Discrete Lebedev’s index transforms

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Abstract

Discrete analogs of the Lebedev transforms with the product of the modified Bessel functions are introduced and investigated. Several expansions of suitable functions and sequences in terms of the series and integrals, involving the modified and incomplete Bessel functions are established.

Keywords: modified Bessel function, Macdonald function, incomplete Bessel function, Struve functions, Fourier series, index transforms

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1 Introduction and preliminary results

In 1962 N.N. Lebedev proved (cf. [2]) the following expansion for all $x > 0$

\[ f(x) = -\frac{4}{\pi^2} \frac{d}{dx} \int_0^\infty \tau \sinh(\pi \tau) K_{2i\tau}^2(x) \int_0^\infty K_{i\tau}(y) [I_{i\tau}(y) + I_{-i\tau}(y)] f(y) dy d\tau, \quad (1.1) \]

where $I_{\nu}(z), K_{\nu}(z)$ are modified Bessel functions of the first and second kind, respectively, and $x^{-1/2}f(x) \in L_1(0,1), \ x^{1/2}f(x) \in L_1(1,\infty)$. It generates the reciprocal pair of the index transforms [5] with the product of the modified Bessel functions

\[ g(\tau) = \int_0^\infty K_{i\tau}(y) [I_{i\tau}(y) + I_{-i\tau}(y)] f(y) dy, \quad (1.2) \]

\[ f(x) = -\frac{4}{\pi^2} \frac{d}{dx} \int_0^\infty \tau \sinh(\pi \tau) K_{2i\tau}^2(x) g(\tau) d\tau. \quad (1.3) \]
The main aim of the present paper is to introduce discrete analogs of the Lebedev transforms (1.2), (1.3) and study their mapping and inversion properties. Precisely, we will investigate the following transformations

\[ a_n = \frac{\pi}{2 \cosh(\pi n/2)} \int_0^\infty K_{in/2}(y) \left[ I_{in/2}(y) + I_{-in/2}(y) \right] f(y) dy, \quad n \in \mathbb{N}, \quad (1.4) \]

\[ a_n = \frac{1}{2} \int_0^\infty K_{in/2}^2(y) f(y) dy, \quad n \in \mathbb{N}, \quad (1.5) \]

\[ f(x) = \frac{\pi}{2} \sum_{n=1}^\infty \frac{a_n}{\cosh(\pi n/2)} K_{in/2}(x) \left[ I_{in/2}(x) + I_{-in/2}(x) \right], \quad x > 0, \quad (1.6) \]

\[ f(x) = \frac{1}{2} \sum_{n=1}^\infty a_n K_{in/2}^2(x), \quad x > 0. \quad (1.7) \]

In the sequel, inversion theorems will be proved for transformations (1.4)-(1.7), employing the theory of the discrete Kontorovich-Lebedev transform recently developed by the author (see [7]).

As is known, the modified Bessel function of the second kind or Macdonald function \( K_\nu(z) \) is represented, for instance, by the integral (cf. [4], Vol. I, Entry 2.4.18.4)

\[ K_\nu(z) = \int_0^\infty e^{-z \cosh(u)} \cosh(\nu u) du, \quad \text{Re} z > 0, \quad \nu \in \mathbb{C}. \quad (1.8) \]

It satisfies the ordinary differential equation

\[ z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - (z^2 + \nu^2) u = 0, \quad (1.9) \]

for which it is the solution that remains bounded as \( z \) tends to infinity on the real line. It has the asymptotic behavior [6]

\[ K_\nu(z) = \left( \frac{\pi}{2z} \right)^{1/2} e^{-z} [1 + O(1/z)], \quad z \to \infty, \quad (1.10) \]

and near the origin

\[ K_\nu(z) = O(z^{-|\nu|}), \quad z \to 0, \quad (1.11) \]

\[ K_0(z) = -\log z + O(1), \quad z \to 0. \quad (1.12) \]

