Tropical Mathematics, Classical Mechanics and Geometry

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Abstract. A very brief introduction to tropical and idempotent mathematics is presented. Applications to classical mechanics and geometry are especially examined.

1. Introduction

Tropical mathematics can be treated as a result of a dequantization of the traditional mathematics as the Planck constant tends to zero taking imaginary values. This kind of dequantization is known as the Maslov dequantization and it leads to a mathematics over tropical algebras like the max-plus algebra. The so-called idempotent dequantization is a generalization of the Maslov dequantization. The idempotent dequantization leads to mathematics over idempotent semirings (exact definitions see below in sections 2 and 3). For example, the field of real or complex numbers can be treated as a quantum object whereas idempotent semirings can be examined as ”classical” or ”semiclassical” objects (a semiring is called idempotent if the semiring addition is idempotent, i.e. $x \oplus x = x$), see [19, 22].
Tropical algebras are idempotent semirings (and semifields). Thus tropical mathematics is a part of idempotent mathematics. Tropical algebraic geometry can be treated as a result of the Maslov dequantization applied to the traditional algebraic geometry (O. Viro, G. Mikhalkin), see, e.g., [17, 41, 42, 47, 49]. There are interesting relations and applications to the traditional convex geometry.

In the spirit of N. Bohr’s correspondence principle there is a (heuristic) correspondence between important, useful, and interesting constructions and results over fields and similar results over idempotent semirings. A systematic application of this correspondence principle leads to a variety of theoretical and applied results [19, 23], see Fig. 1.

The history of the subject is discussed, e.g., in [19]. There is a large list of references.

2. The Maslov dequantization

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers. The so-called max-plus algebra $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}$ is defined by the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

The max-plus algebra can be treated as a result of the Maslov dequantization of the semifield $\mathbb{R}_+$ of all nonnegative numbers with the usual arithmetics. The change of variables

$$x \mapsto u = h \log x,$$

Figure 1. Relations between idempotent and traditional mathematics.
where \( h > 0 \), defines a map \( \Phi_h : \mathbb{R}^+ \to \mathbb{R} \cup \{-\infty\} \), see Fig. 2. Let the addition and multiplication operations be mapped from \( \mathbb{R}^+ \) to \( \mathbb{R} \cup \{-\infty\} \) by \( \Phi_h \), i.e. let

\[
    u \oplus_h v = h \log (\exp(u/h) + \exp(v/h)), \quad u \odot v = u + v,
\]

\[
    0 = -\infty = \Phi_h(0), \quad 1 = 0 = \Phi_h(1).
\]

It can easily be checked that \( u \oplus_h v \to \max\{u, v\} \) as \( h \to 0 \). Thus we get the semifield \( \mathbb{R}_{\max} \) (i.e. the max-plus algebra) with zero \( 0 = -\infty \) and unit \( 1 = 0 \) as a result of this deformation of the algebraic structure in \( \mathbb{R}^+ \).

Figure 2. Deformation of \( \mathbb{R}^+ \) to \( \mathbb{R}^{(h)} \). Inset: the same for a small value of \( h \).
The semifield $R_{\text{max}}$ is a typical example of an idempotent semiring; this is a semiring with idempotent addition, i.e., $x \oplus x = x$ for arbitrary element $x$ of this semiring.

The semifield $R_{\text{max}}$ is also called a tropical algebra. The semifield $R^{(h)} = \Phi_h(R_+)$ with operations $\oplus_h$ and $\odot$ (i.e. $+$) is called a subtropical algebra.

The semifield $R_{\text{min}} = R \cup \{+\infty\}$ with operations $\oplus = \text{min}$ and $\odot = +$ ($0 = +\infty$, $1 = 0$) is isomorphic to $R_{\text{max}}$.

