Random generation of direct sums of finite non-degenerate subspaces

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ABSTRACT. Let $V$ be a $d$-dimensional vector space over a finite field $F$ equipped with a non-degenerate hermitian, alternating, or quadratic form. Suppose $|F| = q^2$ if $V$ is hermitian, and $|F| = q$ otherwise. Given integers $e, e'$ such that $e + e' \leq d$, we estimate the proportion of pairs $(U, U')$, where $U$ is a non-degenerate $e$-subspace of $V$ and $U'$ is a non-degenerate $e'$-subspace of $V$, such that $U \cap U' = 0$ and $U \oplus U'$ is non-degenerate (the sum $U \oplus U'$ is direct and usually not perpendicular). The proportion is shown to be positive and at least $1 - c/q > 0$ for some constant $c$. For example, $c = \frac{7}{4}$ suffices in both the unitary and symplectic cases. The arguments in the orthogonal case are delicate and assume that $\dim(U)$ and $\dim(U')$ are even, an assumption relevant for an algorithmic application (which we discuss) for recognising finite classical groups. We also describe how recognising a classical group $G$ relies on a connection between certain pairs $(U, U')$ of non-degenerate subspaces and certain pairs $(g, g') \in G^2$ of group elements where $U = \text{im}(g - 1)$ and $U' = \text{im}(g' - 1)$.

DEDICATION: To the memory of Joachim Neubüser, a pioneer in Computational Group Theory who envisioned the free software system GAP.

KEYWORDS: non-degenerate, direct sum, finite classical group, proportion

1. Introduction

Let $V = F^d$ be a vector space over a finite field $F$ endowed with a non-degenerate symplectic, unitary or quadratic form. Given positive integers $e, e'$ such that $e + e' \leq d$, for which non-degenerate subspaces of dimensions $e, e'$ exist, we show that a constant proportion of ordered pairs of non-degenerate subspaces of dimensions $e, e'$, respectively, span a non-degenerate $(e + e')$-subspace of $V$. Moreover, we prove that the proportion approaches 1 when $|F|$ approaches infinity.

Theorem 1.1. Let $V = F^d$ be a vector space over a finite field $F$, as in Table 1, equipped with a non-degenerate symplectic, unitary or quadratic form. Let $e, e'$ be positive integers such that $e + e' \leq d$, and let $c$ be a constant, with the type of form, $q$, $e, e'$ and $c$ as in Table 1. Then the proportion of pairs $(U, U')$ of non-degenerate subspaces of dimensions...
e, e' respectively (of fixed but arbitrary type \( \sigma, \sigma' \in \{-, +\} \) in the orthogonal case) that span a non-degenerate \((e + e')\)-subspace (of arbitrary type in the orthogonal case) is at least \(1 - c/q > 0\).

**Table 1.** The form, the field, \(q\), and the constant \(c\) for Theorem 1.1.

| \(X\)   | Case    | Form         | \(|F|\) | \(c\) | Conditions                             |
|--------|---------|--------------|--------|------|----------------------------------------|
| \(U\) | unitary | hermitian    | \(q^2\) | 1.72 | \(q \geq 2, e, e' \geq 1\)              |
| \(Sp\) | symplectic | alternating  | \(q\)   | 1.75 | \(q \geq 2, e, e' \geq 2\) even        |
| \(O^e\) | orthogonal | quadratic    | \(q\)   | 3.125| \(q \geq 4, e, e' \geq 2\) even        |
|        |          |              |         |      | \(q = 3, e, e' \geq 2\) even           |

The condition in the orthogonal case that \(e, e'\) are both even not only simplifies our proof, but it is precisely the assumption that we require for an algorithmic application for recognising finite classical groups, see Section 1.1. In the orthogonal case our methods are not strong enough to prove the result with a sufficiently small value of the constant \(c\) when \(|F| = 2\). Extensive computer experimentation suggests that the result also holds in the case and when \(e, e'\) may not both be even, and also when \(q = 2\).

**Problem 1.2.** Show that Theorem 1.1 holds also in the orthogonal case with \(|F| = 2\) (for some constant \(c < 2\)).

Our strategy for proving Theorem 1.1 is described in Section 2.2. It allows us to reduce to the case \(d = e + e'\), where, after some delicate analysis, we can apply \([7, \text{Theorem } 1.1]\) (which proves Theorem 1.1 in the case \(d = e + e'\)). We paraphrase this result in the symplectic and unitary cases in Theorem 6.1 and in the orthogonal case in Theorem 6.6. When the summands of a direct sum are perpendicular we use the symbol \(\oplus\) instead of \(\oplus\).

**1.1. Algorithmic motivation.** Motivation for proving Theorem 1.1, for us, came from computational group theory, where such non-degenerate subspaces \(U, U'\) are constructed as part of various randomised algorithms for recognising finite classical groups. To justify these algorithms it is necessary to find a lower bound for the probability that \(U, U'\) are disjoint and span a non-degenerate subspace, and this is precisely what Theorem 1.1 does. Thus our main interest in Theorem 1.1 was to justify new algorithms which we are developing for classical groups over finite fields of arbitrary characteristic \([6]\).

Moreover, in the course of our research we discovered that justifying a related probability bound in the analysis of the constructive recognition algorithm in \([2]\) was overlooked in the cases of unitary, symplectic and orthogonal groups in even characteristic, as the proof was given only for groups containing \(\text{SL}(V)\). In Subsection 3.1 we show how to
use Theorem 1.1 to complete the analysis of the algorithm in [2], namely to complete the
proof of [2, Lemma 5.8] and hence of the crucial result [2, Lemma 5.8].

Our new algorithms, and also the algorithms in [2], work with a finite classical group
$G$ with natural module $V$. A major step is to construct a subgroup which is itself a
classical group acting on a much smaller subspace. The basic strategy is to find two
random elements $g$ and $g'$ in $G$ with the following property: $g$ preserves a decomposition
$V = U_g \oplus F_g$ where $g$ is irreducible on $U_g = \text{im}(g - 1)$, $F_g = \ker(g - 1)$, and similarly
for $g'$. It is convenient to write $U = U_g, F = F_g, U' = U_{g'}$ and $F' = F_{g'}$. In the case
of unitary, symplectic and orthogonal groups (which we will informally refer to as the
classical case), the subspaces $U, U'$ and $F, F'$ are non-degenerate. The challenge in all
cases is to prove that with high probability the subspaces $U, U'$ are disjoint, $U + U'$
is non-degenerate, and the subgroup $\langle g, g' \rangle$ induces a classical group on $U + U'$. This
problem can be subdivided into three smaller problems. The first problem is addressed
by the results of this paper, namely to estimate the probability that the subspaces are
disjoint and span a non-degenerate subspace. Second, is the link between the proportion
of pairs of elements $(g, g')$ and the proportion of pairs of suitable subspaces $(U, U')$. We
comment on this in Subsection 3.2. The third problem, given an $(e + e')$-subspace $W$, which
is non-degenerate in the classical case, is to estimate the probability that a pair $(g, g')$ in the relevant classical group on $W$ corresponds to a suitable pair $(U, U')$ spanning
$W$ such that the subgroup $\langle g, g' \rangle$ induces a classical group on $W$. Resolving this last
problem requires deep theory relying on the finite simple group classification. In the
special case where $e = e'$ this problem has been solved in [10], and we plan to tackle the
general case in collaboration also with Lübeck in [6].

The strategy for proving Theorem 1.1, the notation $[\mathcal{V}_e]$ and $[\mathcal{V}_e]_\sigma$, and the links with [2],
are described in Section 2. The algorithmic applications are discussed in more detail in
Section 3. Formulas for the number of non-degenerate $e$-subspaces are given in Section 4
and certain rational functions in $q$ such as $\omega(d, q) = \prod_{i=1}^{d}(1-q^{-i})$ are bounded in Section 5.
Finally, the proof of Theorem 1.1 is detailed in Section 6 for the symplectic, unitary and
orthogonal cases.

2. Strategy for proving Theorem 1.1 and links with [2]

We introduce the notation we will use throughout the paper in Subsection 2.1, and we
explain in our strategy for proving Theorem 1.1, which allows us to build on the work in [7], in Subsection 2.2. More details are given in Subsection 2.3 on the algorithmic
application discussed in Subsection 1.1, in particular we prove [2, Lemma 5.8] for the
classical groups.

2.1. Notation and hypotheses. Let $V = F^d$ be a $d$-dimensional classical space.

(a) Suppose that $V$ admits a non-degenerate form of type $X$ and $F$ is a finite field of
prime power order $q$ or $q^2$ as in Table 1. In particular, if $X = O^e$ then either $d$ is even with
We need to find a lower bound of the form \( \rho(X, V, \mathcal{U}, \mathcal{U}') \) for appropriate \( G^X \)-orbits \( \mathcal{U} \) and \( \mathcal{U}' \).

**2.2. Strategy for the Proof of Theorem 1.1.** Here we explain our strategy for proving Theorem 1.1. Let \( V = \mathbb{F}^d \) and \( G = G^X \) be as in Subsection 2.1 of type \( X \), and let \( \mathcal{U}, \mathcal{U}' \) be \( G \)-orbits of subspaces such that \( \mathcal{U} \subseteq [V]_e^X \) and \( \mathcal{U}' \subseteq [V]_{e'}^X \), as in Subsection 2.1(d), so that to prove Theorem 1.1 we need to find a lower bound of the form \( c/|\mathbb{F}| \) for the proportion \( \rho(X, V, \mathcal{U}, \mathcal{U}') \) of \( X \)-duos in \( \mathcal{U} \times \mathcal{U}' \).
If \( d = \dim(V) = e + e' \) then an \( X \)-duo spans \( V \), and the proportion \( \rho(X, V, U, U') \) is estimated in [7, Theorem 1.1] where the lower bound \( c/|F| \) with \( c \) as in [7, Table 1] is established. Assume henceforth that \( d > e + e' \). In Proposition 2.2 we give a strategy for reducing the general case to the case of dimension \( e + e' \).

Clearly the set of \( X \)-duos in \( U \times U' \) is \( G \)-invariant and, moreover, the group \( G \) acts transitively on \( [V_{e+e'}]^X \) if \( X = U \) or \( Sp \), while if \( X = O^e \), then \( G \) has two orbits on \( [V_{e+e'}]^X \), namely \( [V_{e+e'}]_+ \) and \( [V_{e+e'}]_- \). For a given \( W \in [V_{e+e'}]^X \), the number of \( X \)-duos \( (U, U') \) in \( U \times U' \) such that \( W = U \oplus U' \) depends only on the \( G \)-orbit containing \( W \). Moreover if \( (U, U') \) is an \( X \)-duo in \( U \times U' \) which spans \( W \), then \( U \in [W]^X \) and \( U' \in [W_e']^X \) so that \( (U, U') \) is an \( X \)-duo in \( [W]^X \times [W_e']^X \). This provides a critical link between the proportion \( \rho(X, V, U, U') \) we need to estimate for Theorem 1.1 and the proportion \( \rho(X, W, W, W') \) for the smaller space \( W \) and \( G_W \)-orbits \( W, W' \) in \( [W]^X, [W_e']^X \), respectively which as we mentioned, is estimated in [7, Theorem 1.1].

