LONG-TIME BEHAVIOR OF WEAK SOLUTIONS FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DEGENERATE VISCOSITY

ZHILEI LIANG

Abstract. The long-time regularity and asymptotic of weak solutions are studied for compressible Navier-Stokes equations with degenerate viscosity in a bounded periodic domain in two and three dimensions. It is shown that the density keeps strictly positive from below and above after a finite period of time. Moreover, higher velocity regularity is obtained via a parabolic type iteration technique. Since then the weak solution conserves its energy equality, and decays exponentially to the equilibrium in $L^2$-norm as time goes to infinity. In addition, assume that the initial momentum is zero, the exponential decay rate of the derivatives is derived, and the weak solution becomes a strong one in two dimensional space.

1. Introduction

The time-evolutionary Navier-Stokes equations simulate the motion of un-stationary compressible fluids. It is an important mathematical model in continuous medium mechanics theory and has a wide applications in many fields, such as astrophysics, engineering, and so on. In this paper we focus on the isentropic compressible Navier-Stokes equations (cf. [25, 30])

\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla \rho^\gamma &= \text{div}(\rho \nabla u).
\end{align*}

Here, variables $t > 0$, $x \in \Omega \subseteq \mathbb{R}^n$, $(n = 2, 3)$, the unknown functions $\rho(x,t)$ and $u(x,t)$ are the density and the velocity field respectively; the adiabatic exponent $\gamma$ is assumed to satisfy $\gamma > 1$.

In this paper we are limited to the bounded domain with periodic boundaries, i.e., $\Omega = \mathbb{T}^n$. For completeness, we supplement equations (1.1) with the initial functions

$$\rho(x,0) = \rho_0 \geq 0, \quad \rho u(x,0) = m_0, \quad x \in \Omega.$$  

(1.2)

There are many literatures concerning the existence, regularity, and long-time behavior of solutions for compressible Navier-Stokes equations. We begin with the constant viscous coefficient case and collect some existence and regularity results of weak solutions, without completeness. The progress of one-dimensional problem is satisfactory, see, e.g., [3, 15, 18, 19, 27]. For high dimensions with positive density, we refer to the papers by Hoff [16] and by Serre [35]. However, it is a very challenging problem in mathematics when initial vacuum appears, Lions [30] first proved the global existence of weak solutions when the adiabatic exponent $\gamma \geq \frac{3}{2}$. Feireisl-Novotný-Petzeltová [10] improved Lions’ result to a wider range of $\gamma > \frac{3}{2}$. Recently, Plotnikov-Weigant [34] discussed the global existence of weak solutions in case of $\gamma = \frac{3}{2}$. Jiang-Zhang [21] obtained the existence of weak solutions for all $\gamma > 1$ if

\section*{2010 Mathematics Subject Classification.} 35Q30, 76N10.

\section*{Key words and phrases.} Compressible Navier-Stokes; Degenerate viscosity; Weak solutions; Vanishing vacuum; Asymptotic behavior.
some symmetry assumptions are made. As far as the regularity of solutions is concerned, Desjardins 5 proved that if $\gamma > 3$, the weak solutions satisfy, for some $T > 0$,

$$
\rho \in L^\infty_\text{loc} (0, T; L^\infty(\Omega)) , \ N\mathbf{u} \in L^\infty_\text{loc} (0, T; L^2(\Omega)) , \ \nabla \rho \mathbf{u} , \ \nabla \times (\nabla \times \mathbf{u}) , \ \nabla(\text{div} \mathbf{u} - P) \in L^\infty_\text{loc} (0, T; L^2(\Omega)) .
$$

Choe-Jin 11 discussed similar regularity of weak solutions in case of zero bulk viscosity.

Physically, the dynamics of the viscous fluids near vacuum are better modeled by the Navier-Stokes equations with density-dependent viscosities. It can also be understand mathematically in the derivation of the compressible Navier-Stokes equations from the Boltzmann equation by the Chapman-Enskog expansions, where the viscosity depends on the temperature and thus the density for isentropic flows. Equations (1.1) corresponds to the shallow water model in the case $\gamma = 2$ in dimension two, where $\rho$ stands for the height of the water. Such model describes the horizontal structure of the fluids, and appear often in geophysical flows (cf. [12, 17, 30]). We also regard (1.1) as a special case of the equations

\begin{equation}
\begin{aligned}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0 , \\
\partial_t (\rho \mathbf{u}) + \text{div}(\rho \otimes \mathbf{u}) + \nabla \rho \gamma &= \text{div} \mathbb{S} ,
\end{aligned}
\end{equation}

where the stress tensor

$$
\mathbb{S} = h(\rho)\nabla \mathbf{u} + g(\rho)\text{div} \mathbf{u} \mathbb{I} \quad \text{or} \quad \mathbb{S} = h(\rho)\mathbb{I} \mathbf{u} + g(\rho)\text{div} \mathbf{u} \mathbb{I} ,
$$

$\mathbb{I}$ is the identical matrix in $\mathbb{R}^n$.

For smooth solutions to equations (1.3) on condition that $g(\rho) = \rho h'(\rho) - h(\rho)$, Bresch-Desjardins [6, 7] first developed a new entropy estimate

\begin{align}
\frac{d}{dt} \int \left( \frac{1}{2} \rho |\mathbf{u} + \nabla \varphi(\rho)|^2 + \frac{1}{\gamma - 1} \rho \gamma \right) \\
+ \int (\nabla \varphi(\rho) \nabla \rho \gamma + h(\rho)|\nabla \mathbf{u} - (\nabla \mathbf{u})^\prime|^2) = 0 , \quad \rho \varphi' = h'.
\end{align}

Li-Li-Xin [20] studied the one-dimensional problem, they established the global entropy weak solution to (1.3) and discussed the long-time dynamics: vanishing vacuum states and blow-up phenomena. Strauskraba-Zlotnik [36] studied the global regularity of weak solution and derived exponential decay rate estimates. We also refer to the papers [14, 20, 20, 23, 40, 41, 42] for related results in dimension one, and the papers [13, 22] for high-dimensions with symmetric assumptions. For general high dimensions $n = 2, 3$, Mellet-Vasseur [31] provided a compactness framework so that the weak solutions can be established from a sequence of smooth approximate solutions. On the basis of entropy estimate (1.4) due to Bresch-Desjardins, as well as Mellet-Vasseur type estimates in [31], the global existence of weak solutions to the problem (1.1)-(1.2) are derived by Li-Xin [28] and Vasseur-Yu [37] from different approach.

By weak solutions, we mean

**Definition 1.1.** Function $(\rho, \mathbf{u})$ is called a weak solution to the problem (1.1)-(1.2) if for any fixed $T > 0$

\begin{align}
0 \leq \rho &\in L^\infty_\text{loc} (0, T; L^1(\Omega) \cap L^\gamma(\Omega)) , \\
\nabla \sqrt{\rho} &\in L^\infty_\text{loc} (0, T; L^2(\Omega)) , \ \ \nabla \mathbf{u} \in L^2 (0, T; W^{-1,1}(\Omega)) .
\end{align}

Moreover,

(i). the problem (1.1)-(1.2) is satisfied in the sense of $\mathcal{D}'(\Omega \times [0, T])$, 

(ii). if for some $T > 0$,

$$
\end{align}
(ii). for a.e. \( t > 0 \), the energy inequality
\[
\frac{d}{dt} \int_{\Omega} E(x,t) dx + \int_{\Omega} \Lambda^2 dx \leq 0
\] (1.6)
and the entropy inequality
\[
\frac{d}{dt} \int_{\Omega} \rho |\mathbf{u} + \nabla \ln \rho|^2 dx + \int_{\Omega} \left( |\nabla \rho|^2 + \Lambda^2 \right) dx \leq 0
\] (1.7)
are fulfilled, where the energy density
\[
E = \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right)
\] (1.8)
and the function \( \Lambda \in L^2(\Omega \times (0,T)) \) satisfying
\[
\int_0^T \int_{\Omega} \Lambda \Phi = - \int_0^T \int_{\Omega} \left( \sqrt{\rho} \sqrt{\rho} \mathbf{u} \text{div} \Phi + 2 \sqrt{\rho} \mathbf{u} \Phi \sqrt{\rho} \nabla \sqrt{\rho} \right)
\] (1.9)
for all smooth functions \( \Phi \) valued in \( \mathbb{R}^{n \times n} \).

