A cabling formula for the 2-loop polynomial of knots

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Abstract

The 2-loop polynomial is a polynomial presenting the 2-loop part of the Kontsevich invariant of knots. We show a cabling formula for the 2-loop polynomial of knots. In particular, we calculate the 2-loop polynomial for torus knots.

The Kontsevich invariant is a very strong invariant of knots (which dominates all quantum invariants and all Vassiliev invariants) and it is expected that the Kontsevich invariant will classify knots. A problem when we study the Kontsevich invariant is that it is difficult to calculated the Kontsevich invariant of an arbitrarily given knot concretely. It has recently been shown [17, 7, 5] that the infinite sum of the terms of the Kontsevich invariant with a fixed loop number is presented by using polynomials (after appropriate normalization by the Alexander polynomial). In particular, it is known that the 1-loop part is presented by the Alexander polynomial. The polynomial giving the 2-loop part is called the 2-loop polynomial. The values of the 2-loop polynomial has been calculated so far only for particular classes of knots.

In this paper, we give a cabling formula for the 2-loop polynomial (Theorem 3.1), which presents the 2-loop polynomial of a cable knot (see Figure 1) of a knot $K$ in terms of the 2-loop polynomial of $K$. In particular, we calculate a formula of the 2-loop polynomial for torus knots (Theorem 2.1). This formula and the cabling formula are also obtained independently by Marché [12, 13].

This paper is organized as follows. In Section 1 we review the definition of the 2-loop polynomial. In Section 2 we calculate the 2-loop polynomial of torus knots as the 2-loop part of the cabling formula of the Kontsevich invariant of the trivial knot. In Section 3 we give a cabling formula for the 2-loop polynomial. In Section 4 we show relations to some Vassiliev invariants.

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1It was conjectured by Rozansky [17]. The existence of such rational presentations has been proved by Kricker [7] (though such a rational presentation itself is not necessarily a knot invariant in a general loop degree). Further, Garoufalidis and Kricker [5] defined a knot invariant in any loop degree, from which such a rational presentation can be deduced.

2This follows from the theory of [2] on the MMR conjecture. See also [3, 4] and references therein.

3A table of the 2-loop polynomial for knots with up to 7 crossings is given by Rozansky [18]. The 2-loop polynomial of knots with the trivial Alexander polynomial can often been calculated by surgery formulas [5, 8].
1 The Kontsevich invariant and the 2-loop polynomial

The 2-loop polynomial is a polynomial presenting the 2-loop part of the Kontsevich invariant. In this section, we review its definition and a cabling formula of the Kontsevich invariant.

An open Jacobi diagram is a uni-trivalent graph such that a cyclic order of the three edges around each trivalent vertex of the graph is fixed. Let $A(\ast)$ be the vector space over $\mathbb{Q}$ spanned by open Jacobi diagrams subject to the AS and IHX relations; see Figure 2 for the relations.

The AS relation:

\[
\begin{array}{c}
\begin{array}{c}
\quad \quad \\
\quad \\
\end{array}
\end{array}
\quad = - \begin{array}{c}
\begin{array}{c}
\quad \\
\quad \\
\end{array}
\end{array}
\]

The IHX relation:

\[
\begin{array}{c}
\begin{array}{c}
\quad \\
\quad \\
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\quad \\
\quad \\
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\quad \\
\quad \\
\end{array}
\end{array}
\]

Figure 2: The AS and IHX relations

The Kontsevich invariant $Z^\sigma(K)$ of a framed knot $K$ is defined in $A(\ast)$; for a definition\footnote{In literatures, the Kontsevich invariant is often defined by $Z(K)$ in the space $A(S^1)$. The version $Z^\sigma(K)$ is defined to be the image of $Z(K)$ by the inverse map $\sigma$ of the Poincare-Birkhoff-Witt isomorphism $A(\ast) \rightarrow A(S^1)$.} see e.g. \cite{14}. It is known \cite{10} that the value of the Kontsevich invariant for each knot is group-like, which implies that it is presented by the exponential of some primitive element. That is, $Z^\sigma(K)$ is presented by the exponential of a primitive element, where a primitive element of $A(\ast)$ is a linear sum of connected open Jacobi diagrams.

For example, it is shown \cite{4} that the Kontsevich invariant of the trivial knot, denoted by $\Omega$, is presented by

\[Z^\sigma(\text{the trivial knot}) = \Omega = \exp_\sqcup(\omega),\]

where $\exp_\sqcup$ denotes the exponential with respect to the disjoint-union product, and $\omega$ is
defined by

$$\omega = \frac{1}{2} \log \frac{\sinh(x/2)}{x/2}.$$ 

Here, a label of a power series $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ implies

$$f(x) = c_0 \left( + c_1 \right) + c_2 \left( + c_3 \right) + \cdots,$$

where a label is put on either of the sides of an edge, and the corresponding legs are written in the same side of the edge.\(^5\) Note that $f(x) = \big| f(-x) \big|$ by the AS relation, in the notation of this paper.

