POLYNOMIAL FUNCTIONS OVER BOUNDED DISTRIBUTIVE LATTICES

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Abstract. Let $L$ be a bounded distributive lattice. We give several characterizations of those $L^n \to L$ mappings that are polynomial functions, i.e., functions which can be obtained from projections and constant functions using binary joins and meets. Moreover, we discuss the disjunctive normal form representations of these polynomial functions.

Keywords: Distributive lattice; polynomial function; normal form; functional equation.

1. INTRODUCTION

Let $L$ be a lattice. By a (lattice) polynomial function we simply mean a map $f: L^n \to L$ which can be obtained by composition of the binary operations $\land$ and $\lor$, the projections, and the constant functions; see e.g., page 93 in [2]. If constant functions are not used, then these polynomial functions are usually referred to as term functions.

For a finite lattice $L$, the set of all polynomial functions on $L$ is well understood. Indeed, Kindermann [8] reduces the problem of describing polynomial functions to tolerances, and reasonable descriptions for the latter have been provided in Czedli and Klukovits [4] and Chajda [3].

The goal of the current paper is to present a more direct approach to polynomial functions and provide alternative descriptions, different in nature and flavor, in the case when $L$ is distributive, with 0 and 1 as bottom and top elements. Notably enough, instead of finiteness it suffices to assume that $L$ is a bounded distributive lattice. Functions that are not order-preserving cannot be polynomial functions. Thus, our main result focuses on order-preserving functions.

We shall make use of the following notation. The ternary median term $(x \lor y) \land (x \lor z) \land (y \lor z)$ will be denoted by $\text{med}(x, y, z)$. For $c \in L$, the constant vector $(c, \ldots, c)$ will be denoted by $\overline{c}$. Given a function $f: L^n \to L$, we define $\delta_f: L \to L$ by $\delta_f(x) = f(\overline{x})$ for every $x \in L$. For $K \subseteq \{1, \ldots, n\}$, $a = (a_1, \ldots, a_n) \in L^n$, and $x = (x_1, \ldots, x_n) \in L^n$, let $x^K_a$ be the vector in $L^n$ whose $i$th component is $a_i$ if $i \in K$, and $x_i$, otherwise. Analogously, let $x_{-K}$ denote the vector in $L^{n-|K|}$ obtained from $x$ by removing each component $x_i$ for $i \in K$. Let $f_K^n: L^{n-|K|} \to L$ be the function defined by $f_K^n(x_{-K}) = f(x^K_a)$. If $K = \{k\}$ and $a = \overline{a}$, then we simply write $x^K_a$, $x_{-k}$,
and $f^n_k$. Let $[x]_c$ (resp. $[x]^c$) denote the vector whose $i$th component is 0 (resp. 1), if $x_i \leq c$ (resp. $x_i \geq c$), and $x_i$, otherwise.

The following result reassembles the various characterizations of polynomial functions provided in this paper, and its proof is given in Section 2.

**Main Theorem.** Let $L$ be a bounded distributive lattice and $f: L^n \to L$, $n \geq 1$, be an order-preserving function. The following conditions are equivalent:

(i) $f$ is a polynomial function;

(ii) for every $x \in L^n$ and $1 \leq k \leq n$,

\[ f(x) = \text{med} \left( f(x^1_k), x_k, f(x^2_k) \right); \]

(iii) for every function $g$ obtained from $f$ by substituting constants for variables, $\delta_g$ and $\delta_f$ preserve $\land$ and $\lor$, the sets $\{ f(x) \colon x \in L^n \}$ and $\{ f(a^k) \colon x \in L \}$ for every $a \in L^n$ and $1 \leq k \leq n$, are convex, and for every $x \in L^n$ and $1 \leq k \leq n$,

\[ f(x_1, \ldots, x_{k-1}, f(x, x_{k+1}, \ldots, x_n) = f(x); \]

(iv) $f$ satisfies condition (3) and its dual, where

\[ f(x \land \overline{x}) = f(x) \land c \quad \text{for all } c \in [f(\overline{0}), f(\overline{1})]; \]

