New Schemes for Solving the Principal Eigenvalue Problems of Perron-like Matrices via Polynomial Approximations of Matrix Exponentials†

Desheng Li a,*, Ruijing Wang a

aSchool of Mathematics, Tianjin University, Tianjin 300072, China

Abstract

A real square matrix is Perron-like if it has a real eigenvalue $s$, called the principal eigenvalue of the matrix, and $\text{Re}\, \mu < s$ for any other eigenvalue $\mu$. Nonnegative matrices and symmetric ones are typical examples of this class of matrices. The main purpose of this paper is to develop a set of new schemes to compute the principal eigenvalues of Perron-like matrices and the associated generalized eigenspaces by using polynomial approximations of matrix exponentials. Numerical examples show that these schemes are effective in practice.

Keywords: Perron-like matrix, principal eigenvalue problem, matrix exponential, iterative scheme, combined computation method.

2010 MSC: 15A18, 65F15, 65F10, 34D05.

*Corresponding author

Email addresses: lidsmath@tju.edu.cn (Desheng Li), wrj-math@tju.edu.cn (Ruijing Wang)

†This work was supported by the National Natural Science Foundation of China [11871368]
## Contents

1 Introduction  

2 Preliminaries  
   2.1 Basic notions and notations  
   2.2 Perron-like matrices  
   2.3 On the growth rate of exponential functions of matrices  

3 Fundamental Convergence Results for Perron-like Matrices  
   3.1 Preliminaries  
   3.2 Some fundamental convergence theorems  
   3.3 The case of a semisimple principal eigenvalue  
   3.4 A generalized Perron-like theorem for nonnegative matrices  

4 Computing Principal Eigenvalues and Eigenvectors via Polynomial Approximations of Matrix Exponentials  

5 An Iterative Scheme  
   5.1 An iterative scheme  
   5.2 An iterative scheme with parameters  
   5.3 A remark on the initial matrix  
   5.4 Numerical examples  

6 Determination of the Cyclic Order of $GE_s(A)$  
   6.1 Determining the cyclic order of $GE_s(A)$  
   6.2 A numerical example  

7 A Combined Method for the Computation of Non-semisimple Principal Eigenvalues  
   7.1 A fundamental lemma  
   7.2 A combined method for computing the principal eigenvalue  
   7.3 A numerical example  

8 The Computation of Principal Generalized Eigenspaces  
   8.1 Some fundamental convergence results  
   8.2 An iterative method for the computation of $\hat{Y}(t)$  
   8.3 A numerical example  

2
1. Introduction

The study of numerical eigenvalue problems of matrices has witnessed a long history of more than one hundred years. By far numerous efficient computation methods and algorithms have been developed such as the power method, the Jacobi method, the LR and the QR algorithms, the Lanczos procedure, and the Krylov subspace methods, which were systematically summarized and discussed in many excellent books and monographs (see e.g. [4, 8, 11, 14, 16, 20, 23, 24, 25, 28]). These methods and algorithms enable us to acquire effectively approximate eigenvalues and eigenvectors of a given matrix with prescribed accuracy. The subspace methods even allow one to compute invariant subspaces of matrices. Of course, each method has more or less some limitation or drawbacks. For instance, the convergence of the power method is only guaranteed in a generic sense with respect to the initial vector $x_0$; furthermore, the choice of $x_0$ significantly affects the convergence speed of the iteration sequence. Similar situation also occurs in the subspace methods, where it is often required that the initial subspace $S$ in the iteration procedure satisfies

$$S \cap U^\perp = \{0\}$$  \hspace{1cm} (1.1)

to guarantee the convergence, where $U^\perp$ denotes the orthogonal complement of the target subspace $U$; see Watkins [24, Theorems 5.1.1] and Björck [4, Theorem 3.3.5]. It is known that (1.1) is true in a generic sense with respect to $S$. However in practice, since the target subspace $U$ is in fact unknown, in many cases we have to try our luck whether this requirement is fulfilled by a given subspace $S$. Let us also mention that most of the existing computation methods and algorithms work well only for semisimple matrices whose eigenvalues share the same algebraic and the geometric multiplicities.

In this present work, inspired by the dynamical approach towards the Perron-Frobenius theory in [15], we develop in a self-contained manner some new schemes for computing the principal eigenvalues and the corresponding generalized eigenspaces of a wide class of real matrices which we call Perron-like matrices by using matrix exponentials and their polynomial approximations. This class of matrices contain nonnegative matrices and symmetric ones as typical examples. Similar as in the case of the subspaces methods, our schemes make use of matrix iterations. One advantage of the schemes is that the convergence is always guaranteed as long as the initial matrices are taken nonsingular. Another one is that they can successfully generate the whole generalized principal eigenspace of a Perron-like matrix in the non-semisimple case.

Now we give a more detailed description of our work and the organization of this paper. In Section 2 we make some preliminaries. In Section 3 we prove some convergence results for Perron-like matrices. Denote by $\mathbb{M}_m$ the space consisting of $m \times m$ real matrices. A matrix $A \in \mathbb{M}_m$ is called Perron-like if
it has a real eigenvalue \( s \) (which will be called the *principal eigenvalue* of \( A \)), moreover, \( \Re \mu < s \) for all other eigenvalues \( \mu \). Let \( A \in \mathbb{M}_m \) be a Perron-like matrix with principal eigenvalue \( s \). Given a noningular matrix \( V \in \mathbb{M}_m \) with column vectors \( v_i, i \in \{1, 2, \cdots, m\} := J \), put

\[
X(t) = e^{tA}V.
\]

Let \( y_i = \Pi_1 v_i \), where \( \Pi_1 \) is the projection from \( \mathbb{R}^m \) to the generalized eigenspace \( \text{GE}_s(A) \) associated with \( s \). Denote by \( Q_y(t) \) the vector-valued *characteristic polynomial* of \( y_i \) as defined in (2.6) below. Let \( Q_Y(t) \) be the matrix with column vectors \( Q_y(t) \) \((i \in J)\). We show that there exist \( B_0, \delta > 0 \) such that

\[
\|X(t)\| - \|Q_Y(t)\| \leq B_0 e^{-\delta t}, \quad t \geq 1. 
\]  

(1.2)

Based on this fundamental result, we then verify that there is a matrix \( \Xi \) whose column vectors \( \xi_i \) \((i \in J)\) span a nontrivial invariant subspace of the eigenspace \( E_s(A) \) associated with \( s \) such that

\[
\frac{\|X(t)\|}{\|X(t)\|} - \Xi \leq O(t^{-1}) + B_1 e^{-\delta t}, \quad t \geq 1.
\]

(1.3)

As a consequence, one naturally has

\[
\left| \frac{1}{\|X(t)\|^2} \langle AX(t), X(t) \rangle - s \right| \leq O(t^{-1}) + B_2 e^{-\delta t}, \quad t \geq 1.
\]

(1.4)

If \( s \) is *semisimple*, i.e., \( s \) shares the same algebraic and the geometric multiplicities (hence \( \text{GE}_s(A) = E_s(A) \)), then

\[
E_s(A) = \text{span}\{\xi_i : i \in J\}.
\]

(1.5)

Furthermore, the term \( O(t^{-1}) \) in the above estimates can be removed, and therefore the convergence in (1.3) and (1.4) is actually exponential.

It is interesting to mention that (1.3) also leads to a strengthened version of the Perron theorem for nonnegative matrices; see Section 3.4 for details.

Nowadays there are many effective ways for computing matrix exponentials; see e.g. [7, 13, 17]. One of the simplest ways is to use Taylor expansions which yield polynomial approximations of matrix exponentials. In Section 4 we will give some convergence results parallel to those in (1.2)-(1.4) for polynomial approximations of \( X(t) \). Based on these results, we then design in Section 5 a corresponding iterative scheme. Specifically, for each \( n \in \mathbb{N} \), we define a mapping \( K_n \) on \( \Sigma_1 := \{X \in \mathbb{M}_m : \|X\| = 1\} \) as

\[
K_nX = \frac{T_nX}{\|T_nX\|}, \quad X \in \Sigma_1,
\]

4
where $T_n = \sum_{\ell=0}^{n} \frac{1}{\ell!} A^\ell$. Given a nonsingular matrix $V \in \Sigma_1$, set

$$X_n(0) = V, \quad X_n(k + 1) = K_n X_n(k) \quad (k, n \in \mathbb{N}).$$

We show that there exist positive constants $B_0, B_1$ and $B_2$ such that

$$\|X_n(n) - QY(n)\| \leq B_0 e^{-\delta n} + \left(\frac{\lambda \|A\|}{n + 1}\right)^{n+1} o(1), \quad (1.6)$$

and

$$\|X_n(n) - \Xi\| \leq O(n^{-1}) + B_1 e^{-\delta n} + \left(\frac{\lambda \|A\|}{n + 1}\right)^{n+1} o(1), \quad (1.7)$$

and

$$|\langle AX_n(n), X_n(n) \rangle - s| \leq O(n^{-1}) + B_2 e^{-\delta n} + \|A\| \left(\frac{\lambda \|A\|}{n + 1}\right)^{n+1} o(1), \quad (1.8)$$

where $\lambda = 2e^{2\|A\|}$, and $o(1)$ is an infinitesimal as $n \to \infty$.

As in (1.3) and (1.4), if $s$ is semisimple then the first terms in the righthand sides of (1.7) and (1.8) can be dropped. In view of (1.5) we see that the above iterative scheme readily provides an efficient way for solving the principal eigenvalue problems of Perron-like matrices. The situation in the non-semisimple case seems to be complicated. On one hand, (1.7) and (1.8) as well as the numerical simulation in Section 5 indicate that the convergence can be very slow, which fact may make the scheme to be of little practical sense. On the other hand, the column vectors of $\Xi$ only span a proper subspace of the generalized eigenspace $GE_s(A)$ associated with $s$. In Sections 6 - 8 we develop some new methods to deal with the non-semisimple case.

In Section 5 we design a numerical scheme to determine the cyclic order of $GE_s(A)$, namely, the smallest number $\nu$ such that

$$(A - sI)^\nu (GE_s(A)) = \{0\}.$$

Because the scheme is mainly based on the exponential convergence in (1.6) and a “rough” approximate value $\tilde{s}$ of $s$, it can help us determine the cyclic order of $GE_s(A)$ quickly. This allows us to develop a more efficient combined method to compute the principal eigenvalue $s$, which will be addressed in Section 7.

Section 6 is concerned with the computation of the generalized eigenspace $GE_s(A)$. Let $V \in \mathbb{M}_m$ be a nonsingular matrix with column vectors $v_i \ (i \in J)$. Then one trivially verifies that $GE_s(A) = \text{span}\{y_i : \ i \in J\}$, where $y_i = \Pi_1 v_i$. Therefore the problem of computing $GE_s(A)$ reduces to that of the matrix $Y$ with column vectors $y_i \ (i \in J)$. Based on the simple observation that

$$Y = P(t)Q_Y(t) = P(t)\hat{X}(t) - P(t)\hat{Z}(t),$$
where $\hat{X}(t) = e^{t(A-sI)}V$, $\hat{Z}(t) = e^{t(A-sI)}(V-Y)$, and

$$P(t) = \sum_{k=0}^d (-1)^k \frac{t^k}{k!} (A-sI)^k, \quad \nu - 1 \leq d \leq m - 1,$$

in Section 8 we propose an iterative scheme for computing the matrix $Y$ by using polynomial approximations of matrix exponentials.

Numerical examples will be given in Sections 5-8 to illustrate the efficiency of the computation schemes.

2. Preliminaries

This section is concerned with some preliminaries.

2.1. Basic notions and notations

- **Notations** Throughout the paper $\mathbb{R}$ (resp. $\mathbb{C}$) stands for the field of real (resp. complex) numbers. Let $\mathbb{R}_+ = [0, \infty)$, and $\mathbb{N}$ the set of nonnegative integers. Denote by $\mathbb{R}^m$ the Euclid space consisting of $m$-dimensional column vectors equipped with the usual inner product $\langle \cdot, \cdot \rangle$ and the 2-norm $\| \cdot \|$. Let $\mathbb{M}_m := \mathbb{R}^{m \times m}$ be the space of $m \times m$ real matrices. The same notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ as above will be used to denote the inner product and the 2-norm of $\mathbb{M}_m$, respectively. A matrix $A \in \mathbb{M}_m$ with column vectors $\alpha_i (i \in J)$ will be written as $A = [\alpha_i]_{i \in J}$, where (and below) $J$ denotes the index set $\{1, 2, \cdots, m\}$. It is obvious that

$$\langle A, B \rangle = \sum_{i \in J} \langle \alpha_i, \beta_i \rangle, \quad \forall A = [\alpha_i]_{i \in J}, B = [\beta_i]_{i \in J} \in \mathbb{M}_m.$$  

Denote by $I$ the identity matrix in $\mathbb{M}_m$.

- **Spectrum and invariant subspaces of matrices** Given $A \in \mathbb{M}_m$, denote by $\sigma(A)$ the spectrum of $A$ consisting of the eigenvalues of $A$. The spectral bound $s(A)$ and spectral radius $r(A)$ of $A$ are defined as

$$s(A) = \max \{ \Re \mu : \mu \in \sigma(A) \}, \quad r(A) = \max \{ |\mu| : \mu \in \sigma(A) \}.$$  

One can naturally think of $A$ as a linear transform on either $\mathbb{R}^m$ or $\mathbb{C}^m$. A subspace $Y$ of $\mathbb{R}^m$ (or $\mathbb{C}^m$) is called an invariant subspace of $A$, if $AY \subset Y$.

For each $\mu \in \sigma(A)$, the space

$$\mathcal{GE}_\mu(A) := \{ \xi \in \mathbb{C}^m : (A-\mu I)^k \xi = 0 \text{ for some } k \geq 1 \} \quad (2.1)$$

is a nontrivial invariant subspace of $A$. Consequently

$$\mathcal{GE}_\mu(A) := \{ x, y : u = x + iy \in \mathcal{GE}_\mu(A) \} \quad (2.2)$$
is an invariant subspace of $A$ (in $\mathbb{R}^m$), where $i$ stands for the unit imaginary number in $\mathbb{C}$. We call $\text{GE}_\mu(A)$ the (real) generalized eigenspace (or, singular space) of $A$ associated with $\mu$.

- **The case of real eigenvalues** Now we assume that $\mu$ is a real eigenvalue of $A \in \mathbb{M}_m$. Then $(A - \mu I)^k(x + iy) = 0$ amounts to say that $(A - \mu I)^kx = 0 = (A - \mu I)^ky$. It follows by (2.1) and (2.2) that

\[
\text{GE}_\mu(A) = \{ \xi \in \mathbb{R}^m : (A - \mu I)^k\xi = 0 \text{ for some } k \geq 1 \}
\]

(2.3)

We call each nonzero element $\xi$ in $\text{GE}_\mu(A)$ a generalized eigenvector of $A$. Let $\xi$ be a generalized eigenvector of $A$ corresponding to $\mu$. Then there is an integer $\nu \geq 1$ such that

\[
(A - \mu I)^j\xi \neq 0 \ (0 \leq j \leq \nu - 1), \quad (A - \mu I)^\nu\xi = 0;
\]

(2.4)

furthermore, $w := (A - \mu I)^{\nu-1}\xi$ is an eigenvector of $A$. For convenience, we call $\nu$ the order of $\xi$, denoted by $\text{Ord}(\xi)$. Define

\[
\text{Cord} (\text{GE}_\mu(A)) := \max_{\xi \in \text{GE}_\mu(A)} \text{Ord}(\xi).
\]

Cord $(\text{GE}_\mu(A))$ is called the cyclic order of $\text{GE}_\mu(A)$.

Denote by $E_\mu(A)$ the eigenspace of $A$ spanned by all the eigenvectors of $A$ corresponding to $\mu$. An eigenvector $w \in E_\mu(A)$ is called a dominant eigenvector, if there is a $\xi \in \text{GE}_\mu(A)$ such that

\[
w = (A - \mu I)^{\nu-1}\xi, \quad \text{where } \nu = \text{Cord} (\text{GE}_\mu(A)).
\]

The dominant eigenspace of $A$ associated with $\mu$, denoted by $\text{DE}_\mu(A)$, is the subspace of $E_\mu(A)$ spanned by the dominant eigenvectors of $A$ corresponding to $\mu$. Clearly

\[
\text{DE}_\mu(A) \subset E_\mu(A) \subset \text{GE}_\mu(A).
\]

The validity of the following simple fact is almost obvious.

**Lemma 2.1.** $(A - \mu I)^{\nu-1}\text{GE}_\mu(A) = \text{DE}_\mu(A)$, where $\nu = \text{Cord} (\text{GE}_\mu(A))$.

**Definition 2.2.** We say that $\mu$ is semisimple, if $\text{GE}_\mu(A) = E_\mu(A)$.

It is easy to see that if $\mu$ is semisimple, then

\[
\text{DE}_\mu(A) = E_\mu(A) = \text{GE}_\mu(A).
\]

(2.5)

- **The characteristic polynomial of a generalized eigenvector**
Let $A \in \mathbb{M}_m$, and let $\mu$ be a real eigenvalue of $A$. Given $\xi \in GE_{\mu}(A)$, set

$$Q_\xi(t) := \sum_{k=0}^{\nu-1} \frac{t^k}{k!} (A - \mu I)^k \xi,$$

where $\nu = \text{Ord}(\xi)$. (2.6)

$Q_\xi(t)$ is called the characteristic polynomial of $\xi$. Since $(A - \mu I)^k \xi = 0$ for $k \geq \nu$, we obviously have

$$Q_\xi(t) = \sum_{k=0}^{d} \frac{t^k}{k!} (A - \mu I)^k \xi = e^{t(A - \mu I)} \xi$$

for any integer $d \geq \nu - 1$.

