The Triangle

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Abstract

If we label the vertices of a triangle with 1, 2 and 4, and the orthocentre with 7, then any of the four numbers 1, 2, 4, 7 is the nim-sum of the other three and is their orthocentre. Regard the triangle as an orthocentric quadrangle. Steiner’s theorem states that the reflexions of a point on a circumcircle in each of the three edges of the corresponding triangle are collinear and collinear with its orthontre. This line intersects the circumcircles in new points to which the theorem may be applied. Iteration of this process with the triangle and the points rational leads to a “trisequence” whose properties merit study.
Chapter 1

Preamble

This might have been titled The Triangle Book, except that John Conway already has a project in hand for such a book. Indeed, Conway’s book might well have been completed but for the tragically early death of Steve Sigur. It might also have been finished, had I been in closer proximity to John.

It is a very badly edited version of a paper [roughly §2.1 onwards] that was rejected by the MONTHLY. It displays confusion about what is likely to be the best notation. It may be thought of more as a film script than a book — who’s going to make the film? If John likes to use any of what follows I would be very flattered.

LATER: For more completeness I’ve added Chapters 6 & 7. The former is an almost verbatim copy of the Lighthouse Theorem paper [104]; the latter is an unfinished paper with John Conway, which corrects some errors in the former. They don’t always mix very well. There are repetitions and probably some contradictions!

LATER STILL (2015-08-12) I’ve added a Chapter 8, which is a version of the presentation made at Alberta Math Dialog, Lethbridge. 2015-05-08; MOVES Conference, New York, 2015-08-03; and MathFest, Washington, DC, 2015-08-08. This further duplicates some things, but enables me to repair and replace much of the unsatisfactory and incomplete Chapter 7.

2016-03-12. Just woken up with the realization that I was right to title this The Triangle, because, instead of The Triangle Book, this should be The Triangle Movie. Only that way can we have the infinite number of pictures that we need.

- The Wallace line enveloping the S deltoid.
- The Droz-Farny line enveloping a conic.
- The Lighthouse Theorem generating Morley triangles.
• Feuerbach’s Theorem: the 11-point conic.
• The dual of Feuerbach’s Theorem and the Droz-Farny Theorem.

2016-06-08. See notes on Notation at §7.6

1.1 What is a triangle?

Two women pursuing the same man? Or, dually, two men chasing the same woman. Usually thought of as three points whose pairs determine three straight lines, or, dually, as three straight lines whose pairs intersect in three points. We shall call the points vertices, and the lines edges. More numerically, it can be thought of as three positive real numbers, $a$, $b$, $c$, which satisfy the triangle inequality

$$b + c > a, \quad c + a > b, \quad a + b > c.$$

1.2 What’s new?

What’s new(?) here is the introduction of Quadration and Twinning, which, together with Conway’s Extraversion yield a remarkably general view of the Triangle, sometimes with as many as 32 items for the price of one. Eight vertices, which are also circumcentres, as well as orthocentres. Note that an orthocentre is the perspector of the 9-point circle with the relevant circumcircle.

The triangle now has six pairs of parallel edges which form three rectangles whose twelve vertices are the ends of six diameters of the Central Circle. Also known as the fifty-point circle The other 38 points are the points of contact with the 32 touch-circles and six points of contact with the double-deltoid. Six of the points are associated with the name of Euler, 9+32 with that of Feuerbach, and six with Steiner; the double-deltoid having the appearance of a Star-of-David, and is the envelope of pairs of parallel Wallace (Simson) lines, and homothetic to all of the 144 Morley triangles. There are 32 Gergonne points, 32 Nagel points and 256 radpoints. But I’m getting ahead of myself.

On the other hand, I shall probably never catch up with myself, so let me list here some of the topics that I’d like to cover.
1.3 What’s needed?

A nice notation!! The vertices might be $V_1, V_2, V_4$ with orthocentre $V_7$, circumcentre $V_{\bar{7}}$, etc. But see §7.6.

1.4 Clover-leaf theorems

These usually take the shape of the concurrence of the three radical axes of pairs of circles, chosen from three, in the radical centre.

The original Clover-leaf Theorem arose from what I first called five-point circles but which turned out to be nine-point circles (no, not the traditional nine-point circles) and which I will re-christen medial circles. The radical axis of two medial circles of the same triangle is an altitude of that triangle, so that a medial circle contains a vertex, a midpoint, a diagonal point, two mid-foot points and four altitude points. See Figure 2.4 in Chapter 2.

Other examples of what might be called clover-leaf theorems are proofs (2.) and (3.) of the orthocentre. Again, see Chapter 2. Three edge-circles through the same vertex, which is then the radical centre, and orthocentre or fourth vertex. Three edge-circles of a triangle as in Figure 8.2 so that the radical axes are altitudes again, concurring in the orthocentre.

Clover-leaf theorems may extend into four-leaf clover theorems. For example, add the circumcircle to Figure 8.2 and the 6 radical axes are the 6 edges of the (generalized, quadrated) triangle and the 4 radical centres are the 4 vertices-orthocentres. Another example is adding the 50-point circle to the medial circles as in Figure 2.8 in Chapter 2, §2.2.

1.5 Radical axes of circumcircles

There are $\binom{8}{2} = 28$ of these. Twelve are the 12 edges of the (generalized) triangle. As the circumcircles are all congruent, the other 16 will be the perpendicular bisectors of the segments $V_iV_j$ where $i \in \{1, 2, 4, 8\}$, $j \in \{14, 13, 11, 7\}$, where we may write the latter in hexadecimal as $j \in \{\overline{e}, \overline{d}, \overline{b}, \overline{7}\}$. [Perhaps better is $j \in \{\overline{1}, \overline{2}, \overline{4}, \overline{7}\}$.] Four of these 16 will be perpendicular bisectors of diametral segments joining a twin pair of vertices. These diametral segments are Euler lines.
1.6 Touch-circles

Conway’s extraversion generalizes the incircle to a set of 4 touch-circles which touch the 3 edges of a triangle. And quadration turns a triangle into an orthocentric quadrangle of 4 triangles, giving 16 touch-circles. Finally, twinning doubles this number to 32 touch-circles. Feuerbach tells us that these all touch the 50-point circle (Central Circle).

1.6.1 Interlude: Proof of Feuerbach’s theorem

(See also Altshiller-Court, [6, pp.105,273]; Roger A. Johnson, [114, pp.200,244]).

Area $\Delta = \frac{1}{2}ab \sin C = 2R^2 \sin A \sin B \sin C$

Semiperimeter $s = \frac{1}{2}(a + b + c) = R(\sin A + \sin B + \sin C)$

Inradius $= r = \frac{\Delta}{s} = \frac{2R \sin A \sin B \sin C}{\sin A + \sin B + \sin C}$

50-point radius = $R/2$. Square of distance of incentre, $I$, from 50-pt centre is

$$\left[\frac{1}{2} (b \sin C - R \cos A) - r\right]^2 + \left[\frac{1}{2} \left(\frac{1}{2} a + b \cos C\right) - (s - c)\right]^2$$

We need this to simplify to $(\frac{1}{2}R - r)^2$. Not the right way to go!!

I think we can find a proof which uses the spirit of quadration (1.6 below), edge-circles and 5 or 6 (now 9) proofs that a triangle has an orthocentre (Chap.2 below), etc.

In fact, see §8.5 Hexaflexing, below.

[resume 1.5 Touch-circles]

These 32 touch-circles touch the 12 edges in 96 touch-points. Their joins to appropriate vertices concur in triples at 32 Gergonne points and 32 Nagel points. [No!! We are not so lucky with Nagel points!] Gergonne points are obtained by taking the three touch-points of one (one and only one) touch-circle and joining them to the respective vertices of the relevant triangle. By contrast, the Nagel points are formed using the touch-points of three (all but one) touch-circles out of a set of four, [Here’s the snag! We can’t use any old three. They have to be the three EX-circles.]
Figure 1.1: 32 touch-circles each touch 3 of the 12 edges at one of 8 points, and each touches the 50-point circle (which isn’t drawn, but you can “see” it!)
Figure 1.2: Enlargement of Fig. 1. 32 touch-circles each touch 3 of the 12 edges at one of 8 points, and each touches the 50-point circle (which is drawn in this version),
1.7 Quadration

This is to grant the same status to the orthocentre as to the vertices, so that each of the four points is the orthocentre of the triangle formed by the other three. That is, to regard the triangle as an orthocentric quadrangle. It now has 4 vertices, 6 edges and 3 diagonal points, $D_6, D_5, D_3$.

This may be a good place to give some formulas. We will take the (was 50-point) Centre of the triangle as origin. If the vertices of a triangle are given by vectors $a, b, c$, then the centroid (which we will soon forget!) is $\frac{1}{3}(a + b + c)$, the circumcentre is $a + b + c$, and the orthocentre is $-(a + b + c)$. That is, if $d$ is the orthocentre, then $d = -(a + b + c)$, and $a + b + c + d = 0$

reminding us that each of the four points is the orthocentre of the other three.

Note that $a \cdot b + c \cdot d = a \cdot c + b \cdot d = a \cdot d + b \cdot c$.

Notice that in quadration (and in twinning; see below) the circumradius, $R$, of all four (all eight) triangles is the same. If the angles of a triangle are $A, B, C$, then, by the sine formula, the edges are

$$2R \sin A, \quad 2R \sin B, \quad 2R \sin C.$$  

and the angles and edges of the quadrations are

$$\pi - A, \quad \frac{\pi}{2} - B, \quad \frac{\pi}{2} - C,$$

and

$$2R \sin A, \quad 2R \cos B, \quad 2R \cos C,$$

Note that the triangle inequality appears as

$$\sin B + \sin C > \sin B \cos C + \sin C \cos B = \sin(B + C) = \sin(\pi - A) = \sin A$$

and, for the quadrations

$$\cos B + \cos C > \sin B \cos C + \sin C \cos B = \sin(B + C) = \sin(\pi - A) = \sin A$$

We can contrast and combine this with Conway's extraversion, in which the extraverted triangles have angles and edges

$$-A, \quad \pi - B, \quad \pi - C,$$

and

$$-2R \sin A, \quad 2R \sin B, \quad 2R \sin C,$$

and

$$\pi - A, \quad -B, \quad \pi - C,$$

and

$$2R \sin A, \quad -2R \sin B, \quad 2R \sin C,$$

and

$$\pi - A, \quad \pi - B, \quad -C,$$

and

$$2R \sin A, \quad 2R \sin B, \quad -2R \sin C.$$
1.8 Twinning

If you draw the perpendicular bisectors of each of the six edges of an orthocentric quadrangle, you produce a quadrangle of circumcentres, congruent to the original one; in fact the two tw\textit{ins} are obtained from each other by reflexion in, or rotation through $180^\circ$ about, a common centre, $O$. In fact, if one wishes to select just one out of Clark Kimberling’s six thousand, six hundred-odd triangle centres, then a good case can be made for THE centre, was 50-point centre! Indeed, any other candidate has its twin to compete with.

Figure 1.3: Hello, twins! Hexadecimal subscripts: $a = 10$, $b = 11$, $c = 12$, $d = 13$, $e = 14$.

To the tune of “My Bonny Lies Over the Ocean”

Now that we’ve heard of Quadration,
And Twinning gives Two – symmetry,
And Conway has found Extraversion.
We can bring back that gee-om-met-tree!

Bring back! Oh, bring back!
Oh bring back that gee-om-met-tree, to me!
Bring back! Oh, bring back!
Oh, bring back that gee-om-met-tree !!
More verses about various geometrical objects??

Since then I’ve been invited to preach at Alberta Math Dialog in Lethbridge in May, at the MOVES conference in NYC, and at Mathfest Washington DC in August. So here’s a shot at an

**ABSTRACT**

A triangle has eight vertices, but only one centre

Quadration regards a triangle as an orthocentric quadrangle. Twinning is an involution between orthocentres and circumcentres. Together with variations of Conway’s Extraversion, these give rise to symmetric sets of points, lines and circles. There are eight vertices, which are also both orthocentres and circumcentres. Twelve edges share six midpoints, which with six diagonal points, lie on the was 50-point circle, better known as the 9-point circle. There are 32 circles which touch three edges and also touch the 50-point circle. 32 Gergonne points, when joined to their respective touch-centres, give sets of four segments which concur in eight deLongchamps points, which, with the eight centroids, form two harmonic ranges with the ortho- and circum-centres on each of the four Euler lines. Corresponding points on the eight circumcircles generate pairs of parallel Simson-Wallace lines, each containing six feet of perpendiculars. In three symmetrical positions these coincide, with twelve feet on one line. In the three orthogonal positions they are pairs of parallel tangents to the 50-point circle, forming the Steiner Star of David. This three-symmetry is shared with the 144 Morley triangles which are all homothetic. Time does not allow investigation of the 256 Malfatti configurations, whose 256 radpoints probably lie in fours on 64 guylines, eight through each of the eight vertices.
1.9 Radical axes of touch-circles

On 2015-04-28 Peter Moses told me that The radical centre of the ex-circles is X(10) in [116].

[The Spieker centre, barycentric \((b+c,c+a,a+b)\). So, with quadration & twinning, there are 8 Spieker centres. Lies on IG. And on HM (orthoc & mittenpunkt).]

The radical center of the B- & C-excircles with the incircle is \(b + c, c - a, b - a\) (barycentric).

Call the triangle formed by the cyclic permutation of \(b + c, c - a, b - a\) A'B'C'. The excentral triangle, IaIbIc, is perspective to A'B'C' at X(2), G. Indeed \(-Ia G— / —G A'— = 2\).

1.10 Apollonian interlude

This section was called “Cartesian points and axes” until Peter Moses reminded me that I’d already dealt with many parts of the subject in below, where it can be seen that these points and line are the Isoperimetric Point, the Equal Detour Point, and the Soddy Line! In fact, let the two points be \(X_{175}\) and \(X_{176}\) and the radii of the two circles be \(r_{175}\) and \(r_{176}\), then the perimeters of the three triangles \(X_{175}BC, X_{175}CA, X_{175}AB\) are 

\[
(r_{175} - (s - b)) + a + (r_{175} - (s - c)) = 2r_{175}
\]

and two equal expressions, while the extra detour in travelling from \(B\) to \(C\) via \(X_{176}\) instead of travelling directly is \((s - b) + r_{176}) + (r_{176} + (s - c)) - a = 2r_{176}\), and similarly in detouring through \(X_{176}\) when travelling from \(C\) to \(A\), or from \(A\) to \(B\).

Then I heard Kate Stange’s lecture at the Alberta Number Theory Days in Banff, and realised that while much has been said recently ([122, 87, 88, 89, 90] and about 100 other papers) about Apollonian packings, there is plenty more to be explored.
If we draw circles with centres at the vertices (for which, pro tem, we’ll use the traditional notation $A$, $B$, $C$) and respective radii $s - a$, $s - b$, $s - c$, they will touch one another at the touch points of the edges with the incircle, with which they are orthogonal. [I proposed to call them “vertical circles”, but Peter Moses reminds me that I have already called them “tangent circles” in Chapter 5 below, written some time before the present sections!]

According to Descartes, following Apollonius, there are two circles which touch all three vertical circles. I called their centres the Cartesian points and their join the Cartesian axis, but later (earlier!) found that these are the so-called Soddy circles, centres the isoperimetric point and the equal detour point, $X(175)$ and $X(176)$ in [116, 206], discussed in some detail in Chapter 5 below!!

[I can’t use “Apollonian point” since “Apollonius point” is already used for something else, $X(181)$ in Clark Kimberling’s Encyclopedia [116]. Suppose that the circle circumscribing the three excircles touches them at $A'$, $B'$, $C'$. Then $AA'$, $BB'$, $CC'$ concur at $X(181)$. Its barycentric coordinates are $\{\ldots, b^3 \cos^2[(C - A)/2], \ldots\}$. See [117]. This is a pity, since this is not an example of what has recently received a good deal of attention, namely “Apollonian packings”.

Indeed, look at an Apollonian packing. Draw the segment joining the centres of each pair of circles which touch. We have a triangulation which Kate Stange called an Apollonian city. Now apply much of the enormous amount that is known about triangle geometry to each of the triangles in the triangulation.

By Conway’s extraversion we will have four isoperimetric points, four equal detour points and four Soddy lines. Quadration and twinning give 32 of each of these items.

What coincidences, collinearities and concurrences are there?? Skip ahead to just after Figure 1.10.
Figure 1.4: Eight Cartesian points and four Cartesian axes (quadrature)
That was the result of quadration. Now let’s try extraversion:

Figure 1.5: Eight Cartesian points and four Cartesian axes (extraversion of $V_8$)

Surprise! Surprise! The four Cartesian axes in the extraversion concur!! Will try extraverting the other 3 triangles in the orthocentric quadrangle. NOT A SURPRISE TO THOSE WHO WELL KNOW IT!!!(*) This is the De Longchamps point [X(20) in [116]]. In the notation of §1.6, The orthocentre is $d$, the circumcentre is $-d$, the centroid is $\frac{1}{3}(a + b + c) = -\frac{1}{3}d$, and the De Longchamps point is $-3d$. The four points form a harmonic range.

(*) IN FACT, SEE FIRST PAGE OF CHAPTER 5 BELOW!! No fool like an old fool!
Figure 1.6: Eight Cartesian points and four Cartesian axes (extraversion of $V_1$)
Figure 1.7: Eight Cartesian points and four Cartesian axes (extraversion of $V_2$)
Figure 1.8: Eight Cartesian points and four Cartesian axes (extraversion of $V_4$)
Let us collect together the 16 Cartesian axes:

The four points of concurrence form an orthocentric quadrangle which is homothetic to the original orthocentric quadrangle with ratio $-3$ and centre of similitude the 50-point centre (which is, of course, the 50-point centre of the new (Cartesian?) orthocentric quadrangle).
In Figure 1.10 we have an orthocentric quadrangle, together with its twin, and their Cartesian quadrangles (dashed lines). There are four sets of four equally spaced vertices (dotted lines) which concur and bisect each other at THE centre, $O$.

Figure 1.10: Four quadrangles in perspective

NOT A SURPRISE TO THOSE WHO WELL KNOW IT!!! THE DOTTED LINES (through $O$) ARE EULER LINES!! The 8 points are De Longchamps points of the original 8 triangles. In the notation of §1.6, if we take THE (50-point) centre as origin and the orthocentre as $d$, then the circumcentre is $-d$, the centroid is $\frac{1}{3}(a + b + c) = -\frac{1}{3}d$, and the De Longchamps point is $-3d$, or, as Clark Kimberling ([116]) puts it, the orthocentre of the anticomplementary triangle! A harmonic range:

$$\{\text{Orthocentre, Circumcentre; Centroid, De Longchamps point}\} = -1.$$ 

Reverting to the Apollonian city: for each triangle, the circles of the packing are what I variously called the “vertical circles” or the “tangent circles”. The positions of their centres are well known in terms of Gaussian rational numbers. The incircles form the orthogonal Apollonian packing, so the coordinates of the incentres are similarly well known. But what about the excircles? And more generally, what does Conway’s extraversion bring??

Also what does quadration bring?? Where are the orthocentres? They must be rational points also. Similarly for the circumcentres. And for the 50-point (9-point) centres. So little done! So much to do!
1.11 Nagel points

The joins of the vertices to the touch-points of the excircles with the opposite edges, concur in the Nagel point. I don’t see how to extravert this. But let’s first quadrate it.

![Figure 1.11: Quadration of Nagel points. The four segments illustrate the collinearity of the Nagel point, incentre and centroid.](image)

The results are disappointing. Slopes of joins of Nagel points are:

- $N_1N_8 : \frac{5}{4}$
- $N_2N_4 : -\frac{1}{17}$
- $N_1N_2 : \frac{7}{46}$
- $N_4N_8 : \frac{1}{13}$
- $N_1N_4 : -\frac{3}{22}$
- $N_2N_8 : \frac{1}{21}$

while the slopes of $N_1I_1$, $N_2I_2$, $N_4I_4$, $N_8I_8$ are

- $\frac{5}{12}$
- $-\frac{22}{15}$
- $\frac{28}{45}$
- $\frac{1}{2}$

— not particularly inspiring! Let’s hope for better luck with the . . .
1.12 Gergonne points

Figure 1.12: Extraversion of Gergonne points in triangle $V_1V_2V_4$. The Gergonne point, touch-centre and deLongchamps point are collinear.

Note that the joins $V_1G_{8,1}$, $V_2G_{8,2}$, $V_4G_{8,4}$ concur. In what point?? It's the Nagel point, $N_8$!!!
Again, the joins of the vertices to the Gergonne points concur in the Nagel point, $N_1$.!!

Figure 1.13: Extraversion of Gergonne points in triangle $V_2V_4V_8$.!!
Figure 1.14: Extraversion of Gergonne points in triangle $V_1V_4V_8$. 
Figure 1.15: Extraversion of Gergonne points in triangle $V_1 V_2 V_8$. 

1.12. GERGONNE POINTS
Let’s collect the 16 (32??) Gergonne points into one diagram. The lines connect them to the corresponding touch-centres (without dots) and deLongchamps points (with dots and labels), but the result is not very inspiring!

Figure 1.16: Sixteen Gergonne points. There are 32 if you include their twins.
1.13 Simson-Wallace lines and the (double-) deltoid

A theorem that used to be well known is that the feet of the perpendiculars from a point on the circumcircle to the edges of a triangle are collinear. This is usually known as the Simson line of the point, though it was discovered by William Wallace and doesn’t occur in Simson’s work [50, p. 16], [52, p. 41]. Steiner showed that as the point moves round the circumcircle, the Simson-Wallace line envelops a Steiner deltoid or three-cusped hypocycloid [50, p. 115], [52, p. 44]. This is usually described as the locus of a point on a circle rolling inside a fixed circle of three times its radius. But here we want the curve as a roulette, and it is easier to visualize and to construct if you regard it as the envelope of a diameter of a circle of radius $R$ rolling inside a circle, centre $O$ and radius $3R/2$.

We first give a proof of the Simson-Wallace line theorem (Figure 1.17) and notice the generalization that comes from quadration (Figure 1.18)

![Figure 1.17: The Simson-Wallace line theorem.](image)

Notice that the quadrilaterals $F_6F_3S_8V_2$, $F_5S_8F_6V_4$, $V_1V_4V_2S_8$, $F_5S_8F_3V_1$ are all cyclic, so that $\angle F_6F_3V_2 = \angle F_6S_8V_2 = \angle F_5S_8V_2 - \angle F_5S_8F_6 = \angle F_5S_8V_2 - (\pi - \angle F_5V_4F_6) = \angle F_5S_8V_2 - (\pi - \angle V_1V_4V_2) = \angle F_5S_8V_2 - \angle V_1S_8V_2 = \angle F_5S_8V_1 = \angle F_5F_3V_1$, and the vertically opposite $\angle F_6F_3V_2$ and $\angle F_5F_3V_1$ show that $F_6$, $F_3$ and $F_5$ are collinear.

If you google “Wallace line” you’ll find something closely related to Darwin’s *Origin of Species*, whereas if you google “Wallace Sim(p)son” you’ll find the wife-to-be of the uncrowned King of England.
1.14 Quadration and Twinning of the Simson-Wallace line

Draw radii $V_e S_1$, $V_d S_2$, $V_b S_4$, of the circumcircles of triangles $V_2 V_4 V_8$, $V_4 V_8 V_1$, and $V_8 V_1 V_2$, parallel to the radius $V_7 S_8$ of the circumcircle of triangle $V_1 V_2 V_4$, yielding points $S_1$, $S_2$, $S_4$ on the respective circumcircles. Then the feet of the perpendicularrs from these points onto appropriate edges of the respective triangles give triples of points

\[ \{F_c, F_a, F_b\}, \{F_9, F_5, F_c\}, \{F_3, F_a, F_9\}, \]

all lying on the same, original, Simson-Wallace line, $\{F_6, F_5, F_3\}$.

We know that $V_e V_d V_b V_7$ form a quadrangle congruent to the original $V_1 V_2 V_4 V_8$; note that $S_1 S_2 S_4 S_8$ is also such a congruent quadrangle. In fact $S_1 V_1$, $S_2 V_2$, $S_4 V_4$, $S_8 V_8$ bisect each
other at a point on the Simson-Wallace line. See the second theorem ahead.

Notice that twinning now gives a pair of parallel Simson-Wallace lines.

On three, equally spaced, occasions, this pair will coincide, giving twelve collinear points. These are the feet of the perpendiculars from points on eight circumcircles onto the three edges of the corresponding triangles; note that an edge belongs to just two of the eight triangles.

On three other equally spaced occasions, the pair will be perpendicular to one of the twelve-point lines and are tangents to the 50-point circle at its points of intersection with these twelve-point lines, forming the
1.15 Steiner Star of David

Figure 1.19: The Steiner Star of David
We next notice §2, pp. 44-45:

**Theorem.** The angle between the Simson-Wallace lines of points $S$ and $S'$ on the circumcircle is half the angular measure of the arc $S'S$.

In Figure 1.20 the perpendiculars from $S$ and $S'$ to $V_2V_4$ meet the circumcircle again at $U$ and $U'$. The Simson-Wallace lines of $S$ and $S'$ (shown dashed in the figure) are respectively parallel to $V_1U$ and $V_1U'$. [To see this, note that the quadrilaterals $SV_1UV_2$ and $SF_3F_6V_2$ are both cyclic, so that $\angle SUV_1 = \angle SV_2V_1 = \angle SV_2F_3 = \angle SF_6F_3$, and $UV_1$ is parallel to $F_6F_3$.] The Simson-Wallace line turns at half the angular velocity with which its point is describing the circumcircle, and in the opposite sense.

![Figure 1.20: Angle between Simson-Wallace lines of $S$ and $S' = \angle UV_1U' = \frac{1}{2}\text{arc}UU' = \frac{1}{2}\text{arc}S'S$.](image)

**Theorem.** The Simson-Wallace line of a point $S$ on the circumcircle of triangle $V_1V_2V_4$ bisects the segment $SV_8$, where $V_8$ is the orthocentre of triangle $V_1V_2V_4$.

To see this, look at Figure 1.21. Quadrilaterals $S_8V_1V_4U_6$ and $S_8F_5V_4F_6$ are cyclic, so that $
\angle S_8U_6V_1 = \angle S_8V_4V_1 = \angle S_8V_4F_5 = \angle S_8F_6F_5$

and $V_1U_6$ is parallel to the Simson-Wallace line $F_5F_6$.

[In fact we’ve already proved this further up the page. If the perpendiculars $S_8F_3$ and
$S_8 F_5$ from $S_8$ to the edges $V_1V_2$ and $V_2V_4$ meet the circumcircle $V_1V_2V_4$ again at $U_3, U_5$ respectively, then $V_2U_5$ and $V_4U_3$ are also parallel to the Simson-Wallace line.]
Moreover, if the perpendicular $S_8F_6U_6$ to edge $V_2V_4$ meets the circumcircle $V_2V_4V_8$ at $X_6$, then the segments $V_1V_8$, $V_7V_e$, $U_6X_6$ are equal and parallel so that, in the triangle $S_8X_6V_8$ the Simson-Wallace line passes through the midpoint $F_6$ of edge $S_8X_6$ and is parallel to the edge $X_6V_8$, and so passes through the midpoint, call it $T_8$ for the time being, of the third edge $S_8V_8$.

Now look at triangle $S_8V_7V_8$. The midpoint of edge $V_7V_8$ [the Euler line of triangle $V_1V_2V_4$] where $V_7$, $V_8$ are the circumcentre and orthocentre of triangle $V_1V_2V_4$, is the 50-point centre, $O$. We have just seen that the midpoint of edge $S_8V_8$ is $T_8$, so that $OT_8$ is parallel to the third edge $V_7S_8$ and has length equal to half of $V_7S_8$, that is $\frac{1}{2}R$, and $T_8$ lies on the 50-point circle. This circle is fixed, and serves for all four (quadrated) triangles.
In Figure 1.22, as corresponding points \( S_i \) move round the four circumcircles their common Simson-Wallace line passes through the point \( T_8 \) on the 50-point circle, where \( OT_8 \) is parallel to the radii \( V_{15-i}S_i \) \( (i = 8, 4, 2, 1) \). If the angular velocity of the \( S_i \) is \( \omega \), then that of the Simson-Wallace line is \( -\frac{1}{2}\omega \).

Figure 1.22: The deltoid is a locus and an envelope.

Figure 1.22 shows the 50-point (9-point) circle, centre \( O \) and radius \( \frac{1}{2}R \), together with a concentric circle of radius \( \frac{3}{2}R \) (the Steiner circle). As the points \( S_i \) \( (i = 8, 4, 2, 1) \) move round their respective circumcircles these concentric circles remain fixed. But the circles with centres \( T_8 \) and \( W \) and respective radii \( R \) and \( \frac{1}{2}R \) roll round in contact (at \( T_8, Y \) and \( Z \)) with the fixed circles. The Simson-Wallace line is a diameter of the larger of the two rolling circles. The locus of the ends of the diameter is the deltoid, which is also the envelope of the Simson-Wallace line, since its instantaneous centre of rotation is the contact point \( Y \), and the foot, \( X \), of the perpendicular from \( Y \) to the Simson-Wallace line, lies on the smaller of the two rolling circles. Note that \( \angle T_8XY \) is a right angle in a
1.15. STEINER STAR OF DAVID

To see this analytically, take \( R = 2 \) and the origin at \( O \), the 50-point centre. Then a point on the deltoid is \((2 \sin \theta - \sin 2\theta, 2 \cos \theta + \cos 2\theta)\), or, with \( t = \tan \frac{1}{2} \theta \),

\[
\begin{pmatrix}
\frac{8t^3}{(1 + t^2)^2} \cdot \frac{3 - 6t^2 - t^4}{(1 + t^2)^2}
\end{pmatrix}
\]

\[
\frac{dx}{dt} = \frac{8t^2(3 - t^2)}{(1 + t^2)^3}, \quad \frac{dy}{dt} = \frac{8(t^2 - 3)}{(1 + t^2)^3}, \quad \frac{dy}{dx} = -\frac{1}{t}
\]

and the equation to the Simson-Wallace line is

\[
y = -\frac{x}{t} + \frac{3 - t^2}{1 + t^2}.
\]

Figure 1.23: The deltoid as envelope.
Figure 1.24: The deltoid as locus.

Figure 1.25: The double-deltoid as locus.
Figure 1.26: The double-deltoid as envelope.
1.16 Steiner’s Star of David

It is easy to see, from the way it was generated, that the double deltoid has the symmetry, $S_3$, of an equilateral triangle. There are three special positions of the Simson-Wallace line, and three more, perpendicular to the first three. One of these first three is shown dashed in Figure 1.27. Each is a trisector of the angle between the diagonals (pairs of diameters of the 50-point circle, dotted lines in Figure 1.27) of one of the three rectangles formed by the twelve edges of the triangle.

Figure 1.27: The $S_3$ symmetry of the triangle
1.17 The BEAT

That is: the Best Equilateral Approximating Triangle, or, together with its twin, the Best Equilateral Approximating Twin, which is an alternative name for the Steiner Star of David.
1.18 The Simson-Wallace line and the cardioid

In [101] we read:

Given a circle, a point $P$ on it, and a line: is it possible to inscribe a triangle in the circle so that the given line is the Simson-Wallace line of $P$ w.r.t. the triangle? The answer is that there is an infinity of solutions, provided that the line is “not too far away”.

Take any point $A$ on the circle. Let the circle on $PA$ as diameter cut the given line in $M, N$. If $AM, AN$ meet the circle again in $C$ and $B$, then it is easily shown that $ABC$ is a satisfactory triangle, and that if $BC$ meets the given line at $L$, then $PL$ is perpendicular to $BC$.

The condition for a solution is that it is possible to find $A$ so that the circle on $PA$ intersects the given line. The envelope of such circles, for various $A$, is the cardioid having its cusp at $P$ and vertex $[V]$ diametrically across the given circle. The condition is that the given line must intersect this cardioid.

![Figure 1.28: Finding triangles to fit a given Simson-Wallace line.](image)

It is instructive to invert the figure w.r.t. $P$. The result is a like configuration with the roles of the line $LMN$ and the circle $ABC$ interchanged. The inverse of the cardioid is the parabola, which is the envelope of lines through $A$, perpendicular to $PA$, as $A$ moves along a given line.
In Figure 1.28 $\angle PNA$ and $\angle PMA$ are angles in a semicircle, showing that $PN$, $PM$ are perpendicular to $AB$, $AC$ respectively, and it follows from the main theorem that $PL$ will be perpendicular to $BC$. Figure 1.29 shows the cardioid enveloped by the circles on $PA$ as diameter, as the point $A$ moves round the given circle.

Figure 1.29: The cardioid enveloped by the circles on $PA$ as diameter.

Figure 1.30: Inverse of Figure 1.28.
If we invert Figure 1.28 with respect to $P$, we obtain Figure 1.30. The original given circle inverts into the line $A'B'C'$ and the given line into the circle $L'M'N'$. Notice that the four lines $A'B'C'$, $A'M'N'$, $B'N'L'$, $C'L'M'$ are the respective inverses of the circles $ABC$, $AMN$, $BNL$, $CLM$, which all pass through $P$, and are in fact the circles having $PV$, $PA$, $PB$, $PC$ as diameters, all of which touch the cardioid, so that the four lines touch the parabola which is the inverse of the cardioid, at $V'$, $X$, $Y$, $Z$.

### 1.19 A converse theorem?

Given a point $P$ and a line $LMN$, do the lines through $L$, $M$, $N$ perpendicular to $PL$, $PM$, $PN$ respectively, form a triangle whose circumcircle passes through $P$?

![Figure 1.31: Converse of Simson-Wallace theorem.](image)

From the right angles at $L$, $M$ and $N$ we see that $PANM$, $PNBL$, $PLCM$ are cyclic quadrilaterals, inscribed in circles on $PA$, $PB$, $PC$ as diameters. Hence

\[
\angle LPC = \angle LMC = \angle NMA = \angle NPA,
\]

so that

\[
\angle APC = \angle NPL = \pi - \angle NBL = \pi - \angle ABC
\]

and $P$ lies on the circumcircle of $ABC$.

