Another proof of the positive energy theorem in gravity

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We show that gravitational energy expression simplifies when a new set of coordinates that satisfies a certain asymptotic gauge condition is used. Compared to the ADM formula, positivity of the energy is more transparent in this new construction. Our proof relies on the asymptotic symmetries (specifically supertranslations) of the asymptotically flat spacetimes at spatial infinity in four dimensions.

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I. INTRODUCTION

The positivity of the total energy of an asymptotically flat spacetime is of extreme importance for the stability of the ground state (namely the flat Minkowski spacetime). It is well known that unlike other non-gravitational classical theories, having a positive energy density everywhere does not automatically guarantee a total positive energy of the spacetime in gravity. In fact, there are examples of negative energy spacetimes which are not asymptotically flat: These include the so called AdS (anti-de-Sitter) soliton, Eguchi-Hanson soliton and the AdS-Taub-NUT solitons. In asymptotically flat spacetimes, the negative gravitational potential energy is not expected to dominate the initial “rest” mass, e.g. crudely speaking, the negative gravitational potential energy of the Sun is about a million times smaller than its rest mass.

The positivity of the gravitational energy for asymptotically flat spacetimes in four dimensions was shown satisfactorily first by Schoen and Yau and later by Witten. Schoen and Yau use complicated geometrical tools whereas Witten employs spinors and is inspired by supergravity for which the Hamiltonian is the square of a supercharge. We should note that Parker and Taubes strengthened Witten’s proof with more rigorous mathematical arguments, whereas Nester corrected a technical error in Witten’s calculation. In all of these works, the total energy that one starts with is the celebrated Arnowitt-Deser-Misner (ADM) energy:

$$M_{ADM} = \frac{1}{16\pi} \int_{\Sigma} dS_i (\partial_j h^{ij} - \partial^i \tilde{h})$$,

where $\tilde{h} = h^i_j$. The ADM formula is written in flat Cartesian coordinates, yet the final result, after the integration is carried out at spatial infinity, is gauge (or diffeomorphism) invariant. [This statement, in fact, needs a major refinement: We have implicitly assumed a ‘proper’ asymptotic behavior (or ‘appropriate’ fall-off conditions on $h_{ij}$) at spatial infinity such that the integral is well-defined. If this is not the case, then, even the Minkowski spacetime can have an arbitrary (positive, negative or even divergent) mass in some coordinate system. We show this in the appendix A which somewhat generalizes the earlier work of 12, 13, 14, 15, 16.] However, we stress that the integrand is not gauge invariant. In these coordinates, the two terms in the integrand of (1) can be not only of any sign but also compete with each other, and therefore one cannot see the positivity

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of energy by simply looking at (1); hence the complicated proofs mentioned above. We should note that the proofs of Schoen-Yau and Witten which employ the coordinate dependent integrand in the coordinate independent ADM formula (1) work only in four dimensions and, to our knowledge, no satisfactory generalizations have been given in generic dimensions.

The present work started with the hope of simplifying the proof of the positive gravitational energy theorem and extending it beyond four dimensions. As will be seen shortly, even though simplification can be achieved by expressing the energy in a different gauge, a proof for generic $D$ dimensions is still elusive.

**II. CHOICE OF GAUGE**

We will start with a $D$-dimensional background diffeomorphism invariant expression for the integrand of the total energy and show that, as far as the positivity of the energy is concerned, the ADM expression is not a very convenient choice of gauge. In this gauge, one had to work hard to show that the negative energy contributions in (1) are dominated by the positive ones at the end. Analogous things happen in gauge theories: e.g. in QED, when one uses the Lorenz gauge, one has to deal with propagating ghosts (namely negative norm states) and has to do extra work (see e.g. the Gupta-Bleuler formalism [17, 18]) to decouple these unphysical states. On the other hand, when one uses the Coulomb gauge, one works with only the physical states and the process is devoid of such complications. Guided by this lesson from Quantum Field Theory, we will look for an alternative to the Cartesian gauge of the ADM expression which will surpass the difficulties faced in the proof of the positive gravitational energy theorem. From the onset, it is hardly clear as to what gauge there is to choose. However, we are lucky that we already have the explicitly gauge invariant total energy formula of [19] which reduces to the ADM formula in the relevant limit [See the discussion below for the details.]. We will show that with a proper choice of gauge (the coordinates), one ends up with a more transparent expression from which the positivity of the total energy for the 4-dimensional asymptotically flat spacetimes should follow relatively easily.

