Shape of Deconstruction

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Abstract

We construct a six-dimensional Maxwell theory using a latticized extra space, the continuum limit of which is a shifted torus recently discussed by Dienes. This toy model exhibits the correspondence between continuum theory and discrete theory, and give a geometrical insight to theory-space model building.

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I. INTRODUCTION

Though the idea that the dimension of the space-time may be more than four is very old, the possible existence of the extra space becomes a serious subject of theoretical and phenomenological study on unified theories only recently. In the view of the four-dimensional space-time, every kind of the field in higher dimensions has the corresponding Kaluza-Klein (KK) spectrum, if the extra space is compact. The mass of the excited state is proportional to the inverse of the size of the extra space.

Recently Dienes claimed that the KK spectrum depends on the shape as well as the volume of the extra two-torus[1, 2].[12] The consideration of the shape moduli of the manifold may alter experimental bounds on the compactification scale.

On the other hand, there is another scheme to reexamine the concept of space, which is known as deconstruction[6, 7]. Suppose a number of copies of a four-dimensional theory. Further add a new set of fields, linking pairs of these individual “sites” in the theory space. The resulting whole theory may (or may not) be equivalent to a higher-dimensional theory with discretized, or, latticized extra dimensions. Evidence or signal for this kind of deconstructed dimension would be the discovery of a finite and specific “KK spectrum” in high energy experiments.

A simple latticized model with a large number of sites in “two-dimensional” theory space[8] resembles a continuum compactification. It may be known that the discrete version of the shifted torus which Dienes examined can be (de)constructed naturally. Unfortunately, little attention has been paid to the detail of such formulation. When we wish to distinguish between the discrete and continuum theories by mean of future experiments, we should prepare the several detailed templates of the mass spectra.

In this paper, we analyze the six-dimensional $U(1)$ gauge theory with a latticized torus. We explicitly show the mass spectrum, which becomes the KK spectrum of the continuum theory in the limit of the large number of sites. Our toy model exhibits a clear correspondence between continuum theory and discrete theory and may give a physical and geometrical insight to model building.

In Sec. II, we study the mass spectrum which comes from the $U(1)$ gauge field on a shifted lattice. The one-loop effective potential for the Wilson line in scalar QED is discussed in Sec. III. We close with Sec. IV, where summary and conclusion are given.
II. SHIFTED LATTICE

The discretization of a two-torus is realized by a doubly periodic lattice with $N_u \times N_v$ sites. The lagrangian we first consider is[13]

$$\mathcal{L}_A = \frac{1}{g^2} \sum_{k=1}^{N_u} \sum_{\ell=1}^{N_v} \frac{1}{4} F_{k,\ell \mu \nu}^\mu F_{k,\ell \mu \nu}^\nu - \frac{1}{2} (D^\mu U_{k,\ell}^\dagger D_\mu U_{k,\ell} - (D^\mu V_{k,\ell}^\dagger D_\mu V_{k,\ell} - V(|U_{k,\ell}|, |V_{k,\ell}|)),$$  

(1)

where $g$ is a gauge coupling, $F_{k,\ell \mu \nu}^\mu = \partial_\mu \tilde{A}_{k,\ell}^\nu - \partial_\nu \tilde{A}_{k,\ell}^\mu$ and $\mu, \nu = 0, 1, 2, 3$. The link fields $U_{k,\ell}$ and $V_{k,\ell}$ are transformed as

$$U_{k,\ell} \to W_{k,\ell} U_{k,\ell} W_{k,\ell}^*, \quad V_{k,\ell} \to W_{k,\ell} V_{k,\ell} W_{k+1,\ell+h}^*,$$  

(2)

under $U(1)^{N_uN_v}$ ($|W_{k,\ell}| = 1$ for any $k, \ell$). Here $h$ is an integer. For $h = 0$, a similar model has been analyzed by Lane[8] (see FIG. 1).

