Unified Dark Energy and Dark Matter from Dynamical Spacetime Cosmology

Thesis submitted in partial fulfillment of the requirements for the degree of “DOCTOR OF PHILOSOPHY”

By

DAVID BENISTY

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BEER SHEVA, ISRAEL
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This work was carried out under the supervision of

PROF. EDUARDO GUENDELMAN

The Physics Department

FACULTY OF NATURAL SCIENCES

Approved by the advisor:
Approved by the Dean of the Kreitman School of Advanced Graduate Studies:

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BEER SHEVA, ISRAEL
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1 Research Student’s Affidavit

David Benisty, whose signature appears below, hereby declare that:

- I have written this Thesis by myself, except for the help and guidance offered by my Thesis Advisors.
- The scientific materials included in this Thesis are products of my own research, culled from the period during which I was a research student.

Date: 20.01.2021
Student’s name: David Benisty
Signature:
2 Abstract

A model of unified dark matter and dark energy based on a Dynamical Spacetime Theory (DST) is studied. By introducing a Dynamical Spacetime vector field $\chi_{\mu}$, a conservation of an energy momentum tensor $T^{\mu\nu}_{(\chi)}$ emerges. The action allows for two different potentials, while one represents a dark energy. For constant potentials, the cosmological solution yields a non-singular bouncing solutions that rapidly approaches the $\Lambda$CDM model. The Dynamical Time corresponds to the cosmic time as well. The theory fits with the late time expansion data of the Universe. With higher dimensions a mechanism for inflation and compactification appears, with exponential growth for some dimensions and exponential contraction of the others. By demanding that the Dynamical Spacetime vector field be a gradient of a scalar the DST becomes a theory with diffusive interacting, which asymptotically returns to the $\Lambda$CDM model as a stable point. These formulations lead to scenarios which address our understanding about the origin of the Universe.
3 List of Papers

The following papers are included in the thesis:

1. D. Benisty and E. I. Guendelman,
   “Unified dark energy and dark matter from dynamical spacetime,”
   Phys. Rev. D 98, no. 2, 023506 (2018),
   arXiv:1802.07981

2. F. K. Anagnostopoulos, D. Benisty, S. Basilakos and E. I. Guendelman,
   “Dark energy and dark matter unification from dynamical spacetime: observational constraints and cosmological implications,”
   JCAP 1906, 003 (2019),
   arXiv:1904.05762

3. D. Benisty and E. I. Guendelman,
   “Inflation compactification from dynamical spacetime,”
   Phys. Rev. D 98, no. 4, 043522 (2018),
   arXiv:1805.09314.

4. D. Benisty and E. I. Guendelman,
   “Interacting Diffusive Unified Dark Energy and Dark Matter from Scalar Fields,”
   Eur. Phys. J. C 77, no. 6, 396 (2017),
   arXiv:1701.08667.

5. D. Benisty, E. Guendelman and Z. Haba,
   “Unification of dark energy and dark matter from diffusive cosmology,”
   Phys. Rev. D 99, no. 12, 123521 (2019),
   arXiv:1812.06151.
The following are additional papers which did not fit within the story line of this thesis and are not included.

1. **D. Benisty**, D. Vasak, J. Kirsch and J. Struckmeier, “Low-redshift constraints on covariant canonical Gauge theory of gravity,” Eur. Phys. J. C **81**, no. 2, 125 (2021), arXiv: 2101.07566.

2. **D. Benisty** and D. Staicova, “Testing Late Time Cosmic Acceleration with uncorrelated Baryon Acoustic Oscillations dataset,” Astron. Astrophys. **647**, A38 (2021), arXiv: 2009.10701.

3. **D. Benisty**, “Quantifying the $S_8$ tension with the Redshift Space Distortion data set,” Phys. Dark Univ. **1**, 100766 (2021), arXiv: 2005.03751.

4. **D. Benisty**, E. I. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, “Non Canonical Volume Form Formulation of Modified Gravity Theories and Cosmology,” Eur. Phys. J. Plus **136**, no. 1, 46 (2021), arXiv: 2006.04063.

5. **D. Benisty** and E. I. Guendelman, “The Local Group as a test system for Modified Newtonian Dynamics,” Phys. Dark Univ. **30**, 100708 (2020), arXiv: 2007.13006.

6. **D. Benisty** and E. I. Guendelman, “Quintessential Inflation from Lorentzian Slow Roll,” Eur. Phys. J. C **80**, no. 6, 577 (2020), arXiv: 2006.04129.

7. **D. Benisty** and E. I. Guendelman, “Lorentzian Quintessential Inflation,” Int. J. Mod. Phys. D **29**, no. 14, 2042002 (2020), arXiv: 2004.00339.

8. **D. Benisty**, E. I. Guendelman, E. Nissimov and S. Pacheva, “$\Lambda$CDM as a Noether Symmetry in Cosmology,” Int. J. Mod. Phys. D **26**, 2050104 (2020), arXiv:2003.13146.

9. **D. Benisty**, E. I. Guendelman, E. Nissimov and S. Pacheva, “Quintessential Inflation with Dynamical Higgs Generation as an Affine Gravity,” Symmetry **12**, 734 (2020), arXiv: 2003.04723.

10. **D. Benisty**, E. Guendelman, E. Nissimov and S. Pacheva, “Dynamically Generated Inflationary $\Lambda$CDM,” Symmetry **12**, no. 3, 481 (2020), arXiv: 2002.04110.

11. **D. Benisty**, E. I. Guendelman and E. N. Saridakis, “The Scale Factor Potential Approach to Inflation,” Eur. Phys. J. C **80**, no. 5, 480 (2020), arXiv: 1909.01982.

12. **D. Benisty**, E. I. Guendelman, E. Nissimov and S. Pacheva, “Dynamically generated inflationary two-field potential via non-Riemannian volume forms,” Nucl. Phys. B **951**, 114907 (2020), arXiv: 1907.07625

13. **D. Benisty**, E. Guendelman, E. Nissimov and S. Pacheva, “Dynamically Generated Inflation from Non-Riemannian Volume Forms,” Eur. Phys. J. C **79**, no. 9, 806 (2019), arXiv: 1906.06691.

14. **D. Benisty**, E. I. Guendelman, E. N. Saridakis, H. Stoecker, J. Struckmeier and D. Vasak, “Inflation from fermions with curvature-dependent mass,” Phys. Rev. D **100**, no. 4, 043523 (2019), arXiv: 1905.03731.
15. **D. Benisty**, E. I. Guendelman, “Cosmological Principle in Newtonian Dynamics,” Mod. Phys. Lett. A 35, no. 16, 2050131 (2020), arXiv: 1902.06511.

16. **D. Benisty**, E. I. Guendelman, D. Vasak, J. Struckmeier and H. Stoecker, “Quadratic curvature theories formulated as Covariant Canonical Gauge theories of Gravity,” Phys. Rev. D 98, no. 10, 106021 (2018), arXiv: 1809.10447.

17. **D. Benisty**, E. I. Guendelman, “Two scalar fields inflation from scale-invariant gravity with modified measure,” Class. Quant. Grav. 36, no. 9, 095001 (2019), arXiv: 1809.09866

18. **D. Benisty**, D. Vasak, E. Guendelman and J. Struckmeier, “Energy transfer from space-time into matter and a bouncing inflation from covariant canonical gauge theory of gravity,” Mod. Phys. Lett. A 34, no. 21, 1950164 (2019), arXiv: 1807.03557

19. **D. Benisty**, E. I. Guendelman, “Correspondence between the first and second order formalism by a metricity constraint,” Phys. Rev. D 98, no. 4, 044023 (2018), arXiv: 1805.09667

20. S. Bahamonde, **D. Benisty**, E. I. Guendelman, “Linear potentials in galaxy halos by Asymmetric Wormholes,” Universe 4, no. 11, 112 (2018), arXiv: 1801.08334

21. **D. Benisty**, E. I. Guendelman, “A transition between bouncing hyper-inflation to ΛCDM from diffusive scalar fields,” Int. J. Mod. Phys. A 33, no. 20, 1850119 (2018), arXiv: 1710.10588.

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Conference proceedings:

1. **D. Benisty**, E. I. Guendelman and J. Struckmeier, “Gauge Theory of Gravity Based on the Correspondence Between the 1st and the 2nd Order Formalisms,” Springer Proc. Math. Stat. 335 (2019), 309-316, arXiv: 1808.01978.

2. **D. Benisty**, E. Guendelman, A. Kaganovich, E. Nissimov and S. Pacheva, “Modified Gravity Theories Based on the Non-canonical Volume-Form Formalism,” Springer Proc. Math. Stat. 335 (2019), 239-252, arXiv: 1905.09933.
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5 Introduction

The best explanation for the accelerated expansion of our Universe is the $\Lambda$CDM model, which addresses dark energy ($\Lambda$) and Cold Dark Matter (CDM) components [1–8]. Some models claim that the dark energy is quantum fluctuations of the vacuum, while dark matter is an additional substance that doesn’t interact with light. Yet, the true nature of these two phenomena is still a mystery raising fundamental problems [9]: The cosmological constant problem discusses large disagreement between the vacuum expectation value of the energy momentum tensor which comes from quantum field theory on one side, and the observed value of the dark energy density on the other side [10–14]. This problem is considered to be one from the biggest problems in modern physics. Another fundamental problem is the coincidence between the ratio of dark matter and dark energy in our Universe. In order to solve this problem many approaches emerged [15]. Those models introduce some interaction between dark energy and dark matter [16–18]. Some models claim that the vacuum energy is running with the evolution of the Universe and even may give better data fitting then the usual $\Lambda$CDM model [19–26].

Interaction between dark matter and dark energy was considered in many cases, such as [27–30]. Unification between dark energy and dark matter from an action principle were obtained from scalar fields [31,32] or by other models [33–40]. Others describe the dark matter as an effective scalar field [41–45].

One has to take into consideration the Neutron Stars (NS) merger measurement on 17 August 2017 the [46]. The observation detects the Gravitational Waves from the NS merger and its associated electromagnetic counterparts. Combined analysis of the signals yields an equivalence between the speed of gravity and the speed of light. The equivalence invalidates many modifications to General theory Relativity [47–49].

One elementary way to parametrize dark energy is by a scalar field $\phi$ - i.e. quintessence models [50–62]. A flat quintessential potential does not give a dark matter component. Therefore [63] uses a scalar field model that unifies dark energy and dark matter from one scalar field. However, the model lacks of an action principle. This thesis introduces the action principle based on the model [63] with DST [64]. This generalization reduces to regular dark energy and dark matter components in particular cases, and addresses some extensions. These extensions fit with the late time accelerated expansion of the Universe. Before we present the DST, we will summarize the foundations of the standard model of cosmology.

5.1 FRW cosmology

The dynamics of the Universe is described by the Einstein equations which are the variation of the Einstein-Hilbert action. A simple way to solve the Einstein equations uses generic symmetries. The Friedman-Lemaitre-Robertson-Walker (FLRW) metric [65] is the standard ansatz in cosmology, which is based on the assumption of a homogeneous and isotropic Universe at any point ("The cosmological principle"). The FLRW metric reads:

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right].$$

(5.1)

In this problem the dynamics is associated with the scale factor - $a$. Einstein equations allow us to determine the scale factor provided the matter content of the Universe is specified. The
constant $K$ in the metric (5.1) describes the geometry of the spatial section of spacetime, with closed, flat and open Universes corresponding to $K = +1, 0, -1$, respectively.

The differential equations for the scale factor and the matter density follow from Einstein’s equations:

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}g^\mu_\nu R = \frac{8\pi G}{c^4} T^\mu_\nu,$$  \hspace{1cm} (5.2)

where $R^\mu_\nu$ is the Ricci tensor, $R$ is the Ricci scalar, $G^\mu_\nu$ is the Einstein tensor and $T^\mu_\nu$ is the energy momentum tensor. In the FRW background (5.1) the curvature terms are given by:

$$R^0_0 = \frac{3\ddot{a}}{a},$$  \hspace{1cm} (5.3)

$$R^i_j = \left( \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2K}{a^2} \right) \delta^i_j,$$  \hspace{1cm} (5.4)

$$R = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} \right),$$  \hspace{1cm} (5.5)

where the "dot" denotes a derivative with respect to the cosmic time and the indexes $i, j$ refers to 1, 2, 3. To set up the source term of the energy momentum tensor, the Universe is usually modeled as a perfect fluid. The appropriate energy-momentum tensor is then

$$T^a_\xi = \text{diag}(\rho, -p, -p, -p),$$  \hspace{1cm} (5.6)

where we set $c = 1$. Due to the symmetry properties, the density and the pressure are time dependent. The Friedmann equations read:

$$H^2 \equiv \left( \frac{\ddot{a}}{a} \right)^2 = \frac{8\pi G \rho}{3} - \frac{K}{a^2},$$  \hspace{1cm} (5.7)

$$\dot{H} = -4\pi G (p + \rho) + \frac{K}{a^2},$$  \hspace{1cm} (5.8)

where $H$ is the Hubble parameter. The energy momentum tensor is conserved by virtue of the Bianchi identities, leading to the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0.$$  \hspace{1cm} (5.9)

Equation (5.9) can be derived from Eqs. (5.7) and (5.8), which means that only two of Eqs. (5.7), (5.8) and (5.9) are independent. Eliminating the $K/a^2$ term from Eqs. (5.7) and (5.8), we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p).$$  \hspace{1cm} (5.10)

Hence accelerated expansion occurs for $\rho + 3p < 0$. One can rewrite Eq. (5.7) in the form of $\Omega - 1 = K/(aH)^2$, where $\Omega$ is the fraction between the density $\rho(t)$ and the critical density $\rho_c(t) = 3H^2(t)/8\pi G$. Observations show that the current Universe has a spatially flat geometry with $K = 0$ [66].
5.2 Constant Equation of State

The density $\rho(t)$ and the pressure $p(t)$ are related via an equation of state, which, for a perfect fluid, is characterized by a constant parameter $\omega$:

$$\omega = \frac{p}{\rho},$$

where $\omega$ in our simple model is assumed to be a constant. Then by solving the Einstein equations given in Eqs. (5.7) and (5.8) with $K = 0$, the Friedmann equations yield:

$$a \propto (t - t_0)^{\frac{2}{3(1+w)}},$$

$$\rho \propto a^{-3(1+w)},$$

where $t_0$ is constant. Notice that the above solution is valid for $w \neq -1$. The case of $w = -1$ gives the de-Sitter Space solution:

$$a \propto e^{H_0 t},$$

$$\rho = \text{Const},$$

with a constant Hubble parameter. This solution is apparently the solution for the early and late stages of our Universe. The inflationary paradigm is considered to be a necessary part of the Universe, since it provides a solution to the the horizon, the flatness, and the monopole problems [67–75]. Inflation predicts a short exponential expansion before the reheating the Universe, whereas the late dark energy domination epoch explains the accelerated expansion of the current period of our Universe [1,7,10,14,76,77]. Notably, the early inflationary energy density must be much larger than the late time energy density.

Beside this particular solution, there are two important solutions, that during the evolution of our Universe appear to be a good approximation. The radiation dominated Universe corresponds to $w = 1/3$, whereas the dust dominated Universe to $w = 0$. In those cases, the density read:

$$\text{Radiation: } a \propto (t - t_0)^{1/2}, \quad \rho \propto a^{-4},$$

$$\text{Dust: } a \propto (t - t_0)^{2/3}, \quad \rho \propto a^{-3}.$$
5.3 Cosmology with matter Scalar fields

Quintessence is described by an ordinary scalar field \( \phi \) minimally coupled to gravity:

\[
S = \int \! \! d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} \mathcal{R} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \tag{5.16}
\]

where \( V(\phi) \) is the potential of the field. From the variation with respect to the scalar field, we obtain the Klein Gordon equation:

\[
\Box \phi - V'(\phi) = \frac{1}{\sqrt{-g}} \partial_\mu \left( g^{\mu\nu} \sqrt{-g} \partial_\nu \phi \right) - V'(\phi) = 0 \tag{5.17}
\]

Notice that the d'Alembertian here \( \Box \phi \) is with respect to a particular metric and not for a flat spacetime. In a flat FLRW metric the variation (5.17) gives:

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0. \tag{5.18}
\]

The energy momentum tensor of the field is derived by varying the action (5.16) with respect to \( g^{\mu\nu} \):

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \partial_\mu \vartheta^\phi \partial_\nu \vartheta^\phi - g_{\mu\nu} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \vartheta^\phi \partial_\beta \vartheta^\phi + V(\vartheta^\phi) \right]. \tag{5.19}
\]

In the flat Friedmann background we obtain the energy density and pressure density of the scalar field:

\[
\rho = T^0_0 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = -T^i_i = \frac{1}{2} \dot{\phi}^2 - V(\phi). \tag{5.20}
\]

The Einstein equations yield:

\[
H^2 = \frac{8\pi G}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right], \tag{5.21}
\]

\[
\frac{\dot{a}}{a} = -\frac{8\pi G}{3} \left[ \dot{\phi}^2 - V(\phi) \right]. \tag{5.22}
\]

Notice that the continuity equation (5.9) is derived by combining these equations. The equation of state for the field \( \phi \) ranges in the region \(-1 \leq w_\phi \leq 1\). The slow-roll limit, \( \dot{\phi}^2 \ll V(\phi) \), corresponds to \( w_\phi = -1 \), thus asymptotically one has dark energy/ Cosmological Constant behavior. In the case of a constant potential, stiff matter is obtained and the energy density evolves as \( \rho \propto a^{-6} \). In order to obtain dark matter behavior from the same scalar field, reference [63] extended the idea of the quintessence model.

5.4 Unification of dark matter and dark energy

Reference [63] studies the consequences of modeling dark energy using a scalar field that is of non-Lagrangian type in order to address the coincidence problem. The model offers a unified description of dark energy and dark matter from a single scalar field.

The model explores unification of dark matter and dark energy by direct insertion of a kinetic term into the energy momentum tensor. This scalar is different from quintessence, having an equation of state between \(-1 \) and \( 0 \). To remedy this it is necessary to incorporate a
dynamical term, depending on $\nabla^\mu \phi$, into the equations. For quintessence this is done by including a canonical kinetic term in the Lagrangian; $\Lambda(\phi)$ then becomes the scalar field potential and the total dark energy density includes both potential and kinetic terms. Here we propose the simplest possible alternative, which is the direct insertion of a kinetic term into the energy momentum tensor:

$$\frac{1}{\kappa^2} G_{\mu\nu} = \Lambda(\phi) g_{\mu\nu} - \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi.$$  (5.23)

Now $\Lambda(\phi)$ is not necessarily a constant. From the Einstein equations above we obtain the density and pressure of our scalar

$$\rho_{\text{sca}} = \frac{1}{2} \dot{\phi}^2 + \Lambda(\phi),$$

$$p_{\text{sca}} = -\Lambda(\phi).$$  (5.24)

From the expressions of density and pressure, we know quintessence has the equation of state $-1 \leq w_{\text{qui}} \leq 1$ for $\Lambda \geq 0$, while the scalar field has $-1 \leq w_{\text{sca}} \leq 0$. From the conservation equation we then know that the density of quintessence scales in the range $a^{-6}$ to $a^0$, while for the scalar field the range is restricted to $a^{-3}$ and $a^0$. This property suggests that the scalar field may play the role of both dark matter (scaling approximately as $a^{-3}$) and dark energy (scaling approximately as $a^0$). From the conservation of the energy momentum tensor the model yields:

$$\ddot{\phi} + \frac{3}{2} H \dot{\phi} + \Lambda'(\phi) = 0.$$  (5.25)

For a constant potential $\Lambda(\phi) = \Lambda$, the scalar field solution is $\dot{\phi} \sim a^{-3/2}$. Therefore, Eq. (5.24) gives:

$$\rho_{\text{sca}} = \frac{C}{a^3} + \Lambda,$$

$$p_{\text{sca}} = -\Lambda.$$  (5.26)

where $C$ is an integration constant. The kinetic term gives the dark matter component and the potential gives the dark energy component. This is unlike the standard quintessence model, where the kinetic term gives a stiff equation of state.

This model is not formulated from an action principle. In order to solve this problem we added the DST to the scalar field action. Hence, this theory introduces a vector field $\chi_\mu$ that changes the scalar field dynamics.
6 The basis of DST

6.1 A toy model

In order to understand the formulation of DST, we start with the simplest possible example: The dynamics of a classical particle in a potential $V(x)$ where the energy conservation is assumed from the beginning. The energy reads:

$$\frac{1}{2}mv^2 + V(x) = E = \text{Const}, \quad (6.1)$$

with the speed $v = dx/dt$. The action we assume for this dynamics is:

$$S = \int dt \mathcal{L} = \int dt \frac{dB}{dt} \left( \frac{1}{2}mv^2 + V(x) \right), \quad (6.2)$$

where the dynamical variable $B$ gives the conservation of energy from its variation (6.1). One can differentiate (6.1) with respect to time:

$$m\frac{dv}{dt} = -V'(x), \quad (6.3)$$

which is the Newton’s Second Law.

One can check complete consistency with the other variations. The variation with respect to $x$ gives:

$$m \frac{d}{dt} \left( \frac{dB}{dt} \frac{dx}{dt} \right) = \frac{dV(x)}{dx} \frac{dB}{dt}. \quad (6.4)$$

Since the action (6.2) is time independent, there is a conservation property:

$$\mathcal{H} = \frac{dB}{dt} p_a + \frac{dx}{dt} p_x - \mathcal{L} = \frac{dx}{dt} p_x = mv^2 \frac{dB}{dt} = \text{Const}, \quad (6.5)$$

so that

$$\frac{dB}{dt} = C/v^2. \quad (6.6)$$

Eq. (6.6) with Eq. (6.4) yield:

$$m \frac{d}{dt} \left( \frac{C}{v} \right) = V'(x) \frac{C}{v^2}, \quad (6.7)$$

which gives (6.3) exactly. The conservation of energy functional is implemented by $B$ variation and the additional conservation law determines $B$. This is a classical non-relativistic action. Let’s see how to generalize this idea into a relativistic and covariant theories.

6.2 The DST action

One of the basic features in the standard approach to theories of gravity is the local conservation of an energy momentum tensor. For example, the conservation of energy is derived from the time translation invariance principle. The local conservation of energy momentum tensor can emerge from a variation of a certain vector field. Reference [64] considers a four dimensional case where conservation of a symmetric energy momentum tensor $T^{\mu\nu}_{(\chi)}$ is imposed by introducing a term in the action:

$$\mathcal{S}_{(\chi)} = \int d^4x \sqrt{-g} \chi_{\mu\nu} T^{\mu\nu}_{(\chi)} \quad (6.8)$$
where $\chi_{\mu,\nu} = \partial_\nu \chi_\mu - \Gamma^\lambda_\mu_\nu \chi_\lambda$ and $\Gamma_\mu_\nu$ is the Affine-Connection. We use the metric formalism (or second order formalism) where the connection is assumed to be the Levi-Civita symbol:

$$\Gamma^\rho_\mu_\nu = \left\{ \begin{array}{c} \rho \\ \mu \\ \nu \end{array} \right\} = \frac{1}{2} g^{\rho_\lambda} (g_{\lambda_\mu_\nu} + g_{\lambda_\nu_\mu} - g_{\mu\nu\lambda}). \tag{6.9}$$

The vector field $\chi_\mu$ is called a "Dynamical Spacetime vector", because the energy density of $T^\mu_\nu(\chi)$ is a canonically conjugated variable to $\chi_0$, which is what we expect from a Dynamical Time:

$$\pi_{\chi_0} = \frac{\partial \mathcal{L}}{\partial \dot{\chi}_0} = T^0_0(\chi) \tag{6.10}$$

The variation with respect to $\chi_\mu$ gives a covariant conservation law:

$$\nabla_\mu T^\mu_\nu(\chi) = 0 \tag{6.11}$$

From the variation of the action with respect to the metric, we get a conserved stress energy tensor $G^\mu_\nu$ (in appropriate units), which is well known from Einstein’s equation:

$$G^\mu_\nu = \frac{2}{\sqrt{-g}} \delta \mathcal{L} + \mathcal{L}_m, \quad \nabla_\mu G^\mu_\nu = 0. \tag{6.12}$$

where $G^\mu_\nu$ is Einstein tensor, $\mathcal{L}_\chi$ is the Lagrangian in (6.8) and $\mathcal{L}_m$ is an optional action that involve other contributions. Cosmological solutions with a scalar field behaving as radiation, in the context of gravitational theory with Dynamical Time are discussed in [88].

### 6.3 Diffusive Action

The diffusion equation can be generalized into a curved spacetime by defining a non-conserved stress energy tensor $T^\mu_\nu$ with a current source $j^\mu$ [89,90]:

$$\nabla_\mu T^\mu_\nu = 3\sigma j^\nu \tag{6.13}$$

where $\sigma$ is the diffusion coefficient of the fluid. The current $j^\mu$ is a time-like covariant conserved vector field $j^\mu_\mu = 0$ which describe the conservation of the number of particles in the system. In order to break the conservation of $T^\mu_\nu(\chi)$, we use the action:

$$S_{(\chi,A)} = \int d^4x \sqrt{-g} \chi_{\mu\nu} T^\mu_\nu(\chi) + \frac{\sigma}{2} \int d^4x \sqrt{-g} (\chi_\mu + \partial_\mu A)^2 \tag{6.14}$$

where $A$ is a different scalar field from $\phi$. From a variation with respect to the Dynamical Spacetime vector field $\chi_\mu$, we obtain:

$$\nabla_\nu T^\nu_\mu(\chi) = \sigma (\chi_\mu + \partial_\mu A) = f^\mu, \tag{6.15}$$

where $f^\mu = \sigma (\chi_\mu + \partial_\mu A)$ is a current source for the stress energy momentum tensor $T^\mu_\nu(\chi)$. From the variation with respect to the new scalar $A$, a covariant conservation of the current indeed emerges:

$$\nabla_\mu f^\mu = \nabla_\mu (\chi_\mu + \partial_\mu A) = 0 \tag{6.16}$$
Using different expressions for $T_{\mu\nu}^{(\chi)}$ which depends on different variables, will give the conditions between the Dynamical Spacetime vector field $\chi_\mu$ and the other variables.

A particular case of diffusive energy theories is obtained when $\sigma \to \infty$. In this case, the contribution of the current $f_\mu$ in the equations of motion goes to zero, and from this constraint the vector field becomes to a gradient of the scalar:

$$f_\mu = \sigma (\chi_\mu + \partial_\mu A) = 0 \implies \chi_\mu = -\partial_\mu A$$

(6.17)

The theory (6.14) changes to a theory with higher derivatives:

$$\mathcal{S} = -\int d^4x \sqrt{-g} T_{\mu\nu}^{(\chi)} \nabla_\mu \nabla_\nu A$$

(6.18)

The variation with respect to the scalar $A$ gives $\nabla_\mu \nabla_\nu T_{\mu\nu}^{(\chi)} = 0$ which corresponds to the variations (6.15-6.16). In the DST we obtain four equations of motion from the variation of $\chi_\mu$, which corresponds to a covariant conservation of energy momentum tensor $\nabla_\mu T_{\mu\nu}^{(\chi)} = 0$. By changing the generic four vector to a gradient of a scalar $\partial_\mu \chi$, the number of conditions reduces from four to one and instead of the conservation of energy momentum tensor, we are left with a covariant conservation of the current $f^\nu = \nabla_\mu T_{\mu\nu}^{(\chi)}$.

