Equivalent-Circuit Modeling of Lossless and Lossy Bi-Periodic Scatters by an Eigenstate Approach

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Abstract—The use of an eigenstate-based equivalent-circuit topology is proposed for the analysis and modeling of lossless and lossy bi-periodic scatterers. It significantly simplifies the design of this kind of surfaces, since it reduces the number of elements with respect to other general equivalent circuits. It contains at most only two admittances and one complex turns ratio. The real parts of these admittances can be assumed to be nonnegative, an interesting aspect in the modeling of lossy surfaces such as those present in absorbers. Moreover, due to the capability of decomposition into the eigenexcitations of the structure, the circuit provides an important physical insight. Different cases of scatterers have been analyzed: symmetric and asymmetric, and lossy and lossless. In all these cases, modeling of the circuit admittances has been successfully achieved with a few positive and frequency-independent RLC elements. In the case of structures with symmetries, the turns ratio directly reflects the physical orientation of the scatterer eigenexcitations. Furthermore, in the case of lossy scatterers without symmetries, the resulting equivalent circuit reveals that their eigenexcitations are not linear polarizations, but elliptic polarizations whose properties are described by a complex turns ratio.

Index Terms—Bi-periodic, eigenstates, equivalent circuits, frequency-selective surfaces, lattice network, rasorber, reflectarray, scatterer, transmitarray.

I. INTRODUCTION

NOWADAYS, one of the usual ways to approach the analysis and synthesis of bi-periodic surfaces (2-D periodic, or repeated along two directions) is by means of FEM or FDTD full-wave electromagnetic simulation [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. However, the use of equivalent circuits is also a common tool and is increasingly used [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21]. The reasons for this are manifold. Equivalent circuits provide information about the behavior of the periodic surface in a more understandable and convenient way than the mere parameterization of the simulation results. They allow more sophisticated multilayer structures to be analyzed and designed without resorting to repeated electromagnetic simulations, which are much more costly in terms of CPU time. Moreover, all the existing knowledge of the circuit theory can be used in the design of multilayer surfaces.

On one hand, there are equivalent circuits that simply seek to model the results of the simulations [16], [17], [22]. They can be used with any geometry, no matter how complicated it may be, because their starting point is the result of an electromagnetic simulation obtained by the numerical methods. On the other hand, there are those that seek to provide information on the behavior of the unit cell by the analytical methods, avoiding simulations [18], [19], [23]. Although their subsequent use is much more efficient, their scope of application is limited to geometries which, although interesting, must be relatively simple, since they require simplifications and hypotheses on the fields in their slots or the currents in their conductors. But there is an additional option, which is to extract information from the computational effort of one or a few electromagnetic simulations, to obtain a circuit with a greater field of application than the simulated geometry from which it was obtained [20], [21].

In any case, the choice of circuit topology is also relevant, since choosing one or another topology can facilitate or hinder the modeling. An unsuitable topology can lead to circuit elements with peculiar variations with frequency, or to negative resistances, capacitances, and inductances. These may be correct and valid circuits, but their ability to explain the behavior of the periodic surface is lost and may complicate their use in design tasks.

The authors proposed in [24] a general equivalent-circuit topology for asymmetric two ports that allowed decomposition in eigenstates. Later on, a modified topology based on the same principles but with enhanced properties was presented in [25]. The circuit topology consists of two parallel networks made of a transformer with an admittance at each side. The values of the admittances are the eigenvalues of the
admittance matrix of the two ports, while the turns ratio of the transformer is the ratio between the components of its eigenvectors. Thanks to the eigenstate formulation in which it is based on, the proposed circuit topology has very interesting properties (such as the decomposition of the quasi-even and quasi-odd modes) and was successfully applied to the circuit modeling of asymmetric unit cells of leaky-wave antennas. In spite of the evident potential of this topology for modeling any asymmetric structure, its features had not been exploited for bi-periodic scatterers until now.

This article proposes the reformulation and application of the topology in [25] to model bi-periodic scatterers, with several implications. The first one is the remarkable simplification of the extraction of the eigenexcitations of the unit cell (the crystal axes in the terminology of [21]). On the other hand, this topology leads to circuits in which it is ensured that their immittances have nonnegative real parts. The latter is particularly relevant, given the importance of the transformer is the ratio between the components of its admittance matrix of the two ports, while the turns ratio of the scatterer lies between walls of periodic boundary conditions. These boundary conditions depend on the excitation, i.e., the direction of the wave impinging on the surface. In our case, we will consider the normal incidence, \( \theta_{\text{inc}} = 0 \), and the usual decomposition into the TE and TM polarizations. For the analysis, the \( \phi_{\text{inc}} = 0 \) plane is chosen, so that a first polarization corresponds to the vertical electric field \( \vec{E}_V \) along the \( y \)-axis) and a second polarization to the horizontal electric field \( \vec{E}_H \) along the \( x \)-axis). As is also usual, the thickness of the metallization is small enough to assume zero thickness.

