TWO PAPERS WHICH CHANGED MY LIFE: MILNOR’S SEMINAL WORK ON FLAT MANIFOLDS AND BUNDLES

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Abstract. We survey developments arising from Milnor’s 1958 paper, “On the existence of a connection with curvature zero” and his 1977 paper, “On fundamental groups of complete affinely flat manifolds.”

With warm wishes to Jack Milnor on his eightieth birthday

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For a young student studying topology at Princeton in the mid-1970’s, John Milnor was a inspiring presence. The excitement of hearing him lecture at the Institute for Advanced Study and reading his books and unpublished lecture notes available in Fine Library made a deep impact on me. One heard rumors of exciting breakthroughs in the Milnor-Thurston collaborations on invariants of 3-manifolds and the theory of kneading in 1-dimensional dynamics. The topological significance of volume in hyperbolic 3-space and Gromov’s proof of Mostow rigidity using simplicial volume were in the air at the time (later to be written up in Thurston’s notes [66]). When I began studying geometric...
structures on manifolds, my mentors Bill Thurston and Dennis Sullivan directed me to two of Milnor’s papers [60, 58]. Like many mathematicians of my generation, his papers and books such as Morse theory, Characteristic Classes, and Singular Points of Complex Hypersurfaces, were very influential for my training. Furthermore his lucid writing style made his papers and books role models for exposition.

I first met Jack in person when I was a graduate student in Berkeley and he was visiting his son. Several years later I was extremely flattered when I received a letter from him, where he, very politely, pointed out a technical error in my Bulletin Announcement [38]. It is a great pleasure and honor for me to express my gratitude to Jack Milnor for his inspiration and insight, in celebration of his eightieth birthday.

1. Gauss-Bonnet beginnings

1.1. Connections and characteristic classes. The basic topological invariant of a closed orientable surface $M^2$ is its Euler characteristic $\chi(M)$. If $M$ has genus $g$, then $\chi(M) = 2 - 2g$. Give $M$ a Riemannian metric — then the Gauss-Bonnet theorem identifies $\chi(M^2)$ as $1/2\pi$ of the total Gaussian curvature of $M$, a geometric invariant. This provides a fundamental topological restriction of what kind of geometry $M$ may support.

For example, if the metric is flat — that is, locally Euclidean — then its Gaussian curvature vanishes and therefore $\chi(M) = 0$. Since $M$ is orientable, $M$ must be homeomorphic to a torus.

In 1944 Chern [26] proved his intrinsic Gauss-Bonnet theorem, which expresses the topological invariant $\chi(M^n)$ as the integral of a differential form constructed from the curvature of a Riemannian metric. More generally, if $\xi$ is an oriented $n$-plane bundle over $M^n$, then its Euler number

$$e(\xi) \in H^n(M, \mathbb{Z}) \cong \mathbb{Z}$$

can be computed as an integral of an expression derived from an orthogonal connection $\nabla$ on $\xi$. (Compare Milnor-Stasheff [61].) For example take to $M$ to be a (pseudo-) Riemannian manifold, with tangent bundle $\xi = TM$ and $\nabla$ the Levi-Civita connection. If $\nabla$ has curvature zero, then, according to Chern, $\chi(M) = 0$.

This paradigm generalizes. When $M$ is a complex manifold, its tangent bundle is a holomorphic vector bundle and the Chern classes can be computed from the curvature of a holomorphic connection. In particular $\chi(M)$ is a Chern number. Therefore, if $TM$ has a flat holomorphic connection, then $\chi(M) = 0$. If $M$ has a flat pseudo-Riemannian metric, then a similar Gauss-Bonnet theorem holds (Chern [27]) and
Therefore it is natural to ask whether a compact manifold whose tangent bundle admits a flat linear connection has Euler characteristic zero.

1.2. Smillie’s examples. In 1976, John Smillie [62] constructed, in every even dimension $n > 2$, a compact $n$-manifold such that $TM$ admits a flat connection $\nabla$. However, the torsion of $\nabla$ is (presumably) nonzero. (None of Smillie’s examples are aspherical; it would be interesting to construct a closed aspherical manifold with flat tangent bundle; compare Bucher-Gelander [14].)

Requiring the torsion of $\nabla$ to vanish is a natural condition. When both curvature and torsion vanish, the connection arises from an affine structure on $M$, that is, the structure defined by a coordinate atlas of coordinate charts into an affine space $E$ such that the coordinate changes on overlapping coordinate patches are locally affine. A manifold together with such a geometric structure is called an affine manifold. The coordinate charts globalize into a developing map

$$\tilde{M} \xrightarrow{\text{dev}} E$$

where $\tilde{M} \longrightarrow M$ is a universal covering space. The developing map $\text{dev}$ is a local diffeomorphism (although generally not a covering space onto its image), which defines the affine structure. Furthermore $\text{dev}$ is equivariant with respect to a homomorphism

$$\pi_1(M) \xrightarrow{\rho} \text{Aff}(\mathbb{R}^2)$$

(the affine holonomy representation) where the fundamental group $\pi_1(M)$ acts by deck transformations of $\tilde{M}$. Just as $\text{dev}$ globalizes the coordinate charts, $\rho$ globalizes the coordinate changes. The flat connection on $TM$ arises from the representation $\rho$ in the standard way: $TM$ identifies with the fiber product

$$\xi_{\rho} := \tilde{M} \times_{\rho} \mathbb{R}^n = (\tilde{M} \times \mathbb{R}^n)/\left(\pi_1(M)\right)$$

where $\pi_1(M)$ acts diagonally — by deck transformations on the $\tilde{M}$ factor and via $\rho$ on the $\mathbb{R}^n$-factor. The differential of $\text{dev}$ defines an isomorphism of $TM$ with $\xi_{\rho}$.

We may interpret Smillie’s examples in this description as follows. In each even dimension $n > 2$, Smillie constructs an $n$-manifold $M^n$ and a representation $\pi_1(M) \xrightarrow{\rho} \text{Aff}(\mathbb{R}^n)$ such that $\xi_{\rho}$ is isomorphic to the tangent bundle $TM$. Sections $\xi_{\rho}$ correspond to singular developing maps which may be smooth, but not necessarily local diffeomorphisms.

Despite the many partial results, we know no example of a closed affine manifold with nonzero Euler characteristic.
1.3. **Benzécri’s theorem on flat surfaces.** In dimension two, a complete answer is known, due to the work of Benzécri [10]. This work was part of his 1955 thesis at Princeton, and Milnor served on his thesis committee.

**Theorem** (Benzécri). A closed surface $M$ admits an affine structure if and only if $\chi(M) = 0$.

(Since every connected orientable open surface can be immersed in $\mathbb{R}^2$, pulling back the affine structure from $\mathbb{R}^2$ by this immersion gives an affine structure. With a small modification of this technique, every connected nonorientable open surface can also be given an affine structure.)

Benzécri’s proof is geometric, and starts with a fundamental polygon $\Delta$ for $\pi_1(M)$ acting on $\tilde{M}$. The boundary $\partial \Delta$ consists of various edges, which are paired by homeomorphisms, reconstructing $M$ as the quotient space by these identifications. One standard setup for a surface of genus $g$ uses a $4g$-gon for $\Delta$, where the sides are alternately paired to give the presentation

$$\pi_1(M) = \langle A_1, B_1, \ldots, A_g, B_g \mid A_1B_1A_1^{-1}B_1^{-1} \ldots A_gB_gA_g^{-1}B_g^{-1} = 1 \rangle$$

The developing map $\text{dev}$ immerses $\Delta$ into $\mathbb{R}^2$, and the identifications between the edges of $\partial \Delta$ are realized by orientation-preserving affine transformations.

