New Invariants of Long Virtual Knots

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ABSTRACT

This paper extends the construction of invariants for virtual knots to virtual long knots and introduces two new invariant modules of virtual long knots. Several interesting features are described that distinguish virtual long knots from their classical counterparts with respect to their symmetries and the concatenation product.

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1 Introduction

Virtual knots are a generalization of classical knots introduced by L. Kauffman in 1996 [kauD]. They describe all knots in thickened closed orientable surfaces of any genera, and are in one-to-one correspondence with abstract equivalence classes (or stable equivalence classes) [kk, kauD, CKS2002, Kuper]. They are a supplement to real knots so that all Gauss codes (or Gauss diagrams) can be realized in the category of virtual knots. Consequently they are helpful for the study of some invariants, including the Jones polynomials [kamD, kamDD, kamE, kk, kauD] and finite type invariants, [GPV]. They can be also used in rack and quandle homology theory when describing a 2-cycle as a diagram [CJKS2001a, CKS2001, FRSa, FRSb, FRSc, FRSd, FRSe, Greene]. Some invariants of classical knots can be generalized to those of virtual knots, and the others cannot. Virtual knots also have their own invariants, like the JKSS polynomials (Jaeger, Kauffman, Saleur [JKS] and Sawollek [Saw]), Silver-Williams invariants [SWA], etc. and the quaternionic invariants [BaF, BuF].
A (long) virtual knot diagram is an oriented (long) knot diagram which may have encircled crossings called virtual crossings. Two diagrams are said to be equivalent if they are related by a finite sequence of generalized Reidemeister moves introduced in [kauD]. A (long) virtual knot is the equivalence class of a (long) virtual knot diagram. By closing the ends of a long virtual knot diagram, we obtain a virtual knot diagram. It induces a map from the set of long virtual knots to the set of virtual knots. However, unlike in the classical case, this map is not a bijection or even injective (see [GPV]).

In this paper we introduce four modules $\mathcal{M}_K$, $\hat{\mathcal{M}}_K$, $\check{\mathcal{M}}_K$ and $\hat{\mathcal{M}}_K$ which are invariants of long knots. The modules are left $\mathcal{F}$ modules where $\mathcal{F}$ is the algebra introduced in [BuF, BaF, F]. This has two generators $A, B$ and one relation

$$A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A.$$ 

As in these previous papers, this algebra can be represented onto more tractable algebras, say the quaternions, and invariants, usually polynomials, can be calculated from these. We do this by defining codimension $r$ determinants $\det_\eta^{(r)}(P)$ of a presentation matrix $P$ with representation $\eta$.

The definition of the module $\mathcal{M}_K$ just mimics that of the module of a virtual knot, and $\hat{\mathcal{M}}_K$ is the module of the virtual knot obtained from $K$ by closing the ends, see [F]. The module $\check{\mathcal{M}}_K$ is obtained by putting the input generator equal to zero. The module $\hat{\mathcal{M}}_K$ is obtained by putting the output generator equal to zero. Thus $\check{\mathcal{M}}_K$ and $\hat{\mathcal{M}}_K$ are the definitions that are essentially new in this paper and have an interesting feature. In fact, the determinants of these modules satisfies a product formula with respect to the concatenation product of two long virtual knots.

The second named author, R. Fenn introduced the Budapest switch ([BuF, rFJK]), augmented by $t$, that is the $2 \times 2$ matrix $S$ defined by

$$S = \begin{pmatrix} 1 + i & -tj \\ t^{-1}j & 1 + i \end{pmatrix}$$

where $i, j$ have the usual meanings as quaternions and $t$ is a central variable. The matrix is invertible and satisfies the set theoretic Yang-Baxter equation (cf. [BaF, BuF]). It follows that the first row entries, $1 + i$ and $-tj$, define a representation of the algebra $\mathcal{F}$.