![Diagram](image)

FIG. 1: A schematic view of the two-types of the latticized two-tori. Since the diagrams are similar to FIG. 4 in [7], labels for each site and link are omitted. The left graph corresponds to the case with $h = 0$, while the right one corresponds to the case with $h = 1$.

Then the covariant derivatives are

$$D^\mu U_{k,\ell} = \partial^\mu U_{k,\ell} - i \tilde{A}_{k,\ell}^\mu U_{k,\ell} + i U_{k,\ell} \tilde{A}_{k,\ell+1},$$  

(3)

$$D^\mu V_{k,\ell} = \partial^\mu V_{k,\ell} - i \tilde{A}_{k,\ell}^\mu V_{k,\ell} + i V_{k,\ell} \tilde{A}_{k+1,\ell+h}. $$  

(4)
We assume that the potential $V$ forces each $|U_{k\ell}|$ into $f_u/\sqrt{2}$ and each $|V_{k\ell}|$ into $f_v/\sqrt{2}$. Then $U_{k\ell}$ and $V_{k\ell}$ are expressed as

$$U_{k\ell} = \frac{f_u}{\sqrt{2}} \exp\left(i\tilde{\chi}_{k\ell}/f_u\right), \quad V_{k\ell} = \frac{f_v}{\sqrt{2}} \exp\left(i\tilde{\sigma}_{k\ell}/f_v\right).$$  

We use the Fourier decomposition of the fields:

$$\tilde{A}_{k\ell}^\mu = \frac{1}{\sqrt{N_u N_v}} \sum_{p,q} A_{pq}^\mu \exp\left[2\pi i \left(\frac{pk}{N_v} + \frac{q\ell}{N_u}\right)\frac{1}{N_v}\right],$$  

$$\tilde{\chi}_{k\ell} = \frac{1}{\sqrt{N_u N_v}} \sum_{p,q} \chi_{pq} \exp\left[2\pi i \left(\frac{pk}{N_v} + \frac{q\ell}{N_u}\right)\frac{1}{N_v}\right],$$  

$$\tilde{\sigma}_{k\ell} = \frac{1}{\sqrt{N_u N_v}} \sum_{p,q} \sigma_{pq} \exp\left[2\pi i \left(\frac{pk}{N_v} + \frac{q\ell}{N_u}\right)\frac{1}{N_v}\right].$$

Substituting these component fields into the lagrangian (1), we find that the photon indicated by $A_{pq}^\mu$ acquire the mass:

$$M_{pq}^2 = 4 \left[f_u^2 \sin^2\left(\frac{\pi q}{N_u}\right) + f_v^2 \sin^2\left(\frac{\pi p}{N_v} + \frac{\pi q h}{N_u}\right)\right],$$  

while only one pseudo Nambu-Goldstone boson (PNGB), which is the combination of $\chi_{00}$ and $\sigma_{00}$ survive.

For small $p/N_v$ and $q/N_u$, this mass spectrum becomes a continuum KK spectrum given by

$$M_{pq}^2 \approx 4\pi^2 \frac{1}{\mathcal{V} \tau_2} \left[(p - q\tau_1)^2 + q^2 \tau_2^2\right],$$

This spectrum precisely corresponds to the equation (7) of Dienes’ paper[1]:

$$M_{pq}^2 \approx 4\pi^2 \frac{1}{\mathcal{V} \tau_2} \left[(p - q\tau_1)^2 + q^2 \tau_2^2\right],$$

where

$$\mathcal{V} = \frac{N_u N_v}{f_u f_v}, \quad \tau_2 = \frac{f_u N_v}{f_v N_u}, \quad \tau_1 = -\frac{N_u}{N_v} h.$$  

How much difference is caused when $N_u$ and $N_v$ are finite? For example, the spectra for $(N_u, N_v) = (5, 3)$ and $(21, 13)$ are shown in FIG. 2. Here the horizontal axis indicates the fractional part of $\tau_1$.[14] The vertical axis indicates $M_{pq}\sqrt{\mathcal{V}}/(2\pi)$. The parameters are chosen by Eq. (12) and we take $f_u = f_v$ for simplicity. The curves indicate the mass spectra in the large $N_u, N_v$ limit, or in the corresponding continuum case.
One can see that the lowest (other than zero) mass level is almost on the curve of the continuum theory. For \( N_u = 21 \) and \( N_v = 13 \), the higher levels also become close spectra to that in the continuum limit. Thus it is difficult to find the discreteness of the extra space only by the low mass spectrum, even if the torus is shifted or not.

