ANSWERS OF SOME PROBLEMS ABOUT GRAPH COLORING TEST GRAPHS

TAKAHIRO MATSUSHITA

Abstract. We give answers of two problems about graph coloring test graphs suggested by Kozlov. We prove that a graph whose chromatic number is 2 is a homotopy test graph. We prove there is a graph $T$ with two involutions $\gamma_1$ and $\gamma_2$ where $(T, \gamma_1)$ is a Stiefel-Whitney test graph, but $(T, \gamma_2)$ is not.

1. Introduction

Using topological methods to graph coloring problem is started from Lovász' proof of Kneser conjecture [7]. Lovász defined the neighborhood complexes and showed that the connectivity of the neighborhood complex gives a lower bound of the chromatic number. Babson and Kozlov investigated the Hom complexes in [1], and showed that their topologies are related to chromatic numbers. A test graph is a graph $T$ such that some inequality between the topological invariants of $\text{Hom}(T, G)$ and $\chi(G)$ holds for every graph $G$. Precise definitions we need are prepared in Section 2.

Kozlov suggested the following problems in [4] about test graphs.

Problem 1. Is a graph $G$ with $\chi(G) = 2$ a homotopy test graph? (See Conjecture 6.2.1 of [4])

Problem 2. Does there exist a graph $T$ having two different flipping involutions, $\gamma_1$ and $\gamma_2$, such that $(T, \gamma_1)$ is a Stiefel-Whitney test graph, whereas $(T, \gamma_2)$ is not? (See Section 6.1 of [4])

In this paper, we give the answers of these problems. The following is the answer of Problem 1.

Theorem 1. A graph whose chromatic number is 2 is a homotopy test graph.

Figure 1 describes the graph $T$ and involutions $\gamma_1$ and $\gamma_2$ of $T$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graphT.png}
\caption{graph $T$}
\end{figure}

1The precise statement of the conjecture of [4] is “Every connected bipartite graph is a homotopy test graph.” In some references, a graph $G$ is said to be bipartite if $\chi(G) \leq 2$. But in [4], Kozlov seems to consider that a bipartite graph is a graph whose chromatic number is 2. Indeed, it is obvious that, if $\chi(T) = 1$ or 0, then $T$ is not a homotopy test graph.
The involution $\gamma_1$ is the reflection with respect to the horizontal line, and the involution $\gamma_2$ is the reflection with respect to the vertical line. We prove the following, which is the answer of Problem 2.

**Theorem 2.** Let $T, \gamma_1, \gamma_2$ be the above. Then $(T, \gamma_1)$ is a Stiefel-Whitney test graph but $(T, \gamma_2)$ is not.

The rest of this paper organized as follows. In Section 2, we collect definitions, terminologies, and facts we need in this paper. In Section 3, we give the proof of Theorem 1, and in Section 4, we give the proof of Theorem 2 and some remarks.

### 2. Definitions and Facts

In this section, we recall the definitions of Hom complexes, Stiefel-Whitney test graphs, and homotopy test graphs. For more concrete discussions, we refer [2] and [4] to the reader.

In this paper, we assume that all graphs are nondirected, may have loops, whereas do not have multiple edges. Let $n$ be a nonnegative integer. We write $K_n$ for the complete graph with $n$-vertices. Namely, $V(K_n) = \{1, \cdots, n\}$ and $E(K_n) = \{(x, y) \in V(K_n) \times V(K_n) \mid x \neq y\}$. For a graph $G$, the chromatic number of $G$ is the number $\inf\{n \mid \text{There is a graph map } G \rightarrow K_n, \}$ and is denoted by $\chi(G)$.

A multihomomorphism from a graph $G$ to a graph $H$ is a map $\eta : V(G) \rightarrow 2^{V(H)}$ such that $\eta(v) \times \eta(w) \subset E(H)$ for any $(v, w) \in E(G)$. For multihomomorphisms $\eta$ and $\eta'$ from $G$ to $H$, we write $\eta \leq \eta'$ if $\eta(v) \subset \eta'(v)$ for any $v \in V(G)$. We write $\text{Hom}(G, H)$ for the poset of all multihomomorphisms from $G$ to $H$, called the Hom complex from $G$ to $H$.

An involution of a graph $T$ is a graph map $\gamma : T \rightarrow T$ with $\gamma^2 = \text{id}_T$. An involution $\gamma$ of $T$ is said to be flipping if there is a vertex $v \in V(T)$ with $(v, \gamma(v)) \in E(T)$. A pair $(T, \gamma)$, where $T$ is a graph and $\gamma$ is an involution of $T$, is called a $\mathbb{Z}_2$-graph, and if $\gamma$ is flipping, then $(T, \gamma)$ is said to be flipping $\mathbb{Z}_2$-graph.

