EDGE STATE DYNAMICS ALONG CURVED INTERFACES

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Abstract. We study the propagation of wavepackets along weakly curved interfaces between topologically distinct media. Our Hamiltonian is an adiabatic modulation of Dirac operators omnipresent in the topological insulators literature. Using explicit formulas for straight edges, we construct a family of solutions that propagates, for long times, unidirectionally and dispersion-free along the curved edge. We illustrate our results through various numerical simulations.

1. Introduction

Topological insulators are fascinating materials that are insulating in their bulk but support robust currents along their boundary. From a mathematical point of view, these properties are consequences of the bulk-edge correspondence, an index-like theorem that relates the net conductivity (an analytic index) to the bulk topology (a topological index). For straight interfaces, the currents are explicitly described in terms of edge states: steady waves with ballistic dynamics, confined between regions of distinct topology.

In this work, we construct dynamical analogues of edge states for curved interfaces. Our model is a Dirac operator

\[ H = \begin{bmatrix} \kappa(x) & \varepsilon D_{x_1} - i \varepsilon D_{x_2} \\ \varepsilon D_{x_1} + i \varepsilon D_{x_2} & -\kappa(x) \end{bmatrix} \]  

(1.1)

where \( D_{x_j} = -i \partial_{x_j} \), \( \varepsilon > 0 \) is a small semiclassical parameter and \( \kappa \) is a varying mass term. Such Hamiltonians emerge in the effective theory of honeycomb structures [FLTW16, LTWZ19, Dro19b]; more generally they model the generic dynamics of modes propagating along interfaces between topologically distinct insulators [Dro21b].

Under a transversality condition \(- \nabla \kappa(x) \neq 0 \) when \( \kappa(x) = 0 \) – the set \( \Gamma = \{ x \in \mathbb{R}^2 : \kappa(x) = 0 \} \)

partitions \( \mathbb{R}^2 \) in regions of distinct local topology – see §1.5 for details. A local interpretation of the bulk-edge correspondence suggests that non-trivial currents emerge along \( \Gamma \). This

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Snapshots of the numerically computed dynamical analogue of an edge state – the solution to (1.3) below. The interface is \( y_2 = \tanh(y_1) \) and \( \varepsilon = 10^{-1} \). The state propagates leftwards and dispersion-free along the interface.}
\end{figure}
paper develops the underlying quantitative theory: it provides detailed information on the associated quantum states, such as their speed and profile.

Specifically, we exploit the explicit structure of edge states available when $\kappa(x) = a_1x_1 + a_2x_2$ to construct an infinite-dimensional family of nearly steady solutions to $(\varepsilon D_t + H)\psi = 0$, in the limit $\varepsilon \to 0$. These emerge as the natural channels of conductivity: for long times, they propagate unidirectionally and coherently along $\Gamma$. We show that the curvature of $\Gamma$ plays a key role in limiting the lifetime of these solutions. We illustrate our results via various numerical simulations.

1.1. Simplified main result. Throughout the paper, we assume that $\kappa$ and all its derivatives are bounded: $\kappa \in C^\infty_b(\mathbb{R}^2)$. In this introduction, we require moreover that $y \in \Gamma \Rightarrow \lvert \nabla \kappa(y) \rvert = 1$. (1.2)

This allows us to state a simplified version (Theorem 1) of our main result (Theorem 2). In §3, we replace (1.2) by the more general transversality condition (3.1).

Fix $y_0 \in \Gamma = \kappa^{-1}(0)$ and define $y_t$ by the ODE $\dot{y}_t = \nabla \kappa(y_t)$, where $\nabla \kappa(y)$ denotes the \(\pi/2\)-counterclockwise rotation of $\nabla \kappa(y)$. Under (1.2), $y_t$ is a unit speed parametrization of $\Gamma$. We let $\theta_t$ be the angle between the tangent to $\Gamma$ at $y_t$ and the $x$-axis – see Figure 2. We use the notation $\langle t \rangle = (1 + |t|^2)^{1/2}$.

Theorem 1. Let $\kappa \in C^\infty_b(\mathbb{R}^2)$ satisfy (1.2) and $y_t, \theta_t$ as above. The solution to $$(\varepsilon D_t + H)\Psi_t = 0, \quad \Psi_0(x) = \frac{1}{\sqrt{\varepsilon}} \cdot \exp \left( -\frac{(x - y_0)^2}{2\varepsilon} \right) \left[ e^{-i\theta_0/2} - e^{i\theta_0/2} \right]$$ (1.3) satisfies, uniformly for $\varepsilon \in (0, 1]$ and $t > 0$: $$\Psi_t(x) = \frac{1}{\sqrt{\varepsilon}} \cdot \exp \left( -\frac{(x - y_t)^2}{2\varepsilon} \right) \left[ e^{-i\theta_t/2} - e^{i\theta_t/2} \right] + O_{L^2} (\varepsilon^{1/2} \langle t \rangle) .$$ (1.4)

The initial data (1.3) is a Gaussian concentrated at $y_0$. Theorem 1 shows that the generated solution remains (at leading order, for times $t \ll \varepsilon^{-1/2}$) a Gaussian, concentrated now at $y_t$. This identifies $t \mapsto y_t$ as an exotic quantum trajectory: it is not predicted by the standard results on propagation of semiclassical singularities. See §1.5 for a semiclassical discussion.

![Figure 2. Schematic plot of an interface $\Gamma = \kappa^{-1}(0)$ between topologically distinct regions, together with $y_t$ and $\theta_t$.](image-url)
If $\Gamma$ is not asymptotically straight – for instance if it is a loop – numerical computations confirm that the Gaussian state approximation becomes less and less accurate, see Figure 3. In contrast, if $\Gamma$ is asymptotically straight – as in e.g. the tanh-like interface of Figure 1 – the Gaussian state approximation can work for longer times, see Theorem 4.

We refer to Theorem 2 for a more general version of Theorem 1. It constructs an infinite dimensional family of solutions to $(\varepsilon D_t + H)\Psi_t = 0$ with the same qualitative features as (1.4): coherent states propagating unidirectionally, at unit speed and without dispersion, along $\Gamma$. Our motivation, explained in §1.4 and §1.5 below, is two-fold:

- Identify dynamical analogues of topological edge states along bent interfaces;
- Study a semiclassical system whose matrix-valued symbol has repeated eigenvalues.

1.2. Numerical simulations. We illustrate our results with numerical simulations of the Dirac equation with a Gaussian initial data, for various types of interfaces. The corresponding pictures are snapshots of the dynamics, with the interface marked as a light blue curve.

- Figure 1 and 3 are numerical confirmations of Theorem 1 for tanh-type and circle interfaces, respectively. Figure 3 also verifies that the phase shift after one revolution equals $2\pi/2 = \pi$.

![Figure 3](image-url)

**Figure 3.** Left: numerical solution to $(\varepsilon D_t + H)\Psi_t = 0$ with Gaussian initial state for a circular interface with $\varepsilon = 10^{-2}$ and radius one. The trajectory $y_t$ undergoes curvature effects for all times. This explains a dispersion stronger than for a tanh-type interface. See also Figure 7 and Theorem 4. Right: evolution of the phase of the first coordinate of the numerical solution for each snapshot – corresponding to $-\theta_t/2$ – for different radii of the circle-interface. After a full revolution, the numerical phase difference is about $-\pi$, matching the theoretical prediction $-2\pi/2 = -\pi$. This phase shift interprets as a Berry phase arising from adiabatically varying the parameter $\theta$ in the effective leading order operator $H_{\theta,r}$ (2.1) from 0 to $2\pi$. 
• Figure 4 shows the evolution of other Gaussian states for tanh-type interfaces. The initial data are concentrated like (1.3) but carried by a different vector. If this vector is orthogonal to that in (1.3), the coherence is immediately lost. See Conjecture 1.
• When the more general transversality condition (3.1) holds instead of (1.2), the propagation is coherent in a relaxed sense. Figure 6—a straight interface but a non-linear domain wall—numerically validates Theorem 2.
• Figure 7 illustrates the limits of the dynamical analogues of edge states: for instance, they do not propagate around sharp corners.

We use a Crank-Nicholson scheme to approximate the unitary group $e^{-itH}$, with Fourier spectral spatial discretization. The Matlab code containing the parameters used to obtain our figures can be found on GitHub.1

1.3. Physical motivations. The Dirac equation appears in a wide variety of physical applications. Beyond its original role in the description of relativistic particles, it has emerged as a dominant model in the analysis of topological phases of matter [Vol89, Wit16]. The relativistic Dirac operator ($\kappa = 0$ in our model) displays a generic band crossing; in contrast, adding a mass term opens an energy gap. In our model, the interface is the transition between the two insulating phases $\kappa < 0$ and $\kappa > 0$. These two phases happen to have different topological signatures; this generates unidirectional propagation along the interface.

This asymmetric transport is at the core of most physical applications in the fields of topological insulators and topological superconductors [Ber13, Vol89]. It is the physical manifestation of the quantum Hall effect [BvESB94, ASS90] and its non-magnetic analogues [C+13, Hal88, JS20, HIA19, LD20, S+18]. It also finds numerous applications in fields such as photonics, acoustics, and fluid mechanics [LJS14, PBSM15, RH08, R+13, GJT21]. Broadly speaking, Dirac-type equations often offer the simplest continuum (macroscopic) description of transport in a narrow energy band near the band crossing [Ber13, FC13, Vol89].

1.4. Local topological indices and asymmetric transport. Strikingly, transport at interfaces between distinct topological environments is both asymmetric (a net overall flux propagates in a prescribed direction) and quantized. We discuss here a theory of topological phases that interprets locally the state (1.4) in a topological way. We stress that this interpretation:

- is valid only in the semiclassical regime $\varepsilon \ll 1$;
- is local: our construction works for all $\kappa$, even though in some scenarios $H$ is topologically trivial (for instance when $\Gamma$ is a closed curve).

These considerations use the leading-order approximation $H_y$ of $H$ at a point $y \in \mathbb{R}^2$:

\[
H_y = \begin{bmatrix}
\kappa(y) & \varepsilon D_{x_1} - i\varepsilon D_{x_2} \\
\varepsilon D_{x_1} + i\varepsilon D_{x_2} & -\kappa(y)
\end{bmatrix}, \quad y \notin \Gamma;
\]

\[
H_y = \begin{bmatrix}
-v_y \cdot (x - y) & \varepsilon D_{x_1} - i\varepsilon D_{x_2} \\
\varepsilon D_{x_1} + i\varepsilon D_{x_2} & v_y \cdot (x - y)
\end{bmatrix}, \quad y \in \Gamma,
\]

where $v_y = \nabla \kappa(y)^\perp$ is tangent to $\Gamma$ at $y$. These emerge by replacing $\kappa(x)$ in (1.1) by its leading-order development at $y$: $\kappa(x) \simeq \kappa(y)$ if $y \notin \Gamma$ and $\kappa(x) \simeq \nabla \kappa(y) \cdot (x - y)$ if $y \in \Gamma$. These approximations are reasonable for $|x - y| = O(\varepsilon^{1/2})$: the scale of localization of (1.4).

1https://github.com/slb2604/Semiclassical-edge-states
We observe that $H_y$ has a spectral gap near energy 0 (i.e. it is an insulator) if and only if $y \notin \Gamma$. This identifies $\Gamma$ as the natural channel for conduction of energy. Following [EG02, KRS02], we measure the local conductivity at $y \in \Gamma$ via:

$$I(H, y) = \text{Tr}_{L^2} \left( i[H_y, f(v_y \cdot x)] g'(H_y) \right), \quad y \in \Gamma, \quad (1.5)$$

where $f$ and $g$ are smooth real functions increasing from 0 to 1 with $f'$ and $g'$ compactly supported. Formally,

$$I(H, y) = \frac{d}{dt} \text{Tr}_{L^2} \left( e^{itH_y} f(v_y \cdot x) g'(H_y) e^{-itH_y} \right). \quad (1.6)$$

Looking at $g'$ as a density of probability, $f(v_y \cdot x) g'(H_y)$ measures the probability of a quantum particle to lie in the half-plane $\{v_y \cdot x > 0\}$, per unit energy. Taking the trace in (1.6) corresponds to summing over all states. Hence $I(H, y)$ describes the overall flux moving in the direction of $v_y$, per unit time and energy, at equilibrium.

It turns out that $2\pi \cdot I(H, y) = 1$, see [Bal19] and Remark 1 below. This means that the evolution according to $H_y$ comes with a current propagating in the direction of $v_y$. Since $v_y$ is tangent to $\Gamma$ at $y$, $\Gamma$ emerges intuitively as a natural charge-carrier for $H$. Theorem 1 confirms these heuristics: in the regime $\varepsilon \to 0$, we construct a current propagating along $\Gamma$, with explicit speed and profile.

The quantity (1.5) relates to bulk topological invariants via a universal principle: the bulk-edge correspondence [Hat93, GP13, PSB16, Bal20, Dro21a]. Following the physics literature [Hal88, HIA19], we define a bulk index for $H_y$:

$$B(H, y) = \text{sgn} \left( \kappa(y) \right), \quad y \notin \Gamma. \quad (1.7)$$

When $H$ emerges as an effective Hamiltonian (for instance in graphene), $B(H, y)$ corresponds to the integrated Berry curvature near one of the Dirac point momentum, hence as part of the overall Chern integer [Dro19a]. Direct interpretations of (1.7) as a Chern number include regularization of Dirac operator [Bal19a] and more general bulk-difference invariant [Bal20]. We refer to (1.7) as the local bulk index. It can also be defined by spatially truncating physical space formulas for the global Chern number [Kit06, BR11, PSB16]; or via the spectral localizer [Lor15, LSB20].

Since $\nabla \kappa$ points from negative to positive-index regions, we have for $y \in \Gamma$ and $\delta > 0$ sufficiently small:

$$1 = 2\pi \cdot I(H, y) = B(H, y + \delta \nabla \kappa(y)) - B(H, y - \delta \nabla \kappa(y)).$$

This is a local version of the bulk-edge correspondence: the local conductivity at $y$ is the difference between the local bulk indices across the interface.

