Mass matrices with CP phase in modular flavor symmetry

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Abstract

We study the CP violation and the CP phase of quark mass matrices in modular flavor symmetric models. The CP symmetry remains at $\tau = e^{2\pi i/3}$ by a combination of the $T$-symmetry of the modular symmetry. However, the $T$-symmetry breaking may lead to the CP violation at the fixed point $\tau = e^{2\pi i/3}$. We study such a possibility in magnetized orbifold models as examples of modular flavor symmetric models. These models, in general, have more than one candidates for Higgs modes, while generic string compactifications also lead to several Higgs modes. These Higgs modes have different behaviors under the $T$-transformation. The light Higgs mode can be a linear combination of those modes so as to lead to realistic quark mass matrices. The CP phase of mass matrix does not appear in a certain case, which is determined by the $T$-transformation behavior. Deviation from it is important to realize the physical CP phase. We discuss an example leading to non-vanishing CP phase at the fixed point $\tau = e^{2\pi i/3}$. 
1 Introduction

The origin of the flavor structure including the CP violation is one of important issues to study in particle physics. The four-dimensional (4D) CP symmetry can be embedded into a proper Lorentz symmetry in higher dimensional theory such as superstring theory [1–6]. In such a theory, the CP symmetry can be broken spontaneously at the compactification scale or below it within the framework of 4D effective field theory.

In addition to the CP symmetry, geometrical symmetries of compact space can be sources of the flavor symmetries among quarks and leptons. For example, $D_4$ and $\Delta(54)$ flavor symmetries can be derived from heterotic string theory on orbifolds and magnetized/intersecting D-brane models [7–11]. These non-Abelian discrete flavor symmetries have been used in model building for the quark and lepton flavors from the bottom-up approach [12–21].

The torus $T^2$ and the orbifold $T^2/Z_2$ compactifications have the modular symmetry, which corresponds to the change of basis and is generated by $S$ and $T$ generators. Interestingly, the modular symmetry transforms zero-modes of matter fields. That is, the modular symmetry can also be a source of the flavor symmetry of quarks and leptons. (See for heterotic string theory on orbifolds [22–24] and magnetized D-brane models [25–31].)

Inspired by these extra dimensional models and superstring theory, recently 4D modular flavor symmetric models have been studied in lepton and quark sectors. Indeed, the well-known finite groups $S_3$, $A_4$, $S_4$ and $A_5$ are isomorphic to the finite modular groups $\Gamma_N$ for $N = 2, 3, 4, 5$, respectively [36]. The lepton mass matrices have been given successfully in terms of $\Gamma_3 \simeq A_4$ modular forms [37]. Modular invariant flavor models have also been proposed on the $\Gamma_2 \simeq S_3$ [38], $\Gamma_4 \simeq S_4$ [39] and $\Gamma_5 \simeq A_5$ [40,41]. Other finite groups are also derived from magnetized D-brane models [26]. By using these modular forms, phenomenological studies of the lepton flavors have been done intensively based on $A_4$ [42–46] and $S_4$ [47–49]. The quark mass matrices also have been discussed in order to reproduce observed Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. [50,51]. A lot of references of the modular invariant flavor models are seen in Ref. [21].

We denote the complex structure modulus $\tau$ on $T^2$ as well as its orbifolds. The modulus $\tau$ transforms under the CP symmetry as

$$\tau \rightarrow -\tau^*, \quad (1)$$

because the 4D CP symmetry can be embedded into a proper Lorentz transformation of higher dimensions. Thus, the CP symmetry remains along $\text{Re}\ \tau = 0$. On the other hand, the modulus value $\text{Re}\ \tau = -1/2$ transforms to $\text{Re}\ \tau = 1/2$ under the above CP transformation. However, these modulus values $\text{Re}\ \tau = \pm 1/2$ are equivalent in modular symmetric models by the $T$ transformation. The CP symmetry remains along $\text{Re}\ \tau = \pm 1/2$. This issue was studied explicitly in Ref. [52]. Hence, the CP symmetry and modular flavor symmetries are combined.

1Calabi-Yau compactifications include more moduli and they have larger symplectic modular symmetries [32,33].
so as to construct a larger symmetry \[35, 53–57\]. The CP can be violated at a generic value of the modulus \(\tau\). For example, realization of the Kobayashi-Maskawa CP phase as well as quark masses and mixing angles in magnetized orbifold models was studied in Ref. \[59\].

Indeed, how to fix the modulus value is an important issue, that is, the moduli stabilization problem, although the modulus value is used as a free parameter in many modular flavor symmetric models. The spontaneous CP violation has been studied through the moduli stabilization. (See for early works Refs. \[60–63\].) Recently the CP violation was studied through the moduli stabilization due to the three-form fluxes \[34, 64\]. For example, the modulus stabilization analysis in Ref. \[65\] shows that the modulus can be stabilized at the fixed point \(\tau = \omega\) with a highest provability, where \(\omega = e^{2\pi i/3}\). (See also Refs. \[66–68\].) At the fixed point \(\tau = \omega\) the residual symmetry \(Z_3\) of the modular symmetry remains, while the residual \(Z_2\) and \(Z_4\) symmetries remain at the fixed point \(\tau = i\) for \(PSL(2, \mathbb{Z})\) and \(SL(2, \mathbb{Z})\), respectively. These fixed points are also attractive from the viewpoint of model building in the bottom-up approach. The large flavor mixing angle is simply realized at \(\tau = i\) \[69\] due to the residual \(Z_2\) symmetry. (See also \[70\].) Interestingly, the hierarchy of charged lepton masses are successfully obtained at nearby \(\tau = \omega\) without tuning parameters \[71\]. (See also \[72\].) The challenge to the quark sector is promising. The residual symmetries at the fixed points are also useful to stabilize dark matter candidates \[73\]. Thus, the modular flavor symmetric model presents the phenomenologically special feature at the fixed points. More studies at the fixed points are required to solve the flavor problem such as the CP violation as well as the mass hierarchy.

