Leveraged ETF Investing

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Abstract

It is common knowledge that leverage can increase the potential returns of an investment, at the expense of increased risk. For a passive investor in the stock market, leverage can be achieved using margin debt or leveraged-ETFs. We perform bootstrapped Monte-Carlo simulations of leveraged (and unleveraged) mixed portfolios of stocks and bonds, based on past stock market data, and show that leverage can amplify the potential returns, without significantly increasing the risk for long-term investors.
1 Introduction

Leverage (borrowing to invest) is a way to increase potential returns for an investment, at the expense of increased risk. For a passive investor in the stock market, this can be achieved by taking margin loan from the brokerage, or buying leveraged exchange traded funds (LETFs) [1]. LETFs (that amplify the daily returns of their underlying index) are usually not recommended as long term investments due their decay during fluctuations (even when the index “fluctuates” around a constant value the LETF loses) [2, 3, 4]. Margin leverage is less sensitive to daily fluctuations which makes it interesting to compare both methods. An additional leverage strategy that can be employed is using stock options (e.g. constantly buying call options), but the author does not currently understand them enough to model.

The stock market is chaotic and the price swings are almost uncorrelated day to day. Predicting the future is impossible, but over the long term the market goes up, and this has been consistent for hundreds of years. The daily price change (in percents) \( \Delta P \) can be considered a random variable whose probability distribution is skewed upwards slightly. Given that we have no better knowledge about the future except the past, the best we can do is to assume that \( \Delta P \) in the future is distributed the same as in past. Generating synthetic realizations of the future by drawing from this distribution is called a Monte-Carlo (MC) simulation, which is classically done by assuming some analytic probability distribution for the price changes. The result is a probability distribution for the yield of a portfolio after some investment period (say, 10 years), for which we can calculate risk and reward metrics. Note that for the simulation to be realistic, it need to capture the statistical correlations between different asset classes. One can also draw from the \( \Delta P \) distribution by picking price changes from random days in the actual past data, this is as realistic as possible and captures the correlations between different asset classes. We shall call this bootstrapped Monte-Carlo (BMC) and use it in this work [5].

A mixed stocks/bonds portfolio is beneficial in reducing the fluctuations over time, due to the fact that stocks and bonds are different asset classes that are somewhat anti-correlated [6, 7, 8]. Using backtesting and the Monte-Carlo method it was shown that leveraged stocks/bonds portfolios (risk parity) can boost the risk-adjusted returns [9, 10, 11, 12]. It is important to note that the future is not the same as the past and Monte-Carlo is limited that way [13], but if we want to make the least amount of assumptions about the future, this is the best method we have for quantifying future risk/reward and comparing between different strategies.

In this work we perform BMC simulations ourselves and quantify the performance of leveraged portfolios, using both LETFs and margin. In addition, we do the calculation for both tax-free (IRA account in Israel) and taxable accounts. We show how applying leverage on a mixed stocks/bonds portfolio can greatly amplify the yield without significantly increasing the risk.

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2 Data

In this chapter we present the raw data we use in this work.

2.1 Stocks

For stocks we use the SP500 and NDX100 indices. The autor is mostly interested in the NDX100 (due to its technology bias) which began in 1985, so we will not look at earlier years for both indices for simplicity. The indices data is from Yahoo [14], but SP500-TR (total return, including dividends) data is only from mid-1988, so we confine all the simulation to use data beginning at 1.1.1989. The maximal date of the data is 30.9.20, when the extraction was performed.

The SP500 and SP500-TR data:
Figure 1: Data for the SP500 index (price and total return) for the 1989-2020 period. A tax-free simulation is plotted in red.

We also show a simulation in red (discussed in chapter 3) which follows the SP500 price data and adds (yearly) dividends of 2% (for simplicity we calibrated a constant number for the whole tested period, and it fits pretty good for our purpose).

The NDX100 data is from Yahoo [14], but there is no corresponding NDX100-TR data. The best we found was [15], which begins in 1999, so prior to that we artificially created NDX100-TR as NDX100 + assumed 0.7% dividends:

Figure 2: Data for the NDX100 index (price and total return) for the 1989-2020 period. A tax-free simulation is plotted in red.

Again, a simulation is shown in red that perfectly fits the TR data (green, hidden behind the red). Before 1999 the simulation and data coincide by construction, and after 1999 the constant 0.7% dividend works well. The NDX100 in general has far less dividends so the effect is small.
2.2 Bonds

For bonds we use the long term treasury ETF VUSTX, since its inception is in 1986, prior to 1989. The data is again from Yahoo [14], where the price is the “Close” column and the total return is the “Adj Close” column:

![Figure 3: Data for the long term treasury bond ETF VUSTX (price and total return) for the 1989-2020 period. A tax-free simulation is plotted in red.](image)

First, we can see the significant contribution of the dividends to the total return of holding bonds. The simulation is in red, but the dividend model used is slightly more complicated than a constant percentage we used before. What we use is actually $5\% + 0.5 \times \{\text{1 month LIBOR rate}\}$, where the LIBOR rate is time dependent and taken from [16]:

![Figure 4: Data for the 1-month LIBOR rate for the 1986-2020 period.](image)

We can see it changed significantly over the years, which is why a constant % fit does not work well.
3 Portfolio Evolution Algorithms

In this chapter we describe the algorithms we use in the portfolio evolution simulations, as implemented in [17]. To validate the simulation, we test it against real data.

