CONFORMALLY KÄHLER, EINSTEIN-MAXWELL METRICS ON HIRZEBRUCH SURFACES

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ABSTRACT. In this note we prove that a special family of Killing potentials on certain Hirzebruch complex surfaces, found by Futaki and Ono [20], gives rise to new conformally Kähler, Einstein-Maxwell metrics. The correspondent Kähler metrics are ambitoric [7, 9] but they are not given by the Calabi ansatz [31]. This answers in positive questions raised in [20, 21].

1 INTRODUCTION

In this paper, we study the existence of conformally Kähler, Einstein-Maxwell metrics on compact Hirzebruch complex surfaces.

Definition 1.1. A conformally Kähler, Einstein-Maxwell (cKEM for short) (real) 4-dimensional manifold \((M, J, \tilde{g})\) is a compact complex Kähler manifold \((M, J)\) with a hermitian metric \(\tilde{g}\) for which there exists a function \(f\) such that \(g = f^2 \tilde{g}\) is a Kähler metric, satisfying also the following curvature conditions:

(i) \(\text{Ric}_{\tilde{g}}(J\cdot, J\cdot) = \text{Ric}_{\tilde{g}}(\cdot, \cdot);\)

(ii) \(\text{Scal}(\tilde{g}) = \text{const};\)

where \(\text{Ric}_{\tilde{g}}\) and \(\text{Scal}(\tilde{g})\) denote the Ricci tensor and the scalar curvature of \(\tilde{g}\).

We shall refer to such hermitian metrics as cKEM metrics on \((M, J)\). When \(M\) is a (real) 4-dimensional manifold, a cKEM metric provides a Riemannian signature analogue of a solution to the Einstein-Maxwell equations studied in General Relativity (see [7, 15, 31, 34]).

This class of hermitian metrics on 4-manifolds has been first introduced by C. LeBrun [29], who observed that they extend naturally the more familiar classes of Kähler metrics of constant scalar curvature (cscK for short) much studied since the pioneer work of E. Calabi [12, 13], as well as the Einstein-hermitian 4-manifolds classified in the compact case by Chen-LeBrun-Weber [14]. The theory of cKEM metrics was consequently extended in arbitrary dimension by Apostolov-Maschler [9] who have also formulated the existence problem for such metrics on a compact Kähler manifold in the framework of Calabi’s original approach of finding distinguished representatives for Kähler metrics in a given de Rham class. The point of view in [9] was generalized by A. Landili [28] who showed that the Kähler metrics giving rise to cKEM hermitian structures arise as a special case of a more general notion of weighted constant scalar curvature Kähler metrics to which a great deal of the known machinery in the cscK case can be effectively applied. Finally, additional motivation for studying conformally Kähler Einstein-Maxwell 4-manifolds came from the recent realization by Apostolov-Calderbank [3] that such metrics give rise to extremal Sasaki structures on 5-manifolds [11].
With the above motivation in mind, the existence theory for cKEM metrics is rapidly taking shape. A number of non-trivial examples were constructed on $\mathbb{CP}^1 \times \mathbb{CP}^1$ \cite{30} and on the Hirzebruch complex surfaces $F_k = \mathbb{P}(O \oplus O(k)) \to \mathbb{CP}^1$ \cite{31} by C. LeBrun (we shall let below $F_0 := \mathbb{CP}^1 \times \mathbb{CP}^1$). An extension of these construction to other ruled complex surfaces appears in \cite{26}. LeBrun’s examples on $F_k$ have a large group of automorphisms (actually they are of cohomogeniety one under the action of suitable compact groups). It was shown in \cite{27,21} that any Kähler metric on $F_k$ which is conformal to an Einstein-Maxwell hermitian metric must be invariant under the action of a 2-dimensional torus, i.e. it is toric. Toric cKEM metrics have been studied more generally in \cite{9} and as a consequence of this work it was realized that the existence of a Kähler metric conformal to an Einstein-Maxwell hermitian one in a given Kähler class on $F_k$ can be characterized in terms of the corresponding Delzant image (which is a Delzant trapezoid $\Delta \subset \mathbb{R}^2$) as follows:

(a) there exists an affine linear function $f$ on $\mathbb{R}^2$ which is positive on $\Delta$ and satisfies a non-linear algebraic condition, and
(b) a certain linear functional depending on $f$ is strictly positive on convex piecewise affine linear functions over $\Delta$ which are not affine linear.

