LIMITING WEAK-TYPE BEHAVIOR FOR ROUGH BILINEAR OPERATORS

Moyan Qin, Huoxiong Wu, Qingying Xue

Abstract

Let \( \Omega_1, \Omega_2 \) be functions of homogeneous of degree 0 and \( \vec{\Omega} = (\Omega_1, \Omega_2) \in L^{log}_1(S^{n-1}) \times L^{log}_1(S^{n-1}). \) In this paper, we investigate the limiting weak-type behavior of the bilinear maximal function \( \mathcal{M}_{\vec{\Omega}} \) and the bilinear singular integral \( \mathcal{T}_{\vec{\Omega}} \) associated with rough kernel \( \vec{\Omega}. \) For all \( f, g \in L^1(\mathbb{R}^n), \) we show that

\[
\lim_{\lambda \to 0^+} |\{ x \in \mathbb{R}^n : \mathcal{M}_{\vec{\Omega}}(f, g)(x) > \lambda \}| \leq C \| \Omega_1 \Omega_2 \|_{L^1(S^{n-1})}^{1/2} \left( \frac{n}{2} \prod_{i=1}^2 \| f_i \|_{L^1} \right).
\]

and

\[
\lim_{\lambda \to 0^+} |\{ x \in \mathbb{R}^n : |\mathcal{T}_{\vec{\Omega}}(f, g)(x)| > \lambda \}| \leq C \| \Omega_1 \Omega_2 \|_{L^1(S^{n-1})}^{1/2} \left( \frac{n}{2} \prod_{i=1}^2 \| f_i \|_{L^1} \right).
\]

As consequences, the lower bounds of weak-type norms of \( \mathcal{M}_{\vec{\Omega}} \) and \( \mathcal{T}_{\vec{\Omega}} \) are obtained. These results are new even in the linear case. The corresponding results for rough bi-linear fractional maximal function and fractional integral operator are also discussed.

1 Introduction

It was well-known that the Hardy-Littlewood maximal function was first introduced by Hardy and Littlewood in 1930. It plays very important roles in harmonic analysis, geometric and ergodic theory. As one of the fundamental operators, it enjoys many important properties including the boundedness on \( L^p, \) \( 1 < p \leq \infty, \) and \( L^\infty. \)

The basic properties of the Hardy-Littlewood maximal function were first introduced by

\[
\mathcal{M}_f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,
\]

where \( B(x,r) \) is a ball centered at \( x \) with radius \( r. \)

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\( L^p \) bounded for \( p > 1 \). The same results \([14, 20, 39]\) also hold for the uncentered maximal function which is defined similarly as in (1.1) but the supremum is taken over all arbitrary balls containing \( x \) instead of \( B(x, r) \).

In order to state some other related results, we first introduce some definitions.

Let \( n \in \mathbb{N}, 0 \leq \alpha < n, \mathcal{S}^{n-1} \) be the unit sphere on the Euclidean space \( \mathbb{R}^n, d\sigma(\cdot) \) be the induced Lebesgue measure on \( \mathcal{S}^{n-1} \). Suppose that \( \Omega(x) \in L^1(\mathcal{S}^{n-1}) \) is homogeneous of degree zero, i.e.,

\[
\Omega(tx) = \Omega(x) \quad \text{for any } t > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\}.
\]

Define the maximal type operators with homogeneous kernel \( \Omega \) by

\[
M_{\Omega}^0 f(x) = \sup_{r>0} \frac{1}{|B(x, r)|^{(n-\alpha)/n}} \int_{B(x, r)} |\Omega(x-y)f(y)| \, dy,
\]

and the singular and fractional integral operator \( T_{\Omega, \alpha} \) with homogeneous kernel \( \Omega \) by

\[
T_{\Omega}^\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) \, dy,
\]

where \( B(x, r) \) denotes the ball centered at \( x \) with radius \( r \), and \( \Omega \) satisfies the following vanishing condition

\[
\int_{\mathcal{S}^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.2}
\]

If \( \alpha = 0 \), we denote \( M_{\Omega}^0 \) by \( M_{\Omega} \), and \( T_{\Omega}^0 \) by \( T_{\Omega} \). In particular, if \( \Omega \equiv 1 \), then \( M_1 \) is the Hardy-Littlewood maximal function \( M \) as in (1.1).

It is well-known that \( M_{\Omega} \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \) if \( \Omega \in L^1(\mathcal{S}^{n-1}) \), and is of weak type \((1, 1)\) if \( \Omega \in L \log L(\mathcal{S}^{n-1}) \) (see \([31, \text{p.95}] \) and \([8]\)); \( T_{\Omega} \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \) and is of weak type \((1, 1)\), provided that \( \Omega \in L \log L(\mathcal{S}^{n-1}) \) (see \([2, 36]\)). For \( 0 < \alpha < n \), if \( \Omega \in L^{n/(n-\alpha)}(\mathcal{S}^{n-1}) \), then \( M_{\Omega}^0 \) and \( T_{\Omega}^\alpha \) are of weak type \((1, n/(n-\alpha))\) and of type \((p, q)\) for \( 1 < p < q < \infty \) with \( 1/q = 1/p - \alpha/n \) (see \([31]\)). For more backgrounds and related results of the above operators, we refer the readers to \([1, 7, 12, 14, 21, 29, 34, 35]\) and therein references.

Here we will focus on the best constants problems of weak endpoints estimates for the above operators and related operators, which are less fine problems and have attracted lots of attentions. For example, for \( n = 1 \), Davis \([9]\) obtained the best constant of weak-type \((1, 1)\) for Hilbert transform, and Melas \([32]\) proved that \( ||M||_{L^1 \to L^{1, \infty}} = \frac{1 + \sqrt{n}}{11} \), also see \([15]\) for the uncentered Hardy-Littlewood maximal operator. However, for \( n \geq 2 \), things become more subtle. Stein and Stromberg \([37]\) showed that \( ||M||_{L^1 \to L^{1, \infty}} \) is at worst \( n \), and for \( T_{\Omega} \), even for the well-known Riesz transform, there is no such information.

In 2004, Janakiraman \([26]\) considered the Riesz transform and the singular integral operator \( T_{\Omega} \). It was shown that the constant \( ||T_{\Omega}||_{L^1 \to L^{1, \infty}} \) is at worst \((\log n)||\Omega||_1 \). To explore the lower bounds of \( ||M||_{L^1 \to L^{1, \infty}} \) and \( ||T_{\Omega}||_{L^1 \to L^{1, \infty}} \), in 2006, Janakiraman \([27]\) established the following limiting weak-type behavior for \( T_{\Omega} \):

\[
\lim_{\lambda \to 0^+} \lambda^{1/n} \{x \in \mathbb{R}^n : |T_{\Omega} f(x)| > \lambda \} = \frac{1}{n} ||\Omega||_{L^1} ||f||_{L^1}, \quad \text{for } 0 \leq f \in L^1(\mathbb{R}^n),
\]
if $\Omega$ satisfies the vanishing condition (1.2) and a new regularity condition that

$$\sup_{\xi \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\Omega(\theta) - \Omega(\theta + \delta \xi)| d\sigma(\theta) \leq C n \delta \int_{\mathbb{S}^{n-1}} |\Omega(\theta)| d\sigma(\theta), \quad \text{for } 0 < \delta < \frac{1}{n}. \quad (1.3)$$

Meanwhile, they also obtained the following limiting weak-type behavior of $M$:

$$\lim_{\lambda \to 0^+} \lambda \{ x \in \mathbb{R}^n : M(f)(x) > \lambda \} = \|f\|_{L^1}. \quad (1.4)$$

Subsequently, Janakiraman’s results were essentially improved by Ding and Lai [10,11] in the way that the kernel condition (1.3) was relaxed to a weaker $L^1$-Dini condition, that is, $\Omega \in L^1(\mathbb{S}^{n-1})$ and

$$\int_0^1 \frac{\omega_1(\delta)}{\delta} d\delta < \infty,$$

where $\omega_1(\delta) = \sup_{|h| \leq \delta} \int_{\mathbb{S}^{n-1}} |\Omega(\theta + h) - \Omega(\theta)| d\sigma(\theta)$ for $\delta > 0$. The corresponding results were also obtained for $T^\alpha_\Omega$ and $M^\alpha_\Omega$ with $L^\alpha_\Omega$-Dini kernels. Recently, for $0 < \alpha < n$, Guo et al [18,40] extended the corresponding results of $M^\alpha_\Omega$ and $T^\alpha_\Omega$ to the rough kernels cases.

On the other hand, as a natural generalization of linear case, the multilinear operators have been paid lots of attentions. Multilinear Calderón-Zygmund operators were introduced and first studied by Coifman and Meyer [4], [5], [6], and later on by Grafakos and Torres [16], [17], [19]. Since then, great achievements have been made in the theory of multilinear operators. In particular, Lerner, Ombrosi, Pérez, Torres and Trujillo-González [30] introduced the multilinear maximal operator and useing it to establish the weighted theory for multilinear Calderón-Zygmund operators. Very recently, Tan and Wang [38] studied the limiting weak-type behaviors for multilinear fractional integrals.

In this paper, we aim to establish the corresponding limiting weak type behaviors for multilinear singular integral operators and extend them to rough kernels cases. Before stating our results, we first give the definitions of bilinear maximal operators and bilinear singular integral operators with rough kernels.

**Definition 1.1 (Rough bilinear maximal operators).** For $f_1, f_2 \in L_{\text{loc}}(\mathbb{R}^n)$, $\Omega_1$ and $\Omega_2$ are homogeneous of degree 0. The bilinear maximal operator with rough kernels $\vec{\Omega} = (\Omega_1, \Omega_2)$ is defined by

$$M_{\vec{\Omega}}(f_1, f_2)(x) := \sup_{r > 0} \frac{1}{|B(x, r)|^2} \prod_{i=1}^2 \int_{B(x, r)} |\Omega_i(x - y_i)f_i(y_i)| dy_i.$$

**Remark 1.1.** When $\Omega_1 = \Omega_2 \equiv 1$, $M_{\vec{\Omega}}$ coincides with the bilinear maximal function introduced by Lerner et al. in [30].

**Definition 1.2 (Rough bilinear singular integral operators).** For $f_1, f_2 \in L_{\text{loc}}(\mathbb{R}^n)$, $\Omega_1$ and $\Omega_2$ are homogeneous of degree 0, the bilinear singular integral with rough kernel $\vec{\Omega} = (\Omega_1, \Omega_2)$ is defined by

$$T_{\vec{\Omega}}(f_1, f_2)(x) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\Omega_1(x - y_1)\Omega_2(x - y_2)f_1(y_1)f_2(y_2)}{|(x - y_1, x - y_2)|^2} dy_1dy_2.$$
Remark 1.2. Now we give some comments:

Note that, the following space inclusion relationships are true:
\[ L^1 - \text{Dini} \subset L \log^+ L(S^{n-1}) \sim L \log L(S^{n-1}) \subset L^1(S^{n-1}). \]

Inspired by the above results, it is quite natural to ask the following questions:

Questions: When \( \Omega \) belongs to \( L \log L(S^{n-1}) \), do \( M_\Omega \) and \( T_\Omega \) still preserve the limiting weak-type behaviors? If this is the case, then we can obtain the lower bounds of weak-type norms of these operators. Moreover, one may further ask, how about the rough bilinear operators \( M_\Omega \) and \( T_\Omega \) with each \( \Omega_i \in L \log L(S^{n-1}) ? \)

In this paper, we give a positive answer to these questions. Our main results for \( M_\Omega \) and \( T_\Omega \) are as follows:

**Theorem 1.1.** Let \( \Omega_1, \Omega_2 \in L \log L(S^{n-1}) \). Then for all \( f_1, f_2 \in L^1(\mathbb{R}^n) \), we have

(i) \[ \lim_{\lambda \to 0^+} \lambda \left| \left\{ x \in \mathbb{R}^n : \Omega_1 \Omega_2(f_1, f_2)(x) > \lambda \right\} \right|^2 = \frac{\left\| \Omega_1 \Omega_2 \right\|_{L^{1/2}(S^{n-1})}^2}{\omega_{n-1}^2} \prod_{i=1}^2 \| f_i \|_{L^1}; \]

(ii) \[ \lim_{\lambda \to 0^+} \lambda \left\{ x \in \mathbb{R}^n : \Omega_1 \Omega_2(f_1, f_2)(x) - \frac{\prod_{i=1}^2 \| \Omega_i(x) \|_{L^1}}{\omega_{n-1}^2 |x|^{2n}} \right\} > \lambda \right\|^2 = 0. \]

**Theorem 1.2.** Let \( \Omega_1, \Omega_2 \in L \log L(S^{n-1}) \) satisfy vanishing condition on the unit sphere. Then for all \( f_1, f_2 \in L^1(\mathbb{R}^n) \), we have

(i) \[ \lim_{\lambda \to 0^+} \lambda \left| \left\{ x \in \mathbb{R}^n : T_\Omega(f_1, f_2)(x) > \lambda \right\} \right|^2 = \frac{\left\| \Omega_1 \Omega_2 \right\|_{L^{1/2}(S^{n-1})}^2}{n^2} \prod_{i=1}^2 \| f_i \|_{L^1}; \]

(ii) \[ \lim_{\lambda \to 0^+} \lambda \left\{ x \in \mathbb{R}^n : \left| T_\Omega(f_1, f_2)(x) - \frac{\prod_{i=1}^2 \| \Omega_i(x) \|_{L^1}}{|x|^{2n}} \right| > \lambda \right\} > \lambda \right\}^2 = 0. \]

**Remark 1.2.** Now we give some comments:

- The same reasoning as in Theorem 1.1 and Theorem 1.2 still works for \( M_\Omega \) and \( T_\Omega \).
  Even in this linear case, our results are still new.