(v) for every function $g$ obtained from $f$ by substituting constants for variables, $\delta_g$ and $\delta_f$ preserve $\lor$, and $f$ satisfies (3) and

\[ f(x) = f(x \land \overline{x}) \lor f([x]_c) \quad \text{for all } c \in [f(\overline{0}), f(\overline{1})]; \]

(vi) for every function $g$ obtained from $f$ by substituting constants for variables, $\delta_g$ and $\delta_f$ preserve $\land$ and $\lor$, and $f$ satisfies (4), its dual, and

\[ f(\overline{x}) = c \quad \text{for all } c \in [f(\overline{0}), f(\overline{1})]. \]

Even though not evident at the first sight, note that from the Main Theorem it follows that (v) is equivalent to its dual. Note also that every function satisfying (11) is order-preserving. The equivalence between (i) and (ii) was first established in [11, Theorem 17].

Let $[n]$ stand for $\{1, \ldots, n\}$ and let $2^{[n]}$ denote the set of all subsets of $[n]$. If $\alpha: 2^{[n]} \to L$ is a mapping, then

\[ \bigvee_{I \subseteq [n]} (\alpha(I) \land \bigwedge_{i \in I} x_i) \]

is called a disjunctive normal form over $L$. For a function $f: L^n \to L$, let $\text{DNF}(f)$ denote the set of those maps $\alpha: 2^{[n]} \to L$ for which (6), as an $L^n \to L$ mapping, coincides with $f$. Observe that $\text{DNF}(f) = \emptyset$ if $f$ is not a polynomial function. For $I \subseteq [n]$, let $e_I$ be the characteristic vector of $I$, i.e., the vector in $L^n$ whose $i$th component is 1 if $i \in I$, and 0 otherwise. Define $\alpha_f: 2^{[n]} \to L$, $I \mapsto f(e_I)$.
Lemma 1 (Goodstein [6]). If $L$ is a bounded distributive lattice and $f : L^n \to L$ a polynomial function, then $\alpha_f \in \text{DNF}(f)$. In particular, each polynomial function has a disjunctive normal form representation.

By Lemma 1 for each polynomial function $f : L^n \to L$, we have that
\begin{equation}
(7) \quad f \text{ is uniquely determined by its restriction to } \{0,1\}^n.
\end{equation}

It is noteworthy that, by (7), term functions are exactly those polynomial functions $f : L^n \to L$ for which $\{0\}, \{1\}$, and $\{0,1\}$ constitute subalgebras of $(L, f)$. In addition to the Main Theorem, we prove the following result strengthening Lemma 1.

Proposition 2. Let $L$ be a bounded distributive lattice, $f : L^n \to L$ a polynomial function, and $\alpha : 2^{[n]} \to L$ a mapping. Then $\alpha \in \text{DNF}(f)$ if and only if $\bigvee_{J \subseteq I} \alpha(J) = \alpha_f(I)$ for all $I \subseteq [n]$.

Using Proposition 2, it is straightforward to construct examples of lattices $L$ and polynomial functions $f : L^n \to L$ for which $|\text{DNF}(f)| > 1$, and to provide some technical conditions characterizing those polynomial functions $f$ for which $|\text{DNF}(f)| = 1$. The trivial details are left to the reader.

2. Technicalities and proofs

In this section we provide the proofs of the Main Theorem and Proposition 2. First, we prove the latter.

Proof of Proposition 2. Let $L$ be a bounded distributive lattice, $f : L^n \to L$ a polynomial function, and $\alpha : 2^{[n]} \to L$ a mapping.

Suppose first that $\alpha \in \text{DNF}(f)$. Then, for every $I \subseteq [n]$, $\alpha_f(I) = f(e_I) = \bigvee_{J \subseteq I} \alpha(J)$. Now suppose that $\bigvee_{J \subseteq I} \alpha(J) = \alpha_f(I)$, for all $I \subseteq [n]$, and let $g : L^n \to L$ be the polynomial function such that $\alpha \in \text{DNF}(g)$. Clearly, for every $I \subseteq [n]$, we have $g(e_I) = \bigvee_{J \subseteq I} \alpha(J) = \alpha_f(I) = f(e_I)$. From (7) it follows that $g = f$ and hence, $\alpha \in \text{DNF}(f)$.

