Particle-based, rapid incremental smoother
meets particle Gibbs

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\textbf{Abstract:} The particle-based, rapid incremental smoother (PARIS) is a sequential Monte Carlo technique allowing for efficient online approximation of expectations of additive functionals under Feynman–Kac path distributions. Under weak assumptions, the algorithm has linear computational complexity and limited memory requirements. It also comes with a number of non-asymptotic bounds and convergence results. However, being based on self-normalised importance sampling, the PARIS estimator is biased; its bias is inversely proportional to the number of particles but has been found to grow linearly with the time horizon under appropriate mixing conditions. In this work, we propose the Parisian particle Gibbs (PPG) sampler, whose complexity is essentially the same as that of the PARIS and which significantly reduces the bias for a given computational complexity at the price of a modest increase in the variance. This method is a wrapper in the sense that it uses the PARIS algorithm in the inner loop of particle Gibbs to form a bias-reduced version of the targeted quantities. We substantiate the PPG algorithm with theoretical results, including new bounds on bias and variance as well as deviation inequalities. We illustrate our theoretical results with numerical experiments supporting our claims.

\textbf{Keywords and phrases:} bias reduction, particle Gibbs samplers, sequential Monte Carlo methods, state-space models, joint smoothing.

1. Introduction

\textit{Feynman–Kac formulas} play a key role in various models used in statistics, physics, and many other fields; see [12, 13, 10] and the references therein. Let \( \{ (X_n, \mathcal{X}_n) \} \in \mathbb{N} \) be a sequence of general state spaces, and define, for every \( n \in \mathbb{N} \), \( X_{0:n} := \prod_{m=0}^{n} X_m \) and \( \mathcal{X}_{0:n} := \bigotimes_{m=0}^{n} \mathcal{X}_m \). For a sequence \( \{ M_n \} \in \mathbb{N} \) of Markov kernels \( M_n : X_n \times \mathcal{X}_{n+1} \rightarrow [0, 1] \), an initial distribution \( \eta_0 \in M_1(X_0) \), and a sequence \( \{ g_n \} \in \mathbb{N} \) of bounded measurable potential functions \( g_n : X_n \rightarrow \mathbb{R}_+ \), a sequence \( \{ \eta_{k:n} \} \in \mathbb{N} \) of Feynman–Kac path measures is defined by

\[ \eta_{0:n} : \mathcal{X}_{0:n} \ni A \mapsto \frac{\gamma_{0:n}(A)}{\eta_{0:n}(\mathcal{X}_{0:n})}, \quad n \in \mathbb{N}, \quad \text{(1.1)} \]

where

\[ \gamma_{0:n} : \mathcal{X}_{0:n} \ni A \mapsto \int 1_A(x_{0:n}) \eta_0(dx_0) \prod_{m=1}^{n-1} Q_m(x_m, dx_{m+1}), \quad \text{(1.2)} \]
with
\[ Q_m : \mathcal{X}_m \times \mathcal{X}_{m+1} \ni (x, A) \mapsto g_m(x)M_m(x, A) \] (1.3)
being unnormalised kernels. By convention, \( \eta_{0:0} := \eta_0 \). Note that each \( \eta_{0:n} \) is a probability measure, while \( \gamma_{0:n} \) is not normalised. For every \( n \in \mathbb{N}^* \) we also define the marginal distribution \( \eta_n : \mathcal{X}_n \ni A \mapsto \eta_{0:n}(X_{0:n-1} \times A) \). In the context of nonlinear filtering in general state-space hidden Markov models, \( \eta_{0:n} \) is the joint-smoothing distribution and \( \eta_n \) is the filter distribution; see [12, 9, 10].

For most problems of interest in practice, the Feynman–Kac path or marginal measures are intractable, and so is also any expectation associated with the same. Therefore, \textit{particle approximations} of such measures have been developed. A particle filter approximates the flow \( \{\eta_n\}_{n \in \mathbb{N}} \) of marginals by a sequence of occupation measures associated with a swarm of particles \( \{\xi_n^i\}_{i=1}^N, n \in \mathbb{N} \), each particle \( \xi_n^i \) being a random draw in \( \mathcal{X}_n \). Particle filters revolve around two operations: a \textit{selection step} duplicating/discarding particles with large/small importance weights, respectively, and a \textit{mutation step} evolving randomly the selected particles in the state space. Applying alternatingly and iteratively selection and mutation results in a swarm of \( N \) particles that are both serially and spatially dependent. Feynman–Kac path models can also be interpreted as laws associated with a certain kind of Markovian backward dynamics; this interpretation is useful, for instance, for the smoothing problem in nonlinear filtering [17, 14]. Several convergence results have been proved when the number \( N \) of particles tends to infinity; see, e.g., [12, 18, 13, 10]. A number of non-asymptotic results have also been established, including the bias of the particle approximation of the Feynman–Kac formula and associated chaos propagation. Extensions to backward interpretation can also be found in [17, 14].

In this paper, we focus on the problem of recursively computing smoothed expectations
\[ \eta_{0:n} h_n = \int h_n(x_{0:n}) \eta_{0:n}(dx_{0:n}), \quad n \in \mathbb{N}, \]
where we have introduced the vector notation \( x_{0:n} = (x_0, \ldots, x_n) \in \mathcal{X}_{0:n} := \mathcal{X}_0 \times \cdots \times \mathcal{X}_n \), for additive functionals \( h_n \) in the form
\[ h_n(x_{0:n}) := \sum_{m=0}^{n-1} \hat{h}_m(x_{m:m+1}), \quad x_{0:n} \in \mathcal{X}_{0:n}. \] (1.4)
In nonlinear filtering problems, such expectations appear in the context of maximum-likelihood parameter estimation, for instance, when computing the score function (the gradient of the log-likelihood function) or the EM surrogate; see [5, 3, 25, 6, 26]. In [24], an efficient \textit{particle-based, rapid incremental smoother} (PARIS) was proposed. PARIS has linear computational complexity in the number of particles under weak assumptions and limited memory requirements. An interesting feature of PARIS, which samples on-the-fly from the backward dynamics induced by the particle filter, is that it requires two or more backward draws per particle to cope with the degeneracy of the sampled trajectories and remain numerically stable in the long run, with an asymptotic variance that grows only linearly with time.

In this paper, we highlight a method to reduce the bias of the PARIS estimator of \( \eta_{0:n} h_n \). The idea is to mix the PARIS algorithm (to introduce a conditional PARIS algorithm) and a version of the particle Gibbs algorithm with backward sampling [4,
In Section 3 we introduce the PARISian particle Gibbs (PPG) algorithm, for which we provide an upper bound of the bias which decreases inversely proportionally to the number of particles and exponentially fast with the iteration index (under the assumption that the particle Gibbs sampler is uniformly ergodic).

The paper is structured as follows. In Section 2 we recall the Feynman–Kac model and its backward interpretation and introduce the particle Gibbs. Our presentation is inspired by [16], but differs in that it avoids the use of quotient spaces, while it avoids and its backward interpretation and introduce the particle Gibbs. Our presentation is that the particle Gibbs sampler is uniformly ergodic.

Notation. Let \( \mathbb{R}_+ := [0, \infty) \), \( \mathbb{R}_+^* := (0, \infty) \), \( \mathbb{N} := \{0, 1, 2, \ldots\} \), and \( \mathbb{N}^* := \{1, 2, 3, \ldots\} \) denote the sets of nonnegative and positive real numbers and integers, respectively. We denote by \( \text{Id}_X \) the \( N \times N \) identity matrix. For any quantities \( \{a_{\ell}\}_{\ell=m}^n \) we denote vectors as \( a_{m:n} := (a_m, \ldots, a_n) \) and for any \( (m, n) \in \mathbb{N}^2 \) such that \( m \leq n \) we let \( [m, n] := \{m, m+1, \ldots, n\} \). For a given measurable space \((X, \mathcal{X})\), where \( \mathcal{X} \) is a countably generated \( \sigma \)-algebra, we denote by \( \mathcal{F}(\mathcal{X}) \) the set of bounded \( \mathcal{X}/\mathcal{B}(\mathbb{R}) \)-measurable functions on \( X \). For any \( h \in \mathcal{F}(\mathcal{X}) \), we let \( \|h\|_\infty := \sup_{x \in X} |h(x)| \) and \( \text{osc}(h) := \sup_{(x, x') \in \mathcal{X}^2} |h(x) - h(x')| \) denote the supremum and oscillator norms of \( h \), respectively. Let \( \mathcal{M}(\mathcal{X}) \) be the set of \( \sigma \)-finite measures on \((X, \mathcal{X})\) and \( \mathcal{M}_1(\mathcal{X}) \subset \mathcal{M}(\mathcal{X}) \) the probability measures.

Let \((Y, \mathcal{Y})\) be another measurable space. A possibly unnormalised transition kernel \( K \) on \( X \times Y \) induces two integral operators, one acting on measurable functions and the other on measures; more specifically, for \( h \in \mathcal{F}(\mathcal{X} \otimes Y) \) and \( \nu \in \mathcal{M}_1(\mathcal{X}) \), define the measurable function
\[
Kh : X \ni x \mapsto \int h(x, y) K(x, dy)
\]
and the measure
\[
\nu K : Y \ni A \mapsto \int K(x, A) \nu(dx),
\]
whenever these quantities are well defined. Now, let \((Z, \mathcal{Z})\) be a third measurable space and \( L \) another possibly unnormalised transition kernel on \( Y \times Z \); we then define, with \( K \) as above, two different products of \( K \) and \( L \), namely
\[
KL : X \times Z \ni (x, A) \mapsto \int L(y, A) K(x, dy)
\]
and
\[
K \otimes L : X \times (Y \times Z) \ni (x, A) \mapsto \iint 1_A(y, z) K(x, dy) L(y, dz),
\]
whenever these are well defined. This also defines the \( \otimes \) products of a kernel \( K \) on \( X \times Y \) and a measure \( \nu \) on \( X \) as well as of a kernel \( L \) on \( Y \times X \) and a measure \( \mu \) on \( Y \) as the measures

\[
\nu \otimes K : X \otimes Y \ni A \mapsto \int \int_{A} 1_{A}(x, y) K(x, dy) \nu(dx),
\]

\[
L \otimes \mu : X \otimes Y \ni A \mapsto \int \int_{A} 1_{A}(x, y) L(y, dx) \mu(dy).
\]

2. Feynman–Kac models

In the next sections, we recall Feynman–Kac models, many-body Feynman–Kac models, backward interpretations, and conditional dual processes. Our presentation follows closely [16] but with a different and hopefully more transparent definition of the many-body extensions. We restate (in Theorem 1 below) a duality formula of [16] relating these concepts. This formula provides a foundation for the particle Gibbs sampler described in Section 2.3 and is pivotal for the coming developments.

2.1 Many-body Feynman–Kac models

In the following we assume that all random variables are defined on a common probability space \((\Omega, \mathcal{F}, P)\). The distribution flow \( \{\eta_{m}\}_{m \in \mathbb{N}} \) is intractable in general, but can be approximated by random samples \( \xi_{m} = (\xi_{m}^{i})_{i=1}^{N}, m \in \mathbb{N}, \) of particles, where \( N \in \mathbb{N}^{*} \) is a fixed Monte Carlo sample size and each particle \( \xi_{m}^{i} \) is an \( X_{m} \)-valued random variable. Such particle approximation is based on the recursion \( \eta_{m+1} = \Phi_{m}(\eta_{m}), m \in \mathbb{N}, \) where \( \Phi_{m} \) denotes the mapping

\[
\Phi_{m} : M_{1}(X_{m}) \ni \eta \mapsto \frac{\eta_{Q_{m}}}{\eta_{g_{m}}}
\]

taking on values in \( M_{1}(X_{m+1}) \). In order to describe recursively the evolution of the particle population, let \( m \in \mathbb{N} \) and assume that the particles \( \{\xi_{m}^{i}\}_{i=1}^{N} \) form a consistent approximation of \( \eta_{m} \) in the sense that \( \mu(\xi_{m})h \), where \( \mu(\xi_{m}) := \sum_{i=1}^{N} \delta_{\xi_{m}^{i}}/N \) (where \( \delta_{x} \) denotes the Dirac measure located at \( x \)) is the occupation measure formed by \( \xi_{m} \), which serves as a proxy for \( \eta_{m}h \) for any \( \eta_{m} \)-integrable test function \( h \). (Under general conditions, \( \mu(\xi_{m})h \) converges in probability to \( \eta_{m} \) with \( N \to \infty \); see [12, 10] and references therein.) Then, in order to generate an updated particle sample approximating \( \eta_{m+1} \), new particles \( \xi_{m+1} = \{\xi_{m+1}^{i}\}_{i=1}^{N} \) are drawn conditionally independently given \( \xi_{m} \) according to

\[
\xi_{m+1}^{i} \sim \Phi_{m}(\mu(\xi_{m}^{i})) = \frac{\sum_{i=1}^{N} g_{m}(\xi_{m}^{i})}{\sum_{i'=1}^{N} g_{m}(\xi_{m}^{i'})} M_{m}(\xi_{m}^{i'}, \cdot), \quad i \in [1, N].
\]

Since this process of particle updating involves sampling from the mixture distribution \( \Phi_{m}(\mu(\xi_{m}^{i})) \), it can be naturally decomposed into two substeps: selection and mutation. The selection step consists of randomly choosing the \( \ell \)-th mixture stratum with probability \( g_{m}(\xi_{m}^{i})/\sum_{i'=1}^{N} g_{m}(\xi_{m}^{i'}) \) and the mutation step consists of drawing a new particle \( \xi_{m+1}^{i} \) from the selected stratum \( M_{m}(\xi_{m}^{i}, \cdot) \). In [16], the term many-body Feynman–Kac models...
is related to the law of process \( \{\xi_m\}_{m \in \mathbb{N}} \). For all \( m \in \mathbb{N} \), let \( X_m := X_m^N \) and \( X := X^\otimes N \); then \( \{\xi_m\}_{m \in \mathbb{N}} \) is an inhomogeneous Markov chain on \( \{X_m\}_{m \in \mathbb{N}} \) with transition kernels

\[
M_m : X_m \times X_{m+1} \ni (x_m, A) \mapsto \Phi_m(\mu(x_m))^\otimes N(A)
\]

and initial distribution \( \eta_0 = \eta_0^\otimes N \). Now, denote \( \xi_{0:n} := \prod_{m=0}^n X_m \) and \( \mathcal{X}_{0:n} := \otimes_{m=0}^n \mathcal{X}_m \). (Here and in the following we use a bold symbol to stress that a quantity is related to the many-body process.) The \textit{many-body Feynman–Kac path model} refers to the flows \( \{\gamma_m\}_{m \in \mathbb{N}} \) and \( \{\eta_m\}_{m \in \mathbb{N}} \) of the unnormalised and normalised, respectively, probability distributions on \( \{\mathcal{X}_{0:m}\}_{m \in \mathbb{N}} \) generated by (1.1) and (1.2) for the Markov kernels \( \{M_m\}_{m \in \mathbb{N}} \), the initial distribution \( \eta_0 \), the potential functions

\[
g_m : X_m \ni x_m \mapsto \mu(x_m)g_m = \frac{1}{N} \sum_{i=1}^N g_m(x_m^i), \quad m \in \mathbb{N},
\]

and the corresponding unnormalised transition kernels

\[
Q_m : X_m \times X_{m+1} \ni (x_m, A) \mapsto g_m(x_m)M_m(x_m, A), \quad m \in \mathbb{N}.
\]

### 2.2 Backward interpretation of Feynman–Kac path flows

Suppose that each kernel \( Q_n, n \in \mathbb{N} \), defined in (1.3), has a transition density \( q_n \) with respect to some dominating measure \( \lambda_{n+1} \in M(\mathcal{X}_{n+1}) \). Then for \( n \in \mathbb{N} \) and \( \eta \in M_1(\mathcal{X}_n) \) we may define the \textit{backward kernel}

\[
\hat{Q}_{n,\eta} : \mathcal{X}_{n+1} \times \mathcal{X}_n \ni (x_{n+1}, A) \mapsto \frac{\int 1_A(x_n)q_n(x_n, x_{n+1}) \eta(dx_n)}{\int q_n(x_n^i, x_{n+1}) \eta(dx_n^i)}.
\]  

(2.2)

Now, denoting, for \( n \in \mathbb{N}^+ \),

\[
B_n : \mathcal{X}_n \times \mathcal{X}_{0:n-1} \ni (x_n, A) \mapsto \int \cdots \int 1_A(x_{0:n-1}) \prod_{m=0}^{n-1} \hat{Q}_{m,\eta_m}(x_{m+1}, dx_m),
\]  

(2.3)

we may state the following—now classical—\textit{backward decomposition} of the Feynman–Kac path measures, a result that will play a pivotal role in the following.

