Piecewise linear processes with Poisson-modulated exponential switching times

Antonio Di Crescenzo\textsuperscript{1} | Barbara Martinucci\textsuperscript{1} | Nikita Ratanov\textsuperscript{2} \textsuperscript{1}

\textsuperscript{1}Dipartimento di Matematica, Università di Salerno, Fisciano, Italia
\textsuperscript{2}Facultad de Economía, Universidad del Rosario, Bogotá, Colombia

Correspondence
Nikita Ratanov, Facultad de Economía, Universidad del Rosario, Calle 12c, No. 4-69, Bogotá, Colombia.
Email: nikita.ratanov@urosario.edu.co

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We consider the jump telegraph process when switching intensities depend on external shocks also accompanying with jumps. The incomplete financial market model based on this process is studied. The Esscher transform, which changes only unobservable parameters, is considered in detail. The financial market model based on this transform can price switching risks as well as jump risks of the model.

KEYWORDS
martingale, piecewise linear process, Poisson-modulated exponential distribution, renewal process, risk neutral measure, telegraph process

1 INTRODUCTION AND PROBLEM SETTING

The piecewise linear processes have a long history and still receiving attention in various aspects. This family of processes includes so-called telegraph, which presumes alternating velocities with exponentially distributed time intervals between switchings. The number of switchings in this model is counted by homogeneous Poisson process. This theory has been developing since the seminal paper by Taylor\textsuperscript{1} for almost a century. In the 50s, this model has been studied by Goldstein\textsuperscript{2} and Kac.\textsuperscript{3} See the history and the detailed description in the monograph by Kolesnik and Ratanov.\textsuperscript{4}

The model, which is based on the distribution of the inter-switching times different from exponential, is much less studied. Some examples could be found in previous studies.\textsuperscript{5-9} Another generalisation can be constructed as a piecewise linear process with arbitrary consecutive trends $c_n$ and switching intensities $\lambda_n, n \geq 0, \lambda_n > 0$, see Ratanov\textsuperscript{10,11}

In this paper, we study the piecewise linear processes with exponentially distributed time intervals between successive tendency switchings, but here we assume that the parameter of this distribution depends on exogenous shocks (exogenous impacts and external interventions), arriving at a constant rate. This approach reflects the well posed problem of financial market modelling, when multiple agents are trying to break the trend by interventions, but could affect only the switching rate.

Let $\lambda_n > 0, n \geq 0$, be successive intensities of shocks. That is, on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the sequence of time intervals $\tau_n, \tau_n > 0$, between the consecutive shocks, which are independent and exponentially distributed, $\tau_n \sim \text{Exp}(\lambda_n), n \geq 0$.

We assume shocks do not come explosively, i.e., the process $\tau^{(+n)} = \tau_0 + \ldots + \tau_n$ is a simple point process,

$$\mathbb{P}\left\{ \lim_{n \to \infty} \tau^{(+n)} = \infty \right\} = 1,$$
which is equivalent to
\[ \sum \lambda_n^{-1} = \infty. \] (1.1)

see, eg, Jacobsen.\(^{12}\) This means that there is a finite accumulation of shocks at any finite time interval: \( \mathbb{P}\{N(t) < \infty\} = 1, \) \( \forall t > 0, \) where \( N = N(t), t \geq 0, \) is the corresponding counting process, \( N(t) = \sum_{n=1}^{\infty} 1_{\{\tau_n \leq t\}}, \) and \( 1_A \) is the indicator.

If (1.1) fails, then this process exhibits explosive behaviour, \( \mathbb{P}\{N(t) < \infty\} < 1. \) For example, if \( \lambda_n = (n + 1)^2, \) then \( \mathbb{P}\{N(t) < \infty\} = -2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2 t} < 1, \) see Snyder and Miller.\(^{13}\) Example 6.3.1.

If all \( \lambda_n \)s are equal, \( \lambda_n \equiv \lambda, n \geq 0, \) then \( N = N(t) \) is the homogeneous Poisson process.

Let \( \mu_n, \mu_n > 0, n \geq 0. \) Consider the random variable \( T, T > 0, \) which has the exponential distribution with \( N(t) \)-modulated parameter, \( \mu = \mu_N(t). \) The survivor function of \( T \) is given by
\[ \bar{F}_T(t) := \mathbb{P}\{T > t\} = \mathbb{E}\{e^{-\xi(t)}\}, \] (1.2)
where
\[ \xi(t) := \int_0^t \mu_N(s) ds = \sum_{k=0}^{N(t)-1} \mu_k \tau_k + \mu_N(t) \left( t - \sum_{k=0}^{N(t)-1} \tau_k \right) = \sum_{k=0}^{N(t)-1} (\mu_k - \mu_N(t)) \tau_k + \mu_N(t) t, \quad t \geq 0, \] (1.3)
is the accumulated intensity.

We call such a distribution a Poisson-modulated exponential distribution, PoExp(\( \bar{\lambda}, \bar{\mu} \)). Here, \( \bar{\lambda} = \{\lambda_n\}_{n \geq 0} \) is the sequence of parameters of the underlying process \( N \) counting arrivals of shocks, and \( \bar{\mu} = \{\mu_n\}_{n \geq 1} \) are the sequential parameters of the main exponential distribution.

In this paper, we study the piecewise linear process, which follows two patterns alternating at random instants with Poisson-modulated exponential distribution of inter-switching times.

To begin with, consider the sequence of independent Poisson processes \( N = N_m(t), m \geq 0, \) based on the two alternating sets of parameters \( \lambda^{(0)} = \{\lambda_n^{(0)}\}_{n \geq 0}, \lambda^{(1)} = \{\lambda_n^{(1)}\}_{n \geq 0} \). \( \lambda^{(0)}_n, \lambda^{(1)}_n > 0, n \geq 0, \)
\[ \mathbb{P}\{N_m(t) = n | N_m(s) = n\} = \exp \left( -\lambda^{(m)}_n (t - s) \right), \quad s < t, \quad n \geq 0. \] (1.4)

Here, \( \epsilon_m \in \{0,1\} \) is a sequence of alternating 0 and 1, which indicates the current pattern. Let these processes be nonexplosive, satisfying (1.1),

\[ \sum_{n \geq 0} 1/\lambda^{(i)}_n = \infty, \quad i \in \{0,1\}. \]

Let \( \bar{\mu}^{(0)} = \{\mu_n^{(0)}\}_{n \geq 0} \) and \( \bar{\mu}^{(1)} = \{\mu_n^{(1)}\}_{n \geq 0} \) be two sequences of positive numbers. Consider the sequence of independent random variables \( \{T_m\}_{m \geq 1}, T_m \geq 0, \) with alternating Poisson-modulated exponential distributions, \( T_m \sim \text{PoExp}(\bar{\lambda}^{(m)}, \bar{\mu}^{(m)}), m \geq 1, \)
\[ \mathbb{P}\{T_m > t\} = \mathbb{E}\exp(-\xi_m(t)), \quad \xi_m(t) = \int_0^t \mu^{(m)}_{\lambda, \mu}(s) ds. \]

Consider the point process formed by the times, \( T^{(+,m)} = T_1 + \ldots + T_m, m \geq 1, \) when the patterns are switched, \( T^{(+,0)} = 0. \) Since \( N_m \) are nonexplosive, \( \mathbb{P}\{\forall t > 0 N_m(t) < \infty\} = 1, m \geq 1, \) the process \( T^{(+,m)} \) is nonexplosive too, if
\[ \sum_{m \geq 1} 1/\mu^{(i)}_m = \infty, \quad i \in \{0,1\}. \]

Let \( M = M(t) \in \{0,1, \ldots\} \) be the counting process
\[ M(t) = \max\{m : T^{(+,m)} \leq t\}, \quad t > 0. \] (1.5)

The marginal distributions of the process \( \epsilon = \epsilon(t), t \geq 0, \) indicating the current pattern, are defined by
\[ \mathbb{P}\{\epsilon(t + \Delta t) = i | \epsilon(t) = i, M(t) = m\} = \exp(-\mu^{(i)}_{N_m(t)} \cdot \Delta t) + o(\Delta t), \quad \Delta t \to 0, \quad i \in \{0,1\}, \]
\[ \epsilon(t) = \epsilon_m \in \{0,1\}, \] if \( t \in [T^{(+,m)}, T^{(+,m+1)}]. \) By \( \mathcal{F}_t, \) \( t \geq 0, \) we denote the corresponding filtration.
Process $M = M(t)$ can be treated as a doubly stochastic Poisson process, see Brémaud and Cox. A similar approach is exploited in Daley and Vere-Jones, see there Example 6.3(e), Example 7.3(a), and Example 7.4(e) (bivariate Poisson process). Meanwhile, this model differs from the model of mixed Poisson process (when the parameter $\mu$ of Poisson process is considered as the outcome of a positive $\mathcal{F}_0$-measurable random variable) widely exploited in actuarial applications, see, eg, Grandell and Rolski et al.

The problem of infinite accumulation of arrivals for doubly stochastic Poisson process (when the interarrival times are Poisson-modulated) is more complicated than for a simple point process, see Section 2 for the analysis of this problem.

The transport piecewise linear processes based on a mixed Poisson process have been recently presented in De Gregorio. For the 2D-case of such random motions with jumps (and with constant intensity of switchings) see Garra and others. The similar piecewise linear process $\mathbb{L}$ with the deterministically growing intensities $\mu^{(m)} = mv, v > 0$, has been studied in Di Crescenzo and Martinucci.

In this paper, we study the piecewise linear renewal process $\mathbb{L}(t)$ based on the two alternating sets of tendencies, $\{c^{(0)}(n)\}_{n \geq 0}$ and $\{c^{(1)}(n)\}_{n \geq 0}$, with Poisson-modulated exponential distributions of patterns’ holding times.