Let $K$ be a framed knot with 0 framing. (Throughout this paper, we often mean a framed knot with 0 framing also by a knot, abusing terminology.) The loop expansion of the Kontsevich invariant is given by

$$\log_{\sqcup} Z^\sigma(K) = \frac{1}{2} \log \frac{\sinh(x/2)}{x/2} - \frac{1}{2} \log \Delta_K(e^x) + \sum_{i} \frac{p_{i,1}(e^x)}{\Delta_K(e^x)} + \sum_{i} \frac{p_{i,2}(e^x)}{\Delta_K(e^x)} + \sum_{i} \frac{p_{i,3}(e^x)}{\Delta_K(e^x)} + \text{(terms of (\geq 3)-loop)},$$

where $\log_{\sqcup}$ denotes the logarithm with respect to the disjoint-union product, and $\Delta_K(t)$ is the normalized\(^6\) Alexander polynomial of $K$, and $p_{i,j}(e^x)$ is a polynomial in $e^x$. The 2-loop part is characterized by the polynomial,

$$\Theta'_K(t_1, t_2, t_3) = \sum_i p_{i,1}(t_1)p_{i,2}(t_2)p_{i,3}(t_3).$$

We call its symmetrization,\(^7\)

$$\Theta_K(t_1, t_2, t_3) = \sum_{\{i,j,k\} = \{1,2,3\}} \Theta'_K(t_i^{e_1}, t_j^{e_2}, t_k^{e_3}) \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]/(t_1 t_2 t_3 = 1),$$

\(^5\)Our notation is different from the notation in [5, 8] where a label of an edge is defined by setting a local orientation of the edge that determines the side in which we write the corresponding legs.

\(^6\)We suppose that $\Delta_K(t)$ is normalized, satisfying that $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$.

\(^7\)With respect to the symmetry of the theta graph, of order 12.
the 2-loop polynomial of $K$, which is an invariant of $K$. (Note that this normalization of $\Theta_K(t_1, t_2, t_3)$ is 12 times the usual normalization.) $\Theta_K(t, t^{-1}, 1)$ is a symmetric polynomial in $t^{\pm 1}$ divisible by $t - 1$ (since $\Theta_K(1, 1, 1) = 0$) and, hence, divisible by $(t - 1)^2$. We define the reduced 2-loop polynomial by

$$\hat{\Theta}_K(t) = \frac{\Theta_K(t, t^{-1}, 1)}{(t^{1/2} - t^{-1/2})^2} \in \mathbb{Q}[t^{\pm 1}],$$

which is a symmetric polynomial in $t^{\pm 1}$.

Let us review the cabling formula of the Kontsevich invariant of $K$. Another version of the Kontsevich invariant, called the wheeled Kontsevich invariant $\mathcal{Z}^w(K)$, is defined by

$$\mathcal{Z}^w(K) = \partial_{\Omega}\mathcal{Z}(\sigma(K)),$$

where $\partial_{\Omega} : \mathcal{A}(\ast) \to \mathcal{A}(\ast)$ is the wheeling isomorphism; see [4]. Here, for open Jacobi diagrams $C$ and $D$, $\partial_{C}(D)$ is defined to be 0 if $C$ has more univalent vertices than $D$, and the sum of all ways of gluing all univalent vertices of $C$ to some univalent vertices of $D$ otherwise. We graphically present it by

$$\partial_{C}(D) = \begin{array}{c}
\bullet \\
| \\
\bullet
\end{array} \quad \begin{array}{c}
\bullet \\
| \\
\bullet
\end{array}.$$

Let $\Psi^{(p)} : \mathcal{A}(\ast) \to \mathcal{A}(\ast)$ be the map which takes a diagram with $k$ univalent vertices to its $p^k$ multiple. The $(p, q)$ cable knot of a knot $K$ is the knot given by a simple closed curve on the boundary torus of a tubular neighborhood of $K$ which winds $q$ times in the meridian direction and $p$ times in the longitude direction (see e.g. [11]); for example see Figure 1. The cabling formula of the Kontsevich invariant is given by

**Proposition 1.1 (Le [4], see also [19]).** Let $K$ be a framed knot with 0 framing, and let $K^{(p,q)}$ be the $(p,q)$ cable knot of $K$ (with 0 framing). Then,

$$\mathcal{Z}^w(K^{(p,q)}) = \partial_{\Omega}^{-1}\Psi^{(p)}\partial_{\Omega}\left(\mathcal{Z}^w(K) \sqcup \exp_{\square} \left(\frac{q}{2p} \bigcirc - \frac{q}{48p} \theta\right) \sqcup \exp_{\square} \left(-\frac{pq}{2} \bigcirc + \frac{pq}{48} \theta\right)\right).$$

### 2 The 2-loop polynomial of a torus knot

In this section, we calculate the 2-loop polynomial of a torus knot, picking up the 2-loop part of the cabling formula of the Kontsevich invariant of the trivial knot. The 2-loop part of the Kontsevich invariant for torus knots is also calculated independently by Marché [12, 13].

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8 This is not trivial, since there is another 2-loop trivalent graph, what is called, a “dumbbell diagram”.

9 Proposition 1 is obtained from Theorem 1 of [4] by pulling back by the isomorphism $\mathcal{A}(\ast) \xrightarrow{\partial_{\Omega}} \mathcal{A}(\ast) \xrightarrow{\chi} \mathcal{A}(S^1)$, and by modifying the contribution from the framing of the cable knot, noting that the $(p, q)$ cable knot in the definition of $\mathcal{Z}^w$ has framing $(p - 1)q$.