3.1 Buying papers

We model the purchases of shares as “papers” (for lack of a better name) of a specific stock type (the name of the ETF) which can have an arbitrary value, and not an integer number. This should not matter when the portfolio value is much larger than the cost of single share. Each paper’s value at purchase is logged since it matters for capital gains tax.

Buying and selling stocks costs transaction fee from the brokerage. We take this into account by applying a constant 0.1% fee of the value of each transaction, even though in real brokerages such as Interactive Brokers the fee scheme is not given in a percentage but in a more complicated form. We do not deal with this here and resort to a more simplistic approach, which in fact does not affect the main conclusions in a meaningful way.

The portfolio will be defined by the relative amounts of stocks and bonds. The ideal (or target) portfolio fractions are defined by $f^{ideal}_i$, so they satisfy $\sum_i f^{ideal}_i = 1$. We start off the portfolio by splitting the initial investment and buying $f^{ideal}_i$ from each stock type.

3.2 Price evolution

We simulate how each paper’s value evolves from day to day using closing prices data.

3.2.1 ETF evolution

ETFs follow some index that change in value (in percents) by $dp_{index}$. ETFs also have expenses, called expense ratio $ER$, which is some percentage per year. The fee is charged daily, and since there are 252 trading days a year, the paper value decreases daily by $ER/252$ percents. Overall, the daily price change of the ETF is:

$$dp = dp_{index} - \frac{ER}{252}$$

For the SP500 index we use the VOO ETF that has $ER = 0.03\%$ and for NDX100 we use QQQ ETF that has $ER = 0.2\%$, significantly higher compared to VOO. We do not have an “index” for bonds so we are using the VUSTX price itself as the index, which is an ETF with $ER = 0.05\%$.

3.2.2 Dividends

The dividend rate we use for the stocks and bonds were defined in chapter 2. For simplicity we add them to the cash balance of the account on a daily basis. So if the paper value is $V$ and yearly dividend rate (in percents) is $d$, the cash added daily due to this paper is $Vd/(100 \cdot 252)$.

Examples for how well the simulation of expense ratio and dividends compare to data were shown in chapter 2.

3.2.3 Cash reinvestment

Once a month we reinvest any available cash to buy additional papers, this cash can come either from dividends or from additional periodic investements. We do not simulate the latter in this work, but the option exists in the code [17].

As the portfolio evolves the portfolio fractions change dynamically from $f^{ideal}_i$ to $f_i$. As we reinvest we can try and rebalance by buying papers in different quantities. However, it is most likely that the available cash is not enough to properly rebalance (when the portfolio becomes large enough significant selling will be required to rebalance). We leave the rebalancing step to section 3.4, and simply split the cash by $f^{ideal}_i$ for each stock type.
3.2.4 Leveraged-ETF evolution

The leveraged-ETFs (LETFs) we discuss in this work amplify the daily returns of their underlying index. This comes at a price, the expense ratio is around $\sim 1\%$ (around 30 times higher than of VOO, and 5 times higher than of QQQ), and in addition the loans they take cost as well. In [9] it was estimated that we should use the LIBOR rate as an additional expense for every 100% of leverage. Meaning, if we define the (1-month) LIBOR rate (in percents, yearly) by $LR$, and the ETF leverage factor by $L_{ETF}$, then the additional expense rate is $LR(L_{ETF} - 1)$.

Another important note is that the LETFs are synthetic products that for the most part do not own the actual stocks and track the total return (TR) of the underlying index, meaning the dividends contribution is already priced in. In practice, they do own some stocks and so pay small amount of dividends, but we will neglect this in the simulation and assume no dividends are given.

Overall, the daily price change of the LETF is:

$$dp = dp_{index,TR}L_{ETF} - \frac{ER}{252} - \frac{LR(L_{ETF} - 1)}{252} \tag{2}$$

The leveraged ETFs we simulate are:

- For the SP500 index the 2X LETF is SSO, and the 3X LETF is UPRO (the underlying index of these LETFs is SP500-TR).
- For the NDX100 index the 2X LETF is QLD, and the 3X LETF is TQQQ (the underlying index of these LETFs is NDX100-TR).
- For the bonds, an existing alternative to VUSTX that has LETFs that leverage it is TLT, whose 2X LETF is UBT, and 3X LETF is TMF. We use the VUSTX ETF since it has the longest historic data, and call its fictitious LETFs by VUSTX2 and VUSTX3 (whose underlying index is VUSTX-TR).

The reason we need to simulate the LETFs rather than simply using existing data is because they are relatively new, and we want to “extend” them backwards to 1989. Also, when we perform Monte-Carlo simulations in chapter 4 we are going to generate synthetic realizations of stock histories for which we have to simulate how the LETF would evolve.

Examples We now validate our simulation against real LETF data to show it works well.

