On the Structure of Normal Hausdorff Operators on Lebesgue Spaces

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Dedicated to the memory of my teacher Evgenii Alekseevich Gorin

Received January 18, 2019; in final form, February 25, 2019; accepted May 16, 2019

ABSTRACT. We consider generalized Hausdorff operators and introduce the notion of the symbol of such an operator. Using this notion, we describe, under some natural conditions, the structure and investigate important properties (such as invertibility, spectrum, and norm) of normal generalized Hausdorff operators on Lebesgue spaces over $\mathbb{R}^n$. As an example we consider Cesàro operators.

KEY WORDS: Hausdorff operator, Cesàro operator, symbol of an operator, normal operator, spectrum, spectral representation.

DOI: 10.1134/S0016266319040038

1. Introduction and Preliminaries

Hausdorff operators on Lebesgue spaces originated from some classical summation methods. They were introduced by Hardy [6, Chapter XI] on the unit interval (see also [15]) and by Georgakis and, independently, Liflyand and Moricz on the whole real line [5], [11]. Its natural multidimensional extension is the operator

$$(\mathcal{H}f)(x) = \int_{\mathbb{R}^m} \Phi(u) f(A(u)x) \, du, \quad x \in \mathbb{R}^n,$$

where $A(u)$ is a family of $n \times n$ matrices satisfying the condition $\det A(u) \neq 0$ almost everywhere in the support of $\Phi$. It was introduced by Brown and Moricz [2] on the Lebesgue spaces and by Lerner and Liflyand [8] on the real Hardy space. This class of operators has attracted considerable attention in recent decades. It includes some important examples, such as the Hardy operator, the adjoint Hardy operator, the Cesàro operator, and their multidimensional analogues. The survey articles [10] and [4] contain main results on Hausdorff operators and bibliography up to 2014. To our knowledge, all known results on general Hausdorff operators refer only to their boundedness in various settings (exceptions are the papers [1] and [15], in which some spectra are calculated for the one-dimensional case). But this question, being solved positively, naturally entails other questions, such as whether the operator is invertible and what its inverse is, what the spectrum of this operator is, and a number of others. To deepen the investigations, it is natural to begin with Hausdorff operators on Hilbert spaces (the case of $L^p$ spaces is more difficult because of the lack of the Plancherel theorem; cf. [13]). As is well known, one of the classes of operators on Hilbert spaces most important for applications is that of normal operators (and in, particular, self-adjoint and unitary operators). This work is mainly devoted to this case. We will consider a generalization of a Hausdorff operator and introduce the notion of the symbol of this operator. This notion is crucial for our considerations. Our aim is to describe the structure and investigate important properties (such as invertibility, spectrum, and norm) of generalized normal Hausdorff operators on Lebesgue spaces over $\mathbb{R}^n$. Using the Mellin transform, we prove under the condition of positive definiteness that a normal Hausdorff operator on $L^2(\mathbb{R}^n)$ is unitarily equivalent to the operator of coordinate-wise multiplication by its symbol in the space $L^2(\mathbb{R}^n, \mathbb{C}^{2^n})$. This is a very useful analogue of the spectral theorem for the class of operators under consideration. In the absence of positive definiteness, we use a matrix symbol to describe the structure of a Hausdorff operator in the one-dimensional case. As examples, discrete Hausdorff operators and Cesàro operators are considered. The study of the
one-dimensional Cesàro operator on $L^2$ was pioneered by Brown, Halmos, and Shields [1], and the $L^p$ case was considered in [2]. The results of the work were announced in [14].

**Definition 1** [12]. Suppose given a σ-compact topological space $(Ω, μ)$ endowed with a positive regular Borel measure μ, a locally integrable function Φ on Ω, and a μ-measurable family $(A(u))_{u ∈ Ω}$ of real $n × n$ matrices defined almost everywhere on the support of Φ and satisfying the condition $det(A(u)) ≠ 0$. We define the Hausdorff operator $H = H_{Φ, A}$ with kernel $Φ$ by

$$(H_{Φ, A}f)(x) = ∫_Ω Φ(u)f(A(u)x)dμ(u), \quad \text{where } x ∈ ℝ^n \text{ is a column vector.}$$

As mentioned by Hardy [6, Theorem 217], in the case where $n = 1$ and $Ω = [0, 1]$, the Hausdorff operator possesses some regularity property. The multidimensional version of his result looks as follows.

**Proposition 1.** Let the conditions of Definition 1 be fulfilled. For the transformation $H_{Φ, A}$ to be regular, i.e., such that $H_{Φ, A}f(x) → l$ whenever $f ∈ C(ℝ^n)$ and $f(x) → l$ as $x → ∞$, it is necessary and sufficient that $∫_Ω Φ(u)dμ(u) = 1$.

