Fast algorithms of Bayesian Segmentation of Images

Boris A. Zalesky
Institute of Engineering Cybernetics NAN, Minsk, Belarus *
zalesky@mpen.bas-net.by

29th March 2022

Abstract

The network flow optimization approach is offered for Bayesian segmentation of gray-scale and color images. It is supposed image pixels are characterized by a feature function taking finite number of arbitrary rational values (it can be either intensity values or other characteristics of images). The clusters of homogeneous pixels are described by labels with values in another set of rational numbers. They are assumed to be dependent and distributed according to either the exponential or the Gaussian Gibbs law. Instead traditionally used local neighborhoods of nearest pixels the completely connected graph of dependence of all pixels is employed for the Gibbs prior distributions.

The methods developed reduce the problem of segmentation to the problem of determination of the minimum cut of an appropriate network.

Mathematics Subject Classification 2000: 62, 90, 68

Key words and phrases: image restoration, Ising models, integer programming, quadratic programming, minimum network flow cut algorithm

*address: 220012, Surganov str. 6, Minsk, Belarus
1 Introduction

The segmentation of images and restoration of degraded images are branches of image processing that are now extensively studied for their evident practical importance as well as theoretical interest. There are many approaches to solution of these problems. We consider here methods of the Bayesian segmentation of images, which generalize methods of Bayesian image estimation \([1, 2, 3]\). Because of use of a prior information the methods of Bayesian image estimation would be methods of the first choice for many practical problems but unfortunately they are often difficult to compute. Until recently approximations of the Gibbs estimators have been usually available rather then their exact values. For last decade the significant progress in the Gibbs estimation have been achieved. In particular, the methods of discrete optimization for the high-dimensional Gibbs estimation and segmentation have been obtained \([4, 5, 6, 7, 8, 9]\). The methods developed allowed not only an efficient evaluation of the Gibbs estimates but also their fast exact determination.

In this paper the problem of image segmentation is considered as a problem of cluster-analysis for the case of Gibbs prior distributions of clusters. There are enormous number of papers devoted to the problems of Bayesian cluster-analysis (see \([10, 11, 12, 13]\)). In some of them methods of classification with dependent clusters are presented \([12, 13]\). We consider the problem of classification of observations \(y_1, y_2, \ldots, y_n\) (for instance, image intensities or texture characteristics of images etc.) with the feature function \(f(y_i)\) that takes finite number of rational values. The clusters are identified by appropriate rational labels (note at once that in the presented below models the case of rationally valued feature functions and cluster labels is reduced to the case of functions and labels taking only integer values). The clusters are supposed dependent and distributed according to the Gibbs field (such type models occur frequently in image processing). The labeling Gibbs field is specified on the directed fully connected graph of all pixels (usually only graphs of connected nearest neighbors pixels were considered. Often it was finite d-dimensional regular lattices).

The discrete optimization methods that enable efficient determination of an exact solution of the segmentation problem are described. In the offered methods the problem of identification of the Gibbs classifiers is reduced, first, to the integer optimization problem and then to the problem of finding of the minimum cuts of special networks. The minimum cuts of the networks can be found by fast methods \([9, 14]\) that take into consideration the specific character of the networks and allow execution in concurrent mode.
2 Description of Models

It can be easily seen that in models presented below the case of rationally valued feature functions and cluster labels is reduced to the case of integer classification. So, let an image \( y = (y_1, y_2, \ldots, y_n) \) be with the feature function \( f \) that takes finite number integer values in \( \mathbb{Z}_L = \{0, 1, \ldots, L\} \) and let \( f_i = f(y_i) \). Let \( \mathcal{M} = \{m(0), m(1), \ldots, m(k)\} \), \( (0 \leq m(0) < m(1) < \ldots < m(k) \leq L) \) be a set of allowable cluster labels. Suppose the image \( y \) is partitioned into \( k + 1 \) clusters (the number \( k + 1 \) instead of \( k \) is taken only to simplify formulas) and each cluster is specified by integer number \( j \), \( (0 \leq j \leq k) \), as well as by an appropriate fixed integer label \( m_j \in \mathcal{M} \).

