IN Variant Forms and Automorphisms of Locally Homogeneous Multisymplectic Manifolds

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Abstract

It is shown that the geometry of locally homogeneous multisymplectic manifolds (that is, smooth manifolds equipped with a closed nondegenerate form of degree > 1, which is locally homogeneous of degree k with respect to a local Euler field) is characterized by their automorphisms. Thus, locally homogeneous multisymplectic manifolds extend the family of classical geometries possessing a similar property: symplectic, volume and contact. The proof of the first result relies on the characterization of invariant differential forms with respect to the graded Lie algebra of infinitesimal automorphisms, and on the study of the local properties of Hamiltonian vector fields on locally multisymplectic manifolds. In particular it is proved that the group of multisymplectic diffeomorphisms acts (strongly locally) transitively on the manifold. It is also shown that the graded Lie algebra of infinitesimal automorphisms of a locally homogeneous multisymplectic manifold characterizes their multisymplectic diffeomorphisms.

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1 Introduction

It is well-known that some classical geometrical structures are determined by their automorphism groups; for instance it was shown by Banyaga [3, 4, 5] that the geometric structures defined by a volume or a symplectic form on a differentiable manifold are determined by their automorphism groups, the groups of volume-preserving and symplectic diffeomorphisms respectively, i.e., if \((M_i, \alpha_i), i = 1, 2\) are two paracompact connected smooth manifolds equipped with volume or symplectic forms \(\alpha_i\) and \(G(M_i, \alpha_i)\) denotes the group of volume preserving or symplectic diffeomorphisms, then if \(\Phi: G(M_1, \alpha_1) \to G(M_2, \alpha_2)\) is a group isomorphism, there exists (modulo an additional condition in the symplectic case) a unique \(C^\infty\)-diffeomorphism \(\varphi: M_1 \to M_2\) such that \(\Phi(f) = \varphi \circ f \circ \varphi^{-1}\), for every \(f \in G(M_1, \alpha_1)\) and \(\varphi^*\alpha_2 = c\alpha_1\), with \(c\) a constant. In other words, group isomorphisms of automorphism groups of classical structures (symplectic, volume) are inner, in the sense that they correspond to conjugation by (conformal) diffeomorphisms.

An immediate consequence of the previous theorem is that, if \((M, \alpha)\) is a manifold with a classical structure (volume or symplectic), and a differential form \(\beta\) is an invariant for the group \(G(M, \alpha)\) of volume preserving or symplectic diffeomorphisms, then necessarily \(\beta\) has to be a constant multiple of exterior powers of \(\alpha\). In other words, the only differential invariants of the groups of classical diffeomorphism are multiples of exterior powers of the defining geometrical structure. The infinitesimal counterpart of this result was already known in the realm of Classical Mechanics. In 1947 Lee Hwa Chung stated a theorem concerning the uniqueness of invariant integral forms (the Poincaré-Cartan integral invariants) under canonical transformations [22]. His aim was to use that result in order to characterize canonical transformations in the Hamiltonian formalism of Mechanics; that is, canonical transformations are characterized as those transformations mapping every Hamiltonian system into another Hamiltonian one with respect to the same symplectic structure. Afterwards, this result was discussed geometrically [28] and generalized to presymplectic Hamiltonian systems [12, 18]. The main result there was that in a given a symplectic (resp. presymplectic) manifold, the only differential forms invariant with respect to all Hamiltonian vector fields are multiples of (exterior powers of) the symplectic (resp. presymplectic) form. Since symplectic and presymplectic manifolds represent the phase space of regular and singular Hamiltonian systems, respectively, this result allows one to identify canonical transformations in the Hamiltonian formalism of Mechanics with the symplectomorphisms and presymplectomorphisms group, in each case.

Returning to the general problem of the relation between geometric structures and their group of automorphisms, it is an open question to determine which geometrical structures are characterized by these groups. Apart from symplectic and volume, contact structures also fall into this class [6]. Grabowski proved that similar statements hold for Jacobi and Poisson manifolds [11, 20]. Our main result shows that locally homogeneous multisymplectic manifolds are determined by their automorphisms (finite and infinitesimal).

Multisymplectic manifolds are one of the natural generalizations of symplectic manifolds. A multisymplectic manifold of degree \(k\) is a smooth manifold \(M\) equipped with a closed nondegenerate form \(\Omega\) of degree \(k \geq 2\) (see [9, 10] for more details on multisymplectic manifolds). In particular, multisymplectic manifolds include symplectic and volume manifolds. A diffeomorphism \(\varphi\) between two multisymplectic manifolds \((M_i, \Omega_i), i = 1, 2\), will be called a multisymplectic diffeomorphism if \(\varphi^*\Omega_2 = \Omega_1\). The group of multisymplectic diffeomorphisms of a multisymplectic manifold \((M, \Omega)\) will be denoted by \(G(M, \Omega)\). Multisymplectic structures represent distinguished cohomology classes of the manifold \(M\), but their origin as a geometrical tool can be traced back to the foundations of the calculus of variations. It is well known that the suitable geometric framework to describe (first-order) field theories are certain multisymplectic manifolds (see, for instance [11, 15, 17, 19, 21, 24, 25, 27, 30, 33, 36, 37] and references quoted therein). The automorphism
groups of multisymplectic manifolds play a relevant role in the description of the corresponding system, and it is a relevant problem to characterize them in similar terms as in symplectic geometry (see, for instance, [38]). However many of the multisymplectic structures that arise in applications are natural generalizations of the canonical symplectic structure on a cotangent bundle. Hence they possess a local canonical form, and similar properties to the symplectic case would be expected to hold.

In this paper we show that this is actually the case for an important class of multisymplectic forms; namely, those that satisfy a local homogeneity property similar to the local homogeneity of symplectic structures. In fact it is simple to show that given a symplectic form \( \omega \) on a manifold \( M \), locally there always exists a Euler-like vector field \( \Delta \) such that \( \text{L}(\Delta)\omega = 2\omega \). Given a point \( x_0 \in M \) and an open set \( U \) (which we may consider to be contained on a local chart) a vector field \( \Delta \) defined on \( U \) is said to be an Euler-like vector field if there are local coordinates \( x^i \) defined on \( U \) centered at \( x_0 \) such that \( \Delta = x^i \partial / \partial x^i \).

Then, if \( (M, \Omega) \) is a multisymplectic manifold with \( \Omega \) of degree \( k \), we will say that \( (M, \Omega) \) is locally homogeneous if, for every \( x_0 \in M \) and any open neighborhood \( U \) there exists an Euler–like vector field \( \Delta \) on it such that \( \text{L}(\Delta)\Omega = f\Omega \) (see Section 4, definitions 4.1 and 4.2). Notice that the canonical multisymplectic structures on multicotangent bundles are locally homogeneous and, hence, our study should be of interest, for instance, in the realm of Hamiltonian field theories, where the extended multimomentum phase space is a multicotangent bundle [11, 15, 16, 30, 35].

Then, our main results are:

**Theorem 1.1** Let \( (M_i, \Omega_i) \), \( i = 1, 2 \), be two locally homogeneous multisymplectic manifolds of degree \( k \) and let \( G(M_i, \Omega_i) \) denote their corresponding groups of automorphisms. Let \( \Phi: G(M_1, \Omega_1) \to G(M_2, \Omega_2) \) be a group isomorphism which is also a homeomorphism when \( G(M_i, \Omega_i) \) are endowed with the point-open topology. Then, there exists a \( C^\infty \) diffeomorphism \( \varphi: M_1 \to M_2 \), such that \( \Phi(f) = \varphi \circ f \circ \varphi^{-1} \) for all \( f \in G(M_1, \Omega_1) \) and the tangent map \( \varphi_* \) maps locally Hamiltonian vector fields of \( (M_1, \Omega_1) \) into locally Hamiltonian vector fields of \( (M_2, \Omega_2) \). In addition, if we assume that \( \varphi_* \) maps all infinitesimal automorphisms of \( (M_1, \Omega_1) \) into infinitesimal automorphisms of \( (M_2, \Omega_2) \) then there is a constant \( c \) such that \( \varphi^*\Omega_2 = c\Omega_1 \).

This result generalizes the main theorems in [3] (Theorems 1 and 2), which are in turn generalizations of a theorem by Takens [39]. Contrary to the proof in [3], the proof presented here does not rely on the generalization by Omori [32] of Pursell-Shanks theorem [34], which do not apply to this situation because of the lack of local normal forms for multisymplectic structures. However, we use the following partial generalization of Lee Hwa Chung theorem.

**Theorem 1.2** Let \( (M, \Omega) \) be a locally homogeneous multisymplectic manifold of degree \( k \); then the only differential forms of degree \( k \) invariant under the graded Lie algebra of infinitesimal automorphisms of \( \Omega \) are real multiples of \( \Omega \).

Local properties of multisymplectic diffeomorphisms of locally homogeneous multisymplectic manifolds will play a crucial role throughout the discussion. They stem from a localization property for Hamiltonian vector fields that will be discussed in Lemma 4.4. These local properties are used to prove a further result that is interesting in itself: the group of multisymplectic diffeomorphisms acts transitively on the underlying manifold. This result shows that not all multisymplectic manifolds are locally homogeneous. In fact, R. Bryant [8] showed the existence of a multisymplectic structure.
of degree 3 such that its automorphism group is the exceptional group $G_2$ which is not strongly locally transitive on the underlying manifold (see also [23]).

The paper is organized as follows: in Section 2 we establish some basic definitions and results, mainly related with the geometry of multisymplectic manifolds. Section 3 is devoted to the definition and basic properties of the graded Lie algebra of the infinitesimal automorphisms of multisymplectic manifolds. In Section 4, the definition and some characteristics of locally homogeneous multisymplectic manifolds is stated; in particular, the localization lemma for multisymplectic diffeomorphisms, and the strong local transitivity of the group of multisymplectic diffeomorphisms is proved for locally homogeneous multisymplectic manifolds. In Section 5 we prove the main results on the structure of differential invariants of locally homogeneous multisymplectic manifolds. Finally, in Section 6, these results are used to characterize the multisymplectic transformations, while the proof of the main theorem is completed in Section 7. The paper ends with an appendix where multisymplectic manifolds admitting Darboux type coordinates are analysed (according to [26]), as examples of locally homogeneous multisymplectic manifolds.

