Online convex optimization for constrained control of linear systems using a reference governor*

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Abstract: In this work, we propose a control scheme for linear systems subject to pointwise in time state and input constraints that aims to minimize time-varying and a priori unknown cost functions. The proposed controller is based on online convex optimization and a reference governor. In particular, we apply online gradient descent to track the time-varying and a priori unknown optimal steady state of the system. Moreover, we use a λ-contractive set to enforce constraint satisfaction and a sufficient convergence rate of the closed-loop system to the optimal steady state. We prove that the proposed scheme is recursively feasible, ensures that the state and input constraints are satisfied at all times, and achieves a dynamic regret that is linearly bounded by the variation of the cost functions. The algorithm's performance and constraint satisfaction is illustrated by means of a simulation example.

Keywords: Optimal control, control of constrained systems, dynamic regret, online convex optimization, reference governor

1. INTRODUCTION

Application of online convex optimization (OCO) to the problem of controlling linear dynamical systems subject to time-varying cost functions has recently gained significant interest. In contrast to classical numerical optimization, in the OCO framework the cost functions are allowed to be time-varying and a priori unknown, compare, e.g., Shalev-Shwartz (2012); Hazan (2016) and the references therein for an overview. More specifically, an OCO algorithm has to choose an action $u_t$ at every time instant $t$. Only after the action is chosen, the current cost function $L_t(u_t)$ is revealed to the algorithm, which leads to a cost $L_t(u_t)$ depending on the algorithm’s chosen action.

In the context of controller synthesis, the OCO framework is of interest due to its low computational complexity and ability to handle time-varying and a priori unknown cost functions. These kind of cost functions commonly have to be considered in practice, e.g., due to unknown renewable energy generation or a priori unknown energy prices, compare, e.g., Tang et al. (2017). Therefore, OCO-based controllers have been proposed for linear dynamical systems (Li et al., 2019; Nonhoff and Müller, 2020) subject to disturbances (Agarwal et al., 2019) and in a purely data-driven setting with noisy output feedback (Nonhoff and Müller, 2022b). Such OCO-based control algorithms are typically analyzed theoretically in terms of dynamic regret, a performance measure adopted from the OCO framework. Dynamic regret is able to capture transient performance of the closed loop system and is defined as the difference between the accumulated closed-loop cost of the controller and some benchmark, which is typically defined in hindsight, i.e., with knowledge of all cost functions. Studying the dynamic regret of controllers for dynamical systems has recently gained increasing attention, compare, e.g., Dogan et al. (2021); Didier et al. (2022). However, in the literature on OCO-based control, pointwise in time state and input constraints are only considered in Nonhoff and Müller (2021); Li et al. (2021), and restrictive assumptions or a limited setting are necessary in these works to guarantee constraint satisfaction at all times. In particular, Nonhoff and Müller (2021) invoke controllability arguments with a deadbeat controller which can deteriorate the practical performance, whereas Li et al. (2021) consider constraint satisfaction under disturbances, but limit the setting to disturbance rejection, and, thus, only compare to controllers that drive the system to the origin. Therefore, we propose an OCO-based controller for constrained linear systems and time-varying, a priori unknown cost functions in this work, that does not rely on additional restrictive assumptions.

Another closely related line of research is so-called feedback optimization, where the goal is to control a dynamical system to the solution of a (possibly time-varying) optimization problem. Again, the main focus of research is on linear dynamical systems subject to disturbances (Menta et al., 2018; Lawrence et al., 2018; Colombino et al., 2020), in the data-driven setting (Bianchin et al., 2021), and for nonlinear systems (Hauswirth et al., 2021; Zheng et al., 2020). In contrast to the OCO-based approaches, typically only asymptotic guarantees in the form of stability of the closed-loop system are given. Moreover, in the feedback optimization setting constraints are only considered for...
Assumption 2. \[ f \text{ taken in Kalabić and Kolmanovsky (2014), we design a system under control. Then, comparable to the approach OCO, to track the time-varying optimal steady state of the topic. More specifically, we use online projected gradients designed closed-loop system whenever application of the reference governors modify the reference command to a well-behaved system state, } \]

In order to address the shortcomings discussed above, we combine the OCO-framework with a reference constraints. In order to address the shortcomings discussed above, we combine the OCO-framework with a reference function as well as pointwise in time state and input constraints.

