Robust Trajectory Estimation in Ballistic Phase using Out-of-Sequence High-degree Cubature Huber-based Filtering

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A novel Out-of-Sequence High-degree Cubature Huber-based Filtering (OOS-HCHF) algorithm is presented and utilized to estimate the trajectory of a ballistic target in the ballistic phase. This novel algorithm makes use of the 5th-degree cubature rule to numerically compute Gaussian-weighted integrals, which are propagated through a nonlinear state equation, and then a weighted mean and covariance are taken. As the radar measurements are accentuated with corrupting glint noise which is essentially non-Gaussian and arriving out-of-sequence, usually caused by communication and processing latency, the novel filtering is carefully designed with the consideration of these factors. First, the solution to the OOSM problem is derived in combination with the 5th-degree cubature rule in time update equations. Second, the Huber technique, which is a combined minimum $l_1$ and $l_2$-norm estimation technique, is used to design the measurement update equations. Therefore, the proposed OOS-HCHF could exhibit robustness with respect to deviations from the commonly assumed Gaussian error probability, for which conventional cubature Kalman filtering (CKF) exhibits a severe degradation in estimation accuracy. Furthermore, the out-of-sequence measurements could be incorporated optimally. Finally, in contrast to extended Kalman filtering (EKF), more accurate estimation and faster convergence could be achieved by OOS-HCHF from inaccurate initial conditions. Simulation results are shown to compare the performance of OOS-HCHF with CKF and EKF.

Key Words: Cubature Huber-based Filtering, Trajectory Estimation, Ballistic Phase, Out-of-Sequence Measurement, Glint Noise

Nomenclature

| Abbreviation | Description |
|--------------|-------------|
| KF           | Kalman filtering |
| EKF          | extended Kalman filtering |
| PF           | particle filtering |
| UKF          | unscented Kalman filtering |
| CKF          | cubature Kalman filtering |
| HCKF         | high-degree cubature Kalman filtering |
| HCHF         | high-degree cubature Huber-based filtering |
| GBR          | ground-based radar |
| OOSM         | out-of-sequence measurement |

Subscripts

- $t$: the $t$-th cubature point
- $k$: time $k$
- $0$: initial
- $d$: range
- $\alpha$: azimuth
- $\beta$: elevation

1. Introduction

The trajectory of a ballistic missile can be commonly divided into three phases: the boost, ballistic, and reentry phases. The ballistic phase is an ex-atmospheric, free flight motion which is only governed by the Earth’s gravity. Trajectory tracking during the ballistic phase is a critical step in missile defense, where accuracy would affect the success of interception greatly. However, trajectory estimation during the ballistic phase is rather challenging for several reasons. First, sensor measurements are often corrupted by non-Gaussian noise, which is typically glint noise for target tracking, so the utilization of conventional Gaussian filtering algorithms results in a degradation of tracking accuracy. Second, communication and processing latency introduces the out-of-sequence measurement problem, which must be considered to ensure the estimation accuracy. The definition of the out-of-sequence measurement problem can be given as follows. Assume that a measurement is obtained at time $\tau$, but is available at time $t > \tau$ for various reasons. However, the state has already been updated to time $t$. Now, the problem is how to employ the measurement from time $\tau$ to update the state of time $t$. Standard filtering algorithms such as Kalman filtering (KF), extended Kalman filtering (EKF), and particle filtering (PF) can’t be directly applied to deal with the OOSM. Therefore, several algorithms are developed by the researchers within the framework of KF, EKF, UKF, and PF. Recently, cubature Kalman filtering (CKF) has been proposed in several fields, and is reported as the most numerically stable and accurate algorithm. Furthermore, CKF does not require Jacobians and is therefore applicable to a wide range of problems, including target tracking. In view of the advantages of CKF, one of the key motivations for this paper is to develop a novel algorithm to solve the OOSM problem in the framework of the cubature rule. In contrast to the conventional 3rd-degree algorithm, the 5th-degree cubature rule is utilized to numerically com-
pute Gaussian-weighted integrals accurately. Therefore, the mean and covariance can be estimated more accurately.

