Non-Abelian Monopoles Coupled to Gravity

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ABSTRACT

A static configuration of point charges held together by the gravitational attraction is known to be given by the Majumdar-Papapetrou solution in the Einstein-Maxwell theory. We consider a generalization of this solution to non-Abelian monopoles of the Yang-Mills Higgs system coupled to gravity. The solution is governed by an analog of the Bogomol’nyi equations that had played a central role in the analysis of non-Abelian monopoles.
Under the combined forces of Coulomb repulsion and Newtonian attraction, one can envisage static configurations of any number of point masses at arbitrary locations. This happens when the charges, all of the same sign, are equal in magnitude with the corresponding masses in certain units so that the Newtonian attraction is balanced by the Coulomb repulsion. It is remarkable that such a static configuration is allowed in the framework of general relativity as a solution of the Einstein-Maxwell equations. This was first shown by Weyl[1] assuming axial symmetry and later generalized by Majumdar and Papapetrou[4]. Hartle and Hawking[3] have interpreted this solution as representing an assemblage of extreme Reissner-Nordström black holes. Due to the inherent symmetry between the electric and magnetic fields in the Maxwell theory, one can replace the electrically charged point masses of the Majumdar-Papapetrou solution by magnetic monopoles. In some respects, this suggests an analogy with the case of magnetic monopoles in the framework of spontaneously broken non-Abelian gauge theories. When the Higgs field responsible for breaking the symmetry belongs to the adjoint representation of the gauge group, there exists a ‘no-interaction’ result[4] for a system of monopoles governed by the Bogomol’nyi equations[3] in the Prasad-Sommerfield limit[4]. One is led to a very simple physical interpretation of this result as due to the balance of Coulomb-type repulsive forces among the magnetic charges and Yukawa-type attractive forces due to Higgs fields. A natural question that arises is what happens when such monopoles are coupled to gravity. In this letter, we consider a type of coupling to gravity that admits static solutions that may be viewed as a generalization of the Majumdar-Papapetrou solution to a non-Abelian theory or as an extension of the Bogomol’nyi equations to include gravity. These solutions represent static configurations of non-Abelian monopoles held in place by the gravitational forces rather than the Yukawa forces.

The coupling to gravity of the Yang-Mills Higgs system we consider is given by the following action[4]:

\[
S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{16\pi G v^2} R \phi^2 - \frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2} (D_{\mu} \phi^a)^2 - \frac{\lambda}{4} (\phi^2 - v^2)^2 \right\},
\]

where \( R \) is the Ricci scalar, \( F_{\mu\nu}^a \) is the field strength associated with the gauge field \( A^a_{\mu} \) and \( \phi^a \) is the Higgs field in the adjoint representation. \( D_{\mu} \phi^a \) is the gauge covariant derivative of \( \phi^a \) and \( \phi^2 \) is a short form for \( \phi^a \phi^a \). When \( \phi^2 = v^2 \), the first term gives the usual Einstein
Lagrangian. The rest of the terms represent the standard Yang-Mills Higgs system. We seek a static solution in this system assuming $A_0^a = 0$. We will set our unit of scale to correspond to $4\pi G = 1$.

In the absence of gravity, one can obtain a static solution in closed form in the Prasad-Sommerfield limit ($\lambda \to 0$) by use of Bogomol’nyi equations. In this limit, in terms of the ‘electric’ and magnetic fields

$$E_i^a = D_i \phi^a, \quad B_i^a = \frac{1}{2} \epsilon_{ijk} F_{jk}^a, \quad (2)$$

one deduces the following inequality for the energy functional:

$$E = \frac{1}{2} \int d^3x \left[ (E_i^a)^2 + (B_i^a)^2 \right]$$

$$= \frac{1}{2} \int d^3x (E_i^a \mp B_i^a) (E_i^a \mp B_i^a) \pm \int d^3x E_i^a B_i^a$$

$$\geq \pm \int d^3x E_i^a B_i^a, \quad (3)$$

where the last integral can be written as a surface integral using (2) and the Bianchi identity for $F_{ij}$. Clearly, this becomes an equality only if the Bogomol’nyi equations $E_i^a = \pm B_i^a$ are satisfied. For the gauge group SU(2), a spherically symmetric solution to these equations representing the ’t Hooft-Polyakov monopole (or antimonopole if $-$ sign is chosen for $\pm$) has been found in closed form [5, 6].

