FORMAL THEORY OF CORNERED ASYMPTOTICALLY HYPERBOLIC EINSTEIN METRICS

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Abstract. This paper makes a formal study of asymptotically hyperbolic Einstein metrics given, as conformal infinity, a conformal manifold with boundary. The space on which such an Einstein metric exists thus has a finite boundary in addition to the usual infinite boundary and a corner where the two meet. On the finite boundary a constant mean curvature umbilic condition is imposed. First, recent work of Nozaki, Takayanagi, and Ugajin is generalized and extended showing that such metrics cannot have smooth compactifications for generic corners embedded in the infinite boundary. A model linear problem is then studied: a formal expansion at the corner is derived for eigenfunctions of the scalar Laplacian subject to certain boundary conditions. In doing so, scalar ODEs are studied that are of relevance for a broader class of boundary value problems and also for the Einstein problem. Next, unique formal existence at the corner, up to order at least equal to the boundary dimension, of Einstein metrics in a cornered asymptotically hyperbolic normal form which are polyhomogeneous in polar coordinates is demonstrated for arbitrary smooth conformal infinity. Finally it is shown that, in the special case that the finite boundary is taken to be totally geodesic, there is an obstruction to existence beyond this order, which defines a conformal hypersurface invariant.

1. Introduction

This paper studies the formal existence and expansion of asymptotically hyperbolic (AH) Einstein metrics on manifolds with corner. The setting of interest to us is manifolds with two codimension-one boundary faces, one of which is the conformal infinity for an AH metric, and the other of which is an ordinary embedded hypersurface at which, away from the corner, the metric is regular. Following our earlier study [McK16], we will refer to these spaces as cornered asymptotically hyperbolic (CAH) spaces.

One of the seminal problems in the theory of AH spaces, resolved in [FG85, FG12], was the formal existence of Einstein metrics given a particular conformal infinity \((M^n, [h])\). This study had a significant impact on conformal geometry, and the formal expansion developed in those works has
found myriad applications from geometric analysis to the AdS/CFT conjecture of physics. We will consider the same question of formal existence, but will take $M$ to be a manifold with boundary. This necessitates equipping the Einstein space with a finite boundary, so that the boundary $S$ of $M$ becomes a corner, and we have a boundary condition on the new finite boundary as well as at $M$. For reasons discussed below, we select the CMC umbilic condition on the finite boundary. We will then consider formal existence of a CAH Einstein metric at the boundary $S$ of $M$, viewed as the corner of the $(n + 1)$-space.

AH Einstein spaces with corners have been considered in at least two previous settings. Local regularity at the boundary of AH Einstein metrics was studied in [BH14], and studying a neighborhood of a point on the boundary necessarily introduces a corner where the inner boundary of the neighborhood meets the boundary at infinity. The authors therefore developed doubly weighted function spaces to analyze regularity. Since they were interested primarily in behavior away from the corner, however, they successfully “washed out” the behavior of the metric at the corner itself, and thus said relatively little about the metric there.

The paper [NTU12] reflects an interest in this setting from physicists who wish to study the AdS/CFT correspondence when the conformal field theory is on a space with boundary; see the discussion and references given there. In the first part of that paper, the authors considered a conformally compact manifold $X$ whose infinite boundary, $(M^n, [h])$, was a piece of $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary $S$ and endowed with the conformal class of the flat metric. They then added a finite boundary $Q$ of $X$, intersecting $M$ precisely at $S$, imposing the boundary condition that $Q$ was umbilic with constant mean curvature (CMC umbilic) with respect to the hyperbolic metric $g_+$ on the upper half-space. They concluded that, for $n = 2$, the boundary $S$ was unrestricted, but that for $n = 3$ (or, they posited, larger), $S$ must be a sphere or a plane. They therefore, in the latter part of the paper, considered a particular family of perturbations of the 4-dimensional hyperbolic metric at the corner. They found that the Einstein equations could be solved to first order in the (compactified) distance to the corner for arbitrary $S$.

We will adopt the same CMC umbilic boundary conditions, which are geometrically natural and require a minimum of data beyond the conformal infinity: only a single scalar, the mean curvature. We first, in section 3, undertake a study of the case where the metric $g_+$ has a smooth compactification. We observe that in the case of hyperbolic space, the fact observed in [NTU12] for $n \geq 3$ — that the corner $S$ must be a sphere or a hyperplane — follows in all dimensions $n \geq 2$ from the classical theorem characterizing umbilic hypersurfaces in hyperbolic space. We then obtain a series of conditions on $S$ for smooth CAH Einstein metrics with arbitrary conformal infinity by repeatedly differentiating the condition of umbilicity and applying the Einstein condition. We also explain the absence of the restriction on $S$ for the case $n = 2$ in [NTU12]: the restriction appears at one higher order in
the expansion for a two-dimensional conformal infinity than for any higher
dimension (and in particular, at one higher order than was considered in
that paper). Also in this section, we show that the CMC umbilic boundary
condition at \( Q \) implies that \( Q \) and \( M \) make constant angle with respect to
any compactification of \( g_+ \).

Having confirmed that the requirement of smoothness of \( g_+ \) at the corner
generically imposes severe restrictions on \( S \) in \( (M, [h_i]) \), we turn to a general
theory of formal existence. As a first step, we must find an appropriate
weakening of the smoothness condition. The most obvious condition, and
essentially that which we impose, is instead to require smoothness in polar
coordinates at the corner, a condition which has a long history in problems
with such singularities (see [Kon67, Maz91]). Invariantly, this means blowing
up the corner in the sense of Melrose (throughout this paragraph, see Section
2 for details), obtaining a blown-up space \( \tilde{X} \) and a blowdown map \( b : \tilde{X} \to X \).
We define \( \tilde{M} = b^{-1}(M) \) and \( \tilde{Q} = b^{-1}(Q) \), to each of which \( b \) restricts as
a diffeomorphism; and \( \tilde{S} = b^{-1}(S) \), to which \( b \) restricts as a fibration with
fibers diffeomorphic to \([0,1]\) and the latter with interval fibers. In our prior paper, [McK16],
we considered a class of CAH metrics intermediate between those smooth on
\( X \) and those smooth on \( \tilde{X} \), a restricted subclass of the latter which we called
admissible metrics on \( \tilde{X} \). In that paper, we proved a normal-form theorem,
stated later here as Theorem 2.3. In brief, if \( M \) and \( Q \) make a constant angle
\( \theta_0 \) with respect to compactified metrics and \( g \) is an admissible metric, then
there is a diffeomorphism \( \zeta : [0,\theta_0] \times W \rightarrow \tilde{X} \) (where \( W \) is a neighborhood
of \( S \) in \( M \)) such that

\[
(1.1) \quad \zeta^* g = \frac{d\theta^2 + h_\theta}{\sin^2(\theta)},
\]

where \( h_\theta \) is a smooth one-parameter family of smooth asymptotically hyper-
bolic metrics on \((W, S)\), and \( \zeta \) is unique subject to some technical conditions.
This generalizes the ordinary hyperbolic metric \( g_+ = \frac{dy^2 + |dx|^2}{y^2} \) on the upper
half-space \( \mathbb{H}^{n+1} \), which under the change of variables \( \psi \) given by \( y = \rho \sin \theta \)
and \( x^n = \rho \cos \theta \) takes the form

\[
\psi^* g_+ = \frac{1}{\sin^2(\theta)} \left[ d\theta^2 + \frac{d\rho^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2}{\rho^2} \right].
\]

Now, as mentioned before, the umbilic boundary condition \( K_Q = \lambda g_+|_{\partial Q} \)
(where \( K_Q \) is the scalar second fundamental form) implies that \( Q \) and \( M \)
do make constant angle \( \theta_0 = \cos^{-1}(-\lambda) \), a fact that remains true if \( g_+ \) is
just admissible. In light of this fact and the preceding theorem, it is natural
to break the gauge of the Einstein equations by looking for an Einstein
metric \( g \) on the blowup in the form (1.1). In this gauge, the CMC umbilic
condition at \( Q \) takes the form \( \partial_\theta h_\theta|_{\theta = \theta_0} = 0 \), as shown in Lemma 5.1. Let
Let $M$ be a manifold with boundary $S$. Let $\rho$ be a defining function for $S$ in $M$ (throughout, we will use the same notation for the lift of $\rho$ to $[0, \theta_0] \times M$). We define $M(\theta_0, M)$ to be the set of families $h_\theta(0 \leq \theta \leq \theta_0)$ of smooth AH metrics on $M$ such that $\bar{h}_\theta = \rho^2 h_\theta$ is smooth in $\theta$ (in case $n = 2$ or $n$ is odd) or such that it is smooth in $\theta$ and $\theta^n \log(\theta)$ (if $n \geq 4$ is even). See page 24 for full details. The following is the main theorem of this paper. In the statement, $T = O_g(f)$ for $T$ a tensor field means $|T|_g = O(f)$. Here and throughout the paper, we take $n \geq 2$.

**Theorem 1.1.** Let $M^{n}$ be a manifold with boundary, let $\lambda \in (-1, 1)$, let $\rho$ be a defining function for $S = \partial M$ in $M$, and let $[h]$ be a conformal class on $M$. Set $\theta_0 = \cos^{-1}(-\lambda)$. Then there exists a one-parameter family $h_\theta \in M(\theta_0, M)$ of smooth AH metrics on $M$, such that if $g$ is the normal-form metric

$$g = \csc^2(\theta)[d\theta^2 + h_\theta]$$

on $\tilde{X} = [0, \theta_0]_\theta \times M$, then

(a) $h_0 \in [h]$;
(b) $\partial_\theta \bar{h}_\theta|_{\rho=0} = 0$, where $\bar{h}_\theta = \rho^2 h_\theta$;
(c) the second fundamental form $K_{\tilde{Q}}$ of $\tilde{Q} \setminus S = \{\theta_0\} \times (M \setminus S)$ satisfies $K_{\tilde{Q}} = \lambda g|_{T\tilde{Q}}$; and
(d) the formal Einstein condition

$$\text{Ric}(g) + ng = O_g(\rho^n)$$

is satisfied.

If $\theta_0$ is such that Equation (4.18) has no integral solutions $\nu$ when $s = n$, then in fact we may choose $h_\theta$ so that

$$\text{Ric}(g) + ng = O_g(\rho^\infty).$$

In all cases, if $h_\theta, h'_\theta$ are two families satisfying the above conditions, then $\bar{h}_\theta - \bar{h}'_\theta = O(\rho^n)$. If Equation (4.18) has no integral solutions, as above, then two infinite-order solutions satisfying $\bar{h}_\theta - \bar{h}'_\theta = o(\rho^n)$ satisfy $\bar{h}_\theta - \bar{h}'_\theta = O(\rho^\infty)$.

Notice that the given data is only $(M, [h])$ and $\lambda$, and that we get an Einstein metric in normal form that is unique up to order $n$. In particular, the induced metric $h_0 \in [h]$ is an AH representative of the conformal class that is invariantly defined to order $n$ given only $[h]$ and $\lambda$.

Uniqueness in Theorem 1.1 should hold without [b] but we do not yet have a proof of this.

The proof of Theorem 1.1 involves the complications generally associated with Einstein’s equations: nonlinearity, tensor fields, and of course (as already mentioned) gauge invariance. Before taking up its proof, then, we analyze a simpler linear problem that already raises several of the distinctive issues of analysis in our setting; namely, we consider the formal expansion, at the corner of a constant-angle CAH space, of eigenfunctions of the Laplacian
with inhomogeneous Dirichlet condition at the infinite boundary $\tilde{M}$ and a homogeneous Robin condition at the finite boundary, $Q$. (Besides being a natural boundary condition, this will prove to be of direct relevance to the Einstein problem.) In the usual way, we are using Dirichlet condition to mean prescribing the coefficient at the power of the leading indicial root at $\theta = 0$; see [GZ03]. The problem we wish to solve is

\begin{equation}
\Delta \tilde{u} + s(n-s) \tilde{u} = 0,
\end{equation}

where the expression of the eigenvalue in terms of the spectral parameter $s > \frac{1}{2}$ is traditional (see e.g. [MM87], [GZ03]). Thus, if $s = n$, and we are looking for harmonic functions, then indeed our boundary condition at $\tilde{M}$ is an inhomogeneous Dirichlet condition.

The analysis of the linear problem proceeds in several steps. The key idea, as in general for such constructions, is to determine and then study the \textit{indicial operator} of $\Delta + s(n-s)$, which is an operator on $C^\infty(\tilde{S})$ defined by

\[ I_{s,\nu}(u) = \rho^{-\nu}[(\Delta \rho + s(n-s))(\rho^\nu \tilde{u})]\big|_{\rho=0}, \]

where $\rho$ is a particular defining function for $\tilde{S}$ in $\tilde{X}$ and $\tilde{u}$ is an extension of $u$ to $\tilde{X}$. However, we here meet a significant difference from the usual AH case: whereas the indicial operator is there an algebraic operator, due to the edge structure of $g$ at $\tilde{S}$, it here restricts to a second-order ordinary differential operator on each fiber of $\tilde{S}$, with a regular singularity at $\theta = 0$:

\begin{equation}
I_{s,\nu}(u) = \sin^2(\theta)\partial^2_{\theta}u + (1-n)\sin(\theta)\cos(\theta)\partial_{\theta}u + \nu(\nu+1-n)\sin^2(\theta)u + s(n-s)u
\end{equation}

For any $\nu$, the indicial roots of $I_{s,\nu}$ at $\theta = 0$ are $n-s$ and $s$. The key content of Section 4.2, then, is an analysis of this operator with the “Dirichlet” boundary condition $u(\theta) = o(\theta^{n-s})$ at $\theta = 0$ and the Robin condition $\partial_{\theta}u(\theta_0) + (s-n)\cot(\theta_0)u(\theta_0) = 0$ at $\theta = \theta_0$, for $0 < \theta_0 < \pi$. We study the mapping properties of the Green’s operator, and also study the indicial roots of the Laplacian, or values of $\nu$ for which the indicial operator fails to be injective with the given boundary conditions. The latter is equivalent to studying the singular Sturm-Liouville eigenvalue problem for the operator $L = -\partial^2_{\theta} + (2s-n-1)\cot(\theta)\partial_{\theta} + (s-1)(s-n)$. We estimate the lowest eigenvalue, and can characterize the eigenvalues in general as the roots of an equation involving hypergeometric functions. In the case that $\theta_0 = \frac{\pi}{2}$, we can calculate explicitly that they are $\lambda_k = \nu_k(\nu_k + 1-n)$ for $\nu_k = s + 2k$ ($k \geq 0$). For this reason, we restrict our full analysis to the case $\theta_0 = \frac{\pi}{2}$; although similar ideas would apply in the general case, it would be difficult to be as specific as we can be when we know the eigenvalues explicitly. For that case, in Section 4.3, we formally construct a harmonic function order by order in $\rho$, solving at each order $j$ an equation of the form $I_j f = f_j$. When $2s \notin \mathbb{Z}$, we can construct a unique solution iteratively without ever encountering an indicial root, as $s - (n-s)$ is likewise non-integral. Otherwise, when $j = s + 2k$ is an indicial root, we show that we can proceed by including powers of $\log(\rho)$ in the solution, although uniqueness is lost. In such a case, a formal solution
could be uniquely parametrized by $u|_{\tilde{M}}$ and by $\{v_k\}_{k=0}^\infty$, where $v_k \in C^\infty(S)$ parametrizes the formal freedom at order $s + 2k$.

Depending on $s$, we define $\mathcal{P}_s(\tilde{X})$ to be either smooth functions, or those functions on $\tilde{X}$ that have an asymptotic expansion in $\rho, \theta, \theta^{2s-n} \log(\theta)$ to the first power, and $\rho^{2s-n+2k} \log(\rho)^k (k \geq 0)$, where $\rho$ is a defining function for $\tilde{M} \cap \tilde{S}$ in $\tilde{M}$; see page 24 for a precise definition. Our result is as follows.

**Theorem 1.2.** Let $(\tilde{X}^{n+1}, \tilde{M}, \tilde{Q}, \tilde{S})$ be the blowup of the cornered space $(X, M, Q)$, with $g = b^* g_X$ an admissible metric such that $M$ and $Q$ make constant angle $\frac{\pi}{2}$ with respect to $g_X$. Let $\psi \in C^\infty(\tilde{M})$ and $s > \frac{\pi}{2}$. There exists $F \in \mathcal{P}_s(\tilde{X})$ such that, if $u = \rho^{n-s} \sin^{n-s}(\theta) F$, then $\Delta_g u + s(n-s) u = O(\rho^\infty)$, such that $F|_{\tilde{M}} = \psi$, and such that $\partial_\nu u + s(n-s) \cot(\theta_0) u = 0$ along $Q$, where $\partial_\nu$ is the normal derivative at $Q$. If $2s \notin \mathbb{Z}$, then such $F$ is unique to infinite order. Otherwise, if $F_1$ and $F_2$ are two such functions, then $F_1 - F_2 = O(\rho^{2s-n})$, and $\rho^{n-2s}(u_1 - u_2)|_{\tilde{S}} = v_0 \sin^{2s-n} \Theta$, where $v_0 \in C^\infty(\tilde{S})$ is constant on fibers, and $\Theta$ is the natural angle function on $\tilde{S}$ induced by $g$.

We note that the power $n-s$ of $\rho$ appearing in the definition of $u$ in this theorem is not fully determined by the problem, and could be chosen differently depending on $s$. This is discussed more after the proof of the theorem in Section 4 (In a paper in preparation, we will consider existence and polyhomogeneity for eigenfunctions of the Laplacian, with boundary conditions as above.)

The proof of Theorem 1.2 is conceptually similar to that of the simpler Theorem 1.1, but significant complications arise in the Einstein setting, as mentioned earlier. We define an indicial operator for the Einstein operator $E(g) = \text{Ric}(g) + ng$, as in the scalar case, by $\Gamma(\varphi) = \rho^{-n}(E(g + \rho^2 \varphi) - E(g))|_{\rho=0}$, where $\varphi$ is a section of an appropriate bundle; and as in [GL91], we decompose it into its irreducible parts, in this case seven of them. Once again, and unlike in that paper, the indicial operator is a second-order system of regular singular ordinary differential operators as opposed to algebraic operators. As in the AH case, the part of the indicial operator acting on the trace-free part of the metric perturbations tangent to $S$ is identical with the indicial operator of the scalar Laplacian. We construct the Einstein metric term-by-term in $\rho$. At each order, this gives us a system of second-order regular singular ordinary differential equations to solve, which is overdetermined because of the gauge-broken form 1.1. An additional complication in the analysis comes from the fact that, since the Einstein metric is unique to order $n$, then as observed above and unlike in the case of the usual AH Einstein existence problem, the induced metric $h_0$ in the conformal infinity is uniquely determined and cannot be chosen arbitrarily in the conformal class. These two problems are solved in tandem. For our boundary data, we take $h \in [h]$ to be arbitrary (but AH), and then impose the boundary condition $h_0 = \chi h$, where $\chi \in C^\infty(M)$ is some scalar function to be determined order by order in $\rho$ along with $g$. Thus the induced metric
is determined simultaneously with the metric $g$. At each order, we use four of the seven irreducible parts of the indicial operator to solve uniquely for the perturbation of the metric at that order. We then use the Bianchi identity to show that the remaining three equations are also satisfied. However, this turns out to be true only for a unique choice of the perturbation of $\chi$, and thus we get uniqueness both for $g$ and $\chi h$ (within $[h]$) up to order $n$.

The behavior of the system changes at order $n$: at that order, $\chi$ is no longer determined, but may be chosen freely, and different components of the indicial operator must be used to determine $\varphi$. If this can be done successfully (i.e., if $n$ is not an indicial root), then at higher orders, the system once again acts as at lower powers, and $\chi$ is uniquely determined at each step. The trace-free part of the indicial operator has a set of eigenvalues going to infinity; in the special case that $\lambda = 0$ ($\theta_0 = \frac{\pi}{2}$), the first of these is at order $n$, as mentioned above, and this allows us to identify a conformal hypersurface invariant obstructing smooth existence.

**Theorem 1.3.** Let $M^n$ ($n \geq 2$) be a manifold with boundary $S$, and $\tau$ a smooth metric on $M$. Let $[h]$ be the AH conformal class corresponding to $[\tau]$. There is a generically nontrivial symmetric, trace-free 2-tensor field $K(\tau)$ on $S$, defined by (5.18), whose nonvanishing obstructs the formal existence of a smooth normal-form metric $g = \csc^2(\theta)[d\theta^2 + h_0]$ on $[0, \frac{\pi}{2}] \times M$ satisfying (a)–(c) from Theorem 1.1, and also satisfying $\text{Ric}(g) + ng = O(\rho^{n+1})$.

Moreover, if $\hat{\tau} = \Omega^2 \tau$ for $\Omega \in C^\infty(M)$, then $K(\hat{\tau}) = (\Omega|_S)^2 - n K(\tau)$.

Theorem 1.1 is concerned with smooth formal series in $\rho$. As we will show, the lowest non-negative indicial root $\gamma_0$ in $\rho$ of the Einstein operator satisfies $n - 1 < \gamma_0 < n$ if $\theta_0 > \frac{\pi}{2}$, $\gamma_0 = n$ if $\theta_0 = \frac{\pi}{2}$, and $n < \gamma_0 < \infty$ if $\theta_0 < \frac{\pi}{2}$. Thus, if $\frac{\pi}{2} < \theta_0 < \pi$, then we would expect additional solutions with leading asymptotics $\rho^n$, where $n - 1 < \gamma < n$. If $\theta_0 \leq \frac{\pi}{2}$, then uniqueness would hold mod $O(\rho^n)$ even allowing non-integral powers of $\rho$. If $\theta_0 = \frac{\pi}{2}$, then a term of the form $\rho^n \log(\rho)$ would generically appear, as suggested by Theorem 1.3 above. The form of the solution to higher order depends on indicial roots, which depend on $\theta_0$. Uniqueness fails at order $n$ in every case, however.

The paper is organized as follows.

In Section 2, we define cornered AH spaces and review the results on them from [McK16], including the definition of admissible metrics, the 0-edge structure on CAH spaces, and the normal form.

In Section 3, we study smooth Einstein metrics and deduce the compatibility conditions for smoothness discussed earlier, under the assumption of the CMC umbilic condition at the finite boundary.

In Section 4, we study the scalar Laplacian on constant-angle CAH spaces. After calculating the scalar Laplace operator in coordinates on the blowup, we then compute its indicial operator (1.3), and prove theorems about its eigenvalues and the mapping properties of its Green’s operator. We do this for general $\theta_0$ and arbitrary integral powers of $\log(\theta)$, since although these features are unnecessary for our analysis of the linear scalar problem with
θ₀ = π/2, they will be used in the nonlinear Einstein setting. We then prove Theorem 1.2.

Finally, in Section 5 we study formal existence for arbitrary S, enlarging the class of metrics from those smooth on X to those polyhomogeneous on \( \tilde{X} = [0, \theta_0] \times M \) in the form (1.1); and prove Theorems 1.1 and 1.3. We also discuss an approach to finding formal solutions for which the Einstein tensor vanishes at both \( \tilde{S} \) and \( \tilde{M} \).

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2. Background

Recall that an asymptotically hyperbolic (AH) space is a compact manifold \( X^{n+1} \) with boundary \( \partial X = M^n \), equipped on the interior with a metric \( g \) such that, for any defining function \( \varphi \) of \( M \) in \( X \), the metric \( \varphi^2 g \) extends smoothly to all of \( X \); and such that, for any such defining function, \( |d\varphi|_{\varphi^2 g}^2 = 1 \) along \( M \).

We now review the definition, properties, and blowup of a cornered asymptotically hyperbolic (CAH) metric, as given in [McK16].

**Definition 2.1.** A cornered space is a smooth manifold with codimension-two corners, \( X^{n+1} \), such that

(i) there are submanifolds with boundary \( M^n \subset \partial X \) and \( Q^n \subset \partial X \) of the boundary \( \partial X \), such that \( \emptyset \neq S = M \cap Q \) is the mutual boundary, and is the entire codimension-two corner of \( X \), and such that \( \partial X = M \cup Q \); and

(ii) the corner \( S \subset M \) is a smooth, compact hypersurface in \( M \).

We denote a cornered space by \( (X, M, Q) \), and we set \( \tilde{X} = X \setminus (Q \cup M) \).

Given a cornered space \( (X, M, Q) \), a smooth (resp. \( C^k \)) cornered conformally compact metric on \( X \) is a smooth Riemannian metric \( g_+ \) on \( X \setminus M \) such that, for any smooth defining function \( \varphi \) for \( M \), the metric \( \varphi^2 g_+ \) extends to a smooth (resp. \( C^k \)) metric on \( X \). We call such a metric a cornered asymptotically hyperbolic (CAH) metric if for some (hence any) such defining function \( \varphi \), the condition \( |d\varphi|_{\varphi^2 g_+} = 1 \) holds along \( M \).
A smooth (resp. \( C^k \)) cornered asymptotically hyperbolic (CAH) space is a cornered space \((X, M, Q)\) together with a smooth (resp. \( C^k \)) CAH metric \( g_+ \). We denote such a space by \((X, M, Q, g_+)\). The definition for cornered conformally compact space is analogous.

For a cornered conformally compact space \((X, M, Q, g_+)\), the conformal infinity \([h]\) is the conformal class \([\varphi^2 g_+|_{TM}]\) on \(M\), where \(\varphi\) is a defining function for \(M\). Notice that a consequence of the fact that \(X\) is a manifold with corners is that the boundary components \(M \) and \(Q\) intersect transversely.

For each \(x \in S\), we define \(\theta_0(x)\) to be the angle between \(M\) and \(Q\) at \(X\) with respect to \(\varphi^2 g_+\), where \(\varphi\) is any smooth defining function for \(M\). Plainly \(\theta_0 \in C^\infty(S)\).

