ON THE SIZE AND LOCAL EQUATIONS
OF FIBRES OF GENERAL PROJECTIONS

ZIV RAN

ABSTRACT. For a general birational projection of a smooth nondegenerate projective $n$-fold from $\mathbb{P}^{n+c}$ to $\mathbb{P}^m$, $n < m \leq (n + c)/2$, all fibres have total length asymptotically bounded by $2\sqrt{n+1}$ and the fibres are locally defined by linear and quadratic equations.

INTRODUCTION

Let $X$ be a smooth variety of dimension $n$ in $P = \mathbb{P}^{n+c}$ over $\mathbb{C}$. Let $\Lambda$ be a general linear subspace of dimension $\lambda < c$ in $P$, perforce disjoint from $X$, and let

$$
\pi := \pi_\Lambda : X \to Q := \mathbb{P}^m, m := n + c - \lambda - 1
$$

be projection from $\Lambda$ restricted on $X$. Elements of $Q$ are viewed as $(\lambda + 1)$-dimensional linear subspaces $L \subset P$ containing $\Lambda$. Let

$$
X^\Lambda_k \subset Q
$$

be the $k$-fold locus of $\pi$, i.e. the locus in $Q$ of fibres of $\pi$ that have length $k$ or more. Note that it is $m - n = c - \lambda - 1$ conditions for $L$ to meet $X$, hence $k(m - n)$ conditions for $L$ to meet $X$ $k$ times; thus the expected codimension of $X^\Lambda_k$ in $Q$ is $k(m - n)$.

The study of the projections $\pi_\Lambda$, their fibres and the loci $X^\Lambda_k$ has a long history in classical through modern Algebraic Geometry (some of which is reviewed in [6], [3] and [5]). In the case of small $\lambda$, a real breakthrough in their modern study was obtained fairly recently by Gruson and Peskine [6], who gave complete results in the case where $\Lambda$ is a point. An alternate proof of the Gruson-Peskine theorem, and some partial extensions were given in [9], [10]. In particular, it was shown in [10] in the case where $\Lambda$ is a line, that a generic fibre of given length is reduced, i.e. a collection of distinct points.

The case of projection from a higher-dimensional center has remained more mysterious. Focusing to fix ideas on the case of projection of an $n$-fold to $\mathbb{P}^{n+1}$, Mather’s
work [8], as carried over to the setting of complex algebraic geometry by Alzati and Ot-
taviani [2] and summarized in [3], shows there that the projection has corank at most $\sqrt{n}$ at any point and that the number of distinct points in a fibre is at most $n$. For projections to higher $\mathbb{P}^m$ there are better bounds. A construction of Lazarsfeld [7] shows that there exist many examples of generic projections to $\mathbb{P}^{n+1}$ with points of corank $\sqrt{n}$, asymptotically the largest possible, and consequently, as shown by Beheshti-Eisenbud [3], such projections have schematic fibres of length that grows exponentially with $\sqrt{n}$ (more precisely the length is asymptotically at least $\frac{\sqrt{2\pi}}{\sqrt{m}} 2^{\sqrt{n}}$ for $n \gg 0$). Such fibres are automatically obstructed. The paper [3] also gives a certain bound on an invariant related to the fibre, extending some work of Mather [8].

In this paper we will prove an asymptotically sharp bound on the fibre length of (birational) general projections, which shows in particular that Lazarsfeld’s examples are essentially worst possible; namely, we will prove the following (see Corollary 4 and Corollary 5):

**Theorem.** Let $X$ be a smooth nondegenerate $n$-fold in $\mathbb{P}^{n+c}$ and let $n < m \leq \frac{n+c}{2}$. Then for the general projection of $X$ to $\mathbb{P}^m$, all fibres have total length at most $2 \max(2\sqrt{n} + n - 1, 1 + 2\sqrt{2(n-1)})$ and are locally at each point defined by linear and quadratic equations.

This result comes about as follows. We introduce a numerical measure called *order sequence*, different than intersection length but related to it, and pertaining to the contact or relative position of a subvariety $X$ and a linear space $L$ at an isolated point of their intersection, or more generally the relative position on an ambient variety (such as $\mathbb{P}^{n+c}$) of a subscheme (such as $X$) and a linear series (such as the series of hyperplanes through $L$). The basic properties of order sequences are developed in §2, §3 and §4. Our main technical result is an upper bound on this measure (Theorem 3). It is this upper bound that yields Corollary 4 and Corollary 5. As for the upper bound, it is a consequence of the first-order deformation theory associated to the order sequence, plus a ‘vanishing lemma’ which already appeared in [10] and which in turn is ultimately a reflection of ‘generic smoothness’ in char. 0.

