Properties of $G$-Equivalence of Matrices

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Abstract

The theorem of Hilbert-Burch provides a description of codimension two determinantal varieties and their deformations in terms of their presentation matrices. In this work we use this correspondence to study properties of determinantal varieties, based on methods of singularity theory. We establish the theory of singularities for $n \times p$ matrices extending previous results of Bruce and Tari [10] and Frühbis-Krüger [2]. The main result of this work is the description of equivalent conditions to $G$-finite determinacy of the presentation matrix of Cohen-Macaulay varieties of codimension 2. We apply the results to obtain sufficient conditions for topological triviality of deformations of weighted homogeneous matrices.

1 Introduction

Properties of determinantal varieties have been studied both in geometry and commutative algebra (see [15], [16]). If $M$ is a matrix with entries in $R$ and $t = \min\{n, p\}$, J. Eagon proved in [7] that the ideal $I$ generated by the $t \times t$ minors of $M$ is Cohen-Macaulay.

In particular, in the case where $X$ is a codimension two determinantal variety, we can use the Hilbert-Burch’s theorem to obtain a good description of $X$ and its deformations in terms of its presentation matrix. In fact, if $X$ is a codimension two Cohen-Macaulay variety, then $X$ can be defined by the maximal minors of a $n \times (n + 1)$ matrix. Moreover, any perturbation of a $n \times (n + 1)$ matrix gives rise to a deformation of $X$ and any deformation of $X$ can be obtained through a perturbation of the presentation matrix.

In this work, we use this correspondence to study properties of codimension two Cohen-Macaulay varieties through their presentation matrix. We define a group $G$ acting in the space of the matrices and use this equivalence to classify their singularities.

In [2], A. Frühbis-Krüger considers the case of matrices defining codimension two Cohen-Macaulay singularities to study space curves. In this case, it is
possible to get the normal module and the space $T^1$ of the deformations of first order in terms of presentation matrices. The Tjurina number, $\tau(X)$, of the determinantal variety $X$ is the codimension of the space $T^1$ and coincides with the $G_r$-codimension of $M$. In [4], A. Frühbis-Krüger and A. Neumer obtain a complete list of simple Cohen-Macaulay codimension 2 singularities.

Quadratic matrices have been previously studied by V. I. Arnold in [1]. Moreover, using the $G$-equivalence of matrices, J. W. Bruce, F. Tari and G. J. Haslinger obtain classifications of simple germs of families of symmetric, skew-symmetric and square matrices using the $G$-equivalence ([9], [10], [6]).

In the first part of the paper, we develop the theory of singularities for $n \times p$ matrices, extending the results of [2], [9] and [10] related to the infinitesimal and geometric characterization of finite determinacy and the theorem of versal unfolding.

We give equivalent of conditions to $G$-finite determinacy of the presentation matrix of codimension 2 Cohen-Macaulay varieties with isolated singularity. The corresponding result for the contact group $K$ was obtained by T. Gaffney in [14]. As application of these results we study the $G$-topological triviality of families of matrices.

### 2 Notations and Basic Definitions

Let $Mat_{(n,p)}(\mathbb{C})$ be the set of all $n \times p$ matrices with complex entries, $\Delta_t \subset Mat_{(n,p)}(\mathbb{C})$ the subset formed by matrices that have rank less than $t$, with $1 \leq t \leq \min(n, p)$. It is possible to show that $\Delta_t$ is an irreducible singular algebraic variety, of codimension $(n-t+1)(p-t+1)$ (see [15]). Moreover the singular set of $\Delta_t$ is exactly $\Delta_t - 1$. The set $\Delta_t$ is called generic determinantal variety.

**Definition 2.1.** Let $M = (m_{ij}(x))$ be a $n \times p$ matrix whose entries are complex analytic functions on $U \subset \mathbb{C}^r$, $0 \in U$ and $f = (f_1, \ldots, f_{n+1})$ defined by the $t \times t$ minors of $M$. We say that $X$ is a determinantal variety if $X$ is defined by the equation $f_1 = \ldots = f_{n+1} = 0$ and codimension of $X$ is equal to $(n-t+1)(p-t+1)$.

We can look at a matrix $M = (m_{ij}(x))$ as a map $M: \mathbb{C}^r \rightarrow Mat_{(n,p)}(\mathbb{C})$, with $M(0) = 0$. Then the determinantal variety in $\mathbb{C}^r$ is the set $X = M^{-1}(\Delta_t)$ when codimension of $X$ is equal to $(n-t+1)(p-t+1)$.

Let $\mathcal{O}_r$ be the ring of germs of analytic functions on $\mathbb{C}^r$ and $\mathcal{M}$ its maximal ideal. We denote by $Mat_{(n,p)}(\mathcal{O}_r)$ the set of all matrices $n \times p$ with entries in $\mathcal{O}_r$. This set can be identified with $\mathcal{O}_r^{np}$ the free module of rank $np$.

We concentrate our attention in this paper to codimension 2 determinantal singularities and their deformations. The following proposition follows from the Auslander- Buschsbaum formula and the Hilbert-Burch’s Theorem.

**Proposition 2.1.** ([3], pg. 3994)

1) Let $M$ be a matrix $(n+1) \times n$ with entries in $\mathcal{O}_r$ and $f = (f_1, \ldots, f_{n+1})$ its maximal minors and, by abuse of notation, the ideal generated by them.
If \(\text{codim}(V(f)) \geq 2\) the following sequence
\[
0 \longrightarrow (\mathcal{O}_r)^n \longrightarrow (\mathcal{O}_r)^{n+1} \longrightarrow \mathcal{O}_r/(f) \longrightarrow 0
\]
is exact. Moreover, \(\mathcal{O}_r/(f)\) is Cohen-Macaulay and \(\text{codim}(V(f)) = 2\).

2) If \(X \subset \mathbb{C}^r\) is Cohen-Macaulay, \(\text{codim}(X) = 2\) and \(X = V(I)\), then \(\mathcal{O}_r/I\) has a minimal resolution of the type
\[
0 \longrightarrow \mathcal{O}_r^n \longrightarrow (\mathcal{O}_r)^{n+1} \longrightarrow \mathcal{O}_r/I \longrightarrow 0.
\]

Moreover, there is an unit \(u \in \mathcal{O}_r\) such that \(I = u \cdot f\), where \(f\) is again the ideal of the maximal minors of \(M\).

3. Any deformation of \(M\) is a deformation of \(X\);

4. Any deformation of \(X\) can be generated by a perturbation of the matrix \(M\).

It follows from this proposition that any deformation of a codimension 2 Cohen-Macaulay variety may be given as a perturbation of the presentation matrix. Therefore, we can study these varieties and their deformations using their presentation matrices.

Given a matrix \(M \in \text{Mat}_{n,p}(\mathcal{O}_r)\), let \(C_{ij}(M)\) (respectively \(R_{lk}(M)\)) be the matrix that has the \(i\)-th column (respectively the \(l\)-th row) equal to the \(j\)-th column of \(M\) (respectively the \(k\)-th row) and zeros in any other position.

We denote by \(J(M)\) the submodule generated by the matrices of the form \(\frac{\partial M}{\partial x_i}\) for \(1 \leq i \leq r\) and \(\mathcal{H} = \text{GL}_p(\mathcal{O}_r) \times \text{GL}_n(\mathcal{O}_r)\), where \(\text{GL}_j(\mathcal{O}_r)\) denotes the group of invertible \(j \times j\) matrices with entries in \(\mathcal{O}_r\). Let \(\mathcal{R}\) be the group of changes of coordinates in \((\mathbb{C}^r, 0)\), that is, \(\mathcal{R}\) is the group of analytic diffeomorphism germs in \((\mathbb{C}^r, 0)\).

Let \(M\) be a \(n \times (n+1)\) matrix with entries in the maximal ideal of \(\mathcal{O}_r\), \(n > 1\) and \(r > 1\). We denote by \(f = (f_1, \ldots, f_{n+1})\) the ideal generated by the \(n \times n\) minors of the matrix \(M\) and \(X\) the variety defined by \(f\). The index of each \(f_i\) indicates the column removed from \(M\) to compute the minor. As \(X\) is not in general a complete intersection, there are relations between the component functions of \(f\). Thus, the maximal rank of the jacobian matrix of \(f\) is given by \(d \leq \min\{n+1, r\}\).

### 3 Singularity Theory of Matrices

Given two germs of singularities of matrices, we are interested in studying these germs according to the following equivalence relation.