If the frequencies to be considered do not exceed the cutoff frequency of the first higher order Floquet mode, the scatterer behavior can be represented by a four-port circuit, as shown in Fig. 2(a). The normal incidence under any angle \( \phi \) can be described by a linear combination of the vertical and horizontal polarization incidences, or by the excitation of ports \( \phi \) and \( \theta \) of the circuit in Fig. 2(a), with the corresponding amplitude coefficients.

As explained in [22], the zero-thickness approximation implies the continuity of the electric field for each of the polarizations, so that the \( S \)-parameter matrix has the form

\[
[S] = \begin{bmatrix}
S_{11} & S_{12} & 1 + S_{11} & S_{12} \\
S_{12} & S_{22} & S_{12} & 1 + S_{22} \\
1 + S_{11} & S_{12} & S_{11} & S_{12} \\
S_{12} & 1 + S_{22} & S_{12} & S_{22}
\end{bmatrix}
\]  \( (1) \)

From these \( S \)-parameters, the impedance or the admittance matrix of the two ports in Fig. 2(b) can be obtained. Whereas in [22] the impedance matrix is used, here the use of the admittance matrix is preferred

\[
[Y_4] = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix}
\]  \( (2) \)

whose elements can be written in terms of the \( S \)-parameters of the four ports as

\[
y_{11} = \left( \frac{-2}{\eta_0} \right) \frac{\Delta S + S_{11}}{\Delta S + S_{11} + S_{22} + 1}
\]

\[
y_{12} = \left( \frac{-2}{\eta_0} \right) \frac{S_{12}}{\Delta S + S_{11} + S_{22} + 1}
\]
\[ y_{22} = \left( \frac{-2}{\eta_0} \right) \frac{\Delta S + S_{22}}{\Delta S + S_{11} + S_{22} + 1} \]  
(3)


being \( \Delta S = S_{11}S_{22} - S_{12}^2 \).

A. Eigenstate Circuit Topology

As for any other (lossy or lossless) reciprocal circuit, the two ports in Fig. 2(b) can be represented by the topology proposed in [25], which is based on the eigenstate decomposition. To get the most out of this topology, it is necessary to reformulate the proposal in [25] by means of a simple change in variable, but with important implications. Let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of the matrix in (2) and \( \vec{v}_1 \) and \( \vec{v}_2 \) its eigenvectors. The ratio between the components of \( \vec{v}_1 \) can be written as

\[ p = \frac{y_{11} - y_{22}}{2y_{12}} + \sqrt{\left( \frac{y_{11} - y_{22}}{2y_{12}} \right)^2 + 1}. \]

(4)

Squaring this value leads to

\[ p^2 = 1 + 2 \left( \frac{y_{11} - y_{22}}{2y_{12}} \right) p. \]

(5)

Then, making the change in variable \( p = \tan \phi \)

\[ \tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{2p}{1 - p^2} = \frac{2y_{12}}{y_{22} - y_{11}}. \]

(6)

Now, using (3), the complex angle \( \phi \) can be directly written in terms of the \( S \)-parameters of the unit cell

\[ \tan 2\phi = \frac{2S_{12}}{S_{22} - S_{11}} \]

or

\[ \phi = \frac{1}{2} \arctan \left( \frac{2S_{12}}{S_{22} - S_{11}} \right) \pm k \frac{\pi}{2} \]

(8)

where \( k \) is an arbitrary integer whose only effect is to interchange the order of the eigenvectors and their corresponding eigenvalues.

The change in variable also has effects on the eigenvectors of the circuit which become

\[ \vec{v}_1 = \begin{bmatrix} \sin \phi \\ \cos \phi \end{bmatrix}; \quad \vec{v}_2 = \begin{bmatrix} -\cos \phi \\ \sin \phi \end{bmatrix}. \]

(9)

Obviously these two vectors will be unitary if \( \phi \) is real.

