A Friedlander type estimate for Stokes operators

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Abstract. Let Ω ⊂ R^d be a bounded open connected set with Lipschitz boundary. Let A^N and A^D be the Neumann Stokes operator and Dirichlet Stokes operator on Ω, respectively. Further let λ_1^N ≤ λ_2^N ≤ ... and λ_1^D ≤ λ_2^D ≤ ... be the eigenvalues of A^N and A^D repeated with multiplicity, respectively. Then

\[ \lambda_{n+1}^N < \lambda_n^D \]

for all \( n \in \mathbb{N} \).

1 Introduction

If \( \lambda_1^D \leq \lambda_2^D \leq ... \) are the eigenvalues of the Dirichlet Laplacian and \( \lambda_1^N \leq \lambda_2^N \leq ... \) the eigenvalues of the Neumann Laplacian on a domain Ω with Lipschitz boundary, then

\[ \lambda_{n+1}^N < \lambda_n^D \]  (1)

for all \( n \in \mathbb{N} \). This result has a long history, starting with Payne [Pay] and Pólya [Póly]. In 1991, Friedlander [Fri] proved that \( \lambda_{n+1}^N \leq \lambda_n^D \) for a \( C^1 \) domain, using the Dirichlet-to-Neumann operator. The method was modified and extended by Filonov [Fil] to prove a strict inequality on more general domains. A different and independent proof was obtained by Arendt–Mazzeo [AM2]. On the other hand, Mazzeo [Maz] gave a counter example for manifolds. Under certain geometric conditions on three dimensional domains Hansson [Han] proved (1) for Dirichlet and Neumann eigenvalues of the Heisenberg Laplacian. This was extended to general Heisenberg groups and fairly general domains by Frank–Laptev [FL]. Other results are for powers \((-\Delta)^m\) of the Laplacian by Provenzano [Pro] and a comparison between Robin and Dirichlet eigenvalues by Gesztesy–Mitrea [GM] or for certain operators with mixed boundary conditions by Lotoreichik–Rohleder [LR]. In [BRS] Behrndt–Rohleder–Stadler considered external domains and added a potential to the Laplacian.

The aim of this paper is to prove a similar result for the Stokes operator. Let Ω ⊂ R^d be an open bounded connected set. One can define two versions of the Stokes operator: a Dirichlet version and a Neumann version. The Dirichlet Stokes operator \( A^D \) is defined in the closure \( L_{2,\sigma}(\Omega) \) of solenoidal test functions \( \{ u \in C_c^\infty(\Omega, \mathbb{C}^d) : \text{div} \, u = 0 \} \) in \( L_2(\Omega, \mathbb{C}^d) \). If \( u, f \in L_{2,\sigma}(\Omega) \), then \( u \in D(A^D) \) and \( A^D u = f \) if and only if \( u \in H_0^1(\Omega, \mathbb{C}^d) \), \( \text{div} \, u = 0 \).
and there exists a \( \pi \in L_2(\Omega) \) such that \( f = -\Delta u + \nabla \pi \in H^{-1}(\Omega, \mathbb{C}^d) \). Then \( A^D \) is a positive self-adjoint operator in \( L_{2,\sigma}(\Omega) \) with discrete spectrum. Let \( \lambda_1^D \leq \lambda_2^D \leq \ldots \) be the eigenvalues of \( A^D \) repeated with multiplicity.

The Neumann Stokes operator \( A^N \) is defined in the Hilbert space \( \{ u \in L_2(\Omega, \mathbb{C}^d) : \text{div} \ u = 0 \} \). In Proposition 3.3 we shall give a precise definition of the self-adjoint operator \( A^N \). Loosely speaking, \( u \in D(A^N) \) and \( A^N u = f \) if \( u \in H^1(\Omega, \mathbb{C}^d) \), \( \text{div} \ u = 0 \) and there exists a \( \pi \in L_2(\Omega) \) such that \( f = -\Delta u + \nabla \pi \in H^{-1}(\Omega, \mathbb{C}^d) \) and \( \partial_{\nu} u = (\text{Tr} \ \pi) \nu \), where \( \partial_{\nu} \) is the outward normal derivative. Also \( A^N \) is a positive self-adjoint operator with discrete spectrum. Let \( \lambda_1^N \leq \lambda_2^N \leq \ldots \) be the eigenvalues of \( A^D \) repeated with multiplicity. The main result of this paper is that

\[
\lambda_{n+1}^N < \lambda_n^D
\]

for all \( n \in \mathbb{N} \).

Although the Dirichlet and Neumann Laplacians are self-adjoint operators in the same Hilbert space \( L_2(\Omega) \), this is not the case for the Stokes Dirichlet operator \( A^D \) and Stokes Neumann operator \( A^N \). The Stokes Dirichlet operator \( A^D \) is a self-adjoint operator in the orthogonal complement in \( L_2(\Omega, \mathbb{C}^d) \) of the space

\[
\{ \nabla \pi : \pi \in H^1(\Omega) \},
\]

whilst the Stokes Neumann operator \( A^N \) is a self-adjoint operator in the orthogonal complement in \( L_2(\Omega, \mathbb{C}^d) \) of the space

\[
\{ \nabla \pi : \pi \in H^1_0(\Omega) \}.
\]

Nevertheless, we relate the eigenvalues in (2).

As in Friedlander’s original paper, the main ingredient for our proof is the Dirichlet-to-Neumann operator, adapted to the Stokes problem. We use sesquilinear form methods as in [AM1] and [AM2]. In Section 2 we prove an abstract theorem, Theorem 2.2, on eigenvalue comparison of Dirichlet-type and Neumann-type operators associated with forms, which relies (and expands) on a combination of the papers [AM1], [AM2], [AEKS] and the BSc(Honours) thesis [Gor]. As an immediate application we obtain in Theorem 2.8 the classical result for the Laplacian on a Lipschitz domain. Section 3 begins with the description of the Dirichlet and Neumann versions of the Stokes operator on a Lipschitz domain, as well as of the related Dirichlet-to-Neumann operator. We then apply our abstract eigenvalue theorem, Theorem 2.2, to obtain a Stokes version of the Friedlander eigenvalue theorem.

## 2 An abstract Friedlander comparison theorem

We recall the basic notions of form methods as developed in [AE] and [AEKS], which extend the classical form methods of Kato [Kat] and Lions [Lio]. Since we only use self-adjoint graphs in this paper, we restrict our attention to symmetric forms.

Let \( V, K \) be Hilbert spaces and \( \mathbf{a}: V \times V \to \mathbb{C} \) be a sesquilinear form. The form \( \mathbf{a} \) is called **continuous** if there is an \( M > 0 \) such that \( |\mathbf{a}(u, v)| \leq M \|u\|_V \|v\|_V \) for all \( u, v \in V \). The form \( \mathbf{a} \) is called **positive** if \( \mathbf{a}(u, u) \geq 0 \) for all \( u \in V \) and it is called **symmetric** if
\( a(v,u) = \overline{a(u,v)} \) for all \( u,v \in V \). Let \( j: V \to K \) be a continuous linear map. In Lions [Lio], the map \( j \) is an inclusion map. We say that \( a \) is \( j \)-elliptic if there are \( \omega, \mu > 0 \) such that

\[
\Re a(u,u) + \omega \|j(u)\|_K^2 \geq \mu \|u\|_V^2
\]

for all \( u \in V \). The graph associated with \((a, j)\) is the subspace

\[
\{(j(u), f) \in K \times K : u \in V \text{ and } a(u,v) = (f, j(v))_K \text{ for all } v \in V\}
\]
in \( K \times K \).

There are a number of sufficient conditions to conclude that the graph associated with \((a, j)\) is a self-adjoint graph if \( a \) is continuous and symmetric. For a summary of the terminology in self-adjoint graphs we refer to [AEKS] Section 3 and [BE] Section 2. In this paper we use the following two sufficient conditions.

