Negative scaling dimensions and conformal invariance at the Nishimori point in the 
\[ \pm J \] random-bond Ising model

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We reexamine the disorder-dominated multicritical point of the two-dimensional \( \pm J \)-Ising model, known as the Nishimori point (NP). At the NP we investigate numerically and analytically the behavior of the disorder correlator, familiar from the self-dual description of the pure critical point of the two-dimensional Ising model. We consider the logarithmic average and the \( q \)th moments of this correlator in the ensemble average over randomness, for continuous \( q \) in the range \( 0 < q < 2.5 \), and demonstrate their conformal invariance. At the NP we find, in contrast to the self-dual pure critical point, that the disorder correlators exhibit multi-scaling in \( q \) which is different from that of spin-spin correlators and that their scaling dimension becomes negative for \( q > 1 \) and \( q < 0 \). Using properties on the Nishimori line we show that the first moment (\( q = 1 \)) of the disorder correlator is exactly one for all separations. The spectrum of scaling dimensions at the NP is not parabolic in \( q \).

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I. INTRODUCTION

Conformal field theory (CFT) has been a key ingredient in understanding critical models and phase-transitions in two spatial dimensions. It has helped the exact solution of a number of problems and made it possible to construct a classification of critical points. While so far the success of CFT has been mainly in its application to pure critical systems, one might hope that it will be an equally powerful tool for understanding disorder-dominated critical points in two dimensions, such as those in disordered magnets or the integer quantum Hall effect: It is generally expected that the observation of multi-scaling in \( q \) which is different from that of spin-spin correlators and that their scaling dimension becomes negative for \( q > 1 \) and \( q < 0 \) is possible to construct a classification of critical points. The aim of this paper is to show that it will be an equally powerful tool for understanding critical models and phase-transitions in two spatial dimensions.

One model system which has received renewed attention recently is the \( \pm J \) random-bond Ising model (\( \pm J \) RBIM), which (amongst others) has a special symmetry line in parameter space, known as the Nishimori line. It has a disorder-dominated multicritical point, the Nishimori point (NP), where this line intersects the phase boundary between a ferromagnet and a paramagnet. The NP may be a good candidate for the construction of a consistent CFT of a random critical point since it is one of the simplest model critical points at the outset. The phase transition in the RBIM is also of interest as a version of the quantum Hall plateau transition, since there is an exact mapping from the RBIM to a network model similar that used to represent the latter. Furthermore, owing to extensive numerical and analytic effort invested in studying the \( \pm J \) RBIM and the NP in particular, accurate estimates for various exponents are available, which can be used to test candidate CFTs. At the NP the first few integer \( q \) moments of the spin-spin (\( \sigma \sigma \)-) correlation function have been calculated numerically recently and have been demonstrated to obey the conformal constraints very accurately. The \( q \)-dependence of their scaling dimension has considerable significance, and one can imagine three possibilities. If critical behavior at the NP were that of the percolation transition, the dimensions would be independent of \( q \). By contrast, the critical behavior in a pure system gives dimensions increasing linearly in \( q \). In fact, at the NP scaling dimensions of moments of the \( \sigma \sigma \)-correlator vary non-linearly with \( q \). This behavior is generic in the presence of randomness and referred to as multi-scaling. For the RBIM, Read and Ludwig considered the expectation values of the disorder operator which is the Kramers-Wannier dual to the conventional order operator \( \sigma \). The dual operator will be the main focus of this paper and we will refer to it as \( \mu \)-operator throughout in order to avoid confusion between thermal and configurational disorder. The \( \mu \)-operators are associated with plaquettes of the Ising lattice: the two-point correlator \( \langle \mu(x)\mu(y) \rangle \) in a particular distribution of bonds can be represented by choosing an arbitrary path connecting the pair of plaquettes at \( x \) and \( y \) and reversing the sign of all the bonds crossed by the path. In this way one arrives at a modified system with partition function \( Z' \) (compared to the \( Z \) of the unmodified model) and the the correlator is given by the ratio of the two \( \frac{\langle \mu(x)\mu(y) \rangle}{Z'} = Z' \). In Ref. the authors point out that the \( \mu \)-operator, is, in contrast to \( \sigma \), not bounded from above anymore when antiferromagnetic exchange interaction are present. This opens up the interesting possibility of correlation functions which increase with distance and consequently the possibility of negative scaling dimensions, which in turn is intimately linked to the central problem of non-unitarity in CFT. The fact that \( \mu \)-correlators can increase with distance has been established in Ref. for RBIM’s with equal concentrations of ferromagnetic and antiferromagnetic exchange interactions. Here our concern is instead with behaviour at a critical point. The aim of this paper is...
to explicitly demonstrate conformal invariance with negative scaling dimensions by considering the $\mu\nu$-correlator at the NP. One practical problem in these calculations is to achieve high sampling while keeping the system size sufficiently large. From this perspective making use of the recently developed mapping of the RBIM onto the network model for calculating free energies is crucial since it reduces a formerly exponentially large calculation to one which is only power-law in system size.