All manifolds are real, paracompact, connected and $C^\infty$. All maps are $C^\infty$. Sum over crossed repeated indices is understood.

## 2 Notation and basic definitions

Let $M$ be an $n$-dimensional differentiable manifold. Sections of $\Lambda^m(TM)$ are called $m$-vector fields (or multivector fields of degree $m$) in $M$, and we will denote by $\mathfrak{X}^m(M)$ the set of $m$-vector fields in $M$. Let $\Omega \in \Omega^k(M)$ be a differentiable $k$-form in $M$ ($k \leq n$). For every $x \in M$, the form $\Omega_x$ establishes a correspondence $\hat{\Omega}_m(x)$ between the set of $m$-vectors $\Lambda^m(T_xM)$ and the $(k-m)$-forms $\Lambda^{k-m}(T_x^*M)$ as

$$\hat{\Omega}_m(x) : \Lambda^m(T_xM) \rightarrow \Lambda^{k-m}(T_x^*M)$$

$$v \rightarrow \iota(v)\Omega_x$$

If $v$ is homogeneous, $v = v_1 \wedge \ldots \wedge v_m$, then $\iota(v)\Omega_x = \iota(v_1 \wedge \ldots \wedge v_m)\Omega_x = \iota(v_1) \ldots \iota(v_m)\Omega_x$. Thus, an $m$-vector field $X \in \mathfrak{X}^m(M)$ defines a contraction $\iota(X)$ of degree $m$ of the algebra of differential forms in $M$. We denote

$$\ker^m \Omega = \{X \in \mathfrak{X}^m(M) \mid \hat{\Omega}_m(x)(X) = 0; \text{ for every } x \in M\}$$

The $k$-form $\Omega$ is said to be $m$-nondegenerate (for $1 \leq m \leq k-1$) if, for every $x \in M$, the subspace $\ker \hat{\Omega}_m(x)$ has minimal dimension. Thus:

1. If $\binom{n}{m} \leq \binom{n}{k-m}$, then $\dim(\ker \hat{\Omega}_m(x)) = 0$.

2. If $\binom{n}{m} > \binom{n}{k-m}$, then $\dim(\ker \hat{\Omega}_m(x)) = \binom{n}{m} - \binom{n}{k-m}$, for every $x \in M$.

The form $\Omega$ is strongly nondegenerate if it is $m$-nondegenerate for every $m = 1, \ldots, k-1$. Thus, the $m$-nondegeneracy of $\Omega$ implies that the map $\hat{\Omega}_m : \Lambda^m(TM) \rightarrow \Lambda^{k-m}(T^*M)$ is a bundle monomorphism in the case of item 1, or a bundle epimorphism in the case of item 2. The image of the bundle $\Lambda^m(TM)$ by $\hat{\Omega}_m$ will be denoted by $E_m$. Often, if there is no risk of confusion, we will omit the subindex $m$ and denote $\hat{\Omega}_m$ simply by $\hat{\Omega}$.
**Definition 2.1** Let \( M \) be an \( n \)-dimensional differentiable manifold and \( \Omega \in \Omega^k(M) \). The couple \((M, \Omega)\) is said to be a multisymplectic manifold if \( \Omega \) is closed and 1-nondegenerate. The degree \( k \) of the form \( \Omega \) will be called the degree of the multisymplectic manifold.

Thus, multisymplectic manifolds of degree \( k = 2 \) are the usual symplectic manifolds, and manifolds with a distinguished volume-form are multisymplectic manifolds of degree its dimension. Other examples of multisymplectic manifolds are provided by compact semisimple Lie groups equipped with the canonical cohomology 3-class, symplectic 6-dimensional Calabi-Yau manifolds with the canonical 3-class, etc [23]. Notice that there are no multisymplectic manifolds of degrees 1 or \( n-1 \) because \( \ker \Omega \) is nonvanishing in both cases.

Apart from those already cited, another very important class of multisymplectic manifolds is the *multicotangent bundle* (see also Section 9: appendix on special multisymplectic manifolds): let \( Q \) be a manifold, and \( \pi: \Lambda^k(T^*Q) \to Q \) the bundle of \( k \)-forms in \( Q \). This bundle is endowed with a canonical \( k \)-form \( \Theta \in \Omega^k(\Lambda^k(T^*Q)) \) defined as follows: if \( \alpha \in \Lambda^k(T^*Q) \), and \( U_1, \ldots, U_k \in T_\alpha(\Lambda^k(T^*Q)) \), then

\[
\Theta_\alpha(U_1, \ldots, U_k) = i(\pi_* U_k \wedge \ldots \wedge \pi_* U_1) \alpha
\]

If \((x^i, p_{i_1, \ldots, i_k})\) is a system of natural coordinates in \( W \subset \Lambda^k(T^*Q) \), then

\[
\Theta \big|_W = p_{i_1, \ldots, i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}
\]

Therefore, \( \Omega = -d\Theta \in \Omega^{k+1}(\Lambda^k(T^*Q)) \) is a 1-nondegenerate form. Then the couple \((\Lambda^k(T^*Q), \Omega)\) is a multisymplectic manifold.

Multisymplectic structures of degree \( \geq 3 \) are abundant. In fact, as shown in [31], if \( M \) is a smooth manifold of dimension \( \geq 7 \), then the space of multisymplectic structures of degree \( 3 \leq k \leq n-3 \) is residual. However, there is no local classification of multisymplectic forms, in general, not even in the linear case.

Finally we introduce the Schouten-Nijenhuis bracket. If \( X \in \mathfrak{X}^m(M) \), the graded bracket

\[
L(X) = [d, i(X)] = d i(X) - (-1)^m i(X) d
\]

where, as usual, \( d \) denotes the exterior differential on \( M \), defines a new derivative of degree \( m-1 \). If \( X \in \mathfrak{X}^i(M) \), and \( Y \in \mathfrak{X}^j(M) \), the graded commutator of \( L(X) \) and \( L(Y) \) is another operation of degree \( i+j-2 \) of the same type, i.e., there exists a \((i+j-1)\)-vector denoted by \([X, Y]\) such that,

\[
L([X, Y]) = [L(X), L(Y)].
\]

The \( r \)-bilinear assignement \( X, Y \mapsto [X, Y] \) is called the *Schouten-Nijenhuis bracket* of \( X, Y \). It is a generalization of the Lie bracket for multivector fields (see also [10] [29] for a slightly different definition).

Let \( X, Y \) and \( Z \) be homogeneous multivectors of degrees \( i, j, p \) respectively, then the Schouten-Nijenhuis bracket verifies the following properties characterizing this bracket:

1. \([X, Y] = -(-1)^{(i+1)(j+1)}[Y, X]\).
2. \([X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(i+1)}Y \wedge [X, Z]\).
3. \((-1)^{(i+1)(p+1)}[X, [Y, Z]] + (-1)^{(j+1)(i+1)}[Y, [Z, X]] + (-1)^{(p+1)(j+1)}[Z, [X, Y]] = 0\).

The exterior algebra of multivectors has the structure of an odd Poisson algebra, sometimes also called a Schouten algebra. This allows us to define a structure of an odd Poisson graded manifold on \( M \) whose sheaf of superfunctions is given by the sheaf of multivector fields and the odd Poisson bracket is the Schouten bracket.
The graded Lie algebra of infinitesimal automorphisms of a multisymplectic manifold

From now on, \((M, \Omega)\) will be a multisymplectic manifold of degree \(k\).

A multisymplectic diffeomorphism is a diffeomorphism \(\varphi: M \to M\) such that \(\varphi^*\Omega = \Omega\). A locally Hamiltonian vector field on \((M, \Omega)\) is a vector field \(X\) whose flow consists of multisymplectic diffeomorphisms. It is clear that \(X\) is a locally Hamiltonian vector field if and only if \(L(X)\Omega = 0\), or equivalently, \(i(X)\Omega\) is a closed \((k - 1)\)-form. This fact leads to the following generalization:

**Definition 3.1** Let \(X \in \mathfrak{X}^m(M)\) \((m \geq 1)\).

1. \(X\) is said to be a Hamiltonian \(m\)-vector field if \(i(X)\Omega\) is an exact \((k - m)\)-form; that is, there exists \(\zeta \in \Omega^{k-m-1}(M)\) such that \(i(X)\Omega = d\zeta\) \((1)\)

\(\zeta\) is defined modulo closed \((k - m - 1)\)-forms in \(M\) (they are denoted by \(Z^{k-m-1}(M)\)). The class \(\bar{\zeta} \in \Omega^{k-m-1}(M)/Z^{k-m-1}(M)\) defined by \(\zeta\) is called the Hamiltonian for \(X\), and every element in this class is a Hamiltonian form for \(X\).

2. \(X\) is said to be a locally Hamiltonian \(m\)-vector field if \(i(X)\Omega\) is a closed \((k - m)\)-form. In this case, for every point \(x \in M\), there is an open neighborhood \(W \subset M\) and \(\zeta \in \Omega^{k-m-1}(W)\) such that

\[i(X)\Omega = d\zeta \quad \text{ (on } W)\]

As in the above item, changing \(M\) by \(W\), we obtain the Hamiltonian \(\bar{\zeta} \in \Omega^{k-m-1}(W)/Z^{k-m-1}(W)\) for \(X\), and the local Hamiltonian forms for \(X\).

Conversely, \(\zeta \in \Omega^p(M)\) \((\text{resp. } \zeta \in \Omega^p(W))\) is said to be a Hamiltonian \(p\)-form \((\text{resp. a local Hamiltonian } p\text{-form})\) if there exists a \((k-p-1)\)-vector field \(X \in \mathfrak{X}^{k-p-1}(M)\) \((\text{resp. } X \in \mathfrak{X}^{k-p-1}(W))\) such that \(i(X)\Omega = d\zeta\) \((\text{resp. on } W)\).

We denote by \(\mathcal{H}^p(M)\) the set of Hamiltonian \(p\)-forms in \(M\) and by \(\bar{\mathcal{H}}^p(M)\) the set of Hamiltonian \(p\)-forms modulo closed \(p\)-forms, \(\bar{\mathcal{H}}^p(M) = \mathcal{H}^p(M)/Z^p(M)\). The classes in \(\bar{\mathcal{H}}^p(M)\) will be denoted by \(\bar{\zeta}\), which means by that the class containing the Hamiltonian \(p\)-form \(\zeta\). Let \(\bar{\mathcal{H}}^*(M) = \oplus_{p \geq 0} \bar{\mathcal{H}}^p(M)\).