The eigenvalues can also be expressed as

\[ \lambda_1 = y_{22} + y_{12} \tan \phi; \quad \lambda_2 = y_{11} - y_{12} \tan \phi \]

or, in terms of the \( S \)-parameters, as

\[ \lambda_1 = \left( \frac{-2}{\eta_0} \right) \frac{\Delta S + S_{22} + S_{12}}{\Delta S + S_{11} + S_{22} + 1} \]

(10)

\[ \lambda_2 = \left( \frac{-2}{\eta_0} \right) \frac{\Delta S + S_{11} - S_{12}}{\Delta S + S_{11} + S_{22} + 1}. \]

(11)

Using simple eigenstate theory, the admittance matrix can be expressed in terms of their eigenvalues and eigenvectors as

\[ [Y_d] = \begin{bmatrix} \lambda_1 \vec{v}_1 & 0 \\ 0 & \lambda_2 \vec{v}_2 \end{bmatrix}^{-1} \begin{bmatrix} \vec{v}_1 \vec{v}_2 \end{bmatrix}^{-1} \]

(12)

and, introducing the values of the eigenvectors in (9)

\[ [Y_d] = \lambda_1 \begin{bmatrix} \sin^2 \phi \phi_{\sin \phi}^{\frac{2}{2}} \\ \sin^2 \phi \cos^2 \phi \end{bmatrix} + \lambda_2 \begin{bmatrix} \cos^2 \phi - \sin^2 \phi^{\frac{2}{2}} \\ -\sin^2 \phi \cos^2 \phi \end{bmatrix}. \]

(13)

As more extensively discussed in [25], a circuit topology that achieves this admittance matrix is the one composed of two simple circuits in parallel, both made of an ideal transformer with series admittances at each of its sides. The equivalent circuit for the four ports in Fig. 2(b) is then the one shown in Fig. 3, which makes use of two transformers dependent on \( \phi \) and two different admittances: \( \lambda_1 \) and \( \lambda_2 \). Compared with the transformers of the circuit proposed in [25], the turn ratios \( 1:p \) and \( p:1 \) have been replaced by transformers of ratios \( 1: \tan \phi \) and \( \tan \phi:1 \), or else by transformers of ratios \( \cos \phi: \sin \phi \) and \( \sin \phi: \cos \phi \), respectively.

It is worth noting that the eigenvectors of the equivalent circuit are, directly, what we could refer to as the "eigenexcitations" of the scatterer or unit cell. If the eigenexcitation corresponding to \( \vec{v}_1 \) is considered

\[ \vec{E}_{1}^{inc} = \sin \phi \vec{e}_v + \cos \phi \vec{e}_H \]

(14)

the reflected and transmitted fields are obtained by means of (1) as

\[ \vec{E}_{1}^{ref} = (S_{11} \sin \phi + S_{12} \cos \phi) \vec{e}_v \]

\[ + (S_{21} \sin \phi + S_{22} \cos \phi) \vec{e}_H \]

\[ \vec{E}_{1}^{tr} = ((1 + S_{11}) \sin \phi + S_{12} \cos \phi) \vec{e}_v \]

\[ + (S_{21} \sin \phi + (1 + S_{22}) \cos \phi) \vec{e}_H \]

(15)

whereas the eigenexcitation corresponding to \( \vec{v}_2 \) is

\[ \vec{E}_{1}^{inc} = -\cos \phi \vec{e}_v + \sin \phi \vec{e}_H. \]

(16)

Using (7), it is straightforward to prove that

\[ \vec{E}_{1}^{inc} \times \vec{E}_{1}^{ref} = 0 \]

\[ \vec{E}_{1}^{inc} \times \vec{E}_{1}^{tr} = 0 \]

\[ \vec{E}_{1}^{inc} \times \vec{E}_{1}^{pr} = 0 \]

\[ \vec{E}_{1}^{inc} \times \vec{E}_{1}^{tr} = 0. \]

(17)

That is, the incidence with one of the eigenexcitations only generates the reflected and transmitted fields proportional to
that excitation and no field proportional to the other eigenexcitation. Undoubtedly, this is a similar result to that of the diagonalization of the scattering matrix described in [21]. However, the use of the circuit herein proposed obtains the directions of the eigenexcitations in a much simpler way and solves certain problem not addressed in [21], as will be discussed later in Section III-E.

B. Use of the Equivalent Circuit

One of the important advantages of the proposed circuit is that the admittances it includes always have nonnegative real parts, as was proven in [25]. Therefore, it is expected that it will be possible to model both the admittances by means of positive RLC components.