First, consider the sequence of independent piecewise linear processes

$$l_m(t) = \int_0^t c^{(\epsilon_m)}(N_m(s)) \, ds, \quad \epsilon_m \in \{0, 1\}, \quad m \geq 0,$$

which follows the sequence of tendencies $\{c^{(0)}(n)\}_{n \geq 0}$ or $\{c^{(1)}(n)\}_{n \geq 0}$ with switchings at Poisson random times. Process $\mathbb{L}(t)$ successively follows the two patterns alternating after holding times $T_m$:

$$\mathbb{L}(t) = \mathbb{L}(t) = \int_0^t l_{M(t)}(s) \, ds = \int_0^t \left( \int_0^s c^{(\epsilon)}(N_{M(t)}(u)) \, du \right) \, ds = \sum_{m=1}^{M(t)} l_{M(t)}(t) + l_{M(t)}(t - T^{M(t)}), \quad t \geq 0. \quad (1.6)$$

Bearing in mind applications, we supply $\mathbb{L}(t)$ with jumps. Properties of process $\mathbb{L}$ with jumps and with exponentially distributed time intervals $T_m$ under the arbitrary set of tendencies $c_m$ is recently studied by Ratanov. The processes with renewal restarting points (instead of jumps) is studied by Ratanov. Here, we analyse properties of such processes, which follow the alternating patterns with Poisson-modulated exponential distributions of switching times and with jumps $r^{(i)}(n)$ and $R^{(i)}(n), n \geq 0, i \in \{0, 1\}$, accompanying the tendency switchings and the changes of patterns, respectively.

This approach could be used for financial market modelling when log-returns are determined by the inherent market forces, that is by $\mathbb{L}(t), \{R(n)\}$, and by efforts of “small speculators,” which create the tendency and volatility modulations. Let jumps $R(n)$ accompanying the patterns’ switchings occur as “corrections” of the current trend $c(n)$:

$$c(n)/\mathbb{E}\{R(n)\} < 0.$$

In this case, small players trying to change the trend affect only the volatility and the probability of the next switching of the trend.

We derive coupled integral equations for mean values of the process $\mathbb{L}$ accompanied with jumps (see Section 4). Further, the martingale condition is presented: this process is a martingale if and only if

$$c^{(i)}(n) + \lambda^{(i)}_n\mathbb{E}[R^{(i)}(n)] + \mu^{(i)}_nR^{(i)}(n) \equiv 0, \quad n \geq 0, \quad i \in \{0, 1\}. \quad (1.7)$$

Here, $\mathbb{E}[R^{(i)}(n)]$ and $R^{(i)}(n)$ are the expectations of the random jump values. The same condition characterises martingales in a rather different model, when holding times $T_m$ are independent of the underlying Poisson processes $N_m$, see Ratanov, [Corollary 3.1]. Condition (1.7) looks similar to the martingale condition for a simple jump-telegraph process, see Ratanov:

$$c^{(i)}(n) + \lambda^{(i)}_nR^{(i)}(n) = 0, \quad i \in \{0, 1\}. \quad (1.8)$$

The text is organised as follows. The detailed analysis of Poisson-modulated exponential distributions is presented in Sections 2 and 3. In Section 4, we study the piecewise linear process $\mathbb{L}$ with jumps.

In Section 5, we propose a financial market model based on these processes. This model generalises the simple jump-telegraph market model.
2 | POISSON-MODULATED EXPONENTIAL DISTRIBUTION

In this section, we present some properties of the Poisson-modulated exponential distributions, which we will use later.

Let \( \{\tau_n\}_{n \geq 0} \) be the sequence of independent and exponentially distributed \( \text{Exp}(\lambda_n) \), \( \lambda_n > 0 \), \( n \geq 0 \), random variables, and \( N = N(t) = N(t; \vec{\lambda}) \), \( t \geq 0 \), be the nonexplosive, (1.1), renewal counting process. Denote the probability mass function of \( N(t) \) at \( n \) by \( \pi_n(t) = \mathbb{P}(N(t) = n) \), \( n \geq 0 \). If all \( \lambda_n \) are equal, \( \lambda_n \equiv \lambda \), then \( \pi_n(t) = e^{-\lambda t} / n! \); in the case of distinct \( \lambda_n \) by Ratanov,\textsuperscript{10} [formula (2.4), Proposition 2.1] we have

\[
\pi_n(t) = \Lambda_n a_n(t; \vec{\lambda}),
\]

(2.1)

where \( \Lambda_n = \prod_{k=1}^{n-1} \lambda_k \), \( n \geq 1 \), \( \Lambda_0 = 1 \),

\[
a_n(t) = a_n(t; \vec{\lambda}) = \sum_{k=0}^{n} \kappa_{n,k}(\vec{\lambda}) e^{-\lambda_k t}, \quad t > 0,
\]

(2.2)

and coefficients \( \kappa_{n,k}(\vec{\lambda}) \) are defined by

\[
\kappa_{n,k}(\vec{\lambda}) = \prod_{j=k}^{n} (\lambda_j - \lambda_k)^{-1}, \quad n \geq k, \quad \kappa_{0,0} = 1.
\]

(2.3)

In the nonexplosive case, (1.1),

\[
\sum_{n} \pi_n(t) \equiv 1, \quad \forall t > 0.
\]

In the case when not all \( \lambda \)'s are distinct, the usual changes in notations should be applied, see Ratanov.\textsuperscript{10}

Notice, that coefficients \( \kappa_{n,k} \) satisfy the following known Vandermonde properties: for \( n \geq 1 \)

\[
\sum_{k=0}^{n} \kappa_{n,k}(\vec{\lambda}) \lambda_k^m = 0, \quad 0 \leq m \leq n - 1, \quad \sum_{k=0}^{n} \kappa_{n,k}(\vec{\lambda}) \lambda_k^n = (-1)^n,
\]

(2.4)

(see eg, Kuznetsov\textsuperscript{24} p. 11).

Because of identities (2.4), functions \( a_n = a_n(t) = \sum_{k=0}^{n} \kappa_{n,k}(\vec{\lambda}) \exp(-\lambda_k t) \), satisfy the following conditions: for \( n \geq 1 \)

\[
a_n(0) = 0, \quad \frac{d^m a_n(t)}{dt^m} |_{t=0} = 0, \quad 0 < m \leq n - 1, \quad \text{and} \quad \frac{d^n a_n(t)}{dt^n} |_{t=0} = 1;
\]

(2.5)

\[
a_0(t) = \exp(-\lambda_0 t).
\]

Consider the random variable \( T, T > 0 \), with Poisson-modulated exponential distribution, \( \text{PoExp}(\vec{\lambda}, \vec{\mu}) \). See (1.2) to (1.3).

Let us begin by studying the properties of the accumulated intensity \( \zeta(t) \), based on the nonexplosive counting process \( N(t) \), (1.3). Consider the moment generating function \( \psi(z, t) \) of \( \zeta(t) \),

\[
\psi(z, t) = \mathbb{E}\{e^{-z\zeta(t)}\} = \sum_{n=0}^{\infty} \psi_n(z, t),
\]

(2.6)

where

\[
\psi_n(z, t) = \mathbb{E}\{e^{-z\zeta(t)}1_{N(t) = n}\}, \quad t > 0, \quad n \geq 0.
\]

(2.7)

By definition, we have

\[
\mathbb{P}\{T > t, N(t) = n\} = \psi_n(1, t).
\]

(2.8)

Assume that all linear \( z \)-functions

\[
z \rightarrow \lambda_n + z \mu_n = \lambda_n(z) = \vec{\lambda}_n, \quad n \geq 0,
\]

(2.9)

are distinct, \( \vec{\lambda}_n \neq \vec{\lambda}_k \) (if \( n \neq k \)) one can easily obtain the explicit expression for \( \psi_n \) by means of \( a_k, k \leq n \). Applying Ratanov,\textsuperscript{10} [Theorem 3.1] we have

\[
\psi_n(z, t) = \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\vec{\lambda}) e^{-\lambda_k t} = \Lambda_n a_n(t; \vec{\lambda}),
\]

(2.10)

where \( \kappa_{n,k}(\vec{\lambda}) \) and \( a_n(t; \vec{\lambda}) \) are defined by (2.3) and (2.2) with \( \vec{\lambda} = \vec{\lambda}(z) \) instead of \( \vec{\lambda} \).
We introduce the notation
\[ b_k(\lambda, z\bar{\mu}) := \sum_{n=0}^{\infty} \Lambda_n \kappa_{n,k}(\lambda) < \infty, \quad k \geq 0, \] (2.11)
assuming convergence of the series. From (2.10), one can obtain the representation of \( \psi(z, t) \), which is equivalent to (2.6):
\[ \psi(z, t) = \sum_{n=0}^{\infty} \psi_n(z, t) = \sum_{n=0}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\lambda) e^{-\lambda_k t} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \Lambda_n \kappa_{n,k}(\lambda) \right) e^{-\lambda_k t} = \sum_{k=0}^{\infty} b_k(\lambda, z\bar{\mu}) e^{-\lambda_k t}, \]
if the series in (2.11) converge. This representation is consistent with the identities \( \psi(z, 0) = \mathbb{E}\{e^{-z\delta_t}|_{t=0}\} \equiv 1 \) and \( \psi(0, t) = \mathbb{E}\{e^{-z\delta_t}|_{t=0}\} \equiv 1 \):
\[ \psi(z, 0) = \sum_{k=0}^{\infty} b_k(\bar{\lambda}, z\bar{\mu}) = \sum_{k=0}^{\infty} \Lambda_n \sum_{n=0}^{k} \kappa_{n,k}(\bar{\lambda}) = 1 + \sum_{n=1}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\bar{\lambda}) \equiv 1, \]
which due to (2.5)
gives
\[ \psi(z, 0) = \sum_{k=0}^{\infty} \Lambda_n \sum_{n=0}^{k} \kappa_{n,k}(\lambda) = \sum_{n=1}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\lambda) e^{-\lambda_k t} = \sum_{n=0}^{\infty} \pi_n(t) \equiv 1. \]

Remark 2.1. Notice that equalities (2.6) to (2.7) and (2.10) return us to some known formulae for the \( \lambda_n, \lambda_n = \lambda, \lambda > 0. \)

