An analysis of the Sun–Earth–Asteroid systems based on the two–centre problem∗

Gabriella Pinzari†
Dipartimento di Matematica “Tullio Levi Civita”
Università di Padova
gabriella.pinzari@math.unipd.it

May 26, 2017

Abstract
We propose a new analysis of the two–centre problem particularly suited to be used as a basis to study the dynamics of Sun–Earth–Asteroid systems. Our method, based on a tricky choice of initial coordinates, allows us to evaluate the risk of collisions between the Asteroid and Earth. Moreover, it seems to be fitted to prove the existence of chains of transition tori in the planar Sun–Earth–Asteroid systems.

∗MSC2000 numbers: primary: 37J30, 37J35, 37C29. Keywords: gravitation.
†The author is partially supported by the H2020 Project ERC Starting Grant 677793 StableChaoticPlanetM.
1 Introduction

The Law of Universal Gravitation, according to which, any two masses in the Universe attract each other with a law going as the inverse squared distance, was stated in 1687 by Isaac Newton. Newton was aimed to find a theoretical explanation to the laws discovered by Johannes Kepler between 1609 and 1619. At the same time, he provided the exact solution of the simplest gravitational system: the two–body problem (2BP), or: the problem of Sun and Earth. He also tried to attack the analogous problem with three masses (3BP: Sun, Earth and Moon), and then gave up, calling it a ‘head ache problem’. In 1899 Henri Poincaré proved the non–integrability of 3BP, and this motivated him to introduce the concept of chaos in mathematics, [14]. A major breakthrough came from Kolmogorov–Arnold–Moser (KAM) theory, developed starting with the fundamental papers [9, 10, 1]. Rather than finding an explicit time law for the coordinate functions, KAM theory turned to have, as main objective, that of providing an estimate of ‘stable motions’, from the probabilistic (measure–theoretic) point of view. Clare answers about the ‘metric’ stability of the planetary problem (the problem of one star and any number of smaller masses) have been given in [3, 15, 8, 12, 7].

Much less known as further example of exactly solved gravitational system is the so–called two–centre problem (2CP), solved by Euler in XVIII century. It consists of one particle, attracted by two fixed masses. The model might, at first sight, seem unsatisfactory from the physical point of view, but, as one might argue, may be regarded as good initial approximation to 3BP. Its solution is given in the form of a non–linear system involving elliptic integrals [4, Eq. (53)]. Recently, new analysis of the problem have been worked out, by [16, 5], to which papers we refer for an overview on motivations and complete references. Maybe due to the difficulty of handling, at practical level, Euler’s solutions, 2CP has been not frequently used in the study of 3BP. A study in this direction goes back to [6], who applied 2CP to the restricted 3BP, a model of the true system that will be recalled below.

In this paper, we present an analysis of 2CP that allows us to write at least a first–order (in the ratio ε of the masses of the attracting centers) solution explicitly. Therefore, the method works well when ε is small. The novelty with respect to previous analysis is that the major information on the dynamics generated by 2CP is deferred to the one of its ‘Euler integral’ (the first integral, found by Euler, that determines the integrability of 2CP). More precisely, we prove that, in a suitable set of coordinates, 2CP’s Hamiltonian and its Euler integral have the same trajectories, up to a rescaling of time. Since, in such coordinates, the Euler integral is much simpler than the original two–centre Hamiltonian, at least at the first order in the masses’ ratio, our trick consists in studying the dynamics of the latter function. The method looks particularly amenable in the planar problem, because, in this case, the phase portrait of the Euler integral may be studied exactly, or at least its leading part (this is done Section 2.1.7; see also Figure 1). In the case of the spatial problem, the procedure can still be applied, but the analysis would require to solve a cubic equation (in the planar case, the analogous equation reduces to order two). However, we do not insist to develop the theory of the spatial problem (which certainly is the next step of the research). Rather, aiming highlight the utility of the method from the concrete point of view, we discuss an application to a particular 3BP, namely, the planar Sun–Earth–Asteroid system (SEA). Most of times, SEA is studied, from the theoretical point of view, as a restricted three–body problem. This is a model where the two most massive bodies are constrained on circular, co–planar trajectories having a common centre. A third small body is attracted by the two, without the two are attracted by it. Notwithstanding the important results that have been obtained from the study of the restricted problem, the model is affected by an important limitation: the too low (two) number of degree of freedom causes, by KAM theory, a confinement at all times of action coordinates, in the sense of [11]. Such confinement is not expected to hold for the true system, since the real Hamiltonian has three, four degrees of freedom in the planar, spatial case, respectively.

We propose an alternative analysis of SEA, based on its full Hamiltonian, rather than a model.
We write such Hamiltonian as a small perturbation of $2\mathcal{CP}$. Using our new approach to $2\mathcal{CP}$, we prove, in the planar sea, the existence of stable motions, with the pericentre of the Asteroid performing librations or complete rotations, even in the case that the orbit of the Asteroid around the Sun encloses Earth – a situation, geometrically, at risk of collisions between the two. As a byproduct of the proof, the risk of collision may be evaluated simply looking at the numerical value of the aforementioned Euler integral. Indeed, we find that collisions occur only if (lowest order approximation) of Euler integral takes a suitable value, depending on the mass of Earth and its distance from the sun.

This paper is organized as follows. In Section 2 we discuss the procedure (based in an essential way on a good choice of canonical coordinates) that allows us to find the equivalent Hamiltonian, and we discuss its application to sea. In Section 3 we draw conclusions and foresee perspectives of future work. In particular, we conjecture the existence of chains of transition tori, in the sense of [2], in the planar sea. In order to keep the paper as much readable as possible, we relegate the most technical parts to the appendices.
2 SEA system via 2CP

2.1 The two–centre problem

The two–centre problem is the problem of determining the motions of one moving mass \( m \) gravitationally attracted by two fixed masses \( M, M' \).

Let us fix a orthonormal frame \((i, j, k)\) in \( \mathbb{R}^3 \). After changing the time \( t' := gm^2 M t \), and with \( m := gm^2 M, \varepsilon := M'/M \), where \( g \) is the gravity constant, we write the Hamiltonian as

\[
  h := \frac{|y|^2}{2m} - \frac{1}{|x|} - \frac{\varepsilon}{|x' - x|} \quad x \notin \{0, x'\}
\]  

where \( y, x \) are impulse–position coordinates of the attracted body (\( y = m \dot{x} \), in the new time), while \( |x| \) denotes Euclidean norm. Note that the two attracting centers have been posed at \( 0, x' \), rather than, as more commonly done, at symmetric positions (e.g., \( \pm i \)) with respect to the origin.

For generality, we refer to the spatial problem, namely, \( y, x, x' \in \mathbb{R}^3 \). Later on, we shall reduce to the planar case as a sub–case of the spatial one.

2.1.1 First integrals

In view of the application to sea, we regard \( h \) as a six–degrees of freedom system, i.e., as a function of \((y', y, x', x) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{x = 0, x = x'\}\), which is constant with respect to \( y' \).

In such a enlarged phase space, the vectors \( x' \) and \( C_t = C' + C \) (the total angular momentum vector), where \( C' = x' \times y' \), \( C = x \times y \), are first integrals to \( h \). We then have six conserved quantities, which however are not all mutually Poisson–commuting. It is nevertheless possible to extract, out of such six quantities, the following four commuting ones:

\[
  Z := C_t \cdot k, \quad G := |C_t|, \quad \Theta := \frac{C \cdot x'}{|x'|}, \quad r' = |x'|.
\]  

(2)

Obviously, one expects that two more first integrals, independent of \((2)\), can be chosen. This aspect will be discussed in Section 2.1.4 below.

