Abstract

We continue the enumeration of plane lattice paths avoiding the negative quadrant initiated by the first author in [1]. We solve in detail a new case, the king walks, where all 8 nearest neighbour steps are allowed. As in the two cases solved in [1], the associated generating function is proved to differ from a simple, explicit D-finite series (related to the enumeration of walks confined to the first quadrant) by an algebraic one. The principle of the approach is the same as in [1], but challenging theoretical and computational difficulties arise as we now handle algebraic series of larger degree.

We also explain why we expect the observed algebraicity phenomenon to persist for 4 more models, for which the quadrant problem is solvable using the reflection principle.
1 Introduction

In this paper we continue the enumeration of plane lattice paths confined to non-convex cones initiated by the first author in [1]. Therein the two most natural models of walks confined to the three-quadrant cone $\mathcal{C} := \{(i,j) : i \geq 0 \text{ or } j \geq 0\}$ were studied: walks with steps $\{\to, \uparrow, \leftarrow, \downarrow\}$, and those with steps $\{\nearrow, \searrow, \swarrow, \nwarrow\}$. In both cases, the generating function that counts walks starting at the origin was proved to differ (additively) from a simple explicit D-finite series by an algebraic one. The tools essentially involved power series manipulations, coefficient extractions, and polynomial elimination.

Later, Raschel and Trotignon gave in [13] sophisticated integral expressions for 8 models, which imply that 3 additional models ({$\nearrow, \leftarrow, \swarrow$}, {$\to, \uparrow, \swarrow$}, and {$\to, \nearrow, \leftarrow, \swarrow$}) are D-finite. Their results use an analytic approach inspired by earlier work on probabilistic and enumerative aspects of quadrant walks [5, 12].

In this paper we first extend the results of [1] to the so-called king walks, which take their steps from {$\to, \nearrow, \leftarrow, \swarrow, \searrow, \nwarrow, \downarrow$}. We show that the algebraicity phenomenon of [1] persists: if $Q(x,y;t)$ (resp. $C(x,y;t)$) counts walks starting from the origin that are confined to the non-negative quadrant $Q:= \{(i,j) : i \geq 0 \text{ and } j \geq 0\}$ (resp. to the cone $C$) by the length (variable $t$) and the coordinates of the endpoint (variables $x,y$), then $C(x,y;t)$ differs from the series

$$\frac{1}{3} \left( Q(x,y;t) - Q(1/x,y;t)/x^2 - Q(x,1/y;t)/y^2 \right)$$

by an algebraic series, as detailed in our main theorem below. Moreover, we expect a similar property to hold (with variations on the above linear combination of the series $Q$) for the 7 step sets of Figure 1, related to reflection groups, and for which the quadrant problem can be solved using the reflection principle [7]. However, we also expect the effective solution of these models to be extremely challenging in computational terms, mostly, because the relevant algebraic series have very large degree. This is illustrated by our main theorem below. There, and in the sequel, we use the shorthand $\bar{x} = 1/x$, $\bar{y} = 1/y$, and omit in the notation the dependencies on $t$, writing for instance $Q(x,y)$ instead of $Q(x,y;t)$.

**Theorem 1.** Take the step set $\{-1,0,1\}^2 \setminus \{(0,0)\}$ and let $Q(x,y)$ be the generating function of lattice walks starting from $(0,0)$ that are confined to the first quadrant $Q$ (this series is D-finite and given in [3]). Then, the generating function of walks starting from $(0,0)$, confined to $C$, and ending in the first quadrant (resp. at a negative abscissa) is

$$\frac{1}{3}Q(x,y) + P(x,y), \quad (\text{resp. } -\frac{\bar{x}^2}{3}Q(\bar{x},y) + \bar{x}M(\bar{x},y)), \quad (1)$$

where $P(x,y)$ and $M(x,y)$ are algebraic of degree 216 over $Q(x,y,t)$. Of course, the generating function of walks ending at a negative ordinate follows, using the $x/y$-symmetry.

The series $P$ is expressed in terms of $M$ by:

$$P(x,y) = \bar{x} \left( M(x,y) - M(0,y) \right) + \bar{y} \left( M(y,x) - M(0,x) \right), \quad (2)$$

and $M$ is defined by the following equation:

$$K(x,y) \left( 2M(x,y) - M(0,y) \right) = \frac{2x}{3} - 2t\bar{y}(x+1+\bar{x})M(x,0) + t\bar{y}(y+1+\bar{y})M(y,0) + t(x-\bar{x})(y+1+\bar{y})M(0,y) - t \left( 1 + \bar{y}^2 - 2x\bar{y} \right) \bar{y}M(0,0) - tyM_x(0,0), \quad (3)$$
where $K(x, y) = 1 - t(x + xy + y + \bar{x}y + \bar{x} + \bar{y} + x\bar{y})$. The specializations $M(x, 0)$ and $M(0, y)$ are algebraic each of degree 72 over $\mathbb{Q}(x, t)$ and $\mathbb{Q}(y, t)$, respectively, and $M(0, 0)$ and $M_x(0, 0)$ have degree 24 over $\mathbb{Q}(t)$.

![Figure 1](image_url) The seven step sets to which the strategy of this paper should apply. The first two are solved in [1], the third one in this paper.