### III. THE EFFECTIVE POTENTIAL

We consider the one-loop effect of charged scalar fields and assume that \( N_u \geq 3 \) and \( N_v \geq 3 \). The lagrangian for a complex scalar fields in our theory space is written by

\[
\mathcal{L}_\phi = \sum_{k=1}^{N_u} \sum_{\ell=1}^{N_v} \left[ -(D^\mu \tilde{\phi}_{k\ell})^\dagger D_\mu \tilde{\phi}_{k\ell} \right] + f_u \sum_{k=1}^{N_u} \sum_{\ell=1}^{N_v} \left( \sqrt{2} \tilde{\phi}_{k\ell}^* U_{k\ell} \tilde{\phi}_{k,\ell+1} + \sqrt{2} \tilde{\phi}_{k\ell}^* U_{k,\ell-1}^* \tilde{\phi}_{k,\ell-1} - 2 f_u \tilde{\phi}_{k\ell}^* \tilde{\phi}_{k\ell} \right) + f_v \sum_{k=1}^{N_u} \sum_{\ell=1}^{N_v} \left( \sqrt{2} \tilde{\phi}_{k\ell}^* V_{k\ell} \tilde{\phi}_{k+1,\ell+h} + \sqrt{2} \tilde{\phi}_{k\ell}^* V_{k-1,\ell-h}^* \tilde{\phi}_{k-1,\ell-h} - 2 f_v \tilde{\phi}_{k\ell}^* \tilde{\phi}_{k\ell} \right),
\]

(13)

where

\[
D^\mu \tilde{\phi}_{k\ell} \equiv \partial^\mu \tilde{\phi}_{k\ell} - i \tilde{A}^\mu_{k\ell} \tilde{\phi}_{k\ell}.
\]

(14)

The scalar field can be expanded in the same manner as the gauge field:

\[
\tilde{\phi}_{k\ell} = \frac{1}{\sqrt{N_u N_v}} \sum_{p,q} \phi_{pq} \exp \left[ 2\pi i \left( \frac{p k}{N_u} + \frac{q \ell}{N_v} \right) \right],
\]

(15)

Only the zero-mode fields \( \chi_{00} \) and \( \sigma_{00} \) affect on the mass spectrum induced from the scalar field. The masses are:

\[
M_{pq}^2 = 4 \left[ f_u^2 \sin^2 \left( \frac{\pi q}{N_u} + \frac{\bar{X}}{2 f_u} \right) + f_v^2 \sin^2 \left( \frac{\pi p}{N_v} + \frac{\pi q h}{N_u} + \frac{\bar{\sigma}}{2 f_v} \right) \right],
\]

(16)

where \( \bar{X} \equiv \chi_{00}/\sqrt{N_u N_v} \) and \( \bar{\sigma} \equiv \sigma_{00}/\sqrt{N_u N_v} \).

The one-loop effective potential for \( \bar{X} \) and \( \bar{\sigma} \) is obtained by

\[
\ln \det[-\nabla^2 + M_{pq}^2] = \lim_{\epsilon \to 0} - \frac{1}{(2\pi)^{4-2\epsilon}} \sum_{p,q} \int_0^\infty \frac{dt}{t} \int d^{4-2\epsilon} k \ \exp \left[ -(k^2 + M_{pq}^2) t \right] = \lim_{\epsilon \to 0} - \frac{1}{(4\pi)^{2-\epsilon}} \int_0^\infty \frac{dt}{t} t^{-2+\epsilon} \sum_{p,q} \exp \left[ -M_{pq}^2 t \right],
\]

