LOCAL CUT POINTS AND METRIC MEASURE SPACES WITH RICCI CURVATURE BOUNDED BELOW

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ABSTRACT. A local cut point is by definition a point that disconnects its sufficiently small neighborhood. We show that there exists an upper bound for the degree of a local cut point in a metric measure space satisfying the generalized Bishop–Gromov inequality. As a corollary, we obtain an upper bound for the number of ends of such a space. We also obtain some obstruction conditions for the existence of a local cut point in a metric measure space satisfying the Bishop–Gromov inequality or the Poincaré inequality. For example, the measured Gromov–Hausdorff limits of Riemannian manifolds with a lower Ricci curvature bound satisfy these two inequalities.

1. Introduction

A point $x$ in a metric space is called a local cut point if $U \setminus \{x\}$ is disconnected for some connected neighborhood $U$ of $x$. There exists no local cut point in any $n$-dimensional Alexandrov space with curvature bounded from below for $n \geq 2$. Every Gromov–Hausdorff limit of a sequence of Riemannian manifolds with a uniform lower bound on sectional curvature is such an Alexandrov space. It is therefore natural to ask whether there exists a local cut point in the limit of manifolds with a uniform lower bound on Ricci curvature unless the limit is one-dimensional. We conjecture that the limit has no local cut point.

In this paper, we consider metric measure spaces with “Ricci curvature bounded below”. Let $\text{deg}(x)$ denote the degree of a point $x$, or the supremum of the number of connected components of $U \setminus \{x\}$ for all connected neighborhoods $U$ of $x$. We give an upper bound for the degree of a local cut point. As a consequence, we obtain an upper bound for the number of ends (see Subsection 4.2 for the definition of an end). We also obtain some obstruction conditions for the existence of a local cut point.

Cheeger and Colding [4], [5], [6], [7], and Menguy [18], [19], [20] studied the limits of manifolds with Ricci curvature bounded from below. They constructed examples of limit spaces showing that the local structure is more complicated than that of Alexandrov...
spaces. Recently, Lott and Villani [16], [17], Sturm [25], [26], and Ohta [21] independently introduced the concept of lower bounds on Ricci curvature for metric measure spaces: \( N \)-Ricci curvature \( \geq K \), the curvature-dimension condition \( \text{CD}(K, N) \), and the measure contraction property \( \text{MCP}(K, N) \), respectively. For each \( N \)-dimensional Riemannian manifold \( M \), these three concepts are equivalent to that the Ricci curvature of \( M \) is bounded below by \( K \). Moreover, these are preserved under the measured Gromov–Hausdorff limits.

We note that the measured Gromov–Hausdorff limits of manifolds with Ricci curvature bounded from below satisfy the generalized Bishop–Gromov inequality and a Poincaré inequality of type \((1, 1)\), as explained below.

Let \((X, d, \mu)\) be a complete, locally compact length space equipped with a Borel measure. We denote by \( B_r(x) \) the open ball of radius \( r \) and centered at \( x \in X \). Let \( k \in \mathbb{R}, n \in \mathbb{N}, C \geq 1, 0 < r \leq R \), and \( x \in X \). If \( C = 1 \), then the following \((1.1)\) is the usual Bishop–Gromov inequality with respect to lower bound \((n - 1)k\) of Ricci curvature and upper bound \( n \) of dimension:

\[
\frac{\mu(B_R(x))}{\mu(B_r(x))} \leq C \frac{V_{k, n}(R)}{V_{k, n}(r)},
\]

where \( V_{k, n}(r) \) is the volume of a ball of radius \( r \) in the \( n \)-dimensional, complete, simply connected space of constant curvature \( k \).

We consider a stronger inequality, which is the directionally restricted version of \((1.1)\) ([10, Section 5.1], [5, Appendix 2, (A.2.2.)]); we call the inequality the generalized Bishop–Gromov inequality with constant \( C \) (\( \text{BG}(k, n) \) with \( C \) for short). This inequality is naturally extended to the case that \( n \in \mathbb{R} \) with \( n \geq 1 \). See Definition 3.1 for the precise definition and [14] for slight different definitions. This inequality is preserved under the measured Gromov–Hausdorff limits ([10, Section 5.1], [5, Theorem 1.6, Theorem 1.10]). The measure contraction property \( \text{MCP}((n - 1)k, n) \) implies \( \text{BG}(k, n) \) with \( C = 1 \). For example, the following metric measure spaces satisfy \( \text{MCP}((n - 1)k, n) \).

- \( n \)-dimensional Riemannian manifolds with Ricci curvature bounded below by \((n - 1)k\) equipped with Riemannian measure
- \( n \)-dimensional Alexandrov spaces with curvature bounded below by \( k \) equipped with the \( n \)-dimensional Hausdorff measure
- “Nonbranching” metric measure spaces satisfying the curvature-dimension condition \( \text{CD}((n - 1)k, n) \)
  See Definition 3.3 for the definition of nonbranching.

The non-Euclidean, finite-dimensional, normed linear spaces equipped with the Lebesgue measure satisfy \( \text{MCP}(0, n) \). These spaces, however, can not arise as a Gromov–Hausdorff limit of any Riemannian manifolds with a uniform lower bound on Ricci curvature (see Proposition 3.2).

Applying the method of the proof of Theorem 5.1 in [6], we have the following:
Theorem 1.1. Let $(X, d, \mu)$ be a metric measure space satisfying the generalized Bishop–Gromov inequality $BG(k, n)$ with constant $C$ for some $k \in \mathbb{R}$, $n \geq 1$, and $C \geq 1$. Assume that there exists a local cut point $x$ in $X$. Then we have $\deg(x) \leq C^2 + 1$.

The Cheeger–Gromoll splitting theorem [8] states that, if a Riemannian manifold $M$ of nonnegative Ricci curvature contains a line, then $M$ is isometric to $\mathbb{R} \times N$ for some manifold $N$. Cheeger and Colding [4] extended this to limit spaces of nonnegative Ricci curvature in a generalized sense; see Theorem 2.1 of this paper. For the limits of manifolds with Ricci curvature bounded from below ($C = 1$), Theorem 1.1 ($\deg(x) = 2$) is also proved by using the Cheeger–Colding splitting theorem; see Proposition 4.3 of this paper.

By the splitting theorem, a space of nonnegative Ricci curvature has at most two ends. We note that the splitting theorem for metric measure spaces satisfying $BG(0, n)$ with $C$ does not necessarily hold. As a corollary of Theorem 1.1, we have the following:

Corollary 1.2. Let $(X, d, \mu)$ be a metric measure space satisfying $BG(0, n)$ with $C$ for some $n \geq 1$ and $C \geq 1$. Then the number of ends of $X$ is at most $C^2 + 1$.

We investigate the geometric structure of the neighborhood of a local cut point (Theorem 4.14). As a result, we see that the “three-pronged” space (see Figure 4) does not satisfy $BG(k, n)$ with $C$. We also study the structure of the accumulation of local cut points; we obtain that the convergent sequence of certain local cut points “stands in a line” (Corollary 4.20).

We now assume that a measure $\mu$ satisfies $0 < \mu(B_r(x)) < +\infty$ for all $x \in X$ and all $0 < r < +\infty$. Let $1 \leq p < \infty$. We say that a metric measure space $(X, d, \mu)$ satisfies a Poincaré inequality of type $(1, p)$, if for all $R > 0$ there exists a constant $C_P = C_P(p, R) > 0$ depending only on $p$ and $R$ such that

\begin{equation}
\int_{B_r(x)} |u - u_{B_r(x)}| \, d\mu \leq C_P r \left( \int_{B_r(x)} g^p \, d\mu \right)^{1/p}
\end{equation}

holds for all $x \in X$, $0 < r \leq R$, all measurable functions $u$, and all “upper gradients” $g$ of $u$ (see Section 4), where $u_B := \int_B u \, d\mu := \mu(B)^{-1} \int_B u \, d\mu$.

It follows from Hölder’s inequality that each metric measure space satisfying a Poincaré inequality of type $(1, p)$ also satisfies that of type $(1, q)$ for all $q \geq p$. For instance, the following metric measure spaces satisfy a Poincaré inequality of type $(1, 1)$.

- $n$-dimensional Riemannian manifolds with Ricci curvature bounded below by $(n - 1)k$ equipped with Riemannian measure (Buser [3])
  The constant $C_P$ in (1.2) depends on $k$, $n$, and $R$. If $k = 0$, then $C_P$ depends only on $n$. See [23, Theorem 5.6.5].
- The measured Gromov–Hausdorff limits of manifolds with Ricci curvature bounded from below (Cheeger and Colding [7, Theorem 2.15])
- $n$-dimensional Alexandrov spaces equipped with the $n$-dimensional Hausdorff measure (Kuwae, Machigashira, and Shioya [13, Theorem 7.2])
• Nonbranching metric measure spaces satisfying $\text{CD}(K, N)$ (von Renesse [22, Corollary in page 8], combined with Sturm [26, Lemma 4.1])

• Metric measure spaces with a doubling measure and the “democratic condition” DM (Lott and Villani [17, Theorem 2.5])

The space $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq x_n^2\}$ equipped with the Euclidean distance and the $n$-dimensional Lebesgue measure satisfies a Poincaré inequality of type $(1, p)$ for all $p > n$ ([11, Example 4.2]).

We obtain an obstruction condition for the existence of a local cut point as follows:

**Theorem 1.3.** Let $(X, d, \mu)$ be a metric measure space satisfying a Poincaré inequality of type $(1, p)$ for some $1 \leq p < \infty$. Assume that

$$\liminf_{r \to 0} \frac{\mu(B_r(x))}{r^p} = 0$$

for a point $x \in X$. Then $x$ is not a local cut point.

Although the measured Gromov–Hausdorff limits of manifolds with Ricci curvature bounded from below satisfy a Poincaré inequality of type $(1, 1)$, they do not necessarily satisfy the assumption (1.3) for $p = 1$ (see [5, Proposition 1.22]). Theorem 1.3 is one of the geometric consequences of the Poincaré inequality. See [11, Proposition 4.5] and [12, Theorem 3.13, Theorem 6.15] for related results.

