REMARKS ON FISHER INFORMATION

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Abstract. Some situations are discussed where subquantum oscillations in momentum arise in connection with Fisher information and the quantum potential.

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1. INTRODUCTION

In [6] we derived a formula for Fisher information (FI) based on the thermalization of Grössing [23] for the quantum potential. We want to embellish this here in various ways and indicate connections to other work. For Fisher information we refer to [3, 11, 12, 13] and for related work on thermodynamics and diffusion see e.g. [3, 7, 8, 19, 20, 21, 42].

2. BACKGROUND

We consider first a standard Schrödinger equation (SE) in the form

\[(2A) \ i\hbar \psi_t = -\frac{\hbar^2}{2m}\psi'' + V\psi \text{ where } ' \sim \partial_x \text{ or } \nabla_x \text{ so that for } \psi = R e^{iS/\hbar} \text{ and } P = R^2 = |\psi|^2 \text{ one has} \]

\[
(2.1) \quad (A) \ S_t + \frac{1}{2m}(S')^2 + V - \frac{\hbar^2}{2m} \frac{R''}{R} = 0; \quad (B) \ P_t + \frac{1}{m}(PS')' = 0
\]

For 1-D situations the quantum potential (QP) is written in the form

\[(2B) Q = -\frac{\hbar^2}{2m}(R''/R) = -\frac{\hbar^2}{2m}(\partial_x^2 \sqrt{\rho}/\sqrt{\rho}) \text{ for } \rho = mP. \quad \text{One} \]

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often sets \( p = (1/m)S' \) (momentum) and evidently \( \int R^2 dx = \int Pdx = 1 \). It is seen directly that

\[
(2.2) \quad \frac{\partial^2 \sqrt{\rho}}{\sqrt{\rho}} = \frac{1}{4} \left[ \frac{2\rho''}{\rho} - \left( \frac{\rho'}{\rho} \right)^2 \right]
\]

(where \( \rho \) could be replaced by \( P \)) and hence with \( P = 0 \) for \( |x| \geq x_0 \)

\[
(2.3) \quad \int P \frac{\partial^2 \sqrt{\rho}}{\sqrt{\rho}} dx = \frac{1}{4} \int \left( 2P'' - \frac{(P')^2}{P} \right) dx = -\frac{1}{4} \int \frac{(P')^2}{P} dx = -\frac{1}{4} \text{FI}
\]

(FI means Fisher information). Hence in particular, writing \( Q \) in terms of \( P \),

\[
(2.4) \quad \int PQ dx = \frac{\hbar^2}{8m} \int \frac{(P')^2}{P} dx = \frac{\hbar^2}{8m} \text{FI}
\]

Write now for differential entropy \((2C) \quad \mathfrak{S} = -\int \rho \log(\rho) dx \) and note that

\[
(2.5) \quad \mathfrak{S} = -\int mP \log(mP) dx = -m \int P \left( \log(m) + \log(P) \right) dx = -m \int P \log(P) dx - m\log(m)
\]

since \( \int P dx = 1 \). Consequently

\[
(2.6) \quad \partial_t \mathfrak{S} = -m \int P_t \log(P) + P_t \] dx = -m \int P_t \left( \log(P) + 1 \right) dx
\]

From (2.1) we have \( P_t = -(1/m)PS' \) so

\[
(2.7) \quad \partial_t \mathfrak{S} = -\int (PS')' \log(P) + 1] dx = -\int (PS')' \frac{P'}{P} dx = -\int S'P' dx
\]

**REMARK 2.1.** At this point one often refers to diffusion or hydrodynamical processes and thinks of \( S' = p = m\dot{q} \) where \( \dot{v} = \dot{q} \) represents some collective velocity field related to a free Brownian motion for example with say \( v = -(\hbar/2m)(\nabla P/P) \) involving an osmotic velocity (cf. \[3\] \[19\] \[20\]). Then (2.1B) becomes \((2D) \quad P_t + (Pv)' = 0 \) and it is interesting to note that this form of velocity involving \( \nabla P/P \) or \( \nabla \rho/\rho \) comes up in the form of momentum fluctuations or perturbations in work of Hall-Reginatto \[25\] \[26\] \[27\] \[58\] on the exact uncertainty principle and in Crowell \[7\] in a fluctuation context. Indeed in \[7\] one takes \((2E) \quad \delta \rho = (\hbar/2i)(\nabla \rho/\rho) \) with

\[
(2.8) \quad <\delta \rho >= \int \rho \delta \rho dx = \frac{\hbar}{2i} \int \nabla \rho dx = 0; \quad <(\delta \rho)^2 > = c \int \left[ \frac{(\nabla \rho)^2}{\rho} \right] dx \sim \text{FI}
\]
Thus Q can be written as

\[
Q = -\frac{\hbar^2}{4m} \left[ \frac{1}{2} (\delta p)^2 - \frac{\Delta \rho}{\rho} \right] = \frac{\hbar^2}{4m} \left[ \nabla (\delta p) + \frac{1}{2} (\delta p)^2 \right]
\]

The exact uncertainty principle involves the characterization of quantum fluctuations as being generated by momentum fluctuations \( \nabla P/P \) under certain quite general assumptions. This principle can be applied in many interesting circumstances (cf. [3, 4, 25, 26, 27, 58]).

