PERIODS AND DUALITY SYMMETRIES IN CALABI-YAU COMPACTIFICATIONS

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ABSTRACT

We derive the period structure of several one-modulus Calabi-Yau manifolds. With this knowledge we then obtain the generators of the duality group and the mirror map that defines the physical variable $t$ representing the radius of compactification. We also describe the fundamental region of $t$ and discuss its relation with automorphic functions. As a byproduct of our analysis we compute the non-perturbative corrections of Yukawa couplings.

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1. Introduction

Different string vacua can be related by continuous deformations of some free parameters called moduli. From the compactification point of view moduli deformations can be roughly interpreted as changes in the metric of the internal manifold. In the space-time effective theory moduli correspond to massless scalars with flat potential whose non-zero VEVs signal the continuous deformations. The space of moduli of a given string vacuum is generically symmetric under certain transformations such as the $R \leftrightarrow 1/R$ duality of circle compactifications. Duality symmetries are an important tool in the study of string vacua. For instance, they can be used to infer how non-perturbative effects could modify the low-energy effective theory [1].

The moduli space of Calabi-Yau (CY) compactifications has been studied by several authors [2]. In CY threefolds moduli associated to zero modes of the metric are directly linked to $(1,1)$- and $(2,1)$-harmonic forms. Moduli of type $(1,1)$ and $(2,1)$ correspond respectively to deformations of the Kähler form and the complex structure. A remarkable feature of the moduli space is its factorization into $\mathcal{M}_{(1,1)} \times \mathcal{M}_{(2,1)}$ where $\mathcal{M}_{(1,1)}$ and $\mathcal{M}_{(2,1)}$ are special Kähler manifolds of dimension given respectively by the Hodge numbers $h_{1,1}$ and $h_{2,1}$. $(2,1)$-Moduli have a geometrical meaning in terms of periods of the holomorphic 3-form and it can be shown that their duality symmetries are described by a subgroup of $Sp(2h_{2,1} + 2, \mathbb{Z})$.

On the other hand, $(1,1)$-moduli do not have such a geometrical counterpart and their duality symmetries are rather stringy in character. However, as a consequence of the mirror symmetry [3] that exchanges the rôle of $(1,1)$- and $(2,1)$-moduli it follows that the $(1,1)$-duality is a subgroup of $Sp(2h_{1,1} + 2, \mathbb{Z})$. In practice we are mostly interested in the $(1,1)$-moduli. We know that a “breathing mode” associated to the radius of the internal manifold is always present. Determining the symmetries acting on the corresponding massless field is of the utmost relevance in the analysis of the effective theory.

At present it is not known how to determine systematically the proper subgroup of $Sp(2h_{1,1} + 2, \mathbb{Z}) \times Sp(2h_{2,1} + 2, \mathbb{Z})$ that acts as duality symmetries of the moduli space of
a CY threefold. In a seminal work, Candelas et al [4] found the particular $Sp(2h_{1,1} + 2, \mathbb{Z})$ generators in an specific model. In this note we wish to extend their methods to the analysis of several examples. Our results are interesting in themselves as they can be used to study the effective theory of strings compactified on CY manifolds. They also constitute a further step towards more general developments.

This note is organized as follows. In section 2 we introduce the models we will consider, find their mirror partners and describe some basic properties of their moduli space. In section 3 we review briefly the relation between periods and duality symmetries. In section 4 we derive the differential equations satisfied by the periods of the mirror manifold and then find explicit solutions. Using these results in section 5 we determine the corresponding $Sp(4, \mathbb{Z})$ duality generators. In section 6 we obtain the mirror maps for our models, discuss their relation to automorphic forms and compute the Yukawa couplings. Conclusions are presented in section 7.

2. Models

We will only consider CY threefolds with $h_{1,1} = 1$. Besides the quintic manifold $CP_4(5)$ studied in Ref. [4] there exist other relatively simple CY threefolds with $h_{1,1} = 1$. These are defined as hypersurfaces $H$ in the weighted projective space $WCP_4$. Our notation and the corresponding defining polynomials are given below:

\begin{align*}
A) \quad WCP_4(2, 1, 1, 1, 1)_{-204} : & \quad W_{0A} = X_1^3 + X_2^6 + X_3^6 + X_4^6 + X_5^6 = 0 \\
B) \quad WCP_4(1, 1, 1, 1, 4)_{-296} : & \quad W_{0B} = X_1^8 + X_2^8 + X_3^8 + X_4^8 + X_5^2 = 0 \\
C) \quad WCP_4(2, 1, 1, 1, 5)_{-288} : & \quad W_{0C} = X_1^5 + X_2^{10} + X_3^{10} + X_4^{10} + X_5^2 = 0
\end{align*}

The numbers inside parentheses refer to the weights $n_m$ of the $X_m$ coordinates. Notice that the $W_0$ are quasihomogeneous functions of degree $d = (6, 8, 10)$ for models $(A, B, C)$, e.g. $W_{0A}(\lambda^{n_m} X_m) = \lambda^6 W_{0A}(X_m)$. These manifolds were first discussed in Ref. [5] where their Euler characteristic (shown above as a subscript) was also computed.

Due to the link [6] between $N = 2$ superconformal theories and renormalization group fixed points of Landau-Ginzburg superpotentials the above CY manifolds can also
be described in terms of tensor products of \( N = 2 \) minimal models with diagonal invariants. From the given superpotentials \( W_{0A}, W_{0B}, W_{0C} \), we identify these tensor products respectively as \( A \equiv 4^4 = (1, 4, 4, 4, 4) \), \( B \equiv 6^4 = (6, 6, 6, 6, 6) \), \( C \equiv 3^3 = (3, 8, 8, 8, 8) \); where \((k_1, \cdots, k_r)\) are the levels of the \( N = 2 \) minimal theories. Since a quadratic term corresponds to a trivial \( k = 0 \) theory the last two models only involve four factors.

In the Gepner construction [7] of the tensor models the Hodge numbers are calculated as the number of 27 and \( \overline{27} \) fields. In the above cases the results are \( h_{1,1} = 1 \) and \( h^A_{2,1} = 103 \), \( h^B_{2,1} = 149 \) and \( h^C_{2,1} = 145 \) [8]. These models have then many \((2,1)\)-moduli which can in fact be understood as the coefficients of monomials in the \( X_m \) that can be added to deform the \( W_0 \) while preserving their quasihomogeneity of the given degree.

The \((1,1)\)-modulus of the above manifolds can be interpreted as the radius of the spaces. Our goal is to find the duality symmetries of this modulus. To this end we will consider the associated mirror manifolds that have \( \hat{h}_{1,1} = h_{2,1} \) and \( \hat{h}_{2,1} = h_{1,1} = 1 \). We will then find the generators of duality symmetries of the \((2,1)\)-modulus of the mirror manifolds by exploiting its geometrical interpretation in terms of periods. In virtue of the mirror operation [3] these symmetries translate into symmetries of the \((1,1)\)-modulus of the original manifold.

To obtain the mirrors of our models we use its \( N = 2 \) tensor product description. A \((k_1, \cdots, k_r)\) minimal tensor model has a large group of symmetries given by \( S = Z_{k_1+2} \times \cdots \times Z_{k_r+2} \). Dividing by subgroups of \( S \) leads to new models. In particular, it has been argued [9] that modding by the maximal subgroup of \( S \) yields the mirror of the original model. Discrete symmetries and moddings of \( N = 2 \) tensor products have been studied in detail in Ref. [10]. We now review briefly the basic case of modding by a \( Z_M \) subgroup. \( Z_M \) is generated by a modding vector

\[
\Gamma = (\gamma_1, \cdots, \gamma_r)
\]

satisfying the supersymmetry preserving condition

\[
\sum_{i=1}^{r} \frac{\gamma_i}{k_i + 2} = \text{integer}
\]
The order $M$ is the least integer such that $M(\gamma_1, \cdots, \gamma_r) = 0 \mod (k_1 + 2, \cdots, k_r + 2)$. In terms of the $X_m$ coordinate fields this $Z_M$ modding acts as a phase transformation
\[(X_1, \cdots, X_r) \rightarrow (\sigma^{\gamma_1 n_1} X_1, \cdots, \sigma^{\gamma_r n_r} X_r)\] (4)
where $\sigma = e^{2\pi i / d}$. A multiple modding by $G = Z_{M_1} \times \cdots \times Z_{M_p}$ with $M_b$ divisible by $M_a$ for $b \geq a$ is also possible. Each $Z_{M_a}$ is generated by an independent modding $\Gamma_a$.

