Weak coupling expansion of massless QCD with overlap Dirac operator and axial \( U(1) \) anomaly

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ABSTRACT

We discuss the weak coupling expansion of massless QCD with the Dirac operator which is derived by Neuberger based on the overlap formalism and satisfies the Ginsparg-Wilson relation. The axial \( U(1) \) anomaly associated to the chiral transformation proposed by Lüscher is calculated as an application and is shown to have the correct form of the topological charge density for perturbative backgrounds. The coefficient of the anomaly is evaluated as a winding number related to a certain five-dimensional fermion propagator.
Recently Dirac operators which may describe exactly massless fermion on a lattice are proposed by Neuberger [1] based on the overlap formalism [2, 3] and by Hasenfratz, Lalena and Niedermayer [4] based on the renormalization group method [5, 6]. These Dirac operators are very different from each other but both satisfy the Ginsparg-Wilson relation [7], which ensures that the fermion propagator anti-commutes with $\gamma_5$ at non-zero distances and thus respects chiral symmetry in that sense. Subsequently Lüscher observed [8] that the Ginsparg-Wilson relation implies an exact symmetry of the fermion action and the anomalous behavior of the fermion partition function under its flavor-singlet transformation is expressed in terms of the index of the Dirac operator arised as the Jacobian factor of the path integral measure, providing a clear understanding of the exact index theorem on a lattice in Ref. [4].

Though it was pointed out in Ref. [7] that Dirac operators satisfying the Ginsparg-Wilson relation will necessarily exhibit non-local behavior in the presence of the dynamical gauge fields, a recent perturbative analysis of chiral gauge theory in the overlap formalism [9] suggests some of them will be well controllable as a field theory on a lattice.

In this paper we consider the weak coupling expansion of massless QCD with the Dirac operator which is derived by Neuberger and satisfies the Ginsparg-Wilson relation. Then we apply our results to analyze the axial $U(1)$ anomaly following the formulation of Ref. [8] and confirm that the Jacobian factor is given in the form of the topological charge density for slowly varying perturbative gauge fields, supplementing the original calculation of the same quantity of Ref. [7].

The covariant anomaly in the vacuum overlap formalism has been discussed by Neuberger and Narayanan [11], Randjbar-Daemi and Strathdee [10] and Neuberger [11]. Note also that the axial anomaly has been calculated in the context of the domain-wall QCD by Shamir [12].

We begin with a brief review of the symmetry argument of Ref. [8]. The partition function we consider is of the form

$$Z = \int d\psi d\bar{\psi} e^{-a^2\Sigma\bar{\psi}D\psi}$$

(1)
with a lattice Dirac operator \( D \) satisfying the Ginsparg-Wilson relation

\[
D \gamma_5 + \gamma_5 D = aD \gamma_5 D. \tag{2}
\]

In Ref. [8] Lüscher pointed out that with the aid of the relation (2) the fermion action

\[
a^4 \Sigma \bar{\psi} D \psi \tag{3}
\]

is invariant under the infinitesimal transformation

\[
\psi \rightarrow \psi + i \epsilon T \delta \psi, \quad \bar{\psi} \rightarrow \bar{\psi} + i \epsilon \bar{\delta} \bar{\psi} T \]

where \( T \) is a generator of the rotation in the flavor space and

\[
\delta \psi = \gamma_5 (1 - \frac{1}{2} aD) \psi, \quad \bar{\delta} \bar{\psi} = \bar{\psi} (1 - \frac{1}{2} aD) \gamma_5. \tag{3}
\]

The path integral measure yields the Jacobian factor \(-i \epsilon a \text{tr} \{T \gamma_5 D\}\), which does not vanish only for the flavor singlet chiral rotation, accounting for the index theorem of Ref. [4].

A Dirac operator satisfying the relation (2) was proposed by Neuberger in Ref. [1] and now we review its brief derivation starting from the domain wall fermion of the Shamir type [13] based on Ref. [14], trying to clarify the physics backgrounds of the Dirac operator.

