I. INTRODUCTION

Motivated by the navigation information requirements in the differential interferometric SAR (DinSAR) [1], 3-D mapping [2], and applications requiring high-fidelity position and attitude information [3], this article brings the following main contributions.

1) Employ a Lie group especially tailored for integrating global navigation satellite-based system/inertial navigation system (GNSS/INS) in 3-D navigation, including bias estimation in a Kalman-like smoothing procedure appropriate to Lie groups.

2) Take advantage of the Lie group and the corresponding Lie algebra interplay, allowing an additive noise model evolving in the algebra to translate into a noisy behavior appropriate to rigid bodies. This understanding indicates that Lie-based models embedded into extended Kalman filters (EKF) are bound to accommodate the nonlinearities well and perform better than standard Euler angles or quaternion models [4]. Comparison experiments verify this assertion.

3) Benefit from the postprocessed scenario using the EKF-Lie filter inbuilt into a Rauch-Tung-Striebel (RTS) smoother on Lie groups for improved accuracy (see Lemma II.2). The smoother comes from stochastic principles, and the deterministic approach via observers [5] does not allow a smoother synthesis, as far as the authors are aware.

4) Due to poor heading estimation accuracy, propose a Bayesian calibration procedure tailored to the postprocessing scenario. Heading alignment is recognizably difficult in attitude estimation, e.g., see [5] and [6]. The drone carries a single inertial measurement unit (IMU) lacking gyro-compassing precision and a single GNSS antenna with a known lever arm. No magnetometer is attached due to the radars and drone motor magnetic field interferences.

5) Create synthetic data from actual flights to benchmark and present field tests of a radar bourne terrain imagery application (DinSAR). The application requires highly accurate flight path reconstruction (position and attitude) to render sufficiently well-marked terrain imagery.

6) Quality comparison with industry-standard software favorable to the devised scheme. It indicates the applicability of the Lie-based method developed here, which includes an outlier rejection method for GNSS measurements. These stages complete a full engineering design cycle.

It is known that a strap-down inertial navigation system (INS) based on microelectromechanical systems (MEMS) technology provides position, velocity, and attitude (PVA) information from accelerometers and gyroscopes at high rates but with unqualified errors after a period of discrete-time integration. These low-cost off-the-shelf sensors are inevitably affected by biases and white noise [7], thus, requiring adequate noise modeling. On the other hand, a GNSS can provide position and velocity with strict error bounds but at a lower frequency. Differential GNSS attains high precision, achieving centimeter-level precision using...
code and phase measurements [8]. A GNSS/INS integrated navigation system combines each sensor’s strengths for better PVA estimates.

Possible GNSS/INS integration types have appeared [9], and we adopt the loosely coupled (LC) structure for simplicity. The LC scheme combines the position-ready estimate provided by the GNSS with the IMU measurements.

Kalman like filters have been investigated for GNSS/INS integration algorithms, e.g., see [10], [11], and [12]. To achieve high-order approximations, unscented or particle filters offer alternatives to the EKF. They might provide better estimation but require more computational resources; for this reason, the EKF remains the most common GNSS/INS integration technique, the reference filter within the aerospace industry, cf., [13].

Several parameterizations apply to represent the attitude of a vehicle, such as Euler angles, quaternions, rotation vectors, and rotation matrices, cf., [9]. The challenge in choosing an adequate representation of attitude is that some have singularities or added constraints; see [4] for a thorough discussion.

The kinematic equations for Euler angles involve trigonometric functions, which make the model highly nonlinear, cf., [14]. Quaternions are appealing for attitude representation, cf., [15]. However, they must satisfy a normalization constraint to represent rotations, which is disregarded by the measurement update step of the EKF, cf., [16]. To circumvent such constraint [14], [17] use a multiplicative form of quaternion update. Nevertheless, most of these techniques consider models on Euclidean space and Gaussian distributed driving noise.

More recently, the Lie group theory-based framework has attracted much attention from sensor fusion communities for rigid-body-related data fusion. In robotics, [18] acknowledges that the unknown position of a differential-drive robot distribution displays a banana-shaped distribution, which can be produced with the aid of the exponential map of the SE(2) Lie group.

A discrete EKF on Lie groups (D-LIE-EKF) appears in [19] and [20], generalizing the usual Kalman Filter framework when the system dynamic or measurement model can be cast as a Lie group element. Within D-LIE-EKF, the noise is Gaussian distributed, but acting in the Lie algebra, which induces a concentrated Gaussian distributed in the Lie group, cf., [21].

In this work, we exploit the generalization of EKF on Lie groups to implement an LC Integration of GNSS/MEMS-INS. The double direct isometries Lie group SE(2)(3) (see [22]) is adopted to embed the attitude, velocity, and position states, combined with the translation group T(6) to accommodate the accelerometer and gyroscope biases. Besides, the proposed filter is inbuilt into RTS smoothing on Lie groups, see Lemma II.2 or [23], to benefit from the postprocessing approach to the referred applications. The RTS gathers all information available to leverage state estimates at each time step, attaining higher precision and accuracy requirements than the plain filter result.

In the context of navigation, Lie groups have been applied to industrial unmanned aerial vehicle (UAV) systems [13], real-time UAV helicopter navigation [24], and also land vehicle navigation [25]. However, none of these previous studies aimed at developing a complete navigation system to meet the accuracy requirements such as the DinSAR’s. Most current works focus on land vehicle navigation and filtering solutions only.

A concurrent approach for PVA estimation is based on observers from a vein of nonlinear system theory [26], with extra efforts to deal with inherent disturbances [5]. Although the synthesis of observers often comes with a corresponding convergence analysis, the ongoing assumptions are sometimes hard to verify. The quest for stability also appears in the Lie-based filter synthesis, e.g., see [27], [28]. In the course of using the EKF-Lie filter inbuilt into an RTS smoother, the stability was never an issue, except when low-quality GNSS data were fed into the GNSS/INS scheme. We propose an outlier rejection statistical method based on the Mahalanobis distance and a χ²-test with good results (see Section IV-C).

In addition to the above-mentioned choices, a good alignment process is indispensable to achieving centimeter-level precision. It is well known that the heading alignment hinders low-cost GNSS/INS integration systems. Solutions based on fuzzy [6] or wavelet neural networks [29] are available for long-term applications. Magnetometers are unavailable, the drone flights are short-duration, and corrections along the flight are not needed when the initial alignment is good. Also, because of the postprocessing nature of the applications in focus, one can benefit from a highly accurate flight estimation via the smoothing procedure, a better solution than just filtering. A Bayesian method to optimize heading alignment in such a scenario is developed in Section IV-E.

The strategy inbuilt into the D-LIE-EKF to deal with inherent nonlinearities relies on the interplay between matrix groups and the associated algebra. It is a powerful form of reducing the effect of nonlinearities to a minimum. It is better tailored to deal with non-Euclidean noise models for rigid bodies than other approximate filtering methods in the literature. Together with the statistical method devoted to the head alignment problem, they provide a way to develop optimal estimates given the adopted distributions. Deterministic nonlinear observers cannot mirror this scenario.

Using synthetic data from actual flights, we show that the proposed filter and smoother outperform conventional quaternion and Euler angles-based approaches. In a second comparison, we use a real dataset to show that the Lie group-based filtering inbuilt into the RTS smoothing yields better performances for DinSAR imagery processing than the state-of-the-art commercial software Inertial Explorer.

The rest of this article is organized as follows. Section II-A briefly introduces the mathematical concepts for understanding the Kalman filter algorithm on Lie groups. Section II describes the modeling of dynamic systems and random variables on Lie groups, followed by the description of filtering and smoothing methods. Section III develops
The following isomorphisms are defined:
\[
\begin{align*}
[-]_G : & \quad \mathbb{R}^p \to \mathbb{R}^p \\
X \mapsto [X]_G^\vee \\
\end{align*}
\]
(1)

For brevity, the following notations are used hereafter:
\[
\exp_G^\vee(x) := \exp_G([x]_G^\vee), \quad \log_G^\vee(g) := [\log_G(g)]_G^\vee
\]
(2)

where \( x \in \mathbb{R}^p \), \( g \in G \) and when we write \( g = \exp_G^\vee(x) \) we assume that \( \log_G^\vee(g) = x \), i.e., we work only on the subsets where \( \exp_G^\vee(\cdot) \) and \( \log_G^\vee(\cdot) \) are bijective.