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1. **Setup**

We work over $\mathbb{C}$. To state our results we need some preliminaries. First, notation-
wise, for a point $p$ on an $n$-dimensional subvariety $X \subset \mathbb{P}^{n+c}$, $T_pX$ denotes the Zariski tangent space; for $Z$ is a finite-length scheme, $\ell_p(Z)$ denotes local length at $p$. For a linear space $\Lambda$ of dimension $\lambda$ disjoint from $X$ we let $Q = \mathbb{P}^m$ denote the target space for the projection $\pi$ from $\Lambda$, i.e. the set of linear subspaces $L$ of dimension $\lambda + 1$ containing...
Λ. For $L \in \mathbb{Q}$ and $p \in L \cap X$, set

$$e_p = n + c - \dim(T_pX + T_pL).$$

Note $e_p$ is just the corank of the projection $\pi$ (on $X$) at $p$, i.e. $m - \dim(d\pi_p(T_pX))$. Also $e_p = \dim(\tilde{N}_{L,p} \cap \tilde{N}_{X,p})$ where $\tilde{N}_{L,p}$ denotes the fibre of the conormal bundle in $\mathbb{P}^{n+c}$ to $L$ at $p$ and ditto for $X$.

2. ORDER SEQUENCE AND FILTRATION: GENERAL CASE

The statement involves the notion of order sequence $(d_\bullet)$. We define it here in greater generality than needed for our application here.

This is associated to the following data:

- an ambient space $P$,
- a point $p \in P$,
- a linear system, i.e. a finite-dimensional subspace $V \subset H^0(A)$ for some line bundle $A$ over $P$, with evaluation map $e : V \otimes \mathcal{O}_P \to A$,
- an ideal $\mathcal{I} \subset \mathcal{O}_P$, such that $\mathcal{I}A + e(V)\mathcal{O}_P \subset \mathcal{O}_P$ contains $m^n_P A$ for $N >> 0$.

( The only case needed in this paper is where $P$ is a projective space, $A = \mathcal{O}(1)$, $V$ is the system of hyperplanes containing a linear subspace $L \subset P$ and $\mathcal{I}$ is the ideal of a variety $X \subset P$).

Back to the general case, we denote by $L$ and $X$ respectively the base scheme of $V$, whose ideal is $bs(V)$, i.e. the image of $V \otimes \mathcal{O}_P(-A) \to \mathcal{O}_P$, and the subscheme of $P$ corresponding to $\mathcal{I}$, so that $\mathcal{O}_X = \mathcal{O}_P/\mathcal{I}$. Thus, we are assuming that $p$ is an isolated point of $X \cap L$. To simplify matters we shall also assume that no nonzero element of $V$ vanishes on $X$ (i.e. that $X$ is ‘nondegenerate’ with respect to $V$).

The order sequence is defined as follows. First,

$$d_1 = \max(\text{ord}_p(\tilde{f}_1) : \tilde{f}_1 \in m_P A \text{ and } \exists 0 \neq y_1 \in V \text{ such that } y_1 - \tilde{f}_1 \in \mathcal{I}(A)_p).$$

Thus $d_1$ is the largest order at $p$ of any nonzero element in $e(V) + \mathcal{I}(A)$; equivalently, the largest order at $p$ when restricted on $X$ of any nonzero element of $V$. We set

$$f_1 = \tilde{f}_1|_X = y_1|_X.$$

Assuming $(d_1, y_1), ..., (d_i, y_i)$ are defined, then set

$$d_{i+1} = \text{ord}_p(\tilde{f}_{i+1})$$

as the largest order of any element $f_{i+1} \in e(V) + \mathcal{I}(A)$ not in the $\mathbb{C}$-span of of $\tilde{f}_1, ..., \tilde{f}_i$ modulo $\mathcal{I}(A)$. Thus there is an element $y_{i+1}$ in the restriction of $e(V)$ but not in the $\mathbb{C}$-span of $y_1, ..., y_i$ having order $d_{i+1}$ on $X$, and $d_{i+1}$ is largest with this property. This defines $(d_{i+1}, y_{i+1})$.  