The feature vector \( f = (f_1, f_2, \ldots, f_n) \) of image \( y \) is considered as a random variable with the non-identical exponential

\[
p_{i,\text{exp}}(f_i) = c_i \cdot \exp \{-\lambda_i |f_i - m_i|\}
\]

or with the non-identical Gaussian

\[
p_{i,\text{gaus}}(f_i) = c_i \cdot \exp \{-\lambda_i (f_i - m_i)^2\}
\]

Gibbs distribution (\( \lambda_i \geq 0 \)).

The labels \( m_i \) of clusters are supposed to be dependent random variables distributed according to the Gibbs field. The segmented image \( x = \{x_1, x_2, \ldots, x_n\} \) is specified on the fully connected directed graph \( G = (V, E) \) with the set of vertices \( V = \{0, 1, \ldots, n\} \) and the set of directed arcs \( E = \{(i, j) \mid i, j \in V\} \). It takes values in the space of labels \( \mathcal{M}^V \), i.e. \( x_i \in \mathcal{M}, i \in V \), and is either the exponential or the gaussian form

\[
p_{\text{exp}}(m) = c \cdot \exp \{- \sum_{(i,j) \in E} \beta_{i,j} |m_i - m_j|\},
\]

\[
p_{\text{gaus}}(m) = c \cdot \exp \{- \sum_{(i,j) \in E} \beta_{i,j} (m_i - m_j)^2\},
\]

where \( c \) is the norming quantities (here and below different constant will be denoted by the same letter), the vector \( m = \{m_1, m_2, \ldots, m_n\} \) and parameters \( \beta_{i,j} \geq 0 \).

Remark 1. (i) The Gibbs models considered are extentions of the Ising model.
(ii) Note, that for some image processing problems the prior distribution \( p_{\text{exp}}(x) \) is more preferable than Gaussian Gibbs one because of smaller blurring effect. Moreover, its identification turns out far more computationally efficient.
For the model with the exponential conditional distribution $p_{i,\text{exp}}(f_i)$ and the exponential prior $p_{\text{exp}}(m)$ the Gibbs classifier is equal to

$$
\hat{m}_{\text{exp}} = \arg \min_{m \in \mathcal{M}} \left\{ \lambda_i \sum_{i \in V} |f_i - m_i| + \sum_{(i,j) \in E} \beta_{i,j} |m_i - m_j| \right\},
$$

(1)

and respectively, for the model with the Gaussian conditional distribution $p_{i,\text{gaus}}(f_i)$ and the Gaussian prior $p_{\text{gaus}}(m)$ the Gibbs classifier is of the form

$$
\hat{m}_{\text{gaus}} = \arg \min_{m \in \mathcal{M}} \left\{ \sum_{i \in V} \lambda_i (f_i - m_i)^2 + \sum_{(i,j) \in E} \beta_{i,j} (m_i - m_j)^2 \right\}.
$$

(2)

In spite of clear posing the problem of finding exact values of the Gibbs classifiers $\hat{m}_{\text{exp}}$ and $\hat{m}_{\text{gaus}}$ for large samples is rather complicated. So, for instance, computer images, which are frequent objects of classification, usually consist up to $2^{18}$ variables or more. Nevertheless, it turned out possible to compute efficiently (during polynomial time on the sample size $n$ and number of clusters $k$) both of these classifiers. Theoretically, run-time for the first classifier does not exceed $ckn^3$ and for the second one is less than $c(kn)^3$. For many applied problems real run-time was even of order $O(nk)$. The efficient computation of the classifiers for the mixed model with the exponential conditional distributions $p_{i,\text{exp}}(f_i)$ and the Gaussian prior $p_{\text{gaus}}(m)$ as well as for the mixed model with the Gaussian conditional distributions $p_{i,\text{gaus}}(f_i)$ and the exponential prior $p_{\text{exp}}(m)$ is also available.