**Definition 3.1.** Let \(\mathcal{G} = \mathcal{R} \times \mathcal{H}\) the semi-direct product of \(\mathcal{R}\) and \(\mathcal{H}\). We say that two germs \(M_1, M_2 \in \text{Mat}_{n,p}(\mathcal{O}_r)\) are \(\mathcal{G}\)-equivalent if there exist \((\phi, R, L) \in \mathcal{G}\) such that \(M_1 = L^{-1}(\phi^*M_2)\).
It is not difficult to see that \( G \) is one of Damon’s geometric subgroups of \( K \), hence a consequence of Damon’s result [8] we can use the techniques of singularity theory, for instance, those concerning finite determinacy. The notions of \( G \)-equivalence and \( K_\Delta \)-equivalence, where \( \Delta \) consists of the subvariety of matrices of rank less than the maximal rank [8], coincide for finitely determined germs (see [9]).

The next proposition characterizes the \( G \)-tangent space of a matrix \( M \). The proof is analogous to the proof in [9] in the case of symmetric matrices.

**Proposition 3.1.** 1. The \( R \)-tangent space to the orbit of an element of \( M \in Mat_{(n,p)}(\mathcal{O}_r) \) is the \( \mathcal{O}_r \)-module generated by \( x_j \frac{\partial M}{\partial x_i} \), \( 1 \leq i, j \leq r \).

2. The tangent space to the orbit of \( M \) under the action of the subgroup \( \mathcal{H} = GL_p(\mathcal{O}_r) \times GL_n(\mathcal{O}_r) \) is the \( \mathcal{O}_r \)-module generated by \( C_{ij}(M) \), \( 1 \leq i, j \leq p \), and \( R_{lk}(M) \), \( 1 \leq l, k \leq n \).

It follows from the previous discussion that the \( G \)-tangent space to a germ \( M \) is given by

\[
T_G M = MJ(M) + \mathcal{O}_r \langle R_{lk}(M), C_{ij}(M) \rangle,
\]

where \( 1 \leq l, k \leq n \) and \( 1 \leq i, j \leq p \).

Given \( M \in Mat_{(n,p)}(\mathcal{O}_r) \) we consider the map

\[
g_M = g : Mat_{(p,p)}(\mathcal{O}_r) \times Mat_{(n,n)}(\mathcal{O}_r) \longrightarrow Mat_{(n,p)}(\mathcal{O}_r)
\]
given by \( g(A,B) = BM + MA \).

Then it is possible to rewrite the expression of the tangent space as

\[
T_G M = MJ(M) + \mathcal{O}_r. \text{Im}(g).
\]

This equivalence relation is useful to classify determinantal singularities and to study their deformations.

The next propositions express the normal module \( N_X \) and the space of the first order deformations \( T^1_X \), in terms of matrices, hence we can treat the base of the semi-universal deformation using matrix representation (see [3] for the definitions of \( N_X \) and \( T^1_X \)).

**Proposition 3.2.** ([2], pg. 3996) Let \( M \) be a \( (n + 1) \times n \) matrix with entries in the maximal ideal of \( \mathcal{O}_r \) and \( X \) the germ defined by its maximal minors. The normal module is given by

\[
N_X \cong \frac{Mat_{(n+1,n)}(\mathcal{O}_r)}{\text{Im}(g)}
\]

where \( g \) is the map defined as above.
Proposition 3.3. ([2], pg.3997) With the same notations of the previous lemma
\[ T^1_X = \frac{\text{Mat}_{n+1,n}(\mathcal{O}_r)}{J(M) + \text{Im}(g)}. \]

Our next goal is to write the theorem of finite determinacy for \( n \times p \) families of matrices. In [9] and [10], the geometric characterization of finitely determined germs of square matrices follows the ideas given by Gaffney in [5]. In [2], Anne Frühbis characterizes finitely determined \((n+1) \times n\) matrices using Artin Approximation Theorem. The proofs of the corresponding theorems (3.1) and (3.2) are similar to the proofs the corresponding theorems for Mather’s groups (see [2] and [17]).

Theorem 3.1. [Infinitesimal Criterion for Finite Determinacy] Let \( M \in \text{Mat}_{n,p}(\mathcal{O}_r) \) and \( k \) be a positive integer such that
\[ M^{k+1} \text{Mat}_{n,p}(\mathcal{O}_r) \subseteq M^2J(M) + M\text{Im}(g). \]

Then, \( M \) is \( k \)-finitely determined.

We observe that the action of \( GL_p(\mathbb{C}) \times GL_n(\mathbb{C}) \) on \( \text{Mat}_{n,p}(\mathbb{C}) \) has \( n \) orbits, given by \( \Delta_i \setminus \Delta_{i-1}, 1 \leq i \leq n \), if \( n \leq p \). These orbits determine a stratification of \( \text{Mat}_{n,p}(\mathbb{C}) \), which is a Whitney stratification (see [18]).

Theorem 3.2. [Geometric Criterion of Finite Determinacy] An element \( M : \mathbb{C}^r, 0 \rightarrow \text{Mat}_{n,p}(\mathbb{C}) \) is \( \mathcal{G} \)-finitely determined if and only if \( M \) is transverse to the strata of the stratification of \( \text{Mat}_{n,p}(\mathbb{C}) \) outside the origin.

Corollary 3.1. Let \( M \) be a \( n \times p \) matrix with entries in the maximal ideal of \( \mathcal{O}_r \), defining an isolated singularity. Then \( M \) is \( \mathcal{G} \)-finitely determined.

Proof. Since \( M \) defines an isolated singularity, then \( M(x) \) does not intercept \( \Delta_i \setminus \Delta_{i-1} \) if \( \Delta_i \setminus \Delta_{i-1} \neq \{0\} \). Then \( M \) is transverse to the strata of the stratification outside the origin, and by the geometric criterium, \( M \) is \( \mathcal{G} \)-finitely determined.

A different proof of this result can be found in [2], pg. 3998. The result of Theorem 3.1 can be generalized to matrices on \( \text{Mat}_{n,p}(\mathbb{R}) \) with entries \( \mathbb{C}^\infty \).

4 The main result

The purpose of this section is to prove the following theorem which gives simple geometric and algebraic conditions characterizing \( \mathcal{G} \)-finite determinacy of \( n \times (n+1) \) matrices defining isolated singularities \( X = f^{-1}(0) \) with \( f = (f_1, \ldots, f_{n+1}) \). As before \( f_i \) denotes the \( n \times n \)-minor of \( M \) obtained by removing the \( i \)-th-column of \( M \), \( i = 1, \ldots, n+1 \).

We fix some notations:
a) $E_{ij}$ is the $n \times p$ matrix with 1 at the $(i, j)$ position and zero otherwise;

b) If $q \neq s$ and $\gamma \neq \nu$, we denote by

$$\Delta^{(q,s)} = \frac{\partial f_q}{\partial x_\gamma} \frac{\partial f_s}{\partial x_\nu} - \frac{\partial f_q}{\partial x_\nu} \frac{\partial f_s}{\partial x_\gamma}$$

a 2 × 2-minor of the Jacobian matrix of $f$, where $1 \leq r, s \leq n + 1$ and $1 \leq \nu, \gamma \leq r$.

d) Let $J_f$ be the ideal generated by the 2 × 2 minors of the Jacobian matrix of $f$, i.e.,

$$I_G(M) = J_f + < f_1, \ldots, f_{n+1} > .$$

e) Let $M^j$ be the $n \times n$ matrix obtained removing the $j$-th column of $M$. Indicate by $\text{cof}^j_{\mu-su}(m_{\mu-s})$ the cofactor of the element $m_{\mu-s}$ in $M^j$.

f) Analogously, $M_{kl}^j$ is the $(n - 1) \times (n - 1)$ matrix removing from $M^j$ the $k$-th row and the $l$-th column. We denote by $\text{cof}^j_{kl}(m_{\mu-su})$ the cofactor of the element $m_{\mu-su}$ in $M_{kl}^j$. When $n = 1$, we consider that $\text{cof}^j_{kl}(m_{\mu-su}) = 1$.

**Theorem 4.1.** Let $M$ be the germ of an $n \times (n + 1)$ matrix with entries in the maximal ideal of $O_r$. Then, the following statements are equivalent:

(a) $M$ is $G$-finitely determined and $X$ has isolated singularity;

(b) $X \cap V(J_f) = \{0\}$;

(c) $I_G(M) \supseteq M^k$, for some positive integer $k$.

**Proof.**

To prove that $(a) \implies (b)$, let $\Delta_n \subseteq \text{Mat}_{n,n+1}(\mathbb{C})$ be the set of singular matrices. We note that $\Delta_n$ is an irreducible analytic variety. Also, if we consider the diagram

$$\mathbb{C}^r \xrightarrow{M} \text{Mat}_{n,n+1}(\mathbb{C}) \xrightarrow{\delta} \mathbb{C}^{n+1},$$

then $\Delta_n = \delta_n^{-1}(0)$. Now $M$ is $G$-finitely determined and has isolated singularity, hence $f = \delta_n \circ M$ is a submersion away from zero. Therefore $X \cap V(J_f) = \{0\}$.