Let us start by giving a brief outline of how gravitational charges are defined. (We refer the reader to [19, 20] for details. Even though the discussion in these works is given for asymptotically AdS spacetimes, the formalism can safely be used for asymptotically flat spacetimes as well.) [For other constructions of charges in AdS spaces, see [21].] As usual, assume that the deviation, $h_{\mu\nu}$, of the actual $D$-dimensional static spacetime metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ from an asymptotically flat static metric (or the background) $\bar{g}_{\mu\nu}$ vanishes sufficiently rapidly at infinity. [We would like to stress that for our purposes $\bar{g}_{\mu\nu}$ is flat in all senses but not necessarily equivalent to $\eta_{\mu\nu}$.] One can then find a $D$-dimensional conserved and background gauge invariant charge (corresponding to each background Killing vector $\bar{\xi}^\mu$) which reads

$$Q^\mu(\bar{\xi}) = \frac{1}{4\Omega_{D-2}} \oint_{\Sigma} dS_i \sqrt{-\bar{g}} q^{\mu i}(\bar{\xi}),$$

(2)

where

$$q^{\mu i}(\bar{\xi}) \equiv \bar{\xi}_\nu \nabla^\mu h^{i\nu} - \bar{\xi}_\nu \nabla^i h^{\mu\nu} + \bar{\xi}^i \nabla_\nu h^{\mu\nu} - \bar{\xi}_i \nabla^\mu \bar{\xi}_\nu h^{i\nu} + h^{i\mu} \nabla^i \bar{\xi}_\nu - h^{i\nu} \nabla_\nu \bar{\xi}_i + \bar{\xi}^\mu \nabla^i h - \bar{\xi}^i \nabla^\mu h + h \nabla^i \bar{\xi}_i.$$  

(3)

This integral is to be evaluated at the $(D-2)$-dimensional boundary of a $(D-1)$-dimensional spatial hypersurface. The explicit gauge invariance of (2) under time-independent gauge transformations, which is crucial for our arguments, is shown in [19] [see the appendix A for a subtle issue about ‘large’ gauge transformations]. For a timelike Killing vector $\bar{\xi}^\mu = (-1, 0)$, $Q^0$ defines the total gravitational energy, which further reduces to the ADM formula in flat Cartesian coordinates. For
other types of Killing vectors, one obtains different gravitational charges such as the total angular momentum.

Since the background is assumed to be asymptotically flat and static, one can always choose the spatial boundary to be orthogonal to the time direction and set $\bar{g}_{0i} = 0$. We take the timelike Killing vector $\bar{\xi}^\mu = (-1, 0)$ and use it in $\bar{\xi}^\mu$ by making use of the identities

$$\bar{\nabla}_j \bar{\xi}_0 = -\frac{1}{2} \partial_j \bar{g}_{00}, \quad \bar{\nabla}_j \bar{\xi}_k = 0,$$

(4)

to obtain

$$E \equiv Q_0 = \frac{1}{4\Omega_{D-2}} \oint_{\Sigma} dS_1 \sqrt{-\bar{g}} \left( \bar{\nabla}_j (\bar{h}^{ij} - \bar{g}^{ij} \bar{h}) - \frac{1}{2} (\bar{h}^{ij} - \bar{g}^{ij} \bar{h}) \bar{g}^{00} \partial_j \bar{g}_{00} \right).$$

(5)

In the ADM gauge, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the second term in (5) drops and the covariant derivative in the first one turns into an ordinary derivative yielding (1). However, at this stage we are free to choose other gauges, keeping, of course, $\bar{g}_{0i} = 0$. Here we use this to make the previously advertised gauge choice. Rather than being global, ours will only be an asymptotical one:

$$\bar{\nabla}_j (h^{ij} + \bar{g}^{ij} \bar{h}) \sim O(1/r^{D-2+\epsilon})$$

(6)

for some $\epsilon > 0$ at the boundary $r \to \infty$. Note that the sign between the two terms is plus! Even though, $\bar{\nabla}_j (h^{ij} + \bar{g}^{ij} \bar{h}) = 0$ holds exactly for the Schwarzschild solution presented in the Kerr-Schild coordinates as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{2m}{r} (dt - dr)^2,$$

in general we only need (6) in what follows.

