Symbolic extensions of amenable group actions
and the comparison property

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## Contents

1. Introduction  
   1.1. Motivation  
   1.2. Subject of the paper  
   1.3. Organization of the paper  
2. Preliminaries on actions of countable amenable groups  
   2.1. Group actions, subshifts, symbolic extensions, block codes  
   2.2. An $\epsilon$-modification, $(K, \epsilon)$-invariance, Følner sequence, amenability  
   2.3. The Choquet simplex of invariant probability measures  
   2.4. The ergodic theorem  
   2.5. Entropy  
   2.6. Zero-dimensional systems  
3. Entropy structure and the easy direction of the main theorem  
   3.1. Structures  
   3.2. Superenvelopes  
   3.3. Definition of the entropy structure  
   3.4. Symbolic extensions—the easy direction  
4. Quasitilings and tiling systems  
   4.1. Terminology and facts not requiring amenability  
   4.2. Terminology and facts requiring amenability  
   4.3. Tiled entropy  
5. Quasi-symbolic extensions—the hard direction of the main theorem  
6. The comparison property  
   6.1. Definition of the comparison property  
   6.2. Banach density interpretation of the comparison property  
   6.3. Comparison property of subexponential groups  
7. Encodable tiling systems  
   7.1. Encodable systems of quasitilings  
   7.2. Encodability of tiling systems versus the comparison property  
   7.3. Symbolic extensions for actions of selected groups  
Appendix A  
Acknowledgements  
Bibliography
Abstract

In topological dynamics, the Symbolic Extension Entropy Theorem (SEET) describes the possibility of a lossless digitalization of a dynamical system by extending it to a subshift on finitely many symbols. The theorem gives a precise estimate on the entropy of such a symbolic extension (and hence on the necessary number of symbols). Unlike in the measure-theoretic case, where Kolmogorov–Sinai entropy serves as an estimate in an analogous problem, in the topological setup the task reaches beyond the classical theory of measure-theoretic and topological entropy. Necessary are tools from an extended theory of entropy, the theory of entropy structures developed in [16]. The main goal of this paper is to prove the analog of the SEET for actions of countable amenable groups:

Let a countable amenable group $G$ act by homeomorphisms on a compact metric space $X$ and let $\mathcal{M}_G(X)$ denote the simplex of $G$-invariant probability measures on $X$. A function $E_A$ on $\mathcal{M}_T(X)$ equals the extension entropy function $h^\pi$ of a symbolic extension $\pi : (Y, G) \to (X, G)$, where $h^\pi(\mu) = \sup\{h_\nu(Y, G) : \nu \in \pi^{-1}(\mu)\}$ ($\mu \in \mathcal{M}_G(X)$), if and only if $E_A$ is an affine superenvelope of the entropy structure of $(X, G)$.

Of course, the statement is preceded by the presentation of the concepts of an entropy structure and its superenvelopes, adapted from the case of $\mathbb{Z}$-actions. In full generality we are able to prove a slightly weaker version of SEET, in which symbolic extensions are replaced by quasi-symbolic extensions, i.e., extensions in form of a joining of a subshift with a zero-entropy tiling system. The notion of a tiling system is a subject of several earlier works (e.g. [19], [20]) and in this paper we review and complement the theory developed there. The full version of the SEET (with genuine symbolic extensions) is proved for groups which are either residually finite or enjoy the so-called comparison property. In order to describe the range of our theorem more clearly, we devote a large portion of the paper to studying the comparison property. Our most important result in this aspect is showing that all subexponential groups have the comparison property (and thus satisfy the SEET). To summarize, the heart of the paper is the presentation of the following four major topics and the interplay between them:

- Symbolic extensions,
- Entropy structures,
- Tiling systems (and their encodability),
- The comparison property.
1. Introduction

1.1. Motivation. One of important tasks of the theory of dynamical systems is giving criteria for lossless digitalization of a system. In classical ergodic theory of \( \mathbb{Z} \)-actions, Krieger’s Generator Theorem \([36]\) resolves the problem completely using the Kolmogorov–Sinai entropy: if a measure-automorphism \( T \) of a standard probability space \((X, \Sigma, \mu)\) has finite Kolmogorov–Sinai entropy \( h = h_\mu(T) \) then the system has a finite generating partition, moreover, there exists such a partition into \( l = \lfloor 2^h \rfloor + 1 \) atoms. That is to say, the system is isomorphic to a subshift over \( l \) symbols equipped with some shift-invariant measure. This fact can be interpreted as the possibility of losslessly digitalizing the system, up to a measure-isomorphism, using \( l \) symbols. The result was later generalized by Štefan Šujan \([50]\) to free actions of countable amenable groups (with later improvements by Rosenthal \([47]\) and Danilenko–Park \([14]\)).

In topological dynamics, where one is concerned with a homeomorphism \( T \) acting on a compact metric space \( X \), one can ask an analogous question: how many symbols (or how much entropy) is needed to losslessly encode the system \((X, T)\) in a subshift? In general, it is impossible to represent the system by a topologically conjugate subshift, so instead one considers so-called symbolic extensions, i.e., subshifts \((Y, S)\) which contain \((X, T)\) as a topological factor. Since there are many such extensions, one is interested in optimizing the topological entropy of the symbolic extension. In this way we are lead to the following parameter:

\[
h_{\text{sex}}(X, T) = \inf \{ h_{\text{top}}(Y, S) : (Y, S) \text{ is a symbolic extension of } (X, T) \}.
\]

By analogy to the measure-theoretic case, a naive guess would be that this parameter simply equals the topological entropy of the system \((X, T)\). But it is not the case. The parameter \( h_{\text{sex}}(X, T) \) may assume values much higher than \( h_{\text{top}}(X, T) \), including infinite, in which case symbolic extensions simply do not exist, even though \( h_{\text{top}}(X, T) \) is finite. This phenomenon, first discovered by Mike Boyle in the early 90’s (and published much later in a survey \([6]\)), has lead to the development of the theory symbolic extensions for \( \mathbb{Z} \)-actions (see \([5, 22, 21, 10, 11, 48]\), etc.). Technically, the parameter \( h_{\text{sex}}(X, T) \) is much more sophisticated than \( h_{\text{top}}(X, T) \) and tells us something that topological entropy is incapable of telling: it describes the possibility of losslessly digitalizing the system, in particular it tells us how many symbols are needed for such a digitalization. Nowadays, we have a fairly good understanding of the subject matter and the associated phenomena. We understand why considerations of just topological entropy are insufficient. Insufficient is also observing just the entropy function \( \mu \mapsto h_\mu(T) \) defined on the simplex \( \mathcal{M}_T(X) \) of \( T \)-invariant probability measures on \( X \) associating to each measure its Kolmogorov–Sinai entropy. It are the defects in uniformity of the convergence of measure-theoretic entropy of invariant measures as the resolution improves, that have an essential impact on the entropy of possible symbolic extensions. These defects are captured by the theory of entropy structures and their superenvelopes—objects that have no counterpart in ergodic theory. It is crucial that the way to calculate the topological symbolic extension entropy \( h_{\text{sex}}(X, T) \) is via computing the refined symbolic extension entropy function \( \mu \mapsto h_{\text{sex}}(\mu) \) on invariant measures. By definition, this function equals the pointwise infimum of extension entropy functions \( h^\pi \) defined on \( \mathcal{M}_T(X) \) for each symbolic extension \( \pi : (Y, S) \to (X, T) \), as
follows

\[ h^\pi(\mu) = \sup \{ h_\nu(S) : \nu \in \pi^{-1}(\mu) \} \quad (\mu \in M_T(X)). \]

The key result of the theory of symbolic extensions, the Symbolic Extension Entropy Theorem \[5\] asserts that a function \( E_A \) on \( M_T(X) \) equals \( h^\pi \) in some symbolic extension \( \pi : (Y, S) \to (X, T) \) if and only if it is an affine superenvelope of the entropy structure of the system \((X, T)\) (the definition of such a superenvelope is too complicated to be presented in the introduction and will be provided later). This allows to compute the function \( \mu \mapsto h_{\text{sex}}(\mu) \) as the minimal superenvelope of the entropy structure, and finally, the topological parameter \( h_{\text{sex}}(X, T) \) is obtained as the supremum of \( h_{\text{sex}}(\mu) \) over \( \mu \in M_T(X) \).

A natural direction of generalizing the theory of symbolic extensions is that of actions of countable amenable groups. For such actions we have completely analogous (to that in the \( \mathbb{Z} \)-case) structure of invariant measures (forming a Choquet simplex), with similarly defined notions of entropy (both measure-theoretic and topological), enjoying similar basic properties. Beyond this class, say for arbitrary countable groups, the notion of a symbolic extension still makes sense, but there are serious problems with entropy. Sofic entropy, for instance, may equal minus infinity, or increase when passing to a factor. So, for sofic groups there is no hope to create a theory of symbolic extensions with analogous connections to entropy notions as for \( \mathbb{Z} \). This is why we believe that the realm of topological actions of countable amenable groups is the most natural environment to carry over the theory of symbolic extensions and their entropy.

1.2. Subject of the paper. The goal the authors of this paper have focused on is very simple to formulate: prove an analog of the Symbolic Extensions Entropy Theorem for actions of countable amenable groups. A brief overview of the proof and the tools used for \( \mathbb{Z} \)-actions was rather reassuring: it should be possible to adapt most of them without too much trouble. This optimism however had a relatively short life. In reality, things turned out much more complex than they seemed, leading us to studying many accompanying topics such as quasitilings, tilings and tiling systems and, above all, the mysterious comparison property. Eventually, even though we have acquired quite good insight into these subject matters and nontrivially contributed to their development, we were forced to make some (mild) compromises in the final formulation of the main theorem.

Some steps of the generalization are indeed quite straightforward. For example, most of the notions of the theory of entropy structures, such as uniform equivalence, entropy structure or superenvelope, pass nearly unchanged. Also the proof of the “easy” implication of the main theorem is a relatively painless adaptation from the \( \mathbb{Z} \)-case (practically only one detail needs to be reworked more carefully, but this does not present a serious challenge).

In the opposite “hard” implication the desired generalization becomes much less obvious. The proof of the direction relies on an effective construction of a symbolic extension \((Y, G)\) of a given system \((X, G)\), with an a priori given entropy function on invariant measures on \(Y\) (this function delivered by the theory of entropy structures as a superenvelope). In the \( \mathbb{Z} \)-case such an extension has the form of a topological joining of two subshifts called rows: the first row is the essential encoding of the system, the second row is just a zero entropy encoding of a “dynamical parsing” of each orbit into “pieces” of equal lengths. The second row is easily built and
1. INTRODUCTION

the description of how it is done occupies just a few lines. It is the first row that requires most of the effort in the construction which is divided into two main steps:

(1) From the a priori predicted entropy function (superenvelope) one derives an oracle, a special integer-valued function on pieces of orbits which “prophesies” the number of blocks in the symbolic extension that will correspond to each of these pieces.

(2) Using the oracle one creates the actual first row of the symbolic extension.

Both steps are done with help of the parsing which must be applied beforehand to the elements of the system \((X, T)\). In order to make the decoding (i.e., the topological factor map from \(Y\) to \(X\)) possible, the parsing must be memorized in the symbolic extension, and this is exactly the role played by the second row.

Now, if \(\mathbb{Z}\) is replaced by a general countable amenable group \(G\), we encounter several serious obstacles, which we briefly discuss below.

First of all, the notion of a parsing must be replaced by a much more intricate notion of a tiling system. In the classical case of \(\mathbb{Z}\)-actions, systems of parsings exist as factors in any aperiodic zero-dimensional system, which follows from a marker theorem attributed to Krieger (see [4]). They occur under various names (as Krieger’s markers, Kakutani–Rokhlin partitions or clopen tower partitions, etc.) and have numerous applications, for example in the study of full groups, orbit equivalence and Hopf-equivalence of minimal Cantor systems (see [49] for an exposition on this subject, see also [7, 26, 27, 29]). For amenable group actions, for a long time, quasitilings of Ornstein and Weiss [42] have played a crucial role, and Lindenstrauss’ Pointwise Ergodic Theorem [37] is one of their most important applications. But we have quickly realized that, for building symbolic extensions, quasitilings are rather useless and that we need more precise tilings (we explain why in subsection 7.3). In the long process of building up the foundations for this paper, we have, among many other things, proved in [19] that the Ornstein–Weiss quasitilings can be improved to become tilings. Our tilings have already found numerous applications, see e.g. [15, 24, 25, 50, 57, 58]. Further, in [20] we have proved that quasitilings with arbitrarily good Følner properties exist as factors in any free action of any countable amenable group. The results on tilings from [19] and [20] play a fundamental role in this paper.

Next encountered technical difficulties are associated with building the oracle (i.e., with the step (1) above) and result from lack of subadditivity of certain conditional entropy functions. This problem was resolved using, among other things, a sophisticated behavior (which we needed to establish in the first place) of entropy with respect to tiling systems. Once the oracle is successfully defined, step (2), i.e., building the analog of the first row, is performed in a manner more or less straightforward adapted from the \(\mathbb{Z}\)-case.

The most serious difficulty occurs, somewhat unexpectedly, in building the second row responsible for memorizing the tiling system (in the \(\mathbb{Z}\)-case this is one of the easiest elements of the construction). It turns out that even though a tiling system created in [19] has zero topological entropy, we are unable to encode it as a factor of a symbolic system. Hence we coined a notion of an encodable tiling system and the existence of such tiling systems turns out to be one of the most serious challenges addressed in this paper. We confess, that we have stumbled upon this problem some time ago, and this has delayed the completion of the task undertaken in this work by several years. During these years we studied a new item necessary to put
the pieces of the puzzle together: the *comparison property* of countable amenable groups. And this subject became the second most important theme of this paper. The reader will find out that nearly half of the paper is devoted to or depends on this notion.

Comparison originates in the theory of $C^*$-algebras, but the most important for us “dynamical” version concerns group actions on compact spaces. In this setup it was defined by J. Cuntz (see [13]) and further investigated by M. Rørdam in [45, 46] and by W. Winter in [55]. As in the case of many other properties and notions in dynamical systems, the most fundamental form of comparison occurs in actions of the group $\mathbb{Z}$. In this context comparison is guaranteed for any action on a zero-dimensional compact metric space, which follows from the classical marker property of such actions (see [41]). See also [9] for more on comparison in $\mathbb{Z}$-actions.

For a wider generality, we refer the reader to a recent paper by David Kerr [35], where the notion is defined for other actions including topological and measure-preserving ones. We will focus on a particular case where a countable amenable group acts on a zero-dimensional compact metric space. In fact, this case also plays one of the leading roles in [35].

Unlike for $\mathbb{Z}$-actions, in the case of a general countable amenable group acting on a zero-dimensional compact metric space, it is unknown whether comparison necessarily occurs. There is neither a proof, nor a counterexample, although the problem has been attacked by several specialists for several years. Only a few partial results have been obtained, for instance, it is known (but never published, see [44] and also [52]) that finitely generated groups with a symmetric Følner sequence satisfying Tempelman’s condition (this includes all nilpotent, in particular Abelian, groups) have the comparison property, but beyond this case not much was known. In this paper we succeed in identifying a large class of groups whose any action on a zero-dimensional compact metric space has comparison. Namely, it is the class of *subexponential groups*, i.e., such that every finitely generated subgroup has subexponential growth. This covers all virtually nilpotent groups (which have polynomial growth) but also other, with intermediate growth, the most known example of which is the Grigorchuk group [30]. By a recent result of Breuillard, Green and Tao [8], our result also covers the above mentioned “Tempelman groups”; they turn out to be virtually nilpotent.

We establish a strong connection between comparison and the existence of tiling systems as factors of free zero-dimensional actions. In particular, if a group $G$ enjoys the comparison property then there exists an encodable tiling system of $G$. This opens the possibility to build symbolic extensions. Not counting the (relatively small) class of residually finite amenable groups, we can thus prove the Symbolic Extensions Entropy Theorem (analogous as in the $\mathbb{Z}$-case) for groups which enjoy the comparison property, in particular for all subexponential groups. In case of a general countable amenable group we can prove a slightly deficient version, in which symbolic extensions are replaced by *quasi-symbolic extensions*, defined as extensions in form of topological joinings of subshifts with some (perhaps not encodable) zero entropy tiling system.

### 1.3. Organization of the paper

Section 2 contains rather standard material concerning actions of countable amenable groups, both topological (on compact metric spaces) and measure-theoretic (on standard probability spaces), with special attention paid to subshifts and other zero-dimensional systems, as well as basic
facts about entropy in such actions. The following two sections contain exposi-
tions on concepts less familiar to the potential reader, still not quite new. And so,
in Section 3 we review entropy structures and symbolic extensions (including the
proof of the easy direction of the main Symbolic Extension Theorem). What is
new about these notions is their application (probably for the first time) to actions
of countable amenable groups. But the translation from \( \mathbb{Z} \)-actions is more or less
direct (though not completely trivial). Section 4 treats about Følner systems of
quasitilings and tiling systems, and is mainly a survey of authors’ previous work
[19] and [20]. The section is concluded by the presentation of tiled entropy—a
new approach to dynamical entropy, natural in the context of tiling systems and
necessary to cope with the difficulties encountered in the construction of symbolic
extensions of countable amenable group actions that were not present in the case
of \( \mathbb{Z} \). In Section 5 we prove, in full generality, the hard direction of the Symbolic
Extension Theorem, however in a slightly deficient version in which the extension is
quasi-symbolic. We are able to prove the full version of that theorem (with genuine
symbolic extensions) for two important classes of groups, and Section 6 is devoted
to introducing and studying one of these classes—groups with the comparison prop-
erty. We describe alternative forms of this property, prove various auxiliary facts,
but above all we prove that this property is enjoyed by the large class of subexponential
groups. In Section 7 we show how comparison property allows to encode a
zero entropy tiling system in a subshift on three symbols. This task, which in case
of \( \mathbb{Z} \)-actions can be resolved in one line (using some standard constructions from
topological dynamics), in the general case becomes a complicated issue occupying
several pages. With this tool in hand, we prove the full version of the Symbolic Ex-
tension Theorem for countable amenable groups which either enjoy the comparison
property or are residually finite. At the end of the paper we have put an appendix,
of perhaps independent interest, in which we reduce the alphabet used in Section
7 from three to two symbols.

2. Preliminaries on actions of countable amenable groups

2.1. Group actions, subshifts, symbolic extensions, block codes. Throughout this paper \( G \) denotes a (discrete) countable group with the unity \( e \). By “countable” we will always mean “infinite countable”. For finite groups, everything we address in this paper becomes trivial. Let \( X \) be a compact metric space and let \( \text{Hom}(X) \) denote the group of all homeomorphisms \( \phi : X \rightarrow X \). By an action (more precisely, topological action) of \( G \) on \( X \) we will mean a homomorphism from \( G \) into \( \text{Hom}(X) \), i.e., an assignment \( g \mapsto \phi_g \) such that \( \phi_{gg'} = \phi_g \circ \phi_{g'} \) for every \( g, g' \in G \). It follows automatically that \( \phi_e = \text{Id} \) (the identity homeomorphism) and that \( \phi_{g^{-1}} = (\phi_g)^{-1} \) for every \( g \in G \). Such an action will be denoted by \((X, G)\) (although a group may act on the same space in many different ways, we will usually fix just one such action, hence this notation should not lead to a confusion). Another term used for \((X, G)\) is a topological dynamical system (or briefly a system). From now on, to reduce the multitude of symbols used in this paper, we will write \( \phi_g(x) \) in place of \( \phi_g(x) \). The same applies to subsets \( A \subset X \): \( g(A) \) will replace \( \phi_g(A) \). The action is called free provided that \( g(x) = x \) for at least one \( x \in X \) implies \( g = e \). A Borel measurable set \( A \subset X \) is called invariant if \( g(A) = A \) for every \( g \in G \).

An important example of an action of \( G \) is the shift action on finitely many
symbols. Let \( \Lambda \) be a finite set (usually, we assume that \( \Lambda \) contains more than one
this context the set \( \Lambda \) will be called the alphabet, and let
\[
\Lambda^G = \{ x = (x_g)_{g \in G}, \forall g \in G \ x_g \in \Lambda \}
\]
be equipped with any metric compatible with the product topology. Then \( \Lambda^G \) is a compact metric space and \( G \) acts on it naturally by shifts:
\[
\text{if } x = (x_f)_{f \in G} \text{ and } g \in G \text{ then } g(x) = (x_{fg})_{f \in G}.
\]
The system \((\Lambda^G, G)\) is called the full shift (over \( \Lambda \)) while any non-empty closed invariant subset of \( Y \subset \Lambda^G \) (regarded with the shift action) is called a subshift or a symbolic system. If \((X, G)\) and \((Y, G)\) are actions of the same group on two (not necessarily different) spaces, and there exists a continuous surjection \( \pi : Y \to X \) which commutes with the action (i.e., for any \( g \in G \) and \( y \in Y \), \( \pi \circ g(y) = g \circ \pi(y) \)), then \((X, G)\) is called a topological factor of \((Y, G)\) and \((Y, G)\) is called a topological extension of \((X, G)\). In what follows, we will skip the adjective “topological” and when the acting group is fixed and its action on given spaces is understood, we will also skip it in the denotation of the dynamical systems (i.e., we will use the letters \( X \) and \( Y \) in the meaning of \((X, G)\) and \((Y, G)\)). If \( X \) is a factor of \( Y \), then the above map \( \pi \) will be referred to as the factor map. It has to be remarked that one system, say \( X \), may be a factor of another, say \( Y \), via many different factor maps. Since this may lead to a confusion, we will often use the phrase \( X \) is a factor of \( Y \) via the map \( \pi \). If the factor map is injective, in which case it is a homeomorphism between \( Y \) and \( X \), we will say that the systems are topologically conjugate. From the point of view of topological dynamics, conjugate systems are identical.

By a topological joining of finitely or countably many systems \( X_k \) \((k \in K)\) where \( K = \{1, 2, \ldots, l\} \) or \( K = \mathbb{N} \) we will mean any closed subset \( Z \) of the Cartesian product \( \prod_{k \in K} X_k \), which is invariant under the product (coordinatewise) action, and whose projection on every coordinate is surjective. Such a joining will be sometimes denoted by \( \bigsqcup_{k \in K} X_k \) (although this notation is ambiguous, as there may exist many joinings of the same collection of systems). At least one joining always exists—the product joining. The coordinate projections are factor maps from the joining to the respective coordinate systems. A special case of a countable joining is an inverse limit. We assume that \((X_k)_{k \in \mathbb{N}}\) is a sequence of systems such that, for each \( k \in \mathbb{N} \), \( X_k \) is a factor of \( X_{k+1} \) via a map \( \pi_k \) (referred to as the bonding map). Then the inverse limit of the sequence \((X_k)_{k \in \mathbb{N}}\) is defined as
\[
\lim_{\leftarrow k} X_k = \{ (x_k)_{k \in \mathbb{N}}, \forall k \in \mathbb{N} \ x_k \in X_k, \text{ and } x_k = \pi_k(x_{k+1}) \}.
\]
It is elementary to check that the inverse limit is a countable joining of the systems \( X_k \) \((k \in \mathbb{N})\).

The term “symbolic extension” which appears in the title of this paper is a very natural concept. By a symbolic extension of a system \( X \) we simply mean any symbolic system \( Y \) (over some finite alphabet \( \Lambda \)) which is a (topological) extension of \( X \) (via some factor map \( \pi \)). Symbolic extensions are sometimes also called subshift covers. The criteria for a system \( X \) to admit at least one symbolic extension, and for computing how close the extension can be to \( X \) in terms of information theory, are fairly well understood in case of \( \mathbb{Z} \)-actions. As explained in the Introduction, the goal of this paper is to see to what extent the same criteria apply to actions of general countable amenable groups.
Since we are discussing topological factors, let us mention in this place the specific form of factor maps between two subshifts.

**Definition 2.1.** Let Λ and ∆ be some finite sets (alphabets). By a block code we will mean any function Ξ : Λ^F → ∆, where F is a finite subset of G (called the coding horizon of Ξ).

The Curtis–Hedlund–Lyndon Theorem [32] (which holds for actions of any countable group) states:

**Theorem 2.2.** Let Y ⊂ Λ^G be a subshift (over some finite alphabet Λ). Let ∆ be a finite set. Then ξ : Y → X ⊂ ∆^G is a topological factor map (the image X is then a subshift over ∆) if and only if there exists a finite set F ⊂ G and a block code Ξ : Λ^F → ∆, such that, for all y ∈ Y and g ∈ G we have the equality

(ξ(y))_g = Ξ(g(y)|_F).

The term “block code” refers to both Ξ and ξ, depending on the context, and F is called a coding horizon of ξ (and of Ξ). Clearly, if F is a coding horizon of ξ (and of Ξ), so is any finite set containing F. It will be convenient to assume that coding horizons always contain the unity.

By this opportunity we will introduce another convention often used in symbolic dynamics. Although it is a slight abuse of precision, it is commonly accepted and does not lead to a confusion. If F ⊂ G is a finite set then any element B = (B(f))_f∈F ∈ Λ^F will be called a block (or, if needed, a block over F). Now, if for some g ∈ G the block B' ∈ Λ^Fg satisfies ∀f ∈ F B'(fg) = B(f), then B' will be called a shifted copy of B. Shifted copies of the same block will be often denoted by the same letter. For example, for y ∈ Λ^G, in place of g(y)|_F = B we will write y|_{Fg} = B. This means that ∀f ∈ F yfg = B(f).

With each block B ∈ Λ^F we associate the cylinder set

[B] = {x ∈ Λ^G : x|_F = B}.

The cylinder set is clopen in the full shift Λ^G (following a common practice, we use the term “clopen” in the meaning of “closed and open”). When regarding a subshift X ⊂ Λ^G by a cylinder we will often mean the intersection of a cylinder (as defined above) with X. In this sense, cylinders are clopen in X.

### 2.2. An ε-modification, (K, ε)-invariance, Følner sequence, amenability.

This subsection introduces the key notions of amenability for countable groups. Amenability was introduced by von Neumann [41]. There are many equivalent ways of defining amenability and most of them apply to groups much more general than countable (see e.g. [43]). We use the one which fits us best. It relies of the concept of a Følner sequence introduced by Følner [23]. Note that all notions below depend on comparing cardinalities of certain sets, hence may be considered purely quantitative.

We will use |F| to denote the cardinality of a set F. Given a finite set F ⊂ G and ε > 0, an ε-modification of F is any set F' such that |F ∆ F'| / |F| < ε, where ∆ denotes the symmetric difference of sets. An ε-modification of F which is also a subset of F will be called a (1−ε)-subset of F. If K is another finite subset of G then F is called (K, ε)-invariant if KF is an ε-modification of F. For singletons, instead of “(g, ε)-invariant” we will write “(g, ε)-invariant”. Below we list a few easy but useful facts associated to the notions introduced above.
(1) If $F$ is $(K, \epsilon)$-invariant then it is $(g, 2\epsilon)$-invariant for every $g \in K$.
(2) By the $K$-core of $F$ (denoted by $F_K$) we mean the set $\{f \in F : K f \subset F\} = F \cap \bigcap_{n \in K} g^{-1} F$. If $F$ is $(K, \epsilon)$-invariant then its $K$-core is a $(1 - |K| \epsilon)$-subset of $F$ (see [19] Lemma 2.6).
(3) It follows that if $F$ is $(K, \epsilon)$-invariant then any set $F'$ satisfying $F_K \subset F' \subset K F$ is a $(|K| \epsilon + \epsilon)$-modification of $F$.
(4) A $\delta$-modification of a $(K, \epsilon)$-invariant set is $(K, \epsilon')$-invariant, where $\epsilon' = \frac{|K| \delta + \epsilon}{\delta}$.

Definition 2.3. A sequence of finite subsets of $G$, $(F_n)_{n \in \mathbb{N}}$, is called a Følner sequence if, for every finite set $K$ and every $\epsilon > 0$, the sets $F_n$ are eventually (i.e., except for finitely many indices $n$) $(K, \epsilon)$-invariant. A group which possesses a Følner sequence is called amenable.

An immediate consequence of fact (4) above is that if $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence in $G$ and for each $n$ $F'_n$ is a $\delta_n$-modification of $F_n$, where $\delta_n \to 0$, then $(F'_n)_{n \in \mathbb{N}}$ is a Følner sequence as well.

It is known that if a countable group $G$ is amenable then it possesses a Følner sequence $(F_n)_{n \in \mathbb{N}}$ with the following additional properties:
- $\forall n \in \mathbb{N} \in F_n$,
- $\forall n \in \mathbb{N} F_n \subset F_{n+1}$,
- $\forall n \in \mathbb{N} F_n = F_n^{-1}$ (by convention, $F_n^{-1} = \{f^{-1} : f \in F_n\}$).

In reference to the above three properties of a Følner sequence, we will use the terms centered, nested and symmetric, respectively. The first two properties are easily obtained, for the existence of symmetric Følner sequences see [40] Corollary 5.3.

2.3. The Choquet simplex of invariant probability measures. Let $X$ be a compact metric space and let $\mathcal{M}(X)$ denote the family of all Borel probability measures on $X$. Since we shall consider no measures other than Borel probabilities, from now on “measure” will always mean an element of $\mathcal{M}(X)$. Endowed with the weak-star topology, this set is a metrizable Choquet simplex, that is, it is a nonempty compact convex set which possesses a convex metric, and every its element $\mu$ has a unique representation as the integral average of the extreme points. Clearly, the extreme points of $\mathcal{M}(X)$ are the Dirac measures $\delta_x$ ($x \in X$) and the integral average representation of $\mu$ mentioned above is

$$\mu = \int_X \delta_x \, d\mu(x).$$

One of the standard convex metrics on $\mathcal{M}(X)$ compatible with the weak-star topology is given by the following formula

$$d_s(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} \left| \int f_n \, d\mu - \int f_n \, d\nu \right|,$$

where $(f_n)_{n \in \mathbb{N}}$ is some fixed sequence of continuous functions $f_n : X \to [0, 1]$, linearly dense in the space $C(X)$ of all continuous real functions on $X$ (with the uniform metric).

If $G$ acts on $X$ then it also acts on $\mathcal{M}(X)$: for $g \in G$, the measure $g(\mu)$ is defined by the formula $g(\mu)(A) = \mu(g^{-1}(A))$ (where $A$ is a Borel subset of $X$). We say that $\mu$ is an invariant measure if $\mu = g(\mu)$ for every $g \in G$. An invariant
measure is called ergodic if $\mu(A) \in \{0, 1\}$ for every invariant Borel set $A$. In many aspects concerning invariant measures, actions of amenable groups exhibit the same features as $\mathbb{Z}$-actions. In particular, the following fact holds (follows e.g. from [56]):

**Theorem 2.4.** If a countable amenable group $G$ acts on a compact metric space $X$ then the family of all invariant measures (denoted by $\mathcal{M}_G(X)$) is a metrizable Choquet simplex whose extreme points are exactly the ergodic measures.

The above theorem says that

1. $\mathcal{M}_G(X)$ is a nonempty and weakly-star closed (hence compact) subset of $\mathcal{M}(X)$,
2. $\mathcal{M}_G(X)$ is convex and the collection $\text{ex}\mathcal{M}_G(X)$ of its extreme points coincides with the collection of all ergodic measures,
3. every invariant measure $\mu$ has a unique representation as the integral average of the ergodic measures, i.e., there exists a unique probability distribution $\xi_\mu$ on $\text{ex}\mathcal{M}_G(X)$ such that
   $$\mu = \int_{\text{ex}\mathcal{M}_G(X)} \nu \, d\xi_\mu(\nu).$$

The above formula is referred to as the ergodic decomposition of $\mu$. A way of proving the existence of invariant measures is by investigating empirical measures of the form

$$\mu_{x_n}^F = \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{g(x)},$$

where $F_n$ is a member of the Følner sequence and $x \in X$, and showing that with increasing $n$ such measures accumulate at invariant measures. We skip the details of this standard argument, however, we will need the following refinement:

**Proposition 2.5.** Fix some $\gamma > 0$. If $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence in $G$ then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and any $x \in X$, the measure $\mu_{x_n}^F$ lies within the $\gamma$-neighborhood (in the metric $d_*$) of $\mathcal{M}_G(X)$.

**Proof.** If, for some $\gamma > 0$ and arbitrarily large indices $n$, there existed points $x_n \in X$ such that $d_*(\mu_{x_n}^F, \mathcal{M}_G(X)) > \gamma$ then the sequence of measures $(\mu_{x_n}^F)_{n \in \mathbb{N}}$ would have some accumulation points outside $\mathcal{M}_G(X)$, a contradiction. \qed

Now suppose that a topological dynamical system $X$ is a topological factor of another, $Y$, via a map $\pi$. Then $\pi$ induces a map (which will be denoted by the same letter $\pi$) from $\mathcal{M}_G(Y)$ to $\mathcal{M}_G(X)$, by the formula $\pi(\mu)(A) = \mu(\pi^{-1}(A))$ (where $A \subset X$ is a Borel set). The following fact is well known:

**Proposition 2.6.** The map $\pi : \mathcal{M}_G(Y) \to \mathcal{M}_G(X)$ is a continuous affine surjection which sends extreme points to extreme points.

A factor map $\pi : Y \to X$ such that $\pi : \mathcal{M}_G(Y) \to \mathcal{M}_G(X)$ is injective (i.e., $\pi$ is an affine homeomorphism between the Choquet simplices $\mathcal{M}_G(Y)$ and $\mathcal{M}_G(X)$) is called faithful.
2.4. The ergodic theorem. Let $G$ be a countable amenable group.

**Definition 2.7.** A Følner sequence $(F_n)_{n \in \mathbb{N}}$ in $G$ is called tempered if, for each $n \in \mathbb{N}$, it satisfies the Shulman’s condition:

$$\left| \bigcup_{i=1}^{n} F_{i-1}F_{n+1} \right| \leq C|F_{n+1}|.$$  

It is very easy to see that any Følner sequence $(F_n)_{n \in \mathbb{N}}$ in $G$ contains a tempered subsequence. It suffices to note that for each $n \in \mathbb{N}$ and then sufficiently large $k \in \mathbb{N}$, $F_{n+k}$ is $(\bigcup_{i=1}^{n} F_i, 1)$-invariant, which implies that if, in the above condition, we replace $F_{n+1}$ by $F_{n+k}$, the above condition holds for $C = 2$.

Let $(X, \Sigma, \mu)$ be a standard probability space (roughly, this means that $(X, \Sigma, \mu)$ can be modeled as a compact metric space with a Borel probability measure). By a measure-theoretic action of $G$ we will understand the action on $(X, \Sigma, \mu)$ by measure-automorphisms, i.e., a case in which with each $g \in G$ we have associated a measurable and $\mu$-almost everywhere injective map $\phi_g : X \to X$ such that $\mu(\phi_g^{-1}(A)) = \mu(A)$, for every $A \in \Sigma$. Moreover we require that for all $g, h \in G$, $\phi_{gh} = \phi_g \circ \phi_h$. As in the case of a topological action, we will write $g(x)$ and $g^{-1}(A)$ in place of $\phi_g(x)$ and $\phi_g^{-1}(A)$, respectively. Like in the topological case, a measure-theoretic action of $G$ is called ergodic if $\mu(A) \in \{0, 1\}$ for every invariant set $A \in \Sigma$. The most important for us example of a measure-theoretic action of $G$ occurs when $G$ acts on a compact metric space $X$ by homeomorphisms, $\Sigma$ is the Borel sigma-algebra in $X$ and $\mu \in \mathcal{M}_G(X)$ (then the notion of ergodicity of the action coincides with the, introduced earlier, notion of ergodicity of the measure).

In the context of measure-theoretic actions of countable amenable groups, the pointwise ergodic theorem was proved by E. Lindenstrauss in [37] Theorem 1.2 (see also [1] for the necessity of Shulman’s condition):

**Theorem 2.8.** If $G$ acts by measure-automorphisms on a standard probability space $(X, \Sigma, \mu)$, the action is ergodic, $A \in \Sigma$, and $(F_n)_{n \in \mathbb{N}}$ is a tempered Følner sequence in $G$ then, for $\mu$-almost every point $x \in X$, we have the equality

$$\mu(A) = \lim_{n \to \infty} \frac{1}{|F_n|} \left| \{g \in F_n : g(x) \in A\} \right|.$$  

2.5. Entropy. For actions of countable amenable groups we have well defined notions of topological entropy $h_{\top}(X, G)$, and, for an invariant measure $\mu$, of the measure-theoretic entropy $h_{\mu}(X, G)$ (later denoted by $h(\mu, X)$). Let us briefly recall the basics.

Let $(X, \Sigma, \mu)$ be a standard probability space and let $\mathcal{P}$ be a finite measurable partition of $X$. The Shannon entropy of $\mathcal{P}$ equals

$$H(\mu, \mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log(\mu(P)) \leq \log |\mathcal{P}|.$$  

Now suppose that a countable group $G$ acts on $(X, \Sigma, \mu)$ by measure-automorphisms. Given a finite measurable partition $\mathcal{P}$ of $X$ and a finite set $F \subset G$, by $\mathcal{P}^F$ we will mean the join

$$\mathcal{P}^F = \bigvee_{g \in F} g^{-1}(\mathcal{P}) = \left\{ \bigcap_{g \in F} g^{-1}(P_g) : \forall g \in F \ P_g \in \mathcal{P} \right\}.$$
(which is again a finite measurable partition of $X$). The Shannon entropy of this partition (with respect to $\mu$) will be denoted by $H(\mu, \mathcal{P}^F)$. One of elementary properties of the Shannon entropy, is the following subadditivity property: for any pair of sets $F_1, F_2 \subset G$,

$$H(\mu, \mathcal{P}^{F_1 \cup F_2}) \leq H(\mu, \mathcal{P}^{F_1}) + H(\mu, \mathcal{P}^{F_2}).$$

In fact, strong subadditivity holds (see e.g. [18]):

$$H(\mu, \mathcal{P}^{F_1 \cup F_2}) \leq H(\mu, \mathcal{P}^{F_1}) + H(\mu, \mathcal{P}^{F_2}) - H(\mu, \mathcal{P}^{F_1 \cap F_2}).$$

If now $G$ is amenable (and countable), then one defines the dynamical entropy of $\mathcal{P}$ with respect to $\mu$ by the formula

$$h(\mu, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{|F_n|} H(\mu, \mathcal{P}^{F_n}),$$

where $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence in $G$. Using strong subadditivity, one can prove that the limit defining the dynamical entropy of a partition equals the infimum over all finite subsets $F \subset G$ (see e.g. [18]):

$$h(\mu, \mathcal{P}) = \inf_F \frac{1}{|F|} H(\mu, \mathcal{P}^F).$$

In particular, this shows that the dynamical entropy of a partition (and hence also the Kolmogorov–Sinai entropy defined below) does not depend on the choice of the Følner sequence.

The Kolmogorov–Sinai entropy of the measure-theoretic system $(X, \Sigma, \mu, G)$ is defined as

$$h(\mu, X) = \sup_{\mathcal{P}} h(\mu, \mathcal{P}),$$

where $\mathcal{P}$ ranges over all finite measurable partitions of $X$. The Kolmogorov–Sinai entropy can be infinite, however, this case is of marginal interest for us. The analog of the Kolmogorov–Sinai Theorem holds: if a finite partition $\mathcal{P}$ is a generator (i.e., the smallest sigma-algebra containing the partitions $\mathcal{P}^F$ for all finite sets $F \subset G$ equals $\Sigma$), then the Kolmogorov–Sinai entropy is attained on $\mathcal{P}$:

$$h(\mu, X) = h(\mu, \mathcal{P}).$$

In any case, there exists a refining sequence of finite partitions $(\mathcal{P}_k)_{k \in \mathbb{N}}$, i.e., such that for every $k \in \mathbb{N}$, $\mathcal{P}_{k+1}$ refines $\mathcal{P}_k$ (meaning that every atom of $\mathcal{P}_{k+1}$ is contained in some atom of $\mathcal{P}_k$; we will write $\mathcal{P}_{k+1} \gtrsim \mathcal{P}_k$) and jointly they generate $\Sigma$. Then

$$h(\mu, X) = \lim_k \uparrow h(\mu, \mathcal{P}_k).$$

If $(X, G)$ is a topological dynamical system, then the Kolmogorov–Sinai entropy can be regarded as a function on $\mathcal{M}_G(X)$. In this case, $h(\mu, X)$ will be denoted shortly by $h(\mu)$ and the function $\mu \mapsto h(\mu)$ on $\mathcal{M}_G(X)$ will be called the entropy function.