Remark 2.3. It is clear that if $\xi \in E_{\mu}(A)$ then $Q_\xi(t) \equiv \xi$.

2.2. Perron-like matrices

Definition 2.4. A matrix $A \in \mathbb{M}_m$ is said to be Perron-like, if the spectral bound $s := s(A) \in \sigma(A)$; furthermore, $\Re \mu < s$ for all $\mu \in \sigma(A) \setminus \{s\}$.

For a Perron-like matrix $A$, we call $s := s(A)$ the principal eigenvalue of $A$ with the invariant subspaces $GE_s(A)$, $E_s(A)$ and $DE_s(A)$ being referred to as the principal generalized eigenspace, eigenspace, and dominant eigenspace of $A$, respectively. Correspondingly, each nonzero element in $GE_s(A)$ (resp. $E_s(A)$, $DE_s(A)$) is called a principal generalized eigenvector (resp. eigenvector, dominant eigenvector).

Now let us give several examples of Perron-like matrices.

Example 2.1. A symmetric matrix $A \in \mathbb{M}_m$ is Perron-like with each eigenvalue being semisimple.

Example 2.2. Every nonnegative matrix $A \in \mathbb{M}_m$ is a Perron-like one with $s(A) = r(A)$. This follows from some classical Krein-Rutman type theorems for matrices and bounded linear operators; see e.g. [15, Theorem 4.1].

Example 2.3. If all the off-diagonal entries of a matrix $A \in \mathbb{M}_m$ are nonnegative, then $A$ is a Perron-like matrix. Indeed, in such a case one can pick a positive number $a > 0$ sufficiently large so that $aI + A$ is nonnegative, from which and Example 2.2 one immediately concludes that $A$ is a Perron-like matrix with principal eigenvalue

$$s := s(A) = r(aI + A) - a.$$

Remark 2.5. One can easily give examples of Perron-like matrices that are not covered by Examples 2.1-2.3.
2.3. On the growth rate of exponential functions of matrices

The following lemma contains a well known fundamental fact concerning the growth rate of exponential functions of matrices, which actually holds true for general bounded linear operators and some types of unbounded operators in abstract Banach spaces; see e.g. Henry [12, Theorem 1.5.3]. Here we include a proof for the reader’s convenience.

Lemma 2.6. Let \( A \in \mathbb{M}_m \). Assume that \( s(A) \leq \lambda \). Then for any \( \varepsilon > 0 \), there is a constant \( C > 0 \) such that
\[
\|e^{tA}\| \leq Ce^{(\lambda + \varepsilon)t}, \quad t \geq 0.
\]

Proof. We think of \( A \) as a linear transform (operator) on \( \mathbb{C}^m \). Let \( \mu = \alpha + i\beta \in \sigma(A) \), and \( \xi \in \mathbb{GE}_\mu(A) \). Then \( (A - \mu I)^\nu \xi = 0 \) for some \( \nu \geq 1 \). Hence
\[
e^{tA}\xi = e^{\mu t}e^{t(A-\mu I)}\xi = e^{\mu t} \sum_{k=0}^{\infty} \frac{t^k}{k!}(A-\mu I)^k\xi
= e^{\mu t}Q_\xi(t) = e^{(\alpha + \varepsilon)t}(e^{-\varepsilon t}e^{i\beta t}Q_\xi(t)),
\]
where \( Q_\xi(t) = \sum_{k=0}^{\nu-1} \frac{t^k}{k!}(A-\mu I)^k\xi \). Since \( Q_\xi(t) \) has only a polynomial growth, one easily sees that there is \( C_0 > 0 \) such that \( \|e^{-\varepsilon t}e^{i\beta t}Q_\xi(t)\| \leq C_0\|\xi\| \) for \( t \geq 0 \). Thus by (2.9) we have
\[
\|e^{tA}\| \leq C_0 e^{(\alpha + \varepsilon)t}\|\xi\| \leq C_0 e^{(\lambda + \varepsilon)t}\|\xi\|, \quad t \geq 0.
\]

We infer from the basic knowledge in linear algebra that the space \( E := \mathbb{C}^m \) has a basis \( \{\xi_1, \xi_2, \ldots, \xi_m\} \), where each \( \xi_i \) belongs to some invariant subspace \( \mathbb{GE}_\mu(A) \). For each \( \xi_i \), by (2.10) there is a constant \( C_i > 0 \) such that
\[
\|e^{tA}\xi_i\| \leq C_ie^{(\lambda + \varepsilon)t}\|\xi_i\|, \quad t \geq 0.
\]

Now we write each \( u \in E \) as \( u = \sum_{1 \leq i \leq m} a_i \xi_i \), where \( a_i \in \mathbb{C} \). Define \( \|u\| = \sum_{1 \leq i \leq m} |a_i| \). Then \( \|\cdot\| \) is an equivalent norm on \( E \). Let \( C' = \max_{1 \leq i \leq m} C_i \). Thanks to (2.11), we deduce that
\[
\|e^{tA}u\| = \left\| \sum_{1 \leq i \leq m} a_ie^{tA}\xi_i \right\| \leq C'e^{(\lambda + \varepsilon)t}\|u\| \leq C'e^{(\lambda + \varepsilon)t}\|u\|, \quad \forall u \in E, \ t \geq 0,
\]
from which we see that the operator norm of \( e^{tA} \) (as an operator on \( \mathbb{C}^m \)) is dominated by \( C'e^{(\lambda + \varepsilon)t} \). Consequently the operator norm of \( e^{tA} \) (as an operator on \( \mathbb{R}^m \)) is dominated by \( C'e^{(\lambda + \varepsilon)t} \) as well. The estimate in (2.8) then immediately follows because all the norms on a finite-dimensional Banach space are equivalent. ■
3. Fundamental Convergence Results for Perron-like Matrices

In this section we prove some fundamental convergence results mentioned in the introduction for Perron-like matrices, which serve as the starting point of this work. As an interesting corollary, we also give a generalized Perron-like theorem for nonnegative matrices.

The following notations will be used throughout the paper.

- The notation $O(\varepsilon)$ will stand for a general infinitesimal as $\varepsilon \to 0$ that can be dominated by $C|\varepsilon|$ for some $C > 0$.

- We denote by $X^T$ the transpose of a vector (or, matrix) $X$.

3.1. Preliminaries

For simplicity, we write $E := \mathbb{R}^m$, and let $J = \{1, 2, \ldots, m\}$. Assume that $A \in \mathbb{M}_m$ is a Perron-like matrix with the principal eigenvalue $s := \sigma(A)$.

Set $\sigma_1 = \{s\}$, and $\sigma_2 = \sigma(A) \setminus \{s\}$. Then $E$ has a direct sum decomposition $E = E_1 \oplus E_2$ corresponding to the spectral decomposition $\sigma(A) = \sigma_1 \cup \sigma_2$.

$$E_1 = GE_\sigma(A), \quad E_2 = \bigoplus_{\mu \in \sigma_2} GE_\mu(A). \quad (3.1)$$

Denote by $\Pi_j$ ($j = 1, 2$) the projection from $E$ to $E_j$.

Let $\{v_i\}_{i \in J}$ be a basis of $E$. First, we have the following easy lemma.

**Lemma 3.1.** $\text{span}\{\Pi_j v_1, \ldots, \Pi_j v_m\} = E_j$, $j = 1, 2$.

**Proof.** Suppose, say, that $\text{span}\{\Pi_1 v_1, \ldots, \Pi_1 v_m\} := E_0 \subsetneq E_1$. Then

$$\text{span}\{v_1, \ldots, v_m\} \subset \bigoplus_{j=1,2} \text{span}\{\Pi_j v_1, \ldots, \Pi_j v_m\} \subset E_0 \oplus E_2 \neq E,$$

which leads to a contradiction. $\blacksquare$

Let

$$V = [v_i]_{i \in J}, \quad Y = [y_i]_{i \in J}, \quad Z = [z_i]_{i \in J},$$

where

$$y_i = \Pi_1 v_i, \quad z_i = \Pi_2 v_i.$$

Set

$$Q_Y(t) = [Q_{y_i}(t)]_{i \in J}, \quad t \geq 0,$$

where $Q_{y_i}(t)$ is the (vector-valued) characteristic polynomial of $y_i$ as defined in \cite{27}. Put

$$x_i(t) = e^{tA}v_i \quad (i \in J), \quad X(t) = [x_i(t)]_{i \in J}. \quad (3.2)$$

It is trivial to see that $X(t) = e^{tA}V$. 

3.2. Some fundamental convergence theorems

We now state and prove the first convergence result.

**Theorem 3.2.** There exist $B_0, \delta > 0$ such that

$$\left\| \frac{x_i(t)}{\|X(t)\|} - \frac{Qy_i(t)}{\|QY(t)\|} \right\| \leq B_0 e^{-\delta t}, \quad t \geq 1, \ i \in J. \quad (3.3)$$

**Proof.** Let $A_2 = A|_{E_2}$ be the restriction of $A$ on $E_2$. Clearly $\sigma(A_2) = \sigma_2$. Noticing that $\text{Re}\mu < s$ for all $\mu \in \sigma(A_2)$, one can pick a $\delta > 0$ such that

$$\text{Re}\mu \leq s - 2\delta, \quad \mu \in \sigma(A_2). \quad (3.4)$$

Lemma 2.6 then asserts that there is $C_1 > 0$ such that

$$\|e^{tA_2}\| \leq C_1 e^{(s-\delta)t}, \quad t \geq 0. \quad (3.5)$$

For each $i \in J$, we observe that

$$x_i(t) = e^{tA}v_i = e^{tA}y_i + e^{tA}z_i \quad = e^{st} \sum_{k=0}^{\infty} \frac{(sI-A)^k y_i + e^{tA_2}z_i}{k!} \quad (3.6)$$

For notational simplicity, let us write

$$\nu_i = \text{Ord}(y_i), \quad \text{and} \quad \nu = \text{Cord}(\text{GE}_s(A)).$$

Set $\tilde{x}_i(t) = e^{-st}t^{-(\nu-1)}x_i(t)$. Then by (3.6) we have

$$\tilde{x}_i(t) = p_i(t) + \tilde{z}_i(t), \quad t > 0. \quad (3.7)$$

where

$$p_i(t) = t^{-(\nu-1)}Qy_i(t), \quad \tilde{z}_i(t) = e^{-st}t^{-(\nu-1)}e^{tA_2}z_i. \quad (3.8)$$

By (3.5) one finds that

$$\|\tilde{z}_i(t)\| \leq C_1 e^{-\delta t}t^{-(\nu-1)}\|z_i\| \leq C_1 e^{-\delta t}\|Qy_i\| \leq C_2 \|y_i\|e^{-\delta t}, \quad t \geq 1, \quad (3.9)$$

where $C_2 = C_1 \|Q\|$. Thus if we write

$$\|\tilde{x}_i(t)\| = \|p_i(t)\| + r_i(t),$$

then by (3.7) it is easy to deduce that

$$|r_i(t)| \leq \|\tilde{z}_i(t)\| \leq C_2 \|y_i\|e^{-\delta t}, \quad i \in J. \quad (3.10)$$
Let \( \|\tilde{X}(t)\| := e^{-st}t^{-(\nu-1)}\|X(t)\| \) \((t > 0)\). Obviously
\[
\|\tilde{X}(t)\| = \left( \sum_{i \in J} \|\tilde{x}_i(t)\|^2 \right)^{1/2} = \left( \sum_{i \in J} (\|p_i(t)\| + r_i(t))^2 \right)^{1/2}. \tag{3.11}
\]
It follows by the trigonal inequality of the norm \( \| \cdot \| \) that
\[
g(t) - r(t) \leq \|\tilde{X}(t)\| \leq g(t) + r(t), \quad t \geq 1, \tag{3.12}
\]
where
\[
g(t) = \left( \sum_{i \in J} ||p_i(t)||^2 \right)^{1/2}, \quad r(t) = \left( \sum_{i \in J} |r_i(t)|^2 \right)^{1/2}. \tag{3.13}
\]
We now evaluate \( \left\| \frac{x_i(t)}{\|X(t)\|} - \frac{p_i(t)}{g(t)} \right\| \). For this purpose, let us write
\[
\|\tilde{X}(t)\| = g(t) + R(t). \tag{3.14}
\]
Then by (3.10) and (3.12) it is easy to deduce that
\[
|R(t)| \leq r(t) \leq C_2\|V\|e^{-\delta t}, \quad t \geq 1. \tag{3.15}
\]
Observe that
\[
\left\| \frac{x_i(t)}{\|X(t)\|} - \frac{p_i(t)}{g(t)} \right\| = \left\| \frac{\tilde{x}_i(t)}{\|X(t)\|} - \frac{p_i(t)}{g(t)} \right\| = \left\| \frac{p_i(t)+\tilde{x}_i(t)}{g(t)+R(t)} - \frac{p_i(t)}{g(t)} \right\| \tag{3.16}
\]
\[
= \frac{\|g(t)\tilde{x}_i(t) - R(t)p_i(t)\|}{\|g(t)+R(t)\|g(t)}. \]
Since \( ||p_i(t)|| \leq g(t) \), we have
\[
\|g(t)\tilde{x}_i(t) - R(t)p_i(t)\| \leq g(t) (||\tilde{x}_i(t)|| + |R(t)|). \]
Therefore by (3.10) one gets that
\[
\left\| \frac{x_i(t)}{\|X(t)\|} - \frac{p_i(t)}{g(t)} \right\| \leq \frac{\|\tilde{x}_i(t)||+|R(t)|\|X(t)\|}{g(t)+R(t)} \tag{3.17}
\]
\[
\leq (m+1)C_2\|V\|\|X(t)\|e^{-\delta t}, \quad t \geq 1.
\]
Set \( J_s = \{ i \in J : \nu_i = \nu \} \), and let
\[
w_i = \frac{1}{(\nu - 1)!} (A - sI)^{\nu-1} y_i, \quad i \in J_s. \tag{3.18}
\]
We claim that
\[
\text{DE}_s(A) = \text{span}\{ w_i : i \in J_s \}. \tag{3.19}
\]
Indeed, by Lemma 2.1 and Lemma 3.1 we have
\[
DE_s(A) = (A - sI)^{\nu - 1}GE_s(A) = (A - sI)^{\nu - 1}E_1
\]
\[
= (A - sI)^{\nu - 1}(\text{span}\{y_i : i \in J\})
\]
\[
= \text{span}\{(A - sI)^{\nu - 1}y_i : i \in J\}
\]
\[
= \text{span}\{(A - sI)^{\nu - 1}y_i : i \in J_s\}
\]
\[
= \text{span}\{w_i : i \in J_s\}.
\]
(3.20)

Hence the claim holds true.

For convenience, we assign
\[
w_i = 0, \quad i \in J \setminus J_s.
\]
(3.21)

Then by the definition of \(Q_{y_i}(t)\) one can rewrite
\[
p_i(t) := t^{-(\nu - 1)}Q_{y_i}(t)
\]
as
\[
p_i(t) = w_i + h_i(t), \quad i \in J.
\]
(3.22)

where
\[
h_i(t) = \begin{cases} 
  t^{-(\nu - 1)} \sum_{k=0}^{\nu - 2} \frac{t^k}{k!}(A - sI)^k y_i, & i \in J_s; \\
  t^{-(\nu - 1)} \sum_{k=0}^{\nu - 1} \frac{t^k}{k!}(A - sI)^k y_i, & \text{otherwise},
\end{cases}
\]

In any case it can be easily seen that
\[
\|h_i(t)\| \leq \|y_i\||O(t^{-1})| \leq \|v_i\||O(t^{-1}).
\]
(3.23)

Therefore by (3.14) and (3.15) we deduce that
\[
\lim_{t \to \infty} \|X(t)\| = \lim_{t \to \infty} \|QY(t)\| = \omega \quad \text{(by (3.20))} > 0,
\]
(3.24)

where
\[
\omega = (\sum_{i \in J} \|w_i\|^2)^{1/2} = (\sum_{i \in J_s} \|w_i\|^2)^{1/2}.
\]
(3.25)

Because \(\|X(t)\|, \|QY(t)\| > 0\) for all \(t \geq 1\), (3.24) implies that there is a positive number \(\kappa > 0\) such that
\[
\|X(t)\|, \|QY(t)\| \geq \kappa > 0, \quad t \geq 1,
\]
(3.26)

Thus by (3.17) one concludes that
\[
\left\| \frac{x_i(t)}{\|X(t)\|} - \frac{p_i(t)}{\|QY(t)\|} \right\| \leq C_3\|V\|e^{-\delta t}, \quad t \geq 1,
\]
(3.27)

where \(C_3 = (m + 1)C_2/\kappa\). By (3.28) and (3.13) it can be easily seen that
\[
\frac{p_i(t)}{\|QY(t)\|} = \frac{Q_{y_i}(t)}{\|QY(t)\|},
\]
(3.28)

from which and (3.27) the convergence result in (3.3) immediately follows. □
Notation. In the remaining part of this paper the notation $\delta$ will always stand for the positive constant given in (3.4) for a given Perron-like matrix $A$, which is determined by the spectral gap between the principal eigenvalue $s$ and the other eigenvalues of $A$.

