THE TOPOLOGICAL OBSTRUCTIONS TO THE EXISTENCE OF AN IRREDUCIBLE SO(3) STRUCTURE ON A FIVE MANIFOLD

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Abstract. A nonstandard (maximal) inclusion $SO(3) \subset SO(5)$ associated with the irreducible representation $p_5$ of $SO(3)$ in $\mathbb{R}^5$ is considered. The topological obstructions for admitting the $SO(3)$ structure on the frame bundle over 5-manifold are investigated. The necessary and sufficient conditions are formulated.

1. Introduction

In Ref. [1] we introduced and investigated the irreducible (maximal) $SO(3)$ structure on a 5-dimensional manifold $M$. We found and described the tensor object reducing the structure group of the frame bundle of $M$ from $SO(5)$ to the irreducible $SO(3)$. That paper was mainly devoted to the local analysis of the geometry of manifold $M$ equipped with such a structure. Our motivation for investigation of structures of such kind was the paper [2] of Th. Friedrich, where he listed especially interesting types of special geometries in low dimensions. There are such interesting geometries like $G_2$ structure in dimension 7, $Spin(7)$ structure in dimension 8, $Spin(9)$ structure in dimension 16, $F_4$ structure in dimension 26; Friedrich also adds the $SO(3)$ structure in dimension 5 to this list.

In the case of any structure it is interesting to know under which topological conditions the structure exists on a manifold $M$. For example, it is well known that the $Spin$ structure on an oriented manifold $M$ does exist provided the second Stiefel-Whitney class of the tangent bundle $w_2(TM)$ vanishes.

The main goal of this paper is to prove the following criterion. There exists the maximal $SO(3)$ structure on an oriented 5-dimensional manifold $M$ if and only if there exists the standard $SO(3)$ structure (i.e. the tangent bundle decomposes $TM = E^3 \oplus \theta^2$) and the first Pontryagin class $p_1(TM)$ is divisible by 5. This result is used to construct non-trivial examples of 5-manifolds equipped with the maximal $SO(3)$ structure – see Proposition 1.5 below.

1.1. The irreducible $SO(3)$ structure. To fix the notation let us recall the explicit construction of the unique 5-dimensional representation of $SO(3)$. We identify $\mathbb{R}^5$ with the subspace of $3 \times 3$ real matrices

\[ M^5 = \{ A \in M_{3 \times 3}(\mathbb{R}) : A^T = A, \ tr(A) = 0 \}. \]
The action of $\text{SO}(3)$ on $\mathbb{M}^5$ is given by

$$\rho_5(h)A = h A h^T,$$

$\forall h \in \text{SO}(3), A \in \mathbb{M}.$

Each $\rho_5(h)$ defines the orthogonal transformation of $\mathbb{M}^5$, so the representation $\rho_5$ defines the inclusion

$$\iota_5 : \text{SO}(3) \hookrightarrow \text{SO}(5),$$

which is essentially different from the standard (diagonal) inclusion

$$j : \text{SO}(3) \hookrightarrow \text{SO}(5).$$

The image $\iota_5(\text{SO}(3))$ will be called irreducible or maximal $\text{SO}(3)$ and the usual one $j(\text{SO}(3))$ will be called standard $\text{SO}(3)$.

The irreducible $\text{SO}(3)$ structure on a 5-dimensional manifold $M$ is the reduction of the structure group of the frame bundle $FM$ to the irreducible $\text{SO}(3)$. It was proved in [1] that this reduction is obtained by a pair of tensors $(g, \Upsilon)$ satisfying the following relations.

**Definition 1.1.** The irreducible $\text{SO}(3)$ structure on a 5-dimensional manifold $M$ is a triple $(M, g, \Upsilon)$, where $g$ is a Riemannian metric and $\Upsilon$ is a rank 3 tensor field defining vector bundle morphism

$$\Upsilon : TM \rightarrow \text{End}(TM), \quad v \mapsto \Upsilon_v \in \text{End}(TM).$$

