Ellipsoidal flows in relativistic hydrodynamics of finite systems

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Abstract. A new class of 3D anisotropic analytic solutions of relativistic hydrodynamics with constant pressure is found. We analyse, in particular, solutions corresponding to ellipsoidally symmetric expansion of finite systems into vacuum. They can be utilized for relativistic description of the system evolution in thermodynamic region near the softest point and at the final stage of the hydrodynamic expansion in A+A collisions. The solutions can be used also for testing of numerical hydrodynamic codes solving relativistic hydrodynamic equations for anisotropic expansion of finite systems.

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1. Introduction

The solutions of relativistic hydrodynamics are used typically in cosmology, astrophysics and for an analysis of the processes of ultra-relativistic heavy ion collisions. It is remarkable fact that in these so different fields the Hubble-like solutions play the important role. As for A+A collisions the Hubble-like models becomes to be popular for a description of the experimental data only lately [1] since it was found in RHIC experiments that rapidity distribution is not flat even at these top energies, as it was expected (see, e.g., review in [2]). Thus, one can suppose that the initial state could be more close to spherically symmetric one rather then to boost-invariant in longitudinal direction [3, 4].

The spherically symmetric hydrodynamic solutions with the Hubble velocity distribution, $v = r/t$, has been considered for the first time in Ref. [5]. The equation of state (EoS) was chosen as ultrarelativistic one: $p = c_s^2 \varepsilon$. Some generalization of
these results was proposed in a case of the Hubble flow for EoS of massive gas with conserved particle number in Ref. [6].

It is naturally, however, that, unlike to the Hubble type flows, the velocity gradients in thermal matter formed in A+A collisions should be different in different directions since there is an initial asymmetry between longitudinal and transverse directions in central collisions and, in addition, between in-plane and off-plane transverse ones in non-central collisions. The another important remark is that the Hubble-like hydrodynamic solutions is related to infinite systems while the matter in A+A collisions is occupied essentially finite region.

In this letter we search for analytical solutions of the hydrodynamic equations describing 3D asymmetric relativistic expansions of finite systems.

2. General analysis

Let us start from the equations of relativistic hydrodynamics:

\[ \partial_\nu T^{\mu\nu} = 0, \]  

(1)

where the energy-momentum tensor corresponds to a perfect fluid:

\[ T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p \cdot g^{\mu\nu} \]  

(2)

We can attempt to find a particular class of solutions and therefore have to make some simplifications of (1).

Let us put pressure to be constant,

\[ p = p_0 = \text{const}, \]  

(3)

in other words, we are searching for the solution near the softest point where the velocity of sound \( c_s = 0 \).

Then we find from Eq. (1)

\[ (\varepsilon + p)\partial_\nu (u^\nu u^\nu) + u^\nu \partial_\nu \varepsilon = 0 \]  

(4)

Contracting the latter equation with \( u_\mu \) we obtain:

\[ (\varepsilon + p)\partial_\nu u^\nu + u^\nu \partial_\nu \varepsilon = 0. \]  

(5)

and, thus, the remaining equation to satisfy is

\[ u^\nu \partial_\nu u^\mu = 0, \]  

(6)

which means that flow is accelerationless in the rest systems of each fluid element; this property holds for the known Bjorken (boost-invariant) and Hubble flows.

Finally, the relativistic hydrodynamics at the softest point is described by the equations (5) and (6) for the hydrodynamic velocities \( u^\mu \) and energy density. A serious problem is, however, to find non-trivial solutions for the field \( u^\mu(x) \) of hydrodynamic 4-velocities.
3. Relativistic ellipsoidal solutions

First the solution at \( p = \text{const} \) was obtained in [7] as physically corresponding to a thermodynamic state of the system in the softest point with the velocity of sound \( c_s^2 = 0 \). Such a state could be associated with the first order phase transition. In A+A collisions it corresponds, probably, to transition between hadron and quark-gluon matter at SPS energies. The solution proposed in [7] has the cylindrical symmetry in the transverse plane and the longitudinal boost invariance:

\[
u_\mu = \gamma \left( \frac{t}{\tau}, \frac{x}{r}, \frac{y}{r}, \frac{z}{\tau} \right), \tag{7}\]

where \( \tau = \sqrt{t^2 - z^2} \), \( \gamma = (1 - v^2)^{-1/2} \), \( r \) is transverse radius, \( r = \sqrt{x^2 + y^2} \), and

\[
v = \frac{\alpha}{1 + \alpha \tau} r \tag{8}\]

describes axially symmetric transverse flow.

The above solution has, however, a limited region of applicability since the boost invariance is not expected at SPS energies and can be used only in a small mid-rapidity interval [8], it is not reached even at RHIC energies [2]. But most important is that in non-central collisions there is no axial symmetry and, therefore, one needs in transversely asymmetric solutions to describe the elliptic flows in these collisions, e.g., \( v_2 \) coefficients. Now we propose a new class of analytic solutions of the relativistic hydrodynamics for 3D asymmetric flows.

