Spin polarization induced by magnetic field and the relativistic Barnett effect

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Abstract

First, I study the analogy between the magnetization of a material and the spin polarization of particles in a fluid. Using the relativistic version of the Barnett effect, \textit{i.e.} the magnetization of a material induced by mechanical rotation, the spin polarization induced by thermal vorticity is obtained within a purely classical model, where spin is treated as an intrinsic magnetic moment and rotation is included as a non-inertial effect. I argue that since spin polarization induced by thermal vorticity can be obtained in a classical theory, it cannot be dominated by quantum anomalies.

Second, the spin polarization induced by magnetic field is obtained for a fluid at local thermal equilibrium using statistical quantum field theory. The obtained formula is valid beyond the weak field approximation and when contributions from the non-homogeneity of the magnetic field are small. The exact form of spin polarization is studied for a free Dirac field at global equilibrium, and, like magnetic susceptibility, it oscillates according to the de Haas–van Alphen effect.

Finally, I briefly review how magnetic field contributes to the difference between the spin polarization of $\Lambda$ and $\bar{\Lambda}$ observed in heavy-ion collisions.

1. Introduction

Non-central heavy-ion collisions produce a plasma with large magnetic field \[1\] and angular momentum \[2\]. Both these two factors can be revealed in the measurements of $\Lambda$ spin polarization \[3, 4, 5, 6, 7, 8\]. The impact of

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the vorticity to the spin polarization is predominant compared to the magnetic field because the former survives for a longer time due to the angular momentum conservation. For this reason, many studies have focused in the spin polarization induced by vorticity and it is now a well established phenomenon, see for instance [9, 10] for a review. It is curious how the effect of vorticity is explained in analogy with magnetization as a relativistic Barnett effect [11, 12], while the spin polarization of a relativistic particle induced by magnetic field itself received less attention.

Motivated by the spin physics in heavy-ion collisions, recent studies [13, 14, 15] have included the spin degrees of freedom in the magneto-hydrodynamic equations. The spin polarization induced by magnetic field was derived in Refs. [16, 17, 18] and was analyzed in [19, 20]. Differently from vorticity, a magnetic field produces opposite spin polarization for Λ and ¯Λ. Other effects that might produce a difference in spin polarization in heavy-ion collisions were quantified in [21, 22]. Assessing the intensity of magnetic field for these measurements is also important for estimating the magnitude of the chiral magnetic effect [23].

The previous derivations of the spin polarization induced by magnetic field relied on an analogy with vorticity [17] or they were based on kinetic theory [16, 18]. The main purpose of this work is to derive the spin polarization in a full quantum relativistic framework. Indeed the Quark–Gluon Plasma is composed by strongly interacting quantum fields that can not be described with weakly interacting quasi-particles [24]. Therefore the results of kinetic theory can not be extended inside the plasma as it describes the fields as quasi-particles between the collisions. Despite this, the system is a fluid and the notion of local thermal equilibrium admits a full quantum relativistic description. In this work the spin polarization of a massive spin 1/2 fermion is derived using the Zubarev formalism for the non-equilibrium statistical operator [25, 26, 27, 28, 29, 30, 31, 32] and the exact form of the Wigner function in magnetic field [33, 34].

This paper is structured as follows. In Sec. 2 I review the magnetization and the non-relativistic and relativistic version of the Barnett effect. I also show that even classical theory, i.e. describing the spin as an intrinsic magnetic moment and describing the effect of rotation as a non-inertial effect, predicts a spin polarization along the rotation of the system. In Sec. 3 I derive the local thermal equilibrium in the presence of vorticity and magnetic field and I obtain a formula for spin polarization. In sec. 4 I write the scalar and axial part of the Wigner function of a fermion in external magnetic field.
In sec. 5, I derive the spin polarization vector induced by magnetic field; I compare the full result with a calculation in the non-relativistic limit; I study this polarization at global equilibrium and I discuss the experimental data for heavy-ion collisions.

2. Magnetization and the relativistic Barnett effect

In this section I review the magnetization of a magnetic material induced by an external magnetic field and by a mechanical rotation, which is the Barnett effect, and I derive the relativistic version of the Barnett effect.

Using the classical theory, one can describe a magnetic material as composed by non-interacting particles with magnetic moment $\mu$. Denoting with $n$ the number density of magnetic moments and with $T$ the temperature, the magnetization $M$ induced by a magnetic field $B$ is

$$M = n\mu L \left( \frac{\mu B}{T} \right),$$

where $L$ is the Langevin function

$$L(x) = \coth(x) - \frac{1}{x} \simeq \frac{x}{3} + O \left( x^2 \right).$$

It is well known that the classical magnetization (1) can be derived using classical statistical mechanics. It is sufficient to describe the energy of the magnetic elements in the magnetic field with the Hamiltonian

$$H = -\mu \cdot B = -\mu B \cos \theta.$$ 

Consequently, the partition function of this system is

$$Z = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta e^{\mu B \cos \theta} = 2\pi \int_{-1}^1 dy e^{\mu B \beta y} = 4\pi \frac{\sinh(\mu B \beta)}{\mu B \beta},$$

where $\beta$ is the inverse of temperature $T$. From the partition function one can easily obtain the average magnetic moment in the direction of the magnetic field as

$$\langle \mu \cdot \hat{B} \rangle = \frac{1}{Z} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta |\mu \cos \theta| e^{\mu B \cos \theta} = \frac{1}{Z\beta} \frac{\partial Z}{\partial B} = \frac{1}{\beta} \frac{\partial \log Z}{\partial B},$$

which results in the magnetization (1).
Similarly, magnetic materials can be magnetized by rotating them, a phenomenon known as Barnett effect [11, 12]. Historically, the magnetization induced by rotation was obtained balancing the torques acting on the magnetic moments under rotation [11, 12]. Today, a modern description [35] can be given in terms of non-inertial effects caused by rotation. In order to obtain the result, it is indeed sufficient to observe that in the presence of an angular velocity \( \omega \) the energy of the particle is shifted due to the coupling of angular momentum, denoted by \( \mathbf{J} \), with rotation [36, 37]:

\[
H \rightarrow H - \omega \cdot \mathbf{J}.
\]  

(6)

Note that this is a purely classical effect.

Consider then a molecular magnet as a single particle with charge \( q \) and mass \( m \) revolving in a close orbit about a much more massive nucleus of charge \(-q\). With \( r \) the radius vector and \( \omega \) the angular velocity, the magnetic moment of the molecular system is given by \( \mu = \frac{1}{2} qr \omega^2 \), and its angular momentum is

\[
\mathbf{J} = mr^2 \omega.
\]  

(7)

According to Eq. (6) the energy of the magnetic element is

\[
H = -\mathbf{J} \cdot \omega = -J \omega \cos \theta = -\mu \frac{\omega}{\gamma} \cos \theta,
\]  

(8)

where \( \gamma \) denotes the gyromagnetic ratio \( \gamma = \mu/J \). The comparison of Eqs. (3) and (8) gives the classical Barnett effect, which states that a mechanical rotation of the material induces the same magnetization as an effective magnetic field with intensity

\[
B_{\text{eff}} = \frac{\omega}{\gamma}.
\]  

(9)

This classical description can be easily extended to the quantum world by describing the magnetic elements as quantum particles with spin \( S \) and using the non-relativistic quantum statistical operator at thermal equilibrium

\[
\hat{\rho} = \frac{1}{Z} \exp \left[ -\frac{\hat{H}}{T} + \frac{\mu_0}{ST} \mathbf{B} \cdot \hat{\mathbf{S}} + \frac{\omega}{T} \cdot (\hat{\mathbf{L}} + \hat{\mathbf{S}}) \right],
\]  