As described in more detail in [McK16], we may blow up the cornered space \(X\) along \(S\) as follows, along the lines of [Mel08]. Let \(s \in S\), and set \(N_s S = T_s X/T_s S\), which is a two-dimensional vector space. Set \(NS = \sqcup_{s \in S} N_s S\), and let \(N_+ S \subset NS\) be the vectors pointing into \(X\) (including into \(M, Q\), or \(S\)). Thus, \(N_+ S\) is a bundle with fiber a closed cone in \(\mathbb{R}^2\) and base \(S\). Let \(\tilde{S} = (N_+ S \setminus \{0\})/\mathbb{R}^+\), which is the total space of a fibration over \(S\) with fiber \([0, 1]\). Now define the blow-up space \(\tilde{X}\) by \(\tilde{X} = (X \setminus S) \sqcup \tilde{S}\), and the blow-down map \(b : \tilde{X} \rightarrow X\) by \(b(x) = x\) if \(x \in X \setminus S\) and \(b(s) = \pi(s)\) for \(s \in \tilde{S}\), where \(\pi\) is the basepoint projection. Then \(\tilde{X}\) has a unique smooth structure as a manifold with corners of codimension two such that \(b\) is smooth, \(b|_{\tilde{X}\setminus S} : \tilde{X} \setminus \tilde{S} \rightarrow X \setminus S\) is a diffeomorphism, and \(db|_{\tilde{s}}\) has rank \(n\) for \(s \in \tilde{S}\). We set \(\tilde{M} = b^{-1}(M \setminus S)\) and \(\tilde{Q} = b^{-1}(Q \setminus S)\).

Recall that an edge structure on a manifold with boundary is a fibration of the boundary, and the associated edge vector fields are the vector fields that are tangent to the fibers at the boundary ([Maz91]). An important special case is a 0-structure ([MM87]), for which the boundary fibers are points and the edge vector fields are those that vanish at the boundary. The vector fields in that setting may be viewed as sections of the 0-bundle, \(0TM\), with dual bundle \(0T^*M\). On our blowup space \(\tilde{X}\), the blow-up face \(\tilde{S}\) is the total space of the fibration \(b|_{\tilde{S}} : \tilde{S} \rightarrow S\) with interval fibers, while we can view \(b|_{\tilde{M}} : \tilde{M} \rightarrow M\) as a fibration whose fibers are points. We will refer to the structure defined by these two fibrations as a 0-edge structure, and the associated 0-edge vector fields are the smooth vector fields on \(\tilde{X}\) which are tangent to the fibers at \(\tilde{S}\), and which vanish at \(\tilde{M}\).

The 0-edge vector fields may be easily expressed in appropriate local coordinates. Let \(\theta\) be a defining function for \(\tilde{M}\) whose restriction to each fiber of \(\tilde{S}\) is a fiber coordinate taking values in \([0, \pi]\); let \(\rho\) be any defining function for \(\tilde{S}\); and locally let \(x^s, 1 \leq s \leq n - 1\), be the lifts to \(\tilde{X}\) of functions on \(X\) that restrict to local coordinates on \(S\). Then the vector fields

\[
\sin \theta \frac{\partial}{\partial \theta}, \quad \rho \sin \theta \frac{\partial}{\partial x^s}, \quad \rho \sin \theta \frac{\partial}{\partial \rho}
\]
span the 0-edge vector fields over \( C^\infty(\tilde{X}) \). As in the usual edge case, there is a well-defined vector bundle \( 0^eT\tilde{X} \) whose smooth sections are the 0-edge vector fields. The smooth sections of the dual bundle \( 0^eT^*\tilde{X} \) are locally spanned by

\[
\frac{d\theta}{\sin \theta}, \frac{dx^s}{\rho \sin \theta}, \frac{d\rho}{\rho \sin \theta}.
\]

By a 0-edge metric we will mean a smooth positive definite section \( g \) of \( S^2(0^eT^*\tilde{X}) \). This is equivalent to the condition that locally \( g \) may be written as

\[
g = \begin{pmatrix}
\frac{d\theta}{\sin \theta}, & \frac{dx^s}{\rho \sin \theta}, & \frac{d\rho}{\rho \sin \theta}
\end{pmatrix} G \begin{pmatrix}
\frac{d\theta}{\sin \theta}, & \frac{dx^s}{\rho \sin \theta}, & \frac{d\rho}{\rho \sin \theta}
\end{pmatrix},
\]

where \( G \) is a smooth, positive-definite matrix-valued function on \( \tilde{X} \).

We may now define admissible metrics.

**Definition 2.2.** An admissible metric on \( \tilde{X} \) is a 0-edge metric \( g \) on \( \tilde{X} \) which can be written in the form

\[
g = b^*g_+ + \mathcal{L},
\]

where \( g_+ \) is a smooth cornered asymptotically hyperbolic metric on \( X \) and \( \mathcal{L} \) is a smooth section of \( S^2(0^eT^*\tilde{X}) \) that vanishes on \( \tilde{S} \) and \( \tilde{M} \).

Note that it was shown in [McK16] that \( b^*g_+ \) is, itself, always a 0-edge metric.

Since \( b|_{\tilde{X}\setminus(\tilde{M}\cup\tilde{S})} : \tilde{X}\setminus(\tilde{M}\cup\tilde{S}) \to X\setminus M \) is a diffeomorphism, an admissible \( g \) uniquely determines a smooth metric \( g_X \) on \( X\setminus M \) satisfying \( b^*g_X = g \) on \( \tilde{X}\setminus(\tilde{M}\cup\tilde{S}) \). Since \( \mathcal{L} \) vanishes on \( \tilde{S} \) and \( \tilde{M} \), it is not hard to see that \( g_X \) is a \( C^0 \) CAH metric on \( X \). Thus we will call a metric \( g_X \) on \( X\setminus M \) an admissible metric on \( X \) if \( b^*g_X \) extends to an admissible metric on \( \tilde{X} \).

Observe that an admissible metric \( g_X \) on \( X \) determines a well-defined angle function \( \Theta \) on the blown-up face \( \tilde{S} \), which serves as a smooth fiber coordinate. Let \( \tilde{s} \in \tilde{S} \), with \( s = b(\tilde{s}) \in S \). Then, under one interpretation, \( \tilde{s} \) naturally represents a hyperplane \( P_{\tilde{s}} \) in \( T_{\tilde{s}}X \) containing \( T_{\tilde{s}}S \). The angle \( \Theta(\tilde{s}) \) between \( P_{\tilde{s}} \) and \( T_{\tilde{s}}M \) is well-defined. It can be computed as follows: let \( \varphi \) be any defining function for \( M \), and \( \tilde{g}_X = \varphi^2 g_X \). Let \( \tilde{\nu}_M \in T_{\tilde{s}}M \) be normal to \( T_{\tilde{s}}S \), inward-pointing in \( M \), and unit \( \tilde{g}_X \)-length. Similarly, let \( \tilde{\nu}_{P_{\tilde{s}}} \) be inward-pointing in \( P_{\tilde{s}} \), normal to \( T_{\tilde{s}}S \), and unit length. Then \( \Theta(\tilde{s}) = \cos^{-1}(\tilde{g}_X(\tilde{\nu}_M, \tilde{\nu}_{P_{\tilde{s}}})) \). We could also have defined \( \Theta \) using \( g_+ \), and in particular, it is clear that \( \Theta \in C^\infty(\tilde{S}) \). It is easy to show that this is defined independently of \( \varphi \). Thus, \( \Theta \) is well-defined.

It will be convenient to recall that it was shown in Section 2 of [McK16] that there are coordinates \( (\theta, x^s, \rho) \) in which \( \theta \) is a defining function for \( \tilde{M} \) and restricts at \( \Theta \) to \( \tilde{S} \), in which \( \rho \) is a defining function for \( \tilde{S} \), and in which
\{x^a\} restrict as coordinates to $\tilde{S} \cap \tilde{M}$ and are constant on the fibers of $\tilde{S}$, such that the metric $g$ takes the form (2.2)

$$g_{ij} = \csc^2(\theta) \begin{pmatrix}
1 + O(\rho \sin \theta) & O(\sin \theta) & O(\sin \theta) \\
O(\sin \theta) & \rho^{-2}k_\rho + O(\rho^{-1} \sin \theta) & O(\rho^{-1} \sin \theta) \\
O(\sin \theta) & O(\rho^{-1} \sin \theta) & \rho^{-2} + O(\rho^{-1} \sin \theta)
\end{pmatrix},$$

and (2.3)

$$g^{ij} = \sin^2(\theta) \begin{pmatrix}
1 + O(\rho \sin \theta) & O(\rho^2 \sin \theta) & O(\rho^2 \sin \theta) \\
O(\rho^2 \sin \theta) & \rho^2k_\rho^{-1} + O(\rho^3 \sin \theta) & O(\rho^3 \sin \theta) \\
O(\rho^2 \sin \theta) & O(\rho^3 \sin \theta) & \rho^2 + O(\rho^3 \sin \theta)
\end{pmatrix}.$$

Much more can be said in the case that $M$ and $Q$ make constant angle, i.e., $\Theta$ is constant on $\tilde{S} \cap \tilde{Q}$.

**Theorem 2.3** ([McK16], Corollary 1.5). Let $(X, M, Q, g)$ be an admissible CAH space in which $M$ and $Q$ make constant angle $\theta_0$ with respect to compactifications of $g$; and let $b : \tilde{X} \to X$ be the blowup of $X$. Then for sufficiently small neighborhoods $W$ of $S$ in $M$, there exist a unique neighborhood $\tilde{U}$ of $b^{-1}(W)$ in $\tilde{X}$ and a unique diffeomorphism $\zeta : [0, \theta_0] \times W \to \tilde{U}$ such that $\zeta|_{\{0\} \times W} = \text{id}_W$ and

$$\zeta^* g = \frac{d\theta^2 + h_\theta}{\sin^2(\theta)},$$

where $h_\theta(0 \leq \theta \leq \theta_0)$ is a smooth one-parameter family of smooth asymptotically hyperbolic metrics on $(W, S)$, and such that $b^{-1}(M) = \zeta(\{\theta = 0\})$ and $b^{-1}(Q) = \zeta(\theta = \theta_0)$. Moreover, $\partial_\theta h_\theta|_{\rho = 0} = 0$, where $h_\theta = \rho^2 h_\theta$, and $\rho$ is any defining function for $S$ in $W$.

(Again we use $\rho$ both for the function on $W$ and for its pullback to $[0, \theta_0] \times W$.)

As mentioned in the introduction, this theorem provides the gauge we will use in looking for an Einstein metric.

The following corollary, also proved in [McK16], is a straightforward consequence that will also be of use to us.

**Corollary 2.4.** Let $\tilde{(X, M, Q, \tilde{S})}$, $(X, M, Q)$, and $g$ be as in Theorem 2.3 with again a constant angle $\theta_0$ between $Q$ and $M$, and let $[k]$ be the conformal class induced on $\tilde{S} \cap \tilde{M}$ (thus on $S$ by any compactification of $g$). For any $k \in [k]$ and for sufficiently small $\varepsilon > 0$, there is a neighborhood $\tilde{U}$ of $\tilde{S}$ in $\tilde{X}$ and a unique diffeomorphism $\chi : [0, \theta_0] \times S \times [0, \varepsilon]_{\rho} \to \tilde{U}$ such that $b \circ \chi|_{\{0\} \times S \times \{0\}} = \text{id}_S$ and

$$\chi^* g = \frac{d\theta^2 + h_\theta}{\sin^2(\theta)},$$

where $h_\theta(0 \leq \theta \leq \theta_0)$ is a smooth one-parameter family of smooth asymptotically hyperbolic metrics on $(W, S)$, and such that $b^{-1}(M) = \chi(\{\theta = 0\})$ and $b^{-1}(Q) = \chi(\theta = \theta_0)$. Moreover, $\partial_\theta h_\theta|_{\rho = 0} = 0$, where $h_\theta = \rho^2 h_\theta$, and $\rho$ is any defining function for $S$ in $W$. 

(Again we use $\rho$ both for the function on $W$ and for its pullback to $[0, \theta_0] \times W$.)
where $h_\theta$ is a smooth one-parameter family of smooth AH metrics on $S \times [0, \varepsilon)$ with
\[ h_0 = \frac{d\rho^2 + k_\rho}{\rho^2}, \]
where $k_\rho$ is a smooth one-parameter family of smooth metrics on $S$ with $k_0 = k$, and where $\bar{M} = \chi(\{\theta = 0\})$, $\bar{Q} = \chi(\{\theta = \theta_0\})$, and $\bar{S} = \chi(\{\rho = 0\})$. Moreover, $\partial_\theta(h_\theta|_{\rho=0}) = 0$, where $h = \rho^2h$.

Finally, it will be helpful to review the relationship between two types of conformal classes on a manifold with boundary. Let $M$ be a manifold with boundary $S$. The first type is the usual conformal class, $[\tau]$, where $\tau$ is a smooth metric on $M$. Here, $[\tau]$ is the family of metrics $\tau'$ such that $\tau' = \Omega^2\tau$ for some nonvanishing $\Omega \in C^\infty(M)$. The second type is an AH conformal class, $[h]$, where $h$ is an AH metric on $M$. Here, $[h]$ is precisely the set of metrics $\Psi^2h$, where $\Psi \in C^\infty(M)$ is nonvanishing and $\Psi|_S \equiv 1$.

Observe that there is a one-to-one correspondence between ordinary conformal classes $[\tau]$ and AH conformal classes $[h]$. Given a conformal class $[\tau]$, let $\tau' \in [\tau]$ and let $\varphi \in C^\infty(M)$ be any defining function for $S$ in $M$ such that $|d\varphi|_\tau = 1$ along $S$. Set $h = \varphi^{-2}\tau$. Then the conformal class $[h]$ is independent of the choices of $\varphi$ and of $\tau$. To see this, suppose $\psi$ is some other defining function satisfying $|d\psi|_\tau = 1$ along $S$. Then $\psi^{-2}\tau = \psi^{-2}\varphi^2(\varphi^{-2}\tau)$. But $\Psi = \psi^{-2}\varphi^2$ extends smoothly to all of $M$, and $\Psi|_S \equiv 1$ by the choice of $\varphi$, $\psi$. Thus $[h]$ does not depend on $\varphi$. Similarly, suppose $\tau' = \Omega^2\tau$, and let $\varphi'$ be a defining function for $S$ such that $|d\varphi'|_{\tau'} = 1$ along $S$. Then it is easy to check that if $\varphi = \Omega^{-1}\varphi'$, then $|d\varphi|_{\tau'} = 1$ along $S$; and $\varphi^{-2}\tau' = (\varphi')^{-2}\tau'$. Thus the map taking $[\tau]$ to $[h]$ is well-defined. It is an easy exercise to reverse these steps and show that it is a bijection.

**Notation.** Throughout, $X$ will be of dimension $n+1$, where unless otherwise stated, $n \geq 2$. We use index notation in polar coordinates (such as those given by Theorem 2.3). When doing so, except where stated otherwise, we let $\rho$ be a special defining function for $S$ in $M$ corresponding to $h_0$, i.e., a function such that $h_0 = \frac{d\rho^2 + k_\rho}{\rho^2}$ (see [GL91]). Then the space $[0, \theta_0] \times W$ given by our normal-form theorem decomposes into a product $[0, \theta_0] \times S \times [0, \varepsilon)_\rho$.

The coordinate index 0 will refer to the first factor $\theta$, and $n$ will refer to the last factor $\rho$. The indices $1 \leq s, t \leq n - 1$ will refer to local coordinates on $S$, while the indices $0 \leq i, j \leq n$ will run over all $n+1$ coordinates. Finally, $1 \leq \mu, \nu \leq n$ will run over $S$ and $\rho$.

The metric $g$ will be used to raise and lower indices, except that $g^{ij}$ is the inverse metric, and so also for $h^{ij}$ and any other metrics with raised indices. We write $\tilde{g} = \rho^2 \sin^2(\theta)g$ and $\tilde{h}_\theta = \rho^2h_\theta$, where again $h_\theta$ is as in Theorem 2.3. Note that $\tilde{g}$ is degenerate along $\bar{S}$.

If $A$ is a covariant $k$-tensor, we write $A = O_g(f)$ to indicate that $|A|_g = O(f)$; or equivalently, that if $Y_1, \ldots, Y_k$ are $g$-unit vector fields, then the condition $A(Y_1, \ldots, Y_k) = O(f)$ holds with constant independent of the $Y_i$. 


Similarly, if \( Y \) is a vector field, we write \( Y = O_g(f) \) to indicate \( |Y|_g = O(f) \). Note that this condition is independent of the particular admissible metric \( g \).

### 3. Compatibility Conditions for Smooth Solutions

In this section, we prove basic results about Einstein spaces satisfying our boundary condition \( K_Q = \lambda g|_{TQ} \), such as the fact that \( \lambda \in (-1,1) \) necessarily. We also study compatibility conditions that are imposed on the corner \( S \) if the Einstein metric \( g \) is to have smooth compactification. These prove to be severe with the given boundary conditions.

Suppose a manifold \( M^n \) with boundary \( S \) is given, and is equipped with a Riemannian conformal class \([h]\), which throughout this section will be taken to be a smooth conformal class. In this section, we will investigate, if \((M,[h])\) is to be the conformal infinity of a smooth CAH Einstein space \((X,M,Q,g)\) satisfying \( K_Q = \lambda g|_{TQ} \), what we can deduce from \( M \) and \( S \) about the developments of \( g \) and \( Q \); and what constraints are put on \( S \). Because we will frequently want to use indices on \( g^+ \), and no blowups occur, for convenience we break in this section with the notation used in the rest of this paper and write \( g = g^+ \). Elsewhere, \( g \) will remain an admissible metric on \( \tilde{X} \). Our use of indices will also vary slightly from the remainder of the paper, as we describe below. As mentioned in the introduction, this situation has been analyzed to first order in \([NTU12]\) with the \textit{a priori} assumption that \( g \) is the hyperbolic metric. We remove this assumption to investigate the general case. In this section, we will work directly on \((X,M,Q)\) and not the blowup.

First, we state a classical result.

**Theorem 3.1.** Let \( n \geq 2 \), and let \( Q^n \subset \mathbb{H}^{n+1} \) be an umbilic hypersurface. Then the umbilic coefficient \( \lambda_p = \lambda \) is the same at all points \( p \in Q \). Moreover, as a subset of \( \mathbb{R}^{n+1} \), \( Q \) is either part of a sphere or part of a hyperplane. In particular, \( Q \) falls into one of the following classes:

(i) \( Q \) is part of a geodesic sphere. In this case, \( Q \) is part of a Euclidean sphere entirely contained in \( \mathbb{H}^{n+1} \). In this case, \( |\lambda| > 1 \);

(ii) \( Q \) is a horosphere: either it is part of a Euclidean sphere contained entirely in \( \mathbb{H}^{n+1} \) except for one point, which lies on the boundary \( \mathbb{R}^n \); or it is part of a Euclidean plane contained in \( \mathbb{H}^{n+1} \) and parallel to \( \mathbb{R}^n \). In either case, \( |\lambda| = 1 \);

(iii) \( Q \) is totally geodesic. In this case, \( M \) is either part of a Euclidean hemisphere that meets the boundary \( \mathbb{R}^n \) normally, or part of a Euclidean hyperplane that meets the boundary normally. In either event, \( \lambda = 0 \); or

(iv) \( Q \) is part of a Euclidean sphere or Euclidean hyperplane that intersects the boundary \( \mathbb{R}^n \) non-normally. In the case of a sphere, \( M \) lies in the upper part of a sphere of radius \( r \), such that its center \((y,x)\) satisfies \( 0 < |y| < r \). Either case is called an equidistant hypersurface, because it...
lies at fixed distance from the totally geodesic hypersurface $S$ such that the intersection of $S$ with $\mathbb{R}^n$ is the same as that of the plane or sphere in which $M$ lies. In this situation, $0 < |\lambda| < 1$.

See [Spi99, chap. IV.7] for a discussion and proof. Now this theorem implies that if an umbilic hypersurface $Q$ in hyperbolic space intersects the boundary $M = \mathbb{R}^n$ at infinity, the intersection $S$ must be a hyperplane or a sphere. This duplicates the result found in [NTU12], although it depends on global geometry. On the other hand, and unlike the finding of that paper, this shows that the result must hold even for $n = 2$.

We next prove a lemma in a more general context.

**Lemma 3.2.** Let $(X, g)$ be a Riemannian manifold with a smooth metric $g$, and with embedded hypersurfaces $Q$ and $M$ that intersect transversely in an embedded submanifold $S$. Denote by $K$ the scalar second fundamental form with respect to a fixed unit normal vector. By $K_S$, we will mean the second fundamental form of $S$ considered as a submanifold of $M$, while $K_M$ and $K_Q$ will mean the second fundamental forms of $M$ and $Q$ as submanifolds of $X$. Then

\[
K_S = \frac{K_Q - \langle \nu_Q, \nu_M \rangle K_M}{\langle \nu_Q, \nu_S \rangle} \bigg|_{TS},
\]

where $\nu_S$ is the unit $g$-normal to $S$ in $M$, and $\nu_Q$ and $\nu_M$ are the unit $g$-normal vectors to $Q$ and $M$ in $X$.

**Proof.** Denote by $II$ the vector second fundamental form, with the same conventions as for $K$ in the statement: so, for example, $II_S$ is the vector second fundamental form of $S$ as a submanifold of $(M, g|_{TM})$. We compute each second fundamental form. We will use $P$ to denote an orthogonal projection operator, while $\nabla$ will denote the Levi-Civita connection on $X$.

First, for $p \in S$ and $X, Y \in T_pS$, extended smoothly to a neighborhood,

\[
II_S(X, Y) = P_{TS^\perp}P_{TM} \nabla_X Y = \langle \nabla_X Y, \nu_S \rangle \nu_S.
\]

Next,

\[
II_M(X, Y) = P_{TM^\perp} \nabla_X Y = \langle \nabla_X Y, \nu_M \rangle \nu_M.
\]

Now write $\nu_Q = \nu_Q^S \nu_S + \nu_Q^M \nu_M$ (such a decomposition must be possible at $S$ since $TS \subset TQ$). Then we have

\[
II_Q(X, Y) = \langle \nabla_X Y, \nu_Q \rangle 
= \langle \nu_Q^S \nabla_X Y, \nu_S \rangle + \nu_Q^M \langle \nabla_X Y, \nu_M \rangle (\nu_Q^S \nu_S + \nu_Q^M \nu_M) 
= (\nu_Q^S)^2 \langle \nabla_X Y, \nu_S \rangle + \nu_Q^S \nu_Q^M (\langle \nabla_X Y, \nu_S \rangle \nu_M + \langle \nabla_X Y, \nu_M \rangle \nu_S) 
+ (\nu_Q^M)^2 \langle \nabla_X Y, \nu_M \rangle \nu_M.
\]
Hence,
\[
\langle II_Q(X,Y), \nu_S \rangle = (\nu_Q^S)^2 \langle \nabla_X Y, \nu_S \rangle + \nu_Q^S \nabla^M \langle \nabla_X Y, \nu_M \rangle
\]
\[
= (\nu_Q, \nu_S)^2 \langle II_S(X,Y), \nu_S \rangle
\]
\[
+ \langle \nu_Q, \nu_S \rangle \langle \nu_Q, \nu_M \rangle \langle K_M(X,Y), \nu_M \rangle,
\]
where the last line follows from our prior computations. Now since \(M\) and \(Q\) are transverse, \(\langle \nu_Q, \nu_S \rangle \neq 0\). Moreover, \(\langle II_Q(X,Y), \nu_S \rangle = K_Q(X,Y)(\nu_Q, \nu_S)\).
Thus, we find
\[
K_Q(X,Y) = (\nu_Q, \nu_S)K_S(X,Y) + (\nu_Q, \nu_M)K_M(X,Y),
\]
which yields the claim.

We now continue in the CAH Einstein context, where we get an immediate corollary from the preceding result.

**Corollary 3.3.** Let \((M, [h])\) be a compact manifold with boundary \(S\), equipped with a conformal class \([h]\). Suppose \((X, M, Q, g)\) is a cornered AH Einstein space, such that \(K_Q = \lambda g|_{TQ}\), and with smooth conformal infinity \([h]\). Further, suppose that, for smooth defining functions, the compactified metric \(\bar{g}\) is smooth. Then \(S\) is umbilic in \(M\) with respect to any metric \(h \in [h]\).

**Proof.** It certainly suffices to consider only \(h\), as umbilicity is a conformally invariant condition. For the same reason, it follows that \(Q\) is \(g\)-umbilic. Let \(r\) be a geodesic defining function for \(M\), as in [GL91], and \(\bar{g} = r^2 g\). Because \(g\) is a CAH Einstein metric, \(M\) is \(\bar{g}\)-totally geodesic (see e.g. [Gra00]). Thus, since \(h = \bar{g}|_{TM}\) (where \(r\) is an appropriate defining function), the claim follows directly from (3.1). 

This corollary gives a substantial obstruction to the existence of a smooth CMC-umbilically cornered CAH Einstein space realizing \((M, [h])\) as its conformal infinity. For \(n > 2\), for example, it provides a proof relying only on the boundary geometry that, if \(\tilde{X}\) is hyperbolic space \(\mathbb{H}^{n+1}\) and \(M\) is a subset of \(\mathbb{R}^n\), then the boundary \(S\) of \(M\) must be a sphere or a hyperplane, as these are the only umbilic surfaces in Euclidean space. (This proof does not work if \(n = 2\), since every hypersurface of a 2-space is umbilic.) We may even further characterize the geometry near the boundary as we further expand the umbilic condition.

First we define some helpful coordinates. Fix a metric \(h \in [h]\). On a neighborhood \(V\) near a point of \(S\) in \(M\), we may always choose geodesic normal coordinates \(x^1, \ldots, x^n\) such that \(S = \{x^n = 0\}\) and such that \(\frac{\partial}{\partial x^n} \perp h\) TS, with \(1 = |\frac{\partial}{\partial x^n}|\) and \(\frac{\partial}{\partial x^n}((\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^n})) = 0\) at \(S\). Let \(r\) be a geodesic defining function for \(M\), so that \(h = r^2 g|_{TM}\), such that \(|dr|_\bar{g} = 1\), and such that \(x^0 = r, x^1, \ldots, x^n\) are coordinates for some \(X \supset U \simeq [0, \varepsilon) \times V\), where \(\frac{\partial}{\partial x} \perp \bar{g} TM\).
When working with these coordinates, we will use the Roman indices $0 \leq i, j, k \leq n$ to label coordinates on $X$; the Greek indices $0 \leq \alpha, \beta, \gamma \leq n - 1$ to label coordinates on $Q$; and the Roman indices $1 \leq s, t, u \leq n - 1$ to label coordinates on $S$.

