Axion as a cold dark matter candidate: analysis to third order perturbation for classical axion

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Abstract. We investigate aspects of axion as a coherently oscillating massive classical scalar field by analyzing third order perturbations in Einstein’s gravity in the axion-comoving gauge. The axion fluid has its characteristic pressure term leading to an axion Jeans scale which is cosmologically negligible for a canonical axion mass. Our classically derived axion pressure term in Einstein’s gravity is identical to the one derived in the non-relativistic quantum mechanical context in the literature. We present the general relativistic continuity and Euler equations for an axion fluid valid up to third order perturbation. Equations for axion are exactly the same as that of a zero-pressure fluid in Einstein’s gravity except for an axion pressure term in the Euler equation. Our analysis includes the cosmological constant.

Keywords: cosmological perturbation theory, axions

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1 Introduction

The currently popular concordance cosmology based on perturbed Friedmann world model demands the presence of substantial amount of non-baryonic dark matter. Despite some problems concerning small-scale (less than some kpc) clustering properties, the cold dark matter (CDM) still plays a popular role as the dark matter. Axion, notwithstanding its still hypothetical nature, is known to be one of the prime candidates for the CDM [1–4].

Axion as a coherently oscillating scalar field is known to behave as a pressureless fluid in the background world model [1–4], and to the linear order perturbation in the Newtonian context [5, 6]. The CDM nature of linear order perturbation of the axion in Einstein’s gravity was studied in [7–11]. The CDM nature of axion to the linear order was shown in various gauge conditions: these are the zero-shear gauge [7, 10], the synchronous gauge [8], the uniform-curvature gauge [9], and the axion-comoving gauge [11]. Only in the axion-comoving gauge the axion is shown to behave as the Newtonian CDM in all cosmological scales [11]. The axion, however, has its characteristic pressure term with Solar-System size Jeans scale for the canonical axion mass [5–7, 10, 11], see eq. (5.12). We may increase the Jeans scale to cosmologically relevant one by reducing the axion mass; roles of this extremely low mass axion as a potential variant of warm dark matter with small scale cut off in the density power spectrum deserve further study [12–14].

Previously we have shown that the axion as a classical massive scalar field behaves as the CDM to the second-order perturbations in all cosmological scales [15]. Except for the axion pressure term relevant in the small-scale limit, the relativistic continuity and Euler equations of the axion fluid are shown to be exactly the same as Newtonian equations to the second order in all cosmological scales including the super-horizon scale; in the case of a zero-pressure and irrotational fluid, see [16].

Here, we extend our study to the third-order perturbation in the axion-comoving gauge. The leading nonlinear contribution to the density and velocity power spectra needs the third order perturbation. We will consider the axion as a classical massive scalar field in Einstein’s
gravity. We will use the fully nonlinear perturbation formulation in [17, 18]. We will show that the general relativistic CDM correspondence continues to the third order with due presence of the axion pressure term in the small-scale: for the relativistic continuity and Euler equations of axion, see eqs. (5.4) and (5.5). We show that the axion pressure term classically derived in Einstein’s gravity (to third order) coincides exactly with the one derived from non-relativistic quantum mechanics in the literature [19–22]. For extremely low mass axion where the axion pressure term becomes cosmologically important, we note that the third order pure general relativistic correction terms are also affected by the axion pressure term: see eqs. (5.13) and (5.14).

Readers who want to skip the technical details of deriving the general relativistic continuity and Euler equation for axion fluid based on relativistic nonlinear perturbation analysis can go directly to sections 5 and 6 after reading first two paragraphs of section 4. For the benefit of such readers, below we briefly explain the basic method of analysis in the case of axion where we need fluid-field mixed formulation of nonlinear cosmological perturbation with proper time averaging.

As the axion we consider a classical minimally coupled massive scalar field and take temporal averaging over the coherent oscillation. The Einstein’s equation becomes

\[ \tilde{G}_{ab} = 8\pi G \langle \tilde{T}_{ab} \rangle, \]

where we take the temporal averaging for the scalar field to get the averaged energy-momentum tensor, and the averaged fluid (density, pressure and velocity) quantities; although the average should be applied to the field quantities, here only to show the method we impose it to the energy-momentum tensor. For the energy conservation equation we can use

\[ \langle \tilde{T}^b_{0,b} \rangle = 0, \quad \text{or} \quad \langle \tilde{T}^b_{a;b} \tilde{n}^a \rangle = 0, \]

where the latter expression is the ADM energy-conservation with \( \tilde{n}_a \) the normal-frame four-vector. We note that in the case of axion it is troublesome to use the covariant energy conservation \( \langle \tilde{T}^b_{a;b} \tilde{n}^b \rangle = 0 \) as the fluid four-vector \( \tilde{n}^b \) also involves the scalar field, see eqs. (2.3) and (3.2). For the massive scalar field \( \tilde{T}_{ab} \) involves the scalar field in quadratic combination only, see eq. (3.1). For the equation of motion we do not take average, see eq. (4.7). As the axion-comoving gauge we impose the scalar-type (longitudinal) perturbation part of \( \langle \tilde{T}^0_{ij} \rangle \) equals to zero to all orders in perturbation, see eq. (4.10). We will use the fully nonlinear equations expressed using the fluid quantities [17, 18]; one important point in the analysis of axion is to first express the fluid quantities in terms of the scalar field, and then take average over the quadratic combination of scalar field.

We set \( c \equiv 1 \equiv \hbar \) except for section 5.

