Symmetry group classification and optimal reduction of a class of damped Timoshenko beam system with non-linear rotational moment

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Abstract

We consider a non-linear Timoshenko system of partial differential equations (PDEs) with frictional damping term in rotation angle. The nonlinearity is due to the arbitrary dependence on the rotation moment. A Lie symmetry group classification of the arbitrary function of rotation moment is presented. Optimal system of one-dimensional subalgebras of the non-linear damped Timoshenko system are derived for all the non-linear cases. All possible invariant variables of the optimal systems for the three non-linear cases are presented. The corresponding reduced systems of ordinary differential equations (ODEs) are also provided.

Key words: Timoshenko beam system, Lie symmetry group classification, optimal system, invariant solution.

1 Introduction

The classification of group invariant solutions of differential equations by means of the optimal systems is one of the main applications of Lie group analysis to differential equations. The method was first introduced by Ovsiannikov [1]. The main idea behind the method is discussed in his papers [2] [3] and
also by Chupakin \cite{4} and Ibragimov et al \cite{5} and Olver \cite{6}. We can always construct a family of group invariant solutions corresponding to a subgroup of a symmetry group admitted by a given differential equation. Since there are an infinite number of such subgroups, it is not possible to list all the group invariant solutions. An effective and systematic way of classifying these group invariant solutions is to obtain optimal systems of subalgebras of the symmetry Lie algebra. This leads to non-similar invariant solutions under symmetry transformations.

Timoshenko \cite{7} proposed a beam theory which adds the effect of shear as well as the effect of rotation to the Euler-Bernoulli beam. The Timoshenko model is a major improvement for non-slender beams and for high-frequency response where shear or rotary effects are not negligible \cite{8}. Rivera et al. \cite{9} studied the global stability for the following damped Timoshenko beam system with non-linear rotation moment

\[
\begin{align*}
\rho_1 \phi_{tt} - k(\phi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - (\chi(\psi_x))_x + k(\phi_x + \psi) + d\psi_t &= 0.
\end{align*}
\]

(1.1)

where the functions \(\phi, \psi\) depending on \((t, x) \in (0, \infty) \times (0, L)\) model the transverse displacement of a beam with \((0, L) \in \mathbb{R}\), and rotation angle of a filament, respectively. The constants \(\rho_1, \rho_2, d\) and \(k\) are positive and \(\chi\) is a function of \(\psi_x\) assumed to satisfy

\[
\chi_{\psi_x}(0) = b,
\]

(1.2)

with positive constant \(b\). However, the algebraic properties of the Lie algebra admitted by the (1.1) have not been studied so far. In this paper we perform a Lie symmetry analysis of non-linear damped Timoshenko beam system (1.1). In sections two, the complete Lie group classification is presented using Janet basis. In section three, optimal system of one-dimensional subalgebras of the non-linear damped Timoshenko system are derived for all the non-linear cases. In section four, all possible invariant variables of the optimal systems for the three non-linear cases of \(\chi(\psi_x)\) are presented. Moreover, the corresponding reduced systems of ODEs are also provided. As an illustration, some invariant solutions are given explicitly in three examples by solving the reduced systems of ODEs.

2 Complete Lie group classification

Consider the system of two PDEs with two independent variables \((t, x)\) and two dependent variables \((\phi, \psi)\) and the function \(\chi = \chi(\psi_x)\) given by the system (1.1).

Following is a brief summary of Lie symmetry \cite{10} \cite{11}.
Consider the following symmetry transformation group acting on system of PDEs (1.1).

\[
\begin{align*}
\dot{t} &= t + \epsilon \xi^1(t, x, \varphi, \psi) + O(\epsilon^2), \\
\dot{x} &= x + \epsilon \xi^2(t, x, \varphi, \psi) + O(\epsilon^2), \\
\dot{\varphi} &= \varphi + \epsilon \eta^1(t, x, \varphi, \psi) + O(\epsilon^2), \\
\dot{\psi} &= \psi + \epsilon \eta^2(t, x, \varphi, \psi) + O(\epsilon^2),
\end{align*}
\]

(2.3)

where \( \epsilon \) is the group parameter and \( \xi^1, \xi^2 \) and \( \eta^1, \eta^2 \) are the infinitesimals of transformations for the independent and dependent variables, respectively. The associated Lie point symmetry generator (vector field) of the system (1.1) is of the form

\[
X = \xi^1(t, x, \varphi, \psi) \frac{\partial}{\partial t} + \xi^2(t, x, \varphi, \psi) \frac{\partial}{\partial x} + \eta^1(t, x, \varphi, \psi) \frac{\partial}{\partial \varphi} + \eta^2(t, x, \varphi, \psi) \frac{\partial}{\partial \psi}.
\]

(2.4)

The second prolongation of the generator is given by

\[
X^{[2]} = X + \eta^\mu_i (x^j, u^l, \partial u^l) \frac{\partial}{\partial u^l_{i_1}} + \eta^\mu_{i_1 i_2} (x^j, u^l, \partial u^l_{i_1}, \partial^2 u^l_{i_1}) \frac{\partial}{\partial u^l_{i_1 i_2}},
\]

(2.5)

such that \( j = 1, 2 \) and \((x^1, x^2) = (t, x), (u^1, u^2) = (\varphi, \psi)\), and \((\partial u^1, \partial u^2) = (\partial \varphi, \partial \psi)\), and so on ..., where

\[
\begin{align*}
\eta^\mu_i &= D_i \eta^\mu - \sum_{j=1}^{2} (D_i \xi^j) u^\mu_j, \quad \mu = 1, 2, \\
\eta^\mu_{i_1 i_2} &= D_{i_2} \eta^\mu_{i_1} - \sum_{j=1}^{2} (D_{i_2} \xi^j) u^\mu_{i_1 j}, \\
D_i &= \frac{\partial}{\partial x_i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ij} \frac{\partial}{\partial u^\mu_j} + u^\mu_{i_1 i_2} \frac{\partial}{\partial u^\mu_{i_1 i_2}} + \ldots,
\end{align*}
\]

(2.6)

where \( D_i \) is the total derivative operator.

Using the invariance condition of the system of PDEs (1.1)

\[
\begin{align*}
X^{[2]}(\rho_1 \varphi_{tt} - k(\varphi_x + \psi_x)x)|_{1.1} &= 0, \\
X^{[2]}(\rho_2 \psi_{tt} - \chi_x(\psi_x) + k(\varphi_x + \psi) + d \psi_t)|_{1.1} &= 0,
\end{align*}
\]

(2.7)

and comparing coefficients of the various derivatives of the dependent variables \( \varphi \) and \( \psi \) yields an overdetermined linear PDE system. Carrying out the Janet basis of this over-determined system in the degree reverse lexicographical ordering as \( \psi > \varphi > x > t \) and \( \eta_2 > \eta_1 > \xi_2 > \xi_1 \) by using the command "JanetBasis" involved in the Maple package "Janet" [12], leads to two cases. These two cases arise from the command "Denominators" which returns the functions by which the Janet basis algorithm had to divide. These two cases are given as follows:
2.1 \( \chi(\psi_x) = b\psi_x + \gamma \)

In this case \( \chi(\psi_x) \) is a linear function satisfying the condition (1.2). The Janet basis of the over-determined system is

\[
\begin{bmatrix}
\eta_1^1, \xi_1^1, \eta_2^1, \xi_2^1, \eta_3^1, \xi_3^1, \eta_4^1, \xi_4^1, \eta_5^1, \xi_5^1, \\
\eta_1^{1,\psi}, \xi_1^{1,\psi}, \eta_2^{1,\psi}, \xi_2^{1,\psi}, \eta_3^{1,\psi}, \xi_3^{1,\psi}, \eta_4^{1,\psi}, \xi_4^{1,\psi}, \\
\eta_1^{2,\psi}, \xi_1^{2,\psi}, \eta_2^{2,\psi}, \xi_2^{2,\psi}, \eta_3^{2,\psi}, \xi_3^{2,\psi}, \eta_4^{2,\psi}, \xi_4^{2,\psi},
\end{bmatrix}
\] (2.8)

The solution of this system of determining equations is

\[
\xi^1 = c_1, \quad \xi^2 = c_2, \quad \eta^1 = c_3\varphi + f(t, x), \quad \eta^2 = c_3\psi + g(t, x),
\] (2.9)

where \( f(t, x) \) and \( g(t, x) \) satisfy the following system of PDEs

\[
k f_x + kg + dg_t - bg_{xx} + \rho_2 g_t = 0, \\
\rho_1 f_{tt} - kf_{xx} - kg_x = 0.
\] (2.10)

The corresponding Lie point symmetry generators admitted by the system (1.1) are given as

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \varphi \frac{\partial}{\partial \varphi} + \psi \frac{\partial}{\partial \psi}, \quad X_\infty = f(t, x) \frac{\partial}{\partial \varphi} + g(t, x) \frac{\partial}{\partial \psi},
\] (2.11)

2.2 \( \chi_{\psi_x} \neq 0 \)

The Janet basis of the over-determined system is

\[
\begin{bmatrix}
\eta_1^1, \xi_1^1, \eta_2^1, \xi_2^1, \eta_3^1, \xi_3^1, \eta_4^1, \xi_4^1, \\
\eta_1^{1,\psi}, \xi_1^{1,\psi}, \eta_2^{1,\psi}, \xi_2^{1,\psi}, \eta_3^{1,\psi}, \xi_3^{1,\psi}, \eta_4^{1,\psi}, \xi_4^{1,\psi}, \\
\eta_1^{2,\psi}, \xi_1^{2,\psi}, \eta_2^{2,\psi}, \xi_2^{2,\psi}, \eta_3^{2,\psi}, \xi_3^{2,\psi}, \eta_4^{2,\psi}, \xi_4^{2,\psi},
\end{bmatrix}
\] (2.12)

The solution of the system (2.12) is

\[
\xi^1 = c_1, \quad \xi^2 = c_2, \quad \eta^1 = c_3 + c_4 t + c_5 x + c_6 t x, \quad \eta^2 = F(t)
\] (2.13)

where \( F(t) \) satisfies the following ODE

\[
\rho_2 F''(t) + dF'(t) + kF(t) = -c_6 kt - c_5 k.
\] (2.14)

The characteristic equation of left hand side of the equation (2.14) gives rise to the following three sub-cases
2.2.1 \( d^2 - 4k\rho_2 = 0 \)

The Lie point symmetry generators admitted by the system (1.1) are given by

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial \varphi}, \\
X_4 &= t\frac{\partial}{\partial \varphi}, \quad X_5 = x\frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \psi}, \quad X_6 = tx\frac{\partial}{\partial \varphi} + (2\sqrt{k \kappa} - t)\frac{\partial}{\partial \psi}, \\
X_7 &= e^{-\sqrt{k \kappa} t} \frac{\partial}{\partial \varphi}, \quad X_8 = te^{-\sqrt{k \kappa} t} \frac{\partial}{\partial \psi}.
\end{align*}
\]

