Abstract

Consider the following variant of the set cover problem. We are given a universe $U = \{1, \ldots, n\}$ and a collection of subsets $\mathcal{C} = \{S_1, \ldots, S_m\}$ where $S_i \subseteq U$. For every element $u \in U$ we need to find a set $\phi(u) \in \mathcal{C}$ such that $u \in \phi(u)$. Once we construct and fix the mapping $\phi : U \rightarrow \mathcal{C}$ a subset $X \subseteq U$ of the universe is revealed, and we need to cover all elements from $X$ with exactly $\phi(X) := \bigcup_{u \in X} \phi(u)$. The goal is to find a mapping such that the cover $\phi(X)$ is as cheap as possible.

This is an example of a universal problem where the solution has to be created before the actual instance to deal with is revealed. Such problems appear naturally in some settings when we need to optimize under uncertainty and it may be actually too expensive to begin finding a good solution once the input starts being revealed. A rich body of work was devoted to investigate such problems under the regime of worst case analysis, i.e., when we measure how good the solution is by looking at the worst-case ratio: universal solution for a given instance vs optimum solution for the same instance.

As the universal solution is significantly more constrained, it is typical that such a worst-case ratio is actually quite big. One way to give a viewpoint on the problem that would be less vulnerable to such extreme worst-cases is to assume that the instance, for which we will have to create a solution, will be drawn randomly from some probability distribution. In this case one wants to minimize the expected value of the ratio: universal solution vs optimum solution. Here the bounds obtained are indeed smaller than when we compare to the worst-case ratio.

But even in this case we still compare apples to oranges as no universal solution is able to construct the optimum solution for every possible instance. What if we would compare our approximate universal solution against an optimal universal solution that obeys the same rules as we do? We show that under this viewpoint, but still in the stochastic variant, we can indeed obtain better bounds than in the expected ratio model. For example, for the set cover problem we obtain $H_n$ approximation which matches the approximation ratio from the classic deterministic offline setup. Moreover, we show this for all possible probability distributions over $U$ that have a polynomially large carrier, while all previous results pertained to a model in which elements were sampled independently. Our result is based on rounding a proper configuration IP that captures the optimal universal solution, and using tools from submodular optimization.

The same basic approach leads to improved approximation algorithms for other related problems, including Vertex Cover, Edge Cover, Directed Steiner Tree, Multicut, and Facility Location.

1 Introduction

In a typical online problem part of the input is revealed gradually to an algorithm, which has to react to each new piece of the input by making irrevocable choices. In an online covering problem the online
input consists of a sequence of requests, which have to be satisfied by the algorithm by buying items at minimum total cost.

Some online applications have severe resource constraints, typically in terms of time and/or computational power. Hence even making an online (non-trivial) choice might be too costly. In these settings it makes sense to consider universal algorithms. Roughly speaking, the goal of these algorithms is to pre-compute a reaction to each possible input, so that the online choice can then be made very quickly (say, looking at some pre-computed table). Since the adversary has a lot of power in the universal setting, typically one assumes a stochastic input. In particular, the input is sampled according to some probability distribution \( \pi \), which is either given in input or that can be sampled multiple times at polynomial cost per sample (oracle model).

The most relevant prior work for this paper is arguably due to Grandoni et al. [23] (conference version in [22]). The authors consider the universal stochastic version of some classical NP-hard covering problems such as set cover, non-metric facility location, multicut etc. They provide polynomial-time approximation algorithms for those problems in the independent activation model, where each request \( u \) is independently sampled with some known probability \( p_u \). Crucially, in their work the approximation ratio is obtained by comparing the expected cost of the approximate solution with the expected cost of the optimal offline solution (that knows the future sampled input). For example, in the set cover case they present a polynomial-time algorithm that computes a mapping of expected cost at most \( O(\log(nm))\mathbb{E}[OPT_{off}(X)] \), where the expectation is taken over the sampling of \( X \) according to \( \pi \) and \( OPT_{off}(X) \) is the minimum (offline) cost of a set cover of \( X \). Here \( n \) is size of the universe and \( m \) the number of subsets. For \( m \gg n \) this ratio becomes \( O\left(\frac{\log m}{\log \log m}\right) \) and is tight. They also consider the universal (non-metric) facility location problem in the independent activation model, and provide a \( O(\log n) \) approximation (in the above sense), where \( n \) is the total number of clients and facilities. We remark that their method seems not to lead to any improved approximation factor in the metric version of the problem. We finally mention their \( O(\log^2 n) \) approximation for universal multicut in the independent activation model, where \( n \) is the number of nodes in the graph.

### 1.1 Our Results and Techniques

Comparing with the offline optimum as in [23] might be too pessimistic. And often when we need to optimize under uncertainty we cannot really find a better benchmark, flagship example of it would be online problems. However, stochastic two-stage \([31, 43]\) and stochastic adaptive \([39, 13, 12, 25]\) problems have proven that one can actually compare an approximate solution with an optimum algorithm that is not omnipotent but obeys the same rules of the model as the approximate one. This inspired us to ask the following question:

**Is it also possible in a stochastic universal problem to compare our algorithm with an optimum solution that is restricted by the model in the same way as we are?**

In this paper we show that we can do this indeed. In this way we manage to obtain tighter approximation ratios — which of course are compared to a weaker benchmark, but this benchmark itself can be interpreted as more fair and meaningful — and it also allows us to approach more general problems.

#### 1.1.1 Universal Stochastic Set Cover

We shall describe carefully the Universal Stochastic Set Cover problem in this section so that we will fully present the model. For the remaining problems their full statements will appear in appropriate sections.

In the Universal Stochastic Set Cover problem we are given in input a universe \( U = \{1, 2, ..., n\} \), and a collection \( C \subseteq 2^U \) of \( m \) subsets \( S \subseteq U \), each one with an associated cost \( c(S) \). We need to a priori map each element \( u \in U \) into some set \( \phi(u) \in C \). Then a subset \( X \subseteq U \) is sampled according to some probability distribution \( \pi \) (whose features are discussed later), and we have to buy the sets \( \phi(X) = \bigcup_{u \in X} \phi(u) \) as the cover of \( X \). Our goal is to minimize the expected value of the total cost, i.e., \( \mathbb{E}_{X \sim \pi}[c(\phi(X))] = \mathbb{E}_{X \sim \pi} \left[ \sum_{S \in \phi(X)} c(S) \right] \). One of the most important aspects in our model is that we do not compare ourselves against the expected value of an optimum offline solution for a given scenario,
that is, not against \( E_{X \sim \pi} [OPT_{off} (X)] \). What we compare ourselves with is

\[
\min_{\phi : \forall u \in U \ u \in \phi(u)} E_{X \sim \pi} \left[ \sum_{S \in \phi(X)} c(S) \right],
\]

i.e., the expected cost of an optimal universal mapping.

The results depend on the properties of the sampling process. Here we will focus on the most common models, which are defined as follows:

- **Scenario model**: Here we are given in input all the sets \( X_1, ..., X_N \subseteq U \) such that \( P_{X \sim \pi} [X = X_i] > 0 \) with the associated probability. This model allows for explicit use of all the scenarios in the computations. For the **Universal Stochastic Set Cover** we obtain \( O(\log n) \)-approximation in this case even in the weighted case.

- **Oracle model**: This is the most general model. We have a black-box access to an oracle \( \Pi \) from which we can sample a scenario from distribution \( \pi \). We assume that taking a sample requires polynomial time. In this model we can find an \( O(\log n) \)-approximation for **Universal Stochastic Set Cover** in polynomial time only for the unweighted case; in the weighted case we achieve the same approximation factor in pseudo-polynomial time depending on the largest cost \( \max_{S \in C} c(S) \). We can also show that the same cannot be achieved in polynomial time.

- **Independent activation model**: In this model we assume that every element \( u \in U \) is independently sampled with some given probability \( p_u \). This model does not capture correlations of elements, and therefore sometimes it is not fully realistic. Though it cannot be represented by a polynomial number of scenarios, its nice properties allow one to develop good approximation algorithms for several problems. In this setting we are able to approximate **Universal Stochastic Set Cover** within a factor \( O(\log n) \) in polynomial time even in the weighted case.

Our results are obtained by defining a proper configuration LP (with an exponential number of variables) that captures the optimal mapping. We are able to solve this LP via the ellipsoid method using a separation oracle. Somehow interestingly, our separation oracle has to solve a submodular minimization problem. Then we can round the fractional solution in a standard way.

### 1.1.2 Overview of the results

The robustness of our framework allows us to address universal extensions of several covering problems. After expressing the goal as a true approximation task, we can adapt tools from the rich theory of approximation algorithms.

Here we give an overview of our results. Detailed statements of the theorems appear in appropriate sections.

**Scenario model** In this setting, we are able to construct an LP-based polynomial-time \( O(\log n) \)-approximation to the universal stochastic version of **Set Cover** (Theorem 6), which generalizes to **Non-Metric Facility Location** and **Constrained Set Multicover**. In fact, the latter algorithm achieves an approximation guarantee of exactly \( H_n \). Different rounding procedure leads to a 2-approximation for **Universal Stochastic Vertex Cover**. All these approximation ratios match the best guarantees obtained in the deterministic world. What is more, the exact polynomial time algorithm for **Edge Cover** extends to the scenario model.

**Independent activation model** In this setting, we are able to obtain several results in favour of the \( O(1) \)-approximation for the **Maybecast** problem by Karger and Minkoff [35]. We present a 6.33-approximation for **Universal Stochastic Metric Facility Location** (Theorem 11) and an \( O(\log n) \)-approximation for **Universal Stochastic Multicut** (Theorem 17). As an intermediate result, we obtain a 4.75-approximation for **Universal Stochastic Multicut** on trees.

