Reggeization of $\mathcal{N}=8$ Supergravity and $\mathcal{N}=4$ Yang–Mills Theory

Howard J. Schnitzer
Theoretical Physics Group
Martin Fisher School of Physics, Brandeis University
Waltham, MA 02454

Abstract

We show that the gluon of $\mathcal{N}=4$ Yang–Mills theory lies on a Regge trajectory, which then implies that the graviton of $\mathcal{N}=8$ supergravity also lies on a Regge trajectory. This is consistent with the conjecture that $\mathcal{N}=8$ supergravity is ultraviolet finite in perturbation theory.

1 Introduction

There has been a great deal of interest in the possibility that $\mathcal{N}=8$ supergravity (sugra) has the same degree of divergence as $\mathcal{N}=4$ super Yang–Mills theory (YM), and thus is ultraviolet finite in four-dimensions [1]. In particular the $\mathcal{N}=8$ sugra perturbation expansion is closely related to $\mathcal{N}=4$ YM amplitudes [1, 2]. It has been argued by Green, Russo and Vanhove [3] that the dualities of $M$-theory imply that the four-graviton amplitude of $\mathcal{N}=8$ sugra is ultraviolet finite in four dimensions. Further they argue [4] that even without the duality conjectures, four-dimensional $\mathcal{N}=8$ sugra might be ultraviolet finite at least up to eight loops. Clearly the relationship of these two theories is providing new insights. In this paper we explore the Reggeization program in both theories.

In our work in collaboration with Grisaru and Tsao [5], and with Grisaru [6, 7], we showed that the elementary fields of renormalizable non-Abelian gauge theories in four-dimensions lie on Regge trajectories. In addition, our old preliminary study [7] of $\mathcal{N}=8$ sugra suggested that this might be true there as well. In this paper we extend Ref. [7] to show that the Reggeization of the gluon in $\mathcal{N}=4$ YM implies the Reggeization of the graviton in $\mathcal{N}=8$ sugra.

The calculations leading to this conclusion involves arcane technology (to the present generation of theorists). Therefore we provide an overview of these methods in Sec. 2, which involves (perhaps unfamiliar) concepts of Mandelstam counting, nonsense helicity states, etc. However, since we cannot provide a complete review of these techniques, the reader should refer to the original papers, especially ref. [7] for additional technical details.

In Sec. 3 we apply the formulism of Sec. 2 to the Reggeization of the gluon in $\mathcal{N}=4$ YM and the graviton in $\mathcal{N}=8$ sugra. Concluding remarks are in Sec. 4, while several Appendices collect useful formulae needed in the calculations. Some of the information in the appendices is repeated in the text for clarity of presentation.

1 email: schnitzr@brandeis.edu
Supported in part by the DOE under grant DE-FG02-92ER40706
2 Regge Behavior: An Overview

There exist different ways of finding Regge poles in Lagrangian field theory. One method consists of summing Feynman diagrams for large momentum transfer at fixed $s$ in leading logarithm approximation and recognizing that a Regge trajectory $\alpha(s)$ corresponds to asymptotic behavior $\sim t^{\alpha(s)}$. Another involves solving analyticity-unitarity integral equations for the analytic continuation of the scattering amplitude $f(s, J)$ and looking for Regge poles directly. In $N=8$ sugra the second method is applicable, using as input only knowledge of the Born approximation (as we explain later in this section.) [Given the close relationship of $N=4$ YM scattering amplitudes to those of $N=8$ sugra, summation of leading logarithms in the latter theory may also be possible.] We have used the second method extensively [5]–[7] in the past to find Regge poles in Yang–Mills theories, which gave the same result as diagram summation in all cases where a comparison can be made. We emphasize that the existence and number of Regge trajectories in a neighborhood of small integral or half-integral $J$ is independent of the size of the coupling. As a consequence we have control of local properties in $J$ of Regge trajectories, but not global ones.

One assumes that the (kinematical singularity free) scattering amplitude $F(s, J)$ can be continued to large $Re J$ without encountering singularities in the angular momentum plane, and that it can be continued to the left of $Re J = N$. Here one may encounter singularities such as poles and cuts. In particular if one continues $F(s, J)$ to an integer $J = j \leq N$, one may ask if

$$F(s, J) \equiv f_j(s) \tag{2.1}$$

where the scattering amplitude $f(s, z)$ computed from diagrams has the form

$$f_j(s) = \frac{1}{2} \int_{-1}^{1} dz P_j(z) f(s, z)$$

$$= \sum_{n=0}^{N} b_n(s) \delta_{jn} + \text{analytic in } j, \tag{2.2}$$

where $z = \cos \theta$ and $P_j(z)$ is a Legendre polynomial. [In renormalizable field theories $N \leq 1$, and if $N=8$ sugra is finite, or at most log. divergent, presumably $N \leq 2$.] The presence of kronecker delta terms seems to make the equality (2.1) unlikely, yet one can prove that in certain case that $F(s, j)$ and $f_j(s)$ must coincide. If they do coincide at some value of $j$, one says that $f_j(s)$ “Reggeizes” at $J = j$, and that $F(s, J)$ is analytic in the neighborhood of such $j$.

