DERIVED CATEGORIES AND BIRATIONALITY

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ABSTRACT. We discuss the question of finding conditions on a derived equivalence between two smooth projective varieties $X$ and $Y$ that imply that $X$ and $Y$ are birational. The types of conditions we consider are in the spirit of finding categorical analogues of classical Torelli theorems. We study, in particular, a notion of strongly filtered derived equivalence and study cases where strongly filtered derived equivalence implies birationality. We also consider an open variant of our main question.

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1. INTRODUCTION

It is well-known that the bounded derived category $D(X)$ of a smooth projective variety $X$ over a field $k$ is not a faithful invariant, in the sense that there exist smooth projective varieties $X$ and $Y$ over $k$ with $D(X) \simeq D(Y)$ as $k$-linear triangulated categories but for which $X$ and $Y$ are not isomorphic. The study of such derived equivalent varieties has received a great deal of attention over the last several decades.

The purpose of this paper is to explain various results, questions, and expectations surrounding a phenomenon we first discovered in [21], namely the fact that the derived category together with the codimension filtration on Chow theory has stronger Torelli-type properties than the derived category alone. Question [1.3] below is the main motivation for our work; there are several variants that we also describe in this Introduction.

1.1. A derived equivalence $\Phi: D(X) \to D(Y)$ is induced by a complex $P \in D(X \times Y)$. If $\Phi$ is given by the graph of a birational map then for a dense set of points $x \in X$ the restriction $P_x$ of $P$ to $Y_{k(x)}$ is isomorphic to the skyscraper sheaf associated to a
A derived equivalence \( \Phi: D(X) \to D(Y) \) induces an isomorphism
\[
\Phi_P^*: A^*(X) \to A^*(Y)
\]
which, in general, does not preserve the codimension filtration. If \( \Phi \) is given by a birational isomorphism, then the codimension filtration is, in fact, preserved which leads to the following definition.

**Definition 1.2.** A derived equivalence \( \Phi \) given by a kernel \( P \in D(X \times Y) \) is *filtered* if the isomorphism \( \Phi_P^* \) in (1.1.1) preserves the codimension filtration.

The basic question we consider is the following.

**Question 1.3.** Let \( X \) and \( Y \) be smooth projective varieties over a field \( k \). If there exists a filtered derived equivalence \( \Phi: D(X) \to D(Y) \) does it follow that \( X \) and \( Y \) are birational?

**Remark 1.4.** Note that the condition that \( \Phi \) is filtered is only imposing a condition on the numerical equivalence class of \( P \) and (a priori) does not provide any information about the actual support of \( P \).

1.5. Much of the work on this question has centered on varieties with trivial canonical bundle. When considering more general varieties, it seems natural to impose some stronger conditions. Recall that the Hochschild homology \( HH_*(X) \) and Hochschild cohomology \( HH^*(X) \) are defined as

\[
HH_*(X) = H^{-i}(X, L\Delta_X^* \Delta_X^* \mathcal{O}_X),
\]
\[
HH^i(X) = \text{Ext}^i(L\Delta_X^* \Delta_X^* \mathcal{O}_X, \mathcal{O}_X),
\]
where \( \Delta_X: X \hookrightarrow X \times X \) is the diagonal morphism. Note that we can also write \( HH^i(X) \) as

\[
HH^i(X) \simeq H^i(X, (L\Delta_X^* \Delta_X^* \mathcal{O}_X) \otimes K_X^\vee[-d]).
\]

Using the isomorphisms
\[
\mathcal{H}^i(L\Delta_X^* \Delta_X^* \mathcal{O}_X) \simeq \Omega_X^\vee[-i],
\]
we see that the canonical filtration on \( L\Delta_X^* \Delta_X^* \mathcal{O}_X \) induces spectral sequences
\[
E_2^{pq} = H^q(X, \Omega_X^{-p}) \implies HH_{-p-q}(X).
\]
and
\[
E_2^{pq} = H^q(X, \Omega_X^{-p} \otimes K_X^\vee[-d]) \implies HH^{p+q}(X).
\]
Using the isomorphism $\Omega^{-p} \otimes K^{\vee}_X \simeq \wedge^{d+p} T_X$ we can write this second spectral sequence as

$$E_2^{pq} = H^{q-d}(X, (\wedge^{d+p} T_X)) \Rightarrow HH^{p+q}(X).$$

In particular, these spectral sequences induce filtrations $Fil_{HH^*}$ and $Fil_{HH^*}$ on $HH_*(X)$ and $HH^*(X)$ respectively.

**Definition 1.6.** Let $X$ and $Y$ be smooth projective varieties over a field $k$, and let $\Phi: D(X) \to D(Y)$ be a derived equivalence. We say that $\Phi$ is **strongly filtered** if it is filtered and if the induced isomorphisms

$$\Phi^{HH_*}: HH_*(X) \to HH_*(Y), \quad \Phi^{HH^*}: HH^*(X) \to HH^*(Y)$$

preserve the filtrations.

**Remark 1.7.** This notion of strongly filtered is different from the notion, with the same name, considered in [22].

**Remark 1.8.** In characteristic 0 the HKR isomorphism gives an isomorphism

$$L \Delta_X^* \Delta_X^* \mathcal{O}_X \simeq \oplus_i \Omega^i_X[i],$$

and therefore the spectral sequences (1.5.1) and (1.5.2) degenerate on the $E_1$-page. The associated graded groups are then given by

$$\text{gr}^* HH_n(X) = H\Omega_n(X) := \oplus_{p-q=n} H^q(X, \Omega^p_X)$$

and

$$\text{gr}^* HH^n(X) = HT^n(X) := \oplus_{p+q=n} H^q(X, \wedge^p T_X).$$

In fact the HKR isomorphism defines isomorphisms

$$I_{HKR}^*: HH_*(X) \simeq H\Omega_*(X)$$

and

$$I_{HKR}^*: HH^*(X) \simeq HT^*(X).$$

Under these isomorphisms we have

$$\text{Fil}^* HH_*(X) = \oplus_{p \geq s} \oplus_{p-q=n} H^q(X, \Omega^p_X)$$

and

$$\text{Fil}^* HH^n(X) = \oplus_{q \leq -s} \oplus_{q+p=n} H^p(X, \wedge^q T_X).$$

As shown in [13, §5], it is preferable to modify $I_{HKR}$ by multiplying it by

$$\text{td}^{1/2} \in \oplus_i H^i(X, \Omega^i_X).$$

Multiplication by $\text{td}^{1/2}_X$ on $H\Omega_*(X)$ is an automorphism which preserves the above filtration, and therefore this correcting factor is irrelevant for the purposes of studying the filtration on $H\Omega_*(X)$.

Similarly, the isomorphism $I_{HKR}^*$ should be modified with the automorphism of $HT^*(X)$ given by contraction with $\text{td}^{-1/2}_X$. Again this automorphism preserves the filtration.

**Remark 1.9.** As explained in Proposition 2.10 below, if $K_X = 0$ and the characteristic of $k$ is zero, then any filtered derived equivalence is strongly filtered.
Question 1.10. Let $X$ and $Y$ be smooth projective varieties over a field $k$. If there exists a strongly filtered derived equivalence $\Phi : D(X) \to D(Y)$ does it follow that $X$ and $Y$ are birational?

To phrase our results succinctly it is useful to introduce the following terminology:

Definition 1.11. A smooth projective variety $X$ is of filtered Torelli type (resp. of strongly filtered Torelli type) if, whenever there is a filtered (resp. strongly filtered) derived equivalence $D(X) \to D(Y)$, we have that $X$ and $Y$ are birational.

We can rephrase Question 1.10 as follows.

Question 1.12. Is every smooth projective variety of strongly filtered Torelli type?

Notation 1.13. Given a variety $X$, we will let the symbol $\text{FT}(X)$ (resp. $\text{SFT}(X)$) denote the sentence “$X$ is of filtered Torelli type” (resp. “$X$ is of strongly filtered Torelli type”). Given a positive integer $n$, we will let $\text{FT}(n)$ denote the sentence “for all $X$ of dimension $n$ we have $\text{FT}(X)$”, and we will let $\text{FT}(\leq n)$ denote the sentence “$\text{FT}(m)$ for all $m \leq n$”. We define $\text{SFT}(n)$ and $\text{SFT}(\leq n)$ similarly.

In this paper we present some evidence in favor of an affirmative answers to questions Question 1.3 and Question 1.12. We summarize our main results in the following theorems.

Theorem 1.14. Assume $k$ is of characteristic $0$. Let $n \geq 1$ be an integer and let $X$ be a smooth projective variety over $k$ of dimension $n + 1$.

(i) If the Iitaka dimension of $K_X$ or $-K_X$ is nonzero and the induced fibration is everywhere defined, then $\text{FT}(\leq n)$ implies $\text{FT}(X)$. (Here one only needs $\text{FT}$ for varieties with trivial canonical class.)

(ii) If $H^1(X, \mathcal{O}_X) \neq 0$ and $\text{FT}(\leq n)$ holds then $\text{SFT}(X)$ holds.

Theorem 1.15. Assume $k$ is algebraically closed of characteristic $0$.

1. $\text{FT}(X)$ holds if $X$ is in one of the following classes of varieties.

   (i) Curves and surfaces.
   (ii) Abelian varieties.
   (iii) Threefolds with nonzero Iitaka dimension of $K_X$ or $-K_X$.
   (iv) Kummer varieties of abelian threefolds.

2. $\text{SFT}(X)$ holds for threefolds with $H^1(X, \mathcal{O}_X) \neq 0$.

3. If $X$ is a variety for which $\omega_X$ is torsion and $\tilde{X}$ denotes the canonical cover of $X$, then $\text{FT}(X)$ holds if and only if $\text{FT}(\tilde{X})$ holds.

Remark 1.16. For the purposes of the introduction we have restricted our statements to characteristic $0$, where we have the strongest results. However, some results are known in positive characteristics as well, and in the body of the paper we also occasionally consider fields of arbitrary characteristic.

Remark 1.17. In Section 6 below we also discuss an open variant of Question 1.10.

1.18. Another interesting direction to consider is to replace $\mathbb{A}^n$ by any of the standard Weil cohomology theories. In the language of [21], the various realizations (étale, crystalline, Betti, de Rham) of the Mukai motive.
One way to view this motive is as follows (see for example [28]). Let \( X \) be a smooth projective variety over \( k \) and let \( H^*(X) \) denote the cohomology of \( X \) in any of the standard theories. Assume that \( H^*(X) \) takes values in \( L \)-vector spaces for a field \( L \) of characteristic 0. Let \( L(1) \) denote the realization in our cohomology theory of the standard Tate motive, and let \( E \) denote the ring of Laurent polynomials on \( L(1) \). We view \( E \) as \( \mathbb{Z} \)-graded with \( L(1) \) in degree 2. So \( L(1) \) is a 1-dimensional \( L \)-vector space with additional structure (e.g. Galois action through the cyclotomic character in the étale case). If we fix a basis \( \beta \) for \( L(1) \) then \( E \) is the ring \( \mathbb{L}[\beta, \beta^{-1}] \). This presentation neglects the additional structure on \( L(1) \), but note that the ring \( E \) is canonically graded. In particular, there is a filtration \( \text{Fil}^*E \) given by

\[
\text{Fil}^i_E := L(s) \cdot \text{Sym}^i L(1).
\]

Equivalently \( \text{Fil}^*E \subset L[\beta, \beta^{-1}] \) consists of those Laurent polynomials of degree \( \geq s \).

Define

\[
\tilde{H}(X) := H^*(X) \otimes_L E.
\]

It is an \( L \)-vector space, which in addition inherits a filtration \( \text{Fil}^* \tilde{H}(X) \) from the filtration on \( E \).

Assume further that we have a theory of weights for our cohomology theory \( H^* \) such that \( H^w(X) \) has weight \( w \) for all \( w \) and \( L(1) \) has weight \(-2\). Then \( \tilde{H}(X) \) is an infinite direct sum of pure objects with weight \( w \) piece given by

\[
\tilde{H}(X)_w = \bigoplus_{i \in \mathbb{Z}} H^{w+2i}(X)(i).
\]

Furthermore, the restriction of the filtration \( \text{Fil}^* \tilde{H}(X) \) to \( \tilde{H}(X)_w \) is the filtration given by

\[
\text{Fil}^i_{\tilde{H}(X)_w} = \bigoplus_{i \geq s} H^{w+2i}(X)(i).
\]

Note that through this description, the information packaged in the \( E \)-module \( \tilde{H}(X) \) is the same as what is captured by what we called the Mukai motive in [21], in the case when \( X \) has no odd cohomology.

A derived equivalence \( \Phi: D(X) \to D(Y) \) given by \( P \in D(X \times Y) \) induces an isomorphism

\[
\Phi_P^*: \tilde{H}(X) \to \tilde{H}(Y)
\]

of \( E \)-modules. This will not in general preserve the filtrations. If it does then we say that \( \Phi \) is \( H \)-filtered. We can then ask the following question:

**Question 1.19** (\( H \)-variant of Question 1.10). Let \( X \) and \( Y \) be smooth projective varieties over \( k \). If there exists an \( H \)-filtered derived equivalence \( \Phi: D(X) \to D(Y) \) does it follow that \( X \) and \( Y \) are birational?

1.20. We will not emphasize the cohomological versions of Question 1.10 in this paper, but it should be noted that appropriate \( H \)-versions of the main results also hold, using the same methods.

1.21. Yet another direction to consider is to replace \( A^* \) by cycle groups modulo finer equivalence relations. In particular, we can consider algebraic cycles modulo rational equivalence \( CH^*(X) \) instead of \( A^*(X) \). The property of being filtered with respect to \( CH^* \) is strictly stronger than being filtered with respect to \( A^* \) so the above results
also hold with $CH^*$ instead of $A^*$. However, we have no better results in the case of rational equivalence.

Finally we mention that the consideration of the groups $CH^*$ is closely related to motives in the sense of Voevodsky, and the motives in this sense associated to derived categories of coherent sheaves. We discuss this point of view briefly in section Section 10.

The paper is organized as follows. We begin by collecting a few basic facts about derived equivalences and their kernels in Section 2, giving some easy cases where derived equivalence implies birationality. We then turn to the foundational work needed to prove Theorems 1.14 and 1.15. Most of the ideas here are not new though some of our technical work may be of independent interest. First, in Section 3 we discuss when a complex with support on a closed subscheme is a pushforward from that closed subscheme; we show that for smooth closed subschemes things work as expected, which appears to be new. In the following sections, we exploit two of the basic features of derived equivalences: compatibility with Serre functors and derived Picard groups as in the work of Rouquier. We discuss these two aspects in Sections 4 and 5. In the study of the canonical fibrations of varieties one is naturally led to consider derived equivalences of open varieties, because of the presence of base points. This leads us to formulate a version of Question 1.10 for open varieties using bivariant intersection theory [16, Chapter 17]. We discuss this in Section 6. In Section 7 we then put the foundational work together to prove Theorem 1.14 and in Section 8 we go through and explain the cases where we can give an affirmative answer to Question 1.10, thereby proving Theorem 1.15. Sections 9 and 10 are somewhat independent of the rest of the paper. In Section 9 we give an alternate, more direct, proof of Theorem 1.14 for irregular surfaces. And in Section 10 we discuss the connection with motives in the sense of Voevodsky.

1.22. Conventions. For a noetherian scheme we write $D(X)$ for the bounded derived category of complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves.

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2. Basic results about filtered derived equivalences

2.1. The “generically a sheaf” property.

2.2. Let $k$ be a field and let $X$ and $Y$ be smooth projective varieties over $k$.

Let $P \in D(X \times Y)$ be a complex defining an equivalence

$$\Phi^P : D(X) \rightarrow D(Y).$$

Proposition 2.3. If $\Phi_P$ is a filtered equivalence, then for all $K \in D(X)$ we have equality of ranks

$$\text{rk}(K) = \pm \text{rk}(\Phi_P(K)).$$
Equivalently, either $\Phi_P$ or $\Phi_{P[1]}$ preserves ranks.

**Proof.** It suffices to prove the proposition after replacing $k$ by an algebraically closed field. The rank of $K$ is the image in $\mathbb{A}^0(X)$ of $K$, and likewise for $\text{rk}(\Phi_P(K))$. Since $\Phi_P$ is filtered it induces an isomorphism $Q \simeq \mathbb{A}^0(X) \to \mathbb{A}^0(Y) \simeq Q$. For higher codimension cycle class groups these maps are only defined rationally, but for $\mathbb{A}^0$ these maps are defined integrally; this follows from the integrality of the chern classes appearing in the definition of the map on $\mathbb{A}^0$. Since the same holds for the inverse of $\Phi_P$ this implies that $\Phi_P$ gives an isomorphism on integral Chow groups in codimension 0. The result follows.

**Proposition 2.4.** If $\Phi_P$ is a filtered equivalence and $P$ is generically on $X$ given by a sheaf, then there is an open subset $U \subset X$ such that $P|_{U \times Y}$ is isomorphic to the structure sheaf of the graph of a birational morphism $U \to Y$.

**Proof.** Suppose $P$ is a sheaf over $V \subset X$, and let $v \in V$ be a point with corresponding sheaf $P_v \in D(Y_{\kappa(v)})$. Our assumptions imply that $P_v$ is a sheaf. Let $Z_v \subset Y_{\kappa(v)}$ be the support of $P_v$ and assume $Z_v$ has dimension $\delta$. Let $\{Z_{v,i}\}_{i \in I}$ be the irreducible components of $Z_v$ of dimension $\delta$ and let $r_i$ denote the rank of $P_v$ at the generic point of $Z_{v,i}$. By our assumptions each $r_i$ is a positive integer. If $[P_v]^\delta \in A^\delta(Y_{\kappa(v)})$ denotes the $\delta$-component of the class of $[P_v]$ we then have

$$[P_v]^\delta = \sum_i r_i [Z_{v,i}].$$

Since $r_i > 0$ for all $i$, this class in $A^\delta(Y_{\kappa(v)})$ is nonzero (consider intersection with $H^{d-\delta}$ for an ample class $H$). It follows that we must have $\delta = 0$. We conclude that $P_v$ is supported on a zero-dimensional closed subscheme of $Y_{\kappa(v)}$. Looking at endomorphism rings it follows that $P_v$ is given by the skyscraper sheaf of a $\kappa(v)$-rational point of $Y$.

Applying this to a geometric generic point of $X$ and using generic flatness, we see that there is an open subscheme $U \subset X$ such that $P|_{U \times Y}$ is given by the graph of a morphism $\varphi: U \to Y$. Since $\Phi_P$ is an equivalence, we have that $\varphi$ is a monomorphism, hence unramified, whence by dimension considerations and the smoothness of $X$ and $Y$ we conclude that $\varphi$ is a birational morphism.

**Proposition 2.5.** If $P \in D(X \times Y)$ and $Q \in D(X' \times Y')$ are kernels of Fourier–Mukai equivalences then the action of the exterior product $P \boxtimes Q \in D(X \times X' \times Y \times Y')$, viewed as an equivalence between $X \times X'$ and $Y \times Y'$, fits into a diagram

$$
\begin{array}{ccc}
\mathbb{H}^*(X \times X') & \xrightarrow{\Phi_P \boxtimes \Phi_Q} & \mathbb{H}^*(Y \times Y') \\
\uparrow & & \uparrow \\
\mathbb{H}^*(X) \otimes \mathbb{H}^*(X') & \xrightarrow{\Phi_P \otimes \Phi_Q} & \mathbb{H}^*(Y) \otimes \mathbb{H}^*(Y')
\end{array}
$$

in which the vertical arrows are the Künneth isomorphisms. In particular, a product of filtered equivalences is filtered.

**Proof.** This follows from the fact that the Mukai vector of $P \boxtimes Q$ is $v(P) \boxtimes v(Q)$, which is a consequence of compatibility of chern characters with products and compatibility of Todd classes with direct sums [16, 3.2.3 and 3.2.4].
Lemma 2.6. Let $X$ be a projective scheme over a field $K$ and let $\mathcal{L}$ be an ample invertible sheaf on $X$. Let $P \in D(X)$ be a complex such that there exists an integer $i$ such that for all $n$ sufficiently big the complex $R\Gamma(X, P \otimes \mathcal{L}^\otimes n)$ is concentrated in degree $i$. Then $P \simeq F[-i]$ for a sheaf $F$ on $X$.

If furthermore $R^1(X, P \otimes \mathcal{L}^\otimes n)$ is of dimension 1 for all $n >> 0$ then $P$ isomorphic to a skyscraper sheaf supported at a $K$-rational point of $X$.

Proof. The spectral sequence associated to the canonical filtration takes the form

$$E_2^{pq} = H^q(X, \mathcal{H}^p(P) \otimes \mathcal{L}^\otimes n) \implies H^{p+q}(X, P \otimes \mathcal{L}^\otimes n).$$

For $n$ sufficiently big we have

$$H^q(X, \mathcal{H}^p(P) \otimes \mathcal{L}^\otimes n) = 0$$

for $q > 0$, and the spectral sequence gives

$$H^p(X, P \otimes \mathcal{L}^\otimes n) \simeq H^0(X, \mathcal{H}^p(P) \otimes \mathcal{L}^\otimes n),$$

and this group is nonzero if and only if $\mathcal{H}^p(P) \not\simeq 0$. From this the first part of the lemma follows.