More specifically, we use online projected gradients designed closed-loop system whenever application of the reference governors modify the reference command to a well-behaved system state, \[ x_{t+1} = A_k x_t + Bu_t \] (3a)
\[ y_t = C x_t + D v_t \in \mathcal{Y}, \] (3b)

where \[ C = (C_0 + D_0 K) \] and \[ D = D_0. \] Similarly, the time-varying optimization problem can be equivalently reformulated as
\[ \min_{t=0}^{T} L_t(v_t + K x_t, x_t) \text{ s.t. (3).} \] (4)

Let \[ S_K = (I_n - A_K)^{-1}B \] be the map \[ v \] from an input to the corresponding steady-state of the stabilized system (3) and define the set of all feasible steady-state inputs as \[ S_v := \{ v : (CS_K + D)v \in \mathcal{Y} \}. \]

Assumption 3. The cost functions \[ L_t(u_t, x_t) \] are Lipschitz continuous with Lipschitz constant \[ l_t \] for all \[ t \in \mathbb{N}_{\geq 0}, \] i.e., \[ L_t(u_t, x_t) - L_t(\tilde{u}_t, \tilde{x}_t) \leq l_t \| (u_t, x_t) - (\tilde{u}_t, \tilde{x}_t) \| \] for all \[ (u_t, x_t), (\tilde{u}_t, \tilde{x}_t) \in \mathcal{Z} := \{ (u, x) \in \mathbb{R}^m \times \mathbb{R}^n : C_0 x + D_0 u \in \mathcal{Y} \}, \]

and the steady-state cost functions \[ L_t(v) = L_t(v + K S_K v, S_K v) \] are \( \alpha_t \)-strongly convex and \( l_t \)-smooth \[ \forall t \in \mathbb{N}_{\geq 0} \] and \( v \in S_v. \]

Assumption 3 is a common assumption in the literature on OCO-based control, compare, e.g., Li et al. (2019); Nonhoff and Müller (2022b).

Since the cost functions \[ L_t \] are a priori unknown, we can in general not compute the minimizing input of (4) online. Instead, similar to Nonhoff and Müller (2022b), we adopt the strategy of tracking the a priori unknown and time-varying optimal steady-state reference given by
\[ \eta_t := \arg \min_{r} L_t^*(r) \text{ s.t. } r \in \bar{S}_v. \] (5)

where \( \bar{S}_v \) is a compact, convex subset of \( S_v \) such that \( \bar{S}_v \subseteq \text{int } S_v \). The optimal steady state can be recovered by \( \bar{t}_t := S_K \eta_t. \) Then, we define dynamic regret \( \mathcal{R} \) as the difference between the accumulated closed-loop cost and the optimal steady-state cost as
\[ \mathcal{R} := \sum_{t=0}^{T} L_t(v_t + K x_t, x_t) - L_t^*(\eta_t). \] (6)

The goal is to achieve a bound for the dynamic regret \( \mathcal{R} \) that is linear in the path length given by \( \sum_{t=1}^{T} \| \eta_t - \eta_{t-1} \| \), because Li et al. (2019) show for a similar, unconstrained setting that the best achievable bound is linear in the path length. Additionally, Nonhoff and Müller (2022a) prove that such a linear bound implies asymptotic stability under mild assumptions in the unconstrained case.

