FRUSTRATED TRIANGLES

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Abstract. A triple of vertices in a graph is a frustrated triangle if it induces an odd number of edges. We study the set $F_n \subset [0, \binom{n}{3}]$ of possible number of frustrated triangles $f(G)$ in a graph $G$ on $n$ vertices. We prove that about two thirds of the numbers in $[0, n^{3/2}/2]$ cannot appear in $F_n$, and we characterise the graphs $G$ with $f(G) \in [0, n^{3/2}/2]$. More precisely, our main result is that, for each $n \geq 3$, $F_n$ contains two interlacing sequences $0 = a_0 \leq b_0 \leq a_1 \leq b_1 \leq \cdots \leq a_m \leq b_m \sim n^{3/2}$ such that $F_n \cap (b_t, a_{t+1}) = \emptyset$ for all $t$, where the gaps are $|b_t - a_{t+1}| = (n - 2) - t(t + 1)$ and $|a_t - b_t| = t(t - 1)$. Moreover, $f(G) \in [a_t, b_t]$ if and only if $G$ can be obtained from a complete bipartite graph by flipping exactly $t$ edges/nonedges. On the other hand, we show, for all $n$ sufficiently large, that if $m \in [2 \sqrt{2} n^{3/2}, \binom{n}{3} - 2 \sqrt{2} n^{3/2}]$ and $(n + 1)m$ is even, then $m \in F_n$. Furthermore, we determine the graphs with the minimum number of frustrated triangles amongst those with $n$ vertices and $e \leq n^2/4$ edges.

1. Introduction

Given a graph $G$ on $n$ vertices, let $t_i = t_i(G)$ denote the number of triples of vertices in $G$ inducing $i$ edges for $i = 0, 1, 2, 3$. One of the earliest and best-known results on colorings is due to Goodman [5], who showed that $t_0 + t_3$ is asymptotically minimised by the random graph. Goodman [6] also conjectured the maximum of $t_0 + t_3$ amongst the graphs with a given number of edges, which was later proved by Olpp [1]. More recently, Linial and Morgenstern [1] showed that every sequence of graphs with $t_0 + t_3$ asymptotically minimal is 3-universal. Hefetz and Tyomkyn [7] then proved that such sequences are 4-universal, but not necessarily 5-universal, and moreover that any sufficiently large graph $H$ can be avoided by such a sequence.

The minimum number of triangles in a graph with a given number of edges has also been widely investigated. Erdős [3] conjectured that a graph with $\lfloor \frac{n^2}{4} \rfloor + k$ edges contains at least $k \lfloor \frac{n}{2} \rfloor$ triangles if $k < \frac{n}{2}$, which was later proved by Lovász and Simonovits [1]. More recently, Rozborov [1] determined completely the minimum number of triangles in a graph with a given number of edges using flag algebras. Some bounds for other combinations of $t_0, t_1, t_2, t_3$ were also given in [6, 1], while similar results for three-colored graphs have recently been proved in [1, 2].

In this paper, we are interested in another natural quantity $t_1 + t_3$. This study is further motivated by a phenomenon occurred in several fields of physics called geometrical frustration.
For example, suppose each vertex of a graph is a spin which can take only two values, say up and down, and each edge of the graph is either ferromagnetic (meaning the spins on its end points prefer to be aligned), or anti-ferromagnetic (the spins prefer to be in opposite directions). Now consider a cycle in the graph, and observe that there is no choice of the values of the spins satisfying the preference of every edge of the cycle if and only if there are an odd number of anti-ferromagnetic edges.

We say that a triple of vertices of a graph is a frustrated triangle if it induces an odd number of edges, i.e. either it contains exactly one edge, or it is a triangle. We shall write \( f(G) = t_1 + t_3 \) for the number of frustrated triangles in a graph \( G \). Our aim is to study the set of possible number of frustrated triangles in a graph with \( n \) vertices,

\[
F_n = \{ f(G) : G \text{ is a graph on } n \text{ vertices} \}.
\]

We remark that studies of similar flavor have been done, for example, in [8, 9] where the object of interest is the set of possible number of colors appeared in a complete subgraph of the complete graph on \( N \).

Clearly, \( F_n \subset [0, \binom{n}{3}] \). Note also that \( F_n \) is symmetric about the midpoint \( \frac{n}{2} \), i.e. \( x \in F_n \) iff \( \binom{n}{3} - x \in F_n \). This is because a triple is frustrated in \( G \) iff it is not frustrated in the complement graph \( \overline{G} \), and so \( f(G) + f(\overline{G}) = \binom{n}{3} \).

One might expect \( F_n \) to be the whole interval \( [0, \binom{n}{3}] \). However, this is surprisingly very far from the truth. Our main result specifies subintervals of \( [0, n^{3/2}] \) that are forbidden from \( F_n \), and moreover, we characterise the graphs \( G \) with \( f(G) \in [0, n^{3/2}] \).

Before we state the main theorem, let us introduce the following two sequences which play an important role throughout the paper. For \( 0 \leq t \leq n - 1 \), let \( a_t = a_t^n \) be the number of frustrated triangles in a graph on \( n \) vertices containing \( t \) edges forming a star. For \( 0 \leq t \leq \frac{n}{2} \), let \( b_t = b_t^n \) be the number of frustrated triangles in a graph on \( n \) vertices containing \( t \) edges forming a matching. It is immediate that

\[
a_t = t(n - t - 1) \quad \text{and} \quad b_t = t(n - 2).
\]

Let \( t_{\text{max}} = \max\{ t : b_t < a_{t+1} \} \) be the last \( t \) such that the intervals \([a_t, b_t]\) and \([a_{t+1}, b_{t+1}]\) are disjoint. It is easy to check that

\[
t_{\text{max}} = \max\{ t : t(t+1) < n - 2 \} = \left\lfloor \sqrt{n - \frac{7}{4} - \frac{3}{2}} \right\rfloor \sim \sqrt{n}
\]

exists for \( n \geq 3 \), and

\[
0 = a_0 = b_0 < a_1 = b_1 < a_2 < b_2 < \cdots < a_{t_{\text{max}}} < b_{t_{\text{max}}} < a_{t_{\text{max}} + 1} \sim n^{3/2}.
\]

We can now state the main result.
Theorem 1. Let $G$ be a graph on $n \geq 3$ vertices.

(i) If $f(G) < a_{t_{\text{max}}+1}$ then $f(G) \in [a_t, b_t]$ for a unique $t \leq t_{\text{max}}$.