The CP is not violated at the fixed point \(\tau = \omega\) through the above discussion. That is, the CP symmetry is preserved at \(\tau = \omega\) if the \(T\) symmetry remains. On the other hand, if \(T\) symmetry is broken, the CP violation may occur at the fixed point \(\tau = \omega\). In generic string compactification, there are more than one candidates of Higgs modes, which have the same \(SU(2)_L \times U(1)_Y\) quantum numbers and can couple with quarks and leptons. The torus and orbifold compactifications with magnetic fluxes are interesting compactifications. They can lead to 4D chiral theory, where the generation number is determined by magnetic fluxes \[74, 77\]. These magnetic fluxes determine the number of Higgs modes, which can couple with three generations of quarks and leptons \[78, 79\]. Yukawa couplings are written by theta functions. Realistic quark and leptons mass matrices were studied \[80–83\]. In this paper, we show that such magnetized orbifold models with multi-Higgs modes can break the \(T\)-symmetry, and they can lead to the CP violation even at the fixed point \(\tau = \omega\).

This paper is organized as follows. In section 2, we give a brief review in the CP violation in modular flavor symmetric models, and then study the importance of the \(T\)-symmetry at the fixed point \(\tau = \omega\). In section 3, we study the CP violation in the quark sector of the magnetized orbifold models, which were studied in Ref. \[83\]. In particular, the \(T\)-symmetry is broken and that leads to the CP violation at the fixed point \(\tau = \omega\). Section 4 is our conclusion.

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\[2\] See also for CP in Calabi-Yau compactification \[58\].
Here, at first we give a review on the CP in modular flavor symmetric models, and then study one of key points in the CP violation within the framework of the modular flavor symmetry. We focus on six-dimensional theory, that is, two extra dimensions. Similarly, we can study ten-dimensional theory with six extra dimensions.

2.1 CP symmetry

Here, we briefly review on the CP in modular flavor symmetric models. We use the complex coordinate \( z = y_1 + \tau y_2 \) on two extra dimensions, where \( y_1 \) and \( y_2 \) are real coordinates and \( \tau \) is the complex structure modulus. On \( T^2 \), we identify \( z \sim z + m + n\tau \), where \( m \) and \( n \) are integers. We transform \( z \) as \( z \rightarrow z^* \) or \( z \rightarrow -z^* \) at the same time as the 4D CP transformation. Such a transformation corresponds to a six-dimensional proper Lorentz transformation. Here, we use the latter transformation \( z \rightarrow -z^* \), because it maps the upper half plane \( \text{Im}\tau > 0 \) to the same half plane. Then, the modulus \( \tau \) transforms as Eq. (1) under this CP symmetry. Obviously the line \( \text{Re}\tau = 0 \) is symmetric under Eq. (1). However, the CP symmetry seems to be violated at other points. For example, the line \( \text{Re}\tau = -1/2 \) transforms as

\[
\tau = -1/2 + i\text{Im}\tau \quad \rightarrow \quad 1/2 + i\text{Im}\tau,
\]

and it is not invariant.

The modular symmetry transforms the modulus \( \tau \) as

\[
\tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d},
\]

by \( SL(2,\mathbb{Z}) \) element \( \gamma \),

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

where \( a, b, c, d \) are integers satisfying \( ad - bc = 1 \). The modular symmetry is generated by two elements, \( S \) and \( T \),

\[
S: \tau \rightarrow -\frac{1}{\tau}, \quad T: \tau \rightarrow \tau + 1.
\]

As mentioned above, the line \( \text{Re}\tau = -1/2 \) transforms to the line \( \text{Re}\tau = 1/2 \) under Eq. (1), and is not invariant. However, these lines are transformed each other by the \( T \)-transformation. Thus, the line \( \text{Re}\tau = -1/2 \) is also CP-symmetric by combining the \( T \)-symmetry.

Here, we consider 4D supersymmetric effective theory derived from higher dimensional theory.\(^3\) For example, we study the superpotential terms including quark Yukawa couplings,

\[
W(\tau) = Y_{ij}^{(u)}(\tau)Q_i u_j H^u_\ell + Y_{ij}^{(d)}(\tau)Q_i d_j H^d_\ell,
\]

\(^3\)See e.g. Ref. [84] and references therein.
where $Q_i$, $u_j$, $d_j$ denote superfields corresponding to three generations of left-handed quarks, right-handed up-sector quarks, and right-handed down-sector quarks, respectively, and $H_{\ell}^{u,d}$ are up-sector and down-sector Higgs superfields. Here we have added indexes for Higgs fields $H_{\ell}^{u,d}$, because string compactifications, in general, lead to more than one pairs of Higgs fields. These quark and Higgs fields transform under the modular symmetry,

$$\Phi_i \rightarrow \frac{1}{(cT + d)k_i} \rho(\gamma)_{ij} \Phi_j,$$

where $-k_i$ denote the modular weights of 4D fields and $\rho(\gamma)_{ij}$ is a unitary matrix to represent the modular group. The Yukawa couplings $Y_{ij}^{(u,d)}(\tau)$ depend on the modulus $\tau$, and they are modular forms. Similarly, we can study other terms in the superpotential.

The typical Kähler potential of the modulus field $\tau$ is written by

$$\tilde{K} = - \ln(2\text{Im}\tau),$$

and tree-level Kähler potential of matter field with the modular weight $-k_i$ is written by

$$K_m = \frac{|\Phi_i|^2}{(2\text{Im}\tau)^{k_i}}.$$

The modular symmetry within the framework of supergravity theory requires $e^{\tilde{K}}|W|^2$ to be invariant.

