FROBENIUS-SCHUR INDICATORS OF CHARACTERS IN BLOCKS WITH CYCLIC DEFECT

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Abstract. Let $p$ be an odd prime and let $B$ be a $p$-block of a finite group which has cyclic defect groups. We show that all exceptional characters in $B$ have the same Frobenius-Schur indicators. Moreover the common indicator can be computed, using the canonical character of $B$. We also investigate the Frobenius-Schur indicators of the non-exceptional characters in $B$.

For a finite group which has cyclic Sylow $p$-subgroups, we show that the number of irreducible characters with Frobenius-Schur indicator $-1$ is greater than or equal to the number of conjugacy classes of weakly real $p$-elements in $G$.

1. Introduction and preliminary results

The Frobenius-Schur (F-S) indicator of an ordinary character $\chi$ of a finite group $G$ is

$$\epsilon(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

If $\chi$ is irreducible then $\epsilon(\chi) = 0, \pm 1$. Moreover $\epsilon(\chi) \neq 0$ if and only if $\chi$ is real-valued.

R. Brauer showed how to partition the irreducible characters of $G$ into $p$-blocks, for each prime $p$. Each $p$-block has an associated defect group, which is a $p$-subgroup of $G$, unique up to $G$-conjugacy, which determines much of the structure of the block. If the defect group is trivial, the block contains a unique irreducible character. In the next most complicated case, E. Dade [D] determined the structure of a block which has a cyclic defect group and defined the Brauer tree of the block.

Recall that a $p$-block is said to be real if it contains the complex conjugates of its characters. We wish to determine the F-S indicators of the irreducible characters in a real $p$-block which has a cyclic defect group. In [M2 Theorem 1.6] we dealt with the case $p = 2$; there are six possible indicator patterns, and the extended defect group of the block determines which occurs. In this paper we consider the case $p \neq 2$.

R. Gow showed [G, 5.1] that a real $p$-block has a real irreducible character, if $p = 2$. This is false for $p \neq 2$, as was first noticed by H. Blau in the early 1980’s, in response to a question posed by Gow. His example was for $p = 5$ and $G = 6.S_6$ (Atlas notation). G. Navarro has recently found a solvable example with $p = 3$ and $G = SmallGroup(144, 131)$ (GAP notation). We give examples for blocks with cyclic defect below.

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Now let $B$ be a real $p$-block which has a cyclic defect group $D$. The inertial index of $B$ is a certain divisor $e$ of $p - 1$. Dade showed that $B$ has $e$ irreducible Brauer characters and $e + \frac{|D| - 1}{e}$ ordinary irreducible characters. The latter he divided into $\frac{|D| - 1}{e}$ exceptional characters and $e$ non-exceptional characters.

Suppose that $\frac{|D| - 1}{e} = 1$ (which can only occur when $|D| = p$). Then the choice of exceptional character is arbitrary, and the convention in [F] is to regard $B$ as having no exceptional characters. However, we will see that in this event $B$ has real irreducible characters, all of which have the same F-S indicators. So our convention is to assume that $B$ has a real exceptional character.

The Brauer tree of $B$ is a planar graph which describes the decomposition matrix of $B$. There is one exceptional vertex, representing all the exceptional characters, and one vertex for each of the non-exceptional characters. Two vertices are connected by an edge if their characters share a modular constituent.

J. Green [Gr] showed that all real objects in the Brauer tree lie on a line segment, now called the real-stem of $B$. The exceptional vertex belongs to the real-stem (see Lemma 6 below). So it divides the real non-exceptional vertices into two, possibly empty, subsets. We find it convenient to refer to the corresponding real non-exceptional characters as being on the left or the right of the exceptional vertex. Here is our main theorem:

**Theorem 1.** Let $p$ be an odd prime and let $B$ be a real $p$-block which has a cyclic defect group. Then

(i) All exceptional characters in $B$ have the same F-S indicators.

(ii) On each side of the exceptional vertex, the real non-exceptional characters have the same F-S indicators.

(iii) If $B$ has a real exceptional character then all real irreducible characters in $B$ have the same F-S indicators.

(iv) Suppose that $B$ has no real exceptional characters, and that there are an odd number of non-exceptional vertices on each side of the exceptional vertex. Then the real non-exceptional characters have F-S indicator $+1$ on one side of the exceptional vertex and $-1$ on the other side.

Note that (i) is not a consequence of Galois conjugacy, as there are at least two Galois conjugacy classes of exceptional characters, when $|D| > p$.

In Proposition [13] we show that the F-S indicators of the exceptional characters in $B$ agree with those of the Brauer corresponding block in the normalizer of a defect group. In Theorem [16] we compute this common indicator using the ‘canonical character’ of $B$.

Next recall that an element of $G$ is said to be weakly real if it is conjugate to its inverse in $G$, but it is not inverted by any involution in $G$. Here is an application of Theorem [1] whose statement does not refer to blocks or to modular representation theory:

**Theorem 2.** Let $p$ be an odd prime and let $G$ be a finite group which has cyclic Sylow $p$-subgroups. Then the number of irreducible characters of $G$ with F-S indicator $-1$ is greater than or equal to the number of conjugacy classes of weakly real $p$-elements in $G$. 
We use the notation and results of [NT] for group representation theory, and use [D] and [F, VII] for notation specific to blocks with cyclic defect. When referring to the character tables of a finite simple group we use the conventions of the ATLAS [A]. For other character tables, we use the notation of the computer algebra system GAP [GAP].

2. Examples

We begin with a number of examples which illustrate the possible patterns of F-S indicators in a block which has a cyclic defect group. Throughout $G$ is a finite group and $B$ is a real $p$-block of $G$ which has a cyclic defect group $D$. Also $N_0$ is the normalizer in $G$ of the unique order $p$ subgroup of $D$ and $B_0$ is the Brauer correspondent of $B$ in $N_0$.

Example 1: There are many blocks with cyclic defect group whose irreducible characters all have the same F-S indicators. For blocks with all indicators $+1$, choose $n \geq 2$, a prime $p$ with $n/2 \leq p \leq n$ and any $p$-block of the symmetric group $S_n$. There are numerous blocks with all indicators $-1$ among the faithful $p$-blocks of the double cover $2.A_n$ of an alternating group, with $n/2 \leq p \leq n$ e.g. the four faithful irreducible characters of $2.A_5$ have F-S indicator $-1$ and constitute a 5-block with a cyclic defect group.

Example 2: If $e$ is odd then $B$ has a real non-exceptional character. Now it follows from [D, Part 2 of Theorem 1 & Corollary 1.9] that $B$ has a Galois conjugacy class consisting of $\frac{p-1}{e}$ exceptional characters. So $B$ has a real exceptional character if $\frac{p-1}{e}$ is odd. Thus $B$ always has a real irreducible character if $p \equiv 3 \pmod{4}$.

When $e$ is even and $p \equiv 1 \pmod{4}$, $B$ may have no real irreducible characters. For example SmallGroup(80, 29) = $\langle a, b \mid a^{20}, a^{10} = b^4, a^b = a^7 \rangle$ has such a block, for $p = 5$. It consists of the four irreducible characters lying over the non-trivial irreducible character of $2.A_5$ have F-S indicator $-1$ and constitute a 5-block with a cyclic defect group.

Example 3: $B$ may have a real non-exceptional character but no real exceptional characters. For example SmallGroup(60, 7) = $\langle a, b \mid a^{15}, b^4, a^b = a^2 \rangle$ has such a block, for $p = 5$. It consists of the four irreducible characters lying over a non-trivial irreducible
character of $\langle a^5 \rangle$. This is also an example of part (iv) of Theorem 1: the non-exceptional characters $X.5$ and $X.6$ have F-S indicators $-1$ and $+1$, respectively. Here is the table of character values, with $\alpha = (1 + \sqrt{-15})/2$:

|     | 2   | 2   | 1   | 2   | 2   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1a  | 2a  | 3a  | 4a  | 4b  | 5a  | 6a  | 15a | 15b |
| X.5 | 2   | -2  | -1  | .   | .   | 2   | 1   | -1  | -1  | 1   |
| X.6 | 2   | 2   | -1  | .   | .   | 2   | -1  | -1  | -1  | 2   |
| X.8 | 4   | -2  | .   | -1  | .   | 2   | 1   | -1  | -1  | 2   |
| X.9 | 4   | -2  | .   | -1  | .   | 2   | 1   | -1  | -1  | 2   |

Example 4: There is no apparent relationship between the F-S indicators of the non-exceptional characters in $B$ and in $B_0$. For example, let $B$ be the 5-block $2.A_8$ with $\text{Irr}(B) = \{\chi_{15}, \chi_{19}, \chi_{21}, \chi_{22}\}$. Then the two non-exceptional characters $\chi_{15}$ and $\chi_{19}$ have F-S indicator $+1$ and $-1$, respectively. However $B_0$ is a real block which has no real irreducible characters.