The domain wall fermion of this type is a Wilson fermion in five dimensions with the finite size \( N_5 \) of the fifth space and is described by the action

\[
S = \sum_{s,t} a^4 \sum_{m,n} \bar{\psi}(m,s) D_5(\mu)_{ms,nt} \psi(n,t), \tag{4}
\]

\[
D_5(\mu)_{ms,nt} = \sum_{\mu=1}^4 \gamma_\mu C_\mu(m,n) \delta_{s,t} + B_5(m,n) \delta_{s,t} - \frac{1}{a_5} P_L \delta_{s+1,t} - \frac{1}{a_5} P_R \delta_{s,t+1} + \mu P_L \delta_{s,N} \delta_{t,1} + \mu P_R \delta_{s,1} \delta_{t,N}, \tag{5}
\]

\[
C_\mu(m,n) = \frac{1}{2a} \left[ \delta_{m+\mu,n} U_\mu(m) - \delta_{m,n+\mu} U_\mu^\dagger(n) \right], \quad B_5(m,n) = \frac{1}{a_5} + B(m,n), \tag{6}
\]

\[
B(m,n) = \frac{M_0}{a} + \frac{r}{2a} \sum_\mu \left[ 2 \delta_{m,n} - \delta_{m+n,\mu} U_\mu(m) - \delta_{m,n+\mu} U_\mu^\dagger(n) \right], \tag{7}
\]

where \( m, n \) and \( a \) denote the four dimensional space indices and their lattice spacing while \( s, t \) and \( a_5 \) denote the indices and the lattice spacing of the fifth dimension (\( 1 \leq s, t \leq N_5 \)). In the action Eq. (4), the link variables \( U_\mu(m) \) couple only between the fields \( \bar{\psi}(m,s) \) and \( \psi(m \pm \mu,s) \) so that there is no gauge interaction in the fifth space. The terms proportional to \( 1/a_5 \) come from the kinetic and Wilson terms in the fifth direction.
and the Wilson parameter in that direction is set to be unity. We have omitted the factor $a_5$ in front of the sum $\sum_{s,t}$ so that $\psi$ has the correct dimension $3/2$ for fermions in the four dimensions. The role of the parameter $\mu$ is explained later.

The action Eq. (4) is also regarded as the action of the $N_5$ flavor Wilson fermions in four dimensions with a specific mass matrix. With a suitable choice of $M_0$ the action describes one Dirac fermion with the mass $\mu + \mathcal{O}(e^{-N_5/a})$ and $N_5 - 1$ Dirac fermions with the mass of the order of the inverse lattice spacing $[15]$. Then the following partition function

$$\int d\bar{\psi} d\psi \int d\bar{\psi}_{PV} d\psi_{PV} e^{-a^4 \Sigma \bar{\psi} D_5(0) \psi + a^4 \Sigma \bar{\psi}_{PV} D_5(1/a_5) \psi_{PV}}$$

$$= \det D_5(\mu = 0) / \det D_5(\mu = \frac{1}{a_5})$$

(8)

may describe one massless Dirac fermion regulated à la Pauli-Villars by one Dirac fermion with the mass $1/a_5$ if the $N_5 - 1$ heavy fermions cancel out. Such cancellation allows to take the limit $N_5 \to \infty$. Then taking the limit $a_5 \to 0$, one massless Dirac fermion remains.

Now we compute the determinant $\det D_5(\mu)$ and obtain the final expression of Eq. (8) as a single determinant, which allows the path integral expression like eq. (1). In the chiral basis of the $\gamma$-matrices defined by

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu^\dagger & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_\mu = (-i\sigma_1, -i\sigma_2, -i\sigma_3, 1),$$

(9)

the Dirac field is written in the chiral components as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar{\psi} = (\bar{\psi}_R, \bar{\psi}_L).$$

(10)

We also introduce the notation

$$\psi^\dagger = \bar{\psi} \gamma_4,$$

(11)

although it does not mean hermitian conjugate. Then the action can be written as

$$S = \sum_{s,t} a^4 \sum_{m,n} (\psi_L^\dagger(m,s), \psi_R^\dagger(m,s)) \left[ \begin{pmatrix} C^\dagger & B_5 \\ B_5 & -C \end{pmatrix} \delta_{st} + \begin{pmatrix} 0 & 0 \\ -1/a_5 & 0 \end{pmatrix} \delta_{t,s+1} + \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} \delta_{sN} \delta_{t1} + \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} \delta_{tN} \delta_{s1} \right] \begin{pmatrix} \psi_L(n,t) \\ \psi_R(n,t) \end{pmatrix}.$$