The estimates of the admittances and their variation in frequency, \( \lambda_1 \) and \( \lambda_2 \), can be obtained and modeled with RLC elements, from the electromagnetic simulation of the unit cell (or, when available, by the analytical methods). From these estimates, it is straightforward to obtain the \( S \)-parameters of the scatterer as

\[
\begin{align*}
\tilde{S}_{11} &= -\frac{\hat{\lambda}_1 \eta_0}{2 + \hat{\lambda}_1 \eta_0} \sin^2 \hat{\phi} \\
\tilde{S}_{12} &= \frac{\eta_0 (\hat{\lambda}_2 - \hat{\lambda}_1)}{2 + \hat{\lambda}_1 \eta_0} \sin 2\hat{\phi} \\
\tilde{S}_{22} &= -\frac{\hat{\lambda}_1 \eta_0}{2 + \hat{\lambda}_1 \eta_0} \cos^2 \hat{\phi} - \frac{\hat{\lambda}_2 \eta_0}{2 + \hat{\lambda}_2 \eta_0} \sin^2 \hat{\phi}
\end{align*}
\]

(18)

where \( \hat{\phi} \) can be an estimation or a simple model of the variation with the frequency of \( \phi \), when such variation occurs.

III. DISCUSSION AND APPLICATION EXAMPLES

The behavior of the unit cell depends on its symmetries and loss characteristics. The proposed equivalent circuit reflects these symmetries and characteristics and leads to certain particular configurations that are worth commenting on.

A. Unit Cell With a Symmetry Plane

In the case of a scatterer with a plane of symmetry (with or without losses), it is obvious that there are two linear polarizations, perpendicular to each other, which constitute the eigenexcitations of the unit cell, namely, the polarization along the plane of symmetry and the one perpendicular to it: \( \vec{e}_1 \) and \( \vec{e}_2 \), respectively, as shown in Fig. 4(a). If these polarizations were used for the analysis of the scatterer, \( S_{12} = 0 \) would be obtained. Equation (8) then gives a value of \( \phi = 0 \), and the transformers, which in that case would be of 1:0 and 0:1 ratios, degenerate into open and short circuits. The resulting circuit is shown in Fig. 4(b), where it is apparent that each of the admittances \( \lambda_1 \) and \( \lambda_2 \) is the independent and specific response of the unit cell to each of its eigenexcitations.

When the vertical and horizontal polarizations are used for the analysis, the more general circuit of Fig. 3 should contain the same admittances \( \lambda_1 \) and \( \lambda_2 \). The role of the transformers will be the change in basis from the \( (\vec{e}_V, \vec{e}_H) \) pair of Fig. 1(b) to the \( (\vec{e}_1, \vec{e}_2) \) pair of Fig. 4(a) through the real angle \( \phi_0 \).

A scatterer composed of a simple dipole can serve as an example. Let us consider the rotated dipole analyzed in [22, Section III-B] with \( P = 10 \text{ mm}, W = 0.5 \text{ mm}, L = 7 \text{ mm}, \) and different values of \( \phi_0 \) (related to the parameter \( \alpha \) in [22] as \( \alpha = \pi/2 - \phi_0 \)). Its shape is reproduced in Fig. 5(a). The results of the extraction of the equivalent circuit of Fig. 3 from the HFSS simulations of the unit cell are shown in Fig. 6 for a number of values of \( \phi_0 \). The figures illustrate the ability of the circuit to correctly detect the positioning of the dipole \( (\phi = \phi_0) \), as well as the invariance of the admittances \( \lambda_1 \) and \( \lambda_2 \) with angle \( \phi_0 \). The real parts of the admittances are negligible, as corresponds to a lossless scatterer, and the imaginary part of \( \phi \) is also zero, as expected.

In [22], the electromagnetic simulation of a whole series of dipole angular positions was necessary to obtain a large set of equivalent-circuit element values. However, in our case (like in [21]), a single analysis is necessary at one of the angular positions to obtain a circuit valid for any angle \( \phi = \phi_0 \), and with unique admittances, that can be approximated with positive parameters by zero-pole immittance identification as

\[
\begin{align*}
\hat{\lambda}_1 &= \frac{1}{j \omega L_1 + \frac{1}{j \omega C_1}} \\
\hat{\lambda}_2 &= j \omega C_2
\end{align*}
\]

(19)

with \( L_1 = 9.1 \text{ nH}, C_1 = 6.8 \text{ fF}, \) and \( C_2 = 0.15 \text{ fF} \).

Moreover, for this unit cell composed of a thin dipole, the small capacitance \( C_2 \) is negligible, and thus also the effect of admittance \( \lambda_2 \). The equivalent circuit is then reduced to the one shown in Fig. 5(b), and the unit cell \( S \)-parameters obtained
Fig. 6. Parameters of the equivalent circuit of the dipole proposed in [22, Section III-B] for different rotation angles $\phi_0$, simulated with HFSS. (a) and (b) Angle $\phi$. (c) and (e) Admittance $\lambda_1$. (d) and (f) Admittance $\lambda_2$.

with it are compared in Fig. 7 with the electromagnetic simulation and with the T-network circuit proposed in [22]. As can be seen, the agreement over the entire bandwidth of the fundamental mode is as good as that obtained in [22], but with a simpler circuit, obtained with less computational effort, and without the need for parameterization with the $\phi_0$ angle.

