COUNTING PLANAR CURVES IN $\mathbb{P}^3$ WITH DEGENERATE SINGULARITIES

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Abstract. In this paper, we consider the following question: how many degree $d$ curves are there in $\mathbb{P}^3$ (passing through the right number of generic lines and points), whose image lies inside a $\mathbb{P}^2$, having $\delta$ nodes and one singularity of codimension $k$. We obtain an explicit formula for this number when $\delta + k \leq 4$ (i.e. the total codimension of the singularities is not more than four). We use a topological method to compute the degenerate contribution to the Euler class; it is an extension of the method that originates in the paper by A. Zinger ([28]) and which is further pursued by S. Basu and the second author in [1], [2] and [3]. Using this method, we have obtained formulas when the singularities present are more degenerate than nodes (such as cusps, tacnodes and triple points). When the singularities are only nodes, we have verified that our answers are consistent with those obtained by S. Kleiman and R. Piene (in [15]) and by T. Laarakker (in [18]). We also verify that our answer for the characteristic number of planar cubics with a cusp and the number of planar quartics with two nodes and one cusp is consistent with the answer obtained by R. Singh and the second author (in [20]), where they compute the characteristic number of rational planar curves in $\mathbb{P}^3$ with a cusp. We also verify some of the numbers predicted by the conjecture made by Pandharipande in [21], regarding the enumerativity of BPS numbers for $\mathbb{P}^3$.

CONTENTS

1. Introduction 1
2. Setup and Notation 3
3. Cohomology ring structure of projective fiber bundles 5
4. Intersection Numbers 5
5. Recursive Formulas 6
6. Proof of the recursive formulas 8
   6.1. Proof of Theorem 5.1 and 5.2: computation of $N(A_1^2 A_1)$ when $0 \leq \delta \leq 3$ 9
   6.2. Proof of Theorem 5.3: computation of $N(A_1^2 P A_1)$ when $0 \leq \delta \leq 2$ 21
   6.3. Proof of Theorem 5.4: computation of $N(A_1^2 P A_2)$ when $0 \leq \delta \leq 2$ 23
   6.4. Proof of Theorem 5.5: computation of $N(A_1^2 P A_3)$ when $0 \leq \delta \leq 1$ 30
   6.5. Proof of Theorem 5.6: computation of $N(P A_4)$ 31
   6.6. Proof of Theorem 5.7: computation of $N(P D_4)$ 32
7. Verification with other results and low degree checks 32
   7.1. Verification with S. Kleiman and R. Piene’s result 32
   7.2. Verification with T. Laraakker’s result 33
   7.3. Verification with the second author and R. Singh’s result 33
   7.4. Enumerativity of BPS numbers computed by R. Pandharipande 33
8. Explicit Formulas 34
9. Acknowledgment 36
References 36

1. INTRODUCTION

One of the most fundamental and studied problems in enumerative geometry is the following: what is the number of degree $d$ curves in $\mathbb{P}^2$ that have $\delta$ distinct nodes and that pass through $\frac{d(d+3)}{2} - \delta$ generic points?
A more general question is to enumerate the characteristic number of curves that have more degenerate singularities. To make this precise, let us make the following definition.

**Definition 1.1.** Let \( f : \mathbb{P}^2 \rightarrow \mathcal{O}(d) \) be a holomorphic section. A point \( q \in f^{-1}(0) \) is said to have a singularity of type \( A_k \) or \( D_k \) if there exists a coordinate system \( (x, y) : (U, q) \rightarrow (\mathbb{C}^2, 0) \) such that \( f^{-1}(0) \cap U \) is given by

\[
A_{k \geq 2} : y^2 + x^{k+1} = 0 \quad \text{and} \quad D_{k \geq 4} : y^2 x + x^{k-1} = 0.
\]

In more common terminology, \( q \) is a *simple node* (or just node) if its singularity type is \( A_1 \); a *cusp* if its type is \( A_2 \); a *tacnode* if its type is \( A_3 \) and an *ordinary triple point* if its type is \( D_4 \).

**Remark 1.** We will frequently use the phrase “a singularity of codimension \( k \)”. Roughly speaking, this refers to the number of conditions having that singularity imposes on the space of curves. More precisely, it is the expected codimension of the equisingular strata. Hence, a singularity of type \( A_k \) or \( D_k \) is a singularity of codimension \( k \).

A classical question in enumerative geometry is this: what is \( N_d(A^4) \), the number of degree \( d \) curves in \( \mathbb{P}^2 \), that have \( \delta \) distinct (ordered) nodes, that pass through \( \frac{d(d+3)}{2} - \delta \) generic points? More generally, one can ask what is \( N_d(A^4_{\mathfrak{X}}) \), the number of degree \( d \) curves in \( \mathbb{P}^2 \), that have \( \delta \) distinct (ordered) nodes and one singularity of type \( \mathfrak{X} \), that pass through \( \frac{d(d+3)}{2} - \delta - c_{\mathfrak{X}} \) generic points, where \( c_{\mathfrak{X}} \) is the codimension of the singularity \( \mathfrak{X} \).

The question of computing \( N_d(A^4) \) and \( N_d(A^4_{\mathfrak{X}}) \) has been studied for a very long time starting with Zeuthen ([27]) more than a hundred years ago. It has been studied extensively in the last thirty years from various perspectives by numerous mathematicians including amongst others, Z. Ran ([22], [23]), I. Vainscher ([26]), L. Caporaso and J. Harris ([6]), M. Kazarian ([10]), S. Kleiman and R. Piene ([14]), D. Kernke ([11] and [12]), F. Block ([5]), Y. Tzeng and J. Li ([24], [25]), M. Kool, V. Shende and R. Thomas ([17]), S. Fomin and G. Mikhalkin ([7]), G. Berzzi ([4]) and S. Basu and R. Mukherjee ([1], [2] and [3]).

This problem motivates a natural generalization considered by Kleiman and Piene in [15], where they study the enumerative geometry of nodal curves in a moving family of surfaces (i.e. a fiber bundle version of the earlier question). More recently, this question has been studied further by T. Laarakker in [18].

Let us now state the question more precisely. We define a *planar curve* in \( \mathbb{P}^3 \) to be a curve in \( \mathbb{P}^3 \), whose image lies inside some \( \mathbb{P}^2 \). Let us define

\[
N_{d_{\text{Planar}, \mathbb{P}^3}}(A^4_{\mathfrak{X}}, r, s)
\]

to be the number of planar degree \( d \) curves in \( \mathbb{P}^3 \), intersecting \( r \) lines and passing through \( s \) points, and having \( \delta \) distinct nodes and one singularity of type \( \mathfrak{X} \), where \( r + 2s = \frac{d(d+3)}{2} + 3 - (\delta + c_{\mathfrak{X}}) \) and \( c_{\mathfrak{X}} \) is the codimension of the singularity \( \mathfrak{X} \). The result of S. Kleiman and R. Piene ([15]) can be used to obtain a formula for \( N_{d_{\text{Planar}, \mathbb{P}^3}}(A^4_{\delta}; r, s) \), if \( \delta \leq 8 \) (see section 7.1 for details). In [18], T. Laarakker obtains a formula for \( N_{d_{\text{Planar}, \mathbb{P}^3}}(A^4_{\delta}; r, s) \), for all \( \delta \).

The main result of this paper is as follows:

**Theorem 1.2.** Let \( \mathfrak{X} \) be a singularity of codimension \( c_{\mathfrak{X}} \) and \( \delta \) a non negative integer. We obtain an explicit formula for \( N_{d_{\text{Planar}, \mathbb{P}^3}}(A^4_{\mathfrak{X}}, r, s) \), when \( \delta + c_{\mathfrak{X}} \leq 4 \), provided \( d \geq d_{\text{min}} \), where \( d_{\text{min}} := c_{\mathfrak{X}} + 2\delta \).

In section 7, we verify that when the singularities present are only nodes, our answers agree with the answers obtained by S. Kleiman and R. Piene (in [15]) and by T. Laarakker (in [18]). We also verify some of the numbers predicted by the conjecture made by R. Pandharipande in [21], regarding the enumerativity of the BPS numbers for \( \mathbb{P}^3 \).

Very recently, a stable map version of this question has been studied by the second author, A. Paul and R. Singh (in [19]). In that paper, the authors find a formula for the characteristic number of planar genus zero (rational) degree \( d \)-curves in \( \mathbb{P}^3 \). Building up on the results of that paper, the second author of this paper and R. Singh obtain a formula for the characteristic number of planar genus zero (rational) degree \( d \)-curves in \( \mathbb{P}^3 \) having a cusp (in [20]). In section 7, we also verify that our formula for \( N_{d_{\text{Planar}, \mathbb{P}^3}}(A^4_2; r, s) \) and \( N_{d_{\text{Planar}, \mathbb{P}^3}}(A^4_3A_2; r, s) \) is logically consistent with the formula obtained in [20], when \( d = 3 \) and \( d = 4 \) respectively.
Remark 2. In [14, Theorem 1.2], the authors compute the corresponding numbers \( N(A_4^1) \) for a fixed surface, while in [15] an algorithm is developed to compute \( N(A_3^1) \) for a family of surfaces. It ought to be possible to generalize the algorithm developed in [15] to higher singularities and compute all the numbers obtained by Theorem 1.2 (this has been point out to us by S. Kleiman [16]).

2. Setup and Notation

Let us now describe the setup develop some notation to obtain a formula for the numbers stated in Theorem 1.2. Our basic objects are planar degree degree \( d \) curves in \( P^3 \), i.e. degree \( d \) curves in \( P^3 \) whose image lies inside a \( P^2 \). Let us denote the dual of \( P^3 \) by \( \hat{P}^3 \); this is the space of \( P^2 \) inside \( P^3 \). An element of \( \hat{P}^3 \) can be thought of as a nonzero linear functional \( \eta : C^4 \rightarrow C \) up to scaling (i.e., it is the projectivization of the dual of \( C^4 \)).

Given such an \( \eta \), we define the projectivization of its zero set as \( P^2_\eta := P(\eta^{-1}(0)) \).

Note that this \( P^2_\eta \) is a subset of \( P^3 \).

Next, given a positive integer \( \delta \), let us define

\[ S_\delta := \{ ([\eta], q_1, \ldots, q_\delta) \in \hat{P}^3 \times (P^3)^{\delta} : \eta(q_1) = 0, \ldots, \eta(q_\delta) = 0 \} \]

Clearly \( S_\delta \) is a fiber bundle over \( \hat{P}^3 \) with fiber \( (P^2)^{\delta} \). This is a plane in \( P^3 \) and a collection of \( \delta \) points that lie on that plane. We will often abbreviate \( S_1 \) as \( S \). Let us consider the section of the following line bundle induced by the evaluation map, i.e.,

\[ ev : \hat{P}^3 \times P^3 \rightarrow \gamma_{\hat{P}^3}^* \otimes \gamma_{P^3}^*, \quad \text{given by} \quad \{ ev([\eta],[q]) \} \eta \otimes q := \eta(q), \]

where \( \gamma_{\hat{P}^3}^* \) and \( \gamma_{P^3}^* \) are dual of the tautological line bundles over \( \hat{P}^3 \) and \( P^3 \) respectively (or equivalently \( O_{\hat{P}^3}(1) \) and \( O_{P^3}(1) \) respectively). Note that

\[ S = ev^{-1}(0). \]

Next, let us denote \( D \rightarrow \hat{P}^3 \) to be a fiber bundle over \( \hat{P}^3 \), such that the fiber over each \( [\eta] \in \hat{P}^3 \) is the space of degree \( d \) curves in \( P^2_\eta \). Next, we note that \( \hat{P}^3 \) is naturally isomorphic to \( G(3,4) \). Let us denote \( \gamma_{3,4} \rightarrow G(3,4) \) to be the tautological three plane bundle over the Grassmannian. Hence, via this isomorphism we note that

\[ D \approx P(Sym^d \gamma_{3,4}^*) \rightarrow \hat{P}^3. \]

Hence, \( D \) is a fiber bundle over \( \hat{P}^3 \), whose fibers are isomorphic to \( P^{2(d+3)} \). An element of \( D \) will be denoted by \( ([f],[\eta]) \); this means that \( f \) is a homogeneous degree \( d \)-polynomial defined on \( P^2_\eta \).

Next, given a positive integer \( \delta \), let us define

\[ S_{D_\delta} := \{ ([f],[\eta], q_1, \ldots, q_\delta) \in D \times (P^3)^{\delta} : ([\eta], q_1, \ldots, q_\delta) \in S_\delta \}. \]

Note that \( S_{D_\delta} \) can be considered as pull back bundle of \( D \) via the fiber bundle map \( \pi : S_\delta \rightarrow \hat{P}^3 \), i.e. the following diagram

\[
\begin{array}{ccc}
S_{D_\delta} & \xrightarrow{\pi^*_\delta} & D \\
\downarrow{\pi^*_\delta} & & \downarrow{\pi^D} \\
S_\delta & \rightarrow & \hat{P}^3
\end{array}
\]

commutes. We will abbreviate \( S_{D_1} \) as \( S_D \). Next, let \( X_1, X_2, \ldots, X_\delta \) be subsets of \( S_D \). We define

\[ X_1 \circ X_2 \circ \ldots \circ X_\delta := \{ ([f],[\eta], q_1, \ldots, q_\delta) \in S_{D_\delta} : ([f],[\eta], q_i) \in X_i \ \forall \ i = 1 \ \text{to} \ \delta \quad \text{and} \quad q_i \neq q_j \quad \text{if} \ i \neq j \}. \]

We will make the following abbreviation

\[ X_1^{\delta_1} \circ X_2^{\delta_2} \circ \ldots \circ X_m^{\delta_m} := X_1 \circ \ldots \circ X_1 \circ X_2 \circ \ldots \circ X_2 \circ \ldots \circ X_m \circ \ldots \circ X_m. \]
When \( \delta_i = 1 \), we will omit the superscript. For example,
\[
X_1 \circ X_2^3 \circ X_3 = X_1^1 \circ X_2^3 \circ X_3^1 = X_1 \circ X_2 \circ X_3 \circ X_3.
\]
Next, let \( \mathcal{X} \) be a singularity of a given type. We will also denote \( \mathcal{X} \) to be the space of curves and a marked point such that the curve has a singularity of type \( \mathcal{X} \) at the marked point. More precisely,
\[
\mathcal{X} := \{ ([f], [\eta], q) \in \mathcal{S} : f \text{ has a singularity of type } \mathcal{X} \text{ at } q \}.
\]
For example,
\[
A_2 := \{ ([f], [\eta], q) \in \mathcal{S} : f \text{ has a singularity of type } A_2 \text{ at } q \}.
\]
For example, \( A_1^2 \circ A_2 \) is the space of curves with three ordered points, where the curve has a simple node at the first two points and a cusp at the last point and all the three points are distinct. Similarly, \( A_1^2 \circ A_2^2 \) is the space of curves with three distinct ordered points, where the curve has a simple node at the first two points and a singularity at least as degenerate as a cusp at the last point; the curve could have a tacnode at the last marked point (here \( \hat{X} \) indicates the closure of \( X \)).

Next, consider the following rank two vector bundle \( \pi : W \to \mathcal{S}, \) where the fiber over each point \( ([\eta], q) \) is the tangent space of \( \mathbb{P}_q^2 \) at the point \( q \), i.e.
\[
\pi^{-1}([\eta], q) := T\mathbb{P}_q^2|_q.
\]  
Let \( W_{\mathcal{D}} \to \mathcal{S}_{\mathcal{D}} \) denote the pullback of \( W \) to \( \mathcal{S}_{\mathcal{D}} \) and let \( \mathbb{P}W_{\mathcal{D}} \to \mathcal{S}_{\mathcal{D}} \) denote the projectivization of \( W_{\mathcal{D}} \). We can now define the space of curves having a singularity singularity of certain type together with a direction, i.e. if \( \mathcal{X} \) be singularity of a given type, then define
\[
\hat{\mathcal{X}} := \{ ([f], [\eta], l_\eta) \in \mathbb{P}W_{\mathcal{D}} : f \text{ has a singularity of type } \mathcal{X} \text{ at } q \}.
\]
We can also define the space of curves with a singularity and a specific direction along which certain directional derivatives vanish, i.e.
\[
\mathcal{P}A_k := \{ ([f], [\eta], l_\eta) \in \mathbb{P}W_{\mathcal{D}} : ([f], [\eta], q) \in A_k, \nabla^2 f|_q(v, \cdot) = 0 \quad \forall v \in l_\eta \} \quad \text{if } k \geq 2.
\]
For example, \( \mathcal{P}A_2 \) is the space of curves with a marked point and a marked direction, such that the curve has a cusp at the marked point and the marked direction belongs to the kernel of the Hessian. Note that the projection map \( \pi : \mathcal{P}A_k \to A_k \) is one to one. Next, let us define
\[
\mathcal{P}A_1 := \{ ([f], [\eta], l_\eta) \in \mathbb{P}W_{\mathcal{D}} : ([f], [\eta], q) \in A_1, \nabla^2 f|_q(v, v) = 0 \quad \forall v \in l_\eta \} \quad \text{and}
\mathcal{P}D_4 := \{ ([f], [\eta], l_\eta) \in \mathbb{P}W_{\mathcal{D}} : ([f], [\eta], q) \in D_4, \nabla^3 f|_q(v, v, v) = 0 \quad \forall v \in l_\eta \}.
\]
In other words, \( \mathcal{P}A_1 \) is the space of curves with a marked point and a marked direction, such that the curve has a node at the marked point and the second derivative along the marked direction vanishes. Note that there are two such distinguished directions. Hence, the projection map \( \pi : \mathcal{P}A_1 \to A_1 \) is two to one. Similarly, the projection map \( \pi : \mathcal{P}D_4 \to D_4 \) is three to one.

Next, let \( \mathcal{S}_{\mathcal{D}_3} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} \) denote the fibered product of \( \mathcal{S}_{\mathcal{D}_3} \) and \( \mathbb{P}W_{\mathcal{D}} \) over \( \mathcal{D} \) via the natural forgetful map. It can be considered as a fiber bundle over \( \mathbb{P}^3 \) whose fiber over each point \( [\eta] \in \mathbb{P}^3 \) is
\[
\mathbb{P}(H^0(O(d), \mathbb{P}_q^2)) \times (\mathbb{P}_q^2)^d \times \mathbb{P}(\mathbb{T}\mathbb{P}_q^2).
\]
Let \( \pi : \mathcal{S}_{\mathcal{D}_3} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} \to \mathcal{S}_{\mathcal{D}_3+1} \) denote the projection map. If \( S \) is a subset of \( \mathcal{S}_{\mathcal{D}_3+1} \), then we define
\[
\hat{S} := \{ ([f], [\eta], q_1, \ldots, q_{\delta+1}, l_{q_{\delta+1}}) \in \mathcal{S}_{\mathcal{D}_3} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} : ([f], [\eta], q_1, \ldots, q_{\delta+1}) \in S \} = \pi^{-1}(S).
\]  
Finally, if \( S_1, \ldots, S_\delta \) are subsets of \( \mathcal{S}_{\mathcal{D}} \) and \( T \) is a subset of \( \mathbb{P}W_{\mathcal{D}} \), then we define
\[
S_1 \circ S_2 \circ \ldots \circ S_\delta \circ T := \{ ([f], [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}}) \in \mathcal{S}_{\mathcal{D}} \times_{\mathcal{D}} \mathbb{P}W_{\mathcal{D}} : ([f], [\eta], q_1) \in S_1, \ldots, ([f], [\eta], q_\delta) \in S_\delta, \quad ([f], l_{q_{\delta+1}}) \in T \quad \text{and}
\]
\[
q_1, \ldots, q_\delta, q_{\delta+1} \quad \text{are all distinct} \}.
\]
As an example, \( A_1^2 \circ \mathcal{P}A_2 \) is the space of curves with three distinct ordered points, where the curve has a simple node at the first two points and a cusp at the last point and a distinguished direction at the last marked point, such that the Hessian vanishes along that direction.
3. Cohomology ring structure of projective fiber bundles

We now recapitulate some basic facts about the cohomology ring of the various spaces we will encounter. First, we recall that via the annihilation map, $\mathbb{P}^3$ is isomorphic to $\mathbb{G}(3,4)$. Via this isomorphism, we can think of $a$ (which is actually a generator of $H^*(\mathbb{P}^3)$) as a generator of $H^*(\mathbb{G}(3,4))$. We note that

$$c(\gamma^*_3) = 1 + a + a^2 + a^3.$$ 

Next, using the splitting principle, we conclude that

$$\pi^*$$

Here

$$D$$

Notice that

$$\hat{\lambda}$$

Let

$$\lambda$$

denote the tautological line bundle over the projective bundle $W$ (which is actually a generator of $\mathbb{G}(3,4)$). We are now in a position to define a few numbers. Since we will primarily be dealing with planar degree

$$d$$

in such a case. Let us now define

$$\gamma_D \rightarrow \mathbb{P}(\text{Sym}^d \gamma^*_{3,4})$$

denotes the tautological line bundle and

$$\lambda := c_1(\gamma_D^*).$$

4. Intersection Numbers

Let $\gamma_W \rightarrow \mathbb{P}W$ denote the tautological line bundle over the projective bundle $\mathbb{P}W \rightarrow S$. We denote $\lambda_W := c_1(\gamma_W^*)$ and $H$ to be the standard generator of $H^*(\mathbb{P}^3)$ (i.e. the class of a hyperplane in $\mathbb{P}^3$).

We are now in a position to define a few numbers. Since we will primarily be dealing with planar degree $d$-curves in $\mathbb{P}^3$, we will usually use the prefix $N$ as opposed to the more elaborate $N^\text{Planar,p}$. If there is a chance for confusion, we will use the latter notation.