In order to obtain the group transformations which are generated by the resulting infinitesimal symmetry generators (2.15), we need to solve the following system of first order ODEs

\[
\frac{d\tilde{x}^j(\epsilon)}{d\epsilon} = \xi^j(\tilde{f}(\epsilon), \tilde{x}(\epsilon), \tilde{\varphi}(\epsilon), \tilde{\psi}(\epsilon)), \quad \tilde{x}^j(0) = x^j, \\
\frac{d\tilde{u}^j(\epsilon)}{d\epsilon} = \eta^j(\tilde{f}(\epsilon), \tilde{x}(\epsilon), \tilde{\varphi}(\epsilon), \tilde{\psi}(\epsilon)), \quad \tilde{u}^j(0) = u^j, \quad j = 1, 2.
\]

The one parameter group \( G_i(\epsilon) = e^{\epsilon X_i} \) generated by \( X_i \) for \( i = 1, ..., 8 \), are as follows:

\[
\begin{align*}
G_1(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t + \epsilon, x, \varphi, \psi), \quad G_5(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon x, \psi - \epsilon), \\
G_2(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi, t \varphi - \epsilon), \quad G_6(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon t x, \psi + \epsilon(2\sqrt{k \kappa} - t)), \\
G_3(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi + \epsilon, \psi), \quad G_7(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon e^{-\sqrt{k \kappa} t}), \\
G_4(\epsilon) : (t, x, \varphi, \psi) &\mapsto (t, x, \varphi + \epsilon t, \psi), \quad G_8(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon e^{-\sqrt{k \kappa} t}).
\end{align*}
\]

**Theorem 2.1.** If \( \varphi = f(t, x) \) and \( \psi = g(t, x) \) is a solution of the Timoshenko system (1.1) with \( d^2 - 4k\rho_2 = 0 \), then so is

\[
\begin{align*}
\varphi &= f(t + \epsilon_1, x + \epsilon_2) + \epsilon_3 + \epsilon_4(t + \epsilon_1) + \epsilon_5(x + \epsilon_2) + \epsilon_6(t + \epsilon_1)(x + \epsilon_2), \\
\psi &= g(t + \epsilon_1, x + \epsilon_2) + 2\sqrt{k \kappa} \epsilon_6 - \epsilon_6(t + \epsilon_1) - \epsilon_5 + \epsilon_7 e^{-\sqrt{k \kappa} (t + \epsilon_1)} + \epsilon_8(t + \epsilon_1) e^{-\sqrt{k \kappa} (t + \epsilon_1)},
\end{align*}
\]

where \( \{\epsilon_i\}_{i=1}^8 \) are arbitrary real numbers.

**Proof.** The eight parameters group

\[
G(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8) = G_8(\epsilon_8) \circ G_7(\epsilon_7) \circ G_6(\epsilon_6) \circ G_5(\epsilon_5) \circ G_4(\epsilon_4) \circ G_3(\epsilon_3) \circ G_2(\epsilon_2) \circ G_1(\epsilon_1)
\]

generated by \( X_i \) for \( i = 1, ..., 8 \), can be given by the composition of the transformations (2.17) as follows:

\[
G : (t, x, \varphi, \psi) \mapsto (t + \epsilon_1, x + \epsilon_2, \varphi + \epsilon_3 + \epsilon_4(t + \epsilon_1) + \epsilon_5(x + \epsilon_2) + \epsilon_6(t + \epsilon_1)(x + \epsilon_2), \\
\psi + 2\sqrt{k \kappa} \epsilon_6 - \epsilon_6(t + \epsilon_1) - \epsilon_5 + \epsilon_7 e^{-\sqrt{k \kappa} (t + \epsilon_1)}(\epsilon_7 + \epsilon_8(t + \epsilon_1)),
\]

and this completes the proof. \( \square \)
The one parameter group $G_i(\epsilon) = e^{\epsilon X_i}$ generated by $X_i$ for $i = 1, ..., 8$ are as follows:

$G_6(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon t x, \psi + \epsilon(t + \epsilon_1)(x + \epsilon_2)),$

$G_7(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon\left(\sqrt{\lambda^2 + 4k\rho} - t\right) \cosh\left(\frac{\lambda t}{2\rho}\right)),$

$G_8(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon\left(\sqrt{\lambda^2 + 4k\rho} - t\right) \sinh\left(\frac{\lambda t}{2\rho}\right)).$ (2.21)

**Theorem 2.2.** If $\varphi = f(t, x)$ and $\psi = g(t, x)$ is a solution of the Timoshenko system \((1.1)\) with $d^2 - 4k\rho_2 = \lambda^2$, then so is

$$\varphi = F(t + \epsilon_1, x + \epsilon_2) + \epsilon_3 + \epsilon_4(t + \epsilon_1) + \epsilon_5(x + \epsilon_2) + \epsilon_6(t + \epsilon_1)(x + \epsilon_2),$$

$$\psi = G(t + \epsilon_1, x + \epsilon_2) - \epsilon_6(t + \epsilon_1) - \epsilon_5 + \frac{d}{\lambda} \epsilon_6 + e^{-\frac{(t+\epsilon_1)}{2\rho_2}} \left(\epsilon_7 \cosh\left(\frac{\lambda(t+\epsilon_1)}{2\rho_2}\right) + \epsilon_8 \sinh\left(\frac{\lambda(t+\epsilon_1)}{2\rho_2}\right)\right),$$

(2.22)

where $\{\epsilon_i\}_{i=1}^8$ are arbitrary real numbers.

**Proof.** The eight parameters group

$$G(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8) = G_8(\epsilon_8) \circ G_7(\epsilon_7) \circ G_6(\epsilon_6) \circ G_5(\epsilon_5) \circ G_4(\epsilon_4) \circ G_3(\epsilon_3) \circ G_2(\epsilon_2) \circ G_1(\epsilon_1)$$

generated by $X_i$ for $i = 1, ..., 8$, can be given by the composition of the transformations (2.21) as follows:

$$G : (t, x, \varphi, \psi) \mapsto (t + \epsilon_1, x + \epsilon_2, \varphi + \epsilon_3 + \epsilon_4(t + \epsilon_1) + \epsilon_5(x + \epsilon_2) + \epsilon_6(t + \epsilon_1)(x + \epsilon_2),$$

$$\psi - \epsilon_6(t + \epsilon_1) - \epsilon_5 + \frac{d}{\lambda} \epsilon_6 + e^{-\frac{(t+\epsilon_1)}{2\rho_2}} \left(\epsilon_7 \cosh\left(\frac{\lambda(t+\epsilon_1)}{2\rho_2}\right) + \epsilon_8 \sinh\left(\frac{\lambda(t+\epsilon_1)}{2\rho_2}\right)\right)).$$

(2.23)

and this completes the proof.  

\[ \square \]
\[ d^2 - 4k \rho_2 = -\mu^2, \text{ such that } \mu > 0 \]

The Lie point symmetry generators admitted by the system (1.1) are given by
\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= \frac{\partial}{\partial \varphi}, \\
X_4 &= t \frac{\partial}{\partial \varphi}, & X_5 &= x \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial t}, & X_6 &= tx \frac{\partial}{\partial \varphi} + \left( \frac{4k \rho_2 - \mu^2}{2} - t \right) \frac{\partial}{\partial \varphi}, \\
X_7 &= e^{-\frac{4k \rho_2 - \mu^2}{2} t} \cos\left( \frac{\mu t}{2 \rho_2} \right) \frac{\partial}{\partial \varphi}, & X_8 &= e^{-\frac{4k \rho_2 - \mu^2}{2} t} \sin\left( \frac{\mu t}{2 \rho_2} \right) \frac{\partial}{\partial \varphi}.
\end{align*}
\]

The one parameter group \( G_i(\epsilon) = e^{\epsilon X_i} \) generated by \( X_i \) for \( i = 1, ..., 8 \) are as follows:
\[
\begin{align*}
G_6(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi + \epsilon t, \psi + \epsilon \left( \frac{\mu t}{2} - t \right)), \\
G_7(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon e^{-\frac{\mu t}{2 \rho_2}} \cos\left( \frac{\mu t}{2 \rho_2} \right)), \\
G_8(\epsilon) : (t, x, \varphi, \psi) \mapsto (t, x, \varphi, \psi + \epsilon e^{-\frac{\mu t}{2 \rho_2}} \sin\left( \frac{\mu t}{2 \rho_2} \right)).
\end{align*}
\]

**Theorem 2.3.** If \( \varphi = f(t, x) \) and \( \psi = g(t, x) \) is a solution of the Timoshenko system (1.1) with \( d^2 - 4k \rho_2 = -\mu^2 \), then so is
\[
\begin{align*}
\varphi &= F(t + \epsilon_1, x + \epsilon_2) + \epsilon_3 + \epsilon_4 (t + \epsilon_1) + \epsilon_5 (x + \epsilon_2) + \epsilon_6 (t + \epsilon_1)(x + \epsilon_2), \\
\psi &= G(t + \epsilon_1, x + \epsilon_2) - \epsilon_6 (t + \epsilon_1) - \epsilon_5 + \frac{\sqrt{3}}{\rho_2} \epsilon_6 + e^{-\frac{\mu (t + \epsilon_1)}{2 \rho_2}} \left( \epsilon_7 \cos\left( \frac{\mu (t + \epsilon_1)}{2 \rho_2} \right) + \epsilon_8 \sin\left( \frac{\mu (t + \epsilon_1)}{2 \rho_2} \right) \right).
\end{align*}
\]
where \( \{\epsilon_i\}_{i=1}^8 \) are arbitrary real numbers.

**Proof.** The eight parameter group
\[
G(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8) = G_8(\epsilon_8) \circ G_7(\epsilon_7) \circ G_6(\epsilon_6) \circ G_5(\epsilon_5) \circ G_4(\epsilon_4) \circ G_3(\epsilon_3) \circ G_2(\epsilon_2) \circ G_1(\epsilon_1)
\]
generated by \( X_i \) for \( i = 1, ..., 8 \), can be given by the composition of the transformations (2.25) as follows:
\[
\begin{align*}
G : (t, x, \varphi, \psi) \mapsto (t + \epsilon_1, x + \epsilon_2, \phi + \epsilon_3 + \epsilon_4 (t + \epsilon_1) + \epsilon_5 (x + \epsilon_2) + \epsilon_6 (t + \epsilon_1)(x + \epsilon_2), \\
&\quad \psi - \epsilon_6 (t + \epsilon_1) - \epsilon_5 + \frac{\sqrt{3}}{\rho_2} \epsilon_6 + e^{-\frac{\mu (t + \epsilon_1)}{2 \rho_2}} \left( \epsilon_7 \cos\left( \frac{\mu (t + \epsilon_1)}{2 \rho_2} \right) + \epsilon_8 \sin\left( \frac{\mu (t + \epsilon_1)}{2 \rho_2} \right) \right).
\end{align*}
\]
and this completes the proof.