### 2 Metric and fluid quantities

We consider scalar- and vector-type perturbations in the Friedmann world model with the general background curvature, \( \bar{K} \). Our metric convention valid to the nonlinear order is [23, 24]

\[
\begin{align*}
 ds^2 &= -a^2(1 + 2\alpha) d\eta^2 - 2a^2(\beta_{ij} + B^{(v)}_{ij})d\eta dx^i \\
 &\quad + a^2 \left[ (1 + 2\varphi) \gamma_{ij} + 2\gamma_{ij} + C^{(v)}_{ij} + C^{(v)}_{ji} + 2C^{(t)}_{ij} \right] dx^i dx^j,
\end{align*}
\]
where $a$ is the cosmic scale factor; $\alpha$, $\beta$, $\gamma$, $\varphi$, $B_i^{(v)}$, $C_i^{(v)}$ and $C_{ij}^{(t)}$ are arbitrary functions of space and time with $B_i^{(v)\nu} \equiv 0 \equiv C_i^{(v)\nu}$ for the vector-type perturbations and $C_{ij}^{(t)\nu} \equiv 0 \equiv C_{ij}^{(t)\nu}$ for the tensor-type perturbation; the spatial indices of perturbation variables are raised and lowered by $\gamma_{ij}$ as the metric, and the vertical bar indicates the covariant derivative based on $\gamma_{ij}$ as the metric; for concrete forms of $\gamma_{ij}$, see [18]. The scalar-, vector- and tensor-types decomposition is generally valid considering fully nonlinear perturbations [25]. To the nonlinear order the three-types of perturbations couple with each other in the equation level. In this work we ignore the tensor-type perturbation $C_{ij}^{(t)}$.

The energy-momentum tensor in the energy-frame ($\tilde{g}_{ab} \equiv 0$) is given as

$$\tilde{T}_{ab} = \tilde{\mu} \tilde{u}_a \tilde{u}_b + \tilde{p} \left( \tilde{g}_{ab} + \tilde{u}_a \tilde{u}_b \right) + \tilde{\pi}_{ab},$$

(2.2)

where $\tilde{\mu}$, $\tilde{p}$, $\tilde{u}_a$ and $\tilde{\pi}_{ab}$ are the energy density, pressure, fluid four-vector, and the anisotropic stress, respectively [26–28]; we have $\tilde{u}^c \tilde{u}_c \equiv -1$, $\tilde{\pi}_{ab} = \tilde{\pi}_{ba}$ and $\tilde{\pi}_{ab} \equiv 0 \equiv \tilde{\pi}_{a,b}$. We introduce the perturbed order fluid three-velocity $\tilde{v}_i$ as [17]

$$\tilde{u}_i \equiv a \tilde{\gamma} \tilde{v}_i, \quad \tilde{\gamma} \equiv \frac{1}{\sqrt{1 - \frac{\tilde{v}_c \tilde{v}_c}{1 + 2\varphi}}}.$$

(2.3)

The index of $\tilde{v}_i$ is raised and lowered by $\gamma_{ij}$ as the metric; we ignore the anisotropic stress $\tilde{\pi}_{ab}$ as the minimally coupled scalar field does not support the anisotropic stress.

As the spatial gauge condition we take

$$\gamma \equiv 0 \equiv C_i^{(v)},$$

(2.4)

on the metric tensor. This is the only spatial gauge condition which (together with our temporal gauge condition) allows the remaining variables to be gauge invariant to the fully nonlinear order [16, 17, 23, 24]. With this spatial gauge condition we set $\chi_i \equiv a(\beta_i + B_i^{(v)}) \equiv \chi_i^{(v)}$, and the metric becomes

$$ds^2 = -a^2 (1 + 2\alpha) d\eta^2 - 2a \chi_i d\eta dx^i + a^2 (1 + 2\varphi) \gamma_{ij} dx^i dx^j.$$

(2.5)

This allows us to expand the cosmological perturbations to fully nonlinear order in an exact manner [17, 18].

3 Minimally coupled scalar field

A minimally coupled scalar field is given as

$$\tilde{T}_{ab} = \tilde{\phi}_{,a} \tilde{\phi}_{,b} - \left[ \frac{1}{2} \tilde{\phi}_{,c} \tilde{\phi}_{,c} + \tilde{V}(\tilde{\phi}) \right] \tilde{g}_{ab},$$

(3.1)

In the energy-frame we can show $\tilde{\pi}_{ab} = 0$ [17, 18].

The fluid quantities follow from eqs. (2.2) and (3.1) [see eq. (5.6) of [18]]

$$\tilde{\mu} = \frac{1}{2} \tilde{\phi} \frac{\tilde{\varphi}^2}{\tilde{\phi}} + \tilde{V}, \quad \tilde{p} = \frac{1}{2} \tilde{\phi} \frac{\tilde{\varphi}^2}{\tilde{\phi}} - \tilde{V}, \quad \tilde{v}_i = -\frac{\tilde{\phi}_{,i}}{\tilde{\phi}} \tilde{\gamma},$$

(3.2)
where
\[
\tilde{\phi} \equiv \tilde{\phi}_c \tilde{u}^c = \frac{1}{\gamma} \frac{D\tilde{\phi}}{Dt}, \quad \tilde{\phi}_c = \frac{1}{\sqrt{1 - \frac{\dot{\phi}^2}{a^2(1 + 2\varphi)^2}}}, \quad \tilde{\phi} = \frac{1}{\sqrt{1 - \frac{\dot{\phi}^2}{a^2(1 + 2\varphi)^2}}} \frac{D\tilde{\phi}}{Dt} = \frac{1}{N} \left( \frac{\partial}{\partial t} + \frac{\chi^i}{a^2(1 + 2\varphi)} \nabla_i \right) \tilde{\phi}, \quad N \equiv \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{a^2(1 + 2\varphi)}}. \tag{3.3}
\]