Nonlinear filtering, such as EKF, UKF and CKF, typically operates on the first two moments, and has successfully solved many problems where accurate estimates are required. However, these techniques only work well under the assumption of a Gaussian noise model. As the radar measurements are corrupted by non-Gaussian noise, which is typically glint noise, the filtering should be modified to accommodate the non-Gaussian noise. Particle filtering, which is suitable for non-Gaussian noise models, has shown good performance in several applications. However, the method is computationally complex and time consuming. To address the non-Gaussian noise models, this paper proposes the novel Out-of-Sequence High-degree Cubature Huber-based Filtering algorithm.

The Huber technique is a combined minimum $l_1$ and $l_2$-norm estimation. The proposed filtering exhibits robustness to glint noise by making use of the Huber technique to modify the measurement update equations. The time update equations are developed based on the high-degree cubature rule in consideration of the OOSM problem. Therefore, the novel algorithm is particularly well suited for non-Gaussian measurements and the OOSM problem. Moreover, the algorithm is numerically stable and accurate because of the cubature rule.

This paper is outlined as follows: Section 2 presents the derivation of the Out-of-Sequence High-degree Cubature Huber-Based Filtering algorithm. Section 3 introduces the tracking models for a ballistic target in the ballistic phase. Section 4 shows the simulations and results. Finally, the conclusion is given in Section 5.

2. Out-of-Sequence High-degree Cubature Huber-based Filtering

In this section, the Out-of-Sequence High-degree Cubature Huber-based Filtering algorithm is derived. It consists of time update equations and measurement update equations. The time update equations make use of the 5th-degree cubature rule to numerically compute Gaussian-weighted integrals, which are propagated through a nonlinear state equation, and then a weighted mean and covariance are taken. In addition, the time update equations are also modified to adapt to the out-of-sequence problem. The measurement update equations are based on the Huber technique which can provide robustness against deviations from a non-Gaussian distribution, especially glint noise.

The nonlinear discrete dynamic systems can be described by:

$$
x_{k+1} = f(x_k) + v_k
$$

$$
y_k = h(x_k) + w_k
$$

where $x_k \in R^n$ is the state vector of the dynamic system, $y_k \in R^m$ is the measurement vector, $v_k$ and $w_k$ are process and measurement noise, the subscript $k$ represents the discrete time $t_k$, and $f(\cdot)$ and $h(\cdot)$ are the nonlinear state function and measurement function of the system, respectively.

2.1. Time update equations

Under the Gaussian assumption, the Bayesian filter reduces to the problem of how to compute integrals in which integrands are all in the form “nonlinear function $\times$ Gaussian.” The time update uses the 5th-degree cubature rule to numerically compute Gaussian-weighted integrals. The cubature rule approximates the $n$-dimensional Gaussian-weighted integrals as follows:

The initial set of cubature points are calculated based on the state $\hat{x}_k$, covariance $P_{k|k}$, and the cubature point set $\xi_i$. The cubature points are given as follows

$$x_{k,i} = \hat{x}_k + \sqrt{P_{k|k}}\xi_i, \quad i = 0, 1, \ldots, 2n^2$$

where

$$\begin{bmatrix}
0 & \cdots & 0
\end{bmatrix}^T i = 0
$$

$$\beta_{s_i}^+ = \begin{cases}
1, & i = 1, 2, \ldots, \frac{n(n-1)}{2} \\
n(n-1), & i = \frac{n(n-1)}{2} + 1, \ldots, n(n-1) \\
n(n-1)/2, & i = \frac{n(n-1)}{2} + 1, \ldots, n(n-1)/2 \\
n(n-1)/2, & i = \frac{n(n-1)}{2} + 1, \ldots, 2n(n-1) \\
n(n-1)/2, & i = n(n-1) + 1, \ldots, n(n-1) \\
n(n-1)/2, & i = n(n-1) + 1, \ldots, 2n^2
\end{cases}
$$

The parameters $s_i^+$ and $s_i^-$ are given as

$$s_j^+ = \sqrt{1/2(e_p + e_q)} \quad p < q; \quad p, q = 1, 2, \ldots, n $$

$$s_j^- = \sqrt{1/2(e_p - e_q)} \quad p < q; \quad p, q = 1, 2, \ldots, n
$$

where $e_i$ is the unit vector in $R^n$ with the $i$-th element being 1, the parameter $\beta_\ast = \sqrt{n + 2}$, and $j$ is given as

$$j = \begin{cases}
1, & q = 2, p = 1 \\
(q-1)(q-2)/2 + p, & q > 2, p < q
\end{cases}
$$

