TABULATION OF KNOTS UP TO FIVE TRIPLE-CROSSINGS AND MOVES BETWEEN ORIENTED DIAGRAMS

MICHAŁ JABŁONOWSKI

Abstract. We enumerate and show tables of minimal diagrams for all prime knots up to the triple-crossing number equal to five. We derive a minimal generating set of oriented moves connecting triple-crossing diagrams of the same oriented knot. We also present a conjecture about a strict lower bound of the triple-crossing number of a knot related to the breadth of its Alexander polynomial.

1. Introduction

It is known since at least V.F.R. Jones observation in 1999 (in his Planar Algebras, I c.f. [4]) that any knot and every link has a diagram where, at each of its multiple-points in the plane, exactly three strands are allowed to cross pairwise transversely. For the very recent survey on this topic see [4]. Such triple-point diagrams and moves on them have been studied in several recent papers, such as [1, 2, 3, 6, 7].

The triple-crossing number of a knot $K$, denoted here by $c_3(K)$, is defined in analogy to the classical (double-crossing) number, as the least number of triple-crossings for any triple-crossing diagram of $K$. There are lower bounds for the triple-crossing number, in terms of double-crossing number $c_3(K) \geq \frac{1}{3} c_2(K)$, and if $K$ is alternating then $c_3(K) \geq \frac{1}{2} c_2(K)$ (see [1]).

In [7] the author prove the following bound of the triple-crossing number $c_3$ by the canonical genus $g_c$. Let $K$ be a knot. Then $c_3(K) \geq 2 \cdot g_c(K)$. It follows from this bound that the triple-crossing number is greater or equal to the breadth of the Alexander polynomial $\Delta$, since it is known that $2 \cdot g_c(K) \geq \text{breadth}(\Delta(K))$. We propose a conjecture based on our extensive experiments.

Conjecture 1.1. Let $K$ be a knot, such that $\Delta(K)$ is not monic. Then

$$c_3(K) > \text{breadth}(\Delta(K)).$$

A polynomial is called monic if the coefficient of the highest order term are equal to $\pm 1$. If true, the conjecture immediately gives a sharp enough bound to obtain the exact (unknown) value of the triple-crossing number of many knots (from known upper bounds on the triple-crossing number), such as (giving only for knots with $c_2 \leq 13$):

$9_3$, $9_6$, $9_9$, $9_{16}$, $K11a_{234}$, $K11a_{240}$, $K11a_{263}$, $K11a_{334}$, $K11a_{338}$, $K11a_{355}$, $K11a_{364}$, $K13a_{3092}$, $K13a_{3110}$, $K13a_{3132}$, $K13a_{3377}$, $K13a_{3380}$, $K13a_{4547}$, $K13a_{4558}$, $K13a_{4739}$, $K13a_{4822}$, $K13a_{4828}$, $K13a_{4862}$, $K13a_{4874}$.

Date: March 19, 2022.

2020 Mathematics Subject Classification. 57K10 (primary).

Key words and phrases. minimal triple-crossing diagram, triple-crossing number, Alexander polynomial, tabulation of knots, moves.
In Section 3 of this paper, we also derive a minimal generating set of oriented moves connecting triple-crossing diagrams of the same oriented knot. Later, in Section 4 we enumerate and show tables of minimal diagrams for all prime knots up to the triple-crossing number equal to five. We use the knot names used in the [5] database package, and for 11–14-crossing knots we use the Hoste-Thistlethwaite database.

2. Definitions

The projection of a knot or a link \( K \subset \mathbb{R}^3 \) is its image under the standard projection \( \pi : \mathbb{R}^3 \to \mathbb{R}^2 \) (or into a 2-sphere) such that it has only a finite number of self-intersections, called multiple points, and in each multiple-points each pair of its strands are transverse.

If each multiple-points of a projection has multiplicity three then we call this projection a triple-crossing projection. The triple-crossing is a three-strand crossing with the strand labeled \( T, M, B \), for top, middle and bottom.

The triple-crossing diagram is a triple-crossing projection such that each of its triple points is a triple-crossing, such that \( \pi^{-1} \) of the strand labeled \( T \) (in the neighborhood of that triple point) is on the top of the strand corresponding to the strand labeled \( M \), and the latter strand is on the top of the strand corresponding to the strand labeled \( B \) (see Figure 1).