### 3 Optimal system of one-dimensional subalgebras of the nonlinear damped Timoshenko system

In this section, we give the complete classification of the one-dimensional optimal system for each of the algebras with basis (2.21), (2.22) and (2.23). In order to find the optimal system, one needs to classify
the one-dimensional subalgebras under the action of the adjoint representation. We follow the algorithm explained by Olver [6].

First, we calculate the adjoint representation given by

$$\text{Ad}(\exp(\epsilon X_i)X_j) = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \frac{\epsilon^3}{3!}[X_i, [X_i, [X_i, X_j]]] + ..., $$

and then establish the adjoint table. For each case, we classify the conjugacy classes under the adjoint representation according to the sign of the Killing form.

### 3.1 Optimal system for the case $d^2 - 4k\rho_2 = 0$

The non-zero commutators of the Lie algebra $\mathcal{L}^8$ with basis (2.15) are given by

$$[X_1, X_4] = X_3, \quad [X_1, X_6] = X_5, \quad [X_1, X_7] = -\sqrt{\frac{2}{\rho_2}}X_7, \quad [X_1, X_8] = X_7 - \sqrt{\frac{2}{\rho_2}}X_8,$$

$$[X_2, X_3] = X_3, \quad [X_2, X_6] = X_4.$$

The Lie algebra $\mathcal{L}^8$ is solvable and the Killing form is given by $K = 2 \hat{a}^2 a_7^2$ where $\hat{a} = \sqrt{\frac{2}{\rho_2}}$. The adjoint table is given by

| $\text{Ad}(e^{\epsilon X_i})$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
|-------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $X_1$                         | $X_1$ | $X_2$ | $X_3$ | $X_4 - \epsilon X_3$ | $X_5$ | $X_6 - \epsilon X_5$ | $\epsilon^2 X_7$ | $\epsilon^3 X_8 - \epsilon\epsilon^2 X_7$ |
| $X_2$                         | $X_1$ | $X_2$ | $X_3$ | $X_4 - \epsilon X_3$ | $X_5$ | $X_6 - \epsilon X_5$ | $X_7$ | $X_8$ |
| $X_3$                         | $X_1$ | $X_2$ | $X_3$ | $X_4 - \epsilon X_3$ | $X_5$ | $X_6 - \epsilon X_5$ | $X_7$ | $X_8$ |
| $X_4$                         | $X_1$ | $X_2$ | $X_4 + \epsilon X_3$ | $X_3$ | $X_4$ | $X_5$ | $X_6 - \epsilon X_5$ | $X_7$ | $X_8$ |
| $X_5$                         | $X_1$ | $X_2$ | $X_3$ | $X_4 + \epsilon X_3$ | $X_4$ | $X_5$ | $X_6 - \epsilon X_5$ | $X_7$ | $X_8$ |
| $X_6$                         | $X_1$ | $X_2$ | $X_3$ | $X_4 + \epsilon X_3$ | $X_4$ | $X_5$ | $X_6 - \epsilon X_5$ | $X_7$ | $X_8$ |
| $X_7$                         | $X_1$ | $X_2$ | $X_3$ | $X_4 + \epsilon X_3$ | $X_4$ | $X_5$ | $X_6 - \epsilon X_5$ | $X_7$ | $X_8$ |
| $X_8$                         | $X_1$ | $X_2$ | $X_3$ | $X_4 + \epsilon X_3$ | $X_4$ | $X_5$ | $X_6 - \epsilon X_5$ | $X_7$ | $X_8$ |

The adjoint group is defined by the matrix

$$A = \text{Ad}(e^{-\epsilon_4 X_8}).\text{Ad}(e^{-\epsilon_7 X_7}).\text{Ad}(e^{-\epsilon_6 X_6}).\text{Ad}(e^{-\epsilon_5 X_5}).\text{Ad}(e^{-\epsilon_4 X_4}).\text{Ad}(e^{-\epsilon_3 X_3}).\text{Ad}(e^{-\epsilon_2 X_2}).\text{Ad}(e^{-\epsilon_1 X_1}),$$

which is given by

$$ A = \begin{pmatrix}
1 & 0 & -\epsilon_4 & 0 & -\epsilon_6 & 0 & \hat{a}\epsilon_7 - \epsilon_8 & \hat{a}\epsilon_8 \\
0 & 1 & -\epsilon_5 & -\epsilon_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon_2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \epsilon_1\epsilon_2 & \epsilon_2 & \epsilon_1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \epsilon^{-\hat{a}\epsilon_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \epsilon_1\epsilon^{-\hat{a}\epsilon_1} & \epsilon^{-\hat{a}\epsilon_1}
\end{pmatrix}.$$ 

(3.29)
Theorem 3.1. An optimal system of one-dimensional Lie algebra \( L^8 \) with basis (2.15) is provided by the following generators

\[
\begin{align*}
X^1 &= X_1 + \alpha X_2 + \beta X_6, & \alpha &\in \mathbb{R}, \beta \neq 0, \\
X^2 &= X_1 + \alpha X_2 + \beta X_4, & \alpha, \beta &\in \mathbb{R}, \\
X^3 &= X_2 + \alpha X_6 + \beta X_7 + \gamma X_8, & \alpha &\neq 0, \beta, \gamma &\in \mathbb{R}, \\
X^4 &= X_2 + \alpha X_5 + \beta X_8, & \alpha &\in \mathbb{R}, \beta &\neq 0, \\
X^5 &= \alpha X_2 + \beta X_3 + X_7, & \alpha &\neq 0, \beta &\in \mathbb{R}, \\
X^6 &= X_2 + \alpha X_5, & \alpha &\in \mathbb{R}, \\
X^7 &= \alpha X_5 + X_6 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma &\in \mathbb{R}, \\
X^8 &= \alpha X_4 + X_5 + \beta X_8, & \alpha &\in \mathbb{R}, \beta &\neq 0, \\
X^9 &= \alpha X_4 + \beta X_5 + X_7, & \alpha &\in \mathbb{R}, \beta &\neq 0, \\
X^{10} &= \alpha X_4 + X_5, & \alpha &\in \mathbb{R}, \\
X^{11} &= X_4 + \alpha X_7 + \beta X_8, & \alpha, \beta &\in \mathbb{R}, \\
X^{12} &= \alpha X_3 + X_8, & \alpha &\in \mathbb{R}, \\
X^{13} &= \alpha X_3 + X_7, & \alpha &\in \mathbb{R}, \\
X^{14} &= X_3.
\end{align*}
\]

Proof. Let \( X \) and \( \tilde{X} \) be two elements in the Lie algebra \( L^8 \) with basis (2.15) given by \( X = \sum_{i=1}^8 \tilde{a}_i X_i \) and \( \tilde{X} = \sum_{i=1}^8 \tilde{a}_i X_i \). For simplicity, we will write \( X \) and \( \tilde{X} \) as row vectors of the coefficients on the form \( X = (a_1 \ a_2 \ldots \ a_8) \) and \( \tilde{X} = (\tilde{a}_1 \ \tilde{a}_2 \ldots \ \tilde{a}_8) \). Then in order for \( X \) and \( \tilde{X} \) to be in the same conjugacy class, we must have \( \tilde{X} = X A \), where \( A \) is given by (3.29). So, the theorem is proved by solving the system

\[
\begin{align*}
\tilde{a}_1 &= a_1, \\
\tilde{a}_2 &= a_2, \\
\tilde{a}_3 &= a_3 + \epsilon_1 a_4 + \epsilon_2 a_5 - \epsilon_5 a_2 - \epsilon_4 a_1 + \epsilon_1 \epsilon_2 a_6, \\
\tilde{a}_4 &= a_4 - \epsilon_6 a_2 + \epsilon_2 a_6, \\
\tilde{a}_5 &= a_5 - \epsilon_6 a_1 + \epsilon_1 a_6, \\
\tilde{a}_6 &= a_6, \\
\tilde{a}_7 &= a_1 (\tilde{a}_7 - \epsilon_8) + e^{-\epsilon_1 \tilde{a}} (a_7 + a_8 \epsilon_1), \\
\tilde{a}_8 &= a_1 \epsilon_8 + a_8 e^{-\epsilon_1 \tilde{a}},
\end{align*}
\]

for \( \{\epsilon_i\}_{i=1}^8 \) in term of \( \{a_i\}_{i=1}^8 \) in order to get the simplest values of \( \{\tilde{a}_i\}_{i=1}^8 \). The results are presented for different cases in the tree diagram (1) given in the appendix where it is initiated by the sign of the Killing form and its leafs are given completely. The full details for each leaf are given as follows:

**Case 1** \( a_1 \neq 0, \ a_6 \neq 0 \) : Let \( \epsilon_1 = \epsilon_5 = 0, \epsilon_2 = \frac{-a_5 a_6 - a_3 a_4}{a_1 a_6}, \epsilon_4 = \frac{a_1 a_5 + a_4 a_6 - a_3 a_4}{a_1 a_6}, \epsilon_6 = \frac{a_4}{a_1}, \epsilon_7 = -\frac{a_5 a_6}{\gamma^2 a_1^3} \) and \( \epsilon_8 = -\frac{a_8}{\gamma^4} \) to have \( \tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_7 = \tilde{a}_8 = 0 \). Then the conjugacy class is \( \langle X_1 + \alpha X_2 + \beta X_6 \rangle \) with \( \alpha \in \mathbb{R}, \beta \neq 0 \).

**Case 2** \( a_1 \neq 0, \ a_6 = 0 \) : Let \( \epsilon_1 = \epsilon_2 = \epsilon_5 = 0, \epsilon_4 = \frac{a_4}{a_1}, \epsilon_6 = \frac{a_4}{a_1}, \epsilon_7 = -\frac{a_5 a_6}{\gamma^2 a_1^3}, \epsilon_8 = -\frac{a_8}{\gamma^4} \) to have \( \tilde{a}_3 = \tilde{a}_4 = \tilde{a}_7 = \tilde{a}_8 = 0 \). So the conjugacy class is \( \langle X_1 + \alpha X_2 + \beta X_4 \rangle \) with \( \alpha, \beta \in \mathbb{R} \).