Notice that the edges of triangle $ABC$ are tangents to the parabola having $P$ as focus and $LMN$ as tangent at the vertex $V$. The following theorems are well known to those who well know them (see [63, pp.26–39] for example).
1.20. **THE STEINER LINE THEOREM.**

**Theorem.** The foot of the perpendicular from the focus onto a tangent of a parabola lies on the tangent at the vertex.

That is, given a point $P$ and a line, then, as $L$ moves along the line, the perpendicular to $PL$ through $L$ envelopes a parabola.

**Theorem.** Three tangents to a parabola form a triangle whose circumcircle passes through the focus of the parabola.

**Steiner’s Theorem.** The orthocentre of a triangle circumscribing a parabola lies on the directrix.

These form a sort of dual of the theorem (see the fifth proof in Chapter 2 below) concerning the orthocentre and the circumcircle of a triangle inscribed in a rectangular hyperbola.

Compare the dual of Feuerbach’s theorem. If we consider a family of four-line conics, that is the system of conics touching four lines, in the case where one of the lines is the line at infinity, then the conics will be parabolas.

---

**1.20 The Steiner Line Theorem.**

If, instead of dropping perpendiculars onto the edges from a point on the circumcircle, we reflect the point in the edges then we get (sets of) three points lying on a line parallel to the Wallace line and twice as far from the point(s) on the circumcircle(s). This theorem is due to Steiner. Note that the circumcircles, in general, are reflections of one another in the edges of the triangle (i.e., the orthocentric quadrangle).

Figure 1.27 should be completed by adding the other two special positions of the Simson-Wallace line at angles of $60^\circ$ with the dashed line already exhibited. These each contain 12 points as well as the Centre of the triangle. Then there will be 8 parallels to each of these three lines, passing one through each of the eight vertices. Compare Figure 1.19, which also represents the BEAT twins.
1.21 The Steiner lines and the Droz-Farny theorem.

[The reference to the Droz-Farny theorem points to the fact that the doubled Wallace lines of diametrically opposite points on the circumcircle are perpendicular lines through the orthocentre. See next section, 1.22.]

Take, as before, an arbitrary point, 7B, on the circumcircle of the triangle 124 and reflect it, and the circumcircle, in the edge 24. This gives a point 1A on the circumcircle of the triangle 247. Reflect also in the edges 41 and 12 giving points 2A and 4A on the respective circumcircles of triangles 147 and 271.

Then the points 1A, 2A, 4A lie on a Steiner line, line A say, parallel to the Wallace line of the point 7B and twice its distance from 7B. As we know that the Wallace line bisects the segment joining 7 to 7B the Steiner line A will pass through the vertex (orthocentre) 7.

Now consider the Steiner line of the point 1A which lies on the circumcircle of triangle 247. The reflexions of 1A in the edges 47, 72, and 24 are the points 2B, 4B, 7B, which lie on a line, B say, through vertex 1.

Similarly, for the Steiner line of the point 2A, which lies on the circumcircle of triangle 471. The reflexions of 2A in the edges 71, 14, and 47 are the points 4C, 7B, and 1C. These three points lie on the line C which passes through vertex 2.

Finally, for the Steiner line of the point 4A, which lies on the circumcircle of triangle 712. The reflexions of 4A in the edges 12, 27, and 71 are the points 7B, 1D, and 2D, which lie on the line D which passes through vertex 4.

The previous paragraph should not have started “Finally” since there are still some (how many?) points on circumcircles which reflect into other points on other circumcircles, some of which are new . . . ?

I suspect that, in general, we have an infinite “trisequence”. Each point has 3 followers, of which one is also a predecessor.

In the following table it appears that 7H = 7E and 7I = 7F. Also 7G is diametrically across circle 124 from 7B. So we have some finite cycles. Line O will coincide with line A’ (perpendicular to line A; see Section 1.22 and Fig. 1.34).

This is because, in the notation given below, I chose 7B with \( \theta = \pi/2 \) ! I’ll try again.
Here’s a new table and figure based on a new 7B which is not a rational multiple of \( \pi \) from any of the vertices 1, 2, or 4. Lack-a-day!! That’s not true!! The new \( \theta = \arctan(161/240) = \pi/2 - \beta \) !!

For the old table, see the Apocrypha below.

| New name | Reflect | in edges | to give | lying on | through | slope |
|----------|---------|----------|---------|----------|---------|-------|
| P7       | 7B      | 24, 41, 12 | 1A, 2A, 4A | line A = L7 | 7 | 23/7 |
| P71      | 1A      | 47, 72, 24 | 2B, 4B, 7B | line B = L71 | 1 | –1/7 |
| P72      | 2A      | 71, 14, 47 | 4C, 7B, 1C | line C = L72 | 2 | 97/71 |
| P74      | 4A      | 12, 27, 71 | 7B, 1D, 2D | line D = L74 | 4 | –1 |
| P712     | 2B      | 71, 14, 47 | 4E, 7E, 1A | line E = L712 | 2 | –97/71 |
| P714     | 4B      | 12, 27, 71 | 7F, 1A, 2F | line F = L714 | 4 | 1 |
| P724     | 4C      | 12, 27, 71 | 7G, 1G, 2A | line G = L724 | 4 | –401/79 |
| P721     | 1C      | 47, 72, 24 | 2A, 4H, 7E | line H = L721 | 1 | 1841/887 |
| P741     | 1D      | 47, 72, 24 | 2I, 4A, 7F | line I = L741 | 1 | –7 |
| P742     | 2D      | 71, 14, 47 | 4A, 7G, 1J | line J = L742 | 2 | 41/113 |
| P7124    | 4E      | 12, 27, 71 | 7K, 1K, 2B | line K = L7124 | 4 | 17/31 |
| P7127    | 7E      | 24, 41, 12 | 1C, 2B, 4L | line L = L7127 | 7 | |
| P7147    | 7F      | 24, 41, 12 | 1D, 2M, 4B | line M = L7147 | 7 | |
| P7142    | 2F      | 71, 14, 47 | 4B, 7N, 1K | line N = L7142 | 2 | |
| P7247    | 7G      | 24, 41, 12 | 1O, 2D, 4C | line O = L7247 | 7 | |
| P7241    | 1G      | 47, 72, 24 | 2D, 4C, 7P | line P = L7241 | 1 | |
| P7214    | 4H      | 12, 27, 71 | 7G, 1C, 2Q | line Q = L7214 | 4 | |
|          | 2I      | 71, 14, 47 | 4R, 7R, 1D | line R | 2 | |
|          | 1J      | 47, 72, 24 | 2D, 4R, 7S | line S | 1 | |
|          | 7K      | 24, 41, 12 | 1T, 2T, 4E | line T | 7 | |
|          | 1K      | 47, 72, 24 | 2F, 4E, 7U | line S | 1 | |
|          | 4L      | 12, 27, 71 | 7E, 1V, 2T | line V | 4 | |
|          | 2M      | 71, 14, 47 | 4W, 7F, 1W | line W | 2 | |
|          | 7N      | 24, 41, 12 | 1X, 2M, 4X | line X | 7 | |

etc,
Figure 1.32: Start of new “trisequence”.
1.21. THE STEINER LINES AND THE DROZ-FARNY THEOREM.

Note that if a point lies on a circumcircle, then the midpoint of its join to the corresponding orthocentre lies on the Central Circle (9-pt circle) of the triangle (orthocentric quadrangle). If the figure is scaled so that the radius of the Central Circle is 1, then the midpoints in the following list have coordinates

\[
\left( \frac{r^2 - s^2}{r^2 + s^2}, \frac{2rs}{r^2 + s^2} \right)
\]

where \(r, s\) are given in the right hand column.

| point | coords | orthocentre | midpoint | (r,s) |
|-------|--------|-------------|----------|-------|
| 7B    | (-62,117) | 7 (36,51) | (-13,84) | (6,7) |
| 1A    | (-62,-271) | 1 (36,103) | (-13,-84) | (6,-7) |
| 2A    | (22822.55541)/17^2 | 2 (-204,-77) | (-18067,-16644)/17^2 | (57,146) |
| 4A    | (22.5) | 4 (132,-77) | (77,-36) | (9,-2) |
| 2B    | (9314,1373)/25 | 2 (-204,-77) | (2107,276)/25 | (46,3) |
| 4B    | (-286,149)/289 | 4 (132,-77) | (-74,36) | (2,9) |
| 4C    | (-2014,55541)/17^2 | 4 (132,-77) | (18067,16644) | (73,-57) |
| 1C    | (-806074.356057)/85^2 | 1 (36,103) | (-272987,550116)/85^2 | (413,666) |
| 1D    | (-10.65) | 1 (36,103) | (13,84) | (7,6) |
| 2D    | (50,5) | 2 (-204,-77) | (-77,-36) | (2,-9) |
| 4E    | (-7514,1573)/25 | 4 (132,-77) | (-2107,-276)/25 | (3,46) |
| 7E    | (-806074,-1468707)/85^2 | 7 (36,51) | (-272987,-550116)/85^2 | (413,-46) |
| 7F    | (-10,-219)/289 | 7 (36,51) | (13,-84) | (7,-6) |
| 2F    | (359,149)/289 | 2 (-204,-77) | (79,36) | (9,2) |
| 7G    | (31670,10629)/17^2 | 7 (36,51) | (21037,12684) | (151,42) |
| 1G    | (10799938,-5289871)/17^4 | 1 (36,103) | (6903347,-1656396) | (2646,313) |
| 4H    | (-99465434,-147393187)/1445^2 | 4 (132,-77) | (88976933,-154085556)/1445^2 | (11523,6686) |
| 2I    | (-204,-77) | 2 (36,51) | (36,103) |
| 7I    | 7 (36,51) | 7 (36,103) |
| 1J    | 1 (36,103) | 1 (36,103) |
| 7K    | 7 (36,51) |
| 1K    | 1 (36,103) |
Let the angles of the triangle 124 be $\alpha$, $\beta$ and $\gamma$, so that $\alpha + \beta + \gamma = \pi$.

We will measure arc-lengths on the four circles counter-clockwise starting at 1 on circle 124, so that vertex 2 is at $2\gamma$ and vertex 4 is at $2\alpha + 2\gamma$. Start at 7 for each of the circles 724, 741 and 712, so that vertices 7, 2, and 4 on circle 724 are at 0, $\pi - 2\beta$, and $\pi + 2\gamma$; vertices 7, 4, and 1 on circle 741 are at 0, $\pi - 2\gamma$, and $\pi + 2\alpha$; and vertices 7, 1, and 2 on circle 712 are at 0, $\pi - 2\alpha$, and $\pi + 2\beta$.

Let the point $7B$ (= P7) be at arc-length $\theta$ from vertex 1 on circle 124, so that arc $71A$ on circle 724 is $\pi - 2\beta + 2\gamma - \theta$ and arc $72A$ on circle 741 is $2\pi - (\pi - 2\alpha + \theta) = \pi + 2\alpha - \theta$.

Also arc $74A$ on circle 712 is $2\pi - (\theta - (\pi - 2\alpha)) = 3\pi - 2\alpha - \theta$.

Next we reflect point 1A (= P71) in edges 24, 47 and 72, giving points P7, P712 (= 2B) and P714 (= 4B). These last two points are at arc-lengths $\pi - 2\gamma + 2\beta + \theta$ from 7 on circle 741 and $2\pi - (2\gamma - \theta) - (\pi - 2\beta) = \pi + 2\beta - 2\gamma + \theta$ from 7 on circle 712.

Now reflect point 2A (= P72) in edges 14, 47 and 71, giving points P7, P721 (= 1C) and P724 (= 4C). These last two are at arc-lengths $\theta + \pi - 2\alpha$ from 7 on circle 724 and $\pi - 2\alpha + \theta$ from 7 on circle 714.

Then 4A (= P74), reflected in edges 12, 27 and 71, gives points P7, P741 (= 1D) and P742 (= 2D).

Note: 4H appears to be on the Central Circle, but its distance from the (9-pt) Circle is 85.159353528247..., not 85.

Thanks to Andrew Bremner,

for discovering that there are four 3-cycles; i.e. 12 points which each reflect into two others. They are 3 points on each of the four circumcircles, diametrically opposed to the four vertices 1, 2, 4, and 7 of any orthocentric quadrangle. In our particular numerical example,

the point $(-108,-205)$ is diametrically opposed to the vertex 1 (36,103) on the 7-circle (124-circle), centre $(-36,-51)$. Its reflexions in edges 24, 41, 12 are $(-108,51)$, $(372,51)$, $(-300,51)$, which lie respectively on the 1-circle, 2-circle, and 4-circle. The first of these is diametrically opposite to the point $(36,-257)$ on the 1-circle; the other two are both vertex 1. They all lie on the line $y = 51$ which passes through the vertex 7 (36,51) and is parallel to the edge 24.
Thanks also to Alex Fink,
who observes that every point leads to three 6-cycles. For example, start from a point on the 7-circle and reflect it in the edges 14, 24, 47, 14, 24, 47; a hexagonal path whose edges are parallel to those of a 3-cycle, traversed twice.

He also noted that the six collinearities 12,21,12',21'; 14,41,14',41';
17,71,17',71'; 24,42,24',42'; 27,72,27',72'; 47,74,47',74'; concur in threes at the vertices (–108,–30) (612,231) (–396,231) (–108,–153) of an orthocentric quadrangle, homothetic to, and three times the size of, the original quadrangle 1247. The centre of perspective is, of course, the Centre of the triangle. Moreover

The Central Circle of the Trebled Triangle is the Steiner Circle of the original triangle. And the centroids of the Trebled Triangle are the de Longchamp points of the original triangle (orthocentric quadrangle).

Figure 1.33: Four three-cycles.
In the two-digit labels of the twelve points the first digit refers to the circle on which the point is situated and the second is the vertex to which it is opposite. For example, the point 47' is on the 4-circle, or 127-circle, and is diametrically opposite to the vertex 7.

Figure 1.34: A trebled triangle.
Figure 1.35: Several lines.
Apocrypha

This is the first numerical example, where $\theta = \pi/2$, leading to non-typical coincidences.

| Reflect | in edges | to give | lying on | through | slope |
|---------|----------|---------|----------|---------|-------|
| 7B      | 24, 41, 12 | 1A, 2A, 4A | line A  | 7       | 1     |
| 1A      | 47, 72, 24 | 2B, 4B, 7B | line B  | 1       | $41/113$ |
| 2A      | 71, 14, 47 | 4C, 7B, 1C | line C  | 2       | 7     |
| 4A      | 12, 27, 71 | 7B, 1D, 2D | line D  | 4       | $-7/23$ |
| 2B      | 71, 14, 47 | 4E, 7E, 1A | line E  | 2       | $-7$  |
| 4B      | 12, 27, 71 | 7F, 1A, 2F | line F  | 4       | $7/23$ |
| 4C      | 12, 27, 71 | 7G, 1G, 2A | line G  | 4       | $23/7$ |
| 1C      | 47, 72, 24 | 2A, 4H, 7E | line H  | 1       | $71/97$ |
| 1D      | 47, 72, 24 | 2I, 4A, 7F | line I  | 1       | $97/71$ |
| 2D      | 71, 14, 47 | 4A, 7G, 1J | line J  | 2       | $-1/7$ |
| 4E      | 12, 27, 71 | 7K, 1K, 2B | line K  | 4       | $503/329$ |
| 7E      | 24, 41, 12 | 1C, 2B, 4L | line L  | 7       | $17/31$ |
| 7F      | 24, 41, 12 | 1D, 2M, 4B | line M  | 7       | $73/401$ |
| 2F      | 71, 14, 47 | 4B, 7N, 1K | line N  | 2       | $-9569/10611$ |
| 7G      | 24, 41, 12 | 1O, 2D, 4C | line O  | 7       | $-71/97$ |
| 1G      | 47, 72, 24 | 2D, 4C, 7P | line P  | 1       | $-1$  |
| 4H      | 12, 27, 71 | 7G, 1C, 2Q | line Q  | 4       | $7/601$ |
| 2I      | 71, 14, 47 | 4R, 7R, 1D | line R  | 2       |       |
| 1J      | 47, 72, 24 | 2D, 4R, 7S | line S  | 1       |       |
| 7K      | 24, 41, 12 | 1T, 2T, 4E | line T  | 7       |       |
| 1K      | 47, 72, 24 | 2F, 4E, 7U | line S  | 1       |       |
| 4L      | 12, 27, 71 | 7E, 1V, 2T | line V  | 4       |       |
| 2M      | 71, 14, 47 | 4W, 7F, 1W | line W  | 2       |       |
| 7N      | 24, 41, 12 | 1X, 2M, 4X | line X  | 7       |       |
| etc     | etc       | etc      | etc      | etc     | etc   |
1.21. THE STEINER LINES AND THE DROZ-FARNY THEOREM.

Figure 1.36: Start of a “trisequence”.

[Diagram of geometric figures and points labeled with letters like A, B, C, D, E, F, G, H, K, L, M, N, etc., showing lines and circles intersecting and forming a complex pattern.]
Note that if a point lies on a circumcircle, then the midpoint of its join to the corresponding orthocentre lies on the Central Circle (9-pt circle) of the triangle (orthocentric quadrangle). If the figure is scaled so that the radius of the Central Circle is 1, then the midpoints in the following list have coordinates

\[
\left( \frac{r^2 - s^2}{r^2 + s^2}, \frac{2rs}{r^2 + s^2} \right)
\]

where \(r, s\) are given in the right hand column.

| point | coords | orthocentre | midpoint | (r,s) |
|-------|--------|-------------|----------|------|
| 7B    | (–190,21) | 7 (36,51)    | (–77,36) | (2,9) |
| 1A    | (–190,–175) | 1 (36,103)   | (–77,–36) | (2,–9) |
| 2A    | (230,245)   | 2 (–204,–77) | (13,84)  | (7,6) |
| 4A    | (–106,–91)  | 4 (132,–77)  | (13,–36) | (7,–6) |
| 2B    | (7906,5117)/25 | 2 (–204,–77) | (1403,1596)/25 | (42,19) |
| 4B    | (–80222,–3115)/289 | 4 (132,–77) | (–21037,–12684)/289 | (42,–151) |
| 4C    | (–158,245)  | 4 (132,–77)  | (13,84)  | (6,7) |
| 1C    | (–5114,–2023)/25 | 1 (36,103)   | (–2107,276)/25 | (3,46) |
| 1D    | (–46538,3521)/289 | 1 (36,103)   | (–18067,16644)/289 | (57,146) |
| 2D    | (178,–91)   | 2 (–204,–77) | (13,–84) | (6,–7) |
| 4E    | (–6106,5117)/25 | 4 (132,–77)  | (–1403,1596)/25 | (19,42) |
| 7E    | (–5114,–1827)/25 | 7 (36,51)    | (–2107,–276)/25 | (3,–46) |
| 7F    | (–46538,–48027)/289 | 7 (36,51)    | (–18067,–16644) | (57,–146) |
| 2F    | (101030,–3115)/289 | 2 (–204,–77) | (21037,–12684) | (151,–42) |
| 7G    | (118,–123)  | 7 (36,51)    | (77,36)  | (9,–2) |
| 1G    | (25730,–63055)/289 | 1 (36,103)   | (18067,–16644) | (146,–57) |
| 4H    | (–1499674,–543907)/85² | 4 (132,–77) | (–272987,–550116)/85² | (413,–666) |
| 2I    | 2 (–204,–77) | 2 (–204,–77) | 2 (–204,–77) | 2 (–204,–77) |
| 7I    | 1 (36,103)   | 1 (36,103)   | 1 (36,103) | 1 (36,103) |
| 7K    | (34466,–121563)/5⁴ | 7 (36,51)    | (28483,–44844)/5⁴ | (202,–111) |
| 1K    | (54214,–1931527) | 1 (36,103)   | (157157,–593676) | (621,–478) |
Figure 1.37: The neighborhood of vertex 2 and point 7B
1.22 Doubling Wallace(-Simson) and Droz-Farni.

It is known (see elsewhere) that as a point moves round the circumcircle, the associated Wallace(-Simson) line rotates with half the angular velocity and in the opposite sense. So also does the doubled Wallace(-Simson) line. If we draw such lines for two points on the circumcircle which are diametrically opposed they will form a pair of perpendicular lines through the orthocentre, to which we may apply the Droz-Farni theorem.

In Fig. 1.38 the point 7B generates the douwall line A, and 7B' generates the perpendicular line A'. It appears that line A' is the same as both line O, and line P, passing through 1O, 2D, and 4C on the one hand and 2D, 4C and 7P on the other. Is this true? If so, is it just a coincidence? Evidently 7B' is the same as 7G.
Chapter 2

Five proofs that a triangle has an orthocentre.

First we’ll start with Euclid, who showed that the perpendicular bisectors of the edges of a triangle concurred at a point, the circumcentre, equidistant from the vertices of the triangle. On drawing, through each vertex of a triangle, a parallel to the opposite edge, we produce three parallelograms which form a twice-sized triangle. Since opposite edges of a parallelogram are equal in length, the larger triangle has the vertices of the original triangle for the midpoints of its edges, and the altitudes of the original triangle are the perpendicular bisectors of the edges of the larger triangle, and they consequently concur.

Figure 2.1: Perpendicular bisectors of edges of large triangle are altitudes of small triangle
Second we draw the edge-circles of the triangle, that is, circles having an edge of the triangle for diameter. In Figure 2.2 the circles with diameters $V_2V_4$, $V_4V_1$, $V_1V_2$ cut the other other edges in $D_6$, $D_5$, $D_3$. These points are unique, since the angle in a semi-circle is a right-angle, and $V_1D_6$, $V_2D_5$, $V_4D_3$ are altitudes of the triangle. But these are also radical axes of pairs of the edge-circles, and therefore concur in their radical centre.
Third: this proof also uses edge-circles, though we have to extend our definition of an edge of the triangle $V_1V_2V_4$ to include $V_1V_8$, $V_2V_8$ and $V_4V_8$. First, let the altitudes $V_2D_5$ and $V_4D_3$ meet at $V_8$, so that $V_1D_3V_8D_5$ is cyclic with $V_1V_8$ as a diameter. This new edge-circle, taken with new edge-circles on $V_2V_8$ and $V_4V_8$ as diameters, again have altitudes for radical axes and orthocentre for radical centre. See Figure 8.3.

Fourth: the (original) Clover-leaf Theorem.

In Figure 2.4 the radical centre of the three circles drawn with the medians of a triangle as diameters is the orthocentre of the triangle. The radical axes of pairs of circles are the altitudes, shown dashed.
To prove the Clover-leaf theorem, we show that the radical axis of two of the circles, for example, in Figure 2.5, those with diameters $V_1M_6$ and $V_4M_3$, is an altitude of the triangle. The line of centres (dashed in Figure 2.5) is the join of the midpoints of $V_1M_6$ and $V_4M_3$ and so is midway between $M_3M_6$ and $V_1V_4$, and parallel to them. So the radical axis is perpendicular to $V_1V_4$, and it suffices to show that $V_2$ is on the radical axis. First note that the $V_4M_3$-circle passes through $K$, the midpoint of $V_2D_6$ (we shall call such a point a \textbf{midfoot point}), since $KM_3$ is parallel to $D_6V_1$, making $V_4KM_3$ a right angle. Now the square on the tangent from $V_2$ to the $V_1M_6$-circle equals $V_2M_6 \cdot V_2D_6 = (\frac{1}{2}V_2V_4)(2V_2K) = V_2V_4 \cdot V_2K$ which equals the square on the tangent from $V_2$ to the $V_4M_3$-circle, and indeed $V_2$ is on the radical axis of the two circles.
Fifth, a pair of perpendicular lines $V_1V_8, V_2V_4$ intersects another pair $V_2V_8, V_1V_4$ in the four points $V_1, V_2, V_4, V_8$. These two pairs are rectangular hyperbolas. Any conic passing through these four points is a rectangular hyperbola. In particular, the pair of lines $V_1V_2, V_4V_8$ is a rectangular hyperbola, so that this pair of lines is also perpendicular.

Somewhat more generally, choose axes and scale so that the three vertices of the triangle lie on a rectangular hyperbola, say at $(t_1,1/t_1), (t_2,1/t_2)$ and $(t_3,1/t_3)$.

[This can be done in infinitely many ways; see next paragraph.]

Then the slope of the join of the last two is $\left\{1/t_3 - 1/t_2\right\}/\left\{t_3 - t_2\right\} = -1/t_2t_3$. The equation to the line through the first of the three vertices and having a perpendicular slope is $y - 1/t_1 = t_2t_3(x - t_1)$. This cuts the hyperbola again in the point $(-1/t_1t_2t_3, -t_1t_2t_3)$. The symmetry shows that the other two perpendiculars also pass through the same point.

Moreover, if the circumcircle of the triangle cuts the hyperbola again in the point $(t_4,1/t_4)$ and has equation $x^2 + y^2 - 2gx - 2fy + c = 0$, then $t_1, t_2, t_3, t_4$ are the roots of

$$t^2 + 1/t^2 - 2gt - 2f/t + c \quad \text{or} \quad t^4 - 2gt^3 + ct^2 - 2ft + 1.$$  

The product of the roots is 1, so $t_4 = 1/t_1t_2t_3$ and the 4th point of intersection of the circumcircle is diametrically across the hyperbola from the orthocentre. The centre of the hyperbola is the midpoint of the join of the orthocentre to a point on the circumcircle. The locus of the centres of rectangular hyperbolas drawn through the three vertices of a triangle (i.e., through the four vertices of an orthocentric quadrangle) is midway between the orthocentre and the circumcircle; it is the 50-point circle! This is a special case of Feuerbach’s theorem, which tells us that the locus of the centres of conics passing through 4 points is a conic. In this case, six of the points on the conic are the 3 midpoints and the 3 diagonal points, so that here the conic is indeed a circle.

Sixth, in case five aren’t enough, use the case $n = 2$ of the Lighthouse Theorem (§6.9 in Chapter 6 below).
2.1 What is a 5-point (medial) circle?

The circles of the clover-leaf theorem were discovered as 5-point circles, but we will call them medial circles. In the notation of Figure 2.5, the points \( V_1, M_6, D_6, \) and the midpoints of \( V_2D_3 \) and \( V_4D_5, \) are concyclic. The last two points are the feet of the perpendiculars from \( M_6 \) onto \( V_1V_2 \) and \( V_1V_4; \) call them midfoot points.

A medial circle is a “9-point circle”.

In Figure 2.5 the \( V_1M_6 \)-circle and the \( V_4M_3 \)-circle intersect in two points on the \( A_2D_5 \) altitude, say \( B_i \) for the interior point and \( B_e \) for the exterior point. [Better notation needed for the altitude points.] Similarly, the \( V_2M_5 \)-circle and the \( V_4M_3 \)-circle intersect \( V_1D_6 \) in \( A_i \) and \( A_e \) and the \( V_1M_6 \)-circle and the \( V_2M_5 \)-circle concur at \( C_i \) and \( C_e \) on \( A_4D_3. \)

How does one characterize these points?

We can define them as follows. E.g., \( A_i \) and \( A_e \) are given by the product and the sum of their distances from \( V_1: \)

\[
V_1A_i \cdot V_1A_e = (\frac{1}{2}V_1D_6) \cdot V_1V_4 = V_1D_3 \cdot V_1M_3 = b \cos A \cdot \frac{1}{2}c = \frac{1}{2}bc \cos A = 2R^2 \cos A \sin B \sin C
\]

\[
\text{and} \quad V_1A_i + V_1A_e = \frac{3}{2}V_1D_6 = \frac{3}{2}b \sin C = \frac{3}{2}c \sin B = 3R \sin B \sin C
\]

where \( R \) is the circumradius of \( V_1V_2V_4. \)

Each medial circle passes through two such points on each of two altitudes, making it a 9-point circle, though not a traditional one. These points deserve a name. Call them altitude points. The interior altitude points and the exterior altitude points each form triangles in perspective with \( V_1V_2V_4, \) and with the pedal triangle \( D_6D_5D_3, \) having the orthocentre of \( V_1V_2V_4 \) as perspector. At first glance they appear to be respectively homothetic to \( V_1V_2V_4 \) and to the pedal triangle \( D_6D_5D_3, \) but that is not so. What are the various perspectrices?

Notice that the formulas for the product and the sum of the distances of a pair of altitude points from their vertex, namely

\[
2R^2 \cos A \sin B \sin C \quad \text{and} \quad 3R \sin B \sin C
\]

\[
2R^2 \sin A \cos B \sin C \quad \text{and} \quad 3R \sin A \sin C
\]

\[
2R^2 \sin A \sin B \cos C \quad \text{and} \quad 3R \sin A \sin B
\]

extravert into

[LEFT TO READER FOR THE TIME BEING!]

and quadrate into

[LEFT TO READER FOR THE TIME BEING!]
2.1. WHAT IS A 5-POINT (MEDIAL) CIRCLE?

The degenerate case

If our triangle is right-angled at $V_1$, then the midpoints of $V_1V_2$ and $V_1V_4$ coincide with the midpoints of the joins of $V_2$ and $V_4$ to the orthocentre, $V_8$, which is also the foot of each of the perpendiculars from $V_2$ and $V_4$ onto $V_1V_4$ and $V_1V_2$ respectively, and the midpoint of the join of vertex $V_1$ to itself as orthocentre, so

The 9-point circle becomes a 5-point (medial) circle

![Diagram of a triangle with midpoints and orthocentre labeled]

Figure 2.6: If the triangle $V_1V_2V_4$ is right-angled at $V_1$, then the 9-point circle degenerates into a 5-point (medial) circle $V_1^3 M_6^2 D_6 M_6 M_3^2$

Sixty years ago the present writer showed [100] that a necessary and sufficient condition for a triangle to be right-angled is that the sum of the touch-radii, $r + r_1 + r_2 + r_3$, should be equal to the perimeter, $2s$. Also that another such condition is $rr_3 = r_1r_2$. This was discovered after noting that, for a 3,4,5 triangle, the radii are 1,2,3,6, and that 6 is a perfect number.
The obtuse case

If the angle at $V_1$ is obtuse, then the orthocentre lies outside the triangle, there are only two real altitude points, both of which are exterior to the triangle, on the altitude through $V_1$. Only one pair of medial circles intersect in real points. However, the three altitudes are still radical axes for the pairs of circles. This case does not occur in the context in which we are presently interested.

It is well known (and quadration makes it clear) that

**Theorem** [103] There are three times as many obtuse triangles as there are acute ones.

So that there are only $12 \ (= 6 + 3 \times 2)$ real altitude points in a quadrated triangle, not $24$. 

Figure 2.7: Only one pair of medial circles intersect. The $V_1$-circle doesn’t intersect the other altitudes in real points, but the altitudes (dashed) are still radical axes.
2.2 The 4-leaf clover theorem.

If we throw in the 9-point circle to the clover-leaf theorem we have a four-leaf clover, each of the six pairs of circles has either an altitude or an edge for radical axis, which concur in threes at the vertices and orthocentre of the triangle. [In the language of Quadration, each pair of circles has an edge of the (quadrated) triangle as a radical axis. See next chapter.]

Figure 2.8: The four-leaf clover theorem
Chapter 3

A triangle is an orthogonal quadrangle!

It is reasonable to give equal status to the orthocentre and the vertices of a triangle, since then each point is the orthocentre of the triangle formed by the other three.

3.1 Quadration

We then have 4 vertices, 6 edges, 4 triangles, 12 medians, 12 medial circles. The same 9-point circle serves for all four triangles. Note that the edges serve equal time as edges and altitudes, so that there are twelve altitude points.

3.2 Where did medial circles come from?

Ans: The 260?-point sphere. And where did the 260?-point sphere come from? Obviously the number 260 is arbitrary, and has not been given an updated explanation in what follows.
Figure 3.1: Twelve medial circles and one 9-point circle. There are three circles though each of the 4 vertices; three circles through each of the 6 midpoints, and four through each of the 3 diagonal points; two circles through each of the 12 altitude points, and two through each of the 12 midfoot points.
Chapter 4

Ans: The orthocentric tetrahedron

Although the altitudes of a triangle concur, the altitudes of a tetrahedron do not usually do so. In 1827 Jacob Steiner noted that they are generators of an equilateral hyperboloid. That is, a hyperboloid of one sheet formed by rotating a rectangular hyperbola about its ‘imaginary’ axis. If two altitudes concur, the hyperboloid degenerates into a pair of planes and the other two altitudes also concur, and the tetrahedron is called semi-orthocentric. If two altitudes intersect a third, then all four concur.

Just over two hundred years ago, Gaspard Monge discovered that the six planes (call them midplanes), each passing through a midpoint of an edge of a tetrahedron (any tetrahedron) and perpendicular to the opposite edge, all passed through a single point, now called the Monge point of the tetrahedron. The Monge point is the centre of the equilateral hyperboloid; it specializes to the orthocentre of the tetrahedron in case the tetrahedron is orthocentric, which is a very interesting configuration. See [110], which has a good bibliography; also [39], and [16].

Here we follow [131] and [179, pp. 13–14].

Suppose that a tetrahedron has an orthocentre and take that as origin, $O$. Let the vertices $A, B, C, D$ define vectors $OA = \mathbf{a}, OB = \mathbf{b}, OC = \mathbf{c}, OD = \mathbf{d}$. Then $\mathbf{a}$ is perpendicular to the plane $BCD$, the scalar products
\[
\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (\mathbf{c} - \mathbf{d}) = \mathbf{a} \cdot (\mathbf{d} - \mathbf{b}) = 0,
\]
and by symmetry the six products $\mathbf{a} \cdot \mathbf{b}, \mathbf{a} \cdot \mathbf{c}, \mathbf{a} \cdot \mathbf{d}, \mathbf{b} \cdot \mathbf{c}, \mathbf{c} \cdot \mathbf{d}, \mathbf{d} \cdot \mathbf{b}$, are all equal (to $\sigma$, say).

Let the altitudes through $A, B, C, D$ meet the opposite faces in $O_7, O_6, O_5, O_3$, respectively. Then the scalar product $\mathbf{a} \cdot \mathbf{b}$, for example, is equal to the magnitude of $\mathbf{a}$ times the projection of $\mathbf{b}$ on it. More generally,
\[
\sigma = OA \cdot OO_7 = OB \cdot OO_6 = OC \cdot OO_5 = OD \cdot OO_3
\]
Opposite edges are perpendicular. For example,
\[(b - a) \cdot (d - c) = \sigma - \sigma - \sigma + \sigma = 0\]

Note that this implies that each Monge midplane not only passes through the midpoint of an edge, but it also contains that edge. Moreover, since
\[(b - a)^2 + (d - c)^2 = a^2 + b^2 + c^2 + d^2 - 4\sigma\]
is symmetrical in \(a, b, c, d\), the sum of the squares of the lengths of opposite edges is constant.