**REMARK 2.2.** We note also, going back to (2.7) with \( P' \sim \nabla P \), when (as in certain diffusion processes) \( S' = -(\hbar/2m)(P'/P) \), then (2F) \( \partial_t \mathcal{S} = (\hbar/2m) \int [P'/P] dx \sim \hat{c}FI \) showing that \( \partial_t \mathcal{S} \geq 0 \) exactly how it varies. This feature also arises in studying Ricci flow and information on this is provided in [5, 32, 33, 40, 48, 59]. Let \( P(x) \) be a probability distribution on \( \mathbb{R} \) or \( \mathbb{R}^3 \); we work in \( \mathbb{R} \) for simplicity. Consider a differential entropy \( S = -\int P \log(P) dx \) and assume one wants to specify some value \( \bar{A} = \int A(x) P(x) dx \). Then consider extremizing

\[
\tilde{S} = -\int P \log(P) dx + \lambda (1 - \int P dx) + \alpha (\bar{A} - \int AP dx)
\]

Recall \( \log(1 + x) \sim x \) for \( x \) small and write

\[
\delta \tilde{S} = -\int [(P + \delta P) \log(P + \delta P) - P \log(P)] dx - \lambda \int \delta P dx - \alpha \int A \delta P dx =
\]

\[
= -\int \left\{ (P + \delta P) \log \left[ P + \frac{\delta P}{P} \right] - P \log(P) \right\} dx - \int (\lambda + \alpha A) \delta P dx =
\]

\[
\approx -\int \delta P(x) \left[ \log(P) + 1 + \lambda + \alpha A \right] dx
\]

Since \( \delta P \) is arbitrary one obtains (2G) \( \log(P) + 1 + \lambda + \alpha A = 0 \) or \( P = \exp[-\lambda - \alpha A] = (1/Z) \exp(-\alpha A) \) where \( Z = \exp(1+\lambda) \) and \( \int P dx = 1 \Rightarrow Z = \int \exp[-\alpha A(x)] dx \). Here a suitable candidate for \( A \) would be the total energy of a system and if \( S \) is determined by \( P \) alone with \( P \sim p = (1/m)S' \) as above then \( A \) would have to be the fluctuation energy determined by the quantum potential.

3. **SUBQUANTUM THERMODYNAMICS**

We go here to [23] for a derivation of the SE from vacuum fluctuations and diffusion waves in sub-quantum thermodynamics (cf. also [1, 2, 18, 19, 20, 24, 34, 60] for background). Grössing work in [23] specifies the energy in quantum mechanics arising from sub-quantum fluctuations via nonequilibrium thermodynamics. The ideas are motivated and discussed at great length in e.g. papers 1, 2, and 4 in [23] and we only summarize here following [23]-6. To each particle of nature is attributed an energy \( E = \hbar \omega \)
for some kind of angular frequency $\omega$ and one can generally assume that “particles” are actually dissipative systems maintained in a nonequilibrium steady-state by a permanent source of kinetic energy, or heat flow, which is not identical with the kinetic energy of the particle, but an additional contribution. Thus it is assumed here that (3A) $E_{\text{tot}} = \hbar \omega + [(\delta p)^2 / 2m]$ where $\delta p$ is the additional fluctuating momentum component of the particle of mass $m$. Similarly the particle’s environment is considered to provide detection probability distributions which can be modeled by wave-like intensity distributions $I(x,t) = R^2(x,t)$ with $R$ being the wave’s real valued amplitude; thus one assumes (3B) $P(x,t) = R^2(x,t)$ with $\int P d^n x = 1$ (note $x \sim x$). In [23]-1 it was proposed to merge some results of nonequilibrium thermodynamics with classical wave mechanics in such a manner that the many microscopic degrees of freedom associated with the hypothesized sub-quantum medium can be recast into the more “macroscopic” properties that characterize the wave-like behavior on the quantum level. Thus one considers a particle as being surrounded by a large “heat bath” so that the momentum distribution in this region is given by the usual Maxwell-Boltzmann distribution. This corresponds to a “thermostatic” regulation” of the reservoir’s temperature which is equivalent to saying that the energy lost to the thermostat can be regarded as heat. This leads to emergence at the equilibrium-type probability density ratio (3C) $[P(x,t)/P(x,0)] = \exp[-(\Delta Q)/kT]$ where $T$ is the reservoir temperature and $\Delta Q$ the exchanged heat between the particle and its environment. The conditions (3A)-(3C) are sufficient to derive the SE. Thus first, via Boltzmann, the relation between heat and action is given via an action function $S = \int (E_{\text{kin}} - V) dt$ with $\delta S = \delta \int E_{\text{kin}} dt$ via

\begin{equation}
\Delta Q = 2\omega \delta S = 2\omega [\delta(S)(t) - \delta S(0)]
\end{equation}