Worth mentioning is the Lebedev inequality for the modified Bessel function (see [6], p.219)

\[ |K_{i\tau}(x)| \leq A \frac{x^{-1/4}}{\sqrt{\sinh(\pi \tau)}}, \quad x, \tau > 0, \quad (1.13) \]
where $A > 0$ is an absolute constant. The kernels of discrete transforms (1.4)-(1.7) have representations in terms of the Erdélyi-Kober integrals [6] (see [4], Vol. II, Entries 2.16.3.1, 2.16.3.6)

$$\frac{\pi}{2 \cosh(\pi n/2)} K_{in/2}(x) \left[ I_{in/2}(x) + I_{-in/2}(x) \right] = \int_0^x \frac{K_{in}(t)}{(t^2 - x^2)^{1/2}} \, dt,$$

(1.14)

$$\frac{1}{2} K^2_{in/2}(x) = \int_x^\infty \frac{K_{in}(t)}{(t^2 - x^2)^{1/2}} \, dt.$$  

(1.15)

In the sequel we will also employ the modified Struve functions $M_\nu(z)$, $L_\nu(z)$, having the following relation with the modified Bessel function of the first kind (cf. [3], Entry 11.2.6)

$$M_\nu(z) = L_\nu(z) - I_\nu(z),$$  

(1.16)

and $L_\nu(z) = -ie^{-\pi i\nu/2}H_\nu(iz)$, where $H_\nu(z)$ is the Struve function (see [3], Entry 11.2.2).

## 2 Inversion theorems

We begin with

**Theorem 1.** Let $f$ be a complex-valued function on $\mathbb{R}_+$ which is represented by the integral

$$f(x) = \frac{2x}{\pi} \int_{-\pi}^\pi K_0(x \cosh(u)) \varphi(u) \cosh(u) \, du, \quad x > 0,$$

(2.1)

where $\varphi(u) = \psi(u) \sinh(u)$ and $\psi$ is a $2\pi$-periodic function, satisfying the Lipschitz condition on $[-\pi, \pi]$, i.e.

$$|\psi(u) - \psi(v)| \leq C |u - v|, \quad \forall \, u, v \in [-\pi, \pi],$$

(2.2)

where $C > 0$ is an absolute constant. Then the following inversion formula for transformation (1.4) holds

$$f(x) = \frac{1}{\pi^3} \sum_{n=1}^\infty \sinh(\pi n) \Phi_n(x) a_n, \quad x > 0,$$

(2.3)

where

$$\Phi_n(x) = x \int_{-\pi}^\pi K_0(x \cosh(u)) \sinh(2u) \sin(nu) \, du, \quad x > 0, \quad n \in \mathbb{N}.$$  

(2.4)
Proof. Plugging the right-hand side of the representation (1.14) in (1.4), we change the order of integration to obtain

\[ a_n = \int_0^\infty K_{in}(t) \int_0^\infty \frac{f(y)}{(y^2 - t^2)^{1/2}} \, dy \, dt. \]  

(2.5)

The justification of this interchange comes from the Fubini theorem via the use of Hölder’s inequality, generalized Minkowski’s inequality, asymptotic behavior (1.10)-(1.12) for the modified Bessel function and integral (2.1). In fact, we derive