In the cited papers, a method is described which defines a presentation matrix derived from a diagram of a virtual knot $K$ and this determines an invariant quaternionic module of $K$. The codimension $r$ Study determinants of the presentation matrix are invariants. The following examples are from [BaF, BuF]. For the virtual trefoil (the first figure
in p. 24 of [BuF], the quaternionic module has a presentation matrix
\[
\begin{pmatrix}
-t^2 + 2i & -1 + t(-j + k) + t^{-1}(j + k) \\
-1 + t(-j - k) + t^{-1}(j - k) & -t^{-2} + 2i
\end{pmatrix}
\]
(There is a typo in the (2,2)-entry of the matrix in p. 24 of [BuF]). The determinant is \(1 + 2t^2 + t^4\). For the Kishino knot (the second figure in p. 24 of [BuF]), the determinant is 0 and the codimension 1 determinant is \(1 + (5/2)t^2 + t^4\).

In this paper we shall use this and other representations to define invariant polynomials from the four modules.

We can also repeat this analysis for flat long knots. These are represented like long virtual knots with virtual crossings but instead of standard crossings they have flat crossings which are the projections of standard crossings. These may be interpreted as paths on an oriented surface. Once again we have four modules but these are now left modules over the Weyl algebra, [FT].

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2 Determinants from non-commuting rings

In this section we show how determinants of matrices with entries in a general ring \(R\) can be defined. Here \(R\) is a (possibly non-commutative), associative ring. The definition will depend upon a representation of \(R\) into the ring, \(M_{d,d}\Lambda\) of \(d \times d\) square matrices with entries in a commutative ring \(\Lambda\). For positive codimension determinants it will be useful if \(\Lambda\) supports a greatest common divisor function, written gcd. A good reference for this is [As].

As an illustration, consider the non-commutative ring \(R = \mathbb{H}[t, t^{-1}]\) of Laurent polynomials whose coefficients are quaternions and the variable \(t\) is central. The commutative ring is \(\Lambda = \mathbb{C}[t, t^{-1}]\), the Laurent polynomials in \(t\) with complex coefficients. The representation \(\mu : R \longrightarrow M_{2,2}(\mathbb{C}[t, t^{-1}])\) has \(d = 2\) and is defined by
\[
(\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k)t^s \mapsto \begin{pmatrix}
(\alpha_1 + \alpha_2 i)t^s & (\alpha_3 + \alpha_4 i)t^s \\
(-\alpha_3 + \alpha_4 i)t^s & (\alpha_1 - \alpha_2 i)t^s
\end{pmatrix}
\]
where \(\alpha_1, \ldots, \alpha_4 \in \mathbb{R}\) and \(s \in \mathbb{Z}\). This representation is standard and is usually used in illustrations.

3
The ring $\mathbb{C}[t, t^{-1}]$ has greatest common divisors, see [CF].

Returning now to the general case, let $\eta : R \rightarrow M_{d,d}\Lambda$ be the representation and let $P \in M_{n,m}(R)$. A square submatrix $B$ of $P$ is said to have codimension $r$ if it is obtained by deleting $n - m + r$ rows and $r$ columns if $n \geq m$ or by deleting $m - n + r$ columns and $r$ rows if $m \geq n$. For simplicity assume $m \geq n$. Let $B_1, \ldots, B_s$ be the codimension $r$ submatrices of $P$ of size $(n - r) \times (n - r)$. Consider $\eta(B_1), \ldots, \eta(B_s)$, which are $d(n - r) \times d(n - r)$ matrices whose entries belong to $\Lambda$. The \textit{codimension r \eta-determinant}, of $P$ is the gcd of the usual determinants of these matrices. We denote it by $\det_{\eta}^{(r)}(P)$. If $P$ is square then $\det_{\eta}^{(0)}(P)$ is defined even if $\Lambda$ does not possess greatest common divisors. All determinants are well defined up to multiplication by a unit.

Now suppose that $\mathcal{M}$ is a finitely presented $R$-module with presentation matrix $P \in M_{m,n}(R)$. Then the elements $\det_{\eta}^{(r)}(P)$ will be invariants of the module.

