The framed little 2-discs operad and
diffeomorphisms of handlebodies

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Abstract

The framed little 2-discs operad is homotopy equivalent to a cyclic operad. We show that the
derived modular envelope of this cyclic operad (that is, the modular operad freely generated
in a homotopy invariant sense) is homotopy equivalent to the modular operad made from
classifying spaces of diffeomorphism groups of 3-dimensional handlebodies with marked discs
on their boundaries. A modification of the argument provides a new and elementary proof
of Costello’s theorem that the derived modular envelope of the associative operad is homotopy
equivalent to the ‘open string’ modular operad made from moduli spaces of Riemann surfaces
with marked intervals on the boundary. Our technique also recovers a theorem of Braun that the
derived modular envelope of the cyclic operad that describes associative algebras with involution
is homotopy equivalent to the modular operad made from moduli spaces of unoriented Klein
surfaces with open string gluing.

1. Introduction

The purpose of this paper is to demonstrate how certain interesting spaces in low-dimensional
topology, namely classifying spaces of mapping class groups of 3-dimensional handlebodies
and oriented or unoriented surfaces with boundary, can be built up from relatively simple
cyclic operads via the modular envelope construction. Such models lead to graph complexes
that compute the cohomology of these groups, generalizing the well-known construction of the
ribbon graph complex that computes the cohomology of moduli spaces of Riemann surfaces.

Cyclic operads were introduced by Getzler and Kapranov [9]. Roughly speaking, a cyclic
operad is an operad in which the roles of inputs and outputs are exchangeable. A prototypical
example is the gluing of Deligne–Mumford–Knudsen compactified moduli spaces of genus 0
complex curves with marked points. Modular operads were introduced in [10]. A modular
operad is a cyclic operad together with self-gluings where an input can be contracted with the
output; a prototypical example is given by the compactified moduli spaces of complex curves
of arbitrary genus.

A cyclic operad $O$ generates a modular operad $\text{Mod}_! O$ known as the modular envelope
of $O$. The spaces of the modular envelope can be built as certain colimits of spaces of $O$ over
categories of graphs. This construction has a derived (homotopy invariant) version $\mathbb{L} \text{Mod}_! O$,
in which the colimit is replaced by a homotopy colimit; see Subsection 3.6 for details.

The framed little 2-discs operad $fD_2$ was introduced by Getzler [8] to describe homological
conformal field theories at genus 0. Topologically, (group complete) algebras over $fD_2$ are (up
to homotopy) 2-fold loops on spaces with a circle action [21], and the homology of $fD_2$ is the
operad that describes Batalin–Vilkovisky algebras. Although $fD_2$ is not a cyclic operad on the
nose, it is homotopy equivalent to a cyclic operad. Various homotopy equivalent cyclic models
exist, such as the conformal balls operad of Budney [2] or the compactified moduli spaces of
rational pointed curves with phase parameters from [15, Section 2.4] (see also [11]).

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In this paper, we shall consider a cyclic model for $fD_2$ made from diffeomorphism groups of 3-balls with marked 2-discs on their boundary (the diffeomorphisms fix the discs pointwise). This cyclic model is naturally the genus 0 part $\mathcal{H}bdy_0$ of the modular operad $\mathcal{H}bdy$ made from classifying spaces of diffeomorphism groups of 3-dimensional handlebodies with marked discs on their boundaries and with the operad composition given by gluing along the marked discs. These operads will be constructed precisely in Section 3. Our main theorem says roughly that the handlebody modular operad $\mathcal{H}bdy$ is freely generated (in the derived sense) by its genus 0 part.

**Theorem A.** There is a map of modular operads

$$\text{LMod}_2 \mathcal{H}bdy_0 \longrightarrow \mathcal{H}bdy,$$

which is an isomorphism on $\pi_0$ and a homotopy equivalence on all components except for the component corresponding to a solid torus with zero marked boundary discs.

The idea of the proof is the following. A complete disc system in a handlebody $K$ is a collection of 2-discs that partition $K$ into genus 0 pieces. Contractibility of the space of complete disc systems follows easily from the well-known contractibility of the usual disc complex (the issue with the solid torus stems from non-contractibility of the disc complex in this one exceptional case). Thus, the space of handlebodies is homotopy equivalent to the space of handlebodies equipped with complete disc systems. This space maps to the space of graphs by sending a complete disc system to its dual graph. The modular envelope of the genus 0 part $\mathcal{H}bdy_0$ also maps to the space of graphs. We show the homotopy equivalence of the theorem by a fibrewise comparison over the space of graphs.

Simplified versions of the above argument can in fact prove other interesting theorems of this type. It has long been known that ribbon graphs give orbi-cell decompositions of moduli spaces of Riemann surfaces. In [4, 5], Costello proved the following beautiful incarnation of this idea. Let $\mathcal{A}ss$ denote the associative operad. It has the structure of a cyclic operad, and, up to homotopy, it can be thought of as the framed little 1-discs operad. It is straightforward from the definitions to see that $\text{LMod}_2\mathcal{A}ss$ is the modular operad made from moduli spaces of metric ribbon graphs with gluing at external legs. Let $\mathcal{O}S$ denote the so-called open string moduli space modular operad made from classifying spaces of mapping class groups of oriented surfaces with marked intervals on their boundaries and with the operad composition maps given by gluing surfaces at marked intervals. By Teichmüller theory, the spaces of $\mathcal{O}S$ are homotopy equivalent to moduli spaces of Riemann surfaces with boundary and marked points on the boundary, and this modular operad governs open topological conformal field theories.

**Theorem B.** There is a map of modular operads $\text{LMod}_2\mathcal{A}ss \rightarrow \mathcal{O}S$ that is a $\pi_0$ isomorphism and is a homotopy equivalence on all components except that of the annulus with zero marked boundary intervals.

Costello’s proof uses the geometry of a certain compactification of the moduli spaces of Riemann surfaces. Our proof is instead based on the well-known contractibility of the arc complex.

Our method can also be used to prove an unoriented version of Theorem B. Let $\widetilde{\mathcal{O}S}$ denote the ‘unoriented open string moduli space’ modular operad made from classifying spaces of mapping class groups of unoriented surfaces (the surfaces need not be orientable) with boundary and marked oriented intervals on the boundary and with the operad compositions given by gluing of the internals (compatibly with their orientations). Next, consider the ‘hermitian associative’
operad $\mathcal{A}ss^h$ in $\mathcal{F} op$ that describes associative monoids with an involution $x \mapsto \overline{x}$ such that $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$. This cyclic operad can also be thought of (up to homotopy) as an unoriented version of the framed little 1-discs, meaning that the framings need not all have the same orientation. The derived modular envelope of $\mathcal{A}ss^h$ is the modular operad of moduli spaces of Möbius graphs, which are an unoriented variant of the notion of ribbon graphs.

**Theorem C.** There is a map of modular operads $\mathcal{L} Mod \rightarrow \mathcal{O} \overline{S}$ that is a $\pi_0$ isomorphism and is a homotopy equivalence on all components except those of the annulus or Möbius band with zero marked boundary intervals.

Braun [1] has recently independently introduced the operads $\mathcal{A}ss^h$ and $\mathcal{O} \overline{S}$ and given a very thorough study of their properties. In particular, he gives a proof of Theorem C following Costello’s method.

### 1.1. Graph homology and homotopy colimits

The underlying spaces of the derived modular envelope of a cyclic operad $\mathcal{O}$ are given as homotopy colimits of products of the spaces of $\mathcal{O}$; this provides a Bousfield–Kan homology spectral sequence that computes the homology of a modular envelope.

In [11], Salvatore and the author prove that the cyclic operad $\mathcal{H}bdy_0$ is formal over the reals, in the sense that $C_*(\mathcal{H}bdy_0; \mathbb{R})$ and $H_*(\mathcal{H}bdy_0; \mathbb{R}) = BV$ (the Batalin–Vilkovisky algebra cyclic operad) are quasi-isomorphic as cyclic operads in dg-$\mathcal{V}$ect. Together with Theorem A, this implies that the Bousfield–Kan spectral sequence for the real homology of $\mathcal{H}bdy$ collapses at the $E^2$ page; see Section 7 for more details.

The information contained in the Bousfield–Kan spectral sequence for a derived modular envelope can be reorganized in an interesting way using graph homology. The construction of graph homology was first introduced by Kontsevich [16, 17]; it was subsequently generalized to arbitrary cyclic and modular operads by Getlzer and Kapranov [9, 10], and further explained in [3]. The construction takes a cyclic operad $\mathcal{O}$ in chain complexes as input and produces a chain complex of graphs labelled by the operad as output; the homology of this chain complex is known as the $\mathcal{O}$-graph homology. We recall this construction in Section 7. Kontsevich showed that the graph homology of the Lie algebra cyclic operad $\mathcal{L}ie$ computes the cohomology of (outer) automorphism groups of free groups, and the graph homology of $\mathcal{A}ss$ computes the cohomology of moduli spaces of Riemann surfaces with boundary.