Like any other equivalent circuit, the validity of the one proposed here is limited to the frequency range in which higher order modes do not propagate or in which their attenuation, if they are under cutoff, is high enough. The small discrepancies between the simulation and equivalent circuit observed in Fig. 7, at higher frequencies, is due to the fact that the cutoff frequency of the first higher order Floquet mode is 30 GHz.

It should also be noted that somehow, the symmetry of the unit cell is also lost by the interaction with its neighbors (a case that was not addressed in [21]). This is evident when the size of the scatterer is such that it approaches that of the cell period, $P$. Repeating the analysis for a larger dipole ($L = 8$ mm, $W = 4$ mm) yields the results shown in Fig. 8. It can be seen that the angle $\phi$ no longer exactly agrees with $\phi_0$, and that the admittances vary slightly with the angular position. Fig. 8(a) shows how, for rotations of $0^\circ$ and $45^\circ$, the angle obtained does correspond to the physical position of the dipole at any frequency. This is because, for these two angles, the periodic structure as a whole maintains the symmetry of the unit cell, which is not the case for intermediate angles. Anyway, as proposed in [21], it is possible to model the variation in $\phi$ with $\phi_0$ and the frequency, as well as to parameterize the circuit elements of the admittances with $\phi$.

In all the cases, these will be smooth variations in easy modeling.

B. Lossy Scatterer With a Symmetry Plane

As already mentioned, one of the advantages of the proposed circuit worth noting is that it guarantees admittances with nonnegative real parts. Consider as an example the resistively loaded dipole antenna proposed in [27, Fig. 3] and reproduced in the inset of Fig. 9(d). The structure consists of a fan-shaped metal dipole with a resistor at its center (the black square in the figure) placed on top of an FR4 substrate. There is an air gap between the substrate and a ground plane. The effect of the supporting dielectric is not negligible in this case. Therefore, the whole unit cell with the dipole antenna on its dielectric substrate has been simulated with HFSS. In the simulation, the conductors are considered perfect and with zero thickness, and the resistor as an impedance boundary surface. The losses of the FR4 substrate are also taken into account. To obtain the S-parameters of the scatterer, the effect of the dielectric is de-embedded by simply changing the reference impedance at ports $\odot$ and $\oplus$, shifting the reference planes by the length corresponding to the substrate thickness, and returning to the original reference impedances (vacuum impedance) at all the four ports.
The result, considering the symmetry of the dipole with respect to the vertical plane, leads to $S_{12} = 0$, and therefore $\phi = 0$. Both the polarizations are isolated from each other, and the only relevant parameters are the admittances $\lambda_1$ and $\lambda_2$, whose real and imaginary parts are shown in Fig. 9. A good approximation of these admittances is obtained by means of lossy series resonators

$$\lambda_1 = \frac{1}{R_1 + j\omega L_1 + \frac{1}{j\omega C_1}} + \frac{1}{j\omega L_2 + \frac{1}{j\omega C_2}}$$
$$\lambda_2 = \frac{1}{R_3 + j\omega L_3 + \frac{1}{j\omega C_3}}$$

(20)

with $R_1 = 227 \, \Omega$, $L_1 = 5.53 \, \text{nH}$, $C_1 = 107 \, \text{fF}$, $L_2 = 3.9 \, \text{nH}$, $C_2 = 13.6 \, \text{fF}$, $R_3 = 3.07 \, \Omega$, $L_3 = 1.64 \, \text{nH}$, and $C_3 = 40 \, \text{fF}$. All of them are positive values and correctly reproduce the behavior of the admittances (see Fig. 9) except for the anomalous peak at 11.2 GHz. This peak is thought to be caused by a Fano resonance, which could also be modeled, although it would require the use of two coupled RLC resonators [31], what is beyond the scope of this article.

Using this equivalent circuit and incorporating the supporting dielectric and ground plate as a short-circuited lossy transmission line, the response of the absorber designed in [27] can be reproduced. Fig. 10 shows the comparison between the complete electromagnetic simulation of the structure and the results obtained using the proposed circuit. The agreement between the results is excellent, apart from the aforementioned spike at 11.2 GHz.