Remark 2.1. Suppose \( V = W \) is a non-degenerate 4-dimensional orthogonal space of type \( \tau \in \{-, +\} \), and \( V = U \oplus U' \) is a direct sum of two non-degenerate 2-subspaces. If \( U' \) has type \( \sigma' \) then, perhaps surprisingly, we can say nothing about the type \( \sigma \) of a non-degenerate complement \( U \): the type \( \sigma \) of \( U \) can be \( + \) or \( - \), and for large \( q \) each possibility occurs about half the time! This does not contradict [8, Proposition 2.5.11(ii)] because the sum need not be perpendicular. We stress in the above strategy, that there is no correlation between the subspace type of \( W \) and the subspace types of the orbits \( \mathcal{W} \) and \( \mathcal{W}' \) (which are determined by the subspace types \( \sigma, \sigma' \) of \( U \in \mathcal{U} \) and \( U' \in \mathcal{U}' \), respectively). Thus in Proposition 2.2(b) below, \( \tau, \sigma, \sigma' \) are independent of each other.

Proposition 2.2. Suppose that \( V, G = G^X \) are as in Subsection 2.1 of type \( X \), that \( d, e, e' \) are as in Theorem 1.1 with \( d > e + e' \), and that \( U, U' \) are \( G \)-orbits of subspaces in \( [V]^X, [V_e']^X \), respectively.

(a) If \( X = U \) or \( Sp \), and \( W \in [V_{e+e'}]^X \), then \( U = [V_e]^X, U' = [V_e']^X \), and

\[
\rho(X, V, U, U') = \rho(X, W, [W]^X, [W_e']^X). 
\]

(b) If \( X = O^e \) for some \( e \in \{-, +\} \), and \( U = [V]^e_{\sigma}, U' = [V_e']^e_{\sigma'} \), where \( e, e' \) are even and \( \sigma, \sigma' \in \{-, +\} \), then, choosing \( W_{\tau} \in [V_{e+e'}]_\tau \) for \( \tau \in \{-, +\} \), we have

\[
\rho(O^e, V, U, U') = \sum_{\tau \in \{-, +\}} \rho(O^e, W_{\tau}, [W_{\tau}]^\tau_{e}, [W_{\tau}]^\tau_{e'}). 
\]
Proof. (a) Suppose that $X$ is $U$ or $Sp$. As discussed above, $U = [V]_e^X$, $U' = [V]_e^{X'}$, and $G$ is transitive on $[V]_e^{X}$. Thus each $W \in [V]_e^{X}$ is spanned by the same number of $X$-duos in $U \times U'$, and this number is equal to the number of $X$-duos in $[W]_e^{X} \times [W]_e^{X'}$. Thus, for a chosen $W \in [V]_e^{X}$,

$$
\rho(X, V, U, U') = \frac{|[V]_e^{X}| \cdot |\{X\text{-duos in } [W]_e^{X} \times [W]_e^{X'}\}|}{|[[V]_e^{X}]| \cdot |[[V]_e^{X'}]|} \\
= \frac{|[V]_e^{X}| \cdot |[W]_e^{X}| \cdot |[W]_e^{X'}|}{|[[V]_e^{X}]| \cdot |[[V]_e^{X'}]|} \cdot \rho \left(X, W, \left[\begin{array}{c} W \\ e \end{array}\right]^{X}, \left[\begin{array}{c} W \\ e' \end{array}\right]^{X'}\right).
$$

(b) Now suppose that $X$ is $O_\sigma^e$ and $U = [V]_{e\sigma}^e$, $U' = [V]_{e\sigma'}^e$, for some $\sigma, \sigma' \in \{-, +\}$. The $G$-orbits on $[V]_{e\sigma}^e$ are $[V]_{e\sigma}^e$ and $[V]_{e+e'}\sigma$, and the number of $X$-duos in $U \times U'$ spanning a subspace $W \in [V]_{e\sigma}^e$ depends only on the $G$-orbit containing $W$. Moreover, if $W$ has type $\tau \in \{-, +\}$, then this number is equal to the number of $X$-duos in $[W]_{e\sigma}^\tau \times [W]_{e\sigma'}^\tau$. Thus, choosing $W_\tau \in [V]_{e\sigma}^e$ for $\tau \in \{-, +\}$, each $O_\tau$-duo $(U, U') \in U \times U'$ such that $U \oplus U' = W_\tau$ is an $O_\tau$-duo in $[W]_{e\sigma}^\tau \times [W]_{e\sigma'}^\tau$, where $W = W_\tau$, and conversely each $O_\tau$-duo $(U, U') \in [W]_{e\sigma}^\tau \times [W]_{e\sigma'}^\tau$ satisfies $(U, U') \in U \times U'$. Hence

$$
\rho(O_\tau^e, V, U, U') = \sum_{\tau \in \{-, +\}} \frac{|[V]_{e\sigma}^e| \cdot |\{O_\tau\text{-duos in } [W]_{e\sigma}^\tau \times [W]_{e\sigma'}^\tau\}|}{|[V]_{e\sigma}^e| \cdot |[V]_{e\sigma'}^e|} \\
= \sum_{\tau \in \{-, +\}} \frac{|[V]_{e\sigma}^e| \cdot |[W]_{e\sigma}^\tau| \cdot |[W]_{e\sigma'}^\tau|}{|[V]_{e\sigma}^e| \cdot |[V]_{e\sigma'}^e|} \cdot \rho \left(O_\tau^e, W_\tau, \left[\begin{array}{c} W \tau \\ e \end{array}\right]^\tau, \left[\begin{array}{c} W \tau \\ e' \end{array}\right]^\tau\right). \quad \Box
$$

2.3. Links with [2]. The quantity $\rho(X, V, U, U')$ defined in (1) is equal to the proportion studied in Theorem 1.1 for appropriate subspace families $U, U'$. It is often convenient to count single subspaces rather than subspace pairs, so we note the following easily proved property.

**Lemma 2.3.** Let $U, U'$ be $G^X$-orbits of subspaces as in Proposition 2.2, and let $U \in U$. Then

$$
\rho(X, V, U, U') = \frac{|\{U' \in U' \mid (U, U') \text{ is an } X\text{-duo}\}|}{|U'|}.
$$

**Proof.** Since $U$ is an orbit under each $G^X$ satisfying $\Omega X(V) \triangleleft \Omega X \triangleleft \Omega G(V)$, the number, say $n$, of $U' \in U'$ such that $(U, U')$ is an $X$-duo (that is, $U + U'$ is non-degenerate of dimension $e + e'$) is independent of the choice of $U \in U$. Hence we conclude that

$$
\rho(X, V, U, U') = \frac{n \cdot |U|}{|U| \cdot |U'|} = \frac{n}{|U'|}.
$$

\Box
3. Algorithmic applications of Theorem 1.1

In this section we describe how the main results of this paper will be used in an algorithmic context for recognising classical groups. Conceptually, we wish to construct classical groups of smaller dimension in a given classical group by constructing a subspace duo \((U, U')\) from a pair of elements \((g, g')\) which we call a ‘stingray duo’, and which turns out to generate a classical group on \(U + U'\) with high probability.

3.1. Completing the proof of \([2, \text{Lemma 5.8}]\). First we present several results leading up to Lemma 3.4, which deals with \([2, \text{Lemma 5.8}]\). Our approach is more general as the two subspaces we treat may come from different \(G^X\)-orbits. However we attempt, as far as possible, to use the same notation as in \([2]\) for clarity.

For a subspace \(W\) of \(V\), let \(W^\perp = \{v \in V \mid \beta(v, W) = 0\}\) where \(\beta : V \times V \to F\) is the sesquilinear form preserved by \(V\). When a direct sum \(U \oplus W\) is a perpendicular direct sum, that is when \(U \cap W = 0\) and \(U \subseteq W^\perp\), we write \(U \ominus W\) for emphasis.

**Lemma 3.1.** Let \(U, U'\) be \(G^X\)-orbits of subspaces as in Proposition 2.2, let \(U \in \mathcal{U}, U' \in \mathcal{U}'\), and let \(E = U^\perp \cap (U')^\perp\) and \(W = U + U'\). Then

(a) \(E = W^\perp\);

(b) \(\dim(E) = d - e - e'\) if and only if \(W = U \oplus U'\);

(c) \(W\) is non-degenerate of dimension \(e + e'\) if and only if \(V = E \ominus (U \oplus U')\).

**Proof.** (a) By definition \(E = U^\perp \cap (U')^\perp = (U + U')^\perp = W^\perp\).

(b) By part (a), \(\dim(E) = d - \dim(W)\). Hence \(\dim(E) = d - e - e'\) if and only if \(\dim(W) = e + e'\), which, in turn, is equivalent to \(W = U \oplus U'\).

(c) If \(W\) is non-degenerate of dimension \(e + e'\), then \(W \cap W^\perp = 0\) and \(W = U \oplus U'\). Hence by part (a), \(V = E \ominus (U \oplus U')\). Conversely if \(V = E \ominus (U \oplus U')\), then \(W = U \oplus U'\) has dimension \(e + e'\) and, by part (a), \(W \cap W^\perp = W \cap E = 0\) holds, so \(W\) is non-degenerate. \(\square\)

Lemmas 3.3 and 3.1(c) have the following immediate corollary.

**Corollary 3.2.** Let \(U, U'\) be \(G^X\)-orbits of subspaces as in Proposition 2.2, let \(U \in \mathcal{U}, U' \in \mathcal{U}'\), and let \(E = U^\perp \cap (U')^\perp\), as in Lemma 3.1. Then

\[
\rho(X, V, U, U') = \frac{|\{(U, U') \in U \times U' \mid V = E \ominus (U \oplus U')\}|}{|U| \cdot |U'|} = \frac{|\{U' \in U' \mid V = E \ominus (U \oplus U')\}|}{|U'|}.
\]

Lemma 5.8 of \([2]\) counts group elements rather than subspaces. Further, our proof applies for all fields \(F\) in the symplectic and unitary cases, and for \(|F| \geq 3\) in the orthogonal case. Our next result is a more general version of what is required for \([2]\) because we do not assume that \(\mathcal{U} = \mathcal{U}'\).
Lemma 3.3. Let $U, U'$ be $G^X$-orbits of subspaces as in Proposition 2.2, let $U \in U, U' \in U'$, and for $h \in G^X$ let $E(h) = U^\perp \cap (U')^\perp$. Let

$$X := \{U'h \mid h \in G^X\}, \text{ and } V = E(h) \oplus (U \oplus (U'h)),$$

and $T := \{h \in G^X \mid U'h \in X\}$. Then $|T|/|G^X| = \rho(X, V, U, U')$.

