Brent Everitt

The Combinatorial Topology of Groups

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1

Combinatorial Complexes

By graph and map of graphs, I mean something purely combinatorial or algebraic. Pictures can be drawn, but one has to understand that maps are rigid and not just continuous, maps do not . . . wrap edges around several edges.–John Stallings

1.1 1-Complexes (ie: Graphs)

1.1.1 The category of graphs

Definition 1.1 (1-complex: first go). A 1-complex or graph is a non-empty set $X$ together with an involutary map $i : X \to X$ (ie: $i^2 = \text{id}_X$) and an idempotent map $s : X \to X^0$ (ie: $s^2 = s$) where $X^0$ is the set of fixed points of $i$.

Thus a graph has 0-cells or vertices $X^0$ and 1-cells or edges $X^1 = X \setminus X^0$. From now on we will write $x^{-1}$ for $i(x)$, and say that the edge $e \in X^1$ has start vertex $s(e)$ and terminal vertex $s(e^{-1})$. One thinks of the inverse edge $e^{-1}$ as just $e$, but traversed in the reverse direction (or with the reverse orientation). The edge $e$ is incident with the vertex $v$ if $e \in s^{-1}(v)$. We draw pictures like Figure 1.1 although they are purely for illustrative purposes. If the vertex set has cardinality that of the powerset of the continuum for instance, then there are not enough points on a piece of paper for a picture to fit! Definition 1.1 is quite terse, and it is sometimes useful to spell it out a little more:

Definition 1.2 (1-complex: second go). A 1-complex or graph consists of two disjoint non-empty sets $X^0$ and $X^1$, together with two incidence maps and an inverse map,

$$s, t : X^1 \to X^0 \quad \text{and} \quad -1 : X^1 \to X^1,$$

such that, (i). $e^{-1} \neq e = (e^{-1})^{-1}$ for all $e \in X^1$, and (ii). $t(e) = s(e^{-1})$ for all $e \in X$.

More terminology: an arc is an edge/inverse edge pair, and an orientation for $X$ is a set $\vec{\omega}$ consisting of all the vertices and exactly one edge from each arc. Write $\vec{\omega}$ for the arc containing the edge $e$, so that $e^{-1} = \vec{\omega}$. The graph $X$ is finite when $X^0$ is finite and locally finite when the set $s^{-1}(v)$ is finite for every $v \in X^0$. Thus a finite graph may have infinitely many edges, a situation that possibly differs from that in combinatorics. The cardinality of the set $s^{-1}(v)$ is the valency of the vertex $v$. A pointed graph is a pair $X_v := (X, v)$ for $v \in X$ a vertex.

Fig. 1.1. an edge of a graph with its start and terminal vertices.
Exercise 1.3. Here is another definition of graph more in the spirit of \[\text{§1.2}\]. A 0-complex is a non-empty set \(X\) and a map of 0-complexes is a map \(f : X \to Y\) of sets. The 0-sphere \(S^0\) is the 0-complex with two elements.

A graph \(X\) is a graded set \(X = X^0, X^1\) with \(X^1 \neq \emptyset\), such that

(C1). \(X^0\) is a 0-complex;
(C2). there is an involutory map \(-1 : X \to X\) with fixed point set \(X^0\);
(C3). each \(e \in X^1\) has boundary \(\partial e = (X^e, \alpha_e)\) with \(X^e\) the 0-sphere \(S^0\) and \(\alpha_e : X^e \to X^{(0)}\) a map of 0-complexes.

Here is the exercise: show that all three definitions of graph are equivalent.

The trivial graph has a single vertex and no edges. Figure 1.2 shows some more examples of graphs, including some with countably many edges, which will tend to be more interesting than finite graphs.

A graph map is a set map \(f : X \to Y\) such that the diagram on the left of Figure 1.3 commutes, where \(\sigma_X\) is one of the maps \(s_X\) or \(-1\) for \(X\), and \(\sigma_Y\) similarly, ie: \(fs_X(x) = s_Yf(x)\) and \(f(x^{-1}) = f(x)^{-1}\). Notice that a map can send edges to vertices, and so we call \(f\) dimension preserving if we also have \(f(X^1) \subseteq Y^1\). A map \(f : X_v \to Y_u\) of pointed graphs is a graph map \(f : X \to Y\) with \(f(v) = u\).

The commuting of \(f\) with \(s\) and \(-1\) is a combinatorial version of continuity: an edge incident with a vertex is either mapped to an edge incident with the image of the vertex, or to the image vertex itself. In the second case, if \(e\) is an edge mapped by \(f\) to a vertex \(u\) as on the right in Figure 1.3, then the commuting condition becomes \(fs_X(e) = f(u)\), and in particular the start vertex \(s_X(e)\) must also be mapped to \(u\) (the condition also ensures that \(t_X(e)\) is mapped to \(u\), and not left “hanging”).

Exercise 1.4. Show that graphs and their mappings form a category.

For a fixed vertex \(v \in X\), and \(s_X^{-1}(v) \in X^1\) the edges starting at \(v\), the map \(f : X \to Y\) induces a map \(\tilde{s}_X^{-1}(v) \to s_Y^{-1}(u)\) where \(u = f(v)\). We call this induced map the local continuity of \(f\) at the vertex \(u\).

A graph map \(f : X \to Y\) preserves orientation whenever there are orientations \(\mathcal{O}_X\) for \(X\) and \(\mathcal{O}_Y\) for \(Y\) with \(f(\mathcal{O}_X) \subset \mathcal{O}_Y\). We leave it as an exercise to show that it is always possible to choose orientations for \(X\) and \(Y\) making a map \(f : X \to Y\) orientation preserving (although a map may not be orientation preserving with respect to fixed orientations).
A map \( f : X \to Y \) is an isomorphism if it is dimension preserving and a bijection on the vertex and edge sets.

**Exercise 1.5.** Show that if \( f : X \to Y \) is an isomorphism then the inverse map \( f^{-1} : Y \to X \) is also a graph isomorphism. Show that the set \( \text{Aut}(X) \) of graph isomorphisms \( X \to X \) forms a group under composition.

A group \( G \) acts on a graph \( X \) if there is a homomorphism \( G \to \text{Aut}(X) \). We abbreviate \( \varphi(g)(x) \) to \( g(x) \). An action preserves orientation if there is an orientation \( \emptyset \) for \( X \) with \( g(\emptyset) = \emptyset \) for all \( g \in G \).

**Exercise 1.6.** Let \( X \) be a set and \( ^{-1} : X \to X \) a bijective map without fixed points. Let \( G \to \text{Sym}(X) \) be a group action (here \( \text{Sym}(X) \) is the symmetric group on \( X \)) that commutes with \( ^{-1} \), ie: \( g(x^{-1}) = g(x)^{-1} \) for all \( x \in X \). An inversion is a \( g \in G \) such that \( g(x) = x^{-1} \) for some \( x \), and \( G \) is said to act without inversions if no \( g \in G \) is an inversion (equivalently, no \( G \)-orbit contains both some \( x \) and its inverse \( x^{-1} \)). Show that there exists an \( \emptyset \subset X \) with \( X = \emptyset \cup \emptyset^{-1} \) a disjoint union and \( g(\emptyset) = \emptyset \) if and only if \( G \) acts without inversions on \( X \).

A group \( G \) acts freely if and only if the action is free on the vertices, ie: if \( g \in G \) and \( v \) a vertex with \( g(v) = v \) implies \( g \) is the identity element.

**Exercise 1.7.** If \( G \) acts freely and orientation preservingly on a graph, then show that the action is free on the edges too.

Graph isomorphisms are pretty rigid, and it is useful to have a relation with a bit more “slack”. Thus, a subdivision of an edge replaces it by two new edges and a new vertex as in Figure 1.4, or is the reverse of this process. Write \( X \leftrightarrow X' \) when two graphs differ by the subdivision of a single edge. Two graphs \( X \) and \( Y \) are then homeomorphic, written \( X \approx Y \),

![Fig. 1.4. Subdividing an edge](image)

when there is a finite sequence \( X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_k = Y \) of subdivisions connecting them. It is easy to see that homeomorphism is an equivalence relation for graphs. A topological invariant is a property of graphs that is invariant under homeomorphism in the sense that if \( X \approx Y \), then \( X \) has the property if and only if \( Y \) does.

### 1.1.2 Quotients and subgraphs

The most useful construction in the category of graphs is the quotient:

**Definition 1.8 (quotient relation and quotient graph).** If \( X \) is a graph, then a quotient relation is an equivalence relation \( \sim \) on \( X \) such that

\[
\begin{align*}
& (i). \ x \sim y \Rightarrow s(x) \sim s(y) \text{ and } x^{-1} \sim y^{-1} \\
& (ii). \ x \sim x^{-1} \Rightarrow [x] \cap X^0 \neq \emptyset,
\end{align*}
\]

where \( [x] \) is the equivalence class of \( x \). If \( \sim \) is a quotient relation on a graph \( X \) then define \( s \) and \( ^{-1} \) on the equivalence classes \( X/\sim \) by

\[
\begin{align*}
& (i). \ s[x] = [s(x)], \quad (ii). \ [x]^{-1} = [x^{-1}].
\end{align*}
\]
Notice that edges can be equivalent to vertices, but if an edge is equivalent to its inverse then it must also be equivalent to a vertex. This ensures that in the quotient we have \([e] \neq [e]^{-1}\).

**Proposition 1.9.** If \(\sim\) is a quotient relation then \(X/\sim\) with the maps \(s\) and \(s^{-1}\) defined above is a graph, and the quotient map \(q : X \to X/\sim\) given by \(q(x) = [x]\) is a map of graphs.

The proof is a straightforward exercise. In particular, the fixed points in \(X/\sim\) of the new inverse map \(s^{-1}\) are precisely those equivalence classes \([x]\) where \(x \sim v\) for some \(v \in X^0\). Thus the quotient has vertices the \([v]\) for \(v \in X^0\) (and these classes may include some of the edges of the old graph \(X\) and edges those \([e]\) with \([e] \cap X^0 = \emptyset\).

The two main examples of graph quotients arise by factoring out the action of a group, or by squashing a subgraph down to a vertex. For the first we have the following.

**Proposition 1.10.** Let \(\sim\) be the equivalence relation on \(X\) given by the orbits of the action of a group \(G\). Then \(\sim\) is a quotient relation if and only if the group action is orientation preserving, and we write \(X/G := X/\sim\) for the quotient.

Again, the proof is left as an exercise (see Exercise 1.6). A graph \(X\) is a subgraph of \(Y\) if there is a mapping \(X \hookrightarrow Y\) that is an isomorphism onto its image. Equivalently, it is a subset \(X \subset Y\), such that the maps \(s\) and \(s^{-1}\) give a graph when restricted to \(X\).

Let \(X \subset Y\) be a subgraph and define a relation \(\sim\) on \(Y\) by \(x \sim y\) if and only if \(x = y\) or \(x\) and \(y\) lie in \(X\). Then this is a quotient relation and we write \(Y/X\) for \(Y/\sim\), the quotient of \(Y\) by the subgraph \(X\). It is what results by squashing \(X\) to a vertex.

Extending this a little, if \(X_\alpha\) (\(\alpha \in A\)) is a family of disjoint subgraphs in \(Y\) then define \(\sim\) by \(x \sim y\) iff \(x = y\) or \(x\) and \(y\) lie in the same \(X_\alpha\), and write \(Y/X_\alpha\) (\(\alpha \in A\)), or just \(Y/X_\alpha\), for the corresponding quotient. Note the difference between this, where each \(X_\alpha\) has been squashed to a distinct vertex \(v_\alpha\), and \(Y/(\bigcup X_\alpha)\), where the whole union is squashed to just the one vertex.

**Exercise 1.11.** Recall that an equivalence relation on a set \(X\) is a subset \(S \subset X \times X\) such that, (i). \(S\) contains the diagonal, \((x,x) \in S\) for all \(x \in X\), (ii). \((x,y) \in S\) \(\Rightarrow\) \((y,x) \in S\), and (iii). \((x,y), (y,z) \in S\) \(\Rightarrow\) \((x,z) \in S\). Show that if \(S_\alpha\) (\(\alpha \in A\)) are equivalence relations on \(X\) then so is \(\bigcap S_\alpha\), and hence if \(X\) is any subset of \(X\) we may define the equivalence relation generated by \(Y\) to be the intersection of all equivalence relations \(S\) with \(Y \subset S\).

**Exercise 1.12.** Let \(X_1, X_2\) and \(Y\) be graphs and \(f_i : Y \to X_1 (i = 1, 2)\) dimension preserving maps of graphs. Let \(\sim\) on the disjoint union \(X_1 \bigcup X_2\) be the equivalence relation generated by \(x \sim y\) if and only if there is a \(z \in Y\) with \(x = f_1(z)\) and \(y = f_2(z)\). Show that \(\sim\) is a quotient relation if there are orientations \(\emptyset\) for \(Y\), and \(O_i\) for \(X_i (i = 1, 2)\) with \(f_i(\emptyset) \subseteq O_i\).

### 1.1.3 Balls, spheres, paths and homotopies

The \(I^1\)-graph and \(S^1\)-graph are shown in Figure 1.5. A graph \(X\) is a 1-ball if it is homeomorphic to \(I^1\) (or is trivial), and a 1-sphere if it is homeomorphic to \(S^1\) (or is trivial).

It is easy to see that a 1-ball and 1-sphere have the form shown in Figure 1.6 and that the vertices of a 1-sphere can be labelled \(v_0, \ldots, v_{n-1}\) and the edges \(e_1^\pm, \ldots, e_{n-1}^\pm\) with
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Fig. 1.6. a 1-ball and a 1-sphere.

\[ s(e_i) = v_i \text{ and } t(e_i) = v_{i+1} \quad (i < n - 1) \text{ and } t(e_{n-1}) = v_0. \]

Similarly for a 1-ball, so that it has \textit{end vertices} \( v_0, v_n \) in an obvious sense. The following is easily proved by induction:

\begin{lemma}
A graph \( X \) is a 1-sphere if and only if either \( X \) is trivial or \( X = S^1 \), or there are non-trivial 1-balls \( B_i \) (\( i = 1, 2 \)) with end vertices \( v_{i1}, v_{i2} \), such that \( X = B_1 \cup B_2 / \sim \),

where the equivalence classes of \( \sim \) are \( \{v_{11}, v_{21}\}, \{v_{12}, v_{22}\} \) and the \( \{x\} \) for all other cells \( x \in X_1 \cup X_2 \).

The \textit{standard orientation} \( \mathcal{O} \) for a 1-sphere consists of all the vertices and \( \{e_0, e_1, \ldots, e_{n-1}\} \), \ie the edges taken in a clockwise direction in Figure 1.6. From now on, an orientation preserving map between 1-spheres \textit{preserves the standard orientations} on each.

\begin{exercise}
Let \( X, Y \) be 1-spheres with their standard orientations \( \mathcal{O}_X = \{e_{11}\}, \mathcal{O}_Y = \{e_{21}\} \) and \( f : X \to Y \) an orientation preserving map. If \( f(e_{1i}) = e_{2j} \), then show that \( f(e_{1i+1}) = e_{2j+1} \), or is the vertex \( t(e_{2j}) \). Deduce that if \( f \) is an orientation preserving isomorphism then it is a rotation (in the obvious sense).

There is one particular map of 1-spheres that is not orientation preserving but will be nevertheless useful later on. If \( X \) is a 1-sphere, let \( \iota : X \to X \) be the map interchanging the edges \( e_i \) and \( e_{n-i+1} \) as in Figure 1.7.

\[ \text{Fig. 1.7. the map } \iota : X \to X \text{ for } X \approx S^1. \]

A \textit{path} in \( X \) is a graph mapping \( \gamma : B \to X \) with \( B \) a 1-ball. It is convenient not to insist that the map preserve dimension, but by Exercise 1.15 below, we can always replace the 1-ball by another so that the map is dimension preserving. In any case, the image in \( X \) is a sequence of edges \( e_1 \ldots e_k \) (which we will also call \( \gamma \)), that are consecutively incident
in the obvious way: \( s(e^{-1}) = s(ei + 1) \), and there is no harm in thinking about paths in terms of their images. A path joins the vertices \( s(e1), s(ei + 1) \) that are the images of the end vertices of the 1-ball, and is closed if these end vertices have the same image. If \( \gamma : B \rightarrow X \) is the path \( e1 \cdots ek \) then the inverse path \( \gamma^{-1} : B \rightarrow X \) has edges \( e^{-1}k \cdots e^{-1}1 \).

**Exercise 1.15.** Let \( \gamma : B \rightarrow X \) be a path with the edges of \( B \) labelled \( e^{\pm}1, \ldots, e^{\pm}n \) as in the comments before Lemma 1.13 and image edges \( e_1' \cdots e_k' \) in \( X \). Show there are \( 1 \leq i_1 \leq \cdots \leq i_\ell \leq n \) with \( f(e_{i_j}) = e_1' \) and all other edges mapped to vertices. Thus, \( B \) can be replaced by a 1-ball \( B' \) and dimension preserving map \( \gamma' : B' \rightarrow X \) having the same image path.

**Exercise 1.16.** Show that a closed path \( \gamma : B \rightarrow X \) gives a mapping \( S \rightarrow X \) with \( S \) a sphere.

If \( f : X \rightarrow Y \) is a graph map and \( \gamma : B \rightarrow X \) a path, then there is an induced path in \( Y \) given by the composition \( f \gamma : B \rightarrow Y \). Thus, a graph mapping sends paths to paths. An example is when we have a quotient relation ~ on a graph \( X \) with quotient map \( q : X \rightarrow X/\sim \). Then the quotient relation can be easily extended to paths in \( X \); if \( \gamma, \mu \) are two such, then \( \gamma \sim \mu \) precisely when \( q(\gamma) = q(\mu) \) give the same path in the quotient \( X/\sim \).

The graph \( X \) is connected if any two vertices can be joined by a path. The connected component of \( X \) containing the vertex \( v \) consists of those vertices that can be joined to \( v \), together with all their incident edges. A connected graph has finitely many edges if and only if it is finite and locally finite.

A spur is a path of the form \( ee^{-1} \), i.e., a path that consecutively traverses an edge and its inverse. An elementary homotopy of a path, \( e1 \cdots ei+1 \cdots e_\ell \leftrightarrow e1 \cdots e1(ee^{-1})e_{i+1} \cdots e_\ell \) inserts or deletes a spur so that incidence is preserved as in Figure 1.8.

![Fig. 1.8. elementary homotopy](image-url)

**Proposition 1.18.** The following are equivalent for a graph \( X \):

1. There is at most one reduced path joining any two vertices;
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2. any closed path is homotopically trivial;
3. any non-trivial closed path contains a spur.

A graph satisfying any of the conditions of Proposition 1.18 is called a forest, and a connected forest is a tree.

Proof. (1 $\Rightarrow$ 2): a closed path $\gamma$ is necessarily contained in a component of the graph, thus by assumption there is a unique reduced path connecting any two of its vertices. The path cannot contain just a single vertex $u$ and edge $e$ (which it circumnavigates some number of times), for if so, then the edge $e$ and the trivial path at $u$ are distinct reduced paths from $u$ to $u$. Thus any closed path contains at least two distinct vertices. We show the path $\gamma$ is homotopic to the trivial path based at one of them, say $u$. Let $v$ be another vertex of $\gamma$ and $w$ the reduced path running from $u$ to $v$ as in Figure 1.9 (left). Then $\gamma$ decomposes into two parts, $w_1$ running from $u$ to $v$ and $w_2$ running from $v$ to $u$. If $w_1 \neq w$ then it cannot be reduced, hence must contain a spur. Removing it and continuing, we have a series of homotopies that reduces $w_1$ to $w$ as in Figure 1.9 (middle). Similarly for $w_2$ and $w^{-1}$.

2. Exercise 1.19. Let $X$ be a finite graph, remembering that this means that the vertex set $X^0$ is finite. If each vertex has valency at least two, show that $X$ contains a homotopically non-trivial closed path. Deduce that if $T$ is a finite tree, then $|T^1| = 2(|T^0| - 1)$.

We'll have more to say about trees later. We finish this section by considering how to approximate a graph by a tree: if $X$ is a connected graph, then a spanning tree is a subgraph $T \subset X$ that is a tree and contains all the vertices of $X$ (ie: $T^0 = X^0$). The following exercise shows that under some mild set-theoretic assumptions, spanning trees always exist.
Exercise 1.20. Recall the well-ordering principle from set theory: any set $X$ can be partially ordered $\leq$ (see §3.4) so that for any $x, y \in X$, either $x \leq y$ or $y \leq x$, and for any $S \subset X$ there is a $s \in S$ with $s \leq x$ for all $x \in X$. In particular, choose a well ordering of the edges set of a graph. Choose a basepoint vertex $v_0$, and consider those vertices at distance one from $v_0$, ie: the $v \neq v_0$ with $s(e) = v_0, s(e^{-1}) = v$ for some edge $e$. For each such, choose an edge $e_0$ that is minimal in the well-ordering amongst the edges joining $v_0$ to $v$. Let $T_1$ be the subgraph consisting of $v_0$, its distance 1 neighbours and the edges so chosen.

1. Show that $T_1$ is a tree. Continue the construction inductively: at step $k$, take the tree $T_{k-1}$ constructed at step $k - 1$, and for each vertex $v$ of $X$ a distance 1 from a vertex of $T_{k-1}$, choose a minimal edge as above. Let $T_k$ be the subgraph consisting of $T_{k-1}$ together with the distance 1 vertices and minimal edges. Show that $T_k$ is a tree.

2. Show that $T = \bigcup T_k$, is the required spanning tree.

In a graph the edge set can have a wildly different cardinality from the vertex set, causing difficulties with some arguments. This shortcoming is avoided by spanning trees which have a number of edges that is “roughly” the same as the number of vertices of the graph they span:

Proposition 1.21. Let $X$ be a connected graph and $T \subset X$ a spanning tree. Then

$$|T^1| = \begin{cases} 2|X^0| - 1, & \text{if } X \text{ is finite}, \\ |X^0|, & \text{if } X \text{ is infinite}. \end{cases}$$

Proof. The result for finite graphs is the content of Exercise 1.19. If $X$ is an infinite graph with spanning tree $T$, then the edge set of $T$ must be infinite, as a finite edge set only spans $|T^1| + 1$ vertices. Then $X^0 = T^0 = \bigcup_{e \in T_1} \{ s(e), s(e^{-1}) \}$ has the same cardinality as $T^1$.

Exercise 1.22. Let $T_\alpha \subset X (\alpha \in A)$ be a family of mutually disjoint trees in a connected graph $X$. Show there is a spanning tree $T \subset X$ containing the $T_\alpha$ as subgraphs, and such that $q(T)$ is a spanning tree for $X/T_\alpha$, where $q : X \to X/T_\alpha$ is the quotient map.

A spanning forest is a subgraph $\Phi \subset X$ that is a forest and contains all the vertices of $X$. By considering $q^{-1}(T')$ for some spanning tree $T'$ of the (connected) graph $X/\Phi$, show that any spanning forest can be extended to a spanning tree.

1.2 The category of 2-complexes

1.2.1 2-complexes

Definition 1.23 (2-complex). A combinatorial 2-complex $X$ is a graded set $X = X^0, X^1, X^2$ with $X^1 \neq \emptyset$, such that

(C1). if $X^{(1)} := X^0 \cup X^1$, there are maps $^{-1} : X \to X$ and $s : X^{(1)} \to X^0$ making $X^{(1)}$ a graph;

(C2). each $\sigma \in X^2$ has boundary $\partial \sigma = (X^\sigma, \alpha_{\sigma})$ with $X^\sigma$ a 1-sphere and $\alpha_{\sigma} : X^\sigma \to X^{(1)}$ a dimension preserving map of graphs. The map $^{-1}$ extends to all of $X$, with $\sigma^{-1} \neq \sigma = (\sigma^{-1})^{-1}$ and $\partial^{-1} = (X^\sigma, \alpha_{\sigma})$, with $\partial$ the map given in §1.1.3.

The elements of $X^2$ are the 2-cells or faces and $\alpha_{\sigma}$ is the attaching map of the face $\sigma$ (see Figure 1.11). The underlying graph $X^{(1)}$ is called the 1-skeleton. One thinks of a face as a disc sewn by its boundary onto the 1-skeleton as in Figure 1.11.

We say that the vertex $v$ appears in the boundary of the face $\sigma$ whenever $\alpha_{\sigma}^{-1}(v) \neq \emptyset$. Thus there is a vertex $x$ of $X^\sigma$ mapping to $v$ via the attaching map $\alpha_{\sigma}$ as in Figure 1.11 and indeed there may be several of them. We will call the vertices in $\alpha_{\sigma}^{-1}(v) \subset X^\sigma$ the...
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Fig. 1.11. boundary of a face \( \sigma \) consisting of a 1-sphere and an attaching map, which may wrap the sphere around the face several times. The vertex \( v \) appears in the boundary of \( \sigma \) and the red path wrapping twice around is a boundary path of \( \sigma \) starting at \( v \).

appearances of \( v \) in the boundary of the face \( \sigma \). Similarly for edges. For an appearance \( x \) of \( v \), if we take a path \( \gamma \) consisting of the standard orientation of \( X^\sigma \), then we call its image a boundary path of \( \sigma \) starting at \( v \). We will often write \( \gamma \) for both the path in \( X^\sigma \) and its image under the attaching map. The vertex \( v \) appears a total of \( |\alpha^{-1}_\sigma(v)| \) times in the boundary of \( \sigma \), and each appearance gives rise to a pair of boundary paths starting at \( v \).

Exercise 1.24. There is a more elementary notion of 2-complex where the boundary \((X^\sigma, \alpha_\sigma)\) is identified with its image under the attaching map: we have a graded set \( X = X^0, X^1, X^2 \) with \( X^i \neq \emptyset \), such that

(C1). if \( X^{(1)} := X^0 \cup X^1 \), there are maps \( -1 : X \to X \) and \( s : X^{(1)} \to X^0 \) making \( X^{(1)} \) a graph;

(C2). each \( \sigma \in X^2 \) has boundary \( \partial \sigma \) all cyclic permutations of some fixed closed path \( \gamma_\sigma \);

(C3). the map \( -1 \) has no fixed points in \( X^2 \) and \( \partial \sigma^{-1} \) consists of all cyclic permutations of the inverse path \( \gamma_\sigma^{-1} \).

Show that a 2-complex in our sense gives rise to such a 2-complex, and vice-versa (although we will prefer to keep track of the attaching maps of the faces).

Fig. 1.12. a 2-complex: pictorial version of definition 1.23 (top left); face-centric version with faces sewn on (bottom left) and topologically suggestive version (right).