**Proof.** If $f(x) = 1$, then $H_{Φ, A}f(x) = ∫_Ω Φ(u)dμ(u)$. Thus, $∫_Ω Φ(u)dμ(u) = 1$ is a necessary condition.

To prove sufficiency, first note that if $A(u)$ is invertible, then $x → ∞$ implies $A(u)x → ∞$ (this follows from the inequality $∥A(u)x∥ ≥ ∥x∥∥A(u)^{-1}∥$). But if, in addition, $f ∈ C(ℝ^n)$ and $f(x) → l$, then $f$ is bounded and therefore $H_{Φ, A}f(x) → l$ by the Lebesgue theorem.

Proposition 1 motivates the condition $det(A(u)) ≠ 0$ in Definition 1.

### 2. The Structure of Normal Hausdorff Operators on $L^2$

We need the following lemmas to prove our main result.

**Lemma 1** [12] (cf. [6, (11.18.4)], [2]). Let $|det(A(u))|^{-1/p}Φ(u) ∈ L^1(Ω)$. Then the operator $H_{Φ, A}$ is bounded on $L^p(ℝ^n)$ ($1 ≤ p < ∞$) and

$$||H_{Φ, A}|| ≤ ∫_Ω |Φ(u)||det(A(u))|^{-1/p}dμ(u).$$

This estimate is sharp (see Theorem 1 below).

Below we will consider $L^p(ℝ^n)$ as a subspace of $L^p(ℝ^n)$.

**Corollary 1.** If, under the conditions of Lemma 1, every $A(u)$ maps $ℝ^n_+$ into itself, then the operator $H_{Φ, A}$ is bounded on $L^p(ℝ^n)$ ($1 ≤ p < ∞$).

**Lemma 2** (cf. [2]). Under the conditions of Lemma 1 the adjoint of the Hausdorff operator on $L^p(ℝ^n)$ has the form

$$(H_{Φ, A}^*)f(x) = ∫_Ω Φ(v)|det(A(v))|^{-1}f(A(v)^{-1}x)dμ(v).$$

Thus, the adjoint of a Hausdorff operator is also Hausdorff.

**Lemma 3.** Under the conditions of Definition 1 let $(det(A(u))^{-1/2}Φ(u) ∈ L^1(Ω)$. Then the Hausdorff operator $H_{Φ, A}$ is normal on $L^2(ℝ^n)$ if the $n × n$ matrices $A(u)$ form a commuting family.

**Proof.** Indeed, in view of Lemma 2 this follows from the equalities

$$(H_{Φ, A}H_{Φ, A}^*)f(x) = ∫_Ω ∫_Ω Φ(u)Φ(v)|det(A(v))|^{-1}f(A(v)^{-1}A(u)x)dμ(v)dμ(u),$$

$$(H_{Φ, A}^*H_{Φ, A})f(x) = ∫_Ω ∫_Ω Φ(u)Φ(v)|det(A(v))|^{-1}f(A(u)A(v)^{-1}x)dμ(u)dμ(v),$$

Lemma 1, and Fubini’s theorem.
In the sequel, for $x \in \mathbb{R}^n$, $\alpha = (\alpha_j) \in \mathbb{C}^n$, and $r \in \mathbb{R}$, we put $dx = dx_1 \cdots dx_n$ and $|x|^{r+\alpha} := |x_1|^{r+\alpha_1} \cdots |x_n|^{r+\alpha_n}$.

The following notion is crucial for our investigations.

**Definition 2.** Let the conditions of Definition 1 be fulfilled, and let, for each $u \in \Omega$, $a(u) := (a_1(u), \ldots, a_n(u))$ be the family of eigenvalues (with multiplicities) of the matrix $A(u)$. Suppose that $|\text{det} A(u)|^{-1/p} \Phi(u) \in L^1(\Omega)$. Then we refer to the function

$$\varphi(s) := \int_{\Omega} \Phi(u)|a(u)|^{-1/p-is} \, d\mu(u), \quad s = (s_j) \in \mathbb{R}^n,$$

as the symbol of the Hausdorff operator $\mathcal{H}_{\Phi, A}$ on $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$.

In Definition 2 we assume that $|a(u)|^{-1/p-is} := \prod_{j=1}^n |a_j(u)|^{-1/p-is_j}$, where $|a_j(u)|^{-1/p-is_j} := \exp((1/p - is_j) \log |a_j(u)|)$.

It is clear that $\varphi$ is bounded and continuous on $\mathbb{R}^n$.

As we will see, properties of a Hausdorff operator are closely related to properties of its symbol.