The equation of motion, \( \tilde{\phi}^{\nu} = \partial V / (\partial \tilde{\phi}) \), gives [see eq. (5.10) of [18]]
\[
\tilde{\phi} + \left( 3HN - \kappa N \frac{\chi^i N_i}{a^2 N(1 + 2\varphi)} \right) \tilde{\phi} + N^2 \frac{\partial V}{\partial \tilde{\phi}} + \frac{2\chi^i}{a^2(1 + 2\varphi)} \tilde{\phi}_i - \frac{1}{a^2(1 + 2\varphi)} \left( N^2 \gamma^{ij} - \frac{\chi^i \chi^j}{a^2(1 + 2\varphi)} \right) \tilde{\phi}_{,ij} + \left[ - \frac{N^2}{a^2(1 + 2\varphi)} \left( \frac{N^{ij}}{N} + \varphi_{ij} \right) + \frac{\chi^i}{a^2(1 + 2\varphi)} \chi^j \right] \tilde{\phi}_{,i} = 0. \tag{3.4}
\]

4 Axion

We consider the axion as a massive scalar field with a potential \( \tilde{V}(\tilde{\phi}) = \frac{1}{2} m^2 \tilde{\phi}^2 \). The pseudo nature of axion is not relevant in our cosmological consideration. We consider only the classical nature of axion; we are not capable of participating in the ongoing controversies about the potential roles of quantum nature of the axion as a Bose-Einstein condensation [10, 29–34]; see below eq. (5.10) though. We have
\[
\frac{H}{m} = 2.133 \times 10^{-28} \left( \frac{10^{-5} \text{eV}}{m} \right) \left( \frac{H}{H_0} \right), \tag{4.1}
\]
where \( H_0 \equiv 100 \text{h} \text{kms}^{-1} \text{Mpc}^{-1} \) is the present Hubble parameter with \( H \equiv \dot{a}/a \). As the axion coherently oscillates we strictly ignore \( H/m \) higher order terms.

We decompose the field and fluid quantities to the background and perturbed parts as
\[
\tilde{\phi} \equiv \phi + \delta \phi, \quad \tilde{\mu} \equiv \mu + \delta \mu, \quad \tilde{p} \equiv p + \delta p. \tag{4.2}
\]

We have a background solution [8, 9]
\[
\phi(t) = a^{-3/2} \left[ \phi_{+0} \sin (mt) + \phi_{-0} \cos (mt) \right], \tag{4.3}
\]
where \( \phi_{+0} \) and \( \phi_{-0} \) are the constant coefficients. We take average over time scale of order \( m^{-1} \) for all fluid quantities associated with the axion as a oscillating scalar field. [8]. We have [8, 9]
\[
\mu = \frac{1}{m^2 a^{-3} \left( \phi_{+0}^2 + \phi_{-0}^2 \right)}, \quad p = 0. \tag{4.4}
\]

Thus, for the background the axion evolves exactly the same as a pressureless ideal fluid [1–4]. This conclusion is valid in the presence of both the spatial curvature and the cosmological constant \( \Lambda \) in the background, and in the presence of other fluids. For the nonlinear perturbation, here we consider a single presence of the axion field; considering the presence of other fluids and fields is trivial and those are not affected by the special nature of axion fluid.
4.1 Fluid quantities and equation of motion

Including perturbations we expand [8, 9]

\[ \tilde{\phi} \equiv \phi + \delta \phi \equiv (a^{-3/2} \phi_{+0} + \delta \phi_{+}) \sin(mt) + (a^{-3/2} \phi_{-0} + \delta \phi_{-}) \cos(mt), \]

(4.5)
to fully nonlinear order, where \( \delta \phi_{+} \) and \( \delta \phi_{-} \) are arbitrary functions of space and time. In the analysis we ignore \( H/m \) higher order terms with \( \delta \phi_{\pm} \sim H \delta \phi_{\pm} \).

For the fluid quantities, from eq. (3.2), to the fully nonlinear order, we can show

\[
\begin{align*}
\{ \tilde{\mu}, \tilde{p} \} &= \frac{1}{2} \langle \tilde{\phi}^2 \rangle \pm \frac{1}{2} m^2 \langle \tilde{\phi}^2 \rangle \\
&= \frac{1}{2N^2} \left[ \langle \dot{\phi}^2 \rangle + 2 \frac{\chi^i}{a^2(1 + 2\varphi)} \langle \dot{\phi} \dot{\phi}, i \rangle + \frac{\chi^i \chi^j}{a^4(1 + 2\varphi)^2} \langle \dot{\phi}, i \dot{\phi}, j \rangle \right] \\
&\quad - \frac{1}{2a^2(1 + 2\varphi)} \langle \ddot{\phi} \dot{\phi}, i \rangle \pm \frac{1}{2} m^2 \langle \tilde{\phi}^2 \rangle \\
&= m \frac{1}{4} \left( \frac{1}{N^2} \pm 1 \right) \left[ a^{-3} \left( \phi_{+0}^2 + \phi_{-0}^2 \right) + 2a^{-3/2} \left( \phi_{+0} \delta \phi_{+} + \phi_{-0} \delta \phi_{-} \right) + \delta \phi_{+}^2 + \delta \phi_{-}^2 \right] \\
&\quad + m \frac{\chi^i}{2a^2N^2(1 + 2\varphi)} \left[ a^{-3/2} \left( \phi_{+0} \delta \phi_{-, i} - \phi_{-0} \delta \phi_{+, i} \right) + \delta \phi_{+, i} \delta \phi_{-, i} - \delta \phi_{-, i} \delta \phi_{+, i} \right] \\
&\quad + \frac{\chi^i \chi^j}{4a^4N^2(1 + 2\varphi)^2} \langle \delta \phi_{+, \, i} \delta \phi_{+, \, j} + \delta \phi_{-, \, i} \delta \phi_{-, \, j} \rangle \\
&\quad - \frac{1}{4a^2(1 + 2\varphi)} \langle \delta \phi_{+, i} \delta \phi_{+, j} + \delta \phi_{-, i} \delta \phi_{-, j} \rangle,
\end{align*}
\]

(4.6)
where the upper and lower signs correspond to the \( \tilde{\mu} \) and \( \tilde{p} \) parts, respectively. In the temporal comoving gauge (hypersurface) condition we will take, the \( \tilde{v}_i \) part is fixed by the gauge condition, and later we will identify other variable as the velocity perturbation, see eqs. (4.17) and (5.1). In the axion case where we need proper averaging, the parameter \( \tilde{v}_i \) becomes ambiguous, see paragraph containing eq. (4.10).

