Onset of convective instability within an inclined porous layer with a permeable boundary

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Abstract. The investigation of the thresholds that identify the onset of convective instability in a fluid saturated porous layer is performed. The layer is inclined with respect to the horizontal and the boundaries are held at different uniform temperatures. The layer is also semi-permeable: one boundary is impermeable and the other one is permeable. This configuration yields a basic state characterised by a single convective cell, with zero mass flow rate, that fades for small inclination becoming completely motionless for the horizontal case. Since the central part of this cell is considered, the basic flow is parallel and the basic temperature profile is dominated by conduction. This basic state is perturbed employing small-amplitude disturbances such that the linear stability analysis can be performed. A Squire-type transformation is applied to reduce the complexity of the problem. A numerical procedure is employed to obtain the critical values of the governing parameters.

1. Introduction
The buoyancy driven instability within fluid saturated porous layers is a topic that has been driving the attention of different scientific areas for more than fifty years, from applied mathematics to geophysics or oil, civil and industrial engineering. Natural convection is, in fact, a phenomenon of a key importance when industrial processes are taken into account. The presence of convective cells may have, for instance, undesirable effects on several production processes like the extrusion of plastics or metal casting.

The classical buoyancy driven convection problem for clear fluids is the one that Rayleigh and Bénard studied at the beginning of twentieth century [1, 2]. The analogous of the Rayleigh–Bénard instability problem for fluid saturated porous layers is the so called Horton–Rogers–Lapwood (HRL) problem [3, 4]. The HRL problem involves a horizontal and impermeable layer heated from below where the local momentum balance is Darcy’s law. The results obtained for the classical HRL problem together with those obtained by considering different boundary conditions are resumed in Table 6.1 of the book by Nield and Bejan [5].

The equivalent problem, compare to the HRL one, for a vertical fluid saturated porous layer was investigated by Gill [6]. He found that this configuration, a vertical fluid saturated porous layer with impermeable and isothermal boundaries, is always linearly stable. Barletta [7] found that, when permeable boundaries are considered, the vertical layer may undergo a thermal instability. The role of the porous layer inclination is thus relevant when the onset of thermal instability is studied. This role is extensively analysed by Rees and Bassom [8] for a layer
characterised by impermeable and isothermal boundaries. The onset of thermal instability for inclined porous layers has also been studied in [9, 10, 11].

A semi-permeable porous layer tilted with respect to the horizontal is considered: one boundary is permeable, where an isobaric boundary condition is applied, the other boundary is impermeable. The two boundaries are held at two different uniform temperatures. The aim of this study is understanding the behaviour of the threshold for the onset of instability when the fluid saturated porous layer is inclined.

2. Mathematical model

The fluid saturated porous layer considered is characterised by a thickness \( H \) and infinite width in both the directions \( x \) and \( z \). Two different temperatures are imposed at the boundaries: \( T_0 + \Delta T \) for \( y = 0 \) and \( T_0 \) for \( y = H \). The hydrodynamic boundary conditions are asymmetric: \( y = 0 \) is an impermeable plane while \( y = H \) is permeable, held at constant pressure \( p_0 \), as displayed in Fig. 1 in dimensionless terms. The porous layer is inclined an angle \( \alpha \) with respect to the horizontal. Darcy’s law, together with the Oberbeck-Boussinesq approximation, is employed to model the momentum transport. The fluid is incompressible and no source/sink terms are considered in the local energy balance equation. The governing equations can be written in the following dimensionless form

\[
\begin{align*}
\nabla \cdot \mathbf{u} & = 0, \\
\mathbf{u} & = -\nabla p + RT (\sin \alpha \mathbf{e}_x + \cos \alpha \mathbf{e}_y), \\
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T & = \nabla^2 T, \\
y = 0 & : \quad v = 0, \quad T = 1, \\
y = 1 & : \quad p = 0, \quad T = 0,
\end{align*}
\]

where \( \mathbf{x} = (x, y, z) \) is the Cartesian coordinates vector, \( \mathbf{u} = (u, v, w) \) is the velocity vector, \( p \) is the total head, \( T \) is the temperature, \( \mathbf{e}_x \) and \( \mathbf{e}_y \) are the unit vectors for the \( x \) and \( y \)-direction, and \( t \) is the time. The following scaling is employed to obtain the dimensionless form of the governing equations