\section{3 The Invariant Modules}

Suppose that $R$ is an associative ring and

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a $2 \times 2$ matrix with entries from $R$. If $S$ is invertible and satisfies the set theoretic Yang-Baxter equation in the sense of [BaF, FJK] then $S$ is called a \textit{switch}. The universal case occurs when $R = \mathcal{F}$ and $\mathcal{F}$ has the presentation

$$\mathcal{F} = \langle A, B \mid A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A \rangle$$

and $C, D$ are defined by

$$C = A^{-1}B^{-1}A(1 - A), \quad D = 1 - A^{-1}B^{-1}AB.$$ 

Let $x_0, x_1, \ldots, x_n$ be the semi-arcs of a diagram of a long virtual knot $K$, which appear in this order along $K$. For each positive crossing, we consider a relation

$$S \begin{pmatrix} x_i \\ x_j \end{pmatrix} = \begin{pmatrix} x_{j+1} \\ x_{i+1} \end{pmatrix}$$

where $x_i$ and $x_j$ are incoming semi-arcs and $x_{j+1}$ and $x_{i+1}$ are outgoing semi-arcs such that $x_j$ and $x_{j+1}$ are under-arcs. For each negative crossing, we consider a relation that is the inverse of the positive one.

$$S \begin{pmatrix} x_{j+1} \\ x_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ x_j \end{pmatrix}$$
Of course for a virtual crossing the labeling carries over. See the following diagram, (cf. [BaF, FJK]).

\[
\begin{align*}
  x_{j+1} &= A x_i + B x_j, \quad x_{i+1} = C x_i + D x_j \\
  x_i &= A x_{j+1} + B x_{i+1}, \quad x_j = C x_{j+1} + D x_{i+1}
\end{align*}
\]

The module \( M_K \) is the \( R \)-module generated by \( x_0, x_1, \ldots, x_n \) with the relations associated with positive crossings and negative crossings. There is one more generator than relation.

The module \( \hat{M}_K \) is the quotient of \( M_K \) by an additional relation \( x_0 = x_n \). The number of generators and relations are the same. That is, the presentation matrix is square.

The module \( \hat{o}_M K \) is the quotient of \( M_K \) by an additional relation \( x_0 = 0 \). Again, the presentation matrix is square.

The module \( \hat{n}_M K \) is the quotient of \( M_K \) by an additional relation \( x_n = 0 \). The generator \( x_n \) is the label on the outgoing arc. Again, the presentation matrix is square.

**Theorem 3.1** Suppose \( A, 1 - A \) and \( B \) are invertible. Then the modules \( M_K \), \( \hat{M}_K \), \( \hat{o}_M K \) and \( \hat{n}_M K \) are invariants of a long virtual knot \( K \).

**Proof:** Since \( S \) is invertible, satisfies the set theoretic Yang-Baxter equation, and since \( 1 - A \) is invertible these module are preserved by all generalized Reidemeister moves (see [BaF, FJK]). \( \square \)

As an illustration consider the “fly” long knot, \( F \), pictured below.
The presentation matrices in the four cases are

\[
\mathcal{M} = \begin{pmatrix}
-1 & B & 0 & A & 0 \\
0 & D & -1 & C & 0 \\
0 & A & 0 & B & -1 \\
0 & C & -1 & D & 0
\end{pmatrix}, \quad \tilde{\mathcal{M}} = \begin{pmatrix}
-1 & B & 0 & A & 0 \\
0 & D & -1 & C & 0 \\
0 & A & 0 & B & -1 \\
0 & C & -1 & D & 0 \\
1 & 0 & 0 & 0 & -1
\end{pmatrix},
\]

\[
\mathcal{\hat{M}} = \begin{pmatrix}
-1 & B & 0 & A & 0 \\
0 & D & -1 & C & 0 \\
0 & A & 0 & B & -1 \\
0 & C & -1 & D & 0 \\
1 & 0 & 0 & 0 & -1
\end{pmatrix}, \quad \tilde{\mathcal{\hat{M}}} = \begin{pmatrix}
-1 & B & 0 & A & 0 \\
0 & D & -1 & C & 0 \\
0 & A & 0 & B & -1 \\
0 & C & -1 & D & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

4 The Invariant Polynomials

Let \( K \) be a long knot and let \( \mathcal{M}_K, \tilde{\mathcal{M}}_K, \mathcal{\hat{M}}_K \) and \( \mathcal{\hat{M}}_K \) be the \( \mathcal{F} \)-modules defined in the previous section. Let \( P, \tilde{P}, \mathcal{\hat{P}} \) and \( \mathcal{\hat{P}} \) be the respective presentation matrices. Suppose we now represent the algebra as matrices so that we can define determinantal invariants as described in section 2. Each entry in the \( P \) matrices is a \( d \times d \) matrix for some \( d \).