In general, given a cyclic operad $\mathcal{O}$ in $\mathcal{F} op$, we shall show in Section 7 that the results of Lazarev and Voronov [18] imply that the cohomology of the derived modular envelope $\mathcal{L} Mod(\mathcal{O})$ is computed by the graph homology complex of the cyclic operad $D(C_*\mathcal{O})$, which is the dg-dual of the cyclic operad $C_*\mathcal{O}$. It is well known that $\mathcal{A}ss$ is self-dual: $D(C_*\mathcal{A}ss) \simeq C_*\mathcal{A}ss$. Braun [1] has shown that $\mathcal{A}ss^h$ is also self-dual. Hence, one recovers the statement that the graph homology of $\mathcal{A}ss$ and $\mathcal{A}ss^h$ computes the cohomology of classifying spaces of diffeomorphism groups of oriented and unoriented surfaces respectively (except in the case of a Möbius band or annulus without marked boundary intervals). Moving from dimension 2 to dimension 3, the formality of $\mathcal{H}bdy_0$ and Theorem A imply the following theorem.

**Theorem D.** The $D(BV)$ graph homology computes the real cohomology of the classifying spaces of diffeomorphism groups of 3-dimensional oriented handlebodies (except in the case of a solid annulus with no marked discs).

The operad $D(BV)$ has been studied thoroughly in [24], although the cyclic structure is not discussed there.
2. Graphs, ribbon graphs and Möbius graphs

2.1. Graphs

The language of operads is built on the language of graphs. We therefore begin by laying out some basic definitions of graphs. For us a graph $\gamma$ will be a 1-dimensional finite simplicial complex; we allow graphs with multiple connected components and with isolated (0-valent) vertices, but we require that each component have at least one vertex of valence not equal to 1 (so an edge between two univalent vertices is excluded). A leg is a univalent vertex, an external edge is an edge that meets a leg, and an internal edge is one for which neither end is univalent. We write $E_{\text{int}}(\gamma)$ for the set of internal edges of $\gamma$, and $V(\gamma)$ for the set of non-leg vertices. The rank of a graph is the first Betti number of the graph, and a graph is a tree if it is of rank 0.

Given two graphs $\gamma_1, \gamma_2$, a subgraph collapse $\gamma_1 \to \gamma_2$ is a surjective simplicial map that restricts to a bijection on external edges and such that the inverse image of each vertex in $\gamma_2$ is a connected subgraph of $\gamma_1$. A tree collapse is a subgraph collapse whose vertex pre-images are trees. Note that graph isomorphisms are included in the class of tree collapses. We define the following categories of graphs. Let $G^r_\llbracket \to \rrbracket$ be the category of graphs and subgraph collapses, and let $G^r_\llbracket \to \rrbracket \subset G^r_\llbracket \to \rrbracket$ denote the subcategory of graphs and tree collapses. For reasons that will become clear when we define modular envelopes, we write $\operatorname{Mod}: G^r_\llbracket \to \rrbracket \hookrightarrow G^r_\llbracket \to \rrbracket$ for the inclusion functor. Given a finite set $P$, we write $G^r_\llbracket \to \rrbracket(P) \subset G^r_\llbracket \to \rrbracket(P)$ for the versions of these categories in which the graphs have $|P|$ legs equipped with a bijection to $P$ and the morphisms respect these labellings. Let $*_P \in G^r_\llbracket \to \rrbracket$ denote the corolla with one leg for each element of $P$. We shall often encounter the comma category $(\operatorname{Mod} \downarrow *_P)$ of graphs over $*_P$, which is equivalent to the full subcategory $G^r_\llbracket \to \rrbracket_{\text{conn}}(P)$ of $G^r_\llbracket \to \rrbracket(P)$ spanned by all connected graphs.

2.2. Graphs without bivalent vertices

A vertex is essential if it is not bivalent, or it is bivalent with two legs. A graph is said to be reduced if all vertices are essential. We write $G^r_\llbracket \to \rrbracket_\text{red} \subset G^r_\llbracket \to \rrbracket$ for the full subcategory of reduced graphs, and likewise for each of the other categories of graphs defined above.

There is a reduction functor

$$R: G^r_\llbracket \to \rrbracket \longrightarrow G^r_\llbracket \to \rrbracket_{\text{red}}$$

given by replacing each pair of edges meeting at an inessential bivalent vertex with a single edge. To define $R$ on morphisms, note that any tree collapse factors as a series of (non-loop) edge contractions $\pi_\epsilon: \gamma \to \gamma/\epsilon$. If both endpoints of $\epsilon$ are essential, then there is a unique edge $\overline{e}$ in $R(\gamma)$ such that $R(\gamma/\epsilon) = R(\gamma)/\overline{e}$, and we define $R(\pi_\epsilon) = \pi_{\overline{e}}$. If $\epsilon$ has an inessential endpoint, then $R(\gamma) = R(\gamma/e)$ and $R(\pi_\epsilon)$ is the identity. For $\gamma \in G^r_\llbracket \to \rrbracket_{\text{red}}$, consider the comma category $(R \downarrow \gamma)$ and the full subcategory $(R \downarrow \gamma)_0$ on those objects for which the map to $\gamma$ is an isomorphism in $G^r_\llbracket \to \rrbracket_{\text{red}}$ and there are no bivalent vertices over external edges of $\gamma$, and let

$$J: (R \downarrow \gamma)_0 \hookrightarrow (R \downarrow \gamma)$$

denote the inclusion functor. For the proof of Theorem A, we shall need to relate homotopy colimits over $(R \downarrow \gamma)$ to homotopy colimits over this subcategory.

**Lemma 2.1.** For any object $x \in (R \downarrow \gamma)$, the comma category $(x \downarrow J)$ has a final object.
Proof. Given a graph $\tau \in G_{\text{fr}}$, there is a canonical isotopy class of homeomorphisms $\tau \cong R(\tau)$. Now, for an object $x = (R(\tau) \xrightarrow{\alpha} \gamma)$ of $(R \downarrow \gamma)$, let $E$ denote the set of edges $e$ of $\tau$, which are mapped by $\alpha$ to vertices of $\gamma$ (note that this condition is isotopy invariant and hence well defined). The set $E$ is necessarily a forest in $\tau$. Hence, there is a tree collapse $\tau \to \tau/E$ and an induced tree collapse $R(\tau) \to R(\tau/E)$. By construction there is a unique isomorphism $R(\tau/E) \cong \gamma$ such that the diagram in $G_{\text{fr}}$ red,

\[
\begin{array}{ccc}
R(\tau) & \to & R(\tau/E) \\
\downarrow & & \downarrow \\
\gamma & \to & \gamma
\end{array}
\]

commutes. This diagram gives the desired final object. \qed

**Proposition 2.2.** For any functor $F : (R \downarrow \gamma) \to \mathcal{F}_{op}$, $J$ induces a homotopy equivalence:

$$
\text{hocolim}_{(R\downarrow\gamma)_0} F \cong \text{hocolim}_{(R\downarrow\gamma)} F.
$$

**Proof.** We define a homotopy inverse to the map induced by $J$. Sending $x \in (R \downarrow \gamma)$ to a final object of $(x \downarrow J)$ defines a functor $W : (R \downarrow \gamma) \to (R \downarrow \gamma)_0$ and a natural transformation from $\text{id}_{(R\downarrow\gamma)}$ to $W$. This natural transformation induces a deformation retraction of $\text{hocolim}_{(R\downarrow\gamma)} F$ onto $\text{hocolim}_{(R\downarrow\gamma)_0} F$. \qed

Let $\Delta$ denote the usual category of finite non-empty ordered sets and weakly order-preserving maps. Let $\Delta_{\text{semi}} \subset \Delta$ denote the subcategory of all injective maps.

**Proposition 2.3.** There is an equivalence of categories

$$(R \downarrow \gamma)_0 \simeq \prod_{e \in \text{Eist}(\gamma)} \Delta_{\text{semi}}^{op}.$$

**Proof.** If $R(\tau)$ is isomorphic to $\gamma$, then, in $\tau$, over each edge $e$ of $\gamma$ there is a set $S$ of edges meeting at inessential bivalent vertices. If one chooses an orientation of $e$, then $S$ inherits an ordering and hence determines an object of $\Delta_{\text{semi}}^{op}$. Contracting a subset $E$ of the edges of $S$ determines an injective order-preserving map $S \setminus E \hookrightarrow S$. This determines the equivalence of categories. \qed

2.3. Ribbon graphs and Möbius graphs

Here we shall recall some basic facts about ribbon graphs and a variant known as Möbius graphs.

**Definition 2.4.** A *ribbon graph* is a graph equipped with a cyclic ordering of the half-edges incident at each vertex.

A ribbon graph $\gamma$ can be canonically thickened to an oriented surface $S(\gamma)$ with boundary; legs of the ribbon graph correspond to marked intervals on the boundary of the surface, as shown in Figure 1.
Given an internal edge $e$ in a ribbon graph $\gamma$, one sees that the graph $\gamma/e$ formed by contracting $e$ inherits a ribbon structure from $\gamma$. We thus let $\mathcal{R}ib$ denote the category of ribbon graphs and tree collapses that respect the ribbon structures. Let $\mathcal{R}ib(P)$ denote the category of ribbon graphs with legs labelled by $P$.

Let $\text{Sym}: \mathcal{R}ib \to \mathcal{G}r$ denote the functor that forgets the ribbon structure. The reason for the name will become clear in Subsection 3.5 when we discuss the cyclic operad generated by a non-$\Sigma$ cyclic operad. Observe that $(\text{Sym} \downarrow *P)$ is equivalent to the full subcategory $\mathcal{R}ib_{\text{tree}}(P) \subset \mathcal{R}ib(P)$ of ribbon trees. This subcategory has several connected components, and each component has a final object given by a ribbon corolla.

2.4. Möbius graphs

We now define Möbius graphs, which are a slight variation on ribbon graphs and give models for non-orientable surfaces. A pre-Möbius structure on a graph consists of:

1. a cyclic ordering on the half-edges incident at each vertex;
2. a labelling of the edges by elements of $\mathbb{Z}/2$.