\[
\begin{align*}
\frac{x}{H} & \rightarrow x, \quad \frac{\chi}{\sigma H^2} t \rightarrow t, \\
\frac{H}{\chi} \mathbf{u} & \rightarrow \mathbf{u}, \quad \frac{K}{\chi \mu} (p - p_0) \rightarrow p, \quad \frac{T - T_0}{\Delta T} \rightarrow T,
\end{align*}
\]
where $\chi$ is the average (between fluid phase and solid phase) thermal diffusivity, $\sigma$ is the heat capacity ratio, $K$ is the permeability of the porous layer, and $\mu$ is the dynamic viscosity of the fluid. The Darcy–Rayleigh dimensionless number $R$ in Eq. (1) is defined as

$$ R = \frac{\rho g \beta K H \Delta T}{\chi \mu}, $$

(3)

where $\rho$ is the fluid density, $g$ is the modulus of gravitational acceleration, and $\beta$ is the thermal expansion coefficient of the fluid.

3. Basic state

The stationary basic solution of Eq. (1) employed here for the stability analysis is one characterised by a parallel flow with zero neat mass flow rate

$$ u_B = \left( \frac{R}{2} (1 - 2y) \sin \alpha, 0, 0 \right), \quad \nabla p_B = \left( \frac{R}{2} \sin \alpha, R \left(1 - y\right) \cos \alpha, 0 \right), \quad T_B = 1 - y, $$

(4)

where the subscript “B” denotes the basic state fields. The velocity profile in Eq. (4) describes a single cell of infinite length while the temperature profile describes a pure conduction regime. This single cell arises when the layer has a non negligible inclination with respect to the horizontal, $\alpha > 0$. The horizontal case is, in fact, motionless. The flow configuration presented in Eq. (4) can be produced by considering a porous layer of aspect ratio, width to height, sufficiently high to obtain a shallow layer. The parallel flow in Eq. (4) describes that portion of the porous layer sufficiently far from the $x$–direction boundaries.

4. Linear stability analysis

A pressure–temperature formulation is employed by applying the divergence to the local momentum balance equation, namely

$$ \nabla \cdot \left( \nabla T \right) = \nabla \cdot \left( \nabla T \right), \quad \nabla \cdot \left( \nabla T \right) = \nabla \cdot \left( \nabla T \right), \quad \nabla \cdot \left( \nabla T \right) = \nabla \cdot \left( \nabla T \right), $$

(5)

where the $y$ component of Darcy’s law is employed to redefine the impermeability condition in $y = 0$ in terms of pressure and temperature.

The analysis of which configuration is stable/unstable with respect to the onset of thermal convection can be tackled by perturbing the basic state in Eq. (5). The pressure and the temperature fields are redefined as composed by a basic state plus a perturbation, namely

$$ p = p_B + \varepsilon P, \quad T = T_B + \varepsilon \Theta, $$

(6)
Figure 2. The inclination of the generic convective roll to the reference frame.

where the parameter $\varepsilon$ is arbitrarily small such that, after substituting Eq. (6) to Eq. (5), one can consider only terms $O(\varepsilon)$ to obtain the linearised equations.

$$\nabla^2 P - R \left( \frac{\partial}{\partial x} \sin \alpha + \frac{\partial}{\partial y} \cos \alpha \right) = 0,$$

$$\frac{\partial \Theta}{\partial t} + R \left( 1 - 2y \right) \frac{\partial}{\partial x} \sin \alpha - R \Theta \cos \alpha + \frac{\partial P}{\partial y} = \nabla^2 \Theta,$$

$$y = 0 : \quad \frac{\partial P}{\partial y} = 0, \quad \Theta = 0,$$

$$y = 1 : \quad P = 0, \quad \Theta = 0.$$

The flow pattern of thermal convection induced by buoyancy forces can be reasonably approximated to be roll shaped. One may thus assume the perturbations to have the form of plane waves, namely

$$\{ P(x, y, t), \Theta(x, y, t) \} = \{ f(y), h(y) \} e^{\eta t} e^{i(k_x x + k_z z - \omega t)},$$

where $\eta$ is the growth rate of the perturbation, $k_x$ is the wavenumber in the $x$ direction, $k_z$ is the wavenumber in the $z$ direction, and $\omega$ is the angular frequency. By substituting Eq. (8) into Eq. (7) we obtain two ordinary differential equations, namely

