Metric dimension, doubly resolving set and strong metric dimension for 
\((C_n \square P_k) \square P_m\)

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Abstract
A subset \(Q = \{q_1, q_2, ..., q_l\}\) of vertices of a connected graph \(G\) is a doubly resolving set of \(G\) if for any various vertices \(x, y \in V(G)\) we have \(r(x|Q) - r(y|Q) \neq \lambda I\), where \(\lambda\) is an integer, and \(I\) indicates the unit \(l\)-vector \((1, ..., 1)\). A doubly resolving set of vertices of graph \(G\) with the minimum size, is denoted by \(\psi(G)\). In this work, we will consider the computational study of some resolving sets with the minimum size for \((C_n \square P_k) \square P_m\).

Keywords: cartesian product, resolving set, doubly resolving set, strong resolving set

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1. Introduction and Preliminaries

All graphs considered in this work are assumed to be finite and connected. We use \(d_G(p, q)\) to indicate the distance between two vertices \(p\) and \(q\) in graph \(G\) as the length of a shortest path between \(p\) and \(q\) in \(G\), see [11]. The cartesian product of two graphs \(G\) and \(H\), denoted by \(G \square H\), is the graph with vertex set \(V(G) \times V(H)\) and with edge set \(E(G \times H)\) so that \((g_1, h_1)(g_2, h_2) \in E(G \square H)\), whenever \(g_1 = g_2\) and \(g_1, g_2 \in E(G)\), or \(g_1 = g_2\) and \(h_1, h_2 \in E(H)\). A graphical representation of a vertex \(p\) of a connected graph \(G\) relative to an arranged subset \(Q = \{q_1, ..., q_l\}\) of vertices of \(G\) is defined as the \(k\)-tuple \((d(p, q_1), ..., d(p, q_l))\), and this \(k\)-tuple is denoted by \(r(p|Q)\), where \(d(p, q_j)\) is considered as the minimum length of a shortest path from \(p\) to \(q_j\) in graph \(G\). If any vertices \(p\) and \(q\) that belong to \(V(G) - Q\) have various representations with respect to the set \(Q\), then \(Q\) is called a resolving set for \(G\) [6]. Slater [25] considered the concept and notation of the metric dimension problem under the term locating set. Also, Harary and Melter [12] considered these problems under the term metric dimension as follows. A resolving set of vertices of graph \(G\) with the minimum size or cardinality is called the metric dimension of \(G\) and this minimum size denoted by \(\beta(G)\). Resolving parameters in graphs have been studied in [1, 4, 5, 14, 18, 19, 20, 21, 26].

In 2007 Cáceres et al. [7] considered the concept and notation of a doubly resolving set of graph \(G\). Two vertices \(u, v\) in a graph \(G\) are doubly resolved by \(x, y \in V(G)\) if \(d(u, x) - d(u, y) \neq d(v, x) - d(v, y)\), and we can see that a subset \(Q = \{q_1, q_2, ..., q_l\}\) of vertices of a graph \(G\) is a doubly resolving set of \(G\) if for any various vertices \(x, y \in V(G)\) we have \(r(x|Q) - r(y|Q) \neq \lambda I\), where \(\lambda\) is an integer, and \(I\) indicates the unit \(l\)-vector \((1, ..., 1)\). A doubly resolving set of vertices of graph \(G\) with the minimum size, is denoted by \(\psi(G)\). In 2000 Chartrand et al. showed that for every connected graph \(G\) and the path \(P_2\), \(\beta(G \square P_2) \leq \beta(G) + 1\), see Theorem 7 in [9]. In 2007 Cáceres et al. obtained an upper bound for the metric dimension of cartesian product of graphs \(G\) and \(H\). They showed that for all graphs \(G\) and \(H \neq K_1\), \(\beta(G \square H) \leq \beta(G) + \psi(H) - 1\). In particular, Cáceres et al. [7] showed that for every connected graph \(G\) and the path \(P_2\), \(\beta(G \square P_2) \leq \beta(G) + 1\). Doubly resolving sets have played a special role in the study of resolving sets. Applications of above concepts and related parameters in graph theory and other sciences have a long history.

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in particular, if we consider a graph as a chemical compound then the determination of a doubly resolving set with the minimum size is very useful to analysis of chemical compound, and note that these problems are NP hard, see [3, 8, 10, 15, 16, 17].

The notion of a strong metric dimension problem set of vertices of graph $G$ introduced by A. Sebő and E. Tannier [24], indeed introduced a more restricted invariant than the metric dimension and this was further investigated by O. R. Oellermann and Peters-Fransen [23]. A set $Q \subseteq V(G)$ is called strong resolving set of $G$, if for any various vertices $p$ and $q$ of $G$ there is a vertex of $Q$, say $r$ so that $p$ belongs to a shortest $q - r$ path or $q$ belongs to a shortest $p - r$ path. A strong metric basis of $G$ is indicated by $sdim(G)$ defined as the minimum size of a strong resolving set of $G$.

Now, we use $C_n$ and $P_k$ to denote the cycle on $n \geq 3$ and the path on $k \geq 3$ vertices, respectively. In this article, we will consider the computational study of some resolving sets with the minimum size for $(C_n \square P_k) \square P_m$. Indeed, in Section 3, we define a graph isomorphic to the cartesian product $C_n \square P_k$, and we will consider the determination of a doubly resolving set of vertices with the minimum size for the cartesian product $C_n \square P_k$. In particular, in Section 3, we construct a graph so that this graph is isomorphic to $(C_n \square P_k) \square P_m$, and we compute some resolving parameters with the minimum size for $(C_n \square P_k) \square P_m$.

2. Some Facts

**Definition 2.1.** Consider two graphs $G$ and $H$. If there is a bijection, $\theta : V(G) \rightarrow V(H)$ so that $u$ is adjacent to $v$ in $G$ if and only if $\theta(u)$ is adjacent to $\theta(v)$ in $H$, then we say that $G$ and $H$ are isomorphic.

