DECOMPOSITION IN COXETER-CHAMBERS OF THE CONFIGURATION SPACE OF $d$ MARKED POINTS ON THE COMPLEX PLANE

NOÉMIE COMBE

Abstract. In this paper we investigate a new decomposition in Coxeter-chambers of the $d$-th unordered configuration space of the complex plane, using its natural relation with the space $\text{Pol}_d$ of complex monic degree $d > 0$ polynomials in one variable with simple roots. This decomposition relies on objects called elementa and which are the pieces of a good cover in the sense of Čech of $\text{Pol}_d$. Each elementa is a set of polynomials indexed by a decorated graph, reminiscent of Grothendieck’s dessin’s d’enfant. The main result of this paper is that this new decomposition is invariant under a Coxeter group. This result has two major impacts. Using the good cover in elementa, the explicit calculation of Čech cohomology groups requires an exponential number of incidence relations to study. Therefore, using this new construction by chambers and galleries, the complexity is very much reduced. Secondly, the well known interpretation of the $d$-strand braid group $B_d$, namely as the fundamental group of this configuration space, implies that any braid (which is a path in $\pi_1(\text{Pol}_d)$) can be obtained from a loop in one chamber.

Contents

1. Introduction 1
2. Elementa and diagrams 3
3. Classification of diagrams and adjacency relations 6
3.1. Classification of elementa 6
3.2. Adjacency relations 7
4. Inclusion diagrams 10
4.1. Properties of inclusion diagram 11
4.2. One dimensional faces of the inclusion diagram 12
4.3. Higher dimensional faces of the inclusion diagram 13
5. Decomposition invariant under Coxeter groups 20
5.1. Adjacency, chambers and galleries 20
5.2. Main theorems and their proofs 24
6. Concluding remark 27
Appendix A. The adjacent $Q$-diagrams for $d = 4$ 28
References 29

1. Introduction

In this paper we investigate a decomposition in chambers of the $d$-th unordered configuration space of the complex plane, using its natural relation with the space of complex monic degree...
$d > 0$ polynomials in one variable with simple roots with sum equal to zero (i.e. Tschirnhausen polynomials). Let us denote by $\text{DPol}_d$ this semi-algebraic set of complex dimension $d$. It is well known that the fundamental group of this configuration space is the $d$-strand braid group $B_d$. Using this interpretation, a braid is represented as a path $\gamma(t)$, $t \in (0, 1)$ or rather as the class $[\gamma(t)] \in \pi_1(\text{DPol}_d)$. In spite of a rich and abundant literature concerning the subject of braids, we suggest a new approach giving also a new insight on the space $\text{DPol}_d$. One of the main goals of this article, in the spirit of highlighted problems in \cite{19, 16} is to characterize in a new and more geometric manner the generators for $\Gamma_{0, [n]}$, where $\Gamma_{0, [n]}$ is nothing but the orbifold fundamental group of the moduli space of smooth curves of genus 0 with $n$ unordered marked points. As will be proved in this article, our new decomposition of $\text{DPol}_d$ turns out to have many interesting symmetries, in particular polyhedral symmetries. This decomposition is based on the theory of chambers and galleries. We call Coxeter chamber\cite{13, 31, Chap IX, Sect. 5.2} a fundamental domain, along with reflections hyperplanes. The method we present with those chambers, is in the spirit of previous works \cite{10, 11}. As an advantageous result it thus follows that to define any braid in $B_d$ it is sufficient to consider a path in the Coxeter-chamber.

The present work grew out of the previous works \cite{7, 10, 8, 9}, where we bring into light the existence of a topological stratification, obtained through the notion of drawings $\mathcal{C}_d$ of a polynomial $P$. Such an object is reminiscent of Grothendieck’s dessin’s d’enfant \cite{15} in the sense that we consider the inverse image of the real and imaginary axes under a complex polynomial. The drawing associated to a complex polynomial $[2]$ is, by convention, a system of blue and red curves properly embedded in the complex plane, being the inverse image under a polynomial $P$ of the union of the real axis (colored red) and the imaginary axis (colored blue) $[8]$, $\mathcal{C}_d = P^{-1}(\mathbb{R} \cup \mathbb{R})$. For a polynomial of degree $d$, the drawing contains $d$ blue and $d$ red curves, each blue curve intersecting exactly one red curve. The entire drawing forms a forest whose leaves (terminal vertices) go to infinity in the asymptotic directions of angle $\pi/2d$.

The isotopy classes of drawings, relatively to the $4d$ asymptotic directions, are basic objects for the following construction, we refer to them as “elementa”. Using the Riemann mapping theorem, instead of working with the equivalence classes of drawings in the complex plane we study those graphs in a disk and refer to those combinatorial objects as “diagrams”. Each elementum is indexed by a diagram (i.e. a forest embedded in the disk where edges are oriented, colored in red or blue and the faces in the disk are colored in colors $A, B, C, D$). The elementum define a topological stratification of $\text{DPol}_d$, each elementum is indexed by a diagram $\sigma$ defines a topological stratum $A_{\sigma}$.

As shown in \cite{10, 8} this decomposition allows the construction of a good cover of $\text{DPol}_d$ in the sense of Čech. The number of elementa grows exponentially with the degree $d$. So, to calculate the Čech cohomology groups using such a cover confers an exponential number of incidence relations to study. Therefore, the result we present in this paper has another favorable property: due to the symmetries of the decomposition, the complexity of this problem is very much reduced.

We construct the Čech nerve of the topological stratification (see \cite{10}) using the following relations between elementum. Two elementa are incident if the diagram of one can be obtained from the diagram of the other by a so-called half-Whitehead move (this is a topological operation on the red (or blue) edges of a given diagram modifying one diagram into the other one). This incidence relation on elementa is deeply connected to the topological closure of each topological stratum: it is shown in \cite{7, 8} that the topological closure of a stratum $A_{\sigma}$ indexed by $\sigma$ is given by the combinatorial closure of $\sigma$, which we shall discuss here.

We show that that this structure is reminiscent of translation surfaces and that it contains symmetries. In particular it is invariant under $D_{2d} \oplus \mathbb{Z}_2$ group, where $D_{2d}$ is the dihedral one. The aim of this paper is presentation of the proof of the following theorem:
Theorem 1. Let $d > 3$. The decomposition of $\mathbb{D} Pol_d$ in 4d Coxeter-chambers, induced by the topological stratification of $\mathbb{D} Pol_d$ in elementa, is invariant under the Coxeter group given by the presentation:

$$W = \langle \{s_i\}_{i=1}^{2d+1} | (s_i s_j)^{2d} = Id \rangle.$$

The paper is composed of four main points.

In section 2, we introduce the notion of elementa and diagrams as well as their properties. We discuss the Whitehead moves on the diagrams.

In section 3 we investigate the classification of generic diagrams into four classes $M, F, S$ and $FS$ and show that the elementa are endowed with an incidence relation, which forms a partially ordered set.

In the section 4, we introduce the notion of inclusion diagram of the poset. This study is necessary to display the poset by a geometric structure. Moreover, this inclusion diagram is the nerve in the sense of Čech of the Čech good cover (5), presented in [3]. The construction is based on properties of generic elementa in such way that each vertex corresponds to a generic elementum, each edge corresponds to a elementum of codimension 1 and each 2-face to a elementum of codimension 2 (etc). The incidence relations between the classes are conserved for such diagram. We list and classify the set of generic diagrams into families, and explain how to connected two vertices in the inclusion by an edge according to the Whitehead moves on the diagrams. An explicit construction of the inclusion diagram for low degrees: $d = 2, 3, 4$. The case $d = 4$, is particularly important since it is geometrically very rich. Higher degrees inclusion diagrams are based on the same procedure.

In the section 5 we prove the main statement that the decomposition is invariant under a Coxeter group by presenting a method of construction. The exponential number of faces in the inclusion diagram makes it impossible to draw explicitly the structures for higher degrees than 3. Therefore, using the existence of symmetries in the decomposition and in the inclusion diagram we show that the structure is reminiscent to translation surfaces in the sense that one glues vertices by translation, leading to a connected space.

2. Elementa and diagrams

The isotopy classes $\sigma$ of drawings, relatively to their $4d$ asymptotic directions, are basic for our construction. So, in the following, we call these objects elementa. A set of polynomials indexed by an elementum, denoted $\mathcal{A}_\sigma$, is called a class. An elementum $\sigma$ is characterized by a diagram, having the following properties:

Definition 1. A diagram $\sigma$ is a forest $\Gamma = \{V, E\}$ embedded in the unit disk $\mathbb{D}$ such that:

- the set of vertices:
  $$V = \{v, \bar{v}, \bar{\bar{v}}; v \in \partial \mathbb{D}, \bar{v}, \bar{\bar{v}} \in int(\mathbb{D}), |v| = 4d, |ar{v}| = d, 0 \leq |\bar{\bar{v}}| \leq d - 1\},$$

- the set of edges:
  $$E = \{e_{ij}, \bar{e}_{\alpha,\beta} \mid e_{ij} = <v_i, v_j>, \bar{e}_{\alpha,\beta} = <\bar{v}_\alpha, \bar{v}_\beta>\}$$
  and $\bar{v}_\mu = e_{i,j} \cap e_{k,l}, \mu = (i, j, k, l)$.

This graph verifies the following conditions:

(1) the $4d$ terminal vertices $v$ on $\partial \mathbb{D}$ lie on the $4d$ roots of the unity and are ordered in counterclockwise order. The colors of the terminal vertices alternate from red to blue.

(2) $\deg(v) = 1; \deg(\bar{v}) = 4; 0 \leq \deg(\bar{\bar{v}}) \leq 2d$.