Before continuing to explore the consequences of smoothness, we state a useful result that is true more generally.

**Proposition 3.4.** Let $(X, M, Q)$ be a cornered space, and let $g$ be a smooth CAH Einstein metric on $X$ satisfying $K_Q = \lambda g|_{TQ}$, where $K_Q$ is the second fundamental form of $Q$ with respect to the inward-pointing normal vector. Let $\bar{\nu}_M$ be the $X$-inward $\bar{g}$-unit normal to $M$, and let $\bar{\nu}_S$ be the $M$-inward $h$-unit normal to $S$ in $M$. Then at every point $p \in S$, $\cos(\theta_0(p)) = -\lambda$. In particular, $\lambda \in (-1, 1)$.

Notice that the proof given here depends only on the continuity of $K_Q$ up to $S$; thus, by Lemma 4.3 of [McK16], this proposition remains true if $g$ is only admissible.

**Proof.** Let $\nu$ be the inward-pointing unit normal field on $Q$, and $K$ the second fundamental form of $Q$, both with respect to $g$; and let $\bar{\nu} = \bar{\nu}_Q$ be the inward-pointing unit normal field on $Q$ with respect to $\bar{g}$. Now the umbilic condition is equivalent by Weingarten to

$$\langle \nabla_X \nu, Y \rangle_g = -\lambda \langle X, Y \rangle_g$$

for all $X, Y \in C^\infty(TQ)$. For $r \neq 0$, the unit normal to $Q$ with respect to $\bar{g}$ is given by $\bar{\nu} = r^{-1} \nu$. We wish to compute $\bar{K}(X, Y)$, the second fundamental form of $Q$ with respect to $\bar{g}$.

A straightforward computation shows that for any vector fields $X, Y$, we have

$$\nabla_X Y = \nabla_X Y + r^{-1} \left[ dr(X)Y + dr(Y)X - \langle X, Y \rangle_{\bar{\nu}} \grad_{\bar{g}} r \right].$$

For $q \in Q$ and $X, Y \in TQ|_q$, it follows (taking extensions where necessary) that

$$\bar{K}(X, Y) = -\langle \nabla_X (r^{-1} \nu), Y \rangle_{\bar{\nu}}$$

$$= -r^{-1} \langle \nabla_X \nu + dr(X)\bar{\nu} + dr(\bar{\nu})X - \langle X, \bar{\nu} \rangle \grad_{\bar{g}} r - dr(X)\bar{\nu}, Y \rangle_{\bar{\nu}}$$

$$= r^{-1} \left( r^2 K(X, Y) - dr(\bar{\nu}) \langle X, Y \rangle_{\bar{\nu}} \right).$$

Therefore,

$$\bar{K} = \frac{\lambda - dr(\bar{\nu})}{r} \bar{g}(X, Y)$$

is equivalent to $K_Q = \lambda g$.

Thus, we see that for $r \neq 0$, $Q$ is $\bar{g}$-umbilic with possibly non-constant umbilic coefficient. But we also see that, for $\bar{K}$ to remain smooth up to the
boundary – which it surely must, since \( \tilde{g} \) is a smooth metric – we must have

\[
\frac{dr(\nabla)}{r \to 0} = \lambda.
\]

In particular, since \( \partial_r \) is the inward-pointing unit \( \tilde{g} \)-normal at \( M \), which we denote by \( \bar{\nu}_M \), we find that (denoting \( \bar{\nu} = \bar{\nu}_Q \) for clarity)

\[
(3.4) \quad \langle \bar{\nu}_Q, \bar{\nu}_M \rangle = \lambda, \quad (r = 0),
\]

is equivalent to \( K = \lambda g + O(r^{-1}) \). Since \( \cos(\theta) = -\langle \bar{\nu}_Q, \bar{\nu}_M \rangle \), our claim is established. \( \blacksquare \)

As mentioned, the above result actually holds even for admissible metrics.

We now obtain a result that in general does not. We will henceforth in this section assume that inner products are with respect to \( g \) if not otherwise specified.

**Proposition 3.5.** Let \( X, M, Q, g, \lambda \), and \( K_Q \) be as in Proposition 3.4. Let \( \bar{\nu}_M \) be the \( X \)-inward \( \tilde{g} \)-unit normal to \( M \), and let \( \bar{\nu}_S \) be the \( M \)-inward \( h \)-unit normal to \( S \) in \( M \). Moreover, let \( \bar{\mathcal{R}} \) be the curvature tensor of \( \tilde{g} \). Write \( \bar{K}_S = \eta h \), where \( \bar{K}_S \) is the second fundamental form of \( S \) in \( M \) with respect to \( h \) and \( \bar{\nu}_S \), and \( \eta \in C^\infty(S) \); we can write this by Corollary 3.3. Then for any \( Z \in TS \),

\[
(3.5) \quad Z(\eta) = -\bar{\mathcal{R}}(\bar{\nu}_M, Z, \bar{\nu}_M, \bar{\nu}_S).
\]

**Proof.** Let \( \nu, K \) and \( \bar{\nu} \) be as in the previous proof. We will assume that (3.4) holds, and we multiply through by \( r \) in (3.3) to obtain

\[
(3.6) \quad rK(X, Y) = (\lambda - \langle \bar{\nu}_Q, \partial_r \rangle)\bar{g}(X, Y).
\]

This equation holds on \( Q \) for any \( X, Y \in C^\infty(TQ) \) if and only if \( K = \lambda g \). We will repeatedly differentiate it covariantly to obtain new equations.

Before proceeding, we write \( Q \) locally as the graph of a function,

\[
x^n = \varphi(r, x^1, \ldots, x^{n-1}).
\]

Notice that \( \varphi(0, x^1, \ldots, x^{n-1}) \equiv 0 \) by our choice of coordinates on \( M \). We next define a local frame on \( Q \) by

\[
E_\alpha = \frac{\partial}{\partial x^\alpha} + \frac{\partial \varphi}{\partial x^\alpha} \frac{\partial}{\partial x^n}.
\]

In particular, \( \{E_s\}_{s=1}^{n-1} \) is also a local frame for \( S \) at \( r = 0 \), in fact the coordinate frame.

Define \( f \in C^\infty([0, \varepsilon) \times V) \) by \( f = \varphi - x^n \). Then \( f \) vanishes precisely on \( Q \), and we may write \( \bar{\nu} = \frac{-\text{grad} \varphi}{|\text{grad} \varphi|} \). Using this and the fact that, at \( r = 0 \), the normal \( \bar{\nu} \) may be written as

\[
(3.7) \quad \bar{\nu} = \lambda \partial_r + \sqrt{1 - \lambda^2} \partial_{x^n}
\]

by (3.4), it is straightforward to show that

\[
(3.8) \quad \frac{\partial \varphi}{\partial r} \bigg|_{r=0} = \frac{-\lambda}{\sqrt{1 - \lambda^2}}.
\]
We intend to apply $\nabla^Q_{\xi_0}$, the Levi-Civita connection of $\bar{g}|_{TQ}$, to both sides of (3.7). Doing this once, and utilizing the metric property of the Levi-Civita connection, we obtain
\begin{equation}
(3.9) \quad \overline{K}_{\alpha\beta} + r \nabla^Q_{\xi_0} \overline{K}_{\alpha\beta} = -E_0(\langle \partial_r, \eta \rangle) \bar{g}_{\alpha\beta},
\end{equation}
which should hold for all $r$, along $Q$. Taking $r = 0$, we get
\begin{equation}
(3.10) \quad \overline{K}_{\alpha\beta} = -E_0(\langle \partial_r, \eta \rangle) \bar{g}_{\alpha\beta}
\end{equation}
Now
\begin{equation}
(3.11) \quad E_0(\langle \partial_r, \eta \rangle) = \langle \nabla_{\xi_0} E_0, \eta \rangle - \partial^r (\langle \partial_x^n, \nabla_{\xi_0} \eta \rangle)
\end{equation}
(\text{where } \nabla \text{ is still the Levi-Civita connection for } \bar{g} \text{ on } TX). \text{ Since } E_0 = \partial_r + \frac{\partial^2 \varphi}{\partial r^2} \partial_x^n, \text{ we have } \partial_r = E_0 - \frac{\partial^2 \varphi}{\partial r^2} \partial_x^n. \text{ Hence,}
\begin{equation}
(3.12) \quad \langle \partial_r, \nabla_{\xi_0} \eta \rangle = \langle E_0, \nabla_{\xi_0} \eta \rangle - \partial^r (\langle \partial_x^n, \nabla_{\xi_0} \eta \rangle)
\end{equation}
Moreover,
\begin{equation}
(3.13) \quad \langle \nabla_{\xi_0} \partial_r, \eta \rangle = \langle \nabla_{\xi_0} E_0, \eta \rangle - \langle \nabla_{\xi_0} \left( \frac{\partial \varphi}{\partial r} \partial_x^n \right), \eta \rangle \nonumber
\end{equation}
From (3.11), (3.12), and (3.13), we get
\begin{equation}
(3.14) \quad E_0 \langle \partial_r, \eta \rangle = - \left( \frac{\partial^2 \varphi}{\partial r^2} \langle \partial_x^n, \eta \rangle + \frac{\partial \varphi}{\partial r} E_0 \langle \partial_x^n, \eta \rangle \right).
\end{equation}
Now $1 \equiv |\vec{\eta}|^2 = \vec{\eta}^i (\overset{\rightarrow}{\partial}_i, \overset{\rightarrow}{\partial}_j) \langle \vec{\eta}, \overset{\rightarrow}{\partial}_j \rangle$. We apply $E_0$ to this equation, take $r = 0$, and use the facts that $TS \subset TQ$, so that $\vec{\eta} \perp T\lambda S$; that $\partial_x^n (|\overset{\rightarrow}{\partial}_n|_g) = 0$ on $S$; and that, because $g$ is Einstein, $\partial_\nu g|_{\nu r = 0} = 0$ to conclude that
\begin{equation}
(3.15) \quad \langle \partial_x^n, \eta \rangle E_0 \langle \partial_x^n, \eta \rangle + \langle \partial_r, \eta \rangle E_0 \langle \partial_r, \eta \rangle = 0.
\end{equation}
Combining this with (3.4), (3.8), (3.10), and (3.14), we finally obtain
\begin{equation}
(3.16) \quad \overline{K}_{\alpha\beta} = (1 - \lambda^2)^{3/2} \left( \frac{\partial^2 \varphi}{\partial r^2} \right) \bar{g}_{\alpha\beta} \quad (r = 0).
\end{equation}
By applying Lemma 3.2 the fact that $M$ is totally geodesic, and (3.7), we may conclude that
\begin{equation}
(3.17) \quad \overline{K}_S = (1 - \lambda^2) \left( \frac{\partial^2 \varphi}{\partial r^2} \right) \bar{g}|_{TS}.
\end{equation}
We have thus expressed the second-order term of the development of $Q$ in terms of the geometry of $S$ in $M$. 
We can use the foregoing computations to rewrite (3.9) as
\[ K_{\alpha\beta} + r \nabla^2 E_0 K_{\alpha\beta} = \left( \frac{\partial^2 \varphi}{\partial r^2} \langle \partial x^n, \nu \rangle + \left( \frac{\partial \varphi}{\partial r} \right) E_0 \langle \partial x^n, \nu \rangle \right) g_{\alpha\beta}. \]

To this, we now apply \( \nabla^2 E_0 \) once again. We obtain
\[ 2 \nabla^2 E_0 K_{\alpha\beta} + r (\nabla^2 E_0)^2 K_{\alpha\beta} = \left( \frac{\partial^3 \varphi}{\partial r^3} \langle \partial x^n, \nu \rangle + 2 \left( \frac{\partial^2 \varphi}{\partial r^2} \right) E_0 \langle \partial x^n, \nu \rangle \right) g_{\alpha\beta}. \]

We take \( r = 0 \) and utilize equation (3.15) and find in particular that, at \( r = 0 \),
\[ 2 \nabla^2 E_0 K_{\alpha\beta} = -E_0^2 \langle \langle \partial r, \nu \rangle \rangle \bar{g}_{\alpha\beta}. \]

We next wish to apply the Codazzi-Mainardi equation (see, e.g., [Spi99], Theorem III.1.11). Codazzi states that
\[ \langle R(E_\gamma, E_\alpha)E_\beta, \nu \rangle = \left( \nabla^Q E_\gamma, K \right) (E_\alpha, E_\beta) - \left( \nabla^Q E_\alpha, K \right) (E_\gamma, E_\beta). \]

Hence, taking \( \gamma = 0 \),
\[ \nabla^Q E_0 K_{\alpha\beta} = \nabla^Q E_\alpha K_{0\beta} + \langle R(E_0, E_\alpha)E_\beta, \nu \rangle. \]

We next take \( \alpha = s \) (i.e., \( 1 \leq \alpha \leq n - 1 \)) and \( \beta = 0 \). Hence, we have
\[ \nabla^Q E_0 K_{s0} = \nabla^Q E_s K_{00} + \langle R(E_0, E_s)E_0, \nu \rangle. \]

But along \( S \), \( K \) is known by (3.16), and we find that
\[ \nabla^Q E_s K_{00} = (1 - \lambda^2)^{3/2} \left( \frac{\partial^3 \varphi}{\partial r^3 \partial x^s} \right) \bar{g}_{00}. \]

At \( r = 0 \), \( \bar{g}_{00} = \frac{1}{1-\lambda^2} \), (recall that this is expressed in the \( \{ E_\alpha \} \) frame, not the coordinate frame). This allows us to conclude, via (3.21) and (3.22), that
\[ \nabla^Q E_s K_{s0} = \sqrt{1-\lambda^2} \left( \frac{\partial^3 \varphi}{\partial r^3 \partial x^s} \right) + \langle R(E_0, E_s)E_0, \nu \rangle. \]

We can now substitute \( \nu \) and the definitions of \( E_\alpha \) in terms of the coordinate frame to compute this curvature term in the coordinate basis. (Every appearance of \( \bar{R} \) in index notation will be with reference to local coordinates, not the frame on \( Q \)). We also utilize that, because \( g \) is Einstein, \( M \) is totally geodesic in \( X \) with respect to \( \bar{g} \), and so again by Codazzi, \( \langle R(\partial x^n, \partial x^s)\partial r, \partial x^n \rangle = 0 \). Carrying this straightforward computation out at \( r = 0 \), we find that
\[ \langle R(E_0, E_s)E_0, \nu \rangle = \frac{1}{\sqrt{1-\lambda^2}} R_{0s0n}. \]
Applying these computations to (3.19) and noting that \( \mathcal{F}_{s0} = 0 \) at \( r = 0 \), we conclude that

\[
(3.23) \quad \frac{\partial^3 \varphi}{\partial r^2 \partial x^s} = - \frac{1}{1 - \lambda^2} R_{0s0m}.
\]

In conjunction with (3.17), this yields the claim. ■

**Remark.** Because Euclidean space is flat, this is precisely what we needed to ensure that, in the \( n = 2 \) hyperbolic case, only circles and lines can occur at the corner boundary: if \( g \) is the hyperbolic metric on \( \mathbb{H}^3 \), then one choice for the compactified metric \( \tilde{g} \) is the Euclidean metric itself. In this case, \( R = 0 \), so by (3.5), \( S \) has a constant umbilic coefficient in \((M, \tilde{g}|_{TM})\). But the only umbilic hypersurfaces in Euclidean 2-space with constant umbilic coefficients are circles and straight lines. Notice that in terms purely of the local boundary geometry, this restriction occurs at higher order for \( n \geq 3 \) than for \( n = 2 \), where it is implied by Corollary 3.3.

The arguments of the preceding proposition can be extended to yield further conditions on \( \tilde{g} \) and \( Q \), and perhaps \( S \). For example, let us return to (3.20), this time with \( \alpha = s \) and \( \beta = t \) (\( 1 \leq s, t \leq n - 1 \)). The term \( \nabla^Q_{E_s} K_{0t} \) can be evaluated using our earlier calculations. We get

\[
\nabla^Q_{E_s} K_{0t} = \nabla_{E_s} \left[ (1 - \lambda^2)^{3/2} \left( \frac{\partial^2 \varphi}{\partial r^2} \right) \right] \tilde{g}_{0t} = 0.
\]

Hence, at \( r = 0 \),

\[
\nabla^Q_{E_0} K_{st} = R(E_0, E_s, E_t, \mathbf{v}).
\]

Expanding the right-hand side in the coordinate frame again yields

\[
(3.24) \quad \nabla^Q_{E_0} K_{st} = \lambda(\mathcal{F}_{0st0} - \mathcal{R}_{nstn}).
\]

By (3.19), \( tf_{\tilde{g}} \nabla^Q_{E_0} K_{st} = 0 \); thus we conclude that along \( S \),

\[
\lambda tf_{\tilde{g}|TS} (\mathcal{F}_{0st0} - \mathcal{R}_{nstn}) = 0.
\]

This represents an additional \( \varphi \)-independent condition on \( \tilde{g} \).

We can generalize this process to see the structure of the conditions that would arise as we differentiated more. Covariantly differentiating (3.6) \( k - 1 \) times by \( E_0 \) yields

\[
k(\nabla^Q_{E_0})^{k-1} K_{\alpha\beta} = E^k_0 (\langle \partial_r, \mathbf{v} Q \rangle) \tilde{g}_{\alpha\beta}.
\]

At each step, taking \( \alpha = \beta = 0 \) or \( \alpha = s, \beta = t \) yields new constraints on \( \frac{\partial^k \varphi}{\partial r^k} |_{r=0} \) and also on \( \tilde{g} \), its curvature tensor, and its derivatives. Taking \( \alpha = 0, \beta = s \) yields additional constraints on lower-order (that is, already-developed) \( r \)-derivatives of \( \varphi \), possibly also putting further constraints on \( \tilde{g} \), as in (3.24). In this way, a vastly overdetermined system of equations for \( \tilde{g} \) and \( \varphi \) would be produced. Actually carrying these computations out to higher order becomes quickly unwieldy; in any case, we have already developed extremely restrictive constraints on a smooth solution. In section
we will therefore study formal existence of Einstein metrics on the blowup that are in normal form in the sense of Theorem 2.3. Thus, in particular, the constructed metrics will be smooth on the blowup but not on the base. As we will see, this setting offers precisely the right relaxation of smoothness to allow relatively unique general solutions.

4. The Laplacian and ODE Analysis

In this section, we study the eigenvalue problem for the scalar Laplacian on cornered asymptotically hyperbolic spaces and prove Theorem 1.2. We study the formal asymptotics of eigenfunctions on a CAH space with given boundary conditions. Along the way, we prove results about some ODEs that will also be important in the next section.

Throughout, let \((X, M, Q)\) be a cornered space and \((\tilde{X}, \tilde{M}, \tilde{Q}, \tilde{S})\) its blowup, with \(g\) an admissible metric on \(\tilde{X}\). We will let \(\theta, x, \rho\) be coordinates in which \(g\) takes the form (2.2), with \(k_\rho\) as in that equation. We also introduce the following notation, motivated by [GL91]. We will write \(E^k\) to denote any polynomial of degree less than or equal to \(k\) in \(\sin(\theta)\frac{\partial}{\partial \theta}\) and \(\rho \sin(\theta)\frac{\partial}{\partial x}\) \((1 \leq \mu \leq n)\), with coefficients in \(C^\infty(\tilde{X})\). Finally, we fix \(s > \frac{3}{2}\).

We first compute the Laplace operator of \(g\).

**Lemma 4.1.** Let \((\tilde{X}^{n+1}, \tilde{M}, \tilde{Q}, \tilde{S})\) be the blowup of a cornered space, and \(g\) an admissible metric on \(\tilde{X}\) expressed in the form (2.2). Then for \(u \in C^\infty(\tilde{X} \setminus (\tilde{M} \cup \tilde{S}))\),

\[
\Delta_g u = \sin^2(\theta)\partial_\theta^2 u + (1 - n) \sin(\theta) \cos(\theta) \partial_\theta u + \rho^2 \sin^2(\theta) \partial_\rho^2 u + (2 - n) \rho \sin^2(\theta) \partial_\rho u + \rho^2 \sin^2(\theta) \Delta_k u + \rho \sin(\theta) E^2(u).
\]

**Proof.** Using the formula \(\Delta_g u = g^{-1/2} \partial_i (g^{ij} g^{1/2} \partial_j u)\), this follows easily from (2.2) and (2.3).

We wish to carry out an asymptotic analysis of solutions to a boundary-value problem for the equation \(\Delta_g + s(n - s) u = 0\). To carry out our analysis, we will expand the solution order-by-order in \(\rho\). Consequently, we work in terms of the **indicial operator** of \(P_s = \Delta_g + s(n - s)\). For \(\nu \in \mathbb{R}\), this is the operator \(I_{s, \nu} : C^\infty(\tilde{S}) \to C^\infty(\tilde{S})\) defined by extending \(u \in C^\infty(\tilde{S})\) to \(\tilde{u} \in C^\infty(\tilde{X})\), and then setting \(I_{s, \nu}(u) = \rho^{-\nu} P_s(\rho^\nu \tilde{u})|_{\rho=0}\). The following is immediate from Lemma 4.1.

**Lemma 4.2.** Let \(\nu \in \mathbb{R}\). Then

\[
I_{s, \nu}(u) = \sin^2(\theta) \partial_\theta^2 u + (1 - n) \sin(\theta) \cos(\theta) \partial_\theta u + \nu(n + 1) \sin^2(\theta) u + s(n - s) u.
\]

We now specialize to the constant-angle case, supposing that \(M\) and \(Q\) make constant angle \(\theta_0\) with respect to \(g_X\). We will assume the metric is in the form given by Corollary 2.4; however, we will suppress the
diffeomorphism $\chi$ and will simply regard $(\theta, x, \rho)$ as a parametrization of $\tilde{X} = [0, \theta_0] \times S \times [0, \varepsilon]$ near $\tilde{S}$. Thus, we will take $g$ to be given by

$$
(4.2) \quad g = \frac{1}{\sin^2(\theta)} [d\theta^2 + h_\theta],
$$

where $h_\theta$ is a smooth family of smooth AH metrics on the hypersurface $\tilde{M} = \{0\} \times S \times [0, \varepsilon]$ and where $h_0 = \frac{d\rho^2 + k_\rho}{\rho}$. We impose the inhomogeneous boundary condition $\sin^{s-n}(\theta) u|_{\tilde{M}} = \psi$ at $\tilde{M}$ (where $\psi \in C^\infty(\tilde{M})$) and the homogeneous Robin condition $\partial_\nu u + (s-n) \cot(\theta) u(\theta)|_{\bar{Q}} = 0$ at $\bar{Q}$. Notice that for the metric (1.2), this last is equivalent to $\partial_\nu u(\theta_0) + (s-n) \cot(\theta_0) u(\theta_0) = 0$. Notice also that if $s = n$ or for $\theta_0 = \frac{\pi}{2}$, this reduces to a homogeneous Neumann condition. These boundary conditions are motivated partially by their relevance, in the case $s = n$, to the Einstein problem considered in the next section. The form for $s \neq n$ is motivated by the naturalness of the analysis it leads to; however, the below analysis could easily be modified to study a variety of other boundary conditions.

Notice that, with the constant-angle hypothesis, the indicial operator (4.1) restricts to each fiber as an operator independent of the fiber, if each fiber is identified with the interval $[0, \theta_0]$. Since we will construct solutions order-by-order in $\rho$ using the indicial operator, our problem turns on an analysis of the equation $I_{s, \nu} u = f$, with homogeneous Robin boundary condition at $\theta = \theta_0$ and Dirichlet boundary condition at $\theta = 0$. We turn to an analysis of this equation.

4.1. Function Spaces. It will be useful, before undertaking an analysis of the indicial operator, to define some function spaces.

First we define several spaces of functions on the interval $[0, \theta_0]$, where $\theta_0 \in (0, \pi)$ is fixed. Let $n \geq 2$, $s \geq \frac{n}{2}$, and $k \geq 1$. If $2s \not\in \mathbb{Z}$, define,

$$
(4.3) \quad \mathcal{A}_{n, s, k}(\theta_0) = \theta^{n-s}C^\infty([0, \theta_0]).
$$

Otherwise, if $2s \in \mathbb{Z}$, define

$$
(4.4) \quad \mathcal{A}_{n, s, k}(\theta_0) = \theta^{n-s}C^\infty([0, \theta_0]) \oplus \theta^{n-s} \bigoplus_{i=1}^{k} (\theta^{2s-n} \log \theta)^i C^\infty([0, \theta_0]).
$$

In either case, define

$$
(4.5) \quad \mathcal{A}^0_{n, s, k}(\theta_0) = \{ u \in \mathcal{A}_{n, s, k} : \sin^{s-n}(\theta_0) u(\theta)|_{\theta=0} = 0 \\
\quad \quad \quad = u'(\theta_0) + (s-n) \cot(\theta_0) u(\theta_0) \}.
$$

Next, if $2s \not\in \mathbb{Z}$, define

$$
(4.6) \quad \mathcal{B}_{n, s, k}(\theta_0) = \theta^{n-s+1}C^\infty([0, \theta_0]).
$$

On the other hand, if $2s \in \mathbb{Z}$, define

$$
(4.7) \quad \mathcal{B}_{n, s, 1}(\theta_0) = \theta^{n-s+1}C^\infty([0, \theta_0]) \oplus \theta^{s+1} \log(\theta)C^\infty([0, \theta_0]),
$$
and for $k \geq 2$, define

$$B_{n,s,k}(\theta_0) = B_{n,s,1}(\theta_0) \oplus \theta^{s-n} \bigoplus_{i=2}^{k}(\theta^{2s-n} \log \theta)^i C^\infty([0, \theta_0]).$$

In general, $\theta_0$ will be fixed and clear from the context, and we will refer to these spaces simply as $A_{n,s,k}$ and $B_{n,s,k}$.