**Oracle model** We can generalize most of our results for the scenario model to this setting, with the restriction that in the weighted case we get a pseudo-polynomial running time. This is discussed in Section 3.2.
1.2 Related work

Other universal-like problems have been addressed in the literature. For instance, in the universal TSP problem one computes a permutation of the nodes that is then used to visit a given subset of nodes. This problem has been studied both in the worst-case scenario for the Euclidean plane \[10, 6\] and general metrics \([34, 24, 23]\), as well as in the average-case \([33, 11, 20, 46]\). (For the related problem of universal Steiner tree, see \([35, 33, 24, 20]\).) Jia et al. \([34]\) introduced the universal set cover and universal facility location problems, and studied them in the worst-case: they show that the adversary is very powerful in such models, and give nearly-matching \(\Omega(\sqrt{n})\) and \(\Omega(\sqrt{\log n})\) bounds on the competitive factor. These problems have been later studied by Grandoni et al. in the independent activation model \([25]\), as already mentioned before.

A somewhat related topic is oblivious routing \([11, 50, 8]\) (see, e.g., \([45, 41]\) for special cases). A tight logarithmic competitive result as well as a polynomial-time algorithm to compute the best routing is known in the worst case for undirected graphs \([15, 12]\). For oblivious routing on directed graphs in the worst case the lower bound of \(\Omega(\sqrt{n})\) \([5]\) nearly matches upper bounds in \([29]\) but for the average case. The authors of \([20]\) give an \(O(\log n)\)-competitive oblivious routing algorithm when demands are chosen randomly from a known demand-distribution; they also use “demand-dependent” routings and show that these are necessary.

Another closely related notion is the one of online problems. These problems have a long history (see, e.g., \([9, 17]\), and there have been many attempts to relax the strict worst-case notion of competitive analysis: see, e.g., \([14, 11, 20]\) and the references therein. Online problems with stochastic inputs (either i.i.d. draws from some distribution, or inputs arriving in random order) have been studied, e.g., in the context of optimization problems \([37, 38, 20, 4]\), secretary problems \([19]\), mechanism design \([27]\), and matching problems in Ad-auctions \([30]\).

Alon et al. \([2]\) gave the first online algorithm for set cover with a competitive ratio of \(O(\log m \log n)\): they used an elegant primal-dual-style approach that has subsequently found many applications (e.g., \([3, 10]\)). This ratio is the best possible under complexity-theoretic assumptions \([15]\); even unconditionally, no deterministic online algorithm can do much better than this \([2]\). Online versions of metric facility location are studied in both the worst case \([37, 15]\), the average case \([20]\), as well as in the stronger random permutation model \([22]\), where the adversary chooses a set of clients unknown to the algorithm, and the clients are presented to us in a random order. It is easy to show that for our problems, the random permutation model (and hence any model where elements are drawn from an unknown distribution) are as hard as the worst case.

One can of course consider the (offline) stochastic version of optimization problems. For example, \(k\)-stage stochastic set cover is studied in \([31, 15]\), with an improved approximation factor (independent from \(k\)) later given in \([47]\).

The result for the Oracle model are based on the Sample Average Approximation approach, see \([11]\) for the application most relevant to our work.

As mentioned before, in two-stage stochastic problems \([33, 31]\) and stochastic adaptive problems \([13, 39, 12, 25]\) it is possible to compare a given algorithm with an optimum algorithm which is similarly constrained, and this is what shed a light on the possibility of obtaining results in the same spirit for the universal stochastic optimization.

In two recent papers \([19, 16]\), the authors looked at universal optimization over scenarios, but compared against the average offline optimum, and not the optimum universal solution as we do. All the properties of submodular functions used in our work can be found here \([52]\).

2 Preliminaries

Here we give some basic definitions and properties. Given a universe \(U\), we call a function \(f : 2^U \to \mathbb{R}\) submodular if \(f(A) + f(B) \geq f(A \cap B) + f(A \cup B)\) for each pair of sets \(A, B \subseteq U\). Function \(f\) is monotone if \(f(B) \geq f(A)\) for \(A \subseteq B\). When considering a submodular function, we assume that it is implicitly given in the form of an oracle that can be queried on a specific \(A \subseteq U\) and returns the value \(f(A)\) in constant time.

**Theorem 1** (Iwata et al. \([32]\)). There is an algorithm to minimize a given submodular function \(f : 2^U \to \mathbb{N}\) in polynomial time in \(|U|\) and in the number of bits needed to encode the largest value of \(f\).
Let us introduce a function $g_\pi : 2^U \to \mathbb{R}^+$, $g_\pi(A) = \mathbb{P}[A \cap X \neq \emptyset]$ where $X$ is drawn from the distribution $\pi$. Our framework exploits crucially the fact that $g_\pi$ is a submodular function.

**Lemma 2.** Function $g_\pi$ is submodular and monotone.

**Proof.** Observe that $g_\pi(A) = \sum_{X \subseteq U} \pi(X) \cdot 1[A \cap X \neq \emptyset]$. Function $A \to \mathbb{1}A \cap X \neq \emptyset$ is submodular and a combination of such functions with positive coefficients is submodular. The monotonicity holds trivially by definition. \hfill $\square$

## 3 Universal Set Cover

In this section we present our approximation algorithm for Universal Stochastic Set Cover. We start by presenting an $O(\log n)$ approximation in the scenario and independent activation model (Section 3.1). Then we achieve the same approximation factor in the oracle case (Section 3.2) though in pseudo-polynomial time (polynomial time for the cardinality case). In Section 3.3 we argue that pseudo-polynomial time is indeed needed in order to get a sub-polynomial approximation factor (for the weighted case). In the appendix we present approximation algorithms for some special cases (Section A) and some generalizations (Sections B, C) of Universal Stochastic Set Cover.

### 3.1 The Scenario and Independent Activation Model

We let $n = |U|$ be the number of elements in the universe. Recall that, for $B \subseteq U$, $g_\pi(B) = \mathbb{P}_{X \sim \pi}[B \cap X \neq \emptyset]$. As mentioned before, before, $g_\pi$ is a submodular function over the universe $U$. For our goals it is sufficient that $g_\pi$ can be evaluated in polynomial time. This clearly holds both in the scenario model and in the independent activation model.

We start by expressing our problem as the following integer program.

\[
\begin{align*}
\min \quad & \sum_{S \in \mathcal{C}} c(S) \sum_{B \subseteq S} y_B^S \cdot g_\pi(B) \\
\text{s.t.} \quad & \sum_{B \ni u \ni S} y_B^S \geq 1 \quad \forall u \in U \\
& y_B^S \in \{0, 1\} \quad \forall S \in \mathcal{C} \quad \forall B \subseteq S.
\end{align*}
\]

(1)

Intuitively, $y_B^S = 1$ means that exactly the elements $B$ of $S$ are mapped into $S$, i.e. $B = \{u \in U : \phi(u) = S\}$.

**Lemma 3.** Integer program (CONF-IP-SC) is equivalent to Universal Stochastic Set Cover.

**Proof.** It is easy to translate a mapping $\phi : U \to \mathcal{C}$ into some feasible solution to (CONF-LP-SC). All variables are zeros by default and for each $S \in \mathcal{C}$ such that $\phi^{-1}(S)$ is non-empty, we set $y_{\phi^{-1}(S)}^S = 1$. Note that always $\phi^{-1}(S) \subseteq S$. In that setting the objective value equals the expected cost of the covering.

Let us fix some feasible solution $\{y_B^S\}_{S \in \mathcal{C}, B \subseteq S}$. We know that for each $u \in U$ there is some pair $(B, S)$ so that $u \in B$ and $y_B^S = 1$ (we will call it a covering pair). As long as there are many covering pairs for some $u$, we replace one of them with $(B \setminus \{u\}, S)$. The new solution is still feasible and the objective value is no greater as function $g_\pi$ is monotone. Therefore there exists an optimal solution so that each $u \in U$ admits exactly one covering pair $(B_u, S_u)$. We can define $\phi(u) = S_u$ to obtain a covering with expected cost equal to the value of the objective function. \hfill $\square$

We obtain a linear relaxation (CONF-LP-SC) of (CONF-IP-SC) by replacing the integrality constraints with $y_B^S \geq 0$. (CONF-LP-SC) has an exponential number of variables: in order to solve it we consider its dual, and provide a separation oracle to solve it. Interestingly, our separation oracle uses submodular minimization.

**Lemma 4.** (CONF-LP-SC) can be solved in polynomial time when $g_\pi$ can be evaluated in polynomial time.

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Proof. We show how to solve the dual of (CONF-LP-SC), which is as follows:

\[
\begin{align*}
\text{max} & \quad \sum_{u \in U} \alpha_u & \quad \text{(DP-SC)} \\
\text{s.t.} & \quad \sum_{u \in B} \alpha_u \leq c(S) \cdot g_\pi(B) & \quad \forall S \in \mathcal{C}, \forall B \subseteq S \\
& \quad \alpha_u \geq 0 & \quad \forall u \in U.
\end{align*}
\]

Observe that (DP-SC) has a polynomial number of variables and an exponential number of constraints. In order to solve (DP-SC), it is sufficient to provide a (polynomial-time) separation oracle, i.e. a procedure that, given a tentative solution \( \{\alpha_u\}_{u \in U} \), either determines that it is feasible or provides a violated constraint.