(12)
Thus $\gamma_4 D_5(\mu)$ takes the form
\[
\begin{pmatrix}
C^\dagger & B_5 & 0 & \cdots & \cdots & 0 & \mu \\
B_5 & -C & -1/a_5 & \cdots & \cdots & 0 & 0 \\
0 & -1/a_5 & C^\dagger & B_5 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & 0 & 0 & B_5 & -C
\end{pmatrix}, \quad (13)
\]
where $C^\dagger = \sum_\mu \sigma^\dagger_\mu C_\mu(m, n)$ and $-C = \sum_\mu \sigma_\mu C_\mu(m, n)$. Assuming the four dimensional space consists of $L^4$ sites and the color and flavor groups are $SU(N_c)$ and $SU(N_f)$, each block $C$, $B_5$ etc. above is $q \times q$ matrix with $q = 2N_C N_f L^4$. The determinant of this matrix is evaluated following the technique developed in Ref. [14, 16]. Moving the first $q$ columns to the last, $\gamma_4 D_5$ takes the from
\[
\alpha_i = \begin{pmatrix} A_{2i-1} \\ B_{2i} \\ A_{2i} \end{pmatrix}, \quad \beta_i = \begin{pmatrix} C_{2i-1} \\ B_{2i} \\ C_{2i} \end{pmatrix}. \quad (14)
\]
Decomposing the matrix $[14]$ into the product of the two matrices as
\[
\begin{pmatrix}
\alpha_1 & 0 & \cdots & 0 & \beta_1 \\
\beta_2 & \alpha_2 & 0 & \cdots & 0 \\
0 & \beta_3 & \alpha_3 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \alpha_{N_5}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & \cdots & -v_1 \\
0 & 1 & 0 & \cdots & -v_2 \\
\vdots & 0 & \cdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & \cdots & \cdots & 1 & -v_{N_5-1}
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 & 0 & \cdots & 0 & 0 \\
\beta_2 & \alpha_2 & 0 & \cdots & 0 \\
0 & \beta_3 & \alpha_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \beta_{N_5} & \alpha_{N_5}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & \cdots & -v_1 \\
0 & 1 & 0 & \cdots & -v_2 \\
\vdots & 0 & \cdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & \cdots & \cdots & 1 & -v_{N_5-1}
\end{pmatrix}, \quad v_1 = -\alpha_1^{-1} \beta_1, \quad -\beta_i v_{i-1} - \alpha_i v_i = 0, \quad (15)
\]
its determinant is given by
\[
\prod_{i=1}^{N_5} \det \alpha_i \det(1 - v_{N_5})
\] (16)
and the relations at the end of eq. (15) leads to
\[
v_{N_5} = (-\alpha_{N_5}^{-1} \beta_{N_5}) \cdots (-\alpha_1^{-1} \beta_1).
\] (17)

In the case of our $\gamma_4 D_5$,
\[
\alpha_1 = \cdots = \alpha_{N_5-1} = \begin{pmatrix} B_5 & 0 \\ -C & -1/a_5 \end{pmatrix}, \quad \alpha_{N_5} = \begin{pmatrix} B_5 & 0 \\ -C & \mu \end{pmatrix},
\]
\[
\beta_1 = \begin{pmatrix} -1/a_5 & C^\dagger \\ 0 & B_5 \end{pmatrix}, \quad \beta_2 = \cdots \beta_{N_5} = \begin{pmatrix} \mu & C^\dagger \\ 0 & B_5 \end{pmatrix}
\] (18)
and for the later convenience we introduce the transfer matrix $T$ and the Hamiltonian $H_5$ as
\[-\alpha^{-1} \beta = \gamma_4 T^{-1} \gamma_4, \quad T = e^{-a_5 H_5}, \quad T = \begin{pmatrix} B_5^{-1}/a_5 & B_5^{-1}C \\ B_5^{-1}C^\dagger/a_5 & a_5 C^\dagger B_5^{-1}C + a_5 B_5 \end{pmatrix}
\] (19)
The final expression is
\[
\det \gamma_4 D_5(\mu) = (-1)^{q(N_5-1)} (a_5)^{-qN_5} (\det B_5)^{N_5} 
\det \left\{ \begin{pmatrix} -a_5 \mu & 0 \\ 0 & 1 \end{pmatrix} - T^{-N_5} \begin{pmatrix} 1 & 0 \\ 0 & -a_5 \mu \end{pmatrix} \right\},
\] (20)
where $(-1)^{q(N_5-1)}$ arises when the first $q$ columns are moved into the last column and eq. (18) is given by
\[
\det \frac{1}{2} \{ 1 - \gamma_5 \tanh(\frac{1}{2} N_5 a_5 H_5) \}.
\] (21)
Taking the limit $N_5 \to \infty$,
\[
\tanh(\frac{1}{2} N_5 a_5 H_5) \to \varepsilon(a_5 H_5) = \frac{a_5 H_5}{\sqrt{(a_5 H_5)^2}}.
\] (22)
In the limit $a_5 \to 0$,
\[
a_5 H_5 \to H = \begin{pmatrix} B & -C \\ -C^\dagger & -B \end{pmatrix} = -\gamma_5 X, \quad X = \begin{pmatrix} B & -C \\ C^\dagger & B \end{pmatrix}.
\] (23)
where $X$ is the Wilson-Dirac operator on a lattice. Therefore in this limit, eq. (8) is (up to constant)