We will occasionally be dealing with curves in $\mathbb{P}^2$. In such a case we will use the notation $N^\text{p2}_d$; we will never use $N$ in such a case. Let us now define

$$N(A^i_4 \mathfrak{X}, r, s, n_1, n_2, n_3) := \langle a^{n_1} \lambda^{n_2} \pi_{\delta+1}^* \mathfrak{c}_{n_3} \cap H^r, [A^i_4 \circ \mathfrak{X}] \cap H^r_L \cap H^s_p \rangle, \quad (8)$$

$$N(A^i_4 \mathfrak{P} \mathfrak{X}, r, s, n_1, n_2, n_3, \theta) := \langle a^{n_1} \lambda^{n_2} \pi_{\delta+1}^* \mathfrak{c}_{n_3}^\theta \cap H^r, [A^i_4 \circ \mathfrak{P} \mathfrak{X}] \cap H^r_L \cap H^s_p \rangle \quad \text{and} \quad (9)$$

$$N(A^i_4 \mathfrak{X}, r, s, n_1, n_2, n_3, \theta) := \langle a^{n_1} \lambda^{n_2} \pi_{\delta+1}^* \mathfrak{c}_{n_3}^\theta \cap H^r, [A^i_4 \circ \mathfrak{X}] \cap H^r_L \cap H^s_p \rangle. \quad (10)$$

Here $\pi_i$ denotes the projection onto the $i^{th}$-point.

Next, we note that if $\theta \geq 2$, then

$$N(A^i_4 \mathfrak{P} \mathfrak{X}, r, s, n_1, n_2, n_3, \theta) = -3N(A^i_4 \mathfrak{P} \mathfrak{X}, r, s, n_1, n_2, n_3 + 1, \theta - 1) + N(A^i_4 \mathfrak{P} \mathfrak{X}, r, s, n_1 + 1, n_2, n_3, \theta - 1) - N(A^i_4 \mathfrak{P} \mathfrak{X}, r, s, n_1 + 2, n_2, n_3, \theta - 2) + 2N(A^i_4 \mathfrak{P} \mathfrak{X}, r, s, n_1 + 1, n_2, n_3 + 1, \theta - 2) - 3N(A^i_4 \mathfrak{P} \mathfrak{X}, r, s, n_1, n_2, n_3 + 2, \theta - 2). \quad (11)$$

This is because

$$\lambda^2_W = -c_1(W)\lambda_W - c_2(W) \implies \lambda^2_W = -(3H - a)\lambda_W - (a^2 - 2aH + 3H^2).$$

The Chern classes $c_1(W)$ and $c_2(W)$ are given by eq. (19). Next, we note that

$$N(A^i_4 \mathfrak{X}, r, s, n_1, n_2, n_3) = \frac{1}{\deg(\pi)}N(A^i_4 \mathfrak{P} \mathfrak{X}, r, s, n_1, n_2, n_3, 0), \quad (12)$$
where \( \text{deg}(\pi) \) is the degree of the projection map \( \pi : \mathcal{P}\mathcal{X} \rightarrow \mathcal{X} \). We remind the reader that the degree is one when \( \mathcal{X} = A_{k \geq 2} \), it is two when \( \mathcal{X} = A_1 \) and it is three when \( \mathcal{X} = D_4 \).

We also note that
\[
N(A_1^\delta \hat{X}, n_1, n_2, n_3, \theta) = 0 \quad \text{if} \quad \theta = 0,
\]
\[
N(A_1^\delta \hat{X}, n_1, n_2, n_3, \theta) = N(A_1^\delta \hat{X}, n_1, n_2, n_3) \quad \text{if} \quad \theta = 1 \quad \text{and}
\]
\[
N(A_1^\delta \hat{X}, n_1, n_2, n_3, \theta) = N(A_1^\delta \hat{X}, n_1, n_2 + 1, n_3, \theta - 1) - N(A_1^\delta \hat{X}, n_1, n_2, n_3 + 1, \theta - 2) \quad \text{if} \quad \theta > 1.
\]

Finally, let us define
\[
N(r, s, n_1, n_2) := \langle a^{n_1} \lambda^{n_2}, [D] \cap \mathcal{H}_L \cap \mathcal{H}_p \rangle.
\]

We now note that
\[
\mathcal{H}_L = \lambda + da \quad \text{and} \quad \mathcal{H}_p = \lambda a.
\]

The reason why this is true is explained in [29, Pages 18 and 19]. Now, using the ring structure of \( \mathcal{D} \) (as given by eq. (7)), we can compute \( N(r, s, n_1, n_2) \) by extracting the coefficient of \( a^3 \lambda^{n-1} \) from
\[
(\lambda + da)^r(\lambda a)^sa^{n_1}a^{n_2}.
\]

Hence, \( N(r, s, n_1, n_2) \) can be computed for any \( r, s, n_1 \) and \( n_2 \).

5. Recursive Formulas

We are now ready to state the recursive formulas. We have written a mathematica program to implement these recursive formulas and obtain the final formulas. The program is available on the second author’s homepage

https://www.sites.google.com/site/ritwik371/home

For the convenience of the reader, we have explicitly written down the formulas for \( N(r, s, 0, 0) \) and \( N(A_1^\delta \hat{X}, r, s, 0, 0) \) in Section 8. Note that \( N(r, s, 0, 0) \) is the number of planar degree-\( d \) curves intersecting \( r \) lines and passing through \( s \) points. Our formulas for \( N(A_1^\delta, r, s, 0, 0, 0) \) agree with those obtained by Kleiman and Piene in [15] and by Ties Laarakker in [18].

**Theorem 5.1.** Consider the ring
\[
\mathcal{R} = \mathbb{Z}[a, H, \lambda]/\langle a^4, H^4, \lambda^n + s_1 a \lambda^{n-1} + s_2 a^2 \lambda^{n-2} + s_3 a^3 \lambda^{n-3} \rangle,
\]
where \( s_1, s_2, s_3 \) and \( n \) are as defined in eq. (5) and eq. (6). Let
\[
e := (\lambda + H)(\lambda + da)^r(\lambda a)^s a^{n_1} \lambda^{n_2} H^{n_3}(\lambda + dH)[(\lambda + dH)^2 - (3H - a)(\lambda + dH) + a^2 - 2aH + 3H^2].
\]
Then \( N(A_1, r, s, n_1, n_2, n_3) \) is the coefficient of \( \lambda^{n-1} a^3 H^3 \) in the polynomial \( e \), seen as an element of the ring \( \mathcal{R} \).

**Remark 3.** Theorem 5.1 is true for all \( d \geq 1 \).
Next, we will give a formula for $N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2},n_{3})$, when $1 \leq \delta \leq 3$. First let us make a couple of definitions.

\[
\text{Eul}(\delta,r,s,n_{1},n_{2},0) := (d - 2d^2 + d^3)N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 1,n_{2},0) + (3 - 6d + 3d^2)N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 2,n_{2},0)
\]

\[
\text{Eul}(\delta,r,s,n_{1},n_{2},1) := (d^2 - d)N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 2,n_{2},0) + (3d^2 - 4d + 1)N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 1,n_{2} + 1,0) + (3d - 3)N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 2,0),
\]

\[
\text{Eul}(\delta,r,s,n_{1},n_{2},2) := dN(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 3,n_{2},0) + (2d - 1)N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 2,n_{2} + 1,0) + (3d - 2)N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 1,n_{2} + 2,0) + N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 2,0)
\]

\[
\text{Eul}(\delta,r,s,n_{1},n_{2},3) := N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 3,n_{2} + 1,0) + N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 2,n_{2} + 2,0) + N(A_{1}^{\delta-1}A_{1},r,s,n_{1} + 1,n_{2} + 3,0)
\]

\[
\text{Eul}(\delta,r,s,n_{1},n_{2},n_{3}) = 0 \quad \text{if } n_{3} > 3.
\]  

(16)

We also define

\[
B(\delta,r,s,n_{1},n_{2},n_{3}) := \left(\frac{\delta}{1}\right)B_{1} + \left(\frac{\delta}{2}\right)B_{2} + \left(\frac{\delta}{3}\right)B_{3},
\]

where

\[
B_{1} := \left(N(A_{1}^{\delta-1}A_{2},r,s,n_{1} + 2 + 1,n_{3}) + dN(A_{1}^{\delta-1}A_{2},r,s,n_{1},n_{2},n_{3} + 1) + 3N(A_{1}^{\delta-1}A_{2},r,s,n_{1},n_{2},n_{3},0)\right)
\]

\[
B_{2} := 4\left(N(A_{1}^{\delta-2}A_{3},r,s,n_{1},n_{2},n_{3},0)\right)
\]

\[
B_{3} := \frac{18}{3}\left(N(A_{1}^{\delta-3}A_{4},r,s,n_{1},n_{2},n_{3},0)\right).
\]  

(17)

We are now ready to state the formula for $N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2},n_{3})$.

**Theorem 5.2.** Let $\text{Eul}(\delta,r,s,n_{1},n_{2},n_{3})$ and $B(\delta,r,s,n_{1},n_{2},n_{3})$ be defined as in eq. (16) and eq. (17) respectively. If $1 \leq \delta \leq 3$, then

\[
N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2},n_{3}) = \text{Eul}(\delta,r,s,n_{1},n_{2},n_{3}) - B(\delta,r,s,n_{1},n_{2},n_{3}),
\]

provided $d \geq 2\delta + 1$.

We now state the remaining formulas.

**Theorem 5.3.** If $0 \leq \delta \leq 2$, then

\[
N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2},n_{3},0) = 2N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2},n_{3}),
\]

\[
N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2},n_{3},1) = N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2} + 1,n_{3}) + (d - 6)N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2},n_{3} + 1) + 2N(A_{1}^{\delta}A_{1},r,s,n_{1} + 1,n_{2},n_{3}) - 2\left(\frac{\delta}{2}\right)N(A_{1}^{\delta-2}A_{4},r,s,n_{1},n_{2},n_{3},0),
\]

provided $d \geq 2\delta + 2$.

**Remark 4.** To compute $N(A_{1}^{\delta}A_{1},r,s,n_{1},n_{2},n_{3},\theta)$ when $\theta \geq 2$, we use eq. (11).
Theorem 5.4. Let $0 \leq \delta \leq 2$ and $\theta$ a non negative integer with the following property: if $\delta$ is either 0 or 1, then $\theta$ can be anything, but if $\delta = 2$, then $\theta = 0$. Then,
\[
N(A^\delta_1 PA_2, r, s, n_1, n_2, n_3, \theta) = N(A^\delta_1 PA_1, r, s, n_1 + 1, n_2, n_3, \theta) \\
+ N(A^\delta_1 PA_1, r, s, n_1, n_2 + 1, n_3, \theta) \\
+ (d - 3)N(A^\delta_1 PA_1, r, s, n_1, n_2, n_3 + 1, \theta) \\
- 2\left(\begin{array}{c} \delta \\ 1 \end{array}\right)N(A^{\delta - 1}_1 PA_3, r, s, n_1, n_2, n_3, \theta) \\
- 3\left(\begin{array}{c} \delta \\ 1 \end{array}\right)N(A^{\delta - 1}_1 \hat{D}_4, r, s, n_1, n_2, n_3, \theta) \\
- 4\left(\begin{array}{c} \delta \\ 2 \end{array}\right)N(A^{\delta - 2}_1 PD_4, r, s, n_1, n_2, n_3, \theta),
\]
provided $d \geq 2\delta + 2$.

Remark 5. If $\delta = 2$ and $\theta > 0$, then the formula given by Theorem 5.4 is not valid; there is a further correction term (the interested reader can refer to [3] to see what the extra correction term is). However, to compute $N(A^\delta_1 PA_2, r, s, n_1, n_2, n_3)$ we only need to know what is $N(A^\delta_1 PA_2, r, s, n_1, n_2, n_3, 0)$ and hence for the purposes of this paper, this Theorem is sufficient. We would require $N(A^\delta_1 PA_2, r, s, n_1, n_2, n_3, \theta)$ for $\theta > 0$ if we were computing any of the codimension five (or higher) numbers; in this paper we are computing numbers till codimension four.

Theorem 5.5. If $0 \leq \delta \leq 1$, then
\[
N(A^\delta_1 PA_3, r, s, n_1, n_2, n_3, \theta) = N(A^\delta_1 PA_2, r, s, n_1, n_2 + 1, n_3, \theta) \\
+ 3N(A^\delta_1 PA_2, r, s, n_1, n_2, n_3, \theta + 1) \\
+ dN(A^\delta_1 PA_2, r, s, n_1, n_2 + 1, n_3, \theta) \\
- 2\left(\begin{array}{c} \delta \\ 1 \end{array}\right)N(A^{\delta - 1}_1 PA_4, r, s, n_1, n_2, n_3, \theta),
\]
provided $d \geq 2\delta + 3$.

Theorem 5.6. If $d \geq 4$, then
\[
N(\mathcal{P}A_4, r, s, n_1, n_2, n_3, \theta) = 2N(\mathcal{P}A_3, r, s, n_1, n_2 + 1, n_3, \theta) \\
+ 2N(\mathcal{P}A_3, r, s, n_1, n_2, n_3, \theta + 1) \\
+ 2N(\mathcal{P}A_3, r, s, n_1 + 1, n_2, n_3, \theta) + (2d - 6)N(\mathcal{P}A_3, r, s, n_1, n_2, n_3 + 1, \theta)
\]

Theorem 5.7. If $d \geq 3$, then
\[
N(\mathcal{P}D_4, r, s, n_1, n_2, n_3, \theta) = N(\mathcal{P}A_3, r, s, n_1, n_2 + 1, n_3, \theta) \\
- 2N(\mathcal{P}A_3, r, s, n_1, n_2, n_3, \theta + 1) \\
+ 2N(\mathcal{P}A_3, r, s, n_1 + 1, n_2, n_3, \theta) + (d - 6)N(\mathcal{P}A_3, r, s, n_1, n_2, n_3 + 1, \theta)
\]

We will now prove these recursive formulas.

6. PROOF OF THE RECURSIVE FORMULAS

We are now ready to prove the formulas stated in section 5. We will use a topological method to compute the degenerate contribution to the Euler class. Our method is an extension of the method that originates in the paper by A. Zinger ([28]) and which is further pursued by S. Basu and the second author in [1], [2] and [3].

When there is no cause for confusion, we will sometimes abbreviate $N(A^\delta_1 A_1, r, s, n_1, n_2, n_3)$ and $N(A^\delta_1 \mathcal{P}X, r, s, n_1, n_2, n_3, \theta)$ as $N(A^\delta_1 A_1)$ and $N(A^\delta_1 \mathcal{P}X)$ (for the sake of notational simplicity).
6.1. Proof of Theorem 5.1 and 5.2: computation of \( N(A_1^d A_1) \) when \( 0 \leq \delta \leq 3 \).

We will justify our formula for \( N(A_1^d A_1, r, s, n_1, n_2, n_3) \), when \( 0 \leq \delta \leq 3 \). Recall that in Section 2, we have defined

\[
A_1^d \circ S_D := \{ ([f], [\eta]; q_1, \ldots, q_8, q_{\delta + 1}) \in D \times (\mathbb{P}^3)^{\delta + 1} : \eta(q_i) = 0, \ \forall i \leq 1 \text{ to } \delta + 1, f \text{ has a singularity of type } A_1 \text{ at } q_1, \ldots, q_8, q_{\delta + 1} \text{ all distinct} \}.
\]

Let \( \mu \) be a generic cycle, representing the class

\[
[\mu] = \mathcal{H}_L \cdot \mathcal{H}_p \cdot a^{n_1} \lambda^{n_2} (\pi^*_{\delta + 1} H)^{n_3}.
\]

Here \( \pi_i \) denotes the projection onto the \( i \)-th point. We will often omit writing down \( \pi^*_{\delta + 1} \), if there is no cause for confusion. We now consider sections of the following two bundles that are induced by the evaluation map and the vertical derivative at the last point, namely:

\[
\Psi_{A_0} : A_1^d \circ S_D \longrightarrow \mathcal{L}_{A_0} := \gamma_D \otimes \pi^*_{\delta + 1} \gamma_{p^d D}^d, \quad \{ \Psi_{A_0}(([f], [\eta], q_1, \ldots, q_{\delta + 1})) (f) := f(q_{\delta + 1}) \}
\]

\[
\Psi_{A_1} : \psi_{A_0}^{-1}(0) \longrightarrow \mathcal{A}_1 := \gamma_D \otimes \pi^*_{\delta + 1} W^* \otimes \pi^*_{\delta + 1} \gamma_{p^d D}^d, \quad \{ \psi_{A_1}(([f], [\eta], q_1, \ldots, q_{\delta + 1})) (f) := \nabla f|_{q_{\delta + 1}} \}
\]

We will show shortly that \( \psi_{A_0} \) and \( \psi_{A_1} \) are transverse to zero, provided \( d \geq 2 \delta + 1 \).

Next, let us define

\[
\mathcal{B} := A_1^d \circ S_D - A_1^d \circ S_D.
\]

Hence

\[
\langle e(\mathcal{L}_{A_0}) c(\mathcal{V}_{A_1}), [A_1^d \circ S_D] \cap [\mu] \rangle = N(A_1^d A_1, r, s, n_1, n_2, n_3) + C_{\mathcal{B} \cap \mu},
\]

where \( e \) denote the Euler class and \( C_{\mathcal{B} \cap \mu} \) denotes the contribution of the section to the Euler class from the points of \( \mathcal{B} \cap \mu \).

Let us first explain how to compute the left hand side of eq. (18) (i.e. the Euler class). From equations (15) and (1), we note that

\[
\mathcal{H}_L = \lambda + da, \quad \mathcal{H}_p = \lambda a \quad \text{ and } \quad [\pi^*_{\delta + 1} S_D] = \lambda + \pi^*_{\delta + 1} H.
\]

Next, we need to compute the Chern classes of \( \mathcal{W} \). We note that over \( \mathcal{S} \), we have the following short exact sequence of vector bundles:

\[
0 \longrightarrow \mathcal{W} \longrightarrow \mathbb{T}^3 \longrightarrow \gamma_{p^3}^* \otimes \gamma_{p^3}^* \longrightarrow 0.
\]

Here the first map is the inclusion map and the second map is \( \nabla \eta |_{\mathcal{S}} \). Hence,

\[
c(\mathcal{W}) c(\gamma_{p^3}^* \otimes \gamma_{p^3}^*) = c(\mathbb{T}^3) \implies c_1(\mathcal{W}) = 3H - a \quad \text{ and } \quad c_2(\mathcal{W}) = a^2 - 2aH + 3H^2.
\]

Next, using the splitting principle, we conclude that

\[
e(\gamma_{p^3}^* \otimes \gamma_{p^3}^*) e(\gamma_{p^3}^* \otimes \mathcal{W}^* \otimes \gamma_{p^3}^*) = (\lambda + dH)(\lambda + dH)^2 - c_1(\mathcal{W})(\lambda + dH) + c_2(\mathcal{W}).
\]

Note that we have made an abuse of notation by omitting to write down \( \pi^*_{\delta + 1} \); henceforth we will make this abuse of notation. Now, suppose \( \delta = 0 \). Then, using the ring structure of \( \mathcal{D} \) (as given by eq. (7)) and by extracting the coefficient of \( \lambda^{n-1} a^d H^3 \) from

\[
(\lambda + dH) (\lambda + dH)^2 - c_1(\mathcal{W})(\lambda + dH) + c_2(\mathcal{W}))(\lambda + da)^d (\lambda a)^{n_1} \lambda^{n_2} H^{n_3},
\]

we obtain the Euler class. When \( \delta = 0 \), using eq. (19), we get the formula of Theorem 5.1. When \( \delta > 0 \), we get \( \text{Eul}(\delta, r, s, n_1, n_2, n_3) \) as defined in eq. (16).

Let us now explain how to compute \( C_{\mathcal{B} \cap \mu} \), the degenerate contribution to the Euler class. When \( \delta = 0 \), the boundary \( \mathcal{B} \) is empty and hence we get the result of Theorem 5.1. Let us now consider the case when \( \delta \geq 1 \). Given \( k \) distinct integers \( i_1, i_2, \ldots, i_k \in [1, \delta + 1] \), let us define

\[
\Delta_{i_1, \ldots, i_k} := \{ ([f], [\eta]; q_1, \ldots, q_8, q_{\delta + 1}) \in \mathcal{S}_{D_{\delta + 1}} : q_{i_1} = q_{i_2} = \ldots = q_{i_k} \}
\]

\[
\mathcal{B}(q_{i_1}, \ldots, q_{i_k}) := \mathcal{B} \cap \Delta_{i_1, \ldots, i_k}.
\]
Let us now consider $\mathcal{B}(q_i, q_{\delta+1})$. We claim that

$$\mathcal{B}(q_i, q_{\delta+1}) \approx A_1^{\delta-1} \circ A_1 \quad \forall \ i = 1 \ to \ \delta, \quad (21)$$

where $\mathcal{B}(q_i, q_{\delta+1})$ is identified as a subset of $\mathcal{S}_{D_4}$ in the obvious way (namely via the inclusion of $\mathcal{S}_{D_4}$ inside $\mathcal{S}_{D_{\delta+1}}$, where the $(\delta + 1)^{th}$ point is equal to the $i^{th}$ point). Next, we claim that the contribution from $\mathcal{B}(q_i, q_{\delta+1}) \cap \mu$ is given by

$$\langle e(L_{A_0}), \ [A_1^{\delta-1} \circ A_1] \cap [\mu] \rangle + 3\mathcal{N}(A_1^{\delta-1} \cdot P A_2, r, s, n_1, n_2, n_3, 0). \quad (22)$$

We will explain the reason for both the claims shortly. The expression given by eq. (22) is precisely equal to $B_1$ as defined in eq. (17). Hence, the sum total of the contribution from $\mathcal{B}(q_i, q_{\delta+1})$ for $i = 1 \ to \ \delta$ is $(\delta)B_1$.