The feet of the altitudes are the orthocentres of the faces. E.g., \(BO_7\) is in the plane \(OAB\) and so has shape \(h a + k b\) for some \(h, k\) and
\[(h a + k b) \cdot (d - c) = h\sigma - h\sigma + k\sigma - k\sigma = 0\]
so that \(BO_7\) is perpendicular to \(CD\), and similarly \(CO_7 \perp DB\) and \(DO_7 \perp BC\).

The square of the distance of the point \(\frac{1}{2}(a + b + c + d)\) from \(A\) is
\[\left\{\frac{1}{2}(a + b + c + d) - \frac{1}{2}(c + d)\right\}^2 = \left\{\frac{1}{4}(b + c - d)\right\}^2 = \frac{1}{4}(a^2 + b^2 + c^2 + d^2 + 6\sigma - 6\sigma).
\]
By symmetry, the point \(\frac{1}{2}(a + b + c + d)\) is equidistant from the four vertices of the tetrahedron and so is the circumcentre, \(Q\), say.

The centroid, \(G = \frac{1}{4}(a + b + c + d)\), is the midpoint of \(OQ\), in a somewhat analogous fashion for the Euler line of a triangle. In fact, for the general tetrahedron, the centroid is the midpoint between the Monge point and the circumcentre. The analog of the 9-point circle for the general tetrahedron is the 12-point sphere, which passes through the centroids of the faces of the tetrahedron, and has its centre midway between the Monge point and the centroid. But we will soon see that it is the centroid that is the centre of our 260?-point sphere. For example, the square of the distance of the centroid from the midpoint \(\frac{1}{2}(c + d)\) of \(CD\) is
\[\left\{\frac{1}{4}(a + b + c + d) - \frac{1}{2}(c + d)\right\}^2 = \left\{\frac{1}{4}(a + b - c - d)\right\}^2 = \frac{1}{16}(a^2 + b^2 + c^2 + d^2 + 4\sigma - 8\sigma)\]
and is equal to that from each of the other five midpoints. So we have six points on a sphere.

Of course, for any tetrahedron, the joins of the midpoints of opposite edges concur and bisect each other, but in general the lengths of these three joins are not equal.

### 4.1 Think inside the box!

It helps to visualize properties of tetrahedra if you put them in boxes. Through each of the six edges, draw a plane parallel to the opposite edge. These six planes are the faces of a parallelepiped. In the case of the regular tetrahedron, the box is a cube, of course.
For the more general orthocentric tetrahedron, since opposite edges are perpendicular, the diagonals of the parallelogram faces of the box are perpendicular, so that the faces are rhombuses and the parallelepiped has all twelve edges equal in length. The joins of the midpoints of opposite edges of the tetrahedron have the same length as the edges of the box, and they each have their midpoint at the centre of the box, which is the centroid of the tetrahedron. The midpoints of the edges lie on a sphere with centre at this centroid.

This is our 260-point sphere. There aren't enough letters to cope with the rather large number of points we wish to deal with, so, for the nonce, we introduce the following notation, which is sufficiently different from that used elsewhere in the literature (including other parts of this paper) to avoid confusion. [Probably better to change $V_0$ to $V_8$.] Let the vertices of our orthocentric tetrahedron be $V_0, V_1, V_2, V_4$, and give the label $i + j$ ($1 \leq i + j \leq 6$) to the edge $V_iV_j$ and the label $i + j + k$ ($= 7, 6, 5$ or $3$) to the face $V_iV_jV_k$. For example, the midpoint of the edge $V_iV_j$ is denoted by $E_{i+j}$ and the common foot of the perpendiculars from the other two vertices onto that edge is denoted by $F_{i+j}$. The orthocentre and centroid of the tetrahedron are $O$ and $G$ respectively, and the orthocentre and centroid of the face $V_iV_jV_k$ are $O_{i+j+k}$ and $G_{i+j+k}$. The joins of the midpoints of opposite edges $V_0V_i, V_2V_4$ ($i = 1, 2, 4$; subscripts mod 7) are $E_iE_{7-i}$. They concur and bisect each other at the centroid, $G$. The common perpendiculars of opposite edges are $F_iF_{7-i}(i = 1, 2, 4)$, and they concur at the orthocentre, $O$.

Consider the intersection of the sphere with a face of the tetrahedron (Figure 4.2). It is a circle through the midpoints of the edges of the triangle, that is, its 9-point circle. As there are four faces we appear to have found 36 points on the sphere, but two points on each edge have been counted twice, namely the six midpoints of the edges and the six feet of the common perpendiculars to pairs of opposite edges.
So far we have only 24 points on our sphere. They are $E_i$, $F_i$ ($1 \leq i \leq 6$) and three on each of the four 9-point circles. These turn out to be midfoot points; we will denote the midpoint of $V_iO_j$ by $M_{ij}$, $(i \in \{0, 1, 2, 4\}, j \in \{7, 6, 5, 3\})$. To find more points on the sphere we next consider.

### 4.2 Medial planes of an orthocentric tetrahedron.

The plane through an edge, say $V_1V_4$, and the midpoint, $E_2$, of the opposite edge $V_0V_2$, (call it the $E_5$ medial plane — we economize on notation and use the same name for the plane and the midpoint of the edge through which it passes) cuts the tetrahedron in a triangle, and our sphere in a great circle, since the centre of the sphere, $G$, is the centroid of the tetrahedron, i.e., the midpoint of $E_2E_5$, where $E_5$ is the midpoint of $V_1V_4$. See the left part of Figure 4.3. This circle is a 5-point circle of the triangle, yielding two more midfoot points (black dots), $S_{15}$, $S_{45}$, on the sphere. As there are six such medial planes, we have found 12 points to bring our total back to 36.

The faces and plane sections of an orthocentric tetrahedron are always acute-angled triangles, so that the altitude points are real, and we may include them in our count. They are indicated by small circles; e.g., in the left of Figure 4.3 $T_{15}$ and $T_{65}$, where the medial circle intersects the altitude through $V_1$; and $T_{45}$, $T_{35}$, on the altitude through $V_4$. We now have $4 \times 6$ more points on our sphere, giving a total of 60. In our notation, the first subscript refers to the relevant vertex and the second to the edge, or plane (here number
5). When two points are associated with a vertex \( V_i \), then the interior one has the odious subscript \( i = 0, 1, 2 \) or 4, and the exterior one the evil \( 7 - i = 7, 6, 5 \) or 3.

There are three medial planes through each vertex (1,4,2 through \( V_0 \); 2,6,3 through \( V_2 \); 3,1,5 through \( V_1 \); and 4,5,6 through \( V_4 \), e.g., those through \( V_1 \) intersect in the line \( V_1G \) which in turn intersects the sphere in the medial points \( H_1, H_6 \) which are common to all three planes and circles. The line \( V_4H_4GH_3 \) is omitted from Figure 4.3. Two such points associated with each of 4 vertices brings the total to \( 60 + (2 \times 4) = 68 \).

### 4.3 Midplanes of an orthocentric tetrahedron

We can also consider Monge's midplanes; for example, the plane through an edge, say \( V_1V_4 \) again, which is perpendicular to the opposite edge (see the right of Figure 4.3). This plane contains \( O \), the orthocentre of the tetrahedron, and \( O_6 \) and \( O_3 \), the feet of the perpendiculars from \( V_1 \) and \( V_4 \) onto their opposite faces, and so contains the perpendicular, \( F_5F_2 \), common to the edges \( V_1V_4 \) and \( V_0V_2 \). The intersection with the sphere is not a great circle this time, but it is a circle through the feet of this common perpendicular and also through the midpoint, \( E_5 \), of \( V_1V_4 \), so that it is a medial circle for the triangle \( V_1V_4F_2 \), yielding two midfoot points, \( M_{13} \) and \( M_{46} \), which we have already seen on the 9-point circles. There are three midplanes through each vertex (1,4,2; 2,6,3; etc., as with the medial planes). These concur in the four altitudes \( V_iOO_{7-i} \) which intersect the sphere in eight altitude points, \( K_i, K_{7-i} \) (\( i = 0, 1, 2, 4 \)). This brings our total of points on the sphere to \( 68 + 8 = 76 \).
The number 76 is somewhat arbitrary and has varied upwards and downwards several times during the writing of this paper as we discovered new points or coincidences between old ones. For example, Conway likes to think of the 9-point circle as a 12-point circle with three inscribed rectangles, the fourth corners being the reflexions in the 9-point centre of the feet of the altitudes. This brings our total up to 88. Others would include the points of contact of the three common tangents of the 9-point circle with the deltoid (envelope of the Simson-Wallace line), homothetic with all eighteen Morley triangles (see [134] or [134, pp.72–79]). Now we have a nice round hundred-point sphere. Yet again, Feuerbach’s theorem (due to Brianchon & Poncelet [35]) tells us that the 9-point circle also touches the four touch-circles (incircle and excircles) so this gives us 16 more candidates . . . , but I digress . . . .

4.4 Seeing the sphere

It is hard for most of us to visualize three dimensions. Hopefully, by the time this article appears, there will be a reference to where you can view the sphere on-line, but while we are confined to MONTHLY pages, I will content myself with a stereographic projection from a “north pole” of the sphere onto the tangent plane at the “south pole”, i.e., we will invert with respect to a point on the sphere, so that circles invert into circles. Figure 4.4 shows the (inverses of the) four 9-point circles of the faces of the tetrahedron.
Figure 4.4: Projection of the four 9-point circles, showing the first 24 points of the 260-point sphere. Lower case labels denote the projections of the corresponding upper case points on the sphere.
Figure 4.5 shows the projections of the six medial circles $E_i E_{7-i} F_i$, $i = 1, 2, \ldots 6$, which are the sections of the 260?-point sphere by the medial planes.

Figure 4.5: Projection of the six medial circles in the medial planes, $E_i$ ($1 \leq i \leq 6$) showing the next 12+24+8 points of the 260?-point sphere. The eight medial points, $h_i$ ($0 \leq i \leq 7$), are represented by small circles. $t_{02}$ is a different point from $h_0$. 
4.4. SEEING THE SPHERE

Figure 4.6 shows the projections of the six medial circles $E_i F_i F_{7-i}$, $i = 1, 2, \ldots 6$, which are the sections of the 260?-point sphere by Monge’s midplanes.

Figure 4.6: Projection of the remaining six medial circles, the sections of the 260?-point sphere by Monge’s midplanes, $F_i$, $(1 \leq i \leq 6)$, showing the final eight points, $k_i (0 \leq i \leq 7)$, of the 260?-point sphere, which are represented by small circles; the black dots are the original 24 points on the four 9-point circles.

Why were we interested in the orthocentric tetrahedron?

This is the beginning of our story.
The Pavillet tetrahedron

Axel Pavillet 163] quite recently discovered that if you erect perpendiculars $AA_o$, $BB_o$, $CC_o$ (vertical dashed lines in Figure 5.1) to the plane of the triangle $ABC$, at $A$, $B$, $C$, of respective lengths $s - a$, $s - b$, $s - c$, where $a$, $b$, $c$ are the edge-lengths of the triangle and $s = \frac{1}{2}(a + b + c)$ its semiperimeter, then the three points $A_o$, $B_o$, $C_o$, together with the incentre, $I_o$, of $ABC$, form an orthocentric tetrahedron. Note that $s - a$, $s - b$, $s - c$, are the lengths of the tangents from $A$, $B$, $C$ to the incircle.

This tetrahedron has many interesting properties which will hopefully have been revealed 163] by the time this paper appears. For example, the projection of its orthocentre, $O$, onto the plane $ABC$ is the Gergonne point of the triangle $ABC$; that is, the point of concurrence of the three joins, $AX$, $BY$, $CZ$ of the vertices to the touchpoints $X$, $Y$, $Z$, of the incircle with the opposite edges.

In fact the joins $XA_o$, $YB_o$, $ZC_o$, of these points of contact to the relevant vertices of the Pavillet tetrahedron are three of its altitudes, concurring in its orthocentre, $O$. The plane $A_oB_oC_o$ meets the edges $BC$, $CA$, $AB$ respectively in the points $X'$, $Y'$, $Z'$ which are the harmonic conjugates of $X$, $Y$, $Z$ with respect to $BC$, $CA$, $AB$. The line $X'Y'Z'$ is the polar of the Gergonne point, $G_o$, with respect to the triangle $ABC$, and is called the Gergonne line. This is perpendicular to the projection onto the plane $ABC$ of the fourth altitude of the tetrahedron, which has been called the Soddy line 158] of the triangle $ABC$, though both Apollonius and Descartes might wish to claim priority. This last line contains, in addition to the the incentre and the Gergonne point, the interior and exterior Soddy centres, $S$ and $S'$, and it intersects the Euler line of the triangle $ABC$ in the deLongchamps point 205], $D$, which is the reflexion of the orthocentre of the triangle in its circumcentre, or, alternatively, the orthocentre of the triangle formed by parallels to $BC$, $CA$, $AB$ passing respectively through $A$, $B$, $C$.

The so-called Soddy centres are the centres of the circles which are tangent to each of the
three mutually tangent circles with centres $A$, $B$, and $C$, passing respectively through the touch-points $Y$ & $Z$, $Z$ & $X$, and $X$ & $Y$ of the incircle with the edges of the triangle $ABC$. I will call these circles the tangent circles (compare [69]), since they have the tangents to the incircle from the vertices $A$, $B$, $C$ as radii (lengths $s - a$, $s - b$, $s - c$).

**Theorem.** *The incircle and two Soddy circles of a triangle form a coaxal system with the Soddy line as line of centres and the Gergonne line as radical axis. The circles with centres $X'$, $Y'$, $Z'$, passing respectively through $X$, $Y$, $Z$, are members of the orthogonal coaxal system which has the Gergonne line as line of centres and the Soddy line as radical axis.* [See Figure 5.2] There are extraversions of this theorem; compare §10 below. They are depicted in Figure 5.8.
Figure 5.2: Two orthogonal coaxal systems. The Gergonne and Soddy lines of triangle $ABC$ are the lines of centres of one system and the radical axes of the other. The ‘$o$’ subscripts can be ignored; they are to distinguish from the extraversions in Figures 5.7, 5.8 and 5.9.

In Figure 5.2 only part of the outer Soddy circle is shown. Note that one of the two limiting points on the Soddy line is close to, but not coincident with, the inner Soddy centre. Also included is (an arc of) the circle of the system having the deLongchamps point, $D$, as centre; but no special significance of this circle is yet known to the present writer.
The original draft of this paper considered three cases of triangle: a critical case, and more obtuse, and less obtuse ones. This was influenced by the paper of Veldkamp [206], published a quarter of a century ago in a widely read, and oft-cited, periodical. It unfortunately contains a mistatement relating these angle-dependent cases to the ones we now discuss. The error was eventually corrected by Hajja & Yff [106] in a not so often read journal. With hindsight, our original classification needs to be revised. The situation is confused by the fact that the critical case for the triangle $ABC$ can be defined in a number of equivalent ways, which differ widely in appearance, and only peripherally involve the size of the largest angle.

The following are equivalent definitions of the critical case. The edge-lengths of the triangle are $a, b, c$; the angles $A, B, C$; area $\Delta$; perimeter $2s$; circumradius $R$; inradius $r$; incentre $I_o$; Gergonne point $G_o$.

1. Outer Soddy centre at infinity, outer Soddy circle a straight line;
2. outer Soddy circle coincides with Gergonne line;
3. inner Soddy centre is the midpoint of segment $I_oG_o$;
4. isoperimetric point at infinity (see §8);
5. $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = 2$;
6. $(s-b)^2(s-c)^2 + (s-c)^2(s-a)^2 + (s-a)^2(s-b)^2 = 2\Delta^2$;
7. $2s = 4R + r$;
8. (one of Pavillet’s delightful discoveries) the angle between the fourth altitude (the one through $I_o$) of the Pavillet tetrahedron, and the Soddy line, is $\pi/4$;

and see §9. It is worth noting that these definitions are also equivalent to putting radius $\infty$ into Descartes’s theorem (or putting ‘bend zero’ in Soddy’s Kiss Precise [192]):

$$ \left( \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{\infty} \right)^2 = 2 \times \left( \frac{1}{(s-a)^2} + \frac{1}{(s-b)^2} + \frac{1}{(s-c)^2} + \frac{1}{\infty^2} \right) $$

the other radii, $s-a, s-b, s-c$, being those of the tangent circles.
Figure 5.3: The critical case. The outer Soddy circle coincides with the Gergonne line, \( X'Y'Z' \). The inner Soddy circle, centre \( S_o \), passes through the incentre, \( I \), and the Gergonne point, \( G_o \). The outer Soddy centre is the point at infinity on the Soddy line.

Figure 5.3 shows the critical case, in which the outer Soddy circle is a straight line, coincident with the Gergonne line. One Soddy centre is at infinity, and since these centres form a harmonic range with the Gergonne point, \( G_o \), and the incentre, \( I_o \), the (inner) Soddy centre, \( S_o \), is the midpoint of their join. Note that the Soddy line passes through the deLongchamps point, that is, the reflexion of the orthocentre of \( ABC \) in its circumcentre.

We will rename and redefine the other cases as **external**, with \( 2s < 4R + r \), or **internal**, with \( 2s > 4R + r \). Respectively, the outer Soddy circle is touched externally or internally by the tangent circles. They correspond roughly to the triangle’s being more obtuse or less obtuse than in the critical case, but this correspondence cannot be exact, since there are overlaps, as we shall see in §9. This was the main error in [206].
Figure 5.4 shows the external case. The outer Soddy circle, having $S_e$ as centre, is touched externally by the three tangent circles. The insert shows the detail in the neighborhood of the vertex $A$ and the intersection of the Gergonne line $A'B'C'$ with the Soddy line, which is perpendicular to it. This intersection would be better renamed as the *radical origin* of the triangle $ABC$. The Soddy line contains the Soddy centres, the incentre, the Gergonne point and the deLongchamps point. $S_o$ is not the midpoint of $G_oI_o$ in this case.

Figure 5.4: The external case. The insert shows the inner Soddy circle being touched externally by the $A$-tangent circle and (arcs of) the $B$- and $C$-tangent circles; also portions of the Gergonne line, $X'Y'Z'$, and the (perpendicular) Soddy line, which passes through the incentre, $I_o$, and the Gergonne point, $G_o$, of $ABC$ (but not through $A$).
Figure 5.5 shows the internal case in which the outer Soddy circle is touched internally by the tangent circles. Only an arc of this outer circle is shown. The dashed lines are the Gergonne and Soddy lines. The dotted lines are the lines of centres of the outer Soddy circle with the tangent circles. The three unlabelled points on the Soddy line are the Gergonne point, the inner Soddy centre, and the incentre. They form a harmonic range with the outer Soddy centre. Contrary to appearances, the Gergonne point and the incentre do not lie on the inner Soddy circle.

Figure 5.5: The internal case. The three tangent circles touch the outer Soddy circle internally.

### 5.1 Isoperimetric and equal detour points

In this internal case, with $S$ the centre of the outer Soddy circle ($S_e$ in the figure), and $\rho$ its radius,

- $SA = \rho - (s - a)$,  
- $SB = \rho - (s - b)$,  
- $SC = \rho - (s - c)$,

$SB + SC + BC = \rho - (s - b) + \rho - (s - c) + a = 2\rho = SC + SA + CA = SA + SB + AB$

by symmetry and the perimeters of triangles $SBC$, $SCA$, $SAB$ are equal and $S$ is an **isoperimetric point**. Such a point exists only in the internal case.
The inner Soddy circle is always touched by the tangent circles externally so that, if $S$ is its centre and $\rho$ its radius, then

\[ SA = \rho + s - a, \quad SB = \rho + s - b, \quad SC = \rho + s - c, \]

\[ SB + SC - BC = \rho + s - b + \rho + s - c - a = 2\rho = SC + SA - CA = SA + SB - AB \]

by symmetry, and $S$ is an **equal detour point** in the sense that if you traverse the edges of the triangle by detouring through $S$, the extra distance is the same for each edge. In the external case there are two such equal detour points; in the internal and critical cases, only the inner one. See X(176) and X(175) in \cite{[116]}.

### 5.2 Numbertheoretic interlude

Andrew Bremner has kindly supplied the solution in the critical case for all rational triangles with $2s = 4R + r$. If the area of $ABC$ is $\Delta$, and the perimeter is $a + b + c = 2s$, then $4R = abc/\Delta$ and $r = \Delta/s$, so the critical case has $abc/\Delta + \Delta/s = 2s$, and

\[
\begin{align*}
\Delta^2 + abcs &= 2\Delta s^2 \\
(s - a)(s - b)(s - c) + abc &= 2\Delta s \\
-s^2 + bc + ca + ab &= 2\Delta \\
(bc + ca + ab - s^2)^2 &= 4(s - a)(s - b)(s - c)
\end{align*}
\]

which, in terms of $a$, $b$, $c$ only, simplifies (?) to

\[
5(a^4 + b^4 + c^4) - 4 \left( a^3(b + c) + b^3(c + a) + c^3(a + b) \right) - 2(b^2c^2 + c^2a^2 + a^2b^2) + 4abc(a + b + c) = 0
\]

He observes that this equation defines a quartic curve in projective two-dimensional space with arithmetic genus 3, which is highly singular, and so has geometric genus 0, and is parametrizable! For example,

\[ a = 8u^2(u^2 + 25v^2) \quad b, c = 5(u \pm 5v)^2(u^2 \mp 2uv + 5v^2) \]

or any rational multiple thereof. This solution satisfies the triangle inequality, except that the triangle is degenerate when $u = \pm 5v$. Also,

\[
\frac{b^2 + c^2 - a^2}{2bc} + \frac{7}{25} = \frac{32v^2(6u^2 + 25v^2)}{25(u^4 + 6u^2v^2 + 25v^4)} \geq 0
\]

with equality only if $v = 0$, when the triangle is isosceles with shape $(8,5,5)$. So, except in this case, $\cos A < -\frac{7}{25}$ and angle $A$ is always less than $2 \arcsin 4/5$. Examples of triangles with $2s = 4R + r$ are \((45,40,13), (160,153,25), (325,261,126), (425,416,41)\).
Compare such triangles with those (rational) triangles having an angle \( \arccos \left( -\frac{7}{25} \right) \). The cosine formula gives us \( 25(b^2 + c^2 - a^2) = -14bc \), which may be written

\[
(25a)^2 = (24c)^2 + (25b + 7c)^2
\]

with solution \( a : b : c = 24(p^2 + q^2) : -7p^2 + 48pq + 7q^2 : 25(p^2 - q^2) \) which satisfies the triangle inequality if \( 7q > p > q > 0 \).

The only (shape of) triangle with both \( 2s = 4R + r \), and angle \( \arccos \left( -\frac{7}{25} \right) \), is \((8,5,5)\), given by \( p = 2q \). Other triangles with this angle are \((26,25,3)\), \((30,25,11)\), \((51,38,25)\), \((136,125,29)\). They all have \( 2s < 4R + r \).

Armed with numerical examples, and with help from Peter Moses, we can now clarify the relationship between the three cases and the size of the greatest angle, say \( A \), of the triangle \( ABC \). The external case comprises \( X_1 \), \( Y_1 \), and \( Z_1 \), where \( A \) may be less than, greater than, or equal to \( 2 \arcsin \frac{4}{5} \). \( X_2 \) is the internal case, where \( A \) must exceed \( 2 \arcsin \frac{4}{5} \) (there are no cases \( Y_2 \), \( Z_2 \)). The critical case comprises \( X_3 \) and \( Z_3 \), where \( A \) may not exceed \( 2 \arcsin \frac{4}{5} \) (there is no case \( Y_3 \)).

**X.** \( A < 2 \arcsin \frac{4}{5} \).

- **X1** \( 2s < 4R + r \). Outer Soddy radius negative. Two equal detour points.
  - No isoperimetric point.
  - \((23,22,3)\). \( 48 = 2s < 4R + r = \frac{130}{\sqrt{7}} \approx 49.135 \). \( A = 2 \arcsin \sqrt{\frac{7}{11}} < 2 \arcsin \frac{4}{5} \).

- **X2** \( 2s > 4R + r \). Outer Soddy radius positive. Inner equal detour point only.
  - Isoperimetric point.
  - \((6,5,5)\). \( 16 = 2s > 4R + r = \frac{25}{2} + \frac{3}{2} = 14 \). \( A = 2 \arcsin \frac{3}{4} < 2 \arcsin \frac{4}{5} \).

- **X3** \( 2s = 4R + r \). Outer Soddy radius infinite. Inner equal detour point.
  - Outer detour & isoperimetric points at infinity.
  - \((45,40,13)\). \( 98 = 2s = 4R + r = \frac{650}{7} + \frac{36}{7} = 98 \). \( A = 2 \arcsin \frac{9}{\sqrt{130}} < 2 \arcsin \frac{4}{5} \).

**Y.** \( A > 2 \arcsin \frac{4}{5} \).

- **Y1** \( 2s < 4R + r \). Outer Soddy radius negative. Two equal detour points.
  - No isoperimetric point.
  - \((56,39,25)\). \( 120 = 2s < 4R + r = 130 + 7 = 137 \). \( A = 2 \arcsin \frac{7}{\sqrt{65}} > 2 \arcsin \frac{4}{5} \).

**Z.** \( A = 2 \arcsin \frac{4}{5} \).

- **Z1** \( 2s < 4R + r \). Outer Soddy radius negative. Two equal detour points.
  - No isoperimetric point.
  - \((26,25,3)\). \( 54 = 2s < 4R + r = \frac{325}{6} + \frac{4}{3} = 55\frac{1}{2} \). \( A = 2 \arcsin \frac{4}{5} \).

- **Z3** \( 2s = 4R + r \). Outer Soddy radius infinite. Inner equal detour point.
  - Outer detour & isoperimetric point at infinity.
  - Isosceles triangle \((8,5,5)\). \( 18 = 2s = 4R + r = \frac{50}{3} + \frac{4}{3} = 18 \). \( A = 2 \arcsin \frac{4}{5} \).
There are four (or eight) Pavillet tetrahedra

[In fact, with extraversion, quadration, twinning and reflexion in the plane of the triangle, there are 64 Pavillet tetrahedra. The midpoints of the joins of the orthocentres of a twin pair are the 32 Gergonne points of the triangle (cf. §1.10 and Figure 1.16)]

Figure 5.6: The Pavillet \( a \)-tetrahedron, \( A_aB_aC_aI_a \)
5.3. THERE ARE FOUR (OR EIGHT) PAVILLET TETRAHEDRA

Conway’s extraversion often gives us several items for the price of one. For example, an $a$-flip replaces the angles $A, B, C$ of a triangle by $-A, \pi - B, \pi - C$ respectively, which changes the sign of $a$, and makes the interchanges

$$s \longleftrightarrow s - a \quad \text{and} \quad s - b \longleftrightarrow -(s - c)$$

so if we erect perpendiculars to the plane of $ABC$ of lengths $s, c - s, b - s$ at $A, B, C$ respectively (note that the last two are negative) we arrive at points, say $A_a, B_a, C_a$, which form an orthocentric tetrahedron with $I_a$, the $a$-excentre of $ABC$ (Figure 5.6).

The projection of the orthocentre of this $a$-flip tetrahedron onto the plane $ABC$ is the Gergonne point, $G_a$, the point of concurrence of $AX_a, BY_a, CZ_a$, where $X_a, Y_a, Z_a$ are the respective touchpoints of the $a$-excircle with $BC, CA, AB$. The altitudes of the tetrahedron are $A_aX_a, B_aY_a, C_aZ_a$, which concur at $O_a$, together with $I_aO_a$, which projects onto the Soddy line, $I_aG_a$, in the plane $ABC$. This last contains the deLongchamps point and the centres of the two circles which touch the three tangent circles, centres $A, B, C$, and radii $s, s - c, s - b$ respectively.

So, there are four Pavillet tetrahedra and it may be useful to consider their reflexions in the plane $ABC$ as well, giving a total of eight. There are four manifestations of many of the creatures mentioned above (but only one deLongchamps point). This was noticed nearly half a century ago by Vandeghen [205] who also observed that, if $\odot$ is the circumcentre of $ABC$, then $\odot I_a, \odot I_a, \odot I_b, \odot I_c$ are the Euler lines for the triangles formed by the touchpoints, $X_j, Y_j, Z_j$ ($j = a, b, c$) of the four touchcircles. Figure 5.7 shows four Gergonne lines (dotted), four Soddy lines (dashed), eight Soddy points, four Gergonne points, four touchcentres, . . .
Figure 5.7: Extraversions. Four Soddy lines each contain a touch-centre, $I_j$ ($j = o, a, b, c$); a Gergonne point, $G_j$; two Soddy points, $S_j$ and $S'_j$; and the deLongchamps point, $D$. Four Gergonne lines, $X'_j Y'_j Z'_j$, each contain three harmonic conjugates of touchpoints.
Twenty-four of the points in Figure 5.7 appear in Figure 5.8 as black dots (not all labelled). Each of four Soddy lines contains a touch-centre, $I_j$, and two Soddy centres, $S_j, S'_j$ ($j = o, a, b, c$). The corresponding circles have the Gergonne line perpendicular to the Soddy line as radical axis. Each Gergonne line contains three centres, $X'_j, Y'_j, Z'_j$; and the circles intersect in points on the corresponding Soddy line. The edges of $ABC$ are shown dashed.

Figure 5.8: Four pairs of orthogonal coaxal systems. Three circles intersect in pairs of points on each of the four Soddy lines (lines through $D$). Three circles have centres on each of the four Gergonne lines (perpendicular to the Soddy lines). Contrary to appearances, the $Y'_b$ and $Y'_c$ circles do not intersect on the edge $BC$. 
CHAPTER 5. THE PAVILLET TETRAHEDRON

5.4 L’esprit de l’escalier

No sooner had I submitted this paper that I realized that I should have included a picture of all the extraversions of the Pavillet tetrahedron. Recall that they are

\[ A_oB_oC_oI_o, \quad A_aB_aC_aI_a, \quad A_bB_bC_bI_b, \quad A_cB_cC_cI_c \]

where \( I_o, I_a, I_b, I_c \) are the touch-centres of triangle \( ABC \) and that

\[ \{ A_o, B_o, C_o \}, \quad \{ A_a, B_a, C_a \}, \quad \{ A_b, B_b, C_b \}, \quad \{ A_c, B_c, C_c \} \]

are at respective distances

\[ \{ s - a, s - b, s - c \}, \quad \{ s, -(s-c), -(s-b) \}, \quad \{ -(s-c), s, -(s-a) \}, \quad \{ -(s-b), -(s-a), s \} \]

vertically above the points \( (A, B, C) \).

Had I drawn such a picture I would have noticed that, since \( A_oA_a, B_oB_a, C_oC_a \) are equal and parallel (length \( a \) and perpendicular to plane \( ABC \)) the pairs of corresponding edges \( A_oB_o \) & \( A_aB_a \), and \( A_oC_o \) & \( A_aC_a \), bisect each other, and that their common points, \( E_1 \), \( E_2 \) lie on the respective 260?-point spheres. Also, \( B_oC_o \) is equal and parallel to \( B_aC_a \), and the join of their midpoints is perpendicular to the plane of \( ABC \). Moreover, the right triangles \( A_oI_o \) and \( A_aI_a \) are similar, since the ratio of their edge-lengths is \( s - a : a \), so that \( A_oI_o \) is parallel to \( A_aI_a \). All of these remarks still hold when the letters \( ABC \) and \( abc \) are permuted cyclically.

The three points \( E_1, E_2, E_4 \) in Figure 5.9 indicate points of intersection of three 260?-point spheres (not the same set of spheres in each case). The nine dotted lines are diameters of these spheres, three through each of these points of intersection, three through the 260?-point centre, \( G_o \), and two through each of the other three 260?-point centres, \( G_a, G_b, G_c \). They join pairs of midpoints of opposite edges of tetrahedra.
Figure 5.9: Extraversions of the Pavillet tetrahedron. The ellipses are perspective drawings of touch-circles of the triangle $ABC$. The (large) $c$-circle is only partly accommodated. Small circles indicate the centres, $G_j$ ($j = o, a, b, c$), of the associated 260?-point spheres. The points $A_b$, $B_a$ and $C_o$ are not in the plane $ABC$, but are respectively at distance $s - c$ below, below, and above it.
5.5 So little done; so much to do

1. What are the various perspectrices of the triangles of interior or of exterior altitude points relative to the triangle $ABC$ or to its pedal triangle, $DEF$? All four triangles have the orthocentre as perspector.

2. Given an orthocentric tetrahedron, are there sixteen planes (or some subset thereof), four through each vertex, such that the vertex is an incentre or excentre of a triangle for which the tetrahedron is a Pavillet tetrahedron? ANSWER NO!! Most orthocentric tetrahedra will not serve as Pavillet tetrahedra.

3. For the potentially up to sixteen triangles in 2., are there any incidences among their sixteen Euler lines, or 64 Soddy lines, or 64 Gergonne lines? The Soddy lines concur in fours at sixteen deLongchamps points; do they form any particular configuration?

4. For these sixteen conjectured triangles, are there any collinearities among the more important of their triangle centres?

5. Are there further interesting points on the 260?-point sphere where it intersects the four opposite pairs of parallel, congruent, negatively homothetic faces; and the three rectangular diametral planes; of the octahedron $E_1E_2E_3E_4E_5E_6$?

6. Same question, but for the four faces of the medial tetrahedron, $O_7O_6O_5O_3$.

7. What relations are there between the four (or eight) 260?-point spheres associated with the four (or eight) Pavillet tetrahedra?

8. Characterize the common points of the three circles, centres $X', Y', Z'$, passing respectively through the touchpoints $X, Y, Z$, of the incircle with the edges of $ABC$. These circles form a coaxal system with the Gergonne line as its line of centres and the Soddy line as radical axis, so the common points are on the Soddy line. Do this also for the extraversions based on the three excircles.

9. The coaxal system which is orthogonal to that mentioned in 8. contains the incircle and the two Soddy circles and has the Gergonne line as radical axis. Investigate the member of the system having the deLongchamps point as centre [see Figure 5.2].