(17)
after an appropriate regularization. Using the formula \[10\]
\[
\exp \left[ -4f^2 \sin^2(\theta/2)t \right] = e^{-2f^2t} \sum_{n=-\infty}^{\infty} \cos n\theta I_n(2f^2t)
\]
\[
e^{-2f^2t} \sum_{n=-\infty}^{\infty} e^{in\theta} I_n(2f^2t),
\]
(18)
where \(I_\nu(x)\) is the modified Bessel function, we can rewrite the effective potential as

\[
V_{\text{eff}}(\bar{\chi}, \bar{\sigma}) = -\frac{1}{(4\pi)^2} \sum_{p,q} \left( \sum_{n_u,n_v=-\infty}^{\infty} \right)' e^{in_u\theta_u+in_v\theta_v} I(n_u,n_v),
\]
(19)
where

\[
\theta_u \equiv \frac{2\pi q}{N_u} + \frac{\bar{\chi}}{f_u}, \quad \theta_v \equiv \frac{2\pi p}{N_v} + \frac{2\pi q h}{N_u} + \frac{\bar{\sigma}}{f_v}
\]
(20)
and

\[
I(n_u,n_v) = \int_0^\infty \frac{dt}{t^3} e^{-2f^2t-2f^2t} I_{n_u}(2f^2t) I_{n_v}(2f^2t),
\]
(21)
and in the summation \((\sum)'\), \(n_u = n_v = 0\) is omitted since this term makes no dependence on \(\bar{\chi}\) or \(\bar{\sigma}\).

Carrying out the summation over \(p\) and \(q\), we find that every term satisfying \(n_u \frac{p}{N_u} + n_v \left( \frac{p}{N_v} + \frac{q h}{N_u} \right) \in \mathbb{Z}\) is left. Therefore we find

\[
V_{\text{eff}}(\bar{\chi}, \bar{\sigma}) = -\frac{N_u N_v}{(4\pi)^2} \left( \sum_{p',q'=\infty}^{\infty} \right)' \cos \left[ (q' - \frac{N_v}{N_u} h) \frac{N_u \bar{\chi}}{f_u} + \frac{p' N_v \bar{\sigma}}{f_v} \right] \times I(q' N_u - p' N_u h, p' N_v).
\]
(22)
Its global minima are located at

\[
\frac{N_u}{f_u} \bar{\chi} = 2\pi q, \quad \frac{N_v}{f_v} \bar{\sigma} = 2\pi \left( p + \frac{N_v}{N_u} h q \right) \quad (p, q \text{ are integers}).
\]
(23)
These minima are of course gauge equivalent, because simplest nontrivial Wilson loop elements

\[
\exp \left[ iN_u \frac{\bar{\chi}}{f_u} \right], \quad \exp \left[ iN_v \left( \frac{\bar{\sigma}}{f_v} - h \frac{\bar{\chi}}{f_u} \right) \right],
\]
(24)
become identity at the minima of the effective potential.

An exact expression for \(I(n_u,n_v)\) is complicated as\[10\]

\[
I(n_u,n_v) = \frac{16 f^4}{\sqrt{\pi}} \left( \frac{f^2}{4f^2} \right)^{|n_u|} \frac{\Gamma(5/2 - |n_u|)\Gamma(|n_u| + |n_v| - 2)}{\Gamma(|n_v| - |n_u| + 3)\Gamma(|n_u| + 1)}
\]

\[
\sqrt{\pi} \left( \frac{f^2}{4f^2} \right)^{|n_u|} \frac{\Gamma(5/2 - |n_u|)\Gamma(|n_u| + |n_v| - 2)}{\Gamma(|n_v| - |n_u| + 3)\Gamma(|n_u| + 1)}
\]

