Quantum state transfer with ultracold atoms in optical lattices

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Abstract. Ultracold atoms can be used to perform quantum simulations of a variety of condensed matter systems, including spin systems. These progresses point to the implementation of the manipulation of quantum states and to observe and exploit the effect of quantum correlations. A natural direction along this line is provided by the possibility to perform quantum state transfer (QST). After presenting a brief discussion of the simulation of quantum spin chains with ultracold gases and reminding the basic facts of QST, we discuss how to potentially use the tools of present-day ultracold technology to implement the QST between two regions of the atomic system (the sender and the receiver). The fidelity and the typical timescale of the QST are discussed, together with possible limitations and applications of the presented results.

1. Introduction

An important direction of research in the physics of cold atoms and molecules, achieved in the last decade, is provided by the possibility to prepare in the laboratory artificial systems able to mimic more complex condensed matter systems [1]. As discussed in the papers of the present JPB Special Issue on the implementation of quantum many-body problems with cold atoms and molecules, the present technology has made ultracold
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systems extremely accessible with the possibility to address, even locally in limited and controllable regions of space, most of the system parameters. The combined action of trapping potentials, optical lattices and low temperature cooling techniques, provide a sort of playground to design simulators of a huge class of models. In particular a key ingredient in most of the quantum simulations performed with ultracold trapped gases is the ability to control the geometry of the system and the capability to tailor the one-body external potential felt by the constituents of the gas.

These progresses paved the way from one side to the study of strongly correlated bosonic and fermionic states, both in the continuous space and in the presence of optical lattices [2, 3], and from the other to the control and use of quantum correlations [1]. A typical and important example in the latter direction is provided by atom interferometry with ultracold atoms [4]: it has been experimentally showed that using entangled states one can have a sensitivity of the atom interferometer below the shot-noise limit [5, 6] and the detection of Bell correlations between the spins of ≈ 500 atoms detected in a Bose-Einstein condensate [7] with possible and obvious consequences for the implementation of quantum cryptography algorithms [8].

Moving forward in the direction of the manipulation and use of quantum states, in this paper we address the possibility to perform Quantum State Transfer (QST) in ultracold systems. QST has been deeply investigated in quantum registers and more generally in spin chains: given the fact that ultracold gases can be used to perform a physical emulation of spin systems, a natural direction is to investigate the implementation of possible QST protocols with cold atoms. Two ingredients are instrumental in our study: first, one should use 1D optical lattices and realize an effective low-energy spin Hamiltonian [9]. Moreover, one should be able to identify two regions of sites (say made up by one or two sites each) and control the tunneling from the two regions to the remaining part of the systems. The two regions are typically referred as the sender sites and the receiver sites. The typical lattice unit is in the range 0.5 – 5µm [10]: one may use optical barriers, which has to be of ≈ 1 – 2µm dimension. Controlling such localized potentials, which is experimentally challenging, one may control the tunneling in the direction between the senders and the receivers and the tunneling in the other directions (that we argue to be better negligible). Such control is now within reach, as demonstrated by the recent experiment [11] in which an optical barrier with an 1/e² beam waist at the center of 2.0 ± 0.2µm was used to weakly couple two fermionic gases of ⁶Li atoms, and stable enough to observe Josephson oscillations. Finally, one should have the possibility to monitor the state of the senders and receivers, which requires the use of the techniques of quantum gas microscopy [12].

In this paper we focus our attention on systems composed of 1D spin chains to be employed as a channel for QST. The structure of the paper is the following: After a brief introduction of the possible ways we have nowadays to simulate spin lattices by ultracold atoms, we give a review on the recent progresses obtained regarding the QST of one or two qubits through a spin chain. One of the main ingredient of these QST schemes is the partial decoupling of the sender/receiver sites from the rest of the chain
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by strong local magnetic barriers \[13\, 14\, 15\, 16\, 17\]. In the perspective of a possible implementation with ultracold atoms, one possible strategy is to avoid local on-site energies. To this aim, we propose an alternative scheme for the two-qubit transfer that does not require any local barriers/on-site energies, exploiting a weak coupling instead. We find that an efficient transfer of the fidelity is still achievable. Moreover, numerical data show that the transfer times are comparable with the one obtained within a similar scheme for the single qubit transfer. This result could suggest the remarkable fact that the transfer rate of a many-qubit could be only weakly dependent on the number of qubit and motivates further analytical studies on this purpose.