1.2.2 Examples

Figure 1.12 gives three different versions of a 2-complex that we will call the 2-sphere. The first version (top left) is a straight pictorial version of definition 1.23: the 1-skeleton in the
middle is a graph with two vertices and two edges; there are two faces and the attaching maps are described by labeling the edges of the $X^{\sigma_i}$ with their images in the 1-skeleton. This version is both the most accurate and the most cumbersome.

In the second version (bottom left) we have adopted the convention that parts of the complex with the same label give the same cell and have drawn the complex “face-centrically” with the faces thought of as discs sewn onto the 1-skeleton. The third version is more along the lines of Exercise 1.24 with the face boundaries given by closed paths and their cyclic permutations.

Figure 1.13 gives a very similar example, except that one of the face boundaries goes the other way around the 1-skeleton.

![Fig. 1.13. another 2-complex](image)

Figure 1.14 is the (real) projective plane $\mathbb{RP}^2$, a combinatorial model for the disc with antipodal points on the boundary identified. We have again drawn the complex both face-centrically and \textit{ala} Definition 1.23. Similarly, Figure 1.15 shows various versions of the torus complex.

**Fig. 1.14. projective plane complex**

1.2.3 Maps of 2-complexes

To complete the definition of the category of 2-complexes we need mappings. The principle is the same as for graph maps in §1.1.1: continuity is captured by making maps commute with the various attaching maps of the cells. The definition is complicated slightly as we allow a map to squash a face down to a path.

**Definition 1.25 (maps of 2-complexes).** A map $f : X \to Y$ of 2-complexes is a map $f : X^{(1)} \to Y^{(1)}$ of the underlying graphs such that for each face $\sigma \in X^2$ we have either $f(\sigma)$ is a face $\tau \in Y^2$ or $f(\sigma)$ is a closed path in the graph $Y^{(1)}$. There are the conditions:

(M1). Let $f(\sigma) = \tau$, a face in $Y$, with $\partial \sigma = (X^{\sigma}, \alpha_\sigma)$ and $\partial \tau = (Y^{\tau}, \alpha_\tau)$. Then there is an \textit{orientation preserving} map $\varepsilon = \varepsilon(f, \sigma) : X^{\sigma} \to Y^{\tau}$ making the diagram below left commute;
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Fig. 1.15. various versions of the torus.

Moreover $f(\sigma^{-1}) = \tau^{-1}$ and $\varepsilon(f, \sigma^{-1}) = \varepsilon(f, \sigma)$;

(M2). Let $f(\sigma)$ be the closed path $\gamma : S \to Y$ where $S$ is a 1-sphere. Then there is an orientation preserving map $\varepsilon = \varepsilon(f, \sigma) : X^\sigma \to S$ making the diagram above right commute. Moreover, $f(\sigma^{-1})$ is the inverse path and the path $\gamma : S \to X$ is homotopically trivial.

Thus, if $f(\sigma) = \tau$, then a boundary path for $\sigma$ is mapped to a path that circumnavigates a boundary path for $\tau$, possibly a number of times. See Figure 1.16.

Fig. 1.16. a boundary path for $\sigma$ is mapped via $f$ to a boundary path of $\tau$, possibly repeated.

If a face is mapped to a path then this path must be the image (possibly repeated) of the boundary path of the face, hence closed. Moreover it is homotopically trivial. The motivating example is squashing a face flat: the boundary gets squashed too, into a path of the form $e_1 \ldots e_\ell e_\ell^{-1} \ldots e_1^{-1}$. In this example the result is clearly homotopically trivial, but we will really require this condition in Chapter 2 for the fundamental group $\pi_1$ to be a functor.

Definition 1.26 (map conventions). Let $X$, $Y$ and $Z$ be 2-complexes.

1. When are two maps the same? Let $f, g : X \to Y$ be maps of 2-complexes. Then $f = g$ when $f(x) = g(x)$ for all cells $x \in X$ and if $f(\sigma) = g(\sigma) = \tau$ for faces $\sigma \in X$ and $\tau \in Y$, then $\varepsilon(f, \sigma) = \varepsilon(g, \sigma)$.

2. Forming compositions: Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$, are maps of 2-complexes.

Then the composition $gf$ is formed in the usual way, with the additional proviso that if $\sigma \in X$ is a face with $f(\sigma)$ a face of $Y$ and $gf(\sigma)$ a face of $Z$, then $\varepsilon(gf, \sigma) = \varepsilon(g, f(\sigma)) \varepsilon(f, \sigma)$. 
These conventions have ramifications for commuting diagrams of complexes and maps, which are after all, just statements about maps being the same. For example, when we say that the diagram of maps and complexes on the left commutes,

\[
\begin{array}{ccc}
X & \xrightarrow{f_2} & Y_2 \\
\downarrow{f_1} & & \downarrow{g_1} \\
Y_1 & \xrightarrow{g_2} & Z
\end{array}
\quad \quad \begin{array}{ccc}
X & \xrightarrow{f_2(\sigma)} & Y_2 \\
\downarrow{g_1} & & \downarrow{g_2} \\
Y_1 & \xrightarrow{f_1(\sigma)} & Z_{g_1f_1(\sigma)}
\end{array}
\]

then the compositions \(g_1f_i\) \((i = 1, 2)\) are the same as the dotted map across the middle. If \(\sigma\) is a face of \(X\) that maps to a face \(g_i f_i(\sigma)\) of \(Z\), then the diagram of maps on the right must also commute.

Let \(f : X \to Y\) be a map of 2-complexes, \(u, v\) vertices with \(f(u) = v\), and \(\tau\) a face in \(Y\). If \(\sigma\) is a face of \(X\) that maps to \(\tau\), let \(\varepsilon(f, \sigma) : X^\sigma \to Y^\tau\) be the orientation preserving map from (M1) of Definition 1.27. This map induces a map \(\varepsilon(f, \sigma) : \alpha^{-1}_\tau(u) \subset X^\sigma \to \alpha^{-1}_\tau(v) \subset Y^\tau\) from the appearances of \(u\) in the boundary of \(\sigma\) to the appearances of \(v\) in the boundary of \(\tau\). As this is true for all the \(\sigma\) mapping to \(\tau\), we have,

**Definition 1.27 (local continuity).** Let \(f : X \to Y\) be a map of 2-complexes, \(u, v\) vertices with \(f(u) = v\), and \(\tau\) a face in \(Y\). The *local continuity* of \(f\) at \(v\) is

\[
\Pi \varepsilon(f, \sigma) : \bigcup_{f(\sigma) = \tau} \alpha^{-1}_\sigma(u) \to \alpha^{-1}_\tau(v),
\]

where \(\Pi \varepsilon(f, \sigma)\) is the (disjoint) union of the maps \(\varepsilon(f, \sigma)\) over the faces \(\sigma\) mapping to \(\tau\).

An example is given in Figure 1.17 with the mapping of the “plane” complex to the torus.

**Exercise 1.28.** Show that if the right hand side of the set in Definition 1.27 is empty, then so is the left hand side.

**Definition 1.29 (dimension preserving maps).** A map \(f : X \to Y\) is *dimension preserving* if and only if

1. the graph map \(f : X^{(1)} \to Y^{(1)}\) is dimension preserving;
2. \(f(X^2) \subset Y^2\);
3. if \(\sigma \in X^1\) and \(f(\sigma) \in Y\) are faces then \(\varepsilon(f, \sigma) : X^\sigma \to Y^\tau\) is an orientation preserving isomorphism.

A map is an *isomorphism* if it is dimension preserving, and a bijection on the vertex, edge and face sets. In this case one easily sees that, as for graphs, the inverse map \(f^{-1}\) is also an isomorphism \(f^{-1} : Y \to X\) (just reverse the horizontal arrows in the left commuting diagram of Definition 1.27 (M1)) so that the set of automorphisms \(f : X \to X\) forms a group \(\text{Aut}(X)\) under composition.

**Exercise 1.30.** Show that the 2-complexes of Figures 1.12 and 1.13 are isomorphic.

A group action \(G \xrightarrow{\cong} \text{Aut}(X)\) *preserves orientation* if there is an orientation \(\emptyset\) for \(X\) with \(g(\emptyset) = \emptyset\) for all \(g \in G\). One can then show, just as in Exercise 1.6, that an action of a group preserves orientation if and only if it acts without inversions and no \(g \in G\) sends a face to its inverse. A group acts *freely* on a 2-complex precisely when the action on the underlying graph is free.

A 2-complex \(X\) is a subcomplex of \(Y\) if there is a mapping \(X \hookrightarrow Y\) of 2-complexes that is an isomorphism onto its image.
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Fig. 1.17. Local continuity for a map of 2-complexes: the infinite plane complex maps via \( f \) to the torus; \( \tau \) is the single face of the torus and for the single vertex \( v \), the set \( \alpha_{\sigma}^{-1}(v) \in Y^\tau \) consists of the four vertices marked with little red and blue circles and squares. A fixed vertex \( u \) mapping to \( v \) appears in the boundary of four faces \( \sigma \), with \( \alpha_{\sigma}^{-1}(u) \) consisting of a single vertex in each case. For all other faces \( \sigma \) we have \( \alpha_{\sigma}^{-1}(u) = \emptyset \). The red and blue circles and squares in the \( X^\sigma \) map via the local continuity of Definition 1.27 to the corresponding ones in \( Y^\tau \).

1.2.4 Homotopies and homeomorphisms

As with graphs we can deform paths, simulating in a combinatorial manner the homotopies of paths in topology.

Let \( \gamma = e_1 \ldots e_\ell \) be a path in the 2-complex \( X \). An elementary homotopy either inserts or deletes a spur as in §1.1.3 or inserts or deletes the boundary of a face in the following sense. If \( \sigma \in X^2 \) is a face with boundary \( \partial \sigma = (X^\sigma, \alpha_{\sigma}) \), then by Exercise 1.16 \( \alpha_{\sigma}(X^\sigma) \) is a closed path, say \( e'_1 \ldots e'_k \). The homotopy inserts into (or deletes from) \( \gamma \) the result of completely traversing this closed path, starting at one of its vertices, so that all the incidences match up in the obvious way, ie: so that \( s(e'_j) = t(e'_{j+1}) \) is the vertex \( t(e_i) = s(e_{i+1}) \).

Pictures such as the right hand side of Figure 1.18 should be approached with care. The entire boundary path of \( \sigma \) must be traversed, including any repetitions. Later we will have faces with boundary a closed path that travels a number of times around an edge loop. Any homotopy involving this boundary must then travel the full number of times around the loop.

Two paths are homotopic precisely when there is a finite sequence of elementary homotopies, taking one to the other. A path homotopic to the trivial path is homotopically trivial. For example, two paths running different ways around a face are homotopic as shown in Figure 1.19.

Exercise 1.31. Show that homotopic paths have the same start and end vertices, and thus homotopically trivial paths are necessarily closed. Show that homotopy is an equivalence relation on the paths with common fixed endpoints.

We can also subdivide 2-complexes to get homeomorphic ones, although this will play less of a role than it does with graphs, where graphs homeomorphic to \( S^1 \) were essential to...
Fig. 1.19. homotoping a path across a face. To get from the first picture to the second, insert the boundary of the face $\sigma$; to get from the second to the third, remove the obvious spurs.

the definition of 2-complex. What we want to do is summarized by Figure 1.20 replace an

Fig. 1.20. subdividing a face

existing face $\sigma$ by two new faces $\sigma_1, \sigma_2$ by placing a new edge running between vertices of $\sigma$, or the reverse of this process.

**Exercise 1.32.** Formulate the definition of subdividing a face in the style of Definition 1.23 by using the description of 1-spheres given by Lemma 1.13.

Write $X \leftrightarrow X'$ when the two complexes differ by the subdivision of an edge or face, so that $X$ and $Y$ are then *homeomorphic*, written $X \approx Y$, when there is a finite sequence $X = X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_k = Y$ of subdivisions (of either type) connecting them. It is easy to see that homeomorphism is an equivalence relation and a topological invariant is a well defined property of the equivalence classes. Figure 1.21 shows a series of subdivisions of the 2-sphere.

Fig. 1.21. subdividing a sphere

### 1.3 Quotients of 2-complexes

We often want to squash parts of a complex away, glue complexes together, factor out the action of a group, and so on. In other words, we want to take quotients. We do this much as with graphs by defining an equivalence relation on the cells of the complex and then defining a new complex whose cells are the equivalence classes of the relation. It turns out that there are a number of subtleties complicating the exposition, arising when we want to identify cells of different dimensions.
1.3 Quotients of 2-complexes

1.3.1 Quotients in general

All quotients start with an equivalence relation:

**Definition 1.33 (quotient relation on a 2-complex).** If $X$ is a 2-complex, then a quotient relation is an equivalence relation on the vertices, edges and faces of $X$ such that

(Q1). $\sim$ restricted to the 1-skeleton $X^{(1)}$ is a graph quotient relation as in Definition 1.8 with quotient map $q : X^{(1)} \to X^{(1)}/\sim$, and hence the induced relation on the paths as in §1.1.3

(Q2). if $\sigma, \tau$ are faces with $\sigma \sim \tau$ then $\sigma^{-1} \sim \tau^{-1}$, and if $\sigma$ is a face with $\sigma \sim \sigma^{-1}$, then $[\sigma]$ contains a vertex, edge or path, where $[\sigma]$ is the equivalence class of $\sigma$;

(Q3). if $\sigma$ is a face with $[\sigma] \subset X^2$, then there is a $(X^{[\sigma]}, \alpha_{[\sigma]})$ with $X^{[\sigma]} \approx S^1$ and $\alpha_{[\sigma]} : X^{[\sigma]} \to X^{(1)}/\sim$ dimension preserving, such that for all $\tau \in [\sigma]$ with $\partial \tau = (X^{[\tau]}, \alpha_{[\tau]})$, there is an orientation preserving map $\varepsilon = \varepsilon(f, \tau) : X^{[\tau]} \to X^{[\sigma]}$ making the diagram below left commute:

\[
\begin{array}{ccc}
X^{[\tau]} & \xrightarrow{\alpha_{[\tau]}} & X^{[\sigma]} \\
\varepsilon & & \downarrow \alpha_{[\sigma]} \\
X^{(1)} & \xrightarrow{q} & X^{(1)}/\sim \\
\end{array}
\]

(Q4). if $\sigma$ is a face with $[\sigma] \not\subset X^2$, then there is a homotopically trivial path $\gamma : S \to X^{(1)/\sim}$ such that for all faces $\tau \in [\sigma]$ there is an orientation preserving map $X^{[\tau]} \to S$ making the diagram above right commute.

(Q2) ensures that when we form a quotient we have $[\sigma] \neq [\sigma]^{-1}$. (Q3) says that equivalent faces have boundaries that fold up in the quotient to give the same thing. Similarly, if a face is to be identified with a closed path then (Q4) forces the boundary of the face to be identified with it as well. The homotopically trivial condition is a little obscure at the moment. It’s role will become clearer in Chapter 2

**Definition 1.34 (quotient 2-complex).** If $\sim$ is a quotient relation on the 2-complex $X$ then define the quotient $X/\sim$ as follows:

1. the 1-skeleton $(X/\sim)^{(1)}$ is the quotient graph $X^{(1)}/\sim$ with quotient map $q : X^{(1)} \to X^{(1)}/\sim$

2. the faces $(X/\sim)^2$ are the $[\sigma] \subset X^2$ for $\sigma \in X^2$. Such a face has boundary $\partial [\sigma] = (X^{[\sigma], \alpha_{[\sigma]}})$ as given by Definition 1.33(Q3).

**Proposition 1.35.** If $\sim$ is a quotient relation then $X/\sim$ is a 2-complex and the quotient map $q : X \to X/\sim$ given by $q(x) = [x]$ is a map of 2-complexes.

Proof. That the quotient is a 2-complex is immediate from Definition 1.33 and the commuting diagrams given there are precisely what is needed for $q$ to be a map of 2-complexes. □

1.3.2 Quotients by a group action

When a group $G$ acts on a 2-complex $X$ we can replace $X$ by a complex on which the action of $G$ is trivial, ie: every element of $G$ acts as the identity. We do this by factoring out the group action, and we do this by forming a quotient. Recall from 1.2.2 the group Aut($X$) of automorphisms of the 2-complex $X$, and let a group action $G \xrightarrow{\x} \text{Aut}(X)$ be given.

Let $\sim$ be the equivalence relation on $X$ given by the orbits of the action, so that $x \sim y$ if and only if $y = g(x)$ for some $g \in G$, where $x, y$ are cells of $X$, necessarily of the same dimension (as automorphisms are dimension preserving).
Proposition 1.36 (quotient by a group action). Let \( \sim \) be the equivalence relation on the 2-complex \( X \) given by the orbits of a group action. Then \( \sim \) is a quotient relation if and only if the group action is orientation preserving.

Although we can in principle consider group actions that don’t preserve orientation, as the primary purpose of such actions is to form quotients, we will only consider orientation preserving actions. Compare this with Proposition 1.10, noting how the 2-complex structure imposes no new conditions for \( \sim \) to be a quotient relation. We write \( X/G \) for the quotient complex \( X/\sim \).

Proof. By Exercise 1.6, Proposition 1.10 and basic properties of maps, parts (Q1) and (Q2) of Definition 1.33 are satisfied if and only if the \( G \)-action preserves orientation. As the elements of \( G \) are dimension preserving, part (Q4) never arises. If \( \tau \sim \sigma \) then \( \sigma = g(\tau) \) for some \( g \in G \), so we have an orientation preserving isomorphism \( X^\tau \to X^\sigma \) with the diagram below left commuting:

\[
\begin{array}{ccc}
X^\tau & \xrightarrow{a^\tau} & X^\sigma \\
\downarrow q & & \downarrow \alpha_\sigma \\
X^{(1)} & \xrightarrow{g} & X^{(1)} \\
\end{array}
\]

The triangular diagram on the right commutes by the nature of the quotient map: if \( y = g(x) \) then \( q(x) = q(y) \). Now glue the triangular diagram to the bottom of the square. \( \square \)

Consider as an example Figure 1.22, where the Euclidean plane complex is rolled into an infinite tube by the action of the integers \( \mathbb{Z} \).

Fig. 1.22. let \( \mathbb{Z} \) act on \( X \) (left) with \( 1 \in \mathbb{Z} \) acting as the translation of \( X \) one step to the right, as shown by the red arrow: The quotient \( X/\mathbb{Z} \) is an infinite rolled up tube. If \( m \in \mathbb{Z} \), then its effect on \( X \) is to translate \( m \) steps to the right, whereas its effect on \( X/\mathbb{Z} \) is to rotate a cell \( m \) times around the tube, bringing it back to itself. The induced \( \mathbb{Z} \)-action on \( X/\mathbb{Z} \) is thus trivial.

Exercise 1.37. In the proof of Proposition 1.36 we took \( (X^{[\sigma]}, \alpha_{[\sigma]}s) = (X^\sigma, q\alpha_\sigma) \). Show that we are free to choose instead a different face from \( [\sigma] \): if \( \tau \sim \sigma \) and we take \( (X^{[\sigma]}, \alpha_{[\sigma]}) = (X^\tau, q\alpha_\tau) \) instead, then this new version of \( X/G \) is isomorphic to the old one.

1.3.3 Quotients by a subcomplex

Now for a quotient that involves some serious squashing: if \( Y \subset X \) is a subcomplex we define a new complex where \( Y \) has been compacted down to a single vertex, extending the construction of §1.1.2 from graphs to 2-complexes.
Define \(\sim\) on \(X\) to be the equivalence relation with the following equivalence classes: (i). all the cells in \(Y\) (of whatever dimension) form one class; (ii). every other class has the form \([x] = \{x\}\). Thus, we have \(x \sim y\) if and only if either \(x = y\) or both \(x\) and \(y\) lie in \(Y\).

Exercise 1.38. Let \(X\) be a 1-sphere and \(Y \subset X\) a connected subcomplex. Show that the relation just defined is a (graph) quotient relation with the quotient \(X/Y\) another 1-sphere and the quotient map \(q: X \rightarrow X/Y\) an orientation preserving map.

Proposition 1.39. The relation \(\sim\) is a quotient relation.

Write \(X/Y\) for the corresponding quotient, the quotient of \(X\) by the subcomplex \(Y\): it is what results from collapsing \(Y\) to a vertex and propagating the effects of this on the incidence of cells throughout \(X\), but otherwise leaving the cells of \(X\) unaffected.

Proof. The only part requiring more than a moments thought is the verification of the face conditions (Q3) and (Q4) in Definition 1.33. A face \(\sigma \in Y\) is squashed to a vertex in \(X/Y\), so taking \(S\) to be the trivial graph gives (Q4). If \(\sigma \not\in Y\) then \([\sigma] = \{\sigma\}\) and \(\alpha^{-1}_\sigma(Y) \subset X^{\sigma}\) has connected components \(T_1, \ldots, T_k\). By Exercise 1.38 we can form the successive quotients \(X^{\sigma}/T_1, X^{\sigma}/T_1/T_2, \ldots\), obtaining 1-spheres at each stage. Let \(X^{[\sigma]}\) be the end result of taking the \(k\) quotients and \(\varepsilon(q, \sigma): X^{\sigma} \rightarrow X^{[\sigma]}\) the composition of the quotient maps.

A typical quotient by a subcomplex arises when \(T \subset X\) is a spanning tree for the 1-skeleton and we form \(X/T\) as in Figure 1.23.

![Fig. 1.23. squashing a spanning tree down to a vertex](image)

As in (1.1.2) we write \(X/Y_i (i \in I)\) for the result of squashing each \(Y_i \subset X\) to a vertex \(v_i\), and \(X/(\bigcup Y_i)\) for the result of squashing \(\bigcup Y_i\) to a single vertex \(v\).

1.3.4 Pushouts

The pushout is a pretty general construction which arises whenever a pair of complexes are glued together across a common subcomplex.

Definition 1.40 (pushout). Let \(X_1, X_2\) and \(Y\) be 2-complexes and \(f_i: Y \rightarrow X_i (i = 1, 2)\) maps of 2-complexes. Let \(\sim\) be the equivalence relation on \(X_1 \bigcup X_2\) generated by \(x \sim x'\) if and only if there is a \(y \in Y\) with \(x = f_1(y)\) and \(x' = f_2(y)\). If \(\sim\) is a quotient relation then call the quotient \(X_1 \bigcup X_2/\sim\) the pushout of the maps \(f_i: Y \rightarrow X_i\), and denote it by \(X_1 \coprod_Y X_2\).

Figure 1.24 illustrates a typical pushout.

Exercise 1.41. If \(\sim\) is the relation described in Definition 1.40 show that \(x \sim x'\) iff there are \(x_0, x_1, \ldots, x_k \in X_1 \bigcup X_2\) with \(x_0 = x\) and \(x_k = x'\), and \(y_1, \ldots, y_k \in Y\), such that \(f_1(y_1) = x_0, f_2(y_1) = x_1, f_2(y_2) = x_1, f_1(y_2) = x_2, \ldots\) and so on.
Define \( t_i : X_i \to X_1 \coprod_Y X_2 \) \((i = 1, 2)\) to be the composition \( X_i \leftarrow X_1 \cup X_2 \to X_1 \coprod_X X_2 / \sim \) of the inclusion of \( X_i \) in the union and the quotient map.

**Theorem 1.42 (pushouts exist and are colimits).** Let \( Y, X_1, X_2 \) be 2-complexes, \( \emptyset \subset Y, \emptyset \subset X_1 \) orientations and \( f_i : Y \to X_i \) orientation and dimension preserving maps. Then the quotient \( q : X_1 \cup X_2 \to X_1 \coprod X_2 / \sim \), and hence the \( t_i \), are dimension preserving, and the pushout exists, with the diagram below left commuting.

Moreover the pushout is universal in the sense that if \( Z, t_i' \) are a 2-complex and maps making such a square commute, then there is a map \( h : X_1 \coprod_Y X_2 \to Z \) making the diagram above right commute.

Thus the data \( f_i : Y \to X_i \) forming the input to the pushout gives two sides of a commutative square, and \( X_1 \coprod_Y X_2 \) “pushes out” along the other two sides. Pushouts are thus examples of colimits in the category of 2-complexes.

**Proof.** We show that under the assumptions \( \sim \) is a quotient relation, with the result on the 1-skeleton given by Exercise 1.12, and a similar argument shows that we never have \( \sigma \sim \sigma^{-1} \) for a face \( \sigma \). Exercise 1.41 and the definition of map gives that \( \sigma \sim \tau \implies \sigma^{-1} \sim \tau^{-1} \). As the \( f_i \) are dimension preserving, all the cells in an equivalence class have the same dimension, leaving us with (Q3) of Definition 1.33 to do. Let \((X^\sigma, \alpha_{\{\sigma\}}) = (X^\sigma, q\alpha_{\sigma})\), with \( q\alpha_{\sigma} \) dimension preserving. Let \( \tau \in [\sigma] \) and suppose we are in the special case \( \sigma = f_1(\rho), \tau = f_2(\rho) \) for some face \( \rho \) in \( Y \). Then we get a diagram

\[
\begin{array}{ccc}
X_1 \coprod_Y X_2 & \xrightarrow{f_2} & X_2 \\
\downarrow & & \downarrow \\
X_1 \coprod X_2^{(1)} & \xrightarrow{f_2^{(1)}} & X_2^{(1)} \\
\downarrow & & \downarrow \\
X_1 \cup X_2^{(1)} & \xrightarrow{f_2^{(1)}} & X_2^{(1)} \\
\downarrow & & \downarrow \\
X_1 \coprod_Y X_2 & \xrightarrow{f_2^{(1)}} & X_2^{(1)} \\
\end{array}
\]

with the four squares commuting via the maps \( f_i \) and the inclusions \( X_i \leftarrow X_1 \cup X_2 \). The \( \varepsilon(f_1, \sigma), \varepsilon(f_2, \sigma) \) are orientation preserving isomorphisms and the other two maps along
1.3 Quotients of 2-complexes

The top are the identities. The diagram glued to the bottom commutes by the definition of \( q \). The maps along the top (and their inverses) compose to give an isomorphism \( X_\tau^\sigma \rightarrow X_\tau^\sigma \), and this, together with the outside circuit of the diagram, give condition (Q3).

When \( \sigma \sim \tau \) in general, we have \( \sigma = \sigma_0 = f_1(\rho_1), \sigma_1 = f_2(\rho_1), \ldots, \sigma_{k-1} = f_1(\rho_k), \tau = \sigma_k = f_2(\rho_k) \), and the requirements for a quotient relation can be verified by repeatedly applying the process of the previous paragraph. In particular, for \( \tau \in [\sigma] \) the map \( X^\tau \rightarrow X^{[\sigma]} \) is an orientation preserving isomorphism, so that the quotient map is dimension preserving.

If \( y \in Y \) is a cell then its images under the \( t_i : Y \rightarrow X_i \hookrightarrow \coprod Y X_2 (i = 1, 2) \) are equivalent by the definition of \( \sim \), and so the square commutes.

