Parametrising yields of nuclear multifragmentation

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Abstract

We consider a model where, for a finite disintegrating system, yields of composites can be calculated to arbitrary accuracy. An analytic answer for yields is also known in the thermodynamic limit. In the range of temperature and density considered in this work, the model has a phase transition. This phase transition is first order. The analytic expression for yields of composites, in the thermodynamic limit, does not conform to the expression \( < n_a > = a^{-\tau} f(a^\sigma (T - T_c)) \) where, the usual identification would be that \( T_c \) is the critical temperature and \( \tau, \sigma \) are critical exponents. Nonetheless, for finite systems, we try to fit the yields with the above expression. A minimisation procedure is adopted to get the parameters \( T_c, \tau \) and \( \sigma \). While deviations from the formula are not negligible, one might believe that the deviations are consistent with the corrections attributable to finite particle number effects and might then conclude that one has deduced at least approximately the values of critical parameters. This exercise thus points to difficulties of trying to extract critical parameters from data on nuclear disintegration. An interesting result is that the value of \( T_c \) deduced from the “best” fit is very close to the temperature at which the first order phase transition occurs in the model.

The yields calculated in this model can also be fitted quite well by a...
parametrisation derived from a droplet model.

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I. INTRODUCTION

The following parametrisation, often used to fit nuclear multifragmentation data, gives an elegant expression for yields of composites:

\[ < n_a > = a^{-\tau} f(a^\sigma(T - T_c)) \]  

(1.1)

Here \( a \) is the mass number of the composite, \( T_c \) is the critical temperature, \( \tau \) is the Fisher exponent \([1]\) and the expansion is valid in the neighbourhood of \( T_c \) and for “large” \( a \). Variants of the equation are also used. For example, in the percolation model one replaces \( T - T_c \) by \( p - p_c \) where \( p_c \) is the value of \( p \) at which an infinite cluster first appears \([2]\). The values of \( \tau \) and \( \sigma \) for three dimensional percolation model are 2.18 and 0.45 respectively. In the case of the lattice gas model, also used in a great deal to study nuclear multifragmentation \([3]\), Eq (1.1) is valid not only in the neighbourhood of \( T_c \) but along the entire Kertesz line and extends also to lower than critical density \([3–6]\).

There is also another class of models which have been used to fit multifragmentation data. Typical of these is the Copenhagen statistical multifragmentation model \([7]\). Details vary between different versions but the principal assumption is that, at some larger than normal volume, hot nuclear matter breaks up into different clusters. Nuclear interactions between different clusters are deemed insignificant. The coulomb interaction between different composites can be taken into account at least approximately. Numerous applications of such models have been made with impressive success. The advantage of such models is that quantum effects such as shell effects can be included.

How well does Eq (1.1) work for such models? This is a relevant question in view of the fact that there already exist many applications of such models to fit actual data \([7–9]\). We take a simplified version of such models so that exact answers can be obtained (as opposed to Monte-Carlo sampling which make calculations very long). The story that unfolds is quite interesting and is the subject of this paper. For simplicity, we first consider a model of one kind of particle in the next section. This is then analysed for parametrisation. The method
of analysis is explained in section 3. Results are presented in section 4. After looking at fits to models with isotopic spins with and without the Coulomb force, we return to the model of one kind of particles again, this time, trying a fit with a droplet model. A comparison between different models is presented in section 7. Our conclusions are summarised in section 8.