The quantity $2\pi \cdot I(H, y)$ counts currents algebraically according to their direction of propagation. It is independent of $y$ and stable against large perturbations of $H$; see, e.g. [Bal19a, Bal20] and [PSB16] for similar models. This explains its practical significance: even in the presence of strong perturbations or Anderson localization, there is always $2\pi \cdot I(H, y) = 1$ more current propagating in the direction of $v_y$ rather than $-v_y$ [Bal19b, PSB16]. This clarifies the local topological nature of the quantum state (1.4). Let us stress again that our results hold locally in time: (1.5) is spectral in nature, describing an equilibrium, while (1.4) is relevant for (long, but only transient) times $t \ll \varepsilon^{-1/2}$. 
1.5. **Connection with semiclassical analysis.** What makes the solution (1.4) special? The answer lies in semiclassical territory. In summary (with details provided below): if $C = \Gamma \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^2$, then for times $t \ll \varepsilon^{-1/2}$:

(i) States initially microlocalized at $(y_0, \xi_0) \notin C$ come in pairs propagating in opposite directions;

(ii) States initially microlocalized at $(y_0, \xi_0) \in C$ (i.e. like (1.3), with a potentially different 2-vector) seem to either propagate non-dispersively in the direction of $\nabla \kappa^\perp$, or to disperse; see Figure 4 and Conjecture 1.

This suggests that $\Gamma$—more precisely, its phase-space lift $C$—is the relevant channel for asymmetric propagation.

We now provide a detailed account. We start by writing $H = h(x, \varepsilon D_x)$, where

$$h(x, \xi) = \begin{bmatrix} \kappa(x) & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\kappa(x) \end{bmatrix}.$$

**Theorem 1** constructs solutions to $(\varepsilon D_t + \mathfrak{h}(x, \varepsilon D_x))\phi_t = 0$ for the data

$$\phi_0(x) = \frac{1}{\sqrt{\varepsilon}} \cdot e^{\frac{i}{\varepsilon} x \cdot \xi_0} a \left(\frac{x - x_0}{\sqrt{\varepsilon}}\right), \quad a \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)$$

where $(x_0, \xi_0)$ belongs to the set $C$ defined by

$$C = \{(x, \xi) : \kappa(x) = 0, \xi = 0\} \subset \mathbb{R}^4.$$

The function $\phi_0$ is known in the literature as a semiclassical wavepacket [CR12] with wavefront set $WF_{\varepsilon}(\phi_0) = \{(x_0, \xi_0)\}$—see [Zwo12, §8.4] for definitions and properties of wavefronts. The set $C$ corresponds to semiclassical eigenvalue crossings of $\mathfrak{h}(x, \xi)$: when $(x, \xi) \in C$, $\mathfrak{h}(x, \xi)$ has two degenerate eigenvalues. The systematic study of such semiclassical systems is a delicate problem. In the context of the Landau–Zener effect, which corresponds to a varying crossing energy, we refer to [CdV04] for a derivation of local normal forms, and to [Hag94] for an explicit description of the transition.

This paper focuses on the dynamics of wavepackets localized along $C$ (note that the crossing energy is constant, equal to 0). One could have likewise studied the dynamics of wavepackets semiclassically concentrated at points $(x_0, \xi_0) \notin C$. This is actually a much more standard problem because the eigenvalues of $\mathfrak{h}(x_0, \xi_0)$ are distinct: they are $\pm \lambda(x_0, \xi_0)$, where

$$\lambda(x, \xi) = \sqrt{\kappa(x)^2 + \xi_1^2 + \xi_2^2};$$

we note that $\lambda$ does not vanish away from $C$. We diagonalize $\mathfrak{h}(x, \xi)$ for $(x, \xi)$ near $(x_0, \xi_0)$:

$$\mathfrak{h}(x, \xi) = \mathcal{U}(x, \xi) \begin{bmatrix} -\lambda(x, \xi) & 0 \\ 0 & \lambda(x, \xi) \end{bmatrix} \mathcal{U}(x, \xi)^{-1},$$

where $\mathcal{U}$ is a unitary $2 \times 2$ matrix that depends smoothly on $(x, \xi)$. Thus, after quantization, the system $(\varepsilon D_t + \mathfrak{h}(x, \varepsilon D_x))\psi = 0$ splits semiclassically near $(x_0, \xi_0)$ in two nearly decoupled equations [Teu03, MS09]:

$$\begin{bmatrix} \varepsilon D_t + \begin{bmatrix} -\lambda(x, \varepsilon D_x) & 0 \\ 0 & \lambda(x, \varepsilon D_x) \end{bmatrix} + \mathcal{O}(\varepsilon) \end{bmatrix} \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = 0.$$
According to the classical-to-quantum correspondence, the wavefront set of $\phi_t$ follows the semiclassical trajectories of $\pm \lambda(x, \xi)$ – see e.g. [Zwo12, Theorem 12.5]. These form two branches $(x_t^+, \xi_t^+)$ and $(x_t^-, \xi_t^-)$, that solve respectively
\[
\frac{dx_t^\pm}{dt} = \pm \frac{\partial \lambda}{\partial \xi} (x_t^\pm, \xi_t^\pm), \quad \frac{d\xi_t^\pm}{dt} = \mp \frac{\partial \lambda}{\partial x} (x_t^\pm, \xi_t^\pm). \quad (1.8)
\]

The Hamiltonian trajectories (1.8) never reach $C$ because (a) the energy $\pm \lambda(x_0, \xi_0) \neq 0$ is conserved along them; and (b) $C$ is the zero set of the function $\lambda$. Hence, if $(x_0, \xi_0) \notin C$ then the semiclassical singularities of $\phi_t$ globally evolve according to the classical-to-quantum correspondence: they follow the Hamiltonian trajectories (1.8) and never reach $C$.

Moreover, the two branches in (1.8) point (at $t=0$) in opposite directions: wavepackets concentrated away from $C$ have no preferred direction of propagation. Their contribution to an overall quantum flux cancel out. Hence, $C$ is the only phase-space channel that can support unidirectional waves.

This discussion connects various characterizations of the set $C$:

(i) **Semiclassical:** $C$ is the set of eigenvalue crossings of $\hbar(x, \xi)$;

(ii) **Energetic:** $C$ is the characteristic set of $\hbar(x, \xi)$, i.e. the set of points $(x, \xi)$ such that $\det \hbar(x, \xi) = 0$.

(iii) **Topological:** the local Chern number is not defined on $\Gamma = \kappa^{-1}(0) = \pi(C)$ (with $\pi(x, \xi) = x$) because the eigenvalues of $\hbar(x, \xi)$ are degenerate on $C$.

(iv) **Dynamical:** Among phase-space subsets, $C$ is the only (maximal) candidate that may support unidirectional wavepackets.

Because of (i), the classical-to-quantum correspondence fails. Because of conservation of energy, (ii) suggests that a state semiclassically concentrated along $C$ should remain this way: $C$ acts as a semiclassical waveguide. Theorem 1 provides the corresponding profile and speed. Under global assumptions on $\kappa$, the bulk-edge correspondence predicts a non-vanishing quantum flux between regions of different topology. From (iii), $C$ acts as the natural topological interface in phase-space. According to (iv), it is also the only channel that can support waves contributing to a non-trivial conductivity.

A legitimate criticism to Theorem 1 is that it does not study the dynamics of all initial data localized along $C$: it focuses on those parallel to the two-vector $[e^{-i\theta_0}, e^{i\theta_0}]^\top$. As demonstrated numerically in Figure 4 the data prepared along the orthogonal two-vector $[-e^{i\theta_0}, e^{-i\theta_0}]^\top$ appear to purely disperse along the interface. An investigation of the linear case suggests that the rate of dispersion is $\varepsilon^{-1/4}t^{-1/2}$.

Thus, we conjecture that general initial data semiclassically localized along $C$ transit to the state (1.4). To write a precise statement, we split vectors $[\alpha_1, \alpha_2]^\top \in \mathbb{C}^2$ according to:
\[
[\alpha_1] \\
[\alpha_2] = \lambda_1 \begin{bmatrix} e^{-i\theta_0/2} \\
-e^{i\theta_0/2} \end{bmatrix} + \lambda_2 \begin{bmatrix} e^{-i\theta_0/2} \\
-e^{i\theta_0/2} \end{bmatrix}. \quad (1.9)
\]

We interpret the two terms in (1.9) as projections on the vector from (1.3) and its orthogonal.

**Conjecture 1.** Fix $y_0 \in \Gamma$, $\alpha_1, \alpha_2 \in \mathbb{C}$, and $\lambda_1, \lambda_2$ defined according to (1.9). There exists $\beta < 3/4$ such that under (1.2), the solution $\Psi_t$ to
\[
(\varepsilon D_t + H)\Psi_t = 0, \quad \Psi_0(x) = \frac{1}{\sqrt{\varepsilon}} \cdot \exp \left(-\frac{|x - y_0|^2}{2\varepsilon} \right) \begin{bmatrix} \alpha_1 \\
\alpha_2 \end{bmatrix} \quad (1.10)
\]
Solution to (1.10) for a tanh-like interface with (a) $[\alpha_1, \alpha_2] = [e^{-i\theta_0/2}, e^{i\theta_0/2}]$ and (b) $[\alpha_1, \alpha_2] = [0, e^{-i\theta_0/2}]$. Case (a) corresponds to $[\alpha_1, \alpha_2]$ orthogonal to the vector $[e^{-i\theta_0/2}, -e^{i\theta_0/2}]$ from the initial data of Theorem 1. This generates a purely dispersive wave along the interface. Case (b) corresponds to a linear combination of (1.3) and of Case (a): the solution splits into leftwards-propagating and dispersive components.

The $L^\infty$-remainder in (1.11) is smaller than the leading order term as long as $\varepsilon^{-1/2} \ll \varepsilon^{1-2\beta} \ll t$. Hence, according to this conjecture, $\Psi_t$ is well approximated by the Gaussian term in (1.11) for times $\varepsilon^{1-2\beta} \ll t \ll \varepsilon^{-1/2}$ (with $\beta < 3/4$ ensuring that such times exist). This indicates that dynamical edge states generically emerge from the evolution of initial data localized along $C$. See §3.3 for a more general version of Conjecture 1.

1.6. Organization of the paper. We organize the paper as follows:

- In §2 we review edge state theory for Dirac operators with straight domain walls, i.e. $\kappa(x) = a \cdot x$ in (1.1).
- In §3 we derive the analogues of edge states for weakly curved interface. Specifically, we construct an infinite-dimensional family of solutions to $(\varepsilon D_t + H)\Psi_t = 0$ that propagates along the topological interface $\Gamma$ for times up to $\varepsilon^{-1/2}$. The key ingredient is a local approximation of $H$ by Dirac operators with straight interfaces.
- In §4 we investigate, under a geometric condition of $\kappa$, how the curvature of $\Gamma$ affects the propagation of wavepackets.

Notations.

- We use $\sigma_1, \sigma_2, \sigma_3$ for the standard Pauli matrices:
  $$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$
• A smooth function $f$ on $\mathbb{R}^2$ belongs to $C^\infty_b(\mathbb{R}^2)$ if it is uniformly bounded, together with its derivatives at all order.
• A function $f \in C^\infty_b(\mathbb{R}^2)$ belongs to $\mathcal{S}(\mathbb{R}^2)$ if $x^\alpha \partial_x^\beta f$ is uniformly bounded for any $\alpha, \beta$.
We provide $\mathcal{S}(\mathbb{R}^2)$ with the family of seminorms $|x^\alpha \partial_x^\beta f|_{L^\infty}$.  
• The operators $D_{x_j}$ and $D_t$ are defined by $D_{x_j} = -i\partial_{x_j}$ and $D_t = -i\partial_t$.
• We use the japanese bracket notation: $\langle x \rangle = \sqrt{1 + |x|^2}$.
• We denote by $\ker\gamma(A)$ the kernel of a linear operator $A$ acting on a vector space $\mathcal{V}$.
• If $v \in \mathbb{R}^2$, $v^\perp$ is the counterclockwise $\pi/2$-rotation of $v$.
• $\langle u, v \rangle_{L^2} = \int_{\mathbb{R}^2} \overline{u}v$.
• For $f$ in a normed vector space $\mathcal{X}$, we write $f = \mathcal{O}_X(\varepsilon)$ if $|f|_\mathcal{X} \leq C\varepsilon$ for some constant $C > 0$ independent of $\varepsilon$.
• Given $\alpha \in \mathbb{C}^2$, $\alpha^\perp = -i\sigma_2\alpha$ is the $\pi/2$-rotation of $\alpha$.
• $y_t$ is the solution to the ODE (1.2) with initial data $y_0 \in \Gamma$; $\theta_t$ is the angle between the $y$-axis and $\nabla\kappa(y_t)$; and $r_t = |\nabla\kappa(y_t)|$. See Figure 2.

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2. Edge states and dynamics for straight interfaces

We review here the simplest example of domain wall $\kappa$: we write

$$\kappa(x) = \kappa_{\theta,r}(x) = -r \sin(\theta)x_1 + r \cos(\theta)x_2 = r \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \cdot x$$

with $\theta \in \mathbb{R}$, $r > 0$. The interface $\kappa_{\theta,r}^{-1}(0) = \mathbb{R}v_\theta$ is a straight line, directed by the vector $v_\theta = -[\cos(\theta), \sin(\theta)]^\top$ – see Figure 5. The Hamiltonian is then

$$H_{\theta,r} = \begin{bmatrix} \kappa_{\theta,r}(x) \\ \varepsilon D_{x_1} + i\varepsilon D_{x_2} \end{bmatrix} \begin{bmatrix} \kappa_{\theta,r}(x) \\ \varepsilon D_{x_1} + i\varepsilon D_{x_2} \end{bmatrix}^{-1}.$$

(2.1)

It admits edge states: solutions to $(H_{\theta,r} - \lambda)F_{\theta,r} = 0$ that are localized and harmonic along $\mathbb{R}v_\theta$. Here we review their explicit expression and their dynamical properties.

