A generalisation of the Heckmann - Schucking cosmological solution

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Abstract

An exact solution of the Einstein equations for a Bianchi -I universe in the presence of dust, stiff matter and cosmological constant, generalising the well-known Heckmann-Schucking solution is presented.

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1 Introduction

In spite of the development of the numerical methods for integration of the Einstein equations and an extensive application of the qualitative theory of differential equations to analysis of the behavior of the cosmological models, the construction of exact solutions still represents rather an attractive task. One can underline two important classes of the cosmological solutions known now. One of them is the class of exact solutions of Friedmann-Robertson-Walker isotropic cosmological models, which constitute a basis for comparison of theoretical predictions with observations \footnote{footnote}. Another important exact cosmological solution is the anisotropic Kasner solution for the empty
Bianchi-I universe \[2\], whose importance for theoretical physics is connected with its role in the description of the oscillatory approach to the cosmological singularity \[3\], which in turn appears very promising topic for the study in the string and M-theory context \[4\].

The Heckmann - Schucking \[5\] anisotropic solution for the Bianchi-I universe in the presence of the dust-like matter is of special interest because it constitutes some kind of bridge between these two types of the cosmological solutions: in the vicinity of the cosmological singularity it behaves as a Kasner universe, while at the later stage of the cosmological evolution it behaves as an isotropic flat Friedmann Universe. The Heckmann-Schucking solution has the following form: for the Bianchi-I universe with the metric

\[
\text{ds}^2 = \text{dt}^2 - a^2(t)\text{dx}^2 - b^2(t)\text{dy}^2 - c^2(t)\text{dz}^2
\]

filled with dust whose equation of state is

\[
p = 0
\]

the functions \(a(t), b(t)\) and \(c(t)\) are given by the formulae

\[
\begin{align*}
    a(t) &= a_0 t^{p_1} (t + t_0)^{2/3-p_1}, \\
    b(t) &= b_0 t^{p_2} (t + t_0)^{2/3-p_2}, \\
    c(t) &= c_0 t^{p_3} (t + t_0)^{2/3-p_3},
\end{align*}
\]

where the exponents \(p_1, p_2\) and \(p_3\) are the well-known Kasner exponents \[1\ \[2\] satisfying relations

\[
\begin{align*}
    p_1 + p_2 + p_3 &= 1, \quad (4) \\
    p_1^2 + p_2^2 + p_3^2 &= 1. \quad (5)
\end{align*}
\]

It is easy to see that the solution \[3\] is close to the Kasner solution when \(t \ll t_0\). In the limit \(t \gg t_0\) all the functions \(a(t), b(t)\) and \(c(t)\) are proportional to \(t^{2/3}\), i.e. their behavior coincides with that of the flat Friedmann universe filled with the dust. The energy density of the matter is given by the formula

\[
\varepsilon = \frac{E_0}{t(t + t_0)^3}
\]

where the constant \(E_0 \neq 0\) and does not depend on the choice of the Kasner exponents.
In this paper we generalise the Heckmann-Schucking solution for the case of the Bianchi-I universe filled with the mixture of three perfect fluids: dust, stiff matter and a cosmological constant. The presence of the cosmological constant in combination with other types of matter is of special interest, because the recent discovery of the phenomenon of the cosmic acceleration \[6\] makes the presence of the cosmological constant or some other exotic type of matter which mimics some basic features of the cosmological constant enavoidable for any realistic cosmological model. Another reason which makes the study of the models with the cosmological constant interesting is the fact that inflationary theories of a very early Universe \[7\] contain an effective cosmological constant providing a period of a quasi-exponential expansion at the beginning of the cosmological evolution.

The plan of the paper is the following one: in second section we integrate the Einstein equations for the Bianchi-I model filled with the mixture of three perfect fluids mentioned above; in the third section we consider different limit cases for our solution and rewrite it in the form maximally close to the canonical form of the Heckmann-Schucking solution \[3\].

2 Integration of the Einstein equations for a Bianchi-I cosmology in the presence of matter

We shall look for solutions of the Einstein equations for the Bianchi-I model filled with the mixture of three perfect fluids mentioned above; in the third section we consider different limit cases for our solution and rewrite it in the form maximally close to the canonical form of the Heckmann-Schucking solution \[3\].

\[T^\mu_\nu = \text{diag} (\varepsilon, -p, -p, -p).\] (7)