It follows from Hilbert-Nullstellensatz Theorem that $(b) \iff (c)$.

The proof $(c) \implies (a)$ is harder. It is based on Propositions 4.1 and 4.2 in which we show that the matrices $f_j E_{kl}$ and $J_f E_{kl}$ are on the $G$-tangent space to M, for all $1 \leq l, j \leq n + 1$ and $1 \leq k \leq n$. Then it follows from these conditions that $I_G(M) \text{Mat}_{n,n+1}(\mathbb{C}) \subseteq T_G M$. Hence, if $(c)$ holds, then $T_G M \supset M^k \text{Mat}_{n,(n+1)}$ and the result follows from Theorem 3.1. □
Proposition 4.1. Let $M$ be a $n \times (n + 1)$ matrix with entries in the maximal ideal of $\mathcal{O}_r$. Then $f_j E_{kl} \in TG_M$, for $1 \leq l, j \leq n + 1$ and $1 \leq k \leq n$.

Proof.

If $M = (m_{us})$, we can write each $f_j$, $j = 1, \ldots, n + 1$, expanding by the $k$-th-row of the matrix $M$ for any choice of $k$, $1 \leq k \leq n$, in the following way:

$$f_j = \sum_{s \neq j} m_{ks} \text{cof}_j (m_{ks}).$$

As before, we denote $C_{ls}(M)$, or $C_{ls}$ when the context is clear, the matrix that has $l$-th column equal to $s$-th column of $M$ and zeros in any other position, $1 \leq l, s \leq n + 1$.

For each $l$, we can consider the $n \times (n + 1)$ matrix $A_l = (a_{us})$ defined by

$$A_l = \sum_{s \neq j} \text{cof}_l (m_{ks}) C_{ls} \in TG_M.$$

Then,

i) $a_{us} = 0$, if $s \neq l$;  
ii) $a_{kl} = \sum_{s \neq l} m_{ks} \text{cof}_l (m_{ks}) = f_j$;  
iii) $a_{ul} = \sum_{s \neq j} m_{us} \text{cof}_l (m_{ks}) = 0$, for $u \neq k$, since this is the determinant of a matrix that has two identical rows.

Therefore, $A_l = f_j E_{kl}$ and we get the result. □

Example 4.1. Let $M = (m_{ij}(x))$ be a $3 \times 4$ matrix with entries in the maximal ideal of $\mathcal{O}_r$, and we choose $l = 2$ and $k = 1$ in the previous lemma. To verify that $f_1 E_{12} \in TG_M$, we write

$$A_2 = \sum_{s \neq 1} \text{cof}_1 (m_{1s}) C_{2s} = \text{cof}_1 (m_{12}) C_{22} + \text{cof}_1 (m_{13}) C_{23} + \text{cof}_1 (m_{14}) C_{24}.$$

We note that

i) $\text{cof}_1 (m_{12}) = m_{23} m_{34} - m_{33} m_{24}$,  
ii) $\text{cof}_1 (m_{13}) = -(m_{22} m_{34} - m_{32} m_{24})$,  
iii) $\text{cof}_1 (m_{14}) = m_{22} m_{33} - m_{32} m_{23}$.

Then

i) $a_{12} = m_{12} \text{cof}_1 (m_{12}) + m_{13} \text{cof}_1 (m_{13}) + m_{14} \text{cof}_1 (m_{14}) = f_1$
\[ b) \ a_{22} = m_{22} \text{cof}^j(m_{12}) + m_{23} \text{cof}^j(m_{13}) + m_{24} \text{cof}^j(m_{14}) = 0 \]
\[ c) \ a_{32} = m_{32} \text{cof}^j(m_{12}) + m_{33} \text{cof}^j(m_{13}) + m_{34} \text{cof}^j(m_{14}) = 0 \]

Moreover if \( t \neq 2 \), \( a_{it} = 0 \), \( 1 \leq i \leq 3 \). Therefore \( f_1E_{12} \) belongs to the tangent space of \( M \).

Our next goal is to prove that the \( 2 \times 2 \)-minors of the Jacobian matrix of \( f \) are on the \( \mathcal{G} \)-tangent space of \( M \). For this we need some preliminary lemmas.

**Lemma 4.1.** Let \( M = (m_{us}) \) be a \( n \times (n+1) \) matrix with entries in the maximal ideal of \( \mathcal{O}_r \). Then,
\[
\text{cof}^j(m_{il}) = (-1)^{\alpha} \text{cof}^l(m_{ij}),
\]
where
\[
\alpha = \begin{cases} 
  l - j + 1, & \text{if } l < j \\
  l - j - 1, & \text{if } l > j 
\end{cases}
\]
\( 1 \leq l, j \leq n+1 \) and \( 1 \leq i \leq n \).

**Proof.**
The proof follows directly from the expressions of \( \text{cof}^l(m_{il}) \) and \( \text{cof}^l(m_{ij}) \).

**Lemma 4.2.** We fix \( j, l, \) and \( \gamma \), such that \( j \neq l \), \( 1 \leq j, l \leq n+1 \), and \( 1 \leq \gamma \leq r \). Then,
\[
G_{jl}^{\gamma} = \frac{\partial f_j}{\partial x_\gamma} E_{kl} + (-1)^l \frac{\partial f_l}{\partial x_\gamma} E_{kj} \in TG,
\]
for \( 1 \leq k \leq n \).

**Proof.**
Without loss of generality, we let \( l < j \). We derive the proof in four steps:

**Step 1:** Let \( A = (a_{qv}) \) be the matrix defined by
\[
A = \frac{\partial M}{\partial x_\gamma} \text{cof}^j(m_{kl}) + \sum_{i \neq j} \frac{\partial \text{cof}^j(m_{kl})}{\partial x_\gamma} C_{ti} + \sum_{i \neq j, l} (-1)^{\alpha} \frac{\partial m_{kl}}{\partial x_\gamma} \left( \sum_{u \neq k} R_{ku} \text{cof}^j_k(m_{ul}) \right),
\]
where \( R_{us} \) is the matrix that has \( u \)-th row equal to \( s \)-th row of \( M \) with zeros in any other position and
\[
\alpha = \begin{cases} 
  k + i, & \text{if } i < j \\
  k + i - 1, & \text{if } i > j 
\end{cases}
\]
We now look at each entry of the matrix \( A \). We deal in (i)-(iii) with the case \( q = k \), and in (iv) and (v) with the case \( q \neq k \).

i) At the entry \( (k, j) \), we have
\[
a_{kj} = \frac{\partial m_{kj}}{\partial x_\gamma} \text{cof}^j(m_{kl}) + \sum_{i \neq j} (-1)^{\alpha} \frac{\partial m_{ki}}{\partial x_\gamma} \left( \sum_{u \neq k} m_{uj} \text{cof}^j_k(m_{ul}) \right)
\]
Using lemma 4.1, we can write
\[ a_{kj} = (-1)^{l-j+1} \left( \frac{\partial m_{kj}}{\partial x_{\gamma}} \text{cof}^f(m_{kj}) + \sum_{i \neq j} (-1)^{\alpha} \frac{\partial m_{ki}}{\partial x_{\gamma}} \left( \sum_{u \neq k} m_{uj} \text{cof}^f_{kl}(m_{uj}) \right) \right) \]
\[ = (-1)^{l-j+1} \left( \frac{\partial f_l}{\partial x_{\gamma}} - \sum_{i \neq j} \frac{\partial \text{cof}^f(m_{ki})}{\partial x_{\gamma}} m_{ki} \right). \]

ii) At the entry \((k, l)\), we have
\[ a_{kl} = \frac{\partial m_{kl}}{\partial x_{\gamma}} \text{cof}^f(m_{kl}) + \sum_{i \neq j} \frac{\partial \text{cof}^f(m_{ki})}{\partial x_{\gamma}} m_{ki} + \sum_{i \neq j, l} (-1)^{\alpha} \frac{\partial m_{ki}}{\partial x_{\gamma}} \left( \sum_{u \neq k} m_{ul} \text{cof}^f_{kl}(m_{ul}) \right) \]
\[ = \sum_{i \neq j} \frac{\partial \text{cof}^f(m_{ki})}{\partial x_{\gamma}} m_{ki} + \sum_{i \neq j} \frac{\partial m_{ki}}{\partial x_{\gamma}} \text{cof}^f(m_{ki}) = \frac{\partial f_j}{\partial x_{\gamma}}. \]

iii) If \(t \neq l, j\), at the entry \((k, t)\) we have:
\[ a_{kt} = \frac{\partial m_{kt}}{\partial x_{\gamma}} \text{cof}^f(m_{kl}) + \sum_{i \neq j} \left( -1 ^{\alpha} \frac{\partial m_{ki}}{\partial x_{\gamma}} \left( \sum_{u \neq k} m_{ul} \text{cof}^f_{kt}(m_{ul}) \right) \right). \]

