Optimal asymptotic bounds for spherical designs

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Abstract

In this paper we prove the conjecture of Korevaar and Meyers: for each \( N \geq c_d t^d \) there exists a spherical \( t \)-design in the sphere \( S^d \) consisting of \( N \) points, where \( c_d \) is a constant depending only on \( d \).

Keywords: Spherical designs, Brouwer degree, Marcinkiewicz–Zygmund inequalities, area-regular partitions

AMS subject classification: 52C35, 41A55, 41A63
1 Introduction

Let $S^d$ be the unit sphere in $\mathbb{R}^{d+1}$ with the Lebesgue measure $\mu_d$ normalized by $\mu_d(S^d) = 1$.

A set of points $x_1, \ldots, x_N \in S^d$ is called a spherical $t$-design if

$$\int_{S^d} P(x) \, d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d + 1$ variables, of total degree at most $t$. The concept of a spherical design was introduced by Delsarte, Goethals, and Seidel [12]. For each $t, d \in \mathbb{N}$ denote by $N(d, t)$ the minimal number of points in a spherical $t$-design in $S^d$. The following lower bound

$$N(d, t) \geq \begin{cases} \binom{d+k}{d} + \binom{d+k-1}{d} & \text{if } t = 2k, \\ 2 \binom{d+k}{d} & \text{if } t = 2k + 1, \end{cases}$$

is proved in [12].

Spherical $t$-designs attaining this bound are called tight. The vertices of a regular $t+1$-gon form a tight spherical $t$-design in the circle, so $N(1, t) = t+1$. Exactly eight tight spherical designs are known for $d \geq 2$ and $t \geq 4$. All such configurations of points are highly symmetrical, and optimal from many different points of view (see Cohn, Kumar [8] and Conway, Sloane [11]). Unfortunately, tight designs rarely exist. In particular, Bannai and Damerell [2, 3] have shown that tight spherical designs with $d \geq 2$ and $t \geq 4$ may exist only for $t = 4, 5, 7$ or 11. Moreover, the only tight 11-design is formed by minimal vectors of the Leech lattice in dimension 24. The bound (1) has been improved by Delsarte’s linear programming method for most pairs $(d, t)$; see [22].

On the other hand, Seymour and Zaslavsky [20] have proved that spherical $t$-designs exist for all $d, t \in \mathbb{N}$. However, this proof is nonconstructive and
gives no idea of how big $N(d,t)$ is. So, a natural question is to ask how $N(d,t)$ differs from the tight bound (1). Generally, to find the exact value of $N(d,t)$ even for small $d$ and $t$ is a surprisingly hard problem. For example, everybody believes that 24 minimal vectors of the $D_4$ root lattice form a 5-design with minimal number of points in $S^3$, although it is only proved that $22 \leq N(3,5) \leq 24$; see [6]. Further, Cohn, Conway, Elkies, and Kumar [7] conjectured that every spherical 5-design consisting of 24 points in $S^3$ is in a certain 3-parametric family. Recently, Musin [17] has solved a long standing problem related to this conjecture. Namely, he proved that the kissing number in dimension 4 is 24.

In this paper we focus on asymptotic upper bounds on $N(d,t)$ for fixed $d \geq 2$ and $t \to \infty$. Let us give a brief history of this question. First, Wagner [21] and Bajnok [1] proved that $N(d,t) \leq C_d t^{d/2}$ and $N(d,t) \leq C_d t^{d/3}$, respectively. Then, Korevaar and Meyers [14] have improved these inequalities by showing that $N(d,t) \leq C_d t^{(d^2+d)/2}$. They have also conjectured that $N(d,t) \leq C_d t^d$.

Note that (1) implies $N(d,t) \geq c_d t^d$. Here and in what follows we denote by $C_d$ and $c_d$ sufficiently large and sufficiently small positive constants depending only on $d$, respectively.