**Remarks.**

- If \(m > k\) the previous definitions are void and all \(m\)-vector fields are Hamiltonian.
- There are no Hamiltonian forms of degree higher than \(k - 2\).
- Every \(m\)-vector field \(X \in \ker^m \Omega\) is a Hamiltonian \(m\)-vector field with Hamiltonian the zero class.
- Locally Hamiltonian \(m\)-vector fields \(X\) of degree \(k - 1\) define closed 1-forms \(i(X)\Omega\), which have locally associated a smooth function \(f\) (up to constants) called the local Hamiltonian function of \(X\).

**Lemma 3.2** Let \(\Omega \in \Omega^k(M)\) be a closed \(m\)-nondegenerate form.
1. For every differentiable form $\zeta \in \Omega^{m-1}(M)$ such that \( \binom{n}{m} \leq \binom{n}{k-m} \), there exists a 
(k – m)-locally Hamiltonian multivector field $X$ possessing it as local Hamiltonian form, i.e. 
such that $i(X)\Omega = d\zeta$. As a consequence, the differentials of Hamiltonian $(m-1)$-forms of 
locally Hamiltonian $(k-m)$-vector fields span locally the $m$-multicotangent bundle of $M$, 
$\Lambda^m(T^*M)$.

2. If $\binom{n}{m} \leq \binom{n}{k-m}$ the family of locally Hamiltonian $(k-m)$-vector fields span locally the 
$(k-m)$-multitangent bundle of $M$, which is

$$\Lambda^{k-m}(T_xM) = \text{span}\{ X_x | L(X)\Omega = 0 \ , \ X \in \mathfrak{x}^{k-m}(M) \}.$$ (Proof)

1. Let $\Omega$ be $m$-nondegenerate of degree $k$. The map $\hat{\Omega}_{k-m}$ has its rank in the bundle $\Lambda^m(T^*M)$, 
but $\hat{\Omega}_{k-m}$ is the dual map of $\Omega_m$ (up to, perhaps, a minus sign). And as $\Omega_m$ is a monomorphism, 
then $\pm \hat{\Omega}_m = \hat{\Omega}_{k-m} \colon \Lambda^{k-m}(T^*M) \rightarrow \Lambda^m(T^*M)$ is onto. Then, for every $\zeta \in \Omega^{m-1}(M)$, 
$d\zeta$ defines a section of $\Lambda^m(T^*M)$; hence we can choose a smooth $(k-m)$-vector field $X$ such that 
$\hat{\Omega}_{k-m}(x)(X_x) = d\zeta(x)$, for every $x \in M$.

Taking a family of coordinate functions $x^i$, the same can be done locally for a family of 
$(m-1)$-forms $x^i dx^{i+1} \wedge \ldots \wedge dx^n$, showing in this way that the differentials of Hamiltonians 
$(m-1)$-forms span locally the $m$-multicotangent bundle of $M$.

2. For every $X \in \mathfrak{x}^{k-m}(M)$, $i(X)\Omega \in \Omega^m(M)$. But, taking into account the above item, for every 
x \in M, there exists a neighborhood $U \subset M$ such that $i(X)\Omega|_U = f^i d\zeta_i$, where $f^i \in C^\infty(U)$ 
and $\zeta_i \in \Omega^{m-1}(M)$ with $i(X_i)\Omega|_U = d\zeta_i$ for some locally Hamiltonian $(k-m)$-vector fields 
$X_i$. Therefore $X|_U = f^i X_i + Z$, with $Z \in \ker^{k-m}\Omega$; that is, $i(Z)\Omega = 0$, so $Z$ are also locally 
Hamiltonian $(k-m)$-vector fields and the proof is finished.

Notice that for $m = 1$, if $k \geq 2$, then $n = \binom{n}{1} \leq \binom{n}{k-1}$. Thus if $\Omega$ is 1-nondegenerate, the 
above Lemma states that the differentials of Hamiltonian functions of locally Hamiltonian $(k-1)$- 
vector fields span locally the cotangent bundle of $M$ and that, in its turn, the family of these 
$(k-1)$-Hamiltonian multivector fields span locally the $(k-1)$-multitangent bundle of $M$. However 
the previous Lemma says nothing about the Hamiltonian vector fields. We analyze this question in 
the following Section.

**Proposition 1** 1. A $m$-multivector field $X \in \mathfrak{x}^m(M)$ is a locally Hamiltonian $m$-multivector 
field if and only if $L(X)\Omega = 0$.

2. If $X, Y$ are locally Hamiltonian multivector fields, then $[X, Y]$ is a Hamiltonian multivector 
field with Hamiltonian form $i(X \wedge Y)\Omega$.

(Proof) Item 1 is immediate.

For item 2, if $X, Y$ are multivector fields of degrees $l, m$ respectively we have,

$$L([X, Y])\Omega = L(X)L(Y)\Omega - (-1)^{l+m}L(Y)L(X)\Omega = 0.$$
Furthermore, $i([X,Y])\Omega = L(X)i(Y)\Omega - (-1)^{l+m}i(Y)L(X)\Omega = d(i(X)i(Y)\Omega).$

We denote respectively by $\mathcal{X}_h^m(M)$ and $\mathcal{X}^m_{lh}(M)$ the sets of Hamiltonian and locally Hamiltonian $m$-vector fields in $M$. It is clear by the previous proposition that $\bigoplus_{m \geq 0} \mathcal{X}_h^m(M)$ is a graded Lie subalgebra of the graded Lie algebra of multivector fields. We say that an $m$-vector field is characteristic if it belongs to $\text{ker}^m \Omega$. The set of characteristic $m$-vector fields constitutes a graded Lie subalgebra of $\bigoplus_{m \geq 0} \mathcal{X}_{lh}^m(M)$. Moreover, the characteristic multivector fields define a graded ideal of the graded Lie algebra of Hamiltonian multivector fields. We denote the corresponding quotient graded Lie algebra by $\mathcal{V}_H^m(M,\Omega)$, and

$$\mathcal{V}_H^m(M,\Omega) = \bigoplus_{m \geq 0} \mathcal{V}_H^m(M,\Omega), \quad \mathcal{V}_H^m(M,\Omega) = \mathcal{X}_h^m(M)/\text{ker}^m \Omega.$$ 

Notice that again if $m > k$, $\text{ker}^m \Omega = \Lambda^m(TM)$, hence $\mathcal{V}_H^m(M,\Omega) = 0$ and $\mathcal{V}_H^1(M,\Omega) = \mathcal{X}_{lh}(M)$. Namely, $\mathcal{V}_H^m(M,\Omega) = \bigoplus_{m = 0}^k \mathcal{V}_H^m(M,\Omega)$.

**Definition 3.3** The Lie algebra $\mathcal{V}_H^m(M,\Omega)$ is called the infinitesimal graded Lie algebra of $(M,\Omega)$ or the graded Lie algebra of infinitesimal automorphisms of $(M,\Omega)$.

We can translate this structure of the graded Lie algebra to the corresponding Hamiltonian forms in a similar way to that in symplectic geometry (see [9] for more details on this construction). In fact, we can define a graded Lie bracket on $\mathcal{X}_h^p(M) = \bigoplus_{p \geq 0} \mathcal{X}_h^p(M)$ as follows:

**Definition 3.4** Given $\bar{\xi} \in \mathcal{X}_h^p(M)$, $\bar{\zeta} \in \mathcal{X}_h^m(M)$, let $X_\xi \in \mathcal{X}_h^{k-p-1}(M)$, $Y_\zeta \in \mathcal{X}_h^{k-m-1}(M)$ be their corresponding Hamiltonian multivector fields modulo $\text{ker} \Omega_*$, where $\text{ker} \Omega_* = \bigoplus_{j = 0}^k \text{ker} j \Omega$. The bracket of these Hamiltonian classes (related to the multisymplectic structure $\Omega$) is the $(p+m-k+2)$-Hamiltonian class $\{\xi, \zeta\}$ containing the form,

$$\{\xi, \zeta\} = \Omega(X_\xi,Y_\zeta) = i(Y_\zeta)i(X_\xi)\Omega = i(Y_\zeta)d\xi = (-1)^{(k-p-1)(k-m-1)}i(X_\xi)d\zeta.$$

In the same way as in the symplectic case, the Poisson bracket is closely related to the Lie bracket. Now we have:

**Proposition 2** Given $X_\xi \in \mathcal{X}_h^p(M)$, $Y_\zeta \in \mathcal{X}_h^m(M)$ Hamiltonian multivector fields, let $\bar{\xi} \in \mathcal{H}^{k-p-1}(M)$, $\bar{\zeta} \in \mathcal{H}_h^{k-m-1}(M)$ be the corresponding Hamiltonian classes. Then the Schouten-Nijenhuis bracket $[X_\xi,Y_\zeta]$ is a Hamiltonian $(p+m-1)$-vector field whose Hamiltonian $(k-p-m-2)$-form is $\{\zeta, \xi\}$; that is,

$$X_{\{\zeta, \xi\}} = [X_\xi,Y_\zeta].$$

(Proof) By definition, $i(X_{\{\zeta, \xi\}})\Omega = d\{\zeta, \xi\}$. Furthermore, as a consequence of Proposition 1,

$$i([X_\xi,Y_\zeta])\Omega = di(X_\xi)i(Y_\zeta)\Omega = d\{\zeta, \xi\}.$$

Thus $i(X_{\{\zeta, \xi\}})\Omega = i([X_\xi,Y_\zeta])\Omega$, and therefore $X_{\{\zeta, \xi\}} = [X_\xi,Y_\zeta].$ ■

As a consequence of which, we have:

**Proposition 3** ($\mathcal{H}^p(M), \{ , \}$) is a graded Lie algebra whose grading is defined by $|\bar{\eta}| = k - p - 1$, $\eta$ being a $p$-Hamiltonian form.
Remark 1 The graded Lie algebra \( V^*_H(M, \Omega) \) possesses as elements of degree zero the Lie algebra of locally Hamiltonian vector fields on \((M, \Omega)\), which is the Lie algebra of the ILH-group \([32]\) of smooth multisymplectic diffeomorphisms. This suggests the possibility of embracing in a single structure of supergroup both smooth multisymplectic diffeomorphisms and infinitesimal automorphisms of a multisymplectic manifold \((M, \Omega)\). This can certainly be done by extending to the graded setting some of the techniques used to deal with ILH-Lie groups.