For the second part note that it follows from the assumptions that the Hilbert polynomial of $F$ is constant with value 1. From this it follows that $F$ is a skyscraper sheaf supported at a $K$-point.

□

Corollary 2.7. Let $\Phi_P : D(X) \to D(Y)$ be an equivalence such that for all invertible sheaves $\mathcal{L}$ on $X$ the complex $\Phi_P(\mathcal{L})$ is generically on $Y$ an invertible sheaf concentrated in degree 0. Then $\Phi_P$ is generically on $X$ given by a birational morphism $X \dashrightarrow Y$.

Proof. Let $\eta_Y \in Y$ be the generic point. Then for any invertible sheaf $\mathcal{L}$ on $X$ we have

$$\Phi(\mathcal{L})_{\eta_Y} = R\Gamma(X_{\kappa(Y)}, P_{X \times Y} \otimes \mathcal{L}).$$

concentrated in degree 0 and of rank 1. By Lemma 2.6 we conclude that the restriction of $P$ to $X \times \{\eta_Y\}$ is a sheaf supported on a $\kappa(Y)$-rational point of $X$. By Proposition 2.4 applied to $\Phi^{-1}$ we get the result.

□

2.8. Hodge homology and cohomology.

2.9. As discussed in Paragraph 1.5 over a field $k$ of characteristic 0 and smooth projective $X/k$ we can consider

$$HT^i(X) := \oplus_{p+q=i} H^p(X, \wedge^q T_X)$$

and

$$H\Omega_i(X) := \oplus_{q-p=i} H^p(X, \Omega^q_X).$$

The $k$-vector space $HT^*(X)$ is a ring and $H\Omega^*(X)$ is a graded $HT^*(X)$-module (this is discussed in greater detail in [13]).

There is a decreasing filtration $G^*$ on $H\Omega^*(X)$ given by

$$G^r H\Omega^*(X) := \oplus_{q-p=s, q+p \geq r} H^q(X, \Omega^p_X).$$

Note that if $\widetilde{H}^{[0,1]}_{dR}(X)$ denotes the weight 0 and 1 part of the de Rham realization $\widetilde{H}_{dR}(X)$ of the Mukai motive, then

$$\text{gr}^0 \widetilde{H}^{[0,1]}_{dR}(X) \simeq H\Omega_0(X),$$

$$\text{gr}^1 \widetilde{H}^{[0,1]}_{dR}(X) \simeq H\Omega_1(X),$$

...
where on the left we consider the graded pieces of the Hodge filtration. With this identification the filtration $G^\bullet$ is identified with the filtration induced by the codimension filtration.

The action of $HT^s(X)$ on $\Omega_k(X)$ respects the grading in the sense that it is given by maps

$$HT^s(X) \times \Omega_k(X) \to \Omega_k(X).$$

Furthermore, an element $\alpha \in H^p(X, \wedge^q T_X)$ restricts to a map

$$\alpha \cap (-) : G^r \to G^{r+p-q}.$$ 

**Proposition 2.10.** Suppose $K_X = 0$. Then for any integer $w$

$$\{ \alpha \in HT^s(X) | \alpha \cap G^r \subset G^{r+s-2w} \text{ for all } r \} = \oplus_{p+q=s,q \leq w} H^p(X, \wedge^q T_X).$$

**Proof.** By the preceding observation the right side of (2.10.1) is contained in the left side. Suppose $\alpha = (\alpha_{p,q}) \in \oplus_{p+q=s} H^p(X, \wedge^q T_X) = HT^s(X)$ is an element with $\alpha_{p_0,q_0} \neq 0$ for some $q_0 > w$. By Serre duality, using the assumption that $K_X = 0$, the pairing

$$H^{p_0}(X, \wedge^{q_0} T_X) \times H^{d-p_0}(X, \Omega_{q_0}^0) \to H^d(X, \mathcal{O}_X) \simeq k$$

is a perfect pairing. Let $\beta \in H^{d-p_0}(X, \Omega_{q_0}^0)$ be an element such that $\alpha_{p_0,q_0} \cap \beta \neq 0$ in $H^d(X, \mathcal{O}_X)$. Then $\alpha_{p_0,q_0} \cap \beta$ is also the component of $\alpha \cap \beta$ lying in $H^d(X, \mathcal{O}_X)$. In particular, we find $\beta \in G^{d-p_0+q_0} \Omega_{s-d}(X)$ such that

$$\alpha \cap \beta \in G^d \Omega_{s-d}(X)$$

is nonzero. Now observe that $H^d(X, \mathcal{O}_X) = H^d(X, \mathcal{O}_X)$ and $G^{d+1} \Omega_{s-d}(X) = 0$. Since

$$d-p_0 + q_0 + s - 2w = d + 2(q_0 - w) > d$$

we conclude that $\alpha$ does not lie in the left side of (2.10.1), which implies the lemma. \qed

**Remark 2.11.** In general, the left side of (2.10.1) defines a filtration

$$\text{Fil}^* HT^s(X)$$

on $HT^s(X)$.

### 2.12. Equivariant comparisons.

**Theorem 2.13.** Suppose $X$ and $Y$ are smooth Deligne-Mumford stacks with torsion canonical classes and canonical covers $\pi_X: \tilde{X} \to X$ and $\pi_Y: \tilde{Y} \to Y$. Assume that the order of $\omega_X$ is invertible on $X$. If there is a filtered equivalence

$$\Phi_P : D(X) \to D(Y)$$

then there is an induced equivariant filtered equivalence

$$\Phi_{\tilde{P}} : D(\tilde{X}) \to D(\tilde{Y}).$$
Proof. This works precisely as in [7]. (They discuss quotient varieties, but their proofs hold verbatim in this context.) In particular, the equivariant transform fits into a commutative diagram

\[
\begin{array}{cccc}
D(\tilde{X}) & \xrightarrow{\Phi_{\tilde{P}}} & D(\tilde{Y}) \\
\pi_{\tilde{X}}^* & \downarrow & (\pi_{\tilde{X}})_* \\
D(X) & \xrightarrow{\Phi_P} & D(Y).
\end{array}
\]

Since \(\pi_X\) and \(\pi_Y\) are finite étale, the induced maps on Chow theory fit into a similar diagram

\[
\begin{array}{cccc}
A^*(\tilde{X}) & \xrightarrow{A^*(\Phi_{\tilde{P}})} & A^*(\tilde{Y}) \\
\pi_{\tilde{X}}^* & \downarrow & (\pi_{\tilde{X}})_* \\
A^*(X) & \xrightarrow{A^*(\Phi_P)} & A^*(Y).
\end{array}
\]

In particular, since \(\pi_*\) and \(\pi^*\) preserve codimension filtrations (and ample cones), we see that filtered equivalences lift to equivariant filtered equivalences. \(\square\)

Corollary 2.14. If \(\Phi_P: D(X) \rightarrow D(Y)\) is an equivalence with equivariant lift

\(\Phi_{\tilde{P}}: D(\tilde{X}) \rightarrow D(\tilde{Y})\)

then \(P\) is generically a sheaf if and only if \(\tilde{P}\) is generically a sheaf.

Proof. This can be read off from the transform of a general point. Since

\[\pi_{\tilde{Y}}^* \Phi_P(\kappa(x)) = \Phi_{\tilde{P}}(\pi_X^* \kappa(x)),\]

we see that a general point of \(X\) transforms to a sheaf if and only if a general point of \(\tilde{X}\) transforms to a sheaf, as desired. \(\square\)

3. SUPPORT OF COMPLEXES AND RELATIVIZATION OF EQUIVALENCE

In this section, we prove some foundational results about complexes of coherent sheaves. In particular, we address the question of when a complex supported on a closed subscheme \(X_0 \subset X\) is the pushforward of a complex from \(X_0\). Analogous questions about affine morphisms \(X' \rightarrow X\) have an interesting history. As shown in [30, Theorem 8.1], even for base change by field extensions, it is unusual for a complex with a \(X'\) structure to be pushed forward from \(X'\).

3.1. Let \(k\) be a field and let \(X/k\) be a finite type \(k\)-scheme. Let \(i: X_0 \hookrightarrow X\) be the inclusion of a closed subscheme with \(X_0/k\) smooth. Let \(\mathcal{J} \subset \mathcal{O}_X\) be the ideal sheaf of \(X_0\) in \(X\).

Theorem 3.2. Let \(F \in D(X)\) be a complex such that the multiplication map

\[t: F \otimes^L \mathcal{J} \rightarrow F\]

is the zero map. Then there exists a complex \(M \in D(X_0)\) and an isomorphism \(F \simeq i_* M\).

The proof will be in several steps (Paragraph 3.3)–(Paragraph 3.8).
3.3. We have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X \to i_* \mathcal{O}_{X_0} \to 0.$$  

For any $F \in D(X)$ we get from this by tensoring with $F$ and using the projection formula [33, Tag 0B55]

$$(i_* \mathcal{O}_{X_0}) \otimes^L F \cong i_* \mathcal{L}i^* F$$

a distinguished triangle

$$F \otimes^L \mathcal{F} \xrightarrow{t} F \xrightarrow{q} i_* \mathcal{L}i^* F \xrightarrow{\lambda} F \otimes^L \mathcal{F}[1].$$

If the map $t$ is zero, then there exists a map $\rho: i_* \mathcal{L}i^* F \to F$ such that $\rho \circ q$ is the identity map on $F$. If $\rho': i_* \mathcal{L}i^* F \to F$ is a second such map then $\rho - \rho'$ restricts to the zero map on $F$ and therefore factors as

$$i_* \mathcal{L}i^* F \xrightarrow{\lambda} F \otimes^L \mathcal{F}[1] \xrightarrow{\tau} F$$

for some map $\tau: F \otimes^L \mathcal{F}[1] \to F$ (alternatively this can be seen by noting that $q$ is the inclusion of a direct summand and any two retractions onto this summand differ by such a map $\tau$).

In the case when $t: F \otimes^L \mathcal{F} \to F$ is the zero map we therefore obtain an isomorphism, depending on the choice of $\rho$,

(3.3.1) $$\rho, \lambda): i_* \mathcal{L}i^* F \cong F \oplus F \otimes^L \mathcal{F}[1]$$

identifying $q$ with the inclusion of $F$ into the first factor and $\lambda$ with the projection onto the second factor.

3.4. Next observe that there is a canonical isomorphism

$$\mathcal{L}i^* i_* \mathcal{O}_{X_0} \cong \mathcal{O}_{X_0} \oplus \mathcal{L}i^* \mathcal{F}[1].$$

Indeed the two-term complex

$$\mathcal{F} \to \mathcal{O}_X$$

represents $i_* \mathcal{O}_{X_0}$ and the projection onto $\mathcal{F}[1]$ gives the map to $\mathcal{L}i^* \mathcal{F}[1]$ upon applying $\mathcal{L}i^*$ and the map to $i_* \mathcal{O}_{X_0}$ is adjoint to a map $\mathcal{L}i^* i_* \mathcal{O}_{X_0} \to \mathcal{O}_{X_0}$. Concretely, if $\mathcal{K} \to \mathcal{F}$ is a flat resolution of $\mathcal{F}$ then $\mathcal{L}i^* i_* \mathcal{O}_{X_0}$ is represented by the complex

$$i^* \mathcal{K} \to i^* \mathcal{O}_X,$$

and since $\mathcal{F} \to \mathcal{O}_X$ pulls back to the zero map this complex is isomorphic to

$$\mathcal{O}_{X_0} \oplus i^* \mathcal{K}[1].$$

Now in general if $N \in D(X)$ and $M$ denotes $\mathcal{L}i^* N$ then by the projection formula we have

$$i_* (M) \cong i_* (\mathcal{O}_{X_0} \otimes \mathcal{L}i^* N) \cong (i_* \mathcal{O}_{X_0}) \otimes N.$$

Therefore

$$\mathcal{L}i^* i_* M \cong (\mathcal{L}i^* i_* \mathcal{O}_{X_0}) \otimes \mathcal{L}i^* N \cong M \oplus M \otimes^L \mathcal{L}i^* \mathcal{F}[1].$$

Compatibility of the projection formula with adjunction implies that the map $\mathcal{L}i^* i_* M \to M$ given by the first component of this isomorphism is given by the adjunction map $\mathcal{L}i^* i_* \to \text{id}$. If $S \in D(X_0)$ is a second complex, then a morphism

$$\alpha: i_* M \to i_* S$$

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in $D(X)$ is described by adjunction by a map
\[ Li^*i_*M \simeq M \oplus M \otimes L Li^* \mathcal{F}[1] \to S, \]
or equivalently two maps, which we refer to as components of $\alpha$,
\[ \alpha_0: M \to S, \quad \alpha_{-1}: M \otimes L Li^* \mathcal{F}[1] \to S \]
in $D(X_0)$.

If $\alpha = i_*\beta$ for a map $\beta: M \to S$ then the map $Li^*i_*M \to S$ adjoint to $\alpha$ is given by the adjunction map $Li^*i_*M \to S$ followed by $\beta$. This can be seen by noting that the adjunction map $adj: Li^*i_*M \to id$ is a morphism of functors, which implies that the diagram
\[
\begin{array}{ccc}
Li^*i_*M & \xrightarrow{Li^*\alpha} & Li^*i_*S \\
\downarrow{adj} & & \downarrow{adj} \\
M & \xrightarrow{\beta} & S
\end{array}
\]
commutes, and the fact [33, Tag 0037] that the composition
\[ Li^*i_*M \xrightarrow{Li^*\alpha} Li^*i_*S \xrightarrow{adj} S \]
is the map adjoint to $\alpha$. It follows that in this case $\alpha_0 = \beta$ and $\alpha_{-1} = 0$.

3.5. Returning to our setting of $F \in D(X)$ with $t: F \otimes L \mathcal{F} \to F$ the zero map, let
\[ \varpi: i_*Li^*F \to i_*Li^*F \]
denote the map $q \circ \rho$. In terms of the decomposition (3.3.1) the map $\varpi$ is the projection onto the first factor $F$. The first component of $\varpi$ is then a map
\[ \varpi_0: Li^*F \to Li^*F. \]
Define $F_0$ to be the homotopy colimit of the sequence
\[ Li^*F \xrightarrow{\varpi_0} Li^*F \xrightarrow{\varpi_0} Li^*F \xrightarrow{\varpi_0} \cdots. \]
See for example [5, 2.1 and 2.2] for the definition of this homotopy colimit. Let $\epsilon: F \to i_*F_0$ be the map obtained from the adjunction map $q: F \to i_*Li^*F$ and the natural map $i_*Li^*F \to i_*F_0$ coming from the first term in the sequence.

To prove Theorem 3.2 we show that $\epsilon$ is an isomorphism.

3.6. To verify that $\epsilon$ is an isomorphism we may work locally on $X$. We may therefore assume that $X_0$ is affine.

Let $\hat{X}$ denote the formal scheme obtained by completing $X$ along $X_0$, so that the inclusion $i$ factors as
\[ X_0 \xrightarrow{i} \hat{X} \xrightarrow{\pi} X, \]
and let $\hat{F}$ denote $L\pi^*F$. Since $\pi$ is flat, $\hat{F}$ is an object of $D(\hat{X})$, the bounded derived category of complexes of $\mathcal{O}_{\hat{X}}$-modules with coherent cohomology sheaves. Because $X_0$ is smooth over $k$ and is assumed affine, there exists a morphism
\[ r: \hat{X} \to X_0 \]
such that $r \circ \hat{i}$ is the identity map $X_0 \to X_0$. Fix one such $r$.  

3.7. In this local setting we can prove Theorem 3.2 directly as follows. Let \( M \in D(X_0) \) denote \( Rr_*\hat{F} \) (note that because the cohomology sheaves of \( F \) are annihilated by \( J \) this complex is again bounded with coherent cohomology sheaves), and let \( a: Lr^*M = Lr^*Rr_*\hat{F} \to \hat{F} \) be the adjunction map. We then get a morphism of distinguished triangles

\[
\begin{array}{ccc}
Lr^*Rr_*\hat{F} \otimes \pi_*\hat{J} & \longrightarrow & Lr^*Rr_*\hat{F} \\
\downarrow a & & \downarrow a \\
\hat{F} \otimes \pi_*\hat{J} & \longrightarrow & \hat{F} \\
\end{array}
\]

where \( \hat{J} \) denotes \( L\pi^*J \). Since \( t: \hat{F} \otimes \pi_*\hat{J} \to \hat{F} \) is zero there exists a dotted arrow \( \sigma \) as indicated. On the other hand, \( \hat{i}^*Lr^* = \text{id} \) so we get an induced map

\[
\lambda: \hat{i}_*M \to \hat{F}.
\]

Looking at cohomology sheaves this map is an isomorphism, as is the map

\[
F \longrightarrow \pi_*\hat{F} \longrightarrow \pi_*\hat{i}_*M \simeq \hat{i}_*M.
\]

3.8. We now return to the proof that \( \epsilon \) is an isomorphism in the case when \( X_0 \) is affine and we have chosen a retraction \( \hat{r}: \hat{X} \to X_0 \). By the preceding paragraph Paragraph 3.7 we know that \( F = i_*M \) for \( M \in D(X_0) \) of the form \( L\hat{i}^*N \) for \( N \in D(\hat{X}) \) (in fact take \( N = Lr^*M \)). We describe the map \( \varpi_0 \) in this case.

For this observe that we have

\[
L\hat{i}^*F \simeq \hat{i}^*\hat{F} \simeq (\hat{i}^*\hat{i}_*\mathcal{O}_{X_0}) \otimes^L \hat{i}^*N \simeq M \oplus M \otimes^L \hat{i}^*\mathcal{J}[1].
\]

It follows that there exist maps

\[
\rho_0: \hat{i}^*F \to M, \quad \lambda_0: \hat{i}^*F \to M \otimes^L \hat{i}^*\mathcal{J}[1]
\]

inducing maps as in Paragraph 3.3 upon pushforward by \( i \). In particular, the decomposition (3.3.1) induced by \( \rho_0 \) is induced by a decomposition

\[
\hat{i}^*F \simeq M \oplus M \otimes^L \hat{i}^*\mathcal{J}[1]
\]

in \( D(X_0) \).

The two maps \( i_*\rho_0 \) and \( \rho \) may not agree, but we can describe the map \( \varpi \) in terms of the decomposition defined by \( i_*\rho_0 \) as follows. As noted in Paragraph 3.3 there exists a map

\[
\tau: i_*(M \otimes^L \hat{i}^*\mathcal{J})[1] \to i_*M
\]

such that \( \rho = i_*\rho_0 + \tau \circ \lambda \). It follows that \( \varpi_0 \) restricts to the identity on \( M \) and some map

\[
\tau_0: M \otimes^L \mathcal{J}[1] \to M
\]

on the other factor. In particular, \( \varpi_0 \) fits into a morphism of distinguished triangles

\[
\begin{array}{ccc}
M & \longrightarrow & \hat{i}^*F \\
\downarrow \text{id} & & \downarrow \varpi_0 \\
M & \longrightarrow & \hat{i}^*F \\
\end{array}
\]

\[
\begin{array}{ccc}
M & \longrightarrow & M \otimes^L \mathcal{J}[1] \\
\downarrow 0 & & \downarrow \text{id} \\
M & \longrightarrow & M \otimes^L \mathcal{J}[1] \\
\end{array}
\]

\[
\longrightarrow M[1].
\]

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From this it follows that the natural map $M \to F_0$ is an isomorphism, and therefore $\epsilon$ is an isomorphism as well. This completes the proof of Theorem 3.2. □

Using Theorem 3.2, we can give a criterion for relativization of equivalences.

**Theorem 3.9.** Suppose $X$ and $Y$ are smooth projective varieties, containing open subvarieties $U_X \subset X$ and $U_Y \subset Y$ admitting smooth morphisms $f: U_X \to W$ and $g: U_Y \to W$ to a fixed scheme $W$. Write $Z := U_X \times_W U_Y$ and let $i: Z \to U_X \times U_Y$ be the inclusion morphism. Let $\mathcal{J}$ denote the ideal sheaf of $Z$ in $U_X \times U_Y$.

Suppose $\Phi: D(X) \to D(Y)$ is a derived equivalence with kernel $P$. If $P|_{U_X \times U_Y}$ is set-theoretically supported on the closed subscheme $Z \subset U_X \times U_Y$ then there is a complex $Q \in D(Z)$ such that $P|_{U_X \times U_Y}$ is isomorphic to $i_*Q$.

**Proof.** By Theorem 3.2, it suffices to show that the natural map $\mathcal{J} \otimes P \to P$ is the zero map, or equivalently that the adjoint map

$$(3.9.1) \quad \mathcal{J} \to R\mathcal{H}om(P, P)$$

is zero. This adjoint map can be described as follows.