Now consider two finite measurable partitions of $X$, $\mathcal{P}$ and $\mathcal{Q}$. In this context one defines the conditional Shannon entropy of $\mathcal{P}$ given $\mathcal{Q}$ (with respect to $\mu$) as

$$H(\mu, \mathcal{P}|\mathcal{Q}) = \sum_{B \in \mathcal{Q}} \mu(B) H(\mu_B, \mathcal{P}) = H(\mu, \mathcal{P} \vee \mathcal{Q}) - H(\mu, \mathcal{Q}),$$
where $\mu_B$ is the normalized conditional measure $\mu$ on $B$. Subadditivity still holds for conditional entropy (see formula 1.6.11 in [17]):

$$H(\mu, P^{F_1 \cup F_2} | Q^{F_1 \cup F_2}) \leq H(\mu, P^{F_1} | Q^{F_1}) + H(\mu, P^{F_2} | Q^{F_2}),$$

but strong subadditivity in general fails.

The conditional dynamical entropy of $P$ given $Q$, with respect to $\mu$ is defined analogously, as

$$h(\mu, P|Q) = \lim_{n \to \infty} \frac{1}{|P_n|} H(\mu, P^{F_n} | Q^{F_n}) = h(\mu, P \cup Q) - h(\mu, P).$$

Here also the limit can be replaced by the infimum over all finite sets $F$, which follows from the following three facts:

1. for each finite set $F$, $H(\mu, P^F | Q^F) \geq H(\mu, P^F | Q^G)$, where $Q^G$ is the invariant sigma-algebra generated by $Q$,

2. $h(\mu, P|Q) = \lim_{n \to \infty} \frac{1}{|P_n|} H(\mu, P^{F_n} | Q^{G^1})$ (Abramov-Rokhlin formula [53 Theorem 4.1] or [28 Lemma 1.1]),

3. the conditional entropy $H(\mu, P^F | Q^G)$ is strongly subadditive, which implies that $h(\mu, P|Q) = \inf_{F} \frac{1}{|F|} H(\mu, P^F | Q^G) \leq \inf_{F} \frac{1}{|F|} H(\mu, P^F | Q^F)$ (both infima range over finite sets $F$).

An important consequence of (3) above is the following observation:

**Lemma 2.9.** If $X$ is a topological dynamical system and $P$ and $Q$ are finite partitions such that the boundary of any atom of either $P$ or $Q$ has measure zero for any $\mu \in \mathcal{M}$, where $\mathcal{M} \subset \mathcal{M}_0(X)$, then the function $\mu \mapsto h(\mu, P|Q)$ is upper semicontinuous on $\mathcal{M}$.

**Proof.** The “small boundary property” of $P$ and $Q$ easily implies that, for each finite $F \subset G$, the function $\mu \mapsto \frac{1}{|F|} H(\mu, P^F | Q^F)$ is continuous on $\mathcal{M}$. The infimum of any family of continuous functions on any metric space is upper semicontinuous. 

In case $G = \mathbb{Z}$ and $F_n = \{1, 2, \ldots, n\}$ it is known that the sequences $\frac{1}{n} H(\mu, P^n)$ and $\frac{1}{n} H(\mu, P^n | Q^n)$ are in fact nonincreasing (see e.g., [17 Fact 2.3.1]). This cannot be claimed in the case of a general countable amenable group. We will need to cope with this difficulty later.

Throughout this paper, we will be using the following convention: if $\pi : Y \to X$ is any map (between any spaces) and $P$ is a (finite) partition of $X$, then the lifted partition, $\{\pi^{-1}(P) : P \in P\}$ (which is a (finite) partition of $Y$) will be denoted by the same letter $P$. We will take care to avoid any confusion caused by this convention. Note that if $\pi$ is continuous then lifting partitions preserves measurability and the property of having clopen atoms.

Now consider a topological factor map between two topological dynamical systems, $\pi : Y \to X$. If $\nu \in \mathcal{M}_0(Y)$ and then we can also define the conditional entropy of $\nu$ given $X$, as follows

$$h(\nu, Y|X) = \sup_{Q} \inf_{P} h(\nu, Q|P),$$

where $Q$ ranges over all finite measurable partitions of $Y$, while $P$ ranges over all finite measurable partitions of $X$ (lifted to $Y$). If $h(\mu, X) < \infty$, where $\mu = \pi(\nu) \in \mathcal{M}_0(X)$, then $h(\nu, Y|X)$ is simply the difference $h(\nu, Y) - h(\mu, X)$. 

If, for every $\nu \in \mathcal{M}_G(Y)$, $h(\nu, Y | X) = 0$ then $Y$ is called a principal extension of $X$. A particularly good extension is described in the definition below:

**Definition 2.10.** Let $\pi : Y \to X$ be a topological factor map between topological dynamical systems. We say that $Y$ is an isomorphic extension of $X$ (via the map $\pi$) if the associated map $\pi : \mathcal{M}_G(Y) \to \mathcal{M}_G(X)$ is injective (i.e., the extension is faithful) and, for each $\nu \in \mathcal{M}_G(Y)$ and $\mu = \pi(\nu) \in \mathcal{M}_G(X)$ the measure-preserving actions of $G$ on $(Y, \Sigma_Y, \nu)$ and on $(X, \Sigma_X, \mu)$ ($\Sigma_Y$ and $\Sigma_X$ denote the Borel sigma-algebras in $Y$ and $X$, respectively) are isomorphic in the measure-theoretic sense via the same map $\pi$.

An isomorphic extension is both faithful and principal. For a topological extension $\pi : Y \to X$ to be isomorphic it suffices that there are sets $Y' \subset Y$ and $X' \subset X$ such that $\nu(Y') = \mu(X') = 1$ for every $\nu \in \mathcal{M}_G(Y)$ and $\mu \in \mathcal{M}_G(X)$, and $\pi|_{Y'}$ is a bijection between $Y'$ and $X'$.

Since topological entropy will play in this paper only a marginal role, we reduce its presentation to a necessary minimum. For actions of countable amenable groups the variational principle is valid, hence we can use it instead of a lengthy original definition (in fact one of many possible definitions). So, for our goals the following understanding of topological entropy is completely sufficient:

**Definition 2.11.** Let a countable amenable group $G$ act on a compact metric space $X$. The topological entropy of the system $(X, G)$ equals

$$h_{top}(X, G) = \sup_{\mu \in \mathcal{M}_G(X)} h(\mu, X).$$

### 2.6. Zero-dimensional systems

Let $X$ be, in addition to being compact and metric, also zero-dimensional (equivalently, totally disconnected), i.e., such that there exists a basis of the topology consisting of clopen sets. In such a space there exists a sequence of finite clopen partitions $(\mathcal{P}_k)_{k \in \mathbb{N}}$ (i.e., partitions whose all atoms are clopen) which is jointly refining in the topological sense, that is, denoting $\mathcal{P}_{[1,k]} = \bigvee_{i=1}^k \mathcal{P}_i$, and for a partition $\mathcal{P}$, letting $\text{diam}(\mathcal{P})$ denote the maximal diameter of an atom of $\mathcal{P}$, we have $\text{diam}(\mathcal{P}_{[1,k]}) \to 0$. Note that then the partitions $\mathcal{P}_{[1,k]}$ form a refining sequence also in the previously defined measurable sense. For each $k$, let $\Lambda_k$ be a set of labels bijectively associated to the atoms of $\mathcal{P}_k$, so that $\mathcal{P}_k = \{P_a : a \in \Lambda_k\}$. If now a countable group $G$ acts on $X$ (by homeomorphisms), then we introduce the following notation:

$$\forall k \in \mathbb{N} \forall g \in G \quad (x_{k,g} := a \in \Lambda_k \iff g(x) \in P_a \in \mathcal{P}_k),$$

$$\forall k \in \mathbb{N} \quad \pi_k(x) := (x_{k,g})_{g \in G} \in \Lambda_k^G,$$

$$\pi(x) := (\pi_k(x))_{k \in \mathbb{N}} = (x_{k,g})_{k \in \mathbb{N}, g \in G} \in \prod_{k \in \mathbb{N}} \Lambda_k^G.$$

We will call the double sequence $(x_{k,g})_{k \in \mathbb{N}, g \in G}$ the array-name of $x$. We let $X_k$ denote the image of $X$ by the map $\pi_k$. As easily verified, $X_k$ is a subshift over the alphabet $\Lambda_k$. Because all partitions $\mathcal{P}_k$ are clopen, the maps $\pi_k$ and $\pi$ are continuous. Also, they commute with the action, hence each $X_k$ as well as $\pi(X)$ are factors of $X$. The subshift $X_k$ will be called the $k$th layer of $X$. We will denote by $X_{[1,k]}$ the projection of $X$ onto the first $k$ layers, which is a subshift over the product alphabet $\Lambda_{[1,k]} = \prod_{i=1}^k \Lambda_i$. The natural projections provide bonding maps
between the successive subshifts $X_{[1,k]}$ and allow to identify the image $\pi(X)$ with the inverse limit
\[ \lim_k X_{[1,k]} \]
Because the diameters of the partitions $P_{[1,k]}$ converge to zero, different points in $X$ have different array-names, which means that $\pi$ is injective. In this manner, we conclude that $X$ is topologically conjugate to the above inverse limit of subshifts. We will call it the array representation of $X$ and because we treat conjugate systems as one, we will simply write $X = \lim_k X_{[1,k]}$. From now on, we will imagine any zero-dimensional system in its array representation (we always fix one of many possible such representations). Observe that if $X$ is given the array representation, the partitions $P_k$ can be restored as the symbol partitions:
\[ P_k = \{[a] : a \in \Lambda_k\} \]
where $[a]$ is the one-symbol cylinder at $e$, $\{x \in X : x_{k,e} = a\}$.

3. Entropy structure and the easy direction of the main theorem

Entropy structure for an action of a countable amenable group is defined in exactly the same manner as it is done for $G = \mathbb{Z}$. Let us recall some basic terms from the theory of entropy structures for $\mathbb{Z}$-actions.

3.1. Structures. Let $\mathcal{M}$ be a compact metric set. By a structure on $\mathcal{M}$ we will understand any nondecreasing sequence of commonly bounded nonnegative functions on $\mathcal{M}$, $F = (f_k)_{k \geq 0}$ with $f_0 \equiv 0$. Clearly, the pointwise limit function $f = \lim_k f_k$ exists and is nonnegative and bounded.

Two structures $F = (f_k)_{k \geq 0}$ and $F' = (f'_k)_{k \geq 0}$ are said to be uniformly equivalent if
\[ \forall \epsilon > 0, k_0 \geq 0 \quad \exists k \geq 0 \quad (f'_k > f_{k_0} - \epsilon \text{ and } f_k > f'_k > f_{k_0} - \epsilon). \]
Notice the obvious fact that uniformly equivalent structures have a common limit function.

Let $f$ be a nonnegative bounded function on $\mathcal{M}$. By an upper semicontinuous envelope of $f$ we shall mean the function $\tilde{f}$ defined on $\mathcal{M}$ by any of the following formulas
\[ \tilde{f}(\mu) = \limsup_{\mu' \to \mu} f(\mu') = \inf_{U \ni \mu} \sup\{f(\mu') : \mu' \in U\} = \inf\{g \text{ continuous, } g \geq f\}, \]
where $\mu, \mu' \in \mathcal{M}$ and $U$ ranges over all open neighborhoods of $\mu$. Note that $\tilde{f} \geq f$.
We also define the defect of $f$ as the difference $\ddot{f} = \tilde{f} - f$. The function $f$ is upper semicontinuous if $f = \tilde{f}$ or, equivalently, $\ddot{f} \equiv 0$.

We will say that a structure $F$ has upper semicontinuous differences, if the difference functions $f_{k+1} - f_k$ are upper semicontinuous for every $k \geq 0$.
If $\mathcal{M}$ is a compact convex subset of some locally convex linear space and all functions $f_k$ are affine, then we will say that $F = (f_k)_{k \geq 0}$ is an affine structure.
3.2. Superenvelopes.

**Definition 3.1.** Given a structure $\mathcal{F} = (f_k)_{k \geq 0}$ on a compact domain $\mathcal{M}$, a nonnegative bounded function $E$ on $\mathcal{M}$ is called a superenvelope of $\mathcal{F}$ if $E \geq f_k$ for each $k \geq 0$ and the defects $E - f_k$ tend pointwise to zero. Notice that then $E - f$ (where $f$ is the limit function of $\mathcal{F}$) is upper semicontinuous. A priori a structure may have no bounded superenvelopes. By default, the constant infinity function is added to the collection of superenvelopes of any structure.

We have the following facts (see e.g. \[17\] Lemma 8.1.10, Theorem 8.1.2 (2), Lemma 8.1.12 and Theorem 8.2.5):

**Proposition 3.2.** (1) The infimum of all superenvelopes of a structure $\mathcal{F}$ is a superenvelope of $\mathcal{F}$ (in the extreme case this is the constant infinity function). This minimal superenvelope of $\mathcal{F}$ will be denoted by $E_{\mathcal{F}}$.

(2) Uniformly equivalent structures have the same collection of superenvelopes (hence the same minimal superenvelope).

(3) If $\mathcal{F} = (f_k)_{k \geq 0}$ has upper semicontinuous differences then a function $E$ is its superenvelope if and only if $E - f_k$ is nonnegative and upper semicontinuous for every $k \geq 0$ (in particular $E = E - f_0$ is then upper semicontinuous).

(4) If $\mathcal{F}$ is an affine structure with upper semicontinuous differences, defined on a Choquet simplex, then $E_{\mathcal{F}}$ coincides with the pointwise infimum of all affine superenvelopes of $\mathcal{F}$ (in particular $E_{\mathcal{F}}$ is concave).

We will need the following terminology: Let $\pi : Y \to X$ be a continuous surjection between compact metric spaces. Given a bounded nonnegative function $f$ on $X$, we define its **lift of $f$ against $\pi$** as the composition $f \circ \pi$ (which is a function defined on $Y$). As in the case of partitions, we will denote the lift of $f$ by the same letter $f$. The lifted function is **constant on fibers**, that is, it is constant on the sets $\pi^{-1}(x)$ ($x \in X$). Going in the opposite direction is less obvious. Let now $f$ be a bounded function on $Y$. We define the **push-down of $f$ (along $\pi$)** as the function $f_{\pi}$ on $X$ given by $f_{\pi}(x) = \sup \{ f(y) : y \in \pi^{-1}(x) \}$ ($x \in X$). Lifting reverses the operation of pushing down exclusively for functions constant on fibers.

We have the following facts, some of which are immediate, some are proved in \[17\] Facts A.1.26 and A.2.22:

**Proposition 3.3.** (1) The operation of lifting preserves continuity, upper and lower semicontinuity of a function. If $Y$ and $X$ are convex sets, and $\pi$ is affine, then lifting preserves concavity, convexity and affinity of a function. The same holds for pushing down functions which are constant on fibers.

(2) In general, the pushing down preserves upper semicontinuity of a function. If $Y$ and $X$ are convex sets, and $\pi$ is affine, then pushing down preserves concavity of a function. If, moreover, $Y$ and $X$ are Choquet simplices and $\pi$ sends extreme points of $Y$ to extreme points of $X$, then pushing down also preserves affinity of a function.

3.3. **Definition of the entropy structure.** This section is almost identical as in the $Z$-case \[16\]. The only difference is that the cited theorem about the existence of a principal zero-dimensional extension in the general case (which we
will use below) is incomparably more intricate than that for \( \mathbb{Z} \)-actions. The rest follows the standard scheme and there are no essential differences. The entropy structure for an action of an amenable group \( G \) on a compact metric space \( X \) will be introduced in two steps. At first we will do it in case \( X \) is zero-dimensional, next we will address the general case. In both cases we assume finite topological entropy of \( X \).

### 3.3.1. The zero-dimensional case.

**Definition 3.4.** Let \( X = \lim_{k} X_{[1:k]} \) be a zero-dimensional system in its array representation. Let \( (\mathcal{P}_{k})_{k \in \mathbb{N}} \) denote the associated sequence of symbol partitions. For every \( \mu \in \mathcal{M}_{G}(X) \) define \( h_{k}(\mu) = h(\mu, \mathcal{P}_{[1:k]}) \) (also \( h_{k}(\mu) = h(\mu, X_{[1:k]}) \)), where \( \mu_{k} \) is the image of \( \mu \) on the first \( k \) layers \( X_{[1:k]} \) by the natural projection map \( \pi_{[1:k]} : X \to X_{[1:k]} \). Then the structure \( \mathcal{H} = (h_{k})_{k \geq 0} \) on the Choquet simplex \( \mathcal{M}_{G}(X) \) is called an entropy structure of \( X \).

We have the following crucial fact:

**Theorem 3.5.** The entropy structure is an affine structure with upper semicontinuous differences, converging nondecreasingly to the entropy function.

**Proof.** Affinity of the entropy function is a commonly known fact (see e.g. [17] Theorem 2.5.1]: the same proof applies to actions of all countable amenable groups). Each function \( h_{k} \) is in fact the entropy function on \( X_{[1:k]} \) lifted against the factor map \( \pi_{[1:k]} \). Since the factor map applied to invariant measures is affine, the lifted function is affine, too. The nondecreasing convergence to the entropy function is obvious. For upper semicontinuity of the differences, notice that, for any \( k \geq 0 \) and \( \mu \in \mathcal{M}_{G}(X) \), we have \( h_{k+1}(\mu) - h_{k}(\mu) = h(\mu, \mathcal{P}_{k+1}|\mathcal{P}_{[1:k]}) \) (for \( k = 0 \) there is no conditioning). Both involved partitions are clopen, i.e., their atoms have empty boundary. By Lemma 2.9 the discussed difference function is upper semicontinuous on \( \mathcal{M}_{G}(X) \).

There exist many entropy structures depending on the choice of the array representation, however, all these structures are uniformly equivalent (see below).

### 3.3.2. The general case.

There are many ways of introducing the entropy structure in actions of \( \mathbb{Z} \) on general compact metric spaces (see 16). Most of them (but not all) can be adapted to actions of general countable amenable groups. We choose one which seems to pass in a most direct manner.

By a deep result of Huczek (see 34 Theorem 2]), any action of a countable amenable group \( G \) on a compact metric space \( X \) has a principal zero-dimensional extension \( X' \). Let \( \pi' : X' \to X \) denote the corresponding factor map. We define the entropy structure of a topological dynamical system \( X \) following the idea from 17 Definition 5.0.1]. Recall that we assume finite topological entropy of \( X \).

**Definition 3.6.** The entropy structure of a topological dynamical system \( X \) of finite entropy is defined as any structure \( \mathcal{H} = (h_{k})_{k \geq 0} \) on \( \mathcal{M}_{G}(X) \), such that for any principal zero-dimensional extension \( \pi' : X' \to X \), and any entropy structure \( \mathcal{H}' = (h'_{k})_{k \geq 0} \) on \( \mathcal{M}_{G}(X') \), the structure \( \mathcal{H} = (h_{k})_{k \geq 0} \) lifted against \( \pi' \) is uniformly equivalent to \( \mathcal{H}' \).

Of course, it is a priori not obvious, that such a structure exists. Once the existence of at least one such structure \( \mathcal{H} \) is guaranteed, it becomes obvious that any other structure defined on \( \mathcal{M}_{G}(X) \) is an entropy structure if and only it is uniformly
equivalent to $\mathcal{H}$. In particular, it will be now obvious that if $X$ is zero-dimensional then the entropy structure from Definition 3.4 is consistent with Definition 3.6 and does not depend on the array representation.

The proof of existence is tedious and requires (for example) a concept of entropy via finite families of continuous functions (instead of partitions). We prefer not to copy entire sections from, for example, [16] or [17]. The construction does not depend, in any aspect, on the acting group, and its details play no role in this paper. So, we choose to formulate the existence theorem without a detailed proof and confine ourselves to suitable references.

**Theorem 3.7.** If $X$ has finite topological entropy, then it has an entropy structure which is affine and has upper semicontinuous differences.

**Proof.** Combine [17, Definition 6.2.1] (adapted to a Følner sequence $(F_n)_{n \in \mathbb{N}}$) and [17, Lemma 7.1.2] with part (1) of the proof of [17 Theorem 7.0.1]. The proofs for countable amenable groups are identical. In one place, for upper semicontinuity of a conditional entropy function, Lemma 2.9 (from this paper) is needed. □

In order to obtain a notion which does not depend on any choices, we will replace the “individual” entropy structures by entire uniform equivalence class. Nevertheless, instead of saying that $H$ belongs an entropy structure we will keep saying that it is an entropy structure. The entropy structure defined as a uniform equivalence class is an invariant of topological conjugacy in the following sense:

**Theorem 3.8.** Suppose $X$ and $Y$ are topologically conjugate, say, $\pi : Y \to X$ is the conjugating map. Then a structure $H = (h_k)_{k \geq 0}$ defined on $\mathcal{M}_G(X)$ is an entropy structure of $X$ if and only if $H \circ \pi = (h_k \circ \pi)_{k \geq 0}$ (defined on $\mathcal{M}_G(Y)$) is an entropy structure of $Y$.

**Proof.** Conjugate systems have the same principal zero-dimensional extensions. □

### 3.4. Symbolic extensions—the easy direction

Let us go back to the situation where $Y$ is a topological extension of $X$ via a map $\pi : Y \to X$.

**Definition 3.9.** In this context, on $\mathcal{M}_G(X)$ we define the extension entropy function as the push-down along $\pi$ of the entropy function on $\mathcal{M}_G(Y)$:

$$h^\pi(\mu) = \sup\{h(\nu, Y) : \nu \in \mathcal{M}_G(Y), \pi(\nu) = \mu\}.$$ 

**Definition 3.10.** Given a topological dynamical system $X$, on $\mathcal{M}_G(X)$ we define the symbolic extension entropy function, as the infimum of all extension entropy functions arising from symbolic extensions $Y$ of $X$:

$$h_{sex}(\mu) = \inf\{h^\pi(\mu) : \pi : Y \to X \text{ is a symbolic extension}\}.$$ 

Every symbolic system $Y$ has finite topological entropy (at most $\log |\Lambda|$, where $Y \subset \Lambda^G$), while, by convention, infimum of an empty set equals $+\infty$. Thus, lack of symbolic extensions of $X$ is equivalent to the condition $h_{sex} \equiv \infty$ on $\mathcal{M}_G(X)$ (otherwise $h_{sex}$ is always bounded).

We can also define the topological symbolic extension entropy of $X$, as

$$h_{sex}(X, G) = \inf\{h_{top}(Y, G) : Y \text{ is a symbolic extension of } X\}.$$
It can be proved, using the same methods as in the $\mathbb{Z}$-case (see \ref{thm:Z-case}), that
\[
h_{\text{sex}}(X, G) = \sup_{\mu \in \mathcal{M}_G(X)} h_{\text{sex}}(\mu).
\]
This equality, called the \textit{symbolic extension entropy variational principle}, is the reason why independent study of the topological symbolic extension entropy is of lesser interest. This paper is devoted to proving the analog of the following theorem for $\mathbb{Z}$-actions, known as the \textit{Symbolic Extension Entropy Theorem} \ref{thm:Z-case}.

**Theorem 3.11.** Let $X = (X, T)$ be a topological dynamical system (the $\mathbb{Z}$-action generated by a single homeomorphism $T : X \to X$ of a compact metric space $X$). If the topological entropy of $X$ is infinite then obviously $X$ admits no symbolic extensions. Otherwise we have the following equivalence: Let $E_A$ be a bounded function defined on $\mathcal{M}_T(X)$. There exists a symbolic extension $\pi : Y \to X$ such that $E_A = h^\pi$ if and only if $E_A$ is an affine superenvelope of the entropy structure of $X$. In particular, $h_{\text{sex}} \equiv \mathcal{E}H$, where $\mathcal{H}$ is (belongs to) the entropy structure of $X$ (this includes the infinite case: $\mathcal{E}H \equiv \infty$ if and only if $X$ has no symbolic extensions).

For actions of general countable amenable groups, one implication is relatively easy to prove, and the proof does not differ much from that for $\mathbb{Z}$-actions. As for the other implication, we encounter an (at the moment) inaccessible problem, and we must make a sacrifice: either add an assumption on $G$ or widen the notion of a symbolic extension. In both cases we will need more terminology, thus the formulation will be provided later. For now, we can prove the “easy direction”:

**Theorem 3.12.** Let a countable amenable group $G$ act on a compact metric space $X$. Let $\pi : Y \to X$ be a symbolic extension of $X$. Then the extension entropy function $h^\pi$ is an affine superenvelope of the entropy structure $\mathcal{H}$ of $X$.

**Proof.** One fact used in the proof of the easy direction for $\mathbb{Z}$-actions (that asymptotically $h$-expansive systems have principal symbolic extensions) is uncertain for general countable amenable groups. Thus the proof will change in one place. For the sake of completeness, we present it whole.

Recall that the entropy function on $\mathcal{M}_G(Y)$ is upper semicontinuous, affine and the factor map $\pi$ applied to the sets of invariant measures, $\pi : \mathcal{M}_G(Y) \to \mathcal{M}_G(X)$, is an affine surjection between Choquet simplexes, sending extreme points to extreme points (i.e., ergodic measures to ergodic measures). Now Proposition \ref{prop:extreme-points} implies that the extension entropy function $h^\pi$ is upper semicontinuous and affine. It remains to show that it is a superenvelope of the entropy structure of $X$.

We begin under the additional assumption that the space $X$ is zero-dimensional. Then we can choose the entropy structure $\mathcal{H} = (h_k)_{k \geq 0}$ obtained as $h_k(\mu) = h(\mu, \mathcal{P}_{[1,k]})$, where $(\mathcal{P}_k)_{k \in \mathbb{N}}$ is some jointly refining sequence of finite clopen partitions of $X$. Clearly, for each $k \geq 0$, we have $h^\pi \geq h \geq h_k$. By Proposition \ref{prop:extreme-points} (3), the only thing we need to show is that $h^\pi - h_k$ is upper semicontinuous. For each $k \in \mathbb{N}$ the partition $\mathcal{P}_{[1,k]}$ lifts against $\pi$ to a clopen partition of the symbolic space $Y$. By convention, this lifted partition will be denoted also by $\mathcal{P}_{[1,k]}$. Let $\Lambda$ denote the alphabet of the symbolic system $Y$. The symbolic partition $\mathcal{P}_\Lambda$ of $Y$ generates the entire Borel sigma-algebra, hence the entropy of $h(\nu, Y)$ of every invariant measure $\nu \in \mathcal{M}_G(Y)$ equals $h(\nu, \mathcal{P}_\Lambda)$. Obviously, it also equals $h(\nu, \mathcal{P}_\Lambda \vee \mathcal{P}_{[1,k]})$. On the other hand, $h(\nu, \mathcal{P}_{[1,k]}) = h(\mu, \mathcal{P}_{[1,k]})$,
where $\mu = \pi(\nu)$. So, for every $\nu \in \mathcal{M}_G(Y)$, the difference $h(\nu) - h_k(\pi(\nu))$ equals $h(\nu, \mathcal{P}_\lambda \vee \mathcal{P}_{[1,k]}) - h(\nu, \mathcal{P}_{[1,k]}) = h(\nu, \mathcal{P}_\lambda|\mathcal{P}_{[1,k]})$, and since both partitions are clopen, by Lemma 2.9, the considered difference function is upper semicontinuous on $\mathcal{M}_G(Y)$. Finally, by Proposition 3.3, the push-down of the function $\nu \mapsto h(\nu) - h_k(\pi(\nu))$ is upper semicontinuous. On the other hand, this push-down evaluated at a measure $\mu \in \mathcal{M}_G(X)$ equals $h^\pi(\mu) - h_k(\mu)$ (note that the function $\nu \mapsto h_k(\pi(\nu))$ is constant on fibers, thus it is not affected by pushing down). We have shown that $h^n - h_k$ is upper semicontinuous on $\mathcal{M}_G(X)$. So $h^n$ is indeed an affine superenvelope of $\mathcal{H}$.

Now we will address the general case. For that we will need a simple entropy lemma (which is the same as for $Z$-actions):

**Lemma 3.13.** Consider four topological dynamical systems $X, X', X''$ and $X'''$, where $X'$ and $X''$ are topological extensions of $X$ via factor maps $\pi'$ and $\pi''$, respectively, while $X'''$ is their fiber product:

$$X''' = \{(x', x'') : \pi'(x') = \pi''(x'')\} \subset X' \times X''.$$  
Assume that both $X'$ and $X''$ have finite topological entropy. Then, for any $\mu''' \in \mathcal{M}_G(X''')$ we have

$$h(\mu'''|X''') \leq h(\mu'|X'|, \mu''|X''),$$

where $\mu'$ and $\mu''$ are the projections of $\mu'''$ onto $X'$ and $X''$, respectively.

**Proof.** Clearly, $X'''$ is a topological extension of both $X'$ and $X''$ via the coordinate projections $\text{proj}_1$ and $\text{proj}_2$, respectively. Moreover, it is an extension of $X$ via $\pi' \circ \text{proj}_1 = \pi'' \circ \text{proj}_2$. With this notation, let $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ be finite measurable partitions of $X$, $X'$ and $X''$, respectively. By lifting, we can treat them as partitions of $X'''$. With this notation, we obviously have

$$h(\mu'''|\mathcal{P}') \leq h(\mu'''|\mathcal{P}' \vee \mathcal{P}),$$

which can be written as

$$h(\mu'''|\mathcal{P}' \vee \mathcal{P}'') - h(\mu'''|\mathcal{P}') \leq h(\mu'''|\mathcal{P}' \vee \mathcal{P}) - h(\mu'''|\mathcal{P}).$$

Notice that since $X'''$ is a joining of $X'$ and $X''$, partitions of the form $\mathcal{P}' \vee \mathcal{P}''$ generate the sigma-algebra in $X'''$. This implies, that if $(\mathcal{P}_k)_{k \in \mathbb{N}}$, $(\mathcal{P}'_k)_{k \in \mathbb{N}}$ and $(\mathcal{P}''_k)_{k \in \mathbb{N}}$ are refining sequences of partitions in $X$, $X'$ and $X''$, respectively, then applying the above to $\mathcal{P}_k$, $\mathcal{P}'_k$, $\mathcal{P}''_k$ and letting $k \to \infty$, and because all the terms below are finite, we will obtain

$$h(\mu'''|X''') - h(\mu'|X'|) \leq h(\mu', X').$$

or

$$h(\mu'''|X''') \leq h(\mu', X'|X).$$

The other case is symmetric. $\square$

We return to the main proof. Clearly, all four considered systems have finite topological entropy. Let $\mathcal{H}$ denote an entropy structure of $X$ with upper semicontinuous differences (see Theorem 3.7). Suppose that $Y$ is a symbolic extension of $X$. We pick a principal zero-dimensional extension $\pi' : X' \to X$, and a jointly refining sequence of clopen partitions $(\mathcal{P}'_k)_{k \in \mathbb{N}}$ of $X'$. The sequence $\mathcal{H}' = (h'_k)_{k \geq 0}$, where $h'_0(\mu') = h(\mu', \mathcal{P}'_{[1,k]})$ ($\mu' \in \mathcal{M}_G(X')$) is an entropy structure of $X'$ and, by definition of $\mathcal{H}$, $\mathcal{H}'$ is uniformly equivalent to $\mathcal{H}$ lifted against $\pi$ from $\mathcal{M}_G(X)$ to $\mathcal{M}_G(Y)$. Therefore, $\mathcal{H}'$ is upper semicontinuous.
are both clopen, so \( \in M \) is an upper semicontinuous function on \( P \) partitions \( X \) factor cannot argue directly on extension, we have \( \nu \) push-down \( M \) from \( M \)ous on \( M \)ous on \( M \). From this place onward, everything in this paper has but one goal: proving one more time, we obtain that the push-down along \( \pi \) \( H \) entropy structure \( h \) \( k \)ous on \( M \)ous on \( M \). From this place the proof differs from that in [5] or [17].

4. Quasitilings and tiling systems

From this place onward, everything in this paper has but one goal: proving (in possibly largest generality) the opposite direction of the Symbolic Extensions Entropy Theorem. Subsections 4.1.1 and 4.1.2 are based on the papers [19] and [20].

4.1. Terminology and facts not requiring amenability.

\[1\]From this place the proof differs from that in [5] or [17].
4.1.1. Banach density. Let $G$ be a countable group.

**Definition 4.1.** For a subset $B \subset G$ and a finite set $F \subset G$ we denote
\[
D_F(B) = \inf_{g \in G} \frac{|B \cap Fg|}{|F|} \quad \text{and} \quad \overline{D}_F(B) = \sup_{g \in G} \frac{|B \cap Fg|}{|F|},
\]
\[
\underline{D}(B) = \sup_{F \subset G} D_F(B) \quad \text{and} \quad \overline{D}(B) = \inf_{F \subset G} \overline{D}_F(B),
\]
where $F$ ranges over all finite subsets of $G$. The last two terms are called the lower and upper Banach density of $B$, respectively.

**Remark 4.2.** The notions of upper and lower Banach density have been studied from several points of view. For example, in [3] the reader will find a different definition. It can be shown that that definition is in fact equivalent to ours.

**Definition 4.3.** For two sets $A$ and $B$ of $G$ we define the following quantities
\[
D_F(B, A) = \inf_{g \in G} \frac{1}{|F|}(|B \cap Fg| - |A \cap Fg|), \quad D(B, A) = \sup_{F \subset G} D_F(B, A),
\]
where, as before, $F$ ranges over all finite subsets of $G$. The latter number will be called the Banach density advantage of $B$ over $A$ (which can be negative, but we will never consider such a case).

The following lemma will be repeatedly used in many of our considerations.

**Lemma 4.4.** Let $F, F_1$ be finite subsets of $G$ and let $A, B$ be some arbitrary subsets of $G$. If $F_1$ is $(F, \varepsilon)$-invariant then $D_{F_1}(B, A) \geq D_F(B, A) - 4\varepsilon$.

**Proof.** Given $g \in G$, we have
\[
|B \cap Fhg| - |A \cap Fhg| \geq D_F(B, A)|F|,
\]
for every $h \in F_1$. This implies that
\[
|\{(f, h) : f \in F, h \in F_1, fhg \in B\}| - |\{(f, h) : f \in F, h \in F_1, fhg \in A\}| \geq D_F(B, A)|F||F_1|.
\]
This in turn implies that there exists at least one $f \in F$ for which
\[
|B \cap fF_1g| - |A \cap fF_1g| \geq D_F(B, A)|F_1|.
\]
Since $f \in F$ and $F_1$ is $(F, \varepsilon)$-invariant (and hence so is $F_1g$), we have
\[
|B \cap fF_1g| - |B \cap F_1g| \leq |fF_1 \triangle F_1| = 2|fF_1 \setminus F_1| \leq 2|FF_1 \setminus F_1| \leq 2\varepsilon|F_1|,
\]
and the same for $A$, which yields
\[
|B \cap F_1g| - |A \cap F_1g| \geq (D_{F_1}(B, A) - 4\varepsilon)|F_1|.
\]
To end the proof, it remains to apply the infimum over all $g \in G$ on the left, and divide both sides by $|F_1|$. \square

A set $A \subset G$ is called syndetic (more precisely left syndetic) if there exists a finite set $U \subset G$ such that $UA = G$ (equivalently, for each $g \in G$, $A \cap U^{-1}g \neq \emptyset$). The set $U$ will be referred to as the syndeticity set for $A$, we will also say that $A$ is $U$-syndetic. In noncommutative groups left and right syndeticity are independent notions and throughout this paper right syndeticity will not be used. The following is an easy exercise:
Proposition 4.5. A set $A \subset G$ is syndetic if and only if it has positive lower Banach density. The lower Banach density is at least $\frac{1}{|T|}$, where $U$ is a syndeticity set for $A$.

A set $A \subset G$ is called $F$-separated (more precisely left $F$-separated), where $F$ is another finite subset of $G$, if the sets $Fg$ for $g \in A$ are pairwise disjoint. The upper Banach density of an $F$-separated set is at most $\frac{1}{|T|}$. Every $F$-separated set $A$ is contained in a maximal $F$-separated set $A'$ (i.e., such that $A' \cup \{g\}$ is not $F$-separated for any $g \in G \setminus A'$). Another nearly obvious fact is this:

Proposition 4.6. Any maximal $F$-separated set is $(F^{-1})$-syndetic.

4.1.2. Quasitilings.

Definition 4.7. A quasitiling of $G$ is a countable family $T$ of finite sets $T \subset G$, called the tiles, together with an injective map from $T$ to $G$ assigning to each tile $T$ a point $c_T \in T$ called the center of $T$. The image of this injection, i.e., the set $C(T) = \{c_T : T \in T\}$ will be referred to as the set of centers of $T$. For each tile $T$, the set $S_T = Tc_T^{-1}$ will be called the shape of $T$ (note that every shape contains the unit $e$). The collection of shapes $\{S_T : T \in T\}$ will be denoted by $S(T)$. Given $S \in S(T)$, the set $C_S = \{c_T \in C(T) : S_T = S\}$ will be called the set of centers for the shape $S$. Note that the sets of centers for different shapes are disjoint and their union over all shapes equals $C(T)$. A quasitiling $T$ is proper if the collection of shapes $S(T)$ is finite. From now on, by a quasitiling we shall always mean a proper quasitiling.

Definition 4.8. Let $\varepsilon \in [0, 1)$ and $\alpha \in (0, 1]$, and let $K \subset G$ be a finite set. A quasitiling $T$ is called

1. $(K, \varepsilon)$-invariant if all shapes of $T$ are $(K, \varepsilon)$-invariant;
2. $\varepsilon$-disjoint if there exists a mapping $T \mapsto T^\circ$ ($T \in T$) such that
   - $T^\circ$, a $(1-\varepsilon)$-subset of $T$ and
   - the family $\{T^\circ : T \in T\}$ is disjoint;
3. disjoint if the tiles of $T$ are pairwise disjoint;
4. $\alpha$-covering if $\mathcal{D}(\bigcup T) \geq \alpha$;
5. a tiling if it is a partition of $G$.

We have the following elementary fact:

Proposition 4.9. For any $0 < \varepsilon < 1$, the set of centers $C(T)$ of a $(1-\varepsilon)$-covering quasitiling $T$ is syndetic.

Proof. There exists a finite set $F$ such that for every $g \in G$, $\frac{|Fg \cap \bigcup T|}{|T|} \geq 1 - \varepsilon > 0$. In particular $Fg \cap \bigcup T \neq \emptyset$. Let $T_g$ denote a tile which intersects $Fg$. Since $T$ is proper, the set $V = \bigcup S(T)$ finite. The center $c_g$ of $T_g$ satisfies $T_gc_g^{-1} \in V$ hence there exists $f \in F$ with $fgc_g^{-1} \in V$ and thus $g \in F^{-1}V C(T)$. We have shown that $C(T)$ is $F^{-1}$-$V$-syndetic.

We are about to define dynamical quasitilings. For better differentiation of the notions, the quasitilings defined so far will be referred to as static. A static quasitiling $T$ can be identified with an element of the symbolic space $V^G$ where $V = \{“S” : S \in S(T)\} \cup \{0\}$. Namely, for each $S \in S(T)$ we place the symbol “$S$”
at all the centers \( c \in C_S \), and we place the symbol 0 at all remaining positions. Formally, we can write \( T = \{ T_g : g \in G \} \), where

\[
T_g = \begin{cases} 
  "S" ; & g \in C_S, S \in S(T), \\
  0 ; & g \notin C(T).
\end{cases}
\]

**Definition 4.10.** By a *dynamical quasitiling* with the finite collection of shapes \( S \) we will understand any subshift \( T \subset V^G \), where \( V = \{ "S" : S \in S \} \cup \{ 0 \} \) (the elements of \( T \) are interpreted as static quasitilings \( T \) with \( S(T) \subset S \)).

Every static quasitiling \( T \) generates a dynamical quasitiling \( T \) as its orbit-closure (under the shift action): \( T = \bar{O}(T) = \{ g(\bar{T}) : g \in G \} \).

**Lemma 4.11.** Pick an \( \epsilon \in (0,1) \). If \( T \) is an \( \epsilon \)-disjoint (resp. \( (1-\epsilon) \)-covering) static quasitiling with the collection of shapes \( S(T) \) then every element of \( \bar{O}(T) \) is an \( \epsilon \)-disjoint (resp. \( (1-\epsilon) \)-covering) static quasitiling with the collection of shapes contained in \( S \).