**Example 1** (cf. [2]). Consider the $n$-dimensional Cesàro operator $\mathcal{E}_n$:

$$(\mathcal{E}_n f)(x_1, \ldots, x_n) = \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \ldots, t_n) \, dt_1 \cdots dt_n.$$

This is a bounded Hausdorff operator on $L^p(\mathbb{R}^n)$ (and on $L^p(\mathbb{R}^n_+)$) for $1 < p < \infty$. Here $\Omega = [0, 1]^n$ is endowed with the Lebesgue measure, $\Phi = 1$, and $A(u) = \text{diag}[u_1, \ldots, u_n]$. Its symbol is

$$\varphi(s) = \int_0^1 \cdots \int_0^1 \prod_{j=1}^n u_j^{-1/2-is_j} \, du_1 \cdots du_n = \prod_{j=1}^n \left( \frac{1}{2} - is_j \right)^{-1}, \quad s = (s_j) \in \mathbb{R}^n.$$

**Example 2** (discrete Hausdorff operators). Let $\Omega = \mathbb{Z}_+$, and let $\mu$ be the counting measure. Then Definition 1 takes the form

$$(\mathcal{H}_{\Phi, A} f)(x) = \sum_{k=0}^\infty \Phi(k) f(A(k)x), \quad f \in L^p(\mathbb{R}^n).$$

Assume that $\sum_{k=0}^\infty |\Phi(k)| |\text{det} A(k)|^{-1/p} < \infty$. Then $\mathcal{H}_{\Phi, A}$ is bounded on $L^p(\mathbb{R}^n)$. The symbol of $\mathcal{H}_{\Phi, A}$ is

$$\varphi(s) = \sum_{k=0}^\infty \Phi(k) |\text{det} A(k)|^{-1/p}|a(k)|^{-is},$$

where $s = (s_j) \in \mathbb{R}^n$ and the principal values of the exponential functions are taken. Since this series converges on $\mathbb{R}^n$ absolutely and uniformly, $\varphi$ is uniformly almost periodic.

Note that if $(A(u))_{u \in \Omega}$ is a commuting family of real self-adjoint $n \times n$ matrices, then there is an orthogonal $n \times n$ matrix $C$ and a family of real diagonal matrices $A'(u) = \text{diag}[a_1(u), \ldots, a_n(u)]$ such that $A'(u) = C^{-1} A(u) C$ for $u \in \Omega$. If each $A(u)$ is positive definite, then the corresponding Hausdorff operator $\mathcal{H}_{\Phi, A'}$ on $L^p(\mathbb{R}^n)$ is bounded provided that $\prod_{j=1}^n |a_j(u)|^{-1/p} \Phi(u) \in L^1(\Omega)$.

The next theorem describes, under the assumption of the positive definiteness of $A(u)$, the structure and properties of normal Hausdorff operators on $L^2(\mathbb{R}^n)$.

**Theorem 1.** Let $(A(u))_{u \in \Omega}$ be a commuting family of real nonsingular positive definite $n \times n$ matrices, and let $(\text{det} A(u))^{-1/2} \Phi(u) \in L^1(\Omega)$. Then the Hausdorff operator $\mathcal{H}_{\Phi, A}$ in $L^2(\mathbb{R}^n)$ with symbol $\varphi$ is unitarily equivalent to the operator $M_\varphi$ of coordinate-wise multiplication by $\varphi$ in the space $L^2(\mathbb{R}^n, \mathbb{C}^{2n})$ of $\mathbb{C}^{2n}$-valued functions. More precisely, $\mathcal{H}_{\Phi, A} = \mathcal{U}^{-1} M_\varphi \mathcal{U}$, where $\mathcal{U}$ is a unitary operator between $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n, \mathbb{C}^{2n})$ which does not depend on $\Phi$. In particular, the following assertions hold.

(i) The operator $\mathcal{H}_{\Phi, A}$ is invertible if and only if $\inf |\varphi| > 0$; in this case, the inverse $\mathcal{H}_{\Phi, A}^{-1}$ is unitarily equivalent to the operator $M_{1/\varphi}$ on $L^2(\mathbb{R}^n, \mathbb{C}^{2n})$.  

263
(ii) Let $\mathcal{H}_{\hat{\Phi}}$ and $\mathcal{H}_\Phi$ be two Hausdorff operators over the same measure space $(\Omega, \mu)$ such that $(A(u), B(v))$ is a commuting family of real nonsingular positive definite $n \times n$ matrices $(u$ and $v$ run over the supports of $\Phi$ and $\Psi$, respectively), and suppose that $(\det A(u))^{-1/2}$ and $(\det B(v))^{-1/2}$ are in $L^1(\Omega)$. Then the product $\mathcal{H}_\Phi \mathcal{H}_\psi$ is unitarily equivalent to the operator $M_{\phi\psi}$ on $L^2(\mathbb{R}^n, \mathbb{C}^2)$ (see [3]). Moreover, if we assume that $\Phi(u)$ and $\Psi(v)$ commute and are positive definite, there is an orthogonal $n \times n$ matrix $C$ and a family of diagonal positive definite matrices $A'(u) = \text{diag}(a_1(u), \ldots, a_n(u))$ such that $A'(u) = C^{-1}A(u)C$ for almost all $u \in \Omega$. All functions $a_j(u)$ are positive, and det $A(u) = a_1(u) \cdots a_n(u)$. Consider the unitary operator $\hat{C} f(x) := f(Cx)$ on $L^2(\mathbb{R}^n)$. It is easy to verify that

$$\mathcal{H}_\Phi = \hat{C}^{-1} \mathcal{H}_{\hat{\Phi}} \hat{C}.$$ 

Thus, the operator $\mathcal{H}_\Phi$ is unitarily equivalent to $\mathcal{H}_{\hat{\Phi}}$.