By (2.10) and (2.3) we have
\[ \psi_n(z, t) = \lambda^n z^{-n} \sum_{k=0}^{n} \kappa_{n,k}(\mu) e^{-\lambda_k n!} e^{-\lambda t} = \lambda^n z^{-n} a_n(zt) e^{-\lambda t}, \]
which due to (2.5) gives
\[ \pi_n(t) = \psi_n(z, t)|_{z=0} = \lambda^n e^{-\lambda t} \cdot \lim_{z \to 0} \left[ z^{-n} a_n(zt) \right] = \lambda^n e^{-\lambda t} \lim_{z \to 0} \left[ z^{-n} \frac{\partial^n a_n(t)}{\partial t^n} \right] = (\lambda t)^n e^{-\lambda t}. \]

Further, by (2.9) one has
\[ b_k(\lambda, z\bar{\mu}) = \sum_{n=k}^{\infty} z^{-n} \lambda^n \kappa_{n,k}(\bar{\mu}). \]

Note that the series convergence here and in (2.11) occurs, for example, if
\[ |\mu_n - \mu_m| \geq \nu > 0, \quad n \neq m. \]

Indeed, \( |\kappa_{n,k}(\bar{\mu})| \leq \nu^{-n} \) and
\[ |b_k(\lambda, z\bar{\mu})| \leq \sum_{n=k}^{\infty} \left( \frac{\lambda}{|z|\nu} \right)^n = \frac{(\lambda/|z|\nu)^k}{1 - \lambda/|z|\nu} < \infty \]
for \( |z| > \lambda/\nu. \)

In the linear case, \( \mu_n = \mu + n\nu \) (see Section 3),
\[ \kappa_{n,k}(\lambda) = z^{-n} (-1)^k \nu^{-n} k!(n-k)! \]
and
\[ b_k(\lambda, z\bar{\mu}) = \sum_{n=k}^{\infty} \left( (-1)^k \lambda^n (z\nu)^{-n} k!(n-k)! \right) = \frac{(-\lambda / z\nu)^k}{k!} e^{\lambda / z\nu} < \infty, \quad z \neq 0. \]

We express the distribution of \( T \) in these terms, starting with the following useful formulas.

Proposition 2.1. If all sums \( \lambda_n + \mu_n, n \geq 0, \) are distinct, then
\[ \mathbb{P}\{T > t, N(t) = n\} = \Lambda_n a_n(t; \lambda + \mu), \quad t \geq 0, \]
\[ \mathbb{P}\{T \in dt, N(t) = n\} = \mu_n \Lambda_n a_n(t; \lambda + \mu) dt, \]
where functions \( a_n(t; \lambda + \mu) \) are defined by (2.2).
Equality (2.12) follows from (2.8) and (2.10).

To prove (2.13) note that by Ratanov10 (2.5) the joint distribution of \((\tau_0, \ldots, \tau_{n-1})\) \(\mathbb{1}_{\{N(t)=n\}}\) is given by

\[
\mathbb{P}\{\tau_0 \in ds_0, \ldots, \tau_{n-1} \in ds_{n-1}, N(t) = n\} = \Lambda_n e^{-\lambda \cdot s} \exp\left(-\sum_{k=0}^{n-1} (\lambda_k - \lambda_n) s_k\right) \mathbb{1}_{\xi_n(t)}(s) ds,
\]

where \(\mathbb{1}_{\xi_n(t)}(s) = \begin{cases} 1, & \text{if } s \in \Xi_n(t) \\ 0, & \text{otherwise} \end{cases}\), where \(\Xi_n(t) := \{s = (s_0, \ldots, s_{n-1}) \in \mathbb{R}^n_+ \mid s^{(n)} = s_0 + \ldots + s_{n-1} < t\} \).

By (1.2) to (1.3)

\[
\mathbb{P}\{T \in dt, N(t) = n\}/dt = \mathbb{E}\left\{\mu_{N(t)} \exp(-\xi(t)) \mathbb{1}_{\{N(t)=n\}}\right\} = \mu_n e^{-\mu t} \int_{\Xi_n(t)} \exp\left(-\sum_{k=0}^{n-1} (\mu_k - \mu_n) s_k\right) \cdot \Lambda_n e^{-\lambda \cdot s} \exp\left(-\sum_{k=0}^{n-1} (\lambda_k - \lambda_n) s_k\right) ds.
\]

Applying Ratanov10[(3.5)] we get (2.13).

If some of \(\lambda_n + \mu_n\) are equal, the usual changes should be applied (for details, see Ratanov11).

**Corollary 2.1.** Let the series (2.11) converges.

Under the conditions of Proposition 2.1

- the survivor function \(\overline{F}_T(t) = \mathbb{P}\{T > t\}\) has the form
  \[
  \overline{F}_T(t) = \sum_{k=0}^{\infty} b_k(\hat{\lambda}, \hat{\mu}) e^{-(\lambda_k + \mu_k) t}, \quad t \geq 0;
  \]

- the density function \(f_T(t)\) is given by
  \[
  f_T(t) = \sum_{k=0}^{\infty} (\lambda_k + \mu_k) b_k(\hat{\lambda}, \hat{\mu}) e^{-(\lambda_k + \mu_k) t},
  \]

if the series in (2.15) converges. In particular, \(f_T(0) = 0\).

**Proof.** If (2.11) holds, under the conditions of Proposition 2.1 the survivor function takes the form

\[
\overline{F}_T(t) = \sum_{n=0}^{\infty} \Lambda_n a_n(t; \hat{\lambda} + \hat{\mu}) = \sum_{n=0}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\hat{\lambda} + \hat{\mu}) e^{-(\lambda_k + \mu_k) t}, \quad t \geq 0,
\]

which gives (2.14). By differentiation the density function takes the form (2.15), if the series in (2.15) converges.

On the other hand, by (2.13) we have

\[
f_T(t) = \sum_{n=0}^{\infty} \mu_n \Lambda_n a_n(t; \hat{\lambda} + \hat{\mu}) = \sum_{k=0}^{\infty} \mu_k e^{-(\lambda_k + \mu_k) t} \sum_{n=k}^{\infty} \mu_n \Lambda_n \kappa_{n,k}(\hat{\lambda} + \hat{\mu}).
\]

(2.16)

Representations (2.15) and (2.16) are equivalent. Indeed, by definition (2.3) of \(\kappa_{n,k}\):

\[
\sum_{n=k}^{\infty} \mu_n \Lambda_n \kappa_{n,k}(\hat{\lambda} + \hat{\mu}) = \sum_{n=k}^{\infty} (\mu_n + \Lambda_n) \Lambda_n \kappa_{n,k}(\hat{\lambda} + \hat{\mu}) - \sum_{n=k+1}^{\infty} \Lambda_{n+1} \kappa_{n,k}(\hat{\lambda} + \hat{\mu})
\]

\[
= \sum_{n=k+1}^{\infty} \Lambda_n \kappa_{n-1,k}(\hat{\lambda} + \hat{\mu}) + (\mu_k + \lambda_k) \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\hat{\lambda} + \hat{\mu}) - \sum_{n=k}^{\infty} \Lambda_{n+1} \kappa_{n,k}(\hat{\lambda} + \hat{\mu})
\]

\[
= (\mu_k + \lambda_k) \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\hat{\lambda} + \hat{\mu}),
\]

since (2.11) holds.
Moreover, in this case by (2.11) one can see that
\[
\sum_{k=0}^{\infty} (\lambda_k + \mu_k) b_k(\tilde{\lambda}, \tilde{\mu}) = \sum_{n=0}^{\infty} \Lambda_n \sum_{k=0}^{n} (\lambda_k + \mu_k) \kappa_{n,k}(\tilde{\lambda})
\]
\[= (\lambda_0 + \mu_0) + \lambda_0 \sum_{k=0}^{1} (\lambda_k + \mu_k) \kappa_{1,k}(\tilde{\lambda}) + \sum_{n=2}^{\infty} \Lambda_n \sum_{k=0}^{n} (\lambda_k + \mu_k) \kappa_{n,k}(\tilde{\lambda}).
\]
Therefore, by (2.4) we have
\[
\sum_{k=0}^{\infty} (\lambda_k + \mu_k) b_k(\tilde{\lambda}, \tilde{\mu}) = N^{(\tilde{\lambda})} = \left(\lambda_0 + \mu_0\right) + \sum_{n=1}^{\infty} \Lambda_n \sum_{k=0}^{n} (\lambda_k + \mu_k) \kappa_{n,k}(\tilde{\lambda}).
\]
Hence, \(f_T(0) = \mu_0.\)

The convergence in (2.11) plays the role of a nonexploding condition.

**Proposition 2.2.** Let \(T\) be a Poisson-modulated exponential, \(\text{PoExp}(\tilde{\lambda}, \tilde{\mu})\), random variable.

The distribution of variable \(T\) is proper, that is
\[
P\{T < \infty\} = 1,
\]
if (2.11) holds.

**Proof.** Let (2.11) holds. By integrating in (2.15),
\[
\int_{0}^{\infty} f_T(t) dt = \sum_{k=0}^{\infty} b_k(\tilde{\lambda}, \tilde{\mu}) = \sum_{n=0}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\tilde{\lambda} + \tilde{\mu}) = 1 + \sum_{n=1}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\tilde{\lambda} + \tilde{\mu}) = 1.
\]

The moments of random variable \(T\) can be obtained similarly.