## 3 Computation of the Optimal Classifiers

In the case of two classes ($k = 1$) the Gibbs classifier $\hat{m}_{\text{exp}}$ and $\hat{m}_{\text{gaus}}$ coincide (since any Boolean variable $b$ satisfies the identity $b^2 = |b|$). They can be evaluated by the network flow optimization methods [4, 15]. Moreover, in 1989 Greig, Porteous and Seheult [4] developed a heuristic network flow algorithm that is especially efficient for estimating the Boolean Gibbs estimator. These authors posed also the problem for the case more than 2 clusters. Recently we have described the multiresolution network flow minimum cut algorithm [1] that allows exact computation of Boolean classifiers as well as developed algorithms of computation of the mentioned Gibbs classifiers in general case. It turned out the network flow optimization methods can be used to identify $\hat{m}_{\text{exp}}$ and $\hat{m}_{\text{gaus}}$ even when $k > 1$. 
Identification of $\hat{m}_\text{exp}$

Denote the function to be minimized by

$$U_1(m) = \sum_{i \in V} \lambda_i |f_i - m_i| + \sum_{(i,j) \in E} \beta_{i,j} |m_i - m_j|, \quad (m \in \mathcal{M}^V).$$

The idea of the method is to represent the vector of labels of clusters $m$ by the integer valued linear combination of Boolean vectors and then reduce the problem of integer minimization of the function $U_1(m)$ to the problem of Boolean minimization. The problem of Boolean minimization can be solved by the network flow optimization methods.

Let for arbitrary integers $\mu$ and $\nu$ the indicator function $1_{(\mu \geq \nu)}$ be equal to 1 if $\mu \geq \nu$ and be equal to 0 otherwise. For any $\mu \in \mathcal{M}$ and Boolean variables $x(l) = 1_{(\mu \geq m(l))}$ such that $x(1) \geq x(2) \geq \ldots \geq x(k)$ the identity

$$\mu = m(0) + \sum_{l=1}^k (m(l) - m(l-1)) x(l) \quad (3)$$

is valid, and vice versa, any non-increasing sequence of the Boolean variables $x(1) \geq x(2) \geq \ldots \geq x(k)$ specifies the label $\mu \in \mathcal{M}$ by the formula (3). By analogy, the feature functions $f_i$ are represented as sums

$$f_i = \sum_{\tau=1}^L f_i(\tau)$$

of non-increasing sequence of the Boolean variables $f_i(1) \geq f_i(2) \geq \ldots \geq f_i(L)$.

Let for $l = 1 \div k$ the vector $x(l) = (x_1(l), x_2(l), \ldots, x_n(l))$ be Boolean, the vector $z(l) = (z_1(l), \ldots, z_n(l))$ be with coordinates

$$z_i(l) = \frac{1}{m(l) - m(l-1)} \sum_{\tau=m(l-1)+1}^{m(l)} f_i(\tau), \quad (i = 1 \div n)$$

and the norm of the vector $|x| = \sum_{i=1}^n |x_i|$, then the following proposition is satisfied.

**Proposition 1.** For any integers $\nu \in \mathcal{M}$ and $f_i \in Z_L$ the equality

$$|\nu - f_i| = \left| m(0) - \sum_{\tau=1}^m f_i(\tau) \right| +$$

$$+ \sum_{l=1}^k \left| (m(l) - m(l-1)) 1_{(\nu \geq m(l))} - \sum_{\tau=m(l-1)+1}^m f_i(\tau) \right| + \sum_{l=m(k)+1}^L f_i(\tau)$$

holds true. Therefore, for any feature vector $f \in \mathbb{Z}_L^V$ and the Boolean vector

$$x(l) = (1_{(m_1 \geq l)}, 1_{(m_2 \geq l)}, \ldots, 1_{(m_n \geq l)})$$
the function $U_1(m)$ can be written in the form

$$U_1(m) = \sum_{l=1}^{k} (m(l) - m(l-1)) u(l, x(l)), \quad (4)$$

where for $n$-dimensional Boolean vector $b$ functions

$$u(l, b) = \sum_{i \in V} \lambda_i |z_i(l) - b_i| + \sum_{(i, j) \in E} \beta_{i,j} |b_i - b_j|.$$

Denote by

$$\hat{x}(l) = \arg \min_{b} u(l, b), \quad (l = 1 \div k) \quad (5)$$

Boolean solutions that minimize the functions $u(l, b)$. For two vectors $v$ and $w$ we will write $v \geq w$ if all corresponding pairs of their coordinates satisfy the inequality $v_i \geq w_i$, $(i = 1 \div n)$ and will write $v \not\geq w$ if there exist at least two different pairs such that $v_i \geq w_i$ and $v_j < w_j$. Note that $z(1) \geq z(2) \geq \ldots \geq z(k)$, and what is more, for some integer $1 \leq \kappa \leq k$ their coordinates satisfy the following condition

$$z_i(1) = z_i(2) = \ldots = z_i(\kappa - 1) = 1, 0 \leq z_i(\kappa) \leq 1,$$

$$z_i(\kappa + 1) = \ldots = z_i(k) = 0.$$

It is easy to show that in general the solutions $\hat{x}(l)$ of the (5) are not ordered. Nevertheless, without fail there is at least one non-increasing sequence $\hat{x}(l)$ of solutions of (5).

