ON THE 3-REPRESENTATIONS OF GROUPS AND THE 2-CATEGORICAL TRACES

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Abstract. We 2-categorify the theory of group representations: the 3-representations of groups in 3-categories and the 2-categorical actions of groups on 2-categories. We also 2-categorify the concept of the trace: the 2-categorical trace of a 1-arrow in a 3-category. For a 3-representation ρ of a group G and an element f of G, the 2-categorical trace Tr2ρf is a category. Moreover, the centralizer of f in G acts categorically on Tr2ρf. We construct the 2-categorical action of a finite group induced from that of a subgroup, and show that the 2-categorical trace Tr2 takes induced 2-categorical action into induced categorical action of the initial groupoids. As a corollary, we get the 3-character formula of the induced 2-categorical action.

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1. Introduction

Ganter and Kapranov categorified the theory of group representations and characters in [11]: the 2-representations of groups in 2-categories, the actions of groups on categories and the 2-characters. The theory was developed in [5] [8] [9] [10] [19] etc.. They [11] also categorified the concept of the trace of a linear transformation: the category trace. It is a set associated to any endfunctor on a small category, and is a vector space in the linear case. Moreover a functor commuting this functor defines a linear transformation on this vector space, whose trace defines a join trace. When a group acting on a k-linear category, the join trace of a commuting pair of group elements is the 2-character of the categorical action, a 2-class function, which is an analogy of the character of a group representation on a vector space. Such functions already appear in equivariant Morava $E$-theory [12], the elements of which are naturally described as $n$-class functions, i.e. functions defined on $n$-tuples of commuting elements of a group and invariant under simultaneous conjugation.

It is an active direction to categorify some of algebraic, geometric and analytic concepts in last two decades, e.g. 2-vector spaces, 2-bundles (gerbes), 2-connections and 2-curvatures, etc. They all involve 2-categorical constructions and have various applications, such as geometric definition of elliptic cohomology [1], the 2-gauge theory [3] [4] and two-dimensional Langlands correspondence [13] [18], etc.. It is also believed that higher categorification is necessary for many geometric and physical applications, and there exists a lot of literature. The purpose of this paper is to 2-categorify the theory of group representations and characters: the 3-representations of groups in 3-categories, the 2-categorical actions of groups on 2-categories and the 2-categorical traces, etc.. The problem of investigating representations of groups in higher categories has already mentioned in [11]. 3-categorical constructions already appeared in the theory of 2-gerbes (3-bundles) [6] [7] and the 3-gauge theory [17] [20] [22] by using more general Gray-categories.

A 3-representation of a group $G$ in a 3-category is given by a 1-arrow for each element of $G$, a 2-arrow for each pair of elements of $G$, and a 3-arrow for each triple of elements of $G$. Such 3-arrows must satisfy the 3-cocycle condition. Details are given in section 2. Given a 2-category $V$, we call a 3-representation of $G$ in the associated 3-category $V^*$ a 2-categorical action of $G$ on $V$. It is given by an endfunctor of $V$ for each element of $G$, a pseudonatural transformation between functors for each pair of elements of $G$, and a modification for each triple of elements of $G$.

Recall that for a 2-representation $\rho$ of a finite group $G$ in a 2-category $V$ and an element $f$ of $G$, $\rho_f : x \to x$ is an invertible 1-arrow, where $x$ is an object of $V$. In [11], it is proved that the categorical trace $\text{Tr}_f$ is the set of 2-arrows in $V$ with source to be the unit arrow $1_x$ and target to be $\rho_f$, and the centralizer of $f$ acts on this set naturally. In our case, for a 3-representation $\rho$ of $G$ in a 3-category $C$ and $f \in G$, $\rho_f : x \to x$ is also an invertible 1-arrow in $C$ for a fixed object $x \in C_0$. But the 2-categorical trace $\text{Tr}_f$ is a category whose objects are 2-arrows with source to be the unit arrow $1_x$ and target to be $\rho_f$, and whose morphisms are 3-arrows between them. Moreover, the centralizer of $f$, denoted by $C_G(f)$, acts categorically on
the 2-categorical trace $\mathbb{T}_2\rho f$ in the following way. We can define an endfunctor $\psi_g$ acting on $\mathbb{T}_2\rho f$ for each $g \in C_G(f)$, and if $h, g \in C_G(f)$, a natural transformation $\Gamma_{h,g} : \psi_h \circ \psi_g \to \psi_{hg}$ between functors on the category $\mathbb{T}_2\rho f$. These constructions are given in section 3. To prove the action to be categorical, we have to show the associativity in the definition of categorical action:

\[(1.1) \quad \Gamma_{k,hg}#(\psi_k \circ \Gamma_{h,g}) = \Gamma_{khg}#(\Gamma_{k,h} \circ \psi_g) : \psi_k \circ \psi_h \circ \psi_g \to \psi_{khg},\]

for any $k, h, g \in C_G(f)$, where $\#$ is the composition of natural transformations between functors on the category $\mathbb{T}_2\rho f$. This is the most difficult and technical part of this paper. By applying the 3-cocycle identity repeatedly, we prove that $\{\psi_g, \Gamma_{h,g}\}_{g,h \in C_G(f)}$ gives a categorical action of the centralizer $C_G(f)$ on the category $\mathbb{T}_2\rho f$ in section 6.

Suppose that $C$ is a $k$-linear 3-category. Then $\mathbb{T}_2\rho f$ is a $k$-linear category. If $k, g, f$ are mutually commutative, $\psi_k$ and $\psi_g$ are $k$-linear endfunctors acting on $\mathbb{T}_2\rho f$. We define the 3-character $\chi_{\rho}(f, g, k)$ of the 3-representation $\rho$ to be the join trace of functors $\psi_k$ and $\psi_g$ acting on the $k$-linear category $\mathbb{T}_2\rho f$. It is the trace of the linear transformation induced by the functor $\psi_k$ on the $k$-vector space $\mathbb{T}_2\psi_g$, which is the categorical trace of the functor $\psi_g$ on the category $\mathbb{T}_2\rho f$.

Suppose that $H$ is a subgroup of a finite group $G$ and it 2-categorically acts on the 2-category $\mathcal{V}$. In section 4, we define the induced 2-category $\text{Ind}^G_H(\mathcal{V})$ and 2-categorical action of $G$ on it. In section 5, we calculate the 2-categorical trace of the induced 2-categorical action as

\[(1.2) \quad \mathbb{T}_2(\text{Ind}^G_H(\rho)) = \text{Ind}^\Lambda(G)_H(\mathbb{T}_2(\rho)),\]

where $\Lambda(H)$ and $\Lambda(G)$ are initial groupoids associated to groups $H$ and $G$, respectively. As a corollary, we derive the 3-characters of the induced 2-categorical action, which coincides with the formula in [12] for $n$-characters when $n = 3$. These results are the generalization of induced categorical action and the 2-character formula in [11].

It is interesting to investigate the $m$-representation of a group in a $m$-category, the $m$-cocycle condition and $(m−1)$-categorical trace for positive integer $m > 3$.

2. The 3-representations of groups

2.1. (Small) strict $m$-categories. (Cf. [15]) A globular set $C$ consists of sets and functions

\[
\cdots \cdots \cdots C_{k+1} \overset{s_k}{\underset{t_k}{\longrightarrow}} C_k \overset{s_{k-1}}{\underset{t_{k-1}}{\longrightarrow}} C_{k-1} \cdots \cdots \overset{s_0}{\underset{t_0}{\longrightarrow}} C_0
\]

such that for $k \geq 2$ and $\gamma \in C_{k+1}$, $s_{k-1}s_k(\gamma) = s_{k-1}t_k(\gamma)$ and $t_{k-1}s_k(\gamma) = t_{k-1}t_k(\gamma)$. An element $\gamma$ of $C_k$ is called a $k$-arrow (or a $k$-morphism), and written as $\gamma : s_{k-1}(\gamma) \to t_{k-1}(\gamma)$.

For $k > p \geq 0$, write $C_k \times C_p C_k = \{ (\gamma, \gamma') \in C_k \times C_k ; t_p \circ \cdots \circ t_{k-1}(\gamma) = s_p \circ \cdots \circ s_{k-1}(\gamma') \}$.

A (small) strict $m$-category is a globular set $C$ together with a function (composition) $\#_p : C_k \times C_p C_k \to C_k$ for each $m \geq k > p \geq 0$ and a function (identity) $\iota : C_{k-1} \to C_k ; \gamma \mapsto 1_\gamma := \iota(\gamma)$, such that

1. For each $m > k \geq 0$ and $(\gamma, \gamma') \in C_{k+1} \times C_{k+1}$,

\[
s_k(\gamma \#_k \gamma') = s_k(\gamma), \quad t_k(\gamma \#_k \gamma') = t_k(\gamma'),
\]
and for $m > k > p \geq 0$,
\[
s_k(\gamma #_p \gamma') = s_k(\gamma) #_p s_k(\gamma'), \quad t_k(\gamma #_p \gamma') = t_k(\gamma) #_p s_k(\gamma');
\]
(2) If $m \geq k \geq 0$ and $\gamma \in C_{k-1}$, $s_{k-1}(1_\gamma) = \gamma = t_{k-1}(1_\gamma)$;
(3) For each $m \geq k > p \geq 0$ and $\gamma \in C_k$, then $\gamma #_p^{k-p}(t_p \circ \cdots t_{k-1}(\gamma)) = \gamma = t^k_p(\gamma)$; For any $(\gamma, \gamma')$, $(\gamma', \gamma'' \gamma') \in C_k \times C_p C_k$, we have
\[
(\gamma #_p \gamma') #_p \gamma'' = \gamma #_p (\gamma' #_p \gamma'');
\]
(4) If $m \geq p > q \geq 0$ and $(\vartheta, \vartheta') \in C_p \times C_q C_p$, $1_\vartheta #_q 1_\vartheta' = 1_\vartheta #_q \vartheta'$; If $(\beta, \beta')$, $(\gamma, \gamma') \in C_k \times C_p C_k$ and $(\beta, \gamma)$, $(\beta', \gamma') \in C_k \times C_q C_k$, then
\[
(\beta #_p \beta') #_q (\gamma #_p \gamma') = (\beta #_q \gamma) #_p (\beta' #_q \gamma').
\]
A strict $m$-functor (or a strict functor) is a map of globular sets preserving compositions and identities.

**Remark 2.1.** (1) By definition, we write the composition of $\gamma$ and $\gamma'$ as $\gamma #_p \gamma'$ if $t_p \circ \cdots t_{k-1}(\gamma) = s_p \circ \cdots s_{k-1}(\gamma')$, i.e., we write the composition in terms of the natural order. For simplicity, we only consider strict $m$-categories and strict functors in this paper.

(2) For $q < p < k$, $\vartheta \in C_p$ and $\gamma \in C_k$ with $t_q \cdots t_{p-1}(\vartheta) = s_q \cdots s_{k-1}(\gamma)$, we write the $k$-arrow $\gamma$ left wiksered by $\vartheta$ as
\[
\vartheta #_q \gamma := t^k_p(\vartheta) #_q \gamma,
\]
for simplicity, i.e., omit the identity function $I$. Similarly, we write $\gamma #_q \vartheta$, the $k$-arrow $\gamma$ right wiksered by $\vartheta$. We have
\[
(\vartheta #_q \gamma) #_p (\vartheta #_q \gamma') = \vartheta #_q (\gamma #_p \gamma'),
\]
for $\gamma' \in C_k$, if they are composable, by axioms (3)-(4) above.

### 2.2. 2-categories.

For a 2-category $C$ and any $x, y \in C_0$, $\text{Hom}_C(x, y)$ is a category. The composition $\#_1$ in $C$ is called the *vertical composition* $\phi #_1 \psi : x \xrightarrow{\psi} y \xrightarrow{\phi} C$. We also have the *horizontal composition* $\#_0 : \text{Hom}_C(x, y) \times \text{Hom}_C(y, z) \rightarrow \text{Hom}_C(x, z)$, $(A, B) \mapsto A #_0 B$ as $A #_0 B : x \xrightarrow{A} y \xrightarrow{B} z$ and
\[
\phi #_0 \psi : A #_0 B \Rightarrow C #_0 D,
\]
such that for all composable 1-arrows $A, B, C$ and 2-arrows $\phi, \psi, \omega, \phi', \psi'$, we have
\[
(\varphi #_0 B) #_0 C = \varphi #_0 (B #_0 C),
\]
(2.4)
\[
(\phi #_0 \psi) #_0 \omega = \phi #_0 (\psi #_0 \omega), \quad (\phi #_1 \psi) #_1 \omega = \phi #_1 (\psi #_1 \omega),
\]
\[
(\phi #_1 \psi) #_0 (\phi' #_1 \psi') = (\phi #_0 \phi') #_1 (\psi #_0 \psi'),
\]
and $1_x \in \text{Hom}_C(x,x)$ for any $x \in C_0$ is a unit arrow.

