A NOTE ON IRREDUCIBLE QUADRILATERALS OF $II_1$ FACTORS

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Abstract. Given any quadruple $(N, P, Q, M)$ of $II_1$-factors with finite index, the notions of interior and exterior angles between $P$ and $Q$ were introduced in [1]. We determine the possible values of these angles in terms of the cardinalities of the Weyl groups of the intermediate subfactors when $(N, P, Q, M)$ is an irreducible quadrilateral and the subfactors $N \subset P$ and $N \subset Q$ are both regular. For an arbitrary irreducible quadruple, an attempt is made to determine the values of angles by deriving expressions for the angles in terms of the common norm of two naturally arising auxiliary operators and the indices of the intermediate subfactors of the quadruple. Finally, certain bounds on angles between $P$ and $Q$ are obtained when $N \subset P$ is regular, which enforce some restrictions on the index of $N \subset Q$ in terms of that of $N \subset P$.

1. Introduction

A quadrilateral is a quadruple $(N, P, Q, M)$ of $II_1$-factors such that $N \subset P, Q \subset M, N = P \wedge Q, M = P \vee Q$ and $[M : N] < \infty$; it is called irreducible if $N \subset M$ is irreducible. The Weyl group of a finite index $II_1$-subfactor $N \subset M$ is the quotient group $G := \mathcal{N}_M(N)/\mathcal{U}(N)$, where $\mathcal{N}_M(N)$ denotes the group of unitary normalizers of $N$ in $M$, i.e., $\mathcal{N}_M(N) := \{u \in \mathcal{U}(M) : uNu^* = N\}$. This article concentrates mainly on the analysis of such quadrilaterals from the perspectives of (a) calculating the interior and exterior angles between $P$ and $Q$ as was introduced in [1], (b) understanding the Weyl group of $N \subset M$ in terms of those of $N \subset P$ and $N \subset Q$, and (c) establishing a relationship between the above two aspects.

Unlike the notion of set of angles by Sano and Watatani ([14]), the interior and exterior angles are both single entities and are seemingly more calculable, as we show in Section 2.2 by making some explicit calculations. As an important application of the notion of interior angle, the authors in [1] were able to improve a result of Longo [10] by providing a better bound for the number of intermediate subfactors of a given irreducible subfactor.

A natural question that struck us, after the appearance of [1], was to determine the possible set of values that the interior and exterior angles can attain. This article is devoted to this theme. In general, it looks like a tough nut to crack. However, in the irreducible set up, we see that these angles take some definable values.

In Section 2, we discuss various generalities and formulae related to the interior and exterior angles and employ them to compute angles between two intermediate subfactors associated with a quadruple of crossed product algebras.

In Section 3, our main focus is on irreducible quadrilaterals $(N, P, Q, M)$ for which $N \subset P$ and $N \subset Q$ are both regular. Recall that an unital inclusion of von Neumann algebras $A \subset B$ is said to be regular if $\mathcal{N}_B(A)'' = B$. Jones, in [6], had asked whether an irreducible regular subfactor is always a group subfactor. Making use of a theorem of Sutherland [15] on vanishing of cohomologies, Popa [12] and Kosaki [9] (for properly infinite case) answered Jones’ question in the affirmative, which was announced earlier for the hyperfinite case by Ocnenanu in 1986. Later, Hong gave an explicit realization of the same in [5]. Using Hong’s technique, we deduce (in Theorem 3.5) that an irreducible quadrilateral $(N, P, Q, M)$ with regular $N \subset P$ and $N \subset Q$ can

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be realized as a quadrilateral of crossed product algebras through outer actions of Weyl groups. Using this realization and the calculations of Section 2.2 we provide a direct relationship between the interior and exterior angles between \( P \) and \( Q \) and the Weyl groups of \( N \subset P \) and \( N \subset Q \) in:

**Theorem 3.9** Let \((N, P, Q, M)\) be an irreducible quadrilateral such that \( N \subset P \) and \( N \subset Q \) are both regular. Then, \( \alpha(P, Q) = \pi/2 \), i.e., \((N, P, Q, M)\) is a commuting square, and

\[
\cos(\beta(P, Q)) = \frac{|G|}{|H||K|} - 1 \quad \sqrt{|G : H| - 1}\sqrt{|G : K| - 1},
\]

where \( H, K \) and \( G \) denote the Weyl groups of \( N, P, Q, M \), respectively.

In particular, \((N, P, Q, M)\) is a cocommuting square if and only if \( G = HK \).

Section 4 dwells around the main theme of this article, viz., to determine the possible values of the interior and exterior angles. We first derive expressions for the angles in terms of the common norm \( \lambda \) of two naturally arising auxiliary operators and the indices of the intermediate subfactors of the quadruple (in Proposition 4.5). Then, in the irreducible setup, we exploit these expressions to obtain some definitive values for angles by making use of above relationship between angles and Weyl groups, a theorem of Popa [13] wherein he determines the possible values taken by the set \( \Lambda(M, N) \) of relative dimensions of projections, and relating \( \lambda \) with certain polynomials \( P_n(x), n \geq 0 \), which are near relatives of the Chebyshev polynomials as introduced by Jones in [6]. The results that we prove are:

**Theorem 4.11** Let \((N, P, Q, M)\) be a quadruple with \( N \subset M \) irreducible, \( r := \lceil Q : N \rceil, \quad \tau := \lceil M : N \rceil^{-1} \) and let \( 0 < t \leq 1/2 \) be such that \( t(1 - t) = \tau \). If \( r/\lambda \geq t \), then,

\[
\cos(\alpha(P, Q)) \leq \frac{|P : N||Q : N|(1 - t) - 1}{\sqrt{|P : N| - 1}\sqrt{|Q : N| - 1}}
\]

and

\[
\cos(\beta(P, Q)) \leq \frac{\frac{1}{2} - 1}{\sqrt{|M : P| - 1}\sqrt{|M : Q| - 1}}
\]

And, if \( r/\lambda < t \), then,

\[
\cos(\alpha(P, Q)) = \frac{|P : N||Q : N|\frac{P_k(\tau)}{P_{k-1}(\tau)} - 1}{\sqrt{|P : N| - 1}\sqrt{|Q : N| - 1}}
\]

and

\[
\cos(\beta(P, Q)) = \frac{\frac{P_k(\tau)}{P_{k-1}(\tau)} - 1}{\sqrt{|M : P| - 1}\sqrt{|M : Q| - 1}}
\]

for some \( k \geq 0 \).

**Theorem 4.12** Let \((N, P, Q, M)\) be an irreducible quadrilateral such that \( N \subset P \) and \( N \subset Q \) are both regular and suppose \( |P : N| = 2 \). Then, \( \cos(\beta(P, Q)) = \frac{P_3(m/2)}{\sqrt{P_2(\delta^2/2)P_3(m/2)}} \), where \( m = \lceil M : Q \rceil \in \mathbb{N} \) and, as usual \( \delta := \sqrt{|M : N|} \).

As a ‘geometric’ consequence, in Corollary 4.13 we see that if both \( N \subset P \) and \( N \subset Q \) have index 2, then the exterior angle \( \beta(P, Q) > \pi/3 \).

Finally, while analyzing a quadrilateral intuitively as a picture in the plane (Figure 1), loosely speaking, we realize in Section 5 that the angles impose some sort of rigidity on the lengths of its sides. This could be inferred as a direct consequence of certain bounds on interior and exterior angles that we obtain in:
Theorem 5.1. Let \((N, P, Q, M)\) be a finite index irreducible quadruple such that \(N \subset P\) is regular. Then,
\[
\cos(\alpha(P,Q)) \leq \left(\sqrt{\frac{[P : N] - 1}{[Q : N] - 1}}\right)
\]
and
\[
\cos(\beta(P,Q)) \leq \left(\sqrt{\frac{[P : N] - r}{[Q : N] - r}}\right).
\]

The flow of the article revolves around the results mentioned above, more or less in the same order.

2. Interior and Exterior angles between intermediate subfactors

In this section, we first recall the notions of interior and exterior angles between intermediate subfactors of a given subfactor as introduced by Bakshi et al. in [1] and some useful formulae related to them. This will be followed by some further generalities and explicit calculations related to these angles.