(10)

where the second term in the exponent is the Pauli interaction with an external magnetic field, the last one is the angular momentum-angular velocity coupling and \( \hat{\mu} = \frac{\mu_0}{S} \hat{\mathbf{S}} \) is the magnetic moment of the particle with
\( \mu_0 = q/2m \). Consider the simple case where \( \mathbf{B} \) is parallel to \( \boldsymbol{\omega} \), then the statistical operator can be diagonalized in the basis of the eigenvectors of the spin operator component parallel to \( \mathbf{B} \). The magnetization of the material is given in terms of the expectation value of \( \hat{\boldsymbol{\mu}} \), that is proportional to the spin in the direction of magnetic field:

\[
\langle \hat{\boldsymbol{S}} \cdot \hat{\mathbf{B}} \rangle = \hat{\mathbf{B}} \sum_{\sigma = -S}^{S} \sigma \exp \left[ \frac{\mu_0 B (S + \omega)}{T} \right] = \hat{\mathbf{B}} \frac{\partial}{\partial x} \sum_{\sigma = -S}^{S} e^{x\sigma} = SB_S(x),
\]
(11)

where I defined \( x = (\omega + \mu_0 B/S)/T \), and \( B_S \) is the Brillouin function:

\[
SB_S(x) = \frac{2S + 1}{2} \coth \left( \frac{2S + 1}{2} x \right) - \frac{1}{2} \coth \left( \frac{x}{2} \right) \simeq \frac{S(S + 1)}{3} x + \mathcal{O}(x^2).
\]
(12)

Setting the rotation to zero and considering a case where \( x \ll 1 \), the previous expressions reproduce the Curie law for the magnetization:

\[
M = n \langle \hat{\boldsymbol{\mu}} \cdot \hat{\mathbf{B}} \rangle = \frac{n\mu_0}{S} \langle \hat{\boldsymbol{S}} \cdot \hat{\mathbf{B}} \rangle \simeq n \mu_0 S + 1 \frac{B}{3} x = n \mu_0 \frac{S + 1}{3S} \frac{B}{T},
\]
(13)

which for \( S = 1/2 \) is \( M = n\beta \mu_0^2 B \). The Barnett effect is obtained turning on the rotation. Indeed, the effect of rotation can be read by looking at the argument of the Brillouin function in Eq. (11): the magnetization induced by rotation is the same as the one obtained by an effective magnetic field \( \gamma \) with a gyromagnetic ratio of \( \gamma = \mu_0/S \). The classical result corresponds to the limit \( S \to \infty \) of Eq. (12), that reproduces the Langevin function

\[
\lim_{S \to \infty} SB_S(x) = L(x).
\]
(14)

2.1. The relativistic Barnett effect

Since in the quantum approach the magnetization was obtained evaluating the expectation value of the spin, it is clear that the phenomenon of magnetization is analogous to the one of spin polarization. Despite the fact that the spin of a particle is an inherently quantum property, the main reason that cause the magnetization is classical. Because of that, the spin polarization induced by magnetic field and a parallel rotation can be obtained with the classical theory described above and it is given by

\[
\langle \hat{\boldsymbol{S}} \cdot \hat{\mathbf{B}} \rangle = L(x), \quad x = \frac{\mu}{T} \left( B + \frac{\omega}{\gamma} \right).
\]
(15)
More accurately, for a quantum system in the non-relativistic limit, the spin polarization is given by:

\[
\langle \hat{S} \cdot \hat{B} \rangle = SB_S(x), \quad x = \frac{\omega + \mu_0 B / S}{T}.
\]

(16)

To maintain the right covariant properties, the spin polarization for a relativistic particle is given by a mean spin (four-)vector. Recently, the polarization of the spin of a particle in a relativistic fluid has been studied intensively, see for instance the reviews [9, 10] and reference therein. Around a point \(x\) of the fluid, the spin vector of a particle with momentum \(p\) can be induced by the thermal vorticity of the fluid, which is defined as

\[
\varpi_{\mu \nu} = -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu),
\]

with \(\beta^\mu = u^\mu / T\) and \(u\) is the fluid velocity. For a spin 1/2 particle the resulting spin polarization is [38, 17]

\[
S_{\varpi}^\mu(x, p) = -\frac{1}{8m} (1 - n_F) \epsilon^{\mu \rho \sigma \tau} p_\rho \varpi_{\sigma \tau},
\]

(18)

where \(n_F\) is the Fermi-Dirac thermal distribution function. Being an antisymmetric tensor, the thermal vorticity can be decomposed as

\[
\varpi_{\mu \nu} = u_\nu \frac{a_\mu}{T} - u_\mu \frac{a_\nu}{T} + \epsilon_{\mu \nu \rho \sigma} \frac{\omega^\rho}{T} u_\sigma,
\]

(19)

where \(a\) and \(\omega\) are respectively the acceleration and the angular velocity of the fluid [39]. In a case without acceleration \((a = 0)\), the previous expression reduces to:

\[
S_{\varpi}^\mu(x, p) = \frac{1}{4m} (1 - n_F) \beta \varepsilon_p (\omega^\mu - u^\mu \frac{\omega \cdot p}{\varepsilon_p}),
\]

(20)

where \(\varepsilon_p = p \cdot u\). As I will show in next Sections, the spin polarization induced by a magnetic field is

\[
S_{B}^\mu(x, p) = \frac{1}{4m} (1 - n_F) \beta (qB^\mu - u^\mu \frac{qB \cdot p}{\varepsilon_p}).
\]

(21)

The relativistic Barnett effect is derived comparing the two expressions. The spin polarization induced by rotation is the same as the one induced by a magnetic field with intensity

\[
B_{\text{Eff}}^\mu = \frac{\varepsilon_p}{q} \omega^\mu.
\]

(22)
This is the magnetic field required to produce a synchrotron frequency equals to $\omega$. In the non-relativistic limit this effective magnetic field reduces to

$$B_{\text{Eff}} = \frac{m}{q} \omega = \frac{1}{2\mu_0} \omega,$$

which is the Barnett effect derived above in Eq. (11) for spin $S = 1/2$.

2.2. Spin polarization and gravitational anomaly

The vorticity induced spin polarization is explained above as a relativistic Barnett effect. It is possible to connect the spin polarization (or the magnetization) induced by a magnetic field with the one induced by rotation thanks to the analogy of the Pauli interaction with the energy shift due to rotation: compare the second and third term in Eq. (10). As the Pauli interaction is valid only in the non-relativistic approximation, it is not expected that higher order corrections in angular velocity can be obtained using the effective magnetic field (22). In any case, as shown in [40], the ultimate reason why a spin polarization is generated from rotation is the angular momentum-rotation coupling in Eq. (6), which is a classical non-inertial effect. Indeed, it was possible to obtain the pure classical result (15) for the spin polarization.

It was realized that for a massless field the thermal coefficient giving the linear response of a spin current generated by rotation is tightly connected with the quantum gravitational anomaly coefficient appearing in the divergence of the axial current [41]. Subsequently, it was proposed that the spin polarization induced by vorticity might be caused by the gravitational anomaly [41, 42, 43, 44]. The connection with the quantum anomaly offers new insights in the interplay between quantum field theory, transport and non-inertial effects, but, as argued above, I believe that the cause of this phenomenon is the angular momentum and rotation coupling. If the quantum gravitational anomaly were truly necessary, it could not have been possible to obtain a spin polarization with a classical theory. Furthermore, a direct connection with the gravitational anomaly is only possible for massless fields and breaks down for massive fields, where the spin polarization has been actually measured.

3. Local thermal equilibrium in magnetic field

The main purpose of this work is to obtain the spin polarization of a free fermion in a fluid at local thermal equilibrium in the presence of an external
magnetic field. The spin polarization of a fermion is obtained using [45]

\[
S^\mu(p) = \frac{1}{2} \frac{\int_{\Sigma} d\Sigma \cdot p \; \text{tr}_4 \left[ \gamma^\mu \gamma^5 W_+(x, p) \right]}{\int_{\Sigma} d\Sigma \cdot p \; \text{tr}_4 \left[ W_+(x, p) \right]},
\]

(24)

where \( W_+(x, p) \) is the particle branch of the Wigner function. In heavy-ion collision the predictions for \( \Lambda \) spin polarization are obtained by integrating the previous formula over the freeze-out hypersurface. For a Dirac field in external electromagnetic field the Wigner function is [46]

\[
W(x, p)_{AB} = \frac{1}{(2\pi)^4} \int d^4 y e^{-i p \cdot y} \times \langle \bar{\Psi}_B(x + y/2)U(A, x + y/2, x - y/2)\Psi_A(x - y/2) \rangle,
\]

(25)

where \( U \) is the gauge link

\[
U(A, x_+, x_-) = \exp \left[ -i \int_{x_-}^{x_+} dz^\mu A_\mu(z) \right],
\]

(26)

and the brackets denotes the thermal average with the statistical operator \( \hat{\rho} \):

\[
\langle \hat{X} \rangle = \text{tr} \left( \hat{\rho} \hat{X} \right).
\]

(27)

In what follows, I derive the statistical operator at local thermal equilibrium. Then, I will use it to obtain the Wigner function (25) and finally I will use the Wigner function to obtain the spin polarization with Eq. (24).