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Thus the sequence \((d_1, y_1), \ldots, (d_k, y_k)\) is defined, and \((d_1, \ldots, d_k)\) is called the order sequence associated to the above data. Note that the sequence \((d_\bullet)\) is uniquely determined. It will be denoted \(\text{Ord}_p(V, I)\) or \(\text{Ord}_p(V, X)\). Note

\[ d_1 \geq d_2 \geq \ldots \geq d_k, k = \dim(V). \]

Also, \((y_1, \ldots, y_k)\) is a basis for \(V\) called an \textit{adapted} basis. It is not canonical, however if \(d_r > d_{r+1} = \ldots = d_s > d_{s+1}\) then the span of \(y_{r+1}, \ldots, y_s\) is uniquely determined modulo the span of \(y_1, \ldots, y_r\). Consequently, if we define the \textit{reduced} order sequence \(\text{Ord}_p^r(V, X) = (\hat{d}_\bullet)\) as the sequence of distinct values of \(\text{Ord}_p(V, X)\), Then \(V\) admits a canonical ascending \textit{order filtration} \(F_* V\) (depending on \(p\)) so that \(F_i V\) consists of the elements of order \(\geq \hat{d}_i\) and an adapted basis is simply a basis adapted to this filtration. The reduced order sequence and order filtration are used in the definition of order subsheaf in §4.

\textit{Remarks.} These are nonessential but possibly clarifying.

(i) The corresponding \(f_1, \ldots, f_k\) to \(y_1, \ldots, y_k\) are a set of generators \(\mod I\) - not necessarily minimal - of the ideal \(\mathcal{N} = I + bs(V)\), where \(bs(V)\) is the base ideal of \(V\), with the added property that they also generate the ‘normal cone’ \(\text{gr}^* \mathcal{N} = \bigoplus \mathcal{N} \cap m^i / \mathcal{N} \cap m^{i+1}\).

(ii) The sequence \(\text{Ord}_p(V, X)\) coincides with the ‘vanishing sequence’ (in the sense of Eisenbud-Harris [4]) associated to the restriction of \(V\) on \(X\) (in the case considered in [4] where \(X\) is a smooth curve, the order sequence is strictly decreasing but in higher dimension this is not true).

3. ORDER SEQUENCE AND FILTRATION: PROJECTIVE CASE

We will apply this construction in the case where \(P\) is a projective space, \(L\) is a linear subspace, \(V = H^0(I_L(1))\), the hyperplanes through \(L\), \(A = \mathcal{O}_p(1)\), and \(I\) is the ideal of a subvariety \(X\) (having finite intersection with \(L\)).

In the above situation with \(X, L = \mathbb{P}^{n+c-m} \subset P = \mathbb{P}^{n+c}, p \in L \cap X\) isolated, we define the order sequence \(\text{Ord}_p(L, X)\) as \(\text{Ord}_p(H^0(I_L(1)), X)\). It can be analyzed as follows. We will assume \(X\) is nondegenerate. We begin with the case \(c \geq m\), i.e. \(\dim(X) \leq \dim L\). Let \(y_1, \ldots, y_m\) be linear equations for \(L\) (i.e. essentially, a basis for \(H^0(I_L(1))\), so \(p\) is the origin. (In the case where \(L\) is a fibre for projection from \(\Lambda\), these will be coordinates on the target \(\mathbb{P}^m\)). Then we may choose complementary linear coordinates \(x_1, \ldots, x_{n+c-m}\) - thus in effect choosing a general projection of \(X\) unramifiedly onto a submanifold of \(L\)- so that local analytic equations for \(X\) in \(P\) have the form

\[
\begin{align*}
y_i - f_i(x), & \quad i = 1, \ldots, m, \\
f_{m+1}(x), & \quad f_c(x)
\end{align*}
\]
where \(f_1, ..., f_c\) are analytic functions in \(x_1, ..., x_{n+c-m}\) only, defined near \(p\). In other words, we represent \(X\) locally as a graph over the submanifold of \(L\) defined by \(f_{m+1}, ..., f_c\).

For simplicity we do this only in the projective setting as in §3, as follows. Consider the equations \(y_1, ..., y_m\) that have order 0 at \(p\). Note also that the \(f_i\) are not necessarily minimal generators for \(I_X \cap L \mod L\). This concludes the discussion of the case \(m < c\).