Note that the exponential map can be interpreted as a parameterization for the Lie group in local coordinates around the identity element. This parameterization can be extended to the neighborhood of any element \( \mu \in G \) in a connected Lie group using left translation as follows, \( \mathcal{L}_\mu^\vee(\cdot) := \mu \exp_G^\vee(\cdot), \forall \mu \in \mathbb{R}^p \). The left translation action is illustrated in Fig. 1.

Since the exponential map is locally a diffeomorphism, there exist open neighborhoods of \( \mu \in G \) and \( \epsilon \in \mathbb{R}^p \) for which this parameterization is one-to-one.

A Lie group, in general, is not a commutative structure, which can complicate the algebraic manipulations. However, a concept overcomes this issue: the adjoint representation. There are two adjoint representations. The first one represents the Lie group on its Lie algebra, i.e., the linear map that takes an element of the Lie group to a linear transformation in the Lie algebra. The adjoint representation can be defined as
\[
\operatorname{Ad}_g(y) = [g[y]_G^\vee g^{-1}]_G^\vee
\]
(3)

where \( g \in G \), \( y \in \mathbb{R}^p \). Note that \( \operatorname{Ad}_g(g) \in \mathbb{R}^{p \times p} \) is a linear transformation that can be applied to any vector \( y \in \mathbb{R}^p \).

The second adjoint representation is the representation of the Lie algebra on itself so that each element of the Lie algebra defines a linear transformation in the Lie algebra. This adjoint representation is defined by the Lie bracket\[31\] in the form \([\operatorname{ad}_g(x)]_G^\vee := [[x]_G^\vee, y]_G^\vee \] where \( x, y \in \mathbb{R}^p \).

Since the Lie bracket for matrix Lie groups is the commutator operator, we write
\[
\operatorname{ad}_g(x) = [[x]_G^\vee, y]_G^\vee - [y]_G^\vee [x]_G^\vee.
\]

Furthermore, in many applications of the Lie group, particularly in filtering and smoothing, one is interested in analyzing the behavior of a Lie group element \( g \in G \) as a function of time, and, naturally, its time derivative, \( \dot{g}(t) \).
in the vector space tangent to \( G \) at the element \( g(t) \), i.e., \( g(t) \in T_{g(t)}G \).

As stated before, working in Lie algebra is a common approach to dealing with Lie groups. It turns out that the tangent space at an arbitrary element of the Lie group is isomorphic to the tangent space at identity by performing a left translation so that \( g^{-1}g \in g \). The right-Jacobian matrix, following [31], is given as:

\[
J_r(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \text{ad}_G(x)^k.
\]

Let us consider a local parameterization in the form \( g = \mu \exp_G(\epsilon) \). The relation of the time variation of \( g \) to the time variation of the local coordinates \( x \), represented in the Lie algebra, is given by \( g^{-1}g = [J_r(x)]_G^0 \). Notice that the matrix \( J_r(x) \) is responsible for relating local coordinates variations to Lie group element variations.

B. Random Variables on Lie Group

First, recall that for a random variable (r.v.) \( x \) on the Euclidean space with mean \( \mu \in \mathbb{R}^n \) and covariance \( P = P^\top > 0 \) associated with a pdf \( p(x) \), one has

\[
0_{n \times 1} = \int_{\mathbb{R}^n} (x - \mu)p(x)dx \quad \text{(6a)}
\]

\[
P = \int_{\mathbb{R}^n} (x - \mu)(x - \mu)^\top p(x)dx \quad \text{(6b)}
\]

where the integration is w.r.t. the Lebesgue measure.

This definition can be naturally extended to Lie groups as follows. Given a matrix Lie group \( G \), a random matrix \( X \in G \) with pdf \( p(X) \) has mean \( \mu \in G \) and covariance \( P = P^\top > 0 \) defined by

\[
0_{n \times n} = \int_G \log_G(\mu^{-1}X)p(X)dx \quad \text{(7)}
\]

\[
P = \int_G \log_G(\mu^{-1}X)\log_G(\mu^{-1}X)^\top p(X)dx \quad \text{(8)}
\]

where the integration is w.r.t. the Haar measure [31].

From this standpoint, the concept of concentrated Gaussian distribution (CGD) [23] is used to define a probability density tailored to matrix Lie groups. The group defines the mean, and the covariance is in the Lie algebra. Accordingly, a Gaussian random variable on a matrix Lie group is expressed as follows:

\[
X = \mu \exp_G(\epsilon)
\]

and the pdf of \( X \) takes the form

\[
p(X) := \alpha \exp\left(-\frac{1}{2} \left\| \log_G(\mu^{-1}X) \right\|_{G^{-1}}^2\right)
\]

where \( \alpha \in \mathbb{R} \) is a normalizing factor to ensure \( \int p(X)dx = 1 \).

From (9), \( \epsilon = \log_G(\mu^{-1}X) \) and assuming that \( P \) has small eigenvalues then \( p(\exp_G(\epsilon)) \) concentrates around the group identity. With those working assumptions, the distribution of \( \epsilon \) in the Lie algebra becomes the classical Gaussian distribution, i.e., \( \epsilon \sim \mathcal{N}(0, P) \). The distribution of \( X \) is called a CGD on \( G \), denoted by \( X \sim \mathcal{N}(\mu, P) \). An illustration of the relationship between the neighborhood of the identity element with the Lie Algebra is depicted in Fig. 2 together with the tangent space and its respective Gaussian distribution.

C. Dynamic System

Once an r.v. is defined on a Lie group, a stochastic dynamic system can be modeled such that its states are embedded on a matrix Lie group \( G \). Let \( X \in G \) be the system state. Let the stochastic differential equation be expressed by

\[
dX = X[\Omega(X, u)dt + dW]_G
\]

where \( \Omega : G \times \mathbb{R}^m \to \mathbb{R}^p \) is the left velocity function and \( W \) is a multidimensional Wiener process with covariance \( Q \), i.e., \( W(t_2) - W(t_1) \sim \mathcal{N}(0, (t_2 - t_1)Q) \) with \( t_1 < t_2 \). Also, consider that measurements are available in discrete time instants in the form

\[
y_{k+1} = h(X_k) + v_k
\]

where \( y_k \in \mathbb{R}^m \) is the measurement vector, \( h : G \to \mathbb{R}^m \) is the measurement function, and \( v_k \sim \mathcal{N}(0, R_k) \) is the measurement noise. For small sample time \( \Delta t \), the discrete form of the dynamic system (11) can be approximated as

\[
X_{k+1} = X_k \exp_G(\Omega(X_k, u_k)\Delta t + w_k)
\]

where \( w_k \sim \mathcal{N}(0, Q_k) \) and \( Q_k = Q_\Delta t \).

D. Kalman Filter on Lie Groups

With the definition of r.v.s and stochastic dynamic systems on Lie groups, one can employ the Kalman filtering framework to generate state estimates of a dynamic system state evolving on a Lie group. For convenience, denote \( \hat{\Omega}_k := \Omega(\hat{X}_k, u_k)\Delta t \). The D-EKF on a Lie group [19], [20] is presented next.