Simulation of TMF (assuming TLT-TR as the underlying index) against data, starting at 2011 (approximately when TMF started):
Figure 5: Data for the TLT and TMF ETFs for the 2010-2020 period. A tax-free simulation of TMF (based on total return of TLT) is plotted in red.

Simulation of UPRO against data, starting at 2010 (approximately when UPRO started):

Figure 6: Data for the SP500-TR index and UPRO ETF for the 2010-2020 period. A tax-free simulation of UPRO (based on SP500-TR) is plotted in red.

Simulation of TQQQ against data, starting at 2011 (approximately when TQQQ started):
Figure 7: Data for the NDX100 index and TQQQ ETF for the 2011-2020 period. A tax-free simulation of TQQQ (based on NDX100-TR) is plotted in red.

### 3.3 Margin

The more traditional way to leverage (compared to LETFs) is borrowing money to invest. This method is in principle better than of a daily LETF, because it does not suffer as much from fluctuation of the underlying index.

Interactive Brokers brokerage allows for a 2X leverage on margin (called Reg T margin), meaning you can borrow against 100% of your portfolio. However, during the trading day itself the leverage can change dynamically. If the portfolio value drops, the loan fraction of the total portfolio increases which means higher leverage. IBKR allows for 4X leverage (called maintenance margin) during the trading day, but at the end of the day the original 2X is a maximum. Meaning, any deviation beyond that limit will trigger a “margin call” which means the broker will sell some of the portfolio to bring it back to the 2X leverage limit [18].

Our simulations only deal with end of day data, so we do not care about the higher leverage allowed mid-day. Due to the fact that the leverage changes dynamically, we will target 1.8X leverage, to leave a ∼ 10% buffer to the maximum allowed leveraged. If the margin leverage deviates by 10% from the target, it will trigger a rebalance, which is effectively “margin calling” yourself. The algorithm for this is described in the next section.

The current cost of the loan in Interactive Brokers in 1.59% (yearly). So if the margin debt is $M$, each day the debt is increased by $M \cdot 1.59/252$ percents.

We now compare the performance of the two methods of 2X leverage on a backtest for the NDX100 index. For margin leverage, as the portfolio value change the margin leverage $L$ will change dynamically, so we define the target to be $L = 2$ with 10% allowed buffer prior to rebalance (in practice 2X leaves no buffer, but we simulate this here for fair comparison to the 2X LETF QLD). We also assume the margin simulation is tax-free even though it is not possible to use this method in a tax-free account (at least in Israel), again, for fair comparison.

The two simulations side by side (the yield is defined as the total portfolio size, minus the margin debt, which is roughly 100% in the case of 2X target leverage):
Figure 8: Tax-free simulations for 2X leverage on the NDX100 index for the period 1989-2020, using LETF (red) and margin (black). Margin debt is plotted in dashed black.

The margin leverage as a function of time:

Figure 9: Evolution of the margin leverage in the 2X margin leveraged NDX100 simulation.

We can see the margin leverage $L$ changes dynamically but does not exceed 10% from the target due to active rebalancing, to be discussed in section 3.4.

Same for SP500 index:
3.4 Rebalancing

So far we discussed the simulation of a single stock type. However, a mixed portfolio of stocks and bonds can be beneficial, since the two assets rise over the long term, yet somewhat anti-correlated, so can cover over each other during large price swings. We define a portfolio with some ideal (or target) fractions $f_{i}^{\text{ideal}}$, and as is evolves to $f_{i}$ that deviates too much from the target, we trigger a rebalance that will restore order. In our simulations we picked the deviation trigger as 20% for any stock (e.g. if we pick $f = 10\%$, it will trigger only at 30%, and if $f = 50\%$ it will trigger at 30% or 70%).

Define the total portfolio values as $T$, and each fraction value $T_{i}$ so therefore $f_{i} = T_{i}/T$. We want to transfer money around $T'_{i} = T_{i} + \Delta T_{i}$ to rebalance the portfolio to the ideal fractions:

$$f'_{i} = \frac{T'_{i}}{T} = f_{i}^{\text{ideal}} \quad (3)$$

Therefore:

$$\Delta T_{i} = T \left( f_{i}^{\text{ideal}} - f_{i} \right) \quad (4)$$

If $\Delta T_{i} > 0$ we need to buy the stock, and if $\Delta T_{i} < 0$ we sell. For a portfolio with two stock types (stocks and bonds) naturally $\Delta T_{\text{stocks}} = -\Delta T_{\text{bonds}}$.

In the margin leverage scenario, we have margin debt $M$, so the margin leverage is $L = \frac{T}{T - M}$. As we mentioned in section 3.3, the margin leverage can change dynamically as well, and if it deviates too much (we picked 10% from $L^{\text{ideal}}$) we also trigger a rebalance. If the ideal (or target) leverage is $L^{\text{ideal}}$, then the loan might need to change to $M' = M + \Delta M$, which also changes the total portfolio value $T' = T + \Delta M$:

$$L' = \frac{T'}{T' - M'} = \frac{T + \Delta M}{T - M} = L^{\text{ideal}} \quad (5)$$

Therefore:

$$\Delta M = L^{\text{ideal}} (T - M) - T = T \left( L^{\text{ideal}} \frac{L}{L} - 1 \right) \quad (6)$$