### 6.4 Overview of the Thesis

In this thesis we formulate the DST in the framework of cosmological solutions, which produces a natural unification of dark energy and dark matter. In the first paper "Unified dark energy and dark matter from dynamical spacetime" we formulate the basics for the complete theory and demonstrate simple analytic solutions. The DST introduces a Lagrange multiplier in addition to the quintessential scalar field, that forces the kinetic term to act as dark matter. In the second paper - "Dark energy and dark matter unification from dynamical spacetime: observational constrain and cosmological implications" we extend the solution for the dynamics between dark energy and dark matter, using different potentials and constrain additional parameters with data of the late time accelerated expansion. Including higher dimensions with different scale factors for the same family of theories produces an inflationary scenario, as described in the third paper - "Inflation compactification from dynamical spacetime", which implies a new suggestion for an inflationary solution that wasn’t considered yet. We extended the DST theory to a "diffusive theories" which allows to the dark energy and dark matter components to exchange energy. In the last section we discuss the results. In the fourth paper - "Interacting Diffusive Unified Dark Energy and Dark Matter from Scalar Fields", we solve the complete analytic solution for the simplest version of the theory. In the fifth paper - "Unification of dark energy and dark matter from diffusive cosmology", we analyze the complete combination of the stress energy momentum tensor with regards to diffusive interactions and we show analytically and numerically that all of the solutions yield to the stable $\Lambda$CDM model with new families of solutions. In the last section we discuss the results.
7 The foundations of the model

D. Benisty and E. I. Guendelman,  
“Unified dark energy and dark matter from dynamical spacetime”,  
Phys. Rev. D 98, no. 2, 023506 (2018)  
This paper unifies dark energy and dark matter from one scalar field through an action principle. Introducing the coupling of a Dynamical Spacetime vector field to the energy momentum tensor from [63] gives the ΛCDM model with possible bouncing equation of state. For the ΛCDM solution without the bouncing solution, the Dynamical Spacetime vector is equivalent to the cosmic time.
A unification of dark matter and dark energy based on a dynamical spacetime theory is suggested. By introducing a dynamical spacetime vector field $\chi^{\mu}$ as a Lagrange multiplier, conservation of an energy momentum tensor $T^{\mu\nu}(\chi)$ is implemented. This Lagrangian generalizes the “unified dark energy and dark matter from a scalar field different from quintessence” [Phys. Rev. D 81, 043520 (2010)], which did not consider a Lagrangian formulation. This generalization allows the solutions which were found previously, but in addition to that also nonsingular bouncing solutions that rapidly approach to the $\Lambda$CDM model. The dynamical time vector field exactly coincides with the cosmic time for the $\Lambda$CDM solution and suffers a slight shift (advances slower) with respect to the cosmic time in the region close to the bounce for the bouncing nonsingular solutions. In addition, we introduce some exponential potential which could enter into the $T^{\mu\nu}(\chi)$ stress energy tensor or be coupled directly to the measure $\sqrt{-g}$, giving a possible interaction between dark energy and dark matter, and could explain the coincidence problem.

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I. INTRODUCTION

Dark energy (DE) and dark matter (DM) constitute most of the observable Universe. Yet the true nature of these two phenomena is still a mystery. One fundamental question with respect to these phenomena is the coincidence problem, which is trying to explain the relation between dark energy and dark matter densities. In order to solve this problem, one approach claims that the dark energy is a dynamical entity and hopes to exploit solutions of the scaling or tracking type to remove dependence on the initial conditions. Others left this principle and tried to model the dark energy as a phenomenological fluid which exhibits a particular relation with the scale factor [1], the Hubble constant [2], or even the cosmic time itself [3].

Unifications between dark energy and dark matter from an action principle were obtained from K-essence-type actions [4] or by introducing a complex scalar field [5]. Beyond those approaches, a unified description of dark energy and dark matter using a new measure of integration has been formulated [6–10]. Also, a diffusive interaction of dark energy and dark matter models was introduced in [11,12], and it has been found that diffusive interacting dark energy–dark matter models can be formulated in the context of an action principle based on a generalization of those two-measures theories in the context of quintessential scalar fields [13,14], although these models are not equivalent to the previous diffusive interacting dark energy–dark matter models [11,12].

One has to take now into consideration the measurements on 17 August, 2017, of multimessenger gravitational wave astronomy which are in contradiction to many modified theories of gravity predictions. These observations commenced with the detection of the binary neutron star merger GW170817 and its associated electromagnetic counterparts [15]. Both signals place an exquisite bound on the speed of gravity to be the same as the speed of light. This constraint rejected many modifications to general relativity [16–21] and also many unifications between dark energy and dark matter.

A model, which also continues to be valid after the GW170817 event, for a unification of dark energy and dark matter from a single scalar field $\phi$ was suggested by Gao, Kunz, Liddle, and Parkinson [22]. Their model is close to traditional quintessence and gives dynamical dark energy and dark matter but introduces a modification of the equations of motion of the scalar field that apparently are impossible to formulate in the framework of an action principle. The basic stress energy tensor which was considered in addition to the Einstein equation was
\[ T^{\mu\nu} = -\frac{1}{2} \phi^\nu \phi^\mu + U(\phi)g^{\mu\nu}, \] (1)

where \( \phi \) is a scalar field and \( U(\phi) \) is the potential for that scalar. Assuming homogeneous and isotropic behavior, the scalar field should be only time dependent \( \phi = \phi(t) \). Then the kinetic term \(-\frac{1}{2} \phi^\nu \phi^\mu\) is parameterizing the dark matter, because it contains only energy density with no pressure, and \( U(\phi)g^{\mu\nu}\) is parameterizing the dark energy. The basic requirement for this stress energy tensor is its conservation law \( \nabla_\mu T^{\mu\nu} = 0 \). By assuming a constant potential \( U(\phi) = \text{const} \), the model provides from the potential the traditional cosmological constant and the kinetic term of the scalar field is shown to provide, from the conservation law of the energy momentum tensor, that the kinetic term dependence has a dustlike behavior:

\[ -\frac{1}{2} \nabla_\mu (\phi^\nu \phi^\mu) = 0 \Rightarrow \dot{\phi}^2 \sim \frac{1}{a^2}. \] (2)

This simple case refers to the classical ΛCDM model. The special advantage of this model is a unification of dark energy and dark matter from one scalar field and has an interesting possibility for exploring the coincidence problem.

The lack of an action principle for this model brought us to a reformulation of the unification between dark energy and dark matter ideas put forward by Gao, Kunz, Liddle, and Parkinson [22] in the framework of a dynamical spacetime theory [23,24], which forces conservation of the energy momentum tensor in addition to the covariant conservation of the stress energy momentum tensor that appears in the Einstein equation. In the next section, we explore the equations of motion for these theories. In the third section, we solve analytically the theory for constant potentials which reproduce the ΛCDM model with a bounce, which gives a possibility to solve the initial big bang singularity. In the last section, we solve the theory for an exponential potential which gives a good possibility for solving the coincidence problem.

II. DYNAMICAL SPACETIME THEORY

A. A basic formulation

One of the basic features in the standard approach to theories of gravity is the local conservation of an energy momentum tensor. In the field theory case, it is derived as a result rather than a starting point. For example, the conservation of energy can be derived from the time translation invariance principle. The local conservation of an energy momentum tensor can be a starting point rather than a derived result. Let us consider a four-dimensional case where conservation of a symmetric energy momentum tensor \( T^{\mu\nu}_{(\chi)} \) is imposed by introducing the term in the action

\[ S_{(\chi)} = \int d^4x \sqrt{-g} \chi^\mu \pi_{\chi} T^{\mu\nu}_{(\chi)}. \] (3)

where \( \chi^{\mu
u} = \partial_\mu \chi_\nu - \Gamma^\alpha_\mu_\nu \chi_\alpha \). The vector field \( \chi_\mu \) called a dynamical spacetime vector, because of the energy density of \( T^{\mu\nu}_{(\chi)} \), is a canonically conjugated variable to \( \chi_0 \), which is what we expected from a dynamical time:

\[ \pi_{\chi_0} = \frac{\partial L}{\partial \chi^0} = T^0_0(\chi). \] (4)

If \( T^{\mu\nu}_{(\chi)} \) is independent of \( \chi_\mu \) and having \( \Gamma^\mu_\alpha_\beta \) being defined as the Christoffel connection coefficients (the second-order formalism), then the variation with respect to \( \chi_\mu \) gives a conserved stress energy tensor:

\[ \nabla_\mu T^{\mu\nu}_{(\chi)} = 0. \] (5)

From the variation of the action with respect to the metric, we get a conserved stress energy tensor \( G^{\mu\nu} \) (in appropriate units), which is well known from the Einstein equation:

\[ G^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} [L_\chi + L_m], \quad \nabla_\mu G^{\mu\nu} = 0, \] (6)

where \( G^{\mu\nu} \) is the Einstein tensor, \( L_\chi \) is the Lagrangian in (3), and \( L_m \) is an optional action that involves other contributions.

Some basic symmetries that hold for the dynamical spacetime theory are two independent shift symmetries:

\[ \chi_\mu \rightarrow \chi_\mu + k_\mu, \quad T^{\mu\nu}_{(\chi)} \rightarrow T^{\mu\nu}_{(\chi)} + \Lambda g^{\mu\nu}, \] (7)

where \( \Lambda \) is some arbitrary constant and \( k_\mu \) is a Killing vector of the solution. This transformation will not change the equations of motion, which means also that the process of redefinition of the energy momentum tensor in the action (3) will not change the equations of motion. Of course, such a type of redefinition of the energy momentum tensor is exactly what is done in the process of normal ordering in the quantum field theory, for instance.

B. A connection to modified measures

A particular case of the stress energy tensor with the form \( T^{\mu\nu}_{(\chi)} = L_1 g^{\mu\nu} \) corresponds to a modified measure theory. By substituting this stress energy tensor into the action itself, the determinant of the metric is canceled:

\[ \sqrt{-g} \pi^\mu_L L_1 = \partial_\mu (\sqrt{-g} \pi^\mu) L_1 = \Phi L_1, \] (8)

where \( \Phi = \partial_\mu (\sqrt{-g} \pi^\mu) \) is like a “modified measure.” A variation with respect to the dynamical time vector field will give a constraint on \( L_1 \) to be a constant.
\[ \partial_a L_1 = 0 \Rightarrow L_1 = M = \text{const.} \quad (9) \]

This situation corresponds to the two-measures theory [25–27], where, in addition to the regular measure of integration in the action, \( \sqrt{-g} \) includes another measure of integration which is also a density and a total derivative. Notable effects that can be obtained in this way are the spontaneous breaking of the scale invariance, the seesaw cosmological effects [25], the resolution of the 5th force problem in quintessential cosmology [28], and a unified picture of both inflation and slowly accelerated expansion of the present Universe [29,30]. As we mentioned before in the introduction, the two-measures theory can serve to build unified models of dark energy and dark matter.

Usually, the construction of this measure is from four scalar fields \( \varphi_a \), where \( a = 1, 2, 3, 4 \):

\[ \Phi = \frac{1}{4!} \epsilon_{abcd} \partial_a \varphi^{(a)} \partial_b \varphi^{(b)} \partial_c \varphi^{(c)} \partial_d \varphi^{(d)}, \quad (10) \]

and then we can rewrite an action that uses both of these densities:

\[ S = \int d^4x \Phi L_1 + \int d^4x \sqrt{-g} L_2. \quad (11) \]

As a consequence of the variation with respect to the scalar fields \( \varphi_a \), assuming that \( L_1 \) and \( L_2 \) are independent of the scalar fields \( \varphi_a \), we obtain that for \( \Phi \neq 0 \) it implies that \( L_1 = M = \text{const} \) as in the dynamical time theory with the case of (9).

**III. DE-DM Unified Theory from Dynamical Spacetime**

A suggestion of an action which can produce DE-DM unification takes the form

\[ L = -\frac{1}{2} R + X_{\mu\nu} T_{\mu\nu}^{\phi} - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi). \quad (12) \]

Consisting of an Einstein Hilbert action (8\( \pi G = 1 \)), quintessence, and dynamical spacetime action, when the original stress energy tensor \( T_{\mu\nu}^{\phi} \), is the same as the stress energy tensor (1), Gao and colleagues used

\[ T_{\mu\nu}^{\phi} = \frac{1}{2} \phi^{,\mu} \phi^{,\nu} + U(\phi) g^{\mu\nu}. \quad (13) \]

The action depends on three different variables: the scalar field \( \phi \), the dynamical spacetime vector \( X_{\mu} \), and the metric \( g_{\mu\nu} \). Therefore, there are three sets for the equation of motions. For the solution, we assume homogeneity and isotropy; therefore, we solve our theory with a Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \quad (14) \]

According to this ansatz, the scalar field is just a function of time \( \phi(t) \) and the dynamical vector field will be taken only with a time component \( X_\mu = (x_0, 0, 0, 0) \), where \( x_0 \) is also just a function of time. A variation with respect to the dynamical spacetime vector field \( X_\mu \) will force conservation of the original stress energy tensor, which in Friedmann-Lemaître-Robertson-Walker metric (FRWM) gives the relation

\[ \dot{\phi} + \frac{3}{2} \mathcal{H} \dot{\phi} + U'(\phi) = 0. \quad (15) \]

Compared with the equivalent equation which comes from the quintessence model, this model gives a different and smaller friction term, as compared to the canonical scalar field. Therefore, for an increasing redshift, the densities for the scalar field will increase slower than in the standard quintessence.

The second variation with respect to the scalar field \( \phi \) gives a nonconserved current:

\[ \chi^\mu \chi^{\mu'} U'(\phi) - V'(\phi) = \nabla_\mu j^\mu, \quad (16a) \]

\[ j^\mu = \frac{1}{2} \dot{\phi} (x'^{\mu'} + \chi^{\mu'}) + \phi^{,\mu}, \quad (16b) \]

and the derivatives of the potentials are the source of this current. For constant potentials the source becomes zero, and we get a covariant conservation of this current. In a FLRW metric, this equation of motion takes the form

\[ \dot{\phi}(x_0 - 1) + \dot{\phi}[\ddot{x}_0 + 3\mathcal{H}(\dot{x}_0 - 1)] = \chi^{\mu} U'(\phi)(\dot{x}_0 + 3\mathcal{H}\dot{x}_0) - V'(\phi). \quad (17) \]

Substituting the term of the potential derivative \( U'(\phi) \) from Eq. (15):

\[ [1 - 2\dot{x}_0 - 3\mathcal{H}\dot{x}_0] \dot{\phi} = \left[ \ddot{x}_0 - 3\mathcal{H} + \frac{9}{2} \mathcal{H} \dot{x}_0 + X_0 \mathcal{H} \right] \dot{\phi} + V'(\phi) = 0. \quad (18) \]

The last variation, with respect to the metric, gives the stress energy tensor that is defined by the value of the Einstein tensor:

\[ G^{\mu\nu} = g^{\mu\nu} \left( \frac{1}{2} \dot{\phi} (x'^{\mu'} + \chi^{\mu'}) + \chi^{\mu'} \phi_{,\mu} + \chi^{\mu} \phi_{,\mu} + U'(\phi) \right) - \frac{1}{2} \dot{\phi} (x'^{\mu'} + 2\chi^{\mu'}) + \chi^{\mu'} \phi_{,\mu} + \chi^{\mu} \phi_{,\mu} \right) - \frac{1}{2} \left( \chi^{\mu'} \phi_{,\mu'} + \chi^{\mu} \phi_{,\mu} \right). \quad (19) \]

For the spatially homogeneous, cosmological case, the energy density and the pressure of the scalar field are, respectively,
\[
\rho = \dot{\phi}^2 \left( \dot{\chi}_0 \left( 1 - \frac{3}{2} \frac{H}{\ddot{a}} \right) - \frac{1}{2} \right) + V(\phi) - \phi \ddot{\chi}_0 (U'(\phi) + \dddot{\phi}),
\]

(20a)

\[
p = \frac{1}{2} \dot{\phi}^2 (\dot{\chi}_0 - 1) - V(\phi) - \chi_0 \phi U'(\phi).
\]

(20b)

Substituting the potential derivative \(U'(\phi)\) from Eq. (15) into the energy density term, makes the equation simpler:

\[
\rho = \left( \dot{\chi}_0 - \frac{1}{2} \right) \dot{\phi}^2 + V(\phi),
\]

(21)

which no longer has dependence on the potential \(U(\phi)\) or its derivatives. Those three variations are sufficient for building a complete solution for the theory. Let us see a few simple cases.

IV. THE EVOLUTION OF THE HOMOGENEOUS SOLUTIONS

A. A bouncing \(\Lambda\)CDM solution

In order to compute the evolution of the scalar field and to check whether it is compatible with the observable Universe, we have to specify a form for the potentials. Let us take a simplified case of constant potentials:

\[
U(\phi) = C, \quad V(\phi) = \Omega_\Lambda \phi^3,
\]

(22)

Overall, in the equations of motion, only the derivative of the potential \(U(\phi)\) appears, not the potential itself. Therefore, a constant part of the potential \(U(\phi)\) does not contribute to the solution. However, \(V(\phi)\), as we shall see below, gives the cosmological constant. The conservation of the stress energy tensor from Eq. (15) gives

\[
\dot{\rho}^2 = \frac{2\Omega_m a}{a^3},
\]

(23)

where \(\Omega_m\) is an integration constant which appears from the solution. From the second variation, with respect to the scalar field \(\phi\), a conserved current is obtained, which from Eq. (18) gives the exact solution of the dynamical time vector field:

\[
\dot{\chi}_0 = 1 - \kappa a^{-1.5},
\]

(24)

where \(\kappa\) is another integration constant. Eventually, the densities and the pressure for this potentials are given by (21). By substituting the solutions for the scalar \(\dot{\phi}\) and the vector \(\dot{\chi}_0\) (in units with \(\rho_c = \frac{8\pi G}{3H_0^2} = 1\)), we get

\[
\rho = \Omega_\Lambda \frac{\Omega_\Lambda}{a^{3.5}} + \frac{\Omega_m a}{a^3},
\]

(25a)

\[
p = -\Omega_\Lambda \frac{1}{2} \frac{a}{a^{4.5}},
\]

(25b)

where \(\Omega_\Lambda = \kappa \Omega_m\). Notice that \(\Omega_m\) and \(\Omega_\Lambda\) are integration constants the solution contains and \(\Omega_\Lambda\) is a parameter from the action of the theory. We can separate the result into three different "dark fluids": dark energy (\(\omega = -1\)), dark matter (\(\omega = 0\)), and an exotic part (\(\omega = \frac{1}{3}\)), which is responsible for the bounce (for \(\kappa > 0\)). From Eq. (23), the solution produces a positive \(\Omega_m\), since it is proportional to \(\dot{\phi}^2\). For \(\Omega_\Lambda\), the measurements for the late Universe forces the choice of this parameter to be positive. However, for other solutions (in the context of anti-de Sitter space, for instance), this parameter could be negative from the beginning.

In Fig. 1, we can see the effective potential for different values of \(\Omega_\Lambda\). For \(\Omega_\Lambda = 0\), the solution returns to the known \(\Lambda\)CDM model. However, for \(\Omega_\Lambda < 0\), we obtain a bouncing solution which also returns to the \(\Lambda\)CDM for late time expansion.

In addition to those solutions, there is a strong correspondence between the zero component of the dynamical spacetime vector field and the cosmic time. For \(\Lambda\)CDM, there is no bouncing solution \(\kappa = 0\), and therefore, from Eq. (24), we get \(\dot{\chi}_0 = t\) that implies that the dynamical time is exactly the cosmic time. For bouncing \(\Lambda\)CDM (see Fig. 2), we obtain a relation between the dynamical and the
cosmic time with some delay between the dynamical time and the cosmic time for the early Universe (in the bouncing region). For the late Universe, the dynamical time returns back to run as fast as the cosmic time again. This relation between the dynamical and the cosmic time may have an interesting application in the solution to “the problem of time” in quantum cosmology which will discussed elsewhere. Notice that the dynamical time is a field variable while the cosmic time is a coordinate. The scale parameter evolution depicted in Fig. 3 can show us the initial conditions where the scale parameter is not zero \(a(t) = 0\), because at that point \(a(t)\) is a minimum. In addition, for all cases the initial condition for the scale parameter is not zero \(a(0) \neq 0\). These features imply a bouncing universe solution.

B. Interacting DE-DM

1. Autonomous system method

For studying the evolution of the scalar field in the case of interacting DE-DM, we address more generic potentials. For instance,

\[
U(\phi) = C, \quad V(\phi) = \Omega \Lambda e^{-\beta \phi},
\]

where \(\beta > 0\) (if not, we can perform the transformation \(\phi \rightarrow -\phi\)). In the limit \(\beta \rightarrow 0\), the solution returns to the constant potentials case, and therefore the model is continuously connected to \(\Lambda\)CDM, at least as far as the background evolution is concerned. The first equation of motion (15) gives us the last case (23) or in this form

\[
\ddot{a} = \frac{3}{2} H \dot{a}.
\]

The equation of motion with respect to the scalar field \(\phi\) can be expressed with a new dimensionless parameter:

\[
\dot{\chi} = \dot{a} - 1.
\]

which represents the difference of the rates of change between the zero component of the dynamical spacetime vector and the actual cosmic time. The equation of motion (17) in terms of this variable gets the form

\[
\dot{\phi} = \frac{3}{2} H \phi - \beta V(\phi).
\]

Notice that for \(\beta = 0\) the relation for \(\delta = 2\alpha^{1.5}\) is Eq. (24). The main equations of the dynamical system are given by the following dimensionless quantities:

\[
x = \frac{\dot{\phi}}{\sqrt{6} H}, \quad y = \frac{\sqrt{V(\phi)}}{\sqrt{3} H},
\]

where \(x\) and \(y\) are represent the density parameters of the kinetic (dark-matter-like) and potential (dark-energy-like) terms, respectively. With those three new parameters \((x, y, \delta)\), the equation of motion with respect to the metric is written as

\[
(1 + 2\delta)x^2 + y^2 = 1.
\]

Assuming low values of \(\beta\), the dynamical time and the cosmic time approximately coincide (see Fig. 3), and therefore \(\delta \approx 0\). The phase portrait in that case should not deviate too much from a closed circle. Hence, Eq. (30) can be written by the following autonomous system equations:

\[
\frac{dx}{d\tau} = -\frac{3}{4} (x^2 - 1 + 3y^2),
\]

\[
\frac{dy}{d\tau} = -\frac{y}{4} (-9 + 3x^2 + 9y^2 + 2\sqrt{6} \delta \beta),
\]

where \(\tau = \ln a\). The equation of state \(\omega\) also can be written as

\[
\omega = \frac{1}{2} (1 - x^2 - 3y^2).
\]

The properties of a few fixed points for the exponential potential are presented in Table I. In addition, the phase plane of the autonomous system shown in Fig. 4, with the points that are mentioned in the table. The features of the fixed points can separate to two cases.

| Name | Existence | Stability | Universe          |
|------|-----------|-----------|-------------------|
| A    | All \(\beta\) | Unstable | \(\cdots\)        |
| B    | All \(\beta\) | Stable for \(\beta > \sqrt{3}\) | Dark matter       |
| C    | All \(\beta\) | Asymptotically stable | Dark energy       |
| D    | \(\beta > \sqrt{3}\) | Unstable saddle point | Unified DE-DM     |
One case is when $\beta < \sqrt{\frac{3}{2}}$ and all of the solutions are flowing into a dark energy dominated universe [point $C (x = 0, y = 1)$]. The dark matter dominated universe is an unstable point [point $B (x = 1, y = 0)$] that the universe goes through which corresponds to the dark matter epoch. In any case, point $A [(x = 0, y = 0)]$, which represents no dark matter and no dark energy, does not really exist because of the contradiction to Eq. (31). However, if the initial condition starts close to this point, it is driven into dark energy dominance eventually, as you can see in Fig. 5. Also, for this case the shape of the phase portrait looks as a circle, which ensures our assumption about the identification between the dynamical spacetime and the cosmic time.

In the second case, $\beta > \sqrt{\frac{3}{2}}$ and there are two stable fixed points. One for dark energy ($C$) and one for dark matter ($B$). If the initial conditions are close enough to those points, it will be attracted into them. In addition, a saddle point $D$

---

**FIG. 4.** The phase plane for different values of $\beta$.

**FIG. 5.** The evolution of DE-DM ratios and $e^{\delta} \sim 1$ for small values of $\beta$. 

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\( \left( x = \sqrt{\frac{\beta}{3}}, y = \sqrt{\frac{2\beta - 3}{\sqrt{6}\beta}} \right) \) is obtained. For this point, the ratio between the pressure and the density is \( \omega = -\frac{3}{2} + \frac{1}{\beta} \).

Some solutions are attracted to this point, but eventually they are repelled to the closer point. However, the case of \( \beta > \sqrt{\frac{2}{3}} \) contradicts the assumption that \( \beta \) is small enough in order not to deviate from \( \Lambda \)CDM. Also, for this case the shape of the phase portrait deviates from a circle, which implies a big deviation between the dynamical spacetime and the cosmic time.

This modification, which adds one exponential potential, is not the most general case, since we could also add an additional potential which would enter into the \( T_{\mu \nu}^{\mu} \). We suspect that some form of the potentials which are more general could cause point \( D \) to become stable and will lead us to a more comprehensive understanding of the cosmic coincidence problem, which we will investigate in the future.

2. Evolution of physical quantities

In order to assess the viability of the model, let us see how some physical quantities change vs the redshift \( (z) \). The connection between the cosmic time derivative and a redshift derivative is

\[
\frac{d}{dz} = -\mathcal{H}(z)(z + 1) \frac{d}{d\tau},
\]

which has been obtained from the scale factor dependence on \( \tau \), \( a(\tau) = \frac{\mathcal{H}(\tau)}{\mathcal{H}(z)} \). Figure 5 describes the cosmological energies densities \( \Omega_m \) and \( \Omega_{\Lambda} \) vs the redshift. For the \( \beta = 0 \) case, which refers to \( \Lambda \)CDM model (any time we can set \( \Omega_\chi \) to be zero or small), we can see that in earlier times \( \Omega_{\Lambda} \) becomes dominant; for earlier times, that is, for the very early Universe, \( \Omega_\chi \) (which we have taken to be very small except for the very early Universe) dominates. For different values of \( \beta \), we can see a slight shift from \( \Lambda \)CDM, which should be more dominant in the early Universe. The variable \( \delta \), that measures the difference in the evolution of the dynamical time and the cosmic time, which in the case of \( \beta = 0 \) gives a contribution that can be parametrized by \( \Omega_\chi \), has been taken to be very close to zero in all cases except for the very early Universe, because there a strong impact exists, close to the bounce that replaces now the traditional big bang.

In Fig. 6, we can see the evolution of the equation of state of the whole Universe as a function of the redshift. It behaves as cold dark matter dominated at higher redshifts and dark energy for the lower redshifts. The behavior does not tremendously change for those values of the redshift, but the deviations are measurable.

The set of potentials that were suggested in this section have a nice feature which reduces the dependence on the number of quantities. In this way, a suggestive and convenient parametrization of the solution uses the variable \( \delta = \frac{\mathcal{H}_0}{\mathcal{H}} - 1 \), which contains all the dependence on \( \mathcal{H}_0 \). In the future, it would be interesting to investigate how different potentials would affect the physical quantities of the Universe. However, unlike other models of dark energy and dark matter, even a trivial assumption of constant potentials leads directly to a unification of dark energy and dark matter. In any case, any generalization should assume a constant potential asymptotically.