Let $\mathcal{K} \subset \mathcal{O}_{W \times W}$ be the ideal of the diagonal of $W$. Since $f \times g: U_X \times U_Y \to W \times W$ is smooth we have

$$\mathcal{J} = L(f \times g)^*\mathcal{K}.$$ 

To prove the theorem it suffices to show that

$$(3.9.2) \quad R(f \times g)_*R\mathcal{H}om(P, P) \in D^{\geq 0}(W \times W)$$

and that the map

$$(3.9.3) \quad \mathcal{K} \to R^0(f \times g)_*R\mathcal{H}om(P, P)$$

is zero. Indeed if this holds then the map

$$\mathcal{K} \to R(f \times g)_*R\mathcal{H}om(P, P)$$

is zero as is the adjoint map (3.9.1). Furthermore, since $R(f \times g)_*R\mathcal{H}om(P, P)$ is supported on the diagonal in $W \times W$, it suffices to verify these statements on open subsets of the form $U \times U$, where $U \subset W$ is an open affine subset. We may therefore assume that $W$ is affine, which we do for the rest of the proof.

Write $W = \text{Spec}(R)$ for some ring $R$ so that

$$W \times W = \text{Spec}(R \otimes_k R)$$

Let $I \subset R \otimes_k R$ be the ideal of the diagonal and let $\tilde{I}$ denote the associated quasi-coherent sheaf on $W \times W$ (so $\tilde{I} = \mathcal{K}$).

Let $p: U_X \times U_Y \to U_X$ denote the projection map, so we have a commutative diagram

$$\begin{array}{ccc}
U_X \times U_Y & \xrightarrow{f \times g} & W \times W \\
\downarrow p & & \downarrow \text{pr}_1 \\
U_X & \xrightarrow{f} & W,
\end{array}$$
and let $\mathcal{F}$ denote $\text{pr}_{1,*}\tilde{I}$, a quasi-coherent sheaf on $W$. To prove that
\[ R(f \times g)_* R\mathcal{H}om(P, P) \in D^{\geq 0}(W \times W) \]
it suffices to show that
\[ R\text{pr}_{1,*} R(f \times g)_* R\mathcal{H}om(P, P) \simeq Rf_*Rp_* R\mathcal{H}om(P, P) \in D^{\geq 0}(W), \]
and to prove the vanishing of (3.9.3) it suffices to show that the induced map
\[ (3.9.4) \quad \mathcal{F} \to R^0f_* Rp_* R\mathcal{H}om(P, P) \]
is zero. For this, in turn, it suffices to show that
\[ Rp_* R\mathcal{H}om(P, P) \]
is concentrated in non-negative degrees, and that the map
\[ f^*\mathcal{F} \to R^0p_* R\mathcal{H}om(P, P) \]
is zero. Since $P$ has proper support over $U_X$ the complex $Rp_* R\mathcal{H}om(P, P)$ has coherent cohomology sheaves, and therefore by Nakayama’s lemma it suffices to show that for all points $x \in U_X$, with associated complex $P_x \in D(Y_{\kappa(x)})$, we have
\[ (3.9.5) \quad \text{Ext}^i_{D(Y_{\kappa(x)})}(P_x, P_x) = 0 \quad \text{for} \quad i < 0, \]
and the map
\[ (3.9.6) \quad f^*\mathcal{F}(x) \to \text{Ext}^0_{D(Y_{\kappa(x)})}(P_x, P_x) \]
is zero. To check this we may extend scalars to an algebraic closure of $\kappa(x)$, and may therefore assume that $x$ is a $k$-rational point.

Now observe that we have
\[ \text{Ext}^i_{D(Y)}(P_x, P_x) = \text{Ext}^i_{D(X)}(\kappa(x), \kappa(x)). \]
From this the vanishing of the groups (3.9.5) follows, and furthermore it follows $\text{Ext}^0_{D(Y_{\kappa(x)})}(P_x, P_x)$ is a field. Since $P$ is set-theoretically supported on the diagonal, we know that some power of $J$ acts trivially on $P$, and therefore the image of $f^*\mathcal{F}(x)$ in the ring $\text{Ext}^0_{D(Y_{\kappa(x)})}(P_x, P_x)$ consists of nilpotent endomorphisms. We conclude that (3.9.6) is zero as desired.

\section{Endofunctors and canonical divisors}

4.1. Let $\mathcal{D}$ be a triangulated category and let $\text{End}(\mathcal{D})$ denote the category whose objects are triangulated functors $F: \mathcal{D} \to \mathcal{D}$ and whose morphisms are natural transformations of functors.

Let $\mathcal{S} \subset \text{End}(\mathcal{D})$ be a small subcategory such that the following hold.

(i) Every object $S \in \mathcal{S}$ is an autoequivalence of $\mathcal{D}$. We write $\{S_i\}_{i \in I}$ for the set of objects of $\mathcal{S}$.
(ii) Every morphism in $\mathcal{S}$ is nonzero.
(iii) For all $i, j \in I$ the composition $S_j \circ S_i$ is again in $\mathcal{S}$.
(iv) For all $i \in I$ an inverse equivalence $S_i^{-1}$ is in $\mathcal{S}$. Note that this implies that the identity functor is in $\mathcal{S}$.
Let $S$ be the set of morphisms in $\mathcal{D}$ of the form
$$\alpha_X: S_iX \to S_jX$$
for $\alpha: S_i \to S_j$ a morphism in $\mathcal{D}$ and $X \in \mathcal{D}$ an object.

**Lemma 4.2.** The set $S$ is a multiplicative system in the sense of \cite[Tag 04VC]{[33]}.

**Proof.** Axioms MS1 and MS2 are immediate.

To verify axiom LMS3 let $X \in \mathcal{D}$ be an object and $j \in I$, and consider two morphisms $f, g: S_jX \to Y$ and $\alpha: S_i \to S_j$ in $\mathcal{D}$ such that the compositions
$$f \circ \alpha_X, g \circ \alpha_X: S_iX \to Y$$
are equal. We have to then show that there exists a morphism $\beta: \text{id} \to S_t$ in $\mathcal{D}$ for some $t \in I$ such that $\beta_Y \circ f = \beta_Y \circ g$. Consider the difference $h = f - g: S_jX \to Y$ and fill this into a distinguished triangle
$$S_jX \xrightarrow{h} Y \xrightarrow{u} Z \xrightarrow{v} S_jX[1].$$

By our assumptions there exists $\alpha: S_i \to S_j$ such that $\alpha_X: S_iX \to S_jX$ factors through a morphism $\lambda: S_iX \to Z[-1]$, so $\alpha_X = v[-1] \circ \lambda$. Applying $S_i^{-1}$ to the diagram
$$Z[-1] \xrightarrow{v} S_jX \xrightarrow{h} Y \xrightarrow{\alpha_X} \xrightarrow{\alpha_Y} S_tX \xrightarrow{S_t h} S_tS_j^{-1}Y,$$
we obtain the diagram
$$S_i^{-1}Z[-1] \xrightarrow{v} S_i^{-1} \circ S_jX \xrightarrow{h} S_i^{-1}Y \xrightarrow{\alpha_X} \xrightarrow{\alpha_Y} S_tX \xrightarrow{h} S_j^{-1}Y$$
which implies that the composition
$$S_jX \xrightarrow{h} Y \xrightarrow{\alpha_Y} S_j^{-1}S_iY$$
is zero. From this axiom LMS3 follows. Axiom RMS3 is shown similarly. \qed

We can therefore form the localized category $\mathcal{D}^{-1}\mathcal{D}$ by inverting the morphisms in $S$. Note that in this generality it is not clear that this will be a triangulated category. However, in the cases we are interested in this will be so.

**4.3.** Let $k$ be a field and let $X/k$ be a smooth $k$-scheme. Let $f: X \to Y$ be a dominant morphism to a projective integral $k$-scheme $Y$. Let $\mathcal{O}_Y(1)$ be an ample invertible sheaf on $Y$ and let $\mathcal{L}$ denote the pullback $f^*\mathcal{O}_Y(1)$. For $n \in \mathbb{Z}$ define the functor
$$S_n: \text{D}(X) \to \text{D}(X), \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}^\otimes n.$$

We define $\mathcal{S}$ to be the category of endofunctors of $\text{D}(X)$ whose objects are the $S_n$ and whose morphisms $S_m \to S_n$ are given by sections in $\Gamma(Y, \mathcal{L}^\otimes (n-m))$. 

Lemma 4.4. The axioms (i)-(iv) in Paragraph 4.1 hold.

Proof. This is immediate from the definition.

Proposition 4.5. Let \( \eta \in Y \) be the generic point and let \( X_\eta \) be the generic fiber. Then the restriction \( D(X) \to D(X_\eta) \) induces an equivalence of categories

\[
S^{-1} D(X) \simeq D(X_\eta).
\]

Proof. Let us first show that every object of \( D(X_\eta) \) is in the essential image.

So let \( K \in D(X_\eta) \) be a complex. Since \( X \) is regular, \( K \) is a perfect complex, and by [34, 3.20.1] we can therefore find an open subset \( U \subset Y \) such that \( K \) extends to a perfect complex \( K_U \) on \( X_U := f^{-1}(U) \). To extend \( K_U \) further to an object of \( D(X) \) it suffices by [34, 5.2.2 (a)] to extend each of the cohomology sheaves \( \mathcal{H}(K_U) \), which reduces to problem to the extension problem for coherent sheaves, which is standard. Thus every object of \( D(X_\eta) \) is in the essential image of \( D(X) \).

Next we show the full faithfulness. Let \( F, G \in D(X) \) be two objects and let \( f_U: F_U \to G_U \) be a morphism in \( D(X_U) \), for some \( U \subset Y \) dense open. Since \( X \) is smooth we can represent \( F \) be a strictly perfect complex on \( X \), which we again denote by \( F \), and \( G \) be a bounded below complex of injective quasi-coherent \( \mathcal{O}_X \)-modules. In this case the map \( f_U \) is represented by a map in \( C(\mathfrak{O}_{X_U}) \) of complexes on \( X_U \). By [34, 5.4.1 (a)] there exists an integer \( n \) and a section \( s \in \Gamma(Y, \mathcal{L}^{\otimes n}) \) such that the composition

\[
F_U \otimes \mathcal{L}^{\otimes (-n)} \overset{1 \otimes s^{\otimes n}}\longrightarrow F_U \longrightarrow G_U
\]

extends to a morphism of complexes \( F \otimes \mathcal{L}^{\otimes (-n)} \to G \). From this it follows that

\[
\text{Hom}_{S^{-1} D(X)}(F, G) \to \text{Hom}_{D(X_U)}(F_U, G_U)
\]

is surjective.

Similarly we show that if \( h: F \to G \) is a morphism in \( S^{-1} D(X) \) which induces the zero map over \( X_U \) then \( h \) is zero in \( S^{-1} D(X) \). By the definition of localization of a category [33, Tag 04VB], we can, after replacing \( F \) by \( F \otimes \mathcal{L}^{\otimes n} \) for suitable \( n \in \mathbb{Z} \), assume that \( h \) is given by a morphism \( h: F \to G \) in \( C(X) \) of complexes, where \( F \) is strictly perfect and \( G \) is a bounded below complex of quasi-coherent sheaves, which is homotopic to 0 over \( X_U \). Then by [34, 5.4.1 (c)] we conclude that there exists a nonzero section \( s \in \Gamma(Y, \mathcal{L}^{\otimes n}) \) for \( n \) sufficiently big such that the composition

\[
F \otimes \mathcal{L}^{\otimes (-n)} \overset{s^{\otimes n}}\longrightarrow F \overset{h}\longrightarrow G
\]

is homotopic to 0. This completes the proof of the proposition.

4.6. A variant of the above is the following. As before let \( k \) be a field and \( X/k \) a smooth scheme. Let \( \mathcal{L} \) be an invertible sheaf on \( X \) and for \( n \in \mathbb{Z} \) define

\[
S_n: D(X) \to D(X), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}^{\otimes n}.
\]

Define \( \mathcal{S}_\mathcal{L} \) to be the category of endofunctors of \( D(X) \) whose objects are the functors \( S_n \) and whose morphisms \( S_m \to S_n \) are given by the nonzero sections in \( H^0(X, \mathcal{L}^{\otimes (n-m)}) \). Then again the axioms (i)-(iv) in Paragraph 4.1 hold.
In the case when $f: X \to Y$ is a dominant morphism of projective integral schemes with $Y$ normal then $\Gamma(X, f^*\mathcal{O}_Y(n)) = \Gamma(Y, \mathcal{O}_Y(n))$ and the two constructions agree.

4.7. We will be particularly interested in the category $\mathcal{S}_K$. In this case the $S_n$ are the powers of the Serre functor on the category $D(X)$ shifted by the dimension of $X$.

**Proposition 4.8.** Let $X$ and $Y$ be smooth projective varieties over a field $k$ and let $\Phi: D(X) \to D(Y)$ be an equivalence of triangulated categories. Then the induced functor

\[
\text{End}(D(X)) \to \text{End}(D(Y)), \quad F \mapsto \Phi \circ F \circ \Phi^{-1}
\]

sends $\mathcal{S}_K$ to $\mathcal{S}_Y$.

**Proof.** Following [35], let $L_{\text{perf}}(X)$ (resp. $L_{\text{perf}}$) denote the dg-category of perfect complexes of quasi-coherent sheaves on $X$. The kernel $P$ then defines an equivalence

\[
\bar{\Phi}: L_{\text{perf}}(X) \to L_{\text{perf}}(Y).
\]

Let $S_X: D(X) \to D(X)$ be the Serre functor of $X$. By the uniqueness part of Orlov’s theorem, as well as Toën’s representability result in [35, 8.15] the functor has a lift

\[
\tilde{S}_X: L_{\text{perf}}(X) \to L_{\text{perf}}(X)
\]

which is unique up to equivalence of dg functors (in the sense of [35]).

In fact, $\tilde{S}_X$ is given by $\Delta_X\omega_X \in L_{\text{perf}}(X \times X)$. For integers $n$ and $m$ it therefore makes sense to consider the subspace

\[
\text{Hom}^\prime_{\text{End}(D(X))}(S^n_X, S^m_X) \subset \text{Hom}_{\text{End}(D(X))}(S^n_X, S^m_X)
\]

of morphisms of functors $S^n_X \to S^m_X$ which admit liftings to morphisms of dg functors $\tilde{S}_X^n \to \tilde{S}_X^m$. By [35, 8.9] the set $\text{Hom}^\prime_{\text{End}(D(X))}(S^n_X, S^m_X)$ consists precisely of those morphisms induced by sections of $K_X^{\otimes (m-n)}$.

Now for a lift $\tilde{S}_X$ the functor

\[
\bar{\Phi} \circ \tilde{S}_X \circ \bar{\Phi}^{-1}: L_{\text{perf}}(Y) \to L_{\text{perf}}(Y)
\]

is a dg lift of the Serre functor $S_Y$ of $Y$. From this it follows that (4.8.1) sends

\[
\text{Hom}^\prime_{\text{End}(D(X))}(S^n_X, S^m_X)
\]

to $\text{Hom}^\prime_{\text{End}(D(Y))}(S^n_Y, S^m_Y)$ which implies the lemma. \[\square\]

4.9. For smooth proper $X/k$ and a subcategory $\mathcal{S} \subset \text{End}(D(X))$, we can consider the subcategory $\mathcal{S}^\perp \subset D(X)$, defined to be the smallest full triangulated subcategory containing the objects $K \in D(X)$ such that for every morphism $\alpha: S_i \to S_j$ in $\mathcal{S}$ the induced morphism

\[
\alpha_K: S_i(K) \to S_j(K)
\]

is zero.

**Proposition 4.10.** For $\mathcal{S}_X$ as Paragraph 4.6 let $Z \subset X$ be the base locus of $\{L^{\otimes n}\}_{n>0}$ (the complement of the set of points $x \in X$ for which there exists $n > 0$ and a global section $\alpha \in H^0(X, L^{\otimes n})$ whose image in $L^{\otimes n}(x)$ is nonzero). Then $\mathcal{S}_X^{\perp,>0} \subset D(X)$ is the subcategory $D(X)$ of complexes $K \in D(X)$ whose image in $D(X-Z)$ is 0, where $S_{X,>0} \subset \mathcal{S}_X$ is the subcategory with the same objects but only morphisms $S_m \to S_n$ for $n > m$. 

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Proof. It is immediate that \( S^\perp_{X,>0} \subset D(X \text{ on } Z) \).

To show the reverse inclusion, note that by definition of a full triangulated subcategory for any distinguished triangle in \( D(X) \)
\[
K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow K_1[1]
\]
if two out of \( K_1, K_2, \text{ and } K_3 \) are in \( S^\perp_{X,>0} \) then so is the third. Using this and the standard truncations it then suffices to show that if \( F \) is a sheaf on \( X \) which is zero on \( X-Z \) then \( F \in S^\perp_{X,>0} \). Filtering such an \( F \) further we then reduce to the case when \( F = i_* E \), for \( E \) a vector bundle on a reduced closed subscheme \( i: Z' \subset X \) with \( Z' \subset Z \). For any \( n > 0 \) and \( \alpha \in \Gamma(X, \mathcal{L}^\otimes n) \) the induced map \( F(z) \rightarrow F(z) \otimes \mathcal{L}^\otimes n(z) \) is then zero at all points \( z \in Z \) by definition of \( Z \). Since \( Z' \) is reduced and \( E \) is a vector bundle it follows that the map given by \( \alpha \)
\[
i_* E \rightarrow i_* (E \otimes \mathcal{L}^\otimes n|_{Z'})
\]
is zero which implies that \( F = i_* E \in S^\perp_{X,>0} \). □

Corollary 4.11. Let \( \Phi: D(X) \rightarrow D(Y) \) be a derived equivalence between smooth projective varieties over a field \( k \). Then \( K_X \) is semiample if and only if \( K_Y \) is semiample.

Proof. Indeed Proposition 4.8 and Proposition 4.10 combine to give that the the base locus of the collection \( \{K^\otimes_n X\}_{n>0} \) is empty if and only if the base locus of \( \{K^\otimes_n Y\}_{n>0} \) is empty. □

Following classical works in this direction (see for example [27], [6], [20]) we can use this to obtain compatibility results with canonical varieties.

4.12. Let \( k \) be a field and \( X \) and \( Y \) smooth projective varieties related by a derived equivalence \( \Phi: D(X) \rightarrow D(Y) \). Let \( P \in D(X \times Y) \) be the complex defining \( \Phi \).

Let \( R_X := \oplus_{n \geq 0} H^0(X, K_X^\otimes n) \) (resp. \( R_Y := \oplus H^0(Y, K_Y^\otimes n) \)) be the canonical ring of \( X \) (resp. \( Y \)). As a \( k \)-vector space we have
\[
H^0(X, K_X^\otimes n) = \text{Hom}_{S_K_X}(S_0, S_n).
\]

Note that \( S_n \) is the \( n \)-fold composition of \( S_1 \) with itself, and therefore composition with \( S_m \) defines a map
\[
\text{Hom}_{S_K_X}(S_0, S_n) \rightarrow \text{Hom}_{S_K_X}(S_m, S_{n+m}),
\]
which is the standard isomorphism
\[
H^0(X, K_X^\otimes n) \cong \text{Hom}_X(K_X^\otimes m, K_X^\otimes (n+m)).
\]

This identification allows us to define a map
\[
\text{Hom}_{S_K_X}(S_0, S_m) \times \text{Hom}_{S_K_X}(S_0, S_n)
\]
composition
\[
\text{Hom}_{S_K_X}(S_0, S_{n+m})
\]
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which gives the multiplication rule on $R_X$. In this way the graded ring $R_X$ can be recovered from $S_{K_X}$.

**Corollary 4.13.** $\Phi$ induces an isomorphism of graded rings $\tilde{\tau}: R_X \simeq R_Y$.

4.14. In fact the compatibility with Serre functors gives more information about $P$. Let $Z_X \subset X$ (resp. $Z_Y \subset Y$) be the base locus of $\{K_X^{\otimes n}\}$ (resp. $\{K_Y^{\otimes n}\}$) and set $U_X := X - Z_X$ (resp. $U_Y := Y - Z_Y$). So we have morphisms

$$c_X: U_X \rightarrow \text{Proj}(R_X), \quad c_Y: U_Y \rightarrow \text{Proj}(R_Y).$$

**Proposition 4.15.** Let $x \in U_X(k)$ be a point. Then the support of $P_x \in D(Y)$ is contained in $U_Y$ and lies in the fiber of the morphism $c_Y$.

**Proof.** The skyscraper sheaf $k(x) \in D(X)$ has the following properties:

1. There exists $\alpha: \text{id} \rightarrow S_n$ for some $n > 0$ such that the induced map $\alpha_x: k(x) \rightarrow S_n(k(x))$ is an isomorphism.
2. $S_1(k(x)) \simeq k(x)$.

