Constructing the CKM and PMNS matrices from mixing data

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Abstract

The CKM and PMNS matrices have been constructed based on the latest measurements, largely free from theoretical inputs as well as likely NP effects in the case of the former. To facilitate the construction of the CKM matrix in the PDG representation as well as in view of the comparatively large error in the measured value of the CP violating phase $\delta$, the possibility of its construction from the tree level measured CKM elements has also been explored using the unitarity triangle. In view of the persistent difference between the $|V_{ub}|$ exclusive and inclusive values, we have carried out separate analyses corresponding to these. The PMNS matrix has been constructed by incorporating the constraints due to solar and atmospheric neutrinos as well as by giving full variation to the Dirac-like CP violating phase $\delta$ and considering different values of $s_{13}$ having implications for different models of lepton mass matrices. Taking clue from quark mixing phenomenology, an analogous analysis of the leptonic unitarity triangle allows an estimate of the likely presence of CP violation in the leptonic sector.

In the last few years, several important developments have taken place in the context of phenomenology of Cabibbo-Kobayashi-Maskawa (CKM) matrix\textsuperscript{1} as well as neutrino oscillations. On the one hand, in the case of CKM phenomenology, we have entered the era of precision measurements of some of the CKM parameters like $V_{ud}, V_{us}, V_{cb}, \sin 2\beta$, etc.. In this context, based on elaborate inputs including the loop dominated parameters such as $\epsilon_K, \Delta m_d, \Delta m_s, \Delta m_K$, etc., known to be susceptible to effects of NP, several recent updates of the detailed analyses\textsuperscript{2}-\textsuperscript{5} are available which not only give insight into the CKM phenomenology but also enable one to fix the CKM matrix. On the other hand, in the case of neutrino oscillations, due to the solar neutrino experiments\textsuperscript{6}-\textsuperscript{12}, atmospheric neutrino experiments\textsuperscript{13}-\textsuperscript{15} and the reactor\textsuperscript{16} as well as accelerator\textsuperscript{17} based experiments, there have been continuous improvements in the corresponding neutrino mass
square differences and mixing angles, however these are not adequate to ‘construct’ the Pontecorvo-Maki-Nakagata-Saki (PMNS) matrix \cite{18} without further theoretical inputs.

The quark and neutrino mixing matrices are well known to be related to their corresponding mass matrices which are formulated at the GUT (Grand Unified Theories) scale, therefore a well determined mixing matrix would have deep implications for the GUTs. In the case of quarks, the relationship is fairly straightforward, however the implications of elements of the mixing matrix on those of mass matrix are not that direct. In the case of neutrinos, an additional complication arises if these are Majorana-like, with the corresponding mass matrices given by seesaw mechanism \cite{19}.

For a well determined mixing matrix to provide useful clues for formulating the appropriate mass matrices, several points have to be kept in mind. In case one uses a mixing matrix determined by global input, then one is not sure to what extent the theoretical inputs as well as the NP effects, in case they are present, affect the elements of the mixing matrix, hence correspondingly one would not be able to correlate such effects to the elements of the mass matrices. In this approach, it would also not be an easy exercise to correlate subtle features such as CP violating phase $\delta$ and the angles of the unitarity triangle, found from such a matrix, to the complex nature of the elements of the mass matrices.

To maximize the usefulness of a well determined mixing matrix one needs to adopt a step-wise approach for correlating the elements of such a matrix with those of the mass matrices. In the case of quarks, as a first step one should be able to construct the CKM matrix which is based purely on data and on minimal number of assumptions as well as is free from supposed effects of NP so as to serve as a guiding stone for formulating essentials of mass matrices. The added complications coming from subtle features of CKM phenomenology as well as effects of NP would then be included in the mass matrices as additional effects.