\[
\int_0^\infty |K_{in}(t)| \int_t^\infty \frac{|f(y)|}{(y^2 - t^2)^{1/2}} \, dy \, dt \leq \left( \int_0^\infty \left( \int_t^\infty \frac{|f(y)|}{(y^2 - t^2)^{1/2}} \, dy \right)^p \, dt \right)^{1/p} \\
\times \left( \int_0^\infty K_0^q(t) \, dt \right)^{1/q} \leq \int_0^\infty |f(y)| \, y^{-1/2} \, dy \left( \int_0^1 \frac{1}{(1 - t^2)^{p/2}} \, dt \right)^{1/p} \\
\times \left( \int_0^\infty K_0^q(t) \, dt \right)^{1/q} \leq \left( \int_0^1 \frac{1}{(1 - t^2)^{p/2}} \, dt \right)^{1/p} \left( \int_0^\infty K_0^q(t) \, dt \right)^{1/q} \\
\times \frac{1}{\pi} \int_0^\infty y^{1/p} K_0(y) \, dy \int_{-\pi}^{\pi} |\varphi(u)||\cosh(u)|^{-1/p} \, du \\
= \frac{1}{2\pi} B^{1/p} \left( \frac{1}{2}, 1 - \frac{p}{2} \right) \Gamma^2 \left( \frac{1}{2} \left( 1 + \frac{1}{p} \right) \right) \left( \int_0^\infty K_0^q(t) \, dt \right)^{1/q} \\
\times \int_{-\pi}^{\pi} |\varphi(u)||\cosh(u)|^{-1/p} \, du < \infty,
\]

where \( 1 < p < 2 \), \( q = p/(p - 1) \) and \( B(a, b) \), \( \Gamma(z) \) are Euler’s beta and gamma functions, respectively, [1]. For the same reasons the inner integral with respect to \( y \) in (2.5) can be written, using (2.1), Entry 3.14.1.9 in [1] and particular cases of the hypergeometric function \( {}_0F_1 \) (cf. [5], Vol. II), in the form

\[
\int_t^\infty \frac{f(y)}{(y^2 - t^2)^{1/2}} \, dy = \frac{2}{\pi} \int_{-\pi}^{\pi} \varphi(u) \cosh(u) \int_t^\infty \frac{yK_0(y \cosh(u))}{(y^2 - t^2)^{1/2}} \, dy \, du \\
= \frac{2}{\pi} \int_{-\pi}^{\pi} \varphi(u) \cosh(u) \left[ -\frac{\pi t}{2} {}_0F_1 \left( \frac{3}{2}; \left( \frac{t \cosh(u)}{2} \right)^2 \right) \\
+ \frac{\pi}{2 \cosh(u)} {}_0F_1 \left( \frac{1}{2}; \left( \frac{t \cosh(u)}{2} \right)^2 \right) \right] \, du
\]
\[= \frac{2}{\pi} \int_{-\pi}^{\pi} \varphi(u) \cosh(u) \left[ -\frac{\pi}{2 \cosh(u)} \sinh(t \cosh(u)) \right. \\
+ \frac{\pi}{2 \cosh(u)} \cosh(t \cosh(u)) \left. \right] du = \int_{-\pi}^{\pi} e^{-t \cosh(u)} \varphi(u) du. \]

Therefore, following the same scheme as in the proof of Theorem 5 in [7], we return to (2.5), substituting the latter expression, changing the order of integration owing to the absolute and uniform convergence and employ the formula (see [4], Vol. II, Entry 2.16.6.1)

\[\int_{0}^{\infty} e^{-x \cosh(u)} K_{in}(x) dx = \frac{\pi \sin(nu)}{\sinh(u) \sinh(\pi n)} \] (2.6)

to get finally

\[a_n = \frac{\pi}{\sinh(\pi n)} \int_{-\pi}^{\pi} \varphi(u) \frac{\sin(nu)}{\sinh(u)} du. \] (2.7)

Let \( S_N(x) \) denote a partial sum of the series (2.3). Then, substituting the value of \( a_n \) by integral (2.7) and \( \Phi_n(x) \) by (2.4), it gives

\[S_N(x) = \frac{x}{4\pi^2} \sum_{n=1}^{N} \int_{-\pi}^{\pi} K_0(x \cosh(t)) \sinh(2t) \sin(nt) dt \int_{-\pi}^{\pi} \frac{\varphi(u)}{\sinh(u)} \sin(nu) du. \] (2.8)