To prove the Main Theorem we will need some auxiliary results. We proceed by focusing first on the conditions given in (iv), (v), and (vi).

Lemma 3. Every polynomial function $f : L^n \to L$ satisfies (3) and its dual.

Proof. Let $f : L^n \to L$ be a polynomial function. For any $c \in [f(\overline{0}), f(\overline{1})]$, we have
\begin{align*}
f(x \land \overline{c}) &= \bigvee_{I \subseteq [n]} \left( \alpha_f(I) \land \bigwedge_{i \in I} (x_i \land \overline{c}) \right) = f(\overline{0}) \lor \bigvee_{\substack{I \subseteq [n] \\cap \neg \emptyset}} \left( \alpha_f(I) \land \bigwedge_{i \in I} (x_i \land \overline{c}) \right) \\
&= \left( f(\overline{0}) \lor \bigvee_{\substack{I \subseteq [n] \\cap \neg \emptyset}} \left( \alpha_f(I) \land \bigwedge_{i \in I} x_i \right) \right) \land \overline{c} = f(x) \land \overline{c}.
\end{align*}

Similarly, it follows that $f$ satisfies the dual of (3).
Lemma 4. Let \( f : L^n \to L \) be an order-preserving function. If \( f \) satisfies (3) or its dual, then it satisfies (5). In particular, the set \( \{ f(x) : x \in L^n \} \) coincides with \( [f(\overline{0}), f(\overline{1})] \) and thus is convex.

Proof. If \( f \) satisfies (3), then for any \( c \in [f(\overline{0}), f(\overline{1})] \), we have \( f(\overline{c}) = f(\overline{1} \land c) = f(\overline{1}) \land c = c \) and thus \( f \) satisfies (5). The dual statement follows similarly. The last claim follows immediately from (5).

Lemma 5. Let \( f : L^n \to L \) be an order-preserving function. If \( f \) satisfies (3) and its dual, then \( \delta_f \) preserves \( \land \) and \( \lor \).

Proof. If \( f : L^n \to L \) satisfies (3) and its dual, by Lemma 4 it satisfies (5). Then, for every \( x \in L \), we have \( \delta_f(x) = \text{med}(f(\overline{0}), \delta_f(x), f(\overline{1})) = \delta_f(\text{med}(f(\overline{0}), x, f(\overline{1}))) = \text{med}(f(\overline{0}), x, f(\overline{1})). \) Thus,

\[
\delta_f(x \land y) = \text{med} (f(\overline{0}), x \land y, f(\overline{1})) = \text{med} (f(\overline{0}), x, f(\overline{1})) \land \text{med} (f(\overline{0}), y, f(\overline{1})) = \delta_f(x) \land \delta_f(y).
\]

Similarly, it follows that \( \delta_f \) preserves \( \lor \).

Lemma 6. Let \( f : L^n \to L \) be an order-preserving function. If \( f \) satisfies (3) and its dual, then it satisfies (4) and its dual. Moreover, for every function \( g \) obtained from \( f \) by substituting constants for variables, \( \delta_g \) and \( \delta_f \) preserve \( \land \) and \( \lor \).

Proof. Suppose that \( f \) satisfies (3) and its dual. For any \( x \in L^n \) and any \( c \in [f(\overline{0}), f(\overline{1})] \), we have

\[
f(x \lor \overline{c}) \lor f([x]_c) = (f(x) \lor c) \lor f([x]_c)
\]

and hence, \( f \) satisfies (4). The dual statement follows similarly.

Claim 1. Every function \( g \) obtained from \( f \) by substituting constants for variables satisfies (3) and its dual.

Proof of Claim 1. Let \( K \subseteq [n] \), \( a \in L^n \), and consider \( c \in [f_K^a(\overline{0}), f_K^a(\overline{1})] \). By (4), (3), and the dual of (3), for \( x \in L^n - |K| \), we have

\[
f_K^a(x \lor \overline{c}) = f_K^a(x \lor \overline{c}) \lor f_K^a([x \lor \overline{c}]_c) = (f_K^a(x) \lor c \lor f_K^a([x \lor \overline{c}]_c) = c \lor f_K^a([x \lor \overline{c}]_c) = f_K^a(x) \lor c.
\]

Therefore \( f_K^a \) satisfies the dual of (3). Similarly, we can prove that it also satisfies (3).