The regularity of weak solutions to (1.1) is an important and interesting question. We remark that the inequality sign ”\( \leq \)“ in (1.6) means anomalous dissipation of energy, which is mainly caused by the wildness of the weak solutions. The energy dissipation is one of basic properties of the fluid equations related to its physical origin. It is motivated from Kolmogorov’s theory of turbulence of flow and also reminiscent of the Leray-Hopf weak solutions to the incompressible Navier-Stokes equations. Clearly, if the solution is smooth enough, then energy equality conserves automatically. Yu [39] obtained a sufficient condition so that the weak solutions to the equations (1.1) preserve energy conservation for all positive time. The main idea in [39] is using the commutator estimates developed by DiPerna-Lions [9] to deal with the nonlinear term \( \partial_t (\rho \mathbf{u}) \). For other regularity criterion for energy conservation of weak solutions for compressible fluids, we mention the papers [8, 29, 38] and the references cited therein.

1.1. Main results. The main concern of this current article is the regularity and the long-time asymptotic of the weak solutions to the problem (1.1)-(1.2).

Our main results read in the theorems below.

**Theorem 1.1.** Let \( \Omega \) be a bounded domain with periodic boundaries in \( \mathbb{R}^n \), i.e., \( \Omega = T^n \) with \( n = 2, 3 \). Let \( (\rho, \mathbf{u}) \) be a weak solution to the problem (1.1)-(1.2) in Definition 1.1.

Then there exist positive constants \( C_0, C \), and some large time point \( T < \infty \), which depend only on the initial functions and \( \Omega \), such that
\[
0 < C_0 \leq \rho(x,t) \leq C < \infty, \quad \text{a.e. } (x,t) \in \Omega \times (T, \infty)
\] (1.10)
and
\[
\mathbf{u} \in L^\infty (T, \infty; L^\infty (\Omega)), \quad \sqrt{\rho} \nabla \mathbf{u} \in L^2 (T, \infty; L^2 (\Omega)).
\] (1.11)
Furthermore, the following assertions are valid:

(i). (Energy conservation) For almost all \( t_1 > T \), and all \( t \in [t_1, \infty) \), the solution \( (\rho, \mathbf{u}) \) to (1.1)-(1.2) keeps energy conservation, i.e.,
\[
\int_{\Omega} \mathcal{E}(x,t) dx + \int_{t_1}^t \int_{\Omega} \rho |\nabla \mathbf{u}|^2 dx dt = \int_{\Omega} \mathcal{E}(x,t_1) dx,
\] (1.12)
where \( \mathcal{E} \) is defined in (1.8).

(ii). (Exponential asymptotic) There exist constants \( \sigma \) and \( C \) which rely only on the initial functions and \( \Omega \), such that
\[
\| \mathbf{u} - (\mathbf{u})_a \|^2_{L^2} + \| \rho - \rho_a \|^2_{L^2} \leq Ce^{-\sigma t}, \quad t > T,
\] (1.13)
where
\[ \overline{\rho} = \frac{\int_{\Omega} \rho_0(x) \, dx}{|\Omega|} > 0 \quad \text{and} \quad (u)_a = \frac{\int_{\Omega} m_0(x) \, dx}{\int_{\Omega} \rho_0(x) \, dx}. \] (1.14)

**Remark 1.1.** Inequality (1.10) shows that no concentration or vacuum state can be formed after a finite period of time, regardless of what the initial state is. Moreover, the conservation of energy for weak solutions is automatically established.

In the second theorem we derive higher regularity of the weak solutions, if the initial momentum is assumed to be zero.

**Theorem 1.2.** In addition to the hypotheses in Theorem 1.1, we assume that the initial momentum is zero, namely,
\[ \left| \int_{\Omega} m_0(x) \, dx \right| = 0. \] (1.15)

Then, for some \( T_1 \in (T, \infty) \), there exist constants \( \sigma_1 \in (0, \sigma) \) and \( C \) depending only on the initial functions and \( T_1, \Omega \), such that
\[ \|\nabla \rho\|_{L^2}^2 \leq C e^{-\sigma_1 t}, \quad t > T_1. \] (1.16)

Particularly, \( (\rho, u) \) becomes a strong solution in two-dimensional space, satisfying the improved velocity regularity
\[ u \in L^\infty \left( T_1, \infty; H^1(T^2) \right) \cap L^2 \left( T_1, \infty; W^{2, \frac{2}{3}}(T^2) \right), \] \[ u_t \in L^2 \left( T_1, \infty; L^2(T^2) \right), \] (1.17)
and the long-time asymptotics
\[ \|\nabla \rho\|_{L^2(T^2)}^2 + \|\nabla u\|_{L^2(T^2)}^2 \leq C e^{-\sigma_1 t}, \quad t > T_1. \] (1.18)

**Remark 1.2.** Regularity (1.17) implies that \( (\rho, u) \) satisfies the equations (1.1) for almost everywhere in \( \Omega \times (T_1, \infty) \) in two-dimensional space. In this regard, we call \( (\rho, u) \) a strong solution. However, whether the regularity (1.17) guarantees the uniqueness or not is unclear.

**Remark 1.3.** By slight modification, Theorems 1.1, 1.2 work for general bounded domain with suitable boundary conditions, as long as the existence of weak solutions is known.

1.2. **Methodology.** Let us give a brief, heuristic overview of the proof and explain the underlying motivations.

We are mainly motivated from the one-dimensional results in [26, 36], the global existence of weak solutions in [28, 37], and the regularity criterion in [48, 89].

We first approximate the density in (3.6) by \( \rho_k \) so that for large number \( k \)
\[ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \geq \frac{\rho_k}{|\Omega|} = \frac{1}{|\Omega|} \int_{\Omega} \rho_k(x,t) \, dx. \] (1.19)
(In the proof we assume that \( \frac{1}{|\Omega|} \int_{\Omega} \rho_0(x) \, dx = \int_{\Omega} \rho_0(x) \, dx = 1 \). We remark that the approximate function \( \rho_k \) is to avoid the possible concentration and vacuum. Notice that \( \nabla \rho \in L^\infty(0, T; L^2(\Omega)) \), we have \( \partial \rho_k = \partial \rho \) in \( \{ (x,t) : \frac{1}{k} \leq \rho \leq k \} \) and \( \partial \rho_k = 0 \) for others. Therefore, \( \nabla \rho_k \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; L^2(\Omega)) \). A careful computation shows that, for any fixed \( k < \infty \),
\[ \lim_{t \to \infty} \|\rho_k - \overline{\rho_k}\|_{L^\infty(\Omega)} = 0. \] (1.20)
By (1.20) we see that $\rho_k(x,t)$ stabilize to its average value $\overline{\rho_k}$ when the time $t$ goes to infinity. In particular, there is some finite $T^* = T^*(k) < \infty$ such that

$$\|(\rho_k - \overline{\rho})(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{1}{4}, \text{ for all } t > T^*.$$ 

This, along with (1.19), implies that for $t > T^*$

$$\frac{1}{2} = \frac{3}{4} - \frac{1}{4} \leq \rho_k - \frac{1}{4} \leq \rho_k \leq \frac{5}{4} + \frac{1}{4} = \frac{3}{2},$$

that is,

$$\frac{1}{2} \leq \rho_k \leq \frac{3}{2}, \text{ a.e. } (x,t) \in \Omega \times (T^*, \infty). \quad (1.21)$$

Remember that (3.6), for large $k < \infty$ we deduce from (1.21) that

$$\frac{1}{2} \leq \rho \leq \frac{3}{2}, \text{ a.e. } (x,t) \in \Omega \times (T^*, \infty).$$

Similar argument runs for the general case, and thus (1.10) is proved.

Once (1.10) is obtained, the momentum equations can be regarded as a parabolic system in terms of velocity $u$. Hence, we are motivated to exploit a parabolic type iteration technique as well as a continuity method to conclude (1.11). The former part in (1.11) directly yields $u \in L^4_{\text{loc}}(T, \infty; L^6(\Omega))$. This and (1.10) enable us to deduce that the weak solution conserves its energy equality, following the same step as in [39]. Finally, by means of the inverse operator estimates and basic energy estimates, we show that the solution decays exponentially to its equilibrium in the sense of $L^2(\Omega)$ topology as time goes to infinity.