**Proposition 1.** For every \( n \in \mathbb{N}^+ \) it holds that \( \gamma_{0:n} = \gamma_n \otimes B_n \) and \( \eta_{0:n} = \eta_n \otimes B_n \).

Although the decomposition in Proposition 1 is well known (see, e.g., [14, 16]), we provide a proof in Section 5.1 for completeness. Using the backward decomposition, one can obtain a particle approximation of a given Feynman–Kac path measure \( \eta_{0:n} \) by first sampling, in an initial forward pass, particle clouds \( \{\xi_m\}_{m=0}^n \) from \( \eta_0 \otimes M_0 \otimes \cdots \otimes M_{n-1} \) and then sampling, in a subsequent backward pass, say \( N \) conditionally independent paths \( \{\xi_{i,n}\}_{i=1}^N \) from \( \mathbb{B}_n(\xi_0, \ldots, \xi_n, \cdot) \), where

\[
\mathbb{B}_n : \mathcal{X}_{0:n} \times \mathcal{X}_{0:n} \ni (x_{0:n}, A) \mapsto \int \cdots \int 1_A(x_{0:n}) \left( \prod_{m=0}^{n-1} \hat{Q}_{m,\mu(x_m)}(x_{m+1}, dx_m) \right) \mu(x_n)(dx_n)
\]  

(2.4)
is a Markov kernel describing the time-reversed dynamics induced by the particle approximations generated in the forward pass. (Here and in the following we use blackboard notation to denote kernels related to many-body path spaces.) Finally, \( \mu(\{\tilde{\xi}^i_{0:n}\}) \) is returned as an estimator of \( \eta_{0:n} \) for any \( \eta_{0:n} \)-integrable test function \( h \). This algorithm is in the literature referred to as the forward–filtering backward–simulation (FFBSi) algorithm and was introduced in [21]; see also [7, 17]. More precisely, given the forward particles \( \{\xi_m\}_{m=0}^n \), each path \( \tilde{\xi}^i_n \) is generated by first drawing \( \tilde{\xi}^i_n \) uniformly among the particles \( \xi_n \) in the last generation and then drawing, recursively,

\[
\tilde{\xi}^i_m \sim \tilde{Q}_{m,\mu(\xi_n)}(\tilde{\xi}^i_{m+1}, \cdot) = \sum_{j=1}^{N} \frac{q_m(\tilde{\xi}^j_m, \tilde{\xi}^j_{m+1})}{\sum_{t=1}^{N} q_m(\tilde{\xi}^j_m, \tilde{\xi}^j_{m+1})} \delta_{\tilde{\xi}^i_m}, \tag{2.5}
\]

i.e., given \( \tilde{\xi}^i_{m+1} \), \( \tilde{\xi}^i_m \) is picked at random among the \( \xi_m \) according to weights proportional to \( \{q_m(\tilde{\xi}^j_m, \tilde{\xi}^j_{m+1})\}_{j=1}^N \). Note that in this basic formulation of the FFBSi algorithm, each backward-sampling operation (2.5) requires the computation of the normalising constant \( \sum_{j=1}^{N} q_m(\tilde{\xi}^j_m, \tilde{\xi}^j_{m+1}) \), which implies an overall quadratic complexity of the algorithm. Still, this heavy computational burden can eased by means of an effective accept–reject technique discussed in Section 2.4.

### 2.3 Conditional dual processes and particle Gibbs

The dual process associated with a given Feynman–Kac model (1.1–1.2) and a given trajectory \( \{z_n\}_{n \in \mathbb{N}} \), where \( z_n \in \mathcal{X}_n \) for every \( n \in \mathbb{N} \), is defined as the canonical Markov chain with kernels

\[
M_n(z_{n+1}) : \mathcal{X}_n \times \mathcal{X}_{n+1} \ni (x_n, A) \mapsto \frac{1}{N} \sum_{i=0}^{N-1} \left( \Phi_n(\mu(x_n))^{\otimes i} \otimes \delta_{z_{n+1}} \otimes \Phi_n(\mu(x_n))^{\otimes (N-i-1)} \right) (A), \tag{2.6}
\]

for \( n \in \mathbb{N} \), and initial distribution

\[
\eta_0(z_0) := \frac{1}{N} \sum_{i=0}^{N-1} \left( \eta_0^{\otimes i} \otimes \delta_{z_0} \otimes \eta_0^{\otimes (N-i-1)} \right). \tag{2.7}
\]

As clear from (2.6–2.7), given \( \{z_n\}_{n \in \mathbb{N}} \), a realisation \( \{\xi_n\}_{n \in \mathbb{N}} \) of the dual process is generated as follows. At time zero, the process is initialised by inserting \( z_0 \) at a randomly selected position in the vector \( \xi_0 \) while drawing independently the remaining elements in the same vector from \( \eta_0 \). After this, the process proceeds in a Markovian manner by, given \( \xi_n \), inserting \( z_{n+1} \) at a randomly selected position in \( \xi_{n+1} \) while drawing independently the remaining elements from \( \Phi_n(\mu(\xi_n)) \).

In order to describe compactly the law of the conditional dual process we define the Markov kernel

\[
C_n : \mathcal{X}_{0:n} \times \mathcal{X}_{0:n} \ni (z_{0:n}, A) \mapsto \eta_0(z_0) \otimes M_0(z_1) \otimes \cdots \otimes M_{n-1}(z_n)(A).
\]

The following result elegantly combines the underlying model (1.1–1.2), the many-body Feynman–Kac model, the backward decomposition, and the conditional dual process.
Theorem 1 ([16]). For all $n \in \mathbb{N}$ it holds that
\[
B_n \otimes \gamma_{0:n} = \gamma_{0:n} \otimes C_n.
\] (2.8)

In [16], each state $\xi_n$ of the many-body process maps an outcome $\omega$ of the sample space $\Omega$ into an unordered set of $N$ elements in $X_n$. However, we have chosen to let each $\xi_n$ take on values in the standard product space $X_n^N$ for two reasons: first, the construction of [16] requires sophisticated measure-theoretic arguments to endow such unordered sets with suitable $\sigma$-fields and appropriate measures; second, we see no need to ignore the index order of the particles as long as the Markovian dynamics (2.6–2.7) of the conditional dual process is symmetrised over the particle cloud. Therefore, in Section 5.2, we include our own proof of duality (2.8) for completeness. Note that the measure (2.8) on $X_0:n \otimes X_0:n$ is unnormalised, but since the kernels $B_n$ and $C_n$ are both Markov, normalising the identity with $\gamma_{0:n}(X_0:n) = \gamma_{0:n}(X_0:n)$ yields immediately
\[
B_n \otimes \eta_{0:n} = \eta_{0:n} \otimes C_n.
\] (2.9)

Since the two sides of (2.9) provide the full conditionals, it is natural to take a data-augmentation approach and sample the target (2.9) using a two-stage deterministic-scan Gibbs sampler [4, 11]. More specifically, assume that we have generated a state $(\xi_{0:n}[\ell], \zeta_{0:n}[\ell])$ comprising a dual process with associated path on the basis of $\ell \in \mathbb{N}$ iterations of the sampler; then the next state $(\xi_{0:n}[\ell + 1], \zeta_{0:n}[\ell + 1])$ is generated in a Markovian fashion by, first, sampling $\xi_{0:n}[\ell + 1] \sim C_n(\zeta_{0:n}[\ell], \cdot)$ and, second, sampling $\zeta_{0:n}[\ell + 1] \sim B_n(\xi_{0:n}[\ell + 1], \cdot)$. After arbitrary initialisation (and the discard of possible burn-in), this procedure produces a Markov trajectory $\{(\xi_{0:n}[\ell], \zeta_{0:n}[\ell])\}_{\ell \in \mathbb{N}}$, and under weak additional technical conditions this Markov chain admits (2.9) as its unique invariant distribution. In such case, the Markov chain is ergodic [19, Chapter 5], and the marginal distribution of the conditioning path $\zeta_{0:n}[\ell]$ converges to the target distribution $\eta_{0:n}$. Therefore, for every $h \in F(X'_{0:n}),$ it holds that
\[
\lim_{L \to \infty} \frac{1}{L} \sum_{\ell=1}^{L} h(\zeta_{0:n}[\ell]) = \eta_{0:n} h, \quad \mathbb{P}\text{-a.s.}
\]

2.4 The PARIS algorithm

In the following we assume that we are given a sequence $\{h_n\}_{n \in \mathbb{N}}$ of additive state functionals of type (1.4). This problem is particularly relevant in the context of maximum-likelihood-based parameter estimation in general state-space models, e.g., when computing the score-function (the gradient of the log-likelihood function) via the Fisher identity or when computing the intermediate quantity of the expectation–maximization (EM) algorithm, in which case $\eta_{0:n}$ and $h_n$ correspond to the joint state posterior and an element of some sufficient statistic, respectively; see [8, 17, 14, 26, 24] and the references therein.

Interestingly, as noted in [6, 14], the backward decomposition allows, when applied to additive state functionals, a forward recursion for the expectations $\{\eta_{0:n} h_n\}_{n \in \mathbb{N}}$. More specifically, using the forward decomposition $h_{n+1}(x_{0:n+1}) = h_n(x_{0:n}) + h_n(x_n, x_{n+1})$ and
and, finally, the estimator

\[ \eta_{0:n+1}h_{n+1} = \eta_{n+1}Q_n \eta_n(B_nh_n + \tilde{h}_n). \]  

(2.11)

Since, as we have seen, the marginal flow \( \{\eta_n\}_{n \in \mathbb{N}} \) can be expressed recursively via the mappings \( \{\Phi_n\}_{n \in \mathbb{N}} \), (2.11) provides, in principle, a basis for online computation of \( \{\eta_{0:n}h_n\}_{n \in \mathbb{N}} \). To handle the fact that the marginals are generally intractable we may, following [14], plug particle approximations \( \mu(\xi_{n+1}) \) and \( \tilde{Q}_{n,\mu(\xi_n)} \) (see (2.5)) of \( \eta_{n+1} \) and \( \tilde{Q}_{n,\mu(\eta_n)} \), respectively, into the recursion (2.11). More precisely, we proceed recursively and assume that we at time \( n \) have at hand a sample \( \{(\xi_n^i, \beta_n^i)\}_{i=1}^N \) of particles with associated statistics, where each statistic \( \beta_n^i \) serves as an approximation of \( B_nh_n(\xi_n^i) \); then evolving the particle cloud according to \( \xi_{n+1} \sim M_n(\xi_n, \cdot) \) and updating the statistics using (2.10), with \( \tilde{Q}_{n,\eta_n} \) replaced by \( \tilde{Q}_{n,\mu(\xi_n)} \), yields the particle-wise recursion

\[ \beta_n^{i+1} = \sum_{t=1}^N \frac{q_n(\xi^t_n, \xi^t_{n+1})}{\sum_{\ell=1}^N q_n(\xi^\ell_n, \xi^\ell_{n+1})} \left( \beta_n^i + \tilde{h}_n(\xi_n^i, \xi_{n+1}^i) \right), \quad i \in \{1, N\}, \]  

(2.12)

and, finally, the estimator

\[ \mu(\beta_n) = \frac{1}{N} \sum_{i=1}^N \beta_n^i. \]  

(2.13)

of \( \eta_{0:n}h_n \), where we have set \( \beta_n := (\beta_n^1, \ldots, \beta_n^N), \quad i \in \{1, N\} \). The procedure is initialised by simply letting \( \beta_0^i = 0 \) for all \( i \in \{1, N\} \). Note that (2.13) provides a particle interpretation of the backward decomposition in Proposition 1. This algorithm is a special case of the forward-filtering backward-smoothing (FFBSm) algorithm (see [3, 21, 17, 27]) for additive functionals satisfying (1.4). It allows for online processing of the sequence \( \{\eta_{0:n}h_n\}_{n \in \mathbb{N}} \), but has also the appealing property that only the current particles \( \xi_n^i \) and statistics \( \beta_n^i \) need to be stored in the memory. However, since each update (2.12) requires the summation of \( N \) terms, the scheme has an overall quadratic complexity in the number of particles, leading to a computational bottleneck in applications to complex models that require large particle sample sizes \( N \).

In order to detour the computational burden of this forward-only implementation of FFBSm, the PARIS algorithm [24] updates the statistics \( \beta_n^i \) by replacing each sum (2.12) by a Monte Carlo estimate

\[ \beta_{n+1}^i = \frac{1}{M} \sum_{j=1}^M \left( \beta_{n+1}^{i,j} + \tilde{h}_n(\xi_n^{i,j}, \xi_{n+1}^i) \right), \quad i \in \{1, N\}, \]  

(2.14)

where \( \{(\xi_n^{i,j}, \beta_{n+1}^{i,j})\}_{j=1}^M \) are drawn randomly among \( \{(\xi_n^i, \beta_n^i)\}_{i=1}^N \) with replacement, by assigning \( (\xi_n^{i,j}, \beta_{n+1}^{i,j}) \) the value of \( (\xi_n^i, \beta_n^i) \) with probability \( q_n(\xi_n^i, \xi_{n+1}^i) / \sum_{\ell=1}^N q_n(\xi_n^\ell, \xi_{n+1}^\ell) \).
and the Monte Carlo sample size \( M \in \mathbb{N}^* \) is supposed to be much smaller than \( N \) (say, less than 5). Formally,

\[
\{(\hat{\xi}_i^1, \hat{\beta}_i^1)\}_j^M \sim \left( \sum_{\ell=1}^N q_n(\xi_{i,\ell}^\ell, \xi_{n+1}^\ell) \delta_{i}^{(\xi_{i,\ell}^\ell, \beta_i^\ell)} \right)^{\otimes M}, \quad i \in \llbracket 1, N \rrbracket.
\]

The resulting procedure, summarised in Algorithm 1, allows for online processing with constant memory requirements, since it only needs to store the current particle cloud and the estimated auxiliary statistics at each iteration. Moreover, in the case where the Markov transition densities of the model can be uniformly bounded, i.e., there exists, for every \( n \in \mathbb{N} \), an upper bound \( \bar{\sigma}_n > 0 \) such that for all \((x_n, x_{n+1}) \in X_n \times X_{n+1}\), \( m_n(x_n, x_{n+1}) \leq \bar{\sigma}_n \) (a weak assumption satisfied for most models of interest), a sample \( \{(\xi_{i,j}^1, \beta_{i,j}^1)\} \) can be generated by drawing, with replacement and until acceptance, candidates \( \{(\xi_{i,j}^1, \beta_{i,j}^1)\} \) from \( \{(\xi_{i,j}^1, \beta_{i,j}^1)\}_{i=1}^N \) according to the normalised particle weights \( g_n(\xi_{i,j}^1) / \sum_{\ell \sim \ell'} q_n(\xi_{i,j}^\ell) \}_{i=1}^N \) (obtained as a by-product in the generation of \( \xi_{n+1}^1 \) and accepting the same with probability \( m_n(\xi_{i,j}^1, \beta_{i,j}^1) / \bar{\sigma}_n \). As this sampling procedure bypasses completely the calculation of the normalising constant \( \sum_{\ell=1}^N q_n(\xi_{i,j}^\ell) \) of the targeted categorical distribution, it yields an overall \( \mathcal{O}(MN) \) complexity of the algorithm as a whole; see [17] for details.

