ON *k*-DYCK PATHS WITH A NEGATIVE BOUNDARY

HELMUT PRODINGER

Abstract. Paths that consist of up-steps of one unit and down-steps of *k* units, being bounded below by a horizontal line \(-t\), behave like \(t+1\) ordered tuples of \(k\)-Dyck paths, provided that \(t \leq k\). We describe the general case, allowing \(t\) also larger. Arguments are bijective and/or analytic.

1. Folklore results about \(k\)-Dyck paths

A Dyck path consists of up-steps and down-steps, one unit each, starts at the origin and returns to the origin after \(2n\) steps, and never goes below the \(x\)-axis. The enumeration involves the ubiquitous Catalan numbers [10]. The family of \(k\)-Dyck paths is similarly defined, but the down-steps are now by \(k\) units in one step. Practically every book on combinatorics has something about this; we only give two citations: [1, 4].

The generating function \(y = y(z) = y_k(z)\) of these objects, according to length (the number of steps) can be found by a first return to the \(x\)-axis decomposition:

\[
y = 1 + (zy)^k \cdot z \cdot y = 1 + z^{k+1}y^{k+1},
\]
where the last \(z\) represents the down-step that brings the path back to the \(x\)-axis for the first time. (An equivalent concept is the family of \((k+1)\)-ary trees.)

Let us denote \(x = z^{k+1}\), which also indicates that the path can only return to the \(x\)-axis after \((k+1)n\) steps, and also \(y = 1 + w\). Then the equation becomes amenable to the Lagrange inversion:

\[
w = x(1 + w)^{k+1},
\]
and we can compute the coefficients of \(w\).

\[
[x^n]w = \frac{1}{n} [w^{n-1}(1 + w)^{(k+1)n}] = \frac{1}{n} \binom{(k + 1)n}{n - 1}.
\]

Now we compute

\[
[x^n]y^j = \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} (1 + w)^j
= \frac{1}{2\pi i} \oint \frac{dw(1 + w)^{(n+1)(k+1)}}{w^{n+1}} (1 + w)^j \frac{1 - kw}{(1 + w)^{k+2}}
= [w^n](1 - kw)(1 + w)^{(k+1)-1+j}
= \binom{n(k + 1) - 1 + j}{n} - k \binom{n(k + 1) - 1 + j}{n - 1}
= \frac{j}{(k+1)n + j} \binom{(k + 1)n + j}{n}.\]
2. $k_t$-Dyck paths

Selkirk [9] introduced an extra parameter $t$ to the family of $k$-Dyck paths. The paths might go below the $x$-axis, but never go below the horizontal line $-t$.

The enumeration of $k_t$-Dyck paths is as follows [9]:

**Theorem 1.** For $0 \leq t \leq k$, the number of $k_t$-Dyck paths of length $(k + 1)n$ is given by

$$\frac{t + 1}{(k + 1)n + t + 1} \binom{(k + 1)n + t + 1}{n}.$$

Equivalently, the generating function of $k_t$-Dyck paths by length is given by $y^{t+1}$.

That $y^{t+1}$ has indeed these coefficients was discussed in the previous section. It also enumerates ordered $(t + 1)$-tuples of $k$-Dyck paths, and the bijection in [9] is between these two families of objects. It is to be noted that [5] contains somewhat equivalent statements in the language of $(k + 1)$-ary trees, and instead of a boundary line, the nodes are coloured, and the colour of the root plays a role similar to the $t$ in $k_t$-Dyck paths. I report this information from [9].

In the present note we want to look at this again, but we also want to explain what happens if the condition $0 \leq t \leq k$ is no longer satisfied, i.e., if $t > k$ is allowed. The generating function is no longer $y^{t+1}$, and has to be replaced by something more complicated.

But let us start with some bijective arguments. We draw 3-Dyck paths, but 3 could be replaced by $k$.

**Figure 1.** A $3_3$-Dyck path: down-steps of 3 units and bounded below by the line $-3$.

**Figure 2.** The path from Figure [7] lifted up 3 units, with a sequence of up-steps in the beginning.
We lift up a $k_t$-Dyck paths by $t$ units and add $t$ up-steps in the beginning. The resulting path is a $k$-Dyck path, but does not end on level 0, but rather on level $t$. It is classical that these paths have generating function $z^t y^{t+1}$, thanks to a decomposition that is sketched in Figure 3.

![Figure 3. The path from Figure 2, decomposed.](image)

Removing $t$ extra up-steps, we are at the generating function $y^{t+1}$ again, and the decomposition gives us $t + 1$ (ordered) $k$-Dyck paths. In the example, we get 4 paths:

![Figure 4. Decomposed into 4 paths; the second and third paths are the empty path $\varepsilon$.](image)

Observe that the operation “shifting up the path by $t$ units” makes the origin the first point where the level $t$ is reached. In the beginning, there are the $t$ extra up steps. If $t \leq k$ this is indeed the only option to reach this level for the first time without going below the $x$-axis by using a down-step.