Let $U_1, \ldots, U_{2^n}$ be the open hyperoctants in $\mathbb{R}^n$. Then $L^2(\mathbb{R}^n)$ can be decomposed into an orthogonal sum of subspaces as $L^2(\mathbb{R}^n) = \bigoplus_{j=1}^{2^n} L^2(U_j)$, and each $L^2(U_j)$ is $\mathcal{H}_{\hat{\Phi}}$-invariant. For every $j$, consider the operator $\mathcal{H}_j$ on $L^2(\mathbb{R}^n)$ which coincides with the restriction $\mathcal{H}_{\hat{\Phi}}|L^2(U_j)$ on $L^2(U_j)$ and is equal to zero on the orthogonal complement of $L^2(U_j)$. Then

$$\mathcal{H}_{\hat{\Phi}} = \bigoplus_{j=1}^{2^n} \mathcal{H}_j$$

(the orthogonal sum of operators).

Consider the following modified $n$-dimensional Mellin transform for the $n$-hyperoctant $U_j$ ($j = 1, \ldots, 2^n$, $s \in \mathbb{R}^n$):

$$\mathcal{M}_j f(s) := \frac{1}{(2\pi)^{n/2}} \int_{U_j} |x|^{-1/2+is} f(x) \, dx.$$ 

Each $\mathcal{M}_j$ is a unitary operator between $L^2(U_j)$ and $L^2(\mathbb{R}^n)$. This can be easily deduced from the Plancherel theorem for the $n$-dimensional Fourier transform by using an exponential change of variables (see [3]). Moreover, if we assume that $|y|^{-1/2} f(y) \in L^1(U_j)$, then making use of Fubini’s theorem and integrating with the substitution $x = A'(u)^{-1} y = (y_1/a_1(u), \ldots, y_n/a_n(u))$ yield

$$\mathcal{M}_j \mathcal{H}_j f(s) = \frac{1}{(2\pi)^{n/2}} \int_{U_j} |x|^{-1/2+is} f(x) \, dx \int_\Omega \Phi(u) f(A'(u)x) \, d\mu(u)$$

$$= \frac{1}{(2\pi)^{n/2}} \int_\Omega \Phi(u) \, d\mu(u) \int_{U_j} |x|^{-1/2+is} f(A'(u)x) \, dx$$

$$= \int_\Omega \Phi(u) a(u)^{-1/2-is} \, d\mu(u) \int_{U_j} |y|^{-1/2+is} f(y) \, dy$$

$$= \varphi(s) \langle \mathcal{M}_j f(s), s \in \mathbb{R}^n \rangle.$$ 

It follows by continuity that, for all $f \in L^2(U_j)$,

$$\mathcal{M}_j \mathcal{H}_j f = \varphi \mathcal{M}_j f.$$
So, \( \mathcal{H}_j' \) is unitarily equivalent to the operator \( M'_\varphi \) of multiplication by \( \varphi \) in \( L^2(\mathbb{R}^n) \), and

\[
\mathcal{H}_{\Phi,A'} = \bigoplus_{j=1}^{2^n} M_j^{-1} M'_\varphi M_j.
\]

Consider the operator \( \nu' \) from \( L^2(\mathbb{R}^n) \) to the space \( L^2(\mathbb{R}^n) \oplus \cdots \oplus L^2(\mathbb{R}^n) \) (2\( n \) orthogonal summands), which is isomorphic to the Hilbert space \( L^2(\mathbb{R}^n, \mathbb{C}^{2n}) \), defined by \( \nu' | L^2(U_j) := M_j \) \((j = 1, \ldots, 2^n)\). This operator is unitary, and

\[
\nu'^{-1} M_\varphi \nu' = \bigoplus_{j=1}^{2^n} M_j^{-1} M'_\varphi M_j.
\]

Thus, \( \mathcal{H}_{\Phi,A} = \nu'^{-1} M'_\varphi \nu' \), and the proof of the first statement is complete.

(i) Evidently, the operator \( \mathcal{H}_{\Phi,A} \) is invertible if and only if \( M_\varphi \) is invertible, i.e., if \( \inf |\varphi| > 0 \), and in this case \( \mathcal{H}_{\Phi,A}^{-1} \) is unitarily equivalent to \( M_{1/\varphi} \).