We find it remarkable that this system, when coupled to gravity as in (1), still admits monopole solutions with an ansatz that resembles the Bogomol’nyi equations. This happens when the dimensionless parameter $4\pi G v^2$ becomes unity, i.e. in our units $v = 1$. First, motivated by the Majumdar-Papapetrou solution, we make the following ansatz for the metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = V^2 dt^2 - \frac{1}{V^2} dx^i dx^i, \quad (4)$$

where $V$, independent of $t$, is a function of $x^i$’s. The components of the Ricci curvature are then easily computed:

$$R_{00} = V^3 \partial^2 V - V^2 (\partial V)^2, \quad R_{0i} = 0,$$

$$R_{ij} = \left[ \frac{\partial^2 V}{V} - \frac{(\partial V)^2}{V^2} \right] \delta_{ij} - 2 \frac{\partial_i V \partial_j V}{V^2}, \quad (5)$$
where $\partial$ refers to differentiation along the spatial directions. This yields for the Ricci scalar
the following expression:

$$R = -2V\partial^2 V + 4(\partial V)^2 = 2V^3\partial^2 \left(\frac{1}{V}\right). \quad (6)$$

In the background of the above metric, we wish to minimize the ‘energy functional’

$$\mathcal{E} = \int d^3x \left(\frac{1}{4V^2}R\phi^2 + \frac{1}{4}V^2 F_{ij}^a F_{ij}^a + \frac{1}{2}D_i \phi^a D_i \phi^a\right), \quad (7)$$

obtained from the action (1) ignoring the $\lambda$–term for the moment. The ‘electric’ and magnetic fields are defined to be

$$E_i^a = \frac{1}{V} D_i \chi^a, \quad B_i^a = \frac{V}{2} \epsilon_{ijk} F_{jk}^a, \quad (8)$$

where $\chi^a = V\phi^a$. It is now straightforward to rewrite the above energy functional in terms of $E$ and $B$\[8\]. The result agrees with the previous expression in Eq. (3) and hence obeys the same inequality. The lower bound is still given by a surface integral since the factors of $V$ coming from the above definitions cancel out in the product $E_i^a B_i^a$. Again, the equality holds only if the equations $E_i^a = \pm B_i^a$ are satisfied. In the present case, this implies

$$F_{ij}^a = \pm \frac{1}{V^2} \epsilon_{ijk} D_k \chi^a. \quad (9)$$

This resembles the well studied Bogomol’nyi equations of flat space theories except for the factor $1/V^2$.

Since the above procedure yields a minimum of the energy functional, the Euler-Lagrange equations of motion for $A_i^a$ and $\phi^a$ will be automatically satisfied. It follows from \[8\] that these equations are

$$D_i \left(V^2 F_{ij}^a\right) = gf_{abc} \phi^b D_j \phi^c, \quad (10)$$

$$V^2 D_i (D_i \phi^a) = \frac{1}{2} R \phi^a + \lambda \left(\phi^2 - v^2\right) \phi^a, \quad (11)$$

where $g$ (not to be confused with the determinant of the metric) is the gauge coupling constant and $f_{abc}$'s are the structure constants of the group. It is now easily checked, using \[8\], that Eq. \[10\] is identically satisfied. The ansatz \[8\] requires, as a consequence of the Bianchi identity for $F_{ij}$, that $\chi$ satisfy

$$D_i \left(\frac{1}{V^2} D_i \chi^a\right) = 0. \quad (12)$$
Using this we find that Eq. (11) is also satisfied (again, momentarily ignoring the $\lambda$–term) when the Ricci scalar is given by our previous expression in Eq. (6). Thus the equations of motion for the matter fields are all automatically satisfied.

The gravitational equations are obtained by varying the action (1) with respect to the metric $g_{\mu\nu}$. They take the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{2}{\phi^2}T_{\mu\nu},$$  \hspace{1cm} (13)

where $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = V^2 F_{\mu i}^a F_{\nu j}^a + D_{\mu} \phi^a D_{\nu} \phi^a + g_{\mu\nu} \left\{ \frac{1}{4} V^4 F_{ij}^a F_{ij}^a + \frac{1}{2} V^2 D_i \phi^a D_i \phi^a + \frac{\lambda}{4} (\phi^2 - v^2)^2 \right\},$$  \hspace{1cm} (14)

upto terms involving derivatives of $\phi^2$. Remarkably, these are all satisfied when the following holds:

$$\phi^2 = v^2 = 1, \quad \text{or} \quad \chi^2 = V^2.$$  \hspace{1cm} (15)

This can be easily verified using Eqs. (5), (9) and (12). We note that our neglect of the $\lambda$–term is justified since it vanishes under this condition. All that is required of the scalar potential is that it and its first derivative with respect to $\phi^2$ vanish at $\phi^2 = v^2$.