Let us proceed with the argument in the proof of Theorem 3.2. Note that $\varrho(t)$ (see (3.13)) can be rewritten as $\varrho(t) = \omega + \varrho_{0}(t)$, where $\omega$ is number given in (3.25). By (3.22) and (3.23) one easily checks that

$$|\varrho_{0}(t)| \leq \frac{\left(\sum_{i \in J} \|h_{i}(t)\|^2\right)^{1/2}}{\omega + \varrho_{0}(t)} \leq \|V\|O(t^{-1}).$$  \hspace{1cm} (3.29)

Now we put

$$\xi_{i}(t) = \omega^{-1} w_{i}, \quad i \in J. \hspace{1cm} (3.30)$$

We show that for each $i \in J$,

$$\left\| \frac{p_{i}(t)}{\varrho(t)} - \xi_{i} \right\| = O(t^{-1}) \quad \text{as } t \to \infty. \hspace{1cm} (3.31)$$

Indeed, by (3.23) and (3.29) we have

$$\left\| \frac{p_{i}(t)}{\varrho(t)} - \xi_{i} \right\| = \left\| \frac{w_{i} + h_{i}(t)}{\omega + \varrho_{0}(t)} - \frac{w_{i}}{\omega} \right\| = \left\| \frac{\omega h_{i}(t) - \varrho_{0}(t) w_{i}}{\omega + \varrho_{0}(t)} \right\| \leq \frac{\omega \|h_{i}(t)\| + |\varrho_{0}(t)| \|w_{i}\|}{\omega + \varrho_{0}(t)} \leq \frac{\|h_{i}(t)\| + |\varrho_{0}(t)|}{\varrho(t)} \leq \kappa^{-1} \|V\|O(t^{-1}) \quad \text{as } t \to \infty,$$

where $\kappa$ is the constant given in (3.26). This is precisely what we want.

Combining (3.27) and (3.31), it yields

$$\left\| \frac{x_{i}(t)}{X(t)} - \xi_{i} \right\| \leq O(t^{-1}) + B_{1} e^{-\delta t}, \quad t \geq 1. \hspace{1cm} (3.32)$$

We infer from (3.19) that

$$\text{DE}_{s}(A) = \text{span}\{\xi_{i} : \ i \in J_{s}\}. \hspace{1cm} (3.33)$$

It is also clear that

$$\sum_{i \in J} \|\xi_{i}\|^2 = \sum_{i \in J_{s}} \|\xi_{i}\|^2 = 1. \hspace{1cm} (3.34)$$

Write

$$\psi_{i}(t) = \frac{x_{i}(t)}{\|X(t)\|}, \quad i \in J.$$
Since \( \xi_i \in \text{DE}_s(A) \) for all \( i \in J \) (note that \( \xi_i = 0 \) for \( i \in J \setminus J_s \)), we have
\[
\left| \sum_{i \in J} \langle A \psi_i(t), \psi_i(t) \rangle - s \right| = \left| \sum_{i \in J} \langle A \psi_i(t), \psi_i(t) \rangle - s \sum_{i \in J} \| \xi_i \|^2 \right|
\leq \sum_{i \in J} \| A(\psi_i(t) - \xi_i), \psi_i(t) \| + \sum_{i \in J} \| A(\xi_i, \psi_i(t) - \xi_i) \|
\leq \sum_{i \in J} \| A(\psi_i(t) - \xi_i) \| + \sum_{i \in J} \| A \xi_i \| \| \psi_i(t) - \xi_i \|
\leq 2 \| A \| \sum_{i \in J} \| \psi_i(t) - \xi_i \|
\leq (\text{by (3.32)}) \leq O(t^{-1}) + 2mB_1\| A \| e^{-\delta t}, \quad t \geq 1.
\]

Summarizing the above results, one obtains the following theorem.

**Theorem 3.3.** There exist \( B_1, B_2 > 0 \) and \( \xi_i \in E_s(A) \) (\( i \in J \)) with
\[
\sum_{i \in J} \| \xi_i \|^2 = 1 \quad \text{and} \quad \text{DE}_s(A) = \text{span}\{ \xi_i : i \in J \} \quad (3.35)
\]
such that
\[
\left\| \frac{x_i(t)}{\| X(t) \|} - \xi_i \right\| \leq O(t^{-1}) + B_1e^{-\delta t}, \quad t \geq 1, \quad (3.36)
\]
and
\[
\left| \frac{1}{\| X(t) \|^2} \sum_{i \in J} \langle Ax_i(t), x_i(t) \rangle - s \right| \leq O(t^{-1}) + B_2e^{-\delta t}, \quad t \geq 1. \quad (3.37)
\]

3.3. *The case of a semisimple principal eigenvalue*

Now we consider the particular case where \( s \) is semisimple. In such a case, every \( y_i := \Pi_1v_i \) with \( y_i \neq 0 \) in the above argument is an eigenvector of \( A \). Hence \( Q_{y_i}(t) \equiv y_i \) for \( t \geq 0 \). Consequently (3.3) reads as
\[
\left\| \frac{x_i(t)}{\| X(t) \|} - \xi_i \right\| \leq Be^{-\delta t}, \quad t \geq 1, \quad (3.38)
\]
where
\[
\xi_i = \frac{y_i}{\| Y \|}.
\]
This also leads to a corresponding strengthened version of (3.37). Therefore we have the following theorem.
Theorem 3.4. Suppose $s$ is semisimple. Then there exist $B_1, B_2 > 0$ such that

$$\left\| \frac{x_i(t)}{\|X(t)\|} - \xi_i \right\| \leq B_1 e^{-\delta t}, \quad t \geq 1,$$

and

$$\left\| \frac{1}{\|X(t)\|^2} \sum_{i \in J} (Ax_i(t), x_i(t)) - s \right\| \leq B_2 e^{-\delta t}, \quad t \geq 1.$$

Remark 3.5. Recall that in the semisimple case we have $GE_s(A) = E_s(A)$. Therefore by Lemma 3.1 one concludes that

$$\text{span}\{y_i : i \in J\} = E_s(A).$$

3.4. A generalized Perron-like theorem for nonnegative matrices

As a corollary of Theorem 3.3, we now state a generalized Perron-like theorem for nonnegative matrices. Let

$$\mathbb{R}_+^m = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : x_i \geq 0 \text{ for } i \in J\}.$$ 

A vector $x \in \mathbb{R}^m$ is said to be nonnegative, this means that $x \in \mathbb{R}_+^m$.

Theorem 3.6. Let $A \in \mathbb{M}_m$ be a nonnegative matrix. Then $r := r(A) \in \sigma(A)$, and the principal dominant eigenspace $DE_r(A)$ has a basis consisting of nonnegative eigenvectors.

In particular, if $r$ is semisimple, then the principal eigenspace $E_r(A)$ has a basis consisting of nonnegative eigenvectors.

To prove Theorem 3.6, we need a well known positivity property of exponential functions of nonnegative matrices.

Lemma 3.7. Let $A \in \mathbb{M}_m$ be nonnegative. Then for any $a \in \mathbb{R}$,

$$e^{t(A+at)} \mathbb{R}_+^m \subseteq \mathbb{R}_+^m, \quad t \geq 0.$$ (3.39)

Proof. We include a simple proof for the reader’s convenience.

Let $P = \mathbb{R}_+^m$. Since $e^{t(A+at)} P = e^{at}(e^{tA}P)$ and $e^{at} > 0$, to prove (3.39), it suffices to verify that $e^{tA}P \subseteq P$. So let $x \in P$, and $t \geq 0$. By the nonnegativity of $A$ we see that $A^k x \in P$ (and hence $\frac{t^k}{n!}A^k x \in P$) for all $k \in \mathbb{N}$. Consequently

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}A^k x \in P$$

for $n \in \mathbb{N}$. Since $P$ is closed, setting $n \to \infty$ one immediately concludes that $e^{tA}x \in P$. ■
Proof of Theorem 3.6. We infer from Example 2.2 that $A$ is a Perron-like matrix with $s := s(A) = r(A) := r$. To prove the theorem, there remains to check that $DE_s(A)$ has a basis consisting of nonnegative eigenvectors.

For this purpose, we take

$$v_i = (0, \cdots, 0, 1, 0, \cdots, 0)^T, \quad i \in J.$$

By Lemma 3.7 we have $x_i(t) = e^{tA}v_i \in P := \mathbb{R}_m^+$ for all $t \geq 0$. Consequently $\|x_i(t)\|_P \in P$ for $t \geq 0$. Let $\xi_i$ ($i \in J$) be the vectors given by Theorem 3.3. Then by (3.35) we deduce that $\{\xi_i : i \in J\} \subset P$. This and (3.35) complete the proof of what we desired.

Remark 3.8. As a corollary of Theorem 3.6, we deduce that the principal eigenspace $E_s(A)$ of a nonnegative symmetric matrix $A$ has a basis consisting of nonnegative eigenvectors.

4. Computing Principal Eigenvalues and Eigenvectors via Polynomial Approximations of Matrix Exponentials

Based on the convergence results in Section 3, we now show that the principal eigenvalues and eigenvectors of Perron-like matrices can be computed via polynomial approximations of matrix exponentials.

We employ the same notations as in Section 3.

Let $A \in \mathbb{M}_m$ be a Perron-like matrix with the principal eigenvalue $s := s(A)$, and $\{v_i\}_{i \in J}$ a basis of $E := \mathbb{R}^m$. Let $y_i, x_i(t), V, Y$ and $X(t)$ be the same as in Section 3 and $\xi_i$ ($i \in J$) the vectors given in Theorem 3.3. Set

$$\Xi = [\xi_i]_{i \in J}, \quad q(t) = [q_i(t)]_{i \in J}, \quad (4.1)$$

where $q_i(t) = \frac{Q_i(y_i)}{\|Q_i(y_i)\|}$. Then the convergence results in (3.3), (3.35) and (3.37) can be reformulated as

$$\left\| \frac{X(t)}{\|X(t)\|} - q(t) \right\| \leq B_0 e^{-\delta t}, \quad t \geq 1, \quad (4.2)$$

$$\left\| \frac{X(t)}{\|X(t)\|} - \Xi \right\| \leq O(t^{-1}) + B_1 e^{-\delta t}, \quad t \geq 1, \quad (4.3)$$

and

$$\frac{1}{\|X(t)\|^2} \langle AX(t), X(t) \rangle - s \leq O(t^{-1}) + B_2 e^{-\delta t}, \quad t \geq 1. \quad (4.4)$$
Consider the polynomial approximations of $X(t)$:

$$X_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} A^k V, \quad n \in \mathbb{N}.$$  

For computational convenience, in what follows we put $\|V\| = 1$.

Let $\kappa$ be the positive constant in (3.26), and write

$$\kappa^{-1} e^{2\|A\| t} \left( \frac{e\|A\| t}{n+1} \right)^{n+1} := \vartheta_n(t). \quad (4.5)$$

**Theorem 4.1.** The following estimates hold true for all $n \in \mathbb{N}$:

$$\left\| \frac{X_n(t)}{\|X_n(t)\|} - q(t) \right\| \leq B_0 e^{-\delta t} + \vartheta_n(t) o(1), \quad t \geq 1, \quad (4.6)$$

$$\left\| \frac{X_n(t)}{\|X_n(t)\|} - \Xi \right\| \leq O(t^{-1}) + B_1 e^{-\delta t} + \vartheta_n(t) o(1), \quad t \geq 1, \quad (4.7)$$

and

$$\frac{1}{\|X_n(t)\|^2} \langle AX_n(t), X_n(t) \rangle - s \leq O(t^{-1}) + B_2 e^{-\delta t} + \|A\| \vartheta_n(t) o(1), \quad t \geq 1, \quad (4.8)$$

where $o(1)$ is an infinitesimal as $n \to \infty$ which depends only upon $n$.

**Proof.** We observe that

$$X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k V = X_n(t) + R_n(t)V,$$

where $R_n(t) = \sum_{k=n+1}^{\infty} \frac{t^k}{k!} A^k$. Note that

$$\|R_n(t)V\| \leq \sum_{k=n+1}^{\infty} \frac{t^k}{k!} \|A^k V\| \leq \sum_{k=n+1}^{\infty} \frac{t^k}{k!} \|A\|^k \leq \frac{\|A\|^n}{(n+1)!} e^{t\|A\|}, \quad t \geq 0. \quad (4.9)$$

If we write $\|X(t)\| = \|X_n(t)\| + r_n(t)$, then

$$|r_n(t)| \leq \|X(t) - X_n(t)\| = \|R_n(t)V\|. \quad (4.10)$$

Let

$$\tilde{R}_n(t)V = e^{-st} t^{-(\nu-1)} R_n(t)V, \quad \tilde{r}_n(t) = e^{-st} t^{-(\nu-1)} r_n(t),$$

$$\tilde{R}_n(t)V = e^{-st} t^{-(\nu-1)} R_n(t)V, \quad \tilde{r}_n(t) = e^{-st} t^{-(\nu-1)} r_n(t),$$

18
where $\nu := \text{Cord}(\text{GE}_s(A))$ is the cyclic order of $\text{GE}_s(A)$. Then
\begin{align*}
|\tilde{r}_n(t)| &\leq \left\| \bar{R}_n(t)V \right\| = e^{-st}t^{-(\nu-1)} \left\| R_n(t)V \right\| \\
&\leq \frac{(t\|A\|)^{n+1}}{(n+1)!} e^{\|A\|s} t \\
&\leq \frac{(t\|A\|)^{n+1}}{(n+1)!} e^{2\|A\|t}, \quad t \geq 1.
\end{align*}
(4.11)

Here we have used the simple fact that
\[ |s| \leq r(A) \leq \|A\|. \]

Set
\[ \tilde{X}(t) = e^{-st}t^{-(\nu-1)}X(t), \quad \tilde{X}_n(t) = e^{-st}t^{-(\nu-1)}X_n(t). \]

Then
\begin{align*}
\left\| \frac{X(t)}{\|X(t)\|} - \frac{X_n(t)}{\|X_n(t)\|} \right\| &= \left\| \frac{\tilde{X}(t)}{\|\tilde{X}(t)\|} - \frac{\tilde{X}_n(t)}{\|\tilde{X}_n(t)\|} \right\| \\
&= \left\| \tilde{X}_n(t) \frac{\|X(t)\|}{\|\tilde{X}(t)\|} - \|\tilde{X}(t)\| \frac{\tilde{X}_n(t)}{\|\tilde{X}_n(t)\|} \right\| \\
&\leq (\text{by } (4.10)) \leq 2\left\| \tilde{R}_n(t)V \right\| \frac{\|\tilde{X}(t)\|}{\|X(t)\|}.
\end{align*}
(4.12)

By (3.26) we have $\|\tilde{X}(t)\| \geq \kappa > 0$ for $t \geq 1$. Thus by (4.11) and (4.12) one concludes that
\[ \left\| \frac{X(t)}{\|X(t)\|} - \frac{X_n(t)}{\|X_n(t)\|} \right\| \leq \frac{2}{\kappa} \frac{(t\|A\|)^{n+1}}{(n+1)!} e^{2\|A\|t}, \quad t \geq 1. \]
(4.13)

Combining this with (4.2) we arrive at the following estimates:
\[ \left\| \frac{X_n(t)}{\|X_n(t)\|} - q(t) \right\| \leq B_0 e^{-\delta t} + \frac{2}{\kappa} \frac{(t\|A\|)^{n+1}}{(n+1)!} e^{2\|A\|t}, \quad t \geq 1, \]
(4.14)

and
\[ \left\| \frac{X_n(t)}{\|X_n(t)\|} - \Xi \right\| \leq O(t^{-1}) + B_1 e^{-\delta t} + \frac{2}{\kappa} \frac{(t\|A\|)^{n+1}}{(n+1)!} e^{2\|A\|t}, \quad t \geq 1. \]
(4.15)

Thanks to the classical Stirling’s formula, we have
\[ \lim_{n \to \infty} \frac{n!}{\sqrt{2\pi} n^{n+1}} e^{-n} = \sqrt{2\pi}, \]
by which one finds that
\[ \frac{(t\|A\|)^{n+1}}{(n+1)!} = \left( \frac{e\|A\|t}{n+1} \right)^{n+1} o(1), \]

19
where \( o(1) \) is an infinitesimal as \( n \to \infty \) depending only upon \( n \). The first and second estimates in the theorem now follow from (4.14) and (4.15).

Using (4.7) and a similar argument as in the verification of (3.37), one can obtain the third estimate in Theorem 4.1. We omit the details. ■

In the case where \( s \) is semisimple, by virtue of Theorem 3.4 one can actually remove the terms \( O(t^{-1}) \) in the righthand sides of (4.7) and (4.8). In other words, we have a strengthened version of Theorem 4.1:

**Theorem 4.2.** Assume that \( s \) is semisimple. Then

\[
\left\| \frac{X_n(t)}{\|X_n(t)\|} - \frac{Y}{\|Y\|} \right\| \leq B_1 e^{-\delta t} + \vartheta_n(t)o(1), \quad t \geq 1,
\]

and

\[
\left| \frac{1}{\|X_n(t)\|^2} \langle AX_n(t), X_n(t) \rangle - s \right| \leq B_2 e^{-\delta t} + \|A\|\vartheta_n(t)o(1), \quad t \geq 1.
\]

5. **An Iterative Scheme**

Theoretically the convergence results in Sections 3 and 4 already provide a dynamical way for computing the principal eigenvalue and the principal dominant eigenvectors of a matrix. However, there is the danger of overflow in calculating \( X(t) \) and its polynomial approximations \( X_n(t) \) when \( t \) and \( n \) are large enough. To overcome this drawback, we develop an iterative scheme with double indices corresponding to \( t \) and \( n \) which has a more rapid exponential convergence rate than the one given in Theorem 4.1.