This reduces to check, for each given \( S \in \mathcal{C} \), whether there exists \( B \subseteq S \) such that \( \sum_{u \in B} \alpha_u > c(S) \cdot g_\pi(B) \). In other terms, we wish to determine whether the minimum of function \( h_\pi(B) := c(S) \cdot g_\pi(B) - \sum_{u \in B} \alpha_u \) is negative. Observe that the value of \( h_\pi(B) \) can be computed in polynomial time for a given \( B \). Note also that \( h_\pi \) is submodular: indeed \( g_\pi \) is submodular, costs \( c(S) \) are non-negative by assumption, and \( -\sum_{u \in B} \alpha_u \) is linear (hence submodular). Hence we can minimize \( h_\pi \) over \( B \subseteq S \) in polynomial time via Theorem 1.

Given the optimal solution to (CONF-LP-SC), it is sufficient to round it with the usual randomized rounding algorithm for set cover.

Lemma 5. The optimal solution to (CONF-LP-SC) can be rounded to an integer feasible solution while increasing the cost by a factor \( O(\log n) \) in expected polynomial time.

Proof. In the optimal solution, for each \( S \in \mathcal{C} \), variables \( \{y_B^S\}_{B \subseteq S} \) define a probability distribution. We sample from this distribution (independently for each \( S \)) for \( q = 2 \ln n \) many times. Let \( B_1^S, \ldots, B_q^S \) be the sets sampled for \( S \); we let \( B^S := \cup_i B_i^S \) and tentatively map elements of \( B^S \) into \( S \). In case the same element \( u \) belongs to \( B_i^S \) and \( B_j^S \) for \( i \neq j \), we replace \( B_i^S \) with \( B_j^S \setminus \{u\} \) and iterate: this way each element is mapped into exactly one set. The final sets \( B^S \) induce our approximate mapping.

We can upper bound the expected cost of the solution by \( \sum_{S \in \mathcal{C}} c(S) \sum_{i=1}^q y_i^S \cdot g_\pi(B_i^S) \). And from Markov’s inequality

\[
P\left( \sum_{S \in \mathcal{C}} c(S) \sum_{i=1}^q y_i^S \cdot g_\pi(B_i^S) > 4 \ln n \cdot \text{OPT} \right) < \frac{1}{2}.
\]

The probability that an element \( u \in U \) is not covered with a single sampling over \( y_i^B \) is

\[
\prod_{B \ni u, S \supseteq B} (1 - y_i^S) \leq \prod_{B \ni u, S \supseteq B} e^{-y_i^S} = e^{-\sum_{B \ni u, S \supseteq B} y_i^S} \leq \frac{1}{e^4}.
\]

Therefore, by the independence of the sampling and the union bound, the probability that at least one element is not covered is at most \( n \cdot \frac{1}{e^4} = \frac{1}{n} \).

Altogether this gives a Monte-Carlo algorithm. As usual, this can be turned into a Las-Vegas algorithm with expected polynomial running time by repeating the procedure when some element is not covered or the cost of the solution is greater than \( 4 \ln n \cdot \text{OPT} \).

\footnote{In order to solve the configuration LP, there is an alternative to finding a separation oracle for the dual. We can transform the configuration LP into an optimization program where we need to minimize a sum of Lovász’s extensions \[52\], which are convex functions, over a convex region. This approach would be possibly more efficient, but we have chosen the one above for a simpler presentation.}
The following theorem and corollary are a straightforward consequence of Lemmas \ref{lem:1} and \ref{lem:2}.

**Theorem 6.** Universal Stochastic Set Cover admits a polynomial-time Las Vegas $O(\log n)$ approximation algorithm w.r.t. the optimal universal solution when $g_\pi$ can be evaluated in polynomial time.

**Corollary 7.** Universal Stochastic Set Cover admits a polynomial-time Las Vegas $O(\log n)$ approximation algorithm w.r.t. the optimal universal solution in the scenario case and in the independent activation model.

It turns out that not only the randomized rounding technique but also the dual fitting technique can be adapted to the universal stochastic model. This allows us to improve the above randomized algorithm to a purely deterministic greedy algorithm with slightly improved approximation ratio $H[2]$. The high-level idea is to use a greedy strategy: At each step we select a pair $(B, S)$, $B \subseteq S$, that minimizes $h_S(B) := \frac{c(S)g_\pi(B)}{y_B}$, and remove $B$ from the universe and from all sets. It turns out that finding such a pair can also be reduced to submodular minimization (though in a slightly more complicated way). The analysis then follows using standard arguments for greedy set cover. All the details are given in Section \ref{sec:3} where we apply this alternative approach to the more general Universal Stochastic Constrained Set Multicover problem where each element $u$ has to be covered by at least $r(u)$ distinct sets for given positive integer values $\{r(u)\}_{u \in U}$.

### 3.2 The Oracle Model

We next assume that the expected cost of the optimal solution is at least 1. This is w.l.o.g. (by scaling the minimum set cost to 1) if $\pi$ does not output empty sets.

In this case we are not able to compute $g_\pi(B)$ directly for a given $B$, hence we rather try to estimate its value. In more detail, we sample $N$ sets $A_1, \ldots, A_N$ from $\pi$, for a sufficiently large $N$ to be fixed later. Then we run the algorithm for the scenario case over the sampled scenarios.

We next analyze the above algorithm, starting from the simpler cardinality case (where all sets have cost 1). Let us define

$$\hat{g}(B) = \frac{1}{N} \sum_{i=1}^{N} 1[A_i \cap B \neq \emptyset].$$

Observe that for any feasible integer solution $\{y^S_B\}_{S \subseteq C, B \subseteq S}$ it holds

$$E_{A_i \sim \pi} \left[ \sum_S c(S) \sum_{B \subseteq S} y^S_B \cdot 1[A_i \cap B \neq \emptyset] \right] = \sum_S c(S) \sum_{B \subseteq S} y^S_B \cdot g_\pi(B).$$

We want to keep track of deviation of such random variables. We will exploit Chernoff inequality, which states that for i.i.d. variables $X_1, \ldots, X_N$ over $[0, 1]$ we can estimate

$$\Pr \left[ \frac{1}{N} (X_1 + \cdots + X_N) > (1 + \varepsilon) \mathbb{E}[X] \right] \leq \exp \left( -\frac{\varepsilon^2}{2} N \cdot \mathbb{E}[X] \right),$$

$$\Pr \left[ \frac{1}{N} (X_1 + \cdots + X_N) < (1 - \varepsilon) \mathbb{E}[X] \right] \leq \exp \left( -\frac{\varepsilon^2}{3} N \cdot \mathbb{E}[X] \right).$$

The value of the objective function for a fixed solution $\{y^S_B\}$ is a random variable over $[1, n]$ so after scaling it to interval $[0, 1]$ the expected value is at least $\frac{1}{\pi}$. This gives the following bounds

$$\Pr \left[ \sum_S c(S) \sum_{B \subseteq S} y^S_B \cdot \hat{g}(B) > (1 + \varepsilon) \sum_S c(S) \sum_{B \subseteq S} y^S_B \cdot g_\pi(B) \right] \leq \exp \left( -\frac{\varepsilon^2}{3} \cdot \frac{N}{n} \right).$$

\footnote{Recall that $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is the $n$-th harmonic number.}

\footnote{We remark that the latter algorithm is forced to assign all elements to some set, even the elements that happen not to appear in any $A_i$.}
\[
\mathbb{P} \left[ \sum_{S} c(S) \sum_{B \subseteq S} y_{B}^{S} \cdot \hat{g}(B) < (1 - \varepsilon) \sum_{S} c(S) \sum_{B \subseteq S} y_{B}^{S} \cdot g_{\pi}(B) \right] \leq \exp \left( -\frac{\varepsilon^{2}}{2} \cdot \frac{N}{n} \right)
\]

If we choose \( N \geq \frac{1}{\varepsilon^{2}} \cdot n(\log m + 2 \log n) \), then the probability that any of these two bounds does not hold is at most \( \exp(-n \log m - 2 \log n) \). Observe now that the number of possible assignments in Universal Stochastic Set Cover is \( m^{n} = \exp(n \log m) \) (we omit solutions in which some element is covered many times). By the union bound we can estimate that the probability of any bound for any solution being exceeded is at most \( \frac{1}{m} \). Let us name this event as \( \mathcal{E}_{0} \).

Let \( Opt \) be the optimal integral solution for (CONF-IP-SC). Let also \( \hat{Opt} \) be the optimal integral solution for the set of sampled scenarios, and \( \hat{Apx} \) our approximate solution in the same setting. We let \( c(y', g') \) denote the cost of a solution \( y' \) when in the objective function we replace \( g_{\pi} \) with \( g' \). So for example \( c(\hat{Apx}, g_{\pi}) \) is the actual cost of the approximate solution, \( c(\hat{Opt}, g_{\pi}) \) is the optimal cost etc. One has

\[
\mathbb{E}[c(\hat{Apx}, g_{\pi})|\neg \mathcal{E}_{0}] \leq \frac{1}{1 - \varepsilon} \mathbb{E}[c(\hat{Apx}, \hat{g})|\neg \mathcal{E}_{0}] \leq \frac{\mathbb{E}[c(\hat{Opt}, \hat{g})]|\neg \mathcal{E}_{0}]}{1 - \varepsilon} \cdot \mathbb{E}[c(\hat{Opt}, g_{\pi})|\neg \mathcal{E}_{0}] = \frac{\mathbb{E}[c(\hat{Opt}, g_{\pi})]}{1 - \varepsilon}.
\]

Above in the first and fourth inequality we used the event \( \neg \mathcal{E}_{0} \), in the second one the approximation factor of our algorithm for the scenario case, in the third one the optimality of \( \hat{Opt} \) w.r.t. \( \hat{g} \). Using the fact that \( c(\hat{Apx}, g_{\pi}) \leq n \) deterministically, one obtains

\[
\mathbb{E}[c(\hat{Apx}, g_{\pi})] = \mathbb{P}[\mathcal{E}] \cdot \mathbb{E}[c(\hat{Apx}, g_{\pi})|\mathcal{E}] + \mathbb{P}[\neg \mathcal{E}] \cdot \mathbb{E}[c(\hat{Apx}, g_{\pi})|\neg \mathcal{E}] \\
\leq \frac{1}{n^{2}} + O(\log n)(1 + O(\varepsilon))E[c(\hat{Opt}, g_{\pi})] = O(\log n))E[c(\hat{Opt}, g_{\pi})].
\]

**Theorem 8.** Cardinality Universal Stochastic Set Cover in the oracle model admits a polynomial-time \( O(\log n) \) approximation algorithm.