- Note that the norm \( \left\| \Omega_1 \Omega_2 \right\|_{L^{1/2}(S^{n-1})} \) is finite, which follows from the fact that \( \Omega_1, \Omega_2 \in L \log L(S^{n-1}) \) and Hölder’s inequality.

As a consequence, we obtain the lower bounds of weak norms of \( M_\Omega, T_\Omega, M_\Omega \) and \( T_\Omega \).

**Corollary 1.1.** Let \( \Omega \in L \log L(S^{n-1}) \) and \( \Omega \in L \log L(S^{n-1}) \times L \log L(S^{n-1}) \). Then

(i) \[ \left\| M_\Omega \right\|_{L^1 \to L^{1,\infty}} \geq \left\| \Omega \right\|_{L^1(S^{n-1})/\omega_{n-1}} \text{ and } \left\| M_\Omega \right\|_{L^1 \times L^1 \to L^{1/2,\infty}} \geq \left\| \Omega \Omega_2 \right\|_{L^{1/2}(S^{n-1})/\omega_{n-1}}; \]
Remark 1.3. Let \( \alpha < n \) and \( \Omega_1, \Omega_2 \) are homogeneous of degree 0, the bilinear maximal type operators with rough kernels \( \Omega = (\Omega_1, \Omega_2) \) are defined by

\[
M_\Omega^\alpha(f_1, f_2)(x) := \sup_{r > 0} \frac{1}{|B(x, r)|^{n/(n-\alpha)}} \sum_{i=1}^{2} \int_{B(x, r)} |\Omega_i(x - y_i) f_1(y_i) f_2(y_i)| dy_i.
\]

Remark 1.3. When \( \Omega_1 = \Omega_2 \equiv 1 \), the bilinear fractional maximal operator \( M_\Omega^\alpha (\alpha > 0) \) was independently introduced and studied by Chen, Xue [3] and Moen [33].

Definition 1.4 (Rough bilinear fractional maximal operators). For \( 0 < \alpha < n \), \( f_1, f_2 \in L_{\text{loc}}(\mathbb{R}^n) \), \( \Omega_1 \) and \( \Omega_2 \) are homogeneous of degree 0, the bilinear maximal type operators with rough kernels \( \Omega = (\Omega_1, \Omega_2) \) are defined by

\[
M_\Omega^\alpha(f_1, f_2)(x) := \sup_{r > 0} \frac{1}{|B(x, r)|^{2/(n-\alpha)}} \sum_{i=1}^{2} \int_{B(x, r)} |\Omega_i(x - y_i) f_1(x_i) f_2(y_i)| dy_i.
\]

We summarize our results for \( M_\Omega^\alpha \) and \( T_\Omega^\alpha \) as follows:

Theorem 1.3. Let \( 0 < \alpha < n \) and \( \Omega_1, \Omega_2 \in L^{n/(n-\alpha)}(\mathbb{S}^{n-1}) \). Then for all \( f_1, f_2 \in L^1(\mathbb{R}^n) \), we have

(i) \( \lim_{\lambda \to 0^+} \lambda \left\{ x \in \mathbb{R}^n : M_\Omega^\alpha(f_1, f_2)(x) > \lambda \right\}^{(2(n-\alpha)/n)} = \frac{\|\Omega_1 \Omega_2\|_{L^{n/(n-\alpha)}(\mathbb{S}^{n-1})}}{\omega^{2(n-\alpha)/n} n^{2(n-\alpha)/n}} \prod_{i=1}^{2} \|f_i\|_{L^1}; \)

(ii) \( \lim_{\lambda \to 0^+} \lambda \left\{ x \in \mathbb{R}^n : M_\Omega^\alpha(f_1, f_2)(x) - \frac{\prod_{i=1}^{2} |\Omega_i(x)| \|f_i\|_{L^1}}{\omega^{2(n-\alpha)/n} n^{2(n-\alpha)/n}} > \lambda \right\}^{2(n-\alpha)/n} = 0. \)

Theorem 1.4. Let \( 0 < \alpha < n \) and \( \Omega_1, \Omega_2 \in L^{n/(n-\alpha)}(\mathbb{S}^{n-1}) \). Then for all \( 0 \leq f_1, f_2 \in L^1(\mathbb{R}^n) \), we have

(i) \( \lim_{\lambda \to 0^+} \lambda \left\{ x \in \mathbb{R}^n : \left| T_\Omega^\alpha(f_1, f_2)(x) \right| > \lambda \right\}^{(2(n-\alpha)/n)} = \frac{\|\Omega_1 \Omega_2\|_{L^{n/(n-\alpha)}(\mathbb{S}^{n-1})}}{\omega^{2(n-\alpha)/n} n^{2(n-\alpha)/n}} \prod_{i=1}^{2} \|f_i\|_{L^1}; \)

(ii) \( \lim_{\lambda \to 0^+} \lambda \left\{ x \in \mathbb{R}^n : \left| T_\Omega^\alpha(f_1, f_2)(x) \right| - \frac{\prod_{i=1}^{2} |\Omega_i(x)| \|f_i\|_{L^1}}{|x|^{2(n-\alpha)}} > \lambda \right\}^{2(n-\alpha)/n} = 0. \)
Remark 1.4. In the linear case, similar results for $M_{\Omega}^{\alpha}$ and $T_{\Omega}^{\alpha}$ were proved by Zhao and Guo very recently in [40], while Theorem 1.3 - 1.4 are bilinear version of their results.

Remark 1.5. The above results still hold for the uncentered rough linear and bilinear maximal type operators, with some constant modifications if necessary. Moreover, the same reasoning as in this paper shows that our results can be extended easily to m-linear operators for $m \geq 2$. For simplicity, we only consider the bilinear case.

The organization of this paper is as follows. In Section 2, we will present some basic lemmas, which will be used later. Section 3 will be devoted to give the proofs of Theorems 1.1. In Section 4, the proof of Theorem 1.2 will be given. We need to split and reconstruct the kernels. This is mainly because when we use the splitting method as in the proof of Theorem 1.1, the new kernels may not satisfy the vanishing condition. However, this vanishing kernel condition is necessary to warrant the boundedness of $T_{\Omega}$. In Section 5 we will demonstrate the proofs of Theorem 1.3, 1.4. An extension of Theorem 1.3, 1.4 with power weighted measure will be given in Section 6.

Throughout this paper, the letter $C$, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence, but independent of the essential variables.

2 Preliminaries

We first introduce some basic lemmas which will be used later.

Lemma 2.1. Let $\lambda, \gamma > 0$, $\Phi$ be a homogeneous function of degree 0 on $\mathbb{R}^n$, $S \subset S^{n-1}$ be a measurable set. Then we have

$$\left| \left\{ x \in \mathbb{R}^n : \frac{|\Phi(x)|}{|x|^\gamma} > \lambda, \frac{x}{|x|} \in S \right\} \right| = \frac{\|\Phi\|_{L^{n/\gamma}(S)}}{n\lambda^{n/\gamma}}.$$

Proof. Note that

$$\bigcup_{\theta \in S} \left\{ r\theta : 0 < r < \frac{|\Phi(\theta)|}{\lambda} \right\} = \left\{ x \in \mathbb{R}^n : \frac{|\Phi(x)|}{|x|^\gamma} > \lambda, \frac{x}{|x|} \in S \right\},$$

therefore

$$\left| \left\{ x \in \mathbb{R}^n : \frac{|\Phi(x)|}{|x|^\gamma} > \lambda, \frac{x}{|x|} \in S \right\} \right| = \int_S \int_0^{\frac{|\Phi(\theta)|}{\lambda}} r^{n-1} dr d\sigma(\theta) = \frac{\|\Phi\|_{L^{n/\gamma}(S)}}{n\lambda^{n/\gamma}}.$$

Lemma 2.2. Let $\Omega_1, \Omega_2 \in L \log L(S^{n-1})$, then $M_{\Omega}$ is bounded from $L^1 \times L^1$ to $L^{1/2, \infty}$, and the following norm inequality holds:

$$\|M_{\Omega}\|_{L^1 \times L^1 \to L^{1/2, \infty}} \leq C\|\Omega_1\|_{L \log L(S^{n-1})} \|\Omega_2\|_{L \log L(S^{n-1})}.$$
**Proof.** Let $\Omega \in L \log L(S^{n-1})$ and $f \in L^1$, the famous work of Christ and Rubio de Francia [8] shows that

$$\|M_\Omega f\|_{L^{1,\infty}} \leq C\|\Omega\|_{L \log L(S^{n-1})}\|f\|_{L^1}. \quad (2.1)$$

Since $M_\Omega(f_1, f_2)$ is pointwisely controlled by $M_{\Omega_1} f_1 \cdot M_{\Omega_2} f_2$, then inequality (2.1), together with Hölder’s inequality for weak spaces [14, p.16] yields that

$$\|M_\Omega(f_1, f_2)\|_{L^{1/2,\infty}} \leq C \prod_{i=1}^2 \|M_{\Omega_i} f_i\|_{L^{1,\infty}} \leq C \prod_{i=1}^2 \|\Omega_i\|_{L \log L(S^{n-1})}\|f_i\|_{L^1}.$$

Similarly, we can obtain the weak-type boundedness for $T_\Omega$ as follows.

**Lemma 2.3.** Let $\Omega_1, \Omega_2 \in L \log L(S^{n-1})$ satisfy vanishing condition on $S^{n-1}$. Then $T_\Omega$ is bounded from $L^1 \times L^1$ to $L^{1/2,\infty}$, and enjoys the following norm inequality

$$\|T_\Omega\|_{L^1 \times L^1 \to L^{1/2,\infty}} \leq C\|\Omega_1\|_{L \log L(S^{n-1})}\|\Omega_2\|_{L \log L(S^{n-1})}.$$

Now we state some basic properties about $L \log L$ space.

**Lemma 2.4.** If $\Phi_1(\theta), \Phi_2(\theta) \in L \log L(S^{n-1})$, then they enjoy the following properties:

(i) $\Phi_1(\theta), \Phi_2(\theta) \in L^1(S^{n-1})$, and for $i = 1, 2$, $\|\Phi_i\|_{L^1(S^{n-1})} \leq \|\Phi_i\|_{L \log L(S^{n-1})}$;

(ii) The quasi-triangle inequality is true in $L \log L$ space:

$$\|\Phi_1 + \Phi_2\|_{L \log L(S^{n-1})} \leq 4(\|\Phi_1\|_{L \log L(S^{n-1})} + \|\Phi_2\|_{L \log L(S^{n-1})}).$$

### 3 Proof of Theorem 1.1

#### 3.1 Proof of Theorem 1.1 (i)

**Proof.** Without loss of generality, we can assume $\|f_i\|_{L^1} = 1$ for $i = 1, 2$. The same assumption applies to the rest proofs of our Theorems.