If no margin debt is allowed then $M = 0$ and of course $\Delta M = 0$ always.
During the rebalancing we can change both the component fractions to the ideal fractions, and fix the margin leverage simultaneously:

$$f'_i = \frac{T'_i}{T^0} = \frac{T_i + \Delta T_i}{T + \Delta M} = f^\text{ideal}_i$$

(7)

Therefore:

$$\Delta T_i = T (f^\text{ideal}_i - f_i) + f^\text{ideal}_i \Delta M$$

(8)

3.4.1 Examples

Backtest simulation of a 50%/50% non-leveraged VOO/VUSTX portfolio:

![Graph showing the evolution of fractions](image)

Figure 11: Tax-free simulation for a 50%/50% VOO/VUSTX portfolio in the period 1989-2020.

We can see intuitively that the stock component of the portfolio pulls it upwards, and the bond component reduces drawdown. The evolution of the fractions:
Figure 12: Evolution of the portfolio fractions of VOO and VUSTX in the 50%/50% VOO/VUSTX simulation.

We can see it indeed changes and triggers rebalance at the correct times.

Now move on to 2X leveraged mixed portfolios, with both leverage methods. For SP500 index:

Figure 13: Tax-free simulation for a 50%/50% 2X leveraged mixed SP500/VUSTX portfolios in the period 1989-2020, using LETFs (red) and margin (black).

For NDX100 index:
We can see in this backtest the leveraged portfolio gave higher yields than the non-leveraged portfolio, and also that the margin leverage (at the given margin rate) is more effective than the LETFs, because it is not vulnerable to daily fluctuations as much as LETFs.

3.5 Capital Gains Tax

In the previous section we saw margin leverage can be better than LETF leverage. However, the simulations made assumed no taxes exist. In reality (specific to Israel, at least) an investor with a tax-free IRA account can use LETFs but cannot use margin leverage, so it is only a theoretical exercise. In a normal taxable account, both leverage strategies can be used. But for a taxable account, gains must be tracked because capital gains tax needs to be paid for them. The tax needs to be paid only in the end of the year a paper is sold, so the tax can be deferred for many years which is very beneficial. In case the portfolio is composed of a single stock type, the investment is simply buy-and-hold and there is no need to worry about taxes during the simulation, except at the very end when we sell the entire portfolio. But for a rebalanced portfolio with several stock types, we have to sell papers from time to time, which is a negative effect that we aim to quantify.

3.5.1 Contributions to gains

The gains \( G \) need to be counted, and paid for in the end of the year. Then we reset \( G = 0 \) for the next year. However, if papers are sold at a loss at the end of the year, the losses \( G < 0 \) do not reset but can be carried over to the following years to cancel out future gains.

Gains come from two sources: dividends, and paper sells. All dividends count as gains so have to be tracked on a daily basis \( \Delta G_{\text{div}} = D \).

Rebalancing requires selling papers. Consider a paper with value \( V \) that has profit \( P \) (can be negative in which case it is a loss), the contribution to the gains depends on how much we need to sell \( \Delta T \). If \( \Delta T > V \), then the paper is sold out completely, the yearly gains increase by \( \Delta G = P \), and we move on to the next paper to complete the necessary sell amount. If \( \Delta T \leq V \) then only part of the paper is sold, but the amount sold will count as yearly gains. If the amount sold is larger than the profits then the gains will max out by the profits \( \Delta G = \text{sign}(P) \min(\Delta T, P) \).

To sum up:
\[
\Delta G_{\text{rebal}} = \begin{cases} 
\text{sign} (P) \min(\Delta T, P) & \Delta T \leq V \\
P & \Delta T > V 
\end{cases}
\] (9)

### 3.5.2 Treating end of year taxes

At the end of the year part of the portfolio needs to be sold to generate the cash needed for the tax. If the gains are negative (loss) \( G \leq 0 \), nothing needs to be done, and the loss carries over to the following year.

For positive gains \( G > 0 \), the capital gains tax fraction is \( cg_t = 0.25 \) (Israeli value), so the tax to pay is \( t = G \cdot cg_t \). At the end of the year the portfolio is reduced to \( T' = T - t \).

However, as we sell some of the papers, they generate additional profit or loss, which increase or decrease the total tax that is needed to be paid. Define the profit for the portfolio to be \( P \), so by definition \( P < T \). If we sell a portion \( \Delta T \) the updated tax is therefore:

\[
t' = (G + \Delta T) cg_t
\] (10)

\( \Delta T \) itself is the total amount to be sold so \( \Delta T = t' \). The solution to the equation is:

\[
\Delta T = t' = \frac{cg_t}{1 - cg_t} G = \frac{G}{3} = t^*
\] (11)

So we can see the necessity to pay tax amplifies itself if we have positive yearly gains \( G > 0 \). However, as we said, this enlarged tax is maxed out by the profitable part \( P \) of the portfolio. If \( t^* \leq P \) the solution above is valid, but after we sell off all the profit, the remainder does not incur further tax. So for \( t^* > P \) the tax is paid for \( G + P \):

\[
\Delta T = t' = (G + P) cg_t = \frac{G + P}{4}
\] (12)

To sum up the different cases, as a function of gains \( G \):

\[
\Delta T = \begin{cases} 
G \cdot \frac{cg_t}{1 - cg_t} & G \leq \frac{1 - cg_t}{cg_t} P \\
(G + P) cg_t & G > \frac{1 - cg_t}{cg_t} P 
\end{cases}
\] (13)

For a given portfolio profit \( P \), the amount to sell \( \Delta T \) increases linearly with the yearly gains \( G \), but once the sell amount surpasses the profit \( \Delta T = P \) it continues to increase linearly with \( G \) but with a reduced slope, which makes sense as we discussed.

