Exact One Loop Running Couplings in the Standard Model

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Abstract

Taking the dominant couplings in the standard model to be the quartic scalar coupling, the Yukawa coupling of the top quark, and the SU(3) gauge coupling, we consider their associated running couplings to one loop order. Despite the non-linear nature of the differential equations governing these functions, we show that they can be solved exactly. The nature of these solutions is discussed and their singularity structure is examined. It is shown that for a sufficiently small Higgs mass, the quartic scalar coupling decreases with increasing energy scale and becomes negative, indicative of vacuum instability. This behaviour changes for a Higgs mass greater than 168 GeV, beyond which this couplant increases with increasing energy scales and becomes singular prior to the ultraviolet (UV) pole of the Yukawa coupling. Upper and lower bounds on the Higgs mass corresponding to new physics at the TeV scale are obtained and compare favourably with the numerical results of the one-loop and two-loop analyses with inclusion of electroweak couplings.

It is well understood that removing infinities arising in quantum field theory when evaluating perturbative corrections to classical interactions induces a dependence of the physical couplings on the energy scale \( \mu \) of the process being considered. The most surprising consequence of this dependence is asymptotic freedom, the decrease in the non-Abelian gauge coupling with increasing energy scale. Clearly, the asymptotic behaviour of the running couplings is of great theoretical and phenomenological interest.

The evolution of these running couplings is dictated by a set of non-linear ordinary differential equations in which the various couplings are inextricably linked. This can change a naive expectation of their behaviour based on ignoring this linkage. For example, the quartic scalar self-coupling \( \lambda \) of a pure \( \mathcal{O}(N) \) scalar theory diverges with increasing energy scale, but this need not be true if the coupling is affected by the interaction of this scalar with other fields. In this paper, we demonstrate via an exact solution that this quartic scalar coupling \( \lambda \) in the Standard Model has an asymptotic behaviour and singularity structure that is strongly affected by the top-quark coupling \( g_t \) and the SU(3) gauge coupling \( g_3 \), the dominant couplings of the Standard Model at the weak scale. We show this by explicitly solving in closed form the one-loop equations that govern how \( \lambda \), \( g_t \) and \( g_3 \) evolve with \( \mu \). The importance of having an explicitly analytic solution to a set of non-linear coupling equations has been emphasized in [2]. In fact, the asymptotic behaviour and singularity structure of \( \lambda \) becomes contingent upon the boundary conditions to these equations, and those boundary conditions in turn depend on the Higgs mass \( M_H \). For \( M_H < 168 \text{ GeV} \) we find that \( \lambda \) decreases with increasing \( \mu \) and eventually goes negative; for \( M_H > 168 \text{ GeV} \), \( \lambda \) will be shown to remain positive up to its singularity. This provides an independent way of using the renormalization group equation to analyze the sensitivity of the Standard Model on \( M_H \) that is complementary to numerical approaches [4]. Furthermore, having an exact solution for these running couplings will provide a way of determining the effective potential at leading-log order in the conformal limit of the Standard Model using the method of characteristics [1], which in turn has cosmological implications [5].

The one-loop equations that determine how \( \lambda \), \( g_t \) and \( g_3 \) depend on \( \mu \) are [6]:

\[
\dot{x} = \frac{9}{4}x^2 - 4xz
\]

\[
\dot{y} = 6y^2 + 3xy - \frac{3}{2}z^2
\]

\[
\dot{z} = -\frac{7}{2}z^2
\]
where $x = \frac{g^2}{4\pi^2}$, $y = \frac{\lambda}{4\pi}$, $z = \frac{g^2}{3\pi^2}$ and the dot denotes the derivative with respect to $t = \log(\mu)$. Here $\mu$ is the renormalization-induced mass parameter in the theory.\footnote{It does not appear to be possible to obtain an exact solution by applying computer algebra programs directly to the differential equations for these one loop couplings.}