**Case 3** \( a_1 = 0, \ a_2 \neq 0, \ a_6 \neq 0 \) : Let \( \epsilon_1 = \frac{-a_6}{a_2}, \epsilon_2 = 0, \epsilon_5 = \frac{a_6 a_3 - a_4 a_5}{a_2 a_6}, \) and \( \epsilon_6 = \frac{a_4}{a_2} \) to make \( \tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0 \). Then the
conjugacy class is \( \langle X_2 + \alpha X_5 + \beta X_7 + \gamma X_8 \rangle \), \( \alpha \neq 0, \beta, \gamma \in \mathbb{R} \).

**Case 4** \( a_1 = 0, a_2 \neq 0, a_6 = 0, a_8 \neq 0 \): Let \( \epsilon_1 = -\frac{a_2}{a_6}, \epsilon_2 = 0, \epsilon_5 = \frac{a_2 a_8 - a_6 a_7}{a_2^2 a_6}, \) and \( \epsilon_6 = \frac{a_5}{a_2} \) to make \( \tilde{a}_3 = \tilde{a}_4 = \tilde{a}_7 = 0 \). Then the conjugacy class is of the form \( \langle X_2 + \alpha X_5 + \beta X_8 \rangle \), \( \alpha \in \mathbb{R}, \beta \neq 0 \).

**Case 5** \( a_1 = 0, a_2 \neq 0, a_6 = 0, a_8 = 0, a_7 \neq 0 \): Let \( \epsilon_1 = \frac{\ln \epsilon_2}{\gamma}, \epsilon_2 = 0, \epsilon_5 = \frac{a_4}{a_2} \ln \epsilon_2 + \gamma \), and \( \epsilon_6 = \frac{a_4}{a_2} \) to make \( \tilde{a}_3 = \tilde{a}_4 = 0 \) and \( \tilde{a}_7 = \pm 1 \). Then the conjugacy class after considering appropriate scaling is \( \langle \alpha X_2 + \beta X_5 + X_7 \rangle \), where \( \alpha \neq 0, \beta \in \mathbb{R} \).

**Case 6** \( a_1 = 0, a_2 \neq 0, a_6 = 0, a_8 = 0, a_7 = 0 \): Let \( \epsilon_1 = \epsilon_2 = 0 \) \( \epsilon_5 = \frac{a_2}{a_2} \) and \( \epsilon_6 = \frac{a_4}{a_2} \) to make \( \tilde{a}_3 = \tilde{a}_4 = 0 \). Then the conjugacy class is \( \langle X_2 + \alpha X_5 \rangle \), \( \alpha \in \mathbb{R} \).

**Case 7** \( a_1 = 0, a_2 = 0, a_6 \neq 0 \): Let \( \epsilon_1 = -\frac{a_2}{a_6}, \epsilon_2 = -\frac{a_4}{a_6} \) to make \( \tilde{a}_4 = \tilde{a}_5 = 0 \). Then the conjugacy class is of the form \( \langle \alpha X_3 + X_5 + \beta X_7 + \gamma X_8 \rangle \), \( \alpha, \beta, \gamma \in \mathbb{R} \).

**Case 8** \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 \neq 0, a_8 \neq 0 \): Let \( \epsilon_1 = -\frac{a_2}{a_5} \) and \( \epsilon_2 = \frac{a_4 a_5 - a_8 a_6}{a_5 a_8} \), to get \( \tilde{a}_3 = \tilde{a}_7 = 0 \). Then the conjugacy class is of the form \( \langle \alpha X_4 + \beta X_5 + \gamma X_8 \rangle \), \( \alpha \in \mathbb{R}, \beta \neq 0 \).

**Case 9** \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 \neq 0, a_8 = 0, a_7 \neq 0 \): Let \( \epsilon_1 = \frac{\ln \epsilon_2}{\gamma} \) with \( \epsilon_2 = -\frac{a_4}{a_5} \frac{\ln |a_7| - \gamma}{\gamma} \) to make \( \tilde{a}_5 = 0, \tilde{a}_7 = \pm 1 \). Then the conjugacy class after appropriate scaling is of the form \( \langle \alpha X_4 + \beta X_5 + X_7 \rangle \), \( \alpha \in \mathbb{R}, \beta \neq 0 \).

**Case 10** \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 \neq 0, a_8 = 0, a_7 = 0 \): Let \( \epsilon_2 = -\frac{a_4}{a_5} \) to make \( \tilde{a}_3 = 0 \) and so we have the conjugacy class of the form \( \langle \alpha X_4 + X_5 \rangle \), with \( \alpha \in \mathbb{R} \).

**Case 11** \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 \neq 0 \): Let \( \epsilon_1 = -\frac{a_2}{a_4} \) to make \( \tilde{a}_3 = 0 \). Then the conjugacy class is \( \langle X_4 + \alpha X_2 + \beta X_8 \rangle \), \( \alpha, \beta \in \mathbb{R} \).

**Case 12** \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_8 \neq 0 \): Let \( \epsilon_1 = -\frac{a_2}{a_8} \) to have \( \tilde{a}_7 = 0 \). Then the conjugacy class is \( \langle \alpha X_3 + X_8 \rangle \), \( \alpha \in \mathbb{R} \).

**Case 13** \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_8 = 0, a_7 \neq 0 \): Let \( \epsilon_1 = \frac{\ln |a_7|}{\gamma} \) to have \( \tilde{a}_7 = \pm 1 \) and so the conjugacy class after appropriate scaling is \( \langle \alpha X_3 + X_7 \rangle \), \( \alpha \in \mathbb{R} \).

**Case 14** \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_8 = 0, a_7 = 0 \): Then directly we get a conjugacy class of the form \( \langle X_3 \rangle \).

\[ \square \]

### 3.2 Optimal system for the case \( d^2 - 4k\rho_2 = \lambda^2 \)

The non-zero commutators of the Lie algebra \( \mathcal{L}^8 \) with basis (2.20) are given by

\[
\begin{align*}
[X_1, X_4] &= X_3, & [X_1, X_5] &= X_5, & [X_1, X_7] &= -\frac{d}{2p_2}X_7 + \frac{\lambda}{2p_2}X_8, \\
[X_1, X_8] &= -\frac{d}{2p_2}X_8 + \frac{\lambda}{2p_2}X_7, & [X_2, X_3] &= X_3, & [X_2, X_6] &= X_4.
\end{align*}
\]

(3.32)

The Lie algebra \( \mathcal{L}^8 \) is solvable and the Killing form is given by \( K = 2(\tilde{a}^2 + \tilde{b}^2)\alpha_1^2 \) where \( \tilde{a} = \frac{d}{2p_2} \) and \( \tilde{b} = \frac{\lambda}{2p_2} \). The adjoint table is given by
Table 2:

| $Ad(e^t)$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|
| $X_1$     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
| $X_2$     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
| $X_3$     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
| $X_4$     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
| $X_5$     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
| $X_6$     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
| $X_7$     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
| $X_8$     | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |

$Y_1 = \frac{1}{8}(e^{\epsilon(\hat{a}+b)} + e^{\epsilon(\hat{a}-b)})X_7 + \frac{1}{8}(e^{\epsilon(\hat{a}+b)} - e^{\epsilon(\hat{a}-b)})X_8$ and $Y_2 = \frac{1}{8}(e^{\epsilon(\hat{a}+b)} - e^{\epsilon(\hat{a}-b)})X_7 + \frac{1}{8}(e^{\epsilon(\hat{a}+b)} + e^{\epsilon(\hat{a}-b)})X_8$.

The adjoint group is defined by the matrix

$$A = Ad(e^{-\epsilon X_8}) . Ad(e^{-\epsilon X_7}) . Ad(e^{-\epsilon X_6}) . Ad(e^{-\epsilon X_5}) . Ad(e^{-\epsilon X_4}) . Ad(e^{-\epsilon X_3}) . Ad(e^{-\epsilon X_2}) . Ad(e^{-\epsilon X_1})$$

which is given by

$$A = \begin{pmatrix}
1 & 0 & -\epsilon_4 & 0 & -\epsilon_6 & 0 & -\epsilon_8 & -\epsilon_7 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\epsilon_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

(3.33)

$$\hat{Y}_1 = \frac{1}{8}(e^{-\epsilon_1(\hat{a}+b)} + e^{-\epsilon_1(\hat{a}-b)})$$

and $\hat{Y}_2 = \frac{1}{8}(e^{-\epsilon_1(\hat{a}-b)} - e^{-\epsilon_1(\hat{a}+b)})$.

Theorem 3.2. An optimal system of one-dimensional Lie algebra of $L^8$ with basis $\{X_i\}$ is provided by the following generators



Proof. Let $X$ and $\hat{X}$ be two elements in the Lie algebra $L^8$ with basis $\{X_i\}$ given by $X = \sum_{i=1}^{8} a_i X_i$ and $\hat{X} = \sum_{i=1}^{8} \hat{a}_i X_i$. For simplicity, we will write $X$ and $\hat{X}$ as row vectors of the coefficients on the form
$X = (a_1, a_2 \ldots a_8)$ and $\tilde{X} = (\tilde{a}_1 \tilde{a}_2 \ldots \tilde{a}_8)$. 

Then in order for $X$ and $\tilde{X}$ to be in the same conjugacy class, we must have $\tilde{X} = XA$, where $A$ is given by (3.33). So, the theorem is proved by solving the system

\begin{equation}
\begin{aligned}
\tilde{a}_1 &= a_1, \\
\tilde{a}_2 &= a_2, \\
\tilde{a}_3 &= a_3 + \epsilon_1 a_4 + \epsilon_2 a_5 - \epsilon_5 a_2 - \epsilon_4 a_1 + \epsilon_1 \epsilon_2 a_6, \\
\tilde{a}_4 &= a_4 - \epsilon_6 a_2 + \epsilon_2 a_6, \\
\tilde{a}_5 &= a_5 - \epsilon_6 a_4 + \epsilon_1 a_6, \\
\tilde{a}_6 &= a_6, \\
\tilde{a}_7 &= a_1 (\hat{a} \epsilon_7 - \hat{b} \epsilon_8) + \frac{1}{2} a_7 (e^{-\epsilon_1 (\hat{a} - \hat{b})} + e^{-\epsilon_1 (\hat{a} + \hat{b})}) + \frac{1}{2} a_8 (e^{-\epsilon_1 (\hat{a} - \hat{b})} - e^{-\epsilon_1 (\hat{a} + \hat{b})}), \\
\tilde{a}_8 &= a_1 (\hat{a} \epsilon_8 - \hat{b} \epsilon_7) + \frac{1}{2} a_7 (e^{-\epsilon_1 (\hat{a} - \hat{b})} - e^{-\epsilon_1 (\hat{a} + \hat{b})}) + \frac{1}{2} a_8 (e^{-\epsilon_1 (\hat{a} - \hat{b})} + e^{-\epsilon_1 (\hat{a} + \hat{b})}),
\end{aligned}
\end{equation}

for $\{\epsilon_i\}_{i=1}^8$ in term of $\{a_i\}_{i=1}^8$ in order to get the simplest values of $\{\tilde{a}_i\}_{i=1}^8$. The results are presented for different cases in the tree diagram (2) given in the appendix where it is initiated by the sign of the Killing form and its leaves are given completely. The full details for each leaf are given as follows:

**Case 1** $a_1 \neq 0$, $a_3 \neq 0$ : Let $\epsilon_1 = 0$, $\epsilon_2 = \frac{a_2 a_4 - a_1 a_6}{a_1 a_2}$, $\epsilon_4 = \frac{a_1 a_3 a_6 - a_1 a_4 a_5 + 2a_2^2}{a_1 a_2}$, $\epsilon_5 = 0$, $\epsilon_6 = \frac{a_5}{a_1}$, $\epsilon_7 = -\frac{a_6 a_4 a_5}{(\hat{a} - \hat{b}) a_1}$ and $\epsilon_8 = \frac{b_1 a_3 a_4 a_5}{(\hat{a} - \hat{b}) a_1}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_7 = \tilde{a}_8 = 0$, then we obtain the conjugacy class $(X_1 + \alpha X_2 + \beta X_8)$, where $\alpha \in \mathbb{R}$, $\beta \neq 0$.