10. As the deLongchamps point lies on each of the four Soddy lines (obtained by extraversion; see Figure 5.7) there are in fact four (concentric) circles of the kind mentioned in 9. Investigate their relationship.

11. Examine the critical cases, where the outer Soddy circle is a straight line, for the three extraversions. Compare Figures 5.3 and 5.7.

12. Investigate tetrahedra whose six edges all touch a sphere. It is easy to see that the three sums of the lengths of pairs of opposite edges are equal. They have four ‘tangent spheres’ [69] and a ‘Descartes line’ through the centres of the two spheres which touch
5.6. REFEREES’ COMMENTS.

all four tangent spheres. A particular case is the equilateral pyramid, with $CD = DB = BC \neq AB = AC = AD$, which is also an orthocentric tetrahedron.

Acknowledgements

I am grateful for discussions with several people, including Andrew Bremner, Fred Lunnon, Peter Moses and Axel Pavillet, and perhaps (?) for useful suggestions by the referees.

5.6 Referees’ comments.

Reviewer #1: The paper describes some amazing plane geometry theorems that are new to me, and I thought I knew a fair amount of geometry. I found this interesting, but somewhat overwhelming. There are few proofs given, and those that are in the paper are terse almost to the point of unintelligibility.

I think this paper tries to do too much. The point seems to be to acquaint readers with some “new” geometry theorems, and that is a worthwhile goal for a paper in the Monthly, but I wonder if discussing fewer theorems in a little more detail would better serve the readership.

Reviewer #2: This paper contains some really interesting results in Euclidean triangle geometry. Despite the clear interest of the mathematics, I recommend against publication on the basis of the exposition. I don’t think many Monthly readers will be willing to do the work required to read the paper.

The author makes no attempt to explain why his results are interesting or to put them in any sort of context—he simply launches into technical proofs. This appears to be a deliberate strategy and the paper begins with the words “I’ll tell the story backwards?”. But I don’t think it works. There is nothing that would lure the casual reader into the paper. The audience is limited to those who already know quite a bit about the subject.

A Monthly paper should be accessible to a fairly broad audience of mathematicians, but anyone other than an expert on the kind of Euclidean geometry discussed here would have to do a tremendous amount of background work to get very far into the paper. Of course there’s always the problem of how many definitions to include and which ones can be assumed, but I don’t think the author’s solution is a good one. He is safe in assuming that his readers will know what the medians and orthocenter of a triangle are, but the very first theorem statement on page 2 requires that the reader know what a “radical center” is. The first line of the proof on the top of page 3 refers to the “radical axis”. There is no hint of definition provided. Page 4 uses the words “perspector” and “pedal triangle”. This means that the readership of the paper is effectively limited to those who already know what all these words mean, which is far too restrictive for a Monthly article. Anyone else would have to dig out all the definitions from other sources in order to have
any clue what the author is talking about, and I just don’t see that happening.

The mathematics in the paper appears to be sound, but some of it is difficult to follow. The diagrams are helpful, but some of the more intricate diagrams are quite obscure. For example, it is difficult for me to see what a human being could be expected to learn from looking at Figure 8. It looks like a mass of circles drawn around a triangle and does not provide me with any insight into what is going on.

Despite the fact that I don’t believe the paper is right for the Monthly in its present form, I hope the author will consider reworking the exposition and resubmitting the paper. Euclidean geometry is a highly suitable topic for the Monthly and the results in this paper appear to have the right combination of beauty and surprise.
Chapter 6

The Lighthouse Theorem

This is an almost verbatim copy of [104]. The notation will be at variance with earlier attempts. Some errors are corrected and improvements made in an inchoate paper of Conway & the present writer, given as Chapter 7 below.

6.1 Introduction

The story began in 1993, when Dick Bumby, then Problems Editor of this Monthly, sent me the following problem to referee:

Prove that every prime $p > 7$ of the form $3n + 1$ can be written as

$$p = \sqrt{a^2 + 4762800b^2}$$

for a unique choice of natural numbers $a$ and $b$.

**Paradox 1.** The problem was deemed unsuitable for the Monthly, but here it is.

Fortunately the proposer, Joseph Goggins, had included the motivation for the problem. He was searching for integer-edged triangles whose Morley triangle was also integer-edged. The Morley triangle theorem is still not as well-known as it deserves to be. Its comparatively short history is outlined in §6.26. Figure 8.38 shows its simplest form. The pairs of angle-trisectors of the triangle $ABC$ meet at points which form an equilateral triangle $PQR$, regardless of the shape of the original triangle. Goggins found an infinity of triangles $ABC$ with integer edge-lengths whose Morley triangle also had an integer edge-length.

The combination of geometry and number theory is dear to my heart, and I set to work on the general problem. To trisect angles you need to solve cubic equations, which have three roots. Also, just as angle-bisectors come in pairs, so do (pairs of) trisectors come in triples. I soon discovered that there was not one, but as many as 18 Morley triangles. I
excitedly rang Coxeter, but he claimed not to know that; nor did another geometer who was visiting him. Perhaps they didn’t want to destroy my delight of discovery?

I rang John Conway, and he said, “Funny thing! I was just looking at that last weekend.” So I dashed off a paper [48], added John’s name to it, and sent it to the MONTHLY. A while later the editor, John Ewing, said that he didn’t normally intervene in the refereeing process, but the situation was unusual. One referee had said something like, “O.K., but there are references and other things missing.” A second said, “Isn’t this like a paper I refereed for you a few years ago?” Indeed it was! It had been written by the first referee, John Rigby, who hadn’t resubmitted it.

**Paradox 2.** Rigby knew that there are 18 Morley triangles, and he knew that Morley knew, but few others seemed to know.

Our paper wasn’t resubmitted, either.

**Paradox 3.** But here it is!

Goggins’s real problem was answered by the following:

**Theorem**[33]. *If a rational-edged triangle has a rational Morley triangle, then either the original triangle is equilateral (and 6 of the 18 Morley triangles are rational—in fact, congruent to the original triangle), or it is Pythagorean and belongs to a one-parameter family (and just 2 of the 18 Morley triangles are rational), or it belongs to a two-parameter family of triangles (and all 18 Morley triangles are rational).*

The Pythagorean family has edge-lengths

\[ 2t(3 - t^2)(1 - 3t^2), \quad (1 - t^2)(1 - 14t^2 + t^4), \quad (1 + t^2)^3 \]

for rational \( t \), while the more general solution has edge-lengths

\[ x_i(x_i^2 - 1)(x_i^2 - 9)/(x_i^2 + 3)^3 \quad (i = 1, 2, 3) \]

where \( 3(x_1 + x_2 + x_3) = x_1x_2x_3 \).
6.2. **THE LIGHTHOUSE THEOREM.**

We omitted to mention an observation of Goggins that the lengths of the trisectors are also rational. In fact the reader may like to reconstruct the Conway-Doyle 7-piece jigsaw proof of Morley’s theorem by cutting out an equilateral triangle of edge 1001, three triangles with edge-length triples (1001,1716,1859), (1001,9464,9555), (1001,2695,2464) and three with edge-length triples (9555,2695,12005), (2464,1716,3740), (1859,9464,10985). If you want to be sure that they fit together, use the cosine law to calculate the angles.

**Paradox 4.** Although [33] is a numbertheoretic result, published in *Acta Arithmetica*, it was reviewed in *MR* under the heading Geometry.

But our purpose here is to discuss:

### 6.2 The Lighthouse Theorem.

*Two sets of* \(n\) *lines at equal angular distances, one set through each of the points* \(B, C\), *intersect in* \(n^2\) *points that are the vertices of* \(n\) *regular* \(n\)-*gons. The circumcircles of the* \(n\)-*gons each pass through* \(B\) *and* \(C\).*

That is, they form a **coaxal system** with \(BC\) as **radical axis**. In the exceptional case that the sets of lines are parallel, one \(n\)-gon is at infinity.

**Paradox 5.** Although the theorem is quite striking and an immediate consequence of the following well-known propositions of Euclid, it doesn’t seem to be in the literature:

III.20. The angle at the centre of a circle is twice that at the circumference.

III.21. Angles in the same segment are equal.

III.22. Opposite angles of a cyclic quadrilateral are supplementary.

The first of these also gives: the angle in a semicircle is a right angle; and the last leads to: an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle. And, in the limiting case: the angle between tangent and chord is equal to the angle in the alternate segment. For example, in Figure 6.3, the chord \(PU\) makes two supplementary angles with the tangent at \(P\) (shown dotted); these are equal in size to \(\angle PVU\) and \(\angle PBU\), one in each of the segments into which \(PU\) partitions the circle.

**Informal demonstration of the Lighthouse Theorem for* \(n = 3\).*

In Figure 6.2, each of two lighthouses at \(B\) and \(C\) has one doubly-infinite beam, and each rotates with a uniform angular velocity of one revolution per minute. It’s nighttime. I take photographs every 20 seconds and superimpose them. The locus of the point of intersection of the beams is a circle (Euclid III.21). The point traces out the circle with uniform angular velocity twice that of the lighthouses (Euclid III.20). At 20-second intervals the beams will be equally inclined at angles that are multiples of \(\pi/3\), and the
CHAPTER 6. THE LIGHTHOUSE THEOREM

points on the circle will be at angular distances $2\pi/3$. In other words, they are the vertices of an equilateral triangle (the dashed lines in Figure 6.2).

Paradox 6. I can’t tell from the superimposed photographs whether they were taken every 20 seconds, or every 10 seconds. Indeed, one lighthouse might have been going round twice as fast as the other.

Paradox 7. I can’t tell the senses of rotation from my photographs.

This time, ignorance is bliss, because, in Morley’s theorem, it’s important to choose the intersections of the proximal trisectors of the angles. In this context I will think of the lighthouses as rotating in opposite senses. But now the locus of intersection of their beams is not a circle, but a rectangular hyperbola. In either case the curve passes through the two lighthouses.

Paradox 8. There appear to be a circle and a rectangular hyperbola intersecting in five points.

The Lighthouse Theorem comprises much more than the bald statement that I made earlier. I will also prove:

6.3 The Lighthouse Lemma.

The $\binom{n}{2}$ edges of an $n$-gon arising in the Lighthouse Theorem are parallel to those of any of the other $n-1$ $n$-gons.
6.4. PROOFS OF LIGHTHOUSE RESULTS

The Lighthouse Lemma could be expressed by saying that the $n$-gons are homothetic, but this is so only if $n$ is odd. For example, if $n = 4$, there are two parallel squares and two more at an angle of $\pi/4$ (see Figure 6.23). So, on third thoughts, I include as edges all $(\binom{n}{2})$ joins of vertices to each other (and could include the joins of vertices to themselves, that is the tangents to the circumcircles at the vertices) so that, for $n = 4$, the “diagonals” of one pair of squares are parallel to the “sides” of the other pair.

There is not enough room here to deal with all of the aspects of the Lighthouse Theorem, but I will at least prove the important

**Lighthouse Duplication Theorem.** The points $U, P, V$ are consecutive vertices of one of the $n$ $n$-gons arising in the Lighthouse Theorem (see Figure 6.4). $X$ and $Y$ are the intersections of the next beams with the original ones, making $XY$ an edge of the neighboring $n$-gon, with $XY$ parallel to $UV$ by the Lighthouse Lemma. If $UP$ and $VP$ intersect $XY$ at $Q$ and $R$, respectively, then $\angle RBP = \angle PBC$ and $\angle QCP = \angle PCB$, where $B$ and $C$ are the lighthouses.

It’s time to justify some of our statements.

6.4 Proofs of Lighthouse Results

![Figure 6.3: Proof of the Lighthouse theorem.](image)

*Proof of the (naive) Lighthouse Theorem.* In Figure 6.3 $BP$ and $CP$ are beams from lighthouses at $B$ and $C$, and $U, P, V$ are consecutive vertices of a regular $n$-gon inscribed in the circle $BPC$. Then $\angle PUV = \angle PVU = \pi/n$. Since $UBPV$ is cyclic, the exterior
angle between the directions $UB$ and $BP$ is also $\pi/n$. Similarly for the angle between the directions $VC$ and $CP$. The regular $n$-gon is generated by beams through $B$ and $C$ which are equally spaced by angles that are multiples of $\pi/n$. Generally, the $n^2$ intersections form $n$ regular $n$-gons whose circumcircles pass through $B$ and $C$ and form a coaxal system.

Proof of the Lighthouse Lemma. I write $\beta$ and $\gamma$ for the magnitudes of angles $CBP$ and $BCP$, respectively, and refer to them as the phases of the two lighthouses. Number the beams from each lighthouse from 0 to $n-1$ cyclically towards the baseline $BC$, so that they are counted in opposite senses. In Figure 6.4 $BPY$ and $UBX$ are beams 0 and $n-1$ from lighthouse $B$, while $CPX$ and $VCY$ are beams 0 and $n-1$ from lighthouse $C$.

Now $\angle XBY = \pi/n = \angle XCY$, so that $BCYX$ is cyclic. Then, by Euclid III.21, $\angle YXC = \angle YBC = \beta$, implying that $XY$ makes an angle $\gamma - \beta$ with $BC$. Now the difference between angles $BUV$ and $XBC$ is $(\pi/n + \gamma) - (\pi/n + \beta) = \gamma - \beta$, making $XY$ parallel to $UV$ (and also parallel to the tangent at $P$ to the circle $UBPCV$). Any edge of any $n$-gon thus makes an angle with any other edge of any $n$-gon that is a multiple of $\pi/n$.

Observe that all edges of all $n$-gons make angles with $BC$ that differ from $\gamma - \beta$ only by multiples of $\pi/n$. Our proofs still hold if $\beta$ or $\gamma$ (or both) are changed by a multiple of $\pi/n$, although the pictures look different.
6.4. PROOFS OF LIGHTHOUSE RESULTS

Proof of the Lighthouse Duplication Theorem. The notation of Figure 6.5 is as in Figure 6.4 and the edges $UP$ and $VP$ intersect the edge $XY$ at $Q$ and $R$, respectively. From the Lighthouse Lemma, $\angle QRP = \pi/n$ and is therefore equal to $\angle PBX$. Hence $RXBP$ is cyclic and $\angle RBP = \angle RXP = \angle YXC = \angle YBC = \beta$. Accordingly, $R$ lies on a beam through $B$ with double the phase, $2\beta$. Similarly, $Q$ lies on a beam through $C$ with double the phase, $2\gamma$.

Note. To avoid repetition when I prove Morley’s theorem, it is convenient to observe that $QYCP$ and $BCYX$ are also cyclic; that $\angle PRQ = \angle PQR = \pi/n$; that $\angle BPX = \angle CPY = \beta + \gamma$; that $\angle BPR = \angle CPQ = \pi/n + \gamma$; and that $\angle BRP = \angle CQP = (n-1)\pi/n - \beta - \gamma$. This last angle may be written $(n-2)\pi/n + \alpha$, where $\alpha + \beta + \gamma = \pi/n$.

The Lighthouse Duplication Theorem has many ramifications. The edges of the $n$-gons form $n$ families of $\binom{n}{2}$ parallel lines. Two families intersect in $\binom{n}{2}^2$ points, so that the complete configuration contains $\binom{n}{2}^3$ points, though here some points (for example, the vertices of the $n$-gons) are counted by multiplicity. We now know that several of these points lie on additional beams through the lighthouses, which in turn generate new $n$-gons (or at least $(n/2)$-gons if $n$ is even) whose edges intersect again, and so on indefinitely. Also, as we shall see in the proof of Morley’s theorem, some sets of points lie on new lines, beams from additional lighthouses.

\footnote{The regular $n$-gons are thought of as having every pair of vertices joined, yielding $\binom{n}{2}$ edges.}
6.5 Morley’s Theorem

For $n = 3$ the Law of Small Numbers intervenes, because $\binom{n}{2}$ is no larger than $n$, and the process is in some sense closed, and we have

**The Morley Miracle.** The nine edges of the equilateral triangles of the Lighthouse Theorem for $n = 3$ are the Morley lines of a triangle.

**Paradox 9.** A proof of the complete Morley theorem, without even having a triangle to start with.

In fact the 9 edges of the 3 triangles from one pair of lighthouses form 27 equilateral triangles of which 18 are genuine Morley triangles. The other 9 comprise 3 sets of what Conway has called the **Guy Faux triangles**, one set from each of 3 pairs of lighthouses situated at $B \& C$, $C \& A$, and $A \& B$, where $A$ is a mystery point, yet to be determined.

A good way to count the Morley triangles is to notice that they are each formed from two sides of a GF-triangle together with a third side taken from a different GF-triangle: $3 \times 3 \times 2 = 18$.

**Yet another proof of Morley’s theorem.** I will in fact prove more: what was probably known to Morley, and certainly to Rigby (see also [171]).

**Theorem.** The circumcircles of the nine GF-triangles meet in threes, not only at the vertices of the triangle $ABC$, but also at nine other points.

More precisely, any choice of two GF-circles (that is, circumcircles of GF-triangles), one from each of the three families through $B \& C$, $C \& A$, $A \& B$, pass through two pairs of the six Morley points on one of the 9 Morley lines. These, together with the circle from the third family that passes through the remaining two points on the Morley line, concur in an additional point: a configuration of $27+9+3$ points, 9 circles and 9 lines, each circle passing through $3+3+2$ points and each line through $6+0+0$ points (see Figure 6.9).
Figure 6.6: Proof of Morley’s theorem.
Nothing that Euclid couldn't have done. In Figure 6.6 the labels are as in earlier figures, but in the particular case \( n = 3 \), so that \( PQR \) is a typical potential Morley triangle. The point \( A \) is defined by \( \angle RBA = \beta \) and \( \angle RAB = \pi/3 - (\beta + \gamma) = \alpha \), say. [Warning: it’s best not to connect the edge \( CA \), lest one assume things that, while true, have not yet been established.]

Draw the circle \( BRA \) and let \( PR \) and \( QR \) cut it again at \( S \) and \( T \) respectively. From triangle \( ABR \), \( \angle ARB = \pi - \alpha - \beta \). From the Note following the proof of the Duplication Theorem, \( \angle BRP = \pi/3 + \alpha \), so that \( \angle QRA = 2\pi - \pi/3 - (\pi/3 + \alpha) - (\pi - \alpha - \beta) = \pi/3 + \beta \).

Hence \( RST \) is a GF-triangle generated by lighthouses at \( A \) and \( B \) with phases \( \alpha \) and \( \beta \). By the Duplication Theorem, \( \angle RAQ = \alpha \) and \( \angle RQA = \pi - \alpha - (\pi/3 + \beta) = \pi/3 + \gamma \).

Let \( CQ \) meet \( PR \) in \( F \). The Note also asserts that \( \angle CQP = \pi/3 + \alpha \), whence \( \angle RQF = \pi/3 + \beta \). This means that \( \angle COA = 2\pi - (\pi - \alpha - \beta) - (\pi - \beta - \gamma) = \pi/3 + \beta = \angle CFA = \angle CGA \) and that the GF-circles \( BPC, CGFA, ARB \) concur at \( O \). ■

The point \( O \) is associated with the Morley line \( VGPRFS \) in the sense that the three circles through \( O \) intersect the line in the pairs of points \( \{P, V\}; \{G, F\}; \{R, S\} \).
Figure 6.7 shows the complete configuration of 18 trisectors, 27 Morley points, 9 Morley lines, 18 Morley triangles and 9 GF-triangles. The figure is labelled according to Conway’s scheme, in which the 27 Morley points each receive a three-digit label comprising two numbers 0, 1 or 2 and a star. The (1st, 2nd, 3rd) position of the star indicates the vertex (A, B, C) of the triangle that is not responsible for the point. For example, 1*2 is the intersection of beam 1 from a lighthouse at A with beam 2 from a lighthouse at C. Remember that the beams from the (A, C)-lighthouse pair are numbered starting from 0 for the internal trisector proximal to the edge CA and continuing across CA with the beams counted in opposite senses.
Figure 6.8 is a schematic diagram of the 9 Morley lines and 18 Morley points. The six points on any Morley line come two from each of the three pairs of lighthouses, with two point-labels having a star in the first position, two having it in the second, and two with it in the third. On any Morley line only two of the digits 0,1,2 occur in any one of the three positions. The line-label (large in Figure 6.7) consists of the three digits which do not occur in the three positions. Each line-label has a repeated digit and a different digit, and the sum of the three digits is congruent to 1 modulo 3. For example, the points labelled $^*00$, $^*12$, $^*02$, $^*20$, $0^*$, $2^*$, lie on the Morley line with label 121.

```
       *22
         \  /
          \/
           *22
          /  \
         /    \
        /      \
       /        \
      /          \
     /            \
    /              \
   /                \
  *11
```

Figure 6.8: Schematic diagram of Morley points, lines and triangles, and GF-triangles.

The GF-triangles are found by choosing any vertex, say 0*1 in the bottom row of Figure 6.8 and then taking the only other two points on the Morley lines through that vertex that have stars in the same position (here the B position): $^*20$ and $1^*0$. The other 18 triangles are genuine Morley triangles whose vertices have their stars in the three different positions.
Figure 6.9: Nine Morley lines and nine GF-circles.
Figure 6.9 shows the 9 Morley lines and 9 GF-circles, with more detail shown in Figure 6.10. There are 27 Morley points, 9 points of concurrence of triads of GF-circles, and 3 vertices of the original triangle. Each circle passes through 3+3+2 of these points. Each of the 9 Morley lines passes through 6 Morley points. The label for each of the 9 points of concurrence of triads of GF-circles is a three-digit number that differs from the label of the corresponding Morley line in only one digit: the digit that differs from the other two is changed to the third possibility, so that the sum of the digits of the label of an associated point is congruent to 2 modulo 3, as follows:

| Morley line label | 100 010 001 | 211 121 112 | 022 202 220 |
|------------------|-------------|-------------|-------------|
| associated point label | 200 020 002 | 011 101 110 | 122 212 221 |

Figure 6.10: Enlargement of part of Figure 6.9.
Table 6.1 facilitates the location of all the points, lines and circles. The circumcircle $ABC$ belongs to each of the coaxal systems of GF-circles. The three lines of centres of the systems are the perpendicular bisectors of the edges of $ABC$ and they concur at its circumcentre. If you omit the Morley lines from Figure 6.9 you are left with a pleasing configuration of 9 circles and 12 points with 5 points on each circle, 6 circles through each of 3 points, and 3 circles through each of the remaining 9 points.

| GF-circle | Morley points | Morley lines | associated points |
|-----------|---------------|--------------|-------------------|
| BC0       | *00 *21 *12   | 100 121 112  | 200 101 110       |
| BC1       | *11 *02 *20   | 211 202 220  | 011 212 221       |
| BC2       | *22 *10 *01   | 022 010 001  | 122 020 002       |
| CA0       | 2*1 0*0 1*2   | 211 010 112  | 011 020 110       |
| CA1       | 0*2 1*1 2*0   | 022 121 220  | 122 101 221       |
| CA2       | 1*0 2*2 0*1   | 100 202 001  | 200 212 002       |
| AB0       | 21* 12* 00*   | 211 120 001  | 011 101 002       |
| AB1       | 02* 20* 11*   | 022 202 112  | 122 212 110       |
| AB2       | 10* 01* 22*   | 100 010 220  | 200 020 221       |

| Morley line | GF-circles | Morley points | associated point |
|-------------|------------|---------------|------------------|
| 211         | BC1 CA0 AB0 | *20 *02 0*0 1*2 12* 00* | 011 |
| 121         | BC0 CA1 AB0 | *12 *00 2*0 0*2 00* 21* | 101 |
| 112         | BC0 CA0 AB1 | *00 *21 2*1 0*0 02* 20* | 110 |
| 022         | BC2 CA1 AB1 | *10 *01 1*1 2*0 20* 11* | 122 |
| 202         | BC1 CA2 AB1 | *20 *11 1*0 0*1 11* 02* | 212 |
| 220         | BC1 CA1 AB2 | *11 *02 0*2 1*1 01* 10* | 221 |
| 100         | BC0 CA2 AB2 | *21 *12 2*2 0*1 01* 22* | 200 |
| 010         | BC2 CA0 AB2 | *01 *22 2*1 1*2 22* 10* | 020 |
| 001         | BC2 CA2 AB0 | *22 *10 1*0 2*2 12* 21* | 002 |

Table 6.1: Morley points & lines; GF-circles & associated points.

### 6.6 Conway’s extraversion.

The best insight into the Morley configuration is provided by Conway’s *extraversion*. An “A-flip” of a triangle, for example, replaces angles $A$, $B$, $C$ with $-A$, $\pi - B$, $\pi - C$, respectively. It takes us from THE Morley triangle, 000, with vertices *00, 0*0, 00*, to triangle 011 with vertices *11, 0*1, 01*. Repeated use of A-, B- and C-flips generates the toroidal map of Figure 6.11 where similarly labelled Morley triangles are to be identified.
There are 9 hexagonal regions, bounded by $9 \times 6/2 = 27$ edges, meeting 3 at each of the 18 Morley triangles.

![Image of the 18 Morley triangles related by extraversion.](image)

**Figure 6.11:** The 18 Morley triangles related by extraversion.

### 6.7 The Morley Group

The least obvious of the five abstract groups of order 18 is the semidirect product of $C_3 \times C_3$ with $C_2$. One rarely meets such things in the heat of battle, but here it is: a set of relations is $x^2 = y^3 = z^3 = 1$, $yz = zy$, $yxy = x = zxz$. One way to realize it is with paths in Figure 6.11: $y = AB$, $z = AC$, $x = ABA$. It deserves a proper name.
6.8 Where are the Morley centres?

There is some debate as to what constitutes a “centre” of a triangle \[116\], but for the equilateral triangle, one point, and hence at least one Morley centre (namely 000) is beyond doubt; but 222 and 111 should also be considered. Label the centre of a Morley triangle with the three-digit number that gives the vertices on replacing the digits in turn with a star. For example, 120 is the centre of the Morley triangle with vertices ∗20, 1∗0, 12∗. In Figure 6.12 the Morley lines are dotted and the 18 Morley centres form a “crate”, which, when extended by the 9 GF-centres (smaller unlabelled dots) appears to consist of 27 cuboids, 18 of which have 5 Morley vertices and 3 GF-vertices, and the other 9 have 6 Morley vertices and 2 GF-vertices.
There appear to be 108 body-diagonals that bisect each other in fours at 27 points that form a half-sized “crate”. The 9 GF-centres are collinear in threes—the dashed lines in Figure 6.12 add to the visual confusion.

**Paradox 10.** Figure 6.12 is a drawing of an impossible object.

### 6.9 The Lighthouse Theorem for \( n = 2 \)

For \( n = 2 \), \( \binom{n}{2} < n \), and the ramifications that occur for larger \( n \) do not appear, but even so, the Lighthouse Theorem still has a lot to tell us. Paradox 6 no longer arises, while Paradoxes 7 and 8 are not so apparent, because we expect a rectangular hyperbola and a circle to intersect in four points. In Figure 6.13 two pairs of perpendicular lines through lighthouses at \( E \) and \( F \) intersect at the vertices of two 2-gons, say \( AH \) and \( BC \), segments that are perpendicular to each other. In other words:

**Theorem.** The altitudes of a triangle \( (ABC) \) concur (at the orthocentre \( H \)).

**Paradox 11.** Another triangle theorem without having a triangle.

More symmetrically, each of the 4 points \( A, B, C, H \) is the orthocentre of the triangle formed by the other three. This is a special case of the theorem that any conic through the intersections of two rectangular hyperbolas is a rectangular hyperbola.

But more: if the two 2-gons meet at \( D \), then the Lighthouse Duplication Theorem tells us that \( \angle DEH = \angle HEF \) and \( \angle DFH = \angle HFE \) (i.e., \( EH & AC \) and \( FH & AB \) are the pairs of angle-bisectors at vertices \( E \) and \( F \) of triangle \( DEF \)). The angle-bisectors of triangle \( DEF \) concur in four points, the incentre \( H \) and the excentres \( A, B, C \), provided that \( DH & BC \) are the angle-bisectors of angle \( D \). But \( \angle BDF = \angle BHF \) (\( BDHF \) cyclic) = \( \angle EHC \) (vertically opposite) = \( \angle EDC \) (\( EHDC \) cyclic).

**Paradox 12.** A third triangle theorem without starting from a triangle.

Note that the theorems are quite general. Any triangle \( DEF \) is obtained from lighthouses at \( E \) and \( F \) whose beams are phased to pass simultaneously through \( D \).

Figure 6.13 contains six cyclic quadrangles and three rectangular hyperbolas and the circumcircles of \( ABC, BHC, CHA, AHB \) are all congruent, the last three being reflections of the first in \( BC, CA, AB \) respectively.
6.10. **The Thrice Sixteen Theorem**

Figure 6.14 shows a cyclic quadrangle 0123. The incentres of triangles 123, 023, 013, 012 are the points that are respectively labelled 00, 11, 22, 33. The excentres are labelled with the other twelve two-digit base-4 numbers. The midpoints of the segments joining these in- (& ex-) centres are labelled, somewhat arbitrarily, as follows:

| the point | 04 | 05 | 06 | 14 | 15 | 16 | 24 | 25 | 26 | 34 | 35 | 36 |
|----------|----|----|----|----|----|----|----|----|----|----|----|----|
| is mid of| 10-13 | 10-12 | 22-23 | 00-03 | 00-02 | 21-20 | 01-02 | 11-13 | 00-01 | 11-12 | 01-13 | 02-03 |
| & mid of | 20-23 | 31-33 | 32-33 | 30-33 | 21-23 | 31-30 | 31-32 | 30-32 | 10-11 | 21-22 | 20-22 | 12-13 |

The Thrice Sixteen Theorem states that the $4 \times 4$ incentres of the four triangles 123, 023, 013, 012 lie in fours in a rectangular array on two perpendicular sets of 4 parallel lines. Second, the $4 \times 6$ midpoints of segments joining these centres coincide in twelve pairs, one pair (the dots in Figure 6.14) at each end of six diameters of the **16-point circle** 0123. Third, this circle is the **nine-point circle** for each of the $4 \times 4 = 16$ triangles with vertices \{$ab, ac, ad\}$, where \(a, b, c, d \in \{0, 1, 2, 3\}\) and \(b, c, d\) are distinct. Each of the 16 centres is the orthocentre of one of these triangles, implying that the 16 circumcentres are the reflexions of the 16 incentres in the centre of the 16-point circle. Note that the four centres in any of the four rows or four columns come one each from the four concyclic triangles, so that the first digits of the two-digit labels form a latin square.
Figure 6.14: The Thrice Sixteen Theorem.
Exercise for the reader. Prove the Thrice Sixteen Theorem. Hint: use Euclid III.21 and the Lighthouse Theorem for \( n = 2 \) with pairs of lighthouses at each of the six pairs of points \((0,1), (2,3), (0,2), (3,1), (0,3), (1,2)\) to show that the join 00-11 is perpendicular to 22-32 and many other perpendicularities. Use the Lighthouse Lemma to prove that 00-11 is parallel to 23-32 and many other parallelisms.

Prove also that there are twelve sets of six concyclic points:

\[
\begin{array}{ccc}
\{0,1,33,23,32,22\} & 06 & \{0,2,31,12,33,10\} & 05 & \{0,3,20,10,23,13\} & 04 \\
\{1,0,30,20,31,21\} & 16 & \{1,2,03,33,00,30\} & 14 & \{1,3,23,02,21,00\} & 15 \\
\{2,0,32,13,30,11\} & 25 & \{2,1,01,31,02,32\} & 24 & \{2,3,11,00,10,01\} & 26 \\
\{3,0,21,11,22,12\} & 34 & \{3,1,22,01,20,03\} & 35 & \{3,2,12,03,13,02\} & 36 \\
\end{array}
\]

There is a fourth manifestation of 16: combine these 12 circles with the 16-point circle counted with multiplicity four as the circumcircle of the triangles 123, 230, 301, 012. In Figure 6.14 each of the sextuples \((1,2,3,04,05,06), (2,3,0,14,15,16), (3,0,1,24,25,26), (0,1,2,34,35,36)\) lies on the 16-point circle. In Figure 6.15 the incentre grid is dashed; the lighthouse beams are dotted; and the 16-point circle is drawn with a thick line.
Figure 6.15: Twelve circles and four times one other.
Further manifestations of 16 come from the 4 sets of 4 orthocentric points $a0, a1, a2, a3$ ($a = 0,1,2,3$). Each of the 16 incentres $ab$ is the orthocentre of the triangle $a\bar{b}$, where $\bar{b}$ denotes the 3-element complement of $b$ in the set $\{0,1,2,3\}$. These 16 triangles have a common nine-point circle: the 16-point circle, with centre $N$, say. If $O_{ab}$ is the circumcentre of triangle $a\bar{b}$, then, since $N$ is the midpoint of the segment of the Euler line that joins the orthocentre to the circumcentre, the 16 circumcentres form a $4 \times 4$ grid that is congruent to the grid of orthocentres, being its reflexion in $N$. Moreover the 16 circumcircles are congruent, for each of their radii is twice that of the nine-point radius.

Figure 6.16: Sixteen orthocentres, circumcentres and circumcircles.
The 16 centroids form a similar grid, with one-third the linear dimensions. This is not included in Figure 6.16 which shows the orthocentre grid dashed and the circumcentre grid solid, together with the nine-point centre $N$ and the 16 congruent circumcircles.

**Take five!** (Further exercise for the reader). Draw a figure with five concyclic points, forming \( \binom{5}{3} = 10 \) triangles with five $4 \times 4$ grids of incentres coinciding as 40 pairs.

### 6.11 There are 72 Morley triangles!

[In fact, with *twinning* there are 144.] Does THE Morley triangle, together with its twin, form a Star of David inscribed in the double-deltoid? Not quite? Needs investigation!

Preliminary experiment: The axes of symmetry of the deltoid pass through the 50-point (9-point) centre. The axes of symmetry of THE Morley triangle are parallel to these, but not coincident with them. The angles they make with the edges of the triangle are $|B - C|/3$, $|C - A|/3$, $|A - B|/3$.