The current best model to describe the QGP created in heavy-ion collisions assumes that the system reaches the local thermal equilibrium at a certain initial time. A covariant description of a quantum system that reached the local thermal equilibrium is given by the Zubarev method for the non-equilibrium statistical operator, see for instance [47, 31] for a review. In this approach the statistical operator is obtained by maximizing the total entropy at fixed energy-momentum and charge density in the initial time hypersurface. The local equilibrium statistical operator at later times is obtained by evolving the system and by neglecting the dissipative effects, and it is given by:

\[
\hat{\rho}_{LTE} = \frac{1}{Z} \exp \left[ \int_{\Sigma} d\Sigma_\mu(y) \left( \hat{T}^{\mu \nu}(y)\beta_\nu(y) - \zeta(y)\hat{j}^\mu(y) \right) \right],
\]

(28)

where \( \Sigma \) is the hypersurface at a given time, \( \hat{T}^{\mu \nu} \) is the symmetric stress-energy tensor (SET), \( \hat{j}^\mu \) is the conserved electric current, \( \beta^\mu = u^\mu / T \) is
the four-temperature vector and $\zeta$ is the ratio of chemical potential and temperature.

Furthermore, the magnetic field in heavy-ion collisions can be treated as an external field, as its dynamics approximately decouple from that of the hot nuclear medium \cite{48}. In the presence of an external electromagnetic field the statistical operator (28) remains the same, but the SET is no longer conserved because of the Lorentz force:

$$\partial^\mu \hat{T}_{\mu\nu} = \hat{j}^\lambda F_{\nu\lambda}, \quad \partial^\mu \hat{j}_{\mu} = 0.$$  \hfill (29)

This approach was used to obtain the relativistic magneto-hydrodynamic equations \cite{49, 50} and was recently studied in \cite{32} with the mutual presence of thermal vorticity and electromagnetic field.

Notice that since the statistical operator (28) has the same form with and without an external electromagnetic field, it is important to include the external gauge field inside the operators and solve the equations of motion for the Dirac field in external electromagnetic field. For a free Dirac field interacting with an external gauge field $A^\mu$, the symmetric (and gauge invariant) stress-energy tensor is

$$\hat{T}^{\mu\nu} = \frac{i}{4} \left[ \bar{\psi} \gamma^\mu \partial^\nu \psi - \bar{\psi} \gamma^\mu \partial^\nu \psi + \bar{\psi} \gamma^\nu \partial^\mu \psi - \bar{\psi} \gamma^\nu \partial^\mu \psi \right] - \frac{1}{2} \left( \hat{j}^\mu A^\nu + \hat{j}^\nu A^\mu \right).$$  \hfill (30)

Using the equations of motion, one can check that this SET satisfies the condition in Eq. (29).

The system is at global thermal equilibrium when the statistical operator (28) becomes stationary. This occurs when the operator does not depend on the hypersurface, or equivalently when the divergence of the integrand is vanishing \cite{51}. Taking advantage of Eq. (29), one finds that the global equilibrium is realized when the thermodynamics fields solve the following equations:

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \quad \partial^\mu \zeta = F^{\mu\nu} \beta_\nu.$$  \hfill (31)

The first one is a Killing equation and its solution in flat space is

$$\beta^\mu(y) = b^\mu + \varpi^{\mu\nu} y_\nu,$$  \hfill (32)

where $b$ is a constant time-like vector and $\varpi$ is a constant thermal vorticity, defined in Eq. (17). For a relativistic system the general global thermal equilibrium is realized with a non-homogeneous temperature and with a constant
non-vanishing vorticity. A global equilibrium is possible even in the mutual presence of vorticity and electromagnetic field if the electromagnetic strength tensor satisfies

\[ \beta_\sigma(y) \partial^\sigma F^{\mu\nu}(y) = \omega^\mu_\sigma F^{\sigma\nu}(y) - \omega^\nu_\sigma F^{\sigma\mu}(y). \quad (33) \]

In this case, a general solution for the second Eq. in (31) is given by [32]

\[ \zeta(y) = \zeta_0 - \beta_\sigma(y) A^\sigma(y) - \Phi(y), \quad (34) \]

where \( \Phi \) is necessary to maintain the gauge invariance. For instance a global equilibrium configuration is realized in a constant electromagnetic field such that \( F^{\mu\nu} = k \omega^{\mu\nu} \), with \( k \) a constant. In this configuration the solution (34) becomes

\[ \zeta(y) = \zeta_0 - \beta_\sigma(y) F^{\lambda\sigma} y_\lambda + \frac{1}{2} \omega^{\rho\sigma} y^\rho F^{\lambda\sigma} y_\lambda. \quad (35) \]

In Section 5.2 the spin polarization induced by magnetic field is studied for a system at global equilibrium. Since the focus of this work is the effect of magnetic field, the vorticity is set to zero and the global equilibrium conditions result in a constant homogeneous electromagnetic field, as required by Eq. (33). Considering the case of a vanishing electric field the global thermal equilibrium statistical operator in the presence of an external magnetic field assumes the form [32]

\[ \hat{\rho}_{GTE,B} = \frac{1}{Z} \exp \left\{ -\beta \cdot \hat{P} + \zeta \hat{Q} \right\} \quad (36) \]

where \( \beta \) is a constant time-like vector, \( \zeta \) is also constant, \( \hat{Q} \) is the total electric charge and \( \hat{P} \) is the four-momentum of the system. As in the local equilibrium case, the global statistical operator has the same form with or without an external magnetic field. The effect of magnetic field must be included directly in the field operators of the Dirac field.

Going back to the out of equilibrium case, the Wigner function at local thermal equilibrium is given by

\[ W(x, p)_{LTE} = \frac{1}{Z} \text{tr} \left\{ \exp \left\{ -\int d\Sigma(y) \left( \tilde{T}^{\mu\nu}(y) \beta_\nu(y) - \zeta(y) \tilde{j}^\mu(y) \right) \right\} \tilde{W}(x, p) \right\}. \quad (37) \]

For a fluid in the hydrodynamic regime, such as the QGP, the local equilibrium Wigner function is well approximated by expanding the thermodynamic
fields in the exponent of (37). Since the Wigner function is to be evaluated at the point $x$, the thermodynamic fields and the gauge field are expanded around $x$:

\[
\begin{align*}
\beta_\nu(y) &\approx \beta_\nu(x) + \partial_\lambda \beta_\nu(x)(y - x)^{\lambda}, \\
\zeta(y) &\approx \zeta_\nu(x) + \partial_\lambda \zeta_\nu(x)(y - x)^{\lambda}, \\
A_\nu(y) &\approx A_\nu(x) + \partial_\lambda A_\nu(x)(y - x)^{\lambda}.
\end{align*}
\]

Notice that Eq. (37) depends on the gauge field through the SET (30). For the sake of simplicity, the external electromagnetic field is considered as composed only of a magnetic component, that is

\[
\partial_\mu A_\nu - \partial_\nu A_\mu = -\epsilon_{\mu\nu\rho\sigma} B^\rho u^\sigma.
\]