Eq. (3) can be solved immediately to give
$$z(t) = \frac{z_0}{1 + \frac{2}{7}z_0(t-t_0)} \quad (4)$$
where $z_0 = z(t_0)$ is a boundary value for $z(t)$. Together, Eq. (1) and Eq. (3) give
$$\frac{dx}{dz} = -\frac{2}{7} \left[ \frac{9}{4} \left( \frac{x}{z} \right)^2 - 4 \left( \frac{x}{z} \right) \right] \quad (5)$$
which, if $x = zw$, becomes
$$z \frac{dw}{dz} = -\frac{9}{14}w^2 + \frac{1}{7}w \quad (6)$$
Eq. (6) can be immediately integrated to give
$$\left( \frac{z}{z_0} \right)^{\frac{1}{7}} = \left( \frac{w}{w_0} - \frac{2}{7} \right) \left( \frac{w_0 - \frac{2}{7}w}{w_0} \right) \quad (7)$$
where $w_0 = w(t_0) = \frac{x(t_0)}{z(t_0)}$. If we define $K = z_0^{\frac{1}{7}} \left( \frac{w_0 - \frac{2}{7}}{w_0} \right)$, then Eq. (7) leads to
$$x(t) = \frac{2z(t)}{9 \left[ 1 - Kz(t)^{-\frac{1}{7}} \right]} \quad (8)$$

We now must examine how the scalar couplant, $y$, evolves with $t$. Dividing Eq. (2) by Eq. (3), we obtain
$$\frac{dy}{dz} = -\frac{2}{7} \left[ 6 \left( \frac{y}{z} \right)^2 + 3 \left( \frac{x}{z} \right) \left( \frac{y}{z} \right) - \frac{3}{2} \left( \frac{x}{z} \right)^2 \right] \quad (9)$$
From Eq. (7), we can put
$$w = \frac{x}{z} = \frac{p}{\tau + q} \quad (10)$$
where $\tau = z^{-\frac{1}{7}}$, $p = -\frac{2}{7\lambda}$ and $q = -\frac{1}{\lambda}$, so that if $u = \frac{x}{z}$, then Eq. (9) becomes
$$\frac{\tau du}{d\tau} - 12u^2 - (6w + 7)u = -3w^2 \quad (11)$$
Eq. (11) is a Ricatti equation; the transformation
$$u = -\frac{\tau du}{12v} \quad (12)$$
brings it to the form
$$\frac{d^2v}{d\lambda^2} + \left( \frac{r}{\lambda+1} - \frac{r+6}{\lambda} \right) \frac{dv}{d\lambda} + \frac{r^2}{\lambda(\lambda+1)} \left( \frac{1}{\lambda+1} - \frac{1}{\lambda} \right) v = 0 \quad (13)$$
where $\tau = \lambda q$ and $r = \frac{6p}{q} = \frac{4}{3}$. A rescaling of $v$ so that
$$v = \lambda^\alpha(\lambda + 1)^\beta f \quad (14)$$
leads to
$$\frac{d^2f}{d\lambda^2} + \left( \frac{2\alpha - \frac{22}{3}}{\lambda} + \frac{2\beta + \frac{4}{3}}{\lambda+1} \right) \frac{df}{d\lambda} + \left( 2\alpha\beta + \frac{4}{3}\alpha - \frac{22}{3}\beta + \frac{32}{9} \right) \left( \frac{1}{\lambda} - \frac{1}{\lambda+1} \right) f = 0 \quad (15)$$
provided the conditions
\[
\alpha^2 - \frac{25}{3}\alpha - \frac{16}{9} = 0 \quad (16)
\]
\[
\beta^2 + \frac{1}{3}\beta - \frac{16}{9} = 0 \quad (17)
\]
are used to eliminate terms proportional to \( f' \) and \( \frac{f}{(1+\lambda)^2} \) which arise from Eq. (13). Taking the solutions to Eqs. (16) and (17) to be
\[
\alpha = \frac{25 + \sqrt{689}}{6} \quad (18)
\]
\[
\beta = -\frac{1 - \sqrt{65}}{6} \quad (19)
\]
Eq. (15) is of the form of a hypergeometric differential equation whose independent solutions about \( \lambda = 0 \) when \( | -\lambda | < 1 \), are
\[
f = F(a, b; c; -\lambda) \quad (20)
\]
and
\[
f = (-\lambda)^{1-c} F(a - c + 1, b - c + 1; 2 - c; -\lambda) \quad (21)
\]
provided
\[
ab = 2\alpha\beta + \frac{4\alpha}{3} - \frac{22\beta}{3} + \frac{32}{9} = \frac{1}{18} \left[ 161 - 3\sqrt{65} + 3\sqrt{689} - 13\sqrt{265} \right] \quad (22)
\]
\[
a + b = 2\alpha + 2\beta - 7 = 1 + \frac{1}{3} \left( \sqrt{689} - \sqrt{65} \right) \quad (23)
\]
\[
c = 2\alpha - \frac{22}{3} = 1 + \frac{1}{3} \sqrt{689} \quad (24)
\]
Solving Eqs. (22) and (23) we obtain \( a = 4 + \frac{1}{6}(\sqrt{689} - \sqrt{65}) \) and \( b = -3 + \frac{1}{6}(\sqrt{689} - \sqrt{65}) \).