Immersions of $S^1$ into $\mathbb{R}^2$ are classified up to regular homotopy by their **turning number** (the Whitney-Graustein theorem [74]) which measures the total angle the tangent vector (the velocity) turns as the curve is traversed. Since the restriction $\text{dev}|_{\partial \Delta}$ extends to an immersion of the disc $\Delta$ its turning number (after choosing compatible orientations)

$$\tau(\text{dev}|_{\partial \Delta}) = 2\pi.$$

However, Benzécri shows that for any smooth immersion $[0, 1] \xrightarrow{f} \mathbb{R}^2$ and orientation-preserving affine transformation $\gamma$,

$$|\tau(f) - \tau(\gamma \circ f)| < \pi.$$

Using the fact that $\partial \Delta$ consists of $2g$ pairs of edges which are paired by $2g$ orientation-preserving affine transformations, combining (2) and (3) implies $g = 1$.

Milnor realized the algebraic-topological ideas underlying Benzécri’s proof, thereby initiating the theory of characteristic classes of flat bundles.
2. The Milnor-Wood inequality

2.1. “On the existence of a connection with curvature zero”. In his 1958 paper [60], Milnor shows that a closed 2-manifold \( M \) has flat tangent bundle if and only if \( \chi(M) = 0 \). This immediately implies Benzécri’s theorem, although it doesn’t use the fact that the developing map is nonsingular (or, equivalently, the associated flat connection is torsionfree). In this investigation, Milnor discovered, remarkably, flat oriented \( \mathbb{R}^2 \)-bundles over surfaces \( M \) with nonzero Euler class. In particular the Euler class cannot be computed from the curvature of a linear connection.

Oriented \( \mathbb{R}^2 \)-bundles \( \xi \) over \( M \) are classified up to isomorphism by their Euler class

\[ e(\xi) \in H^2(M; \mathbb{Z}) \]

and if \( M \) is an orientable surface, an orientation on \( M \) identifies \( H^2(M; \mathbb{Z}) \) with \( \mathbb{Z} \). (See Milnor-Stasheff [61] for details.) An oriented \( \mathbb{R}^2 \)-bundle \( \xi \) admits a flat structure if and only if it arises from a representation

\[ \pi_1(M) \xrightarrow{\rho} \text{GL}^+(2, \mathbb{R}) \]

(where \( \text{GL}^+(2, \mathbb{R}) \) denotes the group of orientation-preserving linear automorphisms of \( \mathbb{R}^2 \)). Milnor shows that \( \xi \) admits a flat structure if and only if its Euler number satisfies

\[ |e(\xi)| < g. \tag{4} \]

Since \( e(TM) = \chi(M) = 2 - 2g \), Milnor’s inequality (4) implies that \( g = 1 \).

The classification of \( S^1 \)-bundles is basically equivalent to the classification of rank 2 vector bundles, but is somewhat more general. To any vector bundle \( \xi \) with fiber \( \mathbb{R}^n \) is associated an \( S^{n-1} \)-bundle: the fiber of the \( S^{n-1} \)-bundle over a point \( x \) consists of all directions in the fiber \( \xi_x \approx \mathbb{R}^n \). In particular two \( \mathbb{R}^2 \)-bundles are isomorphic if and only if their associated \( S^1 \)-bundles are isomorphic. Therefore we henceforth work with \( S^1 \)-bundles, slightly abusing notation by writing \( \xi \) for the \( S^1 \)-bundle associated to \( \xi \).

2.2. Wood’s extension and foliations. In 1971, John W. Wood [75] extended Milnor’s classification of flat 2-plane bundles to flat \( S^1 \)-bundles. Circle bundles with structure group \( \text{GL}^+(2, \mathbb{R}) \) have an important special property. The antipodal map associates a direction in a vector space its opposite direction. Since all linear transformations commute with it, the antipodal map defines an involution on any vector bundle or associated sphere bundle \( \xi \). The quotient is the associated \( \mathbb{R}P^1 \)-bundle \( \hat{\xi} \), and the quotient map \( \xi \to \hat{\xi} \) is a double covering. This is
also an oriented $S^1$-bundle, with Euler class

$$e(\hat{\xi}) = 2e(\xi).$$

Wood [75] determines the flat oriented $S^1$-bundles, for an arbitrary homomorphism

$$\pi_1(\Sigma) \rightarrow \text{Homeo}^+(S^1).$$

He proves the Euler number satisfies the following inequality:

(5) $$|e(\rho)| \leq -\chi(\Sigma),$$

(now known as the Milnor-Wood inequality). Furthermore every integer in $[\chi(\Sigma), -\chi(\Sigma)]$ occurs as $e(\rho)$ for some homomorphism $\rho$.

Milnor's proof interprets the Euler class as the obstruction for lifting the holonomy representation $\rho$ from the group $\text{GL}^+(2, \mathbb{R})$ of linear transformations of $\mathbb{R}^2$ with positive determinant to its universal covering group $\tilde{\text{GL}}^+(2, \mathbb{R})$. Suppose $G$ is a Lie group with universal covering $\tilde{G} \rightarrow G$. If $S_g$ is a closed oriented surface of genus $g > 1$, then its fundamental group admits a presentation (1)

Let

$$\pi_1(S_g) \rightarrow G$$

be a representation; then the obstruction $o_2(\rho)$ for lifting $\rho$ to $\tilde{G}$ is obtained as follows. Choose lifts $\tilde{\rho}(A_i), \tilde{\rho}(B_i)$ of $\rho(A_i)$ and $\rho(B_i)$ to $\tilde{G}$ respectively. Then

(6) $$o_2(\rho) := [\tilde{\rho}(A_1), \tilde{\rho}(B_1)] \ldots [\tilde{\rho}(A_g), \tilde{\rho}(B_g)]$$

is independent of the chosen lifts, and lies in

$$\pi_1(G) = \ker(\tilde{G} \rightarrow G).$$

It vanishes precisely when $\rho$ lifts to $\tilde{G}$. When $G = \text{GL}^+(2, \mathbb{R})$, the obstruction class $o_2(\rho)$ is just the Euler class $e(\rho)$.

To identify the element of $\pi_1(\text{GL}^+(2, \mathbb{R}))$ corresponding to $e(\rho)$, Milnor and Wood estimate the translation number of the lifts of generators to $\tilde{G}$, which is based on the rotation number of orientation-preserving circle homeomorphisms. Milnor uses a retraction $\text{GL}^+(2, \mathbb{R}) \rightarrow \text{SO}(2)$ (say, the one arising from the Iwasawa decomposition), which lifts to a retraction

$$\tilde{G} \rightarrow \tilde{\text{SO}}(2) \cong \mathbb{R}$$

and proves the estimate

(7) $$|\theta(\gamma_1 \gamma_2) - \theta(\gamma_1) - \theta(\gamma_2)| < \frac{\pi}{2}.$$
Wood considers a more general retraction $\theta$ defined on $\widetilde{G} = \text{Homeo}^+(S^1)$ and shows a similar estimate, sharpened by a factor of two. Applying this to (6), he shows that if an $m$-fold product of commutators in $\widetilde{G}$ is translation by $a$, then

\begin{equation}
|a| < 2m - 1
\end{equation}

The estimate (8) extends Benzécri’s original estimate (3) in a stronger and more abstract context. This — the boundedness of the Euler class of flat bundles — may be regarded as one of the roots of the theory of bounded cohomology. The fundamental role of the Euler class as a bounded cohomology class was discovered by Ghys [36]. In particular he showed that the bounded Euler class characterizes orientation-preserving actions of surface groups on the circle up to quasi-conjugacy.

For other generalizations of the Milnor-Wood inequality, compare Dupont [33, 34], Sullivan [64], Domic-Toledo [28] and Smillie [63]. For more information, see Burger-Iozzi-Wienhard [17] and the second chapter of Calegari [18]. The question of when a foliation on the total space of a circle bundle over a surface is isotopic to a flat bundle is the subject of Thurston’s thesis [65].