(ii) If $t \leq t_{\text{max}}$ then $f(G) \in [a_t, b_t]$ iff $G$ can be obtained from a complete bipartite graph on $n$ vertices by flipping exactly $t$ pairs of vertices.

Here, to flip a pair of vertices $uv$ means to change its edge/nonedge status, i.e. if $uv$ is an edge, make it a nonedge, and vice versa. We remark that Theorem 1 part (i) is equivalent to the statement: $f(G) \notin (b_t, a_{t+1})$ for all $t \geq 0$. Note also that the intervals have lengths $\lvert (b_t, a_{t+1}) \rvert = (n - 2) - t(t+1)$ and $\lvert [a_t, b_t] \rvert = t(t-1)$.

Theorem 1 is complemented by the following result, which shows that it is close to best possible. Observe that $a_{t_{\text{max}}+1} = (t_{\text{max}} + 1)(n - t_{\text{max}} - 2) \sim n^{3/2}$ and so Theorem 1 deals with the case $f(G) \lesssim n^{3/2}$. Since $F_n$ is symmetric, we automatically have a corresponding result for $f(G) \gtrsim \binom{n}{3} - n^{3/2}$. On the other hand, we prove that every number, up to parity condition, in the large central part of $[0, \binom{n}{3}]$ is realisable as $f(G)$ for some $G$. Note that if $n$ is even, then $f(G)$ must also be even, since adding an edge to a graph changes the ‘frustration status’ of exactly $n - 2$ triples.

Theorem 2.

(i) If $n$ is even and sufficiently large then $F_n$ contains every even integer between $2n^{3/2}$ and $\binom{n}{3} - 2n^{3/2}$.

(ii) If $n$ is odd and sufficiently large then $F_n$ contains every integer between $2\sqrt{2}n^{3/2}$ and $\binom{n}{3} - 2\sqrt{2}n^{3/2}$.

Let us change the direction and turn to the following related natural question. Given the number of vertices and the number of edges, which graphs maximise/minimise the number of frustrated triangles? The method we develop to prove Theorem 1 allows us to partially answer this question. Before we state the result, we shall define some necessary notations. For $0 \leq x \leq \frac{n}{2}$, let $c_x = x(n - x)$ be the number of edges in the complete bipartite graph $K_{x,n-x}$. For an integer $e$, let $g(e) = \min\{|e - c_x| : 0 \leq x \leq \frac{n}{2}\}$ be the distance from $e$ to the sequence $(c_x)$. We are able to determine the minimal graphs when the number of edges is at most $\frac{n^2}{4}$.

Theorem 3. If $G$ is a graph with $n$ vertices and $e$ edges then $f(G) \geq a_{g(e)}$. Moreover, this bound can be achieved when $e \leq \lceil \frac{n^2}{4} \rceil + \lceil \frac{n}{2} \rceil - 1$: in this case the extremal graphs are obtained from a complete bipartite graph by deleting or adding $g(e)$ edges forming a star.

The rest of this paper is organised as follows. In Section 2, we present some preliminary results for the readers to get familiar with frustrated triangles and to motivate the definitions and ideas used to prove the main results. Sections 3 and 4 are devoted to the proofs of Theorems 1
and 2 respectively. In Section 5, we describe an application of Theorems 1. We conclude the paper in Section 6 with some open problems.

2. Preliminaries

We shall start with a coffee time problem which is a special case of Theorem 1. By considering the empty and the complete graphs, we see that 0 and $\binom{n}{3}$ are always in $F_n$. A natural question is that, what is the first nonzero element of $F_n$? If we consider a graph with only one edge $uv$, the frustrated triangles are those triples containing both $u, v$; therefore there are $n - 2$ frustrated triangles. It turns out that $n - 2$ is the answer. Before we give the proof, let us introduce the flipping operation which is an important idea for dealing with frustrated triangles.

Recall that to flip a pair of vertices is to change its edge/nonedge status. For a vertex $v$ of a graph $G$, let $G_v$ denote the graph obtained from $G$ by flipping the pairs $uv$ for all $u \in G \setminus \{v\}$ (see Figure 1), i.e. $V(G_v) = V(G)$ and $E(G_v) = E(G) \cup \{uv : uv \notin E(G)\} - \{uv : uv \in E(G)\}$.

![Figure 1. The flipping operation on vertex v.](image)

By flipping $v$, we mean an operation of changing $G$ to $G_v$. The readers should be warned to note the difference between flipping a pair of vertices and flipping a vertex. We then have the following easy but useful lemma.

**Lemma 4.** For any graph $G$, $f(G)$ is preserved under the flipping operation, i.e. $f(G_v) = f(G)$ for any vertex $v \in G$.

**Proof.** More is true: $\{x, y, z\}$ is frustrated in $G_v$ iff it is in $G$. This is obvious if $v \notin \{x, y, z\}$. If $v \in \{x, y, z\}$ then exactly two pairs of $\{x, y, z\}$ were flipped; therefore the parity of the number of edges induced by $\{x, y, z\}$ stays the same. \(\square\)

Before we show that $n - 2$ is the first nonzero element of $F_n$, it is useful to answer the following question. What are the graphs with no frustrated triangles?

**Proposition 5.** For any graph $G$, $f(G) = 0$ iff $G$ is a complete bipartite graph.
Proof. It is clear that a complete bipartite graph contains no frustrated triangles. Conversely, given a vertex $v$ of a graph $G$ with $f(G) = 0$. Then there is no edge in $G[\Gamma(v)]$ where $\Gamma(v)$ is the neighborhood of $v$; otherwise it would form a frustrated triangle with $v$. Similarly, there is no edge in the nonneighborhood of $v$. Also, if $vx$ is an edge and $vy$ is a nonedge, then $xy$ has to be an edge. Therefore, $G$ is a complete bipartite graph with parts $\Gamma(v)$ and $V \setminus \Gamma(v)$. □

We are now ready to show that there is no graph on $n$ vertices with number of frustrated triangles strictly between 0 and $n - 2$.

**Proposition 6.** For all $n \in \mathbb{N}$, $F_n \cap (0, n - 2) = \emptyset$.

Proof. We apply induction on $n$. There is nothing to check for $n = 3$. Let $G$ be a graph on $n$ vertices. Our aim is to show that $f(G) \notin (0, n - 2)$. Without loss of generality, we may assume that $G$ has an isolated vertex $v$. This is because we can flip each neighbor of $v$ to make it a nonneighbor while preserving the number of frustrated triangles in $G$ by Lemma 4. Let $G' = G - v$ be the graph obtained from $G$ by deleting $v$. Since $v$ is an isolated vertex, we have $f(G) = f(G') + e(G')$.