**LEMMA II.1 (D-LIE-EKF):** The following equations constitute the D-EKF on a Lie group:

\[
\hat{X}_{k+1} = \hat{X}_{k|k} \exp_G(\hat{\Omega}_k)
\]

\[
P_{k+1|k} = \mathcal{F} P_{k|k} \mathcal{F}^\top + J_r(\hat{\Omega}_k) Q_k J_r(\hat{\Omega}_k)^\top
\]

![Fig. 2. Concentrated Gaussian distribution in the neighborhood of the identity element. Notice the curved shape of the distribution on the Lie group.](image-url)
\[ K = P_{k+1|k} \mathcal{H}^T (R_{k+1} + \mathcal{H} P_{k+1|k} \mathcal{H}^T)^{-1} \]  
(14c)

\[ \dot{X}_{k+1|k+1} = \dot{X}_{k+1|k} \exp_{G}^{c} (K (\dot{y}_{k+1} - h (\dot{X}_{k+1|k}))) \]  
(14d)

\[ P_{k+1|k+1} = (I - K \mathcal{H}) P_{k+1|k} \mathcal{H}^T + KR_k + K^T \]  
(14e)

where

\[ \mathcal{F} := \text{Ad}_G (\exp_{G}^{c} (- \dot{\Omega}_k)) + J_e (\dot{\Omega}_k) \mathcal{G}_k \]  
(14f)

\[ \mathcal{G}_k := \frac{\partial}{\partial \epsilon} \left[ \Omega (\dot{X}_{k+1|k} \exp_{G}^{c} (\epsilon)) \right] \bigg|_{\epsilon = 0} \]  
(14g)

\[ \mathcal{H} := \frac{\partial}{\partial \epsilon} \left[ h (\dot{X}_{k+1|k} \exp_{G}^{c} (\epsilon)) \right] \bigg|_{\epsilon = 0}. \]  
(14h)

**Proof:** See Appendix A. \( \square \)

### E. Smoothing on Lie Groups

This section presents the RTS smoother on Lie groups with the D-LIE-EKF built in, as proposed in [23]. Unlike a Kalman recursive filter, the smoother uses all available measurements to compute the state estimates using a forward pass, given by the D-LIE-EKF, followed by a backward pass [32]. Thus, the RTS applies offline in a postprocessing amelioration.

**Lemma II.2 (D-LIE-RTS):** Given the filter solution \( \{ \dot{X}_{k|k}, P_{k|k} \}_{1:T} \), the Rauch–Tung–Striebel recursion on Lie groups for \( k = T, \ldots, 1 \) are

\[ \dot{X}_{k+1|k} = \dot{X}_{k|k} \exp_{G}^{c} (\dot{\Omega}_k) \]  
(15a)

\[ \dot{P}_{k+1|k} = \mathcal{F} P_{k|k} \mathcal{F}^T + J_e (\dot{\Omega}_k) Q J_e (\dot{\Omega}_k)^T \]  
(15b)

\[ G_k = P_{k|k} \mathcal{F}^T P_{k+1|k}^{-1}, \]  
(15c)

\[ \dot{X}_{k+1|k} = \dot{X}_{k|k} \exp_{G}^{c} (G_k \log_{G}^{c} (\dot{X}_{k+1|k}^{-1} X_{k+1|k}^*) )) \]  
(15d)

\[ \dot{P}_{k+1|k} = P_{k|k} + G_k (P_{k+1|k} - P_{k+1|k}) G_k^T. \]  
(15e)

**Proof:** See Appendix B. \( \square \)

**Remark** The D-LIE-EKF presented in close correspondence with the version presented in [19], [20], and [23]. However, we adopt the Joseph form in (44) in the Appendix A, for better numeric stability. In addition, the D-LIE-RTS smoother is derived using the left-error definition instead of the right-error definition presented originally in [23].

### III. INERTIAL NAVIGATION SYSTEM

The section presents the PVA kinematic navigation equations in continuous time [9], [33], and the IMU measurement model.

#### A. Navigation Equations

They are given as

\[ \dot{C}_b^c = C_b^c [\omega_{ib}^b]^x \left[ \omega_{ie}^e \right]^x C_b^c, \]  
(16a)

\[ \dot{p}_{eb}^c = \nu_{eb}^c, \]  
(16b)

\[ \nu_{eb}^c = C_b^e f_{ib}^b - 2 \left[ \omega_{ie}^e \right]^x \nu_{eb}^c + g^e \]  
(16c)

where the position \( p_{eb}^c = [x_{eb}^c, y_{eb}^c, z_{eb}^c]^T \) is coordinated in the ECEF frame, and

\[ \omega_{ie}^e = [\omega_e \cos(L) 0 - \omega_e \sin(L)]^T \]

where \( \omega_e \) is the earth rotation rate, and \( L \) is the latitude; \( \omega_{ib}^b, f_{ib}^b \in \mathbb{R}^3 \) are the angular velocity and specific force, respectively. The gravity may be obtained through a gravity model. This article adopts the model presented in [34] for simplicity.

**Remark** The GNSS navigation solution is given in the ECEF frame. The INS kinematic model is also defined in the ECEF, so the GNSS measurements are used to update the trajectory with no coordinate transformation. If a navigation solution is required in the local frame coordinates, one can quickly transform from ECEF to NED coordinates (see [9]).

#### B. IMU Measurement Model

As pointed out in [35], the INS kinematic model in (16) is exact since there is no model error or uncertainty. Hence, the uncertainty in navigation problems comes from the sensors and the local gravity anomalies.

Gyrosopes and accelerometers are subject to errors that limit the accuracy at which angular rotations or specific forces are measured. Thus, sensor models are essential for the filter and smoothing navigation to achieve reliable results. In particular, we consider the following inertial sensor model:

\[ \ddot{\alpha}_{ib}^b = \alpha_{ib}^b + b_a + \epsilon_a \]  
(17a)

\[ \ddot{f}_{ib}^b = f_{ib}^b + b_a + \epsilon_a \]  
(17b)

where \( \epsilon_a \sim \mathcal{N} (0, \sigma_a^2) \) and \( \epsilon_a \sim \mathcal{N} (0, \sigma_a^2) \) are white noise and are related to the angular random walk (ARW) and velocity random walk parameters of the IMU. In addition, \( b_a, b_a \) are the gyroscope and accelerometer biases, respectively. The biases are modeled as random walks processes in the form

\[ \dot{b}_a = B_a dW_g \]  
(18a)

\[ \dot{b}_a = B_a dW_a \]  
(18b)

where \( W_g \) and \( W_a \) are Wiener processes of appropriate dimensions and \( B_g \) and \( B_a \) are diffusion matrices associated with the IMU’s bias instability.

**Remark** Note that in (17) \( \ddot{f}_{ib}^b \) and \( \ddot{\alpha}_{ib}^b \) are IMU’s noisy values of the specific force and angular velocity, respectively. Note also that the biases are expressed in the body frame.

### IV. INTEGRATION GNSS/INS

#### A. Modeling Navigation and INS Measurements

This work employs the double direct isometries Lie group SE2(3) [22], for embedding the kinematic states in a compound with a translation group T(6), which accommodates both accelerometer and gyroscope biases. The
resulting group structure $G = SE_2(3) \times T(6)$ is

$$X = \begin{bmatrix} C^e_b & v^e & p^e \\ 0_{1 \times 3} & 1 & 0 \\ 0_{1 \times 3} & 0 & 1 \\ 0_{7 \times 5} & b \end{bmatrix}_{12 \times 12}$$

(19)

where $b = \begin{bmatrix} b_a \\ b_g \end{bmatrix} \in \mathbb{R}^6$.

**Lemma IV.1** The velocity function $\Omega : G \times \mathbb{R}^6 \rightarrow \mathbb{R}^{15}$ associated with the navigation equations (16) embedded into the Lie group from (19) is given by

$$\Omega (X, u) = \begin{bmatrix} \tilde{\omega}^h_{ib} - b_g - \omega^h_{ib} \\ \tilde{f}^h_{ib} - b_a - 2C^b_e [\omega^e_e]^x \nu^e_{eb} + C^b_g \nu^g_{eb} \\ C^e_v^b \nu^e_{eb} \\ 0 \\ 0 \end{bmatrix}$$

(20)

where $u = \begin{bmatrix} \tilde{f}^h_{ib} \\ \tilde{\omega}^h_{ib} \end{bmatrix} \in \mathbb{R}^6$ is the IMU’s noisy input measurement. In addition, the process noise is given by

$$dW = \begin{bmatrix} -\sigma_a dw_1 \\ -\sigma_g dw_2 \\ 0 \\ B_a dw_a \\ B_g dw_g \end{bmatrix} = \Gamma d\tilde{W}$$

(21)

where $d\tilde{W} = [dw_1^T dw_2^T dw_a^T dw_g^T]^T$ such that $\tilde{W}$ is a standard 12-dim Wiener process.

**Proof** From (16) and (19), one gets

$$X^{-1} dX = \begin{bmatrix} C^e_b dC^e_b & C^e_b d\nu^e_{eb} & C^e_b d p^e_{ib} \\ 0 & 0 & 0 \\ 0_{7 \times 5} & b \end{bmatrix}_{12 \times 12}$$

$$= [\Omega (X, u) dt + dW]_{G}$$

which implies (20) and (21).