The integrability of \( h \) relies on the existence of a further independent, commuting first integral, found by Euler. It is defined as follows. Let \( e, P \), with \( e \neq 0 \) and \(|P| = 1\), be the eccentricity and the pericentre direction of the Keplerian orbit associated to

\[
  \mathcal{K} = \frac{|y|^2}{2m} - \frac{1}{|x|}.
\]  

(3)

Then Euler’s first integral to \((1)\) is

\[
  G = \mathcal{G}_0 + \varepsilon \mathcal{G}_1
\]  

(4)

where

\[
  \mathcal{G}_0 := |C|^2 - me \dot{x} \cdot P, \quad \mathcal{G}_1 := m \frac{(x' - x) \cdot \dot{x}'}{|x' - x|}.
\]  

(5)

The formula \((5)\), in this precise form, is not standard in the literature, since usually Euler’s integral is written in terms of elliptic coordinates. A derivation of it may be found in Appendix A.1.

Note that \( \mathcal{G}_0 \) is a combination of first integrals to \( \mathcal{K} \), as one should expect, since \( h \) reduces to \( \mathcal{K} \) when \( \varepsilon = 0 \). This also explains why, as one can see from \((5)\), \( \mathcal{G}_0 \) has no singularities for \( x = x' \), while \( \mathcal{G}_1 \) and \( h \) do have.
2.1.2 Kepler maps and $\lambda$–normalization

We write $h = K + \varepsilon U$ where $K$ is the Keplerian term (3), while $U = -\frac{1}{|x_k|}$. We look at systems of canonical coordinates of the form $k = ((\Lambda, u, v), \lambda) \subset \mathbb{R}^{11} \times \mathbb{T}$ under which (for $K < 0$) this term takes the form

$$h_K(\Lambda) = -\frac{m}{2\Lambda^2}$$

with $\lambda$ having the form $\lambda = \ell + \varphi(\Lambda, u, v)$, where

$$\Lambda = \sqrt{ma}, \quad \ell = \text{mean anomaly of } x_k$$

with $a$ the semi–major axis of the ellipse generated by (3). We call such kind of systems partial Kepler maps. Then $h$ becomes

$$h_k((\Lambda, u, v), \lambda) = -\frac{m}{2\Lambda^2} - \frac{\varepsilon}{|x_k'(\Lambda, u, v) - x_k((\Lambda, u, v), \lambda)|}$$

with $x_k$, $x'_k$ denoting $x'$, $x$ written in terms of $k$. Note that, using a terminology introduced in [3], $h_k$ is a ‘properly–degenerate’ Hamiltonian, in the sense that its unperturbed part depends on a number of action coordinates strictly less than the number of degrees of freedom. This is a quite common fact in gravitational problems.

We have to fix a domain for the coordinates so as to exclude collisions. We impose a strong non–collision condition taking $D = \Delta^c \times \mathbb{T}$, where

$$\Delta := \left\{(\Lambda, u, v) : \exists \lambda : x_k'(\Lambda, u, v) = x_k((\Lambda, u, v), \lambda)\right\}.$$  \hspace{1cm} (9)

Note, for next need, that $\Delta$ can be split as a product

$$\Delta = S \times \Pi$$ \hspace{1cm} (10)

where:

- $\Pi$ is the set of $(\Lambda, u, v)$ such that $x_k'(\Lambda, u, v)$ lies on the plane of the orbit $\lambda \rightarrow x_k$, and hence (being orthogonal to $C_k(\Lambda, u, v)$)

$$\Pi = \{(\Lambda, u, v) : \Theta = 0\} ;$$ \hspace{1cm} (11)

- $S$ is defined so that, if $\theta(\Lambda, u, v)$ denotes is the convex angle formed by $x_k'(\Lambda, u, v)$ and $P_k(\Lambda, u, v)$, namely,

$$\theta : \cos \theta = \frac{x'_k \cdot P_k}{r'}$$ \hspace{1cm} (12)

then the distance $r'(\Lambda, u, v) = |x'_k|$ of $x'_k$ from the sun satisfies the equality

$$r' = \frac{a(1 - e^2)}{1 + e \cos \theta}.$$ \hspace{1cm} (13)

Using (12) into (13) and recalling that $a(1 - e^2) = G^2/\varepsilon$ and the definition of $\mathcal{G}_0$ in (5), one sees that

$$S = \{(\Lambda, u, v) : \mathcal{G}_0 = mr'\}.$$ \hspace{1cm} (14)
2.1.3 Streaming ideas

Before, we switch to details, let us try to give an informal account on the ideas we put in play in the next sections. One reasonably expects that, on the collision–less set $\mathcal{D}$ defined above, due to the close–to–be integrable and simultaneously integrable properties of $h_k$ in (8), one can eliminate, in $h_k$, the dependence on $\lambda$ via a convergent perturbative procedure (in the parameter $\varepsilon$), so as to transform, via an $\varepsilon$–close to the identity canonical transformation

$$\kappa = (\Lambda, \lambda, u, v) \to \overline{\kappa} = (\overline{\Lambda}, \overline{x}, \overline{\pi}, \overline{v})$$

(15)

the Hamiltonian $h_k$ in (8) to

$$\overline{\kappa} := h_{\overline{\kappa}} = -\frac{m}{2\Lambda^2} + \varepsilon U + \cdots$$

(16)

where $U$ is the $\overline{\lambda}$–average of $U$, etc. Because of the proper degeneracy mentioned above, there would be, in principle, many ways to obtain (16), depending on how many choices of partial Kepler maps coordinates one has at our disposal. However, whatever is the choice of $\kappa$, one can prove that the function

$$\overline{\mathcal{G}} := \mathcal{G}_{\overline{\kappa}}$$

(17)

corresponding to $\mathcal{G}$ in (4), written in the coordinates at right hand side in (15), is itself $\overline{\lambda}$–independent (details are in Appendix A.3). Now, in the next Section 2.1.4, we shall present a carefully chosen Kepler map such that $h_k$ and $\mathcal{G}_k$ have two effective degrees of freedom (they depend on two angle coordinates only). The trick is that $\kappa$ includes, among its coordinates, many first integrals of $h$ (all of them, but $\mathcal{G}$). In this situation, the corresponding functions $\overline{\kappa}, \overline{\mathcal{G}}$ in (16)–(17) have just one effective degree of freedom and, in addition, Poisson–commute. This implies that, up to rescaling of time (that will be quantified in Section 2.1.5), they have the same trajectories. But since $\mathcal{G}_0$ is $\lambda$–independent and $\overline{\mathcal{G}}$ coincides with $\mathcal{G}_0$ for $\varepsilon = 0$, one definitely has that, for $\varepsilon$ small, the main information on the dynamics of $h$ is nothing else than the one $\mathcal{G}_0$, which, in the aforementioned coordinates has a very simple expression (see (21) below).

Now, it is not simple to prove directly the convergence of the series (16). Therefore, we shall prove it a posteriori, under suitably more stringent assumptions, all of them verified in our application (of course, we expect that the convergence of the series (16) holds in a more general situation). We defer this latter check to the Appendix A.2, being of purely technical nature.