Let \( p^{(r)}_K, \tilde{p}^{(r)}_K, \mathcal{\hat{p}}^{(r)}_K \) and \( \mathcal{\hat{p}}^{(r)}_K \) be the corresponding determinants in the commutative ring \( \Lambda \), with codimension \( r = 0, 1, 2 \ldots \).

Let \( \hat{K} \) be the closure of the long knot \( K \). Then \( \hat{K} \) also has an invariant \( \mathcal{F} \)-module, see [F]. Let \( q^{(r)}_{\hat{K}} \) be the sequence of determinants in \( \Lambda \), corresponding to the presentation.

**Theorem 4.2**

1. \( q^{(r)}_{\hat{K}} = \tilde{p}^{(r)}_K \)

2. \( p^{(r)}_K \) divides \( \tilde{p}^{(r)}_K \).

3. \( p^{(r)}_K \) divides \( \mathcal{\hat{p}}^{(r)}_K \).

**Proof.**

1. This follows since \( \tilde{\mathcal{M}}_K \) is equal to the module of the closure \( \hat{K} \) of \( K \).

2. The module \( \mathcal{M}_K \) has an \( n \times (n + 1) \) matrix \( P \) as a presentation matrix such that the first column corresponds to \( x_0 \) and the last column corresponds to \( x_n \). A codimension \( r \)
submatrix is obtained from $P$ by deleting $r$ rows and $r + 1$ columns. Let $B_1, \ldots, B_s$ be the codimension $r$ submatrices of $P$. Then $p^{(r)}_K$ divides the determinant of all of these after the representation and is the largest, by division, element which does so.

The module $\mathcal{M}_K$ has an $(n+1) \times (n+1)$ presentation matrix $P$ that is obtained from $P$ by adding the row $(1, 0, \ldots, 0)$ to the bottom. The presentation matrix $P$ is simplified to $Q$ that is an $n \times n$ matrix obtained from $P$ by deleting the first column and the bottom row, which is obtained from $P$ by deleting the first column. Since $Q$ is square a codimension $r$ submatrix is obtained from $Q$ by deleting $r$ rows and $r$ columns. It follows that $p^{(r)}_K$ is the gcd of a subset of the values for which $p^{(r)}_K$ is the gcd. The result now follows.

(3) The proof is similar to (2).

**Theorem 4.3** Let $K_1 \cdot K_2$ be the concatenation product of two long virtual knots $K_1$ and $K_2$. For any suitable representation of the fundamental modules

$$\mathcal{H}^0 (K_1 \cdot K_2) = \mathcal{H}^0 (K_1) \mathcal{H}^0 (K_2)$$

and

$$\mathcal{H}^0 (K_1 \cdot K_2) = \mathcal{H}^0 (K_1) \mathcal{H}^0 (K_2).$$

**Proof:** Let $P_1 = (a_0 a_1 \cdots a_n)$ and $P_2 = (b_0 b_1 \cdots b_{n'})$ be the presentation matrices of $\mathcal{M}_{K_1}$ and $\mathcal{M}_{K_2}$ associated with their diagrams. Then $\mathcal{M}_{K_1}$, $\mathcal{M}_{K_2}$ and $\mathcal{M}_{K_1 \cdot K_2}$ have presentation matrices

$$(a_1 \cdots a_n), \quad (b_1 \cdots b_{n'}), \quad \begin{pmatrix} a_1 \cdots a_{n-1} & a_n & 0 \cdots 0 \\ 0 \cdots 0 & b_0 & b_1 \cdots b_{n'} \end{pmatrix}$$

respectively. Thus we have the result for $\mathcal{H}$. The proof for $\mathcal{H}$ is similar. \qed

5 Simplifying the Modules and some Calculations

The presentation matrices defined above can be simplified by the usual rules for manipulating non-commuting relations. That is

1. Interchange any row(column).

2. Multiply any row(column) on the left(right) by a unit.
3 Add any row(column) multiplied on the left(right) to a different row(column).