**Theorem 2.** There is a non-increasing sequence $\hat{x}(1) \geq \hat{x}(2) \geq \ldots \geq \hat{x}(k)$ of solutions of (5).

Some structural properties of the set of solutions $\hat{x}(l)$ are presented in

**Corollary 3.** For integer $1 \leq l' < l'' \leq k$ and the sequence of vectors $z(1) \geq z(2) \geq \ldots \geq z(k)$ the following properties are valid:

(i) If $\hat{x}(l')$ is any solution of (5), then there is a solution $\hat{x}(l'')$ so that $\hat{x}(l') \geq \hat{x}(l'')$, and vice versa, if $\hat{x}(l'')$ is any solution of (5), then there is a solution $\hat{x}(l')$ so that $\hat{x}(l') \geq \hat{x}(l'')$.

(ii) For each $1 \leq l \leq k$ the set of solutions $\{\hat{x}(l)\}$ has the minimal $\underline{x}(l)$ and the maximal $\overline{x}(l)$ elements.

(iii) The set of minimal and maximal elements are ordered, i.e. $\underline{x}(1) \geq \ldots \geq \underline{x}(k)$ and $\overline{x}(1) \geq \ldots \geq \overline{x}(k)$. 

6
Sentence (i) follows immediately from Theorem 2. Sentence (ii) is deduced from (i) considered for \( l' = l'' \), property (iii) follows from (ii) and definition of the minimal and the maximal elements.

For two Boolean vectors \( b' \) and \( b'' \) let the vector \( \overline{b} = b' \lor b'' \), respectively, \( \overline{b} = b' \land b'' \) be with coordinates \( b_i = \min\{b'_i, b''_i\} \), respectively, with coordinates \( b_i = \max\{b'_i, b''_i\} \). If \( \tilde{x}(1), \ldots, \tilde{x}(k) \) is any unordered sequence of solutions of (5) the ordered sequence of solutions can be derived from it by the logical operation \( \land, \lor \) like one-dimensional variational series \( x(1) \geq x(2) \geq \ldots \geq x(k) \). But it is easy to see the sum of ordered solutions \( \tilde{x}(l) \) is a solution of (4).

**Proposition 4.** If \( \tilde{x}(1) \geq \tilde{x}(2) \geq \ldots \geq \tilde{x}(k) \) is a sequence of ordered solutions of (4) then the sum

\[
\hat{m} = m(0) + \sum_{l=1}^{k} (m(l) - m(l-1)) \tilde{x}(l)
\]

minimizes \( U_1(m) \).

The problem of computing Boolean solutions \( \tilde{x}(l) \) is familiar in the discrete optimization [4]. It is equivalent to identification of the minimum cuts for specially built networks. There are fast algorithms to compute them [8, 14].

**Identification of \( \hat{m}_{gaus} \)**

Now denote the function to be minimized by

\[
U_2(m) = \sum_{i \in V} \lambda_i (f_i - m_i)^2 + \sum_{(i,j) \in E} \beta_{i,j} (m_i - m_j)^2, \quad (m \in \mathcal{M}^V).
\]

To find a solution \( \hat{m}_{gaus} \) that minimizes the function \( U_2(m) \) the representation of the vector of cluster labels \( m \) by the integer valued linear combination of Boolean vectors is used once more. Then the problem of integer minimization of the function \( U_2(m) \) is reduced to the problem of Boolean minimization.