**B. GNSS Measurement Model**

The rover GNSS antenna is rigidly fixed relative to the IMU, and the lever arm from the IMU to the GNSS antenna phase center is known, expressed as $I^b$ in the body frame. The GNSS position measurements $p^{GNSS} = [x y z]^T$ are modeled as

$$p^{GNSS} = p^e_{eb} + C^b_g l^b + \nu$$

(22)

where $\nu \sim \mathcal{N}(0, R)$ stands for the GNSS noise that is uncorrelated white noise with covariance $R = \text{diag}(\sigma_x^2, \sigma_y^2, \sigma_z^2)$.

**C. Outlier Rejection**

Outliers are spurious data that differ dramatically from the statistical distribution, leading to erroneous behavior of the filtering algorithm. When adopting GNSS measurements, outliers are inevitably present due to satellite blockage, multipath noise, or cycle slip. To make the proposed algorithm robust against GNSS outliers, the Mahalanobis distance combined with the $\chi^2$-test is employed to define a rejection scheme.

Define the squared normalized residue (SNR) as

$$\zeta_k := \| y_k - \hat{y}_k \|_{\Sigma_k}^2$$

(23)

where $\Sigma_k = R_k + \mathcal{H}_k P_{k+1|k} \mathcal{H}_k^T$, see (14). Assuming that the SNR obeys a chi-square distribution with 3 degrees of freedom, i.e., $\zeta_k \sim \chi^2_3$, a valid GNSS is declared if $\zeta_k \leq \kappa$ where $\kappa$ is a threshold level. Fig. 3 illustrate the region of valid GNSS (green shadow) where the SNR is less than the specified threshold.

However, instead of completely discarding a possible outlier when the threshold is exceeded, another approach, similar to [36], is to mitigate the influence of the respective GNSS measurement. We choose to weight the filter innovation with the following factor:

$$\gamma = \min \left(1, \frac{\kappa}{\zeta} \right).$$

(24)

Accordingly, the update equation (14d) becomes

$$\hat{X}_{k+1|k+1} := \hat{X}_{k+1|k} \exp_G (\gamma K (y_{k+1} - h (\hat{X}_{k+1|k})))$$

(25)

Note that when $\gamma = 1$, the filter is unaltered ($\zeta \leq \kappa$). But if $\zeta > \kappa$, $\gamma$ becomes less than one, scaling down the filter innovation impact on the estimate $\hat{X}_{k+1|k+1}$.

**D. Alignment and Initial Conditions**

A critical factor for achieving accurate navigation is the initialization of the inertial navigation system. Before the take-off, the INS is immobile; the initial velocity is null. The initial position can be obtained with satisfactory accuracy by averaging GNSS measurements in a sufficiently long time window. Likewise, the initial values for the gyroscope biases can be estimated using the average of its measurements during the immobile period.

Moreover, the leveling method [9] can be used to obtain the initial pitch ($\theta_0$) and roll ($\phi_0$) angles as well. The principle of leveling is that when the INS is immobile,
the accelerometer triad only detects gravity acceleration. Hence, the initial pitch and roll angles can be obtained as

$$\theta_0 = \arctan \left( \frac{\tilde{f}_{ib,x}}{\sqrt{\left(\tilde{f}_{ib,y}\right)^2 + \left(\tilde{f}_{ib,z}\right)^2}} \right)$$  \hspace{1cm} (26a)

$$\phi_0 = \arctan_2\left(-\frac{\tilde{f}_{ib,y}}{\tilde{f}_{ib,z}}\right)$$  \hspace{1cm} (26b)

where $\tilde{f}_{ib,x}, \tilde{f}_{ib,y}, \tilde{f}_{ib,z}$ are the average accelerometer output during a time window. Note that the four-quadrant arc tangent function should be used for roll. The accuracy of (26) is determined by the accelerometer biases [33].

Regarding the initial heading alignment, the gyrocompassing method or a magnetometer compass could be used for the initial heading. However, an accurate heading initialization requires expensive navigation-grade gyroscope sensors capable of measuring the earth's rotation rate, whereas magnetometers usually do not attain the required precision, apart from the mentioned hindrances for their use. As a result, it is mandatory to conceive a method relying solely on low-cost MEMs for a satisfactory initialization of the heading value. We propose a Bayesian optimization method to this end, described in the next section.

E. Postprocessed Heading Alignment

To provide a reliable initialization of the heading angle ($\psi_0$) using only low-cost MEM sensors, the Bayesian parametric estimation scheme described in [37, cap.12] is adapted to our scenario. The method applies when all data are collected after the drone flight, represented generically by the data $y_{1:N}$. It consists of evaluating the posterior distribution $p(\psi_0|y_{1:N})$ and taking the most likely value for the initial heading $\psi_0$.

For this purpose, note that

$$p(\psi_0|y_{1:N}) \propto p(y_{1:N}|\psi_0)p(\psi_0)$$  \hspace{1cm} (27)

where $p(\psi_0)$ is some prior distribution. For simplicity, we consider $p(\psi_0) = \mathcal{N}(0, \sigma_{\psi}^2)$, namely, $\psi_0 = 0$ is the a priori heading reference. Moreover, $p(y_{1:N}|\psi_0)$ can be factored in the form

$$p(y_{1:N}|\psi_0) = \prod_{k=1}^{N} p(y_k|y_{1:k-1}, \psi_0).$$  \hspace{1cm} (28)

We assume that the marginal measurement distribution $p(y_k|y_{1:k-1}, \psi_0)$ is a Gaussian distribution of form

$$p(y_k|y_{1:k-1}, \psi_0) = \mathcal{N}\left(\tilde{X}_{k|k-1}^{\psi_0}, R_k + \mathcal{H}_k P_{k|k-1}^{\psi_0} \mathcal{H}_k^T\right)$$  \hspace{1cm} (29)

where $\tilde{X}_{k|k-1}^{\psi_0}$ and $P_{k|k-1}^{\psi_0}$ come from the filtering solution with some fixed $\psi_0$ value. Accordingly, the most likely value for the initial heading can be found by solving $\min_{\psi_0} - \log(p(\psi_0|y_{1:N}))$.

Let $\varphi(\psi_0) = -\log(p(\psi_0|y_{1:N}))$ then one has

$$\varphi(\psi_0) = \sum_{k=1}^{N} -\log(p(y_k|y_{1:k-1}, \psi_0)) - \log(p(\psi_0))$$

Fig. 4 shows an example of the proposed optimization-based heading alignment using $-30^\circ$, $0^\circ$, and $30^\circ$ guesses for the initial heading. The blue curve indicates the log-likelihood function, and the red curve is the proposed approximation.

$$= \sum_{k=1}^{N} \frac{1}{2} \log(2\pi |S_k^{\psi_0}|) + \frac{1}{2} \|z_k^\psi - \bar{X}_{k|k-1}^{\psi_0}\|^2 (S_k^{\psi_0})^{-1} + \frac{1}{2\sigma_{\psi}^2} \|\psi_0\|^2$$  \hspace{1cm} (30)

with $S_k^{\psi_0} = R_k + \mathcal{H}_k P_{k|k-1}^{\psi_0} \mathcal{H}_k^T$ and $z_k^\psi = y_k - h(\bar{X}_{k|k-1}^{\psi_0})$.

**Remark** Note that, for each value of $\psi_0$, (30) provides the respective log-likelihood up to a constant as $p(\psi_0|y_{1:N}) \propto \exp(-\varphi(\psi_0))$. Thus, for a sufficient number of evaluations of different values for $\psi_0$, one can build the distribution $p(\psi_0|y_{1:N})$ using the filtering solution.

Also, we propose the following approximation $p(\psi_0|y_{1:N}) \approx N\left(\hat{\psi}_0^*, \sigma_{\hat{\psi}_0}^2\right)$. This implies that the log-likelihood (30) is approximated quadratic w.r.t $\psi_0$; hence, one can obtain the approximated log-likelihood function by evaluating three distinct points.