Proof. Note that $U'h = U'h'$ if and only if $h'h^{-1} \in G_{U'}^X$. Thus each $U'h \in X$ occurs for exactly $|G_{U'}^X|$ distinct elements $h$ of $G^X$. Further, since $U'$ is a $G^X$-orbit, $|G_{U'}^X| = |G^X|/|U'|$. It follows that $|T| = |X| \cdot |G^X|/|U'|$, and hence

$$\frac{|T|}{|G^X|} = \frac{|X|}{|U'|} = \rho(X, V, U, U'),$$

where the last equality follows from Corollary 3.2. □

For the algorithm in [2], the $G^X$-orbits $U, U'$ are identical, and we simply draw together Theorem 1.1 and Lemma 3.3 for the case where $e = e'$ and $U = U'$. We note that the condition $V = E(h) \oplus (U \oplus (U'h))$ in Lemma 3.3 implies that $\dim(E(h)) = d - e - e'$. The following lemma proves [2, Lemma 5.8] for all the classical groups as described in Section 2.1(b). The bound 0.05 below arises since the smallest value of $1 - c/|\mathbb{F}|$ for $c$ in Table 1 arises for $c = 2.85$ and $q = 3$ by Theorem 6.7.

Lemma 3.4. Suppose that the hypotheses in Section 2.1(a,b,c) hold and $U = U'$. Let $U \in U$ and

$$T = \{h \in G^X \mid V = E(h) \oplus (U \oplus (U'h)) \text{ where } E(h) = U^\perp \cap (Uh)^\perp\}.$$

Then $|T|/|G^X| = \rho(X, V, U, U')$, In particular, for all $X$ and $U$ we have $|T|/|G^X| > 0.05$.

The relation between $|T|/|G^X|$ and $\rho(X, V, U, U)$ in Lemma 3.4 was not appreciated in [2] and hence the authors did not foresee the difficult problem of finding a lower bound for $\rho(X, V, U, U)$ in their proof of the classical case in [2, Lemma 5.8].

Remark 3.5. (a) If we refine the bound in Lemma 3.4 to depend on $X$ and $|\mathbb{F}|$, we have $|T|/|G^X| \geq 1 - c/|\mathbb{F}|$ by Theorem 1.1 where $c$ is given in Table 1. Thus very few random selections are needed in the algorithm in [2] for large $q$.

(b) Recall that our methods are not strong enough to give a useful lower bound for orthogonal groups over a field of order $q = 2$, but we have bounds for all other cases. The omission of this case is not an issue in correcting the proof of [2, Lemma 5.8] since in that paper the analysis is given for fields of even size $q > 4$, see [2, Theorems 1.2 and 1.3, and Remark 1.5]. In relation to the comment in [2, Remark 1.5] about the restriction to $q > 4$ being needed because it relies on results in [10], we note that the cases $q = 3, 4$ are also covered in [10, Theorem 2], and moreover [10, Theorems 5 and 6] are valid for all field sizes, the only exception being orthogonal groups with $q = 2$. Thus the only exclusion in the analyses in both [2] and [10] is for orthogonal groups with $q = 2$. The results for
very small fields given in [10] were not in the preprint available to the authors of [2] at the time of its publication.

3.2. Stingray elements. Let $G = G^X$ be a group as in Section 2.1(b). In this subsection we study elements $g \in G$ for which the image $\text{im}(g - 1)$ is non-degenerate. For such elements $g$, the $G$-conjugacy class $C = g^G$ corresponds to the $G$-orbit $\mathcal{U} = \{\text{im}(g' - 1) \mid g' \in C\}$ of non-degenerate subspaces. Moreover, for two such $G$-conjugacy classes $C$ and $C'$, and corresponding $G$-orbits $\mathcal{U}$ and $\mathcal{U}'$, we establish in Lemma 3.12 that the proportion of $X$-duos in $\mathcal{U} \times \mathcal{U}'$ is equal to the proportion of certain kinds of pairs in $C \times C'$ which we call stingray-duos, see Definition 3.6(d). This connection is crucial for our algorithmic applications. We make the following definitions.

Definition 3.6. Assume that the hypotheses of Section 2.1(a,b) hold with $G = G^X$, and that $d \geq 2$.

(a) For $g \in G$ let $F_g := \ker(g - 1)$ be the 1-eigenspace (fixed-point space) of $g$ in $V$ and let $U_g := \text{im}(g - 1)$. Note that $U_g$ and $F_g$ are $\langle g \rangle$-invariant, and $\dim(U_g) + \dim(F_g) = \dim(V)$.

(b) For a positive integer $e \leq d$, an element $g \in G$ is called an $e$-stingray element if $\dim(U_g) = e$ and $g$ acts irreducibly on $U_g$.

(c) For positive integers $e, e'$ such that $e, e' \leq d$, we say that $(g, g') \in G \times G$ is an $(e, e')$-stingray pair, if $g$ is an $e$-stingray element and $g'$ is an $e'$-stingray element.

(d) An $(e, e')$-stingray pair $(g, g')$ is called an $(e, e')$-stingray duo, if $e \geq e'$ and $(U_g, U_{g'})$ is an $X$-duo as in Section 2.1(d).

Lemma 3.7. Let $G = G^X$ as in Section 2.1(b) and let $g \in G$ be an $e$-stingray element with $U_g, F_g$ as in Definition 3.6. Then

(a) $0 \neq v^g - v \in U_g$, for all $v \in V \setminus F_g$;

(b) if $Z$ is a $\langle g \rangle$-invariant submodule of $V$ then either $g$ acts trivially on $Z$ and $Z \leq F_g$, or $g$ acts non-trivially on $Z$, $U_g \leq Z$ and the restriction $g|_Z$ of $g$ to $Z$ is an $e$-stingray element of $GL(Z)$;

(c) $U_g$ is the unique $\langle g \rangle$-invariant submodule of $V$ on which $g$ acts non-trivially and irreducibly.

Proof. (a) Let $v \in V \setminus F_g$. Then there exist $u \in U_g$ and $f \in F_g$ such that $v = u + f$ and $u \neq 0$. The result follows since $v^g - v = w^g - u \in U_g$, and $v^g - v = 0$ if and only if $v \in F_g$.

(b) Now suppose that $Z$ is a $\langle g \rangle$-invariant submodule of $V$. If $g$ acts trivially on $Z$ then clearly $Z \leq F_g$. Suppose $g$ acts non-trivially on $Z$. Then there exists some $v \in Z$ with $v^g \neq v$ and so $v^g - v \in U_g$ is nonzero. Since $Z$ is $\langle g \rangle$-invariant, and $\langle g \rangle$ is irreducible on $U_g$, it follows that $U_g \leq Z$. Further $Z = U_g \oplus (F_g \cap Z)$ and $g|_Z$ is an $e$-stingray element of $GL(Z)$ as claimed.
(c) Let \( W \) be a \( \langle g \rangle \)-invariant submodule on which \( g \) acts irreducibly and non-trivially. Since \( g \) acts non-trivially on \( W \) it follows from part (b) that \( U_g \leq W \). Then, since \( g \) acts irreducibly on \( W \), this implies that \( U_g = W \). \( \square \)

**Lemma 3.8.** Let \( G = G^X \) be a group as in Section 2.1(b) of type \( X \) and let \( g \in G \) be an e-stingray element with \( U_g, F_g \) as in Definition 3.6. Then

(a) \( V = U_g \oplus F_g \), and in particular, \( U_g \) and \( F_g \) are non-degenerate and \( U_g^\perp = F_g \);

(b) the parity of \( e \) is as given in Table 2, where if \( X = O^e \), then \( U_g \) has minus type.

**Proof.** (a) This observation dates back at least to [11, Corollary p.6]. Let \( \beta : V \times V \rightarrow \mathbb{F} \) denote the non-degenerate sesquilinear form preserved by \( G \). Then, for \( v \in V \) and \( f \in F_g \), we have \( \beta(v, f) = \beta(v^g, f^g) = \beta(v^g, f) \), so \( \beta(v - v^g, f) = 0 \). Hence \( U_g = V(1 - g) \leq F_g^\perp \). However, \( \dim(V) = \dim(U_g) + \dim(F_g) \), so \( \dim(U_g) = \dim(F_g^\perp) \). Hence \( U_g = F_g^\perp \) and so \( U_g^\perp = F_g \). Since \( U_g \cap F_g \) is \( \langle g \rangle \)-invariant and since \( g \) acts irreducibly on \( U_g \), it follows that \( U_g \cap F_g \) is trivial. Thus \( U_g \cap U_g^\perp = 0 \) since \( U_g^\perp = F_g \), and hence \( U_g \) is non-degenerate. Also \( F_g = U_g^\perp \) is non-degenerate and \( V = U_g \oplus F_g \).

(b) By the definition of an e-stingray element, the characteristic polynomial \( c_g(t) \) of \( g \) satisfies \( c_g(t) = (t - 1)^d - ec_h(t) \), where \( h = g|_{U_g} \) denotes the restriction of \( g \) to \( U_g \). Moreover, \( U_g \) is an irreducible \( \langle h \rangle \)-module, so \( c_h(t) \) is a monic irreducible polynomial over \( \mathbb{F} \) of degree \( e \), and in particular \( c_h(0) \neq 0 \). For any polynomial \( f(t) \) over \( \mathbb{F} \) with \( f(0) \neq 0 \), let \( f^{rev}(t) = f(0)^{-1}t^{deg(f)}f(t^{-1}) \), the reverse polynomial of \( f(t) \). Also, if \( X = U \) and \( J \in \text{GL}(V) \), let \( J^{\phi} \) denote the matrix obtained from \( J \) by applying \( q \)th powers to each entry. Now since \( g \in G^X \), we have \( gJg^{\phi} = J \) or \( gJg^T = J \), according as \( X = U \) or \( X \in \{\text{Sp}, O^e\} \) respectively, where in both cases \( J \) is an (invertible) Gram matrix. Therefore \( J^{-1}gJ = g^{-\phi} \) or \( J^{-1}gJ = g^{-T} \), and so \( c_g(t) = c_h^{rev}(t)^{\phi} \) or \( c_g(t) = c_h^{rev}(t) \), respectively. It follows that \( c_h(t) = c_h^{rev}(t) \) or \( c_h(t) = c_h^{rev}(t) \), and (since \( c_h(t) \) is a monic irreducible) that \( e \) is odd, or \( e \) is even, respectively, by [5, Lemma 1.3.11(b) and Lemma 1.3.15(c)]. Thus we obtain the restrictions on the parity of \( e \) in Table 2. Finally, if \( X = O^e \), then \( g|_{U_g} \) irreducible implies that the type of \( U_g \) is minus, see [1, pp.187–188] for example. \( \square \)

We shall be studying \((e, e')\)-stingray pairs in a group \( G = G^X \) as in Section 2.1(b). In the case where \( X = O^e \), we assume that \( e, e' \) are both even so in particular \( d - e \geq 2 \). Thus the assumptions in the following lemma will always hold.

