Embedding Phenomenological Quark-Lepton Mass Matrices into SU(5) Gauge Models

M. Fukugita\textsuperscript{1,2}, M. Tanimoto\textsuperscript{3} and T. Yanagida\textsuperscript{4}

\textsuperscript{1} Institute for Cosmic Ray Research, University of Tokyo, Tanashi, Tokyo 188, Japan
\textsuperscript{2} Institute for Advanced Study, Princeton, NJ 08540, U. S. A.
\textsuperscript{3} Faculty of Education, Ehime University, Matsuyama 790-8577, Japan
\textsuperscript{4} Department of Physics and RESCEU, University of Tokyo, Tokyo 113-0033, Japan

Abstract

We construct phenomenological quark-lepton mass matrices based on $S_3$ permutation symmetry in a manner fully compatible with SU(5) grand unification. The Higgs particles we need are $5, 45$ and their conjugates. The model gives a charge $-1/3$ quark vs charged lepton mass relation, and also a good fit to mass-mixing relations for the quark sector, as well as an attractive mixing pattern for the lepton sector, explaining a large mixing angle between $\nu_\mu$ and $\nu_\tau$, and either large or small $\nu_e - \nu_\mu$ mixing angle, depending on the choice of couplings, consistent with the currently accepted solutions to the solar neutrino problem.
Predictions of the quark-lepton mass spectrum are the least successful aspect of the unified gauge theories. In classical SU(5) grand unification (GUT) with the simplest choice of Higgs scalars, we obtain the mass relations between the charge -1/3 quarks (referred simply to as down quarks) and charged leptons, $m_d = m_e$, $m_s = m_\mu$ and $m_b = m_\tau$. While the last of these relations agrees with experiment [1], the other two are far from the reality. Georgi and Jarlskog (GJ) [2] have shown that the mass degeneracy of $d/e$ and $s/\mu$ can be lifted by introducing 45-plet Higgs to give at the GUT scale

$$m_d = 3m_e, \quad m_s = (1/3)m_\mu, \quad m_b = m_\tau$$

(1)
in reasonable agreement with experiment. No prediction, however, has been given within the SU(5) framework to the charge 2/3 quark spectrum and hence to quark mixing which is usually described in terms of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. By extending the unifying group to SO(10), one may relate the charge 2/3 quark (referred to as up quark) mass to the Dirac mass of neutrinos, and also the quark mixing angles to the lepton mixing angles. The prediction, however, is too tight, and does not seem to be consistent with experiment (e.g. [3]), or else the Higgs content becomes very complicated [4].

There is an empirical approach successful in giving the quark mass-mixing relation at a phenomenological level. It usually assumes some ad hoc symmetry imposed on the mass matrices, as advocated first by Fritzsch [5]. Several simple representations of the quark mass matrices have been known that give mass-mixing relations in fair agreement with experiment [6]. This approach also successfully applies to understanding the relation between lepton mixing and neutrino mass [7] indicated by the solar neutrino problem [8] and by the atmospheric neutrino experiment [9]. In ref.[10] we have proposed specific quark-lepton mass matrices based on S(3)$_L \times$S(3)$_R$ symmetry, or so-called “democratic” principle [11], and small symmetry breaking terms, which
lead to attractive mixing patterns for neutrinos.

One of the problems with the "phenomenological matrix approach" is that the consistency with the unified gauge model is unclear. If one straightforwardly imposes the compatibility with a gauge model, on the other hand, we are usually led to unwanted relations for quark and lepton masses as remnants of prototype gauge models.

In this paper we show that there exists a successful matrix model based on the $S_3$ symmetry approach, which is fully compatible with SU(5) GUT's and at the same time gives predictions for quark and lepton masses and their mixings that agree with experiment [12]. This compatibility reduces the arbitrariness of the matrices, but also inspires us to modify them for the quark sector [13], which brings the predicted quark mixing in good agreement with experiment, especially for the (2,3) sector of flavour. For the neutrino sector all results presented in ref.[10] are retained.