2.1.4 Choice of the Kepler map

We propose a certain partial Kepler map $\kappa$ which includes $\Lambda$, $\ell$ in (7) and, moreover, the functions in (2). The complete set of coordinates is denoted as

$$\kappa = (Z, G, \Theta, R', \Lambda, G, z, \gamma, \theta, r', \ell, g)$$

(18)

where $Z$ is the generalized impulse conjugated to the position coordinate $z$, etc. To define the remaining coordinates, we need the following notations. For $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a vector $w$, $\alpha_w(u, v)$ denotes the positively oriented angle between $u$ and $v$, as seen from $w$ according to the right hand rule. Define ‘nodes’ $n_1$ as $n_0 := k \times C_t$, $n_1 := C_t \times x'$ and $n := x' \times C$ and assume that $C_t$, $C$, $x'$, $x$, $n_0$ and $n$ do not vanish. Then define

$$R' := \frac{y' \cdot x'}{|x'|}, \quad G := |C|, \quad g := \alpha_C(n, C \times P)$$

$$z := \alpha_k(i, n_0), \quad \gamma := \alpha_{C_1}(n_0, n_1), \quad \theta := \alpha_{x'}(n_1, n) .$$

(19)

The coordinates $\kappa$ are canonical, since they can be easily derived from another set of canonical coordinates\(^1\)

$$p = (Z, G, \Theta, R', R, \Phi, z, \gamma, \theta, r', r, \varphi)$$

(20)

\(^1\)With respect to the notations in [13], in (20) we have renamed $C_3 = Z$, $G = G$, $R_1 = R'$, $R_2 = R$, $\zeta = z$, $g = \gamma$, $r_1 = r'$, $r_2 = r$. 

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whose canonical character has been discussed in \cite[Section 2]{13}, up to change the quadruplet \((R, \Phi, r, \varphi)\) in \(P\) with the quadruplet \((\Lambda, G, \ell, g)\) in \(K\), where the two quadruplets are related via the classical (canonical) Delaunay map. Note that, as a common aspect, the coordinates \((Z, G, z)\) depend only on \(C_1\), while \((\Theta, r', \gamma)\) depend on \(C_0\) and \(x'\). Therefore all such six coordinates are first integrals to \(h\) and \(\bar{G}\). This implies that the two couples \((Z, \gamma)\) and the two coordinates \(\vartheta\) and \(R'\) are cyclic in \(h_k\) and \(\bar{G}_k\). Then \(h_k\) and \(\bar{G}_k\) have just two effective degrees of freedom, since they depend only on the six coordinates \((\Lambda, \ell, G, g; r', \Theta)\), including two only angles. Note that: (i) along the \(h_k\)- motions, the angle \(\ell\) is a fast, while \(g\) is slow. Moreover, (ii) \(\mathcal{G}_0\) in (5) is \(\ell\)-independent, being, as said, a first integral to \(K\) in (6). Its expression in terms of \(K\) is

\[
\mathcal{G}_0(r', \Lambda, \Theta, G, g) = G^2 + m\varepsilon \sqrt{1 - \frac{\Theta^2}{G^2}} \sqrt{1 - \frac{G^2}{A^2}} \cos g.
\]

(21)

2.1.5 Renormalizable integrability

Let \(\kappa, \tilde{h}, \bar{G}\) as in (15), (16), (17), with \(K\) chosen as in (18). In particular, (15) becomes

\[
\kappa = (\Theta, R', \Lambda, G, \vartheta, r', \gamma, g) \rightarrow K = (\Theta, R', \Lambda, G, \vartheta, r', \gamma, g)
\]

(22)

(the four coordinates \(Z, G, z, \gamma\) have been neglected). In terms of \(\kappa\), we have \(\tilde{h} = \tilde{h}(r', \Lambda, \Theta, G, g)\), \(\bar{G} = \bar{G}(r', \Lambda, \Theta, G, g)\). Moreover, if \(\mathcal{G}_0\) is as in (21),

\[
\bar{G}(r', \Lambda, \Theta, G, g) = \mathcal{G}_0(r', \Lambda, \Theta, G, g) + O(\varepsilon).
\]

(23)

The commutation of \(\tilde{h}\) and \(\bar{G}\) and an Implicit Function Theorem argument allow to infer that \(\tilde{h}\) may be written as a function of \(\bar{G}\) and the other integrals:

\[
\tilde{h}(r', \Lambda, \Theta, G, g) = \tilde{h}(r', \Lambda, \Theta, G, g).
\]

(24)

We shall use this formula in order to infer informations on the dynamics. In view of the application to the Hamiltonian

\[
\Pi_0(r', \Lambda, \Theta, G, g) = \Pi_0(r', \Lambda, \Theta, G, g) + h_0(r') + g\tilde{h}(r', \Lambda, \Theta, G, g)
\]

(25)

with \(h_0\) depending only on \(r'\). We then may write the solutions of \(\Pi_0\) in terms of the solutions of \(\bar{G}\). If \(\Delta = \mathbb{F} \times \Pi\) denotes the singular set \(\Delta\) in (10) expressed in the coordinates at left hand side of (22) and \(t \rightarrow (\bar{G}(r', \Lambda, \Theta, G, g))\) is any solution to \((\bar{G}(r', \Lambda, \Theta, G, g))\) with initial datum \((\bar{G}_0, \bar{G}_0)\), and \(r_0, \Lambda_0, \Theta_0\) are chosen so that \((r_0, \Lambda_0, \Theta_0, G_0) \notin \bar{F} \times \Pi = \Delta\), then one finds one solution to \(\Pi_0\) by letting

\[
\begin{align*}
\tau'(t) &= \tau_0' \\
\tilde{\lambda}(t) &= \tilde{\lambda}_0 \\
\tilde{\Theta}(t) &= \tilde{\Theta}_0 \\
\bar{R}(t) &= \bar{R}_0 - (h_0' + \varphi\tilde{h}\tilde{r}) + g\tilde{h}G(t - t_0) \\
\bar{t}(t) &= \bar{t}_0 + g(\bar{h}\tilde{r} + \varphi\tilde{h}\tilde{G})\Lambda(t - t_0) \\
\bar{\vartheta}(t) &= \bar{\vartheta}_0 + g(\tilde{h}\tilde{\vartheta} + \bar{h}\bar{G})\Theta(t - t_0)
\end{align*}
\]

(26)

where the derivatives are evaluated at \((r_0, \Lambda_0, \Theta_0, \bar{G}(r_0, \Lambda_0, \Theta_0, G_0, \bar{G}_0))\).
2.1.6 First order solutions in $\varepsilon$

It is possible to prove (see Appendix A.3) the following 'local' version of (24)

$$\hat{U}(\tau, \Lambda, \Theta, G, \varphi) = \hat{U}(\tau, \Lambda, \Theta, G_0(\tau, \Lambda, \Theta, G, \varphi))$$

(27)

where $G_0$ is as in (21). This formula turns to be more useful than (24) from the practical point of view because now the function $\hat{U}$ may be explicitly written (see Appendix A.4). We then have a concrete first order approximation of (26):

$$\begin{align*}
\tau(t) &= \tau_0 \\
\Lambda(t) &= \Lambda_0 \\
\Theta(t) &= \Theta_0 \\
\bar{R}(t) &= \bar{R}_0 - (h'_0 + \varrho \varepsilon \hat{U}_\Lambda + \varrho \varepsilon \hat{U}_G G_0(t)) (t - t_0) \\
\bar{I}(t) &= \bar{I}_0 + \varrho \varepsilon (\hat{U}_\Lambda + \hat{U}_G G_0(t)) (t - t_0) \\
\bar{J}(t) &= \bar{J}_0 + \varrho \varepsilon (\hat{U}_\Omega + \hat{U}_G G_0(t)) (t - t_0)
\end{align*}$$

(28)

where $t \to (G_0(t), \Theta_0(t))$ is any solution to $(\overline{G}, \overline{\varphi}) \to G_0(\tau_0, \Lambda_0, \Theta_0, G, \varphi)$. According to (28), the problem is reduced to study the dynamics of $G_0$ in (21), for $k \in D$. The study of the phase portrait of $G_0$ reduces to study an equation with degree three in $G^2$ (since equation (21) can be written in this way). For such an equation, as well known, one could use Cardano formulae. However, the purpose of this paper is to highlight the utility of the method, rather than to push the analysis to its maximum generality. Therefore, we shall simplify even more the discussion restricting to $\Pi = \{\Theta = 0\}$, in which case the equation to be solved is of degree two, completely explicit.