The equation of motion in eq. (3.4), to the fully nonlinear order, gives

\[
\begin{align*}
\left\{ a^{-3/2} (N - 1) \left[ \mp 3Hm \phi_{+0} + m^2 (1 + N) \phi_{+0} \right] \\
\mp 2m \delta \phi_{\mp} + 3HmN \delta \phi_{\mp} + \delta \phi_{\pm} + 3HN \delta \phi_{\pm} + m^2 \left( N^2 - 1 \right) \delta \phi_{\pm} \\
- \left( \frac{N}{N} + \frac{N}{N} + \frac{\chi^i N_{i}}{a^2 N^2(1 + 2\varphi)} \right) \left[ a^{-3/2} \left( \mp m \phi_{+0} - \frac{3}{2} H \phi_{+0} \right) \mp m \delta \phi_{+} + \delta \phi_{\pm} \right] \\
+ \frac{2\chi^i}{a^2(1 + 2\varphi)} \left\{ \mp m \delta \phi_{+, i} + \delta \phi_{+, i} \right\} - \frac{1}{a^2(1 + 2\varphi)} \left( N^2 \gamma^i_{, i} - \frac{\chi^i \chi^j}{a^2(1 + 2\varphi)} \right) \delta \phi_{\pm, i j} \\
+ \left[ - \frac{N^2}{a^2(1 + 2\varphi)} \left( \frac{N}{N} + \frac{\varphi_{\pm}^i}{1 + 2\varphi} \right) \\
+ \left( 3HN - N \frac{N}{N} - \frac{\chi^k N_{k}}{a^2 N^2(1 + 2\varphi)} \right) \right] \frac{\chi^i}{a^2(1 + 2\varphi)} \\
+ \left( \frac{\chi^i}{a^2(1 + 2\varphi)} \right) + \frac{\chi^k}{a^2(1 + 2\varphi)} \left( \frac{\chi^i}{1 + 2\varphi} \right) \left\{ \sin(mt) \cos(mt) \right\} \times \left\{ \sin(mt) \cos(mt) \right\} = 0,
\end{align*}
\]

(4.7)
where the upper and lower signs correspond to the sin and cos parts, respectively; here we have not ignored \(H/m\)-order terms yet.

### 4.2 Axion-comoving gauge

We can decompose
\[
\hat{v}_i \equiv \hat{\tilde{v}}_i - \hat{\tilde{v}}^{(v)}_i, \quad \hat{\tilde{v}}^{(v)}_i \equiv 0. \tag{4.8}
\]

In the fluid case the comoving gauge takes
\[
\hat{v} \equiv 0, \tag{4.9}
\]
as the temporal hypersurface (slicing) condition. For a scalar field the comoving gauge is the same as \(\delta \phi \equiv 0\), the uniform-field gauge to fully nonlinear order [17, 18].

In the axion case, however, as we have to take proper average over the oscillating field, the concept of \(\hat{v}_i\) and its decomposition become ambiguous, see eq. (3.2). Thus, for the temporal gauge condition we go back to the original energy-momentum tensor and take the scalar part of \(\tilde{T}_0^0\) equal to zero as the comoving gauge. Thus, in the axion-comoving gauge we can impose
\[
\langle \tilde{T}_0^0 \rangle_i \equiv 0, \tag{4.10}
\]
to all perturbation orders. Together with the spatial gauge condition in eq. (2.4) the axion-comoving slicing condition completely fixes the gauge mode. Each perturbation variable in these conditions has a unique gauge-invariant counterpart to all orders in perturbation [16, 17, 23, 24].

Using the inverse metric presented in [18], from eq. (3.1) we can show
\[
\langle \tilde{T}_0^0 \rangle_i = -\frac{1}{a N^2} \left[ \frac{\chi^j}{a^2(1 + 2 \varphi)} \langle \phi_i \phi_j \rangle \right]
\]
\[
= -\frac{1}{2a N^2} \left[ a^{-3/2} m (\phi_{\varphi+0} \delta \phi_{-0} - \phi_{-0} \delta \phi_{\varphi+0}) + m (\delta \phi_{\varphi+0} \delta \phi_{-0} - \delta \phi_{-0} \delta \phi_{\varphi+0}) \right]
\]
\[
+ \frac{1}{a^2(1 + 2 \varphi)} \left( \delta \phi_{\varphi+0} \delta \phi_{\varphi+0} + \delta \phi_{-0} \delta \phi_{-0} \right) \chi^j \right]. \tag{4.11}
\]

As we mentioned below eq. (1.1) we can omit average notation for \(\tilde{T}_{0b}\). Up to this point all equations are valid to fully nonlinear order.

Now, we consider perturbations to the third order. From the gauge condition in eq. (4.10) we can show
\[
\frac{\delta \phi_{\varphi+0}}{\varphi_{\varphi+0}} - \frac{\delta \phi_{-0}}{\varphi_{-0}} = a^{-3/2} \frac{\phi_{\varphi+0}^2 + \varphi_{-0}^2}{\phi_{\varphi+0}^2 \varphi_{-0}^2} \Delta^{-1} \nabla^i \left( \delta \phi_{\varphi+0} \delta \phi_{\varphi+0} \chi^j \right), \tag{4.12}
\]
to the third order, Thus
\[
\tilde{T}_i^0 = -\frac{\mu}{m^2 \varphi_{\varphi+0}^2} \left[ \delta \phi_{\varphi+0} \delta \phi_{\varphi+0} \chi^j - \nabla_i \Delta^{-1} \nabla^j \left( \delta \phi_{\varphi+0} \delta \phi_{\varphi+0} \chi^k \right) \right], \tag{4.13}
\]
which is pure third-order vector-type perturbation. In the fluid formulation eqs. (2.2) and (2.3) give
\[
\tilde{T}_i^0 = (\tilde{\mu} + \tilde{\rho}) \frac{\gamma^2}{N} \tilde{v}_i. \tag{4.14}
\]
By introducing $\hat{v}$ in this fluid relation and using eq. (4.8), we have