**Lemma 3.9 (from [8, Lemma 4.1.1(iv,v)].** Let \( G = G^X \) be a group as in Section 2.1(b) of type \( X \), and let \( U \) be a non-degenerate proper subspace of \( V \). Moreover if \( X = O^e \) assume also that \( \dim(V) - \dim(U) \geq 2 \). Then the group \( G^U \) induced (via restriction) on \( U \) by the setwise stabiliser \( G_U \) is the full isometry group \( GX(U) \).

**Remark 3.10.** Lemma 3.9 follows from [8, Lemma 4.1.1(iv,v)]. However in applying this result we note that the statement of [8, Lemma 4.1.1] involves the hidden assumption...
Table 2. The parity of $e$ for different $X$.

| $X$ | $U$ | $Sp$ | $O^\pm$ | $O^\circ$ |
|-----|-----|------|---------|---------|
| $\Omega X(V)$ | $SU_d(q)$ | $Sp_d(q)$ | $\Omega^\pm_d(q)$ | $\Omega^\circ_d(q)$ |
| parity of $e$ | odd | even | even | even |

$\dim(U) \geq \dim(V)/2$ (see [8, p. 83, Definition]). Thus the conclusion that $G^U$ induces $GX(U)$ follows from part (iv) of [8, Lemma 4.1.1] if $\dim(U) \leq \dim(V)/2$, and from part (v) of [8, Lemma 4.1.1] if $\dim(U) > \dim(V)/2$, noting that, in the latter case, our assumption that $\dim(U) \leq \dim(V) - 2$ when $X = O^\circ$ avoids the exception in [8, Lemma 4.1.1(v)].

3.2.1. Stingray elements and subspaces. Let $G = G^X$ be a group as in Section 2.1(b), let $C$ be a $G$-conjugacy class of $e$-stingray elements as in Definition 3.6(b), and let $q \in G$. Then by Lemmas 3.7(c) and 3.8, $U_g = \im(g - 1)$ is the unique $(g)$-invariant subspace of $V$ on which $g$ acts non-trivially and irreducibly, $U_g$ is non-degenerate, and $\dim(U_g) = e$ with the parity of $e$ as in Table 2. Thus $U = \{U_g \mid g \in C\}$ is a $G$-orbit of non-degenerate subspaces, and so $U \subseteq [V_e]^X$ as described in Section 2.1(c). Clearly $G$ acts transitively via conjugation on $C = g^G$, and the stabiliser of $g$ is $C_G(g)$. Further, $C_G(g)$ leaves both $U_g$ and $F_g = U_g^\perp$ invariant and so $C_G(g) \leq G_{U_g} \leq G$. We next relate $|C|$ and $|U|$.

Lemma 3.11. Let $C, U$ be as above, and let $g \in C$ and $U = U_g$. Then

\begin{equation}
|C| = |U| \cdot |G_U : C_G(g)|,
\end{equation}

and there are precisely $|C|/|U|$ elements $g' \in C$ such that $U_g' = U$.

Proof. It follows from $C_G(g) \leq G_U \leq G$ that

\[|C| = |G : C_G(g)| = |G : G_U| \cdot |G_U : C_G(g)| = |U| \cdot |G_U : C_G(g)|.\]

Therefore the action of $G$ on $C$ preserves the partition of $C$ into classes $C(U')$ for $U' \in U$ where $C(U') := \{g' \in C \mid U_{g'} = U'\}$, and the number of conjugates in each of these classes is $|G_U : C_G(g)|$. \hfill $\Box$

3.2.2. Stingray duos. Suppose that the hypotheses of Section 2.1(a,b) hold, let $G = G^X$, and let $C$ be a $G$-conjugacy class of $e$-stingray elements and $C'$ a $G$-conjugacy class of $e'$-stingray elements such that $e \geq e'$ and $e + e' \leq d$. Thus each $(g, g') \in C \times C'$ is an $(e, e')$-stingray pair, as in Definition 3.6. As defined there, we will say that $(g, g')$ is a stingray duo if $U_g \cap U_{g'} = 0$ and $U_g \oplus U_{g'}$ is non-degenerate. Let $U = \{U_g \mid g \in C\}$ and $U' = \{U_{g'} \mid g' \in C'\}$, so that, as noted in the previous subsection, $U$ is a $G$-orbit contained in $[V_e]^X$, and $U'$ is a $G$-orbit contained in $[V_{e'}]^X$. We denote the proportion of stingray
duos in $C \times C'$ by

\[ (3) \quad \rho_{\text{stingray}}^{\text{duo}} = \frac{|\{(g, g') \in C \times C' \mid (e, e')\text{-stingray duo}\}|}{|C \times C'|}. \]

**Lemma 3.12.** Let $C, C', U, U'$ be as above. Then $\rho_{\text{stingray}}^{\text{duo}} = \rho(X, V, U, U')$.

**Proof.** It follows from Lemma 3.11 that each $U \in U$ is equal to $U_g$ for exactly $|C|/|U|$ elements $g \in C$. Similarly, each $U' \in U'$ is equal to $U_{g'}$ for exactly $|C'|/|U'|$ elements $g' \in C'$. By Definition 3.6(d), a pair $(g, g') \in C \times C'$ is an $(e, e')$-stingray duo if and only if $(U_g, U_{g'})$ is an $X$-duo in $U \times U'$. Hence the number of $(e, e')$-stingray duos in $C \times C'$ is equal to $(|C|/|U|) \cdot (|C'|/|U'|) \cdot Y$, where $Y$ is the number of $X$-duos in $U \times U'$. By (1),

\[ Y = |U| \cdot |U'| \cdot \rho(X, V, U, U'), \]

and hence the number of $(e, e')$-stingray duos in $C \times C'$ is $|C| \cdot |C'| \cdot \rho(X, V, U, U')$. It follows that $\rho_{\text{stingray}}^{\text{duo}} = \rho(X, V, U, U')$. \(\square\)

### 4. Counting non-degenerate subspaces

Let $V$ be a $d$-dimensional classical space over a finite field $\mathbb{F}$. An $e$-dimensional subspace of $V$ is called an $e$-**subspace** and the set of all of non-degenerate $e$-subspaces is denoted by $[V]_e$. In the orthogonal case, $V$ has type $\varepsilon \in \{ -, \circ, + \}$ and the set of non-degenerate $e$-subspaces of subspace type $\tau$ is denoted $[V]_e^\varepsilon$. For our application, $e$ is even so $\tau \in \{ -, + \}$, subspace type equals intrinsic type, and $d$ may be odd or even. We assume that $q$ is odd if $d$ is odd. Hence $V = U \oplus U^\perp$ holds for $U \in [V]_e^\varepsilon$. Furthermore, if $U$ has type $\tau \in \{ -, + \}$, then it follows from [8, Proposition 2.5.11(ii)] that $U^\perp$ has type $\varepsilon \tau$ (even when $d$ is odd).

The order of a classical group can be expressed as a power of $q$ times a rational function in $q$ which approaches 1 as $q \to \infty$. Such a rational function is

\[ (4) \quad \omega(d, q) := \prod_{i=1}^{d}(1 - q^{-i}) \quad \text{for } d \geq 0 \text{ where } \omega(0, q) = 1. \]

The orders of the isometry groups $GU_d(q), Sp_d(q), GO_d^\varepsilon(q)$ given in [8, Table 2.1.C, p.19] are rewritten below in terms of the ‘dominant power of $q$’. In (6) below we identify the symbols $-, \circ, +$ with the numbers $-1, 0, 1$, respectively. Then

\[ (5) \quad |GL_d(q)| = q^{d^2} \omega(d, q), \quad |GU_d(q)| = q^{d^2} \omega(d, -q), \]

\[ (6) \quad |Sp_d(q)| = q^{(d+1)\left\lfloor \frac{d}{2} \right\rfloor}(\frac{d}{2}, q^2), \quad |GO_d^\varepsilon(q)| = 2q^{(d)} \frac{\omega\left(\left\lfloor \frac{d}{2} \right\rfloor, q^2 \right)}{1 + \varepsilon q^{-\left\lfloor \frac{d}{2} \right\rfloor}}. \]

In the unitary case of Proposition 4.1, we assume that $1 \leq e \leq d - 1$, and in the symplectic and orthogonal cases we assume that $e$ is even and $2 \leq e \leq d - 2$.

**Proposition 4.1.** Let $V$ be a non-degenerate $d$-dimensional classical space. Then
The number of non-degenerate \(e\)-subspaces of a unitary space \(V = (\mathbb{F}_q)^d\) is
\[
\left| \begin{bmatrix} V \\ e \end{bmatrix}^U \right| = \frac{q^{2e(d-e)}\omega(d,-q)}{\omega(e,-q)\omega(d-e,-q)}.
\]

The number of non-degenerate \(e\)-subspaces of a symplectic space \(V = (\mathbb{F}_q)^d\) is
\[
\left| \begin{bmatrix} V \\ e \end{bmatrix}^{Sp} \right| = \frac{q^{e(d-e)}\omega(d,q^2)}{\omega(\frac{d}{2},q^2)\omega(\frac{d-e}{2},q^2)}.
\]

Let \(V = (\mathbb{F}_q)^d\) be an orthogonal space of type \(\varepsilon\). For \(e\) even, the number of non-degenerate \(e\)-subspaces of type \(\tau \in \{-, +\}\) is
\[
\left| \begin{bmatrix} V \\ e \end{bmatrix}_\tau^e \right| = \frac{q^{e(d-e)}(1 + \tau q^{-\frac{d}{2}})(1 + \varepsilon \tau q^{-\frac{d}{2}})}{2(1 + \varepsilon q^{-\frac{d}{2}})} \cdot \frac{\omega(\frac{d}{2},q^2)}{\omega(\frac{d}{2} - \frac{e}{2},q^2)}.
\]

Proof. The stabiliser of a non-degenerate subspace \(U\) of \(V\) equals the stabiliser of the decomposition \(V = U \oplus U^\perp\) by [8, Lemma 2.1.5(v)], and the shape of these stabilisers is given in [8, Table 4.1.A]. We use the formulas (5) and (6).