By the induction hypothesis, $f(G') \notin (0, n - 3)$. We shall distinguish two cases.

**Case 1:** $f(G') \geq n - 3$
If $e(G') \geq 1$ then $f(G) = f(G') + e(G') \geq (n - 3) + 1 \geq n - 2$ as required. If $e(G') = 0$ then $G$ is empty and so $f(G) = 0$ as required.

**Case 2:** $f(G') = 0$
By Proposition 5, $G'$ is complete bipartite and so $e(G') = x(n - 1 - x)$ for some $0 \leq x \leq n - 1$. We are done if $G'$ is empty. If $G'$ is not empty, then $e(G')$ is minimised when $x = 1$, i.e. $f(G) = e(G') \geq n - 2$ as required. □

We have just proved that $f(G)$ cannot lie in the gap between the intervals $[a_0, b_0] = \{0\}$ and $[a_1, b_1] = \{n - 2\}$ which is the first case of Theorem 1 part (i). The equation $f(G) = f(G') + e(G')$ in the proof suggests that, in order to understand the possible number of frustrated triangles, we should understand the possible number of edges in a graph with a given number of frustrated triangles. In fact, we will have an analogue of Proposition 5 for $f(G) \lesssim n^{3/2}$, i.e. we will not only know the possible number of edges, but we will also know the possible structure of the graph (see Theorem 1 part (ii)).

Let us now consider the converse of Lemma 4. We write $G \sim H$ if $G$ can be obtained from a graph $H$ by a sequence of vertex flippings. Clearly, $\sim$ is an equivalence relation. Observe that the complete bipartite graphs on $n$ vertices form an equivalence class. Indeed, let $G$ be a complete bipartite graph with parts $A, B$ and let $v \in A$. Then flipping $v$ is equivalent to moving $v$ across from $A$ to $B$, i.e. $G_v$ is the complete bipartite with parts $A \setminus \{v\}$ and $B \cup \{v\}$.
Therefore, another way to state Proposition 5 is: for any graph $G$ on $n$ vertices, $f(G) = 0$ iff $G \sim E_n$ where $E_n$ is the empty graph on $n$ vertices.

This shows that the converse of Lemma 4 is true in the case $f(G) = 0$. Does it hold in general? That is, given two graphs with the same number of frustrated triangles, can we always obtain one from the other by a sequence of vertex flippings? Unfortunately, this is false. As we can see from the proof of Lemma 4, if $G \sim H$ then not only do we have $f(G) = f(H)$ but there is also a bijection $\phi : V(G) \to V(H)$ such that $\{u, v, w\}$ is frustrated iff $\{\phi(u), \phi(v), \phi(w)\}$ is frustrated. It is easy to see that this is also sufficient.

**Proposition 7.** Let $G$ and $H$ be graphs on $n$ vertices. The following statements are equivalent.

(i) $G \sim H$.

(ii) There is a bijection $\phi : V(G) \to V(H)$ such that $\{u, v, w\}$ is frustrated in $G$ iff $\{\phi(u), \phi(v), \phi(w)\}$ is frustrated in $H$.

**Proof.** The implication $(i) \Rightarrow (ii)$ follows from the proof of Lemma 4.

To prove $(ii) \Rightarrow (i)$, suppose $\phi$ is a bijection satisfying $(ii)$. We say that a pair of vertices $ab$ of $G$ and a pair of vertices $xy$ of $H$ agree if both $ab$ and $xy$ are edges, or both $ab$ and $xy$ are nonedges. Let $v$ be a vertex of $G$ and let $U = \{u \in V(G) \setminus \{v\} : uv$ and $\phi(u)\phi(v)$ disagree$\}$. Let $G'$ be the graph obtained from $G$ by flipping each vertex of $U$. We claim that $G'$ and $H$ are isomorphic. Clearly, $uv$ in $G'$ and $\phi(u)\phi(v)$ in $H$ agree for all $u \in V(G') \setminus \{v\}$. Furthermore, by the proof of Lemma 4, $\phi$ still satisfies $(ii)$ after replacing $G$ with $G'$. Hence, for any $u, w \in V(G') \setminus \{v\}$, the graphs $G'[u, v, w]$ and $H[\phi(u), \phi(v), \phi(w)]$ have the same ‘frustration status’. Since two of the pairs agree, the third pairs $uw$ and $\phi(u)\phi(w)$ must also agree. Therefore, $G'$ and $H$ are isomorphic, and so $H$ can be obtained from a $G$ by a sequence of vertex flippings.

Now we shall give an explicit counterexample to the converse of Lemma 4. Let $G$ be a disjoint union of $P_2$ and $P_2$, and let $H$ be a disjoint union of $P_1$ and $P_3$ (see Figure 2) where $P_l$ is a path with $l$ edges.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{counterexample.png}
\caption{Counterexample to the converse of Lemma 4.}
\end{figure}
Then \( f(G) = f(H) = 12 \), but there is no bijection satisfying Proposition 7 part (ii) since, for example, \( G \) contains four vertices inducing no frustrated triangles while \( H \) does not. Therefore, \( G \not\sim H \).

Let us move on and provide some formulae for \( f(G) \) in terms of other graph variables, which also gives us some bounds on \( f(G) \).

**Proposition 8.** Let \( G \) be a graph with \( n \) vertices and \( e \) edges. Write \( p \) for the number of triangles in \( G \), and \( q \) for the number of pairs of independent edges in \( G \). Then

\[(i) \quad f(G) = en - \sum_{v \in G} d_v^2 + 4p \]
\[(ii) \quad f(G) = e(n - e - 1) + 4p + 2q \]
\[(iii) \quad f(G) \in [a_e, b_e] \]

where \( d_v \) is the degree of vertex \( v \).

**Proof.**

(ii) This follows immediately from (i). It is sufficient to show that \( \sum_{v \in G} d_v^2 = e(e + 1) - 2q \). Since a pair of edges is either independent or dependent, we have

\[
\binom{e}{2} = q + \sum_{v \in G} \binom{d_v}{2},
\]

i.e.

\[
e(e - 1) = 2q + \sum_{v \in G} d_v^2 - 2e
\]
as required.

(iii) The upperbound is obvious:

\[ f(G) \leq (\text{number of triples containing at least one edge}) \leq e(n - 2) = b_e, \]

and the lower bound follows from (ii):

\[ f(G) = e(n - e - 1) + 4p + 2q \geq e(n - e - 1) = a_e \]

as required. □

The proof of part (iii) tells us that, amongst the graphs with \( n \) vertices and \( e \leq n/2 \) edges, the \( e \)-matching is the only graph with the maximum number of frustrated triangles and, amongst the graphs with \( n \) vertices and \( e \leq n - 1 \) edges, the \( e \)-star is the only minimal graph.

