Relativistic quantum motion

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Abstract

Using the relativistic quantum stationary Hamilton–Jacobi equation within the framework of the equivalence postulate, and grounding oneself on both relativistic and quantum Lagrangians, we construct a Lagrangian of a relativistic quantum system in one dimension and derive a third order equation of motion representing a first integral of the relativistic quantum Newton’s law. Then, we plot the relativistic quantum trajectories of a particle moving under constant and linear potentials. We establish the existence of nodes and link them to the de Broglie wavelength.

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1. Introduction

Deriving quantum mechanics from an equivalence postulate, Faraggi and Matone showed that the Schrödinger wavefunction must have the form [1–4]

\[ \phi(x) = \left( \frac{\partial S_0}{\partial x} \right)^{-\frac{1}{2}} \left[ \alpha \exp \left( \frac{i}{\hbar} S_0 \right) + \beta \exp \left( -\frac{i}{\hbar} S_0 \right) \right]. \]  

(1)

where \( \alpha \) and \( \beta \) are complex constants and \( S_0 \) is a real function representing the quantum reduced action. They established that the conjugate momentum given by

\[ P = \frac{\partial S_0}{\partial x}, \]  

(2)

never vanishes for bound and unbound states making possible a dynamical approach for the quantum motion of particles. This conjugate momentum is always real even in classically forbidden regions. They showed also, within the framework of differential geometry [1–4], that the quantum stationary Hamilton–Jacobi equation (QSHJE) which leads to the Schrödinger equation is

\[ \frac{1}{2m_0} \left( \frac{\partial S_0}{\partial x} \right)^2 + V(x) - E = \frac{\hbar^2}{4m_0}, \]

\[ \times \left[ \frac{3}{2} \left( \frac{\partial S_0}{\partial x} \right)^2 - \frac{\partial^2 S_0}{\partial x^2} \right] - \left( \frac{\partial S_0}{\partial x} \right)^{-1} \left( \frac{\partial^3 S_0}{\partial x^3} \right). \]

(3)

where \( V(x) \) is the potential and \( E \) the energy. The solution of equation (3) investigated by Floyd [5–8] and Faraggi and Matone [1–3] and Bertoldi et al [4] is given in [9] as

\[ S_0 = \hbar \arctan \left( \frac{a}{\phi} + b \right), \]

(4)

where \( a \) and \( b \) are real constants; \( \phi \) and \( \theta \) are two real independent solutions of the Schrödinger equation. Taking advantage of these results, Bouda and Djama [9] have recently introduced a quantum Lagrangian

\[ L(x, \dot{x}, \mu, \nu) = \frac{1}{2} m \dot{x}^2 f(x, \mu, \nu) - V(x), \]

(5)

from which they derived the quantum law of motion. They stated that the conjugate momentum of the non relativistic and spinless particle is written as

\[ \frac{\hbar}{2} \frac{\partial S_0}{\partial x} = 2(E - V). \]

(6)

From this last equation, they derived the first integral of the quantum Newton’s law (FIQNL).

\[ (E - V)^4 - \frac{m \dot{x}^2}{2} (E - V)^3 + \frac{\hbar^2}{8} \left[ \frac{3}{2} \left( \frac{\dot{x}}{\dot{x}} \right)^2 - \frac{\dot{x}}{\dot{x}} \right] (E - V)^2 \]

\[ - \frac{\hbar^2}{8} \left[ \dot{x}^2 \frac{d^2 V}{dx^2} + \dot{x} \frac{dV}{dx} \right] (E - V) - \frac{3\hbar^2}{16} \left[ \dot{x} \frac{dV}{dx} \right]^2 = 0, \]

(7)

which goes at the classical limit (\( \hbar \to 0 \)) to the classical conservation equation

\[ \frac{m \dot{x}^2}{2} + V(x) = E. \]

(8)
Bouda and Djama [10] have also plotted some trajectories of the particle for several potentials.