For any $0 < \varepsilon \ll 1$, it is easy to see that there exists a real positive number $r_\varepsilon$, such that

$$\int_{B(0,r_\varepsilon)} |f_i(x)| \, dx > 1 - \varepsilon, \quad \text{for } i = 1, 2.$$

Now we set $g_i = |f_i| \chi_{B(0,r_\varepsilon)}$, $h_i = |f_i| \chi_{B(0,r_\varepsilon)^c}$ and $R_\varepsilon = (1 + 1/\varepsilon) r_\varepsilon$. For $\lambda > 0$, we denote

$$E_\lambda = \{ x : M_\Omega(f_1, f_2)(x) > \lambda \};$$

$$E_1^\lambda = \{ x : M_\Omega(g_1, g_2)(x) > \lambda \};$$

$$E_2^\lambda = \{ x : M_\Omega(g_1, h_2)(x) + M_\Omega(h_1, g_2)(x) + M_\Omega(h_1, h_2)(x) > \lambda \}.$$
Since $M_{\Omega_{i}}$ is sublinear and $M_{\Omega_{i}}(g_{1}, g_{2})(x) \leq M_{\Omega_{i}}(f_{1}, f_{2})(x)$, it follows that

$$E_{\lambda}^{1} \subset E_{\lambda} \subset E_{(1-\sqrt{\varepsilon}/2)\lambda}^{1} \cup E_{\sqrt{\varepsilon}\lambda/2}^{2}.$$  

To set up the argument, we first give a decomposition for the rough kernel $\tilde{\Omega}$. Note that $C(S^{n-1})$ is dense in $L \log L(S^{n-1})$, then there exist two continuous functions $\Omega_{1,\varepsilon}, \Omega_{2,\varepsilon}$ on $S^{n-1}$, homogeneous of degree 0 such that

$$\|\Omega_{i} - \Omega_{i,\varepsilon}\|_{L \log L(S^{n-1})} < \varepsilon, \text{ for } i = 1, 2.$$  

We will use the following notations:

$$\Omega_{1} = \Omega_{1,\varepsilon} + \tilde{\Omega}_{1,\varepsilon}, \quad \Omega_{2} = \Omega_{2,\varepsilon} + \tilde{\Omega}_{2,\varepsilon};$$

$$\tilde{\Omega}_{\varepsilon} = (\Omega_{1,\varepsilon}, \Omega_{2,\varepsilon}), \quad \tilde{\Omega}_{1,\varepsilon} = (\Omega_{1}, \tilde{\Omega}_{2,\varepsilon}), \quad \tilde{\Omega}_{1, \varepsilon, 2} = (\tilde{\Omega}_{1,\varepsilon}, \Omega_{2}), \quad \tilde{\Omega}_{\varepsilon, \varepsilon} = (\tilde{\Omega}_{1,\varepsilon}, \tilde{\Omega}_{2,\varepsilon}),$$

and denote

$$E_{\lambda}^{3} = \left\{ |x| > R_{\varepsilon} : |M_{\tilde{\Omega}_{\varepsilon}}(g_{1}, g_{2})(x) - \lambda| \right\};$$

$$E_{\lambda}^{4} = \left\{ x : M_{\tilde{\Omega}_{1,\varepsilon}}(g_{1}, g_{2})(x) + M_{\tilde{\Omega}_{1, \varepsilon, 2}}(g_{1}, g_{2})(x) + 3M_{\tilde{\Omega}_{\varepsilon, \varepsilon}}(g_{1}, g_{2})(x) > \lambda \right\}.$$  

By triangle inequality, it’s easy to check

$$M_{\tilde{\Omega}_{\varepsilon}}(g_{1}, g_{2})(x) \leq M_{\Omega_{1}}(g_{1}, g_{2})(2) + M_{\Omega_{2}}(g_{1}, g_{2})(x) + M_{\tilde{\Omega}_{1,\varepsilon}}(g_{1}, g_{2})(x) + M_{\tilde{\Omega}_{\varepsilon, \varepsilon}}(g_{1}, g_{2})(x).$$

Now we claim that $M_{\tilde{\Omega}}(g_{1}, g_{2})(x)$ can be dominated by

$$M_{\tilde{\Omega}_{\varepsilon}}(g_{1}, g_{2})(x) + M_{\Omega_{1,\varepsilon}}(g_{1}, g_{2})(x) + M_{\Omega_{2,\varepsilon}}(g_{1}, g_{2})(x) + 3M_{\tilde{\Omega}_{\varepsilon, \varepsilon}}(g_{1}, g_{2})(x).$$

In fact, since $M_{\tilde{\Omega}_{\varepsilon}}(g_{1}, g_{2})$ is controlled by

$$M_{\tilde{\Omega}_{\varepsilon}}(g_{1}, g_{2})(x) + M_{\Omega_{1,\varepsilon, \varepsilon}}(g_{1}, g_{2})(x) + M_{\Omega_{2,\varepsilon}}(g_{1}, g_{2})(x) + M_{\tilde{\Omega}_{\varepsilon, \varepsilon}}(g_{1}, g_{2})(x).$$

On the other hand, triangle inequality gives that

$$M_{\Omega_{1,\varepsilon, \varepsilon}}(g_{1}, g_{2})(x) + M_{\tilde{\Omega}_{1,\varepsilon, \varepsilon}}(g_{1}, g_{2})(x)$$

$$= M_{(\Omega_{1,\varepsilon, \varepsilon} - \tilde{\Omega}_{1,\varepsilon, \varepsilon})}(g_{1}, g_{2})(x) + M_{(\tilde{\Omega}_{1,\varepsilon, \varepsilon} - \tilde{\Omega}_{2,\varepsilon})}(g_{1}, g_{2})(x)$$

$$\leq M_{\tilde{\Omega}_{1,\varepsilon}}(g_{1}, g_{2})(x) + M_{\Omega_{2,\varepsilon}}(g_{1}, g_{2})(x) + 2M_{\tilde{\Omega}_{\varepsilon, \varepsilon}}(g_{1}, g_{2})(x),$$

which implies the claim.

Now it’s clear that

$$E_{(1-\sqrt{\varepsilon}/2)\lambda}^{1} \subset E_{(1-\sqrt{\varepsilon})\lambda}^{3} \cup E_{\sqrt{\varepsilon}\lambda/2}^{4} \cup B(0, R_{\varepsilon})$$

and

$$E_{(1+\sqrt{\varepsilon}/2)\lambda}^{1} \subset E_{\lambda}^{1} \cup E_{\sqrt{\varepsilon}\lambda/2}^{4}. $$

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Therefore

\[ |E_{(1+\sqrt{\varepsilon}/2)\lambda}| - |E_{\sqrt{\varepsilon}\lambda}/2| \leq |E_{\lambda}| \leq |E_{\sqrt{\varepsilon}\lambda}/2| + |E_{(1-\sqrt{\varepsilon})\lambda}| + |E_{\sqrt{\varepsilon}\lambda}/2| + |B(0,R_\varepsilon)|. \]

It’s worthy to point out that, when \( \lambda \) and \( \varepsilon \) are small enough, the left side of above inequality is positive.

Here is the main idea of the proof of Theorem 1.1 (i). To estimate \( E_{\lambda} \), we need to give an upper estimate of right side in the above inequality and a lower estimate of left side in the above inequality. We split the proof into four parts. In Part 1 and Part 2, we will give the upper estimates of \( |E_{\sqrt{\varepsilon}\lambda}/2| \) and \( |E_{\sqrt{\varepsilon}\lambda}/2| \). These two terms are small enough since the norm of \( h_i \) and \( \tilde{\Omega}_{i,\varepsilon} \) are less than \( \varepsilon \). Part 3 and Part 4 will be devoted to give the upper estimate and lower estimate of \( |E_{(1-\sqrt{\varepsilon})\lambda}| \) and \( |E_{(1+\sqrt{\varepsilon}/2)\lambda}| \). In these two parts, the good things are that both \( g_1 \) and \( g_2 \) have compact support, \( \Omega_{1,\varepsilon} \) and \( \Omega_{2,\varepsilon} \) are continuous functions. Combining with the upper estimates in Part 1, 2 and 3, we further give the upper estimate for \( E_{\lambda} \) in Part 3. Moreover, the upper estimates in Part 2 and the lower estimate in Part 4 yield the lower estimate for \( E_{\lambda} \) in Part 4.

**Part 1: Upper estimate for \( |E_{\sqrt{\varepsilon}\lambda}/2| \).**

By Lemma 2.2, one may get

\[
\frac{\sqrt{\varepsilon}\lambda}{6} \left( \left\{ x : M_{\tilde{\Omega}}(g_1, h_2)(x) > \frac{\sqrt{\varepsilon}\lambda}{6} \right\} \right)^2 \leq C \prod_{i=1}^{2} \| \Omega_i \|_{L\log L(S^n-1)} \| g_1 \|_{L^1} \| h_2 \|_{L^1},
\]

which implies

\[
\left\{ x : M_{\tilde{\Omega}}(g_1, h_2)(x) > \frac{\sqrt{\varepsilon}\lambda}{6} \right\} \leq C \prod_{i=1}^{2} \| \Omega_i \|_{L\log L(S^n-1)} \frac{\varepsilon^{1/4}}{\lambda^{1/2}}.
\]

Similar estimates hold for \( M_{\tilde{\Omega}}(h_1, g_2) \) and \( M_{\tilde{\Omega}}(h_1, h_2) \).

These estimates together with the definition of \( E_{\sqrt{\varepsilon}\lambda}/2 \) yield that

\[
|E_{\sqrt{\varepsilon}\lambda}/2| \leq C \prod_{i=1}^{2} \| \Omega_i \|_{L\log L(S^n-1)}^{1/2} \frac{\varepsilon^{1/4}}{\lambda^{1/2}} =: C_{\tilde{\Omega}, 1} \frac{\varepsilon^{1/4}}{\lambda^{1/2}}.
\]

(3.1)

**Part 2: Upper estimate for \( |E_{\sqrt{\varepsilon}\lambda}/2| \).**

By Lemma 2.2, one may obtain

\[
\frac{\sqrt{\varepsilon}\lambda}{6} \left( \left\{ x : M_{\tilde{\Omega}_{i,\varepsilon}}(g_1, g_2)(x) > \frac{\sqrt{\varepsilon}\lambda}{6} \right\} \right)^2 \leq C \| \Omega_1 \|_{L\log L(S^n-1)} \| \tilde{\Omega}_{2,\varepsilon} \|_{L\log L(S^n-1)} \prod_{i=1}^{2} \| g_i \|_{L^1}
\]

\[
\leq C \| \Omega_1 \|_{L\log L(S^n-1)} \varepsilon,
\]

(3.2)
which is equivalent to
\[
\left\{ x : M_{\Omega \varepsilon} (g_1, g_2)(x) > \frac{\sqrt{\varepsilon \lambda}}{6} \right\} \leq C \| \Omega \|_{L \log L(S^{n-1})}^{1/2} \varepsilon^{1/4} \lambda^{1/2}.
\]

Similar estimate also holds for \( M_{\Omega \varepsilon, 1} (g_1, g_2) \), and \( M_{\Omega \varepsilon, 2} (g_1, g_2) \) enjoys the property that
\[
\left\{ x : M_{\Omega \varepsilon, 1} (g_1, g_2)(x) > \frac{\sqrt{\varepsilon \lambda}}{6} \right\} \leq C \varepsilon^{1/2} \lambda^{1/2}.
\]

Now, for \( \varepsilon \) small enough \( (\varepsilon \leq \max_{i=1,2} \| \Omega_i \|_{L \log L(S^{n-1})}) \), combining these estimates as in Part 1, we deduce that
\[
|E_{\varepsilon}^3(1 - \sqrt{\varepsilon})| \leq C \left( \sum_{i=1}^{2} \| \Omega_i \|_{L \log L(S^{n-1})}^{1/2} \right)^{1/4} \lambda^{1/2} =: C_{\Omega, 2} \frac{\varepsilon^{1/4} \lambda^{1/2}}{\lambda^{1/2}}. \tag{3.2}
\]

**Part 3: Upper estimate for \( |E_{\varepsilon}^3| \).**

Since \( \Omega_{1, \varepsilon}, \Omega_{2, \varepsilon} \) are continuous on \( S^{n-1} \), then both of them are uniformly continuous on \( S^{n-1} \). Hence, for any \( \varepsilon > 0 \), there exists a real positive number \( d' \), such that, if \( \sigma(\theta_1, \theta_2) < d' \), we have
\[
\| \Omega_{i, \varepsilon}(\theta_1) - \Omega_{i, \varepsilon}(\theta_2) \| < \varepsilon, \quad \text{for } \theta_1, \theta_2 \in S^{n-1}
\]

It’s easy to see that if \( \varepsilon < \sin d'/(1 - \sin d') \), then for \( |x| > R_\varepsilon, |y_i| \leq r_\varepsilon \), it holds that
\[
\sigma \left( \frac{x}{|x|} \cdot \frac{x - y_i}{|x - y_i|} \right) \leq \arcsin \frac{\varepsilon}{1 + \varepsilon} < d'.
\]

Therefore
\[
|\Omega_{i, \varepsilon}(x - y_i) - \Omega_{i, \varepsilon}(x)| = \left| \Omega_{i, \varepsilon} \left( \frac{x - y_i}{|x - y_i|} \right) - \Omega_{i, \varepsilon} \left( \frac{x}{|x|} \right) \right| \leq \varepsilon. \tag{3.3}
\]

See Figure 1 for 2-dimensional case. According to the uniform continuity of \( \Omega_{i, \varepsilon} \), we know \( \varepsilon \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \).

When \( |x| > R_\varepsilon \) and \( y_i \in \text{supp } g_i \), it’s easy to see that \( |x - y| \geq |x| - r_\varepsilon > |x|/(1 + \varepsilon) \), which means that the radius in the supremum of \( M_{\Omega \varepsilon}(g_1, g_2) \) must be greater than \( |x|/(1 + \varepsilon) \). Henceforth by (3.3),
\[
M_{\Omega \varepsilon}(g_1, g_2)(x) \leq \frac{n^2(1 + \varepsilon)^{2n} \prod_{i=1}^{2} (|\Omega_{i, \varepsilon}(x)| + \varepsilon)}{\omega_{n-1}^2 |x|^{2n}}. \tag{3.4}
\]
Figure 1: Suppose $\varepsilon$ is small enough, $|x| > R_{\varepsilon}$ and $|y_i| \leq r_{\varepsilon}$. Despite the fact that $x$ and $x - y_i$ may be far away, when we pull them back to the unit sphere, these two points are close enough.