(a) It follows from Witt’s Theorem, the orbit-stabiliser lemma and (5), that the number of non-degenerate \(e\)-subspaces of \(V\) is
\[
\frac{|GU_d(q)|}{|GU_e(q)||GU_{d-e}(q)|} = \frac{q^d \omega(d,-q)}{q^e \omega(e,-q)q^{(d-e)^2} \omega(d-e,-q)} = \frac{q^{2e(d-e)}\omega(d,-q)}{\omega(e,-q)\omega(d-e,-q)}.
\]

(b) This proof is similar to part (a).

(c) Suppose that \(U\) is a non-degenerate \(e\)-subspace of \(V\) of type \(\tau\). By the preamble to this proposition, \(V = U \oplus U^\perp\) and \(U^\perp\) has type \(\varepsilon\tau\). Hence the stabiliser of \(U\) in \(GO_e^d(q)\) is \(GO_{d,e}^\tau(q)\) by Witt’s Theorem. It follows from the orbit-stabiliser lemma and (6) that \(\left| \begin{bmatrix} V \\ e \end{bmatrix}_\tau^e \right|\) equals
\[
\frac{|GO_{d,e}^\tau(q)|}{|GO_{d,e}^\tau(q) \times GO_{d,e}^{\varepsilon\tau}(q)|} = \frac{\omega(\frac{d}{2})\omega(\frac{d-e}{2})}{\omega(\frac{d}{2} - \frac{e}{2},q^2)} = \frac{q^{e(d-e)}(1 + \tau q^{-\frac{d}{2}})(1 + \varepsilon \tau q^{-\frac{d}{2}})}{2(1 + \varepsilon q^{-\frac{d}{2}})} \cdot \frac{\omega(\frac{d}{2},q^2)}{\omega(\frac{d}{2} - \frac{e}{2},q^2)}.
\]

However, \(\binom{d}{2} - \binom{e}{2} - \binom{d-e}{2} = e(d-e)\) and the result follows. \(\square\)

5. Bounding rational functions in \(q\)

This section derives bounds that are used to estimate functions in our main theorems. The following infinite products provide useful limiting bounds:
\[
\omega(\infty, q) = \prod_{i=1}^\infty \left( 1 - q^{-i} \right) \quad \text{and} \quad \omega(\infty, -q) = \prod_{j=1}^\infty \left( 1 - (-q)^{-j} \right) = \prod_{i=1}^\infty \left( 1 + q^{-(2i-1)} \right)(1 - q^{-2i}).
\]
Lemma 5.1. Suppose \( q > 1 \). Using the notation (4) we have:

\[
1 - q^{-1} - q^{-2} + q^{-5} < \omega(\infty, q) < \cdots < \omega(2, q) < \omega(1, q) < \omega(0, q) = 1 \quad \text{and},
\]

\[
1 = \omega(0, -q) < \omega(2, -q) < \cdots < \omega(\infty, -q) < \cdots < \omega(3, -q) < \omega(1, -q) = 1 + q^{-1}.
\]

Moreover,

\[
\lim_{n \to \infty} \omega(n, q) = \omega(\infty, q) \quad \text{and} \quad \lim_{m \to \infty} \omega(2m, -q) = \lim_{m \to \infty} \omega(2m + 1, -q) = \omega(\infty, -q).
\]

Proof. For bounds such as \( 1 - q^{-1} - q^{-2} + q^{-5} < \omega(\infty, q) \) or \( 1 - q^{-1} - q^{-2} + q^{-5} < \omega(\infty, q) \), see [9, Lemma 3.5]. Since \( \omega(n, q) = \omega(n-1, q)(1-q^{-n}) < \omega(n-1, q) \), the first inequalities follow. Clearly \( \omega(n, q) \to \omega(\infty, q) \) as \( n \to \infty \). The second inequalities use

\[
(1 - q^{-2i})(1 + q^{-(2i+1)}) < 1 < (1 + q^{-(2i-1)})(1 - q^{-2i}).
\]

Not only does this imply that the infinite product \( \omega(\infty, -q) \) converges to a positive limit for all \( q > 1 \), it shows that \( \lim_{m \to \infty} \omega(2m, -q) = \lim_{m \to \infty} \omega(2m + 1, -q) = \omega(\infty, -q) \). □

By virtue of the symmetry \( \binom{n}{k} = \binom{n}{n-k} \), we assume in the following lemma that \( k \leq \lfloor \frac{n}{2} \rfloor \). Similarly, since \( \binom{n-k}{k} = \binom{n}{k} \), we assume \( k \leq \lfloor \frac{n}{2} \rfloor \) whether \( k \) is even or odd.

Lemma 5.2. Suppose \( q > 1 \) and \( k \in \{0, 1, \ldots, n\} \). Using (4), we define

\[
\binom{n}{k}_q := \frac{\omega(n, q)}{\omega(k, q)\omega(n-k, q)} \quad \text{and} \quad \binom{n}{k}_{-q} := \frac{\omega(n, -q)}{\omega(k, -q)\omega(n-k, -q)}.
\]

Then

\[
1 = \binom{n}{0}_q < \binom{n}{1}_q < \binom{n}{2}_q < \binom{n}{3}_q < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor}_q < \frac{1}{\omega(\infty, q)} < \frac{1}{1 - q^{-1} - q^{-2}}.
\]

Further, if \( i := \lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor \rfloor \) and \( j := \lfloor \frac{1}{2} \lceil \frac{n}{2} \rceil \lceil \rfloor \), then \( 2i+1 \leq \lfloor \frac{n}{2} \rfloor \), \( 2j \leq \lfloor \frac{n}{2} \rfloor \) and

\[
\frac{1 - (-q)^{-n}}{1 + q^{-1}} = \binom{n}{1}_{-q} < \binom{n}{3}_{-q} < \cdots < \binom{n}{2i+1}_{-q} < \frac{1}{\omega(\infty, -q)}
\]

and

\[
\frac{1}{\omega(\infty, -q)} < \binom{n}{2j}_{-q} < \cdots < \binom{n}{0}_{-q} = 1.
\]

Proof. Suppose that \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \). Cancelling \( \omega(n-k, q) \) and \( \omega(n-k, -q) \) gives

\[
\binom{n}{k}_q = \frac{\prod_{i=1}^{k} (1 - q^{-(n-k+i)})}{\omega(k, q)} \quad \text{and} \quad \binom{n}{k}_{-q} = \frac{\prod_{i=1}^{k} (1 - (-q)^{-(n-k+i)})}{\omega(k, -q)}.
\]

The inequality \( \binom{n}{k-1}_q < \binom{n}{k}_q \) holds as \( \omega(k-1, q) > 0 \) and \( 1 - q^{-k} < 1 - q^{-(n-k+1)} \). It follows from (9) and Lemma 5.1 that \( \binom{n}{\frac{n}{2}}_q < \frac{1}{\omega(\infty, q)} \). Hence the first chain of inequalities is true. Similar reasoning using \( \omega(k-2, -q) > 0 \) and (7) establishes the remaining
inequalities with the exception of \( \binom{n}{2i+1}_q < \frac{1}{\omega(\infty, -q)} < \binom{n}{2j}_q \). This follows from (9), the definitions of \( i \) and \( j \) in terms of \( n \), and \( \lim_{n \to \infty} \binom{n}{2i+1}_q = \lim_{n \to \infty} \binom{n}{2j}_q = \frac{1}{\omega(\infty, -q)}. \) □

The bounds in Lemmas 5.1 and 5.2 have largest error for small \( q \). For example, 
\[ \omega(\infty, 2) = 0.288 \cdots, \omega(\infty, -2) = 1.210 \cdots, \omega(\infty, 3) = 0.560 \cdots, \omega(\infty, -3) = 1.217 \cdots. \]

In Lemmas 5.3 and 5.4, the functions \( \exp(x), \log(x) \) have the natural base \( e = 2.718 \cdots \).

**Lemma 5.3.** Suppose \( n \geq 0, q > 1 \) and \( \omega_-(n, q) = \prod_{i=1}^{n}(1 + q^{-i}) \). Then 
\[ \frac{1 - q^{-(n+1)}}{1 - q^{-1}} \leq \omega_-(n, q) \leq \exp\left(\frac{q^{-1}(1 - q^{-n})}{1 - q^{-1}}\right) \quad \text{and} \quad 1 - q^{-1} \leq \omega_-(n, -q) \leq 1. \]

**Proof.** The inequalities follow from \( 1 + \sum_{i=1}^{n} a_i \leq \prod_{i=1}^{n}(1 + a_i) \leq \exp(\sum_{i=1}^{n} a_i) \), where each \( a_i \) is positive, and \( (1 - q^{-(2i-1)})(1 + q^{-2i}) < 1 < (1 - q^{-2i})(1 - q^{-(2i+1)}). \) □

**Lemma 5.4.** Suppose \( a, b \in \mathbb{Z} \) and \( q \in \mathbb{R} \) where \( 1 \leq a < b \) and \( q > 1 \). Then 

(a) \[ 1 - \frac{q^{-a}}{\log(q)} < 1 - \frac{q^{-a} - q^{-b}}{\log(q)} \leq \frac{\omega(b, q)}{\omega(a, q)} \leq 1, \text{ and} \]

(b) \[ 1 - q^{-2} \leq 1 - q^{-a-1} \leq \frac{\omega(b, q)}{\omega(a, q)} \leq 1 + q^{-a-1} \leq 1 + q^{-2}. \]

**Proof.** (a) The upper bound of 1 is immediate, and the other bounds follow from 
\[ \frac{\omega(b, q)}{\omega(a, q)} = \prod_{i=a+1}^{b} (1 - q^{-i}) \geq 1 - \sum_{i=a+1}^{b} q^{-i} \geq 1 - \int_{a}^{b} q^{-x}dx = 1 - \frac{q^{-a} - q^{-b}}{\log(q)} > 1 - \frac{q^{-a}}{\log(q)}. \]

(b) Define \( \zeta_n := (1 - (-q)^{-(n+1)})(1 - (-q)^{-n}) \). First consider the upper bounds. We use the fact that \( \zeta_n < 1 \) for odd \( n \) odd by [7, Lemma 4.2(a)]. Write 
\[ \frac{\omega(b, q)}{\omega(a, q)} = \prod_{i=a+1}^{b} (1 - (-q)^{-i}) = A \cdot \left( \prod \zeta_n \right) \cdot B, \]
where the product ranges over all odd \( n \) with \( a + 2 \leq n \leq b \), and \( A = 1 \) if \( a \) is odd and \( A = 1 + q^{-a-1} \) if \( a \) is even, and \( B = 1 \) if \( b \) is odd and \( B = 1 - q^{-b} \) if \( b \) is even. As \( \zeta_n < 1 \) for odd \( n \), we have \( A \cdot \left( \prod \zeta_n \right) \cdot B \leq A \cdot B \leq A \leq 1 + q^{-a-1} \leq 1 + q^{-2}. \)