The condition (a) is characterized in \cite{9} as the vanishing of a Futaki-like invariant on $M$ whereas the condition (b) is referred there as $f$-K-stability of the pair $(\Delta, f)$. It is shown in \cite{9,20} that on $F_0$, (a) holds only for the affine linear functions associated to the explicit solutions found in \cite{30}, thus leading to a complete classification of cKEM metrics on $F_0$ (achieved in \cite{27}). Furthermore, \cite{20} simplifies the search for solutions of (a) by interpreting them as critical points of a volume functional. In particular, \cite{20} identifies all solutions of (a) on the first Hirzebruch surface $F_1$. Their analysis reveal that certain Kähler classes on $F_1$ admit two additional positive affine linear functions $f^+$ and $f^-$ satisfying (a), which do not correspond to the solutions found in \cite{30}. However, even though \cite{21} provides numerical evidence that the condition (b) for those solutions $f^+$ and $f^-$ of (a) holds true, the question of whether or not $f^\pm$ do actually correspond to (new) cKEM metrics on $F_1$ was left open. One of the purposes of this article is to give a positive answer to this question.

**Theorem 1.2.** The first Hirzebruch surface $F_1$ admits conformally Einstein-Maxwell, toric Kähler metrics which are regular ambitoric of hyperbolic type in the sense of \cite{7}. These, together with the metrics of Calabi type constructed by LeBrun in \cite{31} are the only conformally Einstein-Maxwell Kähler metrics on $F_1$, up to a holomorphic homothety.

We note that in \cite{20}, it is shown that similar solutions $f^+_k$ and $f^-_k$ of the condition (a) also arise on any Hirzebruch surface $F_k$, $2 \leq k \leq 4$, but it is unknown if these together with the affine linear functions corresponding to the solutions in \cite{30} are the only solutions. Our method of proof also yields

**Theorem 1.3.** Any Hirzebruch surface $F_k$, with $k = 1, 2, 3, 4$, admits conformally Einstein-Maxwell, toric Kähler metrics which are regular ambitoric of hyperbolic type.

We now explain briefly the main idea of the proof of the results above. It relies on a recent observation from \cite{3} that if $f$ is a positive affine linear function over a Delzant polytope $(\Delta, L)$ in $\mathbb{R}^n$, one can associate to $(\Delta, L, f)$ a different labelled compact convex simple polytope $(\tilde{\Delta}, \tilde{L})$ in $\mathbb{R}^n$, called the $f$-twist transform of $(\Delta, L)$.
Following the theory in [3], we observe in Proposition 5.11 below that \((\Delta, L)\) is \(f\)-\(K\)-stable (i.e. (b) holds with respect to \(f\)) if and only if \((\tilde{\Delta}, \tilde{L})\) is (relatively) \(K\)-stable in the sense originally introduced by Donaldson [17] in the cscK case and by [35] in general. Thus, proving Theorems 1.2 and 1.3 above reduces (via [9, Theorem 5]) to checking that the corresponding \(f\pm k\)-twists \((\tilde{\Delta} \pm k, \tilde{L} \pm k)\) of the Delzant trapezoids associated to \(F_k\) are \(K\)-stable (see Theorem 4.7 for a precise statement). Our key new observation here is that \((\tilde{\Delta} \pm k, \tilde{L} \pm k)\) are in fact equipoised labelled quadrilaterals (which are not trapezoids) in the sense of E. Legendre [33]. It then follows from the latter work that the \(K\)-stability of \((\tilde{\Delta} \pm k, \tilde{L} \pm k)\) can be reduced to checking the positivity of two polynomials of degree \(\leq 4\) over given intervals. This is shown to hold for any equipoised quadrilateral which is not trapezoid in [3, Example 1], thus concluding the proof of the existence. The uniqueness statement in Theorem 1.2 follows from the fact that any cKEM metric must be invariant under the action of a maximal torus in the automorphism group of \(F_k\) [19, 28], and the uniqueness result for toric cKEM metrics established in [9].

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2 CONFORMALLY KÄHLER, EINSTEIN-MAXWELL GEOMETRY

We start by recalling some properties of cKEM metrics, according to C. LeBrun [31] and Apostolov-Calderbak-Gauduchon [7]. We follow the notation from [9, 20] closely.

Let \(\tilde{g}\) be a hermitian metric on a compact complex Kähler manifold \((M, J)\) satisfying Definition 1.1.