5.1. **An iterative scheme**

We continue the argument in Section 4. Let \( A \in \mathbb{M}_m \) be a Perron-like matrix, and set \( T = e^A \). Consider the polynomial approximations of \( T \):

\[
T_n = \sum_{k=0}^{n} \frac{1}{k!}A^k, \quad n \in \mathbb{N}.
\]

For each \( n \in \mathbb{N} \), we have

\[
T = T_n + R_n, \quad \text{where} \quad R_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}A^k.
\]

Similar calculations as in (1.9) yield

\[
\|R_nM\| \leq \left( \frac{\|A\|^{n+1}}{(n+1)!} e^{\|A\|} \right) \|M\|, \quad \forall M \in \mathbb{M}_m.
\]
Let
\[ \Sigma_1 = \{ M \in M_m : \|M\| = 1 \}. \]

Given \( M \in \Sigma_1 \), we write
\[ \|TM\| = \|T_n M\| + r_n(M). \]

Then
\[ |r_n(M)| \leq \|TM\| - \|T_n M\| \leq \|TM - T_n M\| \]
\[ = \|R_n M\| \leq \|A\|^{n+1} \frac{e^{\|A\|}}{(n+1)!}. \]

(5.1)

Define two mappings \( K \) and \( K_n \) on \( \Sigma_1 \) as follows: \( \forall M \in \Sigma_1 \),
\[ KM = \frac{TM}{\|TM\|}, \quad K_n M = \frac{T_n M}{\|T_n M\|}. \]

Using some similar calculations as in (4.12) one can obtain that
\[ \|KM - K_n M\| \leq \frac{2}{\|A\|} \|R_n M\| \leq o_n, \quad M \in \Sigma_1. \]
\[ (5.2) \]

Observing that
\[ \|M\| = \|T^{-1}TM\| \leq \|T^{-1}\| \|TM\|, \quad M \in \Sigma_1, \]
we deduce that
\[ \|TM\| \geq \|T^{-1}\|^{-1} \|M\| \geq e^{-\|A\|}, \quad M \in \Sigma_1. \]
\[ (5.3) \]

Thus by (5.1) and (5.2) it follows that
\[ \|KM - K_n M\| \leq \lambda \frac{\|A\|^{n+1}}{(n+1)!} := o_n, \quad n \in \mathbb{N}, \]
\[ (5.4) \]

where
\[ \lambda = 2e^{2\|A\|} \geq 2. \]
\[ (5.5) \]

Now let \( V = [v_i]_{i \in J} \in \Sigma_1 \) be a nonsingular matrix. Define an iteration sequence with double indices as below:
\[ M_0(0) = V, \quad M_n(k+1) = K_n M_n(k), \quad k, n \in \mathbb{N}. \]
\[ (5.6) \]

Let \( q(t) \) and \( \Xi \) be the same as in Theorem 4.1.

**Theorem 5.1.** Let \( W_n = M_n(n) \). Then for \( n \in \mathbb{N} \), we have
\[ \|W_n - q(n)\| \leq B_0 e^{-\delta n} + \left( \frac{\|A\|}{n+1} \right)^{n+1} o(1), \]
\[ (5.7) \]
\[ \| W_n - \Xi \| \leq O(n^{-1}) + B_1 e^{-\delta n} + \left( \frac{\lambda \| A \|}{n+1} \right)^{n+1} o(1), \]  
(5.8)

and

\[ |\langle AW_n, W_n \rangle - s| \leq O(n^{-1}) + B_2 e^{-\delta n} + \| A \| \left( \frac{\lambda \| A \|}{n+1} \right)^{n+1} o(1), \]  
(5.9)

where \( o(1) \) is an infinitesimal as \( n \to \infty \) depending only upon \( n \).

**Proof.** Define a sequence \( \{ M(k) \}_{k \in \mathbb{N}} \) as follows:

\[ M(0) = V, \quad M(k+1) = KM(k) \quad (k \in \mathbb{N}). \]

Then

\[ M(k) = \frac{TM(k-1)}{\| TM(k-1) \|} = \frac{T}{\| T \|} \left( \frac{TM(k-2)}{\| TM(k-2) \|} \right) \]
\[ = \frac{T^2 M(k-2)}{\| T^2 M(k-2) \|} = \cdots = \frac{T^k M_0}{\| T^k M_0 \|} = e^{kA}V. \]

Hence by virtue of (4.2) and (4.3) we have

\[ \| M(k) - q(k) \| \leq B_0 e^{-\delta k} \]  
(5.10)

and

\[ \| M(k) - \Xi \| \leq O(k^{-1}) + B_1 e^{-\delta k}. \]  
(5.11)

In the sequel we give an estimate for \( \| M_n(k) - M(k) \| \). For notational simplicity, we rewrite \( M_n(k) = \tilde{M}(k) \). Then

\[ \| \tilde{M}(k) - M(k) \| \]
\[ = \| K_n \tilde{M}(k-1) - K M(k-1) \|
\[ \leq \| K_n \tilde{M}(k-1) - K \tilde{M}(k-1) \| + \| K \tilde{M}(k-1) - K M(k-1) \|
\[ \leq (\text{by (5.4)}) \leq o_n + \| K \tilde{M}(k-1) - K M(k-1) \|, \]  
(5.12)

where \( o_n \) is the same as in (5.4). On the other hand,

\[ \| KX - KY \| = \| \frac{\| TY \|}{\| TX \|} (TX - TY) + \frac{\| TY \|}{\| TX \|} TY \| \]
\[ \leq \| TX - TY \| \| TX \| + \| TY \| \| TX \| \| TY \|
\[ \leq \frac{2\| TX - TY \| \| TX \|}{\| TX \|} \leq \frac{2\| T \|}{\| TX \|} \| X - Y \|
\[ \leq (\text{by (5.6)}) \leq \lambda \| X - Y \|, \quad \forall X, Y \in \Sigma_1. \]
Hence by (5.12) one has
\[
\| \tilde{M}(k) - M(k) \| \leq o_n + \lambda \| \tilde{M}(k - 1) - M(k - 1) \| \leq \cdots \\
\leq o_n (1 + \lambda + \lambda^2 + \cdots + \lambda^{k-2}) + \lambda^{k-1} \| \tilde{M}(1) - M(1) \|.
\]
Because \( \tilde{M}(0) = M_n(0) = V = M(0) \), by (5.4) we deduce that
\[
\| \tilde{M}(1) - M(1) \| = \| K_n \tilde{M}(0) - KM(0) \| = \| K_n V - KV \| \leq o_n.
\]
Thereby
\[
\| \tilde{M}(k) - M(k) \| \leq o_n (1 + \lambda + \cdots + \lambda^{k-1}) = \frac{\lambda^k - 1}{\lambda - 1} o_n \leq \lambda^k o_n.
\]
(5.13)
Combining (5.10), (5.11) and (5.13) it yields
\[
\| M_n(k) - q(k) \| \leq B_0 e^{-\delta k} + \lambda^{k+1} \frac{\| A \|^n}{(n+1)!},
\]
and
\[
\| M_n(k) - \Xi \| \leq O(k^{-1}) + B_1 e^{-\delta k} + \lambda^{k+1} \frac{\| A \|^n}{(n+1)!}.
\]
Taking \( k = n \) and using the Stirling’s formula, we immediately arrive at the estimates in (5.7) and (5.8).
As in Theorem 3.3, the estimate in (5.9) is actually a consequence of (5.8). We omit the details of the proof.

As in Theorem 4.2, if the principal eigenvalue is semisimple, then using the convergence result in Theorem 4.2 and repeating the above argument with minor modifications, it can be shown that the iterative scheme developed here has in fact an exponential convergence rate. More precisely, let \( Y = \{ y_i \}_{i \in J} \), where \( y_i = \Pi_1 v_i \), and \( \Pi_1 \) is the projection from \( E = \mathbb{R}^m \) to \( E_1 = GE_s(A) \). Write \( W_n = M_n(n) \). We have

**Theorem 5.2.** Assume that \( s \) is semisimple. Then for \( n \in \mathbb{N} \), we have
\[
\left\| W_n - \frac{Y}{\| Y \|} \right\| \leq B_1 e^{-\delta n} + \left( \frac{\lambda \| A \|}{n+1} \right)^{n+1} o(1),
\]
and
\[
| \langle AW_n, W_n \rangle - s | \leq B_2 e^{-\delta n} + \| A \| \left( \frac{\lambda \| A \|}{n+1} \right)^{n+1} o(1).
\]
(5.14)
(5.15)

**Remark 5.3.** Since \( \text{span}\{ y_i : i \in J \} = E_s(A) \) (see Remark 3.5), in the semisimple case, by (5.14) the iterative scheme given here provides an alternative effective way to compute the whole principal eigenspace \( E_s(A) \) with exponential convergence.
Remark 5.4. Nonnegative irreducible matrices and symmetric matrices are typical examples for which the principal eigenvalues are semisimple. For these two types of matrices one can find a large number of excellent works in the literature on the computation of eigenvalues and eigenvectors; see e.g. [1, 3, 4, 14, 15, 16, 18, 22, 26, 29] etc.

5.2. An iterative scheme with parameters

It can be seen from Theorems 5.1 and 5.2 that the convergence rate of the scheme can be significantly affected by the scale of the matrix $A$, and large scale of $A$ may cause slow convergence speed in the early stage of the iteration. Another risk for a large scale matrix is the overflow in the computation. To overcome these deficiencies, one may introduce a parameter $\gamma > 0$ in the scheme. Specifically, we choose an appropriate $\gamma$ and use the matrix $\gamma A$ in place of $A$.

The iterative scheme is then reformulated as below:

For each $n \in \mathbb{N}$, let

$$K_n(\gamma)X = \frac{T_n(\gamma)X}{\|T_n(\gamma)X\|}, \quad X \in \Sigma_1,$$

where $T_n(\gamma) = \sum_{k=0}^{n} \frac{1}{k!}(\gamma A)^k$. Given a nonsingular matrix $V \in \Sigma_1$, set

$$M_n(0) = V, \quad M_n(k + 1) = K_n(\gamma)M_n(k) \quad (k \in \mathbb{N}). \quad (5.16)$$

Correspondingly, Theorem 5.1 now takes the form:

Theorem 5.5. Let $W_n = M_n(n)$. Then for $n \in \mathbb{N}$ we have

$$\|W_n - q(n)\| \leq B_0(\gamma)e^{-\gamma \delta n} + \left(\frac{\lambda(\gamma)\|A\|}{n+1}\right)^{n+1} o(1),$$

$$\|W_n - \Xi\| \leq O(n^{-1}) + B_1(\gamma)e^{-\gamma \delta n} + \left(\frac{\lambda(\gamma)\|A\|}{n+1}\right)^{n+1} o(1),$$

where $\lambda(\gamma) = 2e^{2\gamma\|A\|}$ (see (5.5)), and

$$\langle AW_n, W_n \rangle - s \leq \gamma^{-1}O(n^{-1}) + \gamma^{-1}B_2(\gamma)e^{-\gamma \delta n} +$$

$$+ \|A\| \left(\frac{\lambda(\gamma)\|A\|}{n+1}\right)^{n+1} o(1).$$

Remark 5.6. As in Theorem 5.2, in the case where $s$ is semisimple, the terms $O(n^{-1})$ and $\gamma^{-1}O(n^{-1})$ in Theorem 5.5 can be removed. We omit the details.
5.3. A remark on the initial matrix

**Remark 5.7.** In the definition of the iteration sequence in (5.6), it is required that $\|V\| = 1$. This is just for the sake of convenience in statement. In practice we can take a non-normalized matrix $V$ as the initial one in the iteration. For such a matrix one easily examines that the matrices $M_n(k)$ given by (5.6) (except $M_n(0)$) and the matrices $q(n)$ and $\Xi$ in Theorem 5.1 are the same as in the case where, instead of $V$, one uses the normalization $V' = V/\|V\|$ of $V$ as the initial matrix. Hence all the results above remain valid.

5.4. Numerical examples

We now give several numerical examples to illustrate the efficiency of our scheme. All the numerical simulations in this work were carried out on the authors’ PC by using Matlab. Detailed information concerning the PC and the softwares is as follows.

- **Computer Configuration**
  Processor: Intel(R) Core(TM) i5-4200M; CPU @ 2.50GHZ 2.49GHZ
  RAM: 8GB
  Operation system: Windows 10 Professional Edition
  System type: 64-bit operating system, x64-based processor

- **Softwares**
  Matlab version: Matlab R2017b

- **Notations**
  Given a Perron-like matrix $A$ and an initial matrix $V$ ($V$ is nonsingular), let $\Xi$ and $W_n := M_n(n)$ be given by Theorem 5.1. Then $W_n$ and $s_n := \langle AW_n, W_n \rangle$ can be regarded as numerical values of $\Xi$ and $s$, respectively. We will employ the following notations:

  $\epsilon(W_n, \Xi)$: Error estimate between $W_n$ and $\Xi$
  $\epsilon(s_n, s)$: Error estimate between $s_n$ and $s$
  $\epsilon$: The total error estimate $\epsilon(W_n, \Xi) + \epsilon(s_n, s)$
Example 5.1. Let

\[ A = \begin{pmatrix}
14 & -21 & 2 & 39 & 63 & 12 & -12 & \frac{9}{2} \\
18 & -12 & 28 & 46 & 31 & -21 & 11 & 11 \\
12 & 2 & 32 & 34 & 17 & 5 & -27 & \frac{7}{5} \\
-26 & 33 & -87 & -119 & -57 & 67 & -11 & \frac{5}{5} \\
-8 & 7 & -13 & -21 & 2 & 8 & 3 & 3 \\
16 & -3 & 57 & 89 & 21 & 17 & 5 & 10 \\
36 & -63 & 117 & 189 & 36 & -36 & 47 & 10
\end{pmatrix}. \]

The principal eigenvalue \( s \) of the matrix is \( s = 2 \), which is a semisimple eigenvalue with geometric multiplicity 5. Theorem 5.5 indicates that the iteration given in (5.6) has an exponential convergence rate for any nonsingular initial matrix \( V \). We also infer from (3.35) and (2.5) that

\[ E_s(A) = \text{span}\{\xi_i : i \in J := \{1, 2, \cdots, 7\}\}, \]

where \( \xi_i \ (i \in J) \) are the column vectors of \( \Xi \).

Taking the initial matrix \( V \) to be the identity matrix \( I \) and using the iterative scheme in (5.6), we obtain the following table concerning the numerical results by using the iterative method developed in this section.

| \( n \) | \( \epsilon(W_n, \Xi) \) | \( \epsilon(s_n, s) \) | \( \epsilon \) |
|---|---|---|---|
| 1 | 0.0785 | 0.0980 | 0.1765 |
| 2 | 0.0076 | 0.0371 | 0.0447 |
| 3 | \( 7.0861 \times 10^{-5} \) | \( 3.7401 \times 10^{-4} \) | \( 4.4487 \times 10^{-4} \) |
| 4 | \( 1.2770 \times 10^{-6} \) | \( 2.4147 \times 10^{-5} \) | \( 2.5424 \times 10^{-5} \) |
| 5 | \( 8.6196 \times 10^{-8} \) | \( 9.1289 \times 10^{-7} \) | \( 9.9909 \times 10^{-7} \) |
| 8 | \( 9.9724 \times 10^{-12} \) | \( 1.0554 \times 10^{-10} \) | \( 1.1552 \times 10^{-10} \) |
| 10 | \( 5.0047 \times 10^{-14} \) | \( 2.4603 \times 10^{-13} \) | \( 2.9607 \times 10^{-13} \) |

Example 5.2. Consider the symmetric matrix

\[ A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}, \]
for which \( s = s(A) = 2 \) with algebraic and geometric multiplicity 3. Taking

\[
V = \begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

and applying the iterative scheme in (5.6), we obtain the following table concerning the error estimates of the numerical results:

| \( n \) | \( \epsilon(W_n, \Xi) \) | \( \epsilon(s_n, s) \) | \( \varepsilon \) |
|---|---|---|---|
| 1 | 0.1252 | 0.0625 | 0.1877 |
| 2 | 0.0151 | 9.1408 \times 10^{-4} | 0.0160 |
| 3 | 5.5105 \times 10^{-6} | 1.2146 \times 10^{-8} | 5.5117 \times 10^{-5} |
| 4 | 1.9435 \times 10^{-6} | 1.5108 \times 10^{-11} | 1.9435 \times 10^{-6} |
| 5 | 2.4565 \times 10^{-11} | 2.2204 \times 10^{-16} | 2.4565 \times 10^{-11} |
| 10 | 1.4687 \times 10^{-16} | 0 | 1.4687 \times 10^{-16} |

The principal eigenvalues of the matrices in the above two examples are semisimple. Numerical results also indicate that in such a case the convergence rate of the iterative scheme developed in this section can be exponential. However, situations seem to be quite different in the case where the principal eigenvalue is not semisimple, and both the theoretical and numerical results demonstrate that the convergence can be very slow.