10 Bar-Natan has also obtained some presentation of the wheeled Kontsevich invariant for torus knots (private communication).
The torus knot $T(p, q)$ of type $(p, q)$ is the $(p, q)$ cable knot of the trivial knot (which is isotopic to $T(q, p)$); for example see Figure 3. It is known, see e.g. [11], that the Alexander polynomial of a torus knot is given by

\[ \Delta_{T(p, q)}(t) = \frac{(t^{pq/2} - t^{-pq/2})(t^{1/2} - t^{-1/2})}{(tp/2 - t^{-p/2})(tq/2 - t^{-q/2})}. \]

**Theorem 2.1.** The 2-loop polynomial of the torus knot $T(p, q)$ of type $(p, q)$ is given by

\[ \Theta_{T(p, q)}(t_1, t_2, t_3) = -\frac{1}{4} \sum_{\{i,j,k\} = \{1,2,3\}} \psi_{p,q}(t_i) \psi_{q,p}(t_j) \Delta_{T(p,q)}(t_k) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]/(t_1t_2t_3 = 1), \]

where $\psi_{p,q}$ is defined by

\[ \psi_{p,q}(t) = \Delta_{T(p,q)}(t) \cdot \left( \frac{t^{pq/2} + t^{-pq/2}}{tp/2 - t^{-p/2}} - q \cdot \frac{tpq/2 + t^{-pq/2}}{tpq/2 - t^{-pq/2}} \right) \]

\[ = \frac{t^{1/2} - t^{-1/2}}{(tp/2 - t^{-p/2})(tq/2 - t^{-q/2})} \left( \frac{t^{pq/2} + t^{-pq/2}}{tp/2 - t^{-p/2}} \cdot \frac{tpq/2 - t^{-pq/2}}{tpq/2 - t^{-pq/2}} - q(t^{pq/2} + t^{-pq/2}) \right). \]

In particular, $\Theta_{T(p, q)}(t_1, t_2, t_3)$ is a polynomial in $t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}$ with integer coefficients of degree

\[ \deg_{t_i}(\Theta_{T(p,q)}(t_1, t_2, t_3^{-1}t_2^{-1})) = (p - 1)(q - 1). \]

**Remark 2.2.** $\psi_{p,q}(t)$ is not a polynomial, but a rational function, while $\Theta_{T(p, q)}(t_1, t_2, t_3)$ is a polynomial. Rozansky [18] suggests that the 2-loop polynomial is a polynomial with integer coefficients; this holds for torus knots by the theorem. He also suggests a conjectural inequality

\[ \deg_{t_1}(\Theta_K(t_1, t_2, t_1^{-1}t_2^{-1})) \leq 2g(K), \]

where $g(K)$ denotes the genus of $K$. Since the genus of $T(p, q)$ equals $(p - 1)(q - 1)/2$ (see e.g. [11]), torus knots give the equality of the above formula.

**Remark 2.3.** The $sl_2$ reduction of the $n$-loop part of the Kontsevich invariant of the Kontsevich invariant is equal to the $n$th line in the expansion of the colored Jones polynomial. Rozansky [16] has calculated it for torus knots.

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11 This value coincides with the value in [12][13]. However, the values of the 2-loop polynomial for some torus knots in Table 2 of [18] have opposite signs to our values. The signs of some values in Table 2 of [18] might not be correct.
For $\alpha, \beta \in A(*)$ we write $\alpha \equiv \beta$ if $\alpha - \beta$ is equal to a linear sum of Jacobi diagrams, either, of ($\geq 3$)-loop, or, having a component of a trivalent graph (i.e., a component with no univalent vertices).

**Proof of Theorem 2.1.** Since the torus knot $T(p, q)$ is obtained from the trivial knot by cabling, we have that

$$Z^w(T(p, q)) \equiv \partial^{-1}_\Omega \Psi(p) \partial_\Omega \left( \Omega \sqcup \exp_{\sqcup} \left( \frac{q}{2p} \right) \right) \sqcup \exp_{\sqcup} \left( -\frac{pq}{2} \right)$$

by Proposition 1.1. The first term of the right hand side is calculated as follows. From the definition of $\partial_\Omega$,

$$\partial_\Omega \left( \exp_{\sqcup} \left( \frac{q}{2p} \right) \right) \sqcup \Omega = \Omega \circlearrowright \exp \left( \frac{q}{2p} \right) \sqcup \Omega$$

Since any component of $\Omega$ has a loop, the ($\leq 1$)-loop part of the right hand side has no edges between the two $\Omega$’s, and, hence, this part is presented by

$$\partial_\Omega \exp_{\sqcup} \left( \frac{q}{2p} \right) \sqcup \Omega.$$ 

Further, its first term is given by

$$\partial_\Omega \exp_{\sqcup} \left( \frac{q}{2p} \right) \equiv \exp_{\sqcup} \left( \frac{q}{2p} \right) \sqcup \Omega_{\ast x},$$

where the equivalence is obtained in the same was as Lemma 6.3 of [H]. The primitive part of the 2-loop part of the right hand side of (11) is equal to a linear sum of diagrams, each of which has precisely one edge between the two $\Omega$’s. Hence, it is presented by

$$\Omega \circlearrowright \exp \left( \frac{q}{2p} \right) \sqcup \Omega.$$ 

Since

$$D = \frac{nx^m}{y} \quad \text{for} \quad D = \frac{x^m}{y},$$

the previous diagram is equivalent to

$$f(x) \frac{f(qx)}{p} ,$$
where \( f(x) \) is given by
\[
f(x) = \frac{d}{dx} \left( \frac{1}{2} \log \frac{\sinh x}{x/2} \right) = \frac{1}{4} \cdot \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} - \frac{1}{2x}.
\]
Hence, the \((\leq 2)\)-loop part of (1) is presented by
\[
\partial_\Omega \left( \exp \left( \frac{q}{2p} \right) \right) \sqcup \Omega \equiv \exp \left( \frac{q}{2p} \right) \sqcup \Omega \sqcup \Omega_{\frac{p}{x}} \sqcup \exp \left( \frac{f(x)}{f\left(\frac{q}{x}\right)} \right).
\]