Dividing by $G$ implies projecting out states in the original spectrum while introducing new twisted states. The resulting spectrum is derived by incorporating these orbifold effects in the Gepner construction. Given our $A, B, C$ models we then look for moddings that produce a new spectrum with $\hat{h}_{2,1} = 1$. In each case we find that such a result is achieved by a maximal subgroup of $S$. The corresponding symmetries and modding generators are given by

\[
\begin{align*}
A) & \quad 1 \ 4^4 \quad \Gamma_1 = (1, 5, 0, 0, 5) \\
& \quad G = Z_6 \times Z_6 \times Z_6 \quad \Gamma_2 = (1, 0, 5, 0, 5) \\
& \quad \Gamma_3 = (1, 0, 0, 5, 5) \\
B) & \quad 6^4 \quad \Gamma_1 = (7, 2, 2, 5) \\
& \quad G = Z_8 \times Z_8 \times Z_8 \quad \Gamma_2 = (7, 2, 5, 2) \\
& \quad \Gamma_3 = (7, 5, 2, 2) \\
C) & \quad 3 \ 8^3 \quad \Gamma_1 = (0, 4, 3, 3) \\
& \quad G = Z_{10} \times Z_{10} \quad \Gamma_2 = (0, 3, 4, 3)
\end{align*}
\]

(5)

From the resulting spectrum it is very simple to identify the primary field associated to the lone $(2, 1)$-modulus of the mirror manifolds. In terms of the $X_m$ coordinate fields these are given respectively by
\[
\begin{align*}
h_A &= -6X_1 X_2 X_3 X_4 X_5 \\
h_B &= -4X_1^2 X_2^2 X_3^2 X_4^2 \\
h_C &= -5X_1^2 X_2^2 X_3^2 X_4^2
\end{align*}
\]

(6)
The numerical factor is conventional. One can check that these are the only monomials (involving more than one $X_m$) of degree $d_A = 6$, $d_B = 8$, $d_C = 10$ that are invariant under
the phase transformations (4) with the \( \Gamma \) generators specified in (5). In cases \( B \) and \( C \), \( \gamma_5 = 0 \) effectively.

In the CY approach the mirror manifolds are obtained by dividing each space defined in (1) by the respective group \( G \). The single \((2,1)\)-modulus can be taken as the coefficient of the only monomial in the \( X_m \) that may be added to \( W_0 \) to give a \( G \)-invariant polynomial of the given degree. We now introduce a modulus \( \psi \) and consider the family of hypersurfaces \( \hat{H} \) defined by the perturbed polynomials

\begin{align*}
A) \quad W_A & = X_1^3 + X_2^6 + X_3^6 + X_4^6 + X_5^6 - 6\psi X_1 X_2 X_3 X_4 X_5 = 0 \\
B) \quad W_B & = X_1^8 + X_2^8 + X_3^8 + X_4^8 + X_5^2 - 4\psi X_1^2 X_2^2 X_3^2 X_4^2 = 0 \\
C) \quad W_C & = X_1^5 + X_2^{10} + X_3^{10} + X_4^{10} + X_5^5 - 5\psi X_1^2 X_2^2 X_3^2 X_4^2 = 0 
\end{align*}

\( \psi \) parametrizes changes in the complex structure of the family of mirror manifolds \( \hat{H}/G \).

For future purposes we need to characterize the holomorphic 3-form \( \Omega \) of our mirror manifolds. For CY \((N − 1)\)-folds defined by an equation \( W = 0 \) in \( WCP_N \), \( \Omega \) can be explicitly constructed as explained in Ref. [5]. First introduce inhomogeneous coordinates, e.g. \( Y_i = X_i X_{N+1}^{-n_i/nN+1} \), \( i = 1, \ldots, N \). Upon this substitution in \( W(\psi) \) the equation \( W(\psi) = 0 \) becomes \( [1 + \hat{W}(\psi)] = 0 \), where \( \hat{W}(\psi) \) is a polynomial in the \( Y_i \). \( \Omega \) can be written as

\[ \Omega(\psi) = \rho(\psi) \frac{dY_1 \wedge \cdots \wedge dY_{N-1}}{\partial \hat{W}(\psi) / \partial Y_N} \]

\( \rho(\psi) \) takes into account the freedom in the normalization of \( \Omega \). A particular \( \rho(\psi) \) in fact represents a choice of gauge for \( \Omega \) [11]. Notice also that condition (3) guarantees \( G \)-invariance of \( \Omega \).

We have seen that for our manifolds having \( \hat{h}_{2,1} = 1 \) the \((2,1)\)-moduli space is just the complex 1-dimensional space of \( \psi \)'s. Some values of \( \psi \) are equivalent. Two points \( \psi \) and \( \psi' \) describe the same model if the change \( \psi \rightarrow \psi' \) in \( W \) can be undone by performing linear transformations of the \( X_m \) coordinate fields. As explained in Ref. [5] transformations \( X_m \rightarrow R_{mn} X_n \) that maintain the form of \( W \) actually correspond to symmetries of the space \( \hat{H}/G \). Hence, they must verify that for each \( g \in G \), \( RgR^{-1} = g' \) for some \( g' \in G \). It
is easy to check that the only transformations satisfying this condition are field rescalings \( X_m \rightarrow \sigma^{q_m n_m} X_m \) with \( q_m \in \mathbb{Z} \) and \( \sigma^d = 1 \). This transformation clearly leaves \( W_0 \) invariant while multiplying the perturbations \( h \) by a phase that can be reabsorbed in \( \psi \). In this way we obtain the symmetries

\[
\psi \rightarrow \alpha \psi \ ; \ \alpha^p = 1
\]

(9)

where \( p = (6, 4, 5) \) for models \((A, B, C)\).

Another important feature of the space of \( \psi \)'s is the existence of singular points, i.e. values of \( \psi \) for which the manifolds defined by (7) become singular. For instance, when \( \psi \rightarrow \infty \) these manifolds degenerate into “pinched” varieties such as that described by \( W_A \rightarrow X_1 X_2 X_3 X_4 X_5 = 0 \). Other singular values of \( \psi \) appear when the holomorphic 3-form fails to be well defined. This occurs when the derivatives \( \partial W/\partial X_m \) all vanish simultaneously. In our examples we find that this condition is satisfied only when \( \psi \) takes values so that

\[
A) \ 4\psi^6 = 1 \\
B) \ \psi^4 = 1 \\
C) \ 4\psi^5 = 1
\]

(10)

Due to the symmetries in (9) all solutions for \( \psi \) are identified. We can thus make the following specific choices of singular points \( \psi_0 \)

\[
A) \ \psi_0 = 2^{-1/3} \\
B) \ \psi_0 = 1 \\
C) \ \psi_0 = 2^{-2/5}
\]

(11)

In the next sections we shall see how these singular points turn out to be very important in the study of modular symmetries.

3. Periods and Symmetries

In this section we review some facts about symmetries of moduli spaces of CY manifolds. Our aim is to establish notation and to introduce some concepts that will be needed in our subsequent analysis.
The (2,1)-moduli space coincides with the space of complex structures thus affording a description in terms of periods of the holomorphic 3-form $\Omega$ [12]. For our mirror manifolds with $\hat{h}_{2,1} = 1$, $\Omega$ can be expanded in a basis of harmonic 3-forms denoted by $(\alpha_1, \alpha_2, \beta_1, \beta_2)$. Thus

$$\Omega = z^1 \alpha_1 + z^2 \alpha_2 - G_1 \beta_1 - G_2 \beta_2$$

(12)

The 3-cycles dual to the harmonic 3-forms are denoted by $(A^1, A^2, B_1, B_2)$ and chosen canonically so that

$$\int_{A^b} \alpha_a = -\int_{B_a} \beta^b = \delta_a^b$$

(13)

with other integrals vanishing.

The coefficients in the expansion (12) are interpreted as periods since they are obtained by integrating $\Omega$ over the canonical 3-cycles. For future convenience we introduce the period vector $P$

$$P = \begin{pmatrix} G_1 \\ G_2 \\ z^1 \\ z^2 \end{pmatrix}$$

(14)

Notice that $P$ is a function of $\psi$. The entries of $P$ are not all independent. In fact, it has been shown [13] that $G_a = \partial G/\partial z^a$, where the prepotential $G(z^1, z^2)$ is a homogeneous function of degree two. Hence, $z^1$ and $z^2$ are only defined projectively, their ratio giving just one independent degree of freedom related to the single (2,1)-modulus $\psi$.

In the previous section we explained how the points $\psi$ and $\alpha \psi$, $\alpha^p = 1$, define the same model. In terms of the period vector the symmetry $\psi \rightarrow \alpha \psi$ is represented by some matrix $S$, this is

$$P \rightarrow SP ; \quad \psi \rightarrow \alpha \psi$$

(15)

This transformation of $P$ necessarily corresponds to a change of homology basis and hence $S \in Sp(4, \mathbb{Z})$. Other symmetries of $P$ are related to monodromy properties of the periods. We will see later that $P(\psi)$ is multivalued about the singular points of $\psi$ discussed in section 1. Transport about these points generates a transformation of $P$

$$P \rightarrow T_{\psi_0}P \quad ; \quad (\psi - \psi_0) \rightarrow e^{2\pi i}(\psi - \psi_0)$$

(16)
This monodromy must reflect a monodromy of the integral homology basis so that $T_{\psi_0} \equiv T \in Sp(4, \mathbb{Z})$.

Since other solutions of (10) are related to $\psi_0$ by $\psi_0 \to \alpha \psi_0$ we deduce that transport about the singular points $\psi_l = \alpha^l \psi_0$, $l = 1, \ldots, p - 1$ is generated by a composition of $S$ and $T$, namely $T_l = S^{-l} T S^l$. Transport about $\psi \to \infty$ is also obtained from $S$ and $T$. This follows because a circuit enclosing $\infty$ and the remaining multi-valued points can be deformed into a cycle enclosing no singularities. In conclusion, the matrices $S$ and $T$ generate a group $\mathcal{D} \in Sp(4, \mathbb{Z})$ of symmetries of $P$. We will refer to $\mathcal{D}$ as the modular or duality group.

