Signal/noise enhancement strategies for stochastically estimated correlation functions

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Correlation functions in Euclidean spacetime

\[ C_{ij}(\tau) = \langle \Omega | \hat{O}_i' e^{-\hat{H} \tau} \hat{O}_j^\dagger | \Omega \rangle \]

\[ = \sum_n Z'_{in} Z_{jn}^* e^{-E_n \tau} \]

\[ \hat{H} | n \rangle = E_n | n \rangle \]

\[ Z'_{in} = \langle \Omega | \hat{O}_i' | n \rangle \]

\[ Z_{jn} = \langle \Omega | \hat{O}_j | n \rangle \]
Extraction of energies

\[ m_{\text{eff}}(\tau) = - \frac{1}{\Delta \tau} \log \frac{\psi'^\dagger C(\tau + \Delta \tau) \psi}{\psi'^\dagger C(\tau) \psi} \]

\[ \approx E_0 + \frac{(\psi'^\dagger Z_1')(Z_1' \psi)}{(\psi'^\dagger Z_0')(Z_0' \psi)} \left[ \frac{1 - e^{-(E_1 - E_0)\Delta \tau}}{\Delta \tau} \right] e^{-(E_1 - E_0)\tau} \]

N-dimensional source vector

exponential suppression at late times

ground state energy

“excited state contamination”
Example: nucleon correlator

\[ E_0 = m_N \]

“Plateau region” can be short; or worse yet, nonexistent
Example: nucleon correlator

Optimized source yields an earlier plateau for ground state, yet the late time uncertainties seem significantly larger.

(data courtesy of W&M)

Source optimization:
— Variational method
— Matrix Prony

\[ E_0 = m_N \]
Source overlap and signal/noise

An investigation of the interplay between excited state contamination and signal/noise
Signal/noise “landscape”

\[ \frac{S}{N} \sim \theta(\psi', \psi) = \frac{|\psi'^\dagger C \psi|}{\sigma(\psi', \psi)} \]

- Signal/noise optimized
- Source optimized?
- second moment of correlator distribution
- unit length source/sink vectors
Behavior of the variance

The variance of a correlator is itself a correlator

\[ \sigma^2(\psi', \psi) = (\psi' \otimes \psi^*) \dagger \sum^2 (\psi \otimes \psi^*) \]

\[ \sum^2 = \langle C \otimes C^* \rangle \]

\[ \sum_{ik;jl}(\tau) = \sum_n \tilde{Z}'_{ik,n} \tilde{Z}^*_{jl,n} e^{-\tilde{E}_n \tau} \]

sum over states with nontrivial valence quantum numbers

N\times N positive matrix

N'^2 \times N^2 matrix
Signal/noise at late times

At sufficiently late times:

\[ \theta(\psi', \psi) \sim \frac{|\psi'^\dagger Z'_0|}{\sqrt{\psi'^\dagger \tilde{Z}'_0 \psi'}} \frac{|Z'^\dagger_0 \psi|}{\sqrt{\psi'^\dagger \tilde{Z}_0 \psi}} e^{-(E_0 - \frac{1}{2} \tilde{E}_0) \tau} \]

Exponential degradation is an inherent and unavoidable property of the system…

… but we retain some control over signal/noise via the interplay between ratios of Z-factors
Toy model: a two state system

\[ \psi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \psi'(\omega, \delta) = \begin{pmatrix} \cos \omega \\ \sin \omega e^{i\delta} \end{pmatrix} \quad \text{(overall phase is irrelevant)} \]

\[ \delta \in \left[-\pi/2, \pi/2\right) \]
\[ \omega \in \left[0, \pi\right) \]

Consider the correlator:
\[ \psi'(\omega, \delta)\dagger C \psi_n \propto e^{-E_n \tau} \]

Pure exponential: NO contamination from the other state!
Toy model: a two state system

Signal/noise, normalized by optimal signal/noise, can be fully parameterized by (the square-root of) a Breit-Wigner-like formula:

\[
\hat{\theta}_n(\omega, \delta) = \frac{1}{\sqrt{R_n + (R_n - 1)x_n(\omega) [x_n(\omega) - 2 \cos(\delta - \delta_n)]}}
\]

\[
x_0(\omega) = \frac{\tan \omega}{\tan \omega_0} \quad x_1(\omega) = \frac{\cot \omega}{\cot \omega_1}
\]

\[
\sqrt{R_n} = \text{enhancement factor} \ (\geq 1)
\]

\[
\omega_n, \delta_n = \text{optimal mixing angles}
\]

System-dependent parameters!
Toy model: a two state system

eigenstate vector  s/n optimized vector

\[ R_n = 10 \]
\[ R_n = 4 \]
\[ R_n = 2 \]
\[ R_n = 1.1 \]

\[ x_n(\omega) \]
Signal/noise optimization

\[ \Xi(\psi', \psi) = \log \theta^2(\psi', \psi) + \xi' (\psi'^\dagger \psi' - 1) \]

\[ \theta(\psi', \psi) = \frac{\left| \psi'^\dagger C \psi \right|}{\sigma(\psi', \psi)} \]

Solution:

\[ \psi'_0 = A'_0(\psi) \sigma^{-2}_\psi C \psi \]

\[ \sigma^2_\psi = \langle C \psi \psi'^\dagger C^\dagger \rangle \]
Further extensions

- S/N optimize both source and sink vectors
- Impose constraints on the sources and sinks
- Include correlations between time slices
Steepest ascent

I: Source optimized vector
II, III: Intermediate vectors
IV: Signal/noise optimized vector
Application to QCD: delta

I: Source optimized
II: Intermediate
III: Signal/noise optimized
Conclusion and future directions

• Proposed a new avenue for correlator optimization
  • many new ideas (see paper), but it remains a bit unclear whether there exists a context where they might be useful
  • idea is general, applicable to systems beyond QCD
  • applicable to excited states

• Many unexplored direction
  • multi-nucleon systems
  • disconnected diagrams
  • three-point functions