Notice that for \(i \neq t\),
\[ \sum_{u \neq k} m_{ut} \text{cof}^f_{kt}(m_{ul}) = 0, \]
since it is the determinant of a \((n-1) \times (n-1)\) matrix with two columns equal to the \(l\)-th column of the matrix \(M\). Then,
\[ a_{kt} = \frac{\partial m_{kt}}{\partial x_{\gamma}} \text{cof}^f(m_{kl}) + (-1)^{k+t} \frac{\partial m_{kt}}{\partial x_{\gamma}} \left( \sum_{u \neq k} m_{ut} \text{cof}^f_{kt}(m_{ul}) \right). \]

Now, it is not hard to verify that \(a_{kl} = 0\), for all \(t \neq l, j\).

iv) Let \(q \neq k\), then at the entry \((q, l)\) we have:
\[ a_{ql} = \frac{\partial m_{ql}}{\partial x_{\gamma}} \text{cof}^f(m_{kl}) + \sum_{i \neq j} \frac{\partial \text{cof}^f(m_{ki})}{\partial x_{\gamma}} m_{qi}. \]

We can write \(\text{cof}^f(m_{ki}) = (-1)^{\beta} \sum_{t \neq i, j} m_{qt} \text{cof}^f_{kt}(m_{qt})\), where
\[ \beta = \begin{cases} 
    k + i, & \text{if } i < j \\
    k + i - 1, & \text{if } i > j.
\end{cases} \]
Then,
\[
a_{ql} = \frac{\partial m_{ql}}{\partial x_{\gamma}} \text{cof}^l(m_{kl}) + \sum_{i \neq j} (-1)^{l,j} \frac{\partial m_{qt}}{\partial x_{\gamma}} \text{cof}^l_k(m_{qt}) m_{qi} + \sum_{i \neq j} (-1)^{l,j} m_{qt} \frac{\partial \text{cof}^l_k(m_{qt})}{\partial x_{\gamma}} m_{qi}.
\]

With similar arguments as in the previous steps, we can show that
\[
S_1 = \sum_{t \neq j} \frac{\partial m_{qt}}{\partial x_{\gamma}} \text{cof}^l(m_{kt}),
\]
and \(S_2 = 0\).

Therefore,
\[
a_{ql} = \sum_{t \neq j, l} \frac{\partial m_{qt}}{\partial x_{\gamma}} \text{cof}^l(m_{kt})
\]
v) The entry \((q, v)\), \(v \neq l\) and \(q \neq k\),
\[
a_{qv} = \frac{\partial m_{qv}}{\partial x_{\gamma}} \text{cof}^l(m_{kl}).
\]

**Step 2:** We consider the \(n \times (n + 1)\) matrix, \(B = (b_{us})\), given by:
\[
B = (-1)^{l-j+1} \sum_{i \neq l} \frac{\partial \text{cof}^l(m_{ki})}{\partial x_{\gamma}} C_{ji}.
\]

Then,
i) \(b_{us} = 0\), if \(s \neq j\);

\[
\text{ii) } b_{uj} = (-1)^{l-j+1} \sum_{i \neq l} \frac{\partial \text{cof}^l(m_{ki})}{\partial x_{\gamma}} m_{ui}
\]

**Step 3:** We consider the \(n \times (n + 1)\) matrix, \(C = (c_{kv})\) given by \(C = A + B\). Then,
i) \(c_{kj} = (-1)^{l-j+1} \frac{\partial f_{l}}{\partial x_{\gamma}}\);

\[
\text{ii) } c_{kl} = \frac{\partial f_{l}}{\partial x_{\gamma}}
\]

\[
\text{iii) } c_{kt} = 0, \text{ for } t \neq l, j;
\]

\[
\text{iv) } c_{ql} = -\sum_{t \neq j, l} \frac{\partial m_{qt}}{\partial x_{\gamma}} \text{cof}^l(m_{kt}), \text{ for } q \neq k.
\]
\( v) \ c_{qj} = \frac{\partial m_{qi}}{\partial x_\gamma} \text{cof}^\ell(m_{kl}) + (-1)^{l-j+1} \sum_{i \neq l} \frac{\partial \text{cof}^\ell(m_{ki})}{\partial x_r} m_{ai}, \) for \( q \neq k. \)

As in item \((iv)\) of the step 1, it is possible to show that

\[ c_{qj} = (-1)^{l-j+1} \sum_{i \neq j, l} \frac{\partial m_{qi}}{\partial x_\gamma} \text{cof}^\ell(m_{kj}). \]

\( vi) \ c_{qv} = \frac{\partial m_{qj}}{\partial x_\gamma} \text{cof}^\ell(m_{kl}), \) \( \forall \ v \neq l, j \) and \( q \neq k. \)

**Step 4:** For each \( q \neq k, 1 \leq q \leq n, \) we consider the matrices

\[ D_q = \sum_{i \neq j, l} \frac{\partial m_{qi}}{\partial x_\gamma} \left( \sum_{u \neq k} (-1)^{\mu} R_{qu} \text{cof}^\ell_{ui}(m_{kl}) \right), \]

where

\[ \mu = \begin{cases} 
    u + i - 1, & \text{if } j < i \text{ and } u > k \text{ or } j > i \text{ and } u > k \\
    u + i, & \text{if } j < i \text{ and } u < k \text{ or } j > i \text{ and } u < k.
\end{cases} \]

We look at each entry of the matrix \( D: \)

\( i) \ d_{ql} = \sum_{i \neq j, l} \frac{\partial m_{qi}}{\partial x_\gamma} \left( \sum_{u \neq k} (-1)^{\gamma} m_{ul} \text{cof}^\ell_{ui}(m_{kl}) \right). \)

Using lemma 4.1 and the expression of the \( \text{cof}^\ell(m_{ki}) \) it is not difficult to see that

\[ \sum_{u \neq k} (-1)^{\gamma} m_{ul} \text{cof}^\ell_{ui}(m_{kl}) = \text{cof}^\ell(m_{ki}). \]

Thus,

\[ d_{ql} = \sum_{i \neq j, l} \frac{\partial m_{qi}}{\partial x_\gamma} \text{cof}^\ell(m_{ki}). \]

\( ii) \ d_{qj} = \sum_{i \neq j, l} \frac{\partial m_{qi}}{\partial x_\gamma} \left( \sum_{u \neq k} (-1)^{\gamma} m_{uj} \text{cof}^\ell_{ui}(m_{kl}) \right). \) Again, using lemma 4.1 and the expression of the \( \text{cof}^\ell(m_{kj}) \) we have

\[ d_{qj} = (-1)^{j-l-1} \sum_{i \neq j, l} \frac{\partial m_{qi}}{\partial x_\gamma} \text{cof}^\ell(m_{kj}) \]

\( iii) \) For \( t \neq j, l, \) we have

\[ d_{qt} = \sum_{i \neq j, l} \frac{\partial m_{qi}}{\partial x_\gamma} \left( \sum_{u \neq k} (-1)^{\gamma} m_{ut} \text{cof}^\ell_{ui}(m_{kl}) \right). \]
Note that for \( i \neq t \), we have
\[
\sum_{u \neq k} (-1)^{\gamma} m_{ut} \text{cof}^j_u(m_{kl}) = 0
\]
since this is the expression of the determinant of a matrix that has two equal columns.

Then,
\[
d_{qt} = \frac{\partial m_{qt}}{\partial x_\gamma} \left( \sum_{u \neq k} (-1)^{\gamma} m_{ut} \text{cof}^j_u(m_{kl}) \right).
\]

Using lemma 4.1 and the expression of the \( \text{cof}^j(m_{kl}) \) we have
\[
d_{qt} = -\frac{\partial m_{qt}}{\partial x_\gamma} \text{cof}^j(m_{kl})
\]

i) \( d_{sv} = 0 \), for \( s \neq q \);

To conclude the proof, it suffices to consider the matrix \( E = (e_{qv}) \) given by
\[
E = C + \sum_{q \neq k} D_q.
\]