Next, let us assume $\delta \geq 2$ and consider $\mathcal{B}(q_{i_1}, q_{i_2}, q_{\delta+1})$. We claim that

$$\mathcal{B}(q_{i_1}, q_{i_2}, q_{\delta+1}) \approx A_1^{\delta-2} \circ A_3 \quad (23)$$

for all distinct pairs $(i_1, i_2)$. We also claim that the contribution from each of the points of $\mathcal{B}(q_{i_1}, q_{i_2}, q_{\delta+1}) \cap \mu$ is 4. We will justify both these claims shortly. Hence the sum total of the contribution as we vary over all $(i_1, i_2)$ is precisely $(\delta)^2 B_2$, where $B_2$ is as defined in eq. (17).

Finally, let us assume $\delta \geq 3$ and consider $\mathcal{B}(q_{i_1}, q_{i_2}, q_{i_3}, q_{\delta+1})$. We claim that

$$\mathcal{B}(q_{i_1}, q_{i_2}, q_{i_3}, q_{\delta+1}) \approx A_1^{\delta-3} \circ A_3 \cup A_1^{\delta-3} \circ D_4 \quad (24)$$

for all distinct triples $(i_1, i_2, i_3)$. Note that $A_1^{\delta-3} \circ A_3 \cap \mu$ is empty, since the sum of their dimensions is one less than the dimension of the ambient space where we are intersecting them. Hence, we get no contribution from $A_1^{\delta-3} \circ A_3 \cap \mu$. Finally, we claim that the contribution from each of the points of $A_1^{\delta-3} \circ D_4 \cap \mu$ is 18. Hence the sum total of the contribution as we vary over all $(i_1, i_2, i_3)$ is precisely $(\delta)^3 B_3$, where $B_3$ is as defined in eq. (17).

Let us now prove the claims regarding transversality and degenerate contributions to the Euler class. We will start by proving transversality. Note that we need to prove $A_1^{\delta+1}$ is a smooth complex submanifold of $\mathcal{S}_{D_{\delta+1}}$ (provided $d \geq 2\delta + 1$). We will prove a stronger statement: we will show that $\overline{A_1^{\delta+1}}$ is a smooth complex submanifold of $\mathcal{S}_{D_{\delta+1}}$ and the sections $\Psi_{A_0}$ and $\Psi_{A_1}$, defined on $\overline{A_1^{\delta}}$, are transverse to zero. Our desired claim follows immediately from this statement since $\overline{A_1^{\delta+1}}$ is an open subset of $\overline{A_1^{\delta+1}}$.

Let us begin by showing that $\Psi_{A_0}$ is transverse to zero if $d \geq 2\delta + 1$. Suppose $\Psi_{A_0}([f], [\eta], q_1, \ldots, q_{\delta+1}) = 0$. Without loss of generality, let us assume that $[\eta]$ determines the plane where the last coordinate is zero, and $q_{\delta+1}$ is the point where only the third coordinate is nonzero and the rest are zero, i.e.

$$\mathbb{P}_n^2 \approx \{ [X, Y, Z, W] \in \mathbb{P}^3 : W = 0 \} \quad \text{and} \quad q_{\delta+1} := [0, 0, 1, 0].$$

Assume that the remaining points are given by $q_i := [X_i, Y_i, Z_i, 0]$ for $i = 1 \ to \ \delta$.

For simplicity, we can assume that all $Z_i$ are nonzero. Furthermore, since all the $q_i$ are distinct, we conclude that $X_i$ and $Y_i$ can not both be zero; for simplicity let us assume $X_i$ is nonzero for each $i$ (from 1 to $\delta$). Consider the homogeneous degree $d$ polynomial, given by

$$\rho_{00} := (X - X_1)^2 (X - X_2)^2 \cdots (X - X_\delta)^2 Z^{d-2\delta}.$$  

We note the following facts about $\rho_{00}$:

$$\rho_{00}(q_{\delta+1}) = 0 \quad \forall \ i = 1 \ to \ \delta, \quad (25)$$

$$\nabla \rho_{00}|_{q_i} = 0 \quad \forall \ i = 1 \ to \ \delta \quad \text{and} \quad (26)$$

$$\rho_{00}(q_{\delta+1}) \neq 0 \quad (27)$$
Now consider the curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{S}_{d+1}$, given by
$$\gamma(t) := ([f + t\rho_0], [\eta], q_1, \ldots, q_{d+1}).$$
Because of eq. (25) and eq. (26), we conclude that this curve lies in $A_1 \circ \mathcal{S}_{d}$. We now note that
$$\{\nabla \Psi_{A_0}([f], [\eta], q_1, \ldots, q_{d+1}) (\gamma'(0)) \} (f) = \rho_{00} (q_{d+1}).$$
Using eq. (27), we conclude that the right hand side of eq. (28) is nonzero, whence $\Psi_{A_0}$ is transverse to zero. Next, let us prove transversality for the section $\Psi_{A_1}$. Consider the polynomials,
$$\rho_{10} := (X - X_1)^2 (X - X_2)^2 \cdots (X - X_\delta)^2 X Z^{d-2\delta-1}$$
and
$$\rho_{01} := (X - X_1)^2 (X - X_2)^2 \cdots (X - X_\delta)^2 Y Z^{d-2\delta-1}.$$
We note that $\rho_{10}$ and $\rho_{01}$ satisfy eq. (25) and eq. (26) with $\rho_{00}$ replaced with $\rho_{10}$ and $\rho_{01}$ respectively. Furthermore,
$$\rho_{10}(q_{d+1}) = 0 \quad \text{and} \quad \rho_{01}(q_{d+1}) = 0.$$  
(29)

Construct the curves
$$\gamma_{10}(t) := ([f + t\rho_{10}], [\eta], q_1, \ldots, q_{d+1}), \quad \gamma_{01}(t) := ([f + t\rho_{01}], [\eta], q_1, \ldots, q_{d+1}).$$
Because of eq. (25) and eq. (26) with $\rho_{00}$ replaced with $\rho_{10}$ and $\rho_{01}$ respectively and eq. (29), these curves lie inside $\Psi_{A_0}^{-1}(0)$. We now note that
$$\{\nabla \Psi_{A_1}([f], [\eta], q_1, \ldots, q_{d+1}) \} (\gamma_{10}(0)) \} (f) = \lambda Z^{d-2\delta-1} \nabla X|_{0,0,1,0}$$
and
$$\{\nabla \Psi_{A_1}([f], [\eta], q_1, \ldots, q_{d+1}) \} (\gamma_{01}(0)) \} (f) = \lambda Z^{d-2\delta-1} \nabla Y|_{0,0,1,0},$$
where $\lambda := (-X_1)^2 \ldots (-X_\delta)^2$.

Since $\nabla X|_{0,0,1,0}$ and $\nabla Y|_{0,0,1,0}$ are two linearly independent vectors of $T\mathbb{P}_{d}^2|_{0,0,1,0}$, we conclude that $\Psi_{A_1}$ is transverse to zero.

Let us now justify the closure and multiplicity claims. We will start by giving the reason for eq. (21) and eq. (22). This follows from the argument given in the proof [2, Lemma 6.3 (1), Page 685] and [2, Corollary 6.6, Page 689]. The proof is the same.

Next, let us justify eq. (23). Without loss of generality, it suffices to justify it when $i_1 := \delta - 1$ and $i_2 := \delta$. Hence, we need to show that
$$\{(f), [\eta], q_1, \ldots, q_{d+1}) \in A_1 \circ \mathcal{S}_{d} : q_{d-1} = q_{d} = q_{d+1}) = A_1^{d-2} \circ A_3.$$
(30)

Before proceeding further, let us make a simple observation. Notice that the left hand side of eq. (30) is the same as
$$\{(f), [\eta], q_1, \ldots, q_{d+1}) \in A_1 \circ \mathcal{S}_{d} : q_{d-1} = q_{d}) = A_1^{d-2} \circ A_3.$$
(31)

Hence, an equivalent way of stating eq. (30) is
$$\{(f), [\eta], q_1, \ldots, q_{d}) \in A_1 \circ \mathcal{S}_{d} : q_{d-1} = q_{d}) = A_1^{d-2} \circ A_3.$$
(32)

Following [2, Equation 6.4, Page 685], we conclude that
$$\left(\{(f), [\eta], q_1, \ldots, q_{d}, q_{d+1}) \in A_1 \circ \mathcal{S}_{d} : q_{d-1} = q_{d} = q_{d+1}) \right) \cap \left(A_1^{d-2} \circ A_2\right) = \emptyset.$$ 
(33)

Equation eq. (33) is saying that if two nodes come together, then the singularity has to be more degenerate than a cusp. Hence, the singularity has to be at least as degenerate as a tacnode (since $A_2 = A_2 \cup A_3$). Hence, the left hand side of eq. (30) is a subset of its right hand side. We will now prove the converse. We will simultaneously prove the following four statements:
$$\{(f), [\eta], q_1, \ldots, q_{d+1}) \in A_1 \circ \mathcal{S}_{d} : q_{d-1} = q_{d} = q_{d+1}) \cup A_1^{d-2} \circ A_3,$$
(34)
$$\{(f), [\eta], q_1, \ldots, q_{d+1}) \in A_1 \circ \mathcal{S}_{d} : q_{d-1} = q_{d} = q_{d+1}) \cap \left(A_1^{d-2} \circ A_3\right) = \emptyset,$$
(35)
$$\{(f), [\eta], q_1, \ldots, q_{d+1}) \in A_1 \circ \mathcal{S}_{d} : q_{d-1} = q_{d} = q_{d+1}) \cap \left(A_1^{d-2} \circ A_4\right) = \emptyset \quad \text{and}$$
(36)
$$\{(f), [\eta], q_1, \ldots, q_{d+1}) \in A_1 \circ \mathcal{S}_{d} : q_{d-1} = q_{d} = q_{d+1}) \cup A_1^{d-2} \circ A_5.$$
(37)
Since $A_1^1 \circ S_D$ is a closed set, eq. (34) implies that the right hand side of eq. (30) is a subset of its left hand side. Before we prove the above four statements, let us explain intuitively the significance of each of the statements.

The first statement, eq. (34) is saying that every tacnode is in the closure of two nodes (we remind the reader that the left hand side of eq. (34) is same as the expression given by eq. (31)). Geometrically, figure 1 explains the meaning of eq. (34).

![Figure 1. Two nodes colliding into a tacnode](image1)

The second statement, eq. (35) is saying that in the closure of three nodes, we get a singularity more degenerate than a tacnode. The third statement, eq. (36) is saying that in the closure of three nodes, we get a singularity more degenerate than an $A_4$ singularity. Finally, eq. (37) is saying that every $A_5$ singularity is in the closure of three nodes. Geometrically, figure 2 explains the meaning of eq. (37).

![Figure 2. Three nodes colliding into an $A_5$-singularity](image2)

We are now ready to prove the above statements. Let us prove the following two claims:

**Claim 6.1.** Let $([f], [\eta], q_1, \ldots, q_5) \in A_4^{\delta - 2} \circ A_3$. Then there exists points

\[
([f_1], [\eta_1], q_1(t), \ldots, q_{\delta - 2}(t); q_{\delta - 1}(t), q_\delta(t), q_{\delta + 1}(t)) \in A_4^{\delta - 2} \circ S_D^3
\]

sufficiently close to $([f], [\eta], q_1, \ldots, q_5; q_\delta, q_5, q_\delta)$, such that

\[
f_i(q_i(t)) = 0, \quad \nabla f_i|_{q_i(t)} = 0 \quad \text{for } i = \delta - 1 \text{ and } \delta.
\]

Furthermore, every such solution satisfies the condition

\[
(f_i(q_{\delta + 1}(t)), \ nabla f_i|_{q_{\delta + 1}(t)}) \neq (0, 0),
\]

i.e. $([f_1], [\eta_1], q_1(t), \ldots, q_{\delta - 2}(t); q_{\delta - 1}(t), q_\delta(t), q_{\delta + 1}(t)) \notin A_4^1 \circ A_1$. In fact, if

\[
([f], [\eta], q_1, \ldots, q_5) \in A_4^{\delta - 2} \circ A_4,
\]

then there does not exist any point

\[
([f_1], [\eta_1], q_1(t), \ldots, q_{\delta - 2}(t); q_{\delta - 1}(t), q_\delta(t), q_{\delta + 1}(t)) \in A_4^{\delta - 2} \circ S_D^3
\]

sufficiently close to $([f], [\eta], q_1, \ldots, q_5; q_\delta, q_5, q_\delta)$, such that

\[
f_i(q_i(t)) = 0, \quad \nabla f_i|_{q_i(t)} = 0 \quad \text{for } i = \delta - 1, \ \delta \text{ and } \delta + 1.
\]

**Claim 6.2.** Let $([f], [\eta], q_1, \ldots, q_5) \in A_4^{\delta - 2} \circ A_5$. Then there exists points

\[
([f_1], [\eta_1], q_1(t), \ldots, q_{\delta - 2}(t); q_{\delta - 1}(t), q_\delta(t), q_{\delta + 1}(t)) \in A_4^{\delta - 2} \circ A_4
\]

sufficiently close to $([f], [\eta], q_1, \ldots, q_5; q_\delta, q_5, q_\delta)$.
Remark 6. We note claim 6.1 proves eq. (34), eq. (35) and eq. (36) simultaneously. We also note that claim 6.2 proves eq. (37).

Remark 7. Before proceeding with the proof, let us make a shorthand notation. We denote

\[ O((|x_1, x_2, \ldots, x_n|)^k) \]

to be a holomorphic function (in the variables \( x_1, \ldots, x_n \)), defined in a neighborhood of the origin in \( \mathbb{C}^n \), whose order of vanishing is at least \( k \) (i.e. all the terms of degree lower than \( k \) are absent in the Taylor expansion of the function around the origin). We say that such an expression is of order \( k \). For example, \( x_1^2 + x_1 x_2 x_3^2 + x_2^2 x_3^3 \) is a term of order 4 and we will denote it by \( O((|x_1, x_2, x_3|)^4) \). Note that we are always dealing with holomorphic functions. Hence, suppose a function (in say one variable) is of type \( O(|x|^n) \), it means, its Taylor expansion is of the type

\[ f(x) = a_2 x^2 + a_3 x^3 + \ldots. \]

It does not mean that there are terms of type \( x^n \) (although the \( |x|^2 \) in the \( O(|x|^2) \) might suggest that). Henceforth, it will be understood that \( O(|x|^n) \) and \( O(x^n) \) mean the same thing in our paper (the latter is the standard notation in one variable).

Proof of claims 6.1 and 6.2: Let us define

\[ \mathbb{C}^2_2 := \{(x, y, z) \in \mathbb{C}^3 : z = 0\}. \]

We will now work in an affine chart where we send the plane \( \mathbb{P}^2_2 \) to \( \mathbb{C}^2_2 \) and the point \( q(t) \in \mathbb{P}^2_2 \) to \( (0, 0, 0) \in \mathbb{C}^2_2 \).

Using this chart, let us write down the Taylor expansion of \( f_t \) around the point \( (0, 0) \), namely

\[ f_t(x, y) = \frac{f_{t0}}{2} x^2 + f_{t1} x y + \frac{f_{t2}}{2} y^2 + \ldots \]

Note that since eq. (38) holds (for \( i = \delta \)), we conclude that \( f_{t00}, f_{t01}, f_{t02} \) are zero.

Next, since \( ([f], [\eta], q_8) \in A_2 \), we conclude that \( f_{t20} \) and \( f_{t21} \) can not both be zero; let us assume \( f_{t02} \neq 0 \). Hence, \( f_t(x, y) \) can be re-written as

\[ f_t(x, y) = A_0(x) + A_1(x) y + A_2(x) y^2 + \ldots \quad \text{where} \quad A_2(0) \neq 0. \]

We will now make a change of coordinates; let us define

\[ \hat{y} := y - B(x) \]

where \( B(x) \) is a function that is to be determined. We claim that there exists a unique holomorphic \( B(x) \) (vanishing at the origin) such that after plugging it in \( f_t(x, y) \) we get

\[ f_t(x, y(x, \hat{y})) = \tilde{A}_0(x) + \tilde{A}_2(x) \hat{y}^2 + \tilde{A}_4(x) \hat{y}^4 + \ldots \]

In other words, we want \( \tilde{A}_1(x) \equiv 0 \). This is possible if \( B(x) \) satisfies the equation

\[ A_1(x) + 2 A_2(x) B(x) + 3 A_3(x) B(x)^2 + \ldots = 0. \]  

(41)

Since \( A_2(0) = \frac{f_{t02}}{f_{t00}} \neq 0 \), \( B(x) \) exists by the Implicit Function Theorem and we can compute \( B(x) \) explicitly as a power series using (41) and then compute \( \tilde{A}_0(x) \). Hence, \n
\[ f_t(x, y(x, \hat{y})) = \varphi(x, \hat{y}) \hat{y}^2 + \frac{B_{t0}^2}{2!} x^2 + \frac{B_{t1}^2}{3!} x^3 + \frac{B_{t2}^2}{4!} x^4 + \mathcal{R}(x) x^5, \]

where

\[ B_{t0}^2 := f_{t00}, \quad B_{t1}^2 := f_{t01} + f_{t10} x, \quad B_{t2}^2 := \frac{f_{t02}}{6} + \frac{f_{t12} f_{t21}}{f_{t02}} + \ldots, \quad \varphi(0, 0) \neq 0 \]

and \( \mathcal{R}(x) \) is a holomorphic function defined in a neighborhood of the origin. Since \( ([f], [\eta], q_8) \in A_3 \), we conclude that \( B_{t0}^2 \) and \( B_{t1}^2 \) are small (close to zero) and \( B_{t2}^2 \) is nonzero. Let us make a further change of coordinates and denote

\[ \hat{\hat{y}} := \sqrt{\varphi(x, \hat{y})}. \]
Note that we can choose a branch of the square root since \( \varphi(0,0) \neq 0 \). Next, for notational convenience, let us now define
\[
\hat{f}_i(x, \hat{y}) := f_i(x, y(x, \hat{y}(\hat{y}))),
\]
i.e. \( \hat{f}_i \) is basically \( f_i \) written in the new coordinates (namely \( x \) and \( \hat{y} \)). Hence,
\[
\hat{f}_i(x, \hat{y}) = \hat{y}^2 + \frac{B_i^f}{2!}x^2 + \frac{B_i^f}{3!}x^3 + \frac{B_i^f}{4!}x^4 + R(x)x^5.
\]
We will now solve eq. (38) for \( i = \delta - 1 \). We note that this amounts to solving for the set of equations
\[
\hat{f}_i(u, v) = 0, \quad \hat{f}_{i_1}(u, v) = 0 \quad \text{and} \quad \hat{f}_{i_2}(u, v) = 0 \quad (u, v) \neq (0,0) \text{ but small},
\]
and requiring \( \hat{f}_i \) to have \( \delta - 2 \) more nodes (all distinct from each other and distinct from \( (0,0) \) and \( (u, v) \)). The solutions to eq. (43) are given by
\[
v = 0, \quad B_2^f = \frac{B_i^f}{12}u^2 + 4u^3R(u) + 2u^4R'(u) \quad \text{and} \quad B_3^f = \frac{B_i^f}{2}u - 18u^2R(u) - 6u^3R'(u).
\]
To see how, we first use the third equation of eq. (43) to get \( v = 0 \). Then we use the second and first equations of eq. (43) to get the value of \( B_2^f \) and \( B_3^f \).

We now require the curve to have \( \delta - 2 \) more nodes. To do that, first construct a degree 4 curve that satisfies eq. (44); we can do that since \( B_4^f \) only depends on the fourth order derivatives of \( f_i \). Call this degree 4 curve \( g \). Let us now assume that the points \( q_1, q_2, \ldots, q_\delta - 2 \) correspond to \( (x_1, y_1), \ldots, (x_\delta - 2, y_\delta - 2) \) under the affine chart we are considering. Define
\[
f_i := g(x, y) \cdot ((x - x_1)^2 + (y - y_1)^2) \cdots ((x - x_\delta - 2)^2 + (y - y_\delta - 2)^2).
\]
This curve \( f_i \) satisfies eq. (43) and has \( \delta - 2 \) nodes. This argument works provided the degree of the curve is at least \( 4 + 2(\delta - 2) \). Hence, solutions to eq. (38) exist, if \( d \geq 4 + 2(\delta - 2) \).

Next, let us prove eq. (39), i.e. we have to show that in a neighborhood of a tacnode, we do not have a curve with three distinct nodes. More precisely, we need to show that there cannot be any solutions to the set of equations
\[
\hat{f}_i(u_1, v_1) = 0, \quad \hat{f}_{i_1}(u_1, v_1) = 0, \quad \hat{f}_{i_2}(u_1, v_1) = 0, \quad \hat{f}_{i_3}(u_1, v_1) = 0,
\]
\[
\hat{f}_i(u_2, v_2) = 0, \quad \hat{f}_{i_1}(u_2, v_2) = 0, \quad \hat{f}_{i_2}(u_2, v_2) = 0, \quad \hat{f}_{i_3}(u_2, v_2) = 0,
\]
\((0,0), (u_1, v_1) \) and \((u_2, v_2) \) all distinct (but small).