In summary, the proposed heading alignment scheme consists of three independent D-LIE-EKF evaluations, each with different heading values $[\psi_1, \psi_2, \psi_3]$. After these runnings, three samples from the log-likelihood are obtained. Thereafter, a parabola $c = m_3 \psi_3^2 + m_2 \psi_2 + m_1$ is fitted to the samples solving $Am = c$ for $m$ where

$$A = \begin{bmatrix} \psi_1^2 & \psi_1 & 1 \\ \psi_2^2 & \psi_2 & 1 \\ \psi_3^2 & \psi_3 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad m = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$  \hspace{1cm} (31)

The best estimate of the initial heading is then chosen as the parabola minimum value, $\hat{\psi}_0^* := -m_2/2m_1$.

Fig. 4 shows an example of the proposed optimization-based heading alignment with $\psi_1 = -30^\circ$, $\psi_2 = 0^\circ$ and $\psi_3 = 30^\circ$ applied to a real dataset. For this case, the best initial heading was $\hat{\psi}_0^* = 3.72^\circ$. The actual log-likelihood (blue curve) was computed by a grid of values for $\psi_0$ from $-60^\circ$ to $+60^\circ$ with $5^\circ$ increment. Notice that the blue curve is close to the red curve (quadratic approximation) near its minimum. In Section V-C, a numeric experiment using
synthetic data illustrates the performance of the proposed heading alignment strategy.

F. GNSS/INS Integration Scheme

The proposed scheme for LC GNSS/INS integration using Lie group consists of four steps. First, the accelerometer data from the IMU during an initial stationary period is used to perform leveling and obtain initial values for pitch and roll as in Section IV-D. Then, the heading alignment method described in Section IV-E is applied to obtain the initial heading value. Next, a fourth pass of the D-LIE-EKF algorithm is performed to obtain a filtered solution with \( \psi^b_0 \) as the initial heading. Finally, the D-LIE-EKF-RTS smoother (or D-LIE-EKS for short) generates the final output. Fig. 5 illustrates these four steps.

V. DATA EXPERIMENTS

A. Settings

This work was especially motivated by the drone-borne DinSAR application described in [1] and [38]. It requires high-fidelity PVA information to provide reliable interferometric results (see Table I). The drone with the radar system is shown in Fig. 6. It consists of a DJI Matrice 600-Pro equipped with a radar system for remote sensing. An INS is mounted independent from the native onboard navigation system, exclusively to provide PVA information for the radar system, consisting of a 6-DOF IMU ADIS16495 from Analog Devices. The IMU is rigidly mounted on the radar antenna so that the attitude measurement from the IMU can be easily transformed into the radar’s orientation information.

In complement, the u-blox ZED F9P GNSS system is installed to provide raw code, and post-processed phase measurements using the open-source package RTKlib [39].

B. Comparisons With Synthetic Data

The objective here is to get realistic and “noiseless” trajectories, such as those produced by one elaborate flight simulator, for performance comparison among Euler, quaternions, and Lie filtering and smoothing schemes. This precedes actual field tests, presented in Section V-E. The starting point is to get a real dataset from drone trajectories, as illustrated in Fig. 7, which is then processed with centimeter-level accuracy. Then, inverse strap-down mechanization similar to [40] but adapted to the Lie group model is implemented to emulate perfect measurements.

More specifically, let \( \{ C^b_2(k), p^f(k), v^r(k) \}_{k=1:N} \) be the reference trajectory generated with a sample rate \( f_s = 1/\Delta t \). For each time instant \( k \), an SE\(_2\)(3) element is built to represent the system state in the form

\[
S_k = \begin{bmatrix}
C^b_2(k) & v^r(k) & p^f(k) \\
0 & I & \end{bmatrix}.
\]  

(32)

Thereafter, the left-velocity vector is computed using

\[
\Omega = \frac{\log_S(S_t^{-1}S_{t+1})}{\Delta t} = \begin{bmatrix}
\Omega_\omega \\
\Omega_f
\end{bmatrix}.
\]  

(33)

From (20), one obtains the angular velocity and the specific force as follows:

\[
\begin{align*}
\omega^b_{ib}(k) &= \Omega_\omega + \Omega_f e \\
f^b_{ib}(k) &= \Omega_f + 2C^b_f(k)[\omega^b_\omega(e)]^x v^r_{ib}(k) - C^b_p(k)g^r
\end{align*}
\]

From such sequences, one can integrate back using zero-order hold or another rule to get time continuous input and state representations, yielding a ground truth trajectory for performance comparisons.

REMARK We choose to generate \( \{ C^b_2(k), p^f(k), v^r(k) \}_{k=1:N} \) with the commercial software Inertial Explorer for GNSS/INS integration, for the sake of independence, but we could choose to process with the proposed scheme equally well.

Finally, perturbations reflecting characteristic errors of actual sensors should be introduced to the emulated measurements, namely, IMU artificial noises are added to form
the simulated IMU measurements
\[ \tilde{\omega}_b^a(k) = \omega^a(k) + \omega^b(k) + N_w^a(k) \]
\[ \tilde{f}_b^a(k) = f^a(k) + \omega^b(k) + N_w^a(k) \]
where \( w^a, w^b \sim \mathcal{N}(0, I) \) and \( N^a, N^b \) are the VRW and ARW parameters of the emulated IMU. Besides, \( b_a, b_g \) are the accelerometer and gyroscope biases simulated using Ornstein-Uhlenbeck processes as follows:
\[ b_a(k+1) = b_a(k) + \tau_a(\beta_a - b_a(k))\Delta t + B_a\sqrt{\Delta t} \epsilon_w(k) \]
\[ b_g(k+1) = b_g(k) + \tau_g(\beta_g - b_g(k))\Delta t + B_g\sqrt{\Delta t} \epsilon_{w^g}(k) \]
where \( \tau_a, \tau_g \) are the biases’ correlation time, \( \beta_a, \beta_g \) are the turn-on constant biases, \( \epsilon_{w^a}, \epsilon_{w^g} \sim \mathcal{N}(0, I) \) and \( B_a, B_g \) are parameters to influence the bias instability. Unlike the random walk process, the Ornstein-Uhlenbeck process is a mean-reverting process, better fitting the IMU biases behavior.

The GNSS data is rendered by down-sampling the reference trajectory to 1 Hz and adding the lever-arm component together with a 3-dimensional white noise as follows:
\[ y(n) = p^r(nF_t) + C_g(nF_t)\epsilon(n) + \epsilon(n) \]  \hspace{1cm} (34)
for \( n = 0, 1, \ldots, \lfloor N/F_t \rfloor, \epsilon \sim \mathcal{N}(0, R) \).

The parameters used in all simulations were chosen to match the ADIS16495 data sheet and the centimeter-level precision of RTK-GNSS and are given in Table II. The covariance matrix \( P_0 \) is formed as a diagonal matrix whose individual elements are chosen according to each state’s expected 99.7\% confidence interval (three standard deviations), as follows:
\[ P_0 = \text{diag}(\left(\frac{1}{3}\right)^2, \left(\frac{1}{3}\right)^2, \left(\frac{5}{3}\right)^2) \]

### Table II

| Parameters & Values | \( N_a \) | \( N_g \) | \( \tau_a \) |
|--------------------|------------|------------|-------------|
| \quad \quad \quad | \quad \quad \quad | \quad \quad \quad | \quad \quad \quad |
| \quad \quad \quad | \quad \quad \quad | \quad \quad \quad | \quad \quad \quad |
| \quad \quad \quad | \quad \quad \quad | \quad \quad \quad | \quad \quad \quad |

C. Heading Alignment Performance

To analyze the accuracy of the heading alignment method described in Section IV-E we perform some experiments with synthetic data using different initial heading values. The optimization-based method is applied in each experiment, and the best estimate of the initial heading is obtained.

The experiments support that the proposed heading alignment can achieve an error less than 2° as required in Table I, when the true heading is between \(-20^\circ\) and \(+20^\circ\) as shown in Fig. 8a.