If $t$ is arbitrarily large, this is no longer true: We can “live” in the strip with boundaries 0 and $t - 1$, end at the highest level, and make one up-step to reach the $k_t$-path. Here is a sketchy drawing explaining this.

So for general $t$ the first part has more freedom. In the next section we will use generating functions to understand the roles of $F$ and $G$ better. In particular, the length of the $F$ may vary now.

### 3. Generating functions

Now we consider paths, living in the strip $-t..h$. We consider generating functions $\varphi_i(z)$, where $i$ marks the level of the endpoint. The recursion $\varphi_i = z\varphi_{i-1} + z\varphi_{i+k}$ is obvious, provided all indices are within the interval $-t..h$. This is best written as a
Figure 5. The path has a first part $F$ and a second part $G$. The length of $F$ will later be called $J$ and analyzed for $k = 1$.

The entry 1 on the righthand side corresponds to the function $\varphi_0$. Let $D_m$ be the determinant of the matrix with $m$ rows (and columns). We find the recursion

$$D_m = D_{m-1} - z^{k+1}D_{m-k-1}, \quad D_0 = D_1 = \cdots = D_k = 1.$$  

The solution is

$$D_m = \sum_{0 \leq l \leq \frac{m}{k}} \binom{m-kl}{l} z^{(k+1)l} (-1)^l.$$  

This can be proved by induction on $m$ or otherwise. For $k = 1$, these polynomials are sometimes called Fibonacci polynomials and appear e. g. in [3]. For $k = 2$, they appear in [8]. A general reference about this method is [7].

Using Cramer’s rule to solve the system, we find

$$\varphi_i = \frac{D_t z^i D_{h-i}}{D_{h+t+1}}, \quad 0 \leq i \leq h$$  

and

$$\varphi_{-i} = \frac{D_h z^i D_{t-i}}{D_{h+t+1}}, \quad 0 \leq i \leq t.$$  

Since we do not need the upper boundary at level $h$, we push it to infinity:

$$\lim_{h \to \infty} \frac{D_t z^i D_{h-i}}{D_{h+t+1}} = y^{i+t+1}.$$
We are only interested in the instance \( i = 0 \), with the result \( y^{t+1} \). So, we obtained for the enumeration of \( k_t \)-Dyck paths:

\[
D_t y_{k,t} = \sum_{0 \leq l \leq \frac{t}{k}} \binom{t - kl}{l} z^{(k+1)t} (-1)^l \cdot y^{t+1}.
\]

This explains once again that for \( 0 \leq t \leq k \) we get the simple result \( y^{t+1} \). In general, the generating function is given by

\[
y_{k,t}(z) = \sum_{0 \leq l \leq \frac{t}{k}} \binom{t - kl}{l} z^{(k+1)t} (-1)^l \cdot \sum_{n \geq 0} \frac{t + 1}{(k+1)n + t + 1} \binom{(k+1)n + t + 1}{n} z^{(k+1)n}.
\]

The number of \( k_t \)-Dyck paths of length \((k+1)n\) is then given by

\[
[z^{(k+1)n}] y_{k,t}(z) = \sum_{0 \leq l \leq \frac{t}{k}} \binom{t - kl}{l} (-1)^l \cdot \sum_{n \geq 0} \frac{t + 1}{(k+1)(n - l) + t + 1} \binom{(k+1)(n - l) + t + 1}{n - l}.
\]

The polynomial \( D_t \) has alternating coefficients, and it is quite likely that there is some sort of an inclusion-exclusion principle underlying.

The polynomial \( D_t \) doesn’t have a combinatorial meaning itself, but we may write

\[
\frac{z^t}{D_t} y_{k,t}(z) = z^t y^{t+1}.
\]

The generating function \( \frac{z^t}{D_t} \) has a combinatorial meaning: In the linear system, first replacing \( t \) by 0 and then \( h \) and \( i \) by \( t - 1 \), leads to \( \frac{z^{t-1}}{D_t} \), and it counts the \( k \)-Dyck paths living in the strip \( 0..h - 1 \), ending at the highest level \( h - 1 \). The function \( \frac{z^t}{D_t} \) differs only by an extra factor \( z \), representing an up-step, touching the level \( h \) for the first time. This is exactly the decomposition as given in Fig. 5.

Paths ending on their highest level appear in [2, 6].