We have moreover a complete algebraic description of all the series needed to reconstruct $P(x, y)$ and $M(x, y)$ from (2) and (3), namely the univariate series $M(0, 0)$ and $M_x(0, 0)$, and the bivariate series $M(x, 0)$ and $M(0, y)$. In particular, both univariate series lie in the extension of $\mathbb{Q}(t)$ (the field of rational functions in $t$) generated in 3 steps as follows: first, $u = t + t^2 + \mathcal{O}(t^3)$ is the only series in $t$ satisfying

$$\begin{align*}
(1 - 3u)^3(1 + u)t^2 + (1 + 18u^2 - 27u^4)t - u &= 0, \\
\text{then } v &= t + 3t^2 + \mathcal{O}(t^3)\text{ is the only series with constant term zero satisfying} \\
(1 + 3v - v^3)u - v(v^2 + v + 1) &= 0, \\
\text{and finally} \\
w &= \sqrt{1 + 4v - 4v^3 - 4v^4} = 1 + 2t + 4t^2 + \mathcal{O}(t^3).
\end{align*}$$

Schematically, $\mathbb{Q}(t) \xrightarrow{\Delta_1} \mathbb{Q}(t, u) \xrightarrow{\Delta_2} \mathbb{Q}(t, v) \xrightarrow{\Delta_3} \mathbb{Q}(t, w)$. Of particular interest is the series $M(0, 0)$: by (1), this is also the series $C_{-1,0}$ that counts by the length walks in $\mathcal{C}$ ending at $(-1, 0)$. It is algebraic, as conjectured in [13], and given by

$$M(0, 0) = C_{-1,0} = \frac{1}{2t} \left( \frac{w(1 + 2v)}{1 + 4v - 2v^3} - 1 \right) = t + 2t^2 + 17t^3 + 80t^4 + 536t^5 + \mathcal{O}(t^6).$$

Due to the lack of space, the extensions of $\mathbb{Q}(x, t)$ generated by $M(x, 0)$ and $M(0, x)$ will only be described in the long version of this paper.

Once the series $C(x, y)$ is determined, we can derive detailed asymptotic results, which refine general results of Denisov and Wachtel [4] and Mustapha [11] (who only obtain the following estimates up to a multiplicative factor).

**Corollary 2.** The number $c_{0,0}(n)$ of $n$-step king walks confined to $\mathcal{C}$ and ending at the origin, and the number $c(n)$ of walks of $\mathcal{C}$ ending anywhere satisfy for $n \to \infty$:

$$
c_{0,0}(n) \sim \frac{2^{29}K^{1/3}}{3^7} \frac{\Gamma(2/3)}{\pi} \frac{8^n}{n^{5/3}},
$$

$$
c(n) \sim \frac{2^{32}K^{1/6}}{3^7} \frac{1}{\Gamma(2/3)} \frac{8^n}{n^{1/3}},
$$

where $K$ is the unique real root of $101^6 K^3 - 601275603K^2 + 92811K - 1$. 

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Outline of the paper
We begin in Section 2 with a general discussion on models of walks with small steps confined to the cone $C$, and on the related functional equations. The main part of the paper, Section 3, is devoted to the solution of the king model. We sketch in the final Section 4 what should be the starting point for the 4 rightmost models of Figure 1.

Some definitions and notation
Let $A$ be a commutative ring and $x$ an indeterminate. We denote by $A[x]$ (resp. $A[[x]]$) the ring of polynomials (resp. formal power series) in $x$ with coefficients in $A$. If $A$ is a field, then $A(x)$ denotes the field of rational functions in $x$, and $A[[x]]$ the field of Laurent series in $x$, that is, series of the form $\sum_{n \geq n_0} a_n x^n$, with $n_0 \in \mathbb{Z}$ and $a_n \in A$. The coefficient of $x^n$ in a series $F(x)$ is denoted by $[x^n]F(x)$.

This notation is generalized to polynomials, fractions, and series in several indeterminates. If $F(x, x_1, \ldots, x_d)$ is a series in the $x_i$’s whose coefficients are Laurent series in $x$, say

$$F(x, x_1, \ldots, x_d) = \sum_{i_1, \ldots, i_d} x_1^{i_1} \cdots x_d^{i_d} \sum_{n \geq n_0(i_1, \ldots, i_d)} a(n, i_1, \ldots, i_d) x^n,$$

then the non-negative part of $F$ in $x$ is the following formal power series in $x, x_1, \ldots, x_d$:

$$[x^{\geq 0}]F(x, x_1, \ldots, x_d) = \sum_{i_1, \ldots, i_d} x_1^{i_1} \cdots x_d^{i_d} \sum_{n \geq 0} a(n, i_1, \ldots, i_d) x^n.$$

We define similarly the negative part of $F$, its positive part, and so on. We denote with bars the reciprocals of variables: that is, $\bar{x} = 1/x$, so that $A[x, \bar{x}]$ is the ring of Laurent polynomials in $x$ with coefficients in $A$.

If $A$ is a field, a power series $F(x) \in A[[x]]$ is algebraic (over $A(x)$) if it satisfies a non-trivial polynomial equation $P(x, F(x)) = 0$ with coefficients in $A$. It is differentially finite (or D-finite) if it satisfies a non-trivial linear differential equation with coefficients in $A(x)$. For multivariate series, D-finiteness requires the existence of a differential equation in each variable. We refer to [8, 9] for general results on D-finite series.

As mentioned above, we usually omit the dependency in $t$ of our series. For a series $F(x, y; t) \in \mathbb{Q}[x, \bar{x}, y, \bar{y}][[t]]$ and two integers $i$ and $j$, we denote by $F_{i,j}$ the coefficient of $x^i y^j$ in $F(x, y; t)$. This is a series in $\mathbb{Q}[[t]]$.

2 Enumeration in the three-quarter plane
We fix a subset $S$ of $\{ -1, 0, 1 \}^2 \setminus \{(0, 0)\}$ and we want to count walks with steps in $S$ that start from the origin $(0, 0)$ of $\mathbb{Z}^2$ and remain in the cone $C := \{(x, y) : x \geq 0 \text{ or } y \geq 0\}$. By this, we mean that not only must every vertex of the walk lie in $C$, but also every edge: a walk containing a step from $(-1, 0)$ to $(0, -1)$ (or vice versa) is not considered as lying in $C$. We often say for short that our walks avoid the negative quadrant. The step polynomial of $S$ is defined by

$$S(x, y) = \sum_{(i,j) \in S} x^i y^j = \bar{y} H_-(x) + H_0(x) + y H_+(x) = \bar{x} V_-(y) + V_0(y) + x V_+(y),$$

for some Laurent polynomials $H_-, H_0, H_+$ and $V_-, V_0, V_+$ (of degree at most 1 and valuation at least $-1$) recording horizontal and vertical displacements, respectively. We denote by
\[ C(x,y,t) \equiv C(x,y) \text{ the generating function of walks confined to } \mathcal{C}, \text{ where the variable } t \text{ records the length of the walk, and } x \text{ and } y \text{ the coordinates of its endpoints:} \]

\[ C(x,y) = \sum_{(i,j) \in \mathcal{C}} \sum_{n \geq 0} c_{i,j}(n)x^iy^nt^n = \sum_{(i,j) \in \mathcal{C}} x^iy^jC_{i,j}(t). \quad (8) \]

Here, \( c_{i,j}(n) \) is the number of walks of length \( n \) that go from \((0,0)\) to \((i,j)\) and that are confined to \( \mathcal{C} \).

