Free Knots are Not Invertible

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1 Introduction. The parity, bracket for free links and the map $\Delta$.

The goal of the present paper is to show that free knots and links are in general not invertible: this fact turns out to be surprisingly non-trivial.

Free links (also known as homotopy classes of Gauss phrases) were introduced by Turaev [5], and regularly studied by Manturov [3, 4] and Gibson [1].

We first briefly recall the basic definitions from [3].

By a 4-graph we mean a topological space consisting of finitely many components, each of which is either a circle or a finite graph with all vertices having valency four.

A 4-graph is framed if for each vertex of it, the four emanating half-edges are split into two sets of edges called (formally) opposite.

A unicursal component of a 4-graph is either a free loop component of it or an equivalence class of edges where two edges $a, b$ are called equivalent if there is a sequence of edges $a = a_0, \ldots, a_n = b$ and vertices $v_1, \ldots, v_n$ so that $a_i$ and $a_{i+1}$ are opposite at $v_{i+1}$.

As an example of a free graph one may take the graph of a singular link. Analogously to 4-graphs we define long 4-graphs; here we allow two vertices $a, b$ to have valency one (the others having valency four) in such a way that the edges $x, y$ incident to $a$ and $b$ should belong to the same unicursal component in the above sense. One may also think of these two edges of the long 4-graph to be noncompact, i.e., we may think that the vertices of valency one are removed and the ends of the edges are taken to infinity.

By a free link we mean an equivalence class of framed 4-valent graphs modulo the following transformations. For each transformation we assume that only one fixed fragment of the graph is being operated on (this fragment is to be depicted) or some corresponding fragments of the chord diagram. The remaining part of the graph or chord diagram are not shown in the picture; the pieces of the chord diagram not containing chords participating in this transformation, are depicted by punctured arcs. The parts of the graph are always shown in a way such that the formal framing (opposite edge relation) in each vertex coincides with the natural opposite edge relation taken from $\mathbb{R}^2$.

The first Reidemeister move is an addition/removal of a loop, see Fig.1

The second Reidemeister move adds/removes a bigon formed by a pair of edges which are adjacent in two edges, see Fig. 2
Figure 1: Addition/removal of a loop on a graph and on a chord diagram

Figure 2: The second Reidemeister move and two chord diagram versions of it
Note that the second Reidemeister move adding two vertices does not impose any conditions on the edges it is applied to: we may take any two two edges of the graph and connect them together as shown in Fig. 2 to get two new crossings.

The third Reidemeister move is shown in Fig. 3.

Note that each of these three moves applied to a framed graph, preserves the number of unicursal components of the graph. Thus, applying these moves to graphs with a unique unicursal cycle, we get to graphs with a unique unicursal cycle.

A free knot is a free link with one unicursal component (obviously, the number of unicursal component of a framed 4-graph is preserved under Reidemeister moves).

Analogously, one defines long free links and long free knots; each free link has one noncompact (long) unicursal component.

Free links are closely connected to flat virtual knots, see, e.g. [2], i.e., with equivalence classes of virtual knots modulo transformation changing over/undercrossing structure. The latter are equivalence classes of immersed curves in orientable 2-surfaces modulo homotopy and stabilization.

Here we introduce the notion of smoothing, we shall often use in the sequel.

Let $G$ be a framed four-valent graph, let $v$ be a vertex of $G$ with four incident half-edges $a, b, c, d,$ s.t. $a$ is opposite to $c$ and $b$ is opposite to $d$ at $v$.

By smoothing of $G$ at $v$ we mean any of the two framed 4-graphs obtained by removing $v$ and repasting the edges as $(a, b), (c, d)$ or as $(a, d) (b, c)$, see Fig. 4.

Herewith, the rest of the graph (together with all framings at vertices except $v$) remains unchanged.

We may then consider further smoothings of $G$ at several vertices.

We are ready to define invariants of free knots in terms of their smoothings.
1.1 The parity axioms

Below we describe an important property of crossings called \textit{parity}, which, if exists, leads to many important consequences in knot theory: one gets some easy ways for establishing minimality of knot diagrams, functorial mappings from knots to knots, constructs powerful invariants, \cite{3,4}.

