Book embedding of 3-crossing-critical graphs with rational average degree between 3.5 and 4

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Abstract. We consider the graphs \( P(m, n) \) with average degree between 3.5 and 4, made up of \( m \) identical pieces together with \( n \) identical pieces glued together in a circular fashion, such that in any drawing of \( P(m, n) \) on the plane, there are exactly three pairwise edges crossings, and when deleting any edge of the graph, the number of crossings of the remaining graph decreases. We then embed \( P(m, n) \) into a book such that the vertices are put on a line called the spine and the edges are put on half-planes called the pages, which have the spine as their common boundary, without crossings. In this paper, we show that the minimal number of pages needed to embed \( P(m, n) \) into a book is three.

1. Introduction

A graph \( G \) is a pair \( (N, E) \) where \( N \) is a non-empty set of vertices and \( E \) is a set of edges. We consider only simple graphs, i.e. without parallel edges and loops. The degree of a vertex \( v \) is the number of edges incident with \( v \). A path from \( v_1 \) to \( v_n \) in graph \( G \) is an alternating vertex and edge sequence \( v_1 e_1 v_2 e_2 v_3 e_3 ... e_n v_n \), where edge \( e_k = v_kv_{k+1} \) (\( 1 \leq k \leq n \)). Since we only consider simple graphs, we denote path from \( v_1 \) to \( v_n \) only by sequence of vertices \( v_1 v_2 v_3 ... v_n \). Paths \( P_1 \) and \( P_2 \) are disjoint if there is no intersection between \( P_1 \) and \( P_2 \). The average degree of a graph \( G \) is the sum of degrees of all vertices of \( G \) divided by the number of vertices of \( G \). A drawing of a graph \( G \) in the plane is a mapping of \( G \) to the plane such that vertices are mapped to distinct points of the plane and edges are mapped into arcs connecting the corresponding points. A crossing in a drawing of graph \( G \) is a point in the plane not corresponding with a vertex of \( G \) where two arcs intersect. The crossing number \( \text{cr}(G) \) of graph \( G \) is the minimal number of crossings among all possible drawings of graph \( G \) in the plane. The crossing number problem was introduced by Tutte [1]. A graph \( G \) with \( \text{cr}(G) = k \) is \( k \)-crossing-critical if deleting any edge \( e \) of \( G \) makes \( \text{cr}(G-e) < k \). Recent studies on crossing-critical graphs are conducted in [2], [3], and [4].

The book with \( k \) pages is the topological space \( B_k \) that consists of a line called the spine and \( k \) half-planes called the pages, all having the spine as their common boundary. A \( k \)-page book embedding of a graph \( G \) is an embedding of \( G \) into book in which the vertices are on the spine, and each edge is contained in one page without crossing. Given a graph \( G \), the minimum \( k \) such that \( G \) can be embedded in a \( k \)-page book is the pagenumber \( \rho(G) \) of graph \( G \) [5]. Figure 1(a) shows the complete bipartite graph \( K_{3,3} \) and Figure 1(b) shows its embedding into 3-page book.
Figure 1. Complete bipartite graph $K_{3,3}$ and its embedding into 3-page book.

The pagenumber problem was introduced by Kainen [6] and attracts attentions since Chung, Leighton and Rosenberg [7] have pointed out its applications to Very Large Scale Integration (VLSI) design. In general, given a graph $G$, it is not easy to determine the pagenumber of $G$; the pagenumber problem is NP-Complete [7].

Pinontoan and Richter [8] introduced the notion of tile which is a small piece of graph with certain conditions and can be glued together in a circular fashion to build an infinite family of graphs. They gave as example the infinite family of the $(\frac{2h+3}{2})$-crossing-critical graphs, with positive integer $h$, having rational average degree $r/3.5, 4$. In this paper, we set $h = 0$ and get 3-crossing-critical graphs $P(m, n)$, for $0 \leq m, n$ and $3 \leq m + n$, which are made up by gluing of $m$ copies of a tile and $n$ copies of other tile, where one of the tile is twisted, in a circular fashion. We embed $P(m, n)$ into book and show that $\rho(P(m, n)) = 3$.