Increasing \( M \) improves the accuracy of the algorithm at the cost of additional computational complexity. As shown in [24], there is a qualitative difference between the cases \( M = 1 \) and \( M \geq 2 \), and it turns out that the latter is required to keep PARIS numerically stable. More precisely, in the latter case, it can be shown that the PARIS estimator \( \mu(\beta_n^1) \) satisfies, as \( N \) tends to infinity while \( M \) is held fixed, a central limit theorem (CLT) at the rate \( \sqrt{N} \) and with an \( n \)-normalised asymptotic variance of order \( \mathcal{O}(1 - 1/(M - 1)) \). As clear from this bound, using a large \( M \) is only to waste the computational work, and setting \( M \) to 2 or 3 typically works well in practice.

3. The Parisian particle Gibbs (PPG) sampler

We now introduce the Parisian particle Gibbs (PPG) algorithm. For all \( n \in \mathbb{N}^* \), let \( Y_n := X_{0:n} \times \mathbb{R} \) and \( \mathcal{Y}_n := X_{0:n} \otimes \mathcal{B}(\mathbb{R}) \). Moreover, let \( Y_0 := X_0 \times \{ \emptyset \} \) and \( \mathcal{Y}_0 := X_0 \otimes \{ \emptyset \} \). An element of \( Y_n \) will always be denoted by \( y_n = (x_{0:n}, b_n) \). The Parisian particle Gibbs sampler comprises, as a key ingredient, a conditional PARIS step, which updates recursively a set of \( \mathcal{Y}_n \)-valued random variables \( \nu_n := (\xi_{0:n}^i, \beta_{0:n}^i), i \in \llbracket 1, N \rrbracket \). Let \( (\nu_n)_{n \in \mathbb{N}} \) denote the corresponding many-body process, each \( \nu_n := \{ (\xi_{0:n}^i, \beta_{0:n}^i) \}_{i=1}^N \) taking on values in the space \( Y_n := Y_n^N \), which we furnish with a \( \sigma \)-field \( \mathcal{Y}_n := \mathcal{Y}_n^\otimes N \). The space \( \mathcal{Y}_0 \) and the corresponding \( \sigma \)-field \( \mathcal{Y}_0 \) are defined accordingly. For every \( n \in \mathbb{N} \), we write \( \xi_{0:n}^i \) for the collection \( \{ \xi_{0:n}^i \}_{i=1}^N \) of paths in \( \nu_n \), and \( \xi_{n,n}^i \) for the collection \( \{ \xi_{n,n}^i \}_{i=1}^N \) of end points of the same.

In the following we let \( n \in \mathbb{N} \) be a fixed time horizon, and describe in detail how the PPG approximates \( \eta_{0:n} h_n \) iteratively. In short, at each iteration \( \ell \), the PPG produces, given an input conditional path \( \xi_{0:n}^{\ell} \), a many-body system \( \nu_n^{\ell + 1} \) by means of a series of conditional PARIS operations; after this, an updated path \( \xi_{0:n}^{\ell + 1} \), serving as input at the next iteration, is generated by picking one of the paths \( \xi_{0:n}^{\ell + 1} \) in
\(\nu_n[\ell + 1]\) at random. At each iteration, the produced statistics \(\beta_n\) (in \(\nu_n\)) provides an approximation of \(\eta_{h_n}\) according to (2.13).

More precisely, given the path \(\zeta_{\nu_n}[\ell]\), the conditional PARIS operations are executed as follows. In the initial step, \(\xi_{[0]}[\ell + 1]\) are drawn from \(\eta_{0}\) defined in (2.7) and \(v_{[0]}[\ell + 1] \leftarrow (\xi_{[0]}[\ell + 1], 0)\) for all \(i \in [1, N]\); then, recursively for \(m \in [0, n]\), assuming access to \(\nu_m[\ell + 1]\), we

1. generate an updated particle cloud \(\xi_{m+1}[\ell + 1] \sim \mathcal{M}_m(\zeta_{m+1}[\ell])\),
2. pick at random, for each \(i \in [1, N]\), an ancestor path with associated statistics \((\tilde{\xi}_{m+1}^i[\ell + 1], \tilde{\beta}_{m+1}^i[\ell + 1])\) among \(\nu_m[\ell + 1]\) by drawing \n\[(\tilde{\xi}_{m}^i[\ell + 1], \tilde{\beta}_{m}^i[\ell + 1]) \sim \sum_{s=1}^{N} \frac{q_m(\xi_{m}^s[\ell + 1], \xi_{m+1}^i[\ell + 1])}{\sum_{s'=1}^{N} q_m(\xi_{m}^{s'}[\ell + 1], \xi_{m+1}^i[\ell + 1])} \delta_{\nu_m^i[\ell + 1]}, \quad i \in [1, N],\]
3. draw, with replacement, \(M - 1\) ancestor particles and associated statistics \(\{(\tilde{\xi}_{m}^i[\ell + 1], \tilde{\beta}_{m}^i[\ell + 1])\}_{j=1}^{M}\) at random from \(\{(\xi_{m}^s[\ell + 1], \beta_{m}^s[\ell + 1])\}_{s=1}^{N}\) according to
\[\{(\tilde{\xi}_{m}^i[\ell + 1], \tilde{\beta}_{m}^i[\ell + 1])\}_{j=1}^{M} \sim \left( \sum_{s=1}^{N} \frac{q_m(\xi_{m}^s[\ell + 1], \xi_{m+1}^i[\ell + 1])}{\sum_{s'=1}^{N} q_m(\xi_{m}^{s'}[\ell + 1], \xi_{m+1}^i[\ell + 1])} \delta_{\xi_{m}^s[\ell + 1], \beta_{m}^s[\ell + 1]} \right)^{\otimes(M-1)},\]
4. set, for all \(i \in [1, N]\), \(\xi_{[m+1]}^i[\ell + 1] \leftarrow (\tilde{\xi}_{m}^i[\ell + 1], \xi_{m+1}^i[\ell + 1])\) and \(v_{[m+1]}[\ell + 1] \leftarrow (\tilde{\xi}_{m}^i[\ell + 1], \tilde{\beta}_{m}^i[\ell + 1])\), where
\[\tilde{\beta}_{m+1}^i[\ell + 1] \leftarrow M^{-1} \sum_{j=1}^{M} \left( \tilde{\beta}_{m}^j[\ell + 1] + h_m(\tilde{\xi}_{m}^j[\ell + 1], \xi_{m+1}^i[\ell + 1]) \right).
\]

This conditional PARIS procedure is summarised in Algorithm 2.

Once the set of trajectories and associated statistics \(\nu_n[\ell + 1]\) is formed by means of \(n\) recursive conditional PARIS updates, an updated path \(\zeta_{\nu_n}[\ell + 1]\) is drawn from \(\mu(\zeta_{\nu_n}[\ell + 1])\). A full sweep of the PPG is summarised in Algorithm 3.

The following Markov kernels will play an instrumental role in the following. For a given path \(\{z_m\}_{m \in \mathbb{N}}\), the conditional PARIS update in Algorithm 2 defines an inhomogeneous Markov chain on the spaces \(\{(Y_m, Y_{m+1})\}_{m \in \mathbb{N}}\) with kernels

\[Y_m \times Y_{m+1} \ni (y_m, A) \mapsto \int M_m(z_{m+1})(x_{m|m}, dx_{m+1}) S_m(y_m, x_{m+1}, A), \quad m \in \mathbb{N},\]
where

\[ S_m : \mathcal{Y}_m \times \mathcal{X}_{m+1} \times \mathcal{Y}_{m+1} \ni (y_m, x_{m+1}, A) \]

\[ \mapsto \int \ldots \int 1_A \left( \left\{ \left( x_{0:m}^i, x_{m+1}^i \right) : \sum_{j=1}^M \left( \tilde{b}_{m}^{i,j} + \tilde{h}_m(x_m, x_{m+1}) \right) \right\} \right) \]

\[ \times \prod_{i=1}^N \left( \sum_{\ell=1}^N \sum_{j=1}^M q_m(x_{m}^\ell, x_{m+1}^j) \delta_{\tilde{s}_{m}^{i}}(d(x_0, \tilde{b}_{m}^{i,1})) \delta_{\tilde{b}_{m}^{i,j}}(x_{m}^{\ell}, x_{m+1}^j) \right) \]

\[ \otimes (M-1) \]

Proposition 2. For all \( n \in \mathbb{N}^* \), \( N \in \mathbb{N}^* \), \( x_{0:n} \in \mathcal{X}_{0:n} \), and \( h \in \mathcal{F}(\mathcal{X}_{0:n}) \),

\[ \int \mathbb{S}_n(x_{0:n}, dy_n) \mu(x_{0:n}) h = \mathbb{E}_n h(x_{0:n}). \]

Finally, we define the Markov kernel induced by the PPG as well as the extended probability distribution targeted by the same. For this purpose, we introduce the extended measurable space \( (\mathcal{E}_n, \mathcal{F}_n) \) with

\[ \mathbb{E}_n := \mathcal{Y}_n \times \mathcal{X}_{0:n}, \quad \mathbb{E}_n := \mathcal{Y}_n \oplus \mathcal{X}_{0:n}. \]

The PPG described in Algorithm 3 defines a Markov chain on \( (\mathbb{E}_n, \mathbb{E}_n) \) with Markov transition kernel

\[ K_n : \mathbb{E}_n \times \mathbb{E}_n \ni (y_n, z_{0:n}, A) \]

\[ \mapsto \int \int 1_A(\tilde{y}_n, z_{0:n}) C_n(z_{0:n}, \tilde{x}_0) S_n(\tilde{x}_0, \tilde{y}_n) \mu(\tilde{x}_0, \tilde{y}_n) (dz_0). \]
Note that the values of $K_n$ defined above do not depend on $y_n$, but only on $(z_{0:n}, A)$. For any given initial distribution $\xi \in M_1(\mathcal{X}_{0:n})$, let $\mathbb{P}_\xi$ be the distribution of the canonical Markov chain induced by the kernel $K_n$ and the initial distribution $\xi$. In the special case where $\xi = \delta_{z_{0:n}}$ for some given path $z_{0:n} \in \mathcal{X}_{0:n}$, we use the short-hand notation $\mathbb{P}_{\xi_{0:n}} = \mathbb{P}_{z_{0:n}}$. In addition, denote by

$$K_n : \mathcal{X}_{0:n} \times \mathcal{X}_{0:n} \ni (z_{0:n}, A) \mapsto \int \int 1_A(\tilde{z}_{0:n}) C_n(z_{0:n}, d\tilde{x}_{0:n}) S_n(x_{0:n}, dy_n) \mu(x_{0:n}) (d\tilde{z}_{0:n})$$

the path-marginalised version of $K_n$. By Proposition 2 it holds that $K_n = C_n B_n$, which shows that $K_n$ coincides with the Markov transition kernel of the backward-sampling-based particle Gibbs sampler discussed in Section 2.3.

Finally, in order to prepare for the statement of our theoretical results on the PPG we need to introduce the following Feynman–Kac path model with a frozen path. More precisely, for a given path $z_{0:n} \in \mathcal{X}_{0:n}$, define, for every $m \in [0, n - 1]$, the unnormalised kernel

$$Q_m(z_{m+1}) : \mathcal{X}_m \times \mathcal{X}_{m+1} \ni (x_m, A) \mapsto \left(1 - \frac{1}{N}\right) Q_m(x_m, A) + \frac{1}{N} g_m(x_m) \delta_{z_{m+1}}(A)$$

and the initial distribution $\eta_0(z_0) : \mathcal{X}_0 \ni A \mapsto (1 - 1/N) \eta_0(A) + \delta_{z_0}(A)/N$. Given these quantities, define, for $m \in [0, n - 1]$, $\gamma_m(z_{0:m}) := \eta_0(z_0) Q_0(z_1) \cdots Q_{m-1}(z_m)$ along with the normalised counterpart $\eta_m(z_{0:m}) := \gamma_m(z_{0:m}) / \gamma_m(z_{0:m}) 1_{\mathcal{X}_{0:m}}$. Finally, we introduce, for $m \in [0, n]$, the kernels

$$B_m(z_{0:m-1}) : \mathcal{X}_m \times \mathcal{X}_{0:m-1} \ni (x_m, A) \mapsto \int \cdots \int 1_A(x_{n-1}) \prod_{m=0}^{n-1} \tilde{Q}_m, \eta_m(z_{0:m}) (x_{m+1}, dx_m),$$

as well as the path model $\eta_{0:m}(z_{0:m}) := B_m(z_{0:m-1}) \otimes \eta_m(z_{0:m})$.

4. Main results

4.1 Theoretical results

This section will be devoted to establishing our main result, namely the exponentially contracting bias bound stated in Theorem 2 below. This result will be proved under the following strong mixing assumptions, which are standard in the literature (see [12, 18, 13, 16]).

### A 4.1 (strong mixing). For every $n \in \mathbb{N}$ there exist $\tau_n, \tilde{\tau}_n, \sigma_n$, and $\bar{\sigma}_n$ in $\mathbb{R}_+^*$ such that

1. $\tau_n \leq g_n(x_n) \leq \tilde{\tau}_n$ for every $x_n \in \mathcal{X}_n$,

2. $\sigma_n \leq m_n(x_n, x_{n+1}) \leq \bar{\sigma}_n$ for every $(x_n, x_{n+1}) \in \mathcal{X}_{n:n+1}$.

Under A 4.1, define, for every $n \in \mathbb{N}$,

$$\rho_n := \max_{m \in [0, n]} \frac{\bar{\tau}_m \bar{\sigma}_m}{\tau_m \sigma_m}$$

and, for every $N \in \mathbb{N}^*$ and $n \in \mathbb{N}$ such that $N > N_n := 1 + 5\rho^2 n/2$,

$$\kappa_{N,n} := 1 - \frac{1 - N_n/N}{1 + 4n(1 + 2\rho^2)/N}.$$
Note that $\kappa_{N,n} \in (0,1)$ for all $N$ and $n$ as above.

**Theorem 2.** Assume A 4.1. Then for every $n \in \mathbb{N}$ there exist $c_n^{bias}$, $c_n^{mse}$, and $c_n^{cov}$ in $\mathbb{R}_+^*$ such that for every $M \in \mathbb{N}^*$, $\xi \in \mathcal{M}_1(X_{0,n})$, $\ell \in \mathbb{N}^*$, $s \in \mathbb{N}^*$, and $N \in \mathbb{N}^*$ such that $N > N_n$,

$$
|\mathbb{E}_\xi [\mu(\beta_n[\ell])(id)] - \eta_{0:n} h_n| \leq c_n^{bias} \left( \sum_{m=0}^{n-1} ||\tilde{h}_m||_\infty \right) N^{-1} \kappa_{N,n}, \tag{4.3}
$$

$$
\mathbb{E}_\xi \left( (\mu(\beta_n[\ell])(id) - \eta_{0:n} h_n)^2 \right) \leq c_n^{mse} \left( \sum_{m=0}^{n-1} ||\tilde{h}_m||_\infty \right)^2 N^{-1}, \tag{4.4}
$$

$$
|\mathbb{E}_\xi [(\mu(\beta_n[\ell])(id) - \eta_{0:n} h_n)(\mu(\beta_n[\ell + s])(id) - \eta_{0:n} h_n)]| \leq c_n^{cov} \left( \sum_{m=0}^{n-1} ||\tilde{h}_m||_\infty \right)^2 N^{-3/2} \kappa_{N,n}. \tag{4.5}
$$

The constants $c_n^{bias}$, $c_n^{mse}$, and $c_n^{cov}$ are explicitly given in the proof. Since the focus of this paper is on the dependence on $N$ and the index $\ell$, we have made no attempt to optimise the dependence of these constants on $n$ in our proofs; still, we believe that it is possible to prove, under the stated assumptions, that this dependence is linear. The proof of the bound in Theorem 2 is based on four key ingredients. The first is the following unbiasedness property of the PARIS under the many-body Feynman–Kac path model.