This morphism satisfies the following conditions

1. it is totally symmetric, i.e. $g(u, \Upsilon_v w) = g(w, \Upsilon_v u) = g(u, \Upsilon_w v)$ for all $u, v, w \in TM$;
2. it is trace free $\text{tr}(\Upsilon_v) = 0$,
3. for any vector $v \in \mathbb{R}^5$ the following identity holds

$$\Upsilon_v^2 v = g(v, v)v.$$

**Remark 1.2.** Let me recall briefly how we obtain the tensor $\Upsilon$ from the irreducible $\text{SO}(3)$ structure (see [1] for details). The explicit realization of the irreducible representation $\rho_5$ is constructed via identification of the point in $\mathbb{R}^5$ with the symmetric, trace-free $3 \times 3$ matrix $A$. The group action is realized by the adjoint transformation. Thus, the determinant $\det A$ is homogeneous invariant polynomial of degree 3; it defines the rank 3 symmetric tensor $\Upsilon$.

**Remark 1.3.** It is worth to note that the tensor $\Upsilon$ alone suffices to reduce the structure group to the irreducible $\text{SO}(3)$. One can read out the Riemannian metric from the identity [4]. The alternative definition of the irreducible $\text{SO}(3)$ structure on a manifold $M$, involving only the bundle morphism $\Upsilon$, is the following.

1. For any vector $v$ the endomorphism $\Upsilon_v$ is trace-free.
2. Any vector $v$ is an eigenvector of $\Upsilon_v^2$. The respective eigenvalue form $g(v, v)$ is quadratic and we assume to be positively defined.
3. The positive, quadratic form, defined in the previous point, provides a Riemannian metric $g$; the tensor $\Upsilon$ is totally symmetric with respect to this metric.

Now, the problem of existence of the irreducible $\text{SO}(3)$ structure arises. The main result of this paper is the following theorem.
Theorem 1.4. Let $M$ be an orientable 5-dimensional manifold. There exists an irreducible $\text{SO}(3)$ structure on $M$ if and only if

$$TM = E^3 \oplus \theta^2, \quad p_1(TM) = 5 \hat{p}, \quad \hat{p} \in H^4(M; \mathbb{Z}),$$

where $\theta^2$ is the 2-dimensional trivial bundle and $p_1(TM)$ is the Pontryagin class.

The above theorem provides wide variety of non-trivial compact examples of manifolds admitting the irreducible $\text{SO}(3)$ structure.

Proposition 1.5. Let $S$ be a complex surface and $M = S \times S^1$, where $S^1$ is a circle.

1. There exist the standard $\text{SO}(3)$ structure on $M$ if and only if $\chi(S) \equiv 0 \mod 2$.

2. There exist the irreducible $\text{SO}(3)$ structure on $M$ if and only if $\chi(S) \equiv 0 \mod 2$, and $\sigma(S) \equiv 0 \mod 5$, where $\sigma(S)$ is the signature of $S$.

Proof. Point 1. We fix an arbitrary Riemannian metric on the tangent bundle $TM$ and we consider the corresponding Stiefel fibration $V \to M$. Its fibre $V$ is the Stiefel manifold consisting of pairs $(v_1, v_2)$ of unit, orthogonal tangent vectors $v_1, v_2 \in TM$. The standard $\text{SO}(3)$ structure on $M$ corresponds to a section of $V$. The only obstruction in the construction of section over $S$ is the 4-th Stiefel-Whitney class $w_4(TM|_S) = w_4(TS)$. The evaluation of the latter class $w_4(TS)$ on the fundamental cycle of $S$ (i.e. “integration” over $S$) gives (see [6] again)

$$< w_4(TS), S > \equiv \chi(S) \mod 2.$$

Thus the Stiefel fibration restricted to $S$, $V|_S \to S$ admits a section $f$ if and only if $\chi(S)$ is even. The section $f \in \Gamma(V|_S \to S)$ can be further prolonged on the whole $M = S \times S^1$. This finishes the proof of point 1.

The second point of the Proposition is the consequence of Theorem 1.4 combined with the Hirzebruch signature formula $\sigma(S) = 3 < p_1(TS), S >$.

Remark 1.6. The above proposition implies, for example, that there exist the irreducible $\text{SO}(3)$ structure on $\widetilde{\mathbb{C}P}^2 \times S^1$, where $\widetilde{\mathbb{C}P}^2$ is the projective space $\mathbb{C}P^2$ with one point blown up.