Although part (iii) looks like what we would like for Theorem 1, it only gives us good bounds when \( e \) is small. For larger \( e \), the lower bound \( a_e \) becomes worse and can even be negative. However, we do not have to apply (iii) to \( G \) directly. By Lemma 4, we can apply (iii) to any graph \( H \) with \( G \sim H \). Therefore, this motivates us to find a graph \( H \) with few edges such that \( G \sim H \), which gives rise to the following crucial definition.

For a graph \( G \), let \( t_G = \min\{e(H) : G \sim H\} \) be the minimum number of edges we can have after some vertex flippings of \( G \). Proposition 8 part (iii) implies that \( f(G) \in [a_t, b_t] \). Therefore, to prove Theorem 1 part (i), it is enough to show that \( t_G \leq t_{\text{max}} \) if \( f(G) < a_{t_{\text{max}} + 1} \).

It is important to note that \( t_G \) is preserved under the flipping operation.

We now observe that \( t_G \) can also be viewed as a measure of how close \( G \) is to being a complete bipartite graph. We say that a pair of vertices \( uv \) is odd with respect to a bipartition \( V(G) = X \cup X^c \) if

- \( uv \notin E(G) \) and \( u, v \) are in different parts,
- \( uv \in E(G) \) and \( u, v \) are in the same part,

i.e. if \( uv \) is not what it should be in the complete bipartite graph between \( X, X^c \). Therefore, flipping the odd pairs w.r.t. \( V(G) = X \cup X^c \) would result in the complete bipartite graph between \( X, X^c \). We have the following equivalent definition for \( t_G \).

**Proposition 9.** For any graph \( G \),

\[ t_G = \min\{\#\text{odd pairs w.r.t. } V(G) = X \cup X^c : X \subset V(G)\}. \]

**Proof.** For \( X \subset V(G) \), we have

\[
\begin{align*}
\text{(the set of odd pairs with respect to } V(G) = X \cup X^c) \\
= E(G[X]) \cup E(G[X^c]) \cup NE(G[X, X^c]) \\
= \text{(the set of edges of the graph obtained from } G \text{ by flipping each vertex in } X) \end{align*}
\]
where \(NE(G[X, X^c])\) is the set of nonedges between \(X, X^c\). □

The proof also gives us another necessary and sufficient condition for \(G \sim H\).

**Corollary 10.** Let \(G\) and \(H\) be graphs on \(n\) vertices. The following statements are equivalent.

- \(G \sim H\).
- There are a bijection \(\phi : V(G) \rightarrow V(H)\) and a subset \(X \subset V(G)\) such that \(uv\) is an odd pair w.r.t. \(V(G) = X \cup X^c\) in \(G\) iff \(\phi(u)\phi(v)\) is an edge in \(H\). □

The definition of \(t_G\) allows us to state a generalisation of Proposition 5,

\[f(G) \in [a_t, b_t] \text{ iff } t_G = t.\]

It is not hard to see that this is false for \(t > t_{\text{max}}\) since \([a_t, b_t]\) and \([a_{t+1}, b_{t+1}]\) overlap. On the other hand, Theorem 1 part (ii) states that this generalisation holds for \(t \leq t_{\text{max}}\).

### 3. Proof of Theorem 1

We shall prove Theorem 1 in the following simple but stronger form.

**Theorem 11.** Let \(G\) be a graph on \(n \geq 3\) vertices and let \(t \geq 0\).

(i) If \(f(G) < a_{t+1}\) then \(t_G \leq t\).

(ii) If \(f(G) > b_{t-1}\) then \(t_G \geq t\).

Note that Theorem 11 part (i) covers a larger range than Theorem 1. Indeed, it describes the structure of graphs \(G\) with \(f(G) \lesssim n^2\), since \(a_{t+1}\) can be as large as \((\frac{n-1}{2})^2\).

Before proving Theorem 11, let us show that it immediately implies Theorem 1.

**Proof of Theorem 1.** (i) Suppose that \(f(G) < a_{t_{\text{max}}+1}\). Then, by Theorem 11 part (i), \(t_G \leq t_{\text{max}}\) and so we are done since \(f(G) \in [a_{t_G}, b_{t_G}]\) by Proposition 8 part (iii). The uniqueness of \(t\) is trivial since \([a_0, b_0], [a_1, b_1], \ldots, [a_{t_{\text{max}}}, b_{t_{\text{max}}}]\) are disjoint.

(ii) \(\Rightarrow\) Suppose that \(f(G) \in [a_t, b_t]\) for some \(t \leq t_{\text{max}}\). By Proposition 9, it is sufficient to show that \(t_G = t\). Since \(t \leq t_{\text{max}}\), we have \(f(G) \in [a_t, b_t] \subset (b_{t-1}, a_{t+1})\), and so \(t_G = t\) by Theorem 11.

\(\Leftarrow\) Suppose that \(G\) can be obtained from a complete bipartite graph by flipping exactly \(t \leq t_{\text{max}}\) pairs of vertices. Equivalently, \(G \sim H\) for some graph \(H\) with \(e(H) = t\) by Corollary 10. Hence, by Proposition 8 part (iii),

\[f(G) = f(H) \in [a_{e(H)}, b_{e(H)}] = [a_t, b_t]\]

as required. □

Now let us prove Theorem 11. Note that part (ii) is easy, and the main content is in part (i).
Proof of Theorem 11. We start by proving part (ii), which is equivalent to the statement: if $t_G \leq t$ then $f(G) \leq b_t$. By Proposition 8 part (iii), $f(G) \leq b_{t_G} \leq b_t$ since $b_t$ is increasing in $t$.

We now prove part (i) by induction on $n$. It is easy to check for $n = 3$. Let $G$ be a graph on $n$ vertices such that $f(G) < a_{t+1}$. Our aim is to find a bipartition of $V(G)$ with at most $t$ odd pairs. We write $a_s$ for $a_s^n$ and $a'_s$ for $a_{s-1}^n$. Since $f(G')$ and $t_G$ are preserved under the flipping operation, we may assume that $G$ has an isolated vertex $v$ (as in the proof of Proposition 6). Let $G' = G - v$, and note that $f(G) = f(G') + e(G')$. Now observe that we are done if $e(G') \leq t$, since $t_G \leq e(G) = e(G')$ by the definition of $t_G$. So we may assume that $e(G') \geq t + 1$, and hence,

$$f(G') = f(G) - e(G') \leq f(G) - (t + 1) < a_{t+1} - (t + 1) = a'_{t+1}.$$ 

Therefore, we may now assume that $f(G') \in [a_s', a_{s+1}']$ for some $s = 0, 1, \ldots, t$. Since $f(G') < a_{s+1}'$, we have, by the induction hypothesis, that $G'$ has a bipartition $V(G') = X \cup Y$ with at most $s$ odd pairs. Without loss of generality, $X$ is the smaller part and we write $x$ for $|X|$. We shall distinguish two cases.