In Theorem 1.2, we obtain better regularity if the initial momentum is assumed to be zero. In particular, in case of (1.15) we first deduce from (1.13) that $e^{\sigma t} \|u\|_{L^2} \leq C$. By this and the entropy inequality (1.7) we successfully proved that $\|
abla \rho\|_{L^2}$ has an exponential decay rate as time tends to infinity. Next, in deriving the higher derivative estimates, we need to overcome the difficulty resulting from the density appeared in the diffusion. An important observation is the Sobolev embedding in dimension two, i.e., $\|\nabla^2 u\|_{L^\frac{2}{3}} \Rightarrow \|\nabla u\|_{L^6}$. This together with the time decay of $\|\nabla \rho(\cdot, t)\|_{L^2}$ guarantee higher velocity regularity in two dimensional case, and consequently, an exponential decay of $L^2$-norm of the velocity derivative.

**Notation:** Refer to [1], we denote the standard Sobolev spaces by

$$L^p = L^p(\Omega), \quad W^{k,p} = \{f : |\partial^i f| \in L^p, \ 0 \leq i \leq k\}, \quad H^k = W^{k,2},$$

and use the simplified conventions

$$\int f = \int f(x)dx, \quad \mathcal{T} = \frac{1}{|\Omega|} \int f,$$

where $|\Omega|$ is the Lebesgue measure of $\Omega$. Throughout this paper, the capital letters $C$, $C_i$ ($i = 1, 2, \ldots$) symbol positive constants which may vary from line to line. In addition, $C(a)$ is used to emphasize that $C$ depends on $a$.

2. Preliminaries

The first lemma is responsible for the existence results of weak solutions to the problem (1.1)-(1.2).

**Lemma 2.1** (See [28, 37]). For any $T < \infty$, the problem (1.1)-(1.2) admits a weak solution $(\rho, u)$ over $(0, T)$ in the sense of Definition (1.7) which satisfies (1.5)-(1.7).

The following embedding inequalities will be frequently used throughout this paper.
Lemma 2.2 (Gagliardo-Nirenberg). Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded domain with smooth boundaries. It holds that for any \( v \in W^{1,q} \cap L^s \)
\[
\|v\|_{L^p} \leq C_1 \|v\|_{L^q} + C_2 \|\nabla v\|_{L^s}^\theta \|v\|_{L^s}^{1-\theta},
\]
where the constant \( C_i (i = 1, 2) \) depends only on \( p, q, s, \theta \), the exponents satisfy
\[
0 \leq \theta \leq 1, \quad 1 \leq q, s \leq \infty, \quad \frac{1}{p} = \theta \left( \frac{1}{q} - \frac{1}{n} \right) + (1 - \theta) \frac{1}{s}
\]
and
\[
\begin{aligned}
\min\{s, \frac{qs}{n-q}\} &\leq p \leq \max\{s, \frac{qs}{n-q}\}, & \text{if } q < n; \\
s &\leq p < \infty, & \text{if } q = n; \\
s &\leq p \leq \infty, & \text{if } q > n.
\end{aligned}
\]
Moreover, \( C_1 = 0 \) if 0-Dirichlet boundary condition or zero average is assumed.

As an application of (2.1), we have
\[
\|v - \overline{v}\|_{L^p} \leq C \|\nabla v\|_{L^\frac{np}{n+p}}, \quad p \in [1, 6]
\]
and
\[
\left( \int_{-r}^r \int |v|^\frac{p}{2} dx dt \right)^\frac{2}{p} \leq C \left( \sup_{t \in [-r, r]} \int |v|^2 dx + \int_{-r}^r \int \left( |v|^2 + |\nabla v|^2 \right) dx dt \right).
\]

Proof. We only prove (2.3). In fact, by (2.1) and Hölder inequality,
\[
\int_{-r}^r \int |v|^\frac{p}{2} dx dt \\
\leq \sup_{t \in [-r, r]} \left( \int |v|^2 dx \right)^\frac{p}{4} \int_{-r}^r \int |v|^2 dx dt \\
\leq C \sup_{t \in [-r, r]} \left( \int |v|^2 dx \right)^\frac{p}{4} \int_{-r}^r \left( \int |v|^6 dx \right)^\frac{1}{4} dt \\
\leq C \sup_{t \in [-r, r]} \left( \int |v|^2 dx \right)^\frac{p}{4} \int_{-r}^r \left( |v|^2 + |\nabla v|^2 \right) dx dt.
\]
Raising the above expression to the power of \( \frac{3}{5} \), applying the Young inequality, we obtain (2.3). \( \square \)

Lemma 2.3. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundaries. Then, for all \( v \in W^{k,p}(\Omega) \cap W^{1,p}_0(\Omega) \) with \( p \in (1, \infty) \) and integer \( k \geq 0 \), there exists a positive constant \( C \) which depends only on \( p, n, k \) such that
\[
\|\nabla^{k+2} v\|_{L^p} \leq C \|\Delta v\|_{W^{k,p}}
\]

Proof. The proof is a classical elliptic regularity theories in [2]. See also [4, Lemma 12]. \( \square \)

Lemma 2.4 (Bogovskii). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, and let \( p \in (1, \infty) \). There is a linear operator \( B = (B^1, ..., B^n) \) from \( L^p \) to \( W^{1,p}_0 \) such that for all \( v \in L^p \) with \( \overline{v} = 0 \),
\[
\text{div} B(v) = v \quad \text{a.e. in } \Omega \quad \text{and} \quad \|\nabla B(v)\|_{L^p} \leq C(p, n, \Omega)\|v\|_{L^p}.
\]

Proof. The detailed proof is available in [32]. \( \square \)
3. Proof of Theorem 1.1

First, for fixed \( t > 0 \), we define
\[
\psi(s) = \begin{cases} 
1, & s \leq t, \\
k(t + \frac{1}{k} - s), & t \leq s \leq t + \frac{1}{k}, \\
0, & s \geq t + \frac{1}{k}.
\end{cases}
\]

Since \( \Omega = \mathbb{T}^n \) is periodic, we test the mass equation (1.1) against \( \psi(t) \) and receive
\[
\int_t^{t+\frac{1}{k}} \int \rho(x, s) = \int \rho_0.
\]

By Lebesgue point theorem, sending \( k \to \infty \) in above equality yields
\[
\int \rho(x, t) = \int \rho_0(x), \quad \text{a.e. } t > 0.
\]

Similar argument runs that
\[
\int \rho u(x, t) = \int m_0(x), \quad \text{a.e. } t > 0.
\]

3.1. Positive bounds of density. We will show that the density is positively bounded from below after a finite time interval. Without loss of generality, we assume
\[
|\Omega| = 1 \quad \text{and} \quad \int_{\Omega} \rho_0 = 1,
\]
and prove that, for some large \( T^* < \infty \),
\[
\frac{1}{2} \leq \rho \leq \frac{3}{2}, \quad \text{a.e. } (x, t) \in \Omega \times (T^*, \infty).
\]

For more general case, the same deduction yields (1.10).

The main task of this subsection is to justify (3.4).

By virtue of Lemma 2.1, the weak solution \((\rho, u)\) satisfies (1.5), that is,
\[
\sup_{t \geq 0} \| \nabla \sqrt{\rho} \|_{L^2} + \int_0^{\infty} \left( \| \nabla \rho^2 \|_{L^2}^2 + \| \sqrt{\rho} \nabla u \|_{L^2}^2 \right) dt \leq C.
\]

Here, and during this subsection, the constant \( C \) is independent of \( t \).