The conjecture of Korevaar and Meyers attracted the interest of many mathematicians. For instance, Kuijlaars and Saff [19] emphasized the importance of this conjecture for $d = 2$, and revealed its relation to minimal energy problems. Mhaskar, Narcowich, and Ward [16] have constructed positive quadrature formulas in $S^d$ with $C_d t^d$ points having almost equal weights. Very recently, Chen, Frommer, Lang, Sloan, and Womersley [9, 10] gave a computer-assisted proof that spherical $t$-designs with $(t + 1)^2$ points exist in $S^2$ for $t \leq 100$.

For $d = 2$, there is an even stronger conjecture by Hardin and Sloane [13] saying that $N(2,t) \leq \frac{1}{2} t^2 + o(t^2)$ as $t \to \infty$. Numerical evidence supporting the conjecture was also given.
In [4], we have suggested a nonconstructive approach for obtaining asymptotic bounds for $N(d, t)$ based on the application of the Brouwer fixed point theorem. This led to the following result:

For each $N \geq C_d t^{2d(d+1)/4d+2}$ there exists a spherical $t$-design in $S^d$ consisting of $N$ points.

Instead of the Brouwer fixed point theorem we use in this paper the following result from the Brouwer degree theory [18, Th. 1.2.6, Th. 1.2.9].

**Theorem A.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping and $\Omega$ an open bounded subset, with boundary $\partial \Omega$, such that $0 \in \Omega \subset \mathbb{R}^n$. If $(x, f(x)) > 0$ for all $x \in \partial \Omega$, then there exists $x \in \Omega$ satisfying $f(x) = 0$.

We employ this theorem to prove the conjecture of Korevaar and Meyers.

**Theorem 1.** For each $N \geq C_d t^d$ there exists a spherical $t$-design in $S^d$ consisting of $N$ points.

Note that Theorem 1 is slightly stronger than the original conjecture because it guarantees the existence of spherical $t$-designs for each $N$ greater than $C_d t^d$.

This paper is organized as follows. In Section 2 we explain the main idea of the proof. Then in Section 3 we present some auxiliary results. Finally, we prove Theorem 1 in Section 4.

## 2 Preliminaries and the main idea

Let $P_t$ be the Hilbert space of polynomials $P$ on $S^d$ of degree at most $t$ such that

$$\int_{S^d} P(x) d\mu_d(x) = 0,$$

equipped with the usual inner product

$$(P, Q) = \int_{S^d} P(x)Q(x)d\mu_d(x).$$
By the Riesz representation theorem, for each point $x \in S^d$ there exists a unique polynomial $G_x \in \mathcal{P}_t$ such that

$$(G_x, Q) = Q(x) \quad \text{for all } Q \in \mathcal{P}_t.$$ 

Then a set of points $x_1, \ldots, x_N \in S^d$ forms a spherical $t$-design if and only if

$$(G_{x_1} + \cdots + G_{x_N} = 0).$$

For a differentiable function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ denote by

$$\frac{\partial f}{\partial x}(x_0) := \left( \frac{\partial f}{\partial \xi_1}(x_0), \ldots, \frac{\partial f}{\partial \xi_{d+1}}(x_0) \right)$$

the gradient of $f$ at the point $x_0 \in \mathbb{R}^{d+1}$.

For a polynomial $Q \in \mathcal{P}_t$ we define the spherical gradient as follows:

$$(\nabla Q(x) := \frac{\partial}{\partial x} Q \left( \frac{x}{|x|} \right),$$

where $| \cdot |$ is the Euclidean norm in $\mathbb{R}^{d+1}$.

We apply Theorem A to the open subset $\Omega$ of a vector space $\mathcal{P}_t$,

$$\Omega := \left\{ P \in \mathcal{P}_t \ \bigg| \int_{S^d} |\nabla P(x)|d\mu_4(x) < 1 \right\}.$$