Therefore, it follows from Lemma 2.1 and (3.4) that

$$|E_{(1 - \sqrt{\varepsilon})\lambda}^3| \leq \left\{ x \in \mathbb{R}^n : \frac{n^2(1 + \varepsilon)^{2n} \prod_{i=1}^{2} (|\Omega_i,\varepsilon(x)| + \tilde{\varepsilon})}{\omega_{n-1}^2|x|^{2n}} > (1 - \sqrt{\varepsilon})\lambda \right\}$$

$$= (1 + \varepsilon)^n \left\| \prod_{i=1}^{2} (|\Omega_i,\varepsilon| + \tilde{\varepsilon}) \right\|_{L^{1/2}((\mathbb{S}^{n-1})^n)}^{1/2} \frac{1}{\omega_{n-1}(1 - \sqrt{\varepsilon})^{1/2}} \frac{1}{\lambda^{1/2}}.$$  

Combining this estimate with (3.1), (3.2), we obtain the upper estimate for $|E_{\lambda}|$.

$$|E_{\lambda}| \leq |E_{(1 - \sqrt{\varepsilon})\lambda}^3| + |E_{(1 - \sqrt{\varepsilon})\lambda/2}^2| + |E_{(1 - \sqrt{\varepsilon})\lambda/2}^4| + |B(0, R_{\varepsilon})|$$

$$\leq (1 + \varepsilon)^n \left\| \prod_{i=1}^{2} (|\Omega_i,\varepsilon| + \tilde{\varepsilon}) \right\|_{L^{1/2}((\mathbb{S}^{n-1})^n)}^{1/2} \frac{1}{\omega_{n-1}(1 - \sqrt{\varepsilon})^{1/2}} \frac{1}{\lambda^{1/2}} + (C_{\tilde{\Omega},1} + C_{\tilde{\Omega},2}) \frac{\varepsilon^{1/4}}{\lambda^{1/2}} + \frac{\omega_{n-1} R_{\varepsilon}^n}{n}.$$  

Now multiplying $\lambda^{1/2}$ on both sides of the above inequality and let $\lambda \to 0^+$, we get

$$\lim_{\lambda \to 0^+} \lambda^{1/2} |E_{\lambda}| \leq \frac{(1 + \varepsilon)^n \left\| \prod_{i=1}^{2} (|\Omega_i,\varepsilon| + \tilde{\varepsilon}) \right\|_{L^{1/2}((\mathbb{S}^{n-1})^n)}^{1/2}}{\omega_{n-1}(1 - \sqrt{\varepsilon})^{1/2}} + (C_{\tilde{\Omega},1} + C_{\tilde{\Omega},2}) \varepsilon^{1/4}. $$

Thus it remains to prove

$$\lim_{\varepsilon \to 0^+} \left\| \prod_{i=1}^{2} (|\Omega_i,\varepsilon| + \tilde{\varepsilon}) \right\|_{L^{1/2}((\mathbb{S}^{n-1})^n)}^{1/2} = \|\Omega_1 \Omega_2\|_{L^{1/2}((\mathbb{S}^{n-1})^n)}^{1/2}. \quad (3.5)$$
As a matter of fact, on the space $L^{1/2}$, there is no triangle inequality, it enjoys the quasi-triangle inequality, which will produce a constant 2. This constant can not be ignored, it’s really a problem. However, if we add a power of 1/2 on the outside of the $L^{1/2}$ norm, this problem degenerates to an absolute value’s triangle inequality with power 1/2. It’s exactly what we need. Hence

$$
\|((\Omega_1,\varepsilon) + \bar{\varepsilon})(\Omega_2,\varepsilon)\|^{1/2}_{L^{1/2}(\Omega)} - \|\Omega_1\Omega_2\|^{1/2}_{L^{1/2}(\Omega)} \\
\leq \|((\Omega_1,\varepsilon) + \bar{\varepsilon})(\Omega_2,\varepsilon) - \|\Omega_1\Omega_2\|^{1/2}_{L^{1/2}(\Omega)}.
$$

Applying this technic again, together with Hölder’s inequality, we get

$$
\|((\Omega_1,\varepsilon) + \bar{\varepsilon})(\Omega_2,\varepsilon)\|^{1/2}_{L^{1/2}(\Omega)} - \|\Omega_1\Omega_2\|^{1/2}_{L^{1/2}(\Omega)} \\
\leq \|((\Omega_1,\varepsilon) + \bar{\varepsilon})(\Omega_2,\varepsilon)\|^{1/2}_{L^{1/2}(\Omega)} + \|\Omega_1\|^{1/2}_{L^1}((\Omega_2,\varepsilon)\|_{L^1} + \varepsilon \omega_{n-1})^{1/2} + \|\Omega_1\|^{1/2}_{L^1}((\Omega_2,\varepsilon)\|_{L^1} + \varepsilon \omega_{n-1})^{1/2}.
$$

By Lemma 2.4 (i), we can deduce

$$
\|((\Omega_1,\varepsilon) + \bar{\varepsilon})(\Omega_2,\varepsilon)\|^{1/2}_{L^{1/2}(\Omega)} - \|\Omega_1\Omega_2\|^{1/2}_{L^{1/2}(\Omega)} \\
\leq (\varepsilon + \varepsilon \omega_{n-1})^{1/2} \left( \sum_{i=1}^{2} \|\Omega_i\|^{1/2}_{L^1} + \varepsilon^{1/2} + (\varepsilon \omega_{n-1})^{1/2} \right).
$$

The right side of this inequality converges to 0 as $\varepsilon \to 0^+$, which implies that inequality (3.5) holds.

Hence, by the arbitrariness of $\varepsilon$, it follows that

$$
\lim_{\lambda \to 0^+} \lambda^{1/2} |E_{\lambda}| \leq \frac{\|\Omega_1\|^{1/2}_{L^{1/2}(\Omega)}}{\omega_{n-1}},
$$

which gives the desired upper estimate.

**Part 4: Lower estimate for $|E^3_{(1+\sqrt{\varepsilon})\lambda}|$.**

When $|x| > R_\varepsilon$, $|y_1|, |y_2| \leq r_\varepsilon$, by (3.3), we have

$$
|\Omega_1,\varepsilon(x)| - \varepsilon \leq |\Omega_1,\varepsilon(x - y_1)|, \quad |\Omega_2,\varepsilon(x)| - \varepsilon \leq |\Omega_2,\varepsilon(x - y_2)|.
$$

However, the following inequality

$$
|\Omega_1,\varepsilon(x)| - \varepsilon |\Omega_2,\varepsilon(x)| - \varepsilon \leq |\Omega_1,\varepsilon(x - y_1)|\Omega_2,\varepsilon(x - y_2)|
$$

may fail, which makes it’s impossibile to give a lower bound of $M_{(\varepsilon)}(g_1, g_2)$ for all $|x| > R_\varepsilon$. To overcome this obstacle, we introduce the following two auxiliary sets.

$$
S_\varepsilon := \{ \theta \in S^{n-1} : |\Omega_1,\varepsilon(\theta)|, |\Omega_2,\varepsilon(\theta)| > \varepsilon \},
$$

$$
V_\varepsilon := \left\{ x \in \mathbb{R}^n : \frac{x}{|x|} \in S_\varepsilon \right\}.
$$
Indeed, using the similar arguments as in Part 3, we obtain

\[ M_{\bar{g}_\epsilon}(g_1, g_2)(x) \geq \frac{n^2(1 - \epsilon)^2 \prod_{i=1}^{2} (|\Omega_{i, \epsilon}| \geq \bar{\epsilon})}{\omega_{n-1}(1 + \epsilon)^{2n}|x|^{2n}}. \]  

(3.8)

Lemma 2.1 together with (3.8) may lead to

\[ |E^3_{(1 + \sqrt{\epsilon}/2)\lambda}| \geq |E^3_{(1 + \sqrt{\epsilon}/2)\lambda} \cap V_{\bar{\epsilon}}| \]

\[ \geq \left\{ x \in V_{\bar{\epsilon}} : \frac{n^2(1 - \epsilon)^2 \prod_{i=1}^{2} (|\Omega_{i, \epsilon}| \geq \bar{\epsilon})}{\omega_{n-1}(1 + \epsilon)^{2n}|x|^{2n}} > \left( 1 + \frac{\sqrt{\epsilon}}{2} \right) \lambda \right\} - |B(0, R_\epsilon)| \]

\[ = \frac{(1 - \epsilon) \left\| \prod_{i=1}^{2} (|\Omega_{i, \epsilon}| \geq \bar{\epsilon}) \right\|_{L^{1/2}(S_{\bar{\epsilon}})}^{1/2} \frac{1}{\lambda^{1/2}} - \frac{\omega_{n-1}R_\epsilon^n}{n}, \]

which, combining with (3.2), further gives that

\[ |E_\lambda| \geq |E^3_{(1 + \sqrt{\epsilon}/2)\lambda}| - |E^4_{\sqrt{\epsilon}/2}| \]

\[ \geq \frac{(1 - \epsilon) \left\| \prod_{i=1}^{2} (|\Omega_{i, \epsilon}| \geq \bar{\epsilon}) \right\|_{L^{1/2}(S_{\bar{\epsilon}})}^{1/2} \frac{1}{\lambda^{1/2}} - C_{\bar{\Omega}, \epsilon}^{1/4} \lambda^{1/2} - \frac{\omega_{n-1}R_\epsilon^n}{n}. \]

Note that when \( \epsilon \) and \( \lambda \) are small enough (\( \lambda \ll \epsilon \)), the right side of above inequality is positive. Multiplying \( \lambda^{1/2} \) on both sides and let \( \lambda \to 0^+ \), we obtain that

\[ \lim_{\lambda \to 0^+} \lambda^{1/2}|E_\lambda| \geq \frac{(1 - \epsilon) \left\| \prod_{i=1}^{2} (|\Omega_{i, \epsilon}| \geq \bar{\epsilon}) \right\|_{L^{1/2}(S_{\bar{\epsilon}})}^{1/2} \frac{1}{\lambda^{1/2}} - C_{\bar{\Omega}, \epsilon}^{1/4} \lambda^{1/2}. \]

Now we are in a position to prove

\[ \lim_{\epsilon \to 0} \left\| \prod_{i=1}^{2} (|\Omega_{i, \epsilon}| \geq \bar{\epsilon}) \right\|_{L^{1/2}(S_{\bar{\epsilon}})}^{1/2} = \left\| \Omega_1 \Omega_2 \right\|_{L^{1/2}(S^{n-1})}, \]

(3.9)

Indeed, using the similar arguments as in Part 3, we obtain

\[ \left\| (|\Omega_{1, \epsilon}| \geq \bar{\epsilon})(|\Omega_{2, \epsilon}| \geq \bar{\epsilon}) \right\|_{L^{1/2}(S_{\bar{\epsilon}})}^{1/2} - \left\| \Omega_1 \Omega_2 \right\|_{L^{1/2}(S_{\bar{\epsilon}})}^{1/2} \]

\[ \leq (\epsilon + \bar{\epsilon} \omega_{n-1})^{1/2} \left( \sum_{i=1}^{2} \left\| \Omega_i \right\|_{L_{\log L}(S^{n-1})}^{1/2} + \epsilon^{1/2} + (\bar{\epsilon} \omega_{n-1})^{1/2} \right). \]
Then inequality (3.8) follows easily from the fact
\[
\lim_{\varepsilon \to 0} \| \Omega_1 \Omega_2 \|_{L^{1/2}(S_\varepsilon)} = \| \Omega_1 \Omega_2 \|_{L^{1/2}((\cup_{\varepsilon>0} S_\varepsilon))} = \| \Omega_1 \Omega_2 \|_{L^{1/2}(\mathbb{R}^{n-1})},
\]
which can be demonstrated by Levi’s theorem.