Let us approximate the density \( \rho \) by
\[
\rho_k = \begin{cases} 
k, & k \leq \rho, \\
\rho, & 1/k \leq \rho \leq k, \\
1/k, & \rho \leq 1/k.
\end{cases}
\]

Due to (3.1) and (3.3), it has \( \rho \in L^\infty(0, \infty; L^1) \), which implies that \( \rho \) is bounded almost everywhere in \( \Omega \times (0, \infty) \). Thus,
\[
\lim_{k \to \infty} \rho_k \to \rho \quad \text{a.e. } \Omega \times (0, \infty).
\]

By Lebesgue Dominated Convergence theorem, one has
\[
\lim_{k \to \infty} \int \rho_k = \int \rho = 1.
\]
Therefore, for some large number $k$,
\[ \frac{3}{4} \leq \int \rho_k \leq \frac{5}{4}. \quad (3.7) \]

It is clear from (3.5) and (3.6) that
\[ \partial \rho_k = \begin{cases} 
\partial \rho, & \text{in } \left\{ (x, t) : \frac{1}{k} \leq \rho \leq k \right\}, \\
0, & \text{in } \left\{ (x, t) : \rho \leq \frac{1}{k} \text{ or } \rho \geq k \right\},
\end{cases} \quad (3.8) \]

where $\partial = \partial_t$ or $\partial = \partial_x$. Moreover, by (3.8) and (2.2), it satisfies that
\[ \rho_k \in L^\infty(0, \infty; H^1) \cap L^2(0, \infty; H^1) \quad (3.9) \]

and that for all $p \in [2, \infty)$
\[ \| \rho_k - \overline{\rho_k} \|_{L^p} \leq C \| \rho_k \|_{L^\infty}^{p-2} \| \rho_k - \overline{\rho_k} \|_{L^2} \]
\[ \leq C \| \nabla \rho_k \|_{L^2} \]
\[ \leq C \| \nabla \rho \|_{L^2(\{x : \frac{1}{k} \leq \rho \leq k\})} \]
\[ \leq C \| \nabla \rho \|_{L^2(\{x : \frac{1}{k} \leq \rho \leq k\})} \]
\[ \leq C \| \nabla \rho \|_{L^2(\{x : \frac{1}{k} \leq \rho \leq k\})}, \quad (3.10) \]

where $\overline{\rho_k}$ is the average function of $\rho_k$, and the constant $C = C(p, k)$ depends on $p$ and $k$.

Integrating (3.10) in time and using (3.5), we get
\[ \int_0^\infty \| \rho_k - \overline{\rho_k} \|_{L^p} \, dt \leq C(p, k). \quad (3.11) \]

We next prove that
\[ \int_0^\infty \left| \frac{d}{dt} \| \rho_k - \overline{\rho_k} \|_{L^p} \right| \, dt \leq C(p, k). \quad (3.12) \]

For this purpose, we compute
\[ \frac{d}{dt} \| \rho_k - \overline{\rho_k} \|_{L^p}^p \]
\[ = p \frac{d}{dt} \| \rho_k - \overline{\rho_k} \|_{L^p}^{p-2} (\rho_k - \overline{\rho_k})^p + p \int (\rho_k - \overline{\rho_k})^{p-2} (\rho_k - \overline{\rho_k}) \partial_t \rho_k. \quad (3.13) \]

By virtue of (3.9), we are allowed to take in (1.9) the test function $\Phi = \rho_k^{-\frac{3}{2}} \tilde{\Phi}$ with any smooth function $\tilde{\Phi}$ to deduce that
\[ \int_0^\infty \int \rho_k^{-\frac{3}{2}} \Lambda \tilde{\Phi} \]
\[ = - \int_0^\infty \int \sqrt{\rho_k} \sqrt{\rho_k} \text{div}(\rho_k^{-1} \tilde{\Phi}) - 2 \int_0^\infty \rho_k^{-\frac{3}{2}} u \nabla \sqrt{\rho_k} \tilde{\Phi} \]
\[ = - \int_0^\infty \int u \text{div}\tilde{\Phi}. \quad (3.14) \]

From (3.14) we see that $\nabla u$ is well defined in $L^2$ in weak sense. By uniqueness, it has
\[ \nabla u = \frac{\Lambda}{\sqrt{\rho_k}} \in L^2(0, \infty; L^2). \quad (3.15) \]
In view of (1.5), (3.8), (3.11), and the mass equation (1.1), it satisfies
\[
\left| \frac{d}{dt} \overline{\rho} \right| \leq \int_{\{x : \frac{1}{2} \leq \rho \leq k\}} |\partial_t \rho|
\]
\[
\leq C \int_{\{x : \frac{1}{2} \leq \rho \leq k\}} (|u \cdot \nabla \rho| + |\rho_k \text{div} u|)
\]
\[
\leq C \left( \|\sqrt{\rho} u\|_{L^2} \|\nabla \sqrt{\rho}\|_{L^2((x : \frac{1}{2} \leq \rho \leq k)})} + \|\rho_k\|_{L^1} \|\sqrt{\rho_k} \nabla u\|_{L^2} \right)
\]
\[
\leq C \left( \left\| \nabla \rho \right\|_{L^2((x : \frac{1}{2} \leq \rho \leq k)})} + \|\sqrt{\rho_k} \nabla u\|_{L^2} \right)
\]
\[
\leq C \left( \left\| \nabla \rho \right\|_{L^2} + \|\sqrt{\rho_k} \nabla u\|_{L^2} \right).
\]
Therefore, substituting (3.16) and (3.18) back into (3.13), integrating the resultant expression, using (3.5) and (3.15), we obtain
\[
\int_0^\infty \left| \frac{d}{dt} \| \rho_k - \rho_k' \|_{L^p} \right| dt 
\leq C \int_0^\infty \left( \| \nabla \rho_k^\gamma \|_{L^2}^2 + \| \sqrt{\rho_k} \nabla u \|_{L^2}^2 \right) dt 
\leq C.
\]
This proves (3.12).

It follows from (3.11) and (3.12) that
\[
\lim_{t \to \infty} \| \rho_k - \rho_k' \|_{L^p} = 0, \quad \forall \ p \in [2, \infty).
\]
(3.19)

By (3.6), it has \( \rho_k - \rho_k' \in L^\infty(0, \infty; L^\infty) \). Thus, sending \( p \to \infty \) in (3.19) yields
\[
\lim_{t \to \infty} \| \rho_k - \rho_k' \|_{L^\infty} \leq \lim_{p \to \infty} \lim_{t \to \infty} \| \rho_k - \rho_k' \|_{L^p} = 0,
\]
which implies, for some large \( T^* = T^*(k) < \infty \),
\[
\| \rho_k - \rho_k' \|_{L^\infty} \leq \frac{1}{4}, \quad \forall \ t \in (T^*, \infty).
\]
(3.20)

By (3.20) one deduces
\[
\rho_k \geq \rho_k' - \frac{1}{4} = \int \rho_k' - \frac{1}{4}
\geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}, \quad \text{a.e. (} x, t \text{) } \in \Omega \times (T^*, \infty),
\]
(3.21)

where the last inequality owes to (3.7). For another hand, from (3.20) and (3.21) we have
\[
\rho_k \leq \rho_k' + \frac{1}{4} = \int \rho_k' + \frac{1}{4}
\leq \frac{5}{4} + \frac{1}{4} = \frac{3}{2}, \quad \text{a.e. (} x, t \text{) } \in \Omega \times (T^*, \infty).
\]
(3.22)

The combination of (3.21) with (3.22) shows that for some large but finite \( k < \infty \), there is a time point \( T^* = T^*(k) < \infty \) such that
\[
\frac{1}{2} \leq \rho_k \leq \frac{3}{2}, \quad \text{a.e. (} x, t \text{) } \in \Omega \times (T^*, \infty).
\]
(3.23)

Utilizing (3.6) once more, we conclude (3.4) from (3.23). The proof is finished.

3.2. Velocity regularity. This subsection aims to improve the velocity regularity. Our approach is using an iteration technique.