**Case 2** $a_1 \neq 0$, $a_6 = 0$ : Let $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_4 = \frac{a_4}{a_1}$, $\epsilon_5 = 0$, $\epsilon_6 = \frac{a_6}{a_1}$, $\epsilon_7 = -\frac{a_7 a_4 a_5}{(\hat{a} - \hat{b}) a_1}$ and $\epsilon_8 = \frac{b_1 a_3 a_4 a_5}{(\hat{a} - \hat{b}) a_1}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_6 = \tilde{a}_7 = \tilde{a}_8 = 0$ and so we obtain the conjugacy class $(X_1 + \alpha X_2 + \beta X_4)$, where $\alpha, \beta \in \mathbb{R}$.

**Case 3** $a_1 = 0$, $a_2 \neq 0$, $a_6 \neq 0$ : Let $\epsilon_1 = -\frac{a_4}{a_6}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_6 a_5}{a_2 a_6}$ and $\epsilon_6 = \frac{a_6}{a_2}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$. Then the conjugacy class is of the form $(X_2 + \alpha X_6 + \beta X_7 + \gamma X_8)$, where $\alpha \neq 0$, $\beta, \gamma \in \mathbb{R}$.

**Case 4** $a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_2 + a_2 \neq 0$, $\frac{a_7 - a_8}{a_7 + a_8} > 0$ : Let $\epsilon_1 = \frac{1}{2} \ln (\frac{\alpha - a_8}{\alpha + a_8})$, $\epsilon_2 = 0$ and $\epsilon_5 = \frac{a_4}{2 a_2 (a_7 + a_8)} \ln (\frac{\alpha - a_8}{\alpha + a_8}) + \frac{a_6}{2 a_2}$ and $\epsilon_6 = \frac{a_6}{2 a_2}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$ and so $(X_2 + \alpha X_5 + \beta X_7)$, where $\alpha \in \mathbb{R}$, $\beta \neq 0$.

**Case 5** $a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 \neq 0$, $\frac{a_7 - a_8}{a_7 + a_8} < 0$ : Let $\epsilon_1 = \frac{1}{2} \ln (\frac{\alpha - a_8}{\alpha + a_8})$, $\epsilon_2 = 0$ and $\epsilon_5 = \frac{a_4}{2 a_2} \ln (\frac{\alpha - a_8}{\alpha + a_8}) + \frac{a_6}{2 a_2}$ and $\epsilon_6 = \frac{a_6}{2 a_2}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$ and so we obtain the conjugacy class $(X_2 + \alpha X_7 + \beta X_8)$, where $\alpha \in \mathbb{R}$, $\beta \neq 0$.

**Case 6** $a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 \neq 0$, $\frac{a_7 - a_8}{a_7 + a_8} = 0$ : Note that in this case $a_8 \neq 0$, so let $\epsilon_1 = \frac{\ln |a_8|}{a_8}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_4}{(a_7 + a_8)^2} \ln |a_8| + \frac{a_6}{a_2}$ and $\epsilon_6 = \frac{a_6}{a_2}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$ and so $(\alpha X_2 + \beta X_5 + \gamma X_7 + X_8)$, where $\alpha \neq 0$, $\beta \in \mathbb{R}$.

**Case 7** $a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 = 0$, $a_8 \neq 0$ : Let $\epsilon_1 = \frac{\ln |a_8|}{a_8}$, $\epsilon_2 = 0$, $\epsilon_5 = \frac{a_4}{(a_7 + a_8)^2} \ln |a_8| + \frac{a_6}{a_2}$ and $\epsilon_6 = \frac{a_6}{a_2}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$ and so the conjugacy class is of the form $(\alpha X_2 + \gamma X_7 + \beta X_5)$, where $\alpha \in \mathbb{R}$.

**Case 8** $a_1 = 0$, $a_2 \neq 0$, $a_6 = 0$, $a_7 + a_8 = 0$, $a_8 = 0$ : Let $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_5 = \frac{a_4}{a_2}$ and $\epsilon_6 = \frac{a_6}{a_2}$ to have $\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0$ and so the conjugacy class is of the form $(\alpha X_2 + \beta X_7 + \gamma X_8)$, where $\alpha, \beta, \gamma \in \mathbb{R}$. 

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Case 10 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 \neq 0, a_7 + a_8 \neq 0, \frac{2\gamma - a_5}{a_7 + a_8} > 0 \): Let \( \epsilon_1 = \frac{1}{2a_5} \ln \left( \frac{2\gamma - a_5}{a_7 + a_8} \right) \) and \( \epsilon_2 = \frac{a_7 - a_6}{2a_5} \ln \left( \frac{2\gamma - a_5}{a_7 + a_8} \right) + \frac{a_8}{a_5} \) to have \( a_5 = a_6 = 0 \) and so the conjugacy class is of the form \( \langle \alpha X_4 + X_5 + \beta X_7 \rangle \), where \( \alpha \in \mathbb{R}, \beta \neq 0 \).

Case 11 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 \neq 0, a_7 + a_8 \neq 0, \frac{2\gamma - a_5}{a_7 + a_8} < 0 \): Let \( \epsilon_1 = \frac{1}{2a_5} \ln \left( \frac{-2\gamma + a_5}{a_7 + a_8} \right) \) and \( \epsilon_2 = -\frac{a_7 - a_6}{2a_5} \ln \left( \frac{-2\gamma + a_5}{a_7 + a_8} \right) - \frac{a_8}{a_5} \) to have \( a_5 = a_6 = 0 \) and so the conjugacy class is of the form \( \langle \alpha X_4 + X_5 + \beta X_8 \rangle \), where \( \alpha \in \mathbb{R}, \beta \neq 0 \).

Case 12 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 \neq 0, a_7 + a_8 \neq 0, \frac{2\gamma - a_5}{a_7 + a_8} = 0 \): Note that in this case \( a_8 \neq 0 \), so let \( \epsilon_1 = \frac{\ln |a_8|}{a - 6} \) and \( \epsilon_2 = -\frac{a_7 - a_6}{(a - 6)a_5} \ln |a_8| - \frac{a_8}{a_5} \), to have \( a_5 = 0 \), and so the conjugacy class is \( \langle \alpha X_4 + \beta X_5 + X_7 + X_8 \rangle \), where \( \alpha \in \mathbb{R}, \beta \neq 0 \).

Case 13 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 \neq 0, a_7 + a_8 = 0, a_8 \neq 0 \): Let \( \epsilon_1 = \frac{\ln |a_8|}{a - 5} \) and \( \epsilon_2 = \frac{a_4}{(a - 5)a_5} \ln |a_8| + \frac{a_8}{a_5} \) to have \( a_5 = 0 \) and so the conjugacy class is of the form \( \langle \alpha X_4 + \beta X_5 + X_7 - X_8 \rangle \), where \( \alpha \in \mathbb{R}, \beta \neq 0 \).

Case 14 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 \neq 0, a_7 + a_8 = 0, a_8 = 0 \): Let \( \epsilon_1 = 0 \) and \( \epsilon_2 = -\frac{a_8}{a_5} \) to have \( a_5 = 0 \) and so we obtain the conjugacy class \( \langle \alpha X_4 + X_5 \rangle \), \( \alpha \in \mathbb{R} \).

Case 15 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 \neq 0 \): let \( \epsilon_1 = -\frac{a_3}{a_4} \) to have \( a_3 = 0 \) and so the conjugacy class is \( \langle X_4 + \alpha X_7 + X_8 \rangle \), where \( \alpha, \beta \in \mathbb{R} \).

Case 16 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_7 + a_8 \neq 0, \frac{2\gamma - a_5}{a_7 + a_8} > 0 \): Let \( \epsilon_1 = \frac{1}{2a_5} \ln \left( \frac{2\gamma - a_5}{a_7 + a_8} \right) \) to have \( a_5 = 0 \) and so the conjugacy class after appropriate scaling is of the form \( \langle \alpha X_4 + X_7 \rangle \), where \( \alpha \in \mathbb{R} \).

Case 17 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_7 + a_8 \neq 0, \frac{2\gamma - a_5}{a_7 + a_8} < 0 \): Let \( \epsilon_1 = \frac{1}{2a_5} \ln \left( \frac{-2\gamma + a_5}{a_7 + a_8} \right) \) to have \( a_7 = 0 \) and so the conjugacy class after appropriate scaling is of the form \( \langle \alpha X_4 + X_8 \rangle \), where \( \alpha \in \mathbb{R} \).

Case 18 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_7 + a_8 \neq 0, \frac{2\gamma - a_5}{a_7 + a_8} = 0 \): Note that \( a_8 \neq 0 \) so let \( \epsilon_1 = \frac{\ln |a_8|}{a - 5} \) to have the conjugacy class \( \langle \alpha X_4 + X_7 - X_8 \rangle \), where \( \alpha \in \mathbb{R} \).

Case 19 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_7 + a_8 = 0, a_8 = 0 \): Let \( \epsilon_1 = \frac{\ln |a_8|}{a - 5} \) to have the conjugacy class \( \langle \alpha X_4 + X_7 - X_8 \rangle \), where \( \alpha \in \mathbb{R} \).