The analogy with quantization is obvious; the parameter $h$ plays the role of the Planck constant. The map $x \mapsto |x|$ and the Maslov dequantization for $R_+$ give us a natural transition from the field $C$ (or $R$) to the max-plus algebra $R_{\text{max}}$. We will also call this transition the Maslov dequantization. In fact the Maslov dequantization corresponds to the usual Schrödinger dequantization but for imaginary values of the Planck constant (see below). The transition from numerical fields to the max-plus algebra $R_{\text{max}}$ (or similar semifields) in mathematical constructions and results generates the so called tropical mathematics. The so-called idempotent dequantization is a generalization of the Maslov dequantization; this is the transition from basic fields to idempotent semirings in mathematical constructions and results without any deformation. The idempotent dequantization generates the so-called idempotent mathematics, i.e. mathematics over idempotent semifields and semirings.

Remark. The term ‘tropical’ appeared in [45] for a discrete version of the max-plus algebra (as a suggestion of Christian Choffrut). On the other hand V. P. Maslov used this term in 80s in his talks and works on economical applications of his idempotent analysis (related to colonial politics). For the most part of modern authors, ‘tropical’ means ‘over $R_{\text{max}}$ (or $R_{\text{min}}$)’ and tropical algebras are $R_{\text{max}}$ and $R_{\text{min}}$. The terms ‘max-plus’, ‘max-algebra’ and ‘min-plus’ are often used in the same sense.
3. Semirings and semifields

Consider a set $S$ equipped with two algebraic operations: addition $\oplus$ and multiplication $\odot$. It is a semiring if the following conditions are satisfied:

- the addition $\oplus$ and the multiplication $\odot$ are associative;
- the addition $\oplus$ is commutative;
- the multiplication $\odot$ is distributive with respect to the addition $\oplus$:

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

and

$$(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$$

for all $x, y, z \in S$.

A unity (we suppose that it exists) of a semiring $S$ is an element $1 \in S$ such that $1 \odot x = x \odot 1 = x$ for all $x \in S$. A zero (if it exists) of a semiring $S$ is an element $0 \in S$ such that $0 \neq 1$ and $0 \oplus x = x$, $0 \odot x = x \odot 0 = 0$ for all $x \in S$. A semiring $S$ is called an idempotent semiring if $x \oplus x = x$ for all $x \in S$. A semiring $S$ with neutral element $1$ is called a semifield if every nonzero element of $S$ is invertible with respect to the multiplication. The theory of semirings and semifields is treated, e.g., in [13].

4. Idempotent analysis

Idempotent analysis deals with functions taking their values in an idempotent semiring and the corresponding function spaces. Idempotent analysis was initially constructed by V. P. Maslov and his collaborators and then developed by many authors. The subject and applications are presented in the book of V. N. Kolokoltsov and V. P. Maslov [18] (a version of this book in Russian was published in 1994).

Let $S$ be an arbitrary semiring with idempotent addition $\oplus$ (which is always assumed to be commutative), multiplication $\odot$, and unit $1$. The set $S$ is supplied with the standard partial order $\preceq$: by definition, $a \preceq b$ if and only if $a \oplus b = b$. If $S$ contains a zero element $0$, then all
elements of $S$ are nonnegative: $0 \preceq a$ for all $a \in S$. Due to the existence of this order, idempotent analysis is closely related to the lattice theory, theory of vector lattices, and theory of ordered spaces. Moreover, this partial order allows to model a number of basic “topological” concepts and results of idempotent analysis at the purely algebraic level; this line of reasoning was examined systematically in [19–32] and [8].

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \to S$, where $X$ is an arbitrary set and $S$ is an idempotent semiring. Functions with values in $S$ can be added, multiplied by each other, and multiplied by elements of $S$ pointwise.

The idempotent analog of a linear functional space is a set of $S$-valued functions that is closed under addition of functions and multiplication of functions by elements of $S$, or an $S$-semimodule. Consider, e.g., the $S$-semimodule $B(X, S)$ of all functions $X \to S$ that are bounded in the sense of the standard order on $S$.