2. Proof of Theorem 1.4

We consider the following 7-dimensional homogeneous spaces

$$V = \text{SO}(5)/j(\text{SO}(3)), \quad B = \text{SO}(5)/\iota_5(\text{SO}(3)),$$

which are the quotients of $\text{SO}(5)$ by the standard and the irreducible $\text{SO}(3)$ respectively. The homogeneous space $V = V_{2,5}$ is the Stiefel manifold of pairs of unit orthogonal vectors in $\mathbb{R}^5$. The second space $B$ is known as the Berger space [5, 3].

In the following proposition we recall one result about homologies of the homogeneous spaces $V$ and $B$. 
Proposition 2.1 (54). The homogeneous spaces $V$, $B$ are simply-connected manifolds with the following integral homologies

$$H_3(V) = \mathbb{Z}_2, \quad H_3(B) = \mathbb{Z}_{10},$$
$$H_i(V) = H_i(B) = \mathbb{Z}, \quad i = 0, 7,$$
$$H_i(V) = H_i(B) = 0, \quad i \neq 0, 3, 7.$$

Proof of Theorem 1.4. The main idea of proof goes as follows. We consider the $\text{SO}(5)$ principal bundle and the associated Berger and Stieffel fibrations. There does not exist the morphism of the latter fibrations. Nevertheless, we consider the Postnikov towers of fibrations and we construct a morphism of these towers up to the 4-th stages. Since the dimension of the base (i.e. the manifold $M$) is 5, this construction efficiently mimic the fibrations morphism.

Let us recall (see [8]) the notation related to the Postnikov towers of fibration. A fibration $E \to M$ provides a tower of fibre spaces $p_j : E_j \to E_{j-1}$ with fibers being the Eilenberg-McLane spaces $K(A_j, n_j)$ and with the following commutative diagram

$$
\begin{array}{ccc}
K(A_3, n_3) & \to & K(A_2, n_2) & \to & K(A_1, n_1) \\
\downarrow & & \downarrow & & \downarrow \\
E_3 & \to & E_2 & \to & E_1 \\
\downarrow & & \downarrow & & \downarrow \\
E & \to & E & \to & M \\
\downarrow & & \downarrow & & \downarrow \\
M & \to & M.
\end{array}
$$

Let $\text{SO}(5) \to \mathcal{P} \to M$ be an $\text{SO}(5)$-principal bundle over a base $M$. Let $B = \mathcal{P} \times_{\text{SO}(5)} B$, $V = \mathcal{P} \times_{\text{SO}(5)} V$ be associated bundles with fibers $B$ and $V$ respectively.

Lemma 2.2. There exist fibration morphisms $\phi_3, \phi_4$, between the respective Postnikov fibrations of $B$ and $V$, which induce isomorphisms of fibers and make the following diagram commutative

$$
\begin{array}{ccc}
\vdots & \to & \vdots \\
K(\mathbb{Z}_2, 4) & \to & K(\mathbb{Z}_2, 4) \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_2, 3) & \to & K(\mathbb{Z}_2, 3) \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_5, 3) & \to & M \\
\downarrow & & \downarrow \\
M, & & M.
\end{array}
$$
Now let us consider an oriented 5-dimensional manifold $M$. We have the fibrations $V$ and $B$ defined above. The existence of the usual $SO(3)$ and the irreducible $SO(3)$ structure is equivalent to the existence of a section of $V$ and $B$ respectively.

The existence of a section of $B$ is equivalent to the existence of a section over $M$ of the 4-th stage in the Postnikov tower $B_4$. The latter statement is a consequence of fact that the base $M$ is 5-dimensional and the fibers of higher fibrations in the Postnikov tower are 4-connected, so the construction of section over $M$ of a higher fibration is unobstructed. The analogous statement is true for the Stieffel fibration $V$.

Using the Lemma 2.2, we deduce that there is one to one correspondence between sections of $B_4$ and the following pairs of sections

$$\Gamma(B_4 \to M) \xrightarrow{i^{-1}} \left\{ \Gamma(B_3^{(1)} \to M), \quad \Gamma(V_3 \to M) \right\}.$$ 

The obstruction to the section of $B_3^{(1)} \to M$ is the characteristic class $\sigma \in H^4(M; \mathbb{Z}_5)$. Analyzing the cohomology groups of oriented grassmannian [6], we deduce that it must be

$$\sigma = \lambda \left[p_1(TM)\right]_5, \quad \lambda \in \mathbb{Z}_5,$$

where $[p_1(TM)]_5$ is the reduction modulo 5 of the Pontrjagin class. Thus, we have the alternative: either $\lambda \neq 0$ or any oriented 5-manifold $M$ such that $TM = E^3 \oplus \theta^2$ admits the irreducible $SO(3)$ structure.