**Definition 2.2.** A vertex $u$ of a graph $G$ is called maximally distant from a vertex $v$ of $G$, if for every $w \in N_G(u)$, we have $d(v, w) \leq d(v, u)$, where $N_G(u)$ to denote the set of neighbors that $u$ has in $G$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then $u$ and $v$ are said to be mutually maximally distant.

**Remark 2.1.** Suppose that $n$ is an even natural number greater than or equal to 4 and $G$ is the cycle graph $C_n$. Then $\beta(G) = 2$, $\psi(G) = 3$ and $sdim(G) = \lceil \frac{n}{2} \rceil$.

**Remark 2.2.** Suppose that $n$ is an odd natural number greater than or equal to 3 and $G$ is the cycle graph $C_n$. Then $\beta(G) = 2$, $\psi(G) = 2$ and $sdim(G) = \lceil \frac{n}{2} \rceil$.

**Remark 2.3.** Consider the path $P_n$ for each $n \geq 2$. Then $\beta(P_n) = 1$, $\psi(P_n) = 2$.

**Theorem 2.1.** Suppose that $n$ is an odd integer greater than or equal to 3. Then the minimum size of a resolving set in the cartesian product $C_n \square P_k$ is 2.

**Theorem 2.2.** Suppose that $n$ is an even integer greater than or equal to 4. Then the minimum size of a resolving set in the cartesian product $C_n \square P_k$ is 3.

**Theorem 2.3.** If $n$ is an even or odd integer is greater than or equal to 3, then the minimum size of a strong resolving set in the cartesian product $C_n \square P_k$ is $n$.

3. Main Results

Some resolving parameters such as the minimum size of resolving sets and strong resolving sets calculated for the cartesian product $C_n \square P_k$, see [7, 22], but in this section we will determine some resolving sets of vertices with the minimum size for $(C_n \square P_k) \square P_m$. Suppose $n$ and $k$ are natural numbers greater than or equal to 3, and $\left\lfloor \frac{n}{k} \right\rfloor$. Now, suppose that $G$ is a graph with vertex set $\{x_1, \ldots, x_n\}$ on layers $V_1, V_2, \ldots, V_k$, where $V_p = \{x_{p-1}+1, x_{p-1}+2, \ldots, x_{p-1}+n\}$ for $1 \leq p \leq k$, and the edge set of graph $G$ is $E(G) = \{x_i, x_j | x_i, x_j \in V_p, 1 \leq i < j \leq n; j - i = 1 \} \cup \{x_i, x_j | x_i \in V_q, x_j \in V_{q+1}, 1 \leq i < j \leq n; \frac{nq}{k} + 1 \leq q \leq k - 1, j - i = n\}$. We can see that this graph is isomorphic to the cartesian product $C_n \square P_k$. So, we can assume throughout this article $V(C_n \square P_k) = \{x_1, \ldots, x_{nk}\}$. Now, in this section we give a more elaborate description of the cartesian product $C_n \square P_k$,
that are required to prove of Theorems. We use $V_p$, $1 \leq p \leq k$, to indicate a layer of the cartesian product $C_n \square P_k$, where $V_p$ is defined already. Also, for $1 \leq e < d \leq nk$, we say that two vertices $x_e$ and $x_d$ in $C_n \square P_k$ are compatible, if $n|d - e$. We can see that the degree of a vertex in the layers $V_1$ and $V_k$ is 3, also the degree of a vertex in the layer $V_p$, $1 < p < k$ is 4, and hence $C_n \square P_k$ is not regular. We say that two layers of $C_n \square P_k$ are congruous, if the degree of compatible vertices in two layers are identical. Note that, if $n$ is an even natural number, then $C_n \square P_k$ contains no cycles of odd length, and hence in this case $C_n \square P_k$ is bipartite. For more result of families of graphs with constant metric, see [3, 13]. The cartesian product $C_4 \square P_3$ is depicted in Figure 1.

![Figure 1. $C_4 \square P_3$](image)

Now, let $m \geq 2$ be an integer. Suppose $1 \leq i \leq m$ and consider $i^{th}$ copy of the cartesian product $C_n \square P_k$ with the vertex set $\{x_1^{(i)}, \ldots, x_{nk}^{(i)}\}$ on the layers $V_1^{(i)}, V_2^{(i)}, \ldots, V_k^{(i)}$, where it can be defined $V_p^{(i)}$ as similar $V_p$ on the vertex set $\{x_1^{(i)}, \ldots, x_{nk}^{(i)}\}$. Also, suppose that $K$ is a graph with vertex set $\{x_1^{(1)}, \ldots, x_{nk}^{(1)}\} \cup \{x_1^{(2)}, \ldots, x_{nk}^{(2)}\} \cup \ldots \cup \{x_1^{(m)}, \ldots, x_{nk}^{(m)}\}$ so that for $1 \leq t \leq nk$, the vertex $x_r^{(i)}$ is adjacent to the vertex $x_{r+1}^{(i)}$ in $K$, for $1 \leq r \leq m - 1$, then we can see that the graph $K$ is isomorphic to $(C_n \square P_k) \square P_m$. For $1 \leq e < d \leq nk$, we say that two vertices $x_e^{(i)}$ and $x_d^{(i)}$ in $i^{th}$ copy of the cartesian product $C_n \square P_k$ are compatible, if $n|d - e$. The graph $(C_4 \square P_3) \square P_2$ is depicted in Figure 2.
Theorem 3.1. Consider the cartesian product $C_n \square P_k$. If $n \geq 3$ is an odd integer, then the minimum size of a doubly resolving set of vertices for the cartesian product $C_n \square P_k$ is 3.

Proof. In the following cases we show that the minimum size of a doubly resolving set of vertices for the cartesian product $C_n \square P_k$ is 3.