A 1-arrow $A : x \to y$ is called strictly invertible or a strict 1-isomorphism, if there exists another 1-arrow $B : y \to x$ such that $1_x = A \#_0 B$ and $B \#_0 A = 1_y$. A 2-category in which every 1-arrow is strictly invertible is called a strict 2-groupoid. A 2-arrow $\varphi : A \Rightarrow B$ is called invertible or a 2-isomorphism if there exists another 2-arrow $\psi : B \Rightarrow A$ such that $\psi \#_1 \varphi = 1_B$ and $\varphi \#_1 \psi = 1_A$. In the strict case, $\psi$ is uniquely determined and called the inverse of $\varphi$.

Let $\mathcal{S}$ and $\mathcal{T}$ be two strict 2-categories. A strict 2-functor $F : \mathcal{S} \to \mathcal{T}$ is an assignment of a 2-arrow $F(x)$ to each 2-arrow $f : x \to y$ such that

- $F(\varphi \#_1 \psi) = F(\varphi) \#_1 F(\psi)$ and $F(1_f) = 1_{F(f)}$ for all composable 2-arrows $\varphi$ and $\psi$ and any 1-arrow $f$;
- $F(g) \#_0 F(f) = F(g \#_0 f)$ for all composable 1-arrows $g$ and $f$, and $F(\varphi) \#_0 F(\psi) = F(\varphi \#_0 \psi)$ for all horizontally composable 2-arrows $\varphi$ and $\psi$.

Let $F_1$ and $F_2$ be two strict 2-functors both from $\mathcal{S}$ to $\mathcal{T}$. A pseudonatural transformation $\rho : F_1 \to F_2$ is an assignment of a 1-arrow $\rho(X)$ in $\mathcal{T}$ to each object $X$ in $\mathcal{S}$ and a 2-isomorphism $\rho(f)$

\begin{equation}
(2.5)
\end{equation}

in $\mathcal{T}$ to each 1-arrow $f : X \to Y$ in $\mathcal{S}$ such that they satisfy two axioms

- The composition of 1-arrows in $\mathcal{S}$:

\begin{equation}
(2.6)
\end{equation}

- It is compatible with 2-arrows:

\begin{equation}
(2.7)
\end{equation}
for any 2-arrow $\varphi : f \Rightarrow g$.

Let $F_1, F_2 : S \rightarrow T$ be two strict 2-functors and let $\rho_1, \rho_2 : F_1 \rightarrow F_2$ be pseudonatural transformations. A modification $\Phi : \rho_1 \Rightarrow \rho_2$ is an assignment of a 2-arrow

$$
\Phi : F_1 \rightarrow F_2
$$

in $T$ to any object $X$ in $S$, which satisfies

$$(2.8)$$

Given a 3-category $C$, we define $C^+$ to be the 2-category with $(C^+)_i = C_{i+1}$ and the $i$-th source and target to be $s_{i+1}$ and $t_{i+1}$, $i = 0, 1, 2$, respectively. Functions $\#_p : C^+_k \times C^+_k \rightarrow C^+_k$ becomes $\#_{p+1} : C_{k+1} \times C_{k+1} \rightarrow C_{k+1}$, $p = 0, 1$, and identities $i : C_k \rightarrow C_k$ are defined obviously. And define $C^{++}$ to be the category with $(C^{++})_i = C_{i+2}$ and the $i$-th source and target to be $s_{i+2}$ and $t_{i+2}$, $i = 0, 1$, respectively. The function $\#_0 : C^{++}_1 \times C^{++}_1 \rightarrow C^{++}_1$ becomes $\#_2 : C_3 \times C_3 \rightarrow C_3$. It is easy to see by definition that $C^+$ is a 2-category and $C^{++}$ is a category.

Given a 2-category $\mathcal{V}$, there exists an associated 3-category $\mathcal{V}^*$ with only one object $\mathcal{V}$, $\mathcal{V}_1^*$ to be all functors from $\mathcal{V}$ to $\mathcal{V}$, $\mathcal{V}_2^*$ to be all pseudonatural transformations between functors and $\mathcal{V}_3^*$ to be all modifications between pseudonatural transformations.

In a 3-category $C$, a 1-arrow $B : x \rightarrow y$ is called an isomorphism if there exists 1-arrow $C : y \rightarrow x$ such that there exists 2-arrows $u : 1_y \Rightarrow C \#_k B$ and $v : 1_x \Rightarrow B \#_k C$. $C$ is called a quasi-inverse to $B$, and vice versa. However, since a $k$-arrow, $k = 2$ or 3, is a $(k-1)$-arrow in $C^+$, we call it an isomorphism if it is invertible strictly.

2.3. The 3-representations of a group in a 3-category. Let $C$ be a strict 3-category and $G$ be a group. $G$ can be viewed as a 3-category with only one object $pt$, $G$ as the set of 1-arrows $pt \rightarrow pt$, all 2-arrows being the identities of 1-arrows, and all 3-arrows being the identities of 2-arrows. A 3-representation of a group $G$ in $C$ is a strong functor $\rho$ from $G$ to $C$. More explicitly, we have

(1) an object $x$ of $C$;
(2) for each $g \in G$, a 1-isomorphism $\rho_g : x \rightarrow x$;
(3) for each $h, g \in G$, a 2-isomorphism $\phi_{h,g} : \rho_h \rho_g \Rightarrow \rho_{h g}$ (here and in the following we write $\rho_h \#_0 \rho_g$ as $\rho_{h g}$ for simplicity), corresponding to the 2-cell

$$
\begin{array}{ccc}
& & \rho_{h g} \\
\rho_h & \swarrow_{\phi_{h,g}} & \rho_g \\
& & \\
\end{array}
$$
(4) for each $g_3, g_2, g_1 \in G$, a 3-isomorphism, called the \textit{associator},

\[
\Phi_{g_3,g_2,g_1} : (\rho_{g_3} \#_0 \phi_{g_2,g_1}) \#_1 \phi_{g_3,g_2g_1} \Rightarrow (\phi_{g_3,g_2} \#_0 \rho_{g_1}) \#_1 \phi_{g_3g_2g_1},
\]

which just exchanges the diagonals of the quadrilateral:

(5) a 2-isomorphism $\phi_1 : \rho_1 \Rightarrow 1_x$; such that the following conditions are satisfied:

- $\phi_{1,g} = \phi_1 \#_0 \rho_g$, $\phi_{g,1} = \rho_g \#_0 \phi_1$.
- the 3-cocycle condition: for any $g_4, \ldots, g_1 \in G$,

\[
\{[\rho_{g_1} \#_0 \Phi_{g_3,g_2,g_1}] \#_1 \phi_{g_4,g_3g_2g_1}\} \#_2 \{[\phi_{g_4} \#_0 \phi_{g_3,g_2} \#_1 \rho_{g_1}] \#_1 \Phi_{g_4,g_3g_2g_1}\} \\
= \{[(\rho_{g_4} \rho_{g_3}) \#_0 \phi_{g_2,g_1}] \#_1 \Phi_{g_4,g_3g_2g_1}\} \#_2 \{[\phi_{g_4,g_1} \#_0 (\rho_{g_2} \rho_{g_1})] \#_1 \Phi_{g_4g_3,g_2g_1}\}.
\]

Equivalently, the composition of the 3-arrows represented by the tetrahedrons in the boundary of the above 4-simplex is identity. This comes from the boundary of the corresponding 4-simplex in the 3-category $G$ is the identity 3-arrow.

\textbf{Remark 2.2.} (1) For simplicity, we assume $\rho_1 = 1_x$ and $\phi_1$ is the identity in this paper.
(2) The 3-cocycle condition is equivalent to the vanishing of 3-curvatures in the lattice 3-gauge theory (the cubical case) in [22], where the 3-gauge theory from the point of view of Gray-categories is investigated. The 3-representations of groups in Gray-categories are much more complicated because of the appearance of the interchanging 3-arrows, which are responsible for the appearance of terms with Peiffer brackets in the formula of the 3-curvature (cf. [22]).

2.4. The 3-cocycle condition. In the LHS of the 3-cocycle condition (2.11), the first 3-arrow is

\[ A_1 = [\rho_{g_4} \# 0 \Phi_{g_3,g_2,g_1}] \# 1 \phi_{g_4,g_3,g_2,g_1}, \]

corresponding to the 3-cell

\[ \text{figure: the 3-arrow } A_1 \]

whose source and target are the 2-arrows

\[ s_2(A_1) = [(\rho_{g_4}\rho_{g_3}) \# 0 \phi_{g_2,g_1}] \# 1 [(\rho_{g_4} \# 0 \phi_{g_3,g_2,g_1}] \# 1 \phi_{g_4,g_3,g_2,g_1} : \rho_{g_4}\rho_{g_3}\rho_{g_2}\rho_{g_1} \rightarrow \rho_{g_4,g_3,g_2,g_1}, \]
\[ t_2(A_1) = [\rho_{g_4} \# 0 \phi_{g_3,g_2} \# 0 \rho_{g_1}] \# 1 [\rho_{g_4} \# 0 \phi_{g_3,g_2,g_1}] \# 1 \phi_{g_4,g_3,g_2,g_1} : \rho_{g_4}\rho_{g_3}\rho_{g_2}\rho_{g_1} \rightarrow \rho_{g_4,g_3,g_2,g_1}, \]

corresponding to the 2-cells

\[ \text{figure: the 2-arrow } s_2(A_1) \quad \text{figure: the 2-arrow } t_2(A_1) \]

respectively.

The second 3-arrow is

\[ A_2 = [\rho_{g_4} \# 0 \phi_{g_3,g_2} \# 0 \rho_{g_1}] \# 1 \Phi_{g_4,g_3,g_2,g_1}, \]
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Let \( \rho_4 \) be the corresponding 3-cell

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
& \rho_4 & \rho_4 \\
\rho_3 & \rho_3 & \\
\rho_2 & \rho_2 & \\
\rho_1 & \rho_1 & \\
\end{array}
\]

with source \( s_2(A_2) = t_2(A_1) \) in (2.13) and target

\[
t_2(A_2) = [\rho_{g_1} \phi_{g_3, g_2} \phi_{g_1} \rho_{g_1}] \phi_{g_4, g_3, g_2} \phi_{g_1} \rho_{g_1} \phi_{g_4, g_2, g_1}.
\]

And the third 3-arrow is

\[
A_3 = [\Phi_{g_4, g_3, g_2} \phi_{g_1}] \phi_{g_4, g_3, g_2} \phi_{g_1},
\]

corresponding to the 3-cell

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
& \rho_4 & \rho_4 \\
\rho_3 & \rho_3 & \\
\rho_2 & \rho_2 & \\
\rho_1 & \rho_1 & \\
\end{array}
\]

with source \( s_2(A_3) = t_2(A_2) \) in (2.14) and target to be the 2-arrow

\[
t_2(A_3) = [\phi_{g_4, g_3} \phi_{g_2} \phi_{g_1}] \phi_{g_4, g_3, g_2} \phi_{g_1} \phi_{g_4, g_2, g_1}.
\]
corresponding to the 2-cell

\[
\begin{array}{c}
\rho_4 \rightarrow \\
\rho_3 \\
\rho_2 \\
\rho_1
\end{array}
\]

(2.16)

\[\text{figure: the 2—arrow } t_2(A_3)\]

Then the composition \(A_1 \#_2 A_2 \#_2 A_3\) of 3-arrows is the LHS of the 3-cocycle condition (2.11) with source \(s_2(A_1)\) in (2.13) and target \(t_2(A_3)\) in (2.15).

In the RHS of the 3-cocycle condition (2.11), the first 3-arrow is

\[
A'_1 = \left[ (\rho_4 \rho_3) \#_0 \phi_{g_2,g_1} \right] \#_1 \Phi_{g_4,g_3,g_2,g_1},
\]

corresponding to the 3-cell

\[\text{figure: the 3—arrow } A'_1\]

whose source is \(s_2(A_1)\) in (2.13) and target is

\[
t_2(A'_1) = \left[ (\rho_4 \rho_3) \#_0 \phi_{g_2,g_1} \right] \#_1 [\phi_{g_4,g_3} \#_0 \rho_{g_2,g_1}] \#_1 [\phi_{g_4,g_3} \#_0 \rho_{g_2,g_1}].
\]
corresponding to the left 2-cells in the following diagram:

\[
\begin{array}{c}
\bullet \\
\leftarrow \rho_{g_1} \\
\rightarrow \rho_{g_2} \\
\leftarrow \rho_{g_3} \\
\rightarrow \rho_{g_4} \\
\bullet
\end{array}
\]

(1)

\[
\begin{array}{c}
\bullet \\
\leftarrow \rho_{g_4} \\
\rightarrow \rho_{g_3} \\
\leftarrow \rho_{g_2} \\
\rightarrow \rho_{g_1} \\
\bullet
\end{array}
\]

(2)

\[
\begin{array}{c}
\bullet \\
\leftarrow \rho_{g_4} \\
\rightarrow \rho_{g_3} \\
\leftarrow \rho_{g_2} \\
\rightarrow \rho_{g_1} \\
\bullet
\end{array}
\]

(3)

By the paste theorem for 2-categories [14], we can interchanging 2-arrows (1) and (2) identically in the left 2-cells above to get the 2-arrow

\[
s_2(A'_2) = [\phi_{g_4,g_3} \#_0(\rho_{g_2}\rho_{g_1})] \#_1[\rho_{g_4,g_3} \#_0\phi_{g_2,g_1}] \#_1\phi_{g_4,g_3,g_2,g_1}
\]

in the right 2-cells above. The last 3-arrow is

\[
A'_2 = [\phi_{g_4,g_3} \#_0(\rho_{g_2}\rho_{g_1})] \#_1\Phi_{g_4,g_3,g_2,g_1}
\]

whose target is exactly the 2-arrow \( t_2(A'_3) \) in (2.15)-(2.16).