2. Quantum simulation of spin chains

The study of magnetic systems is a very active field of research in solid state physics \[18\] and correspondingly it developed the search of synthetic, controllable physical systems having effective magnetic model Hamiltonians with tunable geometry and parameters. Ultracold atomic setups provide a promising possibility to this aim: effective nearest-neighbour interactions for atoms in neighbor wells of an optical lattice may result from super-exchange couplings and second-order tunneling was been observed in superlattices \[19\]. Classical frustrated magnetism in triangular lattices has been simulated with the suitable shaking of optical lattices \[20\].

A very well studied path to simulate spin chains is to use two-component gases: here the two internal degrees of freedom emulate the components of the spins. Since spin interactions can be tuned rather precisely \[11\] one has an important knob to control the parameters of the spin Hamiltonian. Therefore, the realization of controllable Bose-Bose mixtures \[21\] paves the way towards the experimental simulation of spin Hamiltonians.

Another strategy to simulate antiferromagnetic spin chains was discussed in \[22\]. The implementation of an Ising chain in a transverse field is based on the realization of a Mott state in a small applied potential gradient, which exhibits a resonant response when the potential energy drop per lattice spacing is close to the interatomic interaction energy between two atoms. The presence of a small tunneling coupling between neighboring wells turns out to produce an effective Ising Hamiltonian \[22\]. Following the latter suggestion and using a tilted 1D optical lattice, the Ising chain in a transverse field was experimentally implemented \[9\]. The detection of the paramagnetic and the antiferromagnetic phases was carried out by measuring the probability to have an odd occupation of sites \[9\]. Vibrational mode of trapped ions can also be used to simulate interacting spin systems \[23\, 24\, 25\, 26\, 27\].

A simple way to understand how a spin chain Hamiltonian is obtained can be seen by considering ultracold bosons in an optical lattice at half-filling \[28\]: let us consider ultracold bosons in deep optical lattices as described by the Bose-Hubbard Hamiltonian \[2\]

\[
H_{BH} = -t \sum_{i=1}^{N-1} \left( b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i \right) + \frac{U}{2} \sum_{i=1}^{N} n_i (n_i - 1) + V \sum_{i=1}^{N-1} n_i n_{i+1}.
\]
In Eq. (1) $b_i$ is the boson operator in the site $i$ (with $i = 1, \cdots, N$ and $N$ number of sites) and, as usual, the parameter $t$ denotes the hopping strength, and $U$ and $V$ respectively the interaction energy of two particles at the same site and at two nearest neighboring sites [29]. Open boundary conditions are used in Eq. (1).

Denoting the filling by $f$, if $f$ is half-integer (e.g., $f = 1/2$) one can define

$$s_j^z \equiv n_j - f,$$

the eigenvalues of $s_j^z$ are $\pm \frac{1}{2}$. We will later use the notation $s_j^\alpha \equiv (1/2)\hat{\sigma}_i^\alpha$ where $\alpha = x, y, z$ and $\hat{\sigma}_i^\alpha$ are the Pauli matrices. For $t = 0$ the energy per particle is (for $N \to \infty$) $\varepsilon = U f(f - 1)/2 + V f^2$ and therefore one gets, in the limit $U \to \infty$, the effective Hamiltonian

$$H_{XXZ} = -K \sum_{\langle i,j \rangle} (s_i^x s_j^x + s_i^y s_j^y - \Delta s_i^z s_j^z),$$

where

$$K\Delta \equiv V.$$

and

$$K \equiv 2t \left(f + \frac{1}{2}\right).$$

The Glazek-Wilson renormalization group procedure [30] (see as well [31, 32]) provides the XXZ Hamiltonian [1] with renormalized coefficients $K$ and $\Delta$ [28], which are in very good agreement with Bose-Hubbard DMRG numerical results also for $U/t \approx 3$ and for small number of sites [28].

These results show that spin chain Hamiltonians emerge naturally for ultracold atoms in optical lattices: magnetic fields in the $z$-direction are obtained by adding on-site energies [10], while to have transverse fields one may resort to schemes like the one discussed in [22]. Finally, we observe that one can have an XXZ limit by tuning the parameter $\Delta$ to zero and that in general a wide class of spin chain models may be obtained. In the following, we consider mainly an XX spin chain, but our results, with their own peculiarities and modulo analytical and computational difficulties, can be extended to XXZ and Ising in a transverse field models [33].

