Abstract. In this article we try to explore the relation between real conjugacy classes and real characters of finite groups at more refined level. This refinement is in terms of properties of groups such as strong reality and total orthogonality. In this connection we raise several questions and record several examples which have motivated those questions.

1. Introduction

Let $G$ be a finite group. It is a basic statement in the character theory of finite groups that the number of irreducible complex characters is same as the number of conjugacy classes in $G$. Further this statement can be refined to the number of real irreducible characters being equal to the number of real conjugacy classes. However it is very well known that the real characters come from two kinds of representations: orthogonal and symplectic [Se, Section 13.2 Prop. 39]. Schur himself explored this topic and now this is very closely related to Schur index computation. In last 10 years the computation of Schur indices have been almost completed for finite groups of Lie type (for example see [Gr]).

In this article we raise the question to further divide real conjugacy classes in two parts as to match the sizes of partitions with the number of orthogonal characters and symplectic ones. In section 2 we give basic definitions and ask several questions concerning relations between conjugacy classes and characters. In section 3 we define Schur indices of a representation. In section 4 we define canonical involution on a group algebra and mention known results about when the involution restricts to the simple components. In the following section we provide examples which provide strength to the questions raised earlier. This is the main objective of this article. In the section 6 we mention the Lie algebra defined for a group algebra which makes use of real conjugacy classes. Results mentioned in this article are either calculations using computer algebra system GAP or have been collected from various sources related to real conjugacy classes. Some of the questions raised in this article are already known to experts (for example see [GNT]).

2000 Mathematics Subject Classification. 20C15, 20C33.

Key words and phrases. Schur indices, real, strongly real, characters, group algebra, involutions etc.

The second named author thanks IMS for invitation to give a talk in its 76th annual meeting held in Surat.
2. CONJUGACY CLASSES VS. REPRESENTATIONS FOR A GROUP

Let $G$ be a group. An element $g \in G$ is called real if there exists $t \in G$ such that $tg^{-1} = g^{-1}$. An element $g \in G$ is called involution if $g^2 = 1$. An element $g \in G$ is called strongly real if it is a product of two involutions in $G$. Further notice that if an element is (strongly) real then all its conjugates are (strongly) real. Hence reality (i.e. being real) and strong reality are properties of conjugacy classes. The conjugacy classes of involutions and more generally strongly real elements are obvious examples of real classes. However converse need not be true.

Example 2.1. Take $G = Q_8$, the finite quaternion group. Then we see that $iji^{-1} = j^{-1}$ hence $j$ is real but it is not strongly real.

A representation of a group $G$ is a homomorphism $\rho: G \to GL(V)$ where $V$ is a vector space over a field $k$. The representation $\rho$ is called irreducible if $V$ and $\{0\}$ are the only subspaces $W$ of $V$ satisfying $\rho(G).W \subseteq W$. Let $V^*$ denote the dual vector space of $V$. The dual representation of $\rho$ is the representation $\rho^*: G \to GL(V^*)$ given by $\rho^*(g) = {}^t\rho(g^{-1})$, where $^t\rho(g^{-1})$ is the transpose of $\rho(g^{-1})$. To a representation $\rho$ its associated character $\chi: G \to k$ is defined by $\chi(g) = \text{trace}(\rho(g))$. In this section we only consider complex representations (i.e. $k = \mathbb{C}$). It is a classical theorem that the number of conjugacy classes in $G$ is equal to the number of irreducible complex characters [Se, Section 2.5 Theorem 7]. A theorem due to Brauer [JL, Chapter 23] asserts that the number of real conjugacy classes is same as the number of irreducible real characters (i.e. the complex characters which take real values only).

