FULLY DISTRIBUTED CONSENSUS FOR HIGHER-ORDER NONLINEAR MULTI-AGENT SYSTEMS WITH UNMATCHED DISTURBANCES

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Abstract. In this paper, the distributed consensus problem is investigated for a class of higher-order nonlinear multi-agent systems with unmatched disturbances. By the back-stepping technique, a new distributed protocol is designed to solve the consensus problem for multi-agent systems without using the information of the Laplacian matrix and Lipschitz constants. It is proved that the practical consensus of multi-agent systems with unmatched disturbances can be achieved by the proposed protocol. Finally, the validity of the proposed scheme is verified by a simulation.

1. Introduction. In recent years, more and more attention has been paid to the cooperative control of multi-agent systems due to their broad applications in many areas, such as UAVs formation, smart grid synchronization, distributed control of satellite system, etc [10, 35, 20, 9, 15, 32, 37]. In multi-agent system cooperative control, consensus is a basic issue and aiming at that all agents’ states can reach a common value [17, 33, 22, 6, 36, 31, 12, 23].

There are many works that have been done for different types of multi-agent systems, including first-order integrator systems [16, 3], second-order integrator systems [24, 8, 1] and higher-order integrator systems [21, 4]. Because the higher-order multi-agent systems can describe more practical systems, and the first/second-order multi-agent systems are their special cases, the higher-order multi-agent systems have received more attention. The consensus control algorithm for the multi-agent systems was designed in [2] by the finite-time control scheme. By applying the sliding mode control method, Mondal and Su designed a distributed protocol to achieve consensus in [19]. The problem of distributed containment control for higher-order multi-agent systems under directed network topologies was studied in [25]. Gao et al. in [5] designed a distributed observer-based consensus protocol to solve the leader-following consensus problem of higher-order multi-agent systems under switching topologies.

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It is noted that the higher-order multi-agent systems discussed above are all linear systems. However, most multi-agent systems are nonlinear in reality. In addition, practical systems always suffer from many kinds of disturbances, such as the friction in mechanical and electrical systems, electromagnetic interference, environment disturbances and so on. Therefore, disturbances should be taken into account in the research on the consensus issue of the multi-agent systems. Many results on multi-agent systems with disturbances have been reported. For instance, Li et al. in [11] considered a distributed consensus problem of linear heterogeneous multi-agent systems subject to different matching uncertainties and designed a fully distributed continuous adaptive consensus protocol. An event-triggered integral sliding mode control strategy was proposed in [30] to solve the time-varying formation problem for higher-order multi-agent systems which suffer from disturbances. By the Nussbaum-type gain technique and the function approximation capability of neural networks, an adaptive neural networks-based controller for the higher-order nonlinear multi-agent system with a fixed topology was designed in [26]. In [28], Wang et al. applied the output regulation theory to solve an optimization problem for a class of nonlinear multi-agent systems with disturbances.

It is a pity that the information of the Laplacian matrix and Lipschitz constants was used for the protocol design in the above researches, which is unrealistic in practical applications. To the best of our knowledge, the problem of how to design a fully distributed consensus protocol for higher-order nonlinear multi-agent systems with unmatched disturbances has not been solved. Therefore, without using the information of the Laplacian matrix and Lipschitz constants, a consensus protocol is designed for higher-order nonlinear multi-agent systems with unmatched disturbances in this paper.

The main contributions of the paper can be summarized as follows: (1) A higher-order nonlinear multi-agent system with unmatched disturbances is considered. There have been several meaningful anti-disturbance control algorithms for multi-agent systems with disturbances in [11, 30, 26, 28]. However, these literatures only consider matched disturbances. In reality, most of the disturbances in the systems are unmatched. Disturbances were only considered in the input channels in [27, 13]. A higher-order nonlinear multi-agent system with unmatched disturbances was considered in [29], but the nonlinear elements were only synchronized with the input. Therefore a novel protocol is designed in this paper for a class of multi-agent systems with unmatched disturbances, which exist in all channels and include nonlinear elements. (2) Unlike [14, 34], the information of the Laplacian matrix and Lipschitz constants is unavailable in the protocol design.

The rest of the paper is organized as follows: The necessary preliminaries and the problem formulation are given in Section 2. In Section 3, the main results of the paper are presented. The effectiveness of the scheme is illustrated by a simulation example in Section 4 and conclusions are drawn in Section 5.

2. Preliminaries and problem formulation.

2.1. Notations. In this paper, \( \mathbb{R} \) represents the set of all real numbers, \( \mathbb{R}^n \) denotes the \( n \)-dimensional vector space and \( \mathbb{R}^{m \times n} \) stands for the set of \( m \times n \) matrices. Denoting \( A \) as a matrix, \( A > 0 \) (\( A \geq 0 \)) means \( A \) is positive definite (positive semi-definite). \( A^T \) denotes the transpose of \( A \) while \( A^{-1} \) denotes the inverse of \( A \), where \( A \in \mathbb{R}^{n \times n} \). \( \lambda_{\text{min}}(A) \) and \( \lambda_{\text{max}}(A) \) stand for the minimum and maximum
eigenvalues of $A$, respectively. $\|\cdot\|$ is the Euclidean norm for vectors or the induced 2-norm for matrices.

2.2. Graph theory. The communication topology graph of the followers is undirected and denoted by $G = \{V, E, A\}$. For $G$, $\mathcal{V}(G) = \{v_i, \ i = 1, 2, \ldots, n\}$ is the set of vertices, $\mathcal{E}(G) \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set and $A = [a_{ij}]_{n \times n} \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix. For $A$, $a_{ij} > 0$ if and only if $(v_i, v_j) \in \mathcal{E}(G)$ and $a_{ij} = 0$ otherwise, where $(v_i, v_j) \in \mathcal{E}(G)$ means that vertex $v_j$ can obtain the information from vertex $v_i$. The degree of vertex $v_i$ is defined as $\text{deg}(v_i) = d_i = \sum_{j \in N_i} a_{ij}$, where $N_j$ denotes the set of all neighbors for vertex $v_i$. Therefore $\mathcal{D} = \text{diag}(d_1, d_2, \ldots, d_n)$ stands for the degree matrix of graph $G$ and the Laplacian matrix of graph $G$ is defined as $L = \mathcal{D} - A$. If the $i$th follower can obtain the information from the leader directly, $b_i > 0$ and $b_i = 0$ otherwise. A new matrix is defined as $B = \text{diag}\{b_1, b_2, \ldots, b_n\}$.