The organization of this paper is as follows: In Section 2, we recall the definitions of the Hausdorff dimension, length spaces, and the (pointed, measured) Gromov–Hausdorff convergence. In Section 3 we precisely define $\text{BG}(k, n)$ with $C$ and give its basic properties. We prove Theorem 1.1 and Corollary 1.2 in Section 4. Furthermore, we investigate the geometric structure of the neighborhood of local cut points, and also study the structure of the accumulation of local cut points. We prove Theorem 1.3 in Section 5.

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2. Preliminaries

In this section we first recall the definition of the Hausdorff dimension. We then define length spaces and the (pointed, measured) Gromov–Hausdorff convergence. Let $(X, d)$ be a metric space.

2.1. Hausdorff dimension. We refer to [11, Chapter 2] for details. Let $A \subset X$. For $0 \leq s < \infty$ and $0 < \delta \leq \infty$, we define

$$\mathcal{H}_s^\delta(A) = \omega_s \inf \left\{ \sum_i \left( \frac{\text{diam}(U_i)}{2} \right)^s \mid A \subset \bigcup_i U_i, \text{diam}(U_i) \leq \delta \right\},$$
where $\omega_s := \pi^{s/2}/\Gamma(s/2 + 1)$ and $\Gamma$ is the gamma function $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$. If $s$ is a positive integer, then $\omega_s$ equals the volume of a unit ball in $\mathbb{R}^s$. Note that $\mathcal{H}_\delta^s(A)$ is nonincreasing in $\delta$. We define the $s$-dimensional Hausdorff measure of $A$ by
\[
\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(A) = \sup_{\delta > 0} \mathcal{H}_\delta^s(A).
\]
Then $\mathcal{H}^s$ is a Borel regular measure on $X$. The Hausdorff dimension of $A$ is defined as
\[
\dim_H(A) = \inf\{0 \leq s < \infty \mid \mathcal{H}^s(A) = 0\} = \sup\{0 \leq s < \infty \mid \mathcal{H}^s(A) = +\infty\}.
\]

2.2. Length spaces. For a continuous path $\gamma : [0, l] \to X$, we define the length of $\gamma$ by
\[
L(\gamma) = \sup_{0 = t_0 < t_1 < \cdots < t_k = l} \sum_{i=1}^k d(\gamma(t_{i-1}), \gamma(t_i)),
\]
where the supremum is taken over all partitions of $[0, l]$. By the triangle inequality, we have $L(\gamma) \geq d(\gamma(0), \gamma(l))$. We say that a path $\gamma : [0, l] \to X$ is a geodesic if it is locally minimizing and is proportional to arclength, that is, for each $s \in [0, l]$ there exists $\epsilon = \epsilon(s) > 0$ such that
\[
d(\gamma(t), \gamma(t')) = |t - t'|
\]
holds for all $t, t' \in (s - \epsilon, s + \epsilon) \cap [0, l]$. Furthermore, we say that a path $\gamma : (-\infty, \infty) \to X$ is a line if (2.1) holds for all $t, t' \in (-\infty, \infty)$. In this paper, we assume that every path is proportional to arclength.

We say that $X$ is a length space if $d(x, y) = \inf_{\gamma} L(\gamma)$ for all $x, y \in X$, where the infimum is taken over all paths joining $x$ and $y$. A metric space $X$ is a length space if and only if for all $x, y \in X$ and all $\epsilon > 0$ there exists a point $z \in X$ such that $\max\{d(x, z), d(z, y)\} \leq d(x, y)/2 + \epsilon$. If $X$ is a complete, locally compact length space, then all two points in $X$ are joined by a minimal geodesic.

See [2], [10] and references therein for further information.

2.3. (Pointed) Gromov–Hausdorff convergence. Let us recall that the Hausdorff distance between two closed bounded subsets $A$ and $B$ in a metric space $X$ is defined by
\[
d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset U_\epsilon(B), B \subset U_\epsilon(A)\},
\]
where $U_\epsilon(A)$ is the $\epsilon$-neighborhood of $A$. Let $\mathcal{C}$ denote the set of isometry classes of all compact metric spaces. For $X, Y \in \mathcal{C}$, the Gromov–Hausdorff distance between $X$ and $Y$ is defined by
\[
d_{GH}(X, Y) = \inf \{d_H(f(X), g(Y)) \mid \text{all metric spaces } Z \text{ and all isometric embeddings } f : X \to Z, \ g : Y \to Z\}.
\]
Then $d_{GH}$ defines a metric on $\mathcal{C}$. Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that for $\epsilon > 0$ a map $\varphi : X \to Y$ is an $\epsilon$-approximation if the following two conditions hold:
\begin{enumerate}[label=(\roman*)]
\item $|d_X(x, y) - d_Y(\varphi(x), \varphi(y))| < \epsilon$ holds for all $x, y \in X$;
\item the $\epsilon$-neighborhood of $\varphi(X)$ coincides with $Y$.
\end{enumerate}
Let $X_i, X \in \mathcal{C}$ ($i = 1, 2, \ldots$). The sequence $\{X_i\}$ Gromov–Hausdorff converges to $X$ ($d_{GH}(X_i, X) \rightarrow 0$) as $i \rightarrow \infty$ if and only if there exist $\epsilon_i$-approximations from $X_i$ to $X$ (or from $X$ to $X_i$) for some $\epsilon_i \rightarrow 0$. Let $(X_i, x_i), (X, x), i = 1, 2, \ldots$, be pointed metric spaces. We say that $\{X_i\}$ pointed Gromov–Hausdorff converges to $(X, x)$, if for each $R > 0$ there exist $R_i \searrow R, \epsilon_i \searrow 0$, and $\epsilon_i$-approximations $\varphi_i : B_{R_i}(x_i) \rightarrow B_R(x)$ with $\varphi_i(x_i) = x$. A pointed Gromov–Hausdorff limit of $\{(X, r_i^{-1}d, x)\}$ as $r_i \rightarrow 0$ is called a tangent cone at $x$. The Gromov–Hausdorff limit of a sequence of length spaces is also a length space.

We refer to [2], [10] for details.

Let $(M_i, p_i), i = 1, 2, \ldots$, be $n$-dimensional, complete pointed Riemannian manifolds with Ricci curvature $\text{Ric}_{M_i} \geq -(n - 1)$. Gromov’s compactness theorem states that $\{(M_i, p_i)\}$ pointed Gromov–Hausdorff converges to some pointed metric space $(X, x)$. We then have $\dim_H(X) \leq n$ (see Proposition 3.3). The sequence $\{M_i\}$ is said to collapse to $X$ if $\dim_H(X) < n$. In fact, if $\{M_i\}$ collapses, then $\dim_H(X) \leq n - 1$ (Theorem 3.1). It is conjectured that the Hausdorff dimension of the limit $X$ is an integer (Fukaya’s conjecture [9] Conjecture 3.13]). By Gromov’s compactness theorem, for every point $x$ in the limit $X$ there exists a tangent cone at $x$. A tangent cone at each point is not necessarily unique even in the noncollapsed case ([5] Example 8.41)). It does not always have a metric cone structure in the collapsed case ([5] Example 8.95)). We state the splitting theorem for the limit, which holds even in the collapsed case.

**Theorem 2.1** (Cheeger–Colding [4]). Let $(X, x)$ be a pointed Gromov–Hausdorff limit of complete Riemannian manifolds $M_i$ with $\text{Ric}_{M_i} \geq -\delta_i$, where $\delta_i \rightarrow 0$. If $X$ contains a line, then $X$ is isometric to $\mathbb{R} \times Y$ for some length space $Y$.

### 2.4. Measured Gromov–Hausdorff convergence

Fukaya [9] introduced the concept of the measured Gromov–Hausdorff convergence. Let $\mathcal{CM}$ denote the set of all compact metric spaces $X$ equipped with a Borel measure $\mu$ such that $\mu(X) \leq 1$. A directed system $\{(X_\alpha, \mu_\alpha)\}_{\alpha \in \mathcal{A}} \subset \mathcal{CM}$ is said to measured Gromov–Hausdorff converge to $(X, \mu) \in \mathcal{CM}$ if for each $\alpha \in \mathcal{A}$ there exist $\epsilon_\alpha > 0$ and a Borel measurable $\epsilon_\alpha$-approximation $\varphi_\alpha : X_\alpha \rightarrow X$ such that

1. $\lim_{\alpha \in \mathcal{A}} \epsilon_\alpha = 0$;
2. a directed system of push-forward measures $\{(\varphi_\alpha)_*\mu_\alpha\}_\alpha$ converges to $\mu$ vaguely:

$$
\lim_{\alpha \in \mathcal{A}} \int_{X_\alpha} f \circ \varphi_\alpha d\mu_\alpha = \int_X f \, d\mu
$$

holds for all continuous functions $f : X \rightarrow \mathbb{R}$.

We induce a topology on $\mathcal{CM}$ by the measured Gromov–Hausdorff convergence. The topology is Hausdorff ([9] Proposition 2.7]), and the projection from $\mathcal{CM}$ to $\mathcal{C}$ is proper ([9] Proposition 2.10]). We refer to [10] for other topologies on $\mathcal{CM}$.
Let $M_i$, $i = 1, 2, \ldots$, be compact $n$-dimensional Riemannian manifolds with $\text{Ric}_{M_i} \geq -(n - 1)$. Assume that $\{M_i\}$ Gromov–Hausdorff converges to a compact metric space $(X, d)$. Since $(M_i, \text{dvol}_{M_i}/\text{Vol}(M_i)) \in \mathcal{CM}$, where $\text{dvol}_{M_i}$ is the volume element of $M_i$, there exists a measure $\mu$ on $X$ such that $\{(M_i, \text{dvol}_{M_i}/\text{Vol}(M_i))\}$ measured Gromov–Hausdorff subconverges to $(X, \mu)$. In the noncollapsed case, $\mu$ is unique and it coincides with the $n$-dimensional Hausdorff measure, up to constant multiple ([5, Theorem 5.9]). In the collapsed case, $\mu$ is not necessarily unique ([5, Example 1.24, Section 8]).

3. The generalized Bishop–Gromov inequality

In this section, we define the generalized Bishop–Gromov inequality $\text{BG}(k, n)$ with constant $C$, and give basic properties of a metric measure space satisfying $\text{BG}(k, n)$ with $C$.