Assume we have a certain class of knot-like objects which are equivalence classes of \textbf{diagrams} modulo \textit{three Reidemeister moves}. Assume for this class of diagrams (e.g. 4-valent framed graphs) there is a fixed rule of distinguishing between two types of crossings (called even and odd) such that:

1) Each crossing taking part in the first Reidemeister move is even, and after adding/deleting this crossing the parity of the remaining crossings remains the same.

2) Each two crossings taking part in the second Reidemeister move are either both odd or both even, and after performing these moves, the parity of the remaining crossings remains the same.

3) For the third Reidemeister move, the parities of the crossings which do not take part in the move remain the same.

Moreover, the parities of the three pairs of crossings are the same in the following sense: there is a natural one-to-one correspondence between pairs of crossings $A - A', B - B', C - C'$ taking part in the third Reidemeister move, see Fig. 5.

We require that the \textit{parity} of $A$ coincides with that of $A'$, the \textit{parity} of $B$ coincides with that of $B'$ and the \textit{parity} of $C$ coincides with that of $C'$.

We also require that the number of odd crossings among the three crossings in question ($A, B, C$) is even (that is, is equal to 2 or 0).

It turns out that there are \textit{many different} parities for different classes of knots and links.

We should focus on the following two parities:

1) For free knots described by Gauss diagram, we may take a crossing to be \textit{even} if the corresponding chord is linked with an even number of chords or \textit{odd}, otherwise.

2) For free 2-components links, a crossing formed by one component is called \textit{even}, and a crossing formed by two different components is called \textit{odd}.
Figure 5: The third Reidemeister move
The parity axioms in these cases are checked straightforwardly.

1.2 The bracket \( \{ \cdot \} \)

Let \( i > 2 \) be a natural number. Define the set \( \mathbb{Z}_G_i \) to be the \( \mathbb{Z}_2 \)-linear space generated by set of \( i \)-component framed four-valent graphs modulo the following relations:

1) the second Reidemeister moves
2) \( L \cup O = 0 \), i.e., every \( n \)-component link with one split trivial component is equivalent to 0.

For \( i = 1 \), we define \( \mathbb{Z}_G_1 \) analogously with respect to equivalence 1) and not 2).

It can be easily shown that the elements from any \( \mathbb{Z}_G_i \) can be compared algorithmically, namely, each element has a unique minimal representative which can be obtained by applying consecutively decreasing second Reidemeister moves.

Let \( \mathcal{K} \) be the \( \mathbb{Z}_2 \)-linear space generated by free links, and \( \mathcal{K}_i \) be the \( \mathbb{Z}_2 \)-linear space generated by \( i \)-component free links.

We shall construct a map \( \{ \cdot \} : \mathcal{K} \mapsto \{ \mathcal{K} \} \) valued in \( \mathbb{Z}_G \) as follows.

Take a framed four-valent graph \( G \) representing \( \mathcal{K} \). By definition, it has two components. Now, a vertex of \( G \) is called odd if it is formed by two different components, and even otherwise.

The parity axioms can be checked straightforwardly.

Now, we define

\[
\{ G \} = \sum_s G_s,
\]

where we take the sum over all smoothings of all even vertices, and consider the smoothed diagrams \( K_s \) as elements of \( \mathbb{Z}_G \). In particular, we take all elements of \( K_s \) with free loops to be zero.

**Theorem 1.** \( \Box \) The bracket \( \{ K \} \) is an invariant of two-component free links, that is, for two graphs \( G \) and \( G' \) representing the same two-component free link \( K \) we have \( \{ G \} = \{ G' \} \) in \( \mathbb{Z}_G \).

Analogously one defines \( [K] \) as the sum of all one-component summands (from \( \mathbb{Z}_G_1 \)); so, the map \( K \mapsto K \) factors through \( \{ K \} \).

1.3 The map \( \Delta \)

We shall construct a map from \( \mathcal{K}_c \) to \( \mathbb{Z}_G_2 \), \( \Box \), as follows.

In fact, to define the map \( \Delta \), one may require for a free knot to be oriented. However, we can do without.