Now let $Y$ be any manifold with or without boundary, and $V$ a vector bundle over $Y$. We define spaces of one-parameter families of smooth sections $u_{\theta} : Y \to V$ ($0 \leq \theta \leq \theta_0$) as follows. Let $\pi : [0, \theta_0] \times M \to M$ be the projection on the second factor, and $\pi^* V$ be the pullback bundle of $V$ to the product. Thus we let $C^\infty([0, \theta_0] \times M, \pi^* V)$ be the space of smooth sections of the pullback bundle. Then, if $2s \notin \mathbb{Z}$, we set

$$A_{n,s,k}(\theta_0, Y, V) = \theta^{n-s} C^\infty([0, \theta_0] \times M, \pi^* V),$$

and otherwise, we set

$$A_{n,s,k}(\theta_0, Y, V) = \theta^{n-s} C^\infty([0, \theta_0] \times M, \pi^* V) \oplus \theta^{n-s} \bigoplus_{i=1}^{k}(\theta^{2s-n} \log \theta)^i C^\infty([0, \theta_0] \times M, \pi^* V).$$

We similarly define $A_{0,n,s,k}(\theta_0, Y, V)$ and $B_{n,s,k}$ in the obvious fashion, generalizing their scalar counterparts in the same way as for $A_{n,s,k}$. Thus, in particular, $A_{n,s,k}(\theta_0)$ is canonically isomorphic to $A_{n,s,k}(\theta_0, \{0\}, \mathbb{R})$. In each of these function spaces, if $V$ is omitted, then it is taken to be the trivial vector bundle $\mathbb{R} \times Y$.

We next define two families of function spaces on $[0, \theta_0] \times [0, \epsilon)$. For $q \geq 0$ an integer and $2s \in \mathbb{Z}$, we let $E_{n,s,q}$ be the space of functions $\eta$ that can be written

$$\eta(\theta, \rho) = \rho^{s+q} \sum_{i=0}^{\left\lfloor \frac{q}{2} + 1 \right\rfloor} \log(\rho)^i b_i(\theta),$$

where each $b_i \in A_{n,s,1}^0$. Next, we let $F_{n,s,q}$ be the space of functions $\eta$ on $[0, \theta_0] \times [0, \epsilon)$ that can be written

$$\eta(\theta, \rho) = \rho^{s+q} \sum_{i=0}^{\left\lfloor \frac{q+1}{2} \right\rfloor} \log(\rho)^i c_i(\theta),$$

where each $c_i \in B_{n,s,1}$.

Now let $(\tilde{X}, \tilde{M}, \tilde{Q}, \tilde{S})$ be the blowup of a cornered space $X^{n+1}$. Let $(\theta, x, \rho)$, where $x \in S$, be a parametrization of $\tilde{X}$ near $\tilde{S}$ for which $\theta$ and $\rho$ are defining functions for $\tilde{M}$ and $\tilde{S}$, respectively. An example would be the polar decompositions considered in Section 2. Then if $2s \notin \mathbb{N}$, we define

$$P_s(\tilde{X}) = R_s(\tilde{X}) = \rho^{n-s} \theta^{n-s} C^\infty(\tilde{X}).$$
Otherwise, we define
\begin{equation}
\mathcal{R}_s(\bar{X}) = \rho^{n-s}\theta^{n-s}C^\infty(\bar{X}) + \rho^{n-s}\theta^s \log(\theta)C^\infty(\bar{X}).
\end{equation}

Notice that the definition is independent of the choice of \( \theta \). Then, define \( \mathcal{P}_s(\bar{X}) \) to be the space of functions \( u \in C^\infty(\bar{X}) \) that have an asymptotic expansion
\begin{equation}
u_{\theta}(\bar{X}) = a_0(\theta, x, \rho) + \rho^{2s-n} \sum_{j=1}^{\infty} \rho^{2(j-1)} a_j(\theta, x, \rho),
\end{equation}
where each \( a_j \in \mathcal{R}_s(\bar{X}) \).

We next define three families of spaces which will be used only in Section 5, but which are sufficiently related to the above spaces that it will be convenient to have them defined here. First, if \( n \geq 4 \) is even, then we define \( \mathcal{H}_{n,k}(\theta_0, Y, \nu) = \mathcal{A}_{n,n,k}(\theta_0, Y, \nu) \); and we let \( \mathcal{H}_n(\theta_0, Y, \nu) \) be the space of maps \( u \) in \( C^\infty((0, \theta_0] \times M, \pi^*\nu) \cap C^{n-1}([0, \theta_0] \times M, \pi^*\nu) \) that have an infinite asymptotic expansion \( u \sim \sum_{i=0}^{\infty} u_i \) at \( \theta = 0 \), where \( u_i \in (\theta^n \log \theta)^i C^\infty((0, \theta_0] \times M, \pi^*\nu) \). If \( n = 2 \) or \( n \) is odd, then for consistency of notation, we define \( \mathcal{H}_{n,k}(\theta_0, Y, \nu) \) and \( \mathcal{H}_n(\theta_0, Y, \nu) \) to be simply \( C^\infty((0, \theta_0] \times M, \pi^*\nu) \).

Finally, if \( M^n \) is a manifold with boundary, we define the space of metrics \( \mathcal{M}(\theta_0, M) \subset \mathcal{H}_n(\theta_0, M, S^2(\partial^*M)) \) to be the \( h_0 \) in \( \mathcal{H}_n(\theta_0, M, S^2(\partial^*M)) \) such that, for \( \theta \) fixed, \( h_0 \) is a smooth AH metric on \( M \).

4.2. ODE Analysis. It is an elementary exercise in the theory of regular singular operators to see that the indicial roots in \( \theta \) of \( I_{s,\nu} \) at \( \theta = 0 \) are \( n-s \) and \( s \). We will refer to an indicial root (in \( \rho \)) of \( \Delta_g + s(n-s) \) as a value of \( \nu \) for which \( I_{s,\nu} \) is not injective on the space of smooth functions \( u \) on \([0, \theta_0]\) satisfying \( u(\theta) = o(\theta^{n-s}) \) and \( u'(\theta_0) + (s-n) \cot(\theta_0) u(\theta_0) = 0 \). It will be convenient to write \( u(\theta) = \sin^{n-s}(\theta) v(\theta) \). It is then straightforward to compute that, equivalently, an indicial root is a value of \( \nu \) for which \( \lambda_\nu := \nu(\nu + 1 - n) \) is an eigenvalue of the operator
\begin{align*}
L_s &= -\frac{d^2}{d\theta^2} - (n + 1 - 2s) \cot(\theta) \frac{d}{d\theta} + (s-1)(s-n) \\
&= -\sin^{2s-1-n}(\theta) d\theta (\sin^{n+1-2s}(\theta) d\theta) + (s-1)(s-n),
\end{align*}
with the boundary conditions \( v(0) = 0 = v'(\theta_0) \). In the above factoring of the eigenvalues \( \lambda_\nu \), we call \( \nu \) the spectral parameter. It will be useful also to record the relationship
\begin{equation}
I_{s,\nu}(u(\theta)) = \sin^2(\theta)(\lambda_\nu u(\theta) - \sin^{n-s}(\theta) L_s(\sin^{s-n}(\theta) u(\theta))).
\end{equation}
As an operator on \([0, \theta_0]\), \( L_s \) on the interval \((0, \theta_0)\) has a limit point singularity at \( \theta = 0 \), and we impose the homogeneous Neumann condition at \( \theta = 0 \). It will follow from results in this section that its spectrum is discrete; and it is then a standard result in Sturm-Liouville theory that the eigenfunctions, which are smooth, form an orthonormal basis for \( L^2([0, \theta_0]) \).
Proposition 4.5. Let \( L \) satisfy the equation

\[
\int_0^{\theta_0} (L_s u) v (\sin^{n+1-2s}(\theta)) d\theta = \int_0^{\theta_0} u'(\theta) v'(\theta) \sin^{n+1-2s}(\theta) d\theta
\]

and

\[
\int_0^{\theta_0} (L_s u) v (\sin^{n+1-2s}(\theta)) d\theta = \int_0^{\theta_0} u'(\theta) v'(\theta) \sin^{n+1-2s}(\theta) d\theta;
\]

\[
\int_0^{\theta_0} (L_s u) v (\sin^{n+1-2s}(\theta)) d\theta = \int_0^{\theta_0} u(\theta) v(\theta) \sin^{n+1-2s}(\theta) d\theta;
\]

then \( v \) of \( L \) satisfies the equation

\[
(4.11)
\]

\[
\int_0^{\theta_0} (L_s u) v (\sin^{n+1-2s}(\theta)) d\theta = \int_0^{\theta_0} u'(\theta) v'(\theta) \sin^{n+1-2s}(\theta) d\theta
\]

\[
+ (s - 1)(s - n) \int_0^{\theta_0} u(\theta) v(\theta) \sin^{n+1-2s}(\theta) d\theta;
\]

\[
\int_0^{\theta_0} (L_s u) v (\sin^{n+1-2s}(\theta)) d\theta = \int_0^{\theta_0} u(\theta) v(\theta) \sin^{n+1-2s}(\theta) d\theta.
\]

\[
\int_0^{\theta_0} (L_s u) v (\sin^{n+1-2s}(\theta)) d\theta = \int_0^{\theta_0} u(\theta) v(\theta) \sin^{n+1-2s}(\theta) d\theta.
\]

Proof. Equation (4.11) follows using integration by parts after writing \( L_s u(\theta) \) in divergence form as \( L_s u(\theta) = -\sin^{2s-1-n}(\theta) \frac{d}{d\theta} [\sin^{n+1-2s}(\theta) u'(\theta)] + (s - 1)(s - n) \). Equation (4.12) follows from part 4.11 by symmetry.

We next study the eigenvalues of \( L_s \). First, we state a singular Sturm comparison theorem from [Nai12].

Proposition 4.4 (Theorem 3 from [Nai12]). Suppose that \( u \in C^2((a, b)) \) satisfies the equation

\[
(p(t)u')' + q(t)u = 0,
\]

on the interval \((a, b)\), where \( p \in C^1((a, b)) \) and \( q \in C^0((a, b)) \), and \( p(t) \geq 0 \). Suppose further that \( \int_a^b \frac{1}{p(t)u(t)^2} dt = \infty \), and that \( \int_a^b \frac{1}{p(t)u(t)^2} dt = \infty \). Suppose further that \( u(t) \) has exactly \( n - 1 \) zeros on the interval \((a, b)\), where \( n \in \mathbb{N} \).

Now let \( P \in C^1((a, b)) \) and \( Q \in C^0((a, b)) \) be such that \( p(t) \geq P(t) \geq 0 \) and \( Q(t) \geq q(t) \) on \((a, b)\), and that \( Q(t) \neq q(t) \). Suppose that \( v \in C^2((a, b)) \) satisfies the equation

\[
(P(t)v')' + Q(t)v = 0.
\]

Then \( v(t) \) has at least \( n \) zeros in \((a, b)\).

Proposition 4.5. Let \( n \geq 2 \) and \( \theta_0 \in (0, \pi) \). Then the smallest eigenvalue of \( L_s \) for the boundary conditions \( u(0) = 0 \) and \( u'(\theta_0) = 0 \) lies in \([(s - 1)(s - n), s(s + 1 - n)]\) if \( \frac{\pi}{2} < \theta_0 < \pi \);

- is \( s(s + 1 - n) \) if \( \theta_0 = \frac{\pi}{2} \); and

lies in \([(s - 1)(s + 1 - n), s(s + 1 - n)]\) if \( 0 < \theta_0 < \frac{\pi}{2} \).

Remark. In terms of the spectral parameter \( \nu \), this says that \( \lambda_\nu \) is not an eigenvalue for \( \frac{n - 1}{2} - \left| \frac{2s - n - 1}{2} \right| \leq \nu \leq \frac{n - 1}{2} + \left| \frac{2s - n - 1}{2} \right| \), and that taking \( \nu \geq \frac{n - 1}{2} \), the first eigenvalue occurs for \( \nu \) in \( (\frac{n - 1 + 2s - n - 1}{2}, s) \), at \( s \), or in \( (s, \infty) \), respectively.
Proof. It is immediate from Lemma 4.3 that the lowest eigenvalue \( \lambda_0 \) satisfies \( \lambda_0 \geq (s-1)(s-n) \). From the same lemma, it follows that the only solutions \( u \) to \( L_s u = (s-1)(s-n)u \) are the constant functions. These, however, do not satisfy \( u(0) = 0 \). Thus, \( \lambda_0 > (s-1)(s-n) \).

Suppose \( \theta_0 \in (0, \frac{\pi}{2}) \) and that \( \lambda \in ((s-1)(s-n), (s(s+1-n))) \). Suppose that \( u \in C^\infty([0, \theta_0]) \) satisfies our boundary conditions, and that that \( L_s u = \lambda u \).

By considering once again the indicial roots of this equation, we can write

\[ u(\theta) = \sin^{2s-n}(\theta)v(\theta) \]

where \( v(\theta) \) is smooth. The equation then transforms to

\[ v''(\theta) + (2s + 1 - n) \cot(\theta)v'(\theta) + (\lambda + s(n - s - 1))v(\theta) = 0, \]

and since \( u'(\theta_0) = 0 \), we conclude that

\[ v'(\theta_0) = (n - 2s) \cot(\theta_0)v(\theta_0). \]

Then we have

\[ \sin^{2s+1-n}(\theta)v''(\theta) + (2s + 1 - n)\sin^{2s-n}(\theta)\cos(\theta)v'(\theta) = (s(s + 1 - n) - \lambda)\sin^{2s+1-n}(\theta)v(\theta). \]

Thus,

\[
\int_0^{\theta_0} v(\theta) \frac{d}{d\theta} \left[ \sin^{2s+1-n}(\theta)v'(\theta) \right] d\theta = (s(s + 1 - n) - \lambda) \int_0^{\theta_0} \sin^{2s+1-n}(\theta)v(\theta)^2 d\theta.
\]

Integrating by parts, we get

\[
\sin^{2s+1-n}(\theta)v(\theta)v'(\theta) \bigg|_0^{\theta_0} - \int_0^{\theta_0} \sin^{2s+1-n}(\theta)v'(\theta)^2 d\theta = (s(s + 1 - n) - \lambda) \int_0^{\theta_0} \sin^{n+1}(\theta)v(\theta)^2 d\theta.
\]

Applying our boundary condition gives

\[
(n - 2s)\sin^{2s-n}(\theta_0)\cos(\theta_0)v(\theta_0)^2 - \int_0^{\theta_0} \sin^{n+1}(\theta_0)v'(\theta_0)^2 d\theta = (s(s + 1 - n) - \lambda) \int_0^{\theta_0} \sin^{n+1}(\theta)v(\theta)^2 d\theta \geq 0.
\]

Since \( \theta_0 \leq \frac{\pi}{2} \) and \( s > \frac{n}{2} \), it is plain that this equality can hold only if \( v(\theta) \equiv 0 \).

Now suppose that \( \lambda = s(s + 1 - n) \), and that (4.13) holds for some \( v \) satisfying the boundary condition (4.14). It follows immediately that

\[ \sin(\theta)v''(\theta) + (2s + 1 - n)\cos(\theta)v'(\theta) = 0, \]

from which we conclude that \( v'(\theta) = b\sin^{n-1-2s}(\theta) \) for some \( b \). Thus, for some \( a \),

\[ v(\theta) = a + b \int_0^\theta \sin^{n-1-2s}(\phi)d\phi. \]
Since \( v \) is smooth and \( s > \frac{n}{2} \), we conclude that \( b = 0 \). Then \( v \equiv a \) can be nonvanishing and satisfy (4.14) if and only if \( \theta_0 = \frac{\pi}{2} \). In that case, we see that \( u(\theta) = \sin^{2s-n}(\theta) \) is a solution to \( L_s u = \lambda u \).

Before handling the last case, let \( v(\theta) = u'(\theta) = \sin^{2s-n-1}(\theta)\cos(\theta) \). Then by differentiating both sides of the equation \( L_s u = s(s + 1 - n)u \), we find that \( v \) satisfies the equation

\[
-v''(\theta) + (2s - n - 1) \cot(\theta) v'(\theta) + (n + 1 - 2s) \csc^2(\theta) v(\theta) + (s - 1)(s - n)v(\theta) = s(n + 1 - s)v(\theta),
\]

with boundary conditions \( v(0) = 0 = v\left(\frac{\pi}{2}\right) \). Multiplying through by a factor of \(-\sin^{n+1-2s}(\theta)\), we can rewrite this equation as

\[
(s - 1)(s - n)v(\theta) = s(n + 1 - s)v(\theta),
\]

Obviously from its definition, \( v(\theta) \) has no zeros on \((0, \frac{\pi}{2})\). Notice also that

\[
\int_0^\pi \frac{\sin^{2s-1-n}(\theta)}{v(\theta)^2} d\theta = \infty = \int_0^\pi \frac{\sin^{2s-1-n}(\theta)}{u(\theta)^2} d\theta.
\]

Now let \( \lambda > s(n + 1 - s) \), and suppose that \( w(\theta) \) satisfies

\[
(s - 1)(s - n)\csc^{2s-1-n}(\theta)w(\theta) = 0
\]
on \((0, \frac{\pi}{2})\). Then \( w \) has at least one zero on \((0, \frac{\pi}{2})\) by Theorem 4.4 since \( \lambda - (s - 1)(s - n) > 2s - n \).

Now suppose that \( \theta_0 \in (\frac{\pi}{2}, \pi) \). Let \( \lambda_0 > 0 \) be the lowest eigenvalue of \( L_s \), with eigenfunction \( u_0 \) satisfying \( u_0(0) = 0 = u_0'(\theta_0) \). Set \( v_0(\theta) = u_0'(\theta) \). We have already shown that \( \lambda_0 \neq s(s - 1 - n) \). Now differentiating both sides of \( L_s u = \lambda_0 u \), we find that \( v \) satisfies the equation

\[
-v''(\theta) + (2s - n - 1) \cot(\theta) v'(\theta) + (n + 1 - 2s) \csc^2(\theta) v(\theta) + (s - 1)(s - n)v(\theta) = \lambda_0 v(\theta)
\]

with homogeneous Dirichlet conditions at both endpoints. Moreover, \( \lambda_0 \) must be the lowest eigenvalue of this boundary value problem as well, or we could produce a lower eigenvalue to \( L_s u = \lambda u \) by integration. Now as is well known, the lowest eigenfunction of a positive (or boundedly negative) operator with homogeneous Dirichlet boundary values is nonvanishing away from the endpoints. This can be shown, for example, by adapting the proof of Proposition 5.2.4 of \cite{lay11} to the simpler ODE case, using the maximum principle given in Theorem 26.XVIII of \cite{wal98}. So we may conclude that \( v(\theta) \) has no zeros on \((0, \theta_0)\), and in particular on \((0, \frac{\pi}{2})\). Thus, it follows that \( \lambda_0 < s(s + 1 - n) \).

\[\blacksquare\]

We can in fact characterize all of the eigenvalues of \( L_s \).
Proposition 4.6. The eigenvalues of $L_s$ are the values $\lambda_\nu$, where $\nu \geq \frac{n-1}{2}$ runs over the non-negative solutions to the equation

\begin{equation}
\frac{d}{d\theta} \left[ \sin^{2s-n}(\theta) F_{\nu-s+s-n+1}^{2s-n}(\sin^2 \left( \frac{\theta}{2} \right)) \right] \bigg|_{\theta=\theta_0} = 0.
\end{equation}

Here $F_{ab}(x)$ is the hypergeometric function.

Note that by the identity $\frac{d}{dx} F_{a,b}^c(x) = \frac{ab}{c} F_{a+1,b+1}^{c+1}(x)$, the equation (4.17) is equivalent to

\begin{equation}
(2s-n) \cot(\theta_0) \csc(\theta_0) F_{\nu-s+s-n+1}^{2s-n}(\sin^2 \left( \frac{\theta_0}{2} \right)) = \frac{(\nu-s)(\nu+s-n+1)}{2s-n+2} F_{\nu+s-n+2}^{\nu-s+s-n+1}(\sin^2 \left( \frac{\theta_0}{2} \right)).
\end{equation}

Proof. We look for solutions $u(\theta)$ to the equation $L_s u(\theta) = \lambda_\nu u(\theta)$, satisfying $u(0) = 0 = u'(\theta_0)$. As shown before, such a solution will necessarily be $O(2s-n)$. We thus write $u(\theta) = \sin^{2s-n}(\theta) v(\theta)$; it was shown earlier that $v$ then satisfies equation (4.13). We introduce the substitution $x = \sin^2 \left( \frac{\theta}{2} \right)$, and set $v(\theta) = l(x(\theta))$. Then setting $a = s - \nu$, $b = \nu + s - n + 1$, and $c = s - \frac{n}{2} + 1$, equation (4.13) transforms to

\[ x(1-x)l''(x) + (c - (1 + a + b)x)l'(x) - abl(x) = 0. \]

This is the hypergeometric equation, and the solution that is smooth at $x = 0$ is (up to scaling) the hypergeometric function $F_{ab}^c(x)$. Thus, we find $u(\theta) = \sin^{2s-n}(\theta) F_{\nu-s+s-n+1}^{2s-n}(\sin^2 \left( \frac{\theta}{2} \right))$. The claim then follows by differentiating $u$ and requiring that $u'(\theta_0) = 0$. \hfill \qed

In the case $\theta_0 = \frac{\pi}{2}$ we can say much more.

Proposition 4.7. Let $L_s$ be as in Proposition 4.6 and $\theta_0 = \frac{\pi}{2}$. Then the spectrum is given in terms of the spectral parameter by $\text{spec}(L_s) = \{ \lambda_\nu : \nu = s + 2k, \text{ where } k \in \mathbb{Z}_{\geq 0} \}$, and eigenfunctions are given by $w_k(\theta) = \sin^{2s-n}(\theta) C^{s-\frac{n}{2}+k}_{2k}(\cos(\theta))$, where $C_j^a$ are the Gegenbauer polynomials.

Proof. In the equation $L_s u = \lambda_\nu u$, we assume a solution of the form $u(\theta) = \sin^{2s-n}(\theta)v(\cos(\theta))$. This gives the equation

\begin{equation}
\sin^2(\theta)v''(\cos(\theta)) - (2s - n + 2) \cos(\theta)v'(\cos(\theta)) + [\nu (\nu + 1 - n) - s(s + 1 - n)]v(\cos(\theta)) = 0.
\end{equation}

We make the substitution $x = \cos(\theta)$, and get the equation

\[ (1 - x^2)v''(x) - (2s - n + 2)xv'(x) + [\nu (\nu + 1 - n) - s(s + 1 - n)]v(x) = 0. \]

This is the Gegenbauer equation $(1 - x^2)v''(x) - (1 + 2\alpha)xv'(x) + k(k + 2\alpha)v(x) = 0$ with $\alpha = s + \frac{1-n}{2}$ and $k = \nu - s$. Recall that a solution to this equation for each integer $k \geq 0$ is given by the $k$th-degree Gegenbauer
polynomial $C_k^n(x)$; that the polynomials $\{C_k^n(x)\}_{k=0}^\infty$ are an orthonormal basis for $L^2([-1,1],(1-x^2)^{\alpha/2-1})$; and that each polynomial $C_k^n(x)$ has the same parity as $k$, with the even polynomials being nonvanishing at 0. It follows that if we let $u_k(\theta) = \sin^{2s-n}(\theta)C_k^n(\cos(\theta))$, then $u_k$ is a solution of the equation $L_s u = \lambda_{s,k} u$, and that the solutions for $k$ even satisfy the condition that $u_k'(\frac{\pi}{2}) = 0$. By the orthonormal basis property of the Gegenbauer polynomials, it follows that every solution to the equation is one of the $u_k$; and we conclude that precisely the solutions for even $k$ satisfy our boundary conditions. \[\Box\]

We now turn to the mapping properties of $I_{s,\nu}$. Recall that the function spaces mentioned in the following proposition are defined in Section 4.1.

**Proposition 4.8.** Fix $n \geq 2 \in \mathbb{N}$, $s > \frac{n}{2}$, and $\theta_0 \in (0, \pi)$. For $\nu \geq n-s$, and $k \geq 1$, $I_{s,\nu}$ given by \[(4.1)\] maps $A_{n,s,k}^0$ to $B_{n,s,k}$. If $\lambda_{\nu} \notin \text{spec}(L_s)$, then $I_{s,\nu} : A_{n,s,k}^0 \to B_{n,s,k}$ is bijective. Otherwise, $\text{dim ker } I_{s,\nu} = 1$ and the kernel is spanned by a smooth function $\beta_{s,\nu}$; the image in that case is the orthogonal complement in $B_{n,s,k}$ of $\beta_{s,\nu}$ with respect to the measure $\sin^{-(1+n)}(\theta)d\theta$.

When $\nu \notin \text{spec}(L_s)$, the inverse of $I_{s,\nu}$ is given by the Green's operator

\[Gf(\theta) = \sin^{n-s}(\theta)p(\theta) \int_{\theta}^{\theta_0} \frac{q(\phi)f(\phi)}{\sin^{s+1}(\phi)}d\phi + \sin^{n-s}(\theta)q(\theta) \int_{0}^{\theta_0} \frac{p(\phi)f(\phi)}{\sin^{s+1}(\phi)}d\phi,\]

where $q(\theta) = O(\theta^{2s-n})$ is smooth, with $\sin^{s-n}(\theta)q(\theta)$ even in $\theta$; and

- if $s - \frac{n}{2} \notin \mathbb{Z}$ or $\nu \in \{n-s,n-s+1,\ldots,\frac{n}{2} - 1\}$, then $p(\theta)$ is smooth and even in $\theta$; while
- if $s - \frac{n}{2} \in \mathbb{Z}$ and $\nu \notin \{n-s,n-s+1,\ldots,\frac{n}{2} - 1\}$, then $p(\theta) \in C^\infty([0,\theta_0]) + \theta^{2s-n}\log(\theta)C^\infty([0,\theta_0])$, and is even up to order $2s-n$.

If $n$ is odd and $s \geq n$ is an integer, and if $f \in B_{n,s,k}$ is even in $\theta$ through order $s+1$, then $Gf$ is smooth.

**Proof.** For convenience, we set $I = [0, \theta_0]$.

It is straightforward that $I_{s,\nu}$ maps $A_{n,s,k}^0$ into $B_{n,s,k}$. The anomaly in the definition of $B_{n,s,1}$ is because $s$ is an indicial root of $I_{s,\nu}$, so that $I_{s,\nu}(\theta^s u) = O(\theta^{s+1})$ for any $u \in C^\infty(I)$.

