L1-Finite Difference Method for Inverse Source Problem of Fractional Diffusion Equation

Ruiping Zhang, Hao Bai and Fengqun Zhao*
Department of School of Sciences, Xi’an University of Technology, Xi’an, Shaanxi, China

*Corresponding author email: zhaofq@xaut.edu.cn

Abstract. The numerical solution of inverse source problem for time fractional diffusion equation was studied: the time fractional derivative was discretized by L1 algorithm, and the space derivative was approximated by central difference. The L1-finite difference scheme of the inverse source problem was constructed with the accuracy of $O(h^{2-\alpha}+\tau^2)$. By adding a random disturbance to the final value and combining with the discrete scheme of L1-finite difference method, the algorithm flow was given. The effectiveness of L1-finite difference method for solve the inverse problem of time fractional diffusion equation is verified by numerical examples. The method provides an effective reference scheme for the numerical solution of the inverse problem.

1. Introduction

In recent years, the study of fractional dynamics has attracted people's attention [1], and the fractional diffusion equation describing the abnormal diffusion process [2] has also been widely concerned and applied. As the same as the general diffusion problem, the research on the inverse problem of fractional diffusion equation has important theoretical and practical value.

As we know, most of the numerical methods for the inverse problem are based on the regularization method due to its ill posed nature [3-11]. For example, for the inverse source problem of fractional variable coefficients diffusion equation, Wei and Wang [3] proposed an improved quasi boundary value regularization method. To determining unknown source term in time fractional diffusion equation, Nguyen et al [4] proposed a Tikhonov regularization method, and verified the effectiveness of the method by some numerical experiments. By using the generalized and modified generalized Tikhonov regularization method, Ma et al [7] constructed the regularization solution of the finite field time-fraction diffusion equation with variable coefficients, and determined the spatial correlation source term problem. Using the boundary element method and the generalized Tikhonov regularization method, Li and Wei [10] studied the problem of determining a time-dependent source term in the time-space fractional diffusion equation, and obtained a stable numerical approximation of the source term by the generalized cross validation rule of regularization parameter selection.

Finite difference method is one of the most commonly used numerical methods for solving positive problems because of its advantages of simple calculation and small amount of calculation. However, due to the ill posed nature of the inverse problem, the algorithm is often unstable when using finite difference method. In this paper, L1-finite difference method is constructed for the inverse source problem of time fractional diffusion equation(TFDE). By adding random interference term to the final value, stable numerical results are obtained.
2. L1-finite Difference Method for Inverse Source Problem of TFDE

Let the region \( \Omega = \{(x,t) | 0 < x < 1, 0 < t \leq T\} \) be a bounded smooth region, in the paper, we study the inverse source problem of TFDE

\[
\begin{align*}
\frac{D_t^\alpha u(x,t)}{\Gamma(1-\alpha)} &= (a(x)u_x)_x + c(x)u + p(t)f(x) \\
u(0,t) &= k_0(t) \\
u(1,t) &= k_1(t) \\
u(x,0) &= \phi(x) \\
u(x,T) &= g(x)
\end{align*}
\]

(1)

where the coefficients \( a(x), c(x), p(t) \) are known; the boundary value functions \( k_0(t), k_1(t) \) and the initial function \( \phi(x) \) are known; but the source term \( f(x) \) is unknown. Now we know the value of \( u(x,t) \) at time \( T \), that is

\[
u(x,T) = g(x) \quad 0 \leq x \leq 1
\]

(2)

we need to determine \( f(x) \) in problem (1).

In practical problems, only the approximate value \( g^\delta(x_j)(j = 1, 2, \cdots, M) \) of the final value data \( g(x) \) at the internal discrete point in \( 0 \leq x \leq 1 \) can be measured, and it satisfies

\[
|g^\delta(x_j) - g(x_j)| \leq \delta \quad j = 0, 1, 2, \cdots, M.
\]

where \( \delta > 0 \) is a given constant, which we call the error level. We need to construct an algorithm to get the approximate value \( f^\delta(x) \) of \( f(x) \) through the approximate value \( g^\delta(x) \) of the final value data \( g(x) \), and give the error estimate.