With this expansion the statistical operator is approximated by

\[
\hat{\rho}_{LTE} \approx \frac{1}{Z} \exp \left[ -\beta_\nu(x) \hat{P}_\nu + \zeta(x) \hat{Q} + \frac{1}{2} \hat{\nu}_{\mu\nu}(x) \hat{J}_x^{\mu} \\
+ \partial_\lambda \zeta(x) \int d\Sigma_\mu (y - x)^{\lambda} \hat{J}_x^{\mu}(y) - \frac{1}{2} \xi_{\mu\nu}(x) \hat{Q}_{\mu\nu}(x) + \cdots \right],
\]

where

\[
\hat{J}_x^{\mu} = \int d\Sigma_\lambda \left[ (y - x)^{\mu} \hat{T}_{\lambda\nu}(y) - (y - x)^{\nu} \hat{T}_{\lambda\mu}(y) \right]
\]

is the generator of boosts and rotations (the conserved angular momentum operator) and

\[
\hat{Q}_{\mu\nu} = \int d\Sigma_\lambda \left[ (y - x)^{\mu} \hat{T}_{\lambda\nu}(y) + (y - x)^{\nu} \hat{T}_{\lambda\mu}(y) \right]
\]

is a non-conserved symmetric quadrupole like operator. I also introduced the thermal shear

\[
\xi_{\mu\nu} = \frac{1}{2} (\partial_\mu \beta_\nu + \partial_\nu \beta_\mu).
\]

The presence of thermal vorticity, thermal shear and gradients of the chemical potential induces a spin polarization

\[
S_\mu(p) = -\epsilon^{\mu\rho\sigma} p_\rho \int_\Sigma d\Sigma \cdot p n_F (1 - n_F) S_{\rho\sigma} \frac{8 m}{\int_\Sigma d\Sigma \cdot p n_F},
\]

\[
S_{\rho\sigma} = \omega_{\rho\sigma} + 2 i p_\rho p_\sigma \tilde{\xi}_{\lambda\sigma} - \frac{i p_\rho \partial_\sigma \zeta}{2 e_p},
\]

\[11\]
where \( \hat{t} \) is the time direction in the laboratory frame and \( n_F = n_F(\beta(x) \cdot p - \zeta(x)) \) is the Fermi-Dirac distribution function. The first term is the known vorticity induced polarization \[38\], the second is the recently derived shear induced polarization \[52, 53, 54, 55, 18\], and the last one is the contribution from the gradient of the chemical potential that has been studied in \[56, 57, 58\]. In heavy-ion collisions these effects give a larger contribution to spin polarization compared to the contribution of the magnetic field. Indeed the magnetic induced polarization is of opposite sign for \( \Lambda s \) and \( \bar{\Lambda} s \) but the measured polarization is roughly the same.

Notice that it turned out that the magnetic field in \[39\] does not explicitly appear in the statistical operator \[40\]. As in the case of global equilibrium \[36\] the effect of the external magnetic field is included directly in the Dirac field operators. If second order terms in derivatives in the expansion \(38\) of the gauge field were included, then the statistical operator approximation \[40\] would have included a term proportional to \( \partial_{\mu} B_{\nu}(x) \). With the current data it might be possible to observe the effects of magnetic field, but the effects of its gradients are smaller. I therefore left the inclusion of such effects for future analysis. The leading order effect of magnetic field in the Wigner function is obtained using the statistical operator

\[
\hat{\rho}_{LTE} \simeq \frac{1}{Z} \exp \left[ -\beta_{\nu}(x) \hat{P}^{\nu} + \zeta(x) \hat{Q} \right].
\]  

One could also search for an interplay between magnetic field and the thermal vorticity and shear by studying the operators \(41\) and \(42\) in the presence of a magnetic field, but as they are higher order terms I leave those for future works. The spin polarization induced by magnetic field is then obtained using Eq. \(24\) with the Wigner function \(25\) evaluated with the statistical operator \(45\). Because the result is obtained expanding the fields around a fixed point \(x\), the resulting Wigner function is the same as a Wigner function obtained at the global equilibrium \(36\) with a constant four-temperature and magnetic field with values \( \beta^{\mu}(x) \) and \( B^{\mu}(x) \).

Note that there is an important difference between the Quantum Kinetic Theory (QKT) and the Zubarev method discussed here. Whereas the Wigner function \(37\) is evaluated from the statistical operator derived by maximizing the entropy at an initial time, QKT can only be carried out by providing a distribution function. In a underlying quantum field theory the correct distribution function is not derived with a kinetic theory but the distribution function is provided as an educated guess. Instead, the Wigner function
derived here is a consequence of local thermal equilibrium and it is valid in a full quantum relativistic regime. The distribution function might be difficult to guess as for the case of equilibrium with thermal vorticity \[59\].

4. Wigner function in magnetic field

In the previous Section it was found that spin polarization induced by magnetic field is obtained with

\[
S_B^{\mu}(p) = \frac{1}{2} \int_{\Sigma} d\Sigma \cdot p \ A_+(x, p)
\]

(46)

where

\[
F_+(x, p) = \text{tr} \left[ W_+(x, p) \right],
\]

(47)

\[
A_+(x, p) = \text{tr} \left[ \gamma^\mu \gamma^5 W_+(x, p) \right],
\]

are respectively the scalar part and the axial part of the particle branch of the Wigner function \[25\] obtained with the statistical operator \[45\] in external magnetic field.

The Wigner function in external magnetic field is obtained by first solving the Dirac equation in magnetic field and then using that solution to evaluate the Wigner function. The solutions of the Dirac equation in magnetic field are well known, see for instance \[60, 32\]. The equilibrium Wigner function in external magnetic field was already obtained using statistical quantum field theory \[61\], but the spin and magnetic moment part were neglected because in astrophysical situations they were negligible. Recently, the full Wigner function in external magnetic field was obtained with exact methods in \[33, 34\].

Spin polarization only requires the scalar and axial part of the Wigner function. In what follows I omit the \(x\) dependence which is contained inside the temperature, the chemical potential and the magnetic field. The scalar
and axial parts of Wigner function (25) are given by [33]

\[ F_+ (p) = \sum_{n=0}^{\infty} \frac{4m}{(2\pi)^3} \theta(\varepsilon_p) \delta (\varepsilon_p^2 - E_{pz}^{(n)} \xi) e^{-\xi/2} \]
\[ \times n_F (\beta \cdot p - \zeta) (-1)^n [L_n (\xi) - L_{n-1} (\xi)], \]
\[ \mathcal{A}_+^\mu (p) = \sum_{n=0}^{\infty} \frac{4a}{(2\pi)^3} \theta(\varepsilon_p) \delta \left( \varepsilon_p^2 - E_{pz}^{(n)} \xi \right) e^{-\xi/2} \]
\[ \times n_F (\beta \cdot p - \zeta) (-1)^n [L_n (\xi) + L_{n-1} (\xi)], \]

where \( n \) denotes the Landau level, \( L_n (z) \) is a Laguerre polynomial,

\[ \varepsilon_p = p \cdot u, \quad \xi = \frac{2p_T^2}{|qB|}, \quad a^\mu = \varepsilon_p qB^\mu - \frac{qB \cdot p}{|qB|} u^\mu, \]

with \( p_z \) and \( p_T \) denoting respectively the parallel and transverse (respect to the magnetic field) components of momentum, and the energy spectrum is

\[ E_{pz}^{(n)} = \sqrt{2n|qB| + p_z^2 + m^2}. \]

Notice that since the Wigner function (25) is gauge invariant, the solutions written above are gauge independent. With the Wigner function in Eqs. (48) and (49) one can compute the spin polarization at every order of magnetic field. However, it is more convenient to obtain the leading orders as a power series in magnetic field. I will consider two limits. The first, that is more relevant for application, is the weak field approximation when the magnetic field is weaker than the momentum of the particle and than the temperature \( |qB| \ll p_T^2, T^2 \).