A data based CKM mixing matrix can be determined very well in case one has a knowledge of the CP violating phase $\delta$ along with the mixing elements $V_{us}$, $V_{cb}$ and $|V_{ub}|$ which, in the PDG representation are directly related to the three respective mixing angles. The elements $V_{us}$ and $V_{cb}$ are very well determined \cite{2}, whereas the CP violating phase $\delta$ is known with comparatively large error bars \cite{2,3,20}. In the case of $|V_{ub}|$, the exclusive and inclusive measurements have yet to approach a common value, therefore it is desirable \cite{21} to carry out a separate analysis pertaining to these, as has also been done very recently by UTfit Collaboration \cite{4}. In the case of neutrinos, the mixing matrix should preferably be determined from oscillation data, however in the absence of availability of adequate data pertaining to one of the mixing angles one needs to include some input from theoretical models as well.

The purpose of the present paper is the construction of the CKM and PMNS matrices based on the latest measurements, largely free from theoretical inputs as well as likely NP effects in the case of the former. Further, in view of the comparatively large error bars associated with the measured CP violating phase $\delta$ as well as its importance in constructing the matrix, we intend to explore the possibility of its construction from the tree level measured CKM elements. Furthermore, employing the unitarity based
parametrization as well as measured values of $V_{us}$, $V_{cb}$ and the CP violating phase $\delta$ we would like to construct the CKM matrix for both $|V_{ub}|$ exclusive and inclusive values. For the PMNS matrix, we have attempted to construct the matrix using the unitarity based PDG parametrization and presently available data from oscillation experiments. Further, we have attempted to explore the likely range of Dirac-like CP violating phase $\delta$ using $J_i$, the Jarlskog’s rephasing invariant parameter in the leptonic sector.

Most of the present day analyses, related to CKM phenomenology, have been carried out using the Wolfenstein-Buras parametrization [22] of the mixing matrix, however, in the present case we find PDG representation [2] to be more useful as the three angles are directly related to measured CKM elements. Similarly, in the case of neutrino mixing matrix, we adopt the PDG representation. Also, it may be noted that in the PDG representation the unitarity is built-in, however for finding the angles $s_{12}$, $s_{23}$, $s_{13}$ we use the measured values of the elements $V_{us}$, $V_{cb}$, $|V_{ub}|$, without invoking the unitarity constraints on these. For ready reference as well as to facilitate discussion of results, we begin with the quark mixing phenomenon, often expressed in terms of a $3 \times 3$ quark mixing matrix $V_{CKM}$ as

$$
\begin{pmatrix}
  d' \\
  s' \\
  b'
\end{pmatrix} =
\begin{pmatrix}
  V_{ud} & V_{us} & V_{ub} \\
  V_{cd} & V_{cs} & V_{cb} \\
  V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\begin{pmatrix}
  d \\
  s \\
  b
\end{pmatrix},
$$

which in the PDG representation [2] involving angles $\theta_{12}, \theta_{23}, \theta_{13}$ and the phase $\delta$ is given as

$$V_{CKM} =
\begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},
$$

with $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ for $i, j = 1, 2, 3$. In this representation, noting $c_{13} \cong 1$, one can consider $V_{us} \approx s_{12}$, $V_{cb} \approx s_{23}$ and $|V_{ub}| \approx s_{13}$, henceforth $|V_{ub}|$ would be written as $V_{ub}$.

In a similar manner, neutrino mixing PMNS matrix [18] is given by

$$
\begin{pmatrix}
  \nu_e \\
  \nu_\mu \\
  \nu_\tau
\end{pmatrix} =
\begin{pmatrix}
  U_{e1} & U_{e2} & U_{e3} \\
  U_{\mu1} & U_{\mu2} & U_{\mu3} \\
  U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\begin{pmatrix}
  \nu_1 \\
  \nu_2 \\
  \nu_3
\end{pmatrix},
$$

where $\nu_e, \nu_\mu, \nu_\tau$ are the flavor eigenstates and $\nu_1, \nu_2, \nu_3$ are the mass eigenstates. Following PDG representation analogous to the quark mixing case, involving three angles and the Dirac-like CP violating phase $\delta$ as well as the two Majorana phases $\alpha_1, \alpha_2$, the PMNS matrix $U_{PMNS}$ can be written as

$$U_{PMNS} =
\begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}
\begin{pmatrix}
  e^{i\alpha_1/2} & 0 & 0 \\
  0 & e^{i\alpha_2/2} & 0 \\
  0 & 0 & 1
\end{pmatrix}.
$$
The Majorana phases $\alpha_1$ and $\alpha_2$ do not play any role in neutrino oscillations and hence-forth would be dropped from the discussion.

