Memory effect for impulsive gravitational waves

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Abstract
Impulsive gravitational plane waves, which have a δ-function singularity on a hypersurface, can be obtained by squeezing smooth plane gravitational waves with a Gaussian profile. They exhibit (as do their smooth counterparts) the velocity memory effect: after the wave has passed, particles initially at rest move apart with non-vanishing constant transverse velocity. A new effect is that, unlike the smooth case, (i) the velocities of particles originally at rest jump, (ii) the spacetime trajectories become discontinuous along the (lightlike) propagation direction of the wave.

Keywords: gravitational waves, classical general relativity, memory effect

(Some figures may appear in colour only in the online journal)

1. Introduction

The displacement of freely falling particles by a gravitational wave, called the ‘memory effect’ [1–14] has attracted considerable recent attention, due to its potential use for detecting gravitational waves [13, 15, 16].

In our previous papers [11, 12] we studied linearly polarized exact plane waves with a smooth profile and found that after the wave has passed our particles fly apart with constant but non-vanishing velocity, consistently with suggestions by Braginsky, Grishchuk, Thorne, and Polnarev [2–4], and by Bondi and Pirani [5]: instead of a permanent displacement, there will be a velocity memory effect.

In this paper we extend our investigations and derive similar results for impulsive waves [17–24], which has the advantage that explicit calculations are possible. Our new results are consistent with those obtained in the smooth case in [11, 12]. The novelty is the velocity jump
suffered when the δ-function profile wave passed, and there was a discontinuity of the trajectories in the forward direction.

Our paper is organized as follows. After recalling the three main coordinate systems we use, we study what happens when a Gaussian profile is shrunk to a Dirac δ-function. This corresponds to obtaining an impulsive wave by suppressing the inside zone. Carroll symmetry [18, 25], outlined in section 4, plays a distinguished role. Our main section 5, discusses the geodesics in impulsive gravitational waves in various coordinate systems, followed by a numerical study for the Gaussian profile. Section 7 explains the relation to previous work on impulsive waves [22, 23].

2. Plane gravitational waves

2.1. Brinkmann (B) coordinates

Plane gravitational waves are often described in Brinkmann coordinates (B) [26] in terms of which the metric is

\[ g = \delta_{ij} \, dX^i dX^j + 2dU dV + K(U)X^i X^j \, dU^2, \]

where the symmetric and traceless 2 × 2 matrix \( K(U) = (K_{ij}(U)) \) characterizes the profile of the wave. In this paper we consider linearly polarized ‘+’ type waves with

\[ K(U) = \frac{1}{2} A(U) \, \text{diag}(1, -1), \]

where \( A(U) \) is an arbitrary function.\(^4\)

The Brinkmann coordinates \((X^1, X^2, U, V)\) are global, and the transverse spatial distance is simply \(|X - Y| = \sqrt{(X - Y)^2}\).

2.2. Baldwin–Jeffery–Rosen (BJR) coordinates

Another useful description is provided by Baldwin–Jeffery–Rosen coordinates (BJR) [18, 27, 28], for which

\[ g = a_{ij}(u) \, dx^i dx^j + 2du dv, \]

with \( a(u) = (a_{ij}(u)) \) a positive definite 2 × 2 matrix, which is an otherwise arbitrary function of ‘non-relativistic time’, \( u^A \). The BJR coordinates \((x^A) = (x^1, x^2, u, v)\) are typically not global and suffer from singularities [5, 11, 12, 18]. In these coordinates, the transverse spatial distance also involves the transverse metric, \(|x - y| = \sqrt{a_{ij}(u) (x^i - y^i)(x^j - y^j)}\).

Calling \( P(u) \) a square-root of \( a(u) \),

\[ a(u) = P(u)^4 \, P(u), \]

\(^4\)The most general profile is

\[ K_0(U)X^i X^j = \frac{1}{2} A_+ (U) \left( (x^i)^2 - (x^j)^2 \right) + A_\times (U) X^i X^j, \]

where \( A_+ \) and \( A_\times \) are the amplitudes of the + and × polarization states. Although in this paper we focus our investigation at the diagonal case \( A_\times = 0 \), we prefer to keep our general formulae in view of later applications to primordial gravitational waves and cosmic microwave background (CMB).

Our terminology comes from the ‘Eisenhart–Bargmann’ framework [33, 34] where \( u = U \) becomes indeed non-relativistic time and \( v \), respectively. \( V \) are referred to as ‘vertical coordinates’.

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the relation between the two coordinate systems is given by [25, 29]

\[
X = P(u) x, \quad U = u, \quad V = v - \frac{1}{4} x \cdot \dot{a}(u) x.
\]

(2.6)

where the \(2 \times 2\) matrix \(P(u)\) is a solution of the matrix Sturm–Liouville (SL) equations

\[
\dot{P} = KP, \quad \dot{P}^T P - P^T \dot{P} = 0.
\]

(2.7)

Here \(\dot{P} = dP/du\), and the superscript \(T\) denotes transposition. In what follows, we shall agree that \(U = u\) denote the same (‘non-relativistic time’) coordinate and use one or the other notation just to emphasize which coordinate system we are working with. The profiles in the two coordinate systems are related by,

\[
K = \frac{1}{2} P \left( \dot{b} + \frac{1}{2} b^2 \right) P^{-1} \text{ where } b = a^{-1} \dot{a}.
\]

(2.8)

Waves whose profile vanishes outside an interval \(U_i \leq U \leq U_f\) of ‘non-relativistic time’, \(U\) are called sandwich waves. The regions \(U < U_i\), \(U_i \leq U \leq U_f\), \(U_f < U\) are referred to as the before-, inside-, and after-zones, respectively [5]. The before and after-zones are flat; the inside-zone is only Ricci-flat, which requires \(K\) to be traceless. By (2.8) this amounts to

\[
\text{Tr} \left( \dot{b} + \frac{1}{2} b^2 \right) = 0.
\]

(2.9)

Putting now \(\chi = (\text{det} a)^{\frac{1}{2}} > 0\) and \(\gamma = \chi^{-2} a\), eqn. (2.9) leads to another Sturm–Liouville equation,

\[
\ddot{\chi} + \frac{1}{8} \text{Tr} \left( (\gamma^{-1})^2 \right) \chi = 0,
\]

(2.10)

which guarantees that the vacuum Einstein equations are satisfied for an (otherwise arbitrary) choice of the unimodular symmetric \(2 \times 2\) matrix \(\gamma(u)\). The BJR coordinate system is regular as long as \(\chi \neq 0\).