Case 1. First, we show that the minimum size of a doubly resolving set of vertices in $C_n \square P_k$ must be greater than 2. Consider the cartesian product $C_n \square P_k$ with the vertex set $\{x_1, \ldots, x_{n+1}\}$ on the layers $V_1, V_2, \ldots, V_k$, is defined already. Based on Theorem 2.1, we know that $\beta(C_n \square P_k) = 2$. We can show that if $n$ is an odd integer then all the elements of every minimum resolving set of vertices in $C_n \square P_k$ must lie in exactly one of the congruous layers $V_1$ or $V_k$. Without lack of theory if we consider the layer $V_1$ of the cartesian product $C_n \square P_k$ then we can show that all the minimum resolving sets of vertices in the layer $V_1$ of $C_n \square P_k$ are the sets as to form $M_i = \{x_i, x_{i+\lambda}\}$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $N_j = \{x_j, x_{j+\mu}\}$, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. On the other hand, we can see that the arranged subsets $M_i$ of vertices in $C_n \square P_k$ cannot be doubly resolving sets for $C_n \square P_k$ because for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and two compatible vertices $x_{i+\lambda}$ and $x_{i+2\lambda}$ with respect to $x_i$, we have $r(x_{i+\lambda}|M_i) - r(x_{i+2\lambda}|M_i) = -1$, where $I$ indicates the unit 2-vector (1, 1). By applying the same argument we can show that the arranged subsets $N_j$ of vertices in $C_n \square P_k$ cannot be doubly resolving sets for $C_n \square P_k$. Hence, the minimum size of a doubly resolving set in $C_n \square P_k$ must be greater than 2.

Case 2. Now, we show that the minimum size of a doubly resolving set of vertices in the cartesian product $C_n \square P_k$ is 3. For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, let $x_i$ be a vertex in the layer $V_i$ of $C_n \square P_k$ and $x_{c_i}$ be a compatible vertex with respect to $x_i$, where $x_c$ lie in the layer $V_c$ of $C_n \square P_k$. Then we can show that the arranged subsets $M_i = M_{i+c_i} \cup x_c = \{x_i, x_{i+c_i+1}, x_c\}$ of vertices in the cartesian product $C_n \square P_k$ are doubly resolving sets with the minimum size for the cartesian product $C_n \square P_k$. It will be enough to show that for any compatible vertices $x_i$ and $x_d$ in $C_n \square P_k$, $r(x_i|A_i) - r(x_d|A_i) \neq A_i$. Suppose $x_c \in V_p$ and $x_d \in V_q$ are compatible vertices in the cartesian product $C_n \square P_k$, $1 \leq p < q \leq k$. Hence, $r(x_c|M_{i+c_i}) - r(x_d|M_{i+c_i}) = -A_i$, where $\lambda$ is a positive integer, and $I$ indicates the unit 2-vector (1, 1). Also, for the compatible vertex $x_c$ with respect to $x_i$, $r(x_c|x_i) - r(x_c|x_{i+c_i+1}) = \lambda$. So, $r(x_c|A_i) - r(x_c|A_i) \neq A_i$, where $I$ indicates the unit 3-vector $(1, 1, 1)$. Especially, for $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ if we consider the arranged subsets $B_j = N_j \cup x_c = \{x_j, x_{j+c_j}, x_c\}$ of vertices in the cartesian product $C_n \square P_k$, where $x_c$ lie in the layer $V_c$ of the cartesian product $C_n \square P_k$ and $x_c$ is a compatible vertex with respect to $x_j$.

Figure 2. $(C_4 \square P_3) \square P_2$
then by applying the same argument we can show that the arranged subsets $B_j = N_j \cup x_i = \{x_j, x_{i[j+1]}, x_{i}\}$ of vertices in the cartesian product $C_n \square P_k$ are doubly resolving sets with the minimum size for the cartesian product $C_n \square P_k$.

\[\Box\]

**Theorem 3.2.** Consider the cartesian product $C_n \square P_k$. If $n \geq 3$ is an odd integer, then the minimum size of a resolving set of vertices in $(C_n \square P_k) \square P_2$ is 3.

**Proof.** Suppose $V((C_n \square P_k) \square P_2) = \{x^{(1)}_1, ..., x^{(1)}_{nk}\} \cup \{x^{(2)}_1, ..., x^{(2)}_{nk}\}$. Based on Theorem 2.1, we know that if $n \geq 3$ is an odd integer, then the minimum size of a resolving set of vertices in $C_n \square P_k$ is 2. Also, by definition of $(C_n \square P_k) \square P_2$, we can verify that for $1 \leq t \leq nk$, the vertex $x^{(1)}_t$ is adjacent to the vertex $x^{(2)}_t$ in $(C_n \square P_k) \square P_2$, and hence none of the minimal resolving sets of $C_n \square P_k$ cannot be a resolving set for $(C_n \square P_k) \square P_2$. Therefore, the minimum size of a resolving set of vertices in $(C_n \square P_k) \square P_2$ must be greater than 2. Now, we show that the minimum size of a resolving set of vertices in $(C_n \square P_k) \square P_2$ is 3. For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, let $x^{(1)}_i$ be a vertex in the layer $V^{(1)}_i$ of $(C_n \square P_k) \square P_2$ and $x^{(2)}_i$ be a compatible vertex with respect to $x^{(1)}_i$, where $x^{(1)}_i$ and $x^{(2)}_i$ lie in the layer $V^{(1)}_i$ of $(C_n \square P_k) \square P_2$. Based on Theorem 3.1, we know that for $1 \leq i \leq \lceil \frac{n}{2} \rceil$, $1^n$ copy of the arranged subsets $A_i = \{x_i, x_{i[j+1]}, x_{i}\}$, denoted by the sets $A^{(1)}_i = \{x^{(1)}_i, x^{(1)}_{i[j+1]}, x^{(1)}_{i}\}$ of vertices of $(C_n \square P_k) \square P_2$ are doubly resolving sets for the arranged set $\{x^{(1)}_1, ..., x^{(1)}_{nk}\}$ of vertices of $(C_n \square P_k)$, and hence the arranged sets $A^{(1)}_i = \{x^{(1)}_i, x^{(1)}_{i[j+1]}, x^{(1)}_{i}\}$ of vertices of $(C_n \square P_k) \square P_2$ are resolving sets for $(C_n \square P_k) \square P_2$, because for each vertex in the set $\{x^{(2)}_1, ..., x^{(2)}_{nk}\}$ of vertices of $(C_n \square P_k) \square P_2$, we have