So far we have only discussed the $(2,1)$-moduli space of the mirror manifold and how symmetries of $\psi$ imply symmetries of $P$. However, our main goal is to determine the duality symmetries of the $(1,1)$-moduli space of the original manifold. The link between the two spaces is provided by the mirror operation [3]. Indeed, a vector analogous to $P$ can be introduced for the original manifold with $h_{1,1} = 1$. This follows from the existence of a prepotential $\mathcal{F}$ that gives the metric of the special Kähler manifold $\mathcal{M}_{(1,1)}$. $\mathcal{F}$ is a homogeneous function of degree two depending on two variables $w^1, w^2$, defined only projectively. The ratio $w^1/w^2$ corresponds to the $(1,1)$-modulus of the original manifold denoted by $t$. We then define the vector

$$\Pi = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ w^1 \\ w^2 \end{pmatrix}$$

where $\mathcal{F}_a = \partial \mathcal{F}/\partial w^a$. The mirror hypothesis basically states that $P$ and $\Pi$ are equal up to an $Sp(4, \mathbb{Z})$ transformation [9]. As a consequence, duality symmetries of $P$ translate into duality symmetries of $\Pi$. This is an important observation since a priori $\mathcal{F}$, and thus $\Pi$, are only known classically, i.e. without including quantum corrections. Thus, to obtain the duality group of the $(1,1)$-modulus of the original manifold it is enough to first determine the period vector $P$ of the mirror manifold and then derive its transformation properties defined in eqs. (15) and (16).
In principle, the periods can be found by integrating the explicit form of $\Omega$ given in (8) over a basis of 3-cycles. The difficulty with this approach, partially followed in Ref. [4], resides in the identification of canonical 3-cycles as well as in the actual evaluation of the integrals. Fortunately, an alternative route, more suitable to generalizations, can be taken. As noticed in [4] the periods of $\Omega$ obey a differential equation whose knowledge simplifies the analysis considerably. This differential equation, known as the Picard-Fuchs equation, in fact follows from general results in algebraic geometry [14, 15]. These results have been reviewed recently in the Physics and Mathematics literature in connection both with CY manifolds and $N = 2$ topological theories [16, 17, 18, 19, 20, 21, 22, 23]. These issues will be the subject of the next section.

4. Picard-Fuchs Equations and Solutions

In this section we will obtain the Picard-Fuchs (PF) equation satisfied by the periods of $\Omega$ in our models $A, B, C$. We will start by a short review of a method originally due to Dwork [14] and recently discussed by Cadavid and Ferrara [18]. We have included the necessary generalizations to the quasihomogeneous case.

We will consider CY $(N-1)$-folds with a single $(2,1)$-modulus $\psi$ and defined as a quotient space $\hat{H}/G$. Here $\hat{H}$ is a hypersurface in $\mathrm{WCP}_N(n_1, \cdots, n_{N+1})$, described by the equation

$$W(X_m, \psi) = W_0(X_m) + \psi h(X_m) = 0$$ (18)

with $W(X_m, \psi)$ a homogeneous function of degree $d$, i.e. $W(\lambda^{n_m}X_m, \psi) = \lambda^d W(X_m, \psi)$. $G$ is a group of phase symmetries whose elements are specified by modding vectors $\Gamma = (\gamma_1, \cdots, \gamma_{N+1})$ and whose action on the $X_m$ coordinates is given in (4). The holomorphic $(N-1)$-form can be calculated from (8). We denote the periods of $\Omega$ by $\omega_a$, $a = 1, \cdots, b_{N-1}$, where $b_{N-1}$ is the $(N-1)^{th}$-Betti number ($b_1 = 2$ for onefolds and $b_3 = 2h_{2,1} + 2 = 4$ for threefolds with $h_{2,1} = 1$). The $\omega_a$ turn out to be the independent solutions of the matrix equation

$$\frac{dR(\psi)}{d\psi} = R(\psi)M(\psi)$$ (19)

where $R$ and $M$ are $b_{N-1} \times b_{N-1}$ matrices.
Matrix $M$ above can be determined as follows. First introduce the sets

$$I = \{ V = (v_0, v_1, \cdots, v_{N+1}) \mid v_j \in \mathbb{N} \ ; \ dv_0 = \sum_{m=1}^{N+1} n_m v_m \} \quad (20)$$

$$\tilde{I} = \{ V \in I \ ; \ 0 < v_m < d/n_m \ ; \ m = 1, \cdots, N+1 \}$$

The action of $G$ further constrains the elements of both sets by requiring

$$\sum_{m=1}^{N+1} n_m v_m \gamma_m = 0 \text{ mod } d \ ; \ \forall \ \Gamma \in G \quad (21)$$

The set $\tilde{I}$ defines $b_{N-1}$ fundamental monomials of the form

$$\xi_a = X^{V_a} \ ; \ V_a \in \tilde{I} \quad (22)$$

where $X^V \equiv X_0^v_0 X_1^v_1 \cdots X_{N+1}^v_{N+1}$. The next step is to define the covariant derivatives

$$D_m = X_m \frac{\partial}{\partial X_m} + X_0 X_m \frac{\partial W}{\partial X_m} \ ; \ m = 1, \cdots, N+1 \quad (23)$$

and to consider the quantity

$$Q_a = X_0 h(X) \xi_a \quad (24)$$

It can be shown that $Q_a$ can be expanded in terms of fundamental monomials $\xi_b$ modulo covariant derivatives $D_m X^U$, with $U \in I$, $u_0 < v_{a0}$ [14]. Hence we can write schematically

$$Q_a = M_{ab} \xi_b + (DX) \quad (25)$$

$M_{ab}$ is precisely the matrix that we are looking for. Substituting in (19) leads to the Pf equation for $\omega_a$.

To illustrate the application of the method just discussed we will work out two examples in some detail. We will consider first the simpler onefold described by

$$W = X_1^4 + X_2^4 + X_3^2 - 2\psi X_1^2 X_2^2 = 0 \quad (26)$$

which corresponds to a $Z_4$ orbifold of a torus. In this case set $\tilde{I}$ has two elements with corresponding fundamental monomials

$$\xi_1 = X_0 X_1 X_2 X_3 \quad (27)$$

$$\xi_2 = X_0^2 X_1^3 X_2^3 X_3$$
From (26) we recognize $h = -2X_1^2 X_2^2$. Acting with $X_0 h$ on the $\xi$-basis yields

\begin{align*}
Q_1 &= -2X_0^2 X_1^3 X_2^3 X_3 \\
Q_2 &= -2X_0^3 X_1^5 X_2^5 X_3
\end{align*}

(28)

$Q_1$ is already given as $Q_1 = -2\xi_2$. $Q_2$ needs to be reduced. To this end consider the combination

\begin{align*}
4D_1(X_0^2 X_1 X_2^5 X_3) + 4D_2(X_0^2 X_1^5 X_2 X_3) + 4\psi(D_1 + D_2)\xi_2 - (D_1 + D_2)\xi_1 \\
= 32(1 - \psi^2)X_0^3 X_1^5 X_2^5 X_3 + 32\psi\xi_2 - 2\xi_1
\end{align*}

(29)

Hence

$$Q_2 = -\frac{1}{8(1 - \psi^2)}\xi_1 + \frac{2\psi}{(1 - \psi^2)}\xi_2 + (DX)$$

(30)

Matrix $M$ is then given by

$$M = \begin{pmatrix} 0 & -\frac{1}{8(1 - \psi^2)} \\ -2 & \frac{2\psi}{(1 - \psi^2)} \end{pmatrix}$$

(31)

Therefore, the solution of (19) is of the form

$$R = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

(32)

where $\omega_1$ and $\omega_2$ are independent solutions of the PF equation

$$\frac{d^2 \omega}{d\psi^2} + \frac{2\psi}{(\psi^2 - 1)} \frac{d\omega}{d\psi} + \frac{1}{4(\psi^2 - 1)} \omega = 0$$

(33)

We have also studied the onefold described by

$$W = X_1^3 + X_2^5 + X_3^2 - 3\psi X_1 X_4^2 = 0$$

(34)

After a similar analysis we arrive at the PF equation

$$\frac{d^2 \omega}{d\psi^2} + \frac{12\psi^2}{(4\psi^3 - 1)} \frac{d\omega}{d\psi} + \frac{7\psi}{4(4\psi^3 - 1)} \omega = 0$$

(35)

This model corresponds to a $Z_6$ orbifold of a torus. Equations (33) and (35) have been obtained in Ref. [23] in the context of $N = 2$ topological theories.
Let us now turn to threefolds and consider our model $C$. In this case, account taken of constraint (21), set $\tilde{I}$ has four elements. The corresponding fundamental monomials are

$$\xi_1 = X_0 X_1 X_2 X_3 X_4 X_5$$
$$\xi_2 = X_0^2 X_1^3 X_2^3 X_3^3 X_4^3 X_5$$
$$\xi_3 = X_0^3 X_1^2 X_2^7 X_3^7 X_4^7 X_5$$
$$\xi_4 = X_0^4 X_1^4 X_2^9 X_3^9 X_4^9 X_5$$

(36)

Acting with $X_0 h$, $h = -5X_1^2 X_2^2 X_3^2 X_4^2$, gives

$$Q_1 = -5X_0^2 X_1^3 X_2^3 X_3^3 X_4^3 X_5$$
$$Q_2 = -5X_0^3 X_1^5 X_2^5 X_3^5 X_4^5 X_5$$
$$Q_3 = -5X_0^4 X_1^9 X_2^9 X_3^9 X_4^9 X_5$$
$$Q_4 = -5X_0^5 X_1^{11} X_2^{11} X_3^{11} X_4^{11} X_5$$