In this paper, we focus our attention on impulsive waves, which have received extensive attention both from the physical and the mathematical [17–21], and in particular, from distribution-theoretical [22, 23] points of view. Following Penrose, an impulsive gravitational wave is a gravitational wave whose metric is continuous but not \(C^1\) on some (null) hypersurface. Its curvature tensor contains therefore a delta-function [17]. Impulsive waves are sandwich waves whose inside-zone has been suppressed,

\[
U_i = U_f = 0.
\]

(2.11)

The metric is flat both in the before- and the after-zones, \(U < 0\) and \(U > 0\), respectively; their \(\delta\)-function behavior is on the hypersurface \(U = 0\).

Flat metrics can be determined explicitly [12, 18]. Using BJR coordinates we assume that the before-zone \(u < 0\) is described by inertial coordinates and thus \(a_{ij} = \delta_{ij}\). For \(u > 0\) the metric is described in turn by a continuous but not necessarily smooth matrix \(a_{ij}(u)\). Defining \(c_0\) as the right-hand limit of the ‘time’ derivative of the transverse metric in the after-zone \(u > 0\),

\[\text{Impulsive waves should be distinguished from shock waves for which the second derivative of the metric suffers a discontinuity across a (null) hypersurface.}\]
and solving the flatness equation $R_{\text{flat}} = 0$ if the metric is given by $a_{ij} = \delta_{ij}$ in the before-zone, the general formulæ in [12, 18] yield,

$$a(u) = \begin{cases} 
1 & \text{for } u \leq 0, \\
(1 + uc_0)^2 & \text{for } u > 0.
\end{cases}$$

(2.13)

The symmetric $2 \times 2$ matrix $c_0$ in (2.12) characterizes the flat after-zone. Calling $k, \ell$ its (real) eigenvalues we easily find $\chi(u) = \sqrt{(1 + uk)(1 + u\ell)}$. Since the SL equation (2.10) holds both for $u \leq 0$ and $u > 0$, the function $\dot{\chi}$ is necessarily continuous at $u = 0$, hence $\dot{\chi}(0) = 0$ because $\dot{\chi}(u) = 0$ for all $u \leq 0$. Therefore we have $\dot{\chi}(0) = (k + \ell)/(2\chi(0)) = 0$, implying that $c_0$ has two opposite eigenvalues $\pm k$ [18]. In an eigenbasis (which mean in fact polarization), we have therefore

$$c_0 = k \text{ diag}(1, -1)$$

(2.14)

at $u_0 = 0$. We will assume $k \geq 0$ with no loss of generality. If $k = 0$, we would have $a(u) = 1$ for all $u$ and there would be no wave. Henceforth we assume that $k \neq 0$ in the after-zone. The profile, shown in figure 1, is indeed continuous but non-differentiable at $u = 0$,

$$\dot{a}(u) = 2(1 + uc_0)c_0 \theta(u),$$

(2.15)

where $\theta(u)$ is the Heaviside step-function (verifying $\theta(0) = 0$ and $\theta(0^+) = 1$).

The $C^1$ metric (2.4) defined by eqn (2.13) describes therefore a plane impulsive gravitational wave in the sense of Penrose [17].

The transverse matrix $a(u)$ in (2.13) is quadratic in $u$, and we find actually more convenient to use a symmetric square-root $P(u)$ of $a(u)$, namely $a(u) = P(u)^2$, where

$$P(u) = 1 + u \theta(u) c_0$$

(2.16)

is affine in $u$. Its components are shown in figure 2 by dashed black lines.
We note that $\det P$ vanishes exactly once, namely at $u_1 = k - 1 > 0$, signalling that one (but not the other) component of $P(u)$ has a single zero. Realizing that $\det P = 0$ iff $\chi = 0$ [11] indicates that the BJR coordinate system is regular when $u < k - 1$.

Turning to Brinkmann coordinates the profile, obtained by substituting (2.13) and (2.16) in (2.8) is

$$A(U) = 2k \delta(U) \iff c_0 = k \text{diag}(1, -1)$$

(2.17)

confirming their $\delta$-function behavior required by Penrose [17]. The SL equation (2.7) is satisfied when an appropriate regularization is chosen [22, 23]. The amplitude of the wave is recovered as

$$k = \frac{1}{2} \int_{-\infty}^{+\infty} A(U) dU.$$  

(2.18)

Henceforth we will choose $k = 1/2$ for convenience in all figures.

Considering the $\delta$-function profile (2.17) has far-reaching consequences. Let us integrate the SL equation $\dot{P} = KP$ in (2.7) over an interval $U_i < 0 < U_f$ with $U_i$ and $U_f$ chosen arbitrarily in the before- and after-zones, respectively,

$$\dot{P}(U_f) - \dot{P}(U_i) = \int_{U_i}^{U_f} K(u)P(u) du = c_0$$

(2.19)

by (4) and (2.17) and the fact that $P(0) = 1$. Then letting $u_i \to 0^-$ and $u_f \to 0^+$ allows us to conclude that the $\delta$-function at the origin makes therefore $\dot{P}$ jump at $u = U = 0$,

$$\Delta \dot{P} \equiv \dot{P}(0^+) - \dot{P}(0^-) = \dot{P}(0^+) = c_0.$$  

(2.20)

Jumps are characteristic of impulsive waves: if $A$ was smooth and thus bounded then there would be no jump. Such jumps will play a crucial rôle in what follows.

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7 Our equations (2.13) and (2.17) correct a sign error in equations (1) and (2) of Steinbauer et al [22], see [23].