II. MODEL OF ONE KIND OF PARTICLE

Details of the model can be found in [10,11]. They are summarised here for completeness. We first consider just one kind of particles as the thermodynamic properties for this model were studied in detail [10,12–14]. We can have monomers or composites of \( k \) nucleons. The composites have ground state energy \(-W_k + \sigma(T)k^{2/3}\). The first term is the volume energy with \( W = 16 \) MeV. The second term is the surface tension term. The surface tension term is taken to be temperature dependent as in [7]:

\[
\sigma(T) = \sigma_0 \left[ \frac{(T_c^2 - T^2)}{(T_c^2 + T^2)} \right]^{5/4}
\]

with \( \sigma_0 = 18 \) MeV and \( T_c = 18 \) MeV. The internal partition function of a composite of \( k > 1 \) nucleons is:

\[
z_k = \exp \left[ (W_k - \sigma(T)k^{2/3} + T^2k/\epsilon_0)/T \right]
\]  

(2.1)

where we have used the standard Fermi-gas model for excited states (the third term on the right hand side of Eq.(2.1)). For \( k = 1 \) we take \( z_1 = 1 \).

We want to construct the canonical partition function of the system which has total \( A \) nucleons in a volume \( V \) which is much larger than the volume of a nucleus of \( A \) nucleons in the ground state: \( V > V_0 \). We assume when clusters are formed in the volume \( V \), they do not overlap with each other. Thus the available volume free to the clusters is less than \( V \). It is given by \( V_f = V - V_{exc} \). We take \( V_{exc} \) to be \( V_0 = A \times b \) where \( b = 1/16 \, fm^{-3} \). In reality \( V_{exc} \) should also depend on multiplicity but this complication is ignored here. The canonical partition function \( Q_A(T) \) of \( A \) nucleons is then given by

\[
Q_A = \sum \prod_{k \geq 1} \frac{\omega_k^{n_k}}{n_k!}
\]  

(2.2)

where \( \omega_k \) is the partition function of one composite of \( k \) nucleons.
\[ \omega_k = \frac{V_f}{\hbar^3} (2\pi mT)^{3/2} k^{3/2} \times z_k \]  

(2.3)

and the sum rule must be obeyed: \( A = \sum k n_k \). As noted before [10], the partition function \( Q_A \) for \( A \) nucleons can be easily generated in the computer by utilising a recursion relation. Starting with \( Q_0 = 1 \) one can build all higher ones using

\[ Q_p = \frac{1}{p} \sum_{k=1}^{p} k \omega_k Q_{p-k} \]  

(2.4)

The expression for the yield of composites is, of course, of primary interest. This is given by

\[ < n_k > = \omega_k \frac{Q_{A-k}}{Q_A} \]  

(2.5)

Several things are known for this model. The critical temperature for the model is the temperature at which the surface tension vanishes [12] and hence it is at \( T = T_c = 18 \text{MeV} \). The critical volume is at \( V = V_0 \). At temperatures below 18 Mev there is first order phase transition [10,12]. The temperature of phase transition depends upon the density. This temperature was called boiling temperature in [10]. In the temperature range we are concerned with in this paper, there is only first order phase transition. The phase transition temperature is characterised by a sudden jump in specific heat. In finite systems we will take boiling temperature to be the temperature at which the specific heat maximises.

We will try to fit the yields of Eq.(2.5) by the generic formula (Eq.(1.1)). The exact expressions (Eqs.(2.2) to (2.5)) give no clue of a simple parametrisation. Provided \( V \) is reasonably bigger than \( V_0 \) (see however [12]) and \( A \) is also large we can use the grand canonical results to obtain some insight on simple parametrisation. This answer is well-known:

\[ < n_k > = \frac{V_f}{\hbar^3} (2\pi mT)^{3/2} k^{3/2} \exp[(\mu(T) + W + T^2/\epsilon_0)k/T - \sigma(T)k^{2/3}/T] \]  

(2.6)

There is no exact correspondence between Eq.(2.6) and Eq.(1.1). Thus we may at best hope to get an approximate fit. How we do it is detailed in the next section.
III. THE FITTING PROCEDURE

Here we follow very elegant methods given in [5]. For later use in the text, we will give adequate details.

The quality of fit is given by the smallness of $\chi^2$. If the calculated quantity $Y$ is a function of two parameters $Y = Y(a_i, b_i)$ and we are trying to fit it to a function $g(a_i, b_i, \alpha, \beta, \gamma \ldots)$ then $\chi^2 = \sum_i (Y(a_i, b_i) - g(a_i, b_i, \alpha, \beta, \gamma \ldots))^2 / \sum_j Y(a_j, b_j)^2$. Variations of this criterion are also possible. Eq. (1.1) requires us to find “best” values for $\tau, \sigma$ and $T_c$. This is done in several steps.