Now we observe that the existence of a continuous mapping $F : \mathcal{P}_t \to (S^d)^N$, such that for all $P \in \partial \Omega$

$$\sum_{i=1}^N P(x_i(P)) > 0,$$

readily implies the existence of a spherical $t$-design in $S^d$ consisting of $N$ points. Consider a mapping $L : (S^d)^N \to \mathcal{P}_t$ defined by

$$(x_1, \ldots, x_N) \overset{L}{\to} G_{x_1} + \cdots + G_{x_N},$$

and the following composition mapping $f = L \circ F : \mathcal{P}_t \to \mathcal{P}_t$. Clearly

$$(P, f(P)) = \sum_{i=1}^N P(x_i(P))$$
for each $P \in \mathcal{P}_t$. Thus, applying Theorem A to the mapping $f$, the vector space $\mathcal{P}_t$, and the subset $\Omega$ defined by (4), we obtain that $f(Q) = 0$ for some $Q \in \mathcal{P}_t$. Hence, by (2), the components of $F(Q) = (x_1(Q), \ldots, x_N(Q))$ form a spherical $t$-design in $S^d$ consisting of $N$ points.

The most naive approach to construct such $F$ is to start with a certain well-distributed collection of points $x_i$ ($i = 1, \ldots, N$), put $F(0) := (x_1, \ldots, x_N)$, and then move each point along the spherical gradient vector field of $P$. Note that this is the most greedy way to increase each $P(x_i(P))$ and make $\sum_{i=1}^N P(x_i(P))$ positive for each $P \in \partial \Omega$. Following this approach we will give an explicit construction of $F$ in Section 4, which will immediately imply the proof of Theorem 1.

3 Auxiliary results

To construct the corresponding mapping $F$ for each $N \geq C_d t^d$ we extensively use the following notion of an area-regular partition.

Let $\mathcal{R} = \{R_1, \ldots, R_N\}$ be a finite collection of closed sets $R_i \subset S^d$ such that $\bigcup_{i=1}^N R_i = S^d$ and $\mu_d(R_i \cap R_j) = 0$ for all $1 \leq i < j \leq N$. The partition $\mathcal{R}$ is called area-regular if $\mu_d(R_i) = 1/N$, $i = 1, \ldots, N$. The partition norm for $\mathcal{R}$ is defined by

$$\|\mathcal{R}\| := \max_{R \in \mathcal{R}} \text{diam } R,$$

where $\text{diam } R$ stands for the maximum geodesic distance between two points in $R$. We need the following fact on area-regular partitions (see Bourgain, Lindenstrauss [5] and Kuijlaars, Saff [15]):

**Theorem B.** For each $N \in \mathbb{N}$ there exists an area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| \leq B_d N^{-1/d}$ for some constant $B_d$ large enough.

We will also use the following spherical Marcinkiewicz–Zygmund type inequality:

**Theorem C.** There exists a constant $r_d$ such that for each area-regular partition $\mathcal{R} = \{R_1, \ldots, R_N\}$ with $\|\mathcal{R}\| < \frac{r_d}{m}$, each collection of points $x_i \in R_i$ ($i = 1, \ldots, N$), and each algebraic polynomial $P$ of total degree $m,$ the
inequality

\[ \int_{S^d} |P(x)| \, d\mu_d(x) \leq \frac{1}{N} \sum_{i=1}^N |P(x_i)| \leq \frac{3}{2} \int_{S^d} |P(x)| \, d\mu_d(x) \]

holds.

Theorem C follows naturally from the proof of Theorem 3.1 in [16].

**Corollary 1.** For each area-regular partition \( \mathcal{R} = \{R_1, \ldots, R_N\} \) with \( \|\mathcal{R}\| < \frac{r_d}{m+1} \), each collection of points \( x_i \in R_i \) \((i = 1, \ldots, N)\), and each algebraic polynomial \( P \) of total degree \( m \),

\[ \frac{1}{3^{d/2}} \int_{S^d} |\nabla P(x)| \, d\mu_d(x) \leq \frac{1}{N} \sum_{i=1}^N |\nabla P(x_i)| \leq 3 \sqrt{d} \int_{S^d} |\nabla P(x)| \, d\mu_d(x). \]

**Proof.** Since \( |\nabla P| = \sqrt{P_1^2 + \ldots + P_{d+1}^2} \) in \( S^d \), where \( P_j \) are polynomials of total degree \( m + 1 \), Corollary 1 is an immediate consequence of (6) applied to \( P_j \), \( j = 1, \ldots, d + 1 \). \( \square \)

## 4 Proof of Theorem 1

In this section we construct the map \( F \) introduced in Section 2 and thereby finish the proof of Theorem 1.