Given a framed 4-graph \( G \). We shall construct an element \( \Delta(G) \) from \( \mathbb{Z}_G_2 \) as follows. For each crossing \( c \) of \( G \), there are two ways of smoothing it. One way gives a knot, and the other smoothing gives a 2-component link \( G_c \). We take the one giving a 2-component link and write
\[ \Delta(G) = \sum_c G_c \in \mathbb{Z}G_2. \]  \hspace{1cm} (2)

**Theorem 2.** \( \Delta(G) \) is a well defined mapping from \( \mathcal{K}_\infty \) to \( \mathcal{K}_c \).

Analogously, one can consider the map \( \Delta_{\text{odd}} \) where the sum is taken over all odd crossings or \( \Delta_{\text{even}} \) where the sum is taken over all even crossings. These maps are both invariant.

## 2 Invertibility of Long Free Knots

Let us consider the long free knots. They are defined just as free knots, but instead of four-valent framed graphs we consider long four-valent framed graphs. They can be treated as four-valent graphs with two infinite edges.

In this section, we consider oriented long free knots. We define the crossing parity for long free knots just as in the case of the corresponding compact knots: by using parity of the chords of the corresponding Gauss diagrams. We are going to prove the following

**Theorem 3.** Let \( K \) be a framed long four-valent graph with one unicursal component such that:

1) All crossings of \( K \) are odd;
2) There is no room to apply the second decreasing Reidemeister move to \( K \).
3) \( K \) is not isomorphic to itself with the orientation reversed.

Then the long free knot represented by \( K \) is not invertible.

The idea is to modify the bracket for \( [K] \) to make it orientable.

Namely, let us define the bracket \( [G]_{\text{or}} \) for orientable framed four-valent graphs with one unicursal component as follows. We define \( ZG_1^{\text{or}} \) to be the \( \mathbb{Z}_2 \)-linear space of all oriented long four-valent framed graphs with one component modulo the second Reidemeister move.

We take a graph \( G \) and take all smoothings of \( G \) at even crossings; each smoothing of such sort is a long framed four-valent graph; we can naturally endow it with an orientation. Indeed, for every smoothing \( G_s \) has two infinite arcs which coincide with the non-compact arcs of \( G \): the initial one \( a \) and the final one \( b \). Since \( G_s \) is a one-component free knot, we may choose these arcs \( a \) and \( b \) to be the initial and the final arc, respectively.

Now, analogously to Theorem 1 one proves the following

**Theorem 4.** The bracket \( [K]_{\text{or}} \) is an invariant of two-component free links, that is, for two graphs \( G \) and \( G' \) representing the same two-component free link \( K \) we have \( [G]_{\text{or}} = [G']_{\text{or}} \) in \( ZG_1^{\text{or}} \).

Now, theorem 4 naturally yields theorem 6. Indeed, if \( K \) is a oriented long four-valent free graph then the equality \( [K]_{\text{or}} = K \) respects the orientation. Since \( K \) is the minimal representative in its class in \( ZG_1^{\text{or}} \), any other representative \( K' \) of the same long free knot has more crossings. On the other hand,
the same is true about $\tilde{K}$. Since $K$ and $\tilde{K}$ do not coincide as oriented graphs, the corresponding knots are different.

Obviously, there are infinitely many examples satisfying theorem [9]. One example is shown in Fig. 6.

3 Detecting non-invertibility of compact links

As we have seen, the argument of endowing the terms of $\Delta$ (or $\{\cdot\}$) with an orientation works well in the case of long knots, i.e., in the case when we have a reference point. For the case of compact links (or knots), this is not that easy. Before defining the "oriented version" of the bracket we first collect the invertibility "pro" and "contra" arguments in the compact case.

3.1 The invertibility arguments

Let $L$ be an oriented free link, and let $\tilde{L}$ be the free link obtained from $L$ by reversing the orientation of all components of $L$.

Our goal is to construct such free links $L$ for which $\tilde{L} \neq L$.

Here we collect some observations concerning free knots and links.

1. The map $\Delta$ can be treated as a map from oriented free knots to oriented two-component free links. Moreover, $\Delta(K) = \Delta(\tilde{K})$. So, having found an example of a non-invertible free multicomponent link $L$, we may plug in $\Delta$ in order to get a multicomponent free knot $K$ (trying to get $\Delta(K) = L + \ldots$, where the other summands of $\Delta(K)$ are immaterial and $L$ yields non-invertibility of $K$).