The construction of the Lagrangian (5) and the establishment of the fundamental dynamical equations (6) and (7) are important steps to build a deterministic theory which restores the existence of trajectories [9–11]. Nevertheless, such a formalism cannot approach both relativistic velocities cases, and more than one dimension (1D) motion. The aim of this paper is to generalize the dynamical formalism that we have recalled above [9–11] into the 1D relativistic velocity cases. In this purpose let us recall the finding of Bertoldi et al concerning relativistic quantum systems. They stated that the relativistic quantum wavefunction is given by equation (1), where \( \mathcal{S}_0 \) defines the relativistic quantum reduced action, and wrote the relativistic quantum stationary Hamilton–Jacobi equation (RQSHJE) as [3, 4]

\[
\frac{1}{2m_0} \left( \frac{\partial \mathcal{S}_0}{\partial x} \right)^2 - \frac{h^2}{4m_0} \left( \frac{3}{2} \left( \frac{\partial \mathcal{S}_0}{\partial x} \right)^2 \left( \frac{\partial^2 \mathcal{S}_0}{\partial x^2} \right)^2 \right) - \frac{V}{\left( \frac{\partial \mathcal{S}_0}{\partial x} \right)^2} + \frac{1}{2m_0c^2} \left[ m_0^2c^4 - (E - V)^2 \right] \phi(x) = 0, \tag{9}
\]

where \( V \) is the potential, \( E \) is the total energy of the particle of mass equal to \( m_0 \) at rest and \( c \) is the light velocity in vacuum. The solution of equation (9) can be expressed by equation (4), where \( \theta \) and \( \phi \) represent now two real independent solutions of the Klein–Gordon equation

\[
-c^2h^2 \frac{\partial^2 \phi}{\partial x^2} + \left[ m_0^2c^4 - (E - V)^2 \right] \phi(x) = 0. \tag{10}
\]

Taking advantage on these results, we will introduce in the following sections a relativistic quantum formalism with which we study the dynamics of high energy particles. Firstly, in section 2, we present a relativistic quantum Lagrangian from which we derive the relativistic quantum law of motion. Then, in section 3 we study and plot the relativistic quantum trajectories (RQTs) of a particle moving under a constant potential. In section 4, we study the linear potential case. Finally, in section 5, we introduce our definition of the de Broglie wavelength.

### 2. Relativistic quantum motion

Now, we consider a relativistic quantum system. As we have noticed for quantum systems [9], the relativistic quantum reduced action \( \mathcal{S}_0 \) expressed by equation (4) contains two constants more than the usual constant \( E \) appearing in the expression of the classical reduced action. This suggests that the relativistic quantum law of motion is a fourth order differential equation. Then, as in the quantum case [9], we introduce in the expression of the Lagrangian a function \( f \) of \( x \) depending on the constants \( a \) and \( b \) playing the role of hidden parameters. As the relativistic quantum Lagrangian must goes at the classical limit (\( \hbar \to 0 \)) to the relativistic one, we postulate the following form for the Lagrangian

\[
L(x, \dot{x}, a, b) = -m_0c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} f(x, a, b) - V(x), \tag{11}
\]

in which the function \( f(x, a, b) \) satisfies

\[
\lim_{\hbar \to 0} f(x, a, b) = 1. \tag{12}
\]

Since \( L \) depends only on the variables \( x, \dot{x} \) and the constants \( a, b \), the conjugate momentum is given by

\[
P = \frac{\partial L}{\partial \dot{x}} = \frac{m_0c^2 f(x, a, b)}{\sqrt{1 - (\dot{x}^2/c^2)}} f(x, a, b) \tag{13}
\]

The Hamiltonian corresponding to the Lagrangian (11) is

\[
H = P\dot{x} - L. \tag{14}
\]

Substituting equations (11) and (13) into equation (14), one obtains

\[
H(x, P) = \sqrt{m_0^2c^4 + \frac{P^2c^2}{f(x, a, b)}} + V(x), \tag{15}
\]