The map \( \Psi^{(\nu)} \) sends this to
\[
\exp \left( \frac{pq}{2} \right) \sqcup \Omega_{p} \sqcup \Omega_{q} \sqcup \exp \left( \frac{f(px)}{f(qx)} \right).
\]
Further, \( \partial_{\Omega}^{-1} \) sends this (modulo the equivalence) to
\[
\partial_{\Omega}^{-1} \left( \exp \left( \frac{pq}{2} \right) \right) \sqcup \Omega_{p} \sqcup \Omega_{q} \sqcup \exp \left( \frac{f(px)}{f(qx)} \right).
\]
Its first term is graphically shown as
\[
\begin{align*}
\Omega^{-1} & \quad \exp \left( \frac{pq}{2} \right) \sqcup \Omega_{p} \sqcup \Omega_{q} \sqcup \exp \left( \frac{f(px)}{f(qx)} \right).
\end{align*}
\]

The primitive part of the 2-loop part of this diagram is calculated similarly as before; for example, when there is precisely one edge between \( \Omega^{-1} \) and \( \Omega_{px} \), we have the following component,
\[
\begin{align*}
\Omega_{px} \quad -\omega \quad \exp \left( \frac{pq}{2} \right) \quad \equiv -p
\end{align*}
\]
Thus, the primitive part of the 2-loop part of (3) is equal to
\[
\begin{align*}
& \left( \text{the primitive part of the 2-loop part of } \partial_{\Omega^{-1}} \exp \left( \frac{pq}{2} \right) \right) \\
& = pq \quad f(pq) \quad f(pq) \quad = p \\& \quad f(pq) \quad f(pq) \quad -p \quad f(px) \quad f(pqx) \quad \equiv q \quad f(qx) \quad f(pqx).
\end{align*}
\]
where the equality is obtained from Lemma 2.5 below. Hence, the primitive part of the 2-loop part of $Z^w(T(p, q))$ is given by
\[
\begin{align*}
\phi_{p,q}(t) &+ \phi_{q,p}(t) \\
\frac{1}{16} \phi_{p,q}(t) &- \frac{1}{8} \phi_{q,p}(t),
\end{align*}
\]
where we put $t = e^x$ and $\phi_{p,q}$ is defined by $\phi_{p,q}(e^x) = 4(f(px) - qf(pqx))$, that is,
\[
\phi_{p,q}(t) = \frac{tp/2 + t^{-p/2}}{tp/2 - t^{-p/2}} - q \cdot \frac{tpq/2 + t^{-pq/2}}{tpq/2 - t^{-pq/2}}.
\]

Therefore, from the definition of the 2-loop polynomial, we obtain the required formula.

By Corollary 2.4 below, the degree of $\hat{\Theta}_{T(p,q)}(t)$ equals $(p - 1)(q - 1) - 1$. Since $(t^{1/2} - t^{-1/2})^2 \hat{\Theta}_{T(p,q)}(t) = \Theta_{T(p,q)}(t, 1, t^{-1})$ by definition, degree$_1(\Theta_{T(p,q)}(t_1, t_2, t_1^{-1}t_2^{-1}))$ is at least $(p - 1)(q - 1)$. We can show that it is exactly $(p - 1)(q - 1)$ in the same way as the proof of Example 2.6.

\begin{corollary}
The reduced 2-loop polynomial of the torus knot $T(p, q)$ is given by
\[
\hat{\Theta}_{T(p,q)}(t) = \frac{1}{2(t^{1/2} - t^{-1/2})^2} \psi_{p,q}(t) \psi_{q,p}(t)
\]
\[
= \frac{1}{2} \cdot \frac{1}{(tp/2 - t^{-p/2})^2} \cdot \left(\frac{tp/2 + t^{-p/2}}{tp/2 - t^{-p/2}} - q\left(t^{pq/2} + t^{-pq/2}\right)\right)
\times \frac{1}{(t^{q/2} - t^{-q/2})^2} \cdot \left(\frac{tpq/2 - t^{-pq/2}}{t^{q/2} - t^{-q/2}} - p\left(t^{pq/2} + t^{-pq/2}\right)\right).
\]
\end{corollary}

\begin{lemma}
For a scalar $c$,
\[
\partial^{-1} \Omega \exp \left(\frac{c}{2} \left\langle \right\rangle \right) \equiv \exp \left(\frac{c}{2} \left\langle \right\rangle \right) \sqcup \Omega^{-1} \sqcup \exp \left(\frac{c}{2} \left\langle \right\rangle \right).
\]
\end{lemma}

