Sign refinement for combinatorial link Floer homology

ÉTIENNE GALLAIS

Link Floer homology is an invariant for links which has recently been described entirely in a combinatorial way. Originally constructed with mod 2 coefficients, it was generalized to integer coefficients thanks to a sign refinement. In this paper, thanks to the spin extension of the permutation group we give an alternative construction of the combinatorial link Floer chain complex associated to a grid diagram with integer coefficients. In particular we prove that the sign refinement comes from a 2–cohomological class corresponding to the spin extension of the permutation group.

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1 Introduction

Heegaard–Floer homology (Ozsváth–Szabó [9]) is an invariant for closed oriented 3-manifolds which was extended to give an invariant for null-homologous oriented links in such manifolds called link Floer homology (Ozsváth–Szabó [8; 10], Rasmussen [11]). It gives the Seifert genus \( g(K) \) of a knot \( K \) (Ozsváth–Szabó [7]), detects fibered knots (Ghiggini [2] in the case where \( g(K) = 1 \) and Ni [6] in general) and its graded Euler characteristic gives the Alexander polynomial [8; 11]. Recently, a combinatorial description of link Floer homology was given (Manolescu–Ozsváth–Sarkar [4]) and its topological invariance was proved in a purely combinatorial way (Manolescu–Ozsváth–Sarkar–Thurston [5]). The purpose of this paper is to give an alternative description of combinatorial link Floer homology with \( \mathbb{Z} \) coefficients. This point of view was recently used by Audoux [1] to describe combinatorial Heegaard–Floer homology for singular knots.

Let first recall the context of combinatorial link Floer homology: we follow conventions of [5]. A planar grid diagram \( G \) lies in a square on the plane with \( n \times n \) squares where \( n \) is the complexity of \( G \). Each square is decorated with an \( X \), an \( O \) or nothing in such a way that each row and each column contains exactly one \( X \) and one \( O \). We number the \( X \) and the \( O \) from 1 to \( n \) and denote \( X \) the set \( \{X_i\}_{i=1}^n \) and \( O \) the set \( \{O_i\}_{i=1}^n \).
Given a grid diagram $G$, we place it in standard position on the plane as follows: the bottom left corner is at the origin and each cell is a square of length one. We construct a planar link projection by drawing horizontal segments from the $O$ to the $X$ in each row and vertical segments from the $X$ to the $O$ in each column. At each intersection point, the vertical segment is over the horizontal one. This gives an oriented link $\overrightarrow{L}$ in $S^3$ and we say that $\overrightarrow{L}$ has a grid presentation given by $G$.

![Figure 1: Grid presentation of the Hopf link.](image)

We place the grid diagram on the oriented torus $T$ by making the usual identification of the boundary of the square. We endow $T$ with the orientation induced by the planar orientation. Let be the collection of the horizontal circles and the collection of the vertical ones. We associate with $G$ a chain complex $\mathcal{C}$; $\partial$ it is the group ring of $S_n$ over $\mathbb{Z}/2\mathbb{Z}$ where $S_n$ is the permutation group of $n$ elements. A generator $x \in S_n$ is given on $G$ by its graph: we place dots in points $i, x(i)$ for $i = 0, \ldots, n - 1$ (thus the fundamental domain of $G$ is the square minus the right vertical segment and the top horizontal segment).

For $A, B$ two finite sets of points in the plane we define $\mathcal{I}(A, B)$ to be the number of pairs $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$ such that $a_1 < b_1$ and $a_2 < b_2$. Let $\mathcal{J}(A, B) = (\mathcal{I}(A, B) + \mathcal{I}(B, A))/2$. We provide the set of generators with a Maslov degree $M$ given by

$$M(x) = J(x - O, x - O) + 1$$

where we extend $\mathcal{J}$ by bilinearly over formal sums (or differences) of subsets. Each variable $U_{O_i}$ has a Maslov degree equal to $-2$ and constants have Maslov degree equal to zero. Let $M_S(x)$ be the same as $M(x)$ with the set $S$ playing the role of $O$.

We provide the set of generators with an Alexander filtration $A$ given by $A(x) = (A_1(x), \ldots, A_l(x))$ with

$$A_i(x) = J(x - \frac{1}{2}(X + O), X_i - O) - \frac{n_i - 1}{2}$$
where when we number the components of $\tilde{L}$ from 1 to $\ell$, $O_i \subset \mathbb{O}$ (resp. $X_i \subset \mathbb{X}$) is the subset of $\mathbb{O}$ (resp. $\mathbb{X}$) which belongs to the $i$th component of $\tilde{L}$ and $n_i$ is the number of horizontal segments which belongs to the $i$th component. We let $A(U_{O_j}) = (0, \ldots, -1, 0, \ldots, 0)$ where $-1$ corresponds to the $i$th coordinate if $O_j$ belongs to the $i$th component.