The character table of $B$ can be found on p22 of The Atlas. Now $N_0$ is isomorphic to $\text{SmallGroup}(120,7) = \langle a, b \mid a^{15}, b^8, a^b = a^2 \rangle$. Here is the table of character values of its 5-block $B_0$. Again $\alpha = (1 + \sqrt{-15})/2$. In order to save space, we have omitted 4 columns of zero values for the four classes of elements of order 8:

|     | 2   | 3   | 3   | 2   | 3   | 3   | 1   | 2   | 1   | 2   | 2   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1a  | 2a  | 3a  | 4a  | 4b  | 5a  | 6a  | 10a | 12a | 12b | 15a | 15b | 30a | 30b |
| X.11| 2   | -2  | -1  | 2i  | -2i | 2   | 1   | -2  | -i  | i   | -1  | 1   | -1  | 1   | 1   | 1   | 1   | 1   | 1   |
| X.12| 2   | -2  | -1  | -2i | 2i  | 2   | 1   | -2  | i   | -i  | -1  | 1   | -1  | 1   | 1   | 1   | 1   | 1   | 1   |
| X.15| 4   | -4  | -2  | .   | .   | 2   | 1   | -1  | -1  | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| X.16| 4   | -4  | -2  | .   | .   | 2   | -1 | 1   | 1   | 1   | .   | .   | .   | .   | .   | .   | .   | .   | .   |

We note that $B$ has 2 irreducible modules and 2 weights, in conformity with Alperin's weight conjecture. However the irreducible modules are self-dual and the weights are duals of each other. This shows that there is no obvious 'real' version of the weight conjecture for $p$-blocks, when $p \neq 2$.

Consider the inclusion of groups $N_0 < \text{PSL}_2(11) < M_{11}$, where $N_0 \cong 11 : 5$. The principal $11$-blocks each have 5 non-exceptional characters. It is somewhat surprising that the number of real non-exceptional characters in these blocks is 1, 5 and 3, respectively.
Example 5: Finally $B$ may have a real exceptional character but no real non-exceptional characters. For example let $B$ be the 5-block containing the four faithful irreducible characters of SmallGroup(20,1) = $\langle a, b \mid a^5, b^4, a^b = a^{-1} \rangle$. The two exceptional characters have F-S indicators $-1$, but neither of the two non-exceptional characters is real. Here is the character table of $B$, with $\beta = (-1 + \sqrt{5})/2$ and $*\beta = (-1 - \sqrt{5})/2$:

|    | 2  | 2  | 2  | 2  | 1  | 1  | 1  | 1  | 1  |
|----|----|----|----|----|----|----|----|----|----|
| 1a | 2a | 4a | 4b | 5a | 5b | 10a| 10b|



$X.3$ 1 -1 1 1 1 1 1 1
$X.4$ 1 -1 -i 1 1 1 1 1
$X.5$ 2 -2 $\beta$ -i -1 -1 -1 -1
$X.6$ 2 -2 $\beta$ -i -1 -1 -1 -1

3. Miscellaneous results

We need general results from representation theory, some of which are not so well-known. So in this section $p$ is a prime and $B$ is a $p$-block of a finite group $G$.

Let $\chi$ be an irreducible character in $B$, let $x$ be a $p$-element of $G$ and let $y$ be a $p$-regular element of $C_G(x)$. Then

$$\chi(xy) = \sum_\varphi d^{(x)}_{\chi, \varphi}(y),$$

where $\varphi$ ranges over the irreducible Brauer characters in blocks of $C_G(x)$ which Brauer induce to $B$, and each $d^{(x)}_{\chi, \varphi}$ is an algebraic integer, called a generalized decomposition number; if $x = 1$, $\varphi$ is an irreducible Brauer character in $B$ and $d^{(x)}_{\chi, \varphi}$ is simplified to $d_{\chi, \varphi}$. It is an integer called an ordinary decomposition number of $B$.

Brauer [B Theorem (4A)] used his Second Main Theorem to prove the following remarkable ‘local-to-global’ formula for F-S indicators:

$$\sum_\chi \epsilon(\chi)d^{(x)}_{\chi, \varphi}(y) = \sum_\psi \epsilon(\psi)d^{(x)}_{\psi, \varphi},$$

where $\chi$ ranges over the irreducible characters in $B$ and $\psi$ ranges over the irreducible characters in blocks of $C_G(x)$ which Brauer induce to $B$. We have previously used this formula to determine the F-S indicators of the irreducible characters in 2-blocks with a cyclic, Klein-four or dihedral defect group.

Our next result relies on Clifford theory. However it was inspired by (and can be proved using) the notion of a weakly real 2-block, as introduced in [MII]. Suppose that $N$ is a normal subgroup of $G$ and $\phi \in \text{Irr}(N)$, with stabilizer $G_\phi$ in $G$. If $G_\phi \subseteq H \subseteq G$, the Clifford correspondence is a bijection $\text{Irr}(G \mid \phi) \leftrightarrow \text{Irr}(H \mid \phi)$ such that $\chi \leftrightarrow \psi$ if and only if $\langle \chi_H, \phi \rangle \neq 0$ or $\chi = \psi^H$. The stabilizer of $\{\phi, \varphi\}$ in $G$ is called the extended stabilizer of $\phi$, here denoted by $G^*_\phi$. So $|G^*_\phi : G_\phi| \leq 2$, with equality if and only if $\phi \neq \varphi$ but $\phi$
is $G$-conjugate to $\overline{\phi}$. If $G^*_\phi \subseteq H$ it is easy to see that $\chi$ is real if and only if $\psi$ is real. Moreover in this case $\epsilon(\chi) = \epsilon(\psi)$.

We need one other idea. Suppose that $T$ is a degree 2 extension of $G$. Then the Gow indicator $[\mathbb{G}, 2.1]$ of a character $\chi$ of $G$ with respect to $T$ is defined to be

$$\epsilon_{T/\mathbb{G}}(\chi) := \frac{1}{|\mathbb{G}|} \sum_{t \in T/\mathbb{G}} \chi(t^2).$$

Clearly $\epsilon(\chi^T) = \epsilon(\chi) + \epsilon_{T/\mathbb{G}}(\chi)$. Just like the F-S indicator, $\epsilon_{T/\mathbb{G}}(\chi) = 0, \pm 1$, for each $\chi \in \operatorname{Irr}(\mathbb{G})$. Moreover $\epsilon_{T/\mathbb{G}}(\chi) \neq 0$ if and only if $\chi$ is $T$-conjugate to $\overline{\chi}$.

**Lemma 3.** Let $N$ be a normal odd order subgroup of $G$ and let $\phi \in \operatorname{Irr}(N)$. Suppose that $G^*_\phi$ does not split over $G_\phi$. Then there exists $\chi \in \operatorname{Irr}(G \mid \phi)$ such that $\epsilon(\chi) = -1$.

