An explicit calculation to be presented in Appendix A yields (cf. Eq. (A.14)):

$$\left[\mu(\varepsilon \omega_i)\right]^{-1} = 1 + \frac{e^2 N_c}{24} \omega^a_i \omega^a_i + O(\varepsilon^4) = e^{2 \frac{e^2 N_c}{24} \omega^a_i \omega^a_i} + O(\varepsilon^3) = 1 + \frac{e^2 N_c}{24} \omega^a_i \omega^a_i + O(\varepsilon^3/2). \quad (4.27)$$

We are now in a position to write $Z_\tau[U]$ as a path integral:

$$Z_\tau[U] = \int_{U_0}^U [d\mu(U)] \exp\{-A[U]\}, \quad (4.28)$$

where

$$[d\mu(U)] \equiv \prod_{i=1}^{n-1} (\det \chi_i^{-1})^{-1/2} (2\pi\epsilon)^{-D/2} [\mu(\varepsilon \omega_i)]^{-1} d\mu(U_i) \quad (4.29)$$

and

$$A[U] = -\frac{\epsilon}{2} \sum_{i=1}^n \int_{x,y} (\omega_i - \sigma_{i-1})^a_x |\chi_i^{-1}^a_{xy}| (\omega_i - \sigma_{i-1})^b_y. \quad (4.30)$$

Again, one can trade the matrix $\chi_i^{-1}$ for $\chi$ itself by introducing an additional integration over auxiliary momentum variables:

$$Z_\tau[U] = \int \prod_{i=1}^{n-1} d\mu(U_i) \int \prod_{i=1}^n d\pi_i e^{-\epsilon \sum_{i=1}^n \mathcal{L}_i[\pi,U]}, \quad (4.31)$$

with

$$\mathcal{L}_i[\pi,U] \equiv -\frac{\epsilon N_c}{24} \int \omega_i^a \omega_i^a + i \int \pi_i^a (\omega_i - \sigma_{i-1})^a + \frac{1}{2} \int \chi_i^{ab} \pi_i^a \pi_i^b, \quad (4.32)$$

and we have also used Eq. (4.27). To reintroduce the coupling constant $g$ in the previous formulae, it is enough to insert a factor $1/g$ in the r.h.s. of Eq. (4.24) for $\omega_i^a$.

The role of the measure term (i.e., the first term in the r.h.s. of Eq. (4.32)) should be quite clear with the following remark: Assume that we decide to evaluate the path-integral (4.28) by using the “relative angles” $\omega_i$ (in the sense of Eq. (4.24)) as the integration variables. Then, as obvious from Eq. (4.29), the measure terms simply drop out: $[\mu(\varepsilon \omega_i)]^{-1} d\mu(U_i) = d\omega_i$, and Eq. (4.28) reduces to Eq. (4.14) with $W_n[\alpha]$ of Eq. (4.6).
5 Conclusions

We have established the relationships between the two forms of the renormalization group equation towards small $x$ which are available in the literature. These have been obtained in different approaches, and, in spite of some formal similitudes, they rely on different mathematical structures: the $U$- and the $\alpha$-representations of this paper. What we have shown here is that these two representations are equivalent and describe the same basic statistical process: a random walk in the space of Wilson lines, with $\ln 1/x$ playing the role of “time”, $\alpha$ that of the “velocity”, and $U$ being the “position” on the group manifold.

The Langevin description of this random walk provides a natural bridge between the two representations: At step $i$, the gauge field $\alpha_i$ is randomly chosen according to the gaussian probability distribution (4.3), then used to update the SU($N$) group element $U_i$, cf. eq. (4.1), which is in turn used to compute the probability law for selecting the “velocity” $\alpha_{i+1}$ at step $i + 1$, and so on. This Langevin process provides an algorithm for the quantum evolution and stays at the basis of a numerical study of the renormalization group equations which is currently in progress [28]. Note that the results of such Langevin simulations will not only give access to the $x$ dependence of cross sections like the $\gamma^*A$ cross sections of Eq. (1.2), they also provide the basis for an extension to the determination of initial conditions in AA collisions with a calculated $x$ dependence along the lines of [38, 39].