2.3. Problem formulation. Consider a class of higher-order nonlinear multi-agent system including $m$ followers and one leader. The dynamics of all followers are described by

$$
\begin{align*}
\dot{x}_{ij} &= x_{(i+1)j} + f_{ij}(\bar{x}_{ij}, t) + d_{ij}(t) \\
\dot{x}_{nj} &= u_j + f_{nj}(\bar{x}_{nj}, t) + d_{nj}(t) \\
y_j &= x_{ij}, \quad i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, m,
\end{align*}
$$

and the leader’s dynamics is given as:

$$
\begin{align*}
\dot{x}_{i0} &= x_{(i+1)0} + f_{i0}(\bar{x}_{i0}, t) + d_{i0}(t) \\
\dot{x}_{i0} &= f_{n0}(\bar{x}_{i0}, t) + d_{n0}(t) \\
y_0 &= x_{10}, \quad 1 \leq i \leq n-1
\end{align*}
$$

where $x_j = [x_{1j}, x_{2j}, \ldots, x_{nj}]^T \in \mathbb{R}^n$ and $x_0 = [x_{10}, x_{20}, \ldots, x_{n0}]^T \in \mathbb{R}^n$ are the state vectors, $\bar{x}_{ij} = (x_{1j}, x_{2j}, x_{3j}, \ldots, x_{ij})$, $\bar{x}_{i0} = (x_{10}, x_{20}, x_{30}, \ldots, x_{i0})$. $u_j$ is the control input of the $j$th follower to be designed. $y_j$ is the output of the $j$th follower and $y_0$ is the output of the leader. $f_{ij}(\cdot)$ is an unknown nonlinear continuous function and $d_{ij}(\cdot)$ is the external disturbance, where $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$.

**Remark 1.** In reality, many physical systems with unmatched disturbances can be described by multi-agent system (1)-(2). If $d_{ij}(t) = 0$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, multi-agent system (1)-(2) will become a typical higher-order nonlinear system, which has been investigated in many existing literature, such as in [21, 4].

**Remark 2.** Since unmatched disturbances enter the agents through different channels from the control inputs, it is difficult to suppress unmatched disturbances directly by feedback control. Therefore consensus control for multi-agent systems with unmatched disturbances is more challenging than that with only matched disturbances.

**Definition 1.** Multi-agent system (1)-(2) is said to achieve the practical consensus if there exists a constant $\varepsilon > 0$ such that

$$
\lim_{t \to \infty} |x_{ij} - x_{i0}| \leq \varepsilon, \quad i = 1, 2, \ldots, n, j = 1, 2, \ldots, m.
$$
Assumption 1. For multi-agent system (1)-(2), the graph $G$ for all followers is connected, and there is at least one follower connected to the leader, which means that at least one follower can obtain information from the leader directly.

Assumption 2. For all $t \geq 0$, the nonlinear functions satisfy the following Lipschitz condition

$$|f_{ij}(\bar{x}_{ij}, t) - f_{io}(\bar{x}_{i0}, t)| \leq \sum_{g=1}^{i} \rho_g |x_{gj} - x_{g0}|, \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., m,$$

where $\rho_g, g = 1, 2, 3, ..., i$, are positive constants.

Assumption 3. For multi-agent system (1)-(2), all unmatched disturbances are bounded and satisfy

$$|d_{ij}(t)| \leq d_{ij}, \quad i = 1, 2, ..., n, \quad j = 0, 1, ..., m,$$

where $d_{ij}$ are positive constants.

Lemma 2.1. [18] For multi-agent system (1)-(2), if Assumption 1 holds, $\mathcal{L} + \mathcal{B} > 0$.

Lemma 2.2. [7] For any $a > 0, b > 0, \gamma > 0$,

$$|x|^a |y|^b \leq \frac{a}{a+b} \gamma |x|^{a+b} + \frac{b}{a+b} \gamma^{-\frac{a}{b}} |y|^{a+b},$$

where $x$ and $y$ are arbitrary real numbers.

3. Main results. In this section, a new protocol is proposed to solve the practical leader-following consensus for higher-order nonlinear multi-agent systems with unmatched disturbances, where the protocol is designed without the information of the Laplacian matrix and Lipschitz constants.

Remark 3. In the fully distributed control system, each agent can communicate with its neighboring agents and coordinate their actions to achieve a common goal. This control method can still complete the preset task through cooperation when an agent or one communication channel fails. Therefore, this control method can improve the coordination of the multi-agent systems, and have the advantages of low cost, high efficiency, good robustness, good flexibility and so on. In this paper, the consensus gains have been updated by using the norms of the error states based on local information, which means the corresponding consensus protocol is fully distributed.

Define

$$\varphi_{ij} = \sum_{t \in \mathcal{N}_j} a_{ij} (x_{it} - x_{ij}) + b_{ij} (x_{i0} - x_{ij}),$$

and

$$\xi_{2j} = \varphi_{2j} + k_1 \varphi_{1j}$$

$$\xi_{ij} = \varphi_{ij} + k_{i-1} \xi_{(i-1)j}, \quad i = 3, 4, ..., n,$$

then the protocol is proposed as follows

$$u_j = (k + \dot{\theta}_j) \xi_n$$

$$\dot{\theta}_j = \xi_{nj} \sum_{j=1}^{m} a_{ij} (\xi_{nj} - \xi_{nm}) + b_i \xi_{nj}^2 - \tau \dot{\theta}_j,$$

where $\xi_n = [\xi_{n1}, \xi_{n2}, ..., \xi_{nm}]^T$, $k, k_1, k_2, k_3, ..., k_{n-1}$ and $\tau$ are positive constants.
**Theorem 3.1.** Consider multi-agent system (1)-(2), if Assumptions 1-3 hold, the practical consensus can be achieved by the protocol (9).