Hence, calculating the sum via the known identity and invoking the definition of \( \varphi \), equality (2.8) becomes

\[S_N(x) = \frac{x}{4\pi^2} \int_{-\pi}^{\pi} K_0(x \cosh(t)) \sinh(2t) \int_{-\pi}^{\pi} \frac{\varphi(u) + \varphi(-u)}{\sinh(u)} \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} dudt \]
\[= \frac{x}{4\pi^2} \int_{-\pi}^{\pi} K_0(x \cosh(t)) \sinh(2t) \int_{-\pi}^{\pi} \left[ \psi(u) - \psi(-u) \right] \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} dudt. \] (2.9)

Since \( \psi \) is \( 2\pi \)-periodic, we treat the latter integral with respect to \( u \) as follows

\[\int_{-\pi}^{\pi} \left[ \psi(u) - \psi(-u) \right] \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} du \]
\[= \int_{-\pi}^{t+\pi} \left[ \psi(u) - \psi(-u) \right] \frac{\sin((2N+1)(u-t)/2)}{\sin((u-t)/2)} du \]
\[\int_{t-\pi}^{\infty} e^{-x \cosh(u)} K_{in}(x) dx = \frac{\pi \sin(nu)}{\sinh(u) \sinh(\pi n)} \] (2.6)
Moreover, 

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(u - t) \right] \frac{\sin (\frac{(2N + 1)u}{2})}{\sin (u/2)} \, du \]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(t) + \psi(-t) - \psi(u - t) \right] \frac{\sin (\frac{(2N + 1)u}{2})}{\sin (u/2)} \, du.
\]

When \( u + t > \pi \) or \( u + t < -\pi \) then we interpret the value \( \psi(u + t) - \psi(t) \) by formulas

\[
\psi(u + t) - \psi(t) = \psi(u + t - 2\pi) - \psi(t - 2\pi),
\]

\[
\psi(u + t) - \psi(t) = \psi(u + t + 2\pi) - \psi(t + 2\pi),
\]

respectively. Analogously, the value \( \psi(-u - t) - \psi(-t) \) can be treated. Then due to the Lipschitz condition (2.2) we have the uniform estimate for any \( t \in [-\pi, \pi] \)

\[
|\psi(u + t) - \psi(t) + \psi(-t) - \psi(-u - t)| \leq 2C \left| \frac{u}{\sin (u/2)} \right|.
\]

Therefore, owing to the Riemann-Lebesgue lemma

\[
\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(u - t) - \psi(t) + \psi(-t) \right] \frac{\sin (\frac{(2N + 1)u}{2})}{\sin (u/2)} \, du = 0 \quad (2.10)
\]

for all \( t \in [-\pi, \pi] \). Besides, returning to (2.9), we estimate the iterated integral

\[
\int_{-\pi}^{\pi} K_0(x \cosh(t)) \sinh(2t) \left| \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t) \right] \frac{\sin (\frac{(2N + 1)u}{2})}{\sin (u/2)} \, du \right| \, dt
\]

\[
\leq 4C \int_{0}^{\pi} K_0(x \cosh(t)) \sinh(2t) \, dt \int_{-\pi}^{\pi} \left| \frac{u}{\sin (u/2)} \right| \, du < \infty, \quad x > 0.
\]

Consequently, via the dominated convergence theorem it is possible to pass to the limit when \( N \to \infty \) under the integral sign, and recalling (2.10), we derive

\[
\lim_{N \to \infty} \frac{x}{4\pi^2} \int_{-\pi}^{\pi} K_0(x \cosh(t)) \sinh(2t) \int_{-\pi}^{\pi} \left[ \psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t) \right] \, du.
\]
\[
\times \frac{\sin((2N + 1)u/2)}{\sin(u/2)} \frac{dudt}{2} = \frac{x}{4\pi^2} \int_{-\pi}^{\pi} K_0(x \cosh(t)) \sin(2t) \times \lim_{N \to \infty} \int_{-\pi}^{\pi} [\psi(u + t) - \psi(-u - t) - \psi(t) + \psi(-t)] \frac{\sin((2N + 1)u/2)}{\sin(u/2)}dudt = 0.
\]