The only difference in the derivation for the weighted case is that the cost of a solution lies within \([1, W n]\), where \( W \) is the largest set cost (recall that we assume that the minimum set cost is 1). Therefore, it suffices to choose \( N \geq \frac{6}{\varepsilon^{2}} \cdot W n(\log m + 2 \log n + \log W) \) to obtain \( \mathbb{P}[\mathcal{E}] \leq \frac{1}{W n^{2}} \) and the rest of the analysis remains the same.

**Theorem 9.** Universal Stochastic Set Cover in the oracle model admits a pseudo-polynomial-time \( O(\log n) \) approximation algorithm.

We remark that the same reduction from the oracle to the scenario case also works for the other variants of set cover discussed in the appendix. The only difference is in the number \( N \) of samples which are required. The simple technical details are left to the reader.

### 3.3 A Lower Bound for the Oracle Model

The algorithm from Theorem 9 runs in pseudo-polynomial time. We next show that any polynomial-time algorithm for the same setting has approximation ratio \( \Omega(\sqrt{n}) \). Remarkably, this matches (hence strengthening) the best known lower bound for the deterministic version of Universal Stochastic Set Cover, even when comparing with the optimal offline solution.

The basic idea is simple, and can be probably applied to several other problems in the same framework. Intuitively, we consider a lower bound instance for the deterministic version of the problem (where the input is adversarially chosen), and embed it into the unknown probability distribution with super-polynomially small probability. The rest of the probability mass is assigned to a low-cost dummy subproblem. With large probability any polynomial-time approximation algorithm will not be able to see the lower bound instance, and will therefore make blind decisions on how to address it. Thus this approximation algorithm in some sense behaves like an algorithm for the deterministic setting. On the other hand the optimal universal solution can be constructed w.r.t. the whole probability distribution, hence achieving the performance of the optimal offline solution in the deterministic setting.
Theorem 10. Any (possibly randomized) polynomial-time algorithm for Universal Stochastic Set Cover in the weighted case has approximation ratio $\Omega(\sqrt{n})$.

Proof. Let $T = \text{poly}(n, m, \log M)$ be the running time of the considered approximation algorithm, where $n$ is the size of the universe, $m$ the number of sets, and $M$ the largest set weight. Here we assume that $T$ is not a random variable: otherwise a similar argument works with $T$ replaced by, say, $T = 10 \cdot \mathbb{E}[T]$.

Consider the following input instance. The universe is $U = W \cup \{d\}$. The (dummy) element $d$ is covered only by a singleton set $S_d = \{d\}$ of cost 1 (in particular, every feasible solution assigns $d$ to $S_d$). Each element $w_i \in W$ is covered by a singleton set $S_i$ of cost $M/\sqrt{n}$. Furthermore there is a set $S_W = W$ of cost $M$ covering precisely $W$. We choose $M$ large enough so that $T/\sqrt{M} = o(1)$.

There are 3 possible scenarios $X_d = \{d\}$, $X_W = \{W\}$, and $X_w = \{w\}$, where $w$ is an element of $W$ chosen uniformly at random by the adversary. Scenario $X_d$ happens with probability $1 - 1/\sqrt{M}$. Exactly one of the scenarios $X_W$ and $X_w$ happens with the residual probability $1/\sqrt{M}$ according to the following rule. Let $p_W$ denote the probability that, in the cases when the algorithm samples only scenario $X_d$ in the offline stage, then it assigns at least one half of the elements of $W$ to $S_W$ with probability at least 1/2. Note that $p_W$ is well defined also for randomized algorithms. If $p_W \leq 1/2$ the adversary chooses $\mathbb{P}[X_w] = 1/\sqrt{M}$, and otherwise $\mathbb{P}[X_W] = 1/\sqrt{M}$. Note that in any case the algorithm will sample a scenario different from $X_d$ with probability at most $T/\sqrt{M} = o(1)$.

Suppose first that $p_W \geq 1/2$. In this case with probability at least $1/4 - o(1)$ the algorithm assigns $w$ to $S_W$. Therefore the expected cost of the approximate solution is at least

$$(1/4 - o(1)) \cdot \frac{1}{\sqrt{M}} \cdot M = \Omega(\sqrt{M}).$$

A feasible universal solution is $\phi(d) = S_d$ and $\phi(w_i) = S_i$ for all $w_i \in W$. The expected cost of this solution is at most

$$(1 - \frac{1}{\sqrt{M}}) \cdot 1 + \frac{1}{\sqrt{M}} \cdot \frac{M}{\sqrt{n}} = O(\sqrt{M/n}).$$

Suppose next that $p_W < 1/2$. In this case with probability at least $1/2 - o(1)$ the algorithm assigns at least one half of the elements of $W$ to singleton sets $S_i$, hence paying in expectation at least

$$(1/2 - o(1)) \cdot \frac{1}{\sqrt{M}} \cdot \frac{n - 1}{2} \cdot \frac{M}{\sqrt{n}} = \Omega(\sqrt{Mn}).$$

A feasible universal solution is $\phi(d) = S_d$ and $\phi(w_i) = S_W$ for all $w_i \in W$. The expected cost of this solution is at most

$$(1 - \frac{1}{\sqrt{M}}) \cdot 1 + \frac{1}{\sqrt{M}} \cdot M = O(\sqrt{M}).$$

In both cases the approximate solution is a factor $\Omega(\sqrt{n})$ worse than the optimal universal solution.

We remark that, by the above construction, a polynomial dependence on $M$ in the number of samples (hence in the running time) is needed in order to achieve an $o(\sqrt{n})$ approximation ratio.

4 Metric Facility Location in the Independent Activation Model

In the stochastic universal variant of the uncapacitated facility location problem, we are given a set of clients $C$ and a set of facilities $F$. For each client $c \in C$ and facility $f \in F$, there is a cost $d(c, f) \geq 0$ paid if $c$ is connected to $f$; furthermore, there is a cost $a_f \geq 0$ associated with opening facility $f \in F$. In the non-metric version of the problem (considered in Section 3) we let $d$ be arbitrary, while in the metric case (considered in this section) $d$ induces a metric. In the universal solution we need to assign every client $c \in C$ to a facility $\phi(c) \in F$. Then a set $X \subseteq C$ is sampled according to some distribution $\pi$ and we need to open all facilities $\phi(X) := \cup_{c \in X} \{\phi(c)\}$ and connect each $c \in X$ to $\phi(c)$. The goal is to minimize the expected total cost of opening the facilities and connecting clients to facilities, i.e.:

$$\mathbb{E}_{X \sim \pi} \left[ \sum_{f \in \phi(X)} a_f + \sum_{c \in X} d(c, \phi(c)) \right].$$
In this section we present a constant approximation for the independent activation case where each client $c$ is independently chosen into the scenario with probability $p_c$.

**Theorem 11. Universal Stochastic Metric Facility Location admits a deterministic polynomial-time $\frac{k}{e-1}$ approximation algorithm w.r.t. the optimal universal solution in the independent activation model.**

Just by direct modeling of the above formula with the configuration LP we can see that the following is a relaxation of an integer program that solves the problem:

$$
\min \sum_{f \in F} \alpha_f \cdot \sum_{B \subseteq C} y_B \cdot g_\pi(B) + \sum_{c \in C} \sum_{f \in F} g_\pi(c) \cdot d(c, f) \cdot \sum_{B \subseteq C, c \in B} y_B \quad \text{(CONF-LP-FL)}
$$

s.t. $\forall c \in C : \sum_{f \in F, B \subseteq C, c \in B} y_B \geq 1$ and $\forall f \in F, \forall B \subseteq C : y_B \geq 0$.

Similarly to the set cover case, the interpretation of $y_B = 1$ is that facility $f$ serves precisely clients $B$. A similar argument also shows that the associated integer program correctly encodes the input problem. Let $OPT_{CONF-LP-FL}$ be the optimal solution to the above LP. It holds that $OPT_{CONF-LP-FL}$ is a lower-bound on the expected cost of an optimal universal solution.

We remark that we are able to solve (CONF-LP-FL) also in the scenario case, however in that case we are missing a good rounding procedure: this case is left as an interesting open problem.

Consider the following alternative LP:

$$
\min \sum_{f \in F} \alpha_f \cdot \max_{c \in C} (x_c^f) + \sum_{f \in F} \sum_{c \in C} p_c \cdot x_c^f + \sum_{c \in C} \sum_{f \in F} p_c \cdot d(c, f) \cdot (x_c^f + \bar{x}_c^f) \quad \text{(LP-FL)}
$$

s.t. $\forall c \in C : \sum_{f \in F} x_c^f + \bar{x}_c^f \geq 1$ and $\forall f \in F, \forall c \in C : x_c^f, \bar{x}_c^f \geq 0$,

and let $OPT_{LP-FL}$ be the optimum solution cost of (LP-FL).