Suppose now that \( Z, t_1', t_2' \) are as in the statement of the Theorem. After a moment’s thought it is clear what the map \( h : X_1 \coprod Y X_2 \rightarrow Z \) should be: every cell of the pushout has the form \( [x] \) for some \( x \in X_i \), so define \( h[x] = t_i'(x) \). We leave it to the reader to show that this is well defined and gives a map of 2-complexes. That (M1) is satisfied is very similar to the argument above.

Exercise 1.43. With the conditions of Theorem 1.42 show that if \( X_1, X_2 \) are connected then the pushout is connected.

Exercise 1.44 (pointed pushout). Formulate a pointed version of Theorem 1.42 with all the complexes and maps in sight pointed.

Figure 1.25 is a simple example for which the pushout doesn’t exist, i.e., the relation \( \sim \) in the pushout construction is not a quotient relation.

Figure 1.26 is the Stallings fold: the graph \( Y \) is a single edge joining two vertices. Another example, shown in Figure 1.27 is the wedge of a pair of complexes: \( Y \) is now the trivial complex consisting of just a single vertex.

An important example arises when one of the maps \( f_i \) is just an inclusion, so that the initial data consists of two complexes \( X_1 \) and \( X_2 \), and a map \( f \) from a subcomplex of \( X_1 \)
to $X_2$. The pushout (when it exists) is the result of gluing $X_1$ and $X_2$ together via the attaching map $f$ as in Figure 1.28.

Pushouts will really prove their mettle in Chapters 3-4 where the $f_i$ will be covering maps. Figure 1.29 illustrates the kind of initial set-up we will have, and Figure 1.30 the resulting pushout.

Fig. 1.29. initial data for a typical pushout of Chapter 3: $Y$ has six vertices, edges and faces, $X_1$ has three of everything and $X_2$ has two of everything. The boundaries of all faces are hexagonal, but we’ve only shown one in each case. The attaching maps wrap around the 1-skeletons as shown.

### 1.4 Pullbacks and Higman composition

We now come to a pair of constructions which both start with roughly the same kind of data: a complex $Y$, a (finite) family of complexes $X_i$, and a family of maps $f_i : X_i \to Y$. The first of these, the pullback, is dual to the pushout: it is a categorical limit. It is what
we get if we reverse the directions of all the maps in the pushout. Pullbacks, like pushouts, will play a crucial role in the theory of coverings in Chapters 3-4: they will act like a kind of “union” and pushouts like a kind of “intersection”. In §3.4.3 we will be able to be much more precise about what we mean by this.

The other construction, Higman composition, is less well known and can be performed only in very special circumstances. Nevertheless, when possible it will prove extremely powerful, and this makes its inclusion more than worthwhile.

1.4.1 Pullbacks

It is easier to do graphs and then extend to 2-complexes:

Definition 1.45 (pullback of graphs). Let \( Y \) and \( X_1, X_2 \) be graphs and \( f_i : X_i \rightarrow Y \) maps of graphs. The pullback \( X_1 \coprod_Y X_2 \) has vertices (respectively edges) the \( x_1 \times x_2 \) for \( x_1 \in X_0^1 \) (resp. \( x_i \in X_1^i \)), such that \( f_1(x_1) = f_2(x_2) \). The incidence maps are given by \( s(e_1 \times e_2) = s(e_1) \times s(e_2) \) and \( (e_1 \times e_2)^{-1} = e_1^{-1} \times e_2^{-1} \). See Figure 1.31.

![Fig. 1.30. pushout resulting from the set-up in Figure 1.29. There is a single vertex, edge and face in the quotient, and the face has a hexagonal boundary with the attaching map wrapping it around the 1-skeleton six times as shown.](image)

![Fig. 1.31. Construction of the pullback for graphs: the vertices \( u_i \) and \( v_i \) map via the \( f_i \) to \( u \) and \( v \), and the edges \( e_i \) map via the \( f_i \) to \( e \). In the pullback we get vertices \( u_1 \times u_2, v_1 \times v_2 \) joined by an edge \( e_1 \times e_2 \).](image)

We now set up the pullback of 2-complexes by seeing how the boundaries of faces in the \( X_1, X_2 \) behave when we “pullback” their boundaries. Suppose the \( f_i \) are dimension preserving, and that \( \sigma \) is a face of \( Y \) and \( \sigma_i (i = 1, 2) \) faces of the \( X_i \) mapping to \( \sigma \) via the \( f_i \). We get a by now familiar commuting diagram:

\[
\begin{array}{ccc}
X_1^{\sigma_1} & \rightarrow & Y^{\sigma} & \leftarrow & X_2^{\sigma_2} \\
\downarrow \alpha_{\sigma_1} & & & & \downarrow \alpha_{\sigma_2} \\
X_1^{(1)} & \rightarrow & Y^{(1)} & \leftarrow & X_2^{(2)} \\
\end{array}
\]
The \( \varepsilon_i = \varepsilon(f_i, \alpha_i) : X_i^{\sigma_i} \rightarrow Y^\sigma \) are orientation preserving isomorphisms as the \( f_i \) preserve dimension. Let \( \emptyset = \{\varepsilon_1, e_2, \ldots, e_n\} \) be the standard orientation of \( Y^\sigma \), so that the edges of the \( X_i^{\sigma_i} \) can be labelled \( e_{i1}, e_{i2}, \ldots, e_{ik} \), with \( e_j = \varepsilon_i(e_{ij}) \), and the closed path \( \alpha_{\sigma_i}(e_{i1}), \alpha_{\sigma_i}(e_{i2}), \ldots, \alpha_{\sigma_i}(e_{ik}) \) a boundary label for \( \sigma_i \) with \( \alpha_{\sigma_i}(e_{ij}) \) mapping via \( f_i \) to \( \alpha_{\sigma_i}(e_{ij}) \) in \( Y \).

The upshot is that the pullback contains a path of edges
\[
\alpha_{\sigma_1}(e_{11}) \times \alpha_{\sigma_2}(e_{21}), \alpha_{\sigma_1}(e_{12}) \times \alpha_{\sigma_2}(e_{22}), \ldots, \alpha_{\sigma_1}(e_{1k}) \times \alpha_{\sigma_2}(e_{2k}),
\]
and since \( s\alpha_{\sigma_1}(e_{11}) = t\alpha_{\sigma_1}(e_{1k}) \) and \( s\alpha_{\sigma_2}(e_{21}) = t\alpha_{\sigma_2}(e_{2k}) \), this path is closed. The idea is to “sew a face” into the 1-skeleton having boundary this closed path, by taking \( Y^\sigma \) and attaching map \( \alpha_{\sigma_1}e_{1j}^{-1} \times \alpha_{\sigma_2}e_{2j}^{-1} \) that sends \( e_j \) to \( \alpha_{\sigma_1}(e_{1j}) \times \alpha_{\sigma_2}(e_{2j}) \).

**Definition 1.46 (pullback of 2-complexes).** Let \( X_1, X_2 \) and \( Y \) be 2-complexes and \( f_i : X_i \rightarrow Y \) dimension preserving maps. The **pullback** \( X_1 \coprod_Y X_2 \) has \( k \)-dimensional cells the \( x_1 \times x_2 \), for \( x_i \in X_i^k \), such that \( f_1(x_1) = f_2(x_2) \). The incidence maps are given by
\[
s(e_1 \times e_2) = s(e_1) \times s(e_2), \quad (e_1 \times e_2)^{-1} = e_1^{-1} \times e_2^{-1}
\]
and
\[
\partial(\sigma_1 \times \sigma_2) = (Y^\sigma, \alpha_{\sigma_1}e_{1j}^{-1} \times \alpha_{\sigma_2}e_{2j}^{-1}),
\]
where \( f_1(\sigma_1) = \sigma = f_2(\sigma_2) \).

\[\begin{array}{ccc}
\sigma_1 \times \sigma_2 & \subset & X_1 \coprod_Y X_2 \\
v_1 \times v_2 & \longmapsto & \sigma_1 \\
& & \in X_1 \\
\partial \sigma_1 & \longmapsto & \partial \sigma_1 \times \partial \sigma_2 \\
& & \in X_2
\end{array}\]

**Fig. 1.32.** pullback of faces whenever \( f_1(\sigma_1) = f_2(\sigma_2) \).

Notice that unlike the pushout, there is no question of whether the pullback exists or not. Each cell of the pullback has the form \( x_1 \times x_2 \) with the \( x_i \in X_i \), so define
\[
t_i : X_1 \coprod_Y X_2 \rightarrow X_i (i = 1, 2),
\]
by \( t_i(x_1 \times x_2) = x_i \). For \( \sigma_1 \times \sigma_2 \) a face with \( t_i(\sigma_1 \times \sigma_2) = \sigma_i \), we define \( \varepsilon(t_i, \sigma_1 \times \sigma_2) = \varepsilon(f_i, \sigma_i)^{-1} \). We leave it as an exercise to see that the \( t_i \) are dimension preserving maps of 2-complexes.

**Theorem 1.47 (pullbacks are limits).** The diagram below left commutes.

\[
\begin{array}{ccc}
X_1 \coprod_Y X_2 & \xrightarrow{t_2} & X_2 \\
\downarrow t_1 & & \downarrow f_2 \\
X_1 & \xrightarrow{f_1} & Y
\end{array}
\]

Moreover, the pullback is universal in the sense that if \( Z, t_1', t_2' \) are a 2-complex and maps making such a square commute, then there is a map \( h : Z \rightarrow X_1 \coprod_Y X_2 \) making the diagram above right commute.
1.4 Pullbacks and Higman composition

Proof. We have \( f_1 t_1 (x_1 \times x_2) = f_2 t_2 (x_1 \times x_2) \), and writing \( X = X_1 \coprod_Y X_2 \), the face isomorphisms \( X^\sigma_1 \times \sigma_2 \rightarrow Y^\sigma \) are identical for both \( f_i t_i \), and so the square commutes. If \( z \) is a cell of \( Z \), then the commuting of the large square of the righthand diagram that \( f_1 t_1' (z) = f_2 t_2' (z) \), so \( t_1' (z) \times t_2' (z) \) is a cell of the pullback. Define \( h : Z \rightarrow X_1 \coprod_Y X_2 \) to be \( z \mapsto t_1' (z) \times t_2' (z) \), and for the isomorphism \( Z^\sigma \rightarrow X_1^\sigma t_1' (\sigma) \times t_2' (\sigma) = Y f_1 t_1' (\sigma) \), take the composition

\[
Z^\sigma \rightarrow X_1^\sigma t_1' (\sigma) \rightarrow Y f_1 t_1' (\sigma).
\]

We leave it to the reader to check that this is a map of 2-complexes with the required properties. □

![Fig. 1.33. initial data for a typical pullback of Chapters 3-4: \( Y \) has one vertex, edge and face, \( X_1 \) has three of everything and \( X_2 \) has two of everything. The boundaries of all faces are hexagonal with just one shown. The attaching maps wrap around the 1-skeletons as shown.](image)

![Fig. 1.34. Pullback resulting from the set-up in Figure 1.33.](image)

Although pullbacks always exist, unlike pushouts, they are not necessarily connected, unlike pushouts. This simple fact, even for graphs, has surprisingly far-reaching implications as we will see when we study subgroups of free groups. There is a pointed version of all the above which goes some way to fixing this:

**Exercise 1.48 (pointed pullbacks).** Suppose the \( f_i : (X_i)_x \rightarrow Y_y \) are maps of pointed complexes. Then \( x = x_1 \times x_2 \) is a vertex of the pullback. Write \( (X_1 \coprod_Y X_2)_x \) for the connected component containing \( x_1 \times x_2 \). Show that we have a Proposition analogous to Proposition 1.47 for the pointed pullback, with all complexes and maps in sight pointed and connected.

The duality between pullbacks and pushouts can be seen by running the example of §1.3.4 backwards, using the pullback to get back to where we started. In Figure 1.33 we have the same two complexes \( X_1, X_2 \) as in the pushout example, but this time \( Y \) is the end result of that example and the maps \( f_i \) are the \( t_i \) from Proposition 1.42. The resulting pullback, shown in Figure 1.34, is the starting point of the pushout example, and the maps
Definition 1.50 (Higman composition).

The second of our two constructions starts with a (finite) family of maps \( f_i : Y_i \to X \) with the \( Y_i \) disjoint complexes. If there is a certain special configuration of edges in the \( Y_i \), then they can be threaded together into one large complex \( Y \) and a map \( f : Y \to X \). Much as with pullbacks, the pay-off will not be evident until Chapter 3 when we show that if the \( f_i \) are covering maps then so is the new map \( f \).

**Definition 1.49 (handle configuration).** Let \( X, Y_i (i = 1, \ldots, n) \) be 2-complexes and \( f_i : Y_i \to X \) a collection of dimension preserving maps of 2-complexes. Let \( e \in X \) be an edge and \( \{ e_{j1}, e_{j2} \} \) pairs of edges in \( Y_0 = \bigcup Y_i \) such that for each \( j \), the edges \( e_{j1}, e_{j2} \) lie in the fiber \( f_i^{-1}(e) \) for some \( i \), and for each \( i \), the complex \( Y_i \) contains a pair \( \{ e_{j1}, e_{j2} \} \) for some \( j \).

The pairs \( \{ e_{j1}, e_{j2} \} \) form a handle configuration if and only if for every face \( \sigma \in X \) containing \( e \) in its boundary we have,

(i). if \( \tau \in \bigcup_i f_i^{-1}(\sigma) \) and for some \( j \) we have \( \alpha_{\tau}(Y_0^\tau) = e_{j1}^k \) (respectively \( e_{j2}^k \)), then \( k = m \) and there are faces \( \tau_1, \ldots, \tau_{2m} \in \bigcup_i f_i^{-1}(\sigma) \), with \( \tau = \tau_{j1} \) (resp. \( \tau_{j2} \)), and \( \alpha_{\tau_i}(Y_0^\tau_i) = e_{\ell_i}^k \) (\( \ell = 1, \ldots, m \) and \( i = 1, 2 \));

(ii). for any face \( \tau \in \bigcup_i f_i^{-1}(\sigma) \) not of the form (i) and containing \( e_{j1} \) or \( e_{j2} \) in its boundary we have \( \alpha_{\tau}(Y_0^\tau) = (e_{j1}^k e_{j2}^k \gamma_{j1} \gamma_{j2})^k \) with \( \gamma_{j1}, \gamma_{j2} \) not containing \( e_{j1}, e_{j2} \). Moreover, there are faces \( \tau_1, \ldots, \tau_m \in \bigcup_i f_i^{-1}(\sigma) \), with \( \tau = \tau_i \), the \( \alpha_{\tau_i}(Y_0^\tau_i) = (e_{j1}^k e_{j2}^k \gamma_{j1} \gamma_{j2})^k \) with \( \gamma_{j1}, \gamma_{j2} \) not containing \( e_{\ell1}, e_{\ell2} \), and the \( \gamma_{\ell1} \) (respectively the \( \gamma_{\ell2} \)) all mapping to the same path in \( X \).

![Fig. 1.35. Faces around an edge pair in a handle configuration.](image-url)

Whenever Definition 1.49(i) happens the edges \( e_{j1}, e_{j2} \) in the handle configuration start and finish at the same vertex. The faces in Definition 1.49(ii) form a “taco” arrangement as shown in Figure 1.35 whenever a face contains \( e_{j1} \) in its boundary it also contains \( e_{j2} \) and vice-versa.

**Definition 1.50 (Higman composition).** Let \( f_i : Y_i \to X \) (\( i = 1, \ldots, n \)), be dimension preserving and \( \{ e_{j1}, e_{j2} \} \) pairs of edges in \( Y_0 = \bigcup Y_i \). The Higman composition produces a 2-complex \( Y = [Y_1, \ldots, Y_n] \) and a map \( f = [f_1, \ldots, f_n] : Y \to X \) as follows:

(HC1). Delete the edges \( e_{j1}, e_{j2} \) in the handle configurations and replace them by new edges \( e'_{j1}, e'_{j2} \), where for each \( j \), the edge \( e'_{j1} \) connects \( s(e_{j1}) \) to \( t(e_{j+1,1}) \), while \( e'_{j2} \) connects \( s(e_{j2}) \) to \( t(e_{j-1,2}) \), subscripts modulo \( m \) as in Figure 1.36.
1.5 Notes on Chapter 1

(HC2). For the faces $\tau$ in Definition 1.49(i), delete the $\tau_1, \ldots, \tau_m$ and replace them by new faces $\tau'_1, \ldots, \tau'_m$, with $\partial\tau'_j$ using the same $X^{\tau}$ but having attaching map sending it to $e'_j \cdot e'_1 \cdot e'_m \cdot e'_1 \cdots e'_{1-1}$.

(HC3). For the faces in Definition 1.49(ii), delete the $\tau_1, \ldots, \tau_m$ and replace them by new faces $\tau'_1, \ldots, \tau'_m$, with $\partial\tau'_j$ using the same $X^{\tau}$ but having attaching map sending it to the path shown in Figure 1.37.

(HC4). A cell $x$ of $Y$ is either a cell $x$ of one of the $Y_i$ or replaces a cell $x'$ of one of the $Y_i$. Define $f(x)$ to be $f_i(x)$ or $f_i(x')$ as appropriate. The face isomorphisms remain unchanged.

Exercise 1.51. Give an example showing that if the $Y_i$ are all connected then the Higman composition $[Y_1, \ldots, Y_m]$ is not necessarily connected (see also §3.3.3).

We defer further exploration of Higman composition until Chapter 3, where we will show that the Higman composition of a family of coverings $f_i : Y_i \to X$ yields another covering $f : [Y_1, \ldots, Y_m] \to X$.

1.5 Notes on Chapter 1
Topological Invariants
3

Coverings

3.1 Basics

3.1.1 Coverings

Definition 3.1 (covering). A map \( f : Y \to X \) of 2-complexes is a covering if and only if
(C1) \( f \) preserves dimension (see Definition 1.29);
(C2) for every pair of vertices \( u \in Y \) and \( v \in X \), with \( f(u) = v \), the local continuity of \( f \) at \( v \) (see §1.1.1),
\[
s^{-1}_Y(u) \to s^{-1}_X(v),
\]
is a bijection.
(C3) for every pair of vertices \( u \in Y \) and \( v \in X \) with \( f(u) = v \), and every face \( \tau \) of \( X \),
the local continuity of \( f \) at \( v \) (see Definition 1.27),
\[
\prod \varepsilon(f, \sigma) : \bigcup_{f(\sigma) = \tau} \alpha^{-1}_\sigma(u) \to \alpha^{-1}_\tau(v)
\]
is a bijection.

A covering is a kind of “local isomorphism”: if \( f(u) = v \) then \( Y \) looks the same “near” \( u \) as \( X \) does “near” \( v \). Thus (C2) ensures that the configuration of edges around a vertex looks the same both upstairs and downstairs (Figure 3.1 left). Similarly (C3) means that for a face \( \tau \) downstairs containing the vertex \( v \) in its boundary and \( f(u) = v \), this face looks the same near \( v \) as its pre-images do near \( u \). Specifically, if \( v \) appears \( k \) times in the boundary of \( \tau \), so there are \( k \) “wedge-shaped” pieces of \( \tau \) fitting together around \( v \), then there are \( k \) wedge-shaped pieces of face fitting together around \( u \), where these wedges belong to faces \( \sigma \) mapping to \( \tau \) (see Figure 3.1 right).

The terminology cover and lift is used for images and pre-images of a covering map: if \( f(y) = x \), then one says that \( y \) covers \( x \), or that \( x \) lifts to \( y \). The set of all lifts of \( x \), or the set \( f^{-1}(x) \) of all cells covering \( x \), is its fiber. Note that each \( \varepsilon(f, \sigma) \) is an isomorphism as \( f \) preserves dimension, so the local continuity map is a disjoint union of a set of maps, each of which is the restriction of a bijection. In any case, each is individually an injection.

Part 3 of the definition gives in particular that
\[
\sum_{f(\sigma) = \tau} |\alpha^{-1}_\sigma(u)| = |\alpha^{-1}_\tau(v)|, \tag{3.1}
\]
so that \( v \) appears the same number of times in the boundary of \( \tau \) as \( u \) does in the boundaries of all the faces \( \sigma \) in the fiber of \( \tau \) (although be sure to take on board Example 3.7).

It is easy to see that a covering \( f : Y \to X \) can be restricted to a covering \( f : Y^{(1)} \to X^{(1)} \) of the 1-skeletons.
One commonly sees the assumption that in a covering, both the covering complex $Y$ and the covered complex $X$ are connected, but we won’t assume this at the moment. Indeed, we will find it useful in some situations to not assume that a covering complex be connected.

**Exercise 3.2.** Let $f : Y \to X$ be a covering and $Y^o$ a connected component of $Y$. Show that restricting $f$ to $Y^o$ gives a covering. Show that we may not restrict a covering to an arbitrary subcomplex and still get a covering.

**Example 3.3.** Figure 3.2 shows a simple graph covering, with the two vertices in $Y$ covering the single vertex of $X$, and the edges such that the red path in $Y$ covers the red path in $X$. In particular, the two edges of $Y$ both cover the single edge of $X$.

**Example 3.4.** Figure 3.3 extends the graph covering of Example 3.3 to a covering $f_1 : Y_1 \to X$ of 2-complexes. The two faces $\sigma_1, \sigma_2$ both cover the face $\sigma$ of $X$, but the face isomorphisms are different: $\varepsilon_{\sigma_1}$ is the identity map and $\varepsilon_{\sigma_2}$ is a clockwise 1/2-turn.

**Example 3.5.** Figure 3.4 also extends the graph covering of Example 3.3 to a covering $f_2 : Y_2 \to X$ of 2-complexes. In this case $\sigma_1$ covers the face $\sigma$ of $X$ and $\sigma_2$ covers $\sigma^{-1}$. For the face isomorphisms, $\varepsilon_{\sigma_1}$ is the identity map and $\varepsilon_{\sigma_2}$ is a 1/2-turn.
Exercise 3.6. If \( \alpha : Y_1 \to Y_2 \) is the isomorphism of Exercise 1.30 show that it commutes with the coverings of the previous two examples: \( f_1 = f_2 \alpha \).

Example 3.7. Return to the complexes \( Y_1 \) and \( X \) of Example 3.3 but tweak the covering slightly: define \( f'_1 : Y_1 \to X \) by \( f'_1(x) = f_1(x) \) for all cells \( x \in Y_1 \). Where the two coverings differ is in the face isomorphisms: define \( \varepsilon_{F_2} \) to be the identity rather than a \( 1/2 \)-turn. One can then check that the (3.1) is satisfied, but the local continuity maps are not bijections, and so we do not have a covering.

Exercise 3.8 (immersions). Call a map \( f : Y \to X \) an immersion when it preserves dimension and the local continuity maps are injections. Give examples of immersions that are not coverings.

Exercise 3.9. Show that for any \( X \) the identity map \( : X \to X \) is a covering.

3.1.2 Lifting

When we have a covering \( f : Y \to X \) the complexes \( Y \) and \( X \) look the same so long as we restrict our attention to small pieces. If two complexes look the same as each other then we should be able to pull parts of \( X \) back through \( f \) to find parts of \( Y \) mapping to them. Putting these together, when we have a covering we should be able to pull small pieces of \( X \) back through \( f \) and get identical small pieces of \( Y \) covering them. The small pieces turn out to be paths and faces, and this process is called called lifting.

Proposition 3.10 (path, spur and free homotopy lifting). Let \( f : Y \to X \) be a covering with \( f(u) = v \) vertices.

(i). If \( \gamma \) is a path in \( X \) starting at \( v \) then there is a path \( \mu \) in \( Y \) starting at \( u \) and covering \( \gamma \). Moreover, if \( \mu_1, \mu_2 \) are paths in \( Y \) starting at \( u \) and covering the same path in \( X \), then \( \mu_1 = \mu_2 \).

(ii). A path in \( Y \) covering a spur is itself a spur. Consequently, two paths in \( Y \) covering freely homotopic paths are themselves freely homotopic.

Part (i) is called path lifting and part (ii) is spur lifting. Call \( \mu \) the lift of \( \gamma \) at \( u \). Thus a path can be lifted to any vertex that covers its initial vertex to give a covering path, and this lift is unique. As with so many such results, it is the uniqueness of the lift, rather than the existence, that turns out to be most useful.

Proof. The existence of \( \mu \) is easily seen, as in Figure 3.3 since if \( \gamma = e_1 \ldots e_n \), there is an edge \( e'_1 \) covering \( e_1 \) under the bijection \( s_{Y}^{-1}(u) \to s_{X}^{-1}(v) \). This edge \( e'_1 \) must end at a vertex that covers the end vertex of \( e_1 \), as coverings (being maps of complexes) preserve vertex-edge incidences. The process can be repeated starting at this new vertex to give \( \mu \). For the uniqueness, the first edges of the \( \mu_i \) both have initial vertex \( u \) and map to \( e_1 \), hence must be the same edge. Continuing in this manner along the two paths gives their equality. For part (ii), the path in \( Y \) must have the form \( e_1 e_2 \), where the middle vertex is the start of the edges \( e_1^{-1} \) and \( e_2 \). Use the injectivity of \( f \) on the edges starting at this vertex to deduce that \( e_1^{-1} = e_2 \). \( \Box \)
Fig. 3.5. Lifting paths in a covering: individual edges can be lifted via the bijection between the edges starting at a vertex upstairs and the edges starting at a vertex it covers downstairs. Paths are then lifted by repeated edge lifts.

Exercise 3.11. Let $f: Y \to X$ be a covering and $\gamma = \gamma_1 \gamma_2$ a path in $X$, hence $t(\gamma_1) = s(\gamma_2)$. Show that the lift at a vertex $u \in Y$ of $\gamma$ is the path $\mu_1 \mu_2$ consisting of the lift $\mu_1$ of $\gamma_1$ at $u$ followed by the lift $\mu_2$ of $\gamma_2$ at $t(\mu_1)$.

Exercise 3.12. Show that paths cannot necessarily be lifted by an immersion, but when they can, they are unique. Show that spur lifting is a property enjoyed by immersions.

Proposition 3.13 (face lifting). Let $f: Y \to X$ be a covering, $\tau \in X$ a face and $v$ a vertex with $x \in X^\tau$ an appearance of $v$ in the boundary of $\tau$. Let $\gamma$ be a boundary path of $\tau$ starting at $v$ and given by $x$. Finally, let $u$ be a vertex of $Y$ covering $v$ and let $\mu$ be the lift of $\gamma$ to $u$. Then there is a unique appearance $y$ of $u$ in the boundary of some face $\sigma$ covering $\tau$, with the local continuity $\varepsilon(f, \sigma)(y) = x$ and $\mu$ a boundary path of $\sigma$ starting at $u$ and given by $y$.

The boundaries of faces thus lift to the boundaries of faces, as in Figure 3.6. We will call the uniqueness statement of Proposition 3.13 “uniqueness of face lifting”.