Mandelstam [8] has given certain criteria for establishing whether or not kronecker delta singularities are present at a given $j$, based on the following counting argument. Both $f_j(s)$ and $F(s, J)$ have $s$-plane analyticity and unitarity properties which require that they satisfy certain $s$-channel dispersion relations, in which the inhomogeneous terms are the same in both cases, as is the unitarity condition. Where the solutions for $f_j(s)$ and $F(s, J)$ may differ is in the value of possible subtraction constants in the dispersion relations, and in the positions and residues of CDD poles, which correspond to singularities not resolved by analyticity and unitarity. However both $f_j(s)$ and $F(s, J)$ are subject to identical kinematical constraints. If the number of these constraints equals or exceeds the number of free parameters, the amplitudes must coincide. In this case one understands the kronecker delta as the boundary value of an analytic function, e.g.,

$$\frac{\alpha(s) - J}{\alpha(s) - j} \to \delta_{jj}. \tag{2.3}$$
In field theory the free parameters are masses and coupling constants. Since Mandelstam counting is kinematical, it is true to all orders of perturbation theory. However, the Mandelstam procedure must be carried out for each $j$ separately.

The proof of Mandelstam is delicate as it involves the unitarity of the theory, which appears to eliminate non-renormalizable theories, such as massive YM theories without a Higgs mechanism. Nevertheless in this paper we show that the graviton in $\mathcal{N}=8$ sugra Reggeizes as a consequence of the Reggeization of the $\mathcal{N}=4$ YM massless gluon.

We consider the scattering of massless particles with spin $(p_1 + p_2 \rightarrow p_3 + p_4)$ as functions of the Mandelstam variables

$$
\begin{align*}
s &= (p_1 + p_2)^2 = 4q^2 \\
t &= (p_1 - p_3)^2 = -\frac{s}{2} (1 - z) \\
u &= (p_1 - p_4)^2 = -\frac{s}{2} (1 + z)
\end{align*}
$$

where $q$ is the center-of-mass momentum, $z = \cos \theta$, with $\theta$ the scattering angle. One considers the integral equation satisfied by the two-body, kinematical singularity free, helicity amplitudes

$$
\tilde{F}_{\lambda_3 \lambda_4, \lambda_1 \lambda_2}(s, J) = V_{\lambda_3 \lambda_4, \lambda_1 \lambda_2}(s, J)
+ \sum_{\lambda_5 \lambda_6} \int \frac{ds'}{s' - s} \rho(s') \tilde{F}_{\lambda_3 \lambda_4, \lambda_5 \lambda_6}(s', J) \tilde{F}_{\lambda_5 \lambda_6, \lambda_1 \lambda}(s', J)
$$

or as matrices

$$
\tilde{F}_{\mu \lambda} = V_{\mu \lambda} + \int \frac{ds'}{s' - s} \tilde{F}_{\mu \lambda'} \rho_{\lambda' \mu'} \tilde{F}_{\mu' \lambda}
$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_3 - \lambda_4$, and $\rho$ is a phase-space factor. The unitarity condition couples particles of different helicity. In the continuation to small $j$ one reaches a point where $j < |\lambda|, |\mu|$ for some value of $\lambda$ and/or $\mu$. The corresponding amplitudes are unphysical, i.e., states $|\lambda_1, \lambda_2>$ or $|\lambda_3, \lambda_4>$ with $|\lambda|$ or $|\mu| < j$ are “nonsense” states.

$V_{\mu \lambda}(s, J)$ can be obtained to any given order of perturbation theory from diagrams. In particular if we keep only the lowest order diagrams,

$$
V_{\mu \lambda}(s, J) \sim Q_{J - \lambda_m}(z_0(s)) v_{\mu \lambda}(s)
$$

where $\lambda_m = \max (|\lambda|, |\mu|)$ is an integer or half-integer, and $Q$ is a Legendre function of the 2nd kind, noting that $Q$ functions have poles at negative integers. Imagine writing (2.6) for sufficiently large $J$ so that all helicity states are sense, and continuing $J$ to the neighborhood of some small physical value of $j$ where some of the helicity states are nonsense. At such values of $j$ some of the matrix elements of $V_{\mu \lambda}$ are singular since $Q_{J - \lambda}$ develops poles at negative values. Denote $\tilde{F}_{\mu \lambda}$ by

$$
\begin{align*}
\tilde{F}_{ss} \\
\tilde{F}_{ns} \\
\tilde{F}_{ns} \\
\tilde{F}_{nn}
\end{align*}
\begin{cases}
|\lambda|, |\mu| \leq j \\
|\lambda| \leq j, |\mu| > j \\
|\lambda| > j, |\mu| \leq j \\
|\lambda| > j, |\mu| > j
\end{cases}
$$

3
In the neighborhood of \( J \), the integral equation is of the form

\[
\begin{pmatrix}
\tilde{F}_{ss} & \tilde{F}_{sn} \\
\tilde{F}_{ns} & \tilde{F}_{nn}
\end{pmatrix} = \begin{pmatrix}
-v_{ss} \delta J - v_{sn} (J - j)^{-1/2} \\
v_{ns} (J - j)^{-1/2} - v_{nn} (J - j)^{-1}
\end{pmatrix} + \int \frac{ds'}{(s' - s)} \tilde{F} \rho (\tilde{F}) .
\]

The Born approximation quantities \( v_{ss}, v_{sn}, v_{ns}, \) and \( v_{nn} \) are essentially polynomials in \( s \). The solution to (2.9) is

\[
\tilde{F}_{ss}(s, J) = v_{sn}' \left[ K(s) \left( J - j - v(s)K(s) + O(g^4) \right) \right]_{n'n} v_{ns} + \text{regular in } J
\]

where \( K(s) \) is a known integral common to all channels. Here \( \tilde{F}_{ss} \) consists of those helicity amplitudes physical at \( J = j \).