**Case 1: $x \leq t - s$**

With respect to the bipartition $V(G) = X \cup (Y \cup \{v\})$, the number of odd pairs is at most $x + s \leq t$ as required.

**Case 2: $x \geq t - s + 1$**

Since $G'$ contains at most $s$ odd pairs w.r.t. the bipartition $V(G') = X \cup Y$ and $x \leq \frac{n-1}{2}$, we have

$$e(G') \geq x(n - 1 - x) - s = a_x - s \geq a_{t-s+1} - s,$$

and so

$$f(G) = f(G') + e(G') \geq a'_s + a_{t-s+1} - s = a_s + a_{t-s+1} - 2s.$$ 

It is sufficient to show that $a_s + a_{t-s+1} - 2s \geq a_{t+1}$ since it would contradict the fact that $f(G) < a_{t+1}$. The inequality is equivalent to

$$[s(n - 1) - s^2] + [(t - s + 1)(n - 1) - (t - s + 1)^2] - 2s \geq (t + 1)(n - 1) - (t + 1)^2,$$

i.e.

$$(t + 1)^2 - (t - s + 1)^2 \geq s^2 + 2s.$$ 

That is, $2st \geq 2s^2$ which holds for $0 \leq s \leq t$ as required. \hfill \Box

Since $F_n$ is symmetric, we automatically have, by taking complements, the corresponding result for $f(G) \in \left[\left(\begin{array}{c} n \vspace{1pt} \\ 3 \end{array}\right) - \frac{n^3}{2}, \left(\begin{array}{c} n \vspace{1pt} \\ 3 \end{array}\right)\right]$.

**Corollary 12.** Let $G$ be a graph on $n \geq 3$ vertices.

(i) If $f(G) > \left(\begin{array}{c} n \vspace{1pt} \\ 3 \end{array}\right) - a_{t_{\text{max}}+1}$ then $f(G) \in \left[\left(\begin{array}{c} n \vspace{1pt} \\ 3 \end{array}\right) - b_t, \left(\begin{array}{c} n \vspace{1pt} \\ 3 \end{array}\right) - a_t\right]$ for a unique $t \leq t_{\text{max}}$. 

(ii)
(ii) If \( t \leq t_{\text{max}} \) then \( f(G) \in \left[ \left( \binom{n}{3} - b_t, \binom{n}{3} - a_t \right) \right] \) iff \( G \) can be obtained from a disjoint union of two cliques of orders summing to \( n \) by flipping exactly \( t \) pairs of vertices. \( \square \)

4. Proof of Theorem 2

Before we prove Theorem 2, we shall observe that the number of frustrated triangles satisfies the following parity condition.

**Proposition 13.** Let \( G \) be a graph on \( n \) vertices. Then

- \( f(G) \) is even for \( n \) even.
- \( f(G) \) has the same parity as \( e(G) \) for \( n \) odd.

**Proof.** Given an edge \( xy \) of \( G \), let \( V_1 = \{ v \in G \setminus \{x, y\} : vxy \text{ is frustrated} \} \) be the set of vertices forming a frustrated triangle with \( xy \), and let \( V_2 = \{ v \in G \setminus \{x, y\} : vxy \text{ is not frustrated} \} \). Deleting the edge \( xy \) changes the parity of the number of edges induced by \( vxy \). Therefore, \( f(G - xy) = f(G) - |V_1| + |V_2| \). Since \( |V_1| + |V_2| = n - 2 \), we have

\[
f(G - xy) \equiv f(G) + n \mod 2.
\]

If \( n \) is even, we see that \( f(G) \equiv f(G - e_1) \equiv f(G - e_1 - e_2) \equiv \cdots \equiv f(E_n) = 0 \mod 2 \). If \( n \) is odd, we see that \( f(G) \equiv f(G - e_1) + 1 \equiv f(G - e_1 - e_2) + 2 \equiv \cdots \equiv f(E_n) + e(G) = e(G) \mod 2 \).

We also see from the proof that \( |f(G - e) - f(G)| \leq n - 2 \); therefore, \( F_n \) can miss at most \( n - 1 \) consecutive integers. The next corollary follows from Proposition 13 and Theorem 1.

**Corollary 14.** For \( t \leq t_{\text{max}} \), we have

\[ F_n \cap [a_t, b_t] \subset \{ a_t, a_t + 2, \ldots, b_t - 2, b_t \}. \]

**Proof.** If \( n \) is even, this follows immediately from Proposition 13 since \( a_t, b_t \) are even. Let \( n \) be odd and let \( G \) be a graph on \( n \) vertices with \( f(G) \in [a_t, b_t] \) for some \( t \leq t_{\text{max}} \). By Theorem 1 part (ii), \( G \) can be obtained from a complete bipartite graph by flipping exactly \( t \) pairs of vertices. Equivalently, \( G \sim H \) for some graph \( H \) with \( e(H) = t \). Hence, by Proposition 13, we have

\[ f(G) = f(H) \equiv t \equiv a_t \equiv b_t \mod 2 \]

as required. \( \square \)

We shall now prove Theorem 2 part (i). The proof is by construction of graphs consisting of three parts. By modifying the first part of the graphs, we obtain a sequence of even numbers belonging to \( F_n \) in the required interval with gaps at most \( n - 2 \). Then the modification of the second part refines the partition of such interval such that the gaps are now at most \( 2(\sqrt{n} - 1) \). Finally, we modify the third part to obtain all even numbers in the interval.
Proof of Theorem 2 part (i). Let \( n \) be even and sufficiently large. For each even number \( m \in [2n^{3/2}, \left(\frac{n}{3}\right) - 2n^{3/2}] \), our aim is to construct a graph \( G \) on \( n \) vertices with \( f(G) = m \). Since \( F_n \) is symmetric about \( \frac{1}{2} \left( \frac{n}{3} \right) \), it is sufficient to do so for each even number \( m \in [2n^{3/2}, \frac{1}{2} \left( \frac{n}{3} \right)] \).