Moreover, substituting (2.16) and (2.15) in (2.6) yields equation (3) of [22]. Equation (2.13) is also consistent with case 1 of Bini [19].
2.3. Souriau (S) coordinates

Besides the widely used B and BJR coordinates, \((X, U, V)\) and \((x, u, v)\), respectively, one also has yet another coordinate system [18] whose use is particularly convenient in the flat case. Start with a sandwich Brinkmann pp-wave written in BJR coordinates as in (2.4). The most general form of metrics in the flat zones is given by [12, 18]

\[
a(u) = a_0^{\frac{1}{2}} (1 + (u - u_0)c_0)^2 a_0^{\frac{1}{2}}
\]

with \(c_0 = \frac{1}{2} a_0^{-\frac{1}{2}} \dot{a}_0 a_0^{-\frac{1}{2}}\), where \(a_0 = a(u_0)\) and \(\dot{a}_0 = \dot{a}(u_0)\) are initial conditions for some value \(u_0\) chosen within the flat region; here \(a_0^{\frac{1}{2}}\) is a square-root of the matrix \(a_0\). Then the change of coordinates \((x, u, v)\) \rightarrow \((\hat{x}, \hat{u}, \hat{v})\) given by

\[
\hat{x} = (1 + (u - u_0)c_0) a_0^{\frac{1}{2}} x,
\]

\[
\hat{u} = u,
\]

\[
\hat{v} = v - \frac{1}{2} \hat{x} \cdot a_0^{\frac{1}{2}} c_0 (1 + (u - u_0)c_0) a_0^{\frac{1}{2}} x,
\]

brings the metric (2.4) in the flat zone of spacetime we consider to the Minkowski form, namely

\[
g = dx \cdot a(u) dx + 2du dv = d\hat{x} \cdot d\hat{x} + 2d\hat{u} d\hat{v}.
\]

The ‘hatted’ coordinates \((\hat{x}, \hat{u}, \hat{v})\) in terms of which the metric is manifestly flat will be referred to as Souriau coordinates [18]. The inverse of the coordinate change (2.22) is

\[
x = a_0^{-\frac{1}{2}} (1 + (u - u_0)c_0)^{-1} \hat{x},
\]

\[
u = \hat{u}
\]

\[
v = \hat{v} + \frac{1}{2} \hat{x} \cdot c_0 (1 + (u - u_0)c_0)^{-1} \hat{x}.
\]

To comply with our assumptions for impulsive gravitational waves, we will put from now on \(u_0 = 0\) and \(a_0 = 1\) so that the transformation formulae (2.22) and (2.24) become

\[
\hat{x} = P(u)x,
\]

\[
\hat{v} = v - \frac{1}{2} x \cdot c_0 P(u)x \quad \Leftrightarrow \quad x = P^{-1}(u)\hat{x},
\]

\[
v = \hat{v} + \frac{1}{2} \hat{x} \cdot c_0 P^{-1}(u)\hat{x}.
\]

respectively, with \(P\) given as in (2.16), completed with \(\hat{u} = u\) as before.

3. Impulsive wave as a limit of Gaussians

The form (2.17) suggests that the impulsive-wave profile could be obtained, in Brinkmann coordinates, by squeezing a smooth Gaussian profile to a Dirac \(\delta\). Let us indeed consider

\[
A_\lambda(U) = \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 U^2}
\]

normalized as \(\int_{-\infty}^{+\infty} A_\lambda(U) dU = 1\) (consistently with our choice \(k = \frac{1}{2}\) made for figures). Then we can calculate the corresponding matrices \(P_\lambda\) and \(a_\lambda(u)\) numerically.
confirm that, when squeezing the Gaussians to a Dirac \( \delta \)-function by letting \( \lambda \to \infty \), the components of \( P_\lambda(u) \) and of the transverse metric \( a_\lambda(u) = P_\lambda^T(u)P_\lambda(u) \) tend to those of the impulsive wave, namely (2.13) and (2.16).

For finite \( \lambda \) the coordinate transformation (2.6) between B and BJR coordinates is smooth. However in the limit \( \lambda \to \infty \), its \( X \)-part, while still continuous, becomes non-differentiable as shown in figures 1 and 2. For \( V \), it is not even continuous at \( U = 0 \),

\[
V_+ = v_0 \quad \text{and} \quad V_- = v_0 - \frac{1}{2} X_0 \cdot c_0 X_0.
\]

(3.2)

The additional term here corresponds to the ‘gluing’ of Penrose, equation (2) of [17]; see also equation (3) of [22].

4. Interlude: Carroll symmetry

4.1. Isometries in Baldwin–Jeffery–Rosen coordinates

The isometry group of a smooth generic plane gravitational wave has long been known to be a 5-dimensional Lie group [5, 18, 25, 29, 30, 32]. For any transverse matrix \( a(u) \), its action on spacetime is explicitly described in BJR coordinates [18, 25] by

\[
x \to x + H(u)b + c, \quad u \to u, \quad v \to v - b \cdot x - \frac{1}{2} b \cdot H(u)b + f,
\]

(4.1a)

(4.1b)

(4.1c)

with \( b, c \in \mathbb{R}^2 \) and \( f \in \mathbb{R} \), where

\[
H(u) = \int_0^u a^{-1}(w)dw
\]

(4.2)

is a primitive of \( a^{-1}(u) \) which we choose to vanish at \( u = 0 \), say. The isometries of the metric (2.4) form therefore a group isomorphic to the group of matrices

\[
A = \begin{pmatrix}
1 & 0 & c \\
-bf & 1 & f \\
0 & 0 & 1
\end{pmatrix},
\]

(4.3)

identified with the Carroll group in \( 2 + 1 \) dimensions ‘without rotations’ [25]. This group is actually isomorphic to the commutator subgroup \([\text{Carr}(2 + 1), \text{Carr}(2 + 1)]\) of the full Carroll group [31] in \( 2 + 1 \) dimensions.

In the impulsive case, \( H(u) \) be calculated analytically separately both in the before- and in the after-zone starting from the same point \( u_0 = 0 \). Using (2.13) we find \( H(u) = uI \) for \( u \leq 0 \), so that boosts act conventionally in the before-zone, \( x \to x + u b \) for \( u \leq 0 \). However for \( u > 0 \) we have \( H(u) = c_0^{-1}(1 - (1 + u c_0)\theta(u)^{-1}) \), yielding the general expression

\[
H(u) = c_0^{-1}(1 - (1 + u \theta(u) c_0)\theta(u)^{-1}) + u(1 - \theta(u))I = u_+ P^{-1}(u) + u_- I
\]

(4.4)

with \( P(u) \) as in (2.16), using the shorthand \( u_+ = u \theta(u) \), \( u_- = u (1 - \theta(u)) \). Therefore boosts act non-conventionally in the after-zone, namely as
Similarly, translations \( x \rightarrow x + a P^{-1}(a) b \).\(^{(4.5)}\)

The Carroll action \((4.1)\) involves the integral \((4.2)\); this action is defined and is continuous over the entire space: the integration smooths out the breakings of the metric.

Henceforth we restrict ourselves to the Lie algebra of the isometry group.

