Motivic Decomposition of Projective Pseudo-Homogeneous Varieties

Srimathy Srinivasan

IAS
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The category of Chow Motives

- The category \( \text{Chow}(k, \Lambda) \) where \( k \) - field, \( \Lambda \) - coefficient ring
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- What are Hom sets?
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  - $X$ - variety over $k$, $n \in \mathbb{Z}$ and $p \in \text{End}(X)$ a projector or idempotent, i.e., $p^2 = p$
- What are Hom sets? If $X$ is irreducible,
  \[ \text{Hom}_{\text{Chow}(k, \Lambda)}((X, n, p), (Y, m, q)) = q \circ [CH_{\text{dim} X+n-m}X \times Y \otimes_{\mathbb{Z}} \Lambda] \circ p \]
Composition of Morphisms

- How to compose morphisms
  \[ \alpha \in \text{Hom}((X, n, p), (Y, m, q)) \text{ and } \beta \in \text{Hom}((Y, m, q), (Z, r, s)). \]
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\[
\begin{array}{c}
\alpha \in X \times Y \\
p_{12} \downarrow \quad \downarrow \quad p_{23} \\
X \times Y \times Z \\
p_{13}^{-1} \downarrow \\
\alpha \in X \times Y \\
\beta \in Y \times Z \\
\beta \circ \alpha \in X \times Z
\end{array}
\]
Properties of $\text{Chow}(k, \Lambda)$

- $\text{Chow}(k, \Lambda)$ admits tensor product:
  $$(X, n, p) \otimes (Y, m, q) = (X \times Y, n + m, p \times q)$$
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- \( \text{Chow}(k, \Lambda) \) admits direct sum:
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  (X, n, p) \oplus (Y, m, q) = (X \bigsqcup (Y \times \mathbb{P}^{m-n}), n, p + (q \times \alpha_{m-n}))
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  where $\alpha_{m-n} = [pt \times \mathbb{P}^{m-n}] \in \text{End } \mathcal{M}(\mathbb{P}^{m-n})$
How to decompose a motive?

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- Example: $p = [pt \times \mathbb{P}^1] \in \text{End } \mathcal{M}(\mathbb{P}^1)$ is a projector. So get

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- In general,
  \[
  \mathcal{M}(\mathbb{P}^n) \simeq \Lambda \oplus \Lambda(1) \oplus \cdots \oplus \Lambda(n)
  \]
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- We say that **Rost Nilpotence** holds for a variety $X$ over $F$ if for every field extension $E/F$ the kernel of the base change map

$$\text{End}_F(\mathcal{M}(X)) \to \text{End}_E(\mathcal{M}(X_E))$$

$$\alpha \to \alpha_E$$

consists of nilpotents. That is, if $\alpha \in \text{End}_F(\mathcal{M}(X))$ is such that $\alpha_E = 0$, then $\alpha^N = 0$ for some $N > 0$. 
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- Many interesting consequences. One of them - finding projectors
What is known?

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- Not known if RN holds in general
Motivic Decomposition of PHVs

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Is the complete decomposition unique?
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Yes - Krull-Schmidt
Upper Indecomposable summand

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One special summand in complete decomposition of $\mathcal{M}(X)$ - **Upper Indecomposable Summand**

- $M \hookrightarrow \mathcal{M}(X)$ is **upper** if $CH^0(M) := Hom(M, \Lambda) \neq 0$
- Unique as a consequence of KS. Denoted by $U_X$
- Contains lot of information
Suppose $G = SL_3$. Consider

$$\tilde{P} = \left\{ \begin{pmatrix} * & * & * \\ x & y & z \\ * & * & * \end{pmatrix} \mid x^{p^3} = 0, y^{p^3} = 0, z^{p^4} = 0 \right\}$$

Then $\tilde{P}$ is not reduced.
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Underlying reduced scheme is the standard Borel.
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Parabolic Subgroup Schemes

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- A **parabolic subgroup scheme** is a subgroup containing Borel that is not necessarily reduced.
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Notation: $\tilde{P}$ - parabolic subgroup scheme, $P$ - underlying reduced subscheme of $\tilde{P}$
Variety of Unseparated Flags- VUFs

- VUFs are quotients $G/\tilde{P}$ where $\tilde{P}$ is a parabolic subgroup scheme (not necessarily reduced)

Example: $G = \text{SL}_3$. Consider the variety $\tilde{\mathcal{X}}$ in $\mathbb{P}^2 \times \mathbb{P}^2$ given by the equation $\sum_{i=2}^{2} x_i p_i y_i = 0$ where $g \cdot \rightarrow x = g \cdot p_3 \rightarrow x$ and $g \cdot \rightarrow y = (g - t) \cdot p_4 \rightarrow y$. Then $\tilde{\mathcal{P}} = \text{Stab} ([1:0:0] \times [0:0:1]) = \{x^* y^* z^*/x p_3 = 0, y p_3 = 0, z p_4 = 0\}$

$\tilde{\mathcal{X}} = G/\tilde{\mathcal{P}}$ is a VUF
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$\tilde{X} = G/\tilde{P}$ is a VUF
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- VUFs behave very differently from flag varieties
- Nothing much known for their twisted forms over non-algebraically closed fields
Are they related?