(1) At each temperature we find the “best” $\tau$ associated with an attempted fit $< n_a > = a^{-\tau} C$ where $C$ is a constant. This follows from Eq.(1.1) but only at $T = T_c$, hence one can argue that if Eq.(1.1) were exact for yields calculated by Eq.(2.5) we would get null $\chi^2$ and the correct $\tau$ at $T = T_c$. This would determine both $T_c$ and $\tau$. Of course, null $\chi^2$ is not found since “exact” fit is not given by Eq.(1.1) However we can draw a best “$\tau$” vs. $T$ curve for a pure power law. This $\tau$ as a function of $T$ will have a minimum which we label $\tau_{min}$. Since at $T_c$, the fit is strictly a power law, one can accept that temperature as $T_c$ where the $\chi^2$ of the fit is minimum. However, we will determine $T_c$ using the method described in step 3.

(2) Let $z = a^\sigma (T - T_c)$; $f(z)$ has a maximum at some value of $z = \tilde{z}$: $f_{max} = f(\tilde{z})$. For each mass number $a$ the yield $< n_a >$ as a function of temperature has a maximum at some value of temperature, $T_{max}(a)$. At this temperature $< n_a > (max) = a^{-\tau} f_{max}$ where $f_{max}$ is a constant independent of $a$. This relationship allows us to choose “best” values for $\tau$ and $f_{max}$.

(3) The value of $\tau$ found from step (2) is higher than $\tau_{min}$ found at step (1). This means that if we look for $T$ appropriate for $\tau$, two values of $T$ are available from the graph at step (1). The lower value is to be chosen as the value of $T_c$. The scaling property (see steps (4) and (5)) is badly violated for the other choice.

(4) Now that we know $T_c$ and $T_{max}(a)$, the temperature at which the yield of composite
a is maximised, we find, by least squares fit the “best” value of \( \sigma \) from the condition

\[
a^\sigma (T_{\text{max}}(a) - T_c) = \text{const.} \quad \text{for all} \quad a.
\]

(5) The scaling law can now be tested by plotting \( <n_a> a^\tau \) vs. \( a^\sigma (T - T_c) \). Plots for all \( a \)'s should fall on the same graph.

In following steps (1) to (5) the range of \( a \) is to be chosen judiciously. It can not be very small (since Eq.(1.1) applies to “large” \( a \)'s). But \( a \) should also be truncated on the high side (significantly smaller than the size of the dissociating system).

**IV. RESULTS FOR ONE KIND OF PARTICLES**

We present our results in Figs.1 to 3. The sizes of the systems are taken to be \( A = 174 \) and \( A = 240 \). The upper panels of the figures use the freeze-out volume \( V = 3V_0 \) and the lower panels use \( V = 4V_0 \). Both are shown here for completeness. It will suffice here to discuss only the cases with \( V = 3V_0 \).

