SURFACE PENCILS IN EUCLIDEAN 4-SPACE $\mathbb{E}^4$

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Abstract

In the present paper we study the problem of constructing a family of surfaces (surface pencils) from a given curve in 4-dimensional Euclidean space $\mathbb{E}^4$. We have shown that generalized rotation surfaces in $\mathbb{E}^4$ are the special type of surface pencils. Further, the curvature properties of these surfaces are investigated. Finally, we give some examples of flat surface pencils in $\mathbb{E}^4$.

1 Introduction

The problem of constructing a family of surfaces from given curve is (i.e. surface pencils) important for differential geometry. In recent years surface pencils in $\mathbb{E}^3$ was studied many paper with respect to the curves family. In 2004 Wang et al. studied the problem of constructing a surface family from a given spatial geodesic [12]. Further, Lie et al. derived the necessary and sufficient condition for a given curve to be the line of curvature on a surface [10]. However, Kasap et al. generalized the marching-scale functions given in [12] and gave a sufficient condition for a given curve to be a geodesic on a surface [7]. Recently, Bayram et al. extend the method given in [12] to derive the necessary and sufficient condition for a given curve to be both isoparametric and asymptotic on a parametric surface [2]. Also, Ergün et al. considered surface pencil with a common line of curvature in Minkowski 3-space [4]. In 2008, Zhao and Wang proposed a new method for designing developable surface by constructing a surface pencil passing through a given curve which is quite in accord with the practice in industry design and manufacture [13].

In the present paper we extend the surface pencil in 4-dimensional Euclidean space $\mathbb{E}^4$. The object of study in this paper is to extend the correct parametric representation of the surface $M$ for a given curve $\gamma(s)$ in 4-dimensional Euclidean space $\mathbb{E}^4$. This paper consist of 3 sections. The first section is introduction. In the Section 2 we give some basic concepts of surfaces in $\mathbb{E}^4$ which are used in the further sections of this paper. In Section 3, by utilizing the Frenet frame from differential geometry, we derive necessary and sufficient condition for the

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correct representation of the surface patch \(X(s, t)\), where the parameter \(s\) is the arc-length of the curve \(\gamma(s)\). The basis idea is to represent \(X(s, t)\) as a linear combination of the vector functions \(V_2(s)\) and \(V_4(s)\) which are the normal vector and second binormal vector of \(\gamma(s)\) respectively. The surface pencil which we consider in the present paper is parametrized by

\[
X(s, t) = \gamma(s) + A(t)V_2(s) + B(t)V_4(s), \quad t \in J \subset \mathbb{R},
\]

where \(A(t)\) and \(B(t)\) are differentiable functions which are called marching-scale functions. Further, we have calculated the Gaussian, normal and mean curvature of this surface. However, we obtain some equations of the marching-scale functions \(A(t)\) and \(B(t)\) for the case \(M\) becomes a Vranceanu surface. Finally, we give some examples of flat surface pencils in \(\mathbb{E}^4\) given with the spacial marching-scale functions.

2 Basic Concepts

Let \(M\) be a smooth surface in \(\mathbb{E}^4\) given with the patch \(X(u, v) : (u, v) \in D \subset \mathbb{E}^2\). The tangent space to \(M\) at an arbitrary point \(p = X(u, v)\) of \(M\) span \(\{X_u, X_v\}\). In the chart \((u, v)\) the coefficients of the first fundamental form of \(M\) are given by

\[
E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) is the Euclidean inner product. We assume that \(g = EG - F^2 \neq 0\), i.e. the surface patch \(X(u, v)\) is regular.

Let \(\chi(M)\) and \(\chi^\perp(M)\) be the space of the smooth vector fields tangent to \(M\) and the space of the smooth vector fields normal to \(M\), respectively. Given any local vector fields \(X_1, X_2\) tangent to \(M\) consider the second fundamental map \(h : \chi(M) \times \chi(M) \rightarrow \chi^\perp(M)\);

\[
h(X_i, X_j) = \tilde{\nabla}X_i X_j - \nabla X_i X_j \quad 1 \leq i, j \leq 2.
\]

This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal frame field \(\{N_1, N_2\}\) of \(M\), recall the shape operator \(A : \chi^\perp(M) \times \chi(M) \rightarrow \chi(M)\);

\[
A_{N_i}X = - (\tilde{\nabla}X_i, N_i)^T, \quad X_i \in \chi(M).
\]