Then, the cubature points are propagated through the system equation, as follows

$$\hat{x}_{k+1|i,k} = f(\hat{x}_k), \quad i = 0, 1, \ldots, 2n^2$$

Therefore, the predicted state $\hat{x}_{k+1|k}$ and covariance $P_{k+1|k}$ are calculated by

$$\hat{x}_{k+1|k} = \sum_{i=0}^{2n^2} \omega_i \hat{x}_{k+1|i,k}, \quad i = 0, 1, \ldots, 2n^2
$$

$$P_{k+1|k} = \sum_{i=0}^{2n^2} \omega_i (\hat{x}_{k+1|i,k} - \hat{x}_{k+1|k})(\hat{x}_{k+1|i,k} - \hat{x}_{k+1|k})^T
$$

where the weights $\omega_i$ are given as follows

$$\omega_i = \begin{cases}
1/(n+2)^2, & i = 1, 2, \ldots, 2n^2 \\
(4-n)/(2(n+2)^2), & i = 2n^2, 2n^2 + 1, \ldots, 2n^2
\end{cases}
$$
The predicted cubature points $\chi_{k+1|i}$ are then evaluated based on the predicted state $\hat{x}_{k+1|i}$ and covariance $P_{k+1|i}$

$$
\chi_{k+1|i} = \sqrt{P_{k+1|i}} \hat{\xi}_{k+1|i} + \hat{x}_{k+1|i}, \quad i = 0, 1, \ldots, 2n^2
$$

The predicted cubature points $\chi_{k+1|i}$ are propagated through the measurement equation and the corresponding predicted measurement is

$$
\gamma_{k+1|i} = h(\chi_{k+1|i}), \quad i = 0, 1, \ldots, 2n^2
$$

$$
\hat{y}_{k+1|i} = \sum_{i=0}^{2n^2} a_i \gamma_{k+1|i}, \quad i = 0, 1, \ldots, 2n^2
$$

The innovation covariance matrix and cross-covariance matrix need to be evaluated as follows, respectively

$$
P_{yy,k+1} = \sum_{i=0}^{2n^2} a_i (\gamma_{k+1|i} - \hat{y}_{k+1|i})(\gamma_{k+1|i} - \hat{y}_{k+1|i})^T + R_i, \quad i = 0, 1, \ldots, 2n^2
$$

$$
P_{yx,k+1} = \sum_{i=0}^{2n^2} a_i (\chi_{k+1|i} - \hat{x}_{k+1|i})
\cdot (\gamma_{k+1|i} - \hat{y}_{k+1|i})^T, \quad i = 0, 1, \ldots, 2n^2
$$

### 2.2. Time update equations

Typically, the out-of-sequence measurements problem refers to updating the current state at time $t_k$ using the “older”, data sensed at time $t_m$ which arrived at the fusion center at time $t_k$, where $t_m < t_k$. The most common reasons for such out-of-sequence data usually include communication and processing delays. Obviously, these data shouldn’t be discarded, since they still contain information related to the current state, but they do need to be processed accordingly. In general, the time lag of the out-of-sequence measurements is not constant. However, the time lag can be calculated, as the data has a time-stamp. In this section, the time update of the proposed algorithm is modified to incorporate the out-of-sequence measurement optimally.

Since the state at time $t_i$ has not been updated by the measurement, the state and covariance have the following relationship

$$
\hat{x}^-(t_k) = \hat{x}^+(t_k) + \sqrt{P_{x(t_k),x(t_k)}} \xi, \quad i = 0, 1, \ldots, 2n^2
$$

$$
P_{x(t_k),x(t_k)}^-(t_k) = P_{x(t_k),x(t_k)}^+(t_k)
$$

The time delay is calculated and divided into $q$ equal parts as follows

$$
\tau = \frac{t_k - t_m}{q}
$$

Therefore, the time series between $t_m$ and $t_k$ can be given by

$$
T = [t_0, t_1, \ldots, t_{q-1}, t_q] = [t_m, t_m + \tau, \ldots, t_m + (q - 1) \tau, t_k]
$$