**Proposition 2.3.** If the distribution of \(T\) satisfies
\[
\lim_{t \to \infty} t F_T(t) = 0,
\]
and the series \(\sum_{k=0}^{\infty} b_k(\tilde{\lambda}, \tilde{\mu})(\lambda_k + \mu_k)^{-1}\) converges, then the expectation \(\mathbb{E}\{T\}\) exists and

\[
\mathbb{E}\{T\} = \sum_{k=0}^{\infty} b_k(\tilde{\lambda}, \tilde{\mu})(\lambda_k + \mu_k)^{-1} = \sum_{n=0}^{\infty} \Lambda_n \Pi_n^{-1} < \infty,
\]
where \(\Pi_n = \prod_{k=0}^{n} (\lambda_k + \mu_k).\)

Moreover, if for some \(m, m \geq 1,\)
\[
\lim_{t \to \infty} t^m F_T(t) = 0,
\]
and the series \(\sum_{k=0}^{\infty} b_k(\tilde{\lambda}, \tilde{\mu})(\lambda_k + \mu_k)^{-m}\) converges, then
\[
\mathbb{E}\{T^m\} = m! \sum_{k=0}^{\infty} b_k(\tilde{\lambda}, \tilde{\mu})(\lambda_k + \mu_k)^{-m} < \infty.
\]

**Proof.** Because of (2.14) and (2.17)
\[
\mathbb{E}\{T\} = \int_{0}^{\infty} F_T(t) dt = \sum_{k=0}^{\infty} b_k(\tilde{\lambda}, \tilde{\mu})(\lambda_k + \mu_k)^{-1} = \sum_{n=0}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\tilde{\lambda} + \tilde{\mu})(\lambda_k + \mu_k)^{-1}.
\]
Notice that by the Vandermonde properties, see Kuznetsov,24
\[
\sum_{k=0}^{n} \kappa_{n,k}(\tilde{\lambda} + \tilde{\mu})(\lambda_k + \mu_k)^{-1} = \Pi_n^{-1}.
\]
Therefore,
\[
\mathbb{E}(T) = \sum_{n=0}^{\infty} \Lambda_n \Pi_n^{-1} = \sum_{n=0}^{\infty} \left[ \prod_{k=0}^{n-1} \frac{\lambda_k}{\lambda_k + \mu_k} \right] \times \frac{1}{\lambda_n + \mu_n}.
\]

In general, if for some \( m, m \geq 1 \),
\[
\lim_{t \to \infty} t^m \overline{F}_T(t) = 0,
\]
then integrating by parts we obtain
\[
\mathbb{E}\{T^m\} = - \int_{0}^{\infty} t^m d\overline{F}_T(t) = m \int_{0}^{\infty} t^{m-1} \sum_{n=0}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\lambda + \mu)e^{-(\lambda_k + \mu_k)t} dt
\]
\[
= m \sum_{n=0}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\lambda + \mu) \int_{0}^{\infty} t^{m-1} e^{-(\lambda_k + \mu_k)t} dt = m! \sum_{n=0}^{\infty} \Lambda_n \sum_{k=0}^{n} \kappa_{n,k}(\lambda + \mu)(\lambda_k + \mu_k)^{-m}
\]
\[
= m! \sum_{k=0}^{\infty} b_k(\lambda, \mu)(\lambda_k + \mu_k)^{-m} < \infty.
\]

The latter equality follows by (2.11).

We conclude the section with some examples.

**Example 1.**

a. Let \( \lambda_n = \mu_n = \frac{1}{2}(n + 1)^2 \), that is \( \sum_n \lambda_n^{-1} = \sum_n \mu_n^{-1} < \infty \). By (2.3) and (2.11), we have
\[
\kappa_{n,k}(\lambda + \mu) = \left( \prod_{j=0, j \neq k}^{n} [(j + 1)^2 - (k + 1)^2] \right)^{-1} = (-1)^k \frac{2(k + 1) \cdot (k + 1)!}{k!(n-k)!(n+k+2)!},
\]
\[
\Lambda_n = (n!)^2 2^{-n}, \quad \Pi_n = [(n+1)!]^2, \quad b_k(\lambda, \mu) = (-1)^k 2(k+1)^2 \cdot \sum_{n=k}^{\infty} \frac{(n!)^2}{(n-k)!(n+k+2)!} 2^{-n} < \infty
\]
and
\[
\mathbb{E}(T) = \sum_{n=0}^{\infty} \Lambda_n \Pi_n^{-1} = \sum_{n=0}^{\infty} \frac{2^{-n}}{(n+1)^2} < \infty.
\]

b. Let \( \lambda_n = n + 1, \mu_n = 1 \), that is \( \sum_n \lambda_n^{-1} = \infty, \sum_n \mu_n^{-1} = \infty \),
\[
\kappa_{n,k} = \frac{(-1)^k}{k!(n-k)!}, \quad \Lambda_n = n!, \quad \Pi_n = (n+2)!, \quad b_k(\lambda, \mu) = \sum_{n=k}^{\infty} \frac{(-1)^k}{k!(n-k)!} = \infty
\]
and
\[
\mathbb{E}(T) = \sum_{n=0}^{\infty} \Lambda_n \Pi_n^{-1} = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1.
\]

c. Let \( \lambda_n = 1, \mu_n = \frac{1}{n+1} \), that is \( \sum_n \lambda_n^{-1} = \infty, \sum_n \mu_n^{-1} = \infty \),
\[
\kappa_{n,k} = \frac{(-1)^{n-k}(k+1)^{n-1}!}{k!(n-k)!}, \quad \Lambda_n = 1, \quad \Pi_n = n + 2, \quad b_k(\lambda, \mu) = \infty, \quad \mathbb{E}(T) = \infty.
\]

An example with a linearly increasing intensity \( \mu_n \) is given in the next section.

### 3 Example: Poisson-Modulated Exponential Distributions with Linearly Increasing Switching Intensities

All the formulae can be simplified and detailed in the case of a linear increase in switching intensities, \( \mu_n = \mu + n\nu \).

**Theorem 3.1.** Let random variable \( T \) has a Poisson-modulated exponential distribution with a homogeneous underlying Poisson process, \( \lambda_n \equiv \lambda, T \sim \text{PoExp}(\lambda, \mu) \), where \( \mu = (\mu + n\nu)_{n \geq 0} \) with some \( \nu, \nu > 0 \).
The cumulative distribution function of $T$ is given by
\[ F_T(t) = \left(1 - e^{-\mu t - \lambda A(t)}\right) 1_{t \geq 0}. \] (3.1)
and the moment generating function is
\[ \Psi(z) := \mathbb{E}[e^{-zT}] = 1 - ze^{j/\sum_{n=0}^{\infty} (-1)^n(\lambda / \nu)^n} n!/(\lambda + \mu + z + n\lambda). \] (3.2)

Here, we denote $A(t) = \int_0^t a(u)du = t - a(t)/\nu$, $t \geq 0$, where $a(t) = 1 - e^{-\nu t}$.

**Proof.** First, note that in this case
\[ \kappa_{n,k} = \left[ \prod_{j=0}^{n} (j - k)\nu \right]^{-1} = (-1)^k \frac{k!}{(n-k)!} \nu^{-n} \] (3.3)
and
\[ b_k = \sum_{n=k}^{\infty} (\lambda / \nu)^n \frac{(-1)^k}{k! (n-k)!} = \frac{(-1)^k}{k!} (\lambda / \nu)^k e^{\lambda / \nu}. \] (3.4)

By (2.14) and (3.4),
\[ F_T(t) = 1 - F_T(t) = \sum_{k=0}^{\infty} b_k e^{-(\lambda_k + \mu_k)t} = e^{\lambda / \nu} \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(n-k)!} (\lambda / \nu)^k e^{-(\lambda_k + \mu_k)t} = \exp \left( \frac{\lambda}{\nu} \left(1 - e^{-\nu t} \right) - (\lambda + \mu)t \right), \]
which coincides with (3.1).

Therefore,
\[ \Psi(z) = \int_0^{\infty} e^{-z} dF_T(t) = \int_0^{\infty} f(t, A(t)) \exp(-\mu t - \lambda A(t)) \, dt \\
= (\mu + \lambda)I(z) - \lambda I(z + \nu), \]
where
\[ I(z) = \int_0^{\infty} \exp(-(\mu + z)t - \lambda A(t)) \, dt = \frac{1}{\nu} \int_0^{\mu + z + \frac{\lambda}{\nu}} x^{\lambda-1} e^{-\beta x} \, dx, \quad b(z) = \frac{\mu + z + \frac{\lambda}{\nu}}{\nu}, \quad \beta = \frac{\lambda}{\nu}. \]

Because of Gradshteyn and Ryzhik,\textsuperscript{25} formula (3.381.1),
\[ I(z) = \frac{\gamma(b(z), \beta)}{\nu^\beta}. \] (3.6)

Since $b(z + \nu) = b(z) + 1$, by using Gradshteyn and Ryzhik,\textsuperscript{25} formula (8.356.1), from (3.5), we get
\[ \Psi(z) = 1 - \frac{z^{-b(\nu)} \beta^\nu}{\nu} \gamma(b(z), \beta). \]

Formula (3.2) follows by the series representation of the incomplete gamma-function, see Gradshteyn and Ryzhik,\textsuperscript{25} formula (8.354.1).

**Remark 3.1.** The distribution of $T$, given by (3.1) is unimodal, that is, the corresponding density function
\[ f_T(t) = (\mu + \lambda a(t)) \exp(-(\mu + \lambda)t + \lambda a(t)/\nu), \quad t > 0, \]
has a single maximum at point $m$, where $m = 0$ if $\lambda \nu \leq \mu^2$, and
\[ m = \frac{1}{\nu} \ln \left[ \frac{\lambda}{\lambda + \mu} \left( 1 + \frac{\nu}{2(\lambda + \mu)} \right) + \sqrt{\frac{\nu}{\lambda + \mu} \left( 1 + \frac{\nu}{4(\lambda + \mu)} \right)} \right] \]
if $\lambda \nu > \mu^2$. See Figure 1.
FIGURE 1  Density function of the Poisson-modulated distribution, (3.7), with $\lambda = 1.5, \mu = 1, \nu = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

The moments could be computed by differentiating the moment generating function $\Psi(z)$, which is given by (3.2),

$$E\{T_m\} = (-1)^m \Psi^{(m)}(0) = e^{z\nu} m! \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda/\nu)^n}{n!(\lambda + \mu + n\nu)^m}, \quad m = 1, 2, \ldots \quad (3.8)$$

The same result follows from (2.18) and (3.4).