V. DISCUSSION AND FUTURE WORK

In this paper, the unified dark energy and dark matter from a scalar field different from quintessence is formulated through an action principle. Introducing the coupling of a dynamical spacetime vector field to an energy momentum tensor that appears in the action determines the equation of motion of the scalar field from the variation of the dynamical spacetime vector field or effectively from the conservation law of an energy momentum tensor, as in Ref. [22]. The energy momentum tensor that is introduced in the action is related but not, in general, the same as the one that appears in the right-hand side of the gravitational equations, as opposed to the non-Lagrangian approach of Ref. [22], so our approach and that of Ref. [22] are not equivalent. However, in many situations the solutions studied in Ref. [22] can be also obtained here, but there are other solutions, in special nonsingular bounce solutions which are not present in Ref. [22].

In those simple solutions, the dynamical time behaves very close to the cosmic time. In particular, in solutions which are exactly \( \Lambda \)CDM, the cosmic time and the dynamical time exactly coincide with each other. If there is a bounce, the deviation of the dynamical time with respect to the cosmic time takes place only very close to the bounce region. The use of this dynamical time as the time in the Wheeler–de Witt equation should also be a subject of interest.

In principle, we can introduce two different scalar potentials: one coupled directly to \( \sqrt{-g} \) and the other appearing in the original stress energy tensor \( T_{\mu \nu}^{\mu} \). So far,
for the purposes of starting the study of the theory, we have introduced only a scalar potential coupled directly to $\sqrt{-g}$ and shown that this already leads to an interacting dark energy–dark matter model, although the full possibilities of the theory will be revealed when the two independent potentials are introduced.

Possible signatures for this model or for more generalized forms could be identified from the cosmological perturbation theory. For instance, the perturbation for the scalar field is clear. However, the perturbation for the vector field could be represented with more degrees of freedom which can reproduce a different power spectrum for the cosmic microwave background (CMB) anisotropies, for instance. But, more than this, the model that was suggested in the last part was only with an exponential potential. However, many combinations of potentials are applicable for testing the evolution for the energy densities and using data fitting for those models. The benefits for these models are that they still preserve the speed of gravity equal to the speed of light and also that arise from an action principle. Researching those families of solutions with more general potentials could help solve the coincidence problem.

The effects studied in the context of the bouncing solution, which can prevent the initial big bang singularity, could have consequences for the radially falling solutions, since as we have seen the kappa term can introduce a repulsive force that prevents the big bang singularity; there will very likely be a corresponding effect when we study radial collapse of matter, and then the analogous term, that in the homogeneous cosmology solutions prevents the big bang singularity, will in this case prevent the collapse to very high densities. This will, in turn, suppress the structure formation at low redshifts as compared to the expectations from the perturbations observed in the CMB, thus maybe explaining the $\sigma_8$ [31–33]–$\Omega_m$ tension. Notice that this effect on perturbations can take place even for constant potentials, that is, without modifying the standard $\Lambda$CDM homogeneous background, since in the homogeneous background the $\kappa$ term acts only in the very early Universe.

Finally, another direction for research has been started by studying models of this type in the context of higher-dimensional theories, where they can provide a useful framework to study the “inflation-compactification” epoch and an exit from this era to the present $\Lambda$CDM epoch could be further explored [34].

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8 Observational constraints

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This paper explores few potentials with theoretical and observational constraints. The
Autonomous System Method revels stable late-time attractors. For a flat potential, the Hubble
constant is in agreement with Planck 2018 results (within $\sim 1\sigma$).
Dark energy and dark matter unification from dynamical space time: observational constraints and cosmological implications

Fotios K. Anagnostopoulos\textsuperscript{a,1}, David Benisty\textsuperscript{b,c}, Spyros Basilakos\textsuperscript{d,f} and Eduardo I. Guendelman\textsuperscript{b,c,e}

\textsuperscript{a}National and Kapodistrian University of Athens, Physics Department, Panepistimioupoli Zografou, 15772, Athens, Greece
\textsuperscript{b}Physics Department, Ben-Gurion University of the Negev, Sderot David Ben Gurion 1, Beer-Sheva 84105, Israel
\textsuperscript{c}Frankfurt Institute for Advanced Studies (FIAS), Ruth-Moufang-Strasse 1, 60438 Frankfurt am Main, Germany
\textsuperscript{d}Academy of Athens, Research Center for Astronomy and Applied Mathematics, Soranou Efesou 4, 11527, Athens, Greece
\textsuperscript{e}Bahamas Advanced Study Institute and Conferences, 4A Ocean Heights, Hill View Circle, Stella Maris, Long Island, The Bahamas
\textsuperscript{f}National Observatory of Athens, V. Pavlou and I. Metaxa 15236, Penteli, Greece

E-mail: fotis-anagnostopoulos@hotmail.com, benidav@post.bgu.ac.il, svasil@academyofathens.gr, guendel@bgu.ac.il

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\textsuperscript{1}Corresponding author.
Abstract. A recently proposed Dynamical Space-time Cosmology (DSC) that unifies dark energy and dark matter is studied. The general action of this scenario includes a Lagrange multiplier, which is coupled to the energy momentum tensor and a scalar field which is different from quintessence. First for various types of potentials we implement a critical point analysis and we find solutions which lead to cosmic acceleration and under certain conditions to stable late-time attractors. Then the DSC cosmology is confronted with the latest cosmological data from low-redshift probes, namely measurements of the Hubble parameter and standard candles (Pantheon SnIa, Quasi-stellar objects). Performing an overall likelihood analysis and using the appropriate information criteria we find that the explored DSC models are in very good agreement with the data. We also find that one of the DSC models shows a small but non-zero deviation from Λ cosmology, nevertheless the confidence level is close to $\sim 1.5\sigma$.

Keywords: dark energy experiments, dark energy theory, dark matter theory

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A Big Bang Nucleosynthesis (BBN) within DSC

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## 1 Introduction

Almost twenty years after the observational evidence of cosmic acceleration the cause of this phenomenon, labeled as “dark energy” (hereafter DE), remains an open question which challenges the foundations of theoretical physics: the cosmological constant problem — why there is a large disagreement between the vacuum expectation value of the energy momentum tensor which comes from quantum field theory and the observable value of dark energy density [1]–[5]. The simplest model of DE is the so called ΛCDM model that contains non-relativistic matter and cosmological constant. Although, this model fits accurately the present cosmological data it suffers from two fundamental problems, namely the tiny value of the cosmological constant and also the coincidence problem, [6]. Furthermore, there is criticism on the conceptual foundations of the current view of the cosmos, in a sense that there are too many ad hoc hypotheses (e.g. dark Energy, dark matter) needed for “explaining the phenomena”, e.g. [7].

The main argument of the latter article is that whenever a scientific theory encounters difficulties in explaining phenomena, adding auxiliary hypotheses within the body of the theory, is considered bad practice. This is so as it could lead to non-falsifiable theories, [8]. The aforementioned criticism is not founded upon physical considerations so it can not be used right away to construct a cosmological model. However, it motivates the development of alternative cosmological models, which could provide a more natural description of the so called dark sector.
Unification between dark energy and dark matter from an action principle was obtained from scalar fields [11]–[12], by a complex scalar field [13] or other models [14]–[19] including Galileon cosmology [17] or Telleparallel gravity [20]–[21]. Beyond those approaches, a unification of Dark Energy and Dark Matter using a new measure of integration (the so-called Two Measure Theories) has been formulated [23]–[27]. A diffusive interacting of dark energy and dark matter models was introduced in [28]–[29] and it has been found that diffusive interacting dark energy — dark matter models can be formulated in the context of an action principle based on a generalization of those Two Measures Theories in the context of quintessential scalar fields [30]–[31, 32], although these models are not equivalent to the previous diffusive interacting dark energy — dark matter models. In order to overcome the coincidence problem, Gao, Kunz, Liddle and Parkinson [33] suggested a unification of dark energy and dark matter resulting from a single scalar field. Unlike usual quintessence model, here the scalar field behaves either as dark matter or dark energy. Within this framework the unified picture of dark sector introduces a number of modifications in the equations of motion of the aforementioned scalar field. Recently, a Lagrangian formulation was introduced in [34] (see also [35]) toward building the so called Dynamical Space-time Cosmological (DSC) model. In this scenario the gravitational field is described not only by the metric tensor but also by a Lagrange multiplier that is coupled to the energy momentum tensor, a scalar field potential and another potential that describes the interactions between DE and DM. The scalar field $\phi$ plays an important role in the description of the dynamics, since the kinetic term of $\phi$ behaves as DM and the potential is responsible for DE. Therefore, the DSC model provides an elegant alternative in describing the DM and DE dominated eras respectively.

In the current paper we attempt to continue our previous work of [34] in the sense that we study both dynamically and observationally the DSC scenario for a large family of potentials. Specifically, the manuscript is organized as follows. In section II we briefly present the theoretical framework of the Dynamical Space-time Cosmological model and provide the basic cosmological equations. In section III we use a dynamical analysis by studying the critical points of the field equations in the dimensionless variables for a large family of potentials. In section IV we provide the likelihood analysis and the observational data sets that we utilize in order to constraint the free parameters of the DSC model and we compare it with the $\Lambda$CDM model. Finally, in section V we draw our conclusions.

2 Dynamical space-time cosmology

In previous publications two of us (Benisty and Guendelman, [34]) proposed the Dynamical Space-time Cosmological model (DSC) via a space time vector field, and demonstrated the behavior of this scenario toward unifying the dark sector. In this section we briefly present the main features of the DSC model based on first principles. The action that describes the gravitational field equations and unifies the dark sector was first introduced by Benisty and Guendelman [34]:

$$S = \int \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + \chi_{\mu \nu} T^{\mu \nu}_{(\phi)} - \frac{1}{2} g^{\alpha \beta} \phi_{,\alpha} \phi_{,\beta} - V(\phi) \right] d^4x,$$

(2.1)

where $\phi$ is the scalar field and $R$ is the Ricci scalar and $\kappa^2 = 8\pi G = 1$. The vector field $\chi_{\mu}$ is the so called dynamical space time vector, hence the corresponding covariant derivative is $\chi_{\mu;\nu} = \partial\nu \chi_{\mu} - \Gamma^\lambda_{\mu \nu} \chi_\lambda$, where $\Gamma^\lambda_{\mu \nu}$ is the Christoffel symbol. In this context, $T^{\mu \nu}_{(\phi)}$ denotes the
stress energy tensor which was first introduced by Gao and colleagues [33]

\[
T^{\mu \nu}_{(\phi)} = -\frac{1}{2} \phi^\mu \phi^\nu + U(\phi) g^{\mu \nu}. \tag{2.2}
\]

Obviously the action integral contains two different potentials, namely \(U(\phi)\) which is coupled to the stress energy momentum tensor \(T^{\mu \nu}_{(\phi)}\), and \(V(\phi)\) which is minimally coupled into the action. Moreover, the action depends on three different quantities: the scalar field \(\phi\) the dynamical space time vector \(\chi_\mu\) and the metric \(g_{\mu \nu}\).

### 2.1 Equations of motion

There are 3 independent variations for this theory. The first variation is with respect to the dynamical spacetime vector field \(\chi_\mu\) which yields the conservation of the energy momentum tensor \(T^{\mu \nu}_{(\phi)}\):

\[
\nabla_\mu T^{\mu \nu}_{(\phi)} = 0. \tag{2.3}
\]

The second variation with respect to the scalar field \(\phi\) gives a non-conserved current:

\[
\chi^\lambda_\mu U'(\phi) - V'(\phi) = \nabla_\mu j^\mu \tag{2.4a}
\]

\[
J^\mu = \frac{1}{2} \phi,\nu \left( \chi^\mu_\nu; + \chi^\nu_\mu; \right) + \phi^\mu \tag{2.4b}
\]

and the derivatives of the potentials are the source of this current. For constant potentials the source term becomes zero and we get a covariant conservation of the current.

Lastly, varying the action integral with respect to the metric, we derive the gravitational field equations

\[
\frac{1}{\kappa^2} G^{\mu \nu} = g^{\mu \nu} \left( \frac{1}{2} \phi,\alpha \phi,\alpha + V(\phi) + \frac{1}{2} \chi^{\alpha \beta} \phi,\alpha \phi,\beta + \chi^\lambda \phi,\lambda U'(\phi) \right) - \frac{1}{2} \phi^\mu \left[ \left( \chi^\lambda_\lambda + 2 \right) \phi^\nu + \chi^\lambda \phi,\lambda + \chi^\lambda \phi,\nu \right] - \frac{1}{2} \left( \chi^\lambda \phi,\lambda \phi,\nu + \chi^\lambda \phi,\nu \phi,\lambda \right). \tag{2.5}
\]

### 2.2 Homogeneous solution

The (FLRW) Friedman-Lemaitre-Robertson-Walker ansatz is the standard model of cosmology dynamics based on the assumption of a homogeneous and isotropic universe at any point, commonly referred to as the cosmological principle. The symmetry considerations lead to the FLRW metric

\[
ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right]. \tag{2.6}
\]

Herein, \(a(t)\) defines the dimensionless cosmological expansion (scale) factor. For simplicity we consider a homogeneous scalar field \(\phi = \phi(t)\), while the dynamical vector \(\chi_\mu\) is given by the following formula \(\chi_\mu = (\chi_0, 0, 0, 0)\), where \(\chi_0\) is also just a function of time.

Varying the action with respect to the dynamical space time vector field \(\chi_\mu\) we obtain the modified “Klein-Gordon” equation

\[
\dddot{\phi} + \frac{3}{2} H \dot{\phi} + U'(\phi) = 0, \tag{2.7}
\]

where the prime denotes derivative with respect to \(\phi\). Compared with the equivalent equation which comes from quintessence model, this model gives a different and smaller friction term,
as compared to the canonical scalar field. Therefore for increasing redshift, the densities for the scalar field will increase slower than in the standard quintessence.

The second variation, for homogeneous \( \phi \) and \( \chi_\mu = (\chi_0, 0, 0, 0) \) eq. (2.4) becomes

\[
j^\mu = (\dot{\phi}(1 - \chi_0), 0, 0, 0),
\]

hence for FRWL metric we obtain

\[
j^\mu_\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} j^\mu) = \frac{1}{a^3} \partial_\mu (a^3 j^0) = -\ddot{\phi}(\chi_0 - 1) + \dot{\phi} [3H(\chi_0 - 1) - \dot{\chi}_0]
\]

and the source term yields:

\[
\chi^\lambda_\mu U'(\phi) - V'(\phi) = U'(\phi) \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \chi^\mu) - V'(\phi)
\]

\[
= -U'(\phi) \frac{1}{a^3} \partial_\mu (a^3 \chi^0) - V'(\phi) = U'(\phi) [\chi_0 + 3H\chi_0] + V'(\phi).
\]

Therefore, the equation of motion takes the form:

\[
\ddot{\phi}(\chi_0 - 1) + \dot{\phi} [3H(\chi_0 - 1) + \dot{\chi}_0] = U'(\phi) [\chi_0 + 3H\chi_0] - V'(\phi).
\]

For the spatially homogeneous cosmological case the energy density and the pressure of the scalar field read:

\[
\rho = \dot{\phi}^2 \left( \chi_0 \left( 1 - \frac{3}{2} H \right) - \frac{1}{2} \right) + V(\phi) - \dot{\phi} \chi_0 (U'(\phi) + \dot{\phi})
\]

\[
p = \frac{1}{2} \dot{\phi}^2 (\chi_0 - 1) - V(\phi) - \chi_0 \dot{\phi} U'(\phi)
\]

Comparing the stress energy tensor with equations (2.7), we provide the functional forms of the energy density and pressure respectively:

\[
\rho = \left( \chi_0 - \frac{1}{2} \right) \dot{\phi}^2 + V(\phi)
\]

\[
p = \frac{1}{2} \dot{\phi}^2 (\chi_0 - 1) - V(\phi) - \chi_0 \dot{\phi} U'(\phi).
\]

Unlike usual DE models, quintessence and the like, here the vector field \( \chi_0 \) and the potential \( U(\phi) \) modify the density and the pressure of the cosmic fluid. In order to proceed with the analysis we need to know the forms of \( U(\phi) \) and \( V(\phi) \). Below, we consider special forms of the potentials and study the performance of the models at the expansion level.

### 2.2.1 Coupled constant potential into the Lagrange multiplier

Similar to [34] we consider DSC models for which the potential \( U(\phi) \) that is coupled to the stress energy momentum tensor \( T^{\mu\nu}_{(\phi)} \) is constant

\[
U(\phi) = \text{Const}.
\]

The general study of varying \( U(\phi) \) will appear in a forthcoming paper. Substituting the potential into eq. (2.7), using the definition of \( H = \dot{a}/a \) and performing the integration we find

\[
\dot{\phi}^2 = \frac{2C_{m0}}{a^3},
\]
where $C_{m0} > 0$ is the integration constant which can be viewed as the effective dark matter energy density parameter. Introducing the new variable

$$\delta = \dot{\chi}_0 - 1$$  \hspace{1cm} (2.17)

equations eqs. (2.11), (2.13) and (2.14) become

$$\dot{\phi} \left( \frac{3}{2} H \delta + \frac{1}{2} \right) = -V'(\phi).$$  \hspace{1cm} (2.18)

In this context, the energy density and the pressure of the scalar field are given by

$$\rho = \left( \delta + \frac{1}{2} \right) \phi^2 + V(\phi),$$  \hspace{1cm} (2.19a)\n
$$p = \frac{1}{2} \delta \phi^2 - V(\phi).$$  \hspace{1cm} (2.19b)

Furthermore, if we assume $V(\phi) = \Lambda = \text{Const.}$ then the solution of eq. (2.18) is

$$\delta = \frac{1}{2} \xi a^{-3/2},$$  \hspace{1cm} (2.20)

where $\xi$ is a dimensionless integration constant and hence with the aid of (2.16) we obtain

$$\rho = \Lambda + \frac{\xi C_{m0}}{a^{9/2}} + \frac{C_{m0}}{a^3},$$  \hspace{1cm} (2.21a)\n
$$p = -\Lambda + \frac{\xi C_{m0}}{2a^{9/2}}.$$  \hspace{1cm} (2.21b)

In such a case it is trivial to show that the Hubble parameter is given by

$$H(z) = H_0 \left[ \Omega_\Lambda + \Omega_{m0}(1 + z)^3 + \Omega_{\xi0}(1 + z)^{9/2} \right]^{1/2},$$  \hspace{1cm} (2.22)

where $\Omega_{\xi0} = \xi \Omega_{m0}$ and $H_0$ is the Hubble constant, while we normalize the first Friedmann equation by the critical density $\rho_c = 3H_0^2$. $\Omega_\Lambda := \Lambda/\rho_c$, $\Omega_{m0} := C_{m0}/\rho_c$. The current model can be seen as an approximation of the general $U, V$ potentials, namely close to the present era where the potentials vary slowly with time. Therefore, the Hubble expansion eq. (2.22) resembles that of the general case only in the late universe. Moreover, in the case of $\xi < 0$ the latter situation holds for $z < z_{\text{max}}$, where $z_{\text{max}} \simeq (-\Omega_{m0}/\Omega_{\xi0})^{2/3} - 1$. For a barotropic cosmic fluid whose the corresponding equation of state parameter is given by $w_i = p_i/\rho_i$ one can easily recognize three “dark fluids”, namely cosmological constant $[w_\Lambda = -1, V(\phi) = \text{Const.}]$, dark matter ($w_m = 0$), and another fluid with $w_\xi = 1/2$.

Notice that in the case of $\chi_0 = t$, from eq. (2.17) and (2.20) we get:

$$\delta = \xi = 0.$$  \hspace{1cm} (2.23)

Therefore, $\Lambda\text{CDM}$ model is precisely obtained from eq. (2.21a) and (2.21b), namely

$$\rho = \Lambda + \frac{C_{m0}}{a^3}$$  \hspace{1cm} (2.24a)\n
$$p = -\Lambda,$$  \hspace{1cm} (2.24b)

Lastly, it is interesting to mention that for $\Omega_{\xi0} \neq 0$ we can get bouncing solutions as discussed in [34].
2.2.2 Dynamical DM-DE

Here let us concentrate on a more general situation for which \( U(\phi) = \text{Const.} \) and \( V = V(\phi) \). Within this framework, the combination of equations (2.16), (2.18), (2.19a) and (2.19b) provide

\[
\frac{d\delta}{dz} = \frac{V'(\phi)}{(z + 1)^{3/2} \sqrt{\mathcal{C}_{m0} H(\phi, \delta)}} + \frac{3\delta}{2(z + 1)} \quad (2.25a)
\]

\[
\frac{d\phi}{dz} = -\sqrt{2\mathcal{C}_{m0}(z + 1)} H(\phi, \delta) \quad (2.25b)
\]

with the Hubble parameter

\[
H(\phi, \delta) = H_0 \left[ (2\delta + 1)\Omega_{m0} (z + 1)^3 + \Omega_{DE}(\phi) \right]^{1/2}, \quad (2.25c)
\]

where \( \Omega_{DE}(\phi) = V(\phi)/\rho_c \) and \( z = a^{-1} - 1 \) is the redshift. Therefore, in order to derive the evolution of the Hubble parameter we need to solve the system of equations (2.25a), (2.25b).

Suppose that we know the functional form of the potential \( V(\phi) \). First we evaluate eq. (2.25c) at \( z = 0 \) which means that \((2\delta(z = 0) + 1)\Omega_{m0} + \Omega_{DE}(\phi(z = 0)) = 1\). Second, the fact that \( V(\phi) \sim \Lambda \) prior to the present time together with the cosmic sum \( \Omega_{DE,0} + \Omega_{m0} = 1 \) imply \( \delta(z = 0) = 0 \), hence the form of \( V(\phi) \) obeys \( V(\phi) = \Lambda f(\phi) \), where \( f(\phi) = 1 \) at \( z = 0 \). Concerning the types of \( V(\phi) \) potentials involved in the present analysis, we consider the following three cases: exponential with \( V(\phi) = V_0 e^{-\beta \phi} \), cosine with \( V(\phi) = V_0 \cos \beta \phi \) and thawing potential with \( V(\phi) = V_0 e^{-\alpha \phi}(1 + \beta \phi) \), [40]. This family of models has \( \Lambda \text{CDM} \) as an asymptotic solution. Notice that the initial condition for \( \phi \) is chosen appropriately to be compliant with the aforementioned constrain, that is \( \phi(z = 0) = 0 \). Once steps (i) and (ii) are accomplished, we numerically solve the system (2.25a), (2.25b).

3 Dynamical system method

In this section we provide a dynamical analysis by studying the fixed points of the field equations, so that we can investigate the various phases of the current cosmological models. Specifically, for a general potential \( V(\phi) \) we introduce the new dimensionless variables

\[
x = \frac{\dot{\phi}}{\sqrt{6} H}, \quad y = \frac{\sqrt{V(\phi)}}{\sqrt{3} H}, \quad z = \frac{V'(\phi)}{V(\phi)}. \quad (3.1)
\]

In the new system of variables the field equations form an autonomous system which is given by

\[
x' = -\frac{3}{4} x(x^2 + 3y^2 - 1) \quad (3.2a)
\]

\[
y' = -\frac{1}{4} y \left( 3x^2 + 9y^2 - 9 + 2\sqrt{6}xz \right) \quad (3.2b)
\]

\[
z' = -\sqrt{6} z^2 x (\Gamma - 1), \quad (3.2c)
\]

where

\[
\Gamma = \frac{V(\phi)V''(\phi)}{V'(\phi)^2}. \quad (3.3)
\]
The point $(x, y)$

| Name   | Stability                  | Universe         | The point $(x, y)$ |
|--------|----------------------------|------------------|-------------------|
| A      | unstable                   | —                | $(0, 0)$          |
| B      | stable for $\beta > \sqrt{\frac{3}{2}}$ | Dark Matter | $(1, 0)$          |
| C      | asymptotically stable      | Dark Energy      | $(0, 1)$          |
| D      | unstable saddle p.         | unified DE-DM    | $(\sqrt{\frac{3}{2}}\frac{1}{\beta}, \sqrt{\frac{2\beta^2-3}{\sqrt{6}\beta}})$ |

Table 1. Critical points for $V(\phi) \propto e^{-\beta \phi}$.

| Name   | Stability                  | Universe         | The point $(x, y)$ |
|--------|----------------------------|------------------|-------------------|
| A      | unstable                   | —                | $(0, 0)$          |
| B      | unstable                   | Dark Matter      | $(1, 0)$          |
| C      | asymptotically stable      | Dark Energy      | $(0, 1)$          |

Table 2. Critical points for the Cosine potential.

These are the basic variables that we use for mapping the dynamical system. In this case the equation of motion with respect to the metric is written as:

$$(1 + 2\delta)x^2 + y^2 = 1 \quad (3.4)$$

Notice that for $\chi_0 \sim t$, which means $\delta \sim 0$, the phase plane of the system takes the form of a complete circle, where the points $(1, 0)$ and $(0, 1)$ correspond to dark matter and dark energy dominated eras respectively.

Below we provide the results of the dynamical analysis for different types of potentials. The corresponding critical points of the system (3.2a), (3.2b) and (3.2c) are listed in tables I, II and III. In all cases point A with coordinates $(0, 0)$ is ruled out from the constrain (3.4).

### 3.1 Exponential potential ($V_1$)

We continue our work by using the exponential potential. In this case the new variable $z$ (see the last term in eq. (3.1)) becomes constant. The dynamical system includes four critical points, among which one point is stable. Point B with coordinates $(1, 0)$ corresponds to the matter epoch and it is stable when $\beta > \sqrt{\frac{3}{2}}$. Point C with coordinates $(0, 1)$ describes the dark energy dominated era, while point D $(\beta > \sqrt{\frac{3}{2}})$ with coordinates $(\sqrt{\frac{3}{2}}\frac{1}{\beta}, \sqrt{\frac{2\beta^2-3}{\sqrt{6}\beta}})$ is unstable.

### 3.2 Cosine potential ($V_2$)

Now we proceed with the cosine potential $V \propto \cos(\beta \phi)$. Inserting this formula into $\Gamma$ we find

$$z' = \sqrt{6}x(1 + z^2). \quad (3.5)$$

Therefore, for the dynamical analysis we utilize the aforementioned equation together with eqs. (3.2a)–(3.2b). In this case we find three critical points which are not affected by $\beta$. As expected, points $B(1, 0)$ and $C(0, 1)$ describes the dark matter and dark energy dominates eras respectively. Here $B$ is always unstable, while $C$ is asymptotically stable.
### Table 3. Critical points for $V(\phi) \propto e^{-\alpha \phi}(1 + \beta \phi)$.

| Name | Existence | Stability | Universe | The point $(x, y, z)$ |
|------|-----------|-----------|----------|---------------------|
| A    | all $\alpha$ | unstable | —        | $(0, 0)$           |
| B    | all $\alpha$ | $\alpha < \sqrt{3}$ unstable, $\alpha > \sqrt{3}$ saddle point. | Dark Matter | $(1, 0, \alpha)$ |
| C    | all $\alpha$ | stable | Dark Energy | $(0, 1)$         |
| D    | $\alpha > \sqrt{2}$ | $\alpha < \sqrt{2}$ stable focus, $\alpha > \sqrt{2}$ saddle | unified DE-DM | $(\sqrt{\frac{3}{2\alpha}}, \sqrt{\frac{2\alpha - 3}{6\alpha}}, \alpha)$ |

#### 3.3 Thawing potential ($V_3$)

Using the thawing potential $V(\phi) \propto e^{-\alpha \phi}(1 + \beta \phi)$ (3.2c) becomes:

$$z' = \sqrt{6x(z - \alpha)^2}.$$ (3.6)

In this case point $B(1, 0)$ is unstable when $\alpha < \sqrt{3}$ and it is saddle for $\alpha > \sqrt{3}$. The dark energy point $C(0, 1)$ is always stable. Lastly, point D with coordinates $(\sqrt{\frac{3}{2\alpha}}, \sqrt{\frac{2\alpha - 3}{6\alpha}}, \alpha)$ is stable when $\alpha < \sqrt{\frac{5}{6}}$ and it is saddle when $\alpha > \sqrt{\frac{5}{6}}$.