2. For multicomponent links, one may suggest the following order argument. For example, consider a 4-component link $L_1 \cup L_2 \cup L_3 \cup L_4$, where the first component $L_1$ has exactly one intersection point with any other component $L_i, i = 2, 3, 4$, whence any two other components are pairwise disjoint.
We may look at the order of intersection points on the first component $L_1$ according to its orientation: it can be either 2, 3, 4 or 2, 4, 3. Certainly, for the concrete representative (ordered oriented four-valent framed graph) does not coincide with its inverse: if $L$ has the order 2, 3, 4 then the inverse link has the order 2, 4, 3.

Nevertheless, these two links are equivalent: the sequence of Reidemeister moves between these two links is shown in Fig. 7.

We first apply the second Reidemeister move to create two intersection points between the components $L_2$ and $L_3$. Then we perform a third Reidemeister move for the components 1, 2, 3. Finally, we remove the two intersection points between components $L_2$ and $L_3$ by a second Reidemeister move.

Finally, we end up with the link where the ordering of the three intersection points along the orientation of $L_1$ is switched: instead of 2, 3, 4 we get 3, 2, 4, which is the just the same as that for $\bar{L}$.

3. The invariants $\cdot$ and $\{\cdot\}$ which sometimes allow one to reduce the in-
formation about a free knot to some information about its representative graph in the case when \([K] = K\) can not directly be used for the case of orientable free knots (free links). Indeed, the bracket \([K]\) (or, in the case of links, its variant \(\{L\}\)) is defined as a linear combination of non-oriented four-valent framed graphs. Indeed, when we try applying third Reidemeister moves and collecting terms, we will necessarily get to a situation when a smoothing at an even crossing breaks the orientation, and we get two odd crossings where the orientations disagree.\[\text{[2]}\]

So, the right-hand side in the equality \(\{L\} = L\) should be treated as a non-orientable graph (or, at most, as a partially oriented graph) modulo second Reidemeister moves. Our goal will be to use this partial orientability in order to get a genuine orientability.

The main idea of this section is as follows. First, for some category of two-component links, we modify the bracket \(\{L\}\) in order to make it valued in linear combinations of oriented framed graphs modulo second Reidemeister moves compatible with orientation. This category will include only those two-component links with orientable atoms. This will lead to some link \(L\) where \(\bar{L} \neq L\).

Then, by using \(\Delta\), we shall extend this result to some oriented free knots \(K\) where \(\Delta(K) = L + (\ldots) \Delta(\bar{K}) = \bar{L} + (\ldots)\), where the summands \((\ldots)\) will mean some collection of free links which do not affect the non-orientability of \(K\) coming from that of \(L\).

Finally, we shall extend this result for knots and links with orientable atoms.
3.2 Making the Bracket \{·\} Orientable

Let \( \mathcal{L} \) be the category of free two-component links \( L_1 \cup L_2 \) such that the number of crossing points formed by both \( L_1 \) and \( L_2 \) is odd.

Obviously, this property is preserved by Reidemeister moves, so, the category is well-defined.

Now, let \( \mathcal{L}^o \) be the category of links from \( \mathcal{L} \) where the components are ordered: \( L_1 \cup L_2 \) and the component \( L_1 \) is endowed with an orientation.

The main theorem we are going to prove is the following

**Theorem 5.** Let \( L \) be a four-valent framed graph having two unicursal components, one of which is oriented. Assume that:

1. All crossings of \( L \) belong to two different components \( L_1 \) and \( L_2 \)
2. The diagram is irreducible, i.e., no decreasing second Reidemeister move can be applied to it.
3. The diagram is not invertible, i.e., it is not isomorphic to itself with the orientation of \( L_1 \) reversed.
4. There is no isomorphism of \( L_1 \cup L_2 \) onto itself (as framed 4-graphs) which disregards the orientation and interchanges \( L_1 \) and \( L_2 \).

Then the link \( L \) is not invertible.

Let \( \mathbb{Z}_2 \mathcal{L}^o \) be the \( \mathbb{Z}_2 \)-linear space spanned by all links from \( \mathcal{L}^o \). Let \( \mathcal{L} \) be the quotient linear \( \mathbb{Z}_2 \)-space of the space spanned by four-valent framed two-component graphs with one component oriented by the second Reidemeister move.