In the corresponding 2-category \( C^+ \), \( A_1 \) corresponds to the following 2-arrow:

\[
\begin{array}{c}
\bullet \\
\leftarrow \rho_{g_4} \\
\rightarrow \rho_{g_3} \\
\leftarrow \rho_{g_2} \\
\rightarrow \rho_{g_1} \\
\bullet
\end{array}
\]

(3)

\[
\begin{array}{c}
\bullet \\
\leftarrow \rho_{g_4} \\
\rightarrow \rho_{g_3} \\
\leftarrow \rho_{g_2} \\
\rightarrow \rho_{g_1} \\
\bullet
\end{array}
\]

(1)

\[
\begin{array}{c}
\bullet \\
\leftarrow \rho_{g_4} \\
\rightarrow \rho_{g_3} \\
\leftarrow \rho_{g_2} \\
\rightarrow \rho_{g_1} \\
\bullet
\end{array}
\]

(2)
Here the upper and lower boundary 1-arrows in (2.19) in \( C^+ \) represent the 2-arrows in \( C \): the source \( s_2(A_1) \) and target \( t_2(A_1) \) in (2.13), respectively. To draw the picture neatly, we omit the whiskering parts. Then the 3-cocycle condition (2.11) can be expressed simply as an identity of 2-arrows in \( C^+ \) as follows:

\[
\begin{array}{c}
\phi_{g_1,g_2} \circ \phi_{g_2,g_3} = \phi_{g_1,g_3} \\
\phi_{g_1,g_2} \circ \phi_{g_2,g_3} = \phi_{g_1,g_3}
\end{array}
\]

(2.20)

Here \( \bullet \)'s are corresponding 1-arrows in \( C \). Cf. 2-arrows in (2.13), (2.14), (2.15), (2.17) and (2.18) for 1-arrows in (2.20). The 3-cocycle condition (2.20) can be viewed as the commutativity of the 2-arrows in the boundary of the following cube in \( C^+ \):

\[
\begin{array}{c}
\phi_{g_1,g_2} \circ \phi_{g_2,g_3} = \phi_{g_1,g_3} \\
\phi_{g_1,g_2} \circ \phi_{g_2,g_3} = \phi_{g_1,g_3}
\end{array}
\]

(2.21)

Remark 2.3. (1) In the upper boundaries of diagrams in (2.20), the numbers of group elements in the second subscripts of \( \phi_{*,*} \) are increasing: \( g_1, g_2, g_3, g_1 g_2 g_1 \), while in the lower boundaries, the numbers of group elements in the first subscripts of \( \phi_{*,*} \) are increasing: \( g_1, g_2, g_3, g_4 g_3 g_2 g_1 \).

(2) (2.20) or (2.21) is similar to the pentagon condition of bicategories, but here we actually have more complicated whiskering (cf. (2.19)).

Given a group \( G \) and a 2-category \( \mathcal{V} \), we call a 3-representation of \( G \) in the associated 3-category \( \mathcal{V}^* \) a 2-categorical action of \( G \) on \( \mathcal{V} \). In this case, \( \rho_g : \mathcal{V} \to \mathcal{V} \) for each \( g \in G \) is an endfunctor; for each \( h, g \in G \), \( \phi_{h,g} : \rho_h \circ \rho_g \Rightarrow \rho_{hg} \) is a pseudonatural transformation; for each \( g_1, g_2, g_3 \in G \), \( \Phi_{g_1,g_2,g_3} : (\rho_{g_1} \circ \rho_{g_2} \circ \rho_{g_3}) \Rightarrow (\rho_{g_1} \circ \rho_{g_2} \circ \rho_{g_3}) \) is a modification. Here \( \rho_h \circ \rho_g \) is the composition of functors: \( \rho_h \circ \rho_g(w) = \rho_h(\rho_g(w)) \) for \( w \in \mathcal{V} \).

When a 2-category \( \mathcal{V} \) is viewed as a 3-category, a 3-representation in \( \mathcal{V} \) is a 2-representation if the the associator 3-arrow in (2.9) is an identity, and so the 3-cocycle condition (2.11) holds trivially. It coincides with the definition of the 2-representation in the strict sense in [11]. And for a category \( \mathcal{V} \), a 2-representation of \( G \) in the 2-category \( \mathcal{V}^* \) is a categorical action of \( G \) on \( \mathcal{V} \).
3. The 2-categorical traces of 3-representations

3.1. The 2-categorical trace of a 1-isomorphism. Let $C$ be a 3-category. For $x \in C$ and a 1-isomorphism $A : x \to x$, $A$ is an object of the 2-category $\text{Hom}_C(x, x)$. The 2-categorical trace of $A$ is defined as

$$\text{Tr}_2(A) = \text{Hom}_C(1_x, A),$$

which is a category. $\text{Tr}_2(A)$ is a subcategory of $C^{++}$.

Example. Let $C$ be the 3-category of all 2-categories, $x = \mathcal{V}$, a 2-category, and let $A$ be an endfunctor. Then $\text{Tr}_2(A)$ is a category with

- objects: $\text{NT}_C(1_{\mathcal{V}}, A)$, pseudonatural transformations $\chi : 1_{\mathcal{V}} \Rightarrow A$;
- morphisms: modifications $\mathcal{V} \xrightarrow{\chi} \mathcal{V} \xleftarrow{\chi'}$ for $\chi, \chi' \in \text{NT}_C(1_{\mathcal{V}}, A)$.

Let $A : x \to x$ be a 1-isomorphism for $x \in C_0$, and let 1-arrow $C : y \to x$ be a quasi-inverse to 1-arrow $B : x \to y$. Then for a 2-arrow $\chi : 1_x \Rightarrow A$ in $\text{Tr}_2(A)_0$,

$$1_y \xrightarrow{u} C \# A \xrightarrow{\gamma} C \# A \# B \xrightarrow{\gamma B} C \# A \# B$$

defines a functor

$$\Psi(C, B, u) : \text{Tr}_2(A)_0 \to \text{Tr}_2(C \# A \# B)_0,$$

$$(\chi : 1_x \Rightarrow A) \mapsto u \# [C \# A \# B],$$

corresponding to the diagram

![Diagram](https://via.placeholder.com/150)

and for a 3-arrow $\gamma : \chi \Rightarrow \chi'$ in $\text{Tr}_2(A)_1$, we have a morphism

$$\text{Tr}_2(A)_1 \to \text{Tr}_2(C \# A \# B)_1,$$

$$\gamma \mapsto u \# [C \# A \# B],$$

Proposition 3.1. $\Psi(C, B, u) : \text{Tr}_2(A) \to \text{Tr}_2(C \# A \# B)$ is a functor.

Proof. For 2-arrows $\phi, \chi, \tilde{\chi} : 1_x \Rightarrow A$, and 3-arrows $\gamma : \phi \Rightarrow \chi, \tilde{\gamma} : \chi \Rightarrow \tilde{\chi}$, we have the composition $\gamma \# \tilde{\gamma} : \phi \Rightarrow \tilde{\chi}$. Then

$$\Psi(C, B, u)(\gamma \# \tilde{\gamma})$$

= $u \# [C \# A \# B] \# 2 \Psi(C, B, u)(\tilde{\gamma})$

= $u \# [C \# A \# B] \# 2 \Psi(C, B, u)(\tilde{\gamma})$

= $u \# [C \# A \# B] = \Psi(C, B, u)(\gamma \# \tilde{\gamma}),$

by using (2.3) for whiskering. \qed
3.2. The 2-categorical trace $\text{Tr}_2 \rho_f$. Let $\rho$ be a 3-representation of $G$ in a 3-category $C$. Fix an object $x$ in $C$. For $f \in G$, $\rho_f : x \to x$ is an invertible 1-arrow in $C$. $\text{Tr}_2 \rho_f$ defined as above is a category whose objects are 2-arrows with source $1_x$ and target $\rho_f$, and the morphisms are the 3-arrow between them. In the following, we will denote $g^{-1}$ by $g^*$ for simplicity. For $g$ commuting $f$ and a 2-arrow $\chi : 1_x \Longrightarrow \rho_f$ in $(\text{Tr}_2 \rho_f)_{0}$, we define a 2-arrow $\psi_g(\chi) : 1_x \Longrightarrow \rho_f$ by

$$
(3.1) \quad \psi_g(\chi) := u_g\#_1[\rho_g\#_0\chi\#_0\rho_g^*] \#_1[\phi_{g,f}\#_0\rho_g^*] \#_1\phi_{g,f,g}^*.
$$

which is given by the composition of 2-arrows in the following diagram

(3.2)

where $u_g = \phi_{g,g}^{-1} : 1_x \Longrightarrow \rho_g\rho_g^*$. And for a 3-arrow $\Theta : \chi \Longrightarrow \chi'$, $\psi_g(\Theta)$ is $\Theta$ wiskered by corresponding 2-arrows in (3.2), i.e.,

$$
(3.3) \quad \psi_g(\Theta) = u_g\#_1[\rho_g\#_0\Theta\#_0\rho_g^*] \#_1[(\phi_{g,f}\#_0\rho_g^*)\#_1\phi_{g,f,g}^*] : \psi_g(\chi) \Longrightarrow \psi_g(\chi'),
$$

a 3-arrow in the 3-category $C$. Then $\psi_g$ defines an endfunctor $\psi_g$ on $\text{Tr}_2 \rho_f$ by the proof of Proposition 3.1. Namely, we have

$$
\psi_g(\Theta\#_0\Theta') = \psi_g(\Theta)\#_0\psi_g(\Theta')
$$

for any 3-arrow $\Theta' : \chi' \to \chi''$, where $\#_0$ is the composition in the category $C^{++}$ ($\#_0 = \#_2$).

3.3. The adjoint 2-arrows. For a 2-arrow $x \xymatrix{ & y \ar@{.>}[ll]_{\chi_1} \ar@{.>}[rl]^\chi_2 }$ in a 2-category $\mathcal{V}$, we define the adjoint 2-arrow $\phi^\dagger$ to be

(3.4)

This is a 2-arrow with the source and target 1-arrows inverted. This operation will be used later. See also section 2 in [15] for the definition of similar adjoint 2-arrows, but $\phi^{-1}$ in (3.4) is replaced by $\phi$ there.

Proposition 3.2. (1) For a 2-arrows $x \xymatrix{ & y \ar@{.>}[ll]_{\chi_1} \ar@{.>}[rl]^\chi_2 }$ and $x \xymatrix{ & y \ar@{.>}[ll]_{\chi_1} \ar@{.>}[rl]^\chi_2 }$, $(\phi\#_1\psi)^\dagger = \phi^\dagger\#_1\psi^\dagger$.

(2) For 1-arrow $\chi_0 : z \to x$, $(\chi_0\#_0\phi)^\dagger = \phi^\dagger\#_0\chi_0^{-1}$, and for 1-arrow $\widetilde{\chi}_0 : y \to z$, $(\phi\#_0\widetilde{\chi}_0)^\dagger = \widetilde{\chi}_0^{-1}\#_0\phi^\dagger$. 

(3) For a 2-arrow \( y \xrightarrow{\bar{x}_1} z \), \((\phi \#_0 \bar{\phi})^\dagger = \bar{\phi}^\dagger \#_0 \phi^\dagger\), i.e., \( z \xrightarrow{\bar{x}_2} x \).