Let $\rho: G \to GL(V)$ be an irreducible complex representation of $G$ and $\chi$ be the associated character. If $\chi$ takes a complex value then the representation $\rho$ is not isomorphic to its dual $\rho^*$ and the vector space $V$ can not afford a $G$-invariant non-zero bilinear form. If $\chi$ is real then $\rho \cong \rho^*$ and in this case $V$ admits a non-zero $G$-invariant bilinear form. This form can be either symmetric or skew-symmetric depending on whether $V$ is defined over $\mathbb{R}$ or not (see [Se, Section 13.2]). Equivalently the image of $G$ sits inside $O_n$ in the first case and inside $Sp_{2n}$ in the second case. To each character $\chi$ one associates Schur indicator $\nu(\chi)$ which is defined as follows:

$$
\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).
$$

In fact $\nu(\chi) = 0, \pm 1$ and $\nu(\chi) = 0$ if and only if $\chi$ is not real, $\nu(\chi) = 1$ if $\chi$ is orthogonal and symplectic otherwise [Se, Prop. 39]. Hence the real characters come in two classes: orthogonal type and symplectic type. This gives a natural division of real characters. We now have following questions:

Question 2.2. Let $G$ be a finite group.
(1) Can we naturally divide real conjugacy classes of $G$ in two parts so that the number of one part is same as the number of orthogonal representations (we are specially interested in groups of Lie type)?

(2) Is it true that if a group $G$ has no symplectic character (i.e. all self-dual representations are orthogonal) then all real elements are strongly real and vice versa?

A careful look at the examples in section 5 will suggest that these questions are indeed interesting to ask.

We now restrict our attention to those groups in which all elements are real. Such groups are called real or ambivalent. Tiep and Zalesski classify all real finite quasi-simple groups in [TZ]. For real groups all characters are real valued. Two interesting subclasses of the real groups are the following:

- Subclass in which all elements are strongly real. Groups belonging to this class are called strongly real groups.
- Subclass in which all characters are orthogonal. Groups belonging to this class are called totally orthogonal or ortho-ambivalent.

An element $g \in G$ is called rational if $g$ is conjugate to $g^i$ whenever $g$ and $g^i$ generate the same subgroup of $G$. A group is called rational if every element of $g$ is rational.

**Theorem 2.3** ([Sc] 13.1-13.2). A group $G$ is rational if and only if all its characters are $\mathbb{Q}$-valued.

In fact, the number of isomorphism classes of irreducible representations of $G$ over $\mathbb{Q}$ is same as the number of conjugacy classes of cyclic subgroups of $G$.

**Example 2.4.** Let $G = S_n$, the symmetric group. Then every element of $G$ is a product of two involutions and hence strongly real. It is also totally orthogonal. All its character are integer valued. In fact, it is a rational group.

**Example 2.5.** The group $Q_8$ has four 1-dimensional representation which are orthogonal and one symplectic representation. This group is neither strongly real nor totally orthogonal.

Let us denote the class of finite groups which are real by $\mathcal{R}$, the class of real groups which have their Sylow 2-subgroup Abelian by $\mathcal{S}$, the class of strongly real groups by $\mathcal{SR}$ and the class of totally orthogonal groups by $\mathcal{TO}$. Then using the results of [WG] and [Ar] we have the following:

$$\mathcal{S} \subset \mathcal{SR} \subset \mathcal{R} \quad \text{and} \quad \mathcal{S} \subset \mathcal{TO} \subset \mathcal{R}.$$ 

In particular they prove that if $G$ is real then it is generated by its 2-elements and if $G$ is totally orthogonal then it is generated by involutions. In view of above relations we ask following question:
Question 2.6. Find the class $\mathcal{SR} \cap \mathcal{TO}$.

We know that the class $\mathcal{SR} \cap \mathcal{TO}$ contains $\mathcal{S}$. The containment though, is far from being equality. The dihedral group $D_4$ of order 8 is strongly real as well as totally orthogonal but its Sylow 2-subgroup, which is $D_4$ itself, is not Abelian.

