A Comparison of LPV Gain Scheduling and Control Contraction Metrics for Nonlinear Control *

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Abstract: Gain-scheduled control based on linear parameter-varying (LPV) models derived from local linearizations is a widespread nonlinear technique for tracking time-varying setpoints. Recently, a nonlinear control scheme based on Control Contraction Metrics (CCMs) has been developed to track arbitrary admissible trajectories. This paper presents a comparison study of these two approaches. We show that the CCM based approach is an extended gain-scheduled control scheme which achieves global reference-independent stability and performance through an exact control realization which integrates a series of local LPV controllers on a particular path between the current and reference states.

1. INTRODUCTION

In many industrial applications, systems with nonlinear dynamical behavior are required to be operated in a wide range of operating conditions. A widespread approach for this situation is gain-scheduled control using linear parameter-varying (LPV) system representations (Papageorgiou et al., 2000; Rugh and Shamma, 2000; Klatt and Engell, 1998). The underlying idea is to introduce a so-called scheduling variable \( \sigma \) that indicates the current operating point of the system and construct a linear model that describes the local, linearized dynamics of the plant around each operating point. The parameters of the resulting model are dependent on, i.e., vary with \( \sigma \). Next, assuming that \( \sigma \) is an external variable (independent from the inputs) an LPV controller dependent on \( \sigma \) is designed that, by using linear system theory (Becker and Packard, 1994), ensures stability and performance specifications for the LPV model under possible variations of \( \sigma \) in a user specified region of operating conditions \( P \). Finally, a nonlinear control law is obtained by substituting \( \sigma \) with measured information of the operating point of the system. The name of this approach comes from that the changes of the controller parameters are scheduled based on \( \sigma \).

There are many approaches available to construct an LPV model of the plant based on this methodology, see Bachnas et al. (2013) for an overview. Typically, the plant is linearized around a given set of equilibrium points (griding of \( P \)) and the resulting set of LTI models are interpolated over \( P \) or linearization is accomplished over input and state trajectories. Similarly the LPV controller can be obtained by designing LTI controllers separately for finite set of values of \( \sigma \) and then interpolating these LTI controllers on \( P \) or parametrizing an LPV controller and solving the stabilization and performance problem jointly over \( P \). Typically, local equilibrium-independent stability and performance can be assured via these methods, requiring \( \sigma \) to be “sufficiently slow-varying” (Rugh and Shamma, 2000). As we show, this drawback is mainly due to two reasons: the lack of exact realization of the control law synthesized on the local linearized dynamics of the system and that the actual closed-loop trajectory of the system introduces residual terms which are not part of the LPV model for which the control law was designed.

It is important to highlight that next to gain-scheduling based modeling and control, which is often called a local LPV approach, modern LPV control methods are based on directly transforming the nonlinear system model via the so-called global embedding principle, see Tóth (2010); Hoffmann and Werner (2015) for an overview, and then synthesizing an LPV control law that gives stability and performance guarantees over the all possible variations of \( \sigma \). Such methods had been thought superior over gain-scheduling techniques as they provided direct guarantees for the embedded nonlinear system following a differential inclusion concept. However, recent studies indicate that performance issues during reference tracking objectives might still be present due to the considered stability objective of the synthesis. To overcome these issues, an improved stability notion seems to be connected to the differential dynamics (local linearization of the plant) (Scroletti et al., 2015).

Contraction theory which uses similar local linearization models has gained attention in nonlinear analysis (Lohmiller and Slotine, 1998; Forni and Sepulchre, 2014). Related works include velocity linearization (Leith and Leithead, 2000) and Gâteaux derivative (Fromion et al., 2001; Fromion and Scorletti, 2003). Recently, contraction
analysis was extended to constructive nonlinear control design by using a differential version of control Lyapunov function - Control Contraction Metric (CCM) (Manchester and Slotine, 2017, 2018). Further extensions of the CCM based approach include distributed control (Shiromoto et al., 2018), distributed economic model predictive control (MPC) (Wang et al., 2017).

The main contribution of this paper is a comparison study between the CCM based nonlinear control approach and the LPV gain scheduling technique using local linearization and LMI based synthesis. For simplicity, only state feedback control design is considered. We show that CCM based control is an extended LPV gain scheduling approach. First, the so-called differential dynamics in contraction theory can be seen as a local LPV model which takes linearization along any admissible solution rather than an equilibrium family in conventional gain-scheduling. Second, similar parameter-dependent LMI conditions are derived as in local LPV synthesis. One difference is that the CCM based approach explicitly takes the original nonlinear plant into account leading to less conservative results. Furthermore, it uses an exact control realization, which integrates a series of local controllers on a particular path joining the current and reference state trajectory, not resulting in any hidden coupling terms. Based on this, local stability and performance design can be carried onto the entire state space by investigating the curved distance between the state and reference trajectory.

Paper outline. Section 2 gives a brief review of the LPV gain scheduling approach using local linearization, which is mainly adopted from Rugh and Shamma (2000). Section 3 discusses the various connections and extensions between CCM and LPV based approaches. An illustrative example is presented in Section 4.