3. Quantum state transfer

In this Section we give a short overview of the class of QST protocols without dynamical control over the Hamiltonian and/or post-measurement selection procedures. In general, the goal of a QST protocol is to transfer the information encoded in a quantum state of a system from a sender to a receiver, which are located at different spatial positions. The capability of faithfully transferring the quantum information between different locations by means of a quantum channel plays a key role in many quantum information processing tasks, ranging from quantum teleportation to quantum computation and cryptography.
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The elementary unit of quantum information is referred to as qubit and consists of two orthogonal quantum states, conventionally labelled by $|0\rangle$ and $|1\rangle$, which can belong to a great variety of systems where the two-level approximation holds. In the following, we will refer to $n$-QST ($n = 1, 2, ...$) depending on the number of qubits the protocol aims to transfer. Whereas for long-distances 1-QST has been achieved by means of photons \cite{34}, for short-haul transfers it would be more advisable to rely on solid-state implementation, in order to avoid, e.g., information loss at interfaces between, say, a solid-state based quantum processor and a photonic quantum channel.

Since the seminal proposal by Bose \cite{35} to use nearest-neighbor coupled spin-$\frac{1}{2}$ systems as quantum channels for 1-QST, a great amount of work has focused on protocols enhancing the quality of 1-QST. Indeed, in the presence of uniform couplings, the quality of 1-QST falls below the threshold achievable by means of only local operations and classical communication (LOCC) already for spin chains made up of a hundred of spins (for a review see, e.g., \cite{36, 37, 38}).

Among the protocols without dynamical control, it has been shown that perfect length-independent 1-QST can be obtained by engineering the spin-spin couplings so as to induce a linear dispersion relation \cite{39, 40, 41, 42}, which yields a ballistic 1-QST entailing that the time necessary to complete the protocol is proportional to the chain length. This result has been recently extended also including next-to-nearest neighbour couplings \cite{43}. A reliable local modulation involving the entire chain, however, would face several practical difficulties on the experimental side. Ballistic 1-QST can also be achieved under appropriate tuning of the outermost couplings \cite{44, 45}.

A different approach relies on the weak interaction of the sender and receiver spins with a bulk embodied by a uniform chain \cite{46, 47, 48, 49}. Schemes of this kind exploit the appearance of a pair of Hamiltonian eigenstates strongly bi-localized at the outermost weakly-coupled sites, which brings about an effective Rabi-like dynamics \cite{46}. A similar dynamics can be triggered by applying strong magnetic fields on the sender and receiver qubits or on their nearest neighbors \cite{13, 14, 15}. At variance with ballistic QST protocols, a usual drawback of the Rabi-like mechanisms is that they typically require long 1-QST times. To this end, in order to reduce the transfer time, proposals superimposing a staggered interaction pattern of the couplings \cite{50} and a modular partitioning of the staggered quantum channel \cite{51} have been put forward.

Recently, protocols achieving 2-QST have been investigated and it has been found that both in the Rabi-like regime, achieved by applying strong magnetic fields on the first and last spin \cite{16, 17} and, for the ballistic regime, by engineering all of the inter-spin couplings \cite{52}, high-quality 2-QST is achievable.

In this paper we add an original contribution to this topic along this line, by showing that the Rabi-like mechanism yielding high-quality 2-QST can be performed also by adopting the weak coupling protocol. In the following subsection we will briefly illustrate the basic mechanism of the 1- and 2-QST protocols.
3.1. QST in a nutshell