Proof. (C3) of Definition 3.1 gives a unique vertex $y$ in $\bigcup_{f(\sigma) = \tau} \alpha_{\sigma}^{-1}(u)$ mapping to $x$ via the local continuity of $f$, i.e.: there is a face $\sigma$ of $Y$ covering $\tau$, containing $u$ in its boundary, and with the diagram of pointed maps

$$
\begin{array}{c}
Y^\sigma_y & \cong \alpha_{\sigma}^{-1}(u) \\
\alpha_{\sigma} & \downarrow \alpha_{\tau} \\
Y^1_u & \cong X^\tau_\nu
\end{array}
$$

commuting. In particular there is a boundary path of $\sigma$ starting at $u$ that covers $\gamma$, and by the uniqueness of lifts, this must be the path $\mu$. \(\Box\)

The first consequence of lifting justifies the usage of the word “cover”, and is not a priori obvious from the definition:
Corollary 3.14 (surjectivity of coverings). If \( f : Y \rightarrow X \) is a covering with \( X \) connected then \( f \) is a surjective map of 2-complexes, ie: every cell of \( X \) is the image under \( f \) of some cell of \( Y \).

Proof. Path lifting gives the surjectivity on the vertices and edges, and face lifting on the faces: fix a vertex \( u \) of \( Y \), and by connectedness, we can join any vertex \( v \) of \( X \) to \( f(u) \) by a path. Lift this path to \( u \), so that its terminal vertex in \( Y \) maps via \( f \) to \( v \). For an edge \( e \) or face \( \sigma \) of \( X \), let \( v \) be a vertex in the boundary and lift the edge or face to a vertex \( u \) covering \( v \).

Exercise 3.15. Illustrate by an example why the connectedness of \( X \) is necessary in Proposition 3.14.

Another result of being able to find pre-images of paths and faces is that homotopies can be “pulled back” through a covering:

Corollary 3.16 (homotopy lifting). Let \( Y \rightarrow X \) be a covering. Then two paths that cover homotopic paths are themselves homotopic.

Proof. The homotopy between the covered paths is realised by a finite sequence of insertions or deletions of spurs and face boundaries. By spur and face lifting, those sections of the covering paths mapping to the spurs and face boundaries are themselves spurs and face boundaries, while by uniqueness of path lifting, the remaining pieces are identical. Thus the same sequence of elementary homotopies can be realised between the covering paths as between the covered ones.

Exercise 3.17. Let \( f : Y \rightarrow X \) be a covering and \( \gamma_1, \gamma_2 \) paths in \( X \) related by an elementary homotopy, ie: \( \gamma_2 \) is what results by inserting/deleting a spur or face boundary into \( \gamma_1 \). Let \( \mu_1 \) be the lift of \( \gamma_1 \) to some vertex \( u \) of \( Y \) and \( \mu \) the result of lifting to the appropriate vertex the elementary homotopy and performing it on \( \mu_1 \). Show that \( \mu \) is the lift \( \mu_2 \) of \( \gamma_2 \) at \( u \).

Another spin-off of homotopy lifting is the following characterisation of the image of the induced homomorphism between fundamental groups:

Corollary 3.18. Let \( f : Y \rightarrow X \) be a covering with \( f(u) = v \) and

\[
f_* : \pi_1(Y, u) \rightarrow \pi_1(X, v),
\]

the induced homomorphism. Then \( f_* \) is injective, and a closed path \( \gamma \) at \( v \) represents an element of \( f_* \pi_1(Y, u) \) if and only if the lift \( \mu \) of \( \gamma \) to \( u \) is closed.

The injectivity of the induced homomorphism is probably the single most important property of coverings: it means that the fundamental group of the covering space can be identified with a subgroup of the fundamental group of the covered space. The appropriate context in which to develop this idea properly will be the Galois theory of coverings in Chapter 4.

Proof. If two elements of \( \pi_1(Y, u) \) map to the same element of \( \pi_1(X, v) \) then they are represented by closed paths at \( u \) covering homotopic paths at \( v \). Homotopy lifting gives that the paths in \( Y \) are homotopic, and so the two elements of the fundamental group coincide, thus establishing the injectivity of the homomorphism. For the second part, if \( \mu \) is closed then its homotopy class maps via \( f_* \) to the homotopy class of \( \gamma \). Conversely, if \( \gamma \) represents an element in the image of the homomorphism then there is a closed path \( \gamma_1 \) at \( v \), homotopic to \( \gamma \), with \( f(\mu_1) = \gamma_1 \) for \( \mu_1 \) closed at \( u \) (and the lift of \( \gamma_1 \)). By homotopy lifting the lift \( \mu \) of \( \gamma \) is homotopic to \( \mu_1 \), hence has the same endpoints, ie: is closed.
Exercise 3.19. Let \( f : Y \to X \) be a covering with \( f(u) = v \) and \( u' \) the terminal vertex of a path \( \mu \in Y \) starting at \( u \). Show that \( f_*\pi_1(Y, u) = h f_*\pi_1(Y, u')h^{-1} \), where \( h \) is the homotopy class of \( f(\mu) \).

This thread of ideas culminates in (and is subsumed by) the following general lifting result.

Theorem 3.20 (map lifting). If \( f : Y \to X \) is a covering with \( f(u) = v \) and \( g : Z \to X \) a map with \( g(x) = v \) and \( Z \) connected, then there is a map \( \tilde{g} : Z \to Y \) making the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
\downarrow & & \downarrow f \\
\tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{X}
\end{array}
\]

commute if and only \( g_*\pi_1(Z, x) \subset f_*\pi_1(Y, u) \). If \( \tilde{g} \) exists then it is unique.

Think of this result as a generalisation of path lifting: if \( Z \) is a 1-ball then the map \( g : Z \to X \) is a path in \( X \) starting at \( v \). As the fundamental group of a 1-ball is trivial, the condition \( g_*\pi_1(Z, x) \subset f_*\pi_1(Y, u) \) is trivially satisfied. The resulting map \( \tilde{g} : Z \to Y \) is a path in \( Y \) starting at \( u \), and the commuting of the diagram just says that this new path is the lift of the old one.

Proof. The “only if” part can be dispensed with quickly as \( g \) gives \( f_*g_* = g_* \), so that \( g_*\pi_1(Z, x) = f_*g_*\pi_1(Z, x) \subset f_*\pi_1(Y, u) \).

Suppose we have the condition on the fundamental groups, which by Corollary 3.18 means that if \( \gamma \) is a closed path at \( x \in Z \) then the lift \( \mu \) of \( g(\gamma) \) to \( u \) is also closed. We proceed to define a map \( \tilde{g} \) having the required properties: if \( z \) is a vertex of \( Z \), then by connectedness there is a path joining it to \( x \). Take the image of this path by \( g \) and then lift the result via the covering \( f \) to a path at \( u \). Define \( \tilde{g}(z) \) to be the end vertex of the resulting path in \( Y \). Edges and faces are similar: choose a vertex \( z \) in the boundary of the edge or face (for an edge \( e \), choose \( z = s(e) \)) and then lift the image under \( g \) of the edge/face via the covering \( f \) to the vertex \( \tilde{g}(z) \).

If this procedure is well defined, then it is easy to check that we have a map which by definition makes the diagram commute (for the face isomorphism \( Z_2 \to Y \tilde{g}(\sigma) \), take the composition \( \varepsilon(\tilde{g}(\sigma)) = \varepsilon(f, \tilde{g}(\sigma))^{-1} \varepsilon(g, \sigma) \)). To show that \( \tilde{g} \) is well defined on the vertices, suppose that \( \gamma_1, \gamma_2 \) are paths in \( Z \) from \( x \) to the vertex \( z \), so that \( \gamma_1\gamma_2^{-1} \) is a closed path at \( x \). Thus, the lift of its \( g \)-image (which is \( \mu_1\mu_2^{-1} \)) is closed too, and so the image \( \tilde{g}(z) \) does not depend on the choice of the path \( \gamma \). Edge images are well defined as there are no additional choices made. For a face \( \sigma \in Z \) the construction involves a choice of vertex in its boundary, so suppose that \( z_1, z_2 \) are two such. If \( \gamma \) is a boundary path for \( \sigma \), then the lift of \( g(\gamma) \) must pass through both \( \tilde{g}(z_1) \) and \( \tilde{g}(z_2) \) by the well-definedness of \( \tilde{g} \) on the vertices. Applying the uniqueness of face lifting to these two vertices gives what we want.

Finally, if \( \tilde{g}_1, \tilde{g}_2 \) are two maps making the diagram commute, then path and face lifting gives \( \tilde{g}_1(z) = \tilde{g}_2(z) \) for any cell \( x \in Z \). The face isomorphisms of both are the compositions of face isomorphisms of \( g \) and \( f \), and so are identical. Thus \( \tilde{g}_1 = \tilde{g}_2 \). \( \qed \)

3.1.3 Degree

We now come to an important invariant that can be attached to a covering. Looking back at Example 3.13 we have a graph covering of \( X \) where both the fiber of the vertex and the fiber of the edge contain two cells. Extending this covering to one of 2-complexes in Example 3.4 the fiber of the face also contains two cells. The fibers thus all have the same cardinality:
Proposition 3.21 (covering degree). If \( f : Y \to X \) is a covering with \( X \) connected, then any two fibers have the same cardinality.

This common cardinality of the fibers is called the degree of the covering, written
\[
\deg(Y \to X).
\]

The connectedness of \( X \) is easily seen to be essential, for if \( X \) has components \( X_1 \) and \( X_2 \), and \( f_i : Y_i \to X_i \) \((i = 1, 2)\) are coverings of different degree then we can cobble together a new covering \( f : Y = Y_1 \cup Y_2 \to X_1 \cup X_2 \) with \( f|_{Y_i} = f_i \). The cardinality of the fibers now depends on which component of \( X_i \) they lie over. Anyway, the connectedness of \( X \) is used explicitly in the proof:

Proof. If \( v, u \) are vertices of \( X \) and \( \gamma \) a path from \( v \) to \( u \), then lifting \( \gamma \) to a path \( \mu \) at any vertex of the fiber of \( v \) and taking its end vertex \( t(\mu) \), gives a (set) mapping from the fiber of \( v \) to the fiber of \( u \). Interchanging the roles of \( v \) and \( u \) and replacing \( \gamma \) with \( \gamma^{-1} \) gives the inverse of this set map, hence we have a bijection between the fibers of the two vertices.

If \( e \in X \) is an edge, let \( u \) be a vertex in the fiber of \( s(e) \). Then there is a unique edge \( e' \) in the fiber of \( e \) with \( s(e') = u \). It is easy to show that the map \( u \mapsto e' \) is a bijection \( f^{-1}(s(e)) \to f^{-1}(e) \).

Faces are similar: let \( r \in X \) be a face, \( v \in X \) a vertex in its boundary and \( x \in \alpha^{-1}(v) \). If \( u \) is a vertex in the fiber of \( v \) there is a face \( \sigma \in f^{-1}(r) \) and a \( y \in Y^n \) with \( \varepsilon(f, \sigma)(y) = x \). Let \( d : f^{-1}(v) \to f^{-1}(r) \) be the map defined by \( d(u) = \sigma \). Then \( d \) is injective as the attaching map of the face \( d(u) \) sends \( y \) to \( u \). If \( \sigma' \in f^{-1}(r) \) then \( \sigma' = d(u') \) for \( u' = \alpha \varepsilon(f, \sigma)^{-1}(x) \), so \( d \) is a surjection. \( \square \)

Degree plays a similar role for coverings as dimension does for vector spaces or index does for groups. For example, “if \( U \) is a subspace of \( V \) and \( \dim V/U = 1 \), then \( U = V' \), or "if \( H \) is a subgroup of index one in a group \( G \) then \( H = G' \), are arguments whose combinatorial topology version is,

Corollary 3.22. A degree one covering of a connected complex is an isomorphism.

The Corollary follows immediately from the surjectivity of coverings Proposition 3.14, and the definition of degree. This simple little result will play a crucial role in the proof of the Galois correspondence of \[4.3.1].

3.1.4 Lifting and excising simply connected subcomplexes

If \( X \) is 2-complex and \( Z \subset X \) a simply connected subcomplex, then we saw in Chapter 2 that the quotient map \( q : X \to X/Z \) induces an isomorphism \( q_* : \pi_1(X, v) \to \pi_1(X/Z, q(v)) \). If \( f : Y \to X \) is a covering, then \( f^{-1}(Z) \subset Y \) is a collection of isomorphic copies of \( Z \), each simply connected:

Proposition 3.23 (lifting simply connected complexes). Let \( f : Y \to X \) be a covering and \( Z \subset X \) a connected, simply connected subcomplex. Then \( f^{-1}(Z) \subset Y \) is a disjoint union \( f^{-1}(Z) = \bigcup_i Z_i \) with the \( Z_i \) connected, simply connected, and \( f \) maps each \( Z_i \) isomorphically onto \( Z \).

Proof. Let \( v \in Z \) be a vertex with \( \pi_1(Z, v) \) trivial and \( u_i \in Z_i \) a vertex in the fiber \( f^{-1}(v) \) with \( \gamma \) a closed path at \( u_i \). Then \( f(\gamma) \) is a closed path at \( v \), hence homotopically trivial. Homotopy lifting gives \( \gamma \) is homotopically trivial, and thus \( Z_i \) is simply connected. Connectedness follows by path lifting.

Observe that \( f \) restricted to any one of the \( Z_i \) is a covering \( f : Z_i \to Z \). Suppose \( u_1, u_2 \in Z_i \) are vertices with \( f(u_1) = f(u_2) = v \in Z \). If \( \gamma \) is a path in \( Z_i \) from \( u_1 \) to \( u_2 \) then \( f(\gamma) \) is a closed path in \( Z \) at \( v \), hence homotopically trivial. Homotopy lifting gives \( \gamma \) is homotopically trivial. But homotopically trivial paths are necessarily closed, so \( u_1 = u_2 \), the covering \( f : Z_i \to Z \) has degree one and thus is an isomorphism by Corollary 3.22. \( \square \)
Exercise 3.24. In the situation of Proposition 3.23, let \( \sigma \) be a face of \( X \) and \( T_1, \ldots, T_k \subset X^\sigma \) balls such that \( \alpha_\sigma^{-1}(Z) = \bigcup T_i \). Let \( \tau \) be a face of \( Y \) in the fiber of \( \sigma \) and \( S_1, \ldots, S_k \subset Y^\tau \) balls corresponding to the \( T_i \) via the isomorphism \( \varepsilon(f, \sigma) : Y^\tau \rightarrow X^\sigma \). Show that \( \alpha_\tau^{-1} f^{-1}(Z) = \bigcup S_i \) (ie: the \( S_i \) are precisely those parts of \( X^\tau \) that attach into \( f^{-1}(Z) \)) and that each \( S_i \) attaches into a \( Z_i \) for some \( i \).

Theorem 3.25 (excising simply connected complexes). Let \( f : Y \rightarrow X \) be a covering with \( Z \subset X \) a connected, simply connected subcomplex, \( f^{-1}(Z) = \bigcup_i Z_i \) disjoint and with the \( Z_i \) connected, and \( Y/Z_i \) and \( X/Z \) the resulting quotients. Then there is an induced covering \( f' : Y/Z_i \rightarrow X/Z \) making the diagram,

\[
\begin{array}{ccc}
Y & \xrightarrow{q'} & Y/Z_i \\
\downarrow f & & \downarrow f' \\
X & \xrightarrow{q} & X/Z
\end{array}
\]

commute, where \( q, q' \) are the quotient maps.

Proof. Let \( q : X \rightarrow X/Z \) and \( q' : Y \rightarrow Y/Z_i \) be the bottom and top quotient maps, and let \( x' := q'(x) \) be a cell of the quotient \( Y/Z_i \). Define \( f' : Y/Z_i \rightarrow X/Z \) by \( f'(x') = qf(x) \). As each \( Z_i \) is mapped isomorphically onto \( Z \), the map \( f' \) is well defined.

Suppose \( v' \) is a vertex of \( X/Z \) and let \( u' \in Y/Z_i \) be a vertex in the fiber of \( v' \). We need to show that the local continuity maps are bijections. We leave this as an exercise for the edges. It is immediate for the faces if \( v' \) is not the vertex \( q(Z) \), as the faces incident with \( u' \) and \( v' \) are unaffected by passing to the quotient. Suppose then that \( v' = q(Z), u' = q(Z_i) \in Y/Z_i \) for some \( i \) and \( q(\tau) \) is a face of the quotient with \( \alpha_{q(\tau)}^{-1}(u') = \{x_1, \ldots, x_k\} \). Thus there are balls \( T_1, \ldots, T_k \subset X^\sigma \) with the \( T_i \) those parts of \( X^\sigma \) attaching into \( Z \) and \( q(T_i) = x_i \).

We show the surjectivity of local continuity first. Fix \( x_i \) and let \( x_i \in T_i, v = \alpha_\sigma(x_i) \) and \( u \) the unique vertex of \( Z_i \) covering \( v \) (unique as \( Z_i \) covers \( Z \) isomorphically). Applying the covering \( f \) to the triple \( v, u, \sigma \) yields a face \( \tau \) in \( f^{-1}(\sigma) \) and a \( y \in Y^\tau \) attaching to \( u \) and corresponding to \( x_i \) via the isomorphism \( \varepsilon(f, \tau) : Y^\tau \rightarrow X^\sigma \). If \( S_1, \ldots, S_k \) are the balls in \( Y^\tau \) of Exercise 3.24 then we must have \( y \in S_i \). Thus \( y' = q(S_i) \in Y^{q(\tau)} \) attaches to \( u' \) and maps via \( \varepsilon(f', q(\tau)) \) to \( x_i' \).

Now injectivity: let \( y'_1, y'_2 \in Y^{q(\tau_1)}, Y^{q(\tau_2)} \) attach to \( u' \) and map to \( x_i \in X^{q(\sigma)} \) via the \( \varepsilon(f', q(\tau_i)) \). Thus there are balls \( S_{i1}, S_{i2} \subset Y^{\tau_1}, Y^{\tau_2} \) attaching into \( Z_i \) and corresponding to \( T_i \subset X^\sigma \) via the isomorphisms \( \varepsilon(f, \tau_i) \). Let \( x_i \in T_i, v = \alpha_\sigma(x_i) \in Z \) and \( y_1, y_2 \in S_{i1} \) correspond to \( x_i \). Then the \( y_1, y_2 \) attach to a vertex in \( Z_i \) that covers \( v \), and as \( f \) has degree one when restricted to \( Z_i \), they must attach to the same vertex. Applying the covering \( f \) then gives \( y_1 = y_2 \), hence \( y'_1 = y'_2 \) as required. □

Example 3.26. Figure 3.7 shows a degree two covering \( f : Y \rightarrow X \) and \( Z \) (in red) a spanning tree for \( X \) (at the bottom right). A vertex \( v \in X \) is ringed (in blue) and the two vertices of \( X^\sigma \) attaching to it are also ringed. The two balls \( T_1, T_2 \subset X^\sigma \) with \( \alpha_\sigma^{-1}(Z) = T_1 \cup T_2 \) are outlined in blue. The \( Z_1, Z_2 \) are the lifts of \( Z \) to \( Y \), and the balls \( S_{ij} \subset Y^{\tau_i} \) attaching to the \( Z_j \) are outlined in blue. Finally, a vertex \( u \in Y \) in the fiber of \( v \) is ringed in blue and the vertices of the \( Y^{\tau_i} \) attaching to it also (one in each face). The quotients of \( X \) by \( Y \) and \( Z \) by the \( Z_i \) are on the left.

Exercise 3.27. Let \( f : Y \rightarrow X \) be a covering with \( X \) connected and \( Z \subset X \) a connected, simply connected subcomplex with \( f' : Y/Z_i \rightarrow X/Z \) the induced covering of Theorem 3.25. Show that \( \deg(Y \rightarrow X) = \deg(Y/Z_i \rightarrow X/Z) \).
3.2 Actions, intermediate and universal covers

3.2.1 Group actions

Recall from §1.2.3 that a group acts freely on a 2-complex precisely when it acts freely on the vertices, i.e., for any \( g \in G \) and vertex \( v \), if \( g(v) = v \) then \( g \) is the identity. Such group actions give coverings:

**Proposition 3.28 (free actions give covers).** If a group \( G \) acts orientation preservingly and freely on a 2-complex \( X \) then the quotient map \( q : X \to X/G \) is a covering.

Our main supply of free group actions will come from the Galois group of a cover \( Y \to X \) in Chapter 4.

**Proof.** Let \([v]\) be a vertex in \( X/G \) and \( u \in X \) with \( q(u) = [v] \), hence \( u = g(v) \) for some \( g \in G \). An edge of \( X/G \) starting at \([v]\) has the form \([e]\) with \( s(e) = v \). In particular \( g(e) \) has start \( u \) and \( qg(e) = [e] \), giving the surjectivity of (edge) local continuity. If \( e_1, e_2 \) start at \( u \) with \( q(e_1) = q(e_2) \) then \( e_2 = g'(e_1) \) for \( g' \neq 1 \) and \( g'(u) = u \), contradicting the freeness of the \( G \)-action. Thus we have injectivity of (edge) local continuity and \( q : X^{(1)} \to X/G^{(1)} \) a graph covering.

Now let \([\sigma]\) be a face of the quotient containing \([v]\) in its boundary. We have \( \partial[\sigma] = (X^{\sigma}, \alpha[\sigma] = q\alpha_\sigma) \) with \( \alpha^{-1}_v[\sigma] \) those vertices of \( X^{\sigma} \) sent by \( \alpha_\sigma \) into the equivalence class \([v]\). For \( u \in X \) with \( q(u) = [v] \), the set \( \bigcup_{\tau \in [\sigma]} \alpha^{-1}_\tau(u) \) consists of those vertices of \( \bigcup_{\tau \in [\sigma]} X^\tau \) that attach to \( u \), and we require

\[
\Pi \varepsilon(g_\tau, \sigma)^{-1} : \bigcup_{\tau \in [\sigma]} \alpha^{-1}_\tau(u) \to \alpha^{-1}_v[\sigma]
\]

(3.2)

to be a bijection, where \( g_\tau(\sigma) = \tau \). Suppose that \( \tau, \omega \in [\sigma] \) and \( y \in X^\tau, z \in X^\omega \) map via (3.2) to \( x \in X^\sigma \). As \( \alpha_\tau(y) = \alpha_\omega(z) = u \), the elements \( g_\tau, g_\omega \in G \) both map \( \alpha_\sigma(x) \) to \( u \). As the \( \varepsilon(g_\tau, \sigma) \) are injective, we must have \( \tau \neq \omega \), hence \( g_\tau \neq g_\omega \), so \( g_\tau g_\omega^{-1} \) is a non-identity element fixing \( u \), contradicting the freeness of the \( G \)-action. The map (3.2) is thus an injection.

Now let \( x \in X^\sigma \) attach to \( g(v) \in [v] \) and let \( y = \varepsilon(g, \sigma)(x) \in X^{g(\sigma)} \). The attaching map of \( X^{g(\sigma)} \) is the composition \( g^{-1} \alpha_\sigma \varepsilon(g, \sigma)^{-1} \), sending \( y \) to \( v \), hence \( y \) lies in the left hand side of (3.2). \( \Box \)
3.2.2 Intermediate covers

Suppose we have a commuting triangle of complexes and maps,

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{f} & & \downarrow{h} \\
X & \xrightarrow{h} & X
\end{array}
\]

with all three maps coverings. We say that the covers \( Y \xrightarrow{g} Z \xrightarrow{h} X \) are intermediate to \( f : Y \xrightarrow{} X \). We will see in §3.4 that the set of coverings intermediate to a fixed covering \( f : Y \xrightarrow{} X \) has a very nice structure.

**Proposition 3.29.** Let \( Y \xrightarrow{g} Z \xrightarrow{h} X \) be dimension preserving maps of 2-complexes with \( f = hg : Y \xrightarrow{} X \). If any two of \( f, g, h \) are coverings, then so is the third.

**Proof.** We do one of the three cases and leave the other two as an exercise. Suppose then that \( f \) and \( h \) are coverings. We need to show that the local continuity maps for \( g \) in (C2) and (C3) of Definition 3.1 are bijections. Let \( w \in Z, v \in Y \) be vertices with \( g(v) = w \) and \( e \in Z \) an edge with \( s(e) = w \). Injectivity is easiest: if \( e_1, e_2 \in Y \) with \( s(e_i) = v \) and \( g(e_i) = e \) then \( f(e_i) = hg(e_i) = h(e) \). As \( f \) is a covering we get \( e_1 = e_2 \). To find an edge at \( v \) covering \( e \), lift \( h(e) \) to an edge \( e'' \) at \( v \) and let \( e' = g(e'') \). Then both \( e, e' \) start at \( w \) and cover \( h(e) \), so \( e = e' \) as \( h \) is a covering, and \( g(e'') = e \) as required. Local continuity at a face \( \sigma \) with \( w \) lying in its boundary is completely analogous.

**Exercise 3.30.** Let \( Y \) be a graph and \( Y_1, Y_2 \subset Y \) subgraphs of the form,

\[
Y = \begin{array}{c}
Y_1 \\
\text{e} \\
Y_2
\end{array}
\]

(i). If \( Y_1 \) is a tree, \( f : Y \xrightarrow{} X, h : Z \xrightarrow{} X \) coverings with \( X \) having a single vertex, and \( p : Y_2 \xrightarrow{} Z \) a subgraph, then there is an intermediate covering \( Y \xrightarrow{g} Z \xrightarrow{h} X \).

(ii). If \( W \xrightarrow{} Y \) is a covering and \( Y_1 \) a tree, then \( W \) also has the form shown above for some subgraphs \( Y_1', Y_2' \subset W \), and with \( Y_1' \) a tree.

3.2.3 Covers from the “bottom up”

Much of the discussion of coverings so far as been in the abstract: we haven’t seen many actual covers! In this book we will construct specific examples in two ways that can be broadly described as “bottom-up” and “top-down”. The first of these, which we describe in this section, starts with a complex \( X \) and builds upwards to give a covering of it. The other, which is described in §4.1.4 starts with a covering of \( X \) and folds it down into a smaller covering. In both cases how far to build up, or how far to fold down, is governed by a subgroup of a certain group, although as it turns out, a different group in the two cases.

For the bottom-up cover we imitate a standard construction in topology:
Definition 3.31 (1-skeleton of “bottom-up” cover). Let $X$ be a 2-complex and $H \subset \pi_1(X, v)$ a subgroup. Define $X \uparrow H$ to be the following graph:

1. The vertices of $X \uparrow H$ are the equivalence classes of paths starting at $v$ under the following relation: $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1 \gamma_2^{-1}$ represents the homotopy class of an element of $H$ (and so in particular is a closed path). Write $u_\gamma$ for the vertex with representative path $\gamma$.