Let \( G \) be a graph on \( n \) vertices containing \( 2\sqrt{n} \) independent edges. For simplicity, we shall write \( \sqrt{n} \) instead of \( \left\lfloor \sqrt{n} \right\rfloor \). Let \( H \) be a graph obtained from \( G \) by adding all the edges between the isolated vertices of \( G \), i.e. \( H \) is a disjoint union of a complete graph of size \( n - 4\sqrt{n} \) and a matching of size \( 2\sqrt{n} \). Then \( f(G) = 2\sqrt{n}(n - 2) \) and \( f(H) = \left(\frac{n}{3}\right) + \left(\frac{n}{3}\right)(n - r) + 2\sqrt{n}(n - 2) \) where \( r = n - 4\sqrt{n} \). It is sufficient to show that every even number between \( f(G) \) and \( f(H) \) belongs to \( F_n \) since \( f(G) \leq 2n^{3/2} \) and \( f(H) \geq \left(\frac{n - 4\sqrt{n}}{3}\right) \geq \frac{1}{2} \left(\frac{n}{3}\right) \) for sufficiently large \( n \).

We shall break \( G \) into three parts and modify each part separately to obtain new graphs. Let \( V(G) = V_1 \sqcup V_2 \sqcup V_3 \) where

- \( G[V_1] \) is empty with \( |V_1| = r \),
- \( G[V_2] \) and \( G[V_3] \) are matchings of size \( \sqrt{n} \) with \( |V_2| = |V_3| = 2\sqrt{n} \), and
- \( G[V_1, V_2], G[V_2, V_3], G[V_3, V_1] \) are all empty.

Now we add an edge one by one inside \( G[V_1] \) until we obtain \( H \). Each time we add an edge, \( f \) can change by at most \( n - 2 \) by the proof of Proposition 13. Hence, \( F_n \) contains a sequence of even numbers \( f(G) = f(G_1), f(G_2), \ldots, f(G_{\lceil \sqrt{n} \rceil}) = f(H) \) with \( |f(G_i) - f(G_{i+1})| \leq n - 2 \) for all \( i \). Therefore, it is sufficient to show that \( F_n \) contains every even number between \( f(G_i) \) and \( f(G_i) - (n - 2) \) for all \( i \).

Let us fix \( i \). By construction, \( V(G_i) = V_1 \sqcup V_2 \sqcup V_3 \) where \( V_1 \) induces \( i \) edges and each of \( V_2, V_3 \) induces a matching of size \( \sqrt{n} \). We shall modify \( G_i[V_2] \) and \( G_i[V_3] \) to obtain new graphs. Let \( \{x_1y_1, \ldots, x_{\sqrt{n}}y_{\sqrt{n}}\} \) be the matching inside \( V_2 \). First, we delete \( x_2y_2 \) and replace it with \( x_1y_2 \). This decreases \( f \) by 2. Next, we delete \( x_3y_3 \) and replace it with \( x_1y_3 \) which decreases \( f \) by 4 more. We continue similarly (see Figure 3). When we delete \( x_jy_j \) and replace it with \( x_1y_j \), it decreases \( f \) by \( 2(j - 1) \).

![Figure 3. Star accumulation of V2.](image)

Therefore, \( F_n \) contains a decreasing sequence of even numbers \( f(G_i) = f(G_i^1), f(G_i^2), \ldots, f(G_i^{\sqrt{n}}) \) with \( |f(G_i^j) - f(G_i^{j+1})| \leq 2(\sqrt{n} - 1) \) for all \( j \). Moreover, \( f(G_i^{\sqrt{n}}) \) is larger than \( f(G_i) - (n - 2) \)
Since the modification has 

\[ f(G_i^{\sqrt{n}}) = f(G_i) - (2 + 4 + \cdots + 2(\sqrt{n} - 1)) \]

\[ = f(G_i) - (\sqrt{n} - 1)\sqrt{n} \]

\[ \leq f(G_i) - (n - 2) + 2(\sqrt{n} - 1). \]

Hence, it is sufficient to show that \( F_n \) contains every even number between \( f(G_i^j) \) and \( f(G_i^j) - 2(\sqrt{n} - 1) \) for all \( j \). To prove this, let us fix \( j \) and we shall modify the graph \( G_i^j \) as follows. Let \( \{z_1w_1, \ldots, z_{\sqrt{n}}w_{\sqrt{n}}\} \) be the matching inside \( V_3 \). First, we delete \( z_2w_2 \) and replace it with \( w_1w_2 \). This decreases \( f \) by 2. Next, we delete \( z_3w_3 \) and replace it with \( w_2w_3 \) which decreases \( f \) by 2 more. We continue similarly (see Figure 4). When we delete \( z_jw_j \) and replace it with \( w_{j-1}w_j \), it decreases \( f \) by 2.

![Figure 4. Path accumulation of \( V_3 \).](image)

Since the modification has \( \sqrt{n} - 1 \) steps, we conclude that \( F_n \) contains every even number between \( f(G_i^j) \) and \( f(G_i^j) - 2(\sqrt{n} - 1) \) as required. \( \square \)

The proof for part \( (ii) \) is by the same method but with more care. Observe that, for \( n \) odd, the sequence obtained by adding edges to the first part alternates between odd and even numbers. Since the odd and even subsequences have larger gaps than before, of size at most \( 2(n - 2) \), we shall follow the same proof for each subsequence by taking larger matchings for the second and third parts.

**Proof of Theorem 2 part \( (ii) \).** Let \( n \) be odd and sufficiently large. For each number \( m \in [2\sqrt{2}n^{3/2}, \binom{n}{3} - 2\sqrt{2}n^{3/2}] \), our aim is to construct a graph \( G \) on \( n \) vertices with \( f(G) = m \).

Since \( F_n \) is symmetric about \( \frac{1}{2}\binom{n}{3} \), it is sufficient to do so for each \( m \in [2\sqrt{2}n^{3/2}, \frac{1}{2}\binom{n}{3}] \). Let \( G \) be a graph on \( n \) vertices containing \( 2\sqrt{2n} \) independent edges. Let \( H \) be a graph obtained from \( G \) by adding all the edges between the isolated vertices of \( G \), i.e. \( H \) is a disjoint union of a complete graph of size \( n - 4\sqrt{2n} \) and a matching of size \( 2\sqrt{2n} \). Then \( f(G) = 2\sqrt{2n}(n - 2) \) and \( f(H) = \binom{n}{3} + \binom{n}{2}(n - r) + 2\sqrt{2n}(n - 2) \) where \( r = n - 4\sqrt{2n} \). It is sufficient to show that every number between \( f(G) \) and \( f(H) - (n - 2) \) belongs to \( F_n \) since \( f(G) \leq 2\sqrt{2n^{3/2}} \) and \( f(H) - (n - 2) \geq \binom{n - 4\sqrt{2n}}{3} \geq \frac{1}{2}\binom{n}{3} \) for sufficiently large \( n \).