Notations. $|x|$ denotes the Euclidean norm of a vector $x$ and positive (negative) definiteness of a Hermitian matrix $X$ is denoted as $X > 0$ ($X < 0$). $C^k$ denotes the set of $k$th times differentiable. $L_2$ is the space of square-integrable vector signals on $\mathbb{R}_{\geq 0}$, i.e., $\|f\| := \left(\int_0^\infty |f(t)|^2 dt\right)^{1/2} < \infty$. The causal truncation $(\cdot)_T$ is defined by $(\cdot)_T(t) := f(t)$ for $t \leq T$ and 0 otherwise. $L_2$ is the space of vector signals on $\mathbb{R}_{\geq 0}$ whose causal truncation belongs to $L_2$.

A Riemannian metric on $\mathbb{R}^n$ is a smooth matrix function $M(x) > 0$ which defines an inner product $\langle \delta_1, \delta_2 \rangle_x = \delta_1^T M(x) \delta_2$ for any two tangent vector $\delta_1, \delta_2$. A metric is called uniformly bounded if there exist positive constants $a_2 \geq a_1$ such that $a_1 I < M(x) < a_2 I, \forall x \in \mathbb{R}^n$. The derivative of $M(x)$ along a vector field $f$ is defined as $\partial f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} f_i$. The piecewise smooth paths $c : [0, 1] \to \mathbb{R}^n$ with $c(0) = x_0$ and $c(1) = x_1$. The curve length and energy of $c(\cdot)$ is defined by

$$\ell(c) = \int_0^1 \sqrt{\dot{c}_s \cdot c_s(c_s)} ds \quad \text{and} \quad \varepsilon(c) = \int_0^1 \|c_s(c_s)\| ds$$

where $c_s = \partial c / \partial s$, respectively. The geodesic $\gamma(\cdot)$ denotes a path with the minimal length, i.e., $\ell(\gamma) = \inf_{c \in \Gamma(x_0,x_1)} \ell(c)$. The Riemann distance between $x_0$ and $x_1$ is defined as $d(x_0,x_1) = \ell(\gamma)$. For any geodesic $\gamma$, we have $\varepsilon(\gamma) = \ell^2(\gamma)$, see more details in Do Carmo (1992).

2. Gain Scheduling Approach

2.1 Problem Formulation

Consider the following nonlinear plant:

$$\dot{x} = f(x, u, w), \quad z = h(x, u, w) \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^p$ are state, control, disturbance and performance output signals, respectively. The functions $f$ and $h$ are assumed to be smooth and time-invariant. The point $(x_c, u_c, z_c, w_c)$ is said to be an equilibrium point for system (1) if

$$F(x_c, u_c, z_c) = 0, \quad H(x_c, u_c) = 0. \quad (2)$$

To formulate an LPV model of (1) based on a local approach, assume that the equilibrium points are uniquely characterized by $x_c$ and introduce a scheduling variable $\sigma \in \mathbb{R}^{n_{\sigma}}$ that depends on the state, i.e.,

$$\sigma = g(x), \quad (3)$$

where $g$ is a smooth nonlinear vector function. Note that $\sigma$ can also depend on $w$ if it is measurable. Restrict $\sigma(\cdot)$ to belong to the set

$$\mathcal{T} = \{ \sigma \in C^1 : \sigma(t) \in \mathcal{P}, \sigma(t) \in \mathcal{P}, \forall t \geq 0 \} \quad (4)$$

where $\mathcal{P} = \{ \sigma \in \mathbb{R}^{n_{\sigma}} : |\sigma_i| \leq \sigma_i, \forall i \}$ and $\mathcal{P} = \{ p \in \mathbb{R}^{n_{\sigma}} : |p_i| \leq \sigma_i, \forall i \}$ and $i = 1, \ldots, n_{\sigma}$. The following definition describes an equilibrium family parameterized by the scheduling variable $\sigma$.

Definition 1. The set $\{(x_c, u_c, z_c, w_c(\sigma))\}_{\sigma \in \mathcal{P}}$ is said to be an equilibrium family for system (1) if $x_c(\cdot), u_c(\cdot), z_c(\cdot), w_c(\cdot)$ are smooth functions, and $(x_c, u_c, z_c(\sigma))$ is an equilibrium point for all $\sigma \in \mathcal{P}$.

For the sake of simplicity, we only consider the case of state feedback control in this paper. Specifically, we aim to construct gain-scheduled laws:

$$u = \kappa(x, \sigma) \quad (5)$$

in terms of the function $\kappa$, which satisfy the following two objectives:

1) The closed-loop system of (1) and (5) is equilibrium-independent stable, i.e., under $w = 0$, the setpoint $x_c(\sigma)$ is stable for all $\sigma \in \mathcal{P}$.

2) The closed-loop system of (1) and (5) achieves an equilibrium-independent $L_2$-gain bound of $\alpha$. That is, for all $\sigma \in \mathcal{P}, \forall t > 0$ and $w \in L_2$, we have

$$\|z(t, \sigma)\|^2 \leq \alpha^2 \|w(t)\|^2 + \beta(x(0), x_c(\sigma)) \quad (6)$$

where $\beta(x_1, x_2) \geq 0$ with $\beta(x, x) = 0$.