2. Let $e$ be an edge of $X$ and $u_\gamma, u_\mu$ vertices of $X \uparrow H$. Then there is an edge $e'$ of $X \uparrow H$ with start vertex $u_\gamma$ and finish vertex $u_\mu$ if and only if $\gamma e \mu^{-1}$ represents the homotopy class of an element of $H$.

In particular, edges of $X \uparrow H$ arise via the scheme illustrated in Figure 3.9 in this case $\gamma e (\gamma e)^{-1}$ represents the homotopy class of the identity element of $H$.

Figure 3.10 gives complexes $X_i$ ($i = 1, 2, 3$) and the graphs $X_i \uparrow H$ for $H$ the identity subgroup are given in Figure 3.11. Note that for $H$ trivial, paths $\gamma_1 \sim \gamma_2$ if and only if they are homotopic. If $X_1$ is the $S^1$-graph on the left of Figure 3.10 then there is a 1-1 correspondence between the homotopy classes of paths and paths of the form $e \ldots e$ ($k$
times) or $e^{-1}\ldots e^{-1}$ ($k$ times). Thus $X \uparrow H$ has vertices $u_k$ for $k \in \mathbb{Z}$. There is an edge connecting the vertex of the path $e\ldots e$ ($k$ times) to the vertex of the path $e\ldots e (k+1)$ times), to give the infinite 2-valent tree at left in Figure 3.11. Similarly the single-vertexed graph with two edges has $X \uparrow H$ the 4-valent infinite tree.

The last of the three complexes is the torus, which has exactly the same 1-skeleton as $X_2$, with the presence of a face drastically changes the graph. Paths of the form $\gamma_1 e_1 e_2^{-1} e_2^{-1}$ give distinct vertices in $X_2 \uparrow H$ but the same vertex in $X_3 \uparrow H$, forcing the 4-valent tree to bend into the grid shape shown.

**Exercise 3.32.** Let $u_\gamma$ and $u_\mu$ be vertices of $X \uparrow H$. If there is a path $e_1'\ldots e_k'$ in $X \uparrow H$ from $u_\gamma$ to $u_\mu$, then there is a path $e_1\ldots e_k$ in $X$ with the edges $e_i'$ arising from the edge $e_i$ as in Definition 3.31 and $\gamma_1 e_1\ldots e_k \gamma_2^{-1}$ an element of $H$.

**Proposition 3.33.** The graph $X \uparrow H$ is connected. Define $f : X \uparrow H \to X^{(1)}$ by sending a vertex $u_\gamma$ to the terminal vertex of $\gamma$, and $f(e') = e$, where the edge $e'$ arises from $e$ as in Definition 3.31. Then $f$ is a graph covering.

**Proof.** If $u_\gamma$ is a vertex of $X \uparrow H$ with $\gamma = e_1\ldots e_k \in X$, then $e_1'\ldots e_k' \in X \uparrow H$ is a path connecting $u_\gamma$ to $u_\phi = v$, and so $X \uparrow H$ is connected. To see that $f$ is a covering we need to be a dimension preserving map (which we leave to the reader) and for every pair of vertices $u, x$ with $f(u) = x$, it induces a bijection from the edges starting at $u$ to the edges starting at $x$. Suppose then that $u = u_\gamma$ and $e_1, e_2$ are edges of $X \uparrow H$ connecting $u_\gamma$ to vertices $u_1$ and $u_2$. Let $x$ be the terminal vertex of $\gamma$, so that $f(u_\gamma) = x$, and an edge starting at $x$ with $f(e_1) = f(e_2) = e$. Thus there are paths $u_1 \mu_2$ from $v$ to the terminal vertex of $e$ with $\gamma e_1^{-1} (i = 1, 2)$ representing an element of $H$. In particular, $(\gamma e_1^{-1})^{-1} \gamma e_2^{-1}$, which is homotopic to $\mu_1 \mu_2^{-1}$, represents an element of $H$, and so $u_1 = u_2$. The edges $e_i$ both arise by applying the construction of Definition 3.31 to the pair of vertices $u$ and $u_1 = u_2$, and as only one edge can arise this way we have $e_1 = e_2$. The local continuity maps are thus injective. For an edge $e$ starting at $v$, and a vertex $u_\gamma$ in the fiber of $v$ with the path $\gamma$ from $v$ to $x$, there is by definition an edge $e'$ connecting $u_\gamma$ and $u_{\gamma e}$. The local continuity maps are thus surjective. □

**Lemma 3.34.** In the graph covering $f : X \uparrow H \to X^{(1)}$, the boundaries of faces of $X$ lift to closed paths in $X \uparrow H$. More precisely, let $x$ be a vertex of $X$, $\sigma$ a face containing $x$ in its boundary and $\gamma_\sigma$ a boundary path of $\sigma$ starting at $x$. If $u$ is a vertex covering $x$ via the graph covering $f$ and $\mu_\sigma$ is the lift of $\gamma_\sigma$ at $u$, then $\mu_\sigma$ is a closed path in $X \uparrow H$.

The proof is left as an exercise. Thus the boundaries of faces in $X$ give rise to closed paths in $X \uparrow H$, and to construct the 2-skeleton of $X \uparrow H$ we “sew” faces into these closed paths:

**Definition 3.35 ("bottom-up" cover).** Let $X$ be a 2-complex, $H \subset \pi_1(X, v)$ a subgroup and $X \uparrow H$ the graph of Definition 3.31. We add faces in the following way: for each face $\sigma \in X$, fix a boundary label $\gamma_\sigma$ with start vertex $x$ and let $f^{-1}(x) = \{u_i | i \in I\}$ be the fiber of $x$ via the graph covering $f : X \uparrow H \to X^{(1)}$. For the lift of $\gamma_\sigma$ to each $u_i$, define a face $\sigma_i$ with boundary this closed path: $\partial \sigma_i = (\gamma_\sigma, \alpha_\sigma)$. Let $\alpha_\sigma = f^{-1} \alpha_\sigma$.

There are quite a few choices made in this construction. We will see that the complex is independent (upto isomorphism) of these choices in [4.3.2].

**Proposition 3.36.** Let $X \uparrow H$ be the 2-complex of Definition 3.35 and define $f : X \uparrow H \to X$ on the 1-skeleton as in Lemma 3.33 and for each face $\sigma_i$, arising from the face $\sigma$ of $X$ as in Definition 3.35 define $f(\pi_1(X \uparrow H, u_\sigma)) = H \subset \pi_1(X, v)$. 
Proof. We have a path $\gamma'$ in $X \downarrow H$ from the vertex $u_\mu$ to the vertex $u_\nu$ if and only if there is a path $\gamma$ in $X$ from the terminal vertex of $\mu$ to the terminal vertex of $\nu$ with $\mu \gamma \mu^{-1}$ homotopic to an element of $H$. In particular, if $\gamma'$ is a closed path at $u_\mu$, then $\gamma$ is a closed path at $v$ homotopic to an element of $H$, i.e., we have $f_\ast \pi_1(X \downarrow H, u) = H$ as claimed. For $f$ to be a covering we need bijective local continuity on the faces. Thus, let $z \in X$ be a vertex, $\sigma$ a face containing $z$ in its boundary and $y \in X \downarrow H$ a vertex in the fiber of $z$. Suppose also that in the construction of $X \downarrow H$ we chose as boundary path for $\sigma$ the image under $\alpha_\sigma$ of the path $\gamma_\sigma$ circumnavigating $X^\sigma$, and suppose that $\gamma_\sigma = \gamma_0 \gamma_1 \ldots \gamma_k$, where $z$ appears $k$ times in the boundary of $\sigma$ and $\gamma_0 \ldots \gamma_i$ terminates at the $i$-th of these appearances. Lift $\alpha_\sigma(\gamma_0 \ldots \gamma_i)^{-1}$ to $y$ for each $i$. By definition, there is a face of $X \downarrow H$ arising by lifting $\gamma_\sigma$ to the terminal vertices of each of these lifts. We leave it to the reader to show that these are precisely the faces in the fiber of $\sigma$ that $y$ appears in the boundary of, and that it appears exactly $k$ times. ⊓⊔

There is an alternative construction of the complex $X \downarrow H$ that is more group theoretic, at least at the level of the 1-skeleton.

**Definition 3.37 (“bottom-up” cover: version 2).** Let $X$ be a 2-complex and $H \subset \pi_1(X, v)$ a subgroup. Let $X \downarrow H$ be the graph defined as follows:

1. Let $T \subset X$ be a spanning tree and $\{g_i \mid i \in I\}$ be a set of (right) coset representatives for the subgroup $H$ in $\pi_1(X, v)$. For each $i \in I$ let $T_i$ be an isomorphic copy of $T$.
2. We now add edges to $\bigcup T_i$; let $e$ be an edge of $X$ not in $T$ with start vertex $v_1$ and terminal vertex $v_2$. Let $\gamma_e$ be the (reduced) path that travels through $T$ from $v$ to $v_1$, traverses $e$ and then travels through $T$ from $v_2$ to $v$. For $i \in I$ let $u_{i_1}, u_{i_2}$ be the vertices of $T_i$ corresponding to $v_1, v_2$ under the isomorphism $T_i \cong T$. Then there is an edge $e'$ with start $u_{i_1}$ and terminal vertex $u_{j_2}$ if and only if $H g_i \gamma_e = H g_j$. See Figure 3.12.

![Fig. 3.12. Alternative construction of the 1-skeleton of $X \downarrow H$.](image)

Define $f : X \downarrow H \to X^{(1)}$ by $f(u_i) = u$, where $u_i \in T_i$ corresponds to $u$ via the isomorphism $T_1 \cong T_i$, and $f(e') = e$ where $e' \in T_i$ arises from $e \in T$ as above.

**Exercise 3.38.** Show that $X \downarrow H$ is connected and $f : X \downarrow H \to X^{(1)}$ is a covering for which the conclusions of Exercise 3.32 hold.

The remainder of the construction is as in Definition 3.35. We leave it as an Exercise to show that the claims in Proposition 3.36 hold for $X \downarrow H$. Once again there is choice in the construction, and these ambiguities will be ironed out in §4.3.2.

Because the 1-skeletons are given by the cosets of the subgroup $H$, these bottom-up covers are called Schreier coset diagrams.
3.2.4 Universal covers

A complex is always a cover of itself (Exercise 3.9). In this section we show that a complex always has another cover at the other extreme, in that it is as “big” as possible.

**Definition 3.39 (universal covers).** A covering \( f : Y \to X \) is universal if and only if for any covering \( h : Z \to X \) there is a covering \( g : Y \to Z \) making the diagram,

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{f} & & \downarrow{h} \\
X & & 
\end{array}
\]

commute.

Equivalently, \( Y \to X \) is universal when any other covering \( Z \to X \) of \( X \) is intermediate to it.

The construction of a universal cover is a special case of the techniques of the previous section: write \( \tilde{X} \) for the complex \( X \uparrow H \) obtained when \( H \) is the identity subgroup of \( \pi_1(X,v) \).

**Proposition 3.40.** The 2-complex \( \tilde{X} \) is connected, simply connected and the covering \( f : \tilde{X} \to X \) of Proposition 3.36 is universal.

**Proof.** That \( \tilde{X} \) is connected and simply connected is immediate. Let \( h : Y \to X \) be a cover. Map lifting (Theorem 3.20) gives a map \( g : \tilde{X} \to Y \) with \( f = hg \), as the fundamental group of \( \tilde{X} \) is trivial. Proposition 3.29 gives \( g \) is a cover. \( \Box \)

\[ \begin{array}{c}
e_1 \quad \quad \quad \quad e_2 \\
\quad \quad \quad \quad \downarrow{u} \\
e_2 \quad \quad \quad \quad e_1 \\
\end{array} \]

\[ \begin{array}{c}
e_1 \quad \quad \quad \quad e_2 \\
\quad \quad \quad \quad \downarrow{e_1^{-1}} \\
e_2 \quad \quad \quad \quad e_1 \\
\end{array} \]

Fig. 3.13. universal cover of the torus: fix \( \gamma \) for the face \( \sigma \) of \( X \) as shown. The lifts of \( \alpha_\sigma(\gamma) \) to the vertices in the fiber of \( u \) are shown in \( \tilde{X} \) on the right.

**Example 3.41.** Figure 3.13 shows the result of performing this process with the complex \( X_3 \) of Figure 3.10 sewing faces onto the 1-skeleton of Figure 3.11. Figure 3.14 shows the effect on \( \tilde{X} \) of an extra face in \( X \).

**Example 3.42.** Figure 3.15 shows the universal cover of an \( X \) that is itself a degree two cover of the torus of Figure 3.13.

**Exercise 3.43.** Show, using universal coverings, that if \( X \) is a graph and \( \gamma_1, \gamma_2 \) are reduced homotopic paths in \( X \) with the same start vertex, then \( \gamma_1 = \gamma_2 \). [hint: lift the paths to \( \tilde{X} \) and use properties of reduced paths in trees to deduce that these lifts are identical.]

Here is one final result, which we will save up for later (§ 4.3.2):

**Corollary 3.44.** Let \( H \) be a subgroup of \( \pi_1(X,v) \) and let \( x = H \). Then we have an intermediate covering \( \tilde{X}_u \to (X \uparrow H)_z \xrightarrow{f} X_v \) with \( f_*\pi_1(X \uparrow H, x) = H \).
3.2 Actions, intermediate and universal covers

3.2.5 Monodromy

If \( f : Y \to X \) is a covering we will eventually get an action of two different groups on \( Y \) or parts of \( Y \). The more important of these is the Galois group of the covering, which forms the principle subject of Chapter 4.

The less important of these two actions is that of the fundamental group \( \pi_1(X,v) \) on the fiber \( f^{-1}(v) \) of a vertex \( v \in X \). Thus the fundamental group acts as a permutation group on the set of vertices covering \( v \). Such permutation representations of the fundamental groups of 2-complexes will play a key role in the proof of results like Miller’s theorem in a later Chapter.

To define this action, see that it makes sense, and is indeed a homomorphism

\[
\pi_1(X,v) \to \text{Sym}(f^{-1}(v)),
\]

we require no more than the path and homotopy lifting of §3.1.2. So, the “path-lifting action” would probably be a sensible name: the action would then do exactly what it says on the box! However, it is traditional in topology to call this action monodromy, and so we will too.

The definition is illustrated in Figure 3.16: let \( \gamma \) be a closed path at \( v \) representing the element \( g_\gamma \in \pi_1(X,v) \). Let \( \mu \) be the lift of \( \gamma \) at a vertex \( u \in f^{-1}(v) \), and let this lift have end vertex \( x \). Let \( \sigma_\gamma \in \text{Sym}f^{-1}(v) \) be the permutation with \( \sigma_\gamma(u) = x \).

We obviously have a well-defined issue to deal with, so that the permutation \( \sigma_\gamma \) does not depend on our choice of representative \( \gamma \). If \( \gamma' \) is another closed path at \( v \) with \( g_{\gamma'} = g_\gamma \), then the paths \( \gamma, \gamma' \) are homotopic. This homotopy can be lifted, via homotopy lifting, to a homotopy between the lifts \( \mu \) and \( \mu' \) at \( u \), and so the two lifts are homotopic in \( Y \). But homotopic paths have the same endpoints! Thus \( \mu' \) ends at \( x \) as well, and we get \( \sigma_\gamma(u) = x = \sigma_{\gamma'}(u) \).

Fig. 3.14. Complex \( X \) (left) obtained by sewing another face \( \tau \) onto the torus and its universal cover \( \bar{X} \) (right).

Fig. 3.15. Universal cover of an \( X \) that is a degree two cover of the torus. The red boundary path for \( \sigma \) starts at \( u \) whereas the blue boundary path for \( \tau \) starts at \( v \).
By Exercise 3.11 the lift at \( u \) of the path \( \gamma_1 \gamma_2 \) is the path \( \mu_1 \mu_2 \) obtained by lifting \( \gamma_1 \) and then lifting \( \gamma_2 \) to the terminal vertex of \( \mu_1 \). In particular \( \sigma_{\gamma_1 \gamma_2} = \sigma_{\gamma_2} \sigma_{\gamma_1} \), recalling that the product is read from right to left.

**Proposition 3.45.** If \( f : Y \to X \) is a covering then monodromy gives a homomorphism \( \pi_1(X,v) \to \text{Sym}(f^{-1}(v)) \), defined by \( g \gamma \mapsto \sigma^{-1} \gamma \). In particular, a covering of finite degree gives a homomorphism from \( \pi_1(X,v) \) to a finite group.

**Exercise 3.46.** Let \( X \) be a 2-complex with a single vertex and \( H \subset \pi_1(X,v) \) a subgroup with \( X \uparrow H \) and \( f : X \uparrow H \to X \) the covering of §3.2.3. Show that the monodromy action corresponds to the action on the cosets given by \( H g \mapsto H (gh^{-1}) \) for \( h \in \pi_1(X,v) \). Compare with Exercise 4.12 and notice that the action is always well defined.

Suppose that \( X \) has just one vertex \( v \), so that the fiber of \( v \) consists of all the vertices of \( Y \). Monodromy then gives an action of \( \pi_1(X,v) \) on the whole 0-skeleton of \( Y \). The next exercise shows that in general this action cannot be extended any further than this.

**Exercise 3.47.** 1. Let \( X \) be the complex of Figure 3.17. Describe the universal cover \( \tilde{X} \to X \), showing that in particular that it is a covering of degree 6.

2. Show that in \( \tilde{X} \) there exists a pair of vertices joined by an edge but there is no edge joining the images of these two vertices under the monodromy action of \( \pi_1(X,v) \). Thus it is not possible to define an automorphism of \( \tilde{X} \) at this edge. Deduce that there can therefore be no homomorphism from \( \pi_1(X,v) \) to the automorphism group of the 1-skeleton extending the monodromy action on the 0-skeleton.

### 3.3 Operations on coverings

In Chapter 1 we had three constructions arising from a collection of complexes and maps between them: the pushout, pullback and Higman composition. In this section we show that all three are useful ways of creating new coverings from old.

The set-up is as follows: we have a fixed covering \( f : Y \to X \) together with two coverings intermediate to \( f \) as in §3.2.2:

\[
Y \overset{g_1}{\to} Z_1 \overset{h_1}{\to} X \quad \text{and} \quad Y \overset{g_2}{\to} Z_2 \overset{h_2}{\to} X.
\]

We then pushout the covers \( Y \to Z_i \) and pullback the covers \( Z_i \to X \). Throughout this section all complexes are connected.
### 3.3 Operations on coverings

#### 3.3.1 Pushouts of covers

Let \( f : Y \to X \) be a fixed covering of connected 2-complexes, and \( Y \xrightarrow{g_i} Z_i \xrightarrow{h_i} X \) \((i = 1, 2)\) be coverings intermediate to \( f \) with the \( Z_i \) distinct and connected. Thus we have the commuting diagram on the left of Figure 3.18 with all the maps in sight coverings. As the \( g_i \) are dimension preserving, we can by Theorem 1.4.2 form the pushout \( Z_1 \coprod_Y Z_2 \), obtaining in the process maps \( t_i : Z_i \to Z_1 \coprod_Y Z_2 \) \((i = 1, 2)\) as the composition \( Z_i \hookrightarrow Z_1 \cup Z_2 \to Z_1 \coprod Z_2 \sim \) of the inclusion of \( Z_i \) in the disjoint union and the quotient defined in 1.3.4. The universality of the pushout, applied to the maps \( h_i : Z_i \to X \), gives the commuting diagram on the right of Figure 3.18. If \([z] \) is a cell of \( Z_1 \coprod_Y Z_2 \), then \( z \in Z_i \) for some \( i \), so that the map \( h : Z_1 \coprod_Y Z_2 \to X \) sends \([z] \) to \( h_i(z) \in X \).

![Fig. 3.18. two intermediate covers (left) and their pushout (right).](image)

**Proposition 3.48 (pushouts of covers).** The maps

\[
t_i : Z_i \to Z_1 \coprod_Y Z_2, \quad (i = 1, 2) \quad \text{and} \quad h : Z_1 \coprod_Y Z_2 \to X,
\]

are coverings. Thus, the pushout of two intermediate coverings \( Y \to Z_i \to X \) \((i = 1, 2)\) is a connected intermediate covering \( Y \to Z_1 \coprod_Y Z_2 \to X \).

**Proof.** We show that the map \( t_1 g_1 : Y \to Z_1 \coprod_Y Z_2 \) is a covering, and then two applications of Proposition 3.29 give \( t_1 \) and \( h \) are coverings. That \( t_2 \) is a covering is completely analogous. The map \( t_1 g_1 \) is dimension preserving, as \( g_1 \) and \( t_1 \) are, leaving us to show that the local continuity maps are bijections. Suppose then that \( u \) is a vertex of \( Y \) mapping via \( t_1 g_1 \) to the vertex \([v] \) of the pushout, so that there is a vertex \( v \) of \( Z_1 \) with \( v = g_1(u) \) and \([v] = t_1(v) \).

Starting with the surjectivity of the local continuity on edges, suppose we have an edge \([e'] \) of the pushout with start vertex \([v] \). Thus, \( e' \) is an edge in the disjoint union \( Z_1 \cup Z_2 \) with start vertex \( v' \) equivalent to \( v \). The equivalence between \( v \) and \( v' \) is realized by a sequence of lifts and covers (of vertices) through the coverings \( g_i : Y \to Z_i \). The same sequence applied to the edge \( e' \), and using path lifting, yields an edge \( e \in Z_1 \) equivalent to \( e' \), and so \( t_1(e) = [e'] \), and with \( e \) having start vertex \( v \). Lifting \( e \) to \( u \in Y \) gives an edge mapping via local continuity to \([e'] \). Faces work the same: start with an occurrence of \([v] \) in \([\sigma'] \); use face lifting to get an occurrence of \( v \) in a face \( \sigma \in Z_1 \) with \( t_1(\sigma) = [\sigma'] \).

Now to the injectivity of the local continuity on edges, for which we suppose there are edges \( e_1, e_2 \) starting at \( u \in Y \) and mapping via \( t_1 g_1 \) to an edge \([e] \) starting at \([v] \) in the pushout. Thus the edges \( g_1(e_1), g_1(e_2) \) map via \( t_1 \) to \([e] \), hence by \( h_1 \) to \( h[e] \) (starting at \( h[u] \)). Uniqueness of path lifting, applied first to the covering \( h_1 \) and then to \( g_1 \) gives \( e_1 = e_2 \). Again, faces work the same. \( \Box \)

#### 3.3.2 Pullbacks of covers

As in 3.3.1 let \( f : Y \to X \) be a fixed covering of connected 2-complexes, and \( Y \xrightarrow{g_i} Z_i \xrightarrow{h_i} X \) \((i = 1, 2)\) be coverings intermediate to \( f \) with the \( Z_i \) connected. Thus we have the
commuting diagram on the left of Figure 3.19 with all the maps in sight coverings. As the coverings \( h_i \) are dimension preserving, we may, via \([1.4.1]\) form the pullback \( Z_1 \prod_X Z_2 \), obtaining in the process maps \( t_i : Z_1 \prod_X Z_2 \to Z_i \) given by \( t_i : z_1 \times z_2 \mapsto z_i \). The universality of the pullback, Theorem \([1.4.7]\) applied to the maps \( g_i : Y \to Z_i \), gives the commuting diagram on the right of Figure 3.19. The new map \( h : Y \to Z_1 \prod_X Z_2 \) sends a cell \( y \in Y \) to the cell \( g_1(y) \times g_2(y) \in Z_1 \prod_X Z_2 \).

![Fig. 3.19. two intermediate covers (left) and their pullback (right).](image)

**Proposition 3.49 (pullbacks of covers).** The maps 

\[
t_i : Z_1 \prod_X Z_2 \to Z_i \quad (i = 1, 2) \quad \text{and} \quad h : Y \to Z_1 \prod_X Z_2,
\]

are coverings. Thus, the pullback of two intermediate coverings \( Y \to Z_1 \to X \) is an intermediate covering \( Y \to Z_1 \prod_X Z_2 \to X \).

**Proof.** It suffices, by Proposition \([3.29]\) to show that \( h \) is a covering. It is dimension preserving as the \( g_i \) are, and so it remains to show that the various local continuity maps are bijections. This is similar for both edges and faces: suppose that \( v_1 \times v_2 \) is a vertex of the pullback and \( u \) a vertex of \( Y \) with \( h(u) = v_1 \times v_2 \). If two objects at \( u \) (edges starting at \( u \) or appearances of \( u \) in faces) map under \( h \) to a single object at \( v_1 \times v_2 \), then these two map via the \( g_i \) to single objects at \( v_i \in Z_i \). The \( g_i \) are coverings, ensuring that the original two objects coincide, hence injectivity of the local continuity maps.

Surjectivity requires a couple more steps: start with an object at the vertex \( v_1 \times v_2 \) of the pullback. It maps via the \( t_i \) to objects at the \( v_i \in Z_i \), and they in turn map via the \( h_i \) to the same object at \( v = h_i(v_i) \). The path and face lifting provided by the covers \( g_i \) give two objects at \( u \) mapping to this single object at \( u \), one via \( h_1 g_1 \) and the other via \( h_2 g_2 \). But then these two objects map via the covering \( f \) to this single object at \( u \), and so must be the same object. By definition, the image via \( h \) of this single object at \( u \) must be the original object at \( v_1 \times v_2 \) that we started with. \( \square \)

We saw in \([1.4.1]\) that the pullback is not necessarily connected. To get a connected covering we use the pointed version of Exercise \([1.48]\) the \( h_i : (Z_i)_z \to X_z \) are pointed coverings, giving 

\[
t_i h_i : (Z_1 \prod_X Z_2)_z \to X_z,
\]

a pointed covering of connected complexes for \( z = z_1 \times z_2 \).

**Exercise 3.50.** Let \( Y \to X \) be coverings with \( Z \) a forest. Show that the pullback \( X \prod_X Z \) is also a forest.

### 3.3.3 Higman compositions of covers

Let \( f_i : Y_i \to X \) \((i = 1, \ldots, n)\) be dimension preserving maps and \( \{e_{j1}, e_{j2}\} \) \((j = 1, \ldots, m)\) a handle configuration in \( \bigsqcup Y_j \), with \( Y = [Y_1, \ldots, Y_n] \) the Higman composition and \( f = [f_1, \ldots, f_n] : Y \to X \) the map of \([1.4.2]\).
Proposition 3.51 (Higman compositions of covers). If the $f_i$ are covering maps then $f$ is a covering map. Moreover, if the Higman composition is connected then

$$\deg([Y_1, \ldots, Y_n] \to X) = \sum_i \deg(Y_i \to X).$$

Proof. Let $u \in Y$ and $v \in X$ be vertices with $f(u) = v$. If $u$ is not the start or terminal vertex of one of the edges in the handle configuration, then the edges starting at $u$ are completely unaffected by the composition. Otherwise, the edges starting at $u$ are unchanged in number, as Figure 3.36 shows. For local continuity of faces, we have the desired bijection before composition courtesy of one of the covering maps maps $f_i$, with the composition replacing certain occurrences of $u$ in the $\tau_\ell$ by occurrences in the $\tau'_\ell$, while still maintaining the bijection. Thus, $f$ is a covering map. The degree assertion follows from the fact that if $e$ is the edge of Definition 1.49 giving rise to the handle configuration, then the fiber $f^{-1}(e)$ is in bijective correspondence with the disjoint union of fibers $\bigcup_i f_i^{-1}(e)$. □

Fig. 3.20. coverings $f_1, f_2$ and a handle configuration (in red).