Formula (3.8) can be simplified: by 25 [(9.14)] using a generalised hypergeometric function $\mathbf{mFm}(a_1, \ldots, a_m; b_1, \ldots, b_m; z)$,

$$E\{T_m\} = \frac{m! e^{z\nu}}{(\lambda + \mu)^m} mFm(a, \ldots, a; 1 + a, \ldots, 1 + a; -\lambda/\nu),$$

where $a = (\lambda + \mu)/\nu$. In particular, the mean of $T$ is given by

$$E(T) = \frac{e^{z/\nu}}{\lambda + \mu} \Phi \left( \frac{\lambda + \mu}{\nu}, 1 + \frac{\lambda + \mu}{\nu}; -\lambda/\nu \right),$$

where $\Phi$ is a confluent hypergeometric function, see 25 formula (9.210). By 25 formula (8.354.1), $E(T)$ could be written in the equivalent form:

$$E(T) = e^{z/\nu} \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda/\nu)^n}{n!(\lambda + \mu + n\nu)} = e^{z/\nu} \frac{(\lambda/\nu)}{\Gamma \left( \frac{\lambda + \mu}{\nu}, \frac{\lambda}{\nu} \right)},$$

where $\gamma$ is the incomplete gamma-function.

4 | PIECEWISE LINEAR PROCESS WITH TWO ALTERNATING PATTERNS AND A DOUBLE JUMP COMPONENT

Let $N_m = N_m(t), m \geq 0$, be the sequence of independent Poisson processes, which are driven by two alternating sequences of parameters: $\vec{\lambda}^{(0)} = (\lambda^{(0)}_n)_{n \geq 0}$ and $\vec{\lambda}^{(1)} = (\lambda^{(1)}_n)_{n \geq 0}$. That is, $N_m(t) \equiv N(t; \vec{\lambda}^{(\epsilon_m)}_m)$, see the definition in (1.4). Here, $\epsilon_m$ is a sequence of alternating 0 and 1: $(0, 1, 0, 1, 0, \ldots)$ or $(1, 0, 1, 0, \ldots)$.

Let $\{T_m\}_{m \geq 1}$ be the sequence of independent positive random variables, $T_m \geq 0, m \geq 1$, and $M = M(t)$ be the associated process (1.5), that counts arrivals of $T^{(+,m)} := T_1 + \ldots + T_m, T^{(+,0)} = 0$, till time $t, t > 0$. 

Let $\varepsilon = \varepsilon(t) \in \{0, 1\}$ be the process, which indicates the current state as follows: $\varepsilon(t) = \varepsilon_m$ for $T^{(+,m)} \leq t < T^{(+,m+1)}$, $m \geq 0$.

We assume that random variable $T_m$ has Poisson-modulated exponential distribution,

$$T_m \sim \text{PoExp}(\tilde{\lambda}^{(\varepsilon_{m-1})}, \tilde{\mu}^{(\varepsilon_{m-1})}),$$

based on the Poisson process $N_{m-1} = N_{m-1}(t), m \geq 1$. Here, $\tilde{\mu}^{(0)} = \{\mu_n^{(0)}\}_{n \geq 0}$ and $\tilde{\mu}^{(1)} = \{\mu_n^{(1)}\}_{n \geq 0}$ are the two sequences of switching intensities (see (1.2)-(1.3)). The alternating survivor functions of $T_m$, $m \geq 1$, due to (2.12) are given by

$$F^{(i)}(t) = 1 - F^{(i)}(t) = \mathbb{P}\{T_m > t\} = \mathbb{E}\left\{\exp\left(- \int_0^t \lambda_n^{(i)}(s) \, ds\right)\right\} = \sum_{n=0}^{\infty} \Lambda_n^{(i)} \alpha_n(t; \tilde{\lambda}^{(i)} + \tilde{\mu}^{(i)}), \quad t \geq 0,$$

and the corresponding densities are defined by $f^{(i)}(t) = -dF^{(i)}(t)/dt$, $i = \varepsilon_{m-1} \in \{0, 1\}$.

In this section, we study a piecewise linear process, which follows two patterns alternating after the holding times $T_m$. Precisely, we define the piecewise linear process $L$ based on the two sequences of tendencies, $\{c^{(0)}(n)\}_{n \geq 0}$ and $\{c^{(1)}(n)\}_{n \geq 0}$, alternating at the time instants $T^{(+m)}$, such that

$$L(t) = \sum_{m=1}^{M(t)} l_{m-1}(T_m) + I_{M(t)}(t - T^{(+M(t)}) \quad t \geq 0, \quad (4.1)$$

where

$$I_m(t) = \int_0^t c^{(m)}(N_m(s)) ds, \quad m \geq 0, t \geq 0. \quad (4.2)$$

This definition coincides with (1.6). A simulated sample path is presented by Figure 2.

The jump component added to this process consists of two parts. The first one calculates the jumps occurring at the arrival times of the embedded Poisson processes. This corresponds to the case that each tendency switching (inside the time interval $[T^{(+m-1)}, T^{(+m)}]$) is accompanied with a jump of the magnitude $r_m(\cdot)$. The compound Poisson processes $j_m(t), t \geq 0$,

$$j_m(t) = \sum_{n=1}^{N_m(t)} r_m(n-1), \quad m \geq 0,$$

FIGURE 2 A sample path of the piecewise linear process with two patterns of two pairs of alternating velocities. The 0-pattern with $c = 0.5 \text{ and } 2.0; \text{ the } 1\text{-pattern with } c = -1.0 \text{ and } -3.0$. In both cases, the inter-switching times are Poisson-modulated with $\lambda_n = 1.5, \mu_n = 1 + n$ [Colour figure can be viewed at wileyonlinelibrary.com]
presents the summed jump component. Here, \( \{r_m(n)\}, n \geq 0, m \geq 0, \) are independent random jump amplitudes, independent of counting process \( N_m. \) Assume that the distributions of jumps are alternating, such that the processes \( j_m \) with even (odd) \( m \) are identically distributed. Let

\[
j(t) = \sum_{m=1}^{M(t)} j_{m-1}(T_m) + j_{M(t)}(t - T^{(+M(t)})).
\] (4.3)

counts the total number of this type of jumps.

The second jump part is defined by jumps occurring at times \( T^{(+m)}, m \geq 1, \) when the pattern changes. We assume that the jump amplitudes depend on the number of interventions \( N_{m-1}(T_m), \) during the elapsed time \( T_m, \)

\[
J(t) = \sum_{m=1}^{M(t)} R_{m-1}(N_{m-1}(T_m)) .
\] (4.4)

Here, independent random variables \( \{R_m(n)\} \) are the jump magnitudes, which are independent of \( T_m, N_m \) and \( \{r_m(n)\}, n \geq 0, m \geq 0. \) Assume that \( R_m(\cdot) \) are of the alternating distributions.

Summarising, the jump component \( \bar{J}(t) \) is defined by

\[
\bar{J}(t) = j(t) + J(t) = \sum_{m=1}^{M(t)} [R_{m-1}(N_{m-1}(T_m)) + j_{m-1}(T_m)] + j_{M(t)}(t - T^{(+M(t)}), \quad t > 0.
\] (4.5)

Process \( \bar{J} = \bar{J}(t) \) is an alternating renewal process, see, eg, Cox.\textsuperscript{26} The behaviour of the paths of such a process \( X(t) := L(t) + \bar{J}(t) \) is illustrated by Figure 3.

In this paper, we study the piecewise linear process \( L(t) \) accompanied with the double jump component \( \bar{J}(t), \) defined by (4.3) to (4.5). Process \( X(t) := L(t) + \bar{J}(t) \) successively passes through the alternating states determined by the two sets of parameters

\[
\sigma^{(i)} := (\overline{c}^{(i)}, \overline{r}^{(i)}, \overline{R}^{(i)}, \overline{\mu}^{(i)}, \overline{\lambda}^{(i)}), \quad i \in \{0, 1\}.
\] (4.6)

Jump processes \( j(t) \) and \( J(t) \) are of different nature. Under the given trend \( c = c^{(i)}(n) \) the jump \( j(t) \) with the magnitude \( r^{(i)}(n) \) always occurs just after each tendency switching, while the jump with magnitude \( R^{(i)}(n) \) occurs in the case when the process changes the pattern, (4.4).

We analyse the expectation of \( X(t). \) Denote by

\[
\overline{R}^{(i)}(n) := \mathbb{E}\{R^{(i)}(n)\}, \quad \overline{r}^{(i)}(n) := \mathbb{E}\{r^{(i)}(n)\}, \quad m \geq 0, \quad n \geq 0, \quad i = \varepsilon_m,
\]

the expectations of jump values, alternating with respect to \( m, \) that is \( \overline{R}_m(\cdot) \) and \( \overline{r}_m(\cdot). \)

Denote

\[
\rho(n) = \rho^{(i)}_n := \sum_{k=0}^{n-1} r^{(i)}(k), \quad n \geq 1, \quad \rho(0) = 0.
\] (4.7)

\textbf{FIGURE 3} A sample path of the piecewise linear process with two alternating patterns and jumps
Assume that for both the states, \( i \in \{0, 1\} \), the following series converge:

\[
\sum_{n=k}^{\infty} \rho(n) \Lambda_n \kappa_{n,k}(\bar{\mu}) < \infty, \quad k \geq 0.
\]  

\[
\sum_{n=k}^{\infty} \left( c(n) + \mu_n \overline{R(n)} \right) \Lambda_n \kappa_{n,k}(\bar{\mu}) < \infty.
\]  

This condition extends condition (2.11) fixing relations between the sets of “observable” parameters \( c(n), \overline{R(n)} \), \( \rho(n), n \geq 0 \), and “hidden” intensity parameters \( \bar{\lambda}, \bar{\mu} \). Similarly to Equation (2.11), conditions (4.8) are sufficient for “finite accumulation of jumps” at any finite time interval.