It is therefore natural to ask which strongly real groups are totally orthogonal and vice versa. We have made many calculations using GAP and it seems the class $\mathcal{SR} \cap \mathcal{TO}$ is very close to the class $\mathcal{TO}$, though $\mathcal{SR}$ and $\mathcal{TO}$ are not identical. There is a group of order 32 which is strongly real but not totally orthogonal. This group $G$ has the following properties:

1. It is not simple. It has normal subgroups of all plausible orders - 2, 4, 8 and 16.
2. Exponent of $G$ is 4.
3. Derived subgroup of $G$ of order 2.
4. The group $G$ is a semidirect product of $C_2 \times Q_8$ and $C_2$. Here $C_2$ denotes the cyclic group of order 2 and $Q_8$ denotes the quaternion group of order 8.
5. This group has one 4-dimensional character and sixteen 1-dimensional characters. The 4-dimensional character assumes non-zero value only on one non-identity conjugacy class.

It is worth noting that this is the only group of order 63 or smaller which is strongly real and not totally orthogonal. All totally orthogonal groups till this order are strongly real.

3. Schur Indices of Groups

Let $G$ be a finite group. Let $k$ be a field such that $\text{char}(k)$ does not divide $|G|$. The group algebra of $G$ over $k$ is $kG = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in k \right\}$ with operations defined as follows:

$$\sum_{g \in G} \alpha_g g + \sum_{g \in G} \beta_g g = \sum_{g \in G} (\alpha_g + \beta_g)g$$

$$\alpha \left( \sum_{g \in G} \alpha_g g \right) = \sum_{g \in G} \alpha \alpha_g g$$

$$\left( \sum_{g \in G} \alpha_g g \right) \left( \sum_{h \in G} \beta_h h \right) = \sum_{t \in G} \left( \sum_{g \in G} \alpha_g \beta_{g^{-1}t} \right) t$$

It is a classical result of Maschke that $kG$ is a semisimple algebra. Hence by using Artin-Wedderburn theorem one can write it as a product of simple algebras over division algebras, i.e.,

$$kG \cong M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_r)$$
where $D_i$’s are division algebras over $k$ with center, say $L_i$, a finite field extension of $k$. We know that $D_i$ over $L_i$ is of square dimension, say $m_i^2$. Further each simple component $M_{n_i}(D_i)$ corresponds to an irreducible representation of $G$ over $k$. The number $m_i$ is called the Schur index of the corresponding representation. One can write $1 = e_1 + \cdots + e_r$ using above decomposition where $e_i$’s are idempotents. In fact,

$$e_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1})g,$$

where $\chi_i$ is the character of the representation corresponding to $M_{n_i}(D_i)$. For this reason often the Schur index is denoted as $m_k(\chi_i)$. The following are important questions in the subject:

**Question 3.1.**

(1) Find out Schur indices $m_k(\chi_i)$ and $L_i$ for the representations of a group $G$.

(2) Determine the division algebras $D_i$ appearing in the decomposition.

The problem of determining Schur indices for groups of Lie type has been studied extensively in the literature notably by Ohmori [Oh1, Oh2], Gow [Go1, Go2], Turull ([Tu]) and Geck ([Ge]). Geck also gives a table of the all known results on page 21 in [Ge]. However answer to the second question is much more difficult, e.g., Turull does it for $SL_n(q)$ in [Tu].

The group algebra is well studied over field $k = \mathbb{C}, \mathbb{R}, \mathbb{Q}$ or $\mathbb{F}_q$. For example, $CG \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$, as there is only one finite dimensional division algebra over $\mathbb{C}$ which is $\mathbb{C}$ itself (so is over $\mathbb{F}_q$). Moreover the simple components in this decomposition correspond to a finite dimensional representation of $G$ over $\mathbb{C}$. However in the case of $\mathbb{R}$ we know that the finite dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ hence the corresponding Schur index is 1, 2 or 1 respectively. In this case,

$$RG \cong M_{n_1}(\mathbb{R}) \times \cdots \times M_{n_1}(\mathbb{R}) \times M_{n_1}(\mathbb{H}) \times \cdots M_{n_1}(\mathbb{H}) \times M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_1}(\mathbb{C}).$$

In particular we see that the irreducible representations of $G$ are of three kinds which are called orthogonal, symplectic and unitary as they can afford a symmetric bilinear form, an alternating form or a hermetian form respectively. These corrspond to the Schur indicator $\nu(\chi)$ (defined in section 2) being 1, $-1$ or 0 respectively. Hence the question of calculating Schur indices over $\mathbb{R}$ is related to determining the types of representations: orthogonal, symplectic or unitary.