Case 20 \( a_1 = 0, a_2 = 0, a_6 = 0, a_5 = 0, a_4 = 0, a_7 + a_8 = 0, a_8 = 0 \): Then directly we have the conjugacy class \( \langle X_3 \rangle \).

\[ \square \]

### 3.3 Optimal system for the case \( d^2 - 4k \rho = -\mu^2 \)

The non-zero commutators of the Lie algebra \( \mathcal{L}^8 \) with basis \( \{2.24\} \) are given by

\[
\begin{align*}
[X_1, X_4] &= X_3, \\
[X_1, X_6] &= X_5, \\
[X_1, X_7] &= -\frac{d}{2p_2} X_7 - \frac{\mu}{2p_2} X_8, \\
[X_1, X_8] &= -\frac{d}{2p_2} X_8 + \frac{\mu}{2p_2} X_7, \\
[X_2, X_5] &= X_3, \\
[X_2, X_6] &= X_4.
\end{align*}
\]

(3.36)

The Lie algebra \( \mathcal{L}^8 \) is solvable and the Killing form is given by \( K = 2(\mathcal{a}^2 - \hat{b}^2)\mathcal{a}_1^2 \) where \( \mathcal{a} = \frac{d}{2p_2} \) and \( \hat{b} = \frac{\mu}{2p_2} \). The adjoint table is given by
The adjoin group is defined by the matrix

\[ A = Ad(e^{-\epsilon_6 X_8}).Ad(e^{-\epsilon_7 X_7}).Ad(e^{-\epsilon_5 X_5}).Ad(e^{-\epsilon_4 X_4}).Ad(e^{-\epsilon_3 X_3}).Ad(e^{-\epsilon_2 X_2}).Ad(e^{-\epsilon_1 X_1}), \]

which is given by

\[
A = \begin{pmatrix}
1 & 0 & -\epsilon_4 & 0 & -\epsilon_6 & 0 & \hat{a}\epsilon_7 - \hat{b}\epsilon_8 & \hat{a}\epsilon_8 + \hat{b}\epsilon_7 \\
0 & 1 & -\epsilon_5 & -\epsilon_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \epsilon_2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \epsilon_1\epsilon_2 & \epsilon_2 & \epsilon_1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{Y}_1 & -\hat{Y}_2 \\
0 & 0 & 0 & 0 & 0 & 0 & \hat{Y}_2 & \hat{Y}_1 \\
\end{pmatrix}. \quad (3.37)
\]

\[ \hat{Y}_1 = e^{-\epsilon_1 \epsilon_6 \cos(\epsilon_1 \hat{b})}, \quad \hat{Y}_2 = e^{-\epsilon_1 \epsilon_6 \sin(\epsilon_1 \hat{b})} \]

**Theorem 3.3.** An optimal system of one-dimensional Lie algebra \( L^8 \) with basis (2.24) is provided by the following generators

\[
\begin{align*}
X^1 &= X_1 + \alpha X_2 + \beta X_6, & \alpha \in \mathbb{R}, \beta 
eq 0, \\
X^2 &= X_1 + \alpha X_2 + \beta X_4, & \alpha, \beta \in \mathbb{R}, \\
X^3 &= X_2 + \alpha X_6 + \beta X_7 + \gamma X_8, & \alpha \neq 0, \beta, \gamma \in \mathbb{R}, \\
X^4 &= X_2 + \alpha X_5 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma \in \mathbb{R}, \\
X^5 &= \alpha X_4 + \alpha X_6 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma \in \mathbb{R}, \\
X^6 &= X_4 + \alpha X_5 + \beta X_7 + \gamma X_8, & \alpha, \beta, \gamma \in \mathbb{R}, \\
X^7 &= X_5 + \alpha X_8, & \alpha \neq 0, \\
X^8 &= X_5 + \alpha X_7, & \alpha \in \mathbb{R}. \quad (3.38)
\end{align*}
\]

**Proof.** Let \( X \) and \( \tilde{X} \) be two elements in the Lie algebra \( L^8 \) with basis (2.24), given by \( X = \sum_{i=1}^8 a_i X_i \) and \( \tilde{X} = \sum_{i=1}^8 \tilde{a}_i X_i \). For simplicity, we will write \( X \) and \( \tilde{X} \) as row vectors of the coefficients on the form \( X = (a_1 a_2 \ldots a_8) \) and \( \tilde{X} = (\tilde{a}_1 \tilde{a}_2 \ldots \tilde{a}_8) \). Then in order for \( X \) and \( \tilde{X} \) to be in the same conjugacy class, we must have \( \tilde{X} = XA \), where \( A \) is given by (3.37). So, the theorem is proved by solving the
system
\[
\begin{align*}
\tilde{a}_1 &= a_1, \\
\tilde{a}_2 &= a_2, \\
\tilde{a}_3 &= a_3 + \epsilon_1 a_4 + \epsilon_2 a_5 - \epsilon_5 a_2 - \epsilon_4 a_1 + \epsilon_1 \epsilon_2 a_6, \\
\tilde{a}_4 &= a_4 - \epsilon_6 a_2 + \epsilon_2 a_5, \\
\tilde{a}_5 &= a_5 - \epsilon_6 a_1 + \epsilon_1 a_6, \\
\tilde{a}_6 &= a_6, \\
\tilde{a}_7 &= a_1 (\tilde{a} c_7 - \tilde{b} \epsilon_8) + e^{-\epsilon_1 \tilde{b}} (\cos(\epsilon_1 \tilde{b}) a_7 + \sin(\epsilon_1 \tilde{b}) a_8), \\
\tilde{a}_8 &= a_1 (\tilde{b} c_7 + \tilde{a} \epsilon_8) - e^{-\epsilon_1 \tilde{b}} (\sin(\epsilon_1 \tilde{b}) a_7 - \cos(\epsilon_1 \tilde{b}) a_8),
\end{align*}
\] (3.39)

for \(\{\epsilon_i\}_{i=1}^8\) in term of \(\{a_i\}_{i=1}^8\) in order to get the simplest values of \(\{\tilde{a}_i\}_{i=1}^8\). The results are presented for different cases in the tree diagram (3) given in the appendix where it is initiated by the sign of the Killing form and its leaves are given completely. The full details for each leaf are given as follows:

**Case 1** \(a_1 \neq 0, a_6 \neq 0\) : Let \(e_1 = -\frac{a_6}{a_5}, e_2 = -\frac{a_8}{a_6}, e_4 = \frac{a_3 a_6 - a_4 a_5}{a_1 a_6}, e_5 = e_6 = 0\),

\[
e_7 = \frac{1}{a_1 (a^2 + b^2)} e^{\frac{a_5 a_6}{a_6}} \left( (a a_8 - b a_7) \sin \left( \frac{a_5 a_6}{a_6} \right) - (a a_7 + b a_8) \cos \left( \frac{a_5 a_6}{a_6} \right) \right)
\]

and

\[
e_8 = -\frac{1}{a_1 (a^2 + b^2)} e^{\frac{a_5 a_6}{a_6}} \left( (a a_7 + b a_8) \sin \left( \frac{a_5 a_6}{a_6} \right) + (a a_8 - b a_7) \cos \left( \frac{a_5 a_6}{a_6} \right) \right)
\]

to have \(\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_7 = \tilde{a}_8 = 0\). Then we have the conjugacy class \(\langle X_1 + \alpha X_2 + \beta X_3, \rangle\) where \(\alpha, \beta \in \mathbb{R}\).

**Case 2** \(a_1 \neq 0, a_6 = 0\) : Let \(e_1 = e_2 = 0, e_4 = \frac{a_6}{a_1}, e_5 = \frac{\epsilon_1 a_4 - \epsilon_3 a_5}{a_1 (a^2 + b^2)}, e_6 = -\frac{a_3 a_6}{a_1 (a^2 + b^2)}\) to have \(\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = \tilde{a}_7 = \tilde{a}_8 = 0\). Then we have the conjugacy class of the form \(\langle X_1 + \alpha X_2 + \beta X_4, \rangle\) where \(\alpha, \beta \in \mathbb{R}\).

**Case 3** \(a_1 = 0, a_2 \neq 0, a_6 \neq 0\) : Let \(e_1 = -\frac{a_5}{a_6}, e_2 = 0, e_5 = \frac{a_3 a_6 - a_4 a_5}{a_2 a_6}, e_6 = \frac{a_4}{a_2}\) to get \(\tilde{a}_3 = \tilde{a}_4 = \tilde{a}_5 = 0\). Then the conjugacy class is of the form \(\langle X_2 + \alpha X_5 + \beta X_7 + \gamma X_8, \rangle\) where \(\alpha \neq 0, \beta, \gamma \in \mathbb{R}\).

**Case 4** \(a_1 = 0, a_2 \neq 0, a_6 = 0\) : Let \(e_1 = e_2 = 0, e_5 = \frac{a_4}{a_2}, e_6 = \frac{a_3}{a_2}\) to get \(\tilde{a}_3 = \tilde{a}_4 = 0\). Then the conjugacy class is \(\langle X_2 + \alpha X_5 + \beta X_7 + \gamma X_8, \rangle, \alpha, \beta, \gamma \in \mathbb{R}\).

**Case 5** \(a_1 = 0, a_2 = 0, a_6 \neq 0\) : Let \(e_1 = -\frac{a_6}{a_5}, e_2 = -\frac{a_8}{a_6}\) to have \(\tilde{a}_4 = \tilde{a}_5 = 0\) and so we obtain the conjugacy class \(\langle \alpha X_3 + X_6 + \beta X_7 + \gamma X_8, \rangle, \alpha, \beta, \gamma \in \mathbb{R}\).

**Case 6** \(a_1 = 0, a_2 = 0, a_6 = 0, a_4 \neq 0\) : Let \(e_1 = -\frac{a_4}{a_5}\) to have \(\tilde{a}_3 = \tilde{a}_4 = 0\) and so we obtain the conjugacy class \(\langle X_4 + \alpha X_5 + \beta X_7 + \gamma X_8, \rangle, \alpha, \beta, \gamma \in \mathbb{R}\).

**Case 7** \(a_1 = 0, a_2 = 0, a_6 = 0, a_4 = 0 a_5 \neq 0, a_8 \neq 0\) : Let \(e_1 = -\frac{1}{\alpha} \tan^{-1} \left( \frac{a_8}{a_5} \right) \) and \(e_2 = -\frac{a_4}{a_5}\) to have \(\tilde{a}_3 = \tilde{a}_4 = 0\). Then the conjugacy class is of the form \(\langle X_5 + \alpha X_7, \rangle\), \(\alpha \neq 0\).