C. Scatterer With 90° Rotational Symmetry

In search of the special properties, some authors have resorted to scatterers with 90° rotational symmetry. This is...
the case of Swastika-like [1], [2], [8] or multi-semicircle-based scatterers [3]. By far the most common case of this rotational symmetry, and also the simplest, is that of a unit cell with two planes of symmetry: simple, Jerusalem, and elaborate crosses [7], [9], [10], [11], circular and square patches and loops [4], [5], [12], [14], and even more complex geometries [6], [15].

For all these unit cells, $S_{11} = S_{22}$ and $S_{12} = 0$. Therefore, the results of (8) and (7) are mathematically indeterminate, which corresponds to the fact that any pair of linear polarizations perpendicular to each other can be considered as eigenexcitations of this type of scatterers.

The admittances of the equivalent circuit are equal to each other, and their value is obtained from (11) as

$$\lambda_1 = \lambda_2 = \left(\frac{-2}{\eta_0}\right) \frac{S_{11}}{S_{11} + 1}. \quad (21)$$

Similarly, the admittance parameters of (3) are particularized to the values

$$y_{11} = y_{22} = \lambda_1 = \lambda_2$$
$$y_{12} = 0 \quad (22)$$

which means that the equivalent circuit is simplified to the one shown in Fig. 11(b). Therefore, no transformers are needed, and the indeterminacy of (8) and (7) is irrelevant.

As an example, the four-semicircle unit cell proposed in [3, Fig. 1(a)], and reproduced in Fig. 11(a), has been analyzed with parameters $r = 3$, $t = 1$, and $g = 1$ mm. After de-embedding the dielectric substrate, the $S$-parameters of the four ports representing the scatterer have been obtained. It was verified that except for the errors inherent to the numerical simulation, $S_{11} = S_{22}$ and $S_{12} = 0$. The equivalent circuit admittances are shown in Fig. 12, admittances that are easily modeled by two $LC$ series resonators in parallel, as

$$\hat{\lambda}_1 = \hat{\lambda}_2 = \frac{1}{j \omega L_1 + \frac{1}{j \omega C_1}} + \frac{1}{j \omega L_2 + \frac{1}{j \omega C_2}} \quad (23)$$

where $L_1 = 17$ nH, $C_1 = 45$ fF, $L_2 = 5.4$ nH, $C_2 = 12.8$ fF.

**D. Lossless Scatterers**

No matter what symmetry the scatterer exhibits, or the lack of it, when there are no losses, there will always be two eigenexcitations in the form of linear polarizations perpendicular to each other. To verify this, we start from the fact that when there are no losses, the unitarity property of the $S$-matrix in (1) leads to

$$S_{12}(1 + S_{11})^* + S_{22}S_{12}^* + S_{12}^*S_{11} + (1 + S_{22})S_{12}^* = 0$$
$$\Rightarrow \text{Re}(S_{12}) + S_{12}^*S_{11} + S_{22}S_{12}^* = 0. \quad (24)$$

By subtracting from this expression its conjugate

$$S_{12}(S_{11} - S_{22})^* + S_{22}^*(S_{22} - S_{11}) = 0$$
$$\Rightarrow S_{12}(S_{22} - S_{11})^* = S_{12}^*(S_{22} - S_{11}). \quad (25)$$

Hence, $S_{12}(S_{22} - S_{11})^*$ is real. The ratio $2S_{12}/(S_{22} - S_{11})$ will also be real and, in view of (8), so will $\phi$. Consequently, the fields described in (14) and (16) are linear polarizations that constitute the eigenexcitations of the lossless scatterer.

The fact that the lack of losses implies that $\phi$ is real can also be understood from the point of view of the equivalent circuit. As shown in [25] and [24], when the equivalent circuit based on the eigenstates represents a lossless structure, the transformers are of real turns ratios, which, in the circuit of Fig. 3, implies that $\phi$ is real.

An example of a lossless unit cell without any symmetry is the asymmetrically loaded magnetic dipole presented in [23, Fig. 8(a)]. The extraction of the equivalent circuit proposed in this article for this dipole leads to an angle $\phi$ that varies slowly with frequency, and to the two admittances $\lambda_1$ and $\lambda_2$. The impedance $1/\lambda_1$ is negligible and irrelevant to obtain results using the equivalent circuit. The reactance contributed by $\lambda_2$ (the imaginary part of $1/\lambda_2$) is shown in Fig. 13. The figure also shows some simple approximations to these parameters: a constant value for $\phi$ and a parallel $LC$ resonator for the reactance. The numerical values of the capacitance and inductance of the resonator are shown in the figure caption. The transformer, with a fixed 4:7:1 turns ratio, comes from the constant angle $\hat{\phi} = 78^\circ$ with which $\phi$ is approached.