In this article, we will be dealing only with subfactors and quadruples of type \(\text{II}_1\) with finite Jones’ index. Given any such quadruple \(Q \subset M \cup \cup N \subset P\), consider the basic constructions \(N \subset M \subset M_1\), \(P \subset M \subset P_1\) and \(Q \subset M \subset Q_1\). As is standard, we denote by \(e_1\) the Jones projection \(e_M^{M_1}\). It is easily seen that, as \(\text{II}_1\)-factors acting on \(L^2(M)\), both \(P_1\) and \(Q_1\) are contained in \(M_1\). In particular, if \(e_P : L^2(M) \to L^2(P)\) denotes the orthogonal projection, then \(e_P \in M_1\). Likewise, \(e_Q \in M_1\). Thus, we naturally obtain a dual quadruple
\[
P_1 \subset M_1 \cup \cup M \subset Q_1.
\]

2.1. Some useful formulae related to interior and exterior angles. We first list some plausible facts from [1] that make computations of the interior and exterior angles more amenable.

Definition 2.1. [1] Let \(P\) and \(Q\) be two intermediate subfactors of a subfactor \(N \subset M\). Then, the interior angle \(\alpha_N^M(P,Q)\) between \(P\) and \(Q\) is given by
\[
\alpha_N^M(P,Q) = \cos^{-1}\left(\langle v_P, v_Q \rangle_2\right),
\]
where \(v_P := \frac{e_P - e_1}{\|e_P - e_1\|_2}\), \(\langle x, y \rangle_2 := \text{tr}(y^*x)\) and \(\|x\|_2 := (\text{tr}(x^*x))^{1/2}\). And, the exterior angle between \(P\) and \(Q\) is given by \(\beta_M^N(P,Q) = \alpha_{M_1}(P_1, Q_1)\).

We will avoid being pedantic and often drop the superscript \(N\) and the subscript \(M\) when the subfactor \(N \subset M\) is clear from the context. Recall that a (right) Pimsner-Popa basis for a subfactor \(N \subset M\) is a finite collection \(\{\lambda_i : i \in I\}\) in \(M\) satisfying \(\lambda_i e_1 \lambda_i^* = 1\) or, equivalently, \(x = \sum_i E_N(x \lambda_i) \lambda_i^*\) for all \(x \in M\) - see [11] [8] for details.

Theorem 2.2. [1] For a quadruple \((N, P, Q, M)\), let \(\tau_P = \text{tr}(e_P)\), \(\tau_Q = \text{tr}(e_Q)\) and \(\tau = \text{tr}(e_1)\). Then, the interior angle \(\alpha(P,Q)\) satisfies
\[
\cos(\alpha(P,Q)) = \frac{\text{tr}(e_P e_Q) - \tau}{\sqrt{\tau_P - \tau^2} \sqrt{\tau_Q - \tau}}.
\]
which, then, yields that

\[(2.2) \quad \cos(\alpha(P, Q)) = \frac{\sum_{i,j} \text{tr}_M\left(E_N^M(\lambda_i^* \mu_j)\mu_i^* \lambda_j\right) - 1}{\sqrt{|P : N|} - 1 \sqrt{|Q : N|} - 1}\]

for any two Pimsner-Popa bases \(\{\lambda_i\}\) and \(\{\mu_j\}\) of \(P/N\) and \(Q/N\), respectively. And, if the quadruple is extremal, i.e., \(N \subset M\) is extremal, then the exterior angle \(\beta(P, Q)\) satisfies

\[(2.3) \quad \cos(\beta(P, Q)) = \frac{\text{tr}(e_P e_Q) - \tau_P \tau_Q}{\sqrt{\tau_P - \tau_Q} \sqrt{\tau_Q - \tau_P^2}}.\]

The following useful expression for \(\text{tr}(e_P e_Q)\) is quite evident from Equation (2.1) and Equation (2.2); the details can be readily extracted from the proof of [1, Proposition 2.14].

**Lemma 2.3.** Let \(N \subset M\) be a subfactor and \(P\) and \(Q\) be two intermediate subfactors. Then,

\[
\text{tr}_{M_1}(e_P e_Q) = \tau \sum_{i,j} \|E_N(\lambda_i^* \mu_j)\|_2^2
\]

for any two Pimsner-Popa bases \(\{\lambda_i\}\) and \(\{\mu_j\}\) of \(P/N\) and \(Q/N\), respectively, where \(\tau := |M : N|^{-1}\).

The following useful relationship between \(\alpha(P, Q)\) and \(\beta(P, Q)\) was mentioned in [1], following Definition 3.6, without any proof. For the sake of completeness, we include a proof using the planar algebraic technique of Jones (though, only for the extremal case, which will be enough for our requirements).

**Lemma 2.4.** For an extremal subfactor \(N \subset M\) with intermediate subfactors \(P\) and \(Q\), we have

\[
\alpha_M^N(P, Q) = \beta_M^N(P_1, Q_1).
\]

**Proof.** We have a tower

\[N \subset P \subset M \subset P_1 \subset M_1 \subset P_2 \subset M_2\]

where \(P_i \subset M_i \subset P_{i+1} = (M_i, e_{P_i})\) is a basic construction with Jones projection \(e_{P_i} = e_{0,i+1} : L^2(M_i) \to L^2(P_i)\), and \(P_0 := P\) - see [2 §3]. Likewise, we have another tower

\[N \subset Q \subset M \subset Q_1 = (M, e_Q) \subset M_1 \subset Q_2 = (M_1, e_{Q_1}) \subset M_2.\]

From [2, Lemma 4.2], we have \(e_{P_2} = \bullet\)

We have a similar figure for \(e_{Q_2}\) with respect to \(e_Q\). From this pictorial description, it is readily seen through pictures that

\[
\text{tr}(e_P e_Q) = \text{tr}(e_{P_2} e_{Q_2}), \quad \text{tr}(e_P) = \text{tr}(e_{P_2}) \quad \text{and} \quad \text{tr}(e_Q) = \text{tr}(e_{Q_2}).
\]

From Equation (2.1), we have

\[
\cos(\alpha_M^N(P, Q)) = \frac{\text{tr}(e_P e_Q) - \tau_P \tau_Q}{\sqrt{\tau_P - \tau_Q} \sqrt{\tau_Q - \tau_P^2}} \quad \text{and}, \quad \cos(\alpha_M^N(P_2, Q_2)) = \frac{\text{tr}(e_{P_2} e_{Q_2}) - \tau_{P_2} \tau_{Q_2}}{\sqrt{\tau_{P_2} - \tau_{Q_2}} \sqrt{\tau_{Q_2} - \tau_{P_2}}^2}.
\]

Finally, employing the above equalities obtained through pictures, we obtain

\[
\cos(\beta_M^N(P_1, Q_1)) = \cos(\alpha_M^N(P_2, Q_2)) = \cos(\alpha_M^N(P, Q)),
\]

as was desired. \(\square\)

Recall that two subfactors \(N \subset M\) and \(\mathcal{N} \subset \mathcal{M}\) are said to be isomorphic (denoted as \((N \subset M) \cong (\mathcal{N} \subset \mathcal{M})\)) if there exists a *-isomorphism \(\varphi\) from \(M\) onto \(\mathcal{M}\) such that \(\varphi(N) = \mathcal{N}\). Likewise, two quadruples \((N, P, Q, M)\) and \((\mathcal{N}, \mathcal{P}, \mathcal{Q}, \mathcal{M})\) are said to be isomorphic if there is an isomorphism \(\varphi\) between the subfactors \(N \subset M\) and \(\mathcal{N} \subset \mathcal{M}\) such that \(\varphi(P) = \mathcal{P}\) and \(\varphi(Q) = \mathcal{Q}\).
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Remark 2.5. Since Pimsner-Popa bases are preserved by isomorphisms of subfactors, in view of Theorem 2.2 and Lemma 2.3, we observe that an isomorphism between two quadruples preserves interior and exterior angles.

Recall that a quadruple $(N, P, Q, M)$ is said to be a commuting square if $e_P e_Q e_1 = e_q e_P$. It is said to be a cocommuting square if the dual quadruple $(M, P_1, Q_1, M)$ is a commuting square. It is said to be non-degenerate (resp., irreducible) if $\text{span}(P Q) = M$ (resp., $N' \cap M = \mathbb{C}$). Further, it is said to be a parallelogram if $\tau_P \tau_Q = \tau$ or, equivalently, if $[M : P] = [Q : N]$ or $[M : Q] = [P : N]$. And, a quadruple $(N, P, Q, M)$ is said to be a quadrilateral if $P \lor Q = M$ and $P \land Q = N$.

Remark 2.6. Commuting and cocommuting conditions have very natural interpretations in terms of above angles, viz., a quadruple $(N, P, Q, M)$ is a commuting (resp., co-commuting) square if and only if $\alpha(P, Q)$ (resp., $\beta(P, Q)$) equals $\pi/2$ - see [1, §2].