2.1. Conjugation properties. We first show that the Hamiltonians $H_{\theta,r}$ and $H_{0,r}$ are conjugated by a change of frame and gauge. For this purpose, we introduce the operator

$$U_\theta f(x) = U_\theta f(R_\theta x), \quad R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad U_\theta = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}.$$

(2.2)

Lemma 2.1. The Hamiltonian (2.1) is unitarily equivalent to the Hamiltonian $H_{0,r}$ with

$$U_\theta^{-1} H_{\theta,r} U_\theta = H_{0,r}.$$
Proof. Let $\mathcal{R}_\theta$ be the pullback operator by $R_\theta$: $\mathcal{R}_\theta f(x) = f(R_\theta x)$. We note that $\kappa_{\theta,r}(x) = r \cdot R_\theta^{-1} e_2 \cdot x = r (R_\theta x)_2$. Thus $\mathcal{R}_\theta^{-1} \kappa_{\theta,r} \mathcal{R}_\theta = r x_2$. We now use $\mathcal{R}_\theta^{-1} D_x \mathcal{R}_\theta = R_\theta^\top D_x$ to compute partial derivatives involved in $H_{\theta,r}$:

$$\mathcal{R}_\theta^{-1}(D x_1 + i D x_2) \mathcal{R}_\theta = \left[ \begin{array}{c} 1 \\ i \end{array} \right] \cdot R_\theta^\top D_x = R_\theta \left[ \begin{array}{c} 1 \\ i e^{i \theta} \end{array} \right] \cdot D_x = e^{i \theta} (D x_1 + i D x_2).$$

The adjoint identity is

$$\mathcal{R}_\theta^{-1}(D x_1 - i D x_2) \mathcal{R}_\theta = e^{-i \theta} (D x_1 - i D x_2).$$

Grouping these identities, we obtain:

$$\mathcal{R}_\theta^{-1} H_{\theta,r} \mathcal{R}_\theta = \begin{bmatrix} r x_2 & e^{-i \theta} \varepsilon (D x_1 - i D x_2) \\ e^{i \theta} \varepsilon (D x_1 + i D x_2) & -r x_2 \end{bmatrix}$$

where, $s_1, s_2, s_3$ are $2 \times 2$ Hermitian matrices given by

$$s_1 = \begin{bmatrix} 0 & e^{-i \theta} \\ e^{i \theta} & 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & -i e^{-i \theta} \\ i e^{i \theta} & 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_3.$$

An explicit calculation shows that $U_\theta^{-1} s_j U_\theta = \sigma_j$. We conclude that

$$U_\theta^{-1} H_{\theta,r} U_\theta = \sigma_1 \varepsilon D x_1 + \sigma_2 \varepsilon D x_2 + \sigma_3 x_2 = H_{0,r}.$$  \hfill (2.3)

This completes the proof. \hfill \Box

**Remark 1.** The relation (2.2) allows us to calculate the conductivity of $H_{\theta,r}$ in the direction of $v_\theta$, see (1.5): it is equal to 1. Indeed, the conductivity of $H_{0,r}$ (counted positively in the direction of $e_2^\perp = -e_1$) is equal to 1 [Bal19a]. Therefore, using invariance of the trace under conjugation, and the fact that $f$ is a scalar function:

$$1 = \text{Tr}_{L^2} \left( [H_{0,r}, f(-x_1)] g'(H_{0,r}) \right)$$

$$= \text{Tr}_{L^2} \left( [H_{\theta,r}, R_\theta f(-x \cdot e_1) R_\theta^{-1}] g'(H_{\theta,r}) \right) = \text{Tr}_{L^2} \left( [H_{\theta,r}, f(v_\theta \cdot x)] g'(H_{\theta,r}) \right).$$

The Hamiltonian $H_{0,r}$ admits edge states: for any $\xi \in \mathbb{R}$, if

$$F_{0,r}(\xi, x) = \exp \left( \frac{i \xi x_1}{\varepsilon} - \frac{r x_2^2}{2\varepsilon} \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

![Figure 5. Currents propagate along $\Gamma$ at speed $v_\theta$ given by the counterclockwise rotation of $\nabla \kappa$.](image-url)
Lemma 2.1 we deduce that $H$ also admits edge states:

$$F_{\theta,r}(\xi,x) = U_\theta F_{0,r}(\xi,x) = \exp \left( \frac{i\xi(R_{\theta,x})_1}{\varepsilon} - \frac{r(R_{\theta,x})_2^2}{2\varepsilon} \right) \left[ e^{-i\theta/2} \right].$$ (2.4)

2.2. Dynamics of edge states. We review here how edge states give rise to an infinite-dimensional family of ballistic waves for Dirac operators with linear domain walls.

**Proposition 2.2.** For any $f \in \mathcal{S}(\mathbb{R})$, the function

$$\psi_1(x) = \varepsilon^{-1/2} \cdot f(t + (R_{\theta x})_1) \cdot \exp \left( -\frac{r(R_{\theta,x})_2^2}{2\varepsilon} \right) \left[ e^{-i\theta/2} \right]$$ (2.5)

solves the equation $(\varepsilon D_t + H_{\theta,r})\psi_t = 0$.

The functions (2.5) are the ballistic waves generated by edge states: they propagate along the interface $\mathbb{R}v_\theta$ and decay rapidly along $\mathbb{R}v_\theta^1$. Our scaling casts (2.5) as wavepackets:

$$\psi_1(x) = \varepsilon^{-1/2} \cdot a \left( \frac{x - y_t}{\varepsilon} \right), \quad a(y) = e^{-\frac{i}{2}(R_{\theta y})_2^2} f \left( \sqrt{\varepsilon}(R_{\theta y})_1 \right) \left[ e^{-i\theta/2} \right], \quad y_t = tv_\theta$$ (2.6)

with $a$ having a full asymptotic expansion in powers of $\sqrt{\varepsilon}$. This connection will be the basis of our analysis in the context of curved interfaces.

**Proof of Proposition 2.2.** Let $g \in \mathcal{S}(\mathbb{R})$ such that

$$g(\xi) = \frac{1}{2\pi\varepsilon} \int_{\mathbb{R}} e^{\frac{i}{\varepsilon}t\xi} f(t)dt.$$ We introduce

$$\psi_1(x) = \varepsilon^{-1/2} \int_{\mathbb{R}} e^{-\frac{i}{\varepsilon}t\xi} g(\xi) F_{\theta,r}(\xi,x) d\xi.$$ (2.7)

Since $(H_{\theta,r} - \xi) F_{\theta,r}(\xi,\cdot) = 0$, we deduce that

$$\varepsilon D_t \psi_1(x) = -\varepsilon^{-1/2} \int_{\mathbb{R}} \xi e^{-\frac{i}{\varepsilon}t\xi} g(\xi) F_{\theta,r}(\xi,x) d\xi$$

$$= -\varepsilon^{-1/2} \int_{\mathbb{R}} e^{-\frac{i}{\varepsilon}t\xi} g(\xi) H_{\theta,r} F_{\theta,r}(\xi,x) d\xi = -H_{\theta,r} \psi_1(x).$$

This proves that (2.7) is a solution to $(\varepsilon D_t + H_{\theta,r})\psi_t = 0$. Plugging the formula (2.4) for $F_{\theta,r}$ in (2.7), we obtain

$$\psi_1(x) = \varepsilon^{-1/2} \int_{\mathbb{R}} e^{-\frac{i}{\varepsilon}(t + (R_{\theta x})_1)\xi} g(\xi) d\xi \cdot \exp \left( -\frac{r(R_{\theta,x})_2^2}{2\varepsilon} \right) \left[ e^{-i\theta/2} \right]$$

$$= \varepsilon^{-1/2} f(t + (R_{\theta x})_1) \exp \left( -\frac{r(R_{\theta,x})_2^2}{2\varepsilon} \right) \left[ e^{-i\theta/2} \right],$$

by definition of $g$ as the inverse (semiclassical) Fourier transform of $f$. □
3. Dynamical analogues of edge states along curved interfaces

We now consider non-linear domain walls, opening the possibility for curved topological interfaces. We relax (1.2) to a global transversality condition:

$$\inf \left\{ |\nabla \kappa(y)| : \kappa(y) = 0 \right\} > 0. \quad (3.1)$$

We recall that all derivatives of $\kappa$ are uniformly bounded: $\kappa \in C_b^\infty(\mathbb{R}^2)$. We plan to produce a dynamical analogue of edge states: a solution to

$$(\varepsilon D_t + H)\psi = 0, \quad H = \begin{bmatrix} \kappa(x) & \varepsilon D_{x_1} - i\varepsilon D_{x_2} \\ \varepsilon D_{x_1} + i\varepsilon D_{x_2} & -\kappa(x) \end{bmatrix},$$

that propagates for long time along the topological interface $\Gamma = \kappa^{-1}(0)$.

The equation (2.6) motivates the ansatz

$$\psi(t, x) = \varepsilon^{-1/2} a \left( t, \frac{x - y_t}{\sqrt{\varepsilon}} \right),$$

where:

- $a \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)$ has a full expansion in powers of $\varepsilon^{1/2}$;
- $y_0 \in \Gamma$ and $y_t \in \Gamma$ is the solution of the ODE

$$\dot{y}_t = v(y_t), \quad v(y) = \frac{\nabla \kappa(y)}{|\nabla \kappa(y)|} \cdot \mathbf{w}, \quad \mathbf{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} w.$$

The vector $v(y)$ is the local analogue to $v_0$: at each point $y \in \Gamma$, it is the unit tangent vector to $\Gamma$ obtained by rotating counterclockwise $\nabla \kappa(y)$. Since $\kappa(y_0) = 0$, $y_t \in \Gamma$ for any $t$:

$$\frac{d\kappa(y_t)}{dt} = \dot{y}_t \cdot \nabla \kappa(y_t) = v(y_t) \cdot \nabla \kappa(y_t) = 0.$$

Let $\theta_t$ and $r_t$ be such that

$$\nabla \kappa(y_t) = r_t \begin{bmatrix} -\sin(\theta_t) \\ \cos(\theta_t) \end{bmatrix}, \quad \text{so that} \quad v(y_t) = -\begin{bmatrix} \cos(\theta_t) \\ \sin(\theta_t) \end{bmatrix},$$

see Figure 2. With these notations in place, we define $\mathcal{K}_t : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)$ by:

$$\mathcal{K}_t f(x) = r_t^{1/4} f((R_{\theta_t} x)_t) e^{-\frac{i}{2}(R_{\theta_t} x)_t^2} \begin{bmatrix} e^{-i\theta_t/2} \\ -e^{i\theta_t/2} \end{bmatrix}, \quad f \in \mathcal{S}(\mathbb{R}). \quad (3.2)$$

**Theorem 2.** Let $\kappa \in C_b^\infty(\mathbb{R}^2)$ satisfying (3.1) and $y_t$, $\theta_t$ as above. Let $\psi_t$ be the solution to $(\varepsilon D_t + H)\psi_t = 0$ with

$$\psi_0(x) = \frac{1}{\sqrt{\varepsilon}} \cdot \mathcal{K}_0 f \left( \frac{x - y_0}{\sqrt{\varepsilon}} \right), \quad f \in \mathcal{S}(\mathbb{R}). \quad (3.3)$$

Then uniformly for $\varepsilon \in (0, 1]$ and $t > 0$:

$$\psi_t(x) = \frac{1}{\sqrt{\varepsilon}} \cdot \mathcal{K}_t f \left( \frac{x - y_t}{\sqrt{\varepsilon}} \right) + \mathcal{O}_{L^2}(\varepsilon^{1/2}(t)). \quad (3.4)$$

Theorem 2 constructs a solution to $(\varepsilon D_t + H)\psi_t = 0$, propagating dispersion-free along $y_t$, for times $t \ll \varepsilon^{-1/2}$. Under geometric conditions on $\kappa$, we can extend this time of validity; see Theorem 4. These two results focus on maximizing the lifespan of approximate solutions. We can instead focus on improving their accuracy: see Theorem 3 for solutions up to $O(\varepsilon^n)$ for every $n$, but fixed lifetime.
When \( r_t \) is not constant – corresponding to (3.1) holding instead of (1.2) – the state in (3.4) is coherent in a relaxed sense: there may be lateral spreading at scale \( r_t \) (which remains bounded above and below by our assumptions on \( \kappa \)). See the expression (3.2) for \( K_t f \) and Figure 6 for a numerical illustration.

The initial data (3.3) is quite specific: the rescaled amplitude \( K_0 f \) is in the range of \( K_0 \). To obtain a full picture of evolution of states initially microlocalized along \( C \), we need to understand how orthogonal initial data propagate:

\[
\psi_0(x) = \frac{1}{\sqrt{\varepsilon}} \cdot K_0 f \left( \frac{x - y_0}{\sqrt{\varepsilon}} \right) \bigg|_{\perp}.
\]

This suggests a refinement of Conjecture 1. Let \( \Pi : S(\mathbb{R}^2, \mathbb{C}^2) \to S(\mathbb{R}^2, \mathbb{C}^2) \) be the orthogonal projection on the range of \( K_0 \). We observe that \( K_0 \) is an isomorphism to its range; therefore, for any \( a \in S(\mathbb{R}^2, \mathbb{C}^2) \), there exists a unique \( f \in S(\mathbb{R}) \) such that \( \Pi a = K_0 f \).

**Conjecture 2.** There exists \( \beta < 3/4 \) with the following. Let \( a \in S(\mathbb{R}^2, \mathbb{C}^2) \), \( f \in S(\mathbb{R}) \) such that \( \Pi a = K_0 f \), and \( \phi_t \) be the solution to \( (\varepsilon D_t + H)\phi_t = 0 \) with initial data

\[
\phi_0(x) = \frac{1}{\sqrt{\varepsilon}} \cdot a \left( \frac{x - y_0}{\sqrt{\varepsilon}} \right).
\]

Then uniformly in \( \varepsilon \in (0, 1] \), \( t > 0 \):

\[
\phi_t(x) = \frac{1}{\sqrt{\varepsilon}} \cdot K_t f \left( \frac{x - y_t}{\sqrt{\varepsilon}} \right) + O_{L^2} \left( \varepsilon^{1/2} t \right) + O_{L^\infty} \left( \varepsilon^{-\beta} t^{-1/2} \right).
\]

According to Conjecture 2, any function localized (in a semiclassical sense) near \((y_0, 0)\) splits in propagating and dispersive parts, with the analogue of an edge state emerging dynamically. See Figure 4 for a numerical confirmation.