Following [35], where the 1-QST has been first introduced and here generalised to n-QST, the sender prepares an arbitrary state of \( n \) qubits, \(|\phi\rangle_S\) and sets the rest of the chain and of the receivers in the fully polarised state \(|0\rangle_{QC}\) (see Fig. 1). The initial state of the sender block, the quantum channel, and the receiver block thus reads \(|\Psi(0)\rangle = |\phi\rangle_S |0\rangle_{QC} |0\rangle_R\). The system then evolves according to its Hamiltonian \( \hat{H} \) so that, at time \( t \), its state is given by \(|\Psi(t)\rangle = \hat{U}(t) |\Psi(0)\rangle \) with \( \hat{U}(t) = e^{-i\hat{H}t} \). The goal is to exploit such natural dynamics in order to transfer the initial sender’s state \(|\phi\rangle_S\) to the qubits embodying the receiver in a given time \( \tau \), meaning that ideally one aims at obtaining \(|\Psi(\tau)\rangle = |0\rangle_S |0\rangle_{QC} |\phi\rangle_R\) in, possibly, short times. The state of the receiver is evaluated by tracing out the remaining spins, i.e., \( \rho_R(\tau) = \text{Tr}_{S, QC} |\Psi(\tau)\rangle \langle \Psi(\tau)| \), and it is generally a mixed state. One thus aims at making the QST fidelity \( F_\phi(\tau) = \langle \phi| \rho_R(\tau) |\phi\rangle_S \) as large as possible (the fidelity \( F_\phi \) measures how close is the receiver’s state to \(|\phi\rangle_S\)). The fidelity introduced above depends on the specific input \(|\phi\rangle_S\). In order to end up with a state-independent figure of merit for QST protocols, one averages \( F_\phi \) over all possible input states, obtaining thus the average fidelity \( \bar{F}(t) \).

For Hamiltonians conserving the total number of excitations one can obtain simple expressions for the 1- and 2-QST average fidelity, as \(|\Psi(0)\rangle\) is restricted to evolve in the zero- and one-excitation subspaces for 1-QST, whereas for 2-QST also the two-excitation sector takes part in the dynamics. The 1-QST average fidelity, for a \( N \)-site spin system, is simply given by [35]

\[
\bar{F}(t) = \frac{1}{2} + \frac{|f(t)|}{3} + \frac{|f(t)|^2}{6},
\]

where

\[
f(t) = \langle N | e^{-iH_{ch}t} |1\rangle
\]

is the excitation transition amplitude from the first to the last spin (we used the compact notation, referred to as computational basis, \(|n\rangle \equiv |0\rangle_1 \cdots |1\rangle_n \cdots |0\rangle_N\) indicating that only the spin residing on site \( n \) is flipped to \(|1\rangle\)). Note that \( |f(\tau)| = 1 \) entails \( \mathcal{F}(\tau) = 1 \) (perfect QST). Also, the average fidelity is a monotonic function of the transition amplitude and hence the QST performance can be evaluated by just tracking down the excitation transport across the array. For the 2-QST average fidelity, on the other
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hand, one obtains

\[ F(t) = \frac{1}{4} + \frac{5}{54} \text{Re} \left[ f_{s_1}^r + f_{s_2}^r + \frac{7}{5} f_{s_2}^r (f_{s_1}^r)^* + (f_{s_1}^r + f_{s_2}^r) (g_{s_1 s_2}^r)^* \right] + \frac{1}{54} (|f_{s_2}^r|^2 + |f_{s_1}^r|^2)
+ \frac{5}{108} (|f_{s_2}^r|^2 + |f_{s_1}^r|^2) + \frac{1}{36} |g_{s_1 s_2}^r|^2 + \frac{7}{54} \text{Re} \left[ g_{s_1 s_2}^r \right] - \frac{1}{54} \sum_{n=1}^{n \in \mathbb{R}} \left( |g_{s_1 s_2}^{nr_1}|^2 + |g_{s_1 s_2}^{nr_2}|^2 \right)
- \frac{1}{27} \sum_{n=1}^{n \in \mathbb{R}} \text{Re} \left[ (f_{s_2}^n)^* g_{s_1 s_2}^{nr_1} + (f_{s_1}^n)^* g_{s_1 s_2}^{nr_2} \right], \tag{7} \]

where \( f_{n}^{m} = \langle m | e^{-itH_1} | n \rangle \) and \( g_{rs}^{nm} = \langle rs | e^{-itH_2} | nm \rangle \) are the single- and two-particle transfer amplitudes from sites \( n \to m \) and \( \{nm\} \to \{rs\} \), respectively \([16, 17]\). Finally, if the sender’s state is encoded in a qutrit (a system with three levels), the average fidelity is reported in \([53]\).

4. QST in the XX model

Motivated by the quantum simulation of spin models with ultracold atoms as described in Section 2, we focus here on the paradigmatic and simpler case of an XX Hamiltonian.