Denote for brevity \( g_i = f_i - m(0), \ (i \in V), \ a_l = m(l) - m(l-1), \ (l = 1 \div k) \). The vector \( m \) can be represented by the formula (3) as the linear combination \( m(0) + \sum_{l=1}^{k} a_l \tilde{x}(l) \) of Boolean vectors \( \tilde{x}(l) = (x_1(l), x_2(l), \ldots, x_n(l)) \), and the function \( U_2 \) can be written in the form

\[
\sum_{i \in V} \lambda_i \left( g_i - \sum_{l=1}^{k} a_l x_i(l) \right)^2 + \sum_{(i,j) \in E} \beta_{i,j} \left( \sum_{l=1}^{k} a_l (x_i(l) - x_j(l)) \right)^2.
\]
Let \( d_{l,\tau} = a_l a_{\tau}, \ (l, \tau = 1 \div k) \) and \( \beta_{i,i} = 0, (i \in V) \), then \( U_2(m) = \sum_{i \in V} \lambda_i g_i^2 + P(x(1), \ldots, x(k)) \), where the polynomial of Boolean variables \( P(x(1), \ldots, x(k)) \) after cancellation is written as

\[
P(x(1), \ldots, x(k)) = \\
\sum_{i \in V} \sum_{l=1}^{k} \left[ \lambda_i a_l^2 - 2\lambda_i g_i a_l - a_l (m(k) - m(0) - a_l) \sum_{j \in V} (\beta_{i,j} + \beta_{j,i}) \right] x_i(l) + \\
2 \sum_{i \in V} \left[ \lambda_i + \sum_{j \in V} (\beta_{i,j} + \beta_{j,i}) \right] \sum_{1 \leq \tau < l \leq k} d_{l,\tau} x_i(\tau) x_i(l) + \\
\sum_{(i,j) \in E} \beta_{i,j} \left[ \sum_{l=1}^{k} a_l^2 (x_i(l)-x_j(l))^2 + \sum_{l \neq \tau} d_{l,\tau} [(x_i(l)-x_j(\tau))^2 + (x_j(l)-x_i(\tau))^2] \right].
\]

Note that it has the same points of minimum as \( U_2(m) \). Let us consider another polynomial of Boolean variables

\[
Q(x(1), \ldots, x(k)) = \\
\sum_{i \in V} \sum_{l=1}^{k} \left[ \lambda_i a_l^2 - 2\lambda_i g_i a_l - a_l (m(k) - m(0) - a_l) \sum_{j \in V} (\beta_{i,j} + \beta_{j,i}) \right] x_i(l) + \\
2 \sum_{i \in V} \left[ \lambda_i + \sum_{j \in V} (\beta_{i,j} + \beta_{j,i}) \right] \sum_{1 \leq \tau < l \leq k} d_{l,\tau} x_i(\tau) x_i(l) + \\
\sum_{(i,j) \in E} \beta_{i,j} \left[ \sum_{l=1}^{k} a_l^2 (x_i(l)-x_j(l))^2 + \sum_{l \neq \tau} d_{l,\tau} [(x_i(l)-x_j(\tau))^2 + (x_j(l)-x_i(\tau))^2] \right].
\]

such that \( Q(x(1), \ldots, x(k)) \geq P(x(1), \ldots, x(k)) \) and which differs from \( P \) by the term \( \sum_{1 \leq \tau < l \leq k} d_{l,\tau} x_i(\tau) x_i(l) \) in the second line.

Denote by

\[
(q^*(1), q^*(2), \ldots, q^*(k)) = \arg\min_{x(1),x(2),\ldots,x(k)} Q(x(1), \ldots, x(k))
\]

any collection of Boolean vectors that minimizes \( Q(x(1), \ldots, x(k)) \). Without fail \( q^*(1) \geq q^*(2) \geq \ldots \geq q^*(L-1) \). This feature allows expressing solutions of the initial problem as \( \hat{m}_{\text{gaus}} = \sum_{l=1}^{L-1} q^*(l) \).

**Theorem 5.** Any collection \((q^*(1), q^*(2), \ldots, q^*(L-1))\) that minimizes \( Q \) forms the nonincreasing sequence.

The polynomials \( P \) and \( Q \) have the same set of ordered solutions and, therefore, each solution \( \hat{m}_{\text{gaus}} \) is specified by the formula \( \hat{m}_{\text{gaus}} = \sum_{l=1}^{k} q^*(l) \).
Theorem 5 allows determination of the classifier $\hat{m}_{\text{gaus}}$ by the Boolean minimization of the polynomial $Q$. Unlike $P$ this polynomial can be minimized directly by the minimum network cut algorithms [4, 9]. The appropriate network is described in [8].