**Theorem 3.** For every $n \in \mathbb{N}$, $N \in \mathbb{N}^*$, and $\ell \in \mathbb{N}^*$,

$$
\mathbb{E}_{\eta_{0:n}} [\mu(\beta_n[\ell])(id)] = \int \eta_{0:n} \mathbb{C}_n \mathbb{S}_n (db_n) \mu(b_n)(id) = \int \eta_{0:n} \mathbb{S}_n (db_n) \mu(b_n)(id) = \eta_{0:n} h_n.
$$

The proof of Theorem 3 is found in Section 5.3. The second is the uniform geometric ergodicity of the particle Gibbs with backward sampling established in [15].

**Theorem 4.** Assume A 4.1. Then for every $n \in \mathbb{N}$, $(\mu, \nu) \in \mathcal{M}_1(X_{0,n})^2$, $\ell \in \mathbb{N}^*$, and $N \in \mathbb{N}^*$ such that $N > 1 + 5\rho^2 n/2$, $||\mu K_\ell - \nu K_\ell||_{TV} \leq \kappa_{N,n}^\ell$, where $\kappa_{N,n}$ is defined in (4.2).

As a third ingredient, we require the following uniform exponential concentration inequality of the conditional PARIS with respect to the frozen-path Feynman–Kac model defined in the previous section.

**Theorem 5.** For every $n \in \mathbb{N}$ there exist $c_n > 0$ and $d_n > 0$ such that for every $M \in \mathbb{N}^*$, $z_{0:n} \in X_{0:n}$, $N \in \mathbb{N}^*$, and $\varepsilon > 0$,

$$
\int \mathbb{C}_n \mathbb{S}_n (z_{0:n}, db_n) 1 \{ |\mu(b_n)(id) - \eta_{0:n}(z_{0:n}) h_n| \geq \varepsilon \} \leq c_n \exp \left( - \frac{d_n N \varepsilon^2}{2(\sum_{m=0}^{n-1} ||\tilde{h}_m||_\infty)^2} \right).
$$

Theorem 5, whose proof is found in Appendix C.2, implies, in turn, the following conditional variance bound.

**Proposition 3.** For every $n \in \mathbb{N}$, $M \in \mathbb{N}^*$, $z_{0:n} \in X_{0:n}$, and $N \in \mathbb{N}^*$,

$$
\int \mathbb{C}_n \mathbb{S}_n (z_{0:n}, db_n) |\mu(b_n)(id) - \eta_{0:n}(z_{0:n}) h_n|^2 \leq \frac{c_n}{d_n} \left( \sum_{m=0}^{n-1} ||\tilde{h}_m||_\infty \right)^2 N^{-1}.
$$
Using Proposition 3, we deduce, in turn, the following bias bound, whose proof is postponed to Appendix C.4.

**Proposition 4.** For every $n \in \mathbb{N}$ there exists $\tilde{c}^\text{bias}_n > 0$ such that for every $M \in \mathbb{N}^*$, $z_{0:n} \in \mathcal{X}_{0:n}$, and $N \in \mathbb{N}^*$,

$$\left| \int C_n S_n(z_{0:n}, db_n) \mu(b_n)(\text{id}) - \eta_{0:n}(z_{0:n}) h_n \right| \leq \tilde{c}^\text{bias}_n N^{-1} \left( \sum_{m=0}^{n-1} \| \tilde{h}_m \|_{\infty} \right).$$

A fourth and last ingredient in the proof of Theorem 2 is the following bound on the discrepancy between additive expectations under the original and frozen-path Feynman–Kac models. This bound is established using novel results in [20]. More precisely, since for every $m \in \mathbb{N}$, $(x, z) \in \mathcal{X}^2_m$, $N \in \mathbb{N}^*$, and $h \in F(\mathcal{X}_{m+1})$, using A 4.1,

$$|Q_m(z) h(x) - Q_m h(x)| \leq \frac{1}{N} \| g_m \|_{\infty} \| h \|_{\infty} \leq \frac{1}{N} \gamma_m \| h \|_{\infty},$$

applying [20, Theorem 4.3] yields the following.

**Proposition 5.** Assume A 4.1. Then there exists $c > 0$ such that for every $n \in \mathbb{N}$, $N \in \mathbb{N}$, and $z_{0:n} \in \mathcal{X}_{0:n}$,

$$\| \eta_{0:n}(z_{0:n}) h_n - \eta_{0:n} h_n \| \leq c \| \tilde{h}_n \|_{\infty}.$$

Note that assuming, in addition, that $\sup_{n \in \mathbb{N}} \| \tilde{h}_n \|_{\infty} < \infty$ yields an $O(n/N)$ bound in Proposition 5.

Finally, by combining these ingredients we are now ready to present a proof of Theorem 2.

**Proof of Theorem 2.** Write, using the tower property,

$$E^\xi [\mu(\mathbf{b}_n \cdot) \cdot (\text{id})] = E^\xi \left[ E^{z_{0:n} \cdot z_{0:n}} [\mu(\mathbf{b}_n \cdot 0)] \cdot (\text{id}) \right] = \int \xi K^\ell_n C_n S_n (db_n) \mu(b_n)(\text{id}).$$

Thus, by the unbiasedness property in Theorem 3,

$$\left| E^\xi [\mu(\mathbf{b}_n \cdot) \cdot (\text{id})] - \eta_{0:n} h_n \right|$$

$$= \left| \int \xi K^\ell_n C_n S_n (db_n) \mu(b_n)(\text{id}) - \int \eta_{0:n} C_n S_n (db_n) \mu(b_n)(\text{id}) \right|$$

$$\leq \| \xi K^\ell_n - \eta_{0:n} \|_{TV} \text{osc} \left( \int C_n S_n (\cdot, db_n) \mu(b_n)(\text{id}) \right),$$

where, by Theorem 4, $\| \xi K^\ell_n - \eta_{0:n} \|_{TV} \leq \kappa^\ell_{N,n}$. Moreover, to derive an upper bound on the oscillation, we consider the decomposition

$$\text{osc} \left( \int C_n S_n (\cdot, db_n) \mu(b_n)(\text{id}) \right) \leq 2 \left( \left\| \int C_n S_n (\cdot, db_n) \mu(b_n)(\text{id}) - \eta_{0:n} (\cdot) h_n \right\|_{\infty} + \| \eta_{0:n} (\cdot) h_n - \eta_{0:n} h_n \|_{\infty} \right),$$
where the two terms on the right-hand side can be bounded using Proposition 5 and Proposition 4, respectively. This completes the proof of (4.3). We now consider the proof of (4.4). Writing

\[ E_\xi \left[ (\mu(\beta_n[\ell])(id) - \eta_{0:n} h_n)^2 \right] = \int E_\xi K_n^\ell(\mathrm{d}z_{0:n}) C_n z_{0:n}, db_n (\mu(b_n)(id) - \eta_{0:n} h_n)^2, \]

we may establish (4.4) using Proposition 3 and Proposition 5. We finally consider (4.5). Using the Markov property we obtain

\[ E_\xi \left[ (\mu(\beta_n[\ell])(id) - \eta_{0:n} h_n) (\mu(\beta_n[\ell + s])(id) - \eta_{0:n} h_n) \right] = E_\xi \left[ (\mu(\beta_n[\ell])(id) - \eta_{0:n} h_n) (E_{\zeta_0:n[\ell]}[\mu(\beta_n[s])(id)] - \eta_{0:n} h_n) \right], \]

from which (4.5) follows by (4.3) and (4.4).

4.2 The roll-out PPG estimator

In the light of the previous results, it is natural to consider an estimator formed by an average across successive conditional PPG estimators \( \{\mu(\beta_n[\ell])\}_{\ell \in \mathbb{N}} \). To mitigate the bias, we remove a “burn-in” period whose length \( k_0 \) should be chosen proportionally to the mixing time of the particle Gibbs chain \( \{\zeta_0:n[\ell]\}_{\ell \in \mathbb{N}^*} \). This yields the estimator

\[ \Pi(k_0, k), N(h_n) = \frac{(k - k_0) - 1}{k} \sum_{\ell = k_0 + 1}^{k} \mu(\beta_n[\ell])(id). \]  

(4.6)

The total number of particles underlying this estimator is \( C = (N - 1)k \). We denote by \( v = (k - k_0)/k \) the ratio of the number of particles used in the estimator to the total number of sampled particles.

Our final main result provides bounds on the bias and the MSE of the estimator (4.6). The proof is postponed to Appendix C.5.

**Theorem 6. Assume A 4.1. Then for every \( n \in \mathbb{N}, M \in \mathbb{N}^*, \xi \in M_1(\mathcal{X}_{0:n}), \ell \in \mathbb{N}^*, s \in \mathbb{N}^* \), and \( N \in \mathbb{N}^* \) such that \( N > N_n \),

\[ \left| E_\xi [\Pi(k_0, k), N(h_n)] - \eta_{0:n} h_n \right| \leq c_{\text{bias}} \left( \sum_{m=0}^{n-1} ||\hat{h}_m||_\infty \right) \frac{k_0}{(k - k_0)(1 - \kappa_{N,n})N}, \]

(4.7)

\[ E_\xi \left[ (\Pi(k_0, k), N(h_n) - \eta_{0:n} h_n)^2 \right] \leq \left( \sum_{m=0}^{n-1} ||\hat{h}_m||_\infty \right)^2 \left\{ N(k - k_0) \right\}^{-1} \left( c_{\text{mse}}^{\text{cov}} N^{-1/2}(1 - \kappa_{N,n})^{-1} \right). \]

(4.8)
5. Proofs

5.1 Proof of Proposition 1

Using the identity
\[
\eta_0 Q_0 \cdots Q_{n-1} \mathbb{1}_{X_n} = \prod_{m=0}^{n-1} \eta_m Q_m \mathbb{1}_{X_{m+1}}
\]
and the fact that each kernel \( Q_m \) has a transition density, write, for \( h \in F(X_0:n) \),
\[
\eta_{0:n} h = \int \cdots \int h(x_{0:n}) \eta_0(\text{d}x_0) \prod_{m=0}^{n-1} \left( \frac{\eta_m [q_m (\cdot, z_{m+1})]}{\eta_m Q_m \mathbb{1}_{X_{m+1}}} Q_m(z_{m+1}) \right) \left( \frac{q_m (x_m, z_{m+1})}{\eta_m [q_m (\cdot, z_{m+1})]} \right)
\]
\[
= \int \cdots \int h(x_{0:n}) \eta_n(\text{d}x_n) \prod_{m=0}^{n-1} \eta_m(\text{d}x_m) Q_m(z_{m+1}) \left( \frac{q_m (x_m, z_{m+1})}{\eta_m [q_m (\cdot, z_{m+1})]} \right)
\]
\[
= \left( Q_{0,\eta_0} \otimes \cdots \otimes Q_{n-1,\eta_{n-1}} \otimes \eta_n \right) h,
\]
which was to be established.

5.2 Proof of Theorem 1

Lemma 1. For all \( n \in \mathbb{N} \), \( x_n \in X_n \), and \( h \in F(X_{n+1} \otimes X_{n+1}) \),
\[
\int h(x_{n+1}, z_{n+1}) Q_n(x_n, \text{d}x_{n+1}) \mu(x_{n+1})(\text{d}z_{n+1})
\]
\[
= \int h(x_{n+1}, z_{n+1}) \mu(x_n) Q_n(\text{d}z_{n+1}) \mathcal{M}_n(z_{n+1})(x_n, \text{d}x_{n+1}).
\]
(5.2)

In addition, for all \( h \in F(X_0 \otimes X_0) \),
\[
\int h(x_0, z_0) \eta_0(\text{d}x_0) \mu(x_0)(\text{d}z_0) = \int h(x_0, z_0) \eta_0(z_0)(\text{d}x_0) \eta_0(\text{d}z_0).
\]
(5.3)

Proof. Since \( \mu(x_n) Q_n(\text{d}z_{n+1}) = g_n(x_n) \Phi_n(\mu(x_n))(\text{d}z_{n+1}) \), we may rewrite the right-
hand side of (5.2) according to
\[
\int \int h(x_{n+1}, z_{n+1}) \mu(x_n) Q_n(dx_{n+1}) M_n(z_{n+1})(x_n, dx_{n+1})
\]
\[
= g_n(x_n) \frac{1}{N} \sum_{i=0}^{N-1} \int \int h(x_{n+1}, z_{n+1}) \Phi_n(\mu(x_n))(dz_{n+1})
\]
\[
\times \left( \Phi_n(\mu(x_n)) \otimes \delta_{z_{n+1}} \otimes \Phi_n(\mu(x_n)) \otimes (N-i-1) \right)(dx_{n+1})
\]
\[
= g_n(x_n) \frac{1}{N} \sum_{i=1}^{N} \int \int h((x^{1}_{n+1}, \ldots, x^{i-1}_{n+1}, z_{n+1}, x^{i+1}_{n+1}, \ldots, x^{N}_{n+1}), z_{n+1})
\]
\[
\times \Phi_n(\mu(x_n))(dz_{n+1}) \prod_{\ell \neq i} \Phi_n(\mu(x_n))(dx_{n+1}^\ell)
\]
\[
= g_n(x_n) \frac{1}{N} \sum_{i=1}^{N} \int h(x_{n+1}, x^{i}_{n+1}) M_n(x_n, dx_{n+1}).
\]

On the other hand, note that the left-hand side of (5.2) can be expressed as
\[
\int \int h(x_{n+1}, z_{n+1}) Q_n(x_n, dx_{n+1}) \mu(x_{n+1})(dz_{n+1})
\]
\[
= g_n(x_n) \frac{1}{N} \sum_{i=1}^{N} \int h(x_{n+1}, x^{i}_{n+1}) M_n(x_n, dx_{n+1}), \quad (5.4)
\]
which establishes the identity. The identity (5.3) is established along similar lines. \( \square \)

We establish Theorem 1 by induction; thus, assume that the claim holds true for \( n \) and show that for all \( h \in F(\mathcal{X}_{0:n+1} \otimes \mathcal{X}_{0:n+1}) \),
\[
\int \int h(x_{0:n+1}, z_{0:n+1}) \gamma_{0:n+1}(dx_{0:n+1}) B_{n+1}(x_{0:n+1}, dz_{0:n+1})
\]
\[
= \int \int h(x_{0:n+1}, z_{0:n+1}) \gamma_{0:n+1}(dz_{0:n+1}) C_{n+1}(z_{0:n+1}, dx_{0:n+1}). \quad (5.5)
\]
To prove this, we process, using definition (2.4), the left-hand side of (5.5) according to
\[
\int \int h(x_{0:n+1}, z_{0:n+1}) \gamma_{0:n+1}(dx_{0:n+1}) B_{n+1}(x_{0:n+1}, dz_{0:n+1})
\]
\[
= \int \int \gamma_{0:n}(dx_{0:n}) B_{n}(x_{0:n}, dz_{0:n})
\]
\[
\times \int h(x_{0:n+1}, z_{0:n+1}) Q_n(x_n, dx_{n+1}) \mu(x_{n+1})(dz_{n+1}),
\]
where we have defined the function
\[
\tilde{h}(x_{0:n+1}, z_{0:n+1}) := \frac{g_n(z_{n}, z_{n+1}) h(x_{0:n+1}, z_{0:n+1})}{\mu(x_n)[g_n(\cdot, z_{n+1})]}.
\]
Now, applying Lemma 1 to the inner integral and using that
\[ \mu(x_n)Q_n(dx_{n+1}) = \mu(x_n)[q_{n}(\cdot, z_{n+1})] \lambda_{n+1}(dz_{n+1}) \]
yields, for every \( x_{0:n} \) and \( z_{0:n} \),
\[
\begin{align*}
\int\int \hat{h}(x_{0:n+1}, z_{0:n+1}) Q_n(x_n, dx_{n+1}) \mu(x_{n+1})(dz_{n+1}) \\
&= \int\int \hat{h}(x_{0:n+1}, z_{0:n+1}) \mu(x_n)Q_n(dx_{n+1}) M_n(z_{n+1})(x_n, dx_{n+1}) \\
&= \int\int \hat{h}(x_{0:n+1}, z_{0:n+1}) Q_n(z_n, dz_{n+1}) M_n(z_{n+1})(x_n, dx_{n+1}).
\end{align*}
\]
Inserting the previous identity into (5.6) and using the induction hypothesis provides
\[
\begin{align*}
\int\int \hat{h}(x_{0:n+1}, z_{0:n+1}) \gamma_{0:n+1}(dz_{0:n+1}) \mathbb{E}_{n+1}(x_{0:n+1}, dz_{0:n+1}) \\
&= \int\int \gamma_{0:n}(dz_{0:n}) C_n(z_{0:n}, dx_{0:n}) \\
&\quad \times \int\int \hat{h}(x_{0:n+1}, z_{0:n+1}) Q_n(z_n, dz_{n+1}) M_n(z_{n+1})(x_n, dx_{n+1}) \\
&= \int\int \hat{h}(x_{0:n+1}, z_{0:n+1}) \gamma_{0:n+1}(dz_{0:n+1}) C_{n+1}(z_{0:n+1}, dx_{0:n+1}),
\end{align*}
\]
which establishes (5.5).