$$r(x^{(2)}_i, A^{(1)}_i) = (d(x^{(2)}_i, x^{(1)}_i) + 1, d(x^{(2)}_i, x^{(1)}_{i[j+1]} + 1, d(x^{(2)}_i, x^{(1)}_{i}) + 1),$$

so all the vertices of $(C_n \square P_k) \square P_2$ have various representations with respect to the sets $A^{(1)}_i$, and hence the minimum size of a resolving set of vertices in $(C_n \square P_k) \square P_2$ is 3. In the same way for $1 \leq j \leq \lceil \frac{n}{2} \rceil$, if we consider $1^n$ copy the arranged subsets $B_j = \{x_j, x_{j[j+1]}, x_{j}\}$, denoted by the sets $B^{(1)}_j = \{x^{(1)}_j, x^{(1)}_{j[j+1]}, x^{(1)}_{j}\}$ of vertices of $(C_n \square P_k) \square P_2$, where $x^{(1)}_j$ lie in the layer $V^{(1)}_j$ of $(C_n \square P_k) \square P_2$ and $x^{(2)}_j$ is a compatible vertex with respect to $x^{(1)}_j$, then by applying the same argument we can show that the arranged sets $B^{(1)}_j = \{x^{(1)}_j, x^{(1)}_{j[j+1]}, x^{(1)}_{j}\}$ of vertices of $(C_n \square P_k) \square P_2$ are resolving sets for $(C_n \square P_k) \square P_2$.

\[\Box\]

**Lemma 3.1.** Consider the cartesian product $C_n \square P_k$. If $n \geq 3$ is an odd integer, then the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_2$ is greater than 3.

**Proof.** Suppose $V((C_n \square P_k) \square P_2) = \{x^{(1)}_1, ..., x^{(1)}_{nk}\} \cup \{x^{(2)}_1, ..., x^{(2)}_{nk}\}$ and $1 \leq t \leq nk$. For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, let $x^{(1)}_i$ be a vertex in the layer $V^{(1)}_i$ of $(C_n \square P_k) \square P_2$ and $x^{(2)}_i$ be a compatible vertex with respect to $x^{(1)}_i$, where $x^{(1)}_i$ and $x^{(2)}_i$ lie in the layer $V^{(1)}_i$ of $(C_n \square P_k) \square P_2$. Based on proof of Theorem 3.2, we know that the arranged sets $A^{(1)}_i = \{x^{(1)}_i, x^{(1)}_{i[j+1]}, x^{(1)}_{i}\}$ of vertices of $(C_n \square P_k) \square P_2$ cannot be doubly resolving sets for $(C_n \square P_k) \square P_2$, because

$$r(x^{(2)}_i, A^{(1)}_i) = (d(x^{(2)}_i, x^{(1)}_i) + 1, d(x^{(2)}_i, x^{(1)}_{i[j+1]} + 1, d(x^{(2)}_i, x^{(1)}_{i}) + 1).$$

In the same way for $1 \leq j \leq \lceil \frac{n}{2} \rceil$, if we consider the arranged sets $B^{(1)}_j = \{x^{(1)}_j, x^{(1)}_{j[j+1]}, x^{(1)}_{j}\}$ of vertices of $(C_n \square P_k) \square P_2$, where $x^{(1)}_j$ lie in the layer $V^{(1)}_j$ of $(C_n \square P_k) \square P_2$ and $x^{(2)}_j$ is a compatible vertex with respect to $x^{(1)}_j$, then we can show that the arranged sets $B^{(1)}_j$ cannot be doubly resolving sets for $(C_n \square P_k) \square P_2$. Hence the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_2$ is greater than 3.

\[\Box\]

**Theorem 3.3.** Consider the cartesian product $C_n \square P_k$. If $n \geq 3$ is an odd integer, then the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_2$ is 4.