The other one is the opposite case where the magnetic field is very strong. Using the methods developed in [34], one can obtain the Wigner function as a series in magnetic field in the weak field limit. After expanding the Wigner function in powers of the magnetic field and summing over the Landau levels (see Appendix A), its scalar and axial part is approximated.
by
\[ F_+(p) = \frac{4m}{(2\pi)^3} n_F (\beta \cdot p - \zeta) \theta (\varepsilon_p) \delta (p^2 - m^2) \]
\[ + \mathcal{O} (|qB|^2), \]
\[ A^\mu_+ (p) = \frac{4|qB| a^\mu}{(2\pi)^3} n_F (\beta \cdot p - \zeta) \theta (\varepsilon_p) \delta'(p^2 - m^2) \]
\[ + \mathcal{O} (|qB|^2). \]

To the best of my knowledge, this is the first time this result has been derived from quantum field theory without relying on a kinetic theory. It is in agreement with what was previously found using quantum kinetic theory in the Wigner formalism for massless \[16, 62, 63, 64, 18\] and massive \[65, 66, 67, 68, 69, 70, 71, 72\] fermions. Indeed it was found \[66, 67\]

\[ A^\mu_{+KT} = \frac{q}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} p_\nu n_F (\beta \cdot p - \zeta) \theta (\varepsilon_p) \delta'(p^2 - m^2), \]
which reduces to (54) using \( F_{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma} B^\rho u^\sigma \). It is straightforward to check that the integral over momentum of Eq. (54) results in the chiral separation effect:

\[ j^z_{A+} = \int d^4 p A^z_+ (p) = \frac{qB}{2\pi^2} \int_0^\infty dp F_F (\beta E_p - \zeta), \]

where \( E_p = \sqrt{m^2 + p^2} \). This is the same result one obtains using the full solution \[49\], see for instance \[33, 34, 32\].

Instead, in the strong field limit \( |qB| \gg p^2, T^2 \), only the firsts Landau levels are populated. Then, the Wigner function can be approximated by considering only the first terms in the series (48) and (49). For instance, considering only the Lowest Landau Level (LLL) \( n = 0 \), it reads

\[ F_+(p)_{LLL} = \frac{4m}{(2\pi)^3} n_F (\beta E_p^{(0)} - \zeta) \theta (\varepsilon_p - \frac{E_p^{(0)}}{2}) \]
\[ A^\mu_+ (p)_{LLL} = \frac{4a^\mu}{(2\pi)^3} n_F (\beta E_p^{(0)} - \zeta) \theta (\varepsilon_p - \frac{E_p^{(0)}}{2}) \]

5. Spin polarization

The full spin polarization induced by magnetic field is obtained by plugging the Eqs. (48) and (49) into (46) and it is studied below at global equilibrium.
For practical applications and to understand what are the effects of the magnetic field, it is more convenient to use the weak field approximation. Using the Eqs. (53) and (54), the spin polarization (46) becomes

\[
S_B^\mu(p) = \int_\Sigma d\Sigma \cdot p \left( \varepsilon_p qB\cdot u^\mu(qB \cdot p) \right) n_F(\beta \cdot p - \zeta)\theta(\varepsilon_p)\delta(p^2 - m^2)
\]

\[
= \frac{m}{2} \int_\Sigma d\Sigma \cdot p n_F(\beta \cdot p - \zeta)\theta(\varepsilon_p)\delta(p^2 - m^2) + O(|qB|^2).
\]

Notice that the spin vector is orthogonal to the momentum \(p\); this is a property inherited from the Pauli-Lubanski vector [15].

To remove the Dirac deltas and obtain on-shell momenta, I integrate the integrand of the numerator and of the numerator of Eq. (59) in \(\varepsilon_p\). The denominator gives

\[
\int d\varepsilon_p \delta(p^2 - m^2)\theta(\varepsilon_p)n_F(\beta \cdot p - \zeta) = \frac{n_F(\beta E_p - \zeta)}{2E_p}.
\]

In shorthand notation, the numerator becomes

\[
N^\mu = \int d\varepsilon_p |qB| a^\mu n_F\theta\delta(p^2 - m^2) = \int d\varepsilon_p \frac{\theta(\varepsilon_p - E_p)}{2\varepsilon_p} \frac{d}{d\varepsilon_p} \left[ \frac{|qB| a^\mu n_F\theta}{2\varepsilon_p} \right].
\]

Using

\[
\frac{d}{d\varepsilon_p} \left( \frac{|qB| a^\mu}{\varepsilon_p} \right) = \frac{qB \cdot p}{\varepsilon_p^2} u^\mu,
\]

\[
\frac{d}{d\varepsilon_p} n_F = -\beta n_F(1 - n_F),
\]

one obtains

\[
N^\mu = \beta n_F(1 - n_F) |qB| a^\mu \frac{qB \cdot p}{4E_p^2} - n_F \frac{qB \cdot p}{4\varepsilon_p^3} u^\mu,
\]

where the momentum is on-shell: \(\varepsilon_p = E_p\). Notice that the first term is orthogonal to the momentum \(p\):

\[
p_\mu a^\mu = \varepsilon_p \frac{qB \cdot p}{|qB|} - \frac{qB \cdot p}{|qB|} p \cdot u = 0.
\]
whereas the second is not. But, as mentioned above, the spin vector must satisfy \( p_\mu S^\mu(p) = 0 \). Therefore, the second term must be vanishing when integrated over the hypersurface. The final result is

\[
S_B^\mu(p) = \frac{\int_{\Sigma} d\Sigma \cdot p \beta(x) n_F(1 - n_F)(qB^\mu(x) - u^\mu(qB(x)p))}{4m \int_{\Sigma} d\Sigma \cdot p n_F}
\] (60)

where the momentum is on-shell and I denoted

\[
n_F = n_F (\beta(x) \cdot p - \zeta(x))
\]

(61)

with \( n_F(x) = (e^x + 1)^{-1} \). The same procedure described above can be applied for the anti-particle branch of the Wigner function, leading to the result

\[
\bar{S}_B^\mu(p) = -\frac{\int_{\bar{\Sigma}} d\Sigma \cdot p \bar{n}_F(1 - \bar{n}_F)(qB^\mu - u^\mu(qBp))}{4m \int_{\bar{\Sigma}} d\Sigma \cdot p \bar{n}_F},
\]

(62)

where \( \bar{n}_F = n_F (\beta(x) \cdot p + \zeta(x)) \) and \( \bar{\Sigma} \) in heavy-ion collisions denotes the freeze-out hypersurface of the anti-particle. The main difference with the particle spin polarization is of course the overall change in sign. This is in agreement with the chiral kinetic theory results of [16] (in the massless limit) and with [18] for a massive fermion (although the term proportional to \( u^\mu \) was omitted).

In experiments the spin polarization is measured in the rest frame of the particle \( S^* \). This quantity is obtained from Eq. (60) with the back boost:

\[
S^* = S - \frac{p \cdot S}{p_0(p_0 + m)} p.
\]

(63)

Performing this operation, one can see that the part of Eq. (60) proportional to \( u \) does not contribute to \( S^* \).

It is now interesting to compare the result (60) with the spin polarization induced by rotation. From Eq. (44) keeping only the rotational part of the thermal vorticity \( \varpi_{\rho\sigma} = \epsilon_{\rho\sigma\kappa\lambda} \beta \omega^\kappa u^\lambda \), the spin polarization induced by rotation is

\[
S_\omega^\mu(p) = \frac{\int_{\Sigma} d\Sigma \cdot p \beta n_F(1 - n_F)(\varepsilon_\rho \omega^\mu - w^\mu(\omega \cdot p))}{4m \int_{\Sigma} d\Sigma \cdot p n_F}.
\]

(64)

Comparing the Eq. (60) with the Eq. (64), one realizes that the spin polarization induced by rotation can be explained using the relativistic Barnett effect with the effective magnetic field (22) as discussed in Sec. 2.1.
5.1. The non-relativistic case

In the non-relativistic limit, using $\varepsilon_p \approx \mu_0$, the expression (60) reduces to the known result [17]

$$S^\mu_{\text{B,NR}} = \int_\Sigma \frac{d\Sigma \cdot p}{2m} \beta n_F (1 - n_F) \mu_0 (B^\mu \varepsilon_p - u^\mu (B \cdot p)),$$

where the spin polarization is given in terms of the magnetic moment of the particle. The non-relativistic limit can be computed using linear response theory starting from the statistical operator:

$$\hat{\rho}_{\text{B,NR}} = \frac{1}{Z} \exp \left\{ -\frac{\hat{H}}{T} + \hat{\mu} \cdot \mathbf{B} \right\},$$

where $\mu$ is the magnetic moment $\hat{\mu} = 2\mu_0 \hat{S}$. This is essentially the same calculation of spin polarization induced by thermal vorticity leading to Eq. (44). Then, the result (65) is simply obtained replacing $\varpi_{\mu\nu} \rightarrow -2\beta \mu_0 F_{\mu\nu}$. This spin polarization is the non-relativistic limit $\frac{q}{2\varepsilon_p} \approx \mu_0$ of the polarization (60).