$$f'' - k_x^2 f - R h' \cos \alpha - i k_x R h \sin \alpha = 0,$$

$$h'' - \left[ k_x^2 + i k_z R (1 - 2y) \sin \alpha - R \cos \alpha - i \omega \right] h - f' = 0,$$

$$y = 0 : \quad f' = 0, \quad h = 0,$$

$$y = 1 : \quad f = 0, \quad h = 0,$$
where the primes denote the derivatives with respect to \( y \) and \( k^2 = k_x^2 + k_z^2 \). Since we want to investigate the neutrally stable configuration, \( i.e. \) the normal modes with a null growth rate, we have set \( \eta = 0 \). The rolls wavevector \( k \) lies on the plane \((x, z)\), as displayed in Fig. 2, and, in general, it is inclined with respect to the \( x \) direction.

One may now apply a Squire–type transformation to Eq. (9) to reduce the number of parameters, namely

\[
S \cos \phi = R \cos \alpha, \quad \tan \phi = \frac{k_x}{k} \tan \alpha.
\]

The simplified eigenvalue problem is thus

\[
f'' - k^2 f - S' \cos \phi - i k S h \sin \phi = 0, \\
h'' - \left[ k^2 + \frac{i}{2} k S (1 - 2y) \sin \phi - S \cos \phi - i \omega \right] h - f' = 0,
\]

\[
y = 0 : \quad f' = 0, \quad h = 0, \\
y = 1 : \quad f = 0, \quad h = 0.
\]

The transformation in Eq. (10) yields an eigenvalue problem where the relative inclinations between basic flow direction and wavevector of the perturbations are compressed within the definition of the modified angle \( \phi \) and the modified Darcy–Rayleigh number \( S \).

The eigenvalue problem (11) cannot be solved analytically because of the explicit presence of the independent variable \( y \) thus it has to be solved numerically as reported in the next section.

5. Numerical method

The numerical procedure employed to solve the eigenvalue problem (11) is based on the Runge-Kutta method coupled with the shooting method. The Runge-Kutta method is employed to solve the following initial value problem

\[
f'' - k^2 f - S h' \cos \phi - i k S h \sin \phi = 0, \\
h'' - \left[ k^2 + \frac{i}{2} k S (1 - 2y) \sin \phi - S \cos \phi - i \omega \right] h - f' = 0, \\
y = 0 : \quad f' = 0, \quad h = 0, \\
y = 1 : \quad f = 0, \quad h = 0.
\]

In Eq. (12), two initial conditions are added to obtain a well-posed initial value problem. The eigenvalue problem (12) is homogeneous thus the initial condition \( f(0) = 1 \) is imposed to gauge the scale of the eigenfunctions \( f \) and \( h \) that are defined up to an arbitrary constant. The last initial condition \( h' = \xi_1 + i \xi_2 \) is fixed by introducing two unknown parameters \( \xi_1 \) and \( \xi_2 \).

The Runge-Kutta method is employed to obtain the complex valued eigenfunctions \((f, h)\) which, in fact, depend on six real parameters \( (k, \omega, S, \phi, \xi_1, \xi_2) \). The shooting method evaluates \((f, h)\) for the four target conditions

\[
y = 1 : \quad \Re\{f'\} = 0, \quad \Im\{f'\} = 0, \quad \Re\{h\} = 0, \quad \Im\{h\} = 0,
\]

obtaining a system of four equations in the six parameters \( (k, \omega, S, \phi, \xi_1, \xi_2) \). One may thus impose two of these six parameters, \( (k, \phi) \) for instance, to obtain the remaining four unknown parameters \( (S, \omega, \xi_1, \xi_2) \).