**Proof.** The properties of \( \epsilon \)-disjointness and \( (1-\epsilon) \)-covering are shift-invariant, so they pass to \( g(T) \) for all \( g \in G \). Next, \( \epsilon \)-disjointness is easily seen to be a closed property (i.e., inherited by limit points), so it passes to all elements of the orbit closure. It remains to show that if \( T \) is \( (1-\epsilon) \)-covering and for some sequence \( (g_n)_{n \in \mathbb{N}} \) of elements of \( G \) we have the convergence \( g_n(T) \to T' \), then \( T' \) is also \( (1-\epsilon) \)-covering. So, suppose that \( T' \) is not \( (1-\epsilon) \)-covering, i.e., there is a positive number \( a < 1-\epsilon \) such that \( T' \) is at most \( a \)-covering. This means that for every finite set \( F \subset G \) there exists \( g_F \in G \) such that

\[
\left| \bigcup T' \cap Fg_F \right| \leq a|F|.
\]

The convergence \( g_n(T) \to T' \) implies that \( \bigcup T' \cap Fg_F = \bigcup g_n(T) \cap Fg_F \) for all sufficiently large \( n \). We pick one such \( n \) and denote it by \( n_F \). Note that \( \bigcup g_{n_F}(T) = \bigcup Tg_{n_F}^{-1} \). Thus

\[
a|F| \geq \left| \bigcup Tg_{n_F}^{-1} \cap Fg_F \right| = \left| \bigcup T \cap Fg_Fg_{n_F} \right|.
\]

Because this holds for every finite \( F \subset G \), the lower Banach density of \( \bigcup T \) is at most \( a \), i.e., \( T \) is at most \( a \)-covering, a contradiction. \( \square \)

Sometimes, we will be using a convention by which the term “quasitiling” will have a slightly extended meaning. Namely, we will admit that in the set of shapes \( S(T) \) there are repeated terms treated as separate objects. In the symbolic representation, such shapes will be marked by different symbols. The process in which one shapes is marked by multiple (still finitely many) symbols will be referred to as *duplicating (the shapes).* A dynamical quasitiling after duplicating becomes a topological extension of the original. Duplicating does not affect any of the properties listed in Definition 4.8.

### 4.2. Terminology and facts requiring amenability.

#### 4.2.1. Banach density and syndeticity revisited.**If** \( G \) is a countable amenable group with a Følner sequence \( (F_n)_{n \in \mathbb{N}} \) then the upper and lower Banach densities can be evaluated as the limits along a Følner sequence:
Proposition 4.12. For any $A \subset G$, we have
\[ D(A) = \lim_{n \to \infty} D_{F_n}(A) \quad \text{and} \quad D'(A) = \lim_{n \to \infty} D_{F_n}(A). \]
and for any $A, B \subset G$ we also have
\[ D(B, A) = \lim_{n \to \infty} D_{F_n}(B, A). \]

Proof. We will prove the third equality. Then, plugging in $A = \emptyset$ we will get
the first equality and passing to the complement $B^c$ we will get the second equality.
The inequality $\limsup_{n \to \infty} D_{F_n}(B, A) \leq \sup\{D_F(B, A) : F \subset G, F \text{ is finite}\}$ is
obvious. It remains to show that
\[ \liminf_{n \to \infty} D_{F_n}(B, A) \geq \sup\{D_F(B, A) : F \subset G, F \text{ is finite}\}. \]
At the same time this will prove the existence of all three limits.

Let $F \subset G$ be a finite set. Given $\varepsilon > 0$, for any $n$ large enough $F_n$ is $(F, \varepsilon)$-
invariant, hence Lemma 4.13 implies that $\liminf_{n \to \infty} D_{F_n}(B, A) \geq D_F(B, A) - 4\varepsilon$.
Since $\varepsilon > 0$ is arbitrary, it can be ignored. \qed

Corollary 4.13. We have
\[ D(B) - D(A) \leq D(B, A). \]

Proof. Fix a Følner sequence $(F_n)_{n \in \mathbb{N}}$. By the above lemma, we can write
\[ D(B, A) = \lim_{n \to \infty} \inf_{g \in G} \frac{|B \cap F_n g| - |A \cap F_n g|}{|F_n|} \geq \lim_{n \to \infty} \inf_{g \in G} \frac{|B \cap F_n g|}{|F_n|} - \lim_{n \to \infty} \sup_{g \in G} \frac{|A \cap F_n g|}{|F_n|} = D(B) - D'(A). \]
\[ \Box \]

We also have the following:

Proposition 4.14. Upper Banach density is subadditive: for any $A, B \subset G$,
\[ D(A \cup B) \leq D(A) + D(B). \]

Proof. For every $n \in \mathbb{N}$ and $g \in G$, we have
\[ \frac{|(A \cup B) \cap F_n g|}{|F_n|} \leq \frac{|A \cap F_n g|}{|F_n|} + \frac{|B \cap F_n g|}{|F_n|}. \]
Hence,
\[ \sup_{g \in G} \frac{|(A \cup B) \cap F_n g|}{|F_n|} \leq \sup_{g \in G} \frac{|A \cap F_n g|}{|F_n|} + \sup_{g \in G} \frac{|B \cap F_n g|}{|F_n|}. \]
Passing to the limit over $n$ ends the proof. \qed

At some point, we will be needing the following elementary fact (this is basically
Lemma 3.4), where it is formulated using the language of quasitilings).

Lemma 4.15. Let $(A_k)_{k \geq 1}$ and $(g_k)_{k \geq 1}$ be a sequence of subsets of $G$, respectively,
and a sequence of elements of $G$ such that:
1. the union $\bigcup_{k=1}^\infty A_k$ is finite,
2. $A = \bigcup_{k=1}^\infty A_k g_k$ is a disjoint union.

\[ ^2 \text{We remark that in non-amenable groups, instead of taking the limit we would have to apply}
\text{the infimum over all finite sets $F$, which spoils the proof.} \]
For each \( k \) let \( B_k \subset A_k \) and let \( B = \bigcup_{k=1}^{\infty} B_k g_k \). Then
\[
D(B) \geq D(A) \cdot \inf_k \frac{|B_k|}{|A_k|}
\]

Proof. Let \( \alpha = \inf_k \frac{|B_k|}{|A_k|} \). Given \( n \in \mathbb{N} \) and \( g \in G \), denote
\[
A(n, g) = \bigcup\{A_k g_k : A_k g_k \subset F_n\}, \quad \text{and} \quad B(n, g) = \bigcup\{B_k g_k : A_k g_k \subset F_n\}.
\]
Clearly
\[
\frac{|B(n, g)|}{|A(n, g)|} \geq \alpha.
\]

Denote \( K = \bigcup_{k=1}^{\infty} A_k \) (by assumption, this is a finite set). As easily verified, the difference \( F_n g \cap A \setminus A(n, g) \) consists of such sets \( A_k g_k \) which are disjoint from the \( KK^{-1}\)-core of \( F_n g \). Clearly, this core equals \((F_n)_{KK^{-1}} g\), where \((F_n)_{KK^{-1}}\) is the \( KK^{-1}\)-core of \( F_n \). So,
\[
|F_n g \cap B| \geq |B(n, g)| \geq \alpha |A(n, g)| \geq \alpha (|F_n g \cap A| - |F_n \setminus (F_n)_{KK^{-1}}|).
\]

Taking infimum over all \( g \in G \) and dividing both sides by \( |F_n| \), we obtain that
\[
D_{F_n}(B) \geq \alpha D_{F_n}(A) + \epsilon_n
\]
where, by the property (2) in section 4.4.2, \( \epsilon_n \to 0 \) with \( n \). Applying the limit as \( n \) tends to infinity we complete the proof. \( \square \)

4.2.2. Special \( \epsilon \)-quasitilings. We begin with citing some theorems about the existence of special quasitilings in any countable amenable group \( G \) in which we fix a symmetric, centered Følner sequence \((F_n)_{n \in \mathbb{N}}\).

Theorem 4.16. \cite[Lemma 4.1]{19}, see also \cite{42} For any \( \epsilon > 0 \) there exists an integer \( r(\epsilon) \) such that for any \( n \) there exists a static \( \epsilon \)-quasitiling \( T \) of \( G \) with the collection of shapes \( S(T) \subset \{F_{n_1}, F_{n_2}, \ldots, F_{n_{r(\epsilon)}}\} \), where \( n < n_1 < n_2 < \cdots < n_{r(\epsilon)} \).

It is seen that, for large \( n \), the topological entropy of the dynamical quasitiling generated by the above quasitiling \( T \) is small (symbols other than zero appear with small upper Banach density). Since we need this entropy to be zero, we shall use a different combination of results:

Theorem 4.17. \cite[Theorem 6.1]{19} If \( G \) is a countable amenable group then there exists a free action of \( G \) on a zero-dimensional space, which has topological entropy zero.

Theorem 4.18. \cite[Lemma 3.4]{20} Given a free action of \( G \) on a compact metric zero-dimensional space \( X \), \( \epsilon > 0 \) and \( n \in \mathbb{N} \), there exists a dynamical \( \epsilon \)-quasitiling \( T \) of \( G \) with the collection of shapes \( S(T) \subset \{F_{n_1}, F_{n_2}, \ldots, F_{n_{r(\epsilon)}}\} \), where \( n < n_1 < n_2 < \cdots < n_{r(\epsilon)} \) (the dependence \( \epsilon \to r(\epsilon) \) is the same as in the preceding theorem), and which is a topological factor of \( X \).

Theorem 4.19. \cite[Corollary 3.5]{20} The above quasitiling \( T \) can be transformed to a quasitiling \( \hat{T} \) with the following properties
\begin{enumerate}
\item \( \hat{T} \) remains a topological factor of \( X \),
\item \( \hat{T} \) is disjoint and covers the same part of \( G \) as \( T \) (hence \( \hat{T} \) is \((1-\epsilon)\)-covering),
\end{enumerate}
(3) each shape of $\hat{T}$ is a $(1-\epsilon)$-subset of one of the shapes of $\hat{T}$.

The disjoint quasitiling $\hat{T}$ is created from the non-disjoint quasitiling $T$ by replacing the tiles of each $T \in T$ by their subsets. But then the centers may fall outside the new tiles and we need to perform the following adjustment of centers. For each shape $S \in S(T)$ we choose a point $a_S \in S$ (a “new center”). We define a new set of shapes $S' = \{Sa_S^{-1} : S \in S(T)\}$ (notice that each new shape contains the unity, as required). Next, for each $T \in T$ we rewrite each tile $\hat{T} = \hat{S}c \in \hat{T}$ ($\hat{S} \in \hat{S}(T)$ and $c \in C_S$), as $\hat{T} = T' = \hat{S'}c'$ where $\hat{S}' = \hat{S}a_S^{-1}$ and $c = a_sc$. We let $T'$ be the quasitiling with the same (disjoint) tiles as $\hat{T}$ but interpreted as $\hat{T}'$ rather than $\hat{T}$. Clearly, $T'$ is a quasitiling with the set of shapes $\hat{S}'$ and the center sets $C_{\hat{S}}$, $(\hat{S}' \in \hat{S}')$ equal to $a_S C_S$, where $C_S$ is the center set for the old shape $\hat{S} \in S(T)$. Since the quasitiling $\hat{T}$ is disjoint, the new center sets for different shapes in $\hat{S}'$ are disjoint, as required. It is clear that the mapping $\hat{T} \mapsto T'$ is a topological conjugacy between $\hat{T}$ and its image $\hat{T}'$. This map preserves the tiles and just moves the centers to new locations within the tiles.

As far as tilings are concerned, we have at our disposal the following general result:

**Theorem 4.20.** (follows from [19, Theorem 5.2]) For any countable amenable group $G$, any $\epsilon > 0$ and any finite set $K \subset G$, there exists a $(K, \epsilon)$-invariant tiling $T$ of $G$ of entropy zero.

As can be seen from the construction, the above tiling is “made from” the disjoint quasitiling of Theorem 4.10, and its collection of shapes can be divided into $r(\epsilon)$ classes such that each shape in the $i$th class is an $\epsilon$-modification of $F_i$, $(i = 1, 2, \ldots, r(\epsilon))$. However, the algorithm of creating the tiling from the quasitiling is not given by a block code (i.e., it is not a topological factor map), moreover, there is no estimate on the number of shapes of the tiling, which a priori can be uncontrollably large. Zero entropy is due to a relatively small number of configurations of tiles in Følner sets much larger than the tiles.

4.2.3. *Følner systems of quasitilings and tiling systems.* One dynamical quasitiling (or tiling) is insufficient for the construction of a symbolic extension. What we need is a countable joining of a sequence of quasitilings (tilings), with improving disjointness, covering and invariance properties. In case of quasitilings, this is all we are asking for. We make the following definition.

**Definition 4.21.** Let $(\epsilon_k)_{k \in \mathbb{N}}$ be a decreasing to 0 sequence of positive numbers. Let $T = \bigvee_{k \in \mathbb{N}} T_k$ be a topological joining of a sequence of dynamical quasitilings of $G$, such that for every $k \in \mathbb{N}$, $T_k$ is a (dynamical) $\epsilon_k$-quasitiling, and for every $\epsilon > 0$ and finite set $K \subset G$, for $k$ sufficiently large all shapes of $T_k$ are $(K, \epsilon)$-invariant. Such $T$ will be called a *Følner system of quasitilings*. The elements of $T$ will be denoted by $\mathcal{T} = (T_k)_{k \in \mathbb{N}}$ $(\forall_{k \in \mathbb{N}} T_k \in T_k)$. The collection of shapes of $T_k$, $\mathcal{S}(T_k)$, will be abbreviated as $\mathcal{S}_k$.

The term “Følner system of quasitilings” comes from the fact that it is a topological dynamical system and from the observation that the last requirement in the above definition can be formulated differently: the joint collection of shapes $\bigcup_{k \in \mathbb{N}} S_k$, indexed (bijectively, but in an arbitrary order) by natural numbers, is a Følner sequence in $G$. 

The existence of Følner systems of quasitilings in any countable amenable group follows directly from Theorem 4.16, also directly from Theorem 4.18 as we can find such a system as topological factor of any free action of G on a compact metric zero-dimensional space. As a corollary, there exist Følner systems of quasitilings with topological entropy zero. Using Theorem 4.19 we can even require the quasitilings to be disjoint.

We can use Theorem 4.20 to deduce a similar corollary for tilings, but we cannot claim that a Følner system of tilings (or even one dynamical tiling) appears as a topological factor in every free zero-dimensional action of G. We will claim this later under an additional assumption on G.

In case of a Følner system of tilings (rather than quasitilings) we can demand the members of the joining to “interact” with each other in a more specific manner. Two key such interactions are congruency and determinism, as defined below:

**Definition 4.22.** (1) A Følner system of tilings \( T = \bigvee_{k \in \mathbb{N}} T_k \) is *congruent* if for each \( T = (T_k)_{k \in \mathbb{N}} \in T \), for every \( k \in \mathbb{N} \), every tile of \( T_{k+1} \) is a union of some tiles of \( T_k \).

(2) A congruent Følner system of tilings \( T = \bigvee_{k \in \mathbb{N}} T_k \) is *deterministic*, if, for each \( k \in \mathbb{N} \) and every shape \( S' \in S_{k+1} \), there exist sets \( C_S(S') \subset S' \) \( (S \in S_k) \) such that \( S' \) equals the disjoint union

\[
S' = \bigcup_{S \in S_k} \bigcup_{c \in C_S(S')} S_c,
\]

and for each \( T = (T_k)_{k \in \mathbb{N}} \in T \), whenever \( S'c' \) is a tile of \( T_{k+1} \) then the sets \( S \in S_k \) and \( c \in C_S(S') \) are tiles of \( T_k \). We also define \( C_k(S') = \bigcup_{S \in S_k} C_S(S') \).

In the deterministic case, each static tiling \( T_{k+1} \) *determines* the tiling \( T_k \) joined with it, because each of the tiles of \( T_{k+1} \) is partitioned into the tiles of \( T_k \) in a unique way determined by its shape. Clearly, the assignment \( T_{k+1} \to T_k \) is given by a block code. Thus, the joining \( T \) is in fact an inverse limit

\[
T = \lim_{k \to} T_k.
\]

**Remark 4.23.** Any congruent Følner system of tilings \( T = (T_k)_{k \in \mathbb{N}} \) can be easily made deterministic in an inductive process of duplicating the shapes (see end of subsection 4.1.2), as follows: For each \( S' \in S_{k+1} \) there are only finitely many, say \( m(S') \), possible partitions of \( S' \) into tiles from \( S_k \). We duplicate the tile \( S' \) into \( m(S') \) copies (identical as subsets of \( G \), but in the symbolic representation of the tiling we will now associate to them different symbols, say \( "S'_1", "S'_2", \ldots, "S'_{m(S')}" \)) and use them for tiles with the original shape \( S' \) according to how the tile is subdivided by the tiles of \( T_k \). Clearly, this process increases the cardinality of \( S_k \) for each \( k \geq 2 \), but it is a topological conjugacy, so the dynamical properties (for example the topological entropy) of \( T \) remain unchanged.

**Definition 4.24.** A congruent, deterministic, Følner system of tilings will be briefly called a *tiling system*.

The already cited [19] Theorem 5.2 in full strength, translated to the terminology introduced above, states:
Theorem 4.25. If $G$ is a countable amenable group then there exists a tiling system of $G$ with topological entropy zero.

(Determinism is implicit in the proof in a way essential in obtaining topological entropy zero). As mentioned before, in general we cannot claim that an arbitrary free zero-dimensional action of $G$ has a tiling system as a factor.

4.3. Tiled entropy. Throughout this section we assume that $T = \lim_\leftarrow T_k$ is a tiling system of $G$ with topological entropy zero. Recall that the set of shapes of $T_k$ is denoted by $S_k$, and given $T = (T_k)_{k \in \mathbb{N}} \in T$, the set of centers of $T_k$ of the tiles with shape $S \in S_k$ is denoted by $C_S(T_k)$. The set of all centers of $T_k$ is $C(T_k) = \bigcup_{S \in S_k} C_S(T_k)$. We now introduce more notation. For $S \in S_k$, and $s \in S$ by $[S, s]$ we denote the set of elements $T \in T$ for which $s^{-1}$ belongs to $C_S(T_k)$. If $T \in [S, s]$ then $S_k^{-1}$ is the tile of $T_k$ which contains the unit, i.e., the the central tile of $T_k$. The set $[S, e]$ will be abbreviated as $[S]$. Observe that $T \in [S]$ if and only if $T_k(e) = "S"$, so the notation is consistent with that of one-symbol cylinders over the alphabet $V_k = \{ "S" : S \in S_k \} \cup \{ 0 \}$. The family $D_S = \{ [S, s] : S \in S_k, s \in S \}$ is a partition of $T$. Also note that $T \in [S, s]$ if and only if $s^{-1}(T) \in [S]$, i.e., $[S, s] = s([S])$. So, if $\nu$ is a shift-invariant measure on $T$, then $\nu([S, s]) = \nu([S])$ for all $s \in S$.

Recall, that by congruency and determinism of the tiling system, whenever $k' > k$, every shape $S'$ of $T_{k'}$, decomposes in a unique way as a concatenation of tiles of $T_k$ and the set of centers of these tiles is denoted by $C_k(S')$. The subset of $C'_k(S')$ consisting of centers of tiles with a particular shape $S \in S(T_k)$ is denoted by $C_S(S')$ (now the subscript $k$ is not needed; $k$ is determined by $S$).

Let $G$ act on a zero-dimensional compact metric space $X$ given in its array representation $X = \lim_\leftarrow X_{[1,k]}$, and let us assume that $X$ has the tiling system $T$ as a topological factor (in the following chapter, we will replace $X$ by its joining with $T$, so this assumption will be fulfilled). Then we can combine the layers of $X$ (i.e., the subshifts $X_k$) with the layers of $T$ (i.e., the dynamical tilings $T_k$) by replacing each $X_k$ by its topological joining $X_k$ with $T_k$ realized naturally in the common extension $X$. The combined alphabet of $X_k$ is $\Lambda_k = \Lambda_k \times V_k$, while that of $X_{[1,k]}$ is $\Lambda_{[1,k]} = \prod_{i=1}^k \Lambda_i$. In this manner, the system on $X$ is replaced by its topologically conjugate model which is the inverse limit $\bar{X} = \lim_\leftarrow \bar{X}_{[1,k]}$. We will call $X$ the tiled array representation of $X$.

We will use the following notational convention. For $\bar{x} \in \bar{X}$ and $S \in S_k$, the expression $\bar{x}_g = "S"$ means that in the tiling $T_k$ apparent in the $k$th layer of $\bar{x}$, at the position $g$ there occurs a center of a tile with shape $S$. Formally, this means that if $\bar{x}_k$ is the $k$th layer of $\bar{x}$ then

$$\bar{x}_k(g) \in \Lambda_k \times \{ "S" \} \subset \bar{X}_k.$$ 

With this convention, the notions $[S]$ and $[S, s]$ ($S \in S_k$, $s \in S$) may be applied to $X$ in the following way:

$$[S] = \{ \bar{x} \in \bar{X} : \bar{x}_e = "S" \}, \quad [S, s] = s([S]) = \{ \bar{x} \in \bar{X} : \bar{x}_{s^{-1}} = "S" \}.$$ 

4.3.1. Tiled entropy and its monotonicity. We continue to assume that a zero-dimensional dynamical system $X$ has a tiling system $T$ as a topological factor and we denote by $\bar{X}$ the tiled array representation of $X$. 
Definition 4.26. Let $\mathcal{P}$ and $\mathcal{Q}$ be two finite measurable partitions of $\bar{X}$, $|\mathcal{P}| > 1$. Let $\mu$ be a probability measure on $\bar{X}$. By the $k^{th}$ tiled entropy of $\mathcal{P}$ and $k^{th}$ conditional tiled entropy of $\mathcal{P}$ with respect to $\mu$ we will mean the following terms:

$$H_{T_k}(\mu, \mathcal{P}) = \sum_{S \in S_k} \mu([S]) H(\mu|_{[S]}, \mathcal{P}^S), \quad H_{T_k}(\mu, \mathcal{P}|\mathcal{Q}) = \sum_{S \in S_k} \mu([S]) H(\mu|_{[S]}, \mathcal{P}^S|\mathcal{Q}^S),$$

where $\mu([S])$ is the normalized conditional measure $\mu$ on $[S]$.

Alternatively, one can define just the unconditional version and then put

$$H_{T_k}(\mu, \mathcal{P}|\mathcal{Q}) = H_{T_k}(\mu, \mathcal{P} \cup \mathcal{Q}) - H_{T_k}(\mu, \mathcal{Q}).$$

In general, the tiled entropy cannot be easily reduced to a standard notion of conditional entropy (except in some cases, see formula (4.7) below) and indeed requires a separate definition; it resembles a conditional entropy given the partition $\mathcal{D}_{S_k}$, but it takes into account only selected elements of this partition (the cylinders $[S], S \in S_k$) and on each cylinder a different power of $\mathcal{P}$ is considered.

Theorem 4.27. On the simplex of invariant measures $\mathcal{M}_G(\bar{X})$, the sequences of tiled entropies $(H_{T_k}(\mu, \mathcal{P}))_{k \in \mathbb{N}}$ and $(H_{T_k}(\mu, \mathcal{P}|\mathcal{Q}))_{k \in \mathbb{N}}$, converge decreasingly to $h(\mu, \mathcal{P})$ and $h(\mu, \mathcal{P}|\mathcal{Q})$, respectively.

Proof. We begin by showing that $H_{T_{k+1}}(\mu, \mathcal{P}) \leq H_{T_k}(\mu, \mathcal{P})$. The proof for the conditional entropy is identical (we only use subadditivity). Recall that each shape $S' \in S_{k+1}$ decomposes as a disjoint union of tiles of $\mathbb{T}_k$:

$$S' = \bigcup_{S \in S_k} \bigcup_{c \in \mathcal{C}_S(S')} S_c.$$

Consider a point $\bar{x} \in [S]$, i.e., such that in the tiling $T_k$ associated to $\bar{x}$, a tile of shape $S$ occurs centered at $e$. Let $S'e^{-1}$ denote the central tile in the tiling $T_{k+1}$ associated to $\bar{x}$ (i.e., the tile containing $e$). Then $\bar{x} \in [S', e] = e([S'])$. By congruency, $S'e^{-1}$ contains the tile $S$ as its component in the decomposition into the tiles from $S_k$. Equivalently, $S'$ contains $S'c$ in its decomposition, which means that $c \in \mathcal{C}_S(S')$. We conclude that

$$[S] = \bigcup_{S' \in S_{k+1}} \bigcup_{c \in \mathcal{C}_S(S')} [S', c],$$

which is a (disjoint) union of some atoms of the partition $\mathcal{D}_{S_{k+1}}$. Obviously, we have

$$H(\mu|[S], \mathcal{P}^S) \geq H(\mu|[S], \mathcal{P}^S|\mathcal{D}_{S_{k+1}}),$$

and, according to (4.3), whenever $S' \in S_{k+1}$ and $c \in \mathcal{C}_S(S')$, one has $(\mu|[S])[S', c] = \mu([S', c])$. Thus the conditional entropy on the right equals

$$\sum_{S' \in S_{k+1}} \sum_{c \in \mathcal{C}_S(S')} \frac{\mu([S', c])}{\mu([S])} H(\mu|[S', c], \mathcal{P}^S).$$

On the other hand, the term $H(\mu|[S'], \mathcal{P}^S)$ represents the entropy of $\mathcal{P}^S$ restricted to $[S']$ (and with regard to the normalized measure on $[S']$). By the decomposition
of $S'$ (see (1.2)) and subadditivity of entropy (and using invariance of $\mu$ for the first and second equalities), we have

$$H(\mu_{|S'|}, P^{S}) \leq \sum_{S \in S_k} \sum_{c \in C_S(S')} H(\mu_{|S'|}, P^{S_c}) = \sum_{S \in S_k} \sum_{c \in C_S(S')} H(\mu_{\ell(S')}, P^{S}) = \frac{1}{[S']} \sum_{S \in S_k} \sum_{c \in C_S(S')} \mu([S', c]) H(\mu_{[S', c]}, P^{S}).$$

After multiplying both sides by $\mu([S'])$ and summing over $S' \in S_{k+1}$, we obtain

$$H_{T_{k+1}}(\mu, P) \leq \sum_{S' \in S_{k+1}} \sum_{S \in S_k} \sum_{c \in C_S(S')} \mu([S', c]) H(\mu_{[S', c]}, P^{S}) = \sum_{S \in S_k} \sum_{S' \in S_{k+1}} \sum_{c \in C_S(S')} \frac{\mu([S', c])}{\mu([S])} H(\mu_{[S', c]}, P^{S}) = \sum_{S \in S_k} \mu([S]) H(\mu_{[S]}, P^{S}|D_{S_{k+1}}) \leq \sum_{S \in S_k} \mu([S]) H(\mu_{[S]}, P^{S}) = H_{T_k}(\mu, P).$$

We pass to proving the convergence to the appropriate limits. It suffices to prove the unconditional version; the conditional version will follow straightforwardly, by subtraction (recall that $H_{T_k}(\mu, P|Q) = H_{T_k}(\mu, P \vee Q) - H_{T_k}(\mu, Q)$). We will show that given $k \in \mathbb{N}$ and $\delta > 0$, we have

$$\frac{1}{|F_n|} H(\mu, P^{F_n}) \leq H_{T_k}(\mu, P) + \delta$$

for all sufficiently large $n$, and conversely, that given $n \in \mathbb{N}$ and $\delta > 0$ we have

$$H_{T_k}(\mu, P) \leq \frac{1}{|F_n|} H(\mu, P^{F_n}) + \delta$$

for all sufficiently large $k$. This will imply the desired convergence.

Fix $k$ and $\delta$. From now on we will skip the index $k$ in objects associated to the tiling $T_k$ (for instance, the set of tiles of $T$ will be denoted by $S$). Let $\gamma = \frac{\delta}{2 \log |P|}$. Let $n$ be so large that $F_n = (\bigcup S, \bigcup F_{\gamma(n)})$-invariant. We will abbreviate $F_n$ as $F$. Given a tiling $T \in T$ let $F_T$ denote collection of all tiles of $T$ with centers in $F$, i.e.,

$$F_T = \{ S \in S : c \in C_S(T) \cap F \}.$$  

We also let $F_T = \bigcup F_T$. The parameters of the invariance of $F$ were selected so that $F_T$ is a $\gamma$-modification of $F$ ($F_T$ is contained in $\bigcup S F$ and contains the $(\bigcup S)^{-1}$-core of $F$; now use property (3) above Definition 2.4). With $T$ ranging over $T$, there are finitely many possibilities for $F_T$. Let $D$ denote the (obviously finite and measurable) partition of $\bar{X}$ according to which of these possibilities occurs. For each $D \in D$ the (common for all $T \in D$) corresponding family $F_T$ will be denoted by $F_D$. The union $F_D = \bigcup F_D$ is a $\gamma$-modification of $F$. Also, for any $S \in S$, by $C^D_S(F)$ we will denote the (common for all $T \in D$) set of elements of $F$ which are centers of tiles of $T$ of the shape $S$. We can write

$$H(\mu, P^{F}) \leq H(\mu, P^{F}|D) + H(\mu, D) = \sum_{D \in D} \mu(D) H(\mu_D, P^{F}) + H(\mu, D) \leq \sum_{D \in D} \mu(D) H(\mu_D, P^{F_D}) + \mu + H(\mu, D) = L + \eta + H(\mu, D),$$
where \( L = \sum_{D \in \mathcal{D}} \mu(D)H(\mu_D, \mathcal{P}^D) \) and \( \eta = \gamma|F|\log(|\mathcal{P}|) = |F|^\frac{\delta}{2} \). We need to estimate the term \( L \) (we will take care \( H(\mu, \mathcal{D}) \) later). By subadditivity of entropy,

\[
H(\mu_D, \mathcal{P}^D) \leq \sum_{S \in \mathcal{S}} \sum_{c \in C_D^\mathcal{P}(F)} H(\mu_D, \mathcal{P}^{S_c}),
\]

and by invariance of \( \mu \), \( H(\mu_D, \mathcal{P}^{S_c}) \) can be replaced by \( H(\mu_{c(D)}, \mathcal{P}^S) \). So,

\[
L \leq \sum_{D \in \mathcal{D}} \mu(D) \sum_{S \in \mathcal{S}} \sum_{c \in C_D^\mathcal{P}(F)} H(\mu_{c(D)}, \mathcal{P}^S) = \sum_{S \in \mathcal{S}} \sum_{D \in \mathcal{D}} \sum_{c \in C_D^\mathcal{P}(F)} \mu(c(D))H(\mu_{c(D)}, \mathcal{P}^S).
\]

Every set \( c(D) \) with \( D \in \mathcal{D} \) and \( c \in C_D^\mathcal{P}(F) \) is contained in \([S]\). Moreover, every point \( \bar{x} \in [S] \) belongs to the sets \( c(D) \) for exactly \(|F|\) pairs \((D, c)\) with \( D \in \mathcal{D} \), \( c \in C_D^\mathcal{P}(F) \), each time for a different value of \( c \) (although we do not claim that for different pairs \((D, c)\) the sets \( c(D) \) are always different). Indeed, for every \( c \in F \) the point \( c^{-1}(\bar{x}) \) has a tile of shape \( S \) centered at \( c \), hence it belongs to some \( D \in \mathcal{D} \) such that \( c \in C_D^\mathcal{P}(F) \), and then \( \bar{x} \) belongs to \( c(D) \). If \( \bar{x} \) belonged to \( c(D) \) for more than \(|F|\) pairs \((D, c)\) some value of \( c \) would have to repeat, implying that \( c^{-1}(\bar{x}) \) would belong to two different sets \( D \), which is impossible. From these facts we conclude that for each \( c \in F \), the family \( \{c(D) : D \in \mathcal{D} \text{ such that } c \in C_D^\mathcal{P}(F)\} \) is a partition of \([S]\). We will denote it by \( \mathcal{E}^c_{[S]} \).

Since for each \( E \in \mathcal{E}^c_{[S]} \) we have \((\mu_{[S]}|_E) = \mu_E \), the last triple sum can be rearranged, as follows:

\[
\sum_{S \in \mathcal{S}} \sum_{c \in F} \sum_{E \in \mathcal{E}^c_{[S]}} \mu(E)H(\mu_E^S, \mathcal{P}^S) = \sum_{E \in \mathcal{E}^c_{[S]}} \mu([S])H(\mu_{[S]}, \mathcal{P}^S|_{\mathcal{E}^c_{[S]}}) \leq \sum_{E \in \mathcal{E}^c_{[S]}} \sum_{S \in \mathcal{S}} \mu([S])H(\mu_{[S]}, \mathcal{P}^S) = |F|H_T(\mu, \mathcal{P}).
\]

We have obtained

\[
\frac{1}{|F|}H(\mu, \mathcal{P}^F) \leq H_T(\mu, \mathcal{P}) + \frac{\delta}{2} + \frac{1}{|F|}H(\mu, \mathcal{D}).
\]

The partition \( \mathcal{D} \) depends solely on the tiling \( T \) restricted to the set \( F \). We can write this as \( \mathcal{D} \preccurlyeq V^F \) (recall that \( V \) is the alphabet used by the tiling \( T \); here it is identified with the zero-coordinate partition of \( T \)). Thus \( \frac{1}{|F|}H(\mu, \mathcal{D}) \leq \frac{1}{|T|}H(\mu, V^F) \). Because our tiling \( T \) has topological entropy zero, by the choice of large enough \( F = F_n \), this term can be made smaller than \( \frac{\delta}{2} \). This ends the proof of the inequality \([18]\).

For the other inequality, \([18]\), we will use Shearer’s inequality, which is weaker than strong subadditivity and thus holds for unconditional entropy (see e.g. \([18]\)).

Having fixed \( \delta \) and \( n \) we will abbreviate \( F_n \) as \( F \). Let \( k \) be so large that every shape \( S \) of \( T_k \) is \((F, \gamma)\)-invariant where \( \gamma = \frac{\delta}{|T|\log(|\mathcal{P}|)} \). From now on we will skip the index \( k \) in objects associated to the tiling \( T_k \). In the definition of \( H_T(\mu, \mathcal{P}) = \sum_{S \in \mathcal{S}} \mu([S])H(\mu_{[S]}, \mathcal{P}^S) \) we will estimate the term \( H(\mu_{[S]}, \mathcal{P}^S) \). First, we replace it by \( H(\mu_{[S]}, \mathcal{P}^S) \), where \( \tilde{S} \) is the \( F^{-1} \)-core of \( S \) (the set of points \( g \in S \) such
that $F^{-1}g \subset S)$. It follows from the property (2) above Definition 243 that $\tilde{S}$ is a $(1 - |F|\gamma)$-subset of $S$. Thus

$$H(\mu_{[S]}, \mathcal{P}_{\tilde{S}}) \leq H(\mu_{[S]}, \mathcal{P}_S) + |S| |F| \gamma \log |P| + H(\mu_{[S]}, \mathcal{P}_{\tilde{S}}) + |S|\delta.$$ 

It remains to estimate $H(\mu_{[S]}, \mathcal{P}_{\tilde{S}})$. Consider the family $\{Fs : s \in S\}$. Every element of $\tilde{S}$ is contained in precisely $|F|$ sets from this family (it belongs to all $Fs$ with $s \in F^{-1}g$). That is to say, the above family is an $|F|$-cover of $\tilde{S}$, and the Shearer’s inequality applies, yielding

$$H(\mu_{[S]}, \mathcal{P}_{\tilde{S}}) \leq \frac{1}{|F|} \sum_{s \in S} H(\mu_{[S]}, \mathcal{P}_{Fs}) = \frac{1}{|F|} \sum_{s \in S} H(\mu_{[S,s]}, \mathcal{P}_F),$$

by invariance of $\mu$. So,

$$H_\gamma(\mu, \mathcal{P}) \leq \frac{1}{|F|} \sum_{S \in S} \sum_{s \in S} \mu([S,s]) H(\mu_{[S,s]}, \mathcal{P}_F) + \delta$$

(to obtain $\delta$ at the end we have used $|S| \sum_{S \in S} \mu([S]) = 1$). Because the family $\{[S,s] : S \in S, s \in S\}$ is the partition $\mathcal{D}_S$ of $X$, we have obtained:

$$H_\gamma(\mu, \mathcal{P}) \leq \frac{1}{|F|} \sum_{S \in S} H(\mu_{[S]}, \mathcal{P}_F|\mathcal{D}_S) + \delta \leq \frac{1}{|F|} H(\mu, \mathcal{P}_F) + \delta,$$

and the proof is finished. \qed

4.3.2. The language of rectangles. In this section we introduce the key objects in the construction of symbolic extensions, the rectangles. Although for actions of general countable amenable group these objects no longer resemble rectangles (more appropriate would be calling them “stacks”), still, by analogy to $\mathbb{Z}$-actions, we will use the term “$k$-rectangles”.

Let $X$ be the tiled array representation of some zero-dimensional action of $G$, which has the tiling system $T$ as a topological factor. We continue to use the notation from the preceding subsection.

**Definition 4.28.** Given $k \in \mathbb{N}$, by a $k$-rectangle (extended $k$-rectangle) we will mean any block $R \in \hat{\Lambda}^S_{[1,k]}$ (resp. $\hat{R} \in \hat{\Lambda}^{[1,k]} \times \hat{\Lambda}^S_{[k+1]}$), where $S \in \mathcal{S}_k$, which occurs in some $x \in [S] \subset X$. In particular, $R$ (resp. $\hat{R}$) has the symbol “$S$” at the position $e$ of the $k$th layer. In either case, $S$ will be referred to as the shape of the $k$-rectangle $R$ (resp. extended $k$-rectangle $\hat{R}$) and $R$ (resp. $\hat{R}$) will be called a $k$-rectangle (resp. extended $k$-rectangle) over $S$. By $|R|$ (resp. $|\hat{R}|$) we will always mean the size $|S|$ of the shape. The collection of all $k$-rectangles (resp. extended $k$-rectangles) will be denoted by $\mathcal{R}_k$ (resp. $\mathcal{R}_{\hat{k}}$). We will also denote

$$\mathcal{R}_S = \{R \in \mathcal{R}_k : \text{the shape of } R \text{ is } S\} \quad (S \in \mathcal{S}_k),$$

$$\mathcal{R}_S = \{\hat{R} \in \mathcal{R}_{\hat{k}} : \text{the shape of } \hat{R} \text{ is } S\} \quad (S \in \mathcal{S}_k),$$

$$\mathcal{R} = \bigcup_{k \in \mathbb{N}} \mathcal{R}_k \quad \text{and} \quad \mathcal{R} = \bigcup_{k \in \mathbb{N}} \mathcal{R}_{\hat{k}}.$$ 

By congruency and determinism of the sequence of tilings, any $(k+1)$-rectangle $R'$ is a concatenation of several (precisely $|C_k(S')|$) shifted horizontal rectangles, and the projection of $R'$ on the first $k$ layers (denoted by $R'_{[1,k]}$) is a concatenation of $|C_k(S')|$ shifted $k$-rectangles. Although,
the component \(k\)-rectangles (extended \(k\)-rectangles) are, in the general case, not linearly ordered, we will write these concatenations (also ignoring the shifting of the components) as

\[
R' = \hat{R}^{(1)} \hat{R}^{(2)} \ldots \hat{R}^{(q)}, \quad R'_{[1,k]} = R^{(1)} R^{(2)} \ldots R^{(q)}
\]

(\(\hat{R}^{(i)} \in \hat{R}_k, R^{(i)} \in R_k, i = 1, 2, \ldots, q\), \(q = |C_k(S')|, S' \in S_{k+1} \) is the shape of \(R'\)). This will not lead to a confusion, as long as we are only interested in quantitative parameters of the concatenation. (Formally, in writing \(R' = \hat{R}^{(1)} \hat{R}^{(2)} \ldots \hat{R}^{(q)}\)) we make one more imprecision: the concatenation on the right is missing the symbol “\(S'\)” at the position \(e\) and zeros at other positions of the \((k+1)\)st layer. This should cause no confusion.)