$$\det D, \quad D = \frac{1}{a} \left(1 - \gamma_5 \frac{H}{\sqrt{H^2}}\right)$$

which allows the path integral expression over the fermion fields as eq. (1). The Ginsparg-Wilson relation is reduced to $(H/\sqrt{H^2})^2 = 1$ for Dirac operators of the form eq. (24), which is in fact satisfied by definition [14].

Now we discuss the weak coupling expansion of the Dirac operator:

$$aD = 1 + X \frac{1}{\sqrt{X^\dagger X}},$$

$$X_{nm} = \gamma_\mu C_\mu(n,m) + B(n,m) - \frac{1}{a} M_0 \delta_{nm}. \quad (26)$$

The Wilson-Dirac operator $X$

$$X_{nm} = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} e^{ia(qn-pm)} X(q,p) \quad (27)$$

may be expanded as

$$X(q,p) = X_0(p) (2\pi)^4 \delta^4(q-p) + X_1(q,p) + X_2(q,p) + O(g^3), \quad (28)$$

where

$$X_0(p) = \frac{i}{a} \gamma_\mu \sin a p_\mu + \frac{r}{a} \sum_\mu (1 - \cos a p_\mu) - \frac{1}{a} M_0, \quad (29)$$

$$X_1(q,p) = \int \frac{d^4k_1}{(2\pi)^4} (2\pi)^4 \delta^4(q-p-k) g A_\mu(k) V_{1\mu} \left(p + \frac{k}{2}\right), \quad (30)$$

$$X_2(q,p) = \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2\pi)^4 \delta^4(q-p-\sum k_i) \times \frac{g^2}{2} A_\mu(k_1) A_\mu(k_2) V_{2\mu} \left(p + \frac{\sum k_i}{2}\right). \quad (31)$$

The vertex functions are given explicitly as

$$V_{1\mu} \left(p + \frac{k}{2}\right) = i \gamma_\mu \cos a \left(p_\mu + \frac{k_\mu}{2}\right) + r \sin a \left(p_\mu + \frac{k_\mu}{2}\right) \quad (32)$$