Here I report the explicit calculation to check if the replacement actually gives the correct result. The effect of the magnetic field on the Wigner function is obtained by applying the linear response theory to the second term in the exponent of (66):

$$\Delta W_+(x, p) = 2\mu_0 B(x) \int d^3 y \int_0^1 dz \langle \hat{W}_+(x, p) \hat{S}^z (y + iz\beta) \rangle_{c,\beta},$$

where the bracket $\langle \cdots \rangle_{c,\beta}$ denotes a connected thermal average with the statistical operator (66) with $\mathbf{B} = 0$, and the spin operator is obtained from the canonical spin tensor:

$$\hat{S}^z (y) = \frac{i}{2} \bar{\Psi}(y) \gamma^1 \gamma^2 \Psi(y).$$

Expanding the Dirac field in terms of the free creation-annihilation operators and using

$$\langle \hat{a}_{\tau'}^\dagger (k) \hat{a}_\sigma (q) \rangle_\beta = \delta_{\tau\sigma} 2\varepsilon_q \delta^3 (\mathbf{k} - \mathbf{q}) n_F (\beta \cdot k),$$

$$\langle \hat{a}_{\tau'}^\dagger (k') \hat{a}_\sigma^\dagger (q') \rangle_\beta = \delta_{\tau'\sigma'} 2\varepsilon_{q'} \delta^3 (\mathbf{k}' - \mathbf{q'}) (1 - n_F (\beta \cdot k')), $$

\footnote{There is a minus sign between $\varpi$ and $F$, because rotation field and magnetic field are defined with a difference in sign.}
one finds
\[
\Delta W_{ab}(x, p) = 2\mu_0 B(x) \int d^3y \int_0^1 \frac{dz}{(2\pi)^6} \int \frac{d^3k d^3k'}{4\varepsilon_k \varepsilon_{k'}} \delta^4 \left( p - \frac{k + k'}{2} \right) \]
\[
\times \beta n_F(k) (1 - n_F(k')) S^z(k, k')_{ab} e^{i(k-k')(x-y)} e^{z(k-k')\beta}
\]
with
\[
S^z(k, k')_{ab} \equiv \frac{i}{2} (k' + m) \gamma^1 \gamma^2 (k + m).
\]
The traces involved in spin polarization read:
\[
\text{tr} \left[ (k' + m) \gamma^1 \gamma^2 (k + m) \right] = 4(k^2 k'^1 - k^1 k'^2),
\]
\[
\text{tr} \left[ \gamma^\mu \gamma^5 (k' + m) \gamma^1 \gamma^2 (k + m) \right] = -4im(k_0 + k'_0).
\]
Noticing that
\[
\int d^3y e^{i(k-k')(x-y)} = (2\pi)^3 \delta^3(k - k')
\]
one see that the first trace is vanishing and hence there is no contribution from magnetic field to the denominator of the spin polarization formula (24). The numerator instead is
\[
\mathcal{N}^z = \int \Sigma \frac{d^3k}{(2\pi)^3} 2\mu_0 B(x) \frac{1}{(2\pi)^3} \int \frac{d^3k}{4\varepsilon_k} 4m \varepsilon_k \delta^4 (p - k) \beta n_F(k) (1 - n_F(k))
\]
\[
= \frac{4m}{(2\pi)^3} \int \Sigma \frac{d^3k}{(2\pi)^3} \frac{\delta(p_0 - \varepsilon_p)}{2\varepsilon_p} \beta n_F(p) (1 - n_F(p))
\]
\[
= \frac{4m}{(2\pi)^3} \int \Sigma \frac{d^3k}{(2\pi)^3} \frac{\delta(p^2 - m^2)}{\theta(p_0)} \beta n_F(p) (1 - n_F(p)).
\]
The leading order spin polarization resulting from (66) is then
\[
S^z(p) = \frac{1}{2m} \frac{\int \Sigma \frac{d^3k}{(2\pi)^3} \delta(p^2 - m^2) \theta(p_0) \beta n_F(p) (1 - n_F(p))}{\int \Sigma \frac{d^3k}{(2\pi)^3} \beta n_F(p) (1 - n_F(p))}.
\]
As expected this is equal to Eq. (65) in the non-relativistic limit \( \epsilon_p \rightarrow m \).
Figure 1: The weak field approximation of the spin polarization along the magnetic field at global equilibrium \( T \ll m \) as a function of transverse momentum compared with the exact result \( \text{eq. (72)} \) summed up to 100, 500 and 2000 Landau levels (LLs).

Figure 2: The spin polarization along the magnetic field at global equilibrium as a function of magnetic field intensity at low and high temperatures. The spin polarization is evaluated using \( \text{eq. (72)} \) summing up to \( n = 0 \) (LLL), \( n = 1 \) (FLL) and \( n = 200 \) (200 LLs) and using \( \text{eq. (73)} \) for the weak field approximation.
5.2. Study of global equilibrium spin polarization

Consider a global thermal equilibrium with a constant magnetic field. Since there is no coordinate dependence, the integral over the hypersurface can be done in a hypersurface such that its normal vector is $u$:

$$\int_{\Sigma} d\Sigma \cdot p = \varepsilon_p \int d^3x = \varepsilon_p V,$$

and the spin polarization is obtained from

$$S^\mu(p) = \frac{1}{2} \int_{\Sigma} d\Sigma \cdot p A^\mu_+(p) = \frac{1}{2} \frac{A^\mu_+(p)}{\mathcal{F}_+(p)} = \frac{1}{2} \int d\varepsilon_p A^\mu_+(p),$$

(68)

It is convenient to write the exact solution of Wigner function (48) and (49) as

$$F_+(p) = \sum_{n=0}^{\infty} \frac{4m}{(2\pi)^3} \frac{\delta \left( \varepsilon_p - E_{p_z}^{(n)} \right)}{2\varepsilon_p} e^{-\xi/2}$$

$$\times n_F(\beta \cdot p - \zeta)(-1)^n \left[ L_n(\xi) - L_{n-1}(\xi) \right],$$

(69)

$$A^\mu_+(p) = \sum_{n=0}^{\infty} \frac{4a^\mu}{(2\pi)^3} \frac{\delta \left( \varepsilon_p - E_{p_z}^{(n)} \right)}{2\varepsilon_p} e^{-\xi/2}$$

$$\times n_F(\beta \cdot p - \zeta)(-1)^n \left[ L_n(\xi) + L_{n-1}(\xi) \right],$$

(70)

from which it follows that spin polarization at global equilibrium is

$$S^\mu(p) = \frac{1}{2m} \sum_n a^\mu n_F(\beta E_{p_z}^{(n)} - \zeta)(-1)^n \left[ L_n(\xi) + L_{n-1}(\xi) \right],$$

(71)

In the frame where the fluid is at rest, one has $u^\mu = \delta^\mu_0$, $B = B \hat{z}$ and $a^\mu = \left( p^z, 0, 0, E_{p_z}^{(n)} \right)$. The spin polarization along the direction of magnetic field is

$$S^z(p) = \frac{1}{2m} \sum_n E_{p_z}^{(n)} n_F(-1)^n \left[ L_n(\xi) + L_{n-1}(\xi) \right],$$

(72)

In the weak field limit, using (53) and (54),

$$S^z_{\text{Weak field}}(p) = \frac{1}{4} \beta m \frac{qB}{m^2} \left( 1 - n_F(\beta \varepsilon_p - \zeta) \right).$$