The equivalent circuit with these simple approximations succeeds in producing a good agreement with the simulated transmission parameters of the unit cell, as shown in Fig. 14. The interest in this case is focused on the transmission when it is the horizontal polarization which impinges in the
scatterer. The co-polar transmission will correspond to \(1 + S_{22}\) [the parameter \(S_{22}\) according to (1)], while the counter-polar transmission corresponds to the parameter \(S_{21}\) (\(S_{32}\)).

In this case, it should be noted that the circuit obtained in an entirely analytical way in [23], with admittances for the vertical and horizontal polarizations, can be easily reduced to the one proposed here. For this, one can resort to the calculation of the \(S\)-parameters from the circuit in [23] and then to expressions (7) and (11).

### E. Lossy Scatterers Without Symmetry Planes

The scatterers included in the unit cells can be lossy, because they are composed of resistive paints, because they include lumped resistors, or simply because the losses in their conductors are significant.

Even if they are lossy, when they are any of the symmetry properties discussed in the previous sections, expression (8) leads to real values of \(\phi\) and to eigenexcitations in the form of linear polarizations. However, when there are losses and no symmetry, the value obtained for \(\phi\) in (8) is complex: \(\phi = \phi_r + j\phi_i\). This situation is not taken into account, or at least not made explicit, in [21]. There, it is stated that the parameters \(m\) and \(\rho_{rot}\) of expressions [21, (22) and (23)] are real, despite the fact that the elements of the \(S\)-parameter matrix (\(\Gamma_{xx}, \Gamma_{xy},\) and \(\Gamma_{yx}\) in [21]) are, in general, complex.

For lossy scatterers without any symmetry, the eigenexcitations are no longer linear polarizations, but elliptical ones. These elliptical polarizations (see Appendix) will be characterized by the location of their major axes by means of \(\phi_r\), their axial ratios \(|\coth \phi_i|\), and their handedness, determined by the sign of \(r \).

An example of this kind of unit cell is present in [32]. The scatterer is in this case an intricate aluminum layout over a silicon substrate [see the inset in Fig. 15(a)], whose shape and dimensions are given in [32, Fig. 1] and along that reference’s text. The silicon substrate is thick enough so that the simulation of the unit cell by HFSS can be done considering vacuum at the physical input port and silicon at the output port. A simple change in the reference impedance at the output electrical ports \(\|\) and \(\|\) allows the \(S\)-parameters of the four ports of Fig. 2(a) to be obtained. In the electromagnetic simulations, the thickness of the aluminum strips has been taken into account for the computation of losses. But this thickness is small enough to be neglected as far as the equivalent circuit is concerned, since it has been found that the obtained \(S\)-parameters satisfy the relations made explicit in (1).

This bi-periodic surface is designed to operate in the terahertz frequency range, where losses in the aluminum are appreciable enough. As a result, the angle \(\phi\) is complex, and its real and imaginary parts are shown in Fig. 15, where the imaginary part is represented as the axial ratio in decibels, i.e., \(AR(dB) = 20 \log |\coth \theta|\). Note that there are frequencies at which the axial ratio is below 10 dB, so the polarizations of the scatterer eigenexcitations are far from linear. It is also interesting to see how the directions of the semi-major axes, \(\phi_r\), vary with frequency as the different parts of the intricate unit cell layout become relevant.

As in the other cases, the expressions in (11) provide the circuit admittances, which, in this case, have relevant real parts. These admittances are shown in Fig. 16. The five resonances predicted by the authors of [32] are evident and are split between the two admittances. There even appears to be a sixth resonance at around 1.14 THz. It should be noted...
that the real parts in Fig. 16(a) are positive over the entire frequency range.

IV. CONCLUSION

The use of the equivalent circuit based on eigenstates for the analysis and modeling of bi-periodic structures has shown all its advantages. In the case of unit cells whose scatterer element is small compared with the unit cell, the circuit has been able to obtain the element orientation and admittances invariant with the angular position in a simple and explicit way. Consequently, it is able to obtain its response in any of its orientations. When the scatterer element is of similar size to the cell it occupies, it is only necessary to model the variation in the position of its eigenexcitations with the element orientation to obtain a similar result. In any case, the circuit reduces the number of elements compared with a network with the usual \Pi or \Omega topologies, as it contains only two immittances. These are all the advantages shared with the approach in [21].