Fig. 1 shows \( \tau \) vs. \( T \) drawn according to step (1) of section 3. The dotted line is the value of \( \tau \) deduced from step (2). This cuts the curve of step (1) at two temperatures (step (3)). The lower value of the temperature is taken as \( T_c \). In the same Fig. we also plot \( C_v/A \) as a function of \( T \). The peak of this curve corresponds to the first order phase transition, called “boiling” temperature in \([10]\). It is remarkable that the \( T_c \) of Eq.(1.1) is very close to “boiling temperature” in this particular example. In this respect, the model is similar to the behaviour of the lattice gas model where parametrisation Eq.(1.1) works on the lower side of the critical density also provided \( T_c \) is replaced by \( T_b \), the temperature of first order phase transition \([6,5]\) at the given density. In Fig.1 we have also plotted the value of \( \chi^2 \) as a function \( T \) (step 1). In Fig.2 we plot \( \ln <n_a> \) vs. \( \ln a \). Two graphs are shown for each disintegrating systems. The graph with higher values of \( <n_a> \) (shown as diamonds) follows from step (2) of section 3. These are the maximum values of \( <n_a> \) for each \( a \) obtained at corresponding temperatures \( T_{\text{max}}(a) \). The lower values of \( <n_a> \) (shown as stars) are all at the same temperature, namely at \( T_c \) which, for example, is 6.32 MeV for
$A=174$ and is 6.54 MeV for $A=240$. This is how $\tau$ would be estimated from experiments \[15\]. The crucial testing of the scaling law is shown in Fig.3. where we plot $< n_a > a^\tau$ vs. $a^\sigma (T - T_c)$. For the range of $a$ chosen (10 to 40) the results nearly fall on the same graph. Since one does not know \textit{a priori} how much error is due to finite particle number of the disintegrating system, one might be tempted to to conclude that the fit to Eq. (1.1) is adequate. The parameters $\tau, \sigma$ from best fits are 2.72, 1.06 respectively for $A=174$ and 2.78, 1.23 respectively for $A=240$. The deduced $T_c$ are 6.32 MeV and 6.54 MeV which are very different from the critical temperature of 18 MeV for the model but compare remarkably well with the temperatures where the specific heats peak and which correspond to first order phase transition temperatures at the given densities.

Here we want to comment that, as seen from Fig. 1, $\chi^2$ has a minimum at a temperature very close to $T_c$. So one may conclude that the methods of determining $T_c$ using step 1 or 3 yield almost the same result.

These then are the two salient features: (1) Numerical fits of Eq.(1.1) are surprisingly close and (2) interpreting $T_c$ as the critical temperature is wrong although the deduced $T_c$ does correspond to a phase transition temperature. We briefly look at how these features hold for a more realistic model with isotopic spin.

\section*{V. MODELS OF TWO KINDS OF PARTICLES}

We considered the following simplified version. Protons and neutrons are elementary particles. For deuteron, triton, $^3$He and $^4$He, experimental binding energies are used but no excited states are included. For masses greater than 4 a semi-empirical formula is used. Composites of $n$ neutrons and $z$ protons have binding energies $W(n + z) - \sigma(n + z)^{2/3} - a_s (n-z)^2/(n+z)^2 - a_c (n+z)^{1/3}$. Here $W=15.8$ MeV, $\sigma=18$ MeV is the surface tension term (taken here as temperature independent), $a_s = 23.5$MeV is the symmetry energy term and $a_c = .72$ MeV is the Coulomb energy term. For composites $>4$ we also include excited states in the Fermi-gas model (as in section II). We also incorporate Coulomb interaction between composites.
in the Wigner-Seitz approximation [4].

It is however also instructive to consider two kinds of particles with the surface tension and symmetry energy term but with the Coulomb energy switched off. Such matters, in the thermodynamic limit can be self-bound at zero temperature. With Coulomb included however matter would fall apart in the thermodynamic limit even at zero temperature. We show first the results for finite systems with surface and symmetry energy terms but no Coulomb. For brevity we will show results only for the choice $V = 3V_0$. For two kinds of particles, Eq.(1.1) can be used in more than one way. The subscript $a$ in Eq.(1.1) can stand for the mass number $a = n + z$. Then Eq.(1.1) gives the distribution of particles of given mass number, irrespective of $n$ or $z$. The lower panels of Fig. 4 to 6 use that. Or, as is more suitable for experiments, $a$ can stand for $z$, the charge, irrespective of the neutron number $n$. The upper panels of Figs. 4 to 6 use that. We notice that the deduced value of $T_c$ continues to be close to $T_b$ where the specific heat maximises.