For fixed \( t_0 > T^* \), we define the cut-off function of the form
\[
\begin{cases}
0 \leq \xi(t) \in C^1(\mathbb{R}), \\
\xi(t) \equiv 0 \text{ if } t \in (\infty, t_0 - r'], \quad \xi(t) \equiv 1 \text{ if } t \in [t_0 - r, \infty), \\
|\xi'| \leq \frac{2}{|r' - r|}, \quad \frac{t_0 - T^*}{2} \leq r < r' \leq t_0 - T^*.
\end{cases}
\]
(3.24)
If we test the momentum equations (1.1) against $\xi |u|^\beta u$ with $\beta \geq 0$, utilize (1.1) and the Cauchy inequality, integrate over $\Omega \times (-\infty, 2t_0)$, we infer

\[
\frac{1}{2} + \beta \sup_{t \in [t_0 - r, 2t_0]} \int \rho |u|^{2+\beta} + \int_{t_0 - r}^{2t_0} \xi \left( \rho |u|^\beta |\nabla u|^2 + \beta \rho |u|^\beta |\nabla|u|^2 \right) \\
= \frac{1}{2 + \beta} \int_{t_0 - r}^{2t_0} \partial_t \xi \int \rho |u|^{2+\beta} + \int_{t_0 - r}^{2t_0} \xi \int \rho \gamma \text{div} (|u|^\beta u) \\
\leq \frac{1}{2 + \beta} \int_{t_0 - r}^{2t_0} |\partial_t \xi| \int \rho |u|^{2+\beta} \\
+ \frac{1}{2} \int_{t_0 - r}^{2t_0} \xi \left( \rho |u|^\beta |\nabla u|^2 + \beta \rho |u|^\beta |\nabla|u|^2 \right) + C\beta \int_{t_0 - r}^{2t_0} \int \rho^{2\gamma-1} |u|^\beta.
\]

(3.25)

Thanks to (1.10) and (3.24), the (3.25) satisfies

\[
\sup_{t \in [t_0 - r, 2t_0]} \int |u|^{2+\beta} + (1 + \beta) \int_{t_0 - r}^{2t_0} \int \left( |u|^\beta |\nabla u|^2 + |\nabla|u|^{2+\beta} \right) \leq C \left( 2 + \int_{t_0 - r}^{2t_0} \int |u|^{2+\beta} \right).
\]

(3.26)

By Sobolev inequality (2.3), it takes

\[
\left( \int_{t_0 - r}^{2t_0} \int |u|^\frac{5}{3} (2+\beta) \right)^\frac{3}{5} \leq C \sup_{t \in [t_0 - r, 2t_0]} \int |u|^{2+\beta} + \int_{t_0 - r}^{2t_0} \int |\nabla|u|^\frac{\beta+2}{2+\beta} |^2 + \int_{t_0 - r}^{2t_0} \int |u|^\beta + 2.
\]

Making use of (3.26), we estimate the above inequality as

\[
\left( \int_{t_0 - r}^{2t_0} \int |u|^\frac{5}{3} (2+\beta) \right)^\frac{3}{5} \leq C \frac{(2 + \beta)^2}{|r' - r|} \left( 1 + \int_{t_0 - r}^{2t_0} \int |u|^{2+\beta} \right).
\]

(3.27)

Next, we apply the Moser-type iteration to deduce the desired (1.11). Select

\[
2 + \beta = \left( \frac{5}{3} \right)^k, \quad r' = r_k, \quad r = r_{k+1} = \frac{(t_0 - T^*)}{2} \left( 1 + \frac{1}{2^{k+1}} \right).
\]
From (3.27) we receive that, for some constant \( C = C(t_0, T^*) \),
\[
\left( \int_{t_0-r_k+1}^{2t_0} \int |u(\hat{z})|^{k+1} \right) (\hat{z})^{k+1} \\
\leq C \left( 2 \cdot \frac{5}{3} \right)^{2k} \left( \hat{z} \right)^k \left[ 1 + \int_{t_0-r_k}^{2t_0} \int |u(\hat{z})|^k \right] \left( \hat{z} \right)^k \\
\leq C 3^{2k} \left( \hat{z} \right)^k \cdot 2 \left( \hat{z} \right)^k \left[ 1 + \left( \int_{t_0-r_k}^{2t_0} \int |u(\hat{z})|^k \right) \left( \hat{z} \right)^k \right] \\
\leq C 3^{3k} \left( \hat{z} \right)^k \left[ 1 + \left( \int_{t_0-r_k}^{2t_0} \int |u(\hat{z})|^k \right) \left( \hat{z} \right)^k \right],
\]
where the constant \( C \) is independent of \( k \).

We proceed (3.27) in two cases:

**Case 1.** If there exists an infinite sequence to satisfy
\[
1 \geq \left( \int_{t_0-r_k}^{2t_0} \int |u(\hat{z})|^k \right) \left( \hat{z} \right)^k. \tag{3.29}
\]
Then
\[
\|u\|_{L^\infty \left( t_0+T^*, 2t_0; L^\infty \right)} = \lim_{k \to \infty} \left( \int_{t_0+T^*}^{2t_0} \int |u(\hat{z})|^k \right) \left( \hat{z} \right)^k \\
\leq \lim_{k \to \infty} \left( \int_{t_0-r_k}^{2t_0} \int |u(\hat{z})|^k \right) \left( \hat{z} \right)^k \\
\leq 1.
\]

**Case 2.** While if (3.29) is false, there should be a finite number \( k_0 \geq 1 \) such that (3.28) satisfies, for all \( k \geq k_0 \),
\[
\left( \int_{t_0-r_k+1}^{2t_0} \int |u(\hat{z})|^{k+1} \right) (\hat{z})^{k+1} \\
\leq C 3^{3k} \left( \hat{z} \right)^k \left( \int_{t_0-r_k}^{2t_0} \int |u(\hat{z})|^k \right) \left( \hat{z} \right)^k. \tag{3.31}
\]
By deduction, it gives from (3.31) that
\[
\left( \int_{t_0-r_k+1}^{2t_0} \int |u(\hat{z})|^{k+1} \right) (\hat{z})^{k+1} \\
\leq C^a 3^b \left( \int_{t_0-r_k}^{2t_0} \int |u(\hat{z})|^k \right) \left( \hat{z} \right)^k, \quad \forall \ k \geq k_0, \tag{3.32}
\]
with
\[
a \leq \sum_{k=1}^{\infty} \left( \frac{3}{5} \right)^k < \infty, \quad b \leq 3 \sum_{k=1}^{\infty} k \cdot \left( \frac{3}{5} \right)^k < \infty.
\]
Long-time behavior for NS equations

Taking limit \( k \to \infty \) in (3.32) yields

\[
\|u\|_{L^\infty \left( \frac{(t_0+T^*)}{2}, t_0; L^\infty \right)} \leq C \lim_{k \to \infty} \left( \int_{\frac{t_0}{2}}^{2t_0} \int |u(\frac{t}{2})|^k \right) \leq C \left( 1 + \int_{\frac{t_0}{2}}^{2t_0} \int |u|^{2+\beta} \right),
\]

where the last inequality is from a finite many time iteration in (3.28).

Therefore, the combination of (3.30) with (3.33) generates

\[
\|u\|_{L^\infty \left( \frac{(t_0+T^*)}{2}, t_0; L^\infty \right)} \leq C + C \int_{\frac{t_0}{2}}^{2t_0} \int |u|^{2+\beta} .
\]

In terms of (1.5), (1.10), and Sobolev inequality (2.1), we choose \( \beta \in [0, 4] \) in (3.34) and obtain

\[
\|u\|_{L^\infty \left( \frac{(t_0+T^*)}{2}, t_0; L^\infty \right)} \leq C + C \int_{\frac{t_0}{2}}^{2t_0} \int |u|^{2+\beta} \leq C + C t_0 \sup_{t \geq t_0} \int t \leq \int \rho |\nabla u|^2 \leq C.
\]

Finally, repeating the above argument over the intervals \([nt_0, ((n+1)t_0)]\) for \( n = 2, 3, \ldots \), we obtain (1.11), the required.

3.3. Energy conservation. As a result of (1.10) and (1.11), we show that the weak solution \((\rho, u)\) conserves its energy equality for all \( t > T^* \).