In addition to its suitability for practical implementations, the Langevin formulation brings also important conceptual clarifications, as it most prominently exhibits the statistical nature of the underlying physics: The renormalization group at small $x$ is driven by successively integrating out gluonic fluctuations within a small slice in $\ln 1/x$ around a static (and typically large) background field, which represents the quantum modes integrated in previous steps, and which are effectively “frozen”. The ensuing coarse graining in the longitudinal direction is accompanied by a loss of quantum coherence, allowing eventually a description in terms of a classical stochastic process.

The particular structure of the stochastic equations which emerge from the renormalization group has allowed us to avoid discretization ambiguities which generally arise for random walks in inhomogeneous media. We have utilized the so-called Ito discretization, which allows one to construct the new variables in a “causal” manner from the knowledge of the old ones, and we have verified that, with this choice, the Langevin description reproduces indeed the expected renormalization group equations — which here emerge as Fokker-Planck equations — in the continuum limit. Once things are properly formulated within one discretization, other prescriptions can be used as well, and we have briefly elaborated on some alternatives.

The discretization that we have used has also the virtue to greatly facilitate the construction of path-integral representations for the solutions to the Fokker-Planck equations. Such path-integrals can be the starting point for alternative
numerical implementations. It would be interesting in this context to explore the possibility to rewrite this path-integral in a form which is local in the transverse coordinates, possibly after introducing auxiliary fields, as in Ref. [29]. But path-integrals can be also useful for analytical studies. and could give us more insight into the qualitative features of the non-linear physics at small $x$. Although to some extent formal, the explicit path-integral solutions that we have obtained have the merit to exhibit important structural properties, like non-locality (in both longitudinal and transverse coordinates), non-linearity, and colour structure. Less formal analytic results could be obtained by further evaluating the path integrals, e.g., via saddle-point or mean-field approximations. In particular, the self-consistent mean-field approximations discussed in Ref. [8] can be easily reformulated within the present approach.

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A Calculations on Lie group: useful formulae

We collect in this appendix a few identities which are useful in calculations on group manifolds (see, e.g., Refs. [36, 37] for more details).

We consider SU(N) matrices $U$. These depend on $N^2 - 1$ real parameters $\omega^a$, and can be written in the exponential form as:

$$U(\omega) = e^{-i\omega^a t^a} \equiv e^{-i\omega} \quad U^\dagger(\omega) = e^{i\omega}$$

where the $t^a$'s are the infinitesimal generators normalized as $\text{Tr}(t^a t^b) = d_R \delta^{ab}$, with $d_R$ the dimension of the representation considered. When needed, we shall distinguish the adjoint representation by a tilde: $\tilde{U}(\omega) = e^{-i\tilde{\omega}}$.

Consider first the case where these parameters depend on a single variable, the “time” $\tau$. When $\tau$ varies, the matrix $U_\tau$ describes a curve on the group manifold. Consider a variation of the matrix $U$ when $\tau$ increases by a small amount $d\tau$. We can write: $U_{\tau+d\tau} = U_\tau + dU_\tau$, and note that $U_\tau^\dagger dU_\tau = U_\tau^\dagger U_{\tau+d\tau} - 1$ is an element of the Lie algebra that we can write as $U_\tau^\dagger dU_\tau = -iT^a \alpha^a d\tau$ where $\alpha^a(\tau)$ are the
components of a vector tangent to the curve. Clearly \( \text{d}U^\dagger = -iU^\tau T^a \alpha^a \text{d}\tau \), or equivalently

\[
\text{d}U^\dagger = i \alpha^a T^a U^\dagger \text{d}\tau
\]

which allows us to write \( U^\dagger \) as a time ordered exponential:

\[
U^\dagger = T \exp \{ i \int_0^\tau \text{d}\eta \alpha^a(\eta) T^a \} U^\dagger_0
\]

(A.2)

Note that it is the matrix \( U^\dagger \), rather than \( U \) itself, which enters our discussion in the main text. Note also that \( \alpha^a \neq d\omega^a/d\tau \).