Example 3.52. Figure 3.20 shows graphs $X, Y_1, Y_2$ and two coverings $f_i : Y_i \to X$, a handle configuration in $Y_1 \cup Y_2$ and the resulting (disconnected) Higman composition in Figure 3.21.

Fig. 3.21. the (disconnected) Higman composition resulting from the set-up in Figure 3.20

3.4 Lattices of covers

When we introduced pullbacks in Chapter 1 we said that they would act as a kind of union of 2-complexes, with the pushout acting as a kind of intersection. This section makes this
precise: we give the set of coverings intermediate to a fixed covering \( f : Y \to X \) the structure of a poset in which the pullback and pushout give a join \( \lor \) and a meet \( \land \). Thus the intermediate coverings form a lattice (Theorem 3.64 below). The whole business is complicated by the fact that the resulting lattice is slightly too big for what we want it for in Chapter 4. This forces us in 4.4.2 below to consider instead intermediate coverings up to a certain equivalence.

3.4.1 Aside: posets and lattices

We pause and take a brief look at the theory of posets and lattices and some important examples. There are many books on this subject: we have followed [15, Chapter 3].

Partial ordered sets (or posets) formalise the idea of ordering: a poset is a set \( P \) and a binary relation \( \leq \) that is reflexive: \( x \leq x \) for all \( x \in P \); antisymmetric: if \( x \leq y \) and \( y \leq x \) then \( x = y \); and transitive: if \( x \leq y \) and \( y \leq z \) then \( x \leq z \). The motivating example is meant to be the integers \( \mathbb{Z} \) with their usual ordering \( \leq \), and the usual notational conventions from there are used in general: we write \( x < y \) to mean \( x \leq y \) but \( x \neq y \). Elements \( x, y \) with \( x \leq y \) or \( y \leq x \) are comparable, otherwise they are incomparable (a possibility that obviously doesn’t arise with the primordial example \( \mathbb{Z} \)). We say that \( y \) covers \( x \), written \( x \prec y \), when \( x < y \) and if \( x \leq z \leq y \) then either \( z = x \) or \( z = y \).

A morphism (or just map) of posets \( f : P \to Q \) is an order-preserving map of the underlying sets: if \( x \leq y \) in \( P \) then \( f(x) \leq f(y) \) in \( Q \). Notice that this is a one way business: comparable elements are sent to comparable elements, but incomparable elements are allowed to map to comparable ones. An anti-morphism is an order-reversing map: if \( x \leq y \) in \( P \) then \( f(y) \leq f(x) \) in \( Q \).

Bijective morphisms have inverse set maps, although they may not be morphisms. A bijective morphism with order-preserving inverse is an isomorphism: \( x \leq y \) in \( P \) if and only if \( f(x) \leq f(y) \) in \( Q \). Similarly a bijective anti-morphism with order-reversing inverse is an anti-isomorphism.

Posets are often illustrated using their Hasse diagram: a graph whose vertices are the elements of \( P \) and whose edges give the covering relations. Thus, if \( x \prec y \) then the vertex \( y \) is drawn above the vertex \( x \) with an edge connecting them. Two examples of Hasse diagrams (and posets) are given in Figure 3.22.

![Hasse diagram for the poset of subsets of the set \{1, 2, 3\} ordered by inclusion (left) and for a poset with four elements (right) that is not a lattice.](image)

A special place is reserved for those posets which have suprema and infima. If \( x, y \in P \) then \( z \) is an upper bound for \( x \) and \( y \) when both \( x \leq z \) and \( y \leq z \). It is a least upper bound or supremum or join when it is an upper bound such that for any other upper bound \( w \) we have \( z \leq w \). Similarly, \( z \) is a lower bound for \( x \) and \( y \) when both \( z \leq x \) and \( z \leq y \). It is a greatest lower bound or infimum or meet when it is a lower bound such that for any other lower bound \( w \) we have \( w \leq z \).

It is easy to show that if \( x \) and \( y \) have a join then it is unique (hint: any two joins must be \( \leq \) each other) and similarly for the meet. Write \( x \lor y \) for the join and \( x \land y \) for the meet of \( x \) and \( y \).
A poset is a lattice if for every pair of elements \( x \) and \( y \), the join \( x \lor y \) and meet \( x \land y \) exist.

The poset on the left of Figure 3.22 is a lattice, as can be checked directly from the Hasse diagram, but the example on the right is not: if \( x \) and \( y \) are the two elements shown, then they have a join, but no meet.

**Exercise 3.53.** A \( 1 \) in a poset \( P \) is a unique maximal element: for all \( x \in P \) we have \( x \leq 1 \). Similarly a \( 0 \) in \( P \) is a unique minimal element: for all \( x \in P \) we have \( 0 \leq x \). Show that a finite lattice has a \( 0 \) and a \( 1 \).

**Exercise 3.54.** A poset is a meet-semilattice if any two elements have a meet. Dually we have the notion of a join-semilattice. Show that if \( P \) is a finite meet-semilattice with a \( 1 \) then \( P \) is a lattice (dually, if \( P \) is a finite join-semilattice with a \( 0 \) then \( P \) is a lattice).

**Exercise 3.55.** Let \( P \) and \( Q \) be lattices and \( f : P \to Q \) a lattice isomorphism (respectively anti-isomorphism). Show that \( f \) sends joins to joins and meets to meets (resp. joins to meets and meets to joins), i.e.: \( f(x \lor y) = f(x) \lor f(y) \) and \( f(x \land y) = f(x) \land f(y) \).

The most commonly occurring lattice “in nature” is the Boolean lattice on a set \( X \): its elements are the subsets of \( X \) with \( A \subseteq B \) if and only if \( A \subset B \). Meets and joins are just intersections and unions: \( A \land B = A \cap B \) and \( A \lor B = A \cup B \).

**Exercise 3.56.** Let \( X \) be a finite set, \( P \) the Boolean lattice on \( X \) and \( V \) the real vector space with basis \( X \). If \( v = \sum_X \lambda_x x \in V \) define \( |v|^2 = \sum_X \lambda_x^2 \), and let \( \Box^n := \{ v \in V : |v|^2 \leq 1 \} \), the \( n \)-dimensional cube. Embed the underlying set of \( P \) in \( V \) via the map sending \( A \subseteq X \) to \( \sum_{x \in A} x \), and show that the image of \( P \) is the set of vertices of \( \Box^n \), while the vertices and edges of \( \Box^n \) give the Hasse diagram for \( P \).

Another example is the lattice \( L_n(\mathbb{F}) \) of all subspaces of the \( n \)-dimensional vector space over the field \( \mathbb{F} \), with the ordering given by inclusion of one subspace in another. The meet of two subspaces is again their intersection, but this time the union is too small to be their join: the union of two subspaces is not a subspace! Instead we define \( U \lor V = U + V \), their sum, consisting of all vectors of the form \( u + v \) for \( u \in U \) and \( v \in V \). We leave it to the reader to verify that these are indeed infimums and supremums.

Here is one we are particularly interested in,

**Definition 3.57 (lattice of subgroups).** Let \( G \) be a group. The lattice of subgroups \( S(G) \) has as elements the subgroups of \( G \) ordered by inclusion, and with \( H \land K = H \cap K \), \( H \lor K = \langle H, K \rangle \), the subgroup generated by \( H \) and \( K \).

![Fig. 3.23. subgroup lattice for the symmetric group \( S_3 \) with \( \sigma = (1, 2, 3) \) and \( \tau = (2, 3) \).]

**Exercise 3.58.** Show that the set of finite index subgroups of a group \( G \) also forms a lattice, with the same meet and join as \( S(G) \). Show the same with finite index replaced by finitely generated.

**Exercise 3.59.** Show that an isomorphism \( G_1 \to G_2 \) of groups induces an isomorphism of lattices \( S(G_1) \to S(G_2) \).
3.4.2 The poset of intermediate covers

In this section and the next, we construct a lattice whose elements are, more or less, the coverings intermediate to a particular fixed covering \( f \). In the next chapter we’ll see that if we look at this lattice sideways and squint our eyes a little, then it looks the same as the lattice of subgroups of the “group of automorphisms” of the covering \( f \).

Throughout this section \( f : Y \to X \) is a fixed covering of connected 2-complexes and \( Y \overset{g_i}{\longrightarrow} (Z_i)_{x_i} \overset{h_i}{\rightarrow} X \) (\( i = 1, 2 \)) are coverings intermediate to \( f \) with the \( Z_i \) connected.

We call these two intermediate coverings equivalent if and only if there is a isomorphism \( Z_1 \to Z_2 \) making the diagram on the right of Figure 3.24 commute.

This is an equivalence relation on the set of coverings intermediate to \( f \), and we write \( \mathcal{L}(Y \xrightarrow{f} X) \) or just \( \mathcal{L}(Y, X) \) for the set of equivalence classes. The notation here can become very cumbersome, so where possible we will write \( Z \in \mathcal{L}(Y, X) \) to mean the equivalence class represented by the coverings \( Y \to Z \to X \) intermediate to \( f \).

It is possible to do everything in this section and the next in terms of intermediate coverings themselves, and not worry about equivalence at all. Nevertheless, the notion of equivalence will become essential later for the accounting to come out in the wash. Figure 3.25 shows a pair of equivalent graph coverings.

We now turn \( \mathcal{L}(Y, X) \) into a poset: one intermediate covering is “bigger” than another if the first covers the second. Specifically, if \( Z_1, Z_2 \in \mathcal{L}(Y, X) \) then define \( Z_1 \leq Z_2 \) precisely when there is a covering \( Z_2 \to Z_1 \) making the diagram on the left of Figure 3.26 commute.

Presupposing for a minute that this definition makes sense and gives a partial order, we have:

**Definition 3.60 (poset of intermediate covers).** For a fixed covering \( f : Y \to X \) of connected complexes, the set \( \mathcal{L}(Y, X) \) of equivalence classes of connected intermediate
coverings, together with the partial order \( \leq \) defined above is called the **poset of intermediate coverings** (to \( f \)).

It is not hard to check that the order \( \leq \) is well defined: suppose that for \( i = 1, 2 \), we have intermediate coverings \( Z'_i \), equivalent to the \( Z_i \) via isomorphisms \( Z_i \leftrightarrow Z'_i \) and \( Z_i \leq Z_2 \). As isomorphisms are nothing other than degree one coverings, the red map across the middle of the diagram on the right of Figure 3.26 is a covering making the big outside square commute. Thus \( Z'_1 \leq Z'_2 \), and the order doesn’t depend on which representative for the equivalence class we choose.

**Lemma 3.61.** The set \( \mathcal{L}(Y, X) \) is a poset.

**Proof.** Reflexivity and transitivity are immediate, as the identity map is a covering and the composition of coverings is a covering. Only anti-symmetry requires a moments thought: suppose we have \( Z_1, Z_2 \in \mathcal{L}(Y, X) \) with \( Z_1 \leq Z_2 \) and \( Z_2 \leq Z_1 \), so that there are coverings \( Z_1 \rightarrow Z_2 \) making the appropriate diagrams commute. Let \( t \) be the composition \( Z_1 \rightarrow Z_2 \rightarrow Z_1 \) of these two. Then consideration of these commuting diagrams gives \( g_1 = th_1 \), where \( g_1 : Y \rightarrow Z_1 \) is the covering, and so by the surjectivity of \( g_1 \), \( t \) is the identity on \( Z_1 \). But then the covering \( Z_1 \rightarrow Z_2 \) must be injective, ie: of degree 1, and so an isomorphism. Thus \( Z_1 = Z_2 \) in \( \mathcal{L}(Y, X) \).  

### 3.4.3 The lattice of intermediate covers

In the last section we introduced the poset \( \mathcal{L}(Y, X) \) of equivalence classes of connected coverings intermediate to a fixed covering \( Y \rightarrow X \). In this section we show that we have in fact a lattice, with join a pullback and meet a pushout. Because we want all our complexes to be connected and the pullback isn’t necessarily so, everything in sight has to be pointed, and we use the pointed versions of the pushout and pullback in Exercises [1.44](#) and [1.48](#)

Throughout then, \( f : Y_u \rightarrow X_v \) is a fixed pointed covering of connected 2-complexes. All intermediate coverings \( g : Y \rightarrow Z_x \rightarrow X \) are connected and pointed, and \( \mathcal{L}(Y_u, X_v) \) is the poset of equivalence classes of pointed connected intermediate coverings.

We start by showing that we have a meet. Let \( Y_u \rightarrow (Z_1)_{z_1} \rightarrow X_v \) and \( Y_u \rightarrow (Z_2)_{z_2} \rightarrow X_v \) be intermediate to \( f \), and \( Z_1 \coprod_Y Z_2 \) the pushout of the coverings \( g_1 : Y_u \rightarrow (Z_1)_{z_1} \) and \( g_2 : Y_u \rightarrow (Z_2)_{z_2} \). Let \( z = [z_1] = [z_2] \), where \( g(x) = [x] \) is the quotient map arising from the construction of the pushout, and \( (Z_1 \coprod_Y Z_2)_z \) the resulting pointed pushout.

By Proposition [3.48](#) we have a new element of \( \mathcal{L}(Y_u, X_v) \) given by the equivalence class of the intermediate covering,

\[
Y_u \rightarrow (Z_1 \coprod_Y Z_2)_z \rightarrow X_v.
\]

We now need a technical result to ensure that the whole process is well defined: if the \( X_i \) are replaced by equivalent coverings, then the new pushout that results is equivalent to the old one:
**Proposition 3.62.** Let \((V_1)_{y_1}, (V_2)_{y_2} \in \mathcal{L}(Y_u, X_v)\) be equivalent to \((Z_1)_{z_1}, (Z_2)_{z_2}\) via the isomorphisms,

\[s_1 : (Z_1)_{z_1} \to (V_1)_{y_1}, \text{ and } s_2 : (Z_2)_{z_2} \to (V_2)_{y_2} \]

Define a map \(s_1 \amalg s_2 : Z_1 \cup Z_2 \to V_1 \cup V_2\) between the disjoint unions by \(s_1 \amalg s_2|_{Z_1} = s_1\) and \(s_1 \amalg s_2|_{Z_2} = s_2\). Then the map

\[s : (Z_1 \coprod Y Z_2)_z \to (V_1 \coprod Y V_2)_y, \quad (y = [y_1] = [y_2]),\]

defined by \(sq = q'(s_1 \amalg s_2)\), is an isomorphism making these pointed pushouts equivalent, where \(q : Z_1 \cup Z_2 \to Z_1 \coprod Y Z_2\) and \(q' : V_1 \cup V_2 \to V_1 \coprod Y V_2\) are the quotient maps arising in the pushouts.

Thus, the pushout can be extended in a well defined way to equivalence classes of intermediate coverings, so for \((Z_1)_{z_1}, (X_2)_{z_2} \in \mathcal{L}(Y_u, X_v)\) we write

\[(Z_1 \coprod Y Z_2)_z \in \mathcal{L}(Y_u, X_v),\]

for the pushout of these two equivalence classes.

**Proof (of Proposition 3.62).** is a tedious but routine diagram chase. \(\square\)

Now, Proposition 3.48 gives coverings \(t_i : Z_i \to Z_1 \coprod Y Z_2\) so that \((Z_1 \coprod Y Z_2)_z \leq (Z_i)_{z_i}\) (\(i = 1, 2\)) is a lower bound in \(\mathcal{L}(Y_u, X_v)\). If \(V_y\) is any other lower bound then we get coverings \(Y_u \to (Z_i)_{z_i} \to V_y\), and by the universality of the pushout, Proposition 1.42 we have a map \((Z_1 \coprod Y Z_2)_z \to (V_y)\), which by Proposition 3.29 is a covering. Thus \(V_y \leq (Z_1 \coprod Y Z_2)_z\), and the pushout is the meet of the two equivalence classes \((Z_1)_{z_1}\). We write

\[(Z_1)_{z_1} \wedge (Z_2)_{z_2} = (Z_1 \coprod Y Z_2)_z.\]

Now to joins, which are similar. Let \(Y_u \to (Z_1)_{z_1} \to X_v\) and \(Y_u \to (Z_2)_{z_2} \to X_v\) be intermediate to \(f\), and \(Z_1 \coprod X Z_2\) the pullback of the coverings \(h_1 : (Z_1)_{z_1} \to X_v\) and \(h_2 : (Z_2)_{z_2} \to X_v\). Let \(z = z_1 \times z_2\), a vertex of the pullback, and \((Z_1 \coprod X Z_2)\) the pointed pullback consisting of the connected component containing \(z\).

We have a well-definedness result analogous to Proposition 3.62.

**Proposition 3.63.** Let \((V_1)_{y_1}, (V_2)_{y_2} \in \mathcal{L}(Y_u, X_v)\) be equivalent to \((Z_1)_{z_1}, (Z_2)_{z_2}\) via the isomorphisms,

\[s_1 : (Z_1)_{z_1} \to (V_1)_{y_1} \text{ and } s_2 : (Z_2)_{z_2} \to (V_2)_{y_2}.\]

Then the map

\[s : (Z_1 \coprod X Z_2)_{z_1 \times z_2} \to (V_1 \coprod X V_2)_{y_1 \times y_2},\]

defined by \(i(z_1 \times z_2) = i_1(z_1) \times i_2(z_2)\) is an isomorphism making the pointed pullbacks equivalent.

Proposition 3.49 gives coverings \(t_i : (Z_1 \coprod X Z_2)_{z} \to (Z_i)_{z_i}\) for \(z = z_1 \times z_2\), so that the \((Z_i)_{z_i} \leq (Z_1 \coprod X Z_2)_{z}\) (\(i = 1, 2\)) and the pullback is an upper bound in \(\mathcal{L}(Y_u, X_v)\). If \(V_y\) is any other upper bound we get coverings \(V_y \to (Z_i)_{z_i} \to X_v\), and by the universality of the pullback, Proposition 1.47 we have a map \((V_y) \to (Z_1 \coprod X Z_2)_{z}\). Proposition 3.29 again gives the map \(V_y \to (Z_1 \coprod X Z_2)_{z}\) is a covering. Thus \((Z_1 \coprod X Z_2)_{z} \leq V_y\), and the pullback is the join of the two equivalence classes \((Z_1)_{z_1}\). We write

\[(Z_1)_{z_1} \vee (Z_2)_{z_2} = (Z_1 \coprod X Z_2)_{z}.\]

Finally, recall from Exercise 3.53 that a \(\hat{1}\) in a poset is a unique maximal element and a \(\hat{0}\) is a unique minimal element.
Theorem 3.64 (lattice of intermediate coverings). The poset $\mathcal{L}(Y_u, X_v)$ of pointed connected covers intermediate to a fixed covering $f : Y_u \to X_v$ is a lattice with join $(Z_1)_{z_1} \lor (Z_2)_{z_2}$ the pullback $(Z_1 \prod_X Z_2)_{z_{1} \times z_{2}}$, meet $(Z_1)_{z_1} \land (Z_2)_{z_2}$ the pushout $(Z_1 \coprod_{Y} Z_2)_{[z_1]}$, unique minimal element $\hat{0} = X_v$ and unique maximal element $\hat{1} = Y_u$. 

3.5 Notes on Chapter 3
Galois theory arises whenever we have the following situation: $A$ is some object and $\text{Gal}(A)$ is its group of “symmetries”, or Galois group. If $B \subset A$ is a sub-object, then there is a subgroup $H \subset \text{Gal}(A)$ consisting of those symmetries of $A$ that act trivially on $B$. On the other hand, if $H \subset \text{Gal}(A)$ is a subgroup, there is a sub-object $B \subset A$ on which the action of $H$ has been cancelled out.

The first serious theorem in any Galois theory then says that this correspondence between sub-objects of $A$ and subgroups of $\text{Gal}(A)$ is perfect: the sub-objects of $A$ form a lattice, as do the subgroups of $\text{Gal}(A)$, and these two lattices are anti-isomorphic.

In the classical Galois theory, $A$ is an extension $E \subset F$ of fields and $\text{Gal}(A)$ the field automorphisms of $E$ fixing $F$ pointwise. For us, $A$ will be a covering $Y \rightarrow X$ of 2-complexes, and $\text{Gal}(A)$ the automorphisms of $Y$ that permute the fibers of the covering.

Throughout this chapter we make the running assumption that all complexes are connected.

4.1 Galois groups

4.1.1 Automorphisms and Galois groups

**Definition 4.1 (automorphism of a covering).** Let $f : Y \rightarrow X$ be a covering with $Y, X$ connected. A covering automorphism (or Galois automorphism) of $f$ is an isomorphism $a : Y \rightarrow Y$ making the diagram,

$$
\begin{array}{ccc}
Y & \xrightarrow{a} & Y \\
\downarrow{f} & & \downarrow{f} \\
X & & X
\end{array}
$$

commute. If $f : Y_u \rightarrow X_v$ is a pointed covering then $a$ is a pointed isomorphism $a : Y_u \rightarrow Y_u'$ making the diagram commute.

A covering automorphism is thus an automorphism of $Y$ that permutes the fibers of the covering. In topology covering automorphisms are often called deck transformations. Figure 4.1 shows a graph covering with exactly two covering automorphisms. One is the identity $\text{id} : Y \rightarrow Y$, and the other is the automorphism of $Y$ that interchanges the vertices $u_1$ and $u_2$ and interchanges the edges $e_1$ and $e_2$.

Notice that $Y$ has other automorphisms, but they are not covering automorphisms. For example, the map $Y \rightarrow Y$ that fixes the vertices $u_1$ and $u_2$, and sends the edge $e_1$ to $e_2^{-1}$ and $e_2$ to $e_1^{-1}$ is an automorphism, but it is not a covering automorphism as $e_1$ and $e_2^{-1}$ lie in different fibers of the covering.
Fig. 4.1. a simple graph covering: the two vertices of Y cover the single vertex of X and the two arcs of Y similarly. There is a single non-trivial covering automorphism interchanging the vertices $u_1$ and $u_2$ and the edges $e_1$ and $e_2$.

Figure 4.2 extends this example to a covering of 2-complexes as in §3.1.1. The identity automorphism id : $Y \to Y$ is a covering automorphism, as is the automorphism that interchanges the vertices $u_i$, the edges $e_i$ and the faces $\sigma_i$. These are the only two.

Fig. 4.2. The non-trivial automorphism of the covering of Figure 4.1 extends to a unique covering automorphism of the covering of Figure 3.3. The faces $\sigma_1$ and $\sigma_2$ are interchanged by $\alpha$ and the face isomorphism $Y^{\sigma_1} \to Y^{\sigma_2}$ is a $1/2$-turn.

One can check that there is no automorphism of the $Y$ of Figure 4.2 that fixes the vertices and edges but interchanges the faces. In particular notice that distinct covering automorphisms of $Y$ restrict to distinct covering automorphisms of the 1-skeleton of $Y$.

Exercise 4.2. If $f : Y \to X$ is a covering and $a_1, a_2 : Y \to Y$ covering automorphisms, show that their composition $a_2a_1$ and $a_1^{-1}$ are covering automorphisms. Show that the identity map id : $Y \to Y$ is a covering automorphism. Deduce that the set of covering automorphisms forms a group.

Definition 4.3 (Galois group of a covering). The covering automorphisms of the covering $f : Y \to X$ form a group called the Galois group of the covering, denoted

$$\text{Gal}(Y \to X)$$

or just $\text{Gal}(Y, X)$.

If the covering is pointed we write $\text{Gal}(Y_u, X_v)$.

We now come to some basic properties of the action of the Galois group: Recall from §1.2.3 that a group acts freely on a 2-complex $Y$ precisely when it acts freely on the vertices of $Y$.

Lemma 4.4. (i) The action of $\text{Gal}(Y, X)$ on $Y$ is orientation preserving. (ii) The effect of a covering automorphism $a \in \text{Gal}(Y, X)$ is completely determined by the image of a single vertex. In particular, the Galois group acts freely on $Y$.

Proof. (i) Let $a \in \text{Gal}(Y, X)$. If $x$ is an edge or face of $Y$ then both $x$ and $a(x)$ lie in the same fiber of the covering $f$, so that if $a(x) = x^{-1}$ then $f(x) = f(x)^{-1}$, a contradiction. Thus, the Galois group acts without inversions. (ii) Let $a \in \text{Gal}(Y, X)$ with $a(u) = u'$ for $u, u' \in Y$ vertices. If $\gamma$ is a path in $Y$ starting at $u$ (and covering $f(\gamma)$), then $a(\gamma)$ is a path starting at $u'$ and also covering $f(\gamma)$, as $a$ is a covering automorphism. By uniqueness...
4.1 Galois groups

of path lifting, \( a(\gamma) \) must be the lift of \( f(\gamma) \) to \( u' \). Thus the effect of \( a \) on the 1-skeleton is completely determined by \( a(u) = u' \). Faces are similar, using the uniqueness of face lifting. 

The technique used in the proof of Lemma 4.4 is called “cover and lift” (see Figure 4.3): start with the path \( \gamma \); it covers the path \( f(\gamma) \) and this in turn lifts to the image path \( a(\gamma) \).

\[ \text{Fig. 4.3. cover and lift} \]

**Lemma 4.5.** Let \( f : Y \to X \) be a covering and \( f : Y^{(1)} \to X^{(1)} \) its restriction to the 1-skeletons. The restriction of any \( a \in \text{Gal}(Y, X) \) to the one skeletons is a covering automorphism, and this induces an injective homomorphism \( \text{Gal}(Y, X) \to \text{Gal}(Y^{(1)}, X^{(1)}) \).

**Proof.** As \( ab \) restricted to the 1-skeleton is just \( b \) restricted to the 1-skeleton followed by \( a \) restricted to the 1-skeleton, we get a homomorphism. Lemma 4.4 shows that a covering automorphism \( a \) is completely determined by its effect on the 1-skeleton, thus giving injectivity.