2.2. Computation of angles for quadruples of crossed product algebras.

Proposition 2.7. Let $G$ be a finite group acting outerly on a $II_1$-factor $S$. Let $H, K$ and $L$ be subgroups of $G$ such that $L \subseteq H \cap K$ and $H$ and $K$ are non-trivial. Consider the quadruple $(N = S \rtimes L, P = S \rtimes H, Q = S \rtimes K, M = S \rtimes G)$. Then,

\begin{align*}
(2.4) \quad \text{tr}(e_P e_Q) &= \frac{|H \cap K|}{|G|}, \\
(2.5) \quad \cos(\alpha(P, Q)) &= \frac{|H \cap K| - |L|}{\sqrt{|H| - |L|} \sqrt{|K| - |L|}} \\
\text{and} \\
(2.6) \quad \cos(\beta(P, Q)) &= \frac{|G| - 1}{\sqrt{|G : H|} - 1 \sqrt{|G : K|} - 1}.
\end{align*}

In particular, as is well known, $(N, P, Q, M)$ is a commuting (resp., cocommuting) square if and only if $H \cap K = L$ (resp., $G = HK$).

Proof. Note that if $\alpha : G \to \text{Aut}(S)$ denotes the action of $G$ on $S$, then there is a unitary representation $G \ni t \mapsto u_t \in B(L^2(S))$, such that $u_t(x\Omega) = \alpha_t(x)\Omega$ for all $x \in S$ - see [5, §A.4].

Fix left coset representatives $\{h_i : 1 \leq i \leq [H : L]\}$ and $\{k_j : 1 \leq j \leq [K : L]\}$ of $L$ in $H$ and $K$, respectively. Since $E_N(\sum x_i u_i) = \sum x_i u_i$, it follows that $\{h_i : 1 \leq i \leq [H : L]\}$ and $\{k_j : 1 \leq j \leq [K : L]\}$ are (right) orthonormal bases for $P/N$ and $Q/N$, respectively. So, by
Lemma 2.8, we obtain
\[
\text{tr}(ePeQ) = [G : L]^{-1} \sum_{i,j} \text{tr}_M \left( E_N \left( u_i^{-1} u_j \right) u_i^{-1} u_j \right)
\]
\[
= [G : L]^{-1} \sum_{\{i,j\} : h_i L = h_j L} \text{tr}_M \left( u_i^{-1} u_j \right) \quad \text{(since } E_N \left( \sum_{g \in G} x_g u_g \right) = \sum_{l \in L} x_l u_l \text{)}
\]
= [G : L]^{-1} \{ \{(i,j) : k_j \in h_i L\} \}
\]
and note that the map
\[
\{(i,j) : h_i L \cap k_j L \neq \emptyset\} \ni (i,j) \mapsto h_i L = k_j L \in (H \cap K)/L
\]
is a natural bijection; so that, \( \text{tr}(ePeQ) = \frac{|H \cap K|}{[G]} \). Then, from Equation (2.1), we immediately obtain
\[
\cos(\alpha(P,Q)) = \frac{|H \cap K| - |L|}{\sqrt{|H| - |L| / |K| - |L|}}
\]
and, from Equation (2.3), through an elementary simplification, we deduce that
\[
\cos(\beta(P,Q)) = \frac{|G|}{|HK|} - 1 \quad \frac{|G : H| - 1}{\sqrt{|G : K| - 1}}
\]
The commuting and cocommuting conditions follow from Remark 2.10. \(\square\)

Corollary 2.8. Let \( H, K, G \) and \( S \) be as in Proposition 2.7. Consider the quadruple \( (N = S^G, P = S^H, Q = S^K, M = S) \). Then,
\[
\cos(\alpha(P,Q)) = \frac{|G|}{|HK|} - 1 \quad \frac{|G : H| - 1}{\sqrt{|G : K| - 1}}
\]
and
\[
\cos(\beta(P,Q)) = \frac{|H \cap K| - 1}{\sqrt{|H| - 1 / |K| - 1}}
\]
In particular, \( (N, P, Q, M) \) is a commuting square if and only if \( HK = G \). And, it is a cocommuting square if and only if \( H \cap K = \{e\} \).

Proof. Since \( S^G \subset S \) is extremal, by Lemma 2.4 we have
\[
\alpha_M^N(P,Q) = \beta_M^N(P_1, Q_1).
\]
Outhere, we have \( M = S, P_1 = S \rtimes H, Q_1 = S \rtimes K \) and \( M_1 = S \times G \); so, by Proposition 2.7 (taking \( L \) to be the trivial subgroup), we obtain
\[
\cos(\alpha_M^N(S^H, S^K)) = \cos(\beta_M^N(S \rtimes H, S \times K)) = \frac{|G|}{|HK|} - 1 \quad \frac{1}{\sqrt{|G : H| - 1 / |G : K| - 1}}.
\]
On the other hand, by definition, we have \( \beta_M^N(P,Q) = \alpha_M^N(P_1, Q_1) \). Hence, by Proposition 2.7, we obtain
\[
\cos(\beta(P,Q)) = \frac{|H \cap K| - 1}{\sqrt{|H| - 1 / |K| - 1}}.
\]
\(\square\)

It was shown in [1, § 5] that the notion of Sano-Watatani’s set of angles does not agree with the notion of interior angle. Using Corollary 2.8, we add to that list and show that the Sano-Watatani’s set of angles and the interior angle may not be equal even if the former is a singleton.
Example 2.9. Consider the quadruple \((N = R^G, P = R^H, Q = R^K, M = R)\) with the assumption that \(H \cap K = \{e\}\), \(|H \cap K| = 2\), \(H\) and \(K\) are both non-trivial subgroups. Then, the Sano-Watatani’s set of angles \(\text{Ang}_M(P, Q)\) is a singleton and \(\{\alpha(P, Q)\} \neq \text{Ang}_M(P, Q)\).

Proof. From 14 Lemma 5.3 and Proposition 5.2, we have \(\text{Ang}_M(P, Q)\) is a singleton, namely,

\[
\text{Ang}_M(P, Q) = \left\{ \cos^{-1} \left( \frac{|G| - |H||K|}{|K||(|G| - |H|)} \right)^{1/2} \right\}.
\]

And, by Corollary 2.8, we have

\[
\cos(\alpha(P, Q)) = \frac{|G| - 1}{\sqrt{|G| - 1|K| - 1}} = \frac{|G| - 1}{\sqrt{|G||K| - 1}},
\]

where the second equality follows because \(H \cap K = \{e\}\) gives \(|HK| = |H||K|\). Thus, \(\{\alpha(P, Q)\} = \text{Ang}_M(P, Q)\) if and only if

\[
\left( \frac{|G| - |H||K|}{|K||(|G| - |H|)} \right)^{1/2} = \frac{|G| - 1}{|H||K| - 1}.
\]

Note that \(RHS = \left( \frac{|G| - |H||K|}{|G| - |H|} \right)^{1/2} \cdot \left( \frac{|G| - |H||K|}{|K||(|G| - |H|)} \right)^{1/2}\). Hence (2.7) is true if and only if

\[
1 = \frac{|G| - |H||K|}{(|G| - |K||H||K|)},
\]

which is then true if and only if \(|H| = 1\), which is not true since \(H\) is not the trivial subgroup. \(\square\)

We conclude this subsection by deducing the following well known fact.

Example 2.10. Let \(N \subset M\) be a subfactor and \(G\) be a finite group acting outerly (through \(\alpha\)) on \(M\). Then, \((N, N \rtimes G, M, M \rtimes G)\) is a commuting square.

Proof. Note that \(E_N^{M \rtimes G}(\sum_\alpha a_\alpha u_\alpha) = E_N^M(a_e)\). Indeed, for any \(b \in N\), we have

\[
\text{tr}\left( \left( \sum_\alpha a_\alpha u_\alpha \right) b \right) = \text{tr}\left( \sum_\alpha a_\alpha (b) u_\alpha \right) = \text{tr}(a_e b),
\]

and, on the other hand, \(\text{tr}(E_N^M(a_e) b) = \text{tr}(\sum_\alpha a_\alpha (b)) = \text{tr}(a_e b)\).