In the generalized case the portfolio is composed of many individual papers and this needs to be done iteratively over all of them because they will all be profitable or lossy to different extents. This brings additional complications.

In the equations above we assumed we have positive profit \( P > 0 \), but let us say we have total positive gains \( G > 0 \) but the paper we are currently selling is lossy \( P < 0 \). We did not treat this case before.

In this case as we sell more it actually magnifies the loss, the opposite of what we had before, which is beneficial. Meaning

\[
\Delta T = t' = (G - \Delta T) cg_t
\]

The solution is

\[
\Delta T = t' = \frac{cg_t}{1 + cg_t} G = \frac{G}{5} = t^{**}
\] (14)

And again, the lossy sell is maxed out at \( t^{**} = |P| \), and for higher gains we have again:

\[
\Delta T = t' = (G + P) cg_t = \frac{G + P}{4}
\] (15)

Again, summing up the solution but for \( P < 0 \):

\[
\Delta T = \begin{cases} 
G \cdot \frac{cg_t}{1 + cg_t} & G \leq \frac{1 + cg_t}{cg_t} |P| \\
(G + P) cg_t & G > \frac{1 + cg_t}{cg_t} |P| 
\end{cases}
\] (16)
The solution for both profitability cases:

\[ G^* = \frac{1 - \text{sign}(P) \text{cgt}}{\text{cgt}} |P| \]  

\[
\Delta T = \begin{cases} 
G \frac{\text{cgt}}{1 - \text{sign}(P) \text{cgt}} & G \leq G^* \\
(G + P) \text{cgt} & G > G^* 
\end{cases}
\]  

Figure 15: The amount \( \Delta T \) that needs to be sold from a portfolio paper with profit \( P \), to pay for overall portfolio capital gains \( G \), as derived in Eq. (18).

If a specific paper we are looking at has value \( V \) that is larger than the total amount required to sell according to our equations \( V \geq \Delta T \), then all the tax necessary will be sold from that paper alone and we are done. Otherwise, we sell all of it, and continue to the next paper.

When we sell a paper completely, the difference \( \Delta T - V > 0 \) is the tax that remains to be paid. So we continue to the next paper with the gains updated to \( G = (\Delta T - V) / \text{cgt} \).

The order we choose to sell the papers is what we call the tax scheme. If we are free to choose which papers to sell, then the optimized scheme will sort the papers by least profitable to most profitable, and start selling the least profitable paper to minimize total paid tax. If we are not free to choose the order of papers, such as in Israeli brokers, we have to use the FIFO scheme, which sells the papers by the order they were bought.

In the simulation engine we keep of track of all the papers of a specific stock seperately. So when it is time to pay taxes for overall gains \( G \), we actually assign each stock type a separate gain \( fG \) (remember \( \sum f = 1 \)) and let each stock type to handle a separate chuck of gains to pay taxes for. This is a simplifying choice that is made due to our implementation of the simulation.

Summary of the algorithm:

1. We reach the end of year with gains \( G \). If \( G < 0 \) do nothing and continue to the next year with the losses saved for the following year.

2. If \( G > 0 \), we need to pay tax. Each stock type is composed of a set of papers, and is assigned gains \( fG \) to handle.

3. Sort the papers by the chosen tax scheme, where each paper has value and profit \( \{V_i, P_i\} \).

4. Start at paper \( i = 1 \).
5. Calculate the amount that needs to be sold to taxes $\Delta T$ using Eq. (18).

6. If $\Delta T \leq V_i$ we sell an amount $\Delta T$ from paper $i$ ($V'_i = V_i - \Delta T$), all taxes are fully paid. The procedure is done and we reset $G = 0$ for next year.

7. If $\Delta T > V_i$ we fully sell paper $i$ ($V'_i = 0$), update the gains to $G = (\Delta T - V) / cgt$

8. proceed to the next paper $i + 1$ from step (3).

Additnional notes:

- If we are using margin leverage, we can pay the tax by simply borrowing the required money and increasing the margin debt and margin leverage. To avoid complicating the algorithm and have a uniform treatment for both margin and margin-free strategies, we will sell papers to pay the tax, in the same method described in this section. This means the total portfolio size is reduced while the debt does not so the margin leverage increases in the process (until the next rebalancing).

- In any rebalanced taxable portfolio, but especially in a leveraged one, it is possible to reach a situation where rebalances during the year generated some gains, and then the market crashes so severly that at the end of the year selling the entire portfolio might not generate enough cash to pay off the tax. In that case you end up in a problem with the tax authority, so you have to take external loans or flee the country.