The supersymmetric models are CP-symmetric if $|W|^2$ is invariant, i.e.

$$W(\tau) \rightarrow e^{i\chi} \overline{W(\tau)},$$

under the CP transformation with $\tau \rightarrow -\tau^*$ including the CP transformation of chiral matter fields.

### 2.2 CP and T-symmetry

Here, we study the implication of $T$-symmetry from the viewpoint of CP violation. Suppose that the $T$-transformation is represented by $\rho(T)$ in our 4D effective field theory, and $\rho(T)$ satisfies

$$\rho(T^N) = I,$$

that is, the $Z_N^{(T)}$ symmetry. We consider the field basis such that the $T$ transformation is represented by diagonal matrices,

$$T : Q_i \rightarrow e^{2\pi i P(Q_i)/N} Q_i, \quad u_i \rightarrow e^{2\pi i P(u_i)/N} u_i, \quad d_i \rightarrow e^{2\pi i P(d_i)/N} d_i, \quad H_{\ell}^{u,d} \rightarrow e^{2\pi i P(H_{\ell}^{u,d})/N} H_{\ell}^{u,d},$$

(12)
where $P[\Phi_i]$ denote $Z_N^{(T)}$ charges of fields $\Phi_i$. The $T$-invariance of the superpotential requires the following $T$-transformation of Yukawa couplings:

$$T : Y^{(u)}_{ij\ell}(\tau) \to e^{2\pi i P[Y^{u}_{(ij)\ell}]/N} Y^{(u)}_{ij\ell}(\tau), \quad Y^{(d)}_{ij\ell}(\tau) \to e^{2\pi i P[Y^{d}_{(ij)\ell}]/N} Y^{(d)}_{ij\ell}(\tau),$$

where

$$P[Y^{u}_{(ij)\ell}] = -(P[Q_i] + P[u_j] + P[H^u_i]), \quad P[Y^{d}_{(ij)\ell}] = -(P[Q_i] + P[d_j] + P[H^d_i]).$$

We use these $Z_N^{(T)}$ charges satisfying $0 \leq P[Y^{u}_{(ij)\ell}] < N$ and $0 \leq P[Y^{d}_{(ij)\ell}] < N$.

The Yukawa couplings, which are modular forms, can be expanded in terms of $q = e^{2\pi i \tau}$. Since they satisfy the above $T$-transformation behavior, they can be written by

$$Y^{(u)}_{ij\ell}(\tau) = a_0 q^{P[Y^{u}_{(ij)\ell}]/N} + a_1 qq^{P[Y^{u}_{(ij)\ell}]/N} + a_2 q^2 q^{P[Y^{u}_{(ij)\ell}]/N} + \ldots = \hat{Y}^{(u)}_{ij\ell}(q) q^{P[Y^{u}_{(ij)\ell}]/N},$$

$$Y^{(d)}_{ij\ell}(\tau) = b_0 q^{P[Y^{d}_{(ij)\ell}]} + b_1 qq^{P[Y^{d}_{(ij)\ell}]/N} + b_2 q^2 q^{P[Y^{d}_{(ij)\ell}]/N} + \ldots = \hat{Y}^{(d)}_{ij\ell}(q) q^{P[Y^{d}_{(ij)\ell}]/N},$$

where the functions $\hat{Y}^{(u)}_{ij\ell}(q)$ and $\hat{Y}^{(d)}_{ij\ell}(q)$ include only integer powers of $q$, i.e. $q^n$.

Here, let us consider the model with one pair of $H^u$ and $H^d$, which are $T$-invariant. In addition, we set $\text{Re}\tau = -1/2$. In this model, the Yukawa couplings are written by

$$Y^{(u)}_{ij\ell}(\tau) = \hat{Y}^{(u)}_{ij\ell}(q) e^{-\pi i P[Y^{u}_{(ij)\ell}]/N},$$

$$Y^{(d)}_{ij\ell}(\tau) = \hat{Y}^{(d)}_{ij\ell}(q) e^{-\pi i P[Y^{d}_{(ij)\ell}]/N},$$

where we have omitted the indexes for Higgs fields, and

$$P[Y^{u}_{(ij)}] = -(P[Q_i] + P[u_j]), \quad P[Y^{d}_{(ij)}] = -(P[Q_i] + P[d_j]),$$

$$\hat{Y}^{(u)}_{ij}(q) = \hat{Y}^{(u)}_{ij}(q) e^{-2\pi P[Y^{u}_{(ij)}] \text{Im}\tau/N}, \quad \hat{Y}^{(d)}_{ij}(q) = \hat{Y}^{(d)}_{ij}(q) e^{-2\pi P[Y^{d}_{(ij)}] \text{Im}\tau/N}. $$

The Yukawa couplings have phases $e^{-\pi i P[Y^{u,d}_{(ij)}]/N}$, although $\hat{Y}^{(u)}_{ij}(q)$ and $\hat{Y}^{(d)}_{ij}(q)$ are real. However, these phases can be removed by the following rephasing of fields:

$$Q'_i = e^{-\pi i P[Q_i]/N} Q_i, \quad u'_i = e^{-\pi i P[u_j]/N} u_i, \quad d'_i = e^{-\pi i P[d_i]/N} d_i.$$  

Then, this model is CP-invariant. In this discussion, the $T$-symmetry is important.

Similarly, we can study the model with one pair of $H^u$ and $H^d$, which transform non-trivially under the $T$-transformation

$$T : H^{u,d} \to e^{2\pi i P[H^{u,d}]/N} H^{u,d}. $$

When these Higgs fields develop their vacuum expectation values (VEVs), mass matrices have phases, but those are overall phases, and not physical. The Higgs VEVs break the $T$-symmetry
and the $T$-symmetry is broken through the moduli stabilization at $\tau = \omega$, but those are not enough to realize the physical CP phase.