**Proof.** We first show that there exists $\psi \in \operatorname{Irr}(G \mid \phi)$ such that $\epsilon(\psi) = +1$. For let $S$ be a Sylow 2-subgroup of $G$. As $\phi^G$ vanishes on the 2-singular elements of $G$, we have $\phi^G|_S = \frac{|\phi(1)||G|}{|N||S|} \rho_S$, where $\rho_S$ is the regular character of $S$. Now $\frac{|\phi(1)||G|}{|N||S|}$ is an odd integer. So $\langle (\phi^G)|_S, 1_S \rangle$ is odd. Moreover $\phi^G$ is a real character of $G$. So $\langle (\phi^G), \psi \rangle = \langle (\phi^G), \overline{\chi} \rangle$, for each $\psi \in \operatorname{Irr}(G)$. Pairing each irreducible character of $G$ with its complex conjugate, we see that there exists a real-valued $\psi \in \operatorname{Irr}(G \mid \phi)$ such that $\langle \psi|_S, 1_S \rangle$ is odd. Then $\epsilon(\psi) = \epsilon(1_S) = +1$.

Following the discussion before the lemma, we may assume that $G = G^*_\phi$. So $|G : G_\phi| = 2$. Next suppose that $g \in G$ and $\phi^G_\phi(g^2) \neq 0$. Write $g = xy = yx$, where $x$ is a 2-element and $y$ is a 2-regular element. Then $g^2 = x^2y^2$. As $\phi^G_\phi$ vanishes off $N$, we have $x^2 = 1$ and $y^2 \in N$. So $x \in G_\phi$, as $G_\phi$ contains all involutions in $G$. Moreover $y \in N$, as $y$ has odd order. Thus $g \in G_\phi$, whence

$$\epsilon_{G/G_\phi}(\phi^G_\phi) = \frac{1}{|G_\phi|} \sum_{g \in G \setminus G_\phi} \phi^G_\phi(g^2) = 0.$$

Now $\operatorname{Irr}(G_\phi \mid \phi)$ contains no real characters, as $\phi \neq \overline{\phi}$. So $\epsilon(\phi^G) = \epsilon_{G/G_\phi}(\phi^G_\phi) + \epsilon(\phi^G_\phi) = 0$. Equivalently

$$\sum_{\chi \in \operatorname{Irr}(G)} \langle \phi^G, \chi \rangle \epsilon(\chi) = 0.$$

Together with the fact that $\langle \phi^G, \psi \rangle \epsilon(\psi) > 0$, this implies that $\langle \phi^G, \chi \rangle \epsilon(\chi) < 0$, for some $\chi \in \operatorname{Irr}(G)$. Thus $\chi \in \operatorname{Irr}(G \mid \phi)$ and $\epsilon(\chi) = -1$, which completes the proof. $\square$

It is well-known that each $G$-invariant irreducible character of a normal subgroup of $G$ extends to $G$, when the quotient group is cyclic.

**Lemma 4.** Suppose that $N$ is a normal subgroup of $G$ such that $G/N$ is cyclic and of even order. Let $\varphi \in \operatorname{Irr}(N)$ be real and $G$-invariant. Then $\varphi$ has a real extension to $G$ if and only if $\varphi$ has a real extension to $T$, where $N \subseteq T \subseteq G$ and $T/N$ has order 2.
Proof. The ‘only if’ part is obvious. So assume that $\varphi$ has a real extension to $T$. Then both extensions of $\varphi$ to $T$ are real. Let $\omega$ be a generator of the abelian group $\text{Irr}(G/N)$ and let $\chi$ be any extension of $\varphi$ to $G$. Then $\omega^i \chi$, $i \geq 0$ give all extensions of $\varphi$ to $G$. Here $\omega^i = \omega^j$ if and only if $i \equiv j \pmod{|G/N|}$.

As $\chi$ lies over $\varphi$, we have $\chi = \omega^i \chi$, for some $i \geq 0$. Now $\chi \downarrow_T$ is an extension of $\varphi$ to $T$ and $\chi \downarrow_T = (\omega^i \downarrow_T)(\chi \downarrow_T)$. As $\chi \downarrow_T$ is real, it follows that $\omega^i \downarrow_T$ is trivial. So $T \subseteq \ker(\omega^i)$, whence $i \equiv 2j \pmod{|G/N|}$, for some $j \geq 0$. Now $\omega^j \chi = \omega^{i-j} \chi = \omega^i \chi$. So $\omega^i \chi$ is a real extension of $\varphi$ to $G$. $\Box$

Notice that in this context $\varphi$ has a real extension to $T$ if and only if $\epsilon(\varphi) = \epsilon_T/N(\varphi)$.

When $G/N$ has even order, but is not cyclic, and $\varphi$ is a real irreducible character of $N$ which extends to $G$, it is not clear whether there is a sensible sufficient criteria for $\varphi$ to have a real extension to $G$.

Finally we need the following consequence of the first orthogonality relation:

Lemma 5. Let $W \subseteq X \subseteq Y$ be finite abelian groups. Then for $\lambda \in \text{Irr}(Y)$ we have

$$\sum_{x \in X \setminus W} \lambda(x) = \begin{cases} |X| - |W|, & \text{if } X \subseteq \ker(\lambda), \\ -|W|, & \text{if } W \subseteq \ker(\lambda) \text{ but } X \nsubseteq \ker(\lambda), \\ 0, & \text{if } W \nsubseteq \ker(\lambda). \end{cases}$$

4. The Brauer tree and its real-stem

From now on $G$ is a finite group, $p$ is an odd prime and $B$ is a real $p$-block of $G$ which has a cyclic defect group. To avoid trivialities we assume that the defect group is non-trivial.

Dade asserts [D, Theorem 1, Part 2] that each decomposition number in $B$ is either 0 or 1. The Brauer tree of $B$ is a planar graph with edges labelled by the irreducible Brauer character in $B$ and with vertices labelled by the irreducible characters in $B$ (the exceptional characters in $B$ label a single ‘exceptional’ vertex). The edge labelled by an irreducible Brauer character $\theta$ meets the vertex labelled by an irreducible character $\chi$ if and only if the decomposition number $d_{\chi,\theta}$ is not 0.

When $B$ is real, complex conjugation acts on the Brauer tree of $B$, and in particular fixes the exceptional vertex. However, as we have seen in Examples 2,3 and 4 above, $B$ may have no real exceptional characters. So we restate [F, VII,9.2] in the following more precise fashion:

Lemma 6. The subgraph of the Brauer tree of $B$ consisting of the exceptional vertex and those vertices and edges which correspond to real characters and Brauer characters is a straight line segment.

Feit calls this line segment the real-stem of $B$. An easy consequence is:

Corollary 7. The number of real non-exceptional characters in $B$ equals the number of real irreducible Brauer characters in $B$. 

Proof. Suppose that $B$ has $r$ real irreducible Brauer characters. Then the real-stem of the Brauer tree has $r$ edges and $r + 1$ vertices. One of these is the exceptional vertex. So $B$ has $r$ real non-exceptional characters.

Let $\theta$ be a real irreducible $p$-Brauer character of a finite group $G$. As $p$ is odd, the $G$-representation space of $\theta$ affords a non-degenerate $G$-invariant bilinear form which is either symmetric or skew-symmetric. Given the symmetry groups of such forms, we refer to $\theta$ as being of orthogonal or symplectic type. Thompson and Willems \cite{W, 2.8} proved that there is a real irreducible character $\chi$ of $G$ such that $d_{\chi, \theta}$ is odd. Moreover $\theta$ has orthogonal type if $\epsilon(\chi) = +1$ or symplectic type if $\epsilon(\chi) = -1$. This implies that $\epsilon(\psi) = \epsilon(\chi)$, for all real irreducible characters $\psi$ such that $d_{\psi, \theta}$ is odd.

Proof of part (ii) of Theorem 1. Let $X$ and $Y$ be real non-exceptional characters which lie on the same side of the exceptional vertex in the real-stem of $B$. Then by Lemma there is a sequence $X = X_0, X_1, \ldots, X_n = Y$ of real non-exceptional characters and a sequence $\theta_1, \ldots, \theta_n$ of real irreducible Brauer characters such that $d_{X_{i-1}, \theta_i} = d_{X_i, \theta_i}$, for $i = 1, \ldots, n$. The Thompson-Willems result implies that $\epsilon(X_{i-1}) = \epsilon(X_i)$, for $i = 1, \ldots, n$. So $\epsilon(X) = \epsilon(Y)$. This gives part (ii) of Theorem 1.