It follows that $P_x \in D(Y)$ has the analogous properties:

1. There exists $\beta: \text{id} \rightarrow S_n$ for some $n > 0$ such that the induced map $P_x \rightarrow P_x \otimes K_Y^{\otimes n}$ is an isomorphism in $D(Y)$.
2. $P_x \simeq P_x \otimes K_Y$.

Property (i) implies that the support of $P_x$ is contained in $U_Y$.

Property (ii) implies that for all $i \in \mathbb{Z}$ we have $\mathcal{H}^i(P_x) \simeq \mathcal{H}^i(P_x) \otimes K_Y$. By Lemma 4.16 below this implies that the support of $\mathcal{H}^i(P_x)$ is contained in a finite union of fibers of $c_Y$ for all $i$. Thus the support of $P_x$ is contained in a finite union of fibers, and since $\text{End}(P_x) = k$ the support must be connected. It follows that the support of $P_x$ is contained in a single fiber.

**Lemma 4.16.** Let $f: Z \rightarrow U_Y$ be a morphism with $Z$ proper, and let $\mathcal{F}$ be a coherent sheaf on $Z$ such that $\mathcal{F} \otimes f^*K_Y \simeq \mathcal{F}$. Then $f(\text{Supp}(\mathcal{F})) \subset U_Y$ is contained in a finite union of fibers of $c_Y$.

**Proof.** It suffices to prove the lemma after making a base change to an algebraic closure of $k$. Replacing $Z$ be an alteration if necessary we may assume that $Z$ is smooth and proper over $k$ and that $\mathcal{F}$ is supported on all of $Z$.

Let $r$ be the generic rank of $\mathcal{F}$. Then taking determinants we find that

$$\det(\mathcal{F}) \simeq \det(\mathcal{F}) \otimes f^*K_Y^{\otimes r}.$$ 

Therefore $f^*K_Y$ is a torsion line bundle on $Z$, which implies that the image of $Z$ in $\text{Proj}(R_Y)$ is a zero-dimensional subscheme.

4.17. It follows that the support of $P_U$ in $U_X \times U_Y$ is proper over both $U_X$ and $U_Y$. Therefore $P$ induces a functor

$$\Phi_U: D(U_X) \rightarrow D(U_Y).$$

**Proposition 4.18.** The functor $\Phi_U$ is an equivalence of triangulated categories.
Proof. Let \( P^\vee \in D(Y \times X) \) be the complex defining \( \Phi^{-1} : D(Y) \to D(X) \), and let \( P_U^\vee \) be the restriction of \( P^\vee \) to \( U_Y \times U_X \), which defines
\[
\Phi_U^\vee : D(U_Y) \to D(U_X)
\]
We claim that \( \Phi_U \circ \Phi_U^\vee \simeq \text{id}_{D(U_Y)} \) and \( \Phi_U^\vee \circ \Phi_U \simeq \text{id}_{D(U_X)} \).
To see this observe that the restriction of \( P \) to \( U_X \times Y \) is equal to the pushforward of \( P^\vee \) by Proposition 4.15, and similarly for \( P^\vee \). Since the diagram
\[
\begin{array}{ccc}
X \times Y \times X & \to & U_X \times Y \times U_X \\
\downarrow \text{pr}_{13} & & \downarrow \text{pr}_{13} \\
X \times X & \to & U_X \times U_X
\end{array}
\]
is cartesian we conclude that the pushforward of \( p^*_{12}P_U \otimes p^*_{23}P_U^\vee \) along the map
\[
p_{13} : U_X \times U_Y \times U_X \to U_X \times U_X
\]
is isomorphic to \( \Delta_{U_X,\phi_{U_X}} \). It follows that \( \Phi_U^\vee \circ \Phi_U \simeq \text{id}_{D(U_X)} \). The isomorphism
\[
\Phi_U^\vee \circ \Phi_U \simeq \text{id}_{D(U_Y)}
\]
is shown similarly. \( \square \)

4.19. Let \( U_{X,\eta} \) (resp. \( U_{Y,\eta} \)) denote the generic fiber of \( c_X \) (resp. \( c_Y \)). For \( n \in \mathbb{Z} \) let
\[
S_n : D(U_X) \to D(U_X)
\]
by the functor \( \mathcal{F} \mapsto \mathcal{F} \otimes K_{U_X}^n \), and let \( \mathcal{S}_{U_X} \subset \text{End}(D(U_X)) \) be the subcategory whose objects are the functors \( S_n \) and whose morphisms are given by global sections over \( X \) of the powers of the canonical sheaf. Define \( \mathcal{S}_{U_Y} \in \text{End}(D(U_Y)) \) similarly. The functor
\[
\text{End}(D(U_X)) \to \text{End}(D(U_Y)), \quad F \mapsto \Phi_U \circ F \circ \Phi_U^{-1}
\]
then takes \( \mathcal{S}_{U_X} \) to \( \mathcal{S}_{U_Y} \). Indeed since \( D(X) \to D(U_Y) \) is a localization, we have an inclusion
\[
\text{End}(D(U_X)) \hookrightarrow \text{HOM}(D(X),D(U_X))
\]
of categories of triangulated functors, so this compatibility follows from noting that we have a commutative diagram
\[
\begin{array}{cccc}
\text{End}(D(U_X)) & \to & \text{HOM}(D(X),D(U_X)) & \to & \text{End}(D(X)) \\
F \mapsto \Phi_U \circ F \circ \Phi_U^{-1} & & F \mapsto \Phi_U \circ F \circ \Phi_U^{-1} & & F \mapsto \Phi \circ F \circ \Phi^{-1} \\
\text{End}(D(U_Y)) & \to & \text{HOM}(D(Y),D(U_Y)) & \to & \text{End}(D(Y))
\end{array}
\]
and Proposition 4.8.

Theorem 4.20. The functor \( \Phi_U \) induces an equivalence of triangulated categories
\[
\Phi_{U,\eta} : D(U_{X,\eta}) \to D(U_{Y,\eta}),
\]
which is linear with respect to the isomorphism of function fields given by the isomorphism
\[
\tau : \text{Proj}(R_X) \to \text{Proj}(R_Y)
\]
provided by Corollary 4.13.

Proof. This follows from the preceding discussion and Proposition 4.5. \( \square \)
4.21. In fact, if \( x \in U_X \) is a point then the image in \( \text{Proj}(R_Y) \) of the support of \( P_x \) is equal to \( \tau(c_X(x)) \). This can be seen as follows. Let \( z \in \text{Proj}(R_Y) \) be the image of the support of \( P_x \), and let \( \alpha \colon \text{id}_{D(X)} \to S_{X,n} \) be given by a global section of \( K_X^* \otimes^n \) which does not vanish at \( x \). It follows that \( \bar{\tau}(\alpha) \colon \text{id}_{D(Y)} \to S_{Y,n} \) evaluates on \( P_x \) to an isomorphism, and therefore the image of the support of \( P_x \) in \( \text{Proj}(R_Y) \) lies in the basic open set \( D_+(\bar{\tau}(\alpha)) \). The image point of the support of \( P_x \) is then given by the induced homomorphism

\[
(R_Y,\bar{\tau}(\alpha))(0) \to \text{End}(P_x) = k(x)
\]
given by sending a fraction \( \rho/\bar{\tau}(\alpha)^m \) for \( \rho \colon \text{id}_{D(Y)} \to S_{Y,nm} \) to the endomorphism

\[
P_x \xrightarrow{\rho} P_x \otimes K_Y^* \otimes^n \bar{\tau}(\alpha)^{-1} \to P_Y.
\]

From this it follows that the image point of the support of \( P_x \) is precisely \( c_Y^{-1}(\tau(x)) \).

4.22. From this it also follows that we can often generically relativize \( \Phi \) with respect to the canonical morphisms.

**Proposition 4.23.** Suppose the morphisms \( c_X \) and \( \tau^{-1} \circ c_Y \) are smooth over an affine open subset \( V_X \subset \text{Proj}(R_X) \). Let \( U'_X = c_X(V_X) \) and \( U'_Y = c_Y^{-1}(\tau(V_X)) \). Then the restriction \( P|_{U'_X \times U'_Y} \) is in the essential image of the canonical functor

\[
D(U'_X \times V_Y) \to D(U'_X \times U'_Y).
\]

**Proof.** This is an immediate application of Theorem 3.9. \( \square \)

**Remark 4.24.** The assumption that the canonical morphisms are generically smooth holds for example if \( k \) has characteristic 0.

**Corollary 4.25.** The equivalence \( \Phi_{U,\eta} \) is induced by a complex

\[
P_{U,\eta} \in D(U_X,\eta \times \tau_{c_X,\text{Frac}(\text{Proj}(R_Y)),c_Y} U_Y,\eta).
\]

**Proof.** This follows from Proposition 4.23 by localization. \( \square \)

4.26. The above approach can be pushed further in some cases. We will be interested in the case when the powers of the canonical bundle \( K_X^* \) have nonzero global sections but are not necessarily generated by global sections. Fix one such integer \( n \), and let \( H_n \) denote \( H^0(X, K_X^*) \) so we have a rational map

\[
\pi_n \colon X \dashrightarrow \mathbb{P}H_n.
\]

Since \( X \) is normal this map is defined on an open subset \( W_X \subset X \) with complement of codimension \( \geq 2 \). On \( W_X \) we have an inclusion \( \pi_n^* \mathcal{O}_{\mathbb{P}H_n}(1) \hookrightarrow K_{W_X} \). Since \( X \) is regular this extends to an inclusion of invertible sheaves

\[
\pi_n^* \mathcal{O}_{\mathbb{P}H_n}(1) \hookrightarrow K_X^*.
\]

where we abusively write also \( \pi_n^* \mathcal{O}_{\mathbb{P}H_n}(1) \) for the unique invertible sheaf extending the given one on \( W_X \). Let \( W_{X,n} \subset X \) be the maximal open subset over which \( \pi_n^* \mathcal{O}_{\mathbb{P}H_n}(1) \) is generated by \( H_n \). 

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4.27. We can define a canonical complex $\mathcal{C}_{X,n}$ on $X$ with a map of complexes

$$\epsilon_{X,n}: \mathcal{C}_{X,n} \to \pi_n^* \mathcal{O}_{PH_n}(1)$$

which restricts to a quasi-isomorphism over $W_{X,n}$.

The complex $\mathcal{C}_{X,n}$ is the Koszul complex associated to the map $H_n \otimes_k \mathcal{O}_X \to \pi_n^* \mathcal{O}_X(1)$. Precisely, we have

$$\mathcal{C}_{X,n}^i = (\wedge^{i+1} H_n) \otimes_k K_X^{\otimes (in)}$$

for $i \leq 0$ and $\mathcal{C}_{X,n}^i = 0$ for $i > 0$. The differential

$$d_i: \mathcal{C}_{X,n}^i \to \mathcal{C}_{X,n}^{i+1}$$

is given by the usual formula in local coordinates

$$d_i((h_1 \wedge \cdots \wedge h_i) \otimes \ell) = \sum_{j=1}^i (-1)^{j+1} (h_1 \wedge \cdots \hat{h_j} \cdots \wedge h_i) \otimes (\rho(h_i) \otimes \ell),$$

where $\rho: H_n \otimes_k \mathcal{O}_X \to K_X$ is the natural map. The map $\epsilon_{X,n}$ is defined to be the map induced by the natural map $H_n \otimes_k \mathcal{O}_X \to \pi_n^* \mathcal{O}_{PH_n}(1)$. By standard properties of the Koszul complex the restriction of $\epsilon_{X,n}$ to $W_{X,n}$ is a quasi-isomorphism.

**Lemma 4.28.** Let $Q \in D(X)$ be a complex such that

$$Q \otimes L \mathcal{C}_{X,n} \simeq Q.$$ Then the support of $Q$ is contained in $W_{X,n}$.

**Proof.** Let $z \in \text{Supp}(Q)$ be a point in the support. Let $t$ be the largest integer for which $\mathcal{H}^t(Q)_z \neq 0$. Since $\mathcal{C}_{X,n}^i \in D^{\leq 0}(X)$ we then have

$$\mathcal{H}^t(Q \otimes L \mathcal{C}_{X,n})_z \simeq \mathcal{H}^t(Q)_z \otimes_{\mathcal{O}_{X,z}} \mathcal{H}^0(\mathcal{C}_{X,n}).$$

We therefore find that

$$\mathcal{H}^t(Q)_z \otimes_{\mathcal{O}_{X,z}} \mathcal{H}^0(\mathcal{C}_{X,n}) \simeq \mathcal{H}^t(Q)_z.$$

Since we assume that $\mathcal{H}^t(Q)_z$ is nonzero, this implies, by Nakayama’s lemma, that $\mathcal{H}^0(\mathcal{C}_{X,n})_z$ is generated by a single element. It follows that the subsheaf

$$\Sigma_n \subset \pi_n^* \mathcal{O}_{PH_n}(1)$$

generated by the image of $H_n$ is locally free of rank 1 at $z$, which implies that $z \in W_{X,n}$. □

4.29. For integers $n < m$ and a section $\alpha \in H^0(X, K_X^{\otimes (m-n)})$, multiplication by $\alpha$ induces a map $H_n \to H_m$. This map induces a map

$$\gamma_\alpha: \mathcal{C}_{X,n} \to \mathcal{C}_{X,m}$$

of complexes. Define

$$\mathcal{K}os_X \subset \text{End}(D(X))$$

to be the full subcategory whose objects are the functors given by tensor product with the complexes $\mathcal{C}_{X,n}$ and whose morphisms are given by sections of $K_X^{\otimes r}$ as above.
Proposition 4.30. Let $X$ and $Y$ be smooth projective varieties over $k$ and let

$$\Phi: D(X) \rightarrow D(Y)$$

be a derived equivalence given by a kernel $P \in D(X \times Y)$. Then the essential image of $Kos_X$ under the functor

$$End(D(X)) \rightarrow End(D(Y)), F \mapsto \Phi \circ F \circ \Phi^{-1}$$

is equal to $Kos_Y$.

Proof. Let $p: X \times Y \rightarrow X$ and $q: X \times Y \rightarrow Y$ be the projections. The proof of Proposition 4.8 implies that there exist isomorphisms

$$\sigma_n: p^*K_X^{\otimes n} \otimes^L P \simeq P \otimes^L q^*K_Y^{\otimes n}$$

in $D(X \times Y)$, such that for any $n < m$ and $\alpha \in H^0(X, K_X^{\otimes (m-n)})$ the diagram

$$\begin{CD}
p^*K_X^{\otimes n} \otimes^L P @>\alpha>> p^*K_X^{\otimes m} \otimes^L P \\
@V\sigma_n VV @V\sigma_m VV \\
p^*K_Y^{\otimes n} \otimes^L P @>>> p^*K_Y^{\otimes m} \otimes^L P
\end{CD}$$

commutes, where $\tilde{\tau}$ is as in Corollary 4.13. To prove the proposition it suffices to extend these isomorphisms to an isomorphism of complexes

$$\lambda: p^*C_{X,n} \otimes^L P \simeq q^*C_{Y,n} \otimes^L P.$$ 

Indeed if $\Phi_{X,n} \in Kos_X$ (resp. $\Phi_{Y,n} \in Kos_Y$) represents the functor defined by $C_{X,n}$ (resp. $C_{Y,n}$) then such an isomorphism defines an isomorphism

$$\Phi \circ \Phi_{X,n} \simeq \Phi_{Y,n} \circ \Phi.$$ 

For an integer $s$ define $C_{X,n}^{\leq s}$ to be the complex which in degrees $i \leq s$ is the same as $C_{X,n}$ but which has zero terms in degree $> s$. There is then a distinguished triangle for each $s$

$$C_{X,n}^{\leq s}[s] \rightarrow C_{X,n}^{\leq s} \rightarrow C_{X,n}^{\leq s-1} \rightarrow C_{X,n}^{\leq s}[s+1]. (4.30.1)$$

To prove the proposition we construct for each $s$ an isomorphism in $D(X \times Y)$

$$\lambda^{\leq s}: p^*C_{X,n}^{\leq s} \otimes^L P \simeq q^*C_{Y,n}^{\leq s} \otimes^L P,$$

such that the diagram

$$\begin{CD}
C_{X,n}^{\leq s}[s] \otimes^L P @>\lambda^{\leq s}>> C_{Y,n}^{\leq s}[s] \otimes^L P \\
@VV\sigma_n V @VV\sigma_m V \\
C_{X,n}^{\leq s} \otimes^L P @>>> C_{Y,n}^{\leq s} \otimes^L P
\end{CD}$$

commutes.

Lemma 4.31. Let $s$, $i$, and $j$ be integers with $j > s$.

(i) We have

$$\text{Hom}_{D(X \times Y)}(p^*C_{X,n}^{\leq s} \otimes^L P, q^*C_{Y,n}^{\leq s} \otimes^L P[j]) = 0.$$
The restriction map
\[ \text{Hom}_{\mathcal{D}(X \times Y)}(p^*c_{X,n}^s \otimes L P, q^*c_{Y,n}^i \otimes L P[s]) \to \text{Hom}_{\mathcal{D}(X \times Y)}(p^*c_{X,n}^s \otimes L P[s], q^*c_{Y,n}^i \otimes L P[s]) \]

is injective.

Proof. By considering the distinguished triangles (4.30.1) the proof of (i) is reduced to showing that for all integers \( s, i, \) and \( j > s \) we have
\[ \text{Hom}_{\mathcal{D}(X \times Y)}(p^*c_{X,n}^s \otimes L P[s], q^*c_{Y,n}^i \otimes L P[j]) = 0. \]
This follows from noting that elements of this group correspond to morphisms of functors
\[ \Phi \circ \Phi c_{X,n}^s \to \Phi c_{Y,n}^i \circ \Phi, \]
which can be lifted to the dg-categories of complexes of coherent sheaves. Using the isomorphism
\[ \Phi c_{Y,n}^i \circ \Phi \simeq \Phi \circ \Phi c_{X,n}^s \]
and applying \( \Phi^{-1} \) we see that we have to show that there are no nonzero morphisms of functors
\[ \Phi c_{X,n}^s \to \Phi c_{X,n}^i \]
which can be lifted to the dg-category. Here for a complex \( K \in \mathcal{D}(X) \) we write \( \Phi K \) for the endofunctor given by tensoring with \( K \), and similarly for complexes on \( Y \).

Equivalently, we need to show that there are no nonzero morphisms in \( \mathcal{D}(X \times X) \)
\[ \Delta_{X*} c_{X,n}^s \to \Delta_{X*} c_{X,n}^i, \]
which follows from the fact that \( j > s \).

Statement (ii) follows from (i) and consideration of the triangles (4.30.1).

We now construct \( \lambda^{\leq s} \) inductively. For \( s \) sufficiently negative we have \( c_{X,n}^{\leq s} = 0 \) so there is nothing to show. So we assume that \( \lambda^{\leq s} \) has been defined and construct \( \lambda^{s+1} \). For this consider the diagram of distinguished triangles
\[ \begin{array}{ccc}
\mathcal{C}_{X,n}^{s+1}[-(s+1)]P & \to & \mathcal{C}_{X,n}^{(s+1)} \otimes L P \to \mathcal{C}_{X,n}^{s} \otimes L P \to \mathcal{C}_{X,n}^{s+1}[-s]P \\
\downarrow \tau \otimes \sigma & & \downarrow \lambda_n & \downarrow \tau \otimes \sigma \\
\mathcal{C}_{Y,n}^{s+1}[-(s+1)]P & \to & \mathcal{C}_{Y,n}^{(s+1)} \otimes L P \to \mathcal{C}_{Y,n}^{s} \otimes L P \to \mathcal{C}_{X,n}^{s+1}[-s]P,
\end{array} \]
where the right-most inner square commutes by Lemma 4.31(ii). Now define \( \lambda^{(s+1)} \) to be a morphism as indicated by the dotted arrow. Note that in fact such a morphism is unique by Lemma 4.31(i). This completes the proof of Proposition 4.30.

4.32. Let \( W_X \subset X \) be the complement of the base locus of the invertible sheaves \( \pi_n^* \mathcal{O}_{\mathcal{H}_n} \), and define \( W_Y \subset Y \) similarly. We can then repeat the earlier discussion about derived equivalence and the canonical bundle with the categories \( \mathcal{K}_{\text{os}X} \) and \( \mathcal{K}_{\text{os}Y} \). From this we get the following.

**Theorem 4.33.** The analogous statements to Propositions 4.15, 4.18 and 4.23 and Theorem 4.20, where \( U_X \) (resp. \( U_Y \)) is replaced by \( W_X \) (resp. \( W_Y \)), all hold.
4.34. Observe that the induced diagram of Chow groups tensor $\mathbb{Q}$ modulo numerical equivalence

$$
\begin{array}{ccc}
A^*(X) & \longrightarrow & A^*(W_{X,\eta}) \\
\downarrow \Phi & & \downarrow \Phi_{W,\eta} \\
A^*(Y) & \longrightarrow & A^*(Y_{\eta})
\end{array}
$$

commutes. Since the horizontal morphisms are surjective it follows that of $\Phi$ is filtered then $\Phi_{W,\eta}$ is also filtered.