Given two generators $x$ and $y$ and an immersed rectangle $r$ in the torus whose edges are arcs in the horizontal and vertical circles, we say that $r$ connects $x$ to $y$ if $y, x^{-1}$ is a transposition, if all four corners of $r$ are intersection points in $x \cup y$, and if we traverse each horizontal boundary component of $r$ in the direction dictated by the orientation of $r$ induced by $T$, then the arc is oriented from a point in $x$ to the point in $y$. Let $\text{Rect}(x, y)$ be the set of rectangles connecting $x$ to $y$: either it is the empty set or it consists of exactly two rectangles. Here a rectangle $r \in \text{Rect}(x, y)$ is said to be empty if there is no point of $x$ in its interior. Let $\text{Rect}^e(x, y)$ be the set of empty rectangles connecting $x$ to $y$.

![Rectangles](image)

Figure 2: Rectangles. We mark with black dots the generator $x$ and with white dots the generator $y$. There are two rectangles in $\text{Rect}(x, y)$ but only the left one is in $\text{Rect}^e(x, y)$.

The differential $\partial^e: C^- (G) \to C^- (G)$ is given on the set of generators by

$$\partial^e x = \sum_{y \in \mathbb{S}_n} \sum_{r \in \text{Rect}^e(x, y)} U_{O_1}^{O_1(r)} \cdots U_{O_n}^{O_n(r)} y$$

where $O_i(r)$ is the number of times $O_i$ appears in the interior of $r$.

**Theorem 1.1** (Manolescu–Ozsváth–Sarkar [4]) \((C^- (G), \partial^-)\) is a chain complex for $\text{CF}^- (S^3)$ with homological degree induced by $M$ and filtration level induced by $A$ which coincides with the link filtration of $\text{CF}^- (S^3)$.

In [5], the authors define a sign assignment for empty rectangles $S: \text{Rect}^e \to \{ \pm 1 \}$. Then, by considering $C^- (G)$ the group ring of $\mathbb{S}_n$ over $\mathbb{Z}[U_{O_1}, \ldots, U_{O_n}]$ and the
differential \( \partial^- : C^-(G) \to C^-(G) \) given by

\[
\partial^- x = \sum_{y \in \mathcal{O}_n} \sum_{r \in \text{Rect}^i(x,y)} S(r) U_{O_1}^{O_1(r)} \ldots U_{O_n}^{O_n(r)} y
\]

they obtain the following result.

**Theorem 1.2** (Manolescu–Ozsváth–Szabó–Thurston [5]) Let \( \widetilde{L} \) be an oriented link with \( \ell \) components. We number the \( \mathcal{O} \) so that \( O_1, \ldots, O_\ell \) correspond to the different components of \( \widetilde{L} \). Then the filtered quasi-isomorphism type of \( (C^-(G), \partial^-) \) over \( \mathbb{Z}[O_1, \ldots, O_\ell] \) is an invariant of the link.

In this paper, we give a way to refine the complex over \( \mathbb{Z} \) thanks to \( \tilde{S}_n \) the spin extension of \( S_n \) which is a non-trivial central extension of \( S_n \) by \( \mathbb{Z}/2\mathbb{Z} \). In Section 2 we define the spin extension \( \tilde{S}_n \) and make some algebraic calculus. Let \( z \) be the unique non-trivial central element of \( \tilde{S}_n \) and \( \Lambda = \mathbb{Z}[O_1, \ldots, O_n] \). In Section 3 we define a filtered chain complex \( \tilde{C}^-/;z \) where \( \tilde{C}^-/;z \) is the quotient module of the free \( \Lambda \)-module with generating set \( \tilde{S}_n \) by the submodule generated by \( \{z + 1\} \). Finally, in Section 4, we prove that our chain complex defines a sign assignment in the sense of [5] and that \( (\tilde{C}^-/;z) \) is filtered quasi-isomorphic to \( (C^-(G), \partial^-) \) with coefficients in \( \mathbb{Z} \).