(3) the edges $e_{ij}, \bar{e}_{ij}$ carry a coloring red or blue and an orientation;

(4) the edge $e_{ij}$ is colored red (resp blue) if it connects $i \equiv 3 \pmod{4} \quad (\text{or } 0 \pmod{4})$ to $j \equiv 1 \pmod{4} \quad (\text{or } 0 \pmod{4})$.
if we cut along the graph we get a disjoint union of connected components (2-faces) each homeomorphic to an open disk; those 2-faces carry a coloring in four colors: $A, B, C, D$ in trigonometric orientation:

(a) a region is colored $A$ if it contains in its boundary a pair of red and a blue edges of terminal vertices respectively $1 \mod 4$ and $2 \mod 4$. 
(b) a region is colored $B$ if it contains in its boundary a pair of red and a blue edges of terminal vertices respectively $3 \mod 4$ and $2 \mod 4$. 
(c) a region is colored $C$ if it contains in its boundary a pair of red and a blue edges of terminal vertices respectively $3 \mod 4$ and $0 \mod 4$. 
(d) a region is colored $D$ if it contains in its boundary a pair of red and a blue edges of terminal vertices respectively $1 \mod 4$ and $0 \mod 4$.

$\bar{v}$ is incident to four faces $A, B, C, D$; the vertices $\bar{v}$ are incident to faces $A, D, A, D, ...$ (resp. $B, C, B, C, ...$) in trigonometric orientation.

In the following, for simplicity we consider the diagonal $(i, j)$ instead of the edge $e_{i,j}$.

**Definition 2** (Short and long diagonals). Let $\sigma$ be a diagram. A diagonal in a diagram is an edge, denoted by $(i, j)$, connecting a pair of terminal vertices $i, j \in \{1, ..., 4d\}$ of the same parity.

- A diagonal is short if $|i - j| = 2$.
- A diagonal is long if it is not short.

![Figure 1. A long blue diagonal $(i, j)$, with red short diagonals.](image)

**Definition 3.** An elementum is called generic if it does not contain any vertices $\bar{v}$.

**Definition 4** (Codimension). The real codimension of an elementum is the sum of the local indices of all the vertices $\bar{v}$, where the local index at a vertex $\bar{v}$ is equal to $2m - 3$ and $m = \deg(\bar{v})$.

All the generic elementa are of codimension 0.

**Definition 5.** Let $\sigma$ be a diagram.

- A tree bounds a connected component of $\mathbb{D} \setminus \sigma$, if a pair of edges lies in the boundary of this 2-cell.
- Two trees are adjacent if there exists a pair of edges in both trees lying in the boundary of a connected component $\mathbb{D} \setminus \sigma$.
- A pair of diagonals $(i, j)(k, l)$ in a generic diagram $\sigma$ are called successive if the terminal vertices satisfy one of the following conditions: $|i - k| = 2$ or $|i - l| = 2$ or $|j - k| = 2$ or $|j - l| = 2$.

**Proposition 2.** [7, 9] For every $d$, the number of elementa is finite.

Let us introduce a topological operation on the diagrams $\sigma$.

**Definition 6** ([7]). A half-Whitehead move is a topological operation on the diagonals of a diagram, carried out in the following way:
A contracting half-Whitehead move on a diagram $\sigma_0$ is a gluing of $m$ diagonals of the same color in one point such that the new diagram verifies $\cdot$

A smoothing half-Whitehead move is the inverse operation.

For simplicity, a contracting half Whitehead move will be called an incidence relation and denoted by the symbol $\preceq$. We say that $\sigma \preceq \tau$ if there exists a sequence of diagrams $\sigma = \sigma_1 \preceq \sigma_2 \ldots \preceq \sigma_n = \tau$.

Remark 1. Notice, that the contracting Whitehead moves are of two types. The first type is such that one can glue in one point $m$ disjoint diagonals of the same color and occurring in the boundary of a 2-face. The second type is such that one glues together, two vertices $\bar{v}$ from $\sigma$.

Definition 7. Let $\sigma_0$ and $\sigma_1$ be diagrams of the same codimension. A Whitehead move $\delta$ is the composition of a contracting and smoothing half-Whitehead moves starting at $\sigma_0$ and ending on $\sigma_1$. Two signatures differing by a Whitehead move are called adjacent and are denoted $\sigma_0 \leftrightarrow \sigma_1$.

Let us illustrate below a composition of a contracting and a smoothing half-Whitehead move for a pair of diagonals.

$\delta : (\quad \leftrightarrow \quad \times \quad \leftrightarrow \quad \simeq \quad \cdot$.

Figure 2. half-Whitehead move and smoothing half-Whitehead move

Example 1.

• For $d = 2$, we illustrate a Whitehead move (c.f. definition 7) on a pair of red diagonals. The two generic diagrams, illustrated as diagrams on the right and on the left of the figure below, are incident to a codimension 1 diagram. The latter diagram is illustrated in the middle of the figure 3.

Figure 3. Whitehead move on a pair of red diagonals

• The Whitehead move has incidence on the deformation of coefficients of a given polynomial. In this following example, we consider $\text{Im}(P) = 0$:

Theorem 3. Let us consider the set of diagrams $\Sigma_d$ for degree $d > 1$ polynomials. Then, $(\Sigma_d, \preceq)$ is a partially ordered set.

Proof. Transitivity and reflexivity relations are obvious. We prove that the relation is antisymmetric. If one has $\sigma \preceq \tau$ then $\text{codim}(\tau) \geq \text{codim}(\sigma)$. So, if $\sigma \preceq \tau$ and $\tau \preceq \sigma$ then $\sigma$ and $\tau$ have same codimension, and the equality of codimension is possible only if $n = 1$ in the sequence $\sigma_1 \preceq \sigma_2 \ldots \preceq \sigma_n$, so $\sigma = \sigma_1 = \sigma_n = \tau$. □

Definition 8. The combinatorial closure $\overline{\sigma}$ of a diagram $\sigma$ is given by the union of incident sets to $\sigma$: $\overline{\sigma} = \cup_{\sigma \prec \tau} \overline{\tau}$. 
3. Classification of diagrams and adjacency relations

3.1. Classification of elements.

3.1.1. Generic elements. The generic elements can be classified by the diagrams in the following way [7]:

(1) Trees of the diagrams consisting of two short (red and blue) diagonals are of type $M$. An $M$ tree is denoted by $\| i,i+2 \|$, where the number on the first line indicates that there exists a short diagonal $(i+1,i+3)$ crossing the diagonal $(i,i+2)$. A diagram with only $M$ trees is called an $M$-diagram. Note, that there exist only four $M$-diagrams, for any $d$:
   
   (a) $M_1 \leftrightarrow \| \frac{3}{3} \| \| \frac{7}{7} \| \ldots | \frac{4d-1}{4d-3} |$.
   
   (b) $M_2 \leftrightarrow \| \frac{1}{1} \| \frac{7}{7} \| \| \frac{5}{5} \| \| \frac{9}{9} \| \ldots | \frac{4d-3}{4d-1} |$.
   
   (c) $M_3 \leftrightarrow \| \frac{5}{5} \| \frac{7}{7} \| \| \frac{3}{3} \| \| \frac{9}{9} \| \| \frac{1}{1} \| \ldots | \frac{4d-1}{4d-3} |$.
   
   (d) $M_4 \leftrightarrow \| \frac{3}{3} \| \frac{9}{9} \| \| \frac{5}{5} \| \| \frac{1}{1} \| \| \frac{7}{7} \| \ldots | \frac{4d-3}{4d-1} |$.

(2) Trees consisting of one short and one long diagonal are of type $F$. An $F$ tree is denoted by $\| \frac{j}{j} \|$, where $j$ and $i$ are labels of the terminal vertices of the long diagonal; the number $j$, indicates that there exists a short diagonal of the opposite color joining the vertex $j-1$ to $j+1$. Two $F$ diagrams are of opposite $F$ trees if the indexes are switched, for instance $\| \frac{j}{j} \| \text{ and } \| \frac{i}{i} \|$ have opposite orientation. A diagram with only $M$ and $F$ trees is called an $F$-diagram. A diagram of type $F^{\otimes m}$ has exactly $m$ trees of type $F$ (and other of type $M$).

(3) Trees consisting of two long diagonals are of type $S$. An $S$ tree is denoted by $\| \frac{i,j}{k,l} \|$, where the first line gives the coordinates of a long diagonal $(i,j)$, the second line gives the coordinates of the second long diagonal $(k,l)$. A diagram with only $M$ and $S$ trees is called an $S$-diagram. As for the previous family of diagrams, a diagram is of type $S^{\otimes m}$ if there exists $m$ trees of type $S$, the other are $M$ trees. We focus on $S$ trees given by pairs of diagonals $(i,j)$ and $(i+1,j+1)$ or $(i,j)$ and $(i-1,j-1)$ and call the narrow $S$ trees. If we have a diagram $S^{\otimes d-2}$ then all the $S$ trees are narrow.

(4) The combination of $F$, $S$ and $M$ trees gives an $FS$-diagram.

The figure presents, in the case of $d = 4$, an example of $M$, $F$, and $S$ diagrams.
3.1.2. Classification of the diagrams for $d = 3$. In this subsection we consider for the case of $d = 3$ not only the generic diagrams but also the diagrams of non zero codimension.

Diagrams of codimension 0:
In the decomposition by diagrams of $\mathbb{P}\text{Pol}_3$, there exist 22 generic diagrams among which there are four $M$ diagrams, six $S$ diagrams (containing only one $S$ tree) and twelve $F$ diagrams (six with one red long diagonal and six with one long blue diagonal):

- 4 diagrams of type $M$:
- 4 diagrams of type $M$:
- 4 diagrams of type $M$:

Let us notice that in the figures we did not use the standard notations on the terminal vertices, since these are the diagrams directly generated from the computer program.

In addition to generic classes there exist
- 48 diagrams of codimension 1, four families of twelve diagrams which are equivalent up to rotation;
- 30 diagrams of codimension 2;
- 4 diagrams of codimension 3.

Diagrams of codimension 1:

Diagrams of codimension 2:
• 1 family of order 6

• 2 families of order 12

Diagrams of codimension 3:

• 1 family of order 4

Lemma 4. Let $\sigma$ be a generic diagram, having all red (resp. blue) diagonals short. Consider a pair of adjacent $F$ trees, of long blue (resp. red) diagonals $(i, j)(j + 2, k)$ and deform them by a Whitehead move. If $k = i - 2 \mod 4d$, then the number of blue (resp. red) short diagonals is increased by two. Otherwise, the number is increased by one.