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**Answer:** Yes & Yes

I show that their motives are isomorphic in $\text{Chow}(k, \Lambda)$.
A variety $\tilde{X}$ over $k$ is a **projective pseudo-homogeneous variety** for $G$, if $\tilde{X}_k \simeq G/\tilde{P}$, $\tilde{P}$ not necessarily reduced.
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Twisted forms of VUFs
A variety \( \tilde{X} \) over \( k \) is a **projective pseudo-homogeneous variety** for \( G \), if \( \tilde{X}_k \cong G/\tilde{P} \), \( \tilde{P} \) not necessarily reduced.

**Twisted forms of VUFs**

Denote by \( X \) the \( G \)-variety such that \( X_k \cong G/P \) where \( P \) is the underlying reduced scheme of \( \tilde{P} \).
A variety $\tilde{X}$ over $k$ is a \textit{projective pseudo-homogeneous variety} for $G$, if $\tilde{X}_k \simeq G/\tilde{P}$, $\tilde{P}$ not necessarily reduced.

Twisted forms of VUFs

Denote by $X$ the $G$-variety such that $X_k \simeq G/P$ where $P$ is the underlying reduced scheme of $\tilde{P}$.

Call $X$ the \textit{projective homogeneous variety} corresponding to $\tilde{X}$.
A variety $\tilde{X}$ over $k$ is a projective pseudo-homogeneous variety for $G$, if $\tilde{X}_k \simeq G/\tilde{P}$, $\tilde{P}$ not necessarily reduced

Twisted forms of VUFs

Denote by $X$ the $G$-variety such that $X_k \simeq G/P$ where $P$ is the underlying reduced scheme of $\tilde{P}$.

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**Theorem:** $\mathcal{M}(X) \simeq \mathcal{M}(\tilde{X})$
I also show the following

**Theorem**
*Rost nilpotence holds for projective pseudo-homogeneous varieties for \( G \)*

**Corollary**
*Krull-Schmidt holds for projective pseudo-homogeneous varieties for \( G \)*
To prove the main theorem first I prove the following

**Theorem**

Let $X$ be projective $G$-homogeneous variety any field $k$ of any characteristic. Let $Z$ be any geometrically split projective $k$-variety satisfying RN such that the following holds in Chow$(k, \Lambda)$:

1. $U_X \simeq U_Z$
2. $\mathcal{M}(X_L) \simeq \mathcal{M}(Z_L)$ where $L = k(X)$

Then $\mathcal{M}(X) \simeq \mathcal{M}(Z)$. 
Proof of main result

Theorem

\[ M(X) \cong M(\tilde{X}) \]
Proof of main result

**Theorem**

\[ \mathcal{M}(X) \cong \mathcal{M}(\tilde{X}) \]

**Proof.**

- By induction on \( n = \text{rank}(G) \). Trivially true for \( n = 0 \). Assume true for all groups with rank less than \( n \).
- Let \( \text{rank}(G) = n \). Let \( L = k(X) \) and \( G' \) the anisotropic kernel of \( G_L \). Then \( \text{rank}(G') < \text{rank}(G) \).
- \( \mathcal{M}(\tilde{X}_L) = \bigcup_i \mathcal{M}(\tilde{Z}_i)(a_i) \) and \( \mathcal{M}(X_L) = \bigcup_i \mathcal{M}(Z_i)(a_i) \).
- By induction hypothesis, \( \mathcal{M}(\tilde{Z}_i) \cong \mathcal{M}(Z_i) \)
- \( \mathcal{M}(\tilde{X}_L) \cong \mathcal{M}(X_L) \).
- Moreover, \( U_X \cong U_{\tilde{X}} \).
- Applying generic criterion for isomorphic motives, we are done.
Corollary

Let $A$ be a CSA over $k$ of degree $n$ and let $B$ denote the CSA of degree $n$ that is Brauer equivalent to $A^\otimes p$. Then in the category $\text{Chow}(k, \Lambda)$, the motives of twisted flag varieties $X(d_1, d_2, \cdots, d_m, A)$ and $X(d_1, d_2, \cdots, d_m, B)$ are isomorphic. That is,

$$\mathcal{M}(X(d_1, d_2, \cdots, d_m, A)) \cong \mathcal{M}(X(d_1, d_2, \cdots, d_m, B))$$

Taking $m = 1$, we get $\mathcal{M}(\text{SB} d(A)) \cong \mathcal{M}(\text{SB} d(B))$ for twisted Grassmannians. In particular, for the case of Severi-Brauer varieties we have $\mathcal{M}(\text{SB} d(A)) \cong \mathcal{M}(\text{SB} d(B)).$
Examples and Applications

Corollary

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$$\mathcal{M}(X(d_1, d_2, \ldots, d_m, A)) \simeq \mathcal{M}(X(d_1, d_2, \ldots, d_m, B))$$

Taking $m = 1$, we get $\mathcal{M}(SB_d(A)) \simeq \mathcal{M}(SB_d(B))$ for twisted Grassmannians.
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Examples and Applications

Corollary

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Corollary

There exists examples of varieties whose motives are isomorphic when $\Lambda$ is any finite field but not when $\Lambda = \mathbb{Z}$.
Some open questions

- Are the motives of $\tilde{X}$ and $X$ isomorphic even when $G$ is outer?
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- Are the motives of $\tilde{X}$ and $X$ isomorphic even when $G$ is outer?
- Does the Generic criterion for isomorphic motives hold in general i.e., when $X$ and $Z$ are arbitrary varieties?
Thank You