(37)

$Q_2$ and $Q_4$ need to be reduced. For $Q_2$ consider

$$D_1(X_0^2 X_1^5 X_2^5 X_3^5 X_4^5) = 5X_0^2 X_1^5 X_2^5 X_3^5 X_4^5 X_5 - 10\psi X_0^3 X_1^2 X_2^7 X_3^7 X_4^7 X_5$$

(38)

Hence

$$Q_2 = -10\psi \xi_3 + (DX)$$

(39)

Reduction of $Q_4$ is cumbersome but straightforward. The end result for matrix $M$ is

$$M = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{1000(4\psi^5-1)} \\ -5 & 0 & 0 & \frac{-2\psi}{5(4\psi^5-1)} \\ 0 & -10\psi & 0 & \frac{18\psi^3}{(4\psi^5-1)} \\ 0 & 0 & -5 & \frac{-40\psi^4}{(4\psi^5-1)} \end{pmatrix}$$

(40)

Equation (19) is then solved by a matrix $R$ with elements

$$R_{a1} = \omega_a$$
$$R_{a2} = -\frac{1}{5}\omega'_a$$
$$R_{a3} = \frac{1}{50\psi}\omega''_a$$
$$R_{a4} = \frac{1}{250\psi^2}\omega''_a - \frac{1}{250\psi}\omega'''_a$$

(41)
where the $\omega_a$ are the independent solutions of the PF equation

\[
\frac{d^4\omega}{d\psi^4} + \frac{2(16\psi^5 + 1)}{\psi(4\psi^5 - 1)} \frac{d^3\omega}{d\psi^3} + \frac{2(29\psi^5 - 1)}{\psi^2(4\psi^5 - 1)} \frac{d^2\omega}{d\psi^2} + \frac{20\psi^2}{(4\psi^5 - 1)} \frac{d\omega}{d\psi} + \frac{\psi}{4(4\psi^5 - 1)} \omega = 0
\]  (42)

The PF equations for models $A$ and $B$ are obtained \textit{mutatis mutandis}. As for model $C$ in both cases the PF equation takes the form

\[
\frac{d^4\omega}{d\psi^4} + \sum_{j=0}^{3} C_j(\psi) \frac{d^j\omega}{d\psi^j} = 0
\]  (43)

The explicit expressions of the coefficients $C_j(\psi)$ are recorded in Table 1. Equivalent results have been obtained in Ref. [21].

Let us now comment on some generic properties of the PF equations of the various models. They are all Fuchsian equations with the regular singular points located at $\psi = 0, \infty, \alpha_l \psi_0, l = 0, \ldots, p - 1$. All equations can be transformed into generalized hypergeometric equations [24] upon the change of variables $\zeta = (\psi/\psi_0)^p$. Recall that $p = (6, 4, 5)$ for models ($A, B, C$). To obtain solutions around the singular points it is actually simpler to directly apply Frobenius method to the equation in the variable $\psi$. Below we treat model $C$ in some detail.

The solutions of the PF equation (42) around $\psi = 0$ are given by

\[
\omega_j(\psi) = \psi^j F\left(\frac{2j + 1}{10}, \frac{2j + 1}{10}, \frac{2j + 1}{10}; \frac{j + 1}{5}, \frac{j + 2}{5}, \frac{j + 4}{5}, \frac{j + 5}{5}; 4\psi^5\right)
\]

\[j = 0, 1, 3, 4 \quad ; \quad |\psi| < \psi_0\]  (44)

The overbrace indicates that the entry equal to 1 must be dropped. The generalized hypergeometric function $F(a_1,a_2,a_3,a_4;c_1,c_2,c_3;\zeta)$ is defined as [24]

\[
F(a_1,a_2,a_3,a_4;c_1,c_2,c_3;\zeta) = \sum_{l=0}^{\infty} \frac{(a_1)_l(a_2)_l(a_3)_l(a_4)_l}{(c_1)_l(c_2)_l(c_3)_l} \frac{\zeta^l}{l!}
\]  (45)

where $(a)_l = \Gamma(a+l)/\Gamma(a)$. The solutions around $\psi = 0$ for models $A$ and $B$ are given in Table 2.
The indicial equation about $\psi = \psi_0$ has two simple roots $s = 0, 2$ and a double root $s = 1$. We find three independent analytic solutions (with exponents $s = 0, 1, 2$) and one solution with a logarithmic singularity (due to the repeated index $s = 1$). A simple expression for these solutions cannot be given but for our purposes it is necessary to characterize the logarithmic solution that can be written as

$$f(\psi) = g(\psi) \ln(\psi - \psi_0) + \text{analytic} \quad (46)$$

where $g(\psi)$ is itself a solution of the form

$$g(\psi) = \sum_{l=0}^{\infty} a_1(\psi - \psi_0)^{l+1} \quad (47)$$

with $a_1 = -a_0/4$. The rest of the coefficients follows from recurrence relations obtained by substituting (47) in the PF equation (42). In models $A$ and $B$ the results are analogous.

There is just one logarithmic solution of the form (46) with $g(\psi)$ given by an expansion of type (47). In $A$, $a_1 = -5a_0/12$. In $B$, $a_1 = -5a_0/16$. In all cases we can take $a_0 = 1$.

The function $g(\psi)$ will bear heavily in our analysis due to its relation to monodromy properties about $\psi = \psi_0$. Since $f(\psi)$ is one of the independent solutions around $\psi_0$ when continuing any solution $\omega$ near this point there will be a piece proportional to $f$ plus analytic terms, i.e. $\omega = \delta f + \text{analytic}$. We then conclude that $\omega$ will transform as $\omega \to \omega + 2\pi i \delta g$ under transvection about $\psi = \psi_0$.

The indicial equation about $\psi = \infty$ has a quadruple root $s = 1/2$. Hence, three of the independent solutions have logarithmic singularities. These solutions are found in a standard way [25]. They are expressed in terms of the function

$$y(s) = \frac{2^{1/5}}{4\pi^2} (4^{1/5} \psi)^{-s} \sum_{l=0}^{\infty} \frac{\Gamma(l + \frac{s}{5}) \Gamma(l + \frac{s+1}{5}) \Gamma(l + \frac{s+2}{5}) \Gamma(l + \frac{s+3}{5}) \Gamma(l + \frac{s+4}{5})}{\Gamma^4(l + \frac{2s+9}{10})} \frac{1}{(4\psi^5)^l} \quad (48)$$

$$\quad |\psi| > \psi_0 \quad ; \quad 0 \leq \arg \psi \leq \frac{2\pi}{5}$$

Then, a basis of independent solutions around $\psi = \infty$ is given by

$$y_i(\psi) = \frac{d^i y}{d\psi^i} (s_0) \quad ; \quad i = 0, \ldots, 3 \quad (49)$$
where $s_0 = 1/2$. This basis is characterized by its monodromy about $\psi = \infty$. More precisely, under $(1/\psi) \to e^{2\pi i}(1/\psi)$ we have

\[
\begin{align*}
\tilde{y}_0 & \rightarrow \tilde{y}_0 \\
\tilde{y}_1 & \rightarrow \tilde{y}_1 + 2\pi i \tilde{y}_0 \\
\tilde{y}_2 & \rightarrow \tilde{y}_2 + 4\pi i \tilde{y}_1 - 4\pi^2 \tilde{y}_0 \\
\tilde{y}_3 & \rightarrow \tilde{y}_3 + 6\pi i \tilde{y}_2 - 12\pi^2 \tilde{y}_1 - 8i\pi^3 \tilde{y}_0
\end{align*}
\]

(50)

where we have defined $\tilde{y}_i = \psi^{s_0} y_i$. For future purposes we need the explicit form of $y_0$ and $y_1$. Results for all models are recorded in Table 3. In this table $\Psi$ is the Digamma function. To arrive at the expansions for $y_1$ use has been made of Gauss’s multiplication formula for the $\Gamma$-function. In all cases the region of convergence is $|\psi| > \psi_0$; $0 \leq \arg \psi \leq 2\pi/p$.

We will also need the analytic continuation of the solution $y_0$ to $|\psi| < \psi_0$. $y_0$ can be continued as explained in [4]. First its series expansion is converted into a contour integral (a so-called Barnes integral [24]) and then the contour is deformed adequately. The end result can be written as

\[
y_0(\psi) = -\sum_{n=1}^{\infty} \lambda(n) \beta(n) \psi^{s_0(n-1)} ; \quad |\psi| < \psi_0
\]

(51)

where $d$ is the degree of quasihomogeneity and $s_0 = (1, 1/2, 1/2)$ for models $(A, B, C)$. The coefficients $\lambda(n)$ and $\beta(n)$ are given by

\[
\begin{align*}
A) \quad & \lambda(n) = e^{\frac{5i\pi n}{6}} \sin^3 \frac{\pi n}{6} \sin \frac{\pi n}{3} \\
& \beta(n) = \frac{6^n}{6\pi^4(n-1)!} \Gamma^4 \left( \frac{n}{6} \right) \Gamma \left( \frac{n}{3} \right) \\
B) \quad & \lambda(n) = e^{\frac{7i\pi n}{8}} \sin^3 \frac{\pi n}{8} \sin \frac{\pi n}{2} \\
& \beta(n) = \frac{4^n}{8\pi^4(n-1)!} \Gamma^4 \left( \frac{n}{8} \right) \Gamma \left( \frac{n}{2} \right) \\
C) \quad & \lambda(n) = e^{\frac{9i\pi n}{10}} \sin^2 \frac{\pi n}{10} \sin \frac{\pi n}{5} \sin \frac{\pi n}{2} \\
& \beta(n) = \frac{20^n}{10\pi^4(n-1)!} \Gamma^3 \left( \frac{n}{10} \right) \Gamma \left( \frac{n}{5} \right) \Gamma \left( \frac{n}{2} \right)
\end{align*}
\]

(52)
Clearly, the form of $\lambda(n)$ restricts the values of $n$. For instance, in case $C$, $n = (2j + 1) \mod 10$ with $j = 0, 1, 3, 4$. We then recover the expected result that for $|\psi| < \psi_0$, $y_0$ can be written as a combination of the solutions around $\psi = 0$ given in Table 1, specifically

$$y_0(\psi) = - \sum_j \lambda_j \beta_j \omega_j(\psi) \quad |\psi| < \psi_0$$  \hspace{1cm} (53)

where $\lambda_j \equiv \lambda(j s_0 + 1)$ and $\beta_j \equiv \beta(j s_0 + 1)$ are obtained from (52).