This operator is bilinear, self-adjoint and satisfies the following equation:

\[
\langle A_{N_k}X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = c^k_{ij}, \quad 1 \leq i, j, k \leq 2
\]

where \(c^k_{ij}\) are the coefficients of the second fundamental form. The equation \((3)\) is called Gaussian formula and

\[
h(X_i, X_j) = \sum_{k=1}^{2} c^k_{ij} N_k, \quad 1 \leq i, j \leq 2.
\]
The Gaussian curvature and normal curvature of the surface $M$ are given by
\[ K = \frac{1}{W^2} \sum_{k=1}^{2} (c_{11}^k c_{22}^k - (c_{12}^k)^2), \tag{7} \]
and
\[ K_N = \frac{1}{W^2} \left( E (c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1) - F (c_{11}^1 c_{22}^1 - c_{11}^2 c_{22}^2) + G (c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1) \right), \tag{8} \]
respectively. Recall that a surface $M$ is said to be flat (resp. has flat normal bundle) if its Gaussian curvature $K$ (resp. normal curvature $K_N$) vanishes identically. Further, the mean curvature vector of the surface $M$ is defined by
\[ \vec{H} = \frac{1}{2W^2} \sum_{k=1}^{2} (e_{11}^k G + e_{22}^k E - 2e_{12}^k F) N_k. \tag{9} \]
Recall that a surface $M$ is said to be minimal if its mean curvature vector vanishes identically \cite{3}.

### 3 Surface Pencils in $\mathbb{E}^4$

Let $\gamma = \gamma(s) : I \to \mathbb{E}^4$ be a unit speed regular curve in Euclidean 4-space $\mathbb{E}^4$. The corresponding Frenet formulas have the following form:
\[
\begin{align*}
\gamma'(s) &= V_1(s), \\
V_1'(s) &= \kappa_1(s) V_2(s), \\
V_2'(s) &= -\kappa_1(s) V_1(s) + \kappa_2(s) V_3(s), \\
V_3'(s) &= -\kappa_2(s) V_2(s) + \kappa_3(s) V_4(s), \\
V_4'(s) &= -\kappa_3(s) V_3(s),
\end{align*}
\tag{10}
\]
where $V_1(s), V_2(s), V_3(s), V_4(s)$ is the Frenet frame field and $\kappa_1, \kappa_2$ and $\kappa_3$ are the Frenet curvatures of $\gamma(s)$. If the Frenet curvatures are constant then $\gamma(s)$ is called W-curve \cite{5}.

Let $M$ be a local surface given with the regular patch
\[ X(s, t) = \gamma(s) + A(t)V_2(s) + B(t)V_4(s), \quad t \in J \subset \mathbb{R}, \]
where, $A = A(t)$, and $B = B(t)$ are smooth functions, defined in $J \subset \mathbb{R}$ and satisfying
\[
\begin{align*}
(1 - \kappa_1(s) A(t))^2 + (\kappa_2(s) A(t) - \kappa_3(s) B(t))^2 &> 0, \\
A'(t)^2 + B'(t)^2 &> 0.
\end{align*}
\]
For the sake of simplicity let us denote:
\[
\begin{align*}
a(s, t) &= 1 - \kappa_1(s) A(t), \\
b(s, t) &= \kappa_2(s) A(t) - \kappa_3(s) B(t).
\end{align*}
\tag{11}
This surface is one-parameter family of plane curves \( \alpha(t) = (A(t), B(t)) \) lying in the normal plane span \( \{V_2(s), V_4(s)\} \) of \( \gamma \). The surface given with the parametrization (11) is called pencil surface in \( \mathbb{E}^4 \). If \( \gamma(s) \) is a W-curve then \( M \) becomes a generalized rotation surface defined by Ganchev and Milousheva in [6] and see also [1].

We prove the following result.

**Proposition 1** Let \( M \) be a pencil surface given by the parametrization (1). Then the Gaussian curvature of \( M \) is

\[
K = \frac{(a^2 + b^2) (A'B'' - B'A'') \{ A'b\kappa_3 - B'(\kappa_1 a - \kappa_2 b) \} - ((A')^2 + (B')^2) (ab - b_a)^2}{g^2},
\]

where \( a(s, t) \) and \( b(s, t) \) are smooth functions defined in (11).

**Proof.** The tangent space of \( M \) is spanned by the vector fields

\[
X_s = a(s, t)V_1(s) + b(s, t)V_3(s), \quad X_t = A'(t)V_2(s) + B'(t)V_4(s).
\]

Hence, the coefficients of the first fundamental form of the surface are

\[
\begin{align*}
E &= \langle X_s, X_s \rangle = a(s, t)^2 + b(s, t)^2, \\
F &= \langle X_s, X_t \rangle = 0, \\
G &= \langle X_t, X_t \rangle = (A'(t))^2 + (B'(t))^2,
\end{align*}
\]

where \( \langle , \rangle \) is the standard scalar product in \( \mathbb{E}^4 \).