**Proposition 2.1.** Let \( V, K \) be Hilbert spaces and \( a: V \times V \to \mathbb{C} \) be a continuous symmetric sesquilinear form. Let \( j: V \to K \) be a continuous linear map. Suppose at least one of the following conditions is valid.

(a) The form \( a \) is \( j \)-elliptic.

(b) There exists a Hilbert space \( H \) and a compact linear map \( i: V \to H \) such that \( a \) is \( i \)-elliptic.

Then the graph \( A \) associated with \((a, j)\) is a self-adjoint graph which is lower bounded.

Moreover, if \( j \) is compact, then \( A \) has compact resolvent and in particular a discrete spectrum.

**Proof.** (a). This follows from [AE] Theorem 2.1 applied to the Hilbert space \( j(V) \).

(b). See [AEKS] Theorems 4.5 and 4.15.

For the last statement, see [AEKS] Proposition 4.8. \( \square \)

Now we are able to formulate the abstract eigenvalue theorem. We denote by \( A^0 \) the single valued part of a self-adjoint graph \( A \). For an immediate example we refer the reader to the beginning of the proof of Theorem 2.8.

**Theorem 2.2.** Let \( V, H \) and \( K \) be Hilbert spaces. Suppose that \( V \) is embedded in \( H \) and that the inclusion map \( i: V \to H \) is compact. Let \( j: V \to K \) be a compact linear map. Let \( a: V \times V \to \mathbb{C} \) be a positive symmetric continuous \( i \)-elliptic sesquilinear form. Let \( V_D = \ker j \). Let \( A^N \) be the self-adjoint operator in \( V \) associated with \( a \) and let \( A^D \) be the self-adjoint operator in \( V_D \subset H \) associated with \( a|_{V_D \times V_D} \), where the closures are in \( H \). Further, for all \( \lambda \in \mathbb{R} \) define \( b_\lambda: V \times V \to \mathbb{C} \) by

\[
b_\lambda(u,v) = a(u,v) - \lambda (u,v)_H.
\]

Let \( N_\lambda \) be the self-adjoint graph associated with \((b_\lambda, j)\). Suppose

(I) the operator \( A^N \) has no eigenvector in \( V_D \) and

(II) \( \dim \text{span}\{\varphi \in D(N_\lambda) : (N_\lambda^0 \varphi, \varphi)_K = 0\} = \infty \) for all \( \lambda \in (0, \infty) \).
Let $\lambda_1^N \leq \lambda_2^N \leq \ldots$ and $\lambda_1^D \leq \lambda_2^D \leq \ldots$ be the eigenvalues of $A^N$ and $A^D$ repeated with multiplicity, respectively. Then

$$\lambda_{n+1}^N < \lambda_n^D$$

for all $n \in \mathbb{N}$.

**Proof.** Since $a$ is $i$-elliptic there are $\omega, \delta > 0$ such that

$$a(u, u) + \omega \|u\|_H^2 \geq \delta \|u\|_V^2$$

for all $u \in V$. Let $\hat{A}^D$ be the self-adjoint graph in $H$ associated with $(a_{V_D \times V_D}, \nu_{V_D})$. Then $A^D = (\hat{A}^D)^c$, the singular part of $\hat{A}^D$ and $\hat{A}^D$ is positive. Moreover, $\hat{A}^D$ has compact resolvent since $i$ is compact.

For all $\mu \in \mathbb{R}$ define the form $a_\mu: V \times V \to \mathbb{C}$ by

$$a_\mu(u, v) = a(u, v) - \mu (j(u), j(v))_K.$$

Then $a_\mu$ is $i$-elliptic. Let $A_\mu$ be the self-adjoint graph associated with $(a_\mu, i)$. Then $A_\mu$ has compact resolvent by Proposition 2.1. Let $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \ldots$ be the eigenvalues of $A_\mu$, repeated with multiplicity. Then $\lambda_n: \mathbb{R} \to \mathbb{R}$ is a function for all $n \in \mathbb{N}$. Since $\mu \mapsto a_\mu(u)$ is a decreasing function for all $u \in V$, it follows from the mini-max theorem that $\lambda_n$ is a decreasing function for all $n \in \mathbb{N}$.

**Lemma 2.3.** $\lim_{\mu \to -\infty} (I + A_\mu)^{-1} = (I + \hat{A}^D)^{-1}$ in $\mathcal{L}(H)$.

**Proof.** Let $\mu_1, \mu_2, \ldots \in (-\infty, 0)$ and suppose that $\lim_{n \to \infty} \mu_n = -\infty$. We shall prove that $\lim_{n \to \infty} \|(I + A_{\mu_n})^{-1} - (I + \hat{A}^D)^{-1}\|_{\mathcal{L}(H)} = 0$. Suppose not. Then there are $\varepsilon > 0$ and $f_1, f_2, \ldots \in H$ such that

$$\|(I + A_{\mu_n})^{-1} f_n - (I + \hat{A}^D)^{-1} f_n\|_H > \varepsilon \|f_n\|_H$$

for all $n \in \mathbb{N}$. Without loss of generality we may assume that $\|f_n\|_H = 1$ for all $n \in \mathbb{N}$. Passing to a subsequence if necessary, there exists an $f \in H$ such that $\lim f_n = f$ weakly in $H$. Then $\lim (I + \hat{A}^D)^{-1} f_n = (I + \hat{A}^D)^{-1} f$ in $H$ since $(I + \hat{A}^D)^{-1}$ is compact. For all $n \in \mathbb{N}$ set $u_n = (I + A_{\mu_n})^{-1} f_n$.

Let $n \in \mathbb{N}$. Then

$$a(u_n, v) - \mu_n (j(u_n), j(v))_K + (u_n, v)_H = (f_n, v)_H$$

for all $v \in V$. Choosing $v = u_n$ gives

$$a(u_n, u_n) + |\mu_n| \|j(u_n)\|_K^2 + \|u_n\|_H^2 = \text{Re}(f_n, u_n)_H \leq \|f_n\|_H \|u_n\|_H = \|u_n\|_H.$$

So $\|u_n\|_H \leq 1$ and then also $a(u_n, u_n) \leq 1$ and $|\mu_n| \|j(u_n)\|_K^2 \leq 1$. Hence $\lim j(u_n) = 0$ in $K$. Moreover, $\delta \|u_n\|_V^2 \leq (1 + \omega)$ for all $n \in \mathbb{N}$ by (3). Therefore the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $V$. Passing to a subsequence if necessary, there exists a $u \in V$ such that $\lim u_n = u$ weakly in $V$. Then $\lim u_n = u$ strongly in $H$ since the embedding $i$ is compact. Similarly $j(u) = \lim j(u_n)$ in $K$. But $\lim j(u_n) = 0$ in $K$. So $j(u) = 0$ and $u \in V_D$.

Let $v \in V_D$. Taking the limit $n \to \infty$ in (3) gives $a(u, v) + (u, v)_H = (f, v)_H$. So $u \in D(\hat{A}^D)$ is $\hat{A}^D$-elliptic there are $\omega, \delta > 0$ such that

$$\text{Re}(f, u)_H \leq \|f\|_H \|u\|_H = \|u\|_H.$$

This contradicts (4).
Proposition 2.4. If \( n \in \mathbb{N} \), then \( \lim_{\mu \to -\infty} \lambda_n(\mu) = \lambda^D_n \).