The following quantity is interesting:

\[
\lim_{n \to \infty} \sum_{0 \leq l \leq \frac{t}{k}} \frac{(t - kl)(-1)^l}{(k+1)(n - l) + t + 1} \binom{(k+1)(n - l) + t + 1}{n - l} = \sum_{0 \leq l \leq \frac{t}{k}} \binom{t - kl}{l} (-\rho)^l,
\]

with

\[
\rho = \frac{k^k}{(k + 1)^{k+1}}.
\]

This sum is indeed = 1 for \( 0 \leq t \leq k \), but takes smaller values, when \( t \) gets larger in relation to \( k \).
4. ASYMPTOTICS

We want to study the parameter \( J = j \) as in the drawing of Fig. 5. It is given by \( j = h + (k + 1)l \), for some \( l \).

In order to be able to do explicit calculations, we restrict ourselves to the classical case \( k = 1 \) of Dyck paths. Then the recursion of second order admits the solution

\[
D_t(z^2) = D_t(x) = \frac{1 - u^{t+1}}{1 - u} \frac{1}{(1 + u)^t},
\]

with the (classical) substitution \( x = \frac{u}{(1 + u)^2} \), borrowed from [3].

The parameter \( J \) has an automatic contribution of \( t \), which we can add later. The advantage of this is that we now only have generating functions in \( z^2 \), for which we write \( x \).

The probability generating function of interest is

\[
P(x, w) := \frac{[x^n]\frac{1}{D_t(xw)}G(x)}{[x^n]C(x)^{t+1}},
\]

with

\[
C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = 1 + u
\]

the generating function of the Catalan numbers, enumerating Dyck paths by half-length.

The number of paths of length \( 2n + t \) with parameter \( J = 2s + t \) is then given as

\[
t + 2[x^n w^n]P(x, w).
\]

We still need to compute \( G(x) \), the generating function of paths not going below the line \(-t\) and ending on the \( x\)-axis again.

This can be computed by the linear system as a quotient of the usual determinants: Consider paths bounded below by \(-t\), and above by \( i \).

\[
\lim_{i \to \infty} \frac{D_t D_i}{D_{i+h+1}} = \lim_{i \to \infty} \frac{1 - u^{i+1}}{1 - u} \frac{1 - u^{i+1}}{1 - u} \frac{1}{(1 + u)^{t}} \frac{1}{(1 + u)^{i}} \frac{1 - u^{i+t+2}}{1 - u} \frac{1}{(1 + u)^{i+t+1}}
\]

\[
= \frac{1 - u^{i+1}}{1 - u} \frac{1}{(1 + u)^{t}} \frac{(1 + u)^{i+t+1}}{(1 + u)^{i}} = \frac{1 + u}{1 - u} (1 - u^{i+1}).
\]
As a check, we get the product of the two components
\[
\frac{1-u}{1-u^{t+1}}(1+u)^{t} \cdot \frac{1+u}{1-u}(1-u^{t+1}) = (1+u)^{t+1} = C(x)^{t+1},
\]
as it should.

We study now
\[
\frac{1}{D_{t}(x)w}G(x)
\]
using a second variable \( w \) to count the parameter of interest. We compute:
\[
\frac{d}{dw} \frac{1}{D_{t}(x)w} \bigg|_{w=1} = x \frac{d}{dx} \frac{1}{D_{t}(x)} = x \frac{du}{dx} \frac{1}{D_{t}(x)}
\]
\[
= -\frac{u(1+u)^{t}}{(1+u)^{2}} \left( \frac{(1-u)(1+u)^{t}}{1-u^{t+1}} \right)^{2} \frac{d}{du} D_{t}(x)
\]
\[
= -\frac{u(1+u)^{t}}{(1-u)^{2}} \left( \frac{(1-u)(1+u)^{t} - t(1-u)(1+u)}{1-u} \right)
\]
and further
\[
\frac{d}{dw} \frac{1}{D_{t}(x)w} \bigg|_{w=1} \cdot \frac{1+u}{1-u}(1-u^{t+1})
\]
\[
= -\frac{u(1+u)^{t}}{(1-u^{t+1})^{2}} \left( \frac{(1-u)(1+u)^{t} - t(1-u)(1+u)}{1-u} \right) \frac{1+u}{1-u}(1-u^{t+1})
\]
\[
= -\frac{u(1+u)^{t+1}}{1-u^{t+1}} \left( \frac{(1-u)(1+u)^{t} - t(1-u)(1+u)}{(1-u)^{2}} \right).
\]

