Periodic solutions to Klein-Gordon systems with linear couplings

Jianyi Chen\textsuperscript{a,*}, Zhitao Zhang\textsuperscript{b,c}, Guijuan Chang\textsuperscript{a}, Jing Zhao\textsuperscript{a}

\textsuperscript{a} Science and Information College, Qingdao Agricultural University, Qingdao 266109, P. R. China
\textsuperscript{b} Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China;
\textsuperscript{c} School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, P. R. China

Abstract

In this paper, we study the nonlinear Klein-Gordon systems arising from relativistic physics and quantum field theories

\[
\begin{align*}
    u_{tt} - u_{xx} + bu + \varepsilon v + f(t,x,u) &= 0, \\
v_{tt} - v_{xx} + bv + \varepsilon u + g(t,x,v) &= 0
\end{align*}
\]

where \(u, v\) satisfy the Dirichlet boundary conditions on spatial interval \([0, \pi]\), \(b > 0\) and \(f, g\) are \(2\pi\)-periodic in \(t\). We are concerned with the existence, regularity and asymptotic behavior of time-periodic solutions to the linearly coupled problem as \(\varepsilon\) goes to 0. Firstly, under some superlinear growth and monotonicity assumptions on \(f\) and \(g\), we obtain the solutions \((u_\varepsilon, v_\varepsilon)\) with time-period \(2\pi\) for the problem as the linear coupling constant \(\varepsilon\) is sufficiently small, by constructing critical points of an indefinite functional via variational methods. Secondly, we give precise characterization for the asymptotic behavior of these solutions, and show that as \(\varepsilon \to 0\),
\((u_\varepsilon, v_\varepsilon)\) converge to the solutions of the wave equations without the coupling terms. Finally, by careful analysis which are quite different from the elliptic regularity theory, we obtain some interesting results concerning the higher regularity of the periodic solutions.

**Keywords:** Wave equations; Variational method; Klein-Gordon system; Periodic solutions.

**AMS Subject Classification (2010):** 35B10; 35L51; 58E30.

1 **Introduction**

In this paper, we consider the important nonlinear Klein-Gordon system

\[
\begin{align*}
&u_{tt} - u_{xx} + bu + \varepsilon v + f(t, x, u) = 0, \quad t \in \mathbb{R}, \quad x \in [0, \pi] \\
v_{tt} - v_{xx} + bv + \varepsilon u + g(t, x, v) = 0, \quad t \in \mathbb{R}, \quad x \in [0, \pi]
\end{align*}
\]  

(1.1)\textsubscript{a}

satisfying the Dirichlet boundary value conditions on the \(x\)-axis

\[
u(t, 0) = u(t, \pi) = 0, \quad v(t, 0) = v(t, \pi) = 0, \quad t \in \mathbb{R},
\]

(1.1)\textsubscript{b}

and periodic conditions with respect to the time variable \(t\)

\[
u(t + 2\pi, x) = u(t, x), \quad v(t + 2\pi, x) = v(t, x), \quad t \in \mathbb{R}, \quad x \in [0, \pi],
\]

(1.1)\textsubscript{c}

where \(u(t, x)\) and \(v(t, x)\) are the relativistic wave functions generated by the interaction of two mass fields, \(b > 0\) and \(\sqrt{b}\) stands for the mass; \(\varepsilon\) denotes the strength of the fields coupling, and \(\varepsilon\) is assumed to be sufficiently small. The nonlinear forced terms \(f(t, x, u), g(t, x, v)\) are \(2\pi\)-periodic in \(t\). We study the existence, regularity and asymptotic behavior of time-periodic solutions to the linearly coupled problem (1.1)\textsubscript{a,b,c} as \(\varepsilon\) goes to 0.

The coupled Klein-Gordon system in the general form

\[
\begin{align*}
&u_{tt} - u_{xx} = H_u(u, v), \quad v_{tt} - v_{xx} = H_v(u, v),
\end{align*}
\]  

(KG)

is deeply connected with many branches of mathematical physics, such as relativistic physics and quantum field theories. For instance, with the proper choice of the potential function \(H(u, v)\), the system (KG) was used to describe the long-wave dynamics of two coupled one-dimensional periodic chains in the bi-layer materials or the spinless relativistic composite particles (see [1, 18]). Moreover, variations of such systems were also proposed in the work of Klainerman and Tataru [20] as important models to investigate the Yang-Mills equations under the Coulomb gauge condition. The solvability of (KG)
depends upon the nature of the nonlinearities and the type of the boundary conditions. Many interesting theoretical and numerical results can be found in \[2, 15, 19, 27, 33, 34, 38\] and the monograph of Shatah-Struwe \[32\] which contains more extensive references.

It is an important work to study the existence and regularity of time-periodic solutions for the Dirichlet problem of (KG) with the gradient of the potential function \(H(u,v)\) having the interesting coupling form

\[
\nabla H(u,v) = (-bu - \varepsilon v - f(t,x,u), -bv - \varepsilon u - g(t,x,v)).
\]

When \(\varepsilon = 0\), Eq. (1.1) are two copies of nonlinear wave equations

\[
\begin{align*}
u_{tt} - v_{xx} + bu + f(t,x,u) &= 0, \quad t \in \mathbb{R}, \quad 0 < x < \pi, \\
v_{tt} - v_{xx} + bv + g(t,x,v) &= 0, \quad t \in \mathbb{R}, \quad 0 < x < \pi.
\end{align*}
\]

(W1)\hspace{1cm} (W2)

It is well known that even the existence of time-periodic solutions for single wave equation is difficult to study. Since the seminal work \[28\] of P. H. Rabinowitz, several tools in nonlinear analysis are developed by H. Brézis, L. Nirenberg, J. M. Coron, K. C. Chang, J. Mawhin, M. Schechter, S. J. Li et al. to obtain the existence and multiplicity results of the periodic solutions for the scalar wave equations with various type of nonlinearities. We refer to \[5-8, 14, 16, 17, 21, 25, 28, 29, 35\] for one dimensional problem, and \[9-11, 26, 30, 31\] for higher dimensional cases. The existence of solutions with time-period \(T\) to such kinds of wave equations depends upon the nature of the parameter \(b\), period \(T\) and the nonlinearities. All the above results require the crucial condition that \(T/\pi\) is rational and sometimes take \(T = 2\pi\) for simplicity. When \(T\) is an irrational multiple of \(\pi\), we are led to the problems of small divisors which are difficult to deal with (see \[3, 4, 36\] for examples).

As \(\varepsilon \neq 0\), the solvability of the system (1.1)_{a,b,c} is more complicated because of the presence of the linear coupling terms. We need a more delicate analysis to study the behavior of the interaction between the two linear coupling terms. Berkovits and Mustonen \[2\] used the topological degree theory and continuation principle to obtain at least one weak solution \((u,v)\) with time-period \(2\pi\) for the system

\[
\begin{align*}
u_{tt} - v_{xx} + \lambda v + f(t,x,u) &= h_1(t,x), \quad t \in \mathbb{R}, \quad x \in [0,\pi] \\
u_{tt} - v_{xx} + \mu u + g(t,x,v) &= h_2(t,x), \quad t \in \mathbb{R}, \quad x \in [0,\pi]
\end{align*}
\]

\((KG)_{\lambda\mu}\)

where \(\lambda \mu < 0\), \(h_1, h_2 \in L^2([0,2\pi] \times [0,\pi])\) and \(f, g\) satisfy some linear growth conditions. The assumption \(\lambda \mu < 0\) required in \[2\] plays a crucial role in calculating the degree and getting a priori bounds of solutions for the corresponding homotopy equations.
Recently, Yan, Ji and Sun \cite{39} used the change of degree argument to prove the existence of time-periodic weak solutions for some coupled Klein-Gordon systems with variable coefficients when the forced terms satisfy some sublinear conditions.

To the best of our knowledge, little further progress has been made on the study of the existence and regularity of periodic solutions for \((\text{KG})_{\lambda \mu}\) with superlinear forced terms. In the present work, we study such a superlinear problem and consider the situation \(\lambda \mu > 0\) by variational methods. We focus on the case of \(\lambda = \mu\) because the variational structure is required. Our results are in three aspects:

- existence of the time-periodic weak solutions for \((1.1)_{a,b,c}\);
- asymptotic behavior of the weak solutions as \(\varepsilon \to 0\);
- higher regularity of the solutions.

Let \(\Omega = [0, 2\pi] \times [0, \pi]\), we say that \((u, v) \in L^2(\Omega) \times L^2(\Omega)\) is a weak solution to the system \((1.1)_{a,b,c}\) provided that

\[
\int_{\Omega} u (\varphi_{tt} - \varphi_{xx}) + bu \varphi + \varepsilon v \varphi + f(t, x, u) \varphi \, dt \, dx = 0
\]

and

\[
\int_{\Omega} v (\psi_{tt} - \psi_{xx}) + bv \psi + \varepsilon u \psi + g(t, x, v) \psi \, dt \, dx = 0
\]

for all functions \(\varphi\) and \(\psi\) satisfying the conditions \((1.1)_{b,c}\) and belonging to the space \(H\) which is defined by \((2.1)\) in Sect. 2.

Let \(\sigma(L)\) be the set of eigenvalues of the d’Alembert operator \(L = \partial_t^2 - \partial_x^2\) subject to the conditions \((1.1)_{b,c}\), and denote by \(\ker L\) the kernel of the operator \(L\). It is well known that \(\sigma(L)\) consists of the isolate numbers \(\lambda_{jk} = j^2 - k^2\), for \(j \in \mathbb{Z}_+\) and \(k \in \mathbb{Z}\). We see:

(i) \(0 \in \sigma(L)\), and the multiplicity of eigenvalue \(\lambda_{j_0 k_0} = 0\) is infinite since there exists an infinite number of \(j_0\) and \(k_0\) such that \(j_0^2 - k_0^2 = 0\); that is, \(\ker L\) is an infinite dimensional space;

(ii) all the nonzero eigenvalues of \(L\) are of finite multiplicity, and they tend to \(+\infty\) or \(-\infty\);

(iii) for any number \(b\) satisfying \(-b \notin \sigma(L)\), there exists a constant \(\eta > 0\) such that

\[
|\lambda_{jk} + b| \geq \eta, \quad \forall \ j \in \mathbb{Z}_+, \ k \in \mathbb{Z}, \tag{1.2}
\]

noting that the eigenvalues \(\lambda_{jk}\) of the operator \(L\) are isolated.

(a) Existence and the asymptotic behavior of the solutions for \((1.1)_{a,b,c}\)

The first result of this paper is the following theorem concerning the existence of the time-periodic weak solutions for \((1.1)_{a,b,c}\).
Theorem 1.1. Let \( b > 0 \) and \(-b \not\in \sigma(L)\), \( f, g \in C(\Omega \times \mathbb{R}, \mathbb{R}) \) are assumed to be \( 2\pi \)-periodic in \( t \) and satisfy the following superlinear growth and monotonicity conditions (h1)-(h4):

(h1) there exist \( p > 1, q > 1 \), and \( c_0 > 0 \), such that for all \( t, x, \xi \),

\[
|f(t, x, \xi)| \leq c_0(1 + |\xi|^p), \quad \text{and} \quad |g(t, x, \xi)| \leq c_0(1 + |\xi|^q);
\]

(h2) \( f(t, x, \xi) = o(|\xi|) \) and \( g(t, x, \xi) = o(|\xi|) \), as \( \xi \to 0 \) uniformly in \((t, x)\);

(h3) (Ambrosetti-Rabinowitz condition)

\[
(p + 1)F(t, x, \xi) \leq f(t, x, \xi)\xi, \quad \text{and} \quad (q + 1)G(t, x, \xi) \leq g(t, x, \xi)\xi,
\]

for all \( t, x \) and \( \xi \), where \( F(t, x, \xi) = \int_0^\xi f(t, x, s)ds \) and \( G(t, x, \xi) = \int_0^\xi g(t, x, s)ds \);

(h4) (Monotonicity) \( f(t, x, \xi) \) and \( g(t, x, \xi) \) are nondecreasing in \( \xi \).

Then there exists \( \varepsilon_0 > 0 \) such that for \( |\xi| < \varepsilon_0 \), the system (1.1)\(_{a,b,c}\) has at least one nontrivial weak solution \((u, v) \in L^2(\Omega) \times L^2(\Omega)\) with time period \( 2\pi \).

From (h2) we infer \( f(t, x, 0) = 0 \) and \( g(t, x, 0) = 0 \), which implies that \((u, v) = (0, 0)\) is a solution for the system (1.1)\(_{a,b,c}\). We should point out that if \((u, v)\) satisfies (1.1)\(_a\) and \( u \neq 0 \), then we also have \( v \neq 0 \) due to the structure of the system (1.1)\(_a\). In other words, the problem (1.1)\(_{a,b,c}\) possesses no semi-trivial solution of type \((u, 0)\) or \((0, v)\).

Remark 1.2. By virtue of (h1)-(h4), an explicit computation shows some useful facts for proving Theorem 1.1:

(i) \( F(t, x, \xi) \geq 0 \), and \( G(t, x, \xi) \geq 0 \) for all \((t, x, \xi)\).

(ii) There are positive numbers \( c_1 \) and \( c_2 \), such that

\[
F(t, x, \xi) \geq c_1|\xi|^{p+1} - c_2, \quad G(t, x, \xi) \geq c_1|\xi|^{q+1} - c_2, \quad (1.3)
\]

and there are constants \( \bar{r}, c_3 > 0 \), such that

\[
F(t, x, \xi) \geq c_3|\xi|^{p+1}, \quad G(t, x, \xi) \geq c_3|\xi|^{q+1}, \quad \text{for} \quad |\xi| \geq \bar{r};
\]

Furthermore, it follows from (h3) and (h4) that

\[
\lim_{\xi \to +\infty} f(t, x, \xi) = \lim_{\xi \to +\infty} g(t, x, \xi) = +\infty.
\]

(iii) \( F(t, x, \xi)/\xi^2 \to 0, \) and \( G(t, x, \xi)/\xi^2 \to 0 \) uniformly in \((t, x)\) as \( \xi \to 0 \);

moreover, for each \( \nu > 0 \), there exists a positive number \( C_\nu \) such that

\[
|F(t, x, \xi)| \leq \nu \xi^2 + C_\nu |\xi|^{p+1}, \quad \text{and} \quad |G(t, x, \xi)| \leq \nu \xi^2 + C_\nu |\xi|^{q+1}. \quad (1.4)
\]
It is well known that such assumptions as (h1)-(h3) are also of great use in solving the nonlinear elliptic equations and the linearly coupled elliptic systems

\[
\begin{cases}
-\Delta u + u = f(u) + \lambda v, & x \in \mathbb{R}^N, \\
-\Delta v + v = g(v) + \lambda u, & x \in \mathbb{R}^N,
\end{cases}
\]

(see [12, 13, 37] and the references therein). We should point out that, in contrast to the elliptic equations and systems, we may face the following difficulties in the problem of finding periodic solutions for the Klein-Gordon system \((1.1)_{a,b,c}\):

1) The d’Alembert operator \(L = \partial_t^2 - \partial_x^2\) possesses infinitely many eigenvalues going from \(-\infty\) to \(+\infty\), so that the positive part and the negative part of the spectrum of \(L\) are all infinite dimensional spaces, and the functional \(\Phi\) corresponding to \((1.1)_{a,b,c}\) stated in Sect. 2 is neither bounded from above nor from below. Furthermore, because of the kernel of \(L\) is infinite dimensional, the operator \(L\) and its inverse are not compact. This fact gives rise to considerable difficulties in solving the strong indefinite problem \((1.1)_{a,b,c}\).

As the lack of compactness properties, the embedding estimates and methods used in [12, 13, 37] are invalid here.

2) Due to the linear coupling effects, it is hard to obtain the energy estimate and convexity properties of the corresponding functional \(\Phi\) for the system \((1.1)_{a,b,c}\). Many troubles stem from the coupling interplay between the two scalar functions \(u\) and \(v\). The main challenges in constructing the time-periodic solutions for \((1.1)_{a,b,c}\) are to control the energy of \(\Phi\) in some proper working spaces, and to estimate the asymptotic behavior of the components in the kernel of \(L\).

Since the solutions for \((1.1)_{a,b,c}\) obtained in Theorem 1.1 are dependent of \(\varepsilon\), we are interested in considering the asymptotic behavior of these solutions as \(\varepsilon \to 0\). In the next theorem, we prove:

**Theorem 1.3.** Under the conditions of Theorem 1.1, and assume \((u_\varepsilon, v_\varepsilon)\) is a solution of \((1.1)_{a,b,c}\) obtained in Theorem 1.1 for \(|\varepsilon| < \varepsilon_0\). Let \(\varepsilon_n \in (-\varepsilon_0, \varepsilon_0)\) be any sequence with \(\varepsilon_n \to 0\) as \(n \to \infty\). Then, passing to a subsequence, \((u_{\varepsilon_n}, v_{\varepsilon_n})\) converge strongly to \((U_0, V_0)\) in \(L^2(\Omega) \times L^2(\Omega)\) as \(n \to \infty\), where \(U_0\) is a weak solution of (W1), and \(V_0\) is a weak solution of (W2) respectively.