### 3.1. Structure of proof of Theorem 2

We will prove Theorem 2 by establishing the following statements.
(1) Approximate solutions of the Dirac equation solve a hierarchy of transport equations, see Lemma 3.1.

(2) The leading-order transport operator has explicit kernel and a spectral gap away from its kernel, see §3.3.

(3) Solutions to the hierarchy of transport equations exist, see §3.4–§3.5.

(4) Approximate and exact solutions to the Dirac equation are nearly equal, see §3.6.

We will use the notation

\[ W[a]_y(t)(x) = \frac{1}{\sqrt{\varepsilon}} \cdot a \left( \frac{x - y_t}{\sqrt{\varepsilon}} \right) \]

for \( a \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2) \) possibly depending on \( t \) and \( \varepsilon \).

We also introduce the operators \( T_j \) acting on \( \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2) \), defined by:

\[ T_0 = -\dot{y}_t \cdot D_x + \left[ \frac{\nabla \kappa(y_t)x}{D_{x_1} + iD_{x_2}} \begin{bmatrix} D_{x_1} - iD_{x_2} \\ -\nabla \kappa(y_t)x \end{bmatrix} \right], \quad (3.5) \]

\[ T_1 = D_t + \left( \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial^\alpha \kappa(y_t)x^\alpha \right) \sigma_3, \]

\[ T_j = \left( \sum_{|\alpha|=j+1} \frac{1}{\alpha!} \partial^\alpha \kappa(y_t)x^\alpha \right) \sigma_3, \quad j \geq 2. \quad (3.6) \]

### 3.2. Formal approximate solutions via transport equations

We start with the following lemma: solving the hierarchy of transport equations

\[ T_0 a_0 = 0, \quad T_0 a_1 + T_1 a_0 = 0, \quad \ldots, \quad \sum_{\ell=0}^j T_j - \ell a_\ell = 0, \quad j \in [0, m] \quad (3.7) \]

produces approximate solutions to the Dirac equation.

**Lemma 3.1.** For any \( m \in \mathbb{N} \), there exists \( C > 0 \) such that if \( a_0, a_1, \ldots, a_m \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2) \) are solutions of (3.7) and \( a^{(m)} = \sum_{\ell=0}^m \varepsilon^{\ell/2} a_\ell \), then for all \( \varepsilon \in (0, 1) \):

\[ \| (\varepsilon D_t + H)W[a^{(m)}]_y \|_{L^2} \leq C \varepsilon^{m+2} \left( \| D_t a_m \|_{L^2} + \sum_{k=0}^m \| \langle x \rangle^{m+2} a_k \|_{L^2} \right). \]

**Proof of Lemma 3.1.** Fix \( m \in \mathbb{N} \). We observe that for \( a \in \mathcal{S}(\mathbb{R}^2, \mathbb{C}^2) \),

\[ \varepsilon \partial_x W[a]_y = W[\sqrt{\varepsilon} \partial_x a]_y, \quad \varepsilon D_t W[a]_y = W[-\sqrt{\varepsilon} \dot{y}_t \cdot D_x a + \varepsilon D_t a]_y. \quad (3.8) \]

We now write the Taylor–Lagrange identity with integral remainder (note that \( \kappa(y_t) = 0 \)):

\[ \kappa(x) = \left( \sum_{|\alpha|=1} \frac{1}{\alpha!} \partial^\alpha \kappa(y_t)(x - y_t)^\alpha \right) + r_m(x - y_t), \] with

\[ r_m(x) = \frac{1}{(m+1)!} \sum_{|\alpha|=m+2} x^\alpha \int_0^1 (1 - s)^{m+1} \partial^\alpha \kappa(y_t + sx) ds. \]
We deduce that

\[ \kappa(x)W[a]_{y_t}(x) = W \left[ \left( \sum_{|\alpha|=1}^{m+1} \frac{\varepsilon|\alpha|/2}{\alpha!} \partial^\alpha \kappa(y_t)x^\alpha + \varepsilon^{m+2}R_m(x) \right) \right] \}_{y_t} (x), \quad \text{with} \quad (3.9) \]

\[ R_m(x) = \varepsilon^{-m-2}r_m(\varepsilon^{1/2}x) = \frac{1}{(m+1)!} \sum_{|\alpha|=m+2} x^\alpha \int_0^1 (1-s)^{|\alpha|+1} \partial^\alpha \kappa(y_t + s\varepsilon^{1/2}x) \, ds. \]

Since \(|r_m(x)| \leq C|x|^m\), we obtain that \(R_m(x) \leq C|x|^m\) for all \(\varepsilon \in (0, 1]\). From the relations (3.8)-(3.9) and the definition (3.6) of the operators \(T_j\):

\[ \langle \varepsilon D_t + H \rangle W[a]_{y_t} = W \left[ \left( \sum_{j=0}^{m} \varepsilon^{j+1}T_j + \varepsilon^{m+2}R_m \right) \right] \}_{y_t}. \]

In particular, using that \(W[a]_{y_t}\) and \(a\) have the same \(L^2\)-norm,

\[ \left\| \langle \varepsilon D_t + H \rangle W[a]_{y_t} \right\|_{L^2} = \left\| \left( \sum_{j=0}^{m} \varepsilon^{j+1}T_j + \varepsilon^{m+2}R_m \right) a \right\|_{L^2}. \quad (3.10) \]

2. Assume now that \(a_j\) solves the equations (3.7), and plug \(a^{(m)} = \sum_{k=0}^{m} \varepsilon^{k/2}a_k\) for the amplitude in (3.10). Then we obtain:

\[ \left\| \langle \varepsilon D_t + H \rangle W[a^{(m)}]_{y_t} \right\|_{L^2} = \left\| \sum_{j,k=0}^{m} \varepsilon^{j+k+1}T_j a_k + \sum_{k=0}^{m} \varepsilon^{m+2+k}R_m a_k \right\|_{L^2} \]

\[ \leq \sum_{j,k=1}^{m} \varepsilon^{j+k+1} \left\| T_j a_k \right\|_{L^2} + \sum_{k=0}^{m} \varepsilon^{m+2+k} \left\| R_m a_k \right\|_{L^2}. \]

In the second line we used the first sum starts at \(j, k = 1\), since \(j + k \geq m + 1\) and \(j, k \leq m\).

We note that \(T_1\) is the sum of \(D_1t\) and a polynomial of degree 2. For \(j \geq 2, T_j\) is a polynomial of degree \(j + 1\); and \(R_m\) is bounded by \(C|x|^{m+2}\). All coefficients involved depend on derivatives of \(\kappa\); in particular their values at \(y_t\) are uniformly bounded in time. In particular, after extracting \(D_t\), we can bound all multiplicative terms by \(C \langle x \rangle^{m+2}\). We obtain that \(\left\| \langle \varepsilon D_t + H \rangle W[a^{(m)}]_{y_t} \right\|_{L^2}\) is bounded, up to a multiplicative constant, by:

\[ \varepsilon^{m+2} \left\| D_t a_m \right\|_{L^2} + \sum_{j,k=1}^{m} \varepsilon^{j+k+1} \left\| \langle x \rangle^{m+2} a_k \right\|_{L^2} + \sum_{k=0}^{m} \varepsilon^{m+2+k} \left\| \langle x \rangle^{m+2} a_k \right\|_{L^2}. \]

Noting that \(j + k + 1 \geq m + 2\) in the first sum, we conclude that for any \(t\),

\[ \left\| \langle \varepsilon D_t + H \rangle W[a^{(m)}]_{y_t} \right\|_{L^2} \leq C \varepsilon^{m+2} \left( \left\| D_t a_m \right\|_{L^2} + \sum_{k=0}^{m} \left\| \langle x \rangle^{m+2} a_k \right\|_{L^2} \right). \]

This completes the proof. \[\square\]
We will show in the following how to construct solutions \( a_j \) to the hierarchy (3.7), and then bound their derivatives and moments. Together with Lemma 3.1 this will give a rigorous construction of approximate solutions to the Dirac equation.

### 3.3. Spectral analysis of leading order transport operator.

The dominant equation of the hierarchy (3.7) is \( T_0a_0 = 0 \), where \( T_0 \) is defined in (3.5); the other equations are

\[
T_0a_j = - \sum_{\ell=0}^{j-1} T_{j-\ell}a_{\ell}, \quad 1 \leq j \leq m.
\]

Solving these equations amounts to (i) find \( \ker(T_0) \); and (ii) establish a stability estimate (here, a spectral gap) for \( T_0^{-1} \) away from \( \ker(T_0) \). Below we write \( T_0 = \mathcal{L}_{\theta,r} \), where

\[
\mathcal{L}_{\theta,r} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} D_x + \begin{bmatrix} rK_{\theta,r}(x) & D_{x_2} - iD_{x_1} \\ D_{x_1} + iD_{x_2} & -rK_{\theta,r}(x) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} D_x + H_{\theta,r}.
\]

We now focus on the analysis of \( \mathcal{L}_{\theta,r} \) on \( S(\mathbb{R}^2, \mathbb{C}^2) \). We first compute its kernel (Lemma 3.2) and prove it is one to one on the orthogonal complement (Lemma 3.3).

**Lemma 3.2.** For every \( r > 0 \) and \( \theta \in \mathbb{R} \), the nullspace of \( \mathcal{L}_{\theta,r} : S(\mathbb{R}^2, \mathbb{C}^2) \to S(\mathbb{R}^2, \mathbb{C}^2) \) is

\[
\ker_{S(\mathbb{R}^2)}(\mathcal{L}_{\theta,r}) = \left\{ f((R_{\theta}x)_1) e^{-\left( (R_{\theta}x)^2 \right)/2} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix}, \quad f \in S(\mathbb{R}) \right\}.
\]

**Proof.** As in (2.3), \( U_{\theta}^{-1} \mathcal{L}_{\theta,r} U_{\theta} = \mathcal{L}_{0,r} \), with \( U_{\theta} = R_{\theta}U_{\theta} \). Indeed, \( U_{\theta}^{-1}R_{\theta}^{-1}H_{\theta}R_{\theta}U_{\theta} = H_0 \) and

\[
U_{\theta}^{-1} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \cdot D_x U_{\theta} = R_{\theta} e_1 \cdot R_{\theta} D_x = D_{x_1}.
\]

Moreover, if \( S_r f(x) = f(\sqrt{r}x) \), then we have

\[
S_r^{-1} H_{\theta,r} S_r = \sqrt{r} H_{0,1}.
\]

Hence, \( H_{\theta,r} \) and \( H_{0,1} \) are conjugated (up to multiplication by \( \sqrt{r} \)). The identity (3.11) implies that the same holds for \( \mathcal{L}_{\theta,r} \) and \( \mathcal{L}_{0,1} \):

\[
S_r^{-1} U_{\theta}^{-1} \mathcal{L}_{\theta,r} U_{\theta} S_r = \sqrt{r} \mathcal{L}_{0,1}.
\]

Thus, to find the kernel of \( \mathcal{L}_{\theta,r} \), it suffices to find that of \( \mathcal{L}_{0,1} \). We have

\[
\mathcal{L}_{0,1} = \begin{bmatrix} D_{x_1} + x_2 \\ D_{x_1} + iD_{x_2} \\ D_{x_1} - iD_{x_2} \\ D_{x_1} - x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} D_{x_1} + \begin{bmatrix} x_2 \\ iD_{x_2} \\ -iD_{x_2} \\ -x_2 \end{bmatrix}.
\]

We claim that

\[
\ker_{S(\mathbb{R}^2)}(\mathcal{L}_{0,1}) = \left\{ f(x_1) e^{-x_2^2/2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad f \in S(\mathbb{R}) \right\}.
\]

The right inclusion follows from a computation. To prove the left inclusion, we pick \( u \) such that \( \mathcal{L}_{0,1} u = 0 \). We take the Fourier transform in \( x_1 \); this gives \( \mathcal{L}_{0,1}(\xi) \hat{u} = 0 \), where

\[
\mathcal{L}_{0,1}(\xi) = \xi \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ iD_{x_2} \\ -iD_{x_2} \\ -x_2 \end{bmatrix}.
\]

We fix \( \xi \). The operator \( \mathcal{L}_{0,1}(\xi) \) is a linear differential operator; hence the space of decaying solutions to \( \mathcal{L}_{0,1}(\xi)v = 0 \) is at most one-dimensional. Indeed, if \( v_1, v_2 \) are such functions, then
their Wronskian is constant; and they decay. Thus their Wronskian vanishes; this implies that \( v_1, v_2 \) are linearly dependent. We then observe that

\[
L_{0,1}(\xi) e^{-\frac{x_2^2}{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0.
\]

This shows that the kernel of \( L_{0,1}(\xi) \) is one-dimensional. Superposing over \( \xi \) yields (3.14). Applying the equivalence between \( L_{0,1} \) and \( L_{\theta,r} \), we conclude that the kernel of \( L_{\theta,r} \) is precisely made of functions

\[
S_{\theta} R_{\theta} U_{\theta} \left( f(x_1) e^{-\frac{x_2^2}{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = f(\sqrt{r}(R_{\theta} x_1)) e^{\frac{-r(R_{\theta} x_1)^2}{2}} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix}, \quad f \in S(\mathbb{R}).
\]

This corresponds to (3.12), where we rescaled \( f \) by \( \sqrt{r} \) (this preserves the Schwartz class). \( \square \)

We define the space

\[
S_{\theta,r}(\mathbb{R}^2) = \left\{ u \in S(\mathbb{R}^2, \mathbb{C}^2) : u \in \ker_{S(\mathbb{R}^2, \mathbb{C}^2)} (L_{\theta,r})^\perp \right\},
\]

with orthogonality computed with respect to the \( L^2 \)-scalar product. We provide \( S_{\theta,r}(\mathbb{R}^2) \) with the seminorms inherited from \( S(\mathbb{R}^2, \mathbb{C}^2) \).