(73)
In the very strong field limit the spin polarization reaches the asymptotic value

\[ \lim_{B \to \infty} \sqrt{\frac{qB}{mT}} S^z = S^z_{\text{LLL}} = \frac{1}{2} \sqrt{1 + \left( \frac{p_z}{m} \right)^2}, \]  

which corresponds to a situation where only the Lowest Landau Level (LLL) is populated. More accurately, for strong magnetic field the spin polarization can be approximated by stopping the sums in (72) at the first Landau level (FLL):

\[ S^z_{\text{FLL}} = \frac{1}{2} \sqrt{1 + \left( \frac{p_z}{m} \right)^2} \left( 1 + 2 \frac{p_z^2 - |qB|}{|qB|} \frac{n_F(\beta \sqrt{1+p_z^2} + 2|qB| - \zeta)}{n_F(\beta \sqrt{1+p_z^2} - \zeta)} \right). \]  

In Fig. 1 the weak limit approximation (73) is evaluated numerically for small values of the magnetic field as a function of transverse momentum and it is compared with the exact result (72) approximated by truncating the Landau levels sums at a certain Landau level \( N \). The Fig. 1 shows that the weak field limit (73) provides a good approximation of the exact formula (72). The dependence of the spin polarization on the magnetic field intensity is studied in Fig. 2. For very large magnetic field, the spin polarization first approaches the values obtained with the FLL approximation (75) and then the LLL approximation (74). Just like the magnetic susceptibility of a highly degenerate system oscillates as the magnetic field increases according to the de Haas - van Alphen effect [73], the spin polarization is expected to have de Haas - van Alphen oscillations as the magnetic field intensity or the transverse momentum is increased. The oscillations visible in Fig. 1 for the lines obtained including few Landau levels are originated from the Laguerre polynomials in Eq. (72) through the argument \( \xi = 2p_z^2 / |qB| \). Those oscillations disappear for a non-degenerate system such as the one considered in Fig. 1 and 2 but persist for a degenerate gas in strong magnetic field as Fig. 3 shows.

5.3. Heavy-ion collisions

In heavy-ion collisions the magnetic field and the angular momentum of the plasma are aligned. The effect of a magnetic field can be revealed in the difference between the global polarization of \( \Lambda \) and \( \bar{\Lambda} \) particles. The data
Low Temperature $(T \ll m)$, Strong field $(qB >> T^2)$, Degenerate

\[ \beta_m = 3, \mu_m = 30, \rho_T = 0.2, \rho_z = 0.2 \]

Figure 3: The spin polarization along the magnetic field at global equilibrium at low temperature, strong field and high degeneracy has de Haas - van Alphen oscillations. The line is evaluated using (72) summing up to 500 LLs.

\[ \sqrt{S_{NN}} \ [\text{GeV}] \]

Figure 4: The estimate of magnetic field as a function of energy resulting from the difference of global spin polarization [3, 4], only the statistical error is shown.
reported in [3, 4] reveals that this difference is not consistent with zero only at \( \sqrt{S_{NN}} = 7.7 \) GeV. Reminding that for spin 1/2 the polarization vector is related to the spin vector as

\[
P^\mu = 2S^\mu,
\]

adopting the non-relativistic limit of Eq. (44) and Eq. (60), the difference in global polarization is estimated as

\[
P_{\Lambda} - P_{\bar{\Lambda}} \simeq -2\frac{\mu_{\Lambda} B}{T} - \frac{(\nabla \zeta_B \times \mathbf{v}) \cdot \hat{J}}{4m_{\Lambda}},
\]

where \( \zeta_B \) is the baryonic chemical potential divided by temperature, and \( \mu_{\Lambda} \) and \( m_{\Lambda} \) are respectively the magnetic moment and the mass of the \( \Lambda \), while \( \hat{J} \) is the direction of the total angular momentum. The Fig. 4 shows the ratio between the magnetic field and temperature extracted from the experimental data [3, 4] using the formula above neglecting the contribution from the baryonic chemical potential and using \( \mu_{\Lambda} = -0.613\mu_N \) [74] with \( \mu_N \) the nuclear magnetic moment.

The spin polarization is only sensitive to fields at the hadronization hypersurface. It has been showed [75, 76, 77, 78, 19] that the magnetic field decays very rapidly and that the decay is faster at higher energies. This estimate shows that the magnetic field is very small at late stage of the plasma. The values reported in Fig. 4 are in agreement with the calculations of the magnetic field time evolution, possibly overestimating its value at 7.7 GeV [20]. An upper bound of the magnetic field can be obtained from this data [23]. For instance, from the statistical error at 200 GeV, that is \( \Delta(P_{\Lambda} - P_{\bar{\Lambda}}) = 0.052 \times 10^{-2} \), one obtains

\[
\frac{ehB \, 100 \, \text{MeV}}{m_{\pi}^2 \, k_B T} \leq \frac{\Delta(P_{\Lambda} - P_{\bar{\Lambda}})}{0.613} \frac{m_p \, 100 \, \text{MeV}}{m_{\pi}^2} = 4 \times 10^{-3}.
\]

However, the magnetic field is not the only way the spin polarization can be different for baryons and anti-baryons. An other effect is showed in Eq. (77) itself: the contribution of the gradient of the baryonic chemical potential. The effect of an electric field has the same form as the contribution of the gradient of chemical potential, but the electric field has been estimated to

\footnotetext{2}{The freeze-out hypersurface is approximately an hypersurface of constant temperature \( T \approx 155 \) MeV.}
be too small. The presence of a chemical potential also changes the thermal distribution function for $\Lambda$ and $\bar{\Lambda}$, contributing to this difference. It was also found that different freeze-out conditions also contribute [79]. Recently all these effects were studied in [22, 21]. Those simulations showed that all these effects compete in different directions often canceling out each other. After all, the estimate of magnetic field presented in Fig. 4 is a reliable rough estimate. In [80] it was also proposed that the this difference might be caused by the interplay between the axial and helical [81] currents.

6. Discussion and conclusion

I derived the formula for the spin polarization induced by an external magnetic field in a quantum relativistic framework. The leading order correction for a weak magnetic field is given by Eq. (60) and coincides with previous results [16, 17, 18] obtained with different methods. I also derived the full expression given by Eqs. (46), (48) and (49) valid at global equilibrium and at local thermal equilibrium neglecting the gradients of magnetic field. I studied this expression at global equilibrium for different magnetic field intensities and I showed that, just like the magnetic susceptibility, spin polarization in a degenerate gas with strong magnetic field exhibits de Haas - van Alphen oscillations. These effects are derived assuming that the system is at local thermal equilibrium, from which it follows that this is a non-dissipative phenomenon.

While spin polarization was computed for free fields, the framework used to describe the system is a quantum field theory and interactions can be included without breaking the quantum properties of the system. The method used to obtain these results can be extended to higher order in gradients of the magnetic field provided that the magnetic field changes over macroscopic distances. As the full effects of thermal vorticity to the distribution function are difficult to be included without the statistical operator approach [59], the use of this method could be crucial to study the interplay between electromagnetic field and thermal vorticity.

Based on these results I reviewed the relativistic Barnett effect and I showed that the spin polarization induced by vorticity can be obtained with a classical model. While the model is less accurate (the classical theory fails to reproduce the correct gyromagnetic moment), it highlights that the effect of vorticity, that is a rotation, is mainly a classical non-inertial effect. As the classical theory is free of quantum anomalies, this model also suggests that
there is not a tight connection between the spin polarization of a massive particle and the gravitational anomaly in the axial current or any other quantum anomaly.