![Figure 1. A deconstruction/construction of a triple-crossing.](image)

The triple-crossing number of a knot or link \( K \), denoted \( c_3(K) \), is the least number of triple-crossings for any triple-crossing diagram of \( K \). The classical double-crossing number invariant we will denote by \( c_2 \). The minimal triple-crossing diagram of a knot \( K \) is a triple-crossing diagram of \( K \) that has exactly \( c_3(K) \) triple-crossings.

A natural orientation (see an equivalent definition in [3]) on a triple-crossing diagram is an orientation of each component of that link, such that in each crossing the strands are oriented in-out-in-out-in-out, as we encircle the crossing. We begin with an interesting notice.

Lemma 2.1 ([3]). Every orientation of the triple-crossing diagram obtained from an oriented knot is the natural orientation.

3. Oriented moves

Theorem 3.1. Two oriented triple-crossing diagrams of knots are related by a sequence of oriented \( J_R \) and \( J'_R \) moves (see Figure 2) and a spherical isotopy, if and only if they define the same knot type.

In the \( J_R \) move, there can be finitely many triple-crossings (colored here in blue) such that for every triple-crossing the arc, that is nearest the letter \( T \) lies always on top. In the \( J'_R \) move that is the mirror move to \( J_R \), there can be finitely many triple-crossings (colored here in blue) such that for every triple-crossing the arc, that is nearest the letter \( B \) lies always on bottom. The orientations of the blue arcs are determined by the natural orientation property.
Figure 2. A minimal generating set of oriented moves on triple-crossing diagrams of a knot.

Proof. We have the minimal set of unoriented moves $J_R$ and $J'_R$ between unoriented knots, defined by the author in [6] that are identical as our moves in Figure 2 but without decorating arrows (so we leave the names unchanged). From Lemma 2.1 we see that specifying orientation on one strand in any triple-crossing the other strands in that crossing must have determined orientation. Therefore, because in each local diagram of unoriented moves $J_R$ and $J'_R$ there is a strand passing through all other triple-crossings, we have up to four generating moves $J_R$, $J'_R$, $J_S$ and $J'_S$ for oriented diagrams (for the latter pair see Figure 3). But the move $J_S$ can be generated from $J_R$ by a spherical isotopy. First choose any non-outer region adjacent region to the triple-crossings marked $T$, $M$, $B$ and on the left to the triple-crossing then by a spherical isotopy make the region to be the outer (unbounded) region for the knot diagram. Then rotate the diagram by 180 degrees. The same goes with the pair $J'_S$ and $J'_R$.

Corollary 3.2. The set \{\(J_R, J'_R\)\} (presented in Figure 2) is a minimal generating set of oriented moves connecting triple-crossing diagrams of the same oriented knot.

Figure 3. The other pair of oriented moves.
**Table 1.** Relations for the Jones polynomial.

\[
V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) = -t^\frac{3}{2}V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) - tV\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) + \\
-tV\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) - t^\frac{3}{2}V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) - t^\frac{3}{2}V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right)
\]

\[
V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) = -t^{-\frac{3}{2}}V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) - t^{-1}V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) + \\
-t^{-1}V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) - t^{-\frac{3}{2}}V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right) - t^{-\frac{3}{2}}V\left(\begin{array}{c}
\cdot \\
\cdot
\end{array}\right)
\]

\[
V(\underbrace{O\cup \ldots \cup O}_c) = \left(-t^\frac{3}{2} - t^{-\frac{3}{2}}\right)^{c-1}
\]

4. **Knot Tabulation**

After several months of computer computations, following the method described by the author in [6] and implemented in Wolfram Engine 12, we generate the table of knots with the triple-crossing number equal to five.

First, we enumerate all prime, connected, triple-crossing projections, up to spherical isotopy, up to mirror image, and up to moves \(M_1, M_2\) (see Figure 4), with five triple points, with the result of 116 projections (see Table 2).

![Figure 4. Moves M1 and M2.](image)

To identify types of knots, we use the classical polynomial invariants, the Jones polynomial (see Remark 4.1), and later two-variable polynomials where they are needed.