**Lemma 12.** It holds that $OPT_{LP-FL} \leq \frac{k}{e-1} \cdot OPT_{CONF-LP-FL}$

**Proof.** We exploit the following simple inequality (see, e.g., [35] for a proof):

$$
\min \left( 1, \sum_{t \in S} p_t \right) \geq g_\pi(S) = 1 - \prod_{t \in S} (1 - p_t) \geq \left( 1 - \frac{1}{e} \right) \min \left( 1, \sum_{t \in S} p_t \right).
$$

The lower bound in (2) implies that the solution of the following LP is at most $\frac{k}{e-1}$ times bigger than the solution of (CONF-LP-FL):

$$
\min \sum_{f \in F} \sum_{B \subseteq C} y_B \cdot \min \left( 1, \sum_{c \in B} p_c \right) + \sum_{c \in C} \sum_{f \in F} g_\pi(c) \cdot d(c, f) \cdot \sum_{B \subseteq C, c \in B} y_B \quad \text{(3)}
$$

s.t. $\sum_{f \in F, B \subseteq C, c \in B} y_B \geq 1$ \hspace{1cm} $\forall c \in C$

$y_B \geq 0$ \hspace{1cm} $\forall f \in F, \forall B \subseteq C$.

Let $Big$ be the collection of sets $B \subseteq C$ such that $\sum_{t \in B} p_t > 1$, and let $Sml$ be the collection of remaining sets $B \subseteq C$ with $\sum_{t \in B} p_t \leq 1$. Define $x_c^f$ to be the extent to which $c$ was assigned to $f$ via sets from $Big$, and $\bar{x}_c^f$ the extent to which $c$ was assigned to $f$ via sets from $Sml$, i.e., $x_c^f = \sum_{B \in Big, c \in B} y_B$ and $\bar{x}_c^f = \sum_{B \in Sml, c \in B} y_B$. Also, for every client $c$ there is an obvious inequality $\sum_{B \in Big} y_B \geq \sum_{B \in Big, c \in B} y_B = x_c^f$. 

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Now we can lower bound (3):

\[
\sum_{f \in F} a_f \left( \sum_{B \in \text{Big}} y_B^f + \sum_{B \in \text{Sml}} y_B^f \right) + \sum_{c \in C} \sum_{f \in F} p_c \cdot d(c, f) \cdot \sum_{B \subseteq C \subseteq B} y_B^f
\]

\[
= \sum_{f \in F} \left( \sum_{B \in \text{Big}} y_B^f \right) + \sum_{c \in C} \sum_{f \in F} a_f \cdot \sum_{B \subseteq C \subseteq B} y_B^f
\]

\[
\geq \sum_{f \in F} a_f \cdot \max_{c \in C} \left( x_c^f \right) + \sum_{c \in C} \sum_{f \in F} a_f \cdot \bar{x}_c^f + \sum_{c \in C} \sum_{f \in F} p_c \cdot d(c, f) \cdot \left( x_c^f + \bar{x}_c^f \right).
\]

Observe that \( \sum_{f \in F} \sum_{B \subseteq C \subseteq B} y_B^f = \sum_{f \in F} x_c^f + \bar{x}_c^f \geq 1 \) for all \( c \in C \). The claim follows.

Next lemma shows how to round a solution to (LP-FL) into a feasible solution for the original problem.

**Lemma 13.** There is a polynomial-time deterministic algorithm that computes a solution of cost at most \( \text{APX}_{\text{LP-FL}} \leq 4 \text{OPT}_{\text{LP-FL}} \) with respect to the objective function of (LP-FL).

**Proof.** (LP-FL) has a polynomial number of variables and constraints, and so it can be solved in polynomial time: denote by \( (x_c^f, \bar{x}_c^f)_{c \in C, f \in F} \) the corresponding optimal solution. We split the clients into two groups:

\[
C_{\text{big}} := \left\{ c \in C \left| \sum_{f \in F} x_c^f \geq \frac{3}{4} \right\} \right. \quad \text{and} \quad \left. C_{\text{sml}} = \left\{ c \in C \left| \sum_{f \in F} x_c^f < \frac{1}{4} \right\} \right. \right. \}
\]

We assign the two groups to facilities separately. Consider first clients \( C_{\text{big}} \). From the definition we get that \( (4x_c^f)_{c \in C_{\text{big}}, f \in F} \) is a feasible solution to the following LP:

\[
\min \sum_{f \in F} a_f \cdot \max_{c \in C_{\text{big}}} (z_c^f) + \sum_{c \in C_{\text{big}}} \sum_{f \in F} p_c \cdot d(c, f) \cdot z_c^f
\]

\[
\text{s.t.} \, \forall c \in C_{\text{big}} : \sum_{f \in F} z_c^f \geq 1 \quad \text{and} \quad \forall f \in F \forall c \in C_{\text{big}} : z_c^f \geq 0.
\]

The corresponding integer program can be interpreted as a variant of standard uncapacitated metric facility location where the underlying metric \( d \) is distorted by a factor \( p_c \) that depends on client \( c \). A folklore result is a primal-dual 3-approximation algorithm for this problem (this is for example given as an exercise in Vazirani’s book [51]). The idea is to modify the classical primal-dual 3-approximation algorithm so that the dual variable associated to client \( c \) grow at speed \( p_c \) rather than at uniform speed.

We let \( \{z_c^f\}_{c \in C_{\text{sml}}, f \in F} \) be the solution returned by the above rounding algorithm. Thus we have

\[
\sum_{f \in F} a_f \cdot \max_{c \in C_{\text{big}}} (z_c^f) + \sum_{c \in C_{\text{big}}} \sum_{f \in F} p_c \cdot d(c, f) \cdot z_c^f
\]

\[
\leq 3 \cdot \sum_{f \in F} a_f \cdot \max_{c \in C_{\text{big}}} \left( \frac{4}{3} x_c^f \right) + 3 \cdot \sum_{c \in C_{\text{big}}} \sum_{f \in F} p_c \cdot d(c, f) \cdot \frac{4}{3} x_c^f
\]

\[
\leq 4 \cdot \sum_{f \in F} a_f \cdot \max_{c \in C} (x_c^f) + 4 \cdot \sum_{c \in C} \sum_{f \in F} p_c \cdot d(c, f) \cdot x_c^f. \quad (4)
\]

We next consider clients \( C_{\text{sml}} \). Observe that \( (4 \cdot \bar{x}_c^f)_{c \in C_{\text{sml}}, f \in F} \) is a feasible solution to the following LP:

\[
\min \sum_{c \in C_{\text{sml}}} \sum_{f \in F} p_c \cdot (a_f + d(c, f)) \cdot z_c^f
\]

\[
\text{s.t.} \, \forall c \in C_{\text{sml}} : \sum_{f \in F} z_c^f \geq 1 \quad \text{and} \quad \forall f \in F \forall c \in C_{\text{sml}} : z_c^f \geq 0.
\]
The above LP has an optimal integral solution: just assign every client \( c \in C_{\text{sml}} \) to the facility \( f \) that minimizes \( a_f + d(c, f) \). If \( \{ z^f_c \}_{c \in C_{\text{sml}}, f \in F} \) is such an integral solution, then we have that
\[
\sum_{c \in C_{\text{sml}}} \sum_{f \in F} p_c (a_f + d(c, f)) \cdot z^f_c \leq \sum_{c \in C_{\text{sml}}} \sum_{f \in F} p_c (a_f + d(c, f)) \cdot 4 \bar{z}^f_c. \tag{5}
\]

It is then sufficient to return the union of the two mentioned solutions, where \( z^f_c = 1 \) means that \( \phi(c) = f \) (breaking ties arbitrarily in case of multiple assignments of the same client). The cost of the overall solution w.r.t. the original objective function satisfies
\[
\sum_{c \in C} \sum_{t \in T} c_{c, t} \cdot x_{c, t} \leq \sum_{c \in C} \sum_{t \in T} c_{c, t} \cdot \bar{x}_{c, t} \leq 4 \cdot \text{OPT}_{\text{LP-FL}}.
\]

Now Theorem \[13\] follows easily since the cost \( \text{APX}_{\text{CONF-LP-FL}} \) of the approximate solution with respect to the original objective function satisfies
\[
\text{APX}_{\text{CONF-LP-FL}} \leq \text{APX}_{\text{LP-FL}} \leq 4 \cdot \text{OPT}_{\text{LP-FL}} \leq \frac{e}{e - 1} \cdot \text{OPT}_{\text{CONF-LP-FL}}.
\]

where in the first inequality above we used the upper bound in \[2\].

5 Multicut in the Independent Activation Model

In the (classical version of the) Multicut problem we are given an undirected \( n \)-node graph \( G \) with non-negative costs \( c_e \) for all \( e \in E(G) \) and a set of pairs of vertices \( (s_1, t_1), \ldots, (s_k, t_k) \). The goal is to erase a subset of edges \( F \) of minimum cost so that there is no path connecting \( s_c \) with \( t_c \) for any \( c \). Multicut may be considered a covering problem in which the client \( c \) is covered if \( F \) contains some \( (s_c, t_c) \)-cut.