In the example of this paper, the final data \( g^\delta(x_j) \) is obtained by adding a random interference term, namely

\[
g^\delta(x_j) = g(x_j) + \varepsilon g(x_j)(2\text{rand}(j) - 1)
\]

(3)

where \( g(x_j) \) is the analytical solution, \( \text{rand}(j) \) is the uniformly distributed random number on the interval \([0, 1]\), and \( \varepsilon > 0 \) is the relative error level. Then the error level is defined as

\[
\delta = \|g^\delta(x) - g(x)\| \approx \varepsilon \cdot \|g(x)\|.
\]

2.1. Time L1 Semi Discrete Scheme

In order to establish a semi discrete scheme of time for the problem (1), the time interval \([0, T]\) is divided into \( N \) equal parts, i.e. \( 0 = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_N = T \), the time step is \( \tau = T/N, t_n = n\tau, n = 0, 1, 2, \cdots, N \).

Suppose \( 0 < \alpha < 1 \), if \( v(t) \in C^2[0,T], t_n = nh \in [0,T] \), then the Caputo fractional derivative of function \( v(t) \) is

\[
D_t^\alpha v(t_n) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{dv(s)}{ds} \frac{ds}{(t_n - s)^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{dv(t_n - s)}{ds} \frac{ds}{s^\alpha}
\]

\[
\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \frac{v(t_{j+1}) - v(t_j)}{\tau} \int_{t_j}^{t_{j+1}} s^{-\alpha} ds
\]
We call it L1 algorithm [12], which can be recorded as
\[
(D_t^q v(t_n))_{L_1} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n-1} b^{(\alpha)}_j (v_{n-j} - v_{n-1-j})
\]
where \( b^{(\alpha)}_j = (i+1)^{1-\alpha} - i^{1-\alpha} \).

Theorem 1 [13]. if \( \alpha \in (0,2) \), \( \alpha \neq 1 \), \( v(t) \in C^2[0,T], t_\alpha = nh \in [0,T] \), then there is constant \( c_\alpha > 0 \) related to \( \alpha \), the truncation error of L1 algorithm is satisfied
\[
\left| D_t^q v(t_n) - \left( D_t^{q'} v(t_n) \right)_{L_1} \right| \leq c_\alpha \|v^{\alpha'}\|_{2-\alpha}
\]
See reference [13] (theorem 2.3) and reference [14] (lemma 2.2) for proof.

From formula (4), we can obtain the time L1 semi discrete scheme for the inverse problem (1) of TFDE
\[
\begin{align*}
\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n-1} b^{(\alpha)}_j (u(x,t_{n-j}) - u(x,t_{n-1-j})) \\
= (a(x)u_t(x,t_n))_j + c(x)u(x,t_n) + p(t_n)f(x) & \quad n = 1,2,\cdots,N; 0 \leq x \leq 1. \\
u(0,t_n) = k_0(t_n), u(1,t_n) = k_1(t_n) & \quad n = 0,1,2,\cdots,N. \\
u(x,0) = \phi(x) & \quad 0 \leq x \leq 1.
\end{align*}
\]
The local truncation error of the semi discrete scheme is \( O(\tau^{2-\alpha}) \).

2.2. L1-finite Difference Fully Discrete Scheme
The space interval \( \Omega = [0,1] \) is divided into \( M \) equal parts, that is \( 0 = x_0 < x_1 < \cdots < x_j < x_{j+1} < \cdots < x_M = 1 \), the space step is \( h = l/M \), \( x_j = jh, j = 0,1,2,\cdots,M \).

The spatial derivative of function \( u(x,t) \) about \( x \) at \( (x_j,t_n) \) is approximated by central difference, that is
\[
(a(x)u_t)_{l_{j,1/2}} = \frac{a(x)u_{j+1/2} - a(x)u_{j-1/2}}{h} + O(h)
\]
where \( a_{j+1/2} = a(x_{j+1/2}) \). Denoting \( u^0_j \approx u(x_j,t_n), f_j = f(x_j) \), and substituting the above formula into the semi discrete scheme (6), and eliminating the local truncation error, then the L1 finite difference scheme of the inverse problem (1) of TFDE can be obtained
The convergence accuracy of the discrete scheme is $O(\tau^{2-\alpha} + h^2)$.