It is worth mentioning that the ‘distorted’ action \((4.1)\) of the Carroll group can also be derived using S coordinates, in terms of which the metric takes a Minkowski form. The \((2 + 1)\)-dimensional Carroll group is indeed a subgroup of the Poincaré group in \(3 + 1\) dimensions\([25, 34]\); its action on S coordinates is the restriction to Carroll of the usual Poincaré action on Minkowski space\([18]\). Boosts, for example, act conventionally on the ‘hatted’ coordinates,

\[ \hat{x} \rightarrow \hat{x} + a b. \] \(^{(4.6)}\)

Expressing this action in terms BJR coordinates using \((2.22a)\) and \((4.4)\) we recover \((4.1)\).

4.2. The Lie algebra of infinitesimal isometries in Brinkmann coordinates

The infinitesimal symmetries of a plane gravitational wave can also be determined in Brinkmann coordinates \((X^1, X^2, U, V)\). The Killing vectors are indeed of the form

\[ Z = P_1(U) \partial_1 + P_2(U) \partial_2 + h(X, U) \partial_V, \] \(^{(4.7)}\)

where

\[ \ddot{P}_1(U) = \frac{1}{2} \dot{A}(U) P_1(U), \quad \ddot{P}_2(U) = -\frac{1}{2} \dot{A}(U) P_2(U), \] \(^{(4.8a)}\)

\[ h(X^1, X^2, U) = -X^1 \ddot{P}_1(U) - X^2 \ddot{P}_2(U) + \eta \] \(^{(4.8b)}\)

with \(\eta = \text{const.}\) The two SL equations \((4.8a)\) here yield a 4-parameter family of solutions. For our \(\delta\)-function choice \((2.17)\) the integration of these equations is elementary, and yields the general solutions

\[ P_1(U) = [1 + k U \theta(U)] \beta_1 + U \gamma_1, \quad P_2(U) = [1 - k U \theta(U)] \beta_2 + U \gamma_2 \] \(^{(4.9)}\)

with \(\beta_1, \beta_2, \gamma_1, \gamma_2\) integration constants. The last component, \(h(X, U)\), is then determined by \((4.8b)\), namely

\[ h(X, U) = -X \cdot (\theta(U) c_0 \beta + \gamma) + \eta. \] \(^{(4.10)}\)

The infinitesimal isometries of our metric \((2.1)\) form thus a 5-dimensional Lie algebra, which is clearly isomorphic to the Lie algebra of the matrix group \((4.3)\). Remarkably, eqn \((4.9)\) can also be written as

\[ \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = P(U) \beta + U \gamma \] \(^{(4.11)}\)

where, as anticipated by our notations, \(P_1 = P_{11}\) and \(P_2 = P_{22}\) are precisely the components of the diagonal matrix \(P\) in \((2.16)\). In fact, this is not an accident: the coefficients of the Killing vectors are, for any plane gravitational wave, solutions of equation \((2.7)\)\([30, 32]\). What distinguishes our case here from the general one is that, in the impulsive case, eqn \((2.7)\) can be solved explicitly as \((2.16)\), whereas no explicit solution is known for a generic SL equation.
This may well be the reason why the algebraic structure of the symmetry algebra has been identified only recently as that of the Carroll Lie algebra with no rotations in $2 + 1$ dimensions, with $\beta$ and $\gamma$ generating boosts and space translations, respectively, and $\eta$ 'vertical time' translations [18, 25], see footnote 2.

4.3. Geodesics: the Noether and Jacobi constants of the motion

As preparation to the study of free fall in an impulsive plane gravitational wave, let us recall the form of the constants of the motion associated with the symmetries of the problem. These first-integrals will prove crucial to integrate in elementary terms the equations of geodesics of the metric (2.4) in its impulsive guise (2.13).

The isometry group (4.3) generates 5 conserved quantities by Noether’s theorem applied to geodesic motion. For geodesics $(x(s), u(s), v(s))$, they are, in the before- and the after-zones separately, given by [18, 25]

\begin{equation}
\begin{aligned}
p_{\pm} &= a(u) \frac{dx_{\pm}}{ds}, \\
k_{\pm} &= x_{\pm}(u) \frac{du}{ds} - H(u) p_{\pm}, \\
\mu_{\pm} &= \frac{du}{ds},
\end{aligned}
\end{equation}

where the $\pm$ refers to $u \leq 0$ and $u > 0$, respectively. For causal continuous geodesics, where

\begin{equation}
e_{\pm} = \frac{1}{2} g_{\mu\nu} \frac{dx_{\pm}^\mu}{ds} \frac{dx_{\pm}^\nu}{ds} = \text{const.} \leq 0
\end{equation}

we may put $\mu_{\pm} = 1$ and hence use from now on $s = u$ as the curve parameter.

Since the associated conserved quantities (4.12) involve the motion and in particular the velocity, they may jump. The Noether quantities $p_{\pm}$ and $k_{\pm}$ and the 'Jacobi' constants of the motion, $e_{\pm}$, may indeed be different before and after the impulse, as will be shown below.

5. Impulsive waves: geodesics

Now we turn to the geodesics which have a subtle behavior, due, precisely, to the jumps, characteristic of the $\delta$-function profile.

5.1. Geodesics in Baldwin–Jeffery–Rosen and Souriau coordinates

Test particle trajectories identified with the geodesics of the plane GW metric can be determined analytically in both of the flat before- and after-zones $u \leq 0$ and $u > 0$ separately, distinguished by $\mp$ indices. A simple way to find them is to use the conservation laws written in BJR coordinates [11, 12, 18, 25]. From the expression of $p_{\pm}$ in (4.12) and from equation (4.13) we infer that

\begin{equation}
e_{\pm} = \frac{1}{2} p_{\pm} \cdot a^{-1}(u) p_{\pm} + \hat{v}_{\pm}(u),
\end{equation}

hence (4.12) leaves us with

\begin{equation}
x_{\pm}(u) = k_{\pm} + H(u) p_{\pm},
\end{equation}

\begin{equation}
v_{\pm}(u) = \frac{1}{2} p_{\pm} \cdot H(u) p_{\pm} + e_{\pm} u + d_{\pm},
\end{equation}

where we anticipated that the quantities $p_{\pm}, k_{\pm}, d_{\pm},$ and $e_{\pm}$, conserved in their respective zones, may (unlike in the smooth case), be different.
Now, these geodesics are meant to represent worldlines of particles in each zone: they must be continuous functions of \( u \), implying that
\[
\begin{align*}
k_\pm &= x(0) = x_0 \quad \text{and} \quad d_\pm = v_\pm(0) = v_0
\end{align*}
\] (5.3)
since \( H(0) = 0 \). Moreover, we have
\[
\begin{align*}
p_\pm &= \dot{x}_\pm(0) \quad \text{and} \quad \dot{v}_\pm(0) = e_\pm - \frac{1}{2}|p_\pm|^2.
\end{align*}
\] (5.4)