\[
\frac{16 f^4}{\sqrt{\pi}} \left( \frac{f^2}{4f^2} \right)^{|n_u|} \frac{\Gamma(5/2 - |n_u|)\Gamma(|n_u| + |n_v| - 2)}{\Gamma(|n_v| - |n_u| + 3)\Gamma(|n_u| + 1)}
\]
\begin{align}
\times 3F_2\left(|n_u| - |n_v| - 2, |n_u| + |n_v| - 2, |n_u| + 1/2; |n_u| - 3/2, 2|n_u| + 1; -\frac{f_u^2}{f_v^2}\right) \\
+ \frac{32}{\pi} \frac{f_u^5}{f_v^3} \Gamma(|n_u| - 5/2) F_2\left(\frac{1}{2} - |n_v|, \frac{1}{2} + |n_v|; n_u - \frac{1}{2}, -|n_u| - \frac{1}{2}; -\frac{f_u^2}{f_v^2}\right). \quad (25)
\end{align}

For large \(n_u\) and \(n_v\), the integration in (21) can be approximated by using the following evaluation:

\begin{align}
I_\nu(z) &\sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n + 1/2)}{n! \Gamma(n - 1/2)(2z)^n} \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(-1)^n \nu^{2n}}{n!(2z)^n} \\
&\sim \frac{e^z}{\sqrt{2\pi z}} \exp\left(-\frac{\nu^2}{2z}\right) \quad (z \gg 1). \quad (26)
\end{align}

Then the integration results in a rather simple form for large \(n_u\) and \(n_v\):

\begin{align}
I(n_u, n_v) = \frac{32}{\pi} \frac{1}{f_u f_v} \frac{1}{(n_u^2/f_u^2 + n_v^2/f_v^2)^3}. \quad (27)
\end{align}

By using this with the notation (12), we find that the effective potential in the large \(N_u\) and \(N_v\) limit,

\begin{align}
V_{\text{eff}}(\bar{\chi}, \bar{\sigma}) = -\frac{2}{\pi^3 \sqrt{2}} \left(\sum_{p',q'=-\infty}^{\infty} \cos\left[(q' + p'\tau_1) \frac{N_u}{f_u} + p' \frac{N_v}{f_v}\right]ight) \frac{\cos\left[(q' + p'\tau_1)^2/\tau_2 + (p')^2\tau_2^3\right]}{[\tau_1]^2}. \quad (28)
\end{align}

is that in the continuum \(U(1)\) gauge theory with compactification on the shifted two-torus.

When the fractional part of \(\tau_1\) does not vanish, the mass matrix of PNGBs has off-diagonal parts for any \(N\).

**IV. CONCLUSION AND DISCUSSION**

In conclusion, we can build a theory space description of the shifted-torus compactification. The correspondence in the limit of large number of sites is found when the volume of extra space held fixed.

Although the resulting mass spectra contain a finite number of massive states in our models, the lowest non-zero mass is very close to that of the corresponding continuum models. Therefore it is difficult to distinguish the discrete and continuum theories experimentally through the direct production of the lowest KK state. If we can observe the virtual exchange contribution of the KK states to cross sections, we can study the mass spectrum experimentally. For this purpose, we must compute amplitudes along with more realistic theories. This subject is left for future works.
We have also discussed the effective potential for PNGBs in our shifted-lattice model. We will perform more quantitative analysis of the effective potential for simple models of deconstruction. Moreover, the possibility of finite temperature effect on the effective potential will be discussed elsewhere\cite{11}.

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\[12\] See also \cite{3} (for a three-torus), \cite{4} (for the closed string theory) and \cite{5} (for a noncommutative field theory).
\[13\] The quartic term such as $U_{k\ell}V_{k,\ell+1}U_{k+1,\ell+h}^*V_{k\ell}^*$ may be included\cite{6, 7, 8} or generated by radiative correction\cite{9}; we omit such a term in this paper, for mass spectra of vector bosons
are unchanged by the term.

[14] For the symmetry of the figure, the same spectrum is displayed at both $\tau_1 = 0$ and $\tau_1 = 1$. 
FIG. 2: The mass spectra are plotted against the fractional part of $\tau_1$ (see text). (a) for $N_u = 5$ and $N_v = 3$, (b) for $N_u = 21$ and $N_v = 13$. Not all mass levels are exhibited in this case.