For the lower bounds we use the fact that \( \zeta_n > 1 \) for even \( n \) odd by [7, Lemma 4.2(a)]; 
\[ \frac{\omega(b, q)}{\omega(a, q)} = \prod_{i=a+1}^{b} (1 - (-q)^{-i}) = A \cdot \left( \prod \zeta_n \right) \cdot B, \]
where the product ranges over all even \( n \) with \( a + 2 \leq n \leq b \) and \( A = 1 \) if \( a \) is even and \( A = 1 - q^{-a-1} \) if \( a \) is odd and \( B = 1 + q^{-b} \) if \( b \) is odd and \( B = 1 + q^{-b} \) if \( b \) is even. As \( \zeta_n > 1 \) for odd \( n \), we have \( A \cdot \left( \prod \zeta_n \right) \cdot B \geq A \cdot B \geq A \geq 1 - q^{-a-1} \geq 1 - q^{-2}. \) □
Lemma 5.5. Let \( m, a, b \) be positive integers with \( m > a + b \). Then
\[
\frac{\omega(m - a, q)\omega(m - b, q)}{\omega(m - a - b, q)\omega(m, q)} > 1 - \frac{1}{q \log(q)}.
\]

Proof. Suppose first that \( m = a + b + 1 \). It follows from Lemma 5.1 that
\[
\frac{\omega(m - a, q)\omega(m - b, q)}{\omega(1, q)\omega(a + b + 1, q)} > \frac{\omega(b + 1, q)\omega(a + 1, q)}{\omega(1, q)} > \frac{1 - q^{-1} - q^{-2} + q^{-5}}{1 - q^{-1}}.
\]
Moreover, \( \frac{1 - q^{-1} - q^{-2} + q^{-5}}{1 - q^{-1}} \geq 1 - \frac{1}{q \log(q)} \) since \( q - 1 \geq \log(q) \), so the lower bound holds.

Finally, if \( m > a + b + 1 \) then, using Lemma 5.4(a), we obtain
\[
\frac{\omega(m - a, q)}{\omega(m - a - b, q)\omega(m, q)} \geq 1 - \frac{1}{q^m a b \log(q)} \geq 1 - \frac{1}{q^2 \log(q)}.
\]
Moreover, again by Lemma 5.4(a) we obtain \( \frac{\omega(m - b, q)}{\omega(m, q)} \geq 1 \) and thus
\[
\frac{\omega(m - a, q)\omega(m - b, q)}{\omega(m - a - b, q)\omega(m, q)} \geq 1 - \frac{1}{q^2 \log(q)} > 1 - \frac{1}{q \log(q)}.
\]
\[\square\]

6. Transitioning from \( V \) to \( W \) and proving the bounds

The goal of this section is to reduce from \( V \) to the \((e + e')\)-dimensional subspace \( W \) in Proposition 2.2, so we can apply [7, Theorem 1.1]. In Subsections 6.1, 6.2 and 6.3 we shall find lower bounds for the ratios
\[
\frac{|V_{e + e'}^\text{Sp}| \cdot |W_{e'}^\text{Sp}| \cdot |W_{e'}^\text{Sp}|}{|V_e^\text{Sp}| \cdot |V_e^\text{Sp}|}, \quad \frac{|V_{e + e'}^U \cdot |W_e^U| \cdot |W_e^U|}{|V_e^U| \cdot |V_e^U|}, \quad \frac{|V_{e + e'}^\tau | \cdot |W_{e'}^\tau | \cdot |W_{e'}^\tau |}{|V_{e'}^\tau | \cdot |V_{e'}^\tau |}.
\]

This allows us to apply Proposition 2.2(a) in the symplectic and unitary cases and Proposition 2.2(b) in the orthogonal case.

We paraphrase [7, Theorem 1.1] in Theorems 6.1 and 6.7 in a form that is useful for us.

Theorem 6.1 ([7, Theorem 1.1]). Let \( e, e' \) be positive integers with both even in part (a).

(a) Let \( V = (\mathbb{F}_q)^{e + e'} \) be a non-degenerate symplectic space. Then the proportion of pairs \((U, U')\) of non-degenerate subspaces of dimensions \( e \) and \( e' \) respectively that satisfy \( U \cap U' = 0 \), and hence \( V = U \oplus U' \), is at least \( 1 - \frac{5}{5q} > 0 \) for all \( q \geq 2 \).

(b) Let \( V = (\mathbb{F}_q^2)^{e + e'} \) be a non-degenerate unitary space. If \((e, e', q) \neq (1, 1, 2)\), then the proportion of pairs \((U, U')\) of non-degenerate subspaces of dimensions \( e \) and \( e' \) respectively that satisfy \( U \cap U' = 0 \), and hence \( V = U \oplus U' \), is at least \( 1 - \frac{9}{5q^2} > 0 \).
6.1. The symplectic case. In this subsection we establish the following lower bound.

**Theorem 6.2.** Let \( d, e, e' \) be positive even integers where \( d \geq e + e' \). Let \( V = (\mathbb{F}_q)^d \) be a non-degenerate symplectic \( d \)-space over \( \mathbb{F}_q \). Set \( \mathcal{U} = [\nu] \) and \( \mathcal{U}' = [\nu'] \). Then

\[
\rho(\text{Sp}, V, \mathcal{U}, \mathcal{U}') > \left( 1 - \frac{1}{2q^2 \log(q)} \right) \left( 1 - \frac{5}{3q} \right) > 1 - \frac{7}{4q} \quad \text{for all } q \geq 2.
\]

**Lemma 6.3.** Suppose that \( V = (\mathbb{F}_q)^d \) is a non-degenerate symplectic space, and \( W \) is a non-degenerate \((e + e')\)-dimensional subspace. Then \( d, e, e' \) are even, \( d \geq e + e' \) and

\[
\frac{|[\nu + \nu']^\text{Sp}| \cdot |[\nu]^\text{Sp}| \cdot |[\nu']^\text{Sp}|}{|[\nu]^\text{Sp}| \cdot |[\nu']^\text{Sp}|} > 1 - \frac{1}{2q^2 \log(q)}.
\]

**Proof.** For the duration of this proof and, for \( e \) even, we abbreviate \( \omega(e/2, q^2) \) by \( \omega(e) \). Repeated use of Proposition 4.1(b) and extensive cancellation shows

\[
\frac{|[\nu + \nu']^\text{Sp}| \cdot |[\nu]^\text{Sp}| \cdot |[\nu']^\text{Sp}|}{|[\nu]^\text{Sp}| \cdot |[\nu']^\text{Sp}|} = \frac{q^{(e+e')(d-e-e')/2} \omega(d)}{\omega(e+e') \omega(d-e-e')} \frac{q^{2e'e'} \omega(e+e')^2}{\omega(e) \omega(e') \omega(d)} \frac{\omega(e) \omega(d-e-e')}{\omega(e) \omega(d-e-e')} \frac{\omega(e') \omega(d-e-e')}{\omega(e') \omega(d-e-e')} \frac{\omega(e+e') \omega(d-e-e')}{\omega(e+e') \omega(d-e-e')}.
\]

If \( d = e + e' \), then this ratio is 1 and the bound holds trivially. Suppose that \( d > e + e' \). Then

\[
\frac{\omega(e+e')}{\omega(e) \omega(e')} \geq 1 \quad \text{by Lemma 5.2, and applying Lemma 5.5 with } q \text{ replaced by } q^2 \text{ gives}
\]

\[
\frac{\omega(d-e) \omega(d-e')}{\omega(d-e-e') \omega(d)} = \frac{\omega((d-e)/2, q^2) \omega((d-e')/2, q^2)}{\omega((d-e-e')/2, q^2) \omega(d-e'/2, q^2)} > 1 - \frac{1}{2q^2 \log(q)}.
\]

**Proof of Theorem 6.2.** Let \( W \) be a non-degenerate \((e + e')\)-subspace of \( V \). Set \( \mathcal{W} := [\nu] \) and \( \mathcal{W}' := [\nu'] \). It follows from Proposition 2.2(a) that

\[
\rho := \rho(\text{Sp}, V, \mathcal{U}, \mathcal{U}') = \frac{|[\nu + \nu']^\text{Sp}| \cdot |[\nu]^\text{Sp}| \cdot |[\nu']^\text{Sp}|}{|[\nu]^\text{Sp}| \cdot |[\nu']^\text{Sp}|} \cdot \rho(\text{Sp}, W, \mathcal{W}, \mathcal{W}').
\]

However, \( \rho(\text{Sp}, W, \mathcal{W}, \mathcal{W}') > 1 - \frac{5}{3q} \) by Theorem 6.1(a). Therefore Lemma 6.3 implies that

\[
\rho > \left( 1 - \frac{1}{2q^2 \log(q)} \right) \left( 1 - \frac{5}{3q} \right) \quad \text{for all } q \geq 2.
\]

This proves the first inequality. For the second inequality, we write \( \alpha = \frac{5}{3} \) and \( \beta = \frac{1}{2q \log(q)} \) and aim to show that \( (1 - \frac{\alpha}{q})(1 - \frac{\beta}{q}) \geq 1 - \frac{7}{4q} \). This is equivalent to \( \frac{7}{4} > \alpha + \beta - \frac{3\alpha \beta}{q} = \frac{5}{3} + \frac{1}{2q \log(q)} - \frac{5}{6q^2 \log(q)} \). The latter is true for all \( q \geq 2. \)
6.2. The unitary case. In this subsection we verify the following lower bound.