**Example 5.3.** Consider the matrix

\[
A = \begin{pmatrix}
2 & 1 & 0 & 0 & 2 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 3 & 1
\end{pmatrix}
\]

The principal eigenvalue is \( s = 2 \). The algebraic multiplicity of \( s \) is 3, whereas it has a geometric multiplicity 1. Taking \( V = I \) and applying the scheme given in Theorem 5.1 to compute the numerical values \( s_n \) of \( s \), we obtain the following table, from which it can be easily seen that the numerical results and the theoretical one in (5.9) coincide.
Table 3: Numerical results via the scheme in Theorem 5.1

| \(n\) | \(s_n\) | \(\epsilon(s_n, s)\) |
|---|---|---|
| 10 | 2.2160 | 0.2160 |
| 50 | 2.0413 | 0.0413 |
| 100 | 2.0203 | 0.0203 |
| 200 | 2.0101 | 0.0101 |
| 500 | 2.0040 | 0.0040 |

6. Determination of the Cyclic Order of \(\text{GE}_s(A)\)

We have seen from either the theoretical results in Theorems 3.3, 4.1 and 5.1 or the numerical example in Example 5.3 that the situation is more complicated and worse in the case where the principal eigenvalue \(s\) of a matrix \(A\) is non-semisimple. On one hand, the computation methods presented in the previous sections can only allow us to obtain numerical results for the principal dominant eigenspace \(\text{DE}_s(A)\) (recall that all the column vectors of the matrix \(\Xi\) in the aforementioned theorems are principal dominant eigenvectors). On the other hand, the convergence speed can be very slow, which fact may make the methods to be of little practical sense.

The remaining part of this paper is devoted to this bad case where \(s\) is non-semisimple. Our main aim is to present a combined method for the computation of the principal eigenvalue and develop an efficient scheme to compute the whole principal generalized eigenspace \(\text{GE}_s(A)\). To this end, we first address the problem of how to determine the cyclic order \(\text{Cord}(\text{GE}_s(A))\) of the space \(\text{GE}_s(A)\), which is also of independent interest in its own right.

6.1. Determining the cyclic order of \(\text{GE}_s(A)\)

We use the same notations as in Section 5. So let \(v_i\), \(y_i\) and \(z_i\) \((i \in J)\) be the same as therein. Let \(Q_{y_i}(t)\) be the characteristic polynomial of \(y_i\), and \(Q_Y(t) = [Q_{y_i}(t)]_{i \in J}\). We infer from the proof of Theorem 3.2 that

\[
t^{-(\nu-1)} \|Q_Y(t)\| = \omega + O(t^{-1}),
\]

where \(\omega\) is the number in (3.24). Let \(q(t)\) be given as in (4.1).

Assume that \(j \in J\) is a number such that \(y_j \neq 0\). For each \(k \in \{0, 1, \cdots, \nu\}\), where \(\nu = \text{Ord}(y_j)\), put

\[
\psi_k(t, \bar{s}) = t^{2k} \| (A - \bar{s}I)^k q_j(t) \|^2,
\]

(6.2)
where $\bar{s} := s + \Delta s$ denotes an approximate value of $s$. Note that

$$ (A - \bar{s}I)^k = \sum_{i=0}^{k} C_k^i (-\Delta s)^i (A - sI)^{k-i}, \quad (6.3) $$

where $C_k^i$ are the combinatorial numbers. We observe that

$$ (A - sI)^k Q_{y_i}(t) = \sum_{\ell=0}^{\nu-1} \frac{\ell!}{\ell!} (A - sI)^{k+\ell} y_j $$

$$ = \sum_{\ell=0}^{\nu-k-1} \frac{\ell!}{\ell!} (A - sI)^{k+\ell} y_j $$

$$ = t^{\nu-k-1} \left( \frac{1}{(\nu-k-1)!} \bar{w}_j + O(t^{-1}) \right) $$

for $k \leq \nu - 1$, where

$$ \bar{w}_j = (A - sI)^{\nu-1} y_j. \quad (6.5) $$

In what follows we assume that $|t\Delta s| \leq 1$ and argue by cases.

**Case 1:** $k \leq \nu - 1$. In this case by (6.3) and (6.4) we deduce that

$$ (A - \bar{s}I)^k Q_{y_i}(t) = \sum_{i=0}^{k} C_k^i (-\Delta s)^i t^{\nu-k+i-1} \left( \frac{\bar{w}_j}{(\nu-k+i-1)!} + O(t^{-1}) \right) $$

$$ = t^{\nu-k} \sum_{i=0}^{k} (t\Delta s)^i (a_{k,i} \bar{w}_j + O(t^{-1})) $$

$$ = t^{\nu-k} a_{k,0} \bar{w}_j + t^{\nu-k-1} \sum_{i=1}^{k} (t\Delta s)^i a_{k,i} \bar{w}_j + $$

$$ + t^{\nu-k-1} \sum_{i=0}^{k} (t\Delta s)^i O(t^{-1}) $$

$$ = t^{\nu-k} (a_{k,0} \bar{w}_j + O(|t\Delta s|) + O(t^{-1})) $$

as $t \to \infty$ and $t\Delta s \to 0$, where

$$ a_{k,i} = \frac{C_k^i}{(\nu-k+i-1)!}. \quad (6.7) $$

Therefore by (6.1) one easily examines that

$$ t^k (A - \bar{s}I)^k q_j(t) = \omega^{-1} a_{k,0} \bar{w}_j + O(|t\Delta s|) + O(t^{-1}). \quad (6.8) $$

Hence

$$ \psi_k(t, \bar{s}) = \omega^{-2} \|a_{k,0} \bar{w}_j\|^2 + O(|t\Delta s|) + O(t^{-1}). \quad (6.9) $$

**Case 2:** $k = \nu$. Noticing that $(A - sI)^k Q_{y_i}(t) = 0$, as in (6.6) we have

$$ (A - \bar{s}I)^k Q_{y_i}(t) = t^{\nu-k-1} \sum_{i=1}^{k} (t\Delta s)^i (a_{k,i} \bar{w}_j + O(t^{-1})) $$

$$ = t^{\nu-k-1} O(|t\Delta s|), \quad (6.10) $$
Thus by (6.1) we deduce that
\[ t^k (A - \bar{s}I)^k q_j(t) = O(\|t\Delta s\|). \] (6.11)

It then follows by the definition of \( \psi_\nu \) in (6.2) that
\[ \psi_\nu(t, \bar{s}) = O(\|t\Delta s\|^2). \] (6.12)

In practice the function \( q_j(t) \) is in fact unknown. However, Theorem 3.2 indicates that \( \bar{q}_j(t) := \frac{x_j(t)}{\|X(t)\|} \) approaches \( q_j(t) \) exponentially, where \( x_j(t) \) and \( \|X(t)\| \) are the same as in Theorem 3.2. More precisely, we have
\[ \|\bar{q}_j(t) - q_j(t)\| \leq B_0 e^{-\delta t}, \quad t \geq 0. \] (6.13)

Set
\[ \bar{\psi}_k(t, \bar{s}) = t^2 k \| (A - \bar{s}I)^k \bar{q}_j(t) \|^2. \] (6.14)

Combining (6.9), (6.12) and (6.13) together, we obtain the following fundamental result:

**Proposition 6.1.** Assume that \( y_j \neq 0 \). Let \( \nu = \text{Ord}(y_j) \). Then as \( t \to \infty \) and \( t\Delta s \to 0 \), the following assertions hold:
\[ \bar{\psi}_k(t, \bar{s}) = \omega_{-2}^k a_{k,0} \bar{w}_j(t)^2 + O(\|t\Delta s\|) + O(t^{-1}) + O(t^{2k} e^{-\delta t}) \] (6.15)
for \( k \leq \nu - 1 \), where \( \bar{w}_j \) is given by (6.5); and
\[ \bar{\psi}_\nu(t, \bar{s}) = O((\|t\Delta s\|)^2) + O(t^{2\nu} e^{-\delta t}). \] (6.16)

In applications one can use the \( j \)-th column vector \( W_{n,j} \) of the matrix
\[ W_n := M_n(n) \] given in (5.6) to replace \( \bar{q}_j(t) \) in the definition of \( \bar{\psi}_k(t, \bar{s}) \). That is, we define
\[ \bar{\psi}_k(n, \bar{s}) = n^{2k} \| (A - \bar{s}I)^k W_{n,j} \|^2, \quad k, n \geq 0. \] (6.17)

Thanks to Theorem 5.1, Proposition 6.1 is correspondingly modified as below:

**Proposition 6.2.** Assume that \( y_j \neq 0 \). Let \( \nu = \text{Ord}(y_j) \). Then as \( n \to \infty \) and \( n\Delta s \to 0 \), we have
\[ \bar{\psi}_k(n, \bar{s}) = \omega_{-2}^k a_{k,0} \bar{w}_j(t)^2 + O(n|\Delta s|) + O(n^{-1}) + O(n^{2k} e^{-\delta n}) + n^{2k} \left( \frac{\lambda \|A\|}{n+1} \right)^{n+1} o(1), \quad 0 \leq k \leq \nu - 1, \] (6.18)
\[ \bar{\psi}_\nu(n, \bar{s}) = O((n|\Delta s|)^2) + O(n^{2\nu} e^{-\delta n}) + n^{2\nu} \left( \frac{\lambda \|A\|}{n+1} \right)^{n+1} o(1), \] (6.19)
where \( \lambda \) is the constant given in Theorem 5.1.
Now we assign 
\[ \bar{\psi}_{-1}(n, \bar{s}) = \bar{\psi}_0(n, \bar{s}), \]
and define 
\[ \beta_k(n, \bar{s}) = \frac{\bar{\psi}_k(n, \bar{s})}{\bar{\psi}_{k-1}(n, \bar{s})}, \quad 1 \leq k \leq \nu. \] (6.20)
Clearly 
\[ \beta_0(n, \bar{s}) = 1, \quad n \in \mathbb{N}. \] (6.21)
If we choose for each \( n \) an approximate value \( \bar{s} = s_n \) for \( s \) with \( n(s_n - s) \to 0 \) as \( n \to \infty \), then by Proposition 6.2 it can be easily seen that 
\[ \lim_{n \to \infty} \beta_\nu(n, s_n) = 0, \] (6.22)
whereas 
\[ \lim_{n \to \infty} \beta_k(n, s_n) = \nu - k - 1 \geq 1, \quad k = 1, 2, \ldots, \nu - 1. \] (6.23)
These observations form a basis for the computation of the cyclic order of the principal generalized eigenspace.

**The computation scheme for Cord (GE\(_s\)(A)).**

We are now ready to propose a scheme to compute the cyclic order Cord (GE\(_s\)(A)) of the principal generalized eigenspace.

**Step 1.** Pick an appropriately small number \( \varepsilon > 0 \) (say, \( \varepsilon = 1/10 \)).

**Step 2.** Let \( W_n := M_n(n) \), where \( M_n(k) \) is generated by (5.6) with \( V = I \). Denote by \( W_{n,i} \) the \( i \)-th column vector of \( W_n \).

Take a suitably large number \( N \in \mathbb{N} \) and compute \( W_N \) and 
\[ s_N := \langle AW_N, W_N \rangle. \]
Theorem 5.1 asserts that in general 
\[ |s_N - s| = O(N^{-1}). \]
Fix an index \( j \in J \) with 
\[ \|W_{N,j}\| = \max_{i \in J} \|W_{N,i}\|. \] (6.24)
Since 
\[ \|W_N\| = \left( \sum_{i \in J} \|W_{N,i}\|^2 \right)^{1/2} = 1, \]
we deduce that 
\[ \|W_{N,j}\| \geq 1/\sqrt{m}. \]
Further by Theorem 5.1 we see that the \( j \)-th column vector \( \xi_j \) of \( \Xi \) is nonzero provided that \( N \) is sufficiently large. Since each nonzero column vector of \( \Xi \) is a dominated principal eigenvector (see Theorem 3.3), one concludes that 
\[ \text{Cord} (\text{GE}_s(A)) = \text{Ord}(y_j). \] (6.25)

**Step 3.** Take a number \( n \in \mathbb{N} \) with \( n << N \) and compute \( \beta_k(n, s_N) \) for \( k \in J \). Note that 
\[ \beta_k(n, \bar{s}) = \frac{n^2 \|(A - \bar{s}I)^kW_{n,j}\|^2}{\|(A - \bar{s}I)^{k-1}W_{n,j}\|^2}, \quad k \geq 1. \] (6.26)
Suppose that there is a number $k_0 \in J$ such that $\beta_k(n, s_N)$ demonstrates the following dichotomy property:

$$\beta_k(n, s_N) \geq 1 - \varepsilon \quad (0 \leq k < k_0), \quad \beta_{k_0}(n, s_N) < \varepsilon.$$  \hspace{1cm} (6.27)

Then we define $\text{Cord}(GE_s(A)) = k_0$, and the procedure ends.

Otherwise, we re-choose the numbers $N$ and $n$ and repeat the above procedure until the dichotomy phenomenon in (6.27) appears.

**Remark 6.3.** One can fix $j$ and change $N$ and $n$ suitably to see whether the phenomenon in (6.27) becomes stable.

### 6.2. A numerical example

**Example 6.1.** Let $A$ be the matrix given in Example 5.3. The principal eigenvalue is $s = 2$. Since the algebraic multiplicity of $s$ is 3 whereas its geometric multiplicity is 1, it is easy to see that $\text{Cord}(GE_s(A)) = 3$. Now suppose that $\text{Cord}(GE_s(A)) := \nu$ is unknown. Let us try to use the method proposed above to compute this algebraic quantity.

1. Set $\varepsilon = 0.10$.

Let $V = I$. Taking $N = 100, 105, 110$ and performing the iteration in (5.6), we obtain respectively that $s_N = 2.0203, 2.0194, 2.0185$. The number $j = 4$ satisfies (6.24).

Computing $\beta_k(n, s_N)$ for $k \in J$ with different choices of $n (n << N)$, we get the following tables:

| $n$ | $\beta_1(n, s_N)$ | $\beta_2(n, s_N)$ | $\beta_3(n, s_N)$ | $\beta_4(n, s_N)$ | $\beta_5(n, s_N)$ |
|-----|------------------|------------------|------------------|------------------|------------------|
| 5   | 4.0341           | 1.7642           | 0.1045           | 47.2091          | 11.1716          |
| 6   | 4.1960           | 1.5385           | 0.0372           | 57.1796          | 22.6585          |
| 7   | 4.2764           | 1.3296           | 0.1202           | 6.2138           | 41.2745          |
| 8   | 4.2902           | 1.1541           | 0.2415           | 0.7801           | 83.9131          |
| 9   | 4.2511           | 1.0068           | 0.3771           | 0.0721           | 249.3744         |
| 10  | 4.1760           | 0.8821           | 0.5273           | 0.0592           | 79.0894          |
We see that if \( n = 6 \), then
\[
\beta_k(n, s_N) > 1 - \varepsilon \quad (0 \leq k \leq 2), \quad \beta_3(n, s_N) < \varepsilon
\]
for all \( N = 100, 105, 110 \). Hence one concludes that \( \nu = 3 \).

(2) We can also try with \( N = 1000 \). In such a case we obtain that \( s_N = 2.0020 \) and \( j = 4 \). Taking different \( n \ll N \) and computing \( \beta_k(n, s_N) \) for \( k \in J \), it yields the following table:

| \( n \) | \( \beta_1(n, s_N) \) | \( \beta_2(n, s_N) \) | \( \beta_3(n, s_N) \) | \( \beta_4(n, s_N) \) | \( \beta_5(n, s_N) \) |
|---|---|---|---|---|---|
| 4 | 3.7891 | 1.8380 | 0.8602 | 10.6370 | 3.6143 |
| 5 | 4.0698 | 1.7942 | 0.1119 | 42.5756 | 10.9797 |
| 6 | 4.2398 | 1.5744 | 0.0286 | 72.0950 | 22.0971 |
| 7 | 4.3496 | 1.3704 | 0.0919 | 1.0288 | 75.3986 |
| 8 | 4.3177 | 1.0555 | 0.2996 | 10.9190 | 3.6452 |
| 9 | 4.2179 | 0.8536 | 0.0326 | 64.2748 | 22.3722 |
| 10 | 4.1792 | 0.0105 | 7.0344 | 40.2877 |

Table 5: \( N = 105, \ s_N = 2.0194, \ j = 4 \)

| \( n \) | \( \beta_1(n, s_N) \) | \( \beta_2(n, s_N) \) | \( \beta_3(n, s_N) \) | \( \beta_4(n, s_N) \) | \( \beta_5(n, s_N) \) |
|---|---|---|---|---|---|
| 4 | 3.7754 | 1.8266 | 0.8536 | 10.9190 | 3.6452 |
| 5 | 4.0519 | 1.7792 | 0.1081 | 44.8683 | 11.0745 |
| 6 | 4.2179 | 1.5564 | 0.0326 | 64.2748 | 22.3722 |
| 7 | 4.3022 | 1.3499 | 7.0344 | 40.2877 |
| 8 | 4.3199 | 1.1765 | 0.0326 | 64.2748 | 22.3722 |
| 9 | 4.2843 | 1.0311 | 0.1055 | 7.0344 | 40.2877 |
| 10 | 4.2127 | 0.9079 | 0.2150 | 0.8945 | 79.4518 |

Table 6: \( N = 110, \ s_N = 2.0185, \ j = 4 \)
Table 7: \( N = 1000, \ s_N = 2.0020, \ j = 4 \)

| \( n \) | \( \beta_1(n, s_N) \) | \( \beta_2(n, s_N) \) | \( \beta_3(n, s_N) \) | \( \beta_4(n, s_N) \) | \( \beta_5(n, s_N) \) |
|--------|----------------|----------------|----------------|----------------|----------------|
| 5      | 4.4054         | 2.0797         | 0.2263         | 14.7423        | 9.6749         |
| 10     | 4.9526         | 1.4635         | 0.0024         | 6.0700         | 78.3099        |
| 15     | 4.7560         | 1.2232         | 0.0085         | 0.0012         | 186.1411       |
| 20     | 4.5468         | 1.1016         | 0.0155         | 0.0065         | 0.0029         |

Clearly for \( n = 10 \), we have

\[
\beta_k(n, s_N) > 1 - \varepsilon \quad (k \leq 2), \quad \beta_3(n, s_N) < \varepsilon
\]

with \( \varepsilon = 0.01 \). Hence once again we conclude that \( \nu = 3 \).