In fact, we were led to the gauge choice (6) by the observation that the 4-dimensional Schwarzschild solution in its standard spherically symmetric form satisfies this gauge at the boundary:

$$\bar{\nabla}_j (h^{ij} + \bar{g}^{ij} \bar{h}) = \left( - \frac{8m^2}{r(r - 2m)^2}, 0, 0 \right) \sim O(1/r^3).$$

As for the Kerr solution written in the Boyer-Lindquist coordinates as

$$ds^2 = -(1 - f(r, \theta)) dt^2 - 2a f(r, \theta) \sin^2 \theta dt d\varphi + \frac{n(r, \theta) dr^2}{h(r, \theta) (1 - f(r, \theta))} + \frac{n(r, \theta) d\theta^2}{1 - f(r, \theta)} + \left( \frac{h(r, \theta) - a^2 \sin^2 \theta f^2(r, \theta)}{1 - f(r, \theta)} \right) \sin^2 \theta d\varphi^2,$$

where

$$f(r, \theta) = \frac{2mr}{r^2 + a^2 \cos^2 \theta}, \quad n(r, \theta) = r^2 - 2mr + a^2 \cos^2 \theta, \quad h(r, \theta) = r^2 - 2mr + a^2,$$

the background $\bar{g}_{\mu\nu}$ is found by taking $m = 0$ and $a = 0$, and one finds after a rather long calculation...
that

\[ \bar{\nabla}_j (h^{ij} + \bar{g}^{ij} \bar{h}) = - (66a^{10} - 248a^8 mr + 304a^8 r^2 + 200a^6 m^2 r^2 - 872a^6 mr^3 + 64a^4 m^3 r^3 + 558a^6 r^4 + 192a^4 m^2 r^4 - 1232a^4 mr^5 + 512a^2 m^3 r^5 + 472a^4 r^6 + 320a^2 m^2 r^6 - 704a^2 mr^7 + 112a^2 r^8 + 256m^2 r^8 + a^2 [93a^8 - 4a^6(93m - 95r)r + 64a^2 r^4(22m^2 - 23mr + 4r^2) + a^4 r^2 (372m^2 - 1396mr + 523r^2) + 16r^5 (-32m^3 + 60m^2 r - 20mr^2 + 5r^3)] \cos 2\theta + 2a^4 [15a^6 + a^2 r^2 (92m^2 - 140mr + 33r^2) + a^4 (-68mr + 40r^2) + 4r^3 (-8m^3 + 24mr^2 + 5r^3)] \cos 4\theta + [3a^{10} - 12a^8 mr + 4a^6 r^2 + 12a^6 m^2 r^2 - 12a^6 mr^3 + 5a^6 r^4] \cos 2\theta] / (8r^3 (a^2 + r(-2m + r))^2 (a^2 + 2r^2 + a^2 \cos 2\theta)^2), \]

\[ -a^2 \sin 2\theta \left( 15a^6 - 54a^4 mr + 61a^4 r^2 + 48a^2 m^2 r^2 - 120a^2 mr^3 + 96a^2 r^4 + 32m^2 r^4 - 96mr^5 + 56r^6 + a^4 (5a^2 - 10mr + 7r^2) \cos 4\theta + 4a^2 [5a^4 + a^2 r(-8m + 17r) + 2r^2 (-2m^2 - 9mr + 7r^2)] \cos 2\theta) / (4r^3 (a^2 + r(-2m + r))^2 (a^2 + 2r^2 + a^2 \cos 2\theta)^2) \right) ) / O(1/r^3), \]

which shows that the gauge choice (9) is again satisfied.

Unfortunately though, our gauge is not satisfied by the usual \( D > 4 \) dimensional Kerr and Schwarzschild solutions: For example, in \( D = 5 \) instead of the required \( 1/r^3 + \epsilon \) behavior, these geometries have \( 1/r^3 \) decays which renders our asymptotic gauge condition (6) inconvenient. Thus we will stick to four dimensions from now on.