\textit{Proof.} From the definition of $\partial_\Omega$,
\[
\partial_\Omega \left( \exp \left(\frac{c}{2} \left\langle \right\rangle \right) \sqcup \Omega^{-1} \right) = \Omega.
\]
\hfill \blacksquare
Similarly as in the proof of Theorem 2.1, the \((\leq 1)\)-loop part of the right hand side is presented by
\[
\partial \Omega \exp \left( \frac{c}{2} \bigcap \right) \sqcup \Omega^{-1}_{cr} \equiv \exp \left( \frac{c}{2} \bigcap \right).
\]
Further, the primitive part of the 2-loop part of the right hand side of (5) is presented by
\[
\text{Example 2.6. For the \((p, 2)\) torus knot, Theorem 2.1 implies that}
\]
\[
\Theta_{T(p,2)}(t_1, t_2, t_3) = \frac{1}{(t_1 + 1)(t_2 + 1)(t_3 + 1)}
\times \left( \frac{p-1}{2} \left( p t_1^{p-1} + t_1^{p-1} + t_2^{p-1} + t_3^{p-1} \right) \right)
\]
\[
\quad \quad - \frac{t_1^{p-1} - t_1^{-(p-1)}}{t_1 - t_1^{-1}} - \frac{t_2^{p-1} - t_2^{-(p-1)}}{t_2 - t_2^{-1}} - \frac{t_3^{p-1} - t_3^{-(p-1)}}{t_3 - t_3^{-1}}\right).
\]
For example, the coefficients of \(\Theta_{T(7,2)}(t_1, t_2, t_3)\) are as shown in Table 1. Further,
\[
\hat{\Theta}_{T(p,2)}(t) = \frac{t^2}{(t^2 - 1)^2} \left( \frac{p-1}{2} \left( p t^p + t^{-p} \right) - \frac{t^{p-1} - t^{-(p-1)}}{t - t^{-1}} \right)
\]
\[
= \frac{t^3}{(t^2 - 1)^3} \left( \frac{p-1}{2} \left( p^{p+1} - t^{-p} \right) - \frac{p + 1}{2} \left( t^{p-1} - t^{-p+1} \right) \right).
\]

| \(n\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|------|---|---|---|---|---|---|---|
| \(m = 6\) | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
| \(m = 5\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = 4\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = 3\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = 2\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = 1\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = 0\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = -1\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = -2\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = -3\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = -4\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = -5\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |
| \(m = -6\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 |

Table 1: The non-zero coefficients of \(t_1^n t_2^m\) in \(\Theta_{T(7,2)}(t_1, t_2, t_1^{-1} t_2^{-1})\)
Proof. By definition,

\[ \Delta_T(p, 2)(t) = \frac{tp/2 + t^{-p/2}}{t^{1/2} + t^{-1/2}}, \quad \psi_{p, 2}(t) = \frac{tp/2 - t^{-p/2}}{t^{1/2} + t^{-1/2}}; \]

\[ \psi_{2, p}(t) = \frac{1}{(t^{1/2} + t^{-1/2})(tp/2 - t^{-p/2})} \cdot \left( (t + t^{-1}) \cdot \frac{tp - t^{-p}}{t - t^{-1}} - p(t^p + t^{-p}) \right). \]

Hence, when \( \{i, j, k\} = \{1, 2, 3\} \), we have that

\[ \frac{1}{2} \left( \psi_{p, 2}(t_i) \Delta_T(p, 2)(t_k) + \psi_{p, 2}(t_k) \Delta_T(p, 2)(t_i) \right) = \frac{tp/2 - t^{-p/2}}{(t_{i, k}^{1/2} + t_{i, k}^{-1/2})(t_{k, i}^{1/2} + t_{k, i}^{-1/2})}. \]

Therefore,

\[ -\frac{1}{4} \psi_{2, p}(t_j) \cdot \left( \psi_{p, 2}(t_i) \Delta_T(p, 2)(t_k) + \psi_{p, 2}(t_k) \Delta_T(p, 2)(t_i) \right) \]

\[ = \frac{1}{(t_{i, j}^{1/2} + t_{i, j}^{-1/2})(t_{j, k}^{1/2} + t_{j, k}^{-1/2})(t_{k, i}^{1/2} + t_{k, i}^{-1/2})} \cdot \frac{1}{2} \cdot \left( p(tp/2 + t^{-p}) - (t_j + t_j^{-1}) \cdot \frac{tp - t^{-p}}{t_j - t_j^{-1}} \right) \]

\[ = \frac{1}{(t_{i, j}^{1/2} + t_{i, j}^{-1/2})(t_{j, k}^{1/2} + t_{j, k}^{-1/2})(t_{k, i}^{1/2} + t_{k, i}^{-1/2})} \cdot \left( \frac{p - 1}{2} \left( t_j(t_j^p + t^{-p}) - \frac{t_j^{-1} - t_j^{-(p-1)}}{t_j - t_j^{-1}} \right) \right). \]

By Theorem 2.1, we obtain \( \Theta_T(p, 2)(t_1, t_2, t_3) \) as the sum of the above formula over \( \{i, j, k\} = \{1, 2, 3\}, \{2, 3, 1\}, \{3, 1, 2\} \), which gives the required formula. \( \square \)