Proof. (1) \( \phi^\dagger \#_1 \psi^\dagger = (\phi \#_1 \psi)^\dagger \) follows from

\[
\begin{align*}
\begin{array}{c}
y \
\xrightarrow{\chi_1^{-1}} x \xrightarrow{\phi^{-1}} y \xrightarrow{\chi_2^{-1}} x \xrightarrow{\phi^{-1}} y \xrightarrow{\chi_3^{-1}} x \\
\xrightarrow{\chi_1^{-1}} x \xrightarrow{\phi^{-1}} y \xrightarrow{\chi_2^{-1}} x \xrightarrow{\phi^{-1}} y \xrightarrow{\chi_3^{-1}} x
\end{array}
\end{align*}
\]

by \( x \xrightarrow{\chi_2} y \xrightarrow{\chi_2^{-1}} x \xrightarrow{\phi^{-1}} y \xrightarrow{\chi_3} y \) and the paste theorem for 2-categories.

(2) follows from \((\chi_0 \#_0 \phi)^\dagger\) to be

\[
\begin{align*}
\begin{array}{c}
y \
\xrightarrow{\chi_1^{-1}} x \xrightarrow{\phi^{-1}} z \xrightarrow{\chi_0^{-1}} x \xrightarrow{\phi^{-1}} y \xrightarrow{\chi_2^{-1}} x \xrightarrow{\phi^{-1}} z \\
\xrightarrow{\chi_1^{-1}} x \xrightarrow{\phi^{-1}} z \xrightarrow{\chi_0^{-1}} x \xrightarrow{\phi^{-1}} y \xrightarrow{\chi_2^{-1}} x \xrightarrow{\phi^{-1}} z
\end{array}
\end{align*}
\]

and \( \chi_0^{-1} \chi_0 \) to be the identity.

(3) Note that \( \phi \#_0 \bar{\phi} = (\chi_1 \#_0 \bar{\phi}) \#_1 (\phi \#_0 \bar{\chi}_2) \) by using axiom \( \text{(2.2)} \). We see that \((\phi \#_0 \bar{\phi})^\dagger = (\chi_1 \#_0 \bar{\phi})^\dagger \#_1 (\phi \#_0 \bar{\chi}_2)^\dagger = (\bar{\phi}^\dagger \#_0 \chi_1^{-1}) \#_1 (\bar{\chi}_2^{-1} \#_0 \phi^\dagger) = \bar{\phi}^\dagger \#_0 \phi^\dagger\) by using (1) and (2).

\( \square \)

3.4. The categorical action of the centralizer of \( f \) on \( \text{Tr}_2 f \). To construct a categorical action of the centralizer \( C_G(f) \) of \( f \) on the category \( \text{Tr}_2 f \), let us write down the composition of functors \( \psi_h \circ \psi_g : \text{Tr}_2 f \rightarrow \text{Tr}_2 f \) for \( g, h \in C_G(f) \). For fixed \( \chi \in (\text{Tr}_2 f)_0 \) and \( \Theta \in (\text{Tr}_2 f)_1 \), the composition \( \psi_h \circ \psi_g(\chi) = \psi_h(\psi_g(\chi)) \) is given by the following diagram

\[
\begin{align*}
\begin{array}{c}
x \
\xrightarrow{\rho_g} x \xrightarrow{\phi_g} x \xrightarrow{\phi_g^*} x \xrightarrow{\rho_g^*} x \xrightarrow{\phi_g^* \rho_h} x \xrightarrow{\phi_g^* \rho_h} x \xrightarrow{\phi_g^* \rho_h} x \xrightarrow{\phi_g^* \rho_h} x \\
x \
\xrightarrow{\rho_h} x \xrightarrow{\phi_h} x \xrightarrow{\phi_h^*} x \xrightarrow{\rho_h^*} x \xrightarrow{\phi_h^* \rho_g} x \xrightarrow{\phi_h^* \rho_g} x \xrightarrow{\phi_h^* \rho_g} x \xrightarrow{\phi_h^* \rho_g}
\end{array}
\end{align*}
\]

and \( \psi_h \circ \psi_g(\Theta) = \psi_h(\psi_g(\Theta)) \) is defined similarly, by using the definition \( \text{(3.2)} \) of \( \psi \) repeatedly. Recall that \( \rho_{g1} = \rho_g 1_x \) and \( \rho_{h1} = \rho_h 1_x \). The upper half part of \( \text{(3.5)} \) is the same as the lower
half with \( f \) replaced by \( 1_x \) and 2-arrows inverted:

\[
\begin{array}{c}
\phi_{h,1}^{-1} \\
\phi_{h,1} \\
\phi_{g,1}^{-1} \\
\phi_{g,1} \\
\phi_{h,g}^{-1} \\
\phi_{h,g} \\
\phi_{g,h}^{-1} \\
\phi_{g,h} \\
\phi_{h,g}^{-1} \\
\phi_{h,g} \\
\phi_{g,h}^{-1} \\
\phi_{g,h} \\
\phi_{h,g}^{-1} \\
\phi_{h,g} \\
\phi_{g,h}^{-1} \\
\phi_{g,h} \\
\phi_{h,g}^{-1} \\
\phi_{h,g} \\
\phi_{g,h}^{-1} \\
\phi_{g,h} \\
\phi_{h,g}^{-1} \\
\phi_{h,g} \\
\phi_{g,h}^{-1} \\
\phi_{g,h} \\
\end{array}
\]

(3.6)

namely, \( u_h = \phi_{h,h}^{-1} \#_1 [\phi_{h,1}^{-1} \#_0 \rho_{h^*}] \), \( u_g = \phi_{g,g}^{-1} \#_1 [\phi_{g,1}^{-1} \#_0 \rho_{g^*}] \) (\( \phi_{h,1} \) and \( \phi_{g,1} \) are identities by our assumption).

Now let us write down the natural transformation \( \Gamma_{h,g} : \psi_h \circ \psi_g \rightarrow \psi_g \) between functors on the category \( \text{Tr}_2 \rho F \). The lower half of (3.5) is

(3.7)

Here and in the following, for simplicity, we will use the notation \( \rho_{g_1 g_2} := \rho_{g_1 \ldots g_2} \), i.e., we omit the elements of the group between \( g_1 \) and \( g_2 \) in the sequence \( h, g, f, g^*, h^* \).

Note that the associator 3-arrow \( \Phi_{g_3,g_2,g_1} \) in (2.9)- (2.10) can be drawn in the following form:

(3.8)

By definition, the 3-arrow

\[
\hat{\Lambda}_1 = [\rho_h \#_0 \phi_{g,f} \#_0 (\rho_{g^*} \rho_{h^*})] \#_1 [\Phi_{h,g,f,g^*} \#_0 \rho_{h^*}] \#_1 \phi_{h*g^*} \rho_{h^*},
\]

which is the associator \( \Phi_{h,g,f,g^*} \) wiskered by two 2-arrows, changes the diagonal \( \rho_{g^*} \) of the dotted quadrilateral in the diagram (3.7) to the wavy one \( \rho_{h,f} \) in the following diagram:

(3.10)

The 3-arrow

\[
\hat{\Lambda}_2 = [\Phi_{h,g,f} \#_0 (\rho_{g^*} \rho_{h^*})] \#_1 [\phi_{h,f,g^*} \#_0 \rho_{h^*}] \#_1 \phi_{h^*} g^*, h^*,
\]

changes the diagonal \( \rho_{g,f} \) of the dotted-wavy quadrilateral in the diagram (3.10) to the wavy one \( \rho_{h,g} \) in the following diagram:
The 3-arrow

\[ \hat{A}_3 = \{ [\phi_{h,g}\#_0(\rho_f \rho_g^* \rho_{h^*})] \#_1 [\phi_{h,f}\#_0(\rho_g^* \rho_{h^*})] \} \#_1 \Phi_{h,f,g,h^*}^{-1}, \]

the wiskered associator \( \Phi_{h,f,g,h^*}^{-1} \), changes the diagonal \( \rho_{h^*} \) of the dotted quadrilateral in the diagram (3.12) to the wavy one \( \rho_{g^* h^*} \) in the following diagram:

Recall that the upper half part of (3.13) is the same as the lower half with \( f \) replaced by 1 and 2-arrows inverseed. So by the corresponding 3-arrows (in the reverse order), denoted by \( \hat{A}_1, \hat{A}_2, \hat{A}_3 \), the upper half part of (3.13) is changed to

Note that

\[ x \xrightarrow{\rho_h} x \xrightarrow{\rho_g} x \xrightarrow{\rho_{h^*}} x \]

and the part concerning \( \rho_{g^* h^*} \) is also cancelled. As a result, the composition of (3.15) and (3.13), together with 2-arrow \( \chi : 1_x \longrightarrow \rho_f \), gives \( \psi_g \psi_f \). The composition of suitable wiskered 3-arrows \( \hat{A}_1, \hat{A}_2, \hat{A}_3 \) gives a natural transformation \( \Gamma_{h,g} : \psi_h \circ \psi_g \longrightarrow \psi_{hg} \) between functors on the category \( \mathcal{T}_2 \rho_f \). For \( \chi \in (\mathcal{T}_2 \rho_f)_0 \), \( \Gamma_{h,g}(\chi) : \psi_h(\psi_g(\chi)) \longrightarrow \psi_{hg}(\chi) \) is a 3-arrow in \( \mathcal{C} \).

It is not easy to draw 3-arrows \( \hat{A}_1 \)'s in the 3-category \( \mathcal{C} \). But in the 2-category \( \mathcal{C}^+ \), the first 3-arrow \( \hat{A}_1 \) can be drawn as the following 2-arrow:
The upper and lower paths in the above diagram correspond to the 2-arrows in \( C \) in (3.7) and (3.10), respectively, and the 2-arrow \( \hat{\Lambda}_1 \) corresponds to the 3-arrow in \( C \) in (3.9). Since \( \text{Tr}_2 \rho_f \) is a subcategory of \( C^{++} \), diagrams in the 2-category \( C^+ \) are sufficient for our purpose.

In the following, 1-arrows of the form \( \rho_{\mu} \cdots \rho_{a,b} \cdots \rho_{h,g} \), as objects in the 2-category \( C^+ \), are simply written as \( \rho_{a,b} \). For simplicity, we also omit the whiskering part of 1- and 2-arrows in the diagram. The composition of \( \hat{\Lambda}_1, \hat{\Lambda}_2 \) and \( \hat{\Lambda}_3 \), i.e., (3.7) \( \equiv \) (3.10) \( \equiv \) (3.12) \( \equiv \) (3.14), corresponds to the following diagram in the 2-category \( C^+ \):

\[
\begin{array}{c}
\begin{array}{ccc}
\rho_f & \phi_{g,f} & \rho_{g,f} \\
\phi_{h,g} & \phi_{g,h} & \rho_{h,g} \\
\phi_{h,g} & \phi_{h,g} & \rho_{h,g}
\end{array}
\end{array}
\]

As the upper half part of (3.5), (3.6) is changed to (3.15) by the composition of 3-arrows with \( f \) replaced by 1. In \( C^+ \), it is the 2-arrow given by

\[
\begin{array}{c}
\begin{array}{ccc}
\rho_1 & \phi_{g,1} & \rho_{g,1} \\
\phi_{h,1} & \phi_{g,h} & \rho_{h,1} \\
\phi_{h,1} & \phi_{h,1} & \rho_{h,1}
\end{array}
\end{array}
\]

Where \( \hat{\Lambda}_j \) is the 2-arrow corresponding to \( \Lambda_j \) with \( f \) replaced by 1, and \( \hat{\Lambda}_j \) is the 2-arrow adjoint to \( \hat{\Lambda}_j \), defined in (3.3). Recall that the adjoint 2-arrow is the 2-arrow with the source and target 1-arrows inverted. So (3.17) follows from (3.10) by using Proposition 3.2 for the adjoint operation. Then connect (3.17) and (3.16) to get the 2-arrow \( \Gamma_{h,g}(\chi) \) in \( C^+ \):

\[
\begin{array}{c}
\begin{array}{ccc}
\rho_1 & \phi_{g,1} & \rho_{g,1} \\
\phi_{h,1} & \phi_{g,h} & \rho_{h,1} \\
\phi_{h,1} & \phi_{h,1} & \rho_{h,1}
\end{array}
\end{array}
\]

For \( \chi, \chi' \in (\text{Tr}_2 \rho_f)_0 \) and a morphism \( \Theta : \chi \to \chi' \) in \( \text{Tr}_2 \rho_f \) (i.e., a 3-arrow in \( C \)), \( \Gamma_{h,g}(\Theta) \) is also a 3-arrow. We connect (3.16) and (3.17) to get the following diagram in the 2-category \( C^+ \):

\[
\begin{array}{c}
\begin{array}{ccc}
\rho_{g,1} & \phi_{g,1} & \rho_{g,1} \\
\rho_{h,1} & \phi_{h,1} & \rho_{h,1} \\
\rho_{h,1} & \phi_{h,1} & \rho_{h,1}
\end{array}
\end{array}
\]

For \( \chi, \chi' \in (\text{Tr}_2 \rho_f)_0 \) and a morphism \( \Theta : \chi \to \chi' \) in \( \text{Tr}_2 \rho_f \) (i.e., a 3-arrow in \( C \)), \( \Gamma_{h,g}(\Theta) \) is also a 3-arrow. We connect (3.16) and (3.17) to get the following diagram in the 2-category \( C^+ \):
which, by the paste theorem for 2-categories \([14]\), implies that \(\Gamma_{h,g}\) is a natural transformation in the category \(\text{Tr}_{2}\rho_f \subset C^{++}\), i.e.,

\[
\begin{array}{ccc}
\psi_h \circ \psi_g(\chi) & - & \psi_h \circ \psi_g(\chi') \\
\gamma_{h,g}(\chi) & - & \gamma_{h,g}(\chi') \\
\psi_{hg}(\chi) & - & \psi_{hg}(\chi')
\end{array}
\]

is commutative. Here \(\psi_h \circ \psi_g(\chi)\) is the upper boundary of \((3.18)\), while \(\psi_{hg}(\chi')\) is the lower boundary of \((3.18)\). \(\psi_h \circ \psi_g(\Theta)\) and \(\psi_{hg}(\Theta)\) are \(\Theta\) wiskered by the upper and lower boundaries, respectively.