In the PDG representation, as has already been mentioned, the CKM matrix can immediately be constructed in case $V_{us}$, $V_{cb}$, $V_{ub}$ as well as the CP violating phase $\delta$ are well measured. However, in view of the large error bars associated with the phase $\delta$, as a first step towards our analysis, we explore the possibility of its construction from the CKM elements measured at tree level by considering the unitarity triangle which involves these elements. We begin by considering the constraints due to unitarity of the CKM matrix, defined as

\[
\sum_{\alpha=d,s,b} V_{i\alpha}V_{j\alpha}^* = \delta_{ij}, \tag{5}
\]

\[
\sum_{i=u,c,t} V_{i\alpha}V_{i\beta}^* = \delta_{\alpha\beta}, \tag{6}
\]

where Greek indices run over the down type quarks ($d, s, b$) and Latin ones run over the up type quarks ($u, c, t$). Unitarity implies nine relations, three in terms of normalization conditions, given by equation (5), sometimes also referred to as ‘weak unitarity conditions’ and the six non-diagonal relations, given by equation (6), also expressed through the six unitarity triangles in the complex plane. Because of the strong hierarchical nature of the CKM matrix elements as well as the limitations imposed by the present level of measurements, it is difficult to study the implications of normalization relations, therefore, the six relations implied by equation (6) are used to study the implications of unitarity on CKM phenomenology. The usual unitarity triangle, also referred to as the $db$ triangle, is expressed through the relation

\[
V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0. \tag{7}
\]

Using the above expression and by replacing $b$ by $s$ we get the relation for the $ds$ triangle and by replacing $d$ by $s$ we get the relation for the $sb$ triangle. In a similar manner involving the orthogonality of the rows of the CKM matrix, we get the relations for the other three triangles, e.g., rows I and II give the expression for $uc$ triangle, rows I and III give the $ut$ triangle and rows II and III give the $ct$ triangle.

It may be noted that the well known $db$ triangle involves some elements which have not been measured directly at the tree level. In fact, only the $uc$ triangle defined as

\[
V_{ud}V_{cd}^* + V_{us}V_{cs}^* + V_{ub}V_{cb}^* = 0, \tag{8}
\]

has sides which involve CKM elements measured at the tree level. It may be noted that this triangle is highly skewed, with one side being very small as compared to the other two, therefore extracting information about the CP violating phase $\delta$ from this triangle is a non trivial job [23]. Following the procedure given in [23], the phase $\delta$ can be calculated using the Jarlskog’s rephasing invariant parameter $J$, equal to twice the area of any of
the unitarity triangle, through the relation

$$J = s_{12}s_{23}s_{13}c_{12}c_{23}c_{13}^2 \sin \delta.$$  \hfill (9)

In table 1 we present the PDG 2006 values \cite{2} of some of the CKM elements and the CP violating phase $\delta$. For the $uc$ triangle, assuming Gaussian probability density distribution for the CKM matrix elements, the distribution of $J$, shown in figure 1 for the exclusive value of $V_{ub}$, is obtained by Monte Carlo simulations. From this figure we obtain

$$J = (2.865 \pm 0.968) \times 10^{-5},$$  \hfill (10)

from a similar figure, not presented here, for the inclusive value of $V_{ub}$ we can get

$$J = (3.203 \pm 0.943) \times 10^{-5}.$$  \hfill (11)

It may be mentioned that in the figures we have considered only those points for which $J \neq 0$. The large error bars on above mentioned values are essentially due to the fact that the number of points where $J \neq 0$ is much less than the total number of possible points. The above values of $J$ have been found by fitting a Gaussian to the respective figures. Interestingly, we find that the above mentioned $J$ values are inclusive of that found by PDG group through their recent global analysis \cite{2}. The values of $\delta$ corresponding to exclusive and inclusive $V_{ub}$ found through equation (9) are also compatible with the experimentally determined $\delta$ given by PDG 2006. However, it seems that the present level of accuracy of the tree level measured CKM elements is not enough to find $\delta$ with a better accuracy as compared to its measured value, therefore its value given by PDG 2006 is being used for constructing the present CKM matrix.