4 Introduce or delete any zero row.

5 \[ P \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \]

If we apply these rules to the example in section 3 and simplify as far as possible we get,

\[ (D - C \quad D - C), \quad (0), \quad ((C - D)(1 + B^{-1}A)), \quad ((D - C)(1 + B^{-1}A)) \]

for the four presentation matrices. Note that the presentation matrix for \( \widehat{M}_K \) will reduce to zero since the closure of \( B \) is the trivial knot.

Now apply the homomorphism which replaces \( A, B, C, D \) with \( 1 + i, -tj, t^{-1}j, 1 + i \) respectively and use the standard representation. The three codimension zero polynomial invariants are \(|D - C|^2 = 2 + t^{-2}, 0 \) and \(|(C - D)(1 + B^{-1}A)|^2 = (2 + t^{-2})(1 + 2t^{-2})\). All the higher codimension polynomials are 1.

6 Symmetries of Long Virtual Knots

There are various symmetries of the knot diagram which can be applied. Consider reflection in the plane of a knot diagram \( D \). Let \(-D\) denote the resulting diagram. This interchanges plus and minus crossings. Let \( D \) denote the result of reflection in the \( x \)-axis. Finally let \( D^* \) be obtained by reversing the arrow and rotating the result through 180 degrees.

For the fly \( F^* = \overline{F} \). The effect of the other three possibilities on the fly are illustrated below.

![Diagram](image)

The fly reflected: \(-F, \quad \overline{F}, \quad -\overline{F}\)

Using a suitable representation of the fundamental algebra all three can be distinguished from themselves and the original fly as follows.

The resulting polynomials are tabulated as follows. The switch used is given by a representation of the quantum Weyl algebra, with \( A, B, C, D \) the following \(2 \times 2\) matrices.
\[
A = \begin{pmatrix} 1-q & -q^3 + 2q^2 - 1 \\ 0 & 1-q \end{pmatrix}, \quad B = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}
\]

\[
C = \begin{pmatrix} 1 & (\frac{-q^4 + 3q^3 - 2q^2 - 2q + 1}{q}) \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & (\frac{q^3 - 2q^2 + 1}{q}) \\ 0 & 0 \end{pmatrix}
\]

Theorem 6.4 Consider the following three conditions on a switch \( S \). a) \( S = S^\dagger \), b) \( S^2 = 1 \), c) \( SS^\dagger = 1 \), where \( S^\dagger = \begin{pmatrix} D & C \\ C & A \end{pmatrix} \).

If a) then \( A = D \) and \( B = C \) and \( K \) cannot be distinguished from \(-K\).

If b) then the underlying algebra is the Weyl algebra and \( K \) cannot be distinguished from \(-K \) or \( K^* \).

If c) then \( A, B \) commute and \( S \) is \( \begin{pmatrix} 2 & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \), a specialization of the Alexander switch.

Moreover \( K \) cannot be distinguished from \( K^\dagger \).

Proof: Most of the results easily follow by looking at the conditions on the entries of \( S \) and how this affects the calculations of the modules.

For b) the underlying algebra is the Weyl algebra because of results in [FT].

For c) the condition \( SS^\dagger = 1 \) implies

\[
AD + B^2 = \frac{1}{CD} \quad AC = -BA \quad C^2 + DA = 1.
\]

Using the relations

\[
C = A^{-1}B^{-1}A(1-A), \quad D = 1 - A^{-1}B^{-1}AB
\]
gives the result. □

It is well known that the product of two classical knots is a commutative operation. This is not the case for the product of two long virtual knots. For example the products \( F \cdot \bar{F} \) and \( \bar{F} \cdot F \) are distinct. A calculation using the Budapest switch shows that for \( F \cdot \bar{F} \), \( p^{(0)} = 6t^4 + 15t^2 + 6 \) whereas for \( \bar{F} \cdot F \) we have \( p^{(0)} = 3t^4 + 15/2t^2 + 3 \), which is half the previous polynomial. If we are working over the integer quaternions then 2 is not a unit and so this shows the knots are distinct.
This is not perhaps a "killer" example. If we consider 
\[(F \cdot \bar{F}) \cdot F\] and \[F \cdot (F \cdot \bar{F})\]
then in both cases \(p^{(0)} = -12t^8 - 60t^6 - 99t^4 - 60t^2 - 12\) but the values of \(p^{(1)}\) are \(9(t^2 + 1)\) and \(-9/2(2t^2 + 1)(t^2 + 2)\) respectively.