D. Navigation Performance

Three flight scenarios were simulated to evaluate the proposed processing scheme’s performance: helicoidal,
TABLE III
RMSE Comparison for 100 Monte Carlo Simulations Using Filter Only

|      | Rectangular | Circular | Helicoidal |
|------|-------------|----------|------------|
| Roll | D-LIE-EKF   | MEKF     | EULER-EKF  |
| Pitch| D-LIE-EKF   | MEKF     | EULER-EKF  |
| Heading| D-LIE-EKF| MEKF      | EULER-EKF  |
| Longitude| D-LIE-EKF| MEKF      | EULER-EKF  |
| Latitude| D-LIE-EKF | MEKF     | EULER-EKF  |
| Altitude| D-LIE-EKF | MEKF     | EULER-EKF  |

TABLE IV
RMSE Comparison for 100 Monte Carlo Simulations Using Filter and RTS Smoother

|      | Rectangular | Circular | Helicoidal |
|------|-------------|----------|------------|
| Roll | D-LIE-EKS   | MEKS     | EULER-EKS  |
| Pitch| D-LIE-EKS   | MEKS     | EULER-EKS  |
| Heading| D-LIE-EKS| MEKS      | EULER-EKS  |
| Longitude| D-LIE-EKS| MEKS      | EULER-EKS  |
| Latitude| D-LIE-EKS | MEKS     | EULER-EKS  |
| Altitude| D-LIE-EKS | MEKS     | EULER-EKS  |

rectangular, and circular. These profiles are shown in Fig. 7.

For comparisons, for each scenario, the following algorithms were implemented: Lie group-based filter (D-LIE-EKF) inbuilt into the smoother (D-LIE-EKS); multiplicative quaternion-based filter (MEKF) and smoother (MEKS); Euler-based filter (EULER-EKF), and smoother (EULER-EKS).

Thereafter, 100 Monte Carlo realizations were performed, and each algorithm’s respective rmse was computed. All algorithms are initialized with the same parameters for a fair comparison. Table III summarizes the performance for each online processing phase using filtering only. Table IV shows the off-line performance after applying the respective smoother algorithm.

Note that using the Lie group-based algorithm the heading error is lower for the three scenarios in rmse terms. The D-LIE-EKS shows an overall performance gain of about 40% over the MEKS for both helicoidal and circular flight profiles and about 9% for the rectangular profile. This result indicates the superiority of the Lie group approach, which becomes more noticeable for curved trajectories.

E. In Field Drone-Borne DinSAR Performance

This section reports the performance results obtained from a complete experiment of image reconstruction with the drone-borne DInSAR system described in [1]. The digital surface model (DSM) is obtained using the cross-track interferometry information provided by the two C-band antennae and applied in the DInSAR calculation. The controlled experiment was conducted using three trihedral corner reflectors with square sides and edge lengths of 0.6 m to serve as a ground reference. Fig. 9 illustrates the drone-borne DinSAR geometry.

For comparisons, we test the results against commercially produced navigation software specialized for IMU-GNSS integration, the NovAtel Inertial Explorer (IE), using its offline processing mode.

After the flights, the data are processed in two steps for the Lie group-based processing. First, the ground station and rover GNSS receivers are processed using RTKlib [39] to generate centimeter-accuracy position information at 1 Hz. Second, the Lie group filter-smoother algorithm generates the final position, velocity, and attitude solution by

1) The GNSS ground station is placed close to the starting position of the drone, and the GNSS recording is initiated.
2) Then, three flights are carried out, each consisting of the following successive steps: 1) turning on the drone and the radar and waiting 15 min for simultaneous and stationary recording of the ground station and drone GNSS data; 2) taking off and executing the same west-east flight track; and 3) landing and waiting 15 min for simultaneous and stationary recording of the ground station and radar GNSS data. Turning OFF.
3) The GNSS ground station and drone are dismounted, and the acquired data is downloaded for postprocessing.
combining the GNSS information with the 200 Hz IMU measurements. We refer to this solution as D-LIE-EKS.

The GNSS and IMU raw data for the IE processing is fed directly to the software. Although the proposed scheme is LC, we compare it with IE’s loosely coupled (IE-LC) and tightly coupled (IE-TC) solutions.

Now, with the navigation information at hand, the raw radar data are processed on the imaging module, recording each resulting single-look complex (SLC) image. After that, the interferometry is performed, yielding the interferogram, topography subtraction, and phase-to-height conversion.

The output consists of two deformation maps plus the three SLC images, all in slant range geometry. Each interferogram is calculated with 0.047 m resolution in azimuth and 1.228 m resolution in the slant range.

From the close inspection of the reflector’s positioning, we can not detect any noticeable advantages from one or the other procedure, even in the IE-TC mode. After these evaluations, we attempted to spot subtle differences by subtracting images of subsequent flights and looking for terrain inconsistencies.

Fig. 10 shows the relative error of subsequent flights for the three different navigation processing. Notice that the resulting pattern is almost identical for all algorithms.

Now, with the navigation information at hand, the raw radar data are processed on the imaging module, recording each resulting single-look complex (SLC) image. After that, the interferometry is performed, yielding the interferogram, topography subtraction, and phase-to-height conversion.

The output consists of two deformation maps plus the three SLC images, all in slant range geometry. Each interferogram is calculated with 0.047 m resolution in azimuth and 1.228 m resolution in the slant range.

From the close inspection of the reflector’s positioning, we can not detect any noticeable advantages from one or the other procedure, even in the IE-TC mode. After these evaluations, we attempted to spot subtle differences by subtracting images of subsequent flights and looking for terrain inconsistencies.

Fig. 10 shows the relative error of subsequent flights for the three different navigation processing. Notice that the resulting pattern is almost identical for all algorithms.

Table V summarizes the overall error of the three algorithms. We observe that the proposed D-LIE-EKS scheme outperforms both IE-LC and IE-TC in these experiments in terms of rmse.

VI. CONCLUSION

A LC GNSS/INS integration scheme was developed based on Lie group theory, tailored for postprocessing applications seeking the highest accuracy. It incorporates the online Kalman filter inbuilt into the offline RTS smoother to provide estimates of PVA to this goal. The double direct isometry group \( \text{SE}(3) \) compound with the translation group \( \text{T}(6) \) was adopted to model the kinematic and cope with biased navigation system measurements.
The article includes a Bayesian adjustment devised to tackle the heading estimation problem, seeking full trajectory error mitigation. It also contains an outlier test conceived to deal with low-quality GNSS measurements. To our best knowledge, this is the first implementation of LC GNSS/INS integration tailored for postprocessing applications, combining Kalman filtering and RTS smoothing on Lie groups, applied to drone-borne remote sensing applications.

It includes a few numeric experiments based on synthetic data generated using inverse swap-down mechanization. They show that the Lie group approach consistently outperforms classical methods based on quaternion and Euler parametrization of the attitude matrix. The advantage of the Lie group is further highlighted when using heliocoidal and circular trajectories in which, due to the curvy paths, the proposed scheme attains better rms performance.

Furthermore, in a field comparison, the Lie group also exhibits superior performance and better accuracy than the navigation software Inertial Explorer in a DinSAR drone-borne application. Surprisingly, the simpler LC setting of the proposed technique presented superior performance than the IE’s more complex tightly coupled configuration. A favorable quality comparison with industry-standard software in this experiment indicates the novelty and the applicability of the Lie-based method devised here for low-cost and high-precision flight navigation reconstitution.

In a word, this work brings a complete cycle of engineering design, revealing that the Lie group theory of filtering and smoothing forms an appealing framework for high-quality aerial navigation. Finally, as an aside, these experiments point out that the air-borne radar scheme provides an excellent benchmark for low-cost INS evaluations. The scheme circumvents the need for an expensive navigation-grade INS unit in pairing tests, the usual form of calibration and evaluation of such devices.

APPENDIX

A. D-LIE-EKF Equations

PROOF: Lemma II.1. Let \( \hat{X}_t = \hat{X}_{t | k} \exp_G(e_{t | k}) \) with \( e_{t | k} \sim \mathcal{N}(0, P_{t | k}) \) and \( \hat{X}_{t | k} = \hat{X}_{t | k} \exp_G(\hat{\Omega}_t) \). Employing a first-order approximation for \( \hat{\Omega}_k \) around \( \hat{X}_{t | k} \), yields

\[
\hat{\Omega}_k = \Omega (\hat{X}_{t | k} \exp_G(e_{t | k})) \approx \Omega (\hat{X}_{t | k}) + \mathcal{O}(e_{t | k})
\]

where \( \mathcal{O}(e_{t | k}) := \frac{\partial}{\partial e_{t | k}} \left[ \Omega (\hat{X}_{t | k} \exp_G(e_{t | k})) \right] \mid_{e_{t | k} = 0} \).