**Remark 4.1.** By Lemma 2.1 and the Kauffman bracket relations form [1], the Jones polynomial \(V\) for an oriented knot can be calculated from a triple-crossing diagram (regardless of the orientation) by resolving each triple-crossing by the following relations in Table 1.

The number of knots with a specific triple-crossing number is presented in Table 2. Diagrams of the knots, generated by a new algorithm, are presented in Table 3, where the labels are of the form \(tk_n\) (for the triple-crossing number equal to \(k\), and \(n\) the consecutive index in that family). The labels in brackets are the classical Alexander-Briggs-Rolfsen notation of a knot (up to mirror image). The convention here is that the green strand near
a triple-crossing is the bottom strand, and the red one is the upper strand (the loops are always black for simplicity).

**Table 2.** Enumeration of knots and projections.

| crossings | number of projections | number of knots |
|-----------|-----------------------|-----------------|
| 2         | 1                     | 2               |
| 3         | 2                     | 2               |
| 4         | 15                    | 24              |
| 5         | 116                   | 118             |

An sPD code for a given knot can be found below the TikZ code of the corresponding diagram, in the LaTeX source file of this article’s arXiv preprint version. In that archive there is also a text file of sPD codes of all the mentioned triple-crossing projections.

**References**

[1] C. Adams, Triple crossing number of knots and links, *J. Knot Theory Ramifications* **22** (2013), 1350006.
[2] C. Adams, O. Capovilla-Searle, J. Freeman, D. Irvine, S. Petti, D. Vitek, A. Weber and S. Zhang, Multicrossing number for knots and the Kauffman bracket polynomial, *Math. Proc. of Cambridge Philos. Soc.* **164**(1) (2018), 147–178.
[3] C. Adams, J. Hoste and M. Palmer, Triple-crossing number and moves on triple-crossing link diagram, *J. Knot Theory Ramifications* **28** (2019), 1940001.
[4] C. Adams, Multi-Crossing Number of Knots and Links, In *Encyclopedia knot theory*, CRC Press (2021) 63–70.
[5] D. Bar-Natan, *The knot atlas*, [http://katlas.org](http://katlas.org), (2021)
[6] M. Jabłonowski and Ł. Trojanowski, Triple-crossing projections, moves on knots and links, and their minimal diagrams, *J. Knot Theory Ramifications* **29** (2020), 2050015.
[7] M. Jabłonowski, Triple-crossing number, the genus of a knot or link and torus knots, *Topology and its Applications* **285** (2020), 107389.

Table 3: Knots with the triple-crossing number $\leq 5$. 

![t2_1 (3_1)](image1)
![t2_2 (4_1)](image2)
![t3_1 (5_2)](image3)
![t3_2 (6_1)](image4)

![t4_1 (5_1)](image5)
![t4_2 (6_2)](image6)
![t4_3 (6_3)](image7)
| Knot | Description |
|------|-------------|
| $t_{52}$ | $11n_{68}$ |
| $t_{53}$ | $11n_{79}$ |
| $t_{54}$ | $11n_{83}$ |
| $t_{55}$ | $11n_{91}$ |
| $t_{56}$ | $11n_{100}$ |
| $t_{57}$ | $11n_{101}$ |
| $t_{58}$ | $11n_{102}$ |
| $t_{59}$ | $11n_{113}$ |
| $t_{60}$ | $11n_{114}$ |
| $t_{61}$ | $11n_{116}$ |
| $t_{62}$ | $11n_{117}$ |
| $t_{63}$ | $11n_{123}$ |
| $t_{64}$ | $11n_{132}$ |
| $t_{65}$ | $11n_{140}$ |
| $t_{66}$ | $11n_{141}$ |
$t_{594} (12n_{608})$

$14$ MICHAŁ JABŁONOWSKI

$15$

$5$

$94$

$15$

$95$

$13$

$608$

$13$

$838$

$13$

$469$

$13$

$1021$

$13$

$1475$

$13$

$1482$

$13$

$1513$

$13$

$1817$

$13$

$2067$

$13$

$2148$

$13$

$2328$

$13$

$2527$

$13$

$3158$

$13$

$3523$

$13$

$3594$
Institute of Mathematics, Faculty of Mathematics, Physics and Informatics, University of Gdańsk, 80-308 Gdańsk, Poland
Email address: michal.jablonowski@gmail.com