At the classical limit, we see clearly by using equation (12) that the relativistic quantum momentum \( P \) given in equation (13) reduces to its relativistic one expressed as

\[
P = \frac{m_0c}{\sqrt{1 - \dot{x}^2/c^2}}. \tag{16}
\]

Likewise, the Hamiltonian given in equation (15) reduces to its relativistic form

\[
H(x, P) = \sqrt{m_0^2c^4 + \frac{P^2c^2}{f(x, a, b)}} + V(x), \tag{17}
\]

well known in special relativity. For the stationary cases, the Hamiltonian \( H \) corresponds to the total energy \( E \) of the particle. Then, we can write equation (15) as

\[
E = \sqrt{m_0^2c^4 + \frac{P^2c^2}{f(x, a, b)}} + V(x), \tag{18}
\]

which reads

\[
\frac{1}{2m_0} \left( \frac{\partial \mathcal{S}_0}{\partial x} \right)^2 - \frac{1}{f(x, a, b)} + \frac{1}{2m_0c^2} \left[ m_0^2c^4 - (E - V)^2 \right] \phi(x) = 0, \tag{19}
\]

after taking into account the Hamilton–Jacobi definition of the conjugate momentum \( P = \partial \mathcal{S}_0 / \partial x \). By applying the equivalence postulate, already introduced by Faraggi and Matone in previous studies [1–3] and Bertoldi et al [4], to equation (19), we obtain the RQSHJE given in equation (9).

This result suggests strongly that the introduction of a Lagrangian with the form (11) is founded. Now, taking the expression of the function \( f(x, a, b) \) from equation (13) into equation (19), we get

\[
x \frac{\partial \mathcal{S}_0}{\partial x} = E - V(x) - \frac{m_0^2c^4}{(E - V)}. \tag{20}
\]

The last equation represents the relativistic quantum law of motion, so the (RQTs) of a particle moving under any potential \( V(x) \) should be plotted using equation (20). To proceed, one can deduce the expression of the conjugate momentum from equation (4) and replace it into equation (20), then we can integrate this equation to give
the relativistic quantum time equation $x(t)$. Note that at the classical limit $\hbar \to 0$, equation (20) reduces to the relativistic conservation equation

$$E = \frac{m_0c^2}{\sqrt{1 - \dot{x}^2/c^2}} + V(x)$$  \hspace{1cm} (21)$$

since the conjugate momentum takes the form (16). Note also that in the relativistic limit ($c \to \infty$), the kinetic energy $T = E - V - m_0c^2$ satisfies $T \ll m_0c^2$, so, equation (20) reduces to

$$\frac{\partial S_0}{\partial x} = 2T,$$

which is equivalent to equation (6) already established in [9].

Now, deducing $\partial^2 S_0/\partial x^2$ and $\partial^3 S_0/\partial x^3$ from equation (20) and substituting them into the expression of the RQSHJE (equation (9)), we find the first integral of the relativistic quantum Newton’s law (FIRQNL) which reads

$$[(E - V)^2 - m_0^2c^4]^{3} \left[\frac{1 - \dot{x}^2}{c^2} - \frac{m_0^2c^4}{(E - V)^2}\right]$$

$$\frac{\hbar^2}{2} \frac{1}{(E - V)^2 - m_0^2c^4} \left[\frac{\dot{V}}{V} + \frac{\dot{x}^2}{c^2}\right]$$

$$+ \frac{\hbar^2}{2} [(E - V)^2 - m_0^2c^4]^{2} \left[\frac{3}{2} \frac{\dot{x}^2}{c^2} - \frac{\dot{x}}{c}\right]$$

$$- \frac{\hbar^4}{4} \left[4m_0^2c^4 \left(1 - \frac{m_0^2c^4}{(E - V)^2}\right) + 3 \left(E - V + \frac{m_0^2c^4}{E - V}\right) \right]$$

$$\times \left(\frac{\dot{x}}{c} \frac{\dot{V}}{V}\right)^2 = 0$$  \hspace{1cm} (22)$$

As we observe, equation (22) is a third order differential equation in $x$ containing the first and the second derivatives of the potential $V$ with respect to $x$. Then, its solution $x(t, E, a, b, c)$ contains four integration constants which can be determined by the initial conditions. It is clear that if we set $\hbar = 0$, equation (22) reduces to equation (21) representing the relativistic conservation equation. Note also that, after taking the relativistic limit ($c \to \infty$) into equation (22), one gets equation (7) representing the FIQNL.