Example 4.2. Let \( M = (m_{ij}) \) be a \( 2 \times 3 \) matrix with entries in the maximal ideal of \( \mathcal{O}_4 \). Then using the previous result, we will verify
\[
\frac{\partial f_2}{\partial x_\gamma} E_{11} + \frac{\partial f_1}{\partial x_\gamma} E_{12}
\]
belongs to \( TGM \). In fact, we first consider the matrix
\[
A = \frac{\partial M}{\partial x_\gamma} \text{cof}^j(m_{11}) + \frac{\partial \text{cof}^j(m_{11})}{\partial x_\gamma} C_{11} + \frac{\partial \text{cof}^j(m_{13})}{\partial x_\gamma} C_{13} - \frac{\partial m_{13}}{\partial x_\gamma} R_{12}
\]
as in the step 1 of the Lemma 4.2. Then, we look at each entry of \( A \):

a) \( a_{11} = m_{23} \frac{\partial m_{11}}{\partial x_\gamma} + m_{11} \frac{\partial m_{23}}{\partial x_\gamma} - m_{13} \frac{\partial m_{21}}{\partial x_\gamma} - m_{21} \frac{\partial m_{13}}{\partial x_\gamma} = \frac{\partial f_2}{\partial x_\gamma}
\]
b) \( a_{12} = m_{23} \frac{\partial m_{12}}{\partial x_\gamma} - m_{22} \frac{\partial m_{13}}{\partial x_\gamma} = \frac{\partial f_1}{\partial x_\gamma} - \left( m_{12} \frac{\partial m_{23}}{\partial x_\gamma} + m_{13} \frac{\partial m_{22}}{\partial x_\gamma} \right)
\]
c) \( a_{13} = m_{23} \frac{\partial m_{13}}{\partial x_\gamma} - m_{23} \frac{\partial m_{13}}{\partial x_\gamma} = 0
\]
d) \( a_{21} = m_{23} \frac{\partial m_{21}}{\partial x_\gamma} + m_{21} \frac{\partial m_{23}}{\partial x_\gamma} - m_{23} \frac{\partial m_{21}}{\partial x_\gamma} = m_{21} \frac{\partial m_{23}}{\partial x_\gamma}
\]
e) \( a_{22} = m_{22} \frac{\partial m_{22}}{\partial x_\gamma} \)

f) \( a_{23} = m_{23} \frac{\partial m_{23}}{\partial x_\gamma} \)

Now, we define the matrix

\[
B = \frac{\partial \text{cof}_1 (m_{12})}{\partial x_\gamma} C_{22} + \frac{\partial \text{cof}_1 (m_{13})}{\partial x_\gamma} C_{23} = \begin{pmatrix}
0 & m_{12} \frac{\partial m_{23}}{\partial x_\gamma} - m_{13} \frac{\partial m_{22}}{\partial x_\gamma} & 0 \\
0 & m_{22} \frac{\partial m_{23}}{\partial x_\gamma} - m_{23} \frac{\partial m_{22}}{\partial x_\gamma} & 0 
\end{pmatrix},
\]

according to the step 2 of the Lemma 4.2. Then

\[
A + B = \begin{pmatrix}
\frac{\partial f_2}{x_\gamma} & \frac{\partial f_1}{x_\gamma} & 0 \\
m_{21} \frac{\partial m_{23}}{x_\gamma} & m_{22} \frac{\partial m_{23}}{x_\gamma} & m_{23} \frac{\partial m_{23}}{x_\gamma}
\end{pmatrix}.
\]

Finally, by the step 4, we have the matrix

\[
D_2 = -\frac{\partial m_{23}}{\partial x_\gamma} R_{22} = -\begin{pmatrix}
0 & 0 & 0 \\
m_{21} \frac{\partial m_{23}}{\partial x_\gamma} & m_{22} \frac{\partial m_{23}}{\partial x_\gamma} & m_{23} \frac{\partial m_{23}}{\partial x_\gamma}
\end{pmatrix},
\]

and the result follows from adding \( A + B \) with \( D_2 \).

**Proposition 4.2.** The matrices \( \Delta_{(j,t)} E_{kl} \) belong to the \( G \)-tangent space of the matrix \( M \), for \( 1 \leq k \leq n \) and \( 1 \leq j, t, l \leq n + 1 \), \( j \neq t \).

**Proof.**

We show that \( \Delta_{(j,t)} E_{kl} \) are obtained using the matrices of the previous lemma. Let us consider two cases:

i) If \( j \neq l \) and \( t \neq l \), then

\[
\Delta_{(j,t)} E_{kl} = (-1)^{t-j} \frac{\partial f_l}{\partial x_\nu} \left( \frac{\partial f_l}{\partial x_\gamma} E_{kj} + (-1)^{j-t+1} \frac{\partial f_l}{\partial x_\gamma} E_{kl} \right) + \\
\frac{\partial f_l}{\partial x_\gamma} \left( \frac{\partial f_l}{\partial x_\nu} E_{kl} + (-1)^{j-t+1} \frac{\partial f_l}{\partial x_\nu} E_{kj} \right) - \frac{\partial f_l}{\partial x_\gamma} \left( \frac{\partial f_l}{\partial x_\nu} E_{kl} + (-1)^{j-t+1} \frac{\partial f_l}{\partial x_\nu} E_{kj} \right),
\]

then \( \Delta_{(j,t)} E_{kl} \in TGM \).

ii) If \( j = l \), then

\[
\Delta_{(l,l)} E_{kl} = \\
= \frac{\partial f_l}{\partial x_\nu} \left( \frac{\partial f_l}{\partial x_\gamma} E_{kl} + (-1)^{j-t+1} \frac{\partial f_l}{\partial x_\nu} E_{kt} \right) - \frac{\partial f_l}{\partial x_\nu} \left( \frac{\partial f_l}{\partial x_\gamma} E_{kl} + (-1)^{j-t+1} \frac{\partial f_l}{\partial x_\gamma} E_{kt} \right)
\]

Therefore, \( \Delta_{(l,l)} E_{kl} \in TGM \).
5 $\mathcal{G}$-Topological Equivalence of Matrices

As an application of the results of the previous section we study the $\mathcal{G}$-topological triviality of families of matrices.

We concentrate our study on $n \times (n + 1)$ matrices $M$ with entries in $\mathcal{M}_r$. We denote by $f = (f_1, \ldots, f_{n+1})$ the ideal generated by its maximal minors, $X$ the variety defined by $f$ and by $\mathcal{O}_r^0$ the ring of germs at the origin of continuous functions of $\mathbb{K}^r \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The results are applied to deformations of germs of weighted homogeneous matrices.

Definition 5.1. Two matrices $M, N \in \text{Mat}_{(n,n+1)}(\mathcal{O}_r)$ are topologically equivalent (or $C^0$-$\mathcal{G}$-equivalent) if there exist a germ of homeomorphism $\phi : (\mathbb{K}^r, 0) \to (\mathbb{K}^r, 0)$ and invertible matrices $A \in GL_n(\mathcal{O}_0^0)$ and $B \in GL_{n+1}(\mathcal{O}_0^0)$ such that $M = A^{-1}(N \circ \phi)B$.

A control function $\rho : \mathbb{K}^r \to \mathbb{R}$ is a non negative function that satisfies the following condition:

i) $\rho(0) = 0$ and there exist constants $c > 0$ and $\alpha > 0$ such that $\rho(x) \geq c|x|^\alpha$ (i. e., $\rho$ satisfies a Lojasiewicz condition).

Definition 5.2. A matrix $M = (m_{ij}) \in \text{Mat}_{(n,n+1)}(\mathcal{O}_r)$ is $k$-$C^0$-$\mathcal{G}$-determined, if for every matrix $N = (n_{ij}) \in \text{Mat}_{(n,n+1)}(\mathcal{O}_r)$ such that $j^kM(0) = j^kN(0)$ $1 \leq i \leq n$ and $1 \leq j \leq n + 1$, $N$ is $C^0$-$\mathcal{G}$-equivalent to $M$.

Definition 5.3. A one parameter deformation $M \in \text{Mat}_{(n,n+1)}(\mathcal{O}_{r+1})$ of $M_0$ in $\text{Mat}_{(n,n+1)}(\mathcal{O}_r)$ is $C^0$-$\mathcal{G}$-trivial (or $\mathcal{G}$-trivial) if there exists a homeomorphism

$$\Phi : (\mathbb{K}^r \times \mathbb{K}_r^0, 0) \to (\mathbb{K}^r \times \mathbb{K}_r^0, 0)$$

$$(x, t) \mapsto (\phi(x, t), t)$$

such that $\Phi(x, 0) = (x, 0), \phi(0, t) = 0$ and families of matrices $A \in GL_n(\mathcal{O}_{r+1}^0)$ and $B \in GL_{n+1}(\mathcal{O}_{r+1}^0)$ such that $A(x, 0) = I_n$, $B(x, 0) = I_{n+1}$ and $M_0 = A^{-1}(M \circ \Phi)B$.