Proof. Let \( n \in \mathbb{N} \). Let \( \varepsilon > 0 \). By Lemma 2.3 there exists an \( M \in \mathbb{N} \) such that
\[
((I + \tilde{A}^D)^{-1}f, f)_H - \varepsilon \|f\|_H^2 \leq ((I + A_\mu)^{-1}f, f)_H \leq ((I + \tilde{A}^D)^{-1}f, f)_H + \varepsilon \|f\|_H^2
\]
for all \( f \in H \) and \( \mu \in (-\infty, -M] \). Note that the max-min theorem gives
\[
(1 + \lambda_n(\mu))^{-1} = \max_{\dim W = n} \min_{f \in W} \frac{((I + A_\mu)^{-1}f, f)_H}{\|f\|_H^2}
\]
and a similar expression is valid for \( \tilde{A}^D \). So \( (1 + \lambda^D_n)^{-1} - \varepsilon \leq (1 + \lambda_n(\mu))^{-1} \leq (1 + \lambda^D_n)^{-1} + \varepsilon \)
for all \( \mu \in (-\infty, -M] \) and the proposition follows. \( \square \)

The next lemma is a version of the Birman–Schwinger principle.

Lemma 2.5. Let \( \lambda, \mu \in \mathbb{R} \). Then \( \lambda \in \sigma(A_\mu) \) if and only if \( \mu \in \sigma(N_\lambda) \).

Proof. ‘\( \Rightarrow \)’ Suppose \( \lambda \in \sigma(A_\mu) \). Then there exists a \( u \in D(A_\mu) \) such that \( u \neq 0 \) and \( A_\mu u = \lambda u \). Then
\[
a(u, v) - \mu (j(u), j(v))_K = (\lambda u, v)_H \tag{6}
\]
for all \( v \in V \). Hence \( b_\lambda(u, v) = a(u, v) - \lambda (u, v)_H = (\mu j(u), j(v))_K \) for all \( v \in V \). Therefore \( j(u) \in D(N_\lambda) \) and \( j(u), j(v) \in N_\lambda \). It remains to show that \( j(u) \neq 0 \). Suppose that \( j(u) = 0 \). Then it follows from \( \Box \) that \( a(u, v) = (\lambda u, v)_H \) for all \( v \in V \). So \( u \in D(A^N) \) and \( A^N u = \lambda u \). Hence \( u \) is an eigenvector for \( A^N \) and also \( u \in V_D \). By Assumption \( \Box \) this is not possible. So \( j(u) \neq 0 \) and \( \mu \in \sigma(N_\lambda) \).

‘\( \Leftarrow \)’. The proof is similar, even easier. \( \square \)

Lemma 2.6. Let \( n \in \mathbb{N} \). Then \( \lambda_n \) is strictly decreasing.

Proof. We already know that \( \lambda_n \) is decreasing. Let \( \mu_1, \mu_2 \in \mathbb{R} \) with \( \mu_1 < \mu_2 \). Suppose that \( \lambda_n(\mu_1) = \lambda_n(\mu_2) \). Write \( \lambda = \lambda_n(\mu_1) \). Let \( \mu \in [\mu_1, \mu_2] \). Since \( \lambda_n \) is decreasing, it follows that \( \lambda_n(\mu) = \lambda \). So \( \lambda \in \sigma(A_\mu) \). Hence \( \mu \in \sigma(N_\lambda) \) by Lemma 2.5. So \( [\mu_1, \mu_2] \subset \sigma(N_\lambda) \).

This is a contradiction since \( N_\lambda \) has a discrete spectrum by Proposition 2.1. \( \square \)

Lemma 2.7. Let \( \lambda > 0 \). Then \( \sigma(N_\lambda) \cap (-\infty, 0) \neq \emptyset \).

Proof. Let \( N_\lambda^o \) be the single valued part of \( N_\lambda \). Suppose \( \sigma(N_\lambda) \cap (-\infty, 0) = \emptyset \). Then \( N_\lambda^o \) is a positive self-adjoint operator.

Let \( \varphi \in D(N_\lambda^o) \) and suppose that \( (N_\lambda^o \varphi, \varphi)_K = 0 \). Let \( \psi \in D(N_\lambda^o) \). Then
\[
\|\rangle N_\lambda^o \varphi, \psi \rangle_K \rangle \leq \|\langle N_\lambda^o \varphi, \varphi \rangle_K \rangle \|\langle N_\lambda^o \varphi, \varphi \rangle_K \rangle^{1/2}\|\langle \rangle N_\lambda^o \varphi, \psi \rangle_K \rangle^{1/2}\|\langle \rangle N_\lambda^o \varphi, \varphi \rangle_K \rangle = 0.
\]
Hence \( N_\lambda^o \varphi = 0 \) and \( \varphi \in \ker(N_\lambda^o) \). By Assumption \( \Box \)
\[
\dim \dim \ker(N_\lambda^o) = \infty. \quad \Box\]

So \( \dim \ker(N_\lambda^o) = \infty \). But the operator \( N_\lambda^o \) has compact resolvent by Proposition 2.1.

This is a contradiction. \( \square \)
Now we complete the proof of Theorem 2.2. Let $n \in \mathbb{N}$. Choose $\lambda = \lambda_n^D$. Since $\lambda_n$ is strictly decreasing by Lemma 2.6, it follows from Proposition 2.4 that

$$0 \leq \lambda_n^N = \lambda_n(0) < \lambda_n^D = \lambda.$$ 

Let $\mu = \min \sigma(N_\lambda)$. Then $\mu < 0$ by Lemma 2.7. Moreover, $\lambda \in \sigma(A_\mu)$ by Lemma 2.5. Hence there exists a $j \in \mathbb{N}$ such that $\lambda_j(\mu) = \lambda$. Using again Lemma 2.6 and Proposition 2.4, it follows that $\lambda_n(\mu) < \lambda_n^D = \lambda = \lambda_j(\mu)$. Therefore $j \geq n + 1$. Then

$$\lambda_{n+1}^N = \lambda_{n+1}(0) < \lambda_{n+1}(\mu) \leq \lambda_j(\mu) = \lambda = \lambda_n^D,$$

where the strict inequality follows from Lemma 2.6.

The first main example is the classical version of Friedlander’s theorem.

**Theorem 2.8.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and $d \geq 2$. Let $\Delta^N$ and $\Delta^D$ be the Neumann and Dirichlet Laplacians on $\Omega$, respectively. Further let $\lambda_1^N \leq \lambda_2^N \leq \ldots$ and $\lambda_1^D \leq \lambda_2^D \leq \ldots$ be the eigenvalues of $-\Delta^N$ and $-\Delta^D$ repeated with multiplicity, respectively. Then

$$\lambda_{n+1}^N < \lambda_n^D$$

for all $n \in \mathbb{N}$.

**Proof.** Choose $V = H^1(\Omega)$, $H = L^2(\Omega)$ and $K = L^2(\partial \Omega)$. Let $i: V \to H$ be the inclusion map and $j = \text{Tr}: V \to K$. Then $i$ and $j$ are compact. Moreover, $\ker j = H^1_0(\Omega)$. Define $a: V \times V \to \mathbb{C}$ by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v.$$

Then $a$ is a positive symmetric continuous $i$-elliptic sesquilinear form. Further, $-\Delta^N$ is the self-adjoint operator associated with $a$ and $-\Delta^D$ is the self-adjoint operator associated with $a|_{H^1_0(\Omega) \times H^1_0(\Omega)} = a|_{\ker j \times \ker j}$.

It remains to verify Conditions (I) and (II) of Theorem 2.2.

**[I].** Let $u \in D(A^N) \cap V_D$ and $\lambda \in \mathbb{R}$. Suppose that $A^N u = \lambda u$. Then $u \in H^1_0(\Omega)$. Let $\tilde{u} \in H^1(\mathbb{R}^d)$ be the extension of $u$ by zero. Then

$$\int_{\mathbb{R}^d} \nabla \tilde{u} \cdot \nabla v = \int_{\Omega} \nabla u \cdot \nabla(v|_{\Omega}) = (\lambda u, v|_{\Omega})_{L^2(\Omega)} = (\lambda \tilde{u}, v)_{L^2(\mathbb{R}^d)}$$

for all $v \in H^1(\mathbb{R}^d)$. So $-\Delta^N \tilde{u} = \lambda \tilde{u}$, where $\Delta^N$ is the Laplacian on $\mathbb{R}^d$. Since $\tilde{u}$ vanishes on a non-empty open set, it follows from the unique continuation property that $\tilde{u} = 0$. (See for example [RS] Theorem XIII.57.) Then also $u = 0$. So $A^N$ has no eigenvector in $V_D$.