**Remark 2.1** Equations (28) imply that on the manifolds defined by values of $\tau_0, \Lambda_0, \Theta_0, G$ where $\hat{U}_\overline{G}$ vanishes, the coordinates $(\overline{G}, \overline{\varphi})$ do not move. In view of (16) and (27) above, and the Implicit Function Theorem, such zeroes may be approximated by the corresponding zeroes of $\hat{U}_{G_0}$. From the analysis of an $\frac{d}{dt}$-expansion

$$\hat{U} = \frac{1}{a} \left( 1 - \frac{r'^2}{4a^2} \frac{A^3(3\Theta^2 - G_0^2)}{G_0^2} + \cdots \right)$$

(29)

one can see that, as a matter of fact, $\hat{U}_{G_0}$ does vanish for

$$G_0 \sim \pm \sqrt{5} \Theta .$$

(30)

Thus, when $r' \ll a$, (30) is a first approximation of manifolds where $(\overline{G}, \overline{\varphi})$ stay fixed for $\overline{G}$.

2.1.7 Equilibria and phase portrait in the planar case

It is convenient to divide equation (21) by $A^2$ and let $\hat{G}_0 := \frac{\overline{G}}{A^2}, \delta := \frac{\bar{J}}{A}$. Then equation (21) with $\Theta = 0$ becomes

$$\hat{G}_0 = \frac{G^2}{A^2} + \delta \sqrt{1 - \frac{G^2}{A^2} \cos g}.$$ 

(31)
Figure 1: The phase portrait of \( \hat{G}_0 \), for \( 0 < \delta < 1 \) and \( -\delta \leq \hat{G}_0 < 1 \) (Mathematica).

Let us first discuss the equilibria of \( \hat{G}_0 \). This function is even and regular around \((g, G) = (\pi, 0)\) and \((g, G) = (0, 0)\) (note that for \( \Theta \neq 0, G = 0 \) would be a singularity), which are equilibria. However, the character of such equilibria is different accordingly to wether \( \delta \in (2, +\infty) \) or \( \delta \in (0, 2) \): in the former case, \((0, 0)\) and \((\pi, 0)\) are both stable; in the latter case, \((0, 0)\) is unstable, while \((\pi, 0)\) is stable. Correspondingly to such extrema, \( \hat{G}_0 \) takes the values \( \hat{G}_{0\text{sad}} = \delta \) and \( \hat{G}_{0\text{min}} = -\delta \), respectively. To such equilibria, one should add, for \( \delta < 2 \), also the point \((0, \sqrt{1 - \frac{\delta^2}{4}})\), which is stable, and where \( \hat{G}_0 \) takes the maximum possible value \( \hat{G}_{0\text{max}} = 1 + \frac{\delta^2}{4} \). We now study the phase portrait of \( \hat{G}_0 \) in (31). Still, we shall make a further simplification. We take \( 0 < \delta < 1 \) and \( -\delta \leq \hat{G}_0 \leq 1 \), leaving the remaining analysis to the interested reader. The level \( \hat{G}_0 = 1 \) splits as

\[
\{ \hat{G}_0 = 1 \} = \{ G = \Lambda \} \bigcup \{ G = \Lambda | \sin g | \}
\]

For \( -\delta \leq \hat{G}_0 < 1 \), we rewrite equation (31) in terms of \( w := \sqrt{1 - \frac{\delta^2}{4}} \), with \( w \in (0, 1) \). We obtain

\[
w^2 - \delta w \cos g - 1 + \hat{G}_0 = 0 .
\]

Solving for \( w \),

\[
w_{\pm} = \frac{\delta \cos g \pm \sqrt{\delta^2 \cos^2 g + 4 - 4\hat{G}_0}}{2}
\]

we see that \( w_- \) is negative for all \( g \), and hence is to be disregarded, while \( w_+ \) is positive for all \( g \). Moreover, \( w_+ \) does not exceed 1 as soon as

\[
\sqrt{\delta^2 \cos^2 g + 4 - 4\hat{G}_0} < 2 - \delta \cos g .
\]

Solving for \( \cos g \), we obtain

\[
\cos g < \min \left\{ \frac{2}{\delta}, \frac{\hat{G}_0}{\delta} \right\} = \frac{\hat{G}_0}{\delta} .
\]

From this inequality we see that if \( \hat{G}_0 > \delta \), we have a rotational motion of \( g \), while if \( -\delta \leq \hat{G}_0 < \delta \) the motion is librational around \((\pi, 0)\). The situation is depicted in Figure 1.

2.1.8 The separatrix

The separatrix in Figure 1, namely the \( \hat{G}_0 \)-level though \((0, 0)\), having equation \( \hat{G}_0 = \delta \), is precisely the set \( S \) in (14). Therefore, the motions along it, strictly speaking, loose their meaning if replaced within the formulae (28). Note that the homoclinic solution along \( S \) can however be
easily computed:

\[
\begin{align*}
G_0(t) &= \frac{\sigma \Lambda}{\cosh \sigma \Lambda(t-t_0)} \\
\bar{G}_0(t) &= \pm \cos^{-1} \frac{1 - \beta^2}{\sqrt{1 - \frac{\sigma^2}{\cosh^2 \sigma \Lambda(t-t_0)}}}
\end{align*}
\]

with \(\sigma^2 := \delta(2-\delta), \beta^2 := 2-\delta\).

### 2.1.9 Action–Angle coordinates

The previous analysis shows that, by Liouville–Arnold [2], for all \(0 < \delta < 1\) in any of the two regions connected region defined by inequalities \(-\delta < \bar{G}_0 < \delta, \delta < \bar{G}_0 < 1\), it is possible to find a (different) canonical change of coordinates

\[
\lambda = (\bar{R}', \bar{A}, \bar{r}', \bar{\ell}, \bar{\alpha}) \rightarrow \kappa = (\overline{R}', \overline{A}, \overline{r}', \overline{\ell}, \overline{g})
\]

(preserving \(\bar{\Lambda} = \overline{\Lambda}, \bar{r}' = \overline{r}'\)) with \(\bar{\alpha} \in \mathbb{T}\), such that, in their terms, \(G_0\) becomes a function of only \((\overline{r}', \bar{A}, \bar{\Lambda})\). By usual integrability arguments, up to \(\varepsilon\)–variations of the coordinates, one has also \(G\) is so and hence also, by (25), \(H_0\), with all these functions being well defined (collision–free).

### 2.2 Application to the Sun–Earth–Asteroid system

Let us consider the problem of three gravitational masses, \(1, \varepsilon, \mu\), with \(1 \gg \varepsilon \gg \mu\). After the reduction of translation invariance according to the heliocentric method, the three–body problem in \(\mathbb{R}^3\) is governed by the six–degrees–of–freedom Hamiltonian

\[
\overline{H}(\overline{y}', \overline{y}, \overline{x}', \overline{x}) = \frac{||\overline{y}'||^2}{2m'} - \frac{\varepsilon}{||\overline{x}'||} + \frac{||\overline{y}||^2}{2m} - \frac{\mu}{||\overline{x}||}
\]

\[
- \frac{\mu \varepsilon}{||\overline{x}' - \overline{x}||} + \overline{y}' \cdot \overline{y}
\]

where \(m' := (1+\varepsilon)^{-1}, m := (1+\mu)^{-1}\); \(\overline{y}', \overline{y}, \overline{x}', \overline{x} \in \mathbb{R}^3\) are impulse–position coordinates.