Repeat the proof of Morley’s theorem with the lighthouses at phases $\frac{\pi}{6} - \gamma$ and $\frac{\pi}{6} - \beta$ in place of $\beta$ and $\gamma$. This yields 18 Morley triangles for a triangle with base angles $3(\frac{\pi}{6} - \gamma) = \frac{\pi}{2} - C$ and $\frac{\pi}{2} - B$. This is the triangle $BHC$, where $H$ is the orthocentre of $ABC$. The edges of these Morley triangles make angles with $BC$ that differ only by multiples of $\pi/3$ from $(\frac{\pi}{6} - \gamma) - (\frac{\pi}{6} - \beta) = \gamma - \beta$ and so are parallel to the edges of the Morley triangles of $ABC$. With 18 more from each of the triangles $CHA$ and $AHB$ this makes a grand total of 72. In the rational case all 72 have rational edges! Note that while $BC$, $CA$, $AB$ are rational, $AH$, $BH$, $CH$ are not: these, as well as the altitudes and the area, are rational multiples of $\sqrt{3}$.

**Buy three: get two free!**

**Theorem.** If a triangle is inscribed in a rectangular hyperbola, then its orthocentre also lies on the hyperbola, at the opposite end of the diameter of the hyperbola through the fourth point of intersection of the hyperbola with the circumcircle of the triangle.

I prefer synthetic proofs, but this analytic one is so attractive that I can’t resist:

**Proof.** Choose the axes and scale so that the rectangular hyperbola has equation $xy = 1$ and the triangle has vertices $(t, 1/t)$, $(u, 1/u)$, $(v, 1/v)$. The line through the first two has slope $-1/tu$. The line through the third, with perpendicular slope, has equation $y - 1/v = tu(x - v)$, which intersects the hyperbola again in $(-1/tuv, -tuv)$. By symmetry, the altitudes of the triangle concur at this point of the hyperbola. If the equation to the circumcircle of the triangle is $x^2 + y^2 + 2gx + 2fy + c = 0$, then the $x$-coordinates of its points of intersection with the hyperbola are given by $x^4 + 2gx^3 + cx^2 + 2fx + 1 = 0$. The product of the roots is 1, so the fourth point has $x$-coordinate $1/tuv$, at the opposite end
of the diameter of the hyperbola through the orthocentre of the triangle.

In case there was ever any doubt about Paradox 8, here is a list of the 27 Morley points. They lie in threes on 9 GF-circles that pass through pairs of the vertices $ABC$. They also lie in threes on 9 rectangular hyperbolas that pass through pairs of $ABC$. The following table shows what lies on what. Rows of three points are on a GF-circle through the pair of vertices listed at their head. Columns of three points are on a GF-hyperbola through these vertices.

$$
\begin{array}{ccc}
B & C & A \\
*00 & *12 & *21 \\
*11 & *20 & *02 \\
*22 & *01 & *10 \\
\end{array}
\begin{array}{ccc}
C & A & B \\
0*0 & 1*2 & 2*1 \\
1*1 & 2*0 & 0*2 \\
2*2 & 0*1 & 1*0 \\
\end{array}
\begin{array}{ccc}
00* & 12* & 21* \\
11* & 20* & 02* \\
22* & 01* & 10* \\
\end{array}
$$

In fact each 3-by-3 array may be thought of as an affine geometry with 9 points. Two sets of three parallel “lines” are the rows and columns, i.e., the GF-circles and the GF-hyperbolas. The other two sets of three parallel lines are the broken diagonals and correspond to sets of points on beams through $B$ & $C$, or $C$ & $A$, or $A$ & $B$.

6.12 So little done — so much to do!

Nine rectangular hyperbolas, each through 5 points, $\binom{9}{3}$ triangles inscribed in each. Where are the ninety orthocentres? Where do the circumcircles meet the hyperbolas again? Where are the circumcentres? The nine-point circles? The Euler lines? The “buy 3, get 2 free” theorem gives us a good start.

For simplicity, look only at the three hyperbolas associated with the lighthouses at $B$ and $C$. Their centres are all at the midpoint, $M$, of $BC$. Simplify the labels of the Morley points by omitting the star and reading the two digits as a ternary number. For example, *21 is 7. Figure 6.17 (see also Figure 6.18) depicts the nine points whose star is in the first ($A$) position as 0, 1, 2, ..., 8.

Consider just the nine triangles $BCa$ where $0 \leq a \leq 8$. Each is inscribed in hyperbola 0 or 1 or 2 according as $a$ belongs to $\{0, 8, 4\}$ or $\{5, 1, 6\}$ or $\{7, 3, 2\}$. Their nine orthocentres $a'$ lie respectively on these hyperbolas, and the nine sets $\{B, C, a, a'\}$ are each orthocentric:

$$
aa' \perp BC, \quad aB \perp a'C, \quad aC \perp a'B.
$$

The circumcircles of the triangles $Ca$, $Baa'$, $BCa'$ are the reflexions of the circumcircle of $BCa$ in its respective edges $Ca$, $aB$, $BC$, and so each has the same circumradius, twice that of the radius of the common nine-point circle of the four triangles. Each of the nine
points $a'''$ is at the opposite end of the diameter through $a$ of the appropriate GF-circle $BC057$, $BC813$, $BC462$. They form regular hexagons with the original points $a$ inscribed in the GF-circles and are part of a manifestation of the Lighthouse Theorem for $n = 6$. Each is also at the opposite end of the diameter through $a'$ of the appropriate hyperbola $BC084$, $BC516$, $BC732$. These nine diameters concur at $M$. 
Figure 6.17: The three GF-hyperbolas through $B$ and $C$. 
The nine points $a''''$ are not shown in Figures 6.17 or 6.18 though Figure 6.19 contains three specimens. They may be variously described as the second intersection of $aa'$ with the appropriate GF-circle; or as the reflexion of $a'$ in $BC$ (i.e., as being generated by lighthouses whose beams are the reflexions in $BC$ of the original beams through $B$ and $C$); or as the reflexion of $a''$ in the perpendicular bisector of $BC$.

Figure 6.18: Enlargement of part of Figure 6.17.
The four sets of nine points, \{a\}, \{a'\}, \{a''\}, \{a'''\} are the congruent configurations

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 0' & 3' & 6' & 0'' & 1'' & 2'' & 0''' \\
3 & 4 & 5 & 1' & 4' & 7' & 3'' & 4'' & 5'' & 1''' \\
6 & 7 & 8 & 2' & 5' & 8' & 6'' & 7'' & 8'' & 2''' \\
\end{array}
\]

whose rows are points on Beams through B and whose Columns are points on beams through C.

Figure 6.19: Further fun from Figure 6.17.
CHAPTER 6. THE LIGHTHOUSE THEOREM

In Figure 6.19 some of the relations between these configurations can be seen as

1. The triples of points 00'', 55'', 77'' are collinear and perpendicular to BC, with 00', 55', 77' equal in length.
2. The joins 00'', 55'', 77'' are diameters of the GF-circle 057.
3. Each of 0'0'', 5'5'', 7'7'' has M as midpoint.
4. Each of 0''0''', 5''5'''', 7''7''''' is parallel to BC.

Again there are ramifications and more sets of nine points. The astute reader would demand a set \( \{a^{iv}\} \) of reflexions of \( \{a''\} \) in BC or of \( \{a'''\} \) in M, forming a four-group of configurations with \( \{a\}, \{a''\}, \{a'''\} \). The points 0''iv, 5''iv and 7''iv have crept into Figure 6.19 which confines itself to illustrating triangles BC0, BC5 and BC7, with common circumcentre \( \odot \). The Euler lines \( \odot N_00'', \odot N_55'', \odot N_77'' \) are not drawn. The nine-point centres \( N_0, N_5, N_7 \) form an equilateral triangle. The nine-point circles are congruent, each passes through M, and they intersect at angles \( \pi/3 \), forming a pleasing cloverleaf.

Only a tithe. I have considered one-third of the GF-hyperbolas and three-tenths of the triangles inscribed in each. The “buy three, get two free” theorem gives us 180 bonus points from the 90 triangles. What further coincidences, collinearities, concyclicities are there among these points?

6.14 Bifaux and Skewfaux.

When you state the Morley theorem, you must be careful to specify the intersections of the proximal trisectors. What if you make a mistake and use the distal (duplicated) beams? You get a new set of GF-triangles whose edges are the Morley lines of a triangle, \( A'BC \) say, with base angles \( 6\beta \) and \( 6\gamma \), and the original \( A \) is an incentre of \( A'BC \). Or you might get mixed up and choose one proximal and one distal trisector, and arrive at the Morley lines for a triangle \( A''BC \) or \( A'''BC \), with just one base angle doubled, and \( A'' \) and \( A''' \) are the intersections of \( BA \) with \( CA' \) and of \( CA \) with \( BA' \).

6.15 Turning Morley inside-out.

The construction of \( A' \) could be described as the intersection of the “distal treblers” at \( B \) and \( C \), in contrast to the proximal trisectors that are used in Morley’s theorem. Construct \( B' \) and \( C' \) similarly. In Figure 6.20 Arcs mark angles equal to \( A \), Bullets mark angles equal
6.15. TURNING MORLEY INSIDE-OUT.

to $B$, and Circles those equal to $C$. The (reflex) angle $B'AC' = 3A$, angle $C'BA' = 3B$ and angle $A'CB' = 3C$. Let the “proximal treblers” meet at $\alpha'$, $\beta'$ and $\gamma'$, and let $BC$, $B'C'$ meet at $\alpha$, and so forth, as in Figure 6.20.

**Theorem.** (a) $AA', BB', CC'$ concur at the circumcentre of $ABC$.
(b) $A\alpha', B\beta', C\gamma'$ concur at the orthocentre of $ABC$.
(c) The triads of points $(\alpha, \beta, \gamma)$, $(\alpha, \beta', \gamma')$, $(\alpha', \beta, \gamma')$, $(\alpha', \beta', \gamma)$ are each collinear.

**Sketch of proof.** (a) $AB$, $AC$ are angle-bisectors of $\angle A'BC$, $\angle A'CB$, so that $A$ is an incentre of $A'BC$ and $AA'$ is a bisector of $\angle BA'C$ and makes angles $\frac{\pi}{2} - C$, $\frac{\pi}{2} - B$ with $AB$, $AC$ and hence passes through $\odot$, the circumcentre of $ABC$. Similarly for $BB'$, $CC'$.
(b) $\alpha'$ is the reflexion of $A$ in $BC$, so that $A\alpha'$ is perpendicular to $BC$.
(c) $AA'$, $BB'$, $CC'$ concur at $\odot$, making it the perspector of triangles $ABC$, $A'B'C'$, and Desargues tells us that $(\alpha, \beta, \gamma)$ are collinear. Similarly $\odot$ is the perspector of triangles $A'BC$, $AB'C'$, and of two other pairs. ■
6.16 Desargues distended.

The last part of this proof reminds us that we may swap a pair of vertices of two triangles in Desargues’s theorem and produce a new perspectrix. Since any of the 10 points may serve as perspector, there are 40 perspectrices and a configuration of 25 points and 55 lines. Nine lines through each of the original 10 points and six through each of 15 new points. There are four points on each of 15 new lines and three points on each of the 10 old lines and on each of the 30 new perspectrices.

Relabel Figure [6.20] with Alex Fink’s beautiful labelling, as in Figure [6.22].

| Fig 6.20 labels | ⊙ | A | B | C | A' | B' | C' | α | β | γ | α' | β' | γ' |
|-----------------|---|---|---|---|----|----|----|---|---|---|----|----|----|
| Alex’s labels   | 0 | 8 | 7 | 1 | 9  | 3  | 2  | 4 | 5 | 6 | 04 | 05 | 60 |

and see the duality between the Desargues configuration and the Petersen graph (Figure [6.21]), each of which has automorphism group $S_5$.

Figure 6.21: The Petersen graph and Desargues-Petersen labels.

| (a)            | 6  | 8  | 0  | 2  | 4 |
|----------------|----|----|----|----|---|
| (b)            | 012| 234| 456| 678| 890|
| (c)            | 1  | 3  | 5  | 7  | 9 |
| (d)            | 369| 581| 703| 925| 147|
| (e)            | 9023| 1245| 3467| 5689| 7801|
| (f)            | 61  | 57 | 48 | 83 | 79 | 60 | 05 | 91 | 82 | 27 | 13 | 04 | 49 | 35 | 26 |

In the table, rows (a) & (c) are perspectors in the Desargues configuration, and (b) & (d) are the corresponding perspectrices. Vertices (b) & (d) in the Petersen graph are adjacent to the corresponding vertices in rows (a) & (c). Row (e) shows the maximal independent sets of the Petersen graph; they comprise the $5 \cdot \binom{4}{2}$ pairs of points which have a common Desargues line; the edges of the five complete quadrangles in the Desargues configuration. The other $\binom{10}{2} - 30 = 15$ pairs of points, which are independent in the Desargues configuration, correspond to the 15 edges of the Petersen graph. These are given as five triples of pairs in row (f) of the table. The first pair of each triple is a copy of the entries in (a) & (c) and, in the Petersen graph, is the edge perpendicular to the 2nd & 3rd pairs. These pairs can also be read as two-digit labels of the 15 points that amplify the Desargues configuration. For example, read the middle triple as “the joins of 9 & 1 and of 8 & 2 meet in the point whose two-digit label is 05” etc.
With perspector 0, triangles 871 & 932 have perspectrix 4 5 6
triangles 971 & 832 have perspectrix 4 05 60
triangles 831 & 972 have perspectrix 04 5 60
triangles 872 & 931 have perspectrix 04 05 6
which is a description of the relabelled Figure 6.20, shown now as Figure 6.22.
6.17 The Lighthouse Theorem when \( n = 4 \).

When \( n > 3 \), think of the \( n \)-gons as complete graphs on \( n \) vertices, two-dimensional representations of regular \((n-1)\)-dimensional simplexes. There are \( n \) sets of \( \binom{n}{2} \) parallel lines that intersect in \( \binom{n}{2}^3 \) points, though I now count points with multiplicity. The \( n^2 \) vertices are each counted \( \binom{n-1}{2} \) times. If \( n \) is even the “diameters” concur, and for certain \( n \) there are other concurrences, originally enumerated by G. Bol [16] and often rediscovered (see [170] for a good account).

Figure 6.23: The four 4-gons formed from lighthouses with \( n = 4 \).
6.17. **THE LIGHTHOUSE THEOREM WHEN \( N = 4 \).**

Figures [6.23] and [6.24] are for \( n = 4 \). In Figure 6.23 the beams are dotted and the quadrangles solid. In Figure 6.24 the original beams are omitted, the edges of the quadrangles are dotted, and the “duplicated” beams are solid. I number the beams as before and will label the intersections of the lighthouse beams with the hexadecimal numbers 0, 1, \ldots, 9, a, b, \ldots, f=15 which should be thought of as two-digit quaternary numbers \( \bar{2}2 \), where \( \bar{2} \) and \( 2 \) are the numbers of the beams from \( B \) and \( C \). Quadrangle \( q \) (\( 0 \leq q \leq 3 \)) then has vertices whose quaternary digits add to \( q \) modulo 4:

\[
\begin{align*}
q &= 0 \text{ is } 00=0 \text{ 13=7 22=a 31=d}; & q &= 1 \text{ is } 01=1 \text{ 10=4 23=b 32=e}; \\
q &= 2 \text{ is } 02=2 \text{ 11=5 20=8 33=f}; & q &= 3 \text{ is } 03=3 \text{ 12=6 21=9 30=c}.
\end{align*}
\]

Figure 6.24: The duplicate beams through edge-intersections when \( n = 4 \).

Edges of a quadrangle are denoted by concatenating the point-labels in these cyclic orders, starting at the “earliest” end. For instance, if \( q = 2 \) the edges of “length” 1 are 25, 58, 8f and f2, and the two of “length” 2 (the “diagonals”) are 28 and 5f. The Lighthouse Duplication Theorem tells us that the following columns of ten intersections lie on beams through \( B \) with phases \( 2\beta \) and \( 2\beta + \frac{\pi}{2} \) and through \( C \) with phases \( 2\gamma \) and \( 2\gamma + \frac{\pi}{2} \) (hexadecimal numbers are converted back to base-4 numbers, better to show the pattern).

The first eight rows indicate intersections of sides of quadrangles, each of which meets a pair of adjacent sides of the two adjacent quadrangles. Here “side” means the join of two adjacent vertices, and “adjacent” means “neighboring” in the cyclic order 0123. The last two rows contain (duplicates of) the four intersections of diagonals with the opposite diagonal of the opposite quadrangle: the set of four orthocentric points 0a-28, 39-1b, c6-e4,
5f-7d, (the larger dots in Figure 6.24). These constitute an imbedding of the Lighthouse Theorem for \( n = 2 \). Compare the labelling of Figures 6.13 and 6.24:

| Fig 6.13 labels | E | F | H | A | B | C | D |
|------------------|---|---|---|---|---|---|---|
| Fig 6.24 labels  | B | C | 0a-28 | 5f-7d | 39-1b | c6-e4 | — |
| Relabel as       | B | C | I | I_A | I_B | I_C | A |

Figure 6.25 does not show the new label \( A \), which is for the point of intersection of the line joining 5f-7d to 0a-28 with the line joining 39-1b to c6-e4. By the Lighthouse Duplication Theorem these two lines are perpendicular and are the angle-bisectors of \( \angle BAC \) and the beams \( BA \) and \( CA \) have phases \( 4\beta \) and \( 4\gamma \).

The original beams are angle-bisectors of the four triangles \( BCI \), where \( I \) is written collectively for the four incentres of \( ABC \). The original 16 points are the incentres of these triangles, given as follows in the order: incentre, excentre opposite \( I \), excentre opposite \( B \), excentre opposite \( C \):

- Incentres of triangle \( BCI \) \( (I = 0a-28) \) 0 a 2 8
- Incentres of triangle \( BCI_A \) \( (I_A = 5f-7d) \) 5 f 7 d
- Incentres of triangle \( BCI_B \) \( (I_B = 39-1b) \) 3 9 1 b
- Incentres of triangle \( BCI_C \) \( (I_C = c6-e4) \) c 6 e 4

These are, of course, the 16 points of Figure 6.23 similarly labelled.
6.17. THE LIGHTHOUSE THEOREM WHEN $N = 4$.

Figure 6.25: Enlargement of part of Figure 24.
6.18 There’s a Malfatti miracle, too!

In an earlier draft of this paper I asked:

“Is there any significance in the fact that W. E. Philip’s proof of Morley’s theorem [202], [114] and Steiner’s construction for the Malfatti circles both start with the incircle whose centre is lit by beams of phases \( B/n(= \beta) \) and \( C/n(= \gamma) \)?”

Indeed there is! Malfatti [137] asked for three circular cylinders of maximum total volume to be cut from a triangular prism. The complete solution to his problem, begun by Lob & Richmond [132], was obtained only comparatively recently; see [70, Vol.2, pp. 245–247], [82], [83], [78], [159, pp. 145–147], [173] and Guggenheimer’s review of Zalgaller & Los’ [212].

![Figure 6.26: The popular version of the Malfatti problem.](image)

**Paradox 13.** Malfatti misconstrued his own problem.

This is lucky, because his interpretation is a much more interesting problem. He thought that the solution was given by three cylinders, each touching the other two, and each touching two faces of the prism, with a cross-section like Figure 6.26.

**Paradox 14.** This problem is not due to Malfatti.

It occurs in Japanese temple geometry [77, pp. 28, 103–106], [111], [149], [176], where it is often ascribed to Ajima. The isosceles triangle case was considered by Jacques Bernoulli [14]; see Dieudonné’s review of Fiocca [71]. But it was posed 417 years before Malfatti, and in his own country; see the review of Simi & Rigatelli [188].

**Paradox 15.** Almost all writers state that Steiner gave no proof for his solution to the Malfatti problem, but he did give one!
6.18. THERE’S A MALFATTI MIRACLE, TOO!

Steiner’s proof is scattered over several sections of two separate articles [198], [199]. It is indirect, via the more general problem in which the edges of the triangle are replaced by three circles. It refers to figures which are not in the text, but are tucked away at the end of the first volume of Crelle on separate plates from different fascicules.

The literature on Malfatti’s problem is extensive, widely scattered, and not always aware of itself. Many of the published proofs follow Hart’s justification [107] of Steiner’s construction. Others use trigonometry, or are concerned with generalizations to circles, spheres and sections of quadrics. Some references, chronologically since Malfatti, are [1], [41], [168], [181], [42], [43], [40], [189], [44], [39], pp. 154–155], [?], [177], I book III pp. 311–314], [161], [162], [10], IV ch.II Ex.8 pp. 65–69), [174], [167], [148], [173].

Here is a pot-pourri of facts. I’ll hide my proofs because they use what Swinnerton-Dyer calls “the sort of mathematics that no gentleman does in public.”

![Figure 6.27: Construction of the Malfatti circles.](image-url)
Most constructions start with incircles of triangles $BIC$, $CIA$, $AIB$, where $I$ is the incentre of $ABC$. Even when $ABC$ is exceptionally scalene, the radii of these incircles remain surprisingly equal. It is hard to draw a picture which clearly distinguishes the points. In Figure 6.27 vertex $B$ is off left. The three incircles touch $BC$, $CA$, $AB$ at $X$, $Y$, $Z$ respectively and their centres are $A_0$, $B_0$, $C_0$.

Figure 6.28: Detail of the Malfatti construction.
More detail is shown in Figure 6.28 where the perpendiculars (dotted lines) from \( A_0, B_0, \) \( C_0 \) onto \( AI, BI, CI \) indicate their points of contact. Letters in parentheses at the ends of lines are labels of points on those lines, but too distant to appear in the figure. Lines \( B_0C_0, C_0A_0 \) and \( A_0B_0 \) are dashed and they intersect \( AI, BI \) and \( CI \) at \( X', Y' \) and \( Z' \) respectively. Then, surprisingly, \( XX', YY' \) and \( ZZ' \) are the reflexions of \( AI, BI \) and \( CI \) in \( B_0C_0, C_0A_0 \) and \( A_0B_0 \) respectively, and they are not only transverse common tangents to pairs of the incircles, but also transverse common tangents to pairs of the sought-after Malfatti circles. They concur in \( R \), the radical centre of these circles. The quadrilaterals \( YRZA, ZRXB, XRYC \) are inscribable in the sense that they each have an incircle which touches all four sides. These incircles are the Malfatti circles, which touch \( XX', YY', ZZ' \) in pairs at \( X_1, Y_1, Z_1 \) respectively.

A variant construction uses the circle centre \( X \) and radius \( r(1+u)/2 \), where \( r \) is the inradius of the original triangle \( ABC \), and \( u = \tan A/4 \). This circle, just half of which is shown in Figure 6.27, passes through the points of contact of the Malfatti circles with \( BC \) and with each other. Similarly for circles with centres \( Y, Z \) and radii \( r(1+v)/2, r(1+w)/2 \), where \( v = \tan B/4 \) and \( w = \tan C/4 \).

Note that \( u, v, w \) are related, because \( A + B + C = \pi, \tan \frac{A+B}{4} = \tan \frac{\pi-C}{4} \) and \( (u+v)/(1-w) = (1-w)/(1+w) \) or \( 1 + uvw = u + v + w + vw + wu + wv \).

**Paradox 16.** This identity may be written in more than fifty different ways, but never in the one that’s needed at any particular moment.

Andrew Bremner [34] has shown that the radii of all 96 Malfatti circles are rational just if (two of) \( u, v \) and \( w \) are rational.

### 6.19 The light dawns.

The Malfatti miracle comes in four parts. The first part is the relevance of The Lighthouse Theorem. The trick is to concentrate, not on the Malfatti circles, but on the three incircles with which the construction begins. The 16 incentres of the four triangles \( BIC \) are waiting for us in the previous section! They are generated by two 4-beam lighthouses at \( B \) and \( C \) with phases \( \beta = B/4 \) and \( \gamma = C/4 \) and have hexadecimal labels 0 to \( f \) in Figures 6.23, 6.24 and 6.25, where the missing vertex \( A \) is on the verge of making its ghostly appearance.

**Paradox 17.** With no triangle to start from, the Lighthouse Theorem for \( n = 4 \) generates the full set of 32 Malfatti solutions.

It remains to set up two pairs of 4-beam lighthouses, one pair at \( C \) & \( A \) with phases \( \gamma \) & \( \alpha = A/4 \) and another at \( A \) & \( B \) with phases \( \alpha \) & \( \beta \). These have angle-quadrisecting beams of angles \( BCA \) & \( CAB \) and of angles \( CAB \) & \( ABC \) that intersect at the 16 incentres of
the four triangles $CIA$ and the 16 incentres of the four triangles $AIB$. One application of the Lighthouse Duplication Theorem yields the pairs of angle-bisectors of triangle $ABC$, and a second application produces the edges!

6.20 Extraversion again.

Recall that an $A$-flip replaces angles $A, B, C$ by $-A, \pi - B, \pi - C$ respectively. That is, it replaces $(u, v, w)$ by $(-u, 1-v, 1-w)$.

\[ \begin{align*}
4a & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, -v, \frac{w+1}{w-1} \\

2b & \quad C \quad B \quad A \\
& \quad u, \frac{1}{v-1}, \frac{w+1}{w-1} \\

5 & \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1}{1+v}, \frac{1}{w} \\

7a & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1}{v}, \frac{1-w}{1+w} \\

6 & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-w}{1+w}, \frac{v+1}{v-1}, -w \\

4a & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, -v, \frac{w+1}{w-1} \\

2b & \quad C \quad B \quad A \\
& \quad u, \frac{1}{v-1}, \frac{w+1}{w-1} \\

1 & \quad B \quad A \\
& \quad 1-\frac{u}{1+v}, \frac{1}{w} \\

0 & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-w}{1+w}, \frac{v+1}{v-1}, -w \\

3a & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-v}{1-w}, \frac{v-1}{v+1} \\

3b & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-w}{1+w}, \frac{v+1}{v-1}, \frac{1}{w} \\

7c & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-v}{1-w}, \frac{v-1}{v+1}, -w \\

4 & \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1}{v}, \frac{1-w}{1+w} \\

5c & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-v}{1-w}, \frac{v-1}{v+1}, -w \\

6b & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-w}{1+w}, \frac{v+1}{v-1}, -w \\

6c & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-v}{1-w}, \frac{v-1}{v+1}, -w \\

5b & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-w}{1+w}, \frac{v+1}{v-1}, -w \\

5c & \quad C \quad B \quad A \\
& \quad \frac{u+1}{u-1}, \frac{1-v}{1-w}, \frac{v-1}{v+1}, -w \\

\end{align*} \]

Figure 6.29: There are 32 Malfatti solutions.
So start at \((u, v, w)\) in the middle of Figure 6.29 and keep flipping. You’ll find that you’re making a toroidal map like Figure 6.11 but this one has 16 hexagonal regions, bounded by 48 edges, 16 each of \(A\)-flips, \(B\)-flips and \(C\)-flips, which meet three at each of 32 vertices which represent the 32 Malfatti solutions.

Eight ordinary solutions (Table 6.3) are labelled with a bold digit:

\[
\begin{array}{cccc}
0(u, v, w) & 3(u, -\frac{1}{u}, -\frac{1}{w}) & 5(-\frac{1}{w}, v, -\frac{1}{w}) & 6(-\frac{1}{u}, -\frac{1}{v}, w) \\
7(\frac{1}{u}, \frac{1}{v}, \frac{1}{w}) & 4(\frac{1}{u}, -v, -w) & 2(-u, \frac{1}{v}, -w) & 1(-u, -v, \frac{1}{w})
\end{array}
\]

Table 6.3: Tangents of quarter-angles of ordinary solutions.

The other 24 flipped solutions are obtained via an \(A\)-flip, \(B\)-flip or \(C\)-flip and are labelled by respectively appending \(a\), \(b\) or \(c\) to the ordinary solution digit. For example, \(2b\) is a \(B\)-flip away from \(2(-u, \frac{1}{v}, -w)\), and the tangents of its quarter-angles are

\[
\left(\frac{1 - (-u)}{1 + (-u)}, \frac{1 - (-w)}{1 + (-w)}\right) = \left(\frac{1 + u}{1 - u}, \frac{-1}{v} \cdot \frac{1 + w}{1 - w}\right)
\]

### 6.21 The Malfatti group.

What is the group of the 32 solutions? It is generated by \(A\), \(B\), \(C\): notice the following relations:

\[
A^2 = B^2 = C^2 = (ABC)^2 = (BC)^4 = (CA)^4 = (AB)^4 = I, \quad ABC = CBA.
\]

I thank Alex Fink for locating the group. It has no name (nor do most of the other 50 groups of order 32), so let’s give it one. It has Hall-Senior number and Magma number 34. An official set of generator relations is \(g_2^2 = g_4\), \(g_3^2 = g_5\), \(g_1^{-1} g_2 g_1 = g_3 g_4\), \(g_1^{-1} g_3 g_1 = g_3 g_5\) which are realized here by \(g_1 = A\), \(g_2 = AB\), \(g_3 = AC\), \(g_4 = (AB)^2 = (BA)^2\), \(g_5 = (AC)^2 = (CA)^2\). The centre is \(\{0, 3, 5, 6\}\) (the evil numbers). The order 4 elements are \(1a, 1b, 1c, 2a, 2b, 2c, 4a, 4b, 4c, 7a, 7b, 7c\) (the \(A\)-, \(B\)- and \(C\)-flips of the odious numbers). There are 19 elements of order 2. The numbers of subgroups of orders \(\{2, 4, 8, 16\}\) are \(\{19, 31, 30, 7\}\), the 7 maximal subgroups being \(C4 \times C4\) and 6 copies of \(D8 \times C2\). It is a semidirect product of \(C4 \times C4\) with \(C2\).

### 6.22 Which incircles do I draw?

Table 6.4 displays the incentres, \(A_i\), \(B_j\), \(C_k\) of the appropriate respective triangles \(BIC\), \(CIA\), \(AIB\), as triples \(ijk\), where \(i\), \(j\), \(k\) are the hexadecimal numbers which appear in Figures 6.23 to 6.25 and the accompanying list of incentres at the end of §7.
CHAPTER 6. THE LIGHTHOUSE THEOREM

Table 6.4: Incenres appropriate to the 32 solutions.

|   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|---|----|----|----|----|----|----|----|----|
| ordinary | 000 | 280 | 802 | a82 | 028 | 2a8 | 82a | aaa |
| A-flip   | 541 | d43 | 7c1 | fc3 | feb | 7e9 | d6b | 569 |
| B-flip   | 154 | 17c | bfe | bd6 | 3d4 | 3fc | 97e | 956 |
| C-flip   | 415 | ebf | 43d | e97 | 6bd | c3f | 695 |

The pattern becomes memorable if you write the solution number in binary and replace the 1-digits by 2s. Then recall the base-4, beam-number, forms of the hexadecimal numbers. For example, think of solution 6 = 110 as 220. Its incentres are \{8,2,a\} or, in base 4, \{20,02,22\} which are beam numbers from lighthouses at \(BC, CA, AB\), or 2, 2, 0 from \(A, B, C\) respectively.

To find the beam numbers for a flipped solution, for example 6b, fix the \(B\) beam number and advance or decrease the other two beam numbers according as the solution number is evil or odious. As 6 is evil, 2,2,0 become 2+1,2,0+1 = 3,2,1 and the incentres are illuminated by beams 2 & 1, 1 & 3, 3 & 2 from \(B & C, C & A, A & B\) respectively, so that they are the points 21=9, 13=7, 32=e. An odious example is 4c: solution number 4 is 100 in binary, giving beam numbers 2,0,0 which \(C\)-flip to 2−1,0−1,0 = 1,3,0 and illuminate incentres 30,01,13 = c,1,7.

Query. Triangle \(ABC\) has 4 incentres, \(I\). Each of triangles \(IBC, ICA, IAB\) has 4 incircles. There are \(4^4\) choices of a triple of incircles. Where are the 256 radical centres of these triples? Are there coincidences? collinearities? concyclicities?

6.23 In or out, near or far?

What do the solutions look like? The eight ordinary solutions have all their three circles in the interior angles. The flipped solutions have two exterior circles, but keep the appropriate (\(A\)-, \(B\)- or \(C\)-)circle interior, except that it becomes “vertically opposite” (outside the triangle, and denoted by lower case letters in Table 6.5) if the solution number “contains” 4, 2 or 1 respectively [4, 5, 6, 7 contain 4; while 2, 3, 6, 7 contain 2; and 1, 3, 5, 7 contain 1]. Call a circle “Near” if it touches the edges at points nearer to its vertex than the other two circles do. Otherwise it’s “Far”—the proximity of the three circles is as follows—in Table 6.5 I’ve written the solution numbers in binary so that you can see the pattern:
In Figure 6.30 the eight triangles $ABC$ are manifestations of the solutions listed in the first row of Table 6.6. Note the relation of the subscripts to the solution numbers. Other solutions are found by varying the relative sizes of the circles, in a kind of extraversion. For example, if the A-circle is made small, then the solutions are as in the third row of Table 6.6: they are ordinary or A-flip solutions whose numbers contain 4. If the A-circle is enlarged, then the solutions are as in the last row: all solution numbers appear; solutions are ordinary if they contain 1 & 2, A-flips if they do not contain 1 or 2, B-flips if 2 is the
only number contained or missing, and C-flips if 1 is the only number present or missing. Other solutions can be obtained by permuting the sizes of the circles.

\[
\begin{align*}
ABC_{sub} & \quad 000 \quad 111 \quad 222 \quad 012 \quad 444 \quad 104 \quad 240 \quad 421 \\
& \quad 0 \quad 1c \quad 2b \quad 3a \quad 4a \quad 5b \quad 6c \quad 7 \\
\text{A small} & \quad 4 \quad 5a \quad 6a \quad 7a \quad 4a \quad 5 \quad 6 \quad 7 \\
\text{A large} & \quad 3 \quad 5b \quad 6c \quad 0a \quad 7 \quad 1c \quad 2b \quad 4a
\end{align*}
\]

Table 6.6: Malfatti solutions from given circles.

There are more than 32 Malfatti solutions. There are four degenerate solutions, consisting of an incircle taken with multiplicity three, and 24 semi-degenerate solutions consisting of a repeated incircle (4 possibilities) and a third circle touching two sides (3 choices) and touching the incircle either proximally or distally. These solutions are more relevant to Malfatti’s original problem of maximizing the volume of three cylinders cut from a triangular prism \[212\].