**[II].** Let $\lambda \in (0, \infty)$. Let $\omega \in \mathbb{R}^d$ and suppose that $|\omega|^2 = \lambda$. Define $u \in H^2(\Omega)$ by $u(x) = e^{i\omega \cdot x}$. Then $-\Delta u = \lambda u$ as distribution and

$$b_\lambda(u, v) = \int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} u \overline{v}$$

$$= \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} (\Delta u) \overline{v}$$

$$= \int_{\Omega} \text{div}(v \nabla u)$$

$$= \int_{\partial \Omega} \nu \cdot \text{Tr}(v \nabla u) = \int_{\partial \Omega} i(\omega \cdot \nu) \text{Tr} u \overline{v} = (i(\omega \cdot \nu) \text{Tr} u, j(v))_K$$
for all \( v \in C^\infty_0(\Omega) \), where we used the divergence theorem (see [Alt] Theorem A8.8). So by density (see for example [Maz] Theorem 1.1.6/2) one deduces that

\[
\mathfrak{b}_\lambda(u, v) = (i(\omega \cdot \nu) \text{Tr} u, j(v))_K
\]

for all \( v \in H^1(\Omega) \). Hence \( \text{Tr} u \in D(\mathcal{N}_\lambda) \) and \((\text{Tr} u, i(\omega \cdot \nu) \text{Tr} u) \in \mathcal{N}_\lambda\). Moreover,

\[
(\mathcal{N}_\lambda^\circ j(u), j(u))_{L^2(\Omega, \mathbb{C}^d)} = \mathfrak{b}_\lambda(u, u) = \int_{\partial \Omega} i(\omega \cdot \nu) |\text{Tr} u|^2 = i \int_{\partial \Omega} (\omega \cdot \nu) = i \int_{\Omega} \text{div} \omega = 0.
\]

Therefore

\[
\text{Tr} u \in \{ \varphi \in D(\mathcal{N}_\lambda) : (\mathcal{N}_\lambda^\circ \varphi, \varphi)_{L^2(\partial \Omega)} = 0 \}.
\]

It is then obvious that

\[
\text{dim} \text{span}\{ \varphi \in D(\mathcal{N}_\lambda) : (\mathcal{N}_\lambda^\circ \varphi, \varphi)_{L^2(\partial \Omega)} = 0 \} = \infty.
\]

This completes the proof of the theorem.

\[\square\]

### 3 The Stokes operators

In this section we describe the Dirichlet and the Neumann version of the Stokes operator on a Lipschitz domain with the aid of sesquilinear forms. We then apply Theorem 2.2 (abstract eigenvalues comparison theorem) to obtain a Stokes version of Friedlander’s eigenvalues theorem. For independent interest we present a description of the Stokes type Dirichlet-to-Neumann operator.

Let \( \Omega \subset \mathbb{R}^d \) be a bounded open connected set with Lipschitz boundary. We assume that \( d \geq 2 \). We denote by \( \nu \) the outward normalised normal. Let \( H = L^2(\Omega, \mathbb{C}^d) \),

\[
V = \{ u \in H^1(\Omega, \mathbb{C}^d) : \text{div} u = 0 \}
\]

and \( K = L^2(\partial \Omega, \mathbb{C}^d) \). Here and below \( \text{div} \) is the divergence operator acting on distributions. Then the inclusion \( i : V \to H \) is compact. Let \( j = \text{Tr} : V \to K \) be the trace operator. Then also \( j \) is compact. Define the form \( \mathfrak{a} : V \times V \to \mathbb{C} \) by

\[
\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \nabla v.
\]

Then \( \mathfrak{a} \) is continuous and \( i \)-elliptic. Let \( A^N \) be the self-adjoint operator in \( V \subset H \) associated with \( \mathfrak{a} \) and let \( A^D \) be the self-adjoint operator in \( \overline{V_D} \subset H \) associated with \( \mathfrak{a}|_{V_D \times V_D} \), where the closures are in \( H \) and \( V_D = \ker j \). We first characterise \( A^D \) and \( A^N \).

Define

\[
\mathcal{C}^\infty_{c, \sigma}((\Omega, \mathbb{C}^d)) = \{ u \in \mathcal{C}^\infty_0(\Omega, \mathbb{C}^d) : \text{div} u = 0 \},
\]

the space of \textbf{solenoidal test functions}, and let \( L_{2, \sigma}(\Omega) \) be the closure of \( \mathcal{C}^\infty_{c, \sigma}(\Omega) \) in \( L^2(\Omega, \mathbb{C}^d) \).

**Proposition 3.1.**

(a) \textit{The closure of the space} \( V_D \) \textit{in} \( L^2(\Omega, \mathbb{C}^d) \) \textit{is} \( L_{2, \sigma}(\Omega) \).
Lemma 3.2. Let $u, f \in L_{2,\sigma}(\Omega)$. Then $u \in D(A^D)$ and $A^D u = f$ if and only if $u \in H^{-1}(\Omega, \mathbb{C}^d)$, $
abla u = 0$ and there exists a $\pi \in L_2(\Omega)$ such that $f = -\Delta u + \nabla \pi$ in $H^{-1}(\Omega, \mathbb{C}^d)$.

Proof. (a). This follows from [Soh] Lemma II.2.2.3.
(b). The first equality is obvious. The second one follows from Tem Theorem 1.1.4.
(c). Clearly $V_D = \{ w \in H_0^1(\Omega, \mathbb{C}^d) : \nabla w = 0 \}$. Since $A^D$ is the operator associated with $a|_{V_D \times V_D}$ it follows that $u \in V_D$ and $a(u, v) = (f, v)_{L^2(\Omega, \mathbb{C}^d)}$ for all $v \in V_D$. So $u \in H_0^1(\Omega, \mathbb{C}^d)$ and $\nabla u = 0$. Consider $(f + \Delta u) \in H^{-1}(\Omega, \mathbb{C}^d)$. If $v \in H_0^1(\Omega, \mathbb{C}^d)$ and $\nabla v = 0$, then

$$(f + \Delta u)(v) = (f, v)_{L^2(\Omega, \mathbb{C}^d)} + (\Delta u, v)_{H^{-1}(\Omega, \mathbb{C}^d) \times H^1(\Omega, \mathbb{C}^d)} = (f, v)_{L^2(\Omega, \mathbb{C}^d)} - a(u, v).$$

Since $\Omega$ has a Lipschitz boundary, it follows from Soh Lemma II.2.1b that there exists a $\pi \in L_2(\Omega)$ such that $f + \Delta u = \nabla \pi$ in $H^{-1}(\Omega, \mathbb{C}^d)$.

We next consider the operator $A^N$. In order to obtain a description as in Proposition 3.1(c) for $A^D$, we first need a kind of normal derivative. If $\Omega$ is smooth, $u \in H^2(\Omega, \mathbb{C}^d)$, $\pi \in H^1(\Omega)$ and $f \in L_2(\Omega, \mathbb{C}^d)$ with $-\Delta u + \nabla \pi = f$ in $H^{-1}(\Omega, \mathbb{C}^d)$, then

$$
\langle \partial_{\nu} u - (\operatorname{Tr} \pi) \nu, \operatorname{Tr} \Phi \rangle_{H^{-1/2} \times H^{1/2}}
= \int_{\Omega} \nabla u \cdot \nabla \Phi + (\Delta u, \Phi)_{H^{-1/2} \times H^{1/2}} - \int_{\Omega} \pi \div \Phi
= \int_{\Omega} \nabla u \cdot \nabla \Phi - \int_{\Omega} \pi \div \Phi - \int_{\Omega} f \cdot \Phi
$$

for all $\Phi \in H^1(\Omega, \mathbb{C}^d)$. We next define a weak version of this normal derivative.