### 4 Observational constraints

In the following we describe the observational data sets along with the relevant statistics in constraining the DSC models presented in section III.

#### 4.1 Direct measurements of the Hubble expansion

We use the latest $H(z)$ data set compilation, that can be found in [41]. This set contains $N = 36$ measurements of the Hubble expansion in the following redshift range $0.07 \leq z \leq 2.33$. Out of these, there are 5 measurements based on Baryonic Acoustic Oscillations (BAOs), while for the rest, the Hubble parameter is measured via the differential age of passive evolving galaxies.

Here, the corresponding $\chi^2_H$ function reads

$$\chi^2_H(\phi^\nu) = \mathcal{H}C^{-1}_H \mathcal{H}^T,$$ (4.1)

where $\mathcal{H} = \{H_1 - H_0 E(z_1, \phi^\nu), \ldots, H_N - H_0 E(z_N, \phi^\nu)\}$ and $H_i$ are the observed Hubble rates at redshift $z_i$ ($i = 1, \ldots, N$). Notice, that the statistical vector $\phi^\nu$ contains the parameters that we want to fit. The matrix $C$ denotes the covariance matrix. Further considerations regarding the statistical analysis and the corresponding covariance matrices can be found in ref. [43] and references therein.

#### 4.2 Standard candles

The second data-set that we include in our analysis is the binned Pantheon sample of Scolnic et al. [45] and the binned sample of Quasi-Stellar Objects (QSOs), [46]. We would like to note the importance of using the Pantheon SnIa data along with those of QSOs. The latter allows to trace the cosmic history to the redshift range $0.07 < z < 6$. It is important to utilize alternative probes at higher redshifts in order to verify the SnIa results and test any possible evolution of the DE equation of state [48]. Following standard lines, the chi-square function of the standard candles is given by

$$\chi^2_s(\phi^\nu_s) = \mu_s C^{-1}s \mu^T_s,$$ (4.2)
where $\mu_s = \{\mu_1 - \mu_{th}(z_1, \phi'), \ldots, \mu_N - \mu_{th}(z_N, \phi')\}$ and the subscript $s$ denotes SNIa and QSOs. For the SNIa data the covariance matrix is not diagonal and the distance modulus is given as $\mu_i = \mu_{B,i} - M$, where $\mu_{B,i}$ is the apparent magnitude at maximum in the rest frame for redshift $z_i$ and $M$ is treated as a universal free parameter, [45], quantifying various observational uncertainties. It is apparent that $M$ and $h$ parameters are intrinsically degenerate in the context of the Pantheon data set, so we can not extract any information regarding $H_0$ from SNIa data alone. In the case of QSOs, $\mu_i$ is the observed distant modulus at redshift $z_i$ and the covariance matrix is diagonal. In all cases, the theoretical form of the distance modulus reads

$$\mu_{th} = 5 \log \left( \frac{D_L(z)}{\text{Mpc}} \right) + 25, \quad (4.3)$$
where

$$D_L(z) = c(1 + z) \int_0^z \frac{dx}{H(x, \theta^\nu)}$$

(4.4)

is the luminosity distance, for spatially flat FRWL cosmology.

4.3 Joint analysis and model selection

In order to perform a joint statistical analysis of $P$ cosmological probes (in our case $P = 3$), we need to use the total likelihood function

$$L_{\text{tot}}(\phi^\psi) = \prod_{p=1}^P \exp(-\chi_p^2).$$

(4.5)
Figure 3. Observational constraints of the cosine DSC, $V_2 \propto \cos(\beta \phi)$, while we have used $U(\phi) = \text{Const.}$

Consequently the $\chi^2_{\text{tot}}$ expression is given by

$$\chi^2_{\text{tot}} = \sum_{p=1}^{P} \chi^2_P,$$

(4.6)

where the statistical vector has dimension $\psi$, which is the sum of the $\nu$ parameters of the model at hand plus the number $\nu_{\text{hyp}}$ of hyper-parameters of the data sets used, that is $\psi = \nu + \nu_{\text{hyp}}$. The distinction between the hyper-parameters quantifying uncertainties in a data set and the free parameters of the cosmological model is purely conceptual. Regarding the problem of likelihood maximization, we use an affine-invariant Markov Chain Monte Carlo sampler, as described in ref. [36]. We utilize the open-source python package emcee, [37], using
Figure 4. Observational constraints of the thawing DSC, $V_3 \propto e^{-\alpha\phi}(1 + \beta\phi)$, while we have used $U(\phi) = \text{Const.}$

1000 “walkers” and 1500 “states”. The latter setup corresponds to $\sim 10^6$ calls of the total likelihood function. In each call, we need to numerically solve the system of eqs. (2.25) for the redshift range $[0.0, 5.93]$ and also calculate the luminosity distance. This procedure became practical by optimizing critical parts of the calculations using C++ code from ref. [38]. The convergence of the MCMC algorithm is checked with auto-correlation time considerations.

4.4 Statistical results

In order to test the performance of the cosmological models in fitting the data it is imperative to utilize the Akaike Information Criterion (AIC), [49], and Bayesian Information Criterion (BIC), [50]. The AIC criterion is an asymptotically unbiased estimator of the Kullback–
Model $\Omega_{m0}$ $h$ $\alpha$ or $\xi$ $\beta$ $\mathcal{M} \chi_{\min}^2$

$\Lambda$CDM $0.305^{+0.031}_{-0.025}$ $0.6257^{+0.0428}_{-0.0455}$ $0.183^{+0.143}_{-0.125}$ $-0.025$ $0.6257^{+0.0428}_{-0.0455}$ $-0.025$ $-19.397^{+0.034}_{-0.035}$ $84.114$

$V, U_{\text{const.}}$ $0.305^{+0.031}_{-0.025}$ $0.6257^{+0.0428}_{-0.0455}$ $0.183^{+0.143}_{-0.125}$ $-0.025$ $0.6257^{+0.0428}_{-0.0455}$ $-0.025$ $-19.397^{+0.034}_{-0.035}$ $84.114$

$V_1$ $0.277^{+0.024}_{-0.023}$ $0.6885^{+0.0130}_{-0.0128}$ $-0.0127$ $-0.0127$ $-0.593^{+1.367}_{-1.355}$ $-19.390^{+0.014}_{-0.015}$ $88.100$

$V_2($cosine$)$ $0.270 \pm 0.015$ $0.6895^{+0.0128}_{-0.0127}$ $-0.0127$ $-0.0127$ $-0.593^{+1.367}_{-1.355}$ $-19.388 \pm 0.035$ $87.954$

$V_3$ $0.273^{+0.024}_{-0.023}$ $0.6890^{+0.0130}_{-0.0127}$ $1.152^{+1.370}_{-1.352}$ $1$ $-19.389 \pm 0.034$ $87.942$

$\Lambda$CDM $0.281^{+0.016}_{-0.015}$ $0.686 \pm 0.013$ $-0.0127$ $-0.0127$ $-19.403 \pm 0.035$ $85.700$

| Table 4. Observational constraints and the corresponding $\chi_{\min}^2$ for the considered cosmological models. Notice that $\Omega_{\gamma 0} = \xi \Omega_{m0}$. The concordance $\Lambda$ model is included for comparison. |

Leibler information, measuring the loss of information during the fit. Within the standard assumption of Gaussian errors, the AIC estimator is given by [52, 53]

$$AIC = -2 \ln(L_{\max}) + 2\psi + \frac{2\psi(\psi + 1)}{N_{\text{tot}} - \psi - 1}.$$  \hspace{1cm} (4.7)

where $L_{\max}$ is the maximum likelihood of the data set(s) under consideration and $N_{\text{tot}}$ is the total number of data. It is easy to observe that for large $N_{\text{tot}}$, this expression reduces to $AIC \simeq -2 \ln(L_{\max}) + 2\psi$, that is the standard form of the AIC criterion. Following the previous point, it is considered good practice to use the modified AIC criterion in all cases, [51].

On the other hand, the BIC criterion is an estimator of the Bayesian evidence, (e.g. [51–53] and references therein), and is given as

$$BIC = -2 \ln(L_{\max}) + \psi \log(N_{\text{tot}}).$$  \hspace{1cm} (4.8)

The AIC and BIC criteria employ only the likelihood value at maximum. In principle, due to the Bayesian nature of our analysis, the accuracy of the $L_{\max}$ is reduced, meaning that the AIC and BIC values are meant to be used just for illustrative purposes. In practice, however, by using long chains, we obtain $L_{\max}$ values with enough accuracy to use them in order to calculate AIC and BIC. Furthermore, we also compute the Deviance Information Criterion (DIC), that provides all the information obtained from the likelihood calls during the maximization procedure. The DIC estimator is defined as, (see [51, 54])

$$DIC = D(\bar{\psi}) + 2C_B,$$  \hspace{1cm} (4.9)

where $C_B$ is the so called Bayesian complexity that measures the power of data to constrain the parameter space compared to the predictivity of the model which is provided by the prior. In particular, $C_B = D(\bar{\psi}) - D(\bar{\psi}^2)$, where the overline denotes the usual mean value. Also, $D(\bar{\psi}^2)$ is the Bayesian Deviation, where in our case it boils down to $D(\bar{\psi}^2) = -2 \ln(L(\bar{\psi}^2))$.

To proceed with the model selection we need to assign a “probability” to each model following the classical treatment of Jeffreys, [55], that is by using the relative difference of the IC value for a number of models, $\Delta IC_{\text{model}} = IC_{\text{model}} - IC_{\text{min}}$, where the $IC_{\text{min}}$ is the minimum IC value in the set of competing models. Following the notations of ref. [56], $\Delta IC \leq 2$, means that the model under consideration is statistically compatible with the “best” model, while the condition $2 < \Delta IC < 6$ indicates a middle tension between the two models and the condition $\Delta IC \geq 10$ suggests a strong tension.

Utilizing the aforementioned likelihood analysis we summarize our statistical results in table 4.
For the model with constant potentials, we find $\Omega_m = 0.305_{-0.025}^{+0.031}, h = 0.6257_{-0.0455}^{+0.0428}, \xi = 0.183_{-0.125}^{+0.143}$ with $\chi^2_{\text{min}} = 84.114$. The relevant contours are present at figure 1. Interestingly, the $\chi = 0$ value which corresponds to the $\Lambda$CDM limit is outside the $1\sigma$ area.

Regarding the exponential potential ($V_1$), we find $\Omega_m = 0.277_{-0.022}^{+0.024}, h = 0.6885_{-0.0130}^{+0.0130}, \beta = -0.593_{-1.355}^{+1.367}$ with $\chi^2_{\text{min}} = 88.100$ and the contours are in figure 2. Furthermore, the cosmological parameters for the cosine potential ($V_2$) are $\Omega_m = 0.270 \pm 0.015, h = 0.6895_{-0.0127}^{+0.0128}$ and the relevant $\chi^2_{\text{min}} = 87.954$. The contour plots are presented in figure 3. Lastly, for the potential $V_3$ we obtain the contours of figure 4 and the parameter values: $\Omega_m = 0.273_{-0.022}^{+0.024}, h = 0.6890_{-0.0127}^{+0.0130}, \alpha = 1.152_{-1.352}^{+1.370}$ and $\chi^2_{\text{min}} = 87.942$. In most of the cases, the best fit values of the matter energy density are in good agreement with those of Planck 2018, [39]. Considering the result for the flat $\Lambda$CDM, we observe $1\sigma$ compatibility for the $V_i, i = 1, 2, 3$ potentials, while the result for the cosmological model with constant $V, U$ potentials is within $2\sigma$ limits. It is important to note that for $V_2$ and $V_3$ potentials we set $\beta = 1$. However, we have tested that the likelihood analysis provides very similar results for $\beta \sim O(1)$.

We deem interesting to discuss our results with respect to the Hubble constant problem, that is a $\sim 3.7\sigma$ discrepancy between the Hubble constant measured by Riess et al. [57], $(H_0 = 73.48 \pm 1.66 \text{Km/s/Mpc})$ and the relevant value from Planck collaboration, $(H_0^{\text{Planck}} = 67.36 \pm 0.54 \text{ Km/s/Mpc})$, [39], see figure 5. Our results are in agreement (within $1\sigma$) with those provided by the team of Planck, while there is compatibility at $\sim 2\sigma$ level with Riess et al. results. However, the Hubble constant for the constant potentials case is significantly smaller from other relevant results, however due to the large error bar maintains $\sim 1\sigma$ compatibility. As a consistency check we compare our results with the result from the model-independent assessment of the cosmic history obtained by Haridasu et al. [58], namely $H_0^{\text{ind.}} = 68.52 \pm 0.94$ and we report $1\sigma$ compatibility for $V_1, V_2, V_3$ potentials, while the $V, U$ constant potential is within $2\sigma$.

Concerning AIC, BIC and DIC and we present the relevant values at the table 5. In the context of BIC, all models considered are in mild to strong tension with $\Lambda$CDM. As we used binned data sets, we do not anticipate that an information criterion with an explicit dependence from the dataset length could estimate reliably the relative quality of the fits. Further, the BIC criterion is just an asymptotic approximation that is valid while the dataset length tends to infinity. On the other hand, AIC criterion provides a somewhat different view. The model with constant potentials has $\Delta AIC \leq 2$, hence it is nearly indistinguishable from $\Lambda$CDM. The other models ($V_i, i = 1, 2, 3$) are in mild tension with $\Lambda$CDM since they have $2 < \Delta AIC < 6$. In the context of our Bayesian treatment, both AIC and BIC values could

| Model | AIC   | $\Delta$AIC | BIC   | $\Delta$BIC | DIC   | $\Delta$DIC |
|-------|-------|-------------|-------|-------------|-------|-------------|
| const | 92.535| 0.582       | 102.535| 3.020       | 88.567| 0           |
| $V_1$ | 96.521| 4.569       | 106.5210| 7.006       | 95.908| 7.341       |
| $V_2$ | 94.204| 2.252       | 101.770| 2.255       | 93.930| 5.363       |
| $V_3$ | 96.363| 4.411       | 106.363| 6.848       | 95.805| 7.238       |
| $\Lambda$CDM | 91.952| 0           | 99.515| 0           | 91.671| 3.104       |

Table 5. The information criteria AIC, BIC and DIC for various cosmological models along with the corresponding differences $\Delta$IC = IC – I$_{\text{min}}$.
Figure 5. A synopsis of our results regarding the Hubble constant problem. The labels ‘Planck18’ and ‘Riess18’ stand for the relevant results from ref. [39] and [57] respectively.

only serve indicative purposes, as they employ only the value of the likelihood at maximum and not the full set of likelihood values obtained during the sampling procedure. The most interesting observation comes from the DIC criterion, which seems to prefer the cosmological model with constant \((U, V)\) potentials over the concordance model, as \(DIC_\Lambda > DIC_{\text{const}UV}\) and the relevant \(\Delta DIC (\chi^2)\) indicates that the difference is rather significant. However, as we mentioned before, the constant potentials model is an approximation of a more general cosmological model, valid for late universe only. With respect to the other models under consideration, we observe mild-to-strong tension with each of them, with the \(\Lambda CDM\) to be in the second place. A general ascertainment regarding the somewhat similar results of the physically different potentials in the free parameters (e.g. matter energy density and Hubble constant) is that \(\phi\) is very small at late universe, so any \(V(\phi)\) is effectively \(V(\phi) \sim \Lambda\) (where we have set \(8\pi G = 1\)). This is what someone could naively foresee as the field \(\phi\) changes very smoothly across the cosmic history. We expect that a study of the model using the CMB spectrum could discriminate between the different DSC models.

5 Conclusions

We explored a large family of cosmological models in the context of Dynamical Space-time Cosmology (DSC). This scenario unifies naturally the dark sector and it provides an elegant theoretical platform toward describing the various phases of the cosmic expansion. Initially, we performed a standard dynamical analysis and we found that under certain circumstances
DSC model includes stable late-time attractors. Then we tested the class of DSC models against the latest observational data and we placed constraints on the corresponding free parameters. In particular, our observational constraints regarding the Hubble constant are in agreement (within $\sim 1\sigma$) with those of Planck 2018. Moreover, our results are compatible at $\sim 2\sigma$ level with the $H_0$ measurement obtained from Cepheids.

Using the most popular information criteria we found cases for which the DSC model is statistically equivalent with that of $\Lambda$CDM and thus it can be viewed as a viable cosmological alternative. On top of that we found that one of the DSC models, that with $V(\phi) = \text{Const.}$ and $U(\phi) = \text{Const.}$, shows a small but non-zero deviation from $\Lambda$CDM, where the confidence level is close to $\sim 1.5\sigma$. Also, we explicitly checked that our $V_{1,2,3}$ models are able to pass the BBN constraints (see appendix). We argued that the theoretical formulation of ref. [34] could provide competitive cosmological models and thus it deserves further consideration. Finally, in a forthcoming paper we attempt to investigate DSC at the perturbation level for the general case of potentials $U(\phi)$. This will allow us to modify CAMB and thus to confront Dynamical Space-time Cosmology to the Cosmic Microwave Background (CMB) power spectrum from Planck.

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**A Big Bang Nucleosynthesis (BBN) within DSC**

In the appendix we check the various DSC models against BBN. Of course, the complete analysis of this aspect is out of scope of the present study. However, we explicitly checked the compatibility of DSC within the standard BBN using the average bound on the possible variation of the BBN speed-up factor. The latter is defined as the ratio of the expansion rate predicted in a given model versus that of the $\Lambda$CDM model at the BBN epoch, namely $z_{\text{BBN}} \sim 10^9$. Specifically, using the best fit values (see table 4) regarding the cosmological parameters ($\theta^i$) we check the validity of the following inequality, (i.e [59] and references therein)

$$100\% \times \frac{(H_{\Lambda CDM}(z_{\text{BBN}},\Omega_0, h) - H_{i}(z_{\text{BBN}}, \theta^i))^2}{H_{\Lambda CDM}(z_{\text{BBN}},\Omega_0, h)^2} < 10\%.$$  

Notice that $i = 1, 2, 3$ correspond to exponential, cosine and thawing potentials respectively (see section 3). We verify that the latter potentials satisfy the above restriction, hence BBN is safe in the context of DSC cosmology. Concerning the concordance $\Lambda$CDM model we have used that provided by the Planck team [39].

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9 Inflation from higher dimensions

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This article studies the basics of inflation compactification mechanism with the DST. The unification of dark energy, dark matter with the additional bounce of the volume naturally prevents the collapse of the Universe and gives a lower bound for the volume. Consequently, for some values of the anisotropy parameter $E$, the total volume oscillates, the ordinary dimensions increase exponentially with an oscillatory modulation and the extra dimensions decrease correspondingly.
Inflation compactification from dynamical spacetime

David Benisty$^{1,2,3,*}$ and Eduardo I. Guendelman$^{1,3,4,†}$

$^1$Frankfurt Institute for Advanced Studies (FIAS), Ruth-Moufang-Strasse 1, 60438 Frankfurt am Main, Germany
$^2$Goethe-Universität, Max-von-Laue-Strasse 1, 60438 Frankfurt am Main, Germany
$^3$Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
$^4$Bahamas Advanced Study Institute and Conferences, 4A Ocean Heights, Hill View Circle, Stella Maris, Long Island, The Bahamas

A mechanism of inflation from higher dimensions compactification is studied. An early Universe capable of providing exponential growth for some dimensions and exponential contraction for others, giving therefore an explanation for the big size of the observed four-dimensional Universe as well as the required smallness of the extra dimensions is obtained. The mechanism is formulated in the context of dynamical spacetime theory, which produces a unified picture of dark energy, dark matter, and can also provide a bounce for the volume of the Universe. A negative vacuum energy puts an upper bound on the maximum volume, and the bounce imposes a lower bound. So in the early Universe the volume oscillates, but in each oscillation the extra dimensions contract exponentially, and the ordinary dimension expand exponentially. The dynamical spacetime theory provides a natural way to exit from the inflation compactification epoch since the scalar field that drives the vacuum energy can smoothly climb into small positive values of vacuum energy, which is the end of the inflation compactification. A semianalytic solution for a step function potential is also studied, where all of these effects are shown, especially the jump of the vacuum energy effect only on the derivative of dynamical spacetime vector field, and not the volume or its derivatives, which match smoothly.

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I. INTRODUCTION

In many interesting models, a vacuum with a negative cosmological constant is predicted, as in superstrings, supergravity [1], etc. Here, we will present a model which uses extra dimensions and a primordial negative vacuum energy. For cosmology with higher dimensions and in the presence of a noncanonical scalar field, the dynamics is governed by a dynamical spacetime vector field, which is used as a Lagrange multiplier of an energy momentum tensor of the scalar field [2]. A non-Lagrangian approach similar to this was developed by Gao and Collaborators [3].

An interesting feature of the model [2] is that it allows for bouncing solutions. This effect in higher dimensions combined with a negative cosmological constant in the early Universe leads to the existence of an “inflationary phase” for some dimensions and a simultaneous “deflationary phase” for the remaining dimensions, since the volume of the spacetime remains constant or oscillating in the early Universe. For an approximately constant volume, some dimensions will grow exponentially, where others will shrink exponentially. This effect is obtained without invoking exotic matter or quantum effects as a similar inflation compactification scenario was discussed in [4–8].

We discuss how it may be possible to exit from this inflation-compactification era by dynamically increasing the cosmological constant, until it becomes positive and small. This is possible in our model because the scalar field can evolve towards increasing values of vacuum energy, without problems. Finally, the need for trapping the extra dimensions when they become become very small to prevent their reexpansion is studied. One could use, for example, the Casimir effect for periodic extra dimensions for this purpose.

II. THE BASIS OF THE MECHANISM

A. The geometry

For understanding the basics of the mechanism, we review the formalism of cosmology with a higher dimension as developed in [9] for a “classical Kaluza-Klein cosmology for a torus space with a cosmological constant and matter”. The metric we assume is the following:

$$ds^2 = -dt^2 + R(t)^2 \sum dx^i dx^i + R(t)^2 \sum dx^\rho dx^\rho, \quad (1)$$

*benidav@post.bgu.ac.il
†guendel@bgu.ac.il
where
\[ f_D = 1 + \frac{k_D}{4} \sum (x^i)^2, \quad f_d = 1 + \frac{k_d}{4} \sum (y^p)^2. \]  
\[ (2) \]

\( R(t) \) is the scale factor for the \( D \) dimensions \((x^i)\), and \( r(t) \) is the scale factor for the other \( d \) dimensions \((y^p)\). The \( k_D \) and \( k_d \) are the special curvatures. Their Hubble constants are defined as
\[ \mathcal{H}_R = \frac{\dot{R}}{R} \quad \mathcal{H}_r = \frac{\dot{r}}{r}. \]
\[ (3) \]

The complete volume of the Universe is defined as
\[ V = R^D r^d, \]
\[ (4) \]
which allows us to define the “volume expansion parameter”
\[ \mathcal{H} = \frac{\dot{V}}{V}. \]
\[ (5) \]

The connection between the volume expansion parameter and the Hubble parameters, using the definition (4) is
\[ \mathcal{H} = D \mathcal{H}_R + d \mathcal{H}_d. \]
\[ (6) \]

The motivation for defining the total volume is because of the ability to write down one combination of the Einstein equation which has no dependence on the individual scale factor, only through the volume, as we will see below.

### B. Einstein equations

We first consider the case of a stress energy tensor which has for every individual scale factor its own pressure: \( p \) for \( D \) dimensions and \( p' \) for \( d \) dimensions
\[ T^\mu_\nu = \text{diag}(\rho, -p, -p, \ldots, -p', -p', \ldots). \]
\[ (7) \]

Using the identities from the Appendix, we can obtain the solution for the Einstein equation,
\[ \frac{1}{2} D(D - 1) \left[ \frac{\dot{R}^2}{R^2} + \frac{k_D}{R^2} \right] + \frac{1}{2} d(d - 1) \left[ \frac{\dot{r}^2}{r^2} + \frac{k_d}{r^2} \right] \]
\[ + D \dot{R} \dot{r} \frac{\dot{R}}{R r} = 8\pi \rho \]
\[ (8) \]
\[ (D - 1) \frac{\dot{R}}{R} + d \frac{\dot{r}}{r} - d \frac{\dot{R}}{R r} - (D - 1) \left[ \frac{\dot{r}^2}{r^2} + \frac{k_d}{r^2} \right] \]
\[ = -8\pi (\rho + p) \]
\[ (9) \]
\[ D \frac{\ddot{R}}{R} + (d - 1) \frac{\ddot{r}}{r} - D \frac{\ddot{R}}{R r} - (d - 1) \left[ \frac{\dot{r}^2}{r^2} + \frac{k_d}{r^2} \right] \]
\[ = -8\pi (\rho + p'). \]
\[ (10) \]

For simplicity, we set the special curvature for all the dimensions to zero \( k_D = k_d = 0 \). Under the assumption of isotropy of the pressure \( p = p' = (\gamma - 1)\rho \), the relation from Eqs. (8)–(10) gives
\[ D \frac{\ddot{R}}{R} + d \frac{\ddot{r}}{r} = \frac{8\pi \rho}{D + d - 1} [1 - (D + d)\gamma]. \]
\[ (11) \]

The properties of densities as a function of the volume are summarized in Table I. By the definition of the volume (4), the equation could be represented as
\[ \frac{\dot{V}}{V} = \frac{D + d}{D + d - 1} 8\pi (\rho - p). \]
\[ (12) \]

The notation of normalized density gives a dimensionless equation of motion. By integrating the equation and using the dimensionless density \( \Omega := \frac{\rho}{\rho_c} \) we obtain as in [9],
\[ E = \frac{1}{2} \dot{V}^2 - \frac{D + d}{D + d - 1} \Omega V^2, \]
\[ (13) \]

where \( E \) is the anisotropy parameter, which is an integration constant that appears in the solution. The special feature of this equation is that it depends on the total volume and not the separate scale parameters of the individual dimensions.