(b) Regularity results

Another problem that we study is the higher regularity of the solutions to the system \((1.1)_{a,b,c}\). We obtain the \(L^\infty\) bound of the periodic solutions for \((1.1)_{a,b,c}\) basing on some precise descriptions of the energy estimates.
Theorem 1.4. Suppose that the conditions of Theorem 1.1 are satisfied, then there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$, the solution of $(1.1)_{a,b,c}$ lies in $L^\infty(\Omega) \times L^\infty(\Omega)$, where $L^\infty(\Omega)$ is the Lebesgue space equipped with the norm

$$
\|w\|_{L^\infty} = \inf\{C \geq 0 : |w(t,x)| \leq C \text{ for almost every } (t,x) \in \Omega\} < \infty.
$$

Under some more restrictive assumptions on $f$ and $g$, we have

Theorem 1.5. Under the conditions of Theorem 1.1, we assume in addition that $f, g \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$, and $f(t,x,\xi), g(t,x,\xi)$ are strictly increasing in $\xi$ for all $(t,x) \in \Omega$.

Then for sufficiently small $\varepsilon$, the solution of $(1.1)_{a,b,c}$ is continuous on $\Omega$.

Our idea for this theorem is motivated by the works of Rabinowitz [28, 29] and Brézis-Nirenberg [6] which are devoted to the regularity of the solutions for scalar wave equations $w_{tt} - w_{xx} = h(t,x,w)$. The above authors decompose $w$ into a regular term and a null term via the representation theorem (see Lemma 2.13 in [29]). The regular term can be formulated as an integral expression which is continuous in $t$ and $x$. On the other hand, the null term can be controlled by some delicate integral and pointwise estimates, and then the $C^0$-regularity of the solution is guaranteed by a continuity argument.

However, due to the presence of the linear coupling terms, the previous estimates developed in [6, 28, 29] can not be applied directly to study the higher regularity of the solutions for system $(1.1)_{a,b,c}$. Especially, the interplay among the linear coupling null terms is hard to control. To get around this difficulty, a careful calculation and some more precise energy estimates such as (6.3) and (6.4) are required to analyze the interaction between the linear coupling null terms, provided that $\varepsilon$ is sufficiently small.

We conclude this section by illustrating our strategies to tackle the above problems and listing the sketch of the proof of the main results.

First, we prove the existence of the weak solutions for $(1.1)_{a,b,c}$ by constructing the critical points of the functional $\Phi$ defined by (2.2) restricted in some suitable function spaces, via the local linking method introduced by Shujie Li-Jiaquan Liu [23, 24] (see also [14, 22, 40]). The estimates of the components $(u,v)$ in different parts of the function space $E$ are playing crucial roles in solving this problem.

- For $(u,v)$ that belongs to the orthogonal complement of ker $L \times$ ker $L$ in $E$, we can control these components by some compact embedding estimate (2.9);
- For $(u,v) \in$ ker $L \times$ ker $L$, the compact properties will be lost. To overcome this difficulty, we apply the monotonicity technique to analyze the behavior of the nonlinear terms more precisely, as the linear coupling constant $\varepsilon$ is sufficiently small. We require the condition (h4) of Theorem 1.1 to obtain this goal.
Subsequently, we study the asymptotic properties of the solutions constructed in Theorem 1.1 as $\varepsilon \to 0$. We establish Lemma 4.1 to obtain the uniform bound of the solutions $(u_\varepsilon, v_\varepsilon) \in E$ to (1.1)$_{a,b,c}$ for any $|\varepsilon| < \varepsilon_0$, which leads to the strong convergence of $(u_\varepsilon, v_\varepsilon) \to (U_0, V_0)$ in $L^2(\Omega) \times L^2(\Omega)$ as $\varepsilon \to 0$. Then, we verify that $U_0, V_0$ are weak solutions of (W1) and (W2) respectively, by a limiting argument with the aid of some precise energy estimates (4.9)-(4.13) in Sect. 4 to approach it.

Finally, we will improve the regularity of the weak solutions. We carry out the proof by two steps:

**Step 1:** With the help of the presentation theorem for periodic solutions to the scalar wave equations ([5] [29]), we can use some a-priori estimates and comparison methods to achieve the $L^\infty$-estimate of the solutions $(u, v)$ for (1.1)$_{a,b,c}$ constructed by Theorem 1.1. The proof depends on the linear coupling structure of the system. See Sect. 5 for details.

**Step 2:** Relying on the nature of the nonlinearities and the above $L^\infty$-estimate, we can generalize the continuity method ([6] [29]) used in the scalar wave equations to the case of system (1.1)$_{a,b,c}$ taking account of the linear coupling effect. For sufficiently small $\varepsilon$, the integral estimates in [6] are improved to prove the higher regularity of the solutions (see Sect. 6).

We organize the paper as follows. In Sect. 2, we give the functional scheme and define a suitable function space $E$ to work in it, with the aid of Fourier expansion formulations for the functions that satisfy (1.1)$_{b,c}$. Then we introduce a decomposition of $E$ and prepare some basic embedding properties, which enable us to solve (1.1)$_{a,b,c}$ conveniently. Sect. 3 is devoted to proving Theorem 1.1 via the local linking method. One of the major ingredients in the proof is to verify the (PS)$^*$ condition by a compact argument together with a monotonicity technique (see Lemma 3.1 and Lemma 3.2). Then, in Sect. 4, we investigate the limit behavior of the solutions for (1.1)$_{a,b,c}$ and prove Theorem 1.3. At last, we turn to study the further regularity properties of the time-periodic solution for (1.1)$_{a,b,c}$ and prove Theorem 1.4, Theorem 1.5 in Sect. 5 and Sect. 6 respectively.

\section{The variational framework}

In this section, we present the variational framework which will be used to solve the system (1.1)$_{a,b,c}$. First, we define the energy functional and its working space as following.
2.1 Functional Setting

Using the Fourier series, the solutions to the linear equation

\[ \phi_{tt} - \phi_{xx} = h(t, x), \quad 0 < t < 2\pi, \quad 0 < x < \pi, \]

with conditions of \( \phi(t, 0) = \phi(t, \pi) = 0 \) and \( \phi(t + 2\pi, x) = \phi(t, x) \) have a expansion of the form

\[ \phi(t, x) = \sum_{j \in \mathbb{Z}^+, k \in \mathbb{Z}} a_{jk} \sin(jx)e^{ikt}, \quad \text{where} \quad a_{jk} = a_{j, -k}. \]

Then, for \( u(t, x) = \sum_{j \in \mathbb{Z}^+, k \in \mathbb{Z}} u_{jk} \sin(jx)e^{ikt} \) and \( v = \sum_{j \in \mathbb{Z}^+, k \in \mathbb{Z}} v_{jk} \sin(jx)e^{ikt} \), the inner product in \( L^2(\Omega) \) can be formulated by

\[ \langle u, v \rangle = \iint_{\Omega} u(t, x)v(t, x)dt\,dx = \pi^2 \sum_{j \in \mathbb{Z}^+, k \in \mathbb{Z}} u_{jk} v_{jk}, \]

and we can write the following quadratic form as

\[ \langle Lu, v \rangle = \iint_{\Omega} (\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}) \overline{v} \, dt\,dx = \pi^2 \sum_{j \in \mathbb{Z}^+, k \in \mathbb{Z}} (j^2 - k^2)u_{jk} \overline{v}_{jk}. \]

Motivated by \[14\], it is natural to introduce the Hilbert spaces

\[ H = \left\{ u \in L^2(\Omega) : \|u\|^2_H = \pi^2 \sum_{j \in \mathbb{Z}^+, k \in \mathbb{Z}} |j^2 - k^2 + b|u_{jk}|^2 + \pi^2 \sum_{j \in \mathbb{Z}^+, k \in \mathbb{Z}} |u_{jk}|^2 < \infty \right\}, \quad (2.1) \]

and \( E = H \times H \) as our working spaces, where \( E \) is equipped with the norm \( \|(u, v)\|_E = (\|u\|^2_H + \|v\|^2_H)^{1/2} \).

For \( (u, v) \in E = H \times H \), let

\[ \Phi(u, v) = -\frac{1}{2} \langle Lu, u \rangle - b \int_{\Omega} u^2 dt \, dx - \frac{1}{2} \langle Lv, v \rangle - b \int_{\Omega} v^2 dt \, dx - \varepsilon \int_{\Omega} uv dt \, dx - \int_{\Omega} F(t, x, u)dt \, dx - \int_{\Omega} G(t, x, v)dt \, dx. \quad (2.2) \]

In the rest of this paper, we denote by \( \int_{\Omega} \cdot = \iint_{\Omega} \cdot \, dt \, dx \) for convenience.

Thus \( \Phi \) is a \( C^1 \) functional on \( E \), and the Gateaux derivative of \( \Phi \) is

\[ \langle \Phi'(u, v), (\varphi, \psi) \rangle = -\langle Lu, \varphi \rangle - b \int_{\Omega} u \varphi \, dx - \varepsilon \int_{\Omega} v \varphi - \langle Lv, \psi \rangle - b \int_{\Omega} v \psi - \varepsilon \int_{\Omega} u \psi - \int_{\Omega} f(t, x, u)\varphi - \int_{\Omega} g(t, x, v)\psi, \quad \forall \ (u, v), (\varphi, \psi) \in E. \quad (2.3) \]

Then \( (u, v) \) is a weak solution of system (1.1)\( a,b,c \) if and only if \( \Phi'(u, v) = 0 \).
2.2 Local linking structure

The notion of local linking introduced by S.J. Li and M. Willem [22] is a powerful tool to study the existence of critical points for strongly indefinite functionals.

**Definition 2.1.** Let $E$ be a Banach space, and $E = E^1 \oplus E^2$ is a direct sum decomposition of $E$ (noting that both of $E^1$ and $E^2$ may be infinite dimensional spaces). Then $\Phi \in C^1(E, \mathbb{R})$ is said to have a local linking at 0 if for some $r > 0$,

$$\Phi(u) \geq 0, \quad \text{for } u \in E^1, \|u\| \leq r,$$

$$\Phi(u) \leq 0, \quad \text{for } u \in E^2, \|u\| \leq r.$$

In the case of $\dim E^1 = \dim E^2 = \infty$, it is necessary to explore the Galerkin approximation method and some compactness argument to construct the critical points of the functional $\Phi$ in $E$. To this end, we need the following compactness condition which generalize the (PS) condition.

Suppose that $E^1_1 \subset E^1_2 \subset \cdots \subset E^1$, $E^2_1 \subset E^2_2 \subset \cdots \subset E^2$ are two sequences of finite dimensional subspaces such that

$$E^j = \bigcup_{n \in \mathbb{N}} E^j_n, \quad j = 1, 2.$$

For two multi-index $\theta = (\theta^1, \theta^2)$ and $\beta = (\beta^1, \beta^2) \in \mathbb{N}^2$, we denote by $\theta \leq \beta$ if $\theta^1 \leq \beta^1$ and $\theta^2 \leq \beta^2$. A sequence $(\theta_n) \subset \mathbb{N}^2$ is said to be admissible if for each $\theta \in \mathbb{N}^2$, there is an $m \in \mathbb{N}$ such that $\theta_n \geq \theta$ for all $n \geq m$. For $\theta = (\theta^1, \theta^2)$, let $E_{\theta} = E^1_{\theta^1} \oplus E^2_{\theta^2}$ and $\Phi_{\theta} = \Phi|_{E_{\theta}}$.

**Definition 2.2.** The functional $\Phi \in C^1(E, \mathbb{R})$ is said to satisfy the condition (PS)* if every sequence $(u_{\theta_n})$ with $(\theta_n) \subset \mathbb{N}^2$ being admissible such that

$$u_{\theta_n} \in E_{\theta_n}, \quad \sup_{n} \Phi(u_{\theta_n}) < \infty \quad \text{and} \quad \Phi'_{\theta_n}(u_{\theta_n}) \to 0 \quad \text{as} \quad n \to \infty$$

contains a subsequence converging to a critical point of $\Phi$.

We will use the following abstract proposition to solve the system (1.1)_{a,b,c}.

**Proposition A** ([22, 40]). Suppose that $\Phi \in C^1(E, \mathbb{R})$ and

(A1) $\Phi$ satisfies (PS)* condition;

(A2) $\Phi$ has a local linking at 0;

(A3) $\Phi$ maps bounded sets into bounded sets;

(A4) For every $m \in \mathbb{N}$, $\Phi(u) \to -\infty$ as $\|u\| \to \infty$, $u \in E^1_m \oplus E^2$.

Then $\Phi$ has a nontrivial critical point $u_0$ in $E$.

**Remark 2.1.** In [14, 22], it is also pointed out that the critical value corresponding to the critical point $u_0$ obtained in Proposition A satisfying

$$\Phi(u_0) \leq c, \quad \text{where} \quad c = \sup_{u \in E^1_m \oplus E^2} \Phi(u), \quad \text{and} \quad m_1 \text{ is a positive integer.}$$
In order to apply Proposition A to the functional defined by (2.2), we shall introduce the direct sum decomposition of the Banach space $E = H \times H$, where $H$ occurs in (2.1). Some notations are defined as follows:

- $H^+_b$ is the subspace which is spanned by the functions $\sin(jx)e^{ikt}$, where $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$ satisfying $j^2 - k^2 > -b$ and $j \neq |k|$;
- $H^-_b$ is the subspace which is spanned by the functions $\sin(jx)e^{ikt}$, where $j^2 - k^2 < -b$;
- $H^0 = \ker L$ is the subspace which is spanned by the functions $\sin(jx)e^{ikt}$, for $j = |k|$;
- $E^+_b = H^+_b \times H^+_b$, $E^-_b = H^-_b \times H^-_b$, and $E^0 = H^0 \times H^0$;
- $E^1 = E^-_b$, $E^2 = E^+_b \oplus E^0$, we see $\dim E^j = \infty$, for $j = 1, 2$;
- $E^j_m = \text{span}\{e^j_1, \ldots, e^j_m\}$, where $(e^j_n)_{n=1}^\infty$ is a basis for $E^j$, $j = 1, 2$.

We have $E = E^1 \oplus E^2$. Moreover, $E^1_m$, $E^2_m$ are finite dimensional spaces for every $m \in \mathbb{Z}$, and $E^1_1 \subset E^1_2 \subset \cdots \subset E^1$, $E^2_1 \subset E^2_2 \subset \cdots \subset E^2$.

### 2.3 Basic estimates

At the end of this section, we list some basic formulas and properties of the Banach spaces on which we will work.

For $r \geq 1$, we denote by $L^r(\Omega)$ the space of functions $u(t, x)$ with the norm

$$
\|u\|_{L^r} = \left( \int_{\Omega} \left| u(t, x) \right|^r dt dx \right)^{1/r}.
$$

- **Formulas of the norms** $\| \cdot \|_{L^2}$, $\| \cdot \|_H$ and functional $\Phi$

For $u, v \in H$, and

$$
u = \sum_{j \in \mathbb{Z}_+, \ k \in \mathbb{Z}} u_{jk} \sin(jx)e^{ikt}, \quad v = \sum_{j \in \mathbb{Z}_+, \ k \in \mathbb{Z}} v_{jk} \sin(jx)e^{ikt},$$

let $u = u^+ + u^- + y$, $v = v^+ + v^- + z$, where $u^+, v^+ \in H^+_b$, $u^-, v^- \in H^-_b$, and $y, z \in H^0$. By the orthogonality of the subspaces $H^+_b$, $H^-_b$ and $H^0$, we can write the inner product in $L^2(\Omega)$ as $\langle u, v \rangle = \langle u^+, v^+ \rangle + \langle u^-, v^- \rangle + \langle y, z \rangle$, and $\|u\|_{L^2}^2 = \|u^+\|_{L^2}^2 + \|u^-\|_{L^2}^2 + \|y\|_{L^2}^2 = \pi^2 \sum_{j \in \mathbb{Z}_+, \ k \in \mathbb{Z}} |u_{jk}|^2$.