**Proof.** The proof is a recursive process based on the back-stepping design philosophy.

For simplicity, define

\[ \varphi_i = [\varphi_{i1}, \varphi_{i2}, ..., \varphi_{im}]^T \]

\[ u = [u_1, u_2, ..., u_m]^T \]

\[ \bar{x}_i = [x_{i1} - x_{i0}, x_{i2} - x_{i0}, ..., x_{im} - x_{i0}]^T \]

\[ \bar{f}_i = [f_{i1} - f_{i0}, f_{i2} - f_{i0}, ..., f_{im} - f_{i0}]^T \]

\[ \bar{d}_i = [d_{i1}(t) - d_{i0}(t), d_{i2}(t) - d_{i0}(t), ..., d_{im}(t) - d_{i0}(t)]^T, \]

where \( f_{ip} = f_{ip}(\bar{x}_{ip}, t), p = 0, 1, 2, ..., m. \)

If Assumption 3 holds, we can get

\[ \|\bar{d}_i\| = \left\| [d_{i1}(t) - d_{i0}(t), d_{i2}(t) - d_{i0}(t), ..., d_{im}(t) - d_{i0}(t)]^T \right\| \]

\[ \leq m \max \{|d_{i1} - d_{i0}|, |d_{i2} - d_{i0}|, ..., |d_{im} - d_{i0}|\} \]

\[ =: d_i, \]

where \( d_i > 0. \)

**Step 1.** Choose the Lyapunov function as \( V_1 = \frac{1}{2} \varphi_1^T \varphi_1, \) and the time derivative of \( V_1 \) is

\[ \dot{V}_1 = \varphi_1^T \dot{\varphi}_1 \]

\[ = \varphi_1^T [\varphi_2 - (\mathcal{L} + \mathcal{B}) \bar{f}_i - (\mathcal{L} + \mathcal{B}) \bar{d}_i] \]

\[ = \varphi_1^T \varphi_2^* + \varphi_1^T (\varphi_2 - \varphi_2^*) - \varphi_1^T (\mathcal{L} + \mathcal{B}) \bar{f}_i - \varphi_1^T (\mathcal{L} + \mathcal{B}) \bar{d}_i, \]

where \( \varphi_2^* \) is a virtual input.

Choose the virtual input as

\[ \varphi_2^* = -k_1 \varphi_1, \]

and define

\[ \xi_2 = \varphi_2 - \varphi_2^* = \varphi_2 + k_1 \varphi_1. \]

Substituting (13)-(14) into (12), we have

\[ \dot{V}_1 = -k_1 \varphi_1^T \varphi_1 + \varphi_1^T \xi_2 - \varphi_1^T (\mathcal{L} + \mathcal{B}) \bar{f}_i - \varphi_1^T (\mathcal{L} + \mathcal{B}) \bar{d}_i, \]

Note that

\[ -\varphi_1^T (\mathcal{L} + \mathcal{B}) \bar{f}_i \leq \|\varphi_1\| \|\mathcal{L} + \mathcal{B}\| \|\bar{f}_i\| \leq \rho_1 \|\varphi_1\| \|\mathcal{L} + \mathcal{B}\| \|\bar{x}_1\| \]

\[ \leq \rho_1 \lambda_{\text{max}} (\mathcal{L} + \mathcal{B}) \|\varphi_1\| \left\| (\mathcal{L} + \mathcal{B})^{-1} \varphi_1 \right\| \]

\[ \leq \rho_1 \lambda_{\text{max}} (\mathcal{L} + \mathcal{B}) \left\| (\mathcal{L} + \mathcal{B})^{-1} \right\| \|\varphi_1\|^2 \]

\[ \leq \nu_1 \|\varphi_1\|^2, \]

\[ -\varphi_1^T (\mathcal{L} + \mathcal{B}) \bar{d}_i \leq \|\varphi_1\| \|\mathcal{L} + \mathcal{B}\| \|\bar{d}_i\| \leq \|\varphi_1\| \|\mathcal{L} + \mathcal{B}\| \|\varphi_1\| \rho_1 \|\varphi_1\| \]

\[ \leq \frac{\mu_1}{2} \|\varphi_1\|^2 + \frac{1}{2\mu_1} \|\mathcal{L} + \mathcal{B}\|^2 d_i^2, \]
where \( \vartheta_1 = \rho_1 \frac{\lambda_{\max}(L+B)}{\lambda_{\min}(L+B)} \), \( \mu_1 \) and \( \varepsilon_1 \) are two positive constants.

It follows from (15)-(18) that

\[
\dot{V}_1 \leq -\alpha_{1,1} \varphi_1^T \varphi_1 + \beta_1 + \alpha_{2,1} \xi_2^T \xi_2, \tag{19}
\]

where

\[
\alpha_{1,1} = k_1 - \frac{\mu_1}{2} - \frac{\varepsilon_1}{2} - \vartheta_1 \\
\beta_1 = \frac{1}{2\mu_1} \|L + B\|^2 d_1^2 \\
\alpha_{2,1} = \frac{1}{2\varepsilon_1}.
\tag{20}
\]

**Step 2.** Consider the Lyapunov function as \( V_2 = V_1 + \frac{1}{2} \xi_2^T \xi_2 \), and the time derivative of \( V_2 \) is

\[
\dot{V}_2 = \dot{V}_1 + \xi_2^T \dot{\xi}_2 \\
= \dot{V}_1 + \xi_2^T (\dot{\varphi}_2 + k_1 \varphi_1).
\tag{21}
\]

According to (7), one can conclude that

\[
\dot{V}_2 = \dot{V}_1 + \xi_2^T \varphi_3 - \xi_2^T (L + B) \bar{f}_2 - \xi_2^T (L + B) \bar{d}_2 \\
+ k_1 \xi_2^T [\varphi_2 - (L + B) \bar{f}_1 - (L + B) \bar{d}_1]. \tag{22}
\]

Choose the virtual input as

\[
\varphi_3^* = -k_2 \xi_2, \tag{23}
\]

and define

\[
\xi_3 = \varphi_3 - \varphi_3^* = \varphi_3 + k_2 \xi_2. \tag{24}
\]