From (39), one has

\[
e_{k+1 | t} = \hat{X}_{t+1 | k} - X_{k+1}
\]

\[
= \exp_G(G(\hat{\Omega}_k)) \exp_G(\hat{\Omega}_k) \exp_G(\hat{\Omega}_k + w_k)
\]

\[
= \exp_G(G(\hat{\Omega}_k)) \exp_G(\hat{\Omega}_k) \exp_G(\hat{\Omega}_k + e_{t | k} \exp_G(\hat{\Omega}_k))
\]

Assuming that \( e_{t | k} + w_k \) is small, then using the relationship from (40) and the fact that \( \exp_G(x) \exp_G(y) = \exp_G(G(x) + y) \) one gets

\[
e_{k+1 | t} = \exp_G(G(\hat{\Omega}_k)) + J, (\hat{\Omega}_k) w_k
\]

with \( \mathcal{F} = \text{Ad}_G (\exp_G(G(\hat{\Omega}_k))) + J, (\hat{\Omega}_k) w_k \). Therefore, we conclude that \( X_{k+1} \sim \mathcal{N}(\hat{X}_{k+1}, P_{k+1}) \) where

\[
P_{k+1} = \mathcal{F} P_{k | k} \mathcal{F}^T + J, (\hat{\Omega}_k) Q_j, (\hat{\Omega}_k) \mathcal{F}^T.
\]

Now, for the measurement-update step, consider \( X_{k+1} = \hat{X}_{k+1} + \exp_G(\epsilon_{k+1 | k}) \) where \( \epsilon_{k+1 | k} \sim \mathcal{N}(0, P_{k+1}) \). Note that the measurement distribution is \( p(y_{k+1} | X_{k+1}) = \mathcal{N}(y_{k+1}; h(X_{k+1}), R_{k+1}) \) and the prior state distribution is \( p(\hat{X}_{k+1} | y_{k+1}) = \mathcal{N}(\hat{X}_{k+1} | \hat{X}_{k+1}, P_{k+1}) \). Thus, from the Bayes’ rule, one has

\[
p(X_{k+1} | y_{k+1}) = \frac{p(y_{k+1} | X_{k+1}) p(X_{k+1} | y_{k+1})}{p(y_{k+1} | y_{k+1})}
\]

\[
= \exp \left( -\frac{1}{2} \| y_{k+1} - h(X_{k+1}) \|^2_{R_{k+1}^{-1}} - \frac{1}{2} \| \epsilon_{k+1 | k} \|^2_{P_{k+1}^{-1}} \right). \]

Let the posterior distribution be parametrized in the form

\[
p(X_{k+1} | y_{k+1}) = p \left( \hat{X}_{k+1 | k} \exp_G(v) | y_{k+1} \right).
\]

Therefore, the maximum a posteriori (MAP) estimate is \( \hat{X}_{k+1 | k} = \hat{X}_{k+1 | k} \exp_G(v) \) such that \( v^* = \arg \min \ell(v) \) where \( \ell(v) \) is the negative log-likelihood given by

\[
\ell(v) := \frac{1}{2} \| y_{k+1} - h(\hat{X}_{k+1 | k} \exp_G(v)) \|^2_{R_{k+1}^{-1}} + \frac{1}{2} \| v \|^2_{P_{k+1}^{-1}}.
\]

If a linear approximation is adopted, then one can obtain a set of filtering equations similar to the EKF. For this purpose, consider a first-order approximation of (12) in the form

\[
h(\hat{X}_{k+1 | k} \exp_G(v)) \approx h(\hat{X}_{k+1 | k}) + \mathcal{H} v
\]

\[
\text{with } \mathcal{H} = \frac{\partial}{\partial v} \left[ h(\hat{X}_{k+1 | k} \exp_G(v)) \right] \mid_{v=0}.
\]

\[
\ell(v) \approx \frac{1}{2} \| z_{k+1} - \mathcal{H} v \|^2_{R_{k+1}^{-1}} + \frac{1}{2} \| v \|^2_{P_{k+1}^{-1}}.
\]

where \( z_{k+1} := y_{k+1} - h(\hat{X}_{k+1 | k}) \) is the residual. The minimum of (41) is straightforward given by \( v^* = K z_{k+1} \), where \( K = (P_{k+1}^{-1} + \mathcal{H}^T R_{k+1}^{-1} \mathcal{H})^{-1} \mathcal{H} R_{k+1}^{-1} \mathcal{H}^T \) which also can be written as \( K = P_{k+1}^{-1} \mathcal{H} R_{k+1}^{-1} + \mathcal{H} P_{k+1}^{-1} \mathcal{H}^T \) \( K \). Thus

\[
\hat{X}_{k+1} | y_{k+1} = \hat{X}_{k+1 | k} \exp_G(K z_{k+1}),
\]

(42)

Let the state error be \( e_{k+1 | k+1} = \log^y_G(\hat{X}_{k+1 | k+1}) \). From (42), the posterior error follows:

\[
e_{k+1 | k+1} = \log^y_G\left( \hat{X}_{k+1 | k+1} \exp_G(K z_{k+1}) \right)
\]

\[
= \log^y_G(\exp_G(-K z_{k+1} \hat{X}_{k+1 | k+1} | X_{k+1}))
\]

\[
= \log^y_G(\exp_G(-K z_{k+1} \exp_G(\epsilon_{k+1 | k})))
\]

\[
= -K (y_{k+1} - h(\hat{X}_{k+1 | k}) + e_{k+1 | k})
\]

\[
= K (\mathcal{H} e_{k+1 | k} + v_{k+1}) + e_{k+1 | k}
\]

\[
= (I - K \mathcal{H}) e_{k+1 | k} + K R_{k+1} K^T.
\]

(43)

where a linear approximation \( z_{k+1} \approx \mathcal{H} e_{k+1 | k} + v_{k+1} \) is employed. Define \( P_{k+1} | k+1 := \mathbb{E}[\epsilon_{k+1 | k+1} \epsilon_{k+1 | k+1}^T] \). Assuming that \( \mathbb{E}[\epsilon_{k+1 | k+1} v_{k+1}^T] = 0 \), one gets

\[
P_{k+1 | k+1} = (I - K \mathcal{H}) P_{k+1 | k} (\mathcal{H}^T + K R_{k+1} K^T).
\]

(44)
B. Rauch-Tung-Striebel Smoother on Lie Groups

PROOF: Lemma II.2. Following the lines of [37], from a Bayesian perspective, the RTS smoother on Lie groups can be derived as the MAP estimate of the following joint pdf:

\[
p(X_k, X_{k+1}|y_{1:T}) = p(X_k|X_{k+1}, y_{1:T}) p(X_{k+1}|y_{1:T})
\]

\[
= \frac{p(X_k|X_{k+1}) p(X_{k+1}|y_{1:T})}{p(X_k|y_{1:T})} p(X_{k+1}|y_{1:T}).
\]

(45)

Assume that

\[
p(X_{k+1}, X_k|y_{1:k}) = \mathcal{N}_{G \times G}(\hat{X}_{k|k}, P_{k|k})
\]

\[
p(X_k|y_{1:k}) = \mathcal{N}(\hat{X}_{k|k}, P_{k|k})
\]

\[
p(X_{k+1}|y_{1:k}) = \mathcal{N}(X_{k+1}^\delta, P^\delta_{k+1})
\]

\[
p(X_{k+1}|y_{1:k}) = \mathcal{N}(\hat{X}_{k+1|k}, P_{k+1|k})
\]

(46a–d)

where \( P = \begin{bmatrix} P_{k|k} & P_{k|k+1} \end{bmatrix} \) and \( (\hat{X}_{k|k}, P_{k|k})_{k=1:N} \) comes from the D-LIE-EKF. Accordingly, (45) becomes

\[
p(X_k, X_{k+1}|y_{1:T}) \propto \exp\left(-\frac{1}{2} \left[ \begin{array}{c} \log_G(\hat{X}_{k|k}) \\ \log_G(X_{k+1|k}) \end{array} \right] \right)^2 P_{k|k+1}
\]

\[
+ \frac{1}{2} \left\| \log_G(\hat{X}_{k+1|k} X_{k+1}) \right\|^2 P_{k+1|k+1}
\]

\[
- \frac{1}{2} \left\| \log_G(X_{k+1}^\delta X_k) \right\|^2 P_{k|k+1}^{-1}.
\]