3. Motion under a constant potential

First, after using expression (4) of $S_0$, let us write the dynamical equation (20) in the following form

$$\frac{dx}{dt} = \pm \frac{1}{\hbar a W} \left[ E - V(x) - \frac{m_0^2c^4}{E - V(x)} \right] \left[\phi_1^2 + (a\phi_1 + b\phi_2)^2\right],$$

$$\hspace{1cm} (23)$$

where $W$ represents the wronskian of the function $\phi_1$ and $\phi_2$. The ± sign in equation (23) indicates that the motion may be in either direction on the $x$-axis [10]. In the case of a massive particle (not a photon) moving under a constant potential $V = U_0$, we can distinguish two kinds of motions, the classically permitted motion ($E - U_0 \ll m_0c^2$) and the classically forbidden one. Firstly, we review the classically permitted motion. Choosing the two solutions of the Klein–Gordon equation (equation (10)) as

$$\phi_1 = \sin \left( \frac{\sqrt{(E - U_0)^2 - m_0^2c^4}}{\hbar c} x \right),$$

$$\phi_2 = \cos \left( \frac{\sqrt{(E - U_0)^2 - m_0^2c^4}}{\hbar c} x \right),$$

and integrating the dynamical equation (23), we find

$$x(t) = \frac{\hbar c}{\sqrt{(E - U_0)^2 - m_0^2c^4}} \times \arctan \left[ \frac{1}{a} \tan \left( \frac{(E - U_0)^2 - m_0^2c^4}{\hbar (E - U_0)} t \right) - \frac{b}{a} \right] + x_0.$$  \hspace{1cm} (24)$$

This equation represents the time equation of RQTs. As we have mentioned above, $x(t)$ contains four constants since the fundamental equation of motion is a fourth order differential equation. Because the arctangent function is defined on the interval $[-\pi/2, \pi/2]$, equation (24) shows that the particle is contained between

$$- \frac{\hbar c}{\sqrt{(E - U_0)^2 - m_0^2c^4}} \pi 2 + x_0$$

and

$$\frac{\hbar c}{\sqrt{(E - U_0)^2 - m_0^2c^4}} \pi 2 + x_0.$$  \hspace{1cm} (25)$$

This is not possible. It is necessary to readjust the additive integration constant $x_0$ after every interval of time in which the tangent function goes from $-\infty$ to $+\infty$ in such a way to guarantee the continuity of $x(t)$. For this purpose, expression (24) must be rewritten as

$$x(t) = \frac{\hbar c}{\sqrt{(E - U_0)^2 - m_0^2c^4}} \times \arctan \left[ \frac{1}{a} \tan \left( \frac{(E - U_0)^2 - m_0^2c^4}{\hbar (E - U_0)} t \right) - \frac{b}{a} \right] + \frac{\pi \hbar c}{\sqrt{(E - U_0)^2 - m_0^2c^4}} n + x_0.$$  \hspace{1cm} (25)$$

with

$$t \in \left[ \frac{\pi \hbar (E - U_0)}{(E - U_0)^2 - m_0^2c^4} \left( n - \frac{1}{2} \right) ; \right.$$  

$$\frac{\pi \hbar (E - U_0)}{(E - U_0)^2 - m_0^2c^4} \left( n + \frac{1}{2} \right) \right]$$

for every integer number $n$. The purely relativistic trajectory is obtained for $a = 1$ and $b = 0$. Indeed, for these values, equation (25) reduces to the relativistic relation

$$x(t) = \frac{c}{E - U_0} \sqrt{(E - U_0)^2 - m_0^2c^4} t + x_0.$$  \hspace{1cm} (26)$$
In figure 1, we have plotted in $(t, x)$ plane for an electron of energy $(2 + U_0) \text{MeV}$ some trajectories for different values of $a$ and $b$. All these trajectories pass through nodes exactly as we have seen for quantum trajectories of an electron moving under a constant potential [10]. These nodes correspond to the times

$$t_n = \frac{\pi \hbar (E - U_0)}{(E - U_0)^2 - m_0^2c^4} \left( n + \frac{1}{2} \right).$$ \hfill (27)