Unless the above structure is violated by any effects such as non-perturbative effects, the above discussion suggests that we need two or more Higgs VEV directions. We denote them by

\[ v^u_\ell = |v^u_\ell| e^{2\pi i P[v^u_\ell]/N} = \langle H^u_\ell \rangle, \quad v^d_\ell = |v^d_\ell| e^{2\pi i P[v^d_\ell]/N} = \langle H^d_\ell \rangle, \tag{21} \]

where $P[v^u_\ell]$ or $P[v^d_\ell]$ is not integer for a generic VEV. At any rate, the phase structure of mass matrices at $\tau = \omega$ is controlled by the $T$-symmetry. If they satisfy

\[ -\frac{1}{2} \left( P[Y^u_{(i j \ell)}] + P[Q_i] + P[u_j] \right) + P[v^u_\ell] = \text{constant independent of } \ell, \]
\[ -\frac{1}{2} \left( P[Y^d_{(i j \ell)}] + P[Q_i] + P[d_j] \right) + P[v^d_\ell] = \text{constant independent of } \ell, \tag{22} \]

in all of allowed Yukawa couplings with $i, j$ fixed, one can cancel phase of mass matrix elements up to an overall phase by the $Z_N^{(T)}$ rotation. We can compare this condition with the relations (14), where the factor $-1/2$ originates from $\text{Re} \tau = -1/2$. We study this point in the next section by using magnetized orbifold models as an example.

This condition is also applied for the fixed point $\tau = i$. The Yukawa couplings are real in our basis because of $\text{Re} \tau = 0$. Then, the coefficients of $P[Y^u_{(i j \ell)}] + P[Q_i] + P[u_j]$ and $P[Y^d_{(i j \ell)}] + P[Q_i] + P[d_j]$ are zero. That is, if we choose non-trivial phases of Higgs VEVs, which cannot be removed by rephasing, the CP violation occurs. For example, if all of the Higgs VEVs are real in our basis, the CP symmetry remain at the fixed point $\tau = i$. However, the CP can be violated at the fixed point $\tau = \omega$, even if all of the Higgs VEVs are real in our basis. Difference of Higgs VEV phases from their $Z_N^{(T)}$ charges are important. The $Z_N^{(T)}$ charge pattern is the reference to judge whether the non-trivial CP phase appears or not.

### 3 CP phase in magnetized orbifold models

Here we study the CP phase derived from magnetized orbifold models as an example of modular flavor symmetric models.

#### 3.1 Magnetized orbifold models

First, we give a brief review on zero-mode wave functions on magnetized $T^2$ [74]. Higher dimensional fields, e.g. spinor field $\Psi(x, z)$, can be decomposed by

\[ \Psi(x, y) = \sum \chi_i(x) \psi_i(z) + \cdots, \tag{23} \]

where $x$ denotes the 4D coordinate, the first term $\chi_i(x) \psi_i(z)$ corresponds to zero-modes, and the ellipsis denotes massive modes. For simplicity, we explain them by use of $U(1)$ theory. We
introduce $U(1)$ background magnetic flux,

$$F = dA = \frac{\pi iM}{\text{Im}\tau} dz \wedge d\bar{z}, \quad A = \frac{\pi M}{\text{Im}\tau} \text{Im}(\bar{z}dz),$$

(24)

where $M$ must be integer because of the Dirac quantization condition. We consider the Dirac equation for the spinor with $U(1)$ charge $q = 1$. On $T^2$, the spinor $\psi$ has two components, $\psi = (\psi_+, \psi_-)^T$. For $M > 0$, $\psi_+$ has $M$ zero-modes, but $\psi_-$ has no zero-modes. On the other hand, for $M < 0$, $\psi_-$ has $|M|$ zero-modes but $\psi_+$ has no zero-modes. Thus, we can realize a chiral theory. When $M > 0$, the $j$-th zero-mode can be written by

$$\psi^j_M(z, \tau) = \left( \frac{|M|}{\mathcal{A}} \right)^{1/4} e^{i\pi|M|z \frac{\text{Im}\theta}{\text{Im}\tau}} \begin{bmatrix} |M| \\ 0 \end{bmatrix} (|M|z, |M|\tau),$$

(25)

where $\mathcal{A}$ denotes the area of $T^2$ and $\theta$ denotes the Jacobi theta function defined by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (\nu, \tau) = \sum_{\ell \in \mathbb{Z}} e^{\pi i(a+\ell)^2\tau} e^{2\pi i(a+\ell)(\nu+b)}.$$

(26)

Similarly, we can write zero-mode wave functions of $\psi_-$ for $M < 0$. Hereafter, we omit the chirality sign index $\pm$, but we denote the wave function by $\psi_T$. Here we use the normalization,

$$\int d^2z \psi^i_{T|M} (z, \tau) \left( \psi^j_{T|M} (z, \tau) \right)^* = (2\text{Im}\tau)^{-1/2} \delta_{i,j}.$$  

(27)

The Yukawa coupling of zero-modes is written by

$$Y^{ijk} = g \int d^2z \psi^i_{T|M} (z, \tau) \psi^j_{T|jM} (z, \tau) \left( \psi^k_{T|M} (z, \tau) \right)^*$$

$$= g \mathcal{A}^{-1/2} \left| \frac{M_1M_2}{M_1 + M_2} \right|^{1/4} \sum_m \delta_{k,i+j+|M_1M_2|m} \theta \begin{bmatrix} |M_2[i-|M_1j+|M_1M_2|m] \\ |M_1M_2(M_1 + M_2)| / \mathcal{A} \end{bmatrix} (0, |M_1M_2(M_1 + M_2)|),$$

(28)

where $g$ is the 3-point coupling in higher dimensional theory. The gauge invariance requires

$$|M_1| + |M_2| = |M_3|,$$

(29)

for allowed Yukawa couplings. Furthermore, the Kronecker delta $\delta_{k,i+j+|M_1M_2|m}$ implies the coupling selection rule among these modes.