**Proof.** Suppose $V((C_n \square P_k) \square P_2) = \{x^{(1)}_1, ..., x^{(1)}_{nk}\} \cup \{x^{(2)}_1, ..., x^{(2)}_{nk}\}$ and $1 \leq t \leq nk$. Based on Theorem 3.2, we know that if $n \geq 3$ is an odd integer, then $\beta((C_n \square P_k) \square P_2) = 3$ and by Lemma 3.1, we know that the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_2$ is greater than 3. In particular, it is well known that $\beta((C_n \square P_k) \square P_2) \leq \psi((C_n \square P_k) \square P_2)$. Now, we show that if $n \geq 3$ is an odd integer, then the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_2$ is 4. For $1 \leq t \leq \lceil \frac{n}{2} \rceil$, let $x^{(1)}_t$ be a vertex in the layer $V^{(1)}_t$ of $(C_n \square P_k) \square P_2$ and $x^{(2)}_t$ be a
compatible vertex with respect to $x_i^{(i)}$, where $x_i^{(i)}$ lie in the layer $V_k^{(i)}$ of $(C_n \square P_k) \square P_2$. Based on Lemma 3.1, we know that the arranged sets $A_i^{(i)} = \{ x_i^{(i)}, x_{i+1}^{(i)}, \ldots, x_{2t}^{(i)} \}$ of vertices of $(C_n \square P_k) \square P_2$, defined already, cannot be doubly resolving sets for $(C_n \square P_k) \square P_2$. Let, $C_i = A_i^{(i)} \cup x_i^{(2)} = \{ x_i^{(i)}, x_{i+1}^{(i)}, \ldots, x_{2t}^{(i)}, x_i^{(2)} \}$ be an arranged subset of vertices of $(C_n \square P_k) \square P_2$, where $x_i^{(2)}$ lie in the layer $V_k^{(2)}$ of $(C_n \square P_k) \square P_2$ and the vertex $x_i^{(2)}$ is adjacent to the vertex $x_i^{(1)}$. We show that the arranged subset $C_i$ is a doubly resolving set of vertices in $(C_n \square P_k) \square P_2$. It will be enough to show that for any adjacent vertices $x_i^{(1)}$ and $x_j^{(2)}$, $r(x_i^{(1)} | C_i) - r(x_j^{(2)} | C_i) \neq -I$, where $I$ indicates the unit 4-vector $(1, \ldots, 1)$. We can verify that, $r(x_i^{(1)} | A_i^{(i)}) - r(x_j^{(2)} | A_i^{(i)}) = -I$, where $I$ indicates the unit 3-vector, and $r(x_i^{(1)} | A_i^{(i)}) - r(x_j^{(2)} | A_i^{(i)}) = 1$. Thus the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_2$ is 4.

Conclusion 3.1. Consider the cartesian product $C_n \square P_k$. If $n \geq 3$ is an odd integer, then the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_m$ is 4.

Proof. Suppose $(C_n \square P_k) \square P_m$ is a graph with vertex set $\{ x_1^{(1)}, \ldots, x_m^{(1)} \}$ that for every connected graph $G$, applying the same argument in Theorem 3.3, we can see that the arranged sets $A_i^{(i)} = \{ x_i^{(i)}, x_{i+1}^{(i)}, \ldots, x_{2t}^{(i)} \}$ of vertices of $(C_n \square P_k) \square P_m$, defined already, are resolving sets with the minimum size for $(C_n \square P_k) \square P_m$, also by applying the same argument in Theorem 3.3, we can see that the arranged sets $A_i^{(i)} = \{ x_i^{(i)}, x_{i+1}^{(i)}, \ldots, x_{2t}^{(i)} \}$ of vertices of $(C_n \square P_k) \square P_m$, cannot be doubly resolving sets for $(C_n \square P_k) \square P_m$, and hence the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_m$ is greater than 3. Now, let $D_i = A_i^{(i)} \cup x_i^{(m)} = \{ x_i^{(i)}, x_{i+1}^{(i)}, \ldots, x_{2t}^{(i)}, x_i^{(m)} \}$ be an arranged subset of vertices of $(C_n \square P_k) \square P_m$, where $x_i^{(m)}$ lie in the layer $V_k^{(m)}$ of $(C_n \square P_k) \square P_m$. We show that the arranged subset $D_i$ is a doubly resolving set of vertices in $(C_n \square P_k) \square P_m$. It will be enough to show that for every two vertices $x_i^{(r)}$ and $x_i^{(s)}$, $1 \leq t \leq m$, $r(x_i^{(r)} | D_i) - r(x_i^{(s)} | D_i) \neq -I$, where $I$ indicates the unit 4-vector $(1, \ldots, 1)$ and $I$ is a positive integer. For this purpose, let the distance between the two vertices $x_i^{(r)}$ and $x_i^{(s)}$ in $(C_n \square P_k) \square P_m$ is $\lambda$, then we can verify that, $r(x_i^{(r)} | A_i^{(i)}) - r(x_i^{(s)} | A_i^{(i)}) = -I$, where $I$ indicates the unit 3-vector, and $r(x_i^{(r)} | A_i^{(i)}) - r(x_i^{(s)} | A_i^{(i)}) = \lambda$. Thus the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_m$ is 4.

Example 3.1. Consider graph $(C_3 \square P_2) \square P_4$ with vertex set $\{ x_1^{(1)}, \ldots, x_4^{(1)} \} \cup \{ x_5^{(1)}, \ldots, x_8^{(1)} \} \cup \{ x_9^{(1)}, \ldots, x_{12}^{(1)} \} \cup \{ x_{13}^{(1)}, \ldots, x_{16}^{(1)} \}$, we can see that the set $D_1 = \{ x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)} \}$ of vertices of $(C_3 \square P_2) \square P_4$ is one of the minimum doubly resolving sets for $(C_3 \square P_2) \square P_4$, and hence the minimum size of a doubly resolving set of vertices in $(C_3 \square P_2) \square P_4$ is 4.
Remark 3.1. Consider the cartesian product $C_n \square P_k$. If $n \geq 4$ is an even integer, then every pair of various vertices in $C_n \square P_k$ cannot be a resolving set for $C_n \square P_k$.

Lemma 3.2. Consider the cartesian product $C_n \square P_k$. If $n \geq 4$ is an even integer, then the minimum size of a doubly resolving set of vertices for the cartesian product $C_n \square P_k$ is 4.

Proof. In the following cases we show that the minimum size of a doubly resolving set of vertices for the cartesian product $C_n \square P_k$ is 4.