To prove (ii), first note that there exists an orthogonal matrix \( \Phi \) such that both \( A'(u) = C^{-1} A(u) C \) and \( B'(v) := C^{-1} B(v) C \) are diagonal. Then the previous proof shows that \( \nu' \mathcal{H}_{\Phi,A} \nu'^{-1} = M_\varphi \) and \( \nu' \mathcal{H}_{\Phi,B} \nu'^{-1} = M_\psi \) for some unitary operator \( \nu' \) between \( L^2(\mathbb{R}^n) \) and \( L^2(\mathbb{R}^n, \mathbb{C}^{2n}) \) (which depends only on \( n \) and \( C \)), and (ii) follows.

(iii) It is clear that \( \sigma(\mathcal{H}) = \sigma(M_\varphi) \), \( \sigma_p(\mathcal{H}) = \sigma_p(M_\varphi) \), and

\[
M_\varphi - \lambda = \bigoplus_{1}^{2^n} (M'_\varphi - \lambda) := (M'_\varphi - \lambda) \oplus \cdots \oplus (M'_\varphi - \lambda), \quad \lambda \in \mathbb{C}.
\]

Since a finite orthogonal sum of operators is invertible if and only if each summand is invertible (see, e.g., [7, p. 439]), it follows that \( \sigma(M_\varphi) = \sigma(M'_\varphi) \). Moreover,

\[
\sigma_p(M_\varphi) = \sigma_p\left( \bigoplus_{1}^{2^n} M'_\varphi \right) = \bigoplus_{1}^{2^n} \sigma_p(M'_\varphi) = \sigma_p(M'_\varphi).
\]

Next, \( \sigma_r(\mathcal{H}) = \sigma_r(M_\varphi) \), as proved above. Let \( \lambda \notin \sigma_p(M_\varphi) \). Since \( \sigma_r(M'_\varphi) = \emptyset \), we obtain

\[
\text{cl}(\text{Im}(M_\varphi - \lambda)) = \bigoplus_{1}^{2^n} \text{cl}(\text{Im}(M'_\varphi - \lambda)) = \bigoplus_{1}^{2^n} L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n, \mathbb{C}^{2n}),
\]

where \( \text{cl}(S) \) denotes the closure of the subset \( S \subset L^2(\mathbb{R}^n) \) and \( \text{Im}(T) \) denotes the image of the operator \( T \). So, \( \sigma_r(M_\varphi) = \emptyset \), and (iii) follows.

(iv) Since the adjoint of \( \mathcal{H}_{\Phi,A} \) is also of Hausdorff type (with \( A(u)^{-1} \) instead of \( A(u) \)), this statement follows from Definition 2 and Lemma 2.

Finally, the equality \( ||\mathcal{H}_{\Phi,A}|| = \sup |\varphi| \) follows from (iii) and the normality of \( \mathcal{H}_{\Phi,A} \). If, in addition, \( \Phi(u) > 0 \) for \( \mu \)-almost all \( u \), then we have \( \sup |\varphi| = \varphi(0) \), which completes the proof.

**Corollary 2.** Under the assumptions of Theorem 1 the Hausdorff operator \( \mathcal{H}_{\Phi,A} \) is self-adjoint (positive, unitary) on \( L^2(\mathbb{R}^n) \) if and only if its symbol \( \varphi \) is real-valued (respectively, \( \varphi \geq 0 \), \( |\varphi| = 1 \)).

**Proof.** Indeed, the spectral theorem for a normal operator implies that the operator \( \mathcal{H}_{\Phi,A} \) is self-adjoint (positive, unitary) if and only if \( \sigma(\mathcal{H}_{\Phi,A}) = \text{cl}(\varphi(\mathbb{R}^n)) \) is contained in \( \mathbb{R} \) (respectively, in \( \mathbb{R}_+ \), in \( \mathbb{T} = \{|z| = 1\} \)).

**Corollary 3.** Under the assumptions and in the notation of Theorem 1(ii) the operator \( \mathcal{H}_{\Phi,B} \) is the inverse of \( \mathcal{H}_{\Phi,A} \) if and only if \( \varphi \psi = 1 \).

**Corollary 4.** Under the assumptions of Theorem 1 the operator \( \mathcal{H}_{\Phi,A} \) on \( L^2(\mathbb{R}^n_+) \) is unitarily equivalent to the operator of multiplication by \( \varphi \) in \( L^2(\mathbb{R}^n_+) \).

This was established in the course of proving Theorem 1.
Example 3. Consider the Cesàro operator on \(L^2(\mathbb{R}^n)\) (see Example 1). It is normal, and its symbol is \(\varphi(s) = \prod_{j=1}^n (1/2 - is_j)^{-1}\).

The statement (iv) of Theorem 1 implies \(\|\mathcal{C}_n\| = 2^n\) (and by Corollary 4 the same is true on \(L^2(\mathbb{R}_+^n)\)).