(cf. [23] for more details). Next the kinetic energy of the thermostat is $kT/2$ per degree of freedom and the average kinetic energy of an oscillator is $(1/2)\hbar \omega$ so equality of average kinetic energies demands (3D) $kT/2 = \hbar \omega / 2$ or $\hbar \omega = kT = 1/\beta$. Combining (3C), (3.1), and (3D) yields then

\begin{equation}
P(x,t) = P(x,0)e^{-\frac{\hbar}{2}[\delta S(x,t) - \delta S(x,0)]}
\end{equation}

leading to a momentum fluctuation

\begin{equation}
\delta p(x,t) = \nabla(\delta S(x,t)) = -\frac{\hbar \nabla P(x,t)}{2P(x,t)}
\end{equation}

and an additional kinetic energy term

\begin{equation}
\delta E_{\text{kin}} = \frac{1}{2m} \nabla(\delta S) \cdot \nabla(\delta S) = \frac{1}{2m} \left( \frac{\hbar \nabla P}{2P} \right)^2
\end{equation}
The action integral then becomes

\[ A = \int L d^n x dt = \int P(x,t) \left[ \partial_t S + \frac{1}{2m} \nabla S \cdot \nabla S + \frac{1}{2m} \left( \frac{\hbar \nabla P}{P} \right)^2 + V \right] \]

We emphasize here that

\[ (\star \star) \int P(\nabla S \cdot \delta p) d^n x = \int P(\nabla S \cdot \nabla (\delta S)) d^n x = 0 \]

(i.e. the fluctuations terms \( \delta p \) are uncorrelated with the momentum \( p \sim \nabla S \)). Now one uses the Madelung form (3E) \( \psi = R \exp((i/\hbar)S) \) where \( R = \sqrt{P} \) to obtain (3F) \[ \nabla \psi/\psi^2 = [\nabla P/2P]^2 + [\nabla S/\hbar]^2 \] leading to (3.5) in the form

\[ A = \int L dt = \int d^n x dt \left[ |\psi|^2 (\partial_t S + V) + \frac{\hbar^2}{2m} |\nabla \psi|^2 \right] \]

(cf. [?] for \( |\psi|^2 \sim P \)). Then via \( |\psi|^2 \partial_t S = -(ih/2)(\psi^* \dot{\psi} - \dot{\psi}^* \psi) \) one has (3G) \[ L = -(ih/2)(\psi^* \dot{\psi} - \dot{\psi}^* \psi) + (\hbar^2/2m)\nabla \psi \cdot \nabla \psi^* + V \psi \dot{\psi}^* \] leading to the SE (3H) \[ i\hbar \partial_t \psi = -(h^2/2m)\nabla^2 + V] \psi \] along with the “modified” Hamilton-Jacobi equation

\[ \partial_t S + \frac{1}{2m} (\nabla S)^2 + V + Q = 0; \quad Q = -\frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla P}{P} \right)^2 - \frac{\Delta P}{P} \right] = -\frac{\hbar^2}{2m} \frac{\Delta R}{R} \]

Then define (3I) \( u = (\delta p/m) = -(h/2m)(\nabla P/P) \) and \( k_u = -(1/2)(\nabla P/P) = -(\nabla R/R) \) so that \( Q \) can be rewritten as

\[ Q = \frac{mu \cdot u}{2} - \frac{h}{2}(\nabla \cdot u) = \frac{\hbar^2}{2m}(k_u \cdot k_u - \nabla \cdot k_u) \]

Using (3.1) and (3.2) one can also write (3J) \( u = (1/2\omega m)\nabla Q \).