Case 1. Contrary to Theorem 3.1, if $n \geq 4$ is an even integer then every minimum resolving set of vertices in $C_n \square P_k$ may be lie in congruous layers $V_1$, $V_2$, or $V_1 \cup V_2$. Also, based on Theorem 2.2, we know that the minimum size of a resolving set in the cartesian product $C_n \square P_k$ is 3. If $E_1$ is an assigned subset of vertices of $C_n \square P_k$ so that $E_1$ is a minimal resolving set for $C_n \square P_k$ and two elements of $E_1$ lie in the layer $V_1$ and one element of $E_1$ lies in the layer $V_2$, then without loss of generality we can consider $E_1 = \{x_1, x_2, x_3\}$, where $x_j$ is a compatible vertex with respect to $x_1$ and $x_i$ lie in the layer $V_i$ of $C_n \square P_k$. Besides, the arranged subset $E_1 = \{x_1, x_2, x_3\}$ of vertices of $C_n \square P_k$ cannot be a doubly resolving set for $C_n \square P_k$, because $n$ is an even integer and each layer $V_p$, $1 \leq p \leq k$ of $C_n \square P_k$ is isomorphic to the cycle $C_n$, and hence there are two adjacent vertices in each layer $V_p$ of $C_n \square P_k$ say $x_i$ and $x_j$, $i < j$, so that $r(x_i|E_1) - r(x_j|E_1) = -I$, where $I$ indicates the unit 3-vector $(1, 1, 1)$. Thus the arranged subset $E_1 = \{x_1, x_2, x_3\}$ cannot be a doubly resolving set for $C_n \square P_k$. If $E_2$ is an arranged subset of vertices of $C_n \square P_k$ so that $E_2$ is a minimal resolving set for $C_n \square P_k$ and all the elements of $E_2$ lie in exactly one of the congruous layers $V_1$ or $V_2$, then without loss of generality we can consider $E_2 = \{x_1, x_4, x_4+1\}$. Besides, the arranged subset $E_2 = \{x_1, x_4, x_4+1\}$ of vertices in the layer $V_1$ of the cartesian product $C_n \square P_k$ cannot be a doubly resolving set for $C_n \square P_k$, because if we consider two compatible vertices $x_j \in V_p$ and $x_k \in V_q$ in $C_n \square P_k$, $1 \leq p < q \leq k$, then there is a positive integer $\lambda$ so that $r(x_j|E_2) - r(x_k|E_2) = -\lambda I$, where $I$ indicates the unit 3-vector $(1, 1, 1)$, and hence the arranged subset $E_2 = \{x_1, x_4, x_4+1\}$ cannot be a doubly resolving set for $C_n \square P_k$. Thus the minimum size of a doubly resolving set of vertices for the cartesian product $C_n \square P_k$ is greater than 3.

Case 2. Now, we show that the minimum size of a doubly resolving set of vertices in the cartesian product $C_n \square P_k$ is 4. Let $x_1$ be a compatible vertex with respect to $x_1$, where $x_i$ lie in the layer $V_i$ of $C_n \square P_k$ then we can show that the arranged subset $E_1 = E_2 \cup x_1 = \{x_1, x_2, x_3, x_4, x_4+1\}$ of vertices in the cartesian product $C_n \square P_k$ is one of the minimum doubly resolving sets for the cartesian product $C_n \square P_k$. It will be enough to show that for any compatible vertices $x_c$ and $x_d$ in $C_n \square P_k$, $r(x_c|E_3) - r(x_d|E_3) \neq \lambda I$. Suppose $x_c \in V_p$ and $x_d \in V_q$ are compatible vertices in the cartesian product $C_n \square P_k$, $1 \leq p < q \leq k$. Hence, $r(x_c|E_2) - r(x_d|E_2) = -\lambda I$, where $\lambda$ is a positive integer, and $I$ indicates the unit 3-vector $(1, 1, 1)$. Also, for the compatible vertex $x_c$ with respect to $x_1$, $r(x_c|E_3) - r(x_d|E_3) = \lambda$. So, $r(x_c|E_3) - r(x_d|E_3) \neq \lambda I$, where $I$ indicates the unit 4-vector $(1, 1, 1, 1)$.

Theorem 3.4. Consider the cartesian product $C_n \square P_k$. If $n \geq 4$ is an even integer, then the minimum size of a resolving set of vertices in $(C_n \square P_k) \square P_m$ is 4.

Proof. Suppose $(C_n \square P_k) \square P_m$ is a graph with vertex set $\{x^{(1)}_1, ..., x^{(1)}_m\} \cup \{x^{(2)}_1, ..., x^{(2)}_m\} \cup ... \cup \{x^{(m)}_1, ..., x^{(m)}_m\}$ so that for $1 \leq t \leq nk$, the vertex $x^{(t)}_i$ is adjacent to $x^{(r+1)}_i$ in $(C_n \square P_k) \square P_m$, for $1 \leq r \leq m - 1$. Hence, none of the minimal resolving sets of $C_n \square P_k$ cannot be a resolving set for $(C_n \square P_k) \square P_m$. Therefore, the minimum size of a resolving set of vertices in $(C_n \square P_k) \square P_m$ must be greater than 3. Now, we show that the minimum size of a resolving set of vertices in
(\(C_n \Box P_k\) \(\Box P_m\) is 4. Let \(x_1^{(l)}\) be a vertex in the layer \(V_1^{(l)}\) of \((C_n \Box P_k) \Box P_m\) and \(x_1^{(l)}\) be a compatible vertex with respect to \(x_1^{(l)}\), where \(x_1^{(l)}\) lie in the layer \(V_1^{(l)}\) of \((C_n \Box P_k) \Box P_m\). Based on Lemma 3.2, we know that \(l\)th copy of the arranged subset \(E_3 = \{x_1, x_2, x_3, x_t\}\), denoted by the set \(E_3^{(l)} = \{x_1^{(l)}, x_2^{(l)}, x_3^{(l)}, x_t^{(l)}\}\) is one of the minimum doubly resolving sets for the cartesian product \(C_n \Box P_k \Box P_m\). Besides, the vertex \(x_t^{(l)}\) is adjacent to the vertex \(x_t^{(l)}\) in \((C_n \Box P_k) \Box P_m\), and hence the arranged set \(E_3^{(l)} = \{x_1^{(l)}, x_2^{(l)}, x_3^{(l)}, x_t^{(l)}\}\) of vertices of \((C_n \Box P_k) \Box P_m\) is one of the resolving sets for \((C_n \Box P_k) \Box P_m\), because for each vertex \(x_t^{(l)}\) of \((C_n \Box P_k) \Box P_m\), we have
\[
r(x_t^{(l)}|E_3^{(l)}) = (d(x_t^{(l)}, x_1^{(l)}) + i - 1, d(x_t^{(l)}, x_2^{(l)}) + i - 1, d(x_t^{(l)}, x_3^{(l)}) + i - 1, d(x_t^{(l)}, x_t^{(l)}) + i - 1),
\]
so all the vertices \(\{x_1^{(l)}, ..., x_t^{(l)}\} \cup \cdots \cup \{x_m^{(l)}, ..., x_m^{(l)}\}\) of \((C_n \Box P_k) \Box P_m\) have various representations with respect to the set \(E_3^{(l)}\). Thus the minimum size of a resolving set of vertices \(\{x_1^{(l)}, ..., x_t^{(l)}\} \cup \cdots \cup \{x_m^{(l)}, ..., x_m^{(l)}\}\) is 4.