At this stage, the latter constants of integration remain arbitrary as they parametrize independent geodesics in each half-zone. Thus we end up with the parametrized continuous geodesics
\[
\begin{align*}
x_\pm(u) &= H(u) \dot{x}_\pm(0) + x_0, \\
v_\pm(u) &= \frac{1}{2}\dot{x}_\pm(0) \cdot [u - H(u)] \dot{x}_\pm(0) + u \dot{v}_\pm(0) + v_0,
\end{align*}
\] (5.5a, b)
where \( H(u) \) is as in (4.4). We note that the Jacobi constants read
\[
e_\pm = +\frac{1}{2}|\dot{x}_\pm(0)|^2 + \dot{v}_\pm(0).
\] (5.6)
Using our previous notations \( u_\pm \), we can write alternatively
\[
\begin{align*}
x(u) &= u_+ P^{-1}(u) \dot{x}_+(0) + u_- \dot{x}_-(0) + x_0, \\
v(u) &= \frac{1}{2} \dot{x}_+(0) \cdot u_+ (1 - P^{-1}(u)) \dot{x}_+(0) + u_+ \dot{v}_+(0) + u_- \dot{v}_-(0) + v_0,
\end{align*}
\] (5.7)

Henceforth, we limit our investigations at particles initially at rest in the before-zone.

Amongst the previous solutions, what are the physical ones suited to the description of free fall in a plane impulsive GW? Indeed, the worldlines of particles in 3 + 1 dimension should be characterized by 4 initial positions \( x_0, v_0 \), and velocities \( \dot{x}_0, \dot{v}_0 \). So, how could we eliminate one of the spurious velocities, \( \dot{x}_+(0), \) say? An answer is obtained by using the \( S \)-coordinates of section 2.3 proposed by Souriau [18].

As explained before, the metric outside the wave zone can be cast into a canonical Minkowskian form (2.23) in either of the flat zones. The coordinate transformation (2.25) between the \( S \) and BJR coordinates in the after-zone \( u > 0 \) of our impulsive wave is, \( x = (1 + u c_0)^{-1} \hat{x}, \ u = \hat{u}, \ v = \hat{v} + \frac{1}{2} \hat{x} \cdot c_0 (1 + u c_0)^{-1} \hat{x} \) (5.8)
where \( c_0 \neq 0 \). The metric \( g = d\hat{x} \cdot d\hat{x} + 2d\hat{u} d\hat{v} \) no longer involves \( c_0 \) [which got hidden in the transformation]. Formally the same transformation holds therefore in the before-zone characterized by \( c_0 = 0 \), where it is in fact the identity, \( x = \hat{x}, \ u = \hat{u}, \ v = \hat{v} \) wherever \( u \leq 0 \).

Consider now particles initially at rest whose geodesics are clearly given in \( S \) coordinates by affine parametric equations in the before-zone, namely
\[
\begin{align*}
\dot{x}(\hat{u}) &= \hat{x}_0, \\
\dot{v}(\hat{u}) &= \hat{v}_0 + e \hat{u},
\end{align*}
\] (5.9)
where \( e = \text{const} \), is the Jacobi first integral of the geodesic equations and \( \hat{x}_0, \hat{v}_0 \) constants of integration. Now, the after-zone being also flat and indeed Minkowskian when the hatted \( S \) coordinates are used, we argue that the latter have the same parametric form in the after-zone: (5.9) holds for all \( u \). Translating (5.9) to the original BJR coordinates we find, using
\( \dot{x}_0 = x_0, \quad \dot{v}_0 = v_0 - \frac{1}{2} x_0 \cdot c_0 x_0 \) see (5.8), the explicit parametric expression of the geodesics in BJR coordinates\(^8\)

\[
\begin{align*}
\dot{x}(u) & = (1 + u \theta(u)c_0)^{-1} x_0 \\
& = P^{-1}(u)x_0, \\
\dot{v}(u) & = -\frac{1}{2} x_0 \cdot (1 - (1 + u \theta(u)c_0)^{-1}) c_0 x_0 + e u + \dot{v}_0 \\
& = -\frac{1}{2} x_0 \cdot (1 - P^{-1}(u)) c_0 x_0 + e u + \dot{v}_0.
\end{align*}
\]

(5.10)

for all \( u \in (-\infty, k^{-1}) \). Eqn (5.10) allows us to interpret the ‘hatted’ S coordinates as the points of the trajectories at \( u = 0 \)\(^9\) (completed with \( u \) itself).

We see that the geodesic equation (5.10) is a special case of (5.7) where the after-zone initial velocity has been fixed by the initial conditions \( x(0) = x_0 \) and \( \dot{x}(0) = 0 \), namely

\[
\dot{x}(0+) = -c_0 x_0.
\]

(5.11)

The impulsive GW induces a [sort of] ‘percussion’ \([18]\), since

\[
\Delta \dot{x} = \dot{x}(0+) - \dot{x}(0-) = -c_0 x_0, \quad (5.12a)
\]

\[
\Delta \dot{v} = \dot{v}(0+) - \dot{v}(0-) = -\frac{1}{2} |c_0 x_0|^2. \quad (5.12b)
\]

We contend that this canonically determined solution of the geodesic equation is germane to a deterministic description of the scattering of particles initially at rest by a plane impulsive GW. Equation (5.12) provides us with a special instance of the velocity memory effect, which also includes the ‘longitudinal’ velocity, \( \dot{v} \).

Let us note for further record that all BJR trajectories are continuous: this follows from (5.10). In particular, the longitudinal coordinate \( \dot{v}(u) \) suffers no discontinuity.

5.2. Geodesics in Brinkmann coordinates

Now we describe our geodesics, again, independent of section 5.1, in Brinkmann coordinates. The geodesic equations are

\[
\ddot{X}^1 - \frac{1}{2} A(U) X^1 = 0, \quad (5.13a)
\]

\[
\ddot{X}^2 + \frac{1}{2} A(U) X^2 = 0, \quad (5.13b)
\]

\[
\ddot{V} + \frac{1}{4} \dot{A}(U) (X^1)^2 - (X^2)^2 + A(U) (X^1 \dot{X}^1 - X^2 \dot{X}^2) = 0. \quad (5.13c)
\]

In ‘Bargmann terms’ \([12, 34]\) (5.13a) and (5.13b) describe a time-dependent anisotropic ‘oscillator’ in transverse space which, (assuming \( A(U) > 0 \)), is attractive in the \( X^2 \) coordinate and repulsive in the \( X^1 \) coordinate.