A free $\mathbb{Z}_2$-complex is a CW complex $X$ with an involution $\tau$ such that $\tau e \cap e = \emptyset$ for any cell $e$ of $X$. For a free $\mathbb{Z}_2$-complex $X$, we write $\text{ht}_{\mathbb{Z}_2}(X)$ for the number $\sup\{n \geq 0 \mid w_1(X)^n \neq 0\}$ where $w_1(X)$ is the 1st Stiefel-Whitney class of the double cover $X \rightarrow X/\mathbb{Z}_2$. We regard that $\text{ht}_{\mathbb{Z}_2}(X) = -\infty$ if $X$ is empty. For the precise definition and basic properties of the 1st Stiefel-Whitney class of double covers are found in [5] for example.

It is easy to see that if $(T, \gamma)$ is a flipping $\mathbb{Z}_2$-graph, then $\text{Hom}(T, G)$ is a free $\mathbb{Z}_2$-complex with the involution $\eta \rightarrow \eta \circ \gamma$ for $\eta \in \text{Hom}(T, G)$ and for a loopless graph $G$. A flipping $\mathbb{Z}_2$-graph $(T, \gamma)$ is called a Stiefel-Whitney test graph, which is abbreviated to an SWT-graph in this paper, if for any loopless graph $G$, the inequality $\chi(G) \geq \text{ht}_{\mathbb{Z}_2}(\text{Hom}(T, G)) + \chi(T)$ holds, where $\chi(G)$ is the chromatic number of $G$.

For an integer $n \geq -1$, a topological space $X$ is said to be $n$-connected if for any integer $k \leq n$ and for any continuous map $f : S^k \rightarrow X$ can be extended to a map $g : D^{k+1} \rightarrow X$. For a topological space $X$, we write $\text{conn}(X)$ for the number $\sup\{n \geq -1 \mid X \text{ is } n\text{-connected}\}$. We assumed that $\text{conn}(X) = -\infty$ if $X$ is empty. A graph $T$ is called a homotopy test graph, abbreviated to an HT-graph in this paper, if for any graph $G$, the inequality $\chi(G) \geq \text{conn}(\text{Hom}(T, G)) + \chi(T)$ holds.

It is known that, if a flipping $\mathbb{Z}_2$-graph $(T, \gamma)$ is an SWT-graph, then $T$ is an HT-graph. $K_n$ is a SWT-graph for $n \geq 2$ with the involution exchanging 1 and 2 and fixing the other vertices, see [4]. Babson and Kozlov proved in [2] that odd cycles $C_{2r+1}$ are HT-graphs. Moreover, it is known that odd cycles are SWT-graphs with reflection. This fact is proved by Schultz in [8], and reproved in [6] and [9].
3. Proof of Theorem 1

First we remark that a topological space $Y$ which is a retract of an $n$-connected space $X$ is $n$-connected. Indeed, for any integer $k \leq n$ and for any continuous map $f : S^k \to Y$, $i \circ f : S^k \to X$ can be extended to a map $g : S^k \to X$, where $i : Y \hookrightarrow X$ is the inclusion, since $X$ is $n$-connected. Then $r \circ g : S^k \to Y$ is the extension of $f$, where $r : X \to Y$ is the retraction.

Lemma 3. Let $T_1$ and $T_2$ be graphs and suppose $T_1$ is a retract of $T_2$. If $T_1$ is an HT-graph, so is $T_2$.

Proof. Remark that $\chi(T_1) = \chi(T_2)$ in this case, since there are graph maps $T_1 \to T_2$ and $T_2 \to T_1$. Let $G$ be a graph and $n$ an integer with $n \geq -1$. Suppose $\text{Hom}(T_2, G)$ is $n$-connected. Since $\text{Hom}(T_1, G)$ is a retract of $\text{Hom}(T_2, G)$, we have that $\text{Hom}(T_1, G)$ is $n$-connected. Since $T_1$ is an HT-graph, we have the inequality $\chi(G) \geq \text{conn}(\text{Hom}(T_1, G)) + \chi(T_1) \geq n + \chi(T_2)$. Hence we have $T_2$ is an HT-graph. □

Proof of Theorem 1. Let $T$ be a graph with $\chi(T) = 2$. Since $K_2$ is an HT-graph and $T$ retracts to $K_2$, we have that $T$ is a homotopy test graph. □

4. Proof of Theorem 2

Lemma 4. Let $(T_1, \tau_1)$ and $(T_2, \tau_2)$ be flipping $\mathbb{Z}_2$-graphs with $\chi(T_1) = \chi(T_2)$. Suppose there is a $\mathbb{Z}_2$-graph map $f : T_1 \to T_2$. If $(T_1, \tau_1)$ is an SWT-graph, then so is $(T_2, \tau_2)$.