We shall break \( G \) into three parts and modify each part separately to obtain new graphs. Let \( V(G) = V_1 \sqcup V_2 \sqcup V_3 \) where

\[ V_1 = \{z \in V(G) : z \text{ is isolated in } G\} \]

\[ V_2 = \left\{ z \in V(G) : \exists w \in V(G) \text{ such that } z \text{ is adjacent to } w \text{ in } G \right\} \]

\[ V_3 = \left\{ z \in V(G) : \exists w \in V(G) \text{ such that } z \text{ is isolated in } G \text{ and } z \text{ is adjacent to } w \right\} \]
Therefore, $F$ by 2$(G) = f(G_1), f(G_2), \ldots, f(G_j) = f(H)$ with $|f(G_i) - f(G_{i+1})| \leq n - 2$ for all $i$. By Proposition 13, this sequence alternates between odd and even numbers. We claim that it is sufficient to show that $F_n$ contains

$$f(G_i), f(G_i) - 2, f(G_i) - 4, \ldots, f(G_i) - 2(n-2)$$

for all $i$. Indeed, let $m \in [f(G), f(H)-(n-2)]$. Then there is an $i$ such that $f(G_i)$ has the same parity as $m$ and $0 \leq f(G_i) - m \leq 2(n-2)$ since $|f(G_j) - f(G_{j+2})| \leq 2(n-2)$ and $f(G_j), f(G_{j+2})$ have the same parity for all $j$. Hence, $m \in \{f(G_1), f(G_1) - 2, f(G_1) - 4, \ldots, f(G_1) - 2(n-2)\} \subseteq F_n$ as required. The rest of the proof is similar to the previous proof.

Let us fix $i$. By construction, $V(G_i) = V_1 \cup V_2 \cup V_3$ where $V_1$ induces $i$ edges and each of $V_2, V_3$ induces a matching of size $\sqrt{2n}$. We shall modify $G_i[V_2]$ and $G_i[V_3]$ to obtain new graphs. Let $\{x_1y_1, \ldots, x_{\sqrt{2n}}y_{\sqrt{2n}}\}$ be the matching inside $V_2$. First, we delete $x_2y_2$ and replace it with $x_1y_2$. This decreases $f$ by 2. Next, we delete $x_3y_3$ and replace it with $x_1y_3$ which decreases $f$ by 4 more. We continue similarly. When we delete $x_jy_j$ and replace it with $x_1y_j$, it decreases $f$ by $2(j-1)$.

Therefore, $F_n$ contains a decreasing sequence $f(G_i) = f(G_i^1), f(G_i^2), \ldots, f(G_i^{2\sqrt{2n}})$ with $|f(G_i^j) - f(G_i^{j+1})| \leq 2(\sqrt{2n} - 1)$ for all $j$. Moreover, $f(G_i^{2\sqrt{2n}})$ is larger than $f(G_i) - 2(n-2)$ by at most $2(\sqrt{2n} - 1)$. Indeed,

$$f(G_i^{2\sqrt{2n}}) = f(G_i) - (2 + 4 + \cdots + 2(\sqrt{2n} - 1))$$

$$= f(G_i) - (\sqrt{2n} - 1)\sqrt{2n}$$

$$\leq f(G_i) - 2(n-2) + 2(\sqrt{2n} - 1).$$

Hence, it is sufficient to show that $F_n$ contains

$$f(G_i^j), f(G_i^j) - 2, f(G_i^j) - 4, \ldots, f(G_i^j) - 2(\sqrt{2n} - 1)$$

for all $j$. To prove this, let us fix $j$ and we shall modify the graph $G_i^j$ as follows. Let $\{z_1w_1, \ldots, z_{\sqrt{2n}}w_{\sqrt{2n}}\}$ be the matching inside $V_3$. First, we delete $z_2w_2$ and replace it with $w_1w_2$. This decreases $f$ by 2. Next, we delete $z_3w_3$ and replace it with $w_2w_3$ which decreases $f$ by 2 more. We continue similarly. When we delete $z_jw_j$ and replace it with $w_{j-1}w_j$, it decreases $f$ by 2. We are done since the modification has $\sqrt{2n} - 1$ steps. □
5. Proof of Theorem 3

We have seen a special case of Theorem 3 from the proof of Proposition 8 part (iii) that, amongst the graphs with \( n \) vertices and \( e \leq n - 1 \) edges, the \( e \)-star is the only graph with the minimum number of frustrated triangles. We deduce Theorem 3 using Theorem 11 together with this fact.

**Proof of Theorem 3.** By Proposition 9, we see that \( G \) can be obtained from a complete bipartite graph by flipping \( t_G \) pairs of vertices. Therefore, \( e \in [x(n-x)-t_G, x(n-x)+t_G] \) for some \( 0 \leq x \leq \frac{n}{2} \), and hence \( g(e) \leq t_G \) by the definition of \( g \). Now, by Theorem 11 part (i), we have \( f(G) \geq a_{g(e)} \) as required.

Next, we show that there is a graph \( G \) with \( n \) vertices and \( e \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n-1}{2} \rfloor - 1 \) edges such that \( f(G) = a_{g(e)} \). We see that \( g(e) \), the distance from \( e \) to the sequence \( (c_x) \), is at most \( \lfloor \frac{n-1}{2} \rfloor \). By the definition of \( g \), there is \( 0 \leq x \leq \frac{n}{2} \) such that \( g(e) = |e-c_x| \). We shall distinguish two cases.

If \( e = c_x - g(e) \) then let \( G \) be the graph obtained from the complete bipartite graph \( K_{x,n-x} \) by deleting a \( g(e) \)-star with center in the smaller side and leaves in the larger side of \( K_{x,n-x} \). This is possible since \( g(e) \leq \lfloor \frac{n-1}{2} \rfloor \leq n-x \), and if \( x = 0 \) then \( g(e) = 0 \). By Corollary 10, \( G \sim g(e) \)-star, and hence \( f(G) = a_{g(e)} \).

On the other hand, if \( e = c_x + g(e) \) then let \( G \) be the graph obtained from the complete bipartite graph \( K_{x,n-x} \) by adding a \( g(e) \)-star inside the larger side of \( K_{x,n-x} \). This is possible since \( g(e) + 1 \leq \lfloor \frac{n-1}{2} \rfloor + 1 \leq n-x \). By Corollary 10, \( G \sim g(e) \)-star, and hence \( f(G) = a_{g(e)} \).