If $S = \mathbb{R}_{\text{max}}$, then the idempotent analog of integration is defined by the formula

$$I(\varphi) = \int_X \varphi(x) \, dx = \sup_{x \in X} \varphi(x), \quad (1)$$

where $\varphi \in B(X, S)$. Indeed, a Riemann sum of the form $\sum \varphi(x_i) \cdot \sigma_i$ corresponds to the expression $\bigoplus \varphi(x_i) \odot \sigma_i = \max \{ \varphi(x_i) + \sigma_i \}$, which tends to the right-hand side of (1) as $\sigma_i \to 0$. Of course, this is a purely heuristic argument.

Formula (1) defines the idempotent (or Maslov) integral not only for functions taking values in $\mathbb{R}_{\text{max}}$, but also in the general case when any of bounded (from above) subsets of $S$ has the least upper bound.

An idempotent (or Maslov) measure on $X$ is defined by the formula $m_\psi(Y) = \sup_{x \in Y} \psi(x)$, where $\psi \in B(X, S)$ is a fixed function. The integral with respect to this measure is defined by the formula

$$I_\psi(\varphi) = \int_X \varphi(x) \, dm_\psi = \int_X \varphi(x) \odot \psi(x) \, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)). \quad (2)$$
Obviously, if $S = \mathbb{R}_{\min}$, then the standard order is opposite to the conventional order $\leq$, so in this case equation (2) assumes the form
\[
\int_X \varphi(x) \, dm_\psi = \int_X \varphi(x) \odot \psi(x) \, dx = \inf_{x \in X} (\varphi(x) \odot \psi(x)),
\]
where inf is understood in the sense of the conventional order $\leq$.

5. The superposition principle and linear problems

Basic equations of quantum theory are linear; this is the superposition principle in quantum mechanics. The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However, it is linear over the semirings $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$. Similarly, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings. This is V. P. Maslov’s idempotent superposition principle, see [36, 38]. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H \odot X \oplus F$, where $X$, $H$, $F$ are matrices with coefficients in an idempotent semiring $S$ and the unknown matrix $X$ is determined by $H$ and $F$ [2, 5, 6, 9, 10, 14, 15]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbb{R}_{\max}$ and $S = \mathbb{R}_{\min}$, respectively. It is known that principal optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman’s shortest path algorithm corresponds to a version of Jacobi’s algorithm, Ford’s algorithm corresponds to the Gauss–Seidel iterative scheme, etc. [5, 6].

The linearity of the Hamilton–Jacobi equation over $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$, which is the result of the Maslov dequantization of the Schrödinger equation, is closely related to the (conventional) linearity of the Schrödinger equation and can be deduced from this linearity. Thus, it is possible to borrow standard ideas and methods of linear analysis and apply them to a new area.

Consider a classical dynamical system specified by the Hamiltonian
\[
H = H(p, x) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x),
\]
where \( x = (x_1, \ldots, x_N) \) are generalized coordinates, \( p = (p_1, \ldots, p_N) \) are generalized momenta, \( m_i \) are generalized masses, and \( V(x) \) is the potential. In this case the Lagrangian \( L(x, \dot{x}, t) \) has the form

\[
L(x, \dot{x}, t) = \sum_{i=1}^{N} m_i \dot{x}_i^2 / 2 - V(x),
\]

where \( \dot{x} = (\dot{x}_1, \ldots, \dot{x}_N) \), \( \dot{x}_i = dx_i / dt \). The value function \( S(x, t) \) of the action functional has the form

\[
S = \int_{t_0}^{t} L(x(t), \dot{x}(t), t) \, dt,
\]

where the integration is performed along the factual trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle, or the least action principle).

For fixed values of \( t \) and \( t_0 \) and arbitrary trajectories \( x(t) \), the action functional \( S = S(x(t)) \) can be considered as a function taking the set of curves (trajectories) to the set of real numbers which can be treated as elements of \( \mathbb{R}_{\min} \). In this case the minimum of the action functional can be viewed as the Maslov integral of this function over the set of trajectories or an idempotent analog of the Euclidean version of the Feynman path integral. The minimum of the action functional corresponds to the maximum of \( e^{-S} \), i.e. idempotent integral \( \int_{\{paths\}} e^{-S(x(t))} D\{x(t)\} \) with respect to the max-plus algebra \( \mathbb{R}_{\max} \). Thus the least action principle can be considered as an idempotent version of the well-known Feynman approach to quantum mechanics. The representation of a solution to the Schrödinger equation in terms of the Feynman integral corresponds to the Lax–Olejnik solution formula for the Hamilton–Jacobi equation.