To uniform with the standard notation used on the QST literature, we write the XX model of a 1D spin-\( \frac{1}{2} \) chain with open boundary conditions as

\[ \hat{H}(t) = -\sum_{i=1}^{N-1} J_i \left( \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y \right) + \sum_{i=1}^{N} h_i \hat{\sigma}_i^z, \tag{8} \]

where \( \hat{\sigma}^{\alpha} (\alpha = x, y, z) \) are the Pauli matrices. The Hamiltonian described by Eq. (8) conserves the total magnetisation in the \( z \)-direction and hence the formulas reported in the previous Section apply. We observe that in typical experimental realizations one has an external trapping potential (see Fig. 1), whose effect is of course to increase the density of the center of the trap. However, hard wall potentials have been realized \([54]\) and in general the presence of non-shallow boundary conditions is of help for the efficiency of QST. In the following we choose the sender and receiver sites at the end of the systems: this helps as well the QST efficiency, and can be obtained by hard-wall (open) boundary conditions or by the use of strong optical barriers, respectively at the left (right) of the sender (receiver) sites, as clear from Fig. 1. As we commented in Section 4 this requires the capability of controlling laser beams on spatial distances of order of few microns. A similar control is needed to vary the coefficients \( J_i \) at the right (left) of the sender (receiver) sites, i.e., in the direction joining sender and receivers sites, which in turn appears very useful in increasing the QST efficiency.

4.1. Rabi-like 1-QST

In the single-excitation sector, the spectral decomposition of Eq. (8) reads

\[ \hat{H} = \sum_{k=1}^{N} \varepsilon_k |\varepsilon_k\rangle \langle \varepsilon_k|, \]

where \( \varepsilon_k \) is the \( k \)-th energy eigenvalue with corresponding eigenstate
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\[ |\varepsilon_k\rangle = \sum_{n=1}^{N} a_{nk} |n\rangle \]
written in the computational basis. In this representation, the transition amplitude in Eq. (6) is given by

\[ f(t) = \frac{N}{\sum_{k=1}^{N}} e^{-i\varepsilon_k t} a^*_{kN} a_{k1} = \sum_{k=1}^{N} e^{-i\varepsilon_k t} \langle \varepsilon_k | N \rangle \langle 1 | \varepsilon_k \rangle. \]  \hspace{1cm} (9)

The last identity shows that each eigenstate contributes to Eq. (9) through the quantity \[ \langle \varepsilon_k | N \rangle \langle 1 | \varepsilon_k \rangle, \] evolving in time at rate \( \varepsilon_k \).

Various high-quality QST schemes [13, 55, 14, 46, 56] rely on the situation where the edge states \( |1\rangle \) and \( |N\rangle \) have a strong overlap with only two stationary states, say those indexed by \( j = 1, 2 \) (bi-localization). In this case, Eq. (9) can be approximated as

\[ f(t) \simeq e^{-i\frac{\delta \omega t}{2}} a^*_{1N} a_{11} + e^{-i\frac{\delta \omega t}{2}} a^*_{2N} a_{21}, \]  \hspace{1cm} (10)

with \( \delta \omega = \varepsilon_1 - \varepsilon_2 \) (we assumed \( \varepsilon_1 > \varepsilon_2 \)). This implies a Rabi-like dynamics that occurs with a characteristic Rabi frequency given by \( \delta \omega \). Accordingly, \( \tau \sim \delta \omega^{-1} \) showing that the order of magnitude of the transfer time is set by the energy gap between the two bi-localized eigenstates.

The above bi-localization effect is usually achieved by introducing perturbation terms in the Hamiltonian that decouple the outermost spins from the bulk. This can be realized through: (i) application of strong local magnetic fields on the edge spins \( n=1,N \) [13, 55], or (ii) on their nearest-neighbours \( n=2,N-1 \) [14], and (iii) engineering of weak couplings between the edge spins and bulk \( J_1=J_{N-1} \) [46, 56]. While all these models share that a pair of Hamiltonian eigenstates exhibit strong bi-localization on the edge sites, the typical energy gap between such two states – and accordingly the transfer time \( \tau \) – depends on the considered protocol. Calling \( \xi \) the model-dependent perturbation parameter (such as the local magnetic field strength), in (i) the time scales with \( N \) as \( \tau \sim \xi^N \), resulting in a QST time that exponentially increases with the array length, whereas in (ii) and (iii) the time scales as \( O(\xi^2) \) and \( O(\xi^{-2}) \), respectively.