**Theorem 3.1.** \(\{\psi_g, \Gamma_{h,g}\}_{g,h \in C_G(f)}\) is a categorical action of the centralizer \(C_G(f)\) on the category \(\text{Tr}_{2}\rho_f\).

This theorem will be proved in section 6 by checking the associative law \((1.1)\), which is an identity of natural transformations between functors on \(\text{Tr}_{2}\rho_f\). Note that in \((1.1)\), \(s_0(\Gamma_{k,hg}) = \psi_k \circ \psi_{hg} = t_0(\psi_h \circ \Gamma_{g,h})\). So the composition of natural transformations used in \((1.1)\) is in the usual order, not in the natural order we assume in Remark 2.1 \((1)\).

### 3.5. 1-dimensional 3-representations

We fix a field \(k\) of characteristic 0 containing all roots of unity. Let

\(c : G \times G \times G \to k^*\)

be an abelian 3-cocycle, i.e., it satisfies the abelian version of the 3-cocycle condition \([220]\):

\(c(g_3, g_2, g_1)c(g_4, g_3g_2, g_1)c(g_4, g_3, g_2) = c(g_4, g_3, g_2g_1)c(g_4g_3, g_2, g_1)\).

Let \(A\) be a 2-category with only one object, only one 1-arrow and 2-arrows \(A_2 = k\). Let \(\varrho^c\) be a categorical action of \(G\) on \(A\): \(\varrho^c_g\) is the identity functor for each \(g \in G\);

\[\phi_{h,g} : 1_A = \varrho^c_h \varrho^c_g \to \varrho^c_{hg} = 1_A\]

is also the identity pseudonatural transformation for any \(h, g \in G\); and

\[(3.19) \quad \Phi_{g_3, g_2, g_1} : id = (\varrho^c_{g_3} \#_0 \varrho^c_{g_2, g_1}) \#_1 \varrho^c_{g_3, g_2 g_1} = (\varrho^c_{g_3, g_2} \#_0 \varrho^c_{g_1}) \#_1 \varrho^c_{g_3, g_2, g_1} = id,\]

is multiplying \(c(g_3, g_2, g_1)\) for any \(g_3, g_2, g_1 \in G\). Then \(\Phi\) satisfies the 3-cocycle condition \([220]\). A cohomologeous (abelian) 3-cocycle, classified by \(H^3(G, k^*)\), defines an equivalent 3-representation.

For \(f \in G\), \(\text{Tr}_{2}\rho_f\) is a category with only one object, the identity pseudonatural transformation \(\chi_0 : 1_A \to \rho_f = 1_A\), and morphisms \((\text{Tr}_{2}\rho_f)_1 \cong k\) (any 2-arrow in \(A_2 = k\) provides a modification). For \(g \in C_G(f)\), \(\psi_g : \text{Tr}_{2}\rho_f \to \text{Tr}_{2}\rho_f\) is the identity functor by the definition \((3.1)\) and \((3.3)\). And \(\Gamma_{h,g} : \psi_h \circ \psi_g \to \psi_{hg}\) is given by

\[\Gamma_{h,g} = (c(h, g, f, g^{-1})c(h, g, f)c(hf, g, h^{-1})^{-1}c(h, g, g^{-1})^{-1}c(h, g, 1)^{-1}c(h, g, h^{-1})\]

by \((3.9)\) \((3.11)\) \((3.13)\) and the adjoint operation \((3.4)\).
4. The induced 2-categorical action on the induced 2-category

4.1. The induced 2-category. Let $H \subset G$ be a subgroup of a finite group $G$. Let $\rho : H \to \mathcal{V}^*$ be a 2-categorical action of $H$ on a 2-category $\mathcal{V}$. $\text{Ind}^G_H(\mathcal{V})$ is a strict 2-category with

- objects are maps $\vartheta : G \to \mathcal{V}_0$ together with a 1-isomorphism $u_{g,h} : \vartheta(gh) \to \rho_{h^*}\vartheta(g)$ for each $g \in G, h \in H$, satisfying the condition: (1) $u_{g,1} : \vartheta(g) \to \rho_{1^*}\vartheta(g)$ coincides with $\phi^{-1}_{1,\vartheta(g)}$; (2) for each $g \in G, h_1, h_2 \in H$, a 2-isomorphism:

$$
\begin{array}{ccc}
\vartheta(gh_1 h_2) & \xrightarrow{u_{gh_1, h_2}} & \rho_{h_2^*}\vartheta(gh_1) \\
\rho(h_1 h_2)^* \vartheta(g) & \xrightarrow{\phi_{h_2^*, h_1^*}\vartheta(g)} & \rho_{h_2^*}\rho_{h_1^*}\vartheta(g)
\end{array}
$$

- 1-arrows $F : (\vartheta, u) \to (\vartheta', u')$ between objects;
- 2-arrows $\gamma : F \to \tilde{F}$.

The action $(\text{ind}^G_H(\rho))$ on the 2-category $\text{Ind}^G_H(\mathcal{V})$ is given by

$$(\text{ind}^G_H(\rho)) (g) = \vartheta(k^*g), \quad (\text{ind}^G_H(\rho)) u_{g,h} = u_{k^*g,h}.$$ 

for an object $(\vartheta, u)$ in $\text{Ind}^G_H(\mathcal{V})$. And $(\text{ind}^G_H(\rho)) F$ for a 1-arrows $F : (\vartheta, u) \to (\vartheta', u')$ and $(\text{ind}^G_H(\rho)) \gamma$ for a 2-arrow $\gamma : F \to \tilde{F}$ can be defined similarly. In general, each commutative diagram in the definition of the induced category [14] is replaced by a 2-arrow.

We will not write down the definition of the induced 2-category $\text{Ind}^G_H(\mathcal{V})$ explicitly, which is a little bit complicated. Since we only work on finite groups, we can simply identify $\text{Ind}^G_H(\mathcal{V})$ with $\mathcal{V}^m$ as a 2-category, where $m$ is the index of $H$ in $G$. For a strict 2-category $\mathcal{V}$, $\mathcal{V}^m$ is also a strict 2-category:

objects $\mathcal{V}_0^m := \{(x_1, \ldots, x_m) : x_j \in \mathcal{V}_0\}$,

1-arrows $\mathcal{V}_1^m := \{(\gamma_1, \ldots, \gamma_m) : (x_1, \ldots, x_m) \to (y_1, \ldots, y_m) ; \forall \gamma_j : x_j \to y_j\}$,

2-arrows $\mathcal{V}_2^m := \{((\Phi_1, \ldots, \Phi_m) : (\gamma_1, \ldots, \gamma_m) \to (\gamma'_1, \ldots, \gamma'_m) ; \forall \Phi_j : \gamma_j \to \gamma'_j\}$.

The composition is defined as

$$(\ldots, \gamma_j, \ldots) \#_p (\ldots, \gamma'_j, \ldots) := (\ldots, \gamma_j \#_p \gamma'_j, \ldots),$$

$$(\ldots, \Phi_j, \ldots) \#_p (\ldots, \Phi'_j, \ldots) := (\ldots, \Phi_j \#_p \Phi'_j, \ldots),$$

if $t_0(\gamma_j) = s_0(\gamma'_j)$, $t_p(\Phi_j) = s_p(\Phi'_j)$, $p = 0, 1$. The axiom (2.41) for functions $\#_p$ and identity $\iota$ in $\mathcal{V}^m$ obviously holds. The identification $\text{Ind}^G_H(\mathcal{V}) \cong \mathcal{V}^m$ can be obtained by choosing a system of representatives

$\mathcal{R} = \{r_1, \ldots, r_m\}$

of left cosets of $H$ in $G$, and associating to each map $\vartheta : G \to \mathcal{V}_0$ the object $(\vartheta(r_1), \ldots, \vartheta(r_m)) \in \mathcal{V}_0^m$. 
Let \( a_{jk} : \mathcal{V} \to \mathcal{V} \) be functors such that the \( m \times m \) matrix \( F = (a_{jk}) \) has only one nonvanishing entry in each row or column. Then \( F \) is a strict functor from \( \mathcal{V}^m \) to \( \mathcal{V}^m \):
\[
F(\ldots, x_j, \ldots) = \left( \sum_k a_{jk}(x_k), \ldots \right), \quad F(\ldots, \gamma_j, \ldots) = \left( \sum_k a_{jk}(\gamma_k), \ldots \right), \ldots
\]
Here we write \( \sum_k a_{jk}(\ast_k) \) formally, since there exists only one term in this sum. But when the 2-category is \( k \)-linear, such sums exist. If \( \tilde{F} = (\tilde{a}_{jk}) : \mathcal{V}^m \to \mathcal{V}^m \) is another such functor, then we have \( (F\#_0 \tilde{F})_{jk} := \sum_l a_{jl} \tilde{a}_{lk} \). A pseudonatural transformation \( \phi : F \to \tilde{F} \) is given by an \( m \times m \) matrix \( \phi = (\phi_{jk}) \) with \( \phi_{jk} : a_{jk} \to \tilde{a}_{jk} \) to be a pseudonatural transformation between functors on \( \mathcal{V} \).

### 4.2. The induced 2-categorical action

Suppose \( \rho \) is a 2-categorical action of \( H \) on the 2-category \( \mathcal{V} \). For \( f \in G \), we define \((\text{ind}^G_H \rho)_f \) to be a functor from \( \mathcal{V}^m \) to \( \mathcal{V}^m \). Then it can be viewed as an \( m \times m \) matrix whose entries are functor from \( \mathcal{V} \) to \( \mathcal{V} \), i.e., the \((j, i)\)-entry is
\[
(\text{ind}^G_H \rho)_f(j, i) = \begin{cases} \rho_h, & \text{if } fr_i = r_j h, \text{ for } h \in H, \\ 0, & \text{otherwise,} \end{cases}
\]

since \( [\text{ind}^G_H \rho)_f(\vartheta)](r_j) \) equals to \( \vartheta(f^{-1}r_j) \), and \( \vartheta(f^{-1}r_j) = \vartheta(r_i h^{-1}) \to \rho_h \vartheta(r_i) \). It is clear that only one entry in each row or column of the \( m \times m \) matrix \( \text{ind}^G_H \rho \) is nonvanishing. Then,
\[
\text{ind}^G_H \rho(j, i) = \sum (\text{ind}^G_H \rho)_f(j, i),
\]

where \( \gamma_j : x_j \to y_j \) are 1-arrows.