For brevity, with Coulomb included, we show the results for $V = 3V_0$ only. Distribution of particles with a given charge, irrespective of the neutron number $n$ are shown in Figs. 7 to 9. The most noticeable difference with the “no Coulomb” cases is that the deduced $T_c$ is now farther removed from the temperature at which the specific heat maximises. Coulomb energy introduces rather significant changes to the caloric curves. This was noticed before [11]. For example, without the Coulomb, as the particle number increases, the peak of the specific heat becomes narrower and the temperature at which the maximum is obtained slightly shifts to higher value (see Fig. 4). The narrowing and the shifting of the peak seems to be almost exactly compensated by the long range repulsive Coulomb force as more and more charged particles are added (Fig. 4 here and also Fig.(6) in [11] where this is discussed in greater detail). Properties of nuclear matter but where protons carry charges, require further studies.
VI. FIT TO A DROPLET MODEL

We will now try to fit the predicted yields given by the model of section II (one kind of particles) with a well-known droplet model \[1\]. An early application of the model, to heavy-ion collisions, can be found in \[16\]. The model has been revived recently \[17\].

In Fisher’s model, the condensation of a real gas into large drops (clusters) of liquid is modeled. This shares various similar features with multifragmentation models. For instance, the potential energy of large clusters consists entirely of a bulk term and a term associated with the loss of binding energy at the surface. There is no Fermi energy term, as the molecules inside the cluster are assumed to be Boltzmann distributed. The entropy of large clusters is, however, more complicated. As clusters become large, the dominant effect may be ascribed, once again, to a bulk term, and the remainder to a surface term. It was pointed out that liquid clusters may not be restricted to spherical shapes, as is the case in most multifragmentation models. This prohibits the use of one form of surface area to parametrize the surface contribution. In reference \[1\] it is argued that, at low temperatures, the most important configurations will be compact and globular. Their surface areas \(s\) are not much greater than the minimum possible, and are assumed to admit the asymptotic condition that,

\[
\bar{s}(k, \beta)/k \to 0 \quad \text{as} \quad k \to \infty \quad \text{and}
\]

\[
\bar{s}(k, \beta)/\log k \to \infty \quad \text{as} \quad k \to \infty.
\]

(6.1)

Where, \(k\) is the number of molecules occupying the cluster. If, for finite clusters, one introduces this surface area to calculate various surface contributions, one must introduce a correction term which varies as \(\tau \log k\). The sign and magnitude of \(\tau\) is estimated from various other considerations involving other models.

One may thus, very generally, obtain the mean number of clusters of size \(k\) as

\[
<n_k >= Ck^{-\tau} \exp((\mu_g - \mu_l)k/T + c_2k^{2/3}/T)
\]

(6.2)
Here both $\mu_g$ and $\mu_l$ are functions of $T$. At coexistence and also at the critical temperature, they become equal to one another. Also $c_2$ is a function of temperature and at $T_c$ the coefficient $c_2$ goes to zero. Since above $T_c$ there is no distinction between the liquid and the gas phase, one can not speak of droplets. Thus the theory only applies to $T < T_c$. As such the formulation is more limited than the model of Eq. (1.1) which applies to both sides of $T_c$. The following fit was tried. We set $\tau = 2$. let $\alpha = (\mu_g - \mu_l)/T$, $\gamma = c_2/T$. We fit the calculated $< n_k >$ to $Ck^{-2} \exp(\alpha k + \gamma k^{2/3})$ at different temperatures where $\alpha, \gamma$ values at each temperature are varied for best fit. The values of $\alpha, \gamma$ as a function of temperature is shown in Fig. 10 where we also show rather remarkable fit with the values of $< n_k >$ obtained from the model of section II. The values of $\alpha$ and $\gamma$ both go to zero near temperature $T = 6.5$ MeV suggesting that the critical temperature is 6.5 MeV.

**VII. RELATIONSHIP BETWEEN DIFFERENT MODELS**

In this section a grand canonical approximation to the model of section II is constructed. As in the previous section, this will be concentrated in the region below the boiling temperature. A different parametrization of the output from the exact canonical calculation will be obtained. Though this parametrization has not been as throughly investigated as that of the previous section, it will serve to provide a qualitative understanding of the behaviour of the yields below the boiling temperature.