There is an equivalence relation on the set of pre-Möbius structures on a given graph $\gamma$ generated by the following operation: reverse the cyclic order on the half-edges at a vertex and reverse the $\mathbb{Z}/2$ labels on all non-loop edges incident at that vertex.

**Definition 2.5.** A Möbius graph is a graph equipped with an equivalence class of pre-Möbius structures.

A pre-Möbius structure on a graph $\gamma$ determines a canonical thickening to a (not necessarily orientable) surface $S(\gamma)$ with marked directed intervals on the boundary corresponding to the legs; the construction is the same as for ribbon graphs, except that an edge labelled by the non-trivial element $1 \in \mathbb{Z}/2$ now corresponds to a strip that is glued in with a half twist, and a leg at the end of an external edge labelled $1$ now gives a marked boundary interval oriented opposite to the cyclic order at the other end of the edge, as illustrated in Figure 2.

For example, the Möbius graph consisting of a single edge with both ends meeting at a bivalent vertex thickens to the annulus $S^1 \times I$ if the edge is labelled by $0$, and it thickens to a Möbius band if the edge is labelled by $1$. It is straightforward to see that if two
pre-Möbius structures are equivalent, then the corresponding thickenings will be canonically homeomorphic. Thus, a Möbius graph has a well-defined thickening.

As with ribbon graphs, if the source of a tree collapse has a Möbius structure, then the target inherits a well-defined structure and hence there is a category $\mathcal{M}\tilde{\mathcal{O}}$ of Möbius graphs and tree collapses that respect the Möbius structure. Let $\mathcal{M}\tilde{\mathcal{O}}(P)$ denote the category of Möbius graphs with $P$ legs.

As for ribbon graphs, let $\text{Sym}': \mathcal{M}\tilde{\mathcal{O}} \to \mathcal{G}$ denote the functor that forgets the Möbius structure. Observe that $(\text{Sym}' \downarrow *_P)$ is equivalent to the full subcategory $\mathcal{M}\tilde{\mathcal{O}}_{\text{tree}}(P) \subset \mathcal{M}\tilde{\mathcal{O}}(P)$ of Möbius trees. This subcategory has several connected components, and each component has a Möbius corolla as the final object.

3. Definition and homotopy theory of cyclic and modular operads

3.1. A convenient definition for various flavours of operads

Our perspective on cyclic and modular operads is heavily inspired by that of Costello [4]. Below we shall give definitions of cyclic and modular operads that are equivalent to any of the usual definitions but are slightly more convenient for the type of homotopy theory that we will be doing.

Forgetting the labelling of the legs on a graph gives a functor $\mathcal{G}r(P) \to \mathcal{G}$. Given two distinct elements $i, j \in P$, there is a gluing functor

$$\text{glue}_{i,j}: \mathcal{G}r(P) \to \mathcal{G}r(P \setminus \{i,j\})$$

defined on objects by gluing the $i$th leg to the $j$th leg and replacing the resulting pair of edges meeting at a bivalent vertex with a single edge, as shown in Figure 3.

Disjoint union of graphs makes $\mathcal{G}r$ into a symmetric monoidal category. The monoidal product and gluing functors are also defined for $\mathcal{G}r_+$, ribbon graphs and Möbius graphs.

In this paper, the term symmetric monoidal functor shall mean strong symmetric monoidal; that is, we require that the natural transformations $F(a) \otimes F(b) \to F(a \otimes b)$ be isomorphisms (although not necessarily identity morphisms).

**DEFINITION 3.1.** Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category.

1. A cyclic operad in $\mathcal{C}$ is a symmetric monoidal functor $\mathcal{O}: \mathcal{G}r \to \mathcal{C}$ that commutes with all graph gluing functors (up to natural isomorphism):

$$\begin{array}{c}
\mathcal{G}r(P) \xrightarrow{\text{glue}_{i,j}} \mathcal{G}r(P \setminus \{i,j\})
\end{array}$$

$$\xrightarrow{\mathcal{O}}$$

$$\begin{array}{c}
\mathcal{G}r \xrightarrow{\mathcal{O}} \mathcal{C}.
\end{array}$$

2. A modular operad in $\mathcal{C}$ is a symmetric monoidal functor $\mathcal{O}: \mathcal{G}r_+ \to \mathcal{C}$ that commutes with all graph gluing functors.

**Figure 3.** Gluing legs.
(3) A non-$\Sigma$ cyclic operad in $\mathcal{C}$ is a symmetric monoidal functor $\mathcal{O} : \mathcal{D}ib \to \mathcal{C}$ that commutes with all gluing functors.

(4) A Möbius cyclic operad in $\mathcal{C}$ is a symmetric monoidal functor $\mathcal{O} : \mathcal{M}ob \to \mathcal{C}$ that commutes with all gluing functors.

Note that one could easily reformulate the definition of traditional (non-cyclic) operads along the lines of the above definition by using graphs assembled from rooted corollas. As we do not need to talk about non-cyclic operads in this paper, we shall instead use the term operad to generically refer to any of the above flavours.

**Remark 3.2.** Operads are often required to have a unit in the space $\mathcal{O}(*)^2$ associated with a bivalent corolla. However, in this paper we do not impose this requirement.

We now sketch how this definition is equivalent (or very nearly so, in the case of modular operads) to other definitions found in the literature. Observe that commuting with the gluing functors implies that the values of a cyclic or modular operad $\mathcal{O}$ are determined by its values on disjoint unions of corollas. As it is strong symmetric monoidal, its values are in fact determined by what it does on single corollas. Any subgraph collapse can be factored as a sequence of edge contractions (non-loop edge contractions in the case of a tree collapse), so the functor $\mathcal{O}$ is completely determined by its values on corollas and what it does when contracting the edge in a gluing of two corollas (and self-gluings of a single corolla in the case of a modular operad).

Thus, a cyclic operad in our sense is equivalent to giving, for any finite set $P$, an object $\mathcal{O}(P)$ with an action of $\Sigma_P$ and, for any $i \in P, j \in Q$, an operadic composition map

$$i \circ_j : \mathcal{O}(P) \otimes \mathcal{O}(Q) \to \mathcal{O}(P \sqcup Q \setminus \{i, j\}),$$

satisfying certain compatibility conditions. A modular operad is a functor for which contractions of loops is allowed, so it additionally comes with a composition map

$$\circ_{ij} : \mathcal{O}(P) \to \mathcal{O}(P \setminus \{i, j\}),$$

for any $i, j \in P$. We shall sometimes find it convenient to informally define a cyclic or modular operad in these terms.

**Remark 3.3.** Note that our definition of modular operad is not quite equivalent to the original definition given by Getzler and Kapranov [10]. There are two differences. For them, a modular operad is equipped with a grading by genus, whereas our definition does not impose the genus grading and, rather, allows it as an optional additional datum that arises naturally in the case of modular envelopes (see Subsection 3.7). The second difference is that Getzler and Kapranov impose a stability condition requiring that their modular operads have no space of genus $g$ in valence $n$ when $3g - 3 + n < 0$. Our definition in this paper does not impose such a stability requirement. This is an important difference when we consider the framed little 2-discs as the genus 0 part of a modular operad, as the circle in valence 2 is crucial for our purposes.

A cyclic operad, as we have defined above, is an object of the functor category $\text{Fun}(\mathcal{G}r, \mathcal{C})$ equipped with the extra data of the natural isomorphisms making it symmetric monoidal and giving compatibility with the various graph gluing functors (analogous statements apply to the other flavours). A morphism of operads is a natural transformation of functors. If $\mathcal{C}$ has a Quillen model category structure, then we define (weak) homotopy equivalences of operads pointwise. In the context of $\text{dg-Vect}$, we shall sometimes refer instead to quasi-isomorphisms of operads.
The definitions of various types of operads we have given above are designed to be amenable to homotopy theory. If $\mathcal{C}$ is a Quillen model category, then we can equip $\text{Fun}(\mathcal{G}_r, \mathcal{C})$ with the projective model structure in which the weak equivalences and fibrations are detected pointwise (and likewise for the other flavours). We may now perform homotopy theoretic constructions on operads by performing them on the underlying functors. The result is a priori just a functor and not an operad, so we must then show that the resulting functor can be equipped with an appropriate operad structure in a canonical and meaningful way.

### 3.2. Connected graphs as a modular operad

Let $\mathcal{C}at$ denote the category of all small categories with Cartesian product as symmetric monoidal product. An important example of a modular operad is provided by categories of graphs themselves. The rule $P \mapsto \mathcal{G}_{\text{conn}}(P)$ constitutes a modular operad in $\mathcal{C}at$, where the composition maps $\iota_i \circ_j$ and $\iota_i \circ_j$ are defined by gluing the $i$th leg to the $j$th leg and forgetting the resulting bivalent vertex. In terms of Definition 3.1, as a functor $\mathcal{G}_{\text{conn}} \to \mathcal{C}at$, this modular operad sends $\gamma$ to $(\text{Mod} \downarrow \gamma)$.

The full subcategories $\mathcal{G}_{\text{tree}}(P) \subset \mathcal{G}_{\text{conn}}(P)$ of trees constitute a cyclic operad but not a modular operad. Similarly, the categories $\mathcal{R}\text{ib}_{\text{tree}}(P) \simeq (\text{Sym} \downarrow \ast_P)$ and $\mathcal{M}\text{öb}_{\text{tree}}(P) \simeq (\text{Sym}' \downarrow \ast_P)$ of ribbon trees and Möbius trees, respectively, also constitute cyclic operads.