From now on, the induced object will be denoted by the hatted one, e.g., \( \text{ind}^G_H \rho \) is denoted by \( \hat{\rho} \). The composition functor is defined as
\[
(\hat{\rho}_{g_2} \hat{\rho}_{g_1})_{ki} = \begin{cases} \rho_{h_2} \rho_{h_1}, & \text{if } g_1 r_i = r_j h_1, g_2 r_j = r_k h_2, \text{ for } h_1, h_2 \in H, \\ 0, & \text{otherwise.} \end{cases}
\]
\( \hat{\rho}_{g_2} \hat{\rho}_{g_1} \) can be viewed as a product of two \( m \times m \) matrices of functors. On the other hand
\[
(\hat{\rho}_{g_2 g_1})_{ki} = \rho_{h_2 h_1}
\]
since \( (g_2 g_1) r_i = r_k (h_2 h_1) \). Define the pseudonatural transformation, a 2-arrow in \( (\mathcal{V}^m)^\ast \), as
\[
\hat{\phi}_{g_2 g_1} : \hat{\rho}_{g_2} \hat{\rho}_{g_1} \to \hat{\rho}_{g_2 g_1},
\]
to be the \( m \times m \) matrix whose \((k, i)\)-entry is the 2-arrow
\[
(\hat{\phi}_{g_2 g_1})_{ki} = \phi_{h_2, h_1} : \rho_{h_2} \rho_{h_1} \to \rho_{h_2 h_1},
\]
and all other entries vanish. For \( g_1, g_2, g_3 \in G \), the 3-arrow in \( (\mathcal{V}^m)^\ast \)
\[
\hat{\Phi}_{g_3 g_2 g_1} : [\hat{\rho}_{g_3} \#_0 \hat{\phi}_{g_2 g_1}] \#_1 \hat{\phi}_{g_3 g_2 g_1} \to [\phi_{g_3 g_2 g_1} \#_0 \rho_{g_1}] \#_1 \phi_{g_3 g_2 g_1}
\]
is a modification. If \( g_3 r_k = r_l h_3 \) for some \( h_3 \in H \), then \( [\hat{\rho}_{g_3} \#_0 \hat{\phi}_{g_2 g_1}]_{ki} = \rho_{h_3} \#_0 \phi_{h_2, h_1} \) and \( [\hat{\phi}_{g_3 g_2 g_1}]_{ij} = \phi_{h_3, h_2, h_1} \#_0 \rho_{h_1} \) etc. Define \( \hat{\Phi}_{g_3 g_2 g_1} \) to be an \( m \times m \) matrix whose \((l, i)\)-entry is the modification
\[
(\hat{\Phi}_{g_3 g_2 g_1})_{li} = \Phi_{h_3, h_2, h_1} : [\rho_{h_3} \#_0 \phi_{h_2, h_1}] \#_1 \phi_{h_3, h_2, h_1} \to [\phi_{h_3, h_2, h_1} \#_0 \rho_{h_1}] \#_1 \phi_{h_3, h_2, h_1},
\]
a 3-arrow in \( \mathcal{V}^\ast \), and all other entries vanish.
For $g_4 \in G$, write $g_4 r_i = r_i h_4$ for some $h_4 \in H$. The $(t, i)$-entry of the $m \times m$ matrix $[\hat{\rho}_{g_4, 0} \Phi_{g_4, g_2, g_1}] \# [\hat{\rho}_{g_4, g_3, g_2, g_1}]$ is $[\rho_{h_4} \Phi_{h_4, h_3, h_2, h_1}] \# [\hat{\rho}_{h_4, h_3, h_2, h_1}]$, and similarly we obtain other terms in the 3-cocycle condition (2.11) for $\Phi$. So the 3-cocycle condition (2.11) for $\Phi$ follows from the 3-cocycle condition for $\hat{\Phi}$. Therefore $\hat{\rho}$ is a 2-categorical action of $G$ on $V^m \approx \text{Ind}_{H}^{G}(V)$.

5. The 3-character of the induced 2-categorical action

5.1. The 2-categorical trace of the induced 2-categorical action. As above $\rho$ is a 2-categorical action of $H$ on the 2-category $\mathcal{V}$. Let $\mathcal{R}$ be a system of representatives of $G/H$. We have the decomposition

$$\mathcal{R} = \mathcal{R}' \cup \mathcal{R}'',$$

where $\mathcal{R}' := \{ r \in \mathcal{R}; r^{-1}fr \in H \}$, $\mathcal{R}'' := \{ r \in \mathcal{R}; r^{-1}fr \notin H \}$. For a fixed element $f$ of $G$, the decomposition

$$[f]_G \cap H = [h_1]_H \cup \cdots [h_n]_H$$

induces a decomposition

$$\mathcal{R}' = \bigcup_{i=1}^{n} \mathcal{R}_i \quad \text{with} \quad \mathcal{R}_i = \{ r \in \mathcal{R}; r^{-1}fr \in [h_i]_H \}.$$  

For fixed $i$, we pick $r_i \in \mathcal{R}_i$ and write $h_i = r_i^{-1}fr_i$. For $r \in \mathcal{R}_i$, $r^{-1}fr = h^{-1}h_i r$ for some $h \in H$. For $r \in \mathcal{R}_i$, $r^{-1}fr = h^{-1}h_i r$ for all $r \in \mathcal{R}_i$.

It follows from the definition (4.2)-(4.3) that

$$\hat{\rho}_f = \left( \begin{array}{cccc} A_{00} & A_{01} & A_{02} & \cdots & A_{0m} \\ A_{10} & A_{11} & 0 & \cdots & 0 \\ 0 & A_{20} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m0} & 0 & 0 & \cdots & A_{mm} \end{array} \right), \quad A_{ii} = \left( \begin{array}{c} \rho_{h_i} \\ \vdots \\ \rho_{h_i} \end{array} \right)_{m_i \times m_i},$$

where $i = 1, \ldots, m$, $A_{00}$ is an off-diagonal $m' \times m'$ matrix ($m' := |\mathcal{R}'|$), $m_i = |\mathcal{R}_i|$ with $\sum_{i=1}^{n} m_i = m' := |\mathcal{R}'|$. So an object of $\text{Tr}_{2} \hat{\rho}_f$ is a pseudonatural transformation $\chi : 1_{V^m} \to \hat{\rho}_f$ of the form

$$\chi = \left( \begin{array}{c} 0_{m'' \times m''} \\ \vdots \\ D_i \\ \vdots \end{array} \right), \quad D_i = \left( \begin{array}{c} \chi_{m_1+\cdots+m_{i-1}+1} \\ \vdots \\ \chi_{m_1+\cdots+m_i} \end{array} \right).$$

where $\chi_{m_1+\cdots+m_{i-1}+\alpha} : 1_{V} \to \rho_{h_i}$ for $\alpha = 1, \ldots, m_i$. And morphisms in $\text{Tr}_{2} \hat{\rho}_f$ are also diagonal. So we have

$$\text{Tr}_{2} \hat{\rho}_f = \bigoplus_{i=1}^{n} (\text{Tr}_{2} \rho_{h_i})^{m_i}.$$

Lemma 5.1. (Lemma 7.7 in [11]) Left multiplication with $r_i^{-1}$ maps $\mathcal{R}_i$ into a system of representatives of $C_{G}(h_i)/C_{H}(h_i)$.
For \( g \in C_G(f) \) and \( r \in \mathcal{R}_i \), write
\[
(5.4) \quad gr = \tilde{r}h,
\]
for some \( \tilde{r} \in \mathcal{R} \) and \( h \in H \). Also, \( r \) is uniquely determined by \( \tilde{r} \) for fixed \( g \). Then \( \tilde{r}^{-1}f\tilde{r} = h\rho^{-1}g^{-1}fgfh^{-1} = hh_h^{-1} \) by (5.3). Hence \( \tilde{r} \in \mathcal{R}_i \) and so \( \tilde{r}^{-1}f\tilde{r} = h_i \) by the assumption (5.1). It follows that \( h \in C_H(h_i) \). Then
\[
gr = \tilde{r}h \quad \text{gives} \quad (\widehat{\rho}_g)_{\tilde{r}r} = \rho_h, 
fr = rh_i \quad \text{gives} \quad (\widehat{\rho}_f)_{rr} = \rho_{h_i}, 
g^{-1}\tilde{r} = rh^{-1} \quad \text{gives} \quad (\widehat{\rho}_g^{-1})_{r\tilde{r}} = \rho_{h^{-1}},
\]
and all other entries vanish. Thus
\[
(\widehat{\rho}_g\widehat{\rho}_f\widehat{\rho}_g^*)_{\tilde{r}r} = \rho_h\rho_{h_i}\rho_{h^{-1}}
\]
and all other entries vanish in the last \((m' \times m')\)-block (cf. (5.2)).

We denote by \( \psi \) the categorical action of the centralizer \( C_G(f) \) of \( f \) on the category \( \mathbb{T}_R^2\mathcal{R}_i \). By definition, \( \psi \) for \( g \in C_G(f) \) is a functor on it. Recall that for a pseudonatural transformation \( \chi : 1_{\mathcal{V}^m} \to \rho_f \) in (5.3), \( \psi_g(\chi) \) is the pseudonatural transformation
\[
\text{diag}(1_{\mathcal{V}}, \ldots, 1_{\mathcal{V}}) \xrightarrow{(\widehat{\phi}_{g,0})^*} \widehat{\rho}_g \rho_{g^*} \xrightarrow{\rho_{g,h}} \widehat{\rho}_g \rho_{g^*} \xrightarrow{(\widehat{\phi}_{g,f})^*} \rho_{g,0h^*} \xrightarrow{(\widehat{\phi}_{g,f,g})^*} \rho_{g,0h^*} = \rho_f,
\]
where the first \( m'' \) diagonal terms must vanish and other diagonal terms are
\[
(\widehat{\phi}_{g,0g}^*)_{\tilde{r}r} = \phi_{h,h^*}^{-1} : 1_{\mathcal{V}} \to \rho_h\rho_{h_i}^*, 
(\widehat{\phi}_{g,0g^*})_{\tilde{r}r} = (\widehat{\rho}_g)_{\tilde{r}r} \widehat{\chi}_{\tilde{r}r} \# 0 = \rho_h \# 0h_i \# 0h^{-1} : \rho_h\rho_{h_i}^* \to \rho_h\rho_{h_i}\rho_{h^{-1}}, 
(\widehat{\phi}_{g,f}^*)_{\tilde{r}r} = \phi_{h,h_i}^* : \rho_{h_i}\rho_{h_i}^* \to \rho_{h_i}\rho_{h_i} = \rho_{h_i},
(\widehat{\phi}_{g,f,g}^*)_{\tilde{r}r} = \phi_{h,h_i}^* : \rho_{h_i}\rho_{h_i}^* \to \rho_{h_i}\rho_{h_i} = \rho_{h_i},
\]
and all other entries vanish by definitions (4.4)-(4.5). Therefore, \( \psi_g(\chi) \) is a diagonal \( m \times m \) matrix, whose \((\tilde{r}, \tilde{r})\)-entry for \( \tilde{r} \in \mathcal{R}' \) is
\[
(5.5) \quad (\psi_g(\chi))_{\tilde{r}r} = \phi_{h,h_i}^{-1}[\phi_h\# 0h_i \# 0h^{-1}]_1 : 1_{\mathcal{V}} \to \rho_{h_i},
\]
and vanishes for all \( \tilde{r} \in \mathcal{R}' \).

Let \( \psi^{(i)}_h \) be the categorical action of the centralizer \( C_H(h_i) \) of \( h_i \) in \( H \) on the category \( \mathbb{T}_R^2\rho_{h_i} \), which is induced from the 2-categorical action \( \rho \) of \( H \) on \( \mathcal{V} \). By definition, we have a functor \( \psi^{(i)}_h \) for each \( h \in C_H(h_i) \). For \( h \in C_H(h_i) \) and a pseudonatural transformation \( \omega : 1_{\mathcal{V}} \to \rho_{h_i} \), \( \psi^{(i)}_h(\omega) \) is the composition of the following pseudonatural transformations between functors on \( \mathbb{T}_R^2\rho_{h_i} \):
\[
1_{\mathcal{V}} \xrightarrow{\phi_{h,h_i}^{-1}} \rho_h\rho_{h_i} \xrightarrow{\rho_h\# 0h_i \# 0h^{-1}} \rho_h\rho_{h_i} \rho_{h_i}^* \xrightarrow{\phi_{h,h_i}^*} \rho_{h_i}\rho_{h_i}^* \xrightarrow{\phi_{h,h_i}^*} \rho_{h_i}\rho_{h_i}^* = \rho_{h_i}.
\]
Then we see that (5.5) can be written as
\[
(5.6) \quad (\psi_g(\chi))_{\tilde{r}r} = \psi^{(i)}_h(\chi : 1_{\mathcal{V}} \to \rho_{h_i}),
\]
with \( r, \tilde{r} \in R_i \) determined by (5.4).
Note that we have the identification
\[
(\text{Ind}_{C_H(h_i)}^{C_G(h_i)}) \simeq (\text{Tr}_2 \rho_{h_i})^{m_i}, \tag{5.7}
\]
since \(|C_G(h_i)/C_H(h_i)| = m_i\) by Lemma 5.1 and that (5.4) is equivalent to
\[
(r^{-1}_i g r_i) (r^{-1}_i r) = (r^{-1}_i r) h. \tag{5.8}
\]

The coset \(C_G(h_i)/C_H(h_i)\) are represented by \(r^{-1}_i r\) for \(r \in R_i\) by Lemma 5.1 again, and an element of \(C_G(h_i)\) can be written as \(r^{-1}_i g r_i\) for some \(g \in C_G(f)\). As above we denote by \(\hat{\psi}(i)\) be the induced categorical action of \(C_G(h_i)\), the centralizer of \(h_i\) in \(G\), on the category \(\text{Ind}_{C_H(h_i)}^{C_G(h_i)} \simeq (\text{Tr}_2 \rho_{h_i})^{m_i}\). By (4.4) for the induced categorical action in [14], the action of \(\hat{\psi}(i)_{r^{-1}_i g r_i}\) on \((\text{Tr}_2 \rho_{h_i})^{m_i}\) with
\[
\hat{\psi}(i)_{r^{-1}_i g r_i} \cdot \chi_{r^{-1}_i g r_i} = \psi_{h_i}^{(i)}(\chi_{r^{-1}_i g r_i}) : 1 \to \rho_{h_i}, \tag{5.9}
\]
for \(\chi \in (\text{Tr}_2 \rho_{h_i})^{m_i}\). Here we use \(r^{-1}_i r\) as indices of the components of \((\text{Tr}_2 \rho_{h_i})^{m_i}\). Comparing (5.6) with (5.8), we find that the action of \(g \in C_G(f)\) on \((\text{Tr}_2 \rho_{h_i})^{m_i}\) coincides with the induced action of \(r^{-1}_i g r_i \in C_G(h_i)\) on it, and so the action of the centralizer \(C_G(f)\) on \(\text{Tr}_2 \hat{\rho}_f\) decomposes into actions on
\[
\bigoplus_i (\text{Tr}_2 \rho_{h_i})^{m_i} = \bigoplus_i \text{Ind}_{C_H(h_i)}^{C_G(h_i)} \simeq (\text{Tr}_2 \rho_{h_i})^{m_i}. \tag{5.10}
\]

The initial groupoid \(\Lambda(G)\) of a group \(G\) has as objects, the elements of \(G\), and for two such elements \(u\) and \(v\), there is one morphism in \(\Lambda(G)\) from \(u\) to \(v\) for every \(g \in G\) such that \(v = gug^{-1}\). The above result can be summarized as follows.