The CKM matrix for the exclusive as well as inclusive values of $V_{ub}$ can now be easily constructed using the experimentally determined elements $V_{us}, V_{cb}$ and the measured CP violating phase $\delta$. For the exclusive value of $V_{ub}$ with inputs at $1 \sigma$ C.L., the matrix is as

| Parameter | PDG 2006 values | \cite{2} |
|-----------|-----------------|---------|
| $V_{ud}$  | $0.97377 \pm 0.00027$ |         |
| $V_{us}$  | $0.2257 \pm 0.0021$  |         |
| $V_{cd}$  | $0.230 \pm 0.011$    |         |
| $V_{cs}$  | $0.957 \pm 0.017 \pm 0.093$ |         |
| $V_{cb}$  | $(41.6 \pm 0.6) \times 10^{-3}$ |         |
| $V_{ub}$ (excl.) | $(3.84^{+0.67}_{-0.49}) \times 10^{-3}$ |         |
| $V_{ub}$ (incl.) | $(4.40 \pm 0.20 \pm 0.27) \times 10^{-3}$ |         |
| $\delta$ | $(63.0^{+15.0}_{-12.0})^o$ |         |

Table 1: PDG 2006 values of some of the CKM parameters used in the present analysis.
follows,

\[ V_{CKM} = \begin{pmatrix}
0.9736 \text{ to } 0.9747 & 0.2236 \text{ to } 0.2282 & 0.0035 \text{ to } 0.0041 \\
0.2234 \text{ to } 0.2280 & 0.9728 \text{ to } 0.9738 & 0.0409 \text{ to } 0.0423 \\
0.0074 \text{ to } 0.0093 & 0.0402 \text{ to } 0.0416 & 0.9990 \text{ to } 0.9991
\end{pmatrix}. \] (12)

The matrix for the inclusive value of \( V_{ub} \) as well as the average value of \( V_{ub} \), considered by PDG 2006, differs from the above matrix only in the respective \( V_{ub} \) values being 0.0043 to 0.0047 and 0.0040 to 0.0046. It may be noted that \( V_{ub} \ll V_{cb} \ll V_{us} \) ensures that refinements in \( V_{ub} \) would hardly affect the rest of the elements of the CKM matrix as well as our conclusions in this regard.

It needs to be emphasized that this matrix has been constructed by using the unitarity based PDG parametrization, however without incorporating the full constraints due to unitarity, for a detailed discussion regarding this we refer the reader to [24]. Since we have used only the experimental values of \( V_{us} \), \( V_{cb} \), \( V_{ub} \) and the CP violating phase \( \delta \), this matrix is supposedly free from effects of NP. A comparison of its elements with their predictions from models of mass matrices based on GUTs or measurements in different experiments could provide clues to the possibility of NP. The matrix constructed above is fully compatible with the matrix constructed by PDG group using global inputs. In particular, we find that the PDG mean value of the element \( V_{td} \), sensitive to both loop and NP effects, is very near to the present mean value. This value, however, looks to be somewhat higher than the ‘experimentally found’ \( V_{td} \) [2, 25], suggesting the need for further experimental scrutiny in this case.

In the context of mass matrices, such a matrix would provide the possibility of deciphering the appropriate textures which are in principal agreement with the data. This can be achieved by either finding the elements \( V_{us} \), \( V_{cb} \), \( V_{ub} \) as well as the phase \( \delta \) or by finding the entire matrix which should then be compared with the mixing matrix given in equation (12). It may also be added that such an exercise would enable one to find the phase structure of mass matrices which are in broad agreement with the data, hence would throw light on the mechanism as well as NP effects affecting the generation of phases at the GUT scale.