The yields (Fig. 11, Fig. 13) are obtained from the exact expressions Eq. (2.5). For the present analysis they are adequately approximated by Eq. (2.6). The parametrization offered by Eq. (2.6) is of the form

$$< n_k > = \tilde{C}k^{3/2} \exp(\tilde{\alpha} k + \tilde{\gamma} k^{2/3})$$

(7.1)

The above is different from the parametrization of the droplet model where $\tau = 2$. Though the expressions look similar, the fit parameters $\tilde{C}, \tilde{\alpha}, \tilde{\gamma}$ will assume values different from those of $C, \alpha, \gamma$. The interpretation of $\tilde{\alpha}$ is also quite precise in this approximation.
When using Eq.(2.6) to estimate the result of a canonical calculation, the free parameter is the chemical potential \( \mu \). It is usually determined by imposing that the model correctly reproduce the total number of particles composing the system (i.e., \( \sum_k \langle n_k \rangle k = A \)). This is a complicated problem in general. A clue may be obtained by observing the behaviour of \( \mu \) as obtained from the canonically calculated Helmholtz free energy \( F = -T \log Q_A \). A plot of \( \mu \) obtained thus is plotted in Fig. (12) (data points). The behaviour of this \( \mu \) may be estimated by the following simple argument. Far below the boiling temperature the system exists mostly as one large cluster and a few small ones (see Fig. (11)). The large cluster is considered as the liquid state. Far above the boiling temperature the system exists mostly as many small clusters: this is considered as the gas phase. The chemical potential of either system may be estimated by keeping the system in contact with a heat reservoir, adding one particle to the system, and noting the change in free energy, i.e., 

\[
\mu = \left[ \Delta F \right]_{V,T} = F(T, V, A + 1) - F(T, V, A).
\]

On entering the system, the new particle, may apriori attach itself to any of the existing clusters, or simply thermalize as a monomer in the system. It will attach itself to the cluster that minimises the free energy at that temperature and density. There may be more than one unique choice. The resulting change in energy and entropy of the system may be decomposed as the sum of two parts: a kinetic part \( (\Delta E_{kin}, \Delta S_{kin}) \), to do with the cluster’s motion in the environment; and an internal part \( (\Delta E_{in}, \Delta S_{in}) \), to do with the internal motion of the particles constituting the cluster. If the volume is large, one may assume that the clusters form an almost ideal gas. In this case the average kinetic energy of a cluster of size \( k \) is \( (3/2)T \). It does not depend on \( k \) and thus \( \Delta E_{kin} = 0 \). The change in internal energy \( \Delta E_{in} \) may be estimated simply as,

\[
\Delta E_{in} = -W + \frac{T^2}{\epsilon_0} + \sigma(T)\left[(k + 1)^{2/3} - k^{2/3}\right]. \tag{7.2}
\]

The change in internal entropy is given simply as \( \Delta S_{in} = 2T/\epsilon_0 \). The kinetic entropy of an ideal gas of \( n_k \) clusters is given as (see Ref. [21]),

\[
S_{kin} = n_k \left[ \frac{3}{2} \log T + \log V + \frac{3}{2} \log \frac{2\pi m}{\hbar^2} + \frac{3}{2} \log k + \frac{3}{2} \right] - \log n_k! \tag{7.3}
\]
In most cases in nuclear fragmentation, \( n_k \) lies between 0 and 1 (see Fig. (11)). Thus we may ignore the \( n_k! \) term. Thus we get the total change in entropy for the addition of one particle to a cluster of size \( k \) as,

\[
\Delta S = 2T/\epsilon_0 + \frac{3}{2} \log \left(1 + 1/k\right).
\] (7.4)