We first wish to identify independent global solutions to the equation $I_{s,\nu} u = 0$. In fact, it will again be easier to consider the operator $\nu(\nu+1-n) - L_s$ acting on $\sin^{s-n}(\theta)u(\theta)$. As before, we set $x = \sin^2(\frac{\theta}{2})$, and let $l(x)$ be defined by $u(\theta) = \sin^{n-s}(\theta)l(x(\theta))$. Then transforming the equation $(\lambda_{\nu} - L_s)u = 0$ yields the equation

\[x(1-x)l''(x) + \left(\frac{n}{2} + 1 - s - (n+2(1-s))x\right)l'(x) + \left[(s-1)(n-s) + \nu(\nu+1-n)\right]l(x) = 0.\]
Let \( a = n - s - \nu \), let \( b = \nu + 1 - s \), and let \( c = \frac{n}{2} + 1 - s \). Then (4.21) becomes
\[
x(1-x)l''(x) + (c - (1+a+b)x)l'(x) - abl(x) = 0,
\]
which is again the hypergeometric differential equation. The indicial roots of the equation at \( x = 0 \) are \( \gamma = 0 \) and \( \gamma = s - \frac{n}{2} \). It follows that we can find two independent solutions \( u_0 \) and \( u_s \) having the following properties: \( u_s \in C^\infty(I) \) and \( u_s = O(\theta^s) \), while \( u_0(0) \neq 0 \). Furthermore, \( \sin^{s-n}(\theta)u_s(\theta) \) is even in \( \theta \). If \( s - \frac{n}{2} \notin \mathbb{Z} \), then \( u_0 \in C^\infty(I) \), while if \( s - \frac{n}{2} \in \mathbb{Z} \), then it follows by standard hypergeometric theory (for example, paragraph 15.10(c) of [NIS]) that \( u_0 \in C^\infty(I) \) if \( \nu = n - s, n - s + 1, \ldots, \frac{n}{2} - 1 \), and that \( u_0 \in C^\infty(I) + \theta^{2s-n} \log(\theta)C^\infty(I) \) otherwise, with nonzero logarithmic coefficient.

We already know that \( I_{s,\nu} \) is injective if and only if \( \lambda_\nu \notin \text{spec} I_s \). We now show that \( I_{s,\nu} \) is surjective whenever it is injective. Let \( q(\theta) \) be the solution to \( I_{s,\nu}(\sin^{n-s}(\theta)q)_0 = 0 \) with \( \lim_{\theta \to 0^+} q(0)/\theta^{2s-n} = 1 \). Let \( p(\theta) \) be the solution to \( I_{s,\nu}(\sin^{n-s}(\theta)p)_0 = 0 \) with \( p'(0) = 0 \) and \( p(0) = 1 \) (this can be taken to be nonzero by the assumption of injectivity). It follows that \( p \) is some linear combination of \( u_0 \) and \( u_s \) with nontrivial coefficient for \( u_0 \). Now by Abel’s identity, the Wronskian of \( \sin^{n-s}(\theta)p(\theta) \) and \( \sin^{n-s}(\theta)q(\theta) \) is \( c \sin^{n-1}(\theta) \) for some \( c \neq 0 \). It thus follows from standard formulas that the Green’s function for \( I_{s,\nu} \) is given by
\[
\Phi(\theta, \phi) = \begin{cases} 
\frac{\sin^{n-s}(\theta)q(\theta)p(\phi)}{c \sin^{n+s+1}(\phi)} & \theta < \phi, \\
\frac{\sin^{n-s}(\theta)q(\phi)p(\theta)}{c \sin^{n+s+1}(\phi)} & \theta \geq \phi,
\end{cases}
\]
and it is elementary to show that the Green’s operator (4.22)
\[
Gf(\theta) = \sin^{n-s}(\theta)p(\theta) \int_0^\theta \frac{\phi(\phi)f(\phi)}{c \sin^{n+s+1}(\phi)} d\phi + \sin^{n-s}(\theta)q(\theta) \int_{\theta}^0 \frac{p(\phi)f(\phi)}{c \sin^{n+s+1}(\phi)} d\phi
\]
is a right inverse to \( I_{s,\nu} \) when \( f \in B_{n,s,k} \), and a left inverse to \( I_{s,\nu} \) when \( f = I_{s,\nu}u \) with \( u \in A_{0,\nu}^{n,s,k} \). (For the formula in the statement, we may absorb \( c \) into \( q \).) We wish to show that the equation \( I_{s,\nu}u = f \) can be solved whenever \( f \in B_{n,s,k} \), and that in particular, \( G \) maps \( B_{n,s,k} \) to \( A_{0,\nu}^{n,s,k} \). It is easy to check that \( (Gf)'(\theta_0) + (s-n) \cot(\theta_0)(Gf)(\theta_0) = 0 \) and that, for \( f \in B_{n,s,k} \), \( Gf(0) = 0 \). Thus, if \( Gf \) is in \( A_{0,\nu}^{n,s,k} \), it remains to show that \( Gf(\theta) \) is in \( A_{n,s,k} \).

We do this in two steps. We first show that \( G \) maps \( B_{n,s,k} \) into \( A_{n,s,k} + \theta^{n-s+(k+1)(2s-n)} \log(\theta)^k C^\infty(I) \) by asymptotically expanding the integrands in (4.22) and considering the kinds of terms that can arise.

Suppose \( f \in \theta^{n-s+1}C^\infty(I) \). Then since \( q(\phi) = O(\phi^{2s-n}) \), the integral in the first term of (4.22) is smooth. The factor \( p(\theta) \) may contain a term with a factor of \( \theta^{2s-n} \log(\theta) \) if \( s - \frac{n}{2} \notin \mathbb{Z} \). Thus, the first term is in \( A_{n,s,k} \).

Because, in general, \( \theta \int_0^\theta \theta^{-k} d\theta \) is smooth for \( 0 \leq k \leq j \) unless \( k = 1 \), the integral in the second term is smooth unless either \( p(\phi) \) has a \( \phi^{2s-n} \log(\theta) \)
term in it (which could happen if \( s - \tfrac{1}{2} \in \mathbb{Z} \)), or there is a nonvanishing term of order \( \phi^{-1} \) in the expansion of the integrand. In the first case, when \( p(\phi) \) has a term in its expansion of the form \( \phi^{2s-n} \log(\phi) \), the term yields a log at order \( \theta^{s+1} \log(\theta) \) and at higher powers in \( \theta \). When the integrand contains a term of order \( \phi^{-1} \), the second term yields a term of the form \( \theta^s \log(\theta) \). Thus, in this case, \( Gf \in A_{n,s,k} \).

Now suppose that \( f \in \theta^{s+1} \log(\theta)C^\infty(I) \). Then the first integral yields a log at order \( \theta^{s+1} \log(\theta) \), and a possible \( \log(\theta)^2 \) at order \( \theta^{3s+1-n} \log(\theta)^2 \), if \( p(\theta) \) has a term of the form \( \theta^{2s-n} \log(\theta) \).

When the integrand of the second term is expanded in an asymptotic series, we get several nonsmooth terms. First, at lowest order we get a term of the form \( \theta^{s+1} \log(\theta) \), with logs at higher orders as well. Now, it is an elementary result that

\[
\int_0^\theta \theta^j \log(\theta)^2 d\theta = \theta^{j+1} \left( 2(1+j)^{-3} - 2(1+j)^{-2} \log(\theta) + (1+j)^{-1} \log(\theta)^2 \right).
\]

Thus, if \( p(\phi) \) has a term of the form \( \phi^{2s-n} \log(\phi) \), then we get a term of the form \( \theta^{3s+1-n} \log(\theta)^2 \) as well. If \( k = 1 \), we have shown, as desired, that \( Gf \in A_{n,s,k} + \theta^{n-s+(k+1)(2s-n)+1} \log(\theta)^{k+1} C^\infty(I) \). If \( k > 1 \), then in both the cases so far considered, we have seen that \( Gf \in A_{n,s,k} \).

Now suppose that \( k > 1 \) and that \( f(\theta) \in \theta^{n-s}(\theta^{2s-n} \log(\theta))^j C^\infty(I) \) for some \( 2 \leq j \leq k \). Taking account of the possible \( \theta^{2s-n} \log(\theta) \) term in \( p(\theta) \), an integral formula analogous to the above shows that the first term yields a function in \( \theta^{n-s} \otimes \sum_{i=0}^{j+1} (\theta^{2s-n} \log(\theta))^i C^\infty(I) \). This clearly lies in the desired space. It is easy to see that the second term lies in the same space.

Thus, we see that \( Gf \in A_{n,s,k} + \theta^{n-s+(k+1)(2s-n)} \log(\theta)^{k+1} C^\infty([0, \theta_0]) \).

Our second step is now to show that, in fact, the last term does not arise. We have seen that \( I_{s,\nu} Gf = f \in B_{n,s,k} \). But for no \( k \geq 1 \) is \( n-s+(k+1)(2s-n) \) an incidential root of the operator \( I_{s,\nu} \) at \( \theta = 0 \). Thus, if \( 0 \neq u \in C^\infty(I) \), then \( I_{s,\nu}(\theta^{n-s+(k+1)(2s-n)} \log(\theta)^{k+1} u) \) yields a term in \( \theta^{n-s+(k+1)(2s-n)} \log(\theta)^{k+1} C^\infty(I) \), which is not canceled by any other term, and this precludes \( I_{s,\nu} u \) from lying in \( B_{n,s,k} \). Thus, we must in fact have that \( G : B_{n,s,k} \to A_{n,s,k}^0 \).

Now \( I_{s,\nu} \) is injective, and it has a right inverse. Thus, it is a bijection.

It remains to show that if \( I_{s,\nu} \) is not injective, then it is also not surjective. Suppose that \( \ker I_{s,\nu} \neq \emptyset \). Rename \( q \) so that \( 0 \neq q \) satisfies \( \sin^{n-s}(\theta) q(\theta) \in \ker I_{s,\nu} \subseteq A_{n,s,k}^0 \), and note that \( q = O(\theta^{2s-n}) \). Then Lemma 4.3 (4.12) together with (4.10) makes it clear that any \( f \in \text{Im} I_{s,\nu} \) must be orthogonal to \( q \) with respect to measure \( \sin^{-(s+1)}(\theta) d\theta \). Since \( v = \sin(\theta) q \in B_{n,s,k} \) is clearly not orthogonal to \( q \), we conclude that \( I_{s,\nu} \) is not surjective.

If \( q \) is orthogonal to \( f \) with respect to the measure \( \sin^{-(s+1)}(\theta) \), then let \( p \) be any solution of the equation \( I_{s,\nu}(\sin^{n-s}(\theta) p(\theta)) = 0 \) with \( p(\theta) = 1 + o(1) \). Then it is easy to check that with \( p \) and \( q \) reinterpreted according to these definitions, equation (4.22) still gives a solution, of course not unique,
to the equation \( I_{s,v}u = f \), with \( u(\theta)/\sin^{n-s}(\theta)|_{\theta=0} = 0 = u'(\theta_0) + (s - n) \cot(\theta_0)u(\theta_0) \). The hypothesis that \( q \) is orthogonal to \( f \) is necessary in showing that the Robin condition holds. Note that orthogonality to \( q \) with respect to the measure \( \sin^{-(s+1)}(\theta)d\theta \) is equivalent to orthogonality to \( \beta_{s,v} = \sin^{n-s}(\theta)q(\theta) \) with respect to the measure \( \sin^{-(n+1)}(\theta)d\theta \).

We finally turn to the last claim. If \( n \) is odd and \( s \geq n \) is an integer, and if \( f \in \mathcal{B}_{n,s,k} \) is even through order \( s+1 \), then no term of the form \( \theta^{-1} \) can appear in the integrand in (4.20), and no logarithms appear in \( p \) or \( q \). This yields the claim.  

This result will be used, for general \( k \), in our formal solution of the Einstein equations in the next section.

4.3. The Laplacian. We now continue to study the Laplace boundary value problem at hand. Recall that we are letting \( \tilde{X} = [0, \theta_0] \times S \times [0, \varepsilon) \) be the pullback of the blowup of a cornered space by a diffeomorphism as in Corollary 2.4, and are assuming that \( g \) is the pullback of a constant-angle admissible metric on the blowup, in the form (4.1). The goal of this section is to prove Theorem 1.2. This problem requires only the \( k = 1 \) case of Proposition 4.9 since it deals with a linear equation. For this section, we specialize to the case \( \theta_0 = \frac{\pi}{2} \), for which we have a full, explicit solution to the eigenvalue problem. The techniques we use would be of relevance in studying other cases as well, although the behavior would depend crucially on the spectrum, which might display a variety of behaviors in general.

We first undertake some preparations in the case that \( 2s \in \mathbb{Z} \), which we now assume.

Now it is easy to show that for \( v \in C^\infty(\tilde{S}) \),

\[
\Delta_g \left( \rho^\nu \log(\rho)^{k_v} \right) + s(n-s) \left( \rho^\nu \log(\rho)^{k_v} \right) = \\
\rho^\nu \left( \log(\rho)^{k_v} I_{s,v}v + k(1+\nu-k) \log(\rho)^{k_v-1} \sin^2(\theta)v \right) + k(k-1) \log(\rho)^{k_v-2} \sin^2(\theta)v + o(\rho^\nu).
\]

Motivated by this, we define an operator \( J_{s+q} \) on \( \mathcal{E}_{s,q} \) (see page 23) by setting

\[
J_{s+q}(\rho^{s+q} \log(\rho)^{k_v} b) = \rho^{s+q}(\log(\rho)^{k_v} I_{s+q}v(b) + k(1+s+q-k) \log(\rho)^{k_v-1} \sin^2(\theta)b) \]
\[
+ k(k-1) \sin^2(\theta) \log(\rho)^{k_v-2}b,
\]

where \( b \in \mathcal{A}_{s,1} \), and extending linearly. For \( q \) even, we define

\[
\mathcal{E}_{s,q}^0 = \left\{ \eta = \rho^{s+q} \sum_{i=0}^{\frac{d}{2}+1} \log(\rho)^i b_i(\theta) : b_{\frac{d}{2}+1} \in \ker I_{s+q} \right\}.
\]

For consistency of notation, we define \( \mathcal{E}_{s,q}^0 = \mathcal{E}_{s,q} \) when \( q \) is odd.

**Proposition 4.9.** Let \( 2s \in \mathbb{Z} \). For \( q \geq 1 \) odd, \( J_{s+q} : \mathcal{E}_{s,q} \to \mathcal{F}_{s,q} \) is an isomorphism. For \( q \geq 0 \) even, \( J_{s+q} : \mathcal{E}_{s,q}^0 \to \mathcal{F}_{s,q} \) is surjective, with a
one-dimensional kernel spanned by $\rho^{s+q} \sin^{n-s}(\theta)w_{\frac{n}{2}}(\theta)$, where $w_{\frac{n}{2}}$ is as in Proposition 4.4.

**Proof.** That $J_{s+q}$ has the given codomains follows immediately from the definitions of $E_{s,q}, E_{s,0}, F_{s,q}$, and $J$.

For the isomorphism claim, we assume $q = 2j+1$ is odd. We set $J = J_{s+q}$. By Propositions 4.7 and 4.8 $I_{s,s+2j+1} : A_{n,s,1}^0 \to \mathcal{B}_{n,s,1}$ is a bijection. So suppose we wish to solve $J_\eta = f$, where $\eta = \rho^{s+2j+1} \sum_{i=0}^{J+1} \log(\rho)^i b_i(\theta)$ and $f = \rho^{s+2j+1} \sum_{i=0}^{J+1} \log(\rho)^i c_i(\theta)$. We will do this term by term, starting with the highest power of log and working down. We can uniquely solve $I_{s,s+2j+1} b_{j+1} = c_{j+1}$. Set $\eta^{(j+1)} = \rho^{s+2j+1} \log(\rho)^{j+1} b_{j+1}(\theta)$, and note that $f - J_\eta^{(j+1)} = \rho^{s+2j+1} \sum_{i=0}^{j+1} \log(\rho)^i c_i^{(j+1)}(\theta)$, where $c_i^{(j+1)}$ may differ from $c_i$. Now suppose, by way of induction, that we have constructed $\eta^{(l)} \in E_{s,2j+1}$ such that $f - J_\eta^{(l)} = \rho^{s+2j+1} \sum_{i=0}^{l-1} \log(\rho)^i c_i^{(l)}$. Then we can uniquely solve the equation $I_{s,s+2j+1} b_{l-1} = c_l^{(l)}$, and setting $\eta^{(l-1)} = \eta^{(l)} - \rho^{s+2j+1} \log(\rho)^{l-1} b_{l-1}$, we easily get $f - J_\eta^{(l-1)} = \rho^{s+2j+1} \sum_{i=0}^{l-2} \log(\rho)^i c_i^{(l-1)}$. Thus, by induction, we can solve $J_\eta = f$. At each step, the solution is unique, and so we see that $J$ is an isomorphism.

We now address the even case, $q = 2j$. In this case, $I_{s,s+2j}$ has a one-dimensional kernel spanned by $w_{\frac{n}{2}}$, that is, $w_j$. We wish to solve the equation $J_\eta = f$, where we take $\eta = \rho^{s+2j} \sum_{i=0}^{j} \log(\rho)^i b_i(\theta)$ and $f = \rho^{s+2j} \sum_{i=0}^{j} \log(\rho)^i c_i(\theta)$. We cannot necessarily solve the equation $I_{n+2j} b_j = c_j$, since $c_j$ may not be orthogonal to $w_j$, generically (see Proposition 1.8). However, we may solve this using our freedom in the $\log(\rho)^{j+1}$ term. First, notice that $\sin^2(\theta) w_j$ is not orthogonal to $w_j$. Now we define the coefficient $b_{j+1} = \frac{\langle c_j, w_j \rangle}{\langle w_j, w_j \rangle} w_j$, where $\langle , \rangle$ here refers to the $L^2$ norm on $[0, \theta_0]$ with measure $\sin^{-(1+s)}(\theta) d\theta$. (This is finite since $w_j = O(\theta^{2s-n})$ and $c_j = O(\theta^{n+s+1})$.) Set $\eta^{(j+1)} = \rho^{s+2j} \log(\rho)^{j+1} b_{j+1}$. Then $f - J_\eta^{(j+1)} = \rho^{s+2j} \sum_{i=0}^{j} \log(\rho)^i c_i^{(j+1)}$, where $c_i^{(j+1)}$ is orthogonal to $w_j$. We can now proceed by induction as in the odd case; except that at each step $l$, we uniquely add a multiple of $w_l$ to $b_l$ to ensure that $c_l^{(l)}$ is orthogonal to $w_j$. By induction, we may thus solve the equation. However, a solution is unique only up to addition of a multiple of $\rho^{s+2j} w_j$. \hfill \blacksquare

We now return to letting $s > \frac{n}{2}$ be arbitrary.

The following lemma follows easily from Lemma 4.11 and particularly from the fact that $n - s$ and $s$ are indicial roots of $\Delta_\theta + s(n-s)$ at $\theta = 0$.

**Lemma 4.10.** Suppose that $u \in P_s(\tilde{X})$. Then $(\Delta_\theta + s(n-s)) u \in P_s(\tilde{X})$ as well. Moreover, suppose that for any fixed $x \in S$, $u(\theta, x, \rho) \in E_{s,q}^0$ for some $q$ independent of $x$. Then for each fixed $x$, there is some $f \in F_{s,q}$ such that $(\Delta_\theta + s(n-s)) u(\theta, x, \rho) = f(\theta, \rho) + o(\rho^{s+q})$. 

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**FORMAL CORNERED ASYMMETRICALY HYPERBOLIC EINSTEIN METRICS**

33
Proof of Theorem 1.2. We work in the decomposition given by Corollary 2.4. In particular, $\hat{U}, \theta$, and $\rho$ are as in that corollary. We write $\psi = \psi(x, \rho)$. Throughout the proof, primes will refer to a derivative with respect to $\theta$.

Define $u_0 \in \rho^{n-s}\sin^{n-s}(\theta)C^\infty(\hat{U})$ by $u_0(\theta, x, \rho) = \rho^{n-s}\sin^{n-s}(\theta)\psi(x, \rho)$. Now define $f_0 \in \mathcal{B}_{n,s,1}(\overline{\mathcal{X}}, S)$ by $\Delta u_0 + s(n-s)u_0 = \rho^{n-s}f_0$. Then by Proposition 4.8 we can uniquely solve $I_{s,n-s}\varphi_0 = -f_0$ for $\varphi_0 \in A_{n,s,1}(\overline{\mathcal{X}}, S)$. Set $u_0 = \tilde{u}_s + \rho^{n-s}\varphi_0$. Then plainly, $\rho^{s-n}\sin^{s-n}(\theta)u_0 \equiv \psi$, and $(\Delta_g + s(n-s))u_0 = O(\rho^{n-s+1})$. Also, $\partial_\theta u_0(\overline{\mathcal{X}}) = 0$.

Now suppose that either, on the one hand, $2s \not\in \mathbb{Z}$ and $k \in \mathbb{N}$ or, on the other hand, that $0 \leq k < 2s - n$; and suppose, in either case, that $u_k$ has been smoothly and uniquely defined in $\mathcal{R}_s(\overline{\mathcal{X}})$ so that
\[\begin{align*}
(a) & \quad \Delta_g u_k = O(\rho^{n-s+k+1}); \\
(b) & \quad \rho^{s-n}\sin^{s-n}(\theta)u_k|_{\theta=0} = \psi; \quad \text{and} \\
(c) & \quad u_k'(0) = 0.
\end{align*}\]

We wish to find $\varphi_{k+1} \in A_{n,s,1}(\overline{\mathcal{X}}, S)$ so that $u_{k+1} = u_k + \rho^{n-s+k+1}\varphi_{k+1}$ satisfies each of these conditions with $k$ replaced by $k + 1$. Define $f_{k+1} \in \mathcal{B}_{n,s,1}(\overline{\mathcal{X}}, S)$ by $f_{k+1} = \rho^{n-s+k+1}(\Delta_g + s(n-s))u_k|_{\rho=0}$. By Lemma 1.1, $f_{k+1} = O(\rho^{n-s+1})$, and indeed, since $(\Delta_g(\theta^s) + s(n-s)\theta^s) = O(\rho^{n-s+1})$, $f_{k+1} \in \mathcal{B}_{n,1}(\overline{\mathcal{X}}, S)$ as desired. Fix $x \in S$. By Proposition 4.8 we can uniquely solve $I_{s,n-s+k+1}\varphi_{k+1}(\theta, x) = -f_{k+1}(\theta, x)$ in $A_{n,s,1}$, with $\varphi_{k+1}(0, x) = 0$ and $\varphi_{k+1}(\theta_0, x) = 0$. Plainly $\varphi_{k+1}$ depends smoothly on $x$. Thus, $\varphi_{k+1} \in A_{n,s,1}(\overline{\mathcal{X}}, S)$ is determined as desired. Thus, for any $m \in \mathbb{Z}$ (if $2s \not\in \mathbb{Z}$), or for any $m \leq 2s - n$ (otherwise), we can, by induction and Proposition 4.8, construct a function $u_{m-1} \in \mathcal{R}_s(\overline{\mathcal{X}})$, unique through order $n - s + m - 1$, such that $\rho^{s-n}\sin^{s-n}(\theta)u_{m-1}|_{\overline{\mathcal{X}}} = \psi$, such that $u_{m-1}(\overline{\mathcal{X}}) \equiv 0$, and such that $(\Delta_g + s(n-s))u_{m-1} = O(\rho^{n-s+m})$. If $2s \not\in \mathbb{Z}$, we are done, except for the last paragraph below. From here, we therefore assume that $2s \in \mathbb{Z}$.

At order $s$, $I_{s,s}$ is not surjective, so our procedure generically fails for $k = 2s - n - 1$ unless, for each $x$, $f_n(\theta, x)$ is orthogonal to $\sin^s(\theta)$ with respect to the measure $\sin^{-(n+1)}(\theta)d\theta$, i.e., unless it is orthogonal to $\csc^{n+1-s}(\theta)$. To proceed, we must introduce logarithmic terms, and for this we will use Proposition 4.9. Let $f_{2s-n} = \rho^{-s}(\Delta_g + s(n-s))u_{2s-n-1}|_{\rho=0} \in \mathcal{B}_{n,s,1}$ as above, and fix $x$. Then notice that $\rho^s f_{2s-n}(\cdot, x) \in \mathcal{F}_{s,0}$. Thus, by Proposition 4.9 with $q = 0$, there exists a solution $\varphi_{2s-n} \in \mathcal{E}_{s,0}$ to $J_{s} \varphi_{2s-n}(\theta) = -\rho^s f_{2s-n}(\theta, x)$, which however is determined only up to a term that is a multiple of $\rho^s \sin^s(\theta)$. Set $\varphi_{2s-n} = \rho^{-s}\Phi_{2s-n}$, and $u_{2s-n} = u_{2s-n-1} + \rho^s \varphi_s$. Then since this procedure is smooth in $x$, $u_{2s-n}(\theta, x, \rho)$ satisfies $\Delta_g u_{2s-n} + s(n-s)u_{2s-n} = o(\rho^s)$, and the boundary conditions $\rho^{s-n}\sin^{s-n}(\theta)u_{2s-n}|_{\overline{\mathcal{X}}} = \psi$ and $u_{2s-n}(\overline{\mathcal{X}}) = 0$ are satisfied. Notice that $u_{2s-n} \in \mathcal{P}(\overline{\mathcal{X}})$.

We proceed by induction. Suppose that $u_{2s-n+2j} \in \mathcal{P}(\overline{\mathcal{X}})$ has been successfully defined satisfying $\Delta_g u_{2s-n+2j} = o(\rho^{s+2j})$, with both boundary conditions as desired, and containing $(j + 1)$st powers of $\log(\rho)$. Then since $\Delta_g + s(n-s)$ is linear, $(\Delta_g + s(n-s))u_{2s-n+2j}$ will likewise contain at most
(j + 1)st powers of \(\log(p)\). Fix \(x \in S\). Let \(F_{2s-n+2j+1} \in \mathcal{F}_{s,2j+1}\) be such that \((\Delta_g + s(n-s))u_{2s-n+2j}(\theta, x, \rho) = F_{n+2j+1}(\theta, \rho) + o(\rho^{n+2j+1})\). Such an \(F\) exists by Lemma 4.10. We wish to solve the equation \(J_{s+2j+1}\Phi_{2s-n+2j+1} = -F_{2s-n+2j+1}\); by Proposition 4.9 we may do so uniquely, and plainly the solution varies smoothly in \(x\). Then set \(u_{2s-n+2j+1} = u_{2s-n+2j} + \Phi_{2s-n+2j+1}\). Clearly, \(u_{2s-n+2j+1} \in \mathcal{P}(\bar{X})\) satisfies \(\Delta_g u_{n+2j+1} = o(\rho^{n+2j+1})\) and our boundary conditions.