As the Ricci tensor \(\text{Ric}^\tilde{g}\) of the Kähler metric \(g = f^2 \tilde{g}\) also satisfies \(\text{Ric}^\tilde{g}(J\cdot, J\cdot) = \text{Ric}^\tilde{g}(\cdot, \cdot)\), and

\[
\text{Ric}^\tilde{g} = \text{Ric}^g + \frac{2m-2}{f} D^g df + hg,
\]

where \(D^g\) denotes de Levi-Civita connection of \(g\) and \(h\) is a smooth function not given explicitly, the condition (i) in Definition 1.1 is equivalent to the condition that the vector field \(K = J \text{grad} f\) is Killing for both \(g\) and \(\tilde{g}\). Furthermore, condition (ii) in Definition 1.1 reads as

\[
\text{Scal}(\tilde{g}) = f^2 \text{Scal}(g) - 2(2m-1)f \Delta g f - 2m(2m-1) |df|^2 g = c
\]

where \(c\) is a constant, \(\Delta g\) is the Riemannian Laplacian of \(g\) and \(\text{Scal}(g)\) is the scalar curvature of \(g\). We define the function

\[
\text{Scal}_f(g) := f^2 \text{Scal}(g) - 2(2m-1)f \Delta g f - 2m(2m-1) |df|^2 g,
\]
and refer to it as the \((f, 2m)\)-scalar curvature of \(g\). This is a particular case (with \(w = 2m\)) of the notion of \((f, w)\)-scalar curvature

\[\text{Scal}_{(f, w)}(g) := f^2 \text{Scal}(g) - 2(w - 1)f \Delta g - w(w - 1)|df|^2_g\]

studied in \cite{9, 27} for an arbitrary real number \(w\).

Thus, every cKEM metric admits a Killing vector field \(K := J \text{grad} \sigma f\), and we know from \cite{27} Theorem 1 and \cite{19} Theorem 2.1 that every cKEM metric on a compact manifold is invariant under the action of a maximal compact real torus \(T\) inside the reduced automorphism group \(\text{Aut}(M, J)\) of \((M, J)\) with \(K \in t = \text{Lie}(T)\) (see \cite{22} for the definition of \(\text{Aut}(M, J)\)). More precisely:

**Theorem 2.1 \cite{19, 27}**. Let \((M, g, J)\) be a compact Kähler manifold and \(K = J \text{grad} \sigma f\) a Killing vector field with positive Killing potential \(f\). If \(g\) is \(f\)-extremal (i.e. if \(\text{Scal}_f(g)\) is a Killing potential) then \(g\) is invariant under the action of a maximal compact real torus \(T \subset \text{Aut}_f(M, J)\) such that \(K\) and \(J \text{grad} \text{Scal}_f(g)\) belong to \(\text{Lie}(T)\).

3 The Weighted Calabi Problem

Now we fix a maximal compact torus \(T \subset \text{Aut}_f(M, J)\), and a vector field \(K \in t := \text{Lie}(T)\). Let \(\omega_0\) be a \(T\)-invariant Kähler form, and \(\Omega = [\omega_0] \in H^2_{\text{DR}}(M, \mathbb{R})\) be a fixed Kähler class. The problem we are going to study is to find a \(T\)-invariant Kähler metric \(g\) with Kähler form \(\omega_g \in \Omega\), such that \(\tilde{g} = f^{-2} g\) is a cKEM metric, for \(f > 0\) such that \(J \text{grad}_g \sigma f = K\).

Denote by \(K^T_\Omega\) the space of \(T\)-invariant Kähler metrics \(g\) on \((M, J)\) with \(\omega_g \in \Omega\). Then the vector field \(K \in t\) is hamiltonian with respect to \(\omega_g\) (see \cite{22} Chapter 2), i.e.

\[\iota_K \omega_g = -d f_{K,g}\]

for a smooth function \(f_{K,g}\) on \(M\). Such a function is called a Killing potential of \(K\) with respect to \(\omega_g\). We observe that this function is defined up to an additive constant, so we further fix the setting by requiring

\[\int_M f_{K,g} \frac{\omega_g}{m!} = a,\]

where \(a\) is a fixed real constant. We shall denote by \(f_{K,a,g}\) the unique function satisfying the above relations.