1 Since \( A_K \) is Schur stable, the inverse exists and the map is unique.

2 Compare (Nesterov, 2018, Definition 2.1.3 and (2.1.9)).
3. OCO-BASED CONTROL USING A REFERENCE GOVERNOR

3.1 Design of the reference governor

In order to ensure satisfaction of the pointwise in time state and input constraints in (2), we design a reference governor in this section. Reference governors compute a feasible reference command $v_t$ at each time $t$ such that, if $v_t$ is applied constantly to the system (3), then the constraints $y_t \in \mathcal{Y}$ are satisfied for all future time steps. This can be achieved using the maximal output admissible set (MAS) (Gilbert and Tan, 1991).

**Definition 4.** The maximal output admissible set of a system $x_{t+1} = Ax_t$ with constraints $C x_t \in \mathcal{Y}$ is defined as $\mathcal{O}_\infty = \{ x \in \mathbb{R}^n : CA'x \in \mathcal{Y} \forall t \in \mathbb{N}_{\geq 0} \}$.

Note that any MAS $\mathcal{O}_\infty$ is positively invariant by definition, i.e., $A \mathcal{O}_\infty \subseteq \mathcal{O}_\infty$. Moreover, the MAS (or a close inner approximation thereof) can be calculated efficiently if $\mathcal{Y}$ is polytopic, $A$ is at least Lyapunov stable, and the pair $(A, C)$ is observable (Gilbert and Tan, 1991).

Similar to the approach presented in Kalabic and Kolmanovsky (2014), we employ a $\lambda$-contractive set in our reference governor in order to ensure a sufficient rate of convergence and prove bounded dynamic regret.

**Definition 5.** For a system $x_{t+1} = Ax_t$, a set $\mathcal{X} \subseteq \mathbb{R}^n$ is $\lambda$-contractive with some $\lambda \in (0, 1]$ if it is compact, convex, $0 \in \text{int } \mathcal{X}$, and $A \mathcal{X} \subseteq \lambda \mathcal{X}$.

We denote by $e_t := x_t - S_K v_t$ the error between the state $x_t$ and the steady state corresponding to the reference $v_t$. Consider a constant reference $v_t = v$ for all $t \in \mathbb{N}_{\geq 0}$. Then, the error dynamics of the system (3a) and the constraints (3b) written in the error coordinates are

\[ e_{t+1} = x_{t+1} - S_K v_t = A_K x_t - A_K S_K v_t = A_K e_t, \]

\[ y_t = C (e_t + S_K v_t) + Dv_t = C e_t + (CS_K + D) v_t \in \mathcal{Y}. \]

Hence, we choose a $\lambda$ satisfying $\rho(A_K) < \lambda < 1$ and define $\mathcal{O}_\infty^\lambda$ as the MAS of

\[ v_{t+1} = v_t, \]

\[ \chi_{t+1} = \frac{1}{\lambda} A_K \chi_t, \]

\[ \psi_t = C \chi_t + (CS_K + D) v_t \in \mathcal{Y}. \]

Since $\mathcal{Y}$ is compact, convex, and $0 \in \text{int } \mathcal{Y}$ by Assumption 2, $\mathcal{O}_\infty^\lambda$ is closed, convex, and $0 \in \text{int } \mathcal{O}_\infty^\lambda$ (Gilbert and Tan, 1991). Moreover, $\mathcal{O}_\infty^\lambda$ has the following properties.

**Lemma 6.** Let Assumptions 1 and 2 be satisfied and let $\mathcal{E}_\infty^\lambda(v) := \{ e \in \mathbb{R}^n : (v, e) \in \mathcal{O}_\infty^\lambda \}$. For any constant input $v_t = v \in \mathcal{S}_o$ for all $t \in \mathbb{N}_{\geq 0}$, (i) $\mathcal{E}_\infty^\lambda(v)$ is $\lambda$-contractive for (7a), and (ii) $y_t \in \mathcal{Y}$ for all $t \in \mathbb{N}_{\geq 0}$ if $x_0 \in \mathcal{E}_\infty^\lambda(v) \cap \{S_K v \}$.