### 3.3. Left Kan extensions and homotopy left Kan extensions

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories with $\mathcal{C}$ cocomplete. Consider functors

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{C} \\
\downarrow G & & \downarrow \mathcal{B} \\
\end{array}
\]

Recall that the left Kan extension of $F$ along $G$ is a functor $G_!F: \mathcal{B} \to \mathcal{C}$ defined on objects by the colimit

\[
G_!F(b) = \colim_{(G \downarrow b)} F \circ j_b,
\]

where $(G \downarrow b)$ is the comma category of objects in $\mathcal{C}$ over $b$ and $j_b: (G \downarrow b) \to \mathcal{C}$ forgets the morphism to $b$ (to simplify the notation we often omit writing $j_b$). Left Kan extensions possess a universal property: the functor $G_!F$ comes with a natural transformation $F \Rightarrow G_!F \circ P$ that is initial among natural transformations from $F$ to functors factoring through $P$.

If $\mathcal{C}$ is a Quillen model category (such as topological spaces or chain complexes), then there is a homotopy invariant (or, derived) version known as the homotopy left Kan extension $L(G_!F)$; it is given by the formula

\[
L(G_!F)(b) = \hocolim_{(G \downarrow b)} F \circ j_b.
\]

This construction is homotopy invariant in the following sense: a natural transformation $F \Rightarrow F'$ that is a pointwise homotopy equivalence induces a natural transformation $L(G_!F) \Rightarrow L(G_!F')$ that is also a pointwise homotopy equivalence. In fact, this is the left derived functor of left Kan extension with respect to the projective model structure on the functor categories.

There is a homotopy coherent version of the universal property for homotopy left Kan extensions, which the interested reader may work out.
Note that there is a ‘Fubini theorem’ for both ordinary and homotopy colimits:
\[
colim F \cong \colim G_1 F \quad \text{and} \quad \hocolim F \simeq \hocolim G_1 F.
\]

### 3.4. Homotopy colimits, diagrams in \(\mathcal{C}at\) and Thomason’s Theorem

At several points we shall be taking homotopy colimits of diagrams in \(\mathcal{F}op\) obtained from diagrams in \(\mathcal{C}at\) by applying the classifying space functor \(B\) (that is, geometric realization of the nerve) pointwise. Here, we briefly recall a couple of useful tools for this situation.

Given a functor \(F: \mathcal{C} \to \mathcal{C}at\), the Grothendieck construction on \(F\), denoted by \(\mathcal{C} \int F\) is the category in which objects are pairs \((x, y) \in F(x)\), and a morphism \((x, y) \to (x', y')\) consists of an arrow \(f \in \hom_\mathcal{C}(x, x')\) and an arrow \(g \in \hom_{\mathcal{F}}(f, y, y')\). By Thomason’s Theorem [22, Theorem 1.2], there is a natural homotopy equivalence
\[
\hocolim F \cong \int B \mathcal{C} F.
\]

As a special case, if \(\mathcal{C}\) is a group \(G\) (a category with a single object \(*\) and all arrows invertible), then \(BF(*,\ast)\) is space with a \(G\) action and \(B(G \int \mathcal{F})\) is homotopy equivalent to the quotient \((B\mathcal{F})_hG\).

If \(\mathcal{C} = \Delta^{op}_{semi}\) then \(F\) is a semi-simplicial category, \(BF\) is a semi-simplicial space and \(B(\Delta^{op}_{semi} F) \simeq \hocolim F\) is equivalent to the geometric realization of this semi-simplicial space.

### 3.5. From non-\(\Sigma\) or Möbius cyclic operads to cyclic operads

Given a non-\(\Sigma\) cyclic operad \(O: \mathcal{R}ib \to \mathcal{C}\), left Kan extension along \(\text{Sym}: \mathcal{R}ib \to \mathcal{G}r\) gives a cyclic operad \(\text{Sym}_0O\) called the symmetrization of \(O\). The fact that this is indeed a cyclic operad follows easily from the properties of \(O\) and the fact that \(\gamma \mapsto (\text{Sym} \downarrow \gamma) \simeq \coprod_{v \in V(\gamma)} \mathcal{R}ib_{\text{tree}}(*_v)\) is a cyclic operad (where \(*_v\) is the corolla in \(\gamma\) at \(v\)) in \(\mathcal{C}at\). Explicitly, if \(*_P\) is a corolla with a set of legs \(P\), then

\[
\text{Sym}_0O(*_P) = \colim_{\mathcal{R}ib_{\text{tree}}(P)} O.
\]

Likewise, if \(O: \mathcal{M}ob \to \mathcal{C}\) is a Möbius cyclic operad, then it has a symmetrization defined by left Kan extension along \(\text{Sym}': \mathcal{M}ob \to \mathcal{G}r\). In this case, the symmetrization sends \(*_P\) to \(\colim_{\mathcal{M}ob_{\text{tree}}(P)} O\).

For non-\(\Sigma\) or Möbius cyclic operads in a model category such as \(\mathcal{F}op\), there is a derived symmetrization given by homotopy left Kan extension. However, as each component of the categories \((\text{Sym} \downarrow *_P)\) and \((\text{Sym}' \downarrow *_P)\) has a final object, the derived symmetrization is homotopy equivalent to the ordinary symmetrization.

### 3.6. Modular envelopes: from cyclic operads to modular operads

The **modular envelope** of a cyclic operad \(O\) is the modular operad given by the left Kan extension \(\text{Mod}_0O\) along the functor \(\text{Mod}: \mathcal{G}r \to \mathcal{G}r_+\). This should be thought of as the modular operad freely generated by \(O\). As with symmetrizations, one can easily check that this indeed defines a modular operad. Explicitly, the modular envelope sends the \(P\)-legged corolla \(*_P\) to

\[
\text{Mod}_0O(*_P) = \colim_{\mathcal{G}r_{\text{conn}}(P)} O.
\]

The modular operad composition morphisms \(i_{ij}\) and \(\alpha_{ij}\) are induced by the corresponding composition morphisms on the categories \((\text{Mod} \downarrow *_P) \simeq \mathcal{G}r_{\text{conn}}(P)\).

When \(O\) is a cyclic operad in a model category such as \(\mathcal{F}op\), then there is a derived (homotopy invariant) version of the above construction, known as the derived modular
envelope, $\text{LMod}_1\mathcal{O}$. Its value on a corolla $*_P$ is

$$\text{LMod}_1\mathcal{O}(*_P) = \text{holim}_{\mathcal{G}_{\text{conn}}(P)} \mathcal{O}.$$ 

One easily checks that this has the structure of a modular operad.

Unlike for symmetrizations of non-$\Sigma$ or Möbius cyclic operads, the derived modular envelope of a cyclic operad can often be very different from the ordinary version of the modular envelope. Our focus in this paper is on derived modular envelopes.

3.7. Genus grading in modular operads

Often it is the case that a modular operad $\mathcal{O}$ has an internal grading by ‘genus’. This means that the object $\mathcal{O}(*_P)$ is a coproduct

$$\mathcal{O}(P) = \coprod_{g \geq 0} \mathcal{O}_g(p),$$

the composition morphism $i_j$ is of degree 0 in the sense that it sends $\mathcal{O}_g(P) \times \mathcal{O}_h(Q)$ to $\mathcal{O}_{g+h}(P \sqcup Q \setminus \{i, j\})$ and the loop contraction map $\circ_{ij} : \mathcal{O}(P) \rightarrow \mathcal{O}(P \setminus \{i, j\})$ is of degree 1, meaning that it sends the $g$ component to the $g + 1$ component.

Note that the modular envelope and the derived modular envelope of a cyclic operad always have a genus grading. This is because the modular operad of categories of connected graphs, $*_P \mapsto \mathcal{G}_{\text{conn}}(P)$, has a genus grading, with the genus $g$ piece consisting of the component of graphs of rank exactly $g$.

If $\mathcal{O}$ is a modular operad with a genus grading, then the genus 0 part $\mathcal{O}_0$ forms a cyclic operad.

4. Various cyclic and modular operads

4.1. The cyclic operads $\text{Ass}$ and $\text{Ass}^h$

Let $\text{Ass}$ denote the associative cyclic operad in $\mathcal{J}_{\text{op}}$. It is constructed as the symmetrization of the non-$\Sigma$ cyclic operad that is the constant functor sending any ribbon graph to a single point. Thus, $\text{Ass}(*_P)$ is a discrete set of points with one point for each ribbon corolla with $P$ legs, or equivalently, one point for each cyclic ordering of $P$.

The hermitian associative operad, $\text{Ass}^h$, is defined as the symmetrization of the Möbius cyclic operad that sends any graph to a point. Thus, $\text{Ass}^h(*_P)$ can be identified with the set of isomorphism classes of Möbius corollas with $P$ legs.

4.2. The open string moduli space modular operads $\mathcal{OS}$ and $\mathcal{OS}^b$

Here, we construct modular operads from mapping class groups of oriented and unoriented surfaces with marked intervals on their boundary and ‘open string’ gluing.

Let $S$ be a compact connected surface with non-empty boundary and equipped with $n \geq 0$ intervals marked on its boundary and parametrized by $[0, 1]$. We do not require that $S$ be orientable, but if it is equipped with an orientation, then we require that the intervals be parametrized compatibly with the orientation. Such a surface is called admissible if it is neither (1) a disc with strictly fewer than two marked boundary intervals, nor (2) an annulus or Möbius band with zero marked boundary intervals. (We will be forced to consider the annulus and the Möbius band because, although they are not admissible, they can be formed by gluing together admissible surfaces.)