4 Applications

Here we show several numerical tests as well as a result of segmentation of a real 3D US-image of size $196 \times 215 \times 301$.

The original 2D gray-scale image in Figure 1a (see next page) was corrupted by Gaussian random noise (Figure 1b). The results of restoration by the extended Ising model are placed in Figure 1c,1d and by the classical Ising model are depicted in Figure 1e,1f.

In Figure 2a the slice of original 3D US-image of the thyroid gland is depicted. Its contour that was done by expert is drawn in Figure 2a. The corresponding slice of 3D segmentation of the original image by the extended Ising model are placed in Figure 2c,d. The full segmentation of 3D $196 \times 215 \times 301$ image takes about 40min of processor Pentium-III 800.
Figure 1
5 Conclusion

In the paper the Bayesian methods of segmentation and estimation of grayscale and color images are presented. For both of them it is supposed feature function of images and labels of segments take finite number of rational (possibly, different) values and they are distributed according to either the exponential Gibbs or the Gaussian Gibbs distribution. The numerical tests showed the methods developed allow solution problems of practical segmentation and Gibbs estimation of images of large sizes.
References

[1] S. Geman, D. Geman, *Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images*, IEEE Trans. on Pattern Anal. and Mach. Intel. PAMI-6, 6 (1984), 721–741.

[2] B. Gidas, *Metropolis Type Monte Carlo Simulation Algorithm and Simulated Annealing*, Topics in Contemp. Probab. and its Appl., Stochastics Ser., CRC, Boca Raton, Fl, 1995.

[3] B.A. Zalesky, *Stochastic Relaxation for building some classes of piecewise linear regression functions*, Monte-Carlo Meth. and Apl., 6 (2000), no. 2, 141-157

[4] D.M. Greig, B.T. Porteous, A.H. Scheult, *Exact Maximum A Posteriori Estimation for Binary Images*, J. R. Statist. Soc. B., 58 (1989), 271–279.

[5] Yu. Boykov, O. Veksler, R. Zabih, *Fast Approximate Energy Minimization via Graph Cuts*, IEEE Trans. Pattern Anal. and Machine Intel., 23 (2001), no. 11, 1222–1239.

[6] V. Kolmogorov, R. Zabih, *What Energy Functions can be Minimized via Graph Cuts?,* Cornell CS Technical Report, TR2001-1857, 2001.

[7] B.A. Zalesky, *Computation of Gibbs estimates of gray-scale images by discrete optimization methods*, Proceedings of the Sixth International Conference PRIP’2001 (Minsk, May 18-20, 2001), 81-85.

[8] B.A. Zalesky, *Efficient integer-valued minimization of quadratic polynomials with the quadratic monomial \( b_{i,j}^2(x_i - x_j)^2 \)* Dokl. NAN Belarus, 45 (2001), no. 6, 9-11.

[9] B.A. Zalesky, *Network Flow Optimization for Restoration of Images*, Preprint AMS, Mathematics ArXiv, 2001, [math.OC/0106180](http://arxiv.org/abs/math.OC/0106180).

[10] S.A. Aivazyan, B.M. Buchshtaber, I.S. Enyukov, L.L. Meshalkin, *Applied Statistics: Classification and Reducing of Dimension*, Finances and Statistics, Moskow, 1989.

[11] G.J. McLachlan, *Discriminant Analysis and Statistical Pattern Recognition*, John Wiley&Sons, New York, 1992.

[12] V.V. Mottl, I.B. Muchnik, *Hidden Markov Models in Structure Analysis of Signals*, Fizmatlit, Moskow, 1999.
[13] E.E. Juk, Yu.S. Kharin, *Robustness in Cluster Analysis of Multidimensional data*, Belgosuniversitet, Minsk, 1998.

[14] B.V. Cherkassky, A.V. Goldberg, *On Implementing Push-Relabel Method for the Maximum Flow Problem*, Technical Report STAN-CS-94-1523, Department of Computer Science, Stanford University, 1994.

[15] J.C. Picard, H.D. Ratliff, *Minimal cuts and related problems*, Networks, 5 (1975), 357-370.