With each \(k\)-rectangle \(R \in R_k\) (extended \(k\)-rectangle \(\tilde{R} \in \hat{R}_k\)) we will associate its cylinder set

\[
[R] = \{x \in \bar{X} : \bar{x} \in [S], \bar{x}_{[1,k]}|_S = R\}, \quad [\tilde{R}] = \{x \in \bar{X} : \bar{x} \in [S], \bar{x}_{[1,k+1]}|_S = \tilde{R}\},
\]

where \(S \in S_k\) is the shape of \(R\) (and of \(\tilde{R}\)), and \(\bar{x}_{[1,k]}\) (resp. \(\bar{x}_{[1,k+1]}\)) is the projection of \(\bar{x}\) on the first \(k\) (resp. \(k+1\)) layers. For any \(R \in R_k\) we have

\[
(4.6) \quad [R] = \bigcup \{[\tilde{R}] : \tilde{R} \in \hat{R}_k, \tilde{R}_{[1,k]} = R\},
\]

where \(\tilde{R}_{[1,k]}\) is the \(k\)-rectangle obtained by projecting \(\tilde{R}\) on the first \(k\) layers. For a fixed \(S \in S_k\), with a slight abuse of notation (by identifying the \(k\)-rectangles or extended \(k\)-rectangles with their cylinders), we can view \(\hat{R}_S\) and \(R_S\) as partitions of \([S]\), and then \(\hat{R}_S \cong R_S\).

The language of rectangles will play a crucial role in the forthcoming considerations. In particular, a special case of conditional tiled entropy can be conveniently expressed using rectangles. Let \(\mu\) be a probability measure on \(\bar{X}\) and fix some \(k \in \mathbb{N}\). Consider the 4th conditional tiled entropy

\[
H_{T_k}(\mu, \Lambda_{k+1} | \Lambda_{[1,k]}) = \sum_{S \in S_k} \mu([S]) H(\mu|[S], \Lambda_{k+1}^S | \Lambda_{[1,k]}^S)
\]

(where \(\Lambda_{k+1}\) and \(\Lambda_{[1,k]}\) are considered as symbol partitions of \(\bar{X}\)). For each \(S \in S_k\), all points \(\bar{x} \in [S]\) have the symbol “\(S'\)” in row \(k\) at the position \(e\). By determinism of the tiling system, this determines all other symbols from the alphabets \(V_l\) with \(l \leq k\) at all positions within \(S\). In other words, \([S]\) is contained in one atom of the partition \(V_l^S_{[1,k]}\). This implies that on \([S]\), the partitions \(\Lambda_{[1,k]}^S\) and \(\Lambda_{[1,k]}\) are identical. Furthermore, the latter partition coincides with \(\hat{R}_S\) (which is the same as \(R_k\) restricted to \([S]\)). Likewise, the partition \(\Lambda_{[1,k+1]}^S\) coincides on \([S]\) with \(R_S\), which is the same as \(\hat{R}_k\) restricted to \([S]\). We conclude that

\[
H_{T_k}(\mu, \Lambda_{k+1} | \Lambda_{[1,k]}) = \sum_{S \in S_k} \mu([S]) H(\mu|[S], \hat{R}_k | R_k).
\]

This looks very much like a conditional entropy, however, \(\sum_{S \in S_k} \mu([S])\) does not equal 1. It equals \(\mu([C_k])\), where \([C_k]\) is the set of points \(\bar{x}\) which have a tile of \(T_k\) centered at \(e\) (or \(\bar{x}_e = "S'\) for some \(S \in S_k\). If \(\mu_{[C_k]}\) denotes the normalized measure \(\mu\) restricted to \([C_k]\) then \(\mu_{[C_k]}([S]) = \frac{\mu([S])}{\mu([C_k])}\). Moreover, since \(\mu_{[S]}\) is
already normalized, there is no difference between \( \mu_{[S]} \) and \( (\mu_{[C_k]})_{[S]} \). Applying this normalization, we obtain

\[
H_{T_k}(\mu_{[\Lambda_{k+1}]}|_{[\Lambda_{1,k}]}) = \\
\mu((C_k)) \sum_{S \in S_k} \mu_{[C_k]}([S]) H((\mu_{[C_k]}|[S]), \hat{\mathcal{R}}_k|\mathcal{R}_k) = \mu((C_k)) H(\mu_{[C_k]}, \hat{\mathcal{R}}_k|\mathcal{R}_k \lor \mathcal{S}_k),
\]

where \( \mathcal{S}_k, \mathcal{R}_k \) and \( \hat{\mathcal{R}}_k \) are viewed as partitions of \([C_k]\) (indeed, the set \([C_k]\) consists of all points which have the central tile of \(T_k\) centered at \(e\), and the above three partitions classify such points according to the shape of the central tile, the \(k\)-rectangle and the extended \(k\)-rectangle over that tile, respectively). But notice that \( \mathcal{R}_k \supseteq \mathcal{S}_k \), because each \(k\)-rectangle \(R\) carries information about its shape (indeed, the symbol in the \(k\)th layer of \(R\) at the position \(e\) is “\(S\)”, which encodes the shape \(S\) of \(R\)). So, the conditioning with respect to \( \mathcal{S}_k \) can be skipped and we have just proved the following, very useful formula:

\[
(4.7) \quad H_{T_k}(\mu_{[\Lambda_{k+1}]}|_{[\Lambda_{1,k}]}) = \mu((C_k)) H(\mu_{[C_k]}, \hat{\mathcal{R}}_k|\mathcal{R}_k).
\]

Next, with each \(k\)-rectangle we will associate a (usually not invariant) empirical measure:

**Definition 4.29.** Let \(k \in \mathbb{N}\). For each \(k\)-rectangle \(R \in \mathcal{R}_k\) we select one point \(\bar{x}_R\) belonging to the cylinder \([R]\), and we define the **empirical measure associated with** \(R\), as follows:

\[
\mu^R = \frac{1}{|R|} \sum_{g \in S} \delta_g(\bar{x}_R),
\]

where \(S \in \mathcal{S}_k\) is the shape of \(R\).

Although the definition depends on the choice of the point \(\bar{x}_R\), this choice will turn out to be of no importance. This is why we skip \(\bar{x}_R\) in the denotation of \(\mu^R\).

Recall, that one of the key properties of a Følner system of tilings is that the shapes form a Følner sequence, which implies that the measures \(\mu^R\) have the general form \(\mu^F_{\bar{x}}\), as defined prior to Proposition 4.25, and by that proposition, for sufficiently large \(k\), lie in a small neighborhood of \(\mathcal{M}_G(\bar{X})\). The lemma below shows that if the shapes of the \(k\)-rectangles are large enough then the measures \(\mu^R\) (more precisely, their projections \(\mu^R_{[1,k]}\) on \(X_{[1,k]}\)) depend insignificantly on the choice of the points \(\bar{x}_R\).

**Lemma 4.30.** Choose some \(\delta > 0\). Let \(R \in \mathcal{R}_k\), where \(k \in \mathbb{N}\). If the shape \(S\) of \(R\) is a sufficiently far member of a Følner sequence then for any two points \(\bar{x}, \bar{x}' \in [R]\), the empirical measures

\[
\mu^R = \frac{1}{|R|} \sum_{s \in S} \delta_s(\bar{x}) \quad \text{and} \quad \mu'^R = \frac{1}{|R|} \sum_{s \in S} \delta_s(\bar{x}')
\]

satisfy

\[
d_{\star}(\mu^R_{[1,k]}, \mu'^R_{[1,k]}) < \delta.
\]

**Proof.** First of all, notice that \(\mu^R_{[1,k]} = \frac{1}{|R|} \sum_{g \in S} \delta_g(\bar{x}_{[1,k]})\). Let \(K \subset G\), be a finite set such that if \(\bar{x}_{[1,k]}, \bar{x}'_{[1,k]}\) agree on \(K\) then the corresponding Dirac measures \(\delta_{\bar{x}_{[1,k]}}, \delta_{\bar{x}'_{[1,k]}}\) are closer to each other than \(\frac{\delta}{2}\) in \(\mathcal{M}(\bar{X}_{[1,k]})\). If the shape \(S\) is a
It is thus obvious that the sets \( s \) and \( s^{\mu} \) (4.9) with measures that are not necessarily invariant. This is why the statements look very similar, however, the first one deals with invariant measures, while the second one with measures that are not necessarily invariant. This is why the statements are presented separately, with slightly different proofs.

**Corollary 4.31.** Fix some \( \delta > 0 \). If all shapes \( S \in S_k \) \((k \in \mathbb{N})\) are sufficiently far members of a Følner sequence, then, for any invariant measure \( \mu \in \mathcal{M}_G(\bar{X}) \), we have

\[
d_*(\mu_{[1,k]}, \sum_{R \in \mathcal{R}_k} \mu([R])|R|\mu^R_{[1,k]}) < \delta.
\]

**Proof.** We have:

\[
\sum_{R \in \mathcal{R}_k} \mu([R])|R|\mu^R_{[1,k]} = \sum_{S \in \mathcal{S}_k} \sum_{R \in \mathcal{R}_S} \mu([R])|R| \frac{1}{|R|} \sum_{s \in S} \left( \int_{|R|} \frac{1}{|R|} \sum_{s \in S} \delta_s(x) \, d\mu(x) \right)_{[1,k]},
\]

where \( x \) ranges over \([R]\), while the integrated (measure-valued) function is constant.

We now pass to analyzing \( \mu_{[1,k]} \). Given \( S \in \mathcal{S}_k \), \( R \in \mathcal{R}_S \) and \( s \in S \), points \( x \in s([R]) \) are characterized by two properties:

(a) the central tile of the tiling \( T_k \) associated with \( x \) is centered at \( s^{-1} \), and

(b) the \( k \)-rectangle appearing in \( x \) over the central tile is \( R \).

It is thus obvious that the sets \( s([R]) \) with \( S \) ranging over \( \mathcal{S}_k \), \( R \) ranging over \( \mathcal{R}_S \) and \( s \) ranging over \( S \), form a finite measurable partition of \( \bar{X} \). Therefore

\[
\mu_{[1,k]} = \left( \int_{\bar{X}} \delta_{\bar{x}} \, d\mu(\bar{x}) \right)_{[1,k]} = \left( \sum_{S \in \mathcal{S}_k} \sum_{R \in \mathcal{R}_S} \sum_{s \in S} \int_{s([R])} \delta_{\bar{x}} \, d\mu(\bar{x}) \right)_{[1,k]} = \sum_{S \in \mathcal{S}_k} \sum_{R \in \mathcal{R}_S} |R| \left( \int_{[R]} \frac{1}{|R|} \sum_{s \in S} \delta_{\bar{x}} \, d\mu(\bar{x}) \right)_{[1,k]},
\]

(in the last equality it is essential that \( \mu \) is invariant). Comparing the right hand sides of formulas (4.8) and (4.9), we find out that they differ only in having the variable point \( \bar{x} \) ranging over \([R]\) replaced by the constant point \( \bar{x}_R \) (also belonging to
system. extension of a system we mean a topological extension which is a quasi-symbolic

\[ \sum_{S \in \mathcal{S}_k} |R| \mu([R]) = \sum_{S \in \mathcal{S}_k} |S| \mu([S]) = 1. \]

So we are dealing with generalized convex combinations, and by convexity and continuity of the metric \( d_* \) the proof is finished. \( \square \)

**Corollary 4.32.** Fix some \( \delta > 0 \). If all shapes \( S \in \mathcal{S}_k \) \((k \in \mathbb{N})\) are sufficiently far members of a Følner sequence, then, the following holds: Let \( R' \in \mathcal{R}_{k+1} \) be a \((k+1)\)-rectangle and let \( R'_{[1,k]} = R^{(1)} R^{(2)} \ldots R^{(q)} \) be the decomposition of the restriction \( R'_{[1,k]} \) of \( R' \) to the first \( k \) layers into \( k \)-rectangles \((q = |C_k(S')|)\), where \( S' \) is the shape of \( R' \). Then

\[ d_* \left( \mu^{R'_{[1,k]}}, \frac{1}{|R'|} \sum_{i=1}^{q} |R^{(i)}| \mu^{R^{(i)}_{[1,k]}} \right) \leq \delta. \]

**Proof.** In precise terms, the above decomposition of \( R'_{[1,k]} \) means that the set \( C_k(S') \) can be enumerated as \( \{c^{(1)}, c^{(2)}, \ldots, c^{(q)}\} \) and then \( S' = \bigcup_{i=1}^{q} S^{(i)} c^{(i)} \) is the partition of \( S' \in \mathcal{S}_{k+1} \) by the tiles of \( T_k \), and, for each \( i = 1, 2, \ldots, q \), \( R'_{[1,k]} |_{S^{(i)} c^{(i)}} = R^{(i)}. \) With this notation, we have

\[ \mu^{R'} = \frac{1}{|R'|} \sum_{g \in S'} \delta_g(\bar{x}_{R'}) = \frac{1}{|R'|} \sum_{i=1}^{q} |R^{(i)}| \frac{1}{|R^{(i)}|} \sum_{g \in S^{(i)}} \delta_{g^{(i)}}(\bar{x}_{R^{(i)}}). \]

On the other hand,

\[ \frac{1}{|R'|} \sum_{i=1}^{q} |R^{(i)}| \mu^{R^{(i)}} = \frac{1}{|R'|} \sum_{i=1}^{q} |R^{(i)}| \frac{1}{|R^{(i)}|} \sum_{g \in S^{(i)}} \delta_{g^{(i)}}(\bar{x}_{R^{(i)}}). \]

Comparing the right hand sides above we find out that they differ only in having the points \( c^{(i)}(\bar{x}_{R^{(i)}}) \) replaced by \( \bar{x}_{R^{(i)}} \) (selected from the respective cylinders \( [R^{(i)}] \), \( i = 1, 2, \ldots, q \)). But observe that the points \( c^{(i)}(\bar{x}_{R^{(i)}}) \) also belong to the respective cylinders \( [R^{(i)}] \). By Lemma 4.30 for each \( i = 1, 2, \ldots, q \), the measures \( \frac{1}{|R^{(i)}|} \sum_{g \in S^{(i)}} \delta_{g^{(i)}}(\bar{x}_{R^{(i)}}) \) and \( \frac{1}{|R^{(i)}|} \sum_{g \in S^{(i)}} \delta_{g^{(i)}}(\bar{x}_{R^{(i)}}) \) are less than \( \delta \) apart and the assertion follows from convexity of the metric \( d_* \) and the fact that \( \sum_{i=1}^{q} |R^{(i)}| = |R'|. \) \( \square \)

5. Quasi-symbolic extensions—the hard direction of the main theorem

In full generality we can prove the hard direction of the symbolic extension theorem in a somewhat deficient version, where the symbolic extensions are replaced by what we call quasi-symbolic extensions, as defined below:

**Definition 5.1.** By a quasi-symbolic system \( \tilde{Y} \) we mean a topological joining \( Y \vee T \) of a subshift \( Y \) with a zero entropy tiling system \( T \). By a quasi-symbolic extension of a system we mean a topological extension which is a quasi-symbolic system.
Theorem 5.2. Let a countable amenable group $G$ act on a compact metric space $X$ and let $\mathcal{H}$ denote the entropy structure of $X$. Then $E_A$ is a bounded affine superenvelope of $\mathcal{H}$ if and only if there exists a quasi-symbolic extension $\pi : Y \to X$ such that $E_A = h^\pi$.

Proof. The “easy” direction requires just a comment. We need to show that the extension entropy function $h^\pi$ in any quasi-symbolic extension $\pi : Y \to X$, where $Y = Y \upharpoonright T$, is a superenvelope of the entropy structure $\mathcal{H}$ of $X$. We can repeat the proof of Theorem [12] almost unchanged. The only change is that in the first part (for zero-dimensional $X$) we need to replace $Y$ by $\overline{Y}$. But because $Y$ is a principal extension of $X$, we can still use the partition $P_A$ (lifted to $Y$) and the equality $h(\nu, Y) = h(\nu, P_A)$ will hold for all $\nu \in \mathcal{M}_G(Y)$. The rest of the proof passes with no further modifications.

It is the other, “hard” direction, that requires a lot of work. We begin by replacing $X$ with its principal zero-dimensional extension $X'$ provided by [34] Theorem 2.

We choose an array representation $X' = \lim_{k} X_{[1,k]}$, where, for each $k \in \mathbb{N}$, $X_k$ is a subshift over some alphabet $\Lambda_k$ and the alphabet of $X_{[1,k]}$ is $\Lambda_{[1,k]} = \prod_{i=1}^{k} \Lambda_i$.

Next, on $G$ we fix a tiling system $T = \lim_{k} T_k$ of topological entropy zero (whose existence is guaranteed by Theorem [12]). Later we may need to come back to this starting point and replace $T$ by its subsequence (a process which we will call “speeding up the tiling system”), but at the moment we consider $T$ as fixed. We extend $X'$ by joining it (in any case, one can take the direct product) with $T$. We denote this joining by $\tilde{X}$. This is still a principal extension of $X$. Now, since $\tilde{X}$ has $T$ as a topological factor, we can use its tiled array representation, $\tilde{X} = \lim_{k} \tilde{X}_{[1,k]}$, where, for each $k \in \mathbb{N}$, $\tilde{X}_k$ is a subshift over the alphabet $\Lambda_k = \Lambda \times V_k$ (recall that $V_k = \{\text{“S”} : S \in S_k \} \cup \{0\}$), and the alphabet of $\tilde{X}_{[1,k]}$ is $\Lambda_{[1,k]} = \prod_{j=1}^{k} \Lambda_j$.

The entropy structure $\mathcal{H} = (h_k)_{k \in \mathbb{N}}$ of $\tilde{X}$ is given by $h_k(\mu) = h(\mu, \Lambda_{[1,k]})$. Note that since $T$ has topological entropy zero, for each $k \in \mathbb{N}$ we have $h_k = h(\cdot, \Lambda_{[1,k]})$.

Notice also, that $\tilde{h}_k(\mu) = h(\mu_{[1,k]})$, where $\mu_{[1,k]}$ is the projection of $\mu$ on $\tilde{X}_{[1,k]}$, i.e., $\tilde{h}_k$ is in fact a function on $\mathcal{M}_G(\tilde{X}_{[1,k]})$ (lifted to $\mathcal{M}_G(\tilde{X})$).

Clearly, $\tilde{X}$ is a principal zero-dimensional extension of $X$, which, by definition of the entropy structure on $X$, implies that the entropy structure $\mathcal{H}$ lifted to $\mathcal{M}_G(\tilde{X})$ is uniformly equivalent to $\mathcal{H}$. Thus, the lift of $E_A$ (which clearly is a superenvelope of the lift of $\mathcal{H}$), is also a superenvelope of $\mathcal{H}$. If we construct a quasi-symbolic extension of $\tilde{X}$ whose extension entropy function equals (the lift of) $E_A$ then the same extension will be a quasi-symbolic extension of $X$ whose extension entropy function equals $E_A$. From now on, we will skip the “bar” in the denotation of $\mathcal{H}$ and $h_k$.

The construction mimics the one presented in [5] for $\mathbb{Z}$-actions (a corrected and slightly simplified version is given in [17]), but many details have to be reworked. There are three main stages: in the first one, basing on $E_A$ and the entropy structure $\mathcal{H} = (h_k)_{k \in \mathbb{N}}$ we create an oracle—an integer-valued function on rectangles. In stage 2, based on the oracle, we build a quasi-symbolic extension $\tilde{Y}$ of $\tilde{X}$. The last part of the proof, stage 3, is the verification that the corresponding extension entropy function indeed matches $E_A$.

Stage 1. For each $k \in \mathbb{N}$ we have $E_A \geq h_k$, $E_A - h_k$ is affine and upper semi-continuous. Thus the function $E_A - h = \lim_{k} \downarrow (E_A - h_k)$ is also nonnegative,
affine and upper semicontinuous. We can use varbatim [17] Lemma 9.2.6 and find a decreasing sequence of nonnegative affine continuous functions \((g_k)_{k \in \mathbb{N}}\) on \(M_G(\hat{X})\), such that, for each \(k \in \mathbb{N}\), \(g_k\) is constant on fibers of the projection \(M_G(\hat{X}) \to M_G(\hat{X}_{[1,k]})\) (i.e., \(g_k(\mu)\) depends only on the projection \(\mu_{[1,k]}\)) and

1. \(\lim_k g_k = E_A - h\),
2. \(\forall k \ g_k > E_A - h\),
3. \(\forall k \ g_k - g_{k+1} > h_{k+1} - h_k\).

By continuity of \(g_k - g_{k+1}\) and upper semicontinuity of \(h_{k+1} - h_k\), we can find a decreasing to zero sequence of positive numbers \((\delta_k)_{k \in \mathbb{N}}\) such that, for each \(k \in \mathbb{N}\),

\[ g_k - g_{k+1} - 3\delta_k > h_{k+1} - h_k. \]

Our next step is “speeding up” the tiling system \(T\), i.e., replacing it by its subsequence, in order to guarantee some additional properties (note that since \(T\) has topological entropy zero, this will not affect any of the preceding arrangements). Five desired properties can be achieved in this manner:

1. By speeding up we can arrange that for each \(S \in S_{k+1}\), \(|S| > \frac{1}{k}\). This will imply that for every \(S \in S_{k+1}\) and \(t > 0\),

\[ 2^{|S[t]|} \leq 2^{|S|(t+\delta_k)}. \]

2. According to Theorem 4.27 for fixed \(k\), the conditional tiled entropy functions on \(M_G(\hat{X})\), \(H_{\pi_{[1,k]}}(\mu, \Lambda_{k+1} | \Lambda_{[1,k]})\) converge, as \(k' \to \infty\), decreasingly to \(h_{k+1} - h_k\). Since these are continuous functions decreasing to an upper semicontinuous function, and \(g_k - g_{k+1} - 3\delta_k\) is continuous, for large enough \(k'\) we have

\[ g_k - g_{k+1} - 3\delta_k > H_{\pi_{[1,k]}}(\mu, \Lambda_{k+1} | \Lambda_{[1,k]}). \]

Because the partitions \(\Lambda_{k+1}\) and \(\Lambda_{[1,k]}\) do not depend on the tiling system, by speeding up we can arrange that, on \(M_G(\hat{X})\),

\[ g_k - g_{k+1} - 3\delta_k > H_{\pi_{[1,k]}}(\mu, \Lambda_{k+1} | \Lambda_{[1,k]}). \]

3. The functions \(g_k\), being implicitly defined on \(M_G(\hat{X}_{[1,k]})\), can be prolonged to continuous and affine functions on \(M(\hat{X}_{[1,k]})\) (and then lifted to functions on \(M(\hat{X})\) constant on fibers of the projection \(\pi_{[1,k]})\). With the help of Lemma 4.30 and Corollaries 4.31, 4.32 by speeding up the tiling system, we can arrange that for each \(k \in \mathbb{N}\) the following conditions hold:

(a) \(|g_k(\mu^{R'}) - \frac{1}{|R'|} \sum_{i=1}^q |R^{(i)}| g_k(\mu^{R'^{(i)}})| < \delta_k, \)

(b) \(|g_k(\mu) - \sum_{R \in \mathcal{R}_k} \mu(|R|)|R|g_k(\mu^{R'})| < \delta_k, \)

whenever \(\mu \in M_G(\hat{X})\) and \(R' \in \mathcal{R}_{k+1}\) is such that \(R'_{[1,k]} = R^{(1)}R^{(2)} \ldots R^{(q)}, R^{(i)} \in \mathcal{R}_k\) (\(i = 1, 2, \ldots, q\), \(q = |C_k(S')|\), \(S'\) is the shape of \(R'\)).

4. The inequality \(g_k - g_{k+1} - 3\delta_k > H_{\pi_{[1,k]}}(\mu, \Lambda_{k+1} | \Lambda_{[1,k]}\) then holds on a neighborhood of \(M_G(\hat{X})\). According to Lemma 4.26 by speeding up the tiling system, we can arrange that all empirical measures associated to \((k+1)\)-rectangles lie in this neighborhood. Then, for every \((k+1)\)-rectangle \(R'\), we shall have

\[ g_k(\mu^{R'}) - g_{k+1}(\mu^{R'}) - 3\delta_k > H_{\pi_{[1,k]}}(\mu^{R'}, \Lambda_{k+1} | \Lambda_{[1,k]}). \]
5. QUASI-SYMBOLIC EXTENSIONS—THE HARD DIRECTION OF THE MAIN THEOREM

(5) We also need the following to hold for every $k \in \mathbb{N}$: For each concatenation of $k$-blocks, $R^{(1)}R^{(2)} \ldots R^{(q)}$, which occurs as the first $k$ layers of some $(k+1)$-rectangle, $S'$ denoting the shape of that $(k+1)$-rectangle and $|q = C_k(S')|$, we should have

$$\sum_{R' \in \mathcal{R}_{k+1}, R'_{[1,k]} = R^{(1)}R^{(2)} \ldots R^{(q)}} 2^{-|R'|H_{\mathcal{X}_k}(\mathbf{u}_R', \Lambda_{k+1}[\Lambda_{1,k}])} < 2^{|S'|\delta_k}.$$ 

This inequality holds whenever $q$ is sufficiently large; the proof will be provided in a moment. So, this property can be achieved by speeding up the tiling system.

Once (after appropriate speeding up of the tiling system) all the above conditions are satisfied, we can define an oracle, $O : \mathcal{R} \to \mathbb{N}$, as follows: for $R \in \mathcal{R}_k$ we let

$$O(R) = [2^{R|g_k(\mu_R')}].$$

Let us explain, that the oracle “predicts” how many different blocks (of the same shape as $R$) will appear in the elements of the future symbolic extension of $X$ “above” the occurrences of $R$. According to the definition (see Definition 9.2.4), in order to be an oracle, a function $O : \mathcal{R} \to \mathbb{N}$ must satisfy, for every $k \in \mathbb{N}$, and every concatenation $R^{(1)}R^{(2)} \ldots R^{(q)}$, where $R^{(i)} \in \mathcal{R}_k$ $(i = 1, 2, \ldots, q)$, which occurs as the first $k$ layers of some $(k+1)$-rectangle, the following oracle condition:

$$\sum_{R' \in \mathcal{R}_{k+1}, R'_{[1,k]} = R^{(1)}R^{(2)} \ldots R^{(q)}} O(R') \leq O(R^{(1)})O(R^{(2)}) \ldots O(R^{(q)}).$$

**Lemma 5.3.** The function $O(R)$ defined above satisfied the oracle condition.

**Proof.** Fix a concatenation of $k$-blocks, $R^{(1)}R^{(2)} \ldots R^{(q)}$, which occurs as the first $k$ layers of some $(k+1)$-rectangle. If $S'$ denotes the common shape of all such $(k+1)$-rectangles then $q = |C_k(S')|$. We have:

$$\sum_{R'} O(R') = \sum_{R'} [2^{R'|g_k+1(\mu_R')} \leq \sum_{R'} 2^{R'|(g_k+1(\mu_R')+\delta_k)} \leq \sum_{R'} 2^{R'|(g_k(\mu_R') - H_{\mathcal{X}_k}(\mu_R', \Lambda_{k+1}[\Lambda_{1,k}]) - 2\delta_k)} \leq \sum_{R'} 2^{S' |\mathcal{X}_k| R^{(1)}} |g_k(\mu_R^{(1)})| \cdot 2^{-|S'|\delta_k} \sum_{R'} 2^{-|R'|H_{\mathcal{X}_k}(\mu_R', \Lambda_{k+1}[\Lambda_{1,k}])},$$

where in each sum $R'$ ranges as in the oracle condition. In the last line, we see three expressions separated by the multiplication dots. The first expression equals $\prod_{i=1}^q 2^{R^{(i)}}|g_k(\mu_R^{(i)})|$, which, after rounding up the multipliers, equals precisely $O(R^{(1)})O(R^{(2)}) \ldots O(R^{(q)})$ (so is not larger than this product). The last expression (the sum), by (5), does not exceed $2^{|S'|\delta_k}$ which cancels with the central expression $2^{-|S'|\delta_k}$, and the oracle condition is proved.

We return to the missing proof of the property (5).

**Proof of (5).** This is almost literally Lemma 9.2.11, which says that whenever $\Lambda = \Lambda_1 \times \Lambda_2$ is a product alphabet, then for every $n \in \mathbb{N}$ and $\epsilon > 0$ there
exists an \( m_{(n,\epsilon)} \in \mathbb{N} \) such that for every \( q \geq m_{(n,\epsilon)} \) and every \( D \in \Lambda_1^n \) the following holds
\[
\sum_{B \in \Lambda_1^n, \ B_1 = D} 2^{-qH_n(B|B_1)} \leq 2^\eta,
\]
where \( B_1 \) denotes the block appearing in the first row of \( B \). In this formulation (which is meant for the \( Z \)-action of the classical shift), \( H_n(B|B_1) \) stands for \( \frac{1}{n} H(\mu_B, \Lambda^n|\Lambda_1^n) \), with \( \mu_B \) denoting the invariant measure supported by the orbit of the sequence obtained as the infinite concatenation \( \cdots B B \cdots \). We will use this lemma only in case \( n = 1 \), in which only the values of \( \mu_B \) on single symbols play a role. These values are simply the frequencies of the symbols in \( B \), so the “spacial” form of the block \( B \) (i.e., whether it is a linear block over \( \{1, 2, \ldots, q\} \) or a block over a differently looking subset of cardinality \( q \) of some other group) has no meaning.

For given \( R' \in \mathcal{R}_{k+1} \), applying (4.4), we get
\[
|R'\hat{H}_{\Lambda_1^k}(\mu^{R'}, \Lambda_{k+1}|\Lambda_1^k)| = |R'|\mu^{R'}([C_k]) = |R'|\mu^{R'}([C_k])H(\mu^{R'}|B_k)\tilde{R}_k|R_k).
\]
Clearly, \( \mu^{R'}([C_k]) = \frac{|C_k(S')|}{|R'|} = \frac{q}{|R'|} \), so the expression on the right hand side simplifies as \( qH(\mu^{R'}|B_k)\tilde{R}_k|R_k) \). Since the measure \( \mu^{R'}|B_k) \) is applied only to the finite collection of extended \( k \)-rectangles (\( k \)-rectangles, as cylinders, are unions of extended \( k \)-rectangles) on which it is normalized, it can be thought of as a measure \( \mu_B \) on single symbols, where \( B = \hat{R}^{(1)}\hat{R}^{(2)} \cdots \hat{R}^{(q)} \) is the imaginary linearly ordered block over the alphabet \( \tilde{R}_k \) viewed as a subset of the product \( \mathcal{R}_k \times \mathcal{B}_k \), where \( \mathcal{B}_k \) is the family of one layer blocks appearing in the \((k+1)\)st layer of the extended \( k \)-rectangles. Taking for \( D \) the block \( R^{(1)}R^{(2)} \cdots R^{(q)} \), the family of blocks \( B \) with \( B_1 = D \) becomes the family of all \((k+1)\)-rectangles \( R' \) with \( R'_{[1,k]} = R^{(1)}R^{(2)} \cdots R^{(q)} \).

With such an identification, the term \( qH(\mu^{R'}|B_k)\tilde{R}_k|R_k) \) coincides with \( qH_1(B|B_1) \) in the notation of [17] Lemma 9.2.11. The lemma now yields that if \( q = C_k(S') \) is sufficiently large then
\[
\sum_{R' \in \mathcal{R}_{k+1}, R'_{[1,k]} = R^{(1)}R^{(2)} \cdots R^{(q)}} 2^{-|R'|\hat{H}_{\Lambda_1^k}(\mu^{R'}, \Lambda_{k+1}|\Lambda_1^k)} = \sum_{R' \in \mathcal{R}_{k+1}, R'_{[1,k]} = R^{(1)}R^{(2)} \cdots R^{(q)}} 2^{-qH(\mu^{R'}|B_k)\tilde{R}_k|R_k)} = \sum_{B \in \mathcal{R}_k^*, \ B_1 = D} 2^{-qH_1(B|B_1)} \leq 2^{\eta_k}. \]

Obviously, \( 2^{\eta_k} \leq 2^{\|S'\|\delta_k} \), and so (5) is proved. \( \square \)

**Stage 2.** Given an oracle \( O \) we will build a quasi-symbolic extension \( \hat{\pi} : \hat{Y} \to \hat{X} \), where \( \hat{Y} = Y \cup T \) and \( Y \) is a subshift. The mapping \( \hat{\pi} \) will preserve the tiling system (which is a topological factor of both \( \hat{Y} \) and \( \hat{X} \)), i.e., if \( \hat{y} \in \hat{Y} \) and \( \hat{x} = \hat{\pi}(\hat{y}) \) then \( \hat{y} \) and \( \hat{x} \) have the same sequence of tilings \( T = (T_k)_{k \in \mathbb{N}} \) associated to them. The space \( \hat{Y} \) will be obtained as the intersection of spaces \( Y_k \subset Y_k \cup T_k \), each factoring via a map \( \pi_k \) onto \( \hat{X}_{[1,k]} \). These factor maps will be consistent, i.e., \( \pi_k[Y_{k+1}] \) will coincide with \( \pi_k[T_{k+1}] \) composed with the natural projection \( \pi_{[1,k]} : \hat{X}_{[1,k+1]} \to \hat{X}_{[1,k]} \).

Then on the intersection \( \hat{Y} = \bigcap_k Y_k \) we will have \( \hat{\pi}(\hat{y}) \) defined by specifying all its projections: \( ([\hat{\pi}(\hat{y})])_{[1,k]} = \pi_k(\hat{y}) \). It is elementary to see that this map will be a topological factor map from \( \hat{Y} \) onto \( \hat{X} \).
Step 1. We begin the construction of $\tilde{Y}$ and of the map $\bar{\pi}$ by establishing the alphabet $\Lambda$ of the symbolic part $Y_1$ of $\tilde{Y}_1$. This alphabet will remain unchanged in the following steps, as each $\tilde{Y}_k$ will be a subsystem of $\tilde{Y}_1$. Of course, only cardinality of $\Lambda$ matters, and we define it to be the smallest integer $l$ such that, for every $S \in S_1$,

$$|S| \geq \sum_{R \in \mathcal{R}_S} \mathcal{O}(R).$$

With such a choice of $\Lambda$, for every $S \in S_1$ there exists a map assigning to each $R \in \mathcal{R}_S$ a subfamily $\mathcal{F}_S(R) \subset \Lambda^S$ of cardinality $\mathcal{O}(R)$, in such a way that these families are disjoint for different $1$-rectangles $R \in \mathcal{R}_S$. Now, for each $x \in \bar{X}_1$ and $T$ denoting the (first) tiling associated with $x$, we will create a closed subset $\bar{Y}_1(\bar{x}_1) \subset \Lambda^G \times \mathbf{T}$ which will constitute the preimage of $\bar{x}_1$ by the map $\bar{\pi}_1$ (which we are about to define). Namely, we admit $(y, T)$ to belong to $\bar{Y}_1(\bar{x}_1)$ if and only if the first tiling in $T$ equals $T_1$ and, for any tile $S$ of $T_1$, $y|_{S_1} \in \mathcal{F}_S(x_1|_{S_1})$ (note that $x_1|_{S_1}$ is a $1$-rectangle $R \in \mathcal{R}_S$). It is easy to see that the subsets $\bar{Y}_1(\bar{x}_1)$ are disjoint for different elements $x_1 \in \bar{X}_1$ (if $\bar{x}_1$ and $\bar{x}_1'$ differ in having different first tilings, say $T_1 \neq T_1'$, then this difference passes to any elements $\bar{y} \in \bar{Y}_1(\bar{x}_1)$ and $\bar{y}' \in \bar{Y}_1(\bar{x}_1')$; if the first tilings are the same then the first layers of $\bar{x}_1$ and $\bar{x}_1'$ must differ on some tile $S$ of the common first tiling and then any elements $\bar{y} \in \bar{Y}_1(\bar{x}_1)$ and $\bar{y}' \in \bar{Y}_1(\bar{x}_1')$ differ on this tile). We let $\bar{Y}_1 = \bigcup_{x_1 \in \bar{X}_1} \bar{Y}_1(\bar{x}_1)$ and we skip checking that this is a closed shift-invariant set. The functioning of the mapping $\bar{\pi}_1 : \bar{Y}_1 \to \bar{X}_1$ is now obvious: for $y \in \bar{Y}_1$ and $T$ being the sequence of tilings associated with $y$, we let $\bar{\pi}_1(y)$ be the unique $x_1 \in \bar{X}_1$ whose first tiling $T_1$ is the same as the first tiling of $T$, and $y \in \bar{Y}_1(x_1)$. It is fairly easy to see that this map is a block code with coding horizon $\bigcup S_1(\bigcup S_1)^{-1}$.

Step $k+1$. Suppose that for some $k \geq 1$ we have defined $\tilde{Y}_k$ and a topological factor map $\bar{\pi}_k : \bar{Y}_k \to \bar{X}_{[1,k]}$ (a block code with coding horizon $\bigcup S_k(\bigcup S_k)^{-1}$) such that for each $S \in S_k$, with each $k$-rectangle $R \in \mathcal{R}_S$ we have associated a family $\mathcal{F}_S(R) \subset \Lambda^S$ of cardinality $\mathcal{O}(R)$ in such a way that these families are disjoint for different $k$-rectangles $R \in \mathcal{R}_S$ and the preimage of each $x_{[1,k]} \in \bar{X}_{[1,k]}$ consists of all such elements $\bar{y} = (y, T) \in \Lambda^G \times \mathbf{T}$ that the $k$th tilings $T_k$ associated to $\bar{y}$ and to $x_{[1,k]}$ coincide, and, for every tile $S$ of $T_k$ ($S \in S_k$), $y|_{S_c} \in \mathcal{F}_S(x_{[1,k]}|_{S_c})$.

We need to define $\tilde{Y}_{k+1} \subset \tilde{Y}_k$ and the map $\bar{\pi}_{k+1} : \bar{Y}_{k+1} \to \bar{X}_{[1,k+1]}$ which, composed with the natural projection of $\bar{\pi}_{[1,k]} : \bar{X}_{[1,k+1]} \to \bar{X}_{[1,k]}$ coincides with $\bar{\pi}_{k}\bar{Y}_{k+1}$. Here is how we proceed: Consider a shape $S' \in S_{k+1}$ and a concatenation $D = R^{(1)}R^{(2)}\ldots R^{(q)}$ of $k$-rectangles which occurs as the first $k$ layers in some $(k+1)$-rectangle $R^c \in \mathcal{R}_{S'}$ (i.e., $D = R^{(1)}_{[1,k]}$). For each $x_{[1,k+1]} \in \bar{X}_{[1,k+1]}$ and $c \in C_{S'}(x_{[1,k+1]})$ such that the projection $x_{[1,k]}$ of $x_{[1,k+1]}$ satisfies $x_{[1,k]}|_{S'c} = D$, and any $\bar{y} = (y, T) \in \bar{Y}_{k+1}(x_{[1,k+1]})$ we have (in spite of the common tiling $T_k$ for $x_{[1,k]}$ and $T$) the following: if $S^{(i)}c^{(i)} \in \mathcal{O}(\bar{\pi}_k)\mathcal{O}(\bar{\pi}_k)$ blocks belonging to $\Lambda^S$ appearing in the elements of $\bar{\pi}_{k+1}(x_{[1,k+1]})$ “above” each occurrence of any $(k+1)$-rectangle $R^c$ such that $R^c_{[1,k]} = D$ in any $x_{[1,k+1]} \in \bar{X}_{[1,k+1]}$. Note that the families $\mathcal{E}_D$ are disjoint for different concatenations $D$ with a common shape $S'$. By the oracle condition,
with each \((k+1)-\)rectangle \(R'\) satisfying \(R'_{[1,k]} = D\) we can associate a subfamily \(\mathcal{E}_D(R') \subset \mathcal{E}_D\) of cardinality \(\mathcal{O}(R')\) so that these families are disjoint for different \((k+1)-\)rectangles \(R'\) with \(R'_{[1,k]} = D\). By disjointness of the families \(\mathcal{E}_D\) (for different \(D\) with a common shape \(S'\)), there will be no confusion if denote \(\mathcal{F}_D(R')\) by \(\mathcal{F}(R')\). For any \(x_{[1,k+1]} \in \bar{X}_{[1,k+1]}\) we now define the set \(\bar{Y}_{k+1}(x_{[1,k+1]}) \subset \bar{Y}_{k}(x_{[1,k+1]})\) (the preimage of \(x_{[1,k+1]}\) by the future map \(\bar{\pi}_{k+1}\)) by the already familiar rule: \(\bar{Y}_{k+1}(x_{[1,k+1]})\) consists of such elements \(\bar{y} = (y, T) \in \bar{Y}_{k}(x_{[1,k]})\) that the \((k+1)\)st tilings \(\bar{T}_{k+1}\) associated to \(\bar{y}\) and to \(\bar{x}_{[1,k+1]}\) coincide, and, for every tile \(S'c\) of \(\bar{T}_{k+1}\), \(\bar{y}|S'c \in \mathcal{F}(\bar{x}_{[1,k+1]}|S'c)\), where \(D = \bar{x}_{[1,k]}|S'c\). We skip the description of how the map \(\bar{\pi}_{k+1}\) functions; it is fully analogous to the description for \(\bar{\pi}_1\). The coding horizon is now \(\bigcup S_{k+1} \bigcup S_{k+1}^{-1}\).

