Singular solutions of the biharmonic Nonlinear Schrödinger equation

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Abstract

We consider singular solutions of the biharmonic NLS. In the $L^2$-critical case, the blowup rate is bounded by a quartic-root power law, the solution approaches a self-similar profile, and a finite amount of $L^2$-norm, which is no less than the critical power, concentrates into the singularity (“strong collapse”). In the $L^2$-critical and supercritical cases, we use asymptotic analysis and numerical simulations to characterize singular solutions with a peak-type self-similar collapsing core. In the critical case, the blowup rate is slightly faster than a quartic-root, and the self-similar profile is given by the standing-wave ground-state. In the supercritical case, the blowup rate is exactly a quartic-root, and the self-similar profile is a zero-Hamiltonian solution of a nonlinear eigenvalue problem. These findings are verified numerically (up to focusing levels of $10^8$) using an adaptive grid method. We also calculate the ground states of the standing-wave equations and the critical power for collapse in two and three dimensions.

1 Introduction

The focusing nonlinear Schrödinger equation (NLS)

$$i\psi_t(t,x) + \Delta \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi(0,x) = \psi_0(x) \in H^1(\mathbb{R}^d),$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, and $\Delta = \sum_{j=1}^d \partial_j^2$ is the Laplacian, has been the subject of intense study, due to its role in various areas of physics, such as nonlinear optics and Bose-Einstein Condensates (BEC). It is well-known that the NLS (1) possesses solutions that become singular in a finite time [33]. Of special interest is the critical ($\sigma = 2/d$) NLS

$$i\psi_t(t,x) + \Delta \psi + |\psi|^{4/d} \psi = 0, \quad \psi(0,x) = \psi_0(x) \in H^1(\mathbb{R}^d),$$

which models the collapse of intense laser beams that propagate in a bulk Kerr medium.

In this study, we consider the focusing biharmonic nonlinear Schrödinger equation (BNLS)

$$i\psi_t(t,x) - \Delta^2 \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi(0,x) = \psi_0(x) \in H^2(\mathbb{R}^d),$$

1
where $\Delta^2$ is the biharmonic operator. Equation (3) admits waveguide (standing-wave) solutions of the form $\psi(t, x) = \lambda^{2/\sigma} e^{i\lambda t} R(\lambda x)$, where $R$ satisfies the "standing-wave" equation

$$- \Delta^2 R(x) - R + |R|^{2\sigma} R = 0. \quad (4)$$

The BNLS (3) is called “$L^2$-critical”, or simply “critical” if $\sigma d = 4$. In this case, the $L^2$ norm ("power") is conserved under the BNLS dilation symmetry $\psi(t, x) \mapsto L^{-2/\sigma} \psi(t / L^4, x / L)$. The critical BNLS can be rewritten as

$$i\psi_t(t, x) - \Delta^2 \psi + |\psi|^{8/d} \psi = 0, \quad \psi(0, x) = \psi_0(x) \in H^2(\mathbb{R}^d). \quad (5)$$

Correspondingly, the BNLS with $\sigma d < 4$ is called subcritical, and the BNLS with $\sigma d > 4$ is called supercritical. This is analogous to the NLS, where the critical case is $\sigma = 2$.

In [4], Ben-Artzi, Koch and Saut proved that the BNLS (3) is locally well-posed in $H^2$, when $\sigma$ is in the $H^2$-subcritical regime

$$\begin{cases} 
0 < \sigma & d \leq 4, \\
0 < \sigma < \frac{4}{d} & d > 4.
\end{cases} \quad (6)$$

Global existence and scattering of BNLS solutions in the $H^2$-critical case $\sigma = 4/(d-4)$ were studied by Miao, Xu and Zhao [25] and by Pausader [29]. The latter work also showed well-posedness for small data. The $H^2$-critical defocusing BNLS was studied by Miao, Xu and Zhao [24] and by Pausader [27, 28].

The above studies focused on non-singular solutions. In this work, we study singular solutions of the BNLS in the $H^2$-subcritical regime (6).

Our purpose is to characterize these singular solutions: their profile, blowup rate, power concentration, et cetera.

1.1 Summary of results

In this work, we consider singular solutions of the focusing BNLS in the $H^2$-subcritical regime (6). Our purpose is to characterize these singular solutions: their profile, blowup rate, power concentration, et cetera.
1 INTRODUCTION

In some cases, we assume radial symmetry, i.e., that \( \psi(t, x) \equiv \psi(t, r) \) where \( r = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_d^2} \). In these cases, equation (3) reduces to

\[
 i\psi_t(t, r) - \Delta^2_r \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi(0, r) = \psi_0(r),
\]

where

\[
 \Delta^2_r = \partial_t^4 + \frac{2(d-1)}{r} \partial_t^3 + \frac{(d-1)(d-3)}{r^2} \partial_t^2 - \frac{(d-1)(d-3)}{r^3} \partial_r
\]

is the radial biharmonic operator. Specifically, the critical BNLS (5) reduces to

\[
 i\psi_t(t, r) - \Delta^2_r \psi + |\psi|^{8/d} \psi = 0, \quad \psi(0, r) = \psi_0(r).
\]

The paper is organized as follows: In Section 2, we use Noether Theorem to derive conservation laws for the BNLS. We recall that in the critical NLS, the conservation law which follows from invariance of the action integral under dilation leads to the NLS “Variance Identity”, which can be used to prove the existence of singular solutions. In Section 2.1 we use a similar procedure to derive the “Variance Identity” for the critical BNLS, and then generalize it to the supercritical BNLS. However, since it is not clear that the “BNLS variance” is positive definite, this identity does not lead to a proof of the existence of singular solutions.

The ground states of the BNLS standing-wave equation (4) were previously computed only in the one-dimensional case [12], since they were computed using a shooting method, which cannot be easily generalized to multi-dimensions. In Section 3 we use the spectral renormalization method to compute the ground-states of the critical BNLS (5) for one, two and three dimensions. The calculated ground-states provide the first numerical estimate of the critical power for collapse \( P_{cr} = \|R\|_2^2 \) in the two-dimensional and three-dimensional cases, see Section 4. Direct simulations of the critical BNLS suggest that the constant \( P_{cr} \) in Theorem 2 is optimal.

In Section 5 we use rigorous analysis to study the critical BNLS (5). The blowup rate is shown to be lower-bounded by a quartic root, i.e., \( \|\Delta \psi\|_2^{-1/2} \leq C (T_c - t)^{1/4} \). The corresponding bound for the critical NLS is a square root, i.e., \( \|\nabla \psi\|_2^{-1} \leq C (T_c - t)^{1/2} \). We then prove that singular solutions converge to a self-similar profile strongly in \( L^{2+2\sigma} \), for any \( \sigma \) in the \( H^2 \)-subcritical regime [6]. This implies that the singular solutions have the power-concentration property, whereby the amount of power that enters the singularity point is at least \( P_{cr} \). These rigorous results mirror those of the critical NLS.

Let us denote the location of the maximal amplitude of a radially-symmetric solution by

\[
 r_{\text{max}}(t) = \arg \max_r |\psi|.
\]

Singular solutions are called “peak-type” when \( r_{\text{max}}(t) \equiv 0 \) for \( 0 \leq t \leq T_c \), and “ring-type” when \( r_{\text{max}}(t) > 0 \) for \( 0 \leq t < T_c \). In this work, we use asymptotic analysis and numerics to find and characterize peak-type singular solutions of the BNLS equation. Ring-type singular solutions of the BNLS will be studied elsewhere [2, 3].

In Section 6 we use asymptotics and numerics to show that peak-type singular solutions of the critical BNLS collapse with the quasi self-similar profile

\[
 \psi(t, r) \sim \frac{1}{L^{d/2}(t)} R \left( \frac{r}{L(t)} \right) e^{i \int_0^t \frac{1}{L(t')} \d t'}, \quad \lim_{t \to T_c} \frac{L(t)}{t} = 0,
\]
where the self-similar profile is the ground state $R(\rho)$. The blowup rate is shown to be slightly faster than the quartic-root bound. This is analogous to the critical NLS, where the blowup rate of peak-type solutions is slightly faster than the square-root bound, due to the loglog correction (the “loglog law”). It is an open question whether the correction to the BNLS blowup rate is also a loglog term.

In Section 7 we consider peak-type singular solutions of the supercritical BNLS. In this case, asymptotics and numerics show that singular solutions are of the quasi self-similar form

$$\psi(t,r) \sim \frac{1}{L^{2/\sigma(t)}} B \left( \frac{r}{L(t)} \right) e^{i \int \frac{1}{L^{2}(r')} dr'}, \quad \lim_{t \to T_c} L(t) = 0, \quad (10)$$

the blowup rate of $L(t)$ is exactly a quartic-root, and the self-similar profile $B(\rho)$ is different from the ground-state. Rather, as in the supercritical NLS, the self-similar profile $B$ is the zero-Hamiltonian solution of a nonlinear eigenvalue problem. Although $B(\rho)$ is not in $L^2$, it can be the self-similar profile of a collapsing $H^2$ solution, since the collapsing solution is “only” quasi self-similar.

Section 8 presents the numerical methods. The computations of singular BNLS solutions that focus by factors of $10^8$ and more, necessitated the usage of adaptive grids. We develop a modified version of the Static Grid Redistribution method [31, 10], which is more convenient for the biharmonic problem. Calculating the BNLS standing waves in multi-dimensions is done using the Spectral Renormalization Method.

1.2 Discussion

In this study, we use rigorous theory, asymptotic theory and numerics to analyze singular solutions of the BNLS. All the results presented in this work mirror those of the NLS, “up to a change by a factor of 2” in the blowup rate $(1/2 \rightarrow 1/4)$, in the critical value of $\sigma$ $(2/d \rightarrow 4/d)$, et cetera. However, several key features of NLS theory are still missing from BNLS theory. First, the “Variance Identity” for the BNLS cannot be used to prove that singular solutions exist. Second, the critical NLS is invariant under the pseudo-conformal (“lens-transformation”) symmetry which can also be used to construct explicit singular solutions. At this time, it is unknown whether an analogous identity for the critical BNLS exists. Third, in critical NLS theory, the self-similar profile is known to possess a quadratic radial phase term, i.e.,

$$\psi(t,r) \sim \frac{1}{L^{d/2}(t)} R \left( \frac{r}{L(t)} \right) e^{i \tau(t) + \frac{i L d r^2}{4\pi}},$$

This term represents the focusing of the solution towards $r = 0$, and plays a key role in the rigorous and asymptotic theory of the critical NLS. At this time, we do not know the analogous radial phase term for the critical BNLS.

Finally, we note that a similar “up to a factor of 2” connection exists between singular solutions of the nonlinear heat equation, see [17], and the biharmonic nonlinear heat equation, see [7]. For example, the $L^\infty$ norm of singular solutions blows up as $(T_c - t)^{-1/2}$ for the nonlinear heat equation, and as $(T_c - t)^{-1/4}$ for the biharmonic heat equation. The “similarity up to a factor of 2”, however, is not perfect. For example, the self-similar spatial variable is $r/\sqrt{(T_c - t) \log(T_c - t)}$ for the
nonlinear heat equation, and \( r/\sqrt{T_c - t} \) for the biharmonic nonlinear heat equation. Another difference between the equations is that singular solutions are \textit{asymptotically} self-similar for the nonlinear heat equation, and truly self-similar for the biharmonic heat equation. In contrast, the NLS possesses self-similar singular solutions, whereas for the BNLS it is unknown whether singular solutions are truly, or only asymptotically, self-similar.