### 2.1 Interesting step sets

As in the quadrant case [3], we can decrease the number of step sets that are worth being considered \((a \text{ priori}, \text{ there are } 2^8 \text{ of them})\) thanks to a few simple observations:

- Since the cone \( \mathcal{C} \) (as well as the quarter plane \( \mathcal{Q} \)) is \( x/y \)-symmetric, the models defined by \( \mathcal{S} \) and by its mirror image \( \overline{\mathcal{S}} := \{(j,i) : (i,j) \in \mathcal{S}\} \) are equivalent; the associated generating functions are related by \( \overline{\mathcal{C}}(x,y) = \mathcal{C}(y,x) \).

- If all steps of \( \mathcal{S} \) are contained in the right half-plane \( \{(x,y) : x \geq 0\} \), then all walks with steps in \( \mathcal{S} \) lie in \( \mathcal{C} \), and the series \( C(x,y) = 1/(1-tS(x,y)) \) is simply rational. The series \( Q(x,y) \) is known to be algebraic in this case [6].

- If all steps of \( \mathcal{S} \) are contained in the left half-plane \( \{(x,y) : x \leq 0\} \), then confining a walk to \( \mathcal{C} \) is equivalent to confining it to the upper half-plane: the associated generating function is then algebraic, and so is \( Q(x,y) \).

- If all steps of \( \mathcal{S} \) lie (weakly) above the first diagonal \((x = y)\), then confining a walk to \( \mathcal{C} \) is again equivalent to confining it to the upper half-plane: the associated generating function is then algebraic, and so is \( Q(x,y) \).

- If all steps of \( \mathcal{S} \) lie (weakly) above the second diagonal \((x + y = 0)\), then all walks with steps in \( \mathcal{S} \) lie in \( \mathcal{C} \), and \( C(x,y) = 1/(1-tS(x,y)) \) is simply rational. In this case however, the series \( Q(x,y) \) is not at all trivial [3,10]. Such step sets are sometimes called \textit{singular} in the framework of quadrant walks.

- Finally, if all steps of \( \mathcal{S} \) lie (weakly) below the second diagonal, then a walk confined to \( \mathcal{C} \) moves for a while along the second diagonal, and then either stops there or leaves it into the NW or SE quadrant using a South, South-West, or West step. It cannot leave the chosen quadrant anymore and behaves therein like a half-plane walk. By polishing this observation, one can prove that \( C(x,y) \) is algebraic (while \( Q(x,y) = 1 \)).

Symmetric statements allow us to discard step sets that lie in the upper half-plane \( \mathbb{Z} \times \mathbb{N} \), in the lower half-plane \( \mathbb{Z} \times (-\mathbb{N}) \), or weakly below the \( x/y \) diagonal.

In conclusion, one finds that there are exactly 74 essentially distinct models of walks avoiding the negative quadrant that are worth studying: the 79 models considered for quadrant walks (see Tables 1–4 in [3]) except the 5 singular models for which all steps of \( \mathcal{S} \) lie weakly above the diagonal \( x+y = 0 \).

### 2.2 A functional equation

Constructing walks confined to \( \mathcal{C} \) step by step gives the following functional equation:

\[ C(x,y) = 1 + tS(x,y)C(x,y) - t\bar{y}H_{-\alpha}(x)C_{-\alpha}(\bar{x}) - t\bar{x}V_{-\gamma}(y)C_{0,-\gamma}(\bar{y}) - t\bar{x}\bar{y}C_{0,0}1_{(-1,-1) \in \mathcal{S}}, \]
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where the series $C_{-0}(\bar{x})$ and $C_{0,-}(\bar{y})$ count walks ending on the horizontal and vertical boundaries of $C$ (but not at $(0,0)$):

$$C_{-0}(\bar{x}) = \sum_{\begin{subarray}{c}i\geq 0 \\
n\geq 0 \end{subarray}} c_{i,0}(n)x^it^n \in \bar{x}Q[\bar{x}][[t]].$$

$$C_{0,-}(\bar{y}) = \sum_{\begin{subarray}{c}j\geq 0 \\
n\geq 0 \end{subarray}} c_{0,j}(n)y^jt^n \in \bar{y}Q[\bar{y}][[t]].$$

On the right-hand side of the above functional equation, the term $1$ accounts for the empty walk, the next term describes the extension of a walk in $C$ by one step of $S$, and each of the other three terms correspond to a “bad” move, either starting from the negative $x$-axis, or from the negative $y$-axis, or from $(0,0)$. Equivalently,

$$K(x,y)C(x,y) = 1 - t\bar{y}H_-(x)C_{-0}(\bar{x}) - t\bar{x}V_-(y)C_{0,-}(\bar{y}) - t\bar{x}\bar{y}C_{0,0}1_{(-1,-1)\in S},$$

where $K(x,y) := 1 - tS(x,y)$ is the kernel of the equation.