Since \( \partial S / \partial x_i = p_i \), \( \partial S / \partial t = -H(p, x) \), the following Hamilton–Jacobi equation holds:

\[
\frac{\partial S}{\partial t} + H \left( \frac{\partial S}{\partial x_i}, x_i \right) = 0.
\]

(3)

Quantization leads to the Schrödinger equation

\[
-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \hat{H} \psi = H(\hat{p}_i, \hat{x}_i) \psi,
\]

(4)
where $\psi = \psi(x,t)$ is the wave function, i.e., a time-dependent element of the Hilbert space $L^2(\mathbb{R}^N)$, and $\hat{H}$ is the energy operator obtained by substitution of the momentum operators $\hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial x_i}$ and the coordinate operators $\hat{x}_i$: $\psi \mapsto x_i \psi$ for the variables $p_i$ and $x_i$ in the Hamiltonian function, respectively. This equation is linear in the conventional sense (the quantum superposition principle). The standard procedure of limit transition from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function: $\psi(x,t) = a(x,t)e^{iS(x,t)/\hbar}$, and to keep only the leading order as $\hbar \to 0$ (the 'semiclassical' limit).

Instead of doing this, we switch to imaginary values of the Planck constant $\hbar$ by the substitution $\hbar = i\hbar$, assuming $\hbar > 0$. Thus the Schrödinger equation (4) turns to an analog of the heat equation:

$$h \frac{\partial u}{\partial t} = H \left( -h \frac{\partial}{\partial x_i}, \hat{x}_i \right) u, \quad (5)$$

where the real-valued function $u$ corresponds to the wave function $\psi$. A similar idea (the switch to imaginary time) is used in the Euclidean quantum field theory; let us remember that time and energy are dual quantities.

Linearity of equation (4) implies linearity of equation (5). Thus if $u_1$ and $u_2$ are solutions of (5), then so is their linear combination

$$u = \lambda_1 u_1 + \lambda_2 u_2. \quad (6)$$

Let $S = h \ln u$ or $u = e^{S/h}$ as in Section 2 above. It can easily be checked that equation (5) thus turns to

$$\frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^N \frac{1}{2m_i} \left( \frac{\partial S}{\partial x_i} \right)^2 + h \sum_{i=1}^n \frac{1}{2m_i} \frac{\partial^2 S}{\partial x_i^2}. \quad (7)$$

Thus we have a transition from (4) to (7) by means of the change of variables $\psi = e^{S/h}$. Note that $|\psi| = e^{\text{Re}S/\hbar}$, where $\text{Re}S$ is the real part of $S$. Now let us consider $S$ as a real variable. The equation (7) is nonlinear in the conventional sense. However, if $S_1$ and $S_2$ are its solutions, then so is the function

$$S = \lambda_1 \odot S_1 \oplus_h \lambda_2 \odot S_2$$
obtained from (6) by means of our substitution $S = h \ln u$. Here the
generalized multiplication $\circ$ coincides with the ordinary addition and
the generalized addition $\oplus_h$ is the image of the conventional addition
under the above change of variables. As $h \to 0$, we obtain the oper-
ations of the idempotent semiring $R_{\text{max}}$, i.e., $\oplus = \text{max}$ and $\circ = +$, and equation (7) turns to the Hamilton–Jacobi equation (3), since
the third term in the right-hand side of equation (7) vanishes.

Thus it is natural to consider the limit function $S = \lambda_1 \circ S_1 \oplus \lambda_2 \circ S_2$
as a solution of the Hamilton–Jacobi equation and to expect that this
equation can be treated as linear over $R_{\text{max}}$. This argument (clearly,
a heuristic one) can be extended to equations of a more general form.
For a rigorous treatment of (semiring) linearity for these equations see,
e.g., [18][23][43]. Notice that if $h$ is changed to $-h$, then we have that
the resulting Hamilton–Jacobi equation is linear over $R_{\text{min}}$.