$$= \frac{\partial}{\partial p_\mu} X_0 \left(p + \frac{k}{2}\right),$$

$$V_{2\mu} \left(p + \frac{\sum k_i}{2}\right) = -i \gamma_\mu a \sin a \left(p_\mu + \frac{k_\mu}{2}\right) + ar \cos a \left(p_\mu + \frac{k_\mu}{2}\right). \quad (33)$$
Let us assume that \(1/\sqrt{X^\dagger X}\) can be expanded in the following form,
\[
\frac{1}{\sqrt{X^\dagger X}}(p, q) = \left( \frac{1}{\sqrt{X^\dagger X}} \right)_0 (p, q) + Y_1(p, q) + Y_2(p, q) + \cdots. \tag{34}
\]
Then it should satisfy
\[
\int_q \int_s (X^\dagger X)(p, q) \left( \frac{1}{\sqrt{X^\dagger X}} \right)(q, s) \left( \frac{1}{\sqrt{X^\dagger X}} \right)(s, t) = \delta(p - t). \tag{35}
\]
We have made use of the abbreviations as \(\int \frac{\delta(k)}{(2\pi)^3} = f_k\) and \(\delta(q - p) = (2\pi)^4 \delta^4(q - p)\). In the first non-trivial order, the above equation reads
\[
\int_s \left( \sqrt{X^\dagger X} \right)_0 (p, s) Y_1(s, t) + \int_q \int_s \left( X^\dagger X \right)_0 (p, q) Y_1(q, s) \left( \frac{1}{\sqrt{X^\dagger X}} \right)_0 (s, t) + \int_q \left( X^\dagger X \right)_1 (p, q) \left( \frac{1}{X^\dagger X} \right)_0 (q, t) = 0. \tag{36}
\]
Since \(\left( \sqrt{X^\dagger X} \right)_0 (p, q)\) is completely diagonal with respect to the momentum, spinor and color indices,
\[
\left( \sqrt{X^\dagger X} \right)_0 (p, q) = \omega(p) \delta(p - q), \tag{37}
\]
where
\[
a \omega(p) = \sqrt{\sin^2 a p_\mu + \left( \sum_\mu (1 - \cos a p_\mu) - M_0 \right)^2} > 0, \tag{38}
\]
Eq. (36) is reduced to the relation between the elements \(Y_1(p, t)\) and \((X^\dagger X)_1(p, t)\), leading to the solution [7],
\[
Y_1(p, t) = -\frac{1}{\omega(p) \omega(t)} \left\{ \frac{1}{\omega(p) + \omega(t)} \right\} (X^\dagger X)_1(p, t). \tag{39}
\]
Similar procedure yields the expansion of the Dirac operator
\[
D(p, q) = D_0(p) \delta_{pq} + V(p, q) \tag{40}
\]
\[
V(p, q) = \left\{ \frac{1}{\omega(p) + \omega(q)} \right\} \left[ X_1(p, q) - \frac{X_0(p)}{\omega(p)} X_1^\dagger(p, q) \frac{X_0(q)}{\omega(q)} \right] + \left\{ \frac{1}{\omega(p) + \omega(q)} \right\} \left[ X_2(p, q) - \frac{X_0(p)}{\omega(p)} X_2^\dagger(p, q) \frac{X_0(q)}{\omega(q)} \right] + \left\{ \frac{1}{\omega(p) + \omega(q)} \right\} \left\{ \frac{1}{\omega(p) + \omega(t)} \right\} \left\{ \frac{1}{\omega(t) + \omega(q)} \right\} \times 
\]
\[
\left[ -X_0(p) X_1^\dagger(p, t) X_1(t, q) - X_1(p, t) X_0^\dagger(t, q) X_0(q) - X_1(p, t) X_0^\dagger(t, q) X_0(q) X_0(q) + \frac{\omega(p) + \omega(t) + \omega(q)}{\omega(p) \omega(t) \omega(q)} X_0(p) X_1^\dagger(p, t) X_0(t) X_1^\dagger(t, q) X_0(q) \right] + \cdots. \tag{41}
\]
This expansion may be obtained through the integral representation of the inverse square root of $X^\dagger X$:

$$\frac{1}{\sqrt{X^\dagger X}} = \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X^\dagger X}.$$  \hspace{1cm} (42)

Up to the second order, we have

$$\frac{1}{\sqrt{X^\dagger X}} = \int_{-\infty}^{\infty} \frac{dt}{\pi} \frac{1}{t^2 + X^\dagger X_0} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_1 + X_1^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0}$$

$$- \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_2 + X_2^\dagger X_0 + X_1^\dagger X_1 \right) \frac{1}{t^2 + X_0^\dagger X_0}$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_1 + X_1^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_1 + X_1^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0}.$$  \hspace{1cm} (43)

Noting that $(t^2 + X_0^\dagger X_0)^{-1}$ is diagonal in spinor space and it commutes with $X_0$, we obtain

$$aD = aD_0$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2 + X_0^\dagger X_0} \left( t^2 X_1 - X_0 X_1^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0}$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2 + X_0^\dagger X_0} \left( t^2 X_2 - X_0 X_2^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0}$$

$$- \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2 + X_0^\dagger X_0} \left( X_1 \right) \frac{1}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_1 + X_1^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0}$$

$$- \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2 + X_0^\dagger X_0} \left( X_0 X_1^\dagger \right) \frac{1}{t^2 + X_0^\dagger X_0} \left( X_1 \right) \frac{1}{t^2 + X_0^\dagger X_0}$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{X_0} \frac{1}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0} \left( X_0^\dagger X_0 \right) \frac{1}{t^2 + X_0^\dagger X_0} + \cdots.$$  \hspace{1cm} (44)

Going to the momentum space, the integration over the parameter $t$ can be performed explicitly and we obtain Eq. (40).