Noting that $-b \notin \sigma(L)$ and $Ly = 0$ for $y \in H^0$, we have $\langle Lu, v \rangle = \langle Lu^+, v^+ \rangle + \langle Lu^-, v^- \rangle$, and

$$
\langle (L + b)u, u \rangle = \pi^2 \sum_{j \in \mathbb{Z}_+, \ k \in \mathbb{Z}} (j^2 - k^2 + b)|u_{jk}|^2 = \|u^+\|_{H^2}^2 + \|u^-\|_{H^2}^2 + b\|y\|_{L^2}^2.
$$

With the aid of (2.4), we can formulate the energy functional (2.2) as

$$
\Phi(u, v) = -\frac{1}{2}\|u^+\|_{H^2}^2 + \frac{1}{2}\|u^-\|_{H^2}^2 - \frac{b}{2}\|y\|_{L^2}^2 - \frac{1}{2}\|v^+\|_{H^2}^2 + \frac{1}{2}\|v^-\|_{H^2}^2 - \frac{b}{2}\|z\|_{L^2}^2.
$$
\[- \varepsilon \int w v - \int F(t, x, u) - \int G(t, x, v). \quad (2.5)\]

• Some embedding estimates

In view of (1.2), there is \( \eta > 0 \), such that

\[
\| u^+ \|_H^2 = \pi^2 \sum_{j^2 - k^2 > -b \atop j \neq |k|} (j^2 - k^2 + b) |u_{jk}|^2 \geq \eta \| u^+ \|_{L^2}^2, \quad \text{and} \quad \| u^- \|_H^2 \geq \eta \| u^- \|_{L^2}^2. \quad (2.6)
\]

Thus, by (2.1) and (2.6) we have

\[
\| u \|_{L^2}^2 \leq \kappa \| u \|_H^2, \quad \text{where} \quad \kappa = \max\{1/\eta, 1\}. \quad (2.7)
\]

The following properties are well known (see [14] for instance):

\[
\| w \|_{L^r} \leq C \| w \|_H, \quad \text{for} \quad w \in H^+_b \oplus H^-_b \quad \text{and} \quad r \geq 1, \quad (2.8)
\]

where \( C > 0 \) is a constant which only depends on \( r \). Furthermore, the embedding

\[
H^+_b \oplus H^-_b \hookrightarrow L^r(\Omega) \quad \text{is compact, for} \quad r \geq 1. \quad (2.9)
\]

**Remark 2.2.** Let us point out that we lose the compact embedding from \( H^0 \) to \( L^r(\Omega) \) for \( r > 2 \), because of \( 0 \in \sigma(L) \) and \( \dim \ker L = \infty \). Moreover, according to the definition of \( H \) (see (2.1)), we do not obtain that \( \| w \|_{L^r} \leq C \| w \|_H \), for any \( w \in H \), and \( r > 2 \). Thus, we need careful computations to study the behavior of the null components and nonlinear terms appearing in the functional \( \Phi \).

### 3 Existence of weak solutions

In this section, we prove Theorem 1.1. To achieve this goal, we will check that the conditions of Proposition A hold for the functional \( \Phi \) defined by (2.2). We use \( C \) and \( c_\ast \), \( d_\ast \) with quantity subscripts to stand for different constants in the rest of this article.

**Verification of (A1):**

Let \((u_{\theta_n}, v_{\theta_n}) \in E_{\theta_n} := E^1_{\theta_n^1} \oplus E^2_{\theta_n^2}, \) where \( \theta_n = (\theta^1_n, \theta^2_n) \in \mathbb{N}^2 \), the sequence \( \{\theta_n\}_{n=1}^\infty \) is admissible, and \( E^1_{\theta_n^1}, E^2_{\theta_n^2} \) are finite dimensional spaces defined in subsection 2.2.

We suppose \( \{(u_{\theta_n}, v_{\theta_n})\}_{n=1}^\infty \) is a (PS)* sequence of \( \Phi \), that is \( d := \sup_n \Phi(u_{\theta_n}, v_{\theta_n}) < \infty \), and \( \Phi'_{\theta_n}(u_{\theta_n}, v_{\theta_n}) \to 0 \) as \( n \to \infty \), where \( \Phi'_{\theta_n} = (\Phi|_{E_{\theta_n}})' \). First, under the assumptions of Theorem 1.1, we show:

**Lemma 3.1.** Any (PS)* sequence is bounded.
Proof. For simplicity, we denote by \((u, v) = (u_{\theta_n}, v_{\theta_n})\).

**Step 1. Estimates of \(\|u\|_{L^{p+1}}\) and \(\|v\|_{L^{q+1}}\)**

By assumption (h3) and (1.3), a direct calculation shows

\[
\Phi(u, v) - \frac{1}{2}\langle \Phi'_n(u, v), (u, v) \rangle = \frac{1}{2} \int f(t, x, u)u - \int F(t, x, u) + \frac{1}{2} \int g(t, x, v)v - \int G(t, x, v) \geq \frac{p-1}{2} \int F(t, x, u) + \frac{q-1}{2} \int G(t, x, v) \geq c_1\|u\|_{L^{p+1}}^{p+1} + c_1\|v\|_{L^{q+1}}^{q+1} - c_2.
\]

Since \(\{(u, v)\} = \{(u_{\theta_n}, v_{\theta_n})\}\) is a \((PS)^*\) sequence of \(\Phi\), for \(n\) is large enough, we have

\[
c_1\|u\|_{L^{p+1}}^{p+1} + c_1\|v\|_{L^{q+1}}^{q+1} - c_2 \leq d + \|(u, v)\|_E.
\]

Hence, there is \(c_3 > 0\), such that

\[
\|u\|_{L^{p+1}} \leq c_3 + c_3\|(u, v)\|_{L^{p+1}}^{\frac{1}{p+1}}, \text{ and } \|v\|_{L^{q+1}} \leq c_3 + c_3\|(u, v)\|_{L^{q+1}}^{\frac{1}{q+1}}. \tag{3.1}
\]

**Step 2. Estimates of \(\|u^+\|_H, \|v^+\|_H, \|y\|_{L^2} \) and \(\|z\|_{L^2}\)**

We decompose \(u = u^+ + u^- + y, v = v^+ + v^- + z, \) where \(u^+, v^+ \in H_b^+, u^-, v^- \in H_b^-\), and \(y, z \in H^0.\) Noting that \(\|u^2\|_{L^2} = \|u^+\|_{L^2}^2 + \|u^-\|_{L^2}^2 + \|y\|_{L^2}^2\), then by virtue of \(p, q > 1\) and (3.1), we estimate

\[
\|y\|_{L^2}^2 \leq \|u\|_{L^2}^2 \leq c\|u\|_{L^{p+1}}^2 \leq c_4 + c_4\|(u, v)\|_{L^2}^{\frac{2}{2}}, \tag{3.2}
\]

and

\[
\|z\|_{L^2}^2 \leq \|v\|_{L^2}^2 \leq c\|v\|_{L^{q+1}}^2 \leq c_4 + c_4\|(u, v)\|_{L^2}^{\frac{2}{q+1}}. \tag{3.3}
\]

Taking \((\varphi, \psi) = (u^+, v^+)\) in (2.3), from the orthogonality of the subspaces \(H_b^+, H_b^-\) and \(H^0\), we get

\[
\langle \Phi'_n(u, v), (u^+, v^+) \rangle = -\langle (L + b)u^+, u^+ \rangle - \langle (L + b)v^+, v^+ \rangle - 2\varepsilon \int u^+v^+ - \int f(t, x, u)u^+ - \int g(t, x, v)v^+.
\]

In fact of \(\|u^+\|_H^2 = \langle (L + b)u^+, u^+ \rangle, \) and \(\|v^+\|_H^2 = \langle (L + b)v^+, v^+ \rangle\) for \(u^+, v^+ \in H_b^+,\) when \(n\) is large enough we obtain

\[
\|u^+\|_H^2 + \|v^+\|_H^2 \leq o(1) - 2\varepsilon \int u^+v^+ - \int f(t, x, u)u^+ - \int g(t, x, v)v^+. \tag{3.4}
\]

A similar argument as in (3.2) provides that

\[
-2\varepsilon \int u^+v^+ \leq 2|\varepsilon|\|u^+\|_{L^2}\|v^+\|_{L^2} \leq c\|u\|_{L^{p+1}}\|v\|_{L^{q+1}}.
\]

13
By assumption (h1), (3.1) and the Hölder inequality, we get

$$\leq c_5 + c_5 \| (u,v) \|_{E}^{\frac{1}{p+1}} + c_5 \| (u,v) \|_{E}^{\frac{1}{q+1}} + c_5 \| (u,v) \|_{E}^{\frac{1}{p+1} + \frac{1}{q+1}}. \quad (3.5)$$

$$- \int f(t,x,u)u^+ \leq c_0 \int (|u^+| + |u| |u^+|) \leq c_0 \| u^+ \|_{L^2} + c_0 \| u \|_{L^{p+1}} \| u^+ \|_{L^{p+1}}$$

$$\leq c_6 \left(1 + \| (u,v) \|_{E}^{\frac{p}{p+1}} \right) \| u^+ \|_{H}, \quad (3.6)$$

where the last inequality is deduced by (2.7), (2.8) and (3.1). Similarly, we have

$$- \int g(t,x,v)v^+ \leq c \| v^+ \|_{L^2} + c_0 \| v \|_{L^{q+1}} \leq c_6 \left(1 + \| (u,v) \|_{E}^{\frac{q}{q+1}} \right) \| v^+ \|_{H}. \quad (3.7)$$

Inserting (3.5)-(3.7) into the right hand side of (3.4), and by the inequalities that $\| u^+ \|_{H} \leq \| (u,v) \|_{E}$ and $\| v^+ \|_{H} \leq \| (u,v) \|_{E}$, we know

$$\| u^+ \|_{H}^2 + \| v^+ \|_{H}^2 \leq o(1) + c_5 + c_5 \| (u,v) \|_{E}^{\frac{1}{p+1}} + c_5 \| (u,v) \|_{E}^{\frac{1}{q+1}} + c_5 \| (u,v) \|_{E}^{\frac{1}{p+1} + \frac{1}{q+1}}$$

$$+ c_6 \left(1 + \| (u,v) \|_{E}^{\frac{p}{p+1}} + \| (u,v) \|_{E}^{\frac{q}{q+1}} \right) \| (u,v) \|_{E}. \quad (3.8)$$

For $u^-, v^- \in H^1$, analogue to (3.8) we also derive

$$\| u^- \|_{H}^2 + \| v^- \|_{H}^2 = \left\langle \Phi'_{\theta_n}(u,v), (u^-, v^-) \right\rangle + 2\varepsilon \int u^- v^- + \int f(t,x,u)u^- + \int g(t,x,v)v^-$$

$$\leq c'_5 + c'_5 \| (u,v) \|_{E}^{\frac{1}{p+1}} + c'_5 \| (u,v) \|_{E}^{\frac{1}{q+1}} + c'_5 \| (u,v) \|_{E}^{\frac{1}{p+1} + \frac{1}{q+1}}$$

$$+ c_6 \left(1 + \| (u,v) \|_{E}^{\frac{p}{p+1}} + \| (u,v) \|_{E}^{\frac{q}{q+1}} \right) \| (u,v) \|_{E}. \quad (3.9)$$

**Step 3. Bound of $\| (u,v) \|_{E}$**

Observing that

$$\| (u,v) \|_{E}^2 = \| u^+ \|_{H}^2 + \| v^+ \|_{H}^2 + \| u^- \|_{H}^2 + \| v^- \|_{H}^2 + \| y \|_{L^2}^2 + \| z \|_{L^2}^2,$$

and using the estimates of (3.2), (3.3), (3.8), (3.9), we arrive at

$$\| (u,v) \|_{E}^2 \leq c_7 + c_7 \| (u,v) \|_{E}^{\frac{1}{p+1}} + c_7 \| (u,v) \|_{E}^{\frac{1}{q+1}} + c_7 \| (u,v) \|_{E}^{\frac{1}{p+1} + \frac{1}{q+1}} + c_7 \| (u,v) \|_{E}$$

$$+ c_7 \| (u,v) \|_{E}^{\frac{2}{p+1}} + c_7 \| (u,v) \|_{E} + c_7 \| (u,v) \|_{E}^{\frac{p}{p+1} + 1} + c_7 \| (u,v) \|_{E}^{\frac{q}{q+1} + 1}. \quad (3.10)$$

In view of $p, q > 1$, we find all the powers of $\| (u,v) \|_{E}$ in the right hand side of the preceding inequality are less than 2. Hence, there exists $M > 0$ which is independent of $n$, such that $\| (u,v) \|_{E} \leq M$, and we conclude that any (PS)$^*$ sequence $\{ (u_{\theta_n}, v_{\theta_n}) \}$ of $\Phi$ is bounded.

\[\Box\]
In what follows, under the assumptions of Theorem 1.1, we assert that the functional \( \Phi \) satisfies the \((PS)^*\) condition.

**Lemma 3.2.** Let \( \{(u_{\theta_n}, v_{\theta_n})\} \) be a \((PS)^*\) sequence of \( \Phi \), then \( \{(u_{\theta_n}, v_{\theta_n})\} \) contains a subsequence which converges to a critical point of \( \Phi \).

**Proof.** Since \( E \) is a Hilbert space, then Lemma 3.1 guarantees that \( \{(u_{\theta_n}, v_{\theta_n})\} \) converge weakly to some \((u, v) \in E\) along with a subsequence. Decomposing \( u_{\theta_n} = u_{\theta_n}^+ + u_{\theta_n}^- + y_{\theta_n} \),

\[
v_{\theta_n} = v_{\theta_n}^+ + v_{\theta_n}^- + z_{\theta_n},
\]

where \((u_{\theta_n}^+, v_{\theta_n}^+) \in E_b^+, (u_{\theta_n}^-, v_{\theta_n}^-) \in E_b^-, \) and \((y_{\theta_n}, z_{\theta_n}) \in E^0\).

Let \((u^+, v^+) \in E_b^+, (u^-, v^-) \in E_b^-\) and \((y, z) \in E^0\) are the weak limits of \( \{(u_{\theta_n}^+, v_{\theta_n}^+)\} \), \( \{(u_{\theta_n}^-, v_{\theta_n}^-)\} \) and \( \{(y_{\theta_n}, z_{\theta_n})\} \) respectively.

We will conclude that \( \{(u_{\theta_n}, v_{\theta_n})\} \) converge strongly to \((u, v) = (u^+ + u^- + y, v^+ + v^- + z)\), by extracting a subsequence if necessary.

**Strong convergence of \( \{(u_{\theta_n}^+, v_{\theta_n}^+)\} \) and \( \{(u_{\theta_n}^-, v_{\theta_n}^-)\} \) in \( E \)**

For \((u_{\theta_n}^+, v_{\theta_n}^+) \in E_b^+\) and \((u^+, v^+) \in E_b^+\), by virtue of the weak convergence of \( u_{\theta_n}^+ \rightarrow u^+ \) and \( v_{\theta_n}^+ \rightarrow v^+ \) in \( H \), we have \( \langle (L + b)u_{\theta_n}^+, u_{\theta_n}^+ - u^+ \rangle \rightarrow 0 \) and \( \langle (L + b)v_{\theta_n}^+, v_{\theta_n}^+ - v^+ \rangle \rightarrow 0 \) as \( n \rightarrow \infty \). Hence it follows that, for large \( n \),

\[
\| u_{\theta_n}^+ - u^+ \|^2_H + \| v_{\theta_n}^+ - v^+ \|^2_H \\
= \langle (L + b)(u_{\theta_n}^+ - u^+), u_{\theta_n}^+ - u^+ \rangle + \langle (L + b)(v_{\theta_n}^+ - v^+), v_{\theta_n}^+ - v^+ \rangle \\
= \langle (L + b)u_{\theta_n}^+, u_{\theta_n}^+ - u^+ \rangle + \langle (L + b)v_{\theta_n}^+, v_{\theta_n}^+ - v^+ \rangle + o(1). \tag{3.10}
\]

From (2.3), the first two terms in the right hand side of (3.10) can be expressed by

\[
\langle (L + b)u_{\theta_n}^+, u_{\theta_n}^+ - u^+ \rangle + \langle (L + b)v_{\theta_n}^+, v_{\theta_n}^+ - v^+ \rangle \\
= - \langle \Phi'(u_{\theta_n}, v_{\theta_n}), (u_{\theta_n}^+ - u^+, v_{\theta_n}^+ - v^+) \rangle - \int f(t, x, u_{\theta_n})(u_{\theta_n}^+ - u^+) \\
- \varepsilon \int v_{\theta_n}(u_{\theta_n}^+ - u^-) - \varepsilon \int u_{\theta_n}(v_{\theta_n}^+ - v^+) - \int g(t, x, u_{\theta_n})(v_{\theta_n}^+ - v^+). \tag{3.11}
\]

To control (3.11), we denote by \( P_{\theta_n} \) the projection operator from \( E \) to its subspace \( E_{\theta_n} \), and we represent the first term in the right hand side of (3.11) as

\[
\langle \Phi'(u_{\theta_n}, v_{\theta_n}), (u_{\theta_n}^+ - u^+, v_{\theta_n}^+ - v^+) \rangle = \langle \Phi'_{\theta_n}(u_{\theta_n}, v_{\theta_n}), (u_{\theta_n}^+ - P_{\theta_n}u^+, v_{\theta_n}^+ - P_{\theta_n}v^+) \rangle \\
- \langle \Phi'(u_{\theta_n}, v_{\theta_n}), (I - P_{\theta_n})(u^+, v^+) \rangle.
\]

Noting that \((u_{\theta_n}^+ - P_{\theta_n}u^+, v_{\theta_n}^+ - P_{\theta_n}v^+) \in E_{\theta_n}\), then the facts of \( \Phi'_{\theta_n}(u_{\theta_n}, v_{\theta_n}) \rightarrow 0 \) and \((I - P_{\theta_n})(u^+, v^+) \rightarrow 0\) assure that

\[
\langle \Phi'(u_{\theta_n}, v_{\theta_n}), (u_{\theta_n}^+ - u^+, v_{\theta_n}^+ - v^+) \rangle \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty. \tag{3.12}
\]
Observe that \( H^+_b \oplus H^-_b \) embeds compactly into \( L^r(\Omega) \) for \( r \geq 1 \), then \( u^+_\theta_n \rightharpoonup u^+ \), \( v^+_\theta_n \rightharpoonup v^+ \) in \( H \) imply that \( u^+_\theta_n \to u^+ \), \( v^+_\theta_n \to v^+ \) strongly in \( L^r(\Omega) \) for every \( r \geq 1 \). Moreover, since \( \{u_\theta_n\} \), \( \{v_\theta_n\} \) are bounded in \( H \), and from (2.7) we obtain

\[
\begin{align*}
&\varepsilon \int v_\theta_n(u^+_\theta_n - u^+) \leq |\varepsilon| \|v_\theta_n\|_{L^2} \|u^+_\theta_n - u^+\|_{L^2} \to 0, \\
&\varepsilon \int u_\theta_n(v^+_\theta_n - v^+) \leq |\varepsilon| \|u_\theta_n\|_{L^2} \|v^+_\theta_n - v^+\|_{L^2} \to 0, \text{ as } n \to \infty.
\end{align*}
\]

By the assumption (h1) together with Hölder inequality, it follows from (3.1) and the boundedness of \( \|(u_\theta_n, v_\theta_n)\|_E \) that

\[
\begin{align*}
&\int f(t, x, u_\theta_n)(u^+_\theta_n - u^+) \leq c_8 \|u^+_\theta_n - u^+\|_{L^p} + c_8 \|u_\theta_n\|_{L^{p+1}} \|u^+_\theta_n - u^+\|_{L^{p+1}} \to 0, \\
&\int g(t, x, v_\theta_n)(v^+_\theta_n - v^+) \leq c_8 \|v^+_\theta_n - v^+\|_{L^q} + c_8 \|v_\theta_n\|_{L^{q+1}} \|v^+_\theta_n - v^+\|_{L^{q+1}} \to 0,
\end{align*}
\]

as \( n \to \infty \).