Let us try to solve for the above set of equations. Let us first explicitly write down \( \hat{f}_i(x, \hat{y}) \) as
\[
\hat{f}_i(x, \hat{y}) = \hat{y}^2 + \frac{B_2^f}{2!}x^2 + \frac{B_3^f}{3!}x^3 + \frac{B_4^f}{4!}x^4 + \frac{B_5^f}{5!}x^5 + \frac{B_6^f}{6!}x^6 + \ldots
\]
To begin with, we unwind eq. (45) using the expression for \( \hat{f}_i \) as given by eq. (47) and solve for \( B_2^f \) and \( B_3^f \) in terms of \( u_1, v_1, B_4^f, B_5^f \) and \( B_6^f \). We then plug in these values for \( B_4^f \) and \( B_5^f \) in eq. (47) and plug it in eq. (46). Now we can solve for \( B_4^f \) and \( B_5^f \) in terms of \( B_6^f \) and then plugging back those values in the previous expressions for \( B_2^f \) and \( B_3^f \), gives us their values in terms of \( B_6^f \). Doing that, we get
\[
v_1, v_2 = 0, \quad B_2^f = \frac{1}{360}B_6^f u_1^2 u_2^2 + O(|(u_1, u_2)|^5), \quad B_3^f = -\frac{1}{60}B_6^f (u_1 u_2 + u_1 u_2) + O(|(u_1, u_2)|^4),
\]
\[
B_4^f = \frac{1}{30}B_6^f (u_1^2 + u_1 u_2 + u_2^2) + O(|(u_1, u_2)|^3) \quad \text{and} \quad B_5^f = -\frac{1}{5}B_6^f (u_1 + u_2) + O(|(u_1, u_2)|^2),
\]
where \( O(|(u_1, u_2)|^n) \) is as defined in remark 7. Hence, \( B_4^f \) is close to zero, which is a contradiction, since \( ([f], [\eta], q_3) \in A_3 \). Since \( B_5^f \) is also close to zero, we get the last part of the claim 6.1 (i.e. eq. (36)). Finally, we note that the solutions constructed in eq. (48) immediately prove claim 6.2 (in fact these are the only
possible solutions). This finishes the proof of claims 6.1 and 6.2.

Next, we claim that each point of \((A_1^{3-2} \circ A_3) \cap \mu\) contributes 4 to the Euler class in eq. (18). Using eq. (44), we conclude that the multiplicity is the number of small solutions \((x, \hat{y}, u)\) to the following set of equations
\[
\hat{f}_t(x, \hat{y}) := \hat{y}^2 + \frac{B_2^t}{2!} x^2 + \frac{B_4^t}{3!} x^3 + \frac{B_6^t}{4!} x^4 + \mathcal{R}(x) x^5 = \varepsilon_0,
\]
\[
\hat{f}_x(x, \hat{y}) := B_2^t x + \frac{B_4^t}{12} x^2 + \frac{B_6^t}{3!} x^3 + 5x^4 \mathcal{R}(x) + \mathcal{R}'(x) x^5 = \varepsilon_1,
\]
\[
B_2^t = \frac{B_4^t}{12} u^2 + 4u^3 \mathcal{R}(u) + 2u^4 \mathcal{R}'(u) \text{ and } B_3^t = -\frac{B_4^t}{2} u - 18u^2 \mathcal{R}(u) - 6u^3 \mathcal{R}'(u),
\]
where \((\varepsilon_0, \varepsilon_1, \varepsilon_2) \in \mathbb{C}^3\) is small and generic. Let us write \(u := h + x\) and Taylor expand \(\mathcal{R}(x + h)\) and \(\mathcal{R}'(x + h)\) around \(h = 0\), i.e.
\[
\mathcal{R}(x + h) = \mathcal{R}(x) + h \mathcal{R}'(x) + \frac{h^2}{2} \mathcal{R}''(x) + \ldots \text{ and } \mathcal{R}'(x + h) = \mathcal{R}'(x) + h \mathcal{R}''(x) + \ldots \tag{49}
\]
Hence, substituting the values of \(B_2^t, B_3^t, \mathcal{R}(x + h)\) and \(\mathcal{R}'(x + h)\) we conclude that we need to find the number of small solutions \((x, h)\) to the following set of equations
\[
\frac{(x^2 h^2)(B_4^t + O(|(x, h)|))}{24} = \varepsilon_3 \quad \text{and} \quad \frac{(x h)(B_4^t h - B_4^t x + O(|(x, h)|^2))}{12} = \varepsilon_1, \tag{50}
\]
where \(\varepsilon_3 := \varepsilon_0 - \frac{\varepsilon_1^2}{4}\). We claim that we can set \(\varepsilon_1\) to be 0; that is justified in section 6.1.1. Assuming that claim, we use eq. (51) to solve for \(x\) in terms of \(h\) and conclude that
\[
x = h + O(h^2). \tag{52}
\]
This is because \(x = 0\) and \(h = 0\) can not be solutions to eq. (51) (since if we plug it back in eq. (50), we will get 0 and not \(\varepsilon_3\)). Plugging in the value of \(x\) from eq. (52) into eq. (50), we get
\[
\frac{B_4^t}{24} h^4 + O(h^5) = \varepsilon_3. \tag{53}
\]
Equation (53) clearly has 4 solutions.

Finally, we need to justify eq. (24) and the corresponding contribution to the Euler class. More precisely, we are going to show that
\[
\{([f], [\eta], q_1, \ldots, q_{k+1}) \in A_1^{3-3} \circ S_D : q_{k+2} = q_{k+1} = q_\delta = q_{k+1}\} \supset A_1^{3-3} \circ A_3^3 \circ D_4. \tag{54}
\]
Just like eq. (30) is equivalent to eq. (32), we similarly conclude that eq. (54) can be equivalently stated as
\[
\{([f], [\eta], q_1, \ldots, q_{k+1}) \in A_1^{3-3} : q_{k+2} = q_{k+1} = q_\delta\} \supset A_1^{3-3} \circ A_3^3 \circ D_4. \tag{55}
\]
Let us define
\[
W_1 := \{([f], [\eta], q_1, \ldots, q_{k+1}) \in S_{D_{k+1}} : f(q_{k+1}) = 0, \nabla f|_{q_{k+1}} = 0, \nabla^2 f|_{q_{k+1}} \neq 0\} \quad \text{and} \quad W_2 := \{([f], [\eta], q_1, \ldots, q_{k+1}) \in S_{D_{k+1}} : f(q_{k+1}) = 0, \nabla f|_{q_{k+1}} = 0, \nabla^2 f|_{q_{k+1}} = 0\}. \tag{56}
\]
In order to prove eq. (54), it suffices to show that
\[
\{([f], [\eta], q_1, \ldots, q_{k+1}) \in A_1^{3-3} \circ S_D : q_{k+2} = q_{k+1} = q_\delta = q_{k+1}\} \cap W_1 = \left(A_1^{3-3} \circ A_3^3 \circ D_4 \right) \cap W_1 \quad \text{and} \quad \{([f], [\eta], q_1, \ldots, q_{k+1}) \in A_1^{3-3} \circ S_D : q_{k+2} = q_{k+1} = q_\delta = q_{k+1}\} \cap W_2 = A_1^{3-3} \circ D_4. \tag{57}
\]
Note that the right hand side of eq. (57) is a subset of \(W_2\); hence we didn’t write down \(\cap W_2\) on the right hand side of eq. (58). Let us first justify eq. (57). Equations (35) and (36), show that the the left hand side of (57) is a subset of its right hand side. Furthermore, eq. (37) shows that the right hand side of (57) is
a subset of its left hand side; hence eq. (57) is true.

We will now prove eq. (58). Equation (54) shows that the left hand side of eq. (58) is a subset of its right hand side. Hence, what remains is to show that the right hand side of eq. (58) is a subset of its left hand side. Before we start the proof of that assertion, let us give an intuitive idea about the significance of that statement. The statement is saying that every triple point is in the closure of three nodes. To summarize, the

![Figure 3. Three nodes colliding into a triple point](image)

geometric significance of eq. (57) is given by figure 2 while the geometric significance of eq. (58) is given by figure 3. Equation (54) says that these are the only two pictures that can occur.

Let us now prove eq. (58). We will prove the following claim:

**Claim 6.3.** Let \( ([f], [\eta], q_1, \ldots, q_5) \in A_{4}^{4-3} \circ D_4 \). Then, there exists points

\[
([f], [\eta], q_1(t), \ldots, q_6(t); q_{6-3}(t), q_{6-1}(t), q_{6}(t), q_{6+1}(t)) \in A_{4}^{4-3} \circ S_{D_4}
\]

sufficiently close to \( ([f], [\eta], q_1, \ldots, q_5; q_5, q_6, q_6) \), such that

\[
f_i(q_i(t)) = 0, \quad \nabla f_i|_{q_i(t)} = 0 \quad \text{for } i = \delta - 2, \delta - 1 \text{ and } \delta.
\]

**Remark 8.** We note that claim 6.3 implies that the right hand side of eq. (58) is a subset of the left hand side.

**Proof:** Following the setup of the proof of claim 6.1, we will now work in an affine chart, where we send the plane \( \mathbb{P}^2_{\eta} \) to \( \mathbb{C}^2 \) and the point \( q_6(t) \in \mathbb{P}^2_{\eta} \) to \( (0, 0, 0) \in \mathbb{C}^2 \). Using this chart, let us write down the Taylor expansion of \( f_t \) around the point \( (0, 0) \), namely

\[
f_t(x, y) = \frac{f_{x_0}}{2} x^2 + f_{y_0} y^2 + \frac{f_{x_0 x_0} x^2}{6} + \frac{f_{x_0 y} x y}{2} + \frac{f_{y_0 y} y^3}{3} + \ldots
\]

Since \( ([f], [\eta], q_6) \in D_4 \), we conclude that \( f_{x_0}, f_{y_0}, f_{x_0 x_0} \) are all small (close to zero). Let us now construct solutions to eq. (59). Let us assume that the points \( q_{6-1}(t) \) and \( q_{6-2}(t) \) are sent to \( (x_1, y_1, 0) \) and \( (x_2, y_2, 0) \) under the affine chart we are considering. Hence, constructing solutions to eq. (59) is same as constructing solutions to the set of equations

\[
f_t(x_1, y_1) = 0, \quad f_t(x_1, y_1) = 0, \quad f_{t_0}(x_1, y_1) = 0 \quad \text{and}
\]

\[
f_t(x_2, y_2) = 0, \quad f_t(x_2, y_2) = 0, \quad f_{t_0}(x_2, y_2) = 0,
\]

where \( (0, 0), (x_1, y_1) \) and \( (x_2, y_2) \) are all distinct (but close to each other).

Next, let us define

\[
g_t(x, y) := x f_{t_0}(x, y) + y f_{t_0}(x, y) - 2 f_t(x, y).
\]

The quantity \( g_t(x, y) \) is similarly defined with \( f_t \) replaced by \( f \). We note that solving eq. (61) and eq. (62) is equivalent to solving

\[
g_t(x_1, y_1) = 0, \quad f_t(x_1, y_1) = 0, \quad f_{t_0}(x_1, y_1) = 0 \quad \text{and}
\]

\[
g_t(x_2, y_2) = 0, \quad f_t(x_2, y_2) = 0, \quad f_{t_0}(x_2, y_2) = 0,
\]

where \( (0, 0), (x_1, y_1) \) and \( (x_2, y_2) \) are all distinct (but close to each other). We now note that \( g_t(x, y) \) and \( f_t(x, y) \) have exactly the same cubic term in the Taylor expansion. Furthermore, \( g_t(x, y) \) has no quadratic term.

Let us now study the cubic term of the Taylor expansion of \( f \) carefully. Let us assume first \( f_{x_0} \neq 0 \).
Since \([f], [y], q \in D_4\), we conclude that the cubic term factors into three distinct linear factors. Hence, the cubic term can be written as

\[
\frac{f_{30}}{6} (x - A_1(0)y)(x - A_2(0)y)(x - A_3(0)y),
\]

where \(A_1(0), A_2(0)\) and \(A_3(0)\) are all distinct. Note that \(A_1(0), A_2(0)\) and \(A_3(0)\) are explicit expressions involving the coefficients \(f_{ij}\). If \(f_{30} = 0\), then the cubic term will be of the type

\[
\frac{f_{21}}{2} y(x - A_1(0)y)(x - A_2(0)y),
\]

where \(A_1(0)\) and \(A_2(0)\) are distinct and \(f_{21}\) is nonzero. We will assume that \(f_{30} \neq 0\); the case \(f_{30} = 0\) can be dealt with similarly. Hence, \(g_t\) (or equivalently \(f_t\)) can be written as

\[
g_t(x, y) = \frac{f_{30}}{6} (x - A_1 y)(x - A_2 y)(x - A_3 y) + O(||(x, y)||^4),
\]

where \(A_i\) are the same as \(A_i(0)\), but with the \(f_{ij}\) replaced by \(f_{t, ij}\). For notational simplicity, we denoted these quantities by the letter \(A_t\) and not \(A_i(t)\).

Let us now make a change of coordinates

\[
x := \hat{x} + O(||(\hat{x}, \hat{y})||^2) \quad \text{and} \quad y := \hat{y} + O(||(\hat{x}, \hat{y})||^2),
\]

such that

\[
g_t = \frac{f_{30}}{6} (\hat{x} - A_1 \hat{y})(\hat{x} - A_2 \hat{y})(\hat{x} - A_3 \hat{y}).
\]

Hence, \(g_t = 0\) has three distinct solutions, given by \(\hat{x} = A_i \hat{y}\) for \(i = 1, 2\) and \(3\). Converting back in terms of \(x\), we conclude that the solutions to \(g_t(x, y) = 0\) (where \((x, y)\) is small but nonzero) are given by

\[
y = u \quad \text{and} \quad x = A_i u + E_i(u),
\]

where \(E_i(u)\) is a second order term in \(u\) (and \(u\) is small but nonzero).

Next, for notational simplicity we will denote \(f_{102}\) by the letter \(w\). Let us consider the solution \(y := u\) and \(x = A_1 u + E_1(u)\) of the equation \(g_t(x, y) = 0\). Plugging this in \(f_{x,v}(x, y) = 0\) and \(f_{y,v}(x, y) = 0\) and solving for \(A_1 f_{t, 11}\) and \(A_1^2 f_{t, 20}\), we conclude that

\[
A_1 f_{t, 11} = \frac{A_1 f_{30}}{6} (A_1 - A_2)(A_1 - A_3)u - w + O(||(u, w)||^2) \quad \text{and} \quad A_1^2 f_{t, 20} = -\frac{A_1 f_{30}}{3} (A_1 - A_2)(A_1 - A_3)u + w + O(||(u, w)||^2).
\]

Let us now consider a second solution to \(g_t(x, y) = 0\) (where \((x, y)\) is small but nonzero). This will be given by \(y := v\) and \(x := A_2 v + E_2(v)\), where \(v\) is small but nonzero (or the analogous thing with \(A_2\) replaced by \(A_3\)). Using eq. (71) to express the values of \(f_{t, 11}\) and \(f_{t, 20}\) in terms of \(u\) and \(w\) and then using \(f_{t, x}(x, y) = 0\), we conclude that

\[
w = \frac{f_{30}}{6} (A_1^3 - 2A_1^2 A_2 - A_1^2 A_3 + 2A_1 A_2 A_3) u + \frac{f_{30}}{6} (A_1^2 A_3 - A_1^2 A_2) v + O(||(u, w)||^2).
\]

Similarly, using eq. (71) to express the values of \(f_{t, 11}\) and \(f_{t, 20}\) in terms of \(u\) and \(w\) and then using \(f_{t, y}(x, y) = 0\), we conclude that

\[
w = \frac{f_{30}}{6} (A_1^3 - A_1^2 A_2 - A_1 A_2 A_3) u + \frac{f_{30}}{6} (A_1 A_2^2 + A_1 A_2 A_3) v + O(||(u, w)||^2).
\]

Equating the right hand sides of eq. (72) and eq. (73), we conclude that

\[
\frac{f_{30}}{6} A_1 (A_1 - A_2)(A_1 - A_3) u - \frac{f_{30}}{6} A_1 (A_1 - A_2)(A_2 - A_3) v + O(||(u, v, w)||^2) = 0.
\]

From eq. (74), we can further conclude that

\[
A_1 v = \left(\frac{A_1 - A_2}{A_2 - A_3}\right) (A_1 u) + O(||(u, w)||^2).
\]
Finally, substituting the value for $v$ from eq. (75) into $w$ in eq. (72), we get that
\[
w = -\frac{f_{20}}{3} A_1 A_2 (A_1 - A_3) u + O(|u|^2) \quad \Longrightarrow \quad w = -\frac{f_{20}}{3} A_1 A_2 (A_1 - A_3) u + O(|u|^2).
\]
(76)
Plugging the value of $w$ from eq. (76) in eq. (71), we conclude that
\[
\begin{align*}
f_{t_{11}} &= \frac{f_{20}}{6} (A_1 + A_2) (A_1 - A_3) u + O(|u|^2) \quad \text{and} \quad f_{t_{20}} = -\frac{f_{20}}{3} (A_1 - A_3) u + O(|u|^2).
\end{align*}
\]
Hence, solutions to eq. (61) and eq. (62) exist, given by
\[
(x_1, y_1) = (A_1 u + E_1(u), u), \quad (x_2, y_2) = \left( A_2 \frac{(A_1 - A_3)}{(A_2 - A_3)} u + E_2(u), \frac{(A_1 - A_3)}{(A_2 - A_3)} u + E_4(u) \right),
\]
\[
(77)
\]
where $u$ is small and nonzero and the $E_i$ are all second order terms. Furthermore, there are exactly 6 distinct solutions, that corresponds to $(A_1, A_2)$ being replaced with $(A_i, A_j)$, where the $(A_i, A_j)$ are ordered (or alternatively, we can think of this this way; the $(A_i, A_j)$ is unordered as far as the construction of $f_t$ is concerned, but we can permute the values of $(x_1, y_1)$ and $(x_2, y_2)$). This proves claim 6.3 and hence proves eq. (24).

Let us now justify the multiplicity. We claim that each point of $(A_1^{4-3} \circ D_4) \cap \mu$ contributes 18 to the Euler class in eq. (18). As we just explained, there are exactly 6 distinct solutions to eq. (61) and eq. (62); we will call each distinct solution of eq. (61) and eq. (62) a branch of a neighborhood of $A_1^{4-3} \circ D_4$ inside $A_1^6$. Since there are 6 branches, it suffices to show that the multiplicity from each branch is 3 (in which case the total contribution to the Euler class will be 18). Let us now compute the multiplicity from each branch.

Let us consider the branch given by eq. (77). The multiplicity from this branch is the number of small solutions $(x, y, u)$ to the following set of equations
\[
(78)
\]
where $(\varepsilon_0, \varepsilon_1, \varepsilon_2) \in \mathbb{C}^3$ is small and generic and $f_{t_{20}}, f_{t_{11}},$ and $f_{t_{02}}$ are as given in eq. (77). We claim that we can set $\varepsilon_1$ and $\varepsilon_2$ to be zero; this is justified in section 6.1.1. Hence, we need to find the number of small solutions $(x, y, u)$ to the set of equations
\[
(79)
\]
This is same as the number of small solutions $(x, y, u)$ to the set of equations
\[
(80)
\]
where $g_t(x, y)$ is as defined in eq. (63). Let us start by solving only the two equations in eq. (80). Plugging in the values for $f_{t_{20}}, f_{t_{11}},$ and $f_{t_{02}}$ as given in eq. (77) and solving the equation $f_{t_i}(x, y) = 0$, we conclude that
\[
(81)
\]
Similarly, plugging in the values for $f_{t_{20}}, f_{t_{11}},$ and $f_{t_{02}}$ as given in eq. (77) and solving the equation $f_{t_j}(x, y) = 0$, we conclude that
\[
(82)
\]
Multiplying eq. (81) by \((A_1 + A_2)x - 2A_1 A_2 y\) and multiplying eq. (82) by \((-2x + (A_1 + A_2)y)\), we conclude that
\[
(x - A_1 y)(x - A_2 y) \left( (A_1 + A_2 - 2A_3)x - (2A_1 A_2 - A_1 A_3 - A_2 A_3)y \right) + \mathcal{O}(|x, y|^4) = u^2 \mathcal{O}(|(x, y)|^2). \tag{83}
\]
Let us now solve eq. (83). Let us make a change of coordinates
\[
x = \hat{x} + \mathcal{O}((\hat{x}, \hat{y})^2) \quad \text{and} \quad y = \hat{y} + \mathcal{O}((\hat{x}, \hat{y})^2)
\]
such that eq. (83) can be rewritten as
\[
(\hat{x} - A_1 \hat{y})(\hat{x} - A_2 \hat{y}) \left( (A_1 + A_2 - 2A_3)\hat{x} - (2A_1 A_2 - A_1 A_3 - A_2 A_3)\hat{y} \right) = u^2 \mathcal{O}((\hat{x}, \hat{y})^2) \tag{84}
\]
Using eq. (84), we solve for \(\hat{x}\) in terms for \(\hat{y}\) and \(u\) and convert back to \(x\) and \(y\) to conclude that the only possible solutions are given by
\[
x = A_1 y + E_8(y, u) \quad \text{or} \quad x = A_2 y + E_9(y, u) \quad \text{or}
\]
\[
(A_1 + A_2 - 2A_3)x = (2A_1 A_2 - A_1 A_3 - A_2 A_3)y + E_{10}(y, u), \tag{85}
\]
such that \(E_i(y, 0) = \mathcal{O}(|y|^2)\), for \(i = 8, 9\) and 10. Plugging the three solutions obtained in eq. (85) into eq. (81), solving for \(y\) in terms of \(u\) and then plugging that back into eq. (85) to express \(x\) in terms of \(u\), we conclude that the only possible solutions to eq. (80) are given by
\[
(x, y) = \left( A_1 u + \tilde{E}_1(u), u + \tilde{E}_1(u) \right) \quad \text{or} \quad \tag{86}
\]
\[
(x, y) = \left( A_2 \left( \frac{A_1 - A_3}{A_2 - A_3} \right) u + \tilde{E}_2(u), \left( \frac{A_1 - A_3}{A_2 - A_3} \right) u + \tilde{E}_2(u) \right) \quad \text{or} \quad \tag{87}
\]
\[
(x, y) = \left( \frac{2A_1 A_2 - A_1 A_3 - A_2 A_3}{3(A_2 - A_3)} u + \tilde{E}_3(u), \frac{A_1 + A_2 - 2A_3}{3(A_2 - A_3)} u + \tilde{E}_3(u) \right), \tag{88}
\]
where \(\tilde{E}_i(u)\) and \(\tilde{E}_i(u)\) are second order terms (for \(i = 1, 2\) and 3). From eq. (77), we conclude that the solutions in eq. (86) and eq. (87) with \(\tilde{E}_i(u)\) replaced by \(E_i(u)\) and \(\tilde{E}_i(u)\) replaced by 0 (for \(i = 1\) and 2) is a solution to eq. (80). Since the solutions in eq. (86) and eq. (87) are the only solutions to eq. (80), we conclude that \(\tilde{E}_i(u) = E_i(u)\) and \(\tilde{E}_i(u) = 0\) (for \(i = 1\) and 2). Hence, if we plug the solutions obtained from eq. (86) and eq. (87) into \(f_i(x, y)\) (or equivalently \(g_i(x, y)\)), we will get 0 and not \(\varepsilon_0\). Hence, we reject the solutions given by eq. (86) and eq. (87).