3.5.3 Examples

Example 1 Backtest simulation for 50%/50% VUSTX/VOO with taxes in the optimzed scheme:

![Graph showing tax simulation results](image)

Figure 16: Taxed simulation for 50%/50% VOO/VUSTX for the period 1989-2020. Except for the portfolio value (red) we also plot the tracked gains (cyan) and total paid taxes (purple).

As usual, in the plot we have the data for SP500-TR and VUSTX-TR, as well as the total portfolio value in the simulation. In addition, we track the gains as a function of time, and the cummulative taxes paid for those gains at the end of each year. The gains (or losses) are generated upon selling of papers during rebalances, or due to dividends. The “sawtooth” form of the gains mainly comes from dividends that accumulate during the year, and reset at the of each year.

Example 2 Backtest simulation for 50%/50% VUSTX3/UPRO (3X LETFs of bonds and SP500), with taxes in the optimized scheme:
Figure 17: Taxed simulation for 50%/50% 3X leveraged SP500/VUSTX for the period 1989-2020. Except for the portfolio value (red) we also plot the tracked gains (cyan) and total paid taxes (purple).

Since we consider LETFs where we assume the dividends are negligible, the gains/losses are generated solely due to rebalances. In the backtest above no rebalances were triggered until 1996, so there were no gains registered. The plot is focused on the gains so we do not see the final portfolio yield which is around $\sim 200$ at the end of the simulation period, equivalent to $\sim 18\%$ compound annual growth rate (CAGR).

Note that when the gains are negative, they are transferred year to year. In turns out that in this case using the FIFO tax does not give a very different outcome.

Example 3  Backtest of a 50%/50% VUSTX3/TQQQ (3X LETFs of bonds and NDX100):

Figure 18: Taxed simulation for 50%/50% 3X leveraged NDX100/VUSTX for the period 1989-2020. Except for the portfolio value (red) we also plot the tracked gains (cyan) and total paid taxes (purple).

We can see in this example that taxes can hit really hard. The market surge at the year 2000 triggered
many rebalances which in turn generate gains. These massive gains require tax to be paid for them, even though the market came crashing down after than, causing the portfolio to shrink. It important to note that the simulation outcome is highly sensitive to the rebalancing trigger. As we discussed in section 3.4, we trigger a rebalance at 20% deviation from the 50%/50% fractions. Example for the evolution at different rebalancing deviation triggers:

![Figure 19: Taxed simulations for 50%/50% 3X leveraged NDX100/VUSTX for the period 1989-2020, with slightly different rebalancing triggers.](image)

If the simulation were tax-free, there would still be differences, but not as extreme. Backtests are nice, but we cannot depend on them when comparing different strategies for the future. As was just demonstrated, even arbitrary parameters such as the rebalance trigger can vastly impact the result.

4 Bootstrapped Monte-Carlo Simulations

As discussed in the introduction, we predict the yield of a given portfolio in a probabilistic way by assuming the daily price changes of the past represent the future.

4.1 Method

In the bootstrapped Monte-Carlo (BMC) method, we generate possible realizations of multiple year periods of stock histories by stitching together daily price changes (randomly sampled with repetitions) from the past data (defined in chapter 2). Consectutive days in the stock market are mostly uncorrelated (the efficient market drives it to this state), but when we sample days from the past we do it in batches of 5 consecutive days to preserve any real correlations that do appear in the real stock market (sampling single days changes the results but not in a meaningful way). Also, we want to preserve correlations between different assets in the same day (specifically the very important anti-correlation of stocks and bonds) so when we samples a random day from the past, we take the price changes for both stocks and bonds from that day (and also the LIBOR rate, for good measure). Examples of several random 10-year realizations of the SP500:
4.2 Risk/Reward Metrics

Define the portfolio yield as a function of time as \( y(t) \). A long term passive investor needs to define the time horizon of the investment \( t_{\text{end}} \). During that time the investment should be considered locked, “black box”, to be accessed only in the end. Therefore, given the portfolio final yield probability distribution \( y(t_{\text{end}}) \), the statistical measure for reward metric will be defined as the 50% percentile of the distribution (median value), and the risk metric defined as the 5% percentile.

Mostly in the literature, risk is measured during the investment period itself as the volatility of \( y(t) \) (e.g. Sharpe ratio). This is not a rational risk metric, but a psychological risk metric. Nevertheless, we calculate two psychological risk metrics for comparison. One will be the (5% percentile of the) minimal yield during the investment period \( \min_t (y(t)) \) (how low will the portfolio get if you constantly monitor it), and the second will be the (50% percentile of the) maximal drawdown (maximal drop of the portfolio from any new high point reached during a portfolio’s evolution) [19].

As an example, we performed 2000 realizations of 10-year simulation of several portfolios, and saved the outputs above for each of them. The portfolios are 100% SP500 with 1X, 2X and 3X leverage. A histogram of the resulting yields \( y(t_{\text{end}}) \) (left subplot), minimal yields \( \min_t (y(t)) \) (middle subplot) and maximal drawdowns (right subplot).
Figure 21: Probability distributions of the final yield, minimal yield and maximal drawdown, generated using BMC for portfolios of 100% SP500 with 1X, 2X and 3X leverage.