2 Invariance

The BNLS (3) is the Euler-Lagrange equation of the action integral

\[
S = \int \mathcal{L} d\mathbf{x} dt,
\]

where \( \mathcal{L} \) is the Lagrangian density

\[
\mathcal{L} (\psi, \psi^*, \psi_t, \psi_t^*, \Delta \psi, \Delta \psi^*) = \frac{i}{2} (\psi_t \psi^* - \psi_t^* \psi) - |\Delta \psi|^2 + \frac{1}{1 + \sigma} |\psi|^{2(\sigma + 1)}.
\] (11)

Therefore, the conserved quantities of the BNLS can be found using Noether theorem, see Appendix E. As in the standard NLS, invariance of the action integral under phase-multiplications \( \psi \mapsto e^{i\delta} \psi \) implies conservation of the “power” (\( L^2 \) norm), i.e.,

\[
P(t) = P(0), \quad P(t) = \|\psi(t)\|_2^2.
\]

Similarly, invariance under temporal translations \( t \mapsto t + \delta t \) implies conservation of the Hamiltonian

\[
H(t) = H(0), \quad H[\psi(t)] = \|\Delta \psi\|_2^2 - \frac{1}{\sigma + 1} \|\psi\|_2^{2(\sigma + 1)}.
\] (12)

and invariance under spatial translations \( \mathbf{x} \mapsto \mathbf{x} + \delta \mathbf{x} \) implies conservation of the linear momentum, i.e.,

\[
P(t) = P(0), \quad P(t) = \int \text{Im} \{\psi^* \nabla \psi\} \, d\mathbf{x}.
\]

In the critical case \( \sigma \cdot d = 4 \), the action integral is also invariant under the dilation transformation \( \psi(t, \mathbf{x}) \mapsto \lambda^{d/2} \psi(\lambda^2 t, \lambda \mathbf{x}) \). The corresponding conserved quantity is

\[
J(t) = J(0), \quad J(t) = \int \mathbf{x} \cdot \text{Im} \{\psi^* \nabla \psi\} \, d\mathbf{x} + 4tH.
\] (13)

2.1 Towards a variance identity

We recall that the action integral of the critical NLS (2) is invariant under the dilation transformation

\[
\psi_{\text{NLS}}(t, \mathbf{x}) \mapsto \lambda^{d/2} \psi_{\text{NLS}}(\lambda^2 t, \lambda \mathbf{x}).
\]

The corresponding conserved quantity is

\[
J_{\text{NLS}}(t) = J_{\text{NLS}}(0), \quad J_{\text{NLS}} = \int \mathbf{x} \cdot \text{Im} \{\psi^* \nabla \psi\} \, d\mathbf{x} - 2tH_{\text{NLS}}.
\] (14)
In addition, the integral term \( \int x \cdot \text{Im} \{ \psi^* \nabla \psi \} \, dx \) is the time-derivative of the variance, i.e.,

\[
\frac{d}{dt} V_{\text{NLS}}(t) = \int |x|^2 |\psi|^2 \, dx.
\]

Therefore, it follows that

\[
\frac{d^2}{dt^2} V_{\text{NLS}} = 8H_{\text{NLS}}.
\]

In the supercritical NLS, the second derivative of the variance is not related to this conservation law. Nevertheless, direct differentiation shows that

\[
\frac{d}{dt} \int x \cdot \text{Im} \{ \psi^* \nabla \psi \} \, dx = 2H_{\text{NLS}} - \frac{\sigma d - 2}{(\sigma + 1)} \| \psi \|^{2(\sigma+1)}.
\]

Therefore

\[
\frac{d^2}{dt^2} V_{\text{NLS}} = 8H_{\text{NLS}} - \frac{4\sigma d - 4}{2(\sigma + 1)} \| \psi \|^{2(\sigma+1)}. \tag{15}
\]

Since \( V_{\text{NLS}} \geq 0 \), the variance identity (15) shows that solutions of the critical and supercritical NLS, whose Hamiltonian is negative, become singular in a finite time \[34\].

We next extend the analogy between (13) and (14) to the non-critical case. In the case of the BNLS, direct differentiation shows that

\[
\frac{d}{dt} \int x \cdot \text{Im} \{ \psi^* \nabla \psi \} \, dx = 4H - \frac{\sigma d - 4}{2(\sigma + 1)} \| \psi \|^{2(\sigma+1)}.
\]

Therefore, if we define

\[
V_{\text{BNLS}}(t) = V_{\text{BNLS}}(0) + \int_{s=0}^{t} \left( \int x \cdot \text{Im} \{ \psi^* (s, x) \nabla \psi \} \, dx \right) \, ds,
\]

where \( V_{\text{BNLS}}(0) \) is a positive constant, we get the BNLS variance identity

\[
\frac{d^2}{dt^2} V_{\text{BNLS}} = 4H - \frac{\sigma d - 4}{2(\sigma + 1)} \| \psi \|^{2(\sigma+1)}. \tag{16}
\]

In order to use this identity to prove singularity formation, however, one must show that \( V_{\text{BNLS}} \), as defined by (16), has to remain positive. Direct integration by parts gives that

\[
\int x \cdot \text{Im} \{ \psi^* \nabla \psi \} \, dx = \frac{1}{4(d+2)} \int |x|^4 \cdot \text{Im} \{ \nabla \psi^* \Delta \nabla \psi \} \, dx + \frac{1}{16(d+2)} \left( \int |x|^4 |\psi|^2 \, dx \right) \, t.
\]

While the second term on the RHS is a temporal derivative of a positive-definite quantity, the first term on the RHS is not. Therefore, it remains an open question whether \( V_{\text{BNLS}} \) has to be positive.
3 Numerical calculation of standing waves

The BNLS equation (3) admits the standing-wave solutions

\[ \psi(t, x) = \frac{\lambda^2}{\sigma} e^{i\lambda t} R(\lambda x). \]

The equation for the standing-wave profile is

\[ - \Delta^2 R(x) - R + |R|^{2\sigma} R = 0, \]  \hspace{1cm} (17)

where \( R \in H^2 \). For example, in one dimension, equation (17) is given by

\[ - R_{xxxx}(x) - R + |R|^{2\sigma} R = 0, \]  \hspace{1cm} (18)

and in two dimensions by

\[ - (R_{xxxx}(x, y) + 2R_{xxyy} + R_{yyyy}) - R + |R|^{2\sigma} R = 0. \]  \hspace{1cm} (19)

If we impose radial symmetry, eq. (17) reduces to

\[ - \Delta^2_r R(r) - R + |R|^{2\sigma} R = 0, \]  \hspace{1cm} (20a)

where \( \Delta^2_r \) is given in (8). At \( r = 0 \), all the odd derivatives vanish, and so the solution of (8) is subject to the boundary conditions

\[ R'(0) = R''(0) = R(\infty) = R'(\infty) = 0. \]  \hspace{1cm} (20b)

The solution of eq. (20) was computed numerically in the one-dimensional case in [12] as follows.

In the 1D case, eq. (20) can be integrated once, yielding an explicit relation between \( R(0) \) and \( R''(0) \). This parameter-reduction enables the usage of a one-parameter shooting approach. Unfortunately, such a parameter reduction is not possible in higher dimensions. Therefore, in multi-dimensions we compute the ground-states using the Spectral Renormalization method (SRM), which was introduced by Petviashvili in [30], and more recently by Albowitz and Musslimani in [1]. See Section 8.2 for further details.

In Figure 1 we display the results in the critical case, i.e., the ground states of

\[ - \Delta^2 R - R + |R|^{8/d} R = 0. \]  \hspace{1cm} (21)

Figure 1A displays the ground-state of eq. (17) in the critical 1D case, as calculated by the SRM. The solution is in excellent agreement with the solution computed in [12] using the shooting method. Figure 1B and Figure 1C display the ground state in the critical 2D and 3D cases. We note that, while the SRM method that we use does not enforce radial symmetry, the calculated ground states for \( d = 2 \) and \( d = 3 \) are radially symmetric (data not shown). As noted in [12], the ground-states of the BNLS are non-monotonic in \( r \) and change their sign, in contradistinction with the ground-states of the NLS which are monotonically-decreasing and strictly positive.
4 Critical power for collapse

Theorem 2 shows that the critical power for collapse in the critical BNLS (5) is $P_{\text{cr}} = \|R\|_2^2$, when $R$ is the ground-state of (21). The computation of $R$, see Section 3, allows for the numerical calculation of the critical power $P_{\text{cr}}$. The case $d = 1$ was found in [12] to be $P_{\text{cr}}(d = 1) = \int_{x=-\infty}^{\infty} |R(x)|^2 dx \approx 2.9868$.

Using the calculated ground state in the two-dimensional case, see Figure 1B, we now calculate the critical power in the two-dimensional case, giving

$$P_{\text{cr}}(d = 2) = \int_{x,y=-\infty}^{\infty} |R(x,y)|^2 dxdy \approx 13.143.$$  

Similarly, using the calculated ground state in the 3D case, see Figure 1C, gives

$$P_{\text{cr}}(d = 3) = \int_{x,y,z=-\infty}^{\infty} |R(x,y,z)|^2 dxdydz \approx 44.88.$$  

We now ask whether Theorem 2 is sharp, in the sense that for any $\varepsilon > 0$, there exists an initial condition $\psi_0 \in H^2$ such that $\|\psi_0\|_2^2 \leq (1+\varepsilon)P_{\text{cr}}$ and the corresponding solution of the critical BNLS becomes singular. As noted, at present there is no proof that solutions of the BNLS can become singular. Therefore, in particular, it is unknown whether Theorem 2 is indeed sharp. Hence, we will explore this issue numerically.