By Claim 1, each function \( g \) satisfies (3) and its dual. By Lemma 5, \( \delta_g \) preserves \( \land \) and \( \lor \).
As mentioned, the equivalence between (i) and (ii) in the Main Theorem was shown in [11, Theorem 17]. For the sake of self-containment, we provide a simpler proof here.

**Proposition 7** [11]. A function \( f : L^n \rightarrow L \) is a polynomial function if and only if it satisfies (1) for every \( x \in L^n \) and \( 1 \leq k \leq n \).

**Proof.** Clearly, the class of all functions \( f : L^n \rightarrow L \) satisfying (1) for every \( x \in L^n \) and \( 1 \leq k \leq n \), contains all projections, constant functions, and it is closed under taking \( \wedge \) and \( \vee \) of functions. Thus this class contains all polynomial functions.

On the other hand, any function obtained from a function in this class by substituting constants for variables, is also in the class. Thus, if a function \( f : L^n \rightarrow L \) satisfies (1) for every \( x \in L^n \) and \( 1 \leq k \leq n \), then by repeated applications of (1), we can easily verify that \( f \) can be obtained by composition of the binary operations \( \vee \) and \( \wedge \), the projections, and the constant functions. That is, \( f \) is a polynomial function. □

Now we focus on the conditions given in (iii), (v), and (vi) of the Main Theorem. We shall make use of the following general result.

**Lemma 8.** Let \( C \) be a class of functions \( f : L^n \rightarrow L \) \((n \geq 1)\) such that

(i) the unary members of \( C \) are polynomial functions;

(ii) for \( n > 1 \), any unary function obtained from an \( n \)-ary function \( f \) in \( C \) by substituting constants for \( n - 1 \) variables of \( f \) is also in \( C \).

Then \( C \) is a class of polynomial functions.

**Proof.** Let \( C \) be a class of functions satisfying the conditions of the lemma. We show that each \( f : L^n \rightarrow L \) in \( C \) is a polynomial function. By condition (i), the claim holds for \( n = 1 \). So suppose that \( n > 1 \). By Proposition 7, it is enough to show that \( f \) satisfies (1). So let \( a \in L^n \), let \( k \in [n] \), and consider the unary function \( f^{a_k} : L \rightarrow L \) given by \( f^{a_k}(x) = f(a^k_0) \). By condition (ii), we have that \( f^{a_k} \in C \), and hence \( f^{a_k} \) is a polynomial function. By Proposition 7, \( f^{a_k} \) satisfies (1) and hence,

\[
 f(a) = f^{a_k}(a_k) = \text{med} (f^{a_k}(0) , a_k , f^{a_k}(1)) = \text{med} (f(a^k_0) , a_k , f(a^k_1)) .
\]

Since the above holds for every \( a \in L^n \) and \( k \in [n] \), it follows that \( f \) satisfies (1) and thus it is a polynomial function. □

Note that, for \( n = 1 \), (2) reduces to the well-known idempotency equation \( f \circ f = f \); see for instance Kuczma et al. [9, §11.9E].

**Proposition 9.** A unary function \( f : L \rightarrow L \) is a polynomial function if and only if \( \{ f(x) : x \in L \} \) is convex and \( f \) is a solution of the idempotency equation that preserves \( \wedge \) and \( \vee \).