Let $\text{Diff}(S)$ denote the group of diffeomorphisms that fix the marked intervals pointwise and preserve the orientation if $S$ is equipped with an orientation. By Earle and Eells [6], Earle and Schatz [7] and Gramain [12], if $S$ is admissible, then the components of $\text{Diff}(S)$ are
contractible. The mapping class group \( \mathcal{MCG}(S) \) is the group \( \pi_0 \text{Diff}(S) \) of isotopy classes of diffeomorphisms. The identity component of the diffeomorphism group of an annulus or Möbius band is homotopy equivalent to \( S^1 \).

To form a modular operad, we replace the mapping class groups with equivalent groupoids. This construction is a variation of the construction used in [23]. Given a finite set \( P \) (possibly empty), let \( \mathcal{F}(P) \) denote the groupoid in which:

1. objects are admissible oriented surfaces with marked intervals in bijection with \( P \). If \( P = \emptyset \), then we also allow an annulus;
2. morphisms are isotopy classes of orientation-preserving diffeomorphisms that respect the marked intervals.

Similarly, let \( \tilde{\mathcal{F}}(P) \) denote the unoriented version of the above category (when \( P = \emptyset \), then we allow both the annulus and the Möbius band as objects). Sending the corolla \( \ast_P \) to \( \mathcal{S}(P) \) or \( \tilde{\mathcal{S}}(P) \) endows these two families of groupoids each with the structure of a modular operad in \( \mathcal{C}at \) with the composition morphisms induced by gluing pairs of marked intervals together via a direction-reversing identification \([0, 1] \rightarrow [0, 1] \).

For \( P \neq \emptyset \), the components of the space \( B\mathcal{F}(P) \) (or \( B\tilde{\mathcal{F}}(P) \)) have the homotopy type of \( BM\mathcal{MCG}(S) \) for various admissible surfaces \( S \). Note that this statement fails when \( P \) is empty for the components corresponding to the annulus or the Möbius band.

**Definition 4.1.** The oriented open string modular operad \( \mathcal{OS} \) is given by \( \mathcal{OS}(\ast_P) = B\mathcal{S}(P) \). Similarly, the unoriented open string modular operad \( \tilde{\mathcal{OS}} \) is given by \( \tilde{\mathcal{OS}}(\ast_P) = B\tilde{\mathcal{S}}(P) \).

The modular operads \( \mathcal{OS} \) and \( \tilde{\mathcal{OS}} \) have genus gradings; however, *these gradings do not correspond to the genera of the surfaces.* Rather, the genus \( g \) part of \( \mathcal{OS}(\ast_P) \) is the classifying space of the subgroupoid \( \mathcal{F}_g(P) \) of surfaces with first Betti number \( b_1 = g \).

**Proposition 4.2.** There are homotopy equivalences of cyclic operads,

\[
\mathcal{OS}_0 \simeq \text{Ass} \quad \text{and} \quad \tilde{\mathcal{OS}}_0 \simeq \text{Ass}^h.
\]

**Proof.** By definition of the cyclic associative operad,

\[
\text{Ass}(\ast_P) = \text{Sym}_npt(\ast_P) \simeq \mathbb{I}\text{Sym}_npt(\ast_P) = B(\text{Sym} \downarrow \ast_P),
\]

and \( (\text{Sym} \downarrow \ast_P) \) is the groupoid of ribbon corollas with a set of legs \( P \). Sending a ribbon corolla to its thickening defines an equivalence of groupoids \( (\text{Sym} \downarrow \ast_P) \simeq \mathcal{F}_0(P) \). This gives a homotopy equivalence \( \text{Ass}(\ast_P) \simeq \mathcal{OS}_0(\ast_P) \) that is easily seen to be compatible with the cyclic operad structures. The argument of \( \text{Ass}^h \) is the same. \( \square \)

### 4.3. The handlebody modular operad \( \mathcal{Hbdy} \)

Let \( K = K_{g,n} \) be a 3-dimensional compact connected oriented handlebody of genus \( g \geq 0 \) with \( n \geq 0 \) disjoint 2-discs marked on its boundary. The boundary \( \partial K \) is a closed surface of genus \( g \) with \( n \) embedded discs. We call \( K \) admissible if \( (g, n) \neq (0, 0), (0, 1), (1, 0) \). (Note that excluding the case of genus 0 with one marked disc is not strictly necessary, but it will make the exposition somewhat clearer.) We will be forced to consider one exceptional handlebody, namely, the solid torus with zero marked boundary discs; this handlebody is not admissible but it can arise when gluing boundary discs on admissible handlebodies.
If $K$ is admissible, then the components of the diffeomorphism group of $K$ (fixing the marked discs pointwise) are contractible [14]; when $K$ is a solid torus with no marked boundary discs, then the identity component of the diffeomorphism group is homotopy equivalent to $S^1 \times S^1$. Let $\mathcal{H}(K)$ denote the group of isotopy classes of diffeomorphisms of $K$ that fix the discs pointwise, and let $\mathcal{MCG}(\partial K)$ denote the group of isotopy classes of diffeomorphisms of $\partial K$ fixing the discs pointwise. Restriction of diffeomorphisms to the boundary of $K$ is injective on the level of isotopy classes, so one can regard $\mathcal{H}(K)$ as a subgroup $\mathcal{MCG}(\partial K)$ (see [14]). Thus, $\mathcal{H}(K)$ is often called the handlebody subgroup of the mapping class group. Note that for a positive genus the handlebody subgroup depends on the choice of how to realize a given surface as the boundary of a handlebody, but any two choices determine conjugate subgroups. However, when the genus is 0, then the handlebody subgroup is equal to the entire mapping class group.

We now construct a modular operad $\mathcal{H}bdy$ from classifying spaces of handlebody groups with the compositions given by gluing marked discs together. However, as in the previous section we must replace the groups with equivalent groupoids. Given a finite set $P$ (possibly empty), let $\mathcal{H}(P)$ denote the groupoid in which:

1. objects are 3-dimensional compact connected oriented admissible handlebodies with an identification of the marked boundary discs with $\coprod P D^2$. If $P = \emptyset$, then we also allow the (non-admissible) solid torus as an object;
2. morphisms are isotopy classes of orientation-preserving diffeomorphisms that respect the identification of the marked boundary discs.

Fix an orientation-reversing diffeomorphism $D^2 \to D^2$. Given $i \in P$ and $j \in Q$, gluing the $i$th disc to the $j$th disc using the fixed diffeomorphism defines a functor $\circ_{ij}: \mathcal{H}(P) \times \mathcal{H}(Q) \to \mathcal{H}(P \sqcup Q \setminus \{i, j\})$. The self-gluing functor $\circ_{ij}$ is defined similarly.

**Definition 4.3.** The handlebody modular operad $\mathcal{H}bdy$ is the functor $\mathcal{G}r_+ \to \mathcal{T}op$ defined on objects by sending a corolla $*P$ with a set of legs $P$ to the space $B\mathcal{H}(P)$ and defined on morphisms by gluing handlebodies together at the marked discs using the above gluing functors.

The handlebody modular operad clearly has a genus grading. Restricting to the genus 0 part of the handlebody groupoids gives a cyclic operad $\mathcal{H}bdy_0(*P) = B\mathcal{H}_0(P)$.

**Proposition 4.4.** The framed little 2-discs operad $fD_2$ is homotopy equivalent to $\mathcal{H}bdy_0$.

**Proof.** Let $\Pi: \mathcal{T}op \to \mathcal{C}at$ denote the fundamental groupoid functor. As the spaces of $fD_2$ are all $K(\pi, 1)$s (the groups $\pi$ here are ribbon braid groups), there is a homotopy equivalence of operads

$$B\Pi(fD_2) \simeq fD_2.$$  

We define a morphism of operads $\Pi(fD_2) \to \mathcal{H}_0$ in $\mathcal{C}at$ as follows. Write $fD_2(n)$ for the space of $n$ framed little discs in a standard disc. An object of $\Pi(fD_2(n))$ is a configuration of $n$ parametrized discs in a standard disc. Such a configuration determines an object of $\mathcal{H}_0(\text{Legs}(*_{n+1}))$ by thinking of the standard disc as the northern hemisphere on the boundary of a 3-ball and choosing a standard parametrization of the southern hemisphere. It is not hard to see that a path in $fD_2(n)$ determines an isotopy class of diffeomorphisms between the 3-balls corresponding to the endpoints. Hence, we have the desired functor. Moreover, it is not hard to see that this functor is an equivalence of categories and is compatible with the operadic compositions.
5. Proofs of Theorems B and C

The argument we present for Theorems B and C is a simplified version of the proof of Theorem A that we present in the next section.

Regarding $\mathcal{OS}$ as a cyclic operad for a moment, there is an inclusion of cyclic operads, $\mathcal{O}S_0 \hookrightarrow \mathcal{OS}$. As the target is actually a modular operad, the universal property of the homotopy left Kan extension provides a map of modular operads

$$\Phi: \mathbb{LMod}_! \mathcal{OS}_0 \to \mathcal{OS}.$$ 

We will show that this map is an equivalence. Similarly, in the unoriented setting, there is a map $\tilde{\Phi}: \mathbb{LMod}_! \tilde{\mathcal{OS}}_0 \to \tilde{\mathcal{OS}}$ that we will show to be a homotopy equivalence. It suffices to show that these maps are homotopy equivalences when evaluated on corolla $\ast_p$.