But Steiner mentions 48 others! Twenty-four of these he attributes to Gergonne \[81\]. They have two of the circles touching an edge of the triangle “in einem und demselben Punkt” \(X\) in Figure 6.31. The radii of the circles are no longer rational functions of \(u, v\) and \(w\).
6.24 Radpoints, guylines, oddpoints & tielines

[But see Chapter 7 below.]

These are Conway’s names for members of a remarkable configuration. “Radpoints” are radical centres of triads of Malfatti circles, so that there are 32 of them. Their areal (barycentric) coordinates are

\[
\left( \frac{u(1-u^2)}{1+u^2}, \frac{v(1-v^2)}{1+v^2}, \frac{w(1-w^2)}{1+w^2} \right)
\]

where \(u, v, w\) are still the tangents of the quarter-angles of triangle \(ABC\) and they range over the functions shown in Figure 6.29. Denote the shapes of the coordinates by \(I\) (identity), \(R\) (reciprocal), \(S\) (switch), and \(T\) (twist), with lower case letters for their negatives. “Switch” and “twist” are closely related to “flip”.

| function | \(x\) | \(-x\) | \(\frac{1}{x}\) | \(-\frac{1}{x}\) | \(\frac{1-x}{1+x}\) | \(\frac{x-1}{1+x}\) | \(\frac{1+x}{x-1}\) | \(\frac{x+1}{x-1}\) |
|-----------|------|--------|---------------|---------------|----------------|----------------|----------------|----------------|
| radcoord  | \(\frac{x(1-x^2)}{1+x^2}\) | \(\frac{x^2-1}{x^2+1}\) | \(\frac{1-x^2}{2x(1-x)}\) | \(\frac{2x(1-x)}{x(1+x)}\) | \(\frac{x-1}{(x+1)(1+x^2)}\) | \(\frac{2x(x-1)}{(x+1)(x^2+1)}\) | \(\frac{x-1}{(x-1)(1+x^2)}\) | \(\frac{2x(1-x)}{(1-x)(1+x^2)}\) |
| symbol    | \(I_x\) | \(i_x\) | \(R_x\) | \(r_x\) | \(S_x\) | \(s_x\) | \(T_x\) | \(t_x\) |

Table 6.7: Shapes of coordinates of radpoints.

Table [6.8] shows the areal coordinates \(\{x, y, z\}\) of all 32 radpoints, with braces and commas omitted. As the coordinates are homogeneous, I may change all three signs when necessary to produce a majority of upper case letters.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| o | \(I_u I_v I_w\) | \(I_u I_v r_w\) | \(I_u r_v I_w\) | \(i_u R_v R_w\) | \(r_u I_v I_w\) | \(R_u i_v R_w\) | \(R_u R_v i_w\) |
| a | \(i_u S_v S_w\) | \(I_u T_v s_w\) | \(I_u s_v T_w\) | \(I_u T_v t_w\) | \(r_u T_v I_w\) | \(R_u S_v t_w\) | \(R_u t_v S_w\) |
| b | \(S_u i_v S_w\) | \(T_u I_v s_w\) | \(T_u r_v T_w\) | \(S_u R_v t_w\) | \(s_u I_v T_w\) | \(T_u I_v t_w\) | \(t_u R_v S_w\) | \(S_u R_v S_w\) |
| c | \(S_u S_v i_w\) | \(T_u T_v r_w\) | \(T_u s_v I_w\) | \(S_u t_v R_w\) | \(s_u T_v I_w\) | \(t_u S_v R_w\) | \(T_u T_v I_w\) | \(S_u S_v R_w\) |

Table 6.8: Coordinates of the 32 radpoints.

Now it is easy to pick out collinearities. I’ll use the sets of coordinates “III”, “iSS”, “SRt”, etc., with the subscripts omitted, as symbols for “the radical centre of the solutions 0, 0a, 3b” etc., while A, B, C still denote the vertices of the triangle. For example, the triples of points \{A, iSS, RSS\}, \{B, III, IrI\} and \{C, SRt, SRS\} are each collinear, as you can see from the vanishing of the determinants

\[
\begin{vmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
i_u & S_v & S_w & I_u & I_v & I_w & S_u & R_v \\
R_u & S_v & S_w & I_u & r_v & I_w & S_u & R_v \\
\end{vmatrix}
\]
The respective equations to the lines are \( y/S_v = z/S_w, \ x/I_u = z/I_w \) and \( x/S_u = y/R_v \). The triples always comprise a vertex, a radpoint from an evil solution, and a radpoint from an odious solution.

**There are 64 guylines.** This is the second part of the Malfatti miracle:

*There are 64 such collinearities! Sixteen “vertical” ones through each of \( A, B \) and \( C \) and four “nails” through each of the four Nagel points \( N_a, N_b, N_c \).*

**What is a Nagel point?** Choose three of the four incircles of \( ABC \) [if you choose one, as Conway says for “one and only one”, then you get a Gergonne point]. Let them [it] touch \( BC, CA, AB \) at \( A', B', C' \) respectively. Then \( AA', BB', CC' \) concur in a Nagel [Gergonne] point. The Nagel and Gergonne points are listed in Table 6.9 with four representations of their areal coordinates.

\[
\begin{align*}
N_o & \{ s-a, s-b, s-c \} \quad \{ \cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2} \} \quad \{ I-R, I-R, I-R \} \quad \{ \frac{u-v}{u}, \frac{v-w}{v}, \frac{w-u}{w} \} \\
N_a & \{ s, c-s, b-s \} \quad \{ -\cot \frac{A}{2}, 4 \tan \frac{B}{2}, 4 \tan \frac{C}{2} \} \quad \{ I-R, T-S, T-S \} \quad \{ \frac{u-v^2}{2u}, \frac{v-w^2}{2v}, \frac{w-u^2}{2w} \} \\
N_b & \{ c-s, s, a-s \} \quad \{ 4 \tan \frac{A}{2}, -\cot \frac{B}{2}, 4 \tan \frac{C}{2} \} \quad \{ T-S, I-R, T-S \} \quad \{ \frac{2u}{u^2-1}, \frac{2v}{v^2-1}, \frac{2w}{w^2-1} \} \\
N_c & \{ b-s, a-s, s \} \quad \{ 4 \tan \frac{A}{2}, 4 \tan \frac{B}{2}, -\cot \frac{C}{2} \} \quad \{ T-S, T-S, I-R \} \quad \{ \frac{2u}{u^2-1}, \frac{2v}{v^2-1}, \frac{2w}{w^2-1} \} \\
G_o & \{ \frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c} \} \quad \{ \tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2} \} \quad \{ T-S, T-S, T-S \} \quad \{ \frac{u}{1-u^2}, \frac{v}{1-v^2}, \frac{w}{1-w^2} \} \\
G_a & \{ \frac{1}{s}, \frac{1}{c-s}, \frac{1}{b-s} \} \quad \{ 4 \tan \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2} \} \quad \{ T-S, I-R, I-R \} \quad \{ \frac{u}{1-u^2}, \frac{v^2-1}{2v}, \frac{w-1}{2w} \} \\
G_b & \{ \frac{1}{c-s}, \frac{1}{s}, \frac{1}{a-s} \} \quad \{ \cot \frac{A}{2}, 4 \tan \frac{B}{2}, \cot \frac{C}{2} \} \quad \{ I-R, T-S, I-R \} \quad \{ \frac{u^2-1}{2u}, \frac{v^2}{2v}, \frac{w^2-1}{2w} \} \\
G_c & \{ \frac{1}{b-s}, \frac{1}{a-s}, \frac{1}{s} \} \quad \{ \cot \frac{A}{2}, \cot \frac{B}{2}, 4 \tan \frac{C}{2} \} \quad \{ I-R, I-R, T-S \} \quad \{ \frac{u^2-1}{2u}, \frac{v^2-1}{2v}, \frac{w}{1-w^2} \}
\end{align*}
\]

Table 6.9: Coordinates of Nagel and Gergonne points.

The Nagel points are the incentres of the dilated or anticomplementary triangle [with centre \( G \), the centroid, amplify the triangle \( ABC \) by a factor \(-2\); shown dashed in Figure 6.33 with its angle-bisectors dotted]. They are a set of orthocentric points generated by a pair of two-beam lighthouses at two of the vertices of \( A'B'C' \). Their nine-point centre is the orthocentre of \( ABC \). The Gergonne points are not, in general, orthocentric.

Table 6.10 lists the 64 guylines \((0 \leq X \leq 7)\). Each radpoint lies on 4 guylines, one through each of \( A, B \) and \( C \), and one through a Nagel point. This table is easy to memorize, since, in each box, the four pairs of numbers have the same nim-sum [binary addition without carry, i.e., vector addition over GF(2), or XOR]. If this nim-sum is nim-added to the row and column numbers, the total is always 7. Each guyline may be given a three-digit label using the evil digits 0, 3, 5, 6 taken from the row, column, and entry in the table. For example, the line \{C2b6b\} has label 656. Given a line label, its points may be read as “The first digit gives the vertex (A=3, B=5, C=6) or Nagel point (=0), the second
specifies an ordinary (0) or flipped (3,5,6) solution, the third is the evil solution number, the odious one being that which makes the nim-total 7." For example, 536 passes through B (=5) and through two A-flip points (A=3), namely 6a and 7a (since the nim-sum of 5, 3, 6 and 7 is 7).

| Xo (=0) | Xa (=3) | Xb (=5) | Xc (=6) |
|---------|---------|---------|---------|
| No 0o III 7o RRR | Na 0a iSS 4a rTT | Nb 0b SiS 2b TrT | Nc 0c SSi 1c TTr |
| 3o iRR 4o rII | 3a ITT 7a RSS | 3b Srt 1b TIs | 3c StR 2c TsI |
| 5o RiR 2o Irl | 5a RSt 1a ITs | 5b TIT 7b SRS | 5c tSR 4c sTI |
| 6o RRi 1o Iir | 6a RtS 2a IsT | 6b tRS 4b sIT | 6c TTI 7c SSR |
| A 0o III 4o rII | A 0a iSS 7a RSS | A 0b SiS 1b TIs | A 0c SSi 2c TsI |
| 3o iRR 7o RRR | 3a ITT 4a rTT | 3b Srt 2b TrT | 3c StR 1c TTr |
| 5o RiR 1o Iir | 5a RSt 2a IsT | 5b TIT 4b sIT | 5c tSR 7c SSR |
| 6o RRi 2o Irl | 6a RtS 1a ITs | 6b tRS 7b SRS | 6c TTI 4c sTI |
| B 0o III 2o Irl | B 0a iSS 1a ITs | B 0b SiS 7b SRS | B 0c SSi 4c sTI |
| 3o iRR 1o Iir | 3a ITT 2a IsT | 3b Srt 4b sIT | 3c StR 7c SSR |
| 5o RiR 7o RRR | 5a RSt 4a rTT | 5b TIT 2b TrT | 5c tSR 1c TTr |
| 6o RRi 4o rII | 6a RtS 7a RSS | 6b tRS 1b TIs | 6c TTI 2c TsI |
| C 0o III 1o Iir | C 0a iSS 2a IsT | C 0b SiS 4b sIT | C 0c SSi 7c SSR |
| 3o iRR 2o Irl | 3a ITT 1a ITs | 3b Srt 7b SRS | 3c StR 4c sTI |
| 5o RiR 4o rII | 5a RSt 7a RSS | 5b TIT 1b TIs | 5c tSR 2c TsI |
| 6o RRi 7o RRR | 6a RtS 4a rTT | 6b tRS 2b TrT | 6c TTI 1c TTr |

Table 6.10: 32 radpoints in pairs on 64 guylines.
Figure 6.32: 32 radpoints on 48 guylines through 3 vertices.
Figure 6.32 shows the 48 “vertical” guylines. Fortuitously, A 3b 2b and A 6c 4c with equations $y/R_v = z/T_w$ and $y/T_v = z/I_w$ coincide in Figure 6.32 which is drawn with $v = \frac{1}{4}$, $w = \frac{1}{3}$, giving $17y + 50z = 0$ in each case.

Figure 6.33: 32 radpoints on 16 guylines (nails) through 4 Nagel points.

Figure 6.33 shows the remaining 16 guylines: four “nails” through each of the four Nagel points. The triangle $ABC$ is not shown, but the dilated triangle, twice its size, is shown dashed, with its angle-bisectors, shown dotted, concurring in the four Nagel points.
There are 32 oddpoints. The third part of the Malfatti miracle is:

*The 48 “vertical” guylines concur in threes at 32 oddpoints.*

It is easy to see, for example, that the point TII (with coordinates \(\{T_u, I_v, I_w\}\)) lies on each of the three lines \(\{A \text{ III } r\Pi\}, \{B \text{ TTI } Tsl\}, \{C \text{ TIT } TsI\}\). Table 6.11 gives the coordinates of the oddpoints and the labels of the guylines therethrough.

|      | TTT | TTs | TsT | tSS | sTT | StS | SSt | SSS |
|------|-----|-----|-----|-----|-----|-----|-----|-----|
|      | 333 | 336 | 335 | 330 | 333 | 336 | 335 | 330 |
|      | 555 | 556 | 555 | 556 | 553 | 556 | 553 | 550 |
|      | 666 | 666 | 665 | 665 | 663 | 663 | 660 | 660 |
| tRR  | 303 | 305 | 306 | 300 | 300 | 306 | 305 | 303 |
|      | 565 | 565 | 566 | 565 | 565 | 560 | 565 | 563 |
|      | 566 | 566 | 565 | 565 | 563 | 565 | 563 | 565 |
| RtR  | 363 | 363 | 360 | 360 | 366 | 366 | 365 | 365 |
|      | 505 | 503 | 500 | 506 | 506 | 500 | 503 | 505 |
|      | 636 | 633 | 630 | 635 | 633 | 633 | 630 | 635 |
| RRt  | 353 | 363 | 353 | 350 | 355 | 356 | 355 | 356 |
|      | 535 | 503 | 533 | 536 | 535 | 535 | 530 | 536 |
|      | 606 | 600 | 603 | 605 | 605 | 603 | 600 | 606 |

Table 6.11: 48 guylines concur in threes at 32 oddpoints.

6.25 The SHOESTRING duality.

There are very simple relations between the coordinates of the radpoints and of the oddpoints. To get from one radpoint to the other on the same guylines, or to get from one oddpoint to the other on the same tielines (soon to be defined in the fourth part of the miracle) just swap

\[
I \leftrightarrow R \quad \text{and} \quad S \leftrightarrow T
\]

in all three coordinates if the line is through a Nagel or Gergonne point, but only in the \(x\)-, \(y\)- or \(z\)-coordinate if the line is through \(A\), \(B\) or \(C\) respectively. To get from a radpoint to an oddpoint on the same line, swap

\[
I \leftrightarrow S \quad \text{and} \quad R \leftrightarrow T
\]

in all three coordinates if the line is through a Nagel or Gergonne point, but only in the one appropriate coordinate if the line is through a vertex.

There are many other aspects to this duality, which may be mnemonically listed:

| coordinates | points | lines | numbers |
|-------------|--------|-------|---------|
| HERIN      | REcIp  | Rad   | Evl     |
| SOSTG      | IdEN   | NagEl | zERo    |
|            | SwiTch | Odd   | twee    |
|            | TwiST  | GerG. | pENtE   |
|            |        | Tie   | HEx     |
|            |        | Tail  | OdiOuS  |
|            |        |      | Seven   |
|            |        |      | TeSSara |
|            |        |      | TwO     |
|            |        |      | One     |
6.25. THE SHOESTRING DUALITY.

In particular, numerical labels may be given to radpoints and oddpoints by assigning 0 to a coordinate I; 7 to a coordinate S; 3, 5, 6 respectively to \( x-, y-, z \)-coordinates R; and 4, 2, 1 to coordinates T. Radpoints will have evil labels, oddpoints odious ones. The reader will notice many simple relationships between the point labels and the labels of the lines on which they lie.

**There are 64 tielines.** The fourth part of the Malfatti miracle is:

*The oddpoints lie in pairs on 64 tielines, sixteen “vertical” ones through each of A, B, C and four “tails” through each of the four Gergonne points.*

| Xo (7) | Xa (4) | Xb (2) | Xc (1) |
|-------|-------|-------|-------|
| Go 7o SSS 0o TTT | Ga 7a SRR 3a TI | Gb 7b RSR 5b ITI | Gc 7c RRS 6c IIT |
| G(7) | 4o sTT 3o tSS | 4a sII 0a rRR | 4b rTI 6b iSR | 4c rIT 5c iRS |
| 2o TsT 5o StS | 2a TrI 6a SiR | 2b IsI 0b RrR | 2c IrT 3c RiS |
| 1o TTs 6o StS | 1a TrI 5a SRI | 1b ITI 3b RSi | 1c IsI 0c RRT |
| A(4) | 4o sTT 0o TTT | 4a sII 3a TI | 4b rTI 5b ITI | 4c rIT 6c IIT |
| A 7o SSS 3o tSS | A 7a SRR 0a rRR | A 7b RSR 6b iSR | A 7c RRS 5c iRS |
| 2o TsT 6o StS | 2a TrI 5a IsT | 2b IsI 3b RSi | 2c IrT 0c RRt |
| 1o TTs 5o StS | 1a TrI 6a SRI | 1b ITI 0b Rrr | 1c IsI 3c RiS |
| B(2) | 4o sTT 6o StS | 4a sII 5a SRI | 4b rTI 3b RSi | 4c rIT 0c RRt |
| B 7o SSS 5o StS | B 7a SRR 6a SiR | B 7b RSR 0b RrR | B 7c RRS 3c RiS |
| 2o TsT 0o TTT | 2a TrI 3a TI | 2b IsI 5b ITI | 2c IrT 6c IIT |
| 1o TTs 3o tSS | 1a TrI 0a rRR | 1b ITI 6b iSR | 1c IsI 5c iRS |
| C(1) | 4o sTT 5o StS | 4a sII 6a SiR | 4b rTI 0b RrR | 4c rIT 3c RiS |
| C 7o SSS 6o SSS | C 7a SRR 5a SRI | C 7b RSR 3b RSi | C 7c RRS 0c RRt |
| 2o TsT 3o tSS | 2a TrI 0a rRR | 2b IsI 6b iSR | 2c IrTt 5c iRS |
| 1o TTs 0o TTT | 1a TrI 3a TI | 1b ITI 5b ITI | 1c IsI 6c IIT |

Table 6.12: 32 oddpoints in pairs on 64 tielines.

**Help wanted.** In Table 6.12 I have retained the solution numbers, but what, if any, is their relevance? The duality between the radpoints and the oddpoints is clear from Tables 6.8 and 6.11 and Tables 6.10 and 6.12. The radpoints are radical centres, but what relation do the oddpoints have to the solutions I’ve listed them against? From [116, X(1488) and X(2089)] we discover the happy alliterations that SSS is the Second Stevanovic point and that TTT is the Third mid-arc point. At the latter reference you can confirm that these two points are collinear with the Gergonne point, \( G_o \). Perhaps the construction given in [116] generalizes to give all 32 oddpoints?
Let $A'$, $B'$, $C'$ be the first points of intersection of the angle bisectors of triangle $ABC$ with its incircle. Let $A''B''C''$ be the triangle formed by the lines tangent to the incircle at $A'$, $B'$, $C'$. Then $A''B''C''$ is perspective to the intouch triangle of $ABC$, and the perspector is $X(2089)$.

Darij Grinberg, Hyacinthos #8072, 2003-10-01

Another fortuity is the concurrence of three tielines $\{A \ T \ T \ \ast \ T\}$, $\{B \ I \ I \ \ast \ I\}$, $\{C \ R \ R \ \ast \ R\}$ in Figure in a point which is not an oddpoint. In general these lines do not concur.
Figure 6.34: 32 oddpoints on 48 tielines through 3 vertices.
Figure 6.35: 32 oddpoints on 16 tielines (tails) through 4 Gergonne points.
6.26 History and Literature.

The truth is out there. I leave \( n = 5 \) (which, with its \( \binom{5}{2}^3 = 1000 \) points, is particularly beautiful) and larger values of \( n \) to the reader. There’s much, much more to be discovered. A glimpse of the richness for \( n = 6 \) is seen in Figures 6.17 to 6.19. Since 2 and 3 divide 6, there are three imbeddings of Lighthouse-2 and two imbeddings of Lighthouse-3. So there are two complete Morley configurations together with their interaction via \( n = 2 \) with its sets of orthocentric points. Are there further incidences among the 18 GF-circles?

6.26 History and Literature.

The Morley triangle theorem has an interesting history. It’s a very Euclidean theorem, but it was discovered comparatively recently—by Frank Morley around the turn of the previous century, though he didn’t publish anything about it until many years later \[142, 143\]. The first published proofs appeared in 1909: a trigonometric one by Satyanarayana \[180\] and a synthetic one by Naraniengar \[150\]. Naraniengar’s proof, which hinges on a lemma, can be found in Coxeter & Greitzer \[52, pp. 47–50\]. It was rediscovered by Child \[47\] and by others. A direct proof seems to be elusive, though the one given here in Section 3 may qualify. There is an “inside-out” proof due to Bricard \[36\], expounded in \[50, pp. 23–25\]; it uses a lemma that is a corollary to the angle-bisector theorem. A similar proof is due to Bottema \[17, p. 34\]. For other proofs, see \[46, 96, 11, 202, 80, 175, 202\]. There are few hints that there is more than one Morley triangle, but Honsberger \[112, p. 98\] asks the reader to show that Morley’s theorem holds also in the case of the trisection of the exterior angles of a triangle. He gives a proof, for just one more Morley triangle, on pp. 163–164. This is quoted in \[120\], an article drawn to my attention by Coxeter, that is deserving of wider circulation.

There is a statement, rarely remembered or repeated, of the existence of 27 points lying six by six on nine (Morley) lines, at \[114, p. 254, \S 421\]. This is attributed to \[202\] where there are two proofs. The first is quite neat, attributed to W. E. Philip, and given in \[114\]. Figure 3 on \[202, p. 124\] is very like our Figure 6.7. Glanville Taylor & Marr also give some attention to nonequilateral choices of three of the 27 Morley points. Johnson’s next section, his \( \S 422 \), gives the theorem that the incentres of the four triangles of a cyclic quadrangle form the vertices of a rectangle, and its generalization to the first sixteen of the Thrice Sixteen Theorem. This is attributed to Fuhrmann \[76, p. 50\].

I hope that Rigby’s paper \[179\] will eventually see the light of day. As he says there:

\[ \ldots \] I then consulted Morley’s paper on the subject; this contains results that rarely seem to be quoted, so I have included an account of aspects of the theorem that are apparently not generally known, especially the connection with cardioids inscribed in the triangle.
Honsberger [112, p. 93] quotes Morley in this regard:

If a variable cardioid touch the sides of a triangle the locus of its center, that is, the center of the circle on which the equal circles roll, is a set of 9 lines which are 3 by 3 parallel, the directions being those of the sides of an equilateral triangle. The meets of these lines correspond to double tangents; they are also the meets of certain trisectors of angles, internal and external, of the first triangle.

These are, of course, the 9 Morley lines of the triangle. Rigby generalizes this:

**Theorem [179].** The locus of the centres of epicycloids $C(m, n)$ touching a given triangle consists of $(n + m)^2$ lines (with an exception if the triangle is equilateral).

Here the epicycloid $C(m, n)$ is the envelope of the line joining the points with polar coordinates $(1, m\theta)$ and $(1, n\theta)$, where $0 < m < n$ and $m \perp n$. The origin, $O$, is the centre of the epicycloid. E.g., $(m, n) = (1, 2)$ gives the cardioid and $(1, 3)$ the nephroid. If $m$ is allowed to be negative, $C(m, n)$ is a hypocycloid; for example $(m, n) = (-1, 2)$ gives a deltoid and $(-1, 3)$ an astroid. Better known are the connexions with the Steiner deltoid (the envelope of the Simson-Wallace lines of the triangle) and the nine-point circle, their common tangents being parallel to the edges of the Morley triangles. See [11, pp. 345–349] [50, Ex. 7, p. 115], [79, pp. 226–231], [101], [134] pp. 72–79], [79]. A referee has also pointed to [80] and [128, ch. IV pp. 173–194]. An interesting generalization, wherein the angles of the triangle are partitioned in the ratios $1:n:1$, and hence implicating the Lighthouse Theorem for $n + 2$, is given by Fox & Goggins [?].

**Query.** If the lighthouses are rotating with different angular velocities, is there a connexion between the locus of the intersection of the beams with the epicycloids considered by Morley and Rigby?

The story began at about the time that I discovered a “lost notebook” of Conway, and we embarked on writing *The Triangle Book* [49]. In the interim, Steve Sigur has taken over, and, by the time you’re reading this, the book will probably be available. Also, since the story began, news has been spreading, and many people, including D. J. Newman [79] 163–166 and Alain Connes, have taken an interest in Morley’s theorem (see [http://www.cut-the-knot.com/triangle/Morley]).
Chapter 7

More on Morley & Malfatti

This is an unfinished paper by John Conway and myself.

In [104] the second author discussed the geometry of the points and lines associated with the Morley and Malfatti configurations. Here we shall introduce a simple notation which has enabled us to establish the incidences between the points and lines discussed in [104]. In particular, it shows immediately that the “tielines” and “guylines” are identical; we also correct several errors in sign.

It is simplest to define various points and lines by their coordinates, and only later establish their relation to the actual geometry. We start by defining 64 points $\langle ijk \rangle$ (pointed brackets for points), where $i, j, k$ are integers mod 4, by their barycentric (areal) coordinates

$\langle ijk \rangle : \langle f(A + i\pi), f(B + j\pi), f(C + k\pi) \rangle$

where $f(\theta) = \tan\frac{\theta}{4} \cos\frac{\theta}{2}$ and $A, B, C$ represent both the vertices and the angles of our triangle.

It is immediately apparent that the points

$A \langle 0jk \rangle \langle 1jk \rangle \langle 2jk \rangle \langle 3jk \rangle$

are collinear; we call this the guyline $[\ast jk]$ [linear brackets for lines] and define further guylines $[i \ast k]$ and $[i j \ast]$ through $B$ and $C$ respectively. There are 48 such guylines, 16 through each vertex.

It is easy to see that if $t = \tan\frac{\theta}{4}$, then the tangents $t_n$ of $\frac{\theta + n\pi}{4}$ are given by

$t_0 = t \quad t_1 = \frac{1 + t}{1 - t} \quad t_2 = -\frac{1}{t} \quad t_3 = \frac{t - 1}{t + 1}$

If we define $u_n, v_n, w_n$ to be the values of these when $\theta = A, B, C$ respectively, then it is
also easy to see that
\[ \langle ijk \rangle = \left( u_i \frac{1 - u_i^2}{1 + u_i^2}, v_j \frac{1 - v_j^2}{1 + v_j^2}, w_k \frac{1 - w_k^2}{1 + w_k^2} \right) \]

We check that
\[ t_0 \frac{1 - t_0^2}{1 + t_0^2} + t_2 \frac{1 - t_2^2}{1 + t_2^2} = \left( t + \frac{1}{t} \right) \frac{1 - t^2}{1 + t^2} = \frac{1 - t^2}{t} = \frac{2}{\sin \frac{a}{2}} \]
from which it follows that the line joining 000 to 222 passes through
\[ \left\langle \frac{\csc A}{2}, \frac{\csc B}{2}, \frac{\csc C}{2} \right\rangle \]
which are the coordinates of the original Nagel point, \( N_0 \). We call this line a **Nail** (German: Nagel = nail). Similarly
\[ t_1 \frac{1 - t_1^2}{1 + t_1^2} + t_3 \frac{1 - t_3^2}{1 + t_3^2} = \frac{2}{\cos \frac{a}{2}} \]
showing that the line joining \( \langle 111 \rangle \) to \( \langle 333 \rangle \) passes through
\[ \left\langle \frac{\sec A}{2}, \frac{\sec B}{2}, \frac{\sec C}{2} \right\rangle , \]
The original Gergonne point, \( G_0 \). We call this line a **peG**.

### 7.1 Extraversion

All the formulas of triangle geometry remain true if the angles \( A, B, C \) are replaced by
\[ i\pi + A, j\pi + B, k\pi + C \quad \text{if} \quad i+j+k \equiv 0 \]

or \[ i\pi - A, j\pi - B, k\pi - C \quad \text{if} \quad i+j+k \equiv 2 \]
The first author has called this process **extraversion** since it involves turning inside out and produces “extra versions” of triangle constructs.
Figure 7.1: There are 32 Malfatti radical points.
The above operators form the extraversion group, which is generated by the particular operators called the $a$-flip, $b$-flip, and $c$-flip, obtained by respectively taking $ijk = 011, 101, 110$ (see the double-arrows labelled $a$, $b$, $c$ in Figure 7.1). For the Malfatti case, the values of $i$, $j$, $k$ are only relevant mod 4, and the extraversions of 000 are exhibited in Figure 7.1. Note that the opposite edges of the figure are identified, forming a torus, partitioned into 16 hexagonal regions.

### 7.2 Extraverting the Malfatti configuration

In [104] it is shown that 000 is the radical centre of the original three Malfatti circles. Figure 7.1 shows that this is one of 32 triples of extra Malfatti circles, indexed by the triples $ijk$ with $i+j+k$ even, that correspond to the vertices of Figure 7.1. We call these (Malfatti or Ajima) radpoints, and we have shown that the points $A$, $0jk$, $1jk$, $2jk$, $3jk$ are collinear, and since two of these are radpoints, we see that

Each guyline is the join of two radpoints.

The two radpoints on the guyline $[3*3]$ are the ends (303 and 323) of the vertical diagonal of the topmost hexagon in Figure 7.1. By symmetry it follows that

The 48 guylines correspond to the 48 diagonals of the hexagons.

The remaining two points on the typical guyline have $i+j+k$ odd, so are not radpoints; we call them oddpoints. However, they are easily constructed from the radpoints. For instance, $\langle 333 \rangle$ is the intersection of the guylines $[*33]$, $[3*3]$ and $[33*]$, which we have already constructed from the radpoints. We see that these guylines are the three diagonals of the topmost hexagon in Figure 7.1. Extraverting these, we obtain:

The 16 points $\langle ijk \rangle$ with $i+j+k \equiv 1$ correspond to the hexagons.

We call these the 1-points. Each oddpoint is the intersection of three guylines. For the particular 1-point $\langle 333 \rangle$ the guylines are the diagonals of just one hexagon (the topmost one), while for the particular 1-point 111 they are the diagonals of three hexagons. Extraverting this we conclude that the oddpoints fall into two distinct orbits of size 16, namely

1-points, with $i+j+k \equiv 1$, whose guylines are the diagonals of one hexagon;
3-points, with $i+j+k \equiv 3$, whose guylines are the diagonals of three hexagons.
7.2. **EXTRAVERTING THE MALFATTI CONFIGURATION**

In Figure 7.1, \(ijkx\) labels the hexagon whose 1-point is \(ijk\), and whose Nail and peG pass through \(N_x\) and \(G_x\) respectively. The four concepts corresponding to the hexagon \(ijkx\) are:

The 1-point \(\langle ijk \rangle\) on guylines \([*jk], [i * k], [ij*]\);

The 3-point \(\langle i + 2, j + 2, k + 2 \rangle\) “antipodal” to this, through the points
\(\langle *, j + 2, k + 2 \rangle, \langle i + 2, *, k + 2 \rangle, \langle i + 2, j + 2, * \rangle\);

The Nail \([ijkx]\) through \(N_x\) and the points \(\langle ijk \rangle \pm \langle 111 \rangle\);

The peG \([i + 2, j + 2, k + 2, x]\) through \(G_x\) and \(\langle ijk \rangle\) and \(\langle i + 2, j + 2, k + 2 \rangle\).

These four are exemplified in Figure 7.1 by

The 1-point \(\langle 333 \rangle\) on diagonals \([*33], [3 * 3], [33*]\) of the topmost hexagon;

the 3-point \(\langle 111 \rangle\), visualized as the intersection of the diagonals \([*11], [1 * 1], [11*]\) of hexagons 311, 131, 113;

the Nail \([333o]\) through \(N_o\) and the points \(\langle 000 \rangle, \langle 222 \rangle\) furthest from the (topmost) hexagon 333o;

and the invisible peG \([111o]\) through \(G_o\) and the points \(\langle 333 \rangle \langle 111 \rangle\). [This last one is wrong, I think. We remark that the last two points are the vertices of Figure 7.1 furthest from the hexagon.

Figure 7.2 summarizes the situation.
### Figure 7.2: Summary of Figure 7.1

The congruences are mod 4.

| Condition | Value |
|-----------|-------|
| $i+j+k \equiv 0 \text{ or } 2$ | 32 radpoints |
| $i+j+k \equiv 1$ | 16 1-points |
| $i+j+k \equiv 3$ | 16 3-points |
| $i+j+k \equiv 1$ | 16 Nails |
| $i+j+k \equiv 3$ | 16 Gergonne points |
| $i+j+k \equiv 0 \text{ or } 2$ | 4 Nagel points, $N_x$ |
| $i+j+k \equiv 1$ | 4 Gergonne points, $G_x$ |

#### Structural Description

- **3 Vertices**: $A, B, C$
- **48 Guylines**: $jk, i \ast k, ij$
- **32 Radpoints**: $i+j+k \equiv 0 \text{ or } 2$
- **16 1-Points**: $i+j+k \equiv 1$
- **16 3-Points**: $i+j+k \equiv 3$
- **16 Nails**: $i+j+k \equiv 1$
- **16 Gergonne Points**: $i+j+k \equiv 3$
- **4 Nagel Points**: $i+j+k \equiv 0 \text{ or } 2$
7.3 Oddpoints, mid-arc points, Stevanovic points

The next page & a bit are ramblings. But I can now describe the 32 “3rd mid-arc points” — see Section 5, between the Kimberling descriptions of “3rd mid-arc point” and “2nd Stevanovic point”.