**Stage 3.** Once the induction is completed, we have defined both the quasi-symbolic extension \(\bar{Y}\) of \(X\) and the associated factor map \(\bar{\pi} : \bar{Y} \to \bar{X}\). What remains to do is to verify that on \(\mathcal{M}_G(\bar{X})\), \(h^\bar{\pi} = E_A\) (or that \(h^\bar{\pi} - h = E_A - h\)).

**Lemma 5.4.** Fix an invariant measure \(\mu \in \mathcal{M}_G(\bar{X})\) and let \(\mu_{[1,k]}\) denote the projection of \(\mu\) onto \(\bar{X}_{[1,k]}\). Then

\[
h^\bar{\pi}(\mu) - h(\mu) = \lim_{k} \sup_{\nu \in \bar{\pi}^{-1}(\mu)} \mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,k]}).
\]

**Proof.** First observe that for \(\nu \in \bar{\pi}^{-1}(\mu)\), the expressions \(\mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,l]}\) are nonincreasing in both \(k\) and \(l\), hence both iterated limits and the diagonal limit coincide. By Theorem 4.27 the limit in \(k\) (with \(l\) fixed) equals \(h(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,l]}\) = \(h(\nu) - h(\mu_{[1,l]}\), which converges in \(l\) to \(h(\nu) - h(\mu)\).

Because \(\bar{\pi}^{-1}(\mu) \subset \bar{\pi}^{-1}(\mu_{[1,l]}\) and by the “rule of thumb” “lim, sup \(b \geq \sup \lim a\), we have

\[
\lim_{k} \sup_{\nu \in \bar{\pi}^{-1}(\mu)} \mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,k]}\) \geq \sup_{\nu \in \bar{\pi}^{-1}(\mu)} \lim_{k} \mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,k]}\) = \sup_{\nu \in \bar{\pi}^{-1}(\mu)} \lim_{k} \sup_{\nu \in \bar{\pi}^{-1}(\mu)} h(\nu) - h(\mu) = h^\bar{\pi}(\mu) - h(\mu).
\]

On the other hand, if \(l\) is fixed then, since eventually \(k \geq l\), we have

\[
\lim_{k} \sup_{\nu \in \bar{\pi}^{-1}(\mu)} \mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,k]}\) \geq \sup_{\nu \in \bar{\pi}^{-1}(\mu)} \lim_{k} \mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,k]}\) = \cdots \mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,l]}\) = \cdots \mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,l]}\) = \sup_{\nu \in \bar{\pi}^{-1}(\mu)} \lim_{k} \sup_{\nu \in \bar{\pi}^{-1}(\mu)} h(\nu) - h(\mu_{[1,l]}\).
\]

The functions \(\mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,l]}\) are continuous and decrease in \(k\), and we consider the supremum over a fixed compact set. In this situation, the supremum and limit can be exchanged (see e.g. Fact A.1.24), and we can continue as follows:

\[
\cdots = \sup_{\nu \in \bar{\pi}^{-1}(\mu_{[1,l]}\) \lim_{k} \mathcal{H}(\nu, \bar{\Lambda}|\bar{\Lambda}_{[1,l]}\) = \sup_{\nu \in \bar{\pi}^{-1}(\mu_{[1,l]}\) \lim_{k} \sup_{\nu \in \bar{\pi}^{-1}(\mu_{[1,l]}\) h(\nu) - h(\mu_{[1,l]}\).
\]

Since the left hand side does not depend on \(l\), we can apply the limit in \(l\) to the right hand side and the inequality will hold. The function \(\nu \to h(\nu)\) is upper semicontinuous (this is true for symbolic systems and \(\bar{Y}\) differs from the symbolic system \(Y\) by being joined with a zero-entropy system, which does not alter the entropy function). The sets \(\bar{\pi}^{-1}(\mu_{[1,l]}\) decrease in \(l\) to \(\bar{\pi}^{-1}(\mu)\). This implies that \(\sup_{\nu \in \bar{\pi}^{-1}(\mu_{[1,l]}\) h(\nu)\) tends (nonincreasingly with \(l\)) to \(\sup_{\nu \in \bar{\pi}^{-1}(\mu)\) h(\nu)\), while
6. QUASI-SYMBOLIC EXTENSIONS—THE HARD DIRECTION OF THE MAIN THEOREM

$h(\mu_{[1, l]})$ clearly tends to $h(\mu)$. Thus the right hand side (after applying the limit in $l$) becomes $h^\pi(\mu) - h(\mu)$, completing the proof of the lemma. □

The above lemma reduces the problem to finding measures $\nu$ in the preimage $\pi_k^{-1}(\mu_{[1, l]})$ maximizing the conditional tiled entropy $H_{T_k}(\nu, \Lambda|\Lambda_{[1, k]})$. Recall that

$$H_{T_k}(\nu, \Lambda|\Lambda_{[1, k]}) = \sum_{S \in S_k} \nu([S]) H(\nu([S]), \Lambda^S|\Lambda_{[1, k]})$$

Now, on $[S]$ the partition $\Lambda_{[1, k]}^S$ coincides with the partition $R_S$ into $k$-rectangles with the shape $S$, so the above sum splits further as

$$\sum_{S \in S_k} \nu([S]) \sum_{R \in R_S} \nu([S])(|R|)H(\nu([S]), \Lambda^S) = \sum_{S \in S_k} \sum_{R \in R_S} \nu([R])H(\nu([R]), \Lambda^S)$$

(we have used $\nu([S]) = \nu([R])$ and because $R$ depends only on the first $k$ layers; we also have used $\nu([R]) = \mu([R])$ whenever $\pi_k(\nu) = \mu_{[1, l]}$).

Our task is thus to maximize $H(\nu([R]), \Lambda^S)$ for each $S \in S_k$ and $R \in R_S$. By the definition of $\pi_k$, for every $R \in R_S$, the conditional measure $\nu([R])$ is supported by the family of blocks $A_S(R)$ and clearly the largest entropy is achieved when all these blocks have equal masses. In fact, this condition defines, for each $\mu$ on $X$, a measure $\nu$ on $\tilde{Y}_k$ belonging to $\pi_k^{-1}(\mu_{[1, l]})$. We will call this measure $\nu_{\mu_{[1, l]}}^\mu$. Since $|A_S(R)| = O(R)$, we have $H(\nu_{\mu_{[1, l]}}^\mu(R), \Lambda^S) = \log(O(R))$. So,

$$\max_{\nu \in \pi_k^{-1}(\mu_{[1, l]})} H_{T_k}(\nu, \Lambda|\Lambda_{[1, k]}) = \sum_{R \in R_k} \mu([R]) \log(O(R)) = \sum_{R \in R_k} \mu([R])(|R|g_k(\mu^R) + \xi_R) = \cdots$$

where the error term $\xi_R$ ranges between 0 and $\delta_k$. Note that the sum of the coefficients $\mu([R])(|R|)$ over $R \in R_k$ equals 1, so what we see above is a convex combination. Thus, and because $g_k$ only depends on the first $k$ layers, we can continue

$$\cdots = \left(\sum_{R \in R_k} \mu([R])(|R|g_k(\mu^R_{[1, k]})) + \xi_k(\mu) = g_k(\sum_{R \in R_k} \mu([R])(|R|\mu^R_{[1, k]})) + \xi_k(\mu) \cdots$$

where $0 \leq \xi_k(\mu) \leq \delta_k$. By the condition (3b), we can write

$$\cdots = g_k(\mu_{[1, k]}) + \xi_k(\mu) + \xi_k(\mu) + \xi_k(\mu)$$

where $\xi_k(\mu)$ is between 0 and $2\delta_k$. Combining Lemma 5.4 with the properties defining the sequence $(g_k)_{k \in \mathbb{N}}$, we obtain

$$h^\pi(\mu) - h(\mu) = \lim_{k} g_k(\mu) = E_A - h.$$ 

This concludes the proof of Theorem 5.2. □

We end this section by mentioning some obvious consequences. For instance, we obtain a characterization of asymptotic $h$-expansiveness. For our purposes, we define asymptotic $h$-expansiveness by a condition which for $\mathbb{Z}$-actions is known to be equivalent to the original definition by Misiurewicz (see [39], and see [17, 8.4.12] for the equivalence):
Definition 5.5. An action of a countable amenable group $G$ on a compact metric space $X$ is asymptotically $h$-expansive if there exists an entropy structure $\mathcal{H} = (h_k)_{k \in \mathbb{N}}$ which converges to the entropy function $h$ uniformly on $\mathcal{M}_G(X)$.

Observe that in such case, the constant structure $(h)_{k \in \mathbb{N}}$ is also an entropy structure, because it is uniformly equivalent to $\mathcal{H}$. Any other entropy structure must be uniformly equivalent to the constant structure, which in turn implies that it converges to $h$ uniformly. We have proved that in an asymptotically $h$-expansive system every entropy structure converges to $h$ uniformly.

In order to escape distractions from studying our main subject (which are symbolic extensions), we refrain from discussing whether the above definition is equivalent to the original definition adapted to the context of actions of countable amenable groups. We refer to [12] for more details of the adaptation of Misiurewicz’ definition [39] to actions of sofic (including amenable) groups. Our definition reflects the most vital for us features of asymptotic $h$-expansiveness, and allows us to formulate what follows:

Theorem 5.6. An action of a countable amenable group $G$ on a compact metric space $X$ is asymptotically $h$-expansive if and only if it admits a principal quasi-symbolic extension.

Proof. We fix an entropy structure $\mathcal{H} = (h_k)_{k \geq 0}$ of $X$ which has upper semicontinuous differences. If $Y$ is a principal quasi-symbolic extension of $X$ then the extension entropy function equals the entropy function $h$. Theorem 5.2 (the “easy” direction) implies that $h$ is a superenvelope of the entropy structure $\mathcal{H}$ of $X$. So, by Proposition 5.2 (3), the functions $h - h_k$ are upper semicontinuous and obviously they converge nonincreasingly to 0. Such convergence is always uniform, proving that $h_k$ tends to $h$ uniformly, and $X$ is asymptotically $h$-expansive.

For the opposite implication assume that $\mathcal{H}$ converges to $h$ uniformly. As a consequence, for each $k \in \mathbb{N}$, $h - h_k$ is a uniform limit of $h_{k'} - h_k$ as $k' \to \infty$. A uniform limit of upper semicontinuous functions is upper semicontinuous. So, $h - h_k$ is upper semicontinuous (and clearly nonnegative) which implies that $h$ is a bounded superenvelope of $\mathcal{H}$. Also, $h$ is affine on $\mathcal{M}_G(X)$. Now, Theorem 5.2 implies the existence of a quasi-symbolic extension of $X$ with the extension entropy function equal to $h$, which is equivalent to the extension being principal.

6. The comparison property

The remaining part of the paper is almost entirely devoted to isolating and studying a very important property that an action of a countable amenable group $G$ may or may not have. A version of this property can also be associated with the group $G$ itself. For us, its significance lies in the fact that it enables us to create genuine symbolic extensions in place of quasi-symbolic ones. Ironically, for $\mathbb{Z}$-actions the analogous passage occupies in the proof of the Symbolic Extensions Entropy Theorem just one line, and the comparison property is not explicitly invoked (but is implicitly essential).

6.1. Definition of the comparison property. We will understand the comparison property as follows (see also [35]):

Definition 6.1. Let $G$ be a countable amenable group.
(1) Let \( G \) act on a zero-dimensional compact metric space \( X \). For two clopen sets \( A, B \subset X \), we say that \( A \) is subequivalent to \( B \) (and write \( A \preccurlyeq B \)), if there exists a finite partition \( A = \bigcup_{i=1}^{k} A_i \) of \( A \) into clopen sets and there are elements \( g_1, g_2, \ldots, g_k \) of \( G \) such that \( g_1(A_1), g_2(A_2), \ldots, g_k(A_k) \) are disjoint subsets of \( B \). We say that the action admits comparison if for any pair of clopen subsets \( A, B \) of \( X \), the condition that for each invariant measure \( \mu \) on \( X \) we have \( \mu(A) < \mu(B) \), implies \( A \preccurlyeq B \).

(2) If every action of \( G \) on any zero-dimensional compact metric space admits comparison then we will say that \( G \) has the comparison property.

Clearly, \( A \preccurlyeq B \) implies \( \mu(A) \leq \mu(B) \) for every invariant measure \( \mu \), so comparison is "nearly" an equivalence between subequivalence and the inequality for all invariant measures.

Note the obvious fact, that if \( X \) admits comparison and \( Z \) is a zero-dimensional topological factor of \( X \) then \( Z \) also admits comparison.

**Remark 6.2.** Let two clopen sets \( A, B \) satisfy \( \mu(A) < \mu(B) \) for every invariant measure \( \mu \). Because the sets \( A, B \) are clopen, the function \( \mu \mapsto \mu(B) - \mu(A) \) is continuous, and since it is positive on a compact set, it is separated from zero, i.e.,

\[
\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) > 0.
\]

The following definition and the adjacent lemma are not used further in this paper. We provide them for a more complete treatment of the comparison property. Consider the following seemingly weaker property:

**Definition 6.3.** The action of a countable amenable group \( G \) on a zero-dimensional compact metric space \( X \) admits weak comparison if there exists a constant \( C \geq 1 \) such that for any clopen sets \( A, B \subset X \), the condition \( \sup_\mu \mu(A) < \frac{1}{C} \inf_\mu \mu(B) \) (where \( \mu \) ranges over all invariant measures) implies \( A \preccurlyeq B \).

Clearly, comparison implies weak comparison. We will show that these properties are in fact equivalent.

**Lemma 6.4.** Weak comparison implies comparison.

**Proof.** Suppose the action of a countable amenable group \( G \) on a zero-dimensional compact metric space \( X \) admits weak comparison with a constant \( C \). Let two clopen sets \( A, B \) satisfy \( \mu(A) < \mu(B) \) for every invariant measure \( \mu \). By Remark 6.2, \( \inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) > \varepsilon \) for some positive \( \varepsilon \). We order the group (arbitrarily) by natural numbers, as \( G = \{g_1, g_2, \ldots\} \) (or \( G = \{g_1, \ldots, g_n\} \) in case \( G \) is finite). We let \( A_1 = A \cap g_1^{-1}(B) \), and \( B_1 = g_1(A_1) \). For each \( k > 1 \) (or \( 1 < k \leq n \) in the finite case) we set inductively

\[
A_k = A \setminus \left( \bigcup_{i=1}^{k-1} A_i \right) \cap g_k^{-1} \left( B \setminus \bigcup_{i=1}^{k-1} B_i \right),
\]

and \( B_k = g_k(A_k) \). It is not hard to see that the sets \( A_k \) and \( B_k \) are clopen (some of them possibly empty), disjoint subsets of \( A \) and \( B \), respectively and \( \mu(A_k) = \mu(B_k) \) for each \( k \) and every invariant measure \( \mu \). Consider the remainder sets

\[
A_0 = A \setminus \left( \bigcup_{k=1}^{\infty} A_k \right) \quad \text{and} \quad B_0 = B \setminus \left( \bigcup_{k=1}^{\infty} B_k \right),
\]
or in the finite case
\[ A_0 = A \setminus \left( \bigcup_{k=1}^{n} A_k \right) \quad \text{and} \quad B_0 = B \setminus \left( \bigcup_{k=1}^{n} B_k \right). \]

Clearly, for each invariant measure \( \mu \) we have \( \mu(B_0) \geq \varepsilon \). We claim that \( \mu(A_0) = 0 \).

It suffices to consider an ergodic measure. But if \( \mu(A_0) \) was positive, then, by ergodicity, there would exist an \( x \in A_0 \) and \( g = g_k \) (for some \( k \)) such that \( g_k(x) \in B_0 \). This is a contradiction, as, by construction, no orbit starting in \( A_0 \) visits the set \( B_0 \).

Now, by countable additivity of the measures, we obtain, for each invariant measure \( \mu \),
\[
\lim_{k \to \infty} \mu \left( A \setminus \left( \bigcup_{i=1}^{k} A_i \right) \right) = 0.
\]

Clearly, the limit is decreasing. Since the measured sets are clopen, the above measure values viewed as functions on the set of invariant measures are continuous, and thus the convergence is uniform. Let \( \delta > 0 \) be strictly smaller than \( \frac{\varepsilon}{C} \). Then, for \( k \) large enough we have, simultaneously for all invariant measures \( \mu \),
\[
\mu \left( A \setminus \left( \bigcup_{i=1}^{k} A_i \right) \right) \leq \delta \leq \frac{\varepsilon}{C} \mu \left( B \setminus \left( \bigcup_{i=1}^{k} B_i \right) \right).
\]

By the weak comparison assumption, we get
\[ A \setminus \left( \bigcup_{i=1}^{k} A_i \right) \preceq B \setminus \left( \bigcup_{i=1}^{k} B_i \right), \]
which, together with the obvious fact that \( \bigcup_{i=1}^{k} A_i \preceq \bigcup_{i=1}^{k} B_i \), completes the proof of \( A \preceq B \).

**Remark 6.5.** The above proof shows also that every finite group \( G = \{ g_1, g_2, \ldots, g_n \} \) has the comparison property. For such a group we have \( A_0 = A \setminus (\bigcup_{i=1}^{n} A_i) \). The fact that \( A_0 \) has measure 0 for all invariant measures implies that it is empty.

**Remark 6.6.** In the definition of comparison, it suffices to consider only disjoint clopen sets \( A, B \). Indeed, \( \{ A \cap B, A \setminus B \} \) is a clopen partition of \( A \), and \( g_0 = e \) sends \( A \cap B \) inside \( B \), so if \( (A \setminus B) \preceq (B \setminus A) \) then also \( A \preceq B \). Also note that, for any measure \( \mu \), \( \mu(A) < \mu(B) \) if and only if \( \mu(A \setminus B) < \mu(B \setminus A) \).

It is known that many important countable amenable groups, for instance \( \mathbb{Z}, \mathbb{Z}^d \), have the comparison property. However, the following question remains open:

**Question 6.7.** Does every countable amenable group have the comparison property?

Later in this paper we will provide a positive answer in a large class of groups.

### 6.2. Banach density interpretation of the comparison property

Now we provide a characterization of the comparison property of a countable amenable group in terms of Banach density advantage for subsets of the group.
6. THE COMPARISON PROPERTY 47

6.2.1. Passing between clopen subsets of $X$ and subsets of $G$. This subsection contains fairly standard tools, often exploited in symbolic dynamics. We include them for completeness and as an opportunity to introduce our notation. We continue to assume that $G$ is a countable amenable group.

(A) From clopen subsets of $X$ to subsets of $G$. First suppose that $G$ acts on a zero-dimensional compact metric space in which we have two disjoint clopen sets $A$ and $B$. Define a map $\pi_{AB} : X \to \{0, 1, 2\}^G$ by the formula

$$
(\pi_{AB}(x))_g = \begin{cases}
1 & \iff g(x) \in A \\
2 & \iff g(x) \in B \\
0 & \iff (A \cup B)^c,
\end{cases}
$$

respectively ($g \in G$). As easily verified, $\pi_{AB}$ is continuous and intertwines the action on $X$ with the shift action, in other words, it is a topological factor map onto its image $Y_{AB} = \pi_{AB}(X)$, which is a subshift, in which we can distinguish two natural clopen sets, the cylinders $\{1\}$ and $\{2\}$. Notice that $\pi_{AB}^{-1}(\{1\}) = A$ and $\pi_{AB}^{-1}(\{2\}) = B$, hence for every invariant measure $\mu$ on $X$ we have $\mu(A) = \nu(\{1\})$ and $\mu(B) = \nu(\{2\})$, where $\nu = \pi_{AB}(\mu)$ is the image of $\mu$ on $Y_{AB}$ given by $\nu(\cdot) = \mu(\pi_{AB}^{-1}(\cdot))$. It is well known that $\pi_{AB}$ is a surjection onto the set $\mathcal{M}_G(Y_{AB})$ (from now on abbreviated as $\mathcal{M}_{AB}$) of all shift-invariant measures on $Y_{AB}$. For each $x \in X$ we define two subsets of $G$,

$$
\begin{align*}
A_x &= \{g : g(x) \in A\} = \{g : (\pi_{AB}(x))_g = 1\} = \{g : g(\pi_{AB}(x)) \in \{1\}\}, \\
B_x &= \{g : g(x) \in B\} = \{g : (\pi_{AB}(x))_g = 2\} = \{g : g(\pi_{AB}(x)) \in \{2\}\}.
\end{align*}
$$

In the above context we can define new notions:

Definition 6.8. We fix in $G$ a Følner sequence $(F_n)$. The terms

$$
\begin{align*}
\underline{D}(B) &= \liminf_{n \to \infty} \inf_{x \in X} D_{F_n}(B_x), \\
\overline{D}(B) &= \limsup_{n \to \infty} \sup_{x \in X} D_{F_n}(B_x), \\
\underline{D}(B, A) &= \liminf_{n \to \infty} \inf_{x \in X} D_{F_n}(B_x, A_x),
\end{align*}
$$

will be called the uniform lower Banach density of (visits of the orbits in) $B$, uniform upper Banach density of $B$ and uniform Banach density advantage of $B$ over $A$.

A statement analogous to Proposition 4.12 holds:

Lemma 6.9. The values of $\underline{D}(B)$, $\overline{D}(B)$ and $\underline{D}(B, A)$ do not depend on the choice of the Følner sequence, the limits superior and inferior in the definition are in fact limits, and moreover

$$
\begin{align*}
\underline{D}(B) &= \sup_F \inf_{x \in X} D_F(B_x), \\
\overline{D}(B) &= \inf_F \sup_{x \in X} D_F(B_x), \\
\underline{D}(B, A) &= \sup_F \inf_{x \in X} D_F(B_x, A_x),
\end{align*}
$$

where $F$ ranges over all finite subsets of $G$.

Proof. The proof is identical to the proof of Proposition 4.12, with the only difference that Lemma 4.4 applies to the sets $A_x, B_x$ whenever $F_n$ is $(F, \varepsilon)$-invariant, simultaneously for all $x \in X$. \qed
In a moment we will connect the above notions with the values assumed by the invariant measures on \(X\) on the sets \(A\) and \(B\).

\(B\) From subsets of \(G\) to clopen subsets of \(X\). We will now describe the opposite passage: from subsets of \(G\) to clopen subsets of some zero-dimensional compact metric space on which we have a \(G\)-action. Suppose we have two disjoint subsets \(A\) and \(B\) of \(G\). Then they determine an element \(y^{AB}\) of the symbolic space \(\{0,1,2\}^G\), given by the rule

\[
y^{AB}_g = \begin{cases} 
1 & \iff g \in \begin{cases} A \\
B \\
(A \cup B)^c
\end{cases} \\
0
\end{cases}
\]

respectively (\(g \in G\)). The shift-orbit closure of \(y^{AB}\), i.e., the set

\[
Y^{AB} = \{g(y^{AB}) : g \in G\}
\]

is a subshift, which we will call the subshift associated with the sets \(A, B\). The set of its invariant measures, \(\mathcal{M}_G(Y^{AB})\), will be abbreviated as \(\mathcal{M}^{AB}\). In this subshift we will distinguish two clopen sets, \(A = [1]\) and \(B = [2]\). It is almost immediate to see that if we apply the definitions of the preceding paragraph to the shift action on \(Y^{AB}\) and the above sets \(A, B\) then the factor map \(\pi_{AB}\) is the identity, and \(A_{y^{AB}} = \{g : y^{AB}_g = 1\} = A\) and \(B_{y^{AB}} = \{g : y^{AB}_g = 2\} = B\).

**Proposition 6.10.** (1) Suppose \(G\) acts on a zero-dimensional compact metric space \(X\) in which we are given two disjoint clopen sets, \(A, B\). Then

\[
\inf_{\mu \in \mathcal{M}_G(X)} \mu(B) = D(B) = \inf_{x \in X} D(B_x), \\
\sup_{\mu \in \mathcal{M}_G(X)} \mu(B) = \overline{D}(B) = \sup_{x \in X} \overline{D}(B_x), \\
\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) = D(B, A) = \inf_{x \in X} D(B_x, A_x).
\]

(2) Next suppose that \(A\) and \(B\) are disjoint subsets of \(G\). Consider the cylinders \([1]\) and \([2]\) in the subshift \(Y^{AB}\) associated with these sets. Then

\[
\inf_{\mu \in \mathcal{M}^{AB}} \mu([2]) = D(B), \\
\sup_{\mu \in \mathcal{M}^{AB}} \mu([2]) = \overline{D}(B), \\
\inf_{\mu \in \mathcal{M}^{AB}} (\mu([2]) - \mu([1])) = D(B, A).
\]

**Proof.** In (1) we will only show the last line of equalities. The first line will then follow by plugging in \(A = \emptyset\) and the second one by considering the complement of \(B\). First suppose that we have sharp inequality \(\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) > D(B, A)\). By Lemma 6.9 there exists an \(\varepsilon > 0\) such that for every finite set \(F\), \(\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) - \varepsilon > \inf_{x \in X} D_p(B_x, A_x)\). In particular for every set \(F_n\) in an a priori selected Følner sequence, there exists some \(x_n \in X\) and \(g_n \in G\) with

\[
\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) - \varepsilon > \frac{1}{|F_n|}(|B_{x_n} \cap F_ng_n| - |A_{x_n} \cap F_ng_n|).
\]
Note that \(|B_{x_n} \cap F_ng_n| = |\{ f \in F_n : f g_n(x_n) \in B \}|\) (and analogously for \(A\)), thus the right hand side takes on the form
\[
\frac{1}{|F_n|}(\{ f \in F_n : f g_n(x_n) \in B \} - \{ f \in F_n : f g_n(x_n) \in A \}).
\]
The function \(W \mapsto \frac{1}{|F_n|}(\{ f \in F_n : f g_n(x_n) \in W \})\) defined on Borel subsets of \(X\) is equal to the probability measure \(\frac{1}{|F_n|} \sum f \in F_n \delta_{f g_n(x_n)}\). This sequence of measures has a subsequence convergent in the weak-star topology to some \(\mu_0 \in \mathcal{M}_G(X)\). Since the characteristic functions of the clopen sets \(A, B\) are continuous, we have
\[
\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) - \varepsilon \geq \mu_0(B) - \mu_0(A),
\]
which is a contradiction. We have proved that \(\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) \leq \mathcal{D}(B, A)\).

The inequality \(\mathcal{D}(B, A) \leq \inf_{x \in X} \mathcal{D}(B_x, A_x)\) is trivial; both sides differ by changing the order of \(\limsup_n \) and \(\inf_{x}\) and on the left the infimum is applied earlier.

For the last missing inequality, \(\inf_{x \in X} \mathcal{D}(B_x, A_x) \leq \inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A))\), we shall invoke the ergodic theorem (Theorem 2.8). Notice that, given \(\varepsilon > 0\), there exists an ergodic measure \(\mu_0 \in \mathcal{M}_G(X)\) with \(\mu_0(B) - \mu_0(A) < \inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) + \varepsilon\). The ergodic theorem now implies that there exists a subsequence \((F_{n_k})_{k \in \mathbb{N}}\) of the Følner sequence (any tempered subsequence will do), and a point \(x \in X\) (in fact, \(\mu_0\)-almost every point is good) such that both
\[
\lim_{k \to \infty} \frac{1}{|F_{n_k}|} \{| f \in F_{n_k} : f(x) \in A \| = \mu_0(A),
\]
and an analogous formula holds for \(B\). Obviously, \(f(x) \in A\) or \(B\), if and only if \(f \in A_x\) or \(B_x\), respectively. Hence
\[
\lim_{k \to \infty} \frac{1}{|F_{n_k}|} |A_x \cap F_{n_k}| = \mu_0(A) \quad \text{(and similarly for } B_x \text{ and } B)\).
\]
Thus, for each sufficiently large \(k\) we have
\[
\frac{1}{|F_{n_k}|} (|B_x \cap F_{n_k}| - |A_x \cap F_{n_k}|) < \mu_0(B) - \mu_0(A) + \varepsilon < \inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) + 2\varepsilon.
\]
Clearly, the left hand side is not smaller than
\[
\inf_{g \in G} \frac{1}{|F_{n_k}|} (|B_x \cap F_{n_k}g| - |A_x \cap F_{n_k}g|) = \mathcal{D}_{F_{n_k}} (B_x, A_x).
\]
Passing to the limit over \(k\) and then applying infimum over all \(x \in X\) we obtain
\[
\inf_{x \in X} \mathcal{D}(B_x, A_x) \leq \inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) + 2\varepsilon.
\]
Since this is true for every \(\varepsilon > 0\), (1) is proved.

We pass to proving (2). As before, the last equality suffices. From (1) applied to the cylinders \(A = [1]\) and \(B = [2]\) we get
\[
\inf_{\mu \in \mathcal{M}_G} (\mu([2]) - \mu([1])) = \mathcal{D}([2], [1]) = \lim_{n \to \infty} \inf_{y \in Y^{AB}} \inf_{g \in G} \frac{1}{|F_n|} (|B_y \cap F_n g| - |A_y \cap F_n g|).
\]
The above difference \(|B_y \cap F_n g| - |A_y \cap F_n g|\) depends on the block \(y|F_n g\). Notice that we are considering a transitive subshift with the transitive point \(y^{AB}\) (i.e., whose orbit is dense in the subshift), so every block \(y|F_n g\) (for any \(y \in Y^{AB}\) and any \(g \in G\)) occurs also in \(y^{AB}\) as a block \(y^{AB}|F_n g'\) for some \(g'\) (the converse need not be true, unless \(y\) is another transitive point). Thus, for any \(n\), the infimum over
For any finite coding horizon $F$, we recall that $A_{y^{AB}} = A$ and $B_{y^{AB}} = B$. We have proved that
\[
\inf_{\mu \in \mathcal{M}^{AB}} \mu([2]) - \mu([1]) = \lim_{n \to \infty} \inf_{g \in G} \frac{1}{|F_n|} (|B \cap F_n g| - |A \cap F_n g|).
\]
The right hand side is precisely $\mathbb{D}(B, A)$. \qed

Recall the definition of a block code (Definition 2.1). Based on that, we define the following notion:

**Definition 6.11.** Let $X \subset \Lambda^G$ be a subshift. For each $x \in X$ let $A_x \subset G$ and let $\tilde{\varphi}_x : A_x \to G$ be some function. For $X' \subset X$, we will say that the family $\{\tilde{\varphi}_x\}_{x \in X'}$ is determined by a block code if there exists a block code $\Xi : \Lambda^F \to E$, where $E$ is a finite subset of $G$ (and so is $F$), such that if we denote $\varphi_x(g) = \Xi(g(x)|_F)$,
\[(x \in X, g \in G), \text{ then, for each } x \in X', \text{ the mapping from } A_x \to G, \text{ defined by } a \mapsto \varphi_x(a)a,\]
\[(a \in A_x), \text{ coincides with } \tilde{\varphi}_x. \text{ The elements } \varphi_x(a) (\text{belonging to } E) \text{ will be called the multipliers of } \tilde{\varphi}_x.\]

A simple way of checking, that a family $\{\tilde{\varphi}_x\}_{x \in X'}$ is determined by a block code, is finding a finite set $F$ such that, for any $x_1, x_2 \in X'$ and $a_1 \in A_{x_1}, a_2 \in A_{x_2}$,
\[(6.3) \quad a_1(x_1)|_F = a_2(x_2)|_F \implies \tilde{\varphi}_{x_1}(a_1)a_1^{-1} = \tilde{\varphi}_{x_2}(a_2)a_2^{-1}.\]

The following theorem connects the above definition with the relation of sub-equivalence.

**Theorem 6.12.** (1) Let $X \subset \Lambda^G$ be a subshift. Consider the pair of disjoint clopen subsets $A, B \subset X$. Then $A \preceq B$ if and only if there exists a family of functions $\tilde{\varphi}_x : G \to G$, indexed by $x \in X$, determined by a block code, such that for all $x \in X$, $\tilde{\varphi}_x$ restricted to $A_x = \{g : g(x) \in A\}$ is an injection to $B_x = \{g : g(x) \in B\}$.

(2) If, moreover, $X$ is transitive with a transitive point $x^*$, then the above condition $A \preceq B$ is equivalent to the existence of just one function $\tilde{\varphi}_{x^*}$ determined by a block code, whose restriction to $A_{x^*}$ is an injection to $B_{x^*}$.

**Proof.** (1) Firstly, suppose that $A \preceq B$. Let $\{A_1, A_2, \ldots, A_k\}$ be the clopen partition of $A$ and let $g_1, g_2, \ldots, g_k$ be the elements of $G$ such that the sets $B_i = g_i(A_i)$ are disjoint subsets of $B$. Let $E = \{g_1, g_2, \ldots, g_k\}$. Consider the mapping $\xi : X \to E^G$ given by the following rule
\[(\xi(x))_g = \begin{cases} g_i & \text{if } g(x) \in A_i, \ i = 1, 2, \ldots, k, \\ g_1 & \text{otherwise}, \end{cases} (g \in G).\]
Since the sets $A_i$ and $X \setminus A$ are clopen in $X$, the above map is continuous and, as easily verified, it is shift-equivariant. Thus, it is a topological factor map from $X$ into $E^G$. By Theorem 2.2, there exists a block code $\Xi : \Lambda^F \to E$ (with some finite coding horizon $F$) satisfying, for all $x \in X$ and $g \in G$, the equality $$(\xi(x))_g = \Xi(g(x)|_F).$$
For each $x \in X$ we define $\varphi_x : G \to E$ by $\varphi_x(g) = (\xi(x))_g$ and $\tilde{\varphi}_x : G \to G$ by $\tilde{\varphi}_x(g) = \varphi_x(g)g_i$, i.e., the family of maps $\{\tilde{\varphi}_x\}_x$ is determined by the block code $\Xi$. We need to show that, for every $x \in X$, $\tilde{\varphi}_x$ restricted to $A_i$ is an injection to $B_x$.

Throughout this paragraph we fix some $x \in X$ and skip the subscript $x$ in the writing of $A_x, B_x, \varphi_x$ and $\tilde{\varphi}_x$. For $i = 1, 2, \ldots, k$ let $A_i = A \cap \varphi^{-1}(g_i)$. Clearly, $\{A_1, A_2, \ldots, A_k\}$ is a partition of $A$ and for every $a \in A$ we have:

$$a \in A_i \iff \varphi(a) = g_i \iff (\xi(x))_a = g_i \iff a(x) \in A_i, \ (i = 1, 2, \ldots, k).$$

Further, $a(x) \in A_i$ yields $g_ia(x) \in B_i \subset B$, which implies that $g_ia \in B$. Since $g_ia = \varphi(a)a = \varphi(a)$, we have shown that $\tilde{\varphi}$ sends $A$ into $B$. For injectivity of the restriction $\tilde{\varphi}|_A$, observe that if $a_1 \neq a_2$ and both elements belong to the same set $A_i$ then their images by $\tilde{\varphi}$, equal to $g_ia_1$ and $g_ia_2$, respectively, are different by cancellativity. If $a_1 \in A_i$ and $a_2 \in A_j$ with $i \neq j$, then $\tilde{\varphi}(a_1)(x) = g_ia_1(x) \in B_i$ and $\tilde{\varphi}(a_2)(x) = g_ia_2(x) \in B_j$. Since $B_i$ and $B_j$ are disjoint, the elements $\tilde{\varphi}(a_1)$ and $\tilde{\varphi}(a_2)$ must be different.

Now suppose that there exist injections $\tilde{\varphi}_x : A_x \to B_x$ (for all $x \in X$) determined by a block code $\Xi : \Lambda^F \to E = \{g_1, g_2, \ldots, g_k\} \subset G$, where the elements $g_i$ are written without repetitions, i.e., are different for different indices $i = 1, 2, \ldots, k$. That is, denoting, for each $g \in G$,

$$\varphi_x(g) = \Xi(g(x))_F,$$

we obtain maps $\varphi_x$ such that $g \mapsto \varphi_x(g)$ restricted to $A_x$ coincides with $\tilde{\varphi}_x$. Now, for each $i = 1, 2, \ldots, k$ we define

$$A_i = A \cap \Xi^{-1}(g_i) = \{x \in A : \Xi(x)_F = g_i\}.$$

Clearly, $\{A_1, A_2, \ldots, A_k\}$ is a clopen partition of $A$. Let $x \in A_i$ (for some $i = 1, 2, \ldots, k$). Then $e \in A_x$ and thus $\tilde{\varphi}_x(e) \in B_x$, i.e., $\tilde{\varphi}_x(e)(x) \in B$. But $\tilde{\varphi}_x(e) = \varphi_x(e) = \Xi(x)_F = g_i$. We have shown that $g_i(A_i) \subset B$.

It remains to show that the sets $g_i(A_i)$ are disjoint. Suppose that for some $i \neq j$ there exists $x \in X$ belonging to both $g_i(A_i)$ and $g_j(A_j)$. This implies that $g_i$ and $g_j$ both belong to $A_x$, and $\varphi_x(g_i^{-1}) = g_i, \varphi_x(g_j^{-1}) = g_j$. But then

$$\tilde{\varphi}_x(g_i^{-1}) = \varphi_x(g_i^{-1})g_i^{-1} = g_ig_i^{-1} = e \quad \text{and} \quad \tilde{\varphi}_x(g_j^{-1}) = \varphi_x(g_j^{-1})g_j^{-1} = g_jg_j^{-1} = e,$$

which contradicts the injectivity of $\tilde{\varphi}_x$ on $A_x$.

(2) In view of (1), it suffices to show that if a block code $\Xi : \Lambda^F \to E$ determines an injection $\tilde{\varphi}_x : A_x \to B_x$ then it also determines (as usually, by the formulas $\varphi_x(g) = \Xi(g(x))_F$ and $\tilde{\varphi}_x(a) = \varphi_x(a)a$) injections $\varphi_x : A_x \to B_x$ for all $x \in X$. Fix some $x \in X$ and let $a_1 \neq a_2$ belong to $A_x$, i.e., $a_1(x), a_2(x) \in A$. Since $x^*$ is a transitive point, a point $g(x^*)$ (for some $g \in G$) is so close to $x$ that:

(a) $a_1g(x^*), a_2g(x^*) \in A$,

(b) the blocks $g(x^*)|_{F_{a_1} \cup F_{a_2}}$ and $x|_{F_{a_1} \cup F_{a_2}}$ are equal,

(c) $\forall f \in E_{a_1} \cup E_{a_2} \ f(g(x^*)) \in B \iff f(x) \in B$.

By (a), both $a_1g$ and $a_2g$ belong to $A_{x^*}$. Thus $\varphi_{x^*}(a_1g)$ and $\varphi_{x^*}(a_2g)$ are different elements of $B_{x^*}$. But

$$\varphi_{x^*}(a_1g) = \varphi_{x^*}(a_1g)a_1g \quad \text{and} \quad \varphi_{x^*}(a_2g) = \varphi_{x^*}(a_2g)a_2g,$$

which, after canceling $g$, yields

$$\varphi_{x^*}(a_1g)a_1 \neq \varphi_{x^*}(a_2g)a_2.$$
On the other hand, by (b), \( x|_{F_{A_1}} = g(x^\ast)|_{F_{A_1}} \), whence \( a_1(x)|_F = a_1g(x^\ast)|_F \), and
\[
\varphi_x(a_1) = \Xi(a_1(x)|_F) = \Xi(a_1g(x^\ast)|_F) = \varphi_x(\gamma a_1),
\]
which means that \( \tilde{\varphi}_x(\gamma a_1) = \varphi_x(a_1) = \varphi_x(\gamma a_1) \). Analogously, \( \tilde{\varphi}_x(a_2) = \varphi_x(a_2)g(a_2) \). We have shown that \( \tilde{\varphi}_x(a_1) \neq \tilde{\varphi}_x(a_2) \), i.e., \( \tilde{\varphi}_x \) restricted to \( A \) is injective.