**Proof.** (i) Since the MAS is positively invariant, we have by definition of $\mathcal{O}_\infty^\lambda$ that

\[ e_t \in \mathcal{E}_\infty^\lambda(v) \Leftrightarrow (v, e_t) \in \mathcal{O}_\infty^\lambda \Rightarrow \left( v, \frac{1}{\lambda} A_K e_t \right) \in \mathcal{O}_\infty^\lambda \Leftrightarrow e_{t+1} = A_K e_t \in \mathcal{E}_\infty^\lambda(v). \]

(ii) For any $v \in \mathcal{S}_o$, we have $v \in \mathcal{S}_o \Leftrightarrow (CS_K + D)v \in \mathcal{Y} \Leftrightarrow (v, 0) \in \mathcal{O}_\infty^\lambda \Leftrightarrow 0 \in \mathcal{E}_\infty^\lambda(v)$, i.e., $0 \in \mathcal{E}_\infty^\lambda(v)$ for any $v \in \mathcal{S}_o$. By convexity of $\mathcal{O}_\infty^\lambda$, this implies $\lambda \mathcal{E}_\infty^\lambda(v) \subseteq \mathcal{E}_\infty^\lambda(v)$. By (9),

Fig. 1. Block diagram of the OCO-RG scheme

we get for any $v \in \mathcal{S}_o$ and $e_t \in \mathcal{E}_\infty^\lambda(v)$, $e_{t+1} \in \mathcal{E}_\infty^\lambda(v) \Rightarrow e_{t+1} \in \lambda \mathcal{E}_\infty^\lambda(v) \subseteq \mathcal{E}_\infty^\lambda(v)$, i.e., the set $\mathcal{E}_\infty^\lambda(v)$ is positive invariant for the system (7a). Since $x_0 \in \mathcal{E}_\infty^\lambda(v) \cap \{S_K v \}$ implies $e_0 = x_0 - S_K v \in \mathcal{E}_\infty^\lambda(v)$, we have $e_t \in \mathcal{E}_\infty^\lambda(v)$ for all $t \in \mathbb{N}_{\geq 0}$. Moreover, $e_t \in \mathcal{E}_\infty^\lambda(v)$ and the fact that $\mathcal{O}_\infty^\lambda$ is the MAS of system (8) with output (8c), imply $e_t \in \mathcal{E}_\infty^\lambda(v) \Rightarrow y_t = C e_t + (CS_K + D)v \in \mathcal{Y}$. \hfill $\square$

Lemma 6 shows that, if the reference input $v_t$ is kept constant and the error $e_t = x_t - S_K v_t$ together with the reference $v_t$ is contained in $\mathcal{O}_\infty^\lambda$, then we can ensure contraction of the error and satisfaction of the constraints (2).

3.2 OCO-RG scheme

In this section, we introduce the proposed combination of OCO and a reference governor. A block diagram of the proposed approach is shown in Figure 1 and our OCO-RG scheme is given in Algorithm 1. At each time $t \in \mathbb{N}_{\geq 1}$, given access to the previous cost function $L_{t-1}$, we measure the system state $x_t$ and

(1) apply one projected gradient descent step with a suitable step size $\gamma > 0$ in (10) to the optimization problem $\min L_{t-1}(r)$ s.t. $r \in \mathcal{S}_o$ in order to compute an estimate $r_t$ of the previous optimal reference $y_{t-1}$, compare (5). Hence, $r_t$ tracks the optimal steady state reference $y_{t-1}$.

(2) Next, we apply a reference governor that enforces constraint satisfaction by computing a feasible reference command $v_t$ based on the estimate $r_t$ and the measured state $x_t$ in (11). The constraint in (11a) ensures that the error $e_t = x_t - S_K v_t$ between $x_t$ and the steady state corresponding to $v_t$ lies in the set $\mathcal{O}_\infty^\lambda$, which ensures that the error is contractive and constraint satisfaction by Lemma 6.

(3) Finally, we apply $u_t = v_t + K x_t$ as given in (12) to the system (1), receive the current cost function $L_t$, and move to time step $t + 1$.