In addition to (5), one can also choose \( \partial_j \bar{g}_{00} = 0 \) for asymptotically flat spacetimes [24]. Then the total energy expression (5) reduces to

\[ E = -\frac{1}{8\pi} \int_{\Sigma} dS_t \sqrt{-\bar{g}} \partial^t \bar{h}, \]

where this integral is to be evaluated on a large sphere \( S^2 \) at the spatial boundary yielding

\[ E = -\frac{1}{8\pi} \int_{\Sigma} dS_r \sqrt{-\bar{g}} \partial^r \bar{h}. \]

To better appreciate (8) instead of the full fletched gauge invariant (3), one can also directly compute the gauge-fixed 00 component of the linearized Einstein tensor \( G_{00}^L \), from which the total energy follows

\[ \int d^3 x \ T^{00} \bar{\xi}_0 \sqrt{-\bar{g}} \sim \int d^3 x \ G_{00}^L \bar{\xi}_0 \sqrt{-\bar{g}}. \]

The computation of \( G_{00}^L \) is not so hard. We start by noting that the assumptions made on the background so far yield a nontrivial \( \bar{\Gamma}^i_{jk} \) and \( \bar{\Gamma}^\alpha_{\alpha j} = \partial_j (\ln \sqrt{|\bar{g}|}) \), where all remaining background Christoffel symbol components vanish. For the linearized Ricci tensor one finds

\[ R_{00}^L = \frac{1}{2} (-\Box h_{00} - \bar{\nabla}_0 \bar{\nabla}_0 h + 2\bar{\nabla}_\sigma \bar{\nabla}_0 h_{\sigma 0}), \]

which upon explicit calculation simplifies to

\[ R_{00}^L = -\frac{1}{2} \bar{g}^{ij} \bar{\nabla}_i \bar{\nabla}_j h_{00}. \]
Following similar steps, one obtains
\[ R^L = -\Box h + \nabla_\alpha \nabla_\beta h^{\alpha \beta}, \]
where
\[ \nabla_\alpha \nabla_\beta h^{\alpha \beta} = -\frac{1}{\sqrt{-g}} \partial_\ell (\sqrt{-g} \nabla_j h^{ij}) \quad \text{and} \quad \Box h = \tilde{g}^{ij} \nabla_i \nabla_j h. \]

Thus
\[ G^L_{00} = R^L_{00} - \frac{1}{2} g_{00} R^L = \frac{1}{2} g_{00} \left( \tilde{g}^{ij} \nabla_i \nabla_j h + \frac{1}{\sqrt{-g}} \partial_\ell (\sqrt{-g} \nabla_j h^{ij}) \right), \]
and employing our gauge choice (6) yields the suggestive form (25)
\[ G^L_{00} = g_{00} \frac{1}{\sqrt{-g}} \partial_\ell (\sqrt{-g} \partial^i \tilde{h}) \sim T_{00}. \]

Note that, unlike the ADM integrand which is made up of two competing terms with opposite signs, (7) is either positive or negative definite.

By definition, \( \tilde{h} = \tilde{g}^{ij} h_{ij} \) is the trace of a 3 \times 3 positive definite matrix and thus is positive definite. This is so at least in the region where the original full metric \( g_{\mu \nu} \) does not change its signature. For example, for a black hole \( \tilde{h} \) could become negative definite inside the event horizon, however we are only interested in the behavior of \( \tilde{h} \) near the spatial boundary. [See the discussion below as to what goes wrong when \( \tilde{h} \) is negative definite everywhere.] From the very assumption about the convergence properties of the deviation part of the metric, it follows that \( \tilde{h} \rightarrow 0 \) as \( r \rightarrow \infty \). Moreover, being a positive definite quantity, it necessarily follows that its derivative at the spatial boundary is negative which renders (7) positive; i.e. \( \partial^i \tilde{h} < 0 \) and \( E > 0 \). Thus the sign of \( \tilde{h} \) at the spatial boundary determines the sign of the total energy. The crucial point in our argument is that, unlike the ADM case, our integrand does not change sign.

So what is wrong if \( \tilde{h} \) is negative definite (in which case we find \( E < 0 \))? Recall that even for the Schwarzschild solution \( \tilde{h} \rightarrow -\tilde{h} \) also satisfies the (linear) Einstein equations. However, one sign is ruled out by the weak energy condition or by the assumption of the non-existence of naked singularities (see Witten [6]). This is also what happens in our case: If one choice of sign for \( \tilde{h} \) leads to a negative energy, then this points out to the existence of a naked singularity and the sign of \( \tilde{h} \) is to be reversed.