It is convenient to represent the functions \(a(t), b(t)\) and \(c(t)\) in the following form:

\[
\begin{align*}
a(t) &= R(t) \exp(\alpha(t) + \beta(t)), \\
b(t) &= R(t) \exp(\alpha(t) - \beta(t)), \\
c(t) &= R(t) \exp(-2\alpha(t)),
\end{align*}
\] (8)

where \(R(t)\) is the conformal factor while the functions \(\alpha(t)\) and \(\beta(t)\) characterise the anisotropy of the model. The components of the Ricci tensor have
the following form:

\[ R^0_0 = - \left( \frac{\ddot{R}}{R} + 3\dot{R}^2 + 6\dot{\alpha}^2 + 2\dot{\beta}^2 \right), \]  

(9)

\[ R^1_1 = - \left( \frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + 3\frac{\dot{R}}{R}(\dot{\alpha} + \dot{\beta}) + \dot{\alpha} + \dot{\beta} \right), \]  

(10)

\[ R^2_2 = - \left( \frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + 3\frac{\dot{R}}{R}(\dot{\alpha} - \dot{\beta}) + \dot{\alpha} - \dot{\beta} \right), \]  

(11)

\[ R^3_3 = - \left( \frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} - 6\frac{\dot{R}}{R}\dot{\alpha} - 2\ddot{\alpha} \right). \]  

(12)

Due to the isotropy of the energy-momentum tensor (7) one has

\[ R^1_1 = R^2_2 = R^3_3. \]  

(13)

From this equation it is easy to get equations describing for the asymmetry functions \( \alpha(t) \) and \( \beta(t) \):

\[ \ddot{\alpha} + 3\frac{\dot{R}}{R}\dot{\alpha} = 0, \]  

(14)

\[ \ddot{\beta} + 3\frac{\dot{R}}{R}\dot{\beta} = 0. \]  

(15)

The form of Eqs. (14) and (15) coincides with that for the Klein - Gordon equation for the massless scalar field on a Friedmann background, whose energy-momentum tensor coincides with the energy-momentum tensor of the stiff matter [8] with the equation of state

\[ p = \varepsilon. \]  

(16)

From Eqs. (14), (15) it follows immediately that

\[ \dot{\alpha} = \frac{\alpha_0}{R^3}, \]  

(17)

\[ \dot{\beta} = \frac{\beta_0}{R^3}, \]  

(18)

where \( \alpha_0 \) and \( \beta_0 \) are some positive constants.

The 00 component of the Einstein equations has now the form

\[ \frac{\dot{R}^2}{R^2} = \dot{\alpha}^2 + \frac{\dot{\beta}^2}{3} + \frac{4\pi G}{3}\varepsilon. \]  

(19)
We shall consider a universe filled with a mixture of three perfect fluids: dust, the stiff matter obeying the equation of state \( p = -\varepsilon \) and the cosmological constant, whose equation of state is \( p = -\varepsilon \). Choosing convenient normalisation of constants one can represent Eq. (19) for the universe filled with this mixture in the following form:

\[
\frac{\dot{R}^2}{R^2} = \dot{\alpha}^2 + \frac{\dot{\beta}^2}{3} + \Lambda + \frac{M}{R^3} + \frac{S}{R^6},
\]

where \( \Lambda \) is the cosmological term, while the constants \( M \) and \( S \) characterise the quantity of the dust and of the stiff matter in the universe respectively.

After substitution into Eq. (20) the expressions for \( \dot{\alpha} \) and \( \dot{\beta} \) from Eqs. (17) and (18) we come to the following equation for the conformal factor \( R(t) \)

\[
\frac{\dot{R}^2}{R^2} = \Lambda + \frac{M}{R^3} + \frac{S_0}{R^6},
\]

where

\[
S_0 = S + \alpha_0^2 + \frac{\beta_0^2}{3}.
\]

It is possible to integrate Eq. (21) explicitly and the result is the following one:

\[
R^3(t) = \frac{M}{2\Lambda} (\cosh 3\sqrt{\Lambda}t - 1) + \sqrt{\frac{S_0}{\Lambda}} \sinh 3\sqrt{\Lambda}t.
\]

The initial conditions for the solution (23) are chosen in such a way to provide the fulfilling of the relation \( R(0) = 0 \). Substituting the solution (23) into Eqs. (17) and (18) one has after taking integrals:

\[
\alpha(t) = \frac{\alpha_0}{3\sqrt{S_0}} \ln \left( \frac{e^{3\sqrt{\Lambda}t} - 1}{e^{3\sqrt{\Lambda}t} + 2\sqrt{S_0\Lambda-M}} \right),
\]

\[
\beta(t) = \frac{\beta_0}{3\sqrt{S_0}} \ln \left( \frac{e^{3\sqrt{\Lambda}t} - 1}{e^{3\sqrt{\Lambda}t} + 2\sqrt{S_0\Lambda-M}} \right).
\]