**Theorem 4.1.** Let condition (4.8) hold.

The (conditional) expectations \( \mathfrak{M}_i = \mathfrak{M}_i(t) = \mathbb{E}\{ \mathbb{L}(t) + \mathbb{J}(t) | \varepsilon(0) = i \} \), under the given initial set of parameters \( \sigma \) solve the following coupled integral equations:

\[
\mathfrak{M}_0(t) = m(t | \sigma^{(0)}) + \int_0^t f_T^{(0)}(u) \mathfrak{M}_1(t-u) du, \quad t > 0.
\]

\[
\mathfrak{M}_1(t) = m(t | \sigma^{(1)}) + \int_0^t f_T^{(1)}(u) \mathfrak{M}_0(t-u) du,
\]

Here, \( f_T^{(0)} \) and \( f_T^{(1)} \) are the density functions of the alternating distributions of \( T_m \), see (2.15); function \( m(t | \sigma) \) is defined by the initial state \( \sigma = (c, \bar{r}, \bar{R}, \bar{\mu}, \bar{\lambda}) \) as follows:

\[
m(t | \sigma) = \sum_{k=0}^{\infty} \frac{1 - e^{- (\bar{\lambda} + \mu_k) t}}{\bar{\lambda}_k + \mu_k} \sum_{n=k}^{\infty} \Delta(n) \Lambda_n \kappa_{n,k}(\bar{\lambda} + \mu), \quad t \geq 0,
\]

where \( \Delta(n) = \Delta(n | \sigma) = c(n) + \lambda_n \overline{r(n)} + \mu_n \overline{R(n)}, \ n \geq 0. \)

**Proof.** Let \( \varepsilon_0 = 0. \)

The following equality in law holds:

\[
[\mathbb{L}(t) + \mathbb{J}(t)]_{\varepsilon(0) = 0} = [l_0(t) + j_0(t)]_{T > 1} + \{ [l_0(T) + j_0(T) + R_0(N_0(T))] + [\mathbb{L}(t - T) + \mathbb{J}(t - T)] \}_{l(0) = 1}_{T \leq t}.
\]

Here, \( T = T_1 \) is the time of the first pattern’s switching.

Fix the initial state \( \sigma = (c, \bar{r}, \bar{R}, \bar{\mu}, \bar{\lambda}) \). One can write

\[
\mathfrak{M}_0(t) = \mathbb{E}_0 \{ l_0(t) + j_0(t) \} 1_{\{T > 1\}} + \int_0^t f_T^{(0)}(s) \mathbb{E}_0 \{ l_0(s) + j_0(s) + R_0(N_0(s)) \} ds
\]

\[
+ \int_0^t f_T^{(0)}(s) \mathfrak{M}_1(t-s) ds.
\]

(all parameters are of state \( \varepsilon = 0; \mathbb{E}_0 \) is the conditional expectation with respect to the conditional probability \( \mathbb{P}_0\{\cdot\} := \mathbb{P}\{\cdot | \varepsilon(0) = 0\} \). The equation for \( \mathfrak{M}_1(\cdot) \) is similar.

By using (2.11), (2.12) to (2.13) due to (4.7), one can obtain

\[
\mathbb{E}_0 \left[ j(t) \cdot 1_{\{T > 1\}} \right] = \sum_{n=1}^{\infty} \Lambda_n \rho(n) a_n(t, \bar{\lambda} + \bar{\mu}),
\]

\[
\int_0^t f_T^{(0)}(s) \mathbb{E}_0 \{ j_0(s) + R_0(N_0(s)) \} ds = \sum_{n=0}^{\infty} \left( \rho(n) + \overline{R(n)} \right) \mu_n \Lambda_n \int_0^t a_n(s; \bar{\lambda} + \bar{\mu}) ds
\]
Further,
\[
\frac{d}{dt} \mathbb{E}_0 (\Delta(t)1_{T>\tau}) = \sum_{n=0}^{\infty} \mathbb{E}_0 \left( \frac{d\Delta}{dt} 1_{N(t)=n} \mid T > t \right) \mathbb{P}_0 \{ T > t \} + \mathbb{E}_0 \left( \Delta(t) \cdot \frac{d1_{T>\tau}}{dt} \right)
\]

\[
= \sum_{n=0}^{\infty} c^{(0)}(n) \mathbb{P}_0 \{ T > t, N(t) = n \} - f_T^{(0)}(t) \mathbb{E}_0 (\Delta(t)).
\]

Integrating (4.14) by (2.12) we get
\[
\mathbb{E}_0 (\Delta(t)1_{T>\tau}) = \sum_{n=0}^{\infty} c^{(0)}(n) \Lambda_n \int_0^t a_n(s; \lambda + \mu) \, ds - \int_0^t f_T^{(0)}(s) \mathbb{E}_0 (1(s)) \, ds.
\]

Therefore,
\[
\mathbb{E}_0 \{ I_0(t)1_{T>\tau} \} + \int_0^t f_T^{(0)}(s) \mathbb{E}_0 (I_0(s)) \, ds \sum_{n=0}^{\infty} c^{(0)}(n) \Lambda_n \int_0^t a_n(s; \lambda + \mu) \, ds.
\]

Summing up (4.12), (4.13), and (4.15) by using (4.11), we get Equation (4.9) with
\[
m(t \mid \sigma) = \sum_{n=0}^{\infty} \rho(n) \Lambda_n a_n(t; \lambda + \mu) + \sum_{n=0}^{\infty} \Lambda_n \left( (R(n) + \rho(n)) \mu_n + c(n) \right) \int_0^t a_n(s; \lambda + \mu) \, ds.
\]

for both the states \( \sigma \).

To complete the proof, we should convert the last expression into the form of (4.10). Recalling (2.2), function \( m(t \mid \sigma) \) takes the form \( m(t \mid \sigma) = m_0 + m_1(t) \), where \( m_0 \) is constant,
\[
m_0 = \sum_{n=0}^{\infty} \Lambda_n \left( (c(n) + (R(n) + \rho(n)) \mu_n) \sum_{k=0}^{n} (\lambda_k + \mu_k)^{-1} \kappa_{n,k}(\lambda + \mu) \right)
\]

and
\[
m_1(t) = \sum_{k=0}^{\infty} e^{-(\lambda_k + \mu_k)t} \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\lambda + \mu) \left[ \rho(n) - (\lambda_k + \mu_k)^{-1} \left( c(n) + (R(n) + \rho(n)) \mu_n \right) \right]
\]

\[
= \sum_{k=0}^{\infty} e^{-(\lambda_k + \mu_k)t} \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\lambda + \mu) \left[ (\lambda_k + \mu_k) \rho(n) - c(n) - (R(n) + \rho(n)) \mu_n \right].
\]

Then, notice that
\[
\sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\lambda + \mu) \left[ (\lambda_k + \mu_k) \rho(n) - c(n) \right]
\]

\[
= \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\lambda + \mu) \left[ (\lambda_k + \mu_k) - (\lambda_n + \mu_n) + \lambda_n \right]
\]

\[
= - \sum_{n=k+1}^{\infty} \Lambda_n \kappa_{n-1,k}(\lambda + \mu) \rho(n) + \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\lambda + \mu) \rho(n) \lambda_n
\]

\[
= \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\lambda + \mu) \lambda_n R(n) = - \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\lambda + \mu) \lambda_n R(n).
\]

From (4.16) to (4.17), we obtain
\[
m_1(t) = - \sum_{k=0}^{\infty} e^{-(\lambda_k + \mu_k)t} \sum_{n=k}^{\infty} \Lambda_n \kappa_{n,k}(\lambda + \mu) \left( c(n) + \lambda_n R(n) + \mu_n R(n) \right).
\]

Note that by definition \( m(t \mid \sigma) \big|_{t=0} = 0 \). Therefore,
\[
m_0 = - m_1(0)
\]

and
\[
m(t \mid \sigma) = - m_1(0) + m_1(t) = \sum_{k=0}^{\infty} \frac{1 - e^{-(\lambda_k + \mu_k)t}}{\lambda_k + \mu_k} \sum_{n=k}^{\infty} \Delta(n) \Lambda_n \kappa_{n,k}(\lambda + \mu).
\]

Martingale condition (1.7) follows from equations (4.9).
Theorem 4.2. Let condition (4.8) holds.

Process \( X = X(t) \) is a martingale if and only if for both states the parameters of the model satisfy

\[
\Delta(n) := c(n) + \lambda_n r(n) + \mu_n R(n) = 0, \quad n \geq 0. \tag{4.18}
\]

Proof. By renewal character of the process \( X = X(t) \) is a martingale if and only if the expectations vanish, \( \mathbb{E}_n(t) \equiv 0, i \in \{0, 1\} \), or, equivalently, \( m(t \mid \sigma) \equiv 0 \) for both states, \( \sigma \in \{\sigma^{(0)}, \sigma^{(1)}\} \), see (4.9). By (4.10) this is equivalent to

\[
\sum_{n=k}^{\infty} \Lambda_n \Delta(n) k_{n,k}(\bar{\lambda} + \mu) = 0, \quad k = 0, 1, 2, \ldots \tag{4.19}
\]

If \( \Delta(n) = 0 \ \forall \ n, n \geq 0 \), then (4.19) holds.

On the other hand, from (4.19), one can obtain \( \Delta(n) = 0 \ \forall \ n \geq 0 \). Indeed, summing up these equations by using Vandermonde properties (2.4), we have

\[
0 = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \Lambda_n \Delta(n) k_{n,k}(\bar{\lambda} + \mu) = \sum_{n=0}^{\infty} \Delta(n) \Lambda_n \sum_{k=0}^{n} k_{n,k}(\bar{\lambda} + \mu) = \Delta(0) k_{0,0} = \Delta(0).
\]

Hence, \( \Delta(0) = 0 \).