3. Maximal representations

3.1. A converse to the Milnor-Wood inequality. Equality in (5) has special and deep significance. Let $M$ be a closed oriented surface. Then, just as described earlier for affine structures, every hyperbolic structure on $M$ determines a developing pair $(\text{dev}, \rho)$ where

\[
\tilde{M} \xrightarrow{\text{dev}} \mathbb{H}^2 \\
\pi_1(M) \xrightarrow{\rho} \text{Isom}^+(\mathbb{H}^2).
\]

by globalizing the coordinate charts and coordinate charts in an atlas defining the hyperbolic structure. The flat $(\text{Isom}^+(\mathbb{H}^2), \mathbb{H}^2)$-bundle $E_M \to M$ corresponding to $\rho$ has a section $\delta_M$ corresponding to $\text{dev}$, which is transverse to the flat structure on $E_M$. Consequently the normal bundle of $\delta_M \subset E_M$ (by the tubular neighborhood theorem) is isomorphic to the tangent bundle $TM$, and therefore

\[ e(\rho) = e(TM) = \chi(M), \]

proving sharpness in the Milnor-Wood inequality. By conjugating $\rho$ with an orientation-reversing isometry of $\mathbb{H}^2$, one obtains a representation $\rho$ with $e(\rho) = -\chi(M)$.

The converse statement was proved in my doctoral dissertation [37]. Say that a representation is maximal if $e(\rho) = \pm \chi(M)$.
Theorem. Let $\rho \in \text{Hom}(\pi_1(M), \text{PSL}(2, \mathbb{R}))$. Then the following are equivalent:

- $\rho$ is the holonomy of a hyperbolic structure on $M$;
- $\rho$ is an embedding onto a discrete subgroup of $\text{Isom}^+(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R})$;
- For every $\gamma \in \pi_1(M)$ with $\gamma \neq 1$, the holonomy $\rho(\gamma)$ is a hyperbolic element of $\text{PSL}(2, \mathbb{R})$.

3.2. Kneser’s theorem on surface maps. A special case follows from the classical theorem of Kneser [51]:

Theorem. Let $M, N$ be closed oriented surfaces, and $N$ having genus $> 1$. Suppose that $M \xrightarrow{f} N$ is a continuous map of degree $d$. Then

$$d|\chi(N)| \leq |\chi(M)|.$$ 

Furthermore $d|\chi(N)| = |\chi(M)|$ if and only if $f$ is homotopic to a covering space.

The theorem follows by giving $N$ a hyperbolic structure, with holonomy representation $\rho$. Then the composition

$$\pi_1(M) \xrightarrow{f_*} \pi_1(N) \xrightarrow{\rho} \text{PSL}(2, \mathbb{R})$$

has Euler number $d\chi(N)$. Now apply Milnor-Wood and its converse statement to the composition.

3.3. Components of the representation variety. Since the space of hyperbolic structures on $M$ is connected, the Euler class defines a continuous map

$$\text{Hom}(\pi_1(M), \text{PSL}(2, \mathbb{R})) \longrightarrow H^2(M; \mathbb{Z}) \cong \mathbb{Z}.$$ 

Reversing the orientation on $M$ reverses the sign of the Euler number. Therefore the maximal representations constitute two connected components of $\text{Hom}(\pi_1(M), \text{PSL}(2, \mathbb{R}))$. In general the connected components are the fibers of this map (Goldman [38, 40], Hitchin [50]). In particular the space of representations has $4g - 3$ connected components. Each component has dimension $6g - 6$. Only the Euler class 0 component is not a smooth manifold. Furthermore Hitchin relates the component corresponding to Euler class $2 - 2g + k$ to the $k$-th symmetric power of $M$. See Hitchin [50], Bradlow-García-Prada-Gothen [12, 13], as well as my expository article [42].
3.4. **Rigidity and flexibility.** This characterization of maximal representations is a kind of rigidity for surface group representations. Dupont [33, 34], Turaev [72, 73] and Toledo [69] defined obstruction classes $o_2$ for Lie groups $G$ of automorphisms of Hermitian symmetric spaces. In particular Toledo [69] proved the following rigidity theorem:

**Theorem.** (Toledo [69]) Suppose that $\pi_1(M) \xrightarrow{\rho} PU(n,1)$ is a representation. Equality is attained in the generalized Milnor-Wood inequality

$$|o_2(\rho)| \leq \frac{|\chi(M)|}{2}.$$

Then $\rho$ embeds $\pi_1(M)$ as a discrete subgroup of the stabilizer (conjugate to $U(1,1) \times U(n-1)$ of a holomorphic totally geodesic curve $C$ in $H^2_C$. In particular $C/\text{Image}(\rho)$ is a hyperbolic surface diffeomorphic to $M$.

Recently these results have been extended to higher rank in the work of Burger-Iozzi-Wienhard [15, 16, 17].

As maximality of the Euler class in the Milnor-Wood inequality implies rigidity, various values of the Euler class imply various kinds of flexibility [39]. If $\pi = \pi_1(M)$ is the fundamental group of a compact Kähler manifold $M$, and $G$ is a reductive algebraic Lie group, then Goldman-Millson [48] gives a complete description of the the analytic germ of the space of representations $\text{Hom}(\pi, G)$ at a reductive representation $\rho$. Specifically, $\rho$ has an open neighborhood in $\text{Hom}(\pi, G)$ analytically equivalent to the quadratic cone defined by the symmetric bilinear form

$$Z^1(\pi, \mathfrak{g}_{\text{Ad}_\rho}) \times Z^1(\pi, \mathfrak{g}_{\text{Ad}_\rho}) \to H^2(\pi, \mathfrak{g}_{\text{Ad}_\rho})$$

obtained by combining cup product on $\pi$ with Lie bracket

$$\mathfrak{g}_{\text{Ad}_\rho} \times \mathfrak{g}_{\text{Ad}_\rho} \to \mathfrak{g}_{\text{Ad}_\rho}$$

as coefficient pairing.

Consider the special case when $M$ is a closed hyperbolic surface and a representation $\pi \xrightarrow{\rho_0} SU(1,1) \cong SL(2, \mathbb{R})$. We assume that $\rho_0$ has Zariski-dense image, which in this case simply means that its image is non-solvable. In turn this means the corresponding action on $H^2$ fixes no point in $H^2 \cup \partial H^2$. In that case $\rho_0$ is reductive and defines a smooth point of the $\mathbb{R}$-algebraic set $\text{Hom}(\pi, SU(1,1))$. Extend the action to an isometric action on the complex hyperbolic plane $H^3_\mathbb{C}$ via the composition $\rho$ defined by:

$$(9) \quad \pi \xrightarrow{\rho_0} SU(1,1) \hookrightarrow PU(2,1)$$
For $A \in U(1,1)$, taking the equivalence class of the direct sum

$$A \oplus 1 := \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in U(2,1)$$

in $PU(2,1)$ defines an embedding

$$SU(1,1) \hookrightarrow PU(2,1).$$

This representation stabilizes a complex hyperbolic line $H^3_C \subset H^3_C$ inside the complex hyperbolic plane. What are the local deformations of $\rho$ in $\text{Hom}(\pi, PU(2,1))$?

The representation $\rho$ is maximal if and only if $\rho_0$ is maximal, which occurs when $e(\rho) = \pm \chi(M)$. In that case any representation $\pi \rightarrow PU(2,1)$ near $\rho$ stabilizes $H^3_C$, that is, it lies in the subgroup $U(1,1) \subset PU(2,1)$.