Example 2.7. In a similar way as the previous example, we have that

\[ \Theta_T(p, 3)(t_1, t_2, t_3) = \frac{(t_1 - 1)(t_2 - 1)(t_3 - 1)}{(t_1^2 - 1)(t_2^2 - 1)(t_3^2 - 1)} \]

\[ \times \left( (p - 1)(t_1^p + t_1^{-p} + t_2^p + t_2^{-p} + t_3^p + t_3^{-p}) \right. \]

\[ + t_1^{2p} + t_1^{-2p} + t_2^{2p} + t_2^{-2p} + t_3^{2p} + t_3^{-2p} \]

\[ + t_1^{2p} t_2^{2p} + t_2^{2p} t_3^{2p} + t_1^{2p} t_3^{2p} + t_1^{2p} t_2^{-2p} + t_2^{2p} t_3^{-2p} + t_1^{-2p} t_2^{2p} + t_2^{-2p} t_3^{2p} + t_1^{-2p} t_3^{2p} \]

\[ \left( t_1^{3(p-1)/2} - t_1^{-3(p-1)/2} \right) \cdot \left( \frac{2t_1^{p/2} + 2t_1^{-p/2} + t_1^{p/2} t_3^{-p/2} + t_1^{-p/2} t_3^{p/2}}{t_1^{3/2} - t_1^{-3/2}} \right) \]

\[ - \frac{t_1^{2(p-1)/2} - t_1^{-2(p-1)/2}}{t_1^{3/2} - t_1^{-3/2}} \cdot \left( 2t_1^{p/2} + 2t_1^{-p/2} + t_1^{p/2} t_3^{-p/2} + t_1^{-p/2} t_3^{p/2} \right) \]

\[ - \frac{t_2^{3(p-1)/2} - t_2^{-3(p-1)/2}}{t_2^{3/2} - t_2^{-3/2}} \cdot \left( 2t_2^{p/2} + 2t_2^{-p/2} + t_2^{p/2} t_3^{-p/2} + t_2^{-p/2} t_3^{p/2} \right) \]

\[ - \frac{t_3^{3(p-1)/2} - t_3^{-3(p-1)/2}}{t_3^{3/2} - t_3^{-3/2}} \cdot \left( 2t_3^{p/2} + 2t_3^{-p/2} + t_3^{p/2} t_2^{-p/2} + t_3^{-p/2} t_2^{p/2} \right), \]

and

\[ \hat{\Theta}_T(p, 3)(t) = \frac{t^3(p/2 + t^{-p/2})}{(t^3 - 1)^2} \cdot \left( (p - 1)(t^{3p/2} + t^{-3p/2}) - 2 \cdot \frac{t^{3(p-1)/2} - t^{-3(p-1)/2}}{t^{3/2} - t^{-3/2}} \right) \]

\[ = \frac{t^{p/2} + t^{-p/2}}{(t^{3/2} - t^{-3/2})^2} \cdot \left( (p - 1)(t^{3(p+1)/2} + t^{-3(p+1)/2}) - (p + 1)(t^{3(p-1)/2} - t^{-3(p-1)/2}) \right). \]
3 A cabling formula for the 2-loop polynomial

In this section, we give a cabling formula for the 2-loop polynomial. We show the formula by picking up the 2-loop part of the cabling formula of the Kontsevich invariant, modifying the proof of Theorem 2.1. This cabling formula is also obtained independently by Marché [13].

It is known, see e.g. [11], that a cabling formula for the Alexander polynomial is given by

$$\Delta_{K(p,q)}(t) = \Delta_{T(p,q)}(t) \Delta_{K(t^p)}.$$ 

A cabling formula for the 2-loop polynomial is given by

**Theorem 3.1.** Let $K$ be a knot, and let $K'(p,q)$ be the $(p,q)$ cable knot of $K$. Then,

$$\Theta_{K'(p,q)}(t_1, t_2, t_3) = \Theta_{T(p,q)}(t_1, t_2, t_3) + \Theta_{K(t^p_1, t^p_2, t^p_3)}$$

$$+ \frac{1}{2} \Delta_{T(p,q)}(t_1) \Delta_{T(p,q)}(t_2) \Delta_{T(p,q)}(t_3) \sum_{\{i,j,k\} = \{1,2,3\}} \Delta'_{K(t^p_i)} \cdot t^p_i \cdot \phi_{q,p}(t_j) \Delta_{K(t^p_j)} \Delta_{K(t^p_k)}.$$ 

**Proof.** We show the theorem, modifying the proof of Theorem 2.1. By Proposition 1.1, we have that

$$Z_w(K(p,q)) \equiv \partial_{\Omega}^{-1} \Psi(p) \partial_{\Omega} \left( Z_w(K) \sqcup \exp_{\sqcup} \left( \frac{q}{2p} \right) \right) \sqcup \exp_{\sqcup} \left( -\frac{pq}{2} \right),$$

where $Z_w(K)$ is presented by

$$Z_w(K) = \Omega \sqcup \exp_{\sqcup} \left( \frac{1}{2} \log \Delta_{K(e^x)} \right) + \text{(terms of ($\geq 2$)-loop)}.$$ 

The 2-loop part of $Z_w(K)$ contributes to the required formula by $\Theta_{K(t^p_1, t^p_2, t^p_3)}$. We calculate the contribution of the 1-loop part in the following of this proof.