**Theorem 3.5.** Consider the cartesian product \(C_n \Box P_k\). If \(n \geq 4\) is an even integer, then the minimum size of a doubly resolving set of vertices \((C_n \Box P_k) \Box P_m\) is 5.

**Proof.** Suppose \((C_n \Box P_k) \Box P_m\) is a graph with vertex set \(\{x_1, ..., x_n\} \cup \{x_2, ..., x_2\} \cup \cdots \cup \{x_m, ..., x_m\}\) so that for \(1 \leq t \leq nk\), the vertex \(x_t^{(l)}\) is adjacent to \(x_t^{(l)}\) in \((C_n \Box P_k) \Box P_m\), for \(1 \leq r \leq m - 1\). Based on the previous Theorem, we know that the arranged set \(E_3^{(l)} = \{x_1^{(l)}, x_2^{(l)}, x_3^{(l)}, x_t^{(l)}\}\) of vertices of \((C_n \Box P_k) \Box P_m\) is one of the resolving sets for \((C_n \Box P_k) \Box P_m\), so that the arranged set \(E_3^{(l)} = \{x_1^{(l)}, x_2^{(l)}, x_3^{(l)}, x_t^{(l)}\}\) of vertices of \((C_n \Box P_k) \Box P_m\) cannot be a doubly resolving set for \((C_n \Box P_k) \Box P_m\), and hence the minimum size of a doubly resolving set of vertices in \((C_n \Box P_k) \Box P_m\) is greater than 4. Now, let \(E_2 = E_3^{(l)} \cup \{x^{(l)} \cup \{x^{(l)}, x^{(l)}\}, \cdots, x^{(l)} \cup \{x^{(l)}, x^{(l)}\}\}\) be an arranged subset of vertices of \((C_n \Box P_k) \Box P_m\), where \(x_t^{(l)}\) lie in the layer \(V_3^{(l)}\) of \((C_n \Box P_k) \Box P_m\). It will be enough to show that for every two vertices \(x_t^{(l)}\) and \(x_t^{(l)}\), \(1 \leq t \leq nk\), \(1 \leq r < s \leq m\), \(r(x_t^{(l)}|E_3) - r(x_t^{(l)}|E_3) \neq -A\), where \(I\) indicates the unit 5-vector \((1, 1, 1, 1, 1)\) and \(\lambda\) is a positive integer. For this purpose, let the distance between two the vertices \(x_t^{(l)}\) and \(x_t^{(l)}\) in \((C_n \Box P_k) \Box P_m\) is \(A\), then we can verify that, \(r(x_t^{(l)}|E_3) - r(x_t^{(l)}|E_3) = -A\), where \(I\) indicates the unit 4-vector, and \(r(x_t^{(l)}|x^{(l)} - r(x_t^{(l)}|x^{(l)}) = 1\). Therefore, the arranged subset \(E_2\) is one of the minimum doubly resolving sets of vertices in \((C_n \Box P_k) \Box P_m\). Thus the minimum size of a doubly resolving set of vertices \((C_n \Box P_k) \Box P_m\) is 5.

**Example 3.2.** Consider graph \((C_n \Box P_k) \Box P_m\) with vertex set \(\{x_1, ..., x_n\} \cup \{x_2, ..., x_2\} \cup \{x_3, ..., x_3\} \cup \{x_1, ..., x_4\}\), we can see that the set \(E = \{x_1, x_1, x_1, x_4\}\) of vertices of \((C_n \Box P_k) \Box P_m\) is one of the minimum doubly resolving sets for \((C_n \Box P_k) \Box P_m\), and hence the minimum size of a doubly resolving set of vertices in \((C_n \Box P_k) \Box P_m\) is 4.