Coming to the case of neutrinos, adopting the three neutrino framework, several authors have presented updated information regarding the neutrino mass and mixing parameters obtained by carrying out detailed global analyses [26]. The results from solar and atmospheric neutrino experiments allow one to respectively determine the two mixing angles \( s_{12} \) and \( s_{23} \) rather well. However, situation regarding the angle \( s_{13} \) is not well defined, at present only its upper bound [27] is available. The latest situation regarding masses and mixing angles at \( 3\sigma \) C.L. is summarized as follows [26],

\[ \Delta m^2_{12} = (7.1 - 8.9) \times 10^{-5} \text{ eV}^2, \quad \Delta m^2_{23} = (2.0 - 3.2) \times 10^{-3} \text{ eV}^2, \quad (13) \]

\[ \sin^2 \theta_{12} = 0.24 - 0.40, \quad \sin^2 \theta_{23} = 0.34 - 0.68, \quad \sin^2 \theta_{13} \leq 0.040. \quad (14) \]

For the construction of the PMNS matrix, we have adopted the PDG parametrization
which satisfies unitarity by construction, however, we are not constraining the output further by imposing any of the nine unitarity relations, some of which might be amenable to experimental data. It may be noted that in the absence of any definite clues about \( s_{13} \), we have considered those values which have theoretical implications. It is very well recognized that the value of \( s_{13} \) would have deep implications for the neutrino oscillation phenomenology, in particular, very recently, Albright et al. [28] have carried out a very detailed and exhaustive analysis wherein they have studied the implications of the values of \( s_{13} \) on various leptonic and grand unified models of neutrino masses and mixings. Needless to say that the value of \( s_{13} \) would not only have implications for theoretical ideas such as quark-lepton complementarity, etc. [29]-[33] but would also determine future direction of neutrino phenomenology [34]-[36]. Keeping this in mind, we have chosen a few representative values which cover most of these attempts.

Broadly speaking, theoretical implications result in \( s_{13} \) taking values around 0.05, 0.10 and 0.15. Therefore, we have constructed the PMNS matrix by taking above values of \( s_{13} \) and attaching 20% errors along with these. For appropriate construction of the PMNS matrix, we have made use of the 3\( \sigma \) C.L. range of the two well determined mixing angles, \( s_{12} \) and \( s_{23} \), however for \( s_{13} \), we have considered the ranges \( s_{13}=(i) \ 0.05 \pm 0.01 \), \( (ii) \ 0.1 \pm 0.02 \) and \( (iii) \ 0.15 \pm 0.03 \). By giving full variation to the phase \( \delta \), we obtain the following mixing matrices corresponding to the above mentioned \( s_{13} \) ranges,

(i)

\[
U_{PMNS} = \begin{pmatrix}
0.7610 & 0.4893 & 0.4930 & 0.4930 & 0.04 & 0.06 \\
0.2745 & 0.4460 & 0.6984 & 0.5781 & 0.8111 \\
0.2820 & 0.4885 & 0.4521 & 0.6919 & 0.5695 & 0.8195
\end{pmatrix}, \quad (15)
\]

(ii)

\[
U_{PMNS} = \begin{pmatrix}
0.7575 & 0.4871 & 0.6193 & 0.08 & 0.12 \\
0.2414 & 0.4242 & 0.7148 & 0.5756 & 0.8079 \\
0.2478 & 0.5133 & 0.4296 & 0.7085 & 0.5674 & 0.8164
\end{pmatrix}, \quad (16)
\]

(iii)

\[
U_{PMNS} = \begin{pmatrix}
0.7567 & 0.4839 & 0.6152 & 0.12 & 0.15 \\
0.2086 & 0.5468 & 0.4018 & 0.7312 & 0.5718 & 0.8027 \\
0.2130 & 0.5389 & 0.4069 & 0.7256 & 0.5634 & 0.8110
\end{pmatrix}. \quad (17)
\]

In the case of quarks \( s_{13} \ll s_{23} \ll s_{12} \), therefore the elements \( V_{cd}, V_{cs}, V_{td}, V_{ts} \) of the CKM matrix are hardly affected by it, however in the case of neutrinos, although \( s_{13} \) is small as compared to the other two angles yet it is not as small as in the case of quarks, therefore we find that changes in \( s_{13} \) fairly affect the corresponding above mentioned elements of the PMNS matrix. This observation would have implications for the predictions of various models of neutrino mass matrices. It is interesting to note that the present neutrino mixing matrices are fully compatible with a very recent construction of a mixing matrix.
by Bjorken et al. [37] assuming democratic trimaximally mixed $\nu_2$ mass eigenstate as well as with the one presented by Giunti [38].