In a similar way as (2), we have that

$$\partial_{\Omega} \left( Z_w(K) \sqcup \exp_{\sqcup} \left( \frac{q}{2p} \right) \right) \equiv \exp \left( \frac{q}{2p} \right) \sqcup \Omega \sqcup \Omega_{\frac{q}{2p}} \sqcup \exp_{\sqcup} \left( \frac{1}{2} \log \Delta_{K(e^x)} + f(x) + g(x) f \left( \frac{q}{2p} x \right) \right),$$

where $g(x)$ is given by

$$g(x) = \frac{d}{dx} \left( -\frac{1}{2} \log \Delta_{K(e^x)} \right) = -\frac{\Delta'_{K(e^x)} \cdot e^x}{2\Delta_{K(e^x)}}.$$ 

11
\((p, q)\): The non-zero coefficients of \(t_1^n t_2^m\) in \(\Theta_{T(p,q)}(t_1, t_2, t_1^{-1} t_2^{-1})\) in the fundamental domain \(\{0 \leq 2m \leq n\}\) (see \[18\]) for \((p, q)\) with \(p \leq 7, q \leq 4\). The array for each \((p, q)\) is a subset of the full array such as shown in Table 1 and the most left dot is at \((n, m) = (0, 0)\). We can recover the other coefficients for each \((p, q)\) from the presented coefficients by the symmetry of \(\Theta_K(t_1, t_2, t_1^{-1} t_2^{-1})\).

|\(p, q\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |
|---|---|---|---|---|---|---|---|---|
|\(3, 2\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |
|\(5, 2\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |
|\(7, 2\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |
|\(4, 3\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |
|\(5, 3\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |
|\(7, 3\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |
|\(5, 4\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |
|\(7, 4\) | \(-1\) | \(-2\) | \(-9\) | \(-15\) | \(-18\) | \(-20\) | \(-22\) | \(-24\) |

Table 2: The non-zero coefficients of \(t_1^n t_2^m\) in \(\Theta_{T(p,q)}(t_1, t_2, t_1^{-1} t_2^{-1})\) in a fundamental domain \(\{0 \leq 2m \leq n\}\) (see \[18\]) for \((p, q)\) with \(p \leq 7, q \leq 4\). The array for each \((p, q)\) is a subset of the full array such as shown in Table 1 and the most left dot is at \((n, m) = (0, 0)\). We can recover the other coefficients for each \((p, q)\) from the presented coefficients by the symmetry of \(\Theta_K(t_1, t_2, t_1^{-1} t_2^{-1})\).
Calculating its image by $\partial$ contributes to the required formula by term corresponding to the formula (4) is as follows,

$$\Delta(p, q) : \text{The part of non-negative powers in } \hat{\Theta}_{(p, q)}(t)$$

\begin{align*}
(3, 2) & : t \\
(5, 2) & : 3t + 2t^3 \\
(7, 2) & : 6t + 5t^3 + 3t^5 \\
(9, 2) & : 10t + 9t^3 + 7t^5 + 4t^7 \\
(4, 3) & : 3t + 4t^2 + 3t^5 \\
(5, 3) & : 6t + 4t^2 + 6t^4 + 4t^7 \\
(7, 3) & : 10t + 12t^2 + 6t^4 + 12t^5 + 10t^8 + 6t^{11} \\
(8, 3) & : 15t + 12t^2 + 16t^4 + 7t^5 + 15t^7 + 12t^{10} + 7t^{13} \\
(10, 3) & : 21t + 24t^2 + 16t^4 + 25t^5 + 9t^7 + 24t^8 + 21t^{11} + 16t^{14} + 9t^{17} \\
(5, 4) & : 6t + 12t^2 + 9t^3 + 8t^6 + 9t^7 + 6t^{11} \\
(7, 4) & : 15t + 24t^2 + 9t^3 + 18t^5 + 20t^6 + 18t^9 + 12t^{10} + 15t^{13} + 9t^{17} \\
(9, 4) & : 21t + 40t^2 + 27t^3 + 12t^5 + 36t^6 + 30t^7 + 28t^{10} + 30t^{11} + 16t^{14} + 27t^{15} + 21t^{19} + 12t^{23} \\
(6, 5) & : 10t + 24t^2 + 27t^3 + 16t^4 + 15t^5 + 24t^8 + 18t^9 + 15t^{13} + 16t^{14} + 10t^{19} \\
(7, 5) & : 36t + 12t^2 + 20t^3 + 30t^4 + 36t^6 + 24t^8 + 18t^9 + 30t^{11} + 24t^{13} + 18t^{16} + 20t^{18} + 12t^{23} \\
(8, 5) & : 45t + 24t^2 + 14t^3 + 48t^4 + 36t^6 + 30t^7 + 45t^9 + 21t^{11} + 32t^{12} + 36t^{14} + 30t^{17} + 21t^{19} + 24t^{22} + 14t^{27} \\
(9, 5) & : 28t + 60t^2 + 54t^3 + 16t^4 + 36t^6 + 60t^7 + 42t^8 + 40t^{11} + 54t^{12} + 24t^{13} + 40t^{16} + 42t^{17} + 36t^{21} + 24t^{22} + 28t^{26} + 16t^{31}
\end{align*}

Table 3: The parts of non-negative powers in $\Theta_{(p, q)}(t)$ for $(p, q)$ with $p \leq 10$, $q \leq 5$. The remaining part for each $(p, q)$ can recover from the presented part by replacing $t$ with $t^{-1}$.