A similar argument gives the following weak form of parts (i) and (iii) of Theorem 1.

Lemma 8. If $B$ has a real exceptional character and a real non-exceptional character, then all real irreducible characters in $B$ have the same $F$-$S$ indicators.

Notice that if $B$ is the principal $p$-block of a group with a cyclic Sylow $p$-subgroup, and $B$ has an irreducible character with $F$-$S$ indicator $-1$ (e.g. the principal 7-block of $U(3, 3)$) then the lemma implies that $B$ has no real exceptional characters.

5. The Exceptional Characters

We outline some results from \cite{D} using the language of subpairs. See \cite{NT, Chapter 5.9} for a full description of the theory. We then prove results about the local blocks in $B$, in Proposition 10 and the exceptional characters in $B$, in Proposition 11. This allows us to prove parts (i), (iii) and (iv) of Theorem 1.

Recall that $B$ is a $p$-block with a non-trivial cyclic defect group $D$. Write $|D| = p^a$, where $a > 0$, and let $1 \subset D_{a-1} \subset D_{a-2} \subset \cdots \subset D_1 \subset D_0 = D$ be the complete list of subgroups of $D$. So $[D : D_i] = p^i$, for $i = 0, \ldots, a - 1$. Set $C_i = C_G(D_i)$ and $N_i = N_G(D_i)$. So $C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{a-1}$, and $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{a-1}$.

As $p$ is odd, $\text{Aut}(D_i)$ is a cyclic group of order $p^{a-i-1}(p-1)$. So $N_i/C_i$ is a cyclic group whose order divides $p^{a-i-1}(p-1)$. Moreover the centralizer of $D_i$ in $\text{Aut}(D)$ has order $p^i$. So $C_i \cap N_0/C_0$ is a cyclic $p$-group. We note that the unique involution in $\text{Aut}(D)$ inverts every element of $D$.

Fix a Sylow $B$-subpair $(D, b_0)$. So $b_0$ is a $p$-block of $C_0$ such that $b_0^G = B$ and the pair $(D, b_0)$ is uniquely determined up to $G$-conjugacy. Set $b_i := b_0^{C_i}$, for $i = 1, \ldots, a - 1$. Then
by [NT 5.9.3] the lattice of \(B\)-subpairs contained in \((D, b_0)\) is
\[
(1, b) \subset (D_{a-1}, b_{a-1}) \subset \cdots \subset (D_1, b_1) \subset (D, b_0).
\]

Set \(E := N(D, b_0)\), the stabilizer of \(b_0\) in \(N_0\). Then \(e := |E : C_0|\) is called the inertial index of \(B\). Now \(p \nmid e\), by Brauer’s extended first main theorem. So \(e \mid (p - 1)\). Let \(x \in E\). Then \(D_{\ell}^x = D_{\ell}\). As \((D_{\ell}, b_0), (D_{\ell}, b_{\ell}^x) \subset (D, b_0)\), it follows from (2) that \(b_{\ell}^x = b_{\ell}\).

So \(EC_{i'} \subseteq N(D_{\ell}, b_i)\). Conversely let \(n \in N(D_{\ell}, b_i)\). As \((D, b_0)\) and \((D, b_0)^n\) are Sylow \(b_i\)-subpairs (in the group \(C_i\)), there is \(c \in C_i\) such that \(nc_i \in E\). This shows that \(N(D_{\ell}, b_i) \subseteq EC_i\). This recovers Dade’s observation that \(N(D_{\ell}, b_i) = EC_i\).

Now \(E \cap C_i / C_0\) is a subgroup of \(C_i \cap N_0 / C_0\) and a quotient of the group \(E / C_0\). As \(C_i \cap N_0 / C_0\) is a \(p\)-group and \(E / C_0\) has \(p^e\)-order, we deduce that \(E \cap C_i = C_0\). It follows from this \(EC_{i'} / C_i \cong E / C_0\), and in particular \(|EC_{i'} : C_i| = e\).

By [D, Theorem 1, Part 1] \(B\) has \(e\) irreducible Brauer characters, listed as \(\chi_1, \ldots, \chi_e\). Each \(b_i\) has inertial index 1. So \(b_i\) has a unique irreducible Brauer character, denoted \(\varphi_i\).

From the above discussion there are \(|N_i : EC_i| = \frac{|N_i : C_i|}{e}\) distinct blocks of \(C_i\) which induce to \(B\), namely \(b_{\ell}^x\) as \(\tau\) ranges over \(N_i / EC_i\). Also there are \(p^{\ell-1} - p^{\ell-1} / e\) conjugacy classes of \(G\) which contain a generator of \(D_{\ell}\). So \(B\) has \(\frac{p^{\ell-1} - p^{\ell-1} / e}{e}\) subsections \((x, b)\), with \(D_{\ell} = \langle x \rangle\). A consequence of Brauer’s second main theorem [NT 5.4.13(ii)] is that the number of irreducible characters in a block equals the number of columns in the block.

**Lemma 9.** A complete set of columns of \(B\) is
\[
(1, \chi_1), \ldots, (1, \chi_e), \quad (x_{i}^{\sigma_{i}}, \varphi_{i}^{n_{i}}), \quad i = 0, \ldots, a - 1.
\]

Here \(x_{i}\) is a fixed generator of \(D_{\ell}\), \(\sigma_{i}\) ranges over a set of representatives for the cosets of the image of \(N_i / C_i\) in \(\text{Aut}(D_{a-1})\) and \(n_{i}\) ranges over a set of representatives for the cosets of \(EC_i\) in \(N_i\). In particular \(e(B) = e + \frac{p^{\ell-1}}{e}\).

Let \(\Lambda\) be a set of representatives for the \(\frac{p^{\ell-1}}{e}\) orbits of \(E\) on \(\text{Irr}(D)\). Then
\[
\text{Irr}(B) = \{X_1, \ldots, X_e\} \cup \{X_{\lambda} | \lambda \in \Lambda\}.
\]

Also set \(X_0 := \sum_{\lambda \in \Lambda} X_{\lambda}\). Dade refers to the \(X_{\lambda}\) as the exceptional characters of \(B\).

Notice that as \(\ell(b_i) = 1\), \(b_i\) is real if and only if \(\varphi_i\) is real. The next two propositions are relatively elementary.

**Proposition 10.** All the blocks \(b_0, b_1, \ldots, b_{a-1}\) are real or none of them are real.

**Proof.** We have \((b_0)^G = B^0 = B\). So \((D, b_0)\) and \((D, b_0^0)\) are Sylow \(B\)-subpairs, and there is \(n \in N_0\) such that \(b_0^0 = b_0^n\).

Suppose that \(b_j\) is real, for some \(j = 0, \ldots, a - 1\). As \((D_j, b_j^0), (D_j, b_j^n) \subset (D_0, b_0^n)\), it follows from (2) that \(b_j^n = b_j^0 = b_j\). So \(n \in N(D_j, b_j) = EC_j\). Write \(n = ec\), where \(e \in E\) and \(c \in C_j\). Then \(c = e^{-1}n \in C_j \cap N_0\) and \(b_0^n = b_0^0 = b_0^n\). So \(c^2 \in C_j \cap E = C_0\). But \(C_j \cap N_0 / C_0\) has odd order, as it is a \(p\)-group. So \(c \in C_0\), which shows that \(n \in E\). As \(b_0^n = b_0^0\), it follows that \(b_0\) is real.
Now let \( i = 0, \ldots, a - 1 \). Then \((D_i, b_i), (D_i, b_i') \subset (D_0, b_0) = (D_0, b_0^2)\). So \( b_i = b_i^g \), for \( i = 0, \ldots, a - 1 \), using \([2]\). This shows that all \( b_0, \ldots, b_{a-1} \) are real.

We showed in \([M3, 1.1]\) that the number of real irreducible characters in a block equals the number of real columns in the block. Here \((x, \varphi)\) is real if \( x^g = x^{-1} \) and \( \varphi^g = \varphi \), for some \( g \in G \).