4 Locally homogeneous multisymplectic manifolds. The group of multisymplectic diffeomorphisms

As mentioned earlier, in general, it is not true that the locally Hamiltonian vector fields in a multisymplectic manifold span the tangent bundle of this manifold. However, there is a simple property, already mentioned in the introduction, that implies it (among other things).

Definition 4.1 Let \( M \) be a differentiable manifold. Consider \( x \in M \) and a compact set \( K \) such that \( x \in \overset{\circ}{K} \). A local Liouville or local Euler-like vector field at \( x \) with respect to \( K \) is a vector field \( \Delta_x \) on \( M \) such that \( \text{supp} \Delta_x := \{ y \in M \mid \Delta_x(y) \neq 0 \} \subset K \), and there exists a diffeomorphism \( \varphi: \overset{\circ}{\text{supp} \Delta_x} \rightarrow \mathbb{R}^n \) such that \( \varphi_* \Delta_x = \Delta \), where \( \Delta = x^i \frac{\partial}{\partial x^i} \) is the standard Liouville or dilation vector field in \( \mathbb{R}^n \).

Notice that if \( \Delta_x \) is a local Euler-like vector field at \( x \) with respect to \( K \) and \( \lambda \) is a bump function around \( x \) with support contained in \( \overset{\circ}{K} \), then \( \lambda \Delta_x \) is also a local Euler-like vector field at \( x \), but now with respect to \( \text{supp} \lambda \subset K \). Notice also that if \( \Delta_U \) is a vector field defined locally on an open set \( U \) (in this case \( U \subset K \) for some compact \( K \)) such that in some local coordinates it has the form \( x^i \frac{\partial}{\partial x^i} \), then by choosing a bump function around \( x \in U \) with compact support contained in \( U \), the product \( \Delta_{\lambda} = \lambda \Delta_U \) defines a local Euler-like vector field at \( x \) with respect to the compact set \( \text{supp} \lambda \).

Definition 4.2 A differential form \( \Omega \) is said to be locally homogeneous at \( x \in M \) if, for every open set \( U \) containing \( x \), there exists a local Euler-like vector field \( \Delta_x^U \) at \( x \) with respect to a compact set \( K \subset U \) such that

\[
L(\Delta_x^U)\Omega = f \Omega ; \quad f \in C^\infty(U) . \tag{2}
\]

The form \( \Omega \) is locally homogeneous if it is locally homogeneous for all \( x \in M \).

A couple \((M, \Omega)\), where \( M \) is a manifold and \( \Omega \in \Omega^k(M) \) is locally homogeneous is called a locally homogeneous manifold.

It is obvious from the definition of \( \Delta^x_U \) that, out of \( \text{supp} \Delta^x_U \), the function \( f \) vanishes. In many instances it is possible to see that if \( \Omega \) is locally homogeneous at \( x \), then restricting to a smaller open subset, the function \( f \) can be chosen to be constant, \( f = c \).

For instance, multicotangent bundles are locally homogeneous manifolds. In fact, let \( \alpha_0 \in \Lambda^k(T^*Q) \), and consider local natural coordinates \((q^i, p_{i_1,\ldots,i_k})\) in a small neighborhood of \( \alpha_0 \). We define \( r^2 := \sum_i (q^i)^2 + \sum_i (p_{i_1,\ldots,i_k})^2 \), and \( \lambda := \lambda(r^2) \) a bump function. Then consider the local
Liouville vector field $\Delta_\lambda := \lambda \Delta$. Therefore, a straightforward calculation shows that, for the natural multisymplectic form $\Omega \in \Omega^{k+1}(\Lambda^k(T^*Q))$,

$$L(\Delta_\lambda)\Omega = [(k+1)\lambda + 2r^2\lambda']\Omega.$$ 

and the function $f = (k+1)\lambda + 2r^2\lambda'$ vanishes identically outside of $\text{supp}\lambda$, and is constant equal to $k+1$ in a smaller ball around $x$ contained in $\text{supp}\lambda$. Notice that symplectic manifolds, as well as manifolds endowed with volume forms, are locally homogeneous.

At this point, we can show that Hamiltonian vector fields in locally homogeneous multisymplectic manifolds can be localized. This property plays a crucial role in the discussion to follow.

**Lemma 4.3** Let $(M, \Omega)$ be a locally homogeneous manifold. Then, if $\Delta$ is a local Euler-like vector field for $\Omega$ (equation [2]), then its flow leaves invariant the subbundle $E_1 = \text{Im} \Omega_1$.

*Proof* Let $x$ be a point in $M$. As $M$ is locally contractible, let $U$ be a small enough open set and $\Delta^x$ an Euler-like vector field at $x$ with respect to a compact set $K$ contained on it. (We can assume that $U$ is a coordinate chart with coordinates $x^i$ centered at $x$ and adapted to $\Delta^x$ as in Definition [41].) Let $\varphi_s$ be the local flow defined by $\Delta^x$. Notice that the flow $\varphi_s$ is given by $\varphi_s(x) = e^s x$ in the previous local coordinate chart.

The local homogeneity property of $\Omega$ with respect to $\Delta$ implies that $\varphi_s^*\Omega = f_s \Omega$, for some function $f_s$. Notice that $\varphi_s^*\Omega = f_s \Omega \iff L(\Delta)\Omega = f_0 \Omega$, with $f_s = df_s/ds$. Actually, if $L(\Delta)\Omega = f \Omega$, then $f_s(x) = \int_0^s f(\varphi(\tau)) d\tau$ and $f_0 = f$, with $\varphi_s$ the local flow of $\Delta$ such that $\varphi_0(x) = x$.

As a consequence, the flow $\varphi_s$ leaves invariant the subbundle $E_1$. In fact; if $\eta = \text{i}(X)\Omega \in E_1$, then $\varphi^*_s \eta \in E_1$ because

$$\varphi^*_s \eta = \text{i}(\varphi(\varphi_s)_* X) \varphi^*_s \Omega = \text{i}((\varphi_s)_* \varphi^-_* X)(f_s \Omega) = \text{i}(f_s(\varphi^-_* X)) \Omega.$$ 

Moreover, for every $\sigma: \mathbb{R} \to \mathbb{R}$, with appropriate domain and range, the one-parameter family of local diffeomorphisms $\varphi_s = \varphi_{\sigma(s)}$ will preserve $E_1$ too.

*Lemma 4.4* Let $X$ be a locally Hamiltonian vector field on a locally homogeneous multisymplectic manifold $(M, \Omega)$. Let $x_0$ be a point in $M$, then for each open set $U$ containing $x_0$, there exists an open neighborhood $V$ of $x_0$ such that $V \subset V \subset U$, with $V$ compact, and a locally Hamiltonian vector field $X'$ such that $X'$ coincides with $X$ in $V$ and vanishes identically outside of $U$.

*Proof* If $X$ is a locally Hamiltonian vector field, then $\text{i}(X)\Omega = \eta$, with $\eta$ a closed $(k-1)$-form.

We prove the Lemma in two steps:

a) The closed form

$$\text{i}(X)\Omega = \eta,$$

has a special exact form in the coordinate chart $(U, x^i)$.

Because $\Omega$ is locally homogeneous, given $x_0$ and $U$ there exists a local Euler-like vector field at $x_0$ with respect to a compact set $K$ contained in $U$. We will denote by $\Delta$ such vector field and by $\varphi$ its flow. Notice that the vector field $\Delta$ is complete because its support is compact.

Let us consider the smooth isotopy $\rho_t = \varphi_{\ln t}$. The one-parameter family of local maps $\rho_t$ for $t \in [0, 1]$ define a strong deformation retraction from $K$ to $x_0$, i.e., $\rho_1 = \varphi_0 = Id|_K$, and $\rho_0 = \varphi_{-\infty}$.
maps \( \hat{K} \) onto \( x_0 \). Notice that \( \varphi_{-\infty}(x) = \lim_{t \to -\infty} \varphi_t(x) = x_0 \) is well defined because \( \Delta \) being a Euler-like vector field with respect to \( x_0 \) has a unique fixed point \( x_0 \) in the interior of \( K \) which is just its stable manifold.

Now we can obtain the local Poincaré representation of the closed form \( \eta \) in equation 3. Let \( \Delta_t \) be the time-dependent vector field whose flow is given by \( \rho_t \), i.e.,

\[
\frac{d}{dt} \rho_t = \Delta_t \circ \rho_t .
\]  
(4)

Thus,

\[
\frac{d}{dt}(\rho_t^* \eta) = \rho_t^* (L(\Delta_t) \eta) .
\]

Hence,

\[
\rho_t^* \eta - \rho_0^* \eta = \int_0^1 \frac{d}{dt}(\rho_t^* \eta) \, dt = \int_0^1 \rho_t^* (L(\Delta_t) \eta) \, dt = d \int_0^1 \rho_t^*(i(\Delta_t) \eta) \, dt .
\]

Thus,

\[
\eta = d \int_0^1 \rho_t^*(i(\Delta_t) \eta) \, dt ,
\]

because \( d\eta = 0 \) and \( \rho_t^* \eta - \rho_0^* \eta = \eta \). Hence \( \eta = d\zeta \) with \( \zeta = \int_0^1 \rho_t^*(i(\Delta_t) \eta) \, dt \), in the open set \( \hat{K} \).

Observe that we have not used that \( \eta \in E_1 \).

b) Localization of the Hamiltonian vector field \( X \).

We will try to localize the vector field \( X \) by using a bump function \( \lambda \) centered at \( x_0 \), i.e., we shall choose \( \lambda \) such that the closure of \( V = \text{supp} \lambda \) will be a compact set contained in \( \hat{K} \) and with \( x_0 \) in its interior. Unfortunately the vector field \( \lambda X \) is not locally Hamiltonian in general, hence we will proceed by modifying the Hamiltonian form \( \zeta \) of \( X \) instead. We define a new vector field \( \Delta'_t \) by scaling the vector field \( \Delta_t \) by \( \lambda \), i.e.,

\[
\Delta'_t = \lambda \Delta_t .
\]

We denote the flow of \( \Delta'_t \) by \( \rho'_t \). Notice that the family of local maps \( \rho'_t \) will be obtained from \( \rho_t \) by reparametrizing the parameter \( t \). Thus if \( \rho_t \) were leaving the subbundle \( E_1 \) invariant, the same would be true for \( \rho'_t \).