As a consequence, one can draw a neutral stability curve \( S(k) \) for each given value of \( \phi \), as reported in Fig. 3. The minimum of each neutral stability curve defines the threshold condition
6. Neutral stability curves and critical values

The neutral stability curves displayed in Fig. 3 are drawn for different values of the modified angle $\phi$ expressed in radians. The lowest curve is drawn for $\phi = 0$ while the highest curve, the small parabola shaped continuous line, is drawn for $\phi = 1.16$ rad. For angles higher than the threshold value $\phi = 1.10813$ rad, the curves split in two branches. The lower branch of solutions is characterised by closed curves, “islands”, and by values of $S$ that are lower compared to the other branch. These islands shrink in size for increasing values of $\phi$ until they are reduced to a point for $\phi = 1.15063$ rad and then disappear completely for $\phi > 1.15063$ rad. The first curves relative to higher branch of solutions are dashed because they are not relevant for the stability analysis since their minima are higher with respect to the minima of the lower branch of solutions obtained for the same value of $\phi$. The higher branch of solutions becomes relevant when, for $\phi > 1.15063$ rad, the lower branch of solution does not exist anymore. This is the reason why the highest curve, drawn for $\phi = 1.16$ rad, is continuous and not dashed.

The angular frequency $\omega$ relative to both branches is different from zero, except for the horizontal case. In the horizontal case one may obtain from Eq. (10) that $\phi = 0$. One can
conclude that the instability is triggered by travelling normal modes except for the horizontal case.

The minima of the neutral stability curves define the critical values of the governing parameters that identify the threshold for the onset of instability. By employing the procedure presented in Section 5, one obtains a pair \((k_c, S_c)\) for each prescribed value of \(\phi\). A number of critical values are reported in Table 1 for different values of \(\phi\): the discontinuity for the value \(\phi = 1.15063\) rad is due to fact that for this angle the lowest branch disappears. For \(\phi > 1.15063\) rad the focus of the stability analysis jumps on the higher branch: for these values of \(\phi\) this branch becomes the one with the lowest possible values of \(S_c\). The critical values relative to the second branch keep increasing indefinitely with \(\phi\).

It is worth noting how the shapes of the perturbations change along with the angle \(\phi\). Figure 3 displays the lines at constant pressure \(P(x, y, 0) = \text{constant}\), left column, and the lines at constant temperature \(\Theta(x, y, 0) = \text{constant}\), right column, when the critical configurations obtained for different values of \(\phi\) are considered. Each frame of Fig. 4 presents a whole perturbation period \(2\pi/k\). The rolls shape is strongly dependent on the angle \(\phi\).

### Table 1. Critical values of \(S\) and \(k\) for different angles \(\phi\) expressed in radian.

| \(\phi\) | \(k_c\)   | \(S_c\)   |
|---------|----------|----------|
| 0       | 2.32621  | 27.0976  |
| 0.2     | 2.31021  | 27.7920  |
| 0.4     | 2.25794  | 30.0889  |
| 0.6     | 2.15446  | 34.8222  |
| 0.8     | 1.96605  | 44.4307  |
| 1.0     | 1.61979  | 68.1262  |
| 1.05    | 1.49040  | 80.4107  |
| 1.11    | 1.28671  | 106.347  |
| 1.15063 | 0.971417 | 175.350  |
| 1.15063 | 1.97285  | 241.505  |
| 1.16    | 1.98839  | 266.619  |
| 1.17    | 2.03109  | 302.845  |
| 1.18    | 2.11425  | 354.329  |
| 1.19    | 2.24545  | 432.855  |
| 1.2     | 2.47976  | 578.843  |
| 1.205   | 2.71997  | 735.778  |

### 7. Conclusions

A fluid saturated porous layer has been studied with respect to the onset of thermal convection. The layer is tilted to the horizontal direction and it is characterised by isothermal boundaries. Moreover, one boundary is permeable and the other one is impermeable. The linear stability analysis is performed by employing the normal modes method. A reference frame rotation plus a Squire-type transformation are applied to simplify the problem. The Squire-type transformation defines a modified Darcy-Rayleigh number \(S\) and a modified angle \(\phi\). The eigenvalue problem obtained is solved numerically. The most relevant results are reported here:

- The modified angle \(\phi\) has a stabilising influence on the fluid saturated porous layer;
Figure 4. Perturbation pressure isolines, left column, and perturbation temperature isolines, right column, for different values of $\phi$. 
• The lowest values of $S_c = 27.0976$ and is obtained for $\phi = 0$. The corresponding values of $k_c = 2.32621$;
• The values of $S_c$ increase indefinitely for $\phi \to \pi/2$;
• The instability is always activated by travelling modes, except for the case $\phi = 0$;
• The shape of the convective rolls are strongly influenced by the modified angle $\phi$.

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