## 2 Algebraic preliminaries

Let \( S_n \) be the group of bijections of a set with \( n \) elements numbered from 0 to \( n - 1 \). It is given in terms of generators and relations where the set of generators is \( \{\tau_i\}_{i=0}^{n-2} \) with \( \tau_i \) the transposition which exchanges \( i \) and \( i + 1 \) and relations are

\[
\tau_i^2 = 1 \quad 0 \leq i \leq n - 2 \\
\tau_i \tau_j = \tau_j \tau_i \quad |i - j| > 1, \quad 0 \leq i, j \leq n - 2 \\
\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad 0 \leq i \leq n - 3.
\]

**Theorem 2.1** The group given by generators and relations

\[
\tilde{S}_n = \langle \tau_0, \ldots, \tau_{n-2}, z | \quad z^2 = \overline{1}, \quad z \tau_i = \overline{z} \tau_i z, \quad \tau_i^2 = z, \quad 0 \leq i \leq n-2; \\
\tau_i \tau_j = \overline{z} \tau_j \tau_i \quad |i - j| > 1, \quad 0 \leq i, j \leq n - 2; \\
\tau_i \tau_{i+1} \tau_i = \overline{z} \tau_{i+1} \tau_i \tau_{i+1} \quad 0 \leq i \leq n - 3 \rangle
\]

is a non-trivial central extension (\( n \geq 4 \)) of \( S_n \) by \( \mathbb{Z}/2\mathbb{Z} \) called the spin extension of \( S_n \).
Remark 2.2 A proof of this theorem can be found in Karpilovsky [3, Theorem 2.12.3]. To see that it is a non-trivial extension, one can notice the following: let \( \mathbb{Q}_8 \) be the subgroup of \( \mathbb{S}_n \) generated by \( \tilde{t}_0, \tilde{t}_2, z \). Then \( \mathbb{Q}_8 \) is isomorphic to the unit sphere in the space of quaternions intersected with the lattice \( \mathbb{Z}^4 \) by a morphism \( \Phi \) such that \( \Phi(\tilde{t}_0) = i, \Phi(\tilde{t}_2) = j, \Phi(\tilde{t}_0, \tilde{t}_2) = k \) and \( \Phi(z) = -1 \). Therefore \( \mathbb{S}_n \) is non-trivial.

Remark 2.3 Cases \( n = 2 \) and \( n = 3 \) are not interesting in our situation: the only knot which can be represented by a grid diagram of complexity 2 or 3 is the trivial knot. Nevertheless, the group given by generators and relations above still exists: in the case \( n = 2 \), it is isomorphic to \( \mathbb{Z}/4\mathbb{Z} \), in the case \( n = 3 \), it is isomorphic to a subgroup of \( GL(2, \mathbb{C}) \) (see [3, Lemma 2.12.2]).

For \( i < j \), define
\[
\tilde{t}_{i,j} = \tilde{t}_i \cdot \tilde{t}_{i+1} \cdot \ldots \cdot \tilde{t}_{j-2} \cdot \tilde{t}_{j-1} \cdot \tilde{t}_{j-2} \cdot \ldots \cdot \tilde{t}_{i+1} \cdot \tilde{t}_i
\]
and \( \tilde{t}_{j,i} = z^{\tilde{t}_{i,j}} \).

Let \( \varepsilon: \mathbb{S}_n \rightarrow \{0, 1\} \) be the signature morphism.

Lemma 2.4 Let \( \tilde{x} = \tilde{t}_{i_1, i_2, \ldots, i_k} \) be an element in \( \mathbb{S}_n \) and \( x = p(\tilde{x}) \in \mathbb{S}_n \). Then for any \( 0 \leq i \neq j \leq n - 1 \)
\[
\tilde{x}, \tilde{t}_{i,j}, \tilde{x}^{-1} = \varepsilon(z(\tilde{x}), x(j)).
\]

Proof Since \( \tilde{x} = \tilde{t}_{i_1, i_2, \ldots, i_k} \), \( \tilde{x}^{-1} = \varepsilon(z(\tilde{x}), i_1) \). We prove by induction on \( k \geq 1 \) that for any \( i, j \in \{0, \ldots, n - 1\} \) we have \( \tilde{x}, \tilde{t}_{i,j}, \tilde{x}^{-1} = \varepsilon(z(\tilde{x}), x(j)) \).

- **Initialization** Let \( \tilde{x} = \tilde{t}_l \) and \( 0 \leq i < j \leq n - 1 \). So \( \tilde{t}_{l}^{-1} = z^{\tilde{t}_l} \) and \( \varepsilon(x) = 1 \).

There are several cases.