Another piece of information required is the monodromy about $\psi = \psi_0$ of the functions

$$e_l(\psi) \equiv y_0(\alpha^l \psi) = - \sum_j \lambda_j \beta_j \omega_j(\alpha^l \psi) \quad |\psi| < \psi_0$$  \hspace{1cm} (54)

where $\alpha^p = 1$. As explained before, around $\psi = \psi_0$ necessarily

$$e_l(\psi) = \delta_l g(\psi) \ln(\psi - \psi_0) + \text{analytic}$$  \hspace{1cm} (55)

with $g(\psi)$ given in (47). Therefore,

$$e_l(\psi) \rightarrow e_l(\psi) + 2\pi i \delta_l g(\psi)$$  \hspace{1cm} (56)

under transvection about $\psi = \psi_0$. The coefficients $\delta_l$ turn out to be

$$A) \quad \delta_l = \frac{4\sqrt{3}}{\pi^2} \sum_j \alpha^j \lambda_j \quad ; \quad j = 0, 1, 3, 4$$

$$B) \quad \delta_l = \frac{2\sqrt{2}}{\pi^2} \sum_j \alpha^j \lambda_j \quad ; \quad j = 0, 1, 2, 3$$  \hspace{1cm} (57)

$$C) \quad \delta_l = \frac{2}{\pi^2} \sum_j \alpha^j \lambda_j \quad ; \quad j = 0, 1, 3, 4$$

To derive these results we proceed as in Ref. [4]. The starting point is the series expansion of $e_l(\psi)$ obtained from (51). Expanding the $\Gamma$-functions by means of Stirling’s formula and taking derivative we isolate the logarithmically divergent piece. The coefficient of this piece is $\delta_l$ since in the limit $\psi \rightarrow \psi_0$, $\frac{du}{d\psi} \rightarrow \delta_l \ln(\psi - \psi_0) + \text{finite}$.
The sums in (57) are easily evaluated. For $l = 0, \cdots, p - 1$ we obtain

A) $\delta_l = \frac{3\sqrt{3}}{2\pi^2} \alpha^{-l}(1, 1, -3, 2, 2, -3) \quad ; \quad \alpha = e^{2\pi i/6}$

B) $\delta_l = \frac{\sqrt{2}}{\pi^2} \alpha^{-l/2}(1, 1, -3, 3) \quad ; \quad \alpha = e^{2\pi i/4}$

C) $\delta_l = \frac{5}{4\pi^2} \alpha^{-3l}(1, -1, -2, 0, 2) \quad ; \quad \alpha = e^{2\pi i/5}$

The usefulness of these results will be appreciated in the next section.

5. Duality Generators

In this section we will explain how the duality group is obtained starting from the solutions $\omega_a(\psi)$ of the PF equation. We will see that the problems of determining the canonical periods defining $P(\psi)$ in terms of the $\omega_a(\psi)$ and finding the duality generators are solved simultaneously.

The basic idea is to work with a basis of solutions of the PF equation such that the phase transformation $\psi \rightarrow \alpha \psi$ and the transvection about $\psi = \psi_0$ are realized in a simple way. To express $P(\psi)$ in this basis we will exploit the fact that it must transform according to (15) and (16), with $S, T \in Sp(4, \mathbb{Z})$. Implementing $\psi \rightarrow \alpha \psi$ is best achieved by working with solutions around $\psi = 0$ that are defined without restrictions in $\arg \psi$.

Our basis will then be expressed in terms of the $\omega_j$ given in Table 2. In fact, notice that we have already introduced functions $e_l(\psi)$ that transform nicely under $\psi \rightarrow \alpha \psi$, namely $e_l(\psi) \rightarrow e_{l+1}(\psi)$. However, these functions are not quite adequate to our needs. If we choose $e_l(\psi), l = 0, \cdots, 3$ as our basis we will have to expand $e_4(\psi)$ in this basis and in general the coefficients will not be integers. Moreover, the $\delta_l$ factors related to the monodromy matrix $T$ are not even real. Fortunately, both these problems have a common cure.

At this point we recall that we still have a gauge freedom associated to the normalization of $\Omega(\psi)$. In practice this means that we can express $P(\psi)$ in terms of periods

$$\hat{\omega}_j(\psi) = \rho(\psi) \omega_j(\psi)$$

18
where $\rho(\psi)$ is a gauge fixing function. In particular, we may try a new basis $\hat{e}_l(\psi)$ obtained from the analytic continuation of $\hat{y}_0(\alpha^l \psi)$, with $\hat{y}_0(\psi) = \rho(\psi)y_0(\psi)$, and choose $\rho(\psi)$ so that the monodromy coefficients of the $\hat{e}_l(\psi)$ are real. From (58) we easily find the appropriate gauge satisfying this requirement, namely

$$\rho(\psi) = (\psi, \psi^{1/2}, \psi^3)$$  \hspace{1cm} (60)$$

for models $(A, B, C)$. We will then choose a basis

$$\hat{e}_l(\psi) = -\left(\frac{2\pi i}{p}\right)^3 \sum_j \lambda_j \beta_j \hat{\omega}_j(\alpha^l \psi)$$  \hspace{1cm} (61)$$

for $l = 0, \cdots, 3$. The numerical factor is conventional. The monodromy about $\psi = \psi_0$ of this new basis is easily found from previous results. We obtain

$$\hat{e}_l(\psi) \rightarrow \hat{e}_l(\psi) + \hat{\delta}_l \hat{g}(\psi)$$  \hspace{1cm} (62)$$

where

$$A) \quad \hat{g}(\psi) = \frac{\pi^2}{3\sqrt{3}} \psi g(\psi)$$

$$\hat{\delta}_l = (1, 1, -3, 2)$$

$$B) \quad \hat{g}(\psi) = \frac{\pi^2}{2\sqrt{2}} \psi^{1/2} g(\psi)$$

$$\hat{\delta}_l = (1, 1, -3, 3)$$

$$C) \quad \hat{g}(\psi) = \frac{4\pi^2}{25} \psi^3 g(\psi)$$

$$\hat{\delta}_l = (1, -1, -2, 0)$$

(63)

Notice that $\hat{g}(\psi)$ is analytic at $\psi = \psi_0$.

To determine the action on our basis of the transvection about $\psi = \psi_0$ we need to write $\hat{g}(\psi)$ as a combination of the $\hat{e}_l(\psi)$. We obtain

$$A) \quad \hat{g}(\psi) = \hat{e}_0(\psi) - \hat{e}_1(\psi)$$

$$B) \quad \hat{g}(\psi) = \hat{e}_0(\psi) - \hat{e}_1(\psi)$$

$$C) \quad \hat{g}(\psi) = \hat{e}_0(\psi) + \hat{e}_1(\psi)$$

(64)
These results are found by using a trick explained in [4]. First notice that the transformation (62) is equivalent to

$$
\hat{e}_l(\psi) = \frac{\hat{\delta}_l}{2\pi i} \hat{g}(\psi) \ln(\psi - \psi_0) + \text{analytic}
$$

(65)

Next, take $\psi = x$ real, $x > \psi_0$. Discontinuity of $\ln(\psi - \psi_0)$ across the cut from $\psi_0$ to $\infty$ then implies in particular

$$
\hat{\delta}_1 \hat{g}(x) = \hat{e}_1(x - i\epsilon) - \hat{e}_1(x + i\epsilon)
$$

(66)

for $\epsilon$ infinitesimal. We now use

$$
\hat{e}_1(x - i\epsilon) = \hat{e}_0(\alpha(x - i\epsilon))
$$

together with the expansion of $\hat{e}_0(\psi) = \rho(\psi) \gamma_0(\psi)$ for $|\psi| > \psi_0, 0 \leq \arg \psi \leq \frac{2\pi}{p}$, to arrive at

$$
\hat{e}_0(\alpha(x - i\epsilon)) = \rho(\alpha) \alpha^{-s_0} \hat{e}_0(x + i\epsilon)
$$

Substituting back in (66) and analytically continuing to all $\psi$ lead to the final expressions in (64).