Take $t_{q-1}$ and $t_q$ for example. As the predicted state $\hat{x}^-(t_q)$ and covariance $P_{x(t_q),x(t_q)}^-(t_q)$ are known, the predicted state $\hat{x}^-(t_{q-1})$ and covariance $P_{x(t_{q-1}),x(t_{q-1})}^-(t_{q-1})$ at $t_{q-1}$ moment can be calculated as

- Calculate the cubature points:

$$
\hat{x}^-(t_q) = \hat{x}^-(t_k) + \sqrt{P_{x(t_k),x(t_k)}^{-\frac{1}{2}}} \xi, \quad i = 0, 1, \ldots, 2n^2
$$

- Backward propagate the cubature points according to the system equation:

$$
\chi^-(t_q) = f^{-1}(\chi^-(t_q)), \quad i = 0, 1, \ldots, 2n^2
$$

- Calculate the predicted state at $t_{q-1}$ moment:

$$
\hat{x}^-(t_{q-1}) = \sum_{i=0}^{2n^2} a_i \chi^-(t_{q-1}), \quad i = 0, 1, \ldots, 2n^2
$$

- Calculate the covariance at $t_{q-1}$ moment:

$$
P_{x(t_{q-1}),x(t_{q-1})}^-(t_{q-1}) = \sum_{i=0}^{2n^2} a_i (\chi^-(t_{q-1}) - \hat{x}^-(t_{q-1}))(\chi^-(t_{q-1}) - \hat{x}^-(t_{q-1}))^T + Q(t_q, \tau), \quad i = 0, 1, \ldots, 2n^2
$$

Cycle the above steps until the predicted state $\hat{x}^-(t_m)$ and covariance $P_{x(t_m),x(t_m)}^-$ at $t_m$ moment are known, where $Q(t_q, \tau)$ is the system state noise at time $t_i, i = 1, \ldots, q$.

The corresponding predicted measurements are shown by

$$
\chi^-(t_m) = \hat{x}^-(t_m) + \sqrt{P_{x(t_m),x(t_m)}^-} \xi, \quad i = 0, 1, \ldots, 2n^2
$$

$$
\gamma^-(t_m) = h(\chi^-(t_m)), \quad i = 0, 1, \ldots, 2n^2
$$

$$
\hat{y}(t_m) = \sum_{i=0}^{2n^2} a_i \gamma^-(t_m), \quad i = 0, 1, \ldots, 2n^2
$$

Finally, the innovation covariance matrix and cross-covariance matrix are evaluated by

$$
\hat{x}^+_k(t_k) = \hat{x}^+_k(t_k) + \sqrt{P_{x(t_k),x(t_k)}^+} \xi, \quad i = 0, 1, \ldots, 2n^2
$$

$$
P_{x(t_k),x(t_k)}^+(t_k) = \sum_{i=0}^{2n^2} a_i (\chi^+_i(t_m) - \hat{y}(t_m))
\cdot (\gamma_i(t_m) - \hat{y}(t_m))^T + R(t_m), \quad i = 0, 1, \ldots, 2n^2
$$

$$
P_{x(t_k),x(t_k)}^- = \sum_{i=0}^{2n^2} a_i (\chi^+(t_k) - \hat{x}^+_k(t_k))
\cdot (\gamma_i(t_m) - \hat{y}(t_m))^T, \quad i = 0, 1, \ldots, 2n^2
$$

### 2.3. The measurement update using Huber’s technique

In this section, the measurement update problem is solved using the Huber technique. To apply the Huber technique, the measurement update is recast as a regression problem between the measurement and the state prediction. If the true value of the state is written as $x_k$, the state prediction error can be written as

$$
\delta_k = x_k - \hat{x}_k
$$

Then, the state prediction can be expressed as $\hat{x}^+_k = x_k - \delta_k$. The measurement matrix $H_k$ can be given by

$$
H_k = \left[ \text{inv}(P_{x_x,k+1|i}) (P_{x_y,k+1|i}) \right]^T
$$

where $P_{x_x,k+1|i}$ and $P_{x_y,k+1|i}$ can be calculated using Eq. (9) and Eq. (15). For out-of-sequence measurement, the corre-
The regression problem is determined from the derivative of the measurement equation can be approximated by
\[ y_k \approx \hat{y}_k + H_k \delta_k \]  
(32)