7. A Combined Method for the Computation of Non-semisimple Principal Eigenvalues

Theorem 5.1 asserts that \( s_n := \langle AM_n(n), M_n(n) \rangle \) converges to the principal eigenvalue \( s \) of \( A \). If \( s \) is semisimple, we know that the convergence is exponentially fast, and therefore the numerical values of \( s \) within tolerance error can be expected to be computed quickly. However, in the non-semisimple case the convergence may be very slow, as is also demonstrated in Example 5.3.

On the other hand, we have noticed in (5.7) that \( W_n := M_n(n) \) approaches \( q(n) \) swiftly. Using this simple fact, in this section we propose a new combined method for the computation of non-semisimple principal eigenvalues, which turns out to be more efficient.

Since we are only interested in the case of non-semisimple case in this section, in what follows we always assume that

\[
\nu := \text{Cord} \left( \text{GE}_s(A) \right) \geq 2.
\]

We will employ the same notations in the preceding sections. Let \( E := \mathbb{R}^m \).

Denote by \( B_x(r) \) the ball in \( E \) centered at \( x \in E \) with radius \( r > 0 \).

7.1. A fundamental lemma

We begin our work in this section with a fundamental lemma.

**Lemma 7.1.** Let \( \xi \in \text{GE}_s(A) \), \( \xi \neq 0 \). Write \( \nu = \text{Ord} (\xi) \). Given \( x \in E \), define a function \( \phi_x(\tau) \) on \( \mathbb{R} \) as

\[
\phi_x(\tau) = \|(A - \tau I)^\nu x\|^2, \quad \tau \in \mathbb{R}.
\]  

(7.1)
Then there is an \( \varepsilon > 0 \) such that for every \( x \in \overline{B}_\xi(\varepsilon) \), the function \( \phi_x(\tau) \) has a unique minimum point \( \tau_x \). Furthermore, there exist \( L_0, L_1 > 0 \) such that

\[
|\tau_x - s| \leq L_0 \|x - \xi\|, \quad \forall x \in \overline{B}_\xi(\varepsilon) \tag{7.2}
\]

and

\[
\phi_x(\tau_x) \leq L_1 \|x - \xi\|, \quad \forall x \in \overline{B}_\xi(\varepsilon). \tag{7.3}
\]

**Proof.** Since \((A - \tau I)\nu \xi = 0\) if and only if \( \tau = s \), the function \( \phi_\xi(\tau) \) has a unique minimum point \( \tau_\xi = s \) with \( \phi_\xi(s) = 0 \). It is a basic knowledge that \( \phi_\xi'(s) = 0 \). Simple calculations yield

\[
\phi''_\xi(\tau) = \begin{cases} 
2\|\xi\|^2, & \nu = 1 \\
2\nu(\nu - 1) ((A - \tau I)^{\nu - 2} \xi , (A - \tau I)^{\nu} \xi) + 2\nu^2 \| (A - \tau I)^{\nu - 1} \xi \|^2, & \nu > 1.
\end{cases}
\]

In particular, since \((A - s I)^{\nu - 1} \xi \neq 0\), we have

\[
\phi''_\xi(s) = 2\nu^2 \| (A - s I)^{\nu - 1} \xi \|^2 := 3\beta > 0. \tag{7.4}
\]

Fix an \( \eta > 0 \) such that

\[
\phi''_\xi(\tau) \geq 2\beta, \quad |\tau - s| \leq \eta.
\]

Then there is an \( \varepsilon_0 > 0 \) such that

\[
\phi''(\tau) \geq \beta, \quad |\tau - s| \leq \eta, \quad x \in \overline{B}_\xi(\varepsilon_0). \tag{7.5}
\]

We may assume that \( \varepsilon_0 \leq \|\xi\|/2 \). Then it can be easily seen that \( \phi_x(\tau) \to \infty \) as \( |\tau| \to \infty \) uniformly with respect to \( x \in \overline{B}_\xi(\varepsilon_0) \). Thus one can pick a \( T > 0 \) such that

\[
\phi_x(\tau) \geq 2, \quad \forall x \in \overline{B}_\xi(\varepsilon_0), \ |\tau| > T. \tag{7.6}
\]

It can be assumed that \( |s| < T/2 \).

Because \( \phi_\xi(\tau) > 0 \) for \( \eta \leq |\tau - s| \leq T \), we deduce that

\[
\min_{\eta \leq |\tau - s| \leq T} \phi_\xi(\tau) := 3c > 0.
\]

By continuity there is \( \varepsilon_1 > 0 \) such that

\[
\min_{\eta \leq |\tau - s| \leq T} \phi_x(\tau) \geq 2c, \quad x \in \overline{B}_\xi(\varepsilon_1). \tag{7.7}
\]

On the other hand, since \( \phi_\xi(s) = 0 \), one can take a positive number \( \varepsilon < \min(\varepsilon_0, \varepsilon_1) \) such that

\[
\min_{|\tau - s| \leq \eta} \phi_x(\tau) \leq \min(1, c), \quad \forall x \in \overline{B}_\xi(\varepsilon). \tag{7.8}
\]
Combining (7.6), (7.7) and (7.8) we deduce that for each $x \in \overline{B}_\xi(\varepsilon)$,

$$\min_{\tau \in \mathbb{R}} \phi_x(\tau) = \min_{|\tau - s| \leq \eta} \phi_x(\tau).$$  \hfill (7.9)

(7.5) and (7.9) imply that for each $x \in \overline{B}_\xi(\varepsilon)$, the function $\phi_x(\tau)$ has a unique minimum point $\tau_x$ with $|\tau_x - s| \leq \eta$. As $\phi'_x(\tau_x) = 0 = \phi'_\xi(s)$, we have

$$\phi'_x(\tau_x) - \phi'_x(s) = \phi'_\xi(s) - \phi'_x(s).$$

By virtue of the mean value theorem we deduce that $\phi'_x(\tau_x) - \phi'_x(s) = \phi''_x(\theta)(\tau_x - s)$ for some $\theta$ with $|\theta - s| \leq \eta$. Therefore by (7.5) we deduce that

$$\beta |\tau_x - s| \leq |\phi'_\xi(s) - \phi'_x(s)| = |\phi'_x(s)|.$$  \hfill (7.10)

Let $z = x - \xi$. We observe that

$$|\phi'_x(s)| = 2\nu \left| \left\langle (A - sI)^{\nu-1}x, (A - sI)^{\nu}x \right\rangle \right|$$

$$= 2\nu \left| \left\langle (A - sI)^{\nu-1}x, (A - sI)^{\nu}(\xi + z) \right\rangle \right|$$

$$= 2\nu \left| \left\langle (A - sI)^{\nu-1}x, (A - sI)^{\nu}z \right\rangle \right| \leq c_0 \|z\|.$$  \hfill (by (7.2))

Therefore by (7.10) one concludes that $|\tau_x - s| \leq c_0 \beta^{-1} \|z\|$, which is what we desired in (7.2).

We also note that

$$\phi_x(\tau_x) = \phi_x(s) + (\phi_x(\tau_x) - \phi_x(s))$$

$$\leq \phi_x(s) + |\phi_x(\tau_x) - \phi_x(s)|.$$  \hfill (7.11)

On the other hand, it is easy to verify that

$$|\phi_x(\tau_x) - \phi_x(s)| \leq L |\tau_x - s| \leq (by \text{ (7.2)}) \leq c_1 \|z\|$$

for $x \in \overline{B}_\xi(\varepsilon)$, and

$$\phi_x(s) = \left\langle (A - sI)^{\nu}x, (A - sI)^{\nu}x \right\rangle$$

$$= \left\langle (A - sI)^{\nu}z, (A - sI)^{\nu}z \right\rangle \leq c_2 \|z\|^2.$$  \hfill (by (7.11))

Hence by (7.11) we deduce that

$$\phi_x(\tau_x) \leq c_1 \|z\| + c_2 \|z\|^2 \leq (c_1 + c_2 \varepsilon) \|z\|.$$  \hfill (by (7.3))

This completes the proof of (7.3). \quad \blacksquare
7.2. A combined method for computing the principal eigenvalue

Now we propose a combined method for the computation of the principal eigenvalue \( s \) of a Perron-like matrix \( A \). The basic idea is to pick a suitably large \( n \) and solve the minimum point \( \tau_s := \tau_x \) of a suitable function \( \phi_s(\tau) \) as defined by (7.1) with \( x = W_{n,j} \), where \( W_{n,j} \) is a column vector of the matrix \( W_n := M_n(n) \) who has the maximal norm among the column vectors \( W_{n,i} \) \( (i \in J) \) of \( W_n \) (hence \( \| W_{n,j} \| \geq 1/\sqrt{m} \)). For notational convenience, we rewrite

\[ \phi_{W_{n,j}}(\tau) = \phi_n(\tau). \]

Suppose that \( n \) is chosen suitably large, and let \( j \in J \) be such that \( \| W_{n,j} \| = \max_{i \in J} \| W_{n,i} \| \). Then we infer from (6.25) that

\[ \text{Cord}(\text{GE}_{s}(A)) = \text{Ord}(y_j) = \text{Ord}(Q_{y_j}(n)) = \text{Ord}(\xi) := \nu, \]

where \( \xi := q_j(n) = Q_{Y_j}(n) \| Q_{Y_j}(n) \|. \)

In view of Theorem 5.1 we have

\[ \| W_{n,j} - \xi \| \leq B_0 e^{-\delta n} + \left( \frac{\lambda \| A \|}{n+1} \right)^{n+1} o(1), \quad (7.12) \]

which indicates that the function \( \phi_n(\tau) \) should take the form:

\[ \phi_n(\tau) = \|(A - \tau I)^\nu W_{n,j}\|^2, \quad \tau \in \mathbb{R}, \quad (7.13) \]

Before describing the combined computation method, let us make a simple observation on the second order derivative \( \phi_n''(\tau) \) near \( s \). We have

\[ \phi_n''(\tau) = 2\nu(\nu - 1) \langle (A - \tau I)^{\nu - 2} W_{n,j}, (A - \tau I)^{\nu - 1} W_{n,j} \rangle + 2\nu^2 \|(A - \tau I)^{\nu - 1} W_{n,j}\|^2. \]

Let \( \Delta s = \tau - s \). We also assume that \( n|\Delta s| \leq 1 \). Using some similar calculations as in the verifications of (6.8), (6.9), (6.18) and (6.19), we can get that

\[ (A - \tau I)^k W_{n,j} = n^{-k} (\omega^{-1} a_{k,0} \tilde{w}_j + O(n|\Delta s|) + O(n^{-1})) + O(e^{-\delta n}) + \left( \frac{\lambda \| A \|}{n+1} \right)^{n+1} o(1) = O(n^{-k}) \]

for \( 0 \leq k \leq \nu - 1 \), where \( a_{k,0} \) and \( \tilde{w}_j \) are the same as in (6.8) and (6.9); and

\[ (A - \tau I)\nu W_{n,j} = n^{-\nu} O(n|\Delta s|) + O(e^{-\delta n}) + \left( \frac{\lambda \| A \|}{n+1} \right)^{n+1} o(1) = O(n^{-\nu}). \]
Therefore we find that
\[ \phi_n''(\tau) = O(n^{-2(\nu-1)}) \] (7.14)
in a small neighborhood \( U_n \) of \( s \).

We are now in a position to describe our computation scheme.

- **A combined method computing the principal eigenvalues**

  Let \( A \) be a Perron-like matrix. Take \( V = I \) and consider the iteration sequence given by (5.6).

  **Step 1.** Take a number \( N \) suitably large to obtain an approximate value \( s_0 \) for \( s := s(A) \) with error estimate \( O(N^{-1}) \):
  \[
  s_0 := \langle AW_N, W_N \rangle.
  \]

  Meanwhile, pick a number \( j \in J \) with \( \|W_{N,j}\| = \max_{i \in J} \|W_{N,i}\| \).

  **Step 2.** Determine the order \( \nu \) of \( y_j \) via the method in Section 6, which is actually equal to the cyclic order of \( GE_s(A) \).

  **Step 3.** Take a number \( n \ll N \) and solve the initial value problem
  \[
  \frac{d\tau}{dt} = -\left( (\gamma n)^2(\nu-1) \phi_n'(\tau) \right), \quad \tau(0) = s_0
  \] (7.15)
  with parameter \( \gamma \) to obtain an approximate value \( \tau(t_0) \) of the minimum point \( \tau_* \) of the function \( \phi_n(\tau) \) given in (7.13). By virtue of Lemma 7.1 we have
  \[
  |\tau_* - s| = O(\|W_{n,j} - q_j(n)\|). \tag{7.16}
  \]

  On the other hand, since \( s_0 \) is in a small neighborhood of \( s \) (hence \( s_0 \) is close to \( \tau_* \)), by (7.5) and the basic knowledge in the qualitative theory of ODEs we deduce that \( \tau(t) \to \tau_* \) exponentially. This may allow us to get a numerical value \( \bar{s} := \tau(t_0) \) of \( \tau_* \) with \( |\bar{s} - \tau_*| < \epsilon/2 \) quickly, where \( \epsilon \) denotes a tolerance error. Combining this with (7.16) it yields
  \[
  |\bar{s} - s| = \epsilon/2 + O(\|W_{n,j} - q_j(n)\|) \leq \epsilon/2 + B_0 e^{-\delta n} + \left( \frac{\|A\|}{n+1} \right)^{n+1} o(1).
  \]

  Since the second and third terms in the righthand side of the above estimates tend to 0 exponentially as \( n \to \infty \), it is desirable that a smaller choice of \( n \) can produce a numerical value \( \bar{s} \) of \( s \) with desired accuracy.

**Remark 7.2.** The role of the term \( (\gamma n)^2(\nu-1) \) in (7.14) is to speed up the convergence of \( \tau(t) \) to \( \tau_* \), which is due to the simple observation made in (7.13).
Theoretically, the larger the parameter $\gamma$ is taken, the faster the convergence is. But in practice, due to the discretization of the equation and the existence of round errors, large $\gamma$ may cause serious oscillations in the computation of $\tau(t)$.

**Remark 7.3.** The reason why we pick another smaller number $n < N$ in Step 3 is that the function $\phi_n(\tau)$ may have multiple minimum points, with each one being an attractor of the equation in (7.17). For large $n$, these minimum points can be very close to each other. Therefore the initial value $s_0$ in (7.17) may fall into the attraction region of a wrong minimum point. One way to avoid this risk in case the number $n$ in (7.17) is fixed is to enlarge $N$ in Step 1 to obtain a better approximate value $s_0$ so that it is in the attraction region of the right minimum point $\tau_*$. 

### 7.3. A numerical example

**Example 7.1.** We continue our business with the matrix $A$ used in Examples 5.3 and 6.1. Take $N = 100$. Then from Table 3 we get that $s_0 := s_N = 2.0203$. We also infer from the computations involved in Examples 5.3 and 6.1 that $\|W_N,4\| = \max_{i \in J} \|W_{N,i}\|$ (hence $j = 4$) and $\nu = 3$.

Now the equation (7.15) can be expressed as

$$
\frac{d\tau}{dt} = (\gamma n)^4 \cdot \langle (A - \tau I)^2 W_{n,j}, (A - \tau I)^3 W_{n,j} \rangle, \quad \tau(0) = s_0 \quad (7.17)
$$

Taking $\gamma = 0.2$, $n = 20$ and solving (7.17), we obtain the following table of numerical results, where $\epsilon(\bar{s}, s)$ denotes the error estimate between $\bar{s} = \tau(t)$ and the principal eigenvalue $s$: 

---

39
Table 8: Numerical results for $\bar{s} := \tau(t)$ with $\gamma = 0.2$, $n = 20$

| $t$ | $\bar{s}$ | $\epsilon(\bar{s}, s)$ |
|-----|------------|------------------------|
| 0   | 2.020300000000000 | $2.0300 \times 10^{-2}$ |
| 10  | 2.013637560931887  | $1.3638 \times 10^{-2}$ |
| 20  | 2.006802688403743  | $6.8027 \times 10^{-3}$ |
| 30  | 2.002493148980474   | $2.4931 \times 10^{-3}$ |
| 40  | 2.000760076153716   | $7.6008 \times 10^{-4}$ |
| 50  | 2.000216906077931   | $2.1691 \times 10^{-4}$ |
| 60  | 2.000060858938304   | $6.0859 \times 10^{-5}$ |
| 70  | 2.000017020006746   | $1.7020 \times 10^{-5}$ |
| 80  | 2.000004774174110   | $4.7742 \times 10^{-6}$ |
| 90  | 2.0000001358699798   | $1.3587 \times 10^{-6}$ |
| 100 | 2.000000406495274    | $4.0650 \times 10^{-7}$ |

**Remark 7.4.** Since $s = 2$ is an eigenvalue of $A$ with algebraic multiplicity 3, using the iterative method in Section 5.1, one can only obtain a numerical value $\bar{s}$ of $s$ with an error estimate $2.0300 \times 10^{-2}$ by taking $N = 100$. However, the above table indicates that if one uses the combined method developed in this section, the same choice of $N$ can yield a numerical value $\bar{s}$ of $s$ with an error estimate $4.0650 \times 10^{-7}$.