**Proof.** By Proposition 7, every unary polynomial function \( f : L \rightarrow L \) is of the form \( f(x) = \text{med}(f(0), x, f(1)) \) and thus satisfies the conditions stated in the proposition.
Conversely, let \( f : L \to L \) be a solution of the idempotency equation that preserves \( \land \) and \( \lor \) and such that \( \{ f(x) : x \in L \} \) is convex, and let \( x \in L \). If \( x \in [f(0), f(1)] = \{ f(x) : x \in L \} \), then there is \( z \in L \) such that \( x = f(z) \) and hence \( f(x) = f(f(z)) = f(z) = x \). Otherwise, let \( z = \text{med}(f(0), x, f(1)) \in [f(0), f(1)] \). Then, since \( f \) preserves \( \land \) and \( \lor \), it is order-preserving, and we have

\[
\begin{align*}
f(x) &= \text{med}(f(0), f(x), f(1)) = \text{med}(f(f(0)), f(x), f(f(1))) \\
&= f(\text{med}(f(0), x, f(1))) = f(z) = z = \text{med}(f(0), x, f(1)),
\end{align*}
\]

which shows that \( f \) is a polynomial function. \( \square \)

**Proposition 10.** An order-preserving function \( f : L^n \to L \) is a polynomial function if and only if, for every function \( g \) obtained from \( f \) by substituting constants for variables, \( \delta_g \) and \( \delta_f \) preserve \( \land \) and \( \lor \), \( f \) satisfies (2), and the sets \( \{ f(x) : x \in L^n \} \) and \( \{ f(a_k^n) : x \in L \} \), for every \( a \in L^n \) and \( 1 \leq k \leq n \), are convex.

**Proof.** By Lemmas 3 and 5 if \( f \) is a polynomial function, then each \( \delta_g \) and \( \delta_f \) preserve \( \land \) and \( \lor \), and \( \{ f(x) : x \in L^n \} \) is convex. By Proposition 7 every polynomial function \( f \) satisfies (2) and each set \( \{ f(a_k^n) : x \in L \} \), for \( a \in L^n \) and \( 1 \leq k \leq n \), is convex.

To prove the converse claim, consider the class \( C \) of order-preserving functions \( f : L^n \to L \) (\( n \geq 1 \)) satisfying the conditions of the proposition. Clearly, \( C \) satisfies condition (ii) of Lemma 8. By Proposition 9 \( C \) satisfies also condition (i) of Lemma 8 and hence, \( C \) is a class of polynomial functions. \( \square \)

**Proposition 11.** Let \( f : L \to L \) be a function. The following conditions are equivalent:

(i) \( f \) is a polynomial function;

(ii) \( f \) satisfies (3) and preserves \( \lor \);

(iii) \( f \) satisfies (4) and preserves \( \land \) and \( \lor \).

**Proof.** The implication (i) \( \Rightarrow \) (iii) follows from Lemmas 3 and 5. The implication (iii) \( \Rightarrow \) (ii) follows from the fact that if \( f \) satisfies (4) and preserves \( \land \) then it satisfies (3). Finally, to see that the implication (ii) \( \Rightarrow \) (i) holds, observe first that (3) implies (5) by Lemma 4. Since \( f \) preserves \( \lor \), we have that \( f \) satisfies the dual of (3). Moreover, \( f \) is clearly order-preserving, and we have

\[
f(x) = \text{med}(f(0), f(x), f(1)) = f(\text{med}(f(0), x, f(1))) = \text{med}(f(0), x, f(1)),
\]

which shows that \( f \) is a polynomial function. \( \square \)

**Proposition 12.** An order-preserving function \( f : L^n \to L \) is a polynomial function if and only if, for every function \( g \) obtained from \( f \) by substituting constants for variables, \( \delta_g \) and \( \delta_f \) preserve \( \lor \), and \( f \) satisfies (3) and (4).
Proof. By Lemmas 3, 5, 6 and Proposition 10 it follows that the conditions are necessary.

To prove the converse claim, we make use of Lemma 8. Let \( C \) be the class of order-preserving functions \( f : L^n \rightarrow L \ (n \geq 1) \) satisfying the conditions of the proposition. By Proposition 11 \( C \) satisfies condition (i) of Lemma 8. To complete the proof, it is enough to show that (3) and (4) are preserved under substituting constants for variables, since then condition (ii) of Lemma 8 will be also fulfilled. So suppose that \( f : L^n \rightarrow L \) satisfies (3) and (4).