Proof. Consider the first case. Applying a Whitehead move onto this pair of diagonals induces a new diagram, where the new pair of diagonals is $(i, i - 2)(j, j + 2)$, which are both short in the sense of the definition. So, the number of short blue (resp. red) diagonals is increased by two. Concerning the second case, the Whitehead move applied to the pair of diagonals $(i, j)(i - 2, k)$ induces $(i, i - 2)(j, k)$, where $(i, i - 2)$ is a short diagonal. So, the number of short blue (resp. red) diagonals is increased by one.

Corollary 5. Consider a generic diagram $\sigma$ having only short red (resp. blue) diagonals. Then, applying repeatedly Whitehead moves onto pairs of diagonals induces, in a finite number of Whitehead moves, an $M$ diagram.

Lemma 6. Let $\sigma$ be a generic diagram having all red (resp. blue) diagonals short. Consider a pair of non-successive, adjacent blue diagonals in $\sigma$. Then, applying a Whitehead move onto this pair of diagonals:

1. increases by two the number of long blue (resp. red) diagonals, if both diagonals are short,
2. increases by one the number of long blue (resp. red) diagonals, if one of the diagonals is short,
3. gives a new pair of long diagonals if both diagonals are long.

Proof. Let $(i, j)$ and $(l, k)$ be the pair of blue (resp. red) diagonals.

• If both diagonals are short then $|i - j| = |l - k| = 2$, where $k \neq j + 2 \mod 4d$, and $l \neq i - 2 \mod 4d$ (using definition and definition). So, applying the Whitehead move onto the pair $(i, j)$ and $(k, l)$ induces the new pair of diagonals $(i, k)(j, l)$, where $|i - l| \neq 2$ and $|l - j| \neq 2$.

• Let us suppose, without loss of generality, that $(k, l)$ is short. Applying the Whitehead move onto $(i, j)$ and $(k, k + 2)$ gives the pair of diagonals $(i, k + 2)(j, k)$. Since $(i, j)$ and
$(k, l)$ are non-successive, then $l \neq i - 2$ and $k \neq j + 2$. Therefore, $|i - k - 2| \neq 2$ and $|j - k| \neq 2$. Since both diagonals are long, the number of long blue (resp. red) diagonals, is increased by one.

- Let us apply a Whitehead move onto the pair of diagonals $(i, j)$ and $(k, l)$. Since $(i, j)$ and $(k, l)$ are disjoint and belong to a given diagram $\sigma$, their terminal vertices verify $k \equiv i \equiv 1 \mod 4$ and $j \equiv l \equiv 3 \mod 4$ (see definition 1). This pair is first modified by a half-Whitehead move into a pair of diagonals meeting at one point: $(i, k)(j, l)$. This pair is then modified by a smoothing Whitehead move, which induces the unique possible pair of diagonals $(i, l)(j, k)$.

\[\square\]

### 3.2. Adjacency relations.

We recall a few results from [7].

**Theorem 7** (Adjacency theorem). Let us consider a generic diagram. Then, in one Whitehead move

1. $M$-diagrams are connected to $(d - 1)$ $F$-diagrams ([$d \choose 2$] red, blue $F$ diagrams);
2. $F$-diagrams are connected to $M$-diagrams, $S$-diagrams and $SF$-diagrams;
3. $S$-diagrams are connected to $FS$-diagrams or $S$-diagrams

**Proof.**

- Consider an $M$ tree. By lemma 6 we know that in one Whitehead move applied onto a pair of the short diagonals, we obtain a diagram having a pair of long diagonals. This is an $F$-diagram. There exist $d$ blue or red diagonals. Choosing a pair of blue (or red) diagonals gives $d \choose 2$ possibilities. Therefore we have $d(d - 1)$ adjacent $F$-diagrams to an $M$-diagram.

- Consider an $F$ diagram. One Whitehead move applied onto a pair of short red diagonals in a pair of adjacent $F$ trees gives $S$ trees. If there exist no other long diagonals in the diagram, then this is an $S$-diagram. Otherwise, it is a $FS$-diagram. From (1) we know that $F$-diagrams are connected to $M$ diagrams.

- Consider two adjacent trees, one of those trees being an $S$ tree. If the other one is an $M$ tree then applying a Whitehead move to a pair of adjacent diagonals gives an $FS$-diagram (this follows form lemma 6). Otherwise, we have a couple of adjacent $F$ trees.

\[\square\]

**Theorem 8.** [7] Let $\sigma \in \Sigma_d$ be a generic diagram. Then, for every $\sigma$ there exists a path starting at $\sigma$ and ending on an $M$-diagram.

**Proof.** The proof is by induction on the number of long blue diagonals. Let $\sigma$ be generic diagram such that on the left side of a long blue diagonal there exist only short diagonals of red and blue color.

1. **Base case.** Let $\sigma$ have only one long blue diagonal. Then two cases are discussed:
   - (a) The red diagonals are all short.
   - (b) Not all red diagonals are short.

   Consider the first case. Let us apply one of the lemma 4 onto the long blue diagonal and the blue short diagonals on its right side. Since each deformation step increases by one the number of short blue diagonals, so proceeding until there are $d$ short diagonals in the diagram we obtain a diagram having only short blue diagonals. Hence, we have an $M$ diagram. Consider the second case where we have $k$ red long diagonals. Then this long blue diagonal and these $k$ long red diagonals compartment the diagram into $q$ disjoint adjacent polygonal regions. Each polygonal region can be interpreted as an $M$-diagram,
local. So, we have $q$ adjacent locally $M$ diagrams. We apply lemma 4 to the long blue
diagonal in one compartment. Then, after a finite number of deformations using lemma 4
to the long blue diagonal, we have a local $M$-diagram in this compartment. The new
long blue diagonal is now common to the adjacent compartment. We continue to apply
lemma 4 to the new blue diagonal in this adjacent compartment until we obtain locally
an $M$-diagram in this adjacent compartment. We continue in this way in all adjacent
compartments. This gives in final all blue diagonals short. Now we apply the same
procedure to the red long diagonals. We obtain after a finite number of deformations all
red diagonals short. This is an $M$ diagram.

(2) Induction case. Suppose that for a diagram with $m$ long diagonals there exists a path
from the diagram to an $M$ diagram. Let us show that for $m + 1$ long diagonals this
statement is also true. Take a block of adjacent $m$ long diagonals and apply the induction
hypothesis to it. Then there exists a finite number of deformations such that these $m$
long blue and red diagonals are all short, leaving only one long blue diagonal in the
diagram. We can thus apply the case (1) from this discussion.

\[\square\]

Remark 2. The theorem above can be interpreted as the path connectedness of $\text{DPol}_d$, from
which we recover the fact that $\text{DPol}_d$ is connected since, as the complement of a hyperplane
arrangement it is a open subset of $\mathbb{C}^d$.

4. INCLUSION DIAGRAMS

In this section we introduce the inclusion diagram of the topological stratification. This object
describes the geometry of the stratification by elements and constitutes the main tool of the proof
of the main theorem.

4.1. Properties of inclusion diagram. Recall from [7,8] that the thickened elements (denoted $A^+$)
form a good cover, in the sense of Čech.

Definition 9 (Inclusion diagram). Let $(A^+_{\sigma})_{\sigma \in \Sigma_d}$ be a cover of $\text{DPol}_d$ and let $\preceq$
be the incidence relation between two classes. The inclusion diagram $(W, \subset)$ is the nerve of the Čech cover
$A^+_{\sigma} \in \Sigma_d$ satisfying the following conditions:

- an $i$-face in $W_{\sigma}$ is in one-to-one correspondence with a codimension $i$-class $A_{\sigma}$;
- a pair of faces $W_{\tau}, W_{\sigma}$ of the inclusion diagram verify $W_{\tau} \subset W_{\sigma}$, if and only if the
corresponding classes $A_{\tau}, A_{\sigma}$ in $(A^+_{\sigma})_{\sigma \in \Sigma_d}$ verify $A_{\tau} \subseteq A_{\sigma}$;
- if a class $A_{\mu}$ of codimension $k$ is incident to a collection of classes $A_{\sigma_1},...,A_{\sigma_m}$ of smaller
codimension such that $A_{\mu} \cap A_{\sigma_1},...,A_{\sigma_m}$, then the collection of faces $W_{\sigma_1},...,W_{\sigma_m}$ are
joined by a $k$-dimensional face $W_{\mu}$.

Lemma 9 (Edges and 2-faces of the inclusion diagram). Let $(W, \subset)$ be the inclusion diagram
associated to $(A^+_{\sigma})_{\sigma \in \Sigma_d}$. Then:

1. each 1-dimensional face in $W$ is bounded by 2 vertices.
2. each 2-dimensional face in $W$ is bounded by 4 vertices, 4 edges and forms a quadrangle.

Proof. Let us prove the first statement. Consider a diagram $\beta$, of codimension 1. Then, by definition 3
there exists a pair of intersecting diagonals (of the same color). Suppose that the set of terminal vertices of those diagonals is \{i, j, k, l\} where $i < j < k < l$ on the circle. Those numbers $i, j, k, l$ are of the same parity and by definition
verify: $i \equiv k \equiv 1 \mod 4$ and $j \equiv l \equiv 3 \mod 4$ (resp. $i \equiv k \equiv 2 \mod 4$ and $j \equiv l \equiv 0 \mod 4$). Again, from definition 4 we know that in a generic diagram each terminal vertex congruent to $1 \mod 4$ (resp. 2 \mod 4) is attached
by an edge to a terminal vertex which is congruent to 3 mod 4 (resp. 0 mod 4). So, using a smoothing half-Whitehead move the intersection point is smoothed and we obtain two different possible pairs of diagonals: \((i,j)(k,l)\) or \((i,l)(j,k)\), with all the other diagonals of the diagram remaining invariant. So, applying the definition to construct the inclusion diagram, we have that each 1-dimensional face (corresponding to a codimension 1 diagram) in \(W\) is bounded by exactly 2 vertices (corresponding to the diagrams obtained by smoothing the meeting point in \(\beta\)).