The choice of signs in the roots appearing in Eqs. (18) and (19) ensures that \( c - a - b = -\frac{1}{3} - 2\beta \) is positive so that \( F(a, b; c; z) \) is well defined as \( z \to 1^- \) as \( \lim_{z \to 1^-} F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \). As \( t \) increases, so that \(-\frac{z}{q} = \left[ 1 + \frac{q}{z} (t - t_0) \right]^{1/2} \left( 1 - \frac{2z}{q} \right) \) exceeds one in magnitude, we must analytically continue the solutions of Eqs. (20, 21) using the standard results for analytically continuing the hypergeometric function \( F(a, b; c; z) \) into a domain in which \( |z| \geq 1 \). For \( \text{Re} \, z < 1/2 \), one can use the result
\[
F(a, b; c; z) = (1 - z)^a F\left(a, c - b; c; \frac{z}{1 - z}\right)
\]
while for \( |1 - z| < 1 \) we have,
\[
F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F\left(a, b; 1 + a + b - c; 1 - z\right) +
\]
\[
(1 - z)^{c - a - b - \frac{1}{2}} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} F\left(c - a, c - b; 1 - a - b + c; 1 - z\right)
\]
These extensions are valid for the values of \( a, b \) and \( c \) given by Eqs. (22-24). However, for the values of \(-\frac{z}{q} \) we consider, Eqs. (20-21) are valid solutions to Eq. (15). We also note that according to Eq. (8),
\[
x(t) = \frac{2z(t)}{9 \left[ 1 + \frac{z}{q} \right]}
\]
so that \( x(t) \) develops a pole when \(-\frac{z}{q} \to 1 \).
Together Eqs. (12), (14) and (20) show that

\[
y = -\frac{\tau a}{12} F(a, b; c; -\frac{\tau}{q}) + \frac{z\beta}{12} \left[ F(a + 1, b + 1; c + 1; -\frac{\tau}{q}) + C \left( -\frac{\tau}{q} \right)^{1-c} F(a - c + 1 + 2, -\frac{\tau}{q}) \right] + \frac{\tau}{12} \left[ \frac{ab}{c} F(a + 1, b + 1; c + 1; -\frac{\tau}{q}) + C \left( -\frac{\tau}{q} \right)^{1-c} F(a - c + 1 + 2, -\frac{\tau}{q}) \right]
\]

resulting in the expression

\[
y = -\frac{\tau a}{12} \frac{\tau}{q} \left[ F(a, b; c; -\frac{\tau}{q}) + C \left( -\frac{\tau}{q} \right)^{1-c} F(a - c + 1, b - c + 1; 2; -\frac{\tau}{q}) \right] + \frac{\tau}{12} q \left[ \frac{ab}{c} F(a + 1, b + 1; c + 1; -\frac{\tau}{q}) + C \left( -\frac{\tau}{q} \right)^{1-c} F(a - c + 1 + 2, -\frac{\tau}{q}) \right]
\]

indicating that the Yukawa and scalar couplings share an UV pole at \(-\tau/q = 1\). In addition, the scalar couplant can also develop poles when the denominator shared by the last two terms is zero. Hence, the analytic solution in the scalar couplant [Eq. (27)] additionally points to a non-trivial pole structure which builds upon other exact solutions in the literature \[3,4\], thereby increasing our understanding of the behaviour of running couplings in the Standard Model, most specifically in the case of a scalar field in the presence of both a gauge and Yukawa coupling.