Generally a steady state oscillator in nonequilibrium thermodynamics corresponds to a kinetic energy at the sub-quantum level providing the necessary energy to maintain a constant oscillation frequency \( \omega \) and some excess kinetic energy resulting in a fluctuating momentum contribution \( \delta p \) to the momentum \( p \) of the particle (note \( p \sim p \)). Similarly a steady state resonator representing a “particle” in a thermodynamic environment will not only receive kinetic energy from it but in order to balance the stochastic influence of the buffeting momentum fluctuations it will also dissipate heat into the environment. There is a vacuum fluctuation theorem (VFT) from [23]-1 which proposes that larger energy fluctuations of the oscillating system correspond to higher probability of heat dissipated into the environment (rather that absorbed). The corresponding balancing velocity
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is called (after Einstein) the “osmotic” velocity. Thus recalling the stochastic “forward” movement \( u \sim (\delta p/m) \), the current \( \mathbf{J} = P \mathbf{u} \) has to be balanced by \( -\mathbf{u} \), i.e. \( \mathbf{J} = -P \mathbf{u} \). Putting (3I) into the definition of the “forward” diffusive current \( \mathbf{J} \) and recalling the diffusivity \( D = \hbar/2m \) one has (3K) \( \mathbf{J} = P \mathbf{u} = -D \nabla P \) and when combined with the continuity equation \( \dot{P} = -\nabla \cdot \mathbf{J} \) this gives (3L) \( \partial_t P = D \nabla^2 P \). Here (3K) and (3L) are the first and second of the Fick laws of diffusion and \( \mathbf{J} \) is called the diffusion current.

Returning now to (3J) one defines \( \Delta Q = Q(t) - Q(0) < 0 \) and maintaining heat flow as positive one writes \( -\Delta Q \) for heat dissipation and puts this in (3J) to get the osmotic velocity (3M) \( \bar{\mathbf{u}} = -\mathbf{u} = D(\nabla P/P) = -(1/2\omega m)\nabla Q \) with osmotic current (3N) \( \bar{\mathbf{J}} = P \bar{\mathbf{u}} = D \nabla P = -(P/2\omega m)\nabla Q \).

As a corollary to Fick’s second law one has then

\[
\partial_t P = -\nabla \cdot \bar{\mathbf{J}} = -D \nabla^2 P = \frac{1}{2\omega m} [\nabla P \cdot \nabla Q + P \nabla^2 Q]
\]

Next one looks for a thermodynamic meaning for the quantum potential \( Q \). Take first \( Q = 0 \) and look at the osmotic velocity (3M) \( \bar{\mathbf{u}} = -\mathbf{u} = D(\nabla P/P) = -(1/2\omega m)\nabla Q \) with osmotic current (3N) \( \bar{\mathbf{J}} = P \bar{\mathbf{u}} = D \nabla P = -(P/2\omega m)\nabla Q \). As a corollary to Fick’s second law one has then

\[
(3.9) \quad \partial_t P = -\nabla \cdot \bar{\mathbf{J}} = -D \nabla^2 P = \frac{1}{2\omega m} [\nabla P \cdot \nabla Q + P \nabla^2 Q]
\]

Next one looks for a thermodynamic meaning for the quantum potential \( Q \). Take first \( Q = 0 \) and look at the osmotic velocity (3M) \( \bar{\mathbf{u}} = -\mathbf{u} = D(\nabla P/P) = -(1/2\omega m)\nabla Q \) with osmotic current (3N) \( \bar{\mathbf{J}} = P \bar{\mathbf{u}} = D \nabla P = -(P/2\omega m)\nabla Q \). As a corollary to Fick’s second law one has then

\[
(3.10) \quad \partial_t P = \frac{P}{2\omega m} \left[ \nabla^2 Q - \frac{Q^2}{\hbar \omega} \right] \Rightarrow \partial_t P = -\frac{P}{2\omega m} \nabla^2 Q
\]

(via (3P)). Now from (3Q) one has also (3R) \( \partial_t P = -(P/\hbar \omega) \partial_t Q \) so comparison of (3.10) and (3Q) yields (3S) \( \nabla^2 Q = (1/D) \partial_t Q = 0 \). This is nothing but a classical heat equation obtained by the requirement that the quantum potential \( Q = 0 \); it shows that even for free particles both in the quantum and classical case one can identify a heat dissipation process emanating from the particle. A non-vanishing quantum potential then is a means of describing the spatial and temporal dependencies of the corresponding thermal flow in the case that the particle is not free.

Various particular solutions to (3S) are indicated for the case \( Q = 0 \) as well as when \( Q \neq 0 \). Examples are discussed with a view toward resolving a certain “particle in a box” problem of Einstein. In [23]-1 one concentrated on the momentum fluctuations \( \delta p \) generated from the environment to the particle while in [23]-3 one develops the idea of excess energy developed as heat from the particle to its environment (which is described via the
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quantum potential). In fact one can rewrite (3.7) as

$$\partial_t S + \frac{1}{2m} (\nabla S)^2 + V + \frac{\hbar^2}{4m} \left[ \nabla^2 \tilde{Q} - \frac{1}{D} \partial_t \tilde{Q} \right] = 0$$

where $\tilde{Q} = Q/\hbar \omega$. This gives wonderful insight into the nature and role of the quantum potential.