Moreover, we can choose the function \( \lambda \) such that \( \Delta_t(\lambda) = r(t) \) and

\[
r(t) = \begin{cases} 
0, & \text{if } 0 \leq t \leq 1/3 \\
r(t), & \text{which is a positive function such that } \frac{d}{dt}r(t) > 0, \text{ if } 1/3 < t < 2/3 \\
1, & \text{if } 2/3 \leq t \leq 1 .
\end{cases}
\]

Then the flow \( (\rho'_t)^* \) leaves invariant the subbundle \( E_1 = \hat{\Omega}_1(TM) \) and \( (\rho'_t)^* \eta \in E_1 \) for all \( 0 \leq t \leq 1 \). Again, repeating the computation leading to equation 4 using the vector field \( \Delta'_t \) instead, we get

\[
(\rho'_t)^* \eta - (\rho'_0)^* \eta = d \int_0^1 (\rho'_t)^*(\lambda i(\Delta_t) \eta) \, dt .
\]  
(5)

As in the undeformed situation (with \( \lambda = 1 \)), \( \rho'_1 = \text{Id} \), however \( \rho'_0 \) is not a retraction of \( \hat{K} \) onto \( x_0 \). Nevertheless, the \((k - 1)\)-form \( \eta' = d \int_0^1 (\rho'_t)^*(\lambda i(\Delta_t) \eta) \, dt \) is in \( E_1 \), because both \( (\rho'_0)^* \eta \) and \( (\rho'_1)^* \eta \) are in \( E_1 \). Thus there exists a vector field \( X' \) such that

\[
i(X') \Omega = \eta' .
\]
The form $\eta'$ is closed by construction, hence $X'$ is locally Hamiltonian.

Moreover, if $y$ is a point lying in the interior of the set $\lambda^{-1}(1) \subset V$ then, $\rho'_s(y) = \rho_s(y)$. Consequently, from equation \[\eta'(y) = \eta(y)\] and $X'(y) = X(y)$. If, on the contrary, $y$ lies outside the compact set $\hat{V}$, we have $\rho'_s(y) = y$, for all $t$, because $\lambda$ vanishes there, thus $\Delta'_t$ vanishes and the flow is the identity. Then $\eta'(y) = 0$ and $X'(y) = 0$.

A far reaching consequence of the localization lemma is the transitivity of the group of multisymplectic diffeomorphisms. We will first prove the following result:

**Lemma 4.5** Let $(M, \Omega)$ be a locally homogeneous multisymplectic manifold. Then the family of locally Hamiltonian vector fields span locally the tangent bundle of $M$, that is

$$T_x M = \text{span}\{ X_x \mid X \in \mathfrak{X}(M), L(X)\Omega = 0 \} .$$

(*Proof*) We will work locally. Let $U$ be a contractible open neighborhood of a given point $x \in M$. We can shrink $U$ to be contained in a coordinate chart with coordinates $x^i$. The tensor bundles of $M$ restricted to $U$ are trivial. In particular, the subbundle $E_1$ restricted to $U$ is trivial. Let $v \in T_x M$ be an arbitrary tangent vector. Let $\nu = \tilde{\Omega}_1(x)v \in E_1 \subset \Lambda^{k-1}(T^*_x M)$. Consider a vector field $X$ on $U$ such that $X(x) = v$. Then $i(X)\Omega = \eta$ and the $(k-1)$-form $\eta$ is not closed in general. As in Lemma 4.4 we consider a strong deformation retraction $\rho_t$ and the corresponding vector field $\Delta_t$. Now, first we have

$$\int_0^1 \frac{d}{ds}(\rho^*_s \eta) \, ds = \eta,$$

and furthermore,

$$\int_0^1 \frac{d}{ds}(\rho^*_s \eta) \, ds = d \int_0^1 \rho^*_s (i(\Delta_s) \eta) \, ds + \int_0^1 \rho^*_s (i(\Delta_s) d \eta) \, ds . \quad (6)$$

However, as $d \eta = di(X)\Omega = L(X)\Omega$, then

$$i(\Delta_t) d \eta = i(\Delta_t) L(X)\Omega = i([\Delta_t, X])\Omega + L(X)i(\Delta_t)\Omega .$$

Thus, returning to equation (6) we obtain

$$\eta = d \int_0^1 \rho^*_s i(\Delta_s) \eta \, ds + \int_0^1 \rho^*_s (i([\Delta_s, X])\Omega + L(X)i(\Delta_s)\Omega) \, ds .$$

Choosing the vector field $X$ such that its flow leaves $E_1$ invariant, the second term on the r.h.s. of the previous equation will be in $E_1$, so the first term will be in $E_1$ too. Let us define $\eta' = d \left( \int_0^1 \rho^*_s i(\Delta_s) \eta \, ds \right)$, and let us denote by $X'$ the Hamiltonian vector field on $U$ defined by

$$i(X')\Omega = \eta' .$$

Evaluating $\eta'$ at $x$ we find that $\eta'(x) = \eta(x)$, hence $X'(x) = v$. We then localize the vector field $X'$ in such a way that the closure of its support is compact and is contained in $U$. We can then extend this vector field trivially to all $M$, and this extension is locally Hamiltonian. Finally the value of this vector field at $x$ is precisely $v$.

Recall that a group of diffeomorphisms $G$ is said to act $r$-transitively on $M$ if for any pair of collections $\{x_1, \ldots, x_r\}$, $\{y_1, \ldots, y_r\}$ of distinct points of $M$, there exists a diffeomorphism $\phi \in G$
such that \( \phi(x_i) = y_i \). If the group \( G \) acts transitively for all \( r \), then it is said to act \( \omega \)-transitively or transitively for short. The transitivity of a group of diffeomorphisms can be reduced to a local problem because (strong) local transitivity implies transitivity. More precisely, we will say that the group \( G \) acts transitively for all \( x \), then it is said to act \( \omega \)-transitively or transitively for short. The transitivity of a group of diffeomorphisms can be reduced to a local problem because (strong) local transitivity implies transitivity. More precisely, we will say that the group of diffeomorphisms \( G \) is strongly locally transitive on \( M \) if for each \( x \in M \) and a neighborhood \( U \) of \( x \), there are neighborhoods \( V \) and \( W \) of \( x \) with \( V \subset W \subset U \), \( W \) compact, such that for any \( y \in V \) there is a smooth isotopy \( \phi_t \) on \( G \) joining \( \phi \) with the identity, \( \phi_1 = \phi \), \( \phi_0 = \text{id} \), such that \( \phi_1(x) = y \) and \( \phi_t \) leaves every point outside \( W \) fixed. Thus, if \( G \) is strongly locally transitive on \( M \), then \( G \) acts transitively on \( M \) \([7]\).

**Theorem 4.6** The group of multisymplectic diffeomorphisms \( G(M, \Omega) \) of a locally homogeneous multisymplectic manifold is strongly locally transitive on \( M \).

(Proof) By Lemma 4.5 we can construct a local basis of the tangent bundle in the neighborhood of a given point \( x \) made of locally Hamiltonian vector fields \( X_i \). Using Lemma 4.4, we can localize the vector fields \( X_i \) in such a way that the localized Hamiltonian vector fields \( X'_i \) will have common supports. We denote this common support by \( V \) and assume that it will be contained in a compact subset contained in \( U \). However, the vector fields \( X'_i \) will generate the module of vector fields inside the support \( V \), so the flows of local vectors fields cover the same set as the flows of local Hamiltonian vector fields, although the group of diffeomorphisms is locally strongly transitive and the same will happen for the group of multisymplectic diffeomorphisms. ■

**Corollary 1** The group of multisymplectic diffeomorphisms \( G(M, \Omega) \) of a locally homogeneous multisymplectic manifold \((M, \Omega)\) acts transitively on \( M \).

(Proof) The conclusion follows from the results in \([7]\) and Theorem 4.6. ■

**Remark 2** The center of the graded Lie algebra \((\tilde{H}^*(M), \{ , \})\) is a graded Lie subalgebra, whose elements are called Casimirs. We must point out that, on locally homogeneous multisymplectic manifolds, there are no Casimirs of degree 0, i.e., functions commuting with anything, because if this was the case, there would be a function \( S \) such that \( \{S, \eta\} = 0 \), for every Hamiltonian form \( \eta \). In particular, \( S \) commutes with \((k - 1)\)-Hamiltonian forms, but this implies that \( X(S) = 0 \), for every Hamiltonian vector field \( X \). But this is clearly impossible because, for those kinds of multisymplectic manifolds, Hamiltonian vector fields span the tangent bundle by Lemma 4.5.

## 5 Invariant differential forms

In order to prove the main statement in this section, we first establish two lemmas:

**Lemma 5.1** Let \((M, \Omega)\) be a multisymplectic manifold of degree \( k \) and \( \alpha \in \Omega^p(M) \) (with \( p \geq k - 1 \)) a differential form which is invariant under the set of locally Hamiltonian \((k - 1)\)-vector fields, that is, \( L(X)\alpha = 0 \), for every \( X \in \mathfrak{X}_{th}^{k-1}(M) \). Then:

1. For every \( X, Y \in \mathfrak{X}_{th}^{k-1}(M) \),
   \[
   i(X)\Omega \wedge i(Y)\alpha + i(Y)\Omega \wedge i(X)\alpha = 0
   \] (7)
Let \( \Lambda \) be an \( (k-1) \)-form which is invariant under the set of locally Hamiltonian vector fields. Then:

1. If \( p = k-1 \) then \( \alpha = 0 \).

2. If \( p = k \), there exists a unique \( \alpha' \in \mathfrak{X}(M) \) such that

\[
\iota(X)\alpha = \alpha' \iota(X)\Omega,
\]

for every \( X \in \mathfrak{X}_{lh}(M) \).
(Proof) The starting point is the equality 7. Taking $X = Y \not\in \ker^{k-1} \Omega$ (if $X \in \ker^{k-1} \Omega$, then $i(X)\alpha = 0$ by hypothesis), we obtain

$$i(X)\Omega \wedge i(X)\alpha = 0,$$

for every $X \in \mathfrak{X}_{th}^{k-1}(M)$. Therefore we have:

1. If $p = k - 1$, then $i(X)\alpha \in C^\infty(M)$, and according to the first item of Lemma 5.2 (for 1-nondegenerate forms), equation 11 together with 8 leads to the result $i(X)\alpha = 0$, for every $X \in \mathfrak{X}_{th}^{k-1}(M)$. But, taking into account item 2 of Lemma 8.2 (for 1-nondegenerate forms), this also holds for every $X \in \mathfrak{X}^{k-1}(M)$, and we must conclude that $\alpha = 0$.