Let \( i = 0, \ldots, a - 1 \). As \( b_i \) has inertial index 1, it has \( |D| \) irreducible characters. Modifying \([D]\) p26 we use the notation

\[
\text{Irr}(b_i) = \{ X'_{i,\lambda} \mid \lambda \in \text{Irr}(D) \}.
\]

Here \( X'_{i,1} \) is the unique non-exceptional character in \( b_i \), and all characters \( X'_{i,\lambda} \), with \( \lambda \neq 1 \) are exceptional. Suppose that \( b_i \) is real. The columns of \( b_i \) are \( (d, \varphi_i) \), for \( d \in D \). As \( C_i \) acts trivially on the columns, the only real column is \((1, \varphi_i)\). So \( X'_{i,1} \) is the only real irreducible character in \( b_i \).

We will refine the next result in part (i) of Theorem \([I]\)

**Proposition 11.** All exceptional characters in \( B \) are real or none are real.

**Proof.** It follows from Corollary \([7]\) and Lemma \([9]\) that the number of real exceptional characters in \( B \) equals the number of real columns \((x, \varphi)\) with \( x \in D^x \) and \( \varphi \in \text{IBr}(C_G(x)) \).

Suppose that \( B \) has a real exceptional character, and let \((x, \varphi)\) be a real column of \( B \), with \( x \in D^x \). Then \( \langle x \rangle = D_i \), for some \( i = 0, \ldots, a - 1 \). As \( N_i/C_i \) is abelian, the columns \((x', \varphi_i^n)\) are real, for all generators \( x' \) of \( D_i \) and all \( n_i \in N_i \). In particular \((x_i, \varphi_i)\) is a real column. Choose \( n \in N_i \) such that \( x_i^n = x_i^{-1} \) and \( \varphi_i^n = \overline{\varphi}_i \). We may suppose that \( n^2 \in C_i \).

Suppose first that \( b_i \) is real. As \( \varphi_i = \overline{\varphi}_i \), \( n \) fixes \( \varphi_i \) and inverts \( D_i \). So \( nC_i \) is an involution in \( EC_i/C_i \). As \( EC_i/C_i \cong E/C_0 \), we may assume without loss that \( nC_0 \) is an involution in \( E/C_0 \). Now all the blocks \( b_0, \ldots, b_{a-1} \) are real. Hence all \( \varphi_0, \ldots, \varphi_{a-1} \) are real. As \( n \) inverts \( D_j \) and fixes \( \varphi_j \), all columns \((x_j, \varphi_j)\) are real. Thus all columns \((x, \varphi)\), with \( x \in D^x \), are real. So all exceptional characters in \( B \) are real in this case.

Conversely, suppose that \( b_i \) is not real. As \( nC_i \) is the unique involution in \( N_i/C_i \), but \( n \not\in EC_i \), it follows that \( |EC_i : C_i| = e \) is odd. Now \( (D, b_0) \) and \((D, b_0^2)\) are Sylow \( B \)-subpairs, but \( b_0 \neq b_0^2 \). So there is \( m \in N_0 \setminus E \) such that \( b_0^m = b_0 \). As \( m^2 \in E \) and \( |E : C_0| \) is odd, we may choose \( m \) so that \( m^2 \in C_0 \). Then \( mC_0 \) is the unique involution in \( N_0/C_0 \). In particular \( m \) inverts every element of \( D \). Let \( j = 0, \ldots, a - 1 \). Then \((D_j, b_j^m)\) and \((D_j, b_j^g)\) are \( B \)-subpairs contained in \((D, b_0^2)\). So \( b_j^m = b_j^g \) and thus \((d_j, \varphi_j)^m = (d_j^{-1}, \overline{\varphi}_j)\). It follows that all exceptional characters in \( B \) are real in this case also.

Examination of the proof shows that:

**Corollary 12.** All exceptional characters in \( B \) are real if and only if \( b_0 \) is real and \( e \) is even, or \( b_0 \) is not real and \( e \) is odd.

We need some additional notation. Set \( \Lambda_u := \{ \lambda \in \Lambda \mid \ker(\lambda) = D_u \} \), for \( u = 1, \ldots, a \). So \( |\Lambda_u| = \frac{p^u - p^{u-1}}{e} \). Now choose \( \lambda \in \Lambda_u \) and set

\[
\epsilon_u := \epsilon(X_\lambda).
\]
Note that $X_\lambda$ and $X_\mu$ are Galois conjugates, for all $\lambda, \mu \in \Lambda_u$ (this follows from [D, part 2 of Theorem 1 and Corollary 1.9]). So $\epsilon_u$ does not depend on $\lambda$.

Recall our notation $\{\}$ for the irreducible characters $X'_{i,\lambda}$ in $b_i$. As already noted, $X'_{i,1}$ is the only possible real irreducible character in $b_i$. We set

$$\nu_i := \epsilon(X'_{i,1}), \quad \text{for } i = 0, \ldots, a - 1.$$ 

Now let $i = 0, \ldots, a - 1$ and choose $x \in D_i - D_{i+1}$ and $\rho \in N_i$. According to [D, Theorem 1, Part 3] there are signs $\epsilon'_0, \epsilon_0, \epsilon_1, \ldots, \epsilon_e$ and $\gamma_i$ such that

$$d^{(x)}_{X_\lambda, \varphi_i^\rho} = \epsilon'_0 \gamma_i \sum_{\tau \in E C_i / C_i} \lambda(\rho x), \quad d^{(x)}_{X_j, \varphi_i^\rho} = \epsilon_j \gamma_i, \quad \text{for } j = 1, \ldots, e,$$

$$d^{(x)}_{X_{i,1}, \varphi_i^\rho} = \epsilon'_0 \gamma_i \lambda(\rho), \quad d^{(x)}_{X_{i,1}, \varphi_i^\rho} = 1.$$

Here $E C_i / C_i$ is a set of representatives for the cosets of $C_i$ in $E C_i$. Note that Feit uses the notation $\delta_0 = -\epsilon_0$ and $\delta_j = \epsilon_j$, for $j = 1, \ldots, e$. Now let $i = 0, \ldots, a - 1$ and $x \in D_i - D_{i+1}$. Then it follows from [D, Corollary 1.9] that $X_j(x) = |N_i : E C_i| \varphi_i(1) \delta_j \gamma_i$. So $\delta_j \gamma_i$ is the sign of the integer $X_j(x)$.

There is a nice relationship between the signs $\epsilon_0, \epsilon_1, \ldots, \epsilon_e$ and the Brauer tree of $B$. Suppose that $j$ and $k$ are adjacent vertices in the Brauer tree. Then $X_j + X_k$ is a principal indecomposable character of $G$. So it vanishes on $D^\times$, and hence $\delta_j + \delta_k = 0$ (see [F, V11, Section 9]). So suppose that there are $d_j$ edges between the vertex $j$ and the exceptional vertex 0 in the Brauer tree. Then $\delta_j = (-1)^{d_j} \delta_0$. So $\epsilon_j = (-1)^{d_j-1} \epsilon_0$, for $j = 1, \ldots, e$.

We now prove part (i) of our main theorem. But note that this proof does not depend on Propositions [10] and [11].