### 4.1.2 Constructing automorphisms

The explicit construction of automorphisms is achieved by the following result:

**Proposition 4.6.** Let \( f : Y \to X \) be a covering with \( f(u) = v \), and \( u' \) another vertex in the fiber \( f^{-1}(v) \). Then there is a covering automorphism \( a \in \text{Gal}(Y, X) \) with \( a(u) = u' \) if and only if for any closed path \( \gamma \) at \( v \) with lifts \( \mu_1, \mu_2 \) at \( u, u' \), we have

\[ \mu_1 \text{ is closed } \iff \mu_2 \text{ is closed}. \]  

Indeed, the covering automorphism \( a \) is unique by Lemma 4.4 and comes about as follows: use “cover and lift” as in Figure 4.3 to get the effect of \( a \) on the vertices. There is a well-defined issue, and condition \((\dagger)\) is exactly what is needed to resolve it. The effect of \( a \) on the edges is given by path-lifting and on the faces by face lifting.

**Proof.** As automorphisms send closed paths to closed paths and non-closed paths to non-closed paths, the only if direction is clear. On the other hand, if \( x \) is a vertex of \( Y \) and \( \mu \) a path from \( u \) to \( x \), then define \( a(x) \) to be the terminal vertex of the lift of \( f(\mu) \) to \( u' \). If \( e \) is an edge let \( a(e) \) be the lift of \( f(e) \) to the vertex \( a(e) \). If \( \sigma \) is a face and \( x \) some vertex appearing in its boundary, let \( a(\sigma) \) be the lift of \( f(\mu) \) to \( a(x) \).

If \( \mu' \) is another path from \( u \) to \( x \) (so that \( \mu_1 = \mu' \mu^{-1} \) is closed at \( u \)) then \( f(\mu')f(\mu)^{-1} \) is a closed path at \( v \) lifting to \( \mu_1 \), hence by \((\dagger)\), it lifts to a closed path at \( u' \). Thus the lift to \( u' \) of \( f(\mu) \) and \( f(\mu') \) end at the same vertex and so \( a \) is well defined on the vertices. There is also a choice of boundary vertex involved in the definition of \( a \) on the edges and faces: we chose the vertex \( s(e) \) in the boundary of the edge \( e \) and the vertex \( x \) in the boundary of
the face $\sigma$. We show first that an arbitrary choice extends $a$ to a covering automorphism, and then appeal to Lemma 4.4(ii) to see that a different choice gives the same $a$.

We have that $sa(e)$ is by definition equal to $as(e)$, and the uniqueness of the lift of $f(e)^{-1}$ to $ta(e)$ gives $a(e^{-1}) = a(e)^{-1}$. If $\sigma$ is a face, then splicing together the two diagrams provided by the cover $\sigma \mapsto f(\sigma)$ and the lift $a(\sigma) \mapsto f(\sigma)$ gives the required commuting diagram for $\sigma \mapsto a(\sigma)$, and so $a$ is a map. Interchanging the roles of the vertices $u$ and $u'$ gives a map $b : Y \to Y$ with the uniqueness of path and face lifting giving $ab = ba = \text{id}$ on $Y$. It is easy to see that $a$ is dimension-preserving, so that we have an automorphism. Finally, $x$ and $a(x)$ lie in the same fiber of the covering, for any cell $x$, whence $fa = f$. □

**Example 4.7.** Figure 4.4 shows a covering with a pair of vertices (ringed) satisfying (†) of the Proposition with the obvious $1/2$-turn rotation the resulting automorphism.

![Fig. 4.4. vertices satisfying condition (†) of Proposition 4.6 yielding a covering automorphism $a \in \text{Gal}(Y, X)$](image)

### 4.1.3 From covers to subgroups

If $f : Y \to X$ is a covering and $\text{Gal}(Y, X)$ is its Galois group, then in this section and the next we show how an intermediate covering $Y \to Z \to X$ gives rise to a subgroup of $\text{Gal}(Y, X)$ and vice-versa. Indeed, recalling the definition of equivalent intermediate coverings from §3.4.2, equivalent coverings give the same subgroup. Thus, an equivalence class of intermediate coverings gives rise to a subgroup of the Galois group.

Let $Y \to Z_2 \to Z_1 \to X$ be a sequence of coverings intermediate to $f$, and let $a$ be an element of the Galois group $\text{Gal}(Y, Z_2)$, so that the triangle ringed in red in Figure 4.5 commutes.

![Fig. 4.5. from intermediate covers to subgroups of the Galois group.](image)
The unlabeled maps are the coverings. It is easy to see that the triangle ringed in blue must then also commute, so that $a$ can be identified with an element of $\text{Gal}(Y, Z_1)$, and we get a map $\text{Gal}(Y, Z_2) \to \text{Gal}(Y, Z_1)$. The following is then immediate,

**Lemma 4.8.** The map $\text{Gal}(Y, Z_2) \to \text{Gal}(Y, Z_1)$ is an injective homomorphism.

From now on we will just identify $\text{Gal}(Y, Z_2)$ with the subgroup of $\text{Gal}(Y, Z_1)$ consisting of those $a$ that make the red triangle commute in the diagram above. In particular, when $Y \to Z \to X$ is intermediate, we can identify $\text{Gal}(Y, Z)$ with a subgroup of $\text{Gal}(Y, X)$.

Fig. 4.6. the subgroup of the Galois group does not depend on the representative of the equivalence class.

Now suppose we have a pair $Y \to Z_i \to X$, ($i = 1, 2$) of coverings intermediate to $f$ that are equivalent via an isomorphism $Z_1 \to Z_2$. Then in the diagram of Figure 4.6, the red triangle commutes if and only if the blue triangle commutes (just interchange the roles of $Z_1$ and $Z_2$). In particular, an $a \in \text{Gal}(Y, X)$ lies in the subgroup $\text{Gal}(Y, Z_1)$ if and only if it lies in the subgroup $\text{Gal}(Y, Z_2)$.

These two subgroups thus coincide, and we can associate in a well-defined manner a subgroup of $\text{Gal}(Y, X)$ with an equivalence class of coverings intermediate to $f$.

**Example 4.9.** Figure 4.7 is the pair of equivalent coverings from Figure 3.25 with $X$ a bouquet of two loops, $Y$ its universal cover (the infinite 4-valent tree) and $Z_1, Z_2$ shown in the middle of Figure 4.7.

Fig. 4.7. The coverings $Y \to Z_i$ are shown using the little circles and squares. The Galois group $\text{Gal}(Y_u, Z_{x_i})$ of the covering $Y_u \to Z_{x_i}$ acts regularly on the vertices in each fiber. In particular, an automorphism $a \in \text{Gal}(Y_u, Z_{x_1})$ of the lefthand version of $Y$ gives an automorphism $a \in \text{Gal}(Y_u, Z_{x_2})$ of the righthand version and vice-versa.
Subgroups to covers from the “top down”

We constructed covers from the bottom-up in §3.2.3. Here is a reverse process: if \( f : Y \to X \) is a covering and \( \text{Gal}(Y, X) \) its Galois group, we show how a subgroup of \( \text{Gal}(Y, X) \) gives rise to an intermediate covering \( Y \to Z \to X \). Passing to the equivalence class of this covering, we get a subgroup giving rise to an equivalence class of coverings.

**Lemma 4.10.** If \( H_1 \subset H_2 \subset \text{Gal}(Y, X) \) are subgroups then there are coverings

\[
Y \to Y/H_1 \to Y/H_2 \to X
\]

intermediate to \( f \).

**Proof.** By Proposition 1.36 and Lemma 4.4(i) we can form the quotient complexes \( Y/H_i \) \((i = 1, 2)\), as the subgroups are acting orientation preservingly. The action is also free by Lemma 4.4(ii), so the quotient maps \( Y \to Y/H \) are coverings by Proposition 3.28. Two applications of Proposition 3.29 give that \( Y/H_1 \to Y/H_2 \) and \( Y/H_2 \to X \) are coverings. \( \square \)

In particular, letting \( H_1 = H_2 = H \), we can associate to \( H \subset \text{Gal}(Y, X) \) the intermediate covering \( Y \to Y/H \to X \), and by passing to its equivalence class we get, associated to \( H \), an element of the lattice \( \mathcal{L}(Y, X) \) of intermediate covers.

The Galois group and the fundamental group

Let \( f : Y \to X \) be a covering with \( f(u) = v \) and let \( G = \pi_1(X, v) \) and \( H = f_*\pi_1(Y, u) \subset G \). We saw in §4.1.2 that elements of \( G \) can be associated in a well defined way with the vertices of \( f^{-1}(v) \): let \( \gamma, \gamma' \) be closed paths at \( v \) such that \( g_\gamma = g_{\gamma'} \) in \( \pi_1(X, v) \). Lifting the homotopic \( \gamma, \gamma' \) to \( u \) gives \( \mu, \mu' \) homotopic. These must therefore end at the same vertex \( u' \).

When is there a covering automorphism sending \( u \) to \( u' \)?

**Lemma 4.11.** The vertices \( u, u' \) have property (†) of Proposition 4.6 if and only if \( g \in G \) normalizes the subgroup \( H \), i.e: \( g^{-1}Hg = H \).

**Proof.** Write \( g = g_\gamma \) and recall from Corollary 3.18 that a closed path \( \lambda \) at \( v \) represents an element \( h \in H \) if and only if the lift \( \nu \) of \( \lambda \) to \( u \) is closed. The result follows by observing that if \( \mu \) is the lift of \( \gamma \) to \( u \) and \( \nu' \) the lift of \( \lambda \) to \( u' = t(\mu) \), then \( \mu^{-1}\nu'\mu \) is the lift of \( \gamma^{-1}\lambda \gamma \) to \( u \). \( \square \)

![Fig. 4.8. finding the image of a vertex \( x \) under the covering automorphism \( a_g \).](image-url)
reminds us how this covering automorphism comes about: let \( g = g_u \) and lift to \( \mu \) at \( u \in Y \) with \( u' = t(\mu) \). To get the image \( a_g(x) \) of a vertex \( x \in Y \) let \( \nu \) be a path from \( u \) to \( x \) and lift \( f(\nu) \) to \( u' \). Then \( a_g(x) \) is the terminal vertex of this lift.

**Exercise 4.12.** Let \( X \) be a 2-complex with a single vertex and \( H \subset \pi_1(X,v) = G \) a subgroup with \( f : X \rightarrow Y \) the covering of \([3.23]\). Show that for \( k \in \pi_1(X,v) \), and \( Hg \) a vertex of \( X \) \( \cong \) \( Hg \) as in Definition \([3.37]\) we have \( a_k(Hg) = H(kg) \). Compare with Exercise \([3.46]\) and notice that the action is well defined, ie: \( Hg_1 = Hg_2 \Rightarrow H(kg_1) = H(kg_2) \) if and only if \( k \in N_G(H) \).

Here is the result that ties together the fundamental groups of \( X \) and \( Y \) and the Galois group of the covering:

**Proposition 4.13.** The map \( g \mapsto a_{g^{-1}} \) is a surjective homomorphism \( N_G(H) \rightarrow \text{Gal}(Y,X) \) with kernel \( H \), inducing an isomorphism,

\[
N_G(H)/H \cong \text{Gal}(Y,X).
\]

**Proof.** Successively lifting representatives for \( g \) and \( h \), and using the scheme in Figure \([4.3]\) gives \( a_{gh} = a_ga_h \). Writing \( \theta(g) = a_{g^{-1}} \) thus gives \( \theta(gh) = \theta(h)\theta(g) \) and so \( \theta \) is a homomorphism (remember: we read \( gh \) from left to right in \( \pi_1(X,v) \) but \( a_ga_h \) from right to left in the Galois group). The elements of \( H \) are represented by the \( \gamma \) lifting to closed paths at \( u \), hence giving an \( a_{gh} \) that fixes \( u \), and so must be the identity covering automorphism as automorphisms are completely determined by their effect on a single vertex. Thus the kernel is \( H \). If \( a \in \text{Gal}(Y,X) \) sends \( u \) to \( u' \) and \( \mu \) is a path from \( u \) to \( u' \), then \( g = g_{\mu} \) normalizes \( f, \pi_1(Y,u) \), and so \( a_g = a \). \( \square \)

### 4.1.6 Excising simply-connected subcomplexes

We saw in \([3.14]\) that if \( Y \rightarrow X \) is a covering and \( Z \subset X \) a connected, simply connected subcomplex, then \( Z \) can be lifted to \( Y \) to give a new covering, with \( Z \) and its lift “excised”. In this section we show that this process has no effect on the Galois group.

Let \( f : Y_u \rightarrow X_v \) be a covering and \( Z \subset X \) connected and simply-connected with \( v \in Z \). Let \( f^{-1}(Z) = \bigcup_i Z_i \subset Y \) with the \( Z_i \) the connected components, simply connected by Proposition \([3.23]\). Let \( f' : Y/Z_i \rightarrow X/Z \) be the induced covering of Theorem \([3.25]\) with

\[
\begin{array}{ccc}
Y & \xrightarrow{q} & Y/Z_i \\
\downarrow f & & \downarrow f' \\
X & \xrightarrow{q} & X/Z
\end{array}
\]

commuting and \( q, q' \) the respective quotient maps. From Chapter \([2]\) the induced homomorphism

\[
q_* : \pi_1(X,v) \rightarrow \pi_1(X/Z,q(v))
\]

is an isomorphism.

**Proposition 4.14.** The isomorphism \( q_* \) induces an isomorphism

\[
\text{Gal}(Y,X) \cong \text{Gal}(Y/Z_i,X/Z).
\]

**Proof.** “Hit” the commuting diagram above with the \( \pi_1 \)-functor of Chapter \([2]\) to get a commuting diagram of fundamental groups and group homomorphisms with

\[
q'_* : \pi_1(Y,u) \rightarrow \pi_1(Y/Z_i,q'(u)),
\]

surjective. Apply Exercise \([4.15]\) and Proposition \([4.15]\) \( \square \)
Exercise 4.15. Suppose we have the following commutative diagram of groups and homomorphisms,

\[
\begin{array}{c}
H & \xrightarrow{\theta} & H' \\
\downarrow{\psi} & & \downarrow{\psi'} \\
G & \xrightarrow{\varphi} & G'
\end{array}
\]

with \(\theta : H \to H'\) surjective and \(\varphi : G \to G'\) an isomorphism. Let \(N := N_G(\psi(H))\) be the normalizer in \(G\) of the image of \(H\), and \(N' := N_{G'}(\psi'(H'))\) similarly. Show that \(\varphi\) induces an isomorphism

\[
N/\psi(H) \xrightarrow{\cong} N'/\psi'(H').
\]

Example 4.16. Figure 4.9 shows a covering \(Y \to X\) (left), a spanning tree \(Z\) for \(X\) consisting of the single red edge, and the lifts \(\bigcup Z_i\) of \(Z\) to \(Y\) (in red). On the right we have the excised versions \(X/Z\) and \(Y/Z_i\). The Galois group \(\text{Gal}(Y, X) \cong \mathbb{Z}\) is generated by the covering automorphism \(a\) shown. The group is even more transparently \(\cong \mathbb{Z}\) in the righthand version.

Example 4.17. Figure 4.10 revisits Example 3.26: a covering \(Y \to X\) (right), a spanning tree \(Z\) for \(X\) and the lifts \(\bigcup Z_i\) of \(Z\) to \(Y\). On the left we have the excised versions \(X/Z\) and \(Y/Z_i\). The two Galois groups are \(\cong \mathbb{Z}/2\), generated by the \(a\)’s shown.

Fig. 4.9. lattice excision with \(\mathbb{Z} \cong \text{Gal}(Y, X) \xrightarrow{\cong} \text{Gal}(Y/Z_i, X/Z) \cong \mathbb{Z}\).

Fig. 4.10. lattice excision with \(\mathbb{Z}/2 \cong \text{Gal}(Y, X) \xrightarrow{\cong} \text{Gal}(Y/Z_i, X/Z) \cong \mathbb{Z}/2\).
4.2 Galois covers

To get some nice theorems about the Galois group of a cover we must restrict our attention to nice covers. We will see in this section that a nice cover is one that is highly symmetric.

4.2.1 Galois covers

Proposition 4.18. Let \( f : Y \to X \) be a covering and \( v \) a vertex of \( X \). Then the following are equivalent:

(i). For any closed path \( \gamma \) at \( v \), the lifts of \( \gamma \) to each vertex of the fiber \( f^{-1}(v) \) are either all closed or all non-closed.

(ii). The Galois group \( \text{Gal}(Y, X) \) acts regularly on the fiber \( f^{-1}(v) \).

(iii). If \( u \in f^{-1}(v) \), then \( f_*\pi_1(Y, u) \) is a normal subgroup of \( \pi_1(X, v) \).

Proof. The equivalences follow immediately from Proposition 4.6 and Lemma 4.11.

Loosely, a covering is Galois when the vertices of \( Y \) are completely interchangeable with each other, or put another way, \( Y \) looks the same from the viewpoint of any of its vertices. Figure 4.11 shows some examples and Figure 4.12 some non-examples.

\[ \begin{array}{c}
\text{Fig. 4.11. some Galois coverings } Y \to X \text{ with Galois groups } \mathbb{Z}, \text{ the free group of rank 2 and } \mathbb{Z} \times \mathbb{Z}. \\
\end{array} \]

Definition 4.19 (Galois coverings). A covering \( Y \to X \) is Galois (via \( v \)), if it satisfies any (hence all) of the conditions of Proposition 4.18.

Regular is common alternative terminology to Galois, for reasons that the second part of Proposition 4.18 makes clear. As the effect of a covering automorphism on the fiber \( f^{-1}(v) \) is completely determined by the image of a single vertex, a regular action is equivalent to a transitive action when talking about Galois groups.

\[ \begin{array}{c}
\text{Fig. 4.12. some non-Galois coverings of the complex } X \text{ of Figure 4.11.} \\
\end{array} \]

Lemma 4.20. Let \( f : Y \to X \) be a Galois covering via some vertex \( v \) of \( X \). Then,
(i) If \( v' \) is another vertex of \( X \), then the covering is Galois via \( v' \).

(ii) If \( Y \xrightarrow{g} Z \xrightarrow{h} X \) are intermediate coverings and \( x \in h^{-1}(v) \), then \( g : Y \to Z \) is Galois via \( x \).

(iii) the induced covering \( f : Y^{(1)} \to X^{(1)} \) on the 1-skeletons is Galois, and there is an isomorphism,

\[
\text{Gal}(Y, X) \cong \text{Gal}(Y^{(1)}, X^{(1)}).
\]

In light of the first part of the Lemma, we will call a covering \( f : Y \to X \) Galois without reference to a vertex in \( X \).

**Proof (of Lemma 4.20).** For part (i), let \( \lambda \) be a path in \( X \) from \( v' \) to \( v \) and \( \gamma \) a closed path at \( v' \). If \( u_1', u_2' \) are vertices in the fiber \( f^{-1}(v') \) of \( v' \), let \( \nu_1, \nu_2 \) be the lifts of \( \lambda \) to the \( u_i' \). Suppose the \( \nu_i \) finish at vertices \( u_i \), necessarily in the fiber \( f^{-1}(v) \) of \( v \). If \( \mu_1, \mu_2 \) are the lifts of \( \gamma \) to \( u_i' \), then the \( \nu_i^{-1} \mu_i \nu_i \) are the lifts of \( \lambda^{-1} \gamma \lambda \) to the \( u_i \). The question of whether the \( \mu_i \) are closed or not becomes the question of whether the \( \nu_i^{-1} \mu_i \nu_i \) are closed or not. In particular, if the covering is Galois via \( v \) it is Galois via \( v' \).

In the second part suppose that \( \gamma \) is a closed path at \( x \), so that \( h(\gamma) \) is a closed path at \( v \), and let \( u_1, u_2 \) be vertices of \( Y \) in the fiber of \( x \). Let \( \mu_1, \mu_2 \) be the lifts (through \( f \)) of \( h(\gamma) \) to \( u_1, u_2 \). Then they are also the lifts of \( \gamma \) (through \( g \)) to \( u_1, u_2 \), as \( g(\mu_1), g(\mu_2) \) are closed paths at \( x \) covering \( h(\gamma) \), hence must be \( \gamma \). The lifts \( \mu_1, \mu_2 \) are either both closed or both not closed as \( f \) is Galois, and thus \( g \) is Galois also.

For part (iii) we have an injective homomorphism \( \text{Gal}(Y, X) \to \text{Gal}(Y^{(1)}, X^{(1)}) \) from Lemma 4.20 with the image of \( \text{Gal}(Y, X) \) acting transitively on the fiber of some vertex, so \( \text{Gal}(Y^{(1)}, X^{(1)}) \) also acting transitively, hence regularly. Thus the induced covering of graphs is Galois. Suppose \( a \in \text{Gal}(Y^{(1)}, X^{(1)}) \) is a graph covering automorphism and that \( a(u) = u' \) for some vertices \( u, u' \in Y^{(1)} \). Then there is a covering automorphism \( \widehat{a} \in \text{Gal}(Y, X) \) that sends \( u \) to \( u' \) by the regularity of the action of \( \text{Gal}(Y, X) \). It must restrict on the 1-skeleton to \( a \) and so the homomorphism is surjective. \( \square \)

**Proposition 4.21.** Let \( f : Y \to X \) be a Galois covering and \( H \subset \text{Gal}(Y, X) \) a subgroup of index \( [\text{Gal}(Y, X) : H] \). If \( Y \to Y/H \to X \) is the intermediate covering of \( Y, X \) then, \( \deg(Y/H \to X) = [\text{Gal}(Y, X) : H] \).

In particular, taking \( H \) to be the trivial subgroup we get, when \( Y \to X \) is Galois, that the order \( [\text{Gal}(Y, X)] \) of the Galois group is equal to the degree \( \deg(Y \to X) \) of the covering.

**Proof.** If a group \( G \) acts regularly on a set and \( H \) is a subgroup, then the number of \( H \)-orbits is the index \( |G : H| \). The result now follows as the \( H \)-orbits on the fiber (via \( f \)) of a vertex \( v \in X \) are precisely the vertices of \( Y/H \) covering \( v \) via \( Y/H \to X \). \( \square \)

**Proposition 4.22.** If \( f : Y \to X \) is Galois with \( f(u) = v \) and \( g \in \pi_1(X, v) \), then the map \( g \mapsto a_{g^{-1}} \) of Proposition 4.13 induces an isomorphism \( \pi_1(X, v)/f_*\pi_1(Y, u) \cong \text{Gal}(Y, X) \).

If \( Y \xrightarrow{g} Z \xrightarrow{h} X \) is an intermediate covering, we get a subgroup \( h_*\pi_1(Z, x) \subset \pi_1(X, v) \), with \( h_*\pi_1(Z, x)/f_*\pi_1(Y, u) \) mapping via this isomorphism to \( \text{Gal}(Y, Z) \).

**Proof.** The isomorphism follows immediately from Proposition 4.13 as the normalizer is \( \pi_1(X, v) \) and \( H = f_*\pi_1(Y, u) \). The image \( h_*\pi_1(Z, x) \) consists of those \( g \) \in \( \pi_1(X, v) \) with \( \gamma \) lifting to a closed path \( \mu \) at \( x \). Thus, it is those \( g \) such that performing the process of Figure 4.8 we get for any \( x \in Y \), the cells \( x \) and \( a_{\gamma}(x) \) lie in the same fiber of the covering \( g : Y \to Z \). But \( \text{Gal}(Y, Z) \) consists precisely of those \( a \in \text{Gal}(Y, X) \) that permute the fibers of the covering \( g : Y \to Z \). \( \square \)
4.2 Galois covers

Fig. 4.13. A Galois covering of 2-complexes (left) and the induced Galois covering of graphs (right). The left one illustrates the inclusion of groups $1 \hookrightarrow \mathbb{Z}/2$ and the right one the inclusion $2\mathbb{Z} \hookrightarrow \mathbb{Z}$. The Galois groups are both $\mathbb{Z}/2$.

Example 4.23. On the left of Figure 4.13 we have the covering of §4.1.1. It is easy to check that this covering is Galois, either by looking at the lifts of closed paths in $X$, or by observing that $Y$ is simply connected, hence $f_\ast \pi_1(Y, u) = 1$, a normal subgroup of $\pi_1(X, v)$. Lemma 4.20 tells us that the Galois group of this covering is the same as for the induced covering of the 1-skeletons, shown on the right. Indeed, these coverings have degree 2, and so the two Galois groups are isomorphic to $\mathbb{Z}/2$ by Proposition 4.21.

The covering of 2-complexes gives us no more than the covering of the underlying graphs, at least when it comes to their Galois groups. They do illustrate different things though as Proposition 4.22 shows. The 2-complex covering depicts the inclusion of groups $1 \hookrightarrow \mathbb{Z}/2$, and the graph covering the inclusion $2\mathbb{Z} \hookrightarrow \mathbb{Z}$.

4.2.2 Intermediate covers

Let $Y \to X$ be a covering and $Y \to Z \to X$ a covering intermediate to it. When $Z \to X$ is Galois we can define a map $\theta : \text{Gal}(Y, X) \to \text{Gal}(Z, X)$ using the scheme in Figure 4.14. Let $\alpha$ be a covering automorphism in $\text{Gal}(Y, X)$, and $u_1, u_2$ vertices of $Y$ with $\alpha(u_1) = u_2$. In particular, the $u_i$ lie in the fiber $f^{-1}(v)$ of the covering $Y \to X$. They also cover vertices $x_1, x_2$ of $Z$ via the covering $Y \to Z$, with these two lying in the fiber of $v$ via the covering $Z \to X$. As $Z \to X$ is Galois, there is a covering automorphism $\alpha' \in \text{Gal}(Z, X)$ sending $x_1$ to $x_2$.

Proposition 4.24. Let $f : Y \to X$ be a covering with $Y \to Z \to X$ intermediate to it.

(i). If $Z \to X$ is Galois, then the map $\theta : \text{Gal}(Y, X) \to \text{Gal}(Z, X)$ defined by $\theta(\alpha) = \alpha'$ is a homomorphism with kernel $\text{Gal}(Y, Z)$. 

Fig. 4.14. Defining a map $\theta : \text{Gal}(Y, X) \to \text{Gal}(Z, X)$ when $Z \to X$ is Galois.
(ii). If \( Y \to X \) is Galois and \( \text{Gal}(Y, Z) \) is a normal subgroup of \( \text{Gal}(Y, X) \), then \( Z \to X \) is Galois, and \( \theta \) is surjective.

In particular, if \( Y \to X \) is Galois then \( Z \to X \) is Galois if and only if \( \text{Gal}(Y, Z) \) is a normal subgroup of \( \text{Gal}(Y, X) \), in which case we get,

\[
\text{Gal}(Y, X)/\text{Gal}(Y, Z) \cong \text{Gal}(Z, X).
\]

**Proof.** The \( Z \to X \) Galois condition is used in the definition of \( \theta \) and has done its job. That \( \theta \) is well defined is illustrated in Figure 4.15 suppose we choose different vertices \( u_1', u_2' \) in \( Y \) covering \( x_1', x_2' \in Z \) and with \( \alpha(u_1') = u_2' \). If \( \gamma \) is a path from \( u_1 \) to \( u_1' \), then lifting \( f(\gamma) \in X \) to \( u_2 \) gives a path to \( u_2' \) (as \( \alpha(u_1') = u_2' \)). The middle paths in \( X \) are the images of these two paths in \( Y \) under the covering \( Y \to Z \), and these must be the lifts to the \( x_i' \) of \( f(\gamma) \). Cover and lift gives the same covering automorphism in \( \text{Gal}(Z, X) \) sending \( x_1 \) to \( x_2 \) and \( x_1' \) to \( x_2' \).