Firstly we construct the ansatz for normalized 4-velocity:

\[
u_\mu = \left\{ \frac{t}{\sqrt{t^2 - \sum a_i^2(t)x_i^2}}, \frac{a_k(t)x_k}{\sqrt{t^2 - \sum a_i^2(t)x_i^2}} \right\} \tag{9}\]

where the Latin indexes denote spatial coordinates, which are functions of time only. In this case a set of nonequal \( a_i \) induces 3D elliptic flow with velocities \( v_i = a_i(t)x_i/t \) at any time \( t \) the absolute value of velocity is constant, \( \mathbf{v}^2 = \text{const} \), at the ellipsoidal surface \( \sum a_i^2x_i^2 = \text{const} \).

The condition (6) of accelerationless thus reduces to the ordinary differential equation (ODE) for the functions \( a_i(t) \):

\[
\frac{da_i}{dt} = \frac{a_i - a_i^2}{t}, \tag{10}\]

the general solution of which is:

\[
a_i(t) = \frac{t}{t + T_i}, \tag{11}\]

where \( T_i \) is some set of 3 parameters (integration constants) having the dimension of time. The different values \( T_1, T_2 \) and \( T_3 \) results in anisotropic 3D expansion with the ellipsoidal flows.
The solution of Eq. (5) for energy density $\varepsilon$ is found as follows. Taking into account that 
\[ \frac{\partial u^\mu}{\partial \tau} = \sum \frac{a_i}{\tilde{\tau}}, \]
where
\[ \tilde{\tau} = \sqrt{t^2 - \sum a_i^2 x_i^2}, \] (12)
one can get
\[ (\varepsilon + p_0) \sum a_i(t) + t \partial_t \varepsilon + \sum a_i(t) x^i \partial_i \varepsilon = 0. \] (13)
General solution of the equation is
\[ \varepsilon + p_0 = F_\varepsilon \left( x_1 t + T_1, x_2 t + T_2, x_3 t + T_3 \right), \] (14)
where $F_\varepsilon$ is an arbitrary function of its variables. At fixed $T_i$ that define the velocity profile, function $F_\varepsilon$ is completely determined by the initial conditions for the enthalpy density.

If some value, associated with a quantum number or with particle number in a case of chemically frozen evolution is conserved [8], then one should add the corresponding equation to the basic ones. Such an equation has the standard form [9]:
\[ n \partial_\nu u^\nu + u^\mu \partial_\mu n = 0 \] (15)
where $n$ is associated with density of the correspondent conserved value, e.g., with the baryon or particle density. A general structure of this equation is similar to Eq. (13) and, therefore, the solution:
\[ n = \frac{F_n \left( x_1 t + T_1, x_2 t + T_2, x_3 t + T_3 \right)}{(t + T_1)(t + T_2)(t + T_3)}, \] (16)
which looks like as (14) and the function $F_n$ is also an arbitrary function of its arguments and can be fixed by the initial conditions for (particle) density $n$. The corresponding chemical potential is not zero and describes the deviation from chemical equilibrium in relativistic systems. The thermodynamic identities lead to the following expression for the temperature:
\[ T(t, x) = F_T \left( \frac{x_1}{t + T_1}, \frac{x_2}{t + T_2}, \frac{x_3}{t + T_3} \right) \] (17)
where $F_T$ is some function of its arguments that is defined by the initial conditions for $\varepsilon$ and $n$ as well as by EoS $\varepsilon = \varepsilon(n, T)$. If the initial enthalpy density profile is proportional to the particle density profile, $F_n \left( \frac{x_1}{T_1}, \frac{x_2}{T_2}, \frac{x_3}{T_3} \right) \sim F_\varepsilon \left( \frac{x_1}{T_1}, \frac{x_2}{T_2}, \frac{x_3}{T_3} \right)$, then $T = \text{const}$ and so $\mu = \text{const}$.

In a case of chemically equilibrium expansion of the ultrarelativistic gas when the particle number is uncertain and is defined by the conditions and parameters of the thermodynamic equilibrium, e.g., by the temperature $T$, the chemical potential $\mu \equiv 0$. Then it follows that $T = \text{const}$ for such a system and the entropy $s = (\varepsilon(t, x) + p_0)/T$ where $\varepsilon(t, x)$ is defined by (14).
4. Relativistic anisotropic expansion of finite systems

Let us describe some important particular solutions of the equations for relativistic ellipsoidal flows. If one defines the initial conditions on the hypersurface of constant time, say \( t = 0 \), then \( t \) is a natural parameter of the evolution. Such a representation of the solutions, which are similar to the Bjorken and Hubble ones, with velocity field \( v_i = a_i x_i / t \) has property of an infinite velocity increase at \( x \to \infty \). A real fluid, therefore, can occupy only the space-time region where \( |v| < 1 \), or \( \tilde{\tau}^2 > 0 \). To guarantee the energy-momentum conservation of the system during the evolution, all thermodynamic densities have to be zero at the boundary of the physical region, otherwise one should consider the boundary as the massive shell \([10]\). Hence in the standard hydrodynamic approach the enthalpy and particle density must be zero at the surface defined by \( |v(t, x)| = 1 \) at any time \( t \). One of a simple form of such a solution (for the case of particle number conservation) can be obtained from \([15], [16]\) choosing \( F_{\xi, n} \sim \exp \left( -b^2_{\xi} \frac{t^2}{\tilde{\tau}^2} \right) \):