We denote by $c(c_1, \ldots, c_l)$ a positive constant depending only on $c_1, \ldots, c_l$. Let $(X, d, \mu)$ be a complete, locally compact length space equipped with a Borel measure.

3.1. The definition of the generalized Bishop–Gromov inequality. Let $k, n \in \mathbb{R}$ with $n \geq 1$. For $0 \leq r_1 < r_2$, we define

$$V_{k, n}(r_1, r_2) = \alpha_{n-1} \int_{r_1}^{r_2} s_k(t)^{n-1} \, dt,$$

where $\alpha_{n-1} := 2\pi^{n/2}/\Gamma(n/2) = n \omega_n$ and $s_k(t)$, $t \geq 0$, is defined by

$$s_k(t) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k} t) & \text{if } k > 0, \\ t & \text{if } k = 0, \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k} t) & \text{if } k < 0. \end{cases}$$

If $n$ is a positive integer, then $V_{k, n}(r_1, r_2)$ is equal to the volume of an annulus of radius between $r_1$ and $r_2$ in the $n$-dimensional, complete, simply connected space of constant curvature $k$. We denote by $B_r(x)$ and $\overline{B}_r(x)$ the open and closed ball of radius $r$ and centered at $x$, respectively. For a point $x \in X$, let $U$ be a measurable set in $A_{r_1, r_2}(x)$, where $A_{r_1, r_2}(x) = B_{r_2}(x) \setminus \overline{B}_{r_1}(x)$ and $A_{0, r}(x) = B_r(x)$. For $0 \leq s_1 < s_2$ with $s_1 \leq r_1$ and $s_2 \leq r_2$, set

$$S_{s_1, s_2}(x, U) = \{y \in A_{s_1, s_2}(x) \mid d(x, y) + d(y, z) = d(x, z) \text{ for some } z \in U\}$$

(that is, $S_{s_1, s_2}(x, U)$ is the intersection of $A_{s_1, s_2}(x)$ and all minimal geodesics between $x$ and each point in $U$).

**Definition 3.1.** Let $C \geq 1$. We say that a metric measure space $(X, d, \mu)$ satisfies the generalized Bishop–Gromov inequality with constant $C$ ($\text{BG}(k, n)$ with $C$ for short), if for all points $x \in X$ and all $0 \leq r_1 < r_2$, $0 \leq s_1 < s_2$ with $s_1 \leq r_1, s_2 \leq r_2$,

$$\frac{\mu(U)}{\mu(S_{s_1, s_2}(x, U))} \leq C \frac{V_{k, n}(r_1, r_2)}{V_{k, n}(s_1, s_2)}$$

holds for all measurable sets $U$ in $A_{r_1, r_2}(x)$. 

...
BG\(k, n\) with \(C = 1\) induces the usual one. Indeed, if we choose \(r_1 = s_1 = 0, s_2 = r \leq R = r_2,\) and \(U = B_R(x)\), then (3.1) implies
\[
\frac{\mu(B_R(x))}{\mu(B_r(x))} \leq \frac{V_{k, n}(0, R)}{V_{k, n}(0, r)}.
\]

3.2. Examples. We recall metric measure spaces satisfying BG\(k, n\) with \(C = 1\), as shown in the introduction. Every \(n\)-dimensional, normed linear space equipped with the \(n\)-dimensional Lebesgue measure satisfies BG\(0, n\) with \(C = 1\).

**Proposition 3.2.** Let \(X\) be an \(n\)-dimensional, normed linear space. Assume that \(X\) is a Gromov–Hausdorff limit of Riemannian manifolds \(M_i\) with Ricci curvature bounded from below. Then \(X\) is isometric to \(\mathbb{R}^n\).

*Proof.* Expanding the metrics of converging manifolds, we may assume that \(\text{Ric}_{M_i} \geq -\delta_i\), where \(\delta_i \to 0\). Since \(X\) contains \(n\) orthogonal lines, the splitting theorem (Theorem 2.1) completes the proof. \(\square\)

**Definition 3.3** ([25, Definition 2.8]). A metric space \((X, d)\) is said to be nonbranching if for all four points \(y, x_0, x_1, x_2 \in X\) such that \(y\) is a midpoint between \(x_0\) and \(x_1\) and between \(x_0\) and \(x_2\), we have \(x_1 = x_2\). A point \(y\) is called a midpoint between \(x_0\) and \(x_1\) if \(2^{-1}d(x_0, x_1) = d(x_0, y)\) holds.

A metric space \(X\) is nonbranching if and only if for all two minimal geodesics \(\gamma, \gamma' : [0, l] \to X\) with \(\gamma(0) = \gamma'(0)\), we have \(\inf\{t > 0 \mid \gamma(t) \neq \gamma'(t)\} = 0\) or \(\infty\), where \(\inf \emptyset := \infty\). We see that each Alexandrov space with curvature bounded from below is nonbranching.

**Remark 3.4.** For nonbranching metric measure spaces, the curvature-dimension condition CD\((K, N)\) implies the measure contraction property MCP\((K, N)\) (see [26, Lemma 4.1, Theorem 5.4]). Moreover, CD\((K, N)\) is a little weaker condition than having \(N\)-Ricci curvature \(\geq K\).

All \(n\)-dimensional Alexandrov spaces with curvature bounded below by \(k\) equipped with the \(n\)-dimensional Hausdorff measure \(\mathcal{H}^n\) satisfy MCP\(((n - 1)k, n)\) ([21, Proposition 2.8], [26, Theorem 5.7], [14, Lemma 6.1]). For Riemannian manifolds, lower bounds of sectional curvature imply that of Ricci curvature. However, it is an open problem whether those Alexandrov spaces satisfy CD\(((n - 1)k, n)\) or have \(n\)-Ricci curvature \(\geq (n - 1)k\). In particular, we do not know whether all Alexandrov spaces can arise as Gromov–Hausdorff limits of Riemannian manifolds with Ricci curvature bounded from below.

**Remark 3.5.** Let \((X, d, \mu)\) be a measured Gromov–Hausdorff limit of \(n\)-dimensional compact Riemannian manifolds \(M_i\) with \(\text{Ric}_{M_i} \geq (n - 1)k\) and equipped with the normalized Riemannian measure. Set \(m = \dim_H(X)\), which is not necessarily an integer. It is conjectured that there exists a number \(l\) with \(m \leq l \leq n\) such that (3.1) holds for \(\mu = \mathcal{H}^m, n = l,\) and \(C = 1\) ([3, Conjecture 1.34]).
3.3. Basic properties. We begin by recalling the definition of a doubling measure. We say that a measure $\mu$ is doubling if for all $R > 0$ there exists a constant $C_D = C_D(R) \geq 1$ such that $\mu(B_{2r}(x)) \leq C_D \mu(B_r(x))$ holds for all $x \in X$ and all $0 < r \leq R$. If $\mu$ is doubling, then $X$ is proper (that is, all closed bounded subsets are compact). We note that if $(X, d, \mu)$ satisfies $\text{BG}(k, n)$ with $C$, then $\mu$ is doubling.

Although the following lemmas and proposition are somewhat standard, we prove them for the completeness of this paper.

Lemma 3.6. Let $(X, d, \mu)$ satisfy $\text{BG}(k, n)$ with $C = 1$ for some $k, n \in \mathbb{R}$ with $n \geq 1$. Then we have $\mu(S_r(x)) = 0$ for all $x \in X$ and all $r > 0$, where $S_r(x)$ is the sphere of radius $r$ and centered at $x$. In particular, if $X$ does not consist of a single point, then $\mu(\{x\}) = 0$ for all $x \in X$.

Proof. For a sufficiently small $\epsilon > 0$, we have

$$
\mu(S_r(x)) \leq \mu(B_{r+\epsilon}(x)) - \mu(B_{r-\epsilon}(x)) = \mu(B_{r-\epsilon}(x)) \left( \frac{\mu(B_{r+\epsilon}(x))}{\mu(B_{r-\epsilon}(x))} - 1 \right)
$$

$$
\leq \mu(B_{r-\epsilon}(x)) \left( \frac{V_{k,n}(0, r + \epsilon)}{V_{k,n}(0, r - \epsilon)} - 1 \right),
$$

where we have used (3.1) for the last inequality. Letting $\epsilon \to 0$ completes the proof. $\Box$

Let $A \subset X$. We say that $A$ is convex if for all two points $x, y \in A$, all minimal geodesics between $x$ and $y$ are contained in $A$.

Lemma 3.7. Let $(X, d, \mu)$ satisfy $\text{BG}(k, n)$ with $C$ for some $k \in \mathbb{R}$, $n \geq 1$, and $C \geq 1$. Then the following holds:

1. If $k' \leq k$ and if $n' \geq n$, then $(X, d, \mu)$ satisfies $\text{BG}(k', n')$ with $C$.
2. If $a, b > 0$, then $(X, ad, b\mu)$ satisfies $\text{BG}(k/a^2, n)$ with $C$.
3. If $A$ is a convex subset in $X$, then $(A, d, \mu)$ also satisfies $\text{BG}(k, n)$ with $C$.

Proof. Fix $0 \leq r_1 < r_2$, $0 \leq s_1 < s_2$ with $s_1 \leq r_1, s_2 \leq r_2$. (1) follows from facts that $V_{k,n}(r_1, r_2)/V_{k,n}(s_1, s_2)$ is monotone nonincreasing in $k$ and monotone nondecreasing in $n$.