We find Regge poles with trajectories

\[
\alpha(s) = j + (\text{eigenvalues of } v_{nn}) \times K(s) + O(g^4) .
\]

The number of Regge trajectories at \( J = j \) is equal to the rank of the nonsense-nonsense matrix \( v_{nn} \), with trajectories given by (2.11). Eqn. (2.11) has the same accuracy as the leading logarithm approximation in summing diagrams. Therefore complete information about the trajectories (but not the residue) in the neighborhood of \( J \) can be obtained by studying \( v_{nn} \). If we keep only contributions of the \( t- \) and \( u- \) channel poles in Born approximation, this gives the location of the Regge poles for values such that \( \alpha(s_0) = j \), with \( \alpha'(s_0) \) correct to \( O(g^4) \).

Consider (2.10) for \( J \to j \). Consistency for Reggeization to occur at \( j \) requires the matrix factorization

\[
v_{ss} = v_{sn}' (v^{-1})_{n'n} v_{ns}
\]

at \( J = j \). [This should not be confused with the tree factorization of the Born approximation.] One has the following statement [10]. A necessary and sufficient condition for (2.12) to hold is that the rank of the matrix

\[
\begin{pmatrix}
v_{ss} & v_{sn} \\
v_{ns} & v_{nn}
\end{pmatrix} = v
\]

equals that of the nonsense-nonsense matrix \( v_{nn} \). [Recall that the rank \( v_{nn} \) is equal to the number of Regge trajectories at \( J \).]

In the next section we apply the above formalism to gluon-gluon scattering in \( \mathcal{N}=4 \) YM, and graviton-graviton scattering in \( \mathcal{N}=8 \) sugra. We show that the gluon pole at \( J = 1 \) Reggeizes, with rank \( v_{nn} = 1 \) and rank \( v = 1 \), and thus (2.12) is satisfied. As a consequence, we show that these relations in \( \mathcal{N}=4 \) YM also imply that the graviton Reggeizes in \( \mathcal{N}=8 \) sugra. That is (2.12) being satisfied for \( \mathcal{N}=4 \) YM also implies that the analogous factorization condition holds for \( \mathcal{N}=8 \) sugra. In both theories the rank \( v_{nn} = 1 \) and rank \( v = 1 \), though \( \text{dim } v \) differs.
3 Helicity Amplitudes

A. Generalities

In this section we present the relevant information to justify our claim of the Reggeization of the graviton and gluon in $\mathcal{N}=8$ sugra and $\mathcal{N}=4$ YM respectively. Since both theories are supersymmetric, pseudohelicity conservation applies. For the scattering amplitude $F(\lambda_3, \lambda_4; \lambda_1, \lambda_2)$ with helicities $\lambda_1, \cdots, \lambda_4$ one defines the pseudohelicity

$$P(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$$

(3.1)

for the initial state, and similarly for the final state. Supersymmetry requires

$$P(\lambda_1, \lambda_2) = P(\lambda_3, \lambda_4).$$

(3.2)

That is

$$F(\lambda_3, \lambda_4; \lambda_1, \lambda_2) = 0 \text{ if } \lambda_1 + \lambda_2 \neq \lambda_3 + \lambda_4.$$  

(3.3)

As a result, we only need consider $P=0$ states in our discussion of the Reggeization of the graviton and of the gluon.

In order to extract Regge behavior one must deal with kinematical singularity free amplitudes, as these are the ones that satisfy unitarity and analyticity. Given

$$F(\lambda_3, \lambda_4; \lambda_1, \lambda_2) = F_{\mu\lambda}(s, t, u)$$

(3.4)

where $\mu = \lambda_3 - \lambda_4$, $\lambda = \lambda_1 - \lambda_2$, and $s, t, u$ are the Mandelstam variables, the kinematical singularity free amplitudes $\tilde{F}_{\mu\lambda}$ are obtained from

$$F_{\mu\lambda}(s, t, u) = \left[ \sqrt{2} \cos \frac{\theta}{2} \right]^{\lambda+\mu} \left[ \sqrt{2} \sin \frac{\theta}{2} \right]^{\lambda-\mu} \tilde{F}_{\mu\lambda}(s, t, u)$$

$$= (\sqrt{2})^{\lambda+\mu+\lambda-\mu} \left[ \sqrt{-\frac{u}{s}} \right]^{\lambda+\mu} \left[ \sqrt{-\frac{t}{s}} \right]^{\lambda-\mu} \tilde{F}_{\mu\lambda}(s, t, u).$$

(3.5)

A typical singularity free amplitude has the form in Born approximation

$$\tilde{F}_{\mu\lambda} = \frac{a}{t} + \frac{b}{u} + \frac{c}{s}.$$  

(3.6)

Given the kinematical singularity free amplitudes, one obtains the angular momentum projection

$$F_{J\mu} = \frac{1}{2} \int_{-1}^{1} dz C_{J\mu}(z) \tilde{F}_{\mu\lambda}(z)$$

(3.7)

where the matrices $C_{J\mu}$ we need are tabulated in Appendix C, with $t = -\frac{s}{2} (1 - z); \quad u = -\frac{s}{2} (1 + z); \quad z = \cos \theta$. Given the $C_{J\mu}$, one computes the projection (3.7) using

$$\frac{1}{2} \int_{-1}^{1} dz \frac{P_t(z)}{a - z} = Q_t(a).$$

(3.8)
For the continuation to small \( j \) use

\[
Q_{-\ell}(z) = -\pi (\cot \pi \ell) P_{\ell-1}(z) + Q_{\ell-1}(z)
\]  

(3.9)

where

\[
-\pi \cot \pi \ell \sim \frac{1}{\ell - \ell_0}
\]

(3.10)

with \( \ell \) an integer.