4.2. Rabi-like 2-QST

If the QST protocol aims at transferring the quantum information encoded in two qubits, we need to use Eq. (11) for the average fidelity which involves also transition amplitudes in the two-particle sector. A considerable simplification can be achieved by taking into account that the Hamiltonian in Eq. (8) maps into a non-interacting spinless fermion model via the Jordan-Wigner transformation and, hence, its dynamics is fully described by the single-particle basis. This allows to reduce the two-particle transition amplitudes \( g_{nm}^{pq}(t) \) to determinants of matrices whose elements are single-particle transition amplitudes \( f_{ij}^l(t) \), where \( i = \{n, m\} \) and \( j = \{p, q\} \), e.g., [52]:

\[ g_{nm}^{pq}(t) = \begin{vmatrix} f_n^p(t) & f_n^q(t) \\ f_m^p(t) & f_m^q(t) \end{vmatrix}. \]  \hspace{1cm} (11)
In order to achieve an high-fidelity 2-QST via Rabi-like dynamics between the sender and the receiver block, each composed of two qubits, now we should look for eigenstates that are quadri-localized on the first and last two sites. One way to accomplish it has been reported in [16, 17] by means of strong magnetic fields on sites \( n=3 \) and \( n=N-2 \). In the following, we investigate if also weakly coupling the sender and the receiver block to the quantum channel allows for high-quality 2-QST, and which are the conditions to be fulfilled for such a result.

We consider the Hamiltonian (8) with all the couplings uniform but \( J_2=J_{N-2}=J_0 \ll J \equiv 1 \), that is, the sender and receiver block are weakly coupled to the quantum channel. As we commented in Sec. 2, this requires in ultracold systems the possibility to increase the energy barrier between on the right (left) of the sender (receiver) blocks, e.g. with suitably controlled optical barriers. This scheme is expected to be more easily implemented, since the hoppings are be sensibly depending on local applied optical barriers rather than the on-site energies.

The 2-QST properties of the model described by Eq. 8 depend strongly on the number of spins \( N \) of the chain, which, indeed, determines the spectral properties of the energy eigenvalues.

By exploiting perturbation theory, justified by the fact that we set \( J_0 \ll 1 \), we can use the result in Ref. [17] for the average fidelity obtaining

\[
\bar{F}_a(t) = \frac{1}{4} + \frac{10}{54} \text{Re} \left[ f_1^{N-1} \right] + \frac{7}{54} \text{Re} \left[ \left( f_1^{N-1} \right)^2 \right] + \frac{12}{54} |f_1^{N-1}|^2 + \frac{2}{54} |f_1^N|^2 \\
+ \frac{10}{54} |f_1^{N-1}|^2 \text{Re} \left[ f_1^{N-1} \right] - \frac{10}{54} \text{Re} \left[ f_1^{N-1} f_1^N f_2^{N-1} \right] - \frac{7}{54} \text{Re} \left[ f_1^N f_2^{N-1} \right] .
\]

Let us consider the case \( h_i=0 \), for all \( n \), in Eq. (8). For odd \( N \), we always have a zero-energy level and, for \( N=6n \pm 1 \) and \( N=3(2n+1) \ (n=1,2,\ldots) \) there are also two sets of double- and triple-degenerate energy levels at \( E \simeq \pm 2J \), respectively. In such cases the 2-QST is less efficient than the classical transfer, i.e., the fidelity attainable by means of LOCC, which reads \( \bar{F}_{LOCC}=2/5 \) \[57\], and we will not deal with such a case here. For even \( N \), the zero energy level is no longer present and only the two sets of double- and triple-degenerate states remain, respectively for \( N=2(3n+1) \vee 2(3n+2) \) and \( N=6n \). An instance of the 2-QST performance for even \( N \) is shown in Fig. 2 where it is evident that, although both even-\( N \) classes reach high-quality transfer, the faster one is given by lengths of the chain fulfilling \( N=6n \). Notice that with \( J \sim 1nk \), then \( 10^3 Jt \) corresponds to times of seconds, which is a long time for ultracold experiments, but not completely unrealistic. This will be the only case we will analyse in some detail in the following, since for \( N \neq 6n \) the time scale on which the transfer takes place is really well beyond experimental accessibility in cold atom gases.

For such a case, we report in Fig. 3 the transfer time \( \tau \), with the constraint for the average Fidelity \( \bar{F}(t) > 0.97 \), as a function both of the weak coupling value \( J_0 \) and of the length of the chain. An excellent fit of the transfer time is found by setting \( \tau \simeq \sqrt{N}/J_0 \).