It remains to consider the solution given by eq. (88). Plugging in the expression for \(x\) and \(y\) from eq. (88) into \(g_i(x, y)\) gives us
\[
g_i(x, y) = \left( \frac{(A_2 - A_1)^2 (A_4 - A_1)^2}{162(A_2 - A_3)} \right) u^3 + \mathcal{O}(u^4). \tag{89}
\]
From eq. (89), we conclude that \(g_i(x, y) = -2\varepsilon_0\) has 3 solutions. This justifies the multiplicity and concludes the proof of theorem 5.2.

6.1.1. Local degree of a smooth map. It remains to show why we could set \(\varepsilon_2\) to be 0 in eq. (51) and set \((\varepsilon_1, \varepsilon_2)\) to be \((0, 0)\) in eq. (78). Let us first recall the definition of the local degree of a smooth map around a given point. We will follow the discussion and theory developed in [13].

Let us begin with the proposition 2.1.2 of [13]. The statement is as follows:

**Proposition 6.4.** Let \(f \in C^2(\Omega, \mathbb{R}^n)\) where \(\Omega\) is an open subset of \(\mathbb{R}^n\) and let \(b \notin f(\partial \Omega)\). Let \(\rho_0\) be the distance between \(b\) and \(f(\partial \Omega)\) with \(\rho_0 > 0\). Let \(b_1, b_2 \in B(b; \rho_0)\), the ball of radius \(\rho_0\) with center \(b\). If \(b_1, b_2\) both are regular values of \(f\), then \(\deg(f, \Omega, b_1) = \deg(f, \Omega, b_2)\) where \(\deg(f, \Omega, y)\) represent the degree of \(f\) at \(y\) (i.e. the number of solutions to the equation \(f(x) = y\) in \(\Omega\)).
Let us first justify the assertion for eq. (51). Let $U$ be an open ball in $\mathbb{C}^2$ with center $(0,0)$ and radius $r$, where $r$ is sufficiently small and positive real number. Consider the map $\varphi : U \rightarrow \mathbb{C}^2$, given by

$$\varphi(x, h) = (\varphi_1(x, h), \varphi_2(x, h)) := \left(\frac{(x^2h^2)(B_{4r}^f + O(||(x, h)||))}{24}, \frac{(x)(B_{4r}^f h - B_{4r}^f x + O(||(x, h)||^2))}{12}\right).$$

Before proceeding, let us first prove the following claim:

**Claim 6.5.** If $\varepsilon \neq 0$, then the point $(\varepsilon, 0)$ is a regular value of $\varphi$.

**Proof.** Let us assume $\varphi(x, h) = (\varepsilon, 0)$. Using the fact that $\varphi_2(x, h) = 0$, we conclude that $x(h) = h + O(h^2)$.

Plugging in this value of $x$ in $\varphi_1(x, h)$, we conclude that

$$h^4\left(\frac{B_{4r}^f}{24} + O(h)\right) = \varepsilon.$$  \hfill (90)

Note that if $h$ is sufficiently small, then $\frac{B_{4r}^f}{24} + O(h)$ is nonzero, since $B_{4r}^f$ is nonzero. We also note that since \(\varepsilon\) is nonzero, eq. (90) implies that $h$ is nonzero.

Next, let us compute the determinant of the differential of $\varphi$ at $(x(h), h)$. It is given by

$$M := \det \begin{pmatrix} \varphi_{1x} & \varphi_{1h} \\ \varphi_{2x} & \varphi_{2h} \end{pmatrix}_{(x(h), h)} = h^5\left(\frac{(B_{4r}^f)^2}{72} + O(h)\right).$$  \hfill (91)

Using eq. (90) and eq. (91), we conclude that

$$M = h^4 \cdot h\left(\frac{(B_{4r}^f)^2}{72} + O(h)\right) = \varepsilon \left(\frac{(B_{4r}^f)^2}{72} + O(h)\right).$$  \hfill (92)

Since, $B_{4r}^f$ is nonzero, $h$ is small and nonzero and $\varepsilon$ is nonzero, we conclude from eq. (92) that $M$ is nonzero. Hence, $(\varepsilon, 0)$ is a regular value of $\varphi$. \qed

Next, we note that if $S$ is a non empty subset of $\mathbb{C}^2$, then the distance function $d_S : \mathbb{C}^2 \rightarrow \mathbb{R}$ is a continuous function. Hence, the set

$$V := (d_{\varphi(\partial U)} - d_X)^{-1}(0, \infty) = \{(\varepsilon_1, \varepsilon_2) \in \mathbb{C}^2 \mid d_{\varphi(\partial U)}(\varepsilon_1, \varepsilon_2) > d_X(\varepsilon_1, \varepsilon_2)\}$$

is an open subset of $\mathbb{C}^2$, where the function $d_X$ denotes the distance from $x$-axis. Note that $d_X(\varepsilon_1, \varepsilon_2) = |\varepsilon_2|$ and this distance is achieved by taking the distance from the point $(\varepsilon_1, \varepsilon_2)$ to the point $(\varepsilon_1, 0)$ on $x$-axis.

Now, we will show that $V \cap \varphi(U) \neq \emptyset$. Note that $\partial U = \{(x, h) \in \mathbb{C}^2 : |x|^2 + |h|^2 = r^2\}$. Observe that $\partial U$ is compact; so $\varphi(\partial U)$ is compact and hence closed in $\mathbb{C}^2$. Hence, $d_{\varphi(\partial U)}(\varepsilon, 0) = 0$ if and only if $(\varepsilon, 0) \in \varphi(\partial U)$. We conclude that $(\varepsilon, 0) \in V$ if and only if $(\varepsilon, 0) \notin \varphi(\partial U)$. Now, let $(\varepsilon, 0) \in \varphi(\partial U)$. Let us assume $\varphi(x, h) = (\varepsilon, 0)$ with $|x|^2 + |h|^2 = r^2$ and $\varepsilon \neq 0$. We conclude from $\varphi_2(x, h) = 0$ and eq. (90) that

$$x(h) = h + O(h^2) \quad \text{and} \quad h^4\left(\frac{B_{4r}^f}{24} + O(h)\right) = \varepsilon.$$

Now using the fact $|x|^2 + |h|^2 = r^2$, we conclude that $|\varepsilon| = \frac{|B_{4r}^f|}{96}r^4 + O(r^5)$. Hence we get either $\varepsilon = 0$ or $|\varepsilon| = \frac{|B_{4r}^f|}{96}r^4 + O(r^5)$. Note that $\frac{|B_{4r}^f|}{96}r^4 + O(r^5) \neq 0$ as $B_{4r}^f \neq 0$ and $r$ is sufficiently small. So, $(\varepsilon, 0) \in V$ for
all nonzero $\varepsilon$ with $|\varepsilon| < B_1^4/96 + O(r^5)$ (i.e. for all $|\varepsilon|$ sufficiently small). From eq. (53) we concluded that the system $\phi(x, h) = (\varepsilon, 0)$ has solutions in $U$ where $\varepsilon$ is small but nonzero. Hence $(\varepsilon, 0) \in V \cap \phi(U)$ for some nonzero $\varepsilon$ with $|\varepsilon| < B_1^4/96 + O(r^5)$. Hence, $V \cap \phi(U)$ is non empty.

Next, we note that since $\phi : U \to \mathbb{C}^2$ is a non constant holomorphic map, $\phi(U)$ is an open subset of $\mathbb{C}^2$. Hence, $V \cap \phi(U)$ is a non empty open subset of $\mathbb{C}^2$ and has nonzero measure. Using Sard’s Theorem (applied to the function $\phi : U \to \mathbb{C}^2$), we conclude that $V \cap \phi(U)$ contains regular values of $\phi$. Let $(\varepsilon_1, \varepsilon_2) \in V \cap \phi(U)$ be a regular value of $\phi$. Therefore by definition of $V$, $d\phi(\partial U)(\varepsilon_1, \varepsilon_2) \neq 0 \geq 0$. Now, $\phi(\partial U)$ is a closed subset of $\mathbb{C}^2$ and $d\phi(\partial U)(\varepsilon_1, \varepsilon_2) > 0$ together implies that $(\varepsilon_1, \varepsilon_2) \notin \phi(\partial U)$. Hence all the hypothesis of proposition 6.4 are satisfied. We conclude from the proposition that $\deg(\phi, U, (\varepsilon_1, \varepsilon_2)) = \deg(\phi, U, (\varepsilon_1, 0))$, i.e. the number of solutions in $U$ to both the equations $\phi(x, h) = (\varepsilon_1, \varepsilon_2)$ and $\phi(x, h) = (\varepsilon_1, 0)$ are same. This justifies our claim in eq. (51).

Let us now justify the assertion for eq. (78). The argument is similar to the previous argument. We just need to prove the following claim:

Claim 6.6. Let $U \subseteq \mathbb{C}^3$ be a small open neighborhood of $(0, 0, 0)$ and $\phi : U \to \mathbb{C}^3$ be given by

$$\phi(x, y, u) := \left(f_t(x, y), f_{t,x}(x, y), f_{t,y}(x, y)\right),$$

where $f_t$ is as given in eq. (60) and $f_{t,x}, f_{t,y}, f_{t,xy}$ are as given in eq. (77). Let $\hat{U} \subseteq \mathbb{C}$ be a small open neighborhood of 0. If $\varepsilon$ is a generic point of $\hat{U}$, then $(\varepsilon, 0, 0)$ is a regular value of $\phi$.

Proof. Let $(x, y, u) \in U$ such that $\phi(x, y, u) = (\varepsilon, 0, 0)$. We note that

$$\det \begin{pmatrix} f_{t,x}(x, y) & f_{t,xx}(x, y) & f_{t,xy}(x, y) \\ f_{t,y}(x, y) & f_{t,xy}(x, y) & f_{t,yy}(x, y) \\ f_{t,u}(x, y) & f_{t,ux}(x, y) & f_{t,uy}(x, y) \end{pmatrix} = f_{t,u}(x, y) \cdot \det \begin{pmatrix} f_{t,xx}(x, y) & f_{t,xy}(x, y) \\ f_{t,yy}(x, y) & f_{t,yy}(x, y) \end{pmatrix}. \quad (93)$$

This is because $f_{t,x}(x, y)$ and $f_{t,u}(x, y)$ are both equal to zero. We now note that $f_t$ has an $A_1$ singularity at $(0, 0)$; hence determinant of Hessian of $f_t$ does not vanish at $(0, 0)$. Since $(x, y)$ is small, we conclude that the determinant of the Hessian of $f_t$ at $(x, y)$ is nonzero. Hence, if the right hand side of eq. (93) is zero, then $f_{t,u}(x, y)$ has to be zero. We claim this is not possible for a generic $\varepsilon$. To see why this is so, note that the solution to the equation $\phi(x, y, u) = (\varepsilon, 0, 0)$ with $\varepsilon \in \hat{U}$ is given by eq. (88). After plugging the value of $(x, y, u)$ obtained in eq. (88) to the expression of $f_t(x, y)$, we conclude from eq. (89) that

$$f_{t,u}(x, y) = -\frac{3(A_2 - A_1)^2(A_3 - A_1)^2}{324(A_2 - A_3)} u^2 + O(u^3)$$

Note that $f_{t,u}(x, y)$ is a power series of $u$ which is not identically zero in a small open subset of $\mathbb{C}$ containing the origin and hence it has only finitely many zeros. We conclude that $(\varepsilon, 0, 0)$ is a regular value of $\phi$ for all but a finite set of $\varepsilon$; in particular for a generic $\varepsilon$, $(\varepsilon, 0, 0)$ is a regular value of $\phi$. 

6.2. Proof of Theorem 5.3: computation of $N(A_1^d \cap \mathcal{P}A_1)$ when $0 \leq \delta \leq 2$. We will now justify our formula for $N(A_1^d \cap \mathcal{P}A_1, r, s, n_1, n_2, n_3, \theta)$, when $0 \leq \delta \leq 2$. If $\theta = 0$, then the formula follows from eq. (12).

Let us now assume $\theta > 0$. Recall that (as per the definition in section 2)

$$A_1^d \cap \overline{A}_1 := \{([f], [\eta], q_1, \ldots, q_8, l_{q_8+1}) \in \mathcal{S}_{\mathcal{D}} \times D \mathbb{P} \mathcal{W} : f \text{ has a singularity of type } A_1 \text{ at } q_1, \ldots, q_8, ([f], [\eta], l_{q_8+1}) \in \overline{A}_1, q_1, \ldots, q_8+1 \text{ all distinct}\}.$$

Let $\mu$ be a generic cycle, representing the class

$$[\mu] = H_L^* \cdot H_p^* \cdot a^{n_1} \lambda^{n_2}(\pi_{q_8+1}^* H)^{n_3}(\pi_{q_8+1}^* \lambda W)^{\theta}.$$

We now define a section of the following bundle

$$\psi_{\mathcal{P}A_1} : A_1^d \cap \overline{A}_1 \to \mathcal{L}_{\mathcal{P}A_1} := \gamma_p^* \otimes \gamma_w^2 \otimes \gamma_{p,d}^d,$$

given by

$$\{\psi_{\mathcal{P}A_1}([f], q_1, \ldots, q_8, l_{q_8+1})\}(f \otimes v^2) := \nabla^2 f|_{q_8+1}(v, v).$$
We will show shortly that this section is transverse to zero. Next, let us define
\[ B := A_3^\delta \circ A_1 - A_1^\delta \circ A_1. \]

Hence
\[ \langle e(L_{PA_1}), [A_1^\delta \circ A_1] \cap [\mu] \rangle = N(A_1^\delta P A_1, r, n_1, n_2, n_3, \theta) + C_{B \cap \mu}, \]
where as before, \( C_{B \cap \mu} \) denotes the contribution of the section to the Euler class from \( B \cap \mu \). When \( \delta = 0 \), the boundary \( B \) is empty. Hence, plugging in \( C_{B \cap \mu} = 0 \) and unwinding the left hand side of eq. (94) gives us the formula of Theorem 5.3 for \( \delta = 0 \).

Let us now assume \( \delta > 0 \). Given \( k \) distinct integers \( i_1, i_2, \ldots, i_k \in [1, \delta + 1] \), let \( \Delta_{i_1,\ldots,i_k} \) be as defined in the proof of Theorem 5.2. Let us define
\[ \Delta_{i_1,\ldots,i_k} := \pi^{-1}(\Delta_{i_1,\ldots,i_k}), \]
where \( \pi : S_{D_4} \times_D \mathbb{P}W_D \rightarrow S_{D_{3+1}} \) is the projection map. Let us define
\[ B(q_{i_1}, \ldots, q_{i_{k-1}}, l_{q_{k+1}}) := B \cap \Delta_{i_1,\ldots,i_k-1,\delta+1}. \]

Let us now consider \( B(q_1, l_{q_{k+1}}) \). We claim that,
\[ B(q_1, l_{q_{k+1}}) \approx A_1^{\delta-2} \circ A_5 \cup A_1^{\delta-2} \circ D_4. \]

The set \( A_1^{\delta-2} \circ A_5 \cap \mu \) is empty since the sum total of the dimensions of these two varieties is one less than the dimension of the ambient space. Next, we note that the section \( \Psi_{P.A_1} \) vanishes everywhere on \( A_1^{\delta-2} \circ D_4 \); hence it also vanishes on \( A_1^{\delta-2} \circ D_4 \cap \mu \). We claim that the contribution from each of the points of \( B(q_1, q_{i_2}, l_{q_{k+1}}) \cap \mu \) is 6. Hence the total contribution from all the components of type \( B(q_1, q_{i_2}, l_{q_{k+1}}) \) is
\[ 6\left(\binom{\delta}{2}\right) N(A_1^{\delta-2}D_4, n_1, n_2, n_3, \theta). \]

Plugging this in eq. (94) gives us the formula of theorem 5.3.

Let us now justify the transversality, closure and multiplicity claims. We will follow the setup of theorem 5.2. Suppose
\[ \Psi_{PA_1}([f], [\eta], q_1, \ldots, q_3, l_{q_{k+1}}) = 0. \]

As before we assume \( \eta \) determines the plane where the last component is zero and \( q_{k+1} := [0, 0, 0, 1, 0] \). Let us consider \( T\mathbb{P}^2_{[q_{k+1}]} \). Let \( \partial_x \) and \( \partial_y \) be the standard basis vectors for \( T\mathbb{P}^2_{[q_{k+1}]} \) (corresponding to the first two coordinates). Hence
\[ l_{q_{k+1}} = [a\partial_x + b\partial_y] \in T\mathbb{P}^2_{[q_{k+1}]} \]
for some complex numbers \( a, b \) not both of which are zero. Without loss of generality, we can assume \( l_{q_{k+1}} = [\partial_x] \). Let us now consider the polynomial
\[ \rho_{20} := (X - X_1)^2(X - X_2)^2 \cdots (X - X_3)^2X^2Z^{d-2\delta-2} \]
and consider the corresponding curve \( \gamma_{20}(t) \). We now note
\[ \{\nabla \Psi_{PA_1}([f],[\eta],q_1,\ldots,q_3,l_{q_{k+1}})](\gamma_{20}(0))\} (f \otimes \partial_x \otimes \partial_x) = \lambda Z^{d-2\delta-2}X^2Z\mathbb{P}^2_{[0,0,1,0]}(\partial_x, \partial_x). \]

Since \( Z(X)_{[0,0,1,0]}(\partial_x, \partial_x) \) is nonzero, we conclude that the section is transverse to zero.

Next, let us justify the closure claims. Let us start with eq. (95). This statement is saying that when
two nodes collide, we get a tacnode. Hence, the proof of eq. (95) is same as the proof of eq. (23).

Next, let us consider eq. (96). Again, this statement is saying what happens what happens when three nodes collide. Hence, the proof of eq. (96) is same as the proof of eq. (24).

It remains to justify the contribution from the points of $A_1^{δ−2} o D_4 \cap \mu$. We will use the solutions constructed in eq. (77). Using the expression for $f_{l_{40}}$, we note that the multiplicity from each branch is the number of small solutions $u$ to the equation

$$\frac{f_{l_{40}}}{3}(A_1−A_3)u+E_4(u) = \varepsilon.$$  

This is clearly 1. Since there are 6 branches, the total multiplicity is 6. □

6.3. Proof of Theorem 5.4: computation of $N(A_1 f o P A_2)$ when $0 \leq δ \leq 2$. We will justify our formula for $N(A_1 f o P A_2, r, s, n_1, n_2, n_3, \theta)$, when $0 \leq δ \leq 2$. Recall that

$$A_1^δ o P A_1 := \{([f], [η], q_1, \ldots, q_s, l_{q_{s+1}}) \in S_{D_1} \times D P W_D : f \text{ has a singularity of type } A_1 \text{ at } q_1, \ldots, q_s, ([f], [η], l_{q_{s+1}}) \in P A_1, q_1, \ldots, q_{s+1} \text{ all distinct} \}.$$  

Let $μ$ be a generic cycle, representing the class

$$[μ] = H_L^1 : H_p^* : a^{n_1} λ^{n_2}(π_{s+1}^δ H)^{n_3}(π_{s+1}^δ λ W)_δ.$$  

Recall that as per the hypothesis of the Theorem, if $δ = 2$ then $θ = 0$. We now define a section of the following line bundle

$$Ψ_{P A_2} : A_1^δ o P A_1 \longrightarrow L_{P A_2} := γ_δ^p \otimes γ_W^p \otimes (W/λ W)^* \otimes γ_{P D}^δ,$$

given by

$$Ψ_{P A_2}([f], [η], l_{q_{s+1}})) = f(v, v, w) := \nabla^2 f|_{q_{s+1}}(v, w).$$

We will show shortly that this section is transverse to zero. Next, let us define

$$B := A_1^δ o P A_1 − A_1^δ o P A_1.$$  

Hence

$$(e(L_{P A_2}), [A_1^δ o P A_1] \cap [μ]) = N(A_1^δ o P A_2, r, s, n_1, n_2, n_3, \theta) + C_{B P μ}.$$  

(97)

Define $B(q_1, \ldots, q_k, l_{q_{s+1}})$ as before. For simplicity, let us set $(i_1, i_2, \ldots, i_k) := (δ − k, \ldots, δ − 1, δ)$. Before we describe $B(q_1, q_{i_2}, l_{q_{s+1}})$, let us define a few things. Let $ν$ be a fixed nonzero vector that belongs to $l_{q_{s+1}}$. Let us define $W_1, W_2, W_3, W_4$ as

$$W_1 := \{([f], [η], q_1, \ldots, q_s, l_{q_{s+1}}) \in A_1^δ o P A_1 : \nabla^2 f|_{q_{s+1}} \neq 0 \},$$

$$W_2 := \{([f], [η], q_1, \ldots, q_s, l_{q_{s+1}}) \in A_1^δ o P A_1 : \nabla^2 f|_{q_{s+1}} \equiv 0 \},$$

$$W_3 := \{([f], [η], q_1, \ldots, q_s, l_{q_{s+1}}) \in A_1^δ o P A_1 : \nabla^3 f|_{q_{s+1}}(v, v, v) \neq 0 \} \text{ and }$$

$$W_4 := \{([f], [η], q_1, \ldots, q_s, l_{q_{s+1}}) \in A_1^δ o P A_1 : \nabla^3 f|_{q_{s+1}}(v, v, v) = 0 \}.$$  

(98)

We claim that

$$B(q_1, l_{q_{s+1}}) \cap W_1 \approx A_1^{δ−1} o P A_3 \cap W_1,$$  

(99)

$$B(q_1, l_{q_{s+1}}) \cap W_2 \approx A_1^{δ−1} o D_4,$$  

(100)

$$B(q_{i_1}, q_1, l_{q_{s+1}}) \cap W_1 \subset A_1^{δ−2} o P A_5 \cap W_1,$$  

(101)

$$B(q_{i_2}, q_1, l_{q_{s+1}}) \cap (W_2 \cap W_4) \approx A_1^{δ−1} o D_4 \cap W_1 \text{ and }$$

(102)

$$B(q_{i_2}, q_1, l_{q_{s+1}}) \cap (W_2 \cap W_4) \subset A_1^{δ−2} o D_5.$$  

(103)

Notice that equations (101) and (103) say that the left hand side is a subset of the right hand side (unlike the other three equations, which assert equality of sets). We now note that equations (99) and (100), imply that

$$B(q_1, l_{q_{s+1}}) \approx A_1^{δ−1} o P A_3 \cup A_1^{δ−1} o D_4.$$

(104)
while equations (101), (102) and (103) imply that

\[ B(q_{i_1}, q_{i_2}, l_{q_{i_3}+1}) \subset A_1^{\delta-2} \circ PA_1 \cup A_1^{\delta-2} \circ PD_4 \cup A_1^{\delta-2} \circ D_5. \]  

(105)

We claim that the contribution to the Euler class from each of the points of \( A_1^{\delta-1} \circ PA_3 \cap \mu, A_1^{\delta-1} \circ D_4 \cap \mu \) and \( A_1^{\delta-2} \circ PD_4 \cap \mu \) are 2, 3 and 4 respectively.