We recall that in the critical NLS, the necessary condition for collapse $\|\psi\|_2^2 \geq P_{\text{cr}}^{\text{NLS}} := \|R^{\text{NLS}}\|_2^2$ is sharp, since for any $\varepsilon > 0$ the initial condition $\psi_0 = (1+\varepsilon)R^{\text{NLS}}$ becomes singular in a finite time [30]. Therefore, we now check numerically whether for $0 < \varepsilon \ll 1$, the solution of the critical BNLS with the initial condition $\psi_0 = (1+\varepsilon)R(x)$ becomes singular. To do this, we solve the one-dimensional and two-dimensional critical BNLS equations with the initial condition $\psi_0 = \ldots$
5 Blowup rate, blowup profile, and power concentration (critical case)

5.1 Lower-bound for the blowup rate

In [8], Cazanave and Weissler proved that the blowup rate for singular solutions of the critical NLS (2) is not slower than a square-root, i.e., that $\|\nabla \psi\|_2 \geq K(T_c - t)^{-1/2}$. The analogous result for the critical BNLS is as follows:

**Theorem 3.** Let $\psi$ be a solution of the critical BNLS (5) that becomes singular at $t = T_c < \infty$, and let $l(t) = \|\Delta \psi\|_2^{-1/2}$. Then, $\exists K = K(\|\psi_0\|_2) > 0$ such that

$$l(t) \leq K(T_c - t)^{1/4}, \quad 0 \leq t \leq T_c.$$

**Proof.** We follow the proof given by Merle [21] for the critical NLS. For a fixed $t$, $0 \leq t < T_c$, let us define

$$\psi_1(s, \mathbf{x}) = l^{d/2} \psi(t + s \cdot t^4, \mathbf{x} \cdot l).$$
Then, $\psi_1$ is defined for $t + l^4s < T_c \iff s < S_c = l^{-4}(t \cdot (T_c - t))$, and satisfies the BNLS equation

$$i\partial_t \psi_1 + \Delta^2 \psi_1 + |\psi_1|^{8/d} \psi_1 = 0.$$  

Since

$$\|\Delta \psi_1\|^2 = l^4 \|\Delta \psi(t + s \cdot l^4, x \cdot l)\|^2_2,$$

this implies that $\lim_{s \to S_c} \|\Delta \psi_1\|^2_2 = \infty$, i.e., that $\psi_1(s)$ becomes singular as $s \to S_C$. In addition,

$$\|\Delta \psi_1(s = 0, x)\|^2_2 = l^4 \|\Delta \psi(t, x)\|^2_2 = 1. \tag{22a}$$

From the definition of $\psi_1$ and power conservation it follows that

$$\|\psi_1(s = 0, x)\|^2_2 = \|\psi(t, x)\|^2_2 = \|\psi_0(x)\|^2_2. \tag{22b}$$

Using equations (22a) and (22b) and the Cauchy-Schwartz inequality gives

$$\|\nabla \psi_1(s = 0, x)\|^2_2 \leq \|\psi_1(s = 0, x)\|_2 \cdot \|\Delta \psi_1(s = 0, x)\|_2 = \|\psi_0(x)\|^2_2. \tag{22c}$$

Together, the three formulae (22) imply that for any fixed $t \in [0, T_c)$,

$$\|\psi_1(s = 0, x)\|^2_{H^2} \leq \|\psi_0\|^2_2 + \|\psi_0\|_2 + 1. \tag{23}$$

In other words, for each $t$, the initial $H^2$ norm of $\psi_1$ is bounded by a function of $\|\psi_0\|_2$. Specifically, this bound is independent of $t$. From the local existence theory [4], $\psi_1$ exists in $s \in [0, S_M(t)]$, where $S_M = S_M(\|\psi_1(s = 0, x)\|_{H^2})$. Therefore, it follows from (23) that $S_M$ depends on $\|\psi_0\|$, but is independent of $t$.

Since $\psi_1$ blows up at $S_c$ we have that

$$S_M \leq S_c(t) = l^{-4}(t \cdot (T_c - t)),$$

from which the result follows. \hfill \Box

### 5.2 Convergence to a quasi self-similar blowup profile

In [37], Weinstein showed that the collapsing core of all singular solutions of the critical NLS approaches a self-similar profile. We now prove the analogous result for the critical BNLS:

**Theorem 4.** Let $d \geq 2$ and let $\psi(t, r)$ be a solution of the radially-symmetric critical BNLS \[7\] with initial conditions $\psi_0(r) \in H^2_{\text{radial}}$, that becomes singular at $t = T_c < \infty$. Let $l(t) = \|\Delta \psi\|^{-1/2}_2$ and let

$$S(\psi)(t, r) = l^{d/2}(t)\psi(t, l(t)r).$$

Then, for any sequence $t'_k \to T_c$ there is a subsequence $t_k$ such that $S(\psi)(t_k, r) \to \Psi(r)$ strongly in $L^q$, for all $q$ such that

$$\begin{cases} 2 < q \leq \infty & 2 \leq d \leq 4, \\ 2 < q < \frac{2d}{d-4} & 4 < d. \end{cases} \tag{24}$$

In addition, $\|\Psi\|^2_2 \geq \|R\|^2_2$, where $R$ is the ground state of equation (4).\(^1\)

\(^1\) In fact, $q = 2(\sigma + 1)$, where $\sigma$ is in the $H^2$-subcritical regime [4].
5 BLOWUP RATE, BLOWUP PROFILE, AND POWER CONCENTRATION (CRITICAL CASE)

Proof. Let \( t_k \to T_c \) and define

\[ \phi_k(r) = S(\psi)(t_k, r) = l^{d/2}(t_k)\psi(t_k, l(t_k)r). \]

From the definition of \( \phi_k \) it follows that

\[ \|\phi_k\|_2^2 = \|\psi_0\|_2^2, \quad \|\Delta \phi_k\|_2^2 = l^4\|\Delta \psi(t_k)\|_2^2 = 1, \quad H[\phi_k] = l^4H[\psi(t_k)]. \]

Therefore, using Cauchy-Schwartz,

\[ \|\nabla \phi_k\|_2^2 \leq \|\phi_k\|_2 \cdot \|\Delta \phi_k\|_2 = \|\psi_0\|_2. \]

Since \( \|\phi_k\|_{H^2} \) is bounded, it follows that there exists a subsequence of \( \phi_k \) which converges weakly in \( H^2 \) to a function \( \Psi \in H^2_{\text{radial}} \). From the Compactness Lemma 13, see Appendix A, it follows that \( \phi_k \to \Psi \) strongly in \( L^q \), for all \( q \) given by (24).

Next, we prove that \( H[\Psi] \leq 0 \). Since \( \phi_k \rightharpoonup \Psi \) in \( H^2 \), it follows that \( \Delta \phi_k \rightharpoonup \Delta \Psi \), and so \( \|\Delta \Psi\|_2 \leq \lim_{k \to \infty} \|\Delta \phi_k\|_2 = 1 \). Additionally, since \( \phi_k \rightharpoonup \Psi \) for some \( q = 2 + 2s > 2 \), we have that \( \|\Psi\|_2 = \lim_{k \to \infty} \|\phi_k\|_2 \), and so

\[ H[\Psi] \leq \lim_{k \to \infty} H[\phi_k] = \lim_{k \to \infty} l^4H[\psi_0] = 0. \]

In addition, since

\[ 0 = \lim_{k \to \infty} H[\phi_k] = \lim_{k \to \infty} \left( 1 - \frac{1}{\sigma + 1} \|\phi_k\|_2^{2(\sigma+1)} \right), \]

it follows that \( \lim_{k \to \infty} \|\phi_k\|_2^{2(\sigma+1)} = 0 \), so \( \Psi \neq 0 \). Therefore, Corollary 15, see Appendix C, implies that \( \|\Psi\|_2^2 \geq \|R\|_2^2 \). \( \square \)

5.3 Power Concentration

Solutions of critical NLS have the power concentration property, whereby the amount of power that collapses into the singularity is at least \( P_{\text{crNLS}} = \|R_{\text{NLS}}\|_2^2 \), see [37, 23]. In what follows, we prove the analogous results for the critical BNLS.

Corollary 5. Let \( d \geq 2 \), and let \( \psi(t, r) \) be a solution of the radially-symmetric critical BNLS (9) that becomes singular at \( t = T_c < \infty \). Then, \( \forall \epsilon > 0 \),

\[ \liminf_{t \to T_c} \|\psi(t, r)\|_{L^2(r<\epsilon)}^2 \geq P_{\text{cr}}, \]

where \( P_{\text{cr}} = \|R\|_2^2 \).

Proof. The result shall follow directly from Corollary 6. \( \square \)

The following Corollary shows that the rate of power-concentration is not slower than the blowup rate \( l(t) \). The NLS analogue is due to Tsutsumi [35] and Weinstein [37].
Corollary 6. Let $d \geq 2$, let $\psi(t,r)$ be a solution of the radially-symmetric critical BNLS that becomes singular at $t = T_c < \infty$, and let $l(t) = \|\Delta \psi\|^{-1/2}$. Then,

1. For any monotonically-decreasing function $a(t) : [0,T_c) \to \mathbb{R}^+$ such that
   \[ \lim_{t \to T_c} a(t) = 0, \quad \text{and} \quad \lim_{t \to T_c} l/a = 0, \]
   we have that
   \[ \liminf_{t \to T_c} \|\psi(t,r)\|_{L^2(r<a(t))}^2 \geq P_{cr}. \]

2. For any $\epsilon > 0$, $\exists K > 0$ such that
   \[ \liminf_{t \to T_c} \|\psi(t,r)\|_{L^2(r<Kl(t))}^2 \geq (1 - \epsilon)P_{cr}. \]

Proof. See Appendix B.

Since the second part of Corollary 5 is true for all $\epsilon > 0$ and since $Kl(t) \to 0$, Corollary 5 follows.

The next Corollary shows that the power-concentration rate has a quartic-root upper bound. The analogue in NLS theory, which is a square-root upper bound, was proved in [37, 35].

Corollary 7. Let $d \geq 2$ and let $\psi(t,r)$ be a solution of the radially-symmetric critical BNLS that becomes singular at $t = T_c < \infty$. Then,

1. For any monotonically-decreasing $a : [0,T_c) \to \mathbb{R}^+$ such that
   \[ \lim_{t \to T_c} a(t) = 0, \quad \text{and} \quad \lim_{t \to T_c} \frac{(T_c-t)^{1/4}}{a(t)} = 0, \]
   we have that
   \[ \liminf_{t \to T_c} \|\psi(t,r)\|_{L^2(r<a(t))}^2 \geq P_{cr}. \]

2. For any $\epsilon > 0$, $\exists K > 0$ such that
   \[ \liminf_{t \to T_c} \|\psi(t,r)\|_{L^2(r<K(T_c-t)^{1/4})}^2 \geq (1 - \epsilon)P_{cr}. \]

Proof. Theorem 3 implies that $l(t) \leq K(T_c-t)^{1/4}$. Therefore, the result follows immediately from Corollary 6.

Recently, Chae, Hong and Lee [9] used the harmonic analysis method of Bourgain [6] to prove that singular solutions of the critical BNLS for $d \geq 2$ have the power-concentration property

\[ \lim_{t \to T_c} \sup_{x_0 \in \mathbb{R}^d} \|\psi(t,x)\|_{L^2(|x-x_0|<(T_c-t)^{1/4})}^2 > C, \]

where $C$ is a positive constant. This result is more general than Corollaries 5,6,7 in that it does not assume radial symmetry. The proof given here, however, is considerably simpler. More importantly, it shows that $C = P_{cr}$. 


6 Peak-type singular solutions of the critical BNLS

In this, we consider radially-symmetric singular solutions that are “peak-type”, i.e., for which \( r_{\text{max}}(t) \equiv 0 \) for \( 0 \leq t \leq T_c \), where \( r_{\text{max}}(t) = \arg \max_r |\psi| \) is the location of the maximal amplitude.