With initial values of \(y_0\) being taken to be fixed by tree level relation \(y(\phi_0) = \frac{M_H^2}{8\pi^2\phi_0^2}\), \(x_0 = x(\phi_0) = 0.0253\), \(z_0 = z(\phi_0) = 0.0329\) \[1\] (where \(M_H\) denotes the mass of a Standard Model Higgs boson and \(\phi_0 = 246.2\) GeV denotes the vacuum expectation value of the scalar field \(\phi\)), the integration constant \(C\) is obtained from Eq. (25) in terms of \(M_H^2\):

\[
C = \frac{4.862287086 + 0.00003014398478 M_H^2}{1.236892814 - 0.00004219469394 M_H^2} \quad \text{(28)}
\]

where all quantities are in GeV units.

With these initial values, the energy-dependence of the solution (27) for the scalar couplant \(y\) can be examined\[2\]. For sufficiently small Higgs masses, the couplant decreases with increasing energy and eventually turns negative, indicating a vacuum instability in the theory. When the Higgs mass becomes sufficiently large this behavior is seen to dramatically change from being a decreasing function to an increasing function of \(\mu\), ultimately approaching a UV pole. The emergence of a pole in the scalar coupling prior to the pole in the Yukawa coupling indicates that the denominators in the last two terms of (28) become zero prior to \(-\tau/q = 1\). This indicates limits on the applicability of perturbation theory for a sufficiently heavy Higgs mass. This is also consistent with the behaviour of the quartic scalar couplant reported in \[3,4,7\].

The value of \(M_H\) representing the boundary between these two scenarios corresponds to the case where the pole of the Yukawa coupling [and the second term in (27)] coincides with the shared pole of the last two terms in (27). In other words, this bound on \(M_H\) results in a scalar coupling that remains positive until it becomes singular at the same energy scale as the Yukawa coupling. Setting the denominator of the last two terms in (27) to zero at the point \(-\tau/q = 1\) leads to the following expression for \(C\)

\[
C = -\frac{\Gamma(c) \Gamma(1-a) \Gamma(1-b)}{\Gamma(c-a) \Gamma(c-b) \Gamma(2-c)} \quad \text{(29)}
\]

\[2\]We have verified that our analytic results are in agreement with a numerical solution of Eqs. (1)–(3).
corresponding via (28) to $M_H = 168.67\text{GeV}$. Although the electroweak couplings have been ignored in the RG equations (1)–(3), this approximation does not have a significant impact on the numerical value of the boundary value of $M_H$. If one augments the one-loop RG equations to include the electroweak couplings, their numerical solution would lead to a value of $M_H = 160.8\text{GeV}$ above which $\lambda$ remains positive up to its singular value.

Remarkably, the phenomenological implications of the one-loop analytic expression for the scalar coupling are quite consistent with more detailed one-loop and two-loop numerical results including the electroweak gauge couplings. For example, if we assume a 1 TeV scale at which the Standard Model breaks down, then the naive vacuum stability requirement $y(1\text{TeV}) > 0$ leads via (27) to the bound $M_H > 75\text{GeV}$. By comparison, augmenting the one-loop RG equations with electroweak couplings results in $M_H > 72\text{GeV}$ from the numerical solution. Similarly, an upper bound on $M_H$ corresponding to a pole at 1 TeV in the last two terms of (27) results in $M_H < 740\text{GeV}$, while the one-loop numerical solution augmented by electroweak couplings results in $M_H < 744\text{GeV}$. The upper and lower bounds on $M_H$ resulting from our analytic solution are also in good agreement with the full two-loop numerical analyses [4]. In particular, our bounds on $M_H$ would not appear out of place in Figure 2 of [9] or Figure 3 of [10].

The exact solution of the RG equations in the one-loop approximation with the dominant electroweak scale couplings ($g_t$, $\lambda$, and $g_3$) thus appears to reproduce the essential features of Higgs mass-bound phenomenology. We speculate that the underlying singularity structure manifested by the analytic solution (27) is responsible for this concordance.

Finally, we note that knowing the exact behaviour of these one loop couplings would in principle lead to an exact expression for the one loop effective potential in the Standard Model when it is computed using the method of characteristics [1].

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3The value $M_H = 168.67\text{GeV}$ obtained from the solution of Eqs. (28) and (29) has also been verified by numerical exploration of Eq. (25).