Next, we note that for dimensional reasons, the intersection of \( A_1^{\delta-1} \circ PA_3 \) with \( \mu \) is empty. Hence, by eq. (101), the intersection of \( B(q_{\delta-1}, q_\delta, l_{q_{\delta+1}+1}) \cap W_1 \) with \( \mu \) is also empty and hence does not contribute to the Euler class. Finally, let us consider the projection corresponding to the left hand side of eq. (103); this is where we will use \( \theta = 0 \). Let us consider the projection map

\[ \pi : \mathcal{S}_{D_3} \times P^3 W_D \longrightarrow \mathcal{S}_{D_3+1}. \]

We recall that

\[ A_1^{\delta-2} \circ D_3 = \pi^{-1}(A_1^{\delta-2} \circ D_5). \]

Since \( \theta = 0 \), we note that \( \mu \) is the pullback of a class \( \nu \), i.e.

\[ \mu = \pi^*(\nu). \]

Hence, the intersection of \( \mu \) with \( A_1^{\delta-2} \circ D_3 \) is in one to one correspondence with the intersection of \( \nu \) with \( A_1^{\delta-2} \circ D_5 \). But the degree of the cohomology class \( \nu \) is one more than the dimension of the cycle \( A_1^{\delta-2} \circ D_5 \).

Hence, the intersection of \( A_1^{\delta-2} \circ D_3 \) with \( \nu \) is empty and hence, the intersection of \( \mu \) with \( A_1^{\delta-2} \circ D_5 \) is empty.

As a result, by eq. (103), the intersection of \( B(q_{i_1}, q_{i_2}, l_{q_{i_3}+1}) \cap (W_2 \cap W_3) \) with \( \mu \) is also empty. Hence the total contribution from all the components of type \( B(q_{i_1}, q_{i_2}, l_{q_{i_3}+1}) \) equals

\[ 2 \binom{\delta}{1} N(A_1^{\delta-1} \circ PA_3, r, s, n_1, n_2, n_3, \theta) + 3 \binom{\delta}{1} N(A_1^{\delta-1} \circ PD_4, r, s, n_1, n_2, n_3, \theta), \]

while the total contribution from all the components of type \( B(q_{i_1}, q_{i_2}, l_{q_{i_3}+1}) \) equals

\[ 4 \binom{\delta}{2} N(A_1^{\delta-2} \circ PD_4, r, s, n_1, n_2, n_3, \theta). \]

Plugging this in eq. (97) gives us the formula of theorem 5.4.

Let us now prove the claim about transversality. This follows from following the setup of proof of transversality in Theorem 5.3. We consider the polynomial

\[ \rho_{11} := (X - X_1)^2 (X - X_2)^2 \ldots (X - X_\delta)^2 XYZ^{d-2\delta-2} \]

and the corresponding curve \( \gamma_{11}(t) \). Transversality follows by computing the derivative of the section \( \Psi_{PA_2} \) along the curve \( \gamma_{11}(t) \) as before.

Next, let us justify the closure and multiplicity claims. We will start by justifying eq. (104). It suffices to justify eq. (99) and eq. (100). Let us rewrite these two equations explicitly, namely

\[ \{(f), [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}} \} \in A_1^{\delta} \circ PA_1 : q_\delta = q_{\delta+1} \} \cap W_1 = A_1^{\delta-1} \circ PA_3 \cap W_1 \]

and

\[ \{(f), [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}} \} \in A_1^{\delta} \circ PA_1 : q_\delta = q_{\delta+1} \} \cap W_2 = A_1^{\delta-1} \circ D_4. \]

(106)

Since \( \tilde{D}_4 \) is a subset of \( W_2 \), we did not write \( \cap \) on the right hand side of eq. (107).

Let us now start the proof of eq. (106). Let us first explain why the left hand side of eq. (106) is a subset of its right hand side. To see that, first we note that \( PA_1 \) is a subset of \( \tilde{A}_1 \). Since we have shown while proving eq. (23) and eq. (30) that when two nodes collide we get a tacnode in eq. (30), we conclude that

\[ \{(f), [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}} \} \in A_1^{\delta} \circ PA_1 : q_\delta = q_{\delta+1} \} = A_1^{\delta-1} \circ A_3. \]

Hence, we conclude that

\[ \{(f), [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}} \} \in A_1^{\delta} \circ PA_1 : q_\delta = q_{\delta+1} \} \subset A_1^{\delta-1} \circ A_3 \]

\[ \Rightarrow \{(f), [\eta], q_1, \ldots, q_\delta, l_{q_{\delta+1}} \} \in A_1^{\delta} \circ PA_1 : q_\delta = q_{\delta+1} \} \cap W_1 \subset A_1^{\delta-1} \circ A_3 \cap W_1. \]  

(108)
Suppose \( ([f], [\eta], q_1, \ldots, q_6, l_{q_6+1}) \) belongs to the left hand side of eq. (108). Since \( ([f], [\eta], l_{q_6+1}) \) belongs to \( \mathcal{P}A_1 \), we conclude that
\[
\nabla^2 f|_{q_{6}+1}(v, v) = 0 \quad \forall \ v \in l_{q_{6}+1}.
\]
Since \( ([f], [\eta], q_1, \ldots, q_6, l_{q_6+1}) \) is a subset of the right hand side of eq. (108), we conclude that the Hessian \( \nabla^2 f|_{q_{6}+1} \) is not identically zero, but it has a non trivial Kernel. We claim that \( v \) is in the Kernel of the Hessian. To see why, let us assume that the nonzero vector \( \tilde{v} \) is in the Kernel of the Hessian, i.e. \( \nabla^2 f|_{q_{6}+1}(\tilde{v}, \tilde{v}) = 0 \). Let \( w \) be any other vector, linearly independent from \( \tilde{v} \). Since the Hessian is not identically zero and the vector space is two dimensional, we conclude that \( \nabla^2 f|_{q_{6}+1}(v, w) \neq 0 \). Hence, writing the vector \( v := \lambda_1 \tilde{v} + \lambda_2w \) and using \( \nabla^2 f|_{q_{6}+1}(v, v) = 0 \), we conclude that \( \lambda_2 = 0 \). Hence, \( \lambda_2 \) belongs to the Kernel of the Hessian. But we also note that if \( ([f], [\eta], l_{q_6}) \in \mathcal{P}A_3 \) and \( \nabla^2 f|_{q_6}(v, \cdot) = 0 \), then \( \nabla^3 f|_{q_6}(v, v, v) = 0 \). Hence, we can improve eq. (108) and conclude that the left hand side of eq. (106) is a subset of its right hand side.

Let us now prove the converse. We will simultaneously prove the following two statements
\[
\begin{align*}
\{([f], [\eta], q_1, \ldots, q_6, l_{q_6+1}) \in A_1^6 & \circ \mathcal{P}A_1 : q_6 = q_{6+1} \} \supset A_1^{6-1} \circ \mathcal{P}A_3, \quad (109) \\
\{([f], [\eta], q_1, \ldots, q_6, l_{q_6+1}) \in A_1^6 & \circ \mathcal{P}A_2 : q_6 = q_{6+1} \} \cap (A_1^{6-1} \circ \mathcal{P}A_3) = \emptyset \quad \text{and} \quad (110)
\end{align*}
\]
We will prove the following claim:

**Claim 6.7.** Let \( ([f], [\eta], q_1, \ldots, q_6, l_{q_6+1}) \in A_1^{6-1} \circ \mathcal{P}A_3 \). Then there exists points
\[
([f], [\eta], q_1(t), \ldots, q_6-2(t); q_6-1(t), q_6(t), l_{q_6+1}(t)) \in A_1^6 \circ \mathcal{P}A_1
\]
sufficiently close to \( ([f], [\eta], q_1, \ldots, q_6-1; q_6, l_{q_6}) \). Furthermore, every such solution satisfies the condition
\[
\nabla^2 f|_{q_{6}+1}(v, w) \neq 0,
\]
if \( v \) is a nonzero vector that belongs to \( l_{q_{6}+1}(t) \) and \( w \) is a nonzero vector that belongs to \( T^2|_{q_{6}+1} / l_{q_{6}+1}(t) \). In other words,
\[
([f], [\eta], q_1(t), \ldots, q_6-1(t), q_6(t), l_{q_6+1}(t)) \notin A_1^6 \circ \mathcal{P}A_2.
\]

**Remark 9.** We note that claim 6.7 simultaneously proves eq. (109) and eq. (110).

**Proof:** Following the setup of the proofs of claims 6.1 and 6.3, we will now work in an affine chart, where we send the plane \( \mathbb{P}_2^2 \) to \( \mathbb{C}_2^2 \) and the point \( q_6(t) \in \mathbb{P}_2^2 \) to \( (0, 0, 0) \in \mathbb{C}_2^2 \). We also choose coordinates, such that \( \partial_x \in l_{q_{6}+1}(t) \). Using this chart, let us write down the Taylor expansion of \( f_1 \) around the point \( (0, 0) \), namely
\[
f_1(x, y) = f_{1,1}xy + \frac{f_{1,2}}{2}y^2 + \frac{f_{1,3}}{6}x^3 + \frac{f_{1,21}}{2}x^2y + \frac{f_{1,22}}{2}xy^2 + \frac{f_{1,33}}{6}y^3 + \ldots
\]
Since \( ([f], [\eta], l_{q_1(t)}) \in \mathcal{P}A_1 \), we conclude that \( f_{2,0} \) is zero. Next, let us consider the Taylor expansion of \( f \) (not \( f_1 \)). We note that \( ([f], [\eta], l_{q_6}) \in \mathcal{P}A_3 \). This means that \( f_{11} \) and \( f_{02} \) can not both be zero (since that would mean the Hessian is identically zero). If \( f_{02} \neq 0 \), then it implies that \( ([f], [\eta], l_{q_6}) \in A_1 \) (and hence does not belong to \( \mathcal{P}A_3 \)). Hence, \( f_{02} \neq 0 \) and hence we conclude that \( f_{102} \neq 0 \). Finally, since \( ([f], [\eta], l_{q_6}) \in \mathcal{P}A_3 \), we conclude that \( f_{11} \) and \( f_{30} \) are zero; hence \( f_{111} \) and \( f_{300} \) are small (close to zero). We will mainly follow the Proof of claim 6.1. Since \( f_{102} \neq 0 \) we can make the same change of coordinates \( \tilde{y} := y + B(x) \) as in the Proof of claim 6.1 and write \( f_1 \) as
\[
f_1(x, y(x, \tilde{y})) = \varphi(x, \tilde{y})\tilde{y}^2 + \frac{B_{2}^{f_1}}{2!}x^2 + \frac{B_{3}^{f_1}}{3!}x^3 + \frac{B_{4}^{f_1}}{4!}x^4 + \mathcal{R}(x)\tilde{y}^5,
\]
where
\[
B_{2}^{f_1} := \frac{f_{2,1}}{f_{102}}, \quad B_{3}^{f_1} := \frac{f_{3,1}}{f_{102}} - \frac{3f_{1,1}f_{1,21}}{f_{102}} + \frac{3f_{1,1}f_{1,22}}{f_{102}} - \frac{3f_{1,31}}{f_{102}} - \ldots, \quad \varphi(0, 0) \neq 0
\]
and \( \mathcal{R}(x) \) is a holomorphic function defined in a neighborhood of the origin. Since \( ([f], [\eta], q_6) \in \mathcal{P}A_3 \), we conclude that \( B_{2}^{f_1} \) and \( B_{3}^{f_1} \) are small (close to zero) and \( B_{4}^{f_1} \) is nonzero. Let us make a further change of coordinates and denote
\[
\tilde{\tilde{y}} := \sqrt{\varphi(x, \tilde{y})}\tilde{y}.
\]
Hence, there are two solutions to eq. (111) amounts to solving the set of equations
\[ \hat{f}_t = 0, \quad \hat{f}_{tx} = 0 \quad \text{and} \quad \hat{f}_{\hat{y}} = 0, \] (115)
where \((x, \hat{y})\) is small but not equal to \((0,0)\).

We will now construct solutions to eq. (115). The solutions to eq. (115) are given by
\[ \hat{y} = 0, \quad \mathcal{B}_2^f = \frac{B_{11}^f}{12} x^2 + O(|x|^3) \quad \text{and} \quad \mathcal{B}_3^f = -\frac{B_{21}^f}{2} x + O(|x|^2). \] (116)

Now we use the expression of \(\mathcal{B}_2^f, \mathcal{B}_3^f\) and conclude from eq. (116) that
\[ \frac{f_{t_{11}}}{f_{t_{02}}} = -\frac{B_{11}^f}{12} x^2 + O(|x|^3) \quad \text{and} \quad f_{t_{30}} = -3B_{21}^f x + O(|x|^2). \] (117)

Hence, there are two solutions to eq. (117), given by
\[ f_{t_{11}} = \left( \sqrt{-\frac{f_{t_{02}}B_{11}^f}{12}} \right) x + O(|x|^2) \quad \text{or} \quad f_{t_{11}} = -\left( \sqrt{-\frac{f_{t_{02}}B_{11}^f}{12}} \right) x + O(|x|^2), \] (119)
where \(\sqrt{\cdot}\) denotes a branch of the square root. Hence, there are exactly two solutions to eq. (115), given by
\[ x = u, \quad f_{t_{11}} = \pm \left( \sqrt{-\frac{f_{t_{02}}B_{11}^f}{12}} \right) u + O(|u|^2) \] (120)
and \(\hat{y} = 0\) and \(f_{t_{30}}\) as given by eq. (118), where we plug in the expressions for \(x\) and \(f_{t_{11}}\) as given by eq. (120) to express them in terms of \(u\) (the exact expressions in terms of \(u\) are not so important, hence we have not written that out explicitly). This proves claim 6.7. Since eq. (120) are the only solutions and \(\mathcal{B}_1^f \neq 0\), we also conclude that eq. (112) is true.

It remains to compute the multiplicity. We claim the each point of \((A_1^{d-1} \circ \mathcal{P}A_3) \cap \mu\) contributes 2 to the Euler class in eq. (97). Using eq. (120) we conclude that the multiplicity from each branch is the number of small solutions \(u\) to the equation
\[ \left( \sqrt{-\frac{f_{t_{02}}B_{11}^f}{12}} \right) u + O(|u|^2) = \varepsilon \quad \text{and} \quad -\left( \sqrt{-\frac{f_{t_{02}}B_{11}^f}{12}} \right) u + O(|u|^2) = \varepsilon. \]

This number is 1 in each case and hence, the total multiplicity is 2.

Next, let us justify eq. (107). Let us first explain why the left hand side of eq. (107) is a subset of its right hand side. If \(([f], [\eta], q_1, \ldots, q_d, l_{q_{d+1}}) \in W_2\), then it means that \(\nabla^3 f_{q_{d+1}} = 0\). Hence, it means that \(([f], [\eta], l_{q_{d+1}}) \in \mathcal{D}_4\). Hence, the left hand side of eq. (107) is a subset of its right hand side.

Let us now prove eq. (107). Before that, let us introduce a new space. Let us define
\[ \mathcal{D}_4^\# := \{ ([f], [\eta], l_q) \in \mathcal{D}_4 : \nabla^1 f_q (v, v, v) \neq 0 \text{ if } v \in l_q - 0 \}. \]
Note that \(\mathcal{D}_4^\# = \mathcal{D}_4\). We will now simultaneously prove the following two statements:
\[ \{(f, [\eta], q_1, \ldots, q_d, l_{q_{d+1}}) \in A_1^{d} \circ \mathcal{P}A_1 : q_d = q_{d+1} \} \supset A_1^{d-1} \circ \mathcal{D}_4^\# \quad \text{and} \quad (121) \]
\[ \left( \{(f, [\eta], q_1, \ldots, q_d, l_{q_{d+1}}) \in A_1^{d} \circ \mathcal{P}A_2 : q_d = q_{d+1} \} \right) \cap \left( A_1^{d-1} \circ \mathcal{D}_4^\# \right) = \emptyset. \] (122)
We will prove the following claim:

**Claim 6.8.** Let \([f],[η],q_1,\ldots,q_{δ−1},l_{q_δ}\) \(\in A^{δ−1}_1 \circ \bar{D}_4\). Then there exists points
\[
([f],[η],q_1(t),\ldots,q_{δ−2}(t);q_{δ−1}(t),q_δ(t),l_{q_{δ+1}(t)}) \in A^{δ}_1 \circ P A_1
\]
sufficiently close to \(([f],[η],q_1,\ldots,q_{δ−1};q_δ,l_{q_δ})\). Furthermore, every such solution satisfies the condition
\[
\nabla^2f|_{q_{δ+1}}(v,u) \neq 0,
\]
in other words,
\[
([f],[η],q_1(t),\ldots,q_{δ−1}(t),q_δ(t),l_{q_{δ+1}(t)}) \notin A^{δ}_1 \circ P A_2.
\]

**Remark 10.** We note that claim 6.8 proves eq. (121) and eq. (122) simultaneously (since \(\bar{D}^#_4 = \bar{D}_4\)).

**Proof:** Following the setup of the proofs of claims 6.1, 6.3 and 6.7, we will now work in an affine chart, where we send the plane \(\mathbb{P}^2_q\) to \(\mathbb{C}_q^2\) and the point \(q_δ(t) \in \mathbb{P}^2_q\) to \((0,0,0) \in \mathbb{C}_q^2\). We also choose coordinates, such that \(∂x \in l_{q_{δ+1}(t)}\). Using this chart, let us write down the Taylor expansion of \(f_t\) around the point \((0,0)\), namely
\[
f_t(x,y) = f_{t_{11}}xy + \frac{f_{t_{22}}}{2}y^2 + \frac{f_{t_{33}}}{6}x^3 + \frac{f_{t_{12}}}{2}xy^2 + \frac{f_{t_{03}}}{6}y^3 + \ldots
\]
Since \(([f],[η],l_{q_{δ}(t)}) \in \mathcal{P} A_1\), we conclude that \(f_{t_{20}}\) is zero. Next, since \(([f],[η],l_{q_{δ}}) \in \bar{D}_4\), we conclude that \(f_{t_{20}}, f_{t_{11}}\) and \(f_{t_{02}}\) are zero; hence \(f_{t_{11}}\) and \(f_{t_{02}}\) are small (close to zero). Hence, constructing points on the right hand side of eq. (123) amounts to finding solutions to the set of equations
\[
f_t = 0, \quad f_{t_{xz}} = 0 \quad \text{and} \quad f_{t_{yu}} = 0,
\]
where \((x,y)\) is small but not equal to \((0,0)\). Let us define
\[
g_t(x,y) = -2f_t(x,y) + xf_{t_{xz}}(x,y) + yf_{t_{yu}}(x,y).
\]
We note that \(f_t(x,y)\) and \(g_t(x,y)\) have the same cubic term in the Taylor expansion. Furthermore, \(g_t(x,y)\) does not contain any quadratic term. Since \(([f],[η],l_{q_δ}) \in \bar{D}_4\), we conclude that \(f_{t_{30}} \neq 0\). Let
\[
x := \hat{x} + E_1(\hat{x},\hat{y}) \quad \text{and} \quad y := \hat{y} + E_2(\hat{x},\hat{y})
\]
be changes of coordinates (where \(E_1\) and \(E_2\) are second order terms), such that
\[
g_t = \frac{f_{t_{30}}}{3}(\hat{x} - A_1\hat{y})(\hat{x} - A_2\hat{y})(\hat{x} - A_3\hat{y})
\]
There are three solutions to \(g_t = 0\), given by \(\hat{y} = u\) and \(\hat{x} = A_i\hat{u}\), for \(i = 1, 2\) and \(3\). Converting back in terms of \(x\) and \(y\), we conclude that the solutions to \(g_t = 0\) are given by
\[
y = u \quad \text{and} \quad x = A_1u + O(|u|^2).
\]
Let us consider the solution \(x = A_1u + O(|u|^2)\); the other two cases can be dealt with similarly. We plug this solution into the equations \(f_{t_{xz}} = 0\) and \(f_{t_{yu}} = 0\) and solve for \(f_{t_{11}}\) and \(f_{t_{02}}\) in terms of \(u\). Doing that, we get the solutions to eq. (125) are given by
\[
y = u, \quad x = A_1u + O(|u|^2),
\]
\[
f_{t_{11}} = -\frac{f_{t_{30}}}{6}(A_1 - A_2)(A_1 - A_3)u + O(|u|^2) \quad \text{and} \quad f_{t_{02}} = \frac{f_{t_{30}}}{3}A_1(A_1 - A_2)u + O(|u|^2)
\]
and two more similar solutions corresponding to \(x = A_2u + O(|u|^2)\) and \(x = A_3u + O(|u|^2)\). This proves the first assertion of claim 6.8. Furthermore, since \(f_{t_{30}} \neq 0\) and \(A_1, A_2\) and \(A_3\) are distinct, we conclude using eq. (126) that \(f_{t_{11}} \neq 0\); this proves eq. (124).