$$\hat{v} = 0,$$

$$\hat{v}^{(v)}_i = \frac{1}{\mu} \hat{T}^0_i = -\frac{1}{m^2 \phi^2_0} \left[ \delta \phi_{+,i} \delta \phi_{+,j} \chi^j - \nabla_i \Delta^{-1} \nabla^j \left( \delta \phi_{+,j} \delta \phi_{+,k} \chi^k \right) \right].$$

(4.15)

Here we estimate the pure third order terms involving $\chi^i$ in eqs. (4.12), (4.13) and (4.15). It is convenient to have [see eqs. (68) and (70) in [17]]

$$\kappa = \dot{\delta} \sim H \delta, \quad \chi_i = \chi_{i,j} = - \frac{a^2}{\Delta} \delta \phi_{+,i} \delta \phi_{+,k} \chi^k,$$

(4.16)

where $K_{ij}$ is the extrinsic curvature, and $\tilde{\theta}(n)$ is the expansion scalar of the normal-frame four-vector; later we will identify $\kappa$ as the divergence of velocity perturbation, see eq. (5.1).

In the derivation of eq. (4.11) we have ignored $H/m$ higher order terms, and the estimation of $\chi^i$ above reveals that the pure third order terms involving $\chi^i$ in eqs. (4.11), (4.12), (4.13) and (4.15) should be ignored being the same order as the already ignored $H/m$ order terms. Thus, eqs. (4.12), (4.13) and (4.15) become

$$\frac{\delta \phi_+}{\phi_+} - \frac{\delta \phi_-}{\phi_-} = 0, \quad \hat{T}^0_i = 0 = \hat{v}^{(v)}_i,$$

(4.18)

and the axion-comoving gauge implies $\hat{v}_i = 0$ to the third order.

To the third order in the axion comoving gauge eq. (4.6) can be arranged as

$$\left\{ \begin{array}{l}
\tilde{\mu} = \frac{1}{2} N^2 \left[ 1 + \frac{a^2}{\phi^2_{+,0}} \phi^2_{-,0} (\phi_{+,0} \delta \phi_+ + \phi_{-,0} \delta \phi_-) + \frac{a^4}{\phi^2_{+,0}^2} \delta \phi_{+,0}^2 \right] \\
\tilde{p} = -\frac{1}{m^2 a^2 (1 + 2 \varphi)} \frac{a^3}{\phi^2_{+,0}^2} \delta \phi_{+,0} \delta \phi_{+,j}.
\end{array} \right.$$

(4.19)

These are the energy density and pressure supported by the axion fluid to the third-order perturbation. Besides the Einstein’s equations to be presented below, we need an expression of $\delta \tilde{N}$ in terms of the axion field. This relation follows from the equation of motion, and will be presented in eq. (4.25).

4.3 Einstein equations

Equations valid to fully nonlinear perturbation orders are presented in [17, 18]. Here we will again borrow some equations from these works.

As explained in the paragraph containing eq. (1.1) we can use the equations expressed in terms of fluid quantities but have to change fluid quantities to the scalar field before we take average over the oscillating field. After averaging, we will strictly ignore $H/m$-order terms, and will keep only first order in $\Delta/(m^2 a^2)$.

From the covariant energy conservation and the trace of ADM (Arnowitt-Deser-Misner) propagation equations in eqs. (3.8) and (3.4) of [18], respectively, to the third order, we have

$$\tilde{\mu} + \frac{1}{a^2 (1 + 2 \varphi)} \delta \mu_{,k} \chi^k + N (\tilde{\mu} + \tilde{p}) (3H - \kappa) = 0,$$

(4.20)
\[-3 \left[ \frac{1}{N} \left( \frac{\dot{a}}{a} \right) + \frac{a^2}{a^2} + 4\pi G \left( \bar{\mu} + 3 \bar{p} \right) - \frac{\Lambda}{3} \right] + \frac{1}{N} \dot{\kappa} + 2 \frac{\dot{a}}{a} \kappa + \frac{\Delta N}{a^2 N (1 + 2 \varphi)} \right] \\
= \frac{1}{3} \kappa \dot{} - \frac{1}{a^2 N (1 + 2 \varphi)} \left( \chi^i \kappa, i + \frac{\varphi, i}{1 + 2 \varphi} \right) + \kappa^j K_i^j, \quad (4.21) \]

where from eq. (3.10) in [18],

$$K^j_i K^i_j = \frac{1}{a^2 N (1 + 2 \varphi)} \left\{ \frac{1}{2} \chi^{ij} (\chi_{ij} + \chi_{ji}) - \frac{1}{3} \chi^i_i \chi^j_j \right\}$$

where \(\Phi\) is valid to the second order. From eqs. (3.2) and (3.3) in [18], respectively, we have

\[
\frac{\Delta}{\varphi} = -4 \pi G \varphi - \frac{\dot{a}}{a} \kappa + \frac{1}{6} \kappa^2 + \frac{1}{a^2} \left( 4 \varphi \Delta \varphi + \frac{3}{2} \varphi^4 \varphi, i \right) - 3 \varphi (1 - 2 \varphi) \frac{K}{a^2} \\
- \frac{1}{4 a^4} \left[ \frac{1}{2} \chi^{ij} (\chi_{ij} + \chi_{ji}) - \frac{1}{3} \chi^{i}_i \chi^j_j \right], \quad (4.23) \]

valid to the second order.