In addition to our numerical results, we show in this paper that the distribution functions of the $\mu$-operators satisfy strong constraints on the Nishimori line. In particular we consider the $q$th moments of correlation functions and we show that the moments are symmetric in $q$ with respect to $q = 1/2$. Owing to this symmetry the $\mu$-operators for $q = 1$ average exactly to unity, as discussed in the following section. Due to the asymmetry between the spin operator and the disorder operator in the presence of bond disorder, the NP is not microscopically self-dual. Despite this the question arises whether self-duality is restored in the correlation functions asymptotically for large separations. The above exact result on the Nishimori line shows that this does not happen, since the spin-spin correlation function decays with distance. In Sec. II we present results from extensive numerical calculations on the correlations of the disorder $\mu$-operator for the $\pm J$ RBIM. We show that the moments of its correlation function obey the conformal constraints thus reinforcing the idea of an underlying conformal field theory. We find also that the $q$th moment of the $\mu\nu$-correlation function increases with separation $r$ for $q > 1$, establishing the presence negative scaling dimensions and non-uniquity. We test our results against previous calculations in the quasi one-dimensional regime and find excellent agreement. Finally, our results show that multi-scaling at the NP is not parabolic.

II. DISTRIBUTIONS IN THE RBIM ON THE NISHIMORI LINE

We consider the two-dimensional nearest-neighbor Ising model on the square lattice with partition function

$$Z(J, \beta) = \text{Tr}_\sigma \exp \left[ \sum_{ij} \beta J_{ij} \sigma_i \sigma_j \right],$$

where $\sigma_i$ is a classical spin variable taking the values $\pm 1$ and the $\langle ij \rangle$ denote nearest neighbors. The exchange couplings $J_{ij}$ are drawn independently from a probability distribution $P(J)$. For general $P(J$ this model is known as the random-bond Ising model (RBIM). In the conventional notation the correlator of a local operator $O_x$ is

$$\langle O_x O_y \rangle = \frac{1}{Z} \text{Tr}_\sigma O_x O_y \exp \left[ \sum_{ij} \beta J_{ij} \sigma_i \sigma_j \right]. \quad (2.1)$$

It was shown by Kadanoff and Cevalidi, however, that if $O$ represents either the order operator or its dual operator an alternative way of writing Eq. (2.1) is to absorb the product $O_x O_y$ into the Hamiltonian by modifying the set $\{ J_{ij} \} \rightarrow \{ J'_{ij} \}$ so that

$$\langle O_x O_y \rangle = Z'/Z, \quad (2.2)$$

where $Z'$ is the partition function evaluated with the set $\{ J'_{ij} \}$. One important observation when considering averages over bond distributions is that the properties of the average of $\langle O_x O_y \rangle$ crucially depend on whether the modification $O_x O_y$ induces takes the set $\{ J_{ij} \}$ out of the random ensemble or not. To be specific, inserting the order operator $\sigma$ of the Ising model may be viewed as giving the bonds along a semi-infinite path on the lattice an imaginary part while the $\mu$-operator is equivalent to simply reversing the bond signs along a semi-infinite path $\overline{\mu}$. For the $\mu$-operator the modified set $\{ J'_{ij} \}$ is still within the physical ensemble. Its average is therefore bounded below by zero but it can take any positive value in contrast the order parameter which lies between $-1$ and $+1$.

Let us now consider the average of an observable $W(J, \beta)$. $W$ might represent a two-point correlator as in Eq. (2.1) but is generally an arbitrary function over the bond configurations $\{ J \}$ and of temperature. Ensemble averages will be denoted by an over-bar

$$\overline{W} = \int_{-\infty}^{\infty} W(J, \beta) \prod_{ij} P(J_{ij}) dJ_{ij}. \quad (2.3)$$

It was noticed by Nishimori that this average may be rewritten as

$$\overline{W} = \text{Tr}_\tau \int_{0}^{\infty} W(\tau J, \beta) Z(\{ \beta^{-1} \tau A \}, \beta) \prod_{ij} Q(J_{ij}) dJ_{ij} \quad (2.4)$$

where $Z(\{ \beta^{-1} \tau A \}, \beta)$ is an Ising partition function with a set of couplings $\{ \beta^{-1} \tau A_{ij} \}$ and