Let \(\{\lambda_i\}\) be a (right) basis for \(M/N\). Then, from Equation (2.1), we obtain

\[
\cos(\alpha(N \rtimes G, M)) = \frac{\sum_i \text{tr}(E_N^M(\lambda_i^*) u_\alpha \lambda_i) - 1}{\sqrt{|M : N| - 1|G| - 1}} = \frac{\text{tr}\left( \sum_i E_N^M(\lambda_i^*) \lambda_i \right) - 1}{\sqrt{|M : N| - 1|G| - 1}} = 0,
\]

because \(\sum_i \lambda_i E_N(\lambda_i^*) = 1\). Thus, \(\alpha = \pi/2\), and hence, by Remark 2.6 \((N, N \rtimes G, M, M \rtimes G)\) is a commuting square. \(\square\)

3. Weyl group, Quadrilaterals and regularity

In this section we focus on the analysis of irreducible subfactors and quadrilaterals from the perspectives of Weyl group and interior and exterior angles between intermediate subfactors.

Let \(N \subset M\) be a subfactor and let \(U(N)\) (resp., \(U(M)\)) denote the group of unitaries of \(N\) (resp., \(M\)) and \(N_M(N) := \{ u \in U(M) : uNu^* = N \}\) denote the group of unitary normalizers of \(N\) in \(M\). Clearly, \(U(N)\) is a normal subgroup of \(N_M(N)\). For a finite index subfactor \(N \subset M\), one associates the so-called Weyl group, which we shall denote by \(G\), defined as the quotient group \(N_M(N)/U(N)\) (5, 7, 9, 12). It is known that for an irreducible subfactor \(N \subset M\), \(G\) is a finite group with order less than or equal to \(|M : N|\) (see, for instance, 5, 12 as well as 9).
Example 3.1. Let $G$ be a finite group acting outerly on a II$_1$-factor $N$ and $H$ be a normal subgroup of $G$. Then, the Weyl group of the subfactor $N \rtimes H \subset N \rtimes G$ is isomorphic to the quotient group $G/H$.

Proof. Fix a set of coset representatives $\{g_i : 1 \leq i \leq n = |G : H|\}$ of $H$ in $G$. Then, $\{u_{g_i} : 1 \leq i \leq n\}$ forms a two sided orthonormal basis for $(N \rtimes G)/N$ (where $u_{g_i}$’s are as in Proposition 2.7). Clearly, the map $G/H \ni gH \mapsto [u_{g_i}] \in G$ is a bijection. Then, note that
\[
\varphi(g_iH)\varphi(g_jH) = [u_{g_i}][u_{g_j}] = [u_{g_i}u_{g_j}] = [u_{g_i,g_j}].
\]
On the other hand, if $g_i g_j H = g_k H$, then $g_i g_k = g_j h$ for some $h \in H$, which implies that $u_{g_i,g_j} = u_{g_k} u_h$, i.e., $[u_{g_i,g_j}] = [u_{k}]$ in $G$. Thus, $\varphi(g_iH\cdot g_jH) = \varphi(g_kH) = [u_{g_k}] = [u_{g_i,g_j}] = \varphi(g_iH)\varphi(g_jH)$ for all $1 \leq i, j \leq n$. Hence, $K/H \cong G$. \qed

Now, we make some useful observations related to regularity and orthonormal basis determined by the Weyl group. Recall that a subfactor $N \subset M$ is said to be regular if $\mathcal{N}_M(N)' = M$.

Proposition 3.2. Let $P$ be an intermediate II$_1$-factor of a subfactor $N \subset M$. Let $e_P$ denote the canonical Jones projection for the basic construction $P \subset M \subset P_1$ and $\{\lambda_i\}$ be a finite set in $P$. Then, $\{\lambda_i\}$ is a Pimsner-Popa basis for $P/N$ if and only if $\sum_i \lambda_i e_P \lambda_i^* = e_P$.

Proof. If $\{\lambda_i\}$ is a Pimsner-Popa basis for $P/N$, then we know that $\sum_i \lambda_i e_P \lambda_i^* = e_P$ - see , for instance, the proof of [1] Proposition 2.14. This proves necessity.

To prove sufficiency, consider the basic construction $N \subset P \subset N_1$ with Jones projection $e_P$. Recall, from [2], that this tower is isomorphic to the tower $N_{e_P} \subset e_P M_{e_P} = P_{e_P} \subset e_P M_{1} e_P$ via a map $\phi : N_1 \to e_P M_1 e_P$ satisfying $\phi(x) = x e_P$ for all $x \in P$. The Jones projection for the second tower is given by $e_P e_1 = e_1$. Note that $\sum_i \lambda_i e_P e_1 e_P \lambda_i^* = \sum_i \lambda_i e_1 e_P \lambda_i^* = e_P$, Thus, we obtain
\[
\phi\left(\sum_i \lambda_i e_P \lambda_i^* \right) = \sum_i (\lambda_i e_P) e_1 e_P (\lambda_i^* e_P) = e_P = \phi(1).
\]
This implies that $\sum_i \lambda_i e_P \lambda_i^* = 1$ and, hence, $\{\lambda_i\}$ is a Pimsner-Popa basis for $P/N$. \qed

Proposition 3.3. Let $N \subset M$ be an irreducible subfactor and $P = \mathcal{N}_M(N)'$. If $\{g_i : g \in G\}$ denotes a set of coset representatives of $G$ in $\mathcal{N}_M(N)$, then $\{u_{g_i} : g \in G\} \subset \mathcal{N}_P(N)$ and it forms a two sided orthonormal basis for $P/N$.

Proof. Since $N \subset M$ is irreducible, it follows that $P$ is a II$_1$-factor. By definition, we have $\mathcal{N}_P(N) \subset \mathcal{N}_M(N)$. On the other hand, $\mathcal{N}_M(N) \subset P$. So, if $u \in \mathcal{N}_M(N)$, then $u \in \mathcal{N}_P(N)$. Hence, $\mathcal{N}_P(N) = \mathcal{N}_M(N)$. Therefore, we conclude that $N \subset P$ is a regular subfactor and also that the Weyl group of $N \subset P$ is the same as that of $N \subset M$.

Then, since $N \subset P$ is regular and irreducible, we conclude, from [5] Lemma 3.1], that $\{u_{g} : g \in G\}$ forms a two sided orthonormal basis for $P/N$. \qed

Above two propositions yield the following improvement of [5] Lemma 3.1:

Theorem 3.4. Let $N \subset M$ be an irreducible subfactor and $\{u_{g} : g \in G\}$ be a set of coset representatives of $G$ in $\mathcal{N}_M(N)$. Then, the following are equivalent:

1. $|M : N| = |G|$.
2. $\{u_{g} : g \in G\}$ is a two sided orthonormal basis for $M/N$.
3. $N \subset M$ is regular.

Proof. (1) $\Leftrightarrow$ (2) : Let $p := \sum_{g \in G} u_{g} e_1 u_{g}^*$. Then, $\text{tr}(p) = \sum_{g \in G} \text{tr}(u_{g} e_1 u_{g}^*) = |G||M : N|^{-1}$. Thus, $\{u_{g} : g \in G\}$ is an orthonormal basis for $M/N$ if and only if $|M : N| = |G|$.

(2) $\Leftrightarrow$ (3) : This equivalence follows immediately from Proposition 3.2 and Proposition 3.3. \qed
Corollary 3.6. Let $N$ and $N'$ be both regular. Then, the Weyl groups of $N$ and $N'$ are isomorphic.

Corollary 3.8. Let $(N, P, Q, M)$ be an irreducible quadrilateral with $[P : N] = 2 = [Q : N]$. Then, $[M : N]$ is an even integer and the Weyl group of $N$ is isomorphic to the Dihedral group of order $2n$, where $n = [M : P] = [M : Q]$. 

Theorem 3.5. Let $(N, P, Q, M)$ be an irreducible quadrilateral such that $N \subset P$ and $N \subset Q$ are both regular. Then, $G$ acts outerly on $N$ and $(N, P, Q, M) = (N, N \rtimes H, N \rtimes K, N \rtimes G)$, where $H, K$ and $G$ are the Weyl groups of $N \subset P$, $N \subset Q$ and $N \subset M$, respectively.

**Proof.** First, note that $N \subset M$ is regular because
\[ M = P \vee Q = N_P(N)^N \vee N_Q(N)^N \subseteq \{ N_P(N) \cup N_Q(N) \}^N \subseteq N_M(N)^N \subseteq M. \]

Then, since $N \subset M$ is regular and irreducible, Hong [5] had shown that if $N_{-1} \subset N \subset M$ is an instance of downward basic construction with Jones projection $e_{-1}$, then there is a representation $G \ni g \mapsto v_g \in U(N_{-1} \cap M)$ such that $v_g \in N_M(N)$ for all $g \in G$, $M = \{ N, v_g : g \in G \}^N$ and $G \ni g \mapsto Ad_{v_g} \in \text{Aut}(N)$ is an outer action of $G$ on $N$, i.e., $(N \subset M) = (N \subset N \rtimes G)$ - see [5] Lemma 3.3 and Theorem 3.1. Also, for each $g \in G$, the coset $v_g U(N) = g N$. 