One can compare this result with the perturbative analysis of a single excitation
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Figure 2. Average fidelity $\bar{F}(t)$ as a function of time $t$ for $N=22$ and $N=24$, shown in the left and middle panel, respectively. In both cases the weak coupling has been set to $J_0=0.001$. (right panel) Zoom on the time interval where $\bar{F}(t)$ for $N=24$ is maximum (for $N=22$ only the time values on the ascissa change accordingly to those reported in the right panel). The oscillations occurring on a time scale proportional to $J$ are due to the coupling between the two spins embodying the senders (receivers).

Figure 3. (left panel) Transfer time $\tau$ for which $\bar{F}(\tau) > 0.97$ vs. coupling $J_0$ in a log–log plot for fixed values of $N$ yielding $\tau \sim J_0^{-1}$ for $J_0 \ll 1$. (right panel) Transfer time $\tau$ vs $N$ for a fixed value of $J_0$ (here $J_0=0.001$) yielding $\tau \sim \sqrt{N}$ for $J_0 \ll 1$. As a consequence, the transfer time of the 2-QST weak coupling protocol, for $N = 6n$, results to be $\tau \sim \sqrt{N/J_0}$.

propagation done in [18, 15] where a single-spin sender is resonantly coupled with the energy band of the channel. In the present case the sender is composed by two sites with single excitation energies $\epsilon = \pm 2J$. The single excitation is resonant with the edge of the channel band, while the double excitation is resonant with the center of the band. In both cases, it is shown in [18, 15] that the propagation time is also proportional to $\sqrt{N/J_0}$.

Following the analysis of [17], we notice that the term to be maximized in Eq. (12) is $Re \left[ f_1^{N-1} \right]$, which can be expressed as

$Re \left[ f_1^{N-1} \right] = Re \left[ \sum_{k=1}^{N} e^{-i\epsilon_k t} a_{k1} a_{kN-1} \right] \simeq Re \left[ \sum_{i \in k} e^{-i\epsilon_i t} a_{i1} a_{iN-1} \right]$,  

(13)

where in the last step we have exploited the fact that only the two pairs of triple quasi-degenerate levels have a significant overlap with the initial state, labelling that set by $k = \{1, 2, \ldots, 6\}$. By ordering the energy eigenvalues in increasing order, one has
We have shown that also in this case a Rabi-like process dominates the dynamics and an efficient QST can occur also for two-qubit states. Even if the problem of 2-QST is formally more complex than the single qubit case, nevertheless we found that the transmission times are of the same order of magnitude. Numerical results suggest that the entire dynamics is dominated by an evolution similar to that of a single-excitation and, as a consequence, the introduction of an extra qubit does not affect very much the transmission efficiency with respect to the single-qubit case. Also within this protocol, eigenstates localized both on the sender and receivers can be found for chains composed by an even number of spins. This allows us to simplify the average fidelity to give a very accurate estimation
of the transfer times. A drawback of this approach resides in the very large transmission times required, nevertheless, we have shown that for a subclass of even-length chains these times can be broadly compatible with possible experimental times.

This paper provide a step towards further studies of QST in ultracold setups. An interesting future work would be from one side the determination of an effective Hamiltonian description for these class of protocols. This would simplify the large-$N$ analysis and provide a simple model as a tool to optimize and simulate different protocols. A systematic comparison between QST in different spin chains (like the XXZ) would be also desirable, and in particular to compare the QST efficiency in these models of protocols based on the control of the couplings and of the local magnetic fields.

In order to better adhere to the experimental settings, one should consider the presence of a site-dependent local energy in the Hamiltonian, due to the trapping potential the system is confined in. Although much of the analysis presented in this review, including the formal expressions for the dynamics and the fidelity, still hold, new interference term would appear which could have a detrimental effect on the QST performance.

It would also be interesting to consider QST protocols where the sender and the receiver are not located at the edges of the 1D system but in the bulk. Indeed, due to the natural inhomogeneity of the trap, the edge sites are liable to static imperfections, as well as during the initial state preparation. As we discussed, a possibility to avoid this drawback is to isolate the sender and the receiver inside the bulk of the spin chain by effectively decoupling them from the edges by means of two extra optical barriers. Due to the achievable site distance and compared with typical (stable) laser widths, this should be challenging but feasible and it would interesting to perform a more detailed analysis along this line in connection with experimental progresses.

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