The idempotent superposition principle indicates that there exist
important nonlinear (in the traditional sense) problems that are linear
over idempotent semirings. The idempotent linear functional analys-
(is see below) is a natural tool for investigation of those nonlinear infinite-
dimensional problems that possess this property.

6. Convolution and the Fourier–Legendre transform

Let $G$ be a group. Then the space $\mathcal{B}(G, R_{\text{max}})$ of all bounded func-
tions $G \to R_{\text{max}}$ (see above) is an idempotent semiring with respect to
the following analog $\odot$ of the usual convolution:

\[
(\varphi(x) \circ \psi)(g) == \int_G \varphi(x) \circ \psi(x^{-1} \cdot g) \, dx = \sup_{x \in G} (\varphi(x) + \psi(x^{-1} \cdot g)).
\]

Of course, it is possible to consider other “function spaces” (and other
basic semirings instead of $R_{\text{max}}$).

Let $G = \mathbb{R}^n$, where $\mathbb{R}^n$ is considered as a topological group with
respect to the vector addition. The conventional Fourier–Laplace trans-
form is defined as

\[
\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x) \, dx,
\]
where $e^{i\xi \cdot x}$ is a character of the group $G$, i.e., a solution of the following functional equation:
\[ f(x + y) = f(x)f(y). \]

The idempotent analog of this equation is
\[ f(x + y) = f(x) \circ f(y) = f(x) + f(y), \]
so “continuous idempotent characters” are linear functionals of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$. As a result, the transform in (8) assumes the form
\[ \varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_{G}^{\oplus} \xi \cdot x \odot \varphi(x) \, dx = \sup_{x \in G} (\xi \cdot x + \varphi(x)). \tag{9} \]

The transform in (9) is nothing but the Legendre transform (up to some notation) [38]; transforms of this kind establish the correspondence between the Lagrangian and the Hamiltonian formulations of classical mechanics. The Legendre transform generates an idempotent version of harmonic analysis for the space of convex functions, see, e.g., [34].

Of course, this construction can be generalized to different classes of groups and semirings. Transformations of this type convert the generalized convolution $\odot$ to the pointwise (generalized) multiplication and possess analogs of some important properties of the usual Fourier transform.

The examples discussed in this sections can be treated as fragments of an idempotent version of the representation theory, see, e.g., [28]. In particular, “idempotent” representations of groups can be examined as representations of the corresponding convolution semirings (i.e. idempotent group semirings) in semimodules.

7. Idempotent functional analysis

Many other idempotent analogs may be given, in particular, for basic constructions and theorems of functional analysis. Idempotent functional analysis is an abstract version of idempotent analysis. For the sake of simplicity take $S = \mathbf{R}_{\text{max}}$ and let $X$ be an arbitrary set.

The idempotent integration can be defined by the formula (1), see above. The functional $I(\varphi)$ is linear over $S$ and its values correspond to
limiting values of the corresponding analogs of Lebesgue (or Riemann) sums. An idempotent scalar product of functions \( \varphi \) and \( \psi \) is defined by the formula
\[
\langle \varphi, \psi \rangle = \int_X \varphi(x) \odot \psi(x) \, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)).
\]
So it is natural to construct idempotent analogs of integral operators in the form
\[
\varphi(y) \mapsto (K\varphi)(x) = \int_Y K(x, y) \odot \varphi(y) \, dy = \sup_{y \in Y} \{K(x, y) + \varphi(y)\}, \quad (10)
\]
where \( \varphi(y) \) is an element of a space of functions defined on a set \( Y \), and \( K(x, y) \) is an \( S \)-valued function on \( X \times Y \). Of course, expressions of this type are standard in optimization problems.