The distance between two adjacent nodes is on the time axis

$$\Delta t_n = t_{n+1} - t_n = \frac{\pi \hbar (E - U_0)}{(E - U_0)^2 - m_0^2c^4}.$$ \hfill (28)

and on the space axis

$$\Delta x_n = x_{n+1} - x_n = \frac{\pi \hbar c}{\sqrt{(E - U_0)^2 - m_0^2c^4}}.$$ \hfill (29)

These distances are both proportional to $\hbar$ meaning that at the classical limit ($\hbar \to 0$) the nodes becomes infinitely near, and then, all quantum trajectories tend to the purely relativistic one. As explained in [10], this is the reason why in problems for which $\hbar$ can be disregarded, RQTs reduce to the purely relativistic one [9, 10]. It is useful to note that at the classical limit ($\hbar \to 0$), not only do the nodes become infinitely near, but in addition the paths of all RQTs go to the path of the purely relativistic one ($a = 1, b = 0$). Indeed, let us consider an arbitrary point $P(x, t)$ from any RQT. Obviously, this point is situated between two nodes. Now, if we take the orthogonal projection $P_0$ of $P$ on the relativistic trajectory ($a = 1, b = 0$) and compute the distance $P P_0$ we get

$$PP_0 = \sqrt{1 + \frac{(E - U_0)^2}{c^2[(E - U_0)^2 - m_0^2c^4]} |t_P - t_{P_0}|}.$$ \hfill (30)

Since equation (23) indicates that $\dot{x}$ is a monotonous function, then, whatever the point $P$ we have $|t_P - t_{P_0}| < t_{n+1} - t_n$ for every RQT [10]. So, at the classical limit ($\hbar \to 0$), because

$$x(t) = \frac{\hbar c}{2\sqrt{m_0^2c^4 - (E - U_0)^2}} \times \ln \left| \frac{1}{a} \tan \left( \frac{m_0^2c^4 - (E - U_0)^2}{\hbar(E - U_0)} t \right) + \frac{b}{a} \right| + x_0.$$ \hfill (31)

We see clearly from equation (31) that at the finite time

$$- (2n + 1)\pi \hbar \frac{(E - U_0)}{2[(E - U_0)^2 - m_0^2c^4]}$$

the electron crosses an infinite distance and reaches an infinite speed. This is in accordance with standard quantum tunnelling theories which predict infinite velocities and finite reflection times for tunnelling phenomena [12, 13]. In figure 2, we plot for an electron of energy $(U_0 - 2) \text{MeV}$ a RQT in the classically forbidden regions with $a = 0.25$ and $b = 8$. This figure shows clearly how the particle reaches an infinite position at a finite time. In particular, in the classically forbidden regions there are no nodes.

4. The linear potential case

Here, we investigate the motion of a massive particle (electron) under a potential of the form

$$V(x) = gx,$$ \hfill (32)
for which the Klein–Gordon equation takes the form

$$-c^2\hbar^2 \frac{\partial^2 \phi}{\partial x^2} + \left[m_0^2 c^4 - (E - gx)^2\right] \phi(x) = 0. \tag{33}$$

To establish the RQTs for the linear potential case, we integrate equation (23), where $\phi_1$ and $\phi_2$ are, now, two solutions of equation (33).