Hence, combining with (2.9), we obtain by virtue of the definition of \( \varphi \) and \( f \)
\[
\lim_{N \to \infty} S_N(x) = \frac{x}{\pi} \int_{-\pi}^{\pi} K_0(x \cosh(t)) \cosh(u) [\varphi(t) + \varphi(-t)] dt = f(x),
\]
where the integral (2.1) converges since \( \varphi \in C[-\pi, \pi] \). Thus we established (2.3), completing the proof of Theorem 1.

Concerning transformation (1.5), we have

**Theorem 2.** Let \( f \) be a complex-valued function on \( \mathbb{R}_+ \) which is represented by the integral
\[
f(x) = \int_{-\pi}^{\pi} \varphi(u) \left[ \frac{2}{\pi} + x \cosh(u) \right] du, \quad x > 0,
\]
where \( \varphi(u) = \psi(u) \sinh(u) \) and \( \psi \) is a \( 2\pi \)-periodic function, satisfying the Lipschitz condition (2.2). Then the following inversion formula for transformation (1.5) takes place
\[
f(x) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \sinh(\pi n) \Psi_n(x) a_n, \quad x > 0,
\]
where
\[
\Psi_n(x) = \int_{-\pi}^{\pi} \left[ \frac{2}{\pi} + x \cosh(u) \right] \sinh(u) \sin(nu) du, \quad x > 0, \quad n \in \mathbb{N}.
\]

**Proof.** In the same manner as in the proof of Theorem 1 we substitute the right-hand side of (1.15) into (1.5), changing the order of integration. Hence we get
\[
a_n = \int_{0}^{\infty} K_{in}(t) \int_{0}^{t} \frac{f(y)}{(t^2 - y^2)^{1/2}} dy dt.
\]

Noting the integral representation for the modified Struve function (cf. [3], Entry 11.5.4)
\[
M_0(z) = -\frac{2}{\pi} \int_{0}^{\pi/2} e^{-z \cos(\theta)} d\theta,
\]
we easily find the estimate $|M_0(z)| \leq 1$, $\Re z \geq 0$. Hence the motivation of the interchange in (2.14) follows immediately from (2.11) and Fubini’s theorem because

\[
\int_0^\infty |K_m(t)| \int_0^t \frac{|f(y)|}{(t^2 - y^2)^{1/2}} \, dy \, dt \leq \frac{2}{\pi} \int_{-\pi}^{\pi} |\varphi(u)| \, du \int_0^\infty K_0(t) \, dt \int_0^1 \frac{1}{(1 - y^2)^{1/2}} \, dy
\]

\[
+ \int_{-\pi}^{\pi} |\varphi(u)| \cosh(u) \, du \int_0^\infty K_0(t) \, dt \int_0^1 \frac{y}{(1 - y^2)^{1/2}} \, dy
\]

\[
= \int_{-\pi}^{\pi} |\varphi(u)| \left[ \frac{\pi}{2} + \cosh(u) \right] \, du < \infty.
\]

Meanwhile, the inner integral with respect to $y$ in (2.14) can be represented via (1.16), (2.11) and Entries 3.13.1.2, 3.15.1.4 in [1]. Therefore, invoking particular cases of the hypergeometric functions $0F_1$, $1F_2$ (see [3], Vol. III, Entries 7.13.1.6, 7.14.2.78) we find

\[
\int_0^t \frac{f(y)}{(t^2 - y^2)^{1/2}} \, dy = \int_{-\pi}^{\pi} \varphi(u) \left[ 1 - \cosh(u) \right] \int_0^t \frac{I_0(y \cosh(u)) y}{(t^2 - y^2)^{1/2}} \, dy
\]