Further, the fact that \( \tilde{\varphi}_x(a_1g) \in B_x \) yields
\[
B \ni \tilde{\varphi}_x(a_1g)(x^\ast) = \varphi_x(a_1g)a_1g(x^\ast) = \varphi_x(a_1)a_1g(x^\ast).
\]
Since \( \varphi_x(a_1)a_1 \in Ea_1 \), by (c) we get
\[
B \ni \varphi_x(a_1a_1)(x) = \tilde{\varphi}_x(a_1)(x),
\]
and hence \( \tilde{\varphi}_x(a_1) \in B_x \). We have shown that \( \tilde{\varphi}_x \) sends \( A \) injectively to \( B_x \).

\[\square\]

6.2.2. Banach density comparison property of a group.

Definition 6.13. We say that \( G \) has the Banach density comparison property if whenever \( A \subset G \) and \( B \subset G \) are disjoint and satisfy \( D(B, A) > 0 \) then, in the subshift \( Y^{AB} \) there exists an injection \( \tilde{\varphi} : A \to B \) determined by a block code (recall that \( y^{AB} \) is a transitive point in \( Y^{AB} \) and \( A = A_{\gamma A} \), \( B = B_{\gamma A} \), so the above condition is the same as that in Theorem 6.12 (2)).

Remark 6.14. It is immediate to see that any finite group has the Banach density comparison property.

We can now completely characterize the comparison property of a countable amenable group in terms of the Banach density comparison property.

Theorem 6.15. A countable amenable group \( G \) has the comparison property if and only if it has the Banach density comparison property.

Proof. The theorem holds trivially for finite groups, so we can restrict to infinite groups \( G \). Assume that \( G \) has the comparison property and let \( A, B \subset G \) be disjoint and satisfy \( D(B, A) > 0 \). Then, by Proposition 6.10 (2), taking in the subshift \( Y^{AB} \) the clopen sets: \( A = [1] \) and \( B = [2] \), we have \( \inf_{\mu \in M_{\mu}(A)}(\mu(B) - \mu(A)) > 0 \). By the assumption, \( A \preceq B \). Now, a direct application of Theorem 6.12 (2) completes the proof of the Banach density comparison property.

Let us pass to the proof of the opposite implication. Suppose that a countable amenable group \( G \) having the Banach density comparison property acts on a zero-dimensional compact metric space \( X \), in which we have selected two clopen sets \( A \) and \( B \) satisfying, for each invariant measure \( \mu \) on \( X \), the inequality \( \mu(A) < \mu(B) \). By Remark 6.16 we can assume that \( A \) and \( B \) are disjoint; and by Remark 6.2 we have \( \inf_{\mu \in M_{\mu}(A)}(\mu(B) - \mu(A)) > 0 \). This translates to \( \inf_{\mu \in M_{\mu}(X)}(\mu([2]) - \mu([1])) > 0 \) in the factor subshift \( Y_{AB} \). By Proposition 6.11 (1) applied to this subshift, we get \( D([2], [1]) > 0 \).

Since we intend to use the Banach density comparison property and Theorem 6.12 (2), we need to embed \( Y_{AB} \) in a transitive subshift \( Y \) (over the alphabet \( \{0, 1, 2\} \)). We also desire a transitive point \( y^\ast \) which satisfies \( D(B_{y^\ast}, A_{y^\ast}) > 0 \). Below we present the construction of such a transitive subshift.

Choose some positive \( \gamma < \frac{D([2], [1])}{2} \). Fix an increasing (w.r.t. set inclusion) Følner sequence \( (F_n) \) such that \( \bigcup_{n=1}^\infty F_n = G \). By choosing a subsequence we can assume that \( \sum_{i=1}^{n-1} |F_i| < \frac{1}{2^n} |F_n| \) for every \( n \) (in this place we use the assumption
that \( G \) is infinite). Next, we need to find a sequence of blocks \( C_n \in \{0, 1, 2\}^F \) each appearing as \( y_n|_{F_n} \) in some \( Y_n \in Y_{AB} \), such that every \( y \in Y_{AB} \) is a coordinatewise limit of a subsequence \( C_{n_k} \) of the selected blocks. Finally, we need to find a sequence \( \gamma \) of elements of \( G \) such that the sets \( F_n F_n^{-1} F_n g_n \) are disjoint. All the above steps are possible and easy. Once they are completed, \( y^* \) is defined by the rule: for each \( n \) and \( f \in F_n \) we put \( y^*_f g_n = C_n(f) \), and for all \( g \) outside the union \( \bigcup_{n=1}^{\infty} F_n g_n \), we put \( y^*_g = 2 \). We let \( Y \) be the closure of the orbit of \( y^* \).

The following properties hold:

- \( Y \supseteq Y_{AB} \),
- \( D(B_{y^*}, A_{y^*}) \geq \gamma > 0 \).

The first property is obvious by construction: each \( y \in Y_{AB} \) is the limit of a sequence of blocks \( C_{n_k} \), hence it is also the limit of the sequence of elements \( g_{n_k}(y^*) \), and thus it belongs to \( Y \).

We need to prove the latter property. By the definition of \( D([2], [1]) \) in the subshift \( Y_{AB} \), there exist arbitrarily large indices \( n_k \) such that

\[
(6.4) \quad |\{f \in F_{n_k} : y^*_f g = 2\}| - |\{f \in F_{n_k} : y^*_f g = 1\}| \geq \gamma |F_{n_k}|,
\]

for all \( y \in Y_{AB} \) and \( g \in G \). It suffices to show an analogous property for \( y^* \).

Fix some \( g \in G \) and observe the block \( y^* |_{F_{n_k} g} \). The set \( F_{n_k} g \) either does not intersect any of the sets \( F_{n} g_m \) with \( m \geq n_k \) or intersects one of them (say \( F_{n_0} g_m \) with \( m_0 \geq n_k \)).

In the first case, the block \( y^* |_{F_{n_k} g} \) consists mostly of symbols 2, as all symbols different from 2 appear in \( y^* \) only over the intersection of \( F_{n_k} g \) with the union of the sets \( F_i g_i \) with \( i < n_k \), the percentage of such symbols in \( y^* |_{F_{n_k} g} \) is at most

\[
1 \left( \frac{1}{|F_{n_k}|} \sum_{i=1}^{n_k-1} |F_i g_i| \right) = \frac{1}{|F_{n_k}|} \sum_{i=1}^{n_k-1} |F_i| < \frac{1 - \gamma}{2}.
\]

Thus, in this case we have

\[
(6.5) \quad |\{f \in F_{n_k} : y^*_f g = 2\}| - |\{f \in F_{n_k} : y^*_f g = 1\}| \geq \gamma |F_{n_k}|.
\]

In the latter case, we have \( g \in F_{n_k}^{-1} F_{n_0} g_{m_0} \), hence \( F_{n_k} g \subset F_{n_k} F_{n_k}^{-1} F_{n_0} g_{m_0} \subset F_{n_0} F_{n_k}^{-1} F_{m_0} g_{m_0} \). By disjointness of the sets \( F_n F_n^{-1} F_{n_0} g_n \), \( F_{n_k} g \) does not intersect any set \( F_n F_n^{-1} F_{n_0} g_n \) (and hence also \( F_{n_0} g_n \)) with \( n \neq m_0 \). We will compare the block \( y^* |_{F_{n_k} g} \) with the block \( y_{m_0} |_{F_{n_k} g g_{m_0}^{-1}} \). We can write

\[
F_{n_k} g = (F_{n_k} g \cap F_{m_0} g_{m_0}) \cup (F_{n_k} g \setminus F_{m_0} g_{m_0}),
\]

and likewise

\[
F_{n_k} g g_{m_0}^{-1} = (F_{n_k} g g_{m_0}^{-1} \cap F_{m_0}) \cup (F_{n_k} g g_{m_0}^{-1} \setminus F_{m_0}).
\]

By the definition of \( y^* \), the block \( y^* |_{F_{n_k} g \cap F_{m_0} g_{m_0}} \) is identical to \( y_{m_0} |_{F_{n_k} g g_{m_0}^{-1} \cap F_{m_0}} \), while \( y^* |_{F_{n_k} g \setminus F_{m_0} g_{m_0}} \) contains just the symbols 2. Thus the difference

\[
|\{f \in F_{n_k} : y^*_f g = 2\}| - |\{f \in F_{n_k} : y^*_f g = 1\}|
\]

is not smaller than

\[
|\{f \in F_{n_k} : (y_{m_0})_f g g_{m_0}^{-1} = 2\}| - |\{f \in F_{n_k} : (y_{m_0})_f g g_{m_0}^{-1} = 1\}|.
\]

Since \( y_{m_0} \in Y_{AB} \), (6.4) implies that the latter expression is at least \( \gamma |F_{n_k}| \). We have proved (6.5) also in this case.
We have proved that $\mathcal{D}(B_y^*, A_y^*) \geq \gamma > 0$. Now, the Banach density comparison property of $G$ implies that there exists an injection $\hat{\varphi}$ from $A_y^*$ to $B_y^*$ determined by a block code. Thus, by Theorem 6.12 (2), we get $[1] \preccurlyeq [2]$ in the transitive subshift $Y$, and by restriction to a closed invariant set the same holds in $Y_{AB}$, which, by an application of $\pi^{-1}_{AB}$, translates to $A \preccurlyeq B$ in $X$. □

6.2.3. Comparison property via finitely generated subgroups.

**Definition 6.16.** In a group $G$, a set $R$ such that $\bigcup_{n=1}^{\infty} (R \cup R^{-1})^n = G$ is called a generator of $G$. A group having a finite generator is called finitely generated.

**Lemma 6.17.** Let $G$ act on a zero-dimensional compact metric space $X$. Let $A, B \subset X$ be two disjoint clopen sets. Then

$$\sup_{H} \inf_{\mu \in \mathcal{M}_H(X)} (\mu(B) - \mu(A)) = \sup_{H'} \inf_{\mu \in \mathcal{M}_{H'}(X)} (\mu(B) - \mu(A)) = \inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A))$$

where $H$ ranges over all finitely generated subgroups of $G$ and $H'$ ranges over all subgroups of $G$.

**Proof.** The inequality $\leq$ on the left hand side is trivial, while the second inequality $\leq$ follows easily from the fact that every measure invariant under the action of $G$ is invariant under the action of $H'$ for any subgroup $H'$ of $G$.

We need to prove the last missing inequality. By Proposition 6.10 (1), we have $\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) = \mathcal{D}(B, A)$. Then, for any positive $\delta$, there exists a finite set $F$ such that

$$\frac{1}{|F|}(|B_x \cap F g| - |A_x \cap F g|) > \mathcal{D}(B, A) - \delta$$

for every $x \in X$ and all $g \in G$, in particular for all $g \in H$, where $H$ is the subgroup generated by $F$. Thus, for every $x \in X$, we have

$$\inf_{g \in H} \frac{1}{|F|}(|B_x \cap F g| - |A_x \cap F g|) = \mathcal{D}(B, A) - \delta.$$ 

Since $F \subset H$ and $g \in H$, we have $A_x \cap F g = (A_x \cap H) \cap F g$. Note that $A_x \cap H$ equals the set $A_x$ defined for the induced action of $H$ on $X$ (and analogously for $B_x$). Thus, the expression on the left hand side above equals $\mathcal{D}(B_x, A_x)$ evaluated for the action of $H$ on $X$. Now, Lemma 6.13 implies $\mathcal{D}(B_x, A_x) \geq \mathcal{D}(B, A) - \delta$ for every $x \in X$ (where $\mathcal{D}(B, A)$ is evaluated for the action of $H$ on $X$, and $\mathcal{D}(B, A)$ is evaluated for the action of $G$ on $X$), and Proposition 6.11 (1) yields

$$\inf_{\mu \in \mathcal{M}_H(X)} (\mu(B) - \mu(A)) \geq \mathcal{D}(B, A) - \delta = \inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) - \delta.$$ 

After applying the supremum over $H$ on the left we can ignore $\delta$ on the right. □

**Proposition 6.18.** A countable amenable group $G$ has the comparison property if every finitely generated subgroup $H$ of $G$ has it.

**Proof.** Let $G$ act on a zero-dimensional compact metric space $X$ and let $A, B \subset X$ be two disjoint clopen sets satisfying $\mathcal{D}(B, A) > 0$. By the preceding lemma (and by Proposition 6.10 (1) used twice), there exists a finitely generated subgroup $H$ of $G$ such that the inequality $\mathcal{D}(B, A) > 0$ holds also if $\mathcal{D}$ is evaluated for the action of $H$. By the comparison property of $H$, we get that $A \preccurlyeq B$ in this latter action. But this clearly implies the same subequivalence in the action by $G$. □
Remark 6.19. By the proof of Lemma 6.17, if \((H_n)\) is an increasing sequence of subgroups of \(G\) such that \(G = \bigcup_{n=1}^{\infty} H_n\) then
\[
\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A)) = \lim_{n \to \infty} \inf_{\mu \in \mathcal{M}_{H_n}(X)} (\mu(B) - \mu(A)).
\]
Thus, in Proposition 6.18, the assumption can be weakened to the existence of an increasing sequence \((H_n)\) of subgroups of \(G\) such that \(G = \bigcup_{n=1}^{\infty} H_n\), and every \(H_n\) has the comparison property.

Remark 6.20. The converse implication in Proposition 6.18 is a bit mysterious. On the one hand, since there are no examples of countable amenable groups without the comparison property, clearly, there is no counterexample for the implication in question. On the other hand, we failed to deduce the comparison property of a subgroup of \(G\) from the comparison property of the group \(G\).

6.2.4. Subexponential groups.

Definition 6.21. A finitely generated group \(G\) with a generator \(R\) has subexponential growth if \(|(R \cup R^{-1})^n|\) grows subexponentially, i.e.,
\[
\lim_{n \to \infty} \frac{1}{n} \log |(R \cup R^{-1})^n| = 0.
\]
It is very easy to see that subexponential growth of a finitely generated group \(G\) implies subexponential growth of \(|K^n|\) for any finite set \(K \subset G\) and thus does not depend on the choice of a finite generator.

Definition 6.22. A countable group \(G\) (not necessarily finitely generated) is called subexponential if every its finitely generated subgroup has subexponential growth.

It is a standard fact that a group \(G\) is amenable if and only if so is every finitely generated subgroup of \(G\). It is also known that finitely generated groups with subexponential growth are amenable \([2]\), hence every subexponential group is amenable. This is why we can omit the amenability assumption when dealing with subexponential groups. Examples of subexponential groups are: Abelian, nilpotent and virtually nilpotent groups. These examples have polynomial growth, but there are also examples of countable groups with intermediate growth rates \([30]\). By a recent result \([8]\), all finitely generated groups, which admit an increasing sequence of sets \((A_n)_{n \in \mathbb{N}}\) with \(G = \bigcup_{n=1}^{\infty} A_n\) and \(|A_n^2| < C|A_n|\) for some constant \(C > 0\), are virtually nilpotent and hence subexponential. In particular, this applies to finitely generated groups possessing a symmetric Følner sequence \((F_n)\) satisfying Tempelman’s condition \(|F_n^{-1}F_n| \leq C|F_n|\).

6.3. Comparison property of subexponential groups. This section contains our next important result: every subexponential group has the comparison property. The theorem is preceded by a few key definitions and lemmas.

6.3.1. Correction chains. We now introduce the key tool in the proof of the main result. The term \((\phi, E)\)-chain reflects a remote analogy to \((f, \varepsilon)\)-chains in topological dynamics. Throughout this subsection, we let \(A, B\) denote two disjoint subsets of a countable group \(G\).

Definition 6.23. Given a partially defined bijection \(\phi : A' \to B'\), where \(A' \subset A\) and \(B' \subset B\), such that all multipliers \(\phi(a)a^{-1}\) belong to a finite set \(E \subset G\),

by a \((\phi, E)\)-chain of length \(2n\) (or briefly just a chain) we will mean a sequence \(C = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n)\) of \(2n\) different elements alternately belonging to \(A\) and \(B\), such that

for each \(i = 1, 2, \ldots, n\), \(b_i \in Ea_i,\)

and

for each \(i = 1, 2, \ldots, n - 1\), \(b_i \in B'\), \(a_{i+1} \in A'\) and \(b_i = \phi(a_{i+1})\)

(in particular, \(b_i \in Ea_{i+1}\)).

The \((\phi, E)\)-chains starting at a point \(a_1 \in A \setminus A'\) and ending at a point \(b_n \in B \setminus B'\) are of special importance, as they allow one to “correct” the mapping and include \(a_1\) in the domain and \(b_n\) in the range.

**Definition 6.24.** A \((\phi, E)\)-chain \(C = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n)\) will be called a \(\phi\)-correction chain if \(a_1 \in A \setminus A'\) and \(b_n \in B \setminus B'\). With each \(\phi\)-correction chain \(C\) we associate the correction of \(\phi\) along \(C\). The corrected map denoted by \(\phi^C\) is defined on \(A' \cup \{a_1\}\) onto \(B' \cup \{b_n\}\), as follows: for each \(i = 1, 2, \ldots, n\) we let

\[\phi^C(a_i) = b_i,\]

and for all other points \(a \in A'\) we let \(\phi^C(a) = \phi(a)\).

The correction may be visualized as follows (solid arrows in the top row represent the map \(\phi\) and in the bottom row they represent \(\phi^C\); the dashed arrows represent the “\(E\)-proximity relation” \(b \in Ea\)): \[a_1 \rightarrow b_1 \leftarrow a_2 \rightarrow b_2 \leftarrow a_3 \ldots b_{n-1} \leftarrow a_n \rightarrow b_n\]

\[\downarrow\]

\[a_1 \rightarrow b_1 \leftarrow a_2 \rightarrow b_2 \leftarrow a_3 \ldots b_{n-1} \leftarrow a_n \rightarrow b_n\]

(the dashed arrows become solid, the solid arrows are removed from the map). Notice that \(\phi^C\) still has all its multipliers \(\phi^C(a)\) in the set \(E\).

The problem with the correction chains is that the corresponding corrections of \(\phi\) usually cannot be applied simultaneously. The correction chains may collide with each other, i.e., pass through common points and then the corresponding corrections rule each other out. To manage this problem we need to learn more about the possible collisions and then carefully select a family of mutually non-colliding correction chains. The details of this selection are given below.

**Definition 6.25.** Two \(\phi\)-correction chains collide if they have a common point.

Since the starting points of \(\phi\)-correction chains belong to \(A \setminus A'\), the ending points belong to \(B \setminus B'\), other odd points (counting along the chain) belong to \(A'\), other even points belong to \(B'\), where the above four sets are disjoint, and each even point is tied to the following odd point by the inverse map \(\phi^{-1}\), each collision between two \(\phi\)-correction chains, say \(C = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n)\) and \(C' = (a'_1, b'_1, a'_2, b'_2, \ldots, a'_m, b'_m)\), is of one of the following three types:

- **common start:** \(a_1 = a'_1\),
- **common end:** \(b_n = b'_m\),
- all other collisions occur in pairs \((b_i, a_{i+1}) = (b'_j, a'_{j+1})\) for some \(1 \leq i < n\) and \(1 \leq j < m\).

Of course, two chains may have more than one collision. Note that the definition of a \((\phi, E)\)-chain eliminates the possibility of “self-collisions” in one chain.
DEFINITION 6.26. Given a \((\phi, E)\)-chain \(C = (a_1, b_1, a_2, b_2, a_3, \ldots, a_n, b_n)\), the sequence \(n(C) = (p_1, q_1, p_2, q_2, \ldots, p_{n-1}, q_{n-1}, p_n)\), where \(p_i = b_i a_i^{-1} (i = 1, 2, \ldots, n)\) and \(q_i = b_i a_{i+1}^{-1} (i = 1, 2, \ldots, n-1)\), will be called the name of \(C\).

Notice that the name is always a sequence of elements of \(E\), of length \(2n - 1\).

LEMMA 6.27. If two different \(\phi\)-correction chains have the same name (note that their lengths are then equal) and collide with each other then each of them collides also with a strictly shorter \(\phi\)-correction chain.

PROOF. It is obvious that if two \(\phi\)-correction chains with the same name, say

\[ C = (a_1, b_1, a_2, b_2, a_3, \ldots, a_n, b_n), \quad C' = (a'_1, b'_1, a'_2, b'_2, a'_3, \ldots, a'_n, b'_n), \]

have the common start \(a_1 = a'_1\) or the common end \(b_n = b'_n\), or a common pair \((b_i, a_{i+1}) = (b'_i, a'_{i+1})\) with the same index \(i = 1, 2, \ldots, n - 1\), then the chains are equal. The only possible collision between two different \(\phi\)-correction chains with the same name is that they have a common pair \((b_i, a_{i+1}) = (b'_j, a'_{j+1})\) with \(i \neq j\). Let \(i_0\) be the smallest index appearing in the role of \(i\) or \(j\) in the collisions of \(C\) and assume that it plays the role of \(i\) (with some corresponding \(j\)). Then

\[ (a_1, b_1, a_2, b_2, a_3, \ldots, a_{i_0}, b_{i_0}, a_{i_0+1}, b'_{i_0+1}, a'_{j+2}, \ldots, a'_n, b'_n) \]

is a \(\phi\)-correction chain (it has no self-collisions) of length strictly smaller than \(2n\), and clearly it collides with both \(C\) and \(C'\).

We enumerate \(E\) (arbitrarily) as \(\{g_1, g_2, \ldots, g_k\}\). We define

\[ N = \bigcup_{n=1}^{\infty} E^{\times 2n-1}, \]

which means the disjoint union of the \((2n-1)\)-fold Cartesian products of copies of \(E\). This set can be interpreted as the collection of all “potential” names of the correction chains of any partially defined bijection from \(A\) to \(B\) with the multipliers in \(E\). The enumeration of \(E\) induces the following linear order on \(N\):

\[ n < n' \iff |n| < |n'| \lor (|n| = |n'| \land n < n'), \]

where \(|n|\) denotes the length of \(n\) and the last inequality is with respect to the lexicographical order on \(E^{\times |n|}\).

DEFINITION 6.28. A \(\phi\)-correction chain \(C\) is minimal if it does not collide with any other \(\phi\)-correction chain whose name precedes \(n(C)\) in the above defined order on \(N\).

LEMMA 6.29. Minimal \(\phi\)-correction chains do not collide with each other.

PROOF. If two \(\phi\)-correction chains with different names collide, one of them is not minimal. If two \(\phi\)-correction chains with the same name collide, by Lemma 6.27 none of them is minimal.

LEMMA 6.30. Assume that \(E\) is a symmetric set containing the unity \(e\) and let \(a_1 \in A \setminus A'\). If there is a \(\phi\)-correction chain \(C\) of length \(2n\), starting at \(a_1\), then there exists a minimal \(\phi\)-correction chain of length at most \(2n\) contained in the finite set \(E^{s(n)} a_1\) (where \(s(n)\) depends only on \(|E|\) and \(n\)).
PROOF. If $C$ itself is not minimal then it collides with a $\phi$-correction chain $C_1$ with $n(C_1) < n(C)$ in $N$. Clearly, $C_1$ is entirely contained in $E^{2n}a_1$. If $C_1$ is not minimal, then it collides with some $C_2$, whose name precedes that of $C_1$ (and hence also that of $C$). Now, $C_2$ is contained in $E^{6n}a_1$. This recursion may be repeated at most $\sigma_n - 1 = \sum_{i=1}^{n} |E|^{2n} - 1$ times, because this number estimates the number of names preceding $n(C)$. So, before $\sigma_n$ steps are performed, a minimal $\phi$-correction chain must occur. Its length is at most $2n$ and it is entirely contained in $E^{2n\sigma_n}a_1$. □

It is the following lemma, where subexponentiality of the group comes into play. We also exploit the notion of tilings.

**Lemma 6.31.** Let $G$ be a subexponential group. Let $T$ be a tiling of $G$ and let $S$ denote the set of all shapes of $T$. Denote $E = \bigcup_{S \in S} SS^{-1}$. Let $A, B$ be disjoint subsets of $G$ satisfying, for some $\varepsilon > 0$ and every tile $T$ of $T$, the inequality

$$|B \cap T| - |A \cap T| > \varepsilon |T|.$$  

Let $N \geq 1$ be such that for any $n \geq N$,  

$$\frac{1}{n} \log |(E^2)^n| < \log(1 + \varepsilon)$$

(by the subexponentiality assumption, since $E^2$ is finite, such an $N$ exists). Then, for any partially defined bijection $\phi : A' \to B'$ with $A' \subset A$, $B' \subset B$, such that all multipliers $\phi(a)^{-1}$ are in $E$, for every point $a_1 \in A \setminus A'$, there exists a $\phi$-correction chain of length at most $2N$, starting at $a_1$ (and ending in $B \setminus B'$).

**Proof.** For each tile $T$ of $T$ we have

$$\frac{|B \cap T|}{|A \cap T|} \geq \frac{\varepsilon |T|}{|A \cap T|} + 1 \geq 1 + \varepsilon$$

(including the case when the denominator equals 0). Clearly, any $T$-saturated finite set $Q$, i.e., being a union of tiles of $T$, also satisfies

$$\frac{|B \cap Q|}{|A \cap Q|} \geq 1 + \varepsilon.$$

For a set $P \subset G$, we define the $T$-saturation $P^T$ of $P$ as the union of all tiles intersecting $P$:

$$P^T = \bigcup \{T \in T : P \cap T \neq \emptyset \}.$$  

Obviously, $P^T \subset EP$.

Consider a point $a_1 \in A \setminus A'$ (if $A \setminus A' = \emptyset$ then the statement of the theorem holds trivially). Let $T$ be the tile of $T$ containing $a_1$, i.e., $T = \{a_1\}^T$. Since $T$ contains $a_1$ (and thus $|A \cap T| \geq 1$), we have $|B \cap T| \geq 1 + \varepsilon$. There exist $(\phi, E)$-chains of length 2 from $a_1$ to every $b \in B \cap T$. Now, there are two options:

- either at least one of these chains is a $\phi$-correction chain (and then the construction is finished),
- or none of these chains is a $\phi$-correction chain, i.e., $B' \cap T = B \cap T$.

In the latter option we have $|B' \cap T| = |B \cap T| \geq 1 + \varepsilon$, i.e., denoting

$$P_1 = \{a_1\} \quad \text{and} \quad Q_1 = T = P_1^T,$$

we have

$$|B' \cap Q_1| \geq 1 + \varepsilon.$$
From now on we continue by induction. Suppose that for some $n \geq 1$ we have defined a $\mathcal{T}$-saturated set $Q_n$ such that

1. For every $b \in B \cap Q_n$ there exists a $(\phi, E)$-chain of length at most $2n$ from $a_1$ to $b$,
2. $B \cap Q_n = B' \cap Q_n$ (i.e., there are no $\phi$-correction chains starting at $a_1$ and ending in $Q_n$), and
3. $|B' \cap Q_n| \geq (1 + \varepsilon)^n$.

Then we define $P_{n+1} = \phi^{-1}(Q_n) = \phi^{-1}(B' \cap Q_n)$. Bijectivity of $\phi$ implies that $|P_{n+1}| \geq (1 + \varepsilon)^n$. Let $Q_{n+1}$ denote the $\mathcal{T}$-saturation $P_{n+1}^\mathcal{T}$. Every point $b \in B \cap Q_{n+1}$ is of the form $g\phi^{-1}(b')$ with $g \in E$ and $b' \in B' \cap Q_n$, and, by (1), $b'$ can be reached from $a_1$ by a $(\phi, E)$-chain of length at most $2n$. Thus there exists a $(\phi, E)$-chain of length at most $2(n + 1)$ from $a_1$ to every $b \in B \cap Q_{n+1}$. There are two options:

- either at least one of these chains is a $\phi$-correction chain (then the construction is finished),
- or $B \cap Q_{n+1} = B' \cap Q_{n+1}$.

Suppose the latter option occurs. Since $Q_{n+1}$ is $\mathcal{T}$-saturated, we have

$$|B' \cap Q_{n+1}| = |B \cap Q_{n+1}| \geq (1 + \varepsilon)|A \cap Q_{n+1}| \geq (1 + \varepsilon)|P_{n+1}| \geq (1 + \varepsilon)^{n+1}.$$  

Now, (1)–(3) are fulfilled for $n + 1$, so the induction can be continued.

Notice that for each $n$, $Q_n \subset EP_n$ and, by symmetry of the set $E$, $P_{n+1} \subset EQ_n$. As a consequence, we have $Q_{n+1} \subset E^{2n+1}a_1 \subset (E^2)^{n+1}a_1$, and if the latter of the above options occurs, we have

$$|(E^2)^{n+1}| \geq |Q_{n+1}| \geq |B' \cap Q_{n+1}| \geq (1 + \varepsilon)^{n+1},$$

which implies that $n + 1 < N$ by the assumption. So, $n = N - 2$ is the last integer for which nonexistence of $\phi$-correction chains of length $2(n + 1)$ is possible. In the worst case scenario a correcting chain of length $2N$ must already exist.

**Remark 6.32.** It is absolutely crucial in the proof that we are using a tiling, not a quasitiling leaving some part of $G$ uncovered by the tiles. In such case, $a_1$ may be uncovered by the tiles, moreover, we would have no control as to how many elements of $P_{n+1} = \phi^{-1}(Q_n)$ are “lost” in the untiled part of $G$.

**6.3.2. Proof of the comparison property of subexponential groups.**

**Theorem 6.33.** Every subexponential group $G$ has the comparison property.

**Proof.** By Proposition 6.13, it suffices to prove the theorem for finitely generated groups $G$ with subexponential growth, and Theorem 6.15 allows us to focus on the Banach density comparison property. So, let $G$ be a finitely generated group with subexponential growth. Let $A, B \subset G$ be disjoint and satisfy $\bar{D}(B, A) > 0$. All we need is, in the subshift $Y^{AB}$, to construct an injection $\tilde{\varphi} : A \to B$ determined by a block code.

By Lemma 5.1.1, there exists a finite set $F \subset G$ such that $\bar{D}_F(B, A) > 5\varepsilon$ for some positive $\varepsilon$. By Theorem 4.20, there exists an $(F, \varepsilon)$-invariant tiling $T$ of $G$.

We let $S$ denote the set of all shapes of $\mathcal{T}$. By Lemma 4.4, for every shape $S$ of $\mathcal{T}$ we have $\bar{D}_S(B, A) > \varepsilon$, in particular,

$$|B \cap T| - |A \cap T| > \varepsilon|T|,$$

for every tile $T$ of $\mathcal{T}$. Let $E = \bigcup_{S \in \mathcal{S}} SS^{-1}$ and say $E = \{g_1, g_2, \ldots, g_k\}$. 

We will build the desired injection \( \tilde{\varphi} : A \to B \) in a series of steps. The first approximation of \( \tilde{\varphi} \) is the map \( \varphi_1 \) defined on a subset of \( A \) by a procedure similar to that used in the proof of Lemma \( \ref{lem:approximation} \). We let \( A_1 = A \cap g_1^{-1}(B) \), and \( B_1 = g_1(A_1) \subset B \) and then, for each \( j = 2, 3, \ldots, k \) we define inductively

\[
A_j = A \setminus \left( \bigcup_{i=1}^{j-1} A_i \right) \cap g_j^{-1} \left( B \setminus \bigcup_{i=1}^{j-1} B_i \right) \quad \text{and} \quad B_j = g_jA_j \subset B.
\]

On each set \( A_j \) (with \( j = 1, 2, \ldots, k \)), \( \varphi_1 \) is defined as the multiplication on the left by \( g_j \). We let \( A_1' = \bigcup_{i=1}^{k} A_i \subset A \) and \( B_1' = \bigcup_{i=1}^{k} B_i \subset B \) denote the domain and range of \( \varphi_1 \), respectively. The rule behind the construction of \( \varphi_1 \) is as follows: for each \( a \in A \) we first check whether \( g_1a \in B \) and for those \( a \) for which this is true, we assign \( \varphi_1(a) = g_1a \). For other points \( a \) we check whether \( g_2a \in B \) and, unless \( g_2a \) has already been assigned as \( \varphi_1(a') \) (for some \( a' \in A \)) in the previous step, we assign \( \varphi_1(a) = g_2a \). And so on: at step \( i \) we assign \( \varphi_1(a) = g_ia \) if \( g_ia \in B \), unless \( g_ia \) has already been assigned as \( \varphi(a') \) (for some \( a' \in A \)) at steps \( 1, 2, \ldots, i-1 \). We stop when \( i = k \). From this description it is easy to see that \( \varphi_1 \) is an injection from \( A_1' \) into \( B_1' \subset B \). In fact, it is also seen that if \( a_1, a_2 \in A \) and

\[
a_1(y^{AB})|_{E^k} = a_2(y^{AB})|_{E^k},
\]

then either \( \varphi_1(a_1)a_1^{-1} = \varphi_1(a_2)a_2^{-1} \) or both values of \( \varphi_1(a_1) \) and \( \varphi_1(a_2) \) are undefined. Using the criterion \( \ref{lem:oneelement} \) (for a one-element family \( A \)), we conclude that \( \varphi_1 \) restricted to its domain \( A_1' \) is determined by a block code (with the coding horizon \( E^k \)). We remark, that the block code determines some extension of \( \varphi_1 \) to the whole group, but we do not care about the values of the code outside \( A_1' \) and we still treat \( \varphi_1 \) as undefined outside \( A_1' \). If \( A_1' = A \) (which is rather unlikely in infinite groups), then the proof is finished.

Otherwise we continue the construction involving the correction chains and the associated corrections. By Lemma \( \ref{lem:corrector} \) for an appropriate \( N \), every element \( a \in A \setminus A_1' \) is the start of a \( \varphi_1 \)-correction chain of length at most \( 2N \). Next, by Lemma \( \ref{lem:corrector2} \) within \( E^{4N}a \) there is a minimal \( \varphi_1 \)-correction chain of length at most \( 2N \). Finally, by Lemma \( \ref{lem:corrector} \) all minimal \( \varphi_1 \)-correction chains of lengths at most \( 2N \) do not collide with each other. Thus we can perform simultaneous corrections along all \( \varphi_1 \)-correction chains of lengths at most \( 2N \). The corrected map will be denoted by \( \varphi_2 \). For each \( a \in A \setminus A_1' \) perhaps we have not yet included \( a \) in the domain \( A_2' \) of \( \varphi_2 \), but we have included in \( A_2' \) at least one new point from \( E^{4N}a \cap (A \setminus A_1') \). Clearly, \( \varphi_2 \) sends \( A_2' \) into \( B \) and the multipliers of \( \varphi_2 \) are contained in \( E \).

We will now argue why \( \varphi_2 \) is determined by a block code. Notice that given \( a \in A \), finding all \( \varphi_1 \)-correction chains of lengths bounded by \( 2N \) starting at or passing through \( a \) requires examining the values of \( \varphi_1 \) at most in the set \( E^{2N}a \). Then, given such a chain, we can decide whether it is minimal or not by examining all \( \varphi_1 \)-correction chains of lengths bounded by \( 2N \) which collide with it. For this, viewing the values of \( \varphi_1 \) on the set \( E^{4N}a \) suffices. Now suppose that \( a_1, a_2 \in A \) and

\[
a_1(y^{AB})|_{E^{k+4N}} = a_2(y^{AB})|_{E^{k+4N}}.
\]

Since \( E^k \) is the coding horizon for \( \varphi_1 \), we have

\[
a_1(\tilde{\varphi}_1)|_{E^{4N}} = a_2(\tilde{\varphi}_1)|_{E^{4N}}.
\]
where \( \bar{\phi}_1 \) is defined as the symbolic element over the alphabet \( E \cup \{\emptyset\} \) by the rule

\[
(\bar{\phi}_1)_g = \begin{cases} 
\phi_1(g)g^{-1} & \text{if } g \in A'_1, \\
\emptyset & \text{otherwise},
\end{cases}
\]

\((g \in G)\). This implies that \((r_1a_1, s_1a_1, r_2a_1, s_2a_1, \ldots, r_na_1, s_na_1)\) is a (minimal) \( \phi_1 \)-correction chain if and only if \((r_1a_2, s_1a_2, r_2a_2, s_2a_2, \ldots, r_na_2, s_na_2)\) is a (minimal) \( \phi_1 \)-correction chain, whenever \( n \leq N \) and all \( r_i \) and \( s_i \) belong to \( E^{2N} \). Hence either both \( a_1 \) and \( a_2 \) lie on minimal \( \phi_1 \)-correction chains of length at most \( 2N \), or both do not. In the latter case, since \( a_1(y^{AB})|_{E^{k}} = a_2(y^{AB})|_{E^{k}} \), either \( \phi_2(a_1)a_1^{-1} = \phi_1(a_1)a_1^{-1} = \phi_1(a_2)a_2^{-1} = \phi_2(a_2)a_2^{-1} \) or both \( \phi_2(a_1) \) and \( \phi_2(a_2) \) are undefined. In the former case, the lengths and names of the two minimal \( \phi_1 \)-correction chains are the same, moreover \( a_1 \) and \( a_2 \) occupy equal positions in the corresponding chains. This implies that the multipliers \( \phi_2(a_1)a_1^{-1} \) and \( \phi_2(a_2)a_2^{-1} \) (although different than those for \( \phi_1 \)) will both be defined and equal. So, \( \phi_2 \) is indeed determined by a block code.

The above process can be now repeated: the next map \( \phi_3 \) is obtained by performing simultaneous corrections along all minimal \( \phi_2 \)-correction chains of lengths not exceeding \( 2N \). Again, for every \( a \in A \setminus A'_2 \), at least one point from each set \( E^{s(N)}a \) is included in the domain \( A'_3 \) of \( \phi_3 \) (the intersection \((A \setminus A'_2) \cap E^{s(N)}a \) is nonempty as it contains \( a \), and often \( a \) will be the new point included in \( A'_3 \)). By the same arguments as before, the map \( \phi_3 \) is an injection from \( A'_3 \) into \( B \) determined by a block code (with the coding horizon \( E^{k+4N} \)), and the multipliers of \( \phi_3 \) remain in \( E \).

We claim that after a finite number \( m \) of analogous steps all points of \( A \) will be included in the domain of \( \phi_m \), i.e., \( \phi_m \) will be the desired injection \( \bar{\phi} \) from \( A \) into \( B \). Indeed, a point \( a \in A \setminus A'_i \) remains outside the domains of all the maps \( \phi_i \) with \( i \leq m \) only if the number of all other points (except \( a \)) in \((A \setminus A'_1) \cap E^{s(N)}a \) is at least \( m - 1 \) (because in each step at least one new point from this set is included in the domain). This is clearly impossible for \( m > |E^{s(N)}| \), hence the desired finite number \( m \) exists. By induction, all the maps \( \phi_i \) \((i = 1, 2, \ldots, m)\) are determined by block codes (the coding horizon for the code which determines \( \bar{\phi} = \phi_m \) is at most the set \( E^{k+4Nm} \)). This ends the proof. \( \square \)

6.3.3. Two questions. As we have already mentioned, the problem whether all countable amenable groups have the comparison property is rather difficult. On the other hand, based on the experience with subexponential groups, one might hope that other additional assumptions might help as well. We formulate two relaxed, yet still open, versions of Question 6.7.

**Question 6.34.**

1. Do all countable amenable residually finite groups have the comparison property?
2. Do all countable amenable left (right) orderable groups have the comparison property?

7. Encodable tiling systems

7.1. Encodable systems of quasitilings.

**Definition 7.1.** A topological dynamical system will be called **perfectly encodable** if it has an isomorphic symbolic extension. We will say that it is **encodable** if it has a principal symbolic extension.
For $\mathbb{Z}$-actions a full characterization of perfectly encodable systems provided in \cite{11}. In particular, any aperiodic (i.e., free) system with zero entropy is perfectly encodable. In the general case of actions of countable amenable groups an analogous theorem is unknown. The difficulty lies in encoding the zero entropy tiling system of Theorem \cite{4,25}. However, we are able to perfectly encode a zero entropy Følner system of disjoint quasitilings, and the rest of this subsection is devoted to proving this:

**Theorem 7.2.** Let $G$ be a countable amenable group. There exists a perfectly encodable Følner system of disjoint quasitilings $\hat{T}$ of topological entropy zero.

**Proof.** The largest part of the proof is devoted to constructing a perfectly encodable Følner system $T$ of quasitilings which are not yet disjoint, but they support an additional information allowing to create a conjugate disjoint version $\hat{T}$. The construction of $T$ starts with a zero entropy free action of $G$ on a zero-dimensional space $X$ whose existence is guaranteed by Theorem \cite{4,17}. All dynamical quasitilings $T_k (k \in \mathbb{N})$ appearing below are topological factors of $X$, delivered by Theorem \cite{4,18} in particular they have topological entropy zero. For the joining $T$ we choose their “natural joining”, i.e., as they appear joined in $X$. Unfortunately, encodability of a sequence of disjoint quasitilings provided directly by Theorem \cite{4,19} (i.e., \cite{20} Corollary 3.5]) is uncertain and we need to introduce a slight modification in the constructions in \cite{20} of both the $\epsilon$-disjoint and disjoint quasitilings.