Using the volume definition again (4) in the energy equation (13) together with (26), we obtain the first-order differential equations for \( R \) and \( r \) in terms of the volume solution,
\[ \frac{\dot{R}}{R} = \frac{1}{(D + d)V} \left[ \dot{V} + \sqrt{\frac{2Ed}{D}} (D + d - 1) \right] \]
\[ (14a) \]
\[ \frac{\dot{r}}{r} = \frac{1}{(D + d)V} \left[ \dot{V} - \sqrt{\frac{2Ed}{d}} (D + d - 1) \right]. \]
\[ (14b) \]

From those equations, we obtain that the basic condition for the existence of solution is \( E \neq 0 \), because of the appearance of the square root of \( E \). After an integration,
\[ R(t) = V^{\frac{1}{2}} \exp \left[ \frac{1}{D + d} \sqrt{\frac{2Ed(D + d - 1)}{D}} \int dt \right] \]
\[ (15a) \]

| Name            | \( \omega \) | \( \rho \) dependence |
|-----------------|--------------|----------------------|
| stiff           | 1            | \( V^{-2} \)         |
| matter          | 0            | \( V^{-1} \)         |
| radiation       | \( \frac{1}{V^{d-1}} \) | \( V^{-\frac{d-1}{d}} \) |
| dark energy     | -1           | constant             |
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\( r(t) = V^{\frac{1}{d + d}} \exp \left[ -\frac{1}{D + d} \sqrt{2ED(D + d - 1)} \frac{d}{d} \int dt \right]. \)

(15b)

Those equations could be used for obtaining the solution for every individual scale parameter of any particular dimension. After calculating the solution for the total volume from the energy equation, the equations above could give us the evolution of each scale factor. Notice that for any \( E > 0 \), we get an anisotropic evolution.

C. Solutions with constant equation of state

Simple examples of density dependence on the volume could be given under the assumption of a constant equation of state \( \omega = \frac{\rho}{P} \). Substituting the density which the Universe contains into the energy equation, would give the density multiple by the quadratic term of the volume. That means the massless scalar field \( \rho \phi V^2 \) will provide a constant term into the energy equation. However, only a ghost kinetic term of the scalar will shift the potential upwards, and a physical kinetic term of a scalar will push the potential down. The contribution for dark energy will be a parabolic term \( V^2 \) into the effective potential of the energy equation. If the dark energy has negative values, the parabola will provide a barrier, which will prevent high values of \( V \). A positive value of dark energy will provide a nonstable effective potential, which pushes the Universe to infinity. The dark matter gives a linear term in the effective potential of the energy equation with a negative slope.

D. Kasner solution

For a complete vacuum free from density and pressure, the Kasner solution automatically follows from the basic formalism. From the energy equation (13), we get \( V = \sqrt{2Et} \), which means the total volume grows linearly with the time. This case leads to a well-known Kasner vacuum solution [10], which describes an anisotropic universe without matter, with different scale factors as well. Using Eq. (15), we get the powers for the scale factors \( R(t) = r_p, r(t) = r^q \),

\[ p = \frac{1}{D + d} \left( 1 + \sqrt{\frac{d(D + d + 1)}{D}} \right) \]

(16)

\[ q = \frac{1}{D + d} \left( 1 - \sqrt{\frac{D(D + d + 1)}{d}} \right), \]

(17)

which obey the Kasner conditions,

\[ Dp + dq = Dp^2 + dq^2 = 1. \]

(18)

This solution allows us to check that our formalism recovers the well-known vacuum solutions.

III. INFLATION FROM UNIFIED DE-DM

A. Unified dark energy and dark matter solution

A suggestion for an action which produces DE-DM unification takes the form of [2],

\[ \mathcal{L} = -\frac{1}{2} R + \chi_{\mu \nu} T_{\phi}^{\mu \nu} - \frac{1}{2} g^{\mu \nu} \phi_{,\mu} \phi_{,\nu} - V(\phi), \]

(19)

where \( R \) is the Ricci scalar \((8\pi G = 1)\), \( \phi \) is a quintessential scalar field with a potential \( V(\phi) \), and \( \chi_{\mu} \) is a dynamical spacetime vector field which is a Lagrange multiplier enforcing the covariant conservation law of the energy momentum tensor,

\[ \nabla_\mu T^{\mu \nu}_{(\phi)} = 0. \]

(20)

We use the same stress energy tensor as the one postulated by Gao and colleagues [3],

\[ T^{\mu \nu}_{(\phi)} = -\frac{1}{2} \phi^{\nu} \phi^{,\nu} + U(\phi) g^{\mu \nu}, \]

(21)

which they require to be conserved without an action principle. The covariant conservation of this stress energy tensor leads to unified dark energy and dark matter for a constant potential and for interacting DE-DM, a non-constant potential \( U(\phi) \). This action produces very similar effects, but also include additional effects like bouncing, which are not obtained in [3]. The action depends on three different variables: the scalar field \( \phi \), the dynamical space time vector \( \chi_\mu \), and the metric \( g_{\mu \nu} \).

B. Equations of motion

According to this ansatz, the scalar field is just a function of time \( \phi(t) \) and the dynamical vector field will have only the time component \( \chi_\mu = (\chi_0, 0, 0, 0) \), where \( \chi_0 \) is also just a function of time. A variation with respect to the dynamical space time vector field \( \chi_\mu \) will force a conservation of the original stress energy tensor, which implies

\[ \dot{\phi} + \frac{1}{2} \mathcal{H} \phi + U'(\phi) = 0. \]

(22)

Notice that for a standard quintessence, the equation does not contain the factor \( \mathcal{H} \), but only the “volume expansion parameter” \( \mathcal{H} \), which equals to \( 3H \) for the case of the isotopic expansion of \( 3 + 1 \) dimensions. The second variation with respect to the scalar field \( \phi \) gives a non-conserved current,

\[ \chi_\mu U'(\phi) - V'(\phi) = \nabla_\mu j^\mu \]

(23a)

\[ j^\mu = \frac{1}{2} \phi_{,\mu} (\chi^\mu + \chi^{\mu \nu}) + \phi^{,\mu}, \]

(23b)
and the derivatives of the potentials are the source of the current. For constant potentials, the source becomes zero, and we get a covariant conservation of this current. For the metric we presented above, this equation of motion takes the form,

$$\dot{\phi}(\dot{\chi} - 1) + \phi(\mathcal{H}(\dot{\chi} - 1) + \dot{\chi}) = U'(\phi)(\dot{\chi} + \mathcal{H} - 0) - V'(\phi).$$

(24)

For constant potentials, the current (23) is covariantly conserved; a feature which will be used later. The last variation, with respect to the metric, gives the stress energy tensor we know from the Einstein equation,

$$G^\mu_\nu = g^\mu_\nu \left( \frac{1}{2} \phi \phi'^2 + V(\phi) + \frac{1}{2} \chi ^{\alpha \beta} \phi_\alpha \phi_\beta + \chi ^{\lambda \mu} \phi_\lambda \phi_\mu \right)$$

$$- \frac{1}{2} \phi'^2 (\chi^\alpha_\alpha + 2) \phi^\alpha + \chi ^{\alpha \beta} \phi_\alpha \phi_\beta + \chi ^{\lambda \mu} \phi_\lambda \phi_\mu )$$

$$- \frac{1}{2} (\chi ^{\lambda \mu} \phi_\lambda \phi_\mu + \chi ^{\lambda \mu} \phi_\lambda \phi_\mu ) .$$

(25)

For the stress energy tensor from Eqs. (7) and (25), the relation between the energy density and the fields is

$$\rho = \left( \dot{\chi} - 1 / 2 \right) \phi^2 + V(\phi),$$

(26)

which has no dependence on the potential $U(\phi)$ or its derivatives. Those three variations are sufficient for building a complete solution of the theory, using the energy equation (13) and the integration form of the individual scale parameters (15).

### C. Constant potentials solution

In order to compute the evolution of the scalar field, we have to specify a form for the potentials. For a simplified case of constant potentials,

$$U(\phi) = C, \quad V(\phi) = \Omega_{\Lambda}.$$

(27)

The solution for the variation with respect the dynamical time, which is Eq. (22) can be integrated to give

$$\dot{\phi}^2 = \frac{2\Omega_m}{V},$$

(28)

where $\Omega_m$ is an integration constant which represents the effective dark matter ratio. From the second variation, with respect to the scalar field $\phi$ a conserved current is obtained, which from Eq. (24) gives the exact solution of the dynamical time vector field,

$$\dot{\chi} = 1 + \frac{\kappa}{\sqrt{2}}.$$

(29)

where $\kappa$ is another constant of integration. Together with Eqs. (28) and (29) into the density equation (26), the volume dependence of $\Omega := \frac{\rho}{\rho_c}$ is

$$\Omega = \frac{\Omega_{\Lambda} + \Omega_m}{V} + \frac{\Omega_{\kappa}}{V^{1/2}}.$$

(30)

where $\Omega_{\kappa} = \kappa \Omega_m$. Using the energy equation (13) which is the way to solve Einstein equations under the “inflation compactification mechanism”, we obtain the relation,

$$E = \frac{1}{2} \dot{V}^2 + U_{\text{eff}}(V)$$

(31)

with the appropriate effective potential,

$$U_{\text{eff}}(V) = - \frac{\Omega_{\Lambda}}{V} \dot{V}^2 + \Omega_m V + \Omega_{\kappa} \sqrt{V}. $$

(32)

In Fig. 1, we can see the plot of the effective potential for $\Omega_{\Lambda}, \Omega_\kappa < 0$, and $\Omega_m > 0$.

From Eqs. (15), we can see terms with $\sqrt{E}$; therefore, $E > 0$ is a basic condition for the existence of solutions, where $E$ is the measure of the anisotropy of the solution. Only for $E = 0$, we have an isotropic solution. Because of this condition, we can obtain two different cases, represented in Fig. 1: the left case, where all of the effective potential is positive everywhere and the right case, where there is a part with negative values of the potential.

In the left case, if $E = E_{\text{max}}$, we have $V = V_C = \text{const}$, which refers to a constant total volume. But from Eq. (15),

![FIG. 1. The effective potential, for two cases, where $\Omega_{\Lambda}, \Omega_\kappa < 0$ and $\Omega_m > 0$.](image-url)
we obtain that the scale parameter $R(t)$ is exponentially growing and the $r(t)$ is exponentially shrinking,

$$R(t) = V_C^{\frac{1}{2}} \exp \left[ \frac{1}{D + d} \sqrt{\frac{2Ed(D + d - 1)}{D}} t \right]$$

(33a)

$$r(t) = V_C^{\frac{1}{2}} \exp \left[ - \frac{1}{D + d} \sqrt{\frac{2ED(D + d - 1)}{d}} t \right].$$

(33b)

This kind of solutions holds only for the left case, because of the energy condition $E = E_{\text{min}}$ > 0 could only exist if the potential is positive at the minimum. In general, when $E > E_{\text{min}}$, we have an oscillating volume solution. If $E$ is slightly larger than $E_{\text{min}}$, the oscillation will not be so large, and the expansion of the individual scale parameters will be close to an exponentially growing or decreasing, as shown in Fig. 2. On the other hand, if $E$ is much larger than $E_{\text{min}}$, the oscillations will be large also, and the individual scale parameters will grow and shrink modulated by an oscillatory behavior, as shown in Fig. 3. Another important condition is $E < E_{\text{max}}$ as can be seen in Fig. 1.

For proving the existence solutions for a more non-constant potential $V(\phi)$, where dynamically we change from negative to positive values, we study the case of the step function potential.

**D. End of inflation compactification, using a step function potential**

The end of the inflation compactification era will take place when the cosmological constant changes from negative values to positive values since then the effective potential does not prevent the total volume from expanding to infinity. For example, a smooth potential that interpolates from those values is

$$V(\phi) = \frac{\Lambda_{+\infty} - \Lambda_{-\infty}}{2} \tanh(\beta \phi) + \frac{\Lambda_{+\infty} + \Lambda_{-\infty}}{2}.$$  

(34)

FIG. 2. A numerical solution for the volume and the scale factors in a Kaluza Klein universe, with the parameters: $\Omega_\Lambda = -0.04$, $\Omega_m = 0.24$, $\Omega_s = -0.2$, with the initial condition $V = 0.01$.

FIG. 3. A numerical solution of the volume and the scale factors for a Kaluza Klein universe, with the parameters: $\Omega_\Lambda = -0.04$, $\Omega_m = 0.24$, $\Omega_s = -0.2$ with an initial condition $V = 0.5$.

where $\Lambda_{-\infty} > 0$ is the asymptotic value of the potential for $\phi \rightarrow -\infty$ and is chosen to be small. On the other hand, $\Lambda_{+\infty} < 0$ is the asymptotic value of the potential for $\phi \rightarrow +\infty$. For obtaining a partially analytic and more simple solution, we take the limit for $\beta \rightarrow \infty$, which then becomes a step function,

$$V(\phi) = \frac{\Lambda_{+\infty} - \Lambda_{-\infty}}{2} \text{Sign}(\phi) + \frac{\Lambda_{+\infty} + \Lambda_{-\infty}}{2}.$$  

(35)

Notice that there is no problem for the scalar field $\phi$ to increase and go up in the direction of increasing dark energy, since its dynamics is not determined by the potential $V(\phi)$ that determined the value of the dark energy. In general, the scalar field evolution depends on $U'(\phi)$, which is in this case zero, since the potential $U(\phi)$ is a constant. Still by choosing the positive root of Eq. (28), we get the desired effect of increasing dark energy as a function of time. For simplicity, let us define the parameter $\xi$,

$$\xi = \chi_0 - 1,$$  

(36)

which estimates the difference between the dynamical time and the cosmic time. If $\xi = 0$ then $\chi_0 = t$. The variation with respect to the scalar field $\phi$, Eq. (24), takes the form,

$$\dot{\phi} \left( \xi + \frac{1}{2} H\xi \right) = -V'().$$  

(37)

Since $\phi$ is a monotonic function of time, it is better to change the time dependence to the scalar dependence $\frac{d}{d\xi} = \phi \frac{d}{d\phi}$. In this way, the equation is easier to analyze,

$$\frac{2\Omega_m}{V^2} d(\xi V^2) = -dV(\phi).$$  

(38)

For the potential (35), we get the differential equation,
\[
\frac{2\Omega_m}{V} \left( \frac{d}{d\phi} \zeta + \frac{1}{2V} \frac{d}{d\phi} V \right) = -(\Lambda_{+\infty} - \Lambda_{-\infty}) \delta(\phi),
\]

(39)

where in the right-hand side, we obtain a source term, with the piecewise solution,

\[
\xi(\phi < 0) = \frac{\kappa_-}{V_1^2(\phi=0)}, \quad \xi(\phi > 0) = \frac{\kappa_+}{V_1^2(\phi=0)}.
\]

(40)

From continuity of the geometry, we demand \( V_- = V_+ \); otherwise, the geometry is not defined at the junction. From an integration around an infinitesimal region that contains \( \phi = 0 \), we obtain the jump of the \( \zeta \),

\[
\Delta \zeta = -\frac{V(\phi=0)}{2\Omega_m} (\Lambda_{+\infty} - \Lambda_{-\infty}).
\]

(41)

Inserting (40) into (41) gives the discontinuity of \( \kappa \),

\[
\kappa_+ - \kappa_- = -\frac{V(\phi=0)}{2\Omega_m} (\Lambda_{+\infty} - \Lambda_{-\infty}).
\]

(42)

Substituting all the known terms to the energy equation gives

\[
E = \frac{1}{2} V^2 - \frac{D + d}{D + d - 1} V^2 \left[ \frac{2\Omega_m}{V} \left( \zeta + \frac{1}{2} \right) + \frac{\Lambda_{+\infty} - \Lambda_{-\infty}}{2} \text{Sign}(\phi) + \Lambda_{+\infty} + \Lambda_{-\infty} \right].
\]

(43)

Because of the jump of the potential and the field \( \zeta \), we can calculate the jump of \( V^2 \) from the energy equation. The solution gives

\[
\frac{1}{2} \Delta \dot{V}^2 = \frac{D + d}{D + d - 1} \left( \frac{2\Omega_m}{V(\phi=0)} \Delta \zeta + \Lambda_{+\infty} - \Lambda_{-\infty} \right) V^2(\phi=0) = 0,
\]

(44)

where there is no jump in the volume and its first derivative. From Eq. (15), which gives the dependence of the metric components, we obtain that all derivatives of the scale factors are continuous. That leads to the conclusion that even when there is a large discontinuous change in the potential, still the metric and its derivative do not suffer from these discontinuities.

**E. Large times behavior and extra dimensional stabilization**

For obtaining the asymptotic limit of the solutions, let us take the case of the pure vacuum energy \( \Omega = \text{const} \) which is the case of late time expansion; we get an upside down harmonic oscillator, which for large volumes gives the solution,

\[
V(t) = V_0 \exp(\chi t),
\]

(45)

where \( \chi^2 = 2 \frac{D + d}{D + d - 1} \Omega \). The integration form of the scale factors in Eq. (15) leads to

\[
R(t) \sim e^{m_1^2 \chi t} \exp \left[ -\sqrt{2E_d(D + d - 1)} \frac{V(\phi=0)}{D \chi} e^{\chi t} \right]
\]

(46a)

\[
r(t) \sim e^{m_1^2 \chi t} \exp \left[ +\sqrt{2E_d(D + d - 1)} \frac{V(\phi=0)}{D \chi} e^{-\chi t} \right].
\]

(46b)

From the fact that the solution for the integral gives \( e^{-\chi t} \) which decays for large times, we are left with the limit,

\[
R(t) \rightarrow R_0 \exp \left( \frac{\chi}{D + d} t \right)
\]

(47a)

\[
r(t) \rightarrow r_0 \exp \left( \frac{\chi}{D + d} t \right).
\]

(47b)

This represents a restoration of the isotropy in the evolution of all dimensions in the Universe. This has to be avoided, because the extra dimensions should be small also in the late Universe. One way to archive this is to generate a potential for the extra dimensions, which starts to act when the extra dimensions are very small and then freeze the extra dimensions to very small size. This can be obtained, for example, by using the Casimir effect present in periodic extra dimensions [11–13]. The stopping of the extra dimensions can also be used as a particle production mechanism, that can result in the reheating of the Universe by a field independent of the inflaton (our field \( \phi \) which is the extra dimension size. The extra dimension size becomes therefore a curvaton field [14–16].

**IV. DISCUSSION**

In this article, we studied the basics of an inflation compactification mechanism from the interplay of ordinary and higher dimensions. In the case of isotropic pressure, the solution can be obtained for the total volume and with no dependence with the individual scale factors of each dimension. Those can be calculated directly from the total volume dependence and the anisotropy constant \( E \).

For the dynamical spacetime theory produces a unification of dark energy, dark matter, and a bounce of the volume, which naturally prevents the collapse of the Universe and obtains a lower bound for the volume of the Universe. Likewise, the presence of a negative cosmological constant prevents the volume from becoming very big in the early Universe. There is an effective potential that governs the evolution of the volume. In this case, the effective potential is positive and has a minimum, a static solution for the total volume is obtained, and exponential compactification of the extra dimensions occurs. In that case, the ordinary dimensions exponentially increase and
the extra dimensions exponentially decrease. For small values of $E$ higher than the value obtained for the case the volume sits at the minimum, the total volume oscillates and the ordinary dimensions expand exponentially with an oscillatory modulation.

The dynamical spacetime theory provides a natural way to exit from the inflation compactification epoch. The main reason for that is that the theory allows for two different potentials: $U(\phi)$ which drives directly the evolution of the scalar field $\phi$ and $V(\phi)$ which determines the value of the dark energy. It is therefore perfectly possible for the scalar field that drives the vacuum energy to smoothly climb into small positive values of vacuum energy, which is defined as the end of the inflation compactification. A semianalytic solution for the step function potential is also derived. In this limit, the matching of the solution at the value of the scalar field where the vacuum energy jumps, still respects the continuity of all components of the metric and also for its time derivatives.

We have showed that the exponential growth of the total volume breaks the anisotropy and all the scale factors start to expand in a similar fashion. The role of the inflation compactification mechanism, as we have explained, is to push the extra dimensions to very low sizes and the ordinary dimension to very large sizes. However, we cannot extend the model to all future time, since the vacuum energy at the end will restores the isotropy of the expansion of all dimensions. So we have to invoke a mechanism that locks the extra dimensions when they reduce to very small sizes. This could be produced from the known Casimir effect that takes place in the compact extra dimension for example. The stopping of the extra dimensions can be used also as a particle production mechanism, that can result in the reheating of the Universe by a field independent of the inflaton (our field $\phi$) which is the extra dimension size. The extra dimension size becomes therefore a curvaton field.

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APPENDIX: IDENTITIES

The covariant conservation of the energy momentum tensor gives

$$\dot{\rho} + (p + \rho)D \frac{\ddot{R}}{R} + (p' + \rho') \frac{\ddot{r}}{r} = 0.$$  \hspace{1cm} (A1)

The Ricci tensor nonvanishing values, under the metric (1),

$$R_{(0)} = -\left(D \frac{\ddot{R}}{R} + d \frac{\ddot{r}}{r}\right)$$  \hspace{1cm} (A2)

$$R_{DD} = \dot{H}_D + (D\dot{H}_D + d\dot{H}_d)H_D + (D - 1)\frac{k_D}{R^2}$$  \hspace{1cm} (A3)

$$R_{dd} = \dot{H}_d + (D\dot{H}_D + d\dot{H}_d)H_d + (d - 1)\frac{k_d}{r^2}.$$  \hspace{1cm} (A4)

And the Ricci scalar,

\[
R = 2D \frac{\ddot{R}}{R} + 2d \frac{\ddot{r}}{r} + 2Dd\dot{H}_D\dot{H}_d + D(D - 1) \left( H_D^2 + \frac{k_D}{R^2} \right) + d(d - 1) \left( H_d^2 + \frac{k_d}{r^2} \right).
\]  \hspace{1cm} (A5)
10 Diffusion effects from extended DST

D. Benisty and E. I. Guendelman,
“Interacting Diffusive Unified Dark Energy and Dark Matter from Scalar Fields”,
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This paper generalizes the DST by replacing the Dynamical Spacetime vector \( \chi_\mu \) into a gradient \( \partial_\mu \chi \). The replacement introduces a current source for the energy momentum tensor \( T^{\mu\nu}_{(\chi)} \). The current dissipates in an expanding Universe and asymptotically the energy momentum tensor is conserved. The new mechanism predicts an energy transfer between the dark energy and dark matter components.
Interacting diffusive unified dark energy and dark matter from scalar fields

David Benisty\textsuperscript{a}, E. I. Guendelman\textsuperscript{b}

Department of Physics, Ben Gurion University of the Negev, 84105 Beersheba, Israel

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Abstract Here we generalize ideas of unified dark matter–dark energy in the context of two measure theories and of dynamical space time theories. In two measure theories one uses metric independent volume elements and this allows one to construct unified dark matter–dark energy, where the cosmological constant appears as an integration constant associated with the equation of motion of the measure fields. The dynamical space-time theories generalize the two measure theories by introducing a vector field whose equation of motion guarantees the conservation of a certain Energy Momentum tensor, which may be related, but in general is not the same as the gravitational Energy Momentum tensor. We propose two formulations of this idea: (I) by demanding that this vector field be the gradient of a scalar, (II) by considering the dynamical space field appearing in another part of the action. Then the dynamical space time theory becomes a theory of Diffusive Unified dark energy and dark matter. These generalizations produce non-conserved energy momentum tensors instead of conserved energy momentum tensors which leads at the end to a formulation of interacting DE–DM dust models in the form of a diffusive type interacting Unified dark energy and dark matter scenario. We solved analytically the theories for perturbative solution and asymptotic solution, and we show that the $\Lambda$CDM is a fixed point of these theories at large times. Also a preliminary argument as regards the good behavior of the theory at the quantum level is proposed for both theories.

1 Introduction

The best explanation, and fitting with data for the accelerated expansion of our universe, is the $\Lambda$CDM model, which tells us that our universe contains 68% of dark energy, and 27% of dark matter. This model present two big questions: The cosmological constant problem \cite{1–3} – why there is a large disagreement between the vacuum expectation value of the energy momentum tensor which comes from quantum field theory and the observable value of dark energy density – and the coincidence problem \cite{4} – why the observable values of dark energy and dark matter densities in the late universe are of the same order of magnitude.

In order to solve this problem, many approaches emerged \cite{5–8}. One interesting suggestion was a diffusive exchange of energy between dark energy and dark matter made by Calogero \cite{9,10}, Haba et al. \cite{11}, and Szydlowski and Stachowski \cite{12}, with some solution to cosmic problems. The basic notion is that the diffusion equation (or more exactly, the Fokker–Planck equation \cite{13,14}, which describes the time evolution of the probability density function of the velocity of a particle under the influence of random forces), implies a non-conserved stress energy tensor $T_{\mu\nu}$, which has some current source $j_{\mu}$:

$$\nabla_{\mu}T^{\mu\nu} = 3\sigma j_{\nu}$$ \hspace{1cm} (1)

where $\sigma$ is the diffusion coefficient of the fluid. This generalization is Lorentz invariant and fit for curved space-time. The current $j_{\mu}$ is a time-like covariantly conserved vector field and its conservation tells us that the number of particles in this fluid is constant.

However, in the gravitational equations, the Einstein tensor is proportional to the conserved stress energy tensor $\nabla_{\mu}T^{\mu\nu}_{(G)} = 0$, which we labeled with “G” \cite{15,16}. So Calogero came up with what he called $\phi$CDM model, which achieves a conserved total energy momentum tensor appearing in the right hand side of Einstein’s equation. But for the dark energy and dust stress tensors there is some source current for those tensors (however, the sum is conserved):

$$-\nabla_{\mu}T^{\mu\nu}_{(\Lambda)} = \nabla_{\mu}T^{\mu\nu}_{(\text{Dust})} = j^{\nu}, \quad \nabla_{\mu}J^{\mu} = 0.$$ \hspace{1cm} (2)

As Calogero mentioned \cite{9}, the diffusion model introduced in his paper lacks an action principle formulation. Therefore...
we develop from a generalization of two measure theories [17–28] a “diffusive energy theory” which can produce on one hand a non-conserved stress energy tensor \(T^{\mu\nu}_{(\chi)}\), as in (1), and on the other hand the conserved stress energy tensor \(T^{\mu\nu}_{(G)}\) that we know from the right hand side of Einstein’s equation. As we will see, this suggested theory is asymptotically different from the \(\phi\)CDM model, and more close in this limit to the standard \(\Lambda\)CDM.