\[
+ \cosh(u) \int_0^t \frac{L_0(y \cosh(u)) y}{(t^2 - y^2)^{1/2}} \, dy \right] \, du = \int_{-\pi}^{\pi} \varphi(u) \left[ 1 - t \cosh(u) \right] 0F_1 \left( \frac{3}{2}, \left[ \frac{t \cosh(u)}{2} \right] \right)
\]

\[
+ \frac{[t \cosh(u)]^2}{2} 1F_2 \left( 1; \frac{3}{2}, 2; \left[ \frac{t \cosh(u)}{2} \right]^2 \right) \right] \, du = \int_{-\pi}^{\pi} \varphi(u) \left[ \cosh(t \cosh(u)) - \sinh(t \cosh(u)) \right] \, du = \int_{-\pi}^{\pi} e^{-t \cosh(u)} \varphi(u) \, du.
\]

(2.15)

Thus, returning to (2.14) and appealing to (2.6), the discrete transformation (1.5) takes the form (2.7). Now in the same manner as in the proof of Theorem 1 we derive inversion formula (2.12) with the kernel (2.13).

\[\Box\]

The following result establishes the inversion formula for the discrete transformation (1.6). Indeed, we have

**Theorem 3.** Let the sequence $\{a_n\}_{n \in \mathbb{N}}$ be such that the following series converges

\[
\sum_{n=1}^{\infty} |a_n| e^{-\pi n/2} < \infty.
\]

(2.16)
Then the discrete transformation (1.6) can be inverted by the formula

\[ a_n = \frac{1}{\pi^3} \sinh(\pi n) \int_0^\infty \Phi_n(x) f(x) dx, \quad n \in \mathbb{N}, \]

(2.17)

where the kernel \( \Phi_n(x) \) is defined by (2.4) and integral (2.17) converges absolutely.

Proof. In fact, substituting (1.6), (1.14) and (2.4) on the right-hand side of (2.17), we estimate the integral, recalling inequality (1.13), to obtain

\[
\int_0^\infty |\Phi_n(x)f(x)| dx = \int_0^\infty x \left| \int_{-\pi}^\pi K_0(x \cosh(u)) \sinh(2u) \sin(nu) du \right| dx \\
\times \sum_{m=1}^{\infty} a_m \int_0^x \frac{K_{im}(t)}{(x^2 - t^2)^{1/2}} dt dx \\
\leq 4A \int_0^\infty x^{3/4} \int_{-\pi}^\pi K_0(x \cosh(u)) \sinh(2u) du dx \\
\times \sum_{m=1}^{\infty} |a_m| e^{-\pi m/2} \int_0^1 \frac{t^{-1/4}}{(1-t^2)^{1/2}} dt = 2^{15/4} A \sqrt{\pi} \Gamma\left(\frac{3}{8}\right) \Gamma\left(\frac{7}{8}\right) \\
\times \left( \cosh^{1/4}(\pi) - 1 \right) \sum_{m=1}^{\infty} |a_m| e^{-\pi m/2} < \infty,
\]

owing to assumption (2.16). Therefore the interchange of the order of integration and summation is allowed to get

\[
\frac{1}{\pi^3} \sinh(\pi n) \int_0^\infty \Phi_n(x) f(x) dx = \frac{1}{\pi^3} \sinh(\pi n) \sum_{m=1}^{\infty} a_m \int_{-\pi}^\pi \sinh(2u) \sin(nu) \\
\int_0^\infty K_{im}(t) \int_t^\infty \frac{x K_0(x \cosh(u))}{(x^2 - t^2)^{1/2}} dx dt du.
\]

(2.18)

The integral with respect to \( x \) in (2.18) is calculated in the proof of Theorem 1, and we have the formula

\[
\int_t^\infty \frac{x K_0(x \cosh(u))}{(x^2 - t^2)^{1/2}} dx = \frac{\pi e^{-t \cosh(u)}}{2 \cosh(u)}, \quad t > 0.
\]

Hence, recalling (2.6), we derive finally from (2.18)

\[
\frac{1}{\pi^3} \sinh(\pi n) \int_0^\infty \Phi_n(x) f(x) dx = \frac{1}{\pi} \sinh(\pi n) \sum_{m=1}^{\infty} \frac{a_m}{\sinh(\pi m)}
\]
Finally, we demonstrate the inversion theorem for the discrete transform (1.7).