The second partial derivatives of \( X(s, t) \) are expressed as follows

\[
\begin{align*}
X_{ss} &= a_s(s, t)V_1(s) + (\kappa_1 a s, t) - \kappa_2(s)b(s, t)) V_2(s) + b_s(s, t)V_3(s) + \kappa_3(s)b(s, t)V_4(s), \\
X_{st} &= a_t(s, t)V_1(s) + b_t(s, t)V_3(s), \\
X_{tt} &= A''(t)V_2(s) + B''(t)V_4(s)
\end{align*}
\]

Further, the normal space of \( M \) is spanned by the vector fields

\[
\begin{align*}
N_1 &= \frac{1}{\sqrt{(A'(t))^2 + (B'(t))^2}}(-B'V_2 + A'V_4), \\
N_2 &= \frac{1}{\sqrt{a(s, t)^2 + b(s, t)^2}}(-b(s, t)V_1 + a(s, t)V_3).
\end{align*}
\]

Using (5), (14) and (15) we can calculate the coefficients of the second
fundamental form as follows:

\[
c_{11} = \langle X_{ss}(s,t), N_1 \rangle = \frac{A'b\kappa_3 - B'(\kappa_1 a - \kappa_2 b)}{\sqrt{(A'(t))^2 + (B'(t))^2}},
\]

\[
c_{12} = \langle X_{st}(s,t), N_1 \rangle = 0,
\]

\[
c_{22} = \langle X_{tt}(s,t), N_1 \rangle = \frac{A'B'' - B'A''}{\sqrt{(A'(t))^2 + (B'(t))^2}},
\]

\[
(16)
\]

\[
c_{11}^2 = \langle X_{ss}(s,t), N_2 \rangle = \frac{ab_s - ba_s}{\sqrt{a(s,t)^2 + b(s,t)^2}},
\]

\[
c_{12}^2 = \langle X_{st}(s,t), N_2 \rangle = \frac{ab_t - ba_t}{\sqrt{a(s,t)^2 + b(s,t)^2}},
\]

\[
c_{22}^2 = \langle X_{tt}(s,t), N_2 \rangle = 0.
\]

With the help of (7) and (16), we obtain the Gaussian curvature given with the equation (12).

An easy consequence of Proposition 1 is the following.

**Corollary 2** Let \( M \) be a pencil surface given by the parametrization (1). If

\[
A'B'' - B'A'' = 0 \quad \text{and} \quad ab_t - ba_t = 0,
\]

(17)

hold then \( M \) has vanishing Gaussian curvature.

**Proposition 3** Let \( M \) be a pencil surface given by the parametrization (1). Then the mean curvature of \( M \) is

\[
\left\| \overrightarrow{H} \right\|^2 = \frac{1}{4} \left\{ \left( \frac{(ab_s - ba_s)}{E^3} \right)^3 + \frac{1}{G} \left[ \frac{(A'B'' - B'A'')}{G} + \frac{A'b\kappa_3 - b'(\kappa_1 a - \kappa_2 b)}{E} \right]^2 \right\}.
\]

(18)

**Proof.** Substituting (13) and (16) into (9) we get

\[
\overrightarrow{H} = \frac{1}{2EG^{3/2}} \left\{ E(A'B''-B'A'') + G(A'b\kappa_3-B'(\kappa_1 a-\kappa_2 b)) \right\} N_1 + \frac{1}{2EG^{3/2}} (ab_s - ba_s) N_2.
\]

(19)

The norm of the mean curvature vector (19) gives (18). ■

**Proposition 4** Let \( M \) be a pencil surface given by the parametrization (1). Then the normal curvature of \( M \) is

\[
K_N = \frac{(ab_t - ba_t)}{E^2G^2} \left\{ G(A'b\kappa_3 - B'(\kappa_1 a - \kappa_2 b)) - E(A'B'' - B'A'') \right\}.
\]

(20)

**Proof.** Substituting (13) and (16) into (8) we get the result. ■

Now, we consider some special cases of surface pencils.

**Case I:** Let \( M \) be a pencil surface given with the \( W \)-curve

\[
\gamma(s) = (a \cos cs, a \sin cs, b \cos ds, b \sin ds)
\]
as generator. Then $M$ becomes a generalized rotation surface of the form

$$X(s, t) = (f(t) \cos cs, f(t) \sin cs, g(t) \cos ds, g(t) \sin ds)$$  \hfill (21)

where

$$f(t) = a + \frac{1}{k_1} \left( bd^2 B(t) - ac^2 A(t) \right),$$

$$g(t) = b + \frac{1}{k_1} \left(-bd^2 A(t) - ac^2 B(t) \right).$$  \hfill (22)

(see, [6]). For the case $f(t) = r(t) \cos t, g(t) = r(t) \sin t$ and $c = d = 1$ the generalized rotation surface $M$ is called Vranceanu surface [11]. Furthermore, for the case $f(t) = \cos t, g(t) = \sin t$ and $c = d = 1$ the generalized rotation surface $M$ is called Lawson surface [9].