From (22)-(24), it yields that

\[
\dot{V}_2 = \dot{V}_1 - k_2 \xi_2^T \xi_2 + \xi_2^T (\varphi_3 - \varphi_3^*) - \xi_2^T (L + B) \bar{f}_2 - \xi_2^T (L + B) \bar{d}_2 \\
+ k_1 \xi_2^T [\varphi_2 - (L + B) \bar{f}_1 - (L + B) \bar{d}_1]. \tag{25}
\]

Substituting (23)-(24) into (25), we have

\[
\dot{V}_2 = \dot{V}_1 - k_2 \xi_2^T \xi_2 + \xi_2^T \xi_3 - \xi_2^T (L + B) \bar{f}_2 - \xi_2^T (L + B) \bar{d}_2 \\
+ k_1 \xi_2^T \varphi_2 - k_1 \xi_2^T (L + B) \bar{f}_1 - k_1 \xi_2^T (L + B) \bar{d}_1. \tag{26}
\]

Note that

\[
\xi_2^T \xi_3 \leq \|\xi_2\| \|\xi_3\| \\
\leq \frac{\varepsilon_2}{2} \|\xi_2\|^2 + \frac{1}{2\varepsilon_2} \|\xi_3\|^2, \tag{27}
\]

\[
-\xi_2^T (L + B) \bar{f}_2 \leq \|\xi_2\| \|L + B\| \|\bar{f}_2\| \\
\leq \|\xi_2\| \|L + B\| (\rho_1 \|\bar{x}_1\| + \rho_2 \|\bar{x}_2\|) \\
\leq \vartheta_1 \|\xi_2\| \|\varphi_1\| + \vartheta_2 \|\xi_2\| \|\varphi_2\|, \tag{28}
\]

where \( \varepsilon_2 > 0 \) and \( \vartheta_2 = \rho_2 \frac{\lambda_{\max}(L+B)}{\lambda_{\min}(L+B)} \).
Substituting (13)-(14) into (28), we can obtain

\[-\xi_2^T (L + B) \bar{f}_2 \leq \vartheta_1 \| \xi_2 \| \| \varphi_2 \| + \vartheta_2 \| \xi_2 \| \| \xi_2 - k_1 \varphi_1 \| \leq (\vartheta_1 + k_1 \vartheta_2) \| \xi_2 \| \| \varphi_1 \| + \vartheta_2 \| \xi_2 \| ^2]. \tag{29}\]

By using Lemma 2.2, we can find

\[-\xi_2^T (L + B) \bar{f}_2 \leq \frac{\eta_2}{2} (\vartheta_1 + k_1 \vartheta_2) \| \varphi_1 \| ^2 + \frac{\vartheta_1 + k_1 \vartheta_2}{2\eta_2} \| \xi_2 \| ^2 + \vartheta_2 \| \xi_2 \| ^2 \]

\[= \frac{\eta_2}{2} (\vartheta_1 + k_1 \vartheta_2) \| \varphi_1 \| ^2 + \left( \frac{\vartheta_1 + k_1 \vartheta_2}{2\eta_2} + \vartheta_2 \right) \| \xi_2 \| ^2, \tag{30}\]

\[-\xi_2^T (L + B) \bar{d}_2 \leq \| \xi_2 \| \| L + B \| \| \bar{d}_2 \| \leq \| \xi_2 \| \| L + B \| d_2 \]

\[\leq \frac{\mu_2}{2} \| \xi_2 \| ^2 + \frac{\| L + B \| ^2}{2\mu_2} d_2, \tag{31}\]

\[k_1 \xi_2^T \varphi_2 = k_1 \xi_2^T (\xi_2 - k_1 \varphi_1) \leq k_1 \| \xi_2 \| ^2 + k_1^2 \| \varphi_1 \| ^2 \leq k_1 \| \xi_2 \| ^2 + \frac{\omega_2 k_1^2}{2\omega_2} \| \varphi_1 \| ^2 \]

\[= \left( k_1 + \frac{k_1^2}{2\omega_2} \right) \| \xi_2 \| ^2 + \frac{\omega_2 k_1^2}{2\omega_2} \| \varphi_1 \| ^2, \tag{32}\]

\[-k_1 \xi_2^T (L + B) \bar{f}_1 \leq k_1 \| \xi_2 \| \| L + B \| \| \bar{f}_1 \| \leq k_1 \rho_1 \| \xi_2 \| \| L + B \| \| \bar{x}_1 \| \leq k_1 \vartheta_1 \| \xi_2 \| \| \varphi_1 \| \leq \frac{\delta_1 k_1 \vartheta_1}{2} \| \xi_2 \| ^2 + \frac{k_1 \vartheta_1}{2\delta_2} \| \varphi_1 \| ^2, \tag{33}\]

\[-k_1 \xi_2^T (L + B) \bar{d}_1 \leq k_1 \| \xi_2 \| \| L + B \| \| \bar{d}_1 \| \leq k_1 \| \xi_2 \| \| L + B \| d_1 \leq \frac{\sigma_2}{2} \| \xi_2 \| ^2 + \frac{\| L + B \| ^2}{2\sigma_2} d_1^2, \tag{34}\]

where $\eta_2, \mu_2, \omega_2, \delta_2, \sigma_2$ are positive constants.

According to (26)-(34), it can be concluded that

\[\dot{V}_2 \leq -\alpha_{1,2} \varphi_1^T \varphi_1 - \alpha_{2,2} \xi_2^T \xi_2 + \beta_2 + \alpha_{3,2} \xi_3^T \xi_3, \tag{35}\]

where

\[\alpha_{1,2} = \alpha_{1,1} - \frac{\eta_2}{2} (\vartheta_1 + k_1 \vartheta_2) - \frac{\omega_2 k_1^2}{2\omega_2} - \frac{k_1 \vartheta_1}{2\delta_2} \]

\[\alpha_{2,2} = k_2 \alpha_{2,1} - \frac{\varepsilon_2}{2} - \frac{\vartheta_1 + k_1 \vartheta_2}{2\eta_2} - \vartheta_2 - \frac{\mu_2}{2} - k_1 - \frac{k_1^2}{2\omega_2} - \frac{k_1 \vartheta_1}{2\delta_2} - \frac{\sigma_2}{2} \]