- **Case 1:** \( l < i - 1 \) or \( l > j \)
  \[ \tilde{x}, \tilde{t}_{i,j}, z^{\tilde{x}} = z^{\tilde{t}_{i,j}}. \]

- **Case 2:** \( l = i - 1 \)
  \[ \tilde{x}, \tilde{t}_{i,j}, z^{\tilde{x}} = z^{\tilde{t}_{i-1,j}}. \]

- **Case 3:** \( l = i \)
  \[ \tilde{x}, \tilde{t}_{i,j}, z^{\tilde{x}} = z^{\tilde{t}_{i+1,j}}. \]

- **Case 4:** \( i < l < j - 1 \)
  We prove by induction on \( l - i \geq 1 \) for \( i, j \) fixed that \( \tilde{t}_{l}, \tilde{t}_{i,j}, z^{\tilde{x}} = z^{\tilde{t}_{l}, \tilde{t}_{i,j}} \). For \( l = i + 1 \) then we have
  \[
  \tilde{t}_{i+1, i,j}, z^{\tilde{x}} = z^{\tilde{t}_{i+1, i,j}, \tilde{t}_{i,j}}. \\
  = z^{\tilde{t}_{i+1, i,j}, \tilde{t}_{i,j}}. \\
  = z^{\tilde{t}_{i,j}}. \\
  = z^{\tilde{t}_{i,j}}.
  \]
Suppose it is proved until rank \((l - 1) - i\). Then for \(\tilde{x} = \tilde{r}_l\) with \(l < j - 1\) we have
\[
\tilde{x} \tilde{x}_i \tilde{x} = z(\tilde{r}_i, \ldots, \tilde{r}_{l-2}).(\tilde{r}_{l-1}, \tilde{r}_l).\tilde{r}_{l-1,j}, (\tilde{r}_j, \tilde{r}_{l-1}, \tilde{r}_l).\tilde{r}_{l-2}, \ldots, \tilde{r}_l)
\]
\[
= z(\tilde{r}_i, \ldots, \tilde{r}_{l-2}).(\tilde{r}_{l-1}, \tilde{r}_j, \tilde{r}_{l-1}).\tilde{r}_{l-1,j}, (\tilde{r}_j, \tilde{r}_{l-1}, \tilde{r}_l).\tilde{r}_{l-2}, \ldots, \tilde{r}_l)
\]
\[
= z(\tilde{r}_i, \ldots, \tilde{r}_{l-1}).\tilde{r}_j, (\tilde{r}_{l-1}, \ldots, \tilde{r}_l) \text{ by induction}
\]
\[
= z\tilde{r}_j, (\tilde{r}_{l-1}, \ldots, \tilde{r}_l) \text{ by induction}
\]
\[
= z\tilde{r}_{l,j} \text{ by case 2.}
\]

- **Case 5:** \(l = j - 1\)
\[
\tilde{r}_{j-1, \tilde{r}_i, j} \tilde{r}_{j-1} = z(\tilde{r}_i, \ldots, \tilde{r}_{j-3}).\tilde{r}_{j-1, \tilde{r}_i, j} \tilde{r}_{j-2, \tilde{r}_j, j-1, \tilde{r}_{j-1, \tilde{r}_i, j}}(\tilde{r}_{j-2, \tilde{r}_i, j} \ldots, \tilde{r}_l)
\]
\[
= z\tilde{r}_{l,j-1}.
\]

- **Case 6:** \(l = j\)
\[
\tilde{r}_j, \tilde{r}_i, j \tilde{r}_j = z(\tilde{r}_i, \ldots, \tilde{r}_{j-2}).\tilde{r}_j, \tilde{r}_{j-1, \tilde{r}_i, j} (\tilde{r}_{j-2}, \ldots, \tilde{r}_l)
\]
\[
= z\tilde{r}_{l,j+1}.
\]

**Hereditity** Suppose the property is true until rank \(k\). Let \(\tilde{x} = \tilde{r}_{l_1, \ldots, \tilde{r}_{l_k}}\) and \(\tilde{r}_{i,j}\) be two elements in \(\tilde{S}_n\). Denote \(\tilde{y} = \tilde{r}_{l_2, \ldots, \tilde{r}_{l_k}}\). Then \(\tilde{x} \tilde{r}_{i,j} \tilde{x}^{-1} = \tilde{r}_{l_1, \tilde{y}, \tilde{r}_{i,j}, \tilde{y}}^{-1, \tilde{r}_{l_1}} \tilde{x}. \tilde{r}_{i,j} \tilde{x}^{-1}\). By induction hypothesis,
\[
\tilde{y} \tilde{r}_{i,j} \tilde{y}^{-1} = z^{e(y)} \tilde{r}_{y(i), y(j)}.
\]
So, \(\tilde{x} \tilde{r}_{i,j} \tilde{x}^{-1} = \tilde{r}_{l_1, z^{e(y)} \tilde{r}_{y(i), y(j)}, z \tilde{r}_{l_1}} \tilde{x}. \tilde{r}_{i,j} \tilde{x}^{-1}\). By induction hypothesis one more time,
\[
\tilde{x} \tilde{r}_{i,j} \tilde{x}^{-1} = z^{e(y)+1} \tilde{r}_{r_{l_1}, y(i), y(j)} = z^{e(x)} \tilde{r}_{x(i), x(j)}.
\]