In the expressions (24), (25) we have omitted the integration constants, inclusion of which implies only the multiplication of factors \( a(t), b(t) \) and \( c(t) \) by some factors, that is not important for the flat (Bianchi - I) model.
Now, substituting the formulae (23), (24) and (25) into Eq. (8) on e comes to the following cosmological solution:

\[ a(t) = \left( \frac{M}{\Lambda} \right)^{1/3} \left( 1 + \frac{2\sqrt{S_0\Lambda}}{M} \right)^{\frac{\alpha_0 + \beta_0}{3\sqrt{S_0}}} \left( \sinh \frac{3\sqrt{\Lambda t}}{2} \right)^{\frac{1}{3} + \frac{\alpha_0 + \beta_0}{3\sqrt{S_0}}} \times \left( \sinh \frac{3\sqrt{\Lambda t}}{2} + \frac{2\sqrt{S_0\Lambda}}{M} \cosh \frac{3\sqrt{\Lambda t}}{2} \right)^{\frac{1}{3} - \frac{\alpha_0 + \beta_0}{3\sqrt{S_0}}} . \]

\[ b(t) = \left( \frac{M}{\Lambda} \right)^{1/3} \left( 1 + \frac{2\sqrt{S_0\Lambda}}{M} \right)^{\frac{\alpha_0 - \beta_0}{3\sqrt{S_0}}} \left( \sinh \frac{3\sqrt{\Lambda t}}{2} \right)^{\frac{1}{3} - \frac{\alpha_0 - \beta_0}{3\sqrt{S_0}}} \times \left( \sinh \frac{3\sqrt{\Lambda t}}{2} + \frac{2\sqrt{S_0\Lambda}}{M} \cosh \frac{3\sqrt{\Lambda t}}{2} \right)^{\frac{1}{3} + \frac{\alpha_0 - \beta_0}{3\sqrt{S_0}}} . \]

\[ c(t) = \left( \frac{M}{\Lambda} \right)^{1/3} \left( 1 + \frac{2\sqrt{S_0\Lambda}}{M} \right)^{\frac{-2\alpha_0}{3\sqrt{S_0}}} \left( \sinh \frac{3\sqrt{\Lambda t}}{2} \right)^{\frac{1}{3} + \frac{-2\alpha_0}{3\sqrt{S_0}}} \times \left( \sinh \frac{3\sqrt{\Lambda t}}{2} + \frac{2\sqrt{S_0\Lambda}}{M} \cosh \frac{3\sqrt{\Lambda t}}{2} \right)^{\frac{1}{3} - \frac{-2\alpha_0}{3\sqrt{S_0}}} . \] (26)

The solution presented above has rather a cumbersome form. In the next section we consider different limit cases and rewrite Eq. (26) in a more convenient Heckmann-Schucking-like form.

3 Limit cases and the Heckmann-Schucking-like form of the anisotropic cosmological solution

Let us consider the limit case \( \Lambda = 0 \) in Eq. (26), i.e. the case when the cosmological constant is absent. One has the following expressions:

\[ a(t) = \left( \frac{9}{4} M \right)^{1/3} t^{p_1} (t + t_0)^{\frac{3}{2} - p_1} , \]

\[ b(t) = \left( \frac{9}{4} M \right)^{1/3} t^{p_2} (t + t_0)^{\frac{3}{2} - p_2} , \]

\[ c(t) = \left( \frac{9}{4} M \right)^{1/3} t^{p_3} (t + t_0)^{\frac{3}{2} - p_3} . \] (27)
Here we have introduced the parameters
\[ t_0 = \frac{4\sqrt{S_0}}{3M}, \quad (28) \]
\[ p_1 = \frac{1}{3} - \frac{\alpha_0 + \beta_0}{3\sqrt{S_0}}, \]
\[ p_2 = \frac{1}{3} - \frac{\alpha_0 - \beta_0}{3\sqrt{S_0}}, \]
\[ p_3 = \frac{1}{3} + \frac{2\alpha_0}{3\sqrt{S_0}}, \quad (29) \]

The parameter \( t_0 \) plays the same role as in the Heckmann-Schucking solution \[23\], but now it depends not only on the anisotropy parameters \( \alpha_0 \) and \( \beta_0 \) but also on the quantity of the stiff matter in the universe (see Eq. \[22\]). It is easy to see that the parameters \( p_1, p_2 \) and \( p_3 \), which can be called “quasi-Kasner” parameters satisfy the relations
\[ p_1 + p_2 + p_3 = 1, \quad (30) \]
\[ p_1^2 + p_2^2 + p_3^2 = 1 - q^2, \quad (31) \]
where
\[ q^2 = \frac{2}{3} \frac{S}{\alpha_0^2 + \frac{\beta_0^2}{3} + S} = \frac{2S}{3S_0}. \quad (32) \]

The parameter \( q \) reflects the presence of the stiff matter in a universe and was introduced in \[8\]. Apparently, when the stiff matter is absent \( q = 0 \) and the solution \[27\] exactly coincides with the Heckmann-Schucking solution \[23\] for a Bianchi-I universe filled with dust. The presence of the stiff matter simply transforms the relation \[5\] for the Kasner exponents into Eq. \[31\].