### 4.4 Results

Now we assume a flat background curvature, \(K \equiv 0\). From the equation of motion in eq. (4.7) we can show

$$\delta N = (1 - \Phi + \Phi^2) \frac{\Delta}{2m^2 a^2} \Phi, \quad (4.25)$$

where \(\Phi \equiv \delta \phi_{+}/(a^{-3/2} \phi_{+0})\), and we set \(N \equiv 1 + \delta N\). This relation follows from dividing eq. (4.7) by \(\phi_{+0}\) and summing the two relations.

From eq. (4.19) we can show

\[
\delta \rho \mu = - (1 + \Phi) \frac{\Delta}{2m^2 a^2} \Phi - \frac{1}{2m^2 a^2} \Phi^4 \Phi, \\
\Phi = \frac{1}{2} \left( 1 + \frac{\Delta}{4m^2 a^2} \right) \left( \delta - \frac{1}{4} \delta^2 + \frac{1}{8} \delta^3 \right) + \frac{1}{16m^2 a^2} \left( 1 - \frac{3}{2} \delta \right) \delta^i \delta^j. \quad (4.26) \]

Thus, we have

\[
\delta N = \frac{1}{4m^2} \left[ \left( 1 - \frac{1}{2} \delta + \frac{3}{8} \delta^2 \right) \frac{\Delta}{a^2 \delta} - \frac{1}{4} \left( 1 - \frac{1}{2} \delta \right) \frac{\Delta}{a^2 \delta^2} + \frac{1}{8} \frac{\Delta}{a^2 \delta^3} \right] = \frac{1}{2m^2 a^2} \frac{\Delta \sqrt{1 + \delta}}{\sqrt{1 + \delta}}. \]
\[
\frac{\delta p}{\mu} = \frac{1}{4m^2} \left[ -\left( 1 + \frac{1}{2} \delta - \frac{1}{8} \delta^2 \right) \frac{\Delta}{a^2} \delta + \frac{1}{4} \left( 1 + \frac{1}{2} \delta \right) \frac{\Delta}{a^2} \delta^2 - \frac{1}{8} \frac{\Delta}{a^2} \delta^3 - \frac{1}{2a^2} \left( 1 - \delta \right) \delta \cdot \delta \right],
\]

where the compact expression in \( \delta N \) is motivated by the known expression in the literature on non-relativistic Bose-Einstein condensation in Minkowski space-time \([19–22]\), see below eq. (5.10). The ADM momentum conservation in eq. (3.7) of \([18]\) gives

\[
\tilde{p}_{i,j} + (\tilde{\mu} + \tilde{p}) \frac{N^j_i}{N} = -\mu \frac{1}{m^2a^2} (\Phi^j \Phi_i),
\]

and this is consistent with eqs. (4.25) and (4.26).

Now, eqs. (4.20), (4.21) and (4.24), respectively, give

\[
\dot{\delta} - \kappa + \frac{1}{a^2} (1 - 2 \varphi) \delta \cdot \chi = 0,
\]

\[
\dot{\kappa} + 2H \kappa - 4\pi G \mu \delta + \frac{1}{a^2} (1 - 2 \varphi) \kappa \cdot \chi = \frac{1}{3} \kappa^2
\]

\[
- \frac{1}{a^2} (1 - 4 \varphi) \left[ \frac{1}{2} \chi^{ij} (\chi_{ij} + \chi_{ji}) - \frac{1}{5} (\Delta \chi)^2 \right] + \frac{4}{a^4} \left( \chi^{i} \varphi \chi_{i} - \frac{1}{3} \chi^{i} \varphi \Delta \chi \right)
\]

\[
= - \frac{\Delta}{a^2} \delta N = - \frac{\Delta}{2m^2a^2} \frac{\Delta \sqrt{1 + \delta}}{\sqrt{1 + \delta}},
\]

valid to the third order, and

\[
\left( \kappa + \frac{\Delta}{a^2} \chi \right)_{,i} + \frac{3}{4} \frac{\Delta}{a^2} \chi_{i}^{(v)} = \frac{1}{a^2} \left[ (2 \varphi \Delta \chi - \varphi \chi^j)_{,i} + \frac{3}{2} (\varphi \chi^j + \chi_{i} \Delta \varphi) \right],
\]

valid to the second order. We note that compared with the CDM case in \([35]\) the only correction from the axion nature appears in the right-hand-side of eq. (4.30).

From eq. (4.31) we have \( \chi_i^{(v)} = 0 \) to the linear order. To the second order eq. (4.31) can be decomposed as

\[
\kappa + \frac{\Delta}{a^2} \chi = \frac{1}{a^2} \left[ 2 \varphi \Delta \chi - \varphi \chi^j + \frac{3}{2} \Delta^{-1} \nabla^j (\varphi_{,ij} \chi^j + \chi_{i} \Delta \varphi) \right] \equiv \frac{1}{a} X, \]

\[
\frac{\Delta}{a^2} \chi_{i}^{(v)} = \frac{2}{a^2} (\varphi_{,ij} \chi^j + \chi_{i} \Delta \varphi) - \nabla_i \Delta^{-1} \nabla^j (\varphi_{,jk} \chi^k + \chi_{j} \Delta \varphi) \equiv \frac{1}{a} Y_i, \]

\[
\frac{\Delta}{a^2} \chi_{i} = -\kappa_{,i} + \frac{1}{a} X_{,i} + \frac{1}{a} Y_i.
\]

5 Relativistic continuity and Euler equations for axion

Now we have complete relations for axion fluid valid to the third order perturbation in Einstein’s gravity. We will arrange the equations so that we can readily compare with the Newtonian fluid equations.