It remains to solve Eq. (9), or (12), to obtain the final solution. Upto now we have not made any assumption about the gauge group. If we have U(1), we can drop the indices $a, b, \cdots$. Then $\chi^2 = V^2$ implies $\chi = \pm V$ and hence from Eq. (12) we get $\partial^2 (1/V) = 0$. The resulting metric agrees with the well known Majumdar-Papapetrou solution for the Einstein-Maxwell theory[2]. In this case, the scalar field is completely frozen at its vacuum value. The solution corresponds to a collection of point monopoles held together by the gravitational attraction.

Also, we have not made any assumption about the number of monopoles or the symmetry of the solution. There should exist solutions representing static configurations of non-Abelian monopoles, but unlike those of flat space theories, such configurations will be held in place by the gravitational forces rather than the Yukawa forces. This is because the Higgs field is effectively at its vacuum value as can be seen from (15). Balance of forces requires that any such solution has, in our units, its total mass equal to its total monopole charge. Note that the surviving magnetic field at spatial infinity is

$$B_i = B_i^a \phi^a = \pm \frac{1}{V^2} D_i \chi^a \chi^a = \pm \frac{1}{2V^2} \partial_i \left( \chi^2 \right) = \pm \frac{1}{V} \partial_i V,$$  \hspace{1cm} (16)
where we used Eqs. (8), (9) and (15). Now, at spatial infinity, \( V \) is expected to have the behaviour
\[
V \to 1 - \frac{M}{4\pi r}, \quad r \to \infty,
\]
where \( r \) is the radial distance and \( M \) is the total mass of the system. Using this in (16) we obtain the total monopole charge, given by the total magnetic flux emanating from the system, to be equal to \( \pm M \) that agrees in magnitude with the total mass. One can also get this result from the minimum of the energy functional:
\[
\mathcal{E} = \pm \int d^3x E^a_i B^a_i = \int d^3x E^a_i E^a_i = \int d^3x \frac{1}{V^2} D_i \chi^a D_i \chi^a,
\]
which after partial integration using (12) reduces to a surface integral to give the total monopole charge.

Next we consider the gauge group \( SU(2) \) and specialize to the case of spherical symmetry. As in the case of the 't Hooft-Polyakov monopole, we start with the following ansatz for \( \chi^a \) and \( A^a_i \):
\[
\chi^a = \mp \hat{r}^a V(r), \quad A^a_i = \frac{1}{gr} \epsilon^{aib} \hat{r}^b \left( 1 - W(r) \right),
\]
where \( \hat{r} \) is the unit vector along the radial direction. This gives, for the electric and magnetic fields defined in (8),
\[
E^a_i = \mp \delta^{a_i} \frac{W}{r} \mp \hat{r}^a i \left( \frac{V'}{V} - \frac{W}{r} \right),
B^a_i = \frac{V}{g} \left\{ \delta^{a_i} \frac{W'}{r} - \hat{r}^a \hat{r}^i \left( rW' + 1 - W^2 \right) \right\},
\]
where prime denotes differentiation with respect to \( r \). The Bogomol’nyi equations \( E^a_i = \pm B^a_i \) then lead to the following nonlinear system:
\[
\begin{align*}
    r^4 F'' &= e^{-2F}, \\
    \frac{g}{V} &= \frac{1}{r} + F',
\end{align*}
\]
where \( F \) is defined by \( W = e^{-F}/r \). Eq. (21) is a special case of the so-called Lane-Emden equations that play a central role in the internal constitution of stars. To our knowledge, it has not been solved analytically. However, one can show that there exists a class of solutions with the asymptotic behaviour
\[
\begin{align*}
    F &\to Cr + D, \quad r \to \infty, \\
    F &\to -\ln r, \quad r \to 0,
\end{align*}
\]
where $C > 0$ and $D$ are two constants. The expected behaviour of $V$ given by (14) thus follows from (22) and (23) if $C = g$ and $M = 4\pi / g$. This is in accordance with our earlier result that the mass and the monopole charge coincide. However, the behaviour of $V$ as $r \to 0$ is not that simple. Direct use of the asymptotic behaviour of $F$ gives $V = \infty$. We need the next correction to this result to come to any conclusion. It is easily found that the correction to (24) is of the form

$$A\sqrt{r} \sin \left( \frac{\sqrt{7}}{2} \ln r + \delta \right),$$

(25)