So, by Galois correspondence, $P = N \rtimes H'$ and $Q = N \rtimes K'$ for unique subgroups $H'$ and $K'$ of $G$. We assert that $H = H'$ and $K = K'$.

We have $P = N \rtimes H' = \{ N, v_{h'} : h' \in H' \}$ and $Q = N \rtimes K' = \{ N, v_{k'} : k' \in K' \}$. Then, $\{ v_{h'} : h' \in H' \} \subseteq N_P(N)$ and $\{ v_{k'} : k' \in K' \} \subseteq N_Q(N)$. Hence, $H' \cong H$ and $K' \cong K$. 

Corollary 3.6. Let $(N, P, Q, M)$ be an irreducible quadrilateral such that $N \subset P$ and $N \subset Q$ are both regular. Then, the Weyl groups of $N \subset P$ and $N \subset Q$ together generate the Weyl group of $N \subset M$.

**Proof.** Let $G'$ be the subgroup of $G$ generated by $H$ and $K$, then $N \rtimes G' \subseteq N \rtimes G$. Also, since $M = P \vee Q$, we have
\[ N \rtimes G = (N \rtimes H) \vee (N \rtimes K) \subseteq N \rtimes G' \subseteq N \rtimes G. \]

Hence, by Galois correspondence again, we must have $G = G'$, i.e., $G$ is generated by its subgroups $H$ and $K$. 

We have the following partial converse of Corollary 3.6.

**Proposition 3.7.** Let $(N, P, Q, M)$ be an irreducible quadruple such that $N \subset P$ and $N \subset Q$ are both regular. If $N \subset M$ is regular and the Weyl groups of $N \subset P$ and $N \subset Q$ together generate the Weyl group of $N \subset M$, then $M = P \vee Q$.

**Proof.** Fix any set of coset representatives $\{ u_g : g \in G \}$ of $G$ in $N_M(N)$. Since $N \subset M$ is regular, $\{ u_g : g \in G \}$ forms a two sided orthonormal basis for $M/N$, by Theorem 3.4. Note that each $g$ in $G$ is a word in $H \cup K$ and for any pair $g, g' \in G$, we have $[u_{gg'}] = gg' = [u_g][u_{g'}] = [u_{gg'}]$, so that $u_g = vu_{g'}u_{g''}$ for some $v \in U(N)$. Thus, $M = \sum g N u_g \subseteq (\sum h N u_h) \vee (\sum k u_k) = P \vee Q$. 

Following corollary first appeared implicitly in the proof of [14] Theorem 6.2. We include it here, as an application of Theorem 3.5 and Corollary 3.6.

**Corollary 3.8.** Let $(N, P, Q, M)$ be an irreducible quadrilateral with $[P : N] = 2 = [Q : N]$. Then, $[M : N]$ is an even integer and the Weyl group of $N \subset M$ is isomorphic to the Dihedral group of order $2n$, where $n = [M : P] = [M : Q]$. 

Proof. By Goldman’s Theorem (3.4), we know that \((N \subset P) \cong (N \subset N \rtimes \sigma \mathbb{Z}_2)\) and \((N \subset Q) \cong (N \subset N \rtimes \tau \mathbb{Z}_2)\) for some outer actions \(\sigma\) and \(\tau\) of \(\mathbb{Z}_2\) on \(N\) and hence both \(P \cap N\) and \(Q \cap N\) are regular. Then, by Theorem 3.4, we obtain \(|H| = |P : N| = 2\) and \(|K| = |Q : N| = 2\), where \(H\) and \(K\) are as in Theorem 3.5. So, \(H\) and \(K\) are both cyclic of order 2. By Theorem 3.5, \(N \subset M\) is regular and hence \(|M : N| = |G|\) by Theorem 3.3. Also, by Theorem 3.5, \(G\) is a finite group generated by \(H\) and \(K\). Thus, \(G\) is generated by two elements which are both of order 2. Hence, by [10, Theorem 6.8], \(G\) is isomorphic to the Dihedral group of order \(2n\). \(\square\)

We conclude this section with the demonstration of a direct relationship between angles and Weyl groups of intermediate subfactors of an irreducible quadruple. As above, for a quadruple \((N, P, Q, M)\), we denote by \(G, H\) and \(K\) the Weyl groups of \(N \subset M, N \subset P\) and \(N \subset Q\), respectively. First, we deduce the relationship for an irreducible quadrilateral.

**Theorem 3.9.** Let \((N, P, Q, M)\) be an irreducible quadrilateral such that \(N \subset P\) and \(N \subset Q\) are both regular. Then, \(\alpha(P, Q) = \pi/2\), i.e., \((N, P, Q, M)\) is a commuting square, and

\[
\cos (\beta(P, Q)) = \frac{|G|}{\sqrt{|G : H| - 1}} - 1.
\]

In particular, \((N, P, Q, M)\) is a cocommuting square if and only if \(G = HK\).

**Proof.** From Theorem 3.9, we have \((N, P, Q, M) = (N, N \rtimes H, N \rtimes K, N \rtimes G)\). Since \(N = P \cap Q\), \(H \cap K\) must be trivial because \(N \subset N \rtimes (H \cap K) = P \cap Q = N\). The expressions for \(\alpha\) and \(\beta\) then follow from Proposition 2.7 and the fact that \(|H K| = |H||K|\) when \(H \cap K\) is trivial. \(\square\)

More generally, we have the following relationship.

**Theorem 3.10.** Let \((N, P, Q, M)\) be an irreducible quadruple such that \(N \subset P\) and \(N \subset Q\) are both regular. Then,

\[
\cos (\alpha(P, Q)) = \frac{|H \cap K| - 1}{\sqrt{|H| - 1}} - 1
\]

and

\[
\cos (\beta(P, Q)) = \frac{|M : N|}{\sqrt{|M : P| - 1}} - 1.
\]

In particular, \((N, P, Q, M)\) is a commuting square if and only if \(H \cap K\) is trivial if and only \(P \cap Q = N\). And, \((N, P, Q, M)\) is a cocommuting square if and only if \(|G| = |H K| = |M : N|\).

**Proof.** Since \(N \subset M\) is irreducible, \(P \uplus Q\) is a \(II_1\)-factor. Consider the irreducible quadruple \((N, P, Q, P \uplus Q)\). Then, by Theorem 3.5, \((N, P, Q, P \uplus Q) = (N, N \rtimes H, N \rtimes K, N \rtimes G')\) where \(G'\) is the Weyl group of \(N \subset P \uplus Q\). Hence, by Proposition 2.7, we obtain

\[
\cos (\alpha^N_{P \uplus Q}(P, Q)) = \frac{|H \cap K| - 1}{\sqrt{|H| - 1}} - 1.
\]

And, it is known that \(\alpha^N_{P \uplus Q}(P, Q) = \alpha^N_M(P, Q)\) - see [11, Proposition 2.16]. And, since \(P \cap Q = N \rtimes (H \cap K), (N, P, Q, M)\) is a commuting square if and only \(H \cap K\) is trivial, by Remark 2.6.

Also, \(H \cap K\) is trivial if and only if \(P \cap Q = N\).
On the other hand, being irreducible, $N \subset M$ is extremal. So, by Equation (2.33), the exterior angle between $P$ and $Q$ is given by
\[
\cos \left( \beta(P, Q) \right) = \frac{\text{tr}(e_pe_{Q}) - \tau_p \tau_Q}{\sqrt{\tau_P - \tau_p^2} \sqrt{\tau_Q - \tau_Q^2}} = \frac{\left| [M : N]^{-1}|H \cap K| - [M : P]^{-1}|M : Q]^{-1} \right|}{\left[ [M : P]^{-1} - [M : P]^{-2} \sqrt{[M : Q]^{-1} - [M : Q]} \right]} \]
\[
= \frac{[H \cap K][P : N]^{-1}[M : Q] - 1}{\sqrt{[M : P]^{-1} [M : Q]^{-1}}}.
\]
where we have used the equalities $\text{tr}(e_pe_{Q}) = \tau \sum_{h \in H, k \in K} \|E_{N}(u_h^* u_k)\|^2 = [M : N]^{-1}|H \cap K|$ by Lemma 2.3 and the well known formula $[HK] = |H| |K| |H \cap K|^{-1}$.