The case of walks confined to the first (non-negative) quadrant $Q$ has been much studied in the past 15 years. The associated generating function $Q(x,y) \equiv Q(x,y;t) \in Q[x,y][[t]]$ is defined similarly to (8) and satisfies a similarly looking equation:

$$K(x,y)Q(x,y) = 1 - t\bar{y}H_-(x)Q_{-0}(x) - t\bar{x}V_-(y)Q_{0,-}(y) + t\bar{x}\bar{y}Q_{0,0}1_{(-1,-1)\in S},$$

where now

$$Q_{-0}(x) = \sum_{\begin{subarray}{c}i\geq 0 \\
n\geq 0 \end{subarray}} q_{i,0}(n)x^it^n \in Q[x][[t]],$$

$$Q_{0,-}(y) = \sum_{\begin{subarray}{c}j\geq 0 \\
n\geq 0 \end{subarray}} q_{0,j}(n)y^jt^n \in Q[y][[t]].$$

3 The king walks

In this section we focus on the case where the 8 steps of $\{-1,0,1\}^2 \setminus \{(0,0)\}$ are allowed. That is,

$$S(x,y) = (\bar{x} + 1 + x)(\bar{y} + 1 + y) - 1 = x + xy + y + \bar{x}y + \bar{x} + \bar{y} + \bar{x}\bar{y}.$$ 

The functional equation (9) specializes to

$$K(x,y)C(x,y) = 1 - t\bar{y}(x + 1 + \bar{x})C_{-0}(\bar{x}) - t\bar{x}(y + 1 + \bar{y})C_{0,-}(\bar{y}) - t\bar{x}\bar{y}C_{0,0},$$

where we have denoted $C_{-0}(\bar{x}) = C_{-0}(\bar{x}) = C_{0,-}(\bar{x})$ (by symmetry). Equivalently,

$$xyK(x,y)C(x,y) = xy - t(x^2 + x + 1)C_{-0}(\bar{x}) - t(y^2 + y + 1)C_{0,-}(\bar{y}) - tC_{0,0}. $$

(10)

The generating function $Q(x,y)$ of quadrant walks satisfies

$$xyK(x,y)Q(x,y) = xy - t(x^2 + x + 1)Q(x,0) - t(y^2 + y + 1)Q(0,y) + tQ_{0,0}. $$

(11)
3.1 Reduction to an equation with orbit sum zero

A key object in the study of walks confined to the first quadrant is a certain group of birational transformations that depends on the step set. For king walks, it is generated by \((x,y) \mapsto (\bar{x},y)\) and \((x,y) \mapsto (x,\bar{y})\). As in [1], the similarities between the equations for \(C\) and \(Q\), combined with the structure of this group, lead us to define a new series \(A(x,y)\) by

\[
C(x,y) = A(x,y) + \frac{1}{3} (Q(x,y) - \bar{x}^2 Q(\bar{x},y) - \bar{y}^2 Q(x,\bar{y})).
\]  

(13)

Then the combination of (11) and (12) gives

\[
xyK(x,y)A(x,y) = 2xy + \bar{y}A(\bar{x},y) + \bar{x}A(x,\bar{y}) - xyA(\bar{x},\bar{y}) + x\bar{y}A(x,\bar{y}) = 0.
\]  

(14)

Theorem 1 states that \(A(x,y)\) is algebraic. In Section 4 we define an analogous series \(A\) for all models of Figure 1 which we expect to be systematically algebraic.

The proof of Theorem 1 starts as in the case of the simple and diagonal walks in [1]. The first objective, achieved in Section 3.5, is to derive an equation that involves a single bivariate series, essentially \(A - (x)\) (and no trivariate series). In principle, the “generalized quadratic method” of [2] then solves it routinely. But in practise, the king model turns out to be much more difficult to solve than the other two, and raises serious computational difficulties. In what follows, we focus on the points of the derivation that differ from [1]. We have performed all computations with the computer algebra system Maple. The corresponding sessions will be available on the authors’ webpages with the long version of the paper.

3.2 Reduction to a quadrant-like problem

We separate in \(A(x,y)\) the contributions of the three quadrants, again using the \(x/y\)-symmetry of the step set:

\[
A(x,y) = P(x,y) + \bar{x}M(\bar{x},y) + \bar{y}M(\bar{y},x),
\]

where \(P(x,y)\) and \(M(x,y)\) lie in \(\mathbb{Q}[x,y][[t]]\). Note that this identity defines \(P\) and \(M\) uniquely in terms of \(A\). Replacing \(A\) by this expression, and extracting the positive part in \(x\) and \(y\) from the orbit equation (14) relates the series \(P\) and \(M\) by

\[
xyP(x,y) = y (M(x,y) - M(0,y)) + x (M(y,x) - M(0,x)),
\]

which is exactly the same as [1, Eq. (22)], and as Eq. (2) in Theorem 1. We then follow the lines of proof of [1, Sec. 2.3] to obtain the functional equation (3) for \(M\).

3.3 An equation between \(M(0,x)\), \(M(0,\bar{x})\), and \(M(x,0)\)

Next we will cancel the kernel \(K\). As a polynomial in \(y\), the kernel admits only one root that is a formal power series in \(t\):

\[
Y(x) = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2(x + 1 + \bar{x})^2}}{2t(x + 1 + \bar{x})} = (x + 1 + \bar{x})t + \mathcal{O}(t^2).
\]
Note that \(Y(x) = Y(\bar{x})\). We specialize (3) to the pairs \((x, Y(x)), (\bar{x}, Y(\bar{x})), (Y(x), x),\) and \((Y(\bar{x}), \bar{x})\) (the left-hand side vanishes for each specialization since \(K(x, y) = K(y, x)\)), and eliminate \(M(0, Y), M(Y, 0),\) and \(M(\bar{x}, 0)\) from the four resulting equations. We obtain:

\[
(x + 1 + \bar{x}) \left( Y(x) - \frac{1}{Y(x)} \right) (xM(0, x) - 2\bar{x}M(0, \bar{x})) + 3(x + 1 + \bar{x})M(x, 0) - \frac{2\bar{x}Y(x)}{t} \right) + 3M_{1,0} + (2Y(x) - x - \bar{x})M_{0,0} = 0. \tag{15}
\]

### 3.4 An equation between \(M(0, x)\) and \(M(0, \bar{x})\)

Let us denote the discriminant occurring in \(Y(x)\) by

\[
\Delta(x) := (1 - t(x + \bar{x}))^2 - 4t^2(x + 1 + \bar{x})^2 = (1 - t(3x + \bar{x} + 2))(1 + t(x + \bar{x} + 2)) \tag{16}
\]

and introduce the notation

\[
R(x) := t^2M(x, 0) = \frac{\pi t^2}{3} + \left(1 + \frac{x^2}{3}\right)t^3 + \mathcal{O}(t^4),
\]

\[
S(x) := txM(0, x) = x(1 + x)t^2 + 2x(1 + x + x^2)t^3 + \mathcal{O}(t^4). \tag{17}
\]