Since \( \varepsilon \) is arbitrary and small enough, we get
\[
\lim_{\lambda \to 0^+} \lambda^{1/2} |E_\lambda| \geq \frac{\| \Omega_1 \Omega_2 \|_{L^{1/2}(\mathbb{R}^{n-1})}^{1/2}}{\omega_{n-1}}.
\]
Finally, from (3.6) and (3.10), we obtain that
\[
\lim_{\lambda \to 0^+} \lambda |E_\lambda|^2 = \frac{\| \Omega_1 \Omega_2 \|_{L^{1/2}(\mathbb{R}^{n-1})}^2}{\omega_{n-1}^2}.
\]

3.2 Proof of Theorem 1.1 (ii)

Proof. For \( \lambda > 0 \), we set
\[
G_\lambda = \left\{ x : \left| M_{\Omega_1}(f_1, f_2)(x) - \frac{\| \Omega_1(x) \Omega_2(x) \|^2}{\omega_{n-1}^2 |x|^{2n/n^2}} \right| > \lambda \right\};
\]
\[
G_\lambda^1 = \left\{ x : |x| > R_\varepsilon : \left| M_{\Omega_1}(g_1, g_2)(x) - \frac{\| \Omega_1(x) \Omega_2(x) \|^2}{\omega_{n-1}^2 |x|^{2n/n^2}} \right| > \lambda \right\};
\]
\[
G_\lambda^2 = \left\{ x : \left| \frac{\| \Omega_1(x) \Omega_2(x) \|^2}{\omega_{n-1}^2 |x|^{2n/n^2}} - \frac{\| \Omega_1(x) \Omega_2(x) \|^2}{\omega_{n-1}^2 |x|^{2n/n^2}} \right| > \lambda \right\}.
\]

Now we claim that
\[
G_\lambda \subset G_{(1-2\sqrt{\varepsilon})\lambda}^1 \cup G_{\sqrt{\varepsilon}\lambda}^2 \cup E_{\sqrt{\varepsilon}\lambda/2}^2 \cup E_{\sqrt{\varepsilon}\lambda/2}^4 \cup B(0, R_\varepsilon).
\]

To see this, it sufficient to show the complementary set of right side is contained in \( G_\lambda^c \). For any \( x \) in the complementary set of right side, we have the following facts: \( |x| > R_\varepsilon \) and
\[
\left| M_{\Omega_1}(g_1, g_2)(x) - \frac{\| \Omega_1(x) \Omega_2(x) \|^2}{\omega_{n-1}^2 |x|^{2n/n^2}} \right| \leq (1 - 2\sqrt{\varepsilon})\lambda;
\]
\[
M_{\Omega_1}(g_1, h_2)(x) + M_{\Omega_1}(h_1, g_2)(x) + M_{\Omega_1}(h_1, h_2)(x) \leq \frac{\sqrt{\varepsilon}}{2}\lambda;
\]
\[
M_{\Omega_1}(g_1, g_2)(x) + M_{\Omega_1}(g_1, g_2)(x) + 3M_{\Omega_1}(g_1, g_2)(x) \leq \frac{\sqrt{\varepsilon}}{2}\lambda;
\]
\[
\left| \frac{\| \Omega_1(x) \Omega_2(x) \|^2}{\omega_{n-1}^2 |x|^{2n/n^2}} - \frac{\| \Omega_1(x) \Omega_2(x) \|^2}{\omega_{n-1}^2 |x|^{2n/n^2}} \right| \leq \sqrt{\varepsilon}\lambda.
\]
From these inequalities, it’s easy to deduce that

\[ M_{\Omega}(f_1, f_2)(x) \geq M_{\Omega}(g_1, g_2)(x) \geq M_{\Omega}(g_1, g_2)(x) - \frac{\sqrt{\varepsilon}}{2} \lambda \]

\[ \geq \frac{|\Omega_{1,\varepsilon}(x)|\Omega_{2,\varepsilon}(x)|}{\omega^{2n-1}|x|^{2n/2}} - \left(1 - \frac{3\sqrt{\varepsilon}}{2}\right) \lambda \geq \frac{|\Omega_1(x)|\Omega_2(x)|}{\omega^{2n-1}|x|^{2n/2}} - \lambda \]

and

\[ M_{\Omega}(f_1, f_2)(x) \leq M_{\Omega}(g_1, g_2)(x) + \frac{\sqrt{\varepsilon}}{2} \lambda \leq M_{\Omega}(g_1, g_2)(x) + \sqrt{\varepsilon} \lambda \]

\[ \leq \frac{|\Omega_{1,\varepsilon}(x)|\Omega_{2,\varepsilon}(x)|}{\omega^{2n-1}|x|^{2n/2}} + (1 - \sqrt{\varepsilon}) \lambda \leq \frac{|\Omega_1(x)|\Omega_2(x)|}{\omega^{2n-1}|x|^{2n/2}} + \lambda, \]

which are equivalent to

\[ |M_{\Omega}(f_1, f_2)(x) - \frac{|\Omega_1(x)|\Omega_2(x)|}{\omega^{2n-1}|x|^{2n/2}}| \leq \lambda, \]

which implies \( x \in G^{\lambda}_1 \), and the claim (3.11) is proved.

Hence by (3.11), we get

\[ |G^{\lambda}_1| \leq |G^{1}_{(1-2\sqrt{\varepsilon})\lambda}| + |G^{2}_{\sqrt{\varepsilon}\lambda}| + |E^{2}_{\sqrt{\varepsilon}\lambda/2}| + |E^{1}_{\sqrt{\varepsilon}\lambda/2}| + |B(0, R_\varepsilon)|. \tag{3.12} \]

It remains to estimate \( |G^{1}_{(1-2\sqrt{\varepsilon})\lambda}| \) and \( |G^{2}_{\sqrt{\varepsilon}\lambda}| \). We split the proof into three parts.

**Part 1: Upper estimate for \( G^{2}_{\sqrt{\varepsilon}\lambda}| \).**

Applying Lemma 2.1 and the method as in Section 3, we conclude that

\[ |G^{2}_{\sqrt{\varepsilon}\lambda}| = \left( \frac{\sum_{i=1}^{2} \|\Omega_i\|_{L_{\log L}(S^{n-1})}}{\varepsilon^{1/4}\omega^{n-1}1^{1/2}} \right)^{1/2} \leq C \frac{\sum_{i=1}^{2} \|\Omega_i\|_{L_{\log L}(S^{n-1})}}{\varepsilon^{1/4}\omega^{n-1}1^{1/2}} \leq \frac{C}{\varepsilon^{1/4}} \frac{1}{\lambda^{1/2}}. \]

Take \( \varepsilon \) small enough \((\varepsilon \leq \max_{i=1,2} \|\Omega_i\|_{L_{\log L}(S^{n-1})})\), then \( |G^{2}_{\sqrt{\varepsilon}\lambda}| \) enjoys the property that

\[ |G^{2}_{\sqrt{\varepsilon}\lambda}| \leq C \frac{\sum_{i=1}^{2} \|\Omega_i\|_{L_{\log L}(S^{n-1})}}{\varepsilon^{1/4}\omega^{n-1}1^{1/2}} \leq \frac{C}{\varepsilon^{1/4}} \frac{1}{\lambda^{1/2}}. \tag{3.13} \]

**Part 2: Upper estimate for \( |G^{1}_{(1-2\sqrt{\varepsilon})\lambda} \cap V^\circ| \).**

We denote \( I(x) \) and \( II(x) \) by

\[ I(x) = \left( \frac{|\Omega_{1,\varepsilon}(x)| + \varepsilon)(|\Omega_{2,\varepsilon}(x)| + \varepsilon)}{\omega^{n-1}1|x|^{2n/2}(1 + \varepsilon)^{2n}} - \frac{(1 - \varepsilon)^2(|\Omega_{1,\varepsilon}(x)| - \varepsilon)(|\Omega_{2,\varepsilon}(x)| - \varepsilon)}{\omega^{n-1}1|x|^{2n/2}(1 + \varepsilon)^{2n}} \right) ; \]

\[ II(x) = \left( \frac{|\Omega_{1,\varepsilon}(x)| + \varepsilon)(|\Omega_{2,\varepsilon}(x)| + \varepsilon)}{\omega^{n-1}1|x|^{2n/2}(1 + \varepsilon)^{2n}} - \frac{|\Omega_{1,\varepsilon}(x)|\Omega_{2,\varepsilon}(x)}{\omega^{n-1}1|x|^{2n/2}} \right). \]

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Obviously $I(x), II(x)$ are positive. Furthermore, for $x \in \{x : |x| > R_{\varepsilon}, x \in V_{\varepsilon}\}$, from (3.4) and (3.8), it holds that

$$\left| M_{\Omega_{\varepsilon}}(g_1, g_2)(x) - \frac{\Omega_{1,\varepsilon}(x)\Omega_{2,\varepsilon}(x)}{\omega_{n-1}^2|x|^{2n/n^2}} \right| \leq \frac{(|\Omega_{1,\varepsilon}(x)| + \varepsilon)(|\Omega_{2,\varepsilon}(x)| + \varepsilon)}{\omega_{n-1}^2|x|^{2n/n^2}(1 + \varepsilon)^{2n}} - M_{\Omega_{\varepsilon}}(g_1, g_2)(x)$$

$$+ \frac{(\Omega_{1,\varepsilon}(x)| + \varepsilon)(|\Omega_{2,\varepsilon}(x)| + \varepsilon)}{\omega_{n-1}^2|x|^{2n/n^2}(1 + \varepsilon)^{2n}} - \frac{\Omega_{1,\varepsilon}(x)\Omega_{2,\varepsilon}(x)}{\omega_{n-1}^2|x|^{2n/n^2}} \leq I(x) + II(x),$$

then it follows that

$$|G_{(1-2\sqrt{\varepsilon})\lambda} \cap V_{\varepsilon}| \leq \left| \left\{ |x| > R_{\varepsilon} : I(x) + II(x) > (1 - 2\sqrt{\varepsilon})\lambda, x \in V_{\varepsilon} \right\} \right|$$

$$\leq \left| \left\{ |x| > R_{\varepsilon} : I(x) > (1 - 2\sqrt{\varepsilon} - \sqrt{(1 + \varepsilon)^{2n} - 1})\lambda, x \in V_{\varepsilon} \right\} \right| + \left| \left\{ |x| > R_{\varepsilon} : II(x) > \sqrt{(1 + \varepsilon)^{2n} - 1 + \sqrt{\varepsilon}}\lambda, x \in V_{\varepsilon} \right\} \right|.$$ 

It’s sufficient to give the upper estimates for the two terms in the right side.

For $I(x)$, we split it into two terms:

$$I(x) = \frac{(\Omega_{1,\varepsilon}(x)| + \varepsilon)(|\Omega_{2,\varepsilon}(x)| + \varepsilon)}{\omega_{n-1}^2|x|^{2n/n^2}} \left( (1 + \varepsilon)^{2n} - \frac{(1 - \varepsilon)^2}{(1 + \varepsilon)^{2n}} \right)$$

$$+ \frac{2(1 - \varepsilon)^2\varepsilon(|\Omega_{1,\varepsilon}(x)| + |\Omega_{2,\varepsilon}(x)|)}{\omega_{n-1}^2(1 + \varepsilon)^{2n}|x|^{2n/n^2}} =: I_1(x) + I_2(x).$$

When $\varepsilon$ is small enough, Lemma 2.1 gives that

$$\left| \left\{ |x| > R_{\varepsilon} : I_1(x) > (1 - 2\sqrt{\varepsilon} - 2\sqrt{\varepsilon} - \sqrt{(1 + \varepsilon)^{2n} - 1})\lambda, x \in V_{\varepsilon} \right\} \right|$$

$$\leq \left\| (\Omega_{1,\varepsilon} + \varepsilon)(\Omega_{2,\varepsilon} + \varepsilon) \right\|_{L^{1/2}(\mathbb{S}^{n-1})}^{1/2} \left( (1 + \varepsilon)^{4n} - (1 - \varepsilon)^2 \right)^{1/2} \lambda^{1/2}$$

$$\leq C \frac{\left\| \Omega_{1,\varepsilon} \Omega_{2,\varepsilon} \right\|_{L^{1/2}(\mathbb{S}^{n-1})}^{1/2}}{\omega_{n-1}} \frac{\varepsilon^{1/4}}{\lambda^{1/2}}$$

and

$$\left| \left\{ |x| > R_{\varepsilon} : I_2(x) > \sqrt{\varepsilon}\lambda, x \in V_{\varepsilon} \right\} \right| \leq \frac{\sqrt{2}(1 - \varepsilon) \left\| \Omega_{1,\varepsilon} \right\|_{L^{1/2}(\mathbb{S}^{n-1})} \varepsilon^{1/4}}{\omega_{n-1}(1 + \varepsilon)^n}$$

$$\leq C \sum_{i=1}^{2} \frac{\left\| \Omega_{i} \right\|_{L_{\log L}(\mathbb{S}^{n-1})}}{\omega_{n-1}} \frac{1}{\lambda^{1/2}}.$$
These two estimates indicate that
\[
\left| \{ |x| > R_\varepsilon : I(x) > \left( 1 - 2\sqrt{\varepsilon} - \sqrt{\varepsilon_1} - \sqrt{(1 + \varepsilon)^{2n} - 1} \right) \lambda, x \in \mathcal{V}_\varepsilon \} \right| \leq \frac{C_\Omega_4}{\lambda^{1/2}} \left( \varepsilon^{1/2} + \varepsilon_1^{1/4} \right). \tag{3.14}
\]
As for $II(x)$, we split it in the following way:
\[
II(x) = \frac{(\Omega_1(x) + \varepsilon)(\Omega_2(x) + \varepsilon)}{\omega_{n-1}^2 / |x|^{2n}} \left( (1 + \varepsilon)^{2n} - 1 \right) + \frac{\varepsilon \left( |\Omega_1(x)| + |\Omega_2(x)| + \varepsilon \right)}{\omega_{n-1}^2 / |x|^{2n}}
\]
and may get the following desired estimate:
\[
\left| \{ |x| > R_\varepsilon : II(x) > \left( (1 + \varepsilon)^{2n} - 1 + \sqrt{\varepsilon} \right) \lambda, x \in \mathcal{V}_\varepsilon \} \right| \leq \frac{C_\Omega_5 \varepsilon}{\lambda^{1/2}} \left( \varepsilon^{1/4} + \varepsilon_1^{1/4} \right). \tag{3.15}
\]
Combining (3.14) and (3.15), we get
\[
\left| G_{1-2\sqrt{\varepsilon}}(1) \cap \mathcal{V}_\varepsilon \right| \leq \frac{C_\Omega_4 + C_\Omega_5}{\lambda^{1/2}} \left( \varepsilon^{1/4} + \varepsilon_1^{1/4} \right). \tag{3.16}
\]