**Proof of part (i) of Theorem 2** Applying (1), with $\rho \in N_i$ and $x \in D_i - D_{i+1}$, we get

$$\sum_{j=1}^e \epsilon(X_j) \epsilon_j \gamma_i + \sum_{\lambda \in \Lambda} \epsilon(X_\lambda) \epsilon_0 \gamma_i \sum_{\tau \in E C_i / C_i} \lambda(\rho x) = \nu_i.$$

Now set $\sigma := \epsilon_0 \sum_{j=1}^e \epsilon(X_j) \epsilon_j$. So $\sigma$ is independent of $i$, $\rho$ and $x$. Then the above equality transforms to

$$\sum_{u=1}^a \epsilon_u \sum_{\lambda \in \Lambda_u} \sum_{\tau \in E C_i} \lambda(\rho x) = \epsilon_0 \gamma_i \nu_i - \sigma,$$

where the right hand side is independent of $\rho$ and $x$. Let $\rho$ range over a set of representatives for the $|N_i : E C_i|$ cosets of $E C_i$ in $N_i$ and let $x$ range over a set of representatives for the $p^{a-i-1} \epsilon$ orbits of $N_i$ on the generators of $D_i$. Then $\rho x$ will range over all generators of $D_i$. Summing the resulting equalities gives

$$\sum_{u=1}^a \epsilon_u \sum_{\lambda \in \Lambda_u} \sum_{x \in D_i - D_{i+1}} \lambda(x) = \left( p^{a-i-1} \epsilon \right) (\epsilon_0 \gamma_i \nu_i - \sigma).$$
We use \(|\Lambda_u| = \frac{p^u - p^{u-1}}{e}\) and Lemma 5 to transform this equality to

\[
(p^a - p^{a-i}) \sum_{u=1}^{i} \frac{p^u - p^{u-1}}{e} \epsilon_u - p^{a-i-1}p^{i+1} - p^{i} e_{i+1} = \frac{p^{a-i} - p^{a-i-1}}{e} (\varepsilon_0 \gamma_i \nu_i - \sigma).
\]

After cancelling the factor \(\frac{p^{a-i-1}(p-1)}{e}\), we get

\[
(5) \quad \sum_{u=1}^{i} (p^u - p^{u-1}) \epsilon_u - p^{i} e_{i+1} = \varepsilon_0 \gamma_i \nu_i - \sigma.
\]

Here \(\sum_{u=1}^{0} (p^u - p^{u-1}) \epsilon_u\) is taken to be 0, when \(i = 0\). We write down the equalities (5) for \(i = 0, 1, 2, \ldots\) in turn:

\[
\begin{align*}
- \epsilon_1 &= \varepsilon_0 \gamma_0 \nu_0 - \sigma \\
(p-1)\epsilon_1 - p \epsilon_2 &= \varepsilon_0 \gamma_1 \nu_1 - \sigma \\
(p-1)\epsilon_1 + (p^2 - p)\epsilon_2 - p^2 \epsilon_3 &= \varepsilon_0 \gamma_2 \nu_2 - \sigma \\
(p-1)\epsilon_1 + (p^2 - p)\epsilon_2 + (p^3 - p^2)\epsilon_3 - p^3 \epsilon_4 &= \varepsilon_0 \gamma_3 \nu_3 - \sigma \\
&\vdots
\end{align*}
\]

\[
(p-1)\epsilon_1 + (p^2 - p)\epsilon_2 + \cdots + (p^{a-1} - p^{a-2})\epsilon_{a-1} - p^{a-1} \epsilon_a = \varepsilon_0 \gamma_{a-1} \nu_{a-1} - \sigma.
\]

Subtract the first equality from the second to get

\[
p(\epsilon_1 - \epsilon_2) = \varepsilon_0 (\gamma_1 \nu_1 - \gamma_0 \nu_0).
\]

The left hand side equals \(-p, 0\) or \(p\) and the right hand equals \(-2, 0\) or \(2\). As \(p\) is odd, the common value is 0. So \(\epsilon_2 = \epsilon_1\) and \(\gamma_1 \nu_1 = \gamma_0 \nu_0\). Substitute these values back into all equations in (6). Now subtract the first from the third equality to get

\[
p^2(\epsilon_1 - \epsilon_3) = \varepsilon_0 (\gamma_2 \nu_2 - \gamma_0 \nu_0).
\]

Once again both sides are 0. So \(\gamma_2 \nu_2 = \gamma_0 \nu_0\) and \(\epsilon_3 = \epsilon_1\). Proceeding in this way, we get

\[
\epsilon_1 = \epsilon_2 = \cdots = \epsilon_a, \quad \gamma_0 \nu_0 = \gamma_1 \nu_1 = \cdots = \gamma_{a-1} \nu_{a-1}.
\]

\[\square\]

Following the above proof, and the discussion before the proof, we obtain:

**Corollary 13.** Suppose that \(b_0\) is real and let \(D = \langle x \rangle\). Then for each \(i = 0, \ldots, a - 1\) and \(j = 0, \ldots, e\), the integer \(X_j(x^p)X_j(x)\) has sign \(\epsilon(X_{i+1}'\epsilon)(X_{0,1}')\).

There is no apparent relationship between the F-S indicators \(\nu_0, \ldots, \nu_{a-1}\):

**Example:** The 2-nilpotent group \(G = \langle a, b, c \mid a^4, a^2 = b^2, a^b = a^{-1}, c^9, ac = b, b^c = ab \rangle\) has isomorphism type \(3, \text{SL}(2, 3)\). Set \(D = \langle c \rangle\). Then \(D\) is cyclic of order 9, with \(C_0 = D \times \langle a^2 \rangle\) and \(C_1 = G\). Let \(\theta\) be the non-trivial irreducible character of \(C_0/D\), and let \(b_0\) be the 3-block of \(C_0\) which contains \(\theta\). Then \(\theta = X_{0,1}'\) is the unique non-exceptional character
in \( b_0 \). So \( \nu_0 = \epsilon(X'_{0,1}) = +1 \). Set \( b_1 = b_0^G \). Then \( b_1 \) also has a unique non-exceptional character \( X'_{1,1} \). But now \( \nu_1 = \epsilon(X'_{1,1}) = -1 \), as \( X'_{1,1} \) restricts to the non-linear irreducible character of \( \langle a, b \rangle \cong Q_8 \).

This example arises from the fact that the Glauberman correspondence \([NT], 5.12\) does not preserve the F-S indicators of characters.

**proof of part (iii) of Theorem 7.** This is an immediate consequence of Lemma 8 and part (i) of Theorem 11. \( \mathbb{Q} \)

Consider the real-stem of \( B \) as a horizontal line segment with \( s \) vertices and \( s - 1 \) edges, where \( s \geq 1 \). We label the vertices using an interval \([\ell, \ldots, -2, -1, 0, 1, 2, \ldots, r]\) so that 0 labels the exceptional vertex. Thus \( s = r + \ell + 1 \), and there are \( \ell \) real non-exceptional characters on the left of the exceptional vertex, and \( r \) on the right (the choice of left and right is unimportant).

As above, \( X_0 \) is the sum of the exceptional characters in \( B \). Now we relabel the non-exceptional characters in \( B \) so that \( X_i \) is the real non-exceptional character corresponding to vertex \( i \), for \( i = -\ell, \ldots, r \) and \( i \neq 0 \). In view of parts (i) and (ii) of Theorem 11 there are signs \( \epsilon \) such that

\[
\epsilon(X_i) = \begin{cases} 
\epsilon_-, & \text{for } i = -\ell, \ldots, -1; \\
\epsilon_0, & \text{for } i = 0; \\
\epsilon_+, & \text{for } i = 1, \ldots, r.
\end{cases}
\]

Next let \( \sigma \) be a generator of \( D \). It follows from [D, Corollary 1.9] that \( X_0(\sigma) = -\epsilon_0\gamma_0|N_0 : E|\varphi_0(1) \). So \( X_i(\sigma) = (-1)^iX_0(\sigma) \), as \( X_i + X_{i+1} \) is a projective character of \( G \), for \( i = -\ell, \ldots, r - 1 \) (see [F, VII, 2.19(ii)]).

Recall from Section 5 that there are \(|N_0 : E|\) blocks of \( C_0 \) which induce to \( B \); these are the blocks \( b_\sigma \) where \( \sigma \) ranges over \( N_0/E \). We note also that \( X'_{0,1}(\sigma) = \varphi_0(1) \). Now \([B, Theorem(4B)]\) is an immediate consequence of \([B, Theorem(4A)]\). In our context, this states that

\[
\sum_{i=-\ell}^r \epsilon(X_i)X_i(\sigma) = |N_0 : E|\epsilon(X'_{0,1})X'_{0,1}(\sigma).
\]

In view of the previous paragraph this simplifies to

\[
(7) \quad \sum_{i=1}^\ell (-1)^i\epsilon_- + \epsilon_0 + \sum_{i=1}^r (-1)^i\epsilon_+ = -\epsilon_0\gamma_0\nu_0.
\]

We consider a number of cases.