Then (15) reads

\[
\sqrt{\Delta(x)} \left( S(x) - 2S(\bar{x}) + \frac{R(0) - t\bar{x}}{t(x + 1 + \bar{x})} \right) = 3(x + 1 + \bar{x})R(x) + 3R'(0) + \frac{1 - t(x + \bar{x})(x + 2 + \bar{x})}{t(x + 1 + \bar{x})}R(0) - \frac{1 - t(x + \bar{x})}{1 + x + x^2}. \tag{18}
\]

Next, we square this equation and extract the negative part in \(x\). The series \(R(x)\) (mostly) disappears as it involves only non-negative powers of \(x\). This gives an expression for the negative part of \(\Delta(x)S(x)S(\bar{x})\). Using the symmetry of \(\Delta(x)\) in \(x\) and \(\bar{x}\), we then reconstruct an expression of \(\Delta(x)S(x)S(\bar{x})\) that does not involve \(R(x)\), as in [1, Sec. 2.5].

During these calculations, we have to extract the negative and non-negative parts in series of the form \(F(x)/(1 + x + \bar{x})^m\), where \(F(x)\) is a series in \(t\) with coefficients in \(\mathbb{Q}[x, \bar{x}]\). Upon performing a partial fraction expansion, and separating in \(F\) the negative and non-negative parts, we see that the key question is how to extract and express the non-negative part in series of the form \(F(\bar{x})/(1 - \zeta, x)^m\), where \(F(x) \in \mathbb{C}[x][[t]]\) and

\[
\zeta_1 := -\frac{1}{2} + \frac{i\sqrt{3}}{2} \quad \text{and} \quad \zeta_2 := -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\]

are the primitive cubic roots of unity. A simple calculation establishes the following lemma.

**Lemma 3 (Non-negative part at pole \(\rho\)).** Let \(F(x) \in \mathbb{C}[x][[t]]\) and \(\rho \in \mathbb{C}\). Then,

\[
[x^\geq 0] F(\bar{x}) \frac{\rho}{1 - \rho x} = F(\rho) \frac{\rho}{1 - \rho x},
\]

\[
[x^\geq 0] F(\bar{x}) \frac{1}{(1 - \rho x)^2} = F(\rho) \frac{\rho F'(\rho)}{(1 - \rho x)^2} + \frac{\rho F''(\rho)}{1 - \rho x}.
\]

One outcome of the extraction procedure is the following identity:

\[
S(\zeta_1) = S(\zeta_2) = -\frac{R(0) + 3R'(0)}{1 + t} = -t^2 - 11t^4 - 30t^6 + \mathcal{O}(t^8). \tag{19}
\]
Using these results, we finally arrive at an equation relating \( S(x) \) and \( S(\bar{x}) \):

\[
\Delta(x) \left( S(x)^2 + S(\bar{x})^2 - S(x)S(\bar{x}) + \frac{S(x)(xt - R(0)) + \bar{x}S(\bar{x})(\bar{t}t - R(0))}{t(x + 1 + \bar{x})} \right) = (1 + t)S(\zeta_1) \left( 2(x + 1 + \bar{x})R(0) - \frac{(1 - t(\bar{x} + \bar{t}))((t(x + \bar{x}) - 2R(0)))}{t(x + 1 + \bar{x})} \right) + (1 + 4t)(x + \bar{x})R(0) - (t^2 + tR(0) + R(0)^2)(x^2 + \bar{x}^2) + \Delta_0,
\]

(20)

where \( \Delta_0 \) is the coefficient of \( x^0 \) in \( \Delta(x)S(x)S(\bar{x}) \).

### 3.5 An equation for \( M(0, x) \) only

Equation (20) is almost ready for a positive part extraction, except for the mixed term \( S(x)S(\bar{x}) \). To eliminate it, we multiply (20) by \( S(x) + S(\bar{x}) + \frac{x + \bar{x}}{x + 1 + \bar{x}} \). Then we are able to extract the non-negative terms in \( x \). Hereby we repeatedly apply Lemma 3. Additionally, we use \( R(0) = ts'(0) \) and (19). Furthermore, we work with the real and imaginary parts of \( \zeta_1S'(\zeta_1) \) and \( \zeta_2S'(\zeta_2) \). More precisely, we define

\[
(1 + t)^2 \zeta_1S'(\zeta_1) = B_1 + i\sqrt{3}B_2,
(1 + t)^2 \zeta_2S'(\zeta_2) = B_1 - i\sqrt{3}B_2.
\]

(Note that \( B_1 \) and \( B_2 \) here are series in \( t \).) In the end we get a cubic equation in \( S(x) \):

\[
\text{Pol}(S(x), S'(0), S(\zeta_1), B_1, B_2, t, x) = 0,
\]

(21)

where the polynomial \( \text{Pol}(x_0, x_1, x_2, x_3, x_4, t, x) \) is given in Appendix A.

### 3.6 The generalized quadratic method

We now use the results of [2] to obtain a system of four polynomial equations relating the series \( S'(0), S(\zeta_1), B_1, \) and \( B_2 \). Combined with a few initial terms, this system characterizes these four series. Unfortunately, it turned out to be too big for us to solve it completely, by bare hand elimination or using Gröbner bases: we did obtain a polynomial equation for \( S'(0) \) and \( S(\zeta_1) \), but not for the other two series. Instead, we have resorted to a guess-and-check approach, consisting in guessing such equations (of degree 12 or 24, depending on the series), and then checking that they satisfy the system. This guess-and-check approach is detailed in the next subsection. For the moment, let us explain how the system is obtained.