**Case 8** \(a_1 = 0, a_2 = 0, a_6 = 0, a_4 = 0 a_5 \neq 0, a_8 = 0\) : Let \(e_1 = 0 \) and \(e_2 = -\frac{a_4}{a_5}\) to have \(\tilde{a}_3 = \tilde{a}_4 = 0\). Then the conjugacy class is \(\langle X_5 + \alpha X_7, \rangle\), \(\alpha \in \mathbb{R}\).

**Case 9** \(a_1 = 0, a_2 = 0, a_6 = 0, a_4 = 0, a_5 = 0, a_8 \neq 0\) : Let \(e_1 = -\frac{1}{\alpha} \tan^{-1} \left( \frac{a_8}{a_5} \right)\) to have \(\tilde{a}_3 = \tilde{a}_4 = 0\). By appropriate scaling the conjugacy class is of the form \(\langle \alpha X_3 + X_8, \rangle\), where \(\alpha \in \mathbb{R}\).
Case 10. \( a_1 = 0, a_2 = 0, a_6 = 0, a_4 = 0, a_5 = 0, a_8 = 0 \): Let \( \epsilon_1 = 0 \) to have \( a_8 = 0 \) and so we have the conjugacy class 
\( (\alpha X_3 + \beta X_7) \), where \( \alpha, \beta \in \mathbb{R} \).

4 Optimal reductions and invariant solutions

It is known that the invariant solutions for PDEs can be determined by two procedures, which are the invariant form method and the direct substitution method [11]. The idea of looking for group invariant solutions generalize quite naturally to PDEs with any number of independent and dependent variables. A one parameter group that acts nontrivially on one or more independent variables can be used to reduce the number of independent variables by one.

In this section, we focus on the invariant form method which requires that at least one of the infinitesimals \( \xi^1 \) and \( \xi^2 \) does not equal zero [11, 13]. Hence, we solve the invariance surface conditions explicitly by solving the corresponding characteristic equation given by

\[
\frac{dt}{\xi^1(t,x,\varphi,\psi)} = \frac{dx}{\xi^2(t,x,\varphi,\psi)} = \frac{d\varphi}{\eta^1(t,x,\varphi,\psi)} = \frac{d\psi}{\eta^2(t,x,\varphi,\psi)},
\]

(4.40)

to get the corresponding invariants which are used to reduce the number of independent variables by one. The procedure is explained in details in three examples. Moreover, all possible invariant variables of the optimal systems (3.30), (3.34) and (3.38) and their corresponding reductions for the three non-linear cases of \( \chi(\psi_x) \) are given in Table 4, Table 5 and Table 6, respectively.

**Example 4.1. Reduction for case (2.2.1) using invariant form of \( X^3 \).**

Consider the generator \( X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8 \) where \( \alpha \neq 0, \beta, \gamma \in \mathbb{R} \), from the optimal system (3.30). Solving the corresponding characteristic equation

\[
\frac{dt}{0} = \frac{dx}{1} = \frac{d\varphi}{\alpha t x} = \frac{d\psi}{\alpha(\frac{d}{t} - t) + e^{-\frac{d}{t} \rho t} (\beta + \gamma t)},
\]

gives the invariant variables as follows:

\[
\phi(t,x) = Z(\zeta) + \frac{a}{2} t x^2, \quad \psi(t,x) = W(\zeta) + \alpha x(\frac{d}{t} - t) + x(\beta + \gamma t)e^{-\frac{d}{t} \rho t}, \quad \zeta = t.
\]

(4.41)

The reduced system resulting from the invariant variables (4.41) is the system of ODEs of the form

\[
\rho_1 Z'' - k(\gamma \zeta + \beta) e^{-\frac{d}{t} \rho t} \xi - \alpha d = 0,
\]

\[
\rho_2 W'' + d W' + k W = 0.
\]
Solving this system yields the following solution
\[
\varphi(t, x) = \frac{e^{\alpha t}}{\rho_1^2} (d\gamma t + d\beta + 4\gamma \rho_2) e^{-\frac{d^2}{\rho_1^2} t} + \alpha t x + \frac{\alpha d t^2}{\rho_1} + 2c_1 t + 2c_2,
\]
\[
\psi(t, x) = (\gamma t x + \beta x + c_3 t + c_4) e^{-\frac{d^2}{\rho_1^2} t} - \alpha t x + \frac{4c}{\rho_1} x,
\]
where \(d = 2\sqrt{k \rho_2} \).

**Example 4.2.** Reduction for case (2.2.2) using invariant form of \(X^3\).
Consider the generator \(X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8 \) where \(\alpha \neq 0, \beta, \gamma \in \mathbb{R} \), from the optimal system (3.34). Solving the corresponding characteristic equation
\[
\frac{dt}{0} = \frac{dx}{1} = \frac{d\varphi}{\alpha t x} = \frac{d\psi}{\alpha (\frac{d}{k} - t) + e^{-\frac{d^2}{\rho_1^2} t} \left( \beta \cosh \left( \frac{\mu t}{\rho_2} \right) + \gamma \sinh \left( \frac{\mu t}{\rho_2} \right) \right)},
\]
gives the invariant variables as follows:
\[
\phi(t, x) = Z(\zeta) + \frac{a}{2} t x^2, \quad \psi(t, x) = W(\zeta) + \alpha x (\frac{d}{k} - t) + x e^{-\frac{d^2}{\rho_1^2} t} (\beta \cosh \left( \frac{\mu}{\rho_2} \right) + \gamma \sinh \left( \frac{\mu}{\rho_2} \right)), \quad \zeta = t.
\]
The reduced system resulting from the invariant variables (4.42) is the system of ODEs of the form
\[
\rho_1 Z'' - k e^{-\frac{d^2}{\rho_1^2} t} (\beta \cosh \left( \frac{\mu}{\rho_2} \right) + \gamma \sinh \left( \frac{\mu}{\rho_2} \right)) - \alpha d = 0,
\]
\[
\rho_2 W'' + d W' + k W = 0.
\]
Solving this system yields the following solution
\[
\varphi(t, x) = \frac{1}{2\rho_2} \left( (2 \rho_2 k + \beta \lambda^2 + \lambda d) \cosh \left( \frac{\mu t}{\rho_2} \right) + (2 \gamma \rho_2 k + \lambda \beta d + \alpha^2) \sinh \left( \frac{\mu t}{\rho_2} \right) \right) e^{-\frac{d^2}{\rho_1^2} t} + \frac{\alpha d t^2}{\rho_1} + \frac{\alpha}{2} t x^2 + c_1 t + c_2,
\]
\[
\psi(t, x) = \left( (c_3 + \beta x) \cosh \left( \frac{\mu t}{\rho_2} \right) + (c_4 + \gamma x) \sinh \left( \frac{\mu t}{\rho_2} \right) \right) e^{-\frac{d^2}{\rho_1^2} t} - x \alpha t + \frac{4c}{\rho_1} x,
\]
where \(d = \sqrt{k \rho_2 + \lambda^2} \).

**Example 4.3.** Reduction for case (2.2.3) using invariant form of \(X^3\).
Consider the generator \(X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8 \) where \(\alpha \neq 0, \beta, \gamma \in \mathbb{R} \), from the optimal system (3.38). Solving the corresponding characteristic equation
\[
\frac{dt}{0} = \frac{dx}{1} = \frac{d\varphi}{\alpha t x} = \frac{d\psi}{\alpha (\frac{d}{k} - t) + e^{-\frac{d^2}{\rho_1^2} t} \left( \beta \cos \left( \frac{\mu t}{\rho_2} \right) + \gamma \sin \left( \frac{\mu t}{\rho_2} \right) \right)},
\]
gives the invariant variables as follows:

\[ \phi(t, x) = Z(\zeta) + \frac{a}{2}t^2, \quad \psi(t, x) = W(\zeta) + \alpha x(\frac{\zeta}{2} - t) + x e^{-\frac{\mu t}{2p_2}} \left( \beta \cos\left(\frac{\mu t}{2p_2}\right) + \gamma \sin\left(\frac{\mu t}{2p_2}\right) \right), \quad \zeta = t. \]

The reduced system resulting from the invariant variables \((4.43)\) is the system of ODEs of the form

\[ \rho_1 Z'' - k e^{-\frac{\mu t}{2p_2}} \left( \beta \cos\left(\frac{\mu t}{2p_2}\right) + \gamma \sin\left(\frac{\mu t}{2p_2}\right) \right) - \alpha d = 0, \]
\[ \rho_2 W'' + d W' + k W = 0. \]

Solving this system yields the following solution

\[ \begin{align*}
\phi(t, x) &= \left( (\beta \mu^2 - 2 \beta k p_2 - d \gamma \mu) \cos \left(\frac{\mu t}{2p_2}\right) + \left( \gamma \mu^2 - 2 \gamma k p_2 \right) \sin \left(\frac{\mu t}{2p_2}\right) \right) e^{-\frac{\mu t}{2p_2}} \\
&\quad - \beta \frac{\mu}{2p_2} \left( -\alpha k x^2 p_2 + d \right) t \sin \left(\frac{\mu t}{2p_2}\right) - \left( \alpha k d t^2 + 2 c_3 \rho_1 k t + 2 c_2 \right), \\
\psi(t, x) &= \left( (\beta x + c_4) \cos \left(\frac{\mu t}{2p_2}\right) + (\gamma x + c_4) \sin \left(\frac{\mu t}{2p_2}\right) \right) e^{-\frac{\mu t}{2p_2}} + \alpha x \left( \frac{\zeta}{2} - t \right),
\end{align*} \]

where \(d = \sqrt{k p_2 - \mu^2}.\)