We identify \( \delta \) as the density contrast \( \delta \rho/\rho \), and \( \kappa \) as the divergence of velocity perturbation \([16, 35]\)

\[
\kappa \equiv -\frac{1}{a} \nabla \cdot u \equiv -\frac{\Delta}{a} u.
\]
Here we assume the flat background curvature but include the cosmological constant. From eq. (4.32), to the second order, we have

\[
\frac{c}{a} \chi = u + \Delta^{-1} (\nabla X + Y),
\]

\[
X \equiv 2\varphi \nabla \cdot u - u \cdot \nabla \varphi + \frac{3}{2} \Delta^{-1} \nabla \cdot (u \cdot \nabla \varphi + u \Delta \varphi),
\]

\[
Y \equiv 2 \left[ u \cdot \nabla \nabla \varphi + u \Delta \varphi - \Delta^{-1} \nabla \cdot (u \cdot \nabla \varphi + u \Delta \varphi) \right],
\]

with \( \nabla \cdot Y = 0 \); \( \chi \) and \( \varphi \) are metric variables in eq. (2.5), interpreted as the perturbed shear (of the normal four-vector flow) and perturbed spatial curvature, respectively [17, 18].

Equations (4.29) and (4.30) give the general relativistic version of the continuity (mass-conservation) and Euler (momentum-conservation) equations valid to third order perturbations in the axion case. These are

\[
\dot{\delta} + \frac{1}{a} \nabla \cdot u + \frac{1}{a} \nabla \cdot (\delta u) = \frac{1}{a} (\nabla \delta) \cdot \left[ 2\varphi u - \Delta^{-1} (\nabla X + Y) \right],
\]

\[
-\frac{1}{a} \nabla \cdot \left( \dot{u} + \frac{\dot{a}}{a} u \right) + 4\pi G \varphi \delta + \frac{1}{a^2} \nabla \cdot (u \cdot \nabla u) - \frac{\hbar^2 \Delta}{2m^2a^4} \frac{\Delta \sqrt{1 + \delta}}{\sqrt{1 + \delta}}
\]

\[
= \frac{1}{a^2} \left\{ -\frac{2}{3} \varphi u \cdot \nabla (\nabla \cdot u) + 4\nabla \cdot \left[ \varphi \left( u \cdot \nabla u - \frac{1}{3} u \nabla \cdot u \right) \right] 
+ \frac{2}{3} X \nabla \cdot u + u \cdot (\nabla X + Y) - \Delta \left[ u \cdot \Delta^{-1} (\nabla X + Y) \right] \right\}.
\]

In order to close the equations we need \( \varphi \) valid to the linear order. From eq. (4.23) we have

\[
\frac{c^2 \Delta}{a^2} \varphi = -4\pi G \varphi \delta + \frac{\dot{a}}{a} \nabla \cdot u.
\]

Now eqs. (5.3)–(5.6) give closed differential equations for density and velocity perturbations, \( \delta \) and \( u \). These are valid in the presence of the cosmological constant.

In order to derive the axion contribution we have strictly ignored

\[
\frac{\hbar H}{c^2 m}
\]

order terms, and have kept up to only first order in

\[
\frac{\hbar^2 k^2}{c^2 m^2 a^2} = 4.088 \times 10^{-43} \left( \frac{10^{-5} \text{eV}}{m} \right)^2 \left( \frac{1 \text{kpc}}{a_0/k} \right)^2 \left( \frac{a_0}{a} \right)^2.
\]

In this section we have recovered \( c \) and \( \hbar \).

The curvature perturbation variable \( \varphi \) is dimensionless with \( \varphi \sim \delta \Phi/c^2 \) where \( \delta \Phi \) is the perturbed gravitational potential. In the Newtonian limit with \( c \)-goes-to-infinity, the right-hand-sides of eqs. (5.4) and (5.5) vanishes, thus recovering, except for the axion correction term, the well known Newtonian continuity and Euler equations valid to fully nonlinear order [36]; notice that in Einstein’s gravity the pure relativistic correction terms appear from the third order perturbation with linear \( \varphi \) terms.

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The presence of Planck’s constant $\hbar$ in the axion correction term, despite of our classical analysis, can be traced to the presence of the constant in the potential

$$V(\bar{\phi}) = \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \bar{\phi}^2,$$

with a dimension $[\bar{\phi}] = (M/L)^{1/2}$.

The term involving $m$ in eq. (5.5) is the pressure correction term arising from the axion nature of the system. The left-hand-sides of eq. (5.4) and (5.5), except for the axion pressure term, are identical to the Newtonian continuity and Euler equations, respectively [36]. The right-hand-side are third-order terms; these are pure general relativistic corrections in the zero-pressure fluid [35] and are numerically negligible in the current paradigm of concordance cosmology for zero-pressure fluid [37]; the same is true for axion in the super-Jeans scales, see below.

The axion pressure contribution can be written as

$$c_s^2 \frac{\Delta a^2}{a^2} \delta N = \frac{\hbar^2 \Delta}{2m^2a^4} \frac{\Delta \sqrt{1 + \delta}}{\sqrt{1 + \delta}} = \frac{\hbar^2 \Delta}{2m^2a^4} \frac{\Delta \sqrt{\tilde{\rho}}}{\sqrt{\tilde{\rho}}},$$

(5.10)

where $\tilde{\mu} \equiv \tilde{\rho} c^2 = (1 + \delta) \rho c^2$. Equations (4.25), (4.26) and (4.27) show that the perturbed part of lapse function $\delta N$ is only loosely related to the perturbed pressure, see eq. (4.28). We note that this so-called axion pressure term derived from general relativistic third order perturbation coincides (except for the scale factor) exactly with the one derived from Schrödinger equation in the non-relativistic limit in Minkowski space-time [10, 19–22]. Although the authors of [10] emphasized that the result is due to Bose-Einstein condensate nature of the axion, we stress that our result is a purely classical one based on perturbation treatment of a massive scalar field in Einstein’s gravity without assuming non-relativistic limit; the resulting coincidence proves that the classical axion is non-relativistic indeed.