To finish the proof we show the following

**Lemma 2.3.** Let $P$ be an $SO(3)$ principal bundle and $\rho_5$ be the irreducible 5-dimensional representation of $SO(3)$. Then, the first Pontrjagin class of the associated bundle is divisible by 5 i.e.

$$p_1(P \times_{\rho_5} \mathbb{R}^5) \equiv 0 \mod 5.$$

Let $K3$ denote the K3-surface. Let us recall that $\chi(K3) = 24$ and $\sigma(K3) = -16$. The manifold $M = K3 \times S^1$ provides an example of 5-manifold whose tangent bundle decomposes $TM = E^3 \oplus \theta^2$ (see Proposition 1.5) and whose Pontrjagin class $< p_1(TM), K3 > = -3 \cdot 16$ is not divisible by 5. This shows that $\lambda \neq 0$ in (6) and hence the Theorem 1.4 is proved.

**Remark 2.4.** Actually we have proven that the statement of the theorem remains valid after replacing the tangent bundle to a manifold with an arbitrary, oriented bundle of rank 5 over a base of dimension at most 5.

**Proof of Lemma 2.2.** The fibers of $p_1$ are Eilenberg-McLane spaces $K(\mathbb{Z}_5, 3)$ whose cohomologies with $\mathbb{Z}_2$ coefficients are trivial, at least up to gradation 5. Thus the map $p_1$ induces an isomorphism of cohomologies with $\mathbb{Z}_2$ coefficients (up to 5-th gradation). The fibrations $B_3^{(2)} \to B_3^{(1)}$ and $V_3 \to M$ are determined by the characteristic class of the respective fibration. In both cases $B$ and $V$ the generator of the 3-rd homotopy group is the image of the generator of $\pi_3(SO(5)) = \mathbb{Z}$ under the canonical projections $SO(5) \to B$ and $SO(5) \to V$. Thus the characteristic classes in both cases of Berger and Stieffel fibrations coincide. It is known [6] that in the case of Stieffel fibration this obstruction class is equal to the 4-th Stiefel-Whitney class $w_4(TM)$. 

□
Summing up the above considerations, we have proved that the fibration \( B (2) \to B (1) \) is isomorphic to the pull-back via \( p_1 \) of the \( V_3 \) fibration i.e. \( p_1^* (V_3 \to M) \). Thus, there exists the morphism \( \phi_3 \) as in the lemma.

To construct the morphism \( \phi_4 \) we show that the following fibrations over \( B (2) \) are isomorphic:

\[
(B_4 \to B (2)) \cong \phi_3^* (V_4 \to V_3).
\]

These fibrations are determined by the second characteristic elements \( k_B \in H^5 (B (2); \mathbb{Z}_2) \) and \( k_V \in H^5 (V_3; \mathbb{Z}_2) \) respectively; so, it is enough to prove that \( \phi_3^* (k_V) = k_B \). Since the bundles \( B \) and \( V \) are constructed out of the \( SO(5) \) principal bundle, it is enough to determine these characteristic elements for the tautological bundle over the classifying space \( BSO(5) \) i.e. the oriented grassmannian

\[
M = G_5 (\mathbb{R}^\infty).
\]

We consider the spectral sequences of the following fibrations

\[
B \to B (1), \quad B (2) \to B (1), \quad V \to M, \quad V_3 \to M.
\]

We omit the details of calculations which are quite standard. In this calculations we use the following known facts (with \( \mathbb{Z}_2 \) coefficients assumed) [7].