Proposition 5.1. Let $M_0$ be a $n \times (n + 1)$ matrix, defining a codimension 2 Cohen- Macaulay isolated singularity and $M$ a deformation of $M_0$. Suppose that there exists a control function $\rho$ such that

$$\rho \frac{\partial M}{\partial t} = \sum_{i=1}^{r} \xi_i \frac{\partial M}{\partial x_i} + \sum_{i,k=1}^{n+1} L_{ik} C_{ik}(M) + \sum_{s=1}^{n} S_{rs} R_{rs}(M),$$

with $\xi_i(x, t), L_{ik}(x, t), S_{rs}(x, t) \in \mathcal{O}_{r+1}$, satisfying the conditions

$$\frac{|\xi_i(x, t)|}{\rho^2(x, t)} \leq C_1 |x|, \quad \frac{|L_{ik}(x, t)|}{\rho^2(x, t)} \leq C_2 |x|, \quad \frac{|S_{rs}(x, t)|}{\rho^2(x, t)} \leq C_3 |x|. \quad (2)$$

with $\xi_i(0, t) = 0$, $L(x, 0) = Id_{n+1}$ in $S(x, 0) = Id_n$. Then the family $M$ is $C^0$-$\mathcal{G}$-trivial.

$^1j^kM(0)$ denotes the $n \times (n + 1)$ matrix whose entries are the $k$-jets of $m_{ij}$ at zero.
Proof.

To get topological triviality of $M$ we construct continuous vector fields $X$ in $\mathbb{K}^r \times \mathbb{K}$, $W$ in $\mathbb{K}^{n^2} \times \mathbb{K}$ and $Z$ in $\mathbb{K}^{(n+1)^2} \times \mathbb{K}$, lifting $\frac{\partial}{\partial t}$. The condition (2) ensures the uniqueness of the corresponding flow, which gives the family of homeomorphism trivializing $M$.

We start the proof defining $\Phi$. Let $X(x, t)$ be the vector field in $\mathbb{K}^r \times \mathbb{K}$ defined by

\[
\begin{cases}
\frac{\partial}{\partial t} \frac{\xi_i(x, t)}{\rho^2(x, t)} \frac{\partial}{\partial x_i}, & \text{if } x \neq 0 \\
\frac{\partial}{\partial t}, & \text{if } x = 0.
\end{cases}
\]

For each $j$, $X_j$ denotes the $j$-th component of $X$. The vector field $X$ is real analytic along $(\mathbb{K}^r \times \mathbb{K}) \setminus \{(0) \times \mathbb{K}\}$. Furthermore,

\[|X_j(x, t)| = \frac{|\xi_j(x, t)|}{\rho^2(x, t)} \leq C_1|x|,
\]

$1 \leq j \leq r$, so that the vector field $X(x, t)$ satisfies a Lipschitz condition along $\{(0) \times \mathbb{K}\}$. It follows from [15] that this vector field is locally integrable. For more details on the complex case, see [12]. We indicate $\Phi(x, t)$ the corresponding flow.

Now we want to find matrices $A$ and $B$ satisfying the conditions of Definition (5.3), that is, $A \in GL_n(O_{r+1})$ and $B \in GL_{n+1}(O_{r+1})$ such that $A(x, 0) = I_n$, $B(x, 0) = I_{n+1}$ and $AM_0B^{-1} = M \circ \Phi$.

By definition of $\Phi$ and the hypothesis, we have

\[
\frac{\partial}{\partial t}(m_{ij} \circ \Phi)(x, t) = -\sum_{i=1}^{r} \rho_i(x, t) \frac{\partial m_{ij}}{\partial x_i} \Phi(x, t) + \frac{\partial m_{ij}}{\partial t}(x, t) = -\left(\sum_{k=1}^{n+1} L_{jk}(\Phi(x, t)) \frac{\partial}{\partial \rho^2(\Phi(x, t))} m_{ik}(\Phi(x, t)) + \sum_{s=1}^{n} S_{is}(\Phi(x, t)) \frac{\partial}{\partial \rho^2(\Phi(x, t))} m_{sj}(\Phi(x, t))\right).
\]

where $1 \leq i \leq n$ and $1 \leq j \leq n+1$.

Then we want to find matrices $A$ and $C = B^{-1}$ such that

\[
\frac{\partial}{\partial t} (AM_0C)_{ij} = -\left(\sum_{k=1}^{n+1} L_{jk}(\Phi(x, t)) \frac{\partial}{\partial \rho^2(\Phi(x, t))} (AM_0C)_{ik} + \sum_{s=1}^{n} S_{is}(\Phi(x, t)) \frac{\partial}{\partial \rho^2(\Phi(x, t))} (AM_0C)_{sj}\right).
\]

To solve the system (3) it is sufficient to solve the following two systems of differential equations

\[
\frac{\partial a_{ij}}{\partial t} = -\sum_{k=1}^{n+1} L_{jk}(\Phi(x, t)) a_{ik}
\]

(4)

\[
\frac{\partial c_{ij}}{\partial t} = -\sum_{s=1}^{n} S_{is}(\Phi(x, t)) c_{sj}
\]

(5)
with the same initial conditions. In fact, $M_0$ is independent of $t$ and the original system appears multiplying (4) to the right by $M_0 C$, (5) to the left by $A M_0$ and adding the resulting systems.

Consider the following vector field $W = (w_{ij})$ on $\mathbb{K}^{(n+1)^2} \times \mathbb{K}$

$$w_{ij}(x, y, t) = \frac{\partial}{\partial t} + \sum_{k=1}^{n+1} \frac{L_{jk}(\Phi(x, t))}{\rho^2(\Phi(x, t))} y_{ik}.$$  

(6)

In (11) we use $y_{ik}$ to denote the variables of $\mathbb{K}^{(n+1)^2} \times \mathbb{K}$ where $1 \leq j, k \leq n$. By hypothesis,

$$\frac{L_{ij}(\Phi(x, t))}{\rho^2(\Phi(x, t))} \leq C_2 |x|$$

It follows from [12] that the vector field is integrable.

With similar arguments, we obtain that the vector field $Z = (z_{ij})$ in $\mathbb{K}^n \times \mathbb{K}$ defined by

$$z_{ij}(x, y, t) = \frac{\partial}{\partial t} + \sum_{s=1}^{n} \frac{S_{js}(\Phi(x, t))}{\rho^2(\Phi(x, t))} y_{sj}$$

is integrable.

6 Deformations of Weighted Homogeneous Families of Matrices

Our next goal is to prove the topological triviality theorem for deformations of $G$-finitely determined weighted homogeneous matrices $M \in Mat_{(n,n+1)}(\mathcal{O}_r)$. We will show that for deformations of degree greater than the maximum weighted degree of the entries of $M$, the hypothesis of Proposition (5.1) holds, hence they are topologically $G$-trivial.

**Definition 6.1.** Given a set of weights $a = (a_1, ..., a_r) \in \mathbb{N}^r$, for any monomial $x^\alpha = x_1^{a_1} x_2^{a_2} ... x_r^{a_r}$, we define the $a$-filtration of $x^\alpha$ by

$$fil(x^\alpha) = \sum_{i=1}^{r} a_i \alpha_i.$$  

We can define a filtration on the ring $\mathcal{O}_r$, as follows

$$fil(f) = \inf_{\alpha} \left\{ fil(x^\alpha) : \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0) \neq 0 \right\},$$

for all germ $f \in \mathcal{O}_r$, where $|\alpha| = a_1 + a_2 + ... + a_r$.  

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We extend the filtration to the submodule $\Theta_X$ of germs of vector fields tangent to $X$, defining $a \left( \frac{\partial}{\partial x_j} \right) = -a_j$ for all $j = 1, \ldots, n$ so given $\xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_j} \in \Theta_X$, then $fil(\xi) = \inf_j \{ fil(\xi_j) - w_j \}$.

We can extend this definition to the ring of 1-parameter families of germs on $r$ variables, putting $fil(x^{\alpha \beta} t^3) = fil(x^{\alpha})$.

Definition 6.2. A matrix $M \in Mat_{(n,n+1)}(\mathcal{O}_r)$ is called weighted homogeneous of type $(D, a) \in Mat_{(n,n+1)}(\mathbb{N}) \times \mathbb{N}^r$, if

i) $fil(m_{ij}) = d_{ij}$ with respect to $a = (a_1, \ldots, a_r)$;

ii) The following relations are verified

$$d_{ij} - d_{ik} = d_{lj} - d_{lk} \quad \text{for all} \quad 1 \leq i, l \leq n, \quad 1 \leq j, k \leq n + 1.$$

Let $M$ be a $n \times (n+1)$ weighted homogeneous matrix of type $(D; a)$. Let $f = (f_1, \ldots, f_{n+1})$ be the ideal generated by the maximal minors of $M$. The index of $f_u$ indicates the column removed from $M$ to compute the minor. Then it is immediate that $f$ is weighted homogeneous of type $(D_1, \ldots, D_{n+1}; a)$, where $fil(f_u) = D_u$. The converse is also true, that is, if $f \in \mathcal{O}_r$ is a codimension 2 Cohen-Macaulay ideal generated by weighted homogeneous polynomials with respect to some set of weights $a$, then there exists a weighted homogeneous presentation matrix $M$ of $f$ of type $(D, a)$ for some $D \in Mat_{(n,n+1)}(\mathbb{N})$ (see [2] for a proof).