Since \(\min \{ f_{K,a,g} | x \in M \} \) is independent of \(g\) in \(K^T_\Omega\) (see e.g. \cite{9} Lemma 1), following \cite{20}, we define:

\[\mathcal{P}^T_\Omega := \{(K, a) \in t \times \mathbb{R} | f_{K,a,g} > 0 \},\]

(3.1)

\[\mathcal{H}^T_\Omega := \left\{ \tilde{g}_{K,a} = \frac{1}{f^2_{K,a,g}} g | (K, a) \in \mathcal{P}^T_\Omega, g \in K^T_\Omega \right\}.\]

(3.2)

From now on we identify the Kähler metric \(g\) with its Kähler form \(\omega_g\), and we drop the subscript \(g\). Fixing \((K, a) \in \mathcal{P}^T_\Omega\), let

\[\mathcal{H}^T_{\Omega, K,a} := \left\{ \tilde{g}_{K,a} | g \in K^T_\Omega \right\}\]

(3.3)

and
\[ c_{\Omega, \kappa, \alpha} := \left( \int_{M} \frac{g_{\kappa, \alpha}}{f_{K_\alpha}^{2m+1}} \frac{\omega^n}{m!} \right) / \left( \int_{M} \frac{1}{f_{K_\alpha}^{2m+1}} \frac{\omega^n}{m!} \right). \]

It follows from [9] Corollary 1] that \( c_{\Omega, \kappa, \alpha} \) is a constant independent of the choice of \( g \in K_\Omega \).

Also, for each vector field \( H \in \mathfrak{t} \) with Killing potential \( f_{H, b, \kappa} \), we consider

\[ \tilde{\gamma}_{\Omega, \kappa, \alpha}(H) := \int_{M} \left( \frac{g_{\kappa, \alpha} - c_{\Omega, \kappa, \alpha}}{f_{K_\alpha}^{2m+1}} \right) \frac{f_{H, b, \kappa}}{m!} \frac{\omega^n}{m!}, \]

which according to [9] Corollary 1] is a linear functional, independent of the choice of \( (g, b) \in K_\Omega \times \mathbb{R} \).

**Definition 3.1.** The linear map \( \tilde{\gamma}_{\Omega, \kappa, \alpha} : \mathfrak{t} \rightarrow \mathbb{R} \) defined by (3.4) and (3.5) is called the cKEM-Futaki invariant.

**Theorem 3.2 ([9] Corollary 1]).** The vanishing of \( \tilde{\gamma}_{\Omega, \kappa, \alpha} \) is an obstruction to the existence of a cKEM metric in \( \mathcal{H}_{\Omega, \kappa, \alpha} \).

**Remark 3.3.** The main result in [20] gives a useful characterization of the condition \( \tilde{\gamma}_{\Omega, \kappa, \alpha} \equiv 0 \). Indeed, the authors prove that \( \tilde{\gamma}_{\Omega, \kappa, \alpha} \equiv 0 \) if and only if \( (K, \alpha) \) is a critical point of the suitably normalized volume functional acting on \( \mathcal{P}_{\kappa, \alpha} \). The usefulness of their theorem resides in the fact that it allows for a systematic computation of the vanishing of the cKEM-Futaki invariant.

## 4 Toric Kähler Manifolds

From now on, we specialize to the toric case, i.e. we assume that \( \mathbb{T} \subset \text{Aut}_t(M, J) \) is an \( m \)-dimensional torus, where \( m \) is the complex dimension of \( (M, \omega, J) \). We recall that by Theorem 2.1 any cKEM metric \( g \) must be obtained from a toric Kähler metric \( (g, \omega) \). This is the situation studied in [9], by using the Abreu-Guillemin formalism [1, 24].

Let \( (M, \omega, \mathbb{T}) \) be a compact symplectic toric manifold and \( \mu : M \rightarrow \mathfrak{t}^\ast \) its moment map. It is well known [10, 25] that the image of \( M \) by \( \mu \) is a compact simple convex polytope \( \Delta \subset \mathfrak{t}^* \). Furthermore, it is shown in [16] that \( \Delta \) can be given the structure of a **labelled Delzant polytope** \( (\Delta, L) \), i.e. a compact convex simple polytope with \( d \) facets, together with a set \( L = \{ L_1, \ldots, L_d \} \) of non-negative affine linear functions \( L_i \) defining \( \Delta \) by

\[ \Delta := \{ x \in \mathfrak{t}^* : L_i(x) \geq 0, i = 1, \ldots, d \}, \]

and such that \( d L_i \in \mathfrak{t} \) are primitive elements of the lattice \( \Lambda \subset \mathfrak{t} \) of circle subgroups of \( \mathbb{T} \) (integrality condition). It also follows from [16] that the compact symplectic toric manifold \( (M, \omega, \mathbb{T}) \) can be reconstructed from the corresponding labelled integral Delzant polytope \( (\Delta, L) \).