To simplify the presentation we introduce a vector $E(\psi)$ defined by

$$
E = \begin{pmatrix}
\hat{e}_0 \\
\hat{e}_1 \\
\hat{e}_2 \\
\hat{e}_3
\end{pmatrix}
$$

(67)

Under transvection about $\psi = \psi_0$ $E$ transforms as

$$
E \rightarrow T_E E
$$

(68)
The matrix $T_E$ is completely determined from the above results. We find

\[
A) \quad T_E = \begin{pmatrix}
2 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-3 & 3 & 1 & 0 \\
2 & -2 & 0 & 1
\end{pmatrix}
\]

\[
B) \quad T_E = \begin{pmatrix}
2 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-3 & 3 & 1 & 0 \\
3 & -3 & 0 & 1
\end{pmatrix}
\]

\[
C) \quad T_E = \begin{pmatrix}
2 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-2 & -2 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(69)

Notice that these matrices are not symplectic but have determinant one.

As advertised before, our new basis also affords an integral realization of the phase symmetry $\psi \rightarrow \alpha \psi$. We have $\hat{e}_1(\psi) \rightarrow \hat{e}_{i+1}(\psi)$ and for $\hat{e}_4(\psi)$ we find

\[
A) \quad \hat{e}_4(\psi) = -\hat{e}_0(\psi) - \hat{e}_2(\psi)
\]

\[
B) \quad \hat{e}_4(\psi) = -\hat{e}_0(\psi)
\]

\[
C) \quad \hat{e}_4(\psi) = -\hat{e}_0(\psi) - \hat{e}_1(\psi) - \hat{e}_2(\psi) - \hat{e}_3(\psi)
\]

(70)

Then, under $\psi \rightarrow \alpha \psi$ the vector $E(\psi)$ transforms as

\[
E \rightarrow S_E E
\]

(71)

with the matrix $S_E$ given by

\[
S_E = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\sigma_{30} & \sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
\]

(72)

The last row is read off from (70).
So far we have a basis in which transvection and phase transformations are realized by integral matrices of determinant one but not all symplectic. However, there must exist a change of basis from $E(\psi)$ to $P(\psi)$

$$P(\psi) = U E(\psi)$$

such that the matrices

$$S = U S_E U^{-1} \quad ; \quad T = U T_E U^{-1}$$

are both integral and symplectic. To find this change of basis we will first make some justified assumptions.

Under transvection about $\psi = \psi_0$ the periods necessarily transform into themselves plus a piece proportional to $\hat{g}(\psi)$. Moreover, the proportionality constant is an integer (if it were not we could just introduce some appropriate overall normalization factor). Hence, by an $Sp(4, \mathbb{Z})$ transformation we can always bring one of the periods to be equal to $\hat{g}(\psi)$. For definiteness we choose $z^2(\psi) = \hat{g}(\psi)$ so that the fourth row of $U$ follows from (64). As remarked in [4] geometrically $\hat{g}(\psi)$ corresponds to an integral of $\Omega(\psi)$ around a cycle $A^2$ that is unambiguously defined for all $\psi$ near $\psi_0$. Likewise, we could choose another period equal to $\hat{e}_0(\psi)$ since it transforms into itself under transvection about $\psi = \infty$. To see which period we could identify with $\hat{e}_0(\psi)$ notice that under transvection about $\psi = \psi_0$ it transforms as $\hat{e}_0 \rightarrow \hat{e}_0 + \hat{g}$, meaning that $\hat{e}_0(\psi)$ is an integral of $\Omega(\psi)$ around a cycle that necessarily intersects $A^2$. Therefore, we take $\mathcal{G}_2(\psi) = \hat{e}_0(\psi)$. In model $A$ we verified explicitly that integrating $\Omega(\psi)$ obtained from (8) along cycles $A^2, B_2$ defined as in [4] leads to periods $z^2(\psi), \mathcal{G}_2(\psi)$ in complete agreement with our indirect results.

It is very simple to check that with the above choices of $z^2(\psi)$ and $\mathcal{G}_2(\psi)$ the condition $T \in Sp(4, \mathbb{Z})$ requires that the remaining periods $z^1(\psi)$ and $\mathcal{G}_1(\psi)$ be analytic at $\psi = \psi_0$. Therefore, the monodromy matrix $T$ is given by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
in all three models. Geometrically, $z^1(\psi)$ and $G_1(\psi)$ correspond to integrals of $\Omega(\psi)$ around cycles remote from the singular point at which all $\partial W / \partial X_m = 0$ when $\psi = \psi_0$.

Thus far we have determined two rows of the change of basis matrix $U$. One of the eight unknown entries, say $U_{20}$, can be taken to be zero due to an $Sp(4, \mathbb{Z})$ freedom of redefining $z^1(\psi)$ and $G_1(\psi)$ without altering $z^2(\psi)$ and $G_2(\psi)$. Two of the remaining entries are not independent but fixed by the form of $T$, i.e. by requiring that $T \in Sp(4, \mathbb{Z})$. The independent elements are found by imposing that $S \in Sp(4, \mathbb{Z})$. The end results for $U$ and $S$ are given below

$$A) \quad U = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -3 & 0 & 1 & 0 \\ -3 & 4 & 1 & -3 \end{pmatrix}$$

$$B) \quad U = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -2 & -2 & 1 & 2 \\ -4 & 4 & 1 & -3 \end{pmatrix}$$

$$C) \quad U = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & -3 & 1 & 2 \end{pmatrix}$$

Actually, the above matrices are only determined up to the freedom in redefining $z^1(\psi)$ and $G_1(\psi)$ mentioned previously.

As we explained in section 3, the matrices $S$ and $T$ generate the duality group $D$ of the period vector $P(\psi)$. Notice that $T$ is of infinite order whereas $S$ satisfies

$$A) \quad S^6 = 1$$

$$B) \quad S^8 = 1$$

$$C) \quad S^5 = 1$$

(76)

(77)
In example $B$, $S^4 = -1$ due to the branch point at $\psi = 0$ introduced by the gauge $\rho(\psi) = \psi^{1/2}$. Transvection about the singular points $\alpha^l \psi_0$ is given by

$$T_l = S^{-l}TS^l$$

(78)

Transvection about $\psi = \infty$ is also computed from $S$ and $T$. In all three models $T_\infty$ turns out to be

$$T_\infty = (ST)^{-p}$$

(79)

For instance, in example $B$ we have

$$T_\infty T_3 T_2 T_1 T T^0 = 1$$

where $T^0 = -1$ is the transvection about $\psi = 0$. The above equation reflects the fact that a loop enclosing all singularities can be deformed into a loop encircling no singularities. Using (78) and $S^4 = -1$ we indeed find $T_\infty = (ST)^{-4}$. In the next section we will need the explicit results for $(ST)^{-1}$ given below

\[
A) \quad (ST)^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 0 & -4 & -1 & 1 \end{pmatrix}
\]

\[
B) \quad (ST)^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ 2 & -4 & -1 & 1 \end{pmatrix}
\]

(80)

\[
C) \quad (ST)^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 3 & -1 & -1 \end{pmatrix}
\]

$T_\infty$ is easily obtained using (79).

6. Mirror Maps, Automorphic Functions and Yukawa Couplings
The ratio \( t = w^1/w^2 \) of the homogeneous coordinates of the prepotential \( \mathcal{F} \) can be understood as the \((1,1)\)-modulus of the original manifold. The mirror map gives the relation between \( t \) and \( \psi \). Our results in sections 4 and 5 allow a simple derivation of \( t \) as a function of \( \psi \) as we will now explain.

In [4] \( t \) was determined by first obtaining the explicit \( Sp(4, \mathbb{Z}) \) rotation relating \( P \) and \( \Pi \). It was also found that the effect of \( T_\infty \) on \( t \) was an integer shift by the order of the phase symmetry (9). We may take this property as the definition of \( t \). More precisely, we will define \( t \) so that under \((ST)^{-1}\) it transforms as

\[
 t \rightarrow t + 1
\]

In fact, imposing (81) does not fix \( t \) completely. There is a residual ambiguity that just reflects the above translational symmetry. This prescription for finding \( t(\psi) \) has also been advocated in Ref. [21].

Thus, our strategy is to find the, necessarily integer, combinations of the \( z^a, G_a \) that give \( w^1 \) and \( w^2 \) such that their ratio transforms as (81) under \((ST)^{-1}\). The appropriate combinations, up to the ambiguity mentioned, are easily derived from the results in (80). We find

\[
 A) \quad t = \frac{G_1}{G_2} = \frac{1}{3\hat{e}_0}[-\hat{e}_0 + 2\hat{e}_1 + \hat{e}_2 + \hat{e}_3] \\
 B) \quad t = \frac{G_1}{G_2} = \frac{1}{2\hat{e}_0}[-\hat{e}_0 + \hat{e}_1 + \hat{e}_2 + \hat{e}_3] \\
 C) \quad t = -\frac{G_1}{G_2} = -\frac{1}{\hat{e}_0}[\hat{e}_0 + \hat{e}_1 + \hat{e}_3]
\]

The expansions of \( t \) for \( |\psi| < \psi_0 \) follow simply from (61) and the \( \omega_j \) given in Table 2. Notice that the freedom of adding a piece \( mG_2 \) to \( G_1 \) only produces an integer shift in \( t \). This would just imply that the fundamental domain of \( t \) is translated by \( m \) in the Re \( t \) direction as allowed by the axionic symmetry (81).