The zero-mode wave functions transform under the $T$-symmetry as $[25, 31]$

$$T : \psi^j_{T|M} (z, \tau) \rightarrow e^{i\pi j/M} \psi^j_{T|M} (z, \tau),$$

(30)

when $M$ is even. Thus, the $T$-transformation is represented by the diagonal matrix in this basis, and zero-modes have $Z_{2M}^{(T)}$ charges. The zero-mode wave functions transform under the $S$-symmetry as

$$S : \psi^j_{T|M} (z, \tau) \rightarrow (-\tau)^{1/2} e^{i\pi j/4} \sqrt{|M|} \sum_{\ell} e^{2\pi ij/M} \psi^\ell_{T2|M} (z, \tau).$$

(31)

\footnote{See for generic case Ref. [30].}
Note that the transformation of the 4D fields \( \chi_i(x) \) is the inverse of \( \psi_i(z) \) to make \( \Psi(x,y) \) invariant. For example, the 4D fields transform as

\[
T : \chi^j(z) \rightarrow e^{-i\pi j^2/M} \chi^j(z),
\]

under the \( T \)-transformation.

The \( T^2/Z_2 \) orbifold is constructed by identifying \( z \sim -z \) on \( T^2 \). Wave functions on \( T^2/Z_2 \) are classified into \( Z_2 \) even and odd modes,

\[
\psi_{T^2/Z_2^m}(-z) = (-1)^m \psi_{T^2/Z_2^m}(z),
\]

where \( m = 0 \) and \( 1 \) correspond to \( Z_2 \) even and odd modes. The zero-mode wave functions in orbifold models with magnetic fluxes can be written by use of zero-mode wave functions on \( T^2 \) as \[75–77\].

\[
\psi^{j,|M|}_{T^2/Z_2^m}(z) = \mathcal{N}^j \left( \psi^{j,|M|}_{T^2}(z) + (-1)^m \psi^{j,|M|}_{T^2}(-z) \right)
\]

\[
= \mathcal{N}^j \left( \psi^{j,|M|-j,|M|}_{T^2}(z) + (-1)^m \psi^{j,|M|-j,|M|}_{T^2}(-z) \right),
\]

where

\[
\mathcal{N}^j = \begin{cases} 
1/2 & (j = 0, |M|/2) \\
1/\sqrt{2} & \text{(otherwise)} 
\end{cases}
\]

Table II shows the number of zero-modes on magnetized \( T^2/Z_2 \) orbifold \[75–77\]. We can realize three generations by \( Z_2 \) even modes for \( |M| = 4, 5 \) and by \( Z_2 \) odd modes for \( |M| = 7, 8 \).

| \(|M|\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(Z_2\)-even | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 | 6 | 6 | 7 |
| \(Z_2\)-odd | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 |

Table 1: The number of zero-modes on magnetized \( T^2/Z_2 \) orbifold.

Yukawa couplings in magnetized orbifold models can be calculated by

\[
Y^{ijk} = g \int d^2 z \psi^{i,|M_1|}_{T^2/Z_2^m_1}(z, \tau) \psi^{j,|M_2|}_{T^2/Z_2^m_2}(z, \tau) \left( \psi^{k,|M_3|}_{T^2/Z_2^m_3}(z, \tau) \right)^*. 
\]

Their explicit computations are straightforward by use of Eqs. \[28\] and \[34\]. Allowed Yukawa couplings must satisfy

\[
m_1 + m_2 = m_3 \pmod{2},
\]

in addition to Eq. \[29\].
3.2 Quark mass matrix in magnetized orbifold models

Here, we study quark mass matrices in the magnetized orbifold model, which was studied in Ref. [83]. We consider the model that all of left-handed and right-handed quarks are originated from $Z_2$ even zero-modes with $M = 4$, which lead to three zero-modes, that is, three generations. (See for details of model building e.g. Refs. [81,85].) Because of Eqs. (29) and (37), Higgs modes correspond to $Z_2$ even zero-modes with $M = 8$. That means five pairs of $H^u_d$ from Table [1].

These zero-mode wave functions are summarized in Table [2].

| $i$ | $Q_i$ | $u_i, d_i$ | $H^u_d$ |
|-----|------|-----------|--------|
| 0   | $\psi_{T2}^{0,4}$ | $\psi_{T2}^{0,4}$ | $\psi_{T2}^{0,8}$ |
| 1   | $\frac{1}{\sqrt{2}}(\psi_{T2}^{1,4} + \psi_{T2}^{3,4})$ | $\frac{1}{\sqrt{2}}(\psi_{T2}^{1,4} + \psi_{T2}^{3,4})$ | $\frac{1}{\sqrt{2}}(\psi_{T2}^{1,8} + \psi_{T2}^{7,8})$ |
| 2   | $\psi_{T2}^{2,4}$ | $\psi_{T2}^{2,4}$ | $\frac{1}{\sqrt{2}}(\psi_{T2}^{2,8} + \psi_{T2}^{6,8})$ |
| 3   | $\psi_{T2}^{3,8}$ | $\frac{1}{\sqrt{2}}(\psi_{T2}^{3,8} + \psi_{T2}^{5,8})$ |
| 4   | $\psi_{T2}^{4,8}$ | $\frac{1}{\sqrt{2}}(\psi_{T2}^{4,8} + \psi_{T2}^{5,8})$ |

Table 2: Zero-mode wave functions.