Let $k_1 = \text{l.c.m.} \{ D_u | 1 \leq u \leq n + 1 \}$ and $\beta_u = k_1/D_u$. We define

$$N_H^M = \sum_{j=1}^{n+1} |f_j|^{2\beta_j}.$$

Note that $N_H^M$ is a weighted homogeneous control function of type $(2k_1; a)$.

We note that each $\frac{\partial f_q}{\partial x_\gamma}$ is weighted homogeneous of type $(D_q - a_\gamma; a)$ and for each minor $\Delta^{(q,s)}$ of the Jacobian matrix of $f$, there exists an integer $D_{ij}$ such that $\Delta^{(q,s)}$ is weighted homogeneous of type $(D_{qs}; a)$. For $k_2 = \text{l.c.m.}(D_{ij})$ and $\alpha_{qs} = k_2/D_{qs}$ we define

$$N_R^M = \sum_{(q, s)} |\Delta^{(q,s)}|^{2\alpha_{qs}}.$$

This is a weighted homogeneous function of type $(2k_2; a)$. 

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Let \( K = \text{l.c.m.}\{k_1, k_2\} \) and \( c_i = K/k_i \). We define

\[ N_G^c M = N_R^{c_1} M + N_H^{c_2} M. \]

Thus, \( N_G^c M \) is weighted homogeneous of type \((2K; a)\).

Let \( M_t(x) = M_0(x) + t\theta(x) \) be a deformation of \( M_0 \) with \( t \in [0, 1] \), and \( d_{\text{max}} = \max_{ij} \{d_{ij}\} \) and let \( F \) denote the maximal minors of \( M_t \).

We define the control function \( N_R M_t = \sum_{i \in I} |\Delta_t^{(q,s)}|2^{\alpha_{ij}} \), where \( \Delta_t^{(q,s)} \) are the 2 \times 2 minors of the jacobian matrix of \( f_j \) and the \( \alpha_{ij} \) are the same as above. If \( f|\theta(\theta_{ij})| \geq d_{\text{max}} + 1 \), for all \( i, j \), then there exist constants \( C_1 \) and \( C_2 \) such that \( C_1 N_R M_0 \leq N_R M_t \leq C_2 N_R M_0 \). If \( \Theta = (\theta_{ij}) \) and \( f|\theta(\theta_{ij})| = d_{\text{max}} \), then this condition also holds to \( t \) small enough (see [11]).

We can define similarly, the control \( N_H M_t = \sum_{i \in I} |F_i|2^{\beta_i} \), where each \( \beta_i \) is obtained as above and \( F_i \) are the \( i \)th component of \( F \).

We will prove the topological triviality theorem in two parts. First, we show

Proposition 6.1. Let \( M_0 \in \text{Mat}_{(n,n+1)}(\mathcal{O}_r) \) be the germ of a weighted homogeneous matrix of type \((D, a) \in \text{Mat}_{(n,n+1)}(\mathbb{N}) \times \mathbb{N}^r \), satisfying the condition \( N_H(M_0(x)) \geq c|x|^\alpha \) for constants \( c \) and \( \alpha \). Then,

i) Deformations \( M(x, t) = M_0(x) + t\theta(x) \) of \( M_0 \) with \( f|\theta(\theta_{ij})| \geq d_{\text{max}} + 1 \), \( t \in [0, 1] \), \( \forall i, j \), and \( d_{\text{max}} = \max_{ij} \{d_{ij}\} \), are \( C^0 - H \)-trivial.

ii) Moreover, for \( t \) small enough, deformations \( M(x, t) = M_0(x) + t\theta(x) \) of \( M_0 \) with \( f|\theta(\theta_{ij})| = d_{\text{max}} \), \( \forall i, j \), are \( C^0 - H \)-trivial.

Proof.

Let \( M(x, t) = M_0(x) + t\theta(x) \) be a deformation of \( M_0 \) with \( f|\theta(\theta_{ij})| \geq d_{\text{max}} + 1 \), \( t \in [0, 1] \), \( \forall i, j \). To obtain the \( C^0 - H \)-triviality we find \( C^0 \)-map germs \( L : \mathbb{K}^{(n+1)^2} \times \mathbb{K} \to \mathbb{K} \) such that

\[
\frac{\partial M}{\partial t} = \sum_{p=1}^{n+1} \sum_{s=1}^{n+1} L_{ps}(x, t)C_{ps}(M),
\]

where \( C_{rl} = C_{rl}(M) \) is defined in Section 2.

Using the Propositions [4,1] we have

\[
f_{ij}\frac{\partial M}{\partial t} = \sum_{k=1}^{n} \sum_{l=1}^{n+1} \theta_{kl} \left( \sum_{i \neq j} \text{cof}^i(m_{ki})C_{li}(M) \right)
\]

for \( 1 \leq j \leq n + 1 \).
Then, multiplying this equation by $|ft_j|^{2(\beta_j-1)}ft_j$ and adding in $j$, we get

$$N_H \frac{\partial M}{\partial t} = \sum_{j=1}^{n+1} \left( \sum_{k=1}^{n+1} \sum_{l=1}^{n+1} \theta_{kl} \sum_{i \neq j}^{n+1} |ft_j|^{2(\beta_j-1)}ft_jcof_i(m_{ki})C_{li}(M) \right) = \sum_{l=1}^{n+1} \sum_{i=1}^{n+1} \left( \sum_{k=1}^{n+1} \sum_{j \neq i}^{n+1} |ft_j|^{2(\beta_j-1)}ft_jcof_j(m_{ki}) \right) C_{li}(M),$$

We define,

$$L_{li}(x, t) = \sum_{k=1}^{n} \theta_{kl} \sum_{i \neq j} \sum_{k=1}^{n+1} |ft_j|^{2(\beta_j-1)}ft_jcof_j(m_{ki}).$$

Then,

$$\frac{\partial M}{\partial t} = \sum_{l=1}^{n+1} \sum_{i=1}^{n+1} L_{li}(x, t) N_H \frac{M}{M} C_{li}(M).$$

Now

i) $fil \left( \frac{\partial m_{ij}}{\partial t} \right) = fil (\theta_{ij}) \geq d_{\text{max}} + 1$;

ii) $fil (ft_j^{2(\beta_j-1)}ft_j) = 2k_1 - D_j$;

iii) $fil (L_{ji}) \geq 2k_1 - D_j + d_{\text{max}} + 1 + D_j - d_{ki} \geq 2k_1 + 1$, for all $k$.

Then, for each $1 \leq j, i \leq n + 1$, $L_{ji}(x, t) \leq C|x|$ and, therefore, as in the Proposition 5.1 the vector field

$$w_{ji}(x, y, t) = \frac{\partial}{\partial t} + \sum_{j=1}^{n+1} \sum_{i \neq j}^{n+1} \frac{L_{ji}(x, t)}{N_H \frac{M}{M}} y_{ji},$$

is integrable, which implies $C^0 - H$-triviality of $M$.

To prove item ii) let $M(x, t) = M_0(x) + t \Theta(x)$ be a deformation of $M_0$ with $fil (\theta_{ij}) = d_{\text{max}}, \forall i, j$ and $t$ is small enough. Analogously to the proof of item (i), let $p(x, y, t)W(x, y, t)$ be the vector field where $W(x, y, t)$ is defined in the Proposition 5.1 and $p : C^r \times C^{(n+1)^2} \times C \rightarrow C$ is a conic bump function (see 11, Lemma 4), such that the restriction to $C^r \times C^{(n+1)^2} \times \mathbb{C} \setminus \{(0, 0, t)\}$ of $p$ is smooth and

$$p(x, y, t) = \begin{cases} 1, & \text{for all } (x, y, t) \in \overline{U} \\ 0, & \text{in the complement of } V \\ 0 \leq |p(x, y, t)| \leq 1 \text{ in } V - \overline{U} \\ p(0, 0, t) = 0, & \text{for all } t \end{cases}$$
where $V$ and $U$ are neighborhoods of the open set $\{(x, y) \in \mathbb{C} \times \mathbb{C} | |y| < c \rho(x)\}$ on $\mathbb{C}^r \times \mathbb{C}^{(n+1)^2} \times \mathbb{C} \setminus \{(0, 0, t)\}$, where $\rho(x) = [(N_H M_t(x))]^{1/2}$.

$$V = \{(x, y, t) \text{ such that } |y| \leq c_1 \rho(x)\}$$

and $U$ is chosen such that $U \subset \overline{U} \subset V$. Then,

$$|p(x, y, t)W_{ij}(x, y, t)| = \left| \frac{L_{ij}(x, t)}{N_H M_t} \right| |py| \leq \left| \frac{L_{ij}(x, t)}{N_H M_t} \right| N_H M_t^{1/2k_1},$$

which implies the integrability of the field, hence the $C^0 - \mathcal{H}$- triviality.

\[\blacksquare\]

**Proposition 6.2.** Let $M_0 \in \text{Mat}_{(n, n+1)}(\mathcal{O}_r)$ be a weighted homogeneous matrix of type $(D, a) \in \text{Mat}_{(n, n+1)}(\mathbb{N}) \times \mathbb{N}^r$, satisfying the condition $N_R(M_0(x)) \geq c|x|^\alpha$ for constants $c$ and $\alpha$. Then for sufficiently small $x$ and $t$, deformations $M(x, t) = M_0(x) + t\Theta(x)$, with $\text{fil}(\theta_{ij}) \geq d_{\max} + 1$, $\forall i, j$, and $d_{\max} = \max_{ij}\{d_{ij}\}$, are $C^0 - \mathcal{R}$- trivial.
Proof.