Now, repeating the argument given in section 2 for \( \dot{P} \), we show that the transverse velocity, \( \dot{X} \), necessarily jumps at \( U = 0 \). To see this, we assume that the particle is at rest in the

\(^8\)Equation (5.10) holds also for null geodesics, \( e = 0 \).

\(^9\)This also hints at that correspondence between the BJR and S coordinates fails to be one-to-one at points where the trajectories meet—i.e. at caustic points.
before-zone, \( \dot{X}(U) = 0 \) for \( U < 0 \), and integrate (5.13a) and (5.13b) for the impulsive profile \( A = 2k \delta(U) \) see (2.17) over an interval \( U_i < 0 < U_f \) which contains the origin.

Assuming that standard distributional identities hold we get
\[
\dot{X}(U) = k \theta(U) A + U B + C
\]
for some constants \( A, B, C \). Plugging this into (5.13a) and (5.13b) yields
\[
C = A.
\]
Thus
\[
\dot{X} = k \theta(U) A + B,
\]
hence \( B = 0 \) in view of our assumption \( \dot{X}(0-) = 0 \). At last, we find
\[
A = X(0-) = X_0.
\]
It follows from
\[
\dot{X}(U) = \theta(U) c_0 X_0
\]
that the initial velocity of the Brinkmann trajectory jumps
\[
\dot{X}(0+) = +c_0 X_0.
\]
The general form of the spatial trajectory in a flat region is therefore
\[
X(U) = (1 + U \theta(U) c_0) X_0 = P(U) X_0.
\]
The transverse velocity of a particle at rest in the before-zone is, in the afterzone, \( U \)-dependent. More precisely, it depends linearly on the initial position \( X_0 \), as shown by the black dashed lines in figure 3. Consistently with (5.15) and (5.16), the trajectories first focus in the attractive coordinate \( X^2 \); then, after passing the caustic point at \( u_1 = k^{-1} \), they diverge with slopes proportional to their initial positions. The repulsive coordinates \( X^1 \) diverge from

---

10 Note that \( X \) and \( P \) satisfy identical equations. The sign of the \( X \)-jump is the opposite of that in BJR coordinates in (5.12a).
11 The integration of equation (5.13c) for \( V(U) \)—containing multiplication of distributions—would require more elaborate techniques, see, e.g. [22], which go beyond our scope here.
12 Equation (5.14) is consistent with equation (11.19) of [11].
the beginning. The relative distance between two trajectories grows therefore linearly with constant but non-zero relative velocity,

\[
(X(U) - Y(U)) = \left| (X_0 - Y_0) + U c_0 (X_0 - Y_0) \right| \tag{5.17a}
\]

\[
(X'(U) - Y'(U)) = \left| c_0 (X_0 - Y_0) \right| = \text{const.} \neq 0 \tag{5.17b}
\]

since for \( k \neq 0 \) the matrix \( c_0 \) is invertible. We conclude that impulsive wave behave as their smooth counterparts do [11, 12]: no permanent transverse displacement is possible; they exhibit instead the velocity memory effect [2-5, 11, 12].

An even more dramatic effect is a discontinuity suffered by the ‘vertical’ coordinate, \( V \). The Jacobi constant is now \( e = \frac{1}{2} |X|^2 + \frac{1}{2} \delta(U) X(U) \cdot c_0 X(U) \). Integrating this expression between \( U_i < 0 \) and \( U_f > 0 \), and using (5.16) and (5.14), we find

\[
e[U_f - U_i] = \frac{1}{2} U_f X_0 \cdot c_0 X_0 + V(U_f) - V(U_i) + \frac{1}{2} X_0 \cdot c_0 X_0. \tag{5.18}
\]

In the limit \( U_i \to 0^- \) and \( U_f \to 0^+ \) we end thus up with

\[
V(0^+) - V(0^-) = -\frac{1}{2} X_0 \cdot c_0 X_0. \tag{5.19}
\]

5.3. Comparison of the trajectories in Brinkmann and Baldwin–Jeffery–Rosen coordinates

We conclude this section by relating the trajectories in B and in BJR coordinates. The naive expectation might be that this could be achieved by using the transformation formula between the coordinates, (2.6), i.e.

\[
X(U) = P(U) x(u), \tag{5.20}
\]

which is indeed correct in the case of continuous wave profiles for particles initially at rest, [11, 12], for which \( x(u) = x_0 = \text{const.} \) for all \( u \). However, identifying the initial positions, \( x_0 = X_0 \) and combining (5.16) and (5.10) yields instead,

\[
X(U) = (P^T P)(u) x(u) = a(u) x(u). \tag{5.21}
\]

Where does the extra \( P \)-factor come from? The clue is that the delta-function \( \delta(u) \) makes the velocity jump both in B and BJR coordinates—and does it in the opposite way, see in (5.15) and (5.12b), respectively. The extra \( P \) factor takes precise care of these jumps: the first \( P \) in (5.21) straightens the trajectory (5.10) to the trivial one, \( P(u) x(u) = x_0 \), which has zero initial BJR velocity as in the smooth case [11, 12]; then the second \( P(u) \) factor curls it up according to (5.20), yielding \( X(u) \) in (5.16).

Deriving w.r.t. \( u > 0 \) and using (2.20) and (5.12a) confirms also that (5.21) flips over the initial velocities,

\[
\dot{x}(0^+) = -c_0 x_0 \quad \Rightarrow \quad \dot{X}(0^+) = +c_0 X_0. \tag{5.22}
\]

The formula (5.21) allows us to clarify yet another puzzle. Naively, it would seem that (5.10) would show no memory effect, since the transverse-space distance between two arbitrary trajectories \( x(u) \) and \( y(u) \) in (5.10) is constant,

\[
\| x(u) - y(u) \| = \sqrt{(P^{-1} x_0 - P^{-1} y_0) \cdot (P^T P)(P^{-1} x_0 - P^{-1} y_0)} = | x_0 - y_0 | = \text{const.} \tag{5.23}
\]
The error comes from having forgotten that the true distance is not (5.23) but the one between the corresponding Brinkmann trajectories,

\[ |X(U) - Y(U)| = \sqrt{\left| P^2(u) \left( x(u) - y(u) \right) \right|^2} = |P(U)(x_0 - y_0)| \]  

(5.24)

which grows affinely with \( U \), see (5.17).