We denote by $A_{r_1, r_2}^{(d)}(x)$ and $S_{s_1, s_2}^{(d)}(x, U)$ the sets $A_{r_1, r_2}(x)$ and $S_{s_1, s_2}(x, U)$ with respect to a metric $d$, respectively. For all measurable sets $U$ in $A_{r_1, r_2}^{(d)}(x)$, we have

$$
\frac{(b\mu)(U)}{(b\mu)(S_{s_1, s_2}^{(d)}(x, U))} = \frac{\mu(U)}{\mu(S_{s_1, s_2}^{(d)}(x, U))} \leq C \frac{V_{k,n}(r_1/a, r_2/a)}{V_{k,n}(s_1/a, s_2/a)} = C \frac{V_{k/a^2,n}(r_1, r_2)}{V_{k/a^2,n}(s_1, s_2)},
$$

which implies (2). Let $x \in A$ and $U \subset A_{r_1, r_2}(x)$. Since $A$ is convex, $S_{s_1, s_2}(x, U)$ is contained in $A$. We hence have (3). $\Box$

Let $\epsilon > 0$. A set $S \subset X$ is called an $\epsilon$-net if we have $d(x, S) \leq \epsilon$ for all $x \in X$. $S$ is said to be $\epsilon$-separated if we have $d(x, y) \geq \epsilon$ for all two distinct points $x, y \in S$. We see that each maximal $\epsilon$-separated set is an $\epsilon$-net.

Proposition 3.8. Let $(X, d, \mu)$ be as in Lemma 3.4. Then we have $\text{dim}_H(X) \leq n$. 

Proof. We will show that $\mathcal{H}^n(X) = 0$ holds for all $n' > n$. We may assume that $\text{diam}(X) < \infty$. Let $\{x_i\}_{i=1}^N$ be a maximal $\delta$-separated set in $X$ for $\delta > 0$. Then $\{x_i\}_{i=1}^N$ is a $\delta$-net. Note that there exists a constant $c(k, n) > 0$ depending only on $k$ and $n$ such that $V_{k,n}(0, \delta) \geq c(k, n)\delta^n$ holds for all (sufficiently small if $k > 0$) $\delta > 0$. Let $x_{i_0}$ be a point such that $\mu(B_{\delta/2}(x_{i_0}))$ attains $\min_{1 \leq i \leq N} \mu(B_{\delta/2}(x_i))$. Remark that $\{B_{\delta/2}(x_i)\}_{i=1}^N$ is disjoint. Since $(X, d, \mu)$ satisfies $\text{BG}(k, n)$ with $C$, we have

$$
\mu(X) \geq \sum_{i=1}^N \mu(B_{\delta/2}(x_i)) \geq N \mu(B_{\delta/2}(x_{i_0})) \geq N C^{-1} \frac{V_{k,n}(0, \delta/2)}{V_{k,n}(0, \text{diam}(X))} \mu(B_{\text{diam}(X)}(x_{i_0})).
$$

Hence, $N \leq c(k, n)\delta^{-n}$, which implies

$$
\min \left\{ N \in \mathbb{N} \mid X = \bigcup_{i=1}^N B_\delta(x_i) \right\} \\
\leq \max \left\{ N \in \mathbb{N} \mid \{x_i\}_{i=1}^N \text{ is a maximal } \delta\text{-separated set} \right\} \\
\leq c(k, n)\delta^{-n}.
$$

Therefore,

$$
\mathcal{H}_\delta^n(X) = \omega_n \inf \left\{ \sum_{i=1}^\infty \left( \frac{\text{diam}(U_i)}{2} \right)^n \left| X = \bigcup_{i=1}^\infty U_i, \text{diam}(U_i) \leq \delta \right\} \\
\leq \omega_n \inf \left\{ \sum_{i=1}^\infty \left( \frac{\text{diam}(B_{\delta/2}(x_i))}{2} \right)^n \left| X = \bigcup_{i=1}^\infty B_{\delta/2}(x_i) \right\} \\
\leq \omega_n \left( \frac{\delta}{2} \right)^n \min \left\{ N \in \mathbb{N} \mid X = \bigcup_{i=1}^N B_{\delta/2}(x_i) \right\} \\
\leq c(k, n, n') \delta^{n'-n}.
$$

Letting $\delta \to 0$ completes the proof. \hfill \Box

4. Local cut points

In this section, we first prove Theorem 1.1. In Subsection 4.2, we recall the definition of an end, and then prove Corollary 1.2. Furthermore, we investigate the geometric structure of the neighborhood of a local cut point, and also study the structure of the accumulation of local cut points by using the idea of the proof of Theorem 1.1. We now assume that $(X, d)$ is a complete, locally compact length space.

4.1. Local cut points.

Definition 4.1 (local cut point [10, 3.32], r-cut point). We say that a point $x \in X$ is a local cut point if $U \setminus \{x\}$ is disconnected for some connected neighborhood $U$ of $x$. The degree of $x$, denoted by $\deg(x)$, is defined as the supremum of the number of connected
components of $U \setminus \{x\}$ for all connected neighborhoods $U$ of $x$. Let $r > 0$. We say that a point $x \in X$ is an $r$-cut point if the following three conditions hold:

(i) $\overline{B}_r(x) \setminus \{x\}$ is disconnected;
(ii) the number of connected components of $\overline{B}_r(x) \setminus \{x\}$ is equal to $\deg(x)$;
(iii) $O \cap S_r(x)$ is nonempty for all connected components $O$ of $\overline{B}_r(x) \setminus \{x\}$.

If $x$ is a local cut point, then $U \setminus \{x\}$ is disconnected for every sufficiently small neighborhood $U$ of $x$. The end points in a graph (one-dimensional space) are not local cut points. An interior point in a graph is not always a local cut point; see Example 4.2 (3).

Example 4.2. (1) Consider the set $\bigcup_{i=0}^{\infty} \{r \exp(2^{-i} \pi \sqrt{-1}) \in \mathbb{C} | 0 \leq r \leq 2^{-i}\}$ with the induced distance (24, Example 9.1). The origin is a local cut point and its degree is infinite.

(2) Consider the set $\bigcup_{i=1}^{\infty} \{(x, y) \in \mathbb{R}^2 | x^2 + (y - 1/i)^2 = 1/i^2\}$ with the induced distance. The origin is a local cut point and its degree is infinite.

(3) Consider the set $\{(x, 0) | 0 \leq x \leq 1\} \cup \{(0, y) | 0 \leq y \leq 1\} \cup \bigcup_{i=0}^{\infty} \{(x, -x+2^{-i}) | 0 \leq x \leq 2^{-i}\}\subset \mathbb{R}^2$ with the induced distance. The origin is not a local cut point.

(4) Consider the interval $[0, 1] \subset \mathbb{R}$ with the Euclidean distance. The point $1/3 \in [0, 1]$ is not an $r$-cut point for any $r > 1/3$ because (iii) is not satisfied.

(5) Consider the set $\{(x, y) | x^2 + y^2 = 1\} \cup \{(x, 0) | 1 \leq x\} \subset \mathbb{R}^2$ with the induced distance. The point $(-1,0)$ is not an $r$-cut point for any $r \geq \pi$ because (ii) is not satisfied.

Proof of Theorem 1.1. For any positive integer $d$ with $d \leq \deg(x)$, take a sufficiently small $r > 0$. Choose any $d$ connected components $O_1, O_2, \ldots, O_d$ of $\overline{B}_r(x) \setminus \{x\}$ such that the degree of $x$ in $\overline{B}_r(x) \cap (O_1 \cup \cdots \cup O_d)$ is equal to $d$, and $O_i \cap S_r(x)$ is nonempty for all $1 \leq i \leq d$. Let $l$ be a positive number with $0 < l \leq r/2$. For each $1 \leq i \leq d$, we choose a point $x_i \in O_i$ such that $d(x, x_i) = l$. See Figure 1.

For a sufficiently small $0 < \epsilon \ll l$, set $U = B_r(x) \cap O_1$. We claim that each minimal geodesic between every point in $U$ and $x_i$ $(2 \leq i \leq d)$ passes through the local cut point $x$. Suppose that there exists a minimal geodesic $\gamma : [0, l] \to X$ from some point $y$ in $U$ to $x_i$ such that $\gamma$ does not pass through $x$. By the choice of $r$ and $l$, the point $\gamma(t)$ is not contained in $B_r(x)$ for some $t \in [0, l]$. Therefore,

$$d(y, x_i) = d(\gamma(0), \gamma(l)) = l \geq t = d(y, \gamma(t)) \geq d(x, \gamma(t)) - d(x, y) > r - \epsilon.$$ 

On the other hand,

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \epsilon + l \leq \epsilon + \frac{r}{2}.$$ 

This is a contradiction.

We first use the generalized Bishop–Gromov inequality (3.1) with the base point $x_i$ $(2 \leq i \leq d)$. We see that $A_{i, l+\epsilon}(x_i)$ contains $U$ for $2 \leq i \leq d$. Set $S_l = S_{l-\epsilon, l}(x_i, U)$ for
\(2 \leq i \leq d\). We recall that \(S_i\) is the intersection of \(A_{t-i, t}(x_i)\) and all minimal geodesics from \(x_i\) to each point in \(U\), hence to \(x\). Then \(S_i \cap S_j\) is empty for all \(i \neq j\). By applying (3.1) to \(s_1 = l - \epsilon\), \(s_2 = r_1 = l\), \(r_2 = l + \epsilon\), the point \(x_i\), and the set \(U\), it follows for \(2 \leq i \leq d\) that

\[
C^{-1} V_{k, n}(l - \epsilon, l) \leq \frac{\mu(S_{t-i, t}(x_i, U))}{\mu(U)} = \frac{\mu(S_i)}{\mu(U)}.
\]

We now denote \(U' = \bigcup_{i=2}^{d} S_i\). Since \(\bigcap_{i=2}^{d} S_i = \emptyset\), we have \(\sum_{i=2}^{d} \mu(S_i) = \mu(U')\). Summing up (4.1) for all \(2 \leq i \leq d\), we obtain

\[
(d - 1) C^{-1} V_{k, n}(l - \epsilon, l) \leq \sum_{i=2}^{d} \frac{\mu(S_i)}{\mu(U)} = \frac{\mu(U')}{\mu(U)}.
\]

Next, we use (3.1) with the base point \(x_1\). We see that \(A_{t-i, t}(x_1)\) contains \(U'\). Applying (3.1) to \(s_1 = l - \epsilon\), \(s_2 = r_1 = l\), \(r_2 = l + \epsilon\), the point \(x_1\), and the set \(U'\), we obtain

\[
\frac{\mu(U')}{\mu(S_{t-i, t}(x_1, U'))} \leq C \frac{V_{k, n}(l, l + \epsilon)}{V_{k, n}(l - \epsilon, l)}.
\]

Note that each minimal geodesics between \(x_0\) and each point in \(U'\) passes through \(x\) similarly to that mentioned above; hence, \(U\) contains \(S_{t-i, t}(x_1, U')\). Combining (4.2) and (4.3), we have

\[
d \leq C^2 \left( \frac{V_{k, n}(l, l + \epsilon)}{V_{k, n}(l - \epsilon, l)} \right)^2 + 1.
\]

Letting \(\epsilon \to 0\), we have \(d \leq C^2 + 1\). This completes the proof. \(\square\)

For the limits of Riemannian manifolds with Ricci curvature bounded from below, the Cheeger–Colding splitting theorem (Theorem 2.1) implies the same conclusion in Theorem 3.3 below. Since the splitting theorem for metric measure spaces satisfying BG(0, \(n\)) with \(C\) do not hold in general (see Proposition 3.2), the proof of Theorem 3.3 gives another proof that \(\deg(x) = 2\).