There are other geometrical constructions for our oddpoints. Our original 1-point, $\langle 333 \rangle$, is what Kimberling calls the 2nd Stevanovic point, $X_{1488}$ in [116], so the 16 1-points are extraversions of that.

$$\left\langle \frac{\sin A}{1 + \sin A/2}, \frac{\sin B}{1 + \sin B/2}, \frac{\sin C}{1 + \sin C/2} \right\rangle$$

[I still need to get Kimberling’s $X_{1488}$ and $X_{2089}$ constructed, generalized (extraverted), and identified with the 1-points and 3-points, respectively. Presumably, as there are only 16 1-points and 16 3-points, extraversion doesn’t yield 32 different cases. If you separate the operations ‘negative’ and ‘reciprocal’ in $t \rightarrow -\frac{1}{t}$, does this swap 1-points with 3-points?? Here are the (clearly related) barycentrics for $X_{2089}$:

$$\left\langle \frac{\sin A}{1 - \sin A/2}, \frac{\sin B}{1 - \sin B/2}, \frac{\sin C}{1 - \sin C/2} \right\rangle$$

So it looks as though you can get $X_{2089}$ from $X_{1488}$ by negating $A$, $B$ and $C$, i.e., by making an $a$-flip, a $b$-flip, and a $c$-flip, n’est-ce pas?

Let’s answer some questions!

From $t_3 = \frac{t_1}{t_4 + 1}$, we have

$$u_3^1 - u_3^2 = \frac{u - 1}{u + 1} \cdot \frac{4u}{2u^2 + 2} = \frac{2\tan \frac{A}{4}(\tan \frac{A}{4} - 1)}{\sec^2 \frac{A}{4}(\tan \frac{A}{4} + 1)} = \frac{2\sin \frac{A}{4}\cos \frac{A}{4}(\sin \frac{A}{4} - \cos \frac{A}{4})}{\sin \frac{A}{4} + \cos \frac{A}{4}}$$

so the point $\langle 333 \rangle$ is indeed the 2nd Stevanovic point, $X_{1488}$, since the factor $-\frac{1}{2}$ can be omitted by homogeneity. But does this factor sabotage things when we come to extraversions?? Let’s do $\langle 111 \rangle$ first:

From $t_1 = \frac{t_2}{t_4 - 1}$, we have

$$u_1^1 - u_1^2 = \frac{1 + u}{1 - u} \cdot \frac{4u}{2u^2 + 2} = \frac{2\tan \frac{A}{4}(1 + \tan \frac{A}{4})}{\sec^2 \frac{A}{4}(1 - \tan \frac{A}{4})} = \frac{2\sin \frac{A}{4}\cos \frac{A}{4}(\cos \frac{A}{4} + \sin \frac{A}{4})}{\cos \frac{A}{4} - \sin \frac{A}{4}}$$

$$= \frac{\sin \frac{A}{4}(\cos^2 \frac{A}{4} - \sin^2 \frac{A}{4})}{(\cos \frac{A}{4} - \sin \frac{A}{4})^2} = \frac{\sin \frac{A}{2}\cos \frac{A}{2}}{1 - 2\sin \frac{A}{4}\cos \frac{A}{4}} = \frac{1}{2} \cdot \frac{\sin A}{1 - \sin \frac{A}{2}}$$
and we can omit the homogeneous $\frac{1}{2}$ to confirm that $\langle 111 \rangle$ is $X(2089)$ that Kimberling calls the 3rd mid-arc point.

Here are Kimberling’s descriptions, slightly amplified with formulas in our notation. Note that there are no obvious connexions between the constructions and the Malfatti circles.

$X(2089) = \text{3rd MID-ARC POINT} \text{ [my TTT, i.e. } \langle 111 \rangle]\]

Trilinears: $(\cos B/2 + \cos C/2 - \cos A/2) \sec A/2 : (\cos C/2 + \cos A/2 - \cos B/2) \sec B/2 : (\cos A/2 + \cos B/2 - \cos C/2) \sec B/2 = 1/(1 - \sin A/2) : 1/(1 - \sin B/2) : 1/(1 - \sin C/2)$

Barycentrics: $(\sin A)f(A, B, C) : (\sin B)f(B, C, A) : (\sin C)f(C, A, B)$,

\[
\left[ \text{i.e., } \langle \ldots, \frac{v(1+v)}{(1-v)(1+v^2)}, \ldots \rangle \right]
\]

We seem to have 16 examples of Desargues’s theorem.

Let $A', B', C'$ be the first points of intersection of the angle bisectors of triangle $ABC$ with its incircle. Let $A''B''C''$ be the triangle formed by the lines tangent to the incircle at $A'$, $B'$, $C'$. Then $A''B''C''$ is perspective to the intouch triangle of $ABC$, and the perspector is $X(2089)$. (Darij Grinberg, Hyacinthos #8072, 10/01/03)

$X(2089)$ lies on these lines: 1,167 2,178 7,1488

$X(2089) = X(7)$-Ceva conjugate of $X(174)$

$X(2089) = X(173)$-cross conjugate of $X(174)$

### 7.4 Third mid-arc points

Each of the four touch-circles (incircle & three excircles) is cut by three angle-bisectors in two diametrically opposite points, so there are four sets of three pairs of parallel tangents there. These form four sets of eight triangles, each triangle being homothetic to the respective touch-triangle (whose vertices are the points of contact of the edges of the original triangle with the respective touch-circle). So each member of the four sets of eight triangles is in perspective with the respective touch-triangle, and we have 32 perspectors, each of which has a claim to be a “third mid-arc point”. Until my co-author gives me a better notation, I will denote these points by symbols $X_{abc}$ where $X = O, A, B, C$ according to which touch-circle it belongs and $a, b, c$ are $i$ or $o$, for ‘inner’ and ‘outer’, according as the tangents to the touch-circle is nearer to or further from the respective vertices $A, B, C$. For example, $O_{iii}$ is the third mid-arc point as originally defined by Grinberg, and is our point 111.
In fact 16 of the points are our “1-points”:

\[
\begin{align*}
O_{iii} &= 111 & O_{ioo} &= 133 & O_{oi0} &= 313 & O_{ooi} &= 331 \\
A_{ooo} &= 300 & A_{oii} &= 322 & A_{oi0} &= 102 & A_{ioi} &= 120 \\
B_{ooo} &= 030 & B_{oii} &= 012 & B_{oi0} &= 232 & B_{ioi} &= 210 \\
C_{ooo} &= 003 & C_{oii} &= 021 & C_{oi0} &= 201 & C_{ioi} &= 223
\end{align*}
\]

The other 16 have barycentric coordinates

\[
\left< \frac{u}{1 + u^2}, \frac{v}{1 + v^2}, \frac{w}{1 + w^2} \right>
\]

where \(u\) (“0”) extraverts to \(\frac{1+u}{1-u}\) (“1”), \(-\frac{1}{u}\) (“2”), and \(\frac{u-1}{u+1}\) (“3”).

LATER: I discover that this is Kimberling’s X(174), the Yff centre of congruence. Unfortunately, I’m unable to understand the description in ETC: description of X(173), congruent isoscelizers point, is appended below, in case it helps.

“In notes dated 1987, Yff raises a question concerning certain triangles lying within ABC: can three isoscelizers (as defined in connection with X(173), P(B)Q(C), P(C)Q(A), P(A)Q(B) be constructed so that the four triangles P(A)Q(A)A, P(B)Q(B)B, P(C)Q(C)C, ABC are congruent? After proving that the answer is yes, Yff moves the three isoscelizers in such a way that the three outer triangles, P(A)Q(A)A, P(B)Q(B)B, P(C)Q(C)C stay congruent and the inner triangle, ABC, shrinks to X(174).

“Let D be the point on side BC such that \(\text{angle BID} = \text{angle DIC}\), and likewise for point E on side CA and point F on side AB. The lines AD, BE, CF concur in X(174). [Seiichi Kirikami, Jan. 29, 2010]

“Generalization: if I is replaced by an arbitrary point P = p : q : r (trilinears), then the lines AD, BE, CF concur in the point \(K(P) = f(p,q,r,A) : f(q,r,p,B) : f(r,p,q,C)\), where \(f(p,q,r,A) = (q^2 + r^2 + 2qr \cos A)^{-1/2}\). Moreover, if P* is the inverse of P in the circumcircle, then \(K(P*) = K(P)\). [Peter Moses, Feb. 1, 2010, based on Seiichi Kirikami’s construction of X(174)]”

“X(173) = CONGRUENT ISOSCELIZERS POINT

“Let P(B)Q(C) be an isoscelizer: let P(B) on sideline AC and Q(C) on AB be equidistant from A, so that AP(B)Q(C) is an isosceles triangle. Line P(B)-to-Q(C), P(C)-to-Q(A), P(A)-to-Q(B) concur in X(173). (P. Yff, unpublished notes, 1989)

“The intouch triangle of the intouch triangle of triangle ABC is perspective to triangle ABC, and X(173) is the perspector. (Eric Danneels, Hyacinthos 7892, 9/13/03)”

Alternatively, the coordinates are
\[ \left\{ \sin \frac{A}{2}, \sin \frac{B}{2}, \sin \frac{C}{2} \right\} \]

which extravert to

\[ \left\{ -\sin \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2} \right\}, \text{etc.} \]

Note that interchange of “0” and “2”, or of “1” and “3” only changes the sign of the coordinate, so that, for example, “301” is the same as “123”. The 16 points are, until I have a better notation, the “0-points” (= the “2-points”)

\[
\begin{align*}
O_{ooo} &= "000" \\
A_{iii} &= "233" \\
B_{iii} &= "323" \\
C_{iii} &= "332"
\end{align*}
\[
\begin{align*}
A_{oio} &= "022" \\
A_{ioo} &= "211" \\
B_{ioo} &= "301" \\
C_{ioo} &= "310"
\end{align*}
\[
\begin{align*}
O_{iio} &= "202" \\
A_{oio} &= "031" \\
B_{oio} &= "121" \\
C_{oio} &= "130"
\end{align*}
\[
\begin{align*}
O_{iio} &= "220" \\
A_{ooi} &= "013" \\
B_{ooi} &= "103" \\
C_{ooi} &= "112"
\end{align*}
\]

One way to describe these 16 points, is as constituting four quadrangles, one associated with each touch-circle, whose diagonal point triangles each coincide with the original triangle, \(ABC\). We have the following collinearities: [Done in haste: needs checking]

\[
\begin{align*}
AO_{ooo}O_{oii} & \quad BO_{ooo}O_{iio} & \quad CO_{ooo}O_{iio} \\
AO_{iio}O_{iio} & \quad BO_{iio}O_{iio} & \quad CO_{iio}O_{iio} \\
AA_{iii}A_{iio} & \quad BA_{iii}A_{ioo} & \quad CA_{iii}A_{ooi} \\
AA_{oio}A_{ooi} & \quad BA_{oio}A_{ooi} & \quad CA_{oio}A_{ooi} \\
AB_{iii}B_{iio} & \quad BB_{iii}B_{ioo} & \quad CB_{iii}B_{ooi} \\
AB_{oio}B_{ooi} & \quad BB_{oio}B_{ooi} & \quad CB_{oio}B_{ooi} \\
AC_{iii}C_{iio} & \quad BC_{iii}C_{ioo} & \quad CC_{iii}C_{ooi} \\
AC_{oio}C_{ooi} & \quad BC_{oio}C_{ooi} & \quad CC_{oio}C_{ooi}
\end{align*}
\]

7.5 Second Stevanovic point(s)?

I first quote from ETC, and interpolate a

\[ X(1488) = 2\text{nd STEVANOVIC POINT} \begin{bmatrix} \text{my SSS, sh’d’ve been sss, i.e. } \langle 333 \rangle \end{bmatrix} \]

Trilinears: \(f(A, B, C) : f(B, C, A) : f(C, A, B)\), where \(f(A, B, C) = 1/[1 + \sin(A/2)]\)
7.5. SECOND STEVANOVIC POINT(S)?

Barycentrics \((\sin A) f(A, B, C) : (\sin B) f(B, C, A) : (\sin C) f(C, A, B)\), 

\[
\begin{bmatrix}
\text{i.e.,} \\
\langle \ldots, \frac{v(1-v)}{(1+v)(1+v^2)}, \ldots \rangle
\end{bmatrix}
\]

The points \(A'', B'', C''\) in the following construction, have coordinates

\[
\left\langle 0, \frac{v(1-v)}{(1+v)(1+v^2)}, \frac{w(1-w)}{(1+w)(1+w^2)} \right\rangle, \left\langle \frac{u(1-u)}{(1+u)(1+u^2)}, 0, \frac{w(1-w)}{(1+w)(1+w^2)} \right\rangle
\]

and

\[
\left\langle \frac{u(1-u)}{(1+u)(1+u^2)}, \frac{v(1-v)}{(1+v)(1+v^2)}, 0 \right\rangle
\]

so that the first of them, for example, lies on the join of \(I_o\) to the incentre of \(BIC\).

Let \(U\) be the \(A\)-excenter of triangle \(ABC\); let \(A'\) be the incenter of triangle \(UBC\), and define \(B', C'\) cyclically. Let \(A'' = IA' \cap BC\), and define \(B'', C''\) cyclically. The lines \(AA'', BB'', CC''\) concur in \(X(1488)\). (Milorad R. Stevanovic, Hyacinthos #7185, 5/21/03. See also \(X(1130)\) and \(X(1489)\).)

\(X(1488)\) lies on these lines: \(1,166\)  \(7,2089\)  \(57,173\)  \(145,188\)  \(557,1274\)  \(558,1143\)

\(X(1488) = X(1)\)-cross conjugate of \(X(174)\)

In my desperation to generalize the 2nd Stevanovic point I tried “inverting” his construction in the following way. If \(A'\) is the incentre of triangle \(BIC\), let \(A'I_a\) cut \(BC\) in \(A''\) and define \(B'', C''\) cyclically. Then \(AA'', BB'', CC''\) concur. Where? In ETC \(X(483)\), our 000! Are the radpoints some sort of inverses of 32 “2nd Stevanovich points”? One can certainly see the connexion with Steiner’s Malfatti construction, which starts with the incircles of \(BIC, CIA, AIB\). I must get in touch with my coauthor.
7.6 Notation at last!

2016-06-06. This section 7.6, was originally headed “More on Morley” but its original content now appears elsewhere, and has been replaced by recent thoughts on notation.

To avoid coincidences which do not occur in general, we assume that our triangles are scalene, that is, neither right-angled nor isosceles.

The labels of the vertices will be three chosen from the set \( \{1, 2, 4, 7\} \). The fourth member, the nim-sum of those chosen, is the label of the orthocentre. See “quadration” at the beginning of the next chapter (Chap. 8).

On those occasions where it is desirable to distinguish between acute and obtuse triangles, the acute triangle will be \( 124 \) and the related obtuse triangles will be \( 247, 417, 127 \), the vertex with the obtuse angle being \( 7 \) in each case.

After quadration our triangle will have four vertices and six edges. The midpoints of the six edges \( 24, 41, 12, 71, 72, 74 \) are respectively labelled \( 6, 5, 3, \overline{6}, \overline{5}, \overline{3} \) where these last three, ‘bar 6’, ‘bar 5’ and ‘bar 3’ are the negatives of 6, 5 and 3. They are the midpoints of pairs of edges which contain an obtuse angle.

These six midpoints are also the midpoints of the edges \( \overline{71}, \overline{72}, \overline{74}, \overline{24}, \overline{41}, \overline{12} \), of the twin triangles – see 8.3 below.

The diagonal points of the quadrangle \( 1247 \) are the intersections of the pairs of edges \( 17 \& 24, 27 \& 41, 47 \& 12 \) and are denoted by \( D_6, D_5, D_3 \) respectively.

Similarly, the diagonal points of the twin quadrangle \( \overline{1247} \) are the intersections of the pairs of edges \( \overline{17} \& \overline{24}, \overline{27} \& \overline{41}, \overline{47} \& \overline{12} \) and are denoted by \( \overline{D}_6, \overline{D}_5, \overline{D}_3 \) respectively.

It will be noted that the midpoint of the join of any point with its negative is always the same point. We will call this the Centre of the triangle, and denote it by \( 0 \). In particular, the joins of the six points \( 6, 5, 3, D_6, D_5, D_3 \) with their negatives are all diameters of the same circle, often called the nine-point circle, but which we will call the Central circle of the triangle.

It has also been called the fifty-point centre, but it is not advisable to attach a number, since this will be a function of time and taste. In addition to the six midpoints and the six diagonal points, there are 32 points of contact with touch-circles, six points of contact with the double deltoid and with the Steiner Star of David, and eight vertices of the four conics which are enveloped by Droz-Farny lines.
7.6. NOTATION AT LAST!

Figure 7.3: Revised notation
Chapter 8

Eight Vertices, but only One Centre

8.1 Quadration

We’ll start with something we all know: a triangle

has an orthocentre.

That is: the altitudes (perpendiculars from the vertices to the opposite edges) concur.

Here are nine proofs of this.

1. Euclid first showed that the perpendicular bisectors of the edges of a triangle concur; then drew a twice-sized triangle whose perpendicular bisectors were the altitudes of the original triangle.
2. A conic through the four intersections of two rectangular hyperbolas is also a rectangular hyperbola: The pair of thin lines and the pair of thick lines are each perpendicular and form rectangular hyperbolas. Hence the pair of dashed lines form a rectangular hyperbola, and are perpendicular.

3. Draw a rectangular hyperbola through the vertices. Choose axes and scale so that its equation is $xy = 1$. The vertices are $(t, 1/t)$, for $t = t_1$, $t = t_2$, $t = t_3$. The slope of the join of $t_2$ and $t_3$ is $-1/t_2t_3$. The perpendicular through $t_1$ meets the hyperbola again at $t = -1/t_1t_2t_3$, whose symmetry shows that the three perpendiculars concur.

Figure 8.3: Bonus! The circumcircle meets the hyperbola again at $t = 1/t_1t_2t_3$, the opposite end of the diameter of the hyperbola through the orthocentre.
4. The existence of the orthocentre is the case \( n = 2 \) of the Lighthouse Theorem \([AMM, \textbf{114}(2007) \text{97–141}].\)

Two sets of \( n \) (dotted) lines at equal angular distances, \( \frac{\pi}{n} \), one set through each of the points \( B, C \), intersect in \( n^2 \) points that are the vertices of \( n \) regular \( n \)-gons. The \( \binom{n}{2} \) edges of each \( n \)-gon lie in \( n \) equally spaced angular directions.

Figure 8.4: The four 4-gons formed from lighthouses with \( n = 4 \).

5. Proof number 5 uses vectors. Take as origin, \( o \), the intersection of the perpendiculars from the vertices \( b \) and \( c \) onto the opposite edges. That is \( b \) is perpendicular to \( a-c \) and \( c \) is perpendicular to \( b-a \).

\[ b \cdot (a-c) = 0 = c \cdot (b-a) \]
\[ b \cdot a = b \cdot c = c \cdot b = c \cdot a \]
\[ (b-c) \cdot a = 0 \]

and \( a \) is perpendicular to \( b-c \).
The other four proofs are “clover-leaf theorems”. Given three circles, the radical axes of pairs concur in the radical centre. What is the radical axis of a pair of circles? Not very good definition: locus of points whence the tangents to the two circles are equal in length. Better to define the power of a point w.r.t. a circle, that is the square of the distance from the point to the centre of the circle minus the square of the radius of the circle. The power is negative for points inside the circle, positive for points outside the circle, and zero for points on the circle. Then the radical axis of two circles is the locus of points whose powers w.r.t. the two circles are equal. If the circles intersect, it is their common chord. If they are concentric, it is the line at infinity.

6. Draw the three “edge-circles”, the circles having the edges of the triangle as diameters.

![Figure 8.6: Edge-circles](image)

The circles on $AB$, $AC$ as diameters both intersect $BC$ in the point $D$ where $\angle ADB = \angle ADC = \pi/2$ since the angle in a semicircle is a right angle. $AD$ is the radical axis of the circles on $AB$, $AC$ as diameters. The radical axes are the altitudes of the triangle and hence concur.

7. In Fig.8.7 the altitudes from $B$ and $C$ meet at $P$. The circles with $BP$ and $CP$ as diameters cut the edges $AB$, $AC$ respectively at the feet of the altitudes from $C$ and $B$, since the angle in a semicircle is a right angle. It is clear that the radical axis of the $A$– and $B$–circles and that of the $A$– and $C$–circles are altitudes of the triangle. It remains to show that $A$, $P$, $D$ are collinear.

Have I wandered up a cul-de-sac? Does this not lead to a proof?

We shall see, after we’ve introduced quadration, that these three diameters may also be considered as edges of the triangle.
8. Draw the reflexions of the circumcircle in the three edges. From the dashed lines in Fig. 8.8 it will be seen that the line of centres of a pair of such circles is parallel to an edge of the triangle. The radical axis will therefore be perpendicular to that edge, and, since it passes through the opposite vertex, is an altitude of the triangle. Since the radical axes concur, so do the altitudes.
9. This what I call is THE Clover Leaf Theorem. Draw the **medial circles**; that is, the circles having the medians as diameters. Here is one:

![Figure 8.9: A medial circle.](image)

The medial circles pass through some interesting points:

The foot of the altitude, of course; but also what I call the **midfoot points**, half way between a vertex and the foot of an altitude.

Now let’s find the radical axis of two medial circles. It must be perpendicular to the line of centres, which is parallel to an edge.

![Figure 8.10: Line of centres parallel to an edge.](image)
We’ll show that the opposite vertex, $B$ in Fig.8.11, lies on the radical axis.

The radical axes are the altitudes, not the medians! If you like to throw in the nine-point circle, you have the Four Leaf Clover Theorem. The $\binom{4}{2} = 6$ radical axes are the six edges of the orthocentric quadrangle.
8.2 Quadrature

Here’s something else we all know;

(though some of you may not know that you know!)

The vertices and the orthocentre should have equal status.
Each of the four points is the orthocentre of the triangle formed by the other three!

A triangle is an orthocentric quadrangle!

A triangle has (at least!) four vertices and six edges.

**Theorem:** There are three times as many obtuse triangles as acute ones.

(There are four more proofs in *Math. Mag.* 66(1993) 175–179.)

Reminder: The joins of the midpoints of opposite edges of any quadrangle (not just orthocentric ones) concur and bisect each other.

8.3 Circumcentres. Twinning

Let’s look for circumcentres.

Draw the perpendicular bisectors of the edges (there are now six!)

![Figure 8.13: Hello, twins!](image)

We get a configuration congruent to the original orthocentric quadrangle, forming an involution with it, having fixed point $O$.

Our triangle now has **eight** vertices.

They are all orthocentres and all circumcentres!!
There are **twelve edges** which share **six mid-points**.

These, with the **six diagonal points**, form three rectangles; whose six diagonals are diameters of a circle, centre $O$.

You probably call it the nine-point circle.

I call it the **fifty-point circle**. [NOT ANY MORE! See §7.6]

Note that $O$ is THE unique centre of the triangle.

Any other candidate would have a twin with equal right.

![Figure 8.14: Twelve of the 50 points](image)

Notice that, although we now have eight circumcircles, they all have the same radius — twice that of the fifty-point circle.

In fact the circumcircles form pairs of reflexions in the twelve edges:

$2$ reflexions $\times 12$ edges $/ 3$ edges in a triangle $= 8$ circumcircles.

For those of you who prefer to do things analytically, here’s how quadration appears:

A triangle with angles $A$, $B$, $C$ and edge-lengths $2R \sin A$, $2R \sin B$, $2R \sin C$,

quadrates into three more triangles:

\[
\begin{align*}
\pi - A & \quad \pi/2 - C & \quad \pi/2 - B & \quad 2R \sin A & \quad 2R \cos C & \quad 2R \cos B \\
\pi/2 - C & \quad \pi - B & \quad \pi/2 - A & \quad 2R \cos C & \quad 2R \sin B & \quad 2R \cos A \\
\pi/2 - B & \quad \pi/2 - A & \quad \pi - C & \quad 2R \cos B & \quad 2R \cos A & \quad 2R \sin C
\end{align*}
\]
8.4 Touch Circles

You will all be familiar with the incircle and three excircles of a triangle. Their centres, the touch-centres, are where the angle-bisectors concur:

You will be familiar with the incircle and three excircles of a triangle. Their centres, the touch-centres, are where the angle-bisectors concur.

[orthocentric quadrangle!!]

32 touch-circles each touch 3 of the 12 edges at one of 8 points \((8 \times 12/3 = 32)\).

Each touch-circle appears to touch the 50-point circle.

Figure 8.15: The incentre and three excentres form an orthocentric quadrangle whose 50-point centre is the circumcentre of the original triangle.
Figure 8.16: 32 touch-circles each touch 3 of the 12 edges at one of 8 points, and each touches the 50-point circle (which isn’t drawn, but you can “see” it!)
Figure 8.17: Enlargement of Fig.8.16. 32 touch-circles each touch 3 of the 12 edges at one of 8 points, and each touches the 50-point circle (which is drawn in this version).
We have 8 triangles, and hence 32 touch-circles. They all appear to touch the 50-point circle. This is Feuerbach’s theorem, though published by C.-J. Brianchon & J.-V. Poncelet in Recherches sur la détermination d’une hyperbole équilatère, au moyen de quatre conditions données, Ann. des Math., 11(1821) 205–220.

As an undergraduate I learned Feuerbach’s theorem as the 11-point conic, the locus of the poles of a line with respect to a 4-point system of conics, but that is another story. If the 4 points form an orthocentric quadrangle, the conics are rectangular hyperbolas; if the line is the line at infinity, the poles are their centres, which lie on the 9-point circle, the other two points being the circular points at infinity.

I wanted to find a proof of Feuerbach’s theorem that was fit for human consumption. The best that I’ve been able to do is a sort of parody of Conway’s extraversión.

8.5 Hexaflexing

Take one of the \( \binom{4}{2} = 6 \) pairs of touch-circles. The three triangle edges are common tangents. Reflect the triangle in the corresponding angle-bisector. This adds the fourth common tangent, dotted in figure.
Reflect the triangle in all 6 angle-bisectors. If you reflect a line in each of 2 perpendicular lines, you get 2 parallel lines. The 3 pairs of parallel (dotted) lines touch pairs of circles in 12 points, 3 on each of the 4 touch-circles.

Figure 8.19: Three pairs of parallel common tangents to the touch-circles

The 3 points on each of the 4 circles form triangles homothetic to the original triangle:

Figure 8.20: Four triangles homothetic to the original triangle
... and also homothetic to the **medial triangle** (dashed in the following figure). Where is (are) the perspector(s) [centre(s) of perspective]?

They lie on the circumcircles of the homothets, and on the circumcircle of the medial triangle, which is, of course, the 50-point circle. That is, all the 32 touch-circles touch the 50-point circle.

### 8.6 Gergonne points

Join the vertices of a triangle to the touch-points of one the touch-circles to the opposite edges of the triangle. By Ceva’s theorem, the joins concur.
If we join the four Gergonne points to their corresponding touch-centres, the four segments concur in a de LONGCHAMPS POINT.

Figure 8.23: Extraversion of Gergonne points in a triangle. The Gergonne point, touch-centre and deLongchamps point are collinear.

A de Longchamps point lies on the corresponding Euler line

Figure 8.24: A de Longchamps point together with the corresponding centroid separate the circumcentre and orthocentre harmonically.
Here are 32 Gergonne points in one diagram. Lines connect them to corresponding touch-centres (unmarked in figure) and concur in fours at 8 deLongchamps points.

Figure 8.25: Thirty-two Gergonne points.

The 8 de Longchamps points and 8 centroids separate the 8 circumcentres and 8 orthocentres in 8 harmonic ranges which lie in pairs on

Figure 8.26: 4 EULER LINES
8.7 Six more points to make up the fifty.

We will find them as the midpoints of the edges of the BEAT twins.
The Best Equilateral Approximating Triangles.

We start from the Simson Line Theorem. This doesn’t appear in Simson’s work, but is due to Wallace, so we’ll use his name instead.

If, from a point on the circumcircle of a triangle, we drop perpendiculars onto the edges, then their feet are collinear (the Wallace line).

Figure 8.27: The Simson-Wallace line theorem.
As the point moves round the circumcircle, the Wallace line rotates in the opposite sense with half the angular velocity.

Figure 8.28: Angle between Wallace lines of $S$ and $S' = \angle UV_1U'' = \frac{1}{2}\text{arc}UU'' = \frac{1}{2}\text{arc}S'S$. 
The Wallace line bisects the segment joining the orthocentre to the point on the circum-circle.

This bisection point is on the 50-point circle.

The Wallace line is a diameter of a circle of radius $R$ which encloses the 50-point circle and rolls inside the concentric Steiner circle, radius $\frac{3}{2}R$. 
Figure 8.30: The deltoid is a locus and an envelope.

...and, as the point moves round the circumcircle, the Wallace line envelops the STEINER DELTOID

Figure 8.31: The deltoid as envelope.
We’ve found our 3-symmetry!

Let’s try Quadration.

Three perpendiculars from points on each of 4 circumcircles coincide in pairs to give 6 points on the Wallace line.

Figure 8.32: Quadration of the Simson-Wallace line theorem.
Now let’s try Twinning.

Twinning gives 6 points on a parallel Wallace line, the reflexion of the first in the 50-point centre.

Figure 8.33: Quadrature of the Simson-Wallace line theorem.

Figure 8.34: These parallels envelop a double-deltoid.
There are 3 occasions when the parallels coincide, with 12 feet on the same Wallace line.

Figure 8.35: Three-Symmetry

... and three occasions when the parallels are perpendicular to these; the edges of the BEAT twins, forming the Steiner Star of David.

Figure 8.36: The Steiner Star of David
The axes of symmetry are the trisectors of the angles between the diameters through the midpoints and diagonal points, nearer to the former.

\[ |B - C|/3, \quad |C - A|/3, \quad |A - B|/3 \]

These edges are parallel to those of the Morley triangles.

... of which there are $18 \times 8 = 144$. Here are 18 :-
In fact there are 27 equilateral triangles.

Eighteen are genuine Morley triangles.

Conway has given the name Guy Faux triangle to the other nine.

For details see [AMM, 114(2007) 97–141]

which describes, amongst other things, the Lighthouse Theorem for $n = 3$. 
Chapter 9

Notes on the Droz-Farny Theorem

9.1 Preliminary Remarks

The Droz-Farny theorem certainly deserves to reach a wider audience. So also should its several generalizations. Many of the references [22, 68, 85, 115] give generalizations (and perhaps others [165, 166]).

9.2 Statement of theorem

If a pair of perpendicular lines through $H$, the orthocentre of a triangle $ABC$, intersect the edge $BC$ (respectively $CA$ and $AB$) in $X_1$, $X_2$ (respectively $Y_1$, $Y_2$ and $Z_1$, $Z_2$), then the midpoints of $X_1X_2$, $Y_1Y_2$, $Z_1$, $Z_2$ are collinear.

Additionally, though it may not have been in Droz-Farny’s original statement:

The pair of perpendicular lines, the three edges of the triangle, and the Droz-Farny line are all tangent to a parabola.

Note that if two tangents to a parabola are perpendicular, then they intersect on the directrix; moreover, the join of the points of contact passes through the focus.

9.3 Generalizations

1. Beginnings of a projective version:

The pair of perpendicular lines may be replaced by any rectangular hyperbola having the
orthocentre as centre.

To see this, take the pair of perpendicular lines, or asymptotes to the hyperbola, as coordinate axes. Then the equation to any edge of a triangle may be written in intercept form,

\[ \frac{x}{a} + \frac{y}{b} = 1 \]

and the equation to the rectangular hyperbola as \( xy = c^2 \). Then the midpoint of the segment between the asymptotes (axes) is \((a/2, b/2)\) and the midpoint of the chord of the hyperbola is given by

\[ \frac{x}{a} + \frac{c^2}{bx} = 1 \]
\[ bx^2 + ac^2 = abx \]

the sum of whose roots is \( ab/b \), with average \( a/2 \) so that the midpoint of the chord is (also) \((a/2, b/2)\) and the result is independent of the signs of \( a, b \) and \( c^2 \).

See (also?) [68]

2. Semi-affine version:

In place of the midpoints of the three segments one can take any other ratio. [126] See also [115].

It is of interest to investigate the family of pseudo D-F lines that one gets by considering all ratios. In fact they evidently envelop a parabola inscribed in the triangle (the edges of the triangle and the pair of perpendicular lines are all tangents to the parabola). Then one can ask about the family of such parabolas generated by varying the pair of perpendicular lines through the orthocentre. Will every parabola inscribed in the triangle correspond to a suitable pair of perpendicular lines? Probably “yes”. I note in passing that a pair of perpendicular tangents to a parabola meet on the directrix, so the directrix passes through the orthocentre. Where are the focus, axis and vertex?

See (empty!) §9.7 and Appendix (§9.12.) [this all written in bits and cobbled together, so there’s repetition and possibly contradiction!]

What is the locus of the foci of the parabolas? The envelope of the directrices is the orthocentre. What is the locus of the vertices? The envelope of the axes?

### 9.4 Envelope of D-F line

If one varies the pair of perpendicular lines, the resulting D-F lines envelope a conic. This degenerates to a point (the right angle) in the case of a right-angled triangle. Otherwise it is an ellipse or a hyperbola, according as the triangle is acute-angled or obtuse-angled
(othocentre inside or outside the circumcircle). Foci: orthocentre, $H$, and circumcentre, $O$. [That, &/or something later, is wrong!!] Centre: nine-point centre. Major axis: Euler line. Vertices: intersections of Euler line with nine-point circle. Length of major axis: $OH$. Length of conjugate axis: $\sqrt{R^2 - OH^2}$, where $R$ is the circumradius. Asymptotes: diameters of nine-point circle through the points of tangency of the tangents to the nine-point circle from the orthocentre (imaginary if triangle is acute, when orthocentre is inside nine-point circle).

What relations are there, if any, between this conic and the several other conics associated with a triangle?

9.5 Miscellaneous ramblings

I want to write something about the Droz-Farny theorem, and could make use of someone to try things out on. It seems to me that Theorems 131 and 133 in Durell’s Projective Geometry are Feuerbach’s theorem and its dual, which is a projective generalization of the Droz-Farny theorem.

Here’s a start. Feuerbach’s theorem is the ‘11-pt conic’ theorem. Given a line and a 4-point pencil of conics, then the locus of the pole of the line is a conic. This passes through 11 points:

The 3 diagonal points of the quadrangle.
The 6 harmonic conjugates with respect to pairs of points of the quadrangle of the intersection of the line with the edge.
The 2 double points of the involution cut on the line by the conics.

Special cases:
(a) Line is line at infinity. Conjugates are the midpoints of the edges.
(b) Quadrangle is orthocentric. Conics are rectangular hyperbolas.
(c) (a) & (b) together. 11-pt conic is 9-pt circle + circular points at infinity.