Let us first describe the above maps more explicitly. By Thomason’s Theorem

$$\mathbb{LMod}_! \mathcal{OS}_0(\ast_p) \simeq B \left( \mathcal{Gr}_{\text{conn}}(P) \int \mathcal{S}_0(P) \right).$$

An object of the category on the right is a graph $\tau \in \mathcal{Gr}_{\text{conn}}(P)$ and a labelling of each internal vertex $v$ by a surface $S_v \in \mathcal{S}_0(\text{In}(v))$ that is a disc with one marked boundary interval for each half-edge incident at $v$ (note that this is in essence just the datum of a ribbon graph structure on $\tau$). Gluing these labels together according to the edges of $\tau$ gives an object $S \in \mathcal{I}(P)$ and thus defines a functor

$$\phi: \mathcal{Gr}_{\text{conn}}(P) \int \mathcal{S}_0(P) \to \mathcal{I}(P).$$

After taking classifying spaces, this map gives a model for $\Phi$ up to homotopy.

For the proof we replace $\mathcal{S}(P)$ with a homotopy equivalent category, lift $\phi$ to this new category and show that the lift is actually an equivalence of categories.

5.1. Arc systems

Let $S$ be a surface with $n$ marked boundary intervals. An arc in $S$ is an unparametrized curve in $S$ that meets $\partial S$ only at its endpoints, which are required to be disjoint from any of the marked boundary intervals. Given an arc $C$, we may cut $S$ along $C$ to obtain a new (possibly disconnected surface) with $n + 2$ marked boundary intervals coming from the original $n$ marked intervals plus an additional marked interval on either side of the cut, as shown in Figure 4.

**Definition 5.1.** An arc system $\alpha$ in $S$ is a set of disjoint arcs $C_i$ in $S$ such that if one cuts $S$ along all of the $C_i$, then each component of the result is an admissible surface. An arc system $\alpha$ in $S$ is said to be complete if each component of the result of cutting $S$ along the arcs in $\alpha$ is an admissible disc.

Isotopy classes of arc systems form a category in which there is one arrow $\alpha \to \beta$ for each way of forgetting some of the arcs of $\alpha$ and identifying the isotopy class of the remaining arcs

**Figure 4.** Cutting along an arc to produce a surface with marked boundary intervals.
with $\beta$. Let $\mathcal{A}(S)$ denote the category of isotopy classes of non-empty arc systems in $S$, and $\mathcal{A}_0(S)$ denote the category of isotopy classes of complete arc systems (a full subcategory if $S$ is not a disc).

**Proposition 5.2.** For any admissible surface $S$, the spaces $B\mathcal{A}(S)$ and $B\mathcal{A}_0(S)$ are contractible.

**Proof.** The argument of Hatcher [13] shows that $\mathcal{A}(S)$ is contractible (note that our arc systems are slightly different from the arc systems considered by Hatcher as we allow parallel arcs, but it is straightforward to deal with this difference). We will show that the inclusion $\iota: \mathcal{A}_0(S) \hookrightarrow \mathcal{A}(S)$ is a homotopy equivalence on classifying spaces by Quillen’s Theorem A [20] and induction on the complexity of $S$. Let $b_1$ be the first Betti number of $S$ and let $n$ denote the number of boundary components. Order the pairs $(b_1, n)$ lexicographically. If $S$ is a disc ($b_1 = 0$), then $\mathcal{A}_0(S) = \mathcal{A}(S)$. Now suppose that $B\mathcal{A}_0(T) \simeq pt$ for all admissible surfaces $T$ with $(b_1(T), n(T)) < (b_1(S), n(S))$. Given any arc system $\alpha \in \mathcal{A}(S)$, let $\{S_i\}_{i \in I}$ denote the set of connected admissible surfaces resulting from cutting $S$ along the arcs in $\alpha$. Observe that each component has $(b_1(S_i), n(S_i)) < (b_1(S), n(S))$. There is an equivalence of categories,

$$(\iota \downarrow \alpha) \simeq \prod_{i \in I} \mathcal{A}_0(S_i),$$

and hence, by the inductive hypothesis, $B(\iota \downarrow \alpha)$ is contractible. Thus, $\iota$ induces a homotopy equivalence of classifying spaces.

**Remark 5.3.** In the exceptional cases of an annulus or Möbius band with zero marked boundary intervals, the space of complete arc systems is not contractible.

We can regard $S \mapsto \mathcal{A}_0(S)$ as a functor $\mathcal{I} \to \text{Cat}$ or $\tilde{\mathcal{I}} \to \text{Cat}$ (depending on whether $S$ is equipped with an orientation or not), and then form the Grothendieck constructions $\mathcal{I}(P) \int \mathcal{A}_0$ and $\tilde{\mathcal{I}}(P) \int \mathcal{A}_0$. By Proposition 5.2 and Thomason’s Theorem, we have the following lemma.

**Lemma 5.4.** The projections

$$\mathcal{I}(P) \int \mathcal{A}_0 \to \mathcal{I}(P) \quad \text{and} \quad \tilde{\mathcal{I}}(P) \int \mathcal{A}_0 \to \tilde{\mathcal{I}}(P)$$

are homotopy equivalences, except on the components corresponding to an annulus or Möbius band if $P = \emptyset$.

Now observe that $\phi$ lifts to a functor $\phi': \mathcal{I}_{\text{conn}}(P) \int \mathcal{I}_0 \to \mathcal{I}(P) \int \mathcal{A}_0$ by sending an $\mathcal{I}_0$-labelled graph to the surface obtained by gluing the labels and the complete arc system with one arc for each pair of boundary intervals that were glued, as illustrated in Figure 5.

Likewise, $\tilde{\phi}$ lifts to $\tilde{\phi'}: \mathcal{I}_{\text{conn}}(P) \int \tilde{\mathcal{I}}_0 \to \tilde{\mathcal{I}}(P) \int \mathcal{A}_0$. This next lemma now completes the proof of Theorems B and C.

**Lemma 5.5.** The functors $\phi'$ and $\tilde{\phi'}$ are equivalences of categories.
Figure 5. Gluing the labels of an $S_0$-labelled graph to produce a surface with a complete arc system.

Proof. By cutting surfaces along complete arc systems, we see that $\phi'$ and $\tilde{\phi}'$ are both essentially surjective and full. Faithfulness follows from the fact that the automorphism group of any admissible disc (that is, an object of $S_0(P)$ or $\tilde{S}_0(P)$) is trivial. \qed

6. Proof of Theorem A

The argument begins along the same lines as the proof of Theorems B and C in the previous section. We replace surfaces with handlebodies and arcs with discs. The only significant difference is in the analogue of Lemma 5.5, where the analogous functor is not an equivalence of categories, but a somewhat more involved argument shows that it is, nevertheless, still a homotopy equivalence on classifying spaces.

6.1. Outline of the proof

The universal property for the homotopy left Kan extension applied to the inclusion of cyclic operads $Hbdy_0 \hookrightarrow Hbdy$ provides a morphism of modular operads

$$\Theta: \text{LMod}_1Hbdy_0 \rightarrow Hbdy$$

that we will show to be a homotopy equivalence (except on the unmarked solid torus component). As both sides are determined by their values on corollas, it suffices to check that $\Theta$ is a homotopy equivalence whenever one evaluates both modular operads on a corolla $\tau_P$.

We first describe $\Theta$ at the level of categories and groupoids. Recall that $Hbdy(\tau_P) = B\mathcal{H}(P)$, where $\mathcal{H}(P)$ is the groupoid of admissible oriented handlebodies with parametrized boundary discs labelled by $P$ (plus a solid torus if $P = \emptyset$), and $Hbdy_0(\tau_P) = B\mathcal{H}_0(P)$, where $\mathcal{H}_0(P)$ is the subgroupoid of genus 0 handlebodies (that is, balls with labelled discs on their boundary). By Thomason’s Theorem, there is a homotopy equivalence

$$\text{LMod}_1Hbdy_0(\tau_P) \simeq B\left(\mathcal{G}_{\text{conn}}(P)\right)\mathcal{H}_0$$

that glues the genus 0 handlebodies at the vertices together as prescribed by $\tau$ to form an object of the groupoid $\mathcal{H}(P)$.

We lift $\theta$ to a refined functor $\theta'$ mapping into a homotopy equivalent category that is more amenable to comparison, namely, the category of handlebodies equipped with collections of 2-discs that divide them into genus 0 admissible pieces. The source of $\theta'$ projects to the category $\mathcal{G}_{\text{conn}}(P)$ by forgetting the labellings of the vertices. The target category will also project
to $\mathcal{G}_{\text{conn}}(P)$ by taking the dual graph to a complete disc system. We will prove that $\theta'$ is a homotopy equivalence by composing these projections with the graph reduction functor $R: \mathcal{G}_{\text{conn}}(P) \to \mathcal{G}_{\text{conn}}^{\text{red}}(P)$ and making a fibrewise comparison over $\mathcal{G}_{\text{conn}}^{\text{red}}(P)$.

6.2. Disc systems

Let $K = K_{g,n}$ be a connected handlebody of genus $g$ with $n$ discs marked on its boundary. A cutting disc in $K$ is a disc $D$ properly embedded in $K$ (that is, the interior is in the interior of $K$ and the boundary is in the boundary of $K$) with $\partial D$ disjoint from the $n$ marked discs in $\partial K$. We may cut $K$ along $D$ to obtain a (possibly disconnected handlebody) with $n + 2$ marked discs on its boundary coming from the original $n$ discs plus an additional disc on either side of the cut.