Now in case \(c < m\), i.e. \(\dim(X) > \dim(L)\), we may similarly represent \(X\) as a graph over all of \(L\), and this yields local equations for \(X\) of the form \(y_i - f_i(x), i = 1, ..., c\). By suitably choosing \(y_{c+1}, ..., y_m\), we may assume \(x_1, ..., x_{n+c-m}, y_{c+1}, ..., y_m\) are local coordinates on \(X\) and in particular \(y_{c+1}, ..., y_m\) have order 1 at \(p\). Then, replacing \(y_1, ..., y_c\) by an adapted basis for their span, we get an adapted basis \(y_{c+1}, ..., y_m\) of \(H^0(I_X(1))\) and order sequence \(\text{ord}_p(f_1), ..., \text{ord}_p(f_c), 1 = \text{ord}_p(y_{c+1}), ..., 1 = \text{ord}_p(y_m))\).

**Example 1.** Here \(c = 2 = m\). In \(\mathbb{A}^3\) with coordinates \(x, y_1, y_2\) let \(L\) be the \(x\)-axis with equations \(y_1 = y_2 = 0\) and let \(X\) be the curve \(y_1 - x^6, y_2 - x^3\). Then the order sequence at the origin is \((6, 3)\) with adapted basis \((x^6, x^3)\) or \((y_1, y_2)\). \(x\) is a coordinate on either \(X\) or \(L\) an in terms of this the schematic intersection \(X \cap L\) is defined by \(x^3\).

**Example 2.** Here \(c = 2 < m = 3\). In \(\mathbb{A}^4\) with coordinates \(x, y_1, y_2, y_3\) let \(L\) be the \(x\)-axis with equations \(y_1 = y_2 = y_3 = 0\) and let \(X\) be the smooth surface \(y_1 - x^6, y_2 - x^3\). Then note \(y_3, x\) are local coordinates on \(X\) and the order sequence is \((6, 3, 1)\) with adapted basis \((y_1, y_2, y_3)\). Note the schematic intersection \(X \cap L\) has ideal \((y_3, x^3)\) on \(X\) and \(y_1, y_2, y_3\) are nonminimal generators for it.

### 4. Order Subsheaf

The order filtration introduced in §2 can be used to define a natural modification (full-rank subsheaf) at \(p\) of the normal bundle \(N_L\), denoted \(N^{\text{ord}}_{L,p}\) and called the order subsheaf. For simplicity we do this only in the projective setting as in §3 as follows. Consider the order filtration \(0 \subset F_1 \subset ... \subset F_r \subset V = H^0(I_L(1))\) introduced above and let

\[
F_r^\perp \subset ... \subset F_1^\perp \subset V^* = H^0(I_L(1))^*
\]

be the dual filtration. Note that \(V^* \otimes O_L(1) = N_L\). Let \(K_1\) denote the kernel of the natural surjection

\[
N_L \to (V^* / F_1^\perp) \otimes (O_L / m_p^{d_1-1})(1).
\]
Then let $K_2$ denote the kernel of the natural surjection

$$K_1 \rightarrow (F_1^\perp / F_2^\perp) \otimes \mathbb{m}^{d_1 - 1}_{p} / \mathbb{m}^{d_2 - 1}_{p},$$

etc.; then finally $N^{\text{ord}}_{L,p} = K_r$. Also

$$N^{\text{ord}}_L = \bigcap_{p \in X \cap L} N^{\text{ord}}_{L,p}.$$ 

Also, define a subsheaf $N^{\text{ord},1}_L \subset N^{\text{ord}}_L$ analogously, replacing each $d_i - 1$ by $d_i$, and similarly for the local analogues $N^{\text{ord},1}_{L,p}$. These are used only in Remark 7.

5. Results

We continue with the above notations and for a point $p$ in the finite set $X \cap L$ we let $d_{\bullet,p}$ denote the order sequence at $p$. To simplify notation, set

$$a(X, L) = \sum_{p \in L \cap X} \sum_{i=1}^{e_p} \left( n + c - m + d_{i,p} - 2 \right) / d_{i,p} - 2 \right)$$

(2)

**Theorem 3.** Notations as above, assume $X$ is nondegenerate, $\Lambda \subset \mathbb{P}^{n+c}$ is a general linear subspace of codimension $m + 1$ disjoint from $X$. Let $L$ be a codimension-$m$ linear space containing $\Lambda$ such that for each $p \in L \cap X$, the pair $(L, p)$ is general with given order sequence $\text{Ord}_p(L, X)$. Then the following Projection Inequality holds:

$$a(X, L) \leq m.$$ 

Before starting the proof, we give some corollaries. Evidently, the only nonzero terms in the sum defining $a(X, L)$ above are those with $d_{i,p} \geq 2$ (of which there are indeed $e_p$ many by definition of corank). Those terms with $d_i = 2$ contribute 1 each while those with $d_i \geq 3$ contribute at least $n + c - m + 1 = \dim(L) + 1$. Thus, whenever $\dim(L) \geq m$, i.e. $n + c \geq 2m$, we have that $d_i \leq 2$, $\forall i$. In particular, the Theorem yields:

**Corollary 4.** Assume $2m \leq n + c$ and $X$ nondegenerate. Then the fibre $L \cap X$ is locally defined at each point $p$ by equations of order 1 and at most $e_p$ equations of order 2.

**Corollary 5.** Assumptions as above, for a general projection to $\mathbb{P}^{n+1}$, length of any fibre is at most $2 \max(2\sqrt{n} + n - 1, 1 + 2\sqrt{2(n - 1)})$ and the local length at any point is at most $2\sqrt{n}$.

**Proof of Corollary 5** Recall that $m$, the codimension of $L$, is also the dimension of the target projective space for projection from $\Lambda$. So here we are taking a general $\Lambda \subset \mathbb{P}^{n+c}$ of codimension $m + 1 = n + 2$. Then consider an arbitrary fibre $L_0 \cap X$ of projection from $\Lambda$, with order sequences $(d_{\bullet,p} = \text{Ord}_p(L_0, X))$ for $p \in S_0 := (L_0 \cap X)_{\text{red}}$ (a finite set). Let...
Let $L$ be general with the property that $S := (L \cap X)_{\text{red}}$ deforms $S_0$ and $\text{Ord}_p(L, X) = d_{\bullet, p}$ for all $p \in S$. Corollary \cite{4} shows that the local length of $L \cap X$ at $p \in S$ is at most $2e_p$. Mather’s theorem \cite{8} implies that $\sum e_p^2 \leq n$ and hence $|S| \leq n$ and of course $2e_p \leq 2\sqrt{n}$.

Now in [11] it is proven, using a constrained or ‘bordered’ Hessian calculation, that the maximum of the function $g(x) = \sum 2^{x_i}$ on the ball $B = \{(x_1, ..., x_n) : \sum x_i^2 \leq n\}$ in $\mathbb{R}^n$ is achieved when the vector $(x_1, ..., x_n)$ is, up to permutation, of the form either

(i) $(a, ..., a)$, or

(ii) $(a, ..., a, b)$ with $0 < a < 1 < b$ and $2^a/a = 2^b/b$, or

(iii) $(a, b, ..., b)$ with $0 < a < 1 < b$ and $2^a/a = 2^b/b$.

Values $g(x)$ are bounded above as follows:

For $x$ of type (i), we have $a \leq 1$ because $x \in B$, hence $g(x) \leq 2n$.

For $x$ of type (ii),

$$g(x) \leq (n - 1 + 2^{b-a})2^a \leq (n - 1 + 2\sqrt{n})2.$$  

For $x$ of type (iii) we have, using $b^2 \leq n/(n - 1)$ and $n \geq 3$:

$$g(x) \leq (1 + (n - 1)2^{b-a})2^a \leq (1 + (n - 1)2^{n/(n-1)}/2 < 2(1 + (n - 1)2^{1+1/(n-1)}) < 2(1 + (n - 1)2^{3/2}).$$

It follows that in our situation the max in question, with the $x_i$ nonnegative integers, cannot exceed $2\max(2\sqrt{n} + n - 1, 1 + 2\sqrt{2}(n - 1))$. This implies our conclusion. \(\square\)

As mentioned above, a construction due to Lazarsfeld \cite{21, Vol II, Cor. 7.2.18} shows that whenever the cotangent bundle $\Omega_X$ is nef, then for any $\Lambda$ of codimension $n + 2$ disjoint from $X$ there exist points $p \in X$ where the corank $e_p$ is roughly $\sqrt{n}$. In these cases, Beheshti and Eisenbud \cite{3} have shown, using Stirling’s approximation, that $\ell(L \cap X)$ is asymptotically at least $(\sqrt{2/\pi})2\sqrt{n}/\sqrt[3]{n}$. Thus the bound of Corollary \cite{5} is ‘sharp on the dominant term’.