2.2 System Linearization

In this paper, we focus on the gain scheduling approach that is based on (i) deriving an LPV model of (1) based on local linearizations over an equilibrium family and, based on this model, (ii) synthesizing a gain scheduled control law (5). According to this local method, the LPV model is derived as follows:

$$\dot{x}_3 = \begin{bmatrix} A(\sigma) & B_w(\sigma) & B_w(\sigma) \\ C(\sigma) & D(\sigma) & D(\sigma) \end{bmatrix} \begin{bmatrix} x_3 \\ u_3 \\ w_3 \end{bmatrix}, \quad \sigma \in \mathcal{P} \quad (7)$$

where $x_3 = x - x_c(\sigma), u_3 = u - u_c(\sigma), w_3 = w - w_c(\sigma), z_3 = z - z_c(\sigma)$ are deviation variables. The matrices $A, B_w, C, D_w$ are defined as the evaluations of $\partial f / \partial x, \partial f / \partial u, \partial h / \partial x, \partial h / \partial u$ at the $\sigma$ defined equilibrium point.
2.3 Linear Control Design

The aim of control design is to construct a linear state feedback law that measures $x_\sigma$ and $\sigma$ to produce $u_\delta$ such that the gain of external input $w_\delta$ on performance output $z_\delta$ is bounded. The controller takes the form of

$$u_\delta = K(\sigma)x_\delta,$$  \hspace{1cm} (8)

By substituting (8) into (7), we can write the LPV closed-loop dynamics as follows

$$
\begin{bmatrix}
x_\delta \\
z_\delta
\end{bmatrix}
= \begin{bmatrix}
A(\sigma) & B(\sigma) \\
C(\sigma) & D(\sigma)
\end{bmatrix}
\begin{bmatrix}
x_\delta \\
w_\delta
\end{bmatrix}
$$  \hspace{1cm} (9)

with $A = A + B_\sigma K$, $B = B_\sigma$, $C = C + D_\sigma K$, $D = D_\sigma$. Design of (8) is considered in terms of ensuring exponential stability of (9) for all $\sigma \in \mathcal{T}$. The following theorem presents a sufficient condition for this objective.

**Theorem 2.** The unforced closed-loop system

$$\dot{x}_\delta = A(\sigma)x_\delta, \quad \sigma \in \mathcal{T}$$  \hspace{1cm} (10)

is exponentially stable if there exists a $X(\sigma) > 0$ such that

$$\text{He}(X(\sigma)A(\sigma)) + \sum_{i=1}^{n_x} p_i \frac{\partial X(\sigma)}{\partial \sigma_i} < 0$$  \hspace{1cm} (11)

for all $\sigma \in \mathcal{P}$ and $\rho \in \mathcal{P}$. The above theorem implies that $V(x, \sigma) = x_\delta^T X(x_\delta)$ is a parameter-dependent Lyapunov function for system (10). Note that (11) is not convex as it contains cross terms involving $X(\sigma)$ and $K(\sigma, \rho)$. Applying a congruence transformation and substitution as $Y(\sigma) = X^{-1}(\sigma)$ and $L(\sigma) = K(\sigma)Y(\sigma)$ yields a convex formulation:

$$\text{He}(A(\sigma)Y(\sigma) + B_\delta(\sigma)L(\sigma)) - \sum_{i=1}^{n_x} p_i \frac{\partial Y(\sigma)}{\partial \sigma_i} < 0$$  \hspace{1cm} (13)

for all $\sigma \in \mathcal{P}$ and $\rho \in \mathcal{P}$. There are various efficient ways to solve (13) and even alternative methods to synthesize $K$ (Rugh and Shamma, 2000; Hoffmann and Werner, 2015). Under mild conditions, exponential stability of (10) implies a bounded $L_2$ gain for the LPV system (9).

**Theorem 3.** A controller (8) achieves a performance level of $\alpha$ for LPV system (7) if there exists $X(\sigma) > 0$ such that, for all $\sigma \in \mathcal{P}$ and $\rho \in \mathcal{P}$,

$$
\begin{bmatrix}
X(\sigma, \rho) & X(\sigma)B(\sigma) & \alpha^{-1}C'(\sigma) \\
\alpha^{-1}C(\sigma) & \alpha^{-1}D(\sigma) & -I
\end{bmatrix}
\begin{bmatrix}
\dot{X}(\sigma, \rho) & X(\sigma)B(\sigma) & \alpha^{-1}C'(\sigma) \\
\alpha^{-1}C(\sigma) & \alpha^{-1}D(\sigma) & -I
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
$$  \hspace{1cm} (14)

where $X(\sigma, \rho) = \text{He}(X(\sigma)A(\sigma)) + \sum_{i=1}^{n_x} p_i \frac{\partial X(\sigma)}{\partial \sigma_i}$.