Consider next an arbitrary function of \( U^\dagger \). We may write:

\[
\frac{\text{d}f(U^\dagger)}{\text{d}\tau} = (i\alpha^a T^a U^\dagger)_{ij} \frac{\delta}{\delta U^\dagger_{ij}} f[U] \equiv \alpha^a \nabla^a f(U^\dagger),
\]

(A.4)

and therefore

\[
f(U^\dagger) = T \exp \left\{ \int_0^\tau \text{d}\eta \alpha^a(\eta) \nabla^a \right\} f(U^\dagger_0)
\]

(A.5)

The operators \( \nabla^a \) introduced above, commonly referred to as Lie derivatives, satisfy the following relations:

\[
\nabla^a U = -i U T^a , \quad \nabla^a U^\dagger = i T^a U^\dagger , \quad [\nabla^a, \nabla^b] = f^{abc} \nabla^c ,
\]

(A.6)

and form therefore a representation of the Lie algebra.

If one chooses \( \alpha \) to be constant in eqs. (A.3) and (A.5), one gets the following useful formula:

\[
f(e^{i\alpha^a T^a} U^\dagger_0) = e^{\alpha^a \nabla^a} f(U^\dagger_0)
\]

(A.7)

On the group manifold, one may define a metric by \( \text{d}s^2 = \text{Tr} (\text{d}U \text{d}U^\dagger) \) or, in terms of the group parameters \( \omega^a \):

\[
\text{d}s^2 = g_{ab} \omega^a \omega^b = \text{Tr} \left( \frac{\text{d}U}{\text{d}\omega^a} \frac{\text{d}U^\dagger}{\text{d}\omega^b} \right) = (L L^t)_{ab} ,
\]

(A.8)

where we have set

\[
U^\dagger \frac{\partial}{\partial \omega^a} U = -i L_{ab}(\omega) T^b ,
\]

(A.9)

and \( L^t_{ab} = L_{ba} \). From this one gets easily the Haar invariant measure

\[
\text{d}\mu(U) = \mu(\omega) \Pi_a d\omega_a \quad \mu(\omega) = \sqrt{|\text{det} g(\omega)|} .
\]

(A.10)
To calculate $L_{ab}$ we use:

$$U^\dagger \frac{\partial U}{\partial \omega^a} = \int_0^1 ds e^{is\omega} (-iT^a) e^{-is\omega} = -i \int_0^1 ds \ (e^{-is\tilde{\omega}})_{ab} \ T^b . \quad (A.11)$$

Note the emergence of adjoint-representation matrices $\tilde{\omega}$ in the r.h.s.’s of the above formulae. Since matrices in the adjoint representation are real, $(e^{-is\tilde{\omega}})_{ab} = (e^{is\tilde{\omega}})_{ba}$. One can then write

$$\left( L(\omega) L^t(\omega) \right)_{ab} = \int_0^1 ds \int_0^1 d\lambda \ (e^{i(s-\lambda)\tilde{\omega}})_{ab} = \left( \frac{1}{2} \int_{-1}^1 ds \ e^{\frac{i}{2} s \tilde{\omega}} \right)^2_{ab} , \quad (A.12)$$

and obtain $\mu(\omega) = | \det(J_{ab}(\omega)) |$ with

$$J_{ab}(\omega) \equiv \frac{1}{2} \int_{-1}^1 ds \ (e^{\frac{i}{2} s \tilde{\omega}})_{ab} = \sum_{n \geq 0} \frac{(-1)^n (\tilde{\omega}^{2n})_{ab}}{2^n (2n+1)!} = \left( \frac{\sin(\tilde{\omega}/2)}{\tilde{\omega}/2} \right)^2_{ab} . \quad (A.13)$$