**Proposition 4.3.** Let \((X, d)\) be a Gromov–Hausdorff limit of \(n\)-dimensional, complete Riemannian manifolds \((M_i, d_i)\) with \(\text{Ric}_{M_i} \geq -(n - 1)\), where \(d_i\) is the Riemannian distance on \(M_i\). If there exists a local cut point \(x\) in \(X\), then every tangent cone at \(x\) is isometric to \(\mathbb{R}\), in particular \(\deg(x) = 2\).

**Proof.** Take points \(p_i \in M_i\) such that \(\{(M_i, d_i, p_i)\}\) pointed Gromov–Hausdorff converges to \((X, d, x)\). Let us recall that a tangent cone at \(x\) is the pointed limit space of \((X, r_i^{-1}d, x)\) as \(r_i \to 0\). By passing to a subsequence, the tangent cone at \(x\) is itself the pointed limit of rescaled manifolds \((M_j, r_j^{-1}d_j, p_j)\). Then the Ricci curvature of \((M_j, r_j^{-1}d_j)\) is bounded below by \(-(n - 1)r_j^2\).
Because every local cut point is an interior point of some geodesic, the tangent cone at $x$ contains a line (expanded from the geodesic). Applying the splitting theorem (Theorem 2.1), we see that the tangent cone at $x$ is isometric to $\mathbb{R} \times Y$ for some length space $Y$. Since $x$ is a local cut point, it follows that $Y$ consists of a single point.

Remark 4.4. We recall that if $(X, d, \mu)$ satisfies $\text{BG}(k, n)$ with $C$, then $\mu$ is doubling. Although the length space in Example 4.2 (1) has the natural doubling measure, the degree of the origin is infinite.

In the case of graphs, we obtain a better estimate of the degree.

**Proposition 4.5.** Let $(X, d, \mathcal{H}^1)$ be a graph, where $d$ is the usual distance and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. Assume that $(X, d, \mathcal{H}^1)$ satisfies $\text{BG}(k, n)$ with $C$ for some $k \in \mathbb{R}$, $n \geq 1$, and $C \geq 1$. Then we have $\deg(x) \leq C + 1$ for all $x \in X$.

**Proof.** Let $x$ be a local cut point (an interior point) in $X$. Take a point $x_1 \in X$ with $d(x, x_1) \ll 1$, and denote $l = d(x, x_1)$. Let $\gamma : [0, l] \to X$ be the minimal geodesic from $x_1$ to $x$. Set $U = B_\varepsilon(x) \setminus \gamma([0, l])$ for $0 < \varepsilon \ll 1$. We then have $S_{l-\varepsilon, l}(x_1, U) = \gamma((l - \varepsilon, l))$ as $l = d(x, x_1)$ is sufficiently small. By using (3.1) to $s_1 = l - \epsilon$, $s_2 = r_1 = l$, $r_2 = l + \epsilon$, the point $x_1$, and the set $U$, it follows that

$$\frac{(\deg(x) - 1)\varepsilon}{\varepsilon} = \frac{\mathcal{H}^1(U)}{\mathcal{H}^1(S_{l-\varepsilon, l}(x_1, U))} \leq C \frac{V_{k, n}(l, l + \epsilon)}{V_{k, n}(l - \varepsilon, l)}.$$  

(4.4)

Taking $\varepsilon \to 0$, we obtain $\deg(x) \leq C + 1$.  

□
Remark 4.6. In general, the converse of Proposition 4.3 does not necessarily hold.

4.2. The number of ends. We recall the definition of an end.

Definition 4.7. Let $\gamma_1, \gamma_2 : [0, \infty) \to X$ be rays from the base point $x$. Two rays $\gamma_1$ and $\gamma_2$ are said to be cofinal if $\gamma_1(t)$ and $\gamma_2(t)$ lie in the same connected component of $X \setminus B_r(x)$ for all $t, r > 0$ with $t \geq r$. An equivalence class of cofinal rays is called an end of $X$.

Proof of Corollary 4.3. Suppose that the number of ends of $X$ is greater than $C^2 + 1$. For any sequence $\epsilon_i \to 0$, the space $(X, \epsilon_i d, \mu)$ satisfies BG$(0, n)$ with $C$ by Lemma 3.7. For a point $x \in X$, there exists a subsequence $\epsilon_j \to 0$ such that $\{(X, \epsilon_j d, \mu, x)\}$ pointed measured Gromov–Hausdorff converges to some pointed metric measure space $(X_\infty, d_\infty, \mu_\infty, x_\infty)$. The limit space also satisfies BG$(0, n)$ with $C$. Since the number of ends of $X$ is greater than two, the point $x_\infty$ is a local cut point. Then the degree of $x_\infty$ is equal to the number of ends of $X$. This contradicts Theorem 1.1.\hfill $\Box$

4.3. Branch points. In this subsection, we will give an obstruction condition for the existence of a local cut point in a metric measure space $X$ satisfying BG$(k, n)$ with $1 \leq C < \sqrt{2}$. Let us recall that the measured Gromov–Hausdorff limits of manifolds $M_i$ with $\text{Ric}_{M_i} \geq (n - 1)k$ satisfy BG$(k, n)$ with $C = 1$. It is conjectured that the limit space has no local cut point unless the limit is one-dimensional.

Assume that $X$ has a local cut point $x$. Let $x$ be an $r$-cut point. Let $\gamma : [0, l] \to X$ be a minimal geodesic with $\gamma(l) = x$. Assume that $l$ is sufficiently small ($l \leq r/3$ for example). If $\sigma : [0, L] \to X$ is a minimal geodesic such that $\sigma(0) = \gamma(0)$ and $\sigma(L) \in B_r(\gamma(l))$ ($L \geq l$), then $\sigma(l) = \gamma(l)$, that is, $\sigma$ passes through $x$. We then define two kinds of branch points of the geodesic $\gamma$.

Definition 4.8 (branch point). Let $\gamma$ be as above. We say that $\gamma(l)$ is a branch point of $\gamma$ if for all $\epsilon > 0$ there exist two distinct points $x_1, x_2 \in B_\epsilon(\gamma(l))$ such that $d(\gamma(0), x_1) = d(\gamma(0), x_2) > l$.

Note that $\gamma(l)$ is a branch point of $\gamma$ if and only if no neighborhood of $\gamma(l)$ is a segment.

Definition 4.9 (weak branch point [6, Section 5]). Let $\gamma$ be as in Definition 4.8. We say that $\gamma(l)$ is a weak branch point of $\gamma$ if for all $\epsilon > 0$ and all two points $x_1, x_2 \in B_\epsilon(\gamma(l))$ with $d(\gamma(0), x_1), d(\gamma(0), x_2) > l$ there exist minimal geodesics $\sigma_i : [0, l_i] \to X$ ($i = 1, 2$) from $\gamma(l)$ to $x_i$ such that $\sigma_1(s) = \sigma_2(s)$ for some $s > 0$. See Figure 2.

Example 4.10. (1) Let $X = [a, b]$. Every interior point $x \in (a, b)$ is a weak branch point of all paths $\gamma : [0, l] \to X$ with $\gamma(l) = x$ and $l \ll 1$. The point $x$ is not a branch point of any $\gamma$.

(2) Let $X$ be the length space in Figure 2. The intersection $x$ of two spheres is a branch point of all paths $\gamma : [0, l] \to X$ with $\gamma(l) = x$ and $l \ll 1$. However, the point $x$ is not a weak branch point. Although a branch point is not necessarily a weak branch point, we use the term “weak” as in [6, Section 5].
Figure 2. Weak branch point

Assume now that there exists a local cut point in a metric measure space satisfying $BG(k, n)$ with $1 \leq C < \sqrt{2}$. We give an obstruction condition for the existence of a local cut point by observing geodesics which pass through a local cut point.

**Theorem 4.11** (cf. [6, Theorem 5.1]). Let $(X, d, \mu)$ be a metric measure space satisfying the generalized Bishop–Gromov inequality $BG(k, n)$ with constant $C$ for some $k \in \mathbb{R}$, $n \geq 1$, and $1 \leq C < \sqrt{2}$. If there exists a local cut point $x$ in $X$, then it is a weak branch point of all geodesics $\gamma : [0, l] \to X$ with $\gamma(l) = x$ and $l \ll 1$.

The proof is essentially the same as in [6]. We give the proof for the completeness of the paper.

**Proof.** If $x$ has a neighborhood that is a segment, then $x$ is a weak branch point of all paths which go to the point. Assume now that no neighborhood of $x$ is a segment. Let $\gamma : [0, l] \to X$ be a geodesic that branches at $\gamma(l) = x$, where $l$ is sufficiently small. Suppose that $x$ is not a weak branch point: There exist two points $x_1, x_2 \in B_l(\gamma(l))$ with $d(\gamma(0), x_1), d(\gamma(0), x_2) > l$ such that for all geodesics $\sigma_i : [0, l_i] \to X$ ($i = 1, 2$) from $\gamma(l)$ to $x_i$, we have $\sigma_1(s) \neq \sigma_2(s)$ for all $0 < s \leq \min\{l_1, l_2\}$. We may assume that $d(\gamma(l), x_1) = d(\gamma(l), x_2)$, and denote it by $l'$.