### 5.3 Proof of Theorem 3

First, define, for \( m \in \mathbb{N} \),
\[
\mathbf{P}_m : \mathcal{Y}_m \times \mathcal{Y}_{m+1} \ni (y_m, A) \mapsto \int M_m(x_m|x_m, dx_{m+1}) S_m(y_m, x_{m+1}, A). \quad (5.7)
\]
For any given initial distribution \( \psi_0 \in \mathbf{M}_1(\mathcal{Y}_0) \), let \( \mathbb{F}^P_{\psi_0} \) be the distribution of the canonical Markov chain induced by the Markov kernels \( \{\mathbf{P}_m\}_{m \in \mathbb{N}} \) and the initial distribution \( \psi_0 \). By abuse of notation we write, for \( \eta_0 \in \mathbf{M}_1(\mathcal{X}_0) \), \( \mathbb{F}^P_\psi \eta_0 \) instead of \( \mathbb{F}^P_{\psi_0}[\eta_0] \), where we have defined the extension \( \psi_0[\eta_0](A) = \int 1_A(\mathbf{J}(x_0) \eta_0(dx_0), A \in \mathcal{Y}_0 \). We preface the proof of Theorem 3 by some technical lemmas and a proposition.

**Lemma 2.** For all \( n \in \mathbb{N} \) and \( (f_{n+1}, \bar{f}_{n+1}) \in \mathbf{F}(\mathcal{X}_{n+1})^2 \),
\[
\gamma_{n+1}(f_{n+1} B_{n+1} h_n + \bar{f}_{n+1}) = \gamma_n\{Q_n f_{n+1} B_n h_n + Q_n(\bar{h}_n f_{n+1} + \bar{f}_{n+1})\}.
\]

**Proof.** Pick arbitrarily \( \varphi \in \mathbf{F}(\mathcal{X}_{n+1}) \) and write, using definition (2.3) and the fact that \( Q_n \) has a transition density,
\[
\begin{align*}
\int\int \varphi(x_{0:n+1}) \gamma_n(dx_n) Q_n(x_n, dx_{n+1}) \\
&= \int\int \varphi(x_{0:n+1}) \gamma_n[g_{n}(\cdot, x_{n+1})] \lambda_{n+1}(dx_{n+1}) \frac{\gamma_n(dx_n)q_n(x_n, x_{n+1})}{\gamma_n[g_n(\cdot, x_{n+1})]} \\
&= \int\int \varphi(x_{0:n+1}) \gamma_{n+1}(dx_{n+1}) \bar{Q}_{n, \eta_0}(x_{n+1}, dx_n). \quad (5.8)
\end{align*}
\]
Now, by (2.10) it holds that
\[ B_{n+1}h_{n+1}(x_{n+1}) = \int \hat{Q}_{n,\eta_n}(x_{n+1}, dx_n) \left( \bar{h}_n(x_{n:n+1}) + \int h_n(x_{0:n}) B_n(x_n, dx_{0:n-1}) \right); \]
therefore, by applying (5.8) with
\[ \varphi(x_{n:n+1}) := f_{n+1}(x_{n+1}) \left( \bar{h}_n(x_{n:n+1}) + \int h_n(x_{0:n}) B_n(x_n, dx_{0:n-1}) \right) \]
we obtain that
\[ \gamma_{n+1}(f_{n+1}B_{n+1}h_{n+1}) = \int \int \varphi(x_{n:n+1}) \gamma_{n+1}(dx_{n+1}) \hat{Q}_{n,\eta_n}(x_{n+1}, dx_n) \]
\[ = \int \int \varphi(x_{n:n+1}) \gamma_n(dx_n) Q_n(x_n,dx_{n+1}) \]
\[ = \gamma_n(Q_n f_{n+1}B_n h_n + Q_n \bar{h}_n f_{n+1}). \]

Now the proof is concluded by noting that since \( \gamma_{n+1} = \gamma_n Q_n \), \( \gamma_{n+1} \bar{f}_{n+1} = \gamma_n Q_n \bar{f}_{n+1} \).

\[ \square \]

Lemma 3. For every \( n \in \mathbb{N}^* \), \( h_n \in F(Y_n) \), and \( \eta_0 \in M_1(X_0) \) it holds that
\[ \mathbb{E}_{\eta_0}^{P_n}\{h_n(\xi_{n|0}, \ldots, \xi_{n|n}) = \mathbb{S}_n h_n(\xi_{0|0}, \ldots, \xi_{n|n})\}, \quad \mathbb{P}_{\eta_0}^{P_n} - \text{a.s.} \]

Proof. Pick arbitrarily \( v_n \in F(X_{0:n}) \). We show that
\[ \mathbb{E}_{\eta_0}^{P_n}\{v_n(\xi_{0|0}, \ldots, \xi_{n|n}) h_n(\xi_{n|n})\} = \mathbb{E}_{\eta_0}^{P_n}\{v_n(\xi_{0|0}, \ldots, \xi_{n|n}) \mathbb{S}_n h_n(\xi_{0|0}, \ldots, \xi_{n|n})\}, \quad (5.9) \]
from which the claim follows. Using the definition (5.7), the left-hand side of the previous identity may be rewritten as
\[ \int \cdots \int \psi_0(\eta_0)(dy_0) \prod_{m=0}^{n-1} P_m(y_m, dy_{m+1}) h_n(y_n) v_n(x_{0|0}, \ldots, x_{n|n}) \]
\[ = \int \cdots \int \eta_0(dx_{0|0}) \prod_{m=0}^{n-1} M_m(x_{m|m}, dx_{m+1}) S_0(Jx_{0|0}, x_1, dy_1) \]
\[ \times \prod_{m=0}^{n-1} S_m(y_m, x_{m+1}, dy_{m+1}) h_n(y_n) v_n(x_{0|0}, \ldots, x_{n|n}) \]
\[ = \int \cdots \int \eta_0(dx_0) \prod_{m=0}^{n-1} M_m(x_m, dx_{m+1}) S_0(Jx_0, x_1, dy_1) \]
\[ \times \prod_{m=0}^{n-1} S_m(y_m, x_{m+1}, dy_{m+1}) h_n(y_n) v_n(x_0, \ldots, x_n). \]
Thus, we may conclude the proof by using the definition (3.2) of \( \mathbb{S}_n \) together with Fubini’s theorem. \[ \square \]
Lemma 4. For every $n \in \mathbb{N}^*$ and $h_n \in F(Y_n)$ it holds that
\[
\mathbb{E}_{\eta_0}\left[ \left( \prod_{m=0}^{n-1} g_m(\xi_{m|m}) \right) h_n(\nu_n) \right] = \int \gamma_{0:n} S_n(dy_n) h_n(y_n).
\]

Proof. The claim of the lemma is a direct implication of Lemma 3; indeed, by applying the tower property and the latter we obtain
\[
\mathbb{E}_{\eta_0}^P\left[ \left( \prod_{m=0}^{n-1} g_m(\xi_{m|m}) \right) h_n(\nu_n) \right] = \mathbb{E}_{\eta_0}^P\left[ \left( \prod_{m=0}^{n-1} g_m(\xi_{m|m}) \right) s_n h_n(\xi_{0:n}) \right] = \int \cdots \int \eta_0(dx_0) \prod_{m=0}^{n-1} g_m(x_m) M_m(x_m, dx_{m+1}) s_n h_n(x_{0:n}) = \int \gamma_{0:n} S_n(dy_n) h_n(y_n).
\]

Proposition 6. For all $n \in \mathbb{N}^*$, $(N,M) \in (\mathbb{N}^*)^2$, and $(f_n, \tilde{f}_n) \in F(\mathcal{X}_n)^2$,
\[
\int \gamma_{0:n} S_n(dy_n) \left( \frac{1}{N} \sum_{i=1}^{N} \{ b_i^n f_n(x_{n|i}) + \tilde{f}_n(x_{n|i}) \} \right) = \gamma_n B_n h_n + \tilde{f}_n.
\]

Proof. Applying Lemma 4 yields
\[
\int \gamma_{0:n} S_n(dy_n) \left( \frac{1}{N} \sum_{i=1}^{N} \{ b_i^n f_n(x_{n|i}) + \tilde{f}_n(x_{n|i}) \} \right) = \mathbb{E}_{\eta_0}^P\left[ \prod_{m=0}^{n-1} g_m(\xi_{m|m}) \right] \frac{1}{N} \sum_{i=1}^{N} \{ \beta_{n+1}^i f_n(x_{n+1|i}) + \tilde{f}_n(x_{n+1|i}) \} \bigg| \tilde{F}_n \bigg]. \tag{5.10}
\]

In the following we will use repeatedly the following filtrations. Let $\tilde{\mathcal{F}}_n := \sigma(\{\nu_m\}_{m=0}^{n})$ be the $\sigma$-field generated by the output of the PARIS (Algorithm 1) during the first $n$ iterations. In addition, let $\mathcal{F}_n := \tilde{\mathcal{F}}_{n-1} \lor \sigma(\xi_{n|n})$.

We proceed by induction. Thus, assume that the statement of the proposition holds true for a given $n \in \mathbb{N}^*$ and consider, for arbitrarily chosen $(f_{n+1}, \tilde{f}_{n+1}) \in F(\mathcal{X}_{n+1})^2$,
\[
\mathbb{E}_{\eta_0}^P\left[ \prod_{m=0}^{n} g_m(\xi_{m|m}) \right] \frac{1}{N} \sum_{i=1}^{N} \{ \beta_{n+1}^i f_{n+1}(\xi_{n+1|i}) + \tilde{f}_{n+1}(\xi_{n+1|i}) \} \bigg| \tilde{\mathcal{F}}_n \bigg] = \left( \prod_{m=0}^{n} g_m(\xi_{m|m}) \right) \mathbb{E}_{\eta_0}^P\left[ \beta_{n+1}^1 f_{n+1}(\xi_{n+1|1}) + \tilde{f}_{n+1}(\xi_{n+1|1}) \bigg| \tilde{F}_n \bigg],
\]
where we used that the variables \( \{\beta_{n+1}^i f_{n+1}(\xi_{n+1}^{i+1} + f_{n+1}(\xi_{n+1}^{i+1})) \}_{i=1}^N \) are conditionally i.i.d. given \( \tilde{F}_n \). Note that, by symmetry,

\[
\mathbb{E}_{\eta_0}^{P}[\beta_{n+1}^1 | F_{n+1}] = \int \sum_{n} S_n(v_n, \xi_{n+1}^{i+1} | \mathbf{d} y_n) b_{n+1}^i.
\]

Thus, using the induction hypothesis,

\[
\begin{aligned}
\mathbb{E}_{\eta_0}^{P}[\beta_{n+1}^1 | F_{n+1}] &= \int \sum_{n} S_n(v_n, \xi_{n+1}^{i+1} | \mathbf{d} y_n) b_{n+1}^i \\
&= \int \cdots \int \left( \prod_{j=1}^M \sum_{l=1}^N \frac{q_n(\xi_n^l, \beta_{n+1}^l)}{\sum_{l'=1}^N q_n(\xi_n^l, \xi_{n+1}^l)} \delta(\xi_n^l, \beta_{n+1}^l) (d\tilde{x}_{n}^{1,j}, d\tilde{h}_{n}^{1}) \right) \\
&\times \frac{1}{M} \sum_{j=1}^M \left( \tilde{b}_{n}^{1,j} + \tilde{h}_n(\tilde{x}_{n}^{1,j}, \xi_{n+1}^l) \right) \\
&= \sum_{\ell=1}^N \sum_{\ell'=1}^N \frac{q_n(\xi_n^\ell, \xi_{n+1}^\ell)}{\sum_{\ell'=1}^N q_n(\xi_n^\ell, \xi_{n+1}^\ell)} \left( \beta_{n+1}^\ell + \tilde{h}_n(\xi_n^\ell, x_{n+1}) \right).
\end{aligned}
\]

Thus, using the tower property,

\[
\mathbb{E}_{\eta_0}^{P}[\beta_{n+1}^1 f_{n+1}(\xi_{n+1}^{i+1}) | \tilde{F}_n]
\]

\[
= \int \Phi_n(\mu(\xi_{n}^{i})) (dx_{n+1}) f_{n+1}(x_{n+1}) \sum_{\ell=1}^N \frac{q_n(\xi_n^\ell, x_{n+1})}{\sum_{\ell'=1}^N q_n(\xi_n^\ell, x_{n+1})} \left( \beta_{n+1}^\ell + \tilde{h}_n(\xi_n^\ell, x_{n+1}) \right),
\]

and consequently, using definition (2.1),

\[
\begin{aligned}
\left( \prod_{m=0}^n g_m(\xi_{m}|m) \right) \mathbb{E}_{\eta_0}^{P}[\beta_{n+1}^1 f_{n+1}(\xi_{n+1}^{i+1}) | \tilde{F}_n]
&= \left( \prod_{m=0}^{n-1} g_m(\xi_{m}|m) \right) \int \sum_{i=1}^N q_n(\xi_n^i, x_{n+1}) \\
&\times f_{n+1}(x_{n+1}) \sum_{\ell=1}^N \frac{q_n(\xi_n^\ell, x_{n+1})}{\sum_{\ell'=1}^N q_n(\xi_n^\ell, x_{n+1})} \left( \beta_{n+1}^\ell + \tilde{h}_n(\xi_n^\ell, x_{n+1}) \right) \lambda_{n+1}(dx_{n+1}) \\
&= \left( \prod_{m=0}^{n-1} g_m(\xi_{m}|m) \right) \sum_{\ell=1}^N \left( \beta_{n}^\ell Q_n(f_{n+1}(\xi_n^\ell)) + Q_n(h_n f_{n+1})(\xi_n^\ell) \right).
\end{aligned}
\]

Thus, applying the induction hypothesis,

\[
\begin{aligned}
\mathbb{E}_{\eta_0}^{P}\left[ \left( \prod_{m=0}^n g_m(\xi_{m}|m) \right) \frac{1}{N} \sum_{i=1}^N \beta_{n+1}^i f_{n+1}(\xi_{n+1}^{i+1}) \right]
&= \mathbb{E}_{\eta_0}^{P}\left[ \left( \prod_{m=0}^{n-1} g_m(\xi_{m}|m) \right) \frac{1}{N} \sum_{\ell=1}^N \left( \beta_{n}^\ell Q_n(f_{n+1}(\xi_n^\ell)) + Q_n(h_n f_{n+1})(\xi_n^\ell) \right) \right] \\
&= \gamma_n \left( Q_n f_{n+1} B_n h_n + Q_n(h_n f_{n+1}) \right).
\end{aligned}
\]
In the same manner, it can be shown that
\[
\mathbb{E}_{\eta_0}^P \left[ \prod_{m=0}^{n} g_m(\xi_{m|m}) \right] \frac{1}{N} \sum_{i=1}^{N} \tilde{f}_{n+1}(\xi_{n+1|n+1}) = \gamma_n Q_n \tilde{f}_{n+1}. \tag{5.13}
\]

Now, by (5.12–5.13) and Lemma 2,
\[
\mathbb{E}_{\eta_0}^P \left[ \prod_{m=0}^{n} g_m(\xi_{m|m}) \right] \frac{1}{N} \sum_{i=1}^{N} \{ \beta_{n+1}^i f_{n+1}(\xi_{n+1|n+1}) + \tilde{f}_{n+1}(\xi_{n+1|n+1}) \}
\]
\[
= \gamma_n \left( Q_n f_{n+1} B_n h_n + Q_n (h_n f_{n+1} + \tilde{f}_{n+1}) \right)
\]
\[
= \gamma_{n+1} (f_{n+1} B_{n+1} h_{n+1} + \tilde{f}_{n+1}),
\]
which shows that the claim of the proposition holds at time \(n+1\).