### Table 4: Reduction using one-dimensional optimal system \((3.30)\) with \(d = 2\sqrt{k p_2}\)

| Generators in \((3.30)\) | Invariant variables | The reduced system |
|--------------------------|---------------------|---------------------|
| \(X^1 = X_1 + \alpha X_2 + \beta X_6, \alpha \in \mathbb{R}, \beta \neq 0.\) | \(A\) \((k - \alpha^2 \rho_1) Z'' + k W' - \beta \rho_1 \zeta = 0,\) \((\chi'(W') - \alpha^2 \rho_2) W'' - k Z' + \alpha d W' - k W - 3 \beta \rho_2 = 0.\) | \(\rho_1 Z'' - k(\gamma \zeta + \beta) e^{-\frac{\mu t}{2p_2}} - \alpha d = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) |
| \(X^2 = X_1 + \alpha X_2 + \beta X_4, \alpha, \beta \in \mathbb{R}.\) | \(B\) \((k - \alpha^2 \rho_1) Z'' + k W' - \beta \rho_1 = 0,\) \((\chi'(W') - \alpha^2 \rho_2) W'' - k Z' + \alpha d W' - k W = 0.\) | \(\rho_1 Z'' - \beta k \zeta e^{-\frac{\mu t}{2p_2}} = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) |
| \(X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8, \alpha \neq 0, \beta, \gamma \in \mathbb{R}.\) | \(C\) \(\rho_1 Z'' - \frac{\beta}{2} e^{-\frac{\mu t}{2p_2}} \zeta = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) | \(\rho_1 Z'' - \frac{\beta}{2} e^{-\frac{\mu t}{2p_2}} \zeta = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) |
| \(X^4 = X_2 + \alpha X_5 + \beta X_7, \alpha \in \mathbb{R}, \beta \neq 0.\) | \(D\) \(\rho_1 Z'' - \beta \zeta e^{-\frac{\mu t}{2p_2}} = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) | \(\rho_1 Z'' - \frac{\beta}{2} e^{-\frac{\mu t}{2p_2}} \zeta = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) |
| \(X^5 = \alpha X_2 + \beta X_3 + \gamma X_7, \alpha \neq 0, \gamma \in \mathbb{R}.\) | \(E\) \(\rho_1 Z'' - \frac{\beta}{2} e^{-\frac{\mu t}{2p_2}} \zeta = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) | \(\rho_1 Z'' - \frac{\beta}{2} e^{-\frac{\mu t}{2p_2}} \zeta = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) |
| \(X^6 = X_2 + \alpha X_5, \alpha \in \mathbb{R}.\) | \(F\) \(Z'' = 0,\) \(\rho_2 W'' + d W' + k W = 0.\) | \(\rho_2 W'' + d W' + k W = 0.\) |

\(A: \ \phi(t, x) = Z(\zeta) + \frac{a}{2}t^2(x - \frac{\zeta}{2} t), \ \psi(t, x) = W(\zeta) + \alpha x(\frac{\zeta}{2} - t) + x e^{-\frac{\mu t}{2p_2}} \left( \beta \cos\left(\frac{\mu t}{2p_2}\right) + \gamma \sin\left(\frac{\mu t}{2p_2}\right) \right), \ \zeta = x - \alpha t.\)

\(B: \ \phi(t, x) = Z(\zeta) + \frac{a}{2}t^2, \ \psi(t, x) = W(\zeta), \ \zeta = x - \alpha t.\)

\(C: \ \phi(t, x) = Z(\zeta) + \frac{a}{2}t^2, \ \psi(t, x) = W(\zeta) + \alpha x(\frac{\zeta}{2} - t) + x(\beta + \gamma t) e^{-\frac{\mu t}{2p_2}} \left( \beta \cos\left(\frac{\mu t}{2p_2}\right) + \gamma \sin\left(\frac{\mu t}{2p_2}\right) \right), \ \zeta = t.\)

\(D: \ \phi(t, x) = Z(\zeta) + \frac{a}{2}t^2, \ \psi(t, x) = W(\zeta) - x(\alpha - \beta t e^{-\frac{\mu t}{2p_2}}) \left( \beta \cos\left(\frac{\mu t}{2p_2}\right) + \gamma \sin\left(\frac{\mu t}{2p_2}\right) \right), \ \zeta = t.\)

\(E: \ \phi(t, x) = Z(\zeta) + \frac{a}{2}t^2, \ \psi(t, x) = W(\zeta) - \frac{\beta}{2} e^{-\frac{\mu t}{2p_2}} \left( \beta \cos\left(\frac{\mu t}{2p_2}\right) + \gamma \sin\left(\frac{\mu t}{2p_2}\right) \right), \ \zeta = t.\)

\(F: \ \phi(t, x) = Z(\zeta) + \frac{a}{2}t^2, \ \psi(t, x) = W(\zeta) - \alpha x, \ \zeta = t.\)
### Table 5: Reduction using one-dimensional optimal system \((3.34)\) with \(d = \sqrt{4k^2 + \lambda^2}\)

| Generators in \((3.35)\) | Invariant variables | The reduced system |
|---------------------------|---------------------|-------------------|
| \(X^1 = X_1 + \alpha X_2 + \beta X_6\), \(\alpha \in \mathbb{R}, \beta \neq 0\) | \((k - \alpha^2 \rho_1) Z'' + k W' - \beta \rho_1 \zeta = 0, \alpha \in \mathbb{R}, \beta \neq 0, \zeta = x - \alpha t\) | \(\zeta = x - \alpha t\) |
| \(X^2 = X_1 + \alpha X_2 + \beta X_4, \alpha, \beta \in \mathbb{R}\) | \((k - \alpha^2 \rho_1) Z'' + k W' - \beta \rho_1 \zeta = 0, \alpha \in \mathbb{R}, \beta \neq 0, \zeta = t\) | \(\zeta = t\) |
| \(X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8\) | \(\rho_1 Z'' - \frac{2}{\sqrt{d}} \frac{1}{\sqrt{2d}} \left( \beta \cos \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) + \gamma \sinh \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) \right) - \alpha \dot{d} = 0, \rho_2 W'' + d W' + k W = 0\) | \(\zeta = t\) |
| \(X^4 = X_2 + \alpha X_5 + \beta X_7\) | \(\rho_1 Z'' - \frac{2}{\sqrt{d}} \frac{1}{\sqrt{2d}} \left( \cos \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) + \sin \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) \right) = 0, \rho_2 W'' + d W' + k W = 0\) | \(\zeta = t\) |
| \(X^5 = X_2 + \alpha X_5 + \beta X_8\) | \(\rho_1 Z'' - \frac{2}{\sqrt{d}} \frac{1}{\sqrt{2d}} \left( \cos \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) - \sin \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) \right) = 0, \rho_2 W'' + d W' + k W = 0\) | \(\zeta = t\) |
| \(X^6 = \alpha X_2 + \beta X_5 + X_1 + X_8\) | \(\rho_1 Z'' - \frac{2}{\sqrt{d}} \frac{1}{\sqrt{2d}} \left( \cos \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) + \sin \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) \right) = 0, \rho_2 W'' + d W' + k W = 0\) | \(\zeta = t\) |
| \(X^7 = \alpha X_2 + \beta X_5 + X_2 - X_8\) | \(\rho_1 Z'' - \frac{2}{\sqrt{d}} \frac{1}{\sqrt{2d}} \left( \cos \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) - \sin \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) \right) = 0, \rho_2 W'' + d W' + k W = 0\) | \(\zeta = t\) |
| \(X^8 = X_2 + \alpha X_5\) | \(\rho_2 W'' + d W' + k W = 0\) | \(\zeta = t\) |

### Table 6: Reduction using one-dimensional optimal system \((3.38)\) with \(d = \sqrt{4k^2 - \mu^2}\)

| Generators in \((3.38)\) | Invariant variables | The reduced system |
|---------------------------|---------------------|-------------------|
| \(X^1 = X_1 + \alpha X_2 + \beta X_6\), \(\alpha \in \mathbb{R}, \beta \neq 0\) | \((k - \alpha^2 \rho_1) Z'' + k W' - \beta \rho_1 \zeta = 0, \alpha \in \mathbb{R}, \beta \neq 0, \zeta = x - \alpha t\) | \(\zeta = x - \alpha t\) |
| \(X^2 = X_1 + \alpha X_2 + \beta X_4\) | \((k - \alpha^2 \rho_1) Z'' + k W' - \beta \rho_1 \zeta = 0, \alpha \in \mathbb{R}, \beta \neq 0, \zeta = t\) | \(\zeta = t\) |
| \(X^3 = X_2 + \alpha X_6 + \beta X_7 + \gamma X_8\) | \(\rho_1 Z'' - \frac{2}{\sqrt{d}} \frac{1}{\sqrt{2d}} \left( \beta \cos \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) + \gamma \sinh \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) \right) - \alpha \dot{d} = 0, \rho_2 W'' + d W' + k W = 0\) | \(\zeta = t\) |
| \(X^4 = X_2 + \alpha X_5 + \beta X_7 + \gamma X_8\) | \(\rho_1 Z'' - \frac{2}{\sqrt{d}} \frac{1}{\sqrt{2d}} \left( \beta \cos \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) + \gamma \sinh \left( \frac{\pi \sqrt{d}}{2\sqrt{2d}} \right) \right) = 0, \rho_2 W'' + d W' + k W = 0\) | \(\zeta = t\) |
5 Discussion and concluding remarks

The complete Lie symmetry classification of a non-linear Timoshenko system of PDEs with frictional damping term in rotational angle is performed. The classification is related to the arbitrary dependence on the rotation moment \( \chi(\psi_x) \). A Lie symmetry analysis is performed in three cases for non-linear rotational moment. The three cases depend on the sign of the parameter \( d^2 - 4k\rho_2 \). The one-dimensional optimal system is derived for each one of the three cases. All possible invariant forms and their corresponding reductions for each vector field in the optimal systems are found. These reductions to systems of ODEs are given in Table 4, Table 5 and Table 6. They are described by optimal reduction where all non-similar invariant solutions under symmetry transformations can be given from the solution of these reduced system of ODEs.

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Appendix: The Tree diagrams
Figure 1: Tree diagram 1

```
| Case   | a_1 ≠ 0 | a_1 = 0 |
|--------|---------|---------|
| Case 1 | a_6 ≠ 0 | a_6 = 0 |
| Case 2 | a_6 ≠ 0 | a_6 = 0 |
| Case 3 | a_8 ≠ 0 | a_8 = 0 |
| Case 4 | a_7 ≠ 0 | a_7 = 0 |
| Case 5 | a_8 ≠ 0 | a_8 = 0 |
| Case 6 | a_7 ≠ 0 | a_7 = 0 |
| Case 7 | a_5 ≠ 0 | a_5 = 0 |
| Case 8 | a_8 ≠ 0 | a_8 = 0 |
| Case 9 | a_7 ≠ 0 | a_7 = 0 |
| Case 10| a_8 ≠ 0 | a_8 = 0 |
| Case 11| a_8 ≠ 0 | a_8 = 0 |
| Case 12| a_7 ≠ 0 | a_7 = 0 |
| Case 13| a_7 ≠ 0 | a_7 = 0 |
| Case 14| a_8 ≠ 0 | a_8 = 0 |
```
Figure 2: Tree diagram 2
Figure 3: Tree diagram 3

\[
\begin{align*}
\text{Case 10} & : a_8 \neq 0 \\
\text{Case 9} & : a_5 \neq 0 \\
\text{Case 8} & : a_4 \neq 0 \\
\text{Case 7} & : a_3 \neq 0 \\
\text{Case 6} & : a_6 \neq 0 \\
\text{Case 5} & : a_2 \neq 0 \\
\text{Case 4} & : a_6 = 0 \\
\text{Case 3} & : a_1 \neq 0 \\
\text{Case 2} & : a_1 = 0 \\
\text{Case 1} & : a_8 = 0
\end{align*}
\]
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