2. If $p = k$ and $i(X)\alpha = 0$ for all $X$, then $\alpha = 0$. Thus, let us assume that $i(X)\Omega \neq 0$ for some $X$, then the solution of equation 11 is

$$i(X)\alpha = \alpha' X i(X)\Omega,$$

and it is important to point out that the equation 8 for $\alpha$ implies that the function $\alpha' X \in C^\infty(M)$ is the same for every $X, X' \in \mathfrak{X}_{th}^{k-1}(M)$ such that $i(X)\Omega = i(X')\Omega$.

Now, returning to equation 7 we obtain the relation

$$i(Y)\Omega \wedge i(X)\Omega (\alpha' - \alpha' Y) = 0.$$

However, $\alpha' X, \alpha' Y \in C^\infty(M)$ are the unique solution of the respective equations 11 for each $X, Y \in \mathfrak{X}_{th}^{k-1}(M)$; then we have the following options:

- If $i(Y)\Omega \wedge i(X)\Omega \neq 0$ then $\alpha' X = \alpha' Y$.
- If $i(Y)\Omega \wedge i(X)\Omega = 0$ then $X = fY + Z$, where $f \in C^\infty(M)$ and $Z \in \ker^{k-1} \Omega$. Therefore:
  - If $X \in \ker^{k-1} \Omega$ then $Y \in \ker^{k-1} \Omega$. Therefore, taking into account item 2 of Lemma 5.1 the corresponding equations 12 for $X$ and $Y$ are identities, and thus $\alpha' X$ and $\alpha' Y$ are arbitrary functions which we can take to be equal.
  - If $X \not\in \ker^{k-1} \Omega$ then $Y \not\in \ker^{k-1} \Omega$. Therefore, taking into account item 2 of Lemma 5.1 we have

$$i(X)\alpha = i(fY + Z)\alpha = fi(Y)\alpha = f\alpha' Y i(Y)\Omega = \alpha' Y i(fY + Z)\Omega = \alpha' Y i(X)\Omega,$$

which, comparing with 12 gives $\alpha' X = \alpha' Y$.

In any case, $\alpha' X = \alpha' Y$, and as a consequence, the function $\alpha'$ solution to 11 is the same for every $X \in \mathfrak{X}_{th}^{k-1}(M)$.

At this point we can state and prove the following fundamental result:

**Theorem 5.3** Let $(M, \Omega)$ be a locally homogeneous multisymplectic manifold and $\alpha \in \Omega^p(M)$, with $p = k - 1, k$, a differential form which is invariant by the set of locally Hamiltonian $(k - 1)$-vector fields and the set of locally Hamiltonian vector fields; that is, $L(X)\alpha = 0$ and $L(Z)\alpha = 0$, for every $X \in \mathfrak{X}_{th}^{k-1}(M)$ and $Z \in \mathfrak{X}_{th}(M)$. Then we have:

1. If $p = k$ then $\alpha = c \Omega$, with $c \in \mathbb{R}$.
2. If \( p = k - 1 \) then \( \alpha = 0 \).

(Proof)

1. First assume that \( p = k \).

For every \( X \in \mathfrak{X}^{k-1}_{lh}(M) \), according to Lemma 5.2 (item 2), we have

\[
i(X)\alpha = \alpha' i(X)\Omega = i(X)(\alpha' \Omega),
\]

where \( \alpha' \in C^\infty(M) \) is the same function for every \( X \in \mathfrak{X}^{k-1}_{lh}(M) \). However, taking into account item 2 of Lemma 3.2 (for 1-nondegenerate forms), the above equality holds for every \( X \in \mathfrak{X}^{k-1}(M) \), and thus \( \alpha = \alpha' \Omega \).

Then, for every \( Z \in \mathfrak{X}_{lh}(M) \), by hypothesis

\[
0 = L(Z)\alpha = (L(Z)\alpha')\Omega.
\]

Hence, \( L(Z)\alpha' = 0 \), but because of Lemma 4.5 locally Hamiltonian vector fields span the tangent space, thus \( \alpha' = c \) (constant). So

\[
i(X)\alpha = ci(X)\Omega = i(X)(c\Omega),
\]

and taking into account item 2 of Lemma 3.2 (for 1-nondegenerate forms) again, this relation also holds for every \( X \in \mathfrak{X}^{k-1}(M) \). Hence we must conclude that \( \alpha = c\Omega \).

2. If \( p = k - 1 \), the result follows straightforwardly from the first item of the lemma 5.2.

\[\square\]

Remark 3

- Theorem 1.2 is an immediate consequence of Theorem 5.3.

- Another immediate consequence of this theorem is that, if \( \alpha \in \Omega^k(M) \) is a differential form invariant by the sets of locally Hamiltonian \((k-1)\)-vector fields and locally Hamiltonian vector fields, then it is also invariant by the set of locally Hamiltonian \(m\)-vector fields, for \( 1 < m < k - 1 \).

- As is evident, if \( k = 2 \), we have proved (partially) the classical Lee Hwa Chung theorem for multisymplectic manifolds.

6 Characterization of multisymplectic transformations

Now we use the theorems above in order to give several characterizations of multisymplectic transformations in the same way as Lee Hwa Chung’s theorem allows us to characterize symplectomorphisms in the symplectic case [22, 28, 18].

A vector field \( X \) on a multisymplectic manifold \((M, \Omega)\) is said to be a conformal Hamiltonian vector field if there exists a function \( \sigma \) such that

\[
L(X)\Omega = \sigma\Omega.
\]

It is immediate to check that, if \( \Omega^r \neq 0, r > 1 \), then \( \sigma \) must be constant. Then:
Definition 6.1 A diffeomorphism \( \varphi : M_1 \to M_2 \) between the multisymplectic manifolds \( (M_i, \Omega_i) \), \( i = 1, 2 \), is said to be a special conformal multisymplectic diffeomorphism if there exists \( c \in \mathbb{R} \), such that \( \varphi^* \Omega_2 = c \Omega_1 \). The constant factor \( c \) will be called the scale or valence of \( \varphi \).

Therefore we have:

Theorem 6.2 Let \( (M_i, \Omega_i) \), \( i = 1, 2 \), be two locally homogeneous multisymplectic manifolds. A diffeomorphism \( \varphi : M_1 \to M_2 \) is a special conformal multisymplectic diffeomorphism if and only if the differential map \( \varphi_* : \mathfrak{X}(M_1) \to \mathfrak{X}(M_2) \) induces an isomorphism between the graded Lie algebras \( \mathcal{V}_H^*(M_1, \Omega_1), \mathcal{V}_H^*(M_2, \Omega_2) \). Then we have that

\[
\varphi_* X_\xi = \frac{1}{c} X_{\varphi^{-1} \xi} .
\]

In addition, if \( X_1 \in \mathfrak{X}^m_h(M_1) \) is any Hamiltonian (resp. locally Hamiltonian) multivector field with \( \xi_1 \in \Omega^{k-m-1}(M_1) \) a Hamiltonian form for \( X_1 \) (resp. locally Hamiltonian in some \( U_1 \subset M_1 \)), and \( \varphi_* X_1 = X_2 \in \mathfrak{X}^m_h(M_2) \) with Hamiltonian form \( \xi_2 \in \Omega^{k-m-1}(M_2) \) (resp. locally Hamiltonian in \( \varphi(U_1) = U_2 \subset M_2 \)); then

\[
\varphi_* X_1 = \varphi^* \xi_2 + \eta ,
\]

where \( \eta \in \Omega^{k-m-1}(M_1) \) is a closed form. In other words, \( \varphi^* \) induces an isomorphism between classes of Hamiltonian forms.

(Proof) Taking into account Proposition \[11\] we have:

\[ (\Longleftrightarrow) \] For every \( X_1 \in \mathfrak{X}^m_h(M_1) \) (resp. \( X_1 \in \mathfrak{X}^m_h(M_1) \)) we have that \( \varphi_* X_1 = X_2 \in \mathfrak{X}^m_h(M_2) \) (resp. \( X_2 \in \mathfrak{X}^m_h(M_2) \)). In any case \( L(X_2) \Omega_2 = 0 \), then we obtain

\[
0 = \varphi^* L(X_2) \Omega_2 = L(\varphi^{-1} X_2) \varphi^* \Omega_2 = L(X_1) \varphi^* \Omega_2 .
\]

Therefore, by Theorem \[5.3\] we have that \( \varphi^* \Omega_2 = c \Omega_1 \).

\[ (\Longrightarrow) \] Conversely, for every \( X_{\xi_1} \in \mathfrak{X}^m_h(M_1) \) we have \( \mathfrak{i}(X_{\xi_1}) \Omega_1 - d \xi_1 = 0 \). Then, since \( \varphi^* \Omega_2 = c \Omega_1 \), we obtain

\[
0 = \varphi^{-1}(\mathfrak{i}(X_{\xi_1}) \Omega_1 - d \xi_1) = \mathfrak{i}(\varphi_* X_{\xi_1}) \varphi^{-1} \Omega_1 - \varphi^{-1} d \xi_1 X_1
\]

\[
= \frac{1}{c} \mathfrak{i}(\varphi_* X_{\xi_1}) \Omega_2 - d \varphi^{-1} \xi_1 \Longleftrightarrow \mathfrak{i}(\varphi_* X_{\xi_1}) \Omega_2 - d \left( \frac{1}{c} \varphi^{-1} \xi_1 \right) = 0 .
\]

So, \( \varphi_* X_{\xi_1} = X_{\xi_2} \in \mathfrak{X}^m_h(M_2) \) and its Hamiltonian form \( \xi_2 \in \Omega^{k-m-1}(M_2) \) is related with \( \xi_1 \) by equation \[13\]

In an analogous way, using \( \varphi^{-1} \), we would prove that \( \varphi_*^{-1} X_2 \in \mathfrak{X}^m_h(M_1) \), for every \( X_2 \in \mathfrak{X}^m_h(M_2) \).