Note that one has achieved a “thermalization” of the quantum potential (QP) in the form

$$Q = \frac{\hbar^2}{4m} \left[ \nabla^2 \tilde{Q} - \frac{1}{D} \partial_t \tilde{Q} \right]$$

where $\tilde{Q} = Q/\hbar \omega = \alpha Q$ is an expression of heat and $D = \hbar/2m$ is a diffusion coefficient (note $\alpha = \beta$ as in (3D)). In [6] we show that, as a corollary, one can produce a related thermalization of Fisher information (FI) which should have interesting consequences. Thus in (3N) one uses a formula

$$\nabla P = -\frac{1}{2\omega m} \nabla Q = -\frac{1}{\omega \hbar} \nabla Q = -\beta \nabla Q$$

and this leads one to think of $\nabla \log(P) = -\alpha \nabla Q = -\nabla (\alpha Q)$ with a possible solution

$$\log(P) = -\alpha Q + c(t) \Rightarrow P = e^{\alpha Q + c(t)} = \tilde{c}(t)e^{-\alpha Q}$$

(note also (3T) $Q(t) \sim 2\omega \delta S(t)$ and $2/\hbar \delta S(t) \sim (1/kT)Q(t) = \beta Q(t)$).

Now Fisher information (FI) is defined via ($dx \sim dx^3$ for example)

$$F = \int \frac{(\nabla P)^2}{P} dx = \int P \left( \nabla P \right)^2 dx$$

and one can write $Q$ as in (3.7). Consequently (since $\int \Delta P dx = 0$)

$$\int PQ dx = -\frac{\hbar^2}{8m} \int \frac{(\nabla P)^2}{P} dx = -\frac{\hbar^2}{8m} F$$

Then as in [6] one can write formally, first using (3.12) and $\alpha = 1/\omega$ and secondly, using (3.14) and (3.13) (recall also $\alpha = \beta$)

$$F = \int P \left( \nabla P \right)^2 dx = \beta^2 \tilde{c}(t) \int e^{-\beta Q} (\nabla Q)^2 dx$$
In view of the thermal aspects of gravity theories now prevalent it may perhaps be suggested that connections of quantum mechanics to gravity may best be handled thermally. There may also be connections here to the emergent quantum mechanics of [9, 29].

4. REMARKS ON FLUCTUATIONS

We go here to [7] which contains a rich lode of important material on quantum fluctuations and some penetrating insight into physics (but some proofreading seems indicated). We will try to rewrite some of this in a more complete manner. Thus consider a SE for \( \psi = R \exp(iS/\hbar) \) with momentum operator \( \hat{p} \) so that

\[
\hat{p}\psi = p\psi = \left( \nabla S + \frac{\hbar}{i} \nabla R \right) \psi \Rightarrow p \sim <p> + \delta p
\]

which identifies \((\hbar/i)(\nabla R/R)\) as a fluctuation \(\delta p\). Recall now that the SE is (4A) \(i\hbar \psi_t = -(\hbar^2/2m)\Delta \psi + V\psi\) and setting \(\rho = R^2 = \psi^*\psi\) one has

\[
\partial_t \rho = \frac{i\hbar}{2m} [\psi^* \Delta \psi - (\Delta \psi^*)\psi] = \frac{i\hbar}{2m} \nabla \cdot (\psi^* \nabla \psi - \nabla (\psi^*)\psi)
\]

Now in polar form \(\psi = R \exp(iS/\hbar), \rho = R^2\) this becomes (calculating in 1-D for simplicity)

\[
(Re^{iS/\hbar})' = R'e + \frac{RiS'}{\hbar} = \nabla (Re^{iS/\hbar}) = \nabla Re^{iS/\hbar} + \frac{iR\nabla S}{\hbar}e^{iS/\hbar}
\]

\[
\psi^* \nabla \psi - (\nabla \psi^*)\psi = R \nabla R + \frac{iR^2 \nabla S}{\hbar} = \frac{2i\rho \nabla S}{\hbar} \Rightarrow \partial_t \rho = \frac{i\hbar}{2m} \nabla \left( \frac{2i\rho \nabla S}{\hbar} \right) = -\frac{1}{m} \nabla (\rho \nabla S)
\]

This agrees with (2.8) in [7], but (2.7), (2.9), and (2.10) in [7] are somewhat “strange”. Now the quantum potential is

\[
Q = -\frac{\hbar^2}{2m} \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = -\frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \nabla \rho \right)^2 - \frac{\Delta \rho}{\rho} \right] = \frac{\hbar^2}{4m} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \left( \nabla \rho / \rho \right)^2 \right]
\]