By Remark 2.4 again, $(N, P, Q, M)$ is a cocommuting square if and only if $\beta(P, Q) = \pi/2$, i.e., if and only if $[HK] = [M : N]$. Note that $|G| \leq [M : N]$ (see [5, 12]) and $HK \subseteq G$. So, $[HK] = [M : N]$ if and only if $|G| = |HK| = [M : N]$. \[\square\]

Remark 3.11. Sano and Watatani ([14] Theorem 6.1) had proved that an irreducible quadrilateral $(N, P, Q, M)$ with $[P : N] = 2 = [Q : N]$ is always a commuting square. Thus, Theorem 3.9 can also be thought of as a generalization of that result.

4. Possible values of interior and exterior angles

It is a very natural curiousity to know the possible values of interior and exterior angles between intermediate subfactor. As a first attempt in this direction, we make some calculations in the irreducible set up.

Prior to that we recall two auxiliary positive operators associated to a quadruple (from [11, 14]) whose norms are equal, and show that this common entity has a direct relationship with the possible values of interior and exterior angles.

4.1. Two auxiliary operators associated to a quadruple. Consider a quadruple $(N, P, Q, M)$. Let $\{\lambda_i : i \in I\}$ and $\{\mu_j : j \in J\}$ be (right) Pimsner-Popa bases for $P/N$ and $Q/N$, respectively. Consider two positive operators $p(P, Q)$ and $p(Q, P)$ given by
\[
p(P, Q) = \sum_{i,j} \lambda_i \mu_j e_1 \mu_j^* \lambda_i^* \quad \text{and} \quad p(Q, P) = \sum_{i,j} \mu_j \lambda_i e_1 \lambda_i^* \mu_j^*.
\]

Remark 4.1. By [11] Lemma 2.18, $p(P, Q)$ and $p(Q, P)$ are both independent of choices of bases. And, by [11] Proposition 2.22, $Jp(P, Q)J = p(P, Q)$, where $J$ is the usual modular conjugation operator on $L^2(M)$; so that, \[\|p(P, Q)\| = \|p(Q, P)\|\].

Notation 4.2. For a quadruple $(N, P, Q, M)$, let $r := \frac{[P : N]}{[M : Q]} = \frac{[Q : N]}{[M : P]}$ and $\lambda := \|p(P, Q)\| = \|p(Q, P)\|$. \[\]

Recall that for a self adjoint element $x$ in a von Neumann algebra $M$, its support is given by $s(x) := \inf \{p \in \mathcal{P}(M) : px = x = xp\}$. We will need the following useful lemma which follows from [11] Proposition 2.25 & Lemma 3.2.

Lemma 4.3. [11] If $(N, P, Q, M)$ is a quadruple such that $N \subset M$ is irreducible, then $\lambda = [Q : N] \text{tr}(p(P, Q)e_Q)$ and $s(p(P, Q)) = p(P, Q)/\lambda$. In particular, $\text{tr}(s(p(P, Q))) = r/\lambda$ and $p(P, Q)$ is a projection if and only if $\lambda = 1$. 
It turns out that $s(p(P, Q))$ is a minimal projection in $P' \cap Q_1$ which is central as well.

**Proposition 4.4.** Let $(N, P, Q, M)$ be quadruple such that $N \subset M$ irreducible. Then, $\frac{p(P, Q)M}{\lambda}$ (resp., $\frac{p(Q, P)}{\lambda}$) is a minimal projection in $P' \cap Q_1$ (resp., $Q' \cap P_1$) which is also central.

*Proof.* By Lemma 4.3, $\frac{1}{\lambda} p(P, Q)$ is a projection. Further, by [1, Proposition 2.25], we have $p(P, Q) = [P : N] E_{P'}^{N'} (e_Q) \in P' \cap Q_1$. We first show that $\frac{1}{\lambda} p(P, Q)$ is minimal in $P' \cap Q_1$. Consider any projection $q \in P' \cap Q_1$ satisfying $0 \leq q \leq \frac{1}{\lambda} p(P, Q)$. Then, $q = \frac{1}{\lambda} p(P, Q)$. We also have $q e_Q = [M : Q] E_M (qe_Q) e_Q$ (by the Pushdown Lemma [11, Lemma 1.2]). Clearly, $E_M (qe_Q) \in N' \cap M$. Thus, irreducibility of $N \subseteq M$ implies that $qe_Q = te_Q$ for the scalar $t = [M : Q] E_M (qe_Q)$. Therefore,

$$
q = \frac{q}{\lambda} p(P, Q)
= \frac{q}{\lambda} [P : N] E_{P'}^{N'} (e_Q)
= \frac{[P : N]}{\lambda} E_{P'}^{N'} (qe_Q)
= \frac{[P : N]}{\lambda} t E_{P'}^{N'} (e_Q)
= \frac{t}{\lambda} p(P, Q).
$$

Since $q$ and $\frac{1}{\lambda} p(P, Q)$ are projections we conclude that $t^2 = t$. Therefore, $q = 0$ or $p(P, Q)$. Since $q$ was arbitrary, this proves the minimality of $\frac{p(P, Q)}{\lambda}$.

We now prove that $\frac{1}{\lambda} p(P, Q)$ is a central projection in $P' \cap Q_1$. For this, we first show that $e_Q$ is a minimal central projection in $N' \cap Q_1$. Let $u$ be an arbitrary unitary in $N' \cap Q_1$. Then, by the Pushdown Lemma again, we have $ue_Q = [M : Q] E_M (ue_Q) e_Q$. But clearly $E_M (ue_Q) \in N' \cap M = \mathbb{C}$. Thus, $ue_Q u^* = te_Q$ for some scalar $t$. Since $te_Q$ is a non-zero projection, we must have $t = 1$; so that, $ue_Q = e_Q u$ for all $u \in U(N' \cap Q_1)$, thereby implying that $e_Q$ is central in $N' \cap Q_1$. This shows that $v p(P, Q) v^* = [P : N] E_{P'}^{N'} (ve_Q v^*) = [P : N] E_{P'}^{N'} (e_Q) = p(P, Q)$ for all $v \in U(P' \cap Q_1)$ and we are done.

Assertion about $p(Q, P)$ then follows from the fact that $p(Q, P) = J p(P, Q) J$ - see [1, Proposition 2.22]. □

As asserted above, we now present the direct relationship that exists between the values of the interior and exterior angles and the common norm of the above two auxiliary operators.

**Proposition 4.5.** Let $(N, P, Q, M)$ be a finite index quadruple of $II_1$-factors with $N \subset M$ irreducible. Then,

$$
\cos(\alpha(P, Q)) = \frac{(\lambda - 1)}{\sqrt{[P : N] - 1} \sqrt{[Q : N] - 1}}
$$

and

$$
\cos(\beta(P, Q)) = \frac{(\lambda - r)}{\sqrt{[P : N] - r} \sqrt{[Q : N] - r}}.
$$

*Proof.* From Equation (2.7), we have

$$
\cos(\alpha(P, Q)) = \frac{\text{tr}(e p e_Q) - \tau}{\sqrt{\text{tr}(e p) - \tau} \sqrt{\text{tr}(e_Q) - \tau}} = \frac{[M : N] \text{tr}(e p e_Q) - 1}{\sqrt{[P : N] - 1} \sqrt{[Q : N] - 1}}.
$$
It can be shown that \( p(P, Q) = [Q : N]E_{Q_1}^{M_1}(e_P) \)- see the proof of [1], Proposition 2.25]; hence, \( p(P, Q)e_Q = [Q : N]E_{Q_1}^{M_1}(e_P e_Q) \). Thus, \( \text{tr}(p(P, Q)e_Q) = [Q : N]\text{tr}(e_P e_Q) \). This, along with the fact that \( p(P, Q)e_Q = \lambda e_Q \) (because \( N \subset M \) is irreducible - see the proof of [1, Lemma 3.2]), yields \( \text{tr}(e_P e_Q) = \frac{\lambda \text{tr}(e_Q)}{[Q : N]} = \lambda r \), that is \( [M : N]\text{tr}(e_P e_Q) = \lambda \). Thus, we obtain

\[
\cos(\alpha(P, Q)) = \frac{\lambda - 1}{\sqrt{[P : N] - 1}\sqrt{[Q : N] - 1}}.
\]