The legend in the middle subplot indicates the color for each portfolio. On the left plot we have the yield distribution for each portfolio. We can see that increasing the leverage elongates the high yield tail of the distribution, but the low yields are getting heavier as well. The percentiles of the yield distributions are what matters, and the legend indicates the risk/reward metrics (5% and 50% percentiles). We see that increasing the leverage increases both the (rational) risk and the reward. Investing for a 10 year period in the unleveraged portfolio gives less than 5% chance for ending up with less money than was invested (we do not take inflation into account). With the leveraged portfolio, the chance for ending up with less money is much higher. Looking at the two other plots of the psychological risk metric, we see that they too give higher risk to the leveraged portfolios.

Next, we repeat the analysis for 50%/50% portfolios of SP500 and bonds, with 1X, 2X and 3X leverage:

Figure 22: Probability distributions of the final yield, minimal yield and maximal drawdown, generated using BMC for portfolios of 50%/50% portfolios of SP500 and bonds, with 1X, 2X and 3X leverage.

Now the picture is much different. The mixed unleveraged portfolio has about the same reward as before, but with reduced risk. Increasing leverage boosts the reward, and surprisingly the risk almost
does not change (increases but much less than before). The mixed 3X leveraged portfolio beats the nonleveraged 100% SP500 portfolio on both risk and reward, by a vast amount. The leverage does increase the psychological risk metrics, meaning the investment is a much more bumpy ride, but the (rational) end result is much better.

4.3 Numeric Uncertainty

Since we do not have the analytic forms of the probability distribution of the final yield (or the other metrics from section 4.2), but only finite size samples of them, the calculated risk/reward metric have a numeric uncertainty to them. We quantify this uncertainty using bootstrap: the Monte-Carlo sampling yielded a total of $N = 2000$ samples $\{y\}_{i=1}^{N}$ that we have to work with. We perform another Monte-Carlo where in each realization we pick $N$ random samples (with possible repetition) out of $\{y\}_{i=1}^{N}$ and recalculate the metric we are interested in $y_{\text{metric}}$ (such as the reward metric which is 50% percentile). Repeating this $M = 300$ times we end up with a distribution of those metrics $\{y_{\text{metric}}\}_{i=1}^{M}$ and we define the uncertainty in $y_{\text{metric}}$ as the 32%-68% confidence interval (analogous to a ±1σ interval). This interval is what we draw for each portfolio in section 4.4. As $N$ increases the numeric uncertainty decreases, and for our purposes the calculated uncertainty is low enough to reach meaningful conclusions.

4.4 Results

In this section we repeat the Monte-Carlo calculations from section 4.2 for a variety of portfolios with different fractions of stocks and bonds (5% intervals between 0% and 100% stocks). Each color in the following plots deals with two different stock types, where the number on the top right of each point indicates the percent of stocks within the portfolio. The portfolio combinations we explore are SP500/bonds and NDX100/bonds, for an unleveraged, 2X and 3X LETFs portfolios, and 1.8X margin leveraged portfolio. In the tax-free simulations the margin leverage is purely theoretical since it cannot be implemented (at least in Israeli IRA accounts).

All plots share the same y-axis which is the reward metric, on the left y-axis it is the final yield 50% percentile after the investment period, and on the right y-axis it is the associated compound annual growth rate (CAGR). The x-axis of each plot is a different risk metric, on the left plot it is the final yield 5% percentile (the rational risk metric), in the middle plot it is the minimal yield 5% percentile, and on the right plot it is the maximal drawdown 50% percentile.

Our benchmark investment period will be 10 years, and we compare the outcomes between tax-free and taxed investments. We then follow up with the outcomes for 5 and 20 years tax-free investments, as well as a “bearish” scenario.
4.4.1 Tax free (10 years)

First we examine the unleveraged portfolios (in black). We can see that a 100% stock portfolio has a higher risk and reward compared to a 100% bond portfolio. The mixed portfolio does not lie on the interpolation line between these two but rather has a higher risk-adjusted reward [7]. The NDX100 gives higher reward than the SP500, for roughly the same risk. The least risky portfolio (measured by the rational risk metric) is somewhere in the range of 25%-45% stocks allocation.

Moving on to the leveraged portfolios, we can see dramatically higher rewards in the mixed portfolios. Comparing the least risky portfolios in each set, we see that LETF portfolios (in green and red) do increase the (rational) risk but not in a significant way. If our benchmark were a 100% unleveraged stock portfolio, the leveraged mixed portfolios can outperform it at the same risk. Examining the psychological risk metrics, we can see the leveraged portfolios increase those metrics more significantly.
It is also interesting to compare the 100% stock portfolios in terms of reward. For the SP500 the 3X leveraged portfolio beats the unleveraged portfolio, but it is the opposite for NDX100. This is (probably) because the NDX100 data has higher volatility. Both cases are horrible in terms of risk. It is quite remarkable how adding bonds to a leveraged portfolio turns the conclusions on their head. The margin leveraged portfolio (in blue) is interesting because it allows to outperform the unleveraged portfolio in both risk and reward. This is because of its ability to overcome small daily fluctuations, unlike the daily-leveraged ETFs. We note again that for a tax-free account (in Israel) this is not a possible portfolio anyway.