Then, we prove \( \Delta(m) = 0, m = 1, 2, \ldots \), for both the states by induction. This follows by applying (2.4) to the sequential sums, \( m = 1, 2, \ldots \),

\[
0 = \sum_{k=0}^{\infty} (\lambda_k + \mu_k)^m \sum_{n=k}^{\infty} \Lambda_n \Delta(n) k_{n,k}(\bar{\lambda} + \mu) = \sum_{n=0}^{\infty} \Lambda_n \Delta(n) \sum_{k=0}^{n} (\lambda_k + \mu_k)^m k_{n,k}(\bar{\lambda} + \mu).
\]

\[\square\]

Remark 4.1. For some state of the process, let the supports of jump amplitudes \( r(n) \) and \( R(n) \) and the tendency \( c(n) \) be situated in the same semi-line. In this case, by Theorem 4.2, the process \( X \) is not a martingale.

Precisely, the equivalent martingale measure for process \( X \) does not exist in the following cases: In some state of the process, \( i \in \{0, 1\}, \exists n \)

\[
c(n) < 0, \text{ and } \operatorname{supp} \{r(n)\}, \operatorname{supp} \{R(n)\} \subset (-\infty, 0); \tag{4.20}
\]

\[
c(n) > 0 \text{ and } \operatorname{supp} \{r(n)\}, \operatorname{supp} \{R(n)\} \subset (0, +\infty). \tag{4.21}
\]

The problem of existence of equivalent martingale measures is discussed in the next section.

Remark 4.2. Theorem 4.2 seems natural. For instance, if the holding times \( T_m \) are independent of \( N_m \) and exponentially distributed with alternating parameters \( \mu^{(i)} \), condition (4.18) (with \( \mu^{(i)} \) instead of \( \mu_n \)) characterises a martingale for the similar piecewise linear process with double jump component, see Ratanov, Corollary 3.1. Moreover, condition (4.18) is very similar to the martingale condition for the simple jump-telegraph model, (1.8), see previous studies.

In (4.18), the term \( \lambda_n r(n) \) corresponds to the correction of tendency \( c(n) \), which is provoked by jumps occurring at each tendency switching, whereas the term \( \mu_n R(n) \) corresponds to the jumps accompanying the changes of patterns.

5 Market Model

We consider the financial market model based on the piecewise linear stochastic process with jumps \( X = X(t) = \mathbb{L}(t) + \mathbb{J}(t), t \geq 0 \), which is defined on the filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) by (4.1), (4.3), and (4.4). The alternating states of the process \( X \) are described by the two sets of parameters

\[
(\bar{\mathcal{C}}^{(i)}, \bar{\mathbb{R}}_m, \bar{\mu}^{(i)}, \bar{\lambda}^{(i)}), \quad i \in \{0, 1\}, \quad m \geq 0.
\]

Here, \( \bar{\mathcal{C}}^{(i)} = \{c^{(i)}(n)\}_{n \geq 0}, \bar{\mu}^{(i)} = \{\mu_n^{(i)}\}_{n \geq 0} \) and \( \bar{\lambda}^{(i)} = \{\lambda_n^{(i)}\}_{n \geq 0} \) are deterministic, \( \mu_n^{(1)} > 0, \ i \in \{0, 1\}, n \geq 0 \), and independent random jump amplitudes \( \bar{r}_m = \{r_m(n)\}_{n \geq 0} \) and \( \bar{R}_m = \{R_m(n)\}_{n \geq 0} \) are greater than \(-1\),

\[
\operatorname{supp} \{r_m(n)\} \subset (-1, \infty), \quad \operatorname{supp} \{R_m(n)\} \subset (-1, \infty), \quad m \geq 1, n \geq 0.
\]
Consider a market model of two risky primary assets, stock, and bond. The bond price is based on the continuous piecewise linear process

\[ Y(t) = \int_0^t y(u) \, du, \quad t \geq 0, \]

where \( y^{(0)} \geq 0 \) and \( y^{(1)} \geq 0 \) are continuously compounding interest rates depending on the current market state. The bond price dynamics is defined by

\[ B(t) = \exp(Y(t)), \quad t \geq 0. \]  

(5.1)

The stock price is defined by the stochastic exponential of \( X \). Precisely, denote

\[ Z_m(t) := \prod_{n=1}^{N_m(t)} (1 + r_m(n)), \quad m \geq 0, \]  

(5.2)

the stochastic exponential of the independent compound Poisson process \( J_m(t) := \sum_{n=1}^{N_m(t)} r_m(n) \). The stock price is defined by

\[ S(t) = S_0 \exp(\Delta(t) \sum_{m=1}^{M(t)} ([1 + R_{m-1} (N_{m-1}(T_m))] Z_{m-1}(T_m) \cdot Z_{M(t)} (t - T^{(+,M(t))}). \]

(5.3)

Model based on (5.1), (5.3) is characterised by the multiple sources of uncertainty, the Poisson processes \( M \) and \( N_m \), \( m \geq 0 \), which make the model incomplete. Moreover, the model has discontinuities of unpredictable type and size. This makes the model essentially incomplete, see Bardhan and Chao. The model generalises the jump-telegraph model studied by Ratanov. See Kolesnik and Ratanov for the detailed presentation.

On the filtered probability space \((\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathbb{P})\), we define the equivalent measure \( \mathbb{P}_\ast \) by means of the following Girsanov transform.

Consider the numerical (nonrandom) sequences \( R_{m}^{(i)}(n), r_{m}^{(i)}(n) \), such that \( R_{m}^{(i)}(n), r_{m}^{(i)}(n) > -1, n \geq 0, i \in \{0, 1\} \). Let

\[ c_{m}^{(i)}(n) = -\lambda_{m}^{(i)} r_{m}^{(i)}(n) - \mu_{m}^{(i)} R_{m}^{(i)}(n), \quad n \geq 0, i \in \{0, 1\}. \]

(5.4)

Define the piecewise linear process \( \mathbb{L}_{\ast} \) with tendencies \( c_{m}^{(i)}(\cdot) \) (as in Section 4) based on process \( M = M(t) \), counting the states' switching, (4.1)- (4.2),

\[ \mathbb{L}_{\ast}(t) = \int_0^t c_{m}^{(i)}(N_{M(t)}(s)) \, ds = \sum_{m=1}^{M(t)} I_{m,m-1}(T_m) + I_{M(t)} (t - T^{(+,M(t)}). \]

Here,

\[ I_{m,m-1}(t) = \int_0^t c_{m}^{(i)}(N_{m}(s)) \, ds, \quad m \geq 0. \]

The jump process \( \mathbb{J}_{\ast} \) is defined by

\[ J_{m}(t) = \sum_{n=1}^{N_m(t)} r_{m}^{(i)}(n), \quad J_{m}(t) = \sum_{m=1}^{M(t)} j_{m,m-1}(T_m) + j_{M(t)} (t - T^{(+,M(t)}), \]

and

\[ J_{m}(t) = \sum_{m=1}^{M(t)} R_{m}^{(i)}(N_{m-1}(T_m)). \]

See (4.3) to (4.4).

All processes are based on the embedded state process \( \epsilon \) and \( \epsilon_m = \epsilon(T^{(+,m)}) \). Because of (5.4) by Theorem 4.2, the sum \( \mathbb{L}_{\ast}(t) + \mathbb{J}_{\ast}(t), \ t \geq 0, \) is a martingale.
Let \( Z(t) = \mathcal{E}_t(l_\Lambda + J_\varepsilon), \ t > 0 \). On the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), \( t \geq 0 \), define the equivalent measure \( \mathbb{P}_\varepsilon \) by the Radon-Nikodym derivative

\[
\frac{d\mathbb{P}_\varepsilon}{d\mathbb{P}}|_t = e^{\varepsilon(t)} \prod_{m=1}^{M(t)} \left[ \left( 1 + R^{(M-1)}_s(N_{m-1}(T_m)) \right) Z_{s,m-1}(T_m) \right] \cdot Z_{s,M(t)} \left( t - T^{+(M(t))} \right), \quad t > 0,
\]

(5.5)

where \( Z_{s,m} \) are defined as (5.2) (with \( r_m^{(\varepsilon)}(n) \) instead of \( r_m(n) \)).

This means that for any \( A \in \mathcal{F}_T \), where \( \mathcal{F}_T \) is the natural filtration determined by \( X \), we have

\[
\mathbb{P}_\varepsilon(A) = \mathbb{E} \left[ Z(t) \mathbf{1}_A \right], \quad t > 0.
\]

(5.6)

It is easy to see that under measure \( \mathbb{P}_\varepsilon \) defined by (5.5), only (unobservable) intensity parameters \( \overline{\lambda}(i) \) and \( \overline{\mu}(i) \) of process \( X \) are changed. In this circumstances (5.5) to (5.6) may be treated as the Esscher transform \( \mathbb{P}_\varepsilon \sim \mathbb{P} \) (see 29).

The following result serves as a version of Cameron-Martin-Girsanov Theorem for this measure transformation.

**Theorem 5.1.** Under measure \( \mathbb{P}_\varepsilon \), which is defined by (5.5) to (5.6), the underlying state process \( \varepsilon \) is governed by independent \( \text{PoExp} (\overline{\lambda}_s, \overline{\mu}_s) \)-distributed inter-switching times, with the alternating parameters

\[
\lambda_{i\varepsilon}^{(i)} = \lambda_i^{(i)} (1 + R_i^{(\varepsilon)})
\]

(5.7)

and

\[
\mu_{i\varepsilon}^{(i)} = \mu_i^{(i)} (1 + R_i^{(\varepsilon)}), \quad i \in \{0, 1\}.
\]

(5.8)

Notice that under measure \( \mathbb{P}_\varepsilon \) the unobservable intensity parameters \( \overline{\lambda}(0) \) and \( \overline{\mu}(0) \) of \( X \) are transformed in agreement with the traditional results, related to simple jump-telegraph process, see also Bardhan and Chao.