Now, we study quark mass matrices in our model. The mass matrices in the up and down sector of quarks can be written by

$$(M_u)_{ij} = \sum_\ell Y_{ij\ell}^u \langle H^u_\ell \rangle, \quad (M_d)_{ij} = \sum_\ell Y_{ij\ell}^d \langle H^d_\ell \rangle,$$  

(38)

when the Higgs fields develop their VEVs. The Yukawa couplings can be written explicitly by

$$Y_{ij0}^{(u), (d)} = c \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad Y_{ij1}^{(u), (d)} = c \begin{pmatrix} X_3 \\ X_4 \end{pmatrix}, \quad Y_{ij2}^{(u), (d)} = c \begin{pmatrix} \sqrt{2} X_1 \\ \frac{1}{\sqrt{2}}(X_0 + X_2) \end{pmatrix}, \quad Y_{ij3}^{(u), (d)} = c \begin{pmatrix} X_4 \\ X_3 \end{pmatrix}, \quad Y_{ij4}^{(u), (d)} = c \begin{pmatrix} X_2 \\ X_1 \\ X_0 \end{pmatrix},$$  

(39)

where $c$ is an overall constant, and

$$X_0 = \eta_0 + 2\eta_{32} + \eta_{64}, \quad X_1 = \eta_8 + \eta_{24} + \eta_{40} + \eta_{56}, \quad X_2 = 2(\eta_{16} + \eta_{48}),$$

$$X_3 = \eta_4 + \eta_{28} + \eta_{36} + \eta_{60}, \quad X_4 = \eta_{12} + \eta_{20} + \eta_{44} + \eta_{52}. \quad (40)$$
Here, we have used the notation,

\[ \eta_N = \vartheta \left[ \begin{array}{c} \frac{N}{128} \\ 0 \end{array} \right] (0, 128 \tau). \] (41)

Note that each of \( Y_{ij\ell}^{(u),(d)} \) matrices with \( \ell = 0, 1, 2, 3, 4 \) is not a rank-one matrix or an approximate rank-one matrix leading to the realistic quark mass hierarchy except \( \text{Im} \tau \to \infty \). In addition, each of \( Y_{ij\ell}^{(u),(d)} \) matrices with \( \ell = 0, 1, 2, 3, 4 \) has many zero elements, which are originated from the coupling selection rule due to \( \delta_{k,i+j+|M_1M_2|m} \) in Eq. (28). Therefore, a single VEV direction is not realistic.

In particular, we set the modulus value as \( \tau = \omega \) in order to study the quark mass matrices. At this fixed point, the residual \( Z_3 \) symmetry, which is generated by \( ST \), remains. At the compactification energy scale, all of five pairs of Higgs fields are massless. We expect that they generate their mass terms,

\[ \mu(\tau)_{ij} H_u^i H_d^j, \] (42)

below the compactification scale. Then, one pair remains light, and they develop their VEVs. Such mass terms would be generated by non-perurbative effects such as D-brane instanton effects\(^5\). Also coupling terms such as \( Y(\tau)_{ij\ell} H_u^i H_d^d S_\ell \) may be origins of the mass terms when \( S_\ell \) develop their VEVs like the next-to-minimal supersymmetric standard model. Furthermore, higher order terms such as \( Y(\tau)_{ij\ell_1...\ell_n} H_u^i H_d^d S_{\ell_1} \cdots S_{\ell_n} \) may be their origins when \( S_{\ell_1} \) develop their VEVs. These depend on details of the model. Thus, we take a phenomenological approach. That is, we study which VEV directions lead to realistic results in quark mass matrices assuming such direction corresponds to the light mode in the above mass matrices.

Quark masses are hierarchical very much. That means that quark mass matrices are rank-one matrices approximately, and realistic mass matrices deviate slightly from such rank-one matrices. In Ref. [83], VEV directions leading to rank-one mass matrices were studied. We follow those analysis. For example, the \( Z_3 \) symmetry remains at the fixed point \( \tau = \omega \). In particular, VEV directions, which lead to rank-one mass matrices and \( Z_3 \) invariant vacuum, were studied in Ref. [83]. Although the VEV directions in \( Z_3 \) eigenbasis shown in Ref. [83] include non-zero for only one \( Z_3 \) invariant direction and zeros for the other directions, i.e. \( A(1, 0, 0, 0, 0) \), the directions in our basis become

\[
\begin{align*}
\langle H_u^\ell \rangle &= \langle H_d^\ell \rangle = h_\ell, \\
h_\ell &= A(0.6254 e^{0.04567i}, 0.6295 e^{-0.1507i}, 0.2269 e^{0.7397i}, 0.04126 e^{-1.721i}, 0.005421 e^{-3.096i} ).
\end{align*}
\] (43)

It means that even if we consider this \( Z_3 \) invariant Higgs mode is the lightest and only this Higgs field develop its real VEV, this direction is constructed by mixing of Higgs directions with different \( Z_N^{(T)} \) charges, and each of VEV phases is different from its \( Z_N^{(T)} \) charge. Along

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\(^5\)Non-perturbative effects such as D-brane instanton effects may break some part of the modular symmetry [86].
this VEV direction, only the third generations gain masses, but the first and second generations are massless. We take the following VEV directions

\[
\langle H_u^k \rangle = v_u(0.6228e^{0.1169i}, 0.6273e^{-0.1738i}, 0.2425e^{-1.055i}, 0.05774e^{-2.441i}, 0.01186e^{2.088i}),
\]
\[
\langle H_d^k \rangle = v_d(0.6201e^{0.1713i}, 0.6259e^{-0.2009i}, 0.2605e^{-1.215i}, 0.06349e^{-2.538i}, 0.009710e^{1.939i}),
\]
which deviate slightly from the above rank-one directions. When the above directions of Higgs modes are light and they develop their VEVs, we realize the following quark mass matrices:

\[
M_u = m_t \begin{pmatrix}
0.6228e^{0.1167i} & 0.4444e^{-0.3663i} & 0.08799e^{-1.840i} \\
0.4444e^{-0.3663i} & 0.3219e^{-0.8600i} & 0.06626e^{-2.350i} \\
0.08799e^{-1.840i} & 0.06626e^{-2.350i} & 0.01482e^{2.430i}
\end{pmatrix},
\]
\[
M_d = m_b \begin{pmatrix}
0.6201e^{0.1712i} & 0.4433e^{-0.3926i} & 0.09452e^{-2.000i} \\
0.4433e^{-0.3926i} & 0.3253e^{-0.9292i} & 0.06925e^{-2.438i} \\
0.09452e^{-2.000i} & 0.06925e^{-2.438i} & 0.01194e^{2.388i}
\end{pmatrix},
\]

at \( \tau = \omega \) in the orbifold wave function basis of Table 2. These mass matrices lead to the quark mass ratios,

\[
\frac{m_u}{m_t} = 1.64 \times 10^{-5}, \quad \frac{m_c}{m_t} = 6.22 \times 10^{-3},
\]
\[
\frac{m_d}{m_b} = 1.57 \times 10^{-3}, \quad \frac{m_s}{m_b} = 1.32 \times 10^{-2}.
\]

These ratios are obtained at the compactification scale, which may be quite high. When we compare them with experimental values, we have to evaluate renormalization group effects. Renormalization group effects depend on the breaking scale of supersymmetry and \( \tan \beta \). For example, we compare them with mass ratios at the GUT scale by assuming the low-energy supersymmetric model with \( \tan \beta = 5 \) [87]. The Cabibbo-Kobayahi-Maskawa (CKM) matrix is also obtained in our model,

\[
|V_{\text{CKM}}| = \begin{pmatrix}
0.974 & 0.225 & 0.00405 \\
0.225 & 0.974 & 0.0353 \\
0.00719 & 0.0348 & 0.999
\end{pmatrix}.
\]

Furthermore, our model leads to the Jarlskog invariant

\[
J = 2.83 \times 10^{-5}.
\]

These results are shown in Table 3. Thus, our model can realize almost experimental values.

The important point in our results is that we can realize non-vanishing CP phase even at the fixed point \( \tau = \omega \). In section 2, it is found that the CP symmetry remains at the fixed point \( \tau = \omega \) because of the \( T \)-symmetry. Now, let us investigate the \( T \)-symmetry in our model.
from the viewpoint of the CP violation. At \( \tau = \omega \), the Yukawa couplings have the following phases,

\[
X_0 = |X_0|, \quad X_1 = e^{-2\pi i/8}|X_1|, \quad X_2 = -|X_2|, \\
X_3 = e^{-2\pi i/32}|X_3|, \quad X_4 = e^{-2\pi i/32}|X_4|.
\]

Also, the 4D quark fields \( q_j = (Q_j, u_j, d_j) \) with \( j = 0, 1, 2 \) in our model transform

\[
T : q_j \rightarrow e^{-2\pi ij^2/8}q_j,
\]

under \( Z_N^{(T)} \), while the Higgs modes transform as

\[
T : H_{\ell}^{u,d} \rightarrow e^{-2\pi i\ell^2/16}H_{\ell}^{u,d},
\]

with \( \ell = 0, 1, 2, 3, 4 \) under the \( Z_N^{(T)} \) symmetry. For simplicity, we consider the case that only the \( H_0^{u,d} \) and \( H_1^{u,d} \) develop their VEVs. Then, the mass matrices can be written by

\[
M_{u,d} = c \begin{pmatrix} v_0|X_0| & v_1 e^{-2\pi i/32}|X_3| & 0 \\ v_1 e^{-2\pi i/32}|X_3| & v_0 e^{-2\pi i/8}|X_1| & v_1 e^{-2\pi i/32}|X_4| \\ 0 & v_1 e^{-2\pi i/32}|X_4| & -v_0|X_2| \end{pmatrix}.
\]

When the phases of VEVs satisfy Eq. (22), i.e.

\[
(v_0, v_1) = |a|e^{i\phi}(1, e^{-2\pi i/32}),
\]

which are related to the \( Z_N^{(T)} \) charges of \( H_0^{u,d} \) and \( H_1^{u,d} \), the phases in mass matrices can be canceled by the following rephasing of fields,

\[
q_j' = e^{-2\pi ij^2/16}q_j.
\]
which are related to the $Z_N^{(T)}$ charges of $q_j$. Similarly, we can discuss the case that more Higgs modes develop their VEVs. What is important is the difference of the VEV phases from the $Z_N^{(T)}$ charge pattern. On the other hand, if a single mode among $H_u^d$ develops its VEV, we can not realize the physical CP violation as discussed in section 2. In the above example, it corresponds to $v_1 = 0$, and such a case leads to just an overall phase as discussed in section 2. In order to realize non-vanishing CP phase, we need that a linear combination of more than one Higgs modes correspond to the light Higgs mode, and it develop its VEV, where the VEV phases must be different from the $Z_N^{(T)}$ charge pattern. If VEV phases coincide with the $Z_N^{(T)}$ phase pattern, the physical CP does not appear. Thus, the $Z_N^{(T)}$ phase pattern is the reference for the physical CP phase. Note that we need such a linear combination to realize the realistic mass matrices in our model, which deviate slightly from the rank-one mass matrices. The VEV of single mode can not lead to the realistic mass matrices. Phenomenologically, we need Higgs modes along a generic VEV direction leading to rank-one mass matrices and the slight deviation from rank-one mass matrices. When we require such a direction by phenomenological purpose, we can automatically realize the CP violation at the fixed point $\tau = \omega$ for VEV phases different from the $Z_N^{(T)}$ charge pattern. This is an interesting scenario of the CP violation in modular flavor symmetric models.

Also, we comment an obvious example. If the quark mass matrices are diagonal, we can always remove phases by $U(1)^9$ rotation, which may be independent of the $Z_N^{(T)}$ rotation. Such an accidental symmetry may forbid the physical CP phase.