Remark 6. Because a general projection to $\mathbb{P}^m, m > n + 1$ may be followed by a general projection $\mathbb{P}^m \cdots \to \mathbb{P}^{n+1}$ to yield a general projection to $\mathbb{P}^{n+1}$, the bounds of Corollary \cite{5} also hold to projection to $\mathbb{P}^m$ for all $m \geq n + 1$. But these bounds need not be sharp for $m >> n$.

Proof of Theorem \cite{3} We work locally at an isolated point $p \in X \cap L$. We will assume $m \leq c$ as the case $m > c$ is similar. We begin by representing $X$ locally as a graph over a submanifold of $L$, just as in \cite{3}. Let $y_1, ..., y_m$ be affine linear equations for $L$. Then,

\footnote{As the referee points out, the linear bound dominates for $n \leq 37$.}
as discussed above, we may choose local analytic coordinates \( x_1, ..., x_{n+c-m} \) on \( L \) so that local analytic equations for \( X \) have the form

\[
y_1 - f_1(x), ..., y_m - f_m(x), f_{m+1}(x), ..., f_c(x),
\]

with the \( f_i \) functions of the \( x \) coordinates only, vanishing at \( p \), so that \((d_1, ..., d_m) = \text{Ord}_p(L, X)\) is an order sequence for \( X, L, p \) with adapted basis \((f_\bullet)\). Here \( f_{m+1}, ..., f_c \) cut out the projection \( \bar{X} \) on \( L \) which is smooth hence they have linearly independent differentials at \( p \) that cut out the tangent space at \( p \) to \( \bar{X} \). NB \( \bar{X} \) is not the projection of \( X \) from \( \Lambda \) to \( \mathbb{P}^m \).

Now consider a local first-order deformation of \( L \) over \( \mathbb{C}[\epsilon]/(\epsilon^2) \). This corresponds to a subscheme \( \mathcal{L} \) of \( \mathbb{P}^{n+c} \times \text{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) \), flat over \( \text{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) \) and extending \( L \), or equivalently to an element \( v \) of the tangent space of the Grassmannian \( G(n+c-m, n+c) \) at \([L]\), i.e. \( v \in H^0(N_L) \). Equation-wise, \( \mathcal{L} \) is given near \( p \) by deforming each equation \( y_i \) to

\[
y_i' = y_i + \epsilon g_i(x), i = 1, ..., m, \epsilon^2 = 0
\]

(for a global deformation \( g_i \) is linear but this is local). This corresponds to the normal vector field \( v \) (section of \( N_L = \text{Hom}(\mathcal{I}_L, \mathcal{O}_L) \)) defined by \((y_i \mapsto g_i(x), i = 1, ..., m)\).

Now consider a first-order deformation \( u \) of the point \( p \) in the ambient space \( \mathbb{P}^{n+c} \). In coordinates \( u \) is given by a pair \((b, a)\) where \( b = (b_1, ..., b_m) \) is the \( y \)-component and \( a \) is the \( x \) component, i.e. the projection to \( L \). Compatibility of \( u \) and \( v \), i.e. the condition that the pair \((u, v)\) is tangent to the incidence locus of points on subspaces, reads in coordinate form

\[
b_i - g_i(0) = 0.
\]

The condition that \( u \) is tangent to \( X \) reads

\[
b_i - a \cdot \nabla f_i(0) = 0, i = 1, ..., m
\]

\[
a \cdot \nabla f_i(0) = 0, i = m + 1, ..., c.
\]

The condition on \( a \) alone is then

\[
a \cdot \nabla f_i(0) = g_i(0), i = 1, ..., m
\]

\[
a \cdot \nabla f_i(0) = 0, i = m + 1, ..., c.
\]

This then is exactly the condition that \( a \) represent a tangent vector at \( p \) that is compatible with \((g_\bullet(x))\) and tangent to \( X \). Note that these conditions imply that \( g_i(0) = 0 \) whenever \( \text{ord}_p(f_i) > 1 \).