**Corollary 4.35.** Let $X$ and $Y$ be smooth projective varieties over $k$ and let $\Phi: D(X) \rightarrow D(Y)$ be a derived equivalence. If the collection of invertible sheaves $\{\pi_n^* \mathcal{O}_{\mathbb{P}}(1)\}$ on $X$ is base point free then Question 1.3 has an affirmative answer for $X$ and $Y$ if and only if Question 1.3 has an affirmative answer for $X_{\eta}$ and $Y_{\eta}$ (viewed as varieties over $\text{Frac}(R_Y)$).

**Proof.** This follows from the previous discussion. \qed

**Remark 4.36.** In the above we have considered the canonical sheaf and its positive powers. The same arguments can be applied to the dual of the canonical bundle to get results about the base loci of $K_X^{-1}$, the associated rings, etc.

5. **ROUQUIER FUNCTORS**

The purpose of this section is to explain how Rouquier’s work [32] imposes restrictions on the support of kernels of filtered derived equivalences. This is also related to work of Lombardi [24].

5.1. For a projective smooth geometrically connected $k$-scheme $X$ let $\text{Pic}^0_X$ denote the connected component of the identity in the Picard scheme of $X$, and set

$$
\text{Alb}_X := \text{Pic}^0(\text{Pic}^0(X)).
$$

We assume throughout this section that $k$ has characteristic 0 (but see Remark 5.25 below for conditions that give results also in positive characteristic).

This implies, in particular, that $\text{Pic}^0_X$ and $\text{Alb}_X$ are smooth over $k$.

Given a point $x \in X(k)$ we can view $\text{Pic}_X$ as representing the functor of pairs $(\mathcal{L}, \sigma)$ consisting of a line bundle $\mathcal{L}$ on $X$ and a trivialization $\sigma \in \mathcal{L}(x)$ of $\mathcal{L}$ at $x$. For a (scheme-valued) point $y \in X$ we then get an invertible sheaf $\mathcal{M}_y$ on $\text{Pic}_X$ by associating to $(\mathcal{L}, \sigma)$ the line $\mathcal{L}(y)$. This construction defines a morphism

$$
a_x: X \rightarrow \text{Alb}_X.
$$

Let $\mathcal{L}_x^u$ denote the universal line bundle with trivialization $\sigma$ at $x$ over $X \times \text{Pic}_X$. Then $a_x$ can also be described as the map defined by $\mathcal{L}_x^u|_{X \times \text{Pic}_X^0}$, viewed as a family of line bundles on $\text{Pic}_X^0$ parametrized by $X$.

For a second point $x'$ the universal line bundle $\mathcal{L}_{x'}^u$ over $X \times \text{Pic}_X$ satisfies

$$
\mathcal{L}_{x'}^u = \mathcal{L}_x^u \otimes p_*^* \mathcal{M}_{x'}^{-1}.
$$

It follows that $a_x$ is independent of $x$ up to translation on $\text{Alb}_X$.

If $Z \subset X$ is a subscheme of $X$, we say that $Z$ is in a fiber of the Albanese morphism if for some $x$, and hence all $x$, $a_x(Z)$ is a point in $\text{Alb}_X$. Note that this is equivalent to
being in the fiber of the Albanese morphism after base change $k \to \bar{k}$. This makes sense, therefore, even when $X$ does not have a $k$-point.

The main goal of this section is to prove the following:

**Theorem 5.2.** Let $Y$ be a second smooth projective variety and $\Phi: D(X) \to D(Y)$ is a derived equivalence such that the induced isomorphism

$$HT^1(X) \to HT^1(Y)$$

respects the filtration defined in Remark 2.11.

(i) If $P \in D(X \times X)$ denotes the complex defining $\Phi$ then for any point $x \in X$ the complex $P_x \in D(Y_{k(x)})$ is set-theoretically supported in a fiber of the Albanese morphism of $Y$.

(ii) There exists a dense open subset $X' \subset X$ such that for every $x \in X'(k)$ the complex $P_x$ is the derived pushforward of a complex supported on a fiber of the Albanese morphism of $Y$.

Remark 5.3. A strongly filtered derived equivalence satisfies the assumptions of the theorem by Remark 1.8.

The proof occupies the following Paragraph 5.4–Paragraph 5.17. Note that it suffices to prove the theorem after making a base change to an algebraic closure of $k$. For the remainder of this section we therefore assume that $k = \bar{k}$.

5.4. As in the case of our results about the canonical fibration, the key point is that a filtered derived equivalence must preserve certain autoequivalences arising from line bundles.

We say that an autoequivalence

$$\alpha: D(X) \to D(X)$$

satisfies the **Rouquier condition** $R_X$ if the complex $Q_\alpha \in D(X \times X)$ is isomorphic to $\Gamma_{\sigma^*}L$, where $\Gamma_{\sigma}: X \to X \times X$ is the graph $x \mapsto (x, \sigma(x))$ of an automorphism $\sigma$ of $X$ and $L$ is an invertible sheaf on $X$ numerically equivalent to 0.

The key observation of Rouquier in this regard is the following (see [32, Proof of 4.18]).

**Lemma 5.5.** Let $M$ be a scheme and let $Q \in D(X \times X \times M)$ be a complex perfect over $M$. Then there exists an open subset $U \subset M$ characterized by the condition that a point $m \in M$ lies in $U$ if and only if the restriction $Q_m \in D((X \times X)_{k(m)})$ defines a derived equivalence $D(X_{k(m)}) \to D(X_{k(m)})$ and satisfies $R_{X_{k(m)}}$.

**Proof.** For the convenience of the reader we sketch an argument.

The condition that $Q_m$ defines a derived equivalence is an open condition, as follows for example from [21, 3.3].

To show that the condition that $Q_m$ is a derived equivalence that satisfies $R_X$ is open it suffices to show the following:

Let $A' \to A$ be a surjection of Artinian local $k$-algebras with kernel $I$ satisfying $I^2 = 0$, and let $Q_{A'} \in D((X \times X)_{A'})$ be an $A'$-perfect complex such that $Q_A := Q_{A'} \otimes_{A'}^L A$ is isomorphic to $\Gamma_{\sigma^*}L_A$ for an automorphism $\sigma_A: X_A \to X_A$ and invertible sheaf $L_A$.
on $X_A$. Then $Q_{A'} \cong \Gamma_{\sigma A^*} \mathcal{L}_{A'}$ for a lifting $\sigma_A$ of $\sigma_A$ to an automorphism of $X_{A'}$ and $\mathcal{L}_{A'}$ a lifting of $\mathcal{L}_A$ to an invertible sheaf on $X_{A'}$.

For this note first of all that $Q_{A'}$ is a sheaf, by the local criterion for flatness. Furthermore, we have a distinguished triangle in $\text{D}(X_{A'})$

$$\mathcal{L}_A \otimes_A I \to \mathbf{R}p_{1*}Q_{A'} \to \mathcal{L}_A \to \mathcal{L}_A \otimes_A I[1]$$

which implies that $\mathbf{R}p_{1*}Q_{A'}$ is an invertible sheaf $\mathcal{L}_{A'}$ concentrated in degree 0. Here $p_1: X \times X \to X$ is the first projection. By adjunction the isomorphisms

$$\mathcal{L}_{A'} \to \mathbf{R}p_{1*}Q_{A'}$$

defines a map of sheaves

$$p_1^*\mathcal{L}_{A'} \to Q_{A'},$$

which is surjective since this is the case after base change to $A$. Let $J_{A'} \subset \mathcal{O}_{(X \times X)_{A'}}$ be the kernel of the induced map

$$\mathcal{O}_{(X \times X)_{A'}} \to p_1^*\mathcal{L}_{A'}^{-1} \otimes Q_{A'},$$

and let $\gamma: Z \hookrightarrow (X \times X)_{A'}$ be the closed subscheme defined by $J_{A'}$. We then have

$$\mathcal{O}_Z \cong p_1^*\mathcal{L}_{A'}^{-1} \otimes Q_{A'},$$

which implies that $Q_{A'} \cong \gamma_*\mathcal{L}_{A'}|_Z$. Since $Q_{A'}$ is flat over $A'$ we further conclude that $Z$ is flat over $A'$. This implies that the two projections $(i = 1, 2)$

$$Z \xrightarrow{\cong} (X \times X)_{A'} \xrightarrow{p_i} X_{A'}$$

are isomorphisms; that is, $Z$ is the graph of an isomorphism $\sigma': X_{A'} \to X_{A'}$ lifting $\sigma$. □

**Remark 5.6.** The deformation theory calculation in the preceding proof is compatible with the calculation of the tangent space of the deformation functor of $\Delta_{X*}\mathcal{O}_X$ using Hochschild cohomology.

Consider the identity automorphism $(\text{id}_X, \mathcal{O}_X)$ given by the kernel $Q: = \Delta_{X*}\mathcal{O}_X$ in $\text{D}(X \times X)$.

By standard deformation theory the tangent space of the deformation functor of $Q$ is given by

$$\text{Ext}^1_{X \times X}(Q, Q) \cong \text{Ext}^1_X(\mathcal{L}_{\Delta_{X*}\Delta_{X*}\mathcal{O}_X}, \mathcal{O}_X).$$

From the distinguished triangle

$$\Omega^1_X[1] \to \tau_{\geq -1}\mathcal{L}_{\Delta_{X*}\Delta_{X*}\mathcal{O}_X} \to \mathcal{O}_X \to \Omega^1_X[2]$$

we get an exact sequence

$$H^1(X, \mathcal{O}_X) \to \text{Ext}^1_X(\mathcal{L}_{\Delta_{X*}\Delta_{X*}\mathcal{O}_X}, \mathcal{O}_X) \to H^0(X, T_X).$$

Going through the construction of the identification between the tangent space of the deformation space of $Q$ and $\text{Ext}^1_X(\mathcal{L}_{\Delta_{X*}\Delta_{X*}\mathcal{O}_X}, \mathcal{O}_X)$ one sees that this sequence fits into
a commutative diagram of tangent spaces

\[
\begin{array}{ccc}
\text{(deformations of } O_X) & \xrightarrow{\mathcal{L}' \mapsto (\sigma', \mathcal{L}')} & \text{(def. of } (\text{id}_X, O_X)) \\
\cong & \downarrow & \cong \\
H^1(X, O_X) & \xrightarrow{\text{Ext}^1_{X_L}(L \Delta_X^* \Delta_X, O_X, O_X)} & H^0(X, T_X),
\end{array}
\]

where the left and right vertical maps are the standard isomorphisms. It follows that the middle vertical arrow, which is the isomorphism provided by the argument in the proof above, is compatible with the standard identifications of the extremal terms.

5.7. Let \( R^0_X \) be the fibered category which to any \( k \)-scheme \( T \) associates the groupoid of objects \( Q \in D((X \times X)_T) \) of \( T \)-perfect complexes such that for all geometric points \( t \to T \) the fiber \( Q_t \in D((X \times X)_{k(t)}) \) defines an equivalence \( D(X_t) \to D(X_t) \) satisfying \( R_X \) and whose associated automorphism \( X_t \to X_t \) lies in the connected component of the identity in \( \text{Aut}(X) \). Let \( R^0_X \) denote the group scheme

\[ R^0_X = \text{Pic}^0(X) \times \text{Aut}^0(X). \]

Then \( R^0_X \) is a \( \mathbb{G}_m \)-gerbe over \( R^0_X \).

5.8. Consider now a derived equivalence \( \Phi: D(X) \to D(Y) \) with \( Y/k \) also smooth and proper.

**Theorem 5.9** (Rouquier [32]). (i) For any \( T/k \) and \( Q \in D((X \times X)_T) \) in \( R^0_X(T) \) the complex \( \Phi \circ Q \circ \Phi^{-1} \in D((Y \times Y)_T) \) is in \( R^0_Y(T) \).

(ii) The induced functor

\[ R^0_X \to R^0_Y \]

is an equivalence of gerbes.

**Proof.** It suffices to prove the theorem after replacing \( k \) by a field extension, so we may assume that \( k \) is algebraically closed. The choice of a \( k \)-point of \( X \) then defines a trivialization of the gerbe \( R^0_X \to R^0_X \). Let

\[ \mathcal{M}_X \in D((X \times X)_{R^0_X}), \]

be the object obtained by pullback from a trivialization.

For any \( k \)-scheme \( S \) and \( P, Q \in D((X \times X)_S) \) define the convolution

\[ P * Q \in D((X \times X)_S) \]

to be

\[ R_{pr_{13*}}(L_{pr_{12*}} P \otimes L_{pr_{23*}} Q). \]

This is the complex defining the endomorphism of \( D(X_S) \) given by the composition of the endomorphisms given by \( P \) and \( Q \).

Let

\[ m: R^0_X \times R^0_X \to R^0_X \]

be the group law and let

\[ p_1, p_2: R^0_X \times R^0_X \to R^0_X \]

be the two projections. By direct calculation using the condition \( R_X \) one then has

\[ m^* \mathcal{M}_X \cong Lp_1^* \mathcal{M}_X \ast Lp_2^* \mathcal{M}_X \]
in \( D((X \times X)_{R^0_X \times R^0_X}) \).

Let 
\[
\mathcal{M}_X^\Phi \in D((Y \times Y)_{R^0_Y})
\]
be the complex obtained from \( \mathcal{M}_X \) by conjugating by \( \Phi \). By Lemma [5.5] there exists an open subset \( U \subset R^0_X \) whose points are characterized by the condition that \( \mathcal{M}_X^\Phi \) satisfies the condition \( R_Y \). We claim that \( U \) is all of \( R^0_X \). Indeed this open set satisfies the following:

(i) It is nonempty since the fiber over the identity of \( R^0_X \) is the complex representing the identity \( D(Y) \to D(Y) \).

(ii) It is closed under the group law and inversion. This follows from the observation that conjugation by \( \Phi \) is compatible with convolution so we have
\[
m^* \mathcal{M}_X^\Phi \simeq Lp_1^* \mathcal{M}_X^\Phi \otimes Lp_2^* \mathcal{M}_X^\Phi
\]
in \( D((Y \times Y)_{R^0_Y \times R^0_Y}) \).

From this it follows that \( U \) is a connected open subgroup of \( R^0_X \) and therefore equal to all of \( R^0_X \). From this (i) follows.

Statement (ii) follows from (i) and consideration of the inverse of \( \Phi \).

\[\square\]

Remark 5.10. The previous argument using the convolution was explained to us by Christian Schnell.

5.11. By definition we get an equivalence of stacks
\[
\tilde{\tau}: \mathcal{R}^0_X \to \mathcal{R}^0_Y, \quad Q \mapsto \Phi \circ Q \circ \Phi^{-1}
\]
and an induced isomorphism of group schemes
\[
\tau: R^0_X \to R^0_Y.
\]

The map on tangent spaces at the identity of this morphism is a map
\[
T\tau: H^1(X, \mathcal{O}_X) \oplus H^0(X, T_X) \to H^1(Y, \mathcal{O}_Y) \oplus H^0(Y, T_Y).
\]

Our assumption that \( \Phi \) preserves the filtration on \( HT^1 \) implies that
\[
T\tau(H^1(X, \mathcal{O}_X)) \subset H^1(Y, \mathcal{O}_Y).
\]

This implies that the isomorphism \( \tau \) restricts to an isomorphism
\[
\gamma: \text{Pic}^0(X) \to \text{Pic}^0(Y).
\]

Let \( \mathcal{P}ic^0_X \) (resp. \( \mathcal{P}ic^0_Y \)) denote the restriction of \( \mathcal{R}^0_X \) (resp. \( \mathcal{R}^0_Y \)) to \( \text{Pic}^0_X \) (resp. \( \text{Pic}^0_Y \)), so \( \mathcal{P}ic^0_X \) is the \( G_m \)-gerbe of line bundles on \( X \) rationally equivalent to 0, and similarly for \( \mathcal{P}ic^0_Y \).

Then \( \Phi \) induces an isomorphism of stacks
\[
\tilde{\gamma}: \mathcal{P}ic^0_X \to \mathcal{P}ic^0_Y.
\]

5.12. By associating to a line bundle \( \mathcal{L} \) on \( X \) the functor
\[
\otimes \mathcal{L}: D(X) \to D(X)
\]
we get a functor
\[
\epsilon_X: \mathcal{P}ic^0_X(k) \to \text{End}(D(X)).
\]
By construction this functor has the property that the diagram

\[
\begin{array}{ccc}
\Pic^0_X(k) & \xrightarrow{\gamma} & \Pic^0_Y \\
\downarrow{\epsilon_X} && \downarrow{\epsilon_Y} \\
\End(D(X)) & \xrightarrow{\text{ad } \Phi} & \End(D(Y))
\end{array}
\]

commutes, where \( \text{ad } \Phi \) is the map that conjugates by \( \Phi \).

For \( \mathcal{L} \in \Pic^0_X(k) \) write \( T_\mathcal{L}: D(X) \to D(X) \) for \( \epsilon_X(\mathcal{L}) \). From the formula

\[
\Phi(k(x)) \simeq \Phi(T_\mathcal{L}(k(x))) = T_\gamma(\mathcal{L})(\Phi(k(x)))
\]

we conclude that

\[
T_\mathcal{M}(\Phi(k(x))) \simeq \Phi(k(x))
\]

for all \( \mathcal{M} \in \Pic^0_Y(k) \).

Concretely this means that if \( P_x \) denotes the restriction of \( P \) to \( \{x\} \times Y \simeq Y_{k(x)} \) then

\[
P_x \otimes \mathcal{M} \simeq P_x
\]

for all \( \mathcal{M} \in \Pic^0_Y(k) \). Since tensoring with \( \mathcal{M} \) is an exact functor this also implies that

\[
\mathcal{H}^j(P_x) \otimes \mathcal{M} \simeq \mathcal{H}^j(P_x)
\]

for all \( j \in \mathbb{Z} \).

Applying the following lemma to the support of each \( \mathcal{H}^j(P_X) \) we conclude that \( P_x \) is set-theoretically supported in a fiber of the Albanese morphism (note that the support of \( P_x \) is connected since \( \End(P_x) = k(x) \)).

This completes the proof of Theorem 5.2 (i).

**Lemma 5.13.** Let \( f: Z \to Y \) be a morphism of proper \( k \)-schemes and suppose \( \mathcal{F} \) is a coherent sheaf on \( Z \) such that \( f(\text{Supp}(\mathcal{F})) \subset Y \) is connected and such that

\[
\mathcal{F} \otimes f^* \mathcal{M} \simeq \mathcal{F}
\]

for all \( \mathcal{M} \in \Pic^0_Y(k) \). Then \( f(\text{Supp}(\mathcal{F})) \) is contained in a fiber of the Albanese morphism.

**Proof.** This is very similar to the proof of Lemma 4.16.

Replacing \( Z \) by a resolution \( g: Z' \to Z \) and \( \mathcal{F} \) by \( g^* \mathcal{F} \) we see that it suffices to consider the case when \( Z \) is a disjoint union of smooth projective \( k \)-schemes. Furthermore, it suffices to consider each component separately, so to prove the lemma we may assume that \( Z \) is a smooth projective \( k \)-scheme. We may further assume that the support of \( \mathcal{F} \) is all of \( Z \).

Let \( r \) be the generic rank of \( \mathcal{F} \). We then have

\[
\det(\mathcal{F} \otimes f^* \mathcal{M}) \simeq \det(\mathcal{F}) \otimes f^* \mathcal{M}^{\otimes r} \simeq \det(\mathcal{F})
\]

for all \( \mathcal{M} \in \Pic^0_Y(k) \). This implies that

\[
f^* \mathcal{M}^{\otimes r} \simeq 0_Z
\]

for all \( \mathcal{M} \in \Pic^0_Y(k) \).

We conclude that the pullback morphism

\[
(5.13.1) \quad f^*: \Pic_Y^0 \to \Pic_Z
\]
sends the \( k \)-points of \( \text{Pic}^0_Y \), which are dense, to the discrete group scheme \( \text{Pic}^r_Z \). Therefore \((5.13.1)\) is the constant map sending all of \( \text{Pic}^0_Y \) to the identity; that is, the pullback map

\[
f^* : \text{Pic}^0_Y \to \text{Pic}^0_Z
\]

is the trivial map.

If \( z \in Z(k) \) is a point with image \( y \in Y(k) \) we have a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{a_z} & \text{Alb}_Z \\
\downarrow f & & \downarrow 0 \\
Y & \xrightarrow{a_y} & \text{Alb}_Y,
\end{array}
\]

where the right vertical map is 0, being dual to the map of Picard schemes defined by pullback. We conclude that \( f(Z) \subset a_y^{-1}(\text{id}_{\text{Alb}_Y}) \) as desired. \( \square \)

Next we turn to the proof of the scheme-theoretic statement Theorem \([5.2](ii)\).