For \( d, t \in \mathbb{N} \), take \( C_d > (54dB_d/r_d)^d \), where \( B_d \) is as in Theorem B and \( r_d \) is as in Theorem C, and fix \( N \geq C_d t^d \). Now we are in a position to give an exact construction of the mapping \( F : \mathcal{P}_t \to (S^d)^N \) which satisfies condition (3). Take an area-regular partition \( \mathcal{R} = \{R_1, \ldots, R_N\} \) with

\[ \|\mathcal{R}\| \leq B_d N^{-1/d} < \frac{r_d}{54dt} \]

as provided by Theorem B, and choose an arbitrary \( x_i \in R_i \) for each \( i = 1, \ldots, N \). Put \( \varepsilon = \frac{1}{6 \sqrt{d}} \) and consider the function

\[ h_\varepsilon(u) := \begin{cases} u & \text{if } u > \varepsilon, \\ \varepsilon & \text{otherwise.} \end{cases} \]
Take a mapping \( U : \mathcal{P}_t \times S^d \to \mathbb{R}^{d+1} \) such that
\[
U(P, y) = \frac{\nabla P(y)}{h_\varepsilon(|\nabla P(y)|)}.
\]
For each \( i = 1, \ldots, N \) let \( y_i : \mathcal{P}_t \times [0, \infty) \to S^d \) be the map satisfying the differential equation
\[
\frac{d}{ds} y_i(P, s) = U(P, y_i(P, s))
\]
with the initial condition
\[
y_i(P, 0) = x_i,
\]
for each \( P \in \mathcal{P}_t \). Note that each mapping \( y_i \) has its values in \( S^d \) by definition of spherical gradient \( \text{(3)} \). Since the mapping \( U(P, y) \) is Lipschitz continuous in both \( P \) and \( y \), each \( y_i \) is well defined and continuous in both \( P \) and \( s \), where the metric on \( \mathcal{P}_t \) is given by the inner product. Finally put
\[
F(P) = (x_1(P), \ldots, x_N(P)) := (y_1(P, r_d/3t), \ldots, y_N(P, r_d/3t)).
\]
By definition the mapping \( F \) is continuous on \( \mathcal{P}_t \). So, as explained in Section 2, to finish the proof of Theorem 1 it suffices to prove

**Lemma 1.** Let \( F : \mathcal{P}_t \to (S^d)^N \) be the mapping defined by \( \text{(10)} \). Then for each \( P \in \partial \Omega \),
\[
\frac{1}{N} \sum_{i=1}^{N} P(x_i(P)) > 0,
\]
where \( \Omega \) is given by \( \text{(4)} \).

**Proof.** Fix \( P \in \partial \Omega \). For the sake of simplicity we write \( y_i(s) \) in place of \( y_i(P, s) \). By the Newton-Leibniz formula we have
\[
\frac{1}{N} \sum_{i=1}^{N} P(x_i(P)) = \frac{1}{N} \sum_{i=1}^{N} P(y_i(r_d/3t))
\]
\[
= \frac{1}{N} \sum_{i=1}^{N} P(x_i) + \int_{0}^{r_d/3t} \frac{d}{ds} \left[ \frac{1}{N} \sum_{i=1}^{N} P(y_i(s)) \right] ds.
\]