\[\beta_2 = \beta_1 + \frac{\| L + B \| ^2}{2\mu_2} d_2^2 + \frac{\| L + B \| ^2}{2\sigma_2} d_1^2 \]

\[\alpha_{3,2} = \frac{1}{2\varepsilon_2}. \tag{36}\]
Step 8. Assume that we have selected the virtual input as \( \varphi^*_{i+1} = -k_i \xi_i \) such that the time derivative of \( V_i = V_{i-1} + \frac{1}{2} \xi_i^T \xi_i \) satisfies

\[
\dot{V}_i \leq -\alpha_{1,i} \varphi_i^T \varphi_i - \sum_{h=2}^{i} \alpha_{h,i} \xi_h^T \xi_h + \beta_i + \alpha_{i+1,i} \xi_{i+1}^T \xi_{i+1}, \tag{37}
\]

where \( \alpha_{1,i} , \alpha_{2,i} , \alpha_{3,i} , ... , \alpha_{i,i} , \beta_i , \alpha_{i+1,i} > 0 \) and

\[
\xi_{i+1} = \varphi_{i+1} - \varphi^*_{i+1} = \varphi_{i+1} + k_i \xi_i. \tag{38}
\]

It will be shown that inequality (37) also holds for \( i = i + 1 \). To prove this issue, construct the Lyapunov function \( V_{i+1} = V_i + \frac{1}{2} \xi_{i+1}^T \xi_{i+1} \), and the time derivative of \( V_{i+1} \) is

\[
\dot{V}_{i+1} = \dot{V}_i + \xi_{i+1}^T \dot{\xi}_{i+1} \tag{39}
\]

According to (38), we can get

\[
\dot{V}_{i+1} = \dot{V}_i + \xi_{i+1}^T \dot{\xi}_{i+1} + \frac{1}{2} \xi_{i+1}^T \xi_{i+1} \tag{40}
\]

where \( \varphi^*_{i+2} \) is a virtual input.

Choose the virtual input as

\[
\varphi^*_{i+2} = -k_{i+1} \xi_{i+1}, \tag{41}
\]

and define

\[
\xi_{i+2} = \varphi_{i+2} + k_{i+1} \xi_{i+1}. \tag{42}
\]

Substituting (41)-(42) into (40), we have

\[
\dot{V}_{i+1} = \dot{V}_i - k_{i+1} \xi_{i+1}^T \xi_{i+1} + \xi_{i+1}^T \xi_{i+2} - \xi_{i+1}^T (\mathcal{L} + B) \bar{f}_{i+1} \tag{43}
\]

Note that

\[
\xi_{i+1}^T \xi_{i+2} \leq \frac{1}{2} \xi_{i+1}^T \xi_{i+1} + \frac{1}{2} \xi_{i+2}^T \xi_{i+2}, \tag{44}
\]

\[
-\xi_{i+1}^T (\mathcal{L} + B) \bar{d}_{i+1} \leq \xi_{i+1} \mathcal{L} \bar{d}_{i+1} \leq \xi_{i+1} \mathcal{L} \bar{d}_{i+1} \leq \frac{\mu_{i+1}}{2} \xi_{i+1}^2 + \frac{\| \mathcal{L} + B \|^2}{2 \mu_{i+1}} d_{i+1}^2, \tag{45}
\]

where \( \mu_{i+1}, \xi_{i+1} \) are positive constants and \( \bar{d}_{i+1} = \rho_{i+1} \frac{\lambda_{\max}(\mathcal{L} + B)}{\lambda_{\min}(\mathcal{L} + B)} \).

By using (38), we can get

\[
\dot{V}_{i+1} = \theta_{i+1} \xi_{i+1} \phi_{i+1} \tag{46}
\]

where

\[
\phi_{i+1} = -k_1 \varphi_1 + \cdots + \theta_{i+1} \xi_{i+1} \phi_{i+1}. \tag{47}
\]
It can be concluded from (47) that
\[-\xi_{i+1}^T (\mathcal{L} + \mathcal{B}) \tilde{f}_{i+1} \leq \gamma_{1,i+1} \|\xi_{i+1}\| \|\varphi_1\| + \gamma_{2,i+1} \|\xi_{i+1}\| \|\xi_2\| + \cdots + \gamma_{i,i+1} \|\xi_{i+1}\| + \gamma_{i+1,i+1} \|\xi_{i+1}\|^2, \quad (48)\]

where
\[
\gamma_{1,i+1} = \vartheta_1 + \vartheta_2 k_1 + \vartheta_3 k_2 k_1 + \cdots + \vartheta_{i+1} \prod_{p=1}^{i} k_p
\]
\[
\gamma_{2,i+1} = \vartheta_2 + \vartheta_3 k_2 + \vartheta_4 k_3 k_2 + \cdots + \vartheta_{i+1} \prod_{p=2}^{i} k_p
\]
\[\vdots\]
\[
\gamma_{i+1,i+1} = \vartheta_{i+1}. \quad (49)
\]

By using Lemma 2.2, we can get
\[-\xi_{i+1}^T (\mathcal{L} + \mathcal{B}) \tilde{f}_{i+1} \leq \frac{1}{2\eta_{(i+1)1}} \|\varphi_1\|^2 + \frac{\eta_{(i+1)1} \gamma_{1,i+1}^2}{2} \|\xi_{i+1}\|^2 + \frac{\eta_{(i+1)2} \gamma_{2,i+1}^2}{2} \|\xi_{i+1}\|^2 + \cdots + \frac{\eta_{(i+1)i} \gamma_{i,i+1}^2}{2} \|\xi_{i+1}\|^2 + \frac{\eta_{(i+1)i+1} \gamma_{i+1,i+1}^2}{2} \|\xi_{i+1}\|^2, \quad (50)\]

It can be concluded from (50) that
\[-\xi_{i+1}^T (\mathcal{L} + \mathcal{B}) \tilde{f}_{i+1} \leq \frac{1}{2\eta_{(i+1)1}} \|\varphi_1\|^2 + \frac{\eta_{(i+1)2} \gamma_{2,i+1}^2}{2} \|\xi_2\|^2 + \cdots + \frac{\eta_{(i+1)i} \gamma_{i,i+1}^2}{2} \|\xi_i\|^2 + \frac{\eta_{(i+1)i+1} \gamma_{i+1,i+1}^2}{2} \|\xi_{i+1}\|^2 + \vartheta_{i+1} \|\xi_{i+1}\|^2 + \sum_{q=1}^{i} \frac{\eta_{(i+1)q} \gamma_{q,i+1}^2}{2} \|\xi_{i+1}\|^2, \quad (51)\]

where \(\eta_{(i+1)1}, \eta_{(i+1)2}, \eta_{(i+1)3}, \ldots, \eta_{(i+1)i} > 0\).