By applying the congruence transformation to the above formulation, we can synthesize an LPV controller which achieves a minimal $L_2$-gain bound for the closed-loop system. Note that besides of addressing all possible equilibrium points as a continuous variation of $\sigma$, a gridded design of $K$ can be also considered for a finite set of values of $\{\sigma(\iota)\}_{\iota=1}^N$. By this method, local LTI controllers $K(\sigma(\iota))$ are independently designed and then an LPV controller (8) is obtained by applying various interpolation schemes and methods. However, we are not considering such methods here due to their inferior performance and lack of stability guarantees.

2.4 Controller Realization

After design of a gain-scheduled controller (8) in the deviation variables, it is important (i) how (5) is obtained and (ii) how $\sigma$ is computed during online operation of the system, which are together called controller realization. The LPV control realization problem is to construct a gain-scheduled law $u = \kappa(x, \sigma)$ such that

$$u_\delta(\sigma) = \kappa(x_\delta(\sigma), \sigma),$$  \hspace{1cm} (15a)

$$\frac{\partial \kappa}{\partial x}(x_\delta(\sigma), \sigma) = K(\sigma).$$  \hspace{1cm} (15b)

Condition (15a) implies that $(x_\delta(\sigma), \kappa(x_\delta(\sigma), \sigma), u_\delta(\sigma), h(x_\delta(\sigma), \kappa(x_\delta(\sigma), \sigma), w_\delta(\sigma)))$ is an equilibrium point for the closed-loop system and, by (15b), linearization of $u = \kappa(x, \sigma)$ at this equilibrium is the LPV controller (8).

**Remark 4.** Condition (15b) implies that the matrix function $K'(x) = K(g(x))$ is completely integrable. This can be ensured by requiring that each row of $K'$ satisfies the Schwarz condition, i.e., $\frac{\partial K_{ik}}{\partial x_k} = \frac{\partial K_{ik}}{\partial x_i}$. Although this constraint is linear and hence convex in $K$, it is highly nonlinear and not convex jointly in the decision variables $L, Y$ for (13), since $K = LY^{-1}$.

An intuitive choice of control realization in the literature is

$$u = u_\delta(\sigma) + K(\sigma)[x - x_\delta(\sigma)].$$  \hspace{1cm} (16)

Under the assumption that the equilibrium points of (1) are uniquely characterized by $x_\delta$, $\sigma$ can be expressed in terms of $x$ via (3). Using this relation, (16) reads as

$$u = u_\delta(g(x)) + K(g(x))[x - x_\delta(g(x))].$$  \hspace{1cm} (17)

The main “trick” behind of this gain-scheduling approach is that $\sigma$ is treated as a parametric/dynamic uncertainty throughout the design process, but during controller realization is substituted by a function of a measured variable characterizing the operating point changes (Rugh and Shamma (2000)). Although $\sigma$ is implicitly involved via equilibrium parameterizations, linearization of (17) may not satisfy condition (15b) since

$$u_\delta = K(\sigma)x_\delta + \left[\frac{\partial u_\delta(\sigma)}{\partial \sigma} - K(\sigma)\frac{\partial x_\delta(\sigma)}{\partial \sigma}\right]\frac{\partial \kappa}{\partial x}(x_\delta(\sigma), \sigma)x_\delta,$$

with $\sigma = g(x)$, contains additional terms, called hidden coupling terms. These terms may lead to closed-loop instability regardless the fact that exponential stability is achieved in the LPV design stage, which is a well-known disadvantage of local LPV controller design (see Example 8 in Rugh and Shamma (2000)).

2.5 Stability and Performance Assessment

Using the gain scheduling concept, stability and performance guarantees are only given in a local sense, i.e., at the considered equilibrium family. Hence, it is not clear whether the closed-loop system under gain-scheduled control (17) has stability and performance guarantees over a region of the state-space (e.g., transition from one equilibrium to another). The following result reveals that such guarantees do hold when the initial state is sufficiently close to a “slowly varying” reference.
Theorem 5. (Rugh and Shamma (2000)). Assume that the controller (17) does not have hidden coupling terms and the eigenvalues of the linearized closed-loop system have strictly negative real parts for all $\sigma \in \mathcal{P}$. Given positive constants $\rho$ and $T$, if there exist positive constants $\delta(\rho)$ and $\bar{p}(\rho, T)$ satisfying

$$\frac{1}{T} \int_0^T |x(t) - x_c(\sigma(t))| \, dt < \delta,$$

then

$$|x(t) - x_c(\sigma(t))| < \rho, \quad \forall t \geq 0. \quad (19)$$

The basic idea of gain-scheduled control (17) is to track a reference trajectory lying on the equilibrium manifold, i.e., $(x_e, u_e, w_e, z_e)(\sigma(t))$. However, this reference is not an admissible trajectory to the closed-loop system $\dot{x} = f(x, u, w)$ with $\sigma = g(x)$ since simple substitution yields a residual term $E(\sigma)\sigma(t)$ with $E(\sigma) = \frac{\partial \sigma(x)}{\partial x}$. Therefore, the actual linearization of the closed-loop system with $w_1(t) = 0$ is

$$\dot{x} = A(\sigma)x + E(\sigma)\dot{\sigma}. \quad (20)$$

If the rates of parameter variation are not “sufficiently slow”, the residual terms can drive the state away from the small neighborhood of $x_e(\sigma)$, which may violate the local stability design based on gain scheduling. The LPV approach requires excessive simulations or even experiments to verify global stability and performance (Rugh and Shamma, 2000). Due to these issues, global embedding based LPV control has been introduced (see Hoffmann and Werner (2015); Tóth (2010) and references therein). However, recent work indicates that performance issues during reference tracking using global embedding approach might still be present (Scorletti et al., 2015).