Now, it convenient to rewrite the solution \[26\] using the Heckmann-Schucking time parameter \( t_0 \) and the quasi-Kasner exponents. We shall use also the parameter
\[ H_0 = \sqrt{\Lambda}, \quad (33) \]
which is nothing but a Hubble parameter for the DeSitter universe with a cosmological constant \( \Lambda \). Now the solution \[26\] looks like
\[ a(t) = \left( \frac{M}{H_0^2} \right)^{\frac{1}{2}} \left( 1 + \frac{3H_0t_0}{2} \right)^{p_1 - \frac{1}{2}} \left( \sinh \frac{3H_0t}{2} \right)^{p_1} \left( \sinh \frac{3H_0t}{2} + \frac{3H_0t_0}{2} \cosh \frac{3H_0t}{2} \right)^{\frac{1}{2} - p_1}, \]
The solution for the Bianchi-I cosmology in the presence of the cosmological constant, dust and stiff matter, presented in the Heckmann-Schucking-like form (34) is more convenient for the comparison with other solutions. We have already considered the case $\Lambda = 0$ which reproduces the Heckmann-Schucking solution (3) up to change of the relation (31) between the Kasner exponents, which occurs due to the presence of the stiff matter. For completeness we shall write down also the solution for the case when dust component is absent. Taking the corresponding limit in Eq. (34) we easily come to

\[
\begin{align*}
b(t) &= \left( \frac{M}{H_0^2} \right)^{\frac{1}{3}} \left( 1 + \frac{3H_0t}{2} \right)^{p_2 - \frac{1}{3}} \left( \sinh \frac{3H_0t}{2} \right)^{p_2} \left( \sinh \frac{3H_0t}{2} + \frac{3H_0t}{2} \cosh \frac{3H_0t}{2} \right)^{\frac{2}{3} - p_2}, \\
c(t) &= \left( \frac{M}{H_0^2} \right)^{\frac{1}{3}} \left( 1 + \frac{3H_0t}{2} \right)^{p_3 - \frac{1}{3}} \left( \sinh \frac{3H_0t}{2} \right)^{p_3} \left( \sinh \frac{3H_0t}{2} + \frac{3H_0t}{2} \cosh \frac{3H_0t}{2} \right)^{\frac{2}{3} - p_3}. 
\end{align*}
\]

(34)

We shall write down also asymptotics of the solution (34) for large and small times. At large times $t \to \infty$ only the cosmological constant is relevant and a complete isotropisation takes place. The corresponding asymptotics look like

\[
\begin{align*}
a(t) &\sim \exp(H_0t), \\
b(t) &\sim \exp(H_0t), \\
c(t) &\sim \exp(H_0t). 
\end{align*}
\]

(35)

At the beginning of the cosmological evolution when $t \to 0$ only anisotropy and the presence of stiff matter determine the behavior of functions $a(t), b(t)$ and $c(t)$. Their form is now

\[
\begin{align*}
a(t) &\sim t^{p_1}, \\
b(t) &\sim t^{p_2}, \\
c(t) &\sim t^{p_3}, 
\end{align*}
\]

(36)

where the exponents $p_1, p_2, p_3$ satisfy as usual the relations (31), (31). Let us notice that in the model under consideration there are two time scales: $t_0$ and $H_0^{-1}$. One can write down also “intermediate” asymptotics. In the case when $H_0t_0 < 1$ the solution for the time scale

\[
t_0 < t < \frac{1}{H_0},
\]

(37)
looks like
\[ a(t) \sim t^\frac{2}{3}, \quad b(t) \sim t^\frac{2}{3}, \quad c(t) \sim t^\frac{2}{3} \]  
and corresponds to the flat Friedmann universe filled with dust. For the case when \( H_0 t_0 > 1 \) and
\[ \frac{1}{H_0} < t < t_0 \]  
the solution coincides with that from Eq. (36), i.e. describes the flat DeSitter universe.

In conclusion, notice that recently \cite{9} exact solutions for Bianchi - I universes filled by stiff matter or in the presence of the cosmological constant were considered with use of the original technique \cite{5} using analogies between Einstein and Newton cosmologies. The qualitative theory for the Bianchi -I universes filled with different types of matter was developed in \cite{10}.

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