As a result, the total change in free energy and hence \( \mu \), is given as

\[
\mu = \Delta F = -W - \frac{T^2}{\epsilon_0} + \sigma(T) \left[(k + 1)^{2/3} - k^{2/3}\right] - \frac{3}{2} T \log \left(1 + 1/k\right).
\] (7.5)

We note that \( \mu \) becomes progressively more negative with rising \( k \). Thus the added particle prefers to attach to large clusters. In a fragmenting system under the boiling temperature, such a large fragment exists, of about half the size of the system \([10]\). The new particle thus preferentially attaches to this cluster. To illustrate this point more quantitatively, we calculate this \( \mu \) for a system with \( A = 240 \) and a \( V = 4V_0 \). We assume the largest cluster is of size \( A/2 \). No doubt, this size falls gradually with rising temperature \([10]\), with the fall becoming rapid near the boiling temperature. In Fig. (12) we plot the value of this \( \mu \) (solid line) assuming that the largest cluster remains of the same size throughout the temperature range. Above the boiling point the system exists mostly as small clusters, here we assume that the added particle attaches itself to a cluster of size \( k = 2 \). This \( \mu \) is plotted as the dot-dashed line in Fig. (12).

We are interested in obtaining an approximate expression for \( \langle n_k \rangle \) underneath the boiling temperature. Using the expression for \( \mu \) as derived in Eq. (7.5), we obtain the expression for \( \langle n_k \rangle \) as

\[
\langle n_k \rangle = e^{\beta \mu_k} \omega_k = V \left(\frac{2\pi m T}{h^2}\right)^{3/2} k^{3/2}
\]

\[
\times \exp \left[\sigma(T) \left((k_{\text{max}} + 1)^{2/3} - k_{\text{max}}^{2/3}\right) - \frac{3}{2} T \log(1 + 1/k_{\text{max}})\right] \left(k/T - \sigma(T)k^{2/3}/T\right].
\] (7.6)

For most systems, in general, the behaviour of \( k_{\text{max}} \) with \( T \) and \( V \) is difficult to estimate. However, we note that the above equation is precisely of the form of Eq. (7.1). On fitting the data points obtained from Eq. (2.5) we obtain the fit parameters as \( \tilde{C} = 2.73, \tilde{\alpha} = 0.36, \tilde{\gamma} = \)
−3.11 (note that, as in the droplet model, only range of $k$ between 10 to 40, is fitted). A plot of the fit to the values of $n_k$ obtained from a system with $A = 240$, $T = 5\text{MeV}$ and $V = 4V_0$ is shown in Fig. (13). Here the entire region from $k = 2$ to 240 is plotted. Note that both fits coincide extremely well in the region of $k = 10$ to 40.

VIII. SUMMARY AND DISCUSSION

This investigation started out with the following question: suppose “experimental data” are given by the predictions of a theoretical model which, we know, does not conform to Eq. (1.1) exactly. Could we still describe the “data” approximately with the formula? And, if so, what are the significances of the parameters $\tau, \sigma$ and $T_c$? Subsequently we tried a fit with the droplet model and again found that a very adequate fit can be found.

We find that the scaling law is still approximately obeyed. While we can not attach much significance to the extracted values of $\tau$ or $\sigma$ (they are different from those given by the percolation or Ising model, but not terribly so), $T_c$ seems to be a genuinely physical parameter, namely, it reflects the first order phase transition temperature. Coulomb effects tend to somewhat spoil even this correspondence.

Many different fits can be obtained because by necessity the mass number of the composites is limited on the lower as well as on the higher side. We have no control over that since the dissociating systems are extremely finite. This apparently makes even deciding on the order of phase transition very difficult. Fits to data from calculations on percolation model have been made in the past [18]. Since this is a model of continuous phase transition one necessarily concludes that the phase transition is continuous. It is more appropriate to consider the Lattice Gas Model [3] instead which does encompass the Percolation model as a subset [19]. In the Lattice Gas model, if it is assumed that the freeze-out density is less than half the normal density, then the fit of Eq. (1.1) would imply that $T_c$ is indeed a first order phase transition temperature. If, on the other hand, one assumed that the freeze-out density is higher than half the normal density (we consider this highly unlikely), then a fit to
Eq. (1.1) would not imply any usual thermodynamic phase transition [20]. If one depends on theories to decide on what order of phase transition to expect, one is driven towards expecting a first order phase transition. To date all the models which used a Hamiltonian [5–7,10] suggest a first order phase transition.