We shall now show that these are the only extremal graphs. Suppose that \( G \) is a graph with \( n \) vertices and \( e \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n-1}{2} \rfloor - 1 \) edges such that \( f(G) = a_{g(e)} \). We know that \( g(e) \leq \lfloor \frac{n-1}{2} \rfloor \). First, we consider the case \( g(e) = \lfloor \frac{n-1}{2} \rfloor \). This can only happen when \( e = \lfloor \frac{n-1}{2} \rfloor \) or \( \lfloor \frac{n-1}{2} \rfloor + 1 \). If \( e = \lfloor \frac{n-1}{2} \rfloor \) then \( e \leq n-1 \) and so \( G \) is the \( e \)-star. Since \( g(e) = \lfloor \frac{n-1}{2} \rfloor = e \), we see that \( G \) is obtained from \( K_{0,n} \) by adding a \( g(e) \)-star. Similarly, if \( e = \lfloor \frac{n-1}{2} \rfloor + 1 \) then \( e \leq n-1 \) and so \( G \) is the \( e \)-star. Since \( g(e) = \lfloor \frac{n-1}{2} \rfloor + 1 = (n-1) - e \), we see that \( G \) is obtained from \( K_{1,n} \) by deleting a \( g(e) \)-star.

Now we may assume that \( g(e) \leq \lfloor \frac{n-1}{2} \rfloor - 1 \). Therefore, \( f(G) = a_{g(e)} < a_{g(e)+1} \) since \( a_t \) is increasing when \( t \leq \frac{n-1}{2} \). By Theorem 11 part (i), we conclude that \( t_G \leq g(e) \). Recall from the beginning of this proof that \( t_G \geq g(e) \) and so we must have \( t_G = g(e) \). By the definition of \( t_G \), \( G \sim H \) for some graph \( H \) with \( g(e) \) edges. Since \( f(H) = f(G) = f(g(e)\text{-star}) \) and \( g(e) \leq n-1 \), we conclude that \( H \) must be the \( g(e) \)-star. By Corollary 10, we have that \( G \) can be obtained from a complete bipartite graph by flipping \( g(e) \) pairs of vertices forming a star. Since \( g(e) \) is the distance from \( e \) to the sequence of number of edges of a complete bipartite graph, the \( g(e) \) pairs of vertices that we flip must be all edges or all nonedges. \( \Box \)
Let us remark that there are at most two extremal graphs for each \( e \leq \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n-1}{2} \rfloor - 1 \). Indeed, there are at most two \( c_x \)'s such that \( g(e) = |e - c_x| \). The size of the complete bipartite graph is determined by this \( c_x \) while the choice of deleting or adding edges is determined by the sign of \( e - c_x \).

Since \( F_n \) is symmetric, we have, by taking complements, the corresponding result for maximising the number of frustrated triangles amongst the graphs with a fixed number of edges.

**Corollary 15.** If \( G \) is a graph on \( n \) vertices with \( e \) edges then \( f(G) \leq \binom{n}{3} - a_{g(\binom{n}{3})} \). Moreover, the bound can be achieved when \( e \geq \binom{n}{3} - \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n-1}{2} \rfloor + 1 \). In this case, the extremal graphs are obtained from a disjoint union of two complete graphs of orders summing to \( n \) by deleting or adding \( g(\binom{n}{3} - e) \) edges forming a star. \( \square \)

### 6. Open problems

We conclude by mentioning questions and conjectures that would merit further study. Theorem 1 and Theorem 2 describe what \( F_n \) looks like in \([0, \frac{n^3}{2}]\) and \([\frac{2\sqrt{2}}{3} n^{3/2}, \frac{1}{2} \binom{n}{3}]\) respectively. We conjecture that the constant \( 2\sqrt{2} \) in Theorem 2 can be improved to 1.

**Conjecture 16.** For all sufficiently large \( n \), if \( m \in [\frac{n}{3}, \binom{n}{3} - \frac{n^3}{2}] \) and \( (n+1)m \) is even, then \( m \in F_n \).

Let \( t \leq t_{\text{max}} \). Combining Theorem 1 part (ii) and Corollary 10, we have

\[
F_n \cap [a_t, b_t] = \{ f(G) : e(G) = t \}.
\]

Corollary 14 tells us that \( F_n \cap [a_t, b_t] \subset \{ a_t, a_t + 2, \ldots, b_t - 2, b_t \} \). However, it is not true that every number in \( \{ a_t, a_t + 2, \ldots, b_t - 2, b_t \} \) appears in \( F_n \). For example, by considering all graphs with 4 edges, we see that \( \{ f(G) : e(G) = 4 \} = \{ a_4, a_4 + 2, \ldots, b_4 - 2, b_4 \} - \{ a_4 + 2 \} \). Therefore, we ask the following question.

**Problem 17.** Determine the set \([a_t, b_t] - \{ f(G) : e(G) = t \}\) for each \( t \leq t_{\text{max}} \).

We have seen from the proof of Proposition 8 part (iii) that, amongst the graphs with \( n \) vertices and \( e \leq n/2 \) edges, the \( e \)-matching is the only graph with the maximum number of frustrated triangles. Furthermore, Theorem 3 and Corollary 15 partially answer the following question.

**Problem 18.** Given the number of vertices and the number of edges, which graphs on \( n \) vertices maximise/minimise the number of frustrated triangles?

There are several ways one could generalise the definition of frustrated triangles. For instance, we can replace triangle with another subgraph. The most natural generalisation is to cycles.
For $k \geq 3$ and vertices $v_1, v_2, \ldots, v_k$ of a graph $G$, we say that a cyclic ordering $v_1v_2\ldots v_k$ is a frustrated $k$-cycle if
\[ |\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\} \cap E(G)| \]
is odd. Let $f_k(G)$ be the number of frustrated $k$-cycles in a graph $G$. We conjecture the following generalisation of Theorem 11.

**Conjecture 19.** For every $k \geq 3$, $t \geq 0$, and all sufficiently large $n$, the following holds. If $f_k(G) < f_k((t+1)\text{-star on } n \text{ vertices})$, then either $G$ or $\overline{G}$ can be obtained from a complete bipartite graph by flipping at most $t$ edges/non-edges.

Note that when $n$ is odd, this should hold for just $G$ instead of ‘$G$ or $\overline{G}$’.

We could also try to define frustrated triangles in hypergraphs and study the analogue set. One thing to note is that if we wish to proceed along the same method then our new definition should at least give us Lemma 4.

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