4. Canonical Involution on the Group Algebra

One can define an involution $\sigma$ on $kG$ as follows:

$$\sigma \left( \sum \alpha_g g \right) \mapsto \sum \alpha_g g^{-1}.$$ 

This involution is called the canonical involution. We can define a symmetric bilinear form $T: kG \times kG \to k$ by $T(x,y) = \text{tr}(l_x \sigma(y))$ where $l_x$ is the left multiplication operator on $kG$ and $tr$
denotes its trace. We note that $\text{tr}(t_e) = n$ and $\text{tr}(t_g) = 0$ for $g \neq e$. Hence the elements of group $G$ form an orthogonal basis and the form $T \simeq n < 1, 1, \ldots, 1 >$. The following result is proved in [Sc] (chapter 8 section 13): If the form $n < 1, 1, \ldots, 1 >$ is anisotropic over $k$ (for example $k = \mathbb{R}$ or $\mathbb{Q}$) the involution $\sigma$ restricts to each simple component of $kG$. In fact, in the case (ref. [BO] theorem 2) $G$ is real the involution $\sigma$ restricts to each simple component of $kG$.

In general if $kG \cong A_1 \times A_2 \times \cdots \times A_r$ and $k = \mathbb{C}$ then either $\sigma$ restricts to a component $A_i$ or it is a switch involution on $A_i \times A_j$ where $A_i \cong A_j$. However when $k = \mathbb{R}$ the involution $\sigma$ restricts to each component $A_i$, say $\sigma_i$, and is of either first type or second type. It is of the first type when $A_i$ is $M_n(\mathbb{R})$ or $M_m(\mathbb{H})$ and of the second type when the component is isomorphic to $M_l(\mathbb{C})$. Moreover when we tensor this with $\mathbb{C}$ the first type gives the component over $\mathbb{C}$ the one to which $\sigma$ restricts and the second type over $\mathbb{R}$ gives the one which correspond to the switch involution over $\mathbb{C}$.

Let us assume now that the canonical involution $\sigma$ restricts to all components of $kG$, i.e., $(kG, \sigma) \cong \prod_i (A_i, \sigma_i)$ where $A_i$s are simple algebra over $k$ with involution. For example, this happens when $k = \mathbb{R}$ or when $G$ is real. Algebras with involution $(A, \sigma)$ have been studied in the literature (see [KMRT]) extensively for its connection with algebraic groups. They are of two kinds: *involution of the first kind* is one which restricts to the center of $A$ as identity and the *involution of second kind* restrict to the center of $A$ as order 2 element. Further the involution of the first kind are of two types called *orthogonal type* and *symplectic type* the second kind is also called *unitary type*.

A group is called *ortho-ambivalent* with respect to a field $k$ if the canonical involution $\sigma$ restricts to all of its simple components as orthogonal involution (of first kind). The following results are proved in the thesis of Zahinda (ref. [Za] Chapter 2): An ortho-ambivalent group is necessarily ambivalent and in fact it is totally orthogonal (see Proposition 2.4.2 in [Za]). The notion of ortho-ambivalence over $k$ is equivalent to ortho-ambivalence over $\mathbb{C}$. Further the question that which 2 groups are ortho-ambivalent is analysed.

5. Some Examples

Here we write down some examples and some GAP calculations we did.

5.1. Symmetric and Alternating Groups. Conjugacy classes in $S_n$ are in one-one correspondence with partitions of $n$. Every conjugacy class in $S_n$ is strongly real and hence real. All characters of $S_n$ are real and moreover orthogonal.