Let us prove the second statement. Consider a diagram of codimension 2, denoted by \(\omega\). By definition there exist two critical points. Applying the smoothing half-Whitehead move onto one of the critical points gives two different possible diagrams of codimension 1 (this last statement follows from the first point above). So, applying the same arguments to the second critical point, implies that there exist four diagrams of codimension 1, incident to \(\omega\). In other words: there exist \(\{\beta_0, \beta_1, \beta_2, \beta_3\} \prec \omega\), where \(\text{codim}(\beta_i) = 1\) and \(i \in \{0, ..., 3\}\). The smoothing modification applied simultaneously to both critical points, gives four codimension 0 diagrams, all incident to \(\omega\): \(\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \prec \omega\). Applying (1) to every diagram of codimension 1 implies that there exist two diagrams of codimension 0, which are incident to each diagram of codimension 1. So, we obtain the following relations:

\[
\{\sigma_0, \sigma_1\} \prec \beta_0,
\{\sigma_1, \sigma_2\} \prec \beta_1,
\{\sigma_2, \sigma_3\} \prec \beta_2,
\{\sigma_3, \sigma_0\} \prec \beta_3.
\]

The construction of the inclusion diagram \(W\) from definition implies that we have a quadrangle. 

Below, an explicit construction of such inclusion diagrams is given. We start with the case \(d = 2\).

- Relations between the diagrams and Inclusion diagram for \(d = 2\),

In the decomposition of \(D^{\text{Pol}}_2\) by diagrams we have 4 generic diagrams and 4 diagrams of codimension 1 which are illustrated on the figure below.
Figure 6. Relations between diagrams for $d = 2$

We draw an edge between two vertices if the corresponding generic diagrams differ by a Whitehead move. There are no 2-faces, for this low dimensional case. Indeed, a composition of two Whitehead moves (on two pairs of diagonals of the opposite color) gives a diagram which does not satisfy the conditions of definition 1.

Therefore, the inclusion diagram $W$ is a quadrangle (see Fig 7) constituted from:
1. four vertices, corresponding to the generic diagrams,
2. four edges, corresponding to the codimension 1 diagrams

The inclusion diagram is illustrated in figure 7.

Figure 7. Inclusion diagram for $d = 2$

Corollary 10.
- Let $\sigma_0$ and $\sigma_1$ be two generic diagrams. If $\sigma_0$ and $\sigma_1$ are both incident to a diagram of codimension 1, then it is unique.
- Let $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ be four generic diagrams. Then, if they are incident to a diagram of codimension 2, then this codimension 2 diagram is unique.

4.2. One dimensional faces of the inclusion diagrams. Recall that a pair of generic diagrams incident to one diagram of codimension 1 are adjacent, in one Whitehead move. To illustrate this in the inclusion diagram we draw an edge connecting two vertices, (the vertices correspond to the generic diagrams, the edge to the diagram of codimension 1).

We propose to illustrate a few examples of generic diagrams which differ by a Whitehead move.

Example 2 (Whitehead moves between generic diagrams).
- Let $d = 4$. Starting from an $M$ diagram we have two different diagrams $F$ diagrams (see figure 8) in one Whitehead move.
Let $d=5$. Starting from the $S$ diagram, we can obtain an $F$ diagram as in the figure 9:

In order to give a precise geometric description of the inclusion diagram. We introduce some matrix notation.

**Definition 10.** We call matrix of a generic diagram the collection of the $d$ matrices $[i,j]$ and $[j,i]$ corresponding to the $d$ trees of the diagram.

**Remark 3.** Each diagram is associated to one unique matrix.

**Example 3.** Let $d=6$, starting from an $SS$ diagram we can have an $S$ diagram:

This sequence of Whitehead moves for $d=6$ can be written using matricial notation as follows:

$$\begin{bmatrix} 16 & 18 \\ 6 & 4 \end{bmatrix} \leftrightarrow \begin{bmatrix} 16 \\ 6 \end{bmatrix} \| \begin{bmatrix} 18 \\ 4 \end{bmatrix} \leftrightarrow \begin{bmatrix} 16 \| 18 \| 23 \\ 6 \| 4 \| 23 \end{bmatrix} \leftrightarrow \begin{bmatrix} 16 \| 18 \| 23 \\ 6 \| 4 \| 23 \end{bmatrix} \leftrightarrow \begin{bmatrix} 16 \| 18 \| 23 \\ 6 \| 4 \| 23 \end{bmatrix} \leftrightarrow \begin{bmatrix} 16 \\ 6 \end{bmatrix} \| \begin{bmatrix} 18 \\ 4 \end{bmatrix} \leftrightarrow \begin{bmatrix} 16 \| 18 \| 23 \\ 6 \| 4 \| 23 \end{bmatrix} \leftrightarrow \begin{bmatrix} 16 \| 18 \| 23 \\ 6 \| 4 \| 23 \end{bmatrix} \leftrightarrow \begin{bmatrix} 16 \\ 6 \end{bmatrix} \| \begin{bmatrix} 18 \\ 4 \end{bmatrix} .$$

**4.3. Higher dimensional faces of the inclusion diagram.**
4.3.1. General structure of inclusion diagrams. We consider the substructures of the inclusion diagram which correspond to the sets of elements, indexed by linearly ordered diagrams. The greatest element of the linearly ordered set corresponds to the face of the highest dimension of this substructure. This face is unique, since the greatest element of this linearly ordered set is unique.

**Definition 11.** Let \( NC, B, C \) be subsets of \( (\Sigma, \prec) \) endowed with a linear order relation.

1. We call \( NC(d) \) (or \( NC \) in short) the linearly ordered subset of \( (\Sigma, \prec) \) with one upper bound \( \sigma \) having \( d \) blue (resp. red) diagonals intersecting at one point. Two \( NC \) structures are said to be of the same color if the intersecting diagonals of the upper bound diagram are of the same color.
2. We call \( B \) the linearly ordered subset with one upper bound incident to diagrams lying in a pair of \( NC \) structures of the same color.
3. We call \( O \) the linearly ordered subset having one upper bound incident to generic diagrams lying in two \( NC \) structures of the opposite color. This upper bound is a diagram of codimension \( 2d - 4 \) having at least two intersection points of different colors.

We consider those ordered sets in \( W \) and call them sub-structures.

**Proposition 11.** For any \( d > 1 \) there exist four structures \( NC \) which are glued one to another by their \( M \) diagrams, so that each \( M \) diagram is incident to only two \( NC \) structures of the opposite colors and a pair of \( NC \) structures have at most one \( M \) diagram in common.

**Proof.** We know that there exists four \( NC \) structures in the inclusion diagram, where each \( NC \) structure containing among its set of vertices a pair of \( M \)-diagrams, for any \( d > 1 \). The vertices along which the \( NC(d) \) are glued to each other are the \( M \)-diagrams. An \( M \)-diagram is a vertex of valency \( 2(\binom{d}{2}) \), since there exist \( \binom{d}{2} \) possibilities to make one Whitehead move starting form an \( M \) diagram for the red (resp. blue) diagonals (theorem 7). So, this argument shows that an \( M \)-diagram is the intersection of a pair of \( NC \) structures of the opposite color. Suppose, that this common \( M \) diagram is denoted by \( M_1 \). Note, that there still remain two \( M \) diagrams in this pair of \( NC \) structures. We will show that the two remaining \( M \) diagrams are different. Indeed, in one \( NC \) structure only the red diagonals were modified; in the other \( NC \) structure only the blue diagonals were modified. Therefore, in the first \( NC \) structure, the ending \( M \) diagram has different blue diagonals than in \( M_1 \) and in the second \( NC \) structure the \( M \) diagram has different red diagonals than in \( M_1 \). Now, since there exist four \( M \) diagrams, the four \( NC \) structures are glued to each other by their \( M \)-diagrams, and a pair of \( NC \) structures have at most one \( M \)-diagram in common.

4.3.2. Inclusion diagram for \( d = 3 \). The figure 11 illustrates the inclusion diagram for \( d = 3 \), using the matricial notation.

The inclusion diagram contains two distinct parts:

1. The four substructures with black edges which are the \( NC \) substructures. They are the ones connecting a pair of \( M \) diagrams having a common set of short diagonals. These black substructures contain three vertices corresponding to \( F \) diagrams. The 3-face of the black substructure corresponds to the diagram of codimension 3 and it is incident to three 2-faces which correspond to the diagrams of codimension 2 having two inner vertices incident to edges of the same color.

2. The substructure with colored edges. This colored part of the inclusion diagram corresponds to the parts which appear in the construction 12. In particular, the vertices in this colored part are the \( S \) diagrams. The 2-faces in the interior part correspond to...
diagrams of codimension 2, with one inner vertex incident to 4 blue edges and the other one incident to 4 red edges.

Notice that in the case where \( d = 3 \) there are no \( B \) and \( O \) substructures.

This inclusion diagram is resumed by the construction in Figure 12.

- The first quadrangle 2-face connecting the following two \( F \) diagrams and two \( S \) diagrams
  \([\frac{2}{3}, \frac{3}{5}]\), \([\frac{1}{2}, \frac{8}{9}]\) and \([\frac{2}{3}, \frac{8}{9}]\), \([\frac{3}{4}, \frac{10}{11}]\) corresponds in Figure 11 to the blue vertical cycle. The \( F \) diagrams have one long red diagonal.

- The second quadrangle 2-face connecting the following two \( F \) diagrams and two \( S \) diagrams
  \([\frac{9}{3}, \frac{3}{5}]\), \([\frac{2}{3}, \frac{8}{9}]\), \([\frac{3}{4}, \frac{9}{10}]\), \([\frac{4}{3}, \frac{10}{11}]\) corresponds in Figure 11 to the blue horizontal cycle. The \( F \) diagrams have two long blue diagonals.

4.3.3. Inclusion diagram for \( d = 4 \). In this part, we show the construction of the inclusion diagram for \( d = 4 \) which gives a detailed description of non-empty intersections of the topological strata \( \mathcal{T}_\sigma \) in \( \mathcal{D}^{Pol}_4 \).

For convenience, in the case where \( d > 3 \), let us call \( Q \)-diagram the diagram \( S \otimes d - 2 \) and \( Q \)-pieces the union of topological strata \( \mathcal{A}_\sigma \) indexed by adjacent elements to a \( Q \)-diagram and having at least one long diagonal parallel to the ones of the \( Q \)-diagram and the adjacent \( M \)-diagrams.