**Remark 7.5.** As we have mentioned in Remark 7.2, large values of the parameter $\gamma$ in (7.17) may cause oscillations in the numerical computation of the solution $\tau(t)$ of the equation. This can be seen from the following plots of numerical solutions $\tau(t)$ of (7.17) corresponding to different choices of $\gamma$. 


Figure 7.1: Plot of $\tau(t)$ with $n = 20, \gamma = 0.2$.

Figure 7.2: Plot of $\tau(t)$ with $n = 20, \gamma = 0.5$. 
8. The Computation of Principal Generalized Eigenspaces

Let $A \in \mathbb{M}_m$ be a Perron-like matrix with principal eigenvalue $s$. Given a nonsingular matrix $V = [v_i]_{i \in J}$, let $M_n(k)$ $(n, k \geq 0)$ be the iteration sequence defined in (5.6). If $s$ is semisimple, then by Theorem 5.2 $M_n(n)$ converges to $Y_0 := Y/\|Y\|$ exponentially, where $Y = [y_i]_{i \in J}$ with $y_i = \Pi_1 v_i$, and $\Pi_1$ is the projection from $E := \mathbb{R}^m$ to $GE_s(A)$. Since $GE_s(A) = E_s(A) = \text{span}\{y_i : i \in J\}$ (see Remark 3.3), the computation of the invariant subspace $GE_s(A)$ has already been solved (at least in theoretical sense). In the non-semisimple case, we infer from Theorem 4.1 that $W_n := M_n(n)$ converges to a matrix $\Xi$ whose column vectors can only span the principal dominant eigenspace $DE_s(A)$ which is a proper subspace of $GE_s(A)$.

Because in any case we always have

$$GE_s(A) = \text{span}\{y_i : i \in J\}, \quad (8.1)$$

one can select a well-conditioned basis for $GE_s(A)$ from the column vectors of $Y$. Therefore the computation of $GE_s(A)$ can be transformed into that of the matrix $Y$ or related matrices.

8.1. Some fundamental convergence results

For convenience, denote by $MGE_s(A)$ the set of $m \times m$ matrices whose column vectors are in $GE_s(A)$. Since $A(GE_s(A)) \subset GE_s(A)$, we clearly have

$$AM \in MGE_s(A), \quad \forall M \in MGE_s(A). \quad (8.2)$$
In what follows, $C$ denotes a general positive constant which may be different from one to another.

Write $V = Y + Z$. Then $Y \in \text{MGE}_s(A)$, and $Z = [z_i]_{i \in J}$ is a matrix with column vectors $z_i$ in $E_2 := \bigoplus_{\mu \in \sigma(A) \setminus \{s\}} \text{GE}_\mu(A)$.

Let $\tilde{X}(t) := e^{t(A-sI)}V$. We observe that

$$\tilde{X}(t) = e^{t(A-sI)}Y + e^{t(A-sI)}Z = Q_Y(t) + e^{-st}e^{tA}Z := Q_Y(t) + \tilde{Z}(t),$$

where $Q_Y(t) = [Q_{y_i}(t)]_{i \in J}$. The same calculations in (3.9) apply to show that

$$\|\tilde{Z}(t)\| \leq C\|V\|e^{-\delta t}, \quad t \geq 0. \quad (8.3)$$

By the definition of the characteristic polynomials of generalized eigenvectors (see (2.6)) it is easy to see that $Q_{y_i}(t) \in \text{GE}_s(A)$ for all $i \in J$. Consequently $$(A - sI)^k Q_Y(t) = 0, \quad k \geq \nu := \text{Cord}(\text{GE}_s(A)).$$

Hence we deduce that

$$Y = e^{-t(A-sI)}e^{t(A-sI)}Y = e^{-t(A-sI)}Q_Y(t)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{t^k}{k!}(A - sI)^k Q_Y(t)$$

$$= \sum_{k=0}^{d} (-1)^k \frac{t^k}{k!}(A - sI)^k Q_Y(t), \quad t \geq 0 \quad (8.4)$$

for any integer $d$ with $\nu - 1 \leq d \leq m - 1$. Inspired by this observation, we take an integer $d$ with $\nu - 1 \leq d \leq m - 1$ and define

$$P(t) = \sum_{k=0}^{d} (-1)^k \frac{t^k}{k!}(A - sI)^k. \quad (8.5)$$

Then (8.4) can be written as

$$Y = P(t)Q_Y(t), \quad t \geq 0. \quad (8.6)$$

**Theorem 8.1.** There is a constant $C > 0$ such that

$$\|P(t)\tilde{X}(t) - Y\| \leq C\|V\|(1 + t^d)e^{-\delta t}, \quad t \geq 0. \quad (8.7)$$

**Proof.** By (8.3) and (8.6) we deduce that

$$\|P(t)\tilde{X}(t) - Y\| = \|P(t)\tilde{X}(t) - P(t)Q_Y(t)\|$$

$$\leq \|P(t)\|\|\tilde{Z}(t)\| \leq C\|V\|(1 + t^d)e^{-\delta t}, \quad t \geq 0.$$

This is precisely what we desired. □
In practice, one may have only a numerical value $\overline{s} := s + \Delta s$ of $s$. Therefore instead of $\bar{X}(t)$ and the polynomials $P(t)$ defined in (8.5), we need to take into account their approximations:

$$\bar{X}(t) = e^{t(A-\overline{s}I)}V, \quad \bar{P}(t) = \sum_{k=0}^{d} (-1)^k \frac{t^k}{k!}(A-\overline{s}I)^k.$$ \hfill (8.8)

Note that

$$\bar{X}(t) = e^{t(A-\overline{s}I)}V = e^{t(A-sI)}Y + e^{t(A-\overline{s}I)}Z = e^{-t\Delta s}Q_Y(t) + \bar{Z}(t) := \bar{Q}_Y(t) + \bar{Z}(t).$$ \hfill (8.9)

Hence

$$\bar{P}(t)\bar{X}(t) = \bar{P}(t)\bar{Q}_Y(t) + \bar{P}(t)\bar{Z}(t) = Y(t) + \bar{P}(t)\bar{Z}(t),$$ \hfill (8.10)

where

$$\bar{Y}(t) = \bar{P}(t)\bar{Q}_Y(t).$$ \hfill (8.11)

Since $Q_Y(t) \in \text{MGE}_s(A)$ and $\bar{P}(t)$ is a polynomial of $A$, by (8.7) it is easy to see that

$$\bar{Y}(t) = \bar{P}(t)\bar{Q}_Y(t) = e^{-t\Delta s}\bar{P}(t)Q_Y(t) \in \text{MGE}_s(A), \quad t \geq 0.$$ \hfill (8.12)

We may assume in advance that $|\Delta s| < \delta/3$. Then by (8.3) one has

$$\|\bar{Z}(t)\| = \|e^{t(A-\overline{s}I)}Z\| \leq e^{-t\Delta s}\|e^{t(A-sI)}Z\| \leq C\|V\|e^{-\frac{2}{3}\delta t}, \quad t \geq 0.$$ \hfill (8.13)

Thus one deduces that

$$\|\bar{P}(t)\bar{Z}(t)\| \leq \|\bar{P}(t)\|\|\bar{Z}(t)\| \leq C\|V\|(1 + t^d)e^{-\frac{2}{3}\delta t}, \quad t \geq 0.$$ \hfill (8.14)

It follows by (8.10) that

$$\|\bar{P}(t)\bar{X}(t) - \bar{Y}(t)\| \leq C\|V\|(1 + t^d)e^{-\frac{2}{3}\delta t}, \quad t \geq 0.$$ \hfill (8.15)

We also observe that

$$\|\bar{Y}(t) - Y\| = (\text{by (8.6)}) = \|\bar{P}(t)\bar{Q}_Y(t) - P(t)Q_Y(t)\|$$

$$= (\text{by (8.3)}) = \|e^{-t\Delta s}\bar{P}(t)Q_Y(t) - P(t)Q_Y(t)\|$$

$$\leq \|e^{-t\Delta s}\bar{P}(t) - P(t)\||Q_Y(t)||.$$ \hfill (8.15)

Now we assume that $|t\Delta s| \leq 1$. Then

$$\|e^{-t\Delta s}\bar{P}(t) - P(t)\| \leq \|e^{-t\Delta s} - 1\|\bar{P}(t)\| + \|\bar{P}(t) - P(t)\|$$

$$\leq C(1 + t^d)|t\Delta s| + C(1 + t^d)|\Delta s|$$

$$\leq C(1 + t^{d+1})|\Delta s|, \quad t \geq 0.$$
Hence by \(8.15\) we conclude that
\[
\| \bar{Y}(t) - Y \| \leq C(1 + t^{d+1})|\Delta s||Q_Y(t)| \leq C\|V\|(1 + t^{d+\nu})|\Delta s|.
\]

Let us write
\[
\bar{Y}(t) := \bar{P}(t)\bar{X}(t)
\]
and summarize the above results in the following theorem:

**Theorem 8.2.** Suppose \( |\Delta s| < \delta/3 \) and that \( t|\Delta s| \leq 1 \). Then there is a constant \( C > 0 \) depending only upon \( \|A\| \) and \( d \) such that
\[
\| \bar{Y}(t) - \bar{Y}(t) \| \leq C\|V\|(1 + t^{d+\nu})|\Delta s|, \quad t \geq 0.
\]

We infer from \(8.1\) that there are \( i_1, i_2, \ldots, i_\ell \in J \) such that \( \{y_{ik}\}_{1 \leq k \leq \ell} \) forms a basis of \( \text{GE}_s(A) \). Hence
\[
\|a_1 y_{i_1} + \cdots + a_\ell y_{i_\ell}\| \geq \eta > 0, \quad \forall a = (a_1, \cdots, a_\ell)^T \in \mathbb{R}^\ell, \quad \|a\| = 1.
\]

By \(8.17\) we see that for each fixed pair \((t, \Delta s)\), if \( \Delta s \) is sufficiently small so that \( \|\bar{Y}(t) - Y\| \leq \eta/2 \), then one has
\[
\|a_1 \bar{y}_{i_1}(t) + \cdots + a_\ell \bar{y}_{i_\ell}(t)\| \geq \eta/2, \quad \forall a = (a_1, \cdots, a_\ell)^T \in \mathbb{R}^\ell, \quad \|a\| = 1,
\]
where \( \bar{y}_{i}(t) \) denotes the \( i \)-th column vector of the matrix \( \bar{Y}(t) \). This indicates that the vectors \( \bar{y}_{i_k}(t) \) \((1 \leq k \leq \ell)\) are fairly linearly independent and therefore, in view of \(8.12\), form a well-conditioned basis of the space \( \text{GE}_s(A) \).

Unfortunately in practice \( \bar{Y}(t) \) is generally unknown. However, by Theorem \(8.2\) there exists \( t_0 > 0 \) such that
\[
\|\bar{Y}(t) - \bar{Y}(t)\| < \epsilon \quad \text{(8.18)}
\]
for \( t \geq t_0 \) and \( |\Delta s| < \delta/3 \), where \( \epsilon \) denotes tolerance error. Suppose now that \( \Delta s \) is sufficiently small so that we can pick \( t \geq t_0 \) suitably large such that \( \|\bar{Y}(t) - Y\| < 1 \). Then by \(8.18\) one can select a well-conditioned numerical basis for \( \text{GE}_s(A) \) from the column vectors of \( \bar{Y}(t) \).
8.2. An iterative method for the computation of $\tilde{Y}(t)$

The above discussion reduces the computation of the space $GE_s(A)$ to that of the matrix $\tilde{Y}(t)$ for $t$ sufficiently large and $|\Delta s|$ sufficiently small, which can be done by simply computing $\tilde{Y}(t) = \tilde{P}(t)\tilde{X}(t)$. Note that $\tilde{X}(t)$ is a matrix exponential. Although there have been many excellent computation methods for matrix exponentials, here we prefer to develop an iterative scheme as the one in Section 5 to compute the normalization of $\tilde{Y}(t)$, which avoid some technical difficulties (such as overflow for large $t$) in the computation.

- **Some basic estimates** Let us first give some estimates on the lower bounds of the norms of $Q_Y(t)$ and $\tilde{X}(t)$. Since
  \[ Q_Y(t) = \sum_{k=0}^{\nu-1} \frac{t^k}{k!} (A - sI)^k Y = e^{t(A-sI)}Y, \]
we deduce that $\|Q_Y(t)\| > 0$ for all $t \geq 0$. On the other hand, by the first equality in the above equation it is easy to see that
  \[ \|Q_Y(t)\| = t^{\nu-1} \left( \frac{1}{(\nu-1)!} \| (A - sI)^{\nu-1} Y \| + O(t^{-1}) \right) \quad \text{as } t \to \infty. \]
Thus there is a $t_0 > 1$ such that
  \[ \|Q_Y(t)\| \geq c_0 t^{\nu-1}, \quad t > t_0 \]
for some $c_0 > 0$. Take a $c_1 > 0$ with $c_1 < c_0$ such that $\|Q_Y(t)\| \geq c_1$ for $t \in [0, t_0]$. Then
  \[ \|Q_Y(t)\| \geq c_1 > 0, \quad \forall \ t \geq 0. \quad (8.19) \]
Consequently
  \[ \|\tilde{Q}_Y(t)\| = e^{-t\Delta s}\|Q_Y(t)\| \geq c_1 e^{-t|\Delta s|}, \quad \forall \ t \geq 0. \quad (8.20) \]
Further by (8.9) and (8.13) we conclude that
  \[ \|\tilde{X}(t)\| \geq \|\tilde{Q}_Y(t)\| - \|\tilde{Z}(t)\| \geq c_1 e^{-t|\Delta s|} - C\|V\|e^{-\frac{\Delta s}{3t}} \quad (8.21) \]
for $t \geq 0$. Take a $t_1 > 1$ such that $C\|V\|e^{-\frac{\Delta s}{3t}} < c_1/2$ for $t > t_1$. Then since $|\Delta s| < \delta/3$, we have
  \[ \|\tilde{X}(t)\| \geq c_1 e^{-t|\Delta s|} - \frac{1}{2} c_1 e^{-\frac{\Delta s}{3t}} \geq \frac{1}{2} c_1 e^{-t|\Delta s|}, \quad t > t_1. \]
Noticing that $\tilde{X}(t) = e^{t(A-\tilde{s}I)}V \neq 0$ for all $t \geq 0$, as in the case of $Q_Y(t)$, we can pick a number $c_2 > 0$ with $c_2 < c_1/2$ such that
  \[ \|\tilde{X}(t)\| \geq c_2 e^{-t|\Delta s|} \]
Recalling that \( \bar{t} \) for 

\[ \|X(t)\| \geq c_2 e^{-t|\Delta s|}, \quad t \geq 0. \]  

(8.22)

for \( t \in [0, t_1] \). Then

Now let us evaluate \( \|\bar{P} \left( \frac{\bar{x}}{\|\bar{x}\|} \right) - \frac{\bar{y}}{\|\bar{y}\|} \| \), where

\[ \bar{P} = \bar{P}(t), \quad \bar{x} = \bar{X}(t), \quad \bar{y} = \bar{Y}(t), \quad \bar{Q} = \bar{Q}_Y(t). \]

Recalling that \( \bar{X} = \bar{Q}_Y + \bar{Z} \) and \( \bar{Y} = \bar{P}\bar{Q}_Y \), we have

\[
\left\| \bar{P} \left( \frac{\bar{x}}{\|\bar{x}\|} \right) - \frac{\bar{y}}{\|\bar{y}\|} \right\| = \left\| \frac{\bar{y}}{\|\bar{y}\|} - \frac{\bar{P} \bar{x}}{\|\bar{x}\|} \right\| \leq \left\| \frac{\bar{y}}{\|\bar{y}\|} - \frac{\bar{y}}{\|\bar{y}\|} \right\| + \frac{\|\bar{P}\bar{Z}\|}{\|X\|}.
\]

(8.23)

By (8.13) and (8.22) we easily deduce that

\[
\frac{\|\bar{P}\bar{Z}\|}{\|X\|} \leq C\|V\|(1 + t^d)e^{-\frac{\delta}{3}t}, \quad t \geq 0.
\]

(8.24)

If we write \( \|\bar{x}\| = \|\bar{Q}_Y\| + r(t) \), then by \( \bar{X} = \bar{Q}_Y + \bar{Z} \) we see that

\[
r(t) \leq \|\bar{X} - \bar{Q}_Y\| = \|\bar{Z}\| \leq C\|V\|e^{-\frac{\delta}{3}t}.
\]

(8.25)

Thus

\[
\left\| \frac{\bar{y}}{\|\bar{y}\|} - \frac{\bar{y}}{\|\bar{y}\|} \right\| \leq \left\| \frac{1}{\|\bar{y}\|} - \frac{1}{\|\bar{Q}_Y\|} \right\| \|\bar{y}\| \leq \frac{r(t)}{\|\bar{y}\|\|\bar{Q}_Y\|} \|\bar{y}\|
\]

We observe that

\[
\|\bar{Y}\| = \|\bar{P}\bar{Q}_Y\| \leq \|\bar{P}\|\|e^{-t|\Delta s|\bar{Q}_Y}\| \leq C\|Y\|(1 + t^{d+\nu-1})e^{t|\Delta s|}
\]

for \( t \geq 0 \). Hence by (8.26) and the estimates on the lower bounds of \( \|\bar{X}\| \) and \( \|\bar{Q}_Y\| \) we deduce that

\[
\left\| \frac{\bar{y}}{\|\bar{y}\|} - \frac{\bar{y}}{\|\bar{y}\|} \right\| \leq C\|V\|\|Y\|(1 + t^{d+\nu-1})e^{-\frac{\delta}{3}t}, \quad t \geq 0
\]

provided that \( |\Delta s| < \delta/9 \). Combining this with (8.23) and (8.24), we have