The group \(\tilde{S}_n\) has another presentation in terms of generators and relations. Take \(\{z'\} \cup \{\tilde{r}_{i,j}\}_{i \neq j}\) where \(0 \leq i, j \leq n - 1\) as the set of generators with the following relations:

(2–1) \[z', z' = \tilde{1}' \quad z' \tilde{r}_{i,j} = \tilde{r}_{i,j}' z' \quad \tilde{r}_{i,j}' = z' \tilde{r}_{j,i} \quad \tilde{r}_{i,j}' \tilde{r}_{j,i} = z' \quad \text{for any } i, j\]

(2–2) \[z' \tilde{r}_{i,j}' = z' \tilde{r}_{i,j} \quad \text{for any } i, j, k, l \text{ if } \{i, j\} \cap \{k, l\} = \emptyset\]

(2–3) \[\tilde{r}_{i,j}' \tilde{r}_{j,k} = \tilde{r}_{i,j}' \tilde{r}_{j,k} \quad \tilde{r}_{i,j}' \tilde{r}_{j,k} = \tilde{r}_{i,j}' \quad \text{for any } i, j, k\]

**Proof** Let \(\tilde{S}_n\) the group with \(z\) and \(\tilde{r}_i\) as generators and \(\tilde{S}_n'\) the other one. Define \(\phi: \tilde{S}_n \to \tilde{S}_n'\) given on generators by \(\phi(\tilde{r}_i) = \tilde{r}_{i,i+1}'\), \(\phi(z) = z'\). For \(i < j\), let \(\phi(\tilde{r}_{i,j}) = \tilde{r}_{i,j}'\). By definition, (2–1) is verified. Lemma 2.4 gives equations (2–2) and (2–3). So the map \(\phi\) extends to a group isomorphism. \(\square\)
In what follows, we drop the prime exponent and only refer to \( \tau_{i,j} \) and \( z \) (\( \tau_i \) means \( \tau_{i,i+1} \)).

### 3 The chain complex

Let \( G \) be a grid presentation with complexity \( n \) of the link \( \widehat{L} \). Let \( \Lambda \) denote the ring \( \mathbb{Z}[U_{O_1}, \ldots, U_{O_n}] \). We define \( \widetilde{C}^{-}(G) \) to be the free \( \Lambda \)–module with generating set \( \mathcal{S}_n \) quotiented by the submodule generated by \( \{ z^{i+1} \} \) ie

\[
\widetilde{C}^{-}(G) = \mathbb{Z}[\mathcal{S}_n]/ < z + 1 >.
\]

Considered as module, \( \widetilde{C}^{-}(G) \) coincides with the free \( \Lambda \)–module with generating set \( \mathcal{S}_n \). But we can also consider the structure of algebra of \( \widetilde{C}^{-}(G) \) over \( \Lambda \). In this case, one can think of \( \widetilde{C}^{-}(G) \) as the group algebra of \( \mathcal{S}_n \) over \( \Lambda \) where the product is twisted by a non-trivial 2–cocycle (see Section 4).

We endow the set of generators with a Maslov grading \( M \) and an Alexander filtration \( A \) given by \( M(x) = M(\bar{x}) \) and \( A(x) = A(\bar{x}) = A(\bar{x}) \).

Let \( \bar{x} \) be an element of \( \mathcal{S}_n \) and let \( \text{Rect}(\bar{x}) \) be the set of rectangles starting at \( \bar{x} \): by definition it is the set \( \{ \tau_{i,j} \}_{0 \leq i \neq j \leq n-1} \). If we consider the set \( \text{Rect}(\bar{x}, \bar{y}) \) of rectangles connecting \( x \) to \( y \) (where \( y = x \tau_{i,j} \) as in [5], either it is the empty set, or it consists of two rectangles. We interpret the rectangle \( \tau_{i,j} \) in the oriented torus \( T \) as the rectangle whose bottom left corner belongs to the \( i \) th vertical circle. So in the case where \( \text{Rect}(x, y) = \{ r_1, r_2 \} \) the two corresponding rectangles are \( \tau_{i,j} \) and \( \tau_{j,i} \) Let \( r \) be the rectangle of \( \text{Rect}(x, y) \) corresponding to \( \tau \). A rectangle \( \tau \in \text{Rect}(\bar{x}) \) is said to be empty if the corresponding rectangle \( r \in \text{Rect}(x, y) \) is empty. The set of empty rectangles starting at \( \bar{x} \) is denoted \( \text{Rect}^e(\bar{x}) \).