(cf. (2B) and \(\rho = R^2 \) so \(\rho' = 2RR'\) and \(2(R'/R) = \rho'/\rho\); hence in (4.1) one has (4B) \(\delta p \sim (\hbar/2i) (\nabla \rho / \rho)\). There is equivalent material about Fokker-Planck equations in [3] so we omit the discussion in [7]. One notes that (4C) \(<\delta p > \sim \int \rho \delta p dx = (\hbar/2i) \int \nabla p dx = 0 \) whereas \(<(\delta p)^2 > \sim c \int [(\nabla \rho)^2/\rho] dx \sim Fisher information\). Here the quantum potential can also be written as

\[
Q = -\frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \delta p \right)^2 - \frac{\Delta \rho}{\rho} \right] = \frac{\hbar^2}{4m} \left[ \nabla (\delta p) + \frac{1}{2} \left( \delta p \right)^2 \right]
\]
since \((\rho'/\rho)' = (\rho''/\rho) - [(\rho')^2/\rho^2]\) \(\sim \Delta \rho/\rho = \nabla (\delta p) + (\delta p)^2\). Note also that the exact uncertainty principle of Hall-Reginatto (25, 26), developed at length in [3], is based on momentum fluctuations \(p = \nabla + \delta p\) with \(\langle \delta p \rangle = 0\). If one writes \(V(q) = V(<q>) + \nabla_q V(<q>)/\delta q\) as a function of position and recalls that the quantum HJ equation has the form \(4D\) \(S_t + (p^2/2m) + V + Q = 0\) (cf. [3]) then this can be rewritten in terms of fluctuations as

\[
S_t + \frac{1}{2m} <p^2> + V(<q>) + \nabla_q V(<q>)\delta q + Q(p, \delta p);
\]

\[Q = \frac{\hbar^2}{4m} \left(-i\hbar \nabla \cdot \delta p + \frac{1}{2}(\delta p)^2\right)\]

5. EXAMPLES

We go now to [16] with (5A) \(\int dx p(x) = 1\) and (5B) \(I[p] = \int dx F_I(p)\) where \(F_I(p) = p(x)(p')/p^2\). Assume that there are known

\[
\langle A_j \rangle = \int dx A_j(x)p(x) \quad (j = 1, \cdots, M)
\]

Then one uses the principle of extreme physical information (EPI) to find the probability distribution \(p = p_I\) extremizing \(I[p]\) subject to prior conditions \(\langle A_j \rangle\). Jaynes used the Shannon functional \(F = -\int dx \log(p)\) but here one uses the Fisher extremization with

\[
\delta_p \left[I[p] - \alpha < 1 > - \sum \lambda_i < A_i >\right] = 0 \equiv
\]

\[
\equiv \delta_p \left[\int dx \left(F_I[p] - \alpha p - \sum \lambda_i A_i p\right)\right] = 0,
\]

Variation leads to

\[
\delta \int \frac{p'^2}{p^2} dx \approx \int \left[-\frac{p'^2}{p^2} \delta p + \frac{2p'}{p} \delta p'\right] dx \sim \int \left[-\frac{p'^2}{p^2} - \partial_x \left(\frac{2p'}{p}\right)\right] \delta p dx
\]

\[
\int dx \delta p \left[(p)^{-2}(p')^2 + \partial_x \left(\frac{2p'}{p}\right) + \alpha + \sum \lambda_i A_i\right] = 0
\]

which implies, via the arbitrary nature of \(\delta p\)

\[
\left[(p)^{-2}(p')^2 + \partial_x \left(\frac{2p'}{p}\right) + \alpha + \sum \lambda_i A_i\right] = 0
\]

The normalization condition on \(p\) makes \(\alpha\) a function of the \(\lambda_i\) and one lets \(p_I(x, \lambda_i)\) be a solution of (5.4). Then the extreme Fisher information
is (5D) \( I = \int dx p_I^{-1}[p_I']^2 \). Now one can simplify (5.4) via (5E) \( G(x) = +\alpha + \sum_1^M \lambda_i A_i(x) \) and write (5.4) as

\[ (5.5) \quad \left[ \partial_x \log(p_I) \right]^2 + 2 \frac{\partial^2 \log(p_I)}{\partial x^2} + G(x) = 0 \]

Then introduce \( p_I = \psi^2 \) and (5F) \( v(x) = \partial_x \log(\psi(x)) \) so that (5.5) becomes (5G) \( v'(x) = -[(1/4)G(x) + v^2(x)] \) which is a Riccati equation. Setting

\[ (5.6) \quad u(x) = \exp \left[ \int_x^x dx \frac{d\log(\psi)}{dx} \right] = \psi \]

makes (5.5) into a Schrödinger like equation

\[ (5.7) \quad - \frac{1}{2} \psi''(x) - \frac{1}{8} \sum \lambda_i A_i(x) \psi(x) = \frac{\alpha}{8} \psi \]

where (5H) \( U(x) = (1/8) \sum_1^M \lambda_i A_i(x) \) is an effective potential (cf. [14, 15, 16, 17, 41]). Note that \( \psi \) is defined here completely via \( p_I \) so any quantum motion is automatically generated by fluctuation energy (i.e. \( S \sim \delta S \)).