6.1 The critical NLS - review

The critical NLS admits singular solutions that collapse with the universal \( \psi_{\text{R NLS}} \) profile, i.e., \( \psi \sim \psi_{\text{R NLS}} \), where

\[
\psi_{\text{R NLS}}(t, r) = \frac{1}{L_1/\sigma(t)} R_{\text{NLS}}(\rho) e^{i\tau + i\frac{4}{5}L_1 r^2}, \quad \tau = \int_0^t \frac{ds}{L^2(s)}, \quad \rho = \frac{r}{L(t)}.
\] (25)

The self-similar profile \( R_{\text{NLS}} \) is the ground-state solution of

\[-R + \Delta R + |R|^{2\sigma} R = 0.
\]

The blowup rate of \( L(t) \) is given by the \textit{loglog law} \cite{14, 18, 19, 22}

\[
L(t) \sim \left( \frac{2\pi(T_c - t)}{\log |\log(T_c - t)|} \right)^{\frac{1}{2}}, \quad t \to T_c.
\] (26)

Since the blowup rate \cite{26} is slightly faster than a square root \( \lim_{t \to T_c} LL_t = \lim_{t \to T_c} \frac{1}{2} (L^2)_t = 0 \).

Therefore, the phase term \( \frac{4}{5}L_1 r^2 = \frac{LL_1}{8} \rho^2 \) in (25) vanishes as \( t \to T_c \). Hence, the blowup profile reduces to

\[
\psi_{\text{R NLS}}(t, r) = \frac{1}{L^{1/\sigma}(t)} R_{\text{NLS}}(\rho) e^{i\tau}, \quad \tau = \int_0^t \frac{ds}{L^2(s)}, \quad \rho = \frac{r}{L(t)}.
\] (27)

6.2 Informal analysis

We now look for the "corresponding" peak-type singular solutions of the critical BNLS \cite{15}. Theorem \cite{4} suggests that the collapsing core of the singular solution approaches a self-similar form, i.e.,

\[
\psi(t, r) \sim \psi_B(t, r), \quad 0 \leq r \leq \rho_c \cdot L(t),
\]

where

\[
\psi_B(t, r) = \frac{1}{L^{d/2}(t)} B(\rho)e^{i\tau}, \quad \rho = \frac{r}{L},
\] (28)

and \( \rho_c = \mathcal{O}(1) \). Substituting \cite{28} into \cite{9} and requiring that \( [\psi_2] \sim [\Delta \psi] \sim [|\psi|^{d/2} \psi] \) suggests that

\[
\tau(t) = \int_{s=0}^t \frac{1}{L^4(s)} ds.
\]

Let us consider the self-similar profile \( B(\rho) \). In the singular region \( r = \mathcal{O}(L) \) we have that

\[
\Delta^2 \psi \sim \Delta^2 \psi_B \sim \frac{e^{i\tau}}{L^{4+d/2}} \Delta^2 B, \quad |\psi|^{8/d} \psi \sim |\psi_B|^{8/d} \psi_B = \frac{e^{i\tau}}{L^{4+d/2}} |B|^{8/d} B,
\]
and
\[ \psi_t \sim (\psi_B)_t \sim \frac{t^\tau}{L^{d+\sigma/2}} \left\{ iB - L_4 L^3 \left( \frac{d}{2} B + \rho B_{\rho} \right) \right\}. \]
Hence, \( B(\rho) \) satisfies
\[ -B(\rho) - \Delta^2 \rho B + |B|^{8/d} = i \left( \lim_{t \to T_c} L_4 L^3 \right) \left( \frac{d}{2} B + \rho B_{\rho} \right). \] (29)

Theorem 3 shows that the blowup rate of \( L(t) \) is lower-bounded by a quartic-root. In the critical NLS the blowup rate of peak-type solutions is slightly faster than the analogous square-root rate, due to the loglog correction. Hence, we expect that the blowup rate of peak-type critical BNLS solutions is slightly faster than a quartic root, i.e.,
\[ \frac{L(t)}{\sqrt{T_c - t}} \to 0. \] (30)
In that case, \( \lim_{t \to T_c} L_4 L^3 = \lim_{t \to T_c} \frac{1}{4} (L^4)_t = 0 \), and (29) reduces to the standing-wave equation (21). Since the ground-states of (21) attain their maximal amplitudes at \( \rho = 0 \), see Section 3, these are peak-type solutions.

The above informal analysis thus leads to the following Conjecture:

**Conjecture 8.** The critical BNLS admits peak-type singular solutions such that:

1. The collapsing core approaches the self-similar profile
   \[ \psi(t, r) \sim \psi_R(t, r), \quad 0 \leq r \leq \rho_c \cdot L(t), \] (31a)
   where
   \[ \psi_R(t, r) = \frac{1}{L^{d/2}(t)} R(\rho) e^{i \tau(t)}, \quad \rho = \frac{r}{L}, \quad \tau(t) = \int_{s=0}^{t} \frac{1}{L^4(s)} ds, \] (31b)
   and \( R \) is the ground-state of equation (21).

2. The blowup rate of \( L(t) \) is slightly faster than a quartic-root, i.e.,
   \[ \lim_{t \to T_c} \frac{L(t)}{(T_c - t)^p} = \begin{cases} 0 & \quad p = 1/4 \\ \infty & \quad p > 1/4 . \end{cases} \] (32)

In Section 6.3 we provide numerical evidence in support of Conjecture 8.

### 6.3 Simulations

The one-dimensional critical BNLS
\[ i\psi_t(t, x) - \psi_{xxxx} + |\psi|^8 \psi = 0 \] (33)
was solved with the Gaussian initial condition $\psi_0(x) = A_1 e^{-x^2}$ with $A_1 \approx 1.618$, whose power is $\|\psi_0\|_2^2 = 1.1 \cdot P_{cr}(d = 1)$. The maximal amplitude of the solution $\|\psi\|_\infty$ as a function of time is plotted in Fig 3A. The amplitude increases abruptly by a factor of $10^4$ around $T_c \approx 0.0499$, suggesting that the solution becomes singular in a finite time.

The simulation was repeated for the radially-symmetric two-dimensional critical BNLS

$$i\psi_t(t,r) - \frac{1}{r^3} \psi_r + \frac{1}{r^2} \psi_{rr} - 2 \frac{1}{r} \psi_{rrr} - |\psi|^4 \psi = 0,$$

with the Gaussian initial condition $\psi_0(r) = A_2 e^{-r^2}$ with $A_2 \approx 3.034$, whose power is $\|\psi_0\|_2^2 = 1.1P_{cr}(d = 2)$ The amplitude increases abruptly by a factor of $10^8$. around $T_c \approx 0.0606$, see Fig 3B, again suggesting that the solution becomes singular in a finite time.

We next consider the self-similar profile of the collapsing solutions from Figure 3. In order to verify that it is given by (31), we rescale the solutions as

$$\psi_{\text{rescaled}}(t,\rho) = L^{2/\sigma}(t)\psi(t, r = \rho \cdot L), \quad L(t) = \|\psi\|_\infty^{-\sigma/2},$$

with $2/\sigma = d/2$. The rescaled solutions at focusing levels of $L = 10^{-4}$ and $L = 10^{-8}$ are indistinguishable, see Fig 4, indicating that the collapsing core is indeed self-similar according to (31). A predicted, the self-similar profile is very close to the ground-state $R$ in the core region $0 \leq \rho \leq 4$.

We next compute the blowup rate $p$, defined by the relation

$$L \sim \kappa(T_c - t)^p.$$ 

To do that, we perform a least-squares fit of $\log(L)$ with $\log(T_c - t)$, see Figure 5 obtaining a value of $p \approx 0.2516$ for both $d = 1$ and $d = 2$. This value of $p$ is slightly above $1/4$, implying that the quartic-root lower-bound given by Theorem 3 is close to the actual blowup-rate of peak-type singular solutions.

Next, we provide two indications that the blowup rate is faster than $1/4$. First, if the blowup rate is exactly $1/4$, then $\lim_{t \to T_c} L^3 L_t$ should be finite and strictly negative. However, up to focusing
Figure 4: The solutions of Figure 3 rescaled according to \( \text{(3.5)} \), at focusing levels \( L(t) = 10^{-4} \) (blue dotted line) and \( L(t) = 10^{-8} \) (black solid line). Red dashed line is the rescaled ground-state \( |R| \). The three curves are indistinguishable for \( 0 \leq r/L \leq 4 \).

Figure 5: \( L(t) \) as a function of \( (T_c - t) \), on a logarithmic scale, for the solutions of Figure 3 (circles). A) 1D case. Solid line is the fitted curve \( L = 0.742 \cdot (T_c - t)^{0.2516} \). B) 2D case. Solid line is the fitted curve \( L = 0.641 \cdot (T_c - t)^{0.2516} \).
level of $L = 10^{-8}$, $L^3 L_t$ does not appear to converge to a negative constant, but rather to increase slowly towards $0^-$, see Figure 6A. Second, according to the informal analysis in Section 6.2, the blowup rate is faster than a quartic root if and only if the self-similar profile $B(\rho)$ satisfies the standing-wave equation (4), which is indeed what we observed numerically in Figure 4.

**Remark:** In the critical NLS the blowup rate of peak-type solutions is slightly faster than the analogous square-root lower-bound, due to the well-known loglog-correction (26). Figure 5 shows that the blowup rate is slightly faster than a quartic root, and Figure 6A shows that $L^3 L_t \to 0$ very slowly. Together, this suggests that the blowup rate in the critical BNLS is only slightly faster than the analogous quartic root. At present, we do not know if the blowup rate of peak-type solutions of the critical BNLS is a quartic root with a loglog correction. We note, however, that the loglog correction in the critical NLS cannot be determined numerically [13], and can only be derived analytically. Therefore, we expect that the determination of the analogous correction to the $1/4$ blowup rate of the critical BNLS will also have to be done analytically, and not numerically.

## 7 Peak-type singular solutions of the supercritical BNLS

### 7.1 The supercritical NLS - review

In contrast to the extensive theory on singularity formation in the critical NLS, much less is known about the supercritical NLS. Numerical simulations and formal calculations (see, e.g., [33 Chapter 7] and the references therein) suggest that peak-type singular solutions of the supercritical NLS collapse with a universal $\psi_Q$ profile, i.e.,

$$\psi(t,r) \sim \begin{cases} 
\psi_Q(t,r) & 0 \leq r \leq r_c, \\
\psi_{\text{non-singular}}(t,r) & r \geq r_c,
\end{cases} \quad (36a)$$

---

Figure 6: A: $L^3 L_t$ as a function of $1/L$, for the solution of Figure 3A (black solid line) and of Figure 3B (red dashed line). B: same as (A) for the supercritical cases $d = 1, \sigma = 6$ (black solid line) and $d = 2, \sigma = 3$ (red dashed line).
where
\[ \psi_Q(t,r) = \frac{1}{L^{1/\sigma}(t)} Q(\rho) e^{i\tau}, \quad \tau = \int_0^t \frac{ds}{L^2(s)}, \quad \rho = \frac{r}{L(t)}. \] (36b)

Note that the singular region \( r \in [0,r_c] \) is constant in the coordinate \( r \). Therefore, in the rescaled variable \( \rho = r/L(t) \), the singular region \( \rho \in [0,r_c/L(t)] \) becomes infinite as \( L(t) \to 0 \). This is in contradistinction with the critical case, wherein the singular region \( \rho \in [0,\rho_c] \) is constant in the rescaled variable \( \rho \), but shrinks to a point in the original coordinate \( r \).