5.14. First of all, the choice of base point needed to define the Albanese morphism can be avoided as follows.

For a smooth projective variety \( X/k \) let \( T_X \) denote the functor which to any \( k \)-scheme \( T \) associates the set of isomorphism classes of pairs \((s, \sigma)\), where \( s : \text{Pic}^0_{X,T} \to \mathfrak{P}ic^0_{X,T} \) is a trivialization of the \( \mathbb{G}_m \)-gerbe \( \mathfrak{P}ic^0_{X,T} \) and \( \sigma : s_0(e) \simeq [O_X] \) is an isomorphism between the composition

\[
T \xrightarrow{e} \text{Pic}^0_{X,T} \xrightarrow{s} \mathfrak{P}ic^0_{X,T}
\]

and the map \( T \to \mathfrak{P}ic^0_{X,T} \) given by \( \mathfrak{o}_X \).

Observe that any two such sections \((s, \sigma)\) and \((s', \sigma')\) differ by the automorphism of \( \mathfrak{P}ic^0_{X,T} \) obtained by tensoring with a line bundle on \( \text{Pic}^0_{X,T} \) trivialized at 0. In this manner \( T_X \) becomes a torsor under \( \text{Pic}^0_{\text{Pic}^0_X} \). Any point \( x \in X \) defines a pair \((s, \sigma)\), so there is a natural map \( X \to T_X \). Let \( T^0_X \) denote the connected component containing the image of \( X \). Then \( T^0_X \) is a torsor under \( \text{Alb}_X \) receiving a canonical map

\[
c_X : X \to T^0_X.
\]

If we trivialize \( T^0_X \) using a point of \( X \) then this is identified with the usual map from \( X \) to its Albanese.

Note also that we have a canonical isomorphism (this amounts to the fact that the translation action of an abelian variety \( A \) on \( \text{Pic}^0(A) \) is trivial)

\[
\text{Pic}^0(T_X) \simeq \text{Pic}^0(\text{Alb}_X)
\]

and therefore using Eq. \((5.1.1)\) an isomorphism

\[
\text{Pic}^0_{T_X} \simeq \text{Pic}^0_X.
\]

Chasing through these identifications one finds that this is simply given by

\[
c_X^* : \text{Pic}^0_{T_X} \to \text{Pic}^0_X.
\]

5.15. Let \( W \subset T^0_Y \) be the scheme-theoretic image of \( c_Y \), so \( W \) is an integral subscheme of \( T^0_Y \). Let \( W' \subset W \) be a dense open subset such that \( a_y \) restricts to a smooth morphism over \( W' \) (note that here we use the assumption that \( k \) has characteristic 0).
Lemma 5.16. There exists a dense open subset $X' \subset X$ such that for every point $x \in X'(k)$ the support of $P_x$ is set-theoretically contained in $a_y^{-1}(W') \subset Y$.

Proof. Let $Z \subset X \times Y$ be the support of $P$, and let $N \subset X \times T_Y$ be the scheme-theoretic image of $Z$. For every point $x \in X$ the intersection $N \cap \{x\} \times T_Y$ is set-theoretically a point by Theorem 5.2 (i), and therefore if $\tilde{N}$ is the normalization of $N$ the first projection $\tilde{N} \to X$ is an isomorphism. We conclude that $N$ is the graph of a map $h : X \to T_Y$. Since $Z$ dominates $Y$, the image of this morphism must be precisely $W$. Then $h^{-1}(W')$ satisfies the conditions of the lemma. □

5.17. Fix an open subset $X' \subset X$ as in the lemma, and let $x \in X'(k)$ be a point. Let $w' \in W'(k)$ be the image of $\text{Supp}(P_x)$. Let $\mathcal{O}_{W',w'}$ be the local ring of $W'$ at $w'$ and set

$$Y_{(w')} := Y \times_W \text{Spec}(\mathcal{O}_{W',w'}).$$

The complex $P_x$ is set-theoretically supported on the closed fiber of $Y_{(w')}$ and therefore can be viewed as an object of $D(Y_{(w')})$. Furthermore, there exists an integer $n \geq 0$ such that $P_x$ is in the image of the derived category of

$$Y_{(w')} \times_{\text{Spec}(\mathcal{O}_{W',w'})} \text{Spec}(\mathcal{O}_{W',w'}/m^n),$$

where $m \subset \mathcal{O}_{W',w'}$ is the maximal ideal. Therefore the map of rings

$$\mathcal{O}_{W',w'} \to \text{End}_{D(Y_{(w')})}(P_x) \simeq \text{End}_{D(X)}(k(x)) = k$$

factors through $\mathcal{O}_{W',w'}/m^n$ for some $n \geq 0$ and since the target is a field it must factor through $\mathcal{O}_{W',w'}/m = k$. Furthermore, we have

$$\text{Ext}^i_{D(Y_{(w')})}(P_x, P_x) = \text{Ext}^i_{D(X)}(k(x), k(x)),$$

which is zero for $i < 0$. It follows that the map given by multiplication

$$m \to \text{RHom}_{D(Y_{(w')})}(P_x, P_x)$$

is zero. By adjunction this implies that if $\mathcal{F} \subset \mathcal{O}_{Y_{(w')}}$ is the ideal sheaf of the closed fiber then the multiplication map

$$P_x \otimes^L \mathcal{F} \to P_x'$$

is zero. By Theorem 3.2 we conclude that $P_x$ is the image of an object of the derived category of the closed fiber of $Y_{(w')}$. This completes the proof of Theorem 5.2 (ii). □

Remark 5.18. When $\kappa(X) = 0$ (and under our assumption that $k$ has characteristic 0), Kawamata has shown [19, Theorem 1] that the Albanese morphism $c_X : X \to T_X$ is surjective with geometrically connected fibers.

We conclude this section by proving the following stronger result about the scheme-theoretic support of the kernel.

Theorem 5.19. Then there exists dense open subsets $V_X \subset W_X$ and $V_Y \subset W_Y$ and a $\gamma'_1$-linear isomorphism of torsors $t : T_X \to T_Y$ such that the restriction of $P$ to $c^{-1}_X(V_X) \times c^{-1}_Y(V_Y)$ is in the essential image of the functor

$$D(c^{-1}_X(V_X) \times_t T_Y, c^{-1}_Y(V_Y)) \to D(c^{-1}_X(V_X) \times c^{-1}_Y(V_Y)).$$

The proof of Theorem 5.19 will occupy the rest of this section.
Lemma 5.20. Let \( z \in T_X \) be a point and let \( F_z \) denote \( c_X^{-1}(z) \subset X \). Assume that \( F_z \) is reduced. Let \( P_{F_z} \) denote the restriction of \( P \) to \( F_z \times Y \). Then the image of the support of \( P_{F_z} \) under the map

\[
F_z \times Y \xrightarrow{pr_2} Y \xrightarrow{c_Y} T_Y
\]
is a point in \( T_Y \) with residue field \( k(z) \).

**Proof.** It suffices to verify the lemma after making a base change along a field extension \( k \subset L \) and therefore we may assume that \( z \) is a \( k \)-rational point. Let \( s: F_z \to F_z \times X \) be the section of the first projection given by the graph of the inclusion \( F_z \hookrightarrow X \), and let \( \mathcal{O}_s \in D(F_z \times X) \) be the structure sheaf of \( s(F_z) \). Consider the diagram

\[
\begin{array}{ccc}
F_z \times (X \times Y) & \xrightarrow{p} & F_z \\
\downarrow q & & \downarrow f \\
F_z \times X & \xrightarrow{g} & F_z \times Y
\end{array}
\]

Then \( P_{F_z} \) is obtained by applying \( \Phi \) to \( \mathcal{O}_s \in D(F_z \times X) \) and we find that for all line bundles \( \mathcal{M} \) on \( Y \) given by points in \( \text{Pic}^0_Y \) we have

\[
R^0 g_* R\text{Hom}(P_{F_z}, P_{F_z} \otimes \mathcal{M}) \simeq R^0 f_*(\mathcal{O}_s, \mathcal{O}_s \otimes \gamma^{-1}(\mathcal{M})) \simeq \mathcal{O}_{F_z},
\]

where the second isomorphism follows from the fact that any line bundle given by a point of \( \text{Pic}^0(T_X) \simeq \text{Pic}^0(X) \) restricts to the trivial line bundle on \( F_z \). The result now follows from a similar argument to the one used above, applying Lemma 5.13 to each of the cohomology sheaves of \( P_{F_z} \) and noting that the support of \( P_{F_z} \) is connected. \( \square \)

5.21. It follows that if \( \overline{N} \subset T_X \times T_Y \) is the image (which is closed) of the support of \( P \) on \( X \times Y \), then the projections \( \overline{N} \to T_X \) and \( \overline{N} \to T_Y \) are bijections onto their images \( W_X \subset T_X \) and \( W_Y \subset T_Y \). Therefore \( W \) is the graph of a birational isomorphism \( W_X \stackrel{t_W}{\dashrightarrow} W_Y \).

The composition

\[
X \xrightarrow{c_X} W_X \xrightarrow{t_W} W_Y \xleftarrow{c_Y} T_Y
\]
is a rational map from a smooth variety to a torsor under an abelian scheme and therefore is in fact everywhere defined, and we have a morphism \( f: X \to T_Y \) lifting \( t_W \). By the universal property of the Albanese variety this extends to a morphism \( t: T_X \to T_Y \) which is equivariant with respect an isomorphism of abelian varieties \( \delta: \text{Alb}_X \to \text{Alb}_Y \). From this it follows that the diagram
commutes, and that \( t_W \) is a morphism. We will show later that \( \delta = \gamma^t \).

Applying the same argument with the inverse transform we get that \( t_W \) is an isomorphism.

5.22. Let \( V_X \subset W_X \) be a dense open affine subset such that \( c_X^{-1}(V_X) \rightarrow V_X \) is smooth and let \( V_Y \) denote \( t_W(V_X) \). After possibly replacing \( V_X \) by a smaller open subset we can arrange that \( c_Y^{-1}(V_Y) \rightarrow V_Y \) is also smooth.

Let \( P' \) denote the restriction of \( P \) to \( c_X^{-1}(V_X) \times c_Y^{-1}(V_Y) \). Then \( P' \) is set-theoretically supported on the preimage of the graph of \( t_W \)

\[
V_X \longrightarrow V_X \times V_Y, \quad z \mapsto (z, t_W(z)).
\]

By Theorem 3.9 we conclude that \( P' \) is in the image of

\[
D(c_X^{-1}(V_X) \times t_Y c_Y^{-1}(V_Y)).
\]

5.23. Finally let us show that \( \delta = \gamma^t \). For this note that the equivalence of \( G_m \)-gerbes \( \tilde{\gamma} \) in (5.11.2) induces a \( \gamma^t \)-linear isomorphism

\[
\rho : T_X \rightarrow T_Y
\]

sending a trivialization \( s : \text{Pic}^0_X \rightarrow \mathcal{P}ic^0_X \) to the trivialization

\[
\text{Pic}^0_Y \xrightarrow{\gamma^{-1}} \text{Pic}^0_X \xrightarrow{s} \mathcal{P}ic^0_X \xrightarrow{\tilde{\gamma}} \mathcal{P}ic^0_Y.
\]

By the Albanese property the maps \( \rho \) and \( \gamma^t \) are determined by the induced map \( \rho \circ c_X : X \rightarrow T_Y \). Sorting through the definitions this map has the following interpretation. Let \( x \) be a point of \( X \). Then \( \rho \circ c_X(x) \) corresponds to the section \( s_x : \text{Pic}^0_Y \rightarrow \mathcal{P}ic^0_Y \) whose fiber over a point of \( \mathcal{P}ic^0_Y \) given by a line bundle \( \mathcal{M} \) is the set of trivializations of the rank 1 module

\[
\text{Hom}_{\text{D}(Y)}(P_x, P_x \otimes \mathcal{M}) \simeq \text{Hom}_{\text{D}(X)}(\kappa(x), \tilde{\gamma}^{-1}(\mathcal{M})) \simeq \tilde{\gamma}^{-1}(\mathcal{M})(x).
\]

For a dense set of points \( x \) the complex \( P_x \) is supported scheme-theoretically in the fiber of \( Y \) over \( z := t(c_X(x)) \). Writing \( \mathcal{M} \simeq c_Y^t \mathcal{M}' \) for a line bundle \( \mathcal{M}' \) on \( T_Y \), which is possible since \( \text{Pic}^0_{T_Y} \rightarrow \text{Pic}^0_Y \) is an isomorphism, we find that

\[
\text{Hom}_{\text{D}(Y)}(P_x, P_x \otimes \mathcal{M}) \simeq \mathcal{M}'(z) \otimes R^0 c_{Y*} \mathcal{R} \text{Hom}(P_x, P_x) \simeq \mathcal{M}'(z).
\]

From this it follows that \( \rho \circ c_X(x) = z \). This in turn implies that \( \rho = t \) and \( \delta = \gamma^t \).

Remark 5.24. The assumption in Theorem 5.2 that the induced isomorphism

\[
HT^1(X) \rightarrow HT^1(Y)
\]

respect the filtration can sometimes be verified as follows. The failure of this map to respect the filtration is measured by the map on tangent spaces induced by the morphism

\[
\text{Alb}(X) \rightarrow \text{Aut}^0(Y)
\]

arising from Eq. (5.11.1). The source of this map is an abelian variety, so if the target admits no non-constant maps from abelian schemes, for example if \( \text{Aut}^0(Y) \) is an affine algebraic group, then the assumption holds automatically.

Remark 5.25. The characteristic 0 assumption on \( k \) is used in the following ways:
(i) It implies that $\text{Pic}_X^0$ is smooth over $k$.
(ii) It implies that the morphisms $X \to W_X$ and $Y \to W_Y$ are generically smooth.

If one assumes these conditions instead of the characteristic 0 condition on $k$ then the results of this section hold also in positive characteristic.

6. D E R I V E D E Q U I V A L E N C E S W I T H C O M P A C T S U P P O R T

In this section $k$ denotes a field of arbitrary characteristic $p$ (possibly 0).

6.1. The previous results, especially Theorem 4.33, lead to consideration of derived equivalences

$$\Phi: D(U) \to D(V),$$

where $U$ and $V$ are smooth connected quasi-projective schemes over a field $k$, given by a complex $P \in D(U \times V)$ whose support is proper over $U$ and $V$ (we say that such an equivalence $\Phi$ has compact support).

Remark 6.2. The theory of derived equivalences for open varieties, using the notion of the derived category with compact support, is discussed in the context of moduli in the work of [2].

Definition 6.3. A variety $U$ is strongly connected by proper curves if for any closed subvariety $U' \subset U$ of dimension at least 2 and two geometric points $u, v \to U'$, there is an irreducible proper curve $T \to U'$ such that $u$ and $v$ factor through $T$.

Remark 6.4. One source of varieties that are strongly connected by proper curves are varieties $U$ that arise from the deletion of finitely many points from a projective variety. We are presently unaware of any other examples.

Theorem 6.5. Suppose $U$ and $V$ are non-proper smooth varieties and $V$ is strongly connected by proper curves. Given a derived equivalence

$$\Phi: D(U) \to D(V)$$

with compact support, the kernel of $\Phi$ is generically isomorphic to a shift of a graph of a birational isomorphism $U \dashrightarrow V$.

Proof. Since $U$ is not proper, there is an affine curve $C \subset U$ that is Zariski-closed. Let $Z \subset U \times V$ denote the support of the kernel of $\Phi$.

By assumption, $Z$ is proper over $U$ and over $V$. Consider the scheme $Z_{C \times V} \subset C \times V \subset U \times V$.

Fix an irreducible component $Z' \subset Z_{C \times V}$ and let $\Sigma = \text{pr}_2(Z') \subset V$ denote its image in $V$. Note that $Z_{C \times V} \to V$ is proper so $\Sigma$ is closed in $V$.

If $\dim \Sigma > 1$, then any two points of $\Sigma$ lie on a proper irreducible curve $T$ contained in $\Sigma$ (since we assume $V$ is strongly connected by proper curves). The scheme $Z_{C \times T}$ is proper over $k$ (since it is proper over $T$), whence it must have finite image in $C$, since $C$ is affine. In particular, this says that $T$ lies in a finite union of fibers $Z'_{c \times V}$, where $c$ ranges through a finite subset of $C$. Since $T$ is irreducible, we thus have that there is a point $c \in C$ such that $T \subset Z'_{c \times V}$. We conclude that $Z'_{c}$ is independent of $c \in C$, since otherwise we could find $T \subset \Sigma$ that does not lie entirely inside of $Z'_{c}$ (by connecting
two distinct points lying in different fibers of $Z' \to C$). But then, writing $W$ for the common value of $Z'_c$, we have that $C \times W \subset Z_{C \times T}$, whence $Z_{C \times T}$ dominates $C$, which directly contradicts the fact that $Z_{C \times T}$ has finite image in $C$.

We conclude that $\dim \Sigma \leq 1$, which implies that $\dim Z_c = 0$ for general $c \in C$. By [18, Corollary 6.12], we see that $Z$ is generically the graph of a birational isomorphism $\gamma: U \dashrightarrow V$ (and, in fact, the kernel of $\Phi$ is generically isomorphic to a shift of the graph of $\gamma$), as desired. □

**Remark 6.6.** In [8], Bridgeland shows that for an elliptic surface over a curve $X \to C$, certain components of $\operatorname{Pic}_{X/C}$ give rise to elliptic fibrations $Y \to C$ together with relative derived equivalences $D(X) \to D(Y)$. In other words, the kernel of the equivalence is supported on $X \times C Y$, and it is a tautological sheaf for the moduli problem. When $C$ has genus at least 2, one sees that if $X$ and $Y$ do not have isomorphic generic fibers then they cannot be birational.

Given any morphism $T \to C$, we thus get an induced equivalence $D(X_T) \to D(Y_T)$. In particular, choosing $T$ to be $C \times D \setminus \{p\}$ (where $D$ is a high genus curve and $p \in C \times D$ is a closed point), we get an equivalence between two strictly quasi-projective (i.e., non-projective) non-birational varieties that each have the property that any pair of points is contained in an irreducible curve. On the other hand, this property does not hold for subvarieties (since, for example, if we choose a section of $C \times D$ passing through $p$, we end up with an elliptic surface minus a fiber, and all proper curves are now vertical).

Thus, some version of strong connectedness by proper curves is necessary for a derived equivalence with compact support between open varieties to induce a birational isomorphism. On the other hand, these equivalences are not filtered in the sense of Question 6.10 below.

In light of Theorem 6.5, it is natural to consider a formulation of Question 1.10 for open varieties.

6.7. A derived equivalence with compact support induces an isomorphism on Chow groups with $\mathbb{Q}$-coefficients

$$CH^*(U) \to CH^*(V).$$

This can be defined as follows. Let $Z \subset U \times V$ be the support of the kernel $P$, viewed as a subscheme with the reduced structure, and let $p: Z \to U$ (resp. $q: Z \to V$) be the projection. The graph construction of Fulton-MacPherson then associates to $P$ a bivariant class [16, §18.1]

$$\operatorname{ch}^Z_{U \times V}(P) \in CH^*(Z \hookrightarrow U \times V).$$

Set

$$\beta(P):= \operatorname{ch}^Z_{U \times V}(P) \cdot \sqrt{\operatorname{Td}(U \times V)} \in CH^*(Z \hookrightarrow U \times V).$$

We then define

$$\Phi^C_P: CH^*(U) \to CH^*(V)$$

to be the composition

$$CH^*(U) \xrightarrow{p^*} CH^*(U \times V) \xrightarrow{\cap \beta(P)} CH^*(Z) \xrightarrow{q^*} CH^*(V).$$
6.8. Fix a prime $\ell$ invertible in $k$, and let $A^1_\ell(U)$ denote the image of the cycle class map

$$CH^1(U) \to H^{2i}(U_k, Q_\ell(i)),$$

so $A^*(U)$ is the group of cycles modulo $\ell$-adic homological equivalence.

The theory of localized Chern classes in étale cohomology and the bivariant étale theory discussed in [25, 4.24] implies that the map $\Phi^A$ induces a map

$$(6.8.1) \quad \Phi^A: A^1_\ell(U) \to A^1_\ell(V).$$

**Remark 6.9.** The group $A^1_\ell(U)$ does not depend on the choice of $\ell$. To verify this, we may replace $k$ by a field extension, and may therefore assume that $k$ is algebraically closed.

In this case note first that if $j: U \hookrightarrow X$ is a dense open immersion with $U$ smooth, $X$ normal and proper over $k$ and of dimension $d$, and $Z := X - U$ of codimension $r$ in $X$, we have a distinguished triangle

$$i_*\Omega_Z(-d)[-2d] \to \Omega_X(-d)[-2d] \to RJ_\ell Q_{\ell,U} \to i_*R^i\Omega_{\ell}[1],$$

where $\Omega_X$ (resp. $\Omega_Z$) denotes the $\ell$-adic dualizing complex of $X$ (resp. $Z$). Since

$$H^s(Z, \Omega_Z(r - d)[-2d]) = H^{s-2d}(Z, \Omega_Z(r - d)) \simeq H^{2d-s}(Z, Q_{\ell}(d - r))^\vee$$

we see that $H^s(Z, \Omega_Z(r - d)[-2d]) = 0$ for $s < 2r$ and that the cycle class map induces an isomorphism

$$Q_{\ell}^{Irr_{d-r}(Z)} \simeq H^{2r}(Z, \Omega_Z(r - d)[-2d]),$$

where $Irr_r(Z)$ denotes the set of irreducible components of $Z$ of dimension $d - r$.