Next we wish to find \(u_{2s-n+2(j+1)}\), satisfying our boundary conditions, such that \(\Delta_g u_{2s-n+2(j+1)} = o(\rho^{n+2(j+1)})\). But this is exactly the same as the odd case, except that by Proposition 4.9, the solution will be unique only up to a term of the form \(\rho^{s+2(j+1)} \sin^{n-s}(\theta)w_{j+1}\), where \(w_{j+1}\) is as in Proposition 4.7. Hence, by induction, we get an infinite sequence \(\{u_k\}_{k=0}^\infty\) such that \(\Delta_g u_k + s(n-s)u_k = o(\rho^{n-s+k})\), \(u_k|_{\bar{M}} = \psi\), and \(\partial_\theta u_k|_{\theta = \frac{\pi}{2}} = 0\), and such that each member of the sequence \(\{u_k\}\) has at most \([\frac{k+1-n-2s}{2}]\) powers of \(\log(p)\).

Thus, by Borel’s Lemma, as stated in Erd56, there exists a function \(u \in \mathcal{P}(\bar{X})\) such that \((\Delta_g + s(n-s))u = O(\rho^\infty)\), such that \(\partial_\theta u|_{\theta = \frac{\pi}{2}} = 0\), and such that \(\rho^{s-n} \sin^{s-n}(\theta)u|_{\bar{M}} = \psi\).

Notice that, in order to uniquely determine a solution for \(2s \in \mathbb{Z}\), we would need to specify not only \(\psi\), but also a scalar function \(\eta_k \in C^\infty(S)\) at order \(n + 2k\) for all \(k \geq 0\).

We also remark that the power of \(\rho\) stated in the boundary condition – and the lowest appearing in the expansion – has somewhat more flexibility than the power of \(\sin(\theta)\). The expansion can begin at order \(\rho^q \sin^{n-s}(\theta)\) for any \(q\) that is not an indicial root of \(L_s\). For example, when \(2s \notin \mathbb{Z}\), we could simply prescribe that \(\sin^{s-n}(\theta)u|_{\theta = \frac{\pi}{2}} = \psi\). In general, the first \(\log(p)\) term would be expected to appear at the first indicial root that was integrally separated from the starting power. As there is a host of different situations that could be studied along these lines and we do not wish to consider them all here, we leave this problem for now.

5. Einstein Metrics: Formal Existence

In this section, we consider formal existence of CAH Einstein metrics on a cornered space \((X, M, Q)\). This has famously been studied in the AH setting in [FG12], where the boundary data is a conformal class on the conformal infinity \(\bar{M}\). Once again we require a boundary condition at \(Q\), and we continue to follow [NTU12] in requiring that \(Q\) be totally umbilic and of constant mean curvature, so that its second fundamental form \(K_Q\) satisfies \(K_Q = \lambda g|_{TQ}\) away from the corner \(S = Q \cap M\). As we will see, this boundary condition interacts particularly nicely with the geometry of CAH spaces.

We will take as our data a smooth manifold \(M^n\) with boundary \(S\), equipped with an asymptotically hyperbolic conformal class \([\bar{h}]\); and a constant \(\lambda \in (-1, 1)\). Ideally, we would like to realize \(M\) as the infinite boundary of an
appropriate cornered space \((X, M, Q)\), and then to construct an Einstein CAH metric \(g\) satisfying \(\text{Ric}(g) + ng = 0\) to as high an order as possible at the corner, and satisfying \(K_Q = \lambda g|_{TQ}\) along a finite boundary \(Q\). We know by Section 3 that this problem cannot be solved if we take \(g\) to be smooth. We thus look for solutions on a blowup space, smooth in polar coordinates, relaxing the requirement that they be smooth on \(X\).

Motivated by Proposition 3.4 and Theorem 2.3 and by our need to break the gauge in the Einstein equations, we take \(\tilde{X} = [0, \theta_0] \times M\), where \(\theta_0 = \cos^{-1}(-\lambda)\). We look for metrics in the normal form (1.1). We then take \(\tilde{S} = [0, \theta_0] \times S\), \(\tilde{Q} = \{\theta_0\} \times M\), and \(\tilde{M} = \{0\} \times M\). We look for a metric \(g\) on \(\tilde{X} \setminus (\tilde{M} \cup \tilde{S})\), of the form

\[
g = \csc^2(\theta) \left[ d\theta^2 + h_\theta \right], \tag{5.1}
\]

where \(h_\theta\) is a smooth one-parameter family of AH metrics on \(M\) with \(h_0 \in [h]\), and satisfying the Einstein equation to as high an order as possible at \(\tilde{S}\) and the equation \(K_{\tilde{Q}} = \lambda g|_{T\tilde{Q}}\) along \(\tilde{Q}\). Our goal is to prove Theorem 1.1.

Note that there is no loss of generality in our choice of \(\tilde{X}\): although a general cornered space \((X, M, Q)\) might have boundary components \(M\) and \(Q\) of differing topology, our construction is formal and at \(\tilde{S}\), where the topology is determined by \(S = \partial M\) alone.

Throughout this section, it will be convenient to work explicitly with sections of the 0-edge bundle \(0^eT^*\tilde{X}\) and its tensor products, as well as the 0-bundle \(0^0T^*M\). We let \(\{x^\mu\}\) be local coordinates near a point \(p\) of \(M\), so that \((\theta, x^\mu)\) gives a coordinate system on \(\tilde{X}\) near \([0, \theta_0] \times \{p\}\). We define a frame for \(0^eT^*\tilde{X}\) given by

\[
\omega^0 = \frac{d\theta}{\sin(\theta)}, \quad \omega^\mu = \frac{dx^\mu}{\rho \sin(\theta)}
\]

(where \(\rho = x^n\)).

It will be useful to compute the umbilic condition in normal form.

**Lemma 5.1.** For a metric \(g\) on \(\tilde{X}\) in the form (5.1), let \(K_{\tilde{Q}}\) be the second fundamental form of \(\tilde{Q} \setminus \tilde{S}\), and \(\lambda = -\cos(\theta_0)\). Then the condition \(K_{\tilde{Q}} = \lambda g|_{T\tilde{Q}}\) is equivalent to the condition

\[
\partial_\theta h_\theta|_{\theta = \theta_0} = 0.
\]

**Proof.** Plainly, the inward-pointing normal to \(\tilde{Q}\) is given by \(\nu = -\sin(\theta_0) \partial_{\theta_0}\). By Weingarten’s equation,

\[
K_{\mu\nu} = -g_{k\nu} \nabla_{\mu} L^k = \sin(\theta_0) \Gamma_{\mu\theta\nu},
\]

where \(L^k = g_{k\nu} \nu^\nu\).
where \( \Gamma_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) = \frac{1}{2}\partial_\mu g_{\nu\lambda}. \) Since \( g_{\mu\nu} = \csc^2(\theta)h_{\mu\nu} \) and \( \partial_\theta(\csc^2(\theta)) = -2\csc^2(\theta)\cot(\theta), \) the result follows.

As discussed in the introduction, to prove Theorem 1.1, we will choose a conformal representative \( h \in [\hat{h}] \); we will take it to be in AH normal form. Then \( \hat{h}_0 \) in (5.1), as discussed, will be of the form \( \chi h \), where \( \chi \) is a function to be determined. This motivates the following proposition, which we will use to prove the theorem. Recall that \( \mathcal{M}(\theta_0, M) \) is defined in Section 4.1.

**Proposition 5.2.** Let \( n \geq 2 \), and suppose \( S \) is a smooth manifold of dimension \( n-1 \). Given a one-parameter family of metrics \( k_\rho \) on \( S \), there exists a one-parameter family of smooth AH metrics \( \{h_\theta : 0 \leq \theta \leq \theta_0\} \) on \( S \times [0,\varepsilon) \) such that \( h_\theta \in \mathcal{M}(\theta_0, S \times [0,\varepsilon)) \), and letting \( h_\theta = \rho^2h_\theta \), we have

\[
\begin{align*}
(a) \quad \chi|_{\rho=0} &= 1; \\
(b) \quad \partial_\theta h_\theta|_{\theta=\theta_0} &= 0 \text{ for all } \rho; \\
(c) \quad \partial_\theta h_\theta|_{\rho=0} &= 0 \text{ for all } \theta; \\
(d) \quad h_0 &= \chi(d\rho^2 + k_\rho); \text{ and} \\
(e) \quad \text{if } g \text{ is the metric on } [0,\theta_0] \times S \times [0,\varepsilon) \text{ given by}
\end{align*}
\]

\[
(5.2) \quad g = \csc^2(\theta)[d\theta^2 + h_\theta],
\]

then \( \text{Ric}(g) + ng = O_g(\rho^n) \).

Moreover, \( \tilde{h}_\theta \) is even in \( \theta \) to order \( n \).

Finally, if the ordinary differential operator \( I_{n,k} \) given by (4.7) has trivial kernel for all \( k \in \mathbb{N}, \) then in fact \( h_\theta \) may be chosen so that \( \text{Ric}(g) + ng = O_g(\rho^n) \).

We next prove that, assuming Proposition 5.2 is true, Theorem 1.1 follows.

**Proof of Theorem 1.1 using Proposition 5.2.** Let \( h \in [\hat{h}] \), and let \( \psi : S \times [0,\varepsilon) \to W \subset M \) be a diffeomorphism with a neighborhood \( W \) of \( S \) in \( M \) such that \( \psi^*h = \frac{d\rho^2 + k_\rho}{\rho^2} \). Let \( h_\theta \) be a one-parameter family of metrics on \( S \). Then \( \psi \) induces a diffeomorphism \( id \times \psi : [0,\theta_0] \times S \times [0,\varepsilon) \to \tilde{X} = [0,\theta_0] \times M. \)

By Proposition 5.2, there exists a one-parameter family \( \tilde{h}_\theta \in \mathcal{M}(\theta_0, S \times [0,\varepsilon)) \) of AH metrics on \( S \times [0,\varepsilon) \), and a smooth function \( \chi \in C^\infty(S \times [0,\varepsilon)) \) satisfying conditions (a)–(d) and such that \( \tilde{g} = \csc^2(\theta)(d\theta^2 + \tilde{h}_\theta) \) is Einstein mod \( O_g(\rho^n) \); moreover, both \( \tilde{h}_\theta \) and \( \chi \) are uniquely defined mod \( O(\rho^n) \).

Let \( h_\theta = (\psi^{-1})^*\tilde{h}_\theta \in \mathcal{M}(\theta_0, W), \) and \( g = ((id \times \psi)^{-1})^*\tilde{g}; \) then it is clear that \( g = \csc^2(\theta)(d\theta^2 + h_\theta) \), that \( h_0 = ((\psi^{-1})^*\chi)h \in [\hat{h}] \), and that \( \text{Ric}(g) + ng = O_g(\rho^n) \). Moreover, by condition (b) in Proposition 5.2 we conclude that \( \partial_\theta h_\theta|_{\theta=\theta_0} = 0 \). By Lemma 5.1 and because \( \cos(\theta_0) = -\lambda \), it follows that along \( Q \setminus S \), we have \( K_Q = \lambda g|_Q \). Thus, existence is established.

For uniqueness, suppose now that we have another one-parameter family \( h'_\theta \) of AH metrics on \( M \) such that \( g' = \csc^2(\theta)(d\theta^2 + h'_\theta) \) is also Einstein mod \( O_g(\rho^n) \), \( h'_0 \in [\hat{h}] \), and \( K'_Q = \lambda g'|_Q \). Suppose also that \( \partial_\theta(\rho^2h_\theta)|_Q = 0 \). Now
let \( \tilde{h}'_0 = \psi^* h'_0 \), and \( \tilde{\gamma}' = (id \times \psi)^* g' \). Notice that if we write \( h'_0 = \Omega^2 h_0 \), then \( \Omega|_S = 1 \). Then it is easy to see that \( \tilde{\gamma}' \) and \( \tilde{h}'_0 \) satisfy conditions (a) - (e) of Proposition 5.2. Thus, \( \tilde{g} - \tilde{g}' = O_g(\rho^n) \). Pushing forward again, we may conclude that \( g - g' = O_g(\rho^n) \).

Finally, since by hypothesis there are no integral solutions to (4.18) with \( s = n \), it follows by Proposition 4.6 that \( I_{n,\nu} \) has nontrivial kernel for all integral \( \nu \), and the last claim follows.

We now begin working toward a proof of Proposition 5.2. Since we will have no further cause to refer to the setting of Theorem 1.1, for the remainder of this section we will for convenience let \( \tilde{X} = [0, \theta_0] \times S \times [0, \varepsilon] \), let \( \tilde{M} = \{0\} \times S \times [0, \varepsilon] \), let \( M = S \times [0, \varepsilon] \), let \( \tilde{Q} = \{\theta_0\} \times S \times [0, \varepsilon] \), and let \( \tilde{S} = [0, \theta_0] \times S \times \{0\} \).

We begin by computing the following.

**Lemma 5.3.** Let \( S \) be a manifold of dimension \( n - 1 \), let \( \theta_0 \in (0, \pi) \), and let \( g \) be a metric in the normal form (5.2) on \( \tilde{X} = [0, \theta_0] \times S \times [0, \varepsilon] \). Set \( E = \text{Ric}(g) + ng \). Then \( E = \hat{E}_{ij} dx^i dx^j = E_{ij} \omega^i \omega^j \), where

\[
E_{00} = -\frac{1}{2} \sin^2(\theta) \hat{h}^{\mu\nu} \partial_\theta \hat{h}_{\mu\nu} + \frac{1}{2} \sin(\theta) \cos(\theta) \hat{h}^{\mu\nu} \partial_\theta \hat{h}_{\mu\nu} + \frac{1}{4} \sin^2(\theta) \mid \partial_\theta \hat{h} \mid^2_{\hat{h}}
\]

\[
E_{0\sigma} = \frac{1}{2} \sin^2(\theta) \left[ \hat{h}^{\mu\nu} \partial_\theta (\hat{h}_{\mu\nu}) \right]_{\rho\sigma} - n \rho^\mu \partial_\theta \hat{h}_{\rho\sigma\mu} + \rho \left( \nabla^{\mu}(\partial_\theta \hat{h}_{\rho\sigma\mu}) - \nabla_\sigma \left( \nabla^\mu (\partial_\theta \hat{h})_{\rho\mu} \right) \right)
\]

\[
E_{\mu\nu} = -\frac{1}{2} \sin^2(\theta) \partial_\theta^2 \hat{h}_{\mu\nu} + \frac{n-1}{2} \sin(\theta) \cos(\theta) \partial_\theta \hat{h}_{\mu\nu} + \frac{1}{2} \sin^2(\theta) \hat{h}^{\eta\lambda} \partial_\theta (\hat{h}_{\eta\lambda}) \partial_\theta (\hat{h}_{\nu\lambda}) - \frac{1}{4} \sin^2(\theta) \hat{h}^{\eta\lambda} \partial_\theta (\hat{h}_{\eta\lambda}) \partial_\theta (\hat{h}_{\mu\nu}) + \frac{1}{2} \sin(\theta) \cos(\theta) \hat{h}^{\eta\lambda} \partial_\theta (\hat{h}_{\eta\lambda}) \hat{h}_{\mu\nu} + (1-n) \sin^2(\theta) \left( |d\rho|^2_{\hat{h}} - 1 \right) \hat{h}_{\mu\nu} + (n-2) \rho \sin^2(\theta) \nabla_{\mu} \rho_{\nu} + \rho \sin^2(\theta) \nabla^{\eta} \rho_{\eta} \hat{h}_{\mu\nu} + \rho^2 \sin^2(\theta) \text{Ric}(\hat{h})_{\mu\nu}.
\]

Here indices are raised and covariant derivatives taken with respect to \( \hat{h} = \rho^2 \hat{h} \), and \( \hat{h} = \hat{h}_{\mu\nu} dx^\mu dx^\nu \).

**Proof.** Using the form (5.2) of the metric, we compute the Christoffel symbols as follows in coordinates:

\[
\Gamma_{000} = -\csc^2(\theta) \cot(\theta) \quad \Gamma_{\mu\nu0} = \rho^{-2} \csc^2(\theta) \cot(\theta) \hat{h}_{\mu\nu} - \frac{1}{2} \rho^{-2} \csc^2(\theta) \partial_\theta \hat{h}_{\mu\nu} \quad \Gamma_{0\mu0} = 0
\]

\[
\Gamma_{00\sigma} = 0 \quad \Gamma_{0\mu\sigma} = \rho^{-2} \csc^2(\theta) \cot(\theta) \hat{h}_{\mu\sigma} + \frac{1}{2} \rho^{-2} \csc^2(\theta) \partial_\theta \hat{h}_{\mu\sigma}
\]

\[
\Gamma_{\mu\sigma\nu} = -\rho^{-3} \csc^2(\theta) \hat{h}_{\nu(\sigma \rho)} + \rho^{-3} \csc^2(\theta) \hat{h}_{\mu\nu} \rho_{\sigma} + \rho^{-2} \csc^2(\theta) \Gamma_{\mu\nu\sigma},
\]
where $\Gamma$ is the Christoffel symbol of $\bar{h}$. The result now follows from a tedious but straightforward computation using the equation (5.7)

$$R_{ij} = \frac{1}{2} g^{kl} \left( \partial^2_{ik} g_{jk} + \partial^2_{jk} g_{il} - \partial^2_{il} g_{jk} - \partial^2_{ij} g_{kl} \right) + g^{kl} g^{pq} \left( \Gamma_{ilp} \Gamma_{jkq} - \Gamma_{ijp} \Gamma_{klq} \right).$$

We state the following, which will be of use later.

**Lemma 5.4.** If $h_\theta \in H_{n,l}(\theta_0, M, S^2(0 e^* T^* X))$ for some $l \geq 0$ and $\partial_\theta h_\theta |_{\rho=0} = 0$, then for each $j \geq 0$, $\partial^j \rho E(g) |_{\rho=0} \in \oplus_{m_j} (\theta^m \log(\theta)) C^\infty(S, S^2(0 e^* T^* X))$ for some finite $m_j \geq 0$. Moreover, if $h_\theta$ is even in $\theta$ through order $k \geq 2$, then $E(g)$ is even through the same order as a section of $S^2(0 e^* T^* X)$.

**Proof.** We sketch the proof for $E_{00}$ in (5.3). The evenness claim is clear, since the number of factors of $\sin(\theta)$ is the same as the number of derivatives with respect to $\theta$ in each term. For the first claim, inspect each term and notice that at each finite power of $\rho$, there is a bounded power of $\theta^n \log(\theta)$ by the hypothesis on $h_\theta$. The powers of $\theta^n \log(\theta)$ in $h^{-1}$ are bounded at each finite order in $\rho$ due to the hypothesis that $\partial_\theta h_\theta |_{\rho=0} = 0$. The proof for the other components of $E$ is similar.

Our approach to proving Proposition 5.2 will be to construct the metric term by term in powers of $\rho$ by solving the indicial equation, just as for the scalar Laplacian. There are two complications compared to that case: because the operator is nonlinear and acts on sections of a 0-edge bundle, the definition of the indicial operator is more involved and depends on the metric; and because the indicial operator acts differently on different parts of the isotypic decomposition of the metric tensor, there are in effect really several indicial operators. This also occurs in the usual AH case – see e.g. [GL91]. In that case, however, the various parts of the indicial operator are all algebraic, not differential operators.

At each order, we will have to solve a regular singular system of ODEs given by the indicial operators. Because we have gauge-broken the Einstein equations by requiring the metric to be in normal form, the system is overdetermined – we have $\frac{n(n+1)}{2}$ unknowns, but $\frac{(n+1)(n+2)}{2}$ equations. We will therefore follow the usual expedient of using the Bianchi identities to show that the extra equations are automatically satisfied once we have determined the solution using $\frac{n(n+1)}{2}$ equations. It is by the Bianchi equations, as we will see, that $\chi$ will be uniquely determined at each order.

Because of the form of metric (5.2), we will be interested in perturbations $g \mapsto g + \rho^\gamma \varphi$, where $\gamma > 0$ and $\varphi = \varphi_{\mu \nu \omega} \omega^\mu \omega^\nu = \rho^{-2} \csc^2(\theta) \varphi_{\mu \nu} dx^\mu dx^\nu$ is a section of the bundle $T = \{ \eta \in S^2(0 e^* T^* X) : \sin(\theta) \partial_\theta \eta = 0 \}$. A section $\sigma$ of $T$ can be identified with a one-parameter family $\sigma_\theta (0 \leq \theta \leq \theta_0)$ of sections of
Proposition 5.5. The indicial operator $\Gamma^\gamma$ for a metric $g$ in the normal form (5.2) has the form $\Gamma^\gamma(\varphi) = \tilde{I}_{ij}^\gamma(\varphi)\omega^i\omega^j$, where

\begin{align}
2I_{00}^\gamma(\varphi) &= -\sin^2(\theta)\tilde{h}^{\mu\nu}\partial_\theta^2\varphi_{\mu\nu} + \sin(\theta)\cos(\theta)\tilde{h}^{\mu\nu}\partial_\theta\varphi_{\mu\nu} \\
2I_{0\sigma}^\gamma(\varphi) &= \sin^2(\theta) [(\gamma - n) \rho^\nu \partial_\theta \varphi_{\mu\sigma} - (\gamma - 1) \rho_\sigma \tilde{h}^{\mu\nu} \partial_\theta \varphi_{\mu\nu}] \\
2I_{\mu n}^\gamma(\varphi) &= -\sin^2(\theta) \partial_\theta^2 \varphi_{\mu n} + (n - 1) \sin(\theta) \cos(\theta) \partial_\theta \varphi_{\mu n} \\
&\quad + \sin(\theta) \cos(\theta) \partial_\theta (\tilde{h}^{\mu\nu} \varphi_{\mu\nu}) + (\gamma - 2)(\gamma + 1 - n) \sin^2(\theta) \varphi_{\mu n} \\
&\quad + \gamma(2 - \gamma) \sin^2(\theta) \tilde{h}^{\mu\nu} \varphi_{\mu\nu} \\
2\tilde{h}^{\mu\nu} I_{\mu\nu}^\gamma(\varphi) &= -\sin^2(\theta) \partial_\theta^2 (\tilde{h}^{\mu\nu} \varphi_{\mu\nu}) + (2n - 1) \sin(\theta) \cos(\theta) \partial_\theta (\tilde{h}^{\mu\nu} \varphi_{\mu\nu}) \\
&\quad + 2(\gamma - n)(\gamma + 1 - n) \sin^2(\theta) \varphi_{\mu n} - 2\gamma(\gamma - n) \sin^2(\theta) \tilde{h}^{\mu\nu} \varphi_{\mu\nu} \\
2I_{\mu s}^\gamma(\varphi) &= -\sin^2(\theta) \partial_\theta^2 \varphi_{\mu s} + (n - 1) \sin(\theta) \cos(\theta) \partial_\theta \varphi_{\mu s} \\
2\tilde{I}_{sl}^\gamma(\varphi) &= -\sin^2(\theta) \partial_\theta^2 \varphi_{sl} + (n - 1) \sin(\theta) \cos(\theta) \partial_\theta \varphi_{sl} \\
&\quad - \gamma(\gamma + 1 - n) \sin^2(\theta) \varphi_{sl},
\end{align}

where $\varphi_{st} = \varphi_{sl} - \frac{1}{n-1} \tilde{h}^{rq} \varphi_{rq} \tilde{h}_{st}$, and similarly for $\tilde{I}_{st}$.

Note that on each fiber $F$ of $\tilde{S}$, $I$ restricts to an operator $I : C^\infty(F, T) \to C^\infty(F, S^2(0eT^*\tilde{X}))$.

**Proof.** We set $\tilde{g} = g + \rho^\gamma \varphi_{\mu\nu}\omega^\mu\omega^\nu$. Writing $\tilde{g}$ in the form (5.2), we see that the change $g \mapsto \tilde{g}$ is equivalent to the change $\tilde{h}_{\mu\nu} dx^\mu dx^\nu \mapsto (\tilde{h}_{\mu\nu} + \rho^\gamma \varphi_{\mu\nu}) dx^\mu dx^\nu$. We use this expression in equations (5.3) - (5.5) to compute $I_{00}^\gamma$, $I_{0\sigma}^\gamma$, and $I_{\mu\nu}^\gamma$, using the formula $\Gamma^\gamma(\varphi) = \rho^{-\gamma} [E(g + \rho^\gamma \varphi) - E(g)] |_{\rho=0}$. We then specialize
\( \Gamma_{\mu\nu} \) with various choices of \( \mu \) and \( \nu \) to obtain the result. We will carry out only the computation for \( f_{00} \); the rest are similar.

Because \( \partial_{\theta} h_{\theta}\big|_{\rho=0} = 0 \), then in particular \( \partial_{\theta} \tilde{h}_{\theta} = O(\rho) \) and \( \partial_{\theta} \tilde{h}^{-1} = O(\rho) \). It follows that the effect of the perturbation on the inverse metric \( \tilde{h}^{\mu\nu} \) may be ignored, as may the last term in (5.3). Formula (5.8) then follows immediately.