**Lemma 8.3.** Assume that \( |\Delta s| < \delta/9 \). Then there exist \( C = C_V > 0 \) such that

\[
\left\| \bar{P} \left( \frac{\bar{x}}{\|\bar{x}\|} \right) - \frac{\bar{y}}{\|\bar{y}\|} \right\| \leq C(1 + t^{d+\nu-1})e^{-\frac{\delta}{3}t}, \quad t \geq 0.
\]

(8.26)

**• The iterative method** Let \( \bar{A} := A - \bar{s}I \). For each fixed \( n \in \mathbb{N} \), set

\[
T_n = \sum_{k=0}^{n} \frac{1}{k!} \bar{A}^k.
\]

47
Theorem 8.4. Suppose that $|\Delta s| < \delta/9$. Then for $n \in \mathbb{N}$, we have

$$\left\| \frac{\hat{P}(n)S_n}{\|\hat{P}(n)S_n\|} - \frac{\bar{Y}(n)}{\|\bar{Y}(n)\|} \right\| \leq C \|\hat{P}(n)S_n\|^{-1}(1 + n^d) \left( \frac{\lambda \|A\|}{n+1} \right)^{n+1} o(1) +$$

$$+ C \|\bar{Y}(n)\|^{-1}(1 + n^{d+2(\nu-1)}) e^{-\frac{2}{3}n\delta}. \quad (8.28)$$

Proof. Let $\bar{S}_n = \bar{X}(n)/\|\bar{X}(n)\|$. Repeating the same argument in Section 5.1 with the matrix $A$ therein replaced by $\bar{A}$, it can be shown that

$$\|S_n - \bar{S}_n\| \leq \bar{\lambda}^{n+1} o_n, \quad (8.29)$$

where $o_n = \|A\|^{n+1}/(n+1)!$. Therefore

$$\|\hat{P}(n)S_n - \hat{P}(n)\bar{S}_n\| \leq C(1 + n^d)\bar{\lambda}^{n+1} o_n. \quad (8.30)$$

For convenience in statement, in what follows we assign $O_{m \times m} = I$ for the zero matrix $O_{m \times m} \in \mathbb{M}_m$, and assign $0_\mathbb{R} = 1$ for the number $0 \in \mathbb{R}$. Then one trivially verifies that

$$\left\| \frac{M_1}{\|M_1\|} - \frac{M_2}{\|M_2\|} \right\| \leq 2 \left\| \frac{M_1 - M_2}{\|M_2\|} \right\|, \quad \forall M_1, M_2 \in \mathbb{M}. \quad (8.31)$$

Let $F_n = \|\bar{Y}(n)/\|\bar{Q}_V(n)\|\|$. Since $\|\bar{Q}_V(t)\| \leq C(1 + t^{\nu-1})e^{t|\Delta s|}$ for $t \geq 0$, we have

$$\|F_n\| \geq \frac{\|\bar{Y}(n)\|}{C(1 + n^{\nu-1})e^{n|\Delta s|}}, \quad n \in \mathbb{N}. \quad (8.32)$$

Therefore

$$\left\| \frac{\hat{P}(n)\bar{S}_n}{\|\hat{P}(n)\bar{S}_n\|} - \frac{\bar{Y}(n)}{\|\bar{Y}(n)\|} \right\| = \left\| \frac{\hat{P}(n)\bar{S}_n}{\|\hat{P}(n)\bar{S}_n\|} - \frac{\bar{F}_n}{\|\bar{F}_n\|} \right\| \leq 2\|F_n\|^{-1}\left\| \hat{P}(n)\bar{S}_n - \bar{F}_n \right\| \leq C \|\bar{Y}(n)\|^{-1}(1 + n^{d+2(\nu-1)}) e^{-\frac{2}{3}n\delta} \quad (8.33)$$
for all $n \in \mathbb{N}$.

Similarly we have

$$
\left\| \frac{\tilde{P}(n)S_n}{\| P(n)S_n \|} - \frac{\tilde{P}(n)\bar{S}_n}{\| P(n)\bar{S}_n \|} \right\| \leq C \| \tilde{P}(n)S_n \|^{-1} \| \tilde{P}(n)S_n - \tilde{P}(n)\bar{S}_n \| \\
\leq (\text{by (8.30)}) \\
\leq C \| \tilde{P}(n)S_n \|^{-1}(1 + n^d)\bar{\lambda}^{n+1}o_n \\
= C \| \tilde{P}(n)S_n \|^{-1}(1 + n^d) \left( \frac{\epsilon \| \bar{A} \| n^+}{n+1} \right)^{n+1} o(1)
$$

for $n \in \mathbb{N}$. Here we have used the Stirling’s formula. Combining this with (8.31) one immediately obtains the validity of (8.28). ■

**Remark 8.5.** In many cases one can use the method in Section 6 to compute the cyclic order $\nu := \text{Cord}(\text{GE}_s(A))$. Whence $\nu$ is determined, the best choice of the number $d$ is naturally $d = \nu - 1$.

**Remark 8.6.** As we have pointed out in Remark 5.7, in practice one may take an arbitrary nonsingular matrix $V$ as the initial one in the iteration.

- **Computation scheme of** $B_n := \tilde{Y}(n)/\| \tilde{Y}(n) \|$

As we have seen, the set of column vectors of $B_n$ provides a well-conditioned spanning set for the principal generalized eigenspace $\text{GE}_s(A)$. Thanks to Theorem 8.4 it is desirable to get numerical values of $B_n$ within tolerance error quickly by using the iterative method developed here. The computation scheme is as follows.

**Step 1.** Take $V = I$. Use the method in Section 6 to determine the cyclic order $\nu = \text{Cord}(\text{GE}_s(A))$.

**Step 2.** Use the combined method in Section 7 to derive a numerical value $\bar{s}$ of $s$ as accurate as possible.

**Step 3.** Use the iteration sequence defined in (8.27) to compute

$$
\tilde{B}_n := \bar{P}(n)S_n / \| \bar{P}(n)S_n \|,
$$

where $S_n = S_n(n)$. In general $\tilde{B}_n$ approaches $B_n$ exponentially. Hence the desired accuracy may be obtained swiftly by taking a suitably large $n$.

**Remark 8.7.** The invariant subspaces of a matrix can be successively computed via the famous Krylov subspace methods and its combinations with other algorithms; see e.g. Watkins [24] for a systematic discussion on this topic. In Bai
and Demmel [2] the authors developed another method to compute invariant subspaces by using matrix sign functions.

8.3. A numerical example

Example 8.1. Let

$$A = \begin{pmatrix}
-1 & 1 & 0 & 3 & 0 & -\frac{1}{2} & 3 \\
0 & 1 & 2 & 0 & -2 & 1 & 0 \\
3 & -\frac{1}{2} & -1 & -3 & 3 & -\frac{1}{2} & -3 \\
1 & 1 & -2 & 1 & 2 & -1 & -1 \\
0 & -\frac{1}{2} & -2 & 0 & 4 & -\frac{1}{2} & 0 \\
0 & -1 & 1 & 0 & -2 & 3 & -1 \\
-3 & -\frac{1}{2} & 2 & 3 & -2 & \frac{1}{2} & 5
\end{pmatrix}$$

The principal eigenvalue of the matrix is $s = 2$ with an algebraic multiplicity 5 and a geometric multiplicity 3. Taking $V = I = [e_i]_{i \in J}$, where $e_i$ is the $i$-th column vector of $I$, $J = \{1, 2, \cdots, 7\}$, we have

$$Y = [y_i]_{i \in J} = \begin{pmatrix}
-\frac{13}{2} & 0 & -\frac{1}{2} & \frac{15}{2} & \frac{1}{2} & 0 & \frac{15}{2} \\
15 & 1 & 3 & -15 & -3 & 0 & -15 \\
-21 & 0 & -\frac{5}{2} & 21 & \frac{5}{2} & 0 & 21 \\
-14 & 0 & -3 & 15 & 3 & 0 & 14 \\
-21 & 0 & -\frac{5}{2} & 21 & \frac{5}{2} & 0 & 21 \\
9 & 0 & 2 & -9 & -2 & 1 & -9 \\
15 & 0 & 5 & -15 & -5 & 0 & -13
\end{pmatrix}$$

where $y_i = \Pi_1 e_i$, and $\Pi_1$ is the projection from $\mathbb{R}^7$ to $\text{GE}_s(A)$. It is known that $\text{GE}_s(A) = \text{span}\{y_i : i \in J\}$.

• Numerical simulation

(1) Taking $N = 100$ and using the method in Section 5 we can get an approximate value $s_N$ of $s$: $s_N = 2.0203$.

(2) We use the method in Section 6 to determine the cyclic order $\nu := \text{Cord}(\text{GE}_s(A))$ of the space $\text{GE}_s(A)$.

Set $\varepsilon = 0.10$, and take $N = 100$ and $s_N = 2.0203$. Then the number $j$ satisfying (6.24) is $j = 7$. Take different $n \ll N$ and compute $\delta_k(n, s_N)$ defined in (6.20) for $k \in J$, we get the following table:

50
Table 9: Values of $\beta_k(n, s_N)$ with $N = 100, s_N = 2.0203$

| $n$ | $\beta_1(n, s_N)$ | $\beta_2(n, s_N)$ | $\beta_3(n, s_N)$ | $\beta_4(n, s_N)$ | $\beta_5(n, s_N)$ | $\beta_6(n, s_N)$ | $\beta_7(n, s_N)$ |
|-----|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 4   | 3.9192              | 1.2926              | 0.2612              | 14.1965             | 2.5930              | 27.7718             | 87.5299             |
| 5   | 3.8535              | 1.2355              | 0.0228              | 101.9814            | 10.4822             | 4.2120              | 31.0484             |
| 6   | 3.8418              | 1.1058              | 0.0475              | 22.2124             | 23.1030             | 14.1662             | 6.2880              |
| 7   | 3.8398              | 0.9839              | 0.1460              | 2.4422              | 46.0797             | 26.8717             | 19.3435             |
| 8   | 3.8211              | 0.8760              | 0.2686              | 0.2745              | 117.1914            | 42.0806             | 34.7450             |

It can be seen that if $n = 5, 6$ then the dichotomy property in (6.27) is fulfilled with $k_0 = 3$. Hence we deduce that $\nu = 3$.

(3) Now we use the combined method in Section 7 to obtain a refined approximate value $\bar{s}$ of $s$. Let $s_0 = s_N = 2.0203$ and $j = 7$. Consider the initial value problem:

$$\frac{d\tau}{dt} = (\gamma n)^4 \cdot 6 \cdot \langle (A - \tau I)^2 W_{n,j}, (A - \tau I)^3 W_{n,j} \rangle, \quad \tau(0) = s_0,$$

where $W_{n,j}$ is the $j$-th column vector of $W_n := M_n(n)$, and $\{M_n(k)\}_{n,k \in \mathbb{N}}$ is the sequence given by (5.6). Taking $\gamma = 0.20$ and $n = 20$, we obtain that

$$\bar{s} = \tau(150) = 2.000005037291918.$$

(4) Use the iteration sequence defined in (8.27) to compute

$$\bar{B}_n := \bar{P}(n)S_n / \|\bar{P}(n)S_n\|, \quad \text{where } S_n = S_n(n).$$

Let $B_n := \bar{Y}(n)/\|\bar{Y}(n)\|$. Denote by $\epsilon(\bar{B}_n, B_n)$ the error estimate between $\bar{B}_n$ and $B_n$, and by $\epsilon(B_n, Y/\|Y\|)$ the error estimate between $B_n$ and $Y/\|Y\|$. Then we have the following table.

Table 10: Numerical results

| $n$  | $\epsilon(B_n, B_n)$   | $\epsilon(B_n, Y/\|Y\|)$ |
|------|------------------------|---------------------------|
| 20   | $1.9181 \times 10^{-6}$ | 0.0015                    |
| 22   | $3.4588 \times 10^{-7}$ | 0.0020                    |
| 25   | $2.5222 \times 10^{-8}$ | 0.0029                    |
| 28   | $1.5701 \times 10^{-9}$ | 0.0041                    |
| 30   | $2.8581 \times 10^{-10}$| 0.0051                    |
References

[1] P.V. At, Diagonal transformation methods for computing the maximal eigenvalue and eigenvector of a nonnegative irreducible matrix, Linear Algebra Appl. 148 (1991) 93–123. doi:10.1016/0024-3795(91)90089-F

[2] Z.J. Bai, J. Demmel, Using the matrix sign function to compute invariant subspaces, SIAM J. Matrix Anal. Appl. 19 (1) (1998) 205–225. doi:10.1137/S0895479896297719

[3] R. Bellman, On an iterative procedure for obtaining the perron root of a positive matrix, Proc. Amer. Math. Soc. 6 (1955) 719–725. doi:10.2307/2032923

[4] Åke Björck, Numerical Methods in Matrix Computations, Springer, Switzerland, 2015. doi:10.1007/978-3-319-05089-8

[5] A.A. Brauer, A method for the computation of the greatest root of a nonnegative matrix, SIAM J. Numer. Anal. 3 (4) (1996) 546–569. doi:10.1137/0703047

[6] W. Bunse, A class of diagonal transformation methods for the computation of the spectral radius of a nonnegative irreducible matrix, SIAM J. Numer. Anal. 18 (4) (1981) 693–704. doi:10.1137/0718046

[7] M. Fasi, N. J. Higham, An arbitrary precision scaling and squaring algorithm for the matrix exponential, SIAM J. Matrix Anal. Appl. 40 (4) (2019) 1233–1256. doi:10.1137/18M1228876

[8] W. Ford, Numerical Linear Algebra with Applications: Using MATLAB, Academic Press, 2015.

[9] E. Gallopoulos, B. Philippe, A. H. Sameh, Parallelism in Matrix Computations, Springer, Dordrecht-Heidelberg-New York-London, 2016. doi:10.1007/978-94-017-7188-7

[10] H. Gene, Golub, F. Charles, V. Loan, Matrix Computations (4th ed.), The Johns Hopkins University Press, Baltimore, 2013.

[11] C.A. Hall, T.A. Porsching, Computing the maximal eigenvalue and eigenvector of a positive matrix, SIAM J. Numer. Anal. 5 (2) (1968) 269–274. doi:10.1137/0705023

[12] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Vol. 840, Lect. Notes in Math., Springer Verlag, Berlin New York, 1981. doi:10.1007/BFb0089647

[13] M. Hochbruck, C. Lubich, On krylov subspace approximations to the matrix exponential operator, SIAM J. Numer. Anal. 34 (5) (1997) 1911–1925. doi:10.1137/S0036142995280572
[14] A. S. Householder, The Theory of Matrices in Numerical Analysis, New York, 1975. doi:10.2307/2314680

[15] D.S.Li, M. Jia, On the perron-frobenius theory: An elementary dynamical approach via linear odes and generalized krein-rutman type theorems, https://arxiv.org/pdf/1803.02060.pdf

[16] T. Lyche, Numerical Linear Algebra and Matrix Factorizations, Springer Nature Switzerland AG, 2020. doi:10.1007/978-3-030-36468-7

[17] C. Moler, C.V.Loan, Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later, SIAM Review 45 (1) (2003) 3–49. doi:10.1137/1020098

[18] T. Noda, Note on the computation of the maximal eigenvalue of a non-negative irreducible matrix, Numer. Math. 17 (5) (1971) 382–386. doi:10.1007/BF01436087

[19] C. Prakash, An algorithm for computing the Perron root of a nonnegative irreducible matrix, PhD dissertation, North Carolina State University, Raleigh, North Carolina, 2007.

[20] Y. Saad, Iterative Methods for Sparse Linear Systems. 2nd edn, SIAM, Philadelphia, 2003.

[21] P. N. Swarztrauber, A parallel algorithm for computing the eigenvalues of a symmetric tridiagonal matrix, Math. Comput. 60 (202) (1993) 651–668. doi:10.2307/2153107

[22] B.N.Parlett, The Symmetric Eigenvalue Problem,vol.20 (2nd edn.), SIAM, Philadelphia, 1998. doi:10.2307/2007453

[23] D.S.Watkins, Fundamentals of Matrix Computations (2nd ed.), Wiley-Inter Science, NewYork, 2002. doi:10.1002/0471249718

[24] D.S.Watkins, The Matrix Eigenvalue Problem: GR and Krylov Subspace Methods, SIAM, Philadelphia, 2007. doi:10.1137/1.9780898717808

[25] D.S.Watkins, The QR algorithm revisited, SIAM Rev 50 (1) (2008) 133–145. doi:10.1137/100786554

[26] C.M.Wen, T.Z.Huang, A modified algorithm for the perron root of a nonnegative matrix, Appl. Math. Comput. 217 (9) (2011) 4453–4458. doi:10.1016/j.amc.2010.10.048

[27] H.Wielandt, Unzerlegbare, nicht negative matrizen, Math. Z. 52 (1) (1950) 642–648. doi:10.1007/8k02230720

[28] J.H.Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965. doi:10.2307/2003558
[29] K.Wu, H.D.Simon, Thick restart lanczos method for large symmetric eigenvalue problems., SIAM J. Matrix Anal. Appl. 22 (2) (2000) 602-616.

doi:10.1137/S0895479898334605