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Now to prove Lemma 1, we first estimate the value

\[
\left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right|
\]

from above, and then estimate the value

\[
\frac{d}{ds} \left[ \frac{1}{N} \sum_{i=1}^{N} P(y_i(s)) \right]
\]

from below, for each \( s \in [0, r_d/3t] \). We have

\[
\left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right| = \left| \sum_{i=1}^{N} \int_{R_i} P(x_i) - P(x) \, d\mu_d(x) \right| \leq \sum_{i=1}^{N} \int_{R_i} |P(x_i) - P(x)| \, d\mu_d(x)
\]

\[
\leq \frac{\|\mathcal{R}\|}{N} \sum_{i=1}^{N} \max_{z \in S^d: \text{dist}(z, x_i) \leq \|\mathcal{R}\|} |\nabla P(z)|
\]

where \( \text{dist}(z, x_i) \) denotes the geodesic distance between \( z \) and \( x_i \). Hence, for \( z_i \in S^d \) such that \( \text{dist}(z_i, x_i) \leq \|\mathcal{R}\| \) and

\[
|\nabla P(z_i)| = \max_{z \in S^d: \text{dist}(z, x_i) \leq \|\mathcal{R}\|} |\nabla P(z)|,
\]

we obtain

\[
\left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right| \leq \frac{\|\mathcal{R}\|}{N} \sum_{i=1}^{N} |\nabla P(z_i)|.
\]

Consider another area-regular partition \( \mathcal{R}' = \{R'_1, \ldots, R'_N\} \) defined by \( R'_i = R_i \cup \{z_i\} \). Clearly \( \|\mathcal{R}'\| \leq 2\|\mathcal{R}\| \) and so, by (8), we get \( \|\mathcal{R}'\| < r_d/(27d t) \). Applying inequality (7) to the partition \( \mathcal{R}' \) and the collection of points \( z_i \) we obtain that

\[
(12) \quad \left| \frac{1}{N} \sum_{i=1}^{N} P(x_i) \right| \leq 3\sqrt{d} \|\mathcal{R}\| \int_{S^d} |\nabla P(x)| \, d\mu_d(x) < \frac{r_d}{18\sqrt{d} t}
\]
for any $P \in \partial \Omega$. On the other hand, the differential equation (9) implies

$$\frac{d}{ds} \left[ \frac{1}{N} \sum_{i=1}^{N} P(y_i(s)) \right] = \frac{1}{N} \sum_{i=1}^{N} \frac{\left| \nabla P(y_i(s)) \right|^2}{h_\varepsilon(|\nabla P(y_i(s))|)} \geq \frac{1}{N} \sum_{i: |\nabla P(y_i(s))| \geq \varepsilon} |\nabla P(y_i(s))| \geq \frac{1}{N} \sum_{i=1}^{N} |\nabla P(y_i(s))| - \varepsilon.$$

(13)

Since

$$\left| \frac{\nabla P(y)}{h_\varepsilon(|\nabla P(y)|)} \right| \leq 1$$

for each $y \in S^d$, it follows again from (9) that $\left| \frac{dy_i(s)}{ds} \right| \leq 1$. Hence we arrive at

$$\text{dist}(x_i, y_i(s)) \leq s.$$

Now for each $s \in [0, r_d/3t]$ consider the area-regular partition $\mathcal{R}'' = \{R''_1, \ldots, R''_N\}$ given by $R''_i = R_i \cup \{y_i(s)\}$. By (8) we have

$$\|\mathcal{R}''\| < \frac{r_d}{54dt} + \frac{r_d}{3t};$$

so we can apply (7) to the partition $\mathcal{R}''$ and the collection of points $y_i(s)$. This and inequality (13) yield

$$\frac{d}{ds} \left[ \frac{1}{N} \sum_{i=1}^{N} P(y_i(s)) \right] \geq \frac{1}{N} \sum_{i=1}^{N} |\nabla P(y_i(s))| - \frac{1}{6\sqrt{d}} \geq \frac{1}{3\sqrt{d}} \int_{S^d} |\nabla P(x)|d\mu_d(x) - \frac{1}{6\sqrt{d}} = \frac{1}{6\sqrt{d}},$$

(14)

for each $P \in \partial \Omega$ and $s \in [0, r_d/3t]$. Finally, equation (11) and inequalities (12) and (14) imply

$$\frac{1}{N} \sum_{i=1}^{N} P(x_i(P)) > \frac{1}{6\sqrt{d}} \frac{r_d}{3t} - \frac{r_d}{18\sqrt{d}t} = 0.$$

(15)

Lemma 1 is proved.
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