Equations (5.4) and (5.5) can be combined to give

$$\ddot{\delta} + 2a^2 \frac{\dot{\delta}}{a} - 4\pi G \delta + \frac{1}{a^2} [a \nabla \cdot (\delta \mathbf{u})] + \frac{1}{a^2} \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \frac{\hbar^2 \Delta}{2m^2a^4} \frac{\Delta \sqrt{1 + \delta}}{\sqrt{1 + \delta}} = \text{third order terms},$$

(5.11)

where third order terms in the right-hand-side are pure general relativistic terms all involving linear order $\varphi$ term. Besides the axion pressure term, the left-hand-side is valid to fully nonlinear order in Newtonian theory (whereas, this is valid in the Einstein’s theory to the second order only).

Comparing the gravity and pressure terms to the linear order in eq. (5.11) we have the axion Jeans scale [5–7, 10, 11]

$$\lambda_J = \frac{2\pi a}{k_J} = \left( \frac{\pi^3 \hbar^2}{G \rho m^2} \right)^{1/4} = 5.4 \times 10^{14} \text{cm} h^{-1/2} \left( \frac{m}{10^{-5} \text{eV}} \right)^{-1/2},$$

(5.12)

with $\Delta = -k^2$. Examination of eq. (5.11) reveals that the same criteria applies to the third order. The axion pressure term has a role on scale smaller than the Solar System size for a canonical mass $m \sim 10^{-5} \text{eV}$, thus completely negligible in all cosmological scales. This proves the CDM nature of the canonical mass axion in the super-Jeans scale.
Notice that the third order terms arising from the pure Einstein’s gravity in the right-hand-sides of eqs. (5.4), (5.5) and (5.11) include linear order $\varphi$ term. From eqs. (5.4)–(5.6) we have
\[ \dot{\varphi} = \frac{\dot{a}}{a} \frac{h^2 \Delta}{4c^2 m^2 a^2 \delta}, \] (5.13)
to the linear order. The homogeneous solution (which is constant in time) is the zero-pressure fluid part and the inhomogeneous solution is the additional contribution from the axion. From eqs. (5.6) and (5.13) we have the fluid part and the additional axion part of $\varphi$ as
\[ \varphi|_{\text{fluid}} \sim \frac{a^2 H^2}{c^2 \Delta} \delta, \quad \varphi|_{\text{axion}} \sim \frac{h^2 \Delta}{4c^2 m^2 a^2 \delta}. \] (5.14)
Thus, the pure axion part of $\varphi$ is negligible in the super-Jeans scale. We have $\varphi|_{\text{fluid}} = C(x) = \text{constant in time}$ even in the presence of the cosmological constant, see eq. (5.13); $\varphi$ is the curvature perturbation in the comoving gauge which is the well known conserved quantity in the super-sound-horizon scale [39]. Even in sub-Jeans scale the pure axion part of $\varphi$ is simply decaying with $\varphi|_{\text{axion}} \propto a^{-1}$ for $\Lambda = 0$.

Therefore, in the cosmological scales we have proved that, to the third order perturbation, the axion can be identified as the CDM in the general relativistic context. In our previous works we have shown that the leading nonlinear power spectra (of density and velocity perturbations) in Einstein’s gravity (in the comoving gauge) give virtually identical results as in the Newtonian context [37]. In the current paradigm of concordance cosmology the pure general relativistic contributions arising from the third order are entirely negligible in all scales compared with the relativistic/Newtonian results which are identified to the second order perturbation [16].

6 Discussion

In this work we show that, in the super-Jeans scale, the axion behaves as the CDM to the third order perturbation. The axion pressure terms are important only in the cosmologically negligible scale for the canonical mass axion. In the axion-comoving gauge we have taken, in the super-Jeans scale, there exists the relativistic/Newtonian correspondence to the second-order perturbations, and pure relativistic corrections appearing from the third-order are numerically negligible (in the current paradigm of concordance cosmology) [37]. Therefore, in those scales (including the super-horizon scale) the axion is indistinguishable from the CDM as a zero-pressure fluid.

Concerning the axion pressure term, we show that our purely classical perturbation treatment of the axion as a massive scalar field in Einstein’s gravity gives an identical result from quantum mechanical treatment in the non-relativistic limit of the Bose-Einstein condensation [10, 19–22]. Although our result is valid only to third order perturbation, as the analysis is made in fully general relativistic context, the coincidence implies that the axion is non-relativistic indeed. In the regime where the axion pressure term becomes important, say for an extremely low mass axion where the axion Jeans scale becomes cosmologically relevant, besides the axion pressure term appearing from the linear order, the pure general relativistic contributions starting to appear from the third order through $\varphi$ also have axion pressure contribution with decaying nature though, see eq. (5.14).

Our result includes the cosmological constant. Extending our analysis for a realistic cosmological situation in the presence of other components of fluids and fields is trivial. We em-
phasize that the spatial and temporal gauge condition in eqs. (2.4) and (4.10) and the special identification of perturbed velocity variable in eq. (5.1) were essential to get the above results.

One missing contribution we ignored in this work is the simultaneously excited tensor-type (transverse-tracefree) perturbation. In our previous works \cite{35,37} we have ignored the $Y$-term which comes from the vector-type perturbation generated by the nonlinear scalar-type perturbation. In the same sense the tensor-type perturbation is also generated by the nonlinear scalar-type perturbation. These scalar-generated vector- and tensor-type perturbations start to have their roles from the third-order perturbation, and their quantitative effects are studied in \cite{38}.

Whether the axion behaves as a CDM fluid to the fully nonlinear order in Einstein’s gravity is left for future study; the fully nonlinear perturbation equations of the zero-pressure irrotational fluid in Einstein’s gravity are presented in eqs. (56)-(60) in \cite{17}, and our analysis up to eq. (4.11) is valid to fully nonlinear order.

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