Part 3: Upper estimate for $\left| G_{1-2\sqrt{\varepsilon}}(1) \cap \mathcal{V}_\varepsilon \right|$. For $|x| > R_\varepsilon$, it follows from the inequality (3.4) that
\[
\left| M_{\Omega_\varepsilon}(g_1, g_2)(x) \right| \leq \frac{|\Omega_1(x)| |\Omega_2(x)|}{\omega_{n-1}^2 / |x|^{2n}} \leq 2 \frac{\left( |\Omega_1(x)| + \varepsilon (|\Omega_2(x)| + \varepsilon) \right)}{\omega_{n-1}^2 / |x|^{2n}}.
\]
Then by Lemma 2.1, $G_{1-2\sqrt{\varepsilon}}(1) \cap \mathcal{V}_\varepsilon$ is dominated by
\[
\left| G_{1-2\sqrt{\varepsilon}}(1) \cap \mathcal{V}_\varepsilon \right| \leq \frac{\sqrt{2}(1 + \varepsilon)^n}{\omega_{n-1}^2 (1 - 2\sqrt{\varepsilon})^{1/2}} \frac{\| \prod_{i=1}^2 (|\Omega_i| + \varepsilon) \|^2_{L^1(S^{n-1} \setminus S')} 1}{\lambda^{1/2}}.
\]
Let $\theta \in S^{n-1} \setminus S'_\varepsilon$, then at least one of $|\Omega_1(\theta)|$ and $|\Omega_2(\theta)|$ should be not more than $\varepsilon$. So we let
\[
S'_\varepsilon = \{ \theta : |\Omega_1(\theta)| \leq \varepsilon \} \quad \text{and} \quad S''_\varepsilon = \{ \theta : |\Omega_2(\theta)| \leq \varepsilon \}.
\]
It is obvious that $S^{n-1} \setminus S'_\varepsilon \subset S'_\varepsilon \cup S''_\varepsilon$, and
\[
\left\| \prod_{i=1}^2 (|\Omega_i| + \varepsilon) \right\|^{1/2}_{L^1(S^{n-1} \setminus S')} \leq \left\| \prod_{i=1}^2 (|\Omega_i| + \varepsilon) \right\|^{1/2}_{L^1(S'_\varepsilon)} + \left\| \prod_{i=1}^2 (|\Omega_i| + \varepsilon) \right\|^{1/2}_{L^1(S''_\varepsilon)} \leq \sum_{i=1}^2 \left( \|2\varepsilon\|_{L^1(S^{n-1})} \|\Omega_i| + \varepsilon\|_{L^1(S^{n-1})} \right)^{1/2} \leq (2\omega_{n-1}\varepsilon)^{1/2} \sum_{i=1}^2 \left( \|\Omega_i\|_{L \log L(S^{n-1})} + \varepsilon + \varepsilon \omega_{n-1} \right)^{1/2}.
\]
Then for $\varepsilon$ small enough, we get
\[
|G_{(1-2\sqrt{\varepsilon})}\lambda \cap V_{\varepsilon}^\varepsilon| \leq \frac{C_{\tilde{\Omega}}}{\lambda^{1/2}} \leq \frac{C_{\tilde{\Omega}}}{\lambda^{1/2}} \varepsilon^{1/4}.
\] (3.17)

Finally, by (3.1), (3.2), (3.12), (3.13), (3.16) and (3.17), it holds that
\[
|G_{\lambda}| \leq \frac{C_{\tilde{\Omega}}}{\lambda^{1/2}} (\varepsilon^{1/4} + \varepsilon^{1/2}),
\] (3.18)

where $C_{\tilde{\Omega}} = 4 \max(C_{\tilde{\Omega},1,1}, C_{\tilde{\Omega},1,2}, C_{\tilde{\Omega},3,1}, C_{\tilde{\Omega},4,1} + C_{\tilde{\Omega},5,1}, C_{\tilde{\Omega},4,2})$.

Multiplying $\lambda^{1/2}$ on both sides of (3.18), and by the arbitrariness of $\varepsilon$, we have
\[
\lim_{\lambda \to 0^+} \lambda |G_{\lambda}|^2 = 0.
\]

\[\square\]

4 Proof of Theorem 1.2

We use the same notations as in Section 3, and denote
\[
F_\lambda = \{ x : |T_{\tilde{\Omega}}(f_1, f_2)(x)| > \lambda \};
\]
\[
F_\lambda^1 = \{ x : |T_{\tilde{\Omega}}(g_1, g_2)(x)| > \lambda \};
\]
\[
F_\lambda^2 = \{ x : |T_{\tilde{\Omega}}(g_1, g_2)(x)| + |T_{\tilde{\Omega}}(h_1, g_2)(x)| + |T_{\tilde{\Omega}}(h_1, h_2)(x)| > \lambda \}.
\]

Then by triangle inequality, we have
\[
F_\lambda \subset F_\lambda^1 \cup F_\lambda^2 \quad \text{and} \quad F_\lambda^3 \subset F_\lambda^1 \cup F_\lambda^2.
\]

It is quite natural to use $\tilde{\Omega}_{i,\varepsilon}, \tilde{\Omega}_{1,\varepsilon}, \tilde{\Omega}_{2,\varepsilon}$ and $\tilde{\Omega}_{2,\varepsilon}$ as in Theorem 1.1 to define $F_\lambda^3$ and $F_\lambda^4$. In particular, to control $F_\lambda^4$, we need to use the boundedness of $T_{\tilde{\Omega}}$. However, the decomposition may not preserve the vanishing property of the kernel on the unit sphere.

To overcome this difficulty, we make a proper modification and reconstruct the kernel as follows: for $i = 1, 2$, let
\[
\tilde{\Omega}_{i,\varepsilon}' := \tilde{\Omega}_{i,\varepsilon} - \int_{S_{n-1}} \tilde{\Omega}_{i,\varepsilon}(\theta) d\sigma(\theta) \quad \text{and} \quad \Omega_{i,\varepsilon}' = \Omega_i - \tilde{\Omega}_{i,\varepsilon}'.
\]

Obviously $\tilde{\Omega}_{i,\varepsilon}'$ satisfies the vanishing condition, and so does $\Omega_{i,\varepsilon}'$. Furthermore, the $L \log L$ norm of $\tilde{\Omega}_{i,\varepsilon}'$ can be controlled by a constant multiply $\varepsilon$.

Thus we define
\[
\tilde{\Omega}_{i,\varepsilon}' = (\Omega_{i,\varepsilon}', \Omega_{2,\varepsilon}');
\]
\[
\tilde{\Omega}_{1,\varepsilon}' = (\Omega_{1,\varepsilon}', \Omega_{2,\varepsilon}); \quad \tilde{\Omega}_{1,\varepsilon}' = (\Omega_{1,\varepsilon}', \Omega_{2,\varepsilon}); \quad \tilde{\Omega}_{i,\varepsilon}' = (\tilde{\Omega}_{i,\varepsilon}', \tilde{\Omega}_{2,\varepsilon}');
\]
\[
F_\lambda^3 := \left\{ x : |T_{\tilde{\Omega}_{i,\varepsilon}'}(g_1, g_2)(x)| > \lambda \right\};
\]
\[
F_\lambda^4 := \left\{ x : |T_{\tilde{\Omega}_{i,\varepsilon}'}(g_1, g_2)(x)| + |T_{\tilde{\Omega}_{i,\varepsilon}'}(g_1, g_2)(x)| + |T_{\tilde{\Omega}_{i,\varepsilon}'}(g_1, g_2)(x)| > \lambda \right\}.
\]
Then it follows that
\[ F_\lambda \subset F^{2\sqrt{\varepsilon} \lambda / 2} \cup F^{3(1-\sqrt{\varepsilon})\lambda} \cup F^{4\sqrt{\varepsilon} \lambda / 2} \cup \overline{B(0, R_\varepsilon)}, \]
and
\[ F^{3(1+\sqrt{\varepsilon})\lambda} \subset F_\lambda \cup F^{2\sqrt{\varepsilon} \lambda / 2} \cup F^{4\sqrt{\varepsilon} \lambda / 2}. \]  
(4.1)

The same reasoning as in Section 3 yields that
\[ |F^{2\sqrt{\varepsilon} \lambda / 2}| \leq C'_{\Omega,1} \varepsilon^{1/4}, \quad |F^{4\sqrt{\varepsilon} \lambda / 2}| \leq C'_{\Omega,2} \varepsilon^{1/4}, \]  
(4.2)

and
\[ |T_{\Omega,\varepsilon}(g_1, g_2)(x)| \leq \frac{(1+\varepsilon)^2 \prod_{i=1}^{2} (|\Omega_{i,\varepsilon}'(x)| + \varepsilon)}{|x|^{2n}}, \quad \text{for} \ |x| > R_\varepsilon; \]
\[ |T_{\Omega,\varepsilon}'(g_1, g_2)(x)| \leq \frac{2 \prod_{i=1}^{2} (|\Omega_{i,\varepsilon}'(x)| + \varepsilon)}{(1+\varepsilon)^2 |x|^{2n}}, \quad \text{for} \ |x| > R_\varepsilon, \in V_\varepsilon. \]  
(4.3)

From (4.1), (4.2) and (4.3), using the same method as in Section 3, we can obtain
\[ \lim_{\lambda \to 0^+} \lambda |F_\lambda|^2 = \frac{\|\Omega_1 \Omega_2\|_{L^{1/2}(S^{n-1})}}{n^2}, \]
which finishes the proof of Theorem 1.2 (i).

Now we turn to Theorem 1.2 (ii). Let
\[ H_\lambda = \left\{ x : \left| T_{\Omega}(f_1, f_2)(x) - \frac{|\Omega_1(x)\Omega_2(x)|}{|x|^{2n}} \right| > \lambda \right\}; \]
\[ H^1_\lambda = \left\{ |x| > R_\varepsilon : \left| T_{\Omega,\varepsilon}(g_1, g_2)(x) - \frac{|\Omega_{1,\varepsilon}(x)\Omega_{2,\varepsilon}(x)|}{|x|^{2n}} \right| > \lambda \right\}; \]
\[ H^2_\lambda = \left\{ x : \left| \Omega_1(x)\Omega_2(x) - \frac{|\Omega_{1,\varepsilon}(x)\Omega_{2,\varepsilon}(x)|}{|x|^{2n}} \right| > \lambda \right\}. \]

Note that the following inclusion relationship is true:
\[ H_\lambda \subset H^1_{(1-2\sqrt{\varepsilon})\lambda} \cup H^2_{\sqrt{\varepsilon} \lambda} \cup F^{2\sqrt{\varepsilon} / 2} \cup F^{4\sqrt{\varepsilon} / 2} \cup \overline{B(0, R_\varepsilon)}, \]
and \( H_\lambda \) enjoys the following property:
\[ \lim_{\lambda \to 0^+} \lambda |H_\lambda|^2 = 0 \]
The proof of Theorem 1.2 is finished.
5 Proofs of Theorems 1.3 and 1.4

To prove Theorems 1.3 and 1.4, we need the following lemma for \( M_\Omega^\alpha \) and \( T_\Omega^\alpha \).