The approach of [2] instructs us to consider the fractional series \( X \) (in \( t \)), satisfying

\[
\text{Pol}_{x_0}(S(X), S'(0), S(\zeta_1), B_1, B_2, t, X) = 0,
\]

(22)

where \( \text{Pol}_{x_0} \) stands for the derivative of Pol with respect to its first variable. The number and first terms of such series \( X \) depend only on the first terms of the series \( S(x), S'(0), S(\zeta_1), B_1, \)
and $B_2$ (see [2, Thm. 2]). We find that 6 such series exist:

$$X_1(t) = i + 2t^2 + 4t^3 + (36 - 2i)t^4 + O(t^5),$$
$$X_2(t) = -i + 2t^2 + 4t^3 + (36 + 2i)t^4 + O(t^5),$$
$$X_3(t) = \sqrt{t} + t + \frac{3}{2}t^{3/2} + 3t^2 + \frac{51}{8}t^{5/2} + 14t^3 + O(t^{7/2}),$$
$$X_4(t) = -\sqrt{t} - t + \frac{3}{2}t^{3/2} + 3t^2 - \frac{51}{8}t^{5/2} + 14t^3 + O(t^{7/2}),$$
$$X_5(t) = i\sqrt{t} - it^{3/2} + 2it^{5/2} + t^3 - 4it^{7/2} + 2t^4 + O(t^{9/2}),$$
$$X_6(t) = -i\sqrt{t} + it^{3/2} - 2it^{5/2} + t^3 + 4it^{7/2} + 2t^4 + O(t^{9/2}).$$

Note that the coefficients of $X_1$ and $X_2$ (resp. $X_5$ and $X_6$) are conjugates of one another. As discussed in [2], each of these series $X$ also satisfies

$$\text{Pol}_x(S(X), S'(0), S(\zeta_1), B_1, B_2, t, X) = 0,$$  \hfill (23)

where $\text{Pol}_x$ is the derivative with respect to the last variable of $\text{Pol}$, and (of course)

$$\text{Pol}(S(X), S'(0), S(\zeta_1), B_1, B_2, t, X) = 0.$$  \hfill (24)

Using this, we can easily identify two of the series $X_i$: indeed, eliminating $B_1$ and $B_2$ between the three equations (22), (23), and (24) gives a polynomial equation between $S(X), S'(0), S(\zeta_1), t,$ and $X$, which factors. Remarkably, its simplest non-trivial factor does not involve $S(X)$, nor $S'(0)$ nor $S(\zeta_1)$, and reads

$$X^2 - t(1 + X)^2(1 + X^2).$$  \hfill (25)

By looking at the first terms of the $X_i$’s and the other factors, one concludes that the above equation holds for $X_3$ and $X_4$, which are thus explicit.

Let $D(x_1, \ldots, x_4, t, x)$ be the discriminant of $\text{Pol}(x_0, \ldots, x_4, t, x)$ with respect to $x_0$. According to [2, Thm. 14], each $X_i$ is a double root of $D(S'(0), S(\zeta_1), B_1, B_2, t, x)$, seen as a polynomial in $x$. Hence this polynomial, which involves 4 unknown series $S'(0), S(\zeta_1), B_1, B_2,$ has (at least) 6 double roots. This seems more information than we need! In principle, 4 double roots should suffice to give 4 conditions relating the 4 unknown series. However, we shall see that there is some redundancy in the 6 series $X_i$, which comes from the special form of $D$.

We first observe that $D$ factors as

$$D(S'(0), S(\zeta_1), B_1, B_2, t, x) = 27x^2(1 + x + x^2)^2\Delta(x)D_1(S'(0), S(\zeta_1), B_1, B_2, t, x),$$

where $\Delta(x)$ is defined by (16), and $D_1$ has degree 24 in $x$. It is easily checked that none of the $X_i$’s are roots of the prefactors, so they are double roots of $D_1$. But we observe that $D_1$ is symmetric in $x$ and $\tilde{x}$. More precisely,

$$D_1(S'(0), S(\zeta_1), B_1, B_2, t, x) = x^{12}D_2(S'(0), S(\zeta_1), B_1, B_2, t, x + 1 + \tilde{x}),$$

for some polynomial $D_2(x_1, \ldots, x_4, t, s) \equiv D_2(s)$ of degree 12 in $s$. Since each $X_i$ is a double root of $D_1$, each series $S_i := X_i + 1 + 1/X_i$, for $1 \leq i \leq 6$, is a double root of $D_2$. The series $S_i$, for $2 \leq i \leq 6$, are easily seen from their first terms to be distinct, but the first terms of $S_1$ and $S_2$ suspiciously agree: one suspects (and rightly so), that $X_2 = 1/X_1$, and carefully
concludes that $D_2$ has (at least) 5 double roots in $s$. Moreover, since $X_3$ and $X_4$ satisfy (25), the corresponding series $S_3$ and $S_4$ are the roots of $1 + t = tS^2_s$, that is, $S_{3,4} = \pm \sqrt{1 + 1/t}$.

The other roots start as follows:

$$S_2 = 1 + 4t^2 + 8t^3 + \mathcal{O}(t^4), \quad S_{5,6} = \pm \frac{i}{\sqrt{t}} + 1 + t^2 \pm it^{5/2} + \mathcal{O}(t^3).$$

But this is not the end of the story: indeed, $D_2$ appears to be almost symmetric in $s$ and $1/s$. More precisely, we observe that

$$D_2(S'(0), S(\zeta_1), B_1, B_2, s) = s^6 D_3 \left( S'(0), S(\zeta_1), B_1, B_2, ts + \frac{t + 1}{s} \right),$$

for some polynomial $D_3(S'(0), S(\zeta_1), B_1, B_2, t, z) \equiv D_3(z)$ of degree 6 in $z$. It follows that each series $Z_i := tS_i + (1 + t)/S_i$, for $2 \leq i \leq 6$, is a root of $D_3(z)$, and even a double root, unless $tS^2_i = 1 + t$, which precisely occurs for $i = 3, 4$. One finds $Z_{3,4} = \pm 2\sqrt{t(1 + t)}$,

$$Z_2 = 1 + 2t - 4t^2 + \mathcal{O}(t^3), \quad Z_{5,6} = 2t + 2t^3 + \mathcal{O}(t^4).$$

Since $Z_5$ and $Z_6$ seem indistinguishable, we safely conclude that $D_3(z)$ has two double roots $Z_2$ and $Z_3$, and a factor $(z^2 - 4t(1 + t))$. Writing

$$D_3(z) = \sum_{i=0}^{6} d_i z^i,$$

these properties imply, by matching the three monomials of highest degree, that

$$D_3(z) = \left( z^2 - 4t(1 + t) \right)^2 \frac{1}{64d_0^3}.$$ 

Extracting the coefficients of $z^0, \ldots, z^3$ gives 4 polynomial relations between the coefficients $d_i$, resulting in 4 polynomial relations between the 4 series $S'(0), S(\zeta_1), B_1, B_2$. One easily checks that this system, combined with the first terms of these series, defines them uniquely.