It is noted that
\[
\xi_i = \varphi_i + k_{i-1} \xi_{i-1}
\]
\[
= \varphi_i + k_{i-1} \varphi_{i-1} + k_{i-1} k_{i-2} \varphi_{i-2}
\]
\[
\quad \quad \quad \quad \quad \quad \quad = \cdots = \varphi_i + k_{i-1} \varphi_{i-1} + k_{i-1} k_{i-2} \varphi_{i-2} + \cdots + \prod_{p=1}^{i-1} k_p \varphi_1. \quad (52)
\]

From (52), we have
\[
k_i \xi_{i+1}^T \dot{\xi}_i \leq k_i \|\xi_{i+1}\| \|\dot{\varphi}_i\| + k_i k_{i-1} \|\xi_{i+1}\| \|\dot{\varphi}_{i-1}\| + \cdots + \prod_{p=1}^{i} k_p \|\xi_{i+1}\| \|\dot{\varphi}_1\|, \quad (53)
\]
where
\[
\begin{align*}
k_i \|\xi_{i+1}\| \|\dot{\phi}_i\| & = k_i \|\xi_{i+1}\| \|\phi_{i+1} - (\mathcal{L} + \mathcal{B}) \bar{f}_i - (\mathcal{L} + \mathcal{B}) \bar{d}_i\| \\
& \leq k_i \|\xi_{i+1}\| \|\xi_{i+1} - k_i \xi_i\| + k_i \|\xi_{i+1}\| \|\mathcal{L} + \mathcal{B}\| \|\bar{d}_i\| \\
& + k_i \|\xi_{i+1}\| \|\mathcal{L} + \mathcal{B}\| \|\bar{f}_i\|. \\
\end{align*}
\]

(54)

By using Lemma 2.2, it follows that
\[
\begin{align*}
k_i \|\xi_{i+1}\| \|\dot{\phi}_i\| & \leq \frac{k_i}{2a_{i1}} \|\xi_{i+1}\|^2 + \frac{a_{i1} k_i}{2} \|\xi_{i+1}\|^2 + \frac{k_i k_i}{2a_{i2}} \|\xi_i\|^2 + \frac{a_{i2} k_i k_i}{2} \|\xi_{i+1}\|^2 \\
& + \frac{b_i k_i}{2} \|\mathcal{L} + \mathcal{B}\|^2 d_i^2 + \frac{k_i}{2b_i} \|\xi_{i+1}\|^2 \\
& + k_i \|\xi_{i+1}\| \|\mathcal{L} + \mathcal{B}\| \|\bar{f}_i\|, \\
\end{align*}
\]

(55)

where \(a_{i1}, b_i\) are two positive constants.

Note that
\[
\begin{align*}
k_i \|\xi_{i+1}\| \|\mathcal{L} + \mathcal{B}\| \|\bar{f}_i\| & \leq k_i \|\xi_{i+1}\| \|\mathcal{L} + \mathcal{B}\| \left( \vartheta_1 \|\bar{x}_1\| + \vartheta_2 \|\bar{x}_2\| + \cdots + \vartheta_i \|\bar{x}_i\| \right) \\
& = k_i \vartheta_1 \|\xi_{i+1}\| \|\phi_1\| + k_i \vartheta_2 \|\xi_{i+1}\| \|\phi_2\| + \cdots \\
& + k_i \vartheta_i \|\xi_{i+1}\| \|\phi_i\|. \\
\end{align*}
\]

(56)

By using (38), it yields that
\[
\begin{align*}
k_i \|\xi_{i+1}\| \|\mathcal{L} + \mathcal{B}\| \|\bar{f}_i\| & \leq k_i \vartheta_1 \|\xi_{i+1}\| \|\phi_1\| + k_i \vartheta_2 \|\xi_{i+1}\| \|\phi_2\| + \cdots \\
& + k_i \vartheta_i \|\xi_{i+1}\| \|\phi_i\|. \\
\end{align*}
\]

(57)

From (57), one has
\[
\begin{align*}
k_i \|\xi_{i+1}\| \|\mathcal{L} + \mathcal{B}\| \|\bar{f}_i\| & \leq \frac{k_i}{2c_{i1}} \|\varphi_1\|^2 + \frac{c_{i1} k_i \gamma_i^2}{2} \|\xi_{i+1}\|^2 \\
& + \frac{k_i}{2c_{i2}} \|\xi_2\|^2 + \frac{c_{i2} k_i \gamma_i^2}{2} \|\xi_{i+1}\|^2 + \cdots \\
& + \frac{k_i}{2c_{ii}} \|\xi_i\|^2 + \frac{c_{ii} k_i \gamma_i^2}{2} \|\xi_{i+1}\|^2, \\
\end{align*}
\]

(58)

where \(c_{i1}, c_{i2}, c_{i3}, \ldots, c_{ii}\) are positive constants.