The tree level propagator is given by

$$D_0^{-1}(p) = \frac{-i \sum_{\mu} \gamma_\mu \sin ap_\mu}{2 \left( \omega(p) + b(p) \right)} + \frac{a}{2},$$  \hspace{1cm} (45)

$$b(p) = \frac{r}{a} \sum_{\mu} \left( 1 - \cos ap_\mu \right) - \frac{1}{a} M_0.$$  \hspace{1cm} (46)
Since $\omega(p)$ is positive definite, the pole of the propagator occurs when $\omega(p) + b(p) = 0$, which is fulfilled only if $\sin^2 a p_\mu = 0$ and $b(p) < 0$. Therefore for $-2r < M_0 < 0$ the propagator exhibits a massless pole only when $p_\mu = 0$.

Now using the expansion Eq. (H1) we compute $-at\epsilon\gamma_5 D$. (Here the local transformation is treated so that $\epsilon$ is space-dependent.) The non-vanishing contribution comes only from the term containing $X_0(p)X_1^\dagger(p,t)X_0(t)X_1^\dagger(t,q)X_0(q)$. We obtain

$$- at\epsilon\gamma_5 D = a^4 \int_{pq} \epsilon_m e^{i(p-q)\mu} g^2 \text{tr}\{A_\mu(p)A_\nu(-q)\} f_{\mu\nu}(p,q),$$

(47)

$$f_{\mu\nu}(p,q) = \int_k \Omega(p+k,k,k+q) \times \text{tr} \gamma_5 X_0(k+p)\partial_\mu X_0^\dagger(k + p/2)X_0(k)\partial_\nu X_0^\dagger(k + q/2)X_0(k+q),$$

(48)

$$\Omega(p,k,q) = \left\{ \frac{1}{\omega(p) + \omega(q)} \right\} \left\{ \frac{1}{\omega(p) + \omega(k)} \right\} \left\{ \frac{1}{\omega(k) + \omega(q)} \right\} \left\{ \frac{\omega(p) + \omega(k) + \omega(q)}{\omega(p)\omega(k)\omega(q)} \right\}. $$

(49)

Then using the propagator $S(p,p_5)$ defined in five dimensional momentum space $(p_\mu,p_5) \in T^4 \times R$,

$$S(p,p_5)^{-1} = X_0(p) + i \gamma_5 p_5$$

(50)

$f_{\mu\nu}(p,q)$ is rewritten as

$$f_{\mu\nu}(p,q) = 2i \int_{(k,k_5) \in T^4 \times R} \text{tr}\{S(k+p,k_5)\partial_\mu S^{-1}(k+p/2,k_5) \}

S(k,k_5)\partial_\nu S^{-1}(k + q/2,k_5)S(k + q,k_5)\partial_\mu S^{-1}(k,k_5)\}. $$

(51)

Now it is easy to verify the structure $f_{\mu\nu}(p,q) = p_\mu q_\tau \partial_\rho \partial_\tau f_{\mu\nu}(p,q)|_{p,q=0} + O(a)$, and $\partial_\rho \partial_\tau f_{\mu\nu}(p,q)|_{p,q=0} \propto \epsilon_{\rho\mu\tau\nu}$, leading to the anomaly $g^2 c(M_0,r)\epsilon_{\rho\mu\tau\nu}F_{\rho\mu}F_{\tau\nu}$ when inserted in eq. (47) with the coefficient

$$c(M_0,r) = \frac{2i}{2^{25}} \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5} \int_{T^4 \times R} \text{tr}\{L_{\mu_1}L_{\mu_2}L_{\mu_3}L_{\mu_4}L_{\mu_5}\},$$

(52)

$L_\mu = S\partial_\mu S^{-1}$.