Recall that the inclusion diagram contains two subparts: the substructure connecting two \( M \) consecutive diagrams; the complementary substructure. As for two \( F \) diagrams with opposite orientation i.e. \( F^+ = [\frac{1}{2}] \) and \( F^- = [\frac{3}{4}] \) we introduce the notation \( F^+ \) and \( F^- \). Similarly for \( S \) diagrams.
There are eight $Q$-pieces. Each $Q$-piece satisfies the relations in figure ??.

The diagrams contained in the $Q$-piece are described by:

$$F \leftrightarrow S \leftrightarrow F \otimes S \leftrightarrow S \otimes S \leftrightarrow F \otimes S \leftrightarrow S \leftrightarrow F.$$ 

Each $Q$-piece is connected to a consecutive one using the connection piece.

- Two successive $SS$ diagrams ($SS^+$ and $SS^-$) are related by the following commutative diagram:

$$
\begin{align*}
FF^+ & \leftrightarrow SS^+ \\
\uparrow & \uparrow \\
SS^- & \leftrightarrow FF^-,
\end{align*}
$$

where the $FF$ diagram is adjacent to the $SS$ diagram in the following way:
– Two consecutive $S$ diagrams ($S^+$ and $S^-$) are related by the following commutative diagram:

\[
\begin{align*}
F^+ & \leftrightarrow S^+ \\
\updownarrow & \quad \updownarrow \\
S^- & \leftrightarrow F^-.
\end{align*}
\]

• The last substructure is the connection between the four $M$ diagrams. This substructure is represented by

\[
\begin{array}{c}
\text{Figure 13. Four NC(4) structures} \\
\end{array}
\]

The double line between $M_1$ and $M_3$ given in figure 18. The other connection in

\[
\begin{array}{c}
\text{Figure 14. Detail of the connection between $M_1$ and $M_3$} \\
\end{array}
\]

figure 13 are of the same type.

The substructure between the diagrams $M_1$ and $M_3$ contains the following diagrams of codimension 3, 4 and 5:

\[
\begin{array}{c}
\text{Figure 15. Diagrams between $M_1$ and $M_3$} \\
\end{array}
\]
4.3.4. Construction of the decomposition for $d = 4$. We present by induction how to obtain the decomposition, for the first two consecutive $Q$-pieces (see figure 16 for an illustration). On the figure, green circles indicate which vertices are glued together. This is reminiscent of translation surfaces.

- Let us start with the first $Q$-piece having a $Q$-diagram $[1, 11, 3, 9, 2, 8, 3, 9, 2, 8]$. 

  (1) Deform in it the pair of red diagonals $(2, 8), (4, 6)$ in order to obtain $[1, 11] | 3, 9, 2, 8]$. So, one obtains an $FS$ diagram. Let us deform the long blue diagonal in the tree $[1, 7]$ with the short blue diagonal in the $M$ tree $[1, 11]$. This turns it into an $M$ tree and so, we have the $S$ diagram $[1, 11]$. The $F$ diagrams which are obtained from this $S$ diagram by a minimal number of deformation operations are $[1, 7] | 10, 0, 15, 9, 4, 11, 10, 11, 0, 9, 3]$. 

  (2) Deform in $Q$ the pair of red diagonals $(0, 10), (12, 14)$ in order to obtain $[1, 11] | 3, 9, 2, 8]$. So, one obtains an $SF$ diagram. Let us deform the long blue diagonal in the tree $[1, 7]$ with the short blue diagonal in the $M$ tree $[1, 11]$. This turns it into an $M$ tree and so, we have the $S$ diagram $[3, 9]$. The $F$ diagrams which are obtained from this $S$ diagram by a minimal number of deformation operations are $[1, 11] | 15, 9, 2, 8, 9, 3, 12, 3, 9, 3]$. 

- Let us consider the consecutive $Q$-piece.

  (1) The consecutive $Q$ piece contains the $Q$-diagram $[0, 10, 2, 8, 3, 9, 2, 8, 0, 10, 3, 9, 2, 8]$. and an adjacent $S$ diagram $[1, 7, 1, 7]$. obtained by deforming a pair of red diagonals giving the $FS$ diagram $[10, 1, 7, 1, 7]$, or a pair of blue diagonals giving an $FS$ diagram $[15, 9, 1, 7, 1, 7]$. The $S$ diagram $[1, 7, 1, 7]$. is adjacent after one deformation operation to the following $F$ diagrams $[15, 9, 1, 7, 1, 7, 1, 7, 1, 7]$. 

  (2) The $Q$-piece contains the $Q$-diagram $[15, 9, 2, 8, 3, 9, 2, 8, 1, 7, 1, 7]$. and an adjacent $S$ diagram $[15, 9, 2, 8, 3, 9, 2, 8, 1, 7, 1, 7]$. obtained by deforming a pair of blue diagonals giving the $FS$ diagram $[10, 9, 1, 7, 1, 7]$, or a pair of red diagonals giving an $FS$ diagram $[9, 7, 1, 7]$. The $S$ diagram $[15, 9, 2, 8, 3, 9, 2, 8, 1, 7, 1, 7]$. is adjacent after one deformation operation to the following $F$ diagrams $[15, 9, 2, 8, 3, 9, 2, 8, 1, 7, 1, 7]$. 

**Remark 4.** The $S$ diagram $[1, 7, 1, 7]$. is not consecutive with the $S$ diagram $[0, 10, 3, 9, 2, 8, 3, 9, 2, 8]$. since these two diagrams do not share a common set of diagonals. However they belong to consecutive $Q$-pieces and share a common set of $F$ diagrams being adjacent to them in one deformation step. A detailed figure is illustrates this below.

A more general approach is taken by considering the $i$-th pair of consecutive $Q$-pieces. This is obtained by adding $+2i$ to the numbers $\{2, 4, ... 4d\}$ (resp. $\{1, 3, ..., 4d - 1\}$), which correspond to the terminal vertices colored red (resp. blue), modulo $4d$, in the first $Q$-piece. This induces the following consecutive $Q$-pieces. One may apply to these $Q$-pieces the same remark as previously.

(1) Let the $Q$ diagram be $[1+2i, 11+2i, 3+2i, 9+2i, 0+2i, 10+2i, 2+2i, 8+2i]$. Deform in it the pair of red diagonals $(2 + 2i, 8 + 2i), (4 + 2i, 6 + 2i)$ in order to obtain $[1+2i, 11+2i, 3+2i, 9+2i, 0+2i, 10+2i, 2+2i, 8+2i]$. So, one obtains an $FS$ diagram. Let us deform the long blue diagonal in the tree $[9+2i, 3+2i]$ with the short blue diagonal in the $M$ tree $[7+2i, 1+2i]$ . This turns it into an $M$ tree and so, we have the $S$ diagram $[1+2i, 11+2i, 3+2i, 9+2i, 0+2i, 10+2i, 2+2i, 8+2i]$. The $F$ diagrams which are obtained from this $S$ diagram by a minimal number of deformation operations are $[7+2i, 1+2i, 0+2i, 0+2i, 10+2i, 1+2i, 11+2i, 11+2i, 4+2i, 0+2i, 0+2i, 10+2i, 1+2i, 4+2i, 11+2i, 11+2i, 4+2i, 0+2i, 0+2i, 10+2i, 1+2i, 4+2i, 11+2i, 11+2i, 4+2i, 0+2i, 0+2i, 10+2i, 1+2i, 4+2i, 11+2i, 11+2i, 4+2i, 0+2i, 0+2i, 10+2i, 1+2i, 4+2i, 11+2i, 11+2i, 4+2i, 0+2i, 0+2i, 10+2i, 1+2i, 4+2i, 11+2i, 11+2i, 4+2i, 0+2i, 0+2i, 10+2i, 1+2i, 4+2i, 11+2i, 11+2i, 4+2i, 0+2i, 0+2i, 10+2i, 1+2i, 4+2i, 11+2i, 11+2i, 4+2i].$
(2) Deform in $Q$ the pair of red diagonals $(0 + 2i, 10 + 2i)$, $(12 + 2i, 14 + 2i)$ in order to obtain $\begin{vmatrix} 11 + 2i \\ 1 + 2i \end{vmatrix}$ $\begin{vmatrix} 3 + 2i, 9 + 2i \\ 2 + 2i, 8 + 2i \end{vmatrix}$. So, one obtains an $SF$ diagram. Let us deform the long blue diagonal in the tree $\begin{vmatrix} 11 + 2i \\ 1 + 2i \end{vmatrix}$ with the short blue diagonal in the $M$ tree $\begin{vmatrix} 13 + 2i \\ 15 + 2i \end{vmatrix}$. This turns it into an $M$ tree and so, we have the $S$ diagram $\begin{vmatrix} 3 + 2i, 9 + 2i \\ 2 + 2i, 8 + 2i \end{vmatrix}$. The $F$ diagrams which are obtained from this $S$ diagram by a minimal number of deformation operations are $\begin{vmatrix} 15 + 2i \\ 9 + 2i \end{vmatrix}$ $\begin{vmatrix} 2 + 2i, 8 + 2i \\ 0 + 2i \end{vmatrix}$, or a pair of blue diagonals giving an $FS$ diagram $\begin{vmatrix} 15 + 2i \\ 9 + 2i \end{vmatrix}$ $\begin{vmatrix} 2 + 2i, 8 + 2i \\ 0 + 2i \end{vmatrix}$.

(3) The consecutive $Q$-piece contains the $Q$ diagram $\begin{vmatrix} 0 + 2i; 10 + 2i \\ 12 + 2i, 9 + 2i \end{vmatrix}$ $\begin{vmatrix} 1 + 2i, 7 + 2i \\ 2 + 2i, 8 + 2i \end{vmatrix}$ and an adjacent $S$ diagram $\begin{vmatrix} 10 + 2i; 9 + 2i \\ 8 + 2i, 7 + 2i \end{vmatrix}$, or a pair of red diagonals giving the $FS$ diagram $\begin{vmatrix} 15 + 2i; 9 + 2i \\ 8 + 2i, 7 + 2i \end{vmatrix}$.