The dual of Feuerbach’s theorem concerns a point and a range of conics touching 4 fixed lines. The envelope of the polar of the point with respect to the conics is the 11-line conic. The 11 lines are

The 6 harmonic conjugates with respect to pairs of edges of the quadrilateral of the join of the point to the intersection of the pair of edges.
The 3 edges of the diagonal line triangle of the quadrilateral.
The tangents at the point to the two conics which pass through the point.

What’s the connexion with Droz-Farny? I hear you cry. Take one of the four lines to be the line at infinity. The conics are now parabolas. The two parabolas which pass through
the given point have their foci on the circumcircle of the triangle formed by the three lines not at infinity, and their directrices passing through the orthocentre. This last is Steiner’s theorem. Durell A *Concise Geometrical Conics*, p.22. Euler line ?? Simson- Wallace line.

We’re now getting close to the Droz-Farny theorem. Compare Cosmin Pohoata’s Theorem B: Let $P$ be a point in the plane of a given triangle $ABC$. If $A'$, $B'$, $C'$ are the points where the reflexions of lines $PA$, $PB$, $PC$ in a given line through $P$ meet the edges $BC$, $CA$, $AB$, then $A'$, $B'$, $C'$ are collinear.
Let’s try again, with a picture.

Figure 9.1: Droz-Farny (thickline) and Wallace-Simson (dashline) lines.

Query: What are the relative rates of rotation for the
1. Pair of perpendicular lines through $H$,
2. Point $M$ on the circumcircle,
3. Droz-Farny line,
4. Wallace-Simson line of $M$ ??

In connexion with the first of these four, note that the pair of perpendicular lines may be replaced by a rectangular hyperbola having the orthocentre as centre. Now consider rectangular hyperbolas which are close to the perpendiculars (which are their asymptotes). They may lie in the first and third quadrants formed by the perpendicular lines or in the second and fourth.
One way of describing Fig. 9.1 is as follows:

Choose a point $A_3$ on edge $BC$ and draw the circle centre $A_3$ passing through the orthocentre $H$. If this cuts $BC$ in $A_1$ and $A_2$, then $A_3$ is the midpoint of $A_1A_2$ and $\angle A_1HA_2$ is a right angle (angle in a semicircle). Let $M$ be the point of intersection of this circle with the circumcircle of $ABC$. There are two choices for this — take the one on the same side of edge $BC$ as $A$. [Investigate what happens if you make the other choice.]

Let the line $A_1H$ meet the edges $CA$, $AB$ in $B_1$ and $C_1$, and the line $A_2H$ meet the edges $CA$, $AB$ in $B_2$ and $C_2$.

If you can see that circles $B_1HB_2$ and $C_1HC_2$ also pass through $M$, then we have a proof of the Droz-Farny theorem, since the three circles each pass through both $H$ and $M$, and so form a coaxal system with $HM$ as radical axis and the line of centres is the perpendicular bisector of $HM$, passing through the midpoints $A_3$, $B_3$, $C_3$ of $A_1A_2$, $B_1B_2$, $C_1C_2$ — i.e., the Droz-Farny line.

Note that the Wallace line of $M$ also passes through the midpoint of $HM$.

It is now easy to find the envelope of the Droz-Farny line as the pair of perpendiculars through $H$ rotate.

$O$ is the circumcentre. Let the circumradius $OM$ cut the Droz-Farny line at $P$, so that $HP = PM$ and $OP + PH = OP + PM = R$, the circumradius. Moreover the Droz-Farny line is equally inclined to $HP$ and $OP$ so that it is a tangent to the conic with foci $O$ and $H$ and axis of length $R$.

The conic is an ellipse or a hyperbola according as $H$ is internal or external to triangle $ABC$, i.e., according as $ABC$ is acute or obtuse; or, according as the D-F line is the external or internal angle-bisector of $\angle OPH$. 
Let’s try yet again, with another figure [9.2]

In Fig.9.2 $M$ is an arbitrary point on the circumcircle of triangle $ABC$ and $H$ is the orthocentre. The perpendicular bisector of the segment $MH$ cuts the edges $BC$, $CA$, $AB$ of the triangle in $A_3$, $B_3$, $C_3$ respectively. This is a putative Droz-Farny line.

The circles centres $A_3$, $B_3$, $C_3$ passing through $H$, and therefore also through $M$, cut the respective edges $BC$, $CA$, $AB$ in the pairs of points $A_1$, $A_2$; $B_1$, $B_2$; $C_1$, $C_2$. If we can prove that $A_1$, $B_1$, $C_1$ are collinear with $H$ and that $A_2$, $B_2$, $C_2$ are also collinear

Figure 9.2: A converse approach.
with $H$, then we have proved the Droz-Farny theorem. Note that $A_3$, $B_3$, $C_3$ are the midpoints of $A_1A_2$, $B_1B_2$, $C_1C_2$ respectively, since the latter are diameters of the three circles. Moreover angles $A_1HA_2$, $B_1HB_2$, $C_1HC_2$ are angles in semicircles and therefore right-angles, so that it suffices to show, for example, that $A_1HC_2$ is a right angle.

Third time lucky?? Yet another figure! (9.3)

Choose a point $A_3$ arbitrarily on $BC$. The circle centre $A_3$ passing through $H$ cuts $BC$ in $A_1$, $A_2$. The lines $A_1H$, $A_2H$ are perpendicular (angle in a semicircle). They meet the edge $CA$ in $B_1$ and $B_2$ and the edge $AB$ in $C_1$ and $C_2$. Are the midpoints $B_3$ and $C_3$ of $B_1B_2$ and $C_1C_2$ collinear with $A_3$?

Do the 4 circles, centres $O$, $A_3$, $B_3$, $C_3$, all concur??
9.6 There are two tangents from a point to a conic.

In our converse approach (Fig 9.2) there were, in fact, two choices for our point $M$ where the circle centre $A_3$ intersects the circumcircle. Call them $M_1$ and $M_2$.

$M_2$ is the reflexion of $H$ in the edge $BC$. This point is well-known to lie on the circumcircle. Then $BC$ is the perpendicular bisector of $HM_2$ and is a candidate for a Droz-Farny line through $A_3$. Certainly the circles centres $B$ and $C$, passing through $H$, also pass through $M_2$. $OM_2$ intersects $BC$ in $P_2$. $HP_2 + P_2O = M_2P_2 + P_2O = R$, the circumradius. $BC$ is the exterior angle-bisector of $\angle HP_2O$, so that $BC$ is the tangent at $P_2$ to the ellipse with foci $H$ and $O$ and major axis of length $R$.

$A_3BC$ and $A_3B_3C_3$ are the tangents at $P_2$ and $P_1$ to the conic.
9.7 Quadration and the Droz-Farny line.

Let’s find the Droz-Farny lines through $A_3$ for the two triangles $ABC$ and $HBC$. For each triangle, $BC$ is a D-F line.
Figure 9.6: Quadration and the Droz-Farny lines.
Figure 9.7: Enlargement of Fig. 9.6.
9.8 Some background theorems

Carnot’s theorem [38, p.101]. The segment of an altitude from the orthocentre to an edge equals its extension from the edge to the circumcircle.

In fact the circumcircles of $BHC$, $CHA$, $AHB$ are the reflexions of the circumcircle $ABC$ in the edges $BC$, $CA$, $AB$. Indeed

**Theorem R.** [123, p.99][114, §333]

Let $L$ be a line through the orthocentre $H$ of a triangle $ABC$ which cuts $BC$, $CA$, $AB$ in $L_a$, $L_b$, $L_c$. The reflexions of $L$ in the edges of $ABC$, $H_bL_a$, $H_cL_b$, $H_aL_c$, concur at a point, $P$, on the circumcircle. Moreover, the Wallace line of $P$ is parallel to $L$.

The proof is an offshoot of that of the Wallace (Simson) line theorem.

Miquel’s theorem [139][114, §184]. Also known as the Pivot Theorem.

$X$, $Y$, $Z$ are arbitrary points on the respective edges $BC$, $CA$, $AB$ of a triangle $ABC$. Then the circles $AYZ$, $BZX$, $CXY$ pass through a common point, the Miquel point of $XYZ$.

Miquel’s theorem is easy to prove: Suppose that the circles $BZX$, $CXY$ meet again in $M$. Then $\angle ZMX = \pi - B$, $\angle XMY = \pi - C$, so that $\angle YMZ = \pi - A$ and $AZMY$ is cyclic.

The Wallace (aka Simson) Line theorem. This appears elsewhere in this document; see especially §8.7.

**Ceva’s theorem.** $O$ is in the plane of triangle $ABC$. $OA$ meets $BC$ in $A'$, $OB$ meets $CA$ in $B'$, $OC$ meets $AB$ in $C'$. Then the product of the ratios  

$$\frac{BA'}{AC} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = +1$$

and conversely, if the product is 1, the “cevians” concur.
Menelaus’s theorem. If a line in the plane of triangle $ABC$ meets the edges $BC$, $CA$, $AB$ respectively in $A'$, $B'$, $C'$, then the product of the ratios

$$\frac{BA'}{AC} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1$$

and conversely, if the product is -1, then the points $A'$, $B'$, $C'$ are collinear.

Note that combination of Ceva’s and Menelaus’s theorems gives a “pole and polar” relationship between points and lines with respect to a triangle. E.g. The polar of the centroid is the line at infinity.
9.8. SOME BACKGROUND THEOREMS

Parabola theorems.

**Notation.** $S$ is the focus, $A$ the vertex, $X$ the foot of the directrix. $P$ any point on the curve. $PN$ the ordinate from $P$ to the axis. $PM$ perpendicular from $P$ to the directrix. Tangent at $P$ cuts directrix at $R$ and axis at $T$. Normal at $P$ cuts axis at $G$. $SY$ is perpendicular from $S$ to $PT$.

![Figure 9.9: The parabola](image)

**Theorem.** [63, Theorem 9]. (i) $PT$ bisects $\angle SPM; SP = ST = SG$.
(ii) $SM, PT$ bisect each other at right angles.
(iii) $Y$ lies on tangent at vertex.
(iv) $AN = AT; AY = \frac{1}{2}NP; NG = 2AS$.
(v) $SY^2 = SA \times SP$.

**Theorem.** [63, Theorem 10]. The tangents at the ends of a focal chord $PSP'$ meet at $R$, on the directrix. The line through $R$ parallel to the axis bisects $PP'$ at $V$. $\angle PRP' = 90^\circ$. If $RV$ cuts the parabola at $K$, then $RK = KV$ and $PP' = 4SK$.

**Theorem.** [63, Theorem 11]. If $OP, OQ$ are tangents to a parabola, then triangles $SOP, SQO$ are similar; $SO^2 = SP \cdot SQ$; the exterior angle between the tangents is equal to the angle either subtends at the focus; and $\angle SOQ$ equals the angle that $OP$ makes with the axis.

**Theorem.** [63, Theorem 12]. The circumcircle of a triangle circumscribing a parabola passes through the focus and (Steiner’s theorem) the orthocentre lies on the directrix.
Jean-Louis Ayme’s proof of the Droz-Farny theorem.

$H$ is the orthocentre of triangle $ABC$. Two perpendicular lines through $H$, $\mathcal{L}$ and $\mathcal{L}'$, cut $BC$ in $X$ and $X'$, $CA$ in $Y$ and $Y'$, and $AB$ in $Z$ and $Z'$.

Let $\mathcal{C}, \mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c,$ be the circumcircles of triangles $ABC, HXX', HYY', HZZ'$, and their centres be $O, M_a, M_b, M_c$. Let $H_a, H_b, H_c$ be the reflexions of $H$ in $BC$, $CA$ and $AB$, respectively. By Carnot’s theorem, $H_a, H_b, H_c$ lie on $\mathcal{C}$.

Note that, as $XHX'$ is a right angle, the circle $\mathcal{C}_a$ has $XX'$ as a diameter, so that $M_a$ lies on $BC$ and the circle passes through $H_a$ which also lies on the circle $\mathcal{C}$. (Carnot’s theorem.) Similarly $H_b$ is an intersection of $\mathcal{C}$ and $\mathcal{C}_b$ and the perpendicular to $CA$ through $H$.

![Figure 9.10: Start of Ayme's proof.](image-url)
By applying Theorem R to the line $XYZ$ through $H$, we conclude that its reflexions $H_aX$, $H_bY$, $H_cZ$ in the respective edges $BC$, $CA$, $AB$ concur on the circumcircle of $ABC$ at a point $N$, say. Now apply Miquel’s pivot theorem to the triangle $YXN$ with the points $H_a, H_b, H$ on respective edges $XN$, $NY$, $YX$, and we see that the circles $HH_aX = C_a$, $HH_bY = C_b$, and $NH_aH_b = C$ concur in a point $M$, say. Similarly it can be shown that the circles $C_a, C_b, C_c$ all pass through $M$. So the circles $C_a, C_b, C_c$ all pass through $H$ and $M$, and are therefore coaxal, with their centres $M_a, M_b, M_c$ collinear on the Droz-Farny line.

Note that this is the perpendicular bisector of the segment $HM$. $HM$ is the axis of the parabola we’re seeking. The D-F line is the tangent at the vertex. The parallel to the D-F line through $H$ is the directrix. Let this meet $BC$, $CA$, $AB$ in $D_a, D_b, D_c$. Then the perpendiculars to $BC$, $CA$, $AB$ through $D_a, D_b, D_c$ are also tangents to the parabola.

Is there a short proof that $H_aX', H_bY', H_cZ'$ concur at a point, $N'$, say, diametrically across the circumcircle from $N$?

In Figure 9.11, $O$ does not lie on the line $HXYZ$. The points $MHH_cZZ'$ are concyclic (the circle $C_c$). The lines $AHH_a, BHH_b, CHH_c$ are the altitudes of triangle $ABC$. 

Figure 9.11: Conclusion of Ayme’s proof.
The dual of Feuerbach’s 11-point conic theorem is the 11-line theorem [64, Theorem 133]:

\( p \) is a variable line through a fixed point \( L \); \( p' \) is a line conjugate to \( p \) w.r.t. a range of conics touching four fixed lines \( a, b, c, d \); the envelope of \( p' \) is a conic \( \sigma \) which touches the polars of \( L \) w.r.t. the conics of the range.

The conic \( \sigma \) touches the following 11 lines:

If \( h \) is the join of \( L \) and \( ab \) and \( h' \) is the harmonic conjugate of \( h \) w.r.t. \( a, b \), then \( \sigma \) touches \( h' \) and the corresponding five lines through the other five vertices of the quadrilateral \( abcd \); \( \sigma \) also touches the three edges of the diagonal line triangle of \( abcd \), and the two double lines of the involution formed by the pairs of tangents from \( L \) to the range of conics.

We are interested in the case where one of the four lines, say \( d \), is the line at infinity. The conics of the range are then parabolas.

As we vary the pair of perpendicular lines, the locus of the focus of the parabola is the circumcircle of \( ABC \). The directrix passes through the orthocentre \( H \). The Droz-Farny line is the tangent at the vertex of the parabola and we already know that this envelopes the conic with foci \( H \) and \( O \), the circumcentre of \( ABC \), and major (real) axis of length \( R \), the circumradius. Centre is 50-point centre. Axis is diameter of 50-point circle (Euler line).
Here’s a description of the parabola associated with the Droz-Farny line. We’ll use the notation of the last figure, repeated here.

The Droz-Farny line is the tangent to the parabola at its vertex, the midpoint of $MH$; where $H$ is the orthocentre and $M$ is the point on the circumcircle where the circles $C_a, C_b$ and $C_c$ concur.

$M$ is the focus of the parabola, and $MH$ is its axis. The perpendicular to the axis through $H$ is the directrix, parallel to the Droz-Farny line.

The edges $BC, CA, AB$ touch the parabola; where are the points of contact? The perpendicular lines $HXYZ$ and $HX'Y'Z'$ touch the parabola: where? If the points of contact are $P$ and $P'$, then the line $PP'$ passes through the focus, $M$.

The parabola also touches the line at infinity.

Figure 9.12: Parabola associated with Droz-Farny line (to appear!!)
Another drawing

$M$ is the focus of the parabola.

It lies on the circumcircle of $ABC$.

The D-F line is the perpendicular bisector of $MH$, where $H$ is the orthocentre of $ABC$. Let it cut $OM$ in $P$, so that $OP + PH = OP + PM = OM = R$, the circumradius. $MP$ and $HP$ are equally inclined to the D-F line, so that it is a tangent to the ellipse, foci $O$ and $H$ with major axis $R$. In case $H$ is outside $ABC$ (now obtuse) and its circumcircle, then $OP - PH = OM = R$ and the conic is a hyperbola.

D-F line cuts $BC$, $CA$, $AB$ in $X$, $Y$, $Z$.

Circles centres $X$, $Y$, $Z$ passing through $H$ cut $BC$; $CA$; $AB$ in $X_1$, $X_2$; $Y_1$, $Y_2$; $Z_1$, $Z_2$.

$H$, $X_1$, $Y_1$, $Z_1$ are collinear. Also $H$, $X_2$, $Y_2$, $Z_2$. These (dashed) lines are perpendicular, and tangent to the parabola. Synthetic proof wanted. Compare next section (or not! according to taste).
$M$ is a variable point on the circumcircle of $ABC$. The perpendicular bisector of $MH$ is a Droz-Farny line, where $H$ is the orthocentre of $ABC$. In Fig. 9.14, if the coordinates of $H$ are $(a, b)$ and those of $M$ are $(p, q)$, then the D-F line has equation

$$2x(p - a) + 2y(q - b) = p^2 - a^2 + q^2 - b^2$$

Note that the reflexions of $H$ in the edges of $ABC$ lie on the circumcircle, so that the edges are special cases of the D-F line. So also are the perpendicular bisectors of $AH$, $BH$, $CH$, which form the twin triangle of $ABC$ (rotate $ABC$ through an angle $\pi$ about its (one-&-only, “9-point”) CENTRE, the midpoint of $OH$. There is an involution between $O$ and $H$. 
Let’s look at the obtuse case. Interchange $A$ and $H$; i.e., $A$ is the orthocentre of triangle $BHC$. This is an example of quadrature.

Envelope is a hyperbola, foci $A$, $O'$, real axis of length $R$, the circumradius. Note that the edges, $BC$, $CH$ and $HB$ are also tangents to the hyperbola, since the reflexions of $A$ in these edges lie on the circumcircle $BCH$. Since $B$, $C$, $H$ are points on the circumcircle, so also are the perpendicular bisectors of $AB$, $AC$, $AH$ tangents to the hyperbola. Indeed these three are the edges of the TWIN of $HBC$, its reflexion in the CENTRE of triangle $HBC$, which is also the CENTRE of $ABC$ and the centre of the hyperbola, which we may see as follows. The asymptotes of the hyperbola will be the perpendicular bisectors of $AM_1$, $AM_2$, the tangents from $A$ to the circumcircle $BHC$. As tangent is perpendicular to radius, these two bisectors will pass through the midpoints of $AM_1$, $AM_2$ and be parallel to the radii $O'M_1$, $O'M_2$, and hence cut the third edge, $AO'$ of triangles $AM_1O'$, $AM_2O'$ in the midpoint of $AO'$, which is also the midpoint of $OH$ and the CENTRE.
9.10 Interruption — added later — can now throw away?

Or earlier, depending on what edition of this work you are reading. See §9.8

I outline what I hope will be a synthetic proof of the Droz-Farny theorem (＆/or its converse?). Perhaps it’s the same as J.-L. Ayme’s?

Figure 9.16:
Depending on whether one is trying to prove the theorem or its converse, select a point on an edge of the triangle (say $A_3$ on $BC$) or a point on the circumcircle of $ABC$ (say $K$). These are related as follows. If you selected $A_3$ on $BC$, then draw the circle, centre $A_3$, passing through the orthocentre, $H$. This will intersect the circumcircle of $ABC$ in two points, $H_a$ and $K$. $H_a$ is the reflexion of $H$ in the edge $BC$ — it’s well known to those who well know it, that the circumcircle of $HBC$ is the reflexion of the circumcircle of $ABC$ in the edge $BC$. Alternatively, choose $K$ arbitrarily on the circumcircle of $ABC$ and draw the circle through $KHH_a$. By the symmetry already mentioned, its centre will lie on $BC$; call it $A_3$. In either case the circle will intersect $BC$ in $A_1, A_2$, say, and $A_1A_2$ will be a diameter of the circle and $HA_1, HA_2$ will be our perpendicular rays through the orthocentre (shown dotted in Figure 1).

Next, draw the circles $KHH_b$ and $KHH_c$. They form a coaxal system with the circle $KHH_a$. By the symmetry already noted, their centres, $B_3$ and $C_3$ say, will lie on $CA$ and $AB$ respectively, and will be collinear with $A_3$, forming the Droz-Farny line, the perpendicular bisector of $HK$ (shown dashed in Figure 1).

If these two circles respectively cut the edges $CA$ and $AB$ in the points $B_1, B_2$ and $C_1, C_2$, then it remains to be shown that $H, A_1, B_1, C_1$ and $H, A_2, B_2, C_2$ are collinear, and that the two lines are perpendicular. The angles $A_1HA_2, B_1HB_2, C_1HC_2$ are each (right) angles in a semicircle, so it suffices to show, for example, that $HA_1$ is perpendicular to $HB_2$. Because we haven’t made it clear as to which of $A_1, A_2$, etc., is which, we run into difficulty — a difficulty experienced both here and by the author in his proofs; the arguments become diagram dependent, and angles are liable to be confused with their supplements. For example, the specification internal angle-bisector may be ambiguous. In fact, if a line through an angle, $APB$, say, is reflected about either angle-bisector, the result is the same.

### 9.11 Three (or more) new(?) theorems

We have also shown:

**Theorem.** If the pair of perpendicular rays through the orthocentre is allowed to vary, then the locus of the reflexion of the orthocentre in the Droz-Farny line is the circumcircle of $ABC$.

**Theorem.** If the pair of perpendicular rays through the orthocentre is allowed to vary, then the locus of the foot of the perpendicular from the orthocentre onto the Droz-Farny line is the nine-point circle of $ABC$. 
It is also now straightforward to prove:

**Theorem.** The envelope of the Droz-Farny line is a conic. The centre is the nine-point centre of the triangle $ABC$, and the major axis is the segment of the Euler line which is a diameter of the nine-point circle, and has length $R$, the circumradius of $ABC$. The conic is an ellipse or a hyperbola, according as the triangle is acute or obtuse. The conjugate axis has length $\sqrt{R^2 - h^2}$, where $h = HO$, the distance of the orthocentre from the circumcentre. If $h > R$, the (length of the) conjugate axis (of the hyperbola) is imaginary. In this case the asymptotes are the diameters of the nine-point circle through the points of contact of the tangents to the nine-point circle from the orthocentre.

These asymptotes are special cases of the Droz-Farny line. Other examples are the edges of the triangle, and the perpendicular bisectors of the segments $HA$, $HB$ and $HC$. Also the tangents to the nine-point circle at its intersections with the Euler line, i.e., the vertices of the conic.

![Diagram of Droz-Farny lines touching an ellipse](image)

**Figure 9.17:** Droz-Farny lines touching an ellipse. Incomplete. Add perp bisectors of $AH$, $BH$, $CH$, tans to 9-pt circle where it meets the Euler line, and further examples.

For a right triangle the conic degenerates to a point, the right angle itself, through which the Droz-Farny line always passes.
The case of the equilateral triangle deserves special mention. The orthocentre, circum-centre, nine-point centre and incentre all coincide, the Euler line is indeterminate, and the nine-point circle is also the incircle, and is the enveloped conic mentioned in the last of the three theorems above. We have special cases of these earlier theorems:

**Theorem.** Let two perpendicular rays through the centre, $N$, of an equilateral triangle $ABC$ meet the edges $BC; CA; AB$ in $A_1, A_2; B_1, B_2; C_1, C_2$ respectively. Then the midpoints of the segments $A_1A_2, B_1B_2, C_1C_2$ lie on a line tangent to the incircle.

Conversely,

**Theorem.** If a tangent to the incircle of an equilateral triangle $ABC$ meets the edges $BC; CA; AB$ in $X, Y, Z$, respectively, then the circles, centres $X, Y$ and $Z$, passing through the incentre $I$ of the triangle, concur again at a point on the circumcircle of $ABC$. 
9.12 Clarifications(??) of Cosmin Pohoata’s original Note

p.3, l.–6. $\angle BA'C'$ should be $\angle CA'C'$, though I’m suspicious that the angles (or their supplements) may depend on the position of $P$ on the circumcircle (e.g., between $B$ and $C$ or $C$ and $A$ or $A$ and $B$).

p.4, l.3. $\angle BQC = \angle BAC + \angle CPB$ nowhere near in my figure! I suspect that the figure depends on the relationship between $P$ and the triangle — e.g., whether it’s inside the circumcircle or not; &/or in which of the seven regions determined by the triangle edges $P$ lies. For example, in the next line of the text, $\angle C'AB'$ is not equal to $\angle BAC$ in my diagram, but rather to its supplement.
3. Concluding Remarks. Simple enough, right? No, too complicated and wrong!

I attempt here to clarify the author’s “reformulation”. Figure 5 is intended to illustrate that reformulation in the special case of the Droz-Farny line.

$P$ is taken as the special case of the orthocentre of the triangle $ABC$ and the lines through $P$ are shown dashed in Figure 4. They meet the edges of $ABC$ in $A', B', C'$ and $A'', B'', C''$. If the lines through $P$ are perpendicular, then the reflexion of $PA$, say, in the internal bisector of $\angle A'PA''$ will be the same as the reflexion in the external bisector — in fact, even in the general case, this is still so, so there seems to be no need to specify “internal”.

Apologies that this is incomplete. There’s a good paper somewhere, but this isn’t it.

### 9.13 Other details concerning C. P.’s Note

Abstract perhaps too long and rambling.

Introduction is same as Abstract.

Abstract, l.6 ‘overridden’ (spelling), l.7 ‘inesthetic’ (?)

The author does well to avoid the word “side” in connexion with a triangle, but “edge” would be better than ‘sidelines’. I don’t like “altitudines” on p.4. “Altitudes” is the usual word.

The labels $B_3$ and $C_3$ in the author’s Figure 1 should be interchanged.

p.1, l.–2. “intersection” (omit ess)

p.3, l.–6. space in $P, A'$

p.4, l.1. ‘represents’

p.4, l.5. space in ‘in triangle’

p.4, l.–3. ‘are’ for ‘ar’

### 9.14 Appendix

[Not included in submitted report.]

Note that the pair of perpendicular line through $H$, the orthocentre, can be replaced by any rectangular hyperbola, centre $H$.

Note that the pair of perpendicular lines together with the Droz-Farny line and the three edges of the triangle, all touch the same parabola. What can one say about the family of
parabolas produced in this way?

It appears (or does it? is this too wild a guess? I think so.) that the focus of the parabola is the point that I’ve called $K$, the reflexion of $H$ in the D-F line. The directrix is the line through $H$ parallel to the D-F line. The axis is $HK$ and the vertex its intersection with the D-F line.
9.15 Start again with converse. Generalizations.

Our generalized (Quadrated & Twinned) triangle comprises eight triangles:

| triangle | orthocentre | circumcentre |
|----------|-------------|--------------|
| 124      | 7           | 7            |
| 741      | 2           | 2            |
| 472      | 1           | 1            |
| 217      | 4           | 4            |
| 124      | 7           | 7            |
| 741      | 2           | 2            |
| 472      | 1           | 1            |
| 217      | 4           | 4            |

where, to avoid special cases, we assume that the triangles are scalene, that is, neither right-angled nor isosceles, and the labels are chosen so that 7 and $\bar{7}$ are vertices of obtuse angles.
Choose any one of the eight triangles with orthocentre $H$, say ($H = 1$ in Fig. 9.20), and join $H$ to an arbitrary point $P$ on the Centre Circle. Then the line through $P$ perpendicular to $HP$ is a suitable candidate for a Droz-Farny Line in the following sense.

[Note that if $Q$ is on $HP$ produced, so that $HP = PQ$, then $Q$ lies on the appropriate circumcircle.]

If the suggested Droz-Farny Line cuts the edges of the selected triangle in $X$, $Y$, $Z$, and the circles centres $X$, $Y$, $Z$ passing though $H$ cut the respective edges in $X_1$, $X_2$ and $Y_1$, $Y_2$ and $Z_1$, $Z_2$, then, if the subscripts have been chosen suitably, the points $X_1$, $Y_1$, $Z_1$ are collinear, as are $X_2$, $Y_2$, $Z_2$ and the two lines are perpendicular and pass through $H$.

Wanted: a perspicuous synthetic proof. It is immediate that angles $X_1HX_2$, $Y_1HY_2$, $Z_1HZ_2$ are right angles, since they are angles in semicircles.

It is easy to see that the envelope of the Droz-Farny lines for any of the eight triangles is a conic with centre at the Centre of the triangle, foci $H$ and $\bar{H}$, and vertices at the intersections of the appropriate Euler line with the Central Circle. Twins generate the same conic. Acute triangles generate an ellipse. Obtuse triangles generate three hyperbolas whose asymptotes pass through the Centre of the triangle and intersect the Central Circle in six further noteworthy points.

Six of the eight vertices of the triangle lie outside the Central Circle. The points of contact of the six pairs of tangents from these vertices with the Central Circle define six diameters of the Central Circle which are three pairs of asymptotes of the three hyperbolas enveloped by Droz-Farny lines.

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1For example, with orthocentre 1 of triangle 724 and a point $P$ on the Central Circle, and corresponding point $Q$ on the circumcircle 724, the Droz-Farny line is the perpendicular bisector of $Q1$, say $PR$, where $R$ is the intersection of this perpendicular bisector with $Q1$ and $\bar{1}$ the circumcentre of triangle 724. Then $1R \pm 1R = 1R \pm QR = 1Q$, the circumradius of triangle 724, and diameter of the Central Circle, so that $R$ lies on a conic, foci $\bar{1}$ and 1 and major (transverse) axis a diameter of the Central Circle. Moreover, since angle $1RP = angle QRP$, the Droz-Farny line $PR$ is a tangent to the conic.
Figure 9.21: Envelope of Droz-Farny line
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Abstract: Let $F_1$ and $F_2$ denote the Fermat-Toricelli points of a given triangle $ABC$. We prove that the Euler lines of the 10 triangles with vertices chosen from $A$, $B$, $C$, $F_1$, $F_2$ (three at a time) are concurrent at the centroid of triangle $ABC$. 

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Given any point \( O \) in the plane of a triangle \( \Delta \equiv ABC \), let its edges \( a \), \( b \), \( c \) subtend angles \( A' \), \( B' \), \( C' \) at \( O \), and let the distances from \( O \) to the vertices be \( a', b', c' \). Show that the triangle with edges \( aa', bb', cc' \) has angles \( A' - A \), \( B' - B \), \( C' - C \), and find the sextic polynomial relation connecting \( a, b, c, a', b', c' \).

Solution by the proposer. (i) Take, for definiteness, the case in which \( O \) lies inside \( \Delta \), and construct, externally to \( \Delta \), triangle \( PCD \) directly similar to triangle \( OAB \). Then \( \angle OBP = \angle ABC \) and \( BP/BO = BC/BA \), so that triangle \( OBP \) is similar to the given triangle \( ABC \), corresponding edges being in the ratio \( b'/c \). Consequently \( \angle BOP = \angle BAC \) and \( OP = bb'/c \), while also \( PC = aa'/c \). Thus triangle \( OCP \) has edges \( aa'/c, bb'/c, c' \) and angle \( A' - A \) at \( O \), whence the first assertion follows by symmetry.

As an application, we deduce, on taking \( a' = b' = c' = R \), the well known result that, given any circle \( ABC \) having its centre \( O \) on the same side of \( BC \) as \( A \), then the angle subtended by \( BC \) at \( O \) is twice that subtended at \( A \).

(ii) Since \( A' + B' + C' = 2\pi \), we have

\[
1 + 2 \cos A' \cos B' \cos C' = \cos^2 A' + \cos^2 B' + \cos^2 C'.
\]

On substituting \((b'^2 + c'^2 - a'^2)/2b'c'\) for \( \cos A' \), and similar expressions for \( \cos B' \) and \( \cos C' \), we obtain
\[4a^2b^2c^2 + (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2) = a^2(b^2 + c^2 - a^2)^2 + b^2(c^2 + a^2 - b^2)^2 + c^2(a^2 + b^2 - c^2)^2,\]

or, after a little simplification,

\[a^2b^2c^2 + \sum a^4a'^2 + \sum a^2(a^2 - b^2)(a^2 - c^2) = \sum b^2c^2(b^2 + c^2) + \sum (b^2 + c^2)b'^2c'^2.\]

A more symmetric form can be obtained by treating \(O, A, B, C\) on equal footing [compare quadrature – Ed.] and replacing \(a, b, c, a', b', c'\) by \(d_{23}, d_{31}, d_{12}, d_{01}, d_{02}, d_{03}\).

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On the 10th Anniversary of Hyacinthos

Abstract. This paper is a survey of results on reflections in triangle geometry. We work with homogeneous barycentric coordinates with reference to a given triangle ABC and establish various concurrency and perspectivity results related to triangles formed by reflections, in
particular the reflection triangle $P(a)P(b)P(c)$ of a point $P$ in the sidelines of $ABC$, and the triangle of reflections $A(a)B(b)C(c)$ of the vertices of $ABC$ in their respective opposite sides. We also consider triads of concurrent circles related to these reflections. In this process, we obtain a number of interesting triangle centers with relatively simple coordinates. While most of these triangle centers have been catalogued in Kimberling’s Encyclopedia of Triangle Centers (ETC), there are a few interesting new ones. We give additional properties of known triangle centers related to reflections, and in a few cases, exhibit interesting correspondences of cubic curves catalogued in Gibert’s Catalogue of Triangle Cubics (CTC).

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If a homography of points on 2 lines is such that the ideal points correspond, then the join of corresponding points envelopes a parabola. Some consequences of this configuration are investigated, in the affine and Euclidean planes, by analytic and synthetic methods.

Reviewed by H. T. Croft

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  (a) Show that there are an infinite number of triangles inscribed in $C$ with orthocentre $H$.

  (b) Determine the set of points belonging to the interior of all triangles inscribed in $C$ with orthocentre $H$.

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