The \( P = 0 \), kinematic singularity free Born amplitudes for \( N=8 \) sugra and \( N=4 \) YM are listed in Appendices A and B respectively. Throughout, the explicit factors of \( \kappa \) or \( g^2 \) are omitted, though it is obvious how to restore these if needed. The amplitudes relevant for the Reggeization of the gluon or graviton are the \( P=0 \), flavor singlet, or \( a \) amplitudes, tabulated in (D.1)–(D.3) and (E.1)–(E.6) respectively. The angular momentum projections near \( j = 1 \) and \( j = 2 \) are to be found in (D.4)–(D.6) and (E.7)–(E.12) respectively.

We now turn to providing additional information for the relevant \( P=0 \), two-body amplitudes.

**B. \( N=4 \) YM**

The \( P=0 \) states that contribute are \(|1, -1 \rangle \) and \(|1/2^a, -1/2^b \rangle \); \( a = 1 \) to \( 4 \), \( i.e. \), \( \lambda = 2 \) and \( 1 \), with the former nonsense at \( j=1 \) and the latter sense, where the helicity \( 1/2 \) fermion has “flavor” \( a \). The kinematical singularity free amplitudes \( \tilde{F}(1, -1; 1, -1); \tilde{F}(1/2^a, -1/2^b; 1, -1) \) and \( \tilde{F}(1/2^a, -1/2^b; 1/2^c, -1/2^d) \) are to be found in (B.1)–(B.3), which leads to a \( 2\times2 \) scattering matrix for \( P=0 \).

**C. \( N=8 \) sugra**

As a result of the KLT relations [11], the tree amplitudes for the 4-point functions of \( N=8 \) sugra can be expressed in terms of the square of the tree amplitudes of \( N=4 \) YM. The relevant \( P=0 \ N=8 \) states are \(|2, -2 \rangle \), \(|3/2^A, -3/2^B \rangle \) and \(|1[AB], -1_{[CD]} \rangle \), \( A=1 \) to \( 8 \). Schematically the flavor singlet, \( P=0 \) Born amplitudes we need are

\[
\begin{align*}
\tilde{M}(2, -2; 2, -2) & \sim \tilde{F}(1, -1; 1, -1) \tilde{F}(1, -1; 1, -1) \\
\tilde{M}(2, -2; 3/2, -3/2) & \sim \tilde{F}(1, -1; 1, -1) \tilde{F}(1/2, -1/2; 1/2, -1/2) \\
\tilde{M}(3/2, -3/2; 3/2, -3/2) & \sim \tilde{F}(1, -1; 1, -1) \tilde{F}(1/2, -1/2; 1/2, -1/2) \\
\tilde{M}(2, -2; 1, -1) & \sim \tilde{F}(1, -1; 1/2, -1/2) \tilde{F}(1, -1; 1/2, -1/2) \\
\tilde{M}(3/2, -3/2; 1, -1) & \sim \tilde{F}(1, -1; 1/2, -1/2) \tilde{F}(1/2, -1/2; 1/2, -1/2) \\
\tilde{M}(1, -1; 1, -1) & \sim \tilde{F}(1/2, -1/2; 1/2, -1/2) \tilde{F}(1/2, -1/2; 1/2, -1/2)
\end{align*}
\]

(3.11)

where the left-hand side are \( N=8 \) sugra amplitudes, and the right-side are \( N=4 \) YM amplitudes. Eqn. (3.11) implies a \( 3\times3 \ P=0 \) scattering matrix. The structure (3.11) manifests itself in the projections near \( J=2 \) for the sugra amplitudes, and its relationship to the projections near \( J=1 \) for the YM amplitudes. This is evident in comparing (D.4)–(D.6) with (E.7)–(E.11).

Another way of presenting [1] the KLT relations [11] is

\[
\tilde{F} = -\left[ \frac{C_s}{s} + \frac{C_t}{t} + \frac{C_u}{u} \right]
\]

(3.12)
\[ M = - \left[ \frac{C_s}{s} + \frac{C_t}{t} + \frac{C_u}{u} \right] \] \hspace{1cm} (3.13)

which can be confirmed using Appendices A and B.

\section*{D. The helicity matrices}

Generically, for each \( j \) the Born approximation is of the form

\[ F_{\lambda \mu}^J = \begin{pmatrix} v_{ss} \delta_{jj} & v_{sn} (J-j)^{-1/2} \\ v_{ns} (J-j)^{-1/2} & v_{nn} \end{pmatrix} \] \hspace{1cm} (3.14)

which is the first term on the right-hand side of (2.8) given in terms of the submatrices \( v_{ss}, v_{ns} = v_{sn}; v_{nn} \).