**Theorem 3.6.** If \(n\) is an even or odd integer is greater than or equal to 3, then the minimum size of a strong resolving set of vertices for the cartesian product \((C_n \Box P_k) \Box P_m\) is 2n.
Proof. Suppose that $(C_n \Box P_k) \Box P_m$ is a graph with vertex set $\{x^{(1)}_1, ..., x^{(1)}_{nk}\} \cup \{x^{(2)}_1, ..., x^{(2)}_{nk}\} \cup ... \cup \{x^{(m)}_1, ..., x^{(m)}_{nk}\}$ so that for $1 \leq t \leq nk$, the vertex $x^{(t)}_i$ is adjacent to $x^{(t+1)}_i$ in $(C_n \Box P_k) \Box P_m$, for $1 \leq r \leq m - 1$. We know that each vertex of the layer $V^{(1)}_1$ is maximally distant from a vertex of the layer $V^{(m)}_1$ and each vertex of the layer $V^{(m)}_1$ is maximally distant from a vertex of the layer $V^{(1)}_1$. In particular, each vertex of the layer $V^{(m)}_1$ is maximally distant from a vertex of the layer $V^{(1)}_1$ and each vertex of the layer $V^{(1)}_1$ is maximally distant from a vertex of the layer $V^{(m)}_1$, and hence the minimum size of a strong resolving set of vertices for the cartesian product $(C_n \Box P_k) \Box P_m$ is equal or greater than $2n$, because it is well known that for every pair of mutually maximally distant vertices $u$ and $v$ of a connected graph $G$ and for every strong metric basis $S$ of $G$, it follows that $u \in S$ or $v \in S$. Suppose the set $\{x^{(1)}_1, ..., x^{(1)}_{nk}\}$ is an arranged subset of vertices in the layer $V^{(1)}_1$ of the cartesian product $(C_n \Box P_k) \Box P_m$ and suppose that the set $\{x^{(m)}_1, ..., x^{(m)}_{nk}\}$ is an arranged subset of vertices in the layer $V^{(m)}_1$ of the cartesian product $(C_n \Box P_k) \Box P_m$. Now, let $T = \{x^{(1)}_1, ..., x^{(1)}_{nk}\} \cup \{x^{(m)}_1, ..., x^{(m)}_{nk}\}$ be an arranged subset of vertices of the cartesian product $(C_n \Box P_k) \Box P_m$. In the following cases we show that the arranged set $T$, defined already, is one of the minimum strong resolving sets of vertices for the cartesian product $(C_n \Box P_k) \Box P_m$. For this purpose let $x^{(i)}_e$ and $x^{(i)}_d$ be two various vertices of $(C_n \Box P_k) \Box P_m$, $1 \leq i \leq m$, $1 \leq e, d \leq nk$ and $1 \leq r \leq n$.

Case 1. If $i = j$ then $x^{(i)}_e$ and $x^{(i)}_d$ lie in $i^{th}$ copy of $(C_n \Box P_k)$ with vertex set $\{x^{(i)}_1, ..., x^{(i)}_{nk}\}$ so that $i^{th}$ copy of $(C_n \Box P_k)$ is a subgraph of $(C_n \Box P_k) \Box P_m$. Since $i = j$ then we can assume that $e < d$, because $x^{(i)}_e$ and $x^{(i)}_d$ are various vertices.

Case 1.1. If both vertices $x^{(i)}_e$ and $x^{(i)}_d$ are compatible in $i^{th}$ copy of $(C_n \Box P_k)$ relative to $x^{(i)}_r \in V^{(i)}_1 \subset T$ so that $x^{(i)}_r$ belongs to shortest path $x^{(i)}_e - x^{(i)}_d$, say as $x^{(i)}_1, x^{(i)}_2, ..., x^{(i)}_{l-1}, x^{(i)}_l, x^{(i)}_{l+1}, ..., x^{(i)}_d$.

Case 1.2. Suppose both vertices $x^{(i)}_e$ and $x^{(i)}_d$ are not compatible in $i^{th}$ copy of $(C_n \Box P_k)$, and lie in various layers or lie in the same layer in $i^{th}$ copy of $(C_n \Box P_k)$, also let $x^{(i)}_r \in V^{(i)}_1 \subset T$ so that $x^{(i)}_r$ belongs to shortest path $x^{(i)}_e - x^{(i)}_d$, say as $x^{(i)}_1, x^{(i)}_2, ..., x^{(i)}_l, x^{(i)}_{l+1}, ..., x^{(i)}_d$.

Case 2. If $i \neq j$ then $x^{(j)}_e$ lie in $j^{th}$ copy of $(C_n \Box P_k)$ with vertex set $\{x^{(j)}_1, ..., x^{(j)}_{nk}\}$, and $x^{(j)}_d$ lie in $j^{th}$ copy of $(C_n \Box P_k)$ with vertex set $\{x^{(j)}_1, ..., x^{(j)}_{nk}\}$. In this case we can assume that $i < j$.

Case 2.1. If $e = d$ and $x^{(j)}_e \in V^{(j)}_1$ is a compatible vertex relative to $x^{(j)}_d$, then there is the vertex $x^{(j)}_r \in V^{(j)}_1 \subset T$ so that $x^{(j)}_r$ belongs to shortest path $x^{(j)}_e - x^{(j)}_d$, say as $x^{(j)}_1, x^{(j)}_2, ..., x^{(j)}_l, x^{(j)}_{l+1}, ..., x^{(j)}_d$.

Case 2.2. If $e < d$, also $x^{(j)}_e$ and $x^{(j)}_d$ lie in various layers of $(C_n \Box P_k) \Box P_m$ or $x^{(j)}_e$ and $x^{(j)}_d$ lie in the same layer of $(C_n \Box P_k) \Box P_m$ and $x^{(j)}_r \in V^{(j)}_1$ is a compatible vertex relative to $x^{(j)}_r$, then there is the vertex $x^{(j)}_r \in V^{(j)}_1 \subset T$ so that $x^{(j)}_r$ belongs to shortest path $x^{(j)}_e - x^{(j)}_d$, say as $x^{(j)}_1, x^{(j)}_2, ..., x^{(j)}_l, x^{(j)}_{l+1}, ..., x^{(j)}_d$.

Case 2.3. If $e > d$, also $x^{(j)}_e$ and $x^{(j)}_d$ lie in various layers of $(C_n \Box P_k) \Box P_m$ or $x^{(j)}_e$ and $x^{(j)}_d$ lie in the same layer of $(C_n \Box P_k) \Box P_m$ and $x^{(j)}_r \in V^{(j)}_1$ is a compatible vertex relative to $x^{(j)}_r$, then there is the vertex $x^{(j)}_r \in V^{(j)}_1 \subset T$ so that $x^{(j)}_r$ belongs to shortest path $x^{(j)}_e - x^{(j)}_d$, say as $x^{(j)}_1, x^{(j)}_2, ..., x^{(j)}_l, x^{(j)}_{l+1}, ..., x^{(j)}_d$.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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