Turning to the conserved quantities, those associated with the symmetries in section 4 are obtained by the Noether theorem. Since the Killing vector in (4.7) has no component along \( \partial U \), the problematic ‘vertical velocity’ \( dV/dU \) drops out. Nor does any \( \delta \)-function show up, providing us with

\[ \Pi_+ = \dot{X}(0+) = c_0x_0 = -\dot{x}(0+) = p_+ \]  

(5.25a)

\[ \Pi_- = \dot{X}(0-) = 0 = -\dot{x}(0-) = p_- \]  

(5.25b)

which confirms that the S coordinates are in fact the (common) initial positions at \( u = 0 \).

6. Geodesics for Gaussian profile

We have seen in section 3 that the impulsive metric is obtained by shrinking Gaussians; now we turn to their geodesics. We emphasise that neither the before- nor the after-zone is rigorously defined in this case; we use the notation \( u \ll 0 \) and \( u \gg 0 \) merely to indicate ‘far-away regions, where the metric components are very small’.

---

**Figure 4.** In Brinkmann coordinates the geodesics \( (X^1(U), X^2(U), V(U)) \) found numerically for the Gaussian profiles \( A_\lambda \) tend, when \( \lambda \to \infty \), to that in equation (5.16), composed of broken dashed straight lines in black, valid for the \( \delta \)-delta function. In the impulsive case the transverse coordinates are \( C^0 \) but not \( C^1 \), whereas both \( V \) and \( \dot{V} \) jump. Our initial conditions are \( X^0_0 = 1, X^2_0 = 2 \) and \( V_0 = 3 \) at \( u = -\infty \). For all \( \lambda \) there is a unique caustic, namely in the attractive sector \( X^2 \), close to the impulsive value \( u_1 = 2 \).
The equations (5.13) are valid for any profile including Gaussians

\[ A_\lambda = \left( \frac{\lambda}{\sqrt{\pi}} \right) \exp\left[ -\frac{\lambda^2 U^2}{2} \right] \]

in (3.1). However they can only be solved numerically; the results are depicted in figure 4.

For \( U_i \ll 0 \) and \( U_f \gg 0 \) (alternatively for large \( \lambda \)) our trajectories exhibit, once again, the velocity memory effect, as seen by integrating equations (5.13a)-(5.13b) over an interval \([U_i, U_f]\), where these values are defined by the requirement that the Gaussian be very small outside the interval. Then, since the components of \( X \) satisfy the same equations as those of the diagonal matrix \( P \), the proof of (2.19) yields the velocity jump

\[ \Delta \dot{X} = \int_{U_i}^{U_f} \frac{1}{2} A(U) \text{diag}(1,-1)X(U)dU. \]  

### Figure 5
The velocity calculated for a Gaussian tends to non-zero constant value, consistent with the one in the impulsive limit \( c_0 X \) in (5.15) (shown in black dashed lines). The particles are initially at rest; their initial positions are \( x_1^0 = 1, x_2^0 = 2 \).

### Figure 6
Deformation of the initial Tissot circle for squeezed Gaussian profile

\[ A_\lambda = \left( \frac{\lambda}{\sqrt{\pi}} \right) e^{-\lambda^2 u^2} \]

with \( \lambda = 20 \). The only caustic arises for the attractive coordinate \( \chi^2 \) close to the impulsive critical value \( u_1 = k^{-1} = 2 \); the \( \chi^1 \) coordinates diverge apart all the time in the after-zone.

The equations (5.13) are valid for any profile including Gaussians \( A_\lambda = \left( \lambda/\sqrt{\pi} \right) \exp[-\lambda^2 U^2] \) in (3.1). However they can only be solved numerically; the results are depicted in figure 4.
For a $\delta$-function profile this would be $c_0 X_0$; for Gaussian profile it is somewhat different. How much? It on depends on where the approximate sandwich values $U_i$ and $U_f$ are chosen and on how much $X(U)$ varies between them. Letting $\lambda \to \infty$, (6.1) would converge to the $\delta$-function value $c_0 X_0$. For large $\lambda$ the velocities tend rapidly to constant values, as shown in figure 5.

A rigorous study of the behavior of $V(U)$ is more subtle, though: the procedure used for $X$ would require ill-defined multiplication of distributions, whose handling [22, 23] goes beyond the scope of this paper. Here we satisfy ourselves with our plots.

The memory effect can nicely be illustrated using Tissot diagrams borrowed from cartography [11, 12]: one considers a tube of timelike geodesics starting from a circle of radius $R$ (say) in the transverse plane in the before-zone, see figure 6.

So far, we only considered test particles initially at rest in the before-zone, $\dot{X}(U) = 0$ for $u \leq 0$. But this is by no means mandatory: our general solution works for any initial condition in any flat region [35]. Thus we should solve the geodesic equation separately in the before- and in the after-zones with appropriate respective initial conditions and then glue them together at $U = 0$ taking into account the jumping condition (5.15), see figure 7.

### 7. Comparison with other approaches

In [22] Steinbauer presented geodesics in the impulsive case. After correcting some typos, his equations # (14) in [22] are

\[
\begin{align*}
    x^1(u) &= \dot{x}_0^1 \left( \frac{u^+}{1 + u^+} + u^- \right) + x_0^1, \\
    x^2(u) &= \dot{x}_0^2 \left( \frac{u^+}{1 - u^+} + u^- \right) + x_0^2, \\
    v(u) &= \frac{1}{2} \left[ (\dot{x}_0^1)^2 \frac{u^+}{1 + u^+} - (\dot{x}_0^2)^2 \frac{u^+}{1 - u^+} \right] + u \dot{v}_0 + v_0, 
\end{align*}
\]

(7.1)

where $u_\pm$ are as above. It is tacitly assumed that all curves are $C^1$ so that $\dot{x}_0$ is the common left-and-right velocity at $u = 0$. It is shown in dashed black lines in figure 7.

Choosing suitable initial conditions, the Steinbauer solution (7.1) reproduces either the first, or the second half, but not the entire broken trajectory of our (5.7). Choosing $\dot{x}_0 = 0$
would yield indeed the trivial solution $x(u) = x_0$ which is fine in the before-zone, but not in the after-zone. (Note that particles at rest in the before-zone, $\dot{x}_0 = 0$, would remain at rest in the after-zone: no scattering would occur.) Putting instead $\dot{x}_0 = -c_0 x_0$ as in (5.11) would yield our solution (5.7) in the after-zone, but not in the before-zone.