$$Q(J_{ij}) = |P(J_{ij}) + P(-J_{ij})|/|2 \cosh A_{ij}|, \quad (2.5)$$

$$A(J_{ij}) = -\frac{1}{2} \ln[P(J_{ij})/P(-J_{ij})], \quad (2.6)$$

where $\tau_{ij} = \pm 1$. Thus the integral is taken over the positive values of $J_{ij}$ only and the negative part of $P(J_{ij})$ is included through the sum over $\tau_{ij}$. The Nishimori line is defined as the line on which $A(J_{ij}) = \beta J_{ij}$, and hence $Z(\{ \beta^{-1} \tau A \}, \beta) = Z(\{ \tau J \}, \beta)$. Owing to this relation a number of averages can be performed exactly for bimodal and Gaussian distributions (and others) on the Nishimori line. In particular Nishimori calculated the internal energy in these two distributions exactly by considering the average of the free energy, $\overline{W}/Z$. The symmetry on the Nishimori line may also be expressed as a replica symmetry from which further strong constraints on the distribution functions of correlators follow.

Now, for the $q$th moment of a disorder-dependent observable such as the correlation function in Eq. (2.2) on
the Nishimori line one can write
\[ \langle O_x O_y \rangle^q = \text{Tr}_\tau \int_0^\infty \left( \frac{Z'}{Z} \right)^q Z \prod_{i,j} Q(J_{ij}) d\tau_{ij}. \]  
(2.7)

From this it is evident that there appear symmetries in the distribution of such correlation functions if there is a simple relation between \( Z \) and \( Z' \). In particular since any product of \( \mu \)-operators is represented by reversing the sign of a set of bonds, Eq. (2.3) is invariant under the change \( Z \rightarrow Z' \). This can be seen by noticing that taking the trace over \( \tau \) inside the integral takes positive and negative bond strengths with equal weight. In turn, the change \( Z \rightarrow Z' \) is equivalent to \( q \rightarrow 1 - q \) and hence the moments are symmetric with respect to \( q = 1/2 \); the result is \( \langle O_x O_y \rangle^q = 1 \) for \( q = 0, 1 \) readily follows from normalization of Eq. (2.4).

III. NISHIMORI POINT OF THE \( \pm J \)-RBIM

The result of the previous section holds for the general RBIM with a Nishimori line. Here we restrict ourselves to the \( \pm J \) model with the bimodal bond distribution \( P(J_{ij}) = p\delta(J_{ij} - 1) + (1 - p)\delta(J_{ij} + 1) \) for which the Nishimori line is given by \( \exp(-2\beta) = (1 - p)/p \). On this line, the model has a disorder-dominated multicritical point known as the Nishimori point. Its position has been estimated in the past and again, more accurately, recently. Two critical exponents are now known from numerical calculations.

We also focus on the Nishimori point here and we consider the \( q \)-th moments of the \( \mu \)-correlator with separation \( r \). We denote the averaged correlators by \( g(r; q) \). If they are critical with power law decay in the plane then
\[ g(r; q) = \langle \mu_0 \mu_r \rangle^q = A(r/a)^{-\eta(q)}, \]  
(3.1)

where \( A \) and \( a \) are constants and \( \eta(q) \) is a \( q \)-dependent scaling dimension. We would like to answer the following questions: Is this averaged correlator conformally invariant? If so, how are the \( \langle \mu_0 \mu_r \rangle \) distributed, or, equivalently, what is the functional dependence of \( \eta(q) \) on \( q \)?

As shown in the previous section \( g(r; q) = 1 \) for all \( r \) at \( q = 0 \) and \( q = 1 \). The scaling dimension \( \eta(q) \) of \( g \) has then two zeros in \( q \). Furthermore \( \eta(q) \) must be less than or equal to zero for \( q > 1 \) (and \( q < 0 \)) as shown by the following argument: We start from the inequality \( \overline{x^p} \geq (\overline{x})^p \) for any real random \( x \) satisfying \( 0 \leq x \leq \infty \) and any real \( p > 1 \). Applying this to \( x = \langle \mu_0 \mu_r \rangle \) one readily finds that \( \eta(p) \leq pq(1) \) and, since \( \eta(1) = 0 \), the above result follows.

Below we use numerical calculations to show that \( \eta(q) \) is, in fact, non-zero and negative for \( q > 1 \) and we determined its form in detail for a range of \( q \).