Also we show the example with the real VEV direction in our field basis. We take the following VEV directions

$$\langle H_u^u \rangle = v_u(0.7911, -0.6040, 0.09692, -0.00009578, -0.0002417),$$
$$\langle H_d^d \rangle = v_d(0.7471, -0.6512, 0.1333, -0.001655, -0.004176).$$

(56)

When the above directions of Higgs modes are light and they develop their VEVs, we realize the following quark mass matrices:

$$M_u = m_t \left( \begin{array}{ccc} 0.7675 & 0.4170 e^{0.9375\pi i} & 0.034118 e^{-\pi i/4} \\ 0.4170 e^{0.9375\pi i} & 0.2479 e^{-0.1898\pi i} & 0.02733 e^{0.4383\pi i} \\ 0.034118 e^{-\pi i/4} & 0.02733 e^{0.4383\pi i} & -0.006886 \end{array} \right),$$

(57)

$$M_d = m_b \left( \begin{array}{ccc} 0.7247 & 0.4495 e^{0.9374\pi i} & 0.04691 e^{-\pi i/4} \\ 0.4495 e^{0.9374\pi i} & 0.2571 e^{-0.1698\pi i} & 0.02949 e^{0.4498\pi i} \\ 0.04691 e^{-\pi i/4} & 0.02949 e^{0.4498\pi i} & -0.01033 \end{array} \right),$$

(58)

at $\tau = \omega$ in the orbifold wave function basis of Table 2. These mass matrices lead to mass rations, mixing angles and the Jarlskog invariant shown in Table 4. Also, this VEV direction can realize almost experimental values except the ratio $m_u/m_t$. We have concentrated on the modulus value $\text{Re}\tau = \pm 1/2$, in particular the fixed point $\tau = \omega$. This fixed point has the highest provability in the moduli stabilization analysis [65].
Table 4: The mass ratios of the quarks, the values of the CKM matrix elements, and Jarlskog invariant at $\tau = \omega$ under the vacuum alignments of Higgs fields in Eq. (44). Reference values of mass ratios are shown in Ref. [87]. Ones of the CKM matrix elements and the Jarlskog invariant are shown in Ref. [88].

and is phenomenological interesting. The next favorable values in the moduli stabilization analysis [65] are $\text{Re}\tau = \pm 1/4$ and 0. Even at $\text{Re}\tau = -1/4$, the phase structure is controlled by the $T$-symmetry. The Yukawa couplings at this point have the following phase,

$$
X_0 = |X_0|, \quad X_1 = e^{-2\pi i/16}|X_1|, \quad X_2 = -i|X_2|,
$$

$$
X_3 = e^{-2\pi i/64}|X_3|, \quad X_4 = e^{-2\pi i 9/64}|X_4|.
$$

Again for simplicity, we consider the case that only the $H_0^{u,d}$ and $H_1^{u,d}$ develop their VEVs.

$$
M_{u,d} = c \begin{pmatrix}
|v_0| & v_1e^{-2\pi i/64} & 0 \\
v_1e^{-2\pi i/64} & |v_0| & v_1e^{-2\pi i 9/64} \\
0 & v_1e^{-2\pi i 9/64} & |v_0|
\end{pmatrix}.
$$

When the phases of VEVs satisfy

$$
(v_0, v_1) = |a|e^{i\phi} (1, e^{-2\pi i/64}),
$$

which are related to the $Z_N^{(T)}$ charges of $H_0^{u,d}$ and $H_1^{u,d}$, the phases in mass matrices can be canceled by the $Z_N^{(T)}$ rotation. Thus, the $Z_N^{(T)}$ charge pattern is the reference to judge whether the non-trivial CP phase appears or not for $\text{Re}\tau = \pm 1/4$, too.

One can expend our discussions for the modulus value $\text{Re}\tau = \pm 1/n$, although we may not have a clear motivation to set $\text{Re}\tau = \pm 1/n$ from the viewpoint of the moduli stabilization or phenomenology. Also, the CP violation occurs at the $\tau = i$ along the Higgs VEV directions, where VEVs have relatively different phases.
4 Conclusion

We have studied the CP phase of quark mass matrices in modular flavor symmetric models at the fixed point of $\tau$. The CP symmetry remains at $\text{Re}\tau = \pm 1/2$, although $\text{Re}\tau = -1/2$ transforms to $\text{Re}\tau = 1/2$ under CP. Its reason is that these transform each other under the $T$-symmetry. That may suggest that if the $T$-symmetry is broken, the CP is also violated. However, a simple breaking of $T$-symmetry is not enough to lead to the CP violation.

We have studied quark mass matrices in magnetized orbifold models. Our model has five pairs of Higgs fields. In general, string compactification leads to more than one candidates of Higgs modes. We have computed quark mass matrices at the fixed point $\tau = \omega$. This point is favorable from the viewpoint of moduli stabilization and also phenomenologically interesting in the flavor physics. The CP is not violated at the fixed point $\tau = \omega$ if the $T$-symmetry remains. In our model, non-vanishing physical CP phase can be realized. The important point is that Higgs modes mix each other. Such mixing is required by the phenomenological purpose to realize realistic quark mass matrices, which are the approximate rank-one mass matrices. The physical CP phase can appear when the phases of VEVs differ from the $Z_4^{(T)}$ charge.

We have shown a scenario to realize the CP violation in modular flavor symmetric models. It is interesting to apply this scenario to other modular flavor symmetric models including the lepton sector, e.g. at the fixed point $\tau = \omega$. By the phenomenological purpose, we required that the light Higgs modes correspond to linear combinations of Higgs modes. It is important to show this point by computation of the $\mu$ mass matrices theoretically. However, that is beyond our scope. We would study it elsewhere.

We have found that the breaking of $T$-symmetry is important, and shown a scenario of $T$-symmetry breaking leading to the CP violation. It would also be important to study whether another way to break the $T$-symmetry leading to the CP violation.

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