The self-similar profile \( Q \) is the solution of
\[
Q''(\rho) + \frac{d-1}{\rho} Q' - Q + i\frac{\kappa^2}{2} \left( \frac{1}{\sigma} Q + \rho Q' \right) + |Q|^{2\sigma} Q = 0, \\
Q'(0) = 0, \quad Q(\infty) = 0.
\] (36c)

Solutions of (36c) are complex-valued, and depend on the parameter \( \kappa \) and on the initial condition \( Q(0) = Q_0 \). Solutions of (36c) whose amplitude \( |Q| \) is monotonically-decreasing in \( \rho \), and which have a zero Hamiltonian, are called admissible solutions \([33]\). For each choice of \((\sigma,d)\), equation (36c) has a unique admissible solution (up to a multiplication by a constant phase \( e^{i\alpha} \)). This solution is attained for specific real values of \( \kappa \) and \( Q(0) \), which we denote as
\[ \kappa = \kappa_Q(\sigma,d), \quad Q(0) = Q_0(\sigma,d). \] (37)

The blowup rate of \( L(t) \) is a square-root, i.e.,
\[ L(t) \sim \kappa \sqrt{T_c - t}, \quad t \to T_c. \] (38)

Numerical simulations and formal calculations suggest that:

1. The self-similar profile of singular peak-type solutions of the NLS \([11]\) is an admissible solution of (36c). Since \( Q(\rho) \) attains its maximal amplitude at \( \rho = 0 \), the solution is peak-type.

2. The constant \( \kappa \) of the blowup rate (38) is equal to \( \kappa_Q(\sigma,d) \). Hence, in particular, \( \kappa \) is universal (i.e., is independent of the initial condition \( \psi_0 \)).

The admissible solution \( Q(\rho) \) satisfies
\[ |Q(\rho)| \sim C \cdot \rho^{-1/\sigma}, \quad \rho \to \infty. \]

Thus, \( Q \notin L^2(\mathbb{R}) \). Nevertheless, \( Q(\rho) \) can be the self-similar profile of \( H^1 \) solutions, since \( \psi(r,t) \sim \psi_Q \) only for \( r \in [0,r_c] \), see \([5]\).

### 7.2 Informal analysis

As in the supercritical NLS, we expect that singular peak-type solutions of the supercritical BNLS collapse as
\[ \psi(t,r) \sim \begin{cases} \psi_B(t,r) & 0 \leq r \leq r_c, \\ \psi_{\text{non-singular}}(t,r) & r \geq r_c, \end{cases} \] (39)
where \( \psi_B \) is a self-similar profile, to be determined. As in the supercritical NLS, the singular region \( r \in [0, r_c] \), is constant in the coordinate \( r \). Therefore, in the rescaled variable \( \rho = r/L(t) \), the singular region \( \rho \in [0, r_c/L(t)] \) becomes infinite as \( L(t) \to 0 \). This is again in contradistinction with the critical-BNLS case, where the singular region \( \rho \in [0, \rho_c] \) is constant in the rescaled variable \( \rho \), but shrinks to a point in the original coordinate \( r \).

The BNLS is invariant under the dilation symmetry \( r \mapsto r/L, \quad t \mapsto t/L^4, \quad \psi \mapsto \frac{1}{L^2/\sigma} \psi \), where \( L \) is a constant. In the supercritical case \( \sigma d > 4 \), this suggests that

\[
\psi_B(t, r) = \frac{1}{L^{2/\sigma}(t)} B(\rho) e^{i\tau(t)}, \quad \rho = \frac{r}{L}.
\]

Similar arguments as in Section 6.2 show that \( \tau(t) = \int_{s=0}^{t} \frac{1}{L^4(s)} ds \). Therefore, as in the supercritical NLS, we expect the collapsing part of the solution to approach the self-similar profile

\[
\psi_B(t, r) = \frac{1}{L^{2/\sigma}(t)} B(\rho) e^{i\tau(t)}, \quad \rho = \frac{r}{L}, \quad \tau(t) = \int_{s=0}^{t} \frac{1}{L^4(s)} ds. \tag{40}
\]

Theorem 3 showed that in the critical case, if \( L(t) \sim \kappa (T_c - t)^{p} \), then \( p \geq 1/4 \). The following Lemma extends this result to peak-type solutions of the supercritical BNLS.

**Lemma 9.** Let \( \sigma d > 4 \), and let \( \psi \) be a peak-type singular solution of the BNLS that collapses with the \( \psi_B \) profile \( \tag{41} \). If \( L(t) \sim \kappa (T_c - t)^{p} \), then \( p \geq 1/4 \). Furthermore, \( p = 1/4 \) if and only if the self-similar profile \( B(\rho) \) satisfies the equation

\[
-B(\rho) + i\frac{\kappa^4}{4} \left( \frac{2}{\sigma} B + \rho B' \right) - \Delta_\rho B + |B|^{2\sigma} B = 0, \quad \kappa > 0. \tag{41}
\]

**Proof.** If \( \psi \sim \psi_B \), then

\[
\Delta^2 \psi \sim \Delta \psi_B \sim \frac{e^{i\tau}}{L^{4+2/\sigma}} \Delta_\rho B, \quad |\psi|^{2\sigma} \psi \sim |\psi_B|^{2\sigma} \psi_B \sim \frac{e^{i\tau}}{L^{4+2/\sigma}} |B|^{2\sigma} B,
\]

and

\[
\psi_t \sim (\psi_B)_t \sim \frac{e^{i\tau}}{L^{4+2/\sigma}} \left\{ iB - L_t L^3 \left( \frac{2}{\sigma} B + \rho B' \right) \right\}.
\]

Hence, the equation for \( B \) is

\[
-B - i \left( \lim_{t \to T_c} L_t L^3 \right) \left( \frac{2}{\sigma} B + \rho B' \right) - \Delta_\rho^2 B + |B|^{2\sigma} B = 0, \quad \tag{42}
\]

implying that \( L_t L^3 \) should be bounded as \( t \to T_c \). Since \( L^3 L_t \sim -p\kappa^4(T_c - t)^{4p-1} \), it follows that \( p \geq \frac{1}{4} \). If \( p = 1/4 \), then \( L^3 L_t \to -\frac{4^{3}}{\kappa^4} \), and equation \( \tag{42} \) reduces to \( \tag{41} \). \( \Box \)

---

2 Note that in the critical case \( 2/\sigma = d/2 \), hence the self-similar profile \( \tag{41} \) reduces to \( \tag{31} \).
Let us consider the fourth-order nonlinear ODE (11) for the self-similar profile $B(\rho)$. Its solution requires four boundary conditions, and the determination of the parameter $\kappa$. Radial symmetry implies that $B'(0) = B''(0) = 0$. Since the solution is invariant up to rescaling $B(\rho) \to \lambda^{2/\sigma} B(\lambda \rho)$, one can set, with no loss of generality, $B(0) = 1$. Therefore, we require two additional constraints in order to determine $\kappa$ and $B''(0)$. We recall that in the supercritical NLS the “admissible value” of $\kappa$ is determined from the requirements that $Q$ has a zero Hamiltonian and a monotonically-decreasing amplitude, see Section 7.1. For the BNLS, the following informal argument suggests that the zero-Hamiltonian condition should also holds. Indeed, from Hamiltonian conservation it follows that $H[\psi_B]$ is bounded, because otherwise the non-singular region would also have an infinite Hamiltonian. Calculating the Hamiltonian of $\psi_B$, we have

$$H[\psi_B] \sim L^{-4/\sigma + d - 4} \left[ \int_{\rho=0}^{rc/L(t)} \left( |\Delta_\rho B(\rho)|^2 - \frac{1}{1+\sigma} |B|^{2+2\sigma} \right) \rho^{d-1} d\rho \right].$$

From $H^2$-subcriticality, see (6), it follows that $L^{-4/\sigma + d - 4} \to \infty$ as $L \to 0$. Therefore, if $H[\psi_B]$ remains bounded as $t \to T_c$ then

$$H[B] = \int_{\rho=0}^{r_{\infty}} \left( |\Delta_\rho B|^2 - \frac{1}{1+\sigma} |B|^{2+2\sigma} \right) \rho^{d-1} d\rho = 0. \quad (43)$$

WKB analysis of (11) shows that, see Appendix 11

$$B(\rho) \sim c_1 B_1(\rho) + c_2 B_2(\rho) + c_3 B_3(\rho) + c_4 B_4(\rho), \quad \rho \to \infty,$$

where

$$B_1(\rho) \sim \rho^{-\frac{2}{\sigma} - \frac{4}{\sigma d}} \rho,$$

$$B_2(\rho) \sim \frac{1}{\rho^{\frac{2}{\sigma}((\sigma d - 1)}} \exp \left( -i \frac{3}{8} b \rho^{4/3} - i \frac{1}{3b^3} \log(\rho) \right),$$

$$B_3(\rho) \sim \exp \left( \frac{3\sqrt{3}}{8} b \rho^{4/3} \right) \rho^{\frac{2}{\sigma}((\sigma d - 1)}} \exp \left( +i \frac{3}{8} b \rho^{4/3} - i \frac{1}{3b^3} \log(\rho) \right),$$

$$B_4(\rho) \sim \exp \left( - \frac{3\sqrt{3}}{8} b \rho^{4/3} \right) \rho^{\frac{2}{\sigma}((\sigma d - 1)}} \exp \left( +i \frac{3}{8} b \rho^{4/3} - i \frac{1}{3b^3} \log(\rho) \right)$$

and $b = (\kappa^4/4)^{1/3}$. Equation (11) therefore has two algebraically-decaying solutions, $B_1$ and $B_2$, an exponentially-increasing solution $B_3$, and an exponentially-decreasing solution $B_4$. Since $\sigma d > 4$, the exponent $\frac{2}{\sigma}((\sigma d - 1)$ of $B_2$ is larger than the exponent $\frac{2}{\sigma}$ of $B_1$, hence $B_1 \gg B_2$ as $\rho \to \infty$.

**Lemma 10.** Let $B(\rho)$ be a zero-Hamiltonian solution of (11). Then, $c_2 = c_3 = 0$ and

$$B(\rho) \sim c_1 B_1(\rho), \quad \rho \to \infty.$$

Furthermore, $B'' \in L^2$. 