**Proof.** The distribution of the first switching time \( T \) under measure \( \mathbb{P}_\varepsilon \) can be determined by the survivor functions

\[
\mathbb{P}_\varepsilon \{ T > t, N(t) = n | \varepsilon(0) = i \} = \mathbb{E} \left( e^{\varepsilon(t)} \prod_{k=1}^{N(t)} \left( 1 + r_i^{(\varepsilon)}(k) \right) \cdot e^{-\varepsilon(t)} \mathbf{1}_{N(t)=n} | \varepsilon(0) = i \right).
\]

By (2.10) and (2.8), it follows that

\[
\mathbb{P}_\varepsilon \{ T > t, N(t) = n | \varepsilon(0) = i \} = \prod_{k=1}^{n} \left( 1 + r_i^{(\varepsilon)}(k) \right) \Lambda_n a_n(t; \lambda_i^{(i)} + \mu_i^{(i)} - c_i^{(i)}).
\]

(5.9)

Here, \( \lambda_i^{(i)} + \mu_i^{(i)} - c_i^{(i)} \) := \( \{ \lambda_n^{(i)} + \mu_n^{(i)} - c_n^{(i)}(n) \}_{n \geq 0} \). Because of (5.4)

\[
\lambda_n^{(i)} + \mu_n^{(i)} - c_n^{(i)}(n) = \lambda_n^{(i)} (1 + r_n^{(i)}(n)) + \mu_n^{(i)} (1 + R_n^{(i)}(n)).
\]

Now, comparing (5.9) with (2.12), we found that under measure \( \mathbb{P}_\varepsilon^* \) the first switching time \( T \) has Poisson-modulated exponential distribution, \( T \sim \text{PoExp}(\overline{\lambda}_s, \overline{\mu}_s) \), with parameters \( \overline{\lambda}_s, \overline{\mu}_s \), which are defined by (5.7) and (5.8).

The joint distribution of the switching times, \( T_1, T^{+(\varepsilon)}_1, \ldots, T^{+(\varepsilon)_m} \), could be represented similarly. For instance, the joint distribution of \( T_1 \) and \( T_1 + T_2 \) under measure \( \mathbb{P} \) by independence of \( T_1 \) and \( T_2 \) is given by

\[
\mathbb{P} \{ T_1 \in ds, T_1 + T_2 > t | \varepsilon(0) = i \} = \mathbb{P} \{ T_1 \in ds | \varepsilon(0) = i \} \cdot \mathbb{P} \{ T_2 > t - s | \varepsilon(0) = 1 - i \}.
\]
By Proposition 2.1, one can obtain

\[
\mathbb{P} \{ T_1 \in ds, T_1 + T_2 > t \mid \epsilon(0) = i \} = \sum_{n=0}^{\infty} \mu_n^{(i)} \lambda_n^{(i)} a_n(s; \lambda^{(i)} + \mu^{(i)}) \sum_{n=0}^{\infty} \lambda_n^{(1-i)} a_n(t-s; \lambda^{(1-i)} + \mu^{(1-i)}) ds,
\]

(5.10)

see (2.12) to (2.13).

On the other hand, under measure \( \mathbb{P}_s \) by definition (5.5) to (5.6), we have for \( s < t \)

\[
\mathbb{P}_s \{ T_1 \in ds, T_1 + T_2 > t \mid \epsilon(0) = i \} = \mathbb{E} \left[ |Z(t)| \mathbb{1}_{\{T_1 \in ds, T_1 + T_2 > t\} \mid \epsilon(0) = i} \right]
\]

\[
= \mathbb{E} \left[ e^{c_s^{(i)}(N_0(s))} \prod_{k=0}^{N_s(s)} (1 + r_s^{(i)}(k)) \prod_{k=0}^{N_s(t-s)} (1 + r_s^{(1-i)}(k)) \mathbb{1}_{\{T_1 \in ds, T_1 + T_2 > t\} \mid \epsilon(0) = i} \right].
\]

(5.11)

Here, \( f_s^{(i)}(s) := \int_0^s c_s^{(i)}(N_0(u)) du \) and \( f_s^{(1-i)}(t-s) := \int_0^{t-s} c_s^{(1-i)}(N_1(u)) du \) are independent.

By applying again Proposition 2.1, we confirm (5.7) to (5.8).

By the fundamental theorem of market modelling the model is arbitrage-free when the discounted stock price \( \tilde{S}(t) := B(t)^{-1} S(t) \) is a martingale under suitable equivalent measure \( \mathbb{P}_s \) (see previous studies\(^{31-34})).

Note that the discounted stock price \( \tilde{S}(t) \) is of the same form as \( S(t) \) (with the alternating trends \( \tilde{c}^{(i)} = c^{(i)} - y^{(i)}, \ i \in \{0, 1\} \)). Hence, without loss of generality one can assume that the interest rates are zeros, \( y^{(i)} = 0, \ i \in \{0, 1\} \).

Meanwhile, Theorem 5.1 permits to change the intensity parameters arbitrarily. Hence, one can reach martingale condition (4.18) by applying the Esscher transform (5.5) only, if values \( c(n), r(n) \) and \( R(n) \) are not of the same sign (for all states and for all \( n \)).

If conditions (4.20) or (4.21) hold, then model (5.3) has arbitrage opportunities.

**Remark 5.1.** Market model (5.3) might be interpreted as follows.

Assume that the interest rates are zeros. Let

\[
S_0 \mathcal{E}_t(\Delta + J) = S_0 e^{\Delta t} \prod_{m=1}^{M(t)} (1 + R_{m-1} (N_{m-1}(T_m))) , \ t \geq 0,
\]

(5.12)

be the essential component of asset price (5.3), which is determined by inherent market forces. This component takes into account only the jumps accompanying the pattern’s switchings.

We assume that in the both states the tendency \( c(n) \) and the accompanying jump amplitude \( R(n) \) are of the opposite signs:

\[
c(n)/R(n) < 0 , \ \forall n.
\]

(5.13)

This assumption seems natural, since the market model with the stock price defined by (5.12) is arbitrage-free if and only if this condition holds, see previous studies\(^{3,23} \).

The jumps defined by \( z_m, (5.2) \), accompanying each tendency fluctuation inside the current pattern, can be considered as the result of external interventions of small markets players. The stock price follows

\[
S(t) = S_0 \mathcal{E}_t(\Delta + J) \times \prod_{m=1}^{M(t)} z_{m-1}(T_m) \cdot z_{M(t)}(t - T^{(M(t))}) , \ t \geq 0.
\]

If condition (5.13) is fulfilled, then the model is still arbitrage-free, in spite of the efforts of small players. The equivalent martingale measure can be provided by the Esscher transform (5.5) with parameters \( c(n), R(n) \), and \( r_s(n) \) satisfying (5.4) in both the states. To define this transform, we first assume that under the martingale measure the modulation intensity \( \lambda_{sn} \) can be determined neglecting the influence of external interventions, such that \( c(n) + \lambda_{sn} R(n) \) is of the same sign with \( c(n) \):

\[
\lambda_{sn} < \left| c(n)/R(n) \right|.
\]
By (5.7) $\lambda_{an} = \lambda_n(1 + r_s(n))$. Hence, the parameters $r_s(n)$ satisfy the condition

$$-1 < r_s(n) < -1 + \lambda_n^{-1} \left| \frac{c(n)}{r(n)} \right|. $$

Then, the Radon-Nikodym derivative (5.5) provides the martingale measure $P_*$, if (see (4.18))

$$c(n) + \lambda_n r(n) + \mu_n R(n) = 0,$$

which by (5.7) to (5.8) gives

$$R_s(n) = -1 - \frac{c(n) + \lambda_n(1 + r_s(n))r(n)}{\mu_n R(n)} , \quad R_s(n) > -1.$$

**Example 2.** Let us consider the model with deterministic jump values $r^{(i)}(n)$ and $R^{(i)}(n)$. The model is arbitrage-free, if for each $n, n = 0, 1, 2, \ldots$ and $i, i \in \{0, 1\}$, the triplets

$$c^{(i)}(n), \quad r^{(i)}(n) \quad \text{and} \quad R^{(i)}(n)$$

are not of the same sign. In this case martingale measures $P_*$ are defined by transform (5.5) with parameters $c_s(n), r_s(n) \quad \text{and} \quad R_s(n)$ satisfying (5.4) and in both the states

$$c(n) + \lambda_n(1 + r_s(n))r(n) + \mu_n(1 + R_s(n))R(n) \equiv 0.$$ 

Let the Poisson modulation in the model be unobservable. This can occur in the case of constant trends and jumps $R$: in both the states

$$c(n) \equiv c, \quad R(n) \equiv R,$$

with modulated hidden parameters $\lambda(n)$ and $\mu(n)$. We assume also that the parameters’ modulation is not accompanied with jumps: $r(n) \equiv 0$.

In this case, the model is arbitrage-free, if $c/R < 0$. The martingale measures are defined by (5.4) with arbitrary $r_s(n) > -1$ and unique $R_s(n)$, defined by

$$R_s(n) = -1 - \frac{c}{R\mu_n}, \quad n \geq 0.$$

### 6 | CONCLUDING REMARKS

In this paper, we introduce a rather new class of double stochastic piecewise linear processes with a Poisson-modulated exponential distributions of persistent epochs.

The first layer of stochasticity is a usual telegraph process based on an alternating Poisson process $N(t)$, and the second one (driving the change of patterns) is characterised by exponentially distributed holding times with a $N(t)$-modulated parameter.

This class of processes is exploited for the purposes of financial modelling. In particular, we study an incomplete financial market model based on a jump-telegraph process. The dynamics of the considered stochastic process is characterised by two alternating types of tendencies, whose holding times have Poisson-modulated exponential distribution. Moreover, the model includes two different kinds of jumps, one occurring at the tendency switchings and the other at the changes of patterns. A relevant aspect of this model is the presence of external shocks, which affect the rates of the trend switchings. This feature ensures that the model is largely flexible, and thus it is suitable to describe a wide family of financial market scenarios. Specifically, it can be used to describe a financial market model whose log-returns are influenced both by market forces and by efforts of speculators. The main results provided for the piecewise linear process with jumps include the determination of integral equations for the conditional means and the martingale conditions. The applications to the financial market model have been presented in detail, including conditions leading to an arbitrage-free model.
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CONFLICT OF INTEREST

There are no conflicts of interest to this work.

ORCID

Nikita Ratanov https://orcid.org/0000-0002-0242-0549

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