First, owing to (1.10), the exactly same deduction of (3.15) yields for some \( t_1 > T^* \)

\[
\Lambda = \sqrt{\rho \nabla u} \in L^2 \left( t_1, \infty; L^2 \right).
\]

This proves the latter part in (1.11). Moreover, by (1.6), it satisfies

\[
\int \mathcal{E}(x, t) dx + \int_{t_1}^t \int \rho |\nabla u|^2 \leq \int \mathcal{E}(x, t) dx, \quad \forall \ t \geq t_1.
\]

To be continued, we present the following proposition

**Proposition 3.1.** For almost all \( t_1 > T^* \) and for all \( t \in [t_1, \infty) \), we have equality sign in (3.37), that is to say, the desired (1.12) is true, provided that

\[
0 < C \leq \rho(x, t) \leq \overline{C} < \infty, \quad \Omega \times (t_1, \infty),
\]

\[
\begin{align*}
\mathbf{u} & \in L^4 \left( t_1, t; L^6 \right), \\
\|\mathbf{u}(\cdot, t_1)\|_{L^q} & \leq C \quad \text{for some} \quad q_0 > 3.
\end{align*}
\]

**Proof.** The proof of Proposition 3.1 is available [39, Theorem 1.1].
In this connection, to prove (1.12) it suffices to verify (3.38). In fact, the first two conditions (3.38)$_1$ and (3.38)$_2$ are from (1.10) and (1.11) respectively. Using (1.11) once more, we apply the Lebesgue Point theorem and obtain that, for a.e. $t_1 \in (T^*, \infty)$,

$$
\|\mathbf{u}(\cdot, t_1)\|_{L_{q_0}^q}^q = \lim_{\epsilon \to 0} \int_{t_1 - \epsilon}^{t_1 + \epsilon} \int |\mathbf{u}(x, s)|^{q_0} dx ds \leq C.
$$

This confirms (3.38)$_3$.

3.4. **Exponential asymptotics.** We shall prove that the $L^2$ norm of the weak solution $(\rho, \mathbf{u})$ decays exponentially to its equilibrium as time goes to infinity. The following operations are assumed to be carried out over $(T^*, \infty)$.

Denote the material derivative of $f$ by $\frac{df}{dt} = \partial_t + \mathbf{u} \cdot \nabla f$. Owing to (1.14), (1.1)$_1$, and the transport theorem, we compute

$$
\int \nabla \rho \gamma (\mathbf{u} - (\mathbf{u})_a) = - \int (\rho \gamma - \bar{\rho} \gamma) \text{div} \mathbf{u} = \int \rho \left( \frac{\rho \gamma - \bar{\rho} \gamma}{\rho^2} \right) \frac{dt}{dt} = \int \rho \frac{d}{dt} \int_{\rho} \frac{s \gamma - \bar{\rho} \gamma}{s^2} ds = \frac{d}{dt} \int \left( \rho \int_{\rho} \frac{s \gamma - \bar{\rho} \gamma}{s^2} ds \right)
$$

(3.39)

From (1.1)$_2$ one has

$$
\rho(\mathbf{u} - (\mathbf{u})_a)_t + \rho \mathbf{u} \cdot \nabla (\mathbf{u} - (\mathbf{u})_a) + \nabla \rho \gamma = \text{div}(\rho \nabla \mathbf{u}).
$$

(3.40)

Multiplying (3.40) by $\mathbf{u} - (\mathbf{u})_a$ and using (3.39), we obtain

$$
\frac{d}{dt} \int \rho \left( \frac{1}{2} |\mathbf{u} - (\mathbf{u})_a|^2 + \int_{\rho} \frac{s \gamma - \bar{\rho} \gamma}{s^2} ds \right) + \int \rho |\nabla \mathbf{u}|^2 = 0.
$$

(3.41)

Next, for the operator $B$ defined in Lemma 2.3, we have

$$
\int \rho(\mathbf{u} - (\mathbf{u})_a)_t B(\rho - \bar{\rho})
= \frac{d}{dt} \int \rho(\mathbf{u} - (\mathbf{u})_a)B(\rho - \bar{\rho})
- \int \rho_t(\mathbf{u} - (\mathbf{u})_a)B(\rho - \bar{\rho}) + \int \rho(\mathbf{u} - (\mathbf{u})_a)B \text{div}(\mathbf{u})
= \frac{d}{dt} \int \rho(\mathbf{u} - (\mathbf{u})_a)B(\rho - \bar{\rho})
- \int \rho_t(\mathbf{u} - (\mathbf{u})_a)B(\rho - \bar{\rho}) + \int \rho |\mathbf{u} - (\mathbf{u})_a|^2 + (\mathbf{u} - (\mathbf{u})_a)(\rho - \bar{\rho})(\mathbf{u})_a
$$

and

$$
\int \rho \mathbf{u} \nabla (\mathbf{u} - (\mathbf{u})_a) B(\rho - \bar{\rho})
= - \int \text{div}(\rho \mathbf{u})(\mathbf{u} - (\mathbf{u})_a)B(\rho - \bar{\rho}) - \int \rho \mathbf{u} \nabla (\mathbf{u} - (\mathbf{u})_a) \nabla B(\rho - \bar{\rho}).
$$
Taking the last two inequalities and \((\ref{5.11})\) into account, we multiply \((\ref{5.43})\) against \(B(\rho - \bar{p})\) and compute
\[
\frac{d}{dt} \int \rho(u - (u)_a)B(\rho - \bar{p}) + \int \rho^2|u - (u)_a|^2 + \int \rho(\rho - \bar{p})(u - (u)_a)(u)_a - \int \rho u(u - (u)_a)\nabla B(\rho - \bar{p}) - \int (\rho^\gamma - \bar{p}^\gamma)(\rho - \bar{p}) + \int \rho \nabla u \nabla B(\rho - \bar{p}) = 0.
\]  
\[
\frac{d}{dt} A + B = 0,
\]
where
\[
A := \int \rho \left( \frac{1}{2}|u - (u)_a|^2 + \int_\mathbb{R}^2 \frac{s^\gamma - \bar{p}^\gamma}{s^2} ds \right) - \delta \int \rho(u - (u)_a)B(\rho - \bar{p})
\]
and
\[
B = \int \rho|\nabla u|^2 + \delta \int (\rho^\gamma - \bar{p}^\gamma)(\rho - \bar{p}) - \delta \int \rho^2|u - (u)_a|^2 - \delta \int \rho(\rho - \bar{p})(u - (u)_a)(u)_a + \delta \int \rho u(u - (u)_a)\nabla B(\rho - \bar{p}) - \delta \int \rho \nabla u \nabla B(\rho - \bar{p}).
\]
We estimate \(A\) and \(B\) in below.

First, due to \((\ref{1.10})\), Lemma \((\ref{2.4})\) the Poincaré inequality, it satisfies that
\[
C^{-1}|\rho - \bar{p}|^2 \leq \rho \int_\mathbb{R}^2 \frac{s^\gamma - \bar{p}^\gamma}{s^2} ds \leq C|\rho - \bar{p}|^2
\]
and
\[
\left| \int \rho(u - (u)_a) : B(\rho - \bar{p}) \right| \\
\leq C\|\nabla B(\rho - \bar{p})\|_L^2 \\
\leq C\|\nabla B(\rho - \bar{p})\|_L^2 \leq C\|\rho - \bar{p}\|_L^2.
\]
The last two inequalities together with \((\ref{1.10})\) guarantee, for a small \(\delta\), there exists some constant \(C_1 = C_1(\delta) > 1\) so that
\[
C_1^{-1} \left( \|u - (u)_a\|_{L^2}^2 + \|\rho - \bar{p}\|_{L^2}^2 \right) \leq A \leq C_1 \left( \|u - (u)_a\|_{L^2}^2 + \|\rho - \bar{p}\|_{L^2}^2 \right). \tag{3.44}
\]

Next to estimate \(B\). Using \((\ref{1.10})\), Lemma \((\ref{2.4})\) and the Poincaré inequality, one deduces
\[
\int (\rho^\gamma - \bar{p}^\gamma)(\rho - \bar{p}) \geq C^{-1}\|\rho - \bar{p}\|_{L^2}^2,
\]
\[
\|B(\rho - \bar{p})\|_{L^2} \leq C\|\nabla B(\rho - \bar{p})\|_{L^2} \leq C\|\rho - \bar{p}\|_{L^2} \leq C\|\rho\|_{L^2},
\]
\[
\|u - (u)_a\|_{L^2} \leq C\|\nabla u\|_{L^2}.
\]
Remark 3.1. The validity of the last inequality (3.45) can be understand as follows: It gives from (3.41) and (3.42) that $(u)_a = \int_{\mathbb{R}^d} m_0 = \int_{\mathbb{R}^d} u^a$. Let us mollify $u$ by $u(x) = u * \zeta(x)$ with $\zeta$ the Friedrichs mollifier, then, by mean value theorem, we have $u(x) = (u)_a = \int_{\mathbb{R}^d} u^a(x)$ for some $x \in \Omega$. Hence
\[
\left\| u - (u)_a \right\|_{L^2} \\
\leq \left\| u - u^a \right\|_{L^2} + \left\| u^a - (u)_a \right\|_{L^2} + \left\| (u)_a - (u)_a \right\|_{L^2} \\
\leq C \left\| u - u^a \right\|_{L^2} + C \left\| \nabla u^a \right\|_{L^2}.
\]
By (1.10), sending $\epsilon \to 0$ in above inequality gives (3.45).