2 Two measure theories and dynamical time theories

There have been theoretical approaches to gravity theories, where a fundamental constraint is implemented; like in two measure theories where one works, in addition to the regular measure of integration in the action \(\sqrt{-g}\), with yet another measure, which is also a density and which is also a total derivative. In this case, one can use for constructing this measure four scalar fields \(\varphi_a\), where \(a = 1, 2, 3, 4\). Then we can define the density \(\Phi = \varepsilon^{a\beta\gamma\delta}g_{abcd}\partial_a\varphi_b\partial_\beta\varphi_\gamma\partial_\delta\varphi_d\), and we can write an action that uses both of these densities:

\[
S = \int d^4x \Phi L_1 + \int d^4x \sqrt{-g} L_2. \tag{3}
\]

As a consequence of the variation with respect to the scalar fields \(\varphi_a\), assuming that \(L_1\) and \(L_2\) are independent of the scalar fields \(\varphi_a\), we obtain

\[
A^a_\mu \partial_\mu L_1 = 0 \tag{4}
\]

where \(A^a_\mu = \varepsilon^{a\beta\gamma\delta}g_{abcd}\partial_\beta\varphi_b\partial_\gamma\varphi_\delta\partial_\delta\varphi_d\). Since \(\det[A^a_\mu] \sim \Phi^3\) as one easily sees, for \(\Phi \neq 0\), (4) implies that \(L_1 = M = \text{Const}\). This result can expressed as covariant conservation of a stress energy momentum of the form \(T^{\mu\nu}_{(\Phi)} = L_1 g^{\mu\nu}\), and using the second order formalism, where the covariant derivative of \(g^{\mu\nu}\), is zero, we obtain \(\nabla_\mu T^{\mu\nu}_{(\Phi)} = 0\). This implies \(\partial_\mu L_1 = 0\). This suggests generalizing the idea of the two measure theory, by imposing the covariant conservation of a more non-trivial kind of the energy momentum tensor, which we denote \(T^{\mu\nu}_{(\chi)}\) [29]. Therefore, we consider an action of the form

\[
S = S_{(\chi)} + S_{(R)} = \int d^4x \sqrt{-g} \chi_{\mu\nu} T^{\mu\nu}_{(\chi)} + \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \tag{5}
\]

where \(\chi_{\mu\nu} = \partial_\nu \chi_\mu - \Gamma^a_{\mu\nu} \chi_a\). If we assume \(T^{\mu\nu}_{(\chi)}\) to be independent of \(\chi_{\mu\nu}\), and having \(\Gamma^a_{\mu\nu}\) being defined as the Christoffel connection coefficients, then the variation with respect to \(\chi_{\mu\nu}\) gives a covariant conservation: \(\nabla_\mu T^{\mu\nu}_{(\chi)} = 0\).

Notice the fact that the energy density is the canonically conjugated variable to \(\chi^0\), which is what we expect from a dynamical time (here represented by the dynamical time \(\chi^0\)). Some cosmological solutions of (5) have been studied in [30], in the context of spatially flat radiation-like solutions, and considering gauge field equations in curved space time.

For a related approach where a set of dynamical space-time coordinates are introduced, not only in the measure of integration, but also in the lagrangian, see [31].

3 Diffusive energy theory from action principle

Let us consider a four dimensional case, where there is a coupling between the scalar field \(\chi\) and the stress energy momentum tensor \(T^{\mu\nu}_{(\chi)}\):

\[
S_{(\chi)} = \int d^4x \sqrt{-g} \chi_{\mu\nu} T^{\mu\nu}_{(\chi)} \tag{6}
\]

where, \(\mu; \nu\) are covariant derivatives of the scalar field. When \(\Gamma^a_{\mu\nu}\) is being defined as the Christoffel connection coefficients, the variation with respect to \(\chi\) gives a covariant conservation of a current \(f^\mu\):

\[
\nabla_\mu T^{\mu\nu}_{(\chi)} = f^\nu; \quad \nabla_\nu f^\nu = 0, \tag{7}
\]

which is the source of the stress energy momentum tensor. This corresponds to the “dynamical space-time” theory (5), where the dynamical space-time 4-vector \(\chi_{\mu}\) is replaced by a gradient of a scalar field \(\chi\). In the “dynamical space theory” we obtain four equations of motion, by the variation of \(\chi_{\mu}\), which correspond to covariant conservation of the energy momentum tensor, \(\nabla_\mu T^{\mu\nu}_{(\chi)} = 0\). By changing the four vector to a gradient of a scalar \(\delta_\mu \chi\) at the end, what we do is to change the conservation of the energy momentum tensor to the asymptotic conservation of the energy momentum tensor (7) which corresponds to a conservation of a current \(\nabla_\nu f^\nu = 0\). In an expanding universe, the current \(f_\mu\) gets diluted, so then we recover asymptotically a covariant conservation law for \(T^{\mu\nu}_{(\chi)}\) again. Equation (7) has a close correspondence with the one obtained in a “diffusion scenario” for DE–DM exchange [9, 10].

This stress energy tensor is substantially different from the stress energy tensor we all know, which is defined as \(\varepsilon G T^{\mu\nu}_{(G)} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\). In this case, the stress energy momentum tensor \(T^{\mu\nu}_{(\chi)}\) is not conserved (but there is some conserved current \(f^i\), which is the source to this stress energy momentum tensor non-conservation), here there is some conserved stress energy tensor \(T^{\mu\nu}_{(G)}\), which comes from variation of the action according to the metric:

\[
T^{\mu\nu}_{(G)} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_M)}{\delta g^{\mu\nu}}; \quad \nabla_\mu T^{\mu\nu}_{(G)} = 0. \tag{8}
\]
The lagrangian $L_M$ could be the modified term $\chi,\mu;\nu T_{\mu\nu}(\chi)$, but as we will see, it could be added to more action terms. Using different expressions for $T_{\mu\nu}(\chi)$, which depend on other variables, will give the connection between the scalar field $\chi$ and those other variables.

Notice that for the theory the shift symmetry holds, and

$$\chi \rightarrow \chi + C_\chi; T_{\mu\nu}(\chi) \rightarrow T_{\mu\nu}(\chi) + g^{\mu\nu}C_T$$

(9)

will not change any equation of motion. When $C_\chi, C_T$ are some arbitrary constants. This means that if the matter is coupled through its energy momentum tensor as in (9), the process of redefinition of the energy momentum tensor will not affect the equations of motion. Of course, a redefinition of such a type of energy momentum tensor is exactly what is done in the process of normal ordering in quantum field theory, for example.

4 Diffusive energy theory without high derivatives

Another model that does not involve high derivatives is obtained, by keeping $\chi_\mu$ as a 4-vector, which is not gradient, but we introduce the vector field $\chi_\mu$ in another part of the action:

$$S_{(\chi,A)} = \int d^4x \sqrt{-g} \chi_\mu; T_{\mu\nu}(\chi)d^4x$$

$$+ \frac{\alpha}{2} \int d^4x \sqrt{-g}(\chi_\mu + \partial_\mu A)^2d^4x$$

(10)

where $A$ is another scalar field. Then from variation with respect to $\chi_\mu$ we obtain

$$\nabla_\mu T_{\mu\nu}(\chi) = \sigma(\chi_\mu + \partial_\mu A)$$

(11)

as (7), where the source is

$$f_\mu = \sigma(\chi_\mu + \partial_\mu A).$$

(12)

But in contrast to (7), where $f_\mu$ appears as an integration function, here $f_\mu$ appears as a function of the dynamical fields. From the variation with respect to $A$, we indeed see that the current $\chi_\mu + \partial_\mu A$ is conserved, which means again with (7) that $\nabla_\mu \nabla_\nu T_{\mu\nu}(\chi) = 0$, but it does not tell us that all of the equations of motion are the same. Nevertheless, asymptotically, for the late universe, the two theories coincide.

To start with, we discuss a toy model in one dimension describing a system that allows the non-conservation of a certain energy function, which increases or decreases linearly with time, while there is another energy which is conserved. It is of interest to compare with a mechanism that produces non-conserved energy momentum tensors which leads to a formulation of interacting DE–DM models; however, there are crucial differences.

5 A mechanical system with a constant power and diffusive properties

In order to see the applications of the ideas, we start with a simple action of one dimensional particle in a potential $V(x)$. We introduce a coupling between the total energy of the particle $\frac{1}{2}m\dot{x}^2 + V(x)$ and the second derivative of some dynamical variable $B$:

$$S = \int \dot{B} \left[ \frac{1}{2}m\dot{x}^2 + V(x) \right] dr.$$

(13)

In order to see the applications of the ideas, we start with a simple action of a one dimensional particle in a potential $V(x)$. We introduce a coupling between the total energy of the particle $\frac{1}{2}m\dot{x}^2 + V(x)$ and the second derivative of some dynamical variable $B$:

$$S = \int \dot{B} \left[ \frac{1}{2}m\dot{x}^2 + V(x) \right] dr.$$

(14)

The equation of motion according to the dynamical variable $B$ shows that the second derivative of the total energy is zero. In other words, the total energy of the particle is linear in time:

$$\frac{1}{2}m\dot{x}^2 + V(x) = E(t) = Pt + E_0$$

(15)

where $P$ is a constant power which is given to the particle or taken from it, and $E_0$ is the total energy of the particle at time equal to zero.

From the equation of motion according to coordinate $x$ we get a close connection between the dynamical variable $B$ and the coordinate of the particle:

$$m\ddot{x} B + m\dot{x} \frac{d^2B}{dt^2} = V'(x) \frac{d^2B}{dt^2},$$

(16)

which with Eq. (15) gives

$$\frac{\dot{B}}{B} = \frac{2V'(x)}{\sqrt{2m(E(t) - V(x))} - \frac{P}{2(E(t) - V(x))}}.$$  

(17)

To get a feeling of these kinds of theories, let us look at the case of a harmonic oscillator $V(x) = \frac{1}{2}kx^2$. First of all, we see from Eq. (5) and the condition that the right hand side be positive; since the left hand side obviously is positive, we see that there is a boundary time $\tau = -\frac{E_0}{P}$, for which for $P > 0$ we get $t > \tau$, and for $P < 0$ there is a maximal time $t < \tau$. Let us consider the case that the power $P$ is positive. The equations of motion in that case will not oscillate, but they will grow exponentially until the “$Pt$” term present in Eq. (15) dominates, when $(x^2) \propto t$. This is very similar to Brownian motion.
The momenta for this toy model are

\[ \pi_\chi = \frac{\partial L}{\partial \dot{\chi}}, \tag{18} \]
\[ \pi_B = \frac{\partial L}{\partial \dot{B}} - \frac{d}{dt} \frac{\partial L}{\partial B} = -\frac{d}{dt} E(t), \tag{19} \]
\[ \Pi_B = \frac{\partial L}{\partial \dot{B}} = E(t). \tag{20} \]

Using the Hamiltonian formalism (with second order derivative [32,33]) we see that the hamiltonian of the system is

\[ \mathcal{H} = \dot{\pi}_\chi + B\pi_B + B\Pi_B - L \]
\[ = \pi_\chi \sqrt{\frac{2}{m} (\Pi_B - V(\chi))} + B\pi_B = \text{Const.} \tag{21} \]

Since the action in not explicitly dependent on time, the hamiltonian is conserved. So even if the total energy of the particle is not conserved, we have the conserved hamiltonian (21). This notion is equivalent to a non-conserved stress energy tensor \( T_{\mu\nu} \), in addition to the conserved stress energy \( T_{(G)}^{\mu\nu} \), which appears in the Einstein equation.

Notice that this hamiltonian is not necessarily bounded from below. However, there are only mild instabilities in the solutions. For example, for the case \( V(\chi) = 0 \), we get \( \dot{\chi} \propto \dot{B} \propto \dot{t}^2 \). In the case of a harmonic oscillator, where \( V(\chi) = \frac{1}{2} k \chi^2 \), there is an even milder instability at large times: \( \chi \propto t \), which resembles a diffusive behavior, or Brownian motion. This behavior is a mild kind of instability, since no exponential growth appears, only power law growth. The related model in cosmology, as we will see, because of the coupling to an expanding space-time shows dumped perturbations, shows a trend towards a fixed point solution, where it coincides with the standard ΛCDM model. This is because whatever potential instabilities the model may have in a flat background, the expanding space (most notably the de Sitter space) has the counter property of red shifting any perturbation; this effect overcomes and cancels these rather soft instabilities (power law instabilities that may exist for the solution in flat space) as we will see in Sect. 7. The exponential expansion is known to counter all kind of unstable behaviors, for example, it goes against the gravitational instability and a big enough cosmological constant can prevent galaxy formation; our case is much simpler than that, but the basic reason is the same. In this context it is important to notice that in an expanding universe a non-covariant conservation of an energy momentum tensor, which may imply that some energy density is increasing in the locally inertial frame, does not mean a corresponding increase of the energy density in the comoving cosmological frame. For example, a non-covariant conservation of the dust component of the universe, in the examples we study, will produce a still decreasing dust density, although there is a positive flow of energy in the inertial frame. The result of this flow of energy in the local inertial frame is going to be just that the dust energy density decreases a bit slower that the conventional dust in the comoving frame.

Independently of this, we will see how it is possible to construct theories with positive Euclidean action that describe diffusive DE–DM unification.

6 Gravity, “k-essence” and diffusive behavior

Our starting point is the following non-conventional gravity-scalar-field action, which will produce a diffusive type of interacting DE–DM theory:

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} L(\phi, X) \]
\[ + \int d^4x \sqrt{-g} X_{\mu\nu} T_{(X)}^{\mu\nu}, \tag{22} \]

with the following explanations for the different terms: \( R \) is the Ricci scalar which appears in the Einstein–Hilbert action. \( L(\phi, X) \) is the general-coordinate invariant Lagrangian of a single scalar field \( \phi \), which can be of an arbitrary generic “k-essence” type: some function of a scalar field \( \phi \) and the combination \( X = g_{\mu\nu} \partial \phi \partial \phi \) [34–36]:

\[ L(\phi, X) = \sum_{N=1}^{\infty} A_N(\phi) X^N - V(\phi). \tag{23} \]

As we will see, this last action will produce a diffusive interaction between DE–DM type theory. For the ansatz of \( T_{(X)}^{\mu\nu} \) we choose to use some tensor which is proportional to the metric, with a proportionality function \( \Lambda(\phi, X) \):

\[ T_{(X)}^{\mu\nu} = g^{\mu\nu} \Lambda(\phi, X) \Rightarrow S_{(X)} = \int d^4x \Lambda \Box \chi. \tag{24} \]

From the variation of the scalar field \( \chi \) we get \( \Box \Lambda = 0 \), whose solution will be interpreted as a dynamically generated cosmological constant with diffusive source.

We take the simple example for this generalized theory, and for the functions \( L, \Lambda \) we take the first order of the Taylor expansion from (23), or \( L = \Lambda = X (A_1 = 1, A_2 = A_3 = \cdots = 0) \). From the variation according to the scalar field we get a conserved current \( j_{\mu}^{\alpha} = 0 \):

\[ j_\alpha = 2(\Box \chi + 1) \phi_\alpha. \tag{25} \]

For a cosmological solution we take into account only the change as a function of time \( \phi = \phi(t) \). From that we see that the ‘0’ component of the current \( j_0 \) is non-zero. The last variation we should take is according to the metric (using...
the identities in Appendix A), which gives a conserved stress energy tensor:

\[ T_{\mu\nu}^{(G)} = 8\pi(-\Lambda + \chi^{\sigma}A_{,\sigma}) + j^{\mu}\phi^{\nu} - \chi^{\mu}A^{\nu} - \chi^{\nu}A^{\mu}. \]  
(26)

For cosmological solutions the interpretation of the dark energy is by a term proportional to the metric \(-\Lambda + \chi^{\sigma}A_{,\sigma}\), and dark matter dust by the '00' component of the tensor \(j^{\mu}\phi^{\nu} - \chi^{\mu}A^{\nu} - \chi^{\nu}A^{\mu}\). Let us see the solution for the Friedman–Robertson–Walker metric:

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2d\Omega^2 \right]. \]  
(27)

The basic combination becomes \(\mathcal{L} = \Lambda = X = \partial_{\mu}\phi\partial^{\mu}\phi = -\dot{\phi}^2\). Notice that there are high derivative equations, but all such types of equations correspond to conservation laws. For example, we see that the variation of the scalar field (24) will give \(\frac{d}{dt}(2\phi\dot{\phi}a^3) = 0\), which can be integrated to

\[ 2\ddot{\phi} = \frac{C_2}{a^3}, \]  
(28)

which can be integrated again to give

\[ \dot{\phi}^2 = C_1 + C_2 \int \frac{dt}{a^3}. \]  
(29)

The conserved current from Eq. (25) gives the relation

\[ 2\phi(\Box\chi + 1) = \frac{C_3}{a^3}, \]  
(30)

which can also be integrated to give

\[ \dot{\chi} = \frac{1}{a^3} \int a^3dt + \frac{C_4}{a^3} - \frac{C_3}{2a^3} \int \frac{dt}{\phi}, \]  
(31)

which provides the solution for the scalar field \(\chi\). From (26) we get the terms for the DE–DM densities:

\[ \rho_{de} = \dot{\phi}^2 + 2\dot{\chi}\ddot{\phi}, \]  
(32)

\[ \rho_{dm} = \frac{C_3}{a^3}\dot{\phi} - 4\dot{\chi}\ddot{\phi}, \]  
(33)

and the pressure of DE: \(p_{de} = -\rho_{de}\) and DM: \(p_{dm} = 0\). This leads to the Friedman equations with (32) and (33) as source, and there are a few approximations that we want to discuss. The first one is the asymptotic solution.

7 Asymptotic solution and stability of the theory

We can solve asymptotically and by the way show the basic stability of the theory (which should eliminate any concerns related to the formal unboundedness of the action). First we solve for \(\dot{\chi}\); see Eq. (31). We see that the leading term is the fraction \(\frac{1}{a^3} \int a^3dt\). For an asymptotically de Sitter space, where \(a(t) \approx a_0 \exp(H_0t)\), we see that there is a unique asymptotic value:

\[ \lim_{t \to \infty} \dot{\chi} = \frac{1}{3H_0}. \]  
(34)

This is in accordance with our expectations that the expansion of the universe will stabilize the solutions, indeed (30) is basically equivalent to the equation of a particle rolling down a linear potential plus additional negligible terms as \(a(t)\) goes to infinity; the fixed point solution is of course that of constant velocity, when friction \(\times\) velocity = force = 1; since friction = 3H, we obtain Eq. (34).

With this information we can check what the asymptotic value of DE is, from (28), (29) and (32). We see that in this limit, the non-constant part of \(\dot{\phi}^2\) is canceled by \(2\dot{\chi}\ddot{\phi}\), and then asymptotically

\[ \rho_{de} = C_1 + O\left(\frac{1}{a^6}\right); \]  
(35)

with the same analysis for DM density we obtain

\[ \rho_{dm} = \left(C_3\sqrt{C_1} - \frac{2C_2}{3H_0}\right) \frac{1}{a^3} + O\left(\frac{1}{a^6}\right). \]  
(36)

The Friedman equation provides a relation between \(C_1\) and \(H_0\) (the asymptotic value of Hubble constant) which is \(H_0^2 = \frac{8\pi G}{3}C_1\). For negative \(C_2\) we have decaying dark energy, the last term of the contribution for dark energy density is positive (and the opposite). This behavior, where \(C_2 < 0\), has a chance of explaining the coincidence problem, because unlike the standard \(\Lambda\)CDM model, where the dark energy is exactly constant, and the dark matter decreases like \(a^{-3}\), in our case, dark energy can slowly decrease, instead of being constant, and dark matter also decreases, but not as fast as \(a^{-3}\).

As suggested, this behavior can be understood by the observation that in an expanding universe a non-covariant conservation of an energy momentum tensor, which may imply that some energy density is increasing in the locally inertial frame, does not mean a corresponding increase of the energy density in the comoving cosmological frame, here in particular the non-covariant conservation of the dust component of the universe will produce a still decreasing dust density, although, for \(C_2 < 0\), there will be a positive flow of energy in the inertial frame to the dust component, but the
result of this flow of energy in the local inertial frame will be just that the dust energy density will decrease a bit more slowly than conventional dust (but still it decreases).

This is yet another example where potential instabilities are softened or in this case eliminated by the expansion of the universe. It is well known in the case of the Jeans gravitational instability, which is much softer in the expanding universe and also in other situations [37].

Another application of this mechanism could be to use it to explain the particle production, “taking vacuum energy and converting it into particles” as expected from the inflation reheating epoch. Maybe this can be combined with a mechanism that creates standard model particles.

As we see, the expansion of the universe stabilizes the solutions, such that for large times all of them become indistinguishable from ΛCDM, which appears as an attractor fixed point of our theory, showing a basic stability of the solutions at large times. Choosing $C_1$ as positive is necessary, because of the demand that the terms with $\sqrt{C_1}$ will not be imaginary. But for the other constants of integration, there is only the condition $C_1\sqrt{C_1} > \frac{2\pi}{\Omega_1 H_0}$, which gives a positive dust density at large times.

8 $C_2 = 0$ solution

Another special case is when $C_2 = 0$. That means that the dark energy of this universe is constant: we have $\dot{\phi}^2 = C_1$ and $\dot{\phi} = 0$. The equation of motions for the dark energy and dust (32) and (33) are independent on the scalar field $\chi$, and therefore the density of dust in that universe behaves as $\frac{C_1\sqrt{C_1}}{a^2}$. This solution says there is no interaction between dark energy and dark matter. This is precisely the solution of the two measure theory [38–40], with the action

$$ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \int d^4x (\Phi + \sqrt{-g}) \mathcal{L}(X, \phi), $$

which provides a unified picture of DE–DM. For more about two measure theory and related models and solutions for DE–DM, see the discussion in Appendix C. The FRWM for both theories gives the solution

$$ \rho_{DE} = \phi^2 = C_1, $$

$$ \rho_{Dust} = \frac{\sqrt{C_1}C_3}{a^3}. $$

For this trivial case, $C_2 = 0$, there is no diffusion effect between dark matter and dark energy. The current $f^{\mu\nu}$, which is the source of the stress energy tensor $T^{\mu\nu}_{(\rho)}$ (see (7)) is zero, and both stress energy tensors are conserved. This is equivalent to ΛCDM. The exact solution of the case of constant dark energy and dust, using (38) and (39) is [41]

$$ a_0(t) = \left( \frac{C_3}{\sqrt{C_1}} \right)^{\frac{1}{3}} \sinh^{2/3}(\alpha t) $$

(40)

where $\alpha = \frac{3}{2}\sqrt{C_1}$. From comparing to the ΛCDM solution, we can see how the observables values are related to the constant of integration that come from the solution of the theory:

$$ \Omega_\Lambda = \frac{C_1}{H}, \quad \Omega_m = \frac{C_1\sqrt{C_3}}{H}, $$

(41)

where $H$ is the Hubble constant for the late universe. For exploring the non-trivial difusive effect for $C_2 \neq 0$, we use perturbation theory.

9 Perturbative solution

The conclusion from this correspondence is that the diffusion between dark energy and dark matter dust at the late universe is very small, since that is the effect of the $C_2$ term, and therefore we can estimate the solution by perturbation theory. So we obtain two dimensionless terms, which are dependent on time and scale factor, and they tell us the “diffusion rate”:

$$ \lambda_1(t, t_0) = \frac{C_2}{C_1} \int_{t_0}^t \frac{dt}{a^3}, $$

$$ \lambda_2(t, t_0) = \frac{C_2}{\sqrt{C_1}C_3} \chi(t, t_0), $$

(42)

(43)

where the integration is between two close times $t_0$ and $t$. For $C_2 = 0$, both $\lambda_1$ and $\lambda_2$ are equal to zero, and there is no dissipative effect, which as we saw, gives us the ΛCDM model. For any non-zero $\lambda_{1,2} \ll 1$, the stress energy tensor $T^{\mu\nu}_{(\rho)}$ is not conserved, and there is a little diffusion effect.

The use of defining these two dimensionless terms is evident when $C_2$ is small enough for using perturbation theory. By using $\lambda_1$ we can write the scalar field term as $\dot{\phi}^2 = C_1(1 + \lambda)$. The definition for $\lambda_2$ is from the assumption that the leading term in (33), whose scale $\sqrt{C_1}C_3$, is much bigger than the other term $\dot{\chi}\dot{\phi}^2$ (with the $\dot{\chi}C_2$ component, using (28)). The total contributions for the densities, in the context of perturbation theory at the first order, are

$$ \rho_{de} = C_1 \left( 1 + \lambda_1 + \frac{C_3}{\sqrt{C_1}} \lambda_2 \right) + O_2(\lambda_1, \lambda_2), $$

$$ \rho_{dm} = \frac{\sqrt{C_1}C_3}{a^3} \left( 1 - \frac{1}{2} (\lambda_1 + \lambda_2) \right) + O_2(\lambda_1, \lambda_2). $$

(44)

(45)
We can see from those terms that in the deviation from the unperturbed standard solution, the behavior of dark energy and dust are opposite – for increasing dark energy (for example the components are $C_2 < 0; C_1, C_3, C_4 > 0$), the dark matter amount $(\Delta \rho_{dm})$ gets lower. Or in the case of decreasing dark energy, the amounts of dark matter increases (and $C_1, C_2, C_3, C_4 > 0$).

10 Equation of motion and solutions for diffusive energy without higher derivatives

For the second class of theories we proposed in (10)–(12), we can write the diffusive energy action, without high derivatives:

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R + \int \sqrt{-g} \Lambda + \int \sqrt{-g} \chi_{\mu;\nu} T_{(\chi)}^{\mu\nu}$$

$$+ \frac{\sigma}{2} \int \sqrt{-g} (\chi_{\mu} + \partial_{\mu} A)^2$$

(46)

and, as before, the stress energy tensor $T_{(\chi)}^{\mu\nu} = g^{\mu\nu} \Lambda$. From the variation with respect to the vector field $\chi_{\mu}$:

$$\nabla_{\mu} A = f_{\mu} = \sigma (\chi_{\mu} + \partial_{\mu} A).$$

(47)

The variations with respect to the scalars $A$ and $\phi$:

$$f_{\alpha\mu} = 0,$$

(48)

$$j_{\alpha} = 2(\chi_{\lambda\alpha} + 1) \phi_{,\alpha}; j_{\alpha}^{\mu} = 0,$$

(49)

as (7) and (25). Finally for the stress energy tensor, which comes from variation with respect to the metric we obtain

$$T_{(G)}^{\mu\nu} = g^{\mu\nu} \left( -\Lambda + \chi^{\alpha\beta} \Lambda_{\alpha\beta} - \frac{1}{2\sigma} \Lambda^{\alpha\beta} \Lambda_{\alpha\beta} \right)$$

$$+ j^{\mu} \phi^{\nu} - \chi^{\alpha\mu} \Lambda_{\alpha\nu} - \chi^{\nu\mu} \Lambda^\alpha_{\alpha} + \frac{1}{\sigma} \Lambda^{\mu\nu} \Lambda^{\alpha\beta}.$$  

(50)

Both theories, (22) and (46), give rise to similar final equations of motion, besides the variation according to the metric, which asymptotically for large times behave in the same way. The new terms $\frac{1}{2\sigma} \Lambda^{\alpha\beta} \Lambda_{\alpha\beta}$ and $\frac{1}{\sigma} \Lambda^{\mu\nu} \Lambda^{\alpha\beta}$ are negligible at the late universe, since they go as $\frac{1}{\sigma^2}$. For the early universe those terms may be very important, which we will study in future publications.

The modified model of diffusion gives rise to the simpler model when $\sigma$ goes to infinity. Since, in this case, the extra $\frac{\sigma}{2} \int \sqrt{-g} (\chi_{\mu} + \partial_{\mu} A)^2$ term forces $\chi_{\mu} = -\partial_{\mu} A$ (because $(\chi_{\mu} + \partial_{\mu} A)^2 = 0$ and and we are also disregarding light like solutions for $(\chi_{\mu} + \partial_{\mu} A)$, which do not appear relevant to cosmology), i.e. the $\chi_{\mu}$ is a gradient of a scalar. Therefore the theory of dynamical time (5) with a source (46) becomes a diffusive action with high derivatives (6) and (22).