In general, representations $\pi \rightarrow U(1,1)$ can be easily understood in terms of their composition with the projectivization homomorphism $U(1,1) \rightarrow PU(1,1)$. The corresponding map on representation varieties

$$\text{Hom}(\pi, U(1,1)) \rightarrow \text{Hom}(\pi, PU(1,1))$$

is a torus fibration, where the points of the fiber correspond to different actions in the normal directions to $H^3_C \subset H^3_C$ (which are described by characters $\pi \rightarrow U(1)$). In particular $\rho$ defines a smooth point of $\text{Hom}(\pi, U(1,1))$ with tangent space

$$Z^1(\pi, su(1,1)_{Ad\rho_0}) \oplus Z^1(\pi, \mathbb{R})$$

since $u(1)_{Ad\rho_0}$ equals the ordinary coefficient system $\mathbb{R}$ (where $\pi$ acts by the identity).

The general deformation result implies that the analytic germ of $\text{Hom}(\pi, PU(2,1))$ near $\rho$ looks like the Cartesian product of the (smooth) analytic germ of

$$\text{Hom}(\pi, U(1,1)) \times PU(2,1)/U(1,1)$$

with the quadratic cone $\mathcal{Q}_\rho$ in $Z^1(\pi, u(1,1)_{Ad\rho}) \cong \mathbb{C}^{2g}$ defined by the cup-product pairing

$$Z^1(\pi, u(1,1)_{Ad\rho}) \times Z^1(\pi, u(1,1)_{Ad\rho}) \rightarrow H^2(\pi, \mathbb{R}) \cong \mathbb{R}$$

where $u(1,1)_{\rho}$ denotes the $\pi$-module defined by the standard 2-dimensional complex representation of $U(1,1)$. The coefficient pairing (which is derived from the Lie bracket on $su(2,1)$) is just the imaginary part of the indefinite Hermitian form on $u(1,1)$, and is skew-symmetric. In particular the space of coboundaries

$$B^1(\pi, u(1,1)_{Ad\rho}) \subset Z^1(\pi, u(1,1)_{Ad\rho}),$$
is isotropic. Thus we reduce to the symmetric bilinear form obtained from the cohomology pairing

\[ H^1(\pi, u(1,1)_{\text{Ad}_\rho}) \times H^1(\pi, u(1,1)_{\text{Ad}_\rho}) \longrightarrow H^2(\pi, \mathbb{R}) \cong \mathbb{R}. \]

The real dimension of \( H^1(\pi, u(1,1)_{\rho}) \) equals

\[-2\chi(M) = 8(g - 1).\]

By the Signature Theorem of Meyer [57], the quadratic form corresponding to (10) has signature \( 8\varepsilon(\rho_0) \). Thus, near \( \rho \), the \( \mathbb{R} \)-algebraic set \( \text{Hom}(\pi, \text{PU}(2,1)) \) is analytically equivalent to the Cartesian product of a manifold with a cone on \( \mathbb{R}^{8(g-1)} \) defined by a quadratic form of signature \( 8\varepsilon(\rho) \).

Meyer’s theorem immediately gives a proof of Milnor’s inequality (4), since the signature of a quadratic form is bounded by the dimension of the ambient vector space. Furthermore \( \rho \) is maximal if and only if the quadratic form is definite, in which the quadratic cone has no real points, and any small deformation of \( \rho \) must stabilize a complex geodesic.

4. Complete affine manifolds

We return to the subject of flat affine manifolds, and the second [60] of Milnor’s papers on this subject.

4.1. The Auslander-Milnor question. An affine manifold \( M \) is complete if some (and hence every) developing map is bijective. In that case \( \check{M} \) identifies with \( E \), and \( M \) arises as the quotient \( \Gamma \backslash E \) by a discrete subgroup \( \Gamma \subset \text{Aff}(E) \) acting properly and freely on \( E \). The affine holonomy representation

\[ \pi_1(M) \overset{\rho}{\hookrightarrow} \text{Aff}(E) \]

embeds \( \pi_1(M) \) onto \( \Gamma \).

Equivalently, an affine manifold is complete if and only if the corresponding affine connection is geodesically complete, that is, every geodesic extends infinitely in both directions.

A simple example of an incomplete affine structure on a closed manifold is a Hopf manifold \( M \), obtained as the quotient of \( \mathbb{R}^n \setminus \{0\} \) by a cyclic group \( \langle A \rangle \). Here the generator \( A \) must be a linear expansion, that is, an element \( A \in \text{GL}(n, \mathbb{R}) \) such that every eigenvalue has modulus \( > 1 \). Such a quotient is diffeomorphic to \( S^{n-1} \times S^1 \). A geodesic aimed at the origin winds seemingly faster and faster around the \( S^1 \)-factor, although it’s travelling with zero acceleration with respect to
If $M = \Gamma \backslash E$ is a complete affine manifold, then $\Gamma \subset \text{Aff}(E)$ is a discrete subgroup acting properly and freely on $E$. However, in the example above, $\langle A \rangle$ is a discrete subgroup which doesn’t act properly. A proper action of a discrete group is the usual notion of a properly discontinuous action. If the action is also free (that is, no fixed points), then the quotient is a (Hausdorff) smooth manifold, and the quotient map $E \rightarrow \Gamma \backslash E$ is a covering space. A properly discontinuous action whose quotient is compact as well as Hausdorff is said to be crystallographic, in analogy with the classical notion of a crystallographic group: A Euclidean crystallographic group is a discrete cocompact group of Euclidean isometries. Its quotient space is a Euclidean orbifold. Since such groups act isometrically on metric spaces, discreteness here does imply properness; this dramatically fails for more general discrete groups of affine transformations.

L. Auslander [6] claimed to prove that the Euler characteristic vanishes for a compact complete affine manifold, but his proof was flawed. It rested upon the following question, which in [35], was demoted to a “conjecture,” and is now known as the “Auslander Conjecture”:

**Conjecture 4.1.** Let $M$ be a compact complete affine manifold. Then $\pi_1(M)$ is virtually polycyclic.

In that case the affine holonomy group $\Gamma \cong \pi_1(M)$ embeds in a closed Lie subgroup $G \subset \text{Aff}(E)$ satisfying:

- $G$ has finitely many connected components;
- The identity component $G^0$ acts simply transitively on $E$.

Then $M = \Gamma \backslash E$ admits a finite covering space $M^0 := \Gamma^0 \backslash E$ where

$$
\Gamma^0 := \Gamma \cap G^0.
$$

The simply transitive action of $G^0$ define a complete left-invariant affine structure on $G^0$. (The developing map is just the evaluation map of this action.) Necessarily $G^0$ is a 1-connected solvable Lie group and $M^0$ is affinely isomorphic to the complete affine solvmanifold $\Gamma^0 \backslash G^0$. In particular $\chi(M^0) = 0$ and thus $\chi(M) = 0$.

This theorem is the natural extension of Bieberbach’s theorems describing the structure of flat Riemannian (or Euclidean) manifolds; see Milnor [59] for an exposition of this theory and its historical importance. Every flat Riemannian manifold is finitely covered by a flat torus, the quotient of $E$ by a lattice of translations. In the more general case, $G^0$ plays the role of the group of translations of an affine space and the solvmanifold $M^0$ plays the role of the flat torus. The
importance of Conjecture 4.1 is that it would provide a detailed and computable structure theory for compact complete affine manifolds.

Conjecture 4.1 was established in dimension 3 in Fried-Goldman [35]. The proof involves classifying the possible Zariski closures \( A(L(\Gamma)) \) of the linear holonomy group inside \( GL(E) \). Goldman-Kamishima [44] prove Conjecture 4.1 for flat Lorentz manifolds. Grunewald-Margulis [49] establish Conjecture 4.1 when the Levi component of \( L(\Gamma) \) lies in a real rank-one subgroup of \( GL(E) \). See Tomanov [70, 71] and Abels-Margulis-Soifer [2, 3, 4] for further results. The conjecture is now known in all dimensions \( \leq 6 \) (Abels-Margulis-Soifer [5]).