Then, the linear regression problem takes the form
\[ \begin{bmatrix} y_k - h_k(\hat{x}_k) + H_k \hat{x}_k \\ \hat{x}_k \end{bmatrix} = \begin{bmatrix} H_k & I \\ I & -I \end{bmatrix} x_k + \begin{bmatrix} n_k \\ -\delta_k \end{bmatrix} \]  
(33)

With definition of the quantities
\[ T_k = \begin{bmatrix} R_k & 0 \\ 0 & (P_{x,k+1|k}) \end{bmatrix} \]  
(34)

\[ z_k = T_k^{-1/2} \begin{bmatrix} y_k - h_k(\hat{x}_k) + H_k \hat{x}_k \\ \hat{x}_k \end{bmatrix} \]  
(35)

\[ D_k = T_k^{-1/2} \begin{bmatrix} H_k \\ I \end{bmatrix} \]  
(36)

\[ S_k = T_k^{-1/2} \begin{bmatrix} n_k \\ -\delta_k \end{bmatrix} \]  
(37)

the linear regression problem is transformed to
\[ z_k = D_k x_k + S_k \]  
(38)

where the covariance of vector \( S_k \) is simply the identity matrix in this transformed regression problem. The Huber filtering measurement update arises from the minimization of the cost function
\[ J(x_k) = \sum_{i=1}^{n} \rho(\tilde{\epsilon}_i) \]  
(39)

where \( \tilde{\epsilon}_i \) refers to the \( i \)-th component of the normalized residual vector, \( \tilde{\epsilon}_i = (D_k \hat{x}_k - z_k) \), and the function \( \rho \) is known as the "score function" which will usually be defined as
\[ \rho(\tilde{\epsilon}_i) = \begin{cases} \frac{1}{2} \tilde{\epsilon}_i^2 & |\tilde{\epsilon}_i| < \gamma \\ \mu |\tilde{\epsilon}_i| - \frac{1}{2} \mu^2 & |\tilde{\epsilon}_i| \geq \gamma \end{cases} \]  
(40)

where \( \gamma \) is a tuning parameter. The solution of the Huber regression problem is determined from the derivative of the cost function
\[ \sum_{i=1}^{n} \ell(\tilde{\epsilon}_i) \frac{\partial \tilde{\epsilon}_i}{\partial x_k} = 0 \]  
(41)

where \( \ell(\tilde{\epsilon}_i) = \rho'(\tilde{\epsilon}_i) \).

By defining the function \( \psi(\tilde{\epsilon}_i) = \ell(\tilde{\epsilon}_i)/\tilde{\epsilon}_i \) and the matrix \( \Phi = \text{diag}[\psi(\tilde{\epsilon}_i)] \), Eq. (41) can be written in matrix form as
\[ \sum_{i=1}^{n} \ell(\tilde{\epsilon}_i) \frac{\partial \tilde{\epsilon}_i}{\partial x_k} = 0 \]  
(42)

Equation (42) can be expanded to yield
\[ D_k^T \Phi D_k x_k = D_k^T \Phi z_k \]  
(43)

Therefore, the iterative solution to Eq. (43) can be given as
\[ \hat{x}_k^{(j+1)} = (D_k^T \Phi D_k)^{-1} D_k^T \Phi z_k \]  
(44)

where the superscript \( j \) refers to the iteration index. The method can be initialized by using the least-squares solution
\[ \hat{x}_k^{(0)} = (D_k^T D_k)^{-1} D_k^T z_k \]  
(45)

The converged value from the iterative procedure is taken as the corrected state estimation following a measurement update \( \hat{x}_k^\ast \).

Finally, the state estimate error covariance matrix is computed from
\[ P_{k+1|k+1} = (D_k^T \Phi D_k)^{-1} \]  
(46)

using the final value of \( \Phi \) corresponding to the converged state estimate. Note that as \( \gamma \to \infty \), \( \Phi \to I \), the measurement update is equal to the standard high-degree cubature Kalman filtering.

3. Tracking Models of Ballistic Target in Ballistic Phase

After the engine is turned off and the target leaves the atmosphere, a ballistic target enters the ballistic phase of its trajectory—no thrust is applied and no drag is experienced. The motion is governed by gravity alone and other irrelevant factors are neglected. Therefore, the total acceleration is \( \mathbf{a} = \mathbf{a}_G \), where \( \mathbf{a} \) denotes the total acceleration, and \( \mathbf{a}_G \) is the acceleration caused by gravity.