11 Some preliminary ideas on quantization and the boundedness of the Euclidean action

Let us take the action (46); by integration by parts of $\chi_{\mu;\nu} T_{(\chi)}^{\mu\nu}$, and throwing away total derivatives, we obtain the action

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R + \int \sqrt{-g} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}$$

$$- \int \sqrt{-g} \chi_{\mu} \nabla_{\nu} T_{(\chi)}^{\mu\nu} + \frac{\sigma}{2} \int \sqrt{-g} (\chi_{\mu} + \partial_{\mu} A)^2.$$  

(51)

We notice that there are no derivatives acting on the $\chi_{\mu}$ field at this action, and therefor $\chi_{\mu}$ is a Lagrange multiplier. It is legitimate to solve $\chi_{\mu}$ from its equation of motions, and insert the result back into the action. The equation of motion according to the $\chi_{\mu}$ variation is

$$0 = - \nabla_{\nu} T_{(\chi)}^{\mu\nu} + \sigma (\chi_{\mu} + \partial_{\mu} A).$$  

(52)

Solving for $\chi_{\mu}$ and inserting back into the action gives

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R + \int \sqrt{-g} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}$$

$$- \frac{1}{2\sigma} \int \sqrt{-g} (\nabla_{\nu} T_{(\chi)}^{\mu\nu})^2 + \int \sqrt{-g} \partial_{\nu} A \nabla_{\nu} T_{(\chi)}^{\mu\nu}.$$  

(53)

Considering the functional integral quantization for this theory will give a few integrations over the field variables. The functional integral over the scalar $A$ gives rise to a delta function that enforces the covariant conservation of the current $\nabla_{\nu} T_{(\chi)}^{\mu\nu} = f^\nu$. The Euclidean functional integral will be

$$Z = \int D\phi \delta(\nabla_{\nu} f^{\nu})$$

$$\exp \left[ \frac{1}{2\sigma} \int d^4x \sqrt{-g} f_{\mu} f^{\mu} - \int d^4x \sqrt{-g} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right].$$  

(54)

This partition function excludes the Hilbert–Einstein action term, which has its own problems, which are not particularly of interest to this paper. In the full theory we need to include the integration over all the Euclidean geometries.

We can see that the integration measure is positive definite, and the argument of the integrals in the exponents are negative definite in a Euclidean signature space-time sign[+, +, +, +], following the Hawking approach [42]. The terms $f_{\mu} f^{\mu}$ and $\phi_{,\mu} \phi_{,\mu}$ are positive definite, and by choosing the proper sign of $\sigma$, the action is positive definite, and the partition function is convergent. The original theory we formulated in (22) is equivalent to (46) when $\sigma$ goes to minus infinity.

Therefore, this proof is valid for both theories. However, the simple model (22) has to be regularized by first taking
finite and negative $\sigma$, and then letting the $\sigma$ go to minus infinity. This is a preliminary approach, because in the quantum theory, there are many issues concerning how one goes from the Hamiltonian formulation to the path integral formulation, etc. But we see that the quantum theory has a chance to be well defined.

12 Diffusive dark energy and dust by Calogero

The solution for the Calogero suggestion we presented at the beginning, see (1) and (2), leads to the following interdependence between the densities of dark matter and dark energy and the scale parameter:

$$\rho_{de} = C_1 + C_2 \int \frac{dt}{a^3}, \quad (55)$$

$$\rho_{dm} = \frac{C_3}{a^3} - \frac{C_2 l}{a^3}, \quad (56)$$

A complete set of solutions of these differential equations (in the form of Friedman equations) is very complicated, but one phenomenological solution for this theory predicts a DE–DM ratio similar to the observed one [11]. Both approaches (which are described in this paper and in Calogero’s theory) become very similar when the time derivative of the scalar field is low $\dot{\chi} C_2 < 1$. In that case, the dark energy density (32) becomes

$$\rho_{de} = C_1 + C_2 \int \frac{dt}{a^3}. \quad (57)$$

The dark matter dust will reduce to the term (33):

$$\rho_{dm} = \frac{C_3}{a^3} \cdot \dot{\phi} \quad (58)$$

and for those equations it implies diffusion between dark energy and dark matter dust, like Calogero has found. In this model they assumed that the dark energy and the dust are not separately conserved.

We can see that our asymptotic solution does not fit with Calogero’s model, for general $C_2$. As opposed to Eq. (2), in our asymptotic solution (35)–(36) the dark energy density becomes constant, providing a behavior much closer to the standard $\Lambda$CDM model. The main reason for this nonequivalence of those theories is the role of the $\dot{\chi}$ field, which has the effect of making the exchange between dark matter and dark energy less symmetric than in the $\phi$CDM model. In our case, $\dot{\chi}$ makes the decay of DE much lower than in $\phi$CDM, and it keeps the DM evolution still decreasing as $\Lambda$CDM ($a^{-3}$).

13 Discussion, conclusions and prospects

In this paper we have generalized the TMT and the dynamical space-time theory, which imposes the covariant conservation of an energy momentum tensor. We demand that the dynamical space-time 4-vector $\chi_\mu$, which appears in the dynamical space-time theory, be a gradient $\partial_\mu \chi$. We do not obtain the covariant conservation of the energy momentum tensor that is introduced in the action. Instead we obtain current conservation. The current is the divergence of this energy momentum tensor. This current, which drives the non-conservation of the energy momentum tensor, is dissipated in the case of an expanding universe. So we get asymptotic conservation of this energy momentum tensor. Because the four-divergence of the covariant divergence of both the dark matter and dark energy is zero, we can make contact with the dissipative models of [9, 10]. This might give a deeper motivation for these models and allow for the construction of new models.

This energy tensor is not the gravitational energy tensor which appears in the right hand side of the Einstein tensor, in the gravity equations, but the non-covariant conservation of the energy momentum tensor that appears in the action induces an energy momentum transfer between the dark energy and dark matter components, of the gravitational energy momentum tensor, in a way that resembles the ideas in [11]. But one did not provide any action principle to support their ideas. Although the mechanism is similar, our formulation and theirs are not equivalent.

From the asymptotic solution we see that when $C_2 < 0$, unlike the standard $\Lambda$CDM model, where the dark energy is exactly constant, and the dark matter decreases like $a^{-3}$, in our case, dark energy can slowly decrease, instead of being constant, and dark matter also decreases, but not as fast as $a^{-3}$. This special property is different in the $\phi$CMD model, where the exchange between DE and DM is much stronger in the asymptotic limit.

This behavior, where $C_2 < 0$, has a chance of explaining the coincidence problem, because unlike the standard $\Lambda$CDM model, where the dark energy is exactly constant, and the dark matter decreases like $a^{-3}$, in our case, dark energy can slowly decrease, instead of being constant, and dark matter also decreases, but not as fast as $a^{-3}$. This behavior can be understood by the observation that in an expanding universe a non-covariant conservation of an energy momentum tensor, which may imply that some energy density is increasing in the locally inertial frame, does not mean a corresponding increase of the energy density in the comoving cosmological frame. Here in particular the non-covariant conservation of the dust component of the universe will produce a still decreasing dust density, although, for $C_2 < 0$, there will be a positive flow of energy in the inertial frame to the dust component, but the result of this flow of energy in the local inertial frame will be just that the dust energy density will
decrease a bit more slowly than the conventional dust (but it still decreases).

We have seen that in perturbation theory, the behavior of dark energy and dust are different—for increasing dark energy (for example the components are $C_2 < 0; C_1, C_3, C_4 > 0$), the dark matter amount $(\dot{\rho}^3/\dot{\rho}_{dm})$ grows lower. Or in the case of decreasing dark energy, the amounts of dark matter increase (and all the constants of integration are positive).

For another suggestion for diffusive energy action, which does not produce high derivative equations, we have kept the $\chi_{\mu}$ field as a 4-vector (not the gradient of a scalar), but now $\chi_{\mu}$ appears in another term at the action, in addition to a scalar field $A$. The equations of motion produce again a diffusive energy equation, but with the additional contribution of two terms that are negligible for the late universe.

A preliminary argument about the good behavior of the theory at the quantum level is also proposed for both theories. Some additional investigations concerning the quantum theory could be developed by using the WDW equation, in the mini-super space approximation.

Also in the future we will study not only the asymptotic behavior, but the full numerical solution of the dark energy and dark matter components, starting from the early universe, for all the theories we suggested.

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14 Appendix A: Identities

We have

$$\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -\frac{1}{2}(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}),$$

$$\frac{\partial \Gamma^\nu_{\mu\alpha}}{\partial g_{\mu\nu}} = -\frac{1}{2}(g^{\mu\tau}\Gamma^\nu_{\tau\sigma} + g^{\nu\tau}\Gamma^\mu_{\tau\sigma}),$$

$$\frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial g_{\mu\nu,\sigma}} = \frac{1}{4}[g^{\mu\tau}(\delta^\nu_{\alpha}\delta^\sigma_{\beta} + \delta^\nu_{\beta}\delta^\sigma_{\alpha}) + g^{\nu\tau}(\delta^\mu_{\alpha}\delta^\sigma_{\beta} + \delta^\mu_{\beta}\delta^\sigma_{\alpha}) - g^{\tau\sigma}(\delta^\mu_{\alpha}\delta^\nu_{\beta} + \delta^\mu_{\beta}\delta^\nu_{\alpha})],$$

$$T_{\alpha\beta} = \frac{-2}{\sqrt{-g}} \frac{\partial}{\partial g_{ab}} \left( \frac{1}{\sqrt{-g}} \frac{\partial \chi_{\mu\nu\sigma}}{\partial g_{ab}} T_{\chi_{\mu\nu\sigma}}^{\mu\nu\sigma} \right).$$

15 Appendix B

An equivalent expression for (7), when $T_{\chi_{\mu\nu\sigma}}^{\mu\nu\sigma}$ is formulated as a perfect fluid in FRWM space, is

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = \frac{C_2}{a^3},$$

when $C_2 = 0$, the stress energy tensor is conserved, and there is no diffusive effect. For late times, where the scale parameter goes to infinity, we see that the diffusive effect vanishes.

16 Appendix C

TMTs also have many points of similarity with ‘Lagrange multiplier gravity (LMG)’ [43,44]. The Lagrange multiplier field in LMG enforces the condition that a certain function be zero. In the TMT this is equivalent to the constraint that requires some lagrangian to be constant. The two measure models presented here are different from the LMG models of [43,44], and they provide us with an arbitrary constant of integration for the value of a given lagrangian, this constant of integration, if non-zero, can generate spontaneous symmetry breaking of scale invariance, which is present in the theory for example. Recently a lot of interest has been attracted by the so-called “mimetic” dark matter model proposed in [45,46]. The latter employs a special covariant isolation of the conformal degree of freedom in Einstein gravity, whose dynamics mimics cold dark matter as a pressure-less “dust”. Important questions concerning the stability of “mimetic” gravity are studied in Refs. [47,48] where also one formulates a generalized tensor–vector–scalar “mimetic” gravity, which avoids those problems. In [49] the idea is applied to inflationary scenarios.

Most versions of the mimetic gravity, except for [47] appear to be equivalent to a special kind of Lagrange multiplier theory or TMT models that were known before, where we have the simple constraint that the kinetic term of a scalar field be constant. This of course gives identical results to the very special TMT where the lagrangian that couples to the new measure is the kinetic term of this scalar field.

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11 General consideration with diffusion effects

D. Benisty, E. Guendelman and Z. Haba,
“Unification of dark energy and dark matter from diffusive cosmology”
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This paper tests the general combination of $T_{(3)}^{\mu\nu}$ using the $\lambda_1$ and $\lambda_2$ parameters. This formulation of dark energy and dark matter has a direct correspondence with the behavior of the non-Lagrangian formulations of the dark energy and dark matter interactions only in the case $\lambda_2 = 0$. In the other cases, the asymptotic behavior is different and the Dynamical Time approaches $1/3H_0$ asymptotically. All of the solutions have an asymptotically stable $\Lambda$CDM behavior.
Unification of dark energy and dark matter from diffusive cosmology

D. Benisty*

Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel, Frankfurt Institute for Advanced Studies (FIAS), Ruth-Moufang-Strasse 1, 60438 Frankfurt am Main, Germany

E. I. Guendelman†

Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel, Frankfurt Institute for Advanced Studies (FIAS), Ruth-Moufang-Strasse 1, 60438 Frankfurt am Main, Germany

and Bahamas Advanced Study Institute and Conferences, 4A Ocean Heights, Hill View Circle, Stella Maris, Long Island, The Bahamas

Z. Haba‡

Institute of Theoretical Physics, University of Wroclaw, 50-204 Wroclaw, Plac Maxa Borna 9, Poland

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Generalized ideas of unified dark matter and dark energy in the context of dynamical space time theories with a diffusive transfer of energy are studied. The dynamical space-time theories introduce a vector field whose equation of motion guarantees a conservation of a certain energy momentum tensor, which may be related, but in general is not the same as the gravitational energy momentum tensor. This particular energy momentum tensor is built from a general combination of scalar fields derivatives as the kinetic terms, and possibly potentials for the scalar field. By demanding that the dynamical space vector field be the gradient of a scalar the dynamical space time theory becomes a theory for diffusive interacting dark energy and dark matter. These generalizations produce nonconserved energy momentum tensors instead of conserved energy momentum tensors which lead at the end to a formulation for interacting dark energy and dark matter (DE-DM). We solved analytically the theories and we show that the $\Lambda$CDM is a fixed point of these theories at large times. A particular case has asymptotic correspondence to previously studied non-Lagrangian formulations of diffusive exchange between dark energy and dark matter.

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I. INTRODUCTION

Dark energy and dark matter (DE-DM) constitute most of the observable Universe. Yet the true nature of these two phenomena is still a mystery. One fundamental question with respect to those phenomena is the coincidence problem which is trying to explain the relation between dark energy and dark matter densities. In order to solve this problem, one approach claims that the dark energy is a dynamical entity and hope to exploit solutions of scaling or tracking type to remove dependence on initial conditions. Others left this principle and tried to model the dark energy as a phenomenological fluid which exhibits a particular relation with the scale factor [1], Hubble constant [2] or even the cosmic time itself [3].

Interaction between DM and DE was considered in many cases, such as [4]. Unifications between dark energy and dark matter from an action principle were obtained from scalar fields [5–9] or by other models [10–15] including Galileon cosmology [13] or telleparallel modified theories of gravity [16,17]. Beyond those approaches, a unification of dark energy and dark matter using a new measure of integration (the so-called two measure theories) has been formulated [18–22]. A diffusive interaction between dark energy and dark matter was introduced in [23–28] and was formulated in the context of an action principle based on a generalization of those two measures theories in the context of quintessential scalar fields [24,25].

In recent publications [26], diffusion of energy between dark energy into dark matter was discussed. The models of such type are interesting as an approach to solve the coincidence problem. The basis of those models are considering a non-conserved stress energy tensor $T^{\mu\nu}$ with a source current $j^\nu$:

$$\nabla_\mu T^{\mu\nu}_{\text{(Dust)}} = \gamma^2 j^\nu$$  \hspace{1cm} (1)
where $\gamma^2$ is the coupling diffusion coefficient of the fluid. The current $j^\mu$ is a timelike covariant conserved vector field $\vec{f}_\mu = 0$ which describes the conservation of the number of particles in the system. Due to the fact that the Einstein tensor is covariantly conserved $\nabla_\nu G^{\mu\nu} = 0$, we have to introduce on the right-hand side of the Einstein tensor a compensating energy momentum tensor, for two diffusive fluids, where:

$$\nabla_\mu T^{\mu\nu}_{(\text{Dust})} = -\nabla_\mu T^{\mu\nu}_{(\Lambda)} = \gamma^2 j^\nu$$

(2)

so that the total energy momentum tensor is conserved:

$$\nabla_\mu (T^{\mu\nu}_{(\text{Dust})} + T^{\mu\nu}_{(\Lambda)}) = 0$$

(3)

Such models could originate from irreversible diffusive exchange of energy, or have a Lagrangian origin, by introducing an independent stress energy momentum tensor $T^{\mu\nu}_{(\chi)}$ directly in the Lagrangian. The structure of the paper is as follows: In Sec. II we discuss dynamics of exchange of energies between two diffusive fluids, with two different equation of states. Such a system has a universal model independent behavior. In Sec. III we present the Lagrangian model leading to such an interactive energy momentum tensor. In Sec. IV we discuss solutions for the theory which contains more general combinations for the stress energy momentum tensor $T^{\mu\nu}_{(\chi)}$. In Sec. V we are looking for few asymptotic solutions for the theory. In Sec. VI we discuss a special case of a Lagrangian which corresponds to the diffusive model which has been introduced in Sec. II.

II. COUPLED DIFFUSIVE FLUIDS

We assume that stress energy momentum tensors are in the form of ideal fluids, where:

$$T^{\mu\nu}_{(\chi)} = \text{Diag}(\rho, -p, -p, -p)$$

(4)

where $\rho$ is the energy density and $p$ is the pressure. Then Eqs. (1)–(2) read:

$$\dot{\rho}_{\text{dust}} + 3H(1 + \tilde{\omega})\rho_{\text{dust}} = \frac{\gamma^2}{a^2}$$

(5)

and

$$\dot{\rho}_{\Lambda} + 3H(1 + \omega)\rho_{\Lambda} = -\frac{\gamma^2}{a^2}.$$  

(6)

The diffusion constant $\gamma^2$ is always positive. $\omega$ and $\tilde{\omega}$ denote the ratio of the pressure and the density for the corresponding fluids. In order investigate the behavior of the solution, we introduce the dynamical system method for the equations. The dimensionless quantities for the system are defined as [28]:

$$x = \frac{\rho_{\text{dust}}}{3H^2}, \quad y = \frac{\rho_{\Lambda}}{3H^2}, \quad \delta = \frac{\gamma^2}{a^2H\rho_{\text{dust}}}$$

(7)

where $\delta$ describes the strength of the relative diffusion. From Friedmann equations $x + y = 1$. The complete autonomous system method equations are

$$x' = 6x(x - 1)(\omega - \tilde{\omega})$$

(8a)

$$\delta' = \delta(\gamma \delta + 3(x - 1)(\omega - \tilde{\omega})).$$

(8b)

Table I presents the critical points in the system. In order to determine the stability of the system we have to specify the equations of states. For the case of dark matter and dark energy we can choose two cases: the first one: $\omega = -1, \tilde{\omega} = 0$ and the second one $\omega = 0, \tilde{\omega} = -1$. The phase portrait for both cases is presented in Fig. I. The case $\omega = 0, \tilde{\omega} = -1$, which represent the exchange of energy from the dark energy into dark matter include a stable point $A(0, -\frac{2}{3})$ which corresponds to dark energy dominant with diffusion effect. However, the second case $\omega = -1, \tilde{\omega} = 0$, which represent the exchange of energy from the dark matter into dark energy include a stable point $C(0, 0)$ which corresponds to dark energy dominant without diffusion effect.

In this model we have chosen $\omega$ and $\tilde{\omega}$ being constants, whereas in general Lagrangian models $\omega$ and $\tilde{\omega}$ are varying in time. However we expect that $\omega$ and $\tilde{\omega}$ can be approximated by constants for large times. In the next sections we investigate more general dynamics on the basis of the action principle.

| Name | The point | Eigenvalues | Densities fraction |
|------|-----------|-------------|--------------------|
| A    | $(0, \frac{1}{3}(\omega - \tilde{\omega}))$ | $3(3\omega + 1), 3(\omega - \tilde{\omega})$ | 0 |
| B    | $(\frac{1}{3}(\omega + 1), -\frac{2\omega + 1}{3\omega})$ | $\pm \sqrt{\frac{36\omega^2 - 72\omega\tilde{\omega} + 9(\omega - 2)\tilde{\omega} - 3 - 6\omega - 3\tilde{\omega} + 5}{\omega - \tilde{\omega}}}$ | $-\frac{2\omega + 1}{2\omega - 1}$ |
| C    | $(0, 0)$ | $3(\omega - \tilde{\omega}), 3(2\omega + \tilde{\omega} + 1)$ | 0 |
| D    | $\left(\frac{2\omega + 1}{2(\omega - \tilde{\omega})}, 0\right)$ | $-3(2\omega + \tilde{\omega} + 1), \frac{1}{3}(1 + 3\tilde{\omega})$ | $-\frac{2\omega + 1}{2\omega - 1}$ |
II. A LAGRANGIAN WITH DYNAMICAL SPACE-TIME

A. Two measures theories

The two measure theory implies other measure of integration in addition to the regular measure of integration in the action $\sqrt{-g}$. The new measure is also a density and a total derivative. A simple example for constructing this measure is by introducing 4 scalar fields $\varphi_a$, where $a = 1, 2, 3, 4$. The measure reads:

$$\Phi = e^{\phi_i} e_{abcd} \partial_a \varphi_i \partial_b \varphi_d \partial_c \varphi_d.$$  \(9\)

A complete action involving both measures takes the form:

$$S = \int d^4 x \Phi L_1 + \int d^4 x \sqrt{-g} L_2.$$  \(10\)

As a consequence of the variation with respect to the scalar fields $\varphi_a$, under the assumption that $L_1$ and $L_2$ are independent of the scalar fields $\varphi_a$, we obtain that:

$$A_\mu^a \partial_\mu L_1 = 0,$$  \(11\)

where $A_\mu^a = e^{\phi_i} e_{abcd} \partial_\mu \varphi_i \partial_b \varphi_d \partial_c \varphi_d$. Since $\det[A_\mu^a] \sim \Phi^3$ as one easily see then that for $\Phi \neq 0$, Eq. (11) implies that $L_1 = M = \text{Const}$. These kind of contributions have been considered in the two measures theories which are of interest in connection with a unified model of dark energy and dark matter [19].

B. Dynamical time action

The constraint on the term in the action $L_2$ as in the two measure theories (10) could be generalized to a covariant conservation of a stress energy momentum tensor $T^\mu_\nu(\chi)$ which coupled directly in the action [27]:

$$S = \int d^4 x \sqrt{-g} \chi_\rho T^\rho_\mu(\chi)$$  \(12\)

to a vector field $\chi_\mu$ with its covariant derivatives $\chi_{\mu\nu} = \partial_\mu \chi_\nu - \Gamma^\lambda_{\mu\nu} \chi_\lambda$. From the variation with respect to the vector field $\chi_\mu$ gives a constraint on the conservation of the stress energy tensor $T^\mu_\nu(\chi)$:

$$\delta \chi_\mu : \nabla_\mu T^\mu_\nu(\chi) = 0.$$  \(13\)

Similarly as the variation with respect to the scalar field $\varphi_a$ in the Lagrangian (10) yields $\partial_\mu \mathcal{L} = 0$. The correspondence between them is when $T^\mu_\nu(\chi)$ is taken to be as $T^\mu_\nu(\chi) = \rho \chi \mathcal{L}_m$. By introducing the term in the action (12), we get:

$$\int d^4 x \sqrt{-g} \chi_\rho T^\rho_\mu(\chi) = \int d^4 x \sqrt{-g} \chi_\rho \mathcal{L}_m \rho$$

Similarly to the variation (10), the variation with respect to the scalar field gives again $\partial_\mu \mathcal{L}_m = 0$. For dynamical time theories, the variation with respect to the dynamical time vector field yields the same constraint.

The name dynamical time theory (DTT) was considered due to the fact the energy density $T_0^0(\chi)$ is the canonically conjugated variable to the dynamical time $\chi^0$:

$$\pi_x^0 = \frac{\partial L}{\partial \chi^0} = T_0^0(\chi) = \rho(\chi)$$  \(15\)

where $\rho(\chi)$ is the energy density of the original stress energy tensor.

C. Dynamical time action with diffusive source

In order to break the conservation of $T^\mu_\nu(\chi)$ as in the diffusion equation [Eq. (1)], the vector field $\chi_\mu$ should be coupled in a mass like term in the action:

$$S_{(\chi,A)} = \int d^4 x \sqrt{-g} \chi_\rho T^\rho_\mu(\chi) + \frac{\kappa}{2} \int d^4 x \sqrt{-g} (\chi_\rho + \partial_\rho A)^2$$  \(16\)

where $A$ is a scalar field different from $\phi$. From a variation with respect to the dynamical space time vector field $\chi_\mu$ we obtain:

$$\nabla_\mu T^\mu_\nu(\chi) = \kappa (\chi^\mu + \partial^\mu A) = f^\mu,$$  \(17\)

where the current source reads: $f^\mu = \kappa (\chi^\mu + \partial^\mu A)$. From the variation with respect to the new scalar $A$ a covariant conservation of the current indeed emerges:

$$\nabla_\mu f^\mu = \kappa \nabla_\mu (\chi^\mu + \partial^\mu A) = 0.$$  \(18\)

The stress energy tensor $T^\mu_\nu(\chi)$ is substantially different from stress energy tensor that we all know from Einstein equation which is defined as $T^\mu_\nu(\chi) = R^\mu_\nu - \frac{1}{2} g^\mu_\nu R$. In this case, the stress energy momentum tensor $T^\mu_\nu(\chi)$ is a diffusive nonconservative stress energy tensor. However, from a variation with respect to the metric, we get the conserved stress energy tensor as in Einstein equation:

$$T^\mu_\nu(\chi) = \frac{-2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^\mu_\nu}, \quad \nabla_\mu T^\mu_\nu(\chi) = 0.$$  \(19\)
Using different expressions for $T^\mu_\nu(\chi)$ which depends on different variables, gives the conditions between the dynamical space time vector field $X_\mu$ and the other variables.

**D. Higher derivatives action**

A particular case of diffusive energy theories is obtained when $\kappa \rightarrow \infty$. In this case, the contribution of the current $j_\mu$ in the equations of motion goes to zero and yields a constraint for the vector field being a gradient of the scalar:

$$f_\mu = \kappa (\chi_\mu + \partial_\mu A) = 0 \Rightarrow \chi_\mu = -\partial_\mu A. \quad (20)$$

For the rest of the paper we use the notation $\chi$ for the scalar field which is coupled to the stress energy momentum tensor and not $A$ due to earlier publications. The theory (16) is reduced to a theory with higher derivatives:

$$S = -\int d^4x \sqrt{-g} \nabla_\nu T^\mu_\nu(\chi). \quad (21)$$

The variation with respect to the scalar $A$ gives $\nabla_\mu \nabla_\nu T^\mu_\nu = 0$ which corresponds to the variations (17)–(18). In the following paper we use the reduced theory with higher derivative in the action.

**IV. SCALAR FIELD GRAVITY WITH DIFFUSIVE BEHAVIOR**

**A. Dynamical time action with diffusive source**

In this section we consider the following action:

$$\mathcal{L} = -\frac{1}{2} \dot{\chi}^2 \chi + \chi_\mu \dot{T}^\mu_\nu(\chi) - \frac{1}{2} \phi_\mu \phi_\nu - V(\phi) \quad (22)$$

which contains a scalar field with potential $V(\phi)$. The stress energy momentum tensor $T^\mu_\nu(\chi)$ is chosen to be

$$T^\mu_\nu(\chi) = -\frac{\lambda_1}{2} \phi_\rho \phi^\rho - \frac{\lambda_2}{2} \phi^\rho \phi_\rho + \phi^\mu U(\phi), \quad (23)$$

where $\lambda_1$ and $\lambda_2$ are arbitrary constants, and $U(\phi)$ is another potential. In such a case the density and pressure resulting from $T^\mu_\nu(\chi)$ are

$$\rho(\chi) = (\lambda_1 + \lambda_2) \frac{\dot{\phi}^2}{2} + U(\phi), \quad (24)$$

$$p(\chi) = -\lambda_2 \frac{\dot{\phi}^2}{2} - U(\phi). \quad (25)$$

Notice that the starting point was the case of two fluids. But here we discuss the single fluid with a Lagrangian involving two different measures: where the modified measure is generalized by using the dynamical space time vector field $X_\mu$.