On the other hand, being irreducible, \( N \subset M \) is extremal. So, from Equation (2.3), we have

\[
\cos(\beta(P, Q)) = \frac{\text{tr}(e_P e_Q) - \tau_P \tau_Q}{\sqrt{\tau_P - \tau_Q} \sqrt{\tau_Q - \tau^2}} = \frac{\tau \lambda - \tau_P \tau_Q}{\sqrt{\tau_P - \tau_Q} \sqrt{\tau_Q - \tau^2}} = \frac{\lambda - r}{\sqrt{[P : N] - \frac{[P : N]}{[M : Q]} \sqrt{[Q : N] - \frac{[Q : N]}{[M : Q]}}}}.
\]

Then, note that

\[
\left( [P : N] - \frac{[P : N]}{[M : P]} \right) \left( [Q : N] - \frac{[Q : N]}{[M : Q]} \right) = [P : N][Q : N] - \frac{[P : N][Q : N]}{[M : Q]} - \frac{[P : N][Q : N]}{[M : P]} + \frac{[P : N][Q : N]}{[M : P][M : Q]} - [P : N][Q : N] - r[P : N] + r^2 = ([Q : N] - r)([P : N] - r),
\]

and we are done. \( \square \)

**Proposition 4.6.** Let \((N, P, Q, M)\) be a commuting square with \( N \subset M \) irreducible. Then,

\[
\cos(\beta(P, Q)) = \frac{r^{-1} - 1}{\sqrt{[M : P] - 1}\sqrt{[M : Q] - 1}}.
\]

And, if \((N, P, Q, M)\) is a a cocommuting square with \( N \subset M \) irreducible, then

\[
\cos(\alpha(P, Q)) = \frac{r - 1}{\sqrt{[P : N] - 1}\sqrt{[Q : N] - 1}}.
\]

**Proof.** The formula for \( \beta(P, Q) \) is easy and is left to the reader. Cocommuting square implies \( \beta(P, Q) = \pi/2 \). Thus, \( \text{tr}(e_P e_Q) = \tau_P \tau_Q \). Now simply use \( \frac{\tau_P \tau_Q}{\sqrt{\tau_P - \tau_Q} \sqrt{\tau_Q - \tau^2}} = r \) and the formula follows from the definition of \( \alpha(P, Q) \). \( \square \)

4.2. Values of angles in the irreducible setup. In order to determine the values of interior and exterior angles between intermediate subfactors, as is evident from Proposition 4.6, it becomes important to know the possible values of \( r \) and \( \lambda \). Recall, from [13], Popa’s set of relative-dimensions of projections in \( M \) relative to \( N \) given by

\[ \Lambda(M, N) = \{ \alpha \in \mathbb{R} : \exists \text{ a projection } f_0 \in M \text{ such that } E_N(f_0) = \alpha 1_N \} \].

**Lemma 4.7.** For an irreducible quadruple \((N, P, Q, M)\), \( \text{tr}(s(p(P, Q))) = \frac{1}{2} \in \Lambda(M_1, M) \). Also, \( (1 - \frac{1}{2}) \in \Lambda(M_1, M) \).
Proof. Let \( \{ \lambda_i \} \) be a basis for \( P/N \). Then, \( p(P, Q) := \sum \lambda_i e_Q \lambda_i^* \in M_1 \) and clearly \( E_{M_1}^M(p(P, Q)) = \sum \lambda_i E_{M_1}^M(e_Q) \lambda_i^* = \frac{[P : N]}{[M : Q]} \). Thus, \( E_{M_1}^M(p(P, Q)) = \tau \). And, by Lemma 4.3 the operator \( \tau p(P, Q) = s(p(P, Q)) \) is a projection. This completes the proof. \( \square \)

For a commuting square \((N, P, Q, M)\) with \( N \subset M \) irreducible, it is known that \( \lambda = 1 \) (see [1 Proposition 2.20]). Thus, we deduce the following:

**Corollary 4.8.** If \((N, P, Q, M)\) is a commuting square with \( N \subset M \) irreducible, then \( r \in \Lambda(M_1, M) \). And, if \((N, P, Q, M)\) is a parallelogram, then \( \lambda^{-1} \in \Lambda(M_1, M) \).

Consider the polynomials \( P_n(x) \) for \( n \geq 0 \) (introduced by Jones in [6] and) defined recursively by \( P_0 = 1, P_1 = 1, P_{n+1}(x) = P_n(x) - x P_{n-1}(x), n > 0 \). Thus, \( P_2(x) = 1 - x, P_3(x) = 1 - 2x \) and so on. From [6], we know that \( P_k \left( \frac{1}{4 \cos \frac{\pi}{n+2}} \right) > 0 \) for \( 0 \leq k \leq n - 1 \). and \( P_n \left( \frac{1}{4 \cos \frac{\pi}{n+2}} \right) = 0 \).

Furthermore, \( P_k(\epsilon) > 0 \) for all \( \epsilon < 1/4 \) and \( k \geq 0 \). Also, by definition, we have \( \frac{P_{n+1}(\tau)}{P_n(\tau)} = 1 - \frac{P_n(\tau)}{P_{n+1}(\tau)} \) for all \( n \geq 1 \), where, as is standard, \( \tau := [M : N]^{-1} \).

While trying to determine the possible entries of the set \( \Lambda(M, N) \), Popa [13] proved the following theorem:

**Theorem 4.9.** [13] Let \( N \subset M \) be a subfactor of finite index.

1. If \([M : N] = 4 \cos^2 \left( \frac{\pi}{n+2} \right) \) for some \( n \geq 1 \), then
   \[
   \Lambda(M, N) = \{ 0 \} \cup \left\{ \frac{\tau P_{k-1}(\tau)}{P_k(\tau)} : 0 \leq k \leq n - 1 \right\} = \left\{ \frac{P_k(\tau)}{P_{k-1}(\tau)} : 0 \leq k \leq n \right\}.
   \]

2. If \([M : N] \geq 4 \) and \( t \leq 1/2 \) is so that \( t(1-t) = \tau \), then
   \[
   \Lambda(M, N) \cap (0, t) = \left\{ \frac{\tau P_{k-1}(\tau)}{P_k(\tau)} : k \geq 0 \right\}.
   \]

Since a subfactor with index less than 4 does not admit any intermediate subfactor, for any non-trivial quadrilateral \((N, P, Q, M)\), we always have \([M : N] \geq 4 \).

**Proposition 4.10.** Let \((N, P, Q, M)\) be an irreducible quadruple and let \( 0 < t \leq 1/2 \) be such that \( t(1-t) = \tau \). Then, either \( \frac{\tau}{x} \geq t \) or \( \frac{\tau}{x} = \frac{\tau P_{k-1}(\tau)}{P_k(\tau)} \) for some \( k \geq 0 \).

**Proof.** This follows from Theorem 4.9 and Lemma 4.7. \( \square \)

We may thus compute the interior and exterior angles in this specific situation, as follows:

**Theorem 4.11.** Let \((N, P, Q, M)\) be a quadruple with \( N \subset M \) irreducible and let \( 0 < t \leq 1/2 \) be such that \( t(1-t) = \tau \). If \( r/\lambda \geq t \), then,

\[
\cos(\alpha(P, Q)) \leq \frac{[P : N][Q : N](1-t) - 1}{\sqrt{[P : N] - 1} \sqrt{[Q : N] - 1}}
\]

and

\[
\cos(\beta(P, Q)) \leq \frac{1 - \tau}{\sqrt{[M : P] - 1} \sqrt{[M : Q] - 1}}
\]

And, if \( r/\lambda < t \), then,

\[
\cos(\alpha(P, Q)) = \frac{[P : N][Q : N] \frac{P_k(\tau)}{\tau P_{k-1}(\tau)} - 1}{\sqrt{[P : N] - 1} \sqrt{[Q : N] - 1}}
\]

and

\[
\cos(\beta(P, Q)) = \frac{\frac{P_k(\tau)}{\tau P_{k-1}(\tau)} - 1}{\sqrt{[M : P] - 1} \sqrt{[M : Q] - 1}}
\]

for some \( k \geq 0 \).
Proof. First, suppose that \( r/\lambda \geq t \). Thus, \( \lambda \leq r/t \). Observe that \( r/t = [P : N][Q : N](1 - t) \); so, from Equation (4.1), we obtain the first inequality. Also, \( \lambda - r \leq r(1/t - 1) \). Thus, from Equation (4.2), we obtain

\[
\cos(\beta(P, Q)) \leq \frac{r(1/t - 1)}{\sqrt{[P : N] - r\sqrt{[Q : N] - r}} = \frac{(1/t - 1)}{\sqrt{[M : Q] - 1}\sqrt{[M : P] - 1}}
\]