Collecting (3.10)-(3.16), we infer

\[
\|u^+_\theta_n - u^+\|^2_{H} + \|v^+_\theta_n - v^+\|^2_{H} \to 0, \text{ as } n \to \infty,
\]

which means \( \{(u^+_\theta_n, v^+_\theta_n)\} \) converge to \((u^+, v^+)\) strongly in \( E \). Furthermore, for \((u^-_\theta_n, v^-_\theta_n) \in E^-_b\) and \((u^-, v^-) \in E^-_b\), a similar argument allows us to obtain \((u^-_\theta_n, v^-_\theta_n) \to (u^-, v^-)\) strongly in \( E \).

To finish the proof of this lemma, it remains to derive the strong convergence of \((y_\theta_n, z_\theta_n) \to (y, z)\) in \( E \), where \((y_\theta_n, z_\theta_n), (y, z) \in E^0\). Keeping in mind that the embedding from \( E^0 \) to \( L^r(\Omega) \) is not compact, thus the above procedure is invalid to control the components \((y, z) \in E^0\). In order to deal with the difficulties stem from the lack of compactness, we shall employ the monotonicity method to study the asymptotic behavior of \( \{(y_\theta_n, z_\theta_n)\} \).

**Strong convergence of \( \{(y_\theta_n, z_\theta_n)\} \) in \( E \) for small \( \varepsilon \)**

Recall (2.1), it suffices to show that \( \|y_\theta_n - y\|_{L^2} \to 0, \|z_\theta_n - z\|_{L^2} \to 0 \) as \( n \to \infty \), where \((y, z)\) is the weak limit of \( \{(y_\theta_n, z_\theta_n)\} \) in \( E^0\).

From (2.3) and set \((\varphi, \psi) = (y_\theta_n - y, z_\theta_n - z)\), then

\[
\langle Lu_\theta_n, y_\theta_n - y \rangle = \langle Ly_\theta_n, y_\theta_n - y \rangle = 0, \quad \langle Lv_\theta_n, z_\theta_n - z \rangle = \langle Lz_\theta_n, z_\theta_n - z \rangle = 0
\]

imply that

\[
-(\Phi'(u_\theta_n, v_\theta_n), (y_\theta_n - y, z_\theta_n - z)) = \int f(t, x, u_\theta_n)(y_\theta_n - y) + \int g(t, x, v_\theta_n)(z_\theta_n - z)
\]
\[ + b \int u_{\theta_n}(y_{\theta_n} - y) + \varepsilon \int v_{\theta_n}(y_{\theta_n} - y) + b \int v_{\theta_n}(z_{\theta_n} - z) + \varepsilon \int u_{\theta_n}(z_{\theta_n} - z). \] (3.17)

Moreover, an analogue of argument in (3.12) gives
\[
\left\langle n \right\rangle_{\theta_n} - b \int \varepsilon \int u_{\theta_n}(y_{\theta_n} - y) + b \int v_{\theta_n}(z_{\theta_n} - z) = b \int (y_{\theta_n} - y)^2 + b \int (z_{\theta_n} - z)^2 + o(1),
\]
and
\[
\varepsilon \int v_{\theta_n}(y_{\theta_n} - y) + \varepsilon \int u_{\theta_n}(z_{\theta_n} - z) = 2\varepsilon \int (y_{\theta_n} - y)(z_{\theta_n} - z) + o(1), \quad \text{as } n \to \infty.
\]

Moreover, an analogue of argument in (3.12) gives \( \left\langle \Phi'(u_{\theta_n}, v_{\theta_n}); (y_{\theta_n} - y, z_{\theta_n} - z) \right\rangle \to 0 \),
as \( n \to \infty \).

Therefore, the previous three estimates and (3.17) allow us to deduce that
\[
(b - |\varepsilon|)\|y_{\theta_n} - y\|_{L^2}^2 + (b - |\varepsilon|)\|z_{\theta_n} - z\|_{L^2}^2
\]
\[
\leq b\|y_{\theta_n} - y\|_{L^2}^2 + b\|z_{\theta_n} - z\|_{L^2}^2 + 2\varepsilon \int (y_{\theta_n} - y)(z_{\theta_n} - z)
\]
\[
= - \int f(t, x, u_{\theta_n})(y_{\theta_n} - y) - \int g(t, x, v_{\theta_n})(z_{\theta_n} - z) + o(1), \quad \text{as } n \to \infty. \quad (3.18)
\]

To control the right hand side of (3.18), we rewrite
\[
\int f(t, x, u_{\theta_n})(y_{\theta_n} - y) = \int [f(t, x, u_{\theta_n}^+ + u_{\theta_n}^- + y_{\theta_n}) - f(t, x, u_{\theta_n}^+ + u_{\theta_n}^- + y_{\theta_n})](y_{\theta_n} - y)
\]
\[
+ \int [f(t, x, u_{\theta_n}^+ + u_{\theta_n}^- + y) - f(t, x, u_{\theta_n}^+ + u_{\theta_n}^- + y)](y_{\theta_n} - y)
\]
\[
+ \int f(t, x, u_{\theta_n}^+ + u_{\theta_n}^- + y)(y_{\theta_n} - y) := I_1 + I_2 + I_3. \quad (3.19)
\]

We estimate \( I_1, I_2 \) and \( I_3 \) in the sequel.

Noting that \( f(t, x, \xi) \) is nondecreasing in \( \xi \) by the condition (h4) of Theorem 1.1, we have \( I_1 \geq 0 \) immediately.

To estimate \( I_2 \) and \( I_3 \), firstly we check \( y_{\theta_n} \rightharpoonup y \) weakly in \( L^{p+1}(\Omega) \). For \( u_{\theta_n} = u_{\theta_n}^+ + u_{\theta_n}^- \), we have \( u_{\theta_n} \rightharpoonup y_{\theta_n} \) in view of (3.1) and the embedding \( H^1_0 \hookrightarrow L^{p+1}(\Omega) \), we get
\[
\|y_{\theta_n}\|_{L^{p+1}} \leq \|u_{\theta_n}\|_{L^{p+1}} + \|u_{\theta_n}^+\|_{L^{p+1}} + \|u_{\theta_n}^-\|_{L^{p+1}} \leq \|u_{\theta_n}\|_{L^{p+1}} + c\|u_{\theta_n}^+\|_H + c\|u_{\theta_n}^-\|_H
\]
\[
\leq c_3 + c_3\|(u_{\theta_n}, v_{\theta_n})\|_E \leq c_3\|(u_{\theta_n}, v_{\theta_n})\|_E. \quad (3.20)
\]
Then the boundedness of the (PS)* sequence \( \{(u_{\theta_n}, v_{\theta_n})\} \) ensures that \( \{y_{\theta_n}\} \) is bounded in \( L^{p+1}(\Omega) \), and \( \{y_{\theta_n}\} \) possesses a subsequence which converge weakly in \( L^{p+1}(\Omega) \). Recording that \( L^{p+1}(\Omega) \hookrightarrow L^2(\Omega) \) for \( p > 1 \) and \( y \) is the weak limit of \( \{y_{\theta_n}\} \) in \( L^2(\Omega) \), hence by the uniqueness of weak limit, we have \( y_{\theta_n} \rightharpoonup y \) weakly in \( L^{p+1}(\Omega) \), with a subsequence still renamed by \( \{y_{\theta_n}\} \).

By the condition (h1) of Theorem 1.1, we know the operator \( f : \xi \mapsto f(t, x, \xi) \) is continuous from \( L^{p+1}(\Omega) \) to \( L^{\frac{p+1}{p}}(\Omega) \). Since \( u_{\theta_n}^+ \to u^+ \) and \( u_{\theta_n}^- \to u^- \) in \( H \), we have \( u_{\theta_n}^+ + u_{\theta_n}^- \to u^+ + u^- \) strongly in \( L^{p+1}(\Omega) \) via the embedding \( H^+_b \oplus H^-_b \hookrightarrow L^{p+1}(\Omega) \). Therefore,

\[
f(t, x, u_{\theta_n}^+ + u_{\theta_n}^- + y) - f(t, x, u^+ + u^- + y) \to 0 \text{ in } L^{\frac{p+1}{p}}, \quad \text{as } n \to \infty,
\]

and \( f(t, x, u^+ + u^- + y) \) is in \( L^{\frac{p+1}{p}}(\Omega) \). By virtue of \( y_{\theta_n} - y \in L^{p+1} \) and \( L^{\frac{p+1}{p}}(\Omega) = (L^{p+1}(\Omega))^* \), we obtain that \( I_2 \to 0 \) and \( I_3 \to 0 \), as \( n \to \infty \).

With the estimates of \( I_1, I_2, I_3 \) in hand, then passing the limit in (3.19) yields that

\[
\int f(t, x, u_{\theta_n})(y_{\theta_n} - y) \geq 0, \quad \text{as } n \to \infty. \tag{3.21}
\]

Moreover, with a similar computation, we find

\[
\int g(t, x, v_{\theta_n})(z_{\theta_n} - z) \geq 0, \quad \text{as } n \to \infty.
\]

Hence, turning back to (3.18) and choosing \( |\varepsilon| < b \), we have \( \|y_{\theta_n} - y\|_{L^2}^2 + \|z_{\theta_n} - z\|_{L^2}^2 \leq o(1) \) as \( n \to \infty \). Thereby the proof of Lemma 3.2 is completed.

**Proof of (A2):**

The following lemma shows the functional \( \Phi \) satisfying the local linking structure for small \( \varepsilon \).

**Lemma 3.3.** Assume that (h1)–(h4) of Theorem 1.1 are satisfied, then for \( \varepsilon \) is sufficiently small, there exists \( \rho > 0 \), such that

(i) \( \Phi(u, v) \geq 0 \), for \( (u, v) \in E^1 \cap B_\rho \),

(ii) \( \Phi(u, v) \leq 0 \), for \( (u, v) \in E^2 \cap B_\rho \),

where \( E^1 = E^-_b \), \( E^2 = E^+_b \oplus E^0 \), and \( B_\rho = \{(u, v)|\|(u, v)||_E \leq \rho\} \).

**Proof.** (i) For \( (u, v) \in E^-_b = H^-_b \times H^-_b \), by (2.4) we have

\[
\|(u, v)\|^2_E = \|u\|^2_{H^-_b} + \|v\|^2_{H^-_b} = -\langle (L + b)u, u \rangle - \langle (L + b)v, v \rangle.
\]

Thus the energy functional (2.5) can be represented by

\[
\Phi(u, v) = \frac{1}{2}\|(u, v)\|^2_E - \varepsilon \int uv - \int F(t, x, u) - \int G(t, x, v).
\]
By (2.6), we get
\[ \varepsilon \int uv \leq \frac{|\varepsilon|}{2} ||u||_{L^2}^2 + \frac{|\varepsilon|}{2} ||v||_{L^2}^2 \leq \frac{|\varepsilon|}{2\eta} ||u||_H^2 + \frac{|\varepsilon|}{2\eta} ||v||_H^2 = \frac{|\varepsilon|}{2\eta} ||(u, v)||_E^2. \]

Using (1.5) and the embedding \( H_b^- \hookrightarrow L^r(\Omega) \) for \( r \geq 1 \), then for each \( \nu > 0 \), we deduce that
\[ \int F(t, x, u) + \int G(t, x, v) \leq \nu ||u||_{L^2}^2 + C_\nu ||u||_{L^{p+1}}^{p+1} + \nu ||v||_{L^2}^2 + C_\nu ||v||_{L^{q+1}}^{q+1} \]
\[ \leq \frac{\nu}{\eta} ||(u, v)||_E^2 + c_1 ||u||_{H^1}^{p+1} + c_2 ||v||_{H^1}^{q+1}. \]

Putting \( \nu = \varepsilon/2 \), it holds that
\[ \Phi(u, v) \geq \frac{1}{2} \left( 1 - \frac{2|\varepsilon|}{\eta} \right) ||(u, v)||_E^2 - c_1 ||u||_{H^1}^{p+1} - c_2 ||v||_{H^1}^{q+1}. \]

Choosing \( |\varepsilon| < \eta/2 \) and letting
\[ \rho = \min \left\{ \left( \frac{\eta - 2|\varepsilon|}{2c_1 \eta} \right)^{\frac{1}{p-1}}, \left( \frac{\eta - 2|\varepsilon|}{2c_2 \eta} \right)^{\frac{1}{q-1}} \right\}, \]
then for \( (u, v) \in E^1 \) and \( ||(u, v)||_E \leq \rho \), we arrive at
\[ \Phi(u, v) \geq ||u||_{H^1}^{p+1} \left[ \frac{1}{2} \left( 1 - \frac{2|\varepsilon|}{\eta} \right) - c_1 ||u||_{H^1}^{p-1} \right] + ||v||_{H^1}^{q+1} \left[ \frac{1}{2} \left( 1 - \frac{2|\varepsilon|}{\eta} \right) - c_2 ||v||_{H^1}^{q-1} \right] \geq 0. \]

(ii) For \((u, v) \in E_b^+ \oplus E^0\), we split \( u = u^+ + y, \quad v = v^+ + z \), where \( u^+, \quad v^+ \in H_b^+ \), and \( y, \quad z \in H^0 \). Now,
\[ \Phi(u, v) = - \frac{1}{2} \langle (L + b)u^+, u^+ \rangle - \frac{1}{2} \langle (L + b)v^+, v^+ \rangle - \frac{b}{2} \int y^2 - \frac{b}{2} \int z^2 \]
\[ - \varepsilon \int uv - \int F(t, x, u) - \int G(t, x, v). \]

From (2.6), we get
\[ - \int uv \leq \frac{1}{2} ||u^+||_{L^2}^2 + \frac{1}{2} ||v^+||_{L^2}^2 \leq \frac{1}{2\eta} ||u^+||_{H^1}^2 + \frac{1}{2} ||y||_{L^2}^2 + \frac{1}{2} ||v^+||_{H^1}^2 + \frac{1}{2} ||z||_{L^2}^2. \]

Hence, \( F(t, x, u) \geq 0 \) and \( G(t, x, v) \geq 0 \) lead to
\[ \Phi(u, v) \leq - \frac{1}{2} \left( 1 - \frac{|\varepsilon|}{\eta} \right) ||u^+||_{H^1}^2 - \frac{1}{2} \left( 1 - \frac{|\varepsilon|}{\eta} \right) ||v^+||_{H^1}^2 - \frac{1}{2} (b - |\varepsilon|) ||y||_{L^2}^2 - \frac{1}{2} (b - |\varepsilon|) ||z||_{L^2}^2. \]

Then, for \((u, v) \in E^2\), we have \( \Phi(u, v) \leq 0 \) by selecting \( |\varepsilon| < \min\{\eta, b\} \).
Proof of (A3):
Concerning with the bound of $\Phi$ in a bounded set, we have:

**Lemma 3.4.** $\Phi$ maps bounded sets into bounded sets.

*Proof.* Let $R_0 > 0$ and $D = \{(u, v) \in E \mid \|(u, v)\|_E \leq R_0\}$, we claim that there exists a constant $M_0 > 0$, such that $\Phi(u, v) \leq M_0$ for each $(u, v) \in D$.

In fact, we decompose $u = u^+ + u^- + y$, $v = v^+ + v^- + z$, where $u^+, v^+ \in H^+_b$, $u^-, v^- \in H^-_b$, and $y, z \in H^0$. Then combining (2.5), (2.7) with the fact $F(t, x, u) \geq 0$ and $G(t, x, v) \geq 0$, we obtain

$$
\Phi(u, v) \leq \frac{1}{2} \|u^+\|_{H^2}^2 + \frac{1}{2} \|v^-\|_{H^2}^2 - \varepsilon \int uv \leq \frac{1}{2} \|u\|_{H^2}^2 + \frac{1}{2} \|v\|_{H^2}^2 + \frac{|\varepsilon|}{2} \|u\|_{L^2}^2 + \frac{|\varepsilon|}{2} \|v\|_{L^2}^2
$$

$$
\leq \frac{1}{2} \|(u, v)\|_E^2 + \frac{\kappa |\varepsilon|}{2} \|u\|_{H^2}^2 + \frac{\kappa |\varepsilon|}{2} \|v\|_{H^2}^2 \leq \frac{1 + \kappa |\varepsilon|}{2} R_0^2 := M_0
$$

for each $(u, v) \in D$. That is what we desire. \hfill \Box

Proof of (A4):
We move to verify that $\Phi$ holds the last condition of Proposition A.