Consider all minimal geodesics from $x$ to $x_i$ for $i = 1, 2$. By the assumption, the union of all minimal geodesics from $x$ to $x_1$ and that of all minimal geodesics from $x$ to $x_2$ have no intersection except $x$. We will use the generalized Bishop–Gromov inequality (3.1) with the base point $x_i$. Let $0 < \epsilon \ll l'$. We denote by $U$ the connected component of $B_{\epsilon}(x) \setminus \{x\}$ that contains $\gamma$. We see that $A_{\epsilon, l' + \epsilon}(x_i)$ contains $U$. Set $S_i = S_{\epsilon, l' + \epsilon}(x_i, U)$ for $i = 1, 2$. Let us recall that $S_i$ is the intersection of $A_{\epsilon, l'}(x_i)$ and all minimal geodesics from $x_i$ to each point in $U$. Then $S_1 \cap S_2$ is empty by the assumption. For $i = 1, 2$ the generalized Bishop–Gromov inequality (3.1) induces

$$C^{-1} \frac{V_{k, n}(l' - \epsilon, l')}{V_{k, n}(l', l' + \epsilon)} \leq \frac{\mu(S_{\epsilon, l'}(x_i, U))}{\mu(U)} = \frac{\mu(S_i)}{\mu(U)}.$$  \hspace{1cm} (4.5)

Let $U'$ denote the union of connected components of $B_{\epsilon}(x) \setminus \{x\}$ that contain $S_1$ and $S_2$. We see that $A_{\epsilon, l' + \epsilon}(\gamma(0))$ contains $U'$. Since $S_i \subset U'$ holds for $i = 1, 2$, we have
\[ \mu(S_1) + \mu(S_2) \leq \mu(U'). \]

By summing up (4.5) for \( i = 1, 2 \), it follows that
\[
2C^{-1} \frac{V_{k,n}(l', \epsilon, l')}{V_{k,n}(l', l' + \epsilon)} \leq \frac{\mu(S_1) + \mu(S_2)}{\mu(U)} \leq \frac{\mu(U')}{\mu(U)}.
\]

Applying (3.1) with the base point \( \gamma(0) \), we obtain
\[
\frac{\mu(U')}{\mu(S_{l' - \epsilon, l' \gamma(0), U'})} \leq C \frac{V_{k,n}(l, l + \epsilon)}{V_{k,n}(l - \epsilon, l)}.
\]

Since \( U \) contains \( S_{l' - \epsilon, l' \gamma(0), U'} \), it, together with (4.6) and (4.7), follows that
\[
2C^{-1} \frac{V_{k,n}(l', \epsilon, l')}{V_{k,n}(l', l' + \epsilon)} \leq C \frac{V_{k,n}(l, l + \epsilon)}{V_{k,n}(l - \epsilon, l)}.
\]

Taking \( \epsilon \to 0 \), we obtain \( 2 \leq C^2 \), which is a contradiction. \( \square \)

**Remark 4.12.** In [6, Theorem 5.1], Cheeger and Colding studied the limit space which contains a one-dimensional piece and which is not one-dimensional. It follows from Theorem 4.11 that the length space in Figure 2 can not satisfy BG(\( k, n \)) with \( 1 \leq C < \sqrt{2} \) for any \( k, n \), and any measure. This does not follow from [6, Theorem 5.1].

As a corollary of Theorem 4.11, we obtain the particular case of Theorem 1.1.

**Theorem 4.13.** Let \( (X, d, \mu) \) satisfy BG(\( k, n \)) with \( C \) for some \( k \in \mathbb{R}, \ n \geq 1, \) and \( 1 \leq C < \sqrt{2} \). Assume that there exists a local cut point \( x \) in \( X \). Then we have \( \deg(x) = 2 \).

**Proof.** Suppose that \( \deg(x) \geq 3 \) holds. For all geodesics \( \gamma : [0, l] \to X \) with \( \gamma(l) = x \) and \( l \ll 1 \), the local cut point \( x = \gamma(l) \) is not a weak branch point of \( \gamma \). This is because the geodesic \( \gamma \) is extended to two connected components of \( B_r(x) \setminus \{ x \} \) which do not contain \( \gamma \), where \( r \) is sufficiently small. This contradicts Theorem 4.11. \( \square \)
4.4. Local geometric structure. Recall the definition of an $r$-cut point (Definition 4.1). We investigate the geometric structure of the neighborhood of an $r$-cut point in a metric measure space $X$ satisfying $\text{BG}(k, n)$ with $1 \leq C < \sqrt{2}$.

Assume that there exists an $r$-cut point $x$ in $X$. We then have $\text{diam}(O \cap S_r(x)) \leq \text{diam}(B_r(x)) \leq 2r$ for all connected components $O$ of $B_r(x) \setminus \{x\}$. By improving the method of the proof of Theorem 4.11, we obtain a more precise estimate of the diameter as follows:

**Theorem 4.14.** For all $k \in \mathbb{R}$, $n \geq 1$, $1 \leq C < \sqrt{2}$, and all $R > 0$ there exists a constant $\delta = \delta(k, n, C, R) > 0$ depending only on $k$, $n$, $C$, and $R$ such that the following holds: Let $(X, d, \mu)$ be a metric measure space satisfying the generalized Bishop–Gromov inequality $\text{BG}(k, n)$ with constant $C$. If $X$ has an $r$-cut point $x$ with $0 < r \leq R$, then

$$\text{diam}(O \cap S_r(x)) \leq (2 - \delta)r$$

holds for all connected components $O$ of $B_r(x) \setminus \{x\}$.

**Remark 4.15.** The constant $\delta(k, n, C, R)$ in Theorem 4.14 is independent of a metric measure space $(X, d, \mu)$. Moreover, we can calculate the precise value; see Remark 4.19. It follows from Theorem 4.14 that for fixed $k$, $n$, and $C$ the metric space in Figure 4 can not satisfy $\text{BG}(k, n)$ with $1 \leq C < \sqrt{2}$.

**Proof of Theorem 4.14.** For $k \in \mathbb{R}$, $n \geq 1$, $1 \leq C < \sqrt{2}$, and $R > 0$, let $(X, d, \mu)$ be a metric measure space satisfying $\text{BG}(k, n)$ with $C$ and $x \in X$ an $r$-cut point with $0 < r \leq R$. Fix $0 < \delta \ll r$. Suppose that

$$\text{diam}(O \cap S_r(x)) > (2 - \delta)r$$

holds for some connected component $O$ of $B_r(x) \setminus \{x\}$. We take two points $x_1, x_2 \in O \cap S_r(x)$ with $d(x_1, x_2) > (2 - \delta)r$ and then choose minimal geodesics $\sigma_i : [0, r] \to X$ from $x$ to $x_i$ ($i = 1, 2$). Since $\deg(x) = 2$, we denote by $O'$ another connected component of $B_r(x) \setminus \{x\}$. For $0 < \epsilon < \delta r$, denote by $U$ the connected component of $B_r(x) \setminus \{x\}$ which does not contain $\sigma_1$ and $\sigma_2$. We see that $A_{r/3, r/3 + \epsilon}((\sigma_i(r/3)))$ contains $U$ for $i = 1, 2$. 

![Figure 4. “Three-pronged” space](image-url)
Claim 4.16. Each minimal geodesic from \( \sigma_i(r/3) \) to each point in \( U \) passes through \( x \) (\( i = 1, 2 \)).

Proof. Suppose that there exists a minimal geodesic \( \gamma : [0, l] \to X \) from \( \sigma_i(r/3) \) to some point \( y \) in \( U \) such that \( \gamma \) does not pass through \( x \). Then, \( \gamma(t) \not\in B_r(x) \) for some \( t \in [0, l] \) since \( x \) is an \( r \)-cut point. Therefore,

\[
d(y, \sigma_i(r/3)) \geq d(y, \gamma(t)) \geq d(\gamma(t), x) - d(y, x) > r - \epsilon.
\]

On the other hand,

\[
d(y, \sigma_i(r/3)) \leq d(y, x) + d(x, \sigma_i(r/3)) < \epsilon + \frac{r}{3}.
\]

This is a contradiction. \( \square \)

Setting \( S_i = S_{r/3-\delta r-\epsilon, r/3-\delta}(\sigma_i(r/3), U) \) for \( i = 1, 2 \), we have

Claim 4.17. \( S_1 \cap S_2 = \emptyset \).

Proof. By the assumption, we have

\[
(2-\delta)r < d(x_1, x_2) \leq d(x_1, \sigma_1(r/3)) + d(\sigma_1(r/3), \sigma_2(r/3)) + d(\sigma_2(r/3), x_2) = \frac{2r}{3} + d(\sigma_1(r/3), \sigma_2(r/3)) + \frac{2r}{3}.
\]

It follows that \( (2/3 - \delta)r < d(\sigma_1(r/3), \sigma_2(r/3)) \). Suppose that there exists a point \( y \in S_1 \cap S_2 \). We then have

\[
d(\sigma_1(r/3), \sigma_2(r/3)) \leq d(\sigma_1(r/3), y) + d(y, \sigma_2(r/3)) \leq 2\left(\frac{r}{3} - \delta r\right) = \left(\frac{2}{3} - 2\delta\right)r.
\]

This is a contradiction. \( \square \)

Using the generalized Bishop–Gromov inequality \( [3,1] \) with the base point \( \sigma_i(r/3) \), we obtain for \( i = 1, 2 \)

\[
C^{-1} \frac{V_{k,n}\left((1/3 - \delta)r - \epsilon, (1/3 - \delta)r\right)}{V_{k,n}(r/3, r/3 + \epsilon)} \leq \frac{\mu(S_i)}{\mu(U)}.
\]

We set \( U' = S_1 \cup S_2 \). Since \( S_1 \cap S_2 = \emptyset \) (Claim 4.17), we have \( \mu(S_1) + \mu(S_2) \leq \mu(U') \). Therefore, summing up (4.8) for \( i = 1, 2 \) gives

\[
2C^{-1} \frac{V_{k,n}\left((1/3 - \delta)r - \epsilon, (1/3 - \delta)r\right)}{V_{k,n}(r/3, r/3 + \epsilon)} \leq \frac{\mu(S_1) + \mu(S_2)}{\mu(U)} \leq \frac{\mu(U')}{\mu(U)}.
\]

Next, we take an arbitrary point \( x_0 \) in \( O' \cap S_{r/3}(x) \). Such a point exists since \( x \) is an \( r \)-cut point.