The proof for locally Hamiltonian multivector fields is obtained in the same way, working locally on \( U_1 \subset M_1 \) and \( U_2 = \varphi(U_1) \subset M_2 \).

As a consequence of the previous theorem there is another characterization of conformal multisymplectomorphisms.

Corollary 2 Let \( (M_i, \Omega_i) \), \( i = 1, 2 \), be two locally homogeneous multisymplectic manifolds. A diffeomorphism \( \varphi : M_1 \to M_2 \) is a special conformal multisymplectic diffeomorphism if and only if for every \( U_2 \subset M_2 \) and for every \( \xi_2 \in \Omega^p(U_2) \) and \( \zeta_2 \in \Omega^m(U_2) \) \( (p, m < k - 1) \), we have

\[
\varphi^* \{\xi_2, \zeta_2\} = \frac{1}{c} \{ \varphi^* \xi_2, \varphi^* \zeta_2 \} .
\]
(Proof) Let $X_{\zeta_2} \in \mathfrak{X}_h^{k-p-1}(M_2)$ and $Y_{\zeta_2} \in \mathfrak{X}_h^{k-m-1}(M_2)$ be Hamiltonian multivector fields having $\xi_2$ and $\zeta_2$ as Hamiltonian forms in $U_2$.

($\Rightarrow$) We have

$$\varphi^*\{\xi_2, \zeta_2\} = \varphi^*(Y_{\zeta_2})d\xi_2 = i(\varphi_*^{-1}Y_{\zeta_2})\varphi^*d\xi_2.$$  

However, if $\varphi$ is a conformal multisymplectomorphism (of valence $c$), according to Theorem 6.2, $\varphi_*^{-1}Y_{\zeta_2} \in \mathfrak{X}_h^m(M_1)$ and

$$i(\varphi_*^{-1}Y_{\zeta_2})\Omega_1 = \frac{1}{c}d\varphi^*\zeta_2 .$$

that is, $\varphi_*^{-1}Y_{\zeta_2} = \frac{1}{c}Y_{\varphi^*\zeta_2}$. Therefore, because of equation (15) we conclude

$$\varphi^*\{\xi_2, \zeta_2\} = i(\varphi_*^{-1}Y_{\zeta_2})\varphi^*d\xi_2 = \frac{1}{c}i(Y_{\varphi^*\zeta_2})\varphi^*d\xi_2 = \frac{1}{c}\{\varphi^*\xi_2, \varphi^*\zeta_2\}$$

($\Leftarrow$) Assuming that equation (14) holds and again using the definition of Poisson bracket, it can be written as

$$\varphi^*i(Y_{\zeta_2})d\xi_2 = i(\varphi_*^{-1}Y_{\zeta_2})\varphi^*d\xi_2 = \frac{1}{c}i(Y_{\varphi^*\zeta_2})\varphi^*d\xi_2 ,$$

for every $\xi_2$. Hence we conclude that $\varphi_*^{-1}Y_{\zeta_2} = \frac{1}{c}Y_{\varphi^*\zeta_2} \in \mathfrak{X}_h^m(M_1)$, for every $Y_{\zeta_2} \in \mathfrak{X}_h^m(M_2)$, and again because of Theorem 6.2 $\varphi$ is a special conformal multisymplectomorphism.

7 Proof of the main Theorem

We will prove now Theorem 1.1.

**Theorem 1.1.** Let $(M_i, \Omega_i)$, $i = 1, 2$, be two locally homogeneous multisymplectic manifolds and $G(M_i, \Omega_i)$ will denote their corresponding groups of automorphisms. Let $\Phi: G(M_1, \Omega_1) \to G(M_2, \Omega_2)$ be a group isomorphism which is also a homeomorphism when $G(M_i, \Omega_i)$ are endowed with the point-open topology. Then, there exists a $C^\infty$ diffeomorphism $\varphi: M_1 \to M_2$, such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$ for all $f \in G(M_1, \Omega_1)$ and the tangent map $\varphi_*$ maps locally Hamiltonian vector fields of $(M_1, \Omega_1)$ into locally Hamiltonian vector fields of $(M_2, \Omega_2)$. In addition, if we assume that $\varphi_*$ maps all infinitesimal automorphisms of $(M_1, \Omega_1)$ into infinitesimal automorphisms of $(M_2, \Omega_2)$, then there is a constant $c$ such that $\varphi^*\Omega_2 = c\Omega_1$.

(Proof) Let $M_i$, $i = 1, 2$, be two locally homogeneous multisymplectic manifolds and let $\Phi$ be a group isomorphism from $G(M_1, \Omega_1)$ to $G(M_2, \Omega_2)$, which is in addition a homeomorphism if we endow $G(M_i, \Omega_i)$ with the point-open topology. Then, Corollary 1 implies that the group $G(M_i, \Omega_i)$ acts transitively on $M_i$, $i = 1, 2$. Hence, by the main theorem in [42] there exists a bijective map $\varphi: M_1 \to M_2$ such that $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$. Moreover, the map $\varphi$ is a conformal multisymplectic diffeomorphism if $\Phi$ verifies the conditions in Theorem 1.1 as the following argument shows.

- $\varphi$ is a homeomorphism.
  - Let $\mathcal{A}(M)$ be the class of fixed subsets of $G(M, \Omega)$, i.e.,
    $$\mathcal{A}(M) = \{ \text{Fix}(f) \mid f \in G(M, \Omega) \}, \quad \text{Fix}(f) = \{ x \in M \mid f(x) = x \}.$$  
  - Let $\mathcal{B}(M)$ be the class of complements of elements of $\mathcal{A}(M)$, which is
    $$\mathcal{B}(M) = \{ B = M - A \mid A \in \mathcal{A}(M) \}.$$
Hence, $B(M)$ is a class of open subsets of $M$. If $B \in B(M)$ we can construct a multisymplectic diffeomorphism $g$ such that $B$ is the interior of $\text{supp}(g)$. In fact, for any point $x \in M$ and a neighborhood $U$ of $x$, it follows from Lemma 4.4 that there exists $B \in B(M)$ such that $x \in B \subset U$. Thus, $B(M)$ is a basis for the topology of $M$. Moreover, if $f \in G(M_1, \Omega_1)$, then $\text{Fix} (\varphi \circ f \circ \varphi^{-1}) = \varphi(\text{Fix}(f))$, and if $g \in G(M_2, \Omega_2)$, then $\text{Fix} (\varphi^{-1} \circ g \circ \varphi) = \varphi^{-1}(\text{Fix}(g))$. Hence, $\varphi, \varphi^{-1}$ take basic open sets (in $B(M)$) into basic open sets. Thus, they are both continuous, i.e., $\varphi$ is a homeomorphism.

- $\varphi$ is a smooth diffeomorphism.

To prove this we adapt the proof in [39] and [3] to our setting. To prove that $\varphi, \varphi^{-1}$ are $C^\infty$ it is enough to show that $h \circ \varphi \in C^\infty(M_1)$ for all $h \in C^\infty(M_2)$ and $k \circ \varphi^{-1} \in C^\infty(M_2)$ for all $k \in C^\infty(M_1)$.

Let $x \in M_1$ and $U$ be an open neighborhood of $x$, which is the domain of a local coordinate chart $\psi: U \to \mathbb{R}^n$. According to Lemma 4.5, there exist Hamiltonian vector fields $X_i$, with compact supports on $U$, which are a local basis for the vector fields on an open neighborhood of $x$ contained in $U$. Let $\phi^i_t$ be the $1$-parameter group of diffeomorphisms generated by $X_i$. Now let $X$ be any locally Hamiltonian vector field on $M_1$, which we localize on a neighborhood of $x$ in such way that its compact support will be contained in $U$. We will denote the localized vector field again by $X$. Let $\phi_t$ be a $1$-parameter group of multisymplectic diffeomorphisms generated by $X$ (which exists because $X$ is complete). For each $t$, $\Psi_t := \Phi(\phi_t) = \varphi \circ \phi_t \circ \varphi^{-1}$ is a $C^\infty$ multisymplectic diffeomorphism. The evaluation map,$$
abla : \mathbb{R} \times M_2 \longrightarrow M_2$$

is continuous. Moreover $\Psi_0 = \text{Id}$ and $\Psi_{t+s} = \Psi_t \circ \Psi_s$. Therefore, the map $\Psi$ is a continuous action of $\mathbb{R}$ on $M_2$ by $C^\infty$ diffeomorphisms. By the Montgomery-Zippin theorem, since $\mathbb{R}$ is a Lie group, this action is $C^\infty$, i.e., $\Psi$ is smooth in both variables $t$ and $x$. Therefore, the $1$-parameter group of multisymplectic diffeomorphisms $\Psi_t$ has an infinitesimal generator, i.e., a $C^\infty$ locally Hamiltonian vector field $X_\Psi$ such that,$$
abla \frac{d}{dt} \Psi_t = X_\Psi \circ \Psi_t.$$Given $h \in C^\infty(M_2)$, its directional derivative $X_\Psi(h)$ is a $C^\infty$ function. For any $x \in M_1$ we have,$$X_\Psi(h)(\varphi(x)) = \frac{d}{dt} h(\Psi_t(\varphi(x))) \bigg|_{t=0} = \frac{d}{dt} (h \circ \varphi)(\phi_t(x)) \bigg|_{t=0}.$$Therefore, if $X$ is any of the Hamiltonian vector fields $X_i$ above, for all $y$ in a small neighborhood of $x$, the preceding formula gives$$X_i(h \circ \varphi)(y) = \frac{d}{dt} (h \circ \varphi)(\phi^i_t(y)) \bigg|_{t=0} = (X_i)_\Psi(h)(\varphi(y)).$$This formula shows that $h \circ \varphi$ is a $C^1$-map and that for every locally Hamiltonian vector field $X$,$$(X_\Psi(h)) \circ \varphi = X(h \circ \varphi). \quad (16)$$To compute higher partial derivatives, we just iterate this formula using the vector fields $X_i$; for instance,$$(X_j)_\Psi((X_i)_\Psi(h)) \circ \varphi = X_j(X_i(h \circ \varphi)) .$$Since the Hamiltonian vector fields $X_i$ are a local basis for the vector fields on an open neighborhood of $x$, we have proved that $h \circ \varphi \in C^\infty(M_1)$.
• $\varphi_*$ maps locally Hamiltonian vector fields into locally Hamiltonian vector fields.