7 Long flat virtual knots

We now repeat the previous analysis for long flat knots. For a full discussion of the following see [FT]. A switch \(S\) can be used provided it satisfies \(S^2 = id\). The conditions for this are contained in the following theorem.

**Theorem 7.5** Suppose \(A, 1 - A\) and \(B\) are invertible and

\[S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\]

is a \(2 \times 2\) matrix with entries satisfying

\[A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A = 1\]

and \(C, D\) are defined by

\[C = A^{-1}B^{-1}A(1 - A), \quad D = 1 - A^{-1}B^{-1}AB.\]

Then \(S^2 = 1\) if and only if \(u = B, v = B^{-1}A^{-1}\) satisfy \(uv - vu = 1\). (The elements \(u, v\) are the generators of the Weyl algebra.) \(\square\)

We now repeat the construction considered earlier and arrive at modules \(\mathcal{WM}_F, \mathcal{WM}_F, \mathcal{WM}_F^\circ\) and \(\mathcal{WM}_F^n\) which are invariants of a long flat knot \(F\).

Again by analogy we can find tractible invariants given representations onto finite matrices. Many examples are given in [FT].

The following is an example with ring \(\mathbb{Z}_2[a, x, y]\).

\[u = \begin{pmatrix} x & a \\ 0 & x \end{pmatrix}, \quad v = \begin{pmatrix} y & 0 \\ 1/a & y \end{pmatrix}\]

Consider the "flat fly" denoted by \(FF\) and its reflection in the \(x\)-axis illustrated below.
Using the representation above, setting \( y = x \) and \( a = 1 \), the codimension zero polynomials are

| \( K \) | \( p^{(0)}(K) \) | \( \hat{p}^{(0)}(K) \) | \( \bar{p}^{(0)}(K) \) | \( \bar{p}^{(0)}(K) \) |
|------|----------------|----------------|----------------|----------------|
| \( d(FF) \) | \( x^2 + 1 \) | 0 | \( x^8 + x^6 + x^2 + 1 \) | \( x^8 + x^6 + x^2 + 1 \) |
| \( d(\overline{FF}) \) | \( x^6 + 1 \) | 0 | \( x^8 + x^6 + x^2 + 1 \) | \( x^8 + x^6 + x^2 + 1 \) |

This shows that \( FF \) and \( \overline{FF} \) are both non-trivial and distinct.

It is interesting to note Turaev’s descent map \( d \) of long flat knots to long virtual knots as an alternative method to show that \( FF \) is non-trivial. This lifts flat knots by turning the first time a crossing is met to an overcrossing. For example \( FF \) and \( \overline{FF} \) are converted as shown in the following diagram.

Then, using the same switch as in the previous example, the polynomials for \( d(FF) \) and \( d(\overline{FF}) \) are also

| \( K \) | \( p^{(0)}(K) \) | \( \hat{p}^{(0)}(K) \) | \( \bar{p}^{(0)}(K) \) | \( \bar{p}^{(0)}(K) \) |
|------|----------------|----------------|----------------|----------------|
| \( d(FF) \) | \( x^2 + 1 \) | 0 | \( x^8 + x^6 + x^2 + 1 \) | \( x^8 + x^6 + x^2 + 1 \) |
| \( d(\overline{FF}) \) | \( x^6 + 1 \) | 0 | \( x^8 + x^6 + x^2 + 1 \) | \( x^8 + x^6 + x^2 + 1 \) |

Our invariants are consequence of biquandles \([FJK]\). When we deform a given virtual knot or a given long virtual knot into a braid form, it is easier to calculate the biquandle, the quaternionic module and the determinant invariants. For braiding of virtual knots and long virtual knots, refer to \([SkamA, SkamB, kaulam]\).
As H. Aslaksen: *Quaternionic determinants*, Math. Intel. **18** (1996), 1-19

BaF A. Bartholomew and R. Fenn: *Quaternionic invariants of virtual knots and links*, preprint.

BuF S. Budden and R. Fenn: *The equation \([B, (A - 1)(A, B)] = 0\) and virtual knots and links*, Fund. Math. **184** (2004), 19–29.