Let $g \in A_n$. Then if $Z_{S_n}(g) \subset A_n$ then $g^{S_n} = g^{A_n} \cup (xgx^{-1})^{A_n}$ where $x$ is an odd permutation. In case $Z_{S_n}(g) \not\subset A_n$ then $g^{S_n} = g^{A_n}$. In [Pal], Parkinson classified real elements in $A_n$. Let $n = n_1 + n_2 + \cdots + n_r$ be a partition and $C$ be a conjugacy class corresponding to that in $S_n$ contained in $A_n$. Then, $C$ is real in $A_n$ if and only if
(1) each $n_i$ distinct,
(2) each $n_i$ odd and
(3) $\frac{1}{2}(n-r)$ is odd.

And hence the number of conjugacy classes in $A_n$ is equal to the number of real even partitions + twice the number of non-real even partitions. In fact (see [Br, Pa]), $A_n$ is ambivalent if and only if $n = 1, 2, 5, 6, 10, 14$.

For example in the case of $n = 7$, the partitions are given by $1^7, 1^62, 1^43, 1^32^2, 1^223, 13^2, 2^23, 34, 25, 124, 16, 71^25, 1^34, 12^3$. Out of which $1^7, 1^43, 1^32^2, 13^2, 2^23, 124, 71^25$ correspond to elements in $A_7$. By above criteria the only non-real class corresponds to the partition 7. Hence there are 7 real conjugacy classes in $A_7$ out of total $7 + 2\cdot1 = 9$ conjugacy classes. Using GAP we verified following statements about $A_7$.

(1) All but one conjugacy classes are real.
(2) All real conjugacy classes are strongly real.
(3) All real characters are orthogonal.

We summarise some GAP calculations below for $A_n$:

| $n$ | total classes | real classes | real | orthogonal | symplectic | unitary |
|-----|---------------|--------------|------|-------------|------------|---------|
| 5   | 5             | 5            | 5    | 0           | 0          | 0       |
| 6   | 7             | 7            | 7    | 0           | 0          | 0       |
| 7   | 9             | 7            | 7    | 0           | 2          | 0       |
| 8   | 14            | 10           | 10   | 0           | 4          | 0       |
| 9   | 18            | 16           | 16   | 0           | 2          | 0       |
| 10  | 24            | 24           | 24   | 0           | 0          | 0       |
| 14  | 72            | 72           | 72   | 0           | 0          | 0       |

In [Su] section 3, Suleiman proved that in alternating groups an element is real if and only if it is strongly real. Moreover in $A_n$ every real character is orthogonal, i.e., the Schur index of $A_n$ is 1. This follows from a work of Schur which is quoted in a paper of Turull (see [Tu2], Theorem 1.1). This result of Schur says that the Schur index of every irreducible representation of $A_n$ (for each $n$) is 1. Thus $A_n$ has no symplectic characters. Hence for $A_n$,

$|\text{strongly real classes}| = |\text{real classes}| = |\text{real characters}| = |\text{orthogonal characters}|$.

Here the first equality is from Suleiman, second is obvious and third one is the Schur index computation by Schur himself.
Now one can ask similar questions for the covers of these groups namely $\tilde{S}_n$ and $\tilde{A}_n$. We summarise some GAP calculations below for $\tilde{A}_n$:

| n | total classes | real classes | real orthogonal characters | real symplectic characters | real unitary characters |
|---|---------------|--------------|----------------------------|----------------------------|------------------------|
| 4 | 7             | 3            | 2                          | 2                          | 4                      |
| 5 | 9             | 9            | 2                          | 5                          | 4                      |

We summarise some GAP calculations below for $\tilde{S}_n$:

| n | total classes | real classes | real orthogonal characters | real symplectic characters | real unitary characters |
|---|---------------|--------------|----------------------------|----------------------------|------------------------|
| 4 | 8             | 6            | 6                          | 0                          | 2                      |
| 5 | 12            | 8            | 6                          | 7                          | 4                      |

Here we see an example of group, say the Schur cover of $A_5$ for which there are only 2 strongly real classes and all 9 real classes, while it has 5 orthogonal characters and remaining 4 symplectic ones.