Equation (16) shows that $X_{\Psi} = \varphi_*X$ and because $\Psi_t$ is a flow of multisymplectic diffeomorphisms, then $\varphi_*X$ is another locally Hamiltonian vector field. Thus, $\varphi_*$ maps every locally Hamiltonian vector field into a locally Hamiltonian vector field.

• $\varphi$ is a special conformal multisymplectic diffeomorphism.

Finally we show that, with the additional hypothesis stated in Theorem 1.1, then $\varphi^*\Omega_2 = c\Omega_1$. In fact, if in addition we assume that the tangent map $\varphi_*$ maps all infinitesimal automorphisms of $(M_1, \Omega_1)$ into infinitesimal automorphisms of $(M_2, \Omega_2)$, then as a consequence of Theorem 6.2 we have that $\varphi^*\Omega_2 = c\Omega_1$. (It is important to point out that this conclusion cannot be reached unless this new hypothesis is assumed, since the starting set of assumptions allows us to prove only that $\varphi_*$ maps locally Hamiltonian vector fields into locally Hamiltonian vector fields. However, this result cannot be extended to Hamiltonian $m$-multivector fields, with $m > 1$.)

8 Conclusions

We show that locally homogeneous multisymplectic forms are characterized by their automorphisms (finite and infinitesimal). As pointed out in the introduction it is remarkable that a Darboux-type theorem for multisymplectic manifolds is not known, although a class of multisymplectic manifolds with a local structure defined by Darboux type coordinates has been characterized [10]. This obliges us to use a proof that does not rely on normal forms.

The statement in Theorem 1.1 can be made slightly more restrictive assuming that we are given a bijective map $\varphi: M_1 \to M_2$ such that it sends elements $f \in G(M_1, \Omega_1)$ to elements $\varphi \circ f \circ \varphi^{-1} \in G(M_2, \Omega_2)$. Then throughout the proof of the theorem in Section 7 we show that $\varphi$ is $C^\infty$. The generalization we present here uses the transitivity of the group of multisymplectic diffeomorphisms and is a simple consequence of theorems by Wechsler [42] and Boothby [7]. However, we do not know yet if the continuity assumption for $\Phi$ can be dropped and replaced by weaker conditions as in the symplectic and contact cases. To answer these questions, it would be necessary to describe the algebraic structure of the graded Lie algebra of infinitesimal automorphisms of the geometric structure as in the symplectic and volume cases [2]. A necessary first step in this direction will be to describe the extension of Calabi’s invariants to the multisymplectic setting.

We wish to stress that in the analysis of multisymplectic structures beyond the symplectic and volume manifolds, it is necessary to consider not only vector fields, but the graded Lie algebra of infinitesimal automorphisms of arbitrary order. Only the Lie subalgebra of derivations of degree zero is related to the group of diffeomorphisms, but derivations of higher degrees are needed to characterize the invariants.

Finally, we wish to remark that Theorem 5.3 (which plays a relevant role in this work) is just a partial geometric generalization for multisymplectic manifolds of Lee Hwa Chung’s theorem. A complete generalization would have to characterize invariant forms of every degree. Our guess is that, in order to achieve this, additional hypotheses must be considered; namely: strong non-degeneracy of the multisymplectic form and invariance by locally Hamiltonian multivector fields of every order. Nevertheless, it is important to point out that the hypotheses that we assume here (1-nondegeneracy, local homogeneity and invariance by locally Hamiltonian vector fields and
locally Hamiltonian \((k - 1)\)-multivector fields) are sufficient for our aim. This is a relevant fact since, for example, in the jet bundle description of classical field theories (the regular case), the Lagrangian and Hamiltonian multisymplectic forms are just 1-nondegenerate, and in the analysis of field equations, only locally Hamiltonian \((k - 1)\)-multivector fields are relevant \cite{13,14,16,33}.

9 Appendix: Locally special multisymplectic manifolds

We have seen that multicotangent bundles are locally homogeneous manifolds. It is remarkable that a Darboux-type theorem for multisymplectic manifolds, in general, is not known, although a class of multisymplectic manifolds with a local structure defined by Darboux type coordinates was characterized in \cite{10}. In \cite{26}, multisymplectic manifolds admitting Darboux-type coordinates (or what is equivalent, the existence of normal forms for the multisymplectic structure) have been classified, and this is a sufficient condition which guarantees that property. In fact, this is a large class of multisymplectic manifolds having the property of being locally homogeneous.

In order to introduce these manifolds, some previous concepts are required (see \cite{10,26} for details concerning the definitions and the subsequent results). First, given a multicotangent bundle \(\Lambda^k(T^*Q)\), let us denote by \(\Lambda_{\ast}^k(T^*Q)\) the subbundle made of those \(m\)-forms in \(Q\) vanishing when applied to \(r\) vector fields in \(Q\). Let \(\rho: \Lambda^k(T^*Q) \rightarrow Q\) and \(\rho_r: \Lambda_{\ast}^k(T^*Q) \rightarrow Q\) be the natural projections, and \(\Theta_Q \in \Omega^k(\Lambda^k(T^*Q))\), \(\Theta_Q^r \in \Omega^k(\Lambda_{\ast}^k(T^*Q))\) the tautological \(m\)-forms in these bundles.

Then, following the terminology introduced by Tulczyjew in \cite{40,41}, we define:

**Definition 9.1** A special multisymplectic manifold is a multisymplectic manifold \((M, \Omega)\) (of degree \(k\)) such that:

1. \(\Omega = d\Theta\), for some \(\Theta \in \Omega^{k-1}(M)\).
2. There exist a diffeomorphism \(\phi: M \rightarrow \Lambda^{k-1}(T^*Q)\), with \(\dim Q = n \geq k - 1\), (or \(\phi: M \rightarrow \Lambda_{\ast}^{k-1}(T^*Q)\)), and a fibration \(\pi: M \rightarrow Q\) such that \(\rho \circ \phi = \pi\) (resp. \(\rho_r \circ \phi = \pi\), and \(\phi^* \Theta_Q = \Theta\) (resp. \(\phi^* \Theta_Q^r = \Theta\)).

((\(M, \Omega\) is said to be multisymplectomorphic to a bundle of forms).

It is important to point out that every special multisymplectic manifold has a local chart of Darboux coordinates around every point; that is, we have coordinates \((x^i; p_{i_1\ldots i_{k-1}})_{i=1\ldots n}\) such that

\[
\Theta = p_{i_1\ldots i_{k-1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}} , \quad \Omega = dp_{i_1\ldots i_{k-1}} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}} .
\]  

(17)

If \((M, \Omega)\) is a multisymplectic manifold of degree \(k\) and \(W\) a distribution in \((M, \Omega)\), for every \(x \in M\) and \(l < k\), we define the vector space

\[
W(x)^{1,l} = \{ v \in T_x M \mid i(v \wedge w_1 \wedge \ldots \wedge w_l)\Omega_x = 0, \ \text{for every } w_1, \ldots, w_l \in W(x) \} .
\]

Then, \(W\) is said to be an \(l\)-isotropic distribution if \(W(x) \subset W(x)^{1,l}\), for every \(x \in M\). Therefore:

**Definition 9.2** Let \((M, \Omega)\) be a multisymplectic manifold of degree \(k\), and \(W\) a 1-isotropic involutive distribution of \((M, \Omega)\).
1. The triple \((M, \Omega, W)\) is a multisymplectic manifold of type \((k, 0)\) if, for every \(x \in M\), we have that:

(a) \(\dim W(x) = \dim \Lambda^{k-1}(T_x M/W(x))^*\).

(b) \(\dim (T_x M/W(x)) > k - 1\).

2. A multisymplectic manifold of type \((k, r)\) \((1 \leq r \leq k - 1)\) is a quadruple \((M, \Omega, W, E)\) such that \(E\) is a “generalized distribution” on \(M\) (in the sense that, for every \(x \in M\), \(E(x) \subset T_x M/W(x)\) is a vector subspace) and, for every \(x \in M\), denoting by \(\pi_x: T_x M \to T_x M/W(x)\) the canonical projection, we have that:

(a) \(i(v_1 \wedge \ldots \wedge v_r) \Omega_x = 0\), for every \(v_i \in T_x M\) such that \(\pi_x(v_i) \in E(x)\) \((i = 1, \ldots, r)\).

(b) \(\dim W(x) = \dim \Lambda^{k-1}_r(T_x M/W(x))^*\), where the horizontal forms are considered with respect to the subspace \(E(x)\).

(c) \(\dim (T_x M/W(x)) > k - 1\).

And the fundamental result is:

**Proposition 4** Every multisymplectic manifold of type \((k, 0)\) (resp. of type \((k, r)\)) is locally multisymplectomorphic to a bundle of \((k-1)\)-forms \(\Lambda^{k-1}(T^* Q)\) (resp. \(\Lambda^{k-1}_r(T^* Q)\)), for some manifold \(Q\); that is, to a canonical multisymplectic manifold. Therefore, there is a local chart of Darboux coordinates around every point.

**Definition 9.3** Multisymplectic manifolds which are locally multisymplectomorphic to bundles of forms are called locally special multisymplectic manifolds.

Obviously, every special multisymplectic manifold is a locally special multisymplectic manifold.

**Remark 4** As is evident, locally special multisymplectic manifolds have local Euler-like vector fields; in particular, the local vector fields \(x^i \frac{\partial}{\partial x^i} + p_{i_1 \ldots i_k} \frac{\partial}{\partial p_{i_1 \ldots i_k}}\). Thus, the corresponding multisymplectic forms are locally homogeneous.

A far reaching consequence of all of this is the transitivity of the group of multisymplectic diffeomorphisms since, if \((M, \Omega)\) is a locally special multisymplectic manifold, then the family of locally Hamiltonian vector fields span locally the tangent bundle of \(M\). In fact, taking into account the local expression \[\Omega\] of \(\Omega\), we have that the local vector fields \(\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_{i_1 \ldots i_k}} \right\}\) are locally Hamiltonian.

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