Recall that the definitions and constructions described above can be extended to the case of idempotent semirings which are conditionally complete in the sense of the standard order. Using the Maslov integration, one can construct various function spaces as well as idempotent versions of the theory of generalized functions (distributions). For some concrete idempotent function spaces it was proved that every ‘good’ linear operator (in the idempotent sense) can be presented in the form (10); this is an idempotent version of the kernel theorem of L. Schwartz; results of this type were proved by V. N. Kolokoltsov, P. S. Dudnikov and S. N. Samborskiï, I. Singer, M. A. Shubin and others. So every ‘good’ linear functional can be presented in the form \( \varphi \mapsto \langle \varphi, \psi \rangle \), where \( \langle \cdot, \cdot \rangle \) is an idempotent scalar product.

In the framework of idempotent functional analysis results of this type can be proved in a very general situation. In [25, 28, 30, 32] an algebraic version of the idempotent functional analysis is developed; this means that basic (topological) notions and results are simulated in purely algebraic terms (see below). The treatment covers the subject from basic concepts and results (e.g., idempotent analogs of the well-known theorems of Hahn-Banach, Riesz, and Riesz-Fisher) to idempotent analogs of A. Grothendieck’s concepts and results on topological tensor products, nuclear spaces and operators. Abstract idempotent versions of the kernel theorem are formulated. Note
that the transition from the usual theory to idempotent functional analysis may be very nontrivial; for example, there are many non-isomorphic idempotent Hilbert spaces. Important results on idempotent functional analysis (duality and separation theorems) were obtained by G. Cohen, S. Gaubert, and J.-P. Quadrat. Idempotent functional analysis has received much attention in the last years, see, e.g., [18–32] and works cited in [19]. Elements of "tropical" functional analysis are presented in [18].

8. The dequantization transform, convex geometry and the Newton polytopes

Let $X$ be a topological space. For functions $f(x)$ defined on $X$ we shall say that a certain property is valid almost everywhere (a.e.) if it is valid for all elements $x$ of an open dense subset of $X$. Suppose $X$ is $\mathbb{C}^n$ or $\mathbb{R}^n$; denote by $\mathbb{R}_+^n$ the set $x = \{(x_1, \ldots, x_n) \in X \mid x_i \geq 0 \text{ for } i = 1, 2, \ldots, n \}$. For $x = (x_1, \ldots, x_n) \in X$ we set $\exp(x) = (\exp(x_1), \ldots, \exp(x_n))$; so if $x \in \mathbb{R}^n$, then $\exp(x) \in \mathbb{R}_+^n$.

Denote by $\mathcal{F}(\mathbb{C}^n)$ the set of all functions defined and continuous on an open dense subset $U \subset \mathbb{C}^n$ such that $U \supset \mathbb{R}_+^n$. It is clear that $\mathcal{F}(\mathbb{C}^n)$ is a ring (and an algebra over $\mathbb{C}$) with respect to the usual addition and multiplications of functions.

For $f \in \mathcal{F}(\mathbb{C}^n)$ let us define the function $\hat{f}_h$ by the following formula:

$$\hat{f}_h(x) = h \log |f(\exp(x/h))|,$$

where $h$ is a (small) real positive parameter and $x \in \mathbb{R}^n$. Set

$$\hat{f}(x) = \lim_{h \to +0} \hat{f}_h(x),$$

if the right-hand side of (12) exists almost everywhere.

We shall say that the function $\hat{f}(x)$ is a dequantization of the function $f(x)$ and the map $f(x) \mapsto \hat{f}(x)$ is a dequantization transform. By construction, $\hat{f}_h(x)$ and $\hat{f}(x)$ can be treated as functions taking their values in $\mathbb{R}_{\max}$. Note that in fact $\hat{f}_h(x)$ and $\hat{f}(x)$ depend on the restriction of $f$ to $\mathbb{R}_+^n$ only; so in fact the dequantization transform is
constructed for functions defined on $\mathbb{R}^n_+$ only. It is clear that the dequantization transform is generated by the Maslov dequantization and the map $x \mapsto |x|$. Of course, similar definitions can be given for functions defined on $\mathbb{R}^n$ and $\mathbb{R}^n_+$. If $s = 1/\hbar$, then we have the following version of (11) and (12):
\[
\hat{f}(x) = \lim_{s \to \infty} (1/s) \log |f(e^{sx})|.
\]
(12')