We don’t try to simplify this any further, but expand it around \( u = 1 \), which corresponds to the singularity \( z = \frac{1}{4} \), which is relevant for Dyck-paths. We are in the regime called “sub-critical”, compare [4], where the singular expansion of numerator and denominator is of the same type:
\[
\frac{a_{0} + a_{1}\sqrt{1-4z} + \ldots}{b_{0} + b_{1}\sqrt{1-4z} + \ldots},
\]
whence the asymptotic expansion of the parameter \( J \) of interest is given by \( \frac{a_{1}}{b_{1}} \). One does not need to switch back to the \( x \)-world, since this quotient can also be obtained via \( \frac{a_{1}}{b_{1}} \), in
\[
\frac{a_{0} + a_{1}(1-u) + \ldots}{b_{0} + b_{1}(1-u) + \ldots}.
\]
In our instance, we are led to
\[
\frac{\frac{1}{3}2^{t}(t-1) - \frac{1}{6}2^{t}(t-1)(t+1)(1-u) + \ldots}{2^{t+1} - 2^{t}(t+1)(1-u) + \ldots},
\]
and the quotient of interest is
\[
\frac{\frac{1}{6}2^{t}(t-1)(t+1)}{2^{t}(t+1)} = \frac{t(t-1)}{6}.
\]
Multiplying this by 2, in order to switch from half-length to length, and adding the fixed contribution $t$ leads to the average value of $J$:

$$t + 2 \frac{t(t - 1)}{6} = \frac{t(t + 2)}{3}.$$ 

In this subcritical regime it is also relatively easy to determine the discrete limiting distribution. We start from

$$\frac{1}{D_t(xw)} G(x) \frac{C(x)}{C(x)^{t+1}};$$

the quotient $G(x)/C(x)^{t+1}$ does not depend on the parameter, and can thus be replaced by its limit at the singularity $x = \frac{1}{4}$, or, easier at $u = 1$:

$$\lim_{u \to 1} \frac{G(x)}{C(x)^{t+1}} = \frac{2(t + 1)}{2^t + 1} = \frac{t + 1}{2^t}.$$ 

Hence, the discrete limiting distribution is given by the (probability) generating function

$$p_t(u) = \frac{t + 1}{2^t} \frac{1 - u}{1 - u^{t+1}}(1 + u)^t.$$ 

One can even read off coefficients explicitly:

$$[x^m]p_t(u) = \frac{t + 1}{2^t} \frac{1}{2\pi i} \oint \frac{dx}{x^{m+1}} p_t(u)$$

$$= \frac{t + 1}{2^t} \frac{1}{2\pi i} \oint \frac{du(1 - u)(1 + u)^{2m+2}}{u^{m+1}} \frac{1 - u}{1 - u^{t+1}}(1 + u)^t$$

$$= \frac{t + 1}{2^t} [u^m] \frac{(1 - u)^2}{1 - u^{t+1}}(1 + u)^{2m-1+t}$$

$$= \frac{t + 1}{2^t} \sum_{\lambda \geq 0} \left[ \binom{2m - 1 + t}{m - \lambda(t + 1)} - 2 \binom{2m - 1 + t}{m - 1 - \lambda(t + 1)} + \binom{2m - 1 + t}{m - 2 - \lambda(t + 1)} \right].$$

This quantity might be interpreted as the probability that in an (almost) infinitely long path the parameter $\frac{J - t}{2}$ has value $m$.

**Figure 7.** Limiting distribution: $t = 7, t = 10, t = 14

**Acknowledgments.** I thank Nancy Gu for valuable discussions. Most of this work was developed while I visited the Technical University in Graz, Austria. I thank my host Peter Grabner for his hospitality.
ON k-DYCK PATHS WITH A NEGATIVE BOUNDARY

REFERENCES

[1] Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. volume 281, pages 37–80. 2002. Selected papers in honour of Maurice Nivat.

[2] Mireille Bousquet-Mélou and Yann Ponty. Culminating paths. *Discrete Math and Theoretical Computer Science*, 10:125–152, 2008.

[3] N. G. de Bruijn, D. E. Knuth, and S. O. Rice. The average height of planted plane trees. In *Graph theory and computing*, pages 15–22. 1972.

[4] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, 2009.

[5] Nancy S. S. Gu, Helmut Prodinger, and Stephan Wagner. Bijections for a class of labeled plane trees. *European J. Combin.*, 31(3):720–732, 2010.

[6] Benjamin Hackl, Clemens Heuberger, Helmut Prodinger, and Stephan Wagner. Analysis of bidirectional ballot sequences and random walks ending in their maximum. *Annals of Combinatorics*, 20:775–797, 2016.

[7] Helmut Prodinger. Analytic methods. In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 173–252. CRC Press, Boca Raton, FL, 2015.

[8] Helmut Prodinger. On some questions by Cameron about ternary paths – a linear algebra approach. 2019.

[9] Sarah J. Selkirk. MSc-thesis: *On a generalisation of k-Dyck paths*. Stellenbosch University, 2019.

[10] Richard P. Stanley. *Catalan numbers*. Cambridge University Press, New York, 2015.

H. PRODINGER, DEPARTMENT OF MATHEMATICAL SCIENCES, MATHEMATICS DIVISION, STELLENBOSCH UNIVERSITY, PRIVATE BAG X1, 7602 MATHIEUX, SOUTH AFRICA

E-mail address: hproding@sun.ac.za