According to the statement (iii) of Theorem 1, \(\sigma_r(\mathcal{C}_n) = \emptyset\), \(\sigma_p(\mathcal{C}_n) = \emptyset\), and \(\sigma(\mathcal{C}_n)\) equals the closure of the range of \(\varphi\). Consider the circle \(S := \{(1/2 - it) : t \in \mathbb{R}\} = \{z \in \mathbb{C} : |z - 1| = 1\} \). We have \(\sigma(\mathcal{C}_n) = \{z_1 \ldots z_n : z_j \in S, j = 1, \ldots, n\}\). Let \(z_j \in S\), \(z_j = 1 + ei\theta_j\) \((\theta_j \in [-\pi, \pi])\). Then \(|z_j| = 2 \cos(\theta_j/2)\) and \(\arg(z_j) = \theta_j/2\) \((j = 1, \ldots, n)\). We set \(z = z_1 \ldots z_n\), \(r := |z|\), and \(\theta := \arg(z)\). Then \(\theta = (\theta_1 + \cdots + \theta_n)/2\) and \(r = 2^n \prod_{j=1}^n \cos(\theta_j/2)\). Using the identity

\[
\prod_{j=1}^n \cos(\alpha_j) = \frac{1}{2^{n-1}} \sum_{\varepsilon \in \{-1,1\}^n} \cos(\varepsilon \cdot \alpha) = \frac{1}{2^{n-1}} \sum_{\varepsilon' \in \{-1,1\}^{n-1}} \cos(\alpha_1 + \varepsilon' \cdot \alpha'),
\]

where \(\alpha = (\alpha_j)_{j=1}^n = (\theta_j/2)_{j=1}^n\), \(\alpha' = (\theta_j/2)_{j=2}^n\), and \(\cdot\) denotes dot product, we conclude that

\[
\sigma(\mathcal{C}_n) = \{re^{i\theta} : r \leq 2(2^{n-1} - 1 + \cos \theta), \theta \in [-\pi, \pi]\}.
\]

(By Corollary 4 the same is true for the Cesàro operator on \(L^2(\mathbb{R}_+^n)\)).

In particular, for the bivariate Cesàro operator, we have

\[
\sigma(\mathcal{C}_2) = \{re^{i\theta} : r \leq 2(1 + \cos \theta), \theta \in [-\pi, \pi]\},
\]

i.e., \(\sigma(\mathcal{C}_2)\) is the region in \(\mathbb{C}\) bounded by a cardioid.

Results of Example 3 are consistent with classical results of [1], where the case \(n = 1\) and the space \(L^2(\mathbb{R}_+)\) were considered (see p. 137 therein; in [1] the Cesàro operator on \(L^2(\mathbb{R}_+)\) was denoted by \(\mathcal{C}_\infty\)).

Example 4. Consider the \((C, k)\) mean of a function \(f \in L^2(\mathbb{R})\):

\[
(C, k)f(x) = \frac{k}{x^k} \int_0^x (x - t)^{k-1} f(t) \, dt \quad (k > 0).
\]

This is an operator of Hausdorff type with \(n = 1\), \(\Omega = [0, 1]\) endowed with the Lebesgue measure, \(\Phi(u) = k(1 - u)^{k-1}\), and \(A(u) = u\). Its symbol is

\[
\varphi(s) = \int_0^1 k(1 - u)^{k-1} u^{-1/2 - is} \, du = \frac{\Gamma(k + 1) \Gamma(1/2 - is)}{\Gamma(k + 1/2 - is)}.
\]

Using [16, Sec. 12.13, Example 1], we can rewrite this as

\[
\varphi(s) = \prod_{l=1}^\infty \frac{l(k + l - 1/2 - is)}{(k + l)(l - 1/2 - is)}.
\]

Therefore, the spectrum \(\sigma((C, k))\) is the curve given by the complex equation

\[
z = \prod_{l=1}^\infty \frac{l(k + l - 1/2 - is)}{(k + l)(l - 1/2 - is)}, \quad s \in \mathbb{R} \cup \{\infty\}.
\]

Moreover, the last representation of \(\varphi\) implies also that \(\max_\mathbb{R} |\varphi| = \varphi(0)\). It follows in view of Theorem 1 that

\[
\|\mathcal{(C, k)}\| = \sqrt{\pi} \frac{\Gamma(k + 1)}{\Gamma(k + 1/2)}
\]

(since \(\Phi \geq 0\), this follows also from Theorem 1 (v)).

If, in addition, \(k \in \mathbb{N}\), then we have \(\|\mathcal{(C, k)}\| = k! 2^k / 1 \cdot 3 \cdots (2k - 1)\) and

\[
\varphi(s) = k! \prod_{j=0}^{k-1} \left( j + \frac{1}{2} - is \right)^{-1}.
\]