The map $\Psi^{(p)}$ sends this to

$$\exp\left(\frac{pq}{2}\right) \bigcup \Omega_{px} \bigcup \Omega_{qx} \bigcup \exp\left(\begin{array}{c}
\frac{1}{2} \log \Delta_K(p^x) f(px) + g(px) f(qx) \\
\end{array}\right),$$

Calculating its image by $\partial^{-1}$ in a similar way as in the proof of Theorem 2.11 the error term corresponding to the formula (4) is as follows,

$$g(px) \quad f(qx) \quad -p \quad g(px) \quad f(pqx)$$

$$= \frac{1}{4} \quad g(px) \quad \phi_{q,p}(t) \quad \phi_{q,p}(t) \quad \frac{1}{2} \quad g(px)$$

This contributes to the required formula by

$$\sum_{\{i,j,k\} = (1,2,3)} \frac{\Delta'_K(t^i) \cdot t^p}{2\Delta_K(t^i)} \cdot \Delta_K(t_j) \phi_{q,p}(t_j) \Delta_K(t_j) \Delta_K(t_j) \Delta_K(t_k).$$

Noting that $\Delta_K(t) = \Delta_T(t) \Delta_K(t^p)$, we obtain the required formula. □

A cabling formula for the reduced 2-loop polynomial is given by
Corollary 3.2. For the notation in Theorem 3.1,
\[ \hat{\Theta}_{K(p,q)}(t) = \hat{\Theta}_{T(p,q)}(t) + \frac{(tp/2 - tp/2)^2}{(t^{1/2} - t^{-1/2})^2} \cdot \hat{\Theta}_{K}(t) \]
\[ - \frac{tp}{(t^{1/2} - t^{-1/2})^2} \cdot \Delta_{T(p,q)}(t) \Delta_{K}(tp) \Delta_{K}'(tp) \psi_{q,p}(t). \]

Proof. The required formula is obtained from the formula of Theorem 3.1 by putting \( t_1 = t, \ t_2 = 1/t, \) and \( t_3 = 1. \)

4 Relations to Vassiliev invariants

In this section we show some relations to Vassiliev invariants of degree 2, 3.

A leading part of the Kontsevich invariant is presented by
\[ \log_{\text{Jac}} Z_{\sigma}(K) - \omega = \frac{v_2(K)}{2} - \frac{1}{3} \sum_{\{i,j,k\} = \{1,2,3\}} f_i(x) f_j(-x) f_k(0) \]
where the degree of a Jacobi diagram is half the number of univalent and trivalent vertices of the diagram, and \( v_2, v_3 \) are \( \mathbb{Z} \)-valued primitive Vassiliev invariants of degree 2, 3 respectively (see [14]). Since \( \frac{1}{6} \sum_{\{i,j,k\} = \{1,2,3\}} f_i(x) f_j(-x) f_k(0) \) has 1-loop, \( v_2(K) \) can be presented by the Alexander polynomial; in fact, from the formula of the loop expansion,
\[ v_2(K) = -\left( \text{the coefficient of } x^2 \text{ in the expansion of } \Delta_{K}(e^x) \right) \]
\[ = -\frac{1}{2} \Delta''_{K}(1). \]

Further, since \( \frac{1}{6} \sum_{\{i,j,k\} = \{1,2,3\}} f_i(x) f_j(-x) f_k(0) \) has 2-loop, \( v_3(K) \) can be presented by the 2-loop polynomial; in fact, we have

Proposition 4.1.
\[ v_3(K) = \frac{1}{2} \hat{\Theta}_{K}(1). \]

Proof. Let us consider the map
\[ \begin{array}{c}
\begin{array}{c}
 f_3(0) \\
 f_2(0) \\
 f_1(0)
\end{array}
\end{array} \quad \mapsto \quad \begin{array}{c}
\begin{array}{c}
 f_3(x) \\
 f_2(x) \\
 f_1(x)
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
 f_3(x) \\
 f_2(x) \\
 f_1(x)
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
 f_3(x) \\
 f_2(x) \\
 f_1(x)
\end{array}
\end{array}
\]
\[ \quad \mapsto \quad \frac{1}{6} \sum_{\{i,j,k\} = \{1,2,3\}} f_i(x) f_j(-x) f_k(0). \]

This map takes the 2-loop part of \( \log_{\text{Jac}} Z_{\sigma}(K) \) to \( \frac{1}{12}(e^{x/2} - e^{-x/2})^2 \hat{\Theta}_{K}(e^x) \), whose coefficient of \( x^2 \) equals \( \frac{1}{12} \hat{\Theta}_{K}(1) \). Since \( \begin{array}{c}
\begin{array}{c}
 f_3(x) \\
 f_2(x) \\
 f_1(x)
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
 f_3(x) \\
 f_2(x) \\
 f_1(x)
\end{array}
\end{array} \]
by the AS and IHX relations, the above
maps takes this diagram to $\frac{2}{3}x^2$. Hence, $\frac{1}{6}v_3(K) = \frac{1}{12}\hat{\Theta}_K(1)$, which implies the required formula.

\[\square\]

**Example 4.2.** A cabling formula for $v_3$ is given by

$$v_3(K^{(p,q)}) = p^2 \cdot v_3(K) + \frac{1}{12}p(p^2 - 1)q \cdot \Delta''_K(1) + \frac{1}{144}p(p^2 - 1)q(q^2 - 1).$$

**Proof.** From Proposition 4.1 and Corollary 3.2 putting $t = 1$, we have that

$$v_3(K^{(p,q)}) = v_3(T(p,q)) + p^2 \cdot v_3(K) - \frac{p}{2}\Delta''_K(1)\phi'_{q,p}(1).$$

The required formula follows from it, by using

$$v_3(T(p,q)) = \frac{1}{2}\hat{\Theta}_{T(p,q)}(1) = \frac{1}{144}p(p^2 - 1)q(q^2 - 1),$$

$$\phi'_{q,p}(1) = \frac{1}{6}q(1 - p^2).$$

For the value of the first formula, see also [19].

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