In the universal stochastic setting we are also given a probability distribution \( \pi \) over the subsets of \( C = [1, k] \) and the solution is a mapping \( \phi : C \rightarrow 2^E \) such that \( \phi(c) \) forms a \( (s_c, t_c) \)-cut. The expected cost of the solution induced by a mapping \( \phi \) equals \( \mathbb{E}_{X \sim \pi} \left[ \sum_{e \in \phi(X)} c_e \right] \), where \( \phi(X) = \{ e \in E : e \in \phi(c) \text{ for some } c \in X \} \).

We express the problem with a configuration integer program. Let \( \mathcal{P}_c \) denote the family of all paths connecting \( s_c \) with \( t_c \) and let \( y^c_B = 1 \) mean that \( B = \{ c \in C : e \in \phi(c) \} \).

\[
\min \sum_{e \in E} c_e \sum_{B \subseteq C} y^c_B \cdot g_\pi (B) \tag{CONF-IP-MC} \tag{6}
\]
\[
\text{s.t.} \sum_{e \in P \cdot B \subseteq C} y^c_B \geq 1 \quad \forall c \in C \quad \forall P \in \mathcal{P}_c.
\]
\[
y^c_B \in \{0, 1\} \quad \forall e \in E \quad \forall B \subseteq C.
\]

We next use (CONF-IP-MC) to denote the linear relaxation of (CONF-IP-MC). Likewise for Facility Location, we will show that in the independent activation model we can reduce the universal stochastic setting to the rent-or-buy setting, where each edge \( e \) can be either bought for price \( c_e \) to serve all the clients or be rented by client \( c \) for price \( c_e \cdot p_e \). We define variables \( x^c \) to indicate that \( e \) has been bought and variables \( \bar{x}_e^c \) to express the event of \( e \) being rented by \( c \). This transition simplifies the linear program CONF-LP-MC to the following one by sacrificing an approximation factor of \( \frac{\bar{e}_{X}^e}{e_{X}^e} \).

\[
\min \sum_{e \in E} c_e \cdot x^c + \sum_{e \in E} c_e \sum_{c \in C} p_e \cdot \bar{x}_e^c \tag{LP-MC}
\]
\[
\text{s.t.} \sum_{e \in P} (x^c + \bar{x}_e^c) \geq 1 \quad \forall c \in C \quad \forall P \in \mathcal{P}_c.
\]
\[
x^c, \bar{x}_e^c \geq 0 \quad \forall e \in E \quad \forall c \in C.
\]

The proofs of the following two lemmas and corollary are similar to derivations in Section \[13\] so we placed them in Appendix \[13\].

\[4\]In the facility location case we did not use the rent-or-buy interpretation explicitly: we do that here in order to give a different viewpoint.

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Lemma 14. If \( \pi \) is an independent activation distribution, then the optimal value of \((\text{LP-MC})\) is at most \( \frac{c}{\log n} \) times larger than the optimal value of \((\text{CONF-LP-MC})\).

Lemma 15. If \( G \) is a tree, then one can round a fractional solution to \((\text{LP-MC})\) to an integral one of cost at most 3 times larger. The procedure runs in polynomial time.

Corollary 16. Universal Stochastic Multicut on trees admits a \( \frac{c}{\log n} \)-approximation w.r.t. the optimal universal solution in the independent activation model.

In order to solve the problem on general graphs, we will embed the graph into a tree that approximately preserves the structure of cuts. The following construction has been introduced by Räcke [12].

We call a tree \( T \) a decomposition tree of \( G \) if

1. there is a bijection between the leaves of \( T \) and the vertices of \( G \),
2. each edge \( e \) in \( T \) has capacity \( c^T(e) \) equal to the weight of the cut it induces on \( V(G) \) (we call this cut \( m_T(e_T) \)).

For an edge \( e \in E(G) \) and a decomposition tree \( T \) we define the relative load of \( e \) as

\[
\text{rload}_T(e) = \left( \sum_{e_i \in E(T)} c^T(e_i) \right) / c_e.
\]

The main result in [12] concerns the relation between multicommodity flows in \( G \) and \( T_i \). As our LP formulation is slightly more sophisticated we need to exploit this result in more detail. Section 2.1 in [12] describes how to find (in polynomial time) a convex combination of decomposition trees \( \{\lambda_i T_i\}_{i=1}^q \) for a graph \( G \), such that \( M := \max_{e \in E(G)} |\sum_{i=1}^q \lambda_i \text{rload}_{T_i}(e)| = O(\log n) \).

Theorem 17. Universal Stochastic Multicut in the independent activation model admits a polynomial-time \( O(\log n) \)-expected approximation w.r.t. the optimal universal solution.

Proof. Lemma [15] implies that a fractional solution to \((\text{LP-MC})\) over \( T_i \) can be rounded to an integer solution with an increase of the cost by most a factor 3. Observe that each tree edge on a path between terminals corresponds to some cut between these terminals so any solution to Universal Stochastic Multicut on a decomposition tree induces a solution on the original graph of at most the same cost.

Let \( L_i \) denote the optimal value of \((\text{LP-MC})\) over \( T_i \) and \( L \) denote the same for \( G \). We are going to show that \( \sum_{i=1}^q \lambda_i L_i = O(L \log n) \). In particular this means that \( \min_{i=1}^q L_i = O(L \log n) \). After rounding the fractional solution on \( T_i \) of the smallest value, we will obtain an integral solution for \( G \) of cost \( O(L \log n) \), what entails an \( O(\log n) \) approximation.

Let us consider the dual linear program of \((\text{LP-MC})\).
\[
\sum_{i=1}^{q} \lambda_i r_{load_{T_i}}(e) \text{ times in } (\alpha). \] If we consider vectors \((\beta^T_{c_i})\) given by flow routed between terminals of client \(c\), then we conclude the same for constraint \([13]\).

After scaling \((\alpha)\) down times \(M = \max_{e \in E(G)} \left[ \sum_{i=1}^{q} \lambda_i r_{load_{T_i}}(e) \right]\) we obtain a feasible solution to (DP-MC) for \(G\). This means that \(L\) is no less than \(\sum_{i=1}^{L_i} L_i\) divided by \(M\). As we know from \([12]\) that \(M = O(\log n)\), the claim follows.

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A Vertex Cover and Edge Cover

In this section we consider the universal stochastic versions of Vertex Cover and Edge Cover, which are both special cases of Universal Stochastic Set Cover. In particular, Universal Stochastic Vertex Cover is the special case induced by letting the edges of an undirected graph be the elements of the universe, and the vertices of the same graph be the sets (covering all the edges incident to them). Note that costs are placed on the nodes. Observe also that each vertex might be assigned only a subset of the edges incident to it. One obtains the Universal Stochastic Edge Cover problem by interchanging the roles of edges and vertices in the above construction (in particular, each edge is a set that covers its two endpoints). We recall that standard Vertex Cover admits a 2 approximation while standard Edge Cover can be solved exactly in polynomial time. Here we show that the same can be achieved in the universal stochastic setting.

Let us start with Universal Stochastic Vertex Cover. We first need the following lemma. Recall that in a set cover instance the frequency \( f \) of an element \( u \) is the number of sets containing \( u \).

**Lemma 18.** Under the assumption that each element has frequency at most \( f \), any feasible solution to linear program (CONF-IP-SC) (see Section 3) can be rounded to an integer solution of cost at most \( f \) times larger in deterministic polynomial time.

**Proof.** Fix a solution \( \{y_B^S\} \) \( S \subseteq B \subseteq S \). As for each \( u \in U \) we have \( \sum_{B \subseteq B \subseteq S} \sum_{S \subseteq S} y_B^S \geq 1 \), then there is a set \( S_u \) such that \( \sum_{B \subseteq B \subseteq S_u} y_B^{S_u} \geq \frac{1}{f} \) (if there are many such sets, we pick an arbitrary one). We want to argue that such an assignment provides an integer solution of desirable cost. Recall that the cost
of the solution equals \( \sum_{S \subseteq C} c(S) \sum_{B \subseteq S} y^S_B \cdot g_\pi(B) \). Let \( B_S = \{ u \in S : S_u = S \} \). We need to prove that for each set \( S \) it holds

\[
g_\pi(B_S) \leq f \cdot \sum_{B \subseteq S} y^S_B \cdot g_\pi(B), \quad \text{or equivalently}
\]

\[
\sum_{X \subseteq U} \pi(X) \cdot 1 \{ X \cap B_S \neq \emptyset \} \leq f \cdot \sum_{B \subseteq S} y^S_B \cdot \left( \sum_{X \subseteq U} \pi(X) \cdot 1 \{ X \cap B \neq \emptyset \} \right).
\]

We will count the contribution of each set \( X \subseteq U \) to the left and the right side of the inequality. If \( X \cap B_S = \emptyset \) then \( X \) does not influence the left side and may add something to the right side. Otherwise it adds exactly \( \pi(X) \) to the left side. In this case, let \( u \) be any element in \( X \cap B_S \). The contribution of \( X \) to the right side equals \( \pi(X) \cdot f \cdot \sum_{B \subseteq S, B \cap u \neq \emptyset} y^S_B \) which is at least \( \pi(X) \cdot f \cdot \sum_{B \subseteq S, u \in B} y^S_B \geq \pi(X) \cdot f \cdot \frac{1}{2} = \pi(X) \). The claim follows by summing over all sets \( X \).

As the linear program (CONF-IP-SC) can be solved in polynomial time (see Lemma [4]), we obtain an algorithm that matches the approximation ratio for deterministic Vertex Cover.

**Theorem 19.** Universal Stochastic Vertex Cover admits a polynomial-time deterministic 2-approximation w.r.t. the optimal universal solution when \( g_\pi \) can be evaluated in polynomial time.