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FIGURES

FIG. 1. \( \tau, C_v/A \) (left-hand scale) and \( \chi^2 \) (right-hand scale) plotted against temperature in the model of one kind of particles. The different panels are for different choices of \( A \) and \( V \).

FIG. 2. \( \ln < n_a > vs.lna \). The solid line is the best fit to \( \ln < n_a > \) at each \( T_{\text{max}}(a) \) presented by diamonds. The dotted line joining stars represents the distribution at \( T_c \). The different panels are for different choices of \( A \) and \( V \).

FIG. 3. The scaling behavior in the mass range \( (10 \leq a \leq 40) \)

FIG. 4. Same as Fig. 1, but in a model of two kinds of particles. Coulomb interaction has been switched off. The upper and lower panels are with respect to charge and mass distributions.

FIG. 5. Upper panels: \( \ln < n_z > vs.lnz \), and lower panels: \( \ln < n_a > vs.lna \); in the model of two kinds of particles with Coulomb interaction switched off.

FIG. 6. Scaling behavior in the charge range \( (7 \leq z \leq 17) \) (upper panels) and mass range \( (10 \leq a \leq 40) \) (lower panels) in two kinds of particles model without Coulomb interaction.

FIG. 7. \( \tau_z \) and \( C_V/A \) as a function of temperature in a model of two kinds of particles, with Coulomb interaction included.

FIG. 8. Same as upper panels of Fig. 5, but with Coulomb interaction.

FIG. 9. Same as upper panels of Fig. 6, but with Coulomb interaction.

FIG. 10. The parameters of droplet model \( \alpha \) and \( \gamma \) as a function of temperature for a system with \( A = 240 \) and \( V = 4V_0 \). The right panels show the fit of the droplet model to the yields obtained in the model of one kind of particles described in section II. On the graph one can not distinguish between fitted points and the actual points from canonical calculations.
FIG. 11. $\langle n_a \rangle$ vs. $a$ on a logarithmic plot. The right panel expands the region $10 \leq a \leq 40$.

FIG. 12. $\mu$ vs. $T$ for a system with $A = 240$ and $V = 4V_0$. Data points represent results from a canonical calculation (see section II). Solid line represents $\mu$ for addition to the largest cluster, dot-dashed line is $\mu$ for addition to a small cluster (see section VII for details).

FIG. 13. Fits to $\langle n_a \rangle$ vs. $a$ from two different models: open circles are from an exact canonical calculation; the solid line represents the fit by a droplet model; the dotted line represents the fit from the grand canonical approximation.
$\ln a$

$\ln a$

$A = 174$

$V = 3V_0$

$A = 240$

$V = 3V_0$

$A = 174$

$V = 4V_0$

$A = 240$

$V = 4V_0$
$\sigma(T-T_c)$

$A=174$
$V=3V_0$

$A=240$
$V=3V_0$

$A=174$
$V=4V_0$

$A=240$
$V=4V_0$
\[ \ln \langle n_z \rangle = \alpha \ln z + \beta \]

at \( T_{\text{max}}(z) \)

\[ \text{best fit} \]

\[ \times \text{ at } T_c \]

\[ \ln \langle n_a \rangle = \gamma \ln a + \delta \]

at \( T_{\text{max}}(a) \)

\[ \text{best fit} \]

\[ \times \text{ at } T_c \]
\[ \ln \langle n_z \rangle \text{ at } T_{\text{max}}(z) \]

- \( V = 3 V_0 \)
- \( N = 104, Z = 70 \)
- \( N = 146, Z = 94 \)
from calculation
droplet model fit
grandcanonical approximation