3. CCM BASED NONLINEAR CONTROL

In this section, we will show that the CCM based nonlinear control framework (Manchester and Slotine, 2017, 2018) extends the local LPV-based gain scheduling approach in the following two aspects: (i) it uses an exact control realization which does not result in any hidden coupling term, and (ii) local stability and performance design can be carried onto the entire state space by investigating the curved distance between the state and reference point.

3.1 Problem Formulation

Instead of stabilizing system (1) around an equilibrium manifold, the CCM approach considers a more general tracking control problem where the target trajectory $(x^*, u^*)(\cdot)$ is a forward-complete solution of the nominal system

$$\dot{x} = F(x, u) = f(x, u, 0), \quad z = H(x, u) := h(x, u, 0), \quad (21)$$

with $x^*(\cdot)$ piecewise differentiable and $u^*(\cdot)$ at least piecewise continuous. We adopt stronger notations for stability and performance from Manchester and Slotine (2018).

Definition 6. System (21) is said to be reference independent if $x^*(\cdot)$ is a universal exponentially stable under a feedback controller if, for any target trajectory $(x^*, u^*)(\cdot)$ and any initial condition $x(0)$, there exist a convergence rate $\lambda > 0$ and a $C > 0$ such that the closed-loop state trajectory $x(t)$ satisfies

$$|x(t) - x^*(t)| \leq Ce^{-\lambda t}|x(0) - x^*(0)|. \quad (22)$$

Definition 7. A control system for system (1) achieves universal $L_2$ bound of $\alpha > 0$ if for any target trajectory $(x^*, u^*)(\cdot)$, any initial condition $x(0)$ and input $w \in \mathcal{L}_2^e$, the following inequality holds for all $T > 0$:

$$\|(z - z^*)_T\|^2 \leq \alpha^2 \|(w)_T\|^2 + \beta(x(0), x^*(0)) \quad (23)$$

where $\beta(x_1, x_2) \geq 0$ with $\beta(x, x) = 0$.

Compared with the notation of incremental stability used in Scorletti et al. (2015), universal stability is weaker since it only requires the convergence between the state and reference state trajectory rather than any pair of state trajectories.

The control design objective is to construct state feedback laws that depend explicitly on known, but otherwise arbitrary parameters $(x^*, u^*)$ defined by a target trajectory:

$$u = \kappa(x, x^*, u^*) \quad (24)$$

such that the closed-loop system is universal exponentially stable and it achieves an universal $L_2$-gain bound of $\alpha$ for all $(x^*, u^*)(\cdot) \in \mathcal{T}$.

3.2 System Linearization

In the CCM based approach, Jacobian linearization is extended to a general admissible solution $\sigma(\cdot) = (x(\cdot), u(\cdot), w(\cdot))$ of system (1). Other related linearization techniques include velocity linearization (Leith and Leithead, 2000) and Gâteaux derivative (Fromion et al., 2001).

We can construct the linearized systems (called differential dynamics) as follows:

$$\begin{bmatrix} \dot{\delta}_x \\ \dot{\delta}_u \end{bmatrix} = \begin{bmatrix} A(\sigma) & B_u(\sigma) & B_w(\sigma) \\ C(\sigma) & D_u(\sigma) & D_w(\sigma) \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_u \end{bmatrix} \quad (25)$$

where the matrices $A, B_u, B_w, C, D_u, D_w$ are defined in a similar way as in the LPV approach. Here, the variables $\delta_x, \delta_u, \delta_w, \delta_\sigma$ are the virtual displacement between neighboring solutions (Lohmiller and Slotine, 1998) or the tangent vector of the solution manifold (Forni and Sepulchre, 2014), which are only locally defined near the trajectory $\sigma(\cdot)$. Note that evaluated along any trajectory, the differential dynamics can be seen as an LPV system.

3.3 Linear Control Design

The control synthesis searches for a differential controller:

$$\delta_u = K(\sigma)\delta_x, \quad (26)$$

which makes the unforced closed-loop dynamics

$$\dot{\delta}_x = A(\sigma)\delta_x := [A(\sigma) + B_u(\sigma)K]\delta_x \quad (27)$$

exponentially stable. A sufficient condition is

$$\text{He}(M(x)A(\sigma)) + 2M(x) + \partial f(\sigma)M(x) < 0. \quad (28)$$

This implies that the control contraction metric (CCM) $V(x, \delta_x) = \delta_\sigma^T M(x) \delta_\sigma$ is a differential version of the control Lyapunov function. By introducing the dual CCM $W(x) = M^{-1}(x)$, (28) can be rewritten as a $\sigma$-dependent LMI:

$$\text{He}(AW + B_uL) + 2\lambda W - \delta f W < 0. \quad (29)$$

with $L(\sigma) = K(\sigma)M(x)$. The above condition is convex but infinite dimensional as the decision variables are sets of smooth matrix functions. Finite dimensional LMI approximations can be constructed by projecting the decision
variables into the linear combinations of a finite basis functions (e.g., polynomials up to certain order), and solve the problem either by gridding over $\sigma$, or by the sum-of-squares relaxation (Parrilo, 2003). For control-affine systems, under slightly stronger conditions, the pointwise LMI (29) depends only on $x$, dramatically reducing its dimension (see details in Manchester and Slotine (2017)). The closed-loop differential dynamics can be written as

$$\begin{bmatrix} \delta_x \\ \delta_w \end{bmatrix} = \begin{bmatrix} A(\sigma) & B(\sigma) \\ C(\sigma) & D(\sigma) \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_w \end{bmatrix}$$

(30)

where $A, B, C, D$ are defined in a similar way as in the LPV case. Performance verification can be done via the following $\sigma$-dependent LMI:

$$\begin{bmatrix} M & MB & \alpha^{-1}C' \\ B'M & -I & \alpha^{-1}D' \\ \alpha^{-1}C & \alpha^{-1}D & -I \end{bmatrix} < 0$$

(31)

where $M(\sigma) = \text{He}[M(x)A(\sigma)] + \partial f(\sigma)M(x)$. The performance synthesis can be achieved by the dual transformation (Manchester and Slotine, 2018).

**Remark 8.** In a small neighborhood of the equilibrium family, the Lyapunov matrix $X(\sigma)$ from the LPV control synthesis (11) can be seen as a CCM since $\sigma$ only depends on $x$. Note that both the CCM and the Lyapunov matrix could be dependent on the measurable $w$ but not on $u$, otherwise the control realization would involve algebraic loops. Unlike the LPV case, it is possible for the CCM based approach to have a differential gain $K$ that depends on $u$, whose control realization does not contain any algebraic loop (see the next section).

**Remark 9.** The main difference between the local LPV and CCM based control synthesis comes from the ways to handle the scheduling variable. In the LPV approach, the main “trick” for the convex formulation is to treat $\sigma = g(x)$ as an external parameter which is described by a coarse model $\sigma(\cdot) \in T$. The CCM based approach considers a detailed description for $\sigma(\cdot)$ (i.e., the nonlinear system (1)). Although the synthesis problem becomes more complicated than in the LPV case, it can lead to less conservative results. For instance, a non-uniform metric $M(x)$ can be found even if both the parameter $\sigma$ and its variations are unbounded, e.g., the system $\dot{x} = -x + x^3$ admits a non-uniform contraction metric $M(x) = 1 + 3x^2$. For performance verification, the CCM approach may require $\sigma(\cdot) \in T$. For example, if the model growth in differential dynamics is unbounded, finite differential $L_2$-gain bound can only be computed for compact state sets (Manchester and Slotine, 2018).

### 3.4 Controller Realization

As discussed in Remark 4, it is difficult to construct a completely integrable differential controller (26), i.e., there exists a gain-scheduled law $\kappa(\cdot)$ whose Jacobian is $K$. Unlike the LPV approach, the CCM based control realization only investigates a significantly weaker condition - the path-integrability of $K$.

Given a CCM $M(x)$ and a differential feedback gain $K(\sigma)$ satisfying LMI (29), the closed-loop differential dynamics (27) is exponentially stable, i.e.,

$$\frac{d}{dt}(\delta_x'(M(x))\delta_x) = \delta_x'M\delta_x + 2\delta_x'M(A + BK)\delta_x \leq -2\lambda\delta_x'M\delta_x.$$  

(32)

Integrating the above inequality along any smooth path $c(t, \cdot) \in \Gamma(x^*(t), x(t))$ yields

$$\frac{d}{dt}\epsilon(c(t, \cdot)) \leq -2\lambda\epsilon(c(t, \cdot))$$

(33)

if the following integral equation has a unique solution: $\kappa_p(c, u^*, t, s) := u^* + \int_s^t K(c(s), \kappa_p(c, u^*, t, s), w)c_x(s)ds.$

(34)

Note that the smooth path of control signal $\kappa_p$ satisfies

$$\frac{\partial \kappa_p}{\partial s} = K(c, \kappa_p, w), \kappa_p(c, u^*, t, 0) = u^*$$

(35)

which is much weaker than the LPV realization requirement (15). The existence and uniqueness of $\kappa_p$ is established in Manchester and Slotine (2017). A smooth feedback law which achieves the universal exponential stability is then given as follows:

$$u(t) = \kappa_p(\gamma, u^*(t), t, 1)$$

(36)

where $\gamma(t, \cdot)$ is the geodesic joining $x^*(t)$ and $x(t)$.

**Remark 10.** The computation of $\gamma$ can be formulated as a simple nonlinear MPC problem which can be efficiently solved by the pseudospectral approach in Leung and Manchester (2017). Although there may exist multiple geodesics between $x^*$ and $x$, it is proved that the above controller is smooth almost everywhere on $\mathbb{R}^n$ and continuous at $x = x^*$ (Manchester and Slotine, 2017).