Substituting (58) into (55), we have
\[
\begin{align*}
k_i \|\xi_{i+1}\| \|\dot{\phi}_i\| & \leq \frac{k_i}{2a_{i1}} \|\xi_{i+1}\|^2 + \frac{a_{i1} k_i}{2} \|\xi_{i+1}\|^2 + \frac{k_i k_i}{2a_{i2}} \|\xi_i\|^2 + \frac{a_{i2} k_i k_i}{2} \|\xi_{i+1}\|^2 \\
& + \frac{b_i k_i}{2} \|\mathcal{L} + \mathcal{B}\|^2 d_i^2 + \frac{k_i}{2b_i} \|\xi_{i+1}\|^2 \\
& + \frac{k_i}{2c_{i1}} \|\varphi_1\|^2 + \frac{c_{i1} k_i \gamma_i^2}{2} \|\xi_{i+1}\|^2 + \frac{k_i}{2c_{i2}} \|\xi_2\|^2 + \frac{c_{i2} k_i \gamma_i^2}{2} \|\xi_{i+1}\|^2 \\
& + \cdots + \frac{k_i}{2c_{ii}} \|\xi_i\|^2 + \frac{c_{ii} k_i \gamma_i^2}{2} \|\xi_{i+1}\|^2. \\
\end{align*}
\]

(59)

From (53)-(59), we can see that
\[
\begin{align*}
k_i \xi_{i+1} \mathcal{L} \xi_i & \leq \sum_{p=1}^i \frac{b_p}{2} \|\mathcal{L} + \mathcal{B}\|^2 \prod_{q=p}^i k_q d_p^2 \\
& + \chi_1 \|\varphi_1\|^2 + \chi_2 \|\xi_2\|^2 + \cdots + \chi_i \|\xi_i\|^2 + \chi_{i+1} \|\xi_{i+1}\|^2, \\
\end{align*}
\]

(60)
where

\[
\chi_1 = \frac{k_1 \prod_{p=1}^{i} k_p}{2a_{12}} + \sum_{p=1}^{i} \frac{i \prod_{q=p}^{i} k_q}{2c_{p1}}
\]

\[
\chi_2 = \frac{k_2 \prod_{p=2}^{i} k_p}{2a_{22}} + \frac{i \prod_{p=1}^{i} k_p}{2a_{11}} + \sum_{p=2}^{i} \frac{i \prod_{q=p}^{i} k_q}{2c_{p2}}
\]

\[
\vdots
\]

\[
\chi_i = \frac{k_i k_{i-1}}{2a_{i2}} + \frac{k_i k_{i-1}}{2a_{(i-1)1}} + \frac{k_i}{2c_{ii}}, \quad (61)
\]

and

\[
\chi_{i+1} = \sum_{p=1}^{i} \frac{a_{p1} \prod_{q=p}^{i} k_q}{2} + \sum_{p=1}^{i} a_{p2} k_p \prod_{q=p}^{i} k_q + \sum_{p=1}^{i} \frac{i \prod_{q=p}^{i} k_q}{2} + \frac{k_i}{2a_{i1}}
\]

\[
+ \sum_{p=1}^{i} \left[ \frac{i \prod_{q=p}^{i} k_q}{2} \sum_{q=1}^{p} \left( \theta_q + \theta_{q+1} k_q + \cdots + \theta_{p} \prod_{o=q}^{p-1} k_o \right)^2 c_{pq} \right]. \quad (62)
\]

Substituting (44), (45), (51) and (60) into (43) leads to

\[
\dot{V}_{i+1} \leq -\alpha_{1,(i+1)} \varphi_1 T_1 \varphi_1 - \sum_{h=2}^{i+1} \alpha_{h,(i+1)} \xi_h^T \xi_h + \beta_{i+1} + \alpha_{(i+1)\cdots(i+1)} \xi_{i+2}^T \xi_{i+2}, \quad (63)
\]

where

\[
\beta_{i+1} = \beta_i + \frac{\|L + B\|^2}{2 \epsilon_{i+1}} d_{i+1}^2 + \sum_{p=1}^{i} \frac{b_p \|L + B\|^2}{2} \prod_{q=p}^{i} k_q d_{i+1}^2
\]

\[
\alpha_{1,(i+1)} = \alpha_{1,i} - \frac{1}{2 \eta_{(i+1)1}} - \chi_1
\]

\[
\alpha_{2,(i+1)} = \alpha_{2,i} - \frac{1}{2 \eta_{(i+1)2}} - \chi_2
\]

\[
\vdots
\]

\[
\alpha_{i,(i+1)} = \alpha_{i,i} - \frac{1}{2 \eta_{(i+1)i}} - \chi_i
\]

\[
\alpha_{(i+1),(i+1)} = k_{i+1} - \frac{\epsilon_{i+1}}{2} - \frac{\mu_{i+1}}{2} - \chi_{i+1} - \left( \theta_{i+1} + \sum_{q=1}^{i} \frac{\eta_{(i+1)q}}{2} \gamma_{q,i+1} \right)
\]

\[
\alpha_{(i+2),(i+1)} = \frac{1}{2 \epsilon_{i+1}}, \quad (64)
\]

and \(k_1, k_2, \ldots, k_{i+1}\) are positive constants such that

\[
\alpha_{1,(i+1)}, \alpha_{2,(i+1)}, \ldots, \alpha_{(i+1),(i+1)} > 0.
\]
Step n. Choose the Lyapunov function as

$$V_n = \frac{1}{2} \varphi_1^T \varphi_1 + \sum_{h=2}^{n} \frac{1}{2} \xi_h^T \xi_h + \frac{1}{2} \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right)^2,$$

and the time derivative of $V_n$ satisfies

$$\dot{V}_n \leq - \alpha_{1,n} \varphi_1^T \varphi_1 - \sum_{h=2}^{n-1} \alpha_{h,n} \xi_h^T \xi_h + \alpha_{n,n-1} \xi_n^T \xi_n + \beta_n$$

$$- \xi_n^T (\mathcal{L} + \mathcal{B}) u + \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right) \hat{\theta}_j,$$

where $\alpha_{1,n}, \alpha_{2,n}, ..., \alpha_{(n-1),(n-1)}$ are positive constants, and $\theta = \frac{\alpha_{n,(n-1)}}{\lambda_{\min}(\mathcal{L} + \mathcal{B})}$.