**Lemma 3.3.** For every \( \theta \in \mathbb{R} \) and \( r > 0 \), the operator \( L_{\theta,r} \) acting on \( S_{\theta,r}(\mathbb{R}^2) \) is one to one, with inverse \( L_{\theta,r}^{-1} \) bounded on \( S_{\theta,r}(\mathbb{R}^2) \).

**Proof.** 1. We recall that \( L_{\theta,r} \) and \( \sqrt{r} L_{0,1} \) are conjugated by operators bounded on \( S(\mathbb{R}^2, \mathbb{C}^2) \), see (3.13). Thus, it suffices to prove the lemma for \( L_{0,1} \) only.

We introduce the annihilation and creation operators \( \mathfrak{a} \) and \( \mathfrak{a}^* \), as well as its associated quantum harmonic oscillator \( \mathfrak{h} = \mathfrak{a}^* \mathfrak{a} \) and quantum states \( \varphi_n \):

\[
\mathfrak{a} = x_2 + \partial_{x_2}, \quad \mathfrak{a}^* = x_2 - \partial_{x_2}, \quad \mathfrak{h} = -\partial^2_{x_2} + x_2^2 - 1,
\]

\[
\varphi_0(x_2) = \frac{1}{\pi^{1/4}} e^{-\frac{x_2^2}{2}}, \quad \varphi_n(x_2) = \frac{(\mathfrak{a}^*)^n}{2^{n/2} \sqrt{n!}} \varphi_0(x_2).
\]

The quantum states \( \varphi_n \) form a complete orthonormal basis of eigenvectors of \( \mathfrak{h} \): for every \( n \), \( \| \varphi_n \|_{L^2} = 1 \) and \( \mathfrak{h} \varphi_n = 2n \varphi_n \). Moreover they satisfy the creation and annihilation relations: \( \mathfrak{a} \varphi_0 = 0 \) and for \( n \in \mathbb{N} \),

\[
\mathfrak{a}^* \varphi_n = \sqrt{2n + 2} \varphi_{n+1}, \quad \mathfrak{a} \varphi_{n+1} = \sqrt{2n + 2} \varphi_n. \tag{3.15}
\]

Introduce

\[
\tilde{L}_{0,1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} L_{0,1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \mathfrak{a}^* \\ \mathfrak{a} & 2D_{x_2} \end{bmatrix}, \tag{3.16}
\]

and the associated space \( \tilde{S}_{0,1}(\mathbb{R}^2) \) – defined similarly as \( S_{0,1}(\mathbb{R}^2) \):

\[
\tilde{S}_{0,1}(\mathbb{R}^2) = \left\{ u \in S(\mathbb{R}^2) : u \in \ker_{S(\mathbb{R}^2)} (\tilde{L}_{0,1})^\perp \right\}
\]

\[
= \left\{ u \in S(\mathbb{R}^2) : \forall x_1 \in \mathbb{R}, \int_{\mathbb{R}^2} u_1(x) \varphi_0(x_2) dx_2 = 0 \right\}. \tag{3.17}
\]

The lemma boils down to prove that \( \tilde{L}_{0,1} \) is invertible on \( \tilde{S}_{0,1}(\mathbb{R}^2) \).
2. Let $\mathcal{W}$ be the Fréchet space of functions $w \in C^\infty(\mathbb{R} \times \mathbb{N}, \mathbb{C}^2)$ such that $w_1(\cdot, 0) = 0$, equipped with the seminorms

$$N_{\alpha,\beta,\gamma}(w) = \sup_{n, \xi} \left| \langle n \rangle^{2\alpha} \langle \xi \rangle^\beta \partial_\xi^\gamma w(\xi, n) \right|, \quad \alpha, \beta, \gamma \in \mathbb{N}.$$ 

We define $S : \tilde{\mathcal{S}}_{0,1}(\mathbb{R}^2) \to \mathcal{W}$ by

$$Su(\xi, n) = \int_{\mathbb{R}^2} e^{-i\xi x_1} \begin{bmatrix} u_1(x) \varphi_{n+1}(x_2) \\ u_2(x) \varphi_n(x_2) \end{bmatrix} dx, \quad u \in \tilde{\mathcal{S}}_{0,1}(\mathbb{R}^2), \ n \in \mathbb{N}, \ \xi \in \mathbb{R}.$$ 

We first observe that $S : \tilde{\mathcal{S}}_{0,1}(\mathbb{R}^2) \to \mathcal{W}$ is continuous. Indeed, if $u \in \tilde{\mathcal{S}}_{0,1}(\mathbb{R}^2)$ and $\alpha, \beta, \gamma \in \mathbb{N}$, we have

$$(2n+1)\langle \xi \rangle^\beta D_\xi^n Su(\xi, n) = Sv(\xi, n), \quad v(x) = \langle h \rangle^\alpha \langle x \rangle^\beta (-i\xi)^\gamma u(x).$$

Moreover, $v \in \mathcal{S}(\mathbb{R}^2)$ when $u \in \mathcal{S}(\mathbb{R}^2)$. The Cauchy–Schwarz inequality yields

$$N_{\alpha,\beta,\gamma}(Su) = \sup_{n, \xi} |Sv(\xi, n)| \leq \sup_{n} \left| \int_{\mathbb{R}^2} \begin{bmatrix} u_1(x) \varphi_{n+1}(x_2) \\ u_2(x) \varphi_n(x_2) \end{bmatrix} dx \right| dx \leq 2 \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |v(x)|^2 dx \right)^{1/2} dx_1,$$

where we used $\|\varphi_n\|_{L^2} = 1$. The RHS is controlled by Schwartz semi-norms of $v = \langle h \rangle^\alpha \langle x \rangle^\beta D_\xi^n u$, thus of $u$. Hence $u \equiv 0$ and $S$ is continuous.

Moreover, $S$ is invertible. The range of $S$ is $\mathcal{W}$: if $w \in \mathcal{W}$ then we have $Su = w$ with

$$u(x) = \frac{1}{2\pi} \int e^{i\xi x_1} \sum_{n=0}^\infty \begin{bmatrix} \varphi_{n+1}(x_2) u_1(\xi, n) \\ \varphi_n(x_2) u_2(\xi, n) \end{bmatrix} d\xi,$$

using the Fourier inversion formula and orthogonality relations for the $\varphi_n$. We now show that $S$ is one-to-one. If $u \in \tilde{\mathcal{S}}_{0,1}(\mathbb{R}^2)$ is such that $Su \equiv 0$ then

$$\forall x_1 \in \mathbb{R}, \ n \in \mathbb{N}, \int_{\mathbb{R}} \begin{bmatrix} u_1(x) \varphi_{n+1}(x_2) \\ u_2(x) \varphi_n(x_2) \end{bmatrix} dx_2 = 0 \quad (3.18)$$

from the Fourier inversion formula. Since $\varphi_n$ forms an orthonormal basis of $L^2(\mathbb{R})$, $(3.18)$ implies that $u_2 \equiv 0$ and $u_1(x) = c(x_1) \varphi_0(x_2)$. From $u \in \tilde{\mathcal{S}}_{0,1}(\mathbb{R}^2)$ and (3.17), $u_1 \equiv 0$. Hence $S$ is invertible.

3. Because of the closed graph theorem, invertible continuous operators between Fréchet spaces have continuous inverses. Hence the inverse of $S$ is continuous from $\mathcal{W}$ to $\tilde{\mathcal{S}}_{0,1}(\mathbb{R}^2)$. Hence, to prove the lemma it suffices to show that $\tilde{S} \tilde{L}_{0,1}^{-1} : \mathcal{W} \to \mathcal{W}$ is continuously invertible. But $\tilde{S} \tilde{L}_{0,1}^{-1}$ is actually a simple multiplication operator: using that $D_{\xi_1}$ corresponds to $\xi$ in Fourier space and $a, a^*$ are shift operators – see (3.15) – in Hermite space, we have:

$$\tilde{S} \tilde{L}_{0,1}^{-1} w(\xi, n) = \begin{bmatrix} 0 & \sqrt{2n+2} \\ \sqrt{2n+2} & 2\xi \end{bmatrix} w(\xi, n). \quad (3.19)$$

This is a continuous operator on $\mathcal{W}$; and (3.19) yields a formula for $\tilde{L}_{0,1}^{-1}$:

$$\tilde{L}_{0,1}^{-1} = S^{-1} \begin{bmatrix} 2\xi & -\sqrt{2n+2} \\ -\sqrt{2n+2} & 0 \end{bmatrix} S.$$ 

This completes the proof. \qed
3.4. Solving the dominant equation. We now focus on solving the hierarchy of equations (3.7), starting with the first two:

\[ T_0a_0 = 0, \quad T_0a_1 + T_1a_0 = 0. \]

Below we abuse notation: we allow functions in \( S(\mathbb{R}) \) or \( S(\mathbb{R}^2, \mathbb{C}^2) \) to also depend smoothly on time, and we consider the operator \( \mathcal{K}_t \) from (3.2) on functions depending on \( t \). For instance, we write (3.20) as

\[ a_0(t, x) = \mathcal{K}_t f_0(t, x) = i t^{1/4} f_0(t, (R_\theta, x)_1) e^{-\frac{r_t(R_\theta, x)_1^2}{2}} \left[ e^{i\theta_t/2} - e^{-i\theta_t/2} \right] \tag{3.20} \]

Since \( T_0 = L_{\theta, r_t} \), Lemma 3.2 implies that for any \( f_0 \in S(\mathbb{R}) \) (potentially depending on \( t \)), (3.20) solves the equation \( T_0a_0 = 0 \).

3.5. Solving the subleading equation. The subleading equation in the hierarchy (3.7) is \( T_0a_1 + T_1a_0 = 0 \) where \( T_0 = L_{\theta, r_t} \) and

\[ T_1 = D_t + \sum_{|\alpha|=2} \frac{\partial^\alpha \mathcal{K}(y_t)}{\alpha!} x^\alpha \sigma_3. \tag{3.21} \]

Given \( a_0 \) satisfying (3.20), we regard \( T_0a_1 + T_1a_0 = 0 \) as an equation with unknown \( a_1 \in S(\mathbb{R}^2, \mathbb{C}^2) \). According to Lemma 3.3, a solution exists if for any \( t \in \mathbb{R}, T_1a_0(t, \cdot) \in S_{\theta_t, r_t}(\mathbb{R}^2) \). We now look for \( f_0 \) such that this holds.

We note that \( T_1a_0 \in S_{\theta_t, r_t}(\mathbb{R}^2) \) if and only if for every \( t \in \mathbb{R} \) and \( g \in S(\mathbb{R}) \):

\[ \int_{\mathbb{R}^2} g((R_\theta, x)_1) e^{-\frac{r_t(R_\theta, x)_1^2}{2}} \left[ e^{i\theta_t/2} - e^{-i\theta_t/2} \right] \cdot T_1a_0(t, x) dx = 0. \tag{3.22} \]

We make the substitution \( x \mapsto R_{\theta_t}^\top x \) and pick functions \( g \) approaching delta distributions to obtain that (3.22) is equivalent to:

\[ \forall t, x_1 \in \mathbb{R}, \quad \int_{\mathbb{R}} e^{-\frac{r_t x_1^2}{2}} \left[ e^{i\theta_t/2} - e^{-i\theta_t/2} \right] \cdot (T_1K_t f)(t, R_{\theta_t}^\top x) dx_2 = 0. \tag{3.23} \]

Lemma 3.4. If \( f(t, \cdot) \in S(\mathbb{R}) \) depends smoothly on \( t \), then

\[ \int_{\mathbb{R}} e^{-\frac{r_t x_1^2}{2}} \left[ e^{i\theta_t/2} - e^{-i\theta_t/2} \right] \cdot (T_1K_t f)(t, R_{\theta_t}^\top x) dx_2 = 2 \sqrt{\frac{\pi}{r_t}} D_t f(t, x_1). \tag{3.24} \]

Proof. We note the identities

\[ \left\langle \begin{bmatrix} e^{-i\theta_t/2} \\ -e^{i\theta_t/2} \end{bmatrix}, \sigma_3 \begin{bmatrix} e^{-i\theta_t/2} \\ -e^{i\theta_t/2} \end{bmatrix} \right\rangle = 0, \quad \left\langle \begin{bmatrix} e^{-i\theta_t/2} \\ -e^{i\theta_t/2} \end{bmatrix}, \begin{bmatrix} -\dot{\theta}_t e^{-i\theta_t/2} \\ \dot{\theta}_t e^{i\theta_t/2} \end{bmatrix} \right\rangle = 0. \tag{3.25} \]

Therefore, using the expressions (3.21) for \( T_1 \) and (3.2) for \( K_t \), we have:

\[ \left[ \begin{bmatrix} e^{i\theta_t/2} \\ -e^{-i\theta_t/2} \end{bmatrix} \cdot T_1K_t f(t, x) = 2D_t \left( r_t^{1/4} f(t, (R_\theta, x)_1) e^{-\frac{r_t(R_\theta, x)_1^2}{2}} \right) \right. \]

\[ \left. = \frac{2}{i} e^{-\frac{r_t(R_\theta, x)_1^2}{2}} \left( \frac{\partial}{\partial t} + (\dot{R}_\theta, x)_1 \frac{\partial}{\partial x_1} - \frac{\dot{r}_t(R_\theta, x)_1^2}{2} - r_t(R_\theta, x)_2 (\dot{R}_\theta, x)_2 \right) r_t^{1/4} f(t, (R_\theta, x)_1). \right] \]
We deduce that
\[
\left[ e^{i\theta_t/2} - e^{-i\theta_t/2} \right] \cdot T_1 K_t f(t, R^\top_{\theta_t} x) = -2ie^{-\frac{\dot{r}_t x_2^2}{2}} \left( \frac{\partial}{\partial t} + \dot{\theta}_t x_2 \frac{\partial}{\partial x_1} - \frac{\dot{r}_t x_2^2}{2} - r_t x_2 (\dot{R}_{\theta_t} R^\top_{\theta_t} x_2) \right) r_t^{1/4} f(t, x_1). \tag{3.26}
\]