**Theorem 5.1.** Let \(V\) be a \(k\)-linear 2-category. The 2-categorical trace \(\text{Tr}_2\) takes induced 2-categorical action into induced categorical action of the associated initial groupoids, i.e., (5.10) holds.

5.2. **The 3-character formula.** Let \(g\) be a categorical action of a finite group \(H\) on the \(k\)-linear category \(W\). For commuting \(g\) and \(f\) in \(H\), the 2-character \(\chi_{\psi}(f, g)\) of categorical action \(\psi\) is the trace of the linear transformation induced by the functor \(\psi_g\) on the \(k\)-vector space \(\text{Tr}_2 \rho_f\) (we assume it is finite dimensional).

**Theorem 5.2.** (Corollary 7.6 in [14]) Let \(g\) be a categorical action of a subgroup \(H\) of a finite group \(G\) on a \(k\)-linear category \(W\). Suppose that \(\text{Tr}_2 \rho_h\) is finite dimensional for each \(h \in H\). Then the 2-character of the induced categorical action of \(G\) is given by
\[
\chi_{\text{ind}}(f, g) = \frac{1}{|H|} \sum_{s \in G, s^{-1} f g s^{-1} \in H \times H} \chi_{\psi}(s f s^{-1}, s g s^{-1}) \tag{5.10}
\]
for \(g \in C_G(f)\).

Now let \(\rho\) be a 2-categorical action of a finite group \(H\) on a \(k\)-linear 2-category \(V\). \(\text{Tr}_2 \rho_f\) is a \(k\)-linear category and \(\psi\) defines a categorical action on it. If \(k, g, f \in H\) mutually commutative, we define the 3-character \(\chi_{\rho}(f, g, k)\) of the 2-categorical action \(\rho\) to be the join trace of functors \(\psi_g\) and \(\psi_k\) acting on the \(k\)-linear category \(\text{Tr}_2 \rho_f\), i.e., \(\chi_{\psi}(g, k)\), the trace of the linear transformation induced by the functor \(\psi_k\) on the \(k\)-vector space \(\text{Tr}_2 \psi_g\) (we assume it is finite dimensional).
Theorem 5.3. Let $H$ be a subgroup of a finite group $G$ and let $\rho$ be a 2-categorical action of $H$ on the 2-category $\mathcal{V}$. Let $\psi$ be the categorical actions of the centralizers on the 2-categorical trace. Suppose that $\text{Tr} \psi_h$ is finite dimensional for each $h \in H$. Then the 3-character of the induced 2-categorical action of $G$ is given by

$$\chi_{\text{ind}}(f, g, k) = \frac{1}{|H|} \sum_{s \in G} \sum_{s^{-1}(f,g,k)s \in H \times H \times H} \chi_{\rho}(s^{-1}fs, s^{-1}gs, s^{-1}ks)$$

for $f, g$ and $k$ mutually commutative.

Proof. By the decomposition (5.9) of the action of $C_G(f)$ on $\text{Tr}_2\hat{\rho}_f$ and (5.6)-(5.8), we have

$$\chi_{\text{ind}}(f, g, k) = \sum_{i=1}^{m'} \chi_{\psi_i}(r_i^{-1}gr_i, r_i^{-1}kr_i).$$

Now apply Theorem 5.2 to the categorical action $\psi(i)$ of $C_G(h_i)$, given by (5.8), induced from the categorical action $\psi(i)$ of $C_H(h_i)$ on $\text{Tr}_2\rho_{h_i}$ to get

$$\chi_{\text{ind}}(f, g, k) = \sum_{i=1}^{m'} \frac{1}{|C_{H}(h_i)|} \sum_{t \in C_{G}(h_i)} \chi_{\psi_i}(t^{-1}r_i^{-1}gr_it, t^{-1}r_i^{-1}kr_it).$$

Note that

$$\chi_{\psi_i}(t^{-1}r_i^{-1}gr_it, t^{-1}r_i^{-1}kr_it) = \chi_{\rho}(h_i, t^{-1}r_i^{-1}gr_it, t^{-1}r_i^{-1}kr_it)$$

by the definition of 3-character for 2-categorical action $\rho$ of group $H$. Here $\psi(i)$ is the categorical action of $C_H(h_i)$ on $\text{Tr}_2\rho_{h_i}$ induced from the 2-categorical action $\rho$ of $H$, and $t^{-1}r_i^{-1}fr_it = t^{-1}h_it = h_i$ by definition. Also the decomposition of the action of $C_G(f)$ on $\text{Tr}_2\hat{\rho}_f$ in section 5.1 is independent of the choice of $h_i \in [h_i]_H$, the conjugate class of $h_i$ in $H$. Therefore,

$$\chi_{\text{ind}}(f, g, k) = \sum_{h \in H} \frac{1}{|h|_H |C_H(h)|} \sum_{s^{-1}fs = h} \chi_{\rho}(h, s^{-1}gs, s^{-1}ks).$$

We have used the fact that $h_i = s^{-1}fs = s^{-1}r_i^ih_ir_i^{-1}s$ implies that $r_i^{-1}s \in C_G(h_i)$. Note that for $s \in G$, $s^{-1}gs, s^{-1}ks \in H$ if and only if $s^{-1}gs, s^{-1}ks \in C_H(h_i)$ since $g$ and $k$ commute with $f = s^{-1}hs$. The 3-character formula (5.11) follows. \qed
6. The centralizer of $f$ acts categorically on $\Gamma_{T_2 \rho f}^+$

6.1. The natural transformation $\Gamma_{k,h,g}^*(\psi_k \circ \Gamma_{h,g})$. For $\chi \in (\Gamma_{T_2 \rho f})_0$, by using the definition of the composition in (6.3) twice, $\psi_k \circ \psi_h \circ \psi_g(\chi)$ is the composition of the following 2-arrows:

Let us calculate the 3-arrow $[\Gamma_{k,h,g}^*(\psi_k \circ \Gamma_{h,g})](\chi) : \psi_k \circ \psi_h \circ \psi_g(\chi) \Rightarrow \psi_{khg}(\chi)$ for $\chi \in T_{T_2 \rho f} \subset C^{++}$. We consider the lower half part of (6.1) first. The 3-arrow

$$\Lambda_1 = \#_1[\rho_k \circ_0 \Phi_{h,g,f} \circ_0 (\rho_h \circ_0 \rho_k)] \#_1 \cdot$$

the associator $\Phi_{h,g,f}$ wiskered by 2-arrows $\#_1$ which we do not write down explicitly, changes the diagonal $\rho_{gf}^*$ of the above dotted quadrilateral in (6.1) to the wavy $\rho_{hf}$ in the following diagram:

The 3-arrow

$$\Lambda_2 = \#_1[\rho_k \circ_0 \Phi_{h,g,f} \circ_0 (\rho_g \circ_0 \rho_h \circ_0 \rho_k)] \#_1 \cdot$$

changes the diagonal $\rho_{gf}$ of the above dotted-wavy quadrilateral to the wavy one $\rho_{hg}$ in the following diagram:
The 3-arrow

\[(6.6) \Lambda_3 = \diamond \#_1[\rho_k \#_0 \Phi_{\rho_k}^{-1}\varphi_{h,g,h^*} \#_0 \rho_k] \#_1\diamond \]

changes the diagonal $\rho_{hg^*}$ of the above dotted quadrilateral to the wavy one $\rho_{g^*h^*}$ in the following diagram:

\[(6.7) \]

Note that the diagrams (6.3), (6.5) and (6.7) are exactly the diagrams (3.10), (3.12) and (3.14) by adding

\[(x x x x x) \]

from below, respectively. By definition, the composition $\Lambda_1 \#_2 \Lambda_2 \#_2 \Lambda_3$ is the 3-arrow $[\psi_k \circ \Gamma_{h,g}](\chi) = \psi_k \circ \psi_h \circ \psi_g(\chi)$ corresponding to the lower half part of (6.1).

The 3-arrow

\[(6.8) \]

changes the diagonal $\rho_{hh^*}$ of the above dotted-wavy quadrilateral in (6.7) to the wavy one $\rho_{hf}$ in the following diagram:

\[(x x x x x) \]

The 3-arrow

\[(6.9) \]

changes the diagonal $\rho_{hh^*}$ of the above dotted-wavy quadrilateral in (6.7) to the wavy one $\rho_{hf}$ in the following diagram:
changes the diagonal $\rho_{hf}$ of the above dotted quadrilateral to the wavy one $\rho_{kh}$ in the following diagram:

\[
(6.9)
\]

The 3-arrow

\[\Lambda_6 = \diamond \#^1 [G_{k,hg}] \quad \psi_k \circ \psi_{hg}(\chi) \Rightarrow \psi_{khg}(\chi)\]

gives the corresponding diagram in 2-categories: the commutativity of $\phi_{k,hf}$ and $\phi_{g,kh}$. Let $D_f$ be the corresponding diagram in $C^+$ with $f$ replaced by 1, by using adjoint operations as in (3.17). Then the 2-arrow in $C^+$ corresponding to the morphism $[\Gamma_{k,hg} \# (\psi_k \circ \Gamma_{hg})(\chi)]$ in $\mathcal{T}_{f} \rho_f$ is

\[
(6.11)
\]

in the 2-category $C^+$. "=" above follows from the paste theorem for 2-categories: the commutativity of $\phi_{khf}$ and $\phi_{f,kh}$.
6.2. The natural transformation $\Gamma_{kh,g}\#(\Gamma_{k,h} \circ \psi_g)$. To calculate $\Gamma_{k,h} \circ \psi_g$, we fix the part in the lower half part of (6.1), which corresponds to $\psi_g$. The 3-arrow

$$\tilde{\Lambda}_1 = \Diamond \#_{\dag} \Phi_{k,h,g}^\ast,h^\ast \#_{\dag} \Diamond$$

changes the 1-arrow $\rho_{hg^\ast}$ in the lower part of (6.1) to the wavy one $\rho_{kh^\ast}$ in the following diagram:

The 3-arrow

$$\tilde{\Lambda}_2 = \Diamond \#_{\dag} [\Phi_{k,h,gg}^\ast \#_{\dag} (\rho_h^\ast, \rho_k^\ast)] \#_{\dag} \Diamond$$

changes the diagonal $\rho_{hg^\ast}$ of the above dotted-wavy quadrilateral to the wavy one $\rho_{kh^\ast}$ in the following diagram:

The 3-arrow

$$\tilde{\Lambda}_3 = \Diamond \#_{\dag} \Phi_{k,g}^{-1} \#_{\dag} \Diamond$$

changes the diagonal $\rho_{h^\ast g}$ of the above dotted quadrilateral to the wavy one $\rho_{kh^\ast}$ in the following diagram:

(6.16)
The composition $\tilde{\Lambda}_1 \#_2 \tilde{\Lambda}_2 \#_2 \tilde{\Lambda}_3$ is the 3-arrow $\Gamma_{k,h} \circ \psi_g : \psi_k \circ \psi_h \circ \psi_g(\chi) \Longrightarrow \psi_{kh} \circ \psi_g(\chi)$ corresponding to the lower half part of (6.1).