It remains to check the base case \(n = 0\), which holds trivially true as \(\beta_0 = 0\), \(B_0 h_0 = 0\) by convention, and the initial particles \(\xi_{0|0}\) are drawn from \(\eta_0\). This completes the proof.

Proof of Theorem 3. The identity \(\int \eta_{0:n}(dz_{0:n}) S_n(x_{0:n}, db_n) \mu(b_n)(id) = \eta_{0:n} h_n\) follows immediately by letting \(f_n \equiv 1\) and \(\tilde{f}_n \equiv 0\) in Proposition 6 and using that \(\gamma_{0:n}(X_{0:n}) = \gamma_{0:n}(X_{0:n})\). Moreover, applying Theorem 1 yields
\[
\int \eta_{0:n} C_n S_n(db_n) \mu(b_n)(id) = \int \int \eta_{0:n}(dx_{0:n}) C_n(z_{0:n}, dx_{0:n}) \int S_n(x_{0:n}, db_n) \mu(b_n)(id)
\]
\[
= \int \int \eta_{0:n}(dx_{0:n}) \mathbb{E}_n(x_{0:n}, dz_{0:n}) \int S_n(x_{0:n}, db_n) \mu(b_n)(id)
\]
\[
= \int \eta_{0:n} S_n(db_n) \mu(b_n)(id).
\]

Finally, the first identity holds true since \(K_n\) leaves \(\eta_{0:n}\) invariant.

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A. Numerical results

Given measurable spaces \((X, \mathcal{X})\) and \((Z, \mathcal{Z})\), a hidden Markov model (HMM) is a bivariate stochastic process \(\{(X_m, Z_m)\}_{m \in \mathbb{N}}\) where \((X_m, Z_m)\) takes on values in \((X \times Z, \mathcal{X} \otimes \mathcal{Z})\) and where \(\{X_n\}_{n \in \mathbb{N}}\) (the state sequence) is a (possibly unnormalised) Markov chain which is only partially observed through the observation process \(\{Z_m\}_{m \in \mathbb{N}}\). We denote by \(\{M_n\}_{n \in \mathbb{N}}\) and \(\chi\) the Markov transition kernels and initial distribution of \(\{X_n\}_{n \in \mathbb{N}}\), respectively. Conditionally on the unobserved state sequence, the observations are assumed to be independent and such that the conditional distribution of each \(Z_m\) only depends on the corresponding state \(X_m\) and has a density \(g_m(X_m, \cdot)\) with respect to some dominating measure. HMMs are used today in a variety of scientific and engineering disciplines; see [2, 9, 10]. Inference in HMMs typically involves the computation of conditional distributions of unobserved states given observations. Of particular interest are the sequences of filter distributions, where the filter at time \(m \in \mathbb{N}\), denoted \(\eta_m\), is defined as the conditional distribution of \(X_m\) given \(Z_{0:m} := (Z_0, \ldots, Z_m)\), and the joint-smoothing distributions, where the joint-smoothing distribution at time \(m\), denoted \(\eta_{0:m}\), is defined as the joint conditional distribution of the states \(X_{0:m} = (X_0, \ldots, X_m)\) given \(Z_{0:m}\). Consequently, \(\eta_m\) is the marginal of \(\eta_{0:m}\) with respect to the last state \(X_m\). Given a sequence \(\{z_m\}_{m \in \mathbb{N}}\) of fixed observations, it can be shown (see [9, Section 3] for details) that \(\{\eta_{0:m}\}_{m \in \mathbb{N}}\) forms a Feynman–Kac model as defined in Section 1 with Markov kernels \(\{M_m\}_{m \in \mathbb{N}}\) potential functions \(g_m := g(\cdot, z_m), m \in \mathbb{N}\), on \(X\).

We now numerically assess the proposed algorithm on two different models, namely (i) a linear Gaussian state-space model (for which the filter and joint-smoothing distribution flows are available in a closed form) and (ii) a stochastic volatility model proposed in [22].

**Linear Gaussian state-space model (LGSSM).** We first consider a linear Gaussian HMM

\[
X_{m+1} = AX_m + Q \epsilon_{m+1}, \quad Z_m = BX_m + R \zeta_m, \quad m \in \mathbb{N},
\]

where \(\{\epsilon_m\}_{m \in \mathbb{N}}\) and \(\{\zeta_m\}_{m \in \mathbb{N}}\) are sequences of independent standard normally distributed random variables. The coefficients \(A, Q, B,\) and \(R\) are assumed to be known and equal to 0.97, 0.60, 0.54, and 0.33, respectively. Using this parameterisation, we generate, by simulation, a record \(z_{0:n}\) of observations with \(n = 1000\).

In this setting, we aim at computing smoothed expectations of the state one-lag covariance \(h_n(x_{0:n}) := \sum_{m=0}^{n-1} x_m x_{m+1}\). In the linear Gaussian case, the disturbance smoother (see [9, Algorithm 5.2.15]) provides the exact values of the smoothed sufficient statistics, which allows us to study the bias of the estimator for a given computational budget \(C\).

To illustrate the bias bounds provided by Theorems 2 and 6, we calculate the bias after different numbers \(k\) of iterations on the basis of 1000 independent replicates of the
algorithm for $N \in \{10, 25, 50, 100\}$ with $C = Nk = 500$. Figure 1 displays the evolution of the bias of the roll-out estimator (given by (4.6)) in the cases where $k_0 = k - 1$ and $k_0 = \lfloor k/2 \rfloor$. Moreover, for comparison the estimated bias of the PARIS algorithm (Algorithm 1) with $N = C$ particles is shown. For each $N$, we fit a curve of type $e^{ak+b}$ to the produced bias estimates to illustrate the exponentially (with $k$) decreasing PPG bias bound established by Theorems 2 and 6. We also note that the most economical strategy in terms of smallest possible bias for a given computational cost $kN$ is to use a large $N$ and a small $k$. Indeed, we find that for $N \in \{50, 100\}$, only two iterations of the PPG are sufficient to be less biased than the reference estimator PARIS with $N = C$ particles. The preceding remark is also supported by Figure 2, which displays, for three different total budgets $C$, the distribution of estimates of $\eta_{0:n|h_n}$ using the PARIS as well as three different configurations of the PPG corresponding to $k \in \{2, 4, 10\}$ (and $N = C/k$) with $k_0 = k - 1$. The reference value is shown as a red-dashed line and the mean value of each distribution is shown as a black-dashed line. Each boxplot is based on 1000 independent replicates of the corresponding estimator. We observe that in this example, all configurations of the PPG are less biased than the equivalent PARIS estimator. We also observe that a larger $k$ does not lead to smaller bias in all configurations.

**Stochastic Volatility.** As a second example, we consider the stochastic volatility model

$$X_{m+1} = \phi X_m + \sigma \epsilon_{m+1}, \quad Z_m = \beta \exp(X_m/2)\zeta_m, \quad m \in \mathbb{N},$$

(A.2)

where $\{\epsilon_m\}_{m \in \mathbb{N}}$ and $\{\zeta_m\}_{m \in \mathbb{N}}$ are as in the previous example and the model parameters $\phi$, $\beta$, and $\sigma$, are set to 0.975, 0.63, and 0.16, respectively. Again, an observation record $z_{0:n}$ of length $n = 1000$ is generated through simulation. The reference value is calculated using the PARIS with $N = 10000$ particles. Similarly to the LGSSM example, the bias

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Output of the PPG roll-out estimator for the LGSSM. The curves describe the evolution of the bias with increasing $k$ for different particle sample sizes $N$. The left and right panels correspond to $k_0 = k - 1$ and $k_0 = \lfloor k/2 \rfloor$, respectively.}
\end{figure}
Fig 2: PARIS and PPG outputs for the LGSSM. The different panels correspond to the different budgets $C \in \{100, 500, 1000\}$ and for each panel, yellow boxes correspond to PPG outputs produced using $k \in \{10, 4, 2\}$ iterations and $N \in \{C/10, C/4, C/2\}$ particles.

decay with respect to $k$ is shown in Figure 3 and the comparison with the PARIS algorithm in Figure 4, and the same remarks made for the LGSSM model apply to the stochastic volatility model. In Figure 4, we observe that for small budgets ($N = 100$), the choice of taking a large number $k$ of steps leads to a higher bias than for the PARIS estimator, indicating that in this case the chain has not yet mixed properly.

Fig 3: Output of the PPG roll-out estimator for the stochastic volatility model. The curves describe the evolution of the bias with increasing $k$ for different particle sample sizes $N$. The left and right panels correspond to $k_0 = k - 1$ and $k_0 = \lfloor k/2 \rfloor$, respectively.
Fig 4: PARIS and PPG outputs for the stochastic volatility model. The different panels correspond to the different budgets $C \in \{100, 500, 1000\}$ and for each panel, yellow boxes correspond to PPG outputs produced using $k \in \{2, 4, 10\}$ iterations and $N \in \{C/2, C/4, C/10\}$ particles.
B. Algorithms

Data: $\{(\xi^i_n, \beta^i_n)^N_{i=1}\}$
Result: $\{(\xi^i_{n+1}, \beta^i_{n+1})^N_{i=1}\}$
draw $\xi_{n+1} \sim M_n(\xi^i_n)$;
for $i \leftarrow 1$ to $N$ do
  | draw $\{(\tilde{\xi}^i_{n,j}, \tilde{\beta}^i_{n,j})^M_{j=1}\} \sim \left(\sum_{\ell=1}^N \frac{q_n(\xi^\ell_n, \xi^i_{n+1})}{\sum_{\ell'=1}^N q_n(\xi^\ell_n, \xi^i_{n+1})} \delta(\xi^\ell_n, \beta^\ell_n)\right)^\otimes M$;
  | set $\beta^i_{n+1} \leftarrow \frac{1}{M} \sum_{j=1}^M \left(\tilde{\beta}^i_{n,j} + \tilde{h}_n(\tilde{\xi}^i_{n,j}, \xi^i_{n+1})\right)$;
end

Algorithm 1: One update of the PARIS algorithm

Data: $\nu_n, \zeta_{n+1}$
Result: $\nu_{n+1}$
draw $\xi_{n+1} \sim M_n(\zeta_{n+1})(\xi^i_n)$;
for $i \leftarrow 1$ to $N$ do
  | draw $\{(\tilde{\xi}^i_{0,n}, \tilde{\beta}^i_{0,n})^M_{i=1}\} \sim \left(\sum_{\ell=1}^N \frac{q_n(\xi^\ell_n, \zeta^i_n)}{\sum_{\ell'=1}^N q_n(\xi^\ell_n, \zeta^i_n)} \delta(\xi^\ell_n, \beta^\ell_n)\right)^\otimes (M-1)$;
  | draw $\{(\tilde{\xi}^i_{n,j}, \tilde{\beta}^i_{n,j})^M_{j=1}\} \sim \left(\sum_{\ell=1}^N \frac{q_n(\xi^\ell_n, \zeta^i_{n+1})}{\sum_{\ell'=1}^N q_n(\xi^\ell_n, \zeta^i_{n+1})} \delta(\xi^\ell_n, \beta^\ell_n)\right)^\otimes (M-1)$;
  | set $\beta^i_{n+1} \leftarrow \frac{1}{M} \sum_{j=1}^M \left(\tilde{\beta}^i_{n,j} + \tilde{h}_n(\tilde{\xi}^i_{n,j}, \zeta^i_{n+1})\right)$;
  | set $\xi^i_{0:n+1} \leftarrow (\tilde{\xi}^i_{0:n}, \xi^i_{n+1})$ and $\nu^i_{n+1} \leftarrow (\xi^i_{0:n+1}, \beta^i_{n+1})$;
end

Algorithm 2: One conditional PARIS update

Data: $\zeta_{0:n}$
Result: $\nu_n, \zeta_{0:n}'$
draw $\xi_0 \sim \eta_0(\zeta_0)$;
set $\nu_0 \leftarrow \{((\xi_0^i, 0))^N_{i=1}\}$;
for $m \leftarrow 0$ to $n-1$ do
  | set $\nu_{m+1} \leftarrow \text{CondPaRIS}(\nu_m, \zeta_{m+1})$;
end
draw $\zeta_{0:n}' \sim \mu(\zeta_{0:n})$;

Algorithm 3: One iteration of the PARISian particle Gibbs (PPG)
C. Additional Proofs

C.1 Proof of Proposition 2

First, note that, by definitions (3.1) and (3.2),

\[ H_n(x_{0:n}) := \int \mathbb{S}_n(x_{0:n}, dy_n) \mu(x_{0:n}|n) h \]

\[ = \int \cdots \int \left( \frac{1}{N} \sum_{j=1}^{N} h(x_{0:n-1}^j, x_n^j) \right) \]

\[ \times \prod_{m=0}^{n-1} \prod_{i_{m+1}=1}^{N} \int \sum_{j=1}^{N} \frac{q_m(x_m^j, x_{m+1}^j)}{\sum_{j'=1}^{N} q_m(x_m^{j'}, x_{m+1}^{j'})} \delta_{x_0^j, x_0^{j+1}} (dx_0^j) \]

\[ = \frac{1}{N} \sum_{j=1}^{N} \sum_{j_{k-1}=1}^{n-1} \prod_{i_{k+1}=1}^{j_k} \sum_{j_k'=1}^{N} q_{n+k}(x_k^{j_k}, x_{k+1}^{j_k+1}) \alpha_{k,n}(x_0, \ldots, x_{k-1}, x_k^{j_k}, \ldots, x_n^{j_n}) \]

with

\[ \alpha_{k,n}(x_0, \ldots, x_{k-1}, x_k^{j_k}, \ldots, x_n^{j_n}) \]

\[ = \int \prod_{m=0}^{k-1} \prod_{i_{m+1}=1}^{N} \sum_{j_{m+1}=1}^{N} \frac{q_m(x_m^j, x_{m+1}^j)}{\sum_{j'=1}^{N} q_m(x_m^{j'}, x_{m+1}^{j'})} \delta_{x_0^j, x_0^{j+1}} (dx_0^j) \]

\[ \times \prod_{i_k=1}^{N} \sum_{j_{k-1}=1}^{N} q_{k-1}(x_k^{j_{k-1}}, x_k^{j_k}) \delta_{x_{k-1}^{j_{k-1}}, x_{k-1}^{j_{k-1}}} (dx_{k-1}^{j_{k-1}}) \]

and since \( x_{0:k-1}|k-1 = (x_{0:k-2}|k-1, x_{k-1}^{j_{k-1}}) \), it holds that

\[ \int \prod_{i_k=1}^{N} \sum_{j_{k-1}=1}^{N} q_{k-1}(x_k^{j_{k-1}}, x_k^{j_k}) \delta_{x_{k-1}^{j_{k-1}}, x_{k-1}^{j_{k-1}}} (dx_{k-1}^{j_{k-1}}) \]