One of the key issues in the context of neutrino oscillations is the possibility of observing CP violation in the leptonic sector. In this context, an important parameter in the leptonic sector, parallel to quark mixing case, is the Jarlskog’s rephasing invariant parameter $J_l$. In most of the recent analyses of neutrino oscillation phenomenology [39, 40], it is usual to calculate the upper limit of $J_l$, however, in the present analysis, following the quark mixing analogy, we attempt to calculate the most probable range of $J_l$ keeping in mind the present uncertainties regarding the mixing angle $s_{13}$ and the Dirac-like CP violating phase $\delta$. To this end, considering each element to be Gaussian and following the same procedure as in the quark case, using the first two rows of the matrix given in equation (16), we obtain $J_l$ as

$$J_l = 0.0192 \pm 0.0079,$$

the distribution of $J_l$ has been plotted in figure 2. It may be noted that similar calculations can also be carried out using the matrices given in equations (15) and (17), however, the values of $J_l$ are not much different from the one given in equation (18). Also, we can calculate $J_l$ using other triangles as well, yielding more or less the same value. Since the input values have been taken at 3$\sigma$ C.L., therefore the calculated value of $J_l$ can be considered a fairly reasonable estimate of likely CP violation in the leptonic sector. We find the corresponding $\delta$ in the range $15^\circ - 165^\circ$. Although it cannot be predicted as a signal for CP violation, yet it does suggests that in case $s_{13} \sim 0.1$, CP violation in the leptonic sector may be there.

To summarize, we have carried out the construction of the CKM and PMNS matrices based on the latest measurements, largely free from theoretical inputs as well as likely NP effects in the case of the former. In view of the comparatively large error bars associated with the measured CP violating phase $\delta$, we have first explored the possibility of its construction from the tree level measured CKM elements. In the context of CKM matrix, employing the unitarity based parametrization, we have constructed the matrix by using the precisely measured $V_{us}, V_{cb}$, the CP violating phase $\delta$ found from data as well as $V_{ub}$ exclusive and inclusive values.

The same procedure is followed for the construction of the PMNS matrix, however, in this case we have carried out the construction for different values of $s_{13}$, having implications for different models of lepton mass matrices, as well as by giving full variation to the CP violating phase $\delta$. The Jarlskog’s rephasing invariant parameter in the leptonic sector $J_l$ has been calculated by finding the area of the leptonic unitarity triangle, its comparatively large value suggests that CP violation may be there in the leptonic sector.

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References

[1] N. Cabibbo, Phys. Rev. Lett. 10 (1963) 531; M. Kobayashi, T. Maskawa, Prog. Theor. Phys. 49 (1973) 652.

[2] W.-M. Yao et al., J. Phys. G 33 (2006) 1, updated results available at http://pdg.lbl.gov/.

[3] J. Charles et al., CKMfitter Group, Eur. Phys. J. C41 (2005) 1, updated results available at http://ckmfitter.in2p3.fr/.

[4] M. Bona et al., UTfit Collaboration, JHEP 0610 (2006) 081, updated results available at http://www.utfit.org/.

[5] E. Barberio et al., Heavy Flavor Averaging Group (HFAG), hep-ex/0603003, updated results available at http://www.slac.stanford.edu/xorg/hfag/.

[6] R. Davis, Prog. Part. Nucl. Phys. 32 (1994) 13.

[7] B. T. Cleveland et al., Astrophys. J. 496 (1998) 505.

[8] J. N. Abdurashitov et al., SAGE Collaboration, Phys. Rev. C60 (1999) 055801.

[9] C. M. Cattadori, GNO Collaboration, Nucl. Phys. Proc. Suppl. 110 (2002) 311.

[10] W. Hampel et al., GALLEX Collaboration, Phys. Lett. B447 (1999) 127.

[11] S. Fukuda et al., Super-Kamiokande Collaboration, Phys. Lett. B539 (2002) 179.