Claim 4.18. Each minimal geodesic from \( x_0 \) to each point in \( U' \) passes through \( x \).
Proof. Suppose that there exists a minimal geodesic $\gamma : [0, l] \to X$ from $x_0$ to some point $y$ in $U'$ such that $\gamma$ does not pass through $x$. Then, $\gamma(t) \notin B_r(x)$ for some $t \in [0, l]$. Therefore,

$$d(x_0, y) \geq d(x_0, \gamma(t_0)) \geq d(\gamma(t_0), x) - d(x, x_0) \geq r - \frac{r}{3} = \frac{2r}{3}.$$ 

On the other hand,

$$d(x_0, y) \leq d(x_0, x) + d(x, y) \leq \frac{r}{3} + \delta r + \epsilon.$$

This is a contradiction. \hfill $\square$

Set $S_0 = S_{r/3 - \delta, r/3}(x_0, U')$. The generalized Bishop–Gromov inequality \eqref{eq:4.9} with the base point $x_0$ implies

\begin{equation}
\frac{\mu(U')}{\mu(S_0)} \leq C \frac{V_{k,n}((1/3 + \delta)r, (1/3 + \delta)r + \epsilon)}{V_{k,n}(r/3 - \epsilon, r/3)}.
\end{equation}

Since $U$ contains $S_0$, inequalities \eqref{eq:4.9} and \eqref{eq:4.10} give

\begin{equation}
2 \leq C^2 \frac{V_{k,n}(r/3, r/3 + \epsilon)}{V_{k,n}(r/3 - \epsilon, r/3)} \frac{V_{k,n}((r/3 + \delta)r, (r/3 + \delta)r + \epsilon)}{V_{k,n}((r/3 - \delta)r - \epsilon, (r/3 - \delta)r)}.
\end{equation}

After letting $\epsilon \to 0$, a sufficiently small $\delta > 0$ implies a contradiction. This completes the proof of Theorem 4.14. \hfill $\square$

Remark 4.19. In the case $k = 0$, we have

$$\delta(0, n, C, R) = \frac{1}{4} \left( \frac{2}{C^2} \right)^{1/(n-1)} - 1.
$$
Indeed, the right-hand side of (4.11) is equal to
\[
C^2 \frac{(r/3 + \epsilon)^n - (r/3)^n}{(r/3)^n - (r/3 - \epsilon)^n} \frac{[(1/3 + \delta)r + \epsilon] - [(1/3 + \delta)r] - [r - \epsilon] - [r - \epsilon]}{[(1/3 - \delta)r] - [(1/3 - \delta)r]},
\]
which converges to \( C^2 [(1/3 + \delta)/(1/3 - \delta)]^{n-1} \) as \( \epsilon \to 0 \). Therefore, it suffices to determine a positive number \( \delta \) such that \( C^2 [(1/3 + \delta)/(1/3 - \delta)]^{n-1} < 2 \).

4.5. Convergence of local cut points. We observe the structure of the accumulation of local cut points in a metric measure space satisfying \( \text{BG}(k, n) \) with \( 1 \leq C < \sqrt{2} \) by using Theorem 4.14.

Let \( (X, d, \mu) \) satisfy \( \text{BG}(k, n) \) with \( C \) for some \( k \in \mathbb{R}, n \geq 1 \), and \( 1 \leq C < \sqrt{2} \). Assume that there exist three \( r \)-cut points \( x_1, x_2, x_3 \in X \) \( (r > 0) \) such that \( d(x_i, x_j) \) is sufficiently small for all \( 1 \leq i, j \leq 3 \). We recall that \( \text{deg}(x_i) = 2 \) (Theorem 4.13). Denote by \( O_1, O_1' \) the connected components of \( \overline{B}_r(x_1) \setminus \{x_1\} \). Assume then that \( O_1' \) contains \( x_2 \) and \( x_3 \) and we have \( d(x_1, x_2) < d(x_1, x_3) \) without loss of generality. For \( i = 2, 3 \) denote by \( O_i \) the connected component of \( \overline{B}_r(x_i) \setminus \{x_i\} \) which contains \( x_1 \) and \( O_i' \) another one. By the definition, \( O_2 \cap O_3 \) is nonempty. We say that \( \{x_1, x_2, x_3\} \) stands in a line if \( O_2 \cap O_3 \) is nonempty.

As a corollary of Theorem 4.14 we have the following:

**Corollary 4.20.** Let \( (X, d, \mu) \) be a metric measure space satisfying \( \text{BG}(k, n) \) with \( C \) for some \( k \in \mathbb{R}, n \geq 1 \), and \( 1 \leq C < \sqrt{2} \). Let \( \delta = \delta(k, n, C, R) \) be the constant in Theorem 4.14 for \( R > 0 \). Assume that there exist \( r \)-cut points \( x_1, x_2, x_3 \in X \) with \( 0 < r \leq R \). If \( d(x_i, x_j) < \delta r/6 \) holds for all \( 1 \leq i, j \leq 3 \), then \( \{x_1, x_2, x_3\} \) stands in a line.

**Proof.** Suppose that \( O_2 \cap O_3' \) is empty; see Figure 6. Since \( x_2 \) is an \( r/3 \)-cut point, there exist at least two connected components of \( O_2 \cap S_{r/3}(x_2) = (O_1 \cap S_{r/3}(x_2)) \cup (O_3' \cap S_{r/3}(x_2)) \). Denote by \( O \) one of connected components of \( O_1 \cap S_{r/3}(x_2) \) and by \( O' \) one of connected components of \( O_3' \cap S_{r/3}(x_2) \).

Since \( x_2 \) is an \( r/3 \)-cut point, all minimal geodesics from each point in \( O \) to each point in \( O' \) pass through \( x_1 \) and \( x_3 \). Therefore, we have
\[
d(O, O') = d(x_1, O) + d(x_1, x_3) + d(x_3, O').
\]

Since \( d(x_i, x_j) < \delta r/6 \) holds by the assumption, we have
\[
d(x_1, O) \geq \frac{r}{3} - d(x_2, x_1) > \frac{r}{3} - \frac{\delta r}{6} = \left( 1 - \frac{\delta}{2} \right) \frac{r}{3}.
\]

Similarly, we have
\[
d(x_3, O') > \left( 1 - \frac{\delta}{2} \right) \frac{r}{3}.
\]

\[
d(x_1, x_3) > \left( 1 - \frac{\delta}{2} \right) \frac{r}{3}.
\]
Hence, relations (4.12), (4.13) and (4.14) imply
$$\text{diam}(O_2 \cap S_{r/3}(x_2)) \geq d(O, O')$$
$$= d(x_1, O) + d(x_1, x_3) + d(x_3, O')$$
$$> \left(1 - \frac{\delta}{2}\right)\frac{r}{3} + \left(1 - \frac{\delta}{2}\right)\frac{r}{3}$$
$$= \left(2 - \delta\right)\frac{r}{3}.$$  
This contradicts Theorem 4.14.

Assume now that there exists a sequence \(\{x_i\}_{i=1}^{\infty}\) of \(r\)-cut points in \(X\) such that \(d(x_i, x_j)\) is sufficiently small for all \(i\) and \(j\). Denote by \(O_i, O'_i\) the connected components of \(\overline{B}_r(x_i) \setminus \{x_i\}\). Assume that \(O_i\) contains \(x_j\) for all \(j < i\). We say that \(\{x_i\}_{i=1}^{\infty}\) stands in a line if \(O'_i \cap O'_{i+1}\) is nonempty for all \(i\).

Corollary 4.20 implies

Corollary 4.21. Let \((X, d, \mu)\) be as in Corollary 4.20. For each \(r > 0\), the set of all \(r\)-cut points in \(X\) is closed.

Proof. Assume that there exist a sequence \(\{x_i\}_{i=1}^{\infty}\) of \(r\)-cut points in \(X\), which converges to a point \(x\) in \(X\). By Corollary 4.20 \(\{x_i\}_{i=N}^{\infty}\) stands in a line for a sufficiently large \(N\). Hence the limit \(x\) is a local cut point. Since each \(x_i\) is an \(r\)-cut point, so is \(x\).

Remark 4.22. Let \(\{x_i\}_{i=1}^{\infty}\) be a sequence of \(r_i\)-cut points, where \(r_i \to 0\). It is then not necessarily that a limit point of \(\{x_i\}\) is a local cut point. Consider the sequence \(\{1/i\}_{i=1}^{\infty} \subset [0, 1]\).

In fact, Corollary 4.21 holds without the assumption \(\text{BG}(k, n)\) with \(1 \leq C < \sqrt{2}\).
Proposition 4.23. Let \((X, d)\) be a complete, locally compact length space. Assume that for \(r > 0\) there exists a sequence \(\{x_i\}_{i=1}^{\infty}\) of \(r\)-cut points in \(X\) which converges to a point \(x \in X\). Then, for some sufficiently large \(N\), we have \(\deg(x_i) = 2\) for all \(i \geq N\). Moreover, \(\{x_i\}_{i=N}^{\infty}\) stands in a line. (Hence, the limit point \(x\) is also an \(r\)-cut point. We do not always have \(\deg(x) = 2\).)

Proof. Suppose that for each \(N\) there exists \(i \geq N\) such that \(\deg(x_i) \geq 3\). We may assume that \(\deg(x_i) \geq 3\) for all \(i\), and \(d(x, x_i) > d(x, x_j)\) for all \(i < j\). For each \(i\), there exists a connected component \(O_i\) of \(\overline{B}_r(x_i) \setminus \{x_i\}\) which does not contain \(x\) and \(x_{i-1}\). Since \(O_i \cap S_r(x_i)\) is nonempty, we have a contradiction to the locally compactness at \(x\). Thus, it follows that \(\deg(x_i) = 2\) for all sufficiently large \(i\).