4.2. The Kostant-Sullivan theorem. Although Conjecture 4.1 remains unknown in general, the question which motivated it was proved by Kostant and Sullivan [52].

Theorem (Kostant-Sullivan). Let \( M \) be a compact complete affine manifold. Then \( \chi(M) = 0 \).

Their ingenious proof uses an elementary fact about free affine actions and Chern-Weil theory. The first step is that if \( \Gamma \subset Aff(E) \) is a group of affine transformations acting freely on \( E \), then the Zariski closure \( A(\Gamma) \) of \( \Gamma \) in \( Aff(E) \) has the property that every element \( g \in A(\Gamma) \) has 1 as an eigenvalue. To this end suppose that \( \Gamma \subset Aff(E) \) acts freely. Then solving for a fixed point

\[
\gamma(x) = L(\gamma)(x) + u(\gamma) = x
\]

implies that \( L(\gamma) \) has 1 as an eigenvalue for every \( \gamma \in \Gamma \). Thus every element \( \gamma \in \Gamma \) satisfies the polynomial condition

\[
\det(L(\gamma) - I) = 0
\]

which extends to the Zariski closure \( A(\Gamma) \) of \( \Gamma \) in \( Aff(E) \).

Next one finds a Riemannian metric (or more accurately, an orthogonal connection) to which Chern-Weil applies. Passing to a finite covering, using the finiteness of \( \pi_0(A(\Gamma)) \), we may assume the holonomy group lies in the identity component \( A(\Gamma)^0 \), which is a connected Lie group. Since every connected Lie group deformation retracts to a maximal compact subgroup, the structure group of \( TM \) reduces from \( \Gamma \) to a maximal compact subgroup \( K \subset A(\Gamma)^0 \). This reduction of structure group gives an orthogonal connection \( \nabla \) taking values in the Lie algebra \( \mathfrak{k} \) of \( K \). Since every compact group of affine transformations fixes a point, we may assume that \( K \subset GL(E) \). Since every element of \( A(\Gamma) \) (and hence \( K \)) has 1 as an eigenvalue, every element of \( \mathfrak{k} \) has determinant zero. Thus the Pfaffian polynomial (the square root of the determinant) vanishes on \( \mathfrak{k} \). Since the curvature of \( \nabla \) takes values
in \( \mathfrak{g} \), and the Euler form is the Pfaffian of the curvature tensor, the Euler form is zero. Now apply the Chern-Gauss-Bonnet theorem [26]. Integrating over \( M \) gives \( \chi(M) = 0 \), as claimed.

4.3. “On fundamental groups of complete affinely flat manifolds”. In his 1977 paper [60], Milnor set the record straight caused by the confusion surrounding Auslander’s flawed proof of Conjecture [41]. Influenced by Tits’s work [68] on free subgroups of linear groups and amenability, Milnor observed, that for an affine space \( \mathbb{E} \) of given dimension, the following conditions are all equivalent:

- Every discrete subgroup of \( \text{Aff}(\mathbb{E}) \) which acts properly on \( \mathbb{E} \) is amenable.
- Every discrete subgroup of \( \text{Aff}(\mathbb{E}) \) which acts properly on \( \mathbb{E} \) is virtually solvable.
- Every discrete subgroup of \( \text{Aff}(\mathbb{E}) \) which acts properly on \( \mathbb{E} \) is virtually polycyclic.
- A nonabelian free subgroup of \( \text{Aff}(\mathbb{E}) \) cannot act properly on \( \mathbb{E} \).
- The Euler characteristic \( \chi(\Gamma \mathbb{E}) \) (when defined) of a complete affine manifold \( \Gamma \mathbb{E} \) must vanish (unless \( \Gamma = \{1\} \) of course).
- A complete affine manifold \( \Gamma \mathbb{E} \) has finitely generated fundamental group \( \Gamma \).

(If these conditions were met, one would have a satisfying structure theory, similar to, but somewhat more involved, than the Bieberbach structure theory for flat Riemannian manifolds.)

In [60], Milnor provides abundant “evidence” for this “conjecture”. For example, the infinitesimal version: Namely, let \( G \subset \text{Aff}(\mathbb{E}) \) be a connected Lie group which acts properly on \( \mathbb{E} \). Then \( G \) must be an amenable Lie group, which simply means that it is a compact extension of a solvable Lie group. (Equivalently, its Levi subgroup is compact.) Furthermore, he provides a converse: Milnor shows that every virtually polycyclic group admits a proper affine action. (However, Milnor’s actions do not have compact quotient. Benoist [9] found finitely generated nilpotent groups which admit no affine crystallographic action. Benoist’s examples are 11-dimensional.)

However convincing as his “evidence” is, Milnor still proposes a possible way of constructing counterexamples:

“Start with a free discrete subgroup of \( \text{O}(2,1) \) and add translation components to obtain a group of affine transformations which acts freely. However it seems difficult to decide whether the resulting group action is properly discontinuous.”
This is clearly a geometric problem: As Schottky showed in 1907, free groups act properly by isometries on hyperbolic 3-space, and hence by diffeomorphisms of $\mathbb{E}^3$. These actions are not affine.

One might try to construct a proper affine action of a free group by a construction like Schottky’s. Recall that a Schottky group of rank $g$ is defined by a system of $g$ open half-spaces $H_1, \ldots, H_g$ and isometries $A_1, \ldots, A_g$ such that the $2g$ half-spaces

$$H_1, \ldots, H_g, A_1(H_1^c), \ldots, A_g(H_g^c)$$

are all disjoint (where $H^c$ denotes the complement of the closure $\bar{H}$ of $H$). The slab

$$\text{Slab}_i := H_i^c \cap A_i(H_i)$$

is a fundamental domain for the action of the cyclic group $\langle A_i \rangle$. The ping-pong lemma then asserts that the intersection of all the slabs

$$\Delta := \text{Slab}_1 \cap \cdots \cap \text{Slab}_g$$

is a fundamental domain for the group $\Gamma := \langle A_1, \ldots, A_g \rangle$. Furthermore $\Gamma$ is freely generated by $A_1, \ldots, A_g$. The basic idea is the following. Let $B_i^+ := A_i(H_i^c)$ (respectively $B_i^- := H_i$) denote the attracting basin for $A_i$ (respectively $A_i^{-1}$). That is, $A_i$ maps all of $H_i^c$ to $B_i^+$ and $A_i^{-1}$ maps all of $A_i(H_i)$ to $B_i^-$. Let $w(a_1, \ldots, a_g)$ be a reduced word in abstract generators $a_1, \ldots, a_g$, with initial letter $a_i^\pm$. Then

$$w(a_1, \ldots, a_g)(\Delta) \subset B_i^\pm.$$ 

Since all the basins $B_i^\pm$ are disjoint, $w(A_1, \ldots, A_g)$ maps $\Delta$ off itself. Therefore $w(A_1, \ldots, A_g) \neq 1$.

Freely acting discrete cyclic groups of affine transformations have fundamental domains which are parallel slabs, that is, regions bounded by two parallel affine hyperplanes. One might try to combine such slabs to form “affine Schottky groups”, but immediately one sees this idea is doomed, if one uses parallel slabs for Schottky’s construction: parallel slabs have disjoint complements only if they are parallel to each other, in which case the group is necessarily cyclic anyway. From this viewpoint, a discrete group of affine transformations seems very unlikely to act properly.

5. Margulis spacetimes

In the early 1980’s Margulis, while trying to prove that a nonabelian free group can’t act properly by affine transformations, discovered that discrete free groups of affine transformations can indeed act properly!