This expression shows that $c(M_0,r)$ is invariant by a momentum-dependent, continuous change of the scale of the propagator $S$: $S(p) \to \Omega(p)S(p)$ as long as $\Omega(p) \neq 0$ in
$T^4 \times R$. It allows to replace $S$ and $S^{-1}$ in eq. (52) with $V$ and $V^{-1}$ where

$$V(p, p_5) = N(p, p_5) S(p, p_5), \quad N(p, p_5) = \sqrt{p_5^2 + \omega(p)^2}. \quad (53)$$

$V$ is a mapping from $T^4 \times R$ to $S^5$ and $n_5$ monotonously increases from $-1$ to $1$ as $p_5$ increases from $-\infty$ to $\infty$. This allows one to interpret that $c(M_0, r)$ is the winding number of the mapping, $V'$, from $T^4$ to $S^4 \subset S^5$ derived from $V$ by fixing $p_5$ and it is an integer. For $M_0 > 0$, $n_0(\propto b)$ is positive definite on $T^4 \times R$ and the image of $T^4$ by $V'$ does not cover $S^4 \subset S^5$, leading to the value $c(M_0, r) = 0$. Further studies along this line leads to the value $c(M_0, r) = 1/16\pi^2$ for $-2r < M_0 < 0$ [19]. (Note that our convension for the gamma matrices is as $\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 = -1$.)

This value is also obtained by evaluating the difference $\Delta c = c(\epsilon, r) - c(-\epsilon, r)$ in the limit $\epsilon \to 0$ [19]. A continuous and momentum dependent deformation of $M_0$ does not change $c(M_0, r)$ as long as $N \neq 0$ in $T^4 \times R$, stemming from the fact of $c(M_0, r)$ being the winding number. Here the surfaces $M_0 = \pm \epsilon$ are deformed to $m_\pm(p)$ defined on $T^4$, where $m_\pm(p) = 0$ for $\sum_{\mu=1}^4 p_\mu^2 \gtrsim \delta^2$ and $m_\pm(p) = \pm \epsilon$ for $\sum_{\mu=1}^4 p_\mu^2 \lesssim \delta^2$. Then the difference $\Delta c$ arises from the integration in the vicinity of the center of the Brillouin zone in eq. (52). For sufficiently small $\delta$ (still larger than $\epsilon$), $S$ and $S^{-1}$ may be replaced by their continuum expressions, leading to

$$\Delta c = 2 \int_{|p|<\delta} \frac{2\epsilon}{N^3} \to -\frac{1}{16\pi^2} \quad (54)$$

by taking the limit $\epsilon \to 0$ for a finite small $\delta$.

Thus we obtain the final result of the axial $U(1)$ anomaly as

$$- \text{art} \left\{ \varepsilon \gamma_5 D \right\} = \frac{g^2}{32\pi^2} N_f \int d^4 x \varepsilon(x) \varepsilon_{\rho\mu\tau\nu} F_{\rho\mu}(x) F_{\tau\nu}(x). \quad (55)$$

The same analysis can be done in two dimensions where the anomaly comes from the second term in eq. (41). The result is

$$- \text{art} \left\{ \varepsilon \gamma_5 D \right\} = \frac{g}{2\pi} N_f \int d^2 x \varepsilon(x) \varepsilon_{\mu\nu} F_{\mu\nu}(x). \quad (56)$$

We have discussed the weak coupling expansion of the lattice QCD with a Dirac operator satisfying the Ginsparg-Wilson relation using the expression of Ref. [1]. We
confirmed that the anomalous behavior of the fermion partition function under the axial $U(1)$ transformation of Ref. [8] is expressed in the form of the topological charge density for slowly varying perturbative gauge fields, which supplements the earlier calculation in Ref. [7].

Narayanan, Vranas and Singleton Jr. have studied numerically the relation between the index of the Dirac operator $\text{Tr} \left( \frac{H}{\sqrt{H^2}} \right)$ and the topological charge of lattice gauge field [20]. An interesting possibility for studying this type of Dirac operator in numerical simulation was considered recently by Chiu [21].

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The following relation may be also useful;

\[ \frac{1}{\sqrt{A+B}} = \frac{1}{\Gamma(1/2)} \int_0^{\infty} dt t^{-1/2} e^{-tA} e^{tA} e^{-t(A+B)}. \]  

One of us (AY) would like to thank Tomohiko Takahashi for the discussion on this point.

For example, when we sweep the $T^4$ with the hypersurface of constant $b$ starting from the center of the Brillouin zone, the region within the first coner of the Brillouin zone covers $S^4 \subset S^5$ completely by the mapping $V'$ and the image of the remaining region covers and uncovers certain region in the upper half side of $S^4$ centered by the north pole in a oscillatory way as its inverse image goes away from the center of the Brillouin zone, and finially converges to the north pole.
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