We remark that
\[
\dot{R}_{\theta_t} \cdot R^\top_{\theta_t} x = \dot{\theta}_t \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \dot{\theta}_t \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}. \tag{3.27}
\]

We deduce that (3.26) becomes:
\[
\left[ e^{i\theta_t/2} - e^{-i\theta_t/2} \right] T_1 K_t f(t, R^\top_{\theta_t} x) = -2ie^{-\frac{\dot{r}_t x_2^2}{2}} \left( \frac{\partial}{\partial t} + \dot{\theta}_t x_2 \frac{\partial}{\partial x_1} - \frac{\dot{r}_t x_2^2}{2} + r_t \dot{\theta}_t x_2 x_1 \right) r_t^{1/4} f(t, x_1).
\]

We plug this identity in (3.24) to obtain:
\[
-2i \int_\mathbb{R} e^{-r_t x_2^2} \left( \frac{\partial}{\partial t} + \dot{\theta}_t x_2 \frac{\partial}{\partial x_1} - \frac{\dot{r}_t x_2^2}{2} + r_t \dot{\theta}_t x_2 x_1 \right) dx_2 \cdot r_t^{1/4} f(t, x_1). \tag{3.28}
\]

We now perform the integrals over $x_2$. The function $x_2 e^{-r_t x_2^2}$ has vanishing integral; moreover an integration by parts shows that
\[
\sqrt{\frac{\pi}{r_t}} = \int_\mathbb{R} e^{-r_t x_2^2} dx_2 = 2r_t \cdot \int_\mathbb{R} x_2^2 e^{-r_t x_2^2} dx_2.
\]

Hence (3.28) reduces to:
\[
-2i \sqrt{\frac{\pi}{r_t}} \left( \frac{\partial}{\partial t} - \frac{\dot{r}_t}{4r_t} \right) r_t^{1/4} f(t, x_1). \tag{3.29}
\]

We finally observe that in the sense of differential operators,
\[
\left( \frac{\partial}{\partial t} - \frac{\dot{r}_t}{4r_t} \right) r_t^{1/4} = \frac{\partial}{\partial t}.
\]

Using this identity in (3.29) completes the proof. \hfill \square

From (3.23) and Lemma 3.4, we obtain the transport equation for $f_0$: $D_t f_0 = 0$. Hence, $f_0$ depends on $x_1$ only, and we write $f_0(t, x_1) = f_0(x_1)$. Therefore, if
\[
a_0(t, x) = r_t^{1/4} f_0((R_{\theta_t} x)_1) e^{-\frac{r_t x_2^2}{2}} \left[ e^{-i\theta_t/2} - e^{i\theta_t/2} \right] = K_t f_0(x) \tag{3.30}
\]

for some $f_0 \in \mathcal{S}(\mathbb{R})$, then $T_1 a_0(t, \cdot) \in \mathcal{S}_{\theta_t, r_t}(\mathbb{R}^2)$ for every $t \in \mathbb{R}$; hence the equation $T_0 b_1 + T_1 a_0 = 0$ has a unique solution $b_1$ such that $b_1(t, \cdot) \in \mathcal{S}_{\theta_t, r_t}(\mathbb{R}^2)$ for every $t \in \mathbb{R}$. We obtain the general solution to $T_0 a_1 + T_1 a_0 = 0$ by adding an element of $\ker(L_{\theta_t, r_t})$: $a_1 = b_1 + K_t f_1$:
\[
a_1(t, x) = b_1(t, x) + r_t^{1/4} f_1(t, (R_{\theta_t} x)_1) e^{-\frac{r_t x_2^2}{2}} \left[ e^{-i\theta_t/2} - e^{i\theta_t/2} \right], \quad f_1(t, \cdot) \in \mathcal{S}(\mathbb{R}). \tag{3.31}
\]
3.6. **Proof of Theorem 2.** We are now in a position to prove Theorem 2. We start with a classical result based on Duhamel’s formula.

**Lemma 3.5.** Let $\psi_t \in \mathcal{S}(\mathbb{R}^2)$ be a solution to $(\varepsilon D_t + H)\psi_t = 0$. Then for any $v_t \in \mathcal{S}(\mathbb{R}^2)$,

$$
\|v_t - \psi_t\|_{L^2} \leq \|v_0 - \psi_0\|_{L^2} + \frac{1}{\varepsilon} \int_0^t \| (\varepsilon D_s + H) v_s \|_{L^2} ds.
$$

**Proof.** Let $w_t = v_t - \psi_t$ and $r_t = (\varepsilon D_t + H)v_t$. Then, $(\varepsilon D_t + H)w_t = r_t$. By Duhamel’s formula,

$$
v_t - \psi_t = w_t = e^{-itH/\varepsilon}w_0 + \frac{1}{\varepsilon} \int_0^t e^{-i(t-s)H/\varepsilon} r_s ds = e^{-itH/\varepsilon}(v_0 - \psi_0) + \frac{1}{\varepsilon} \int_0^t e^{-i(t-s)H/\varepsilon} r_s ds.
$$

We bound both sides in $L^2$, using that $e^{-itH}$ is unitary:

$$
\|v_t - \psi_t\|_{L^2} \leq \|v_0 - \psi_0\|_{L^2} + \frac{1}{\varepsilon} \int_0^t \| (\varepsilon D_s + H) v_s \|_{L^2} ds.
$$

This completes the proof. □

**Proof of Theorem 2.** 1. Let $f_0 \in \mathcal{S}(\mathbb{R})$. Let $a_0$ as in (3.20), $b_1$ is as in (3.31) and $a^{(1)} = a_0 + \varepsilon^{1/2}a_1$. We apply Lemma 3.1 with $m = 1$:

$$
\| (\varepsilon D_t + H) W[a^{(1)}] y_t \|_{L^2} \leq C\varepsilon^{3/2} \left( \| D_t b_1 \|_{L^2} + \| \langle x \rangle^3 a_0 \|_{L^2} + \| \langle x \rangle^3 b_1 \|_{L^2} \right). \tag{3.32}
$$

2. We now bound the right-hand-side of (3.32), starting with $\langle x \rangle^3 a_0$ in $L^2$. We write $a_0 = \mathcal{K}_{\theta, r} f_0$, where

$$
\mathcal{K}_{\theta, r} = r^{1/4} f((R_\theta x)_{1}) e^{-\frac{r(R_\theta x)^2}{2}} \begin{bmatrix} e^{-i\theta/2} \\ -e^{i\theta/2} \end{bmatrix}.
$$

We note that we have the identity $\mathcal{K}_{\theta, r} = D_r U_\theta \mathcal{K}_{0,1}$, where $D_r$ is a partial dilation operator and $U_\theta$ was introduced in (2.2):

$$
D_r g(x) = r^{1/4} g \left( x_1, \sqrt{r} x_2 \right), \quad U_\theta g(x) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} g(R_\theta x). \tag{3.33}
$$

The operator $\mathcal{K}_{0,1}$ is bounded from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)$; $U_\theta$ is uniformly bounded from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}(\mathbb{R}^2, \mathbb{C}^2)$ for $\theta \in \mathbb{R}$; and $D_r$ is bounded uniformly on $\mathcal{S}(\mathbb{R}^2)$ for $r$ in compact subsets of $(0, \infty)$. Moreover, $r_t = \| \nabla \kappa(y_t) \|$ lives in a compact subset of $(0, \infty)$, because of $\kappa \in C_b^\infty(\mathbb{R}^2)$ and (3.1). We deduce that $a_0 \in \mathcal{S}(\mathbb{R}^2)$, with uniform-in-time bounds on its seminorms. In particular, $\| \langle x \rangle^3 a_0 \|_{L^2}$ is uniformly bounded.

For later use, we observe that $\partial_t a_0$ is also uniformly bounded in $\mathcal{S}(\mathbb{R}^2)$. Indeed, from (3.33), we have

$$
\partial_t a_0 = \dot{r}_t \partial_r D_r U_\theta \mathcal{K}_{0,1} f_0 + \dot{\theta}_t D_r \partial_\theta U_\theta \mathcal{K}_{0,1} f_0. \tag{3.34}
$$

The operators $\partial_\theta U_\theta$ and $\partial_r D_r$ are uniformly bounded on $\mathcal{S}(\mathbb{R}^2)$ – the latter because $r_t$ lives in a compact subset of $(0, \infty)$. The quantities $\dot{r}_t$ and $\dot{\theta}_t$ are uniformly bounded:

$$
|\dot{r}_t| = \frac{\langle \nabla \kappa(y_t), \nabla^2 \kappa(y_t) \rangle}{2|\nabla \kappa(y_t)|} \leq |\nabla \kappa(y_t)| \cdot |\nabla^2 \kappa(y_t)| \leq C;
$$

$$
|\dot{\theta}_t| = \frac{|\nabla \kappa(y_t)|}{\sqrt{\rho(y_t)}} \leq C.
$$
and likewise,

\[ |\dot{\theta}| = \left| \frac{d}{dt} \frac{\nabla \kappa(y_t)}{r_t} \right| \leq \frac{1}{r_t} + \frac{|\nabla^2 \kappa(y_t)|}{r_t} \leq C. \]

Therefore, we deduce from (3.34) that \( \partial_t a_0 \) is uniformly bounded in \( \mathcal{S}(\mathbb{R}^2) \).

3. We now control in \( L^2 \) the terms \( D_t b_1 \) and \( \langle x \rangle^3 b_1 \) that appear in (3.32). We use (3.13) to write \( b_1 \) as:

\[ b_1(t, \cdot) = -L_{\theta(t)}^{-1} a_0 - \sqrt{r_t} S_{r_t}^{-1} U_{\theta(t)}^{-1} L_0^{-1} U_0 S_{r_t} a_0. \tag{3.35} \]

As in Step 2, all operators involved in (3.35) are uniformly bounded in \( \mathcal{S}(\mathbb{R}^2) \), and we deduce that \( b_1 \in \mathcal{S}(\mathbb{R}^2) \) uniformly in time. Also similarly to (3.34), taking time derivatives produces quantities such as \( \dot{r}_t, r_t^{-1/2}, \dot{\theta}_t \) (all uniformly bounded); operators such as \( \partial_t \mathcal{D}_{r_t}, \partial_t \mathcal{D}_{r_t^{-1}}, \partial_u U_0 \) and \( \partial_t U_{\theta(t)} \), all uniformly bounded on \( \mathcal{S}(\mathbb{R}^2) \); and the function \( \partial_t a_0 \) also bounded uniformly in \( \mathcal{S}(\mathbb{R}^2) \). We deduce that \( b_1, \partial_t b_1 \) are uniformly in \( \mathcal{S}(\mathbb{R}^2) \). Hence, \( \| \langle x \rangle^3 b_1 \|_{L^2} \) and \( \| \partial_t b_1 \|_{L^2} \) are uniformly bounded.

4. Going back to (3.32), we have for any \( t \):

\[ \|(\varepsilon D_t + H) W[a^{(1)}]_{y_t}\|_{L^2} \leq C \varepsilon^{3/2}. \tag{3.36} \]

Let \( \psi_t \) be the solution to \((\varepsilon D_t + H) \psi_t = 0\) with initial data \( \psi_0 = a_0(0, \cdot) \); and \( v_t = W[a^{(1)}]_{y_t} \). We note that \( v_0 - \psi_0 = \varepsilon^{1/2} b_1(0, \cdot) \) and that \( v_t \) satisfies the bound (3.36). Thanks to Lemma 3.5, we get

\[ \|v_t - \psi_t\|_{L^2} \leq \varepsilon^{1/2} \|b_1(0, \cdot)\|_{L^2} + C \varepsilon^{1/2} t. \]

Therefore,

\[ \psi_t = W[a^{(1)}]_{y_t} + \mathcal{O}_{L^2}(\varepsilon^{1/2} \langle t \rangle) = W[a_0]_{y_t} + \mathcal{O}_{L^2}(\varepsilon^{1/2} \langle t \rangle). \]

This completes the proof. \( \square \)

3.7. **Subsequent equations.** We now focus on deriving a version of Theorem 2 that favors accuracy over lifetime. This requires to solve higher-order transport equations.

The base case is the result of \(3.4-3.5\), summarized as follows:

**\((H_1)\)** For any \( f_0 \in \mathcal{S}(\mathbb{R}) \), there exists \( b_1 \) such that for any \( f_1(t, \cdot) \in \mathcal{S}(\mathbb{R}) \) if \( a_0 = \mathcal{K}_t f_0 \) and \( a_1 = b_1 + \mathcal{K}_t f_1 \), then \( a_0 \) and \( a_1 \) solve (3.7) with \( m = 1 \), i.e.

\[ \sum_{\ell=0}^{j} T_{j-\ell} a_{\ell} = 0, \quad 0 \leq j \leq 1. \]

To construct \( a_0 \) and \( a_1 \), we had to enforce a condition on \( f_0 \). Likewise, to construct \( a_m \) we will enforce a condition on \( f_{m-1} \).

Our inductive assumption is, for \( m \geq 1 \):

**\((H_m)\)** For any \( f_0 \in \mathcal{S}(\mathbb{R}) \), there exist \( b_1, f_1, \ldots, b_{m-1}, f_{m-1}, b_m \in \mathcal{S}(\mathbb{R}) \) depending smoothly on \( t \), such that for any \( f_m \in \mathcal{S}(\mathbb{R}) \), if \( a_0 = \mathcal{K}_t f_0 \) and \( a_{\ell} = b_{\ell} + \mathcal{K}_t f_\ell \) then

\[ \sum_{\ell=0}^{j} T_{j-\ell} a_{\ell} = 0, \quad 0 \leq j \leq m. \]
We proved \((H_1)\) in §3.5. We now assume that \((H_{m-1})\) holds and we prove \((H_m)\) for \(m \geq 2\). Because of Lemma 3.1, this boils down to constructing \(a_m = b_m + \mathcal{K}_t f_m\) such that:

\[
T_0(b_m + \mathcal{K}_t f_m) + T_1 a_{m-1} + \cdots + T_m a_0 = 0,
\]

where:

\[
(3.37)
\]

- The operators \(T_k\) are defined in (3.6);
- The amplitudes \(a_0, \ldots, a_{m-2}\) are fully specified by \((H_{m-1})\);
- The amplitude \(a_{m-1} = b_{m-1} + \mathcal{K}_t f_{m-1}\), with \(b_{m-1}\) given by \((H_{m-1})\) and \(f_{m-1} \in \mathcal{S}(\mathbb{R})\) remains be selected.