![Fig. 4.15. \( \theta \) is well defined (left) and \( Y \to X \) Galois implies \( Z \to X \) is Galois (right).](image)

Splice together two copies of Figure 4.14 to see that \( \theta \) is a homomorphism and \( a \) is in the kernel if and only if \( x \) and \( a(x) \) cover the same vertex of \( Z \) for all \( x \); such \( a \) are precisely the subgroup \( \text{Gal}(Y, Z) \) as in (4.1.3)

The second part is illustrated on the right of Figure 4.15. Start at the bottom of the picture and move clockwise around it. Suppose the closed path \( \gamma \) in \( X \) lifts to a closed path at \( x_1 \) in \( Z \), and then lift this to a path \( \mu \in Y \) starting at \( u_1 \) and finishing at \( u_2 \). Let \( x_2 \) be some other vertex of \( Z \) covering \( v \) and \( u_1' \) a vertex of \( Y \) covering \( x_2 \). As \( Y \to X \) is Galois there is a covering automorphism \( b \in \text{Gal}(Y, X) \) with \( b(u_1) = u_1' \), and as \( Y \to Z \) is Galois (Lemma 4.20(ii)), there is an \( a \in \text{Gal}(Y, Z) \) with \( a(u_1) = u_2 \). If the image path \( b(\mu) \) ends at the vertex \( u_2' \), then the conjugate \( bab^{-1} \) maps \( u_1' \) to \( u_2' \). By the normality condition, this automorphism is also in \( \text{Gal}(Y, Z) \), and so \( u_2' \) must cover \( x_2 \). Finally, the image under the covering \( Y \to Z \) of \( b(\mu) \), a closed path at \( x_2 \), is the lift of \( \gamma \) to \( x_2 \). Repeating the argument with the roles of \( x_1 \) and \( x_2 \) interchanged, we get the lift of \( \gamma \) to \( x_1 \) is closed if and only if the lift to \( x_2 \) is closed. Thus \( Z \to X \) is Galois.

If \( a' \in \text{Gal}(Z, X) \) sends \( x_1 \) to \( x_2 \), and \( u_1, u_1' \) are vertices of \( Y \) covering them, then there is an \( a \in \text{Gal}(Y, X) \) mapping \( u_1 \) to \( u_1' \), as \( Y \to X \) is Galois. Thus, \( \theta \) is surjective.

**4.2.3 Universal covers and excision**

In Chapter 3 we had said quite a bit about coverings before we gave our first non-trivial example of a covering of an arbitrary complex \( X \). This example was the universal cover \( \tilde{X} \to X \) of 3.2.4 and it will now give our first non-trivial example of a Galois covering of an arbitrary complex \( X \):
Proposition 4.25. If \( X \) is a 2-complex, then the universal cover \( \tilde{X}_u \to X_v \) is a Galois covering with
\[
\text{Gal}(\tilde{X}, X) \cong \pi_1(X, v).
\]
Moreover, if \( \tilde{X}_u \to Z_x \xrightarrow{g_x} X_v \) is intermediate, then the subgroup \( g_x \pi_1(Z, x) \subset \pi_1(X, v) \) maps via this homomorphism to \( \text{Gal}(\tilde{X}, Z) \).

Proof. By Proposition 4.18 the universal cover \( \tilde{X} \) is simply connected and so \( f_* \pi_1(\tilde{X}, u) \) is the trivial subgroup, hence is normal, so by Proposition 4.22 the covering is Galois. The rest is then an immediate application of Proposition 4.22 \( \square \)

Exercise 4.26. Show that if \( G \) is a group, then there is a covering \( Y \to X \) of 2-complexes with \( \text{Gal}(Y, X) \cong G \).

Returning now to the excision of §4.1.6, we show that this process sends Galois covers to Galois covers. Recall that \( f : Y_u \to X_v \) is a covering and \( Z \subset X \) connected and simply-connected with \( v \in Z \). Also \( f^{-1}(Z) = \bigcup Z_i \subset Y \) with the \( Z_i \) connected and simply connected and \( f' : Y/Z_i \to X/Z \) the induced covering of §3.1.2.

Proposition 4.27. The induced covering \( f' : Y/Z_i \to X/Z \) is Galois if and only if \( f : X \to Z \) is Galois.

Proof. In the proof of Proposition 4.14 we “hit” the commuting diagram of 2-complexes with \( \pi_1 \), although we didn’t show the result. Here it is:
\[
\begin{array}{ccc}
\pi_1(Y, u) & \xrightarrow{q_*} & \pi_1(Y/Z, q'(u)) \\
\downarrow f_* & & \downarrow f'_*
\end{array}
\]
\[
\pi_1(X, v) \xrightarrow{g_*} \pi_1(X/Z, q(v))
\]
with the top homomorphism surjective and the bottom an isomorphism. This gives the image \( f'_* \pi_1(Y/Z, q'(u)) \subset \pi_1(X/Z, q(v)) \) equaling the image \( q_* f_* \pi_1(Y, u) \). In particular, \( \pi_1(Y, u) \subset \pi_1(X, v) \) is normal if and only if \( \pi_1(Y/Z, q'(u)) \subset \pi_1(X/Z, q(v)) \) is normal. \( \square \)

4.3 Galois correspondences

In its purest form, the Galois correspondence says that one lattice is the same as another turned upside down. The two lattices concerned are the lattice of covers intermediate to a fixed cover \( Y \to X \), and the lattice of subgroups of the Galois group of \( Y \to X \).

4.3.1 The Galois correspondence

We return to the set-up of the last section of Chapter 3 where \( f : Y_u \to X_v \) is a fixed pointed covering of 2-complexes, connected as always, with \( \mathcal{L} = \mathcal{L}(Y_u, X_v) \) the lattice of equivalence classes of pointed connected intermediate covers and \( \text{Gal}(Y_u, X_v) \) the Galois group. Let \( S = S(Y_u, X_v) \), be the lattice of subgroups of \( \text{Gal}(Y_u, X_v) \) as in Definition 3.57.

Theorem 4.28 (Galois correspondence). If \( Y_u \to X_v \) is a Galois covering of 2-complexes, then the map
\[
\Phi : \mathcal{L} \to S,
\]
that associates to the equivalence class of \( Y_u \to Z_x \to X_v \), the subgroup \( \text{Gal}(Y_u, Z_x) \) is a lattice anti-isomorphism. Its inverse \( \Psi \) is the map that associates to the subgroup \( H \subset \text{Gal}(Y_u, X_v) \) the equivalence class of \( Y_u \to Y/H_{q(u)} \to X_v \). In particular,
(i).
\[ \text{Gal}(Y_u, Y/H_{q(u)}) = H \text{ and } Y_u \to Z_x \to X_v \text{ is equivalent to } Y_u \to Y/\text{Gal}(Y,Z)_{q(u)} \to X_v; \]

(ii).
\[ \text{equivalence classes of covers } Y_u \to Z_x \to X_v \text{ with } Z_x \to X_v \text{ Galois correspond to normal subgroups of the Galois group } \text{Gal}(Y_u, X_v); \]

(iii).
\[ [\text{Gal}(Y_u, X_v) : \text{Gal}(Y_u, Z_x)] = \deg(Z_x \to X_v); \]

(iv).
\[ \text{given the subgroup } H \subset \text{Gal}(Y_u, X_v), \text{ we have} \]
\[ \deg(Y/H_{q(u)} \to X_v) = [\text{Gal}(Y_u, X_v), H]. \]

**Proof.** We show that both \( \Phi \) and \( \Psi \) are order-reversing bijections, recalling the order on \( \mathcal{L} \) from (3.4.2) and that the order on \( \mathcal{S} \) is inclusion of subgroups. Suppose we have two equivalence classes of intermediate coverings with representatives \( Z_x, Z'_x \in \mathcal{L} \), and \( Z_x \leq Z'_x \).

By Lemma 4.8 we get \( \text{Gal}(Y_u, Z'_x) \subset \text{Gal}(Y_u, Z_x) \), i.e.: \( \Phi(Z'_x) \leq \Phi(Z_x) \). On the other-hand, if \( H_1 \subset H_2 \) in \( \mathcal{S} \), then Lemma 4.10 gives \( Y_u \to Y/H_1 \to Y/H_2 \to X_v \), i.e.: \( \Psi(H_2) \leq \Psi(H_1) \). Thus \( \Phi \) and \( \Psi \) are lattice anti-morphisms.

We now show that the composition \( \Psi \Phi \) is the identity map on the lattice \( \mathcal{L} \). Let \( Y_u \to Z_x \to X_v \) be intermediate with \( Y_u \to Z_x \) Galois by Lemma 4.20. We also have the intermediate covering \( Y_u \to Y_u/\text{Gal}(Y_u, Z_x) \to Z_x \) with \( Y_u/\text{Gal}(Y_u, Z_x) \to Z_x \) degree 1 by Proposition 4.21 hence an isomorphism by Corollary 3.22.

![Fig. 4.16](image)

**Fig. 4.16.** The composition \( \Psi \Phi = \text{id}_L \).

What results is the diagram of Figure 4.16 with the whole square commuting as \( Z \) and \( Y/\text{Gal}(Y,Z) \) are intermediate to \( Y \to X \), and the left triangle commuting as \( Y/\text{Gal}(Y,Z) \) is intermediate to \( Y \to Z \). The right triangle thus commutes as well, and the map across the middle is an isomorphism. Thus, the intermediate coverings \( Y_u \to Z_x \to X_v \) and \( Y_u \to Y_u/\text{Gal}(Y_u, Z_x) \to X_v \) are equivalent, and so we have \( \Psi \Phi = \text{id}_L \).

Now for \( \Phi \Psi \): if \( H \subset \text{Gal}(Y_u, X_v) \) with \( q : Y \to Y/H \) the quotient map, then \( qa = q \) for any \( a \in H \), so \( H \subset \text{Gal}(Y_u, (Y/H)_{q(u)}) \) with the covering \( Y_u \to Y/H_{q(u)} \) intermediate, hence Galois. Proposition 4.21 gives the index of \( H \) in \( \text{Gal}(Y_u, (Y/H)_{q(u)}) \) as the degree of the covering \( Y/H_{q(u)} \to Y/H_{q(u)} \), i.e.: \( H = \text{Gal}(Y_u, (Y/H)_{q(u)}) \), and we have \( \Phi \Psi = \text{id}_S \). The anti-morphisms \( \Phi \) and \( \Psi \) are thus anti-isomorphisms.

The claims of part (i) are direct applications of \( \Phi \Psi = \text{id}_L \) and \( \Psi \Phi = \text{id}_S \). The correspondence in (ii) between intermediate Galois coverings and normal subgroups follows from Proposition 4.24. Part (iii) is just part (i) applied to Proposition 4.21 and part (iv) is just Proposition 4.21.

**Example 4.29.** Figure 4.17 shows a covering \( Y_u \to X_v \) with the Galois group \( \text{Gal}(Y_u, X_v) \) of order four, generated by the 1/4-turn rotation \( \alpha \) shown. Thus \( \text{Gal}(Y_u, X_v) \cong \mathbb{Z}_4 \). The fiber of the vertex \( v \) consists of the the four vertices at the center of \( Y \). In particular the Galois group acts transitively (hence regularly) on this fiber and the covering is Galois.

The subgroup lattice is very simple, just \( 1 \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \), and so by the Galois correspondence there is a single equivalence class of intermediate coverings \( Y_u \to Z_x \to X_v \) with the degree of the covering \( Z \to X \) the index \([\mathbb{Z}_4 : \mathbb{Z}_2] = 2\), and \( Z \) the quotient of \( Y \) by the automorphism \( \alpha^2 \) (see Figure 4.18).
4.3 Galois correspondences

Fig. 4.17. Galois covering $Y \to X$ of degree four and covering automorphism $a$.

$$Z = Y/\langle a^2 \rangle =$$

Fig. 4.18. Quotient $Y/Z_2$ corresponding to the single equivalence class of intermediate coverings.

Returning to generalities, when two lattices are anti-isomorphic, then the join of two elements in one corresponds to the meet in the other, and vice-versa. Following this through with $\Phi(-) = \text{Gal}(Y_u, -)$ we get the following,

**Corollary 4.30.** Let $Y_u \to X_v$ be Galois with $Y_u \to Z_x \to X_v$ and $Y_u \to Z'_y \to X_v$ representatives of elements in the lattice $\mathcal{L}(Y_u, X_v)$. Then,

$$\text{Gal}(Y_u, (Z \prod_X Z')_z) = \text{Gal}(Y_u, Z_x) \cap \text{Gal}(Y_u, Z'_y),$$

$$\text{Gal}(Y_u, (Z \prod_Y Z')_z) = \langle \text{Gal}(Y_u, Z_x), \text{Gal}(Y_u, Z'_y) \rangle,$$

where in the first case $z = x \times y$ in the formation of the pullback $(Z \prod_X Z')_z$, and in the second case $z = [x] = [y]$ in the formation of the pushout $(Z \prod_Y Z')_z$.

Similarly the anti-isomorphism $\Psi(-) = (Y/-)_{q(u)}$ sends joins to meets and meets to joins, giving

**Corollary 4.31.** Let $Y_u \to X_v$ be Galois and $H_1, H_2 \subset \text{Gal}(Y_u, X_v)$. Then the intermediate coverings,

$$Y_u \to Y/(H_1, H_2)_{q(u)} \to X_v \text{ and } Y_u \to (Y/H_1 \prod_Y Y/H_2)_{q(u)} \to X_v,$$

$$Y_u \to Y/(H_1 \cap H_2)_{q(u)} \to X_v \text{ and } Y_u \to (Y/H_1 \prod_X Y/H_2)_w \to X_v,$$

are equivalent, where the $q$ are the appropriate quotient maps and $w = q_1(u) \times q_2(u)$ with $q_i : Y \to Y/H_i (i = 1, 2)$ the the quotient map.
Here is another simple application of the Galois correspondence. In Exercise 4.26 we showed that any group could be realized as the Galois group of some covering $Y \to X$ of 2-complexes. If the reader will allow us the liberty of “borrowing” a theorem from a later chapter, then we can realize the group as the Galois group of a covering of graphs:

**Corollary 4.32 (Inverse Galois theorem).** Let $G$ be a group. Then there is a Galois covering of graphs $Y \to X$ with $\text{Gal}(Y, X) \cong G$.

**Proof.** The theorem to be borrowed is that there is a free group $F$ and a normal subgroup $H \triangleleft F$ with $G \cong F/H$. As the fundamental groups of graphs are free, choose a graph $X$ with $\pi_1(X, v) \cong F$. Consider the universal cover $\tilde{X} \to X$ with $\text{Gal}(\tilde{X}, X) \cong \pi_1(X, v)$, by Proposition 4.25. Identifying $H$ with the corresponding subgroup of $\text{Gal}(\tilde{X}, X)$ under the isomorphism above, we get an intermediate covering $\tilde{X} \twoheadrightarrow \tilde{X}/H \twoheadrightarrow X$, with $\text{Gal}(\tilde{X}, \tilde{X}/H) = H$. In particular, Proposition 4.24 gives $\tilde{X}/H \to X$ is a Galois covering with Galois group isomorphic to $G$. □

**Exercise 4.33 (completely irregular covers = malnormal subgroups).** A subgroup $H \subset G$ is *malnormal* whenever $g \in G \setminus H$ implies $gHg^{-1} \cap H = \{1\}$. Let $f : Y \to X$ be a covering with $f(u) = v$, and let $\gamma$ be a homotopically non-trivial closed path at $v$ whose lift to $u$ is also closed. Call $f$ *completely irregular* if the lift of $\gamma$ to every other vertex of $f^{-1}(v)$ is non-closed.

Show that equivalence classes of covers $Y_u \to Z_x \to X_v$ with $Z_x \to X_v$ completely irregular correspond to malnormal subgroups of the Galois group $\text{Gal}(Y_u, X_v)$.

### 4.3.2 Galois correspondence for the universal cover

There is a more familiar version of the Galois correspondence using the fact that the universal cover $\tilde{X} \to X$ is Galois, and translating the various ingredients of Theorem 4.28 into this special setting. The aim is to remove all mention of the top cover, and just focus on coverings of $X$.

Let $X$ be a 2-complex and $v \in X$ a vertex. Two coverings $f_i : Y_i \to X$ ($i = 1, 2$) with $f_i(y_i) = v_i$ are equivalent when there is an isomorphism $(Y_1)_{y_1} \to (Y_2)_{y_2}$ making $Y_1 \xrightarrow{\cong} Y_2$ commute. Let $\mathcal{L} = \mathcal{L}(X_v)$ be the set of equivalence classes of coverings.

**Exercise 4.34.** Show that two coverings are equivalent in the sense just defined if and only if the intermediate coverings $\tilde{X}_u \to (Y_i)_{y_i} \to X_v$ are equivalent, and so there is an isomorphism of lattices $\mathcal{L}(X_v) \to \mathcal{L}(\tilde{X}_u, X_v)$.

Identifying $\mathcal{L}(\tilde{X}_u, X_v)$ with $\mathcal{L}(X_v)$, the Galois correspondence sends (the equivalence class of) the cover $f : Y_y \to X_v$ to the subgroup $\text{Gal}(\tilde{X}_u, Y_y) \subset \text{Gal}(\tilde{X}_u, X_v)$. Proposition 4.23 in turn gives an isomorphism
ϕ : Gal(\(\overline{X}_u, X_v\)) → \(\pi_1(X, v)\),

which sends the subgroup \(Gal(\overline{X}_u, Y_u)\) to \(f_*\pi_1(Y, y) \subset \pi_1(X, v)\).

Let \(S = S(X_v)\) be the lattice of subgroups of the fundamental group \(\pi_1(X, v)\) ordered, as usual, by inclusion. A translation of Theorem 4.28 then gives:

**Corollary 4.35 (Galois correspondence for \(\overline{X}\): first go).** If \(X\) is a 2-complex, then the map

\[ \Phi : \mathcal{L} \rightarrow S, \]

that associates to the equivalence class of the covering \(f : Y \rightarrow X\), the subgroup \(f_*\pi_1(Y, y) \subset \pi_1(X, v)\), is a lattice anti-isomorphism. Its inverse \(Ψ\) is the map that associates to the subgroup \(H \subset \pi_1(X, v)\) the equivalence class of \(\overline{X}/\varphi^{-1}(H)q(u)\) \(\rightarrow X_v\) where \(q : \overline{X} \rightarrow \overline{X}/\varphi^{-1}(H)\) is the quotient map.

Moreover, equivalence classes of Galois covers \(Y \rightarrow X\) correspond to normal subgroups of \(\pi_1(X, v)\).

The mantra is thus, “covers of \(X\) correspond to subgroups of the fundamental group of \(X\)”.

Corollary 4.38 is not quite perfect: our aim was to remove all reference to the top covering (\(\overline{X}\) in this case) and focus entirely on \(X\), but our top-down approach to associating a cover to a subgroup has hard wired \(\overline{X}\) into the picture. The solution is to return to the bottom-up construction of 3.2.3.

**Proposition 4.36 (bottom-up=top-down).** Let \(H \subset \pi_1(X, v)\) be a subgroup and \((X \uparrow H)_x \rightarrow X_v\) the bottom-up cover of \(X\) constructed in 3.2.3 Then the coverings

\[(X \uparrow H)_x \rightarrow X_v\text{ and }\overline{X}/\varphi^{-1}(H)q(u)\rightarrow X_v\]

are equivalent.

**Proof.** By Corollary 3.44 the cover \((X \uparrow H)_x \rightarrow X_v\) corresponds to \(H \subset \pi_1(X, v)\), hence is equivalent to \(\overline{X}/\varphi^{-1}(H)q(u)\rightarrow X_v\) by Corollary 4.38. \(\Box\)

**Corollary 4.37 (Galois correspondence for \(\overline{X}\): second go).** If \(X\) is a 2-complex, then the map

\[ \Phi : \mathcal{L} \rightarrow S, \]

that associates to the equivalence class of the covering \(f : Y \rightarrow X\), the subgroup \(f_*\pi_1(Y, y) \subset \pi_1(X, v)\), is a lattice anti-isomorphism. Its inverse \(Ψ\) is the map that associates to the subgroup \(H \subset \pi_1(X, v)\) the equivalence class of \((X \uparrow H)_x \rightarrow X_v\).

Moreover, equivalence classes of Galois covers \(Y \rightarrow X\) correspond to normal subgroups of \(\pi_1(X, v)\).

### 4.3.3 Lattice excision

We have seen (3.4.1) that the excision of simply-connected subcomplexes has no effect on Galois groups. Another consequence of the Galois correspondence is that these excisions have no effect on lattices of intermediate coverings.

Recalling the set-up, let \(f : Y \rightarrow X\) be a covering and \(Z \subset X\) connected and simply-connected with \(v \in Z\). Let \(f^{-1}(Z) = \bigcup Z_i \subset Y\) a disjoint union with the \(Z_i\) connected and simply connected. Finally, \(q : X \rightarrow X/Z, q' : Y \rightarrow Y/Z_i\) are the quotient maps and \(f' : Y/Z_i \rightarrow X/Z\) the induced covering.
Corollary 4.38 (lattice excision). There is an isomorphism of lattices

$$\mathcal{L}(Y_u, X_v) \cong \mathcal{L}((Y/Z_i)_q(u), X/Z_q(v)),$$

that sends the equivalence class of $$Y_u \to W_x \xrightarrow{g} X_v$$ to the equivalence class of $$(Y/Z_i)_q(u) \to (W/Z'_i)_{q'(x)} \to (X/Z)_q(v)$$, with $$g^{-1}(Z) = \bigcup Z'$$, and Galois coverings $$W_x \xrightarrow{g} X_v$$ to Galois coverings $$(W/Z'_i)_{q'(x)} \to (X/Z)_q(v)$$.

Fig. 4.19. lattice excision applied to Example 4.29

Proof. We show the result first in the case that $$Y \to X$$ is Galois, leaving off the pointings for clarity. By Proposition 4.14 the Galois groups $$\text{Gal}(Y, X)$$ and $$\text{Gal}(Y/Z_i, X/Z)$$ are isomorphic and by Exercise 3.59 this induces a lattice isomorphism $$\mathcal{S}(Y, X) \to \mathcal{S}(Y/Z_i, X/Z)$$ between the subgroup lattices of the two Galois groups. Proposition 4.27 gives the induced covering $$Y/Z_i \to X/Z$$ is Galois. Now apply the Galois correspondence twice,

$$\mathcal{L}(Y, X) \to \mathcal{S}(Y, X) \to \mathcal{S}(Y/Z_i, X/Z) \to \mathcal{L}(Y/Z_i, X/Z),$$

(‡) to give a composition of an isomorphism and two anti-isomorphisms, with net effect the isomorphism we seek.

It is possible to find the image of $$Y \to W \to X$$ by brute force. Alternatively, commuting diagrams like that in the proof of Proposition 4.27 give the images under the isomorphism $$q_* : \pi_1(X, v) \to \pi_1(X/Z, q(v))$$ of the subgroups $$f_* \pi_1(Y, u)$$ and $$g_* \pi_1(W, x)$$ to be $$f'_* \pi_1(Y/Z_i, q'(u))$$ and $$g'_* \pi_1(W/Z'_i, q'(x))$$. In particular, and by Proposition 4.22 the isomorphism $$\text{Gal}(Y, X) \to \text{Gal}(Y/Z_i, X/Z)$$, and hence the isomorphism $$\mathcal{S}(Y, X) \to \mathcal{S}(Y/Z_i, X/Z)$$, sends the subgroup $$\text{Gal}(Y, W)$$ to $$\text{Gal}(Y/Z_i, W/Z'_j)$$. Following this through in (‡) gives

$$(Y \to W \to X) \mapsto \text{Gal}(Y, W) \mapsto \text{Gal}(Y/Z_i, W/Z'_j) \mapsto (Y/Z_i \to W/Z'_j \to X/Z).$$

We’ve already observed that $$q_*g_* \pi_1(W, x) = g'_* \pi_1(W/Z'_j, q'(x))$$, hence Galois covers correspond to Galois covers.
Suppose now that $Y \to X$ is an arbitrary cover and let $h : \tilde{X} \to X$ be the universal cover of $\{3,2,4\}$ Thus, we have intermediate coverings

$$\tilde{X} \to Y \to X,$$

hence intermediate coverings $\tilde{X}/Z'_j \to Y/Z_i \to X/Z$, with $\bigcup Z'_i = h^{-1}(Z) \subset \tilde{X}$. It is clear that $\mathcal{L}(Y, X)$ is a sublattice of $\mathcal{L}(\tilde{X}, X)$ and $\mathcal{L}(Y/Z_i, X/Z)$ is a sublattice of $\mathcal{L}(\tilde{X}/Z'_j, X/Z)$. Applying lattice excision to the Galois cover $\tilde{X} \to X$ gives an isomorphism $\mathcal{L}(\tilde{X}, X) \to \mathcal{L}(\tilde{X}/Z'_j, X/Z)$ and it is easy to check $\mathcal{L}(Y, X)$ is sent to $\mathcal{L}(Y/Z_i, X/Z)$. 

In particular there are homeomorphisms

$$\left(W_1 \prod_{X} W_2\right)/Z_k \to \left(W_1/Z_{1j}\right) \prod_{X/Z} (W_2/Z_{2j}),$$

$$\left(W_1 \prod_{Y} W_2\right)/Z_k \to \left(W_1/Z_{1j}\right) \prod_{Y/Z_i} (W_2/Z_{2j}).$$

where $W_1, W_2$ are intermediate to $Y \to X$, and the $Z_{1j}, Z_{2j}, Z_k$ are the components of the preimages of $Z$ via the various coverings.

**Example 4.39.** We revisit Example 4.29 with the covering $Y \to X$ shown in Figure 4.19. A spanning tree $Z$ for $X$ is given in red and the lifts $\bigcup Z_i$ in $Y$. We saw in Example 4.29 that the trees were just a distraction when it came to intermediate covers: with the Galois group $\cong \mathbb{Z}_4$ there was just a single equivalence class of intermediate cover. Indeed the same effect could have been achieved by ignoring all the red trees and focusing on the covering of a single loop by the central square.

### 4.4 Notes on Chapter 4
Generators and Relations
The Topological Dictionary
Amalgams
The Arboreal Dictionary
Ends
Appendix

10.1 Comparison with “proper” topology

10.1.1 The category of CW complexes

10.1.2 A functor
Hints for the Exercises
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