**Theorem 6.4.** Suppose that \(d, e, e'\) are positive integers where \(d \geq e + e'\). Let \(V = (\mathbb{F}_q^d)^d\) be a non-degenerate hermitian \(d\)-space. Set \(U = \left[\begin{smallmatrix} e \\ e' \end{smallmatrix} \right]\) and \(U' = \left[\begin{smallmatrix} V \\ e' \end{smallmatrix} \right]\). If \(q \geq 2\), then
\[
\rho(U, V, U, U') \geq 1 - \frac{1.72}{q}.
\]

**Lemma 6.5.** Suppose that \(V = (\mathbb{F}_q^d)^d\) is a non-degenerate hermitian space, and \(W\) is a non-degenerate \((e + e')\)-dimensional subspace. Then \(d \geq e + e'\) and
\[
\frac{|\left[\begin{smallmatrix} V \\ e + e' \end{smallmatrix} \right] \cdot |\left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]| \cdot |\left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]|}{|\left[\begin{smallmatrix} V \\ e \end{smallmatrix} \right] \cdot |\left[\begin{smallmatrix} V \\ e' \end{smallmatrix} \right]|} \geq \frac{(1 - q^{-1})(1 - q^{-2})}{1 + q^{-2}}.
\]

*Proof.* The inequality \(d \geq e + e'\) is clear. For the duration of this proof we abbreviate \(\omega(e, -q)\) by \(\omega(e)\) and we suppress the superscript \(U\) in notation such as \(\left[\begin{smallmatrix} V \\ e \end{smallmatrix} \right]^U\). Repeated use of Proposition 4.1(a) and extensive cancellation shows
\[
\frac{|\left[\begin{smallmatrix} V \\ e + e' \end{smallmatrix} \right] \cdot |\left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]| \cdot |\left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]|}{|\left[\begin{smallmatrix} V \\ e \end{smallmatrix} \right] \cdot |\left[\begin{smallmatrix} V \\ e' \end{smallmatrix} \right]|} = \frac{q^{2(e+e')(d-e-e')}\omega(d)}{\omega(e+e')\omega(d-e-e')} \cdot \frac{q^{2(e+e')}\omega(e+e')^2}{\omega(e)^2\omega(e')^2} \cdot \frac{\omega(e)\omega(d-e)}{q^{2(d-e)}\omega(d)} \cdot \frac{\omega(e')\omega(d-e')}{\omega(d-e-e')\omega(d)}
\]

If \(d = e + e'\), then this ratio is 1 and the bound holds trivially. Suppose that \(d > e + e'\). Then \(\frac{\omega(e+e')}{\omega(e)\omega(e')} \geq \frac{1 - (q^{-1})^{e+e'}}{1 + q^{-1}} \geq \frac{1 - q^{-2}}{1 + q^{-1}}\) by Lemma 5.2. Applying the upper and lower bounds from Lemma 5.4(b) gives
\[
\frac{\omega(d-e)\omega(d-e')}{\omega(d-e-e')\omega(d)} \geq \frac{1 - q^{-2}}{1 + q^{-2}}.
\]

In summary, we have
\[
\frac{|\left[\begin{smallmatrix} V \\ e + e' \end{smallmatrix} \right] \cdot |\left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]| \cdot |\left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]|}{|\left[\begin{smallmatrix} V \\ e \end{smallmatrix} \right] \cdot |\left[\begin{smallmatrix} V \\ e' \end{smallmatrix} \right]|} \geq \frac{1 - q^{-2}}{1 + q^{-1}} \cdot \frac{1 - q^{-2}}{1 + q^{-1}} = \frac{(1 - q^{-1})(1 - q^{-2})}{1 + q^{-2} - 2q^{-4}}
\]
as claimed. \(\square\)

*Proof of Theorem 6.4.* The bound holds when \(d = e + e'\) as [7, Theorem 4.1] shows that
\[
\rho(U, V, U, U') \geq 1 - \frac{2}{q^2} \geq 1 - \frac{1}{q} > 1 - \frac{1.72}{q}.
\]

Suppose, henceforth that \(d > e + e'\). Let \(W\) be a non-degenerate \((e + e')\)-subspace of \(V\). We suppress the superscript \(U\) in notation such as \(\left[\begin{smallmatrix} V \\ e \end{smallmatrix} \right]^U\). Set \(W := \left[\begin{smallmatrix} W \\ e \end{smallmatrix} \right]\) and \(W' := \left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]\). It follows from Proposition 2.2(a) that
\[
\rho(U, V, U, U') = \frac{|\left[\begin{smallmatrix} V \\ e + e' \end{smallmatrix} \right] \cdot |\left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]| \cdot |\left[\begin{smallmatrix} W \\ e' \end{smallmatrix} \right]|}{|\left[\begin{smallmatrix} V \\ e \end{smallmatrix} \right] \cdot |\left[\begin{smallmatrix} V \\ e' \end{smallmatrix} \right]|} \cdot \rho(U, W, W, W').
\]

(10)
Suppose first that \((e, e', q) = (1, 1, 2)\). It follows from the first displayed formula in the proof of Lemma 6.6 that

\[
\frac{|V_1^e| \cdot |V_2^e| \cdot |W_0^q|}{|V_1^e| \cdot |V_2^e|} = \frac{\omega(2, -q)}{\omega(1, -q)^2} \cdot \frac{\omega(d - 1, -q)^2}{\omega(d - 2, -q)\omega(d, -q)} = \frac{1 - q^{-2}}{1 + q^{-1}} \cdot \frac{1 - (-q)^{-d + 1}}{1 - (-q)^{-d}} = \frac{(1 - q^{-1})(1 - (-q)^{-d + 1})}{1 + q^{-3}} = \frac{1}{3}.
\]

However \(\rho(U, W, W') \geq 1 - \frac{2}{q^2} = \frac{1}{2}\) by [7, Theorem 4.1]. The result now follows from (10) as \(\rho(U, V, U, U') \geq \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} > 1 - \frac{172}{q^2}\).

Suppose next that \((e, e', q) \neq (1, 1, 2)\). Then \(\rho(U, W, W') \geq 1 - \frac{9}{5q^2}\) by Theorem 6.1(b). Thus (10) and Lemma 6.5 show \(\rho(U, V, U, U') \geq \frac{(1 - q^{-1})(1 - q^{-2})(1 - \frac{q^{-2}}{2})}{1 + q^{-2}}\). To prove the stated bound, we must prove that \(\frac{(1 - q^{-1})(1 - q^{-2})(1 - \frac{q^{-2}}{2})}{1 + q^{-2}} \geq 1 - \frac{172}{q}\). This, in turn, is equivalent to showing

\[
1.72 \geq \frac{q\left[(1 + q^{-2}) - (1 - q^{-2})(1 - q^{-2})(1 - \frac{q^{-2}}{2})\right]}{1 + q^{-2}} = 1 + \frac{19q^{-1} - 14q^{-2} - \frac{9q^{-3}}{2} + \frac{9q^{-4}}{4}}{1 + q^{-2}}.
\]

holds for all \(q \geq 2\). The right side approaches 1 quite rapidly as \(q \to \infty\), so the upper bound holds if \(q\) is 'large'. It is not hard to see that the right side attains a maximum value of \(\frac{43}{25} = 1.72\) when \(q = 3\). This completes the proof. \(\square\)

### 6.3. The orthogonal case.

We use Theorem 1.1 of [7] which we paraphrase below in the context we need.

**Theorem 6.6.** Suppose that \(e, e'\) are positive even integers and \(W = (\mathbb{F}_q)^{e+e'}\) is a non-degenerate orthogonal space of type \(\pm\). If \(q \geq 3\), then the proportion of pairs \((U, U')\) where \(U\) is a non-degenerate \(e\)-subspace of \(W\) of type \(\sigma\) and \(U'\) is a non-degenerate \(e'\)-subspace of \(W\) of type \(\sigma'\), which satisfy \(U \cap U' = 0\), is at least \(1 - c/q\) where \(c = 43/16\).

Our goal in this subsection is to prove the following theorem.

**Theorem 6.7.** Suppose that \(e, e'\) are even positive integers and \(d \geq e + e'\). Let \(V\) be a non-degenerate orthogonal \(d\)-space of type \(\varepsilon\) over \(\mathbb{F}_q\), where \(\varepsilon \in \{+, 0, -\}\). Let \(\sigma, \sigma'\) \(\in \{-, +\}\) and let \(U = [V]_{\sigma}, U' = [V]_{\sigma'}\) be the set of non-degenerate subspaces of dimensions \(e, e'\) of types \(\sigma, \sigma'\), respectively. Then, if \(q \geq 3\), the proportion of pairs \((U, U') \in U \times U'\) for which \(U \cap U' = 0\) and \(U + U'\) is non-degenerate satisfies

\[
\rho(O^e, V, U, U') \geq 1 - \frac{2.85}{q} = 0.05 \text{ if } q = 3, \text{ and } \rho(O^e, V, U, U') \geq 1 - \frac{25}{8q} \text{ if } q \geq 4.
\]

**Remark 6.8.** As discussed in Section 3.2, our algorithmic application finds group elements \(g, g'\) in the isometry group of \(V\) where \(g\) acts irreducibly on \(U := \text{im}(g - 1)\) and \(g'\) acts irreducibly on \(U' := \text{im}(g' - 1)\). In the orthogonal case \(e = \dim(U) \geq 2\) implies that \(e\)
must be even and the type $\sigma$ of $U$ is minus, see Lemma 3.8(b). Therefore in the algorithmic application the random elements $g, g'$ correspond to non-degenerate subspaces $U$ and $U'$ of even dimension and minus type. By contrast, in Theorem 1.1 (and Theorem 6.7) the non-degenerate subspaces $U, U'$ may have types $\sigma, \sigma' \in \{-, +\}$ as they need not arise from group elements $g, g' \in GO(V)$ as in our algorithmic application.

Let $V = (F_q)^d$ be a non-degenerate orthogonal $d$-space of type $\varepsilon$. As $e, e'$ are even, so is $e + e'$ and for these dimensions the subspace type equals the intrinsic type, so we can unambiguously abbreviate to type. Let $U = [V]^\varepsilon_\sigma$ denote the set of non-degenerate $e$-subspaces of $V$ of type $\sigma$, and similarly $U' = [V]^\varepsilon_{\sigma'}$. We note first that Theorem 6.7 follows immediately from Theorem 6.6 if $d = e + e'$, since in this case Theorem 6.6 implies that, for all $q \geq 3$,

$$\rho(O^e, V, U, U') \geq 1 - \frac{43}{16q} > \max \left\{ 1 - \frac{2.85}{q}, 1 - \frac{25}{8q} \right\}. $$

Thus we may, and shall, assume that $d \geq e + e' + 1$, so that $[V^e_{e+e'}]_\tau$ is non-empty for each $\tau \in \{-, +\}$. Fix $W_\tau \in [V^e_{e+e'}]_\tau$ for $\tau \in \{-, +\}$. If $q \geq 3$, then it follows from Theorem 6.6 and Proposition 2.2(b) that

$$\rho(O^e, V, U, U') \geq \left( 1 - \frac{43}{16q} \right) \sum_{\tau \in \{-, +\}} \frac{|[V^e_{e+e'}]_\tau| \cdot |[W^\tau_{e+e'}]|}{|[V^e_{e}]_\sigma| \cdot |[W^\tau_{e+e'}]|}.$$ 

Arguing as in the proof of Lemma 6.3 we temporarily set $\omega(n) = \omega(|n/2|, q^2)$. By Proposition 4.1(c), for $e$ even, the number of $e$-subspaces of $V$ of type $\sigma$ equals

$$\left( \begin{array}{c} V \\ e \end{array} \right)_\sigma \begin{array}{c} = q^{e(d-e)} \kappa(e, d - e, \varepsilon, \sigma) \frac{\omega(d)}{\omega(e) \omega(d - e)} \text{ where} \\
\end{array}$$

$$\kappa(e, d - e, \varepsilon, \sigma) = \frac{(1 + \sigma q^{-e/2})(1 + \varepsilon q^{-d/2} + e/2)}{2(1 + \varepsilon q^{-d/2})}. $$

Using the formula (11) for the other factors in the product above, we obtain a complicated expression for $\frac{|V^e_{e+e'}||W^\tau_{e+e'}|}{|V^e_{e+e'}||W^\tau_{e+e'}|}$ which admits two simplifications. First, the powers of $q$ all cancel, and second the following factor which is independent of $\tau \in \{+, -\}$ can be estimated:

$$\frac{\omega(e + e')}{\omega(e) \omega(e')} \geq 1 \text{ and } \frac{\omega(d - e) \omega(d - e')}{\omega(d - e) \omega(e)} > 1 - \frac{1}{2q^2 \log(q)}$$

where the first inequality follows from Lemma 5.2, and the second follows from applying Lemma 5.5 with $q$ replaced by $q^2$ (as in the last line of the proof of Lemma 6.3). Using
the above lower bound gives the following (strict) lower bound for $\rho(O^e, V, U, U')$:

$$
\left(1 - \frac{43}{16q}\right) \left(1 - \frac{1}{2q^2 \log(q)}\right) \sum_{\tau \in \{+, -\}} \frac{\kappa(e + e', d - e - e', \varepsilon, \tau) \kappa(e, e', \tau, \sigma) \kappa(e', e, \tau, \sigma')}{\kappa(e, d - e, \varepsilon, \sigma) \kappa(e', d - e', \varepsilon, \sigma')}.
$$

The following technical lemma helps us bound the above sum.