7 PEAK-TYPE SINGULAR SOLUTIONS OF THE SUPERCRITICAL BNLS
Proof. The exponentially increasing solution $B_3$ must vanish identically if the integrals are to converge, hence $c_3 = 0$. Next, the $\rho^{4/3}$ phase term in $B_2$ implies that

$$\left| B''_2 \right|^2 \sim \rho^{-\frac{4}{3\sigma}(\sigma d - 1) + \frac{2}{3}}.$$ 

Hence, the integral

$$\left\| B''_2 \right\|_2^2 \sim \int \rho^{-\frac{4}{3\sigma}(\sigma d - 1 - \sigma)} \rho^{d-1} d\rho$$

diverges in the $H^2$-subcritical regime $\sigma(d - 4) < 4$, i.e., $B''_2 \notin L^2$. Since, in addition,

$$B''_1 \in L^2, \quad B_1 \in L^{2+2\sigma}, \quad B_2 \in L^{2+2\sigma},$$

the Hamiltonian can be finite only if $c_2 = 0$, in which case $B'' \in L^2$. □

Lemma 10 shows that the condition $H[B] = 0$ imposes the two constraints $c_2 = 0$ and $c_3 = 0$. Hence, zero-Hamiltonian solutions of equation (41) satisfy the five boundary conditions

$$B(0) = 1, \quad B'(0) = 0, \quad B''(0) = 0, \quad c_2 (B''_0, \kappa) = c_3 (B''_0, \kappa) = 0.$$ 

Therefore they form a discrete set of solutions and of values of $\kappa$. We conjecture that the additional condition of monotonicity of $|B|$ will lead to a unique admissible solution $B$ and a unique value of $\kappa$.

**Corollary 11.** Let $B(\rho)$ be a zero-Hamiltonian solution of (41). Then, $\|B\|_2 = \infty$. Nevertheless, $\lim_{t \to T_c} \|\psi_B\|_{L^2_r(r < r_c)} < \infty$.

Proof. Since $B(\rho) \sim c_1 B_1(\rho)$,

$$\|B_1\|_2^2 \sim C \int_{\rho=0}^{\infty} \rho^{-4/3\sigma + d-1} d\rho \sim C \rho^{-4/3\sigma} \bigg|_{\rho=0}^{\infty} = \infty.$$ 

Following the arguments of [5], the profile $\psi_B$ satisfies

$$\|\psi_B\|_{L^2_r(r < r_c)}^2 \sim L^{d-4/\sigma}(t) \cdot \int_{\rho=0}^{r_c/L(t)} |B(\rho)|^2 \rho^{d-1} d\rho \sim L^{d-4/\sigma}(t) \cdot \left( C \rho^{-4/3\sigma} \bigg|_{\rho=0}^{r_c/L(t)} \right) = O(1).$$ 

□

In summary, we conjecture the following:

**Conjecture 12.**

Let $\psi$ be peak-type singular solution of the supercritical BNLS. Then,
1. The collapsing core approaches the self-similar profile \( \psi_B \), i.e.,
\[
\psi(t, r) \sim \psi_B(t, r), \quad 0 \leq r \leq r_c,
\]
where
\[
\psi_B(t, r) = \frac{1}{L^{2/\sigma(t)}} B(\rho) e^{i \tau(t)}, \quad \rho = \frac{r}{L}, \quad \tau(t) = \int_{s=0}^{t} \frac{1}{L^4(s)} ds.
\]

2. The self-similar profile \( B(\rho) \) is the solution of
\[
-B(\rho) + i \kappa^4 \left( \frac{2}{\sigma} \rho B + \rho B' \right) - \Delta \rho B + |B|^{2\sigma} B = 0,
\]
\[
B(0) = 1, B'(0) = B''(0) = 0, \quad H[B] = 0,
\]
where \( \kappa > 0 \) and \( H[B] \) is the Hamiltonian of \( B \), see (43).

3. In particular, \( B(\rho) \neq R(\rho) \).

4. Equation (44c) has a unique “admissible solution” with a unique “admissible value” of \( \kappa = \kappa(\sigma, d) \), such that \( |B(\rho)| \) is monotonically decreasing. Additionally, \( B(\rho) \sim \rho^{-2/\sigma-i4/\kappa^4} \) as \( \rho \to \infty \).

5. The admissible solution is the self-similar profile \( B \) of \( \psi_B \), see (44b).

6. The blowup rate of singular peak-type solutions is exactly a quartic root, i.e.,
\[
L(t) \sim \kappa \sqrt[4]{T_c - t}, \quad \kappa > 0.
\]

7. The coefficient \( \kappa \) of the blowup rate of \( L(t) \) is equal to the value of \( \kappa \) of the admissible solution \( B \), i.e.,
\[
\kappa := \lim_{t \to T_c} \frac{L(t)}{\sqrt[4]{T_c - t}} = \kappa(\sigma, d).
\]

In particular, \( \kappa \) is universal (i.e., it does not depend on the initial condition).

In Section 7.3 we provide numerical evidence in support of Conjecture 12.
Figure 7: Maximal amplitude of peak-type singular solutions of the supercritical BNLS.

Figure 8: The solutions of Figure 7 rescaled according to (3.5), at the focusing levels $1/L = 10^4$ (blue solid line) and $1/L = 10^8$ (black dashed line). The magenta dotted line is the rescaled ground-state $R$. 
Figure 9: The solutions of Figure 7, rescaled according to (3.5), at focusing the level $1/L = 10^8$ (circles). Solid lines are the fitted curves $y = 0.63 \cdot (r/L)^{-0.33}$ (left) and $y = 0.85 \cdot (r/L)^{-0.66}$ (right).

7.3 Simulations

The radially-symmetric BNLS (7) was solved in the supercritical case $d = 1, \sigma = 6$ with the initial condition $\psi_0(x) = 1.6e^{-x^2}$, and in the supercritical case $d = 2, \sigma = 3$ with the initial condition $\psi_0(r) = 3e^{-r^2}$. In both cases, the solutions blowup at a finite time, see Figure 7.

To check whether the solutions collapse with the self-similar profile (44), the solution was rescaled according to (3.5). The rescaled solutions at the focusing levels $L = 10^{-4}$ and $L = 10^{-8}$ are indistinguishable in both the one-dimensional case (Figure 8A) and the two-dimensional case (Figure 8B), providing numerical support that the solution collapses with the $\psi_B$ profile (44). As predicted, the self-similar profile is different than the ground-state $R$. Indeed, Figure 9 shows that as $\rho \to \infty$, the self-similar profile of $\psi$ decays as $\rho^{-2/\sigma}$, which is in agreement with the decay rate of $B_1(\rho)$.

We next verify that the solution converges to the asymptotic profile for $r \in [0, r_c]$, i.e., for $\rho \in [0, r_c/L(t)]$. To do this, we plot in Figure 10 the rescaled solution at focusing levels of $1/L = 10, 100, 1000, 10000$, as a function of $\log(r/L)$. The curves are indistinguishable at $r/L = \mathcal{O}(1)$, but bifurcate at increasing values of $r/L$. These “bifurcations positions” are marked by circles in Figure 10 and their $r/L$ values are listed in Table 1. The “bifurcation positions” are linear in $1/L$, indicating that the region where $\psi \sim \psi_B$ is indeed $\rho \in [0, r_c/L(t)]$, which corresponds to $r \in [0, r_c]$.

To compute the blowup rate $p$, we performed a least-squares fit of $\log(L)$ with $\log(T_c - t)$, see Figure 11. The resulting values are $p \approx 0.2502$ in the $d = 1, \sigma = 6$ case and $p \approx 0.2504$ in the $d = 2, \sigma = 3$ case. Next, we provide two indications that the blowup rate is exactly $1/4$, i.e.,

$$L(t) \sim \kappa \sqrt{T_c - t}, \quad \kappa > 0.$$ 

First, if the blowup rate is exactly a quartic root, then $L^3 L_t \to \frac{-\kappa^4}{4} < 0$. Indeed, Figure 6B shows
Figure 10: Convergence to a self-similar profile. The solutions of Figure 7 rescaled according to (35), as a function of \( \log(r/L) \), at the focusing levels \( L = 10^{-1} \) (dashed blue line), \( L = 10^{-2} \) (dash-dotted red line), \( L = 10^{-3} \) (dotted green line), \( L = 10^{-4} \) (solid black line) and \( L = 10^{-8} \) (solid magenta line). The circles mark the approximate position where each curve bifurcates from the limiting profile, see also Table [1].
Figure 11: $L(t)$ as a function of $(T_c - t)$, on a logarithmic scale, for the solutions of Figure 7 (circles). Solid lines are the fitted curves $L = 1.048 \cdot (T_c - t)^{0.2502}$ (A) and $L = 0.931 \cdot (T_c - t)^{0.2504}$ (B).

that in the case $d = 1$, $\sigma = 6$, $L^3 L_t \to -0.289$, implying that

$$\kappa(d = 1, \sigma = 6) \approx \sqrt[4]{4 \cdot 0.289} \approx 1.037.$$  \hfill (45)

In the case $d = 2$, $\sigma = 3$, $L^3 L_t \to -0.171$, implying that

$$\kappa(d = 2, \sigma = 3) \approx \sqrt[4]{4 \cdot 0.171} \approx 0.909.$$  \hfill (46)

Since $L^3 L_t$ converges to a finite, negative constant, this shows that the blowup rate is exactly $1/4$.

Second, according to Lemma 9, if $L^3 L_t \to 0$ the self-similar profile $B(\rho)$ should not satisfy the standing-wave equation (44). In Figure 8 we saw that the rescaled self-similar BNLS solutions and the corresponding ground-states differ considerably (compare with Figure 4), indicating that the blowup rate is exactly $1/4$. Therefore, the numerical results again support Conjecture 12.

Finally, we verified that the value of $\kappa$ in the blowup rate (44) is universal. We solve the BNLS in the case $d = 1$, $\sigma = 6$ with the initial condition $\psi_0(x) = 2e^{-x^2}$. In this case, the calculated value of $\kappa(d = 1, \sigma = 6)$ is $\kappa = \lim_{t \to T_c} \sqrt[4]{-4L_t L^3} \approx 1.037$, which is equal, to first 3 significant digits, to the previously obtained value, see (45), for the initial condition $\psi_0(x) = 1.6e^{-x^2}$. Similarly, in the case $d = 2$, $\sigma = 3$, we solve the equation with the initial condition $\psi_0(x) = 3e^{-x^2}$. The calculated value of $\kappa(d = 2, \sigma = 3)$ is $\kappa = \lim_{t \to T_c} \sqrt[4]{-4L_t L^3} \approx 0.913$, which is equal, to first 2 significant digits, to the previously obtained value, see (46), for the initial condition $\psi_0(x) = 3e^{-x^2}$.
Figure 12: The grid-spacing $\Delta r_m$ obtained using the SGR method of [10] for a peak-type singular solution of the BNLS. A) The grid generated the original method of [10] at focusing level of $L = 10^{-6}$. The Singular and non-singular regions are well-resolved, but the transition region $\Delta r_m$ displays a discontinuity. At this point, the finite difference operator becomes ill-conditioned. B) same as (A), after adding the new penalty function $w_3$, at focusing level $L = 10^{-12}$. Even at this much larger focusing level, the transition region is now well resolved.

8 Numerical methods

8.1 Adaptive mesh construction using the SGR method

In this study, we computed singular solutions of the BNLS equation (7). These solutions become highly-localized, so that the spatial scale-difference between the singular region $r - r_{\text{max}} = \mathcal{O}(L)$ and the exterior regions can be as large as $\mathcal{O}(1/L) \sim 10^{10}$. In order to resolve the solution at both the singular and non-singular regions, we use an adaptive grid.