For \( \mathcal{N}=4 \) YM, we obtain the helicity matrix

\[ v = \begin{pmatrix} v_{nn} & v_{sn} \\ v_{ns} & v_{ss} \end{pmatrix} \] \hspace{1cm} (3.15)

from the \( P=0 \) states near \( j = 1 \). Explicit values are in (D.7)–(D.9), which we repeat for convenience

\[ v_{nn} = 4 \]
\[ v_{ns} = v_{sn} = \frac{16}{\sqrt{3}} \]
\[ v_{ss} = \frac{64}{3} . \] \hspace{1cm} (3.16)

From (3.15) and (3.16) we have

\[ \text{rank } v_{nn} = 1 \]
\[ \text{rank } v = 1 \]
\[ v_{ss} = v_{sn} (v_{nn})^{-1} v_{ns} . \] \hspace{1cm} (3.17)

Eqn. (3.17) combined with Mandelstam counting implies that the gluon must Reggeize.

For \( \mathcal{N}=8 \) sugra, the analogous helicity matrix for \( P=0 \) is [using \( V \) to avoid confusion with (3.15)]

\[ V = \begin{pmatrix} V_{nn} & V_{sn} \\ V_{ns} & V_{ss} \end{pmatrix} \] \hspace{1cm} (3.18)

near \( j = 2 \). Explicit values are in (F.1)–(F.4), with the KLT relations [11] evident in (F.1)–(F.4).

Using (F.1)–(F.4), (3.12) and (3.15)–(3.17) we find that

\[ \text{rank } V_{nn} = 1 \]
\[ \text{rank } V = 1 \] \hspace{1cm} (3.19)

We conclude that the graviton must Reggeize, as a consequence of the Reggeization of the gluon!
4 Concluding Remarks

In this paper we have shown that the gluon of $\mathcal{N}=4$ YM theory lies on a Regge trajectory. The factorization condition for this to hold, (3.17), also implies that the graviton of $\mathcal{N}=8$ sugra lies on a Regge trajectory, c.f. (F.5)–(F.7). In Ref. [7] we only verified that the rank $V_{nn} = 1$, but did not check the factorization condition for $V$, since we presumed that $\mathcal{N}=8$ sugra was non-renormalizable. Here we verify that the factorization condition for $\mathcal{N}=4$ YM, (3.17) then also implies $\det V_{nn} = 1$ and rank $V = 1$, leading to the conclusion that the graviton must Reggeize, which is consistent with the speculation that $\mathcal{N}=8$ sugra is ultraviolet finite. It should be emphasized that this is not a holographic result, as both theories are considered in perturbation theory.

The renormalizability vs. non-renormalizability of a field theory is not a trivial issue for the Reggeization program, since at face value there are an infinite number of free parameters for a non-renormalizable theory, and thus Mandelstam counting would not apply. An example is massive YM theory without the Higgs mechanism. It is known that the factorization condition (2.12) fails in this case [9], and thus there the gluon does not Reggeize. However, $\mathcal{N}=8$ sugra appears to evade the difficulties exampled by that massive YM example.

The computation presented in this paper is equivalent to the leading logarithm approximation in the summation of an infinite set of diagrams. It is therefore reasonable to expect that a leading logarithm summation of diagrams in $\mathcal{N}=8$ graviton-graviton scattering will reproduce our results. The Regge trajectories, computed in perturbation theory, will not lead to recurrences in weak-coupling, but rather are analogous to the Regge trajectories of potential scattering. All orders in perturbation theory continued to strong-coupling, may well produce Regge recurrences.

The other fundamental fields of the Lagrangians of $\mathcal{N}=8$ sugra and $\mathcal{N}=4$ YM should Reggeize as well, as a consequence of the unbroken SU(8) and SU(4) flavor symmetries (respectively) of the theories.

Further exploration of the possible implications of an ultraviolet finite $\mathcal{N}=8$ sugra promises to be a fruitful enterprise. It is an important issue of principle to know whether or not $\mathcal{N}=8$ sugra is a finite quantum theory of gravity distinct from string theory. Given that $\mathcal{N}=8$ sugra contains Regge poles, we speculate that is is not distinct.

Acknowledgements

We thank Albion Lawrence for stimulating conversations. We are also appreciative of our old collaboration with Marc Grisaru on this subject.
Appendix A

$\mathcal{N}=8$ sugra: Kinematical Free Amplitudes: $P = 0$

$$\tilde{M}(2,-2;2,-2) = \frac{s^2}{16} \left[ \frac{1}{t} + \frac{1}{u} \right], \quad (A.1)$$

$$\tilde{M} \left( \frac{-3A}{2}, \frac{-3}{2B}; 2, -2 \right) = -\tilde{M} \left( 2, -2; \frac{3}{2B}, \frac{-3A}{2} \right)$$

$$= -\frac{\delta^A_{B} s^2}{16} \left[ \frac{1}{t} + \frac{1}{u} \right], \quad (A.2)$$

$$\tilde{M} \left( \frac{3A}{2}, \frac{-3}{2B}; \frac{3}{2C}, \frac{-3D}{2} \right) = \frac{1}{8} \left( \frac{s^3}{u} \right) \left[ \frac{\delta^A_{B} \delta^D_{C}}{s} + \frac{\delta^A_{B} \delta^D_{C}}{t} \right], \quad (A.3)$$

$$\tilde{M}(1^{AD}, -1_{BC}; 2, -2) = \tilde{M}(2, -2; 1^{AD}, -1^{BC})^*$$

$$= -\frac{s^2}{16} \left( \frac{1}{t} + \frac{1}{u} \right) \delta^A_{BC}, \quad (A.4)$$