*Revision of the construction in \cite{20}.* We need to recall some portions of the proofs of \cite{20}, Lemma 3.4 and Corollary 3.5. The first one contains a construction of a factor map $x \mapsto T_x$ where $x \in X$ (X is a free zero-dimensional system) and $T_x$ is an $\epsilon$-quasitiling with the set of shapes $S = \{F_{n_1}, \ldots, F_{n_r}\}$ (throughout, $(F_n)_{n \in \mathbb{N}}$ denotes a nested and symmetric Følner sequence starting with $F_1 = \{\epsilon\})$. The tiles are distributed over $G$ in the reversed order: at first we distribute (for all $x \in X$) tiles with the largest shape $F_{n_r}$, then those with the shape $F_{n_{r-1}}$ and so on, until the smallest shape $F_{n_1}$. In each step $j = r, \ldots, 1$ we proceed as follows: we cover the space $X$ by finitely many clopen sets $U_{j,1}, \ldots, U_{j,m_j}$ such that, for each $i$, the images $g(U_{j,i})$ are pairwise disjoint for different $g \in F_{n_j}$. Next, we proceed by an (inner) induction over $i = 1, 2, \ldots, m_j$ (each step of the resulting double induction is indexed by a pair $(j, i)$). At step $(j, i)$, we accept as tiles of $T_x$ these sets of the form $F_{n_j}g$ which satisfy:

\begin{enumerate}
  \item $g(x) \in U_{j,i}$ and
  \item $F_{n_j}g \cap V_{j,i}$ is a $(1-\epsilon)$-subset of $F_{n_j}g$, with $V_{j,i}$ abbreviating a complicated formula describing the union of all tiles accepted in all preceding steps (i.e., in steps $(j', i')$, where either $j' > j$ or $j' = j$ and $i' < i$).
\end{enumerate}

Later, in the proof of \cite{20} Corollary 2.5], the tiles of the disjoint quasitiling $\hat{T}$, are exactly the above sets $F_{n_j}g \setminus V_{j,i}$. This is all we need to recall from \cite{20}. Just observe, that the construction associates to each tile of $T_x$ a double index $(j, i) (j = r, r-1, \ldots, 1, i = 1, 2, \ldots, m_j)$ which introduces a partial order among the tiles, such that if two different tiles are not disjoint then one strictly precedes another, and later the disjoint tiles are obtained by subtracting from each tile the union of all preceding tiles. The problem which we must solve now is that the partial order among the tiles in $T_x$ depends not only on $T_x$ but also on $x$. Thus, even if we prove that the $\epsilon$-disjoint quasitilings $\mathcal{T} = \{T_x : x \in X\}$ created for a decreasing to zero sequence $(\epsilon_k)_{k \in \mathbb{N}}$ constitute an encodable system of quasitilings
\( T = \bigvee_{k \in \mathbb{N}} T_k \), this will not imply encodability of the corresponding system of disjoint quasitilings \( T \).

To resolve the problem at a minimized cost of changes in the original construction, we need to do three things:

1. Choose the initial free system \( X \) to be minimal (this is always possible, because each free system has a minimal subsystem which is also free).
2. For each \( j = r, r - 1, \ldots, 1 \) construct the cover \( U_{j,1}, U_{j,2}, \ldots, U_{j,m_j} \) in a more specific way: Choose the first clopen set \( U_{j,1} \) arbitrarily (yet so that the sets \( g(U_{j,1}) \) are disjoint for \( g \in F_{n_j} \)). By minimality, there are finitely many elements \( g_{j,1} = e, g_{j,2}, \ldots, g_{j,m_j} \) of \( G \) such that \( g_{j,1}(U_{j,1}), g_{j,2}(U_{j,1}), \ldots, g_{j,m_j}(U_{j,1}) \) cover \( X \) (we may assume that all sets in this cover are indispensable). Now, for each \( i = 1, \ldots, m_j \), define

\[
U_{j,i} = g_{j,i}(U_{j,1}) \setminus \left( \bigcup_{i' = 1}^{i-1} g_{j,i'}(U_{j,1}) \right).
\]

It is clear that the sets \( U_{j,i} \) (\( i = 1, \ldots, m_j \)) have the required properties (each of them is clopen, has disjoint images under \( g \in F_{n_j} \), and jointly they cover \( X \)).

3. Apply the following duplicating of shapes of \( T \): replace each symbol “\( S \)” (\( S \in S \)) by two symbols “\( S_p \)” and “\( S_n \)” (the symbolic representation of \( T \) after duplicating will use an alphabet of cardinality \( 2r(e) \)). For each \( j \) and \( S = F_{n_j} \) place the symbols “\( S_p \)” at centers of all tiles with shape \( S \) associated with the index \((j,1)\). Otherwise (for indices \((j,i), i > 1\)) place the symbols “\( S_n \)”.

The rest of the construction is unchanged. What we have gained is captured in the lemma below.

**Lemma 7.3.** The dynamical quasitilings \( \hat{T} \) (we mean the version obtained via the above revision of the construction including the duplicating of shapes) and its disjoint version \( \hat{T} \) are topologically conjugate.

**Proof.** The revision enables one to recognize, for each \( j = 1, 2, \ldots, r \), which tiles with the shape \( S = F_{n_j} \) are associated with the double index \((j,1)\). Call them primary tiles (the subscripts \( p \) and \( a \) stand for “primary” and “non-primary”). The association of the indices \((j,i)\) to non-primary tiles is also possible: if \( c \) is the center of a non-primary tile of \( T_z \) with shape \( S = F_{n_j} \) then we examine all the elements \( g_{j,2}^{-1}c, g_{j,3}^{-1}c, \ldots, g_{j,m_j}^{-1}c \). The term \( i \) in the double index \((j,i)\) associated with the considered tile \( Sc \) can be determined as the smallest index \( i \) for which \( g_{j,i}^{-1}c \) is a center of a primary tile (we skip the elementary verification that this works). Once the indices \((j,i)\) are determined for all tiles (and thus the partial order among the tiles), the disjoint version \( \hat{T} \) is also determined: given \( x \in X \), \( \hat{T}_z \) is obtained by subtracting, from each tile of \( T_z \), all its predecessors (we may also need to perform an adjustment of centers, as described in subsection 4.2.2). This is clearly a finite horizon block code, so \( \hat{T} \) is a topological factor of \( T \). In order for \( \hat{T} \) to be conjugate to \( T \) it suffices to apply to \( \hat{T} \) duplication of shapes, by which each tile of \( \hat{T} \) will “remember” the shape of the tile of \( T \) from which it was created. We omit more formal details of this easy step. \( \square \)
Here the revision ends, but we continue establishing properties (independent of the above revision) of the quasitilings provided by Theorem 4.18. Let $K \subset G$ be a finite set. Easy examples show that there exists no pair $(F, \epsilon)$, where $F \subset G$ is finite and $\epsilon > 0$, such that $(F, \epsilon)$-invariance and $\epsilon$-disjointness of a general quasitiling guarantees $K$-separating of its set of centers. However, the quasitilings provided by Theorem 4.18 are specific and thus we can prove what follows:

**Lemma 7.4.** Let $K \subset G$ be a finite set and let $\epsilon > \frac{1}{2}$. If the shapes $F_{ni}, \ldots, F_{nr}$ of the $\epsilon$-quasitilings $T_x$ $(x \in X)$ constructed in the proof of Theorem 4.18 (i.e., of Lemma 3.4) are $(K^{-1}K, \epsilon)$-invariant, then $C(T_x)$ is $K$-separated.

**Proof.** Consider two tiles $T \neq T'$ of $T_x$. If they are disjoint then $|T \cap T'| = 0$. If not, then one of them, say $T' = S'c'$, strictly precedes the other, say $T = Sc$, in the partial order associated with the indices $(j, i)$. In this case $T \setminus T'$ is a $\frac{1}{2}$-subset of $T$, i.e., $|T \cap T'| < \frac{1}{2}|T|$. Moreover, since the Følner sequence $(F_n)_{n \in \mathbb{N}}$ is nested and $j' \geq j$, we also have $S \subset S'$. We can thus write

$$|T \cap T'| = |Sc' \cap S'c'| = |(S'(S'c')^{-1}c)^{-1}| = |S' \cap c(c')^{-1}(S')^{-1}| = |S^{-1} \cap c(c')^{-1}S^{-1}|.$$  

Suppose that $Kc$ and $Kc'$ are not disjoint. Then $c(c')^{-1} \in K^{-1}K$, and since, by symmetry of the Følner sequence, $S^{-1}$ is $(K^{-1}K, \frac{1}{2})$-invariant, it is $(c(c')^{-1}, 1)$-invariant (see observation (1) above Definition 2.3). i.e., $|S^{-1} \cap c(c')^{-1}S^{-1}| > \frac{1}{2}|S^{-1}| = \frac{1}{2}|T|$. We have arrived at a contradiction. \hfill $\Box$

We are in a position to start the actual construction of a perfectly encodable Følner system of disjoint quasitilings $T$. We fix a decreasing to zero sequence $(\epsilon_k)_{k \in \mathbb{N}}$ with $\epsilon_k < \frac{1}{2}$. We will inductively construct a Følner system of (non-disjoint) quasitilings $T = \bigvee_{k \in \mathbb{N}} T_k$ so that for each $k \in \mathbb{N}$, $T_k$ is a dynamical $\epsilon_k$-quasitiling obtained via the above revised construction, with the collection of shapes $S_k \subset \{F_{n_{1,k}}, F_{n_{2,k}}, \ldots, F_{n_{r(\epsilon_k),k}}\}$, where $n_{1,k} < n_{2,k} < \cdots < n_{r(\epsilon_k),k} < n_{1,k+1}$ and the dependence $\epsilon \mapsto r(\epsilon)$ is the same as in Theorem 4.18. Due to duplication, each shape $S \in S_k$ will correspond to two symbols, “$S_\ell$” and “$S_r$”. At the same time we will construct a decreasing sequence of subshifts $Z_k$ on three symbols together with a consistent sequence of topological factor maps $\pi_k : Z_k \to T_{[1,k]} = \bigvee_{j=1}^k T_j$. The meaning of “consistency” is the same as in the proof of Theorem 4.2. $\pi_{k+1}$ is composed with the natural projection $\pi_{[1,k]} : T_{[1,k+1]} \to T_{[1,k]}$ coincides with the restriction of $\pi_k$ to $Z_{k+1}$. The intersection $Z = \bigcup_{k \in \mathbb{N}} Z_k$ will be a symbolic extension of the entire system of quasitilings $T = \bigvee_{k \in \mathbb{N}} T_k$. Later we will show that this extension is in fact isomorphic. This will prove perfect encodability of $T$. By Lemma 7.3, the disjoint version $\hat{T} = \bigvee_{k \in \mathbb{N}} \hat{T}_k$, being conjugate to $T$, will also be perfectly encodable. The construction of $\hat{T}$ follows now.

**Step 1.** We let $T_1$ be the dynamical quasitiling whose only element is the tiling by singletons. This is an $\epsilon_1$-quasitiling (regardless of $\epsilon_1$) whose only shape is $F_1 = \{e\}$ (i.e., $S_1 = \{\{e\}\}$). We let $Z_1 = \{-1,0,1\}^G$ (the full shift on three symbols). Clearly, $Z_1$ is a topological extension of $T_1$.

**Step 2.** Define $m = \lceil \log_3 (2r(\epsilon_2)) \rceil + 1$. Fix a set $U_2 \subset G$ of cardinality $m$ and containing the unit. Theorem 4.18 provides a zero entropy dynamical disjoint $\epsilon_2$-quasitiling $T_2$ with the collection of shapes $S_2 \subset \{F_{n_{1,2}}, F_{n_{2,2}}, \ldots, F_{n_{r(\epsilon_2),2}}\}$, where

\[3\text{The number of symbols can be reduced to two, though not without some effort, see the Appendix.}\]
We define the positions of the symbols 0 in $z$. The quasitiling and non-primary tiles. We also assume that we have constructed a subshift duplicated alphabet (of cardinality $2^S$) in a symbolic extension of dynamical there exist finite sets $U$. Step $i$ allows to fully reconstruct the set of centers $C$, while $\mathcal{S}$ tells us whether the tile is primary or not. We have deduced that in the symbolic representation of $T$, $T(c) = "S_i"$. In this manner $z$ allows to fully reconstruct $T$ (with the duplicated alphabet) using a block code with coding horizon $U_2$. It is clear that the set denoted by $\pi^{-1}(T_{[1,2]})$ is indeed the preimage of $T_{[1,2]}$ by the above mapping $\pi$.

**Step $k+1$.** Given $k \geq 2$ suppose that for each $2 \leq l \leq k$ we have selected a minimal dynamical $\ell_l$-quasitiling $T_l$ with the collection of shapes $S_l = \{F_{n_{l,1},}, F_{n_{l,2},}, \ldots, F_{n_{l,(\ell_l)}},\}$, where $n_{l,(\ell_l),} < n_{l,1} < n_{l,2} < \cdots < n_{l,(\ell_l)}$, represented as a subshift over the duplicated alphabet (of cardinality $2^r(\ell_l)$) allowing to differentiate between primary and non-primary tiles. We also assume that we have constructed a subshift $Z_k$ on three symbols, and a topological factor map $\pi_k : Z_k \to T_{[1,k]}$. We assume that there exist finite sets $U_k \subset V_k$ with $\cup_{U_k} \leq \frac{1}{2^k - 1}$ such that for each $T_k \in T_k$ the set of centers $C(T_k)$ is $V_k$-separated and the factor map $\pi_k$ is given by a block code with coding horizon $U_k$ (at step 2 we have taken $V_2 = U_2$). Moreover, we require certain structure of the fibers (preimages of points) of $\pi_k$, captured in the conditions (1)-(3) below. Given a $k$-tuple $T_{[1,k]} = (T_1, T_2, \ldots, T_k) \in T_{[1,k]}$ and $c \in C(T_k)$ consider the restriction $T_{[1,k]}(c)$. Since each $T_i$ is symbolic ($l \leq k$), $T_{[1,k]}(c)$ is also symbolic and

---

4The “trit” (analogue of “bit” but with three values) of information carried by the index $i \in \{-1,0,1\}$ is, at this step, superfluous, but will be essentially used in the following steps.
this restriction is in fact a (shifted) block on finitely many symbols over the domain $U_k$. Let $D_k$ denote the (finite) family of all such blocks

$$D_k = \{ T_{[1,k]} | U_i^c : T_{[1,k]} \in T_{[1,k]}, c \in C(T_k) \}.$$  

1. For every $D \in D_k$ there are exactly three different blocks $B_{D,-1}^{(k)}$, $B_{D,0}^{(k)}$ and $B_{D,1}^{(k)}$ belonging to $\{-1, 0, 1\}^U$ such that whenever $D = T_{[1,k]} | U_i^c$ for some $T_{[1,k]} \in T_{[1,k]}$ and $c \in C(T_k)$, and $z \in \pi_k^{-1}(T_{[1,k]})$ then $z | U_i^c = B_{D,i}^{(k)}$ for some $i \in \{-1, 0, 1\}$.

2. For any fixed $T_{[1,k]}$, all independent choices of the above indices $i$ for different centers $c \in C(T_k)$ are represented in the elements $z \in \pi_k^{-1}(T_{[1,k]})$ (it is essential that the sets $U_i^c$ are pairwise disjoint).

3. The restrictions of all elements $z \in \pi_k^{-1}(T_{[1,k]})$ to the complement of the set $U_k C(T_k)$ (called the background of $T_k$) are equal.

We now need to construct $T_{k+1}$, $Z_{k+1}$ and define $\pi_{k+1}$. Since $T_k$ is minimal, it is transitive, say $T_k = \dot{O}(T_k^*)$. By Proposition 4.39 there exists a finite set $U$ containing $c$, such that the set $C(T_k^*)$ of all centers of $T_k^*$ is $U^1$-syndetic. Since $U^1$-syndeticity is clearly an invariant and closed property, the same holds for each $T_k \in T_k$, that is to say, in every shifted set $T$ there exists at least one center $c$ of some tile $T$ of $T_k$. We define $m = \lfloor \log_2(3 \cdot |U| 2r(\epsilon_{k+1})) \rfloor + 1$. Further, there exists a (much larger) finite set $U \supset U$ such that for each $T_k \in T_k$, in every shifted copy $Ug$ there are at least $m$ centers of $T_k$ (it suffices that $U$ contains $m$ disjoint shifted copies of $U$). We define $U_{k+1} = U_kUU$. We also choose a finite set $V_{k+1} \supset U_{k+1}$ with $\frac{|U_{k+1}|}{|V_{k+1}|} \leq \frac{1}{3}$.

The revised version of Theorem 4.18 combined with Lemma 7.4 provides a zero entropy minimal dynamical $\epsilon_{k+1}$-quasitiling $T_{k+1}$ with at most $r(\epsilon_{k+1})$ shapes belonging to the Folner sequence $S_{k+1} \subset \{ F_{n_{r(\epsilon_k),k}}, F_{n_{r(\epsilon_k+1),k}}, \ldots, F_{n_{r(\epsilon_k+1),k+1}} \}$, where $n_{r(\epsilon_k),k} < n_{1,k+1} < n_{2,k+1} < \cdots < n_{r(\epsilon_k+1),k+1}$, and such that for every $T_{k+1} \in T_{k+1}$ the set of centers $C(T_{k+1})$ is $V_{k+1}$-separated. The quasitiling is represented as a subshift over the duplicated alphabet \{“$S_k$”: $S \in S_{k+1}, s \in \{p,n\}$\}, allowing to determine the primariness of the tiles.

There are at most $|U| 2r(\epsilon_{k+1})$ triples $(u,S,s)$, where $u \in U$, $S \in S_{k+1}$ and $s \in \{p,n\}$, while there are at least $2m^{-1} \geq 3 \cdot |U| 2r(\epsilon_{k+1})$ words of length $m$, over the alphabet \{-1, 0, 1\} (i.e., functions from \{1,2,…,m\} → \{-1,0,1\}), in which 0 occurs exactly once, at the first position. Thus, to every triple $(u,S,s)$ one can disjointly associate a family $\{ W_{u,s,s,-1}, W_{u,s,s,0}, W_{u,s,s,1} \}$ of three different such words.

For each $(k+1)$-tuple $T_{[1,k+1]} = (T_1, T_2, \ldots, T_k, T_{k+1}) \in T_{[1,k+1]}$ we will now select a subset of $\pi_k^{-1}(T_{[1,k]})$ where $T_{[1,k]} = (T_1, T_2, \ldots, T_k)$, which will constitute the preimage $\pi_k^{-1}(T_{[1,k+1]})$. Recall that all elements $z \in \pi_k^{-1}(T_{[1,k+1]})$ are equal on the background of $T_k$, while on every set $U_k c$ ($c \in C(T_k)$) there occur three possible blocks $B_{D,i}^{(k)} (i \in \{-1, 0, 1\})$, where $D = T_{[1,k]} | U_i^c$. We will soon restrict these possibilities in a way that depends on $T_{k+1}$.

We enumerate the set $U$ as $\{g_1, g_2, \ldots, g_{|U|}\}$ starting with the elements of $U$, i.e., so that $U = \{g_1, g_2, \ldots, g_{|U|}\}$. Let $c_0 \in C(T_{k+1})$, i.e., for some $S \in S_{k+1}$, $S_{k+1}$ is a tile of $T_{k+1}$. In $U_0$ there is at least one center of $T_k$. We let $c_1$ be the first one in the enumeration of $U_0$ as $\{g_1 c_0, g_2 c_0, \ldots, g_{|U|} c_0\}$. We denote by $u$ the element...
it only remains to determine $T$ is closed and shift invariant. $T$ of the projection of the desired image $Z$ Clearly, by construction, $Z$ is primary or not, plus one extra trit of information for future use. This will be achieved by encoding (within $z|U_{k+1,c_0}$) one of the three words $\{W_{u,S,s,-1}, W_{u,S,s,0}, W_{u,S,s,1}\}$. 

To this end, we simply require that the indices $i$ in the blocks $B^{(k)}_{D_j,i}$, where $D_j = T|U_{k+1,j} \cup U_{k+1,c_0}$, $j = 1, \ldots, m$ follow one of the words $W_{u,S,s,-1}$ or $W_{u,S,s,0}$, or $W_{u,S,s,1}$. Formally, we require that:

$$\exists i' \in \{-1,0,1\}, \forall j = 1, \ldots, m \quad z|U_{k,c} = B^{(k)}_{D_j,U,u,S,i'(j)},$$

Roughly speaking, on the set $\bigcup_{j=1}^m U_k c_j$ we have reduced the number of possibilities from $3^m$ (represented by all possible configurations of the indices $i$) to just 3 (represented by the new index $i'$). Since each $c_j$ belongs to $U_{k+1} \subset U_k c_0$, the above restrictions affect $z$ only on the set $U_k U_k U_{c_0} = U_{k+1} c_0$. As the set $C(T_{k+1})$ is $U_{k+1}$-separated, there is no collision between the above restrictions introduced for different centers $c_0 \in C(T_{k+1})$. For fixed $T|_{T_{k+1}}$ we allow all independent choices of the indices $i'$ for different centers $c_0 \in C(T_{k+1})$ to be represented in the elements $z \in \pi^{(k)}_{k+1}(T|_{T_{k+1}})$.

Additionally, we introduce two “background rules”:

1. The “small background”: if $c$ is a center of $T_k$ within $U_{k+1} c_0$ other than any $c_j$ ($j = 1, \ldots, m$), then we require that for all $z \in \pi^{(k)}_{k+1}(T|_{T_{k+1}})$, $x|U_{k,c} = B^{(k)}_{D,1}$ (where $D = T|_{T_{k}} U_{k,c}$). With this rule, the block $z|U_{k+1,c_0}$ may assume one of only three possible forms (corresponding to the new index $i'$). The collection of these three blocks depends only on the restriction $D' = T|_{T_{k+1}} U_{k+1,c_0}$, hence we can denote these three blocks as $B^{(k+1)}_{D',-1}, B^{(k+1)}_{D',0}$ and $B^{(k+1)}_{D',1}$.

2. The “large background”: if $c$ is a center of $T_k$ outside $U_{k+1} C(T_{k+1})$, we also require that for all $z \in \pi^{(k)}_{k+1}(T|_{T_{k+1}})$, $z|U_{k,c} = B^{(k)}_{D,1}$, where $D = T|_{T_{k}} U_{k,c}$.

This concludes the definition of $\pi^{(k)}_{k+1}(T|_{T_{k+1}})$. We let

$$Z_{k+1} = \bigcup \{\pi^{(k)}_{k+1}(T|_{T_{k+1}}) : T|_{T_{k+1}} \in T|_{T_{k+1}}\}.$$  

Clearly, by construction, $Z_{k+1} \subset Z_k$. We skip the elementary verification that $Z_{k+1}$ is closed and shift invariant.

We will now describe the functioning of the code $\pi_{k+1}$. Let $z \in Z_{k+1}$. Clearly, $z \in Z_k$, and by the inductive assumption, we can determine the image $T|_{T_{k+1}} = \pi_k(z)$ by a block code with the coding horizon $U_k$. The $k$-tuple $T|_{T_{k+1}}$ will play the role of the projection of the desired image $T|_{T_{k+1}}$ onto the first $k$ coordinates, and it only remains to determine $T_{k+1}$ given $T|_{T_{k}}$. This will automatically guarantee consistency of $\pi_{k+1}$ with the preceding maps $\pi_l$ ($l \leq k$). In particular, we can locate all centers $c \in C(T_{k})$, and, for every such center we can determine the block $D = T|_{T_{k}} U_{k,c}$. Next, for every such pair $c$ and $D$ we check whether $z|U_{k,c} = B^{(k)}_{D,0}$. If yes, then we denote $c$ by $c_1$ and we know that the center $c_0$ of a tile of $T_{k+1}$ lies within $U^{-1} c_1$, say $c_0 = u^{-1} c_1$. We need to determine three pieces of data:
u, the shape $S \in S_{k+1}$ of the tile of $T_{k+1}$ centered at $c_0$, and its primariness. In $\hat{U}c_1$ we can easily locate the first $m - 1$ (other than $c_1$) centers of $T_k$ in the ordering of $\hat{U}c_1$ as \{$g_1c_1, g_2c_1, \ldots, g_Uc_1$\}, and call them $c_2, c_3, \ldots, c_m$. By the rules of creating $Z_{k+1}$, the blocks $z|_{U_kc_1}$ will have only two forms, either $B_{D_1, -1}^{(k)}$ or $B_{D_2, 1}^{(k)}$, where $D_j = T_{[1,k]}|_{U_kc_1}$. The indices $-1, 1$, together with the initial 0, will form a word $W \in \{-1, 0, 1\}^{[1,2,\ldots,m]}$ equal to one of the words $W_{u,s,s,i'}$ for a unique combination of parameters $u \in U$, $S \in S(T_{k+1})$, $s \in \{p, n\}$, $i' \in \{-1, 0, 1\}$. Now we can determine $c_0$ as $u^{-1}c_1$ and we know that $T_{k+1}$ has a tile (primary or not, according to the value of $s$) centered at $c_0$ with the shape $S$, i.e., that $T_{k+1}(c_0) = “Ss”$. In this manner, we have recognized the tile and its primariness by viewing the set $U_{k+1} = U_k\hat{U}U$ shifted to the center of this tile. The trit of information carried by the index $i'$ is, at this step, superfluous, but clearly crucial in further steps.

Once the induction is completed, we define $Z$ as the decreasing intersection of the subshifts $Z_k$ ($k \in \mathbb{N}$). It is clear that $Z$ is a symbolic extension of the countable joining $T = \bigvee_{k \in \mathbb{N}} T_k$. The factor map $\pi : Z \to T$ is defined as the limit of the blocks codes $\pi_k$: by consistency this limit exists (each code $\pi_k$ allows to determine the first $k$ layers of the image by $\pi$).

We shall now argue that $Z$ is an isomorphic extension of $T$ by showing that the factor map $\pi$ is injective except on a set of universal measure zero (i.e., of measure zero for all invariant measures on $Z$). It suffices to show that $\pi^{-1}(T)$ ($T = (T_1, T_2, \ldots) \in T$) is a singleton except when $T$ belongs to some set of universal measure zero on $T$. A way to prove it is by showing that the set of all elements $T \in T$ which have multiple preimages by $\pi$, i.e., the set

$$A = \{ T : |\pi^{-1}(T)| > 1 \}$$

(which is clearly Borel-measurable in $T$) has universal invariant measure zero. An element $T \in T$ is in $A$ if there exists $g \in G$ and two elements $z, z' \in \pi^{-1}(T)$ with $z_g \neq z'_{g}$. Thus $A = \bigcup_{g \in G} A_g$, where $A_g = \{ T : \exists z, z' \in \pi^{-1}(T) \; z_g \neq z'_{g} \}$. It suffices to prove that for every $g \in G$, $A_g$ has universal measure zero. Because $A_g = g(A_e)$, we can consider only $g = e$. Let $T \in A_e$ and let $z, z'$ be as in the definition of $A_e$. For each $k$ we then have $z, z' \in \pi_k^{-1}(T_{[1,k]})$, where $T_{[1,k]}$ is the projection of $T$ on the first $k$ coordinates. Thus, $z \neq z'$ is possible only when $e$ does not belong to the background of $T_k$, i.e., when $e \in U_kC(T_k)$. Thus, for each $k$:

$$A_e \subset B = \{ T = (T_1, T_2, \ldots) : e \in U_kC(T_k) \}.$$ 

Given $T \in T$, we have, in the notation of Proposition 6.10 (with $x$ replaced by $T$), the following equality:

$$B_T = \{ g : gT \in A_e \} \subset U_kC(T_k).$$

By left invariance and subadditivity of upper Banach density in amenable groups (see Sections 4.1.1, 4.1.2), for each $k \geq 2$ we have

$$\overline{D}(B_T) \leq \overline{D}(U_kC(T_k)) \leq |U_k| \overline{D}(C(T_k)) \leq |U_k| \frac{1}{|V_k|} \leq \frac{1}{k - 1}.$$
We have proved that for any \( T \in T \), \( \overline{D}(B_T) \leq \frac{1}{k-1} \) for every \( k \geq 2 \), i.e., that \( \overline{D}(B_T) = 0 \). By Proposition 6.10, we obtain
\[
\sup_{\mu \in M_G(T)} (A_\mu) \leq \sup_{\mu \in M_G(T)} (B) = 0.
\]
This ends the proof that the extension \( \pi : Z \to T \) is isomorphic, and thus the entire proof of Theorem 7.2 is completed. \( \square \)

7.2. Encodability of tiling systems versus the comparison property.
Encodable Følner systems of disjoint quasitilings is just a step in our pursuit towards facing the true challenge which is the creation of encodable tiling systems. Only such systems will allow us to built genuine symbolic extensions. As the theorem below shows, the comparison property is crucial in this aspect.

**Theorem 7.5.** Let \( G \) be a countable amenable group. A Følner system of disjoint quasitilings \( \mathbf{T} \) admitting comparison has a tiling system as a topological factor.

Before the proof, let us draw some corollaries. The first one is not very useful for us, but perhaps has an interest of its own. The second one is absolutely crucial for the rest of this paper. Recall that in large parts of Section 4 we have been dealing with zero-dimensional systems which had a system of tilings as a topological factor. They were artificially created by joining an arbitrary zero-dimensional system with a tiling system. We can now characterize, in terms of comparison, these free zero-dimensional systems which have a system of tilings as a topological factor, without needing to join them with anything.

**Corollary 7.6.** Let a countable amenable group \( G \) act freely on a compact metric zero-dimensional space \( X \). Then the action has a tiling system as a topological factor if and only if it admits comparison. The forward implication holds without assuming that the action is free.

**Proof.** By Theorem 4.19, the action admits a Følner system of disjoint quasitilings \( \mathbf{T} \) as a topological factor. If the action on \( X \) admits comparison, so does \( \mathbf{T} \). Now, Theorem 7.5 implies that \( \mathbf{T} \) (and hence also \( X \)) has a tiling system \( \mathbf{T} \) as a topological factor. Since this is just a corollary, and we are anxious to give the proof of Theorem 7.5 we save the somewhat lengthy proof of the forward direction for later. \( \square \)

**Corollary 7.7.** Suppose \( G \) has the comparison property. Then there exists an encodable zero entropy tiling system \( \mathbf{T} \) of \( G \).

**Proof.** By Theorem 7.2, there exists a (perfectly) encodable Følner system of disjoint quasitilings \( \mathbf{\tilde{T}} \) of topological entropy zero (for that, the comparison property is not used yet). In the proof of Theorem 7.2, \( \mathbf{\tilde{T}} \) is obtained as a factor of some free action on a zero-dimensional space \( X \). By the assumed comparison property, the action on \( X \) admits comparison, and thus so does \( \mathbf{\tilde{T}} \). Now, Theorem 7.5 implies that there exists a tiling system \( \mathbf{T} \) which is a topological factor of \( \mathbf{\tilde{T}} \). Clearly, the isomorphic symbolic extension of \( \mathbf{T} \) (which exists by perfect encodability) is also a principal symbolic extension of \( \mathbf{\tilde{T}} \) and thus \( \mathbf{\tilde{T}} \) is encodable. \( \square \)

**Proof of Theorem 7.7.** Firstly, assuming comparison, we will show that \( \mathbf{\tilde{T}} \) has, as a topological factor, a Følner system of tilings \( \mathbf{T} \) (which is not necessarily congruent let alone deterministic; we will take care of ensuring these properties
and that \( T = \bigvee_{k \in \mathbb{N}} \hat{T}_k \) then, for arbitrarily large indices \( k \in \mathbb{N} \), the quasitilings \( \hat{T}_k \) are conjugate to some dynamical tilings \( \hat{\mathcal{T}}_k \) which have only slightly worse invariance properties. This will imply that \( \hat{T} \) has the joining \( T = \bigvee \hat{T}_k \) (where \( k \) ranges over the respective subsequence) as a topological factor and that \( T \) is a Følner system of tilings.

Fix a finite set \( K \subset G \) and \( \epsilon > 0 \). Let \( \delta \) be so small that

\[
\frac{2\delta}{1 - \delta} < \frac{\epsilon}{2|K|}.
\]

For some \( k \in \mathbb{N} \) the dynamical quasitiling \( \hat{T}_k \) is \((K, \hat{\mathcal{T}})\)-invariant, disjoint and \((1 - \delta)\)-covering. We denote by \( \hat{\mathcal{S}}_k \) the collection of all shapes used by this quasitiling. By choosing \( k \) large enough, we can also assume that each shape \( S \in \hat{\mathcal{S}}_k \) has cardinality so large that the interval \((\frac{1}{1 - \delta} |S|, \frac{2\delta}{1 - \delta} |S|)\) contains an integer \( i_S \). Since \( k \) is fixed from now on, we will skip it in the denotation. In each shape \( S \in \hat{\mathcal{S}} \) we select (arbitrarily) a subset \( S_c \subset S \) of cardinality \( i_S \). For consistency with the notation of \((6.1)\) and \((6.2)\) we will denote the elements \( \hat{T} \in \hat{\mathcal{T}} \) also by the letter \( x \). Given \( x = \hat{T} \in \hat{T} \), we now observe two subsets of \( G \):

\[
A_x = G \setminus \bigcup_{(S, c) \in \hat{T}} \hat{\mathcal{T}} \quad \text{and} \quad B_x = \bigcup_{(S, c) \in \hat{T}} B_{Sc}.
\]

Clearly, \( D(A_x) = 1 - D(\bigcup \hat{\mathcal{T}}) < \delta \). Using Lemma \((4.14)\) we easily get \( D(B_x) > (1 - \delta) \cdot \frac{2\delta}{1 - \delta} = 2\delta \). By Corollary \((4.13)\) \( D(B_x, A_x) > \delta \). Define two subsets of \( \hat{T} \):

\[
A = \{ x : e \in A_x \} \quad \text{and} \quad B = \{ x : e \in B_x \}.
\]

Since one can determine whether \( e \in A_x \) (and likewise, whether \( e \in B_x \)) from the symbolic representation of \( x = \hat{T} \) by viewing the symbols in the bounded horizon \( \bigcup_{S \in \hat{\mathcal{S}}} S^{-1} \) around \( e \), both sets \( A \) and \( B \) are clopen (and obviously disjoint). The notation \( A_x, B_x \) is now consistent with \((6.1)\) and \((6.2)\) for the sets \( A, B \), respectively, hence, by Proposition \((6.10)\) \((1)\) (the last equality) we obtain \( D(B, A) \geq \delta > 0 \). The fact that the action on \( \hat{T} \) admits comparison implies that \( A \preceq B \).

By Theorem \((6.12)\) \((1)\), there exists a family of injections \( \hat{\varphi}_x : A_x \to B_x \) indexed by \( x \in \hat{T} \) determined by a block code. We are in a position create, basing on the quasitilings \( \hat{T} \), the desired tilings \( \hat{T} \). Given \( x = \hat{T} \in \hat{T} \), we define a transformation of the tiles \( Sc \in \hat{T} \) as follows:

\[
\Phi_x(Sc) = Sc \cup \hat{\varphi}_x^{-1}(BSc) \subset Sc \cup A_x
\]

(recall that \( BSc \) is a part of the set \( B_x \), so its preimage by \( \hat{\varphi}_x \) is a part of \( A_x \)). We will call the set \( \hat{\varphi}_x^{-1}(BSc) \) the \textit{added set}. We define the center of the new tile \( \Phi_x(Sc) \) as \( c \). The shape of the new tile equals

\[
\Phi_x(Sc)c^{-1} = S \cup \hat{\varphi}_x^{-1}(BSc)c^{-1}.
\]

Note that

\[
\hat{\varphi}_x^{-1}(BSc)c^{-1} \subset E^{-1}(BSc)c^{-1} \subset E^{-1}S,
\]

which is a finite set (recall that \( E \) is the finite set of multipliers used by \( \hat{\varphi}_x \), common for all \( \varphi_x \) for all \( x \in \hat{T} \)). Since \( \hat{S} \) is finite, the set \( \hat{S} \) of all new shapes is also finite.
As the quasitiling \( \hat{T} \) is disjoint, \( \hat{\varphi}_x \) restricted to \( A_x \) is injective, and the image of \( A_x \) is contained in \( B_x = \bigcup_{S_c \in \hat{T}} B_{S_c} \), it is clear that the new quasitiling
\[
\hat{T} = \{ \Phi_x(S_c) : S_c \in \hat{T} \}
\]
is a tiling (disjoint and covering \( G \) completely).

Further, for any tile \( S_c \) of \( x = \hat{T} \) the added set \( \hat{\varphi}_x^{-1}(B_{S_c}) \) has cardinality at most \( |B_S| = i_S < \frac{\pi}{2}K \). Thus
\[
|K\Phi_x(S_c)| \leq |K S_c| + |K| \cdot \frac{\varepsilon}{2|K|} |S| = |K S| + \varepsilon \frac{1}{2} |S|.
\]
We can assume (at the beginning of the proof) that \( e \in K \), and then \((K, \hat{\varphi})\)-invariance of \( S \) is equivalent to the inequality \( |K S| < (1 + \frac{\varepsilon}{2})|S| \). Thus
\[
|K\Phi_x(S_c)| < (1 + \varepsilon)|S| \leq (1 + \varepsilon)|\Phi_x(S_c)|,
\]
and so \( \Phi_x(S_c) \) is \((K, \varepsilon)\)-invariant. Summarizing, we have constructed a mapping \( \hat{T} \to \hat{T} \) from \( \hat{T} \) into tilings with a finite set \( \hat{\mathcal{S}} \) of \((K, \varepsilon)\)-invariant shapes.

We need to show that the above is a topological factor map. To do so, we can use the criterion (6.3), i.e., we need to indicate a finite set \( J \subset G \), such that for any \( \hat{T}, \hat{T}' \in \hat{T} \) and \( g \in G \),
\[
(7.1) \quad \hat{T}|_{Jg} = \hat{T}'|_{Jg} \implies \hat{T}_g = \hat{T}'_g.
\]
We claim that the set \( J = \{ e \} \cup FE^{-1}R \) is good, where \( F \) is the finite coding horizon of \( \hat{\varphi}_x \) (common for all \( x \in \hat{T} \)), \( E \) is the set of multiplies, and \( R = \bigcup \hat{\mathcal{S}} \).

In order to verify this claim, assume that with so defined \( J \) the left hand side of (7.1) holds for some \( \hat{T}, \hat{T}' \in \hat{T} \) and \( g \in G \). Since \( g \in Jg \), we have \( \hat{T}_g = \hat{T}'_g \). If this common entry is 0 then \( g = c \) is a center of some tile in the set \( \hat{T} \) nor \( \hat{T}' \), and then \( g \) is not a center of any tile in \( \hat{T} \) nor \( \hat{T}' \), i.e., \( \hat{T}_g = \hat{T}'_g = 0 \). If the common entry is some “S” with \( S \in \hat{\mathcal{S}} \) then we know that \( g = c \) is a center of some tile in both \( \hat{T} \) and \( \hat{T}' \), their shapes have the same common part \( S \) and may differ only in having different added sets. The added sets equal \( \hat{\varphi}_x^{-1}(B_{S_c})c^{-1} \) and \( \hat{\varphi}_{x'}^{-1}(B_{S_c})c^{-1} \), where \( x \) and \( x' \) stand for \( \hat{T} \) and \( \hat{T}' \), respectively. We need to show that
\[
\hat{\varphi}_x^{-1}(B_{S_c})c^{-1} = \hat{\varphi}_{x'}^{-1}(B_{S_c})c^{-1}.
\]
Since \( FE^{-1}Rc = FE^{-1}Rg \subset Jg \), the left hand side of (7.1) implies \( \hat{T}|_{FE^{-1}Rc} = \hat{T}'|_{FE^{-1}Rc} \). Recall that the family \( \{ \hat{\varphi}_x \}_{x \in \hat{T}} \) is determined by a block code with coding horizon \( F \). We deduce that \( \hat{\varphi}_x \) agrees with \( \hat{\varphi}_{x'} \) on the set \( E^{-1}Rc \), which contains \( E^{-1}S_c \), which contains \( E^{-1}B_{S_c} \). But \( E^{-1}B_{S_c} \) contains the union \( \hat{\varphi}_x^{-1}(B_{S_c}) \cup \hat{\varphi}_{x'}^{-1}(B_{S_c}) \).