Denote by $\partial \hat{f}$ the subdifferential of the function $\hat{f}$ at the origin.
If $f$ is a polynomial we have
\[
\partial \hat{f} = \{ v \in \mathbb{R}^n \mid (v, x) \leq \hat{f}(x) \ \forall x \in \mathbb{R}^n \}.
\]
It is well known that all the convex compact subsets in $\mathbb{R}^n$ form an idempotent semiring $\mathcal{S}$ with respect to the Minkowski operations: for $\alpha, \beta \in \mathcal{S}$ the sum $\alpha \oplus \beta$ is the convex hull of the union $\alpha \cup \beta$; the product $\alpha \odot \beta$ is defined in the following way: $\alpha \odot \beta = \{ x \mid x = a + b \}$, where $a \in \alpha, b \in \beta$, see Fig. 3. In fact $\mathcal{S}$ is an idempotent linear space over $\mathbb{R}_{\max}$.

Of course, the Newton polytopes of polynomials in $n$ variables form a subsemiring $\mathcal{N}$ in $\mathcal{S}$. If $f, g$ are polynomials, then $\partial(\hat{f}\hat{g}) = \partial\hat{f} \odot \partial\hat{g}$; moreover, if $f$ and $g$ are “in general position”, then $\partial(\hat{f} + \hat{g}) = \partial\hat{f} \oplus \partial\hat{g}$. For the semiring of all polynomials with nonnegative coefficients the dequantization transform is a homomorphism of this “traditional” semiring to the idempotent semiring $\mathcal{N}$.

**Theorem 8.1.** If $f$ is a polynomial, then the subdifferential $\partial \hat{f}$ of $\hat{f}$ at the origin coincides with the Newton polytope of $f$. For the semiring

\[
\text{Figure 3. Algebra of convex subsets.}
\]
of polynomials with nonnegative coefficients, the transform $f \mapsto \partial \hat{f}$ is a homomorphism of this semiring to the semiring of convex polytopes with respect to the Minkowski operations (see above).

Using the dequantization transform it is possible to generalize this result to a wide class of functions and convex sets, see [31].

9. Dequantization of set functions and measures on metric spaces

The following results are presented in [33].

**Example 9.1.** Let $M$ be a metric space, $S$ its arbitrary subset with a compact closure. It is well-known that a Euclidean $d$-dimensional ball $B_\rho$ of radius $\rho$ has volume

$$\text{vol}_d(B_\rho) = \frac{\Gamma(1/2)^d}{\Gamma(1 + d/2)} \rho^d,$$

where $d$ is a natural parameter. By means of this formula it is possible to define a volume of $B_\rho$ for any real $d$. Cover $S$ by a finite number of balls of radii $\rho_m$. Set

$$v_d(S) := \lim_{\rho \to 0} \inf \rho < \rho \sum_{m} \text{vol}_d(B_{\rho_m}).$$

Then there exists a number $D$ such that $v_d(S) = 0$ for $d > D$ and $v_d(S) = \infty$ for $d < D$. This number $D$ is called the Hausdorff-Besicovich dimension (or HB-dimension) of $S$, see, e.g., [35]. Note that a set of non-integral HB-dimension is called a fractal in the sense of B. Mandelbrot.

**Theorem 9.2.** Denote by $\mathcal{N}_\rho(S)$ the minimal number of balls of radius $\rho$ covering $S$. Then

$$D(S) = \lim_{\rho \to 0} \log_{\rho}(\mathcal{N}_\rho(S)^{-1}),$$

where $D(S)$ is the HB-dimension of $S$. Set $\rho = e^{-s}$, then

$$D(S) = \lim_{s \to +\infty} (1/s) \cdot \log \mathcal{N}_{exp(-s)}(S).$$
So the HB-dimension $D(S)$ can be treated as a result of a dequantization of the set function $N_\rho(S)$.