**Lemma 3.5.** For every $m \in \mathbb{N}$, and $(u, v) \in E^1_m \oplus E^2$, we have $\Phi(u, v) \to -\infty$, as $\|(u, v)\|_E \to \infty$ and $\varepsilon$ is sufficiently small.

*Proof.* Let $u = u^+ + u^- + y$, $v = v^+ + v^- + z$, for $(u^+, v^+) \in E^+_b$, $(y, z) \in E^0$, and $(u^-, v^-) \in E^1_m = \text{span}\{e^1_1, \ldots, e^1_m\}$, where $(e^1_n)_{n=1}^\infty$ is a basis for $E^1 = E^1_b$.

With the aid of (2.6), the coupled term in (2.5) can be controlled by

$$
-\varepsilon \int uv \leq \frac{|\varepsilon|}{2} \|u\|_{L^2}^2 + \|v\|_{L^2}^2 \leq \frac{|\varepsilon|}{2 \eta} (\|u^+\|_{H^2}^2 + \|u^-\|_{H^2}^2 + \|v^+\|_{H^2}^2 + \|v^-\|_{H^2}^2) + \frac{|\varepsilon|}{2} (\|y\|_{L^2}^2 + \|z\|_{L^2}^2).
$$

To deal with the nonlinear forced terms in (2.5), we utilize (1.3) and the embeddings $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega), L^{q+1}(\Omega) \hookrightarrow L^2(\Omega)$ for $p, q > 1$, to get

$$
- \int F(t, x, u) - \int G(t, x, v) \leq -c_1 \|u\|_{L^{p+1}}^{p+1} - c_1 \|v\|_{L^{q+1}}^{q+1} + c_2 \leq -c_3 \|u\|_{L^2}^{p+1} - c_3 \|v\|_{L^2}^{q+1} + c_2.
$$

Noting the dimension of $E^1_m$ is finite, and the norms in the function space $E^1_m$ are equivalent, then for $(u^-, v^-) \in E^1_m$, $\|u^-\|_{H^2}^{p+1} \leq c \|u^-\|_{L^2}^{p+1} \leq c \|u\|_{L^2}^{p+1}$, and $\|v^-\|_{H^2}^{q+1} \leq c \|v\|_{L^2}^{q+1}$.

Inserting the preceding estimates into (2.5), we have

$$
\Phi(u, v) \leq -\frac{1}{2} \left(1 - \frac{|\varepsilon|}{\eta}\right) (\|u^+\|_{H^2}^2 + \|v^+\|_{H^2}^2) - \frac{1}{2} (b - |\varepsilon|) (\|y\|_{L^2}^2 + \|z\|_{L^2}^2)
$$

$$
+ \frac{1}{2} (1 + \frac{|\varepsilon|}{\eta}) \|u^-\|_{H^2}^2 - c_4 \|u^-\|_{H^2}^{p+1} + \frac{1}{2} (1 + \frac{|\varepsilon|}{\eta}) \|v^-\|_{H^2}^{q+1} - c_4 \|v^-\|_{H^2}^{q+1} + c_2.
$$
As \( \| (u, v) \|_E = (\| u \|_H^2 + \| v \|_H^2)^{1/2} \to \infty \), then: (i) \( \| u^+ \|_H^2 + \| v^+ \|_H^2 + \| y \|_{L^2}^2 + \| z \|_{L^2}^2 \to \infty \), or (ii) \( \| u^- \|_H^2 + \| v^- \|_H^2 \to \infty \) holds.

If (i) satisfies, then there exists a positive number \( C > 0 \), such that

\[
\frac{1}{2} (1 + \frac{1}{\eta}) \| u^- \|_H^2 - c_4 \| u^- \|_{H^2}^{p+1} + \frac{1}{2} (1 + \frac{1}{\eta}) \| v^- \|_H^2 - c_4 \| v^- \|_{H^2}^{q+1} \leq C,
\]

for \( p, q > 1 \). Hence, it follows that \( \Phi(u, v) \to -\infty \) by selecting \( |\varepsilon| < \min \{ \eta, b \} \), as \( \| u^+ \|_H^2 + \| v^+ \|_H^2 + \| y \|_{L^2}^2 + \| z \|_{L^2}^2 \to \infty \).

If (ii) holds, then

\[
\frac{1}{2} (1 + \frac{1}{\eta}) \| u^- \|_H^2 - c_4 \| u^- \|_{H^2}^{p+1} + \frac{1}{2} (1 + \frac{1}{\eta}) \| v^- \|_H^2 - c_4 \| v^- \|_{H^2}^{q+1} \to -\infty
\]

by virtue of \( p, q > 1 \). We derive that \( \Phi(u, v) \to -\infty \) as \( \| u^- \|_H^2 + \| v^- \|_H^2 \to \infty \), for \( |\varepsilon| < \min \{ \eta, b \} \). The conclusion of Lemma 3.5 is thereby obtained.

Now, we have proved that the functional \( \Phi \in C^1(E, \mathbb{R}) \) satisfy the conditions (A1)–(A4) of the Proposition A, which ensure us to construct a nontrivial critical point \( (u, v) \) of \( \Phi \) in \( E \). Hence, we finish the proof of Theorem 1.1.

\[\square\]

4 Asymptotic behavior of the solutions as \( \varepsilon \to 0 \)

In the following, we use \( c_i \) and \( C \) to denote positive constants which are independent of \( \varepsilon \), and whose value may differ from line to line.

Let \( (u_\varepsilon, v_\varepsilon) \) be the solution of (1.1) \(_{a,b,c} \) obtained in Theorem 1.1 for \( |\varepsilon| < \varepsilon_0 \), we know

\[
\langle (L + b)u_\varepsilon, u_\varepsilon \rangle + \varepsilon \int u_\varepsilon v_\varepsilon + \int f(t, x, u_\varepsilon)u_\varepsilon = 0,
\]

\[
\langle (L + b)v_\varepsilon, v_\varepsilon \rangle + \varepsilon \int u_\varepsilon v_\varepsilon + \int g(t, x, v_\varepsilon)v_\varepsilon = 0.
\]

Then, by (2.2), (1.3) and (h3) of Theorem 1.1, we have

\[
\Phi(u_\varepsilon, v_\varepsilon) = \frac{1}{2} \int \left[f(t, x, u_\varepsilon)u_\varepsilon - F(t, x, u_\varepsilon)\right] + \frac{1}{2} \int \left[g(t, x, v_\varepsilon)v_\varepsilon - G(t, x, v_\varepsilon)\right]
\geq \frac{p-1}{2} \int F(t, x, u_\varepsilon) + \frac{q-1}{2} \int G(t, x, v_\varepsilon)
\geq \frac{(p-1)c_1}{2} \int |u_\varepsilon|^{p+1} + \frac{(q-1)c_2}{2} \int |v_\varepsilon|^{q+1} - 2c_2\pi^2.
\]

(4.1)

On the other hand, we deduce from Remark 2.1 together with Lemma 3.4 and Lemma 3.5 that, there exists a positive number \( c_3 \) independent of \( \varepsilon \), such that

\[
\Phi(u_\varepsilon, v_\varepsilon) \leq c_3, \quad \text{for any } \varepsilon \in (-\varepsilon_0, \varepsilon_0).
\]

(4.2)
Combining with (4.1), (4.2) and by virtue of \( p, q > 1 \), we get
\[
\|u_\varepsilon\|_{L^{p+1}} + \|v_\varepsilon\|_{L^{q+1}} \leq C, \quad \text{for any } \varepsilon \in (-\varepsilon_0, \varepsilon_0).
\] (4.3)

Let \( \varepsilon_n \in (-\varepsilon_0, \varepsilon_0) \) be any sequence with \( \varepsilon_n \to 0 \) as \( n \to \infty \). Subsequently, we will prove that \((u_{\varepsilon_n}, v_{\varepsilon_n})\) converge strongly to some \((U_0, V_0)\) in \( L^2(\Omega) \times L^2(\Omega) \) as \( n \to \infty \), by passing to a subsequence, and justify that \( U_0, V_0 \) are weak solutions of the scalar wave equations (W1) and (W2) respectively.

At first, we decompose \( u_{\varepsilon_n} = u^+_{\varepsilon_n} + u^-_{\varepsilon_n}, v_{\varepsilon_n} = v^+_{\varepsilon_n} + v^-_{\varepsilon_n} + z_{\varepsilon_n} \), where \((u^+_{\varepsilon_n}, v^+_{\varepsilon_n}) \in E^+_b, (u^-_{\varepsilon_n}, v^-_{\varepsilon_n}) \in E^-_b, \) and \((y_{\varepsilon_n}, z_{\varepsilon_n}) \in E^0, \) and show the next lemma concerning with the asymptotic behavior of \((u_{\varepsilon_n}, v_{\varepsilon_n})\).

**Lemma 4.1.** Passing to a subsequence of \( \varepsilon_n \to 0 \) as \( n \to \infty \), we have
(i) \((u^+_{\varepsilon_n}, v^+_{\varepsilon_n})\) converge strongly to some \((U^+_0, V^+_0)\) in \( E^+_b; \)
(ii) \((u^-_{\varepsilon_n}, v^-_{\varepsilon_n})\) converge strongly to some \((U^-_0, V^-_0)\) in \( E^-_b; \)
(iii) \((y_{\varepsilon_n}, z_{\varepsilon_n})\) converge strongly to some \((U^0_0, V^0_0)\) in \( E^0; \)
(iv) \( u_{\varepsilon_n} \rightharpoonup U_0 \) weakly in \( L^{p+1}(\Omega) \), and \( v_{\varepsilon_n} \rightharpoonup V_0 \) weakly in \( L^{q+1}(\Omega) \), where \( U_0 = U^+_0 + U^-_0 + U^0_0, V_0 = V^+_0 + V^-_0 + V^0_0. \)

**Proof.** First, we establish the uniform bound for \( \{(u_{\varepsilon_n}, v_{\varepsilon_n})\} \) in \( E \).

Recording (4.3) and the embedding properties of \( L^r(\Omega) \hookrightarrow L^2(\Omega) \) for \( r \geq 2 \), we have
\[
\|u_{\varepsilon_n}\|_2^2 + \|z_{\varepsilon_n}\|_2^2 \leq \|u_{\varepsilon_n}\|_2^2 + \|v_{\varepsilon_n}\|_2^2 \leq c_4 \|u_{\varepsilon_n}\|_{L^{p+1}} + c_4 \|v_{\varepsilon_n}\|_{L^{q+1}} \leq C. \] (4.4)

For \((u^+_{\varepsilon_n}, v^+_{\varepsilon_n}) \in E^+_b, \) by (h1), (2.8), (4.3) and the orthogonality of \( H^+_b, H^-_b, H^0, \)
\[
\|u^+_{\varepsilon_n}\|_H^2 + \|v^+_{\varepsilon_n}\|_H^2 = \langle (L + b)u_{\varepsilon_n}, u^+_{\varepsilon_n} \rangle + \langle (L + b)v_{\varepsilon_n}, v^+_{\varepsilon_n} \rangle
\]
\[
= -2\varepsilon_n \int u^+_{\varepsilon_n} v^+_{\varepsilon_n} - \int f(t, x, u_{\varepsilon_n}) v^+_{\varepsilon_n} - \int g(t, x, v_{\varepsilon_n}) v^+_{\varepsilon_n}
\leq c_5 (\|u_{\varepsilon_n}\|_{L^2} \|v_{\varepsilon_n}\|_{L^2} + \|u_{\varepsilon_n}\|_{L^2} + \|u_{\varepsilon_n}\|_{L^{p+1}} \|u^+_{\varepsilon_n}\|_H + \|v_{\varepsilon_n}\|_{L^2} + \|v_{\varepsilon_n}\|_{L^{q+1}} \|v^+_{\varepsilon_n}\|_H)
\leq C(1 + \|(u^+_{\varepsilon_n}, v^+_{\varepsilon_n})\|_E),
\] (4.5)

Similarly, we get
\[
\|u^-_{\varepsilon_n}\|_H^2 + \|v^-_{\varepsilon_n}\|_H^2 = -\langle (L + b)u_{\varepsilon_n}, u^-_{\varepsilon_n} \rangle - \langle (L + b)v_{\varepsilon_n}, v^-_{\varepsilon_n} \rangle
\leq C(1 + \|(u^-_{\varepsilon_n}, v^-_{\varepsilon_n})\|_E), \quad \text{for } (u^-_{\varepsilon_n}, v^-_{\varepsilon_n}) \in E^-_b.
\] (4.6)

Therefore, summing up (4.4)-(4.6), we arrive at
\[
\|(u_{\varepsilon_n}, v_{\varepsilon_n})\|_{L^2}^2 \leq C(1 + \|(u_{\varepsilon_n}, v_{\varepsilon_n})\|_E),
\]
which implies that
\[ \| (u_{\varepsilon_n}, v_{\varepsilon_n}) \|_E \leq C_0, \quad \text{where } C_0 > 0 \text{ is independent of } \varepsilon_n. \] (4.7)

Then, \( \{ (u_{\varepsilon_n}, v_{\varepsilon_n}) \} \) converge weakly to some \((U_0, V_0) \in E\), by passing to a subsequence of \( \varepsilon_n \to 0 \) as \( n \to \infty \). Spilt \( U_0 = U_0^+ + U_0^- + U_0^0 \), \( V_0 = V_0^+ + V_0^- + V_0^0 \), where \( (U_0^+, V_0^+) \in E_b^+, \) \( (U_0^-, V_0^-) \in E_b^- \) and \( (U_0^0, V_0^0) \in E^0 \). We assert that \((U_0^+, V_0^+), (U_0^-, V_0^-) \) and \((U_0^0, V_0^0) \) satisfy (i), (ii), (iii) of this lemma.

Since \( \Phi'(u_{\varepsilon_n}, v_{\varepsilon_n}) = 0 \), and by the orthogonality of \( H_b^+, H_b^-, H^0 \), it follows that
\[
\| u_{\varepsilon_n}^+ - U_0^+ \|_H^2 = \langle (L + b)(u_{\varepsilon_n}^+ - U_0^+), u_{\varepsilon_n}^+ - U_0^+ \rangle \\
= \langle (L + b)u_{\varepsilon_n}, u_{\varepsilon_n}^+ - U_0^+ \rangle - \langle (L + b)U_0, u_{\varepsilon_n}^+ - U_0^+ \rangle \\
= -\varepsilon_n \int v_{\varepsilon_n}(u_{\varepsilon_n}^+ - U_0^+) - \int f(t, \varepsilon_n)(u_{\varepsilon_n}^+ - U_0^+) - \langle (L + b)U_0, u_{\varepsilon_n}^+ - U_0^+ \rangle.
\]

By the weak convergence of \( u_{\varepsilon_n}^+ \to U_0^+ \) as \( n \to \infty \), we can proceed as in the proof of (3.10), (3.13) and (3.15) in Lemma 3.2 to show that \( u_{\varepsilon_n}^+ \to U_0^+ \) strongly in \( E_b^+ \), as \( n \to \infty \). Then (i) holds. We can prove (ii) in the same way.

For \( y_{\varepsilon_n} \in H^0 \), \( z_{\varepsilon_n} \in H^0 \),
\[
-\langle \Phi'(u_{\varepsilon_n}, v_{\varepsilon_n}), (y_{\varepsilon_n} - U_0^0, z_{\varepsilon_n} - V_0^0) \rangle \\
= \langle Lu_{\varepsilon_n}, y_{\varepsilon_n} - U_0^0 \rangle + b \int u_{\varepsilon_n}(y_{\varepsilon_n} - U_0^0) + \varepsilon_n \int v_{\varepsilon_n}(y_{\varepsilon_n} - U_0^0) + \int f(t, \varepsilon_n)(y_{\varepsilon_n} - U_0^0) \\
+ \langle Lv_{\varepsilon_n}, z_{\varepsilon_n} - V_0^0 \rangle + b \int v_{\varepsilon_n}(z_{\varepsilon_n} - V_0^0) + \varepsilon_n \int u_{\varepsilon_n}(z_{\varepsilon_n} - V_0^0) + \int g(t, \varepsilon_n)(z_{\varepsilon_n} - V_0^0).
\]

By virtue of \( \Phi'(u_{\varepsilon_n}, v_{\varepsilon_n}) = 0 \) and \( \langle Lu_{\varepsilon_n}, y_{\varepsilon_n} - U_0^0 \rangle = \langle Lv_{\varepsilon_n}, z_{\varepsilon_n} - V_0^0 \rangle = 0 \), we have
\[
b \int u_{\varepsilon_n}(y_{\varepsilon_n} - U_0^0) + \varepsilon_n \int v_{\varepsilon_n}(y_{\varepsilon_n} - U_0^0) + b \int v_{\varepsilon_n}(z_{\varepsilon_n} - V_0^0) + \varepsilon_n \int u_{\varepsilon_n}(z_{\varepsilon_n} - V_0^0) \\
= -\int f(t, \varepsilon_n)(y_{\varepsilon_n} - U_0^0) - \int g(t, \varepsilon_n)(z_{\varepsilon_n} - V_0^0).
\]

By an analogue proof of (3.18) – (3.21) in Lemma 3.2, we deduce that
\[ \| y_{\varepsilon_n} - U_0^0 \|_{L^2}^2 + \| z_{\varepsilon_n} - V_0^0 \|_{L^2}^2 \to 0, \quad \text{as } n \to \infty, \]
and (iii) is satisfied.