We endow \( \widetilde{C}^{-}(G) \) with a differential \( \widetilde{d}^{-} \) given on elements of \( \mathcal{S}_n \) by:

\[
\widetilde{d}^{-}\bar{x} = \sum_{\tau \in \text{Rect}^e(\bar{x})} U_{O_1(\tau)} \cdots U_{O_n(\tau)} \bar{x} \tau
\]

where \( O_k(\tau) \) is the number of times \( O_k \) appears in the interior of \( r \).

**Proposition 3.1** The differential \( \widetilde{d}^{-} \) drops the Maslov degree by one and respect the Alexander filtration.

**Proof** It is a straightforward consequence of calculus done in [5]. \qed
Figure 3: Rectangles. Black dots represent \( x \) and white dots \( y \). The two hatched regions correspond to rectangles \( \bar{r}_{0,2} \in \text{Rect}(\bar{x}) \) and \( \bar{r}_{2,0} \in \text{Rect}(\bar{x}) \). The rectangle \( \bar{r}_{0,2} \) is an empty rectangle while \( \bar{r}_{2,0} \) is not.

**Proposition 3.2** The endomorphism \( \tilde{\partial}^{-} \) of \( \widetilde{C}^{-}(G) \) is a differential, i.e.

\[
\tilde{\partial}^{-} \circ \tilde{\partial}^{-} = 0.
\]

**Proof** Let \( \bar{x} = s(x) \in \bar{\mathcal{G}}_n \), viewed as a generator of \( \widetilde{C}^{-}(G) \). Then

\[
\tilde{\partial}^{-} \circ \tilde{\partial}^{-}(\bar{x}) = \sum_{\bar{r}_2 \in \text{Rect}^c(\bar{x}, \bar{r}_1)} \sum_{\bar{r}_1 \in \text{Rect}^c(\bar{x})} U_{O_1}^O(\bar{r}_1) + O_1(\bar{r}_2) \ldots U_{O_n}^O(\bar{r}_1) + O_n(\bar{r}_2) \bar{x} \bar{r}_1 \bar{r}_2.
\]

There are different cases which are illustrated by Figure 4.

**Cases 1,2** The rectangles corresponding to \( \bar{r}_{i,j} \) and \( \bar{r}_{k,l} \) give the elements \( \bar{z}_1 = \bar{x} \bar{r}_{k,j} \bar{r}_{i,j} \) and \( \bar{z}_2 = \bar{x} \bar{r}_{i,j} \bar{r}_{k,j} \). By equation (2–2) contribution to \( \tilde{\partial}^{-} \circ \tilde{\partial}^{-}(\bar{x}) \) is null.

**Case 3** Supports of the rectangles have a common edge. The two corresponding elements are \( \bar{z}_1 = \bar{x} \bar{r}_{i,j} \bar{r}_{j,k} \) and \( \bar{z}_2 = \bar{x} \bar{r}_{i,k} \bar{r}_{i,j} \) with \( i < j < k \). By equation (2–3), \( \bar{z}_1 = z \bar{z}_2 \) and so the contribution is null. Other cases work in a similar way.

**Case 4** The vertical annulus is of width 1 and corresponds to \( \bar{z}_1 = U_{O_m} \bar{x} \bar{r}_{i,j} \bar{r}_{i} \) (it is a consequence of the condition on rectangles to be empty).

To this vertical annulus corresponds the horizontal annulus of height 1 which contains \( O_m \). This horizontal annulus contributes for \( U_{O_m} \bar{x} \bar{r}_{i,k} \bar{r}_{k,l} = U_{O_m} \bar{x} \) for a pair \( k < l \in \{0, \ldots, n-1\} \). So, the contribution of each vertical annulus is canceled by the corresponding horizontal annulus. The global contribution to \( \tilde{\partial}^{-} \circ \tilde{\partial}^{-}(\bar{x}) \) is null.
4 Sign assignment induced by the complex

In this section we prove that the chain complex $\tilde{C}^{-}(G)$ coincides with the chain complex $C^{-}(G)$ over $\mathbb{Z}$ after a choice of a sign assignment.