CJKS2001a J. S. Carter, D. Jelsovsky, S. Kamada and M. Saito: *Quandle homology groups, their Betti numbers, and virtual knots*, J. Pure Appl. Algebra **157** (2001), 135–155.

CKS2001 J. S. Carter, S. Kamada and M. Saito: *Geometric interpretations of quandle homology*, J. Knot Theory Ramifications **10** (2001), 345–386.

CKS2002 J. S. Carter, S. Kamada and M. Saito: *Stable equivalence of knots on surfaces and virtual knot cobordisms*, J. Knot Theory Ramifications **11** (2002), 311–322.

CF R. H. Crowell and R. H. Fox: *An introduction to knot theory*, Ginn and Co, (1963).

F R. Fenn: *Quaternion algebras and invariants of virtual knots and links I*, to appear in JKTR

FKJ R. Fenn, M. Jordan, and L. Kauffman: *Biquandles and virtual links*, Topology Appl. **145** (2004), 157–175.

FT R. Fenn and V. Turaev: *Weyl Algebras and Knots*, J. Geometry and Physics **57** (2007), 1313-1324

FRSa R. Fenn, C. Rourke and B. Sanderson: *An introduction to species and the rack space*, “Topics in Knot Theory” (M. E. Bozhuyu, ed.), Kluwer Academic, pp. 33–55, (1993).

FRSb R. Fenn, C. Rourke and B. Sanderson: *Trunks and classifying spaces*, Appl. Categ. Structures **3** (1995), 321–356.

FRSc R. Fenn, C. Rourke and B. Sanderson: *James bundles and applications*, preprint (1996), available at [http://www.maths.warwick.ac.uk/~cpr/ftp/james.ps](http://www.maths.warwick.ac.uk/~cpr/ftp/james.ps)

FRSd R. Fenn, C. Rourke and B. Sanderson: *The rack space*, TAMS, 359 (2007) 701-740
preprint (2003), available at arXiv: math.GT/0304228

**FRSe** R. Fenn, C. Rourke and B. Sanderson: *James bundles*, Proc. London Math. Soc. 89 (2004), 217–240.

**GPV** M. Goussarov, M. Polyak, and O. Viro: *Finite-type invariants of classical and virtual knots*, Topology 39 (2000), 1045–1068.

**Greene** M. T. Greene: *Some results in geometric topology and geometry*, Ph.D. Dissertation, Warwick (1997).

**JKS** F. Jaeger, L. H. Kauffman, and H. Saleur: *The conway polynomial in $R^3$ and in thickened surfaces: A new determinant formulation*, J. Combin. Theory Ser. B 61 (1994), 237–259.

**kamD** N. Kamada: *The crossing number of alternating link diagrams of a surface*, Proceedings of Knots 96, World Scientific Publishing Co., 1997, 377–382.

**kamDD** N. Kamada: *Span of the Jones polynomial of an alternating virtual link*, Algebr. Geom. Topol. 4 (2004), 1083–1101.

**kamE** N. Kamada: *A relation of Kauffman’s f-polynomials of virtual links*, Topology and its Application 146–147 (2005), 123–132.

**kk** N. Kamada and S. Kamada: *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications 9 (2000), 93–106.

**SkamA** S. Kamada: *Braid presentation of virtual knots and welded knots*, Osaka J. Math., to appear, available at arXiv: math.GT/0008092

**SkamB** S. Kamada: *Invariants of virtual braids and a remark on left stabilizations and virtual exchange moves*, Kobe J. Math. 21 (2004), 33–49.

**kauD** L. H. Kauffman: *Virtual knot theory*, Europ. J. Combinatorics 20 (1999) 663–690.

**Kuper** G. Kuperberg: *What is a virtual link?,* Algebr. Geom. Topol. 3 (2003), 587–591.

**Saw** J. Sawollek: *On Alexander-Conway polynomials for virtual knots and links*, preprint (1999), available at arXiv: math.GT/9912173
SWb D. S. Silver and S. G. Williams, *Alexander groups and virtual links*, preprint.

SWA D. Silver and S. Williams: *Polynomial invariants of virtual links*, J. Knot Theory Ramifications 12 (2003), 987–1000.