Conformal field theory predicts that the correlation functions in Eq. (3.1) decay on a cylinder with circumference \( M \) according to
\[ \langle \mu_0 \mu_r \rangle^q = [(M/a\pi) f(r/M)]^{-\eta(q)}, \]  
(3.2)

where
\[ f(r/M) = \sin(\pi x/M) \text{ or } \sinh(\pi y/M) \]  
(3.3)

and \( r \equiv (x, y) \) with \( x \) and \( y \) the coordinate separations around the circumference and along the cylinder, respectively. In addition one can also consider the logarithmic average (typical value) which is related to the spectrum in \( q \) through the derivative with respect to \( q \) at \( q = 0 \). If Eq. (3.2) holds then the typical value decays on the cylinder as
\[ \ln(\mu_0 \mu_r) = -\left( \frac{\partial \eta(q)}{\partial q} \bigg|_{q=0} \right) \ln[(M/a\pi) f(r/M)]. \]  
(3.4)

Using the mapping of the RBIM onto a network model of free fermions described in detail in Ref. [4] and exploiting the efficiency of the transfer-matrix algorithm in the network formalism we calculate the correlation functions in Eq. (3.2) and Eq. (3.4). The logarithmic average has small fluctuations compared to the direct averages and can be used as a sensitive test of whether the two-point correlators are actually conformally invariant.

Fig. (6) shows the dependence of the typical value of the \( \mu \)-correlator on cylinders with circumferences between \( M = 6 \) and \( M = 22 \) with separation along the cylinder \( x = 0 \), calculated using \( 10^4 \) samples. The statistical errors here are between 0.5 and 1.0 percent for all system sizes and hence the error bars in the plot appear smaller than the symbols. The data fall accurately onto a single straight line with slope \( k = -0.6911(17) \). The vertical dashed line is the point \( r = M \) where, roughly, the correlator crosses over from two-dimensional to quasi 1D behavior. The typical correlation function hence convincingly obeys the prediction from conformal invariance.

The direct averages in Eq. (3.2) have extremely large fluctuations and are dominated by rare events. Therefore, compared to computing the logarithmic average many more samples are needed. In the following we have typically used \( 10^6 \) disorder realizations. We calculate the average correlation functions on a cylinder now with the separation \( r \) running around the circumference \( y = 0 \) for \( q \)-values between \( q = 0 \) and \( q = 2.5 \). Four of the direct averages for \( q = 0.40, 0.60, 1.00, 1.28 \) and \( M = 22 \) are displayed in Fig. (7). Again, error bars are smaller than the symbol sizes but now there is a strong system size dependence: the statistical errors range from 0.5 percent for the smallest \( q \) to 4.0 percent for \( q = 1.28 \). The functional form dictated by Eq. (3.2) is, again, obeyed accurately and \( \eta(q) \) can be read off as the slope of the fitted lines. In Fig. (8) the slopes for various values of \( q \) are plotted versus \( q \). As expected the curve is symmetric with respect to \( q = 1/2 \) and has a zero at \( q = 1 \), crossing to negative values for \( q > 1 \). The fact that we reproduce \( \eta(1) = 0 \) accurately provides a strong check of our numerical calculations, since this is a property that emerges only after averaging over random bond realizations.

Our error margin for \( \eta(q) \) shown in Fig. (8) is less than three times the symbol sizes for \( q > 1 \) but smaller than
and it has been proposed and confirmed numerically for functions of fermions in a random gauge potential the inverse participation ratios of the ground state wave-functions at the integer quantum Hall transition.

Parabolic spectra stem, in two dimensions, from quantities whose logarithms have the distribution of a free massless field, and it can be shown that this would be compatible with the constraints on the Nishimori line. However, while for $q < 1.2$ the $q$-dependence of $\eta$ on $q$ seems indeed compatible with a parabola, at larger $q$ the curve $\eta(q)$ versus $q$ deviates significantly and an exactly parabolic spectrum at the Nishimori point can thus be excluded. This is shown in Fig. 1 where the dashed line is a parabola fitted to $\eta(q)$ between $q = 0$ and $q = 1$.

IV. CONCLUSIONS

We have demonstrated the presence of negative scaling dimensions at the multi-critical point in the $\pm J$ random-bond Ising model by considering the the Kramers-Wannier dual operator. Since the latter it is not bounded from above in the presence of antiferromagnetic bonds its correlation functions can increase with separation in this case which we have demonstrated numerically for the $\pm J$ random-bond Ising model in two dimensions. Furthermore at the disorder dominated multi-critical point in this model we find that the $q$th moments of the two-point correlator in the ensemble distribution behave according to the prediction from conformal invariance and that, for $q > 1$, the exponents $\eta(q)$ become negative. We have calculated the multiscaling spectrum $\eta(q)$ and we support our observations using symmetry properties of the Nishimori line on which the critical point is located and which forces $\eta(0) = \eta(1) = 0$. 

\[ \ln \langle \frac{\partial^2 f}{\partial q^2} \rangle + k \ln(\pi y/M) \]

\[ \ln \langle \frac{\partial f}{\partial q} \rangle \]

\[ \ln(\sin \pi y/M) \]
FIG. 3. The scaling dimension of the $q$-moment of the $\mu\mu$-correlator as a function of $q$. The dashed line shows the approximate parabolic spectrum valid for $0.2 < q < 1.2$. For larger $q$ the deviation is significant.

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