5.2. $GL_n(q)$. The group $GL_n(q)$ has the property that all real elements are strongly real (ref. [Wo]). It doesn’t have irreducible symplectic representations, i.e., all self-dual irreducible representations are orthogonal ([Dp] Theorem 4). In [GS1] and [GS2] the precise number of the real elements are calculated. In [Ma], Macdonald gives an easy way to enumerate conjugacy classes.

**Theorem 5.1** (Macdonald). Conjugacy classes in $GL_n(q)$ are in one-one correspondance with a sequence of polynomials $u = (u_1, u_2, \ldots)$ satisfying:

1. a partition of $n$, $\nu = 1^{n_1}2^{n_2}\cdots$, i.e., $|\nu| = \sum_i i n_i = n$,
2. $u_i(t) = a_{n_i} t^{n_i} + \cdots + a_1 t + 1 \in \mathbb{F}_q[t]$ for all $i$ with $a_{n_i} \neq 0$.

Hence the number of conjugacy classes in $GL_n(q)$ is

$$\sum_{\{\nu:|\nu|=n\}} c_\nu = \sum_{\{\nu:|\nu|=n\}} \prod_{n_i > 0} (q^{n_i} - q^{n_i-1}).$$

**Theorem 5.2** ([GS1]). Real conjugacy classes in $GL_n(q)$ are in one-one correspondance with a sequence of polynomials $u = (u_1, u_2, \ldots)$ satisfying

1. a partition of $n$, $\nu = 1^{n_1}2^{n_2}\cdots$, i.e., $|\nu| = \sum_i i n_i = n$,
2. $u_i(t) = a_{n_i} t^{n_i} + \cdots + a_1 t + 1 \in \mathbb{F}_q[t]$ for all $i$ with $a_{n_i} \neq 0$.
3. $u_i(t)$ self-reciprocal.
Hence the number of real conjugacy classes in $GL_n(q)$ is

$$\sum_{\nu:|\nu|=n} \prod_{i>0} n_{\nu,n_i}$$

where $n_{\nu,n_i}$ is the number of polynomials $u_i(t)$ of above kind of degree $n_i$ over field $\mathbb{F}_q$.

Hence in $GL_n(q)$ the number of strongly real elements is same as the number of orthogonal characters.

5.3. $SL_2(q)$. In this case if $q$ is even all $q + 1$ classes are real as well as strongly real. If $q$ is odd, there are exactly 2 strongly real classes. In the case $q \equiv 1 \pmod 4$ all $q + 4$ classes are real and if $q \equiv 3 \pmod 4$ only $q$ out of $q + 4$ are real (in fact, exactly unipotent ones are not real). Hence we can say that the groups $SL_2(q)$ are ambivalent if and only if $q$ is a sum of (at most) two squares.

In the case $q$ is even all characters are orthogonal. However if $q$ is odd there is always a symplectic character. One can refer to the calculations of Schur indices in [Sh]. Hence one can conclude that the group $SL_2(q)$ is ortho-ambivalent if and only if $q$ is even.

5.4. $SL_n(q)$. The real and strongly real classes are calculated in [GST]. Turull calculated Schur indices of characters of $SL_n(q)$ in [Tu] over $\mathbb{Q}$ and also determined the division algebras appearing in the decomposition of group algebra. The following is known from the work of Ohmori, Gow, Zelevinsky, Turull, Geck etc. For the group $SL_n(q)$ if $n$ is odd or $n \equiv 0 \pmod 4$ or $|n|^2 > |p-1|^2$ then all real characters are orthogonal. In the case $2 \leq |n|^2 \leq |p-1|^2$ there are symplectic representations.

5.5. Orthogonal Group. In [Wo], Wonenburger proved that every element of orthogonal group is a product of two involutions. Hence these groups are strongly real. In [Go3], Gow proved (Theorem 1) that all characters of $O_n(q)$ are orthogonal.