$$\tilde{M} \left( \frac{3A}{2}, \frac{-3}{2B}; 1_{CF}, -1^{DE} \right) = -\tilde{M} \left( 1^{CF}, -1^{DE}; \frac{3}{2A}, \frac{-3B}{2} \right)^*$$

$$= \frac{1}{8} \left( \frac{s^3}{u} \right) \left\{ \frac{1}{t} \left[ \delta^A_{C} \delta^D_{F} - \delta^A_{C} \delta^D_{G} \right] - \frac{1}{s} \delta^A_{B} \delta^D_{CF} \right\}, \quad (A.5)$$

where throughout (*) is an SU(8) conjugation which raises and lowers indices only:

$$\tilde{M}(1^{AH}, -1_{BC}; 1_{DE}, -1^{FG}) = -\frac{s^2}{4} \left\{ \frac{1}{u} \frac{1}{4!} \varepsilon^{AHFGMNOP} \varepsilon_{BCDEMNOP} + \frac{1}{s} \delta^A_{BC} \delta^D_{EF} + \frac{1}{t} \delta^A_{DE} \delta^B_{FG} \right\} \quad (A.6)$$

with

$$\delta^A_{CD} = \delta^A_{C} \delta^B_{D} - \delta^B_{D} \delta^C_{A}, \quad (A.7)$$

$$\delta^A_{BC} = \delta^A_{D} \delta^B_{EF} \delta^C_{D} \delta^E_{EF} + \delta^B_{D} \delta^C_{A} \delta^D_{EF}$$

$$= \frac{1}{5!} \varepsilon^{ABCGHKLM} \varepsilon_{DEFGHKLM}, \quad (A.8)$$

$$\delta^A_{FGHKL} = \frac{1}{3!} \varepsilon^{ABCDMN} \varepsilon_{FGHKLMNO} \quad (A.9)$$

where the flavor indices are $A = 1$ to 8.
Appendix B

$\mathcal{N}=4$ YM: Singularity-Free Amplitudes

Pseudohelicity (0)

\[
\tilde{F} (1,-1;1,-1) = s \left[ \frac{\alpha}{t} + \frac{\beta}{u} \right] \quad (B.1)
\]

\[
\tilde{F} \left( \frac{1}{2}^a, -\frac{1}{2}^b; 1, -1 \right) = \delta^a_b \ s \left[ \frac{\alpha}{t} + \frac{\beta}{u} \right] \quad (B.2)
\]

\[
\tilde{F} \left( \frac{1}{2}^a, -\frac{1}{2}^b; \frac{1}{2}^c, \frac{1}{2}^d \right) = 2 \left[ s \delta^a_c \delta^d_b + t \delta^a_b \delta^d_c \right] \left[ \frac{\alpha}{t} + \frac{\beta}{u} \right] \quad (B.3)
\]

where

\[
\alpha = f_{ikn} f_{nj\ell} \\
\beta = f_{ijn} f_{njk} \quad (B.4)
\]

are products of the structure constants of the gauge group, and the flavor indices are $a = 1$ to 4.
Appendix C  The functions $C^J_{\lambda\mu}$

\[
C^J_{44} = \frac{(J+1)(J+2)(J+3)(J+4)}{(2J-5)(2J-3)(2J-2)(2J+1)} \ P_{J-4}
\]
\[
+ \frac{4(J+2)(J+3)(J+4)}{(2J-3)(2J-1)(2J+1)} \ P_{J-3} + \frac{28(J-3)(J+2)(J+3)(J+4)}{(2J-5)(2J-1)(2J+1)(2J+3)} \ P_{J-2} + \cdots,
\]
\[
(C.1)
\]

\[
C^J_{33} = \frac{(J+1)(J+2)(J+3)}{(2J-3)(2J-1)(2J+1)} \ P_{J-3}
\]
\[
+ \frac{3(J+2)(J+3)}{(2J-1)(2J+1)} \ P_{J-2} + \frac{15(J-2)(J+2)(J+3)}{(2J-3)(2J+1)(2J+3)} \ P_{J-1} + \cdots,
\]
\[
(C.2)
\]

\[
C^J_{34} = \sqrt{(J+4)(J-3)} \ \left\{ \frac{(J+1)(J+2)(J+3)}{(2J-5)(2J-3)(2J-1)(2J+1)} \ P_{J-4}
\]
\[
+ \frac{3(J+2)(J+3)}{(2J-3)(2J-1)(2J+1)} \ P_{J-3}
\]
\[
+ \frac{7(J+2)(J+3)(2J-7)}{(2J-5)(2J-1)(2J+1)(2J+3)} \ P_{J-2} + \cdots \right\},
\]
\[
(C.3)
\]

\[
C^J_{24} = \sqrt{(J-2)(J-3)(J+3)(J+4)} \ \left\{ \frac{(J+1)(J+2)}{(2J-5)(2J-3)(2J-1)(2J+1)} \ P_{J-4}
\]
\[
+ \frac{2(J+2)}{(2J-3)(2J-1)(2J+1)} \ P_{J-3}
\]
\[
+ \frac{4(J-6)(J+2)}{(2J-5)(2J-1)(2J+1)(2J+3)} \ P_{J-2} + \cdots \right\},
\]
\[
(C.4)
\]