Using the proof of the Proposition \ref{prop:4.2}, for each \( k, l \) fixed we have

\[
\Delta^{(q,s)} E_{kl} = (-1)^{l-q} \frac{\partial f_i}{\partial x_\gamma} G^\gamma_{sq} + \frac{\partial f_s}{\partial x_\gamma} G^\nu_{ql} - \frac{\partial f_l}{\partial x_\gamma} G^\nu_{sl},
\]

(7)

if \( q, s \neq l \) and

\[
\Delta^{(l,s)} E_{kl} = \frac{\partial f_i}{\partial x_\gamma} G^\nu_{sl} - \frac{\partial f_j}{\partial x_\gamma} G^\nu_{jl}
\]

(8)

for \( j \neq t, 1 \leq k \leq n, 1 \leq j, t, l \leq n + 1 \) and \( 1 \leq \gamma, \nu \leq r \).

Now multiplying the equation (7) by \( |\Delta^{(q,s)}|^{2(\alpha_{qs} - 1)} \Delta^{(q,s)} \) and adding in
\( q \neq s \), multiplying the equation (8) by \( |\Delta^{(l,s)}|^{2(\alpha_{ls} - 1)} \Delta^{(l,s)} \) and adding in \( s \), we get

\[
\begin{align*}
\sum_{q \neq s} |\Delta^{(q,s)}|^2 \Delta^{(q,s)} E_{kl} &= \sum_{q > s} |\Delta^{(q,s)}|^2 \Delta^{(q,s)} \left( (-1)^{l-q} \frac{\partial f_i}{\partial x_\gamma} G^\gamma_{sq} + \frac{\partial f_s}{\partial x_\gamma} G^\nu_{ql} - \frac{\partial f_l}{\partial x_\gamma} G^\nu_{sl} \right), \\
\sum_{s \neq l} |\Delta^{(l,s)}|^2 \Delta^{(l,s)} E_{kl} &= \sum_{s \neq l} |\Delta^{(l,s)}|^2 \Delta^{(l,s)} \left( \frac{\partial f_i}{\partial x_\gamma} G^\nu_{sl} - \frac{\partial f_j}{\partial x_\gamma} G^\nu_{jl} \right).
\end{align*}
\]

Then it follows that

\[
N_{RM} E_{kl} = \sum_{\nu=1}^{r} \sum_{\gamma > \nu} \rho_{qs} \left( (-1)^{l-q} \frac{\partial f_i}{\partial x_\gamma} G^\gamma_{sq} + \frac{\partial f_s}{\partial x_\gamma} G^\nu_{ql} - \frac{\partial f_l}{\partial x_\gamma} G^\nu_{sl} \right) + \sum_{s \neq l} \rho_{ls} \left( \frac{\partial f_i}{\partial x_\gamma} G^\nu_{sl} - \frac{\partial f_j}{\partial x_\gamma} G^\nu_{jl} \right)
\]

where \( \rho_{qs} = |\Delta^{(q,s)}|^{2(\alpha_{qs} - 1)} \Delta^{(q,s)} \) and \( \rho_{ls} = |\Delta^{(l,s)}|^{2(\alpha_{ls} - 1)} \Delta^{(l,s)} \).

Using the Lemma \ref{lem:4.1} and the matrices above, we have

\[
N_{RM} \frac{\partial M}{\partial t} = \sum_{\nu=1}^{r} \xi_{\nu} \frac{\partial M}{\partial x_\nu} + \sum_{i=1}^{n+1} B_{ki} C_{li} + \sum_{u \neq k} S_{ul} R_{ku}
\]

(9)

where

\[
\begin{align*}
\xi_{\nu} &= \sum_{\gamma \neq \nu} \sum_{k,l} \frac{\partial m_{kl}}{\partial t} \left[ \sum_{s=1}^{n+1} \sum_{q > s} \rho_{qs} \left( \frac{\partial f_s}{\partial x_\gamma} \text{cof}^q(m_{kl}) - \frac{\partial f_l}{\partial x_\gamma} \text{cof}^s(m_{kl}) \right) \right] \\
B_{ki} &= \sum_{\nu=1}^{r} \sum_{\gamma > \nu} \left[ \sum_{k,l} \frac{\partial m_{kl}}{\partial t} \left( \sum_{s=1}^{n+1} \sum_{q > s} \rho_{qs} \left( \frac{\partial f_s}{\partial x_\gamma} \text{cof}^s(m_{kl}) - \frac{\partial f_l}{\partial x_\gamma} \text{cof}^s(m_{kl}) \right) \right) + \sum_{s \neq l} \rho_{ls} \left( \frac{\partial f_i}{\partial x_\gamma} \text{cof}^s(m_{kl}) \right) \right]
\end{align*}
\]
\[ S_{ul} = \sum_{\gamma > \nu} \sum_{k,l} \frac{\partial m_{kl}}{\partial t} \left[ \sum_{s > q} \rho_{qs} \left( (-1)^{s-q} \frac{\partial f_i}{\partial x_\gamma} \sum_{i \neq s,q} \frac{\partial m_{ki}}{\partial x_\nu} \cof_{ki}^s (m_{ul}) + \frac{\partial f_s}{\partial x_\gamma} \sum_{i \neq q,l} \frac{\partial m_{ki}}{\partial x_\nu} \cof_{ki}^s (m_{ul}) \right) \right] \]

Then,

\[ \text{fil} (\xi_\nu) \geq 2k_2 + 1, \quad (10) \]
\[ \text{fil} (B_{ki}) \geq 2k_2 + 1, \quad (11) \]
\[ \text{fil} (S_{ul}) \geq 2k_2 + 1. \quad (12) \]

Finally, by proposition 5.1 follows the \( C^0 - R \)-triviality.

**Theorem 6.1.** Let \( M_0 \in \text{Mat}_{n,n+1}(\mathcal{O}_r) \) be a germ of weighted homogeneous matrix of type \((D, a) \in \text{Mat}_{n,n+1}(\mathbb{N}) \times \mathbb{N}_r \) satisfying the condition \( N_G(M_0(x)) \geq c|x|^\alpha \) for constants \( c \) and \( \alpha \).

a) Deformations \( M(x, t) = M_0(x) + t\Theta(x) \), with \( \text{fil}(\theta_{ij}) \geq d_{\text{max}} + 1, \forall i, j \) and \( d_{\text{max}} = \max_{ij} \{d_{ij}\} \), are \( C^0 - G \)-trivial.

b) Deformations \( M(x, t) = M_0(x) + t\Theta(x) \), with \( \text{fil}(\theta_{ij}) \geq d_{\text{max}}, \forall i, j \), and \( d_{\text{max}} = \max_{ij} \{d_{ij}\} \), are \( C^0 - G \)-trivial for \( t \) sufficiently small.

**Proof.**

a) Since the group \( C^0 - G \) is the semi-direct product of the groups \( C^0 - R \) and \( C^0 - H \), the vector fields are defined as in cases \( R \) and \( H \), and the control function \( N_G M \) is defined by \( N_G M = N_{C1}^R M + N_{C2}^H M \) where \( c_1 \) and \( c_2 \) are constants such that \( N_G M \) is weighted homogeneous.

b) By Theorem 4.1 if \( M_0 \) is \( G \)-finitely determined, then \( N_G M_0(x) \) satisfies the condition of the item a), and we get the result.

**Example 6.1.** Let \( M_0 = \begin{pmatrix} z & y & x^3 \\ x^2 & z & y \end{pmatrix} \) weight homogeneous with weights \((3, 8, 7)\). Since \( \mathcal{M}^3 \text{Mat}_{n,n+1}(\mathbb{C}) \subset TG M_0 \) we have that \( M_0 \) is \( G \)-finitely determined. Therefore, all deformations \( M \) of \( M_0 \) with filtration of degree greater than 8 are \( C^0 - G \)-trivial. Moreover, if \( t \) sufficiently small deformations of degree exactly 8 also are \( C^0 - G \)-trivial.
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