In order to eliminate small numbers from denominators, one rescales time, Hamiltonian and coordinates, via

\[
H(y', y, x', x) := \varepsilon^{-1} \overline{H}(\mu y', \mu y, x', x) = H_0 + \varrho f
\]

where \(H_0 := h_0 + \rho h, f := \frac{\varepsilon y'}{2m'} + \varepsilon y' \cdot y\) with \(h_0 := -\frac{\mu}{2\varepsilon^2}||y'||^2\). Let us write such functions in terms of \(\kappa\), without changing them the names. Since \(f\) still still possesses \(C_1\) as a first integral, but no longer \(x'\), we have that \(f\) is a function of \(R', \Lambda, G, \Theta, \ell, g, \vartheta\) and, moreover, it depends parametrically, on \(G\). The manifolds \(\Pi_k := \{(\Theta, \vartheta) = (0, 0)\}\) and \(\Pi_\ell := \{(\Theta, \vartheta) = (0, \pi)\}\) are invariant to the \(H\)–flow. Motions on \(\Pi, \Pi_k, \Pi_\ell\) correspond to have, at all times, the orbits of Earth and the Sun on the same instantaneous plane, with a suitable choice of the mutual inclination \((0, \pi)\) of \(C_1, C_2\). We reduce the coordinates \(\kappa\) in (22). Then \(H_0\) is carried to \(\overline{H}_0\) in (25), while \(f\) to a suitable \(\bar{f}\). The dynamics of \(\overline{H}_0\) on \(\Pi\) (and hence, in particular, on its sub–manifolds \(\Pi_k, \Pi_\ell\)) has been discussed in the previous paragraph. We consider the motions corresponding to the cases (a), (b) or (c), and aim to extend (many of) them to sea. We put the system in the coordinates \(\lambda\) in (37), letting \(\overline{H}(\overline{R}', \bar{A}, \bar{r}', \bar{\ell}, \bar{\alpha}) = H_0(\bar{A}, \bar{r}') + \varrho^2 \bar{f}(\overline{R}', \bar{A}, \bar{r}', \bar{\ell}, \bar{\alpha})\) the relative Hamiltonian.
Via normal form theory (see Appendix B), we conjugate $\tilde{H}$ to (omitting to write the dependence on $G$)

$$
\tilde{H} = \tilde{H}_0(\tilde{\Lambda}, \tilde{A}, \tilde{r}') + \varrho^2 \tilde{H}_1(\tilde{R}', \tilde{\Lambda}, \tilde{A}, \tilde{r}') + \varrho^2 \tilde{f}(\tilde{R}', \tilde{\Lambda}, \tilde{A}, \tilde{r}', \tilde{\ell}, \tilde{a})
$$

(40)

where $\tilde{f}$ is of order $2^{-c'/\varrho}$, while, up to higher orders,

$$
\tilde{H}_1 \sim \frac{(\tilde{R}')^2}{2m'} + \frac{\Phi(\tilde{\Lambda}, \tilde{A}, \tilde{r}')^2}{2m'r'^2}
$$

(41)

with $\Phi(\tilde{\Lambda}, \tilde{A}, \tilde{r}')$ a suitable regular function. At this point, one integrates the term $\tilde{H}_0 + \varrho^2 \tilde{H}_1$ with respect to $(\tilde{R}', \tilde{r}')$. Since, by construction, $\tilde{H}_0$ is regular, and because of (41), the integration of such term is analogous to a two–body system, and one finds, for low energies, an action–angle couple $(\Lambda', \ell')$ such that $\tilde{H}_0 + \varrho^2 \tilde{H}_1$, after the integration, would depend on $(\tilde{\Lambda}, \tilde{\Lambda}', \tilde{A})$ only. An application of KAM theory allows to infer the existence of quasi–periodic motions with three frequencies, with a residual set having an exponentially small density.
3 Conclusions and perspectives

We proposed a new analysis of the two-centre Hamiltonian \( h \) in (1) based on the Euler integral \( \mathcal{G} \) in (4). We introduced a ‘ad–hoc’ system of canonical coordinates (18) which includes all its first integrals but \( \mathcal{G} \). Accordingly, we wrote \( h \) as an effective, properly degenerate, two–degrees of freedom system, with a fast angle \( \ell \) and a slow one \( g \). Eliminating (in the regime where the two attracting centres have much different masses ratio \( \varepsilon \)) the fast angle \( \ell \) via perturbative methods, we obtained a new one degrees of freedom Hamiltonian \( \tilde{h} \) in (16) and proved that the dynamics of \( \tilde{h} \) is completely determined by the one of \( \mathcal{G} \), written in the new coordinates, the function \( \mathcal{G} \) in (17).

The result carries an important consequence: at the lowest order in \( \varepsilon \), the common motions of \( h \) and \( \mathcal{G} \) are determined by the simple one–dimensional Hamiltonian \( \mathcal{G}_0 \) in (21). In the case of the planar problem, the phase portrait of \( \mathcal{G}_0 \) is explicitly, rigorously computable (Section 2.1.7 and Figure 1). It shows that Liouville–Arnold action–angle coordinates \( (A, a) \iff (G, g) \) do exist for all \( r' < a \) and \(-mr' \leq \mathcal{G}_0 < \Lambda^2\), apart for a zero measure set \( S \), where collisions are possible.

We applied the result to the SEA system, regarding it as a perturbation of 2CP. A suitable normal form theory and KAM theory allows to infer the existence of quasi–periodic motions with three frequencies, with a residual set having an exponentially small density. As a byproduct of the proof, the risk of Earth–Asteroid collision may be excluded in all cases where \( \mathcal{G}_0 \) is sufficiently far away from \( mr' \), where \( m \) is the Earth mass, in suitable units, and \( r' \) is distance from the sun.

We conjecture that closely to the collision sets \( S \)'s, SEA possesses chains of transition tori in the sense of [2].

Acknowledgements

Figure 1 was produced with Mathematica. The author thanks M. Guzzo for his encouragement.
A  Technical details to Section 2

A.1  The Euler integral

The formulae in (4)–(5) are not standard in the literature. For sake of completeness, and for the reader’s facility, we report their derivation here.

In Section A.1.1, we check that, writing the two–centre Hamiltonian in the more usual ‘symmetric’ form

\[ h_{\text{sim}} = \frac{|y|^2}{2} - \frac{m_+}{|x + x_0|} - \frac{m_-}{|x - x_0|} \]  \hspace{1cm} (42)

then the Euler integral to \( h_{\text{sim}} \) is given by

\[ G_{\text{sim}} = |x \times y|^2 + (x_0 \cdot y)^2 + 2x \cdot x_0 (\frac{m_+}{|x + x_0|} - \frac{m_-}{|x - x_0|}) \]  \hspace{1cm} (43)

In Section A.1.2, we shall check that, when \( h \) is written in the form (1), then \( G_{\text{sim}} \) reduces to \( G \) in (4)–(5).

Observe, incidentally, that, in the symmetric case, when the two stars merge, e.g., \( x_0 = 0 \), \( G_{\text{sim}} \) reduces to \( |C|^2 \).