**Corollary 20.** Universal Stochastic Vertex Cover admits a polynomial-time deterministic 2-approximation w.r.t. the optimal universal solution in the scenario model and in the independent activation model.

It turns out that even some exact algorithms can be translated into the universal stochastic paradigm.

**Theorem 21.** Universal Stochastic Edge Cover can be solved exactly in deterministic polynomial time when \( g_\pi \) can be evaluated in polynomial time.

**Proof.** Let us consider an edge \( e = uv \) with cost \( c_e \). Covering both \( u, v \) with \( e \) costs \( c(e) \cdot g_\pi(\{u, v\}) = c(e) \cdot \mathbb{P}_{\pi \sim \pi}(u \in X \lor v \in X) \). Covering only one vertex, e.g. \( u \), with \( e \) costs \( c(e) \cdot \mathbb{P}_{\pi \sim \pi}(u \in X) \). We construct a new deterministic instance of Edge Cover where each edge \( uv \) gets its cost multiplied by \( g_\pi(\{u, v\}) \) and we add loops around \( u, v \) with costs as above – all these costs can be evaluated in polynomial time.

To get rid of the loops we create additional vertices \( a, b \), we connect them with a 0-cost edge, and we transform every loop \( vv \) into an edge \( va \) with the same cost. Covering \( a \) and \( b \) is free so taking the \( va \) edge is equivalent to covering \( v \). This is a standard instance of Edge Cover which is in P.

**Corollary 22.** Universal Stochastic Edge Cover can be solved exactly in deterministic polynomial time in the scenario model and in the independent activation model.

## B Non-Metric Facility Location

**Theorem 23.** Universal Stochastic Non-Metric Facility Location admits a Las-Vegas polynomial-time \( O(\log |C|) \) approximation algorithm w.r.t. the optimal universal solution when \( g_\pi \) can be evaluated in polynomial time.

**Proof.** We consider an integer programming formulation for Universal Stochastic Non-Metric Facility Location that resembles CONF-IP-SC for Universal Stochastic Set Cover (see Section [3]), where there is an additional additive term in the objective function responsible for the connection costs. The corresponding linear relaxation and its dual are as follows.

\[
\begin{align*}
\min & \quad \sum_{f \in F} c(f) \sum_{B \subseteq C} y^f_B \cdot g_\pi(B) + \sum_{f \in F} \sum_{u \in C} \left( d(u, f) \sum_{B \ni u} y^f_B \right) \\
\text{s.t.} & \quad \sum_{B \ni u} \sum_{f \in F} y^f_B \geq 1, \\
& \quad y^f_B \geq 0 \quad \forall_{u \in C} \\
& \quad y^f_B \geq 0 \quad \forall_{f \in F} \forall_{B \subseteq C}
\end{align*}
\]

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\[
\max \sum_{u \in C} \alpha_u \\
\text{s.t. } \sum_{u \in B} \alpha_u \leq c(f) \cdot g_\pi(B) + \sum_{u \in B} d(u, f) \quad \forall f \in \mathcal{F} \forall B \subseteq C \\
\alpha_u \geq 0 \quad \forall u \in C
\]

The dual admits a separating oracle alike in Lemma 4 with the submodular function to be minimized \(h_f(B) = c(f) \cdot g_\pi(B) + \sum_{u \in B} d(u, f) - \sum_{u \in B} \alpha_u\). The rounding procedure is analogous to that from Lemma 5. \(\square\)

**Corollary 24.** Universal Stochastic Non-Metric Facility Location in the scenario model and in the independent activation model admits a Las-Vegas polynomial-time \(O(\log |C|)\) approximation algorithm w.r.t. the optimal universal solution.

### C Constrained Set Multicover

The Constrained Set Multicover problem is the generalization of Set Cover where each element \(u\) of the universe must be covered by a given positive integer number \(r(u)\) of distinct sets. In the universal stochastic setting a solution is a mapping \(\phi: U \rightarrow 2^S\) such that \(|\phi(u)| = r(u)\). We can express its expected cost as \(\mathbb{E}_{X \sim \pi} \sum_{S \in \phi(X)} c(S)\), where \(\phi(X) = \bigcup_{u \in X} \phi(u)\).

We will adapt the greedy approach for the deterministic Constrained Set Multicover (see Section 13.2.1 in [50]). For every pair \((B, S), S \in S, B \subseteq S\) we define its cost-effectiveness

\[
c(B, S) = \frac{c(S) \cdot g_\pi(B)}{|B|}.
\]

We are going to maintain a family of sets \((R_s)_{s \in S}\) initiated with identity \(R_S = S\). The set \(R_S\) represents those elements that could be covered by \(S\) in the future. As long as there are some not sufficiently covered elements we choose a pair \((B, S), S \in S, B \subseteq R_S\) minimizing \(c(B, S)\). When pair \((B, S)\) is picked, we update \(R_S := R_S \setminus B\). Moreover, when an element \(u\) gets covered \(r(u)\) times, we erase it from all \(R_S\) sets.

Observe that for each \(u \in U\) and \(S \in S\) there is at most one pair \((B, S)\) that covers \(u\) in the solution. Therefore there is a family of \(r(u)\) distinct sets \(S\) with this property and we return it as \(\phi(u)\). If the solution contains multiple pairs \((B_i, S)\) for a single \(S\), then by subadditivity \(g_\pi(\bigcup_{i=1}^n B_i) \leq \sum_{i=1}^n g_\pi(B_i)\) and we can replace these sets with their union.

At first we argue that this routine can be implemented in polynomial time. The only non-trivial part is minimizing the cost-effectiveness.

**Lemma 25.** If we can evaluate \(g_\pi\) in polynomial time, then we can find a feasible pair minimizing \(c(B, S)\) in polynomial time.

**Proof.** We iterate through all \(S \in S\) and for each of them we want to minimize the function \(h_S(B) = \frac{c(S) \cdot g_\pi(B)}{|B|}\) defined over \(R_S\). As the nominator is a submodular function, the problem reduces to Lemma 20. \(\square\)

**Lemma 26.** Let \(f\) be a submodular function over universe \(U\) with integer values in \([0, M]\) and \(f(\emptyset) \geq 0\). The minimum of \(\tilde{f}(X) = \frac{f(X)}{|X|}\) over \(\emptyset \neq X \subseteq U\) can be found in polynomial time in \(|U|\) and in \(\log_2 M\).

**Proof.** We begin with an observation that if two values of \(\tilde{f}\) differ, then they differ by at least \(\frac{1}{M}\). To see this, consider \(1 \leq a_1 < a_2 \leq M, 1 \leq b_1, b_2 \leq n\), such that \(\frac{a_1}{b_1} < \frac{a_2}{b_2}\). Equivalently we can write \(a_1 b_2 < a_2 b_1\). As the last equations concerns integers we can observe that \(a_1 b_2 + \frac{b_1}{b_2} \leq a_2 b_1\), what implies \(\frac{a_1}{b_1} + \frac{1}{b_2} \leq \frac{a_2}{b_2}\). This proves the observation.

We will take advantage of the binary search. It suffices to check if there is a non-empty set satisfying \(\frac{f(X)}{|X|} < c\) for a constant \(c\). It follows from the observation above, that we require at most \(\log_2 (Mn^2)\)
iterations to converge and we have analogous bound on the binary length of $c$. To answer the question for a given $c$, we may equivalently ask whether the minimum of the function $f_c(X) = f(X) - c |X|$ over $X \subseteq U$ is negative (note that $f_c(\emptyset) = f(\emptyset) \geq 0$ so it does not influence the answer). The function $f_c$ is submodular and the encoding length of the function value is $O(\log M + \log n)$, so the question can be answered in polynomial time due to Theorem $1$.

**Theorem 27.** Universal Stochastic Constrained Set Multicover admits an $H_n$-approximation w.r.t. the universal optimal solution when $g_x$ can be evaluated in polynomial time.

**Proof.** Consider the following linear relaxation of the problem:

$$\min \sum_{S \in \mathcal{S}} c(S) \sum_{B \subseteq S} y^B_S \cdot g_x(B) \quad \text{(CONF-LP-CSM)}$$

subject to:

1. $\sum_{B \subseteq S: u \in B} y^B_S \geq r(u)$ for all $u \in U$
2. $\sum_{B \subseteq S} y^B_S \leq 1$ for all $S \in \mathcal{S}$
3. $y^B_S \geq 0$ for all $S \in \mathcal{S}, B \subseteq S$.

Here the variables $y^B_S$ have the usual interpretation. Note that the second constraint is now needed to avoid that a set is used to cover multiple times the same element. The reader might easily check that integral solutions to (CONF-LP-CSM) are in one-to-one correspondence with feasible solutions to the original problem.