It remains to compute the multiplicity. We claim the each point of \((A^{δ−1}_1 \circ \bar{D}^#_4) \cap μ\) contributes 3 to the Euler class in eq. (97). Using eq. (126) we conclude that the multiplicity from each branch is the number of small solutions \(u\) to the equation
\[
-\frac{f_{t_{30}}}{6}(A_1 - A_2)(A_1 - A_3)u + O(|u|^2) = ε.
\]
Proof: sufficiently close to \((B_4)\) where coordinates \(\hat{x}\) of the origin. Let us make a further change of coordinates and denote

To see why this is so, we simply note that eq. (130) is exactly the same as how we justified eq. (128). Let us start with the proof of (130). Let us note that the only solutions to the set of equations eq. (130) are as defined in eq. (132). Let \(f_4(t)\) be a polynomial in two variables \(x\) and \(y\). Hence, \(\hat{f}_4(x,\hat{y}) = \hat{y} - f_4(x,\hat{y})\) where \(B_4^f\) are as defined in eq. (132). \(\varphi(0, 0) \neq 0\) and \(R(x)\) is a holomorphic function defined in a neighborhood of the origin. Let us make a further change of coordinates and denote

\[\hat{y} := \sqrt{\varphi(x, \hat{y})}\]

as in the Proof of claim 6.7. Let us denote the polynomial \(f_4\) by \(\hat{f}_4\) which is a polynomial in two variables \(x\) and \(\hat{y}\). Hence, \(\hat{f}_4(x, \hat{y}) = \hat{y}^2 + \frac{B_4^f}{2!}x^2 + \frac{B_4^f}{3!}x^3 + \frac{B_4^f}{4!}x^4 + \frac{B_4^f}{5!}x^5 + \frac{B_4^f}{6!}x^6 + R(x)x^7\), which does not exist any solutions to the set of equations

\[\hat{f}_4(u_1, v_1) = 0, \quad \hat{f}_{\hat{y}}(u_1, v_1) = 0 \quad \text{and} \quad \hat{f}_4(u_2, v_2) = 0, \quad \hat{f}_{\hat{y}}(u_2, v_2) = 0\]

where \((u_1, v_1)\) and \((u_2, v_2)\) and \((0, 0)\) are all distinct, but close to each other.

We now note that the only solutions to the set of equation eq. (130) and eq. (131) is given by \(v_1, v_2 = 0\),

\[B_4^f = \frac{1}{300}B_6^f u_1^2 u_2 + O((u_1, u_2)^5), \quad B_5^f = -\frac{1}{60}B_6^f (u_1^2 u_2 + u_1 u_2^2) + O((u_1, u_2)^4), \quad B_5^f = \frac{1}{30}B_6^f (u_1^2 + 4u_1 u_2 + u_2^2) + O((u_1, u_2)^3) \quad \text{and} \quad B_5^f = -\frac{1}{3}B_6^f (u_1 + u_2) + O((u_1, u_2)^2)\]

To see why this is so, we simply note that eq. (130) and eq. (131) are the same as eq. (45) and eq. (46); hence, the argument is exactly the same as we justified eq. (48) is the solution to eq. (130) and eq. (131).

We now note that \(v_1, v_2\) are both zero; hence \(u_1\) and \(u_2\) are both nonzero, but small. Hence, \(B_5^f\) is close to zero. This is a contradiction, since \(([f], [\eta], l_{q_{4}}) \in \mathcal{P} A_4\).

Next, let us prove (102). We will prove the following claim:

Claim 6.10. Let \(([f], [\eta], q_1, \ldots, q_{4}, l_{q_{4}}) \in A_1^{d-2} \cap \mathcal{P} D_4\). Then there exists points \(([f], [\eta], q_1(t), \ldots, q_{4-3}(t), q_{4-2}(t), q_{4-1}(t), q_4(t), l_{q_{4+1}(t)}) \in A_1^d \cap \mathcal{P} A_1\)

This number is 1 and hence, the total multiplicity is 3. Finally, we note that since \(\mu\) is a generic cycle all points of \((A_1^{d-1} \cap D_1) \cap \mu\) will actually belong to \((A_1^{d-1} \cap D_1^\#) \cap \mu\). □
sufficiently close to \((\{f\}, \{y\}, q_1, \ldots, q_{\delta - 2}; q_{\delta - 1}; 1_{l_{q_{\delta - 1}}})\). Furthermore, every such solution satisfies the condition
\[
\nabla^2 f|_{l_{q_{\delta + 1}(t)}}(v, w) \neq 0,
\]
if \(v\) is a nonzero vector that belongs to \(l_{q_{\delta + 1}(t)}\) and \(w\) is a nonzero vector that belongs to \(T(\mathbb{R}^2)_{y|_{\delta + 1}(t)}/l_{q_{\delta + 1}(t)}\). In other words,
\[
(\{f\}, \{y\}, q_1(t), \ldots, q_{\delta - 3}(t); q_{\delta - 2}(t), q_{\delta - 1}(t), q_\delta(t), l_{q_{\delta + 1}(t)}) \not\in A_1^t \cap \mathcal{P}A_2.
\]

**Proof:** Following the setup of the proof of claim 6.8, let us write down the Taylor expansion of \(f_t\) around the point \((0, 0)\), namely
\[
f_t(x, y) = f_{t_{11}}(x, y) + f_{t_{12}}(x, y) + f_{t_{21}}(x, y) + f_{t_{22}}(x, y) + f_{t_{13}}(x, y) + f_{t_{23}}(x, y) + \ldots
\]
Since \((\{f\}, \{y\}, l_{q_{\delta}(t)}) \in \mathcal{P}A_1\), we conclude that \(f_{t_{20}}\) is zero. Next, since \((\{f\}, \{y\}, l_{q_{\delta}}) \in \mathcal{P}D_4\), we conclude that \(f_{t_{11}}, f_{t_{21}}, f_{t_{30}}\) are zero; hence \(f_{t_{11}}, f_{t_{21}}, f_{t_{30}}\) are small (close to zero). Constructing points on the right-hand side of eq. (133) amounts to finding solutions to the set of equations
\[
f_t(x_1, y_1) = 0, \quad f_t(x_1, y_1) = 0, \quad f_t(x_1, y_1) = 0 \quad \text{and} \quad (135)
\]
\[
f_t(x_2, y_2) = 0, \quad f_t(x_2, y_2) = 0, \quad f_t(x_2, y_2) = 0, \quad (136)
\]
where \((0, 0), (x_1, y_1)\) and \((x_2, y_2)\) are all distinct (but close to each other). As before, we define
\[
g_t(x, y) := x f_t(x, y) + y f_t(y, x) - 2 f_t(x, y).
\]
We note that \(g_t\) has no quadratic term and has the same cubic term as \(f_t\). The cubic term of \(f\) can be written as either \(\frac{f_{t_{21}}}{6}(y - A_1(x)) (y - A_2(x)) (y - A_3(x))\) or it can be written as \(\frac{f_{t_{21}}}{2}(f_{t_{21}} + f_{t_{22}})\) (if \(f_{t_{21}} \neq 0\)). We will assume the former case; the latter case can be dealt with similarly. Hence, we can write \(g_t\) as
\[
g_t(x, y) = \frac{f_{t_{21}}}{6}(y - A_1(x)) (y - A_2(x)) (y - A_3(x)) + E(x, y),
\]
where \(E\) is a fourth order term. Let us assume that \(A_3\) is close to zero. We also note that since \(f_{t_{21}} \neq 0\), hence \(A_1\) and \(A_2\) are both nonzero. Using the equation \(g_t = 0\), let us consider the solution
\[
x = u \quad \text{and} \quad y = A_1 u + O(|u|^2).
\]
Let us now use \(f_t(x, y) = 0\) and solve for \(f_{t_{11}}\) in terms of \(u\). Doing that, we get
\[
f_{t_{11}} = \frac{f_{t_{21}}}{6}(A_2^2 - A_1 A_2 - A_1 A_3 + A_2 A_3) u + O(|u|^2).
\]
Plugging in this value of \(f_{t_{11}}\) into the equation \(f_t\), and solving for \(f_{t_{21}}\), we get that
\[
f_{t_{21}} = \frac{f_{t_{21}}}{6}(-2A_1 + 2A_2 + 2A_3 - \frac{2A_2A_3}{A_1}) u + O(|u|^2).
\]
Let us now try to produce a second node. We will justify shortly that \(x = v\) and \(y = A_2 v + O(|v|^2)\) is a not a possible solution. Hence, let us consider \(x = v\) and \(y = A_3 v + O(|v|^2)\). Plugging this into \(f_t(x, y) = 0\) and solving for \(u\) in terms of \(v\), we conclude that
\[
u = \left(\frac{A_1(A_3 - A_2)}{(A_1 - A_2)(A_1 - 2A_3)}\right) v + O(|v|^2).
\]
Plugging in this value for \(u\) into \(f_t(x, y) = 0\) and solving for \(A_3\), we conclude that
\[
A_3 = O(|v|).
\]
Plugging in the value of \(A_3\) into \(u\) and then plugging that back into \(f_{t_{11}}\) and \(f_{t_{21}}\), we conclude that
\[
u = \frac{A_2}{A_2 - A_1} v + O(|v|^2), \quad f_{t_{11}} = -\frac{f_{t_{21}}}{6} A_1 A_2 v + O(|v|^2) \quad \text{and} \quad f_{t_{21}} = \frac{f_{t_{21}}}{3} A_2 v + O(|v|^2).
\]
There are ways to construct such solutions (interchange \((A_1, A_3)\), with \((A_2, A_3)\)). Furthermore, we can permute the nodal points. From the expression for \(f_{t_{11}}\), we see that the order of vanishing is 1; hence the total multiplicity is 4.
It remains to show why we reject the solution \( x = v \) and \( y = A_2v + O(|v|^2) \). If we take that solution, then we plug it in \( f_{\nu} = 0 \), then solving for \( u \) (in terms of \( v \)), we conclude that
\[
u = \left( \frac{A_3 - A_2}{A_1 - A_2} \right) v + O(|v|^2)
\]
Plugging this into \( f_{\nu} \), we conclude that
\[
f_{\nu} = \frac{1}{3} \left( \frac{(A_1 - A_2)^2(A_3 - A_2)}{A_1} \right) v^2 + O(|v|^2).
\]
This is clearly nonzero, if \( v \) is small and nonzero. Hence, we reject the solution corresponding to \( x = v \) and 
\( y = A_2v + O(|v|^2) \). This completes the proof.

Finally, let us justify eq. (103). This follows from eq. (127). This completes the proof of Theorem 5.4. □

6.4. **Proof of Theorem 5.5: computation of \( N(A_1^3 \mathcal{P} A_3) \) when \( 0 \leq \delta \leq 1 \).** We will justify our formula for \( N(A_1^4 \mathcal{P} A_3, r, s, n_1, n_2, n_3, \theta) \), when \( 0 \leq \delta \leq 1 \). Recall that
\[
A_1^4 \circ \mathcal{P} A_2 := \{(f, \eta), q_1, \ldots, q_8, l_{q_{4+1}} \} \in \mathcal{S}_{D_4} \times_{\mathcal{P}} \mathbb{P}_W : f \text{ has a singularity of type } A_1 \text{ at } q_1, \ldots, q_8,
\]
\((f, \eta, l_{q_{4+1}}) \in \mathcal{P} A_2, q_1, \ldots, q_{8+1} \text{ all distinct}\).

Let \( \mu \) be a generic cycle, representing the class
\[
[\mu] = H_r^* \cdot H_p^* \cdot a^{n_1} \lambda^{n_2} (n_3^{\delta+1} H)^{n_3} (n_3^{\delta+1} \lambda W)^{\theta}.
\]
We now define a section of the following bundle
\[
\Psi_{\mathcal{P} A_3} : A_1^4 \circ \mathcal{P} A_2 \longrightarrow \mathbb{L}_{\mathcal{P} A_3} := \gamma_D^* \otimes \gamma_W^* \otimes \gamma_{\mathcal{P} A_3}^*, \quad \text{given by}
\]
\[
\{ \Psi_{\mathcal{P} A_3}(f, \eta, q_1, \ldots, q_8, l_{q_{4+1}}) \}(f \otimes \gamma_W^* \otimes \gamma_{\mathcal{P} A_3}^*) := \nabla^3 f_{q_{4+1}}(v, v, v).
\]
Analogous to [2, Lemma 6.1], we conclude that for \( d \geq 4 \),
\[
\mathcal{P} A_2 = \mathcal{P} A_2 \cup \mathcal{P} A_3 \cup \underline{D}_4.
\]
Furthermore, analogous to [2, Lemma 6.3] we conclude that for \( d \geq 4 \),
\[
\underline{A}_1^4 \circ \mathcal{P} A_2 = (A_1^4 \circ \mathcal{P} A_2) \cup A_1^4 \circ (\mathcal{P} A_2 - \mathcal{P} A_3) \cup \underline{A}_1^4 - 1 \circ (\Delta \mathcal{P} A_4 \cup \Delta \underline{D}_5).
\]
Let us define
\[
\mathcal{B} := \underline{A}_1^4 \circ \underline{P} A_2 - A_1^4 \circ (\mathcal{P} A_2 \cup \underline{P} A_3).
\]
We will show shortly that the section \( \Psi_{\mathcal{P} A_3} \) vanishes on the points of \( A_1^4 \circ \mathcal{P} A_3 \) transversally. Hence,
\[
(\mathcal{L}_{\mathcal{P} A_3}, [A_1^4 \circ \mathcal{P} A_2] \cap [\mu]) = N(A_1^4 \mathcal{P} A_3, n_1, n_2, n_3, \theta) + C_{\mathcal{B} \gamma_\eta}.
\]
We now give an explicit description of \( \mathcal{B} \). Let us first define
\[
\mathcal{B}_0 := \{ (f, \eta, q_1, \ldots, q_8, l_{q_{4+1}}) \in \mathcal{B} : q_1, q_2 \ldots q_{8+1} \text{ all distinct} \}.
\]
In other words, \( \mathcal{B}_0 \) is that component of the boundary, where all the points are still distinct. By eq. (137), we conclude that
\[
\mathcal{B}_0 = \underline{A}_1^4 \circ \underline{D}_4.
\]
If we intersect \( \mathcal{B}_0 \) with \( \mu \) then we will get a finite set of points. Since the representative \( \mu \) is generic, we conclude that the third derivative along \( v \) will not vanish, i.e. the section \( \Psi_{\mathcal{P} A_3} \) will not vanish on those points. Hence, \( \mathcal{B}_0 \cap \mu \) does not contribute to the Euler class.

Next, let us consider the components of \( \mathcal{B} \) where one (or more) of the \( q_i \) become equal to the last point \( q_{8+1} \). Define \( \mathcal{B}(q_1, \ldots, q_8, l_{q_{4+1}}) \) as before. Analogous to the proof of [2, Lemma 6.3], we conclude that
\[
\mathcal{B}(q_1, l_{q_{4+1}}) \approx A_1^{4-1} \circ \mathcal{P} A_4 \cup A_1^{4-1} \circ \underline{D}_5.
\]
Furthermore, analogous to the proof of [2, Corollary 6.13, Page 700], we conclude that the contribution to the Euler class from each of the points of \( A_1^{4-1} \circ \mathcal{P} A_4 \cap \mu \) is 2. Finally, we note that the section \( \Psi_{\mathcal{P} A_3} \) does not
vanish on $A_{1}^{d-1} \circ D_{5} \cap \mu$, since $\mu$ is generic. Hence, the total contribution from all the components of type $B(q_{i}, l_{q_{i}+1})$ equals

$$2 \left( \frac{d}{3} \right) N(A_{1}^{d-1} \mathcal{P} A_{4}, n_{1}, n_{2}, n_{3}, \theta).$$

Plugging in this in eq. (139) gives us the formula of theorem 5.5.

It just remains to prove the transversality claim. This follows from following the setup of proof of transversality in Theorem theorem 5.4. We consider the polynomial

$$\rho_{30} := (X - X_{1})^{2}(X - X_{2})^{2} \cdots (X - X_{d})^{2} X^{d-2d-3}$$

and the corresponding curve $\gamma_{30}(t)$. Transversality follows by computing the derivative of the section $\Psi_{\mathcal{P} A_{3}}$ along the curve $\gamma_{30}(t)$ as before. \hfill $\Box$

6.5. Proof of Theorem 5.6: computation of $N(\mathcal{P} A_{4})$. We will now justify our formula for $N(\mathcal{P} A_{4}, r, s, n_{1}, n_{2}, n_{3}, \theta)$. Let $\mu$ be a generic cycle, representing the class

$$[\mu] = \mathcal{H}_{\nu}^{*} \cdot \mathcal{H}_{p}^{*} \cdot a^{n_{1}} \lambda^{n_{2}} (\pi^{*} H)^{n_{3}} (\pi^{*} \lambda_{W})^{0}.$$

Let $v \in \gamma_{W}$ and $w \in \pi^{*} W / \gamma_{W}$ be two fixed nonzero vectors. Let us introduce the following abbreviation:

$$f_{ij} := \nabla^{i+j} f |_{q(v, \cdots, v, w, \cdots, w)}.$$ 

We now define a section of the following bundle

$$\Psi_{\mathcal{P} A_{4}} : \overline{\mathcal{P} A_{3}} \rightarrow L_{\mathcal{P} A_{4}} := \gamma_{\mathcal{P} A_{3}}^{2} \otimes \gamma_{W}^{4} \otimes \mathcal{P} / \gamma_{W}^{*} \otimes \gamma_{W}^{2d},$$

$$\{\Psi_{\mathcal{P} A_{4}}([f], l_{q}) | f^{(\otimes 2)} \otimes v^{(\otimes 4)} \otimes w^{(\otimes 2)} := f_{02} A_{4}^{f}, \quad \text{where} \quad A_{4}^{f} := f_{40} - \frac{3f_{21}^{2}}{f_{02}}. \quad (141)$$

Analogous to [2, Lemma 6.1], we conclude that

$$\overline{\mathcal{P} A_{3}} = \mathcal{P} A_{3} \cup \overline{\mathcal{P} A_{4}} \cup \overline{\mathcal{P} D_{4}}. \quad (142)$$

Hence, let us define

$$B := \overline{\mathcal{P} A_{3}} - \mathcal{P} A_{3} \cup \mathcal{P} A_{4}.$$

We will show shortly that the section $\Psi_{\mathcal{P} A_{4}}$ vanishes on the points of $\mathcal{P} A_{4}$ transversally. Hence,

$$\langle e(L_{\mathcal{P} A_{4}}), \overline{\mathcal{P} A_{3}} | [\mu] \rangle = N(A_{4}^{f} \mathcal{P} A_{4}, r, s, n_{1}, n_{2}, n_{3}, \theta) + C_{B \cap \mu}. \quad (143)$$

Let us now study the boundary $B$. By eq. (142), we conclude that

$$B \cap \mu = \overline{\mathcal{P} D_{4}} \cap \mu.$$ 

Since the representative $\mu$ is generic, we conclude that the directional derivative $f_{21}$ will not vanish on those points. Since $f_{02} = 0$ on $B$, we conclude that

$$f_{02} A_{4}^{f} = f_{02} f_{40} - \frac{3f_{21}^{2}}{f_{02}} \neq 0$$

if $f_{21} \neq 0$. Hence, the section $\Psi_{\mathcal{P} A_{4}}$ will not vanish on $B \cap \mu$. Hence, the total boundary contribution is zero and eq. (143) gives us the formula of theorem 5.6.

It remains to prove the claim regarding transversality. This follows from following the setup of proof of transversality in Theorem 5.5. We consider the polynomial

$$\rho_{40} := X^{d} Z^{d-4}$$

and the corresponding curve $\gamma_{40}(t)$. Transversality follows by computing the derivative of the section $\Psi_{\mathcal{P} A_{4}}$ along the curve $\gamma_{40}(t)$ as before. \hfill $\Box$
6.6. **Proof of Theorem 5.7:** computation of $N(\mathcal{P}D_4)$. We will now justify our formula for $N(\mathcal{P}D_4, r, s, n_1, n_2, n_3, \theta)$.

Let $\mu$ be a generic cycle, representing the class $[\mu] = \mathcal{H}_L^* \cdot \mathcal{H}_p^* \cdot a^{n_1} \lambda^2 ((\pi^*H)^{n_2}((\pi^*\lambda W)^{n_3})$.