Next, we show that \(\{x_i\}_{i=N}^{\infty}\) stands in a line for some sufficiently large \(N\). Suppose that \(\{x_i\}_{i=N}^{\infty}\) does not stand in a line for any large \(N\). By taking a subsequence, we may assume that \(O'_i \cap O'_{i+1} = \emptyset\) for \(i = 1, 3, 5, \ldots\). Since \(O'_i \cap S_r(x_i)\) is nonempty, we have a contradiction to the locally compactness at \(x\).

In general, Proposition 4.23 does not necessarily hold without the locally compactness of \(X\).

Example 4.24. Consider the set \(\bigcup_{i=1}^{\infty} \{2/i, y\} \cap \{(x, 0) \mid 0 \leq x \leq 2\} \cup \{(x, 0) \mid 0 \leq x \leq 2\} \subset \mathbb{R}^2\) with the induced distance. Although the point \((2/i, 1/i)\) is a 1-cut point for every \(i \in \mathbb{N}\), the limit point \((0, 0)\) is not a local cut point.

5. The Poincaré inequality

In this section we prove Theorem 1.3. We begin by recalling the definition of a Poincaré inequality of type \((1, p)\). Let \((X, d)\) be a metric space.

Definition 5.1 (upper gradient \([12]\)). Let \(u\) be a function on \(X\). A Borel function \(g : X \to [0, \infty]\) is called an upper gradient of \(u\) if for all paths \(\gamma : [0, l] \to X\) (proportional to arclength), we have

\[
|u(\gamma(l)) - u(\gamma(0))| \leq \int_0^l g(\gamma(t)) \, dt.
\]

Every function has an upper gradient \(g \equiv \infty\), and hence upper gradients are never unique. For a Lipschitz function \(u : X \to \mathbb{R}\), we define |\(\nabla u\)| : \(X \to \mathbb{R}\) by

\[
|\nabla u|(x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)}
\]

if \(x\) is not isolated, and |\(\nabla u\)|(\(x\)) = 0 if \(x\) is isolated.

Proposition 5.2. If \(u\) is a Lipschitz function, then |\(\nabla u\)| is an upper gradient of \(u\).

Proof. Let \(\gamma : [0, l] \to X\) be a path. The function \(u \circ \gamma\) is Lipschitz and hence is differentiable almost everywhere by Rademacher’s theorem. Since \(\gamma\) is proportional to arclength,

\[
|(u \circ \gamma)'(t)| = \lim_{s \to t} \frac{|u(\gamma(t)) - u(\gamma(s))|}{|t - s|} \leq \lim_{s \to t} \frac{|u(\gamma(t)) - u(\gamma(s))|}{d(\gamma(t), \gamma(s))} \leq |\nabla u|(\gamma(t))
\]
holds for all differentiable points \( t \in [0, l] \). Therefore
\[
|u(\gamma(l)) - u(\gamma(0))| \leq \int_0^l |(u \circ \gamma)'(t)| \, dt \leq \int_0^l |\nabla u(\gamma(t))| \, dt,
\]
which completes the proof. \(\square\)

Assume that \( (X, d, \mu) \) is a complete, locally compact length space equipped with a Borel measure such that \( 0 < \mu(B_r(x)) < +\infty \) holds for all \( x \in X \) and all \( 0 < r < +\infty \). We denote
\[
u_B = \int_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu
\]
for \( B \subset X \).

**Definition 5.3 (Poincaré inequality).** Let \( 1 \leq p < \infty \). We say that \( (X, d, \mu) \) satisfies a Poincaré inequality of type \((1, p)\), if for all \( R > 0 \) there exists a constant \( C_p = C_p(p, R) > 0 \) depending only on \( p \) and \( R \) such that
\[
\int_{B_r(x)} |u - u_{B_r(x)}| \, d\mu \leq C_p r \left( \int_{B_r(x)} g^p \, d\mu \right)^{1/p}
\]
holds for all \( x \in X \), all \( 0 < r \leq R \), all measurable functions \( u \), and all upper gradients \( g \) of \( u \).

In our setting \( (X, \mu) \) is a length space, a Poincaré inequality of type \((1, p)\) is derived from a “weak” Poincaré inequality of type \((1, p)\) if we assume that \( \mu \) is doubling (see [11] for details).

**Remark 5.4.** It follows from Hölder’s inequality that each metric measure space satisfying a Poincaré inequality of type \((1, p)\) also satisfies that of type \((1, q)\) for all \( q \geq p \).

Keith and Zhong [15] proved the following: Let \( p > 1 \). If \( \mu \) is doubling and if \( (X, d, \mu) \) satisfies a Poincaré inequality of type \((1, p)\), then there exists \( \epsilon > 0 \) such that \( (X, d, \mu) \) satisfies a Poincaré inequality of type \((1, q)\) for all \( q > p - \epsilon \).

**Proof of Theorem 1.3.** The proof is by contradiction; suppose that there exists a local cut point \( x_0 \) satisfying [13]. Fix a sufficiently small \( r > 0 \) such that \( B_r(x_0) \setminus \{x_0\} \) is disconnected. We choose two connected components \( O_1, O_2 \) of \( B_r(x_0) \setminus \{x_0\} \). For sufficiently large numbers \( N \in \mathbb{N} \), we define functions \( u_N : B_r(x_0) \to \mathbb{R} \) as follows. We set \( U_i = O_i \cap (B_r(x_0) \setminus B_1/\mu(u_i)(x_0)) \) for \( i = 1, 2 \), and define
\[
u_N(x) = \begin{cases} (-1)^i + 1/\mu(O_i) & \text{on } U_i, \\ (-1)^i + 1/Nd(x_0, x) & \text{on } O_i \setminus U_i, \\ 0 & \text{on } B_r(x_0) \setminus (O_1 \cup O_2). \end{cases}
\]
The function \( u_N \) is Lipschitz; hence, \( |\nabla u_N| \) is an upper gradient of \( u_N \) (Proposition 5.2). Since \( (X, \mu) \) satisfies a Poincaré inequality of type \((1, p)\), for \( R > r \) there exists a constant
$C_P = C_P(p, R) > 0$ such that

\begin{equation}
(5.1) \quad \int_{B_r(x_0)} |u_N - (u_N)_{B_r(x_0)}| \, d\mu \leq C_P r \left( \int_{B_r(x_0)} |\nabla u_N|^p \, d\mu \right)^{1/p}.
\end{equation}

We first estimate the left-hand side of \((5.1)\) from below. We have

\[
\int_{B_r(x_0)} |u_N - (u_N)_{B_r(x_0)}| \, d\mu \\
\geq \int_{B_r(x_0)} |u_N| \, d\mu - |(u_N)_{B_r(x_0)}| \\
= \frac{1}{\mu(B_r(x_0))} \left[ \left( \frac{\mu(U_1)}{\mu(O_1)} + \int_{O_1 \setminus U_1} Nd(x_0, x) \, d\mu(x) + \frac{\mu(U_2)}{\mu(O_2)} + \int_{O_2 \setminus U_2} Nd(x_0, x) \, d\mu(x) \right) \\
- \left| \frac{\mu(U_1)}{\mu(O_1)} + \int_{O_1 \setminus U_1} Nd(x_0, x) \, d\mu(x) - \frac{\mu(U_2)}{\mu(O_2)} - \int_{O_2 \setminus U_2} Nd(x_0, x) \, d\mu(x) \right| \\
\geq \frac{1}{\mu(B_r(x_0))} \left[ \left( \frac{\mu(U_1)}{\mu(O_1)} + \int_{O_1 \setminus U_1} Nd(x_0, x) \, d\mu(x) + \frac{\mu(U_2)}{\mu(O_2)} + \int_{O_2 \setminus U_2} Nd(x_0, x) \, d\mu(x) \right) \\
- \left| \frac{\mu(U_1)}{\mu(O_1)} - \frac{\mu(U_2)}{\mu(O_2)} \right| + \int_{O_1 \setminus U_1} Nd(x_0, x) \, d\mu(x) + \int_{O_2 \setminus U_2} Nd(x_0, x) \, d\mu(x) \right] \\
= \frac{1}{\mu(B_r(x_0))} \left( \frac{\mu(U_1)}{\mu(O_1)} + \frac{\mu(U_2)}{\mu(O_2)} - \left| \frac{\mu(U_1)}{\mu(O_1)} - \frac{\mu(U_2)}{\mu(O_2)} \right| \right).
\]

Since \(N\) is sufficiently large, \(\mu(U_i)/\mu(O_i)\) is approximately equal to one \((i = 1, 2)\). Therefore, we see that the left-hand side of \((5.1)\) is bounded below by the positive number \(\mu(B_r(x_0))^{-1}\) which is independent of \(N\).

Next, we estimate the right-hand side of \((5.1)\) from above. Note that

\[ |\nabla u_N| = \begin{cases} N & \text{on } (O_1 \setminus U_1) \cup (O_2 \setminus U_2), \\ 0 & \text{otherwise,} \end{cases} \]

and \((O_1 \setminus U_1) \cup (O_2 \setminus U_2)\) is contained in \(B_{\max\{1/N\mu(O_1), 1/N\mu(O_2)\}}(x_0)\). It follows that

\[
C_P r \left( \int_{B_r(x_0)} |\nabla u_N|^p \, d\mu \right)^{1/p} = C_P r \left[ \mu(B_r(x_0))^{-1} N^p \mu((O_1 \setminus U_1) \cup (O_2 \setminus U_2)) \right]^{1/p} \\
\leq C_P r \left[ \mu(B_r(x_0))^{-1} N^p \mu\left( B_{\max\{1/N\mu(O_1), 1/N\mu(O_2)\}}(x_0) \right) \right]^{1/p},
\]

which converges to zero as \(N \to \infty\), by the assumption \([1.3]\). This is a contradiction. \(\square\)

Let \(\alpha > 0\). Consider the metric measure space \(\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq x_n^{2\alpha}\}\) equipped with the Euclidean distance and the \(n\)-dimensional Lebesgue measure \(\mathcal{L}^n\). The origin \(o = (0, 0, \ldots, 0)\) is a local cut point. We have \(\mathcal{L}^n(B_r(o)) = \)
c(n) r^{\alpha(n-1)+1}. It follows from Theorem 1.3 that, if the metric measure space satisfies a Poincaré inequality of type \((1, p)\), then we have

\[ \alpha \leq \frac{p-1}{n-1}. \]

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