A.1.1 Derivation of (43)

For part of the proof, we use the canonical coordinates \( p \) in (20) (for uniformity of notations, we shall replace the symbols \( x', y', r', R' \) in (19) and (20) with \( x_0, y_0, r_0, R_0 \), respectively). As said, the \( p \)'s have in common with \( k \)'s in (18) almost all the coordinates, apart for the two quadruplets \( (R, \Phi, r, \varphi) \) (for the \( p \)'s) and \( (\Lambda, G, \ell, g) \) (for the \( k \)'s). The definition of the former is\(^2\) (within the same notations as in Section 2.1.4)

\[ R = \frac{y \cdot x}{|x|}, \quad \Phi = |C|, \quad r = |x|, \quad \varphi = \alpha_{C}(n, k \times x). \]  \hspace{1cm} (44)

In terms of \( p \), the scalar product \( x_0 \cdot x \) takes the form

\[ x_0 \cdot x = -r_0r \sqrt{1 - \frac{\Theta^2}{\Phi^2}} \cos \varphi \]

and so \( h_{\text{sim}} \) in (1) becomes

\[ h_{\text{sim}} = \frac{R^2}{2} + \frac{\Phi^2}{2r^2} \frac{m_+}{r_+} - \frac{m_-}{r_-} \]  \hspace{1cm} (45)

where

\[ r^2_\pm := r^2_0 \pm 2r_0r \sqrt{1 - \frac{\Theta^2}{\Phi^2}} \cos \varphi + r^2. \]

\( h_{\text{sim}} \) has now two degrees of freedom, exactly as in the classical discussion, which goes along the ‘elliptic coordinates’

\[ \lambda = \frac{r_+ + r_-}{2r_0}, \quad \mu = \frac{r_+ - r_-}{2r_0}. \]

We then define a change of canonical coordinates \( (R, \Phi, r, \varphi) \rightarrow (p_{\lambda}, p_{\mu}, \lambda, \mu) \) where \( \lambda, \mu \) are as above, while their conjugated momenta \( p_{\lambda}, p_{\mu} \) are found taking the inverse of

\[ r_+ = r_0(\lambda + \mu) \quad r_- = r_0(\lambda - \mu) \]  \hspace{1cm} (46)

\(^2\)The angle \( \varphi \) in (44) corresponds to \( \varphi + \frac{\pi}{2} \) of [13, equation (2.10)].
and then squaring and summing, or subtracting. This gives

\[ r = r_0 \sqrt{\lambda^2 + \mu^2} - 1 \quad \varphi = \cos^{-1} \left( -\frac{\lambda \mu}{\sqrt{\lambda^2 + \mu^2} - 1} \right) \]  

(47)

Then one considers the generating function

\[
S(\Phi, \Theta, R_0, R, \lambda, \mu, \hat{r}_0, \hat{\theta}) = \Theta \hat{\theta} + R_0 \hat{r}_0 + R_0 \sqrt{\lambda^2 + \mu^2} - 1 + \int_\Phi \cos^{-1} \left( -\frac{\lambda \mu}{\sqrt{\lambda^2 + \mu^2} - 1} \right) d\Phi .
\]

The transformation generated by \( S \) leaves the coordinates \( \Theta, r_0 \) unvaried (therefore, we shall not change their names), while shifts in an inessential way (since they do not appear into \( h_{\text{sim}} \)) the coordinates \( \theta, R_0 \). Taking the derivatives with respect to \( \lambda, \mu \), one finds

\[
\begin{align*}
\lambda = & \frac{r_0 \lambda R}{\sqrt{\lambda^2 + \mu^2} - 1} - \frac{\mu \sqrt{(1 - \mu^2)(\lambda^2 - 1)\Phi^2 - (\lambda^2 + \mu^2 - 1)\Theta^2}}{(\lambda^2 + \mu^2 - 1)(\lambda^2 - 1)}, \\
\mu = & \frac{r_0 \mu R}{\sqrt{\lambda^2 + \mu^2} - 1} + \frac{\lambda \sqrt{(1 - \mu^2)(\lambda^2 - 1)\Phi^2 - (\lambda^2 + \mu^2 - 1)\Theta^2}}{(\lambda^2 + \mu^2 - 1)(1 - \mu^2)}
\end{align*}
\]

whence, taking the inverse with respect to \( R, \Phi \)

\[
\begin{align*}
R = & \frac{\lambda(\lambda^2 - 1)\rho + \mu(1 - \mu^2)\rho}{r_0(\lambda^2 - \mu^2)\sqrt{\lambda^2 + \mu^2} - 1}, \\
\Phi^2 = & \frac{(\lambda \rho^2 - \mu \rho^2)(\lambda^2 - 1)(1 - \mu^2)}{(\lambda^2 - \mu^2)} + \frac{\lambda^2 + \mu^2 - 1}{(1 - \mu^2)(\lambda^2 - 1)}\Theta^2
\end{align*}
\]

Replacing these expressions and the one for \( r_+, r_- \), \( r \) in (46), (47) into the Hamiltonian \( h_{\text{sim}} \) in (45), one finds the classical expression

\[
h_{\text{sim}} = \frac{p_\lambda^2(\lambda^2 - 1)}{2r_0^2(\lambda^2 - \mu^2)} + \frac{p_\mu^2(1 - \mu^2)}{2r_0^2(\lambda^2 - \mu^2)} + \frac{\Theta^2}{2r_0^2(\lambda^2 - \mu^2)} \left( \frac{1}{1 - \mu^2} + \frac{1}{\lambda^2 - 1} \right) - \frac{(m_+ + m_-) \lambda - (m_+ - m_-) \mu}{r_0^2(\lambda^2 - \mu^2)}.
\]  

(48)

Then one sees that Hamilton–Jacobi equation

\[ h_{\text{sim}} - E = 0 \]

splits as

\[ \mathcal{F}(\mu)(p_\mu, \mu, \Theta, E, r_0) - \mathcal{F}(\lambda)(p_\lambda, \lambda, \Theta, E, r_0) = 0 \]  

(49)

where

\[
\begin{align*}
\mathcal{F}(\mu) = & p_\mu^2(1 - \mu^2) + \frac{\Theta^2}{1 - \mu^2} + 2(m_+ - m_-) \mu + 2r_0^2 \mu^2 E, \\
\mathcal{F}(\lambda) = & -p_\lambda^2(\lambda^2 - 1) - \frac{\Theta^2}{\lambda^2 - 1} + 2(m_+ + m_-) \lambda + 2r_0^2 \lambda^2 E.
\end{align*}
\]
Equation (49) implies then that \( F(\mu)(p, \mu, \Theta, E, r_0) = G_{\text{sim}}^{(\mu)}(\Theta, E, r_0) \) is actually independent of \((p, \mu)\); \( F(\lambda)(p, \mu, \Theta, E, r_0) = G_{\text{sim}}^{(\lambda)}(\Theta, E, r_0) \) is actually independent of \((p, \lambda)\), and, a fortiori, since the partial derivatives of \( G_{\text{sim}}^{(\mu)} \), \( G_{\text{sim}}^{(\lambda)} \) depend explicitly on \( \mu, \lambda \), there must exists a \( G_{\text{sim}} \in \mathbb{R} \) such that

\[
F(\mu) = F(\lambda) = G_{\text{sim}}.
\]

Therefore,

\[
G_{\text{sim}} = \frac{1}{2} (F(\mu) + F(\lambda))
\]

\[
= \frac{p_\mu^2}{2} (1 - \mu^2) - \frac{p_\lambda^2}{2} (\lambda^2 - 1) + \frac{\Theta^2}{2} \left( \frac{1}{1 - \mu^2} - \frac{1}{\lambda^2 - 1} \right)
\]

\[
+ m_+ (\lambda + \mu) + m_- (\lambda - \mu)
\]

\[
+ 2r_0^2 (\lambda^2 + \mu^2) E .
\]

After some elementary computations, one finds the expression of \( G_{\text{sim}} \) in terms of the coordinates \( p \) is

\[
G_{\text{sim}} = \Phi^2 + r_0^2 (1 - \frac{\Theta^2}{\Phi^2}) (-R \cos \varphi + \frac{\Phi}{r} \sin \varphi)^2
\]

\[
- 2rr_0 \cos \varphi \sqrt{1 - \frac{\Theta^2}{\Phi^2} (\frac{m_+}{r_+} - \frac{m_-}{r_-})}.
\]

While, in terms of the coordinates \((y_0, x_0), (y, x)\), \( G_{\text{sim}} \) has the expression in (43).