Lemma 3.2. Let $u \in H^1(\Omega, \mathbb{C}^d)$, $\pi \in L_2(\Omega)$ and $f \in L_2(\Omega, \mathbb{C}^d)$. Suppose that $-\Delta u + \nabla \pi = f$ in $H^{-1}(\Omega, \mathbb{C}^d)$. Then there exists a unique $F \in H^{-1/2}(\partial \Omega, \mathbb{C}^d)$ such that

$$
\langle F, \varphi \rangle_{H^{-1/2}(\partial \Omega, \mathbb{C}^d) \times H^{1/2}(\partial \Omega, \mathbb{C}^d)} = \int_{\Omega} \nabla u \cdot \nabla \Phi - \int_{\Omega} \pi \div \Phi - \int_{\Omega} f \cdot \Phi
$$

for all $\varphi \in H^{1/2}(\partial \Omega, \mathbb{C}^d)$ and $\Phi \in H^1(\Omega, \mathbb{C}^d)$ with $\operatorname{Tr} \Phi = \varphi$.

Proof. Let $\Phi \in H^1(\Omega, \mathbb{C}^d)$ and suppose that $\operatorname{Tr} \Phi = 0$. Then

$$
\int_{\Omega} \nabla u \cdot \nabla \Phi - \int_{\Omega} \pi \div \Phi - \int_{\Omega} f \cdot \Phi = \langle -\Delta u, \Phi \rangle_{H^{-1} \times H^1} + \langle \nabla \pi, \Phi \rangle_{H^{-1} \times H^1} - \int_{\Omega} f \cdot \Phi
= \langle -\Delta u + \nabla \pi - f, \Phi \rangle_{H^{-1}(\Omega, \mathbb{C}^d) \times H^1(\Omega, \mathbb{C}^d)} = 0.
$$
Hence there exists a unique \( F : H^{1/2}(\partial \Omega, \mathbb{C}^d) \to \mathbb{C} \) such that

\[
F(\varphi) = \int_{\Omega} \nabla u \cdot \nabla \Phi - \int_{\Omega} \pi \text{div} \Phi - \int_{\Omega} f \cdot \Phi
\]

for all \( \varphi \in H^{1/2}(\partial \Omega, \mathbb{C}^d) \) and \( \Phi \in H^1(\Omega, \mathbb{C}^d) \) with \( \text{Tr} \Phi = \varphi \). Obviously \( F \) is anti-linear and

\[
|F(\text{Tr} \Phi)| \leq (\|u\|_{H^1(\Omega, \mathbb{C}^d)} + \|\pi\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega, \mathbb{C}^d)})\|\Phi\|_{H^1(\Omega, \mathbb{C}^d)}
\]

for all \( \Phi \in H^1(\Omega, \mathbb{C}^d) \). Since \( \text{Tr} : H^1(\Omega, \mathbb{C}^d) \to H^{1/2}(\partial \Omega, \mathbb{C}^d) \) has a continuous right-inverse, the lemma follows.

We denote the element \( F \in H^{-1/2}(\partial \Omega, \mathbb{C}^d) \) in Lemma 3.2 by \( \partial_\nu(u, \pi) \). So

\[
\langle \partial_\nu(u, \pi), \text{Tr} \Phi \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega, \mathbb{C}^d)} = \int_{\Omega} \nabla u \cdot \nabla \Phi - \int_{\Omega} \pi \text{div} \Phi - \int_{\Omega} f \cdot \Phi
\]

for all \( u \in H^1(\Omega, \mathbb{C}^d) \), \( \pi \in L^2(\Omega) \), \( f \in L^2(\Omega, \mathbb{C}^d) \) and \( \Phi \in H^1(\Omega, \mathbb{C}^d) \) with \( -\Delta u + \nabla \pi = f \) in \( H^{-1}(\Omega, \mathbb{C}^d) \).

Let \( u \in H^1(\Omega, \mathbb{C}^d) \), \( \pi \in L^2(\Omega) \), \( f \in L^2(\Omega, \mathbb{C}^d) \) and \( c \in \mathbb{C} \) with \( -\Delta u + \nabla \pi = f \) in \( H^{-1}(\Omega, \mathbb{C}^d) \). Then \( -\Delta u + \nabla (\pi + c \mathbbm{1}_\Omega) = f \) in \( H^{-1}(\Omega, \mathbb{C}^d) \) and

\[
\partial_\nu(u, \pi + c \mathbbm{1}_\Omega) = \partial_\nu(u, \pi) - c \nu
\]

by a straightforward calculation.

We also need to characterise the range \( \text{Tr} V \) of the form domain under the trace map. Note that \( \text{Tr} : H^1(\Omega, \mathbb{C}^d) \to H^{1/2}(\partial \Omega, \mathbb{C}^d) \). If \( u \in V \), then the divergence theorem gives

\[
0 = \int_{\Omega} \text{div} u = \int_{\partial \Omega} \nu \cdot \text{Tr} u.
\]

Define

\[
L_{2,0}(\partial \Omega, \mathbb{C}^d) = \{ \varphi \in L^2(\partial \Omega, \mathbb{C}^d) : \int_{\partial \Omega} \varphi \cdot \nu = 0 \}.
\]

Then \( \text{Tr}(V) \subset H^{1/2}(\partial \Omega, \mathbb{C}^d) \cap L_{2,0}(\partial \Omega, \mathbb{C}^d) \). We next show that one actually has an equality.

**Lemma 3.3.** \( \text{Tr}(V) = H^{1/2}(\partial \Omega, \mathbb{C}^d) \cap L_{2,0}(\partial \Omega, \mathbb{C}^d) \).

**Proof.** We only have to show ‘\( \supset \)’. Let \( \varphi \in H^{1/2}(\partial \Omega, \mathbb{C}^d) \cap L_{2,0}(\partial \Omega, \mathbb{C}^d) \). There exists a \( u \in H^1(\Omega, \mathbb{C}^d) \) such that \( \text{Tr} u = \varphi \). Then \( \int_{\partial \Omega} \text{div} u = \int_{\partial \Omega} \nu \cdot \text{Tr} u = \int_{\partial \Omega} \nu \cdot \varphi = 0 \). Hence by [Soh] Lemma II.2.1.1a there exists a \( w \in H^1_0(\Omega, \mathbb{C}^d) \) such that \( \text{div} w = \text{div} u \). Set \( v = u - w \). Then \( v \in H^1(\Omega, \mathbb{C}^d) \) and \( \text{div} v = 0 \). So \( v \in V \). Moreover, \( \text{Tr} v = \text{Tr} u = \varphi \).

Although we do not need the next density lemma, we state it for completeness.