\[
\epsilon + p_0 = \frac{C_\epsilon}{\prod_i (t + T_i)} \exp \left( -b^2_\epsilon \frac{t^2}{\tilde{\tau}^2} \right),
\]

\[
n = \frac{C_n}{\prod_i (t + T_i)} \exp \left( -b^2_n \frac{t^2}{\tilde{\tau}^2} \right),
\]

where \( \tilde{\tau} \) is defined by \([12]\), and the constants \( C_\epsilon, C_n, b_\epsilon \) and \( b_n \) are determined by the initial conditions as described in the previous section. As one can see, the enthalpy density tends to zero when \( |x| \) becomes fairly large approaching the boundary surface defined by \( |v(t, x)| = 1 \), in the other words, when \( \tilde{\tau} \to 0 \). Thus the physically inconsistent situation when massive fluid elements move with the velocity of light at the surface \( \tilde{\tau} = 0 \) is avoided. Of course, in such a solution one has to put a constant pressure to be zero, \( p_0 = 0 \).

On the other hand, the solution \([15], [16]\) can describe effectively finite systems at the hypersurface \( \tilde{\tau} = \text{const.} \) To show this one can substitute by the definition \( t^2 = \tilde{\tau}^2 + \sum a_i^2 x_i^2 \) in the solution, and get:

\[
\epsilon + p_0 = \frac{C_\epsilon}{\prod_i (t + T_i)} \exp \left( -b^2_\epsilon \sum a_i^2 x_i^2 / \tilde{\tau}^2 \right),
\]

\[
n = \frac{C_n}{\prod_i (t + T_i)} \exp \left( -b^2_n \sum a_i^2 x_i^2 / \tilde{\tau}^2 \right),
\]

supposing that \( x_i \) are spatial coordinates on the hypersurface. This describes effectively finite hydrodynamic flow with spatial radii \( R_i \approx \tau / \xi \) at \( \tau \gg T_k \).

As it follows from an analysis of the behavior of the thermodynamic values in the previous section, the temperature is constant if \( b_\epsilon = b_n \), otherwise one can choose the temperature approaching zero at the system’s boundary, e.g., for EoS
which is linear in temperature, the latter has the form

\[ T = \text{const}, \quad b_\varepsilon = b_n, \]

\[ T \sim e^{-(b_\varepsilon^2 - b_n^2)\frac{x^2}{\tilde{\tau}^2}} \to 0, \quad |v(x)| \to 1, \quad b_\varepsilon > b_n. \]  

(22)

Note that in the region of non-relativistic velocities, \( v^2 = \sum a_i^2 x_i^2 \ll 1 \), the space distributions of the thermodynamical quantities \((18),(19)\) as the Gaussian profile:

\[ \varepsilon + p_0 \simeq \prod_i (t + T_i) e^{-b_\varepsilon^2 \sum a_i^2 x_i^2}, \]

\[ n \simeq \prod_i (t + T_i) e^{-b_n^2 \sum a_i^2 x_i^2}. \]  

(23)

The forms of the solutions \((23)\) are similar to what was found in Ref. [11] as the elliptic solutions of the non-relativistic hydrodynamics equations. In this sense the solution proposed could be considered as the generalization (at vanishing pressure) of the corresponding non-relativistic solutions allowing one to describe relativistic expansion of the finite system into vacuum.

One can note that the case of equal flow parameters \( T_i = 0 \) and \( b_\varepsilon = b_n = 0 \) induces formally Hubble-like velocity profile.

It is worthily to emphasize that the physical solutions with non-zero constant pressure have a limited region of applicability at least in time-like direction: if one wants to continue the solutions to asymptotically large times, then \((\varepsilon + p_0)_{t \to \infty} \approx \frac{\varepsilon}{p_0} \to 0\), and this results in non-physical asymptotical behavior \( \varepsilon \to -p_0 \), unless we set \( p_0 = 0 \). Therefore, it is naturally to utilize such kind of solutions in a region of the first order phase transition, characterized by the constant temperature and soft EoS, \( c_s^2 = \partial p / \partial \varepsilon \approx 0 \), or at the final stage of the evolution that always corresponds to the quasi-inertial flows.

5. Conclusions

A general analysis of the flows at soft EoS \( p = \text{const} \) in the relativistic hydrodynamics is done. A new class of analytic solutions for 3D relativistic expansion with anisotropic flows is found. These solutions can describe the relativistic expansion of the finite systems into vacuum. They can be utilized for a description of the matter evolution in central and non-central ultra-relativistic heavy ion collisions, especially during deconfinement phase transition and the final stage of evolution of hadron systems. Also, the solutions can serve as a test for numerical codes describing 3D asymmetric flows in the relativistic hydrodynamics.
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