Before proving Proposition 5.2, we define some notation that will be useful. Notice that, by the product structure of \( \tilde{X} \), there is a natural decomposition \( \tilde{0}eT \tilde{X} \approx 0eT[0,\theta_0] \oplus 0eTS \oplus 0eT[0,\varepsilon] \), where the summands on the right have their obvious meanings. Similarly, there is a natural decomposition \( \tilde{0}TM \approx 0TS \oplus 0T[0,\varepsilon] \). For a section \( T \in C^\infty(\tilde{X},S^2(0eT^* \tilde{X})) \), we let \( T|_{TS} \) be the restriction of \( T \) to the middle factor in the above three-way decomposition. Now if \( k \) is a metric on \( S \), then \( \rho^2 k \) is a metric on \( 0TS \). For \( T \in C^\infty(\tilde{X},S^2(0eT^* \tilde{X})) \) and \( k \) a metric on \( S \), we will define the notation

\[
\tilde{t}_k T := \tilde{t}_\rho k(T|_{TS}),
\]

so that \( \tilde{t}_k T \) is a section in \( \text{csc}^2(\theta)C^\infty(\tilde{X},S^2(0TS)) \). In components, this takes the usual form

\[
(\tilde{t}_k T)_{st} = T_{st} - \frac{1}{n-1} k^{pq} T_{pq} k_{st}.
\]

Similarly, if \( T \) is a section of \( S^2(0T^*M) \), then \( \tilde{t}_k T \) will refer to \( \tilde{t}_k(T|_{TS}) \). We will also use the notation \( \tilde{t}_k T \) in its usual sense when \( T \) is an ordinary symmetric two-tensor.

**Proof of Proposition 5.2.** We will construct a solution order-by-order in \( \rho \). At each step, we will solve the regular singular system of ODEs given by the operators (5.8) - (5.13). As mentioned earlier, we will actually use only some of the equations to solve for \( \varphi \), and will show that the others are satisfied by our solution using the Bianchi identity.

We are determining \( h_{\theta} \) in (5.2); although we will work instead with \( \tilde{h}_{\theta} = \rho^2 h_{\theta} \). Our boundary condition at \( \theta = \theta_0 \) is that \( \partial_{\theta} \tilde{h}_{\theta}\big|_{\theta=\theta_0} = 0 \). At \( \tilde{M} \), our boundary condition is that \( \tilde{h}_{\theta} = \chi(d\rho^2 + k_{\rho}) \), where \( \chi \) is as-yet an undetermined function.

We assume for now that \( n > 2 \).

We define \( \tilde{h}_{\theta}^{(0)} = d\rho^2 + k_{\rho} \), and \( g^{(0)} = \text{csc}^2(\theta)(d\theta^2 + \rho^{-2}\tilde{h}_{\theta}^{(0)}) \). It is straightforward to show using Lemma 5.3 that \( E^{(0)} := \text{Ric}(g^{(0)}) + ng^{(0)} = O_g(\rho) \) — only terms on the last two lines of (5.5) are nonvanishing. We similarly define \( \chi^{(0)} \in C^\infty(M) \) by \( \chi^{(0)} \equiv 1 \).

We will now proceed by induction. Let \( 1 \leq \gamma \leq n-1 \), and suppose for purpose of induction that we have a metric \( g^{(\gamma-1)} = \text{csc}^2(\theta) \left[d\theta^2 + \rho^{-2}\tilde{h}_{\theta}^{(\gamma-1)}\right] \) and a smooth function \( \chi^{(\gamma-1)} \in C^\infty(\tilde{M}) \) such that

1. \( \partial_{\theta} \tilde{h}_{\theta}^{(\gamma-1)}\big|_{\rho=0} = 0 \);
2. \( \tilde{h}_{\theta}^{(\gamma-1)} \in \mathcal{H}_{n,l}(\theta_0,M,S^2T^*M) \) for some \( l \);
(iii) \( \chi \in C^\infty(M) \);
(iv) \( \partial_\theta \tilde{h}_\theta^{(\gamma-1)}|_{\theta=\theta_0} = 0 \);
(v) \( \chi^{(\gamma-1)}|_{\rho=0} = 1 \);
(vi) \( \tilde{h}_\theta^{(\gamma-1)}|_{\theta=0} = \chi^{(\gamma-1)}(d\rho^2 + k_\rho) \);
(vii) \( E^{(\gamma-1)} := E(g^{(\gamma-1)}) = O(\rho^\gamma) \);
(viii) \( h_\theta^{(\gamma-1)} \) is even in \( \theta \) through order \( n \) if \( n \) is even, or infinite order if \( n \) is odd;
(ix) \( \rho^{-\gamma} tf_{k_0} E^{(\gamma-1)}|_\rho=0 \in B_{n,m}(\theta_0, S, S^2(0T^*M)) \) for some \( m \); and
(x) \( \chi^{(\gamma-1)} \) and \( h_\theta^{(\gamma-1)} \) satisfying conditions (i) - (vi) are uniquely defined modulo \( O(\rho^\gamma) \).

We wish to show that we can construct a function \( \chi^{(\gamma)} \) and a family of metrics \( h_\rho^{(\gamma)} \) such that these conditions are all satisfied with \( \gamma \) everywhere replaced by \( \gamma + 1 \). (Conditions (viii) - (ix) will be used at several points in the induction step.) Put differently, we wish to show that we may uniquely define perturbations \( \varphi^{(\gamma)} \in \csc^2(\theta) \mathcal{H}_{n,l'}(\theta_0, S, S^2(0T^*M)) \) (for some \( l' \)) and \( \psi^{(\gamma)} \in C^\infty(S) \) such that, taking \( \tilde{h}_0^{(\gamma)} = \tilde{h}^{(\gamma-1)} + \rho^\gamma \tilde{\varphi}^{(\gamma)} \), the desired conditions are satisfied. Actually, condition (vi) will be satisfied only through order \( \gamma \), due to the impact on higher-order terms on the right-hand side of changing \( \chi \) at order \( \gamma \); we will restore (vi), however, without affecting any other conditions by adding one more perturbation that is independent of \( \theta \). We will henceforth refer just to \( \varphi \) and \( \psi \), leaving the \( (\gamma) \) implicit.

Define \( f = \rho^{-\gamma} E^{(\gamma-1)}|_{\rho=0} \in C^{n-1}(\tilde{S}, S^2(0T^*\tilde{X})) \). It is easy to see that we will have completed the induction if we can find \( \varphi \) and \( \psi \) such that

1. \( I^\gamma(\varphi) = -f \), where \( I^\gamma \) is the indicial operator defined above;
2. \( \varphi_{nn}|_{\theta=0} = \psi \);
3. \( \tilde{h}^{\mu\nu} \varphi_{\mu\nu}|_{\theta=0} = n \psi \);
4. \( \varphi_{nn}|_{\theta=0} = 0 \);
5. \( \varphi_{st}|_{\theta=0} = 0 \);
6. \( \partial_\theta \varphi|_{\theta=0} = 0 \);
7. \( \varphi \) is even in \( \theta \) through order \( n \) for even \( n \), or infinite order if \( n \) is odd;
8. \( \tilde{\varphi} \in \mathcal{H}_{n,l'}(\theta_0, S, S^2T^*M) \) for some \( l' \);
9. \( \psi \in C^\infty(M) \); and
10. \( \rho^{-(\gamma+1)} tf_{k_0} E(g^{(\gamma)})|_{\rho=0} \in B_{n,m'}(\theta_0, S, S^2(0T^*M)) \) for some \( m' \),

with \( \varphi \) and \( \chi \) determined uniquely by conditions (i) - (vi).

Fix \( x \in \tilde{M} \cap \tilde{S} \approx S \), which we regard as determining a fiber. We will determine \( \varphi \) and \( \psi \) on the fiber \([0, \theta_0] \times \{x\} \times \{0\} \). Since our constructions will all depend smoothly on \( x \), we suppress it when convenient and write \( \varphi \) as a function of \( \theta \) alone. We first regard \( \psi(x) \) as a free parameter and show that, for any choice of \( \psi(x) \), \( \varphi(x, \theta) \) is uniquely determined. Thus, for now regard \( \psi(x) \) as given, and \( \chi^{(\gamma)} = \chi^{(\gamma-1)} + \rho^\gamma \psi \).
We first determine $\hat{\varphi}_{st} = t f_{00} \varphi$. Our boundary condition at $\theta = \theta_0$ is, as noted above, $\partial_\theta \hat{\varphi}|_{\theta = \theta_0} = 0$. Our boundary condition at $\theta = 0$ is $\hat{\varphi}_{st} = 0$. Now we wish to solve $\hat{I}_{st}(\varphi) = -\hat{f}_{st}$. By (5.13), $\hat{I}$ acts as a scalar on $\hat{\varphi}$. Moreover, $\hat{I}$ is merely $-\frac{1}{\theta}$ times the indicial operator of the scalar Laplacian, by Lemma 1.2. Now by (13), $\hat{f}_{st} \in \mathcal{B}_{n,m}(\theta_0, S, S^2(0 T^* M))$, and so by Propositions 4.5 and 4.8, the equation $\hat{I}_{st}(\varphi) = -\hat{f}_{st}$ has a unique solution $\hat{\varphi}_{st}$ in $\mathcal{A}_{n,m}(\theta_0, S, S^2(0 T^* M))$. By induction and (5.5), $\hat{f}_{st}$ is even in $\theta$ through order $n$ if $n$ is even, or infinite order otherwise. Thus, by the form (14.20) of the Green’s operator, we also may conclude that if $n$ is even, then $\hat{\varphi}_{st}$ is also even to the order $n$ in $\theta$.

Now suppose $n$ is odd. Because $\hat{f}_{st}$ is even to infinite order, and in particular through order $n + 2$, it follows by Proposition 13 that $\hat{\varphi}$ is smooth and even to infinite order, and contains no logarithmic terms. Whether $n$ is even or odd, then, $\hat{\varphi}_{st} \in \mathcal{H}_{n,m}(\theta_0, S, S^2 T^* M)$.

We next determine the trace $h^{\mu\nu} \varphi_{\mu\nu}$, which for convenience we denote $\ell(\theta)$ for the remainder of this proof. Because of the overdetermined nature of our system, it would appear a priori possible to use either $\hat{I}_{00}$ or $h^{\mu\nu} \hat{I}_{\mu\nu}$ to do this. However, $\hat{I}_{00}$ is simpler because it involves only the trace of $\varphi$, whereas $h^{\mu\nu} \hat{I}_{\mu\nu}$ involves both the trace and $\varphi_{nm}$. (Because $\hat{h}$ at $\rho = 0$ is independent of $\theta$, we may regard $\hat{I}_{00}$ as giving a differential equation for $\ell$.) We thus proceed with $\hat{I}_{00}$. As usual, we wish to solve the equation $\hat{I}_{00}(\varphi) = -f_{00}$, subject to the conditions $\ell(0) = n \psi(x)$ and $\ell'(0) = 0$. We claim that $f_{00} = O(\theta^4)$: by the induction hypothesis, $\partial_\theta \hat{h}^{(\gamma - 1)} = O(\theta)$, so the last term in (5.3) is $O(\theta^4)$. The other two terms may be written $-\sin^2(\theta)(\hat{h}^{\mu\nu} \partial_\theta^2 \hat{h}_{\mu\nu} - \cot(\theta) \hat{h}^{\mu\nu} \partial_\theta \hat{h}_{\mu\nu})$. But since $\hat{h}^{(\gamma - 1)}$ is even in $\theta$ to order $n$, the term in parenthesis is $O(\theta^2)$, as claimed. Moreover, since $\partial_\theta \hat{h}_{\theta}$ is even through order $n$ or infinity in $\theta$ (depending on parity), it follows from (5.3) that $f_{00}$ is as well. It is easy to verify that a solution to the equation $\hat{I}_{00} \ell = -f_{00}$ satisfying $\ell(0) = n \psi(x)$ and $\ell'(0) = 0$ is given by

$$\ell(\theta) = 2 \left( \cos(\theta) \int_0^{\theta_0} \csc^3(\phi) f_{00}(\phi) d\phi + \int_0^\theta \cot(\phi) \csc(\phi) f_{00}(\phi) d\phi \right) - \int_0^{\theta_0} \csc^3(\phi) f_{00}(\phi) d\phi + n \psi(x).$$

The solution is easily shown to be unique: the homogeneous equation is linear, with general solution $a \cos(\theta) + b$. Given the requirements that $\ell(0) = 0 = \ell'(0)$, we may deduce that $a = 0 = b$. Furthermore, since $f_{00}$ is even in $\theta$ through order $n$ or infinity, the solution $\ell(\theta)$ is as well. This may be easily verified by differentiating the above solution formula, obtaining $\ell'(\theta) = -\sin(\theta) \int_0^{\theta_0} \csc^3(\phi) f_{00}(\phi) d\phi$. Similarly, Lemma 5.4 implies that $f_{00} \in \mathcal{H}_{n',p}(\theta_0, S)$ for some $l'$, and since $\int \theta^p \log(\theta)^q d\theta = O(\theta^{p+1} \log(\theta)^q)$, we conclude that $\ell(\theta) \in \mathcal{H}_{n,l'}(\theta_0, S)$ as well.
Next we wish to determine \( \varphi_{nn} \), for which we use \( I^\alpha_m \). We get the equation

\[(\gamma - n)\partial_\theta \varphi_{nn} = (\gamma - 1)\ell'(\theta) - f_0n(\theta),\]

with conditions \( \varphi_{nn}(0) = \psi(x) \) and \( \varphi'_{nn}(\theta_0) = 0 \). Now \( \ell'(\theta_0) = 0 \) by construction. Moreover, by (5.14) and our inductive hypothesis – according to which \( \partial_\theta \bar{h}^{(\gamma-1)}|_{\theta = \theta_0} = 0 \) – we see that \( f_0n(\theta_0) = 0 \) as well. Thus, the right hand side vanishes at \( \theta = \theta_0 \), and our boundary condition at \( \theta_0 \) is satisfied automatically. We can therefore integrate to uniquely determine \( \varphi_{nn} \) subject to the condition that \( \varphi_{nn}(0) = \chi(x) \). By construction of \( \ell(\theta) \) and by (5.4), the right-hand side of the above equation is odd through order \( n - 1 \) or through order infinity; therefore, \( \varphi_{nn}(\theta) \) is even through order \( n \) or infinity (depending on parity). A similar argument as for the trace also shows that \( \varphi_{nn} \in \mathcal{H}_{n,l}(\theta_0, S) \).

We have only to determine \( \varphi_{ns} \), which is to say, \( \left( \frac{\partial}{\partial \rho} \bigtriangledown \varphi \right) \big|_{TS} \). To do this, we use \( I_{0s} \). We get the equation

\[(\gamma - n)\partial_\theta \varphi_{ns} = -f_{0s}(\theta).\]

As in the previous case, the boundary condition at \( \theta_0 \) is automatically satisfied, and we can integrate to get a unique solution satisfying our conditions; parity is preserved as desired, and \( \varphi_{ns} \in \mathcal{H}_{n,m}(\theta_0, S, T^* M) \).

We have determined \( \varphi \), and thus have constructed a \( g^{(\gamma)} \) so that \( E^\gamma_{00}, E^\gamma_0r, \) and \( E^\gamma_{st} = O_g(\rho^{\gamma+1}) \). However, it remains to analyze \( \bar{h}^{\mu \nu} E^\mu_{\nu r}, E^\gamma_{ns}, \) and \( E^\gamma_{ns}, \) since their corresponding indicial operators were not used in our construction. (We will henceforth omit the \( (\gamma) \) from \( E \) for clarity.) For this, we will use the contracted Bianchi identities, which state that \( 2\nabla^1 E_{ij} = \nabla_j E^i_i; \) or working now in the coordinate frame, that

\[0 = B_i := 2g^{ik} \partial_k \bar{E}_{ij} - g^{ik} \partial_i \bar{E}_{jk} - 2g^{ik} g^{dl} \Gamma_{jkq} \hat{E}_{il}.\]

We apply this to \( g^{(\gamma)} \) using our earlier computations (5.6) of Christoffel symbols. Still working in the coordinate frame, we find

\[
B_0 = \sin^2(\theta)\partial_\theta \bar{E}_{00} + 2\rho^2 \sin^2(\theta) \bar{h}^{\mu \nu} \partial_\nu \bar{E}_{0\mu} - \rho^2 \sin^2(\theta) \bar{h}^{\mu \nu} \partial_\nu \bar{E}_{\mu \nu} \\
+ 2(1 - n) \sin(\theta) \cos(\theta) \hat{E}_{00} + \sin^2(\theta) \bar{h}^{\mu \nu} \partial_\nu (\bar{h}_{\mu \nu}) \hat{E}_{00} \\
+ 2(2 - n)\rho \sin^2(\theta) \rho^2 \bar{E}_{00} - 2\rho^2 \sin^2(\theta) (\hat{h}^{\mu \nu} \bar{h}_{\rho \lambda}) \Gamma_{\mu \nu \rho} \hat{E}_{0\lambda};
\]

and

\[
B_\sigma = 2\sin^2(\theta) \partial_\theta \bar{E}_{0\sigma} - \sin^2(\theta) \partial_\sigma \bar{E}_{00} + 2\rho^2 \sin^2(\theta) \bar{h}^{\mu \nu} \partial_\nu \hat{E}_{\mu \sigma} \\
- \rho^2 \sin^2(\theta) \bar{h}^{\mu \nu} \partial_\sigma \bar{E}_{\mu \nu} + 2(1 - n) \sin(\theta) \cos(\theta) \hat{E}_{0\sigma} \\
+ \sin^2(\theta) \bar{h}^{\mu \nu} \partial_\nu (\bar{h}_{\mu \nu}) \hat{E}_{0\sigma} + 2(2 - n)\rho \sin^2(\theta) \rho^2 \hat{E}_{0\lambda} \\
+ 2\rho^2 \sin^2(\theta) (\bar{h}^{\mu \nu} \bar{h}_{\rho \lambda}) \Gamma_{\mu \nu \rho} \hat{E}_{0\lambda},
\]

where \( \Gamma \) is the Christoffel symbol of \( \bar{h} \).
We evaluate $B_0 \mod O(\rho^{\gamma+1})$, using the fact that we already know the following:

\[
\begin{align*}
\hat{E}_{00} &= O(\rho^{\gamma+1}) & \hat{E}_{0\mu} &= O(\rho^\gamma) & \hat{E}_{st} &= O(\rho^{\gamma-1}) \\
\hat{E}_{ns} &= O(\rho^{\gamma-2}) & \hat{E}_{nn} &= O(\rho^{\gamma-2}) & \bar{h}^{\mu\nu} \hat{E}_{\mu\nu} &= O(\rho^{\gamma-2}).
\end{align*}
\]

The first row are all $O(\rho^{\gamma+1})$, as desired, but the second row are one order lower. Putting these into the equation for $B_0$ and setting it equal to 0 yields

\[
\bar{h}^{\mu\nu} \partial_\theta \hat{E}_{\mu\nu} = O(\rho^{\gamma-1}).
\]

This says that $\rho^{1-\gamma} \bar{h}^{\mu\nu} \hat{E}_{\mu\nu}|_{\rho=0}$ is a constant, say $\frac{c}{2}$. We need $c = 0$. By definition of the indicial operator, the equation $\rho^{1-\gamma} \bar{h}^{\mu\nu} \hat{E}_{\mu\nu}|_{\rho=0} = \frac{c}{2}$ is equivalent to saying $2\bar{h}^{\mu\nu} I_{\mu\nu}(\varphi) + 2 \bar{h}^{\mu\nu} f_{\mu\nu} = c$. The left-hand side of this latter equation, of course, is already determined up to choice of $\psi$, since $\varphi$ is. Notice in (5.11) that $\bar{h}^{\mu\nu} I_{\mu\nu}$ depends on $\varphi_{nn}$ and $\bar{h}^{\mu\nu} \varphi_{\mu\nu}$, which in turn we have determined using the operators (5.9) and (5.8), respectively. Neither of these operators has a zeroth-order part, and so adding $\delta$ to $\psi$ adds $\delta$ to $\varphi_{nn}$ and $n\delta$ to $\bar{h}^{\mu\nu} \varphi_{\mu\nu}$, by our boundary conditions (2) and (3). Now using equation (5.11), but shifting it to the coordinate frame, we see that adding $\delta$ to $\psi$ adds $2(1-n)(\gamma-n)(\gamma+1)\delta$ to $2 I_{\mu\nu}(\varphi)$. Thus, since $\gamma \neq n$, there is a unique choice of $\psi(x)$ such that $c = 0$; and so we find that $\chi^{(\gamma)} = \chi^{(\gamma-1)} + \rho^{\gamma}\psi$ is uniquely determined up through order $\gamma$ so that $\bar{h}^{\mu\nu} E_{\mu\nu} = O_2(\rho^{\gamma})$; and there remains no further freedom in our system.

It remains to analyze $\hat{E}_{ns}$. We next look at $B_s$. We find that

\[
2\rho \partial_\theta \hat{E}_{ns} + 2(2-n) \hat{E}_{ns} = O(\rho^{\gamma-1}).
\]

Now write $\hat{E}_{ns} = \xi_s \rho^{\gamma-2}$, which we may do by the above computations. Putting this into our equation, we find

\[
(\nu - n) \xi_s = O(\rho^{\gamma-1}),
\]

as desired.

Before proceeding to $\hat{E}_{nn}$, we note that we can write $\hat{E}_{\mu\nu} = \alpha \rho^{\gamma-2} \rho_{\mu} \rho_{\nu} + 2 \xi_{(\mu} \rho_{\nu)} + \frac{1}{n} \ell \bar{h}_{\mu\nu} + \eta_{\mu\nu}$, where $\rho^\mu \xi_\mu = 0 = \bar{h}^{\mu\nu} \eta_{\mu\nu}$ and $0 = \rho^\mu \eta_{\mu\nu}$, and finally $\eta_{\mu\nu} = \eta(\mu\nu)$. Then every term here except $\alpha \rho^{\gamma-2}$ is $O(\rho^{\gamma-1})$ by our earlier analysis. Now using the Bianchi identity $B_n = 0$, we find

\[
2\rho \partial_\theta \hat{E}_{nn} + 2(2-n) \hat{E}_{nn} = O(\rho^{\gamma-1}).
\]

Since $\hat{E}_{nn} = \alpha \rho^{\gamma-2}$, we find that $(\gamma - n) \alpha = O(\rho)$, and thus, since $\gamma \neq n$, we conclude that $\hat{E}_{nn} = O(\rho^{\gamma-1})$.

Thus, $E(g^{(\gamma)}) = O_2(\rho^{\gamma+1})$. Now $\hat{h}_\theta^{(\gamma)}$ lies in $\mathcal{H}_{n,l'}(\theta_0, \mathcal{S}, S^2 T^* M)$ for some $l'$, is even to order $n$ in $\theta$, and satisfies our boundary conditions. Also it is unique subject to these conditions. It remains only to show that (10) obtains. This is trivial if $n$ is odd; so let $n$ be even.
Consider equation (5.5). Let \( v \) be any term on the right hand side except for the first two and except for
\[
w = \frac{1}{2} \sin(\theta) \cos(\theta) \bar{h}^{\nu \lambda} \partial_\theta (\bar{h}_{\nu \lambda}) \bar{h}_{\mu \nu};
\]
then since \( \bar{h}_\theta \in A_{n,v}(\theta_0, S, S^2 T^* M) \), it follows easily that for every \( j \geq 0 \), we have
\[
\partial_\rho^j v|_{\rho=0} \in B_{n,m'}(\theta_0, S, S^2 (0 T^* M)) \quad \text{(some } n').
\]
For example, take \( v = \frac{1}{2} \sin^2(\theta) \bar{h}^{\nu \lambda} \partial_\theta (\bar{h}_{\mu \nu}) \bar{h}_{\nu \lambda} \). The lowest order at which \( \log(\theta) \) can appear in \( \partial_\theta \bar{h}_\theta \) is at power \( \theta^{n-1} \). Since there is a factor of \( \sin^2(\theta) \), \( \log(\theta) \) therefore does not appear in \( v \) before order \( \theta^{n+1} \) (and in fact \( \theta^{n+2} \), since \( \partial_\theta \bar{h}_\theta = O(\theta) \) as well). Similarly, for \( i > 1 \), \( \log(\theta)^i \) never appears before order \( \theta^{in} \), due to the factor of \( \sin^2(\theta) \) and the hypothesis that \( \bar{h}_\theta \in A_{n,v}(\theta_0, S, S^2 T^* M) \). The remaining terms are similar. Likewise, if \( v \) is the sum of the first two terms, we have the same result, because \( n \) is an indicial root at \( \theta = 0 \) of the operator \( -\sin^2(\theta) \partial_\theta^2 + (n-1) \sin(\theta) \cos(\theta) \partial_\theta \).

Now let \( \bar{h}_\theta \in A_{n,v}(\theta_0, S, S^2 (0 T^* M)) \) and extend it to a section in \( C^\infty(\tilde{M}, S^2 T^* \tilde{M}) \) by making it constant in \( \theta \). Now replace \( \bar{h}_\theta \) by \( \bar{h}_\theta + b \). Since \( b \) is independent of \( \theta \), this obtains condition (vi) without compromising our other conditions.

And so by induction, we may construct \( g^{(n-1)} \) such that \( E(g^{(n-1)}) = O_g(\rho^n) \), satisfying the desired boundary condition at \( \tilde{M} \) to order \( \rho^n \) and to
infinite order at $\tilde{Q}$. This completes the proof for $n > 2$, except for the claim
that if $I_{n,n,\nu}$ is injective for all integral $\nu$, then we may solve the system
to infinite order. As observed above, of course, $I_{n,n,\nu}$ is simply twice the
tracefree part of the indicial operator of the Einstein problem; the absence
of integral indicial roots simply means that we will be able to solve the
tracefree equation $\hat{I}_{st}^\gamma(\varphi) = -\hat{f}_{st}$ for any integral $\gamma$ (which, of course, are the
only $\gamma$ we will encounter on our induction). Meanwhile, if $\gamma \notin \mathbb{Z}$ is some
value for which $\hat{I}_{\gamma}$ actually does fail to be injective, we may nevertheless
simply choose the coefficient of $\rho_{\gamma}$ in the expansion of $\bar{h}$ to be
0, without affecting our ability to expand indefinitely. In fact, we
must so choose the coefficients of $\rho_{\gamma}$ for non-integral $\gamma$, as our metric is supposed to be smooth.
Hence there is no loss of uniqueness.

Thus, our induction can proceed indefinitely, using the above arguments,
with a single problem: at order $\gamma = n$, several crucial coefficients in the
indicial operators vanish, causing our above induction-step arguments for
components other than the tracefree tangential component to fail. Thus,
we now provide an argument that at $\gamma = n$ we may (non-uniquely) extend
$\bar{h}(n-1)$ to $\bar{h}(n)$, so long as $n$ is not an indicial root; the above arguments
then go through once more at every higher order, so we can complete our
induction. Note that the only loss of uniqueness occurs at order $n$, so given
a single scalar choice at that order, uniqueness otherwise remains to infinite
order.