Since we do not have access to any cost function at time $t = 0$, we apply an initial reference $r_0 = r_0$ and set $\alpha_0 = 1$. In order to ensure safe operation, i.e., satisfaction of the state and input constraints at all times, we assume that the initial reference $r_0$ is feasible.

**Assumption 7.** The initial reference $r_0 \in \mathcal{S}_o$ satisfies $(r_0, x_0 - S_K r_0) \in \mathcal{O}_\infty^\lambda$.

The proposed algorithm is illustrated in Figure 2. Note that $r_t \in \mathcal{S}_o$ for all $t \in \mathbb{N}_{\geq 0}$ due to the projection in (10) and Assumption 7. Hence, $(r_t, S_K r_t)$ is a feasible steady state of system (3) at all times.

At each time $t$, we have to compute one gradient step and a projection in (10), and solve one scalar optimization prob-
Algorithm 1: OCO-RG scheme

Given a step size $0 < \gamma \leq \frac{2}{\alpha_t+\varepsilon}$, a stabilizing feedback $K \in \mathbb{R}^{n \times \delta}$, and an initial reference $v_0$:

At $t = 0$: Set $\alpha_0 = 1$, $v_0 = r_0$ and apply $u_0 = v_0 + Kx_0$.
At each time $t \in \mathbb{N}_{\geq 1}$:

**OCO:** $t = \Pi_{S^e_v} (r_{t-1} - \gamma \nabla L_{t-1}^e (r_{t-1}))$, \hspace{1cm} (10)

**RG:**

\[
\begin{align*}
\alpha_t &= \max_{\alpha \in [0,1]} \alpha \text{ s.t. } (v_t, x_t - S_K v_t) \in \mathcal{O}_\infty^\lambda \\
&= v_{t-1} + \alpha (r_t - v_{t-1}), \hspace{1cm} (11) \\
\text{Control Input:} & \hspace{1cm} u_t = v_t + K x_t. \hspace{1cm} (12)
\end{align*}
\]

Lemma 9. Suppose Assumptions 1, 2, and 7 are satisfied. There exists $\epsilon \in (0,1]$ such that $\alpha_t \geq \epsilon$ for all $t \in \mathbb{N}_{\geq 0}$.

**Proof.** Fix any $t \in \mathbb{N}_{\geq 1}$. Assumption 7 and Lemma 8 imply \((v_{t-1}, x_{t-1} - S_K v_{t-1}) \in \mathcal{O}_\infty^\lambda\) and, thus,

\[
\begin{align*}
(v_{t-1}, x_{t-1} - S_K v_{t-1}) &\in \mathcal{O}_\infty^\lambda \\
\iff v_{t-1} &\in \mathcal{E}_\lambda^\omega (v_{t-1}) \Rightarrow A_K v_{t-1} \in \lambda \mathcal{E}_\lambda^\omega (v_{t-1}) \\
\iff (v_{t-1}, \frac{1}{\lambda} A_K (x_{t-1} - S_K v_{t-1})) &\in \mathcal{O}_\infty^\lambda \\
\iff (v_{t-1}, \frac{1}{\lambda} (x_t - S_K v_{t-1})) &\in \mathcal{O}_\infty^\lambda.
\end{align*}
\]

because $A_K S_K v_{t-1} = S_K v_{t-1} - B v_{t-1}$. Let $\mathcal{Y}$ be a compact subset of int $\mathcal{Y}$ such that $(CS_K + D) \bar{S}_v \subseteq \mathcal{Y}$, and let $\mathcal{O}_\infty^\lambda$ be the MAS of (8) with $\mathcal{Y}$ replaced by $\mathcal{Y}$ in (8c). Then, $\mathcal{O}_\infty^\lambda \subseteq \mathcal{O}_\infty^\lambda$ by definition, and $(r, 0) \in \mathcal{O}_\infty^\lambda$ for all $r \in \bar{S}_v$ because $r \in \bar{S}_v \Rightarrow C \cdot r + (CS_K + D) r \in \mathcal{Y} \Rightarrow (r, 0) \in \mathcal{O}_\infty^\lambda$. Hence, there exists $\delta > 0$ such that $(r \in \mathcal{O}_\infty^\lambda \subseteq \mathcal{O}_\infty^\lambda$ for all $r \in \bar{S}_v$ and $r \in \mathcal{B}$, where $\mathcal{B} \subseteq \mathbb{R}^m$ is the unit ball. We proceed by a case distinction.