Now assume \((L, p)\) is general with given order sequence \( \text{Ord}_p(L, X) \), and consider a deformation of \((p, L)\) that preserves \( \text{Ord}_p(L, X) \). Then \((L, p)\) must deform to a nearby
pair \((L', p')\) having similar position with respect to \(X\) at \(p'\), so \(X\) is locally given by similar equations

\[
y'_1 - f'_1, \ldots, y'_m - f'_m, f'_{m+1}, \ldots, f'_c
\]
as in (3), and because the order sequence is preserved, the \(f'_i\) must have the same order \(d_i\) as \(f_i\), albeit with respect to \(p'\). Now we have, since \(e^2 = 0\) (which implies \(\epsilon g_i(x + ea) = \epsilon g_i(x)\)), that for all \(i \leq m\),

\[(7) \quad y'_i - f'_i(x + ea) = y_i + \epsilon g_i(x + ea) - f_i(x + ea) = y_i + \epsilon g_i(x) - f_i(x) - ea \cdot \nabla f_i(x)
\]

where \(\text{ord}_p(\nabla f_i) \geq d_i - 1\). Because \(y'_i - f'_i(x + ea)\) must have order \(d_i\) at least, the terms of order \(< d_i\) in \(g_i(x)\) must cancel out with the \(ea \cdot \nabla f_i(x)\), while \(\text{ord}_p(\nabla f_i) \geq d_i - 1\) it follows that \(g_i(x)\) cannot have any term of order \(< d_i - 1\), in other words

\[(8) \quad \text{ord}_p(g_i) \geq d_i - 1, \forall i \leq m
\]
is a necessary condition on the deformation given by the \(g_i\) to preserve the order sequence.

We denote the subsheaf of \(N_L\) defined by the conditions (8) for given \(p\) by \(N^\text{orp}_{L,p}\) and set \(N^\text{orp}_L = \bigcap_p N^\text{orp}_{L,p}\). This is called the order-preserving subsheaf.

Remark 7 (Incidental remark). Though unimportant for the proof, one may wonder, what about sufficiency? Indeed sufficiency is irrelevant for the purpose of proving the Theorem: all we need is that there is some subsheaf, say \(N^1_L\) of \(N^\text{ord}_L\), containing \(N^\text{ord,1}\), such that a local first-order deformation of \(L\) preserves order sequence iff it is locally in \(N^1_L\) near each \(p \in L \cap X\). As the argument below will show, the exact nature of \(N^\text{orp}_L\) is immaterial. That said, note as to the sufficiency question that by (7), given \(g_i(x)\) of order \(\geq d_i - 1\), the function \(y'_i - f'_i(x + ea)\) as above will have order \(d_i\) for given \(a\) whenever the term of order \(d_i - 1\) in \(g_i\) is cancelled by the like term in \(a \cdot \nabla f_i(x)\). Here \(a\) must satisfy the conditions (6), meaning that the point \(p\) is moving compatibly with the motion of \(L\) and remains on \(X\) (which is not moving). So given \((g_\bullet(x))\), the sufficient condition at \(p\) to preserve order sequence is that there should exist \(a\) satisfying the conditions (6) such that

\[(g_i(x)) \equiv (a \cdot \nabla f_i(x)) \mod m_p^{d_i}.
\]
Again, as we saw, the conditions on \(a\) mean that \(a\) is the \(L\)-projection of a tangent vector at \(p\) that is compatible with the deformation \((g_\bullet(x))\) and tangent to \(X\). \(\square\)
Returning to the main argument, note the exact diagram

\[
\begin{array}{c}
0 \\
\uparrow \\
0 \rightarrow N^\text{ord}_L \rightarrow N_L \rightarrow A^\text{ord} \rightarrow 0 \\
| \\
| \\
| \\
0 \rightarrow N^\text{orp}_L \rightarrow N_L \rightarrow A^\text{orp} \rightarrow 0 \\
\uparrow \\
0
\end{array}
\] (9)

where \( A^\text{ord}, A^\text{orp} \) are torsion sheaves supported at \( L \cap X \). Also \( A^\text{ord} \) decomposes as direct sum of cyclic torsion sheaves, of total length at least \( a(X, L) \).

Now note the following:

**Lemma 8** (Generalized vanishing lemma). Suppose \((\Lambda, L, \mathbb{P}^M) \simeq (\mathbb{P}^{M-m-1}, \mathbb{P}^{M-m}, \mathbb{P}^M)\), and let \( N^0 \subset N_L \) be a modification on a finite subset disjoint from \( \Lambda \). Assume the image of

\[
H^0(L, N^0) \rightarrow H^0(\Lambda, N^0|_\Lambda) = H^0(\Lambda, N_L|_\Lambda)
\]

contains the image of

\[
H^0(\Lambda, N_\Lambda) \rightarrow H^0(\Lambda, N_L|_\Lambda).
\]

Then \( H^1(L, N^0(-1)) = 0 \).