Going back to our examples, for the Schwarzschild solution one easily finds \( E = m \), when \( \tilde{h} = 2m/(r-2m) \) is plugged into (8). As for the Kerr solution things are a little more complicated since
\[ \tilde{h} = \frac{a^2 \cos^2 \theta}{r^2} + \frac{2mr - a^2 \sin^2 \theta}{a^2 + r(r-2m)} + \frac{a^2(a^2 + 2mr + 2r^2 + (a^2 - 2mr) \cos 2\theta)}{r^2(a^2 + 2r^2 + a^2 \cos 2\theta)}. \]

However, substituting this into (8) one correctly finds \( E = m \) again.

The fact that the masses of the Kerr and Schwarzschild black holes can be obtained using (8) instead of the ADM expression is quite remarkable. Moreover, the mass of any spacetime which asymptotically approaches either one of these two examples can be given by (8) and will necessarily be positive as well. For more general spacetimes, we would now like to prove the existence of a “gauge potential” \( \zeta_\mu \) at the spatial boundary \( r \rightarrow \infty \) such that the new deviation part of the metric obtained by \( h_{\mu \nu} \rightarrow h_{\mu \nu} + \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu \) satisfies the asymptotic gauge condition (6) in four dimensions, for which the energy becomes (8). It is well established that for asymptotically flat spacetimes the deviation part \( h_{ij} \) should decay like \( h_{ij} = O(1/r^\alpha) \) where \( 1/2 < \alpha \), otherwise the
energy definition becomes meaningless \([12, 13, 14, 15]\). So according to this, our gauge choice (in flat Cartesian coordinates)
\[
\partial_i h^{ij} + \partial_j h \sim O(1/r^{1+\alpha}),
\]
does not satisfy the \(O(1/r^{2+\epsilon})\) requirement. However, one can still employ a gauge transformation to take care of this. Under a gauge transformation, the left hand side of (6) (in flat Cartesian coordinates) implies that the gauge potential satisfies
\[
\partial^2 \tilde{\zeta}^i + 3 \partial^i \partial^j \tilde{\zeta}_j \sim O(1/r^{1+\alpha}),
\]
which requires that \(\tilde{\zeta}^i \sim O(r^{1-\alpha})\). Such a gauge potential \(\tilde{\zeta}^i\) always exists: In fact, suppose the right hand side has the weakest decay given by the special form \(cx^i/r^{2+\alpha}\). Then
\[
\tilde{\zeta}^i = x^i\left(c_1 - \frac{c_2}{r^2} + \frac{c}{4\alpha(\alpha - 3)r^\alpha}\right),
\]
with \(c_1\) and \(c_2\) arbitrary real constants. Choosing \(c_1 = 0\), we have the required fall-off conditions. This is exactly what one needs to have an invariant mass definition under coordinate transformations at the asymptotically flat end of the spacetime. Such transformations are called ‘supertranslations’ \([22]\) and their required behavior was examined long time ago by the ‘asymptotic symmetries theorem’ \([15]\). This completes our proof that one can always choose (6) without changing the mass.

In conclusion, starting with a diffeomorphism invariant integrand for the energy and choosing a more suitable gauge, we have given an alternative yet simpler total energy expression (8) for the 4-dimensional asymptotically flat static spacetimes, from which positivity of gravitational energy follows.

APPENDIX A: THE MASS OF THE MINKOWSKI SPACE

Here we explain in detail how ‘large’ coordinate transformations can give mass to even a flat spacetime. We will write the standard Minkowski metric in two different coordinate systems and compare one’s mass relative to the other one.