**Definition 6.1.** A disc system in $K$ is a set of disjoint cutting discs $D_i$ in $K$ such that if one cuts $K$ along all of the $D_i$, then each component of the result is an admissible handlebody.

A disc system in $K$ is said to be complete if the result of cutting $K$ along the discs in the system consists of only genus 0 pieces.

Given a disc system $\alpha$ in $K$, there are two associated groups. The first is the group $\text{Stab}(\alpha) \subset \mathcal{H}(K)$ of isotopy classes of diffeomorphisms of $K$ fixing the marked boundary discs and disc system $\alpha$ (more precisely, the diffeomorphism must send $\alpha$ to itself without permuting the discs of the system). To define the second group, let $K_\alpha$ denote the result of cutting $K$ along the discs of $\alpha$, and write $\{K_{\alpha,j}\}$ for the components of $K_\alpha$; we regard each component $K_{\alpha,j}$ as a handlebody with marked boundary discs. The second group is

$$\prod_j \mathcal{H}(K_{\alpha,j}).$$

**Proposition 6.2.** Gluing the components of $K_\alpha$ back together induces a homomorphism

$$\prod_j \mathcal{H}(K_{\alpha,j}) \to \text{Stab}(\alpha).$$

This homomorphism is surjective, and its kernel is the free abelian subgroup freely generated by the elements that simultaneously Dehn twist in opposite directions around each of the cuts.

As with arc systems, isotopy classes of non-empty disc systems in $K$ form a category $\mathcal{D}(K)$ in which there is an arrow $\alpha \to \beta$ for each way of identifying $\beta$ with a subset of the discs of $\alpha$ up to isotopy. Let $\mathcal{D}_0(K)$ denote the category of isotopy classes of complete disc systems; it is a subcategory of $\mathcal{D}(K)$ if $K$ is not a ball.

**Proposition 6.3.** If $K$ is admissible, then $B\mathcal{D}(K) \simeq B\mathcal{D}_0(K) \simeq \ast$.

**Proof.** Contractibility of $B\mathcal{D}(K)$ is the content of McCullough [19, Theorem 5.3]; the proof given there does not mention the setting in which $K$ has a non-zero number of marked boundary discs, but the same argument works without modification. Alternatively, one can adapt the argument of Hatcher [13] to discs. Contractibility of $B\mathcal{D}_0(K)$ then follows from an induction completely analogous to the proof of Lemma 5.2. \qed
(In the exceptional case of a solid torus, neither $B\mathcal{D}(K)$ nor $B\mathcal{D}_0(K)$ is contractible).

We can regard $K \mapsto \mathcal{D}_0(K)$ as a functor $\mathcal{D}_0: \mathcal{H}(P) \to \mathcal{C}at$. One can then form the Grothendieck construction $\mathcal{H}(P) \int \mathcal{D}_0$. Proposition 6.3 and Thomason’s Theorem then give the following lemma.

**Lemma 6.4.** The projection $\mathcal{H}(P) \int \mathcal{D}_0 \to \mathcal{H}(P)$ is a homotopy equivalence (except on the component corresponding to a solid torus when $P = \emptyset$).

The functor $\theta: \mathcal{G}_0 \to \mathcal{H}(P)$ lifts to $\theta': \mathcal{G}_0 \to \mathcal{H}(P)$ as follows. An object in the source category is a graph with vertices labelled by genus 0 handlebodies. An object of the target is a handlebody equipped with a complete disc system.

The functor $\theta$ was defined by gluing the genus 0 handlebodies at the vertices together as prescribed by the graph; we define $\theta'$ by the same procedure, except that we now remember where the gluings were performed to obtain a complete disc system in the resulting handlebody.

### 6.3. From disc systems to dual graphs

A complete disc system $\alpha$ in a handlebody $K$ with boundary discs labelled by $P$ determines a *dual graph* $\Gamma(\alpha) \in \mathcal{G}_0$ as follows. Let $K_\alpha$ denote the result of cutting $K$ along the discs of $\alpha$. To build the graph $\Gamma(\alpha)$, put a vertex for each connected component of $K_\alpha$ and an edge for each cutting disc $D$ of $\alpha$; the endpoints of the edge corresponding to $D$ are the vertices corresponding to the components on either side of $D$. Finally, for each $p \in P$ add a leg (an edge terminating in a univalent vertex) with label $p$ at the vertex corresponding to the component on which the corresponding boundary disc lies. Sending a complete disc system to its dual graph defines a functor

$$\Gamma: \left(\mathcal{H}(P) \int \mathcal{D}_0\right) \to \mathcal{G}_0.$$

Consider $\mathcal{L}\pi pt: \mathcal{G}_0 \to \mathcal{F}op$, where $\pi: \mathcal{G}_0 \to \mathcal{G}_0$ is the projection functor and $pt$ is the singleton-valued constant functor. The functor $\theta'$ induces a natural transformation

$$\theta'': \mathcal{L}\pi pt \to \mathcal{L}\Gamma pt.$$

Note that this transformation is not generally a pointwise homotopy equivalence of functors. Let us examine these functors more closely.

**Lemma 6.5.** Let $\tau$ be a graph in $\mathcal{G}_0$. Choose a handlebody $K \in \mathcal{H}(P)$ and a disc system $\alpha$ in $K$ such that the dual graph $\Gamma(\alpha)$ is isomorphic to $\tau$. Then the following conditions are satisfied:

1. $\mathcal{L}\pi pt(\tau) \simeq B(\pi \downarrow \tau) \simeq \prod_j B\mathcal{H}(K_{\alpha,j})$, and edge contractions correspond to gluings;
2. $\mathcal{L}\Gamma pt(\tau) \simeq B(\Gamma \downarrow \tau) \simeq B\text{Stab}(\alpha)$. Contracting an edge in $\tau$ corresponds to the morphism $\alpha \to \alpha'$ that forgets the corresponding cutting disc, and the induced map is given by including the stabilizer of $\alpha$ into the stabilizer of $\alpha'$;
3. under these homotopy equivalences, the map $\theta'': \mathcal{L}\pi pt(\tau) \to \mathcal{L}\Gamma pt(\tau)$ corresponds to the map of Proposition 6.2.
Proof. For (1) and (2), the first equivalence follows from the definitions. The equivalence \( B \text{Stab}(\alpha) \cong B(\Gamma \downarrow \tau) \) is induced by the inclusion \( J \) of the automorphism group of the object \( \overline{\alpha} = (\alpha, \Gamma(\alpha) \cong \tau) \in (\Gamma \downarrow \tau) \). This inclusion is a homotopy equivalence of classifying spaces by Quillen’s Theorem A as the comma category \((\alpha \downarrow J)\) has a final object. The second equivalence in (1) is shown by a variation of the same reasoning. Statement (3) is a matter of unwinding the definitions.

6.4. Integrating out the bivalent vertices: from graphs to reduced graphs

To prove that \( \theta': \mathcal{G}_{\text{conn}}(P) \xrightarrow{\theta'} \mathcal{H}(P) / \mathcal{D}_0 \) is a homotopy equivalence, we shall examine it fibrewise over \( \mathcal{G}_{\text{red}}(P) \) via the graph reduction functor \( R: \mathcal{G}_{\text{conn}}(P) \to \mathcal{G}_{\text{red}}(P) \) that removes the inessential bivalent vertices.

Observe that, for any operad \( \mathcal{O} \) in a symmetric monoidal category \((\mathcal{C}, \otimes)\), the object \( \mathcal{O}(\ast_2) \) is a monoid in \( \mathcal{C} \); the product is induced by the edge contraction shown in Figure 6.

In the case of \( B \mathcal{K}_0 \), the topological monoid associated with a bivalent corolla is homotopy equivalent to the groupoid \( \mathcal{K}_0(\ast_2) \) as the groupoid \( \mathcal{K}_0(\ast_2) \) is equivalent to \( \mathbb{Z} \) (generated by the Dehn twist around the equator of a 3-ball with marked boundary discs at the north and south poles). Thus, the space \( B \mathcal{K}(P) \) is equipped with \( |P| \) circle actions. Given an internal edge \( e \) in a graph \( \tau \), let \( S^1_e \) denote the circle acting on \( \mathbb{L} \pi_1 \text{pt}(\tau) \) via the diagonal of the two circle actions coming from the two endpoints of \( e \), as depicted in Figure 7.

**Lemma 6.6.** There is a homotopy equivalence, \( L\mathcal{R}_1 \mathbb{L} \pi_1 \text{pt}(\tau) \cong \mathbb{L} \pi_1 \text{pt}(\tau) \times_{\prod_e S^1_e} E(\prod_e S^1_e) \).

**Proof.** Let \( k = |E_{\text{int}}(\tau)| \). By Propositions 2.2 and 2.3, \( L\mathcal{R}_1 \mathbb{L} \pi_1 \text{pt}(\tau) \) is homotopy equivalent to the realization of a \( k \)-fold semi-simplicial space. Furthermore, by inspection one sees that this \( k \)-fold semi-simplicial space is exactly the iterated 2-sided bar construction for each of the circles \( S^1_e \) acting as above.
Lemma 6.7. For $\tau \in \mathcal{G}_{\text{red}}$ there is a homotopy equivalence $L_R L_\Gamma pt(\tau) \simeq L_\Gamma pt(\tau)$, and this is natural in $\tau$.