1. The Steenrod algebra structure of cohomologies of the Eilenberg-McLane space \( K (\mathbb{Z}_2, 3) \).
2. The non trivial cohomologies of the Berger space \( B \) and the Stiefel manifold \( V \) located in gradation 3 and 4. The Steenrod operation \( Sq^1 \) gives rise to the isomorphism of these spaces. The latter property is the consequence of the fact that \( Sq^1 \) coincides to the Bockstein homomorphism.
3. The transgression operation \( \tau \) in the spectral sequence commutes with the Steenrod squaring operations \( Sq^j \).
4. The characteristic elements \( k_B \) and \( k_V \) map to zero in cohomologies of the total spaces of fibrations, i.e. \( H^5 (B; \mathbb{Z}_2) \) and \( H^5 (V; \mathbb{Z}_2) \) respectively.

Using the above facts we can unequally determine characteristic elements \( k_B \) and \( k_V \). Since the Steenrod algebra structure of cohomologies of the Berger space \( B \) and the Stiefel manifold \( V \) coincide, the respective characteristic elements also coincide; more precisely, they are related by \( \phi_3^* \) which gives rise to the isomorphism of 5-th cohomologies (with \( \mathbb{Z}_2 \) coefficients)

\[
k_B = \phi_3^* k_V.
\]

Thus the morphism \( \phi_4 \) can be constructed.

Proof of Lemma 2.3. Let \( P \) be a principal \( SO(3) \) bundle, \( \rho_3 \) and \( \rho_5 \) be irreducible representation of \( SO(3) \) in dimension 3 and 5 respectively. We show the following relation among the Pontrjagin classes of the associated bundles

\[
p_1 (P \times_{\rho_3} \mathbb{R}^5) = 5 \cdot p_1 (P \times_{\rho_5} \mathbb{R}^3).
\]

Since the Pontrjagin classes are torsion-free, it is enough to verify the thesis of the lemma in the de’Rham cohomologies. Let us choose a basis \( (E_1, E_2, E_3) \) of the Lie algebra \( so(3) \) satisfying the standard commutation relations \( [E_1, E_2] =
\]
The representations $\rho_3, \rho_5$ map this basis to the following matrices (see [1]):

$$\rho_3(E_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \rho_3(E_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \rho_3(E_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. $$

$$\rho_5(E_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_5(E_2) = \begin{pmatrix} 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \rho_5(E_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}. $$

The following identities hold

$$\det (\lambda I + r_1 \rho_3(E_1) + r_2 \rho_3(E_2) + r_3 \rho_3(E_3)) = \lambda^3 + \lambda \left( r_1^2 + r_2^2 + r_3^2 \right),$$

$$\det (\lambda I + r_1 \rho_5(E_1) + r_2 \rho_5(E_2) + r_3 \rho_5(E_3)) = \lambda^5 + \lambda^3 \cdot 5 \cdot \left( r_1^2 + r_2^2 + r_3^2 \right) + \lambda (\ldots).$$

We choose an $\mathfrak{so}(3)$ connection $\Gamma$ on the principal bundle $\mathcal{P}$. The local curvature form of $\Gamma$ reads

$$K = r^1 E_1 + r^2 E_2 + r^3 E_3, \quad r^j \in \Omega^2. $$

The differential form representing the Pontrjagin class $p_1$ is constructed from the invariant polynomial of the curvature (see [3]). Using the identities (8) we get the relation (7).

□

References

[1] Bobieński M., Nurowski P., Irreducible $SO(3)$ geometry in dimension five, (in preparation).

[2] Th Friedrich, On types of non-integrable geometries, Rend. Circ. Mat. Palermo, Serie II, Suppl. 71 (2003), 99-113.

[3] Goethe S., Kitchloo N., Shankar, K., Diffeomorphism type of the Berger space $SO(5)/SO(3)$, Amer. J. Math., 126, (2004), 395–416.

[4] Husemoller D., Fibre bundles, Springer-Verlag, New York, 1994.

[5] Kitchloo N., Shankar, K., On complexes equivalent to $S^3$-bundles over $S^4$, Internat. Math. Res. Notices, 8, (2001), 381–394.

[6] Milnor J. W., Stasheff J. D., Characteristic classes, Ann. of Math. Stud. 76, Princeton Univ. Press, Princeton, NJ, 1974.

[7] Mosher R. E., Tangora M. C., Cohomology operations and applications in homotopy theory, Harper & Row Publishers, New York, 1968.

[8] Spanier E. H., Algebraic topology, Springer-Verlag, New York.

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