For the sake of comparison, we present, with the help of (2.14), our parametrized $C^0$ geodesics in a coordinate-wise form similar to the Steinbauer expression (7.1),

$$
\begin{align*}
\dot{x}_1^1(u) &= \frac{u_+}{1 + k u_+} + u_- \dot{x}_1^1(0) + x_0^1, \\
\dot{x}_2^2(u) &= \frac{u_+}{1 - k u_+} + u_- \dot{x}_2^2(0) + x_0^2, \\
v(u) &= \frac{1}{2} k u_+^2 \left[ \frac{\dot{x}_1^1(0)^2}{1 + k u_+} - \frac{\dot{x}_2^2(0)^2}{1 - k u_+} \right] + u_+ \dot{v}_+(0) + u_- \dot{v}_-(0) + v_0,
\end{align*}
$$

where the left and right velocities, $\dot{x}_\pm(0)$ and $\dot{v}_\pm(0)$, were carefully distinguished. Note that rewriting the geodesic equation in BJR coordinates as

$$
\ddot{x}_1 + 2 \frac{\dot{P}_{11}}{P_{11}} \dot{x}_1 = 0, \quad \ddot{x}_2 + 2 \frac{\dot{P}_{22}}{P_{22}} \dot{x}_2 = 0, \quad \ddot{v} - P_{11} \dot{P}_{11} (\ddot{x}_1)^2 - P_{22} \dot{P}_{22} (\ddot{x}_2)^2 = 0
$$

yields an easy check of (7.2). The solution (7.2) is valid for $u < u_1 = k^{-1}$ only; trajectories, depicted in dashed black lines, strongly diverge when $u \uparrow k^{-1} (= 2)$ as can be seen in figure 7.

Comparison with (7.1) shows that this solution corresponds to (7.2) with $k = 1$ and the $C^1$-assumption of unique initial velocities,

$$
\dot{x}_+(0) = \dot{x}_-(0) = 0, \quad \dot{v}_+(0) = \dot{v}_-(0) = 0.
$$

Moreover, choosing an appropriate initial condition at $u = 0$ and letting $\lambda \to \infty$ in the Gaussian, the Steinbauer solution (7.1) is obtained for all $u$, as shown in figure 7. The velocity $\dot{x}_0$ is the common left and right-side limit. We propose therefore to drop the $C^1$ property of transverse-space trajectories and use (7.2) in the before- and after-zone separately, with different initial velocities at $u = 0 \pm$, namely

$$
\dot{x}_0 = \begin{cases} 
0 & \text{in the before-zone } u \leq 0 \\
-c_0 x_0 & \text{in the after-zone } u > 0
\end{cases}
$$

dictated by the ‘jumping conditions’ (5.11) and yielding our desired solution—which, however, cannot be obtained by shrinking Gaussian profiles.

8. Conclusion

Using the ‘hatted’ S-coordinates (2.22) is particularly convenient to determine the geodesics, since the latter are simple straight lines in Minkowski space; the nontrivial behavior is hidden in the transformation formula (2.25). In flat zones the B and S-coordinates coincide and therefore the only problem is matching correctly the geodesics—allowing us to avoid ill-posed multiplication of distributional functions. This matching is analogous to similar problems in continuous mechanics, and can be achieved by considering the ‘jump structure’ using the waves with a smooth (Gaussian) profile with appropriate parameters, close to the distributional limit.

This scheme leads to a very short, clear and physically justified form of GW memory effects.
Both in the flat before- and after-zones the motion is thus along straight lines with constant velocity. This should not come as a surprise when remembering that a 4D GW spacetime can also be seen as the ‘Bargmann description’ of a non-relativistic system in \(2 + 1\) dimensions [33, 34]. For a sandwich wave the flat before- and after-zones describe a free non-relativistic particle in transverse space. Our finding is therefore ... a confirmation of Newton’s first law\(^{13}\). The \(\delta\)-function profile causes jumps and breakings which correspond to the work done by the wave on the particle.

The impulsive wave metric is obtained by shrinking Gaussians. However, the coordinate-change formula (2.6) between B and BJR coordinates fails for geodesics. Solving the geodesic equations separately in B and in BJR coordinates, (5.16) and (5.10), respectively, their comparison yields instead (5.21).

By shrinking Gaussians, we get smooth transverse trajectories which match our exact solution (7.2) in the after-zone \(u > 0\) but not in the before-zone, see figure 7; they yield instead, for all \(u\), the smooth Steinbauer solutions (7.1) with no velocity jump. However the velocity jump (5.12b) is mandatory; it can be taken into account by working in the flat zones separately and then gluing the solutions respecting the jump conditions (2.20), (5.12), respectively, (5.15) and (5.19).

The most dramatic effect, which arises only in the impulsive case and is absent for smooth profile, is the discontinuity suffered by the lightlike coordinate \(V(U)\)\(^{14}\). Our (5.19) is indeed consistent with (3.2) when we remember that the BJR coordinate \(v(u)\) is continuous, as noted before. Alternatively, the B-trajectories are the images of those trivial straight ones in Minkowski space, given in (5.26).

One may protest that discontinuous worldlines are unphysical. We agree, however, we also argue that approximating real smooth wave profiles (as Gaussians with ‘width’ \(\lambda^{-1}\)) is a mathematical idealization in itself. Those continuous trajectories do exhibit, as illustrated by figures 3 and 4, sharp increases of the \(V\) coordinate, which do tend to (5.19) when \(\lambda \to \infty\)\(^{15}\).

The situation is reminiscent of geometrical optics: approximating the real situation with a sharp change of the refractive index leads to the Snell law and to the recently proposed spin Hall effect of light [38] which are ‘unphysical idealizations’,—which yield, nevertheless, predictions in agreement with observations. In conclusion, we believe that the jump is physical—even if it is very small: let alone the millions-of-kms-long arms of future LISA might be too short when compared to the distances from the sources.

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\(^{13}\) In contrast, Kulczycki and Malec [36] found circular trajectories in some Friedmann–Lemaître–Robertson–Walker background.

\(^{14}\) In our previous papers the behavior of \(V(0)\) was neglected by the practical reason that it is quadratic and therefore irrelevant for eventual observations of GWs: even the first-order velocity effect is very small [13].

\(^{15}\) Podolský and Ortaggio [37] had also found discontinuous trajectories in (anti-)de Sitter space-time.
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