**Lemma 5.1.** For \( 0 < \alpha < n \) and \( \Omega_1, \Omega_2 \subset L^n/(n-\alpha)(\mathbb{S}^{n-1}) \), \( M_\Omega^\alpha \) and \( T_\Omega^\alpha \) are bounded from \( L^1 \times L^1 \) to \( L^{n/(n-\alpha),\infty} \), and the following norm inequalities hold:

\[
\|M_\Omega^\alpha\|_{L^1 \times L^1 \to L^{n/(n-\alpha),\infty}} \leq C \|\Omega_1\|_{L^n/(n-\alpha)(\mathbb{S}^{n-1})} \|\Omega_2\|_{L^n/(n-\alpha)(\mathbb{S}^{n-1})};
\]

\[
\|T_\Omega^\alpha\|_{L^1 \times L^1 \to L^{n/(n-\alpha),\infty}} \leq C \|\Omega_1\|_{L^n/(n-\alpha)(\mathbb{S}^{n-1})} \|\Omega_2\|_{L^n/(n-\alpha)(\mathbb{S}^{n-1})}.
\]

**Proof.** Let \( \Omega \in L^n/(n-\alpha)(\mathbb{S}^{n-1}) \) and \( f \in L^1 \), it was shown in [31] that

\[
\|M_\Omega^\alpha f\|_{L^n/(n-\alpha)} \leq C \|\Omega\|_{L^n/(n-\alpha)(\mathbb{S}^{n-1})} \|f\|_{L^1};
\]

\[
\|T_\Omega^\alpha f\|_{L^n/(n-\alpha)} \leq C \|\Omega\|_{L^n/(n-\alpha)(\mathbb{S}^{n-1})} \|f\|_{L^1}.
\]

Then Lemma 5.1 follows from the fact that

\[
M_\Omega^\alpha (f_1, f_2)(x) \leq M_\Omega^\alpha f_1(x) \cdot M_\Omega^\alpha f_2(x), \quad |T_\Omega^\alpha (f_1, f_2)(x)| \leq T_\Omega^\alpha |f_1(x)| \cdot T_\Omega^\alpha |f_2(x)|
\]

and Hölder’s inequality for weak spaces [14, p.16].

**Proof of Theorem 1.3.** Define

\[
\tilde{E}_\lambda = \left\{ x : M_\Omega^\alpha (f_1, f_2)(x) > \lambda \right\};
\]

\[
\tilde{E}_\lambda^2 = \left\{ x : M_\Omega^\alpha (g_1, h_2)(x) + M_\Omega^\alpha (h_1, g_2)(x) + M_\Omega^\alpha (h_1, h_2)(x) > \lambda \right\};
\]

\[
\tilde{E}_\lambda^3 = \left\{ |x| > R_\varepsilon : M_\Omega^\alpha (g_1, g_2)(x) > \lambda \right\};
\]

\[
\tilde{E}_\lambda^4 = \left\{ x : M_\Omega^\alpha_{\varepsilon,1} (g_1, g_2)(x) + M_\Omega^\alpha_{\varepsilon,2} (g_1, g_2)(x) + 3M_\Omega^\alpha_{\varepsilon,3} (g_1, g_2)(x) > \lambda \right\},
\]

where \( \Omega_{\varepsilon,1}, \Omega_{\varepsilon,2}, \Omega_{\varepsilon,3} \) are selected to be the same as in Section 3, the only difference is that we use the inequality \( \|\Omega_{\varepsilon,0}\|_{L^n/(n-\alpha)(\mathbb{S}^{n-1})} < \varepsilon \) to replace \( \|\Omega_{\varepsilon,0}\|_{L^\log L(\mathbb{S}^{n-1})} < \varepsilon \). By the same reasoning, we have

\[
\tilde{E}_\lambda^3 (1+\sqrt{\varepsilon/2}) \setminus \tilde{E}_\lambda^3 \subset \tilde{E}_\lambda \subset \tilde{E}_\lambda^3 \cup \tilde{E}_\lambda^3 (1-\sqrt{\varepsilon}) \cup \tilde{E}_\lambda^4 \cup B(0, R_\varepsilon).
\]

(5.1)

Then the same steps as in Section 3 give that

\[
|\tilde{E}_\lambda^2| \leq C_{\Omega,\alpha,1} \frac{\varepsilon^{n/(4(n-\alpha))}}{\chi_{n/2(n-\alpha)}}, \quad |\tilde{E}_\lambda^4| \leq C_{\Omega,\alpha,2} \frac{\varepsilon^{n/(4(n-\alpha))}}{\chi_{n/2(n-\alpha)}},
\]

(5.2)

and

\[
M_\Omega^\alpha (g_1, g_2)(x) \leq \frac{n^{2(n-\alpha)/n} (1+\varepsilon)^{2(n-\alpha)}}{\omega_{n-1}^{2(n-\alpha)/n} |x|^{2(n-\alpha)}}, \quad \text{for } |x| > R_\varepsilon;
\]

(5.3)

\[
M_\Omega^\alpha (g_1, g_2)(x) \leq \frac{n^{2(n-\alpha)/n} (1+\varepsilon)^{2(n-\alpha)}}{\omega_{n-1}^{2(n-\alpha)/n} |x|^{2(n-\alpha)}}, \quad \text{for } |x| > R_\varepsilon, x \in V_\varepsilon.
\]

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Then we have
\[ \lim_{\lambda \to 0^+} \lambda |\tilde{E}_\lambda|^{2(n-\alpha)/n} = \frac{\|\Omega_1\Omega_2\|_{L^{n/2(n-\alpha)}(\mathbb{R}^{n-1})}}{\omega_{n-1}^{2(n-\alpha)/n}}. \]

The proof of Theorem 1.3 (i) is finished.

It remains to prove Theorem 1.3 (ii), we set
\[ \tilde{G}_\lambda = \left\{ x : \left| M_\Omega^\alpha(f_1, f_2)(x) - \frac{|\Omega_1(x)\Omega_2(x)|}{\omega_{n-1}^{2(n-\alpha)/n}|x|^{2(n-\alpha)/n}} \right| > \lambda \right\}; \]
\[ \tilde{G}_\lambda^1 = \left\{ |x| > R_\varepsilon : \left| M_\Omega^\alpha(g_1, g_2)(x) - \frac{|\Omega_{1,\varepsilon}(x)\Omega_{2,\varepsilon}(x)|}{\omega_{n-1}^{2(n-\alpha)/n}|x|^{2(n-\alpha)/n}} \right| > \lambda \right\}; \]
\[ \tilde{G}_\lambda^2 = \left\{ x : \left| \frac{|\Omega_1(x)\Omega_2(x)|}{\omega_{n-1}^{2(n-\alpha)/n}|x|^{2(n-\alpha)/n}} - \frac{|\Omega_{1,\varepsilon}(x)\Omega_{2,\varepsilon}(x)|}{\omega_{n-1}^{2(n-\alpha)/n}|x|^{2(n-\alpha)/n}} \right| > \lambda \right\}. \]

It’s easy to check that the following inclusion relationship holds:
\[ \tilde{G}_\lambda \subset \tilde{G}_\lambda^1 \subset \tilde{G}_\lambda^2 \subset \tilde{E}_\varepsilon^{\varepsilon/2} \subset \tilde{E}_\varepsilon^{\varepsilon/2} \cup B(0, R_\varepsilon). \]

We may apply the same reasoning as in Section 3 to obtain
\[ \lim_{\lambda \to 0^+} \lambda |\tilde{G}_\lambda^2|^{2(n-\alpha)/n} = 0. \]

The only difference is the powers of \( \omega_{n-1}, n, |x| \) and \( 1 + \varepsilon \), so we omit the details. This finishes the proof of Theorem 1.3.

**Proof of Theorem 1.4.** Let \( \Omega_{\varepsilon}, \tilde{\Omega}_{\varepsilon,1}, \tilde{\Omega}_{\varepsilon,2}, \tilde{\Omega}_{\varepsilon,\varepsilon} \) be the same kernels as in the proof of Theorem 1.3, and set
\[ \tilde{F}_\lambda = \left\{ x : \left| T_{\tilde{\Omega}}^\alpha(f_1, f_2)(x) \right| > \lambda \right\}; \]
\[ \tilde{F}_\lambda^2 = \left\{ x : \left| T_{\tilde{\Omega}}^\alpha(g_1, h_2)(x) \right| + \left| T_{\tilde{\Omega}}^\alpha(h_1, g_2)(x) \right| + \left| T_{\tilde{\Omega}}^\alpha(h_1, h_2)(x) \right| > \lambda \right\}; \]
\[ \tilde{F}_\lambda^3 = \left\{ |x| > R_\varepsilon : \left| T_{\tilde{\Omega}_{\varepsilon,\varepsilon}}^\alpha(g_1, g_2)(x) \right| > \lambda \right\}; \]
\[ \tilde{F}_\lambda^4 = \left\{ x : \left| T_{\tilde{\Omega}_{\varepsilon,1}}^\alpha(g_1, g_2)(x) \right| + \left| T_{\tilde{\Omega}_{\varepsilon,2}}^\alpha(g_1, g_2)(x) \right| + \left| 3 T_{\tilde{\Omega}_{\varepsilon,\varepsilon}}^\alpha(g_1, g_2)(x) \right| > \lambda \right\}. \]

Then we have
\[ \tilde{F}_\lambda^{\varepsilon_1/2} \setminus \tilde{F}_\lambda^{\varepsilon_2/2} \subset \tilde{F}_\lambda \subset \tilde{F}_\lambda^{\varepsilon_1/2} \cup \tilde{F}_\lambda^{\varepsilon_2/2} \cup \tilde{F}_\lambda^{\varepsilon_3/2} \cup B(0, R_\varepsilon). \]

Using Lemma 5.1, we can show that
\[ |\tilde{F}_\lambda^{\varepsilon_2/2}| \leq C_{\Omega,\alpha,1} \varepsilon^{n/4(n-\alpha)} \lambda^{n/2(n-\alpha)}, \quad |\tilde{F}_\lambda^{\varepsilon_3/2}| \leq C_{\Omega,\alpha,2} \varepsilon^{n/4(n-\alpha)} \lambda^{n/2(n-\alpha)}, \]
and
\[
|T_{\Omega_\varepsilon}^\alpha (g_1, g_2)(x)| \leq \frac{(1 + \varepsilon)^{2(n-\alpha)} \prod_{i=1}^2 (|K_{1,\varepsilon}(x)| + \varepsilon)}{|x|^{2(n-\alpha)}}, \quad \text{for } |x| > R_\varepsilon;
\]
\[
|T_{\Omega_\varepsilon}^\alpha (g_1, g_2)(x)| \leq \frac{2 \prod_{i=1}^2 (|K_{1,\varepsilon}(x)| + \varepsilon)}{(1 + \varepsilon)^{2(n-\alpha)} |x|^{2(n-\alpha)}}, \quad \text{for } |x| > R_\varepsilon, \varepsilon \in V_\varepsilon.
\]

With these estimates in hand, it’s easy to get
\[
\lim_{\lambda \to 0^+} \lambda^{-2(n-\alpha)/n} \|\tilde{F}_{\lambda}\|_{L^{n/2(n-\alpha)}(\mathbb{R}^{n-1})} = \frac{\|\Omega_1 \Omega_2\|}{n^{2(n-\alpha)/n}},
\]
which finishes the proof of Theorem 1.4 (i).

To prove Theorem 1.4 (ii), we denote
\[
\bar{H}_\lambda = \left\{ x : \left| T_{\Omega_\varepsilon}^\alpha (f_1, f_2)(x) - \frac{\Omega_1(x)\Omega_2(x)}{|x|^{2(n-\alpha)}} \right| > \lambda \right\};
\]
\[
\bar{H}_\lambda^1 = \left\{ |x| > R_\varepsilon : \left| T_{\Omega_\varepsilon}^\alpha (g_1, g_2)(x)) - \frac{\Omega_1,\varepsilon(x)\Omega_2,\varepsilon(x)}{|x|^{2(n-\alpha)}} \right| > \lambda \right\};
\]
\[
\bar{H}_\lambda^2 = \left\{ x : \left| \frac{\Omega_1(x)\Omega_2(x) - \Omega_1,\varepsilon(x)\Omega_2,\varepsilon(x)}{|x|^{2(n-\alpha)}} \right| > \lambda \right\}.
\]

Then Theorem 1.4 (ii) follows easily from the same steps as in the proof of Theorem 1.3. \(\square\)

6 Results with power weighted measure

Eﬀorts have been made in generalizing the Lebesgue measure to other measures. For example, [25] showed that for \(X = (0, \infty)\) with the Euclidean metric and \(d\nu(x) = 1_{x(x>0)} x dx\), it holds that
\[
\lim_{\lambda \to 0^+} \lambda\nu (\{ x \in X : M_{\nu} f(x) > \lambda \}) = \frac{1}{4} \|f\|_{L^1(X, d\nu)}.
\]

Let us first introduce some background about power weighted measure. Power weighted measure in \(\mathbb{R}^n\) is a special measure deﬁned by \(d\mu(x) = |x|^\beta dx\), where \(\beta \in \mathbb{R}\). As a measure, it was shown that \(d\mu\) is doubling when \(\beta > -n\); as a weight, \(|x|^\beta\) is an \(A_p\) weight if and only if \(-n < \beta < n(p - 1)\) for \(1 < p < \infty\). Note that the power weighted measure of a measurable set in \(\mathbb{R}^n\) is given by \(\mu(E) = \int_E d\mu(x)\).