**Remark 11.** It is important for the CCM $M$ to be independent on $u$, otherwise the computation of $\gamma$ and $\kappa_p$ will form an algebraic loop. However, unlike the LPV case, the above control realization admits the differential control gain $K$ to be dependent on $u$.

Here we give a geometric interpretation about the interconnection between the path of control signal $\kappa_p$ and the local LPV controller (16). Let $0 = s_0 < s_1 < \cdots < s_N = 1$ with $s_{j+1} - s_j$ be sufficiently small. For any frozen time $t$, the integral equation (34) gives

$$\kappa_p(s_{j+1}) \approx \kappa_p(s_j) + K(\gamma(s_j), \kappa_p(s_j), w)[\gamma(s_{j+1}) - \gamma(s_j)].$$

For simplicity, only the dependence of $s$ is considered. Note that $K(s_{j+1})$ is an LPV controller (16) that stabilizes the state $\gamma(s_{j+1})$ around $\gamma(s_j)$, as shown in Fig. 1. Based on this observation, $\kappa_p$ integrates a series of local LPV controllers (26) along a particular path $\gamma$ and the CCM based gain scheduling law (36) is the corresponding control action to the measured state $x(t) = \gamma(t, 1)$.

**Remark 12.** From (34), the CCM based controller (36) does not contain any hidden coupling term and serves as an exact realization for the differential controller (26). Moreover, it can be also applied to those approaches using incremental analysis (Leith and Leithead, 2000; Fromion and Scorletti, 2003) where exact control realization for general nonlinear systems is still an open problem (Scorletti et al., 2015).

### 3.5 Stability and Performance Assessment

As shown in Section 2, it is rather intuitive to design and implement gain-scheduling based controllers. However, it
Consider the following nonlinear system
\[ x(t) + a_1 x(t) + a_2 x(t) + w(t) = 0. \]
where \( a_1 \) and \( a_2 \) are constants. The system is exponentially stable at the equilibrium point \( x^* \) if
\[ \lim_{t \to \infty} e^{-\gamma t} \| x(t) \| = 0, \]
for some \( \gamma > 0 \).

Theorem 13. If there exists a uniformly bounded metric \( a_1 I \leq M(x) \leq a_2 I \) for which (29) holds for all \( x, u, w \), then system (1) is exponentially stable at the equilibrium point \( x^* \). Theorem 14. The controller (36) achieves an universally exponentially stable at rate \( \lambda \) and overshoot \( C = \sqrt{\frac{a_2}{a_1}} \).

The controller (36) is universal exponentially stable with rate \( \lambda \) and overshoot \( C = \sqrt{\frac{a_2}{a_1}} \).

To derive a gain-scheduling controller, we consider placing both the closed-loop eigenvalues at \(-2\), leading to
\[ K(\sigma) = \begin{bmatrix} 1 & -3 & -\sigma \end{bmatrix}. \]

Then, the control law (16) with the choice of \( \sigma = e^{-w} \) corresponds to the nonlinear controller
\[ u = u_c(w) + x_1 - (3 + e^{-w})(x_2 - w) \]
which is referred as Gain-Scheduled Controller (GSC) 1.

The differential dynamics of the closed-loop system can be represented by
\[ \dot{x}_e = A(x)\delta_x = \begin{bmatrix} -1 & -1 & 0 \\ 1 & a(x_2, w) \\ 0 & 0 & 0 \end{bmatrix} \delta_x \]
where \( a(x_2, w) = e^{-x_2} - e^{-w} - 3 \). Since \( A(x) \) has positive eigenvalues and \( x_2 < -\ln(4 + e^{-w}) \), the closed-loop system is unstable in this region.

By implementing the scheduling law according to the equilibrium relation \( \sigma = e^{-x_2} \), (17) gives the gain scheduled controller
\[ u = x_1 + e^{-x_2} - 1. \]

The resulting closed-loop system is globally exponential stable, however the differential dynamics have eigenvalues \( \lambda_{1,2} = -1/2 \leq \sqrt{3/2i} \) with larger real parts than the specified ones \( \lambda_{1,2} = -2 \). This mismatch is caused by the hidden coupling terms:
\[ K_0(\sigma) \triangleq \frac{\partial u_c(\sigma)}{\partial x_e(\sigma)} - \frac{\partial x_e(\sigma)}{\partial \sigma}, \]
This can be compensated since \( \sigma \) is a scalar, which leads to the controller GSC 2 as follows
\[ u = x_1 + e^{-x_2} - 1 - \int_{w}^{x_2} K_h(\sigma) d\sigma \]
Linearization of the closed-loop system with the control map (45) at the time-varying reference \( x_e(w(t)) = [0, w(t)] \) yields
\[ \dot{x}_e = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -3 & 0 \end{bmatrix} x_e - \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]
Although it is globally exponential stable, zero error for time-varying references cannot be achieved.