Let \( k \in [n] \) and \( a \in L \). To see that \( f_k^a \) satisfies (3), just note that if \( x \in L^{n-1} \) and \( c \in [f_k^a(\emptyset), f_k^a(\top)] \) then, by (4),

\[
f_k^a(x \land \top) = f_k^a(x \land x) = f_k^a(x) \land c.
\]

To see that \( f_k^a \) satisfies (4), let \( x \in L^n \) and \( c \in [f_k^a(\emptyset), f_k^a(\top)] \). If \( a \leq c \), then \( f([x_k^a] \land \top) = f_k^a([x^a] \land \top) \) and \( f([x_k^a]c) \leq f_k^a([x^a]c) \). If \( a \geq c \), then \( f([x_k^a] \land \top) \leq f_k^a([x^a] \land \top) \) and \( f([x_k^a]c) = f_k^a([x^a]c) \). In both cases, by (4), we get

\[
f_k^a([x^a] \land \top) \leq f_k^a([x^a] \land \top) \land f_k^a([x^a]c) \leq f_k^a([x^a] \land \top).
\]

Since the above holds for every \( x \in L^{n-1} \), we have that \( f_k^a \) satisfies (4).

Proposition 13. An order-preserving function \( f : L^n \rightarrow L \) is a polynomial function if and only if, for every function \( g \) obtained from \( f \) by substituting constants for variables, \( \delta_g \) and \( \delta_f \) preserve \( \land \) and \( \lor \), and \( f \) satisfies (4), its dual, and (7).

Proof. By Lemmas 3, 5, and 6 it follows that the conditions are necessary.

To prove the converse claim, we make use of Lemma 8. Let \( C \) be the class of order-preserving functions \( f : L^n \rightarrow L \ (n \geq 1) \) satisfying the conditions of the proposition. By Proposition 11 \( C \) satisfies condition (i) of Lemma 8. To complete the proof, it is enough to show that (3), its dual, and (4) are preserved under substituting constants for variables, since then condition (ii) of Lemma 8 will be also fulfilled. So suppose that \( f : L^n \rightarrow L \) satisfies (3), its dual, and (4).

Let \( k \in [n] \) and \( a \in L \). To see that \( f_k^a \) satisfies (4), just note that if \( c \in [f_k^a(\emptyset), f_k^a(\top)] \) then, by (4) and its dual,

\[
f_k^a(\top) = f_k^a(\top) \land f_k^a(\top) = f_k^a(\top) \land c.
\]

The proof that \( f_k^a \) satisfies (4) follows exactly the same steps as in the proof of Proposition 12. The dual claim follows similarly.

We can now provide a proof of the Main Theorem.

Proof of the Main Theorem. The equivalences \((i) \iff (ii) \iff (iii)\) are given by Propositions 7 and 10. The equivalences \((i) \iff (iv) \iff (v) \iff (vi)\) follow immediately from Lemmas 3, 4, 5, 6 and Propositions 12 and 13.
3. Concluding remarks

By the equivalence \((i) \iff (iii)\), polynomial functions \(f: L^n \to L\) with \(f(\emptyset) = 0\) and \(f(\top) = 1\) coincide exactly with those for which \(\{f(x): x \in L^n\} = L^n\). These are referred to as \textit{discrete Sugeno integrals} and were studied in [10] where equivalence \((i) \iff (iv)\) of the Main Theorem was established for this particular case when \(L\) is an interval of the real line. Also, the implication \((i) \implies (v)\) of the Main Theorem reduces to that established by Benvenuti, Mesiar, and Vivona [1] when \(L\) is an interval of the real line, since in this case the conditions that \(\delta_g\) and \(\delta_f\) preserve \(\lor\) become redundant. Condition \((3)\) and its dual, when strengthened to all \(c \in L\), are referred to as \textit{\(\land\)-homogeneity} and \textit{\(\lor\)-homogeneity}, respectively; see [7]. These were used by Fodor and Roubens [5] to axiomatize certain classes of aggregation functions over the reals.

Given the nature of statements \((iii)-(vi)\), it is natural to ask whether the equivalences between these and \((i)\) continue to hold over non-distributive lattices. The reader can easily verify that (already for unary polynomial functions) this is not the case. Also, the property of being order-preserving is a consequence of \((ii)\) of the Main Theorem, but this is not the case for \((iii)-(vi)\).

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