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Appendix A. Weak field limit

In this appendix I derive the results for the weak field approximation of the Wigner function following [34]. To obtain this limit, one takes advantage of the following identities

\[ L_1 = \sum_{n=0}^{\infty} (-1)^n L_n \left( \frac{2p_T^2}{|qB|} \right) e^{2\sigma n |qB|} = \frac{2p_T^2}{1 + e^{2\sigma |qB|}}, \quad (A.1) \]

and

\[ L_2 = \sum_{n=0}^{\infty} (-1)^n L_{n-1} \left( \frac{2p_T^2}{|qB|} \right) e^{2\sigma n |qB|} = -e^{2\sigma |qB|} \frac{2p_T^2}{1 + e^{2\sigma |qB|}}, \quad (A.2) \]

where \( \sigma = \pm \). Combined together the two identities give

\[ \sum_{n=0}^{\infty} (-1)^n \left[ L_n \left( \frac{2p_T^2}{|qB|} \right) - L_{n-1} \left( \frac{2p_T^2}{|qB|} \right) \right] e^{2\sigma n |qB|} = L_1 - L_2 = \exp \left[ \frac{2p_T^2}{|qB|(1 + e^{-2\sigma |qB|})} \right], \quad (A.3) \]

and

\[ \sum_{n=0}^{\infty} (-1)^n \left[ L_n \left( \frac{2p_T^2}{|qB|} \right) + L_{n-1} \left( \frac{2p_T^2}{|qB|} \right) \right] e^{2\sigma n |qB|} = L_1 + L_2 = \frac{1 - e^{2\sigma |qB|}}{1 + e^{2\sigma |qB|}} \exp \left[ \frac{2p_T^2}{|qB|(1 + e^{-2\sigma |qB|})} \right], \quad (A.4) \]
Reminding that \( \xi = 2p_T^2/|qB| \), consider then the scalar part of the Wigner function

\[
\mathcal{F}_+(p) = \sum_{n=0}^{\infty} \frac{4m}{(2\pi)^3} \frac{\delta (\varepsilon_p - E_{p_z}^{(n)})}{2\varepsilon_p} n_F (\beta \cdot p - \zeta) \times (-1)^n \left[ L_n(\xi) - L_{n-1}(\xi) \right] e^{-\xi/2}.
\] (A.5)

The first part of each term in this series can be expanded as a series in magnetic field as follow

\[
F \left( p_z^2 + 2n|qB| \right) = \sum_{l=0}^{\infty} \left( 2n|qB| \right)^l \frac{F^{(l)}(p_z^2)}{l!}.
\] (A.6)

where I denoted

\[
F^{(l)}(z) = \frac{d^l}{dz^l} F(z).
\] (A.7)

Since each coefficient of this expansion can be obtained as the limit

\[
F \left( p_z^2 + 2n|qB| \right) = \lim_{s \to 0} \sum_{l=0}^{\infty} \left( \frac{d^l}{ds^l} e^{2s|qB|n} \right) \frac{F^{(l)}(p_z^2)}{l!},
\] (A.8)

one obtains

\[
\mathcal{F}_+(p) = \lim_{s \to 0} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{d^l}{ds^l} e^{2s|qB|n} \right) \frac{F^{(l)}(p_z^2)}{l!} \times (-1)^n \left[ L_n(\xi) - L_{n-1}(\xi) \right] e^{-\xi/2}.
\] (A.9)

Exchanging the order of the summations, one finds the identity \( A.3 \) and

\[
\mathcal{F}_+(p) = \lim_{s \to 0} \sum_{l=0}^{\infty} \frac{F^{(l)}(p_z^2)}{l!} e^{-\xi/2} \times \frac{d^l}{ds^l} \sum_{n=0}^{\infty} (-1)^n \left[ L_n(\xi) - L_{n-1}(\xi) \right] e^{2s|qB|n}
\] (A.10)

\[
= \lim_{s \to 0} \sum_{l=0}^{\infty} \frac{F^{(l)}(p_z^2)}{l!} e^{-\xi/2} \frac{d^l}{ds^l} e^{2s|qB|n}.
\]
Keeping terms up to $|qB|^3$, the scalar part of Wigner function is approximated by

$$
\mathcal{F}_\pm(p) \simeq \lim_{s \to 0} \sum_{l=0}^{\infty} \frac{F^{(l)}(p_z^2)}{l!} e^{-\xi/2} \frac{dl}{ds} \left[ e^{s p_z^2 - \frac{1}{2} p_T^2 |qB|^2 s^3} + \mathcal{O}(|qB|^3) \right]
$$

$$
\simeq \sum_{l=0}^{\infty} \frac{F^{(l)}(p_z^2)}{l!} \frac{p_T^2 + \mathcal{O}(|qB|^2)}{l!} = F(p_z^2 + p_T^2) + \mathcal{O}(|qB|^2)
$$

$$
= \frac{4m}{(2\pi)^3} \delta (\varepsilon_p - E_p) \frac{n_F(\beta \cdot p - \zeta)}{2\varepsilon_p} + \mathcal{O}(|qB|^2)
$$

$$
= \frac{4m}{(2\pi)^3} \delta (\varepsilon_p - E_p^2) \theta(\varepsilon_p)n_F(\beta \cdot p - \mu)
$$

$$
= \frac{4m}{(2\pi)^3} \delta (p^2 - m^2) \theta(\varepsilon_p)n_F(\beta \cdot p - \mu), \quad (A.11)
$$

where I used the properties of the Dirac delta and I denoted $E_p = \sqrt{m^2 + p^2}$.

The weak field limit of the axial part of the Wigner function is obtained in a similar fashion. Define the function

$$
A^\mu(p_z^2 + n|qB|) = \frac{4a^\mu \theta(\varepsilon_p)}{(2\pi)^3} \delta \left( \varepsilon_p^2 - E_{p_z}^{(n)} \right) n_F(\beta \cdot p - \zeta), \quad (A.12)
$$

such that as shown for the scalar part

$$
A^\mu_+(p_z^2) = \sum_{n=0}^{\infty} \frac{4a^\mu \theta(\varepsilon_p)}{(2\pi)^3} \delta \left( \varepsilon_p^2 - E_{p_z}^{(n)} \right) n_F(\beta \cdot p - \zeta)
$$

$$
\times (-1)^n \left[ L_n(\xi) + L_{n-1}(\xi) \right] e^{-\xi/2}
$$

$$
= \lim_{s \to 0} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{dl}{ds} e^{2s|qB|n} \right) \frac{A^{\mu(l)}(p_z^2)}{l!}
$$

$$
\times (-1)^n \left[ L_n(\xi) + L_{n-1}(\xi) \right] e^{-\xi/2}. \quad (A.13)
$$

Exchanging the order of the summations and using (A.4), the axial part becomes

$$
A^\mu_+(p_z^2) = \lim_{s \to 0} \sum_{n=0}^{\infty} \frac{A^{\mu(l)}(p_z^2)}{l!} e^{-\xi/2}
$$

$$
\times \frac{dl}{ds} \left. \frac{1 - e^{2s|qB|n}}{1 + e^{2s|qB|n}} \right|_{|qB| = |qB|} \frac{2p_T^2}{e^{2s|qB|n} + e^{2s|qB|n}}. \quad (A.14)
$$
Up to the first order in $|qB|$, the expansion of the axial part is

$$\mathcal{A}_\mu^+(p) \simeq \lim_{s \to 0^-} -\sum_{l=0}^{\infty} \frac{A^{\mu(l)}(p^2)}{l!} \frac{d^l}{ds^l} \left[ e^{p^2 s} |qB| + \cdots \right]$$

$$= -\sum_{l=0}^{\infty} \frac{A^{\mu(l)}(p^2)}{(l-1)!} p_T^{2(l-1)} |qB| + \mathcal{O}(|qB|^2).$$  \hspace{1cm} (A.15)

Noticing that the series is the derivative of the function $A^\mu$:

$$\mathcal{A}_\mu^+(p) = -|qB| \frac{d}{dp_T^2} A^\mu(p^2 + p_T^2) + \mathcal{O}(|qB|^2),$$  \hspace{1cm} (A.16)

that the derivative of

$$A^\mu(p^2 + p_T^2) = 4a^\mu \theta(\varepsilon_p) \frac{(2\pi)^3}{n_F} \delta (\varepsilon_p^2 - E_p^2) n_F (\beta \cdot p - \zeta)$$  \hspace{1cm} (A.17)

acts only on the delta function

$$\frac{d}{dp_T^2} \delta (\varepsilon_p^2 - E_p^2) = \frac{dE_p}{dp_T^2} \frac{d}{dE_p} \delta (\varepsilon_p^2 - E_p^2)$$

$$= \frac{1}{2E_p} \frac{d}{dE_p} \delta (\varepsilon_p^2 - E_p^2) = -\delta' (p^2 - m^2),$$  \hspace{1cm} (A.18)

one obtains

$$\mathcal{A}_\mu^+(p) = \frac{4|qB| a^\mu}{(2\pi)^3} n_F (\beta \cdot p - \zeta) \theta(\varepsilon_p) \delta'(p^2 - m^2).$$  \hspace{1cm} (A.19)

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