266
Example 5. Let $\mathcal{H}_\varphi$, and $\varphi$ be as in Example 2. If $p = 2$ and $a_j(k) > 0$, then Theorem 1 implies $\sigma(\mathcal{H}_\varphi) = \text{cl}(\varphi(\mathbb{R}^n))$. Assume, in addition, that $A(k) = A^k$, where $A \not= E$ is an $n \times n$ matrix with positive eigenvalues $\lambda_1, \ldots, \lambda_n$ (with multiplicities taken into account). Let $z \in \overline{D} := \{ |z| \leq 1 \}$. The function $F(z) := \sum_{k=0}^{\infty} \Phi(k)(\det A)^{k/2}z^k$ belongs to the commutative Banach algebra $\mathcal{A}^+$ of functions on $\overline{D}$ with absolutely convergent Taylor series, and $\varphi(s) = F(\exp(-i \sum_j s_j \log \lambda_j))$. Now Theorem 1 implies $\sigma(\mathcal{H}_\varphi;A) = F(T)$. It follows also that $\|\mathcal{H}_\varphi;A\| = \sup_T |F|$ and $\mathcal{H}_\varphi;A$ is invertible if and only if $\inf_T |F| > 0$. Since the last condition is equivalent to $\inf_{\overline{D}} |F| > 0$, under this condition the function $G := 1/F$ belongs to the algebra $\mathcal{A}^+$ as well (this follows from Gelfand’s theory).

Let $G(z) = \sum_{k=0}^{\infty} b(k)z^k$ be the Taylor expansion for $G \ (z \in \overline{D})$. Then the inverse of $\mathcal{H}_\varphi;A$ is the Hausdorff operator

$$\mathcal{H}_\varphi^{-1}A f(x) := \sum_{k=0}^{\infty} b(k)(\det A)^{k/2}f(A^kx).$$

Indeed, in this case, all conditions of Theorem 1 (ii) for the pair $\mathcal{H}_\varphi;A$, $\mathcal{H}_\varphi^{-1}A$ are fulfilled, and the symbol of the last operator is

$$\psi(s) = \sum_{k=0}^{\infty} b(k)a(k)^{-is} = G\left(\exp\left(-i \sum_j s_j \log \lambda_j\right)\right) = \frac{1}{F(\exp(-i \sum_j s_j \log \lambda_j))} = \frac{1}{\varphi(s)}.$$

Thus, the result follows from Corollary 3.

The following theorem shows that if the matrices $A(u)$ are not positive definite, then the symbol is a matrix function. In the sequel, for the case $n = 1$, we will write $a(u)$ instead of $A(u)$ and put $\Omega_\pm := \{ u \in \Omega : a(u) \geq 0 \}$.

Definition 3. Let $n = 1$, and let $|a(u)|^{-1/p}\Phi(u) \in L^1(\Omega)$. We define the matrix symbol of a one-dimensional Hausdorff operator $\mathcal{H}_{\varphi,a}$ by

$$\phi = \begin{pmatrix} \varphi_+ & \varphi_- \\ \varphi_- & \varphi_+ \end{pmatrix},$$

where $\varphi_+$ (or $\varphi_-$) denotes the symbol of the one-dimensional Hausdorff operator with the same $\Phi$ and $a$ as $\mathcal{H}_{\varphi,a}$ but with $\Omega_+$ (respectively, with $\Omega_-$) instead of $\Omega$.

Note that the (scalar) symbol $\varphi$ of $\mathcal{H}_{\varphi,a}$ equals $\varphi_+ + \varphi_-$. We put also $\varphi^* := \varphi_+ - \varphi_-$.  

Theorem 2. Let $|a(u)|^{-1/p}\Phi(u) \in L^1(\Omega)$. A one-dimensional Hausdorff operator $\mathcal{H}_{\varphi,a}$ on $L^2(\mathbb{R})$ with matrix symbol $\phi$ is normal and unitarily equivalent to the operator $M_\phi$ of multiplication by $\phi$ in the space $L^2(\mathbb{R}, \mathbb{C}^2)$. In particular,

(i) $\sigma(\mathcal{H}_{\varphi,a}) = \text{cl}(\varphi(\mathbb{R}) \cup \varphi^*(\mathbb{R}))$;

(ii) $\|\mathcal{H}_{\varphi,a}\| = \max(\sup_{\mathbb{R}} |\varphi|, \sup_{\mathbb{R}} |\varphi^*|)$.

Proof. We set $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$. We will consider the spaces $L^2(\mathbb{R}_\pm)$ as subspaces of $L^2(\mathbb{R})$. We define the operator

$$H_{11}f(s) = \int_{\Omega_+} \Phi(u)f(a(u)x) d\mu(u)$$

on $L^2(\mathbb{R}_-)$ and let $H_{22}$ be the operator given by the same formula but acting on $L^2(\mathbb{R}_+)$. Then $\varphi_+$ is the symbol of both $H_{11}$ and $H_{22}$. Similarly, we define the operator

$$H_{12}f(s) = \int_{\Omega_-} \Phi(u)f(a(u)x) d\mu(u)$$

acting between $L^2(\mathbb{R}_-)$ and $L^2(\mathbb{R}_+)$ and $H_{21}$ given by the same formula but acting between $L^2(\mathbb{R}_-)$ and $L^2(\mathbb{R}_+)$. Then $\varphi_-$ is the symbol of both $H_{12}$ and $H_{21}$. For $f \in L^2(\mathbb{R})$, we put $f_\pm := f 1_{\mathbb{R}_\pm} (1_{E} \mid E \mid_{\mathbb{R}_\pm})$. 