Next observe that this implies that if $X$ is a proper normal variety then the image of $CH^1(X)$ in the $H^2(X, \Omega_X(1 - d)[-2d])$ is independent of $\ell$. Indeed by [14, 4.1] we can find a proper generically finite morphism $f: X' \to X$ with $X'$ projective and smooth. We then obtain a commutative diagram

$$\begin{array}{ccc}
CH^1(X') & \longrightarrow & H^2(X', Q_\ell(1)) \\
\downarrow f_* & & \downarrow f_* \\
CH^1(X) & \longrightarrow & H^2(X, \Omega_X(1 - d)[-2d]),
\end{array}$$

where the left vertical map is surjective. We conclude that $A^1_\ell(X)$ is a quotient of $NS(X')$. Let $U \subset X$ be a dense open smooth subset with complement of codimension $2$ in $X$. By the preceding paragraph the restriction map

$$H^2(X, \Omega_X(1 - d)[-2d]) \to H^2(U, Q_\ell(1))$$

is injective and the kernel of the composition

$$H^2(X', Q_\ell(1)) \xrightarrow{f^*} H^2(X, \Omega_X(1 - d)[-2d]) \longrightarrow H^2(U, Q_\ell(1))$$

is the image of $Q_{\ell}^{Irr_{d-1}(f^{-1}(X-U))}$. It follows that $A^1_\ell(X)$ is the quotient of $NS(X')$ by the subgroup generated by the divisors in $X'$ which are contracted in $X$.

This argument also gives the independence of $\ell$ for an arbitrary smooth $U$ as follows. Choose a dense open immersion $j: U \hookrightarrow X$ with $X/k$ proper and normal, and let $i: Z \hookrightarrow
be the complement of $U$. Then it follows from the preceding discussion that $A^1_\ell(U)$ is the cokernel of the map

$$Q^{Ir_d-1}(Z) \to A^1_\ell(X)$$

which is independent of the choice of $\ell$.

In what follows we write $\text{NS}(U)$ for the group $A^1_\ell(U)$. Note that the preceding argument also shows that $A^1_\ell(U)$ is finitely generated.

**Question 6.10** (Open version of Question [1.3]). Let $U$ and $V$ be smooth connected quasi-projective schemes over a field $k$, and let $\ell$ be a prime invertible in $k$. If there exists a derived equivalence $\Phi: D(U) \to D(V)$ with compact support such that the induced map (6.8.1) preserves the codimension filtration, does it follow that $U$ and $V$ are birational?

**Lemma 6.11.** Let $U/k$ be a quasi-projective smooth scheme such that an ample class is zero in $\text{NS}(U)$. Then $U$ is quasi-affine.

**Proof.** Choose a compactification $j: U \hookrightarrow X$ with $X$ normal and proper over $k$, and such that our vanishing ample class is given by the restriction of an ample line bundle $\mathcal{L}$ on $X$. The group $\text{NS}(U)$ is the quotient of $\text{NS}(X)$ by the subgroup of $\text{NS}(X)$ generated by divisors $D \subset X$ supported on $X - U$. It follows that there exists an ample divisor $D \subset X - U$ with class that of $\mathcal{L}$. The complement of this divisor is then an affine open subset containing $U$. □

**Remark 6.12.** Note that this applies, in particular, to non-proper curves.

**Lemma 6.13.** Suppose $U$ and $V$ are smooth connected quasi-affine schemes and that there exists an equivalence (6.1.1) with compact support. Then $U$ and $V$ are birational.

**Proof.** By our assumptions, $\Phi$ can be lifted to a functor between the dg categories of quasi-coherent modules on $U$ and $V$. We can obtain the coordinate rings $\Gamma(U, \mathcal{O}_U)$ and $\Gamma(V, \mathcal{O}_V)$ from these categories by taking the 0-th Hochschild homology group, and therefore the equivalence $\Phi$ induces an isomorphism of these rings. By [33, Tag 01P9] this implies that $U$ and $V$ are birational. □

6.14. It is also natural to consider actions on cohomology in the open setting.

If $i: Z \hookrightarrow U \times V$ is the inclusion, then as discussed in [25, 3.11] the class $\beta(P) \in CH^s(Z \hookrightarrow U \times V)$ defines a class

$$\beta_\ell(P) \in \bigoplus_s H^{2s}(Z, i^!Q_\ell(s)) \simeq \bigoplus_s H^{2(s-2d)}(Z, \Omega_Z(s-2d)).$$

Define

$$\tilde{H}_c(U, Q_\ell) := \bigoplus_s H^{2s}_c(U, Q_\ell(s)).$$

Since the projections $p: Z \to U$ and $q: Z \to V$ are proper, we can then consider the composition

$$\begin{align*}
\tilde{H}_c(U, Q_\ell) \xrightarrow{p^*} \tilde{H}_c(Z) &\xrightarrow{\beta_\ell(P)} \bigoplus_s H^{2s}_c(Z, \Omega_Z(s)) \\
&\xrightarrow{\bigoplus_s H^{2s}_c(V, q_!\Omega_Z(s)) \xrightarrow{q_!\Omega_Z(-d) \to Q_\ell[2d]} H_c(V, Q_\ell)}.
\end{align*}$$
A standard verification shows that this is an isomorphism, which we will denote by

\[ (6.14.1) \quad \Phi^H_p : \tilde{H}_c(U, Q_\ell) \to \tilde{H}_c(V, Q_\ell). \]

There is a natural decreasing filtration on \( \tilde{H}_c(U, Q_\ell) \) whose \( r \)-th step is given by

\[ \oplus_{s \geq r} H_c^{2s}(U, Q_\ell(s)). \]

As in the case of cycle class groups, we refer to this as the codimension filtration.

**Question 6.15.** Let \( U \) and \( V \) be smooth quasi-projective varieties over a field \( k \), and let \( \Phi : D(U) \to D(V) \) be a derived equivalence with compact support. Suppose that the induced isomorphism

\[ \tilde{H}_c(U, Q_\ell) \to \tilde{H}_c(V, Q_\ell) \]

preserves the codimension filtration. Does it follow that \( U \) and \( V \) are birational?

### 7. Proof of Theorem 1.14

If the Iitaka dimension of \( K_X \) or \( -K_X \) is positive then let \( X_\eta^\circ \) (resp. \( Y_\eta^\circ \)) be the generic fiber of the (anti)-canonical fibration. By Theorem 4.33 we can view \( X_\eta^\circ \) and \( Y_\eta^\circ \) as varieties over the same field, related by a derived equivalence with compact support. If the base locus of the (anti)-canonical fibration is generically empty, then this reduces Question 1.10 to the lower dimensional case.

To handle the case of \( H^1(X, \mathcal{O}_X) \neq 0 \), we use the Albanese morphisms and Theorem 5.19 to again reduce to the lower-dimensional case.

Note that in both cases we use the argument of Paragraph 4.34 to conclude that the property of being filtered passes to the generic fiber. \( \square \)

### 8. Known Cases

In this section we summarize the cases in which we have an affirmative answer to Question 1.10, thereby proving Theorem 1.15. Throughout this section we consider varieties over a field \( k \). Though many of our results are restricted to characteristic 0 we make that explicit as there are some partial results known also in positive characteristic. Throughout this section \( k \) denotes an algebraically closed field.

#### 8.1. \( \text{FT}(X) \) holds for \( X \) an abelian torsor.

Indeed in this case a theorem of Orlov [26, 3.2] implies that a complex \( P \in D(X \times Y) \) defining a derived equivalence \( D(X) \to D(Y) \) is generically on \( X \) a sheaf. By Proposition 2.4 this implies that it is generically the graph of an isomorphism.

#### 8.2. \( \text{FT}(X) \) holds for \( X \) a curve.

By the methods of section Section 4 using the canonical bundle, one gets that Question 1.10 has an affirmative answer for curves of genus \( \neq 1 \), and the genus one case is covered by the case of abelian torsors.
8.3. FT($X$) holds for $X$ a surface. Many cases of this are discussed in [10].

We show that (1.10) holds for surfaces over algebraically closed fields of characteristic not equal to 2 or 3, using the classification of surfaces. (In Section 3 below we give a different proof, not using the classification, for irregular surfaces in characteristic 0.) Before doing so it is useful to record some general observations about non-minimal surfaces with Kodaira dimension $\geq 0$.

Let $X$ be such a surface with minimal model $\pi: X \to X'$, let $E \subset X$ be the exceptional locus, and let $Z \subset X'$ denote $\pi(E)$ (a finite set of points). Let $\text{Aut}(X, E)$ (resp. $\text{Aut}(X', Z)$) denote the group scheme of automorphisms $\alpha: X \to X$ (resp. $\alpha': X' \to X'$) such that $\alpha(E) = E$ (resp. $\alpha'(Z) = Z$). If $S$ is a scheme and $\alpha: X_S \to X_S$ is an element of $\text{Aut}(X, E)(S)$ with induced morphism

$$\gamma_\alpha: X_S \to (X' \times X')_S, \quad x \mapsto (\pi(x), \pi(\alpha(x))),$$

then the sheaf $\gamma_\alpha^*O_{X_S}$ is a coherent sheaf on $(X' \times X')_S$ flat over $S$ and the formation of this sheaf commutes with arbitrary base change $S' \to S$ (this follows from [33, Tag 08ET] and the calculation of the derived pushforward in the fibers over points of $S$). It follows that the first projections $\text{Spec}(\gamma_\alpha^*O_{X_S}) \to X'$ is an isomorphism and that the composition

$$X_S \xrightarrow{\alpha} X_S \xrightarrow{\pi} X'_S,$$

factors through an isomorphism $\alpha': X'_S \to X'_S$. This construction defines a morphism

$$\text{Aut}(X, E) \to \text{Aut}(X', Z),$$

which is an inclusion on points. We claim that $\text{Aut}(X, E)$ does not receive any non-constant morphism from an abelian variety. To see this it suffices to show that this is the case for $\text{Aut}^0(X', Z)$, the connected component of the identity of $\text{Aut}(X', Z)$. For this note that $\text{Aut}^0(X', Z)$ necessarily fixes each of the points of $Z$, and for any of these points $z \in Z$ we get a representation of $\text{Aut}^0(X', Z)$ on $O_{X', z}/m^2_z$ for all $n \geq 1$. Furthermore, for $n$ sufficiently big this representation is faithful proving that $\text{Aut}^0(X', Z)$ is a subscheme of a general linear group.

We conclude, in particular, that any filtered derived equivalence $D(X) \to D(Y)$ satisfies the conditions of Theorem 5.2 by Remark 5.24. We now turn to the proof that FT($X$) holds.

- For rational surfaces and surfaces of general type we get the result immediately from Corollary 4.13. More precisely, we can detect rationality from the vanishing of all (positive) plurigenera, and we can calculate a birational model for surfaces of general type from the canonical ring.
- Next we treat the case of nonrational ruled surfaces.

Let $C$ be the normalization of the image of the Albanese morphism of $X$, so we have a canonical map $\pi: X \to C$. Since $X$ is not rational, $C$ is a curve of genus $g \geq 1$ and $X$ contains finitely many exceptional curves of the first kind. The automorphism group scheme $\text{Aut}(X)$ necessarily permutes these exceptional curves, and therefore the connected component of the identity $\text{Aut}^0(X)$ fixes each such exceptional curve. It follows that if $\pi: X \to X'$ is a minimal model, then, as in the discussion above, we have an inclusion

$$\text{Aut}^0(X) \hookrightarrow \text{Aut}^0(X', Z),$$
where $Z$ denotes the image in $X'$ of the contracted curves.

If $X$ is not minimal, in which case $Z$ is nonempty, then it follows from the same argument as above that $\text{Aut}^0(X', Z)$ does not admit a nonconstant morphism from an abelian variety, the assumptions of Theorem \ref{thm:5.2} hold, and we get $\text{FT}(X)$ from Theorem \ref{thm:5.19} (using also Remark \ref{rem:5.25}) and the case of curves.

If $X$ is minimal we proceed as follows. Write $X = P(E)$ for a rank 2 vector bundle on the curve $C$. Since $C$ has positive genus and the fibers of $X \to C$ are isomorphic to $P^1$, it follows that any automorphism of $X$ over a general $k$-scheme descends to an automorphism of $C$, and we have an exact sequence of automorphism group schemes

$$1 \to \text{Aut}(X/C) \to \text{Aut}(X) \to \text{Aut}(C),$$

where $\text{Aut}(X/C)$ denotes the group scheme of automorphisms of $X/C$.

We claim that $\text{Aut}(X/C)$ does not receive any non-constant morphism from an abelian variety. To see this, let $\mathcal{L}$ denote the determinant of $\mathcal{E}$. For any automorphism $\alpha: X \to X$ over $C$, we can consider the functor $I$ on the étale site of $C$ which to any $U \to C$ associates the set of isomorphisms $\tilde{\alpha}: \mathcal{E} \to \mathcal{E}$ inducing $\alpha|_U$ and the identity on $\mathcal{L}|_U$. Then $I$ is a $\mu_2$-torsor on $C$. Since $H^1(C, \mu_2)$ is finite, it follows that any element of $\text{Aut}^0(X/C)$ lifts to an element of $\text{Aut}_C(\mathcal{E})$, the group scheme of automorphisms of $\mathcal{E}$. This group scheme can be realized as the preimage of $G_m \subset A^1$ under the scheme map associated to the determinant map

$$H^0(C, \mathcal{E}^\vee \otimes \mathcal{E}) \to H^0(C, \mathcal{O}_C) \simeq k.$$ 

It follows that $\text{Aut}^0(\mathcal{E})$ is an affine group scheme, and therefore so is

$$\text{Aut}^0(X/C) \simeq \text{Aut}_C(\mathcal{E})/G_m.$$

In particular, this group admits no nonconstant morphism from an abelian varieties. If the genus of $C$ is greater than or equal to 2, which implies that $\text{Aut}(C)$ is finite, we conclude that $\text{Aut}^0(X)$ does not admit a nonconstant map from an abelian variety, and therefore the assumptions of Theorem \ref{thm:5.2} hold, and the proof in this case follows from Theorem \ref{thm:5.19} Remark \ref{rem:5.25} and the case of curves.

To conclude the case of non-rational ruled surfaces, we have to consider the case when the genus of $C$ is 1. Let $\Phi: D(X) \to D(Y)$ be a filtered derived equivalence, and let

$$\tau: R^0_X \to R^0_Y,$$

be the map defined in Eq. \ref{eq:5.11.1}. We claim that the induced map $H^{1}(X) \to H^{1}(Y)$ preserves the filtrations so that the assumptions of Theorem \ref{thm:5.2} hold.

First note that $Y$ is again ruled over a genus 1 curve. The statement that $Y$ is a ruled surface follows from preservation of Kodaira dimension and plurigenera, and the genus of the base curve can be read off from the dimension any realization of the Mukai motive as in \cite{21}. If $Y$ is not minimal then $\text{Aut}^0(Y)$ is affine, by the discussion above, and therefore we conclude that the assumptions of Theorem \ref{thm:5.2} hold. So it suffices to consider the case when $Y$ also is
minimal and fibered $\pi_Y: Y \to C_Y$ over a genus 1 curve. We claim that the composition

$$\text{Pic}^0(X) \xrightarrow{\tau} \text{Aut}^0(Y) \xrightarrow{} \text{Aut}^0(C_Y)$$

is zero. For this it suffices to show that the composition of maps of tangent spaces at the identity

$$H^1(X, \mathcal{O}_X) \xrightarrow{} H^0(Y, T_Y) \xrightarrow{} H^0(Y, \pi_Y^* T_{C_Y}) = H^0(C_Y, T_{C_Y})$$

is zero. By an argument similar to that proving Proposition 2.10 and using the assumption that $\Phi$ is filtered, for this it suffices to show that if $\alpha \in H^0(Y, T_Y)$ is a class with nonzero image in $H^0(C_Y, T_{C_Y})$ then

$$(8.3.1) \cap \alpha: H^1(Y, \Omega^1_Y) \to H^1(Y, \mathcal{O}_Y)$$

is nonzero. For this observe that if $\bar{\alpha} \in H^0(C_Y, T_{C_Y})$ is the image of $\alpha$ then the composition

$$H^1(C_Y, \Omega^1_{C_Y}) \xrightarrow{\pi^*} H^1(Y, \Omega^1_Y) \xrightarrow{\cap \alpha} H^1(Y, \mathcal{O}_Y) \simeq H^1(C_Y, \mathcal{O}_{C_Y})$$

is equal to the map capping with $\bar{\alpha}$. Since $\Omega^1_{C_Y}$ and $T_{C_Y}$ are both trivial, for any nonzero class $\bar{\alpha}$ the map

$$\cap \bar{\alpha}: H^1(C_Y, \Omega^1_{C_Y}) \to H^1(C_Y, \mathcal{O}_{C_Y})$$

is nonzero, and therefore Eq. (8.3.1) is also nonzero.

We conclude that $\tau$ maps $\text{Pic}^0(X)$ to $\text{Aut}^0(Y/C)$, which by the same argument as above is affine. It follows that in fact $\tau$ sends $\text{Pic}^0(X)$ to $\text{Pic}^0(Y)$ and the assumptions of Theorem 5.2 are verified. The proof in this case is now completed by applying Theorem 5.19 and the case of curves.

- For varieties with positive Kodaira dimension we use Corollary 4.13 and Corollary 4.35.

- This leaves surfaces of Kodaira dimension 0. Consider first the case of such a surface $X$ which is not minimal and a derived equivalence $D(X) \to D(Y)$ given by a kernel $P \in D(X \times Y)$. Let $\pi_X: X \to X'$ and $\pi_Y: Y \to Y'$ be the minimal models, obtained by contracting $(-1)$-curves. Let $n > 0$ be an integer such that $K_X^\otimes n \simeq \mathcal{O}_{X'}$ and $K_Y^\otimes n \simeq \mathcal{O}_{Y'}$. Then the base locus of $K_X^\otimes n$ is precisely the $(-1)$-curves on $X$, and similarly for $K_Y^\otimes n$. Let $U \subset X$ (resp. $V \subset Y$) be the complement of the $(-1)$-curves. Note that $U$ (resp. $V$) is the complement of a finite set of points in $X'$ (resp. $Y'$). By Proposition 4.18 we get an induced equivalence with compact support

$$\Phi: D(U) \to D(V),$$

and therefore by Theorem 6.5 $X$ and $Y$ are birational.

We may thus assume that $X$ is minimal. Abelian surfaces were already considered above, $K3$ surfaces were settled in [21, 6.1] in characteristic at least 5, and the case of Enriques surfaces was resolved in [17] in characteristic at least 5.
Finally, consider the case of a minimal hyperelliptic surface $X$. By [11, 10.27] the canonical cover of $X$ is a product of two genus 1 curves and therefore we get our result from the case of abelian torsors and Theorem 2.13.

8.4. Threefolds. Let $k$ be of characteristic 0 and let $X/k$ be a smooth projective threefold.

First, suppose $X$ is the Kummer variety associated to an abelian threefold $A$. Recall the McKay correspondence of Bridgeland–King–Reid [9]: there is a derived equivalence

$$D(X) \to D([A/\mu_2])$$

between $X$ and the stack quotient $[A/\mu_2]$, and moreover the kernel is generically a sheaf.

An equivalence

$$\Phi: D(X) \to D(Y)$$

thus induces an equivalence

$$\psi: D([A/\mu_2]) \to D(Y).$$

Applying Theorem 2.13, we get an induced equivalence

$$\varphi: D(A) \to D(\tilde{Y})$$

of canonical covers. We conclude that $Y$ is an abelian variety and that the kernel of $\varphi$ is a sheaf, which implies that the kernel of $\psi$ is a sheaf, and this in turn implies that the kernel of $\Phi$ is a sheaf. Applying Proposition 2.4 gives the desired result.

If $H^1(X,\mathcal{O}_X) \neq 0$ then using the Albanese morphism we see from Theorem 5.19 that SFT($X$) follows from FT($\leq 2$).

Similarly suppose either $K_X$ or $-K_X$ have nonzero Iitaka dimension. We then get a rational map

$$f: X \dashrightarrow B$$

defined on an open set $U \subset X$ with complement $Z \subset X$ of codimension $\geq 2$.

If the generic fiber of $f$ is proper over $k(\eta)$ then we obtain FT($X$) from FT($\leq 2$), already shown.

If the generic fiber $U_\eta$ is not proper but of dimension $\leq 1$ then FT($X$) follows from Theorem 4.33 and Lemma 6.13.