We generate the adaptive grids using the Static Grid Redistribution (SGR) method, which was first introduced by Ren and Wang [31], and later simplified and improved by Gavish and Ditkowsky [10]. Using this approach, the solution is allowed to propagate (self-focus) until it becomes under-resolved. At this stage, a new grid, with the same number of grid-points, is generated using De’Boors ‘equidistribution principle’, wherein the grid points $\{r_m\}$ are spaced such that a certain weight function $w_1[\psi]$ is equidistributed, i.e., that

$$\int_{r=r_m}^{r_{m+1}} w_1[\psi(r)] \, dr = \text{const},$$

see [31] [10] for details.

The algorithms of [31] [10] keeps a recursive set of grids, the coordinates of each is given in the reference frame of the previous grid. This enables the simulation to reach high focusing levels, where the grid-points position cannot be stored at the physical reference frame due to loss of significant digits. In this study, however, we implement a simplified version of the method of [10], in which
we dispense with the above hierarchy of grids and store the grid-points position in terms of the original reference frame. On this highly non-uniform grid, we use a standard non-uniform finite-difference approximation of the radial biharmonic operator \[^8\]. The approximation is third-order accurate, with a seven-point stencil. Thus, we are limited by the standard machine accuracy\[^3\] to focusing levels of no more than \( L \approx 10^{-12} \). However, the resulting gain in software (and numerical) simplicity justifies this limitation.

The method in \[^10\] allows control of the fraction of grid points that migrate into the singular region, preventing under-resolution at the exterior regions. This is done by using a weight-function \( w_2 \), which penalizes large inter-grid distances. However, we found that this numerical mechanism, while necessary, is insufficient for our purposes. In order to understand the reason, let us consider the grid-point spacings \( \Delta r_m = r_{m+1} - r_m \). Using the method of \[^10\] with both \( w_1 \) and \( w_2 \) causes a very sharp bi-partition of the grid points – to those inside the singular region, whose spacing is determined by \( w_1 \) and is \( \Delta r_m = \mathcal{O}(L) \), and to those outside the singular region, whose spacing is determined by \( w_2 \) and is \( \Delta r_m = \mathcal{O}(1) \), see Figure \[^12A\]. Inside each of these regions, the finite difference approximation we use is well conditioned. However, at the transition between these two regions, the finite-difference stencil, seven-points in width, spans grid-spacings with \( \mathcal{O}(1/L) \) scale-difference — leading to under-resolution which completely violates the validity of the finite-difference approximation.

In order to overcome this limitation, we improve the algorithm of \[^10\] by adding a third weight function

\[
w_3(r_m) = \sqrt{1 + \frac{|\Delta^2 r_m|}{\Delta r_m}},
\]

which penalizes the second-difference \( \Delta^2 r_m = \Delta r_{m+1} - \Delta r_m \) operator of the grid locations, allowing for a smooth transition between the singular region and the non-singular region, see Fig \[^12B\].

On the sequence of grids, the equations are solved using a Predictor-Corrector Crank-Nicholson scheme, which is second-order in time.

### 8.2 The Spectral Renormalization Method

Here, we describe the adaptation of the Spectral Renormalization method (SRM) to the standing-wave BNLS equation \[^17\]. Denoting the Fourier transform of \( R(x) \) by \( \mathcal{F}[R](k) \), equation \(^{17}\) transforms to

\[
\mathcal{F}[R](k) = \frac{1}{k^4 + 1} \mathcal{F}[|R|^{2\sigma} R],
\]  

where \( k^4 = |k|^4 \), leading to the fixed-point iterative scheme

\[
\mathcal{F}[R_{m+1}] = \frac{1}{k^4 + 1} \mathcal{F}[|R_m|^{2\sigma} R_m], \quad m = 0, 1, \ldots .
\]

Typically this iterative scheme diverges either to \( \infty \) or to 0. In order to avoid this problem, we renormalize the solution as follows. Multiplying equation \(^{17}\) by \( \mathcal{F}[R]^* \) and integrating over \( k \)

\[^4\] At the time of writing, the standard machine accuracy is IEEE 64-bit floating points, with relative machine error of \( 10^{-16} \). Using 128-bit floating points will, for all practical purposes, eliminate the above restriction.
A Compactness Lemma

Here we provide an extension of the Compactness Lemma for $H^1_{\text{radial}}$ functions [32], to the case of $H^2_{\text{radial}}$:

Lemma 13. (Compactness Lemma) Let $d \geq 2$ and let $\sigma > 0$ be in the $H^2$-subcritical regime (6). Then, the embedding $H^2_{\text{radial}}(\mathbb{R}^d) \to L^{2(\sigma+1)}(\mathbb{R}^d)$ is compact, i.e., every bounded sequence $u_n' \in H^2_{\text{radial}}(\mathbb{R}^d)$ has a subsequence $u_n$ which converges strongly in $L^{2(\sigma+1)}(\mathbb{R}^d)$.

Proof. If $\|u_n'\|_{H^2} \leq M$, then the sequence $u_n'$ has a subsequence $u_n$ which converges weakly to $u$ in $H^2$. Since the limit of radial functions is a radial function, $u \in H^2_{\text{radial}}$. In addition, since for any bounded domain $\Omega$, the embedding $H^2(\Omega) \to L^2(\Omega)$ is compact, there is a subsequence which
converges strongly to \( u \) in \( L^2(\Omega) \), i.e., \( \lim_{n \to \infty} \int_{\Omega} |u_n - u|^2 \, dx = 0 \). From the Gagliardo-Nirenberg inequality on the bounded domain \( \Omega \), see [15, 16, 26],
\[
\|f\|_{L^2(\sigma+1)(\Omega)}^{2(\sigma+1)} \leq B_{\sigma,d,\Omega} \|\Delta f\|_{L^2(\Omega)}^{\sigma d/4} \cdot \|f\|_{L^2(\Omega)}^{2(\sigma+1)-\sigma d/4} = B_{\sigma,d,\Omega} \|\Delta f\|_{L^2(\Omega)}^{\sigma d/4} \cdot \|f\|_{L^2(\Omega)}^{2(1-\frac{d}{4})}
\]
and since \( 1 > \frac{d}{4} \) in the \( H^2 \)-subcritical case, it follows that \( u_n \to u \) strongly in \( L^2(\sigma+1)(\Omega) \), so that
\[
\lim_{n \to \infty} \int_{\Omega} |u_n - u|^{2(\sigma+1)} \, dx = 0.
\]

Next, Strauss radial Lemma [32] for \( H^1 \) functions gives that \( \forall \rho_\epsilon > 1 \) and \( n \),
\[
\int_{|x| > \rho_\epsilon} |u_n|^{2(\sigma+1)} \, dx \leq \frac{C}{\rho_\epsilon^{(\sigma-1)d}},
\]
so that \( \forall \epsilon \exists \rho_\epsilon \) s.t. \( \forall n \)
\[
\int_{|x| > \rho_\epsilon} |u_n|^{2(\sigma+1)} \, dx \leq \epsilon.
\]

Finally, since
\[
\|u_n - u\|_{L^2(\sigma+1)(\mathbb{R}^d)} \leq \|u_n - u\|_{L^2(\sigma+1)(|x| < \rho_\epsilon)} + \|u_n\|_{L^2(\sigma+1)(|x| > \rho_\epsilon)} + \|u\|_{L^2(\sigma+1)(|x| > \rho_\epsilon)}
\]
the convergence in \( \mathbb{R}^d \) is obtained.

\[ \square \]

B Proof of Corollary 6

**Proof.** From the proof of Theorem 4
\[
\|\psi(t_k, r)\|_{L^2(\sigma<r<a(t_k))} = \|\phi_k(r)\|_{L^2(\sigma<r<1/l(t_k))},
\]
and since \( \lim_{k \to \infty} a(t_k)/l(t_k) = \infty \), we have that \( \forall M > 0 \)
\[
\liminf_{k \to \infty} \|\phi_k(r)\|_{L^2(r<M)} \leq \liminf_{k \to \infty} \|\phi_k(r)\|_{L^2(\sigma<r<1/l(t_k))}.
\]
Since \( \phi_k(r) \to \Psi \) \( L^2(M) \), it follows that \( \phi_k(r) \to \Psi \) \( L^2(M) \), and so
\[
\|\Psi\|_{L^2(r<M)}^2 \leq \liminf_{k \to \infty} \|\phi_k(r)\|_{L^2(r<M)}^2
\]
This is true \( \forall M \), and so
\[
P_{cr} \leq \|\Psi\|_{L^2}^2 \leq \liminf_{k \to \infty} \|\psi(t_k, r)\|_{L^2(\sigma<r<a(t_k))}^2.
\]
For the second result, since \( \|\Psi\|_{L^2} \geq P_{cr} \), it follows that for all \( \epsilon > 0 \) there exist \( K > 0 \) such that
\[
\|\Psi\|_{L^2(r>K)} \geq (1 - \epsilon)P_{cr}.
\]
Therefore, since
\[ \| \psi(t_k, r) \|_{L^2(r < K \cdot l(t_k))}^2 = \| \phi_k(r) \|_{L^2(r < K)}^2, \]
a similar argument as in the previous section gives
\[ (1 - \epsilon) \frac{c}{P_{cr}} \leq \| \Psi \|_{L^2(r < K)}^2 \leq \lim_{k \to \infty} \inf \| \psi(t_k, r) \|_{L^2(r < K \cdot l(t_k))}^2. \]

\[ \square \]

C Gagliardo Nirenberg inequality for $H^2$ functions

In $L^2$-critical case $\sigma_d = 4$, which is always in the $H^2$-subcritical regime [4], the appropriate Gagliardo-Nirenberg inequality in $H^2$ is [15, 16, 26]:

**Lemma 14.** (Gagliardo-Nirenberg inequality) Let $\sigma_d = 4$, and let $f \in H^2(\mathbb{R}^d)$, then
\[ \| f \|_{2(\sigma+1)}^{2(\sigma+1)} \leq B_{\sigma,d} \| \Delta f \|_2^2 \| f \|_2^{2\sigma}. \quad (49) \]

We note that the ground-state $R$ of equation (4) is the minimizer of the Gagliardo-Nirenberg inequality [12], and that its $L^2$ norm, the critical power, satisfies
\[ P_{cr} = \| R \|_2^2 = \left( \frac{\sigma + 1}{B_{\sigma,d}} \right)^{1/\sigma}. \]

Hence, the Gagliardo-Nirenberg inequality implies the following Corollary:

**Corollary 15.** Let $f \in H^2$ and $\sigma_d = 4$, then
\[ H[f] \geq \left[ 1 - \left( \frac{\| f \|_2^2}{P_{cr}} \right)^\sigma \right] \| \Delta f \|_2^2, \]
so that
\[ \| f \|_2^2 \leq P_{cr} \implies H[f] \geq 0. \quad (50) \]

D WKB analysis of eq. (44c).