**Proof.** The proof is identical to the proof of the Vanishing Lemma ([9, Lemma 5.8]). \( \square \)

**Remark.** In fact the map (11) is clearly surjective, so the assumption is actually equivalent to the map (10) being surjective. Intuitively the assumption as stated means that any given 1-parameter deformation of \( \Lambda \) is ‘covered’ by a compatible \( N^0 \) deformation of \( L \), where compatibility means having the same image in \( H^0(\Lambda, N_L|_\Lambda) \) (note that \( N_L|_\Lambda \) is a quotient of \( N_\Lambda \)). \( \square \)

Now as in [9], using \( \text{char}(\mathbb{C}) = 0 \), the hypothesis that \( \Lambda \) is general with respect to \( X \) and that \( L \) is general with given order sequences at each point \( p \in L \cap X \) shows that the hypotheses of Lemma 8 are satisfied for \( N^\text{orp}_L \) in place of \( N^0 \). Briefly, this results from the fact that the scheme \( Z \) parametrizing triples \( z = (\Lambda, L, p) \) with given order sequence-in fact already the corresponding reduced scheme \( Z_{\text{red}} \) projects generically surjectively to the Grassmannian \( G = G(\lambda, n + c) \) parametrizing \( \Lambda \), where the map \( Z \rightarrow G \) is the composite of the natural map of \( Z \) to the flag variety of pairs \( (\Lambda, L) \) and the projection of the latter to \( G \). Then by generic smoothness in char. 0 this induces a surjection on Zariski tangent spaces \( T_z Z_{\text{red}} \rightarrow T_\Lambda G \), a fortiori \( T_z Z \rightarrow T_\Lambda G = H^0(\Lambda, N_\Lambda) \) is surjective. But by definition of order preserving and generality of \( L \) with given order sequence, we
have a diagram

\[ T \mathcal{Z} \rightarrow H^0(\Lambda, N_\Lambda) \]
\[ \downarrow \quad \downarrow \]
\[ H^0(N_{\text{ord}}) \rightarrow H^0(N_L|_\Lambda) \]

(12)

where the right vertical arrow is obviously surjective. Thus, the hypotheses of Lemma 8 are satisfied. Consequently by the Lemma, the cohomology sequence of the bottom row of (9) twisted by -1 yields

\[ m = h^0(N_L(-1)) \geq \ell(A_{\text{ord}}) \geq \ell(A_{\text{ord}}) \geq a(X, L). \]

This proves the theorem. \qed

**Example 9.** This is an example of a 1-st order deformation preserving order sequence. Consider the situation of Example 1. Deforming \( L \) by \( \epsilon_1 = 6\epsilon x^5, \epsilon_2 = 3\epsilon x^2 \), i.e. deforming to \( y_1 = 6\epsilon x^5, y_2 = 3\epsilon x^2 \), the deformation of \( L \) has the same order sequence with adapted basis \((x + \epsilon)^6, (x + \epsilon)^3\) at the point on \( X \) with \( x \)-coordinate \(-\epsilon \), i.e. \((-\epsilon, 0, 0)\), which is a deformation of the origin on \( X \).

**REFERENCES**

1. User Andreas 677727, *Max of an exponential sum*, Stackexchange (2019), https://math.stackexchange.com/q/3286625 (sourced 2019-07-23).
2. A. Alzati and G. Ottaviani, *The theorem of Mather on generic projections in the setting of algebraic geometry*, Man. math. 74 (1992), 391–412.
3. R. Beheshti and D. Eisenbud, *Fibers of generic projections*, Comp. math 146 (2010), 435–456.
4. D. Eisenbud and J. Harris, *Limit linear series: Basic theory*, Inventiones math. 85 (1986), 337–371.
5. Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Pure and applied mathematics, a Wiley-Interscience series of texts, monographs and tracts, John Wiley and sons, 1978.
6. L. Gruson and C. Peskine, *On the smooth locus of aligned Hilbert schemes. The k-secant lemma and the general projection theorem*, Duke math. J. 162 (2013), 553–578, arxiv.org/1010.2399.
7. Robert Lazarsfeld, *Positivity in algebraic geometry*, Springer, 2004.
8. J. Mather, *Generic projections*, Ann. Math. 98 (1973), 226–245.
9. Z. Ran, *Unobstructedness of filling secants and the Gruson-Peskine general projection theorem*, Duke math. J. 164 (2015), 739–764, arxiv.org/1302.0824.
10. __________, *Superficial fibres of generic projections*, J. Inst. Math. Jussieu 17 (2018), no. 2, 419–425, doi:10.1017/S1474748016000050.

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