This leaves the case when $B$ has dimension 1. In this case we claim that the generic fiber $U_\eta$ is strongly connected by proper curves. For this let $t, s \in U_\eta$ be two closed points and let $T \subset X$ (resp. $S \subset X$) be the closure of $t$ (resp. $s$). Then $S$ and $T$ have codimension 2 in $X$ and the intersections $S \cap Z$ and $T \cap Z$ are finite sets of points. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of $S \cup T$. By Bertini’s First Theorem [15, 3.4.10] there exists an integer $m \geq 1$ such that for a general section $H \in H^0(X, \mathcal{I}(m))$ the intersection $H \cap X$ is irreducible of dimension 2 and $H \cap Z$ has codimension $\geq 2$ in $H$, and therefore is a finite set of closed points. In particular, the generic fiber of $H \cap X$ is an irreducible proper curve in $U_\eta$ containing $s$ and $t$. We conclude from this that FT($X$) holds by Theorem 4.33 and Theorem 6.5.
9. AN ALTERNATE PROOF FOR IRREGULAR SURFACES

In this section, we give an answer to Question 1.3 for all irregular surfaces, without using the classification of surfaces. The ideas that went into this proof resurfaced in [23] as a universal reconstruction theorem. This proof also applies to transforms whose action on line bundles yields complexes satisfying certain cohomological conditions (reminiscent of various forms of perversity). We do not pursue this here.

Given a $k$-scheme $X$, let $\mathcal{P}(X)$ denote the $k$-linear category of invertible sheaves on $X$. Write $\text{Arr}(X)$ for the set of all non-zero arrows of $\mathcal{P}(X)$. If $X$ is integral, the set $\text{Arr}(X)$ is a multiplicative system in the sense of [33, Tag 04VB].

**Proposition 9.1.** Given a regular connected separated $k$-scheme $X$, the localization $\text{Arr}(X)^{-1} \mathcal{P}(X)$ has a single object with endomorphism $k$-algebra $\kappa(X)$.

**Proof.** Since any two invertible sheaves $L$ and $M$ on $X$ can be connected by a tent

$$
\begin{array}{ccc}
N & \xleftarrow{f} & L \\
& \swarrow & \downarrow \\
L & & M
\end{array}
$$

of non-zero arrows, we see that $\text{Arr}(X)^{-1} \mathcal{P}(X)$ is connected. On the other hand, we can compute the endomorphism ring of $\mathcal{O}_X$ using tents

$$
\begin{array}{ccc}
& f & \\
\mathcal{O} & \xleftarrow{g} & \mathcal{O}
\end{array}
$$

We get a map of $k$-algebras

$$
\varepsilon: \text{End}_{\text{Arr}(X)^{-1} \mathcal{P}(X)}(\mathcal{O}_X) \to \kappa(X)
$$

by sending the tent $(f, g)$ above to the rational section $g/f: \mathcal{O}_X \to \mathcal{O}_X$. Given any rational function $\varphi$ on $X$, we get a divisor $Z \subset X$ of zeros of $\varphi$ and a divisor $P \subset Z$ of poles of $X$. By the basic theory of linear equivalence, we have that $\mathcal{O}(-Z) \cong \mathcal{O}(-P)$. Writing this invertible sheaf as $L$, we get an induced tent

$$
\begin{array}{ccc}
P & \xleftarrow{f} & L \\
& \swarrow & \downarrow \\
\mathcal{O} & & \mathcal{O}
\end{array}
$$

whose image under the map defined above is $\varphi$ (up to scaling). Thus, $\varepsilon$ is surjective.

Now suppose given two tents

$$
\begin{array}{ccc}
f & \xleftarrow{g} & L \\
& \swarrow & \downarrow \\
\mathcal{O} & & \mathcal{O}
\end{array}
$$
such that \( g/f = g'/f' \) in \( k(X) \).

We get an equality of divisors

\[
Z(g) - Z(f) = Z(g') - Z(f'),
\]

giving an equality

\[
Z(g) - Z(g') = Z(f) - Z(f').
\]

Write

\[
Z(g) - Z(g') = Z(f) - Z(f') = B - A
\]

for effective divisors \( A \) and \( B \) on \( X \). Unwinding the arithmetic gives equalities of effective divisors

\[
Z(f) + A = Z(f') + B
\]

and

\[
Z(g) + A = Z(g') + B.
\]

Let

\[
L'' = \mathcal{O}(-Z(f) - A) = \mathcal{O}(-Z(f') - B).
\]

The divisor equalities just described give a commuting diagram

\[
\begin{array}{ccc}
\mathcal{O} & \xleftarrow{f''} & L'' \\
\downarrow{f'} & & \downarrow{g'} \\
\mathcal{O} & \xleftarrow{L'} & \mathcal{O}
\end{array}
\]

with \( f, f', f'' \in \text{Arr}(X) \). This shows that \( \varepsilon \) is injective, completing the proof. \( \square \)

**Corollary 9.2.** Suppose \( X \) and \( Y \) are regular connected separated \( k \)-schemes. If there is a faithful functor

\[
f: \mathcal{P}(X) \to \mathcal{P}(Y)
\]

then there is an inclusion \( k(X) \to k(Y) \) over \( k \).

**Proof.** Since \( f \) is faithful it induces an injective map \( \text{Arr}(f): \text{Arr}(X) \to \text{Arr}(Y) \). Thus, there is an induced functor

\[
\text{Arr}(X)^{-1} \mathcal{P}(X) \to \text{Arr}(Y)^{-1} \mathcal{P}(Y).
\]

The resulting map of endomorphism rings gives an injection \( k(X) \to k(Y) \), as desired. \( \square \)

**Theorem 9.3.** Let \( X \) and \( Y \) be smooth projective surfaces over a field \( k \) of characteristic 0. Assume that \( H^1(X, \mathcal{O}_X) \neq 0 \).
(i) If there is a rank-preserving equivalence

\[ \Phi_D : D(X) \to D(Y) \]

then \( X \) and \( Y \) are isogenous (i.e., there are inclusions \( k(X) \hookrightarrow k(Y) \hookrightarrow k(X) \)).

(ii) If there exists a filtered equivalence \( \Phi_P \) then the kernel \( P \) is generically given by a sheaf (up to a shift) and induces a birational isomorphism \( X \to Y \).

**Proof.** By [29, Corollary B] \( h^1(X, \mathcal{O}_X) = h^1(Y, \mathcal{O}_Y) \). Suppose \( P_L : = \Phi(L) \in D(Y) \) has rank 1 and satisfies \( \dim \text{Ext}^1(P_L, P_L) = h^1(Y, \mathcal{O}_Y) \). Let \( H^i_L \) be the \( i \)th cohomology sheaf of \( P_L \). The usual spectral sequence (see for example [6, Proof of 2.2]) gives

\[ E_2^{p,q} = \bigoplus_i \text{Ext}^p(H^i_L, H^{i+q}_L) \implies \text{Ext}^{p+q}(P_L, P_L) \simeq \text{Ext}^{p+q}(\mathcal{L}, \mathcal{L}) = H^{p+q}(X, \mathcal{O}_X). \]

Since \( X \) is a surface we have \( E_2^{p,q} = 0 \) for \( p \geq 3 \). This can be seen by noting that for any two coherent sheaves \( \mathcal{F} \) and \( \mathcal{G} \) on \( X \) the support of \( \text{Ext}^j(\mathcal{F}, \mathcal{G}) \) has dimension \( \leq d - j \), and then using the cohomological dimension bounds for noetherian topological spaces. In particular, the map \( E_2^{1,0} \to E_2^{3,-1} \) is zero, and therefore

\[ \sum_i \text{ext}^1(H^i_L, H^i_L) \leq \text{ext}^1(P_L, P_L). \]

On the other hand, if \( F \) is a sheaf of positive rank, the trace map splits the inclusion to yield an injection \( H^1(Y, \mathcal{O}_Y) \to \text{Ext}^1(F, F) \). It follows that a most one \( H^i \) has positive rank. Since \( P \) has rank 1, we conclude that exactly one \( H^i \) has rank 1 and the rest have rank 0. Moreover, the cohomology sheaf of rank 1 must sit in an even degree \( d(L) \).

We claim that \( d(L) \) is independent of \( L \). Indeed, note that any two invertible sheaves \( L \) and \( M \) fit into a tent of invertible sheaves

\[
\begin{array}{ccc}
N & \xleftarrow{\Phi(\iota)} & \Phi(M) \\
L & \downarrow & \Phi(\iota) \\
M & \leftarrow & \Phi(L)
\end{array}
\]

To show that \( d(L) = d(M) \), it thus suffices to do so under the additional assume that there is an injective map \( \iota : L \to M \). In this case, the cokernel \( M/L \) has rank 0, whence the third vertex \( C \) of the triangle

\[
\Phi(\iota)
\]

has rank 0. In particular, the restriction \( C_{\kappa(Y)} = 0 \in D(\kappa(Y)) \). By the preceding argument, we have that \( \Phi(L)_{\kappa(Y)} \cong \kappa(Y)[-d(L)] \) and \( \Phi(M)_{\kappa(Y)} \cong \kappa(Y)[-d(M)] \). The only way \( C_{\kappa(Y)} \) can be zero is if \( \Phi(\iota)_{\kappa(Y)} \) is an isomorphism. After shifting, we may assume that \( d(L) = 0 \) for all \( L \). We make that assumption for the remainder of the proof.
We can thus define a faithful $k$-linear functor 
\[ \Phi^{(1)}: \mathcal{P}(X) \to \mathcal{P}(Y) \]
by sending an invertible sheaf $L$ to the object $H^0(\Phi(L))^{\vee}$ and a map $\iota: L \to M$ to the map 
\[ (\Phi(\iota))^{\vee}: H^0(\Phi(L))^{\vee} \to H^0(\Phi(M))^{\vee}. \]

Corollary 9.2 yields an injection $k(X) \hookrightarrow k(Y)$. Applying this to the inverse transform yields the desired result.

Now assume that $X$ is projective. Fix an ample invertible sheaf $\mathcal{O}_X(1)$. The above argument shows that for any invertible sheaf $L$ on $X$, the complex 
\[ \Phi_P(L) = R(\text{pr}_2)_*(\text{pr}_1^* L \otimes P) \in D(Y) \]
has the property that its restriction to the generic point of $Y$ is quasi-isomorphic to a sheaf of rank 1 (as a complex concentrated in degree 0). Consider the restriction $P_k(Y) \in D(X_k(Y)).$ For all $i$ and all $s$ sufficiently large, the hypercohomology spectral sequence degenerates to yield 
\[ H^i(X_k(Y), P_k(Y)(s)) = H^0(X_k(Y), \mathcal{H}^i(P_k(Y))(s)). \]
On the other hand, the preceding calculation of $\Phi_P(\mathcal{O}_X(s))$ shows that the left side is 0 unless $i = 0$, in which case it has dimension 1 as a $k(Y)$-vector space. Since $\mathcal{O}_X(1)$ is ample, we conclude that $\mathcal{H}^i(P_k(Y)) = 0$ for all $i \neq 0$, and that the Hilbert polynomial of $\mathcal{H}^0(P_k(Y))$ with respect to $\mathcal{O}_X(1)$ is the constant function 1. This implies that $\mathcal{H}^0(P_k(Y))$ is an invertible sheaf supported on a section of the structure map $X_k(Y) \to \text{Spec } k(Y)$. We conclude that $P$ is generically given by a sheaf. Applying Proposition 2.4 gives the result.

10. A MOTIVIC PERSPECTIVE

The original definition of the Mukai motive obtained by taking the direct sum of various Tate twists of the cohomology groups may appear a bit ad hoc. However, it arises naturally in the context of the motivic realizations of dg-categories as discussed in [4, §3] and [31].

10.1. Let us briefly discuss the basic machinery relevant for this paper.

Let $k$ be a field and let $\mathcal{SH}_k^\otimes$ denote the stable motivic $\infty$-category defined in [31, 2.39]. The $\infty$-category $\mathcal{SH}_k^\otimes$ is a stable presentable symmetric monoidal $\infty$-category. In particular, its associated homotopy category is a triangulated category. More generally, we can consider $\mathcal{SH}_X^\otimes$ for a smooth $k$-scheme $X$. We can also consider the rational versions of these categories, denoted $\mathcal{SH}_X^\otimes_{\mathbb{Q}}$. For a morphism $f: X \to Y$ one has functors 
\[ f_*, f!: \mathcal{SH}_X^\otimes \to \mathcal{SH}_Y^\otimes, \quad f^*, f^!: \mathcal{SH}_Y^\otimes \to \mathcal{SH}_X^\otimes. \]

There is a realization functor 
\[ M_k: \text{Sm}_k \to \mathcal{SH}_k^\otimes \]
from the category of smooth \(k\)-schemes, viewed as a symmetric monoidal category with products over \(k\). For a smooth \(k\)-scheme \(X\) we write \(M_k(X)\) for the associated object of \(\mathcal{SH}_k^\otimes\).

There is a notion of Tate twist in \(\mathcal{SH}_k^\otimes\); see [12, 1.4.5]. For \(X/k\) we write \(1_X \in \mathcal{SH}_X^\otimes\) for the unit object, and for \(n \in \mathbb{Z}\) we write \(1_X(n)\) for the \(n\)-th Tate twist.

The object \(M_k(X)\) has the following interpretation using the six functor formalism developed in [12]. In general, for \(p: X \to \text{Spec}(k)\) not necessarily smooth, one has (by definition; see [12, 1.1.34])
\[
M_k(X) = p_!(1_X),
\]
where \(1_X \in \mathcal{SH}_X^\otimes\) is the unit object and \(p_!\) is the left adjoint to \(p^*\). Moreover, for \(p: X \to \text{Spec}(k)\) smooth and proper we have
\[
M_k(X) = p_*1_X(d)[2d],
\]
by [12, 2.4.42].

There is also a realization functor for dg-categories. To explain this, we first need the commutative algebra object \(B_{U_X} \in \mathcal{SH}_X^\otimes\), for \(X/k\) smooth. This is discussed in [4, 3.1.1]. The precise interpretation, in terms of homotopy invariant algebraic K-theory, is not relevant for us. The most important feature is provided by [4, 3.35] wherein it is shown that
\[
B_{U_X} \simeq Q[1_X(1)[2]^\pm],
\]
where the right side is the free Laurent polynomial algebra on the object \(1_X(1)[2]\).

Let \(\text{dgcat}^{\text{idem}}_k\) denote the \(\infty\)-category of differential graded \(k\)-linear categories which are idempotent complete and triangulated (see [4, 2.1]). Then it is shown in [4, 3.2.2] that there is a symmetric monoidal functor
\[
M^\text{dg}_k: \text{dgcat}^{\text{idem}}_k \to \text{Mod}_{B_{U_k}},
\]
where the target denotes the category of \(B_{U_k}\)-modules in \(\mathcal{SH}_k^\otimes\). Furthermore, it is shown in [4, 3.13] that for a finite type morphism \(p: X \to \text{Spec}(k)\) we have a canonical isomorphism in \(\text{Mod}_{B_{U_k}}\)
\[
M^\text{dg}(\text{Perf}(X)) \simeq p_*(B_{U_X}),
\]
where \(\text{Perf}(X)\) denotes the dg-category of perfect complexes on \(X\). If \(p\) is further assumed smooth and proper then this gives an isomorphism
\[
M^\text{dg}(\text{Perf}(X)) \simeq (M_k(X)(d)[2d]) \otimes B_{U_k}.
\]

Tensoring with \(Q\), this gives an isomorphism
\[
M^\text{dg}(\text{Perf}(X))Q \simeq \oplus_{s \in \mathbb{Z}} M_k(X)Q(s)[2s].
\]
In other words, the motive \(M^\text{dg}(\text{Perf}(X))Q\) is the 2-periodization of \(M_k(X)Q\).

In particular, if \(X\) and \(Y\) are smooth projective varieties over \(k\) and \(\Phi: D(X) \to D(Y)\) is a derived equivalence, then using Orlov’s canonical lift to an equivalence \(\text{Perf}(X) \to \text{Perf}(Y)\) we get an isomorphism of motives
\[
(10.1.1) \quad \oplus_s M_k(X)Q(s)[2s] \to \oplus_s M_k(Y)Q(s)[2s],
\]
which is induced by a morphism
\[
M_k(X)Q \to \oplus_s M_k(Y)Q(s)[2s].
\]
Remark 10.2. In [28] a more direct construction of the isomorphism (10.1.1) is given.

10.3. Let $D(\mathcal{SH}_{k,Q}^\otimes)$ denote the triangulated category associated to $\mathcal{SH}_{k,Q}^\otimes$, and let

$$DM_k \subset D(\mathcal{SH}_{k,Q}^\otimes)$$

denote the thick triangulated subcategory generated by objects of the form $M_k(X)_Q(p)$ for $X/k$ smooth of finite type and $p \in \mathbb{Z}$. This is is the subcategory of constructible Beilinson motives in the sense of [12, Part IV]. By [12, 11.1.13] the constructible Beilinson motives are precisely the compact objects in $D(\mathcal{SH}_{k,Q}^\otimes)$.

If $X$ and $Y$ are smooth projective varieties of dimension $d$ and $\Phi: D(X) \to D(Y)$ is a derived equivalence, then because the motive $M_k(X)_Q$ is compact with characteristic 0, we have $M_k(X)_Q \simeq CH^{d-s}(X \times Y)_Q$.

10.4. In order to speak about filtered derived equivalences in the motivic context we need to assume the existence of a motivic $t$-structure $\mu$ in the sense of [3] (wherein it is shown that the existence of this $t$-structure implies the standard conjectures!). For this we assume that $k$ has characteristic 0. Let $Mot_k$ denote the heart of this $t$-structure and for $i \in \mathbb{Z}$ let

$$\mu^i: DM_k \to Mot_k$$

be the $i$-th cohomology functor. Beilinson shows that the motivic $t$-structure $\mu$, if it exists, is unique and satisfies the following properties:

(i) Any of the standard realization functors $r: DM_k \to D(K)$, where $K$ is a field of characteristic 0 (e.g. $K = Q_\ell$ for $\ell$ invertible in $k$ and $\ell$-adic realization functor), is $t$-exact and compatible with monoidal structures (this follows from [3, 1.6])

(ii) If $X$ is smooth and projective then $M(X) \simeq \oplus_s \mu^i(M(X))[-i]$ and the objects $\mu^i(M_k(X))$ are semisimple $[3, 1.4$ and 1.5$]$

(iii) Every object $M \in Mot_k$ carries a finite increasing filtration $W_iM$ such that the successive quotients $W_iM/W_{i-1}M$ are semisimple and each irreducible constituent is a direct summand of some $\mu^i(M(X))(a)$ for $X$ a smooth projective variety and $i = s - 2a [3, 1.7]$

(iv) For a realization functor $r: DM_k \to D(K)$ the restriction $Mot_k \to Vec_K$, which we denote again by $r$, is faithful.

10.5. Continuing with the setup and notation of Paragraph 10.3 we get by applying $\mu^0$ to $\rho$ an isomorphism

$$\rho_{even}: \oplus_s \mu^i H^{2s}(M(X))(s) \to \oplus_s \mu^i H^{2s}(M(Y))(s),$$
and by applying $\mu H^1$ an isomorphism

$$\bar{\rho}_{\text{odd}} : \bigoplus_s \mu H^{1+2s}(M(X))(s) \to \bigoplus_s \mu H^{1+2s}(M(Y))(s).$$

For an integer $j$ let

$$\rho^\Delta : H^j(M(X)) \to H^j(M(Y))$$

be the map given by taken the $j$-th diagonal entry of $\bar{\rho}_{\text{even}}$ and applying a Tate twist, if $j$ is even, and the $j$-th diagonal entry of $\bar{\rho}_{\text{odd}}$ and applying a Tate twist, if $j$ is odd. Let

$$\rho^\Delta : M(X) \simeq \bigoplus_j \mu H^j(M(X))[-j] \to \bigoplus_j \mu H^j(M(Y))[-j] \simeq M(Y)$$

denote $\sum_j \rho^\Delta_j [-j]$.

**Theorem 10.6.** Assume that a motivic $t$-structure exists and that $X$ and $Y$ are smooth projective varieties over a field $k$ of characteristic 0. Then for any filtered derived equivalence $\Phi$ the induced map of motives $\rho^\Delta : M(X) \to M(Y)$ is an isomorphism.

**Remark 10.7.** Loosely this result can be interpreted as saying that under the (very strong) assumption that a motivic $t$-structure exists, Orlov’s conjecture [28, Conjecture 1] holds for filtered derived equivalences.

**Proof.** The point is that the composition

$$(\text{Chow motives w.r.t rational equivalence}) \xrightarrow{M} DM_k \xrightarrow{\sum \mu H^i} \text{Mot}_k$$

factors through numerical Chow motives (see [1, 21.1.5]). It follows that if $\Phi$ is filtered then the maps $\bar{\rho}_{\text{even}}$ and $\bar{\rho}_{\text{odd}}$ preserve the filtrations given by the grading, and therefore the diagonal entries of these maps are isomorphisms.

**Remark 10.8.** The conjectural expectation that the property of a morphism of motives being an isomorphism only depends on the numerical Chow motive is discussed in [1, 11.5.1.1 and 11.5.2.1].
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