### A.1.2 Derivation of (4)–(5)

Let \( h \) be as in (1). We preliminarily rescale \( h \), letting

\[
\hat{h}(\hat{y}, \hat{x}, \hat{x}') = m^{-1} h(m \hat{y}, \hat{x}, \hat{x}')
\]

\[
= \frac{|\hat{y}|^2}{2} - \frac{m^{-1} \varepsilon}{|\hat{x}|} - \frac{m^{-1} \varepsilon}{|\hat{x}' - \hat{x}|} . \tag{50}
\]

Letting further

\[
\hat{y}' = \frac{1}{2} (y_0 - \hat{y}) \quad \hat{y} = y \quad \hat{x}' = 2x_0 \quad \hat{x} = x_0 + \hat{x}
\]

we approach the Hamiltonian \( h_{\text{sim}} \) in (42), with masses

\[
m_- = \varepsilon m^{-1} \quad m_+ = m^{-1}
\]

and \((y, x)\) replaced by \((\hat{y}, \hat{x})\). But \( h_{\text{sim}} \) admits the integral \( G_{\text{sim}} \) in (43), and hence, applying the inverse transformations of (51) and (50) we find that \( h \) has the first integral

\[
\hat{G} \overset{m}{=} \frac{1}{m} (|x - x'|^2 \times y)^2 + \frac{1}{4m} (x' \cdot y)^2
\]

\[
+ x' \cdot (x - x') \left( \frac{1}{|x|} - \frac{\varepsilon}{|x' - x|} \right)
\]

After multiplying by \( m \), we rewrite this integral as

\[
\hat{G} = G_0 + \varepsilon G_1 + m \frac{|x'|^2}{2} h \tag{52}
\]

where

\[
G_0 := |C|^2 - x' \cdot L \quad G_1 := m \frac{(x' - x) \cdot x'}{|x' - x|}.
\]
with
\[ C := x \times y \quad L = y \times C - m \frac{x}{|x|}. \]
Since the last term in (52) is itself an integral for h, we can neglect it and conclude that the function
\[ G := G_0 + \varepsilon G_1 \]
is an integral to h. We recognize that C, L are the angular momentum and the eccentricity vector associated to h, respectively. This is exactly what we had to check, after recalling that L is related (in our units) to e and P via \( L = meP \).

A.2 On the convergence of the series (16)
Since two different partial Kepler maps \( \kappa = ((\Lambda, u, v), \lambda), \kappa' = ((\Lambda, u', v'), \lambda') \) are linked by a relation of the form
\[ \Lambda = \Lambda, (u, v) = F(\Lambda, u', v'), \lambda = \lambda' + \psi(\Lambda, u', v') \] (53)
the character of the series (16) does not depend on the choice of \( \kappa \). Therefore, we choose \( \kappa \) as in Section 2.1.4. We prove that the series (16) converges in the domain defined by the following inequalities
\[ \Theta = 0, \quad 0 < \delta < 1, \quad G_0 \in [-\delta, \delta) \cup (\delta, 1) \] (54)
which is enough for our purposes. With this choice, \( h_\kappa \) and \( G_\kappa \) depend, as already remarked, only on the two angles \( \lambda \) and \( g \). Let us discuss the question using Liouville–Arnold theorem. Regarding \( h_\kappa \) and \( G_\kappa \) as functions of \( (\Lambda, G, \ell, g) \), we look at level sets
\[ \mathcal{M}_{h, G, \varepsilon} = \{ (\Lambda, G, \ell, g, \varepsilon) : h_\kappa(\Lambda, G, \ell, g, \varepsilon) = h, \ G_\kappa(\Lambda, G, \ell, g) = G \}. \]
For \( \varepsilon = 0 \), \( h_\kappa \) reduces to \( h_K \) in (6), while \( G \) reduces to \( G_0 \) in (31). Therefore, \( \mathcal{M}_{h, G_0, 0} \) is the product \{\( h_\kappa(\Lambda) = h \}\} \times \{G_0 = G\}, which, by the discussion in Section 2.1.7 and the choice (54) of the domain, are compact. Then, \( \mathcal{M}_{h, G_0, \varepsilon} \) remains compact for small \( \varepsilon \), because collisions are excluded. Then, action–angles coordinates \( \alpha = (\bar{\alpha}, \bar{\Lambda}, \bar{\alpha}, \alpha) \) can be found in each connected component of \( \mathcal{D} \). In such coordinates, both \( h_\kappa \) and \( G_\kappa \) would depend on \( (\bar{\alpha}, \bar{\Lambda}) \) only. Moreover, by its definition, \( A \) is \( \varepsilon \)–close to
\[ A_0 = \frac{1}{2\pi} \int G dg \] (55)
where G solves \( G_0(\bar{\Lambda}, G, g) = G \). This function corresponds to \( \bar{\Lambda}\sqrt{1 - w^2} \), where has been computed in Section 2.1.7. From this expression, one sees that \( \partial_{\bar{\Lambda}} A_0 \neq 0 \) (being the integral of a positive function, it is strictly increasing), therefore also \( \partial_{\bar{\Gamma}} A \neq 0 \). By Implicit Function Theorem, one can invert A as a function of \( \bar{\alpha}, \bar{\Lambda}, G \). This allows to write \( h_\alpha(\bar{\alpha}, \bar{\Lambda}, G, \bar{\Gamma}) =: \bar{h}(\bar{\alpha}, \bar{\Lambda}, G) \), which corresponds to the sum of series (16).

A.3 \( \bar{\alpha} \)–independence of \( \bar{G} \) and \( (\bar{U}, \bar{G}_0) \) commutation
In this section we state an abstract result that allows to prove (i) that \( \bar{U} \) commutes with \( \bar{G}_0 \) and (ii) that \( \bar{G} \) is \( \bar{\alpha} \)–independent. Note that (i) easily implies (27).
(i) and (ii) follow from the corresponding theses of lemma below, taking \( H := \bar{h} \) in (15), \( \bar{J} := \bar{G} \) in (17) and \( (I, \varphi, p, q) = \bar{\kappa} \).
Lemma A.1 Let \((I, \varphi, p, q)\), with \((I, \varphi)\), \((p, q)\) pairwise conjugate, canonical coordinates on the phase space \(\mathcal{P} = V^1 \times T^1 \times \mathcal{B}\), where \(V^1 \subset \mathbb{R}^1, \mathcal{B} \subset \mathbb{R}^{2n}\) open and connected. Let \(\mathcal{H} : \mathcal{P} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}\) a \(\varphi\)-independent function of the form

\[
\mathcal{H}(I, p, q, \varepsilon) = \mathcal{H}_0(I) + \varepsilon \mathcal{H}_1(I, p, q) + \cdots
\]  

(56)

analytic in \(\varepsilon\). Then

(i) \(\mathcal{J}_0\) Poisson–commutes with \(\mathcal{H}_1\);

(ii) any first integral \(\mathcal{J}(I, \varphi, p, q, \varepsilon)\) to \(\mathcal{H}(I, p, q, \varepsilon)\), analytic in \(\varepsilon\), with \(\mathcal{J}_0(I, \varphi, p, q) := \mathcal{J}(I, \varphi, p, q, 0)\) independent of \(\varphi\), is \(\varphi\)-independent for all \(\varepsilon \in (-\varepsilon_0, \varepsilon_0)\).

Proof We start with (ii). Let

\[
\mathcal{J}(I, \varphi, p, q, \varepsilon) = \mathcal{J}_0(I, p, q) + \varepsilon \mathcal{J}_1(I, \varphi, p, q) + \cdots .
\]

We prove that for all \(j \geq 1\), \(\mathcal{J}_j\) is \(\varphi\)-independent. By assumption, \(\mathcal{J}_0\) is \(\varphi\)-independent. Assuming, inductively, that, for a given \(k \geq 1\) and any \(0 \leq j \leq k - 1\), \(\mathcal{J}_j\) is \(\varphi\)-independent (so that the case \(k = 1\) corresponds with the assumption), we prove that \(\mathcal{J}_k\) is so. Writing the commutation relation of \(\mathcal{H}\) and \(\mathcal{J}\), and picking the term proportional to \(\varepsilon^k\) one as

\[
\left\{ \mathcal{H}_0, \mathcal{J}_k \right\} + \sum_{j=1}^{k} \left\{ \mathcal{H}_j, \mathcal{J}_{k-j} \right\} = 0
\]

Since \(\mathcal{H}_0\) depends only on \(I\), the first term in this identity has vanishing \(\varphi\)-average, while, by the inductive assumption, the second term is \(\varphi\)-independent. Then one has, identically,

\[
\left\{ \mathcal{H}_0, \mathcal{J}_k \right\} = \sum_{j=1}^{k} \left\{ \mathcal{H}_j, \mathcal{J}_{k-j} \right\} = 0
\]

Therefore, \(\mathcal{J}_k\) is \(\varphi\)-independent. The thesis (i) follows from this identity, with \(k = 1\).