Around the same time, David Fried and I were also working on these questions, and reduced Milnor’s question in dimension three to what
seemed at the time to be one annoying case which we could not handle. Namely, we showed the following: Let $E$ be a three-dimensional affine space and $\Gamma \subset \text{Aff}(E)$. Suppose that $\Gamma$ acts properly on $E$. Then either $\Gamma$ is polycyclic or the restriction of the linear holonomy homomorphism

$$\Gamma \to \text{GL}(E)$$

discretely embeds $\Gamma$ onto a subgroup of $\text{GL}(E)$ conjugate to the orthogonal group $O(2,1)$.

In particular the complete affine manifold $M^3 = \Gamma \backslash E$ is a complete flat Lorentz 3-manifold after one passes to a finite-index torsionfree subgroup of $\Gamma$ to ensure that $\Gamma$ acts freely. In particular the restriction $L|_\Gamma$ defines a free properly discrete isometric action of $\Gamma$ on the hyperbolic plane $H^2$ and the quotient $\Sigma^2 := H^2/L(\Gamma)$ is a complete hyperbolic surface with a homotopy equivalence

$$M^3 := \Gamma \backslash E \cong H^2/L(\Gamma) =: \Sigma^2.$$ 

Already this excludes the case when $M^3$ is compact, since $\Gamma$ is the fundamental group of a closed aspherical 3-manifold (and has cohomological dimension 3) and the fundamental group of a hyperbolic surface (and has cohomological dimension $\leq 2$). This is a crucial step in the proof of Conjecture 4.1 in dimension 3.

That the hyperbolic surface $\Sigma^2$ is noncompact is a much deeper result due to Geoffrey Mess [56]. Later proofs and a generalization have been found by Goldman-Margulis [47] and Labourie [53]. (Compare the discussion in §5.3.) Since the fundamental group of a noncompact surface is free, $\Gamma$ is a free group. Furthermore $L|_\Gamma$ embeds $\Gamma$ as a free discrete group of isometries of hyperbolic space. Thus Milnor’s suggestion is the only way to construct nonsolvable examples in dimension three.

5.1. Affine boosts and crooked planes. Since $L$ embeds $\Gamma_0$ as the fundamental group of a hyperbolic surface, $L(\gamma)$ is elliptic only if $\gamma = 1$. Thus, if $\gamma \neq 1$, then $L(\gamma)$ is either hyperbolic or parabolic. Furthermore $L(\gamma)$ is hyperbolic for most $\gamma \in \Gamma_0$.

When $L(\gamma)$ is hyperbolic, $\gamma$ is an affine boost, that is, it has the form

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

in a suitable coordinate system. (Here the $3 \times 3$ matrix represents the linear part, and the column 3-vector represents the translational part.) $\gamma$ leaves invariant a unique (spacelike) line $C_\gamma$ (the second coordinate line in (11)). Its image in $E^3_1/\Gamma$ is a closed geodesic $C_\gamma/\langle \gamma \rangle$. Just as
for hyperbolic surfaces, most loops in $M^3$ are freely homotopic to such closed geodesics. (For a more direct relationship between the dynamics of the geodesic flows on $\Sigma^2$ and $M^3$, compare Goldman-Labourie [45]).

Margulis observed that $C_\gamma$ inherits a natural orientation and metric, arising from an orientation on $E$, as follows. Choose repelling and attracting eigenvectors $L(\gamma)^\pm$ for $L(\gamma)$ respectively; choose them so they lie in the same component of the nullcone. Then the orientation and metric on $C_\gamma$ is determined by a choice of nonzero vector $L(\gamma)^0$ spanning $\text{Fix}(L(\gamma))$. this vector is uniquely specified by requiring that:

- $L(\gamma)^0 \cdot L(\gamma)^0 = 1$;
- $(L(\gamma)^0, L(\gamma)^-, L(\gamma)^+)$ is a positively oriented basis.

The restriction of $\gamma$ to $C_\gamma$ is a translation by displacement $\alpha(\gamma)$ with respect to this natural orientation and metric.

Compare this to the more familiar geodesic length function $\ell(\gamma)$ associated to a class $\gamma$ of closed curves on the hyperbolic surface $\Sigma$. The linear part $L(\gamma)$ acts by transvection along a geodesic $c_{L(\gamma)} \subset H^2$. The quantity $\ell(\gamma) > 0$ measures how far $L(\gamma)$ moves points of $c_{L(\gamma)}$.

This pair of quantities $\left(\ell(\gamma), \alpha(\gamma)\right) \in \mathbb{R}_+ \times \mathbb{R}$ is a complete invariant of the isometry type of the flat Lorentz cylinder $E/\langle \gamma \rangle$. The absolute value $|\alpha(\gamma)|$ is the length of the unique primitive closed geodesic in $E/\langle \gamma \rangle$.

A fundamental domain is the parallel slab

$$(\Pi_{C_\gamma})^{-1} \left( p_0 + [0, \alpha(\gamma)] \gamma^0 \right)$$

where

$E \xrightarrow{\Pi_{C_\gamma}} C_\gamma$

denotes orthogonal projection onto

$C_\gamma = p_0 + \mathbb{R}\gamma^0$.

As noted above, however, parallel slabs can’t be combined to form fundamental domains for Schottky groups, since their complementary half-spaces are rarely disjoint.

In retrospect this is believable, since these fundamental domains are fashioned from the dynamics of the translational part (using the projection $\Pi_{C_\gamma}$). While the effect of the translational part is properness, the dynamical behavior affecting most points is influenced by the linear part: While points on $C_\gamma$ are displaced by $\gamma$ at a polynomial rate, all other points move at an exponential rate.

Furthermore, parallel slabs are less robust than slabs in $H^2$: while small perturbations of one boundary component extend to fundamental
domains, this is no longer true for parallel slabs. Thus one might look for other types of fundamental domains better adapted to the exponential growth dynamics given by the linear holonomy $L(\gamma)$.

Todd Drumm, in his 1990 Maryland thesis [29], defined more flexible polyhedral surfaces, which can be combined to form fundamental domains for Schottky groups of 3-dimensional affine transformations. A crooked plane is a PL surface in $\mathbb{E}$, separating $\mathbb{E}$ into two crooked half-spaces. The complement of two disjoint crooked halfspace is a crooked slab, which forms a fundamental domain for a cyclic group generated by an affine boost. Drumm proved the remarkable theorem that if $S_1, \ldots, S_g$ are crooked slabs whose complements have disjoint interiors, then given any collection of affine boosts $\gamma_i$ with $S_i$ as fundamental domain, then the intersection $S_1 \cap \cdots \cap S_g$ is a fundamental domain for $\langle \gamma_1, \ldots, \gamma_g \rangle$ acting on all of $\mathbb{E}$.

Modeling a crooked fundamental domain for $\Gamma$ acting on $\mathbb{E}$ on a fundamental polygon for $\Gamma_0$ acting on $\mathbb{H}^2$, Drumm proved the following sharp result:

**Theorem** (Drumm [29, 30]). Every noncocompact torsionfree Fuchsian group $\Gamma_0$ admits a proper affine deformation $\Gamma$ whose quotient is a solid handlebody.

(Compare also [31] [25].)

---

**Figure 1.** A crooked plane, and a family of three pairwise disjoint crooked planes.
5.2. **Marked length spectra.** We now combine the geodesic length function $\ell(\gamma)$ describing the geometry of the hyperbolic surface $\Sigma$ with the Margulis invariant $\alpha(\gamma)$ describing the Lorentzian geometry of the flat affine 3-manifold $M$.