In accordance with (3.45), (1.10), (1.11), we deduce
\[
B \geq C \left\| \nabla u \right\|_{L^2}^2 + C \delta \left\| \rho - \nabla \right\|_{L^2}^2 \\
- C \delta \left\| \nabla u \right\|_{L^2}^2 - C \delta \left\| \nabla u \right\|_{L^2} \left\| \rho - \nabla \right\|_{L^2} \\
\geq C \left\| \nabla u \right\|_{L^2}^2 + C \delta \left\| \rho - \nabla \right\|_{L^2}^2 \\
- C \delta \left\| \nabla u \right\|_{L^2}^2 - C \delta \left( \delta \left\| \nabla u \right\|_{L^2} + \delta \left\| \rho - \nabla \right\|_{L^2} \right) \\
\geq C \left( 1 - \delta \right) \left\| \nabla u \right\|_{L^2}^2 + C \delta \left( 1 - \delta \right) \left\| \rho - \nabla \right\|_{L^2}^2 \\
\geq C \left( \left\| \nabla u \right\|_{L^2}^2 + \left\| \rho - \nabla \right\|_{L^2}^2 \right),
\]
where the constant $C_2 = C_2(\delta) > 0$, as long as $\delta \in (0, 1)$ is taken small enough.

Therefore, with (3.43) and (3.46) in hand, we may choose $\sigma C_1 \leq \frac{1}{2} C_2$ and multiply (3.46) by $e^{\sigma t}$, to deduce
\[
\frac{d}{dt} \left( e^{\sigma t} \left( \left\| u - (u)_a \right\|_{L^2}^2 + \left\| \rho - \nabla \right\|_{L^2}^2 \right) \right) \\
+ C_3 e^{\sigma t} \left( \left\| \nabla u \right\|_{L^2} + \left\| \rho - \nabla \right\|_{L^2} \right) \leq 0
\]
for some $C_3 = \frac{1}{2} C_1^{-1} C_2$. Integration of the above inequality yields
\[
e^{\sigma t} \left( \left\| u - (u)_a \right\|_{L^2}^2 + \left\| \rho - \nabla \right\|_{L^2}^2 \right) + \int_{t_1}^{\infty} e^{\sigma t} \left\| \nabla u \right\|_{L^2}^2 dt \\
\leq C e^{\sigma t_1} \left( \left\| \nabla u \right\|_{L^2}^2 + \left\| \rho - \nabla \right\|_{L^2}^2 \right) (t_1) \\
\leq C,
\]
for some $t_1 > T^*$ and for any $t \geq t_1$. This yields the desired (1.13). Thus, the proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2

4.1. Exponential decay of $\left\| \nabla \rho \right\|_{L^2}$. In case of zero initial momentum assumption (1.15), we show that the $L^2$-norm of the derivative of density decays exponentially to zero.

Thanks to (1.10) and (4.1),
\[
e^{\sigma t} \left\| \nabla u \right\|_{L^2}^2 + \int_{t_1}^{\infty} e^{\sigma t} \left\| \nabla u \right\|_{L^2}^2 dt \leq C, \quad t > t_1.
\]
Recalling (1.5) and (1.10), for some small $\delta \in (0, 1)$ one has
\[
\delta \left\| \nabla \rho \right\|_{L^2}^2 - C(\delta) \left\| \nabla u \right\|_{L^2}^2 \leq \int \rho |u + \nabla \ln \rho|^2 \leq \left\| \nabla \rho \right\|_{L^2}^2 + C \left\| u \right\|_{L^2}^2 \\
and \int \int \left( |\nabla \rho|^2 + \Lambda^2 \right) \geq \int \int |\nabla \rho|^2 \geq C_4 \left\| \nabla \rho \right\|_{L^2}^2.
\]
With the last two inequalities, we multiply (4.7) by $e^{\sigma_1 t}$ and deduce

$$
\delta e^{\sigma_1 t} \|
abla \rho \|^2_{L^2} + C_4 \int_{t_1}^\infty e^{\sigma_1 t} \|
abla \rho \|^2_{L^2} dt \\
\leq C(\delta) e^{\sigma_1 t} \|
abla \rho \|^2_{L^2} + \sigma_1 \int_{t_1}^\infty e^{\sigma_1 t} \left( \|
abla \rho \|^2_{L^2} + C \|
abla \|^2_{L^2} \right).
$$  (4.2)

Select $0 < \sigma_1 < \min\{\sigma, C_4\}$ in (4.2), and utilize (4.1), to discover

$$
e^{\sigma_1 t} \|
abla \rho \|^2_{L^2} + \int_{t_1}^\infty e^{\sigma_1 t} \|
abla \rho \|^2_{L^2} dt \leq C(\delta),
$$

which yields the required (1.16).

4.2. Higher regularity in dimension two. In the rest of this paper, we show that $(\rho, u)$ becomes a strong solution to (1.1) in two dimensional case when time is large.

In terms of (1.5), (1.10), (1.11), and the mass equation (1.1), it satisfies that for all $t > T^*$

$$
\begin{align*}
\rho &\in L^\infty(t, \infty; L^\infty) \cap L^2(t, \infty; H^1) \cap H^1(t, \infty; L^2), \\
u &\in L^\infty(t, \infty; L^\infty) \cap L^2(t, \infty; H^1).
\end{align*}
$$

Using (4.3), we multiply (1.12) by $u_t$ and integrate it over $(t, \infty)$, to find

$$
\frac{1}{2} \frac{d}{dt} \int \rho |\nabla u|^2 + \int \rho |u_t|^2 + \\
- \int \rho u \cdot \nabla u \cdot u_t - \int u_t \cdot \nabla \rho \gamma + \frac{1}{2} \int \rho_t |\nabla u|^2 \\
\leq \frac{1}{4} \int \rho |u_t|^2 + C \int \left( |\nabla \rho|^2 + |\nabla u|^2 \right) + \frac{1}{2} \int \rho_t |\nabla u|^2.
$$

By Lemma 2.3 and the momentum equations, it satisfies

$$
\begin{align*}
\|\nabla^2 u\|_{L^\frac{6}{5}} &\leq C \|\Delta u\|_{L^\frac{6}{5}} \\
&\leq C \left( \|\text{div}(\rho \nabla u)\|_{L^\frac{6}{5}} + \|\rho \nabla \nabla u\|_{L^\frac{6}{5}} \right) \\
&\leq C \left( \|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} \right) + C_4 \|\nabla \rho\|_{L^2} \|\nabla u\|_{L^6}.
\end{align*}
$$

(4.5)

Remember that the Sobolev inequality in dimension two, it has

$$
\|\nabla u\|_{L^6} \leq C \left( \|\nabla^2 u\|_{L^\frac{6}{5}} + \|\nabla u\|_{L^2} \right),
$$

which combining with (1.16) shows, for some large $\bar{t}_1 \geq t_1$,

$$
C_4 \|\nabla \rho\|_{L^2} \|\nabla u\|_{L^6} \\
\leq C_4 e^{-\sigma_1 t} \left( \|\nabla^2 u\|_{L^\frac{6}{5}} + \|\nabla u\|_{L^2} \right) \\
\leq \frac{1}{2} \left( \|\nabla^2 u\|_{L^\frac{6}{5}} + \|\nabla u\|_{L^2} \right), \quad \forall \ t > \bar{t}_1.
$$

Insert the last inequality into (4.5) brings us to

$$
\|\nabla^2 u\|_{L^\frac{6}{5}} \leq C \left( \|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \rho\|_{L^2} \right).
$$