Suppose first that \( \epsilon_0 \neq 0 \). Then \( \epsilon_- = \epsilon_0 = \epsilon_+ \), by part (iii) of Theorem 11. So (7) becomes

\[
(8) \quad -\epsilon_0\gamma_0\nu_0\epsilon_0 = \begin{cases} 
(-1)^\ell, & \text{if } s \text{ is odd}; \\
0, & \text{if } s \text{ is even}.
\end{cases}
\]

In particular \( b_0 \) is not real if \( s \) is even. As \( e \) is odd when \( s \) is even, this already follows from Corollary 12.
Suppose then that $\epsilon_0 = 0$. Now \((7)\) evaluates as

\[
-\epsilon_0 \gamma_0 \nu_0 = \begin{cases} 
\epsilon_-, & \text{if } \ell \text{ is odd and } r \text{ is even.} \\
\epsilon_- + \epsilon_+, & \text{if } \ell \text{ and } r \text{ are both odd.} \\
\epsilon_+, & \text{if } \ell \text{ is even and } r \text{ is odd.} \\
0, & \text{if } \ell \text{ and } r \text{ are both even.}
\end{cases}
\]

proof of part (iv) of Theorem 1. The hypothesis is that $\epsilon_0 = 0$, at least one of $\epsilon_-, \epsilon_+$ is not zero and $\ell \equiv r \equiv 1 \pmod{2}$. Now $B$ has $e$ non-exceptional characters, of which $\ell + r$ are real-valued. So $e \equiv \ell + r$ is even. Then $b_0$ is not real, according to Corollary 12. This in turn implies that $\nu_0 = 0$. So $\epsilon_- + \epsilon_+ = 0$, according to \((9)\). We conclude that $\epsilon_- \epsilon_+ = -1$, which gives the conclusion of (iv).

\[\square\]

6. Passing from $B$ to its canonical character

Let $i = 0, \ldots, a - 1$. Then $N_i$ contains the normalizer $N_0$ of $D$ in $G$. So by Brauer’s first main theorem there is a unique $p$-block $B_i$ of $N_i$ such that $B_i^G = B$. As $(B_i)^G = B^o = B$, the uniqueness forces $B_i^o = B_i$. Now $B_i$ has defect group $D$ and inertial index $e = |EC_i : C_i|$. So $\ell(B_{a-1}) = e$ and $k(B_{a-1}) = e + \frac{b_0 - 1}{2}$. We first consider the block $B_{a-1}$ of the largest subgroup $N_{a-1}$. Following [D, Section 7], write

\[
\text{IBr}(B_{a-1}) = \{\tilde{X}_1, \ldots, \tilde{X}_e\}, \quad \text{Irr}(B_{a-1}) = \{\tilde{X}_1, \ldots, \tilde{X}_e\} \cup \{\tilde{X}_\lambda \mid \lambda \in \Lambda\},
\]

and set $\tilde{X}_0 = \sum \tilde{X}_\lambda$.

**Proposition 14.** The exceptional characters in $B$ and $B_{a-1}$ have the same F-S indicators.

**Proof.** Suppose first that $|\Lambda| \geq 2$. According [D, (7.2)] there is a sign $d$ such that

\[
(\tilde{X}_\lambda - \tilde{X}_\mu)^G = d(X_\lambda - X_\mu), \quad \text{for all } \lambda, \mu \in \Lambda.
\]

It follows that $\langle \tilde{X}_\lambda, X_\lambda \rangle$ or $\langle \tilde{X}_\mu, X_\lambda \rangle$ is odd. So in view of part (i) of Theorem 1 the conclusion holds in this case.

From now on we suppose that $|\Lambda| = 1$. Then $E$ has a single orbit on $\text{Irr}(D)^\times$, which forces $|D| = p$ and $e = p - 1$. As $X_0$ is the unique exceptional character in $B_{a-1}$, it is real valued. Then it follows from part (iii) of Theorem 1 that all real irreducible characters in $B_{a-1}$ have the same F-S indicators.

Now by [D, (7.3), (7.8), first two paragraphs of p40], there is a sign $\epsilon'_0$ such that

\[
(\tilde{X}_0 - \sum_{i=1}^{p-1} \tilde{X}_i)^G = \epsilon'_0 \sum_{i=0}^{p-1} \epsilon_i X_i.
\]

Here $\epsilon_0, \ldots, \epsilon_{p-1}$ are as introduced earlier and $X_0$ can be chosen to be real, as $p$ is odd. Taking inner-products of characters, and reading modulo 2, we see that $\langle \tilde{X}_0^G, X_0 \rangle$ is odd, for some real $\tilde{X}_i$. So $\epsilon(\tilde{X}_i) = \epsilon(X_0)$. Then by the previous paragraph $\epsilon(\tilde{X}_0) = \epsilon(X_0)$.

\[\square\]

**Proposition 15.** All exceptional characters in $B_0, \ldots, B_{a-1}$ and $B$ have the same F-S indicators.
Then according to W. Reynolds [NT, 5.8.14], for (10) $\chi$ in $\text{Irr}(D)$ of (4) for the irreducible characters in $\phi_i$ centralizes $D$, which lies over $b_i$. Moreover $\overline{b}_i$ has cyclic defect group $\overline{D}$ of characters commute, this block is dominated by $B_i$ in inertial index as $|\phi_i|$. Now $b_i$ has the unique irreducible Brauer character $\phi_i$, and we can and do identify $\phi_i$ with the unique irreducible Brauer character in $\overline{b}_i$. Then the inertia group of $\overline{b}_i$ in $\overline{N}_i$ is the inertia group of $\phi_i$ in $\overline{N}_i$, which is $\overline{EC}_i$.

According to [D] Section 4, there is a unique $p$-block of $N_i$, denoted here by $\overline{B}_i$, which lies over $\overline{b}_i$. Moreover $\overline{B}_i$ has cyclic defect group $\overline{D}$. As inflation and induction of characters commute, this block is dominated by $B_i$. Now $B_i$ and $\overline{B}_i$ have the same inertial index as $|EC_i : C_i| = |\overline{EC}_i : \overline{C}_i|$. So by inflation $\text{IBr}(\overline{B}_i) = \text{IBr}(B_i)$. In particular $\overline{B}_i$ is the unique block of $\overline{N}_i$ that is dominated by $B_i$. Also by inflation $\text{Irr}(\overline{B}_i) \subseteq \text{Irr}(B_i)$.

As $|\overline{D}| < |D|$, all exceptional characters in $\overline{B}_0, \ldots, \overline{B}_{a-1}$ have the same F-S indicators, by our inductive hypothesis. But the inclusion $\text{Irr}(\overline{B}_i) \subseteq \text{Irr}(B_i)$ identifies the exceptional characters in $\overline{B}_i$ with exceptional characters in $B_i$. It now follows from part (i) of Theorem 14 that all exceptional characters in $B_0, \ldots, B_{a-1}$ have the same F-S indicators.

Recall that $b_0$ has a unique irreducible Brauer character $\phi_0$. This is the canonical character of $B$, in the sense of [NT] 5.8.3. For the next theorem, we simplify the notation of (4) for the irreducible characters in $b_0$ by writing $\chi_{\lambda}$ in place of $X'_{0,\lambda}$, for all $\lambda \in \text{Irr}(D)$. Then according to W. Reynolds [NT] 5.8.14, for $c \in C_0$ we have

$$\chi_{\lambda}(c) = \begin{cases} 
\lambda(c_p)\phi_0(c_p), & \text{if } c_p \in D, \\
0, & \text{if } c_p \notin D.
\end{cases}$$

Then $\text{Irr}(b_0) = \{\chi_{\lambda} \mid \lambda \in \text{Irr}(D)\}$. Notice that $\chi_1$ is the unique irreducible character in $b_0$ whose kernel contains $D$.

**Theorem 16.** Suppose that $B$ has a real exceptional character. Then $N_0/C_0$ has a unique subgroup $T/C_0$ of order 2, and all exceptional characters in $B$ have F-S indicator equal to the Gow indicator $e_{T/C_0}(\chi_1)$.

**Proof.** Recall that $B$ has a real exceptional character if $b_0$ is real and $e$ is even, or if $b_0$ is not real and $e$ is odd. In both these cases $|N_0 : C_0|$ is even. As $N_0/C_0$ is also cyclic, it has a unique subgroup $T/C_0$ of order 2.