for some constants $A$ and $\delta$. Use of this suggests that $V$ goes to zero as $\sqrt{r}$, as $r \to 0$. However, it does so in an oscillatory way. The area of any sphere surrounding the origin, $4\pi r^2 / V^2$, vanishes periodically and tends to zero in this limit. The Ricci scalar given by (3) goes as $-4V^2 / r^2$ and diverges whenever the area vanishes, the geometry being singular for those values of $r$. What is then the nature of the singularity at $r = 0$? In the case of the Majumdar-Papapetrou solution for the metric, the area tends to a constant and one approaches the event horizon of the black hole in this limit [3]. In our case the origin does have an event horizon, but with zero surface area. This suggests that the singularity at the origin is close to being naked. There are no other horizons to dress this singularity since $1/V$ behaves smoothly, and hence $V$ does not vanish, for finite values of $r$.

It seems that the above features are specific to $4\pi G v^2$ being unity and to get a better understanding, one needs move away from this critical value. We may test the sensitivity of our conclusions by modifying the equation for $F$ as follows:

$$r^{4\gamma} F'' = e^{-2\gamma F},$$

(26)

where $\gamma$ is somehow related to $4\pi G v^2$ and when unity yields the critical case discussed earlier. It is now easy to note that the asymptotic behaviour of $F$ as $r \to \infty$ is unchanged when $\gamma$ is greater than 1/2, but for $r \to 0$ one finds

$$F \to -\left( 2 - \frac{1}{\gamma} \right) \ln r - \frac{1}{2\gamma} \ln \left( 2 - \frac{1}{\gamma} \right), \quad r \to 0.$$  

(27)

This shows that near origin $V$ approaches zero as $r$,

$$V \to \frac{g\gamma}{1 - \gamma} r, \quad r \to 0.$$  

(28)
This is what is expected for a black hole with a finite event horizon at the origin. This sounds attractive, but we have not been able to derive this case from an action principle.

The constant $D$ of Eq. (23) is not determined by the data at spatial infinity, leading to a one parameter family of solutions. It appears to be a measure of the inverse size of the core containing non-Abelian structure. As it tends to $\infty$ the core collapses to the origin and the solution coincides with that of Majumdar and Papapetrou for a single monopole. This case, where $W$ is zero and $1/V = 1 + 1/gr$, can be verified to follow directly from Eqs. (20). It lies entirely within the U(1) subgroup of SU(2) defined by the direction of the Higgs field.

To summarize, we have analyzed a type of gravitational coupling to non-Abelian monopoles and obtained a solution that may be viewed as a generalization of the Majumdar-Papapetrou solution or as an extension of the Bogomol’nyi equations. The case of conventional coupling of gravity to non-Abelian monopoles have been discussed in the literature[10] drawing support from numerical treatments. The advantage of the present approach is that it admits analytical treatment to a large extent. Perhaps, this is possible only in the critical case discussed in this letter and moving away from criticality may require numerical analysis. Secondly, the present approach opens up the possibility of analyzing multimonopole solutions in the presence of gravity and it will be interesting if such solutions exist and the known techniques are applicable in analyzing them. It will also be interesting if there exists a stationary generalization analogous to the one that exists for the Majumdar-Papapetrou solution[11].

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[7] In our conventions, $g_{\mu\nu}$ has the signature $(+ - - -)$. Indices $\mu, \nu, \cdots$ run from 0 to 3 and $i, j, \cdots$ from 1 to 3. Indices $a, b, \cdots$ run over the adjoint representation of the gauge group. Repeated indices are assumed to be summed.

[8] According to G. W. Gibbons and S. W. Hawking, Phys. Rev. D15, 2752 (1977), one should add to the gravitational action a surface integral of the trace of the extrinsic curvature. In our case, we need to include this surface term (times $\phi^2$) in the action (1) since the energy functional, as we defined it, will not otherwise yield the usual result for the total energy of the system. This surface term cancels a similar term that arises when rewriting the energy functional in terms of $E$ and $B$.

[9] Eq. (21) arises when one considers an isothermal gas sphere in gravitational equilibrium. For a detailed discussion of this and related equations see ‘An Introduction to the Study of Stellar Structure’, S. Chandrasekhar, Dover Publications (1957), chapter IV. The present equation agrees with Eq. (383) in this chapter after the identifications $x = r$ and $\psi = 2F - \ln 2$. We do not, however, agree with the asymptotic behaviour quoted in Eq. (449) to be compared with Eq. (23) of this letter.

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