**Lemma 3.4.** The range \( \text{Tr}(V) \) is dense in \( L_{2,0}(\partial \Omega, \mathbb{C}^d) \).

**Proof.** Let \( \varphi \in L_{2,0}(\partial \Omega, \mathbb{C}^d) \). Since \( H^{1/2}(\partial \Omega, \mathbb{C}^d) \) is dense in \( L^2(\partial \Omega, \mathbb{C}^d) \), there is a sequence \( (\varphi_n)_{n \in \mathbb{N}} \) in \( H^{1/2}(\partial \Omega, \mathbb{C}^d) \) such that \( \lim \varphi_n = \varphi \) in \( L^2(\partial \Omega, \mathbb{C}^d) \). Then \( \lim_{\partial \Omega} \varphi_n \cdot \nu = \lim_{\partial \Omega} \varphi \cdot \nu = 0 \). Note that \( \nu \in L^\infty(\partial \Omega, \mathbb{C}^d) \subset L^2(\partial \Omega, \mathbb{C}^d) \). Hence as above there is a sequence \( (\nu_n)_{n \in \mathbb{N}} \) in \( H^1(\partial \Omega, \mathbb{C}^d) \) such that \( \lim \nu_n = \nu \) in \( L^2(\partial \Omega, \mathbb{C}^d) \). Then \( \lim_{\partial \Omega} \nu_n \cdot \nu = \sigma(\partial \Omega) \neq 0 \), so we may assume that \( \int_{\partial \Omega} \nu_n \cdot \nu \neq 0 \) for all \( n \in \mathbb{N} \). For all \( n \in \mathbb{N} \) define \( \tilde{\varphi}_n = \varphi_n - c_n \nu_n \), where \( c_n = \left( \int_{\partial \Omega} \nu_n \cdot \nu \right)^{-1} \int_{\partial \Omega} \varphi_n \cdot \nu \). Then \( \tilde{\varphi}_n \in H^{1/2}(\partial \Omega, \mathbb{C}^d) \cap L_{2,0}(\partial \Omega, \mathbb{C}^d) = \text{Tr}(V) \) by Lemma 3.3. Moreover, \( \lim \tilde{\varphi}_n = \varphi \) in \( L^2(\partial \Omega, \mathbb{C}^d) \).
Now we are able to characterise the operator $A^N$.

**Proposition 3.5.**

(a) The closure of the space $V$ in $L_2(\Omega, \mathbb{C}^d)$ is \[ \{ u \in L_2(\Omega, \mathbb{C}^d) : \text{div} \ u = 0 \}. \]

(b) $V^\perp = \{ \nabla \pi : \pi \in H_0^1(\Omega) \}$, where the orthogonal complement is in $L_2(\Omega, \mathbb{C}^d)$.

(c) Let $u, f \in \overline{V}^{(H)}$. Then $u \in D(A^N)$ and $A^N u = f$ if and only if $u \in H^1(\Omega, \mathbb{C}^d)$, $\text{div} \ u = 0$ and there exists a $\pi \in L_2(\Omega)$ such that $f = -\Delta u + \nabla \pi$ in $H^{-1}(\Omega, \mathbb{C}^d)$ and $\partial_v(u, \pi) = 0$.

**Proof.** (a). See [MNW] Lemma 2.1.

(b). By definition of $V$ we obtain that $V^\perp$ is the closure of $\{ \nabla \pi : \pi \in C^\infty_c(\Omega) \}$ in $L_2(\Omega, \mathbb{C}^d)$. Obviously the latter is the closure of the space $\{ \nabla \pi : \pi \in H_0^1(\Omega) \}$ in $L_2(\Omega, \mathbb{C}^d)$. By [Maz] Corollary 1.1.11 and the Dirichlet Poincaré inequality the space $\{ \nabla \pi : \pi \in H_0^1(\Omega) \}$ is closed in $L_2(\Omega, \mathbb{C}^d)$.

(c). $\Rightarrow$. By definition $u \in V$ and $a(u, v) = (f, v)_{L_2(\Omega, \mathbb{C}^d)}$ for all $v \in V$. So $u \in H^1(\Omega, \mathbb{C}^d)$ and $\text{div} \ v = 0$. Consider $(f + \Delta u) \in H^{-1}(\Omega, \mathbb{C}^d)$. If $v \in H_0^1(\Omega, \mathbb{C}^d)$ and $\text{div} \ v = 0$, then

$$(f + \Delta u)(v) = (f, v)_{L_2(\Omega, \mathbb{C}^d)} + (\Delta u, v)_{H^{-1}(\Omega, \mathbb{C}^d) \times H_0^1(\Omega, \mathbb{C}^d)} = (f, v)_{L_2(\Omega, \mathbb{C}^d)} - a(u, v) = 0.$$

Since $\Omega$ has a Lipschitz boundary, it follows from [Soh] Lemma II.2.1.1b that there exists a $\pi \in L_2(\Omega)$ such that $f + \Delta u = \nabla \pi$ in $H^{-1}(\Omega, \mathbb{C}^d)$.

If $v \in V$, then

$$\langle \partial_v(u, \pi), \text{Tr} v \rangle_{H^{-1/2}(\partial \Omega, \mathbb{C}^d) \times H^{1/2}(\partial \Omega, \mathbb{C}^d)} = \int_\Omega \nabla u \cdot \nabla v - \int_\Omega \pi \text{div} \ v - \int_\Omega f \cdot \nu$$

$$= a(u, v) - (f, v)_{L_2(\Omega, \mathbb{C}^d)} = 0.$$

So by Lemma [3.3] one deduces that $\langle \partial_v(u, \pi), \varphi \rangle_{H^{-1/2}(\partial \Omega, \mathbb{C}^d) \times H^{1/2}(\partial \Omega, \mathbb{C}^d)} = 0$ for all $\varphi \in H^{1/2}(\partial \Omega, \mathbb{C}^d) \cap L_{2,0}(\partial \Omega, \mathbb{C}^d)$.

Fix $\varphi_0 \in H^{1/2}(\partial \Omega, \mathbb{C}^d)$ such that $(\varphi_0, v)_{L_2(\partial \Omega, \mathbb{C}^d)} \neq 0$. Let $c \in \mathbb{C}$ be such that

$$\langle \partial_v(u, \pi), c \nu, \varphi_0 \rangle_{H^{-1/2}(\partial \Omega, \mathbb{C}^d) \times H^{1/2}(\partial \Omega, \mathbb{C}^d)} = 0.$$

Then it follows from (7) that

$$\langle \partial_v(u, \pi + c 1_\Omega), \varphi \rangle_{H^{-1/2}(\partial \Omega, \mathbb{C}^d) \times H^{1/2}(\partial \Omega, \mathbb{C}^d)} = 0$$

for all $\varphi \in \{ \varphi_0 \} \cup \left( H^{1/2}(\partial \Omega, \mathbb{C}^d) \cap L_{2,0}(\partial \Omega, \mathbb{C}^d) \right)$ and then by linearity for all

$$\varphi \in \text{span} \left( \{ \varphi_0 \} \cup \left( H^{1/2}(\partial \Omega, \mathbb{C}^d) \cap L_{2,0}(\partial \Omega, \mathbb{C}^d) \right) \right) = H^{1/2}(\partial \Omega, \mathbb{C}^d).$$

So $\partial_v(u, \pi + c 1_\Omega) = 0$. Replacing $\pi$ by $\pi + c 1_\Omega$ completes the proof of the implication $\Rightarrow$.

$\Leftarrow$. Let $u \in H^1(\Omega, \mathbb{C}^d)$ and $f \in \overline{V}^{(H)}$. Suppose that $\text{div} \ u = 0$ and there exists a $\pi \in L_2(\Omega)$ such that $f = -\Delta u + \nabla \pi$ and $\partial_v(u, \pi) = 0$. Then $u \in V$ and

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega \pi \text{div} \ v - \int_\Omega f \cdot \nu = \langle \partial_v(u, \pi), \text{Tr} v \rangle_{H^{-1/2}(\partial \Omega, \mathbb{C}^d) \times H^{1/2}(\partial \Omega, \mathbb{C}^d)} = 0$$


for all \( v \in H^1(\Omega, \mathbb{C}^d) \). Hence if \( v \in V \), then \( \text{div} \, v = 0 \) and
\[
\mathbf{a}(u, v) = \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f \cdot \bar{v} = (f, v)_{L^2(\Omega, \mathbb{C}^d)}.
\]
So \( u \in \text{dom}(A^N) \) and \( A^N u = f \).