For a long term (10 year) investor happy with the (rational) risk of 100% stocks (SP500 or NDX100) unleveraged portfolio, switching to a 3X leveraged portfolio with around 35%-45% stocks allocation can significantly boost the reward, while reducing the (rational) risk at the same time. If the same investor does not wish to take on more psychological risk than what the 100% unleveraged stocks portfolio offers, then switching to a 2X leveraged portfolio (40%-60% SP500 allocation and 35%-45% NDX100 allocation) can deliver higher reward with lower risk (both rational and psychological).

4.4.2 With taxes (10 years)

Figure 25: SP500/bonds portfolio metrics for a 10 years taxed investment.
We can see that taxes reduce the reward (and increase the risk), but if one wishes to invest in a taxable account, it is still meaningful to ask how leverage can improve the performance of the portfolio. We can see that the benefits of leverage using LETFs did not change, meaning that the tax events generated by rebalancing were not significant (compared to the unleveraged case). On the other hand, taxes significantly hurt the performance of margin leverage. For high stock allocation (over 80% SP500 or over 60% NDX100) the risk is so high that the yield can even go negative (capital gains taxes need to be paid but the portfolio is null). In lower stock allocations 10%-50% the 1.8X margin leveraged portfolios beats the 2X LETF portfolio on both reward and risk (rational and psychological).

The conclusions from the previous section on the benefits of the 3X leveraged portfolio remain unchanged, even when taxes are taken into account.

### 4.4.3 Tax free (5 years)

![Figure 27: SP500/bonds portfolio metrics for a 5 years tax-free investment.](image-url)
final yield 5% percentile
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
final yield 50% percentile

We can see that as the investment period is shorter (than 10 years), the LETF portfolios have a higher (rational) risk. However, a leveraged mixed portfolio can still beat the unleveraged 100% stock portfolio on both risk and reward.

4.4.4 Tax free (20 years)

Figure 29: SP500/bonds portfolio metrics for a 20 years tax-free investment.
In an even longer investment horizon, the opposite happens and the leveraged portfolios have an even lower (rational) risk and can outperform even the least risky unleveraged portfolio, on both risk and reward. 

Note that the reward for different investments periods (5, 10 and 20 years) is vastly different when looking at the final yield, but it corresponds to roughly the same CAGR, which is expected.

### 4.4.5 Bearish scenario

The optimistic results we got for the yields are only as good as the data they were generated from. Meaning, if the daily price change $\Delta P$ is a probability distribution whose mean is negative, the investment will not bring profits. The data we used in this work is from the period 1989-2020. Narrowing the scope around the period 1999-2009 that contains the two major stock crashes of the last 30 years would give much worse results, where stocks do not give positive yields (bonds still do).

We define a “bearish” scenario using the period 1997-2011. Results for tax free 10 years investments for the bearish scenario:

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Figure 30: NDX100/bonds portfolio metrics for a 20 years tax-free investment.

Person 31: SP500/bonds portfolio metrics for a 10 years tax-free investment, using data from the period 1997-2011 as a “bearish” scenario.
We can see that in the non-leveraged portfolios, 100% stocks are worse than 100% bonds on both risk and reward, and adding up to \( \sim 35\% \) stocks is approximately neutral compared to 100% bonds. This is expected in the “bearish” scenario. Moving to the leveraged LETF portfolios, we can see higher reward are possible, albeit with higher risk. However, it can still beat a 100% stocks portfolio, meaning give higher reward at the same risk.

5 Conclusions

In this work we performed bootstrapped Monte-Carlo simulations of leveraged (and unleveraged) mixed portfolios of stocks and bonds, based on past stock market data (stocks represented by the SP500 and NDX100 indices, and bonds by the long term treasury ETF VUSTX). We showed that leverage (using margin debt or leveraged-ETFs) can significantly amplify the yield of an investment, without significantly increasing the risk relevant for long-term investors. The rational risk we refer to is low yield at the end of the investment period. However, the leveraged portfolios are more volatile and therefore have higher psychological risks.

Our results show that for a long term (> 10 year term) investor that is happy to take on the risk of 100% stocks (SP500 or NDX100) unleveraged portfolio, switching to a 3X leveraged portfolio (using leveraged-ETFs) with around 35%-45% stocks allocation (SP500 or NDX100) can significantly boost the reward, while reducing the (rational) risk at the same time.

If the same investor does not wish to take on more psychological risk than what the 100% unleveraged stocks portfolio offers, then switching to a 2X leveraged portfolio (40%-60% SP500 allocation and 35%-45% NDX100 allocation) can deliver higher reward with lower risk (both rational and psychological). We simulate both tax-free and taxable account, and show the conclusions do not change for a taxable accounts, although the yields are obviously lower due to the capital gains taxes (25% tax on profits). In a taxable account leverage can also be achieved using margin debt, and we show that a 1.8X margin leveraged portfolio (which can be practically implemented) slightly outperforms the 2X LETF portfolio, on both risk and reward.

Overall, comparing the two stocks indices, NDX100 outperforms SP500 on both risk and reward.
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