(47)

Parametrize \( X_k = \hat{X}_{k|k} \exp_G(\delta_k) \) and \( X_{k+1} = \hat{X}_{k+1|k} \exp_G(\delta_{k+1}) \). Thereafter, taking the negative logarithm of (47) yields

\[
\ell(\delta_k, \delta_{k+1}) = \frac{1}{2} \left\| \begin{array}{c} \log_G\left[ \exp_G(z_k) \exp_G(\delta_{k+1}) \right] \end{array} \right\|^2 P_{k|k+1}
\]

\[
- \frac{1}{2} \left\| \log_G(\exp_G(z_k) \exp_G(\delta_{k+1})) \right\|^2 P_{k|k+1}^{-1}
\]

\[
+ \frac{1}{2} \left\| \delta_{k+1} \right\|^2 (P_{k+1}^{-1})^{-1}.
\]

(48)

where \( z_k := \log_G(\hat{X}_{k+1|k} \hat{X}_{k+1}) \). Assuming that both \( z_k \) and \( \delta_{k+1} \) are small, one has

\[
\ell(\delta_k, \delta_{k+1}) \approx \frac{1}{2} \left\| \begin{array}{c} \delta_k \\ z_k + \delta_{k+1} \end{array} \right\|^2 P_{k|k+1}
\]

\[
- \frac{1}{2} \left\| \delta_{k+1} \right\|^2 (P_{k+1}^{-1})^{-1}
\]

\[
= \frac{1}{2} \left\| A \delta + b \right\|^2 Y
\]

(49)

where

\[
\delta := \begin{bmatrix} \delta_k \\ \delta_{k+1} \end{bmatrix}, \quad A := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad b := \begin{bmatrix} z_k \\ z_{k+1} \end{bmatrix}
\]

\[
\mathcal{W} = \text{blkdiag}(P^{-1}, -P^{-1}_{k+1|k}, (P^*_{k+1|k})^{-1})
\]

The optimal solution of (49) is straightforward given by

\[
\delta^* = (A^T \mathcal{W} A)^{-1} A^T \mathcal{W} b.
\]

(50)

Finally, the smoothed state estimate is \( \hat{X}_k^s = \hat{X}_{k|k} \exp_G(G_k z_k) \) and the final covariance is obtained from the first block diagonal of \( (A^T \mathcal{W} A)^{-1} \) (see [23]).

ACKNOWLEDGMENT

The authors would like to thank the Radaz Indústria e Comércio de Produtos Eletrônicos S.A. for supporting this work.

REFERENCES

[1] L. Moreira et al., “A drone-borne multiband DInSAR: Results and applications,” in Proc. IEEE Radar Conf., 2019, pp. 1–6.

[2] M. Nagai, T. Chen, R. Shibasaki, H. Kumagai, and A. Ahmed, “UAV-borne 3-D mapping system by multisensor integration,” IEEE Trans. Geosci. Remote, vol. 47, no. 3, pp. 701–708, Mar. 2009.

[3] J. N. Gross, Y. Gu, M. B. Rhudy, S. Gururajana, and M. R. Napolitano, “Flight-test evaluation of sensor fusion algorithms for attitude estimation,” IEEE Trans. Aerosp. Electron. Syst., vol. 48, no. 3, pp. 2128–2139, Jul. 2012.

[4] N. A. Chaturvedi, A. K. Sanyal, and N. H. McClamroch, “Rigid-body attitude control,” IEEE Control Syst. Mag., vol. 31, no. 3, pp. 50–51, Jun. 2011.

[5] J. Keigbocadi, M. Hosseini-Pishrobat, J. Faraji, and M. N. Langehagh, “Design and experimental evaluation of immersion and invariance observer for low-cost attitude-heading reference system,” IEEE Trans. Ind. Electron., vol. 67, no. 9, pp. 7871–7878, Sep. 2020.

[6] H. Nourmohammadi and J. Keigbocadi, “Fuzzy adaptive integration scheme for low-cost SINS/GPS navigation system,” Mech. Syst. Signal Process., vol. 99, pp. 434–449, 2018.

[7] J. A. Farrell, Aided Navigation Systems: GPS and High Rate Sensors. New York, NY, USA: McGraw-Hill, 2008.

[8] R. B. Langley, “RTK Gps,” GPS World, vol. 9, pp. 70–75, 1998.

[9] P. D. Groves, Principles of GNSS, Inertial, and Multisensor Integrated Navigation Systems, 2nd ed. Norwood, MA, USA: Artech House, 2013.

[10] J. L. Crassidis, “Sigma-point Kalman filtering for integrated GPS and inertial navigation,” IEEE Trans. Aerosp. Electron. Syst., vol. 42, no. 2, pp. 750–756, Apr. 2006.

[11] E. Edwan, “A new loosely coupled DCM based GPS/INS integration method,” Navigation-US, vol. 59, no. 2, pp. 93–106, 2012.

[12] Q. Zhang, X. Meng, S. Zhang, and Y. Wang, “Singualar value decomposition-based robust curvature Kalman filtering for an integrated GPS/SINS navigation system,” J. Navigation, vol. 68, no. 3, pp. 549–562, 2015.

[13] A. Barrau and S. Bonnabel, “Invariant Kalman filtering,” Annu. Rev. Control, Robot., Auton. Syst., vol. 1, pp. 237–257, 2018.

[14] E. J. Lefferts, F. L. Markley, and M. D. Shuster, “Kalman filtering for spacecraft attitude estimation,” J. Guid. Control Dyn., vol. 5, no. 5, pp. 417–429, 1982.
Marcos R. Fernandes was born in Apucarana, Brazil, in 1993. He received the B.S. degree in control and automation engineering from the Federal University of Technology – Paraná (UTFPR), Paraná, Brazil, in 2017, and the M.Sc. degree in electrical engineering in 2019 from the State University of Campinas, São Paulo, Brazil, where he is currently working toward the Ph.D. degree in estimation and filtering. His theoretical research interests include stochastic filtering, Bayesian estimation theory, inertial navigation, and GNSS processing.

Giorgio M. Magalhães was born in Fortaleza, Brazil, in 1993. He received the B.Sc. degree in electronics engineering from the Military Institute of Engineering, Rio de Janeiro, Brazil, in 2015 and the M.Sc. degree in electrical engineering from the State University of Campinas, Campinas, Brazil, in 2021.

Since 2016, he has been working in the industry, specializing in developing radar systems. His expertise encompasses digital signal processing, object detection, tracking, and classification, as well as sensor fusion. His research interests include applying Lie group concepts to radar signal processing, object tracking, and sensor fusion.

Yusef Rafael Cárdenas Zúñiga was born in Lima, Peru, in 1971. He received the B.Sc. degree in electronics engineering from the PUCP, San Miguel, Peru, in 1997 and the M.Sc. and Ph.D. degrees in electrical engineering from the State University of Campinas, Campinas, Brazil, in 2001 and 2007, respectively.

Since 2007, he has been working in the industry, focusing on developing multtarget-multisensor (MTMS) tracking for radar systems. His research interests include applying Lie group concepts to MTMS tracking, localization, and navigation.

João B. R. do Val was born in São Paulo State, Brazil, in 1955. He received the B.S. and M.S. degrees from the State University of Campinas, Sao Paulo, Brazil, in 1977 and 1981, respectively, and the Ph.D. degree from the Imperial College of Science and Technology in London, London, U.K., in 1985, all in electrical engineering. He also received the Diploma of Imperial College (DIC) in 1985.

In 1978, he joined the School of Electrical Engineering, State University of Campinas, where he is currently a Professor of Control Theory. He was a Visiting Scholar with the Decision and Control Laboratory (Coordinated Science Laboratory), University of Illinois at Urbana-Champaign and Institut d’Economie et de Management Université de Nantes, France. His research interests include stochastic systems and control, jump processes, filtering, radar tracking, and navigation.