We start with the observation that the phenomenological matrices given in [13] (for quarks) and [10] (for leptons) are compatible with the Yukawa couplings in the presence of a 5-plet Higgs of SU(5); an introduction of a 45-plet, which is necessary to lift unwanted mass degeneracy, requires only a minimum modification of the symmetry breaking matrix for the up quark sector. We write the Higgs coupling

$$L_{\text{Yukawa}} = Y(5_H)_{Uij} \mathbf{10}_i \mathbf{10}_j 5^*_H + Y(45_H)_{Uij} \mathbf{10}_i \mathbf{10}_j 45^*_H + \text{perturbations}$$

where bald face symbols with suffix $H$ denote Higgs scalars of a specified multiplet, and those with suffix $i$ or $j$ (refer to flavour) are SU(5) matter fields, $5_i = (d^c_1, d^c_2, d^c_3, e^-, \nu_e)_{Li}$ and $10_j = (u^c_1, \ldots, u^c_i, \ldots, d^c_i, \ldots, e^+)_{Lj}$. We write the down quark and charged lepton sectors as $D/E$ as they are unified. The last term of eq. (2) is an effective neutrino coupling where the neutrino is taken to be of the Majorana type. We suppose that the main part of the mass term arises from $5_H$; $45_H$ gives only perturbations.
We postulate that the main part of the mass matrices is $S_3$ permutation symmetry invariant, i.e., $S_3^{10} \times S_3^5$ in our context (10 and 5 refer to representations of fermions). The choice of the Yukawa coupling matrix $Y(5^*_H)_{D/Eij}$ is then unique:

$$Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{3}$$

Two matrices are allowed for $Y(5^*_H)_{Uij}$ from the invariance under $S_3^{10} \times S_3^5$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{4}$$

We take the 1:1 combination of the two to give form (3) for simplicity and for agreement with the democracy argument as in [11, 13]. We also assume for $\kappa(5_H^5_H)_{ij}$ the first matrix of (4) for the reason explained in [10].

We break $S_3^{10} \times S_3^5$ symmetry with an extra Yukawa coupling to $5^*_H$ and a coupling to $45^*_H$ for down-quark and charged-lepton sectors, and $5_H$ and $45_H$ for the up-quark sector. Namely, our mass matrices are

$$M_D = [Y(5^*_H)_D + aY(45^*_H)_D] \langle \phi_{5_H}^* \rangle,$$  

$$M_E = [Y(5^*_H)_D - 3aY(45^*_H)_D] \langle \phi_{5_H}^* \rangle,$$  

$$M_U = [Y(5_H)_U + bY(45_H)_U] \langle \phi_{5_H} \rangle,$$

where $a = \langle \phi_{45_H}^* \rangle_D / \langle \phi_{5_H}^* \rangle_D$ and $b = \langle \phi_{45_H} \rangle_U / \langle \phi_{5_H} \rangle_U$. Or, more explicitly, we write

$$M_D = \frac{K}{3} \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -\epsilon_D & 0 & 0 \\ 0 & \epsilon_D & 0 \\ 0 & 0 & \delta_D \end{bmatrix} \right) \tag{5'}$$

for the down-quark sector. At this level the symmetry breaking term of (5') may be $5^*$ or $45^*$. The Higgs leading to symmetry breaking is either new Higgs that develops a small vacuum expectation value or the same Higgs giving the main mass term but with small Yukawa couplings. We do not distinguish these two possibilities here.
For the charged lepton sector,

\[ M_E = \frac{K}{3} \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -\epsilon_L & 0 & 0 \\ 0 & \epsilon_L & 0 \\ 0 & 0 & \delta_L \end{bmatrix} \right). \] (6')

If all symmetry breaking terms in the second matrices of (5') and (6') come from \( 5_H \) (i.e., \( a = 0 \)) we are led to unwanted down quark-charged lepton mass degeneracy. This problem can be avoided by assuming that \( \delta \) elements are generated from the coupling to a \( 45^*_H \)-plet Higgs scalar, while \( \epsilon \) terms are from \( 5^*_H \). We have then

\[ \epsilon_D = \epsilon_L = \epsilon \] (8)

and

\[ \delta_D = -\delta_L / 3 = \delta, \] (9)

with real \( \epsilon \) and \( \delta \) [14]. We obtain

\[ m_b \simeq K(1 + \delta/9), \quad m_s \simeq 2K\delta/9, \quad m_d \simeq -K\epsilon^2(1 + \delta)/6\delta, \] (10)

\[ m_\tau \simeq K(1 - 3\delta/9), \quad m_\mu \simeq -2K\delta/3, \quad m_e \simeq K\epsilon^2(1 - 3\delta)/18\delta, \] (11)

after diagonalization of the matrices. The masses of (10) and (11) satisfy the SU(5) GJ mass relation (1) when \( \delta \ll 1 \). Now all parameters in the down-quark and charged-lepton sectors are determined solely by \( e, \mu \) and \( \tau \) masses.