Recently, under the power weighted measure, Hou and Wu [23] gave similar results for fractional maximal functions and fractional integrals associated with power weighted measure \(d\mu\) deﬁned by
\[
M_{\mu}^\alpha f(x) = \sup_{r>0} \frac{1}{\mu(B(x, r))^{(n-\alpha)/n}} \int_{B(x, r)} |f(y)| d\mu(y)
\]

{22}
and
\[ T^\alpha_\mu f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\mu(B(x, |x-y|))^{(n-\alpha)/n}} d\mu(y). \]

The main results in [23] are as follows:

**Theorem A.** ([23]) For \( d\mu(x) = |x|^\beta \, dx, \beta \geq 0 \) and \( f \in L^1(\mathbb{R}^n, d\mu) \), it holds that
\[
\lim_{\lambda \to 0^+} \lambda^{n/(n-\alpha)} \mu \left( \left\{ x \in \mathbb{R}^n : M^\alpha f(x) > \lambda \right\} \right) = \frac{\omega_{n-1}}{(n + \beta)\mu(e_1, 1)} \| f \|_{L^1(\mathbb{R}^n, d\mu)}^{\alpha/(n-\alpha)},
\]
\[
\lim_{\lambda \to 0^+} \lambda^{n/(n-\alpha)} \mu \left( \left\{ x \in \mathbb{R}^n : M^\alpha f(x) - \frac{\| f \|_{L^1(\mathbb{R}^n, d\mu)}}{\mu(B(x, |x|))^{(n-\alpha)/n}} > \lambda \right\} \right) = 0,
\]
where \( e_1 = (1, 0, \cdots, 0) \) is the unit vector on \( \mathbb{R}^n \). Similar results were also given for \( T^\alpha_\mu \) when \( f \geq 0 \).

For some other related works, we refer the readers to [22, 24, 38].

Motivated by the above works, it’s natural to ask whether we can obtain the similar results for multilinear fractional type operators with rough kernel. In this section, we give a confirmative answer to this question.

We begin with one definition and some lemmas.

**Definition 6.1.** Suppose \( 0 < \alpha < n \), \( d\mu(x) = |x|^\beta, \beta \geq 0 \), \( \Omega_1, \Omega_2 \) are homogeneous of degree 0. The fractional maximal functions and fractional integrals associated with rough kernels \( \Omega_1, \Omega_2 \) and the power weighted measure \( d\mu \) by
\[
M^\alpha_{\Omega_1, \mu} (f_1, f_2)(x) := \sup_{r > 0} \frac{1}{\mu(B(x, r))^{2(\alpha-n)/n}} \prod_{i=1}^2 \int_{B(x,r)} |\Omega_i(x-y_i)f_i(y_i)| d\mu(y_i),
\]
\[
T^\alpha_{\Omega_1, \mu} (f_1, f_2)(x) := \int_{\mathbb{R}^n} \frac{\Omega_1(x-y_1)\Omega_2(x-y_2)f_1(y_1)f_2(y_2)}{\mu(B(x, |x-y_1, x-y_2|))^{2(\alpha-n)/n}} d\mu(y_1)d\mu(y_2).
\]

We need the following lemmas:

**Lemma 6.1.** For all \( \beta \geq 0, x \in \mathbb{R}^n \) and \( r > 0 \), the following properties holds:

(i) \( \mu(B(x, r|x|)) = \mu(e_1, r)|x|^{n+\beta}; \)

(ii) \( \mu(B(0, r)) \leq \mu(B(x, r)). \)

**Proof.** Obviously, (i) can be deduced by variable substitution.

So we focus on (ii). If \( |x| > 2r \), then for all \( y \in B(0, r) \), \( |x-y| > r > |y| \), thus
\[
\mu(B(0, r)) = \int_{B(0,r)} |y|^\beta dy \leq \int_{B(0,r)} |x-y|^\beta dy = \int_{B(x,r)} |y|^\beta dy = \mu(B(x, r)).
\]

If \( |x| < 2r \), then \( E := B(0, r) \cap B(x, r) \) is not an empty set. Now for all \( y \in B(0, r) \setminus E \), we have \( |x-y| > r > |y| \). See Figure 2 for 2-dimensional case.
Thus

\[ \mu(B(0, r)) = \int_{B(0, r)} |y|^\beta \, dy = \int_E |y|^\beta \, dy + \int_{B(0, r) \setminus E} |y|^\beta \, dy \]

\[ \leq \int_{|y| < r, |x-y| < r} |y|^\beta \, dy + \int_{B(0, r) \setminus E} |x-y|^\beta \, dy \]

\[ = \int_{|y'| < r, |x-y'| < r} |x-y'|^\beta \, dy + \int_{B(0, r) \setminus E} |x-y|^\beta \, dy \]

\[ = \int_{B(0, r)} |x-y|^\beta \, dy = \mu(B(x, r)). \]

The proof of Lemma 6.1 is finished. \( \square \)

**Remark 6.1.** Note that, for all \( x_1, x_2 \in \mathbb{R}^n \) satisfying \( |x_1| \leq |x_2| \), the inequality \( \mu(B(x_1, r)) \leq \mu(B(x_2, r)) \) holds.

We need the weighted version of Lemma 2.1 as follows:

**Lemma 6.2.** For fixed \( \lambda, \gamma > 0 \), \( \Phi \) is homogeneous of degree \( 0 \), \( S \subset \mathbb{S}^{n-1} \) is a measurable set. Then we have

\[ \mu \left( \left\{ x \in \mathbb{R}^n : \frac{|\Phi(x)|}{|x|^\gamma} > \lambda, \frac{x}{|x|} \in S \right\} \right) = \frac{||\Phi||_{L^{(n+\beta)/\gamma}(S)}}{(n+\beta)\lambda^{(n+\beta)/\gamma}}. \]

We need the following endpoint weak type estimates of \( M_{\Omega, \mu}^\alpha \) and \( T_{\Omega, \mu}^\alpha \).
Lemma 6.3. If $0 < \alpha < n$ and $\Omega_1, \Omega_2 \in L^{n/(n-\alpha)}(\mathbb{S}^{n-1})$, then $M^\alpha_{\Omega, \mu}$ and $T^\alpha_{\Omega, \mu}$ are bounded from $L^1(\mathbb{R}^n, d\mu) \times L^1(\mathbb{R}^n, d\mu)$ to $L^{n/2(n-\alpha)\infty}(\mathbb{R}^n, d\mu)$.

Proof. Define the convolution associated with $\mu$ by

$$f \ast g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)d\mu(y).$$

By Lemma 6.1, we have

$$M^\alpha_{\Omega, \mu}(f_1, f_2)(x) \leq \sup_{r > 0} \int_{B(x, r)^2} \frac{\prod_{i=1}^2 |f_i(y_i)\Omega_i(x - y_i)|d\mu(y_i)}{\mu(B(x, \max(|x - y_1|, |x - y_2|)))^{2(n-\alpha)/n}}$$

$$\leq \int_{\mathbb{R}^{2n}} \frac{\prod_{i=1}^2 |f_i(y_i)\Omega_i(x - y_i)|d\mu(y_i)}{\mu(B(x, \max(|x - y_1|, |x - y_2|)))^{2(n-\alpha)/n}}$$

$$\leq \int_{\mathbb{R}^{2n}} \frac{\prod_{i=1}^2 |f_i(y_i)\Omega_i(x - y_i)|d\mu(y_i)}{\mu(B(0, |(x - y_1, x - y_2)|/\sqrt{2}))^{2(n-\alpha)/n}}$$

$$\leq C \prod_{i=1}^2 \frac{\Omega_i(x)}{|(n-\alpha)(n+\beta)/n|} * |f_i|(x),$$

and

$$|T^\alpha_{\Omega, \mu}(f_1, f_2)(x)| \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\prod_{i=1}^2 |\Omega_i(x - y_i)||f_i(y_i)|d\mu(y_i)}{\mu(B(x, |(x - y_1, x - y_2)|))^{2(n-\alpha)/n}}$$

$$\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\prod_{i=1}^2 |\Omega_i(x - y_i)||f_i(y_i)|d\mu(y_i)}{\mu(B(0, |(x - y_1, x - y_2)|))^{2(n-\alpha)/n}}$$

$$= C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\prod_{i=1}^2 |\Omega_i(x - y_i)||f_i(y_i)|d\mu(y_i)}{|(x - y_1, x - y_2)|^{2(n-\alpha)(n+\beta)/n}}$$

$$\leq C \prod_{i=1}^2 \frac{\Omega_i(x)}{|(n-\alpha)(n+\beta)/n|} * |f_i|(x).$$

On the other hand, Hölder’s inequality, together with Young’s inequality for weak type
spaces \[14, p.23\] and Lemma 6.2 yields that
\[
\left\| \prod_{i=1}^{2} \frac{\left| \Omega_i(\cdot) \right|}{\cdot \cdot \cdot} \frac{\left| f_i \right| (x)}{(n-\alpha)(n+\beta)/n} \right\|_{L^{n/2(n-\alpha),\infty}(\mathbb{R}^n, d\mu)} \leq C \prod_{i=1}^{2} \left\| \frac{\left| \Omega_i(\cdot) \right|}{\cdot \cdot \cdot} \frac{\left| f_i \right| (x)}{(n-\alpha)(n+\beta)/n} \right\|_{L^{n/(n-\alpha),\infty}(\mathbb{R}^n, d\mu)} \leq C \prod_{i=1}^{2} \left\| \frac{\left| \Omega_i(\cdot) \right|}{\cdot \cdot \cdot} \frac{\left| f_i \right| (x)}{(n-\alpha)(n+\beta)/n} \right\|_{L^{n/(n-\alpha),\infty}(\mathbb{R}^n, d\mu)} = C \prod_{i=1}^{2} \left\| \Omega_i \right\|_{L^{n/(n-\alpha)}(\mathbb{S}^{n-1}, d\sigma)} \left\| f_i \right\|_{L^{1}(\mathbb{R}^n, d\mu)}.
\]

Therefore the \(L^{n/2(n-\alpha),\infty}(\mathbb{R}^n, d\mu)\) norms of \(M_{\Omega,\mu}^{\alpha} (f_1, f_2)\) and \(T_{\Omega,\mu}^{\alpha} (f_1, f_2)\) can be controlled by \(C \prod_{i=1}^{2} \|\Omega_i\|_{L^{n/(n-\alpha)}(\mathbb{S}^{n-1}, d\sigma)} \|f_i\|_{L^{1}(\mathbb{R}^n, d\mu)}\). Thus Lemma 6.3 is proved. \(\square\)

By Lemma 6.1 - 6.3, and applying the method in Section 5, one may obtain

**Theorem 6.1.** Let \(0 < \alpha < n\) and \(\Omega_1, \Omega_2 \in L^{n/(n-\alpha)}(\mathbb{S}^{n-1})\). Then for all \(f_1, f_2 \in L^{1}(\mathbb{R}^n, d\mu)\), we have

(i) \(\lim_{\lambda \to 0^+} \lambda \mu \left( \left\{ x : M_{\Omega,\mu}^{\alpha} (f_1, f_2)(x) > \lambda \right\} \right)^{2(n-\alpha)/n} = \frac{A}{\mu(B(0, 1))^2(n-\alpha)/n};\)

(ii) \(\lim_{\lambda \to 0^+} \lambda \mu \left( \left\{ x : M_{\Omega,\mu}^{\alpha} (f_1, f_2)(x) - \frac{\prod_{i=1}^{2} |\Omega_i(x)| \|f_i\|_{L^{1}(\mathbb{R}^n, d\mu)}}{\mu(B(x, |x|))} > \lambda \right\} \right)^{2(n-\alpha)/n} = 0;\)

(iii) \(\lim_{\lambda \to 0^+} \lambda \mu \left( \left\{ x : T_{\Omega,\mu}^{\alpha} (|f_1|, |f_2|)(x) > \lambda \right\} \right)^{2(n-\alpha)/n} = \frac{A}{\mu(B(0, \sqrt{2}|x|))} 2(n-\alpha)/n;\)

(iv) \(\lim_{\lambda \to 0^+} \lambda \mu \left( \left\{ x : T_{\Omega,\mu}^{\alpha} (|f_1|, |f_2|)(x) - \frac{\prod_{i=1}^{2} |\Omega_i(x)| \|f_i\|_{L^{1}(\mathbb{R}^n, d\mu)}}{\mu(B(x, \sqrt{2}|x|))} > \lambda \right\} \right)^{2(n-\alpha)/n} = 0,\)

where
\[A = \frac{\|\Omega_1 \Omega_2\|_{L^{n/2(n-\alpha)}(\mathbb{S}^{n-1})}}{(n + \beta)^{2(n-\alpha)/n}} \prod_{i=1}^{2} \|f_i\|_{L^{1}(\mathbb{R}^n, d\mu)}.\]

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