[12] Q. R. Ahmad et al., SNO Collaboration, Phys. Rev. Lett. 89 (2002) 011301; S. N. Ahmad et al., Phys. Rev. Lett. 92 (2004) 181301.

[13] Y. Fukuda et al., SuperKamiokande Collaboration, Phys. Rev. Lett. 81 (1998) 1562.

[14] A. Surdo, MACRO Collaboration, Nucl. Phys. Proc. Suppl. 110 (2002) 342.

[15] M. Sanchez, Soudan 2 Collaboration, Phys. Rev. D68 (2003) 113004.

[16] K. Eguchi et al., KamLAND Collaboration, Phys. Rev. Lett. 90 (2003) 021802; Phys. Rev. Lett. 94 (2005) 081801.

[17] M. H. Ahn et al., K2K Collaboration, Phys. Rev. Lett. 90 (2003) 041801.

[18] B. Pontecorvo, Zh. Eksp. Teor. Fiz. (JETP) 33 (1957) 549; ibid. 34 (1958) 247; ibid. 53 (1967) 1717; Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. 28 (1962) 870.
[19] P. Minkowski, Phys. Lett. B67 (1977) 421; T. Yanagida, in Proc. of Work- shop on Unified Theory and Baryon number in the Universe, eds. O. Sawada and A. Sugamoto, KEK, Tsukuba (1979); M. Gell-Mann, P. Ramond, R. Slansky, in Super- gravity, eds P. van Nieuwenhuizen and D. Z. Freedman (North Holland, Amsterdam 1980); P. Ramond, Sanibel talk, retroprinted as hep-ph/9809459. S. L. Glashow, in Quarks and Leptons, Carg’ese lectures, eds M. Levy, (Plenum, 1980, New York) p. 707; R. N. Mohapatra, G. Sen-janovic, Phys. Rev. Lett. 44 (1980) 912.

[20] A. Hocker et al., Eur. Phys. J. C21 (2001) 225.

[21] Private communication from A. Ali and I. Bigi.

[22] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945; A. J. Buras, M. E. Lautenbacher, G. Ostermaier, Phys. Rev. 50 (1994) 3433.

[23] M. Randhawa, M. Gupta, Phys. Lett. B516 (2001) 446.

[24] G. Ahuja, M. Gupta, S. Verma, M. Randhawa, hep-ph/0608074

[25] M. Okamoto, PoS LAT2005 013 (2005).

[26] J. W. F. Valle, hep-ph/0608101, hep-ph/0603223 M. Maltoni, T. Schwetz, M. A. Tortola, J. W. F. Valle, New J. Phys. 6 (2004) 122; A. Strumia, F. Vissani, hep-ph/0606054

[27] M. Appolonio et al., CHOOZ Collaboration, Eur. Phys. J. C27 (2003) 331.

[28] C. H. Albright, M. C. Chen, hep-ph/0608137

[29] H. Minakata, A. Yu. Smirnov, Phys. Rev. D70 (2004) 073009.

[30] A. S. Joshipura, A. Yu. Smirnov, Nucl. Phys. B750 (2006) 28.

[31] S. Goswami, A. Yu. Smirnov, Phys. Rev. D72 (2005) 053011.

[32] H. Fritzsch, Z. Z. Xing, Phys. Lett. B634 (2006) 514.

[33] B. Kayser, hep-ph/0605005.

[34] R. N. Mohapatra et al., hep-ph/0510213

[35] M. Lindner, A. Merle, W. Rodejohann, Phys. Rev. D73 (2006) 053005.

[36] H. Minakata, S. Uchinami, New J. Phys. 8 (2006) 143.

[37] J. D. Bjorken, P. F. Harrison, W. G. Scott, Phys. Rev. D74 (2006) 073012.

[38] C. Giunti, hep-ph/0611125.

[39] M. Fukugita, M. Tanimoto, Phys.Lett. B515 (2001) 30.

[40] M. Obara, Z. Z. Xing, hep-ph/0608280
Figure 1: Probability density distribution of $J$ for the $uc$ triangle in the quark case.

Figure 2: Probability density distribution of $J_l$ for the triangle corresponding to first two rows of the PMNS matrix.