If we would take the same form as (5') also for the up-quark sector [13], i.e., the symmetry breaking term necessarily limited to \( 5^* \), we were led to \( V_{23} \simeq 0.015 \) compared with experiment \( 0.036 - 0.042 \), whereas \( V_{12} \) is successfully predicted. We note, however, in our scheme (5-7) that the breaking terms for up- and down-quark sectors may not necessarily be the same form. In fact, \( 45_H \) Higgs, when coupled to \( 10_i \times 10_j \), should give matrix elements different from (5'). That is, it gives rise to flavour off-diagonal elements, rather than diagonal due to the antisymmetric nature of \( 45_H \). Therefore, we
take for (7),

\[ M_U = \frac{K'}{3} \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right) + \begin{bmatrix} -\epsilon_U & 0 & \delta_U \\ 0 & \epsilon_U & \delta_U \\ -\delta_U & -\delta_U & 0 \end{bmatrix} \) \] 

(12)

where \( \delta_U \) comes from \( 45_H \). \( \delta_U \) in the (1,3) matrix element may generally differ from that in (2,3), or simply it may even vanish. We take a parallelism with the down quark sector that \( 45_H^* \) couples to the third generation matrix elements: we found that the simple choice taken here gives resulting mixing angles in agreement with experiment.

For this mass matrix the mass eigenvalues are (\( \delta_U \) and \( \epsilon_U \) being real)

\[ m_t \simeq K'(1 - 2\delta_U^2/9), \quad m_c \simeq 2K'\delta_U^2/9, \quad m_u \simeq -K'\epsilon_U^2/6\delta_U, \] 

(13)

and resulting quark mixing angles read

\[ V_{12} \simeq \sqrt{m_s/m_d} - \sqrt{m_u/m_c} \] 

(14)

and

\[ V_{23} \simeq m_s/\sqrt{2m_b} - \sqrt{m_c/m_t}. \] 

(15)

The explicit analytical expression is more complicated for \( V_{13} \), and we do not bother the reader by writing it here.

Let us now carry out a numerical analysis. The parameters for the down-quark and charged-lepton sectors are fixed by \( m_e, m_\mu, m_\tau \) to be \( K = 1.13 \) GeV, \( \epsilon = 0.019 \) and \( \delta = -0.093 \). Here, we should use masses at GUT energy scale, and our input parameters are taken from two-loop calculations of ref. [15] as presented in Table. 1, where we also compare predicted down-quark masses with those expected at GUT mass scale. The agreement of the prediction with “experiment” is good, though \( m_d \) is somewhat smaller [16]. For the up-quark sector we take \( K' = 129 \) GeV, \( \epsilon_U = -0.00072 \) and \( \delta_U = -0.103 \) to fit the central values of \( u, c, t \) masses in Table 1.
The resulting mixing matrix is

\[
|V_{\text{quark}}| = \begin{bmatrix}
0.975 & 0.220 & 0.0086 \\
0.220 & 0.975 & 0.036 \\
0.016 & 0.033 & 0.999
\end{bmatrix}
\] (16)

which is compared with experimental values \(|V_{12}| = 0.217 - 0.224, |V_{23}| = 0.036 - 0.042\) and \(|V_{13}| = 0.002 - 0.005\). We emphasize that an excellent agreement is achieved with experiment for \(V_{23}\), whereas a factor of two disagreement has been taken to be a problem with the democratic matrix [13]. Our solution seems to be the simplest among others [6]. While the predicted \(V_{13}\) is somewhat larger than experiment, we do not pursue this problem further here [we can bring \(V_{13}\) in a good agreement with experiment without conflicting with our principles, if \(\epsilon_D\) is put on the (1,3) and (3,1) components, in addition to (1,1) and (2,2), of (5'), see [17]].