Next, suppose that \( N \subseteq M \) is irreducible and \( r/\lambda < t \). By Proposition 4.10, we have \( \lambda = \frac{r}{\tau}P_k(\tau) \), for some \( k \geq 0 \). Observe that \( r/\tau = [P : N][Q : N] \). Thus, by Equation (4.1), we obtain

\[
\cos(\alpha(P, Q)) = \frac{[P : N][Q : N]P_k(\tau)}{\tau^{[P : N] - 1}[P : N] - 1}
\]

and, by Equation (4.2), we obtain

\[
\cos(\beta(P, Q)) = \frac{r}{\sqrt{[P : N] - r\sqrt{[Q : N] - r}}} = \frac{P_k(\tau)}{\sqrt{[M : P] - 1}[M : Q] - 1}
\]

\[\square\]

**Theorem 4.12.** Let \((N, P, Q, M)\) be an irreducible quadrilateral such that \( N \subset P \) and \( N \subset Q \) are both regular and suppose \( [P : N] = 2 \). Then, \( \cos(\beta(P, Q)) = \frac{P_2(m/2)}{\sqrt{P_2(\delta^2/2)P_3(m/2)}} \), where \( m = [M : Q] \in \mathbb{N} \) and, as usual, \( \delta := [M : N] \).

**Proof.** Let \( H, K \) and \( G \) denote the Weyl groups of \( N \subset P, N \subset Q \) and \( N \subset M \), respectively. We have \( m = [M : Q] = \frac{|G|}{|Q : N|} \in \mathbb{N} \) (by Theorem 3.5 and Theorem 3.4) and \( |H| = 2 \). Thus, by Theorem 3.9, we obtain

\[
\cos(\beta(P, Q)) = \frac{\frac{|G|}{|H||K|} - 1}{\sqrt{\frac{|G|}{|H|}} - 1}\sqrt{\frac{|G|}{|K|}} - 1 \quad = \quad \frac{m - 1}{\sqrt{\frac{\delta^2}{2}} - 1}\sqrt{\frac{m}{2}} - 1 = \frac{P_2(m/2)}{\sqrt{P_2(\delta^2/2)P_3(m/2)}}
\]

as was desired. \[\square\]

**Corollary 4.13.** Let \((N, P, Q, M)\) be an irreducible quadrilateral such that \( [P : N] = 2 = [Q : N] \). Then, \( \alpha(P, Q) = \pi/2 \) and \( \beta(P, Q) = \cos^{-1}\left(\frac{P_2(m/2)}{\sqrt{P_2(\delta^2/2)P_3(m/2)}}\right) \), where \( m = [M : P] = [M : Q] \) is an integer. In particular, \( \beta(P, Q) > \pi/3 \).

**Proof.** This follows from Theorem 4.12 and the fact that \( P_2(\delta^2/2) = P_2(m) = P_3(m/2) \), where \( [M : P] = [M : Q] = m \). \[\square\]

**Corollary 4.14.** Let \((N, P, Q, M)\) be an irreducible quadrilateral such that \( N \subseteq P \) and \( N \subseteq Q \) are both regular with \( [P : N] = 2 \) and \( [Q : N] = 3 \). Then, \( \alpha(P, Q) = \pi/2 \) and \( \beta(P, Q) > \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \).

**Proof.** By Theorem 3.9, we have \( \alpha(P, Q) = \pi/2 \) and taking \( m = |G : K| \), we obtain

\[
\cos^2(\beta(P, Q)) = \frac{m^2/4 - m + 1}{3m^2/2 - 5m/2 + 1} < 1/6,
\]
where the inequality follows from a routine comparison using the fact that \( m \geq 2 \).

\[ \Box \]

5. Certain bounds on angles and their implications

In this section, we observe that when one leg of an irreducible quadruple is assumed to be regular, it enforces certain bounds on interior and exterior angles, which then imposes some bounds on the index of the other leg.

**Theorem 5.1.** Let \((N, P, Q, M)\) be an irreducible quadruple such that \(N \subset P\) is regular. Then,

\[
\cos(\alpha(P,Q)) \leq \left( \frac{|P : N| - 1}{|Q : N| - 1} \right)
\]

and

\[
\cos(\beta(P,Q)) \leq \left( \frac{|P : N| - r}{|Q : N| - r} \right).
\]

**Proof.** For any \( u \in N_M(P) \) we have \( uE_Q(u^*) \in N' \cap M = C \). To see this, let \( n \in N \) be arbitrary. Then, \( uE_Q(u^*).n = uE_Q(u^*) = uE_Q(u^*nu.u^*) = n.uE_Q(u^*) \).

Let \( uE_Q(u^*) = t \in C \). If \( t \neq 0 \) we get \( E_Q(u^*) = tu^* \) so that \( u \in Q \), which implies that \( t = uE_Q(u^*) = uu^* = 1 \). Thus, \( uE_Q(u^*) \in \{0,1\} \) for all \( u \in N_M(P) \).

Using Theorem 3.3, fix an orthonormal basis \( \{u_i\} \subset N_P\) for \( P/N \). Consider the auxiliary operator \( p(P,Q) = \sum u_iE_Q(u_i^*) \). Since \( N \subset M \) is irreducible, we have \( p(P,Q)e_Q = \lambda e_Q \) (see [11, Lemma 3.2]), where \( \lambda = \|p(P,Q)\| \). Also, \( p(P,Q)e_Q = \sum u_iE_Q(u_i^*) = \sum u_iE_Q(u_i^*)e_Q \in C e_Q \), which yields \( \sum u_iE_Q(u_i^*) = \lambda \). In particular, we obtain \( 0 \leq \lambda \leq |N_P(N)/U(N)| = |P : N| \). Thus, \( \lambda - 1 \leq |P : N| - 1 \) and the lower bound for \( \alpha \) follows from Proposition 4.5.4.

We also have \( \lambda - r \leq |P : N| - r \). So the lower bound for \( \beta \) follows from Proposition 4.5.4. \( \Box \)

**Remark 5.2.** Note that in above proof, we also observed that \( \|p(P,Q)\|=1 \) is an integer less than or equal to \( |P : N| \).

Above theorem imposes some immediate bounds on \( |Q : N| \) in terms of \( |P : N| \), as follows.

**Corollary 5.3.** Let \((N, P, Q, M)\) be as in Theorem 5.1. Then, we have the following:

1. If \( \alpha(P,Q) \leq \pi/3 \), then \( |Q : N| \leq 4|P : N| - 3 \).
2. If \( \alpha(P,Q) \leq \pi/4 \), then \( |Q : N| \leq 2|P : N| - 1 \).
3. If \( \alpha(P,Q) \leq \pi/6 \), then \( |Q : N| \leq \frac{4}{3}|P : N| - 1/3 \).

**Proof.** If \( \alpha(P,Q) \leq \pi/3 \), then \( \cos(\alpha(P,Q)) \geq \cos(\pi/3) = 1/2 \). So, \( \frac{|P : N| - 1}{|Q : N| - 1} \geq 1/4 \) and hence \( |Q : N| \leq 4|P : N| - 3 \). Others follow similarly. \( \Box \)

As a consequence, when we intuitively try to visualize an irreducible quadruple \((N, P, Q, M)\) with \( |P : N| = 2 \) as a 4-sided structure in plane (as in Figure 1), then it seems that the smaller is the interior angle between \( P \) and \( Q \) the shorter is the length (or index) of \( N \subset Q \). This assertion is supported by the following observations:

If \( |Q : N| > 7/3 \), then it follows from above corollary that \( \alpha(P,Q) > \pi/6 \). Likewise, If \( |Q : N| > 3 \), then \( \alpha(P,Q) > \pi/4 \). And, if \( |Q : N| > 5 \), then \( \alpha(P,Q) > \pi/3 \). In particular, since \( P \) is a minimal subfactor, i.e., \( N \subset P \) admits no intermediate subfactor, in the last scenario, \( Q \) cannot be a minimal subfactor because, by [11], we know that the interior angle between two minimal intermediate subfactors is always less than \( \pi/3 \). Also, observe that if \( \alpha(P,Q) \leq \pi/4 \), then \( |Q : N| \leq 3 \) and hence \( N \subset Q \) must be a Jones’ subfactor.

The reader can get a better feeling of above assertion by making similar calculations for an irreducible quadruple \((N, P, Q, M)\) such that \( P \subset N \) is regular with \( |P : N| = n \), for arbitrary \( n \).
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