Since, as we have shown, \( \hat{\varphi}_x \) and \( \hat{\varphi}_{x'} \) agree on this union, we conclude that \( \hat{\varphi}_x^{-1}(B_{S_c}) = \hat{\varphi}_{x'}^{-1}(B_{S_c}) \). We have shown that the tiling \( \hat{T} \) is a topological factor of \( \hat{T} \).

Restoring the indices \( k \), and applying the above to all quasitilings \( \hat{T}_k \) \((k \in \mathbb{N}) \), we create the desired Følner system of tilings \( \hat{T} \) as a topological factor of \( \hat{T} \). The next step in the proof is turning this system into a congruent and deterministic one, i.e., into a tiling system \( \hat{T} \). Only congruency is essential, because determinism can be easily achieved later using Remark 4.224 (by duplicating the shapes). Now, passing from the Følner system of tilings \( \hat{T} \) to a congruent one is described in the proof of (19) Lemma 5.1, and here we only briefly sketch the method. Recall that \( \hat{T} = \bigcup_{k \in \mathbb{N}} \hat{T}_k \). We let \( \hat{T}_1 = \phi_1(\hat{T}_1) = \hat{T}_1 \) and then, in an inductive procedure,
Lemma 4.4 implies that for every $\phi$ eventually we create a map $\phi$ is already congruent), we extend this map to $\phi$ as follows: given $T_{[k+1]} = (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_k, \tilde{T}_{k+1}) \in T_{[k+1]}$, for each tile $\tilde{T}$ of $\tilde{T}_{k+1}$ we define its modification $\tilde{T}$ as the union all tiles of $\tilde{T}_k$ whose centers lie in $\tilde{T}$, where $\tilde{T}_k$ is the $k$th term in $\phi_{[k]}(\tilde{T}_{[k]}) = \phi_{[k]}(\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_k) = (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_k)$.

The tiling consisting of the new tiles $T$ is denoted by $\tilde{T}_{k+1}$ and is added as the last term in the definition of $\phi_{[k+1]}(\tilde{T}_{[k+1]}) = (\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_k, \tilde{T}_{k+1})$. We may need to apply adjustment of centers of $\tilde{T}_{k+1}$ in case some of them falls outside the new tiles, but this can be done using a topological conjugacy (see subsection 3.2.2). Eventually we create a map $\phi$ sending each $T = (\tilde{T}_k)_{k \in \mathbb{Z}} \in T$ to a congruent system of tilings $\tilde{T} = (\tilde{T}_k)_{k \in \mathbb{Z}}$. We define $\tilde{T}$ as the image $\phi(T)$. A careful verification that $\tilde{T}$ remains a Følner systems of tilings and that $\phi$ is a topological factor map is given in [19] and it is pointless to copy it here.

**Proof of the missing implication in Corollary 4.6** Suppose $G$ acts on a zero-dimensional compact metric space $X$ (we do not assume freeness of the action) and that it admits a tiling system $T = \bigvee_{k \in \mathbb{N}} T_k$ as a topological factor. Let $A, B$ be disjoint clopen subsets of $X$ such that $\mu(B) > \mu(A)$ for all invariant measures $\mu$ on $X$. We need to show that $A \not\leq B$.

As we have observed in Remark 6.2 the infimum $\inf_{\mu \in \mathcal{M}_G(X)} (\mu(B) - \mu(A))$ is positive. Proposition 6.11 (1) implies that $\mathcal{D}(B, A) \geq 6\varepsilon$,

for some $\varepsilon > 0$. By Lemma 6.9 there exists a finite set $F \subset G$ satisfying, for every $x \in X$, $\mathcal{D}_F(B_x, A_x) \geq 5\varepsilon$. For some $k$, the set of shapes $S = S_k$ of $\tilde{T} = \tilde{T}_k$ consists of $(F, \varepsilon)$-invariant sets. Recall that $\tilde{T}$ is a topological factor of $X$ via a map $x \mapsto \tilde{T}_x$.

Lemma 4.4 implies that for every $S \in S$ and $x \in X$, we have $\mathcal{D}_S(B_x, A_x) \geq \mathcal{D}_F(B_x, A_x) - 4\varepsilon > 0$,

which yields $|A_x g^{-1} \cap S| < |B_x g^{-1} \cap S|$ for every $g \in G$.

We will now build an auxiliary symbolic factor $\hat{X}$ of $X$ carrying the minimum information about both the sets $A, B$ and the dynamical tiling. Namely, we define a factor map $\pi : X \to \hat{X} \subset \hat{V}^\mathbb{Z}$, where $\hat{V} = \{0, 1, 2\} \times V$ (as usually, $V = \{\text{“}S\text{”} : S \in S\} \cup \{0\}$ is the alphabet of the symbolic representation of the dynamical tiling $T$), as follows

$$\pi(x)_g = \begin{cases} (1, \text{“}S\text{”}) & \text{if } g \in A_x, Sg \in \tilde{T}_x \\ (2, \text{“}S\text{”}) & \text{if } g \in B_x, Sg \in \tilde{T}_x \\ (0, \text{“}S\text{”}) & \text{if } g \notin A_x \cup B_x, Sg \in \tilde{T}_x \\ (1, 0) & \text{if } g \in A_x, Sg \notin \tilde{T}_x \\ (2, 0) & \text{if } g \in B_x, Sg \notin \tilde{T}_x \\ (0, 0) & \text{if } g \notin A_x \cup B_x, Sg \notin \tilde{T}_x. \end{cases}$$

Clearly, the subshift $\hat{X}$ factors onto $T$ and $\tilde{T}_x = T_x$ whenever $x \in \pi^{-1}(\hat{x})$. Denote $\hat{A} = \{1, \cdot\}$ and $\hat{B} = \{2, \cdot\}$. We have $A = \pi^{-1}(\hat{A})$ and $B = \pi^{-1}(\hat{B})$. 

Thus it suffices to show that $\hat{A} \approx \hat{B}$ in $\hat{X}$. By Theorem 6.12 (1), the proof will be ended once we will have constructed a family of injections $\tilde{\varphi}_x : \hat{A}_x \to \hat{B}_x$ indexed by $\hat{x} \in \hat{X}$ and determined by a block code.

By the definition of $\pi$ we have, that if $\hat{x} = \pi(x)$ then $A_x = \hat{A}_x$ and $B_x = \hat{B}_x$, and the inequality $|A_x g^{-1} \cap S| < |B_x g^{-1} \cap S|$ translates to $|\hat{A}_x g^{-1} \cap S| < |\hat{B}_x g^{-1} \cap S|$ (for each $\hat{x} \in \hat{X}$, $S \in \mathcal{S}$ and $g \in G$). In other words, in every block $g(\hat{x})$ there are more symbols 2 than 1 (we just consider the first entries in the pairs which constitute the symbols). Since $\mathcal{S}$ is finite and for each $S \in \mathcal{S}$ there are only finitely many blocks $C \in \hat{V}^\mathcal{S}$, we have globally a finite number of possible blocks $C$ appearing in the role $g(\hat{x})|_S$ (with $\hat{x} \in \hat{X}$, $g \in G$ and $S \in \mathcal{S}$). For every block $C$ in this finite collection we select arbitrarily an injection $\varphi_C : \{s \in S : C(s) = (1,\cdot)\} \to \{s \in S : C(s) = (2,\cdot)\}$, where $S$ is the domain of $C$.

Fix some $\hat{x} \in \hat{X}$ and $a \in \hat{A}_x$. Let $x$ be the tile of $\mathcal{T}_x$ containing $a$ and let $C = c(\hat{x})|_S$. We define

$$\tilde{\varphi}_x(a) = \varphi_C(ac^{-1})c.$$  

Since $C(ac^{-1}) = \hat{x}_a = (1,\cdot)$, $\varphi_C(ac^{-1})$ is defined and satisfies $C(\varphi_C(ac^{-1})) = (2,\cdot)$, and thus $\hat{x}_c(\varphi_C(ac^{-1})) = (2,\cdot)$, i.e., $\tilde{\varphi}_x(a) \in \hat{B}_x$. Notice that $\tilde{\varphi}_x(a)$ belongs to the same tile of $\mathcal{T}_x$ as $a$. Injectivity of so defined $\tilde{\varphi}_x$ is very easy. Consider $a_1 \neq a_2 \in \hat{A}_x$. If both elements belong to the same tile of $\mathcal{T}_x$, then their images are different by injectivity of $\varphi_C$, where $C = c(\hat{x})|_S$. If they belong to different tiles, their images also belong to different tiles, hence are different. The last thing to check is the condition (6.3), which will establish that the family $\{\tilde{\varphi}_x\}_{\hat{x} \in \hat{X}}$ is determined by a block code. We claim that the horizon $E = \bigcup_{S \in \mathcal{S}} SS^{-1}$ is good. Indeed, suppose, for some $\hat{x}_1, \hat{x}_2 \in \hat{X}$ and $a_1 \in \hat{A}_{\hat{x}_1}, a_2 \in \hat{A}_{\hat{x}_2}$, that

$$a_1(\hat{x}_1)|_E = a_2(\hat{x}_2)|_E.$$  

Let $S$ be the central (i.e., containing the unit) tile of $\mathcal{T}_{a_1(\hat{x}_1)}$. Then the second entry of the pair constituting the symbol $(a_1(\hat{x}_1))_c$ equals “S”. Since $c \in \bigcup_{S \in \mathcal{S}} S^{-1} \subset E$, by (7.2) we obtain that the second entry of the symbol $(a_2(\hat{x}_2))_c$ also equals “S”, so that $S$ is the central tile of $\mathcal{T}_{a_2(\hat{x}_2)}$. Further, since $S \subset E$, by (7.2) we have $a_1(\hat{x}_1)|_S = a_2(\hat{x}_2)|_S$ and hence $c a_1(\hat{x}_1)|_S = c a_2(\hat{x}_2)|_S$. That is, these two restrictions define the same block $C \in \hat{V}^\mathcal{S}$. This implies that both $\tilde{\varphi}_{\hat{x}_1}(a_1)$ and $\tilde{\varphi}_{\hat{x}_2}(a_2)$ are defined with the help of the same injection $\varphi_C$, and

$$\tilde{\varphi}_{\hat{x}_1}(a_1) = \varphi_C(a_1 c_1^{-1})c_1, \quad \tilde{\varphi}_{\hat{x}_2}(a_2) = \varphi_C(a_2 c_2^{-1})c_2,$$

where $c_1$ is the center of the tile of $\mathcal{T}_{\hat{x}_1}$ containing $a_1$, and $c_2$ is the center of the tile of $\mathcal{T}_{\hat{x}_2}$ containing $a_2$. By shift equivariance of the dynamical tiling, we easily see that $c_1 = c a_1$ and $c_2 = c a_2$, which yields

$$\tilde{\varphi}_{\hat{x}_1}(a_1)^{-1} c^{-1} = \varphi_C(\hat{x}_1)^{-1} c = \tilde{\varphi}_{\hat{x}_2}(a_2)^{-1}.$$  

This is exactly the condition (6.3) and the proof is finished. \hfill \square

Combining Theorem 6.33 with Corollary 7.4 we obtain:

**Corollary 7.8.** If $G$ is a subexponential group then every action of $G$ on a zero-dimensional compact metric space has a tiling system as a topological factor.

We conclude this section with a question. Let us say that a countable amenable group $G$ has the tiling property if any free action of $G$ on a zero-dimensional compact metric space has a tiling system as a topological factor. In such case, by Theorem 7.6
any free action on a zero-dimensional compact metric space admits comparison. It is easy to see that the property of having a tiling system as a factor cannot be extended (without modifying the definition) to non-free actions. However, there are a priori no obvious reasons why admitting comparison could not be extended. Thus the following question is very natural:

**Question 7.9.** Is it true that if $G$ has the tiling property (which depends on free actions only) then it also has the comparison property (which depends on all actions; of course in both cases we restrict our attention to zero-dimensional compact metric spaces).

### 7.3. Symbolic extensions for actions of selected groups.

#### 7.3.1. What goes wrong in general countable amenable groups.

It is clear that in order to create a purely symbolic extension of $X$ (or of $\bar{X}$) with $h^* = E_A$, in place of a quasi-symbolic one, at least using the techniques known to us, one would need to use, in the construction of Section 5 an encodable tiling system $T$ of topological entropy zero. Having an encodable tiling system $T$ with entropy zero and its principal symbolic extension $Z$ at our disposal, we can use $T$ in the construction of Section 5 and then turn the resulting quasi-symbolic extension $\bar{Y} = Y \vee T$ into purely symbolic by extending it, in a natural way, to $Y \vee Z$.

Let us make it clear that attempting to build a purely symbolic extension as in Section 5 using a Følner system of disjoint quasitilings instead of a tiling system, is simply not going to work. There are two ways to explain that. The first one reveals the technical obstacles, the second one actually kills the idea.

1. A disjoint quasitiling $T_k$ leaves a small portion of the group, say a set $B$ of small upper Banach density, uncovered. When building a symbolic preimage $y$ of $\bar{x}_{[1,k]}$ (or even a quasi-symbolic preimage $\bar{y}$ which would include a copy of the quasitiling associated to $\bar{x}_{[1,k]}$), we would have to decide where the information about $\bar{x}_{[1,k]}|_B$ should be encoded in $y$ (or $\bar{y}$). Any attempt to do so, eventually turns out to be equivalent to trying to distribute the set $B$ amongst the tiles of the quasitiling $T_k$ so that to each tile $T$ we associate a portion of $B$ of cardinality relatively small compared to $|T|$. And we need to do it using a finite horizon block code. So, in fact, we are trying to factor the dynamical quasitiling onto a tiling. In view of Theorem 7.2 if we were able to factor each quasitiling onto a tiling, we could also build an encodable tiling system.

2. Just observe that if we were able to build (using no matter what technique) purely symbolic extensions with $h^* = E_A$ then a tiling system with topological entropy zero (which exists, by Theorem 4.17) should admit a principal symbolic extension. So, we are back dealing with the problem of existence of encodable tiling systems. By the way, we have just proved that if the group $G$ admits one encodable zero entropy tiling system then all zero entropy tiling systems are encodable.

The problem with existence of an encodable tiling system is rather serious. In spite of many efforts, we failed to prove it for general countable amenable groups. In fact, we failed to construct any tiling system that would admit any symbolic extension. Let us give the reader a glimpse into the obstacles. In [19, Theorem 5.2] we do create a tiling system from a Følner system of quasitilings. The main step is [19, Theorem 4.3] in which we factor a disjoint $\epsilon$-quasitiling $T$ onto a tiling. The framework of the proof is identical as in the proof of Theorem 7.5 above: the uncovered part $B$ of the group is distributed amongst the tiles of $T$ so that to each
tile $T$ we “attach” a portion of $B$ of cardinality smaller than $\epsilon |T|$. Moreover, the attached portion is contained in $FT$ for some finite set $F$. However, without the comparison property, the “algorithm of attaching” is not governed by a block code, so the tiling is not a topological factor of the quasitiling. We use a version of Hall’s Marriage Lemma \[31\] and deciding to which tile a given element $b \in B$ should be attached requires examining the quasitiling of the entire group. Although this method allows to build a tiling system from a Følner system of disjoint quasitilings, it provides no tools to prove its encodability. This really looks paradoxical, because, as we prove later, the resulting tiling still has topological entropy zero, so the number of possible configurations of the tiles and the uncovered areas in some large Følner set $F_n$ is relatively small. Thus we should be able to encode the “algorithm of attaching” using a small number of symbols and a small percentage of the space available in $F_n$. This would lead to a tiling obtainable from $T$ by a block code with coding horizon $F_n$. However, the Følner set $F_n$ does not tile the group $G$. In any covering of $G$ by shifted copies of $F_n$ these copies overlap and the encoded information may simply conflict with each other in the overlapping areas. For a non-conflictive encoding of the “algorithm of attaching” we need a disjoint Følner tiling of some higher order. Unfortunately, we only have at our disposal quasitilings of higher order. Any such quasitiling, in spite of covering a subset of $G$ of lower Banach density extremely close to 1 may leave uncovered areas containing huge portions of the set $B$ and we would have no indication as to where (i.e., to which tiles) the elements of these portions should be attached. The situation is better in groups having a symmetric Følner sequence $(F_n)_{n \in \mathbb{N}}$ satisfying so-called Tempelman’s condition, which guarantees a bounded proportion between $|F_{n-1}F_n|$ and $|F_n|$. One can show that then there exists a symmetric Følner sequence $(F_n)_{n \in \mathbb{N}}$, also satisfying the Tempelman’s condition, and such that $(F_{2n}^2)_{n \in \mathbb{N}}$ is also a Følner sequence (see \[44\]). The group can be covered by shifted copies of $F_{2n}^2$ so that the corresponding shifted copies of $F_n$ are disjoint. In such case, one can encode the “algorithm of attaching” occurring within each shifted copy of $F_{2n}^2$ using the space available in the disjoint shifted copies of $F_n$. This idea works indeed, see \[44\]. However, as it turns out (see \[33\]) all groups satisfying the Tempelman’s condition are subexponential, hence, by Theorem 6.33, fall in the category of groups with the comparison property and so the result of \[44\] is swallowed by the (more general) Theorem 7.10 provided later in this section. Finally, one could hope to encode the “algorithm of attaching” occurring within each “new” tile (of the tiling) using the space available in the “old” tile (of the quasitiling). That is, we could try to encode the new shape using some finite number of symbols, in form of a block over a small portion of the old tile. Unfortunately, this simple idea also fails, because we have no control over the number of shapes of the tiling that are created from one shape of the quasitiling. It is true that every tile of the tiling build from a tile $T$ of the quasitiling (by attaching to it small portions of $B$) is contained in $FT$ for some finite set $F$ (independent of $T$), but we have no control over the size of $F$. Even by attaching just one element of $B \cap FT$ to $T$ at a time, we can produce a number of new tiles much larger than $l^{l|T|}$ (where $l$ is some a priori assumed cardinality of the alphabet used for the coding) so that encoding the new shape within $T$ becomes impossible. This is not contradicted by topological entropy zero, because the relatively small number of configurations of
tiles may occur in a Følner set $F_n$ much larger than all the shapes of the tiling (and we have already discussed why this is useless).

7.3.2. Groups with the comparison property. A class of countable amenable groups, in which we can claim the full version of the symbolic extension theorem is that of groups with the comparison property.

**Theorem 7.10.** Suppose $G$ is a countable amenable group with the comparison property. Then, for every action of $G$ on a compact metric space $X$, we have the equivalence: a function $E_A$ on $M_G(X)$ is a bounded and affine superenvelope of the entropy structure $H$ of $X$ if and only if there exists a symbolic extension $\pi : Y \to X$ such that $h^\pi = E_A$ on $M_G(X)$.

**Proof.** Only one implication needs a proof, and the proof is now straightforward. Let a superenvelope $E_A$ be given. By Corollary 7.7 there exists an encodable tiling system $\tilde{T}$ of $G$ with entropy zero, having a principal (hence also of entropy zero) symbolic extension $Z$. We use $\tilde{T}$ to create a quasi-symbolic extension $\bar{Y} = Y \vee \tilde{T}$ of $X$ with the extension entropy function matching $E_A$, as in Theorem 5.2. Finally, we extend $\bar{Y} = Y \vee \tilde{T}$ to a symbolic system $Y \vee Z$, without changing the extension entropy function. \qed

7.3.3. Residually finite groups. Note that full comparison property of the group $G$ is not necessary for the above proof to work. What we need is just one encodable tiling system of entropy zero. The knowledge about tiling options in general countable amenable groups is very limited. It is unknown whether all such groups are monotileable, i.e., for some Følner sequence $(F_n)_{n \in \mathbb{N}}$ and every $n \in \mathbb{N}$, admit a tiling with just one shape $F_n$.

An example of monotileable groups are residually finite (see e.g. [38] for definition) countable amenable groups. Monotileability of such groups is shown in [54, Theorem 1]. Moreover, in such groups there exists a tiling system consisting of one-shape tilings. Such tiling system obviously has entropy zero and is encodable (the proof of Theorem 7.2 applies with $r(\epsilon_k) = 1$ for every $k \in \mathbb{N}$). We remark, that it is an open problem whether all residually finite groups have the comparison property. This is why the extension of the full version of our symbolic extension theorem to this class is, in the present state of knowledge, independent of Theorem 7.10.

**Theorem 7.11.** Suppose $G$ is a residually finite countable amenable group. Then, for every action of $G$ on a compact metric space $X$, we have the equivalence: a function $E_A$ on $M_G(X)$ is a bounded and affine superenvelope of the entropy structure $H$ of $X$ if and only if there exists a symbolic extension $\pi : Y \to X$ such that $h^\pi = E_A$ on $M_G(X)$.

**Appendix A**

It is known since a long time that any $\mathbb{Z}$-action with topological entropy zero can be extended to a zero entropy subshift with two symbols (combine [6, Theorem 7.4] with e.g. [17, Theorem 7.2.3]). Since one symbol produces only the trivial subshift, two is clearly the necessary minimum. Two symbols have this nice feature that symbolic elements in $\{0,1\}^G$ can be identified with subsets of the group. In any construction of a symbolic extension (of some $\mathbb{Z}$-action) over two symbols, at some place, more or less explicitly, it is used that for any finite set $F \subset \mathbb{N}$, $F = F + n$ if
and only if \( n = 0 \). It is not necessarily so in more abstract groups and this section is added just to cope with this slight difficulty. The goal is to prove, in anticipation of possible questions, that in the symbolic extension in Theorem \([7,2] \), we can, although not without some extra effort, reduce the number of symbols from three to two. We will use this opportunity to develop a small (nevertheless somewhat excessive for our goal) “theory of recognizability”.

A.1. Recognizability. Throughout this section we assume that \( G \) is an infinite group with unit \( e \).

**Definition 0.12.** A finite set \( A \subset G \) has recognizable origin if the only \( g \in G \) such that \( Ag = A \) is the unit \( e \).

This property has the following interpretation: for every shifted copy of \( A \), (i.e., for any set of the form \( Ag, g \in G \)) we know exactly where its origin (i.e., the element \( g \)) is.

**Lemma 0.13.** Any one-element set has recognizable origin. Let \( A \subset G \) be a finite set of cardinality at least 2. Let \( g \notin AA^{-1} \). Then \( B = A \cup \{g\} \) has recognizable origin. (Note that \( g \notin A \), so \( |B| = |A| + 1 \).)

**Proof.** The first statement is trivial. Suppose that \( B \) does not have recognizable origin, i.e., there exists \( h \in G \), \( h \neq e \), such that \( Bh = B \). We have \( Ah \cup \{gh\} = A \cup \{g\} \) and since \( gh \neq g \), we get \( g \in Ah \) (and also \( g \in Ah^{-1} \)). On the other hand, since \( |A| \geq 2 \), we get \( A \cap Ah \neq \emptyset \) implying \( h \in A^{-1}A \) (and \( h^{-1} \in A^{-1}A \)) and thus \( g \in AA^{-1}A \), a contradiction. \( \square \)

**Remark 0.14.** It follows from the proof that \( g \) can be selected from any a priori given infinite subset of \( G \).

We remark that if \( |A| = 1 \), there may be no \( g \notin A \) such that \( A \cup \{g\} \) has recognizable origin. This happens if all elements of the group are of order 2 (i.e., \( g^2 = e \) for all \( g \in G \)). This is why the lemma produces sets with recognizable origin of all possible finite cardinalities except 2.

**Definition 0.15.** Let \( \{A_1, A_2, \ldots, A_k\} \) be a collection of finite sets of equal cardinalities. We say that the collection is recognizable with recognizable origins if for any \( 1 \leq i, j \leq k \) and \( g \in G \), the only possibility that \( A_i g = A_j \) is when \( i = j \) and \( g = e \).

This property has the following interpretation: given any set of the form \( A_i g \), one can recognize which of the sets from the collection has been shifted and how (in particular, the sets \( A_i \) must all be different).

**Lemma 0.16.** Let \( \{A_i, 1 \leq i \leq k\} \) be a collection of finite sets of equal cardinalities larger than or equal to 2. Then there exist elements \( g_i \notin A_i \) (\( i = 1, 2, \ldots, k \)) such that the collection \( \{B_i, 1 \leq i \leq k\} \) where \( B_i = A_i \cup \{g_i\} \), is recognizable with recognizable origins.

**Proof.** We choose \( g_1 \notin A_1A_1^{-1}A_1 \), so that \( B_1 \) has recognizable origin. From here on we proceed by induction. Suppose that for some \( 1 \leq i \leq k - 1 \) we have selected \( g_1, g_2, \ldots, g_i \notin A_i \) so that the collection \( \{B_1, \ldots, B_i\} \) is recognizable with recognizable origins. We select \( g_{i+1} \notin A_{i+1}A_{i+1}^{-1}A_{i+1} \) (so that \( B_{i+1} \) has recognizable origin) and moreover, we choose \( g_{i+1} \) so it does not belong to any of the sets \( g_jA_j^{-1}A_{i+1} \cup A_ig_j^{-1}A_{i+1} \) (\( 1 \leq j \leq i \)). Suppose that the collection \( \{B_1, \ldots, B_{i+1}\} \) is
not recognizable. The only possibility is that $B_{i+1} = B_jg$ for some $j = 1, 2, \ldots, i$ and $g \in G$. That is,

$$A_{i+1} \cup \{g_{i+1}\} = A_jg \cup \{g\}.$$  

One option is that $g_{i+1} = g_jg$ and $A_{i+1} = A_jg$. This leads to $g = g_j^{-1}g_{i+1}$ and $g \in A_j^{-1}A_{i+1}$, hence $g_{i+1} \in g_jA_j^{-1}A_{i+1}$, which is impossible. Otherwise, we have $g_{i+1} \in A_jg$ and $g_jg \in A_{i+1}$ leading to $g \in A_j^{-1}g_{i+1} \cap g_j^{-1}A_{i+1}$, and hence $g_{i+1} \in A_jg_j^{-1}A_{i+1}$, which is also impossible. \hfill \Box

**Remark 0.17.** It follows from the proof that the elements $g_1, g_2, \ldots, g_k$ can be selected from any a priori given infinite subset of $G$.

**Definition 0.18.** Let $\{A_i, 1 \leq i \leq k\}$ be a collection of finite sets of equal cardinalities larger than or equal to 2, each containing the unit. A family of their shifted copies

$$\{A_i g_{i,j} : 1 \leq i \leq k, j \in \mathbb{N}\}$$

is said to be **fully recognizable** if, for any $i_0 \in \{1, 2, \ldots, k\}$, the inclusion

$$A_{i_0}g \subset \bigcup \{A_i g_{i,j} : 1 \leq i \leq k, j \in \mathbb{N}\}$$

is possible only if $i_0 = i$ and $g = g_{i,j}$ for some $j \in \mathbb{N}$.

This property has the following interpretation: in the above union we can recognize all component sets together with their origins (in particular, the collection $\{A_i, 1 \leq i \leq k\}$ must be recognizable with recognizable origins).

**Lemma 0.19.** If the collection $\{A_1, A_2, \ldots, A_k\}$ is recognizable with recognizable origins and the following two sets

$$\bigcup \{A_i^{-1}A_i^{-1}A_i' : 1 \leq i', i, i'' \leq k\}$$

(later called the margin of full recognizability) and

$$\{g_{i',j'}(g_{i'',j''})^{-1} : 1 \leq i', i'' \leq k, j', j'' \in \mathbb{N}, (i', j') \neq (i'', j'')\}$$

are disjoint, then the family $\{A_i g_{i,j} : 1 \leq i \leq k, j \in \mathbb{N}\}$ is fully recognizable.

**Proof.** Suppose that $A_{i_0}g$ is contained in the union $\bigcup \{A_i g_{i,j} : 1 \leq i \leq k, j \in \mathbb{N}\}$. Either $A_{i_0}g$ matches one component sets $A_i g_{i,j}$ or not. If it does, then by recognizability with recognizable origins of the collection $\{A_1, A_2, \ldots, A_k\}$, we have $i_0 = i$ and $g = g_{i,j}$, as required. If it does not, then $A_{i_0}g$ intersects two different sets $A_i g_{i',j'}$ and $A_i g_{i'',j''}$ (with $(i', j') \neq (i'', j'')$), hence the sets $A_i^{-1}A_i^{-1}A_i'$ and $A_i^{-1}A_i^{-1}A_i''$ are not disjoint (both contain $g$), which, after elementary rearrangements leads to $g_{i',j'}(g_{i'',j''})^{-1} \in A_i^{-1}A_i A_i^{-1}A_i''$. By assumption, this cannot happen. \hfill \Box

**A.2. Reduction of the number of symbols.**

**Theorem 0.20.** Let $G$ be a countable amenable group. There exists a perfectly encodable Følner system $T$ of disjoint quasitilings of topological entropy zero which has an isomorphic symbolic extension on two symbols.

**Proof.** In the proof of Theorem 0.2 we need to change only the steps 1 and 2. In step 1 we let $X_1$ be the full shift over two symbols $X_1 = \{0, 1\}^G$. It is step 2, which requires the essential modification.
Step 2. We define \( M = 3 \cdot 2r(\varepsilon_2) \). Now, we choose a family \( \mathcal{U} \) consisting of \( M \) sets, which is recognizable with recognizable origins (Lemma 0.16 guarantees that such a family exists). We let \( U_2 \) be the margin of full recognizability for the family \( \mathcal{U} \) (see Lemma 0.19 note that \( U_2 \supseteq \bigcup \mathcal{U} \)). Then, as in the proof of Theorem 4.18 we let \( T_2 \) be a zero entropy \( \varepsilon_2 \)-quasitiling with the collection of shapes \( S_2 \subset \{ F_{n_{1,2}}, F_{n_{2,2}}, \ldots, F_{n_{r(\varepsilon_2),2}} \} \), \((n_{1,2} < n_{2,2} < \cdots < n_{r(\varepsilon_2),2}) \) and such that for every \( T_2 \in \mathcal{T}_2 \) the set of centers \( C(T_2) \) is \( U_2 \)-separated. We use the revised version of Theorem 4.18 and thus, for each tiling \( T_2 \in \mathcal{T}_2 \) and each shape \( S \in \mathcal{S}(\mathcal{T}_2) \), we can determine the primariness of the tiles of \( T_2 \) with the shape \( S \) (by observing the symbols “\( S_p \)” versus “\( S_n \)”.

To every symbol “\( S_p \)” , where \( S \in S_2 \) and \( s \in \{ p, n \} \), one can disjointly associate a family of three different sets \( \{ B_{S,S,-1}^{(2)}, B_{S,S,0}^{(2)}, B_{S,S,1}^{(2)} \} \subset \mathcal{U} \). We allow \( \varepsilon \in Z_1 \) to be a member of \( \pi_2^{-1}(\mathcal{T}_{[1,2]}) \) (where \( \mathcal{T}_{[1,2]} \in \mathbb{T}_{[1,2]} \)) if the following holds:

1. Whenever \( T_2(c) = \pi_p \) (i.e., \( c \in C(T_2) \) is the center of a primary or non-primary tile \( Sc \) of \( T_2 \)) then we require that \( z|_{U_2c} = 1 B_{S_{S_i},c}^{(2)}|_{U_2c} \) for some \( i \in \{ -1, 0, 1 \} \) (it is essential that the sets \( U_2c \) contain the sets \( B_{S_{S_i},c}^{(2)} \) and are disjoint for different \( c \in C(T_2) \)).

2. All independent choices of the above indices \( i \) for different centers \( c \in C(T_2) \) are represented in the elements \( z \in \pi_2^{-1}(\mathcal{T}_{[1,2]}) \).

3. We define the background of \( T_2 \) as the complement of \( U_2C(T_2) \), and we require that \( z_g = 0 \) for every \( z \in \pi_2^{-1}(\mathcal{T}_{[1,2]}) \) and all \( g \) in this background.

The map \( \pi_2 \) now functions by a slightly different rule: By Lemma 0.19 and since, for every \( T_2 \in \mathcal{T}_2 \), the set of centers \( C(T_2) \) is \( U_2 \)-separated, any \( z \in Z_2 \) equals the characteristic function of a fully recognizable family of shifted copies of members of the collection \( \mathcal{U} \). This allows to locate all center sets \( c \in C(T_2) \) and recognize the sets \( B_{S_{S_i},c}^{(2)} \) attached to them (with determining the parameters \( S, s \) and \( i \)), using a block code with coding horizon \( U_2 \).

This ends the description of the modification of step 2. From now on we have, as before, three blocks admitted to encode every shape and all further steps of the construction remain unchanged.

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Bibliography

1. M. A. Akcoglu and A. del Junco, *Convergence of averages of point transformations*, Proc. Amer. Math. Soc. **49** (1975), 265–266.
2. George M. Adelson-Velsky and Yu. A. Šreider, *The Banach mean on groups*, Uspehi Mat. Nauk (N.S.) **12** (1957), 131–136.
3. Mathias Beiglböck, Vitaly Bergelson, and Alexander Fish, *Sunset phenomenon in countable amenable groups*, Adv. Math. **223** (2010), 416–432.
4. Mike Boyle, *Lower entropy factors of sofic systems*, Ergodic Theory Dynam. Systems **3** (1983), 541–557.
5. Mike Boyle and Tomasz Downarowicz, *The entropy theory of symbolic extensions*, Invent. Math. **156** (2004), 119–161.
6. Mike Boyle, D. Fiebig, U. Fiebig, *Residual entropy, conditional entropy and subshift covers*, Forum Mathematicum **14** (2002), 713–758.
7. Mike Boyle and David Handelman, *Orbit equivalence, flow equivalence and ordered cohomology*, Israel J. Math. **95** (1996), 169–210. MR 1418293
8. Emmanuel Breuillard, Ben Green, and Terence Tao, *The structure of approximate groups*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 115–221.
9. Julian Buck, *Smallness and comparison properties for minimal dynamical systems*, arXiv:1306.6681 (2013).
10. D. Burguet, *C^2* surface diffeomorphisms have symbolic extensions, Invent. math. **186**, 191-236.
11. D. Burguet and Tomasz Downarowicz *Uniform generators, symbolic extensions with an embedding, and structure of periodic orbits*, J Dyn Diff Equat (2018), https://doi.org/10.1007/s10884-018-9674-y
12. Nhan-Phu Chung and Guohua Zhang *Weak expansiveness for actions of sofic groups*, J. Funct. Anal. **268** (2015), 3534–3565.
13. Joachim Cuntz, *Dimension functions on simple C^*-algebras*, Math. Ann. **233** (1978), 145–153.
14. A. Danilenko and K. Park, *Generators and Bernoullian factors for amenable actions and cocycles on their orbits*, Ergod. Th. Dynam. Sys. **22** (2002), 1715–1745.
15. Dou Dou, *Minimal subshifts of arbitrary mean topological dimension*, Discrete Contin. Dyn. Syst. **37** (2017), 1411–1424.
16. T. Downarowicz, *Entropy structure*, J. Anal. Math., **96** (2005), 57–116.
17. Tomasz Downarowicz, *Entropy in dynamical systems*, New Mathematical Monographs vol 18, Cambridge University Press, 2011.
18. Tomasz Downarowicz, Bartosz Frej and Pierre-Paul Romagnoli, *Shearer's inequality and Infimum Rule for Shannon entropy and topological entropy*, Contemporary Math. Proceedings of Max Planck Institute (Dynamics and Numbers) **669** (2016), 63–75.
19. Tomasz Downarowicz, Dawid Huczek and Guohua Zhang, *Tilings of amenable groups*, J. Reine Angew. Math. (2016), https://doi.org/10.1515/crelle-2016-0025.
20. Tomasz Downarowicz and Dawid Huczek, *Dynamical quasitilings of amenable groups*, Bulletin Polish Acad. Sci. Math. **66** (2018), 45–55.
21. Tomasz Downarowicz, Alejandro Maass, *Smooth interval maps have symbolic extensions*, Invent. math. **176** (2009), 617-636.
22. Tomasz Downarowicz, Sheldon Newhouse, *Symbolic extension entropy in smooth dynamics*, Invent. Math. **160** (2005), 453–499.
23. On groups with full Banach mean value, Math. Scand. **3** (1955), 243–254.
24. Bartosz Frej and Dawid Huczek, *Minimal models for actions of amenable groups*, Groups Geom. Dyn. **11** (2017), 567–583.
25. , *Faces of simplices of invariant measures for group actions*, Monatsh. Math. **185** (2018), 61–80.
26. Thierry Giordano, Ian F. Putnam, and Christian F. Skau, *Topological orbit equivalence and C*-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.
27. , *Full groups of Cantor minimal systems*, Israel J. Math. **111** (1999), 285–320.
28. Eli Glasner, Jean-Paul Thouvenot and Benjamin Weiss, *Entropy theory without a past*, Ergodic Th. Dynam. Sys. **20** (2000), 1355–1370.
29. Eli Glasner and Benjamin Weiss, *Weak orbit equivalence of Cantor minimal systems*, Internat. J. Math. **6** (1995), 559–579.
30. Rostislav I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), 939–985.
31. Philip Hall, *On Representatives of Subsets*, J. London Math. Soc. **10** (1935), 26–30.
32. G. A. Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Math. Systems Theory **3** (1969), 320–375.
33. Wen Huang, Xiangdong Ye, and Guohua Zhang, *Local entropy theory for a countable discrete amenable group action*, J. Funct. Anal. **261** (2011), 1028–1082.
34. Dawid Huczek, *Zero-dimensional extensions of amenable group actions*, arXiv:1503.02827
35. David Kerr, *Dimension, comparison and almost finiteness*, arXiv:1710.00393 (2017).
36. W. Krieger, *On the subsystems of topological Markov chains*, Erg. Th. Dyn. Syst. **2** (1882), 195–202.
37. Elon Lindenstrauss, *Pointwise theorems for amenable groups*, Invent. Math. **146** (2001), 259–295.
38. Wilhelm Magnus, *Residually finite groups*, Bull. Amer. Math. Soc. **75** (1969), 305–316.
39. Michal Misiurewicz, *Topological conditional entropy*, Studia Math. **55** (1976), 175–200.
40. I. Namioka, *Følner’s conditions for amenable semi-groups*, Math. Scand. **15** (1964), 18–28.
41. John von Neumann, *Zur allgemeinen Theorie des Masses*, Fund. Math. **13** (1929), 73–116.
42. Donald S. Ornstein and Benjamin Weiss, *Ergodic theory of amenable group actions. I. The Rohlin lemma*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), 161–164.
43. Alan L. T. Paterson, *Amenability*, Mathematical Surveys and Monographs, vol. 29, American Mathematical Society, Providence, RI, 1988.
44. Maxence Phalempin, *Representation of congruent sequences of tilings on amenable groups*, 2016 (unpublished) Internship report–University of Rennes.
45. Mikael Rørdam, *On the structure of simple C*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. **107** (1992), 255–269.
46. , *The stable and the real rank of absorbing C*-algebras*, Internat. J. Math. **15** (2004), 1065–1084.
47. A. Rosenthal, *Finite uniform generators for ergodic, finite entropy, free actions of amenable groups*, Prob. Th. Rel. Fields **77** (1988), 147–166.
48. Jacek Serafin, *A faithful symbolic extension*, Commun. Pure Appl. Math. **11** (2012), 1051–1062.
49. Konstantin Slutskiy, *Lecture notes on topological full groups of cantor minimal systems*, [http://homepages.math.uic.edu/~kslutsky/papers/Topological-full-groups.pdf](http://homepages.math.uic.edu/~kslutsky/papers/Topological-full-groups.pdf).
50. Yuhei Suzuki, *Almost finiteness for general étale groupoids and its applications to stable rank of crossed products*, International Mathematics Research Notices, rny187, https://doi.org/10.1093/imrn/rny187
51. Stefan Šušanj, *Generators for amenable group actions*, Mh. Math. **95** (1983), 67–79.
52. Gábor Szabó, *Private communication*, 2017.
53. T. Ward and Q. Zhang, *The Abramov-Rokhlin entropy addition formula for amenable group actions*, Monatsh. Math. **114** (1992), 317–329.
54. , *Monotileable amenable groups*, Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, **202**, Amer. Math. Soc., Providence, RI, 2001, 257–262.
55. Wilhelm Winter, *Decomposition rank and Z-stability*, Invent. Math. **179** (2010), 229–301.
56. V. S. Varadarajan, *Groups of automorphisms of Borel spaces*, Trans. Amer. Math. Soc. **109** (1963), 191–220.
57. Ruifeng Zhang, *Topological pressure of generic points for amenable group actions*, J. Dyn. Diff. Equat. **30** (2018), 1583–1606.
58. Dongmei Zheng, Ercai Chen, and Jiahong Yang. *On large deviations for amenable group actions*, Discrete Contin. Dyn. Syst. **36** (2016), 7191–7206.