Substituting (9) into (65), we can get

$$\dot{V}_n \leq - \alpha_{1,n} \varphi_1^T \varphi_1 - \sum_{h=2}^{n-1} \alpha_{h,n} \xi_h^T \xi_h + \alpha_{n,n-1} \xi_n^T \xi_n + \beta_n$$

$$- \xi_n^T (\mathcal{L} + \mathcal{B}) \left( k + \hat{\theta}_j \right) \xi_n + \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right) \hat{\theta}_j$$

$$= - \alpha_{1,n} \varphi_1^T \varphi_1 - \sum_{h=2}^{n-1} \alpha_{h,n} \xi_h^T \xi_h + \alpha_{n,n-1} \xi_n^T \xi_n + \beta_n$$

$$- \xi_n^T (\mathcal{L} + \mathcal{B}) \left( k + \theta \right) \xi_n - \tau \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right) \hat{\theta}_j,$$

where

$$- \tau \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right) \hat{\theta}_j = - \tau \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right)^2 + \tau \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right) \theta$$

$$\leq - \tau \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right)^2 + \frac{\tau \gamma_1}{2} \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right)^2 + \frac{\tau}{2 \gamma_1} \theta^2$$

$$= - \left( \tau - \frac{\tau \gamma_1}{2} \right) \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right)^2 + \frac{\tau}{2 \gamma_1} \theta^2,$$

and

$$- \xi_n^T (\mathcal{L} + \mathcal{B}) \left( k + \theta \right) \xi_n \leq - \left( k + \theta \right) \lambda_{\min}(\mathcal{L} + \mathcal{B}) \xi_n^T \xi_n$$

$$\leq - k \lambda_{\min}(\mathcal{L} + \mathcal{B}) \xi_n^T \xi_n - \alpha_{n,n-1} \xi_n^T \xi_n.$$

Substituting (67)-(68) into (66), we can obtain

$$\dot{V}_n \leq - \alpha_{1,n} \varphi_1^T \varphi_1 - \sum_{h=2}^{n} \alpha_{h,n} \xi_h^T \xi_h - \tau_1 \sum_{j=1}^{m} \left( \theta - \hat{\theta}_j \right) \hat{\theta}_j + \beta,$$

where

$$\alpha_{n,n} = k \lambda_{\min}(\mathcal{L} + \mathcal{B}) > 0$$

$$\tau_1 = \tau - \frac{\tau \gamma_1}{2} > 0$$

$$\beta = \beta_n + \frac{\tau}{2 \gamma_1} \theta^2 > 0.$$
Remark 4. In protocol (9), the parameters \( k \) and \( k_i \) are related to the convergence rate. It can be seen that the convergence rate will be faster if \( k \) and \( k_i \) are larger. However, the control signals always have upper bound due to the presence of saturation properties. Therefore, \( k \) and \( k_i \) cannot be chosen arbitrarily large.

4. Simulation example. In this section, a numerical example is employed to demonstrate the effectiveness of the proposed protocol.

Consider the following third-order system, including one leader and four followers. The dynamics of agents are described by

(leader 0)
\[
\begin{align*}
\dot{x}_{10} &= x_{20} + \sin (0.1x_{10}) + \sin (0.1t) \\
\dot{x}_{20} &= x_{30} + \sin (0.1x_{10} + 0.1x_{20}) + \sin (0.2t) \\
\dot{x}_{30} &= \sin (0.1x_{10} + 0.1x_{20} + 0.1x_{30}) + \sin (0.3t),
\end{align*}
\]

(follower 1)
\[
\begin{align*}
\dot{x}_{11} &= x_{21} + \sin 0.1x_{11} + \cos 0.12t \\
\dot{x}_{21} &= x_{31} + \sin (0.1x_{11} + 0.1x_{21}) + \cos 0.22t \\
\dot{x}_{31} &= u_1 + \sin (0.1x_{11} + 0.1x_{21} + 0.1x_{31}) + \cos 0.32t,
\end{align*}
\]

(follower 2)
\[
\begin{align*}
\dot{x}_{12} &= x_{22} + \sin 0.1x_{12} + \cos 0.14t \\
\dot{x}_{22} &= x_{32} + \sin (0.1x_{12} + 0.1x_{22}) + \cos 0.24t \\
\dot{x}_{32} &= u_2 + \sin (0.1x_{12} + 0.1x_{22} + 0.1x_{32}) + \cos 0.34t,
\end{align*}
\]
(follower 3)
\[
\begin{align*}
\dot{x}_{13} &= x_{23} + \sin 0.1x_{13} + \cos 0.16t \\
\dot{x}_{23} &= x_{33} + \sin (0.1x_{13} + 0.1x_{23}) + \cos 0.26t \\
\dot{x}_{33} &= u_3 + \sin (0.1x_{13} + 0.1x_{23} + 0.1x_{33}) + \cos 0.36t,
\end{align*}
\]
(78)

(follower 4)
\[
\begin{align*}
\dot{x}_{14} &= x_{24} + \sin 0.1x_{14} + \cos 0.18t \\
\dot{x}_{24} &= x_{34} + \sin (0.1x_{14} + 0.1x_{24}) + \cos 0.28t \\
\dot{x}_{34} &= u_4 + \sin (0.1x_{14} + 0.1x_{24} + 0.1x_{34}) + \cos 0.38t,
\end{align*}
\]
(79)

and its undirected communication topology is shown in Figure 1.

\[\text{Figure 1. The topology of the multi-agent system}\]

The initialization conditions for all agents are as follows: \(x_{10}(0) = 1.5, x_{11}(0) = -2,\)
\(x_{12}(0) = 5.5, x_{13}(0) = 3, x_{14}(0) = -2.5, x_{20}(0) = x_{21}(0) = x_{22}(0) = x_{23}(0) = x_{24}(0) = 0\) and \(x_{30}(0) = x_{31}(0) = x_{32}(0) = x_{33}(0) = x_{34}(0) = 0.\) The protocol (9) is adopted and the corresponding parameters are taken as: \(k = 10, k_1 = 6, k_2 = 8,\)
\(\tau = 1.\) The simulation results are shown in Figures 2-4. The position states of the agents are described in Figure 2, Figure 3 shows the velocity states of the agents, and the acceleration states of the agents are presented in Figure 4. It can be seen that the practical consensus of the multi-agent system can be achieved.

\[\text{Figure 2. Position states of agents}\]
5. Conclusion. This paper has offered an effective solution to the problem of leader-following consensus of higher-order nonlinear multi-agent systems which suffer from unmatched disturbances. By the adaptive control strategy and the backstepping technique, a new consensus control scheme has been presented. The practical consensus of the higher-order nonlinear multi-agent system was realized by the proposed consensus algorithm in the presence of unmatched disturbances. Finally, a numerical example was given and the effectiveness of the proposed protocol was illustrated.

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