(4) The $Q$-piece contains the $Q$ diagram $\begin{vmatrix} 0 + 2i; 10 + 2i \\ 12 + 2i, 9 + 2i \end{vmatrix}$ and an adjacent $S$ diagram $\begin{vmatrix} 10 + 2i; 9 + 2i \\ 8 + 2i, 7 + 2i \end{vmatrix}$, or a pair of blue diagonals giving the $FS$ diagram $\begin{vmatrix} 15 + 2i; 9 + 2i \\ 8 + 2i, 7 + 2i \end{vmatrix}$.

The $S$ diagram $\begin{vmatrix} 15 + 2i; 9 + 2i \\ 8 + 2i, 7 + 2i \end{vmatrix}$ is adjacent after one deformation operation to the following $F$ diagrams $\begin{vmatrix} 7 + 2i; 11 + 2i \\ 6 + 2i, 10 + 2i \end{vmatrix}$ $\begin{vmatrix} 3 + 2i; 9 + 2i \\ 0 + 2i, 10 + 2i \end{vmatrix}$.  

![Figure 16](image_url)

**Figure 16.** The first two consecutive $Q$-pieces for $d = 4$

4.3.5. *Intersection of substructures.* In the case of $d > 3$, there are three different substructures $NC, B$ and $O$. These do not appear in the case of $d = 3$. We present explicitly the substructures of the inclusion diagram, for $d = 4$.

**Definition 12.** We call *vertical* (resp. *horizontal*) *bridge* (denoted by $B$) the substructure connecting two opposite $F$ diagrams, where the $F$ diagram contains a single $F$ tree having one long blue (resp. red) diagonal.
The figure 17 presents a substructure of the inclusion diagram $W$ which consists of a quadrangle of vertices $M_1, ..., M_4$ where the edges represent the $NC$ structure and the vertices $M_1, ..., M_4$ correspond to the four $M$ diagrams. This quadrangle is divided into four regions by a green and red line. These lines represent respectively the vertical and horizontal bridges and connect two $F$ diagrams oppositely oriented. The intersection of the red and green line is denoted by $S$, and corresponds to the common vertex to the vertical and horizontal bridges.

There exist 8 horizontal bridge structures and 8 vertical bridge structures. So, the vertical bridges are labeled from 1 to 8. A bridge substructure is drawn in the figure 18.

5. Decomposition invariant under Coxeter groups

5.1. Adjacence, chambers and galleries. The aim of this section is to introduce a new type of decomposition built from the *elementa* and thus different from a topological stratification. We show that this new decomposition is invariant under some Coxeter group.

In order to prove the main statement, we study the adjacency relations between the $M$, $F$, $FS$ and $S$ diagrams. We introduce a method relying on the symmetry groups of those diagrams and adjacency of diagrams. Note that a diagram can be differently interpreted as a superimposition.
of two concentric regular $2d$-gons where the red diagram is a rotation of $\frac{\pi}{d}$ about the center of the polygon of the blue diagram. The symmetry group of regular polygons are the dihedral groups.

Via this method of construction, we show explicitly the existence of chambers and galleries in the decomposition. The method of construction is as follows:

1. (a) The starting point of this procedure is to consider the diagram $S^d$, which we call a $Q$-diagram for simplicity. This is a superimposition of two concentric monochromatic diagrams (one blue, one red). The blue and red diagrams have $d - 2$ long diagonals and the red diagram is the blue diagram turned through $\frac{\pi}{d}$ relative to the center of the diagram.

(b) The blue (resp. red) diagram has a mirror line being parallel to the long diagonals of the diagram. Indexing the long diagonals from 1 to $d - 2$, note that if $d$ is even then this mirror line lies between the diagonals $\frac{d-2}{2}$ and $\frac{d}{2}$, at equidistance from them, and parallel to them. If $d$ is odd then, the mirror lies on the diagonal $\frac{d-1}{2}$.

(c) There exists a finite group acting independently on both monochromatic diagrams, which is of order of $2d$ defined by $\langle r | r^{2d} = Id \rangle$, $r = \frac{\pi}{2d}$. The order of the group of rotations acting on the $Q$-diagram is also of order $2d$. Thus rotating the $Q$-diagram by an angle $\pi$ (i.e. $2d$ rotations by an angle $\frac{\pi}{2d}$) gives the identity.

We construct, via Whitehead moves repeatedly applied on this diagram, a sequence of its adjacent strata. This leads to the second step.

2. Apply Whitehead moves onto the $Q$-diagram such that in each of the new generic diagrams there remains at least one long blue (or red) diagonal parallel to the mirror line. In one Whitehead move, from an $F$ diagram the last generic diagrams obtained are the $M$ diagrams. The union of all the strata indexed by those diagrams defines a $Q$-piece of the stratification. There are $2d$ such $Q$-pieces, due to the fact that there exist $2d$ rotations of a $Q$-diagram about an angle of $\frac{\pi}{2d}$.

3. A $Q$-piece is said to be adjacent to another one if they have the same blue (resp. red) monochromatic diagrams and red (resp. blue) diagrams which differ by a rotation of $-\frac{\pi}{2d}$ about the center of the polygon.

REMARK 5. Consider a diagram given by the concentric superimposition of a blue and a red diagram, which lies in a $Q$-piece. Using elementary geometry, it is clear that a rotation through $-\frac{\pi}{2d}$ of the red diagram about the center of the polygon is the equivalent as using a mirror line in the blue diagram, parallel to the long diagonals in the $Q$-piece to do the transformation.

EXAMPLE 4. Detail of a $Q$-piece for $d = 4$ is given in figure 19.

To prove that the stratification is invariant under a Coxeter group, we describe the construction using an algorithm. We define each diagram by a certain type of matrix which encodes the end vertices of its diagonals.

A “$Q$” diagram is associated to a matrix $[L_0 \ L_1]$ where $L_0$ (resp. $L_1$) describes the monochromatic blue (resp. red) diagram in terms of pairs of terminal vertices connected by long blue (resp. red) diagonals; the matrix satisfies the following condition:

- the pair of terminal vertices of two crossing diagonals lie in the same column of the matrix;
- the first and the last columns of the matrix contains the pair of terminal vertices of the shortest long diagonals;
- two adjacent columns contain the terminal vertices of diagonals bounding the same region in $D \setminus \sigma$. 
Lemma 12. Consider the $2d$ regular polygon formed from the blue (resp. red) terminal vertices in a diagram. Let $r$ be the rotation through $\frac{\pi}{2d}$ about the center of the polygon. Let $(i, j)$ be a diagonal of the polygon. So, $r^k : (i, j) \mapsto (i + 2k, j + 2k) \mod 4d$.

Proof. Let us proceed by induction on the angle $\frac{k\pi}{2d}$.

1) Base case. Let $k = 1$. Let us rotate by $r = \frac{\pi}{2d}$ the diagonal $(i, j)$. Since $i, j \in \{1, 3, \ldots, 4d - 1\}$ (resp. $\{2, 4, \ldots, 4d\}$) are the vertices of a regular $2d$-gon, then the rotation maps the diagonal $(i, j)$ to the diagonal $(i + 2, j + 2)$.

2) Induction case. Suppose that for a given $k$ in $\{1, \ldots, 2d\}$ the statement is true: the diagonal $(i, j)$ rotated by an angle $r^k = \frac{k\pi}{2d}$ is mapped onto the diagonal $(i + 2k, j + 2k)$.

Let us show that for $k + 1$ the statement is true. We rotate about an angle $\frac{(k+1)\pi}{2d}$ the diagonal $(i, j)$. By induction hypothesis rotating by $\frac{k\pi}{2d}$ maps $(i, j)$ onto $(i + 2k, j + 2k)$. Rotating the diagonal $(i + 2k, j + 2k)$ by an angle of $\frac{(k+1)\pi}{2d}$ maps it by (1) onto the diagonal $(i + 2(k + 1), j + 2(k + 1))$. $\square$

Lemma 13. Let $Q_1$ and $Q_2$ be two “$Q$”-diagrams, lying in a pair of adjacent $Q$-pieces. Then, their associated matrices verify

$$Q_1 = \begin{bmatrix} L_0 \\ L_1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} L_1 \\ L_2 = L_1 - 1 \end{bmatrix},$$

where $L_1 - 1$ means that all the indexes of the end vertices of the diagonals in $L_2$ are those of $L_1$, minus one, modulo $4d$.

Proof. Suppose that $Q_1 = \begin{bmatrix} L_0 \\ L_1 \end{bmatrix}$. By hypothesis, since $Q_2$ lies in an adjacent $Q$-piece, it is the superimposition of the blue (resp. red) monochromatic diagram of $Q_1$ and of the red monochromatic diagram in $Q_1$, rotated about $\frac{k\pi}{2d}$. Suppose, that $L_1$ is the monochromatic diagram, common
to \(Q_1\) and \(Q_2\). From the lemma \ref{lem:rotation} we have that \(L_1 = L_0 - 1\) in \(Q_2\). Now, since the monochromatic diagrams differ by a rotation about \(\pi/4d\) we have that \(L_1 = L_0 - 1\) and thus \(L_2 = L_0 - 1\), modulo \(4d\).

**Example 5.** In the case of \(d = 3\), the adjacent \(Q\)-diagrams are illustrated in figure \ref{fig:decomposition} for \(d = 3\).

![Decomposition for \(d = 3\).](attachment:image.png)

The adjacent \(Q\)-diagrams for \(d = 4\) is given in appendix, Fig \ref{fig:decomposition22}.

In the following part, one Whitehead move corresponds to the modification of one generic diagram into another one, through an operation on no more than a pair of diagonals of the same color.

**Lemma 14.** Let \(Q_1\) and \(Q_2\) be two “\(Q\)”-diagrams, lying in adjacent \(Q\)-pieces. Then, these diagrams are glued to each other via a \(F^{\otimes d-2}\) diagram.