**Case 1:** $\|r_t - v_{t-1}\| > (1 - \lambda) \delta$. Let $r_t^\delta := \frac{r_t - v_{t-1}}{\|r_t - v_{t-1}\|}$ be a feasible candidate solution at time $t$ because $0 < \alpha_t \leq 1$. Since $\alpha_t$ is maximized in (11), we get

\[
\begin{align*}
1 &\geq \alpha_t \geq \alpha_t^\delta := \frac{(1 - \lambda) \delta}{\|r_t - v_{t-1}\|} \geq (1 - \lambda) \frac{\delta}{\Delta} := \epsilon > 0,
\end{align*}
\]

where $\Delta > 0$ is a finite constant satisfying $\Delta \geq \max_{v_{t-1}, v_{t-2} \in \bar{S}_v} \|v_{t-1} - v_{t-2}\|$, which exists by compactness of $\bar{S}_v$.

**Case 2:** $\|r_t - v_{t-1}\| < (1 - \lambda) \delta$. Let $r_t^\delta := \frac{r_t - v_{t-1}}{\|r_t - v_{t-1}\|}$ be a feasible candidate solution at time $t$ because $\|r_t, x_t - S_K r_t \| \in \mathcal{O}_\infty^\lambda$, i.e., $v_t = r_t$ and $\alpha_t = 1 \geq \epsilon$ is a feasible solution to (11a) at time $t$.

Combining the above inclusion to the constraints in (11), we get that $v_t^\delta = v_{t-1} + \alpha_t^\delta (r_t - v_{t-1})$ is a feasible candidate solution at time $t$ because $0 < \alpha_t \leq 1$. Since $\alpha_t$ is maximized in (11), we get

\[
\begin{align*}
1 &\geq \alpha_t \geq \alpha_t^\delta := \frac{(1 - \lambda) \delta}{\|r_t - v_{t-1}\|} \geq (1 - \lambda) \frac{\delta}{\Delta} := \epsilon > 0,
\end{align*}
\]

where $\Delta > 0$ is a finite constant satisfying $\Delta \geq \max_{v_{t-1}, v_{t-2} \in \bar{S}_v} \|v_{t-1} - v_{t-2}\|$, which exists by compactness of $\bar{S}_v$.

Finally, we are ready to analyze closed-loop performance.

**Theorem 10.** Let Assumptions 1-3, and 7 be satisfied. The closed loop achieves

\[
\mathcal{R} \leq K_0 + K_{0, \theta} \sum_{t=1}^{T} \|v_t - \eta_{t-1}\|,
\]

where $K_0 = K_{0, \theta} \|x_0 - \theta_0\| + K_{0, \eta} \|v_0 - \eta_0\|$ and $K_{0, \theta}, K_{0, \eta}, K_0 > 0$ are constants independent of $T$.