A.4 The function \(\hat{U}\)

In this appendix, we write relation (27) explicitly. This is not used in the paper, but, in view of equations (28) and Remark 2.1, may turn to be useful in applications.

As a matter of fact, there are infinite ways of representing \(\hat{U}\) as a function of \(r', \Lambda, \Theta, g_0\). Indeed, since \(\hat{U}(r', \Lambda, \Theta, G, g)\) and \(g_0(r', \Lambda, \Theta, g)\) commute, if \(G(g_0, r', \Lambda, \Theta, g)\) solves

\[
G_0(r', \Lambda, \Theta, G, g) = g_0
\]

then the function \(\hat{U}(r', \Lambda, \Theta, G(g_0, r', \Lambda, \Theta, g), g)\) is \(g\)-independent, and one can take, for any fixed \(g_0\),

\[
\hat{U}(r', \Lambda, \Theta, g_0) = \hat{U}(r', \Lambda, \Theta, G(g_0, r', \Lambda, \Theta, g_0), g_0) .
\]  

(57)

A convenient choice is to take \(g_0 = \frac{\pi}{2}\), in which case, as one sees from (21), \(G(g_0, r', \Lambda, \Theta, g) \equiv \sqrt{g_0}\). Then (57) becomes

\[
\hat{U}(r', \Lambda, \Theta, g_0) = \hat{U}(r', \Lambda, \Theta, \sqrt{g_0}, \frac{\pi}{2}) .
\]
More explicitly, using the expression of $U$ in terms of $k$, which is

$$
U(r', \Lambda, \Theta, G, g) = -\frac{1}{2\pi} \int_0^{2\pi} d\ell \left[ r'^2 + 2r'ar'\rho \sqrt{1 - \frac{\Theta^2}{G^2} \cos(g + \nu) + a^2 r'^2} \right]^{-1/2}
$$

where $a$ is as in (7) and

- $e = \sqrt{1 - \frac{G^2}{\Lambda^2}}$ is the eccentricity;
- $\zeta$ is the eccentric anomaly, solving Kepler’s equation $\zeta - e \sin \zeta = \ell$;
- $\rho := |x_k|/a = 1 - e \cos \zeta$;
- $\nu$ is the true anomaly, defined by $\nu = \arg(\cos \zeta - e, \sqrt{1 - e^2 \sin \zeta})$;

we find

$$
\hat{U}(r', \Lambda, \Theta, G_0) := -\frac{1}{2\pi} \int_\mathcal{T} d\zeta \left( 1 - E(\Lambda, G_0) \cos \zeta \right) \left[ r'^2 - 2ar'\mathcal{I}(\Lambda, \Theta, G_0) \sin \zeta + a^2 (1 - E(\Lambda, G_0) \cos \zeta)^2 \right]^{-1/2}
$$

(58)

where

$$
E(\Lambda, G_0) := \frac{\sqrt{\Lambda^2 - G_0}}{\Lambda} \quad \mathcal{I}(\Lambda, \Theta, G_0) := \frac{\sqrt{G_0 - \Theta^2}}{\Lambda} .
$$

Note that negative values for the expressions under the square roots are not a problem, since $\hat{U}$ is even in $r'$, $a$, $E$, $\mathcal{I}$ separately.
### B Normal Form Theory

Let $\hat{D} = \hat{B} \times T^2$, with $\hat{B} \subset \mathbb{R}^4$ compact and $T := \mathbb{R}/(2\pi \mathbb{Z})$ be a real domain for the coordinates $((\hat{r}', \hat{\Lambda}, \hat{A}, \hat{R}'), \hat{\ell}, \hat{\alpha})$ where $\hat{h}$ is regular, and let $\hat{D}_c \supset \hat{D}$ be a suitable complex, compact domain such that $\hat{h}$ has an holomorphic extension on $\hat{D}_c \supset \hat{D}$. Let us denote:

$$\omega((\hat{r}', \hat{\Lambda}, \hat{A})) := \partial_{(\hat{r}', \hat{\Lambda}, \hat{A})} \hat{H}_0$$

$$\langle \hat{f} \rangle_{\hat{\ell}, \hat{\alpha}} := \frac{1}{(2\pi)^2} \int_{[0, 2\pi]^2} \hat{f} \hat{d}\hat{\ell} \hat{d}\hat{\alpha}$$

$$\|g\| := \sup_{\hat{D}_c} |g| , \quad \mathcal{X} := \sup_{\hat{D}_c} |\hat{R}'| .$$

The following result is known, even though not completely standard. The peculiarity of the Normal Form Lemma is that no small divisors condition is needed. This is possible because the coordinate $\hat{R}'$ is not an angle. Roughly, this circumstance allows for a careful choice of the integration constant in the homological equation that allows to 'de–singularize' the small denominators. Details will be published elsewhere. In the text, it has been applied with $N \sim \varrho^{-1}$, as it is possible, since $\omega((\hat{\Lambda}, \hat{A}))$ is of order $\varrho$, while $\omega_{\hat{r}'}$ is of order $1$.

**Lemma B.1** There exists a constant $c$ such that, for any $N \in \mathbb{N}$ such that the following inequalities are satisfied

$$cN\mathcal{X}\|\frac{\omega_{\hat{\Lambda}}}{\omega_{\hat{r}'}}\| < 1 , \quad cN\mathcal{X}\|\frac{\omega_{\hat{A}}}{\omega_{\hat{r}'}}\| < 1 , \quad cN\mathcal{X}\|\frac{1}{\omega_{\hat{r}'}}\||\hat{f}\| < 1$$

one can find a $\varrho$–close to the identity canonical transformation

$$(\hat{r}', \hat{\Lambda}, \hat{A}, \hat{R}', \hat{\ell}, \hat{\alpha}) \rightarrow (\tilde{r}', \tilde{\Lambda}, \tilde{A}, \tilde{R}', \tilde{\ell}, \tilde{\alpha})$$

that carries $\hat{H}$ to

$$\tilde{H} = \hat{H}_0 + \tilde{H}_1 + \tilde{f}$$

where $\tilde{H}_1, \tilde{f}$ satisfy

$$\|\tilde{H}_1 - \langle \hat{f} \rangle\| \leq c\mathcal{X}\|\frac{1}{\omega_{\hat{r}'}}\||\hat{f}\|^2 , \quad \|\hat{f}\| \leq \frac{\varrho^2}{2^{N+1}} \|\hat{f}\| . \quad (59)$$

The formula in (41) follows from the former inequality in (59) and $\langle y'_\lambda \cdot y_\lambda \rangle_{\hat{\ell}, \hat{\alpha}} = 0$, so that

$$\langle \hat{f} \rangle_{\hat{\ell}, \hat{\alpha}} = \frac{\langle |y'_\lambda|^2 \rangle_{\hat{\ell}}}{2m'r'^2} = \frac{\hat{R}^2}{2m'r'^2} + \frac{\langle (G - G_\lambda)^2 \rangle_{\hat{\ell}}}{2m'r'^2} . \quad (60)$$
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