As $\rho \to \infty$, the nonlinear term in (44c) becomes negligible, and (44c) reduces to
\[ -B(\rho) - \Delta_\rho B + i b^3 \left( \frac{\rho}{2} B + \rho B_\rho \right) = 0, \quad b^3 = \frac{\kappa^4}{4}, \quad (51) \]
where
\[ \Delta_\rho = -\frac{(d-1)(d-3)}{\rho^2} \partial_\rho^2 + \frac{(d-1)(d-3)}{\rho^2} \partial_\rho^2 + \frac{2(d-1)}{\rho} \partial_\rho^3 + \partial_\rho^4. \]
In order to apply the WKB method, we substitute \( B(\rho) = \exp(w(\rho)) \), and expand

\[
w(\rho) \sim w_0(\rho) + w_1(\rho) + \ldots .
\]

Substituting \( w_0(\rho) = \alpha \rho^p \) and balancing terms shows that \( p = 4/3 \), and that the equation for the leading-order, the \( \mathcal{O}(\rho^{4/3}) \) terms, is

\[
(w_0')^3 = \left(\frac{4\alpha}{3}\right)^3 \rho = ib^3 \rho.
\]

Therefore,

\[
\alpha = \frac{3}{4} be^{i\pi/4} = \frac{3}{4} b \cdot \left\{-i, \frac{\sqrt{3} + i}{2}, \frac{-\sqrt{3} + i}{2}\right\}.
\]

The equation for the next order, the \( \mathcal{O}(1) \) terms, is

\[
1 + 2dib^3 = ib^3 \left(\frac{2}{\sigma} + 3pw_1'\right),
\]

implying that

\[
w_1 = \frac{1}{3} \left(\frac{2}{\sigma}(1 - \sigma d) - \frac{1}{ib^3}\right) \log \rho.
\]

The next-order terms are \( \mathcal{O}(\rho^{-4/3}) = o(1) \) and can be neglected. We therefore obtain the three solutions

\[
B_2(\rho) \sim \frac{1}{\rho^{2\sigma/(\sigma d - 1)}} \exp \left(-i \frac{3}{4} bp^{4/3} - i \frac{1}{3b^3} \log(\rho)\right),
\]

\[
B_3(\rho) \sim \frac{\exp \left(\frac{3\sqrt{3}}{8} bp^{4/3}\right)}{\rho^{2\sigma/(\sigma d - 1)}} \exp \left(+i \frac{3}{8} bp^{4/3} - i \frac{1}{3b^3} \log(\rho)\right),
\]

\[
B_4(\rho) \sim \frac{\exp \left(-\frac{3\sqrt{3}}{8} bp^{4/3}\right)}{\rho^{2\sigma/(\sigma d - 1)}} \exp \left(+i \frac{3}{8} bp^{4/3} - i \frac{1}{3b^3} \log(\rho)\right).
\]

Since (51) is a fourth order ODE, another solution is required. To obtain the fourth solution, we substitute \( w_0 \sim \beta \log(\rho) \) in (51) and obtain that the equation for the leading-order, the \( \mathcal{O}(1) \) terms, is

\[
-1 + ib^3 \left(\frac{2}{\sigma} + \beta\right) = 0,
\]

and that the next order is \( o(1) \) and can be neglected. The fourth solution is therefore

\[
B_1(\rho) \sim \rho^{-\frac{2}{\sigma} - \frac{1}{5\sigma}}.
\]
The Lagrangian density of the BNLS is
\[
L(\psi, \psi^*, \psi_t, \psi_t^*, \Delta \psi, \Delta \psi^*) = \frac{i}{2} \left( \psi \psi^* \psi_t - \psi \psi^* \psi_t^* \right) - |\Delta \psi|^2 + \frac{1}{1 + \sigma} |\psi|^2(\sigma + 1).
\]

We cite here Noether’s Theorem, as given in [33]:

**Theorem 16 (Noether Theorem).** If the action integral \(\int \int L \, dx \, dt\) is invariant under the infinitesimal transformation

\[
t \mapsto \tilde{t} = t + \delta t(x, t, \psi),
\]
\[
x \mapsto \tilde{x} = x + \delta x(x, t, \psi),
\]
\[
\psi \mapsto \tilde{\psi} = \psi + \delta \psi(x, t, \psi),
\]

then

\[
\int \left[ \frac{\partial L}{\partial \psi_t'} (\psi_t' \delta t + \nabla \psi \cdot \delta x - \delta \psi') + \frac{\partial L}{\partial \psi_t^*} (\psi_t^{*'} \delta t + \nabla \psi^* \cdot \delta x - \delta \psi^*) - L \delta t \right] dx
\]

is a conserved quantity.

For example, the BNLS action integral is invariant under the phase-multiplication \(\psi(t, x) \mapsto e^{i\varepsilon} \psi(t, x)\). In this case, \(\delta t = 0, \delta x = 0, \delta \psi = i \psi\), and so Theorem [16] implies that the integral

\[
\int \left[ \frac{i}{2} \psi^* (\psi_t' \cdot 0 + \nabla \psi^* \cdot 0 - i \psi) - \frac{i}{2} \psi (\psi_t^{*'} \cdot 0 + \nabla \psi^* \cdot 0 + i \psi^*) - L \cdot 0 \right] dx
\]

i.e., the power, is a conserved quantity. Other conservation laws can be found in a similar manner.

**References**

[1] MJ Ablowitz and ZH Musslimani, *Spectral renormalization method for computing self-localized solutions to nonlinear systems*, Opt. Lett., 30 (2005), pp. 2140–2142.

[2] G. Baruch, G. Fibich, and N. Gavish, *Singular standing ring solutions of nonlinear partial differential equations*, Submitted for publication, (2009).

[3] G. Baruch, G. Fibich, and E. Mandelbaum, *Ring-type singular solutions of the biharmonic nonlinear Schrödinger equation*, Preprint, (2009).

[4] M. Ben-Artzi, H. Koch, and J.-C. Saut, *Dispersion estimates for fourth order Schrödinger equations*, C. R. Acad. Sci. Paris Sér. I Math., 330 (2000), pp. 87–92.
REFERENCES

[5] L. BERGÉ AND D. PESME, *Time dependent solutions of wave collapse*, Phys. Lett. A, 166 (1992), pp. 116–122.

[6] J. BOURGAIN, *Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity*, Int Math Res Notices, 1998 (1998), pp. 253–283.

[7] C. J. BUDD, J. F. WILLIAMS, AND V. A. GALAKTIONOV, *Self-similar blow-up in higher-order semilinear parabolic equations*, SIAM Journal of Applied Mathematics, 64 (2004), pp. 1775–1809.

[8] T. CAZENAVE AND F.B. WEISSLER, *The Cauchy problem for the nonlinear Schrödinger equation in $H^1$*, Manuscripta Math, 61 (1988), pp. 477–498.

[9] M. CHAE, S. HONG, J. KIM, S. LEE, AND C. W. YANG, *On mass concentration for the $L^2$-critical nonlinear Schrodinger equations*, Comm. in Partial Diff. Equations, 34 (2009), pp. 486–505(20).

[10] A. DITKOWSKY AND N. GAVISH, *A grid redistribution method for singular problems*, J. Comp. Phys., 228 (2009), pp. 2354–2365.

[11] G. FIBICH AND A. GAETA, *Critical power for self-focusing in bulk media and in hollow waveguides*, Opt. Lett., 25 (2000), pp. 335–337.

[12] GADI FIBICH, BOAZ ILAN, AND GEORGE PAPANICOLAOU, *Self-focusing with fourth-order dispersion*, SIAM J. Applied Math., 62 (2002), pp. 1437–1462.

[13] G. FIBICH AND G.C. PAPANICOLAOU, *Self-focusing in the perturbed and unperturbed nonlinear Schrödinger equation in critical dimension*, SIAM J. Applied Math., 60 (1999), pp. 183–240.

[14] G.M. FRAIMAN, *Asymptotic stability of manifold of self-similar solutions in self-focusing*, Sov. Phys. JETP, 61 (1985), pp. 228–233.

[15] E. GAGLIARDO, *Proprieta di alcune classi di funzioni in piu variabili*, Ricerche di Math., 7 (1958), pp. 102–137.

[16] ———, *Ulteriora proprieta di alcune classi di funzioni in piu variabili*, Ricerche di Math., 8 (1959), pp. 24–51.

[17] Y. GIGA AND R.V. KOHN, *Asymptotically self-similar blow-up of semilinear heat equations*, Comm. on Advances Pure and Appl. Math., 38 (1985), pp. 297–319.

[18] M.J. LANDMAN, G.C. PAPANICOLAOU, C. SULEM, AND P.L. SULEM, *Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension*, Phys. Rev. A, 38 (1988), pp. 3837–3843.

[19] B.J. LEMESURIER, G.C. PAPANICOLAOU, C. SULEM, AND P.L. SULEM, *Local structure of the self-focusing singularity of the nonlinear Schrödinger equation*, Physica D, 32 (1988), pp. 210–226.
REFERENCES

[20] Elad Mandelbaum, Singular solutions of the biharmonic nonlinear Schrödinger equation., master’s thesis, Department of applied Mathematics, Shool of Mathematical Sciences, Tel Aviv University, 2009.

[21] F. Merle, Lower bounds for the blow-up rate of solutions of the Zakharov equation in dimension two, Comm. Pure Appl. Math., 49 (1996), pp. 765–794.

[22] F. Merle and P. Raphael, Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation, Geom. Funct. Anal., 13 (2003), pp. 591–642.

[23] F. Merle and Y. Tsutsumi, $L^2$ concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity, J. Diff. Equ., 84 (1990), pp. 205–214.

[24] Changxing Miao, Guixiang Xu, and Lifeng Zhao, Global wellposedness and scattering for the defocusing energy-critical nonlinear Schrodinger equations of fourth order in dimensions $d \geq 9$, arXiv preprint, (2008).

[25] ———, Global well-posedness and scattering for the focusing energy-critical nonlinear Schrödinger equations of fourth order in the radial case, J. of Differ. Equations, 246 (2009), pp. 3715 – 3749.

[26] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa, 13 (1959), pp. 115–162.

[27] B. Pausader, Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case, Dynamics of PDE, 4 (2007), pp. 197–225.

[28] Benoit Pausader, The cubic fourth-order Schrödinger equation, Journal of Functional Analysis, 256 (2009), pp. 2473 – 2517.

[29] ———, The focusing energy-critical fourth-order Schrödinger equation with radial data, Discrete and continuous dynamical systems, 24 (2009), pp. 1275–1292.

[30] V. I. Petviashvili, Equation of an extraordinary soliton, Sov. J. Plasma Phys., 2 (1976), pp. 469–472.

[31] W. Ren and X.P. Wang, An iterative grid redistribution method for singular problems in multiple dimensions, J. Comput. Phys., 159 (2000), pp. 246–273.

[32] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys., 55 (1977), pp. 149–162.

[33] C. Sulem and P.L. Sulem, The Nonlinear Schrödinger Equation, Springer, New-York, 1999.

[34] V.I. Talanov, Focusing of light in cubic media, JETP Lett., 11 (1970), pp. 199–201.

[35] Y. Tsutsumi, Rate of $L^2$ concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power, Nonlinear-Anal, 15 (1990), pp. 719–724.
[36] M.I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys., 87 (1983), pp. 567–576.

[37] ———, *The nonlinear Schrödinger equations - singularity formation, stability and dispersion*, Contemporary Mathematics, 99 (1989), pp. 213–232.
$\log(\|\psi\|_{\text{max}})^2$ vs. $\log(L)$
$R_{\text{max}} = 5 \times 10^{-6}$