**Theorem 4.** Let the sequence \( \{a_n\}_{n \in \mathbb{N}} \in l_1 \), i.e. the series

\[
\sum_{n=1}^{\infty} |a_n| < \infty \quad (2.19)
\]

converges. Then for the discrete transformation (1.7) the following inversion formula holds

\[
a_n = \frac{1}{\pi^2} \sinh(\pi n) \int_{0}^{\infty} \Psi_n(x)f(x)dx, \quad n \in \mathbb{N}, \quad (2.20)
\]

where the kernel \( \Psi_n(x) \) is defined by (2.13) and integral (2.20) converges absolutely.

**Proof.** Indeed, similarly to the proof of Theorem 3 we recall (1.7), (1.15), (2.13) to find the estimate

\[
\int_{0}^{\infty} |\Psi_n(x)f(x)| \, dx = \int_{0}^{\infty} \left| \int_{-\pi}^{\pi} \left[ \frac{2}{\pi} + x \cosh(u)M_0(x \cosh(u)) \right] \sinh(u) \sin(nu) \, du \right| \, dx \\
\sum_{m=1}^{\infty} a_m \int_{x}^{\infty} \frac{K_{im}(t)}{(t^2 - x^2)^{1/2}} \, dt \, dx \leq 2 \int_{0}^{\infty} \int_{0}^{\pi} \left[ \frac{2}{\pi} + x \cosh(u) \right] \sinh(u) \sin(nu) \, du \\
\sum_{m=1}^{\infty} |a_m| \int_{1}^{\infty} \frac{K_{0}(xt)}{(t^2 - 1)^{1/2}} \, dt \, dx = 2 \sum_{m=1}^{\infty} |a_m| \int_{1}^{\infty} \frac{1}{t(t^2 - 1)^{1/2}} \int_{0}^{\infty} K_{0}(x) \int_{0}^{\pi} \left[ \frac{2}{\pi} + \frac{x}{t} \cosh(u) \right] \\
\left| M_0 \left( \frac{x \cosh(u)}{t} \right) \right| \sinh(u) \, du \, dx \leq 2 \sum_{m=1}^{\infty} |a_m| \int_{1}^{\infty} \frac{1}{t(t^2 - 1)^{1/2}} \int_{0}^{\infty} K_{0}(x) \int_{0}^{\pi} \left[ \frac{2}{\pi} + \frac{x}{t} \cosh(u) \right] \\
\times \sinh(u) \, du \, dx = \left[ 2\pi \sinh^2 \left( \frac{\pi}{2} \right) + \sinh^2(\pi) \right] \sum_{m=1}^{\infty} |a_m| < \infty.
\]

Therefore, taking into account (2.15), it says
Thus we derive via (2.6)

\[
\frac{1}{\pi^2} \sinh(\pi n) \int_0^\infty \Psi_n(x)f(x)dx = \frac{1}{\pi^2} \sinh(\pi n) \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \sinh(u) \sin(\nu u) \int_0^\infty K_{im}(t) \\
\times \int_0^t \left[ \frac{2}{\pi} + x \cosh(u)M_0(x \cosh(u)) \right] \frac{dxdtdu}{(t^2 - x^2)^{1/2}} \\
= \frac{2}{\pi} \sinh(\pi n) \sum_{m=1}^{\infty} \frac{a_m}{\sinh(\pi m)} \int_0^\pi \sin(\nu u) \sin(\nu u)du = a_n.
\]

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