We want to construct the bracket map \( \{·\} : \mathbb{Z}_2 \mathcal{L}^o \to \mathcal{L} \) To do that, we introduce the parity 1.2.1 for links from \( \mathcal{L}^o \): a crossing for a two-component link \( L_1 \cup L_2 \) is even if it is formed by one component \( L_1 \) or \( L_2 \), and it is odd if it is formed by the two components \( L_1 \cup L_2 \).

Then for a link \( L \in \mathbb{Z}_2 \mathcal{L}^o \) we take its bracket \( \{·\} \) as described above and modify it as follows. First, the bracket \( \{·\}_2 \) will contain only two-component summands. Note that every summand of \( \{·\} \) has at least two components: some components correspond to the former \( L_1 \), and the others correspond to the former \( L_2 \). So, we are interested in the case when the summand has exactly two components, that is, smoothings at vertices of \( L_1 \) lead to one component and smoothings at vertices of \( L_2 \) lead to the other component.

Note that if we just take \( \{·\}_2 \) to be the sum of these two-component summands regardless any orientation, it becomes an invariant of free links, because this map just factors through the usual \( \{·\} \) map.

Now, we would like to endow the summands of \( \{·\}_2 \) with an orientation of the component \( L_1 \) (by abusing notation we denote by \( L_1 \) the component consisting of edges belonging to \( L_1 \)).

Let \( s \) be a smoothing, and let \( L_1^s \) be the result of applying this smoothing to \( L_1 \) (we agreed that it gives one component). The number of crossings between
the new $L_1$ and $L_2$ is the same as that between the old $L_1$ and $L_2$, because we do not smooth odd vertices which form crossings between $L_1$ and $L_2$.

So, we have some $2n + 1$ crossings on the new $L_1$ with pieces of orientation of the original component $L_1$ on it. These orientations may disagree since the way of smoothing of the original link $L$ does not agree with the orientation of $L_1$, in general.

In total, we have $2n + 1$ ways of orienting the link $L_1$; assume some $2l + 1$ of them give one orientation $o$ and the remaining $2(n - l)$ ones give the opposite orientation $\bar{o}$ of $L_1$.

Now, we choose the orientation $o$ for $L_1$ in the given summand. So, we endowed the terms of $\{L\}_2$ with an orientation of one component $L_1$. From now on we consider $\{L\}_2$ as a sum of two-component framed graphs with one component oriented.

**Theorem 6.** The bracket $\{\cdot\}_2$ with one-component orientation described above, is an invariant of two-component links from $\mathcal{L}^o$.

**Proof.** One should just repeat the invariance proof for the bracket $\{\cdot\}$ in its usual non-oriented version and see that the orientation of the $L_1$ components for all pairs of cancelling terms agree.

For the $\Omega_1$ move, there is nothing to prove since the only crossing in question gets smoothed and does not affect the orientation.

The same happens for the $\Omega_2$ move with two crossings on the same component and for the $\Omega_3$ move applied to three crossings lying on the same component.

Now, for a move $\Omega_2$ which is applied to two crossings lying in $L_1$ and $L_2$, these two crossings contribute into the orientation of $L_1$; namely, if we had some $2n - 1$ crossings formed by $L_1$ and $L_2$ before the move, we get $2n + 1$ crossings after the move. But the two orientation coming from initial component $L_1$ coming from these two crossings agrees for the smoothed $L_1^*$, so the rule for choosing the orientation for $L_1^*$ remains the same.

Finally, when we apply the third Reidemeister move referring to two components, we have to check several cases. If this move applies to two pieces of $L_2$ and one piece of $L_1$ then the two crossings between $L_2$ and $L_1$ contribute the same orientation to $L_1$ (in the LHS as well as in the RHS, because they are consequent crossings on the same arc), so the choice of the orientation remains the same in both sides of the equation.

So, we are left with the case when $L_1$ occurs twice and $L_2$ occurs once, and the only “even” point in our Reidemeister move belongs to $L_1$.

The two variants of this move are drawn in Fig. 9.

In the top picture we see that the summands from the first pair in the LHS and RHS contain two crossings with opposite orientations each, so the total number of crossings contributing to each orientation is the same for these two summands.

The second in the second pair are identical.