This expression allows in particular the computation of $\mu(\omega)$ to quadratic order in $\omega$, as required in Sect. 4.3:

$$\mu(\omega) = \det J(\omega) = e^{\text{Tr} \ln J(\omega)} = e^{-\frac{N}{24} \omega^a \omega^a} + O(\omega^3) . \quad (A.14)$$

Integrals over the group manifold are typically of the form:

$$\int d\mu(U) \ f(U) = \int \Pi_a d\omega_a \ \mu(\omega) f(e^{-i\omega}) . \quad (A.15)$$

A particularly useful tool is the group $\delta$-function defined by

$$\int d\mu(U) f[U] \delta_G[U,V] = f[V] \quad (A.16)$$

with obvious invariance properties, like $\delta_G[U,V] = \delta_G[UV^\dagger, 1] = \delta_G[1, U^\dagger V]$, which follow directly from the invariance of the Haar measure $d\mu(U)$. It will useful to relate this group $\delta$-function to the ordinary $\delta$-function. To do so, consider the integral:

$$f(e^{-i\theta}) = \int d\mu(U) f(U) \delta_G(U, e^{-i\theta}) = \int \Pi_a d\omega_a \ \mu(\omega) f(e^{-i\omega}) \delta_G(e^{-i\omega}, e^{-i\theta}) \quad (A.17)$$

On the other hand, we have

$$f(e^{-i\theta}) = \int \Pi_a d\omega_a f(e^{-i\omega}) \delta(\omega - \theta) , \quad (A.18)$$

with $\delta(\omega - \theta) \equiv \Pi_a \delta(\omega^a - \theta^a)$. Combining these two equations, we obtain the following identity:

$$\mu(\omega) \delta_G(e^{-i\theta}, e^{-i\omega}) = \delta(\omega - \theta) \quad (A.19)$$
Further useful formulæ involve Lie derivatives. First we may use e.q. (A.11) together with the properties (A.6) of Lie derivatives to obtain:

\[
\frac{\partial U}{\partial \omega^a} = L_{ab} \nabla^b U \quad \frac{\partial U^\dagger}{\partial \omega^a} = L_{ab} \nabla^b U^\dagger.
\]  

(A.20)

From these relations, and again e.q. (A.11), one easily deduces the following useful identities:

\[
\nabla^a \left. f(U) \right|_{U=1} = \frac{\partial}{\partial \omega^a} \left. f(U(\omega)) \right|_{\omega=0},
\]  

(A.21)

\[
\frac{1}{2} \left( \nabla^a \nabla^b + \nabla^b \nabla^a \right) \left. f(U) \right|_{U=1} = \frac{\partial^2}{\partial \omega^a \partial \omega^b} \left. f(U(\omega)) \right|_{\omega=0}.
\]  

(A.22)

By using these formulæ and the relation obtained above for the group \( \delta \)-function, one can also derive the following identities:

\[
\mu(\omega) \nabla^a \delta_G(U(\omega), 1) = \frac{\partial}{\partial \omega^a} \delta(\omega),
\]  

(A.23)

\[
\mu(\omega) \chi^{ab} \nabla^a \nabla^b \delta_G(U(\omega), 1) = \chi^{ab} \frac{\partial^2}{\partial \omega^a \partial \omega^b} \delta(\omega),
\]  

(A.24)

for any symmetric \( \chi^{ab} \).

All the previous formulæ are easily extended to the case where the matrices \( U \) are fields \( U(x) \) in the transverse coordinates \( x \). Thus for instance, Lie derivatives become

\[
\nabla^a_x f[U] = \left( -iU(x) t^a \right)_{ij} \frac{\delta}{\delta U_{ij}(x)} f[U].
\]  

(A.25)

We have also:

\[
\nabla^a_x U_y = -iU_y t^a \delta_{xy}, \quad \nabla^a_x U_y^\dagger = i t^a U_y^\dagger \delta_{xy}, \quad [\nabla^a_x, \nabla^b_y] = f^{abc} \nabla^c_y \delta_{xy},
\]  

(A.26)

where \( \delta_{xy} \equiv \delta^{(2)}(x - y) \).
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