Consider the usual four dimensional Minkowski spacetime metric written in spherical coordinates \(ds^2 = -dt^2 + dr^2 + r^2d\Omega^2\). Now make a coordinate transformation taking \(t = f(\tau, \rho)\) and \(r = h(\tau, \rho)\). [A more restricted version of this was given beforehand in \([12, 13]\).] The metric becomes
\[
ds^2 = (\dot{h}^2 - \dot{f}^2) d\tau^2 + 2(\dot{h}h' - f \dot{f}') d\tau d\rho + (h^2 - f'^2) d\rho^2 + h^2 d\Omega^2, \tag{A1}
\]
where prime and dot denote differentiation with respect to \(\rho\) and \(\tau\), respectively. Needless to say, this is a flat metric with all the Riemann tensor components vanishing. However does this metric have zero ADM mass? The conventional version of the positive energy theorem assigns zero mass uniquely to the flat Minkowski space. Here we will see that (A1) can have any energy. Setting \(f = \tau\) and \(h = \rho\) as the background and calculating the mass using (11) or (3), one finds
\[
M = \frac{1}{2\rho}[-\rho^2 f'^2 + (h - \rho h')^2]. \tag{A2}
\]
In principle, depending on the choice of \(f\) and \(h\), \(M\) can be any real number. For example, when \(f = \tau\), choosing \(h = \rho H(\tau) + \kappa\sqrt{\rho}\), with \(H\) an arbitrary function of \(\tau\) and \(\kappa\) any real number, yields \(M = \kappa^2/8\). In fact, if we choose a more general \(h\) as \(h = \rho + \kappa \rho^{1-s}\), \(M\) will be divergent if \(s < 1/2\) and will vanish for \(s > 1/2\). Likewise, when \(h = \rho\), choosing \(f = F(\tau) + \kappa\sqrt{\rho}\), with \(F\) an arbitrary function of \(\tau\), one finds a negative \(M = -\kappa^2/8\). In fact, one can examine the effect of such ‘large’ gauge transformations on the mass of any physically relevant solutions \([23]\).
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[1] G.T. Horowitz and R.C. Myers, Phys. Rev. D 59, 026005 (1999) [arXiv:hep-th/9808079].
[2] R. Clarkson and R.B. Mann, Phys. Rev. Lett. 96, 051104 (2006) [arXiv:hep-th/0508109]; Class. Quant. Grav. 23, 1507 (2006) [arXiv:hep-th/0508200].
[3] R.B. Mann and C. Stelea, Phys. Lett. B 632, 537 (2006) [arxiv:hep-th/0508186].
[4] H. Cebeci, Ö. Sarıoğlu and B. Tekin, Phys. Rev. D 73, 064020 (2006) [arXiv:hep-th/0602117].
[5] R. Schoen and S.T. Yau, Commun. Math. Phys. 65, 45 (1979); Commun. Math. Phys. 79, 231 (1981).
[6] E. Witten, Commun. Math. Phys. 80, 381 (1981).
[7] S. Deser and C. Teitelboim, Phys. Rev. Lett. 39, 249 (1977).
[8] M.T. Grisaru, Phys. Lett. B 73, 297 (1978).
[9] T. Parker and C.H. Taubes, Commun. Math. Phys. 84, 223 (1982).
[10] J.A. Nester, Phys. Lett. A 83, 241 (1981).
[11] R. Arnowitt, S. Deser and C.W. Misner, Phys. Rev. 117, 1595 (1960); in Gravitation: An Introduction to Current Research, ed. by L. Witten (Wiley, NY, 1962); “The dynamics of general relativity,” arXiv:gr-qc/0405109.
[12] V.I. Denisov and V.O. Solovev, Theor. and Math. Phys. 56, 832 (1983), English translation, original pagination 301314.
[13] H.L. Bray and P.T. Chrusciel, “The Penrose inequality,” arXiv:gr-qc/0312047.
[14] R. Bartnik, Commun. Pure Appl. Math. 39, 661 (1986).
[15] P.T. Chrusciel, Class. Quant. Grav. 3, L115 (1986); Commun. Math. Phys. 120, 233 (1988).
[16] N. O’Murchadha, J. Math. Phys. 27, 2111 (1986).
[17] S.N. Gupta, Proc. Phys. Soc. (London), A63, 681 (1950).
[18] K. Bleuler, Helv. Phys. Acta 23, 567 (1950).
[19] S. Deser and B. Tekin, Phys. Rev. D 67, 084009 (2003) [arXiv:hep-th/0212292]; Phys. Rev. Lett. 89, 101101 (2002) [arXiv:hep-th/0205318].
[20] L.F. Abbott and S. Deser, Nucl. Phys. B 195, 76 (1982).
[21] M. Henneaux and C. Teitelboim, Commun. Math. Phys. 98, 391 (1985).
[22] R.M. Wald, General Relativity, (The Univ. of Chicago Press, Chicago, 1984).
[23] Ö. Sarıoğlu and B. Tekin, work in progress.
[24] At this point, one might mistakenly think that we have restricted the background to be just the flat Cartesian Minkowski spacetime, however this is not correct at all.
[25] There is a caveat here. To actually obtain (9), we have assumed that the gauge choice $\nabla_j (h^{ij} + \bar{g}^{ij} \bar{h}) = 0$ is valid everywhere in the bulk unlike ours which is only asymptotical. However this does not change the final result since the neglected terms do not contribute to the energy.