2. Infinite Family of 3-crossing-critical graphs with rational average degree between 3.5 and 4.

In this section we recall the notion of tile which was introduced by Pinontoan and Richter [8] and define the infinite family of 3-crossing-critical graphs having rational average degree between 3.5 and 4.

2.1. Tiles

A tile is a 3-tuple $(G, L, R)$ where $G$ is a connected graph, the left-wall $L$ and the right-wall $R$ are finite sequence of vertices of $G$. A tile drawing of tile $T = (G, L, R)$ is the drawing of $G$ on the unit square $[0, 1] \times [0, 1]$ such that the vertices of $L$ occur in the line $\{0\} \times [0, 1]$ in the decreasing order, the vertices of $R$ occur in the line $\{1\} \times [0, 1]$ also in the decreasing order, and only the vertices $L \cup R$ occur on the boundary $\{0, 1\} \times [0, 1]$. The tile crossing number $tc_r(T)$ of tile $T$ is the smallest number of crossings in any tile drawing of $T$. A tile $T$ is planar if $tc_r(T) = 0$, otherwise non-planar. The twist $T'$ of $T = (G, L, R)$ is the tile obtained by reversing the order of $R$.

Figure 2 shows the tile $S = ((\{a, b, c, d, e, f\}, \{ab, ae, bc, bd, cf, de, ef\}), (a, c), (e, f))$, the tile $T = ((\{x, y, z, u, v\}, \{xz, xu, yz, uz, vz, yv\}), (x, y), (u, v))$, the twist $S' = ((\{a, b, c, d, e, f\}, \{ab, ae, bc, bd, cf, de, ef\}), (a, c), (f, e))$ of $S$, and the twist $T' = ((\{x, y, z, u, v\}, \{xz, xu, yz, uz, vz, yv\}), (x, y), (v, u))$ of $T$. Note that the tiles $S$ and $T$ are planar, whereas their twists $S'$ and $T'$ respectively are not planar with tile crossing number $tc_r(S') = 3$ and $tc_r(S') = 3$.

Figure 2. Tiles $S$ and $T$ together with their twists $S'$ and $T'$. 
A tile $S = (G_1, L_1, R_1)$ is compatible with tile $T = (G_2, L_2, R_2)$ if $|R_1| = |L_2|$. If $S = (G_1, L_1, R_1)$ is compatible with $T = (G_2, L_2, R_2)$, then catenation of tiles $S$ and $T$ is $ST = (G_1 \cup G_2, L_1, R_2)$ where $R_1$ is identified with $L_2$. A tile is self-compatible if it is compatible to itself. If $T$ is a self-compatible tile, then $T^n$ is the catenation of $n$ copies of $T$ in a linear fashion and $o(T^n)$ is the catenation of $n$ copies of $T$ where the last and the first copy of $T$ also put in catenation to get a circular fashion of arrangement. Let $T$ be a self-compatible tile, then $\Theta(T^n) = o(T^n \cdot T')$, the twist circular catenation of $n$ copies of $T$. Note that $T^n$ is a tile, whereas $o(T^n)$ and $\Theta(T^n)$ are graphs.

Figure 3 shows the catenation $ST$ of tiles $S$ and $T$ where the vertices $e$ and $f$ of $S$ are identified with respectively the vertices $x$ and $y$ of $T$, and catenation $ST'T$ of tiles $S$, twist $T'$ of $T$, and tile $T$. In Figure 3, we can see that there are three pair-disjoint paths in $ST'T$, namely $\{aeuv, cfzu\}$, $\{aeuv, cbdf'xz'\}$, and $\{abdez'xzu\}$.