**Proof.** Lipschitz continuity from Assumption 3 together with $\theta_t = S_K \eta_t$ yields

\[
\begin{align*}
\mathcal{R} \leq K_0 + K_{0} \sum_{t=1}^{T} \|v_t - \eta_{t-1}\|,
\end{align*}
\]
Rearranging yields
\[
\mathcal{R}(v_t + Kx_t, x_t) - L_t(\eta_t + K\theta_t, \theta_t) = I_L(T\sum_{t=0}^T \left[\left[\eta_t + K\theta_t\right] - \frac{v_t + Kx_t}{x_t}\right]) + I_L(T\sum_{t=0}^T \left[\left[KS_K + L_m(v_t - \eta_t)\right] - \frac{x_t - S_Kv_t}{x_t}\right]) \\
\leq k_x \sum_{t=0}^T \left|\left|\sum_{i=0}^{t} x_i\right| - S_Kv_t\right| + k_x \sum_{t=0}^T \left|\left|v_t - \eta_t\right|\right|, \quad (15)
\]
where we defined \(k_v := l_L\left[\left\|\left(KS_K + L_m\right)v_t\right\|\right]\) and \(k_x := l_L\left[\left\|K^T, I_n\right\|\right]\). In the following, we proceed to bound the sums in (15) separately. First, note that the optimal steady-state input \(\eta_t\), its estimate \(\hat{\eta}_t\), and the modified reference input \(v_t\) are only defined for \(t \in \mathbb{N}_0\). Thus, we set without loss of generality \(\eta_t = v_t - r_t = r_0\). Moreover, we make use of the following standard result from convex optimization.

Lemma 11. (Nesterov, 2018, Theorem 2.2.14) Suppose Assumptions 2 and 3 are satisfied and let \(\gamma \in \left[0, \frac{1}{2\epsilon_x + \frac{1}{\alpha_t}}\right]\). Then,
\[
\left|\Pi_{S_\eta}(r - \gamma \nabla L_t^\theta(r)) - \eta_t\right| \leq \kappa \left|\left|r - \eta_t\right|\right| \quad (16)
\]
holds for any \(r \in S_\eta\), where \(\kappa = 1 - \gamma \alpha_t \in [0, 1]\).

Using (16), the triangle inequality, and \(r_t = \eta_t\), we get
\[
T \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right| \leq \kappa \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right| + \kappa \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right|. \quad (17)
\]
Rearranging yields
\[
\sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right| \leq \frac{1 - \kappa}{1 - \kappa} \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right|. \quad (17)
\]
Next, using the triangle inequality we get
\[
\sum_{t=0}^T \left|\left|v_t - \eta_t\right|\right| \leq \sum_{t=0}^T (1 - \alpha_t) \left|\left|v_t - \eta_t - 1\right|\right| + \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right| \leq \sum_{t=0}^T \left|\left|v_t - \eta_t - 1\right|\right| + \left(1 - \epsilon\right) \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right|, \quad (17)
\]
where we used \(1 \geq \alpha_t \geq \epsilon > 0\) and \(v_t = \eta_t - 1\). Again, rearranging yields
\[
\sum_{t=0}^T \left|\left|v_t - \eta_t\right|\right| \leq \frac{1}{\epsilon - \kappa} \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right|, \quad (18)
\]
where \(\epsilon_k := \epsilon(1 - \kappa)\). Moreover, \(v_t = \eta_t - 1\) and (18) imply
\[
\sum_{t=0}^T \left|\left|v_t - v_t - 1\right|\right| \leq 2 \sum_{t=0}^T \left|\left|v_t - \eta_t\right|\right| + 2 \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right| \leq \frac{2 + \epsilon_k}{\epsilon - \kappa} \sum_{t=0}^T \left|\left|\hat{\eta}_t - \eta_t - 1\right|\right|. \quad (19)
\]
It remains to bound the first sum in (15). First, consider the error dynamics \(e_t = x_t - S_Kv_t\) given by \(e_t = A_K^i(v_t - 1) + S_K(v_t - 1)\). Applying this equation repeatedly yields
\[
e_t = A_K^i e_0 + \sum_{i=0}^{t-1} A_K^i S_K(v_{t-i-1} - v_{t-i} - 1). \quad (20)
\]
Since \(A_K\) is Schur stable, there exist constants \(c \geq 1, \sigma \in (0, 1)\) such that \(\|A_K^i\| \leq c\sigma^i\). Hence, applying the closed-loop error dynamics (20) yields
\[
T \sum_{t=0}^T \left|\left|x_t - S_Kv_t\right|\right| \leq \sum_{t=0}^T \left|\left|A_K^i\right|\right| \left|\left|c\right|\right|^{\epsilon_t} + \sum_{t=0}^T \left|\left|S_K\right|\right| \sum_{t=0}^T \left|\left|A_K^i\right|\right| \left|\left|v_{t-i-1} - v_{t-i}\right|\right|. \quad (20)
\]
Inserting (18) and (21) into (15), and using \(v_{t-1} = v_0\) proves the result (14).}