267
for all \( \lambda \) functions valued \((\text{Int} u r n, t h i s \ i s \ q u i v a l e n t t o \ i n f \)

This proves the first statement of the theorem.

(ii) This follows from (i) and the normality of \( f \).

Consider the unitary operator \( \mathcal{U} := \text{diag}[\mathcal{M}_2, \mathcal{M}_1] \) between the space \( L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+) \) (which is isomorphic to \( L^2(\mathbb{R}) \)) and \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C}^2) \). Then

\[
\mathcal{H}_{\Phi,a} = \mathcal{U}^{-1} \begin{pmatrix} \mathcal{M}_{\phi^+} & \mathcal{M}_{\phi^-} \\ \mathcal{M}_{\phi^-} & \mathcal{M}_{\phi^+} \end{pmatrix} \mathcal{U} = \mathcal{U}^{-1} \mathcal{M}_\phi \mathcal{U}.
\]

This proves the first statement of the theorem.

(i) For \( \lambda \in \mathbb{C} \), the operator \( \lambda - \mathcal{H}_{\Phi,a} \) is unitarily equivalent to the operator \( M_{\lambda - \phi} \) on \( L^2(\mathbb{R}, \mathbb{C}^2) \).

The matrix \( \lambda - \phi \) is invertible if and only if \( \Delta(s) := ((\lambda - \phi))_2 - \phi_- (s)^2 \neq 0 \) for all \( s \in \mathbb{R} \). In this case,

\[
(\lambda - \phi)^{-1} = \frac{1}{\Delta} \begin{pmatrix} \lambda - \phi^+ & -\phi_- \\ -\phi_+ & \lambda - \phi^- \end{pmatrix}.
\]

So, \( \lambda - \mathcal{H}_{\Phi,a} \) is invertible if and only if \( \Delta(s) \neq 0 \) for all \( s \in \mathbb{R} \) and \( M_{\lambda - \phi}^{-1} \) acts on \( L^2(\mathbb{R}, \mathbb{C}^2) \). The functions \( M_{\lambda - \phi}^{-1}(f,0)^T = ((\lambda - \phi_+) / \Delta)f, (\phi_- / \Delta)f) \) and \( M_{\lambda - \phi}^{-1}(0,f)^T \) belong to \( L^2(\mathbb{R}, \mathbb{C}^2) \) for all \( f \in L^2(\mathbb{R}) \) if and only if \( \sup |(\lambda - \phi_+)/\Delta| < \infty \) and \( \sup |\phi_- / \Delta| < \infty \) (the superscript \( T \) denotes transposition). The last two inequalities are equivalent to \( \sup |(\lambda - \phi_+)/\Delta + \phi_- / \Delta| < \infty \).

In turn, this is equivalent to \( \inf |(\lambda - ((\phi_+ + \phi_-)(\mathbb{R}) \cup (\phi_+ - \phi_-)(\mathbb{R}))) > 0 \), or \( \lambda \notin \overline{((\phi_+ + \phi_-)(\mathbb{R}) \cup (\phi_+ - \phi_-)(\mathbb{R}))} \).

(ii) This follows from (i) and the normality of \( \mathcal{H}_{\Phi,a} \) (Lemma 3).

**Corollary 5.** Let operators \( \mathcal{H}_{\Phi,a} \) and \( \mathcal{H}_{\Psi,b} \) meet the conditions of Theorem 2.

1. The operator \( \mathcal{H}_{\Phi,a} \) is self-adjoint (positive, unitary) on \( L^2(\mathbb{R}) \) if and only if \( \phi_\pm \) are real-valued (respectively, nonnegative, \( |\phi| = 1 \) and \( |\phi^*| = 1 \)).

2. The operator \( \mathcal{H}_{\Phi,a} \) is invertible if and only if \( \inf |\phi| > 0 \) and \( \inf |\phi^*| > 0 \); in this case, \( \mathcal{H}_{\Phi,a}^{-1} \) is unitarily equivalent to the operator \( M_{\phi^{-1}} \) on \( L^2(\mathbb{R}^n, \mathbb{C}^2) \).

3. The operator \( \mathcal{H}_{\Phi,a} \mathcal{H}_{\Psi,b} \) is unitarily equivalent to the operator \( M_{\phi \psi} \) on \( L^2(\mathbb{R}^n, \mathbb{C}^2) \) (\( \phi \) and \( \psi \) are the matrix symbols of \( \mathcal{H}_{\Phi,a} \) and \( \mathcal{H}_{\Psi,b} \), respectively).

**Acknowledgments**

The author thanks the referee for quickly and carefully reading the manuscript and comments that have helped to improve the presentation.

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