We next turn to a Stokes version of the Dirichlet-to-Neumann operator. For all \( \lambda \in \mathbb{R} \) define \( b_\lambda : V \times V \to \mathbb{C} \) by
\[
b_\lambda(u, v) = \mathbf{a}(u, v) - \lambda (u, v)_H.
\]
Let \( N_\lambda \) be the self-adjoint graph in \( L_2(\partial \Omega, \mathbb{C}^d) \) associated with \( (b_\lambda, j) \). We call \( N_\lambda \) the Stokes Dirichlet-to-Neumann graph. The singular part \( N_\lambda^s \) is a self-adjoint operator in the Hilbert space \( D(N_\lambda) \), where the closure is in \( L_2(\partial \Omega, \mathbb{C}^d) \). Note that \( D(N_\lambda) \subset j(V) = \text{Tr} \, V = L_2,0(\partial \Omega, \mathbb{C}^d) \) by Lemma 3.4. We next characterise \( N_\lambda^s \) in the case where \( \lambda \) is not an eigenvalue of \( A^D \).

**Proposition 3.6.** Let \( \lambda \in \mathbb{R} \setminus \sigma(A^D) \). Let \( \varphi, \psi \in L_2,0(\partial \Omega, \mathbb{C}^d) \). Then the following are equivalent.

(i) \( \varphi \in D(N_\lambda^s) \) and \( N_\lambda^s \varphi = \psi \).

(ii) There exist \( u \in H^1(\Omega, \mathbb{C}^d) \) and \( \pi \in L_2(\Omega) \) such that

- \( \varphi = \text{Tr} \, u \),
- \( \text{div} \, u = 0 \),
- \( -\Delta u + \nabla \pi = \lambda u \) in \( H^{-1}(\Omega, \mathbb{C}^d) \), and
- \( \psi = \partial_\nu(u, \pi) \).

**Proof.** (i) \( \Rightarrow \) (ii). By definition there exists a \( u \in V \) such that \( \text{Tr} \, u = \varphi \) and \( b_\lambda(u, v) = \langle \psi, \text{Tr} \, v \rangle_{L_2(\partial \Omega, \mathbb{C}^d)} \) for all \( v \in V \). Consider \( (-\Delta u - \lambda u) \in H^{-1}(\Omega, \mathbb{C}^d) \). If \( v \in H^1_0(\Omega, \mathbb{C}^d) \) and \( \text{div} \, v = 0 \), then
\[
(-\Delta u - \lambda u)(v) = b_\lambda(u, v) = \langle \psi, \text{Tr} \, v \rangle_{L_2(\partial \Omega, \mathbb{C}^d)} = 0.
\]
Since \( \Omega \) has a Lipschitz boundary, it follows from \text{[Soh]} Lemma II.2.1.1b that there exists a \( \pi \in L_2(\Omega) \) such that \( -\Delta u - \lambda u = -\nabla \pi \) in \( H^{-1}(\Omega, \mathbb{C}^d) \).

Let \( v \in V \). Then
\[
\langle \psi, \text{Tr} \, v \rangle_{L_2(\partial \Omega, \mathbb{C}^d)} = b_\lambda(u, v) = \int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} u \cdot \overline{v} = \int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} u \cdot \overline{v} - \int_{\Omega} \nabla \pi \text{div} \, v = \langle \partial_\nu(u, \pi), \text{Tr} \, v \rangle_{H^{-1/2} \times H^{1/2}}.
\]
So \( \langle \partial_\nu(u, \pi) - \psi, \tau \rangle_{H^{-1/2} \times H^{1/2}} = 0 \) for all \( \tau \in \text{Tr} \, V = H^{1/2}(\partial \Omega, \mathbb{C}^d) \cap L_2,0(\partial \Omega, \mathbb{C}^d) \) by Lemma 3.3. Now it follows as at the end of the proof of the implication ‘\( \Rightarrow \)’ in the proof of Proposition 3.3(e) that there exists a \( c \in \mathbb{C} \) such that \( \partial_\nu(u, \pi + c 1_\Omega) = \psi \).

(ii) \( \Rightarrow \) (i). Let \( u \) and \( \pi \) be as in (ii). Note that \( u \in V \). Let \( v \in V \). Then
\[
b_\lambda(u, v) = \int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} u \cdot \overline{v} - \int_{\Omega} \nabla \pi \text{div} \, v = \langle \partial_\nu(u, \pi), \text{Tr} \, v \rangle_{H^{-1/2} \times H^{1/2}} = \langle \psi, \text{Tr} \, v \rangle_{H^{-1/2} \times H^{1/2}} = \langle \psi, \text{Tr} \, v \rangle_{L_2(\partial \Omega, \mathbb{C}^d)}
\]
and (i) follows. \( \square \)
Now we are able to state and prove the main theorem of this paper. For an explicit description of the Neumann Stokes operator and Dirichlet Stokes operator we refer to Propositions 3.5(c) and 3.1(c).

**Theorem 3.7.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded open connected set with Lipschitz boundary. Let \( A^N \) and \( A^D \) be the Neumann Stokes operator and Dirichlet Stokes operator on \( \Omega \), respectively. Further let \( \lambda_1^N \leq \lambda_2^N \leq \ldots \) and \( \lambda_1^D \leq \lambda_2^D \leq \ldots \) be the eigenvalues of \( A^N \) and \( A^D \) repeated with multiplicity, respectively. Then

\[
\lambda_{n+1}^N < \lambda_n^D
\]

for all \( n \in \mathbb{N} \).

**Proof.** We have to verify Conditions (I) and (II) of Theorem 2.2.

(I). Let \( u \in D(A^N) \cap \ker j \) and suppose that \( u \) is an eigenvector for \( A^N \) with eigenvalue \( \lambda \in \mathbb{R} \). Then \( u \in H^1_0(\Omega, C^\infty) \) and \( \operatorname{div} u = 0 \) in \( \Omega \). Let \( \tilde{u} \in H^1(\mathbb{R}^d, C^\infty) \) be the extension by zero of \( u \). Then \( \nabla \tilde{u} = \nabla u \), the extension by zero of \( \nabla u \). So \( \tilde{u} = 0 \) in \( \mathbb{R}^d \). Let \( v \in H^1(\mathbb{R}^d, C^\infty) \) and suppose that \( \operatorname{div} v = 0 \) in \( \mathbb{R}^d \). Then

\[
\int_{\mathbb{R}^d} \nabla \tilde{u} \cdot \nabla v = \int_{\Omega} \nabla u \cdot \nabla (v|_{\Omega}) = a(u, v|_{\Omega}) = (\lambda u, v|_{\Omega})_{L^2(\Omega, C^\infty)} = (\lambda \tilde{u}, v)_{L^2(\mathbb{R}^d, C^\infty)}.
\]

So \( \tilde{u} \in D(A_{\mathbb{R}^d}) \), the Stokes operator on \( \mathbb{R}^d \), and \( A_{\mathbb{R}^d} \tilde{u} = \lambda \tilde{u} \). Then \( -\Delta \tilde{u} = \lambda \tilde{u} \) by Lemma III.2.3.2. But \( \tilde{u} \) vanishes on a non-empty open set. So \( \tilde{u} = 0 \) by the unique continuation property, see for example [RS] Theorem XIII.57. Hence \( u = 0 \) and \( u \) is not an eigenvector.