As explained at the beginning of this subsection, we have at the moment only been able to derive from this system polynomial equations (of degree 24) for $S'(0)$ and $S(\zeta_1)$. For the other two, we had to resort to a guess-and-check approach, which we now describe.

### 3.7 Guess-and-check

**Guessing.** Returning to the functional equation (10) it is easy to extract a simple recurrence for the polynomials $c_n(x, y)$ that count walks of length $n$ by the position of their endpoint. We implemented this recurrence in the programming language C using modular arithmetic and the Chinese remainder theorem to compute the explicit values of this sequence up to $n = 2000$. Then we were able to guess polynomial equations satisfied by $S'(0)$, $S(\zeta_1)$, $B_1$, and $B_2$, using the gfun package in MAPLE [14]. Of course, those obtained for $S'(0)$ and $S(\zeta_1)$ coincide with those that we derived from the system of the previous subsection. Details on the corresponding equations are shown below.

| Generating function | Degree in $GF$ | Degree in $t$ | Number of terms |
|---------------------|---------------|---------------|-----------------|
| $S'(0)$             | 24            | 12            | 323             |
| $S(\zeta_1)$       | 24            | 32            | 823             |
| $B_1$               | 12            | 26            | 229             |
| $B_2$               | 24            | 60            | 477             |
The algebraic structure of $S'(0), S(\zeta_1), B_1,$ and $B_2$. We begin with the simplest series, $B_1$, of (conjectured) degree 3. Let $P(F,t)$ be its guessed monic minimal polynomial. Using the Subfields command of Maple for several fixed values of $t$, one conjectures that the extension $Q(t,B_1)$ possesses a subfield $Q(t,u)$ of degree 4 over $Q(t)$. Maple gives a possible generator $u$ for fixed values of $t$, but how can we choose $u$ for a generic $t$? Indeed, the value of $u$ given by Maple for fixed $t$ has no reason to be canonical. But the factorisation of $P(F,t)$ over $Q(t,u)$, of the form $P_3(F)P_9(F)$ (with $P_i$ of degree $i$), with coefficients in $Q(t,u)$, is canonical. Hence the cubic $P_3(F)$ factors into a product of three polynomials of degree 1, which coincides with (7), given the Definition (17) of $S(x)$. For each $i$, $B_i$ is defined as a subfield of $Q(t,u)$ generated by $u$ and the series $S_i(0)$, and $B_2$ is defined as a subfield of $Q(t,u)$ generated by $u$ and the series $S_2(0)$, 3 by (6). In particular,

\[
S'(0) = \frac{1}{2} \left( \frac{w(1+2v)}{1+4v-2v^4} - 1 \right),
\]

which coincides with (7), given the Definition (17) of $S(x)$.

Now that we have guessed rational expressions of $S'(0)$, $S(\zeta_1)$, $B_1$, and $B_2$ in terms of $t$, $v$, and $w$, the 4 equations obtained in Section 3.6 are readily checked to hold, using the minimal polynomials of $v$ and $w$.

3.8 Back to $S(x)$ and $R(x)$

For $S(x)$ we start with Equation (21), with all one-variable series replaced by their expressions in terms of $t$, $v$, and $w$. We eliminate $w$ and $v$ using resultants to arrive at an equation of degree 72 over $Q(t,x)$ for $S(x) = txM(0,x)$. 

---

1 For this section, we have greatly benefited from the help of Mark van Hoeij (https://www.math.fsu.edu/~hoeij/), who explained us how to find subextensions of $Q(t,B_1)$, and “simple” series in these extensions.
We can simplify (21) by working with the depressed equation, i.e., removing the quadratic term by a suitable change of variable. Indeed, defining $T(x)$ by

$$S(x) = T(x) + \frac{3xS'(0) - 2x^2 - 1}{3(x^2 + x + 1)},$$

we find that $T(x)$ satisfies a cubic equation with no quadratic term, involving $t$ and $v$ but not $w$. That is, $T(x)$ has degree 36 over $\mathbb{Q}(t, x)$, instead of 72 for $S(x)$.

Introducing $T(x)$ also helps understanding the algebraic structure of $R(x)$. Returning to (18), we recall that $R(0) = tS'(0)$ and use (19) to express $R'(0)$ in terms of $t, v$, and $w$. The left-hand side simply reads $\sqrt{\Delta(x)}(T(x) - 2T(x))$, and is found to be an element of $w\mathbb{Q}(t, x, T(x))$. In the end, $R(x)$ has degree 72 and belongs to the same extension of $\mathbb{Q}(t, x)$ as $S(x)$. This ends the proof of our main result, Theorem 1.

4 More models

For each of the 7 step sets $S$ of Figure 1, we are able to define a series $A(x, y)$ that

- satisfies the same equation as $C(x, y)$ (see (9)), but with a different constant term,
- satisfies an orbit sum identity similar to (14).

Explaining where this series comes from would require us to introduce the group associated to a step set. For the sake of conciseness, we simply define $A(x, y)$ without further justification.