\[
C^J_{23} = \sqrt{(J-2)(J+3)} \ \left\{ \frac{(J+1)(J+2)}{(2J-3)(2J-1)(2J+1)} \ P_{J-3} + \frac{2(J+2)}{(2J-1)(2J+1)} \ P_{J-2}
\]
\[
+ \frac{5(J-3)(J+2)}{(2J-3)(2J+1)(2J+3)} \ P_{J-1} + \cdots \right\},
\]
\[
(C.5)
\]
\[ C_{22}^J = \frac{(J-1)J}{(2J+1)(2J+3)} P_{J+4} + \frac{2(J-1)}{(2J+1)} P_{J+1} \]
\[ + \frac{6(J-1)(J+2)}{(2J-1)(2J+3)} P_J + \frac{2(J+2)}{(2J+1)} P_{J-1} \]
\[ + \frac{(J+1)(J+2)}{(2J-1)(2J+1)} P_{J-2} + \cdots, \]  
\text{(C.6)}

\[ C_{11}^J = \frac{J}{(2J+1)} P_{J+1} + P_J + \frac{(J+1)}{(2J+1)} P_{J-1} + \cdots, \]  
\text{(C.7)}

\[ C_{12}^J = \sqrt{(J-1)(J+2)} \left\{ \frac{-J}{(2J+1)(2J+3)} P_{J+2} - \frac{1}{(2J+1)} P_{J+1} \right. \]
\[ + \frac{-3}{(2J-1)(2J+3)} P_J + \frac{1}{(2J+1)} P_{J-1} \]
\[ + \frac{(J+1)}{(2J-1)(2J+1)} P_{J-2} + \cdots \} \]  
\text{(C.8)}
Appendix D

a) $\mathcal{N}=4$ flavor singlet amplitudes

\[ \tilde{F}(1,-1;1,-1) = s \left[ \frac{\alpha}{t} + \frac{\beta}{u} \right] \]  \hspace{1cm} (D.1)

\[ \tilde{F}(1/2,-1/2;1,-1) = 4s \left[ \frac{\alpha}{t} + \frac{\beta}{u} \right] \]  \hspace{1cm} (D.2)

\[ \tilde{F}(1/2,-1/2;1/2,-1/2) = 8[s + 4t] \left[ \frac{\alpha}{t} + \frac{\beta}{u} \right] \]  \hspace{1cm} (D.3)

with $\alpha, \beta$ in (B.4).

b) Angular momentum projections near $J=1$

Using (2.9) and Appendix C

\[ F_{22}^J(1,-1;1,-1) = \frac{v_{nn}(\alpha - \beta)}{(J - 1)} \]  \hspace{1cm} (D.4)

\[ F_{12}^J(1/2,-1/2;1,-1) = \frac{v_{sn}(\alpha - \beta)}{\sqrt{J - 1}} \]  \hspace{1cm} (D.5)

\[ F_{11}^J(1/2,-1/2;1/2,-1/2) = v_{ss}(\alpha - \beta)\delta_{J1} \]  \hspace{1cm} (D.6)

where $(\alpha - \beta)$ belongs to the adjoint representation of the group, and

\[ v_{nn} = 4 \]  \hspace{1cm} (D.7)

\[ v_{sn} = v_{ns} = \frac{16}{\sqrt{3}} \]  \hspace{1cm} (D.8)

\[ v_{ss} = \frac{64}{3} \]  \hspace{1cm} (D.9)

Note that (2.11) is satisfied, and thus

\[ \det \mathbf{v} = 0 \]  \hspace{1cm} (D.10)

where the helicity matrix $\mathbf{v}$ is defined in (2.13).
Appendix E

a) $\mathcal{N}=8$ flavor singlet amplitudes

\[
\tilde{M}(2, -2; 2, -2) = -\frac{s^2}{16} \left[ \frac{1}{t} + \frac{1}{u} \right] \\
\tilde{M}(2, -2; 3/2, -3/2) = -\frac{s^2}{2} \left[ \frac{1}{t} + \frac{1}{u} \right] \\
\tilde{M}(3/2, -3/2; 3/2, -3/2) = 2 \left( \frac{s^3}{u} \right) \left[ \frac{8}{s} + \frac{1}{t} \right] \\
\tilde{M}(2, -2; 1, -1) = -\frac{7s^2}{2} \left[ \frac{1}{t} + \frac{1}{u} \right] \\
\tilde{M}(3/2, -3/2; 1, -1) = \frac{7s^3}{u} \left[ \frac{4}{t} \right] \\
\tilde{M}(1, -1; 1, -1) = -28s^2 \left[ \frac{15}{u} + \frac{28}{s} + \frac{1}{t} \right].
\]

b) Angular momentum projections near $J=2$

The structure of (E.7)–(E.14) below reflects that $\mathcal{N}=8$ tree amplitudes can be expressed in terms of the squares of $\mathcal{N}=4$ YM tree amplitudes.

\[
M_{44}^J(2, -2; 2, -2) = \frac{3s}{8} \frac{v_{nn}v_{nn}}{(J - 2)} \\
M_{43}^J(2, -2; 3/2, -3/2) = \frac{-3\sqrt{2}s}{8} \frac{v_{nn}v_{ns}}{(J - 2)} \\
M_{33}^J(3/2, -3/2; 3/2, -3/2) = \frac{-3s}{4} \frac{v_{nn}v_{ns}}{(J - 2)} \\
M_{42}^J(2, -2; 1, -1) = \frac{-21is}{32} \sqrt{\frac{6}{5}} \frac{(v_{ns})(v_{ns})}{\sqrt{J - 2}} \\
M_{32}^J(3/2, -3/2; 1, -1) = \frac{-21s}{16} \sqrt{\frac{3}{5}} \frac{(v_{ns})(v_{ss})}{\sqrt{J - 2}} \\
M_{22}^J(1, -1; 1, -1) = \frac{-21s}{320} \frac{(v_{ss})(v_{ss})}{\delta_{J2}}
\]
Appendix F