As a consequence of (i), (ii), (iii), we have \( \{ (u_{\varepsilon_n}, v_{\varepsilon_n}) \} \) converge strongly to \((U_0, V_0) \) in \( E \). Particularly,
\[ (u_{\varepsilon_n}, v_{\varepsilon_n}) \to (U_0, V_0) \text{ strongly in } L^2(\Omega) \times L^2(\Omega), \quad \text{as } n \to \infty. \] (4.8)
On the other hand, (4.3) implies that there exists some $U_1 \in L^{p+1}(\Omega)$, such that \{u_{\varepsilon_n}\} possesses a subsequence which converge weakly to $U_1$ in $L^{p+1}(\Omega)$. By virtue of $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega)$ for $p > 1$, and noting the fact that the weak limit of \{u_{\varepsilon_n}\} is unique, we have $U_1 = U_0$. Thus, $u_{\varepsilon_n} \rightharpoonup U_0$ weakly in $L^{p+1}(\Omega)$. Similarly, we get $v_{\varepsilon_n} \rightharpoonup V_0$ weakly in $L^{q+1}(\Omega)$. Hence, (iv) holds. \hfill \Box

Consequently, we assert

**Lemma 4.2.** For any $\varphi \in H \cap L^{p+1}(\Omega)$, $\psi \in H \cap L^{q+1}(\Omega)$, then passing to a subsequence of $\varepsilon_n \to 0$ as $n \to \infty$, we have

\[
\langle (L + b)u_{\varepsilon_n}, u_{\varepsilon_n} - \varphi \rangle \to \langle (L + b)U_0, U_0 - \varphi \rangle,
\]

(4.9)

\[
\langle (L + b)v_{\varepsilon_n}, v_{\varepsilon_n} - \psi \rangle \to \langle (L + b)V_0, V_0 - \psi \rangle;
\]

(4.10)

\[
\langle f(t, x, \varphi), u_{\varepsilon_n} - \varphi \rangle \to \langle f(t, x, \varphi), U_0 - \varphi \rangle,
\]

(4.11)

\[
\langle g(t, x, \psi), v_{\varepsilon_n} - \psi \rangle \to \langle g(t, x, \psi), V_0 - \psi \rangle;
\]

(4.12)

\[
\langle \varepsilon_n u_{\varepsilon_n}, u_{\varepsilon_n} - \varphi \rangle \to 0,
\]

and $\langle \varepsilon_n u_{\varepsilon_n}, v_{\varepsilon_n} - \psi \rangle \to 0.$

(4.13)

**Proof.** To reach (4.9), we write

\[
\langle (L + b)u_{\varepsilon_n}, u_{\varepsilon_n} - \varphi \rangle - \langle (L + b)U_0, U_0 - \varphi \rangle = \langle (L + b)u_{\varepsilon_n}, u_{\varepsilon_n} - U_0 \rangle + \langle (L + b)(u_{\varepsilon_n} - U_0), U_0 - \varphi \rangle := A_1 + A_2.
\]

(4.14)

From the orthogonality of $H^+_b$, $H^-_b$, $H^0$, it follows that

\[
A_1 = \langle (L + b)u_{\varepsilon_n}^+, u_{\varepsilon_n}^- - U_0^+ \rangle + \langle (L + b)u_{\varepsilon_n}^-, u_{\varepsilon_n}^- - U_0^- \rangle + b\langle y_{\varepsilon_n}, y_{\varepsilon_n} - U_0^0 \rangle.
\]

Then, by (4.7), Lemma 4.1 and using Hölder inequality, we infer

\[
|A_1| \leq \|u_{\varepsilon_n}^+\|_H \|u_{\varepsilon_n}^- - U_0^+\|_H + \|u_{\varepsilon_n}^-\|_H \|u_{\varepsilon_n}^- - U_0^-\|_H + b\|y_{\varepsilon_n}\|_{L^2} \|y_{\varepsilon_n} - U_0^0\|_{L^2} \to 0,
\]

as $n \to \infty$. Furthermore, since $\varphi \in H$, we deduce that $A_2 \to 0$ in the same way. Thus, (4.9) holds, and we can obtain (4.10) similarly.

We come to prove (4.11). By condition (h1) of Theorem 1.1, we have

\[
|\langle f(t, x, \varphi), u_{\varepsilon_n} - U_0 \rangle| \leq c_0 \int |u_{\varepsilon_n} - U_0| + c_0 \int |\varphi|^p |u_{\varepsilon_n} - U_0|.
\]

(4.15)

Then (4.8), (iv) of Lemma 4.1 and $|\varphi|^p \in L^{\frac{p+1}{p}}(\Omega) = (L^{p+1}(\Omega))^*$ show the right hand side of (4.15) go to zero as $n \to \infty$, which gives (4.11). Furthermore, (4.12) also holds for any $\psi \in H \cap L^{q+1}(\Omega)$.

At last, by (4.7) and Hölder inequality, it follows that

\[
|\langle \varepsilon_n u_{\varepsilon_n}, u_{\varepsilon_n} - \varphi \rangle| + |\langle \varepsilon_n u_{\varepsilon_n}, v_{\varepsilon_n} - \psi \rangle|
\]

24
\[ \leq |\varepsilon_n| \|v_{\varepsilon_n}\|_{L^2} \|u_{\varepsilon_n} - \varphi\|_{L^2} + |\varepsilon_n| \|u_{\varepsilon_n}\|_{L^2} \|v_{\varepsilon_n} - \psi\|_{L^2} \leq C|\varepsilon_n| \to 0, \]

as \( n \to \infty \). Hence, we arrive at (4.13).

We are ready for the proof of Theorem 1.3:

Proof. We are suffice to prove that

\[ \langle (L + b)U_0 + f(t, x, U_0), \omega \rangle = 0, \quad \forall \, \omega \in H \cap L^{p+1}(\Omega), \quad (4.16) \]

\[ \langle (L + b)V_0 + g(t, x, V_0), \chi \rangle = 0, \quad \forall \, \chi \in H \cap L^{q+1}(\Omega). \quad (4.17) \]

As \((u_{\varepsilon_n}, v_{\varepsilon_n})\) solves the problem (1.1)_{a,b,c} with linear coupling constant \( \varepsilon = \varepsilon_n \), then

\[ \langle (L + b)u_{\varepsilon_n} + f(t, x, u_{\varepsilon_n}) + \varepsilon_n v_{\varepsilon_n}, \ u_{\varepsilon_n} - \varphi \rangle = 0, \quad \forall \, \varphi \in H \cap L^{p+1}(\Omega). \quad (4.18) \]

By (h4) of Theorem 1.1, we know

\[ \langle f(t, x, u_{\varepsilon_n}) - f(t, x, \varphi), \ u_{\varepsilon_n} - \varphi \rangle \geq 0. \]

Hence, we derive from (4.18) that

\[ \langle (L + b)u_{\varepsilon_n} + f(t, x, \varphi) + \varepsilon_n v_{\varepsilon_n}, \ u_{\varepsilon_n} - \varphi \rangle \leq 0, \quad \forall \, \varphi \in H \cap L^{p+1}(\Omega). \quad (4.19) \]

By (4.9), (4.11) and (4.13) of Lemma 4.2, then passing to the limit in (4.19), we have

\[ \langle (L + b)U_0 + f(t, x, \varphi), U_0 - \varphi \rangle \leq 0, \quad \forall \, \varphi \in H \cap L^{p+1}(\Omega). \quad (4.20) \]

Choosing \( \varphi = U_0 - \lambda \omega \) with \( \lambda > 0 \) in (4.20), then dividing by \( \lambda \) and letting \( \lambda \to 0 \), we obtain

\[ \langle (L + b)U_0 + f(t, x, U_0), \omega \rangle \leq 0, \]

and noting \( \omega \in H \cap L^{p+1}(\Omega) \) is chosen arbitrarily, we infer that \( U_0 \) satisfies

\[ \langle (L + b)U_0 + f(t, x, U_0), \omega \rangle = 0, \quad \forall \, \omega \in H \cap L^{p+1}(\Omega). \]

Thereby, (4.16) is concluded.

In the same manner, we obtain (4.17) by virtue of (4.10), (4.12) and (4.13). This concludes the proof of Theorem 1.3.

5 Higher regularity of the solutions

In this section, we prove the solutions \((u, v) \in E\) obtained in the previous section enjoy the higher regularity, providing that \( \varepsilon \) is sufficiently small.
5.1 A representation theorem for single wave equation

To proceed, we collect some facts that are useful in improving the regularity of the weak solutions for (1.1). Let \( \Omega = [0, 2\pi] \times [0, \pi] \), and \( \ker L \) is the kernel of the d’Alembert operator \( L = \partial_t^2 - \partial_x^2 \). Denote \( R(L) \) is the range of \( L \), then we have \( R(L) = (\ker L)^\perp \). The following representation theorem plays an important role in the study of the regularity theory of scalar wave equations.

**Proposition 5.1.** (See [5]) Given \( h \in L^1(\Omega) \cap R(L) \) satisfying \( h(t + 2\pi, x) = h(t, x) \), then the solution of the wave equation

\[
\begin{align*}
Lw &\equiv w_{tt} - w_{xx} = h(t, x), \quad (t, x) \in \Omega, \\
w(t, 0) = w(t, \pi) = 0, \quad w(t + 2\pi, x) = w(t, x),
\end{align*}
\]

(W)

can be presented in the form of \( w = w_0 + w_1 \), where \( w_0 \in \ker L \), and \( w_1 \in (\ker L)^\perp = R(L) \).

More precisely, we have

\[
w_0(t, x) = p(t + x) - p(t - x),
\]

for some \( p \in L^1 \) which is \( 2\pi \)-periodic and satisfying \( \int_0^{2\pi} p(\tau)d\tau = 0 \), and

\[
w_1(t, x) = -\frac{1}{2} \int_x^\pi d\xi \int_{t+x-\xi}^{t-x+\xi} h(\tau, \xi)d\tau + \frac{\pi - x}{2\pi} \int_0^{\pi} d\xi \int_{t-\xi}^{t+\xi} h(\tau, \xi)d\tau.
\]

Estimates of the component \( w_1 \) in \( R(L) = (\ker L)^\perp \):

From Proposition 5.1, the \( L^\infty \)-norm of \( w_1 \in R(L) \) can be controlled by

\[
\|w_1\|_{L^\infty} \leq c\|h\|_{L^1}, \quad \text{for} \ h \in L^1 \cap R(L).
\]

Furthermore, if \( h \in L^\infty(\Omega) \cap R(L) \), then we have

\[
\|w_1\|_{C^{0,1}} \leq c\|h\|_{L^\infty},
\]

where \( \| \cdot \|_{C^{0,1}} \) is the norm of the Lipschitz space \( C^{0,1}(\Omega) \).

An Integral Formula concerning with the component in \( \ker L \):

If \( p(s), q(s) \) are \( L^1 \) functions with period \( 2\pi \) and satisfy \( \int_0^{2\pi} p(s)ds = \int_0^{2\pi} q(s)ds = 0 \), then a computation as in [28] shows that

\[
\int_\Omega p(t + x)q(t - x)dtdx = 0.
\]

We also require the next property to characterize the range of the operator \( L \):
A function \( h \) belongs to \( R(L) \) if and only if
\[
\int_{0}^{\pi} [h(t + x, x) - h(t - x, x)] dx = 0. \tag{5.4}
\]
In other words, the sufficient and necessary condition for the solvability of linear wave equation (W) is that the function \( h(t, x) \) satisfies (5.4). We see [5, 25] for more details.

### 5.2 Proof of Theorem 1.4

Let \((u,v) \in E\) be a solution of (1.1) constructed by local linking method. We decompose it into \((u,v) = (u_1 + y, v_1 + z)\), where \(u_1, v_1 \in H_b^+ \oplus H_b^- \equiv R(L)\), and \(y, z \in H^0 \equiv \ker L\). We apply Proposition 5.1 to represent \(y, z\) by
\[
y = p(t + x) - p(t - x) \quad \text{and} \quad z = q(t + x) - q(t - x),
\]
for some \(2\pi\)-periodic functions \(p, q \in L^1([0,2\pi])\) such that
\[
\int_{0}^{2\pi} p(\tau) d\tau = \int_{0}^{2\pi} q(\tau) d\tau = 0.
\]

**The \( L^\infty \)-regularity of the components in \( H_b^+ \oplus H_b^- \):**

Since \((u,v)\) satisfy the system
\[
\begin{align*}
  u_{tt} - u_{xx} + bu + \varepsilon v + f(t, x, u) &= 0, \\
  v_{tt} - v_{xx} + bv + \varepsilon u + g(t, x, v) &= 0,
\end{align*}
\]
we shall derive the \( L^\infty \)-estimate of \(u_1, v_1 \in H_b^+ \oplus H_b^-\) from Proposition 5.1. We have:

**Lemma 5.2.** Assume that \((u,v)\) is a solution of (1.1) obtained in Theorem 1.1, then there exists \(d_1, d_2 > 0\), such that \(\|u_1\|_{L^\infty} \leq d_1\) and \(\|v_1\|_{L^\infty} \leq d_2\).

**Proof.** By (2.8), \(p > 1\) and (h1) of Theorem 1.1, we have
\[
\int |f(t, x, u)| \leq c_0 \int (1 + |u|^p) \leq c_1 (1 + \|u\|_E^p) < \infty,
\]
which means \(f \in L^1(\Omega \times \mathbb{R})\). Noting that (2.8) also implies \(u, v \in L^1(\Omega)\), then we can use the estimate (5.1) to obtain a number \(d_1 > 0\), such that
\[
\|u_1\|_{L^\infty} \leq c \|bu + \varepsilon v + f(t, x, u)\|_{L^1} \leq d_1 < \infty. \tag{5.5}
\]
Similarly, we have
\[
\|v_1\|_{L^\infty} \leq c \|bv + \varepsilon u + g(t, x, v)\|_{L^1} \leq d_2 < \infty, \tag{5.6}
\]
for some \(d_2 > 0\). \(\square\)
The $L^\infty$-regularity of the components in $H^0 \equiv \ker L$.

For the terms of $y, z$ in $H^0$, the proof of $y, z \in L^\infty(\Omega)$ is a difficult task since the a-prior estimate (5.1) is invalid for $y$ and $z$. To this end, we set

$$N_1 = \|p\|_{L^\infty(\Omega)}, \quad N_2 = \|q\|_{L^\infty(\Omega)}.$$  \hspace{1cm} (5.7)

Without loss of generality, we may suppose that there are $s_1$ and $s_2$, such that $p(s_1) > N_1 - 1$ and $q(s_2) > N_2 - 1$. We shall derive the upper bounds of $N_1$ and $N_2$.

Let $\tilde{f}(t, x) = f(t, x, u(t, x))$ and

$$h_1(t, x) = bu(t, x) + \varepsilon v(t, x) + \tilde{f}(t, x),$$  \hspace{1cm} (5.8)

where $u(t, x) = u_1(t, x) + p(t + x) - p(t - x)$, $v(t, x) = v_1(t, x) + q(t + x) - q(t - x)$.

We prepare the following two lemmas to prove Theorem 1.4.