The matrix form of discrete format (7) is

$$AU^n = rp(t_n)F + F_1^n + F_2^n, \quad n = 1, 2, \ldots, N$$

where

$$r = \Gamma(2-\alpha)h^n, \quad d_j = 1 + \frac{r}{h^2} (a_{j+\frac{1}{2}} + a_{j-\frac{1}{2}}) - rc(x_j),$$

$$A = \begin{bmatrix}
  d_1 & -\frac{r}{h^2} a_{\frac{3}{2}} & 0 & \cdots & 0 \\
  -\frac{r}{h^2} a_{\frac{3}{2}} & d_2 & -\frac{r}{h^2} a_{\frac{5}{2}} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  -\frac{r}{h^2} a_{M-\frac{3}{2}} & 0 & \cdots & -\frac{r}{h^2} a_{M-\frac{1}{2}} & d_{M-1}
\end{bmatrix}, \quad U^n = \begin{bmatrix}
  u^n_1 \\
  u^n_2 \\
  \vdots \\
  u^n_{M-1} \\
  u^n_M
\end{bmatrix}, \quad F = \begin{bmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_{M-1} \\
  f_M
\end{bmatrix}, \quad F_1^n = \begin{bmatrix}
  \frac{r}{h^2} a_{\frac{1}{2}} u^n_0 \\
  0 \\
  \vdots \\
  0 \\
  \frac{r}{h^2} a_{\frac{1}{2}} u^n_M
\end{bmatrix}$$

In formula (8), $U^N = G = [g(x_0), g(x_1), \ldots, g(x_N)]^T$ is known, and $F = [f_0, f_1, \ldots, f_{M-1}, f_M]^T$ is the quantity to be solved. So the equation of $F$ is

$$rp(t_n)F + F_1^N + F_2^N = AG$$

In which $U^{N-1}, F_1^N$ are obtained by iteration $AU^n = rp(t_n)F + F_1^n + F_2^n, \quad n = 1, 2, \ldots, N$.

3. Numerical Results and Analysis

Example 1

Suppose the inverse source problem (1) with exact solution coefficients $a(x) = 1, c(x) = 0, \quad p(t) = (2\pi t)^2 + 2t^{2-\alpha}/(3-\alpha), \quad$ boundary conditions $k_0(t) = 0, k_1(t) = 0, \quad$ initial condition $\phi(x) = 0, \quad$ and final condition $g(x) = \sin(2\pi x)$. We need to get the approximate value $f^N(x)$ of the source term $f(x)$ from the approximate value $g^N(x)$ of the final value data $g(x)$. The exact solution $f(x) = \sin(2\pi x), \quad 0 \leq x \leq 1$. By using the L1-finite difference method of the inverse source problem of TFDE obtained in this
paper, when \( h = 0.05, \tau = 0.01, \varepsilon = 0.01, \alpha = 0.3 \), we can get the comparison between the numerical solution and the accurate solution \( f(x) \), and the error situation, as shown in Fig.1. When \( h = 0.05, \tau = 0.01, \varepsilon = 0.01, \alpha = 0.8 \), the calculation results are shown in Fig. 2.

![Figure 1. Comparison of numerical solution and exact solution of \( f(x) \) when \( h = 0.05, \tau = 0.01, \varepsilon = 0.01, \alpha = 0.3 \)](image1)

![Figure 2. Comparison of numerical solution and exact solution of \( f(x) \) when \( h = 0.05, \tau = 0.01, \varepsilon = 0.01, \alpha = 0.8 \)](image2)

It can be seen from Fig. 1 and Fig. 2 that, when \( h = 0.05, \tau = 0.01, \varepsilon = 0.01, \alpha = 0.3 \), the error can reach \( 10^{-8} \); when \( \alpha = 0.8 \), the error can reach \( 10^{-7} \). This shows that the L1-finite difference method for the inverse source problem of TFDE is effective.

![Figure 3. Error curve of \( f(x) \) with \( \alpha \) under \( h = 0.05, \tau = 0.01, \varepsilon = 0.01 \)](image3)
Example 2
Consider the inverse source problem (1) with coefficients $a(x) = x^2 + 1, c(x) = -(x + 1)$, $p(t) = e^{-t}$, boundary conditions $k_0(t) = 0, k_1(t) = 0$, initial condition $\phi(x) = 0$. When the source term $f(x) = (x(1-x))^\alpha \sin(5\pi x)$, we first use the L1-finite difference method to find the numerical solution $u(x,t)$ of the positive problem. Then $u(x,T) = g(x)$ is known, and then the inverse source problem is solved by this method, which further verifies the effectiveness of our method.