In view of Proposition 13 we may assume that $G = N_0$. So $B = B_0$, $D$ and $C_0$ are normal subgroups of $G$ and $E$ is the stabilizer of $b_0$ in $G$. Then $\Lambda$ is a set of representatives for the orbits of $N_0$ on $\text{Irr}(D)^\times$. Set $E^*$ as the stabilizer of $\{b_0, b_0'\}$ in $G$. Clifford correspondence defines a bijection between the irreducible characters of $E^*$ which lie over $b_0$ and the irreducible characters in $B$. This bijection preserves reality, and hence F-S indicators. So from now on we assume that $G = E^*$. 

As \( \chi_1 \) is invariant in \( E \) and \( E/C_0 \) is cyclic, \( \chi_1 \) has \( e \) extensions to \( E \), which we denote by \( \eta_1, \ldots, \eta_e \). Then \( X_i := \eta_i^G \), for \( i = 1, \ldots, e \), give the \( e \) non-exceptional characters in \( B \). Moreover \( X_\lambda := \chi_\lambda^G \), for all \( \lambda \in \Lambda \), give the exceptional characters in \( B \).

Following Corollary 2, there are three cases we must consider:

**Case 1:** \( b_0 \) is real, \( e \) is even and \( B \) has real non-exceptional characters. Then according to part (iii) of Theorem 1, all real irreducible characters in \( B \) have the same F-S indicators. We choose notation so that \( X_1 \) is real. As \( X_1 \downarrow_T \) is a real extension of \( \chi_1 \) to \( T \), it follows that \( \epsilon(X_1) = \epsilon(X_1 \downarrow_T) = \epsilon_{T/C_0}(\chi_1) \). This concludes Case 1.

**Case 2:** \( b_0 \) is real, \( e \) is even but \( B \) has no real non-exceptional characters. As \( \chi_1 \) does not extend to a real character of \( E \), it does not extend to a real character of \( T \), according to Lemma 4. So \( \epsilon_{T/C_0}(\chi_1) = -\epsilon(\chi_1) \), by the definition of the Gow indicator.

Now consider the notation used in the proof of part (i) of Theorem 1. Here \( C_i = C_0 \) and \( \varphi_i = \varphi_0 \) and \( X'_i1 = \chi_1 \), for \( i = 0, \ldots, a - 1 \). If \( \lambda \in \Lambda \) then \( (X_\lambda) \downarrow_{C_0} = \sum_{\tau \in G/C_0} \chi_{\lambda \tau} \).

So \( d_{X_\lambda, \varphi_i}^{(e)} = \sum_{\tau \in G/C_0} \lambda(\tau x) \), for all \( x \in D^X \). This means that \( \varepsilon_0 \gamma_i = 1 \), for \( i = 0, \ldots, a - 1 \). Now in (6), the term \( \sigma \) is 0, as none of \( X_1, \ldots, X_e \) are real. So the first equation in (6) simplifies here to \( -\epsilon(X_\lambda) = \epsilon(\chi_1) \), for all \( \lambda \in \Lambda \). So \( \epsilon(\chi_\lambda) = \epsilon_{T/C_0}(\chi_1) \), for all \( \lambda \in \Lambda \), by the previous paragraph and Proposition 15.

**Case 3:** The final case is that \( b_0 \) is not real and \( e \) is odd. As \( B \) has an odd number \( e \) of non-exceptional characters, at least one of them must be real valued. So we assume that \( X_1 \) is real. Then, just as in Case 1, all real irreducible characters in \( B \) have the same F-S indicators.

As \( |E : C_0| \) is odd and \( |G : E| = 2 \), we have \( G/C_0 = E/C_0 \times T/C_0 \). Now \( T/C_0 \) conjugates \( \text{Irr}(b_0) \) into \( \text{Irr}(b_0) \). So \( \chi_1 \) is \( T \)-conjugate to \( \chi_1 \). In particular \( \chi_1 \uparrow_T \) is irreducible and real valued. Now \( \chi_1 \downarrow_T = (\chi_1) \uparrow_T \), by Mackey’s theorem.

Now from above \( \epsilon(\chi_\lambda) = \epsilon(X_1) \), for all \( \lambda \in \Lambda \). Also \( \epsilon(\chi_\lambda) = \epsilon((X_1) \downarrow_T) \), as both are real valued. Finally \( \epsilon((X_1) \downarrow_T) = \epsilon_{T/C_0}(\chi_1) \), by the definition. This completes Case 3.

Finally, we apply the proof to ordinary characters as stated in the Introduction:

**Proof of Theorem 2** Let \( x \) be a weakly real \( p \)-element of \( G \) of maximal order and set \( Q := \langle x \rangle \) and \( N := N_G(Q) \). Let \( \lambda \) be a faithful linear character of \( Q \). Then \( N_\lambda = C_N(x) \) and \( N_\lambda^* = C_N^*(x) \). So \( N_\lambda^* \) does not split over \( N_\lambda \). By Lemma 3 there exists \( \chi \in \text{Irr}(N \mid \lambda) \) such that \( \epsilon(\chi) = -1 \).

Let \( \tilde{B} \) be the \( p \)-block of \( N \) which contains \( \chi \) and let \( D \) be a defect group of \( \tilde{B} \). Then \( Q \subseteq D \) and \( N_G(D) \subseteq N \). In particular \( B := \tilde{B}^G \) is defined and \( B \) has defect group \( D \). So \( Q = D_i, N = N_i \) and \( \tilde{B} = B_i \) for some \( i \geq 0 \), in cyclic defect group notation.

Notice that \( \lambda \) is non-trivial. So \( D \nsubseteq \ker(\chi) \). This means that \( \chi \) is an exceptional character in \( B_i \). So all exceptional characters in \( B_i \), and hence also in \( B \), are symplectic. The number of exceptional characters in \( B \) is \( \frac{|D|^{i-1}}{e} \), where \( e \) is the inertial index of \( B \). The number of weakly real \( p \)-conjugacy classes of \( G \) is equal to the number of \( N \)-orbits on \( Q^X \), which equals \( \frac{|D|^{i-1} \cdot \text{sign}(\chi)}{|N : C_D|} \). As \( |D_i| \leq |D| \) and \( e \leq |N_i : C_i| \), we conclude that the
number of symplectic irreducible characters of $G$ is not less than the number of weakly real $p$-conjugacy classes of $G$. □

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References

[Al] J. L. Alperin Weights for Finite Groups, Proc. Symp. Pure Math. 47 (1987) 369–379.
[A] J. H. Conway, R. T. Curtis et al. ATLAS of finite groups: maximal subgroups and ordinary characters for simple groups, Clarendon Press, Oxford, 1985.
[B] R. Brauer, Some applications of the theory of blocks of characters of finite groups III, J. Algebra 3 (1966) 225–255.
[F] W. Feit, The representation theory of finite groups, North-Holland Math. Library 25. 1982.
[D] E. C. Dade, Blocks with cyclic defect groups, Ann. Math. 84 (1966) 20–48.
[GAP] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.8; 2017, (https://www.gap-system.org).
[G] R. Gow, Real valued and 2-rational group characters, J. Algebra 61 (1979) 388–413.
[Gr] J. A. Green, Walking around the Brauer Tree. J. Austral. Math. Soc. 17 (1974) 197–213.
[M1] J. Murray, Strongly real 2-blocks and the Frobenius-Schur indicator, Osaka J. Math. 43 (1) (2006) 201–213.
[M2] J. Murray, Components of the involution module in blocks with a cyclic or Klein-four defect group J. Group Theory 11 (1) (2008) 43–62.
[M3] J. Murray, Real subpairs and Frobenius-Schur indicators of characters in 2-blocks, J. Algebra 322 (2) (2009) 489–513.
[NT] H. Nagao, Y. Tsushima, Representations of finite groups, Academic Press, 1987.
[W] W. Willems, Duality and forms in representation theory, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), Progr. Math., 95, Birkhuser, Basel, 1991, 509–520.

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