3.1. State model

Assuming that the Earth and the ballistic target can be represented as point masses at their geo-centers, and the gravitational force of other objects can be neglected, the gravitational acceleration \( \mathbf{a}_G \) is the solution of a two-body problem, so
\[ \mathbf{a}(\mathbf{r}) = \mathbf{a}_G(\mathbf{r}) = -\frac{\mu \mathbf{r}}{||\mathbf{r}||^3} \]  
(47)

where \( \mathbf{r} \) is the state vector from the Earth’s center to the target and \( \mu \) is the Earth’s gravitational constant.

In the Earth-centered inertial coordinate frame system (ECI-CS), the state vector \( \mathbf{x}_k \) is defined as
\[ \mathbf{x}_k = [\mathbf{r}, \mathbf{v}, \mathbf{a}]^T = [x, y, z, v_x, v_y, v_z, a_x, a_y, a_z]^T \]  
(48)

where the state vector includes the position, velocity, and acceleration of the target.

According to Eq. (47), the state model is as follows:
\[ \mathbf{r} = [x, y, z]^T \]  
(49)

\[ \mathbf{v} = \mathbf{r} = [v_x, v_y, v_z]^T \]  
(50)

\[ \mathbf{a} = \dot{\mathbf{v}} = [a_x, a_y, a_z]^T = -\frac{\mu \mathbf{r}}{||\mathbf{r}||^3} \]  
(51)

3.2. Measurement model

Ground-Based Radar (GBR) is used for ballistic target
Fig. 1. Sensor coordinate system.

tracking, and the measurements provided by the radar consist of range \( d \), azimuth \( \alpha \), and elevation \( \beta \) (as shown in Fig. 1). These measurements are generally modeled in the following form of additive noise:

\[
\begin{align*}
\tilde{d} &= d + w_d \\
\tilde{\alpha} &= \alpha + w_\alpha \\
\tilde{\beta} &= \beta + w_\beta
\end{align*}
\]

where \((d, \alpha, \beta)\) denotes the error-free true target position in the sensor spherical coordinates, and \(w_d, w_\alpha, \) and \(w_\beta\) are random measurement errors of range, azimuth, and elevation, respectively.

So, the measurement vector is defined as follows:

\[
y_k = \begin{bmatrix} d_k, \alpha_k, \beta_k \end{bmatrix}^T
\]

Then the measurement equation is

\[
y_k = h(x_k) + w_k
\]

where \(x_k\) is the state vector, and \(w_k\) is the measurement noise of the GBR. \(y_k\) consists of range \(d_k\), azimuth \(\alpha_k\), and elevation \(\beta_k\), so the measurement model is presented as

\[
y_k = \begin{bmatrix} d(x_k) \\
\alpha(x_k) \\
\beta(x_k)
\end{bmatrix} = \begin{bmatrix}
\sqrt{x_r^2 + y_r^2 + z_r^2} \\
\arcsin \frac{z_r}{d} \\
\arctan \frac{2(y_r, x_r)}
\end{bmatrix} + w_k
\]

where \([x_r, y_r, z_r]^T\) is the position of the target in the Sensor Coordinate System, and

\[
\begin{bmatrix} x_r \\
y_r \\
z_r
\end{bmatrix} = C_f \begin{bmatrix} x - x_c \\
y - y_f \\
z - z_c
\end{bmatrix}
\]

where \(C_f\) is the coordinate transform matrix from ECI-CS to the Sensor Coordinate System, and \([x_c, y_f, z_c]^T\) is the position of the sensor in ECI-CS.

4. Simulation

To verify the effectiveness of the proposed algorithm, simulations are conducted, and the trajectory of a ballistic target in the free flight portion is shown in Fig. 2. The simulation parameters are given in Table 1, including the sensor noise, time delay, measuring frequency, initial position error, and velocity error.