**Lemma 5.3.** Assume that $(u(t, x), v(t, x))$ is a solution of (1.1) in $\ker L$ obtained in Theorem 1.1, then there exists a number $d_3 > 0$, such that

$$\int_0^\pi \tilde{f}(s_1 - x, x)dx \leq d_3 - b\pi p(s_1) - 2\varepsilon\pi q(s_1).$$  \hspace{1cm} (5.9)

**Proof.** Noting that $Lu + h_1(t, x) = 0$, then putting $t = s_1$ in (5.4), we have

$$\int_0^\pi h_1(s_1 + x, x)dx = \int_0^\pi h_1(s_1 - x, x)dx.$$  \hspace{1cm} (5.10)

Since $p(s_1 + 2x) - p(s_1) \leq 1$ and $u_1(s_1 + x, x) \leq d_1$, it follows from (h4), (5.5) and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ that

$$\tilde{f}(s_1 + x, x) \leq f(s_1 + x, x, d_1 + 1) \leq d_4,$$

for some $d_4 > 0$. Therefore, by virtue of $\int_0^\pi q(s_1 + 2x)dx = 0$, we deduce from (5.5), (5.6) and the above inequality that

$$\int_0^\pi h_1(s_1 + x, x)dx \leq \int_0^\pi \left[ bd_1 + |\varepsilon|d_2 + \varepsilon q(s_1 + 2x) - \varepsilon q(s_1) + d_4 \right]dx$$

$$\leq d_5 - \varepsilon\pi q(s_1).$$  \hspace{1cm} (5.11)

On the other hand, it follows from (5.5), (5.6) that

$$u_1(s_1 - x, x) \geq -d_1 \quad \text{and} \quad v_1(s_1 - x, x) \geq -d_2.$$

Furthermore, by the fact of $\int_0^\pi p(s_1 - 2x)dx = \int_0^\pi q(s_1 - 2x)dx = 0$, we have

$$\int_0^\pi h_1(s_1 - x, x)dx \geq b\pi p(s_1) + \varepsilon\pi q(s_1) + \int_0^\pi \tilde{f}(s_1 - x, x)dx - b\pi d_1.$$  \hspace{1cm} (5.12)

Hence, combining with (5.10)-(5.12), we arrive at (5.9). \hfill \square
To proceed further, we denote by
\[ \Lambda_1 = \left\{ x \in [0, \pi] : p(s_1 - 2x) \leq \frac{N_1}{2} \right\} \quad \text{and} \quad \Lambda_2 = \left\{ x \in [0, \pi] : q(s_2 - 2x) \leq \frac{N_2}{2} \right\}. \]

We have

**Lemma 5.4.** Assume that \((u(t, x), v(t, x))\) is a solution of (1.1)_{a,b,c} obtained in Theorem 1.1, then \(\text{meas}\Lambda_1 \geq \pi/3\), \(\text{meas}\Lambda_2 \geq \pi/3\) and there exists a number \(d > 0\), such that
\[
\int_{\Lambda_1} f \left( s_1 - x, x, \frac{N_1}{2} - d_1 - 1 \right) dx + \int_{\Lambda_2} g \left( s_2 - x, x, \frac{N_2}{2} - d_2 - 1 \right) dx \leq d. \quad (5.13)
\]

**Proof.** (1) At first, we prove that \(\text{meas}\Lambda_1 \geq \pi/3\). In view of
\[
p(s_1 - 2x) \geq \begin{cases} 
-N_1, & \text{for } x \in \Lambda_1, \\
N_1/2, & \text{for } x \in \Lambda_1^c,
\end{cases}
\]
we get
\[
0 = \int_0^\pi p(s_1 - 2x) dx = \int_{\Lambda_1} p(s_1 - 2x) dx + \int_{\Lambda_1^c} p(s_1 - 2x) dx \geq -N_1 \text{meas}\Lambda_1 + \frac{N_1}{2} (\pi - \text{meas}\Lambda_1),
\]
which indicates \(\text{meas}\Lambda_1 \geq \pi/3\).

Similarly, we also have \(\text{meas}\Lambda_2 \geq \pi/3\).

(2) Now we turn to the proof of (5.13). To this end, we represent the left hand side of (5.9) in the form of
\[
\left( \int_{\Lambda_1} + \int_{\Lambda_1^c} \right) \tilde{f} \left( s_1 - x, x \right) dx := \tilde{Q}_1 + \tilde{Q}_2.
\]

It follows from (5.7) that \(p(s_1) - p(s_1 - 2x) \geq N_1/2 - 1\), for \(x \in \Lambda_1\). Then by (h4) and (5.5), we know
\[
\tilde{Q}_1 \geq \int_{\Lambda_1} f \left( s_1 - x, x, \frac{N_1}{2} - d_1 - 1 \right) dx.
\]

On the other hand, recording \(u_1(s_1 - x, x) + p(s_1) - p(s_1 - 2x) \geq -d_1 - 1\), then we deduce from (h1) and (h4) that
\[
\tilde{f} \left( s_1 - x, x \right) \geq f \left( s_1 - x, x, -d_1 - 1 \right) \geq -c_0 - c_0 d_1^p,
\]
which gives
\[
\tilde{Q}_2 \geq - \int_{\Lambda_1^c} (c_0 + c_0 d_1^p) dx \geq -d_6.
\]
Thus, coming back to (5.9), we get
\[
\int_{\Lambda_1} f \left( s_1 - x, x, \frac{N_1}{2} - d_1 - 1 \right) \, dx \leq \int_{0}^{\pi} \tilde{f} (s_1 - x, x) \, dx - \tilde{Q}_2 \\
\leq d_7 - b \pi p(s_1) - 2 \varepsilon \pi q(s_1). \tag{5.14}
\]

Applying a similar procedure, we also have
\[
\int_{\Lambda_2} g \left( s_2 - x, x, \frac{N_2}{2} - d_2 - 1 \right) \, dx \leq d_7' - b \pi q(s_2) - 2 \varepsilon \pi p(s_2). \tag{5.15}
\]

Choosing $|\varepsilon| < b/2$, we infer from (5.7) that $|p(s_2)| \leq p(s_1) + 1$, $|q(s_1)| \leq q(s_2) + 1$, which assure $b \pi p(s_1) + 2 \varepsilon \pi p(s_2) \geq -2 \varepsilon \pi$ and $b \pi q(s_2) + 2 \varepsilon \pi q(s_1) \geq -2 \varepsilon \pi$. Then, adding up (5.14) and (5.15), we obtain (5.13).

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** Without loss of generality, we may assume that $N_1 > 2d_1 + 2$ and $N_2 > 2d_2 + 2$. Thus, we follow from (h2) and (h4) to derive
\[
\int_{\Lambda_1} f \left( s_1 - x, x, \frac{N_1}{2} - d_1 - 1 \right) \, dx \geq 0, \quad \int_{\Lambda_2} g \left( s_2 - x, x, \frac{N_2}{2} - d_2 - 1 \right) \, dx \geq 0.
\]

Now, we claim that $N_1 < +\infty$. If not, from the fact of $f(s_1 - x, x, \xi) \to +\infty$ as $\xi \to +\infty$, for $x \in \Lambda_1$, we deduce that for any $\gamma > 0$, there are $\beta > 0$ and a subset $D$ of $\Lambda_1$ with $\text{meas} D < \pi/6$, such that
\[
f(s_1 - x, x, \beta) \geq \gamma, \quad \text{for } x \in D^c.
\]

Hence, for $N_1 > 2(\beta + d_1)$, we infer from the above inequality and (h4) that
\[
\int_{\Lambda_1 \cap D^c} f \left( s_1 - x, x, \frac{N_1}{2} - d_1 - 1 \right) \, dx \geq \gamma \text{meas}(\Lambda_1 \cap D^c) \geq \frac{1}{6} \gamma \pi. \tag{5.16}
\]

On the other hand, by (h2), (h4) and $N_1 > 2d_1 + 2$, we have
\[
\int_{\Lambda_1 \cap D} f \left( s_1 - x, x, \frac{N_1}{2} - d_1 - 1 \right) \, dx \geq 0. \tag{5.17}
\]

Then, with the aid of (5.13), (5.16) and (5.17), we show that
\[
\frac{1}{6} \gamma \pi \leq \int_{\Lambda_1} f \left( s_1 - x, x, \frac{N_1}{2} - d_1 - 1 \right) \, dx \leq d,
\]
which is impossible when we fix $\gamma > 6d/\pi$.

Therefore, we conclude that $y$ is essentially bounded. With a similar argument, we have $z \in L^\infty$. The proof of Theorem 1.4 is thereby completed. \qed
6 Proof of Theorem 1.5

This section is devoted to the proof of the continuity of the solutions for (1.1)\textsubscript{a,b,c}.

Let \((u, v) \in L^\infty(\Omega) \times L^\infty(\Omega)\) be a solution of (1.1)\textsubscript{a,b,c} provided by Theorem 1.4. Splitting it into \((u, v) = (u_1 + y, v_1 + z)\), with \(u_1, v_1 \in H^+_b \oplus H^-_b \equiv (\ker L)^\perp\), and \(y, z \in H^0 \equiv \ker L\).

6.1 Continuity of the regular terms

It follows from (h1) and the fact of \(u, v \in L^\infty\) that \(bu + \varepsilon v + f(t, x, u) \in L^\infty\) and \(bv + \varepsilon u + g(t, x, v) \in L^\infty\). Hence, noting that \((u, v) = (u_1 + y, v_1 + z)\) solves the system of (1.1)\textsubscript{a,b,c}, we infer from (5.2) that \(u_1, v_1 \in C^{0,1}(\Omega)\), which means

\[
|u_1(t, x) - u_1(\tau, \zeta)| + |v_1(t, x) - v_1(\tau, \zeta)| \leq C (|t - \tau| + |x - \zeta|),
\]

for all \((t, x), (\tau, \zeta) \in \Omega\). Consequently, we achieve that \(u_1(t, x)\) and \(v_1(t, x)\) are continuous on \(\Omega\).

6.2 Continuity of the null terms

We turn to prove that \(y, z \in C(\Omega)\). Regarding that \(y, z \in \ker L\) can be represented in the form of \(y = p(t + x) - p(t - x)\) and \(z = q(t + x) - q(t - x)\), where \(p, q \in L^1([0, 2\pi])\) are \(2\pi\)-periodic and satisfy \(\int_0^{2\pi} p(\tau) d\tau = \int_0^{2\pi} q(\tau) d\tau = 0\), then it suffices to show that \(p\) and \(q\) are continuous, that means

\[
p(t + h) - p(t) \to 0, \quad q(t + h) - q(t) \to 0, \quad \text{as} \quad h \to 0, \quad \text{for a.e.} \quad t \in [0, 2\pi].
\]

To reach our goal, we denote by \(\hat{p}_h(t) = p(t + h) - p(t)\), \(\hat{q}_h(t) = q(t + h) - q(t)\), for fixed \(|h| < 1/4\), and let

\[
M_1 = \|\hat{p}_h\|_{L^\infty(\Omega)}, \quad M_2 = \|\hat{q}_h\|_{L^\infty(\Omega)}.
\]

Then \(M_1 \leq 2N_1 < \infty\) and \(M_2 \leq 2N_2 < \infty\), where \(N_1 = \|p\|_{L^\infty(\Omega)}\), and \(N_2 = \|q\|_{L^\infty(\Omega)}\). Moreover, without loss of generality, we may assume that there exist \(s_1\) and \(s_2\), such that

\[
\hat{p}_h(s_1) > M_1(1 - |h|), \quad \hat{q}_h(s_2) > M_2(1 - |h|).
\]

We intend to prove \(M_1 \to 0, M_2 \to 0,\) as \(|h| \to 0\).

For simplicity, we denote by \(\hat{p}(t) = \hat{p}_h(t)\) and \(\hat{q}(t) = \hat{q}_h(t)\). The following notations and estimates are in order. Define

\[
\phi(\tau) = \min_{(t, x) \in \Omega, \ |s| \leq 2N_1} \left[ f(t, x, u(t, x) + s + \tau) - f(t, x, u(t, x) + s) \right],
\]

31
\[
\psi(\tau) = \min_{(t,x) \in \Omega, |s| \leq 2N_2} \left[ g(t, x, v_1(t, x) + s + \tau) - g(t, x, v_1(t, x) + s) \right];
\]

We shall prove that
\[
\int_0^\pi \left[ \phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x)) + \psi(\hat{q}(s_2) - \hat{q}(s_2 - 2x)) \right] dx \leq C|h|,
\]
and
\[
\int_{\Sigma_1} \phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x)) dx + \int_{\Sigma_2} \psi(\hat{q}(s_2) - \hat{q}(s_2 - 2x)) dx \leq C|h|,
\]
where \( C > 0 \) is a number independent of \( h \), and
\[
\Sigma_1 = \left\{ x \in [0, \pi] : \hat{p}(s_1) - \hat{p}(s_1 - 2x) \geq \frac{M_1}{2} \right\}, \quad \Sigma_2 = \left\{ x \in [0, \pi] : \hat{q}(s_2) - \hat{q}(s_2 - 2x) \geq \frac{M_2}{2} \right\}.
\]

6.2.1 Proof of (6.3)

We carry out the argument by three steps.

**Step 1:** We establish the following lemma to study the behavior of the integral
\[
J = \int_0^\pi \left[ b\hat{p}(s_1) - b\hat{p}(s_1 - 2x) + \varepsilon\hat{q}(s_1) - \varepsilon\hat{q}(s_1 - 2x) - \hat{f}(s_1 - x, x) + f(s_1 - x, x, u_1(s_1 - x, x) + p(s_1 + h) - p(s_1 + h - 2x)) \right] dx.
\]

**Lemma 6.1.** Assume that \((u(t, x), v(t, x))\) is a solution of (1.1)_{a,b,c} obtained in Theorem 1.1, then there exists a number \( C > 0 \) independent of \( h \), such that
\[
J \leq C|h| - \varepsilon\pi\hat{q}(s_1).
\]

**Proof.** We denote by
\[
J_1 = \int_0^\pi \left[ b\hat{u}_1(s_1 - x, x) + b\hat{p}(s_1 + h) - b\hat{p}(s_1 + h - 2x) + \varepsilon\hat{v}_1(s_1 - x, x) + \varepsilon\hat{q}(s_1 + h) - \varepsilon\hat{q}(s_1 + h - 2x) + f(s_1 - x, x, u_1(s_1 - x, x) + p(s_1 + h) - p(s_1 + h - 2x)) \right] dx,
\]
and
\[
J_2 = \int_0^\pi \left[ b\hat{u}_1(s_1 + x, x) + b\hat{p}(s_1 + h + 2x) - b\hat{p}(s_1 + h) + \varepsilon\hat{v}_1(s_1 + x, x) + \varepsilon\hat{q}(s_1 + h + 2x) - \varepsilon\hat{q}(s_1 + h) + f(s_1 + x, x, u_1(s_1 + x, x) + p(s_1 + h + 2x) - p(s_1 + h)) \right] dx.
\]

Since \((u, v)\) is a solution of \( Lu + h_1(t, x) = 0 \), and recalling the notation of the function \( h_1(t, x) \) occurs in (5.8), then we note that
\[
\int_0^\pi h_1(s_1 + h - x, x) dx = \int_0^\pi h_1(s_1 + h + x, x) dx \quad \text{and}
\]
\[ \int_0^\pi h_1(s_1 - x, x)dx = \int_0^\pi h_1(s_1 + x, x)dx, \]

by taking into account (5.4) with \( t = s_1 + h \) and \( t = s_1 \) respectively. Then we have

\[ J = \left( J_1 - \int_0^\pi h_1(s_1 + h - x, x)dx \right) + \left( \int_0^\pi h_1(s_1 + h + x, x)dx - J_2 \right) \]
\[ + \left( J_2 - \int_0^\pi h_1(s_1 + x, x)dx \right) \]
\[ := \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3. \] (6.6)

**Behavior of \( \tilde{J}_1 \) and \( \tilde{J}_2 \):**

We begin with the study of the asymptotic behavior of the first term in the right hand side of (6.6). It is plain to check that

\[ \tilde{J}_1 = \int_0^\pi \left[ bu_1(s_1 - x, x) - bu_1(s_1 + h - x, x) + \varepsilon v_1(s_1 - x, x) - \varepsilon v_1(s_1 + h - x, x) \right. \]
\[ - f(s_1 + h - x, x, u_1(s_1 + h - x, x) + p(s_1 + h) - p(s_1 + h - 2x)) \]
\[ + f(s_1 - x, x, u_1(s_1 - x, x) + p(s_1 + h) - p(s_1 + h - 2x)) \] \] dx.

By virtue of (6.1) and \( f \in C^1(\Omega \times \mathbb{R}) \), we get

\[ |\tilde{J}_1| \leq \int_0^\pi \left[ C|h| + C|u_1(s_1 - x, x) - u_1(s_1 + h - x, x)| \right]dx \leq C|h|, \] (6.7)

where \( C > 0 \) is independent of \( h \). Moreover, a similar calculation shows that

\[ |\tilde{J}_2| \leq \int_0^\pi \left[ bu_1(s_1 + h + x, x) - bu_1(s_1 + x, x) + \varepsilon v_1(s_1 + h + x, x) - \varepsilon v_1(s_1 + x, x) \right. \]
\[ + f(s_1 + h + x, x, u_1(s_1 + h + x, x) + p(s_1 + h + 2x) - p(s_1 + h)) \]
\[ - f(s_1 + x, x, u_1(s_1 + x, x) + p(s_1 + h + 2x) - p(s_1 + h)) \] \] dx
\[ \leq C|h|. \] (6.8)

**Behavior of \( \tilde{J}_3 \):**

Using the notations of \( \hat{p}(t) = p(t + h) - p(t) \) and \( \hat{q}(t) = q(t + h) - q(t) \), we can represent

\[ \tilde{J}_3 = \int_0^\pi \left[ b\hat{p}(s_1 + 2x) - b\hat{p}(s_1) + \varepsilon\hat{q}(s_1 + 2x) - \varepsilon\hat{q}(s_1) \right. \]
\[ + f(s_1 + x, x, u_1(s_1 + x, x) + p(s_1 + h + 2x) - p(s_1 + h)) \]
\[ - f(s_1 + x, x, u_1(s_1 + x, x) + p(s_1 + 2x) - p(s_1)) \] \] dx.