The 3-arrow

$$(6.17) \quad \tilde{\Lambda}_4 = \diamond \#_1 [\Phi_{kh,gf,g} \#_0 (\rho_k \circ \rho_k^* )] \#_1 \diamond$$

changes the diagonal $\rho_{g^*}$ of the above dotted quadrilateral in (6.16) to the wavy one $\rho_{k^*}$ in the following diagram:

The 3-arrow

$$(6.18) \quad \tilde{\Lambda}_5 = \diamond \#_1 [\Phi_{kh,g,f} \#_0 (\rho_g \circ \rho_k^* )] \#_1 \diamond$$

changes the diagonal $\rho_{f^*}$ of the above dotted quadrilateral to the wavy one $\rho_{k^*}$ in the following diagram:

At last, the 3-arrow

$$(6.19) \quad \tilde{\Lambda}_6 = \diamond \#_1 [\Phi_{kh,g,f}^{-1} \#_0 (\rho_f \circ \rho_k^* )] \#_1 \diamond$$

changes the diagonal $\rho_{kg^*}$ of the above dotted quadrilateral to the wavy one $\rho_{g^*} \circ \rho_k^*$ in the following diagram:

The composition $\tilde{\Lambda}_4 \#_2 \tilde{\Lambda}_5 \#_2 \tilde{\Lambda}_6$ is the 3-arrow $\Gamma_{kh,g}(\chi) : \psi_{kh} \circ \psi_g(\chi) \Longrightarrow \psi_{khg}(\chi)$ corresponding to the lower half part of (6.1).
The composition of $\tilde{A}_1 \#_2 \cdots \#_2 \tilde{A}_6$ in the 2-category $C^+$ is the following diagram $D_f^r := (6.20)$

\[
\begin{array}{c}
\rho_f \\
\phi_{g,f} \quad \rho_{g f} \quad \phi_{g f, g^*} \quad \phi_{h g^*} \quad \phi_{h^* h^*} \quad \phi_{h h^*} \quad \phi_{k h^*} \quad \phi_{k h k^*} \quad \phi_{k k^*}
\end{array}
\]

Let $D_f^r$ be the corresponding diagram in $C^+$ with $f$ replaced by 1, by using adjoint operations as in (6.17). Then the 2-arrow in $C^+$ corresponding to the morphism $[\Gamma_{kh,g}(\Gamma_{k,h} \circ \psi_g)](\chi)$ in $\text{Tr}_2 \rho_f$ is

\[(6.21) \quad \begin{array}{c}
D_f^r \xrightarrow{\chi} D_f^r.
\end{array}\]

6.3. The proof of the associativity. Let us show the identity (1.1), i.e., that diagrams $D_f^r \xrightarrow{\chi} D_f^r$ in (6.12) and $D_f^r \xrightarrow{\chi} D_f^r$ in (6.21) are identical in the 2-category $C^+$, by using the 3-cocycle identity (2.20) repeatedly.

Apply the 3-cocycle identity (2.20) to the dotted diagram in (6.20) with $g_4 = k, g_3 = h, g_2 = g f, g_1 = g^*$ to get wavy arrows in the following diagram (6.22)

\[
\begin{array}{c}
\rho_f \\
\phi_{g,f} \quad \rho_{g f} \quad \phi_{g f, g^*} \quad \phi_{h g^*} \quad \phi_{h^* h^*} \quad \phi_{h h^*} \quad \phi_{k h^*} \quad \phi_{k h k^*} \quad \phi_{k k^*}
\end{array}
\]

Note that $\tilde{A}_3$ in (6.15) and $\tilde{A}_6$ in (6.19) are the inverse of associators. Apply the 3-cocycle identity, the inverse version of (2.20) (the lower and upper boundaries are exchanged), to the
above dotted diagram with \( g_4 = k_f, g_3 = g^*, g_2 = h^*, g_1 = k^* \) to get wavy arrows in the following: (6.23)

\[
\begin{align*}
\rho f & \to \rho_{gf} \to \rho_{gg^*} \to \rho_{h^*g^*} \to \rho_{h^*h^*} \to \rho_{kk^*} \\
\phi_{k,f} & \to \phi_{gf,g^*} \to \phi_{h,g^*} \to \phi_{h^*h^*} \to \phi_{kk^*}
\end{align*}
\]

where \( \hat{\Lambda} \) is the inverse of a wiskered associator. Note that the commutative cube in (2.21) implies the following identity.

(6.24)

The LHS is the back, bottom and right faces (2-arrow inversed) of the cube in (2.21), while the RHS is the left (2-arrow inversed), top and front faces of the cube. Apply (6.24) to the dotted-wavy diagram in (6.23) with \( g_4 = k, g_3 = h_f, g_2 = g^*, g_1 = h^* \) to get wavy arrows in the following: (6.25)

\[
\begin{align*}
\rho f & \to \rho_{gf} \to \rho_{gf,g^*} \to \rho_{h^*g^*} \to \rho_{h^*h^*} \to \rho_{kk^*} \\
\phi_{k,f} & \to \phi_{gf,g^*} \to \phi_{h,g^*} \to \phi_{h^*h^*} \to \phi_{kk^*}
\end{align*}
\]
Apply the 3-cocycle identity (2.20) to the above dotted diagram with \( g_4 = k, g_3 = h, g_2 = g, g_1 = f \) to get wavy arrows in the following diagram \( \tilde{\mathcal{D}}_f := (6.26) \)

With \( f \) replaced by \( 1 \), by using adjoint operations as in (3.17), the diagram \( \tilde{\mathcal{D}}_1 \) corresponding to the upper half part is identically changed to the following diagram \( \tilde{\mathcal{D}}_1 := (6.27) \)

Where \( \Xi_j \) is the adjoint of \( \Xi_j, j = 1, 2 \). Then the whole diagram (6.24) is identically changed to

\[ \tilde{\mathcal{D}}_f \xrightarrow{\gamma} \tilde{\mathcal{D}}_f, \]

namely,

\[ (6.28) \]

\[ \cdots \rho_1 \xrightarrow{\gamma} \rho_f \xrightarrow{\gamma} \cdots \]

\[ \cdots \rho_g \xrightarrow{\gamma} \rho_k \xrightarrow{\gamma} \cdots \]

\[ \cdots \rho_k \xrightarrow{\gamma} \cdots \]

\[ \cdots \rho_k \xrightarrow{\gamma} \cdots \]

\[ \cdots \rho_k \xrightarrow{\gamma} \cdots \]
Note that (6.26) is exactly $\mathcal{D}_f^j$ in (6.11) with two extra 2-arrows $\Xi_1$ and $\Xi_2$. But the 2-arrows $\Xi_1$ and $\Xi_1^\dagger$ are the associators corresponding to the 3-arrows changing

$$
\begin{tikzpicture}
\node (A) at (0,0) {$X$};
\node (B) at (1,0) {$X$};
\node (C) at (2,0) {$X$};
\node (D) at (3,0) {$X$};
\node (E) at (0,-1) {$X$};
\node (F) at (1,-1) {$X$};
\node (G) at (2,-1) {$X$};
\node (H) at (3,-1) {$X$};
\draw[->] (A) to node[above] {$\rho_k$} (B);
\draw[->] (B) to node[above] {$\rho_k$} (C);
\draw[->] (C) to node[above] {$\rho_k$} (D);
\draw[->] (D) to node[above] {$\rho_k$} (E);
\draw[->] (E) to node[below] {$\rho_k$} (F);
\draw[->] (F) to node[below] {$\rho_k$} (G);
\draw[->] (G) to node[below] {$\rho_k$} (H);
\end{tikzpicture}
$$

$$
\begin{tikzpicture}
\node (A) at (0,0) {$X$};
\node (B) at (1,0) {$X$};
\node (C) at (2,0) {$X$};
\node (D) at (3,0) {$X$};
\node (E) at (0,-1) {$X$};
\node (F) at (1,-1) {$X$};
\node (G) at (2,-1) {$X$};
\node (H) at (3,-1) {$X$};
\draw[->] (A) to node[above] {$\rho_k$} (B);
\draw[->] (B) to node[above] {$\rho_k$} (C);
\draw[->] (C) to node[above] {$\rho_k$} (D);
\draw[->] (D) to node[above] {$\rho_k$} (E);
\draw[->] (E) to node[below] {$\rho_k$} (F);
\draw[->] (F) to node[below] {$\rho_k$} (G);
\draw[->] (G) to node[below] {$\rho_k$} (H);
\end{tikzpicture}
$$

to

$$
\begin{tikzpicture}
\node (A) at (0,0) {$X$};
\node (B) at (1,0) {$X$};
\node (C) at (2,0) {$X$};
\node (D) at (3,0) {$X$};
\node (E) at (0,-1) {$X$};
\node (F) at (1,-1) {$X$};
\node (G) at (2,-1) {$X$};
\node (H) at (3,-1) {$X$};
\draw[->] (A) to node[above] {$\rho_k$} (B);
\draw[->] (B) to node[above] {$\rho_k$} (C);
\draw[->] (C) to node[above] {$\rho_k$} (D);
\draw[->] (D) to node[above] {$\rho_k$} (E);
\draw[->] (E) to node[below] {$\rho_k$} (F);
\draw[->] (F) to node[below] {$\rho_k$} (G);
\draw[->] (G) to node[below] {$\rho_k$} (H);
\end{tikzpicture}
$$

and we have

$$
\begin{tikzpicture}
\node (A) at (0,0) {$X$};
\node (B) at (1,0) {$X$};
\node (C) at (2,0) {$X$};
\node (D) at (3,0) {$X$};
\node (E) at (0,-1) {$X$};
\node (F) at (1,-1) {$X$};
\node (G) at (2,-1) {$X$};
\node (H) at (3,-1) {$X$};
\draw[->] (A) to node[above] {$\rho_k$} (B);
\draw[->] (B) to node[above] {$\rho_k$} (C);
\draw[->] (C) to node[above] {$\rho_k$} (D);
\draw[->] (D) to node[above] {$\rho_k$} (E);
\draw[->] (E) to node[below] {$\rho_k$} (F);
\draw[->] (F) to node[below] {$\rho_k$} (G);
\draw[->] (G) to node[below] {$\rho_k$} (H);
\end{tikzpicture}
$$

So $\Xi_1$ and $\Xi_1^\dagger$ are cancelled. More precisely, as a 3-arrow, $\Xi_1^\dagger \#_0 \chi \#_0 \Xi_1$ is $(\Xi_1^\dagger \#_0 \chi) \#_1 (\Xi_1 \#_0 \chi) = (\Xi_1^\dagger \#_1 \Xi_1) \#_0 \chi = 1_{\rho_{kg}} \#_0 \chi$, which corresponds to

$$
(\phi_{k,hg}^{-1} \#_0 \phi_{h,g}^{-1}) \#_0 (\phi_{h,g} \#_0 \phi_{k,hg}) \#_0 \chi = (\phi_{k,hg}^{-1} \#_0 \phi_{h,g}^{-1}) \#_0 \chi \#_0 (\phi_{h,g} \#_0 \phi_{k,hg})
$$

in the 2-category $\mathcal{C}^+$, i.e., $\tilde{\mathcal{D}}_1 \xrightarrow{\chi} \tilde{\mathcal{D}}_j^f$ in (6.28) is identical to

$$
\begin{tikzpicture}
\node (A) at (0,0) {$\cdots$};
\node (B) at (1,0) {$\rho_1$};
\node (C) at (2,0) {$\chi$};
\node (D) at (3,0) {$\rho_f$};
\node (E) at (0,-1) {$\cdots$};
\node (F) at (1,-1) {$\rho_{hg}$};
\node (G) at (2,-1) {$\phi_{h,g}^{-1}$};
\node (H) at (3,-1) {$\rho_{hg}$};
\node (I) at (0,-2) {$\cdots$};
\node (J) at (1,-2) {$\phi_{k,hg}^{-1}$};
\node (K) at (2,-2) {$\phi_{k,hg}$};
\node (L) at (3,-2) {$\cdots$};
\draw[->] (A) to node[above] {$\cdots$} (B);
\draw[->] (B) to node[above] {$\chi$} (C);
\draw[->] (C) to node[above] {$\cdots$} (D);
\draw[->] (E) to node[below] {$\cdots$} (F);
\draw[->] (F) to node[below] {$\phi_{h,g}^{-1}$} (G);
\draw[->] (G) to node[below] {$\cdots$} (H);
\draw[->] (I) to node[below] {$\cdots$} (J);
\draw[->] (J) to node[below] {$\phi_{k,hg}$} (K);
\draw[->] (K) to node[below] {$\cdots$} (L);
\end{tikzpicture}
$$

Similarly, the 2-arrows $\Xi_2$ (6.29) and $\Xi_2^\dagger$ in (6.27) are also cancelled. The resulting diagram is exactly the diagram $\mathcal{D}_1^f \xrightarrow{\chi} \mathcal{D}_j^f$ in (6.12). We complete the proof of Theorem 3.1.

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