Next consider the dual of (CONF-LP-CSM):

$$\max \sum_e r(u) w_e - \sum_S z_S \quad \text{(DP-CSM)}$$

subject to:

1. $\sum_{e \in B} w_e - z_S \leq c(S) \cdot g_x(B)$ for all $S \in \mathcal{S}, B \subseteq S$
2. $w_u \geq 0, z_S \geq 0$ for all $u \in U \forall S \in \mathcal{S}$

It is convenient to imagine each element $u$ as a set of copies $u(1), \ldots, u(r(u))$ which are covered by distinct sets, where $u(i)$ is the $i$-th copy of $u$ to be covered by the greedy algorithm. Let us define $\text{price}(u, i)$ to be the cost-effectiveness of the pair $(B, S)$ that covered $u(i)$. For $u \in U, S \in \mathcal{S}$ we define $j_u^S$ to be the number of the copy of $u$ covered by a pair $(B, S)$ (for some $B \subseteq S$) or $r(u)$ if $u$ has not been covered by any such pair. Recall that there can be at most one pair satisfying this condition. We also define

$$\alpha_u = \text{price}(u, r(u))$$
$$\beta_S = \sum_{u \in U} \text{price}(u, r(u)) - \text{price}(u, j_u^S)$$

We have $\text{price}(u, i) \leq \text{price}(u, i + 1)$ so $\text{price}(u, j_u^S) \leq \text{price}(u, r(u))$ and $\beta_S \geq 0$. Observe that the total cost we pay in the algorithm equals

$$\sum_{u \in U} \sum_{i=1}^{r(u)} \text{price}(u, i) = \sum_{u \in U} r(u) \alpha_u - \sum_S \beta_S,$$

that is the objective function of the dual linear program (DP-CSM) for variables $(\alpha, \beta)$. In order to show that the obtained solution is a $H_n$-approximation, we need to prove that $\left(\frac{\alpha}{n \cdot H_n}, \frac{\beta}{H_n}\right)$ is a feasible solution to (DP-CSM), which is equivalent to proving that $\forall S \in \mathcal{S}, B \subseteq S$ it holds

$$\sum_{u \in B} \text{price}(u, j_u^S) \leq c(S) \cdot g_x(B) \cdot H_n.$$
We fix a pair \( S \in S, B \subseteq S \). The summand \( \text{price}(u, j^S_u) \) is the cost-effectiveness of the set covering \( u \) in the moment it got removed from \( R_S \). Let us order the elements of \( B \) in the order they were removed from \( R_S \): \( u_1, u_2, \ldots, u_k \). Observe that \( u_i \) could be covered at that moment by \((B_i, S)\) where \( B_i = u_1, \ldots, u_k \), so
\[
\text{price}(u_i, j^S_u) \leq \frac{c(S) \cdot g_v(B_i)}{|B_i|} \leq \frac{c(S) \cdot g_v(B)}{k - i + 1},
\]
\[
\sum_{u \in B} \text{price}(u, j^S_u) \leq c(S) \cdot g_v(B) \cdot \sum_{i=1}^{|B|} \frac{1}{k - i + 1} \leq c(S) \cdot g_v(B) \cdot H_n.
\]
This shows that \((\frac{\mu^v}{n^v}, \frac{\sigma^v}{n^v})\) is a feasible solution to linear program (DP-CSM). Therefore the cost of the solution is no greater than \( H_n \) times the optimum of (DP-CSM) which equals the optimum of (CONF-LP-CSM). The claim follows. \( \square \)

**Corollary 28.** Universal Stochastic Constrained Set Multicover admits an \( H_n \)-approximation w.r.t. the universal optimal solution in the scenario model and in the independent activation model.

### D Multicut: proofs of Lemmas 14 and 15

**Proof of Lemma 14** We want to transform the following LP
\[
\min \sum_{c \in E} \sum_{B \subseteq C} y^c_B \cdot g_v(B) \tag{9}
\]
\[
\text{s.t. } \sum_{c \in P} \sum_{B \subseteq C} y^c_B \geq 1 \quad \forall_{c \in C} \forall_{P \in P_c}. \\
\sum_{c \in E} y^c_B \geq 0. \quad \forall_{c \in E \forall_{B \subseteq C}}.
\]
Similarly to Section B let us inject the bound from [35]:
\[
g_v(B) = 1 - \prod_{j \in B} (1 - p_j) \geq \left(1 - \frac{1}{e}\right) \min \left(1, \sum_{j \in B} p_j \right)
\]
into the linear program \(9\). Here is what we obtain:
\[
\min \sum_{c \in E} \sum_{B \subseteq C} y^c_B \cdot \min \left(1, \sum_{j \in B} p_j \right) \tag{10}
\]
\[
\text{s.t. } \sum_{c \in P} \sum_{B \subseteq C} y^c_B \geq 1 \quad \forall_{c \in C} \forall_{P \in P_c}. \\
\sum_{c \in E} y^c_B \geq 0. \quad \forall_{c \in E \forall_{B \subseteq C}}.
\]
Observe that the value of the optimal solution to \(10\) is at most \( \frac{1}{e - 1} \) times larger than the optimum of \(9\). On the other hand every integral solution to \(10\) translates into an integral solution to \(9\) of the same cost because \( \min \left(1, \sum_{j \in B} p_j \right) \geq g_v(B) \).

Let Big be a collection of all the sets \( B \) such that \( \sum_{j \in B} p_j > 1 \), and let Smal be a collection of all the sets \( B \) such that \( \sum_{j \in B} p_j \leq 1 \). We can rewrite the objective function of \(10\) as
\[
\sum_{c \in E} \sum_{B \subseteq C} y^c_B \cdot \min \left(1, \sum_{j \in B} p_j \right) = \sum_{c \in E} c_e \left( \sum_{B \in \text{Big}} y^c_B + \sum_{B \in \text{Smal}} y^c_B \left( \sum_{j \in B} p_j \right) \right)
\]
\[
= \sum_{c \in E} c_e \left( \sum_{B \in \text{Big}} y^c_B \right) + \sum_{c \in E} \sum_{c \in C} c_e \left( \sum_{B \in \text{Smal}} \sum_{c \in E \in B} y^c_B \right) \tag{11}
\]
Let $x^e_c = \sum_{B \in \text{Big}, e \in B} y^e_B$ and $\bar{x}^e_c = \sum_{B \in \text{Sml}, e \in B} y^e_B$. Also, we have an obvious inequality:

$$\forall e \in C: \sum_{B \in \text{Big}} y^e_B \geq \sum_{B \in \text{Big}, e \in B} y^e_B = x^e_c,$$

and so (11) is greater than

$$\sum_{e \in E} c_e \cdot \max_{e \in C} (x^e_c) + \sum_{e \in E} c_e \sum_{e \in C} p_c \cdot \bar{x}^e_c.$$ 

The condition $\sum_{e \in E} \sum_{B \in \text{Big}} y^e_B \geq 1$ translates into $\sum_{e \in E} (x^e_c + \bar{x}^e_c) \geq 1$. Observe that for each edge $e$ we can replace all variables $x^e_c$ with $x^c_e = \max_{e \in C} (x^e_c)$; the objective does not change and the conditions remain satisfied. Finally, we obtain a new LP with the value of the optimal solution at most $\frac{e}{c}$ times the optimum of (11), which is what we have claimed.

$$\min_{e \in E} \sum_{e \in E} c_e \cdot x^e + \sum_{e \in E} c_e \sum_{e \in B} p_c \cdot \bar{x}^e_c$$

$$\text{s.t. } \sum_{e \in P} (x^e_c + \bar{x}^e_c) \geq 1 \quad \forall e \in E \forall p \in P_c.$$ 

$$x^e_c, \bar{x}^e_c \geq 0 \quad \forall e \in E \forall e \in C.$$  

\[\square\]

**Proof of Lemma** 15. Let us consider a restricted version of **Universal Stochastic Multicut** with an additional assumption that the graph $G$ is a tree. In this case each family $P_c$ consists of only one path — let us denote it by $P_c$. We will take advantage of that fact in order to round the linear program (12). At first, note that it has a polynomial number of variables and constraints so it can be solved in polynomial time. Define $(x^e_c, \bar{x}^e_c)_{e \in C, e \in E}$ to be the optimal solution and denote its value as $OPT_{LP}$. We split $C$ into two groups:

$$C_{\text{big}} := \left\{ c \in C \left| \sum_{e \in P_c} x^e_c \geq \frac{2}{3} \right\} \right.$$  

and $C_{\text{sml}} = \left\{ c \in C \left| \sum_{e \in P_c} x^e_c \geq \frac{1}{3} \right\} \right.$.

Observe that $(\frac{2}{3} \cdot x^c_e)_{e \in E}$ is a feasible solution to the linear relaxation of the standard **Multicut** problem on trees with the cut constraints given by $C_{\text{big}}$. This problem admits a 2-approximation with respect to the LP optimum (21). Therefore, we can construct an integral solution $(\bar{x}^e_c)_{e \in E}$ for (12) satisfying constraints from $C_{\text{big}}$ of cost not exceeding $2 \cdot 2 \cdot \sum_{e \in E} c_e \cdot x^e$.

On the other hand $(3 \cdot \bar{x}^e_c)_{e \in C_{\text{sm}}}$ forms a feasible solution to the following LP.

$$\min \sum_{e \in E} c_e \sum_{c \in C} p_c \cdot \bar{x}^e_c$$

$$\text{s.t. } \sum_{e \in P_c} \bar{x}^e_c \geq 1 \quad \forall e \in C_{\text{sm}}$$ 

$$\bar{x}^e_c \geq 0 \quad \forall e \in E \forall e \in C_{\text{sm}}.$$ 

It is easy to see that the optimal solution is integral and can be obtained by assigning each client $c$ to the cheapest edge along the path $P_c$ — denote this solution as $(\bar{z}^e_c)_{e \in C, e \in E}$ where $\bar{z}^e_c = 0$ for $c \in C_{\text{big}}$.

The total cost of the constructed solution is

$$\sum_{e \in E} c_e \cdot z^e + \sum_{e \in E} c_e \sum_{e \in C} p_c \cdot \bar{x}^e_c \leq 2 \sum_{e \in E} c_e \left(\frac{3}{2} \cdot x^e_c\right) + \sum_{e \in E} c_e \sum_{e \in C} p_c \cdot (3 \cdot \bar{x}^e_c) = 3 \cdot OPT_{LP}.$$  

\[\square\]

**Corollary** 16 directly follows from the above two lemmas.