The expansions of \( t \) for \( |\psi| > \psi_0 \) require the analytic continuation of the periods. There is a simple way of performing this continuation. We already know a basis around \( \psi = \infty \), namely \( \hat{y}_i(\psi) = \rho(\psi)y_i(\psi) \) with \( y_i(\psi) \) given in (49). Furthermore, the monodromy

25
of this basis about $\psi = \infty$ is also known. By comparing $T_\infty$ computed from (79) and (80) with this monodromy we can find how the periods are expanded in the $\hat{y}$-basis. Clearly, $G_2 = (\frac{2\pi i}{p})^3 \hat{y}_0$ and for $G_1$ we find that it is proportional to $\hat{y}_1$ as expected since $\hat{y}_1$ roughly transforms into itself plus $\hat{y}_0$ under transvection about $\psi = \infty$. Actually, we also find that $G_1$ has a piece proportional to $\hat{y}_0$ that cannot be determined from the monodromy data only. However, this piece is irrelevant since it can be cancelled against a term proportional to $G_2$ that we could have added to $G_1$ in (82). In all examples the final result for $t$ can be written as

$$t = \frac{p}{2\pi i} \frac{y_1}{y_0}$$  \hspace{1cm} (83)

where we have used $\hat{y}_1/\hat{y}_0 = y_1/y_0$, i.e. $t$ is gauge invariant. Notice that $t$ transforms as $t \to t + p$ under $T_\infty$. However, other generators such as $S$ or $T$ do not have a simple action on $t$.

The explicit expressions of $t(\psi)$ for $|\psi| > \psi_0$ are obtained from the expansions in Table 3. In all three models, as well as in the quintic hypersurface studied in [4], the result takes the general form

$$t = \frac{1}{2\pi i} \left\{ -\ln((c\psi)^p) + \frac{1}{y_0} \sum_{l=0}^{\infty} \frac{(dl)! (c\psi)^{-pl}}{(n_1 l)! \cdots (n_5 l)!} [d\Psi(dl+1) - n_1 \Psi(n_1 l+1) - \cdots - n_5 \Psi(n_5 l+1)] \right\}$$  \hspace{1cm} (84)

where

$$\tilde{y}_0 = \sum_{l=0}^{\infty} \frac{(dl)! (c\psi)^{-pl}}{(n_1 l)! \cdots (n_5 l)!}$$  \hspace{1cm} (85)

Recall that $d$ is the degree of quasihomogeneity of the defining polynomial, the $n_m$ are the weights of the $X_m$ coordinates and $p$ is the order of the phase symmetry $\psi \to \alpha \psi$. Constant $c$ can be found from

$$(c\psi_0)^p = \frac{d^d}{n_1^{n_1^*} \cdots n_5^{n_5^*}}$$  \hspace{1cm} (86)

Then, $c = (6, 16, 20)$ in models $(A, B, C)$ as shown in Table 3. Notice that the large $\psi$ limit of $t$ is

$$t \to -\frac{p}{2\pi i} \ln(c\psi)$$  \hspace{1cm} (87)

since $d = n_1 + \cdots + n_5$. 26
Knowing \( t(\psi) \) we can determine the fundamental region \( \mathcal{T} \) of \( t \) as the image of the fundamental region of \( \psi \) given by the wedge \( 0 \leq \arg \psi \leq \frac{2\pi}{p} \). Roughly speaking, \( \mathcal{T} \) consists of two adjacent triangles with vertices at \( t(\infty) = i\infty \), \( t(\psi_0) \), \( t(0) \) and \([t(0) + 1]\) as shown schematically in Figure 1. The vertices correspond to fixed points of the duality group. \( t(0) \) must be a fixed point of the \( S \) generator since \( \psi = 0 \) is fixed under the phase symmetry \( \psi \to \alpha \psi \). From (82) and (72) we can verify that this is indeed the case. On the other hand, \( t(\psi_0) \) must be fixed under the action of \( T \) since this transvection leaves \( \psi_0 \) invariant. To check this consider model \( C \) for definiteness. Under \( T \) we have
\[
t \to -\frac{G_1}{G_2 + z^2}
\]
but \( z^2(\psi_0) = 0 \) and thus \( t(\psi_0) \to t(\psi_0) \) as claimed. Notice also that \( t(\infty) \) is fixed under \( T_\infty \).

In all models \( t \) has values at the vertices given by
\[
t(\psi_0) = ia \\
t(0) = -\frac{1}{2} + ib
\]
where \( a \) and \( b \) are constants with \( a > b \). Constant \( b \) is evaluated from the expansion of \( t \) for \( |\psi| < \psi_0 \). We obtain
\[
b = \frac{1}{2}(\sqrt{3}, 1 + \sqrt{2}, \sqrt{5 + 2\sqrt{5}})
\]
for models \((A, B, C)\). For constant \( a \), a rough numerical computation gives
\[
a \sim (1.42, 1.70, 1.98)
\]
These results follow from either the expansion of \( t \) for \( |\psi| > \psi_0 \) or the expansion of \( t \) for \( |\psi| < \psi_0 \) since they both converge as \( \psi \to \psi_0 \).

The boundary arc joining \( t(\psi_0) \) to \( t(0) \) is the image of the interval \([0, \psi_0]\). At \( t(\psi_0) \) this arc is tangent to the imaginary axis as expected from the fact that \( t(0) \) is a fixed point of a generator of infinite order. We can check this result from the behavior of \( t(\psi) \) near \( \psi_0 \) encoded in the relation
\[
\frac{t(\psi) - t(\psi_0)}{(\psi - \psi_0)} = -i \frac{ap^3 \hat{g}_0}{(2\pi)^4 \hat{g}_0(\psi_0)} \ln(\psi - \psi_0)
\]
where \( \tilde{g}_0 \) defined by \( \tilde{g}(\psi) = \tilde{g}_0(\psi - \psi_0) + \cdots \) is found from (63). The above expression follows from (82) and (65). We can study the behavior of \( t(\psi) \) near \( \psi = 0 \) in a similar manner. We find that at \( t(0) \) the angle between the arc and the imaginary axis is \( \pi/p \) as it should since \( t(0) \) is fixed under a symmetry of order \( p \).

We now turn to a discussion of the relation of \( t \) to automorphic functions [26]. To this purpose it is instructive to study the simpler onefold models. To obtain \( t(\psi) \) in this case it is convenient to adopt the approach taken in [16, 22, 23]. Let us consider for example the \( Z_6 \) model. Defining \( t = \omega_1/\omega_2 \), with \( \omega_1 \) and \( \omega_2 \) two independent solutions of the PF equation (35) we find that \( t \) satisfies

\[
\{t, \psi\} = -\frac{\psi(20\psi^3 - 41)}{2(4\psi^3 - 1)^2}
\]  

(93)

where \( \{f, x\} = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \) is the Schwarzian derivative. Introducing the variable

\[
k = \frac{4\psi^3}{4\psi^3 - 1}
\]

(94)

then leads to

\[
\{t, \kappa\} = \frac{3}{8(1 - \kappa)^2} + \frac{23}{72\kappa(1 - \kappa)} + \frac{4}{9\kappa^2}
\]  

(95)

The solution of the above differential equation is known to be the absolute modular invariant of \( PSL(2, \mathbb{Z}) \), i.e. \( \kappa(t) = J(t) \) [27]. Hence, the mirror map is implicitly given by

\[
J(t) = \frac{4\psi^3}{4\psi^3 - 1}
\]

(96)

Recall that \( J(t) = \frac{1}{1728}[q^{-1} + 744 + 196884q + \cdots] \), where \( q = e^{2\pi i t} \) is the uniformizing variable. From (96) we also conclude that the duality group acting on \( t \) is \( PSL(2, \mathbb{Z}) \) as expected for a torus compactification.

In our threefold models \( t \) should also represent the inverse of an automorphic function of the duality group acting on this variable. In general, automorphic functions will be
written as rational functions of $\psi^p(t)$. A Fourier series of $\psi^p$ in powers of $q = e^{2\pi it}$ can be obtained from the expansion of $t$ for large $\psi$. Below we give the first terms of the series

\begin{align*}
A) \quad (6\psi)^6 &= \frac{1}{q} + 2772 + 5703858 q + 14332453152 q^2 + \cdots \\
B) \quad (16\psi)^4 &= \frac{1}{q} + 15808 + 178476448 q + 2473876932608 q^2 + \cdots \\
C) \quad (20\psi)^5 &= \frac{1}{q} + 179520 + 2539812000 q + 4599352920320000 q^2 + \cdots \\
\end{align*}

$\psi^p$ is clearly invariant under $t \rightarrow t + 1$. Being an automorphic function, it must also be invariant under the remaining duality generators. Rather than invariant functions, for the purpose of constructing duality invariant effective actions it is necessary to know the equivalent of modular forms, i.e. functions that transform in a specific way under the duality group. Of particular interest [1] is the generalization of the Dedekind cusp form $\eta(t)$.

Let us now consider the computation of the Yukawa coupling of the $(1, 1)$-field. This coupling is given by [4]

$$\kappa_{\psi\psi\psi} = \frac{1}{G_2^2} \kappa_{\psi\psi\psi} \left( \frac{d\psi}{dt} \right)^3$$

(98)

$\kappa_{\psi\psi\psi}$ is in turn given by

$$\kappa_{\psi\psi\psi} = W_3(\psi) \rho^2(\psi)$$

(99)

$W_3$ is the solution of the differential equation

$$\frac{dW_3}{d\psi} + \frac{1}{2} C_3(\psi) W_3 = 0$$

(100)

where $C_3(\psi)$ is the coefficient appearing in the PF equation. We find

\begin{align*}
A) \quad \kappa_{\psi\psi\psi} &= c \frac{\psi^3}{1 - 4\psi^6} \\
B) \quad \kappa_{\psi\psi\psi} &= c \frac{\psi}{1 - \psi^4} \\
C) \quad \kappa_{\psi\psi\psi} &= c \frac{\psi^7}{1 - 4\psi^5} \\
\end{align*}

(101)
The normalization of $\kappa_\psi \psi \psi$ is fixed by the condition that in the classical (large radius) limit

$$\kappa_{ttt} \to m_0$$

where $m_0$ is the intersection number of the $(1, 1)$-form of the original hypersurface $\mathcal{H}$ [5].