As noted by Margulis, $\alpha(\gamma) = \alpha(\gamma^{-1})$, and more generally

$$\alpha(\gamma^n) = |n|\alpha(\gamma).$$

The invariant $\ell$ satisfies the same homogeneity condition, and therefore

$$\frac{\alpha(\gamma^n)}{\ell(\gamma^n)} = \frac{\alpha(\gamma)}{\ell(\gamma)}$$

is constant along hyperbolic cyclic subgroups. Hyperbolic cyclic subgroups correspond to periodic orbits of the geodesic flow $\phi$ on the unit tangent bundle $U\Sigma$. Periodic orbits, in turn, define $\phi$-invariant probability measures on $U\Sigma$. Goldman-Labourie-Margulis [46] prove that, for any affine deformation, this function extends to a continuous function $\Upsilon_\Gamma$ on the space $C(\Sigma)$ of $\phi$-invariant probability measures on $U\Sigma$. Furthermore when $\Gamma_0$ is convex cocompact (that is, contains no parabolic elements), then the affine deformation $\Gamma$ acts properly if and only if $\Upsilon_\Gamma$ never vanishes. Since $C(\Sigma)$ is connected, nonvanishing implies either all $\Upsilon_\Gamma(\mu) > 0$ or all $\Upsilon_\Gamma(\mu) < 0$. From this follows Margulis’s **Opposite Sign Lemma**, first proved in [54, 55] and extended to groups with parabolics by Charette and Drumm [20]:

**Theorem** (Margulis). If $\Gamma$ acts properly, then all of the numbers $\alpha(\gamma)$ have the same sign.

For an excellent treatment of the original proof of this fact, see the survey article of Abels [1].

5.3. **Deformations of hyperbolic surfaces.** The Margulis invariant may be interpreted in terms of deformations of hyperbolic structures as follows [17, 11].

Suppose $\Gamma_0$ is a Fuchsian group with quotient hyperbolic surface $\Sigma_0 = \Gamma_0 \backslash H^2$. Let $g_{\text{Ad}}$ be the $\Gamma_0$-module defined by the adjoint representation applied to the embedding $\Gamma_0 \hookrightarrow O(2,1)$. The coefficient module $g_{\text{Ad}}$ corresponds to the Lie algebra of right-invariant vector fields on $O(2,1)$ with the action of $O(2,1)$ by left-multiplication. Geometrically these vector fields correspond to the infinitesimal isometries of $H^2$.

A family of hyperbolic surfaces $\Sigma_t$ smoothly varying with respect to a parameter $t$ determines an **infinitesimal deformation**, which is a cohomology class $[u] \in H^1(\Gamma_0, g_{\text{Ad}})$. The cohomology group $H^1(\Gamma_0, g_{\text{Ad}})$ corresponds to **infinitesimal deformations** of the hyperbolic surface $\Sigma_0$. 


In particular the tangent vector to the path $\Sigma_t$ of marked hyperbolic structures smoothly varying with respect to a parameter $t$ defines a cohomology class $[u] \in H^1(\Gamma_0, \mathfrak{g}_{\text{Ad}})$.

The same cohomology group parametrizes affine deformations. The translational part $u$ of a linear representations of $\Gamma_0$ is a cocycle of the group $\Gamma_0$ taking values in the corresponding $\Gamma_0$-module $\mathcal{V}$. Moreover two cocycles define affine deformations which are conjugate by a translation if and only if their translational parts are cohomologous cocycles. Therefore translational conjugacy classes of affine deformations form the cohomology group $H^1(\Gamma_0, \mathcal{V})$. Inside $H^1(\Gamma_0, \mathcal{V})$ is the subset $\text{Proper}$ corresponding to proper affine deformations.

The adjoint representation $\text{Ad}$ of $O(2,1)$ identifies with the orthogonal representation of $O(2,1)$ on $\mathcal{V}$. Therefore the cohomology group $H^1(\Gamma_0, \mathcal{V})$ consisting of translational conjugacy classes of affine deformations of $\Gamma_0$ can be identified with the cohomology group $H^1(\Gamma_0, \mathfrak{g}_{\text{Ad}})$ corresponding to infinitesimal deformations of $\Sigma_0$.

**Theorem.** Suppose $u \in Z^1(\Gamma_0, \mathfrak{g}_{\text{Ad}})$ defines an infinitesimal deformation tangent to a smooth deformation $\Sigma_t$ of $\Sigma$.

- The marked length spectrum $\ell_t$ of $\Sigma_t$ varies smoothly with $t$.
- Margulis’s invariant $\alpha_u(\gamma)$ represents the derivative
  
  \[ \left. \frac{d}{dt} \right|_{t=0} \ell_t(\gamma) \]

- (Opposite Sign Lemma) If $[u] \in \text{Proper}$, then all closed geodesics lengthen (or shorten) under the deformation $\Sigma_t$.

Since closed hyperbolic surfaces do not support deformations in which every closed geodesic shortens, such deformations only exist when $\Sigma_0$ is noncompact. This leads to a new proof [17] of Mess’s theorem that $\Sigma_0$ is not compact. (For another, somewhat similar proof, which generalizes to higher dimensions, see Labourie [53].)

The tangent bundle $TG$ of any Lie group $G$ has a natural structure as a Lie group, where the fibration $TG \xrightarrow{\Pi} G$ is a homomorphism of Lie groups, and the tangent spaces

\[ T_xG = \Pi^{-1}(x) \subset TG \]

are vector groups. The deformations of a representation $\Gamma_0 \xrightarrow{\rho_0} G$ correspond to representations $\Gamma_0 \xrightarrow{\rho} TG$ such that $\Pi \circ \rho = \rho_0$. In our case, affine deformations of $\Gamma_0 \xhookrightarrow{} O(2,1)$ correspond to representations
in the tangent bundle $TO(2,1)$. When $G$ is the group $G(\mathbb{R})$ of $\mathbb{R}$-points of an algebraic group $G$ defined over $\mathbb{R}$, then

$$TG \cong G(\mathbb{R}[\epsilon])$$

where $\epsilon$ is an indeterminate with $\epsilon^2 = 0$. (Compare [41].) This is reminiscent of the classical theory of quasi-Fuchsian deformations of Fuchsian groups, where one deforms a Fuchsian subgroup of $\text{SL}(2,\mathbb{R})$ in

$$\text{SL}(2,\mathbb{C}) = \text{SL}(2,\mathbb{R}[i])$$

where $i^2 = -1$.

5.4. Classification. In light of Drumm’s theorem, classifying Margulis spacetimes $M^3$ begins with the classification of hyperbolic structures $\Sigma^2$. Thus the deformation space of Margulis spacetimes maps to the Fricke space $\mathfrak{F}(\Sigma)$ of marked hyperbolic structures on the underlying topology of $\Sigma$.

The main result of [46] is that the positivity (or negativity) of $\Upsilon_{\Gamma}$ on $\mathcal{C}(\Sigma)$ is necessary and sufficient for properness of $\Gamma$. (For simplicity we restrict ourselves to the case when $L(\Gamma)$ contains no parabolics — that is, when $\Gamma_0$ is convex cocompact.) Thus the proper affine deformation space $\text{Proper}$ identifies with the open convex cone in $H^1(\Gamma_0, V)$ defined by the linear functionals $\Upsilon_\mu$, for $\mu$ in the compact space $\mathcal{C}(\Sigma)$. These give uncountably many linear conditions on $H^1(\Gamma_0, V)$, one for each $\mu \in \mathcal{C}(\Sigma)$. Since the invariant probability measures arising from periodic orbits are dense in $\mathcal{C}(\Sigma)$, the cone $\text{Proper}$ is the interior of half-spaces defined by the countable set of functional $\Upsilon_\gamma$, where $\gamma \in \Gamma_0$.