**Proposition 5** For the marching-scale functions

$$A(t) = -\frac{ak_1(r(t) \cos t - a) + bk_1(r(t) \sin t - b)}{a^2 + b^2},$$

$$B(t) = \frac{bk_1(r(t) \cos t - a) - ak_1(r(t) \sin t - b)}{a^2 + b^2}.$$

the pencil surface $M \subset \mathbb{E}^4$ becomes a Vranceanu surface.

**Proof.** Let $M$ be a pencil surface given by the parametrization (1). Substituting $f(t) = r(t) \cos t, g(t) = r(t) \sin t$ and $c = d = 1$ into (22) we get the result. \hfill \blacksquare

**Corollary 6** Let $M$ be a pencil surface given by the parametrization (7). If the smooth functions $A(t)$ and $B(t)$ are given by

$$A(t) = -\frac{ak_1(\lambda e^{\mu t} \cos t - a) + bk_1(\lambda e^{\mu t} \sin t - b)}{a^2 + b^2},$$

$$B(t) = \frac{bk_1(\lambda e^{\mu t} \cos t - a) - ak_1(\lambda e^{\mu t} \sin t - b)}{a^2 + b^2}.$$

then $M$ becomes a flat Vranceanu surface in $\mathbb{E}^4$.

**Case II:** Suppose $M$ is a pencil surface given with the marching-scale functions $A(t) = B(t) = t$.

A standard ruled surface $M$ in a 4-dimensional Euclidean space $\mathbb{E}^4$ is defined by

$$M : X(s, t) = \gamma(s) + t\beta(s),$$  \hfill (23)

where

$$\beta(s) = \sum_{i=2}^{4} \beta_i V_i(s),$$  \hfill (24)

is the unit vector in $\mathbb{E}^4$ and $V_i(s)$ the Frenet vector of the unit speed curve $\gamma(s)$ [5].

By the use of (23), (24) with (1) we get the following result.
Corollary 7 Let $M$ be generalized standard ruled surface given with the parametrization
\[ X(s, t) = \gamma(s) + \frac{t}{\sqrt{2}} (V_2(s) + V_4(s)). \] (25)
Then $M$ is a pencil surface with the marching scale functions $A(t) = B(t) = t$.

Corollary 8 Let $M$ be a pencil surface given with the parametrization (25). Then the Gaussian and normal curvatures of $M$ are given by
\[ K = -\frac{(\kappa_2 - \kappa_3)^2}{((1 - t\kappa_1)^2 + t^2(\kappa_2 - \kappa_3)^2)^2}, \]
and
\[ K_N = \frac{(\kappa_2 - \kappa_3)(t(\kappa_1^2 + \kappa_2^2 - \kappa_3^2) - \kappa_1)}{2((1 - t\kappa_1)^2 + t^2(\kappa_2 - \kappa_3)^2)^2}. \]
respectively.

As a consequence of Corollary 8 we obtain the following result.

Corollary 9 If $\kappa_2 = \kappa_3$ then generalized standard ruled surface given with the parametrization (25) is a flat surface with flat normal bundle.

Case III: Suppose $M$ is a pencil surface given the parametrization
\[ A(t) = r(t) \cos t, \quad B(t) = r(t) \sin t, \] (26)
then we obtain the following result.

Proposition 10 Let $M$ be a pencil surface given with the parametrization (26). If $M$ is a flat surface satisfying (17) then one of the following case is occurs;

i) The profile curve $\gamma(s)$ is a planar and
\[ r(t) = \frac{1}{c_1 \sin t - c_2 \cos t}. \]

ii) The profile curve $\gamma(s) \subseteq E^4$ has curvatures $\kappa_1, \kappa_2$ and $\kappa_3$ with
\[ \kappa_3(s) = \frac{c_1 \kappa_2(s)}{c_2 + \kappa_1(s)} \quad \text{and} \quad r(t) = \frac{1}{c_1 \sin t - c_2 \cos t}. \]

iii) The profile curve $\gamma(s)$ is a circle and
\[ r(t) = \frac{1}{c_1 \sin t - c_2 \cos t}. \]

iv) The profile curve $\gamma(s)$ has constant first curvature
\[ \kappa_1(s) = \frac{1}{c_1} \quad \text{and} \quad r(t) = \frac{c_1}{c_1 \cos t}. \]

Here $c_1$ and $c_2$ are real constants.
Proof. Let $M$ be a pencil surface given with the parametrization (26). Then substituting (26) into the (17) we get the following system of differential equations
\begin{align*}
2(r')^2 - r r'' + r^2 &= 0, \\
\kappa_1\kappa_3r^2 + (r'\kappa_2 - r\kappa_3) \cos t - (r'\kappa_3 + r\kappa_2) \sin t &= 0.
\end{align*}
Solving this system of equations with the help of Maple programme we get the required results.

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