Let $\gamma = n$, then, and assume once more that conditions (1) - (x) hold. Let
$f$ be as before. We again wish to find $\varphi$ and $\psi$ such that (1) - (11) hold, except in this case not uniquely. In particular, $\psi$ will remain undetermined
in this argument (and parametrizes our freedom). Thus, let $\psi \in C^\infty(S)$
be arbitrary. We have already seen, by the above remarks, that we may
uniquely find $\varphi_{st}$ such that $\hat{I}_{st}(\varphi) = -\hat{f}_{st}$. The same arguments given in the
previous case establish that $\varphi_{st}$ is even in $\theta$ to order $n$, or to infinite order if
$n$ is odd, and is also smooth if $n$ is odd. Thus, $\varphi_{st} \in H_{n,n}(t_0,S,S^2T^*M)$.

Once again, we next determine the trace, $\ell = \hat{h}^{\mu\nu}\varphi_{\mu\nu}$. However, this
time we use the trace of the indicial operator; that is, we wish to solve the
equation $\hat{h}^{\mu\nu}I_{\mu\nu}(\varphi) = -\hat{h}^{\mu\nu}f_{\mu\nu}$. Notice in (5.11) that, for $\gamma = n$ only, the
operator $\hat{h}^{\mu\nu}I_{\mu\nu}$ is uncoupled from $\varphi_{nn}$. It is easy to see that the equation
$\hat{h}^{\mu\nu}P_{\mu\nu}\ell = -\hat{h}^{\mu\nu}f_{\mu\nu}$, with initial conditions $\ell'(t_0) = 0$ and $\ell(0) = n\psi(x)$ has
the unique solution

$$\ell(\theta) = n\psi(x) - 2\int_0^{t_0} \sin^{2n-1}(\phi) \int_0^{\theta} \csc^{2n-1}(\beta)\hat{h}^{\mu\nu}f_{\mu\nu}(\beta)d\beta d\phi.$$ 

If $n$ is odd, then $\hat{h}^{\mu\nu}f_{\mu\nu}$ is even to infinite order, and thus $\ell$ is smooth and
even to infinite order as well. If $n$ is even, $\ell(\theta)$ is smooth and even through
(at least) order $n$, as desired.

To determine the $\varphi_{nn}$ component, we can no longer use $I_{00}$. However,
as we have already determined $\ell$, we can use $I_{nn}$, as given in (5.10). It is
straightforward to see that the unique solution to \( I^n_{mn} \varphi_{mn} = -f_{nn} \) satisfying 
\[ \varphi_{nn}(0) = \psi \] 
and \[ \varphi'_{nn}(\theta_0) = 0 \] is \( \varphi_{nn}(\theta) = \psi + u(\theta) \), where \( u(\theta) \) is the unique solution to \( I_{n,n-2} u = 2f_{nn} + \sin(\theta) \cos(\theta)\ell'(\theta) - n(n-2)\sin^2(\theta)\ell(\theta) + (n-2)\sin^2(\theta)\psi(x) \) satisfying \( u(0) = 0 = u'(0) \), and where where \( I_{n,n-2} \) is the indicial operator for the scalar Laplacian analyzed in Proposition 4.8. Since \( I_{n,n-2} \) is a bijection by that proposition, the existence and uniqueness of \( u \) follow, and the desired parity and smoothness properties for \( \varphi_{nn} \) follow from the same proposition and from the already-determined properties of \( f \) and of \( \ell \).

Finally, \( \varphi_{sn} \) may be easily and uniquely determined from the equation 
\[ I^n_{sn} \varphi_{sn} = -f_{sn}, \] 
using (5.12). Thus, we have uniquely determined \( \varphi^{(n)} \) subject to our freedom in choosing \( \psi \).

It remains now to use the Bianchi identities again, this time to show that \( E_{00}, E_{0n}, \) and \( E_{0s} \) vanish to the desired orders in \( \rho \). Letting \( E = \hat{E}_{ij}dx^i dx^j \) be the Einstein tensor of \( g^{(n)} \), we know the following by construction:
\[
\hat{E}_{00} = O(\rho^n) \\
\hat{E}_{0\mu} = O(\rho^{n-1}) \\
\hat{E}_{nn} = O(\rho^{n-1}) \\
\hat{E}_{ss} = O(\rho^{n-1}) \\
\hat{h}^{\mu\nu} \hat{E}_{\mu\nu} = O(\rho^{n-1}).
\]
This time, the last entry in the first row and the entire second row are all \( O_g(\rho^{n+1}) \), as we would like, but the first two are only \( O_g(\rho^n) \) \emph{a priori}.

We introduce functions \( \alpha(\theta,x,\rho) \) and \( \beta(\theta,x,\rho) \) defined by \( \hat{E}_{00} = \alpha \rho^n \) and \( \hat{E}_{0n} = \beta \rho^{n-1} \). Using the Bianchi identity \( B_0 = 0 \) with (5.14) now yields
\[
\partial_\theta \hat{E}_{00} + 2(1-n)\cot(\theta)\hat{E}_{00} + 2\rho^2 \partial_\rho \hat{E}_{0n} + 2(2-n)\rho \hat{E}_{0n} = O(\rho^{n+1})
\]
which, at \( \rho = 0 \), gives us the equation
\[
(5.16) \quad \alpha'(\theta) + 2(1-n)\cot(\theta)\alpha(\theta) + 2\beta(\theta) = 0,
\]
where for notational convenience we regard \( \alpha \) and \( \beta \) as functions only of \( \theta \) when \( \rho = 0 \). Similarly, using (5.15) and the equation \( B_n = 0 \), we find
\[
(5.17) \quad 2\beta'(\theta) + 2(1-n)\cot(\theta)\alpha(\theta) - n\alpha(\theta) = 0
\]
Now, \( \alpha \) and \( \beta \) satisfy \( \alpha(\theta_0) = 0 = \beta(\theta_0) \), which follows from our induction hypothesis, our construction of \( \varphi \), and examination of equations (5.3) and (5.4). Thus, by the uniqueness of solutions to first-order ODEs, we conclude that \( \alpha \equiv \beta \equiv 0 \). Hence, in fact \( \hat{E}_{00} = O(\rho^{n+1}) \) and \( \hat{E}_{0n} = O(\rho^n) \). It is now completely straightforward to show, using \( B_s \), that \( \hat{E}_{0s} = O(\rho^n) \), and we omit the details. Thus, \( E(g^{(n)}) = O_g(\rho^{n+1}) \) as desired. The proof that (10) obtains is identical to the case for \( \gamma < n \), and thus we omit it. Since the arguments given for \( \gamma < n \) also work for \( \gamma > n \), the claim that a solution \( g \) exists satisfying \( E(g) = O(\rho^{\infty}) \) follows by induction and Borel’s lemma. This concludes the case \( n > 2 \).

If \( n = 2 \), the above proof needs slight modification. At the first step, we define \( \hat{h}^{(0)}_{\theta} = d\rho^2 + k_0 \), which is constant in \( \theta \) and also in \( \rho \); and also define \( \chi^{(0)} = 1 \). It follows that \( E^{(0)} = O_g(\rho^2) \), since the only term in the Einstein equations that does not vanish is the Ricci term in (5.3). We need only
solve one more equation, the first-order perturbation, to be done. We set \( \bar{h}^{(1)} = \bar{h}^{(0)} + \rho \bar{\phi} \). The equation we wish to solve is \( P^1(\varphi) = 0 \), with boundary conditions \( \partial_{\rho} \bar{\varphi} |_{\theta = 0} = 0 \) and \( \bar{\varphi} |_{\theta = 0} = \chi^{(1)} \partial_{\rho} \bar{h} \), where \( \chi^{(1)} = \chi^{(0)} + \rho \psi \).

Now \( E^{(0)}_{00} \equiv 0 \) and \( E^{(0)}_{0s} \equiv 0 \), so the above analysis of equations (5.8) and (5.9) goes through without problem; this determines \( \bar{h}^{\mu\nu} \varphi_{\mu\nu}, \varphi_{\tau\nu}, \) and \( \varphi_{ns} \). It remains to determine \( \bar{\varphi}_{st} \). But notice that when \( n = 2 \) and \( \gamma = 1 \), any constant is a solution to \( I_{st}(\varphi) = 0 \); so we may simply set \( \bar{\varphi}_{st} = \chi \log \bar{h} \partial_{\rho} \bar{h} |_{\rho = 0} \).

We have thus determined \( g^{(1)} \), subject to the freedom in \( \bar{\psi} \); the Bianchi analysis goes through as before, determining \( \psi \). Finally, if \( n = 2 \) is not an indicial root, the order-\( n \) analysis above allows construction to higher order, as before.

Notice that it is clear from the above proof that the Taylor coefficients of \( \bar{h}_{\theta} \) in \( \rho \) are, through order \( n - 1 \), universal functions of \( \theta \) and of \( k_{\theta} \), the derivatives of \( k_{\rho} \) at \( \rho = 0 \), and their tangential derivatives. Moreover, the \( j \)th Taylor coefficient function depends only of \( \partial_{\rho}^j k_{\rho} |_{\rho = 0} \) for \( j \leq i \).

It seems apparent that, by including appropriate powers of \( \log(\rho) \) as in the proof of Theorem 1.2, the above construction could be extended to infinite order for any \( \theta_0 \); but we do not here undertake the calculations demonstrating this.

In general, of course, one might want a solution \( g \) satisfying \( \text{Ric}(g) = O_g(\rho^n \sin^\infty(\theta)) \), or even \( O_g(\rho^\infty \sin^\infty(\theta)) \). We here sketch an approach that would yield such a metric, although we omit details. First, one could use the above theorem to obtain a metric \( \bar{g} \) satisfying \( \text{Ric}(\bar{g}) + n\bar{g} = O_g(\rho^m) \) (where \( m \) is as high as possible, or possibly \( \infty \)). Then, pulling back by the diffeomorphism \( \psi \) between \( [0, 1]_r \times M \) and \( [0, \theta_0]_\rho \times M \) induced by defining \( r = 2 csc(\theta_0) \cot(\theta_0) \), it is easy to show, using calculations from [McK16], that the pullback metric \( \psi^* \bar{g} \) is in the usual AH normal form \( \frac{dr^2 + \bar{g}_{rr}}{r^2} \) (but with each \( \bar{g}_{rr} \) an AH metric), and that it still satisfies \( \text{Ric}(\psi^* \bar{g}) + n\psi^* \bar{g} = O(\rho^m) \). Let \( \bar{g}' = \psi^* \bar{g} \). We now can perform the inductive Fefferman-Graham construction (as in [FG12]) at \( \bar{M} \) to show that one can find a perturbation \( \Phi \), vanishing at \( \bar{M} \), and satisfying \( \text{Ric}(\bar{g}' + \Phi) + n(\bar{g}' + \Phi) = O(r^\infty) \). In the Fefferman-Graham construction, the indicial operators are simply multiplication operators, so at each order in \( r \), the perturbation \( \Phi \) will be \( O(\rho^m) \). Therefore, by our analysis of the indicial operators above, we conclude that, letting \( \bar{g}' = \bar{g}' + \Phi \), we will in fact obtain \( \text{Ric}(\bar{g}') + ng' = O(\rho^m r^\infty) \). Finally, since \( \Phi = O_\bar{g}(\rho^m) \), we can then take a cutoff function \( \eta \) that is \( 1 \) near \( \bar{M} \) and \( 0 \) near \( \bar{Q} \), and set \( g = (\psi^{-1})^* (\bar{g}' + \eta \Phi) \). By construction, then, \( g \) will satisfy our boundary conditions at \( \bar{Q} \) and \( \bar{M} \), and will also satisfy \( \text{Ric}(g) + ng = O_g(\rho^m \sin^\infty(\theta)) \).

If \( \theta_0 > \frac{\pi}{2} \), then by Proposition 1.3 there will be an indicial root \( \gamma_0 \) for \( I_{st}^{\bar{g}} \) between \( n \) and \( n - 1 \), and uniqueness in Proposition 5.2 will not be quite to order \( n \) without the requirement of smoothness; we will expect additional solutions with leading asymptotics at order \( \rho^m \).
We will now focus on the proof of Theorem 1.3. Suppose that \( \theta_0 = \frac{\pi}{2} \).

By Propositions 4.7, 4.8, and 5.5, we know that \( n \) is an indicial root, and that we can solve the tracefree part of the Einstein equations to order \( O_g(\rho^{n+1}) \) by a smooth perturbation only if the tracefree tangential part of \( \rho^{-n}E(g^{(n-1)})\big|_{\rho=0} \) is orthogonal to \( w_0 = \sin^n(\theta) \) with respect to the measure \( \sin^{-(n+1)}(\theta)d\theta \). This suggests a way to define the obstruction tensor promised in Theorem 1.3. Suppose \( M^n \) is a manifold with boundary \( S \), and equipped with a metric \( \tau \). Near \( S \), we can uniquely define a diffeomorphism \( \eta : S \times [0, \varepsilon)_{\rho} \hookrightarrow M \) so that \( \eta^*\tau = d\rho^2 + \kappa_\rho \), and so that \( \eta|_{S \times \{0\}} = id_S \). Then by Proposition 5.2 there is a metric \( g \) in the normal form (5.2) on \( \tilde{X} \), where \( \tilde{X} = [0, \frac{\pi}{2}] \times S \times [0, \varepsilon) \), and a function \( \chi \in C^\infty(S \times [0, \varepsilon)) \) such that \((a) - (e)\) hold. Now notice that for any section \( 0T \in C^\infty(\tilde{S}, S^2(0^cT^*\tilde{X})) \) satisfying \( T = O_g(\sin^2(\theta)) \), we can get a well-defined corresponding section \( T \in C^\infty(S, S^2T^*\hat{X}) \) by setting \( T = \rho^2(0T) \).

Now observe that \( E(g) \in C^\infty(S^2(0^cT^*\tilde{X})) \), with \( E(g) = O_g(\rho^n) \). In particular, \( \rho^{-n}\text{tf}_{k_0} E(g)\big|_{\rho=0} \in C^\infty(S, S^2T^*\tilde{X}) \). Moreover, \( E(g) = O_g(\sin^2(\theta)) \).

This follows easily from equations (5.3) - (5.5), remembering that \( \partial_\theta h_\theta = O(\theta) \) by evenness in \( \theta \). Thus, \( \rho^2(\rho^{-n}\text{tf}_{k_0} E(g)\big|_{\rho=0}) \in C^\infty(S, S^2T^*\tilde{X}) \). For shorthand, we write \( \rho^{-n}\text{tf}_{k_0} E(g)\big|_{\rho=0} \in C^\infty(S, S^2T^*\tilde{X}) \). Then we define a smooth symmetric tracefree tensor \( \hat{K}(\tau) \) on \( S \) by

\[
\hat{K}(\tau) = \left( \rho^{-n}\text{tf}_{k_0} E(g)\big|_{\rho=0}, w_0 \right)_{\sin^{-(n+1)}(\theta)d\theta} \in S^2T^*S,
\]

where \( E \) here refers to \( \text{tf}_{k_0} E \).

**Proof of Theorem 1.3.** We first must show that \( \hat{K}(\tau) \) is well defined. First, the integral (5.18) converges, since \( E_{\mu
u} = O(\theta) \) by (5.5). Next, although \( \hat{h}_\theta \) is only determined mod \( O(\rho^n) \), perturbations of the form \( \hat{h}_\theta \mapsto \hat{h}_\theta + \rho^n\hat{\varphi} \) satisfying \( \hat{\varphi}|_{\theta=0} = 0 \) and \( \partial_\theta \hat{\varphi}|_{\theta=0} = 0 \) leave \( \left( \text{tf}_{k_0} \rho^{-n}E(g)\big|_{S,T,S}, w_0 \right)_{\sin^{-(n+1)}(\theta)d\theta} \)

unchanged, since \( n \) is an indicial root of \( \hat{I}_{st}^n \) and, by Propositions 4.7, 4.8, and 5.5, the image of \( \hat{I}^n \) is orthogonal to \( \sin^n(\theta) \). Thus, \( \hat{K}(\tau) \) is well defined.

Next, we must show that the conformal transformation law holds. Suppose \( \hat{\tau} = \Omega^2\tau \), where \( \Omega \in C^\infty(M) \). Let \( \hat{\eta} : S \times [0, \varepsilon)_{\hat{\rho}} \hookrightarrow M \) be a diffeomorphism onto a neighborhood of \( S \) so that \( \hat{\eta}^*\hat{\tau} = d\hat{\rho}^2 + \hat{k}_{\hat{\rho}} \) and so that \( \hat{\eta}|_{S \times \{0\}} = id_S \).

Let \( \hat{\eta}_\theta \) and \( \hat{\chi} \) satisfy \((a) - (e)\) in Proposition 5.2 in particular with \( \hat{h}_{\theta_0} = \hat{\chi}(d\hat{\rho}^2 + \hat{k}_{\hat{\rho}}) \). Similarly, we take \( \hat{g} = \csc^2(\theta)[d\theta^2 + \hat{h}_\theta] \).

Set \( \hat{\eta}_\theta = \hat{\eta}^*(\eta^{-1})^*h_\theta \), as well as \( \hat{\chi} = \hat{\eta}^*(\eta^{-1})^*\chi \) and \( \hat{\rho} = \hat{\eta}^*(\eta^{-1})^*\rho \). Similarly set \( \hat{\Omega} = \hat{\eta}^*\Omega \). Now plainly

\[
\hat{\rho}^2\hat{h}_0 = \hat{\chi}\hat{\eta}^*\tau = \hat{\Omega}^{-2}\hat{\chi}(d\hat{\rho}^2 + \hat{k}_{\hat{\rho}}) = \hat{\Omega}^{-2}\hat{\chi}\hat{\rho}^{-2}\hat{h}_0.
\]
This implies that \( \tilde{h}_0 = \frac{\hat{\rho}^2 \hat{\chi}}{\hat{\rho}^2 \hat{\chi}} \tilde{h}_0 \). Since \( \tilde{h}_0 \) and \( \hat{h}_0 \) are both AH metrics on \( S \times [0, \varepsilon) \), we must therefore have

\[
\frac{\hat{\rho}}{\tilde{\rho}} \bigg|_{\hat{\rho} = 0} = \hat{\Omega}_{\tilde{\rho} = 0}.
\]

Now

\[
\rho^2 \tilde{h}_0 = \frac{\hat{\rho}^2}{\rho^2} \rho^2 \tilde{h}_0 = \frac{\hat{\rho}^2}{\rho^2} \hat{\Omega}^{-2} \hat{\chi} (d\rho^2 + \hat{k}_\rho),
\]

where \( \frac{\hat{\rho}^2 \Omega^{-2} \hat{\chi}}{\rho^2} \bigg|_{\hat{\rho} = 0} = 1 \). Thus, by the uniqueness statement in Proposition 5.2, \( \tilde{h}_\theta = \tilde{h}_0 \mod O(\tilde{\rho}^n) \). Set \( \tilde{g} = \csc^2(\theta) |d\theta^2 + \tilde{h}_\theta| = \tilde{\eta}(\eta^{-1})^* g \). It then follows from the discussion in the first paragraph of this proof and the fact that \( \eta|_{\rho = 0} = \tilde{\eta}|_{\tilde{\rho} = 0} = \id \) that

\[
K(\tau) = \left< \rho^{2-n} tf_{k_0} E(g)|_{\rho = 0}, w_0 \right>_{\sin^{-2}(n+1)(\theta)d\theta}
= \left< \rho^{2-n} tf_i E(\tilde{g})|_{\rho = 0}, w_0 \right>_{\sin^{-2}(n+1)(\theta)d\theta}
= \left< \rho^{2-n} tf_{k_0} E(\tilde{g})|_{\rho = 0}, w_0 \right>_{\sin^{-2}(n+1)(\theta)d\theta}
= \rho^{n-2} \left< \rho^{2-n} tf_{k_0} E(\tilde{g})|_{\rho = 0}, w_0 \right>_{\sin^{-2}(n+1)(\theta)d\theta}
= \left< \hat{\Omega}^{\chi} \rho^{-2} K(\chi) \right>_{\rho = 0}
= \left< \hat{\Omega}_{S}^{\chi} \rho^{-2} K(\chi) \right>_{\rho = 0},
\]

which is the desired result.

We need finally to show that \( K \) is generically nontrivial. We will do this by showing that

\[
(5.19) \quad K(\tau) = c \left< tf_{k_0} \partial_{\rho}^n k_{\rho}|_{\rho = 0} + K'(\tau),
\]

where \( c \neq 0 \) and \( K'(\tau) \) depends only on \( \partial_{\rho}^j k_{\rho}|_{\rho = 0} \) for \( j < n \). To proceed, let \( \kappa = \partial_{\rho}^j k_{\rho}|_{\rho = 0} \), and extend it to be a section in \( C^\infty(\tilde{S}, S^2 T^* S) \) by taking it to be constant in \( \theta \). Define \( \tilde{\varphi} \in \mathcal{H}_a(\theta_0, S, S^2 T^* S) \) by \( \partial_{\rho}^n \tilde{h}_{\tilde{\varphi}} = \kappa_{\tilde{\varphi}} + \partial_{\rho}^n ((\chi - 1)k_{\rho})\bigg|_{\rho = 0} + \tilde{\varphi} \), where the second term, like \( \kappa \), is extended to be constant in \( \theta \). Notice that the second term depends only on \( \partial_{\rho}^j k_{\rho}|_{\rho = 0} \) for \( j < n \). Plainly, we have \( \tilde{\varphi}|_{\theta = 0} = 0 \) and \( \partial_{\theta} \tilde{\varphi}|_{\theta = 0} = 0 \). It follows then that the inner product \( \left< \rho^{2-n} tf_{k_0} E|_{\rho = 0}, w_0 \right> \) is independent of \( \tilde{\varphi} \), by the discussion in the first paragraph of this proof. We wish to find the coefficient of \( \kappa_{\tilde{\varphi}} \) in \( \left< \rho^{2-n} tf_{k_0} E, w_0 \right> \), and in particular to show that it is nonvanishing.
Consider now the last term of $E_{\mu\nu}$ in (5.5), which is given by $u_{\mu\nu} = \rho^2 \sin^2(\theta) \text{Ric}(\tilde{h})_{\mu\nu}$. Recall that the expression for $\text{Ric}(\tilde{h})_{\mu\nu}$ is

\begin{equation}
\text{Ric}(\tilde{h})_{\mu\nu} = \frac{1}{2} \tilde{h}^{\mu\lambda}(\partial_{\lambda} \tilde{h}_{\nu\eta} + \partial_{\eta} \tilde{h}_{\nu\lambda} - \partial_{\nu} \tilde{h}_{\lambda\eta} - \partial_{\mu} \tilde{h}_{\eta\lambda}) + \tilde{h}^{\mu\lambda} \tilde{h}^{\sigma\tau}(\Gamma_{\mu\lambda\sigma} \Gamma_{\nu\eta\tau} - \Gamma_{\mu\nu\sigma} \Gamma_{\eta\lambda\tau})
\end{equation}

Now because $\tilde{h}_{\theta\theta}|_{\rho=0} = d\rho^2 + k_0$, we see that the third term contributes to $u$ a term of the form $-\frac{1}{2(n-2)!} \sin^2(\theta) \rho^n \partial^n_{\rho} \tilde{h}_{\mu\nu}$. No other term of (5.20) contributes a multiple of $\partial^n_{\rho} \tilde{h}_{\mu\nu}$ at order $\rho^n$. Thus, in particular, $u$ contributes a term of the form $-\frac{1}{2(n-2)!} \sin^2(\theta) \rho^n \kappa_{st}$ (as well as terms involving $\tilde{\varphi}_{st}$ and lower orders of $k_\rho$) to $\tilde{E}_{st}$.

Consider next the second-last term of (5.5), which is $\rho^2 \sin^2(\theta)\nabla^\eta \rho_{\eta\mu\nu}$. Because of the factor of $\rho$, it does not make any contribution of the form $\rho^n \kappa_{st}$ to $\tilde{E}_{st}$.

Next consider the third-last term, $v_{\mu\nu} = (n-2)\rho \sin^2(\theta) \nabla^\mu \rho_{\nu\nu}$. It is easy to compute that this takes the form

\[ v_{\mu\nu} = \frac{n-2}{2} \sin^2(\theta) \rho (\partial_{\rho} \tilde{h}_{\mu\nu} - 2\partial_{(\mu} \tilde{h}_{\nu)\rho}) + O(\rho). \]

The third-last term thus contributes a term of the form $\frac{n-2}{2(n-1)!} \sin^2(\theta) \rho^n \kappa_{st}$ to $\tilde{E}_{st}$.

We claim that no other term of (5.5) contributes a term involving $\kappa$ to $E$ at order $\rho^n$. The first two terms do not, because $\kappa$ does not depend on $\theta$. The next three terms do not because $\partial_{\theta} \tilde{h} = O(\rho)$; and the next, because $|d\rho|^2 = 1 = O(\rho)$. Thus, the only contributions of $\kappa_{st}$ to $\tilde{E}_{st}$ are those already calculated from the seventh and ninth terms; their sum is $\frac{1}{2(n-1)!} \rho^n \sin^2(\theta) \kappa_{st}$. But it is plain that the coefficient of $\rho^n$ here is not orthogonal to $\csc(\theta)$; indeed,

\[ \frac{-1}{2(n-1)!} \langle \sin^2(\theta) \kappa_{st}, w_0 \rangle \sin^{-(n+1)}(\theta) d\theta = \frac{-1}{2(n-1)!} \int^\pi_0 \sin(\theta) \kappa_{st} d\theta \]

\[ = \frac{-1}{2(n-1)!} \kappa_{st}. \]

Thus, (5.19) holds with $c = \frac{-1}{2(n-1)!}$. Now $\tau$ (thus $k_\rho$) may be changed at order $\rho^n$ independently of any lower orders; and we have seen above that changing $\varphi$ does not alter the inner product $\langle \rho^{2-n} E|_{\rho=0}, w_0 \rangle \sin^{-(n+1)}(\theta) d\theta$. Thus, for generic choices of $t_\rho k_\rho \partial^\rho_{\varphi} k_\rho|_{\rho=0} = \tilde{\kappa}$, $\kappa(\tau)$ is nonvanishing. \[\blacksquare\]

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