**Definition 4.1** A sign assignment is a function $S: \text{Rect}^{\circ} \to \{\pm 1\}$ such that

- (Sq) for any distincts $r_1, r_2, r_1', r_2' \in \text{Rect}^{\circ}$ such that $r_1 \ast r_2 = r_1' \ast r_2'$ we have
  
  $S(r_1).S(r_2) = -S(r_1').S(r_2').$

- (V) if $r_1, r_2 \in \text{Rect}^{\circ}$ are such that $r_1 \ast r_2$ is a vertical annulus then
  
  $S(r_1).S(r_2) = -1,$

- (H) if $r_1, r_2 \in \text{Rect}^{\circ}$ are such that $r_1 \ast r_2$ is a horizontal annulus then
  
  $S(r_1).S(r_2) = +1.$
Let \( s: \mathcal{G}_n \to \widetilde{\mathcal{G}}_n \) be a section of the map \( p \) that is \( p \circ s = \text{id}_{\mathcal{G}_n} \).

\[
1 \xrightarrow{i} \mathbb{Z}/2\mathbb{Z} \xrightarrow{s} \widetilde{\mathcal{G}}_n \xrightarrow{p} \mathcal{G}_n \xrightarrow{1}
\]

To define the sign assignment we need the 2–cocycle \( c \in C^2(\mathcal{G}_n, \mathbb{Z}/2\mathbb{Z}) \) associated to the map \( s \) given by

\[
(4-1) \quad s(x).s(y) = (i \circ c(x, y))s(x.y).
\]

The cohomological class of \( c \) measures how \( s \) fails to be a group morphism. In particular, it is non-trivial \((n \geq 4)\) since \( \widetilde{\mathcal{G}}_n \) is a non-trivial central extension of \( \mathcal{G}_n \) by \( \mathbb{Z}/2\mathbb{Z} \).

We say that a rectangle \( r \) is horizontally torn if given the coordinates \((i_{bl}, j_{bl})\) of its bottom left corner and \((i_{tr}, j_{tr})\) of its top right corner then \( i_{bl} > i_{tr} \). Otherwise, \( r \) is said to be not horizontally torn.

**Lemma 4.2** The complex \((\widetilde{\mathcal{C}}^-(G), \tilde{\partial}^-)\) induces a sign assignment in the sense of **Definition 4.1**: for all \((x, y) \in \mathcal{G}_n^2\) and all \( r \in \text{Rect}^o(x, y)\)

\[
(4-2) \quad S(r) = \varepsilon(r).c(x^{-1}.y, x)
\]

where \( \varepsilon(r) = +1 \) if \( r \) is a rectangle not horizontally torn and \( \varepsilon(r) = -1 \) otherwise.

**Remark** The sign assignment in the sense of **Definition 4.1** is unique up to a 1–coboundary: if \( S_1 \) and \( S_2 \) are two sign assignments then there exists an application \( f: \mathcal{G}_n \to \{ \pm 1 \} \) such that for all rectangles \( r \in \text{Rect}^o(x, y) \), \( S_1(r) = f(x).f(y)S_2(r) \).

It is a consequence of the fact that the central extension corresponds to a 2–cohomological class in \( H^2(\mathcal{G}_n, \mathbb{Z}/2\mathbb{Z}) \) (compare with [5, Theorem 4.2]). Here, we construct explicitly a map \( s: \mathcal{G}_n \to \widetilde{\mathcal{G}}_n \) such that \( p \circ s = \text{id} \) which means making a choice of a representative of this class, another choice must differ by a 1–coboundary.

**Proof** Since \( c \) is 2–cocycle we have \( \delta c = 1 \) i.e. for all \((x, y, z) \in \mathcal{G}_n^3\)

\[
\delta c(x, y, z) = c(y, z).c(x, y, z).c(x, y, z).c(x, y) = 1.
\]

By definition we have \( c(x, 1) = c(1, x) = 1 \) and \( c(\tau_{i,j}, \tau_{i,j}) = -1 \). Let’s prove that \( S \) satisfy properties (Sq), (V) et (H).

(Sq) Let any four distincts rectangles \( S, r_1, r_2, r_1', r_2' \in \text{Rect}^o \) such that \( r_1 \ast r_2 = r_1' \ast r_2' \).