The Majorana neutrino coupling to Higgs is taken freely from the other sectors, since therein \(5^*_i 5^*_j\) does not appear. The only requirement is that the matrix should respect \(S_{10}^3 \times S_5^3\) in its main part, and we take the first matrix of (4). We may take the \(S_{10}^3 \times S_5^3\) breaking terms to be

\[
M^{(1)}_{\nu} = \begin{bmatrix}
0 & \epsilon_\nu & 0 \\
\epsilon_\nu & 0 & 0 \\
0 & 0 & \delta_\nu
\end{bmatrix},
\] (17)

or

\[
M^{(1)}_{\nu} = \begin{bmatrix}
-\epsilon_\nu & 0 & 0 \\
0 & \epsilon_\nu & 0 \\
0 & 0 & \delta_\nu
\end{bmatrix}.
\] (18)

This is exactly the same as the model we have presented in [10], and the lepton (neutrino) mixing matrix reads

\[
|V_\ell| = \begin{bmatrix}
0.998 & 0.045 & 0.050 \\
0.066 & 0.613 & 0.787 \\
0.005 & 0.789 & 0.614
\end{bmatrix},
\] (19)

or

\[
|V_\ell| \approx \begin{bmatrix}
0.737 & 0.674 & 0.050 \\
0.386 & 0.479 & 0.787 \\
0.555 & 0.562 & 0.614
\end{bmatrix},
\] (20)
according as we take (17) or (18). The second mixing matrix virtually agrees with the one given by Fritzsch and Xing [7] derived from different principles. The mass eigenvalues are $K_{\nu} \pm \epsilon_{\nu}$, and $K_{\nu} + \delta_{\nu}$, hence describing three neutrinos almost degenerate in mass. We note that the mixing matrices are predominantly determined by charged lepton masses; neutrino masses change the elements little. The case with (17) describes large mixing for $\nu_\mu - \nu_\tau$, $\sin^2 2\theta_{\mu\tau} \simeq 0.93$ or the $\nu_\mu$ survival fraction of 54% in the atmospheric neutrino experiment, and small mixing for $\nu_e - \nu_\mu$ ($\sin^2 2\theta_{e\mu} \simeq 8 \times 10^{-3}$) consistent with the small angle solution of the MSW explanation for the solar neutrino problem [8]. The case with (18) predicts large mixing for both $\nu_\mu - \nu_\tau$ and $\nu_e - \nu_\mu$, the latter being consistent with either MSW large angle solution or mixing angle required in solar neutrino oscillation in vacuum, independent of the neutrino mass difference squared. The (1,3) elements of (19) and (20) come out to be consistent with the CHOOZ experiment [18], which yields roughly $< 0.2 - 0.3$ for this element when the mass of $\nu_\tau$ is in the range Super-Kamiokande indicates.

In conclusion, we have shown that one could successfully embed phenomenological quark-lepton mass matrices obtained by the democracy principle into the SU(5) scheme. The choice of matrices is not yet unique, but the interlocking of the two principles tightly constrains the allowed form, and reduces the number of parameters of the model; the matrix for charged leptons is no longer independent of that for down quarks. In addition we have found the matrices in better agreement with experiment after tweaking to reconcile with the SU(5) than other empirical matrices constructed without referring to gauge models. The approach reconciling empirical matrices with gauge theory as we have done in this paper might perhaps give a guiding principle to understand the Higgs sector of gauge theories, which otherwise appears too arbitrary.

**Note added:** we have learned that Mohapatra and Nussinov [19] have recently proposed a gauge model for $S_3$ symmetric mass matrices embedded into $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. 

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experiment, keeping the simplicity of the model, since the introduction of phases modifies the model only at a quantitative level.

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Table 1. Input quark-lepton mass parameters and the prediction of our model (mass in MeV units). “exp” means the experimental value expected at GUT energy scale, as given by a two loop analysis of [15]. The input masses given here are used to predict the CKM matrix (16) and the lepton mixing matrix (19) or (20).

| m_e  | m_µ  | m_τ  | m_d  | m_s  | m_b  | m_u  | m_c  | m_t  |
|------|------|------|------|------|------|------|------|------|
| “exp.” | 0.325 | 68.60 | 1171 | 1.3 ± 0.2 | | 26.5^{+3.4}_{-3.7} | 1000 ± 40 | 1.0 ± 0.2 | 302^{+25}_{-27} | 129^{+196}_{-40} × 10^3 |
| pred. | input | input | input | 0.67 | 24.4 | 1120 | input | input | input |