**Example 9.3.** Let $\mu$ be a set function on $M$ (e.g., a probability measure) and suppose that $\mu(B_\rho) < \infty$ for every ball $B_\rho$. Let $B_{x,\rho}$ be a ball of radius $\rho$ having the point $x \in M$ as its center. Then define $\mu_x(\rho) := \mu(B_{x,\rho})$ and let $\rho = e^{-s}$ and

$$D_{x,\mu} := \lim_{s \to +\infty} (1/s) \cdot \log(\mu_x(e^{-s})).$$

This number could be treated as a dimension of $M$ at the point $x$ with respect to the set function $\mu$. So this dimension is a result of a dequantization of the function $\mu_x(\rho)$, where $x$ is fixed. There are many dequantization procedures of this type in different mathematical areas. In particular, V.P. Maslov’s negative dimension (see [39]) can be treated similarly.

**10. Dequantization of geometry**

An idempotent version of real algebraic geometry was discovered in the report of O. Viro for the Barcelona Congress [47]. Starting from the idempotent correspondence principle O. Viro constructed a piecewise-linear geometry of polyhedra of a special kind in finite dimensional Euclidean spaces as a result of the Maslov dequantization of real algebraic geometry. He indicated important applications in real algebraic geometry (e.g., in the framework of Hilbert’s 16th problem for constructing real algebraic varieties with prescribed properties and parameters) and relations to complex algebraic geometry and amoebas in the sense of I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, see [12,48]. Then complex algebraic geometry was dequantized by G. Mikhalkin and the result turned out to be the same; this new ‘idempotent’ (or asymptotic) geometry is now often called the tropical algebraic geometry, see, e.g., [17,23,24,29,41,42].

There is a natural relation between the Maslov dequantization and amoebas. Suppose $(\mathbb{C}^*)^n$ is a complex torus, where $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ is the group of nonzero complex numbers under multiplication. For $z =$
Figure 4. Tropical line and deformations of an amoeba.

\((z_1, \ldots, z_n) \in (\mathbb{C}^*)^n\) and a positive real number \(h\) denote by \(\text{Log}_h(z) = h \log(|z|)\) the element

\((h \log |z_1|, h \log |z_2|, \ldots, h \log |z_n|) \in \mathbb{R}^n.\)

Suppose \(V \subset (\mathbb{C}^*)^n\) is a complex algebraic variety; denote by \(\mathcal{A}_h(V)\) the set \(\text{Log}_h(V)\). If \(h = 1\), then the set \(\mathcal{A}(V) = \mathcal{A}_1(V)\) is called the amoeba of \(V\); the amoeba \(\mathcal{A}(V)\) is a closed subset of \(\mathbb{R}^n\) with a non-empty complement. Note that this construction depends on our coordinate system.

For the sake of simplicity suppose \(V\) is a hypersurface in \((\mathbb{C}^*)^n\) defined by a polynomial \(f\); then there is a deformation \(h \mapsto f_h\) of this polynomial generated by the Maslov dequantization and \(f_h = f\) for \(h = 1\). Let \(V_h \subset (\mathbb{C}^*)^n\) be the zero set of \(f_h\) and set \(\mathcal{A}_h(V_h) = \text{Log}_h(V_h)\). Then there exists a tropical variety \(\text{Tro}(V)\) such that the subsets \(\mathcal{A}_h(V_h) \subset \mathbb{R}^n\) tend to \(\text{Tro}(V)\) in the Hausdorff metric as \(h \to 0\). The tropical variety \(\text{Tro}(V)\) is a result of a deformation of the amoeba \(\mathcal{A}(V)\) and the Maslov dequantization of the variety \(V\). The set \(\text{Tro}(V)\) is called the skeleton of \(\mathcal{A}(V)\).

Example 10.1. For the line \(V = \{(x, y) \in (\mathbb{C}^*)^2 \mid x + y + 1 = 0\}\) the piecewise-linear graph \(\text{Tro}(V)\) is a tropical line, see Fig.4(a). The amoeba \(\mathcal{A}(V)\) is represented in Fig.4(b), while Fig.4(c) demonstrates the corresponding deformation of the amoeba.
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