Moreover, and unlike [21], the proposed circuit and its decomposition into eigenstates have been able to deal with lossy structures without any further difficulties. The result is a circuit with two immittances that have positive real parts. The modeling of these admittances, in the cases considered so far, has always been feasible with constant and positive RLC elements. This has considerably simplified the handling of unit cells with lossy elements, such as those found in rasorbers.

A number of examples have been shown of how the circuit is simplified and configured according to the symmetry properties of the unit cell it represents. The characteristics of these symmetries are directly transferred and made explicit in the final appearance of the circuit.

The proposed circuit is very general and thus, in principle, valid for any arbitrary planar geometry. A higher geometrical complexity of the modeled structure could only lead, in the worst case, to a more complicated model for the behavior of the admittances and transformers of the circuit.

It should also be noted that in the cases of lossy and nonsymmetrical scatterer elements, the circuit contains transformers with complex turns ratios. But the circuit is then able to provide some insight into the occurrence of the complex transformers, which in the end reflects the different characters of the unit cell eigenexcitations. These cease to be linear polarizations and it is the circuit deduction procedure itself that allows us to discover that they become nonorthogonal-in-power elliptic polarizations.

APPENDIX

When $\phi$ is complex, so are the eigenvectors (9). Writing $\phi$ in terms of its real and imaginary parts, $\phi = \phi_r + j \phi_i$, the eigenvectors are

$$\vec{v}_1 = \begin{bmatrix} \sin \phi_r \cosh \phi_i + j \cos \phi_r \sinh \phi_i \\ \cos \phi_r \cosh \phi_i - j \sin \phi_r \sinh \phi_i \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -\cos \phi_r \cosh \phi_i + j \sin \phi_r \sinh \phi_i \\ \sin \phi_r \cosh \phi_i + j \cos \phi_r \sinh \phi_i \end{bmatrix}. \quad (26)$$

Let us define two new (real and unit) vectors that correspond to expressions (9) when $\phi$ is replaced by its real part

$$\vec{u}_1 = \begin{bmatrix} \sin \phi_r \\ \cos \phi_r \end{bmatrix}; \quad \vec{u}_2 = \begin{bmatrix} -\cos \phi_r \\ \sin \phi_r \end{bmatrix}. \quad (27)$$

With these new vectors, the eigenvectors can be expressed as

$$\vec{v}_1 = \vec{u}_1 \cosh \phi_i - j \vec{u}_2 \sinh \phi_i,$n$$ $$\vec{v}_2 = j \vec{u}_1 \sinh \phi_i + \vec{u}_2 \cosh \phi_i. \quad (28)$$
For convenience, a parameter $r$ can be defined as $r = \coth \phi_i$. Then, vectors $\hat{u}_1$ and $\hat{u}_2$ can be renormalized to obtain unit eigenvectors in the case of complex $\phi_i$ into these two eigenexcitations. For instance, an electric field $E$ orthogonal when the transformer turns ratios are not real [24].

These two vectors represent two elliptical polarizations with the same axial ratio, $|r|$, with the same handedness (sign of $r$) and whose semi-major axes are perpendicular to each other. While one of them will have its semi-major axis in the $\hat{u}_1$ direction, the other will have it in the $\hat{u}_2$ direction, as shown in Fig. 17.

Had these elliptic polarizations a different handedness, they would be orthogonal. But sharing the same handedness, they do not obey the power orthogonality relation as the linear polarizations do. This fact runs parallel to the fact that the excitations of the proposed equivalent circuit are not power orthogonal when the transformer turns ratios are not real [24].

Note that any arbitrary field always admits a decomposition into these two eigenexcitations. For instance, an electric field $\vec{E}$ can be written as

$$\vec{E} = c_1 \hat{u}_1 + c_2 \hat{u}_2$$

(30)

where

$$c_1 = \sqrt{r^2 + 1} \left( \frac{r \hat{E} \cdot \hat{u}_1 - j \hat{E} \cdot \hat{u}_2}{r^2 - 1} \right)$$

$$c_2 = \sqrt{r^2 + 1} \left( j \hat{E} \cdot \hat{u}_1 + r \hat{E} \cdot \hat{u}_2 \right).$$

(31)

In view of the expressions of these two coefficients, the decomposition into eigenexcitations will not be possible when the axial ratio $|r|$ is very close to unity, since both the coefficients would tend to infinity. But this cannot happen. From (8), it follows that:

$$\phi_i = \frac{1}{2} \arg \left( \frac{j(S_{22} - S_{11}) - 2S_{12}}{j(S_{22} - S_{11}) + 2S_{12}} \right).$$

(32)

Therefore, $|\phi_i| \leq \pi/2$, and the axial ratio has as lower bound: $|r| \geq \coth(\pi/2) = 1.09$.

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