Consider now a situation with one function \( A_i \) (5I) \( \bar{A} = \int pA dx \). Then from (5.2)-(5.5)

\[ (5.8) \quad p^{-2}(p')^2 + \partial_x \left( \frac{2p'}{p} \right) + \alpha + \lambda A = 0 \]

Now, following [16], one translates the Legendre structure of thermodynamics into a Fisher context. Thus from (5D), integrating by parts implies

\[ (5.9) \quad \frac{\partial I}{\partial \lambda} = \int dx \frac{\partial p_I}{\partial \lambda} \left[ -p_I^{-2} \left( \frac{\partial p_I}{\partial x} \right)^2 - \frac{\partial}{\partial x} \left( \frac{2}{p_I} \frac{\partial p_I}{\partial x} \right) \right] \]

where \( p_I \) is a solution of (5.8). Comparing (5.8) to (5.9) one has

\[ (5.10) \quad \frac{\partial I}{\partial \lambda} = \int dx \frac{\partial p_I}{\partial \lambda} [\alpha + \lambda A] \]

Then on account of normalization (\( \int p_I dx = 1 \))

\[ (5.11) \quad \frac{\partial I}{\partial \lambda} = \lambda \frac{\partial}{\partial \lambda} \int dx p_I A(x) \equiv \frac{\partial I}{\partial \lambda} = \lambda \frac{\partial}{\partial \lambda} <A> \]

which is a generalized Fisher-Euler theorem. Here the term \( \int dx \alpha \partial_x p_I \sim \alpha \partial_x \int dx p_I = 0 \) via \( \int p_I dx = 1 \). The thermodynamic counterpart of (5.11) is the derivative of the entropy with respect to mean values. Thus \( I = I(\lambda), \ p_I = p_I(\lambda), \) and via normalization \( \alpha = \alpha(\lambda) \). Thus \( \lambda \) and \( <A> \) play reciprocal roles within thermodynamics and one introduces a generalized thermodynamic potential as a Legendre transform of I, namely

\[ (5.12) \quad \Lambda = I(<A>) - \lambda <A> \]
Then, using (5.11)
\[
\frac{\partial \Lambda}{\partial \lambda} = \frac{\partial I}{\partial <A>} - \lambda \frac{\partial <A>}{\partial \lambda} - <A> = -<A>
\]
and one has a summary collection of formulas
\[
\Lambda = I - \lambda <A>;
\frac{\partial \Lambda}{\partial \lambda} = -<A>;
\frac{\partial I}{\partial <A>} = \lambda;
\frac{\partial \lambda}{\partial <A>} = \frac{\partial^2 I}{\partial <A>^2};
\frac{\partial <A>}{\partial \lambda} = -\frac{\partial^2 \Lambda}{\partial \lambda^2}
\]
and we recall (5.11) in the form (5J) \( \frac{\partial I}{\partial \lambda} = \lambda \frac{\partial <A>}{\partial \lambda} \). Thus the Legendre transform structure of thermodynamics has been translated into the Fisher context (see here also Chap. 4 of [13] for more on this and for additional general information we cite e.g. [10, 11, 12, 35, 36, 37, 38, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 54, 55, 56, 57]).

On the other hand with \( \bar{A} \) the sole constraint consider
\[
\tilde{H} = FI + \alpha(\bar{A} - \int pAdx)
\]
Then one finds directly
\[
\delta \tilde{H} = \int dx \delta p \left[ \left( \frac{p'}{p^2} \right)^2 + \partial_x \left( \frac{2p'}{p} \right) + \alpha A \right]
\]
This means (as in (5.8))
\[
\frac{(p')^2}{p^2} + \partial_x \left( \frac{2p'}{p} \right) + \alpha A = 0
\]
Now we can write \( \partial_x (2p'/p) = (2p''/p) - [2(p'/p)^2/p^2] \) which means, via (4.5), that
\[
2 \frac{\Delta p^{1/2}}{p^{1/2}} + \alpha A = 0
\]
and via (2B) this means that the extreme probability \( p_I \) directly determines a quantum potential \( Q \) via
\[
Q = -\frac{\hbar^2}{2m} \frac{\Delta p_I^{1/2}}{p_I^{1/2}} \Rightarrow Q = \frac{\hbar^2}{4m} (\alpha A)
\]
However, although \( \bar{A} \) has not been specified we seem to have a result that a constraint \( \bar{A} \) for which the Fisher thermodynamic procedure works with \( I[p] \) as defined, requires \( A \) to satisfy (5.19). This is in fact tautological since we are dealing with a situation where fluctuations based on \( p_I \) are the only source of energy and one will have (5K) \( \tilde{F} = \int PQdx = \)
\[ \int p_t Q \, dx = \frac{\hbar^2}{4m} \int dx \frac{\alpha A}{\alpha A} = \frac{\hbar^2}{4m} [\alpha \dot{A}] \]