Suppose \( \tau_{i,j} = \tau_1 \in \text{Rect}^o(\mathcal{X}) \) corresponds to \( r_1 \) and \( \tau_{k,l} = \tau_2 \in \text{Rect}^o(\mathcal{X}, \tau_{i,j}) \) corresponds to \( r_2 \). Then \( \tau_{i,j} = \tau_{k,l} \in \text{Rect}^o(\mathcal{X}) \) corresponds to \( r_1' \) and \( \tau_{i,j} = \tau_{k,l} \in \text{Rect}^o(\mathcal{X}) \) corresponds to \( r_2' \).
A consequence of the above proposition and [5, Theorem 1.2] is the following.

\[ \text{Rect}^\circ (\mathbf{x}, \tau_{k,l}) \text{ corresponds to } r'_2. \]  
There are several cases to verify, as for the proof of \( \tilde{\delta}^{-} \circ \tilde{\delta}^{-} = 0 \) but all cases can be verified in a similar way. We verify the case \( i < j < k < l \). We calculate \( \delta c(\tau_{k,l}, \tau_{i,j}, x) \) and \( \delta c(\tau_{l,k}, \tau_{i,j}, x) \). With equalities \( c(\tau_{i,j}, \tau_{k,l}, x) = c(\tau_{k,l}, \tau_{i,j}, x) \) and \( c(\tau_{i,j}, \tau_{l,k}) = -c(\tau_{l,k}, \tau_{i,j}) \) we get

\[ S(r_1).S(r_2) = -S(r'_1).S(r'_2). \]

(V) Let \( r_1, r_2 \in \text{Rect}^\circ \) such that \( r_1 \neq r_2 \) is a vertical annulus. Suppose that \( \tilde{r}_1 = \tilde{r}_i \in \text{Rect}^\circ (\mathbf{x}) \) corresponds to \( r_1 \) and \( \tilde{r}_2 = \tilde{r}_i \in \text{Rect}^\circ (\mathbf{x}, \tau_{i,j}) \) corresponds to \( r_2 \). We calculate \( \delta c(\tau_{i,j}, \tau_{i,l}, x) \) and with equalities \( c(x, 1) = 1, c(\tau_{i,l}, \tau_{i,j}) = -1 \) we get

\[ S(r_1).S(r_2) = -1. \]

(H) Let \( r_1, r_2 \in \text{Rect}^\circ \) such that \( r_1 \neq r_2 \) is a horizontal annulus (of height one). Suppose \( \tilde{r}_1 = \tilde{r}_{i,j} \in \text{Rect}^\circ (\mathbf{x}) \) corresponds to \( r_1 \) and \( \tilde{r}_2 = \tilde{r}_{i,j} \in \text{Rect}^\circ (\mathbf{x}, \tau_{i,j}) \) corresponds to \( r_2 \). We calculate \( \delta c(\tau_{i,j}, \tau_{i,j}, x) \) and with equalities \( c(x, 1) = 1, c(\tau_{i,j}, \tau_{i,j}) = -1 \) we get

\[ S(r_1).S(r_2) = +1. \]

\[ \square \]

**Proposition 4.3**  
The filtered chain complex \( (\tilde{C}^-, (\tilde{\delta}^-)) \) is filtered isomorphic to the filtered chain complex \( (\tilde{C}^- (G), \tilde{\delta}^-) \).

**Proof**  
The map \( s: \mathbb{S}_n \rightarrow \mathbb{\tilde{S}}_n \) extends linearly with respect to \( \mathbb{Z}[U_1, \ldots, U_n] \) uniquely to a map \( s: C^- (G) \rightarrow \tilde{C}^- (G) \) which is an isomorphism of modules. It commutes with the differentials \( i.e. s \circ \tilde{\delta}^- = \tilde{\delta}^- \circ s \) where the sign assignment \( S \) is given by equation (4–2). By definition, \( s \) respects the Alexander filtration and the Maslov grading. So \( s \) defines a filtered isomorphism between the complexes \( (C^- (G), \tilde{\delta}^-) \) and \( (\tilde{C}^- (G), \tilde{\delta}^-) \).  

\[ \square \]

A consequence of the above proposition and [5, Theorem 1.2] is the following.

**Corollary 4.4**  
Let \( \mathbf{\widetilde{L}} \) be an oriented link with \( \ell \) components. We number the discs \( O_1, \ldots, O_\ell \) so that \( O_1, \ldots, O_\ell \) correspond to the different components of \( \mathbf{\widetilde{L}} \). Then the filtered quasi-isomorphism type of \( (\tilde{C}^- (G), \tilde{\delta}^-) \) over \( \mathbb{Z}[U_{O_1}, \ldots, U_{O_\ell}] \) is an invariant of the link.

**Remark**  
The proof of this theorem can also be done by adapting the original proof in [5], sometimes with slightly simplified arguments.
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Laboratoire de Mathématiques Jean Leray (LMJL), UFR Sciences et Techniques
2 rue de la Houssinière - BP 92208, 44 322 Nantes Cedex 3, France
Etienne.Gallais@univ-nantes.fr
http://www.math.sciences.univ-nantes.fr/~gallais/

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