6. The Lie Algebra $\mathcal{L}(G)$

Since $kG$ is an associative algebra we can define $[x,y] = xy - yx$ which makes it a Lie algebra. The subspace $\mathcal{L}(G)$ generated by $\{ \hat{g} = g - g^{-1} \mid g \in G \}$ is Lie subalgebra. This Lie algebra associated to a finite group is studied in [CT] and called Plesken Lie algebra. They prove,

**Theorem 6.1.** The Lie algebra $\mathcal{L}(G)$ admits the decomposition:

$$\mathcal{L}(G) \cong \bigoplus_{\chi \in \mathbb{R}} o(\chi(1)) \oplus \bigoplus_{\chi \in \mathbb{H}} sp(\chi(1)) \oplus \bigoplus_{\chi \in \mathbb{C}} gl(\chi(1))$$

where the sums are over different kind of irreducible characters (with obvious meaning) and the last sum is 'd meaning we have to take only one copy of $gl(\chi(1))$ for $\chi$ and $\chi^{-1}$. 
They also prove that the Lie algebra $\mathcal{L}(G)$ is semisimple if and only if $G$ has no complex characters and every character of degree 2 is of symplectic type, i.e., $G$ is ambivalent and every non-linear character is symplectic. They also classify when $\mathcal{L}(G)$ is simple.

7. Group Algebra and Real Characters

We have $\mathbb{R}G$ a group algebra with involution $\sigma$ which restricts to each simple component of it. On one hand we see that $Z(\mathbb{R}G) \cong \bigoplus g \mathbb{R}c_g$ where sum on the right hand side is over conjugacy classes and $c_g = \sum_{t \in G} g^t$ and on other hand we have $Z(\mathbb{R}G) \cong \bigoplus Z(M_n(D)) \cong \bigoplus_{\chi \in \mathbb{R}} \mathbb{R} \oplus \bigoplus_{\chi \in \mathbb{C}} \mathbb{C} \oplus \bigoplus_{\chi \in \mathbb{H}} \mathbb{R}$. We know that center of a semisimple algebra is an étale algebra and $\sigma$ restricts to it. In fact, in the first situation we have $Z(\mathbb{R}G) \cong \bigoplus_{g \in G} \mathbb{R}c_g \oplus (\mathbb{R}c_g \oplus \mathbb{R}c_{g^{-1}})$ where on the first component $\sigma$ restricts as trivial involution and on the second component it becomes as a switch involution (hence of the second kind). In the second isomorphism we know that $\sigma$ restricts to the trivial map on the $\mathbb{R}$ and $\mathbb{H}$ components and is of the second kind on $\mathbb{C}$ components. Hence counting the components where $\sigma$ restricts as first kind gives us the number of real conjugacy classes is same as the number of real plus symplectic representations.

However applying the same trick to $\mathbb{Q}G$ doesn’t give the corresponding result regarding rational representations. As it will also count the odd degree field extensions of $\mathbb{Q}$. It will be interesting to find such a proof.

References

[Ar] Armeanu, Ion, “About ambivalent groups”, Ann. Math. Blaise Pascal 3 (1996), no. 2, 17-22.
[BO] Boulagouaz, M.; Oukhtite, L., “Involution of semisimple group algebras”, Arab. J. Sci. Eng. Sect. C Theme Issues 25 (2000), no. 2, 133-149.
[Br] Berggren, J. L. “Finite groups in which every element is conjugate to its inverse”, Pacific J. Math. 28 1969 289–293. Arab. J. Sci. Eng. Sect. C Theme Issues 25 (2000), no. 2, 133-149.
[CT] Cohen, Arjeh M.; Taylor, D. E., “On a certain Lie algebra defined by a finite group”, Amer. Math. Monthly 114 (2007), no. 7, 633–639.
[Dp] Prasad, Dipendra, “On the self-dual representations of finite groups of Lie type”, J. Algebra 210 (1998), no. 1, 298-310.
[Ge] Geck, Meinolf, “Character values, Schur indices and character sheaves”, Represent. Theory 7 (2003), 19-55 (electronic).
[GNT] Guralnick, Robert M.; Navarro, Gabriel; Tiep, Pham Huu, “Real class sizes and real character degrees”, Math. Proc. Cambridge Philos. Soc. 150 (2011), no. 1, 47-71.
[Go1] Gow, R., “Schur indices of some groups of Lie type”, J. Algebra 42 (1976), no. 1, 102-120.
[Go2] Gow, R., “On the Schur indices of characters of finite classical groups”, J. London Math. Soc. (2) 24 (1981), no. 1, 135-147.
REAL ELEMENTS AND SCHUR INDICES OF A GROUP