**Proof.** We induct on \(d\), where \(d > 3\) (lower degrees are irrelevant since the \(Q\) diagrams do not exist).

- **Base case \(d = 4\).** The \(Q\) diagram \([L_0]_1\) has two long blue diagonals and two long red diagonals. Suppose, that one of the \(S\) is given by \([i]\). The pairs of red and blue long diagonals bound the same 2-face in \(D\), which implies that the Whitehead operation can be applied to a pair of diagonals. Suppose, without loss of generality, that we modify the red diagonals. The second pair of (blue) long diagonals remains fixed. Deforming the pair of long red diagonals induces in the new diagram a pair of short red diagonals. This gives a new diagram of type \(F^{\otimes 2}\). We have thus showed the existence of a sequence connecting a \(Q\) diagram to a \(F^{\otimes 2}\). Now, we show that starting from \(F^{\otimes 2}\) there exists a sequence of Whitehead moves giving the adjacent \(Q\) diagram of matrix \([L_0]_1\)\. Consider the pair of short red diagonals, lying in the boundary of a 2-face in \(D\) and different from the previous one. Deform it by a Whitehead move. One obtains a diagram \(SF_1\), where the \(S\) is given by \([i]\). Deform the remaining pair of red short diagonals. This gives a
diagram $S^d$ with matrix $[L_{L_{d+1}}]$.  

- **Induction case.** Assume that the statement is true for a pair of $S^d$ diagrams lying in adjacent $Q$-pieces. We will prove that for two adjacent $S^d$ diagrams with $d - 1$ pairs of $S$ trees there exists a sequence of deformations containing a diagram of type $F^d$. Consider the degree $d + 1$ diagram $S^{d+1}$. Since the assumption of the lemma is true for $S^{d+2}$, then there exists a sequence of deformations from $S^{d+1}$ to a diagram $S^{d+2}$, where $S$ is a tree given by the crossing of the shortest long diagonals in the diagram. It remains to deform the pair of diagonals (one long, the other one short) lying in the boundary of a 2-face in $\mathbb{D}$, so as to obtain $F^{d+1}$. Now starting from $F^{d+1}$, there exists by induction hypothesis a sequence of Whitehead moves giving the diagram $S^{d+2}$. Finally, one more Whitehead move induces the path from the $F$ tree to the $S$ one. This gives the $S^{d+1}$ diagram. 

$\square$

**Remark 6.** We digress on the generic diagrams obtained in one Whitehead move from a $Q$-diagram. Then, one deformation of the $Q$ diagram gives an $SF$ diagram. A second Whitehead move gives an $S$ diagram with narrow $S$ trees. Let us enumerate all the three possible cases. Notice that a $Q$ diagram has two $S$ trees made of the shortest long diagonals in the diagram.

1. $S^{d-2}$ is modified in one Whitehead move into a diagram $S^{d-1}F$ (or to $FS^{d-3}$). This is obtained by modifying a pair of diagonals (one in $M$ and one in adjoining $S$, both being of the same color). From this diagram, using a Whitehead move which deforms the long diagonal in $F$ with the diagonal in $M$ of the corresponding color, we obtain $S^{d-3}M$, all the $S$ trees being aligned in the disk.

2. $S^{d-2}$ is deformed in one deformation step to $S^{d-4}FF$ (or to $FFS^{d-4}$). This is obtained by modifying a pair of long diagonals: one being the shortest long diagonal of the diagram, the other one being adjoining to it. Deforming once more the pair of long diagonals in $FF$ reduces to the trees two $MM$. So the resulting diagram is $S^{d-4}MM$.

3. $S^{d-2}$ is deformed in one deformation step to $S...SFFS...S$ where the number of $S$ is $d - 4$. This is obtained by deforming a pair of adjoining long diagonals (which are not the shortest long diagonals of the diagram). Deforming the remaining long diagonals in $FF$ gives a pair of trees $MM$. This gives the diagram $S...SMM...S$.

All the other diagrams are obtained iterating the procedure described above onto each of the remaining collection of narrow $S$ trees in the diagram. Applying this method continuously, we obtain diagrams with only one narrow $S$ tree and in particular we can obtain the $F$ diagrams constructed from the shortest long diagonals.

### 5.2. Main theorems and their proofs.

**Lemma 15.** The group of symmetries of a $Q$-piece is $\langle s_1 | s_1^2 = 1 \rangle$.

**Proof.** Let us describe the construction of the $Q$-piece. Recall that the monochromatic diagrams in a $Q$-diagram have a mirror line, parallel to the long diagonals. We show that for any diagram obtained by a composition of Whitehead moves operating on $\{d^2 - 2 + 1, \ldots, d - 2\}$ there exists its reflection diagram about the mirror, where the Whitehead moves operate on $\{1, \ldots, \frac{d^2 - 2}{2}\}$.

Let us apply the Whitehead move separately on the blue and red diagrams of the $Q$-diagram. Then, the composition of Whitehead moves applied on the set of blue (or red) diagonals $\{\frac{d^2 - 2}{2} + 1, \ldots, d - 2\}$ (or $\{1, \ldots, \frac{d^2 - 2}{2}\}$) of the $Q$-diagram gives a new diagram.
A rotation of \( \pi \) of this new monochromatic blue (or red) diagram through the center of the polygon gives a diagram which is its reflection about the mirror line.

So, all deformations on \( \{1, \ldots, \frac{d-2}{2}\} \) have a corresponding deformation \( \{\frac{d-2}{2} + 1, \ldots, d-2\} \).

Since any diagram of the \( Q \)-piece is obtained from this procedure, a \( Q \)-piece is invariant under an automorphism group of order 2.

**Lemma 16.** If \( d \) is even, then the \( Q \)-piece has two reflections, in the sense of Coxeter groups.

*Proof.* From lemma [13], it is known that in a \( Q \)-piece there exists one reflection hyperplane, coming across the \( Q \)-diagram. Now, consider \( F \) diagrams in the \( Q \)-piece such that there is one \( F \) tree of long diagonal \((i, j)\) where \( j = i + 6 \mod 4d \). Its reflection gives an \( F \) diagram with one \( F \) tree of long diagonal \((i + 2d, j + 2d) \mod 4d \). Notice that the short diagonals of the opposite color are the same if \( d \) is even. We modify \((i, j)\) by a Whitehead move with the diagonal \((i + 2, i + 4) \mod 4d\), so that it becomes \((i, i + 2)\). By symmetry, we proceed on \((i + 2d, j + 2d)\) and \((i + 2 + 2d, i + 4 + 2d)\), so that the Whitehead operation gives the pair \((i + 2d, i + 2 + 2d)\) \((i + 4 + 2d, i + 6 + 2d) \mod 4d\). Hence, we obtain the same \( M \) diagrams. Therefore, there is a second reflection hyperplane coming across the \( M \) diagrams in the \( Q \)-piece.

**Theorem 17.** Let \( Z \) be a couple of adjacent \( Q \)-pieces and let \( Y \) be the stratification. Then \( p : Y \rightarrow Z \) is a Galois covering, with Galois group of order \( d \).

*Proof.* Let us define \( p : Y \rightarrow Z \) where \( Y = \bigcup_{\alpha \in G} Z^\alpha \) and \( G \) is a cyclic group of finite order \( d \), where \( Y \) is connected. Each inverse image \( p^{-1}(A_\sigma) \) of \( A_\sigma \in Z \) is constituted from classes, having diagrams which are equivalent up to a rotation of \( \sigma \). So, any action \( \rho \in \text{Aut}_Z(Y) \) on \( \sigma \) gives a diagram \( \sigma' \) which belongs to \( p^{-1}(A_\sigma) \). We show that for \( Y \times Z = \{(z, z') \in Y \times Y, p(z) = p(z')\} \), the map

\[
\phi : G \times Y \rightarrow Y \times Z, \quad (g, z) \mapsto (z, gz)
\]

is a homeomorphism.

- The map is a bijection. First let us show that the map is injective. Consider \( \phi(g, z) = \phi(g', z') \) i.e. \((z, gz) = (z', g'z') \in Y \times Z Y\). Then, \( z = z' \) and \( g' = g \) and in particular \((g, z) = (g', z') \in G \times Y\), so the map is injective. The map is surjective since for every \((z, gz) \in Y \times Z Y\), there exists at least one element in \((g, z) \in G \times Y\) such that \( \phi((g, z)) = (z, gz) \).

- The map is bicontinuous because the group \( G \) continuously acts on \( Y \).

So, the map \( p : Y \rightarrow Z \) satisfies the definition of a Galois covering for an order \( d \) Galois group.

**Lemma 18.** Let \( S_1 = \{L_{0+2d}^{i+1}, L_{0-2d+1}\} \) and \( S_2 = \{L_{0-2d}^{i+1}, L_{0+2d-1}\} \) be two \( S \) diagrams. Then, \( S_1 = S_2 \).

*Proof.* Consider any diagonal \((i, j) \in \mathbb{L}_0\), where \( i, j \) are integers of the same parity modulo \( 4d \). We have \( i \equiv j \mod 4d \) if and only if \( i + 2d \equiv j \mod 4d \). Proceeding similarly for \( j \) we have that \( S_1 = S_2 \).

**Stratification by the \( Q \)-procedure** Let \( G_1 \) be the left part of the \( Q \)-piece about the reflection axis and \( G_2 \) the right part of the \( Q \)-piece. Let \( S_1 \) resp. \( S_2 \) be the diagram in the left (resp. right) part of the \( Q \)-piece, such that \( S_1 = \{L_{0+2d}^{i+1}, L_{0-2d+1}\}, S_2 = \{L_{0-2d}^{i+1}, L_{0+2d-1}\} \). Let us discuss the matrix relations between each pair of adjacent \( Q \)-piece by induction.