Proof. Suppose that $K$ is a handlebody with marked boundary discs, $\alpha$ is a complete disc system in $K$ and $D \subset K$ is a disc of $\alpha$ that is parallel to either one of the marked boundary discs or another disc of $\alpha$. Then $D$ corresponds to an edge with an inessential endpoint in the dual graph $\Gamma(\alpha)$, and forgetting $D$ produces a complete disc system $\overline{\alpha}$ whose dual graph is obtained from $\Gamma(\alpha)$ by contracting the edge corresponding to $D$. An element of $\mathcal{H}(K)$ acts trivially on the dual graph of $\alpha$ if and only if it acts trivially on the dual graph of $\overline{\alpha}$. It follows from this and Lemma 6.5 that $L_\Gamma pt$ sends each arrow of $(R \downarrow \tau)_0$ to a homotopy equivalence. The result now follows from Propositions 2.2 and 2.3.

Combining the above two lemmas with Proposition 6.2 and Lemma 6.5 now yields the key lemma of the proof.

Lemma 6.8. The natural transformation $\theta''$ induces a homotopy equivalence of functors, $L_R L_\pi pt \simeq L_R L_\Gamma pt$.

We can now complete the proof. Up to homotopy, the map $\Theta: L\text{Mod}_3 \mathcal{H}bdy_0(*_P) \to \mathcal{H}bdy(*_P)$ factors as a sequence of equivalences

$$
\begin{align*}
L\text{Mod}_3 \mathcal{H}bdy_0(*_P) & \simeq \text{colim}_{\mathcal{G}_\text{conn}(P) \downarrow \mathcal{H}_0} pt \\
& \simeq \text{colim}_{\mathcal{G}_\text{conn}(P)} L_R L_\pi pt \\
& \simeq \text{colim}_{\mathcal{G}_\text{conn}(P)} L_R L_\Gamma pt \\
& \simeq B \left( \mathcal{H}(P) \int \mathcal{D}_0 \right) \simeq B \mathcal{H}(P) \simeq \mathcal{H}bdy(*_P)
\end{align*}
$$

(of course, when $P$ is empty, the composition fails to be a homotopy equivalence on the component corresponding to a solid torus).

\[\square\]

7. Cohomology of modular envelopes

7.1. The Bousfield–Kan spectral sequence for a modular envelope

Expressing a modular operad $\mathcal{M}$ as the derived modular envelope of a cyclic operad $\mathcal{O}$ gives, in particular, a description of the individual spaces of $\mathcal{M}$ in terms of homotopy colimits of products of spaces of $\mathcal{O}$. This leads to a spectral sequence computing the homology of $\mathcal{M}$.

In general, let $F: \mathcal{C} \to \mathcal{C}ook$ be a functor. There is a Bousfield–Kan homology spectral sequence

$$E^2_{p,q} = \text{colim}^p H_q(F; k) \implies H_{p+q}(\text{hocolim} F; k),$$

where $\text{colim}^p$ is the $p$th left derived functor of $\text{colim}_\phi$. The functor $F$ is said to be formal over a field $k$ if there is a zigzag of natural transformations between the functors $C_* (F; k)$ and $H_* (F; k)$ from $\mathcal{C}$ to the category $\text{dg-Vect}$ of chain complexes over $k$ (where the values of the second functor are given the zero differential).
Proposition 7.1. If $F : \mathcal{C} \to \mathcal{Top}$ is a functor that is formal over $k$, then the Bousfield–Kan spectral sequence for $H_*(hocolim F; k)$ collapses at the $E^2$ page.

Proof. The space $hocolim F$ is the geometric realization of the bisimplicial set

$$(p, q) \mapsto \bigoplus_{\sigma \in N_p \mathcal{C}} S_q F(\sigma(0)),$$

where $N_p \mathcal{C}$ is the set of $p$-simplices in the nerve of $\mathcal{C}$, $S_q (-)$ is the set of singular $q$-simplices, and $\sigma(0)$ is the first vertex of $\sigma$. Hence, the homology of $hocolim F$ is computed by the total complex of the associated double complex

$$B_{p,q} = \bigoplus_{\sigma \in N_p \mathcal{C}} C_q(F(\sigma(0)); k).$$

The Bousfield–Kan homology spectral sequence comes from the horizontal filtration on this double complex. If $F$ is formal, then the double complex $B$ is quasi-isomorphic to the double complex $B'$ obtained by replacing $C_*(F; k)$ with $H_*(F; k)$. The spectral sequences for the horizontal filtrations on $B$ and $B'$ are isomorphic from the $E^2$ page onwards. As the vertical differential in $B'$ is identically zero, the $B'$ spectral sequence collapses at $E^1$.

Let $K_{g,n}$ be a 3-dimensional compact connected oriented handlebody of genus $g$ with $n$ marked boundary discs. By Theorem A,

$$BDiff(K_{g,n}) \simeq hocolim_{\mathcal{G} r_{g,n}} \mathcal{H}bdy_0,$$

where $\mathcal{G} r_{g,n}$ is the full subcategory of $\mathcal{G} r_{\text{conn}}(n)$ consisting of graphs of rank $g$. This gives a Bousfield–Kan homology spectral sequence

$$E^2_{p,q} = \colim_{\mathcal{G} r_{g,n}} H_q(\mathcal{H}bdy_0; k) \Rightarrow H_{p+q}(BDiff(K_{g,n}); k).$$

(1)

The main result of Giansiracusa and Salvatore [11] is that the cyclic operad $\mathcal{H}bdy_0$ is formal over $\mathbb{R}$. Hence, we obtain the following corollary.

Corollary 7.2. When $k = \mathbb{R}$, the above spectral sequence (7.1) collapses at the $E^2$ page.

7.2. Graph homology and modular envelopes

Let us recall the definition of the graph homology complexes $CT^\mathcal{O}_*(q, n)$ for a cyclic operad $\mathcal{O}$ in the category $\text{dg-}\mathbf{Vect}$ of chain complexes over a field $k$; see [3, 18] for further details. Note that in these papers bivalent vertices in the graphs are not permitted; we shall need a slight modification in which such vertices are allowed.

If $V$ is a finite-dimensional vector space, then $\text{Det}(V)$ denotes the 1-dimensional vector space given by the top exterior power $\Lambda^{\dim(V)} V$. The orientation line of a graph $\gamma$ is defined as

$$\text{Or}(\gamma) := \text{Det}(k^{E_{\text{int}}(\gamma)}) \otimes \text{Det}(H^1(\gamma)),$$

regarded as a graded vector space concentrated in degree $|E_{\text{int}}(\gamma)| - 1$. For a non-loop internal edge $e$, let $\gamma/e$ denote the graph resulting from contracting $e$. There is an isomorphism $(e \wedge -) : \text{Or}(\gamma/e) \cong \text{Or}(\gamma)$.

Let $\mathcal{O}$ be a cyclic operad. For each pair of non-negative integers $(g, n)$ (excluding $(0, 0), (1, 0)$ and $(0, 1)$) there is a graph complex for $\mathcal{O}$ made from graphs of rank $g$ with $n$ ordered legs. It is defined as

$$CT^\mathcal{O}_*(q, n) := \bigoplus_{\gamma} \mathcal{O}(\gamma) \otimes \text{Or}(\gamma),$$
where the summation is over isomorphism classes of graphs in $\mathcal{F}_{g,n}$. The differential on this chain complex is the sum of the internal differential $d_O$ from $O$ acting on the first factor and an edge contracting differential $d_E$ acting on the second factor by the formula

$$d_E(\gamma \otimes (e \wedge \omega)) = \sum_e \gamma/e \otimes \omega,$$

where the summation runs over all non-loop internal edges. It is easy to see that a quasi-isomorphism of cyclic operads $O \to O'$ induces a quasi-isomorphism of the associated graph complexes.

**Theorem 7.3.** There is an isomorphism, $H^\ast(\mathbb{L}\text{Mod}_O(\ast_n)_g) \cong H^{DO}_{3g-4-\ast}(g,n)$.

Here $DO$ is the dg-dual cyclic operad (see [9] or [18] for the definition), and $\mathbb{L}\text{Mod}_O(\ast_n)_g$ denotes the genus $g$ component of the space $\mathbb{L}\text{Mod}_O(\ast_n)$ in the canonical genus grading of the modular envelope.

**Proof.** Consider the canonical projection $\pi: \mathbb{L}\text{Mod}_O(\ast_n)_g \to B\mathcal{F}_{g,n}$. The hypercohomology of the pre-sheaf $\mathcal{F}: U \mapsto C^\ast(\pi^{-1}U)$ on $B\mathcal{F}_{g,n}$ is isomorphic to the cohomology of $\mathbb{L}\text{Mod}_O(\ast_n)_g$. By the results of Lazarev and Voronov [18], the hypercohomology of $\mathcal{F}$ in degree $\ast$ is isomorphic to the graph homology of $DO$ in degree $3g-4-\ast$ (in their exposition, the graphs are not allowed to have bivalent vertices, but the adaptation to include bivalent vertices is entirely straightforward).

Combining Theorem A and the formality of the cyclic operad $\mathcal{H}bdy_0$ from [11] with the above theorem then gives the following corollary.

**Corollary 7.4.** There is an isomorphism, $H^\ast(B\text{Diff}(K_{g,n}); \mathbb{R}) \cong H^{DBV}_{3g-4-\ast}(g,n)$, where $\mathcal{B}V = H_\ast(\mathcal{H}bdy_0; \mathbb{R})$ is the Batalin-Vilkovisky cyclic operad.

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