For the first four step sets $S$ of Figure 1, the series $A(x, y)$ is defined by (13) (with $Q(x, y)$ counting quadrant walks with steps in $S$) as we have seen. For the next two step sets,

$$C(x, y) = A(x, y) + \frac{1}{5} \left( Q(x, y) - \bar{x}^2 y Q(\bar{x} y, y) + \bar{x}^3 Q(\bar{x} y, \bar{x}) + y^3 Q(\bar{y} y, \bar{x}) - x y^2 Q(x, x y) \right).$$

Finally, for the seventh one,

$$C(x, y) = A(x, y) + \frac{1}{7} \left( Q(x, y) - \bar{x}^2 y Q(\bar{x} y, y) + \bar{x}^4 y Q(\bar{x} y, \bar{x} y) - \bar{x}^4 Q(\bar{x}, \bar{x} y) - y^4 Q(\bar{y} y, \bar{x}) + x^2 y^3 Q(x y, x^2 y) - x^2 y^2 Q(x, x^2 y) \right).$$

In all cases, the series $A(x, y)$ satisfies the following variant of (9):

$$K(x, y) A(x, y) = P_0(x, y) - t \bar{y} H_{-}(x) A_{-0}(\bar{x}) - t \bar{y} V_{-}(y) A_{0-}(\bar{y}) - t \bar{y} A_{00} \mathbb{I}_{(-1, -1) \in S},$$

where $K(x, y) = 1 - t S(x, y)$ as before, and $P_0(x, y)$ is a Laurent polynomial. This equation is easily obtained by combining the equations for $C(x, y)$ and $Q(x, y)$.

Finally, the vanishing orbit sum, which is (14) for the first four models, reads

$$x y A(x, y) - \bar{x} y^2 A(\bar{x} y, y) + \bar{x}^2 y A(\bar{x}, \bar{y}) - \bar{x} \bar{y} A(\bar{y}, \bar{x}) + x y^2 A(y, y) - x^2 \bar{y} A(x, x y) = 0$$

for the next two, and

$$x y A(x, y) - \bar{x} y^2 A(\bar{x} y, y) + \bar{x}^3 y^2 A(\bar{x} y, \bar{x} y) - \bar{x}^3 y A(\bar{x}, \bar{x} y) + x y^2 A(x y, \bar{y}) + x^3 y^2 A(x y, x^2 y) - x^3 \bar{y} A(x, x^2 y) = 0$$

for the last one. We conjecture that the series $A(x, y)$ is systematically algebraic (this is now proved for the first three models). To support this conjecture, we have tried to guess (using the gfun package [14] in MAPLE), for the 4 models for which it is still open, a polynomial
equation for the series $A_{-1,0}$, which, in all cases, coincides with the generating function $C_{-1,0}$ of walks ending at $(-1,0)$ (for the second model we consider $A_{-2,0}$ instead, since $A_{-1,0}=0$ due to the periodicity of the model). This series has degree 4 (resp. 8, 24) in the three solved cases. We could not guess anything for the 4th model (using the counting sequence for such walks up to length $n = 4000$), but we discovered equations of degree 24 for each of the next three.

We believe that it would be worth exploring if the guiding principles of the present paper apply to these 4 other models. In all cases, we expect to face a system of quadrant-like equations rather than a single one. We plan to investigate at least some of these models.

To conclude, we recall that the 4 small step models that are algebraic for the quadrant problem are conjectured to be algebraic for the three-quadrant cone as well [1, Fig. 5]. In this case, the series $A(x,y)$ simply coincides with $C(x,y)$, as the orbit sum of $xyC(x,y)$ vanishes.

A Final polynomial equation for $S(x)$ in the king model

The polynomial $\text{Pol}$ involved in the cubic Equation (21) defining $S(x)$ is:

\[
\begin{align*}
\text{Pol}(x_0, x_1, x_2, x_3, x_4, t, x) &= -3(x^2 + x + 1)^2(x^4t + 2xt + x + t)(3x^2t + 2xt - x + 3t)x_0^3 \\
&+ 3(x^2 + x + 1)(x^2t + 2xt + x + t)(3x^2t + 2xt - x + 3t)(3x_1 - 2x - 1)x_0^2 \\
&+ [3x^2(x^2 + x + 1)^2(2x_4x + x - x_3) - 3x^2(t + 1)^2(x + x + 1)x_2^2 \\
&+ 6x(t + 1)(x^2 + x + 1)(x^4 + 2xt^2 + x^2 + t)x_1x_2 \\
&+ 3x(t + 1)(x^2 + x + 1)(x^4 + x^2 + x^2 t + x t - x + t)x_2 - 3(x^8 - 8x^7 + 18x_8) \\
&+ 10x_0^2 + 20x_0^2 + 4x_5t + 25x_4t^2 + 20x_3t^2 - 4x^4 + 4x^3t + 10x_2^2 + 2x_2^2 + t^2) x_1^2 \\
&- 3(x^8 - 11x^7t - x^7t - 32x^6t - 9x^6t - 53x^5t^2 - 6x^5t - 55x^4t^2 + 3x^5 - 15x^4t \\
&- 39x^3 + 6x^3t - 16x^2t^2 - x^3 + 5x^2t - 5xt^2 - x + t^2) x_1 - 12x^2 + 30x^2t^2 - 63xt \\
&- 51x^2t - 60x^2t^2 + 3x^2 - 12x^5t - 54x^4t^2 - 36x^3t^2 + 3x^4 - 6x^3t - 21x^2t^2 - 6xt^2 \\
&- 3x^2] x_0 + x^2(x^2 + x + 1) [(2x_4x^2 - 6x_4x - 2x_4) x_1^2 - (x^2 + 2)x_1 + 3x_4x^2 \\
&+ (2x_1 - 1)(3x_4x + x_3 + x_2 + x_1 + x_2) x_1 + 3x^2(t + 1)^2(x^2 + x + 1)(x_1 - x_2) \\
&- 3x^2(t + 1) x_1 x_1 - t(x^2 + t^2)(x + x + 1) x_1 + t(x^4 + x^2 + 1)(x + 1)(x_2 + 1) \\
&+ 3xt(x^2 + x + 1)^2(x_1 - x)(t(x^2 - x + 1) x_1^2 + (x^2 - 5xt - x + t) x_1 + t(x^2 - x + 1))
\end{align*}
\]

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