Fig. 4 shows the numerical solutions $g(x)$ and $u(x,t)$ of the positive problem when $h = 0.01, \tau = 0.01, \alpha = 0.3$; Fig. 5 shows the comparison between the numerical solution and the accurate solution $f(x)$ of the inverse problem and the error curve when $h = 0.01, \tau = 0.01, \varepsilon = 0.01, \alpha = 0.3$. Fig. 6 and Fig. 7 are the calculation results when $\alpha = 0.8$.

It can be seen that, with $h = 0.01, \tau = 0.01, \varepsilon = 0.01$, when $\alpha = 0.3$ and $\alpha = 0.8$, the error order of $f(x)$ can reach $10^{-9}$. This not only shows that the method in this paper is very effective for solving the inverse source problem of time-sharing diffusion equation, but also for solving the positive problem.
4. Conclusion

At present, for the numerical solution of inverse problems, most of the methods are improved and innovated on the basis of regularization methods, including Tiknonov regularization, Landweber iterative method, etc. In this study, the inverse source problem of TFDE is studied by using L1-finite difference method without any regularization, and the stable and high-precision numerical solution of the source term related to space is obtained. This method has low complexity and good stability. It can be further used to solve the inverse problems, such as initial data, unknown coefficients and so on.

References

[1] Wei L. L. Fully Discrete Local Discontinuous Galerkin Methods for the Fractional Partial Differential Equations with Higher Derivatives [D]. Xi'an: Xi'an Jiaotong University, 2012. (in Chinese)

[2] Carreras B. A., Lynch V. E., Zaslavsky G. M. Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence model[J]. Physics of Plasmas, 2001, 8(12):5096-5103.

[3] Wei T., Wang J.G.. A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation[J]. Applied Numerical Mathematics, 2014, 78:95-111.

[4] Nguyen H. T., Le D. L., Nguyen V. T.. Regularized solution of an inverse source problem for a time fractional diffusion equation[J]. Applied Mathematical Modelling, 2016, 40(19-20): 8244-8264.

[5] Cheng Q, Xiong X. An iterative method for an inverse source problem of a time-fractional diffusion equation[J]. Computational Mathematics, 2017, 39(03): 295-308 (in Chinese).

[6] Sun L. Studies on determination of source and coefficient for the time-fractional diffusion equations[D]. Lanzhou: Lanzhou University, 2017 (in Chinese).

[7] Ma Y K , Prakash P , Deiveegan A . Generalized Tikhonov methods for an inverse source problem of the time-fractional diffusion equation[J]. Chaos Solitons & Fractals, 2018, 108:39-48.
[8] Zhang S, Qiang J, Liu X, Liu H, Zhu X. Inverse problems of pollution source identification based on Bayesian-DE [J]. Journal of Shandong University (Engineering Science), 2018, 48(01): 131-136 (in Chinese).

[9] Yan X. B. Inverse source problem for a fractional diffusion equation and a diffusion wave equation [D]. Lanzhou: Lanzhou University, 2018 (in Chinese).

[10] Li Y. S., Wei T.. An inverse time-dependent source problem for a time–space fractional diffusion equation [J]. Applied Mathematics and Computation, 2018, 336(1): 257-271.

[11] Jin Z W. Research on pollution source inversion based on fractional order anomalous diffusion equation [D]. Harbin: Harbin Institute of Technology, 2014 (in Chinese).

[12] Guo B, Pu X, Huang F. Fractional partial differential equations and their numerical solutions [M]. Beijing: Science Press, 2011 (in Chinese).

[13] Diethelm, K. Generalized compound quadrature formulae for finite-part integrals [J]. IMA Journal of Numerical Analysis, 1997, 17(3): 479-493.

[14] Diethelm K. An algorithm for the numerical solution of differential equations of fractional order [J]. Electronic Transactions on Numerical Analysis. Numer. Anal., 1997, 5:1-6.