For CCM based control design, we choose the following differential state feedback control
\[ \delta_u = K(x)\delta_x \]
with \( K(x) = \begin{bmatrix} 1 & -(3 + e^{-x_2}) \end{bmatrix} \). This leads to an exponentially stable closed-loop differential dynamics with the same eigenvalues as the LPV controller. Thus, we can obtain a constant CCM for the closed-loop system, which implies that the geodesic between \( x^* \) and \( x \) is a straight line (i.e., \( \gamma(s) = (1 - s)x^* + sx \)). Further, the CCM controller can be computed as
\[ u = u + \int_0^1 K(\gamma(s))(x - x^*) ds \]
where the target trajectory is \( x^*(t) = [0, w(t)]^T, u^*(t) = e^{-w(t)} - 1 + w(t) \). The closed-loop system
\[ \frac{d}{dt} x_2 - w = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -3 & 0 \end{bmatrix} x_2 - w \]
is globally exponential stable at \( x^*(t) \).
Comparison studies are carried out on tracking control of piecewise-constant setpoints and a time-varying reference. As shown in Fig. 2, the closed-loop system under GSC 1 is not stable when the state $x_2$ enters into certain regions. The system under GSC 2 can follow piecewise-constant setpoints but cannot achieve zero error for the time-varying reference. The CCM based controller can track both time-varying setpoints and reference.

5. CONCLUSION

In this paper, we investigated the apparent connection between contraction theory based nonlinear controller design and the gain-scheduling approach which corresponds to LPV control based on local linearization of the nonlinear system. We show that the CCM based control is an extended LPV gain scheduling approach as it yields a control realization without any hidden coupling term and achieves global reference-independent stability.

REFERENCES

Bachas, A.A., Tóth, R., Mesbah, A., and Ludlage, J. (2013). A review on data-driven linear parameter-varying modeling approaches: A high-purity distillation column case study. J. Process Control, 24(4), 272–285.

Becker, G. and Packard, A. (1994). Robust performance of linear parametrically varying systems using parametrically-dependent linear feedback. Syst. Control Lett., 23(3), 205–215.

Do Carmo, M.P. (1992). Riemannian Geometry. Springer, Boston, USA.

Forni, F. and Sepulchre, R. (2014). A differential lyapunov framework for contraction analysis. IEEE Trans. Autom. Control, 3(59), 614–628.

Fromion, V., Monaco, S., and Normand-Cyrot, D. (2001). The weighted incremental norm approach: from linear to nonlinear $H_\infty$ control. Automatica, 37(10), 1585–1592.

Fromion, V. and Scorletti, G. (2003). A theoretical framework for gain scheduling. Int. J. Robust Nonlin. Control, 13(10), 951–982.

Hoffmann, C. and Werner, H. (2015). A survey of linear parameter-varying control applications validated by experiments or high-fidelity simulations. IEEE Trans. on Control Syst. Techno., 23(2), 416–433.

Klatt, K.U. and Engell, S. (1998). Gain-scheduling trajectory control of a continuous stirred tank reactor. Comput. & Chem. Eng., 22(4), 491–502.

Leith, D.J. and Leithead, W.E. (2000). Survey of gain-scheduling analysis and design. Int. J. Control, 73(11), 1001–1025.

Leung, K. and Manchester, I.R. (2017). Nonlinear stabilization via control contraction metrics: A pseudospectral approach for computing geodesics. In Proc. Amer. Control Conf. (ACC), 1284–1289. Seattle, WA.

Lohmiller, W. and Slotine, J.J.E. (1998). On contraction analysis for non-linear systems. Automatica, 34(6), 683–696.

Manchester, I.R. and Slotine, J.J.E. (2017). Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design. IEEE Trans. Autom. Control, 62(6), 3046–3053.

Manchester, I.R. and Slotine, J.J.E. (2018). Robust control contraction metrics: A convex approach to nonlinear state-feedback $H_\infty$ control. IEEE Control Syst. Lett., 2(3), 333–338.

Papageorgiou, G., Glover, K., D’Mello, G., and Patel, Y. (2000). Taking robust LPV control into flight on the VAAC Harrier. In Proc. Conf. on Decision and Control (CDC), 4558–4564. Sydney, Australia.

Parrilo, P.A. (2003). Semidefinite programming relaxations for semialgebraic problems. Math. Program., 96(2), 293–320.

Rugh, W.J. (1991). Analytical framework for gain scheduling. IEEE Control Syst. Mag., 11, 74–84.

Rugh, W.J. and Shamma, J.S. (2000). Research on gain scheduling. Automatica, 36, 1401–1425.

Scorletti, G., Fromion, V., and De Hillerin, S. (2015). Toward nonlinear tracking and rejection using LPV control. In Proc. IFAC Workshop on Linear Parameter Varying Systems, 13–18. Grenoble, France.

Shiromoto, H.S., Revay, M., and Manchester, I.R. (2018). Distributed nonlinear control design using separable control contraction metrics. IEEE Trans. Control Netw. Syst. doi:10.1109/TCNS.2018.2885270.

Tóth, R. (2010). Modeling and Identification of Linear Parameter-Varying Systems. Lecture Notes in Control and Information Sciences, Vol. 403. Springer, Heidelberg.

Wang, R., Manchester, I.R., and Bao, J. (2017). Distributed economic MPC with separable control contraction metrics. IEEE Control Syst. Lett., 1(1), 104–109.