(II). Let \( \lambda > 0 \). Let \( \omega \in \mathbb{R}^d \) and suppose that \( |\omega|^2 = \lambda \). Let \( b \in \mathbb{R}^d \) be such that \( |b| = 1 \) and \( b \cdot \omega = 0 \). Define \( \tau \in H^2(\Omega) \) by \( \tau(x) = e^{i\omega \cdot x} \). Then \( -\Delta \tau = |\omega|^2 \tau = \lambda \tau \). Let \( u = \tau b \in H^1(\Omega, C^\infty) \). Then \( \operatorname{div} u = i \tau (b \cdot \omega) = 0 \), so \( u \in V \). Let \( v \in V \). Then

\[
\mathbf{b}_\lambda(u, v) = \int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} u \cdot v
\]

\[
= \int_{\Omega} \nabla \tau \cdot \nabla (b \cdot v) - \lambda \int_{\Omega} \tau (b \cdot v)
\]

\[
= -\int_{\Omega} (\Delta \tau) (b \cdot v) + \int_{\partial \Omega} (\partial_\nu \tau) \operatorname{Tr} (b \cdot v) - \lambda \int_{\Omega} \tau (b \cdot v)
\]

\[
= \int_{\partial \Omega} (\partial_\nu \tau) b \cdot \operatorname{Tr} v = \int_{\partial \Omega} i (\omega \cdot \nu) (\operatorname{Tr} \tau) b \cdot \operatorname{Tr} v.
\]

Therefore \( j(u) \in D(\mathcal{N}_\lambda) \) and \( (j(u), i (\omega \cdot \nu) (\operatorname{Tr} \tau) b) \in \mathcal{N}_\lambda \). Moreover,

\[
(\mathcal{N}_\lambda^o j(u), j(u))_{L^2(\Omega, C^\infty)} = \mathbf{b}_\lambda(u, u)
\]

\[
= i \int_{\partial \Omega} (\omega \cdot \nu) |b|^2 |\operatorname{Tr} \tau|^2 = i |b|^2 \int_{\partial \Omega} (\omega \cdot \nu) = i |b|^2 \int_{\Omega} \operatorname{div} \omega = 0.
\]

Consequently

\[
\dim \text{span}\{\varphi \in D(\mathcal{N}_\lambda) : (\mathcal{N}_\lambda^o \varphi, \varphi)_{L^2(\Omega, C^\infty)} = 0\} = \infty
\]

as required. \( \square \)
We finish with a small extension of the previous theorem.

**Example 3.8.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open connected set with Lipschitz boundary. Let $H = L_2(\Omega, \mathbb{C}^d)$,
\[ V = \{ u \in H^1(\Omega, \mathbb{C}^d) : \text{div} \, u = 0 \} \]
and $K = L_2(\partial \Omega, \mathbb{C}^d)$. Then the inclusion $i : V \to H$ is compact. Let $j = \text{Tr} : V \to K$ be the trace operator. Then $j$ is compact. Fix $\alpha \in (-1, 1]$. Define the form $a : V \times V \to \mathbb{C}^d$ by
\[ a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \alpha \int_{\Omega} (\nabla u)^T \cdot \nabla v, \]
where the superscript $T$ denotes the transpose. So Theorem 3.7 corresponds to the case $\alpha = 0$. Obviously $a$ is continuous and if $|\alpha| < 1$, then $a$ is $i$-elliptic. If $\alpha = 1$, then
\[ \text{Re} \, a(u) = \frac{1}{2} \int_{\Omega} |\nabla u + (\nabla u)^T|^2 \]
for all $u \in V$. Since $\Omega$ has a Lipschitz boundary, Korn’s second inequality states that there exists a $C > 0$ such that
\[ \| u \|_{H^1(\Omega, \mathbb{C}^d)}^2 \leq C \left( \int_{\Omega} |\nabla u + (\nabla u)^T|^2 + \| u \|_{L_2(\Omega, \mathbb{C}^d)}^2 \right) \]
for all $u \in H^1(\Omega, \mathbb{C}^d)$. For an easy proof, see [Nit] Section 3. Hence the form $a$ is also $i$-elliptic if $\alpha = 1$. The associated operator $A^N$ is the Neumann Stokes operator studied in [MMW] and $A^D$ is the (Dirichlet) Stokes operator. We next show that the conditions in Theorem 2.2 are valid.

[II]. Let $u \in D(A^N) \cap \ker j$ and suppose that $u$ is an eigenvector for $A^N$ with eigenvalue $\lambda \in \mathbb{R}$. Then as before $u \in H^1_0(\Omega, \mathbb{C}^d)$ and $\text{div} \, u = 0$ in $\Omega$. Let $\tilde{u} \in H^1(\mathbb{R}^d, \mathbb{C}^d)$ be the extension by zero of $u$. Again $\nabla \tilde{u} = \tilde{\nabla} u$, the extension by zero of $\nabla u$, and $\text{div} \, \tilde{u} = 0$ in $\mathbb{R}^d$. Let $v \in H^1(\mathbb{R}^d, \mathbb{C}^d)$ and suppose that $\text{div} \, v = 0$ in $\mathbb{R}^d$. Then
\[ \int_{\mathbb{R}^d} \left( \nabla \tilde{u} + \alpha (\nabla \tilde{u})^T \right) \cdot \nabla \alpha = \int_{\Omega} \left( \nabla u + \alpha (\nabla u)^T \right) \cdot \nabla (v|_{\Omega}) = a(u, v|_{\Omega}) \]
\[ = (\lambda u, v|_{\Omega})_{L_2(\Omega, \mathbb{C}^d)} = (\lambda \tilde{u}, v)_{L_2(\mathbb{R}^d, \mathbb{C}^d)}. \]
Since $\text{div} \, \tilde{u} = 0$, one deduces that
\[ \int_{\mathbb{R}^d} (\nabla \tilde{u})^T \cdot \nabla v = 0. \]
Therefore $\tilde{u} \in D(A_{\mathbb{R}^d})$, the Stokes operator on $\mathbb{R}^d$, and $A_{\mathbb{R}^d} \tilde{u} = \lambda \tilde{u}$. Now one can argue as in the proof of Theorem 3.7 that $u = 0$ and hence $u$ is not an eigenvector.

[III]. Let $\lambda > 0$. Let $\omega \in \mathbb{R}^d$ and suppose that $|\omega|^2 = \lambda$. As before let $b \in \mathbb{R}^d$ and $\tau \in H^2(\Omega)$ be such that $|b| = 1$, $b \cdot \omega = 0$ and $\tau(x) = e^{i\omega \cdot x} \tau = -\lambda \tau$. Let $u = \tau b$. Then $\text{div} \, u = i\tau (b \cdot \omega) = 0$, so $u \in V$. Let $v \in V$. Then
\[ a_{\lambda}(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \alpha \int_{\Omega} (\nabla u)^T \cdot \nabla v - \lambda \int_{\Omega} u \cdot \nabla v \]
\[
\begin{align*}
&\int_{\Omega} \nabla \tau \cdot \nabla (b \cdot v) + \alpha \sum_{k,l=1}^{d} \int_{\Omega} i \omega_k \tau b_l \partial_l v_k - \lambda \int_{\Omega} \tau (b \cdot v) \\
&= -\int_{\Omega} (\Delta \tau) (b \cdot v) + \int_{\partial \Omega} (\partial_{\nu} \tau) \text{Tr} (b \cdot v) \\
&\quad + \alpha \sum_{k,l=1}^{d} \int_{\Omega} \omega_k \omega_l \tau b_l v_k + i \alpha \sum_{k,l=1}^{d} \int_{\partial \Omega} \omega_k (\text{Tr} \tau) \nu_l b_l \text{Tr} v_k - \lambda \int_{\Omega} \tau (b \cdot v) \\
&= \int_{\partial \Omega} (\partial_{\nu} \tau) b \cdot Tr v + i \alpha \int_{\partial \Omega} (\text{Tr} \tau) (\nu \cdot b) \omega \cdot Tr v,
\end{align*}
\]

where we used in the last step that \(b \cdot \omega = 0\). Consequently \(j(u) \in D(\mathcal{N}_\lambda)\) and \((j(u), (\partial_{\nu} \tau) b + i \alpha (\text{Tr} \tau) (\nu \cdot b) \omega) \in \mathcal{N}_\lambda^\circ\). Moreover,

\[\mathcal{N}_\lambda^\circ (j(u), j(u))_{L_2(\Omega, C_d)} = b_\lambda (u, u) = i \int_{\partial \Omega} (\omega \cdot \nu) |b|^2 |\text{Tr} \tau|^2,\]

where we used once more that \(b \cdot \omega = 0\). Then argue as in the proof of Theorem 3.7.

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