[Go3] Gow, R., “Real representations of the finite orthogonal and symplectic groups of odd characteristic”, J. Algebra 96 (1985), no. 1, 249-274.

[GS1] Gill, Nick; Singh, Anupam, “Real and Strongly Real Classes in $SL_n(q)$”, to appear in the Journal of Group Theory.

[GS2] Gill, Nick; Singh, Anupam, “Real and Strongly Real Classes in $PGL_n(q)$ and quasi-simple Covers of $PSL_n(q)$”, to appear in the Journal of Group Theory.

[JL] James, Gordon; Liebeck, Martin, “Representations and characters of groups”, Second edition, Cambridge University Press, New York, 2001.

[KMRT] Knus, Max-Albert; Merkurjev, Alexander; Rost, Markus; Tignol, Jean-Pierre, “The book of involutions”. With a preface in French by J. Tits. American Mathematical Society Colloquium Publications, 44. American Mathematical Society, Providence, RI, 1998.

[Le] David W. Lewis, “Involutions and Anti-Automorphisms of Algebras”, Bull. London Math. Soc. 38 (2006) 529-545.

[Ma] Macdonald, I. G., “Numbers of conjugacy classes in some finite classical groups”, Bull. Austral. Math. Soc. 23 (1981), no. 1, 23-48.

[Oh1] Ohmori, Zyozyu, “On the Schur indices of certain irreducible characters of finite Chevalley groups”, Hokkaido Math. J. 28 (1999), no. 1, 39-55.

[Oh2] Ohmori, Zyozyu, “On the Schur indices of certain irreducible characters of reductive groups over finite fields”, Osaka J. Math. 25 (1988), no. 1, 149-159.

[Pa] Parkinson, Claire, “Ambivalence in alternating symmetric groups”, Amer. Math. Monthly 80 (1973), 190-192.

[Sc] Scharlau, Winfried, “Quadratic and Hermitian forms”, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 270. Springer-Verlag, Berlin, 1985.

[Se] Serre, Jean-Pierre, “Linear representations of finite groups”, Translated from the second French edition by Leonard L. Scott. Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, New York-Heidelberg, 1977.

[Sh] Shahabi Shojaei, M. A., “Schur indices of irreducible characters of $SL(2,q)$”, Arch. Math. (Basel) 40 (1983), no. 3, 221-231.

[Su] Suleiman, I. “Strongly real elements in Sporadic groups and Alternating groups”, Jordan Journal of Mathematics and Statisticscs (JJMS) l(2), 2008, pp. 97-103.

[Tu] Turull, Alexandre, “The Schur indices of the irreducible characters of the special linear groups”, J. Algebra 235 (2001), no. 1, 275-314.

[Tu2] Turull, A., “The Schur Index of projective characters of symmetric and alternating groups”, The Annals of Mathematics, Second Series, Vol. 135, No. 1 (Jan., 1992), pp. 91-124.
[TZ] Tiep, Pham Huu; Zalesski, A. E., “Real conjugacy classes in algebraic groups and finite groups of Lie type”, J. Group Theory 8 (2005), no. 3, 291-315.

[WG] Wang, K. S.; Grove, L. C., “Realizability of representations of finite groups”, J. Pure Appl. Algebra 54 (1988), no. 2-3, 299–310.

[Wo] Wonenburger, Maria J., “Transformations which are products of two involutions”, J. Math. Mech. 16 1966 327-338.

[Za] Obed N. Zahinda, “Ortho-ambivalence des groupes finis”, Ph.D. Thesis submitted at Universite Catholique de Louvain, Mai 2008.