\[ = \sum_{j_{k-1}=1}^{N} \sum_{j_k'=1}^{N} q_{k-1}(x_k^{j_{k-1}}, x_k^{j_k}) \delta_{x_{k-1}^{j_{k-1}}, x_{k-1}^{j_{k-1}}} (dx_{k-1}^{j_{k-1}}) \]

\[ \int \prod_{i_k=1}^{N} \sum_{j_{k-1}=1}^{N} q_{k-1}(x_k^{j_{k-1}}, x_k^{j_k}) \delta_{x_{k-1}^{j_{k-1}}, x_{k-1}^{j_{k-1}}} (dx_{k-1}^{j_{k-1}}) \]

\[ \times \sum_{j_k'=1}^{N} q_{k}(x_k^{j_{k-1}}, x_k^{j_k}) \delta_{x_{k-1}^{j_{k-1}}, x_{k-1}^{j_{k-1}}} (dx_{k-1}^{j_{k-1}}) \]

\[ \times \sum_{j_k''=1}^{N} q_{k}(x_k^{j_{k-1}}, x_k^{j_k}) \delta_{x_{k-1}^{j_{k-1}}, x_{k-1}^{j_{k-1}}} (dx_{k-1}^{j_{k-1}}) \]

\[ \times \sum_{j_k'''=1}^{N} q_{k}(x_k^{j_{k-1}}, x_k^{j_k}) \delta_{x_{k-1}^{j_{k-1}}, x_{k-1}^{j_{k-1}}} (dx_{k-1}^{j_{k-1}}) \]
Therefore, we obtain

\[ a_{k,n}(x_0, \ldots, x_{k-1}, x_{k}^{j_k}, \ldots, x_n^{j_n}) = \int \prod_{m=0}^{k-2} \prod_{m+1=1}^{N} \prod_{m+1}^{N} \sum_{j_m=1}^{N} q_m(x_m^{j_m}, x_m^{j_m+1}) \delta_{x_m^{j_m}}(d_x^{j_m+1}) \]

\[ \times \prod_{j_{k-1}=1}^{N} \sum_{j_{k-1}=1}^{N} q_{k-1}(x_{k-1}^{j_k}, x_k^{j_k}) \beta(x_{0:k-2}^{j_k-1}, x_{k-1}^{j_k-1}, x_k^{j_k+1}, \ldots, x_n^{j_n}). \]

Now, changing the order of summation with respect to \( j_{k-1} \) and integration on the right hand side of the previous display yields

\[ a_{k,n}(x_0, \ldots, x_{k-1}, x_{k}^{j_k}, \ldots, x_n^{j_n}) = \sum_{j_{k-1}=1}^{N} \sum_{j_{k-1}=1}^{N} q_{k-1}(x_{k-1}^{j_k}, x_k^{j_k}) a_{k-1,n}(x_0, \ldots, x_{k-2}, x_{k-1}^{j_k-1}, \ldots, x_n^{j_n}). \]

Thus,

\[ H_{k,n}(x_{0:n}) = \frac{1}{N} \sum_{j_n=1}^{N} \cdots \sum_{j_k=1}^{N} \prod_{k=1}^{n-1} q_{k}(x_{k}^{j_k}, x_{k+1}^{j_k+1}) \]

\[ \times \sum_{j_{k-1}=1}^{N} \sum_{j_{k-1}=1}^{N} q_{k-1}(x_{k-1}^{j_k}, x_k^{j_k}) a_{k-1,n}(x_0, \ldots, x_{k-2}, x_{k-1}^{j_k-1}, \ldots, x_n^{j_n}) \]

\[ = \frac{1}{N} \sum_{j_n=1}^{N} \cdots \sum_{j_k=1}^{N} \prod_{k=1}^{n-1} q_{k}(x_{k}^{j_k}, x_{k+1}^{j_k+1}) a_{k-1,n}(x_0, \ldots, x_{k-2}, x_{k-1}^{j_k-1}, \ldots, x_n^{j_n}) \]

\[ = H_{k-1,n}(x_{0:n}), \]

which establishes the recursion. Therefore, \( H_n \equiv H_{0,n} \) and we may now conclude the proof by noting that \( \mathbb{E}_n h \equiv H_{0,n} \).

### C.2 Proof of Theorem 5

In order to establish Theorem 5 we will prove the following more general result, of which Theorem 5 is a direct consequence.

**Proposition 7.** For every \( n \in \mathbb{N} \) and \( M \in \mathbb{N}^* \) there exist \( c_n > 0 \) and \( d_n > 0 \) such that
for every $N \in \mathbb{N}^*$, $z_{0:n} \in X_{0:n}$, $(f_n, \tilde{f}_n) \in \mathcal{F}(X_n)^2$, and $\varepsilon > 0$,

\[
\int \mathbb{C}_n \mathbb{S}_n(z_{0:n}, dB_n) \times 1 \left\{ \frac{1}{N} \sum_{i=1}^{N} \{ h'_n(x_{i|n}) + \tilde{f}_n(x_{i|n}) \} - \eta_n(z_{0:n})(f_n B_n(z_{0:n-1}) h_n + \tilde{f}_n) \geq \varepsilon \right\} \leq c_n \exp \left( -\frac{d_n N \varepsilon^2}{2 \kappa_n^2} \right),
\]

where

\[
\kappa_n := \| f_n \|_{\infty} \sum_{m=0}^{n-1} \| \tilde{h}_m \|_{\infty} + \| \tilde{f}_n \|_{\infty}. \tag{C.1}
\]

To prove Proposition 7 we need the following technical lemma.

**Lemma 5.** For every $n \in \mathbb{N}$, $(f_{n+1}, \tilde{f}_{n+1}) \in \mathcal{F}(X_{n+1})^2$, $z_{0:n+1} \in X_{0:n+1}$, and $N \in \mathbb{N}^*$,

\[
\gamma_{n+1}(z_{0:n+1})(f_{n+1} B_{n+1}(z_{0:n}) h_{n+1} + \tilde{f}_{n+1}) = \left( 1 - \frac{1}{N} \right) \gamma_n(z_{0:n}) \{ Q_n f_{n+1} B_n(z_{0:n-1}) h_n + Q_n(\tilde{h}_n f_{n+1} + \tilde{f}_{n+1}) \}
\]

\[
+ \frac{1}{N} \gamma_n(z_{0:n}) g_n \left( f_{n+1}(z_{n+1}) B_{n+1}(z_{0:n}) h_{n+1}(z_{n+1}) + \tilde{f}_{n+1}(z_{n+1}) \right).
\]

**Proof.** Since Lemma 2 holds also for the Feynman–Kac model with a frozen path, we obtain

\[
\gamma_{n+1}(z_{0:n+1})(f_{n+1} B_{n+1}(z_{0:n}) h_{n+1} + \tilde{f}_{n+1}) = \gamma_n(z_{0:n}) \{ Q_n f_{n+1} B_n(z_{0:n}) h_n + Q_n(\tilde{h}_n f_{n+1} + \tilde{f}_{n+1}) \}.
\]

Thus, the proof is concluded by noting that for every $x_n \in X_n$ and $h \in \mathcal{F}(X_{n+1})$,

\[
Q_n(z_{n+1}) h(x_n) = \left( 1 - \frac{1}{N} \right) Q_n h(x_n) + \frac{1}{N} g(x_n) h(x_n, z_{n+1}).
\]

\[\square\]

Finally, before proceeding to the proof of Proposition 7, we introduce the law of the PARIS evolving conditionally on a frozen path $\tilde{z} = \{ \tilde{z}_m \}_{m \in \mathbb{N}}$. Define, for $m \in \mathbb{N}$ and $z_{m+1} \in X_{m+1}$,

\[
P^\z_m(z_{m+1}) : Y_m \times Y_{m+1} \ni (y_m, A) \mapsto \int M^\z_m(z_{m+1})(x_{m|m}, dx_{m+1}) S^\z_m(y_m, x_{m+1}, A).
\]

For any given initial distribution $\psi_0 \in M_1(Y_0)$, let $P^\z_\psi$ be the distribution of the canonical Markov chain induced by the Markov kernels $\{ P^\z_m(z_{m+1}) \}_{m \in \mathbb{N}}$ and the initial distribution $\psi_0$. By abuse of notation we write $P^\z_{\psi_0}$ instead of $P^\z_{\psi_0(\eta_0(\tilde{z}_0))}$, where the extension $\psi_0|\eta_0$ is defined in Section 5.3.
Proof of Proposition 7. We proceed by forward induction over \( n \). Let the \( \sigma \)-fields \( \tilde{\mathcal{F}}_n \) and \( \mathcal{F}_n \) be defined as in the proof of Theorem 3, but for the conditional PARIS dual process. Then, under the law \( \mathbb{P}_{\eta_0}^{P,z} \), reusing (5.11),

\[
\mathbb{P}_{\eta_0}^{P,z} \left[ \beta_n f_n(\xi_1^n) + \tilde{f}_n(\xi_1^n) \mid \tilde{\mathcal{F}}_{n-1} \right]
\]

\[
= \mathbb{P}_{\eta_0}^{P,z} \left[ \mathbb{P}_{\eta_0}^{P,z} \left[ \beta_n f_n(\xi_1^n) + \tilde{f}_n(\xi_1^n) \mid \mathcal{F}_{n-1} \right] \right]
\]

\[
= \mathbb{P}_{\eta_0}^{P,z} \left[ f_n(\xi_1^n) \sum_{\ell=1}^{N} \frac{q_{n-1}(\xi_{n-1}^{\ell},\xi_1^n)}{\sum_{\ell'=1}^{N} q_{n-1}(\xi_{n-1}^{\ell'},\xi_1^n)} \left( \beta_n^{\ell} + \tilde{h}_n(\xi_{n-1}^{\ell},\xi_1^n) \right) + \tilde{f}_n(\xi_1^n) \mid \tilde{\mathcal{F}}_{n-1} \right].
\]

Using (2.6), we get

\[
\mathbb{P}_{\eta_0}^{P,z} \left[ \beta_n f_n(\xi_1^n) + \tilde{f}_n(\xi_1^n) \mid \mathcal{F}_{n-1} \right]
\]

\[
= \left( 1 - \frac{1}{N} \right) \sum_{\ell=1}^{N} \frac{\beta_n^{\ell} Q_{n-1} f_n(\xi_{n-1}^{\ell}) + Q_{n-1}(\tilde{h}_n f_n + \tilde{f}_n)(\xi_{n-1}^{\ell})}{\sum_{\ell'=1}^{N} q_{n-1}(\xi_{n-1}^{\ell'},\xi_1^n)}
\]

\[
+ \frac{1}{N} \sum_{\ell=1}^{N} \frac{q_{n-1}(\xi_{n-1}^{\ell},z_n)}{\sum_{\ell'=1}^{N} q_{n-1}(\xi_{n-1}^{\ell'},\xi_1^n)} \left( \beta_n^{\ell} + \tilde{h}_n(\xi_{n-1}^{\ell},z_n) \right) + \tilde{f}_n(z_n). \tag{C.2}
\]

In order to apply the induction hypothesis to each term on the right-hand side of the previous identity, note that

\[
B_n(z_{0:n-1})h_n(z_n) = \frac{\eta_n(z_{0:n-1}) q_{n-1}(\cdot,z_n) \{ B_{n-1}(z_{0:n-2}) h_{n-1}(\cdot) + \tilde{h}_n(\cdot,z_n) \}}{\eta_n(z_{0:n-1}) q_{n-1}(\cdot,z_n)}.
\]

Therefore, using Lemma 5 and noting that \( \gamma_n(z_{0:n}) 1_{\mathcal{X}_n} / \gamma_{n-1}(z_{0:n}) 1_{\mathcal{X}_{n-1}} = \eta_n(z_{0:n-1}) q_{n-1} \) yields

\[
\eta_n(z_{0:n}) (f_n B_n(z_{0:n-1}) h_n + \tilde{f}_n) = \frac{1}{N} \left( f_n(z_n) B_n(z_{0:n-1}) h_n(z_n) + \tilde{f}_n(z_n) \right)
\]

\[
+ \left( 1 - \frac{1}{N} \right) \eta_n(z_{0:n-1}) \{ Q_{n-1} f_n B_{n-1}(z_{0:n-2}) h_n + Q_{n-1}(\tilde{h}_n f_n + \tilde{f}_n) \}. \tag{C.3}
\]

By combining (C.2) with (C.3), we decompose the error according to

\[
\frac{1}{N} \sum_{i=1}^{N} \{ \beta_i f_n(\xi_{n|i}^i) + \tilde{f}_n(\xi_{n|i}^i) \} - \eta_n(z_{0:n}) (f_n B_n(z_{0:n-1}) h_n + \tilde{f}_n)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \{ \beta_i f_n(\xi_{n|i}^i) + \tilde{f}_n(\xi_{n|i}^i) \} - \mathbb{E}_{\eta_0}^{P,z} \left[ \beta_n f_n(\xi_1^n) + \tilde{f}_n(\xi_1^n) \mid \tilde{\mathcal{F}}_{n-1} \right]
\]

\[
+ \mathbb{E}_{\eta_0}^{P,z} \left[ \beta_n f_n(\xi_1^n) + \tilde{f}_n(\xi_1^n) \mid \tilde{\mathcal{F}}_{n-1} \right] - \eta_n(z_{0:n}) (f_n B_n(z_{0:n-1}) h_n + \tilde{f}_n)
\]

\[
= I_N^{(1)} + \left( 1 - \frac{1}{N} \right) I_N^{(2)} + \frac{1}{N} I_N^{(3)}. \tag{C.4}
\]
Lemma 4] yields, using also the bounds

\[ \sum_{\ell=1}^N \{ \beta_n^\ell f_n(\xi_n^\ell) + \tilde{f}_n(\xi_n^\ell) \} - \mathbb{E}_{P_{\eta_0}} \left[ \beta_n^1 f_n(\xi_n^1) + \tilde{f}_n(\xi_n^1) \mid \tilde{f}_{n-1} \right], \]

\[ \sum_{\ell=1}^N \{ \beta_n^\ell Q_{n-1} f_n(\xi_n^\ell) + Q_{n-1}(\tilde{h}_{n-1} f_n + \tilde{f}_n)(\xi_n^\ell) \} \]

\[ \frac{\eta_{n-1}(z_{0:n-1})}{\eta_{n-1}(z_{0:n-1})} \{ Q_{n-1} f_n B_n(z_{0:n-1}) h_n + Q_{n-1}(\tilde{h}_{n-1} f_n + \tilde{f}_n) \} \]  \[ \frac{\eta_{n-1}(z_{0:n-1})}{\eta_{n-1}(z_{0:n-1})} \{ Q_{n-1} f_n B_n(z_{0:n-1}) h_n + Q_{n-1}(\tilde{h}_{n-1} f_n + \tilde{f}_n) \} \]

(C.5)

and

\[ \left( \beta_n^\ell + \tilde{h}_{n-1}(\xi_n^\ell) \right) \]

\[ - \beta_n^\ell f_n(z_n) - \eta_{n-1}(z_{0:n-1}) g_{n-1}(\xi_n^\ell) \eta_{n-1}(z_{0:n-1}) \{| Q_{n-1} f_n B_n(z_{0:n-1}) h_n + Q_{n-1}(\tilde{h}_{n-1} f_n + \tilde{f}_n) \} \]

(C.6)

The proof is now completed by treating the terms \( I_N^{(1)}, \) \( I_N^{(2)}, \) and \( I_N^{(3)} \) separately, using Hoeffding’s inequality and its generalisation in [17, Lemma 4]. Choose \( \varepsilon > 0; \) then, by Hoeffding’s inequality,

\[ \mathbb{P}_{P_{\eta_0}} \left( \left| I_N^{(1)} \right| \geq \varepsilon \right) \leq 2 \exp \left( -\frac{1}{2} \frac{\varepsilon^2}{\kappa_n^2} N \right). \]

(C.7)

To treat \( I_N^{(2)}, \) we apply the induction hypothesis to the numerator and denominator, each normalised by \( 1/N, \) yielding, since \( \| Q_{n-1} h \|_\infty \leq \tau_{n-1} \| h \|_\infty \) for all \( h \in F(\mathcal{X}_{n-1} \otimes \mathcal{X}_n), \)

\[ \mathbb{P}_{P_{\eta_0}} \left( \left| \frac{1}{N} \sum_{\ell=1}^N \{ \beta_n^\ell Q_{n-1} f_n(\xi_n^\ell) + Q_{n-1}(\tilde{h}_{n-1} f_n + \tilde{f}_n)(\xi_n^\ell) \} \right| \geq \varepsilon \right) \]

\[ \leq C_{n-1} \exp \left( -\frac{d_{n-1} \varepsilon^2}{\tau_{n-1}^2 \kappa_n^2} N \right) \]

and

\[ \mathbb{P}_{P_{\eta_0}} \left( \left| \frac{1}{N} \sum_{\ell=1}^N g_{n-1}(\xi_n^\ell) - \eta_{n-1}(z_{0:n-1}) g_{n-1} \right| \geq \varepsilon \right) \leq C_{n-1} \exp \left( -\frac{d_{n-1} \varepsilon^2}{\tau_{n-1}^2 \kappa_n^2} N \right) \]

Combining the previous two bounds with the generalised Hoeffding inequality in [17, Lemma 4] yields, using also the bounds

\[ \mathbb{E}_{P_{\eta_0}} \left[ \beta_n^1 f_n(\xi_n^1) + \tilde{f}_n(\xi_n^1) \mid \tilde{f}_{n-1} \right], \]

\[ \sum_{\ell=1}^N \{ \beta_n^\ell Q_{n-1} f_n(\xi_n^\ell) + Q_{n-1}(\tilde{h}_{n-1} f_n + \tilde{f}_n)(\xi_n^\ell) \} \]

\[ \sum_{\ell=1}^N g_{n-1}(\xi_n^\ell) \]

\[ \leq \kappa_n \]
Finally, combining the bounds (C.7–C.9) completes the proof.