Since the operator \(\mathcal{K}_t\) parametrizes the kernel of \(T_0\), (3.37) is equivalent to

\[
T_0 b_m = \beta_{m-1} - T_1 \mathcal{K}_t f_{m-1}, \quad \beta_{m-1} = -T_1 b_{m-1} - T_2 a_{m-2} - \cdots - T_m a_0.
\]

Note that \((H_{m-1})\) fully prescribes \(\beta_{m-1}\).

As in §3.5, to solve (3.38), it suffices that for any \(t\), \((\beta_{m-1} - T_1 \mathcal{K}_t f_{m-1})(t, \cdot)\) is in the kernel of \(T_0\). This is equivalent to

\[
\forall t, x_1 \in \mathbb{R}, \quad \int_{\mathbb{R}} e^{-\frac{r_s^2}{4}} \left[ e^{i\theta_t/2} \cdot (\beta_{m-1} - T_1 \mathcal{K}_t f_{m-1})(t, R^T_{\theta_t} x) \right] dx_2 = 0.
\]

Thanks to Lemma 3.4, this is equivalent to:

\[
D_t f_{m-1}(t, x_1) = \frac{1}{2} \sqrt{\frac{r_t}{\pi}} \int_{\mathbb{R}} e^{-\frac{r_s^2}{2}} \left[ e^{i\theta_t/2} \cdot \beta_{m-1}(t, R^T_{\theta_t} x) \right] dx_2,
\]

and hence – setting \(f_{m-1}(0, x_1) = 0\):

\[
f_{m-1}(t, x_1) = \int_0^t \frac{1}{2} \sqrt{\frac{r_s}{\pi}} e^{-\frac{r_s^2}{2}} \left[ e^{i\theta_s/2} \cdot \beta_{m-1}(s, R^T_{\theta_s} x) \right] ds.
\]

When \(f_{m-1}\) is given by this formula, the equation (3.38) admits a solution \(b_m(t, \cdot) \in \mathcal{S}(\mathbb{R}^2, \mathcal{C}^2)\).

This completes the proof of \((H_m)\). The following result summarizes our findings:

**Theorem 3.** Fix \(T > 0\) and \(n \in \mathbb{N}\). If \(a_j \in \mathcal{S}(\mathbb{R}^2)\) are constructed as above, then \((\varepsilon D_t + H)\phi_t = 0\) has a solution of the form

\[
\phi_t(x) = \frac{1}{\sqrt{\varepsilon}} \cdot \mathcal{K}_t f \left( \frac{x - y_t}{\sqrt{\varepsilon}} \right) + \sum_{j=1}^{n} \frac{\varepsilon^{-j+1}}{2} a_j \left( t, \frac{x - y_t}{\sqrt{\varepsilon}} \right) + \mathcal{O}_{L^2}(\varepsilon^{n+1}),
\]

uniformly for \(\varepsilon \in (0, 1]\) and \(t \in [0, T]\).

According to Theorem 3, after adequately correcting the initial data (3.3) we obtain approximate solutions concentrated near \(y_t\) at arbitrary accuracy in \(\varepsilon\). Correcting the initial data is necessary: otherwise the subleading amplitude (which is of order \(\varepsilon^{1/2}\)) likely contains a dispersive part, hence cannot remain fully concentrated near \(y_t\).

**Remark 2** (Timescale of validity of error estimates). Including higher order correctors as in (3.41) does not extend the timescale of validity \(\varepsilon^{-1/2}\) of the approximation solution. Indeed, the \(n\)-th corrector is of order \(\varepsilon^{n+1/2} t^n\) – the term \(t^n\) corresponds to \(n\) recursive integrations in (40). After applying Lemma 3.5, this yields that the constant implicitly involved in the remainder \(\mathcal{O}_{L^2}(\varepsilon^{n+1/2})\) of (3.41) grows like \(T^{n+1}\): it is small only for \(T \ll \varepsilon^{-1/2}\).
Proof of Theorem 3. Fix $n \in \mathbb{N}$, $T > 0$ and $f_0 \in \mathcal{S}(\mathbb{R})$. We pick $a_j$ solving (3.7) for $0 \leq j \leq n+1$ (constructed above) with $f_{n+1} = 0$, and we define

$$a^{(n)} = \sum_{j=0}^{n+1} \varepsilon^{j/2} a_j, \quad v_t(x) = W[a^{(n)}]_{y_t}(x) = \frac{1}{\sqrt{\varepsilon}} \sum_{j=0}^{n+1} \varepsilon^{j/2} a_j \left( t, \frac{x-y_t}{\sqrt{\varepsilon}} \right).$$

(3.42)

By construction, the functions $a_j$ are smooth in $t$ and Schwartz in $x$. In particular, they satisfy uniform Schwartz-class bounds for $t$ in compact intervals. Hence, thanks to Lemma 3.1, we have uniformly in $t \in [0,T]$:

$$\| (\varepsilon D_t + H) v_t \|_{L^2} \leq C \varepsilon^{n+1}.$$

Let $\phi_t$ be the solution to $(\varepsilon D_t + H) \phi_t = 0$ with $\phi_0 = v_0$ – see (3.42) with $t = 0$. Thanks to Lemma 3.5:

$$\| v_t - \phi_t \|_{L^2} \leq C \varepsilon^{n+1/2}.$$

In other words, $v_t = \phi_t + \mathcal{O}_{L^2}(\varepsilon^{n+1/2})$.

4. The effect of curvature

It is natural to wonder which quantities affect the lifetime of our quantum state. For instance, when $\kappa$ is linear, the interface is straight and the edge states have infinite lifetime. If $\kappa$ is asymptotically linear, the interface is asymptotically straight and we expect an extended time of validity. In contrast, numerical simulations indicate that circular interfaces come with gradual dispersion: see Figure 3.

This suggests that an integrated curvature limits the lifespan. Curvature however cannot be the only limiting factor: as Figure 6 shows, even straight interfaces can generate dispersion. To isolate the effects of curvature, we consider in this section domain walls $\kappa$ that satisfy a geometric condition:

$$y \in \kappa^{-1}(0) \Rightarrow \left| \nabla \kappa(y) \right| = 1, \quad \nabla^2 \kappa(y) \cdot \nabla \kappa(y) = 0.$$

(4.1)

Example of $\kappa$ satisfying (4.1) include:

- $\kappa(x) = \omega \cdot x$ with $|\omega| = 1$, for a straight interface;
- $\kappa(x) = \sqrt{x_1^2 + x_2^2} - 1$, for a circle.

The condition (4.1) is not geometrically restrictive: given $\Gamma$, we can always find $\kappa$ with $\Gamma = \kappa^{-1}(0)$, satisfying (4.1) – see §4.2. This condition excludes scenarios such as those giving rise to Figure 6. Under (4.1), $\dot{\theta}_t$ is the curvature of $\Gamma$ at $y_t$; and in a suitable frame, the Hessian of $\kappa$ along $\Gamma$ depends only on $\dot{\theta}_t$:

$$\left\langle R_{\theta_t}^T x, \nabla^2 \kappa(y_t) R_{\theta_t}^T x \right\rangle = \dot{\theta}_t x_1^2.$$

(4.2)

Theorem 4. Under (4.1), the solution (1.4) to $(\varepsilon D_t + H) \Psi_t = 0$ of Theorem 1 satisfies, uniformly in $t > 0$ and $\varepsilon \in (0,1]$:}

$$\Psi_t(x) = \frac{1}{\sqrt{\varepsilon}} \cdot \exp \left( -\frac{(x-y_t)^2}{2\varepsilon} \right) \left[ e^{i\theta_t/2} + e^{-i\theta_t/2} \right] + \mathcal{O}_{L^2}(\varepsilon^{1/2} + \varepsilon t(1 + \Theta_t)), \quad \Theta_t = \int_0^t \dot{\theta}_s^2 ds.$$

(4.3)
When $\Gamma$ is asymptotically straight (i.e. it has $L^2$-curvature), the remainder in (4.3) remains small for $t \ll \varepsilon^{-1}$; our quantum state is longer-lived. In contrast, if $\Gamma$ is a closed loop then $\Theta_t$ grows linearly and our state is only close to the exact solution for $\varepsilon t^2 \ll 1$, that is $t \ll \varepsilon^{-1/2}$: there is no improvement over Theorem 1. Thus, such states – which are not globally topological – have a shorter lifetime.

Theorem 4 highlights effective limitations of dynamical edge states: they do not survive in strongly curved environments; see Figure 7. This means that our results rely on $\kappa$ being sufficiently regular. Other limitations include cross-type or knot-type interfaces, for which $\kappa$ degenerates quadratically; see Figure 8. Such scenarios form interesting open problems.

4.1. **Proof of Theorem 4.** The proof of Theorem 4 relies on the precise calculation of the corrector $a_1 = b_1 + \mathcal{K}_t f_1$ involved in §3.4-3.5.

**Lemma 4.1.** In the setup of Theorem 4, the subleading amplitude $a_1 = b_1 + \mathcal{K}_t f_1$ satisfies

$$b_1(t,x) = \frac{1 - x_1^2}{2} x_2 e^{-\frac{x_1^2}{2}} \begin{bmatrix} e^{-i\theta t/2} \\ -e^{i\theta t/2} \end{bmatrix} \hat{\theta}_t, \quad f_1(t,x_1) = \frac{2x_1 - x_3^2}{2} e^{-\frac{x_1^2}{2}} \Theta_t$$

(4.4)

**Proof of Lemma 4.1.** The proof relies on the hierarchy of transport equations studied in §3.4. We use the notations introduced there, keeping in mind that $r_t = 1$ here.

We first compute $b_1$. From the initial condition (1.3),

$$a_0(0,x) = e^{-\frac{x_1^2}{2}} \begin{bmatrix} e^{-i\theta_0/2} \\ -e^{i\theta_0/2} \end{bmatrix}.$$  

Hence $f_0(x_1) = e^{-x_1^2/2}$. Moreover $b_1$ is the unique solution in $\ker(T_0)^\perp$ to $T_0 b_1 + T_1 a_0 = 0$. With $q_t(x) = \langle x, \nabla^2 \kappa(y_t) x \rangle$, this equation reads

$$T_0 b_1 = -e^{-\frac{x_1^2}{2}} \left( D_t + \frac{q_t(x)}{2} \sigma_3 \right) \begin{bmatrix} e^{-i\theta t/2} \\ -e^{i\theta t/2} \end{bmatrix} = \frac{1}{2} e^{-\frac{x_1^2}{2}} \left( \hat{\theta}_t - q_t(x) \right) \begin{bmatrix} e^{-i\theta t/2} \\ e^{i\theta t/2} \end{bmatrix},$$

(4.5)

where we used the identities (3.25). To find $b_1$, we use the operators $\mathcal{R}_\theta$ and $\mathcal{U}_\theta$ introduced in (2.2) and we look for $b_1$ of the form

$$b_1 = \mathcal{R}_\theta U_\theta \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$
Figure 8. Left: interface $\kappa(x) = x_1 x_2$; right: an interface consisting of two rings parametrized by $|x + e_1|/|x - e_1| = 1$ both with $\varepsilon = 2 \cdot 10^{-2}$. While the direction of propagation can be heuristically predicted using the bulk-edge correspondence, establishing a rigorous theory remains an open problem.

We take advantage of the relation $T_0 = L_{\theta_{1,1}} = \mathcal{R}_{\theta_1} U_{\theta_1} L_{0,1} U_{\theta_1}^{-1} \mathcal{R}_{\theta_1}^{-1}$ (see (3.11) and the beginning of the proof of Lemma 3.2) and apply the operator $U_{\theta_1}^{-1} \mathcal{R}_{\theta_1}^{-1}$ to the equation (4.5). We deduce that $c_1$ and $c_2$ must solve:

$$L_{0,1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} e^{-\frac{x_2^2}{\pi}} \left( \dot{\theta}_t - q_t \left( R_{\theta_1}^\top x \right) \right) U_{\theta_1}^{-1} \begin{bmatrix} e^{-i\theta_t/2} \\ e^{i\theta_t/2} \end{bmatrix} = \frac{1}{2} e^{-\frac{x_2^2}{\pi}} \left( \dot{\theta}_t - q_t \left( R_{\theta_1}^\top x \right) \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We now use the operator $\tilde{L}_{0,1}$ of (3.16) and get:

$$\begin{bmatrix} 0 \\ a \end{bmatrix} \begin{bmatrix} c_1 - c_2 \\ c_1 + c_2 \end{bmatrix} = e^{-\frac{x_2^2}{2}} \left( \dot{\theta}_t - q_t \left( R_{\theta_1}^\top x \right) \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From $a^*(c_1 + c_2) = 0$, we obtain $c_1 = -c_2$ because $a^*$ has trivial kernel. Thus,

$$b_1(t, x) = c_1(t, R_{\theta_1} x) U_{\theta_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1(t, R_{\theta_1} x) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad ac_1(t, x) = \frac{1}{2} e^{-\frac{x_2^2}{\pi}} \left( \dot{\theta}_t - q_t \left( R_{\theta_1}^\top x \right) \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We now use (4.2): $q_t \left( R_{\theta_1}^\top x \right) = \dot{\theta}_t x_1^2$. Hence $c_1$ satisfies the equation

$$ac_1(t, x) = \frac{1}{2} e^{-\frac{x_2^2}{\pi}} \dot{\theta}_t.$$

From the condition $b_1 \in \ker(T_0)^\perp$ we deduce that $c_1(t, x_1, \cdot) \perp e^{-x_2^2/2}$ for every $(t, x_1)$. Therefore, $c_1$ is explicitly given by:

$$c_1(t, x) = \frac{1}{2} \frac{-x_1^2}{x_2} e^{-\frac{x_2^2}{\pi}} \dot{\theta}_t. \quad (4.6)$$

This yields the identity (4.4) for $b_1$.