In this paper, we do not present the analytic solutions of equation (33), and in order to plot the RQTs, we approach the problem by numeric methods. We first integrate numerically equation (33) to obtain two independent solutions $\phi_1$ and $\phi_2$, then, we plot the RQTs from equation (23). We opt in the two steps for the Euler integration method. The RQTs for the linear potential are presented in figure 3. We choose $E = 2$ MeV, and $g = 1$ kg m s. In [10], we chose $g = 10^{-9}$ kg m s, which is very small compared with $g = 1$ kg m s. We take this last value for the relativistic problem to render the quantity $gx$ of the same order as $E$, so that the Klein–Gordon equation does not reduce to the Schrödinger equation.

As we can notice from figure 3, the nodes are also present for the linear potential case. The distance between two adjacent nodes in figure 3 increases as the velocity decreases when it approaches the turning point (point where the velocity vanishes). This note will be exposed in section 5. Here, we do not investigate the classically forbidden regions. We would like to stress that as for the linear potential in quantum cases, we check that the positions of the nodes on the $x$-axis are related to the zeros of the solution of the Schrödinger equation $\phi_1$ present in the denominator of the rapport in the expression (4) of the reduced action $S_0$. This fact indicates that the RQTs are like the QTs and it is obvious that the QTs are a limit of the RQTs when $c \rightarrow \infty$.

5. De Broglie wavelength

One of the most important ideas that sections 3 and 4 bring is the existence of nodes through which all RQTs pass, even the purely relativistic one. In this section, we link the distance between two adjacent nodes to the de Broglie wavelength

$$\lambda = \frac{\hbar}{p} \tag{34}$$

In equation (34), $p$ is the relativistic momentum. For a particle moving under a constant potential

$$p = \sqrt{(E - U_0)^2 - m_0^2 c^4}. \tag{35}$$

By replacing equation (35) in equation (34), we get

$$\lambda = \frac{\hbar}{\sqrt{(E - U_0)^2 - m_0^2 c^4}}. \tag{36}$$

The distance between two nodes in the case of a constant potential is

$$\Delta x_n = \frac{\pi \hbar c}{\sqrt{(E - U_0)^2 - m_0^2 c^4}}. \tag{37}$$

From equations (36) and (37) we get

$$\Delta x_n = \frac{\lambda}{2}. \tag{38}$$

Thus, the de Broglie wavelength represents the double of the distance between two adjacent nodes. As we have presented in [10], we can generalize this definition for other potentials. Indeed, if we compute the mean value of $\partial S_0 / \partial x$ between two adjacent nodes, and taking into account equations (25), (28) and (29) we find

$$\left(\frac{\partial S_0}{\partial x}\right) = \frac{1}{\Delta x_n} \int_{x_{(n)}}^{x_{(n+1)}} \frac{\partial S_0}{\partial x} \, dx = \frac{S_0[x(t_{n+1})] - S_0[x(t_n)]}{\Delta x_n} = \frac{\sqrt{(E - U_0)^2 - m_0^2 c^4}}{c}, \tag{39}$$

which is equal to $p$ (equation (35)). We propose to define a new wavelength after substituting $p$ by

$$p = \left|\frac{\partial S_0}{\partial x}\right|. \tag{40}$$

Then for any potential, after using relation (4) of the reduced action $S_0$ to calculate the mean value of the quantum momentum, we can write

$$p = \frac{\pi \hbar}{\Delta x}, \tag{41}$$

with $\Delta x$ is the distance between two adjacent nodes. If we substitute (41) into (34), we find

$$\Delta x = \frac{\lambda}{2}. \tag{42}$$

This relation links the distance separating two adjacent nodes and the de Broglie wavelength whatever the potential under which the particle moves.
6. Conclusions

To conclude, we would like to stress that we exposed in this paper an original approach for relativistic quantum mechanics. It is a generalization of the one exposed in [9–11]. So, we have derived the fundamental relativistic quantum Newton’s law expressed in equations (20) and (22). In addition, we have plotted the RQTs of particles moving under a constant potential and a linear potential in both classically and forbidden regions. For the classically permitted regions, we established the existence of some nodes that we linked successfully to the de Broglie wavelength.

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