2.2. The infinite family $P(m, n)$
Pinontoan and Richter [8] defined an infinite family of the $\binom{2h+3}{2}$-crossing-critical graphs, with $h$ is positive integer. We set $h$ to zero and define, for $0 \leq m, n$ and $3 \leq m + n$, the graph $P(m, n) = \Theta(S^nT')$ where $S$ and $T$ are the tiles in Figure 2, i.e. the twist circular catenation of $m$ copies of tile $S$ and $n$ copies of tiles $n$. More precisely, for $0 \leq m, n$ and $3 \leq m + n$, the graph $P(m, n)$ has vertices $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, ..., a_m, b_m, c_m, d_m, a_{m+1}, b_{m+1}, c_{m+1}, d_{m+1}, a_{m+2}, b_{m+2}, c_{m+2}, d_{m+2}, (1 \leq k \leq m)$, $a_{k+1}, c_{k+1}, d_{k+1}, d_{k+1} (1 \leq k \leq m - 1)$, $x_{k}z_{k}, y_{k}z_{k} (1 \leq k \leq n)$, $x_{k}y_{k}, y_{k+1}z_{k+1}, z_{k}y_{k+1}, z_{k}y_{k+1} (1 \leq k \leq n - 1)$, $d_{m}x_{1}, d_{m}y_{1}, z_{1}c_{1},$ and $y_{n}a_{1}$. The twist can be put as $S'$ the twist of $S$ or $T'$ the twist of $T$ as pictured in Figure 2.

Figure 4 shows $P(2, 4)$ with the twist $S'$ of $S$ and with the twist $T'$ of $T$. 

![Figure 3. Catenations ST and ST'T.](image3)

![Figure 4. Graph P(2, 4) with twist S' and with twist T'.](image4)
2.3. $P(m, n)$ is 3-crossing-critical with rational average degree between 3.5 and 4

As shown in Subsection 2.1, there are three pair-disjoint paths in $ST'T$. Similarly, there are three pair-disjoint paths in $SS'S$, $TT'T$, and $TS'S$. Hence, using the tile mechanism for calculation crossing numbers established in [8], we conclude that $cr(P(m, n)) = 3$.

Consider the $S'$ of tile $S$ in Figure 2. Obviously, deleting the edges $ae$, $cf$, and $bd$ decreases the number of crossings. If the edge $ab$ is deleted, then the edge $cf$ can be rerouted only to cross $ab$ and does not cross $bd$, and hence there are only two crossings. Similarly, by symmetry, deleting other edges of $S'$ will decrease the number of crossings. Thus, deleting any edge in $P(m, n)$ will decrease $cr(P(m, n))$. Therefore, $P(m, n)$ is 3-crossing-critical graph.

The average degree of $P(m, n)$ is

$$\frac{4(2m + 3(n + 1)) + 3(2m)}{4m + 3n + 3} = \frac{14m + 12n + 12}{4m + 3n + 3}.$$ 

Given any rational number $r \in (3.5, 4)$, we have $4rm + 3rn + 3r = 14m + 12n + 12$ which is equivalent to

$$m = \frac{(3n + 3)(4 - r)}{4r - 14}.$$ 

Hence, for any rational number $r \in (3.5, 4)$, there are many choices for $n$ to get integer $m$. Therefore, $P(m, n)$ is 3-crossing-critical graph with rational average degree in the interval (3.5, 4).

3. Book Embeddings of $P(m, n)$

In this section, we prove the main result.

**Theorem.** Let $0 \leq m, n$ and $3 \leq m + n$. Then $p(P(m, n)) = 3$.

**Proof.** Let $0 \leq m, n$ and $3 \leq m + n$. We showed that $cr(P(m, n)) = 3$ and so $P(m, n)$ is not planar and hence it needs at least three pages to embed into a book. Thus $p(P(m, n)) \geq 3$.

Now we show that $p(P(m, n)) \leq 3$ by drawing the embedding of $P(m, n)$ into a 3-page book, as follows. Put on the page 1 the following edges: $a_kb_k, b_kc_k, b_kd_k (1 \leq k \leq m), a_kx_{k+1}, d_kx_{k+1} (1 \leq k \leq m - 1), x_kz_k, y_kz_k (1 \leq k \leq n), x_ky_{k+1}, z_ky_{k+1} (1 \leq k \leq n - 1), a_kx_1$, and $d_kx_1$. On page 2, put the following edges $c_kx_{k+1}, d_kx_{k+1} (1 \leq k \leq m - 1), y_kz_{k+1}, z_ky_{k+1} (1 \leq k \leq n - 1), c_ky_1, d_ky_1, c_1z_n$, and $a_1z_n$. Finally, put the edges $c_1x_{n+1}$ and $a_1y_{n+1}$ on page 3. Therefore $p(P(m, n)) = 3$. □

Figure 5 shows a 3-page book embedding of graph $P(2, 4)$.
Figure 5. Book embedding of $P(2, 4)$.

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