1. Take any diagram in \( G_1 \) of the first \( Q \)-piece, associated to \( \{L_{0+2d}^{i+1}, L_{0-2d+1}\} \) and its rotated diagram by an angle \( \pi \) which is \( \{L_{0+2d+1}^{i+1}, L_{0-2d-1}\} \) in \( G_2 \). Then, applying lemma [13], the consecutive diagrams are respectively of type \( \{L_{0+2d}^{i+1}, L_{0-2d+1}\} \) and \( \{L_{0-2d}^{i+1}, L_{0+2d-1}\} \).
(2) Take any diagram in the subgraph $G_1$ of the $i$-th $Q$-piece, associated to $\begin{bmatrix} L_0 \end{bmatrix}_{L_0-1}$ and its rotated diagram by an angle $\pi$ which is $\begin{bmatrix} L_0-i+1+2d \end{bmatrix}_{L_0-i+2d}$, in $G_2$. Then, applying lemma 13 the consecutive diagrams are respectively of type $\begin{bmatrix} L_0 \end{bmatrix}_{L_0-i}$ and $\begin{bmatrix} L_0 \end{bmatrix}_{L_0-(i+1)+2d}$.

(3) For the $2d-1$-th $Q$-piece, we have $\begin{bmatrix} L_0-(2d-1) \end{bmatrix}_{L_0-2d}$ and $\begin{bmatrix} L_0+2d-(2d-2) \end{bmatrix}_{L_0+2d-1-(2d-1)}$. The consecutive diagrams are respectively $\begin{bmatrix} L_0 \end{bmatrix}_{L_0-2d}$ and $\begin{bmatrix} L_0+2d-2d \end{bmatrix}_{L_0+2d-1-2d}$.

By Lemma 15 we have $L_0-2d = L_0 + 2d \mod 4d$. So, diagrams in $G_1$ of the $2d-1$-th $Q$-piece are glued by a connection to the $G_2$, of the first $(2d-1)$ $Q$-piece. This switches the positions of the parts $G_1$ and $G_2$ in the $Q$-pieces numbered from 2 to $4d-1$, compared to the positions of $G_1$ and $G_2$ in the $Q$-pieces numbered from 1 to $2d-1$. So, the identity is obtained after $4d$ consecutive $Q$-pieces.

Below we present a part of the collection of glued $Q$-pieces, the connections are represented by thin vertical double arrows, deformations by a thick horizontal double arrow. The final structure is obtained by glueing the top and the bottom of Fig 21 along the long and thin horizontal arrows.

---

\[
\begin{array}{c}
\begin{bmatrix} L_0 \end{bmatrix}_{L_0-1} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} Q_1 \end{bmatrix} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} L_0+2d \end{bmatrix}_{L_0+2d-1+2d} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\begin{bmatrix} L_0-1 \end{bmatrix}_{L_0-2} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} Q_2 \end{bmatrix} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} L_0-1+2d \end{bmatrix}_{L_0-2+2d} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\begin{bmatrix} L_0-2 \end{bmatrix}_{L_0-3} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} Q_3 \end{bmatrix} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} L_0-2+2d \end{bmatrix}_{L_0-3+2d} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\vdots \quad \quad \quad \quad \quad \quad \vdots \\
\begin{bmatrix} L_0-(2d-2) \end{bmatrix}_{L_0-(2d-1)} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} Q_{2d-1} \end{bmatrix} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} L_0+2d-(2d-2) \end{bmatrix}_{L_0+2d-1-(2d-1)} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
\begin{bmatrix} L_0-(2d-1) \end{bmatrix}_{L_0-2d} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} Q_{2d} \end{bmatrix} \leftrightarrow \ldots \leftrightarrow \begin{bmatrix} L_0+2d-2d \end{bmatrix}_{L_0+2d-1-2d} = \begin{bmatrix} L_0 \end{bmatrix}_{L_0-1} \\
\end{array}
\]

---

**Figure 21.** Detail of the collection of glued $Q$-pieces.

**Theorem 19.** The decomposition of $D_{Pol}$ in $4d$ ($d>2$) Coxeter-chambers, is induced by the topological stratification of $D_{Pol}$ in elements and is invariant under the the Coxeter group given by the presentation:

\[ W = \langle (s_i)^{2d+1} \rangle (s_i s_j)^{2d} = Id \rangle. \]

**Proof.** Let us recall from lemma 15 that each $Q$-piece is invariant under an automorphism group of order 2. Moreover, it follows from the proposition 17 that the stratification is invariant under a cyclic group of order $2d$. We show that the stratification is invariant under the Coxeter group $W = \langle (s_i)^{2d+1} \rangle (s_i s_j)^{2d} = Id \rangle$, using the $Q$-algorithm.

Between any pair of adjacent $Q$-pieces there exists a reflection hyperplane. Let us consider the pair of $i$-th and $(i + 1)$ adjacent $Q$-pieces, where $i \in \mathbb{Z}_{2d}$.

The blue (resp. red) monochromatic diagrams in those adjacent $Q$-pieces are the same. The red monochromatic diagrams in the adjacent $Q$-piece are symmetric about the vertical reflection axis of the blue (resp. red) monochromatic diagram. Thus the diagrams in the adjacent $Q$-piece
have the same relations as in the initial \(Q\)-piece, up to symmetry about a reflection axis. The next \(Q\)-piece, indexed \((i + 2)\), is obtained from the diagrams in the \(i\)-th \(Q\)-piece by a rotation of \(\frac{\pi}{2d}\). Therefore, we have \(2d\) reflection hyperplanes. This implies that there exist \(2d\) horizontal reflection hyperplanes and one vertical reflection hyperplane. 

\[\square\]

**Corollary 20.** For \(d > 2\), there exist \(4d\) chambers in this stratification such that one \(Q\)-piece is the union of two chambers.

**Corollary 21.** Let \(W\) be the inclusion diagram. Then \(W\) is invariant under the Klein group \(Z_2 \times Z_2\).

We now state the following conjecture:

**Conjecture:** The poset of diagrams can be realized as the poset of cells of a CW-complex.

### 6. Concluding Remark

In this paper, we have investigated a new stratification of the \(d\)-th unordered configuration space of the complex plane space, using its natural relation with the space of complex monic degree \(d > 0\) polynomials in one variable with simple roots. The decomposition of \(P_d\) has been done considering *drawings of polynomials* - objects reminiscent of Grothendieck’s *dessin’s d’enfant*. These objects are decorated graphs, properly embedded in the complex plane, obtained by taking the inverse image under a polynomial \(P\) of the real and imaginary axes. A detailed study of those objects has been given in section 2. A stratum, (called *elementa*) of the decomposition is defined as a set of polynomials having drawings belonging to the same isotopy class, relatively to the asymptotic directions and which is described as a diagram (for simplicity diagrams are represented as circular diagrams). An important result of the stratification is that the topological closure of a stratum indexed by the diagram \(\sigma\) is given by the combinatorial closure of the diagram \(\sigma\) [7], which motivated our investigations in this paper concerning the combinatorial closure of an elementa. The study of this stratification lead to the introduction of the notion of an inclusion diagram which is a precious tool describing geometrically not only the neighborhood of each stratum but the entire stratification itself. It is also known that this inclusion diagram is the nerve in the sense of Čech, of the cover in [7, 8]. We showed that the stratification for \(d > 2\) is invariant under a Coxeter group with \(4d\) chambers, \(2d\) horizontal reflection hyperplanes and one vertical reflection hyperplane. To conclude, the paths and loops in \(P_d\) have been defined in a new way, using diagrams. Moreover, from the deep relation \(\pi_1(P_d) = B_d\) [5, 17, 18] we showed that a braid can be defined, using a sequence of diagrams, obtained one from another via Whitehead move. From the result, showed in this paper, a braid can be defined as a path (loop) in one chamber. This result is also very advantageous for the calculation of the Čech cohomology of this space, since it reduces significantly the complexity.
Appendix A. The adjacent $Q$-diagrams for $d = 4$

Figure 22. Decomposition for $d=4$
References

[1] E. Artin, Theory of braids, Ann. of Math. 48, No 1 (1947), 101-126.
[2] N. A’Campo, Signatures of monic polynomials, arXiv:1702.05885 [math.AG]
[3] J. S. Birman, K. H. Ko, S. J. Lee, A new approach to the and conjugacy problem in the braid groups, Advances in Mathematics bf 139, (1996), 322-353.
[4] N. Bourbaki. Lie groups Lie algebras Elements of Mathematics, Chapters 7-9 Springer Berlin (2005)
[5] E. Brieskorn, Sur les groupes de tresses, Sem. Bourbaki, 401, novembre 1971 (Lecture Notes in Math., No 317, 1973, 21-44).
[6] E. Čech, Théorie générale de l’homologie quelconque Fund. Math. (1932) vol 19, pp149-183
[7] N.C. Combe, On a new cell decomposition of a complement of the discriminant variety: application to the cohomology of braid groups, PhD Thesis (2018).
[8] N.C. Combe Čech cover of the complement of the discriminant variety Arxiv: 1808.
[9] N.C. Combe, V. Jugé Counting bi-colored A’Campo forests, Arxiv : 1702.07672
[10] N. Combe, Geometric classification of real ternary octahedral quartics, Discrete Computational Geometry, vol 60, Issue 2, Springer (2018)
[11] N. Combe, On Coxeter algebraic varieties : the geometry of $CB_n$ quartics, Math. Semesterberichte, (September 2018)
[12] H. S. M. Coxeter, W. O.J Moser, Generators and relations for discrete groups ,Third edition, (1972) Springer-Verlag Berlin Heidelberg New York.
[13] P. Deligne. Les immeubles des groupes de tresses généralises Inv. math. vol. 17 (1972), 273-302
[14] F. A. Garside, The braid group and other groups, Quart. J. Math. Oxford Ser. (2) 20 1969 235-254
[15] A. Grothendieck. Esquisse d’un Programme, [archive] (1984). https://fr.wikipedia.org/wiki/Alexander Grothendieck
[16] Y. Ihara Automorphisms of pure sphere braid groups and Galois representations, Birkhauser, Progress in Mathematics 87 ,Basel 1991, 353-373.
[17] E. Fadell, L. Neuwirth, Configurations spaces, Math Scand,10 (1962), 111-118.
[18] R. Fox, L. Neuwirth, The braid groups, Math Scand, 10 (1962), 119-126.
[19] P. Lochak, L. Schneps, The Grothendieck-Teichmüller group and automorhisms of braid groups, LMS LNM 200, Cambridge Univ. Press ,1994.

Sorbonne Université, 4 Place Jussieu, 75005 Paris
E-mail address: noemiecombe11@gmail.com