The zero level sets $\Upsilon^{-1}_\gamma(0)$ correspond to affine deformations where $\gamma$ does not act freely. Therefore $\text{Proper}$ defines a component of the subset of $H^1(\Gamma_0, V)$ corresponding to affine deformations which are free actions.

Actually, one may go further. An argument inspired by Thurston [67], reduces the consideration to only those measures arising from multicurves, that is, unions of disjoint simple closed curves. These measures (after scaling) are dense in the Thurston cone $\mathcal{ML}(\Sigma)$ of measured geodesic laminations on $\Sigma$. One sees the combinatorial structure of the Thurston cone replicated on the boundary of $\text{Proper} \subset H^1(\Gamma_0, V)$. (Compare Figures 2 and 3.)
Two particular cases are notable. When Σ is a 3-holed sphere or a 2-holed cross-surface (real projective plane), then the Thurston cone degenerates to a finite-sided polyhedral cone. In particular properness is characterized by 3 Margulis functionals for the 3-holed sphere, and 4 for the 2-holed cross-surface. Thus the deformation space of equivalence classes of proper affine deformations is either a cone on a triangle or a convex quadrilateral, respectively.

When Σ is a 3-holed sphere, these functionals correspond to the three components of ∂Σ. The halfspaces defined by the corresponding three Margulis functionals cut off the deformation space (which is a polyhedral cone with 3 faces). The Margulis functionals for the other curves define halfspaces which strictly contain this cone.

When Σ is a 2-holed cross-surface these functionals correspond to the two components of ∂Σ and the two orientation-reversing simple closed
curves in the interior of $\Sigma$. The four Margulis functionals describe a polyhedral cone with 4 faces. All other closed curves on $\Sigma$ define halfspace strictly containing this cone.

In both cases, an ideal triangulation for $\Sigma$ models a crooked fundamental domain for $M$, and $\Gamma$ is an affine Schottky group, and $M$ is an open solid handlebody of genus 2 (Charette-Drumm-Goldman [21, 22, 23]). Fig. 2 depicts these finite-sided deformation spaces.

For the other surfaces where $\pi_1(\Sigma)$ is free of rank two (equivalently $\chi(\Sigma) = -1$), infinitely many functionals $\Upsilon_\mu$ are needed to define the deformation space, which necessarily has infinitely many sides. In these cases $M^3$ admits crooked fundamental domains corresponding to ideal triangulations of $\Sigma$, although unlike the preceding cases there is no single ideal triangulation which works for all proper affine deformations. Once again $M^3$ is a genus two handlebody. Fig. 3 depicts these infinite-sided deformation spaces.

5.5. An arithmetic example. These examples are everywhere. As often happens in mathematics, finding the first example of generic behavior can be quite difficult. However, once the basic phenomena are recognized, examples of this generic behavior abound. The following example, taken from [22], shows how a proper affine deformation sits inside the symplectic group $\text{Sp}(4, \mathbb{Z})$.

Begin with a 2-dimensional vector space $L_0$ over $\mathbb{R}$ with the group of linear automorphisms $\text{GL}(L_0)$. Let $V$ denote the vector space of symmetric bilinear maps $L_0 \times L_0 \rightarrow \mathbb{R}$ with the induced action of $\text{GL}(L)$. Identifying $V$ with symmetric $2 \times 2$ real matrices, the negative of the determinant defines an invariant Lorentzian inner product on $V$. In particular this defines a local embedding $\text{GL}(L_0) \rightarrow \text{O}(2, 1)$.

Let $L_\infty := L^*$ denote the vector space dual to $L_0$ and $W := L_0 \oplus L_\infty$ the direct sum. Then $W$ admits a unique symplectic structure $\omega$ such that $L_0$ and $L_\infty$ are Lagrangian subspaces and the restriction of $\omega$ to $L_0 \times L_\infty$ is the duality pairing. Let $\text{Sp}(4, \mathbb{R})$ denote the group of linear symplectomorphisms of $(W, \omega)$. It acts naturally on the homogeneous space $\mathcal{L}(W, \omega)$ of Lagrangian 2-planes $L$ in $(W, \omega)$.

The Minkowski space $E$ associated to $V$ consists of Lagrangians $L \in \mathcal{L}(W, \omega)$ which are transverse to $L_\infty$. This is a torsor for the Lorentzian vector space $V$ as follows. $V$ consists of symmetric bilinear forms on $L_0$, and these can be identified with self-adjoint linear maps

$$L_0 \rightarrow L_\infty \cong (L_0)^*.$$
A 2-dimensional linear subspace of $V$ which is transverse to $L_\infty$ is the graph $L := \text{graph}(f)$ of a linear map $L_0 \xrightarrow{f} L_\infty$. Moreover $L$ is Lagrangian if and only if $f$ is self-adjoint. Furthermore, since $V$ is a vector space, it acts simply transitively on the space $E$ of such graphs by addition. In terms of $4 \times 4$ symplectic matrices ($2 \times 2$ block matrices using the decomposition $W = L_0 \oplus L_\infty$), these translations correspond to shears:

$$
(12) \quad \begin{bmatrix} I_2 & f_2 \\ 0 & I_2 \end{bmatrix}
$$

where the corresponding symmetric $2 \times 2$ matrix corresponding to $f$ is denoted $f_2$. The corresponding subgroup of $\text{Sp}(4, \mathbb{R})$ consists of linear symplectomorphisms of $(W, \omega)$ which preserve $L_\infty$, and act identically both on $L_\infty$ and on its quotient $W/L_\infty$.

As the translations of $E$ are represented by shears in block upper-triangular form $[12]$, the linear isometries are represented by the block diagonal matrices arising from $\text{SL}(L_0)$. More generally, the Lorentz similarities of $E$ correspond to $\text{GL}(L_0)$ as follows. A linear automorphism $L_0 \xrightarrow{g} L_0$ induces a linear symplectomorphism $g \oplus (g^\dagger)^{-1}$ of $W = L_0 \oplus L_\infty$:

$$
g \oplus (g^\dagger)^{-1} = \begin{bmatrix} g & 0 \\ 0 & (g^\dagger)^{-1} \end{bmatrix}.
$$

These linear symplectomorphisms can be characterized as those which preserve the Lagrangian 2-planes $L_0$ and $L_\infty$. Furthermore $g$ induces an isometry of $E$ with the flat Lorentzian structure if and only if $\text{Det}(g) = \pm 1$.

Here is our example. The level two congruence subgroup $\Gamma_0$ is the subgroup of $\text{GL}(2, \mathbb{Z})$ generated by

$$
\begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}
$$

and the corresponding hyperbolic surface is a triply punctured sphere. For $i = 1, 2, 3$ choose three positive integers $\mu_1, \mu_2, \mu_3$ (the coordinates of the translational parts). Then the subgroup $\Gamma$ of $\text{Sp}(4, \mathbb{Z})$ generated by

$$
\begin{bmatrix} -1 & -2 & \mu_1 + \mu_2 - \mu_3 & 0 \\ 0 & -1 & 2\mu_1 & -\mu_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & -\mu_2 & -2\mu_2 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}
$$

defines a affine deformation of a $\Gamma_0$. 
By the main result of Charette-Drumm-Goldman [21], this affine deformation is proper with a fundamental polyhedron bounded by crooked planes. The quotient 3-manifold $M^3 = \Gamma \backslash \mathbb{E}$ is homeomorphic to a genus two handlebody. Fig. 4 depicts the intersections of crooked fundamental domains for this group (when $\mu_1 = \mu_2 = \mu_3 = 1$) with a spacelike plane. Note the parallel line segments cutting off fundamental domains for the cusps of $\Sigma$; the parallelism results from the parabolicity of the holonomy around the cusps.

Figure 4. A proper affine deformation of level two congruence subgroup of $\text{SL}(2, \mathbb{Z})$. 
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