By (6.2) and the definition of \( M_1 \), we obtain

\[ \hat{p}(s_1 + 2x) \leq M_1 < \hat{p}(s_1) + M_1|h|, \]

33
which indicates that

\[ p(s_1 + h + 2x) - p(s_1 + h) < p(s_1 + 2x) - p(s_1) + M_1|h|. \]  

(6.9)

In addition, from the facts of \( q \) is \( 2\pi \)-periodic and \( \int_0^{2\pi} q(\tau)d\tau = 0 \), we deduce that

\[ \int_0^{\pi} \hat{q}(s_1 + 2x)dx = \int_0^{\pi} [q(s_1 + h + 2x) - q(s_1 + 2x)]dx = 0. \]

Thus, the above estimates give

\[ \int_0^{\pi} [b\hat{p}(s_1 + 2x) - b\hat{p}(s_1) + \varepsilon\hat{q}(s_1 + 2x) - \varepsilon\hat{q}(s_1)]dx \leq \pi b M_1|h| - \varepsilon \pi \hat{q}(s_1). \]  

(6.10)

Moreover, we infer from (6.9), (h4) and \( f \in C^1(\Omega \times \mathbb{R}) \) that

\[ \int_0^{\pi} \left[ f(s_1 + x, x, u_1(s_1 + x, x) + p(s_1 + h + 2x) - p(s_1 + h)) \right. \\
\left. - f(s_1 + x, x, u_1(s_1 + x, x) + p(s_1 + 2x) - p(s_1)) \right]dx \leq CM_1 \pi |h|. \]  

(6.11)

Therefore, making use of (6.10) and (6.11), we have

\[ \tilde{J}_3 \leq \pi b M_1|h| - \varepsilon \pi \hat{q}(s_1) + CM_1 \pi |h|. \]  

(6.12)

Now, inserting the estimates (6.7), (6.8) and (6.12) into (6.6), we obtain (6.5). \( \square \)

**Step 2:** We require the next lemma concerning with the properties of the integrals

\[ R_1 = \int_0^{\pi} \left[ f(s_1 - x, x, u_1(s_1 - x, x) + p(s_1 + h) - p(s_1 + h - 2x)) \right. \\
\left. - f(s_1 - x, x, u_1(s_1 - x, x) + p(s_1) - p(s_1 - 2x)) \right]dx, \]

\[ R_2 = \int_0^{\pi} \left[ g(s_2 - x, x, v_1(s_2 - x, x) + q(s_2 + h) - q(s_2 + h - 2x)) \right. \\
\left. - g(s_2 - x, x, v_1(s_2 - x, x) + q(s_2) - q(s_2 - 2x)) \right]dx. \]

**Lemma 6.2.** Under the assumptions of Theorem 1.5, we have \( R_1 + R_2 \leq C|h| \) for \( \varepsilon \) is sufficiently small, where \( C > 0 \) is independent of \( h \).

**Proof.** Noting that \( \int_0^{\pi} \hat{p}(s_1 - 2x)dx = \int_0^{\pi} \hat{q}(s_1 - 2x)dx = 0 \), then by a direct computation we observe that

\[ R_1 = J - \int_0^{\pi} \left[ b\hat{p}(s_1) - b\hat{p}(s_1 - 2x) + \varepsilon\hat{q}(s_1) - \varepsilon\hat{q}(s_1 - 2x) \right]dx \]
\[ J - b\pi \hat{p}(s_1) - \varepsilon\pi \hat{q}(s_1). \]

Hence, with the help of Lemma 6.1 we have
\[ R_1 \leq -b\pi \hat{p}(s_1) - 2\varepsilon\pi \hat{q}(s_1) + C|h|. \]  
\[ (6.13) \]

Similar to the derivation of the inequalities (6.5) and (6.13), we are able to get
\[ R_2 \leq -b\pi \hat{q}(s_2) - 2\varepsilon\pi \hat{p}(s_2) + C|h|. \]  
\[ (6.14) \]

Selecting \(|\varepsilon| < b/2\), we deduce from (6.2) and the definitions of \(M_1, M_2\) that
\[ b\hat{p}(s_1) + 2\varepsilon \hat{p}(s_2) \geq bM_1(1 - |h|) - 2|\varepsilon|M_1 > -bM_1|h|, \]
\[ b\hat{q}(s_2) + 2\varepsilon \hat{q}(s_1) \geq bM_2(1 - |h|) - 2|\varepsilon|M_2 > -bM_2|h|. \]

By virtue of the above two inequalities, then adding up (6.13) and (6.14) will yield that \(R_1 + R_2 \leq C|h|\) for \(|\varepsilon| < b/2\). That is what we desire. \[ \Box \]

**Step 3:** Now, we are in a position to prove (6.3).

Under the assumptions of Theorem 1.5, we know \(\phi(\tau), \psi(\tau)\) are strictly increasing in \(\tau\) and \(\phi(0) = 0, \psi(0) = 0\). Moreover, it follows from \(f, g \in C^1\) that \(\phi(\tau), \psi(\tau)\) are Lipschitz continuous on any bounded intervals.

By the definitions of \(R_1, R_2, \phi\) and \(\psi\), we conclude that
\[ \phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x)) \leq f(s_1 - x, x, u_1(s_1 - x, x) + p(s_1 + h) - p(s_1 - 2x + h)) \]
\[ - f(s_1 - x, x, u_1(s_1 - x, x) + p(s_1) - p(s_1 - 2x)). \]

Then integrating the above in \(x\) leads to
\[ \int_0^\pi \phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x))dx \leq R_1. \]

And a similar computation shows that \(\int_0^\pi \psi(\hat{q}(s_2) - \hat{q}(s_2 - 2x))dx \leq R_2. \)

Therefore, we derive (6.3) from Lemma 6.2. \[ \Box \]

**6.2.2 Proof of (6.4)**

We denote by \(\Sigma_1^c, \Sigma_2^c\) the complement spaces of \(\Sigma_1, \Sigma_2\) respectively.

For any \(x \in [0, \pi]\), it follows from (6.2) and the choice of \(M_1\) that
\[ \hat{p}(s_1) - \hat{p}(s_1 - 2x) \geq M_1(1 - |h|) - M_1 = -M_1|h|. \]
Combining the above inequality with the facts of \( \phi(0) = 0 \), \( \phi \) is strictly increasing and Lipschitz continuous, we conclude that

\[
\phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x)) \geq \phi(-M_1|h|) - \phi(0) \geq -C|h|, \quad \text{for } x \in \Sigma_c^c.
\]

Integrating the above in \( x \) on \( \Sigma_c^c \), we have

\[
-\int_{\Sigma_c^c} \phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x))dx \leq C|h|. \tag{6.15}
\]

On the other hand, concerning with the behavior of \( \psi(\hat{q}(s_2) - \hat{q}(s_2 - 2x)) \) restricted on \( \Sigma_c^c \), we also get

\[
-\int_{\Sigma_c^c} \psi(\hat{q}(s_2) - \hat{q}(s_2 - 2x))dx \leq C|h|. \tag{6.16}
\]

Hence, the inequalities of (6.3), (6.15) and (6.16) ensure that

\[
\int_{\Sigma_1} \phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x))dx + \int_{\Sigma_2} \psi(\hat{q}(s_2) - \hat{q}(s_2 - 2x))dx
\]

\[
= \left( \int_0^\pi - \int_{\Sigma_1^c} \right) \phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x))dx + \left( \int_0^\pi - \int_{\Sigma_2^c} \right) \psi(\hat{q}(s_2) - \hat{q}(s_2 - 2x))dx
\]

\[
\leq C|h|,
\]

where \( C > 0 \) is independent of \( h \). Thus, we arrive at (6.4). \( \square \)

### 6.2.3 Complete the proof of Theorem 1.5

Finally, we examine the continuity of \( p \) and \( q \) by controlling the values of \( \phi(M_1/2) \) and \( \psi(M_2/2) \).

Recalling the definitions of \( \Sigma_1, \Sigma_2 \) and the facts that \( \phi(\tau), \psi(\tau) \) are strictly increasing in \( \tau \), we deduce that \( \phi(M_1/2) \leq \phi(\hat{p}(s_1) - \hat{p}(s_1 - 2x)) \) for \( x \in \Sigma_1 \), and \( \psi(M_2/2) \leq \psi(\hat{q}(s_2) - \hat{q}(s_2 - 2x)) \) for \( x \in \Sigma_2 \). Thus, we infer from (6.4) that

\[
\int_{\Sigma_1} \phi(M_1/2)dx + \int_{\Sigma_2} \psi(M_2/2)dx \leq C|h|. \tag{6.17}
\]

We next claim that \( \text{meas}\Sigma_1 > 0 \) and \( \text{meas}\Sigma_2 > 0 \).

Indeed, we follow from \( \int_0^{2\pi} \hat{p}(\tau)d\tau = 0 \) that

\[
0 = \int_0^\pi \hat{p}(s_1 - 2x)dx = \int_{\Sigma_1} \hat{p}(s_1 - 2x)dx + \int_{\Sigma_1^c} \hat{p}(s_1 - 2x)dx. \tag{6.18}
\]
On the other hand, according to the definitions of $M_1$, $s_1$ and $\Sigma_1^c$, we deduce

$$\hat{p}(s_1 - 2x) \geq \begin{cases} -M_1, & \text{for } x \in \Sigma_1, \\ \hat{p}(s_1) - \frac{M_1}{2} > \frac{M_1}{2} - M_1|h|, & \text{for } x \in \Sigma_1^c. \end{cases} \quad (6.19)$$

Then, comparing with (6.18) and (6.19), we arrive at

$$0 \geq -M_1 \text{meas}\Sigma_1 + \left(\frac{1}{2} - |h|\right)M_1(\pi - \text{meas}\Sigma_1) = -(\frac{3}{2} - |h|)M_1 \text{meas}\Sigma_1 + \left(\frac{1}{2} - |h|\right)M_1\pi,$$

which implies that

$$\text{meas}\Sigma_1 > \frac{(1 - 2|h|)\pi}{3 - 2|h|} > \frac{\pi}{5}, \text{ as } |h| < \frac{1}{4}.$$

A similar argument allows us to obtain $\text{meas}\Sigma_2 > \pi/5$, for $|h| < 1/4$.

Therefore, we make use of (6.17) and the preceding estimates for $\text{meas}\Sigma_1$, $\text{meas}\Sigma_2$ to get

$$\phi(M_1/2) + \psi(M_2/2) \leq C|h|.$$

Furthermore, noting that $\phi(M_1/2) > 0$, $\psi(M_2/2) > 0$ and $\phi(\tau)$, $\psi(\tau)$ are strictly increasing in $\tau$, we conclude that $M_1 \to 0$ and $M_2 \to 0$. Hence, it follows that $y = p(t + x) - p(t - x) \in C(\Omega)$ and $z = q(t + x) - q(t - x) \in C(\Omega)$. We finish the proof of Theorem 1.5. \hfill \Box

References

[1] I.S. Akhatov, V.A. Baikov, K.R. Khusnutdinova, Non-linear dynamics of coupled chains of particles, J. Appl. Maths. Mechs. 59 (3) (1995) 353-361.

[2] J. Berkovits, V. Mustonen, On nonresonance for systems of semilinear wave equations, Nonlinear Anal. 29 (6) (1997) 627-638.

[3] M. Berti, P. Bolle, Sobolev periodic solutions of nonlinear wave equations in higher spatial dimensions, Arch. Ration. Mech. Anal. 195(2) (2010) 609-642.

[4] J. Bourgain, Construction of periodic solutions of nonlinear wave equations in higher dimension, Geom. Funct. Anal. 5 (4) (1995) 629-639.

[5] H. Brézis, J. M. Coron, L. Nirenberg, Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz, Comm. Pure Appl. Math. 33 (5) (1980) 667-689.

[6] H. Brézis, L. Nirenberg, Forced vibrations for a nonlinear wave equation, Comm. Pure Appl. Math. 31 (1) (1978) 1-30.
[7] K.C. Chang, Solutions of asymptotically linear operator equations via Morse theory, Comm. Pure Appl. Math. 34 (5) (1981) 693-712.

[8] K.C. Chang, S.P. Wu, S.J. Li, Multiple periodic solutions for an asymptotically linear wave equation, Indiana Univ. Math. J. 31 (5) (1982) 721-731.

[9] J.Y. Chen, Z.T. Zhang, Infinitely many periodic solutions for a semilinear wave equation in a ball in $\mathbb{R}^n$, J. Differ. Equ. 256 (4) (2014) 1718-1734.

[10] J.Y. Chen, Z.T. Zhang, Existence of infinitely many periodic solutions for the radially symmetric wave equation with resonance, J. Differ. Equ. 260 (7) (2016) 6017-6037.

[11] J.Y. Chen, Z.T. Zhang, Existence of multiple periodic solutions to asymptotically linear wave equations in a ball, Calc. Var. Partial Differ. Equ. 56 (3) (2017) Art.58.

[12] Z.J. Chen, W.M. Zou, Standing waves for linearly coupled Schrödinger equations with critical exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (3) (2014) 429-447.

[13] Z.J. Chen, W.M. Zou, On linearly coupled Schrödinger systems, Proc. Amer. Math. Soc. 142 (1) (2014) 323-333.

[14] Y.H. Ding, S.J. Li, Periodic solutions of a superlinear wave equation, Nonlinear Anal. 29 (3) (1997) 265-282.

[15] D.Y. Fang, R.Y. Xue, Global existence of small solutions for cubic quasi-linear Klein-Gordon systems in one space dimension, Acta Math. Sin. 22 (4) (2006) 1085-1102.

[16] S.G. Ji, Periodic solutions for one dimensional wave equation with bounded nonlinearity, J. Differ. Equ. 264 (9) (2018) 5527-5540.

[17] S.G. Ji, Y. Li, Time periodic solutions to the one-dimensional nonlinear wave equation, Arch. Ration. Mech. Anal. 199 (2) (2011) 435-451.

[18] K.R. Khusnutdinova, D.E. Pelinovsky, On the exchange of energy in coupled Klein-Gordon equations, Wave Motion 38 (1) (2003) 1-10.

[19] D. Kim, Global existence of small amplitude solutions to one-dimensional nonlinear Klein-Gordon systems with different masses, J. Hyperbolic Differ. Equ. 12 (4) (2015) 745-762.

[20] S. Klainerman, D. Tataru, On the optimal regularity for Yang-Mills equations in $\mathbb{R}^{4+1}$, J. Amer. Math. Soc. 12 (1) (1999) 93-116.
[21] S.J. Li, A. Szulkin, Periodic solutions for a class of nonautonomous wave equations, Differ. Integral Equ. 9 (6) (1996) 1179-1212.

[22] S.J. Li, M. Willem, Applications of local linking to critical point theory, J. Math. Anal. Appl. 189 (1) (1995) 6-32.

[23] S.J. Li, J.Q. Liu, Morse theory and asymptotic linear Hamiltonian system, J. Differ. Equ. 78 (1) (1989) 53-73.

[24] J.Q. Liu, S.J. Li, An existence theorem for multiple critical points and its application (Chinese), Kexue Tongbao (Chinese) 29 (17) (1984) 1025-1027.

[25] H. Lovicarova, Periodic solutions of a weakly nonlinear wave equation in one dimension, Czech. Math. J. 19 (2) (1969) 324-342.

[26] J. Mawhin, Periodic solutions of some semilinear wave equations and systems: a survey, Chaos, Solitons and Fractals 5 (9) (1995) 1651-1669.

[27] K. Nishihara, Y. Wakasugi, Global existence of solutions for a weakly coupled system of semilinear damped wave equations, J. Differ. Equ. 259 (8) (2015) 4172-4201.

[28] P. Rabinowitz, Periodic solutions of nonlinear hyperbolic partial differential equations, Comm. Pure Appl. Math. 20 (1) (1967) 145-205.

[29] P. Rabinowitz, Free vibrations for a semilinear wave equation, Comm. Pure Appl. Math. 31 (1) (1978) 31-68.

[30] M. Schechter, Rotationally invariant periodic solutions of semilinear wave equations, Abstr. Appl. Anal. 3 (1-2) (1998) 171-180.

[31] M. Schechter, Monotonicity methods for infinite dimensional sandwich systems, Discrete Contin. Dyn. Syst. 28 (2) (2010) 455-468.

[32] J. Shatah, M. Struwe, Geometric Wave Equations, Courant Lecture Notes in Mathematics, Amer. Math. Soc., Providence, R. I., 2000.

[33] H. Sunagawa, On global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimension, J. Differ. Equ. 192 (2) (2003) 308-325.

[34] B. Wang, A. Iserles, X.Y. Wu, Arbitrary-order trigonometric Fourier collocation methods for multi-frequency oscillatory systems, Found. Comput. Math. 16 (1) (2016) 151-181.
[35] Q. Wang, C.G. Liu, A new index theory for linear self-adjoint operator equations and its applications, J. Differ. Equ. **260** (4) (2016) 3749-3784.

[36] E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, Comm. Math. Phys. **127** (3) (1990) 479-528.

[37] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.

[38] X.Y. Wu, B. Wang, W. Shi, Efficient energy-preserving integrators for oscillatory Hamiltonian systems, J. Comput. Phys. **235** (2013) 587-605.

[39] L. Yan, S.G. Ji, L.L. Sun, Asymptotic bifurcation results for coupled nonlinear wave equations with variable coefficients, J. Differ. Equ. **269** (9) (2020) 7157-7170.

[40] Z.T. Zhang, Variational, Topological, and Partial Order Methods with their Applications, Springer, Berlin, 2013.