The last term $I_N^{(3)}$ is treated along similar lines; indeed, by the induction hypothesis, since $\|q_{n-1}\|_\infty \leq \bar{\tau}_{n-1} \bar{\sigma}_{n-1}$,

$$
\mathbb{P}_{\eta_0} \left( \left| \frac{1}{N} \sum_{\ell=1}^{N} q_{n-1}(\xi_{n-1}^\ell, z_{n}) \left( \beta_{n-1} + h_{n-1}(\xi_{n-1}^\ell, z_{n}) \right) \right| \geq \varepsilon \right) 
\leq c_{n-1} \exp \left( -d_{n-1} \left( \frac{\varepsilon}{\bar{\tau}_{n-1} \bar{\sigma}_{n-1} \sum_{m=0}^{n-1} \|h_m\|_\infty} \right)^2 N \right)
$$

and

$$
\mathbb{P}_{\eta_0} \left( \left| \frac{1}{N} \sum_{\ell=1}^{N} q_{n-1}(\xi_{n-1}^\ell, z_{n}) - \eta_{n-1}(x_{0:n-1}) \eta_{n-1}(z_{0:n-1}) \right| \geq \varepsilon \right) 
\leq c_{n-1} \exp \left( -d_{n-1} \left( \frac{\varepsilon}{\bar{\tau}_{n-1} \bar{\sigma}_{n-1} \sum_{m=0}^{n-1} \|h_m\|_\infty} \right)^2 N \right).
$$

Thus, since

$$
\sum_{\ell=1}^{N} \frac{q_{n-1}(\xi_{n-1}^\ell, z_{n})}{\sum_{\ell'=1}^{N} q_{n-1}(\xi_{n-1}^{\ell'}, z_{n})} \left( \beta_{n-1} + h_{n-1}(\xi_{n-1}^\ell, z_{n}) \right) \leq \sum_{m=0}^{n-1} \|h_m\|_\infty
$$

and $\eta_{n-1}(x_{0:n-1}) \eta_{n-1}(z_{0:n-1}) \geq \tau_{n-1}$, the generalised Hoeffding inequality provides

$$
\mathbb{P}_{\eta_0} \left( |I_N^{(3)}| \geq \varepsilon \right) \leq c_{n-1} \exp \left( -d_{n-1} \left( \frac{\tau_{n-1} \varepsilon}{2 \bar{\tau}_{n-1} \bar{\sigma}_{n-1} \|f_n\|_\infty \sum_{m=0}^{n-1} \|h_m\|_\infty} \right)^2 N \right).
$$

Finally, combining the bounds (C.7–C.9) completes the proof.

\[ \square \]

### C.3 Proof of Proposition 3

The statement of Proposition 3 is implied by the following more general result, which we will prove below.

**Proposition 8.** For every $n \in \mathbb{N}$, $M \in \mathbb{N}^*$, $N \in \mathbb{N}^*$, $x_{0:n} \in \mathcal{X}_{0:n}$, $(f_n, \tilde{f}_n) \in F(X_n)^2$, and $p \geq 2$, it holds that

$$
\int \mathbb{C}_n \mathbb{S}_n(x_{0:n}, \mathbf{d}_n) \left| \frac{1}{N} \sum_{i=1}^{N} \left[ b_{n_i} f_n(x_{n_i|n}) + \tilde{f}_n(x_{n_i|n}) \right] - \eta_n(x_{0:n}) (f_n B_n(x_{0:n-1}) h_n + \tilde{f}_n) \right|^p 
\leq c_n (p/d_n)^{p/2} N^{-p/2} \kappa_n^p,
$$
where \( c_n > 0, d_n > 0 \) and \( \kappa_n \) are defined in Proposition 7 and (C.1), respectively.

Before proving Proposition 8, we establish the following result.

**Lemma 6.** Let \( X \) be an \( \mathbb{R}^d \)-valued random variable, defined on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), satisfying \( \mathbb{P}(\|X\| \geq t) \leq c \exp(-t^2/(2\sigma^2)) \) for every \( t \geq 0 \) and some \( c > 0 \) and \( \sigma > 0 \). Then for every \( p \geq 2 \) it holds that \( \mathbb{E}[\|X\|^p] \leq cp^{p/2}\sigma^p \).

**Proof.** Using Fubini’s theorem and the change of variable formula,

\[
\mathbb{E}[\|X\|^p] = \int_0^\infty pt^{p-1}\mathbb{P}(\|X\| \geq t) \, dt = cp^{p/2-1}\sigma^p\Gamma(p/2),
\]

where \( \Gamma \) is the Gamma function. It remains to apply the bound \( \Gamma(p/2) \leq (p/2)^{p/2-1} \) (see [1]), which holds for \( p \geq 2 \) by [2, Theorem 1.5]. \( \square \)

**Proof of Proposition 8.** By combining Proposition 7 and Lemma 6 we obtain

\[
N \int C_n S_n(z_{0:n}, db_n) \frac{1}{N} \sum_{i=1}^N \left| b_i f_n(x_{n|n}^i) + \hat{f}_n(x_{n|n}^i) - \eta_n(z_{0:n}) f_n B_n(z_{0:n-1}) h_n + \tilde{f}_n \right|^2 \leq c_n (p/d_n)^{p/2} N^{-p/2} \left( \|f_n\|_\infty \sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty + \|\tilde{f}_n\|_\infty \right)^p,
\]

which was to be established. \( \square \)

### C.4 Proof of Proposition 4

Like previously, we establish Proposition 4 via a more general result, namely the following.

**Proposition 9.** For every \( n \in \mathbb{N} \), there exists \( \kappa_n \) such that for every \( M \in \mathbb{N}^* \), \( N \in \mathbb{N}^* \), \( z_{0:n} \in X_{0:n} \), and \( (f_n, \hat{f}_n) \in \mathcal{F}(X_n)^2 \),

\[
\left| \int C_n S_n(z_{0:n}, db_n) \frac{1}{N} \sum_{i=1}^N \left| b_i f_n(x_{n|n}^i) + \hat{f}_n(x_{n|n}^i) - \eta_n(z_{0:n}) f_n B_n(z_{0:n-1}) h_n + \tilde{f}_n \right| \right| \leq c_n^{\text{bias}} \kappa_n N^{-1},
\]

where \( \kappa_n \) is defined in (C.1).

We preface the proof of Proposition 9 by a technical lemma providing a bound on the bias of ratios of random variables.

**Lemma 7.** Let \( \alpha \) and \( \beta \) be (possibly dependent) random variables defined on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and such that \( \mathbb{E}[\alpha^2] < \infty \) and \( \mathbb{E}[\beta^2] < \infty \). Moreover, assume that there exist \( c > 0 \) and \( d > 0 \) such that \( |\alpha/\beta| \leq c \), \( \mathbb{P}\text{-a.s.} \), \( |a/b| \leq c \), \( \mathbb{E}[(\alpha-a)^2] \leq c^2 d^2 \), and \( \mathbb{E}[(\beta-b)^2] \leq d^2 \). Then

\[
|\mathbb{E}[\alpha/\beta] - a/b| \leq 2c(d/b)^2 + c|\mathbb{E}[\beta - b]|/b + |\mathbb{E}[\alpha - a]|/b.
\] (C.10)
Proof. Using the identity
\[ \mathbb{E}[\alpha/\beta] - a/b = \mathbb{E}[(\alpha/\beta)(b - \beta)^2]/b^2 + \mathbb{E}[(\alpha - a)(b - \beta)]/b^2 + a\mathbb{E}[b - \beta]/b^2 + E[a - a]/b, \]
the claim is established by applying the Cauchy–Schwarz inequality and the assumptions of the lemma according to
\[ |\mathbb{E}[\alpha/\beta] - a/b| \leq c\mathbb{E}[(\beta - b)^2]/b^2 + \mathbb{E}[(\alpha - a)^2] \leq 1/2 + |a||\mathbb{E}[b - \beta]|/b^2 + |\mathbb{E}[\alpha - a]|/b. \]
\[ \square \]

Proof of Proposition 4. We proceed by induction and assume that the claim holds true for \( n - 1 \). Reusing the error decomposition (C.4), it is enough to bound the expectations of the terms \( \Gamma_N^{(2)} \) and \( \Gamma_N^{(3)} \) given in (C.5) and (C.6), respectively (since \( \mathbb{E}_{\eta_0}^{\pi}[\Gamma_N^{(1)}] = 0 \)). This will be done using the induction hypothesis, Lemma 7, and Proposition 8. More precisely, to bound the expectation of \( \Gamma_N^{(2)} \), we use Lemma 7 with \( \alpha \leftarrow \alpha_n, \beta \leftarrow \beta_n, \alpha \leftarrow a_n, \) and \( b \leftarrow b_n \), where
\[ \alpha_n := \frac{1}{N} \sum_{\ell=1}^{N} (\beta_n^{\ell} Q_n - f_n(h_n^{\ell} + \hat{f}_n^\ell)(\xi_n^{\ell} - 1)), \quad \beta_n := \frac{1}{N} \sum_{\ell=1}^{N} g_n^{\ell}(\xi_n^{\ell} - 1), \]
\[ a_n := \eta_n(\zeta_0^{\ell} Q_n - f_n(h_n^{\ell} + \hat{f}_n^\ell)(\xi_n^{\ell} - 1)), \quad b_n := \eta_n g_n^{\ell}(\zeta_0^{\ell} Q_n - f_n(h_n^{\ell} + \hat{f}_n^\ell)(\xi_n^{\ell} - 1)). \]
For this purpose, note that \( |\alpha_n/\beta_n| \leq \kappa_n \) and \( |a_n/b_n| \leq \kappa_n \), where \( \kappa_n \) is defined in (C.1). On the other hand, using Proposition 8 (applied with \( p = 2 \)), we obtain
\[ \mathbb{E}_{\eta_0}^{\pi}[\alpha_n - a_n]^2 \leq d_n^2 \kappa_n^2 \quad \text{and} \quad \mathbb{E}_{\eta_0}^{\pi}[\beta_n - b_n]^2 \leq d_n^2, \]
where \( d_n^2 := c_n \tau_n^{-1}/(d_n N) \). Using the induction assumption, we get
\[ |\mathbb{E}_{\eta_0}^{\pi}[\alpha_n - a_n] - \xi_n^{\ell} N^{-1} \tau_n^{-1} k_n| \quad \text{and} \quad |\mathbb{E}_{\eta_0}^{\pi}[\beta_n - b_n]| \leq \xi_n^{\ell} N^{-1} \tau_n^{-1} k_n. \]
Hence, the conditions of Lemma 7 are satisfied and we deduce that
\[ |\mathbb{E}_{\eta_0}^{\pi}[\Gamma_N^{(2)}]| = \mathbb{E}_{\eta_0}^{\pi}[\alpha_n/\beta_n] - a_n/b_n| \leq 2\kappa_n \frac{c_n \tau_n}{d_n N} \sum_{\ell=1}^{N} \|\hat{h}_\ell\|_{\infty} + 2\kappa_n \xi_n^{\ell} N^{-1} \tau_n^{-1} k_n. \]
The bound on \( |\mathbb{E}_{\eta_0}^{\pi}[\Gamma_N^{(2)}]| \) is obtained along the same lines. \( \square \)

C.5 Proof of Theorem 6

We first consider the bias, which can be bounded according to
\[ |\mathbb{E}_t[\Pi(\kappa_0, k)\cdot N(f)] - \eta_{0:n} h_n| \leq (k - k_0)^{-1} \sum_{\ell=k_0+1}^{k} |\mathbb{E}_t[\mu(\beta_n[\ell])\cdot (id) - \eta_{0:n} h_n| \]
\[ \leq (k - k_0)^{-1} N^{-1} c_n^\text{bias} \left( \sum_{m=0}^{k-1} \|\hat{h}_m\|_{\infty} \right) \sum_{\ell=k_0+1}^{k} N_{\ell,n}, \]
from which the bound (4.7) follows immediately.

We turn to the MSE. Using the decomposition

$$
\mathbb{E}_\xi[(\Pi_{(k_0,k),N}(f) - \eta_{0:n}h_n)^2] \leq (k - k_0)^{-2} \left\{ \sum_{\ell=k_0+1}^{k} \mathbb{E}_\xi[(\mu(\beta_n[i])(id) - \eta_{0:n}h_n)^2] 
+ 2 \sum_{\ell=k_0+1}^{k} \sum_{j=\ell+1}^{k} \mathbb{E}_\xi[(\mu(\beta_n[i])(id) - \eta_{0:n}h_n)(\mu(\beta_n[j])(id) - \eta_{0:n}h_n)] \right\},
$$

the MSE bound in Theorem 2 implies that

$$
\sum_{\ell=k_0+1}^{k} \mathbb{E}_\xi[(\mu(\beta_n[i])(id) - \eta_{0:n}h_n)^2] \leq c_{\text{mse}} \left( \sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty \right)^2 N^{-1}(k - k_0).
$$

Moreover, using the covariance bound in Theorem 2, we deduce that

$$
\sum_{\ell=k_0+1}^{k} \sum_{j=\ell+1}^{k} \mathbb{E}_\xi[(\mu(\beta_n[i])(id) - \eta_{0:n}h_n)(\mu(\beta_n[j])(id) - \eta_{0:n}h_n)] 
\leq c_{\text{cov}} \left( \sum_{m=0}^{n-1} \|\tilde{h}_m\|_\infty \right)^2 N^{-3/2} \left( \sum_{\ell=k_0+1}^{k} \sum_{j=\ell+1}^{k} \kappa_{N,n}^{(j-\ell)} \right).
$$

Thus, the proof is concluded by noting that

$$
\sum_{\ell=k_0+1}^{k} \sum_{j=\ell+1}^{k} \kappa_{N,n}^{(j-\ell)} \leq (k - k_0)/(1 - \kappa_{N,n}).
$$