Note that in general Fisher information \( \tilde{F} = I[p] \) as in (5C) is an action term which can be added to a classical Hamiltonian (as in Section 2) in order to quantize it and thus \( \tilde{F} \) is a “natural” constraint ingredient. Fixing \( \dot{A} \) would mean fixing the contribution of the quantum potential (QP) \( Q \) or fixing the amount of quantization allowed. In some way this would also correspond to restraining the probability in order to achieve a fixed amount of quantization.

**REMARK 5.1.** Suppose we extremize \( \tilde{S} \) as in (2.10) with \( A \sim E = \text{total energy} \) to arrive at a probability (5L) \( P = (1/Z) \exp(-\gamma E) \) where \( E \sim E(x,t), \) \( Z = \int \exp(-\gamma E) \, dx. \) Then compute the Fisher information for this \( P \) which will be based upon (cf. (2.2))

\[ P' = \frac{1}{Z} (-\gamma E') e^{-\gamma E}; \quad P'' = -\frac{\gamma}{Z} [E'' - \gamma (E')^2] e^{-\gamma E} \]

\[ \frac{P'}{P} = -\gamma E'; \quad \frac{P''}{P} = -\gamma E'' + \gamma^2 (E')^2 \]

\[ Q = \frac{\gamma \hbar^2}{8m} [\gamma (E')^2 - 2E''] \]

\[ FI = \frac{8m}{\hbar^2} \int P Q \, dx = \frac{\gamma^2}{Z} \int (E')^2 e^{-\gamma E} \, dx \]

On the other hand extremizing \( FI \) as in (5.15), with fluctuations as the only source of energy, leads to \( P = p_t \) defined via \( A \) as in (5.18) which yields in particular \( Q = (\hbar^2/4m)[\alpha A] \) as in (5.19). Now recall for \( P \) as in Section 3

\[ \frac{P(x,t)}{P(x,0)} = e^{\frac{\alpha}{\hbar^2}} \text{ from (3C)} \]

\[ \Delta Q = Q(t) - Q(0) = 2\omega[(\delta S)(t) - (\delta S)(0)] \text{ from (3.1) and (2/\hbar)\delta S(t) \sim \beta Q(t) \text{ (cf. (3T))}} \]

\[ \delta p = \nabla (\delta S(x,t)) = -(\hbar/2)(\nabla P/P) \text{ as in (3.3) and from (3.13)} \]

\[ (\nabla P/P) = (1/\omega \hbar) \nabla Q = \beta \nabla Q \]

\[ \delta E_{\text{kin}} = (\hbar^2/8m)(\nabla P/P)^2 = (1/8m \omega^2)(\nabla Q)^2 \]

\[ P = \hat{c}(t) \exp(-\alpha Q) \text{ from (3.14) with } \alpha = 1/\omega \hbar \sim 1/kT \sim \beta \text{ this can also be seen from (3.2) as } P \propto \exp[-\beta Q(t)] \]

For \( E = \text{total energy} \) as in (5L) it is clear that to apply this here we must think of \( E = \hbar \omega + \delta E_{\text{kin}} \) and \( \hbar \omega \) will only enter as a constant. The thermalization fluctuation energy for probabilities appears as \( \hat{c} \exp[-\delta S(x,t)] \sim \hat{c} \exp[-\beta Q(t)] \) as indicated so \( \delta E_{\text{kin}} \) is expressed in terms of \( Q. \) This means
that (5.24) involving \((E')^2\) (or \((\nabla E)^2\)) for FI corresponds to the \((\nabla Q)^2\) formula of (3.18). In other words the fluctuation energy is equivalent to the the thermal energy (modulo constants); see here also [3, 22, 28].

We omit here mention of many related topics and work involving Fisher information, thermodynamics, and entropy based on results of Abe, Frieden, Garbaczewski, Gellman, Hall, Kaniadakis, Naudts, Pennini, A. Plastino, A.R. Plastino, Reginatto, Soffer, and Tsallis in particular; some references can be found in [3, 8, 11, 12, 13, 21, 42].
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