Proof. Let $G$ be a loopless graph. Since there is $\mathbb{Z}_2$-map $\text{Hom}(T_2, G) \to \text{Hom}(T_1, G)$, we have that $\text{ht}_{\mathbb{Z}_2}(\text{Hom}(T_2, G)) \leq \text{ht}_{\mathbb{Z}_2}(\text{Hom}(T_1, G))$. Then we have the inequality

$$\chi(G) \geq \text{ht}_{\mathbb{Z}_2}(\text{Hom}(T_1, G)) + \chi(T_1) \geq \text{ht}_{\mathbb{Z}_2}(\text{Hom}(T_2, G)) + \chi(T_2)$$

since $(T_1, \tau_1)$ is an SWT-graph and $\chi(T_1) = \chi(T_2)$. This implies that $(T_2, \tau_2)$ is an SWT-graph. □

Corollary 5. Let $T$ and $\gamma_1$ be those of the statement of Theorem 1. Then $(T, \gamma_1)$ is an SWT graph.

Proof. There is a $\mathbb{Z}_2$-graph map $C_5 \to (T, \gamma_1)$, where the involution of $C_5$ is a reflection. Since $\chi(C_5) = 3 = \chi(T)$ and $C_5$ is an SWT-graph, we have that $(T, \gamma_1)$ is an SWT-graph by Lemma 4. □

To prove Theorem 2, we only prove the following.

Proposition 6. $(T, \gamma_2)$ is not an SWT-graph.

Proof. To prove this, we only prove that the 1st Stiefel-Whitney class of $\text{Hom}(T, K_3)$ with the involution induced by $\gamma_2$ is not equal to 0. Indeed, if $w_1(\text{Hom}(T, G))$ is not equal to 0, then we have $\chi(K_3) = 3 < \text{ht}_{\mathbb{Z}_2}(\text{Hom}(T, K_3)) + \chi(T)$ and $(T, \gamma_2)$ is not an SWT-graph.

Define a graph map $f : T \to K_3$ by Figure 2. To prove $w_1(\text{Hom}(T, K_3)) \neq 0$, we only prove that $f$ and $f \circ \gamma_2$ is in the same component of $\text{Hom}(T, K_3)$.

![Figure 2](image-url)
Figure 3.
First we remark the following.

(∗) Let $\varphi$ and $\psi$ be graph maps from a loopless graph $G$ to a graph $H$. If there is $v \in V(G)$ such that $\varphi(x) = \varphi(x)$ for any $x \in V(G) \setminus \{v\}$. Then $\varphi$ and $\psi$ are contained in the same connected component of $\text{Hom}(G, H)$.

Indeed, under the assumption of (∗), the map $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}, x \mapsto \{\varphi(x), \psi(x)\}$ forms a multihomomorphism with $\varphi \leq \eta$ and $\psi \leq \eta$.

See Figure 3 (the previous page). See the forth and the back of each arrow in Figure 3 are satisfies the condition (∗) above. So Figure 3 implies that $f$ and $f \circ \gamma_2$ are in the same component in $\text{Hom}(T, K_3)$. □

These complete the proof of Theorem 2.

Finally we remark the following. Recall that, if the flipping $\mathbb{Z}_2$-graph $(T, \gamma)$ is an SWT-graph, then $T$ is an HT-graph. Theorem 2 implies that the inverse does not hold.

**Corollary 7.** There is a flipping $\mathbb{Z}_2$-graph $(T, \gamma)$ such that $T$ is an HT-graph but $(T, \gamma)$ is not an SWT-graph.

**Proof.** Let $T, \gamma_1, \gamma_2$ be those of Theorem 4. Then since $(T, \gamma_1)$ is an SWT-graph, $T$ is an HT-graph. But $(T, \gamma_2)$ is not an SWT-graph. □

**References**

[1] Eric Babson and Dmitry N. Kozlov, *Complexes of graph homomorphisms*, Israel J. Math, 152 (2006) 285-312
[2] Eric Babson and Dmitry N. Kozlov, *Proof of the Lovász conjecture*, Ann. of Math. (2), 165 (2007)(3):965-1007
[3] Anton Dochtermann and Carsten Schultz, *Topology of Hom complexes and test graphs for bounding chromatic number* Israel J. Math, 187 (2012) 371-417
[4] Dmitry N. Kozlov, *Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes*. In geometric combinatorics, volume 13 of IAS/Park City Math. Ser., pages 249-315. Amer. Math. Soc., Providence, RI, 2007.
[5] Dmitry N. Kozlov, *Combinatorial Algebraic Topology*. Algorithms and Computation in Mathematics, Vol. 21. Springer, Berlin (2008)
[6] Dmitry N. Kozlov, *Cobounding odd cycle colorings*, Electron. Res. Announc. Amer. Math. Soc. 12 (2006), 53-55
[7] L. Lovasz, *Kneser conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A, 25 (1978) (3):319-324
[8] Carsten Schultz, *A short proof of $w_n^c(\text{Hom}(C_{2r+1}, K_{n+2})) = 0$ for all $n$ and a graph colouring theorem by Babson and Kozlov*, Israel J. Math, 170 (2009):125-134.
[9] Carsten Schultz, *Graph colorings, spaces of edges and spaces of circuits*, Adv. Math., 221 (2009)(6):1733-1756.