As before, let $v \in \gamma_W$ and $w \in \pi^*W/\gamma_W$ be two fixed nonzero vectors. Define a section of the following bundle

\[ \Psi_{PD_4} : \mathcal{P}A_3 \rightarrow \mathcal{L}_{PD_4} := \gamma_D^0 \otimes (W/\gamma_W)^{r_2} \otimes \gamma_D^{r_2} \]

given by

\[ \{ \Psi_{PD_4}([f], [l]) (f \otimes w^{r_2}) := \nabla^2 f_{l,q}(w, w) \}. \] (144)

We recall eq. (142), namely

\[ \overline{\mathcal{P}A_3} = \mathcal{P}A_3 \cup \overline{\mathcal{P}A_4} \cup \overline{\mathcal{P}D_4}. \] (145)

We now define

\[ \mathcal{B} := \overline{\mathcal{P}A_3} - (\mathcal{P}A_3 \cup \overline{\mathcal{P}D_4}). \]

We will show that the section $\Psi_{PD_4}$ vanishes on the points of $\mathcal{P}D_4$ transversally. Hence,

\[ (e(\mathcal{L}_{PD_4}), [\overline{\mathcal{P}A_3}] \cap [\mu]) = N(\mathcal{P}D_4, r, s, n_1, n_2, n_3, \theta) + \mathcal{C}_{\mathcal{B} \cap \mu}. \] (146)

By definitions, the section $\Psi_{PD_4}$ does not vanish on $\mathcal{P}A_4 \cap \mu$. Hence, the total boundary contribution is zero and eq. (143) gives us the formula of theorem 5.7.

It remains to prove the claim regarding transversality. This follows from following the setup of proof of transversality in Theorem 5.6. We consider the polynomial

\[ \rho_{02} := Y^2 Z^{d-2} \]

and the corresponding curve $\gamma_{02}(t)$. Transversality follows by computing the derivative of the section $\Psi_{PD_4}$ along the curve $\gamma_{02}(t)$ as before. \( \square \)

7. **Verification with other results and low degree checks**

Let us make a few low degree checks. We will abbreviate $N(A_1^{d+1}, r, s, 0, 0)$ as $N(A_1^{d+1}, r, s)$.

7.1. **Verification with S. Kleiman and R. Piene’s result**. Let us start by verifying the numbers predicted by the algorithm of S. Kleiman and R. Piene in [15]. Let us explain how to obtain the formula for $N(A_1^{d+1}, r, s)$ using [15, Algorithm 2.3, Page 5]. Let us first define four polynomials (called Bell polynomials), given by

\[ P_1(a_1) := a_1, \quad P_2(a_1, a_2) := a_1^2 + a_2, \quad P_3(a_1, a_2, a_3) := a_1^3 + 3a_1a_2 + a_3 \quad \text{and} \quad P_4(a_1, a_2, a_3, a_4) := a_1^4 + 6a_1^2a_2 + 3a_3^2 + 4a_1a_3 + a_4. \]

We define the following cycles in $\mathcal{S}_{D_1}$, namely

\[ v := \lambda + dH, \quad w_1 := a - 3H \quad \text{and} \quad w_2 := a^2 - 2aH + 3aH^2. \] (147)

Note that $v = c_1(\mathcal{L}_{A_0})$ and $w_i = c_1(T^*W)$, where $\mathcal{L}_{A_0}$ and $W$ are the bundles defined in section 6.1. The algorithm [15, Algorithm 2.3, Page 5] produces polynomials $b_i(v, w_1, w_2)$ of degree $i + 2$ (from $i = 1$ to 8). Let us write down the expressions explicitly,

\[ b_1(v, w_1, w_2) = v^3 + v^2 w_1 + vw_2, \quad b_2(v, w_1, w_2) = -7v^4 - 13v^3 w_1 - 6v^2 w_2 - 7v^2 w_2 - 6vw_1 w_2, \]
\[ b_3(v, w_1, w_2) = 138v^5 + 394v^4 w_1 + 376v^3 w_1^2 + 138v^3 w_2 + 120v^2 w_1^3 + 256v^2 w_1 w_2 + 120v^2 w_2^2 \quad \text{and} \quad b_4(v, w_1, w_2) = -4824v^6 - 19134v^5 w_1 - 28842v^4 w_1^2 - 3888v^4 w_2 - 19572v^3 w_1 \]
\[ -12438v^3 w_1 w_2 - 5040v^2 w_1^4 - 13596v^2 w_1^2 w_2 + 936v^2 w_2^2 \]
\[ -5040vw_1 w_2^2 + 936vw_1 w_2^2. \] (148)

The numbers $N(A_1^{d+1}, r, s)$ will be computed from the polynomials $P_5 + 1$ by intersecting cycles in $\mathcal{S}_{D_{d+1}}$. Let $\pi_i : \mathcal{S}_{D_{d+1}} \rightarrow \mathcal{S}_D$, denote the $i^{th}$ projection map. Then

\[ N(A_1, r, s) = [b_1] \cdot \mathcal{H}_L^* \cdot \mathcal{H}_p^* , \]
where the right hand side is an intersection number on $S_{D_2}$. Note that we plug in the values for $v, w_1$ and $w_2$ from eq. (147) in eq. (148), use eq. (15) for $H_L$ and $H_p$, and the ring structure as given by eq. (7) to compute the intersection number. Next, let us explain how to compute $N(A^1_2, r, s)$. This is given by

$$N(A^1_2, r, s) = (\pi^*_1 b_1) \cdot (\pi^*_2 b_1) \cdot H^r_L \cdot H^s_p + b_2 \cdot H^r_L \cdot H^s_p.$$  \hfill (149)

The first number on the right hand side of eq. (149) is an intersection number on $S_{D_2}$, while the second one is an intersection number on $S_{D_1}$. Similarly,

$$N(A^3_1, r, s) = \left( (\pi^*_1 b_1) \cdot (\pi^*_2 b_1) \cdot (\pi^*_3 b_1) + 3(\pi^*_1 b_1) \cdot (\pi^*_2 b_1) + b_3 \right) \cdot H^r_L \cdot H^s_p$$

and

$$N(A^4_1, r, s) = \left( (\pi^*_1 b_1) \cdot (\pi^*_2 b_1) \cdot (\pi^*_3 b_1) \cdot (\pi^*_4 b_1) + 6(\pi^*_1 b_1) \cdot (\pi^*_2 b_1) \cdot (\pi^*_3 b_1) + 3(\pi^*_1 b_2) \cdot (\pi^*_2 b_2) + 4(\pi^*_1 b_1) \cdot (\pi^*_2 b_3) + b_4 \right) \cdot H^r_L \cdot H^s_p.$$

We have written a mathematica program to implement this formula and verified that the answers agree with our formula.

7.2. Verification with T. Laraakker’s result. Next we note that in [18, Appendix A, Page 32], T. Laraakker has explicitly written down the formulas for $N(A^{d+1}_1, 0, 0)$. We have verified that our formulas agree with his.

7.3. Verification with the second author and R. Singh’s result. We now verify some of the numbers obtained by R. Mukherjee and R. Singh in [20]. In [20], the authors compute $C^\text{planar, } \mathbb{P}^3_d(r, s)$, the number of planar genus zero degree $d$ curves in $\mathbb{P}^3$ intersecting $r$ lines and passing through $s$ points having a cusp (where $r + 2s = 3d + 1$). Let us compare this with $N_d(A^1_2, A_2, r, s)$, the number of planar degree $d$ curves in $\mathbb{P}^3$, passing through $r$ lines and passing through $s$ points, that have $\delta$ (ordered) nodes and one cusp (where $r + 2s = \frac{d(d + 3)}{2} + 1 - \delta$). For $d = 3$, and $\delta = 0$, this number should be the same as the characteristic number of genus zero planar cubics in $\mathbb{P}^3$ with a cusp, i.e. $C_d(r, s)$. We have verified that is indeed the case. We tabulate the numbers for the readers convenience:

$$C_3(10, 0) = 17760, \quad C_3(8, 1) = 2064, \quad C_3(6, 2) = 240 \quad \text{and} \quad C_3(4, 3) = 24.$$

These numbers are the same as $N_d(A^1_2, A_2, r, s)$ for $d = 3$ and $\delta = 0$.

Next, we note that when $d = 4$ and $\delta = 2$, the number $\frac{1}{3!} N_d(A^1_2, A_2, r, s)$ is same as the characteristic number of genus zero planar quartics in $\mathbb{P}^3$ with a cusp, i.e. $C_d(r, s)$. We have verified that fact. The numbers are

$$C_4(13, 0) = 10613184, \quad C_4(11, 1) = 760368, \quad C_4(9, 2) = 49152 \quad \text{and} \quad C_4(7, 3) = 2304.$$

These numbers are the same as $\frac{1}{3!} N_d(A^1_2, A_2, r, s)$ for $d = 4$ and $\delta = 2$. We have to divide out by a factor of $\delta!$ because in the definition of $N_d(A^1_2, A_2, r, s)$, the nodes are ordered.

7.4. Enumerativity of BPS numbers computed by R. Pandharipande. We will now verify some of the numbers predicted by the conjecture made by Pandharipande in [21], regarding the enumerativity of the BPS numbers for $\mathbb{P}^3$. Let $N_d^g(r, s)$ denote the genus $g$ Gromov-Witten invariant of $\mathbb{P}^3$ (corresponding to the insertion of $r$ lines and $s$ points) and let $E^d_d(r, s)$ denote the corresponding BPS invariant as given by [21, Equations 5 and 9, Pages 493 and 494]. The numbers $E^d_d(r, s)$ are conjectured to be integers. Even if the conjecture is true, it is not always clear if the the BPS numbers have an enumerative significance. We will now give some evidence for the enumerativity of some of the BPS number.

Let us consider the case $g = 2$ and $d = 4$. It is far from clear that $E^d_d(r, s)$ is enumerative when $d = 4$, because the moduli space of curves has more than the expected dimension (see the remark in [21] just after Theorem 3, Page 494). We claim that $E^d_d(r, s)$ is enumerative when $d = 4$. To see how, we first note that every degree 4, genus 2 curve lies inside some $\mathbb{P}^2$ (this follows from the Castelnuovo bound, [9, Page 527]). Since the genus of a smooth degree 4 curve is 3, we conclude that the corresponding enumerative invariant is equal to the characteristic number of planar degree 4 curves in $\mathbb{P}^3$ with one node. We have verified that $E^d_d(r, s)$ is indeed equal to $N_d(A_1, r, s)$ for all $r$ and $s$ when $d = 4$. We tabulate the numbers for the readers convenience

$$N_4(A_1, 16, 0) = 255300, \quad N_4(A_1, 14, 1) = 15498, \quad N_4(A_1, 12, 2) = 792 \quad \text{and} \quad N_4(A_1, 10, 3) = 27.$$  \hfill (150)
The degree four, genus two BPS numbers are directly tabulated in [8, Page 43] and are seen to be equal to the above numbers listed in eq. (150).

8. Explicit Formulas

For the convenience of the reader, we write down some explicit formulas.

\[
N(r, s, 0, 0) = \begin{cases}
\frac{1}{12}d(d^2 - 1)(d^2 + 2d + 2) \left(2d^2 + 4d + 6\right) \left(2d^3 + 6d^2 + 13d + 3\right) & \text{if } s = 0, \\
\frac{1}{3}d(d^2 - 1)(d^2 + 2d + 2) \left(2d^2 + 8d + 3\right) & \text{if } s = 1, \\
\frac{1}{12}d(d - 1)(d + 4) & \text{if } s = 2, \\
1 & \text{if } s = 3.
\end{cases}
\]

\[
N(A_1, r, s, 0, 0) = \begin{cases}
\frac{1}{108}d(d^2 - 1)^2(d + 2)(d + 3) \left(2d^4 + 4d^3 + d^2 - 10d - 6\right) & \text{if } s = 0, \\
\frac{1}{12}d(d - 1)^2(d + 3) \left(2d^3 + 6d^2 - 9d^2 - 3d - 2\right) & \text{if } s = 1, \\
d(d - 1)^2 \left(d^2 + 3d - 6\right) & \text{if } s = 2, \\
3(d - 1)^2 & \text{if } s = 3.
\end{cases}
\]

\[
N(A_2, r, s, 0, 0) = \begin{cases}
\frac{1}{172}d(d^2 - 1)(d^2 + 2s) \left(2d^6 + 12d^5 + 11d^4 - 30d^3 - 49d^2 - 18\right) & \text{if } s = 0, \\
\frac{1}{12}d(d - 1)(d - 2) \left(2d^5 + 12d^4 + d^3 - 54d^2 + 9d + 6\right) & \text{if } s = 1, \\
4d(d - 1)(d - 2) \left(d^2 + 3d - 8\right) & \text{if } s = 2, \\
12(d - 1)(d - 2) & \text{if } s = 3.
\end{cases}
\]

\[
N(A_3, r, s, 0, 0) = \begin{cases}
\frac{1}{105}d(d - 1)(d - 2) \left(50d^8 + 408d^7 + 539d^6 - 2556d^5 - 6625d^4\right. & \\
\left.+762d^3 + 10050d^2 - 11232d + 8208\right) & \text{if } s = 0, \\
\frac{1}{12}d(d - 2)(d - 1) \left(50d^8 + 258d^7 - 485d^6 - 2241d^5\right. & \\
\left.+2172d^2 + 1512d - 658\right) & \text{if } s = 1, \\
\frac{2}{7}d(d - 2)(d + 5) \left(25d^2 - 96d + 84\right) & \text{if } s = 2, \\
2(25d^2 - 96d + 84) & \text{if } s = 3.
\end{cases}
\]

\[
N(A_4, r, s, 0, 0) = \begin{cases}
\frac{5}{27}(d - 1)(d - 3) \left(6d^6 + 50d^5 + 41d^4 - 445d^6 - 715d^5\right. & \\
\left.+1529d^4 + 2720d^3 - 7902d^2 + 7164d - 2160\right) & \text{if } s = 0, \\
\frac{7}{3}(d - 3)(6d^7 + 26d^6 - 105d^5 - 231d^4 & \\
\left.+765d^3 - 107d^2 - 762d + 360\right) & \text{if } s = 1, \\
20d(d - 3)(3d - 5) \left(d^2 + 3d - 12\right) & \text{if } s = 2, \\
60(d - 3)(3d - 5) & \text{if } s = 3.
\end{cases}
\]

\[
N(D_4, r, s, 0, 0) = \begin{cases}
\frac{5}{36}d(d - 1)(d - 2)^2(d + 4) \left(2d^7 + 12d^6 - d^5 - 66d^4 + 91d^3\right. & \\
\left.+234d^2 - 270d + 108\right) & \text{if } s = 0, \\
\frac{5}{7}(d - 2)^2 \left(2d^6 + 12d^5 - 15d^4 - 102d^3 + 85d^2 + 90d - 48\right) & \text{if } s = 1, \\
15d(d - 2)^2 \left(d^2 + 3d - 12\right) & \text{if } s = 2, \\
45d^2 & \text{if } s = 3.
\end{cases}
\]

\[
N(A_1^2, r, s, 0, 0) = \begin{cases}
\frac{1}{108}d(d^2 - 1)(d^2 - 4) \left(6d^8 + 30d^7 - 25d^6 - 255d^5 - 142d^4\right. & \\
\left.+333d^3 + 629d^2 + 18d + 198\right) & \text{if } s = 0, \\
\frac{1}{12}d(d - 1)(d - 2) \left(6d^3 + 30d^2 - 55d^3 - 297d^4 + 190d^3\right. & \\
\left.+537d^2 - 69d - 78\right) & \text{if } s = 1, \\
d(d - 1)(d - 2) \left(d^2 + 3d - 8\right) \left(3d^2 - 3d - 11\right) & \text{if } s = 2, \\
3(d - 1)(d - 2) \left(3d^2 - 3d - 11\right) & \text{if } s = 3.
\end{cases}
\]
\[ N(A_1 A_2, r, s, 0, 0) = \begin{cases} \frac{1}{2}d(d-1)(d-2)(d-3)(6d^3 + 60d^6 + 155d^7 - 186d^8 \\ -1288d^5 - 1422d^4 + 641d^3 + 1512d^2 - 2034d + 1836) & \text{if } s = 0, \\
\frac{1}{2}(d^2 - 1)(d-2)(d-3)(6d^6 + 36d^5 - 37d^4 - 338d^3 \\ + 123d^2 + 438d - 144) & \text{if } s = 1, \\
4d(d-2)(d-3)(d+5)(3d^3 - 6d^2 - 11d + 18) & \text{if } s = 2, \\
12(d-3)(3d^3 - 6d^2 - 11d + 18) & \text{if } s = 3. 
\end{cases} \]

\[ N(A_1 A_3, r, s, 0, 0) = \begin{cases} \frac{1}{15}d(d-1)(d-3)(50d^11 + 358d^{10} - 489d^9 - 6967d^8 \\ -3139d^7 + 40955d^6 + 40482d^5 - 112250d^4 - 131080d^3 \\ + 436176d^2 - 402480d + 120960) & \text{if } s = 0, \\
\frac{1}{15}(d-3)(50d^6 + 158d^5 - 1471d^7 - 2389d^6 + 14857d^5 \\ + 2359d^4 - 41156d^3 + 7912d^2 + 41808d - 19440) & \text{if } s = 1, \\
2d(d-3)(d^2 + 3d - 12)(25d^3 - 71d^2 - 122d + 280) & \text{if } s = 2, \\
6(d-3)(25d^3 - 71d^2 - 122d + 280) & \text{if } s = 3. 
\end{cases} \]

\[ N(A_1^3, r, s, 0, 0) = \begin{cases} \frac{1}{120}d(d-1)(d-2)(18d^{12} + 108d^{11} - 315d^{10} - 2664d^9 \\ + 470d^8 + 21919d^7 + 19103d^6 - 58136d^5 - 106948d^4 \\ + 7039d^3 + 129360d^2 - 165798d + 110700) & \text{if } s = 0, \\
\frac{1}{120}(d-1)(d-2)(18d^{10} + 54d^9 - 567d^8 - 1179d^7 + 6383d^6 \\ + 7774d^5 - 25775d^4 - 20197d^3 + 26955d^2 + 20802d - 8640) & \text{if } s = 1, \\
d(d-2)(d+5)(9d^6 - 54d^5 + 9d^4 + 423d^3 \\ - 458d^2 - 829d + 1050) & \text{if } s = 2, \\
3(9d^6 - 54d^5 + 9d^4 + 423d^3 - 458d^2 - 829d + 1050) & \text{if } s = 3. 
\end{cases} \]

\[ N(A_2 A_3, r, s, 0, 0) = \begin{cases} \frac{1}{3}(d-1)(d-3)(6d^{13} + 36d^{12} - 159d^{11} - 1124d^{10} + 1209d^9 \\ + 12169d^8 + 664d^7 - 52991d^6 - 39896d^5 + 127254d^4 \\ + 129112d^3 - 452904d^2 + 413280d - 120960) & \text{if } s = 0, \\
(d-3)(6d^{11} + 12d^{10} - 249d^9 - 236d^8 + 3653d^7 + 367d^6 \\ - 20186d^5 + 6389d^4 + 38600d^3 - 7828d^2 - 42896d + 19680) & \text{if } s = 1, \\
12d(d-3)(d^2 + 3d - 12)(3d^5 - 12d^4 - 30d^3 \\ + 125d^2 + 82d - 280) & \text{if } s = 2, \\
36(d-3)(3d^5 - 12d^4 - 30d^3 + 125d^2 + 82d - 280) & \text{if } s = 3. 
\end{cases} \]
The ideas of this paper originated while the second author had discussions with Martijn Kool and Ties Laarakker regarding the papers [3] and [17]. During the discussion we wondered if one can count planar curves in $\mathbb{P}^3$ with singularities. As shown by Ties Laarakker in [18], one can adapt the techniques in [17] to count $\delta$-nodal planar curves in $\mathbb{P}^3$. On the other hand, the discussions also led us to conclude that by adapting the methods of [1], [2] and [3], we can enumerate planar curves in $\mathbb{P}^3$ with singularities that are more degenerate than nodes. The result is this present paper. The second author is therefore very grateful to Martijn Kool and Ties Laarakker for the discussions and fruitful exchange of ideas that resulted in this paper. The second author would like to acknowledge the External Grant he has obtained, namely MATRICS (File number: MTR/2017/000439) that has been sanctioned by the Science and Research Board (SERB). Both the authors are grateful to Anantadulal Paul and Rahul Singh for several fruitful discussions.

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\[
N(A^4_{1, r, s, 0, 0}) = \begin{cases} 
\frac{1}{36}(d-1)(d-3)(18d^{15} + 90d^{14} - 747d^{13} - 3843d^{12} + 11660d^{11} \\
+ 6314d^{10} - 75352d^9 + 486678d^8 + 73143d^7 + 1773729d^6 + 1150606d^5 \\
- 4123550d^4 - 3282032d^3 + 12893256d^2 - 11795040d + 3404160) & \text{if } s = 0, \\
\frac{1}{4}(d-3)(18d^{13} + 18d^{12} - 945d^{11} - 261d^{10} + 18590d^9 - 4254d^8 \\
- 164328d^7 + 80206d^6 + 653953d^5 - 362481d^4 - 1051128d^3 \\
+ 245636d^2 + 1215312d - 554880) & \text{if } s = 1, \\
3d(d-3)(d^2 + 3d - 12)(9d^7 - 45d^6 - 135d^5 + 801d^4 \\
+ 691d^3 - 4671d^2 - 1386d + 7880) & \text{if } s = 2, \\
9(d-3)(9d^7 - 45d^6 - 135d^5 + 801d^4 + 691d^3 \\
- 4671d^2 - 1386d + 7880) & \text{if } s = 3.
\end{cases}
\]
COUNTING PLANAR CURVES IN $\mathbb{P}^3$ WITH DEGENERATE SINGULARITIES

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