There are three independent sets of equations of motions: $\chi$, $\phi$ and the metric $g_{\mu\nu}$. The variation with respect to the field $\chi$ yields:

$$\nabla_\mu \nabla_\nu T^\mu_\nu(\chi) = 0 \quad (26)$$

The variation with respect to the field $\phi$ gives a non-conserved current $j^\mu$:

$$j^\mu = \frac{\lambda_1}{2} (\chi^{\alpha\beta} + \chi^{\mu\nu}) \phi_\alpha + (1 + \lambda_2) \dot{\phi}^\mu, \quad (27)$$

with the nonconservation law:

$$\nabla_\mu j^\mu(\chi) = V'(\phi) - \nabla \chi U'(\phi). \quad (28)$$

The Einstein equations derived from the variation with respect to the metric take the form:

$$G^\mu_\nu = g^{\rho\nu} \left( -\chi_{,\alpha\beta} T^\alpha_\beta(\chi) + \frac{1}{2} \phi_\alpha \phi_\alpha + V(\phi) \right)$$

$$- \phi_\mu \phi_\nu + \chi_{,\alpha\beta} \frac{\partial T^\alpha_\beta(\chi)}{\partial g_{\mu\nu}}$$

$$+ \nabla_\delta (\chi^{\mu\nu} T^\delta_\nu(\chi) + \chi^{\mu\nu} T^\delta_\nu(\chi) - \chi^{\mu\nu} T^\delta_\nu(\chi)) \quad (29)$$

where the derivative of the energy momentum tensor $T^\mu_\nu(\chi)$ with respect to $g_{\mu\nu}$ yields:

$$\chi_{,\alpha\beta} \frac{\partial T^\alpha_\beta(\chi)}{\partial g_{\mu\nu}} = -\frac{\lambda_1}{2} \chi^{(\alpha\beta} \phi^\gamma) \Box \phi + \left( \frac{\lambda_1}{2} + \lambda_2 \right) \phi_\alpha \phi^\gamma \Box \chi$$

$$+ \frac{\lambda_1}{2} \chi^{(\alpha\beta} \phi^\gamma \phi_\gamma - \lambda_2 \phi_\alpha \phi_\beta \phi_\gamma - \lambda_2 \chi^{(\alpha} \phi_\beta \phi_\gamma \phi_\delta + \frac{\lambda_1}{2} \phi_\delta \chi^{\alpha\beta} \phi_\gamma \phi_\delta$$

$$+ \frac{\lambda_1}{2} \chi^{(\alpha\beta} \phi_\gamma \phi_\delta + \chi^{(\alpha} \phi_\beta \phi_\delta \phi_\gamma + \chi^{(\alpha} \phi_\beta \phi_\gamma \phi_\delta \phi_\delta.$$
Integrating Eq. (26) once, we express it in the form:

$$(\lambda_1 + \lambda_2) \ddot{\phi} + U'(\phi) \dot{\phi} + 3H\lambda_1 \phi^2 = \frac{\sigma}{a^3}$$

(31)

where $\sigma$ is an integration constant.

For dark energy dynamics we can assume that $U(\phi) = \text{const}$. Then the solution for Eq. (31) is

$$\dot{\phi}^2 = \dot{\phi}_0^2 a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}} + \frac{\sigma}{\lambda_1 + \lambda_2} a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}} \int_0^t ds a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}}.$$  

(32)

In addition for the same theoretical reason we assume that $V(\phi) = \text{Const}$. Then the current conservation law (28) has the solution:

$$\left(\frac{\lambda_2}{2} + \lambda_2 \frac{3}{2} H \lambda_2 \right) \dot{\phi} + (1 - 3H\dot{\phi}) \dot{\phi} = \frac{\sigma}{\dot{\phi}a^{3}}$$

(33)

where $\tilde{\sigma}$ is another integration constant. Now from the stress energy momentum tensor the total energy density term is

$$\rho = \frac{3}{2} H(\lambda_1 - 2\lambda_2)\dot{\phi}^2 + \frac{1}{2} \dot{\phi}^2 (1 - 2(\lambda_1 + \lambda_2)\dot{\phi})$$

$$+ \ddot{\phi} \phi ((\lambda_1 + \lambda_2)\dot{\phi}) + V,$$

and the total pressure is

$$p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \lambda_1 \ddot{\phi} \dot{\phi}^2 + \lambda_2 \dot{\phi} \ddot{\phi} - V.$$  

(35)

V. ASYMPTOTIC SOLUTIONS

We are not able to find the exact solutions for the Einstein Eq. (29) together with the equations for the scalar fields $\chi$ [Eq. (31)] and $\phi$ [Eq. (33)]. So we are looking for asymptotic solutions.

A. A power law solution

We assume a power law solution for a large time:

$$a \sim t^n. $$

(36)

Then from Eq. (31) the solution for the scalar field $\phi$ derivative is

$$\dot{\phi} = \sqrt{\frac{2\sigma}{3\sigma(\lambda_1 - \lambda_2) + \lambda_1 + \lambda_2 a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}}}},$$

(37)

where $\phi_0$ is an arbitrary integration constant.

The solution for the scalar field $\chi$ is

$$\dot{\chi} = Ct$$

with the constant:

$$C = \frac{2\lambda_2}{-6\alpha \lambda_2 + \lambda_1 - 2\lambda_2}.$$  

(39)

By inserting the solutions (37) and (38) into Einstein equation we obtain:

$$\rho = \frac{a_1}{a^3} + \frac{a_2 t}{a^3} + V$$

(40)

where the constants are

$$a_1 = \frac{18\alpha^2 \lambda_2(2\lambda_2 - \lambda_1)}{2(\lambda_2 - 2\lambda_2(3\alpha + 1))}$$

(41)

$$a_2 = \frac{(6\alpha + 2)\lambda_1 \lambda_2 + 2(3\alpha + 1)(\lambda_2 - 1)\lambda_2 + \lambda_1}{2(\lambda_2 - 2\lambda_2(3\alpha + 1))}.$$  

(42)

We get an asymptotic solution if the potential $V = 0$ and the power of the scale factor is one:

$$a \sim t.$$  

(43)

This solution is the same as the one obtained in the model of Einstein equation with relativistic diffusion exchange of energy [28].

B. Exponential solution

We insert the exponential solution $a(t) \sim e^{H_0 t}$ in Eq. (32). Then we get:

$$\dot{\phi}^2 = \dot{\phi}_0^2 a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}} - \sigma_0 H_0 \frac{\lambda_1 + \lambda_2}{3\lambda_2} \frac{1}{a^3}$$

(44)

if we impose $\frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} > 0$. Then from Eq. (33) we obtain the asymptotic solution:

$$\dot{\chi} = \frac{1}{3H_0} + \mathcal{O}\left(\frac{1}{a^3}\right).$$

(45)

With those solutions the density is given by:

$$\rho = H_0(3\lambda_2 - 1) \sigma \frac{\lambda_1 + \lambda_2}{6\lambda_2} \frac{1}{a^3} + V$$

$$+ \frac{1}{2} \dot{\phi}_0^2 (1 - 2\lambda_2) a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}}.$$  

(46)

This particular solution corresponds to a slowly varying dark energy ($V + \frac{1}{2} \dot{\phi}_0^2 (1 - 2\lambda_2) a^{\frac{3\lambda_1}{\lambda_1 + \lambda_2}}$) approaching a constant value $V$, for $\lambda_1$ and $\lambda_2$ being positive, and $\lambda_1 \ll \lambda_2$. In the case of negative $\lambda_1$ but still $|\lambda_1| \ll |\lambda_2|$, we get slowly growing vacuum energy, which corresponds to an asymptotically super accelerating universe.
VI. $\lambda_2 = 0$ CASE

Solution (44) does not make sense for $\lambda_2 = 0$. Therefore this case should be treated separately. This special choice of the energy momentum has been explored by Gao, Kunz, Liddle and Parkison as a unification of dark energy and dark matter [29] without using a Lagrangian formulation. These authors proposed as a unification of dark energy and dark matter:

$$T_{\mu \nu} = -\frac{\lambda_1}{2} \phi^{\mu \nu} + \phi^{\mu \nu} U(\phi)$$

as the right-hand side of Einstein tensor. The action that produces asymptotically the same model using dynamical time theories was obtained in Ref. [30]. Here we explore the asymptotic solution with diffusive behavior. Under the assumption that all of the potentials are constant Eq. (31) has the solution:

$$\dot{\phi}^2 = \frac{\phi_{(0)}^2}{a^4} + \frac{\sigma}{\lambda_1 a^4}.$$  

Then, the integral of Eq. (33) is

$$\dot{\chi}(t) = \dot{\chi}(0) - \frac{2}{\lambda_1} t + \int dt' \frac{2\sigma}{\lambda_1 \phi a^4}$$

with the asymptotic behavior:

$$\dot{\chi}(t \to \infty) \to -\frac{2t}{\lambda_1}.$$  

Notice that this asymptotic behavior is essentially different from the previous cases. Then the total density reads:

$$\rho = V + \frac{\alpha_1}{a^3} + \frac{\alpha_2}{a^4},$$

where the coefficients are

$$\alpha_1 = \frac{5\phi_{(0)}^2 \lambda_1 + \lambda_1 \sigma_0 + 3\sigma t}{2\lambda_1},$$

$$\alpha_2 = -\frac{2\sigma}{3\phi_{(0)} H_0 \lambda_1} \left(3\phi_{(0)}^2 H_0 \lambda_1 + 3H_0 \sigma t + \sigma\right).$$

Additional symmetry for this case is obtained:

$$\chi \to \chi + c t$$

or in terms of the dynamical time ($\chi^0 \Rightarrow \dot{\chi}$)

$$\chi^0 \to \chi^0 + c.$$  

In the previous cases $\dot{\chi}$ is asymptotically a constant, equal to $\frac{1}{\phi_0}$. In the special case of $\lambda_2 = 0$ there cannot be any particular choice for asymptotic value of $\chi$, because the symmetry will change it to any other arbitrary constant. One can calculate the conserved quantity associated with the symmetry (55) and it is the analogous of particle number.

A remarkable result is the correspondence between the solution (51) and the solutions for the DM-DE interaction system from Sec. II. For $\omega = 0$ the dust density equation yields:
with the solution:
\[ \rho_{\text{dust}} = C_1 a^3 + \gamma^2 t a^3, \quad (57) \]
where \( C_1 \) is an integration constant. For interacting dark energy, that satisfies \( \omega = -1 \), the energy density reads:
\[ \rho = C_1 + \gamma^2 t a^3 + O\left(\frac{1}{a^6}\right), \quad (58) \]
whereas \( \rho_\Lambda \)
\[ \rho_\Lambda = C_2 - \gamma^2 \int \frac{dt}{a^3}. \quad (59) \]
The \( C_2 \) is another integration constant. Asymptotically, the total density gives:
\[ \rho = \frac{C_1 + \gamma^2 t}{a^3} + C_2 + O\left(\frac{1}{a^6}\right), \quad (60) \]
which corresponds to the density (51), and the last term \( \frac{\gamma^2}{a^6} \) becomes negligible. Hence, the integration constants equal to the integration constants from the Lagrangian case:
\[ C_1 = \frac{5\phi_0^2 + \sigma\chi_0}{2}, \quad (61) \]
\[ \gamma^2 = \frac{3\sigma}{2\lambda_1}, \quad C_2 = V. \quad (62) \]
This correspondence does not hold for the whole history of the universe, however asymptotically the models (our Lagrangian model and the previously studied non-Lagrangian models) fit each other for the case \( \lambda_2 = 0 \) and approach \( \Lambda \)CDM for late times. Of course that the solutions will have to be studied and this will be a main goal for further investigations.

One can see that both models with exactly the same homogeneous solution where \( \tilde{\sigma} = 0 \). In this case \( \alpha_2 = 0 \) [see Eq. (53)] and the corresponding relations between the constants of the models present in Eqs. (61)–(62).

In order to assess the viability of the model, let us see how some physical quantities change versus the redshift \( z \) for both models. The connection between the cosmic time derivative and the redshift derivative reads:
\[ \frac{d}{dt} = -H(z)(z + 1) \frac{dz}{dz}, \quad (63) \]
which is obtained from the dependence of scale factor on the redshift \( a = \frac{1}{z+1} \). The numerical solution of the partial densities for the simplest case appear in Fig. 2. Even this simple case describes a diffusive interaction between dark energy dark matter from an action principle. However, the presence of the coupling constant \( \tilde{\sigma} \) yields to additional part \( \sim a^{-4.5} \) which could resolve the singularity problem as discussed in Ref. [30]. But in any case, all the solutions approach \( \Lambda \)CDM model for the late universe.

VII. CONCLUSIONS

We have extended the results of our earlier papers concerning the DM-DE interaction in the context of two measures models and the dynamical time theories. The extension consists in a general choice of the conserved noncanonical energy-momentum tensor. The energy momentum tensor is more general than the one proposed by Gao, Kunz, Liddle, and Parkinson [29] as well as the dark energy dark matter unification obtained in the two measures limit, which corresponds to the case where the conserved noncanonical energy-momentum tensor is proportional to the metric tensor [19,20].
The constants \( \lambda_1 \) and \( \lambda_2 \) parametrize the more general choice considered here. \( \lambda_2 = 0 \) corresponds to the case considered by Gao, Kunz, Liddle, and Parkinson in their non-Lagrangian formalism. In our Lagrangian formulation, for this type of energy momentum tensor, as additional shift symmetry for the dynamical time appears and at the same time the dynamical time behaves asymptotically as the cosmic time. Diffusive type is obtained when the dynamical space time vector is taken to be the gradient of a scalar, then instead of a conservation law of the energy momentum introduced in the action, we obtain a nonconservation of this energy momentum tensor of the diffusive type, which leads then to an interacting DE/DM scenario. This formulation of DE-DM have a direct correspondence with the behavior of non Lagrangian formulations of DE/DM interactions only in the case \( \lambda_2 = 0 \). In the other cases, the asymptotic behavior is different and in particular the dynamical time does not behave as cosmic time asymptotically, in fact as the cosmic time increases, the dynamical time approaches the finite value \( \frac{1}{H} \) in an asymptotically de Sitter space. In all cases we do not need to introduce the dark matter in the initial Lagrangian, it appears dynamically. As a result of the dynamic evolution in our model we obtain an asymptotically \( \Lambda \)CDM solution.

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Erratum: Unification of dark energy and dark matter from diffusive cosmology
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D. Benisty, E. I. Guendelman, and Z. Haba

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Modifications for the paper are studied. We find that the variation with respect to scalar $\phi$ has a typo that changes some coefficient that does not change the statement of the paper. The variation with respect to the scalar field $\phi$,

$$\frac{\lambda_1}{2} \ddot{\chi} + (1 - 3H\dot{\chi})\lambda_2 = \frac{\dot{\sigma}}{\phi a^3},$$

should be modified for Eq. (33),

$$\lambda_1 \ddot{\chi} - \lambda_2 (\ddot{\chi} - 3H\dot{\chi}) + 1 = \dot{\sigma}/(\phi a^3).$$

This modification changes the asymptotic values of the scalar from the value

$$\dot{\chi} = \frac{1}{3H_0} + O\left(\frac{1}{a^3}\right),$$

to the value [Eq. (45)]

$$\dot{\chi} = \frac{1}{3\lambda_2 H_0} + O\left(\frac{1}{a^3}\right).$$

(45)

Therefor the asymptotic density equations have different coefficient constants. The asymptotic density term changes from

$$\rho = H_0 (3\lambda_2 - 1)\dot{\sigma} \frac{\lambda_1 + \lambda_2}{6\lambda_2} \frac{1}{a^3} + V + \frac{1}{2} \phi_0^2 (1 - 2\lambda_2) a^{-\frac{\lambda_1}{\lambda_2}},$$

to [Eq. (46)]

$$\rho = H_0 \sigma_0 (\lambda_1 + \lambda_2) \frac{1}{3\lambda_2} a^{-\frac{\lambda_1}{\lambda_2}} + V - \frac{1}{2} \phi_0^2 a^{-\frac{\lambda_1}{\lambda_2}}.$$  (46)

The sentence after Eq. (46) changes to the following:

“This particular solution corresponds to a slowly varying dark energy ($V - \frac{1}{2} \phi_0^2 a^{-\frac{\lambda_1}{\lambda_2}}$) approaching a constant value $V$, for $\lambda_1$ and $\lambda_2$ being positive, and $\lambda_1 \ll \lambda_2$.”

$\lambda_2 = 0$ case.—As a consequence of the modification in Eq. (33), the particular case of $\lambda_2 = 0$ yields the modified Eq. (49):

$$\dot{\chi}(t) = \dot{\chi}(0) - \frac{1}{\lambda_1} t + \int dt \frac{\dot{\sigma}}{\lambda_1 \phi a^3}$$

(49)

instead of the incorrect one:

$$\dot{\chi}(t) = \dot{\chi}(0) - \frac{2}{\lambda_1} t + \int dt \frac{2\dot{\sigma}}{\lambda_1 \phi a^3}.$$
The asymptotic behavior reads [Eq. (50)]
\[ \dot{\chi}(t \to \infty) \to -t \lambda_1 \]  
 Instead of the wrong one:
\[ \dot{\chi}(t \to \infty) \to -2t \lambda_1. \]

The coefficients in the density equation read [Eqs. (52) and (53)]
\[ \alpha_1 = \sigma \dot{\chi}_0 + \frac{\sigma t}{\lambda_1} + 3 \dot{\phi}_0^2 \]  
\[ \alpha_2 = \frac{\sigma}{\sqrt{\lambda_1 (\lambda_1 \dot{\phi}_0^2 + \sigma t)}} (3H_0 (\lambda_1 \dot{\phi}_0^2 + \sigma t) + 2 \sigma) \]
Instead of the incorrect ones:
\[ \alpha_1 = 5 \dot{\phi}_0^2 \lambda_1 + \lambda_1 \sigma \chi_0 + 3 \sigma t \]  
\[ \alpha_2 = -\frac{2 \sigma}{3 \dot{\phi}_0 H_0 \lambda_1} (3 \dot{\phi}_0^2 H_0 \lambda_1 + 3 H_0 \sigma t + \sigma). \]

The sentence after Eq. (53) should be modified with the \( \lambda_2 \) coefficient:
“\( \dot{\chi} \) is asymptotically a constant, equal to \( 1/(3 \lambda_2 H_0) \).”

Therefore the asymptotic density term contains the coefficients [Eqs. (61) and (62)]
\[ C_1 = \frac{3 \dot{\phi}_0^2 + \sigma \dot{\chi}_0}{2}, \]  
\[ \gamma^2 = \frac{\sigma}{\lambda_1}, \quad C_2 = V \]
Instead of the incorrect ones:
\[ C_1 = \frac{5 \dot{\phi}_0^2 + \sigma \dot{\chi}_0}{2}, \]  
\[ \gamma^2 = \frac{3 \sigma}{2 \lambda_1}, \quad C_2 = V. \]

We thank Pedro Labraña for finding the basic typo.
12 Discussion

This thesis is based on the papers: [91–95]. We formulate the DST and solve it for cosmological solutions, yielding a unification of dark energy and dark matter. The DST introduces a Lagrange multiplier that forces the kinetic part to mimic a dark matter component.

The scalar field model of dark energy uses the slow roll assumption in order to get effective dark energy behavior. The potential is more dominant than the kinetic part $\phi^2 \ll V(\phi)$. For the case of a constant potential the kinetic term gives a stiff equation of state ($\rho = p$) to the matter fields, with a contribution of $\rho_\phi \sim 1/a^6$ to the Friedmann equation. From the measurements of the late time expansion, we don’t see that component. Therefore, [63] formulates a modified energy momentum tensor. The modified energy momentum tensor yields a dark matter contribution ($p = 0$) for a constant potential. This elegant model produces an interaction between dark energy and dark matter components with a corresponding potential. However, this model lacks an action principle.

In [91] we used the DST to formulate the unification of dark energy and the dark matter components. The energy momentum tensor that introduced in the DST is the same one that was introduced in [63]. From the variation with respect to the Dynamical Time Vector, the conservation of the energy momentum tensor $T^{\mu\nu}_\chi$ forces the kinetic term of the scalar field to behaves as a dark matter component $\rho_\phi \sim 1/a^3$. In many situations the solutions studied in [63] can be also obtained here. However, there are other solutions such as non-singular bounce solutions which are not present in [63].

For the constant potentials the Dynamical Time corresponds to the cosmic time. In particular, for exact $\Lambda$CDM solution the cosmic time and the Dynamical Time coincide. The solution yields the $\Lambda$CDM background with a bouncing solution that solves the singularity problem. For an exponential potential, the solution behaves differently for the early Universe, but approaches $\Lambda$CDM for the late Universe. Moreover, [91] proves the stability of the constant and the exponential potential.

Reference [92] investigates many potentials with constraints from the late Universe data. In this work we performed the autonomous system method and showed that under certain circumstances the DST includes stable late-time attractors. The asymptotic solution approaches $\Lambda$CDM model for those potentials. The observational constraints regarding the Hubble constant are in agreement (within $\sim 1 \sigma$) with those of Planck [96]. In addition, the results are compatible at $\sim 2 \sigma$ level with the $H_0$ measurement obtained from Cepheids [1,2,97–99]. On top of that [92] finds that one of the models with constant potentials have the smallest deviation from $\Lambda$CDM, where the confidence level is close to $\sim 1.5 \sigma$. In addition, [92] explicitly checked the compatibility of DST within the standard BBN using the average bound on the possible variation of the BBN speed-up factor. Reference [92] shows that the deviation from the Hubble rate of $\Lambda$CDM for the radiation dominant era is not larger then 10%. Therefore, the BBN production is still applicable with those potentials.

Reference [93] studied the solution of DST with higher dimensions. Reference [93] studied inflation solutions from the interplay of ordinary and higher dimensions. In the case of isotropic pressure, the solution is obtained from the total volume without dependence on an individual scale factor. The evolution for the total volume is calculated directly for the anisotropy constant $E$. The DST naturally prevents the collapse of the Universe from the additional bounce and obtains a lower bound for the volume of the Universe. Likewise the presence of
a negative cosmological constant prevents the volume from becoming very large in the early Universe.

There is an effective potential that governs the evolution of the total volume. In the case the effective potential includes a minimum, the total volume oscillates. Consequently, the ordinary dimensions increase and the extra dimensions decrease. The DST provides a natural way to exit from the inflation compactification epoch by one potential, which drives directly the evolution of the scalar field and a different potential which determines the value of the dark energy. As a consequence, the scalar field may drive the vacuum energy smoothly to rise into a small positive value, defined as the end of the inflation compactification.

In the second part of the thesis we extend the DST to the Diffusive Action, that breaks the conservation of the stress energy tensor $T^\mu_\nu$, and introduces a current source $j^\mu \ [89, 90]$. However, the Einstein tensor (and the corresponding matter fields) are covariantly conserved.

Reference [94] extends the DST theory into a Diffusive Action which allows the dark energy and dark matter to exchange energies. This formulates an action principle to the diffusion interaction that was introduced in [89] by hand. From the asymptotic solution [94] one finds that for some values of the diffusion constant, unlike the standard $\Lambda$CDM model (which has a constant dark energy density and the dark matter decreases like $a^{-3}$) dark energy can slowly decrease, instead of being constant, and dark matter also decreases, but not as fast as $a^{-3}$. This behavior explains the coincidence problem.

While [94] considers one type of the stress energy momentum tensor $T^\mu_\nu$, the last paper in this thesis, [95], analyzes the complete combination of the stress energy momentum tensor with regards to diffusive interactions. Reference [95] shows analytically and numerically that all of the solutions yield the stable $\Lambda$CDM model with new families of solutions. The general consideration of $T^\mu_\nu$ includes the constants $\lambda_1$ and $\lambda_2$. The choice $\lambda_2 = 0$ corresponds to the original papers in the introduction of this thesis [91–93]. Asymptotically the solutions approach $\Lambda$CDM model and the Dynamical Time approaches the finite value $1/3H_0$. In addition, the case $\lambda_2 = 0$ corresponds to the behavior of non-Lagrangian formulations of dark energy and dark matter that suggested in [89].

In summary, the scalar field model of dark energy has a successful description of our Universe without a dark matter component. However under the DST an elegant unification between dark energy and dark matter components emerges naturally. The model is proven to be stable and fits with the data of the late time expansion. Compactification with higher dimensions yields inflationary scenarios. By breaking the conservation of the energy momentum tensor $T^\mu_\nu$ diffusion effects are obtained. For these reasons, it seems that DST reveals the capabilities of the scenario and makes it a good candidate for the description of Nature.
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תקציר
מודל שלחומראפל מאוחד עם אנרגיה אפלה המבוססת על תורת זמנים דינמיitm. הממדים שונים על ידי חזרה של אינפלייה המתקיימת במגנטודינמי. הפולה והעובר של פוטנציאל可视化 מתחרים עם מסלול הקוסמולוגים והפרטים המ芰ים. מייצג את האפסון האדום. בנוסף, הפרטים הקוסמולוגים של הפרטים המtiğiים משתנים. את המגזר של אלментים של קוסמולוגיה יסודית מתאימה בארוחת המגזרים של הקוסمواית והחזרה. התיאוריה מתאימה לאינפלייה המתקיימת במגזר הקוסמולוגי. התיאוריה מתאימה לאינפלייה המתקיימת במגזר הקוסמולוגי. התיאוריה מתאימה לאינפלייה המתקיימת במגזר הקוסמולוגי. התיאוריה מתאימה לאינפלייה המתקיпит.
הצהרת תلمידה המחקר עם הגשה עבודה

הדוקטורט לשבט

אני ההוגה מתכונת בקשת להיבחר בתוכנית לתואר דוקטורט ולחזור את התוכן פרסומתי של דוקטורט בפקודת מנהלת הקולנוע.

ההכרה במפורטenegro יובל-120022132 הוגה מאית מחקר ואית תמיכה מחוקקת מנהלת תחום מחקר.

תאריך: 20.01.2021
שם התلمידה: דוד ברנוביץ
חתימה:

חתימה:
העבידה שעשתה בהדרכת פרופ' אדריאד גנולמן
במרחקלあたり יישוע
בפקולטה לפיסיקה

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uni05F3
העבודהנעשתהבהדרכתפרופ
במחלקהלמדעיהטבע
בפקולטהלפיסיקה
איחוד חומרא פלואנרגיה אפלה במישרין
מישרין
מניח

דוד חיבניסטי
דוקטור לפילוסופיה
מגיש מחקר
לשם מילוי חלק שני
ה디ריצי
תמונה
’דוקטור לפילוסופיה’

נ marché בשמה של כמה מתוך שלakit
וхаיב

אישור המנהלה:
אישור דיק ביט הספר הלימודים
מחבר מחקר המקדימים
לע"ש קריטריון

נואר 2021

בר שבע
איחוד חומרא פלואו אינרגיה אפופל מרחוק זמן דינמי

כמית

דוד חיבניסטי

מגיש

איחוד חומרא פלואו אינרגיה אפופל מרחוק זמן דינמי

כמית

דוד חיבניסטי

מגיש

דוד חיבניסטי

שופט

חרט לקפל חלקי של הדרישות לקבצל תואר "דוקטור פילוסופיה"

והם לסינרט אוניברסיטי בנג'י בנג'ב

ינואר 2021

באר שבע