5. NUMERICAL EXAMPLE

In this section, we illustrate the performance of the proposed OCO-RG scheme by numerical simulation. We consider the problem of tracking a time-varying and a priori unknown reference while minimizing control effort. In particular, the linear system (1) is generated randomly by sampling each entry of \(A \in [\mathbb{R}^{5 \times 5}\) from a uniform distribution over the interval \([-1, 1]\). We obtained an unstable system with \(\rho(A) \approx 1.62\) and set \(B = [0 \ldots 0 1]^T\) \(\in [5 \times 1]\). We consider box constraints of the form \(|x_i| \leq 1, i \in \{1, \ldots, 5\}\), and \(|u| \leq 1\). The stabilizing controller \(K\) is chosen such that the eigenvalues of \(A_K\) are given by \(\text{eig}(A) = (0.1, 0.15, \ldots, 0.3)\). Finally, we let \(\lambda = 0.05, S_\theta = 0.05 S_\eta \subseteq \text{int} S_\eta\), and compute \(O_K\) using multi-parametric programming (Herenc et al., 2013). The cost functionals are given by \(L(t, x) \leq \frac{1}{2} \left|\left|x - \bar{x}\right|\right|^2 + q_t \left|\left|u_t\right|\right|^2\), where \(\bar{x} = x_t + 0.2 \sin(\frac{\pi t}{100})\), and both \(\bar{x}_t \in [-1, 1]\). Finally, we set \(\gamma = 0.1\) and initialize the system and the algorithm with \(x_0 = 0\), and \(r_0 = 0\). The results are illustrated in Figure 3. The top plot of Figure 3 shows the first state of the closed-loop system \(x_t\) together with the optimal steady state \(\theta_t\) and the corresponding constraints. It can be seen that the closed loop follows the optimal steady state closely, for sudden changes as well as for slow, continuous changes induced by the sine term in the definition of \(x_t\). Moreover, the middle plot of Figure 3 illustrates the optimal steady-state input \(\eta_t + K\theta_t\), the estimate \(r_t + Kx_t\), the applied input \(u_t = v_t + Kx_t\), and the input constraints. Since the reference command \(r_t\) together with the stabilizing feedback \(Kx_t\) would violate the input constraints, the reference governor modifies the reference such that \(u_t\) satisfies the constraints at all times. The bottom plot of Figure 3 shows the parameter \(\alpha_t\).\(\alpha_t > 0\) at all times \(t\). More specifically, in this simulation the lowest value of \(\alpha_t\) is approximately \(6 \cdot 10^{-5}\). The maximum computation time of the proposed
In this paper, we propose an algorithm for controlling dynamical systems subject to time-varying and a priori unknown cost functions as well as pointwise in time state and input constraints by combining the online convex optimization framework with a reference governor. In particular, we make use of a λ-contractive set to ensure constraint satisfaction at all times as well as a sufficient convergence rate for proving that the closed loop’s dynamic regret is bounded linearly in the variation of the cost functions. Future works includes obtaining theoretical guarantees for the practically important case of disturbances.

6. CONCLUSION

In this paper, we propose an algorithm for controlling dynamical systems subject to time-varying and a priori unknown cost functions as well as pointwise in time state and input constraints by combining the online convex optimization framework with a reference governor. In particular, we make use of a λ-contractive set to ensure constraint satisfaction at all times as well as a sufficient convergence rate for proving that the closed loop’s dynamic regret is bounded linearly in the variation of the cost functions. Future works includes obtaining theoretical guarantees for the practically important case of disturbances.

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