(4.6)
With the help of $(4.8)$ and $(4.9)$, we estimate the last integral in $(4.4)$ as
\[
\int \rho_i |\nabla \mathbf{u}|^2 = \int \rho \mathbf{u} : \nabla \mathbf{u}^2 \\
\leq C \|\nabla \mathbf{u}\|_{L^\infty} \|\nabla^2 \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^3} \\
\leq \delta \|\nabla^2 \mathbf{u}\|_{L^2}^2 + C(\delta) \|\nabla \mathbf{u}\|_{L^2}^2 \\
\leq C\delta (\|\mathbf{u}\|_{L^2} + \|\mathbf{v}\|_{L^2})^2 + C(\delta) \|\nabla \mathbf{u}\|_{L^2}^2.
\] (4.7)

If we substitute $(4.7)$ into $(4.4)$, and choose $\delta$ suitably small, we get
\[
\frac{d}{dt} \int \rho |\nabla \mathbf{u}|^2 + \int \rho |\mathbf{u}_t|^2 \leq C \int (|\nabla \rho|^2 + |\nabla \mathbf{u}|^2).
\] (4.8)

Hence, integrating $(4.8)$ after multiplied by $e^{\nu t}$, utilizing $(4.1)$, $(1.10)$, $(1.10)$, we obtain
\[
e^{\nu t} \|\nabla \mathbf{u}(\cdot, t)\|_{L^2} + \int_{\tilde{t}_1}^t e^{\nu s} \|\partial_s \mathbf{u}\|_{L^2}^2 ds \leq C, \quad t > \tilde{t}_1.
\] (4.9)

The combination of $(4.9)$ with $(1.10)$ generates $(1.18)$.

As a result of $(4.6)$ and $(4.9)$, we have
\[
\mathbf{u} \in L^\infty(t, \infty; H^1) \cap L^2(t, \infty; W^{2,\frac{1}{2}}) \cap H^1(t, \infty; L^2), \quad t > \tilde{t}_1.
\]

This proves the desired $(1.17)$.

The proof of Theorem $(1.2)$ is thus completed.

References

[1] R. Adams, Sobolev spaces, New York: Academic Press (1975).

[2] S. Agmon, A. Douglas, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, Comm. Pure Appl. Math., 12 (1959), 623-727; II, Comm. Pure Appl. Math. 17 (1964), 35-92.

[3] S. N. Antontsev; A. V. Kazhikhov; V. N. Monakhov, Boundary Value Problems in Mechanics of Non-Homogeneous Fluids. Amsterdam, New York: North-Holland, 1990.

[4] Y. Cho; H. Choe; H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl., 83 (9) (2004), 243-275.

[5] B. Desjardins, Regularity of weak solutions of the compressible isentropic Navier Stokes equations, Commun. Partial Differ. Equa., 22 (1997), 977-1008.

[6] D. Bresch; B. Desjardins, Some diffusive capillary models of Korteweg type, C. R. Math. Acad. Sci. Paris, Section Mecanique, 332(11), (2004), 881-886.

[7] D. Bresch; B. Desjardins, Existence of global weak solution for 2D viscous shallow water equations and convergence to the quasi-geostrophic model, Comm. Math. Phys., 238(1-2) (2003), 211-223.

[8] R. Chen; Z. Liang; W. Wang; R. Xu, Energy equality in compressible fluids with physical boundaries, SIAM J. Math. Anal., 52 (2020), 1363-1385.

[9] R. DiPerna; P. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989), 511-547.

[10] E. Feireisl; A. Novotný; H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, J. Math. Fluid Mech., 3 (2001), 358-392.

[11] H. Choe; B. Jin, Regularity of weak solutions of the compressible Navier-Stokes equations J. Korean Math. Soc., 40 (2003), 1031-1050.

[12] P. Gent, The energetically consistent shallow water equations, J. Atmos. Sci., 50 (1993), 1323-1325.

[13] Z. Guo; Q. Jin; Z. Xin, Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, SIAM J. Math. Anal., 39 (2008), 1402-1427.

[14] B. Haspot, Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D, Mathematische Nachrichten, 291(14-15) (2018), 2188-2203.

[15] D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, Trans. Amer. Math. Soc., 303 (1987), 169-181.

[16] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional, compressible flow with discontinuous initial data. J. Differ. Equ. 120 (1995), 215-254.
[17] J. Gerbeau; B. Perthame, Derivation of viscous Saint-Venant system for laminar shallow water, numerical validation, Discrete Contin. Dyn. Syst. Ser B, 1 (2001), 89-102.

[18] S. Jiang, Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains, Comm. Math. Phys., 200 (1999), 181-193.

[19] S. Jiang, Remarks on the asymptotic behaviour of solutions to the compressible Navier-Stokes equations in the half-line, P. Roy. Soc. Edinb. A, 132 (2002), 627-638.

[20] S. Jiang; Z. Xin; P. Zhang, Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity, Methods Appl. Anal., 12 (2005), 229-252.

[21] S. Jiang; P. Zhang, On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations, Comm. Math. Phys., 215 (2001), 559-581.

[22] Q. Jiu; Z. Xin, The Cauchy problem for 1D compressible flows with density-dependent viscosity coefficients, Kinet. Relat. Mod., 1(2) (2008), 313-330.

[23] M. Kang; A. Vasseur, Global smooth solutions for 1D barotropic Navier-Stokes equations with a large class of degenerate viscosities, J. Nonlinear Sci., 30(4) (2020), 1703-1721.

[24] O. Ladyzenskaja; V. Solonnikov; N. Ural’tseva, Linear and quasi-linear equations of parabolic type. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[25] L. Landau; E. Lifshitz, Electrodynamics of Continuous Media, 2nd edn. Pergamon, New York (1984)

[26] H. Li; J. Li; Z. Xin, Vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations, Comm. Math. Phys., 281(2) (2008), 401-444.

[27] J. Li; Z. Liang, Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier-Stokes system in unbounded domains with large data, Arch. Rational Mech. Anal., 220 (2016), 1195-1208.

[28] J. Li; Z. Xin, Global existence of weak solutions to the barotropic compressible Navier-Stokes flows with degenerate viscosities, http://arxiv.org/abs/1504.06826v2

[29] Z. Liang, Regularity criterion on the energy conservation for the compressible Navier-Stokes equations, P. Roy. Soc. Edinb. A, (2020), 1-18.

[30] P. Lions, Mathematical Topics in Fluid Mechanics, Volume 1-2, Oxford Science Publication, Oxford, (1996,1998).

[31] A. Mellet; A. Vasseur, On the barotropic compressible Navier-Stokes equations, Commun. Partial Differ. Equ., 32 (2007), 431-452.

[32] A. Novotný; I. Straˇnskákraba, Introduction to the Mathematical Theory of Compressible Flow. Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 27, 2004.

[33] Q. Nguyen; P. Nguyen; B Tang, Energy inequalities for compressible Navier-Stokes equations, Nonlinearity, 32(11) (2019), 4206-4231.

[34] P. Plotnikov; W. Weigant, Isothermal Navier–Stokes Equations and Radon Transform, SIAM J. Math. Anal., 47(1) (2015), 626-653.

[35] D. Serre, Solutions faibles globales des quations de Navier-Stokes pour un fluide compressible, C. R. Acad. Sci. Paris. I Math. 303(13) (1986), 639-642.

[36] I. Straˇnskákraba; A. Zlotnik, Global properties of solutions to 1D-viscous compressible barotropic fluid equations with density dependent viscosity, Z. angew. Math. Phys., 54 (2003), 593-607.

[37] A. Vasseur; C. Yu, Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations, Invent. Math., 206(3) (2016), 935-974.

[38] Y. Ye; Y. Wang; W. Wei, Energy equality in the isentropic compressible Navier-Stokes equations allowing vacuum, arXiv: 2108.09425

[39] C. Yu, Energy conservation for the weak solutions of the compressible Navier-Stokes equations, Arch. Ration. Mech. Anal., 225(3) (2017), 1073-1087.

[40] T. Yang; Z. Yao; C. Zhu, Compressible Navier-Stokes equations with density dependent viscosity and vacuum, Commun. Partial Differ. Equ., 26 (2001), 965-981.

[41] T. Yang; H. Zhao, A vacuum problem for the one-dimensional compressible Navier-Stokes equations with density-dependent viscosity, J. Differ. Equ., 184 (2002), 163-184.

[42] T. Yang; C. Zhu, Compressible Navier–Stokes equations with degenerate viscosity coefficient and vacuum, Comm. Math. Phys., 230 (2002), 329-363.

School of Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China

Email address: zhilei0592@gmail.com