In the bottom picture, the first summand in the LHS has two crossings contributing the same to the orientation of $L_1$; so, their impact cancels, as well
Figure 9: The behaviour of orientations under the third Reidemeister move

as that for the first summand of the RHS. For the second summand in the LHS, we have two crossings with opposite orientations, and the same in the RHS. So, the orientations of the corresponding summands in the LHS and in the RHS are the same.

Now, we are ready to prove theorem 5. Indeed, if a link $L = L_1 \cup L_2$ satisfies the conditions of Theorem 5 then $\{L\}_2 = L$, and if $\bar{L}$ denotes the two-component link with the orientation of $L_1$ reversed then $\{\bar{L}\}_2 = \{\bar{L}\}_2 = \bar{L}$, and since $L$ and $\bar{L}$ are not isomorphic as four-valent framed oriented graphs, they are not equivalent as framed two component links with one component oriented.

As an example of a link $L$ satisfying the conditions of Theorem 5 we may consider the link shown in Fig. 10.

To see that it is indeed non-invertible, let us enumerate the points on $L_1$ along the orientation of $L_1$: $A_1, \ldots, A_{11}$ and count whether the distances with respect to $A_1$ between adjacent crossings of $A_2$. Thus we get a sequence of numbers defined modulo 11 and up to sign: denote the distance between $A_i$ and $A_{i+1}$ along the component $L_2$ by $\beta_i$, $i$ is taken modulo 11. The numbers $\beta_i$ are defined up to sign because the component $L_2$ is not oriented. The (cyclic) sequence is $3, 3, 3, 4, 6, 7, 6, 2, 6, 9, 6$ for one orientation of $L_2$. This sequence has only one fragment of three consecutive equal numbers: $3, 3, 3$. If we take the other orientation of $L_2$, we shall get $8, 8, 8, 5, 2, 5, 9, 5, 4, 5, 7$ (with three consecutive 8’s). None of these two sequences coincides with the cyclic sequences obtained by inverting the orientation of $L_1$: they will have fragments $6, 3, 3, 3, 4$ and $7, 8, 8, 5$.

Finally, if we change the roles of $L_1$ and $L_2$ we shall get four other cyclic sequences, e.g., $8, 6, 8, 5, 4, 3, 7, 7, 8, 4, 4$ (and similar) which have no three con-
sequent equal numbers.

So, $L$ (with an orientation of $L_1$ fixed) is an example of a two-component free link with unordered components such that $L$ is not equivalent to $L$.

4 A Non-Invertible Free Knot

Consider the Gauss diagram $K$ shown in Fig. 11.

**Statement 1.** The free knot $K$ represented by the diagram shown in Fig. 11 is not equivalent to its inverse.

**Proof.** Consider the knot $\bar{K}$ obtained from $K$ by reversing the orientation.

By construction, we have $\Delta(\bar{K}) = (\Delta(K))$.

Thus, if we show that $\Delta(K)$ is not invertible as a 2-component free link then we see that $K$ is not invertible either.

Let us extend the map $\{\cdot\}_2$ to all two-component free links. This map is already defined for those links where two components have an odd intersection. We extend it just by 0 to the remaining two-component links.

Now, $\Delta(K)$ is a $\mathbb{Z}_2$-linear combination of 2-component free links. Moreover, the chord diagram $C(K)$ has exactly one chord which is linked with all the other chords. The result of smoothing along this chord leads to the link $L$ from the previous example.

Note, that for this particular $K$ we have $\Delta(K) = L + \sum_i L_i$ where $L$ is the link $L$ from the previous example and all links $L_i$ have at least one crossing belonging to one component.
Now, if we take \( \{ \Delta(K) \}_2 \), we get exactly one summand \( (L) \) which is represented by a diagram with 11 crossings and can not be represented by a diagram with a fewer number of crossings, and all the diagrams \( L_i \) have strictly less than 11 crossings. Analogously, \( \{ \Delta(\bar{K}) \}_2 = \bar{L} + \sum \bar{L}_i \).

Now, since \( L \) is not equivalent to \( \bar{L} \) as elements from \( \mathcal{L} \) and neither of \( L \) or \( \bar{L} \) is equivalent to none of \( L_i \) or \( \bar{L}_i \), we see that \( \{ \Delta(\bar{K}) \}_2 \neq \{ \Delta(K) \}_2 \), so \( K \) is not equivalent to \( \bar{K} \).

\[ \square \]

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