4.1. Simulation scene and simulation conditions

The sensor measurement errors are drawn from the mixture of zero-mean Gaussian probability distributions, defined by the probability density function

\[
p(\xi) = \begin{cases}
\frac{1 - \epsilon}{\sigma_1 \sqrt{2\pi}} \exp \left(\frac{-(\xi/\sigma_1)^2}{2}\right) \\
\frac{\epsilon}{\sigma_2 \sqrt{2\pi}} \exp \left(\frac{-(\xi/\sigma_2)^2}{2}\right), & 0 \leq \epsilon \leq 1
\end{cases}
\]

where \(\sigma_1\) and \(\sigma_2\) are the standard deviations of the individual Gaussian distributions and \(\epsilon\) is a perturbing parameter representing the error model contamination.

4.2. Simulation results

In the simulation section, different filter algorithms are utilized to compare their performance in trajectory estimation, including EKF, HCK, HCHF, and OOS-HCHF. EKF is the best known nonlinear filtering for trajectory estimation and is based on the first-order Taylor-series linearization. HCKF, which is based on the fifth degree spherical–radial cubature rule during the time update stage, can improve the estimation accuracy compared to EKF. HCHF differs from HCKF by using the Huber technique to modify the measurement equation. The OOS-HCHF is proposed in this paper. To verify the effectiveness of the proposed algorithm, two simulation conditions are utilized. The simulation results in Fig. 3 come from 100 Monte Carlo simulation runs using four different parameters.

Table 1. Simulation parameters.

| Parameter | Parameter value |
|-----------|----------------|
| Range measurement noise of GBR | \(\sigma_1 = 0.15\) m, \(\sigma_2 = 5\sigma_1, \epsilon = 0.05\) |
| Azimuth measurement noise of GBR | \(\sigma_1 = 2.4 \times 10^{-3}\) rad, \(\sigma_2 = 5\sigma_1, \epsilon = 0.05\) |
| Elevation measurement noise of GBR | \(\sigma_1 = 2.4 \times 10^{-3}\) rad, \(\sigma_2 = 5\sigma_1, \epsilon = 0.05\) |
| Measurement time delay | 80 ms |
| Measuring frequency | 10 Hz |
| Initial position error | \([2000 2000 2000]^T\) m |
| Initial velocity error | \([100 100 100]^T\) m/s |

Fig. 2. Trajectory of ballistic target in free flight portion.
algorithms. The word “delay” means that the measurements are fused 80 ms later. The word “ideal” means that the measurements are fused without delay. Figure 3 shows that the OOS-HCHF performs almost as well as the HCHF-ideal. This means that the OOS problem is solved optimally using OOS-HCHF. However, the accuracy of HCHF decreases without considering the OOS problem. The results of HCKF are worse than both the HCHF-delay and OOS-HCHF, as the glint noise degrades the performance of HCKF. Finally, the results of EKF are worst, because neither the OOS problem nor the glint noise are treated properly. Figure 3 indicates that the proposed OOS-HCHF gets the best estimation results. The covariance estimates of these algorithms are shown in Fig. 4, and the results are consistent with Fig. 3.

The relationship between estimated errors of position and velocity and the perturbing parameter $\varepsilon$ is shown in Figs. 5 and 6, respectively. For the case $\varepsilon = 0$, the estimated errors of OOS-HCHF and HCKF are close to one another, especially the estimated velocity error. The reason is that when the perturbing parameter $\varepsilon = 0$, the noise is Gaussian. As $\varepsilon$ increases, the HCKF errors increase much more rapidly than the OOS-HCHF errors. The reason is the robustness of HCHF to thickly tailed measurement error distributions. Figure 7 presents the estimated position error changing with respect to time delay. When the delay time is short, OOS-HCHF has accuracy similar to HCHF, but as delay time increases, the precision of HCHF declines dramatically. On the other hand, the precision of OOS-HCHF only has a small decline. Table 2 and Table 3 show in detail the relationship between accuracy and the perturbing parameter, and accuracy and delay time, respectively.
5. Conclusion

In this paper, OOS-HCHF is derived and applied to tracking a ballistic missile in the free flight portion. By making use of the state-transition inverse matrix to modify the time update equations, and the Huber technique to modify the measurement update equations, OOS-HCHF exhibits robustness with respect to deviations from the commonly assumed Gaussian error probability, and reduces the influence of time delay. A free flight model is used for formulating the state equation. Simulation results indicated the performance of OOS-HCHF exceeds HCKF, HCHF, and SEKF for cases with glint noise and time delay.

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