We now focus on $f_1$. It solves the transport equation (3.39):

$$D_t f_1(t, x_1) = \frac{1}{2 \sqrt{\pi}} \int_R e^{-\frac{x_2^2}{4}} \begin{bmatrix} e^{i\theta_t/2} \\ -e^{-i\theta_t/2} \end{bmatrix} : \beta_1(t, R_{\theta_1}^\top x) \, dx_2,$$
where by (3.38) \( \beta_1 = -T_1 b_1 - T_2 a_0 \). In view of (3.6), \( T_2 \) is carried by \( \sigma_3 \) and we deduce from (3.25) that

\[
-e^{i\theta_t/2} \beta_1(t, x) = 2D(t)(c_1(t, R_\theta x)) = 2 \left( D_t + R_\theta x \cdot D_x \right) c_1(t, R_\theta x).
\]

Using (3.27), we obtain:

\[
-e^{i\theta_t/2} \beta_1(t, R_\theta^T x) = 2 \left( D_t + R_\theta R_\theta^T x \cdot D_x \right) c_1(t, x) = 2 \left( D_t + \dot{\theta}_t \left[ \frac{x_2}{-x_1} \right] \cdot D_x \right) c_1(t, x),
\]

hence the transport equation for \( f_1 \):

\[
D_t f_1(t, x_1) = -\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\frac{x_2^2}{2}} (D_t c_1(t, x) + \dot{\theta}_t (x_2 D_{x_1} - x_1 D_{x_2}) c_1(t, x)) \, dx_2.
\]  (4.7)

Thanks to the explicit formula (4.6) for \( c_1 \), we have:

\[
\int_{\mathbb{R}} e^{-\frac{x_2^2}{2}} D_t c_1(t, x) \, dx_2 = 1 - \frac{x_1^2}{2} e^{-\frac{x_1^2}{2}} \int_{\mathbb{R}} x_2 e^{-\frac{x_2^2}{2}} \, dx_2 \cdot D_t \dot{\theta}_t = 0.
\]

We deduce from integrating (4.7) and using the condition \( f_1(0, x_1) = 0 \) that

\[
f_1(t, x_1) = -\frac{1}{\sqrt{\pi}} \int_0^t \dot{\theta}_s \int_{\mathbb{R}} e^{-\frac{x_2^2}{2}} (x_2 \partial_{x_1} - x_1 \partial_{x_2}) c_1(s, x) \, dx_2 \, ds
\]

\[
= -\frac{1}{\sqrt{\pi}} \int_0^t \dot{\theta}_s \int_{\mathbb{R}} x_2 e^{-\frac{x_2^2}{2}} (\partial_{x_1} - x_1) c_1(s, x) \, dx_2 \, ds
\]  (4.8)

where we have performed an integration by parts in \( x_2 \). We now compute the integrals that appear in (4.8) using (4.6). The integral on the LHS corresponds to integrating an odd function, hence produces 0. Regarding the one on the RHS, we observe

\[
(\partial_{x_1} - x_1) c_1(t, x) = x_2 (x_1^2 - 2x_1) e^{-\frac{x_1^2}{2}} \dot{\theta}_t.
\]

Therefore, the RHS of (4.8) becomes:

\[
\frac{1}{\sqrt{\pi}} \int_0^t \dot{\theta}_s \int_{\mathbb{R}} x_2 e^{-\frac{x_2^2}{2}} (\partial_{x_1} - x_1) c_1(s, x) \, dx_2 \, ds = (x_1^2 - 2x_1) e^{-\frac{x_1^2}{2}} \int_0^t \dot{\theta}_s^2 \, ds = \frac{2x_1 - x_1^3}{2} \int_0^t \dot{\theta}_s^2 \, ds.
\]  (4.9)

Plugging (4.9) in (4.8), we conclude that

\[
f_1(t, x_1) = \frac{x_1^3 - 2x_1}{2} e^{-\frac{x_1^2}{2}} \dot{\theta}_t, \quad \text{where} \quad \Theta_t = \int_0^t \dot{\theta}_s^2 \, ds.
\]

This completes the proof of Lemma 4.1.

\[ \square \]

**Proof of Theorem 4.** We set \( a^{(2)} = a_0 + \varepsilon^{1/2}a_1 + \varepsilon b_2 \), with \( a_0, a_1, b_2 \) solutions of

\[
T_0 a_0 = 0, \quad T_0 a_1 + T_1 a_0 = 0, \quad T_0 b_2 + T_1 a_1 + T_2 a_0 = 0;
\]

see §3.4-3.7 for their construction. Thanks to Lemma 3.1, we have:

\[
\| (\varepsilon D_t + H) W [a^{(2)}]_1 \|_{L^2} \leq C \varepsilon^2 \left( \| D_t b_2 \|_{L^2} + \| \langle x \rangle^4 a_0 \|_{L^2} + \| \langle x \rangle^4 a_1 \|_{L^2} + \| \langle x \rangle^4 b_2 \|_{L^2} \right).
\]
From the explicit expression (3.30) for \( a_0 \), \( \| \langle x \rangle^4 a_0 \|_{L^2} \) is uniformly bounded. From the explicit expression (4.4) for \( a_1 \), \( \| \langle x \rangle^4 a_1 \|_{L^2} \) is bounded by \( 1 + \Theta_t \). It remains to bound \( \| \langle x \rangle^4 b_2 \|_{L^2} \) and \( \| D_t b_2 \|_{L^2} \). By construction, recalling that \( r_t = 1 \):

\[
b_2(t, \cdot) = -L_{\theta_1}^{-1}(T_1 a_1 + T_2 a_0).
\]

The explicit expressions for \( a_0 \) and \( a_1 \) allow us to bound Schwartz-class seminorms of \( T_1 a_1 + T_2 a_0 \) by \( 1 + \Theta_t \) (the term \( \partial_t \Theta_t = (\partial_t \theta_1)^2 \) is uniformly bounded). Arguing as in (3.35), we deduce that Schwartz-class seminorms of \( b_2(t, \cdot) \) and \( D_t b_2(t, \cdot) \) are bounded by \( 1 + \Theta_t \). In particular:

\[
\| D_t b_2 \|_{L^2} + \| \langle x \rangle^4 b_2 \|_{L^2} \leq C(1 + \Theta_t).
\]

We deduce that

\[
\left\| (\varepsilon D_t + H) \mathcal{W}[a^{(2)}]_{y_t} \right\|_{L^2} \leq C \varepsilon^2 (1 + \Theta_t).
\]

We note that at \( t = 0 \), \( \Psi_t \) and \( \mathcal{W}[a^{(2)}]_{y_t} \) coincide up to \( \mathcal{O}_{L^2}(\varepsilon^{1/2}) \). Thus, applying Lemma 3.5, we conclude that

\[
\left\| \Psi_t - \mathcal{W}[a^{(2)}]_{y_t} \right\|_{L^2} \leq C \varepsilon^{1/2} + C \varepsilon t (1 + \Theta_t).
\]

This completes the proof of Theorem 4. \( \square \)

### 4.2. Geometric setup.

We prove here the geometric facts stated above. First, if \( \Gamma \) is a nodal set, then we can find a function \( \kappa \) satisfying (4.1) with \( \kappa^{-1}(0) = \Gamma \).

**Lemma 4.2.** If \( \Gamma = \tilde{\kappa}^{-1}(0) \) for a function \( \tilde{\kappa} \in C_0^\infty(\mathbb{R}^2) \) satisfying the transversality condition (3.1), then we can find \( \kappa \in C_0^\infty(\mathbb{R}^2) \) satisfying (4.1) such that \( \Gamma = \kappa^{-1}(0) \).

**Proof of Lemma 4.2.** Without loss of generality, we may assume that \( |\nabla \tilde{\kappa}(y)| = 1 \) along \( \Gamma \). We aim to construct \( \rho \in C_0^\infty(\mathbb{R}^2) \) with \( |\nabla \rho|_{\infty} < 1 \) such that if

\[
\kappa = \tilde{\kappa} - \rho \tilde{\kappa}^2 - \frac{2}{\tilde{\kappa}^2} \rho \left( 1 - \frac{\tilde{\kappa} \rho}{2} \right) \quad (4.10)
\]

then \( \kappa \) satisfies (4.1). Under the condition \( |\kappa \rho|_{\infty} < 1 \), \( \kappa^{-1}(0) = \tilde{\kappa}^{-1}(0) = \Gamma \). Moreover,

\[
\nabla \kappa = \nabla \tilde{\kappa} (1 - \rho \tilde{\kappa}) - \frac{\tilde{\kappa}^2}{2} \nabla \rho;
\]

hence if \( y \in \Gamma \) then \( \nabla \kappa(y) = \nabla \tilde{\kappa}(y) \). Also

\[
\nabla^2 \kappa = \nabla^2 \tilde{\kappa} (1 - \rho \tilde{\kappa}) - \rho \nabla \tilde{\kappa} \nabla \tilde{\kappa}^\top - \tilde{\kappa} \nabla \rho \nabla \kappa^\top - \tilde{\kappa} \nabla \kappa \nabla \rho^\top - \frac{\tilde{\kappa}^2}{2} \nabla^2 \rho.
\]

So, if \( y \in \Gamma \) then \( \nabla^2 \kappa(y) = \nabla^2 \tilde{\kappa}(y) - \rho(y) \nabla \tilde{\kappa}(y) \nabla \tilde{\kappa}^\top(y) \). We deduce that for \( y \in \Gamma \),

\[
\langle \nabla \kappa(y), \nabla^2 \kappa(y) \nabla \kappa(y) \rangle = \langle \nabla \tilde{\kappa}(y), \nabla^2 \tilde{\kappa}(y) \nabla \tilde{\kappa}(y) \rangle - \rho(y) \langle \nabla \tilde{\kappa}(y), \nabla \tilde{\kappa}(y) \nabla \tilde{\kappa}(y)^\top \rangle - \nabla \tilde{\kappa}(y) \nabla \tilde{\kappa}(y)^\top \rho(y)
\]

We now pick \( \tilde{\rho} \in C^\infty(\mathbb{R}^2) \), such that \( \rho(y) = \langle \nabla \tilde{\kappa}(y), \nabla^2 \tilde{\kappa}(y) \nabla \tilde{\kappa}(y) \rangle \) for \( y \in \Gamma \). Then, with

\[
\rho(y) = \frac{\tilde{\rho}(y)}{1 + \tilde{\rho}(y)^2 \tilde{\kappa}(y)^2}.
\]
we still have $\rho(y) = \langle \nabla \tilde{k}(y), \nabla^2 \tilde{k}(y) \nabla \tilde{k}(y) \rangle$ for $y \in \Gamma$; $\rho \in C^\infty_0(\mathbb{R}^2)$; and finally,

$$|\rho\tilde{k}| = \frac{|\hat{\rho}\tilde{k}|}{1 + \hat{\rho}^2\tilde{k}^2} \leq \frac{1}{2}. $$

The function $\kappa$ given by (4.10) now satisfies the requirements of the lemma. Indeed, by construction we have for $y \in \Gamma$:

$$|\nabla \kappa(y)| = |\nabla \tilde{k}(y)| = 1, \quad \langle \nabla \kappa(y), \nabla^2 \kappa(y) \nabla \kappa(y) \rangle = 0. \quad (4.11)$$

We can then write $|\nabla \kappa|^2 = 1 + \alpha \kappa$ for some smooth function $\alpha$. Taking the gradient on both sides produces the identity:

$$2\nabla^2 \kappa \cdot \nabla \kappa = \alpha \nabla \kappa + \kappa \nabla \alpha.$$ 

In particular, pairing with $\nabla \kappa^{-1}$ gives

$$2 \langle \nabla \kappa^{-1}, \nabla^2 \kappa \cdot \nabla \kappa \rangle = \kappa \langle \nabla \kappa^{-1}, \nabla \alpha \rangle. $$

Specializing at $y \in \Gamma$ produces

$$\langle \nabla \kappa(y)^{-1}, \nabla^2 \kappa(y) \cdot \nabla \kappa(y) \rangle = 0,$$

which together with the second identity of (4.11) yields $\nabla^2 \kappa(y) \nabla \kappa(y) = 0$ when $y \in \Gamma$.

We now prove the useful relation (4.2).

**Proof of (4.2).** We recall that $R_{\theta t}^* e_1 = -\dot{y}_t = -\nabla \kappa(y_t)^{-1}$ and $R_{\theta t}^* e_2 = -\dot{y}_t^{-1} = \nabla \kappa(y_t)$. Therefore, proving (4.2) boils down to showing

$$\langle \dot{y}_t, \nabla^2 \kappa(y_t) \dot{y}_t \rangle = \theta_t, \quad \langle \dot{y}_t, \nabla^2 \kappa(y_t) \dot{y}_t^{-1} \rangle = 0, \quad \langle \dot{y}_t^{-1}, \nabla^2 \kappa(y_t) \dot{y}_t \rangle = 0. \quad (4.12)$$

The last two identities are direct consequences of $\nabla^2 \kappa(y) \nabla \kappa(y) = 0$ for $y \in \kappa^{-1}(0)$. For the first identity in (4.12), we note that

$$\begin{cases}
\cos(\theta_t) = -\langle \dot{y}_t, e_1 \rangle = \langle \nabla \kappa(y_t), e_2 \rangle \\
\sin(\theta_t) = -\langle \dot{y}_t, e_2 \rangle = -\langle \nabla \kappa(y_t), e_1 \rangle.
\end{cases}$$

Taking time-derivatives, and the identity $\sin(\theta_t)e_2 + \cos(\theta_t)e_1 = -\dot{y}_t$, we deduce that

$$\begin{cases}
\dot{\theta}_t \sin(\theta_t) = -\langle \nabla^2 \kappa(y_t) \dot{y}_t, e_2 \rangle \\
\dot{\theta}_t \cos(\theta_t) = -\langle \nabla^2 \kappa(y_t) \dot{y}_t, e_1 \rangle \Rightarrow \dot{\theta}_t = \langle \nabla^2 \kappa(y_t) \dot{y}_t, \dot{y}_t \rangle.
\end{cases}$$

This completes the proof of (4.2). \hfill \Box

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