For weighted projective spaces this topological invariant can be shown to be equal to the degree of quasihomogeneity divided by the product of the weights [28]. Then, $m_0 = (3, 2, 1)$ in models $(A, B, C)$.

It was first conjectured [4] and later proven [19] that $\kappa_{ttt}$ can be expressed as

$$\kappa_{ttt} = \sum_{k=1}^{\infty} \frac{m_k k^3 q^k}{1 - q^k} = m_0 + m_1 q + (8m_2 + m_3)q^2 + \cdots$$  \hspace{1cm} (102)

where $m_k$ is the number of rational curves of degree $k$ in $\mathcal{H}$. Given our expansions for $t(\psi)$ and $G_2(\psi)$ it is straightforward to compute $\kappa_{ttt}$. We have checked that its expansion is of the form (102). Table 4 shows the first coefficients $m_k$.

Except for model $C$ where we have taken the correct normalization $m_0 = 1$ the values obtained agree with recent results [21].

To end this section we discuss briefly the structure of the prepotential $F(t)$. From the explicit results for $\Pi$ we can compute $F = \frac{1}{2} w^a F_a$ and verify that it has the expected behavior

$$F(t) = -\frac{m_0}{6} t^3 + f_2 t^2 + f_1 t + F_{\text{loop}} + \text{non-perturbative}$$  \hspace{1cm} (103)

where the non-perturbative terms involve powers of $e^{2\pi it}$. As explained in Ref. [4], $F_{\text{loop}}$ is due to a sigma-model four-loop correction and its form must be given by a universal constant times the Euler characteristic $\chi$. Indeed, in all models we find

$$F_{\text{loop}} = \frac{i\zeta(3)}{2(2\pi)^3} \chi$$  \hspace{1cm} (104)

in agreement with the results of [4].

7. Conclusions

In this paper we have tackled the problem of finding the duality symmetries of Calabi-Yau compactifications. We focused our attention on the more tractable case of
one-modulus manifolds. In particular, we considered three specific models and explained how the $Sp(4, \mathbb{Z})$ duality generators could be found systematically.

The starting point in our analysis was the Picard-Fuchs differential equation satisfied by the periods of the associated mirror manifolds. We obtained explicit solutions of this equation that were then used to construct simultaneously the symplectic basis of periods and the duality generators. With these results we then derived the relation between the $(1,1)$-modulus $t$ of the original manifold and the $(2,1)$-modulus $\psi$ of the mirror partner. Our prescription for computing the mirror map $t(\psi)$ was based on the existence of the axionic symmetry $t \rightarrow t + 1$ together with the property that $t$ can be written as a ratio of periods in the symplectic basis. We found a general expression for $t(\psi)$ with parameters that follow directly from the equation defining the mirror manifolds.

$t$ is the interesting physical variable, its real and imaginary parts correspond respectively to the antisymmetric tensor field and the radius of compactification. In general, the generators of the duality group act on $t$ in a complicated way. Nonetheless, knowing the mirror map $t(\psi)$ allowed us to study the basic features of the $t$ fundamental domain $\mathcal{T}$. From the shape of $\mathcal{T}$ we deduce the existence of a symmetry relating large and small radius. Equivalently, we may say that there is a minimum value for the radius of compactification given by $\text{Im} \ t(0)$. Other generic properties of the fundamental region will likely hold in more complicated models. For instance, the vertices of $\mathcal{T}$ are given by $t(0), t(\psi_0)$ and $t(\infty)$ that correspond respectively to the Gepner point $\psi = 0$ and the singular points $\psi = \psi_0, \psi = \infty$. Furthermore, the internal angles of $\mathcal{T}$ are determined by the order of the symmetries that leave fixed these special points.

The results for $t(\psi)$ can be applied to determine other properties of the models such as non-perturbative corrections to the prepotentials and the Yukawa couplings. We computed the first terms in the expansion of the Yukawa couplings. From the Physics point of view these corrections are needed in the analysis of effective theories. They are also related to the numbers of rational curves of the manifold [4, 19]. For these numbers we obtained values in agreement with recent work [21]. We also verified that the four-loop sigma-model correction to the prepotential has a universal form.
We remarked the fact that $t$ can be interpreted as the inverse of an automorphic function of the duality group and $t(\psi)$ can be inverted to find the basic building blocks of such functions. The $Sp(4,\mathbb{Z})$ duality generators that we have found in principle can be used to construct modular forms following a method developed in Ref. [29]. It would be interesting to investigate how the forms obtained in that approach are related to the automorphic functions and Yukawa couplings derived from the mirror maps. Work along these lines is in progress.

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### Table 1. Coefficients in the Picard-Fuchs Equation

| Model | $C_0$ | $C_1$ | $C_2$ | $C_3$ |
|-------|-------|-------|-------|-------|
| A     | $\frac{4\psi^2}{4\psi^6 - 1}$ | $\frac{60\psi^3}{4\psi^6 - 1}$ | $\frac{2(50\psi^6 - 1)}{\psi^3(4\psi^6 - 1)}$ | $\frac{2(20\psi^6 + 1)}{\psi(4\psi^6 - 1)}$ |
| B     | $\frac{1}{16(\psi^4 - 1)}$ | $\frac{5\psi}{\psi^4 - 1}$ | $\frac{29\psi^2}{2(\psi^4 - 1)}$ | $\frac{8\psi^3}{\psi^4 - 1}$ |
| C     | $\frac{\psi}{4(4\psi^5 - 1)}$ | $\frac{20\psi^2}{4\psi^5 - 1}$ | $\frac{2(29\psi^5 - 1)}{\psi^2(4\psi^5 - 1)}$ | $\frac{2(16\psi^5 + 1)}{\psi(4\psi^5 - 1)}$ |

### Table 2. Solutions around $\psi = 0$

| Model | $j$ | $\omega_j(\psi)$ |
|-------|-----|------------------|
| A     | 0,1,3,4 | $\psi^j F\left(\frac{j+1}{6}, \frac{j+1}{6}, \frac{j+1}{6}, \frac{j+1}{6}; \frac{j+2}{6}, \frac{j+3}{6}, \frac{j+5}{6}, \frac{j+6}{6}; 4\psi^6\right)$ |
| B     | 0,1,2,3 | $\psi^j F\left(\frac{2j+1}{8}, \frac{2j+1}{8}, \frac{2j+1}{8}, \frac{2j+1}{8}; \frac{j+1}{4}, \frac{j+2}{4}, \frac{j+3}{4}, \frac{j+4}{4}; \psi^4\right)$ |
| C     | 0,1,3,4 | $\psi^j F\left(\frac{2j+1}{10}, \frac{2j+1}{10}, \frac{2j+1}{10}, \frac{2j+1}{10}; \frac{j+1}{5}, \frac{j+2}{5}, \frac{j+4}{5}, \frac{j+5}{5}; 4\psi^5\right)$ |
$$y_0 = \psi^{-1} F\left(\frac{1}{6}; \frac{1}{3}, \frac{2}{3}, \frac{5}{6}; 1, 1, 1; \frac{1}{4\psi^6}\right)$$

$$y_1 = -y_0 \ln(6\psi) + \psi^{-1} \sum_{l=0}^{\infty} \frac{(6l)!}{(2l)! (l!)^4 (6\psi)^{6l}} [3\Psi(6l + 1) - \Psi(2l + 1) - 2\Psi(l + 1)]$$

$$y_0 = \psi^{-\frac{4}{5}} F\left(\frac{1}{8}; \frac{3}{8}, \frac{5}{8}, \frac{7}{8}; 1, 1, 1; \frac{1}{\psi^4}\right)$$

$$y_1 = -y_0 \ln(16\psi) + \psi^{-\frac{4}{5}} \sum_{l=0}^{\infty} \frac{(8l)!}{(4l)! (l!)^4 (16\psi)^{4l}} [2\Psi(8l + 1) - \Psi(4l + 1) - \Psi(l + 1)]$$

$$y_0 = \psi^{-\frac{5}{9}} F\left(\frac{1}{10}; \frac{3}{10}, \frac{7}{10}, \frac{9}{10}; 1, 1, 1; \frac{1}{4\psi^5}\right)$$

$$y_1 = -y_0 \ln(20\psi) + \psi^{-\frac{5}{9}} \sum_{l=0}^{\infty} \frac{(10l)!}{(5l)! (2l)! (l!)^3 (20\psi)^{5l}} [10\Psi(10l + 1) - 5\Psi(5l + 1) - 2\Psi(2l + 1) - 3\Psi(l + 1)]$$

**Table 3.** Solutions around $\psi = \infty$

| Model | $m_0$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|-------|-------|-------|-------|-------|-------|
| A     | 3     | 7884  | 6028452 | 11900417220 | 34600752005688 |
| B     | 2     | 29504 | 128834912 | 1423720546880 | 23193056024793312 |
| C     | 1     | 231200 | 12215785600 | 1700894366474400 | 350154658851324656000 |

**Table 4.** Coefficients in the expansion of the Yukawa couplings.