**Lemma 6.9.** Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\delta \neq \pm 1$. Then

$$
\frac{1}{2} \sum_{\tau \in \{-, +\}} \frac{(1 - \tau \alpha)(1 - \tau \beta)(1 - \tau \gamma)}{1 + \tau \delta} = 1 + \alpha \beta + \alpha \gamma + \alpha \delta + \beta \gamma + \beta \delta + \gamma \delta + \alpha \beta \gamma \delta
$$

Proof. Let $\Lambda$ be the stated sum. Adding fractions and cancelling “odd” terms gives

$$
\Lambda = \frac{(1 + \alpha)(1 + \beta)(1 + \gamma)(1 + \delta) + (1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta)}{2(1 - \delta)(1 + \delta)}
$$

$$
= \frac{1 + \alpha \beta + \alpha \gamma + \alpha \delta + \beta \gamma + \beta \delta + \gamma \delta + \alpha \beta \gamma \delta}{1 - \delta^2}. \quad \Box
$$

**Lemma 6.10.** Suppose that $d \geq e + e' + 1$ where $e, e'$ are both even and $d$ may be odd. Suppose further that the types $\sigma, \sigma', \varepsilon$ satisfy $\sigma, \sigma' \in \{-, +\}$ and $\varepsilon \in \{-, o, +\}$. Then

$$
K := \sum_{\tau \in \{-, +\}} \frac{\kappa(e + e', d - e - e', \varepsilon, \tau) \kappa(e, e', \tau, \sigma) \kappa(e', e, \tau, \sigma')}{\kappa(e, d - e, \varepsilon, \sigma) \kappa(e', d - e', \varepsilon, \sigma')}.
$$

satisfies $K \geq \frac{(1 - q^{-3})(1 - 2q^{-2} - 3q^{-3} - q^{-5})}{(1 + q^{-2})^2}$ and hence $K \geq 1 - \frac{9}{2q^2}$ for $q \geq 3$.

**Proof.** Let $K$ be as in the statement. Using the definition $\kappa$ and cancelling (8 of the 15 factors) gives

$$
K = \frac{1}{2} \sum_{\tau \in \{-, +\}} \frac{(1 + \varepsilon q^{-|[\frac{d}{2}] + \varepsilon e']})(1 + \tau \sigma q^{-[\frac{d}{2}]})(1 + \tau \sigma' q^{-[\frac{d}{2}]})}{(1 + \tau q^{-|[\frac{d}{2}] + \varepsilon e']})(1 + \varepsilon q^{-|[\frac{d}{2}] + \varepsilon e']})(1 + \varepsilon \sigma q^{-[\frac{d}{2}] + \varepsilon e'])} \cdot \Lambda,
$$

where

$$
\Lambda = \frac{1}{2} \sum_{\tau \in \{-, +\}} \frac{(1 + \varepsilon q^{-|[\frac{d}{2}] + \varepsilon e']})(1 + \tau \sigma q^{-[\frac{d}{2}]})}{1 + \tau q^{-[\frac{d}{2} + \varepsilon e']}},
$$

(13)
Suppose that \( d \geq e + e' + 2 \). This implies \( \left| \frac{d}{2} \right| > \frac{e + e'}{2} \), whence \( -\left| \frac{d}{2} \right| < -\frac{e'}{2} \leq -1 \), and so \( -\left| \frac{d}{2} \right| + \frac{e'}{2} \leq -2 \); and similarly \( -\left| \frac{d}{2} \right| + \frac{e'}{2} \leq -2 \). Using Lemma 6.9 gives

\[
\Lambda = \frac{1 + \varepsilon \sigma q^{-\left| \frac{d}{2} \right| + \frac{e'}{2} + \varepsilon \sigma' q^{-\left| \frac{d}{2} \right| + \frac{e'}{2}} - \varepsilon q^{-\left| \frac{d}{2} \right| + \sigma \sigma' q^{-\left| \frac{d}{2} \right| + \frac{e' + e'}{2}} - \sigma q^{-\frac{e'}{2} - e - \varepsilon \sigma \sigma' q^{-\left| \frac{d}{2} \right| - \frac{e' + e'}{2}}}}{1 - q^{-e - e'}}
\]

\[
\geq 1 - q^{-2} - q^{-2} - q^{-3} - q^{-3} - q^{-3} - q^{-5} = 1 - 3q^{-2} - 3q^{-3} - q^{-5}.
\]

Upon closer inspection, \( \varepsilon \sigma \), \( \varepsilon \sigma' \) and \( \sigma \sigma' \) cannot all be \( -1 \), so that the sharper bound \( \Lambda \geq 1 - 2q^{-2} - 3q^{-3} - q^{-5} \) holds.

Inserting this lower bound for \( \Lambda \) into the expression for \( K \) we obtain

\[
K \geq \frac{1 + \varepsilon q^{-\left| \frac{d}{2} \right|}}{(1 + \varepsilon \sigma q^{-\left| \frac{d}{2} \right| + \frac{e'}{2}})(1 + \varepsilon \sigma' q^{-\left| \frac{d}{2} \right| + \frac{e'}{2}})} . \frac{1 - 2q^{-2} - 3q^{-3} - q^{-5}}{1}.
\]

This proves the first bound when \( d \geq e + e' + 2 \). To prove that this lower bound is at least \( 1 - \frac{9}{2q} \) for all \( q \geq 3 \), we rearrange this inequality and show that it is equivalent to proving that the following is true for all \( q \geq 3 \):

\[
\frac{9}{2} \geq \frac{q^2((1 + q^{-2})^2 - (1 - q^{-3})(1 - 2q^{-2} - 3q^{-3} - q^{-5}))}{(1 + q^{-2})^2} = \frac{4 + 4q^{-1} + q^{-2} - q^{-3} - 3q^{-4} - q^{-6}}{(1 + q^{-2})^2}.
\]

As \( q \to \infty \), the the right side \( \to 4 \). Indeed the values of the right side lie between 4 and \( \frac{9}{2} \) for all \( q \geq 3 \). This proves the second bound when \( d \geq e + e' + 2 \).

Now we consider the remaining case where \( d = e + e' + 1 \). Here \( d \) is odd so \( \left| \frac{d}{2} \right| = \frac{e + e'}{2} \). If \( \tau = -\varepsilon \), then one summand in the expression for \( \Lambda \) in (13) is zero because the factor \( 1 + \varepsilon \tau q^{-\left| \frac{d}{2} \right| + \frac{e' + e'}{2}} \) is zero. Hence

\[
K = \frac{1 + \varepsilon q^{-\left| \frac{d}{2} \right|}}{2(1 + \varepsilon \sigma q^{-\left| \frac{d}{2} \right| + \frac{e'}{2}})(1 + \varepsilon \sigma' q^{-\left| \frac{d}{2} \right| + \frac{e'}{2}})} . \frac{(1 + q^0)(1 + \varepsilon \sigma q^{-\frac{e'}{2}})(1 + \varepsilon \sigma' q^{-\frac{e'}{2}})}{1 + \varepsilon q^{-\frac{e' + e'}{2}}}.
\]

Hence \( K = 1 \) and both of the stated bounds also hold in this final case. \( \square \)

**Proof of Theorem 6.7.** As discussed after Remark 6.8, Theorem 6.7 follows from Theorem 6.6 if \( d = e + e' \), so assume that \( d \geq e + e' + 1 \). Let \( K \) be as in Lemma 6.10. It follows
from the display before Lemma 6.9 that

$$\rho(O^c,V,U,U') > \left(1 - \frac{43}{16q}\right) \left(1 - \frac{1}{2q^2 \log(q)}\right) K.$$ 

Setting $q = 3$ in the first lower bound in Lemma 6.10 shows that $\rho(O^c,V,U,U') > 0.05$. Thus setting $c = 2.85$, we have $\rho > 0.05 = 1 - \frac{c}{3}$. Similarly, if $q \in \{4, 5, 7, 8, 9\}$ then again using the first lower bound on $K$, we find that $\rho(O^c,V,U,U') > 1 - \frac{3.125}{q}$ holds. For $q \geq 11$ we use the second lower bound $K \geq 1 - \frac{9}{2q^2}$ in Lemma 6.10. This shows

$$\rho(O^c,V,U,U') > \left(1 - \frac{43}{16q}\right) \left(1 - \frac{9}{2q^2}\right) = \left(1 - \frac{\alpha}{q}\right) \left(1 - \frac{\beta}{q}\right) \left(1 - \frac{\gamma}{q}\right)$$

where $\alpha = \frac{43}{16}$, $\beta = \frac{1}{2q \log(q)}$ and $\gamma = \frac{9}{2q}$. The inequality $(1 - \frac{\alpha}{q})(1 - \frac{\beta}{q})(1 - \frac{\gamma}{q}) \geq 1 - \frac{3.125}{q}$ is equivalent to

$$3.125 \geq q \left(1 - \left(1 - \frac{\alpha}{q}\right) \left(1 - \frac{\beta}{q}\right) \left(1 - \frac{\gamma}{q}\right)\right) = \alpha + \beta + \gamma - \frac{\alpha \beta + \beta \gamma + \gamma \alpha}{q} + \frac{\alpha \beta \gamma}{q^2}.$$ 

The stronger condition $3.125 \geq \alpha + \beta + \gamma + \frac{\alpha \beta \gamma}{q}$ does indeed hold for all $q \geq 11$. Hence $\rho(O^c,V,U,U') \geq 1 - \frac{3.125}{q}$ holds for all $q \geq 4$. 

\[\square\]

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