\( \mathcal{N}=8 \) helicity matrices

Equations (E.7)–(E.12) can be written as in (2.8), but using \( V_{ss}, V_{sn}, \) and \( V_{nn} \) to distinguish these from the analogous matrices \( v_{ss}, v_{sn}, \) and \( v_{nn} \) of \( \mathcal{N}=4 \) YM. [c.f. Appendix D.]

\[
V_{nn} = \frac{3s}{8} v_{nn} \begin{bmatrix} v_{nn} & -i\sqrt{2} v_{ns} & -2 v_{ss} \\ -i\sqrt{2} v_{ns} & -2 v_{ss} & \end{bmatrix} \tag{F.1}
\]

from (E.7)–(E.9)

\[
V_{sn} = -\frac{21}{32} \sqrt{\frac{3}{5}} s v_{ns} \begin{bmatrix} i\sqrt{2} & v_{ns} \\ 2 v_{ss} & \end{bmatrix} \tag{F.2}
\]

from (E.10)–(E.11).

\[
V_{ss} = \frac{-(21)^2}{320} s v_{ss} v_{ss} \tag{F.3}
\]

from (E.12). The 3×3 matrix \( V \) is

\[
V = \begin{bmatrix} V_{nn} & V_{sn} \\ V_{sn} & V_{ss} \end{bmatrix} \tag{F.4}
\]

One has

\[
\text{det } V_{nn} = 0 ; \quad \text{rank } V_{nn} = 1 \tag{F.5}
\]

as a consequence of the factorization condition (2.12) for \( \mathcal{N}=4 \) YM, i.e.,

\[
v_{nn} v_{ss} - v_{ns} v_{sn} = 0 . \tag{F.6}
\]

Thus there is only one flavor singlet Regge trajectory at \( J=2 \) in \( \mathcal{N}=8 \) sugra.

From (F.1)–(F.4) and (F.6) one finds

\[
\text{rank } V = 1 \tag{F.7}
\]

which implies that the graviton in \( \mathcal{N}=8 \) sugra Reggeizes as a consequence of the Reggeization of the gluon in \( \mathcal{N}=4 \) YM.
References

[1] Z. Bern, L. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, “On the relationship between Yang–Mills theory and gravity and its implication for ultraviolet divergences,” Nucl. Phys. B530 (1998) 401, hep-th/9802162; Z. Bern, L. Dixon and R. Roiban, “Is N=8 supergravity ultraviolet finite?”, Phys. Lett. B644 (2007) 265, hep-th/0611086; N. Berkovits, “New higher-derivative $R^4$ theorems,” hep-th/0609006.

[2] N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, “Perturbative gravity and twistor space,” hep-th/0606268; “Similarities of gauge and gravity amplitudes,” hep-th/0608007.

[3] M.B. Green, J.H. Schwarz and L. Brink, “$\mathcal{N}=4$ Yang–Mills and $\mathcal{N}=8$ supergravity as limits of string theories,” Nucl. Phys. B198 (1982) 474; M.B. Green, J.G. Russo and P. Vanhove, “Non-renormalization conditions in type II string theory and maximal supergravity,” hep-th/0610299.

[4] M.B. Green, J.G. Russo and P. Vanhove, “Ultraviolet properties of maximal supergravity,” hep-th/0611273.

[5] M.T. Grisaru, H.J. Schnitzer and H-S. Tsao, “The Reggeization of Yang–Mills gauge mesons in theories with a spontaneously broken symmetry,” Phys. Rev. Lett. 20 (1973) 811; “Reggeization of elementary particles in renormalizable gauge theories: vectors and spinors,” Phys. Rev. D8 (1973) 4498; “The Reggeization of elementary particles in renormalizable gauge theories: scalars,” Phys. Rev. D9 (1974) 2864.

[6] M.T. Grisaru and H.J. Schnitzer, “Reggeization of gauge vector mesons and unified theories,” Phys. Rev. D20 (1979) 784; “Reggeization of elementary fermions in arbitrary renormalizable gauge theories,” Phys. Rev. D21 (1980) 1952.

[7] M.T. Grisaru and H.J. Schnitzer, “Dynamical calculation of bound-state supermultiplets in $\mathcal{N}=8$ supergravity,” Phys. Letters 108B (1981) 196; “Bound states in $\mathcal{N}=8$ supergravity and $\mathcal{N}=4$ supersymmetric Yang–Mills theories,” Nucl. Phys. B204 (1982) 267.

[8] S. Mandelstam, Phys. Rev. B137 (1965) 949.

[9] D. Dicus, D.Z. Freedman and V.L. Teplitz, , Phys. Rev. D4 (1971) 2320.

[10] M.T. Grisaru and H-S. Tsao (unpublished).

[11] H. Kawai, D.C. Lewellen and S-H.H. Tye, “A relation between tree amplitudes of closed and open strings,” Nucl. Phys. B269 (1986).