Now, let \( (M, g, J, \mathbb{T}) \) be a compact toric Kähler manifold and \( \mu : M \rightarrow \mathfrak{t}^* \) its moment map. According to [24], on the dense open subset \( M^0 := \mu^{-1}(\Delta^0) \) (where \( \Delta^0 \) denotes the interior of \( \Delta \)), the toric Kähler structure \( (g, J, \omega) \) can be written in moment-angle coordinates \( (x, t) \) as:

\[ g = \langle dx, G(x), dx \rangle + \langle dt, J H(x), dt \rangle, \quad J dt = -(G(x), dx), \quad J dx = -(H(x), dt), \]

\[ \omega = \langle dx \wedge dt \rangle, \quad f dt = -(G(x), dx), \quad f dx = -(H(x), dt). \]
where $H$ is a smooth positive definite $S^2 t^*$-valued function on the moment image $\Delta^0$ and $G = H^{-1}$ is its pointwise inverse, a smooth $S^2 t$-valued function. Furthermore, $G = \text{Hess}(u)$ is the Hessian of a real function $u \in C^\infty(\Delta^0)$, called symplectic potential of $(g, J, \omega)$.

We denote by $S(\Delta, L)$ the set of symplectic potentials of globally defined $T$-invariant $\omega$-compatible Kähler metrics $(g, J)$ on $(M, \omega, T)$. By the theory in [1, 2] (see also [8, Proposition 1] and [18]), $S(\Delta, L)$ consists of smooth strictly convex functions $u \in C^\infty(\Delta^0)$, whose inverse Hessian

$$H^u = (H^u_{ij}) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^{-1}$$

is smooth on $\Delta$, positive definite on the interior of any face and satisfies, for every $y$ in the interior of a facet $F_i \subset \Delta$ with inward normal $e_i = dL_i$, the following boundary conditions [8, Proposition 1] (see also [4, Proposition 1]):

$$H^u_{ij}(e_i, \cdot) = 0 \text{ and } dH^u_{ij}(e_i, e_i) = 2e_i.$$  

**Remark 4.1.** $S(\Delta, L)$ can be introduced independent of the integrality condition on $(\Delta, L)$, as in [17].

In [11], Abreu computed the scalar curvature of the metric (4.1) associated to a symplectic potential $u \in S(\Delta, L)$ to be the pull-back by the moment map of the smooth function on $\Delta$

$$S(u) = - \sum_{ij=1}^m \frac{\partial^2 H^u}{\partial x_i \partial x_j}.$$  

Notice that in the toric setting the space of Killing potentials of elements in $t$ with respect to $(g, \omega)$ is in one-to-one correspondence with affine linear functions (pulled-back by $\mu$) on $t^\ast$. The extremal affine linear function $\zeta_{(\Delta, L)}$ is the $L^2$-projection (with respect to the euclidean measure) of $S(u)$ to the finite dimensional space of affine linear functions on $t^\ast$. In fact, $\zeta_{(\Delta, L)}$ is independent of the symplectic potential $u \in S(\Delta, L)$ (see [17]) and may also be defined as the solution of a linear system depending only on $(\Delta, L)$.

Any solution $u \in S(\Delta, L)$ of

$$S(u) = - \sum_{ij=1}^m \frac{\partial^2 H^u}{\partial x_i \partial x_j} = \zeta_{(\Delta, L)}$$

gives rise to an extremal Kähler metric and (4.4) is know as the Abreu equation. The cscK case reduces to the special situation when $\zeta_{(\Delta, L)}$ is constant.

In the case when $(M, \omega, J, T)$ is a toric Kähler manifold and $f$ is an affine linear function on $t^\ast$ which is positive on $\Delta$, the scalar curvature of $\tilde{g} = f^{-2} g$ is computed in [11] to be

$$S_{\tilde{g}}(u) = - f^{2m+1} \sum_{ij=1}^m \left(\frac{1}{f^{2m-1}} H^u_{ij}\right).$$

Closely related to the discussion above, it is proved in [9] that the $L^2$-projection of (4.5) to the space of affine linear functions on $t^\ast$ is independent of $g$ (i.e. of
where \( \zeta_{(\Delta, L, f)} \) is defined in terms of \((\Delta, L, f)\).

Solutions to the problem above are called \((f, 2m)\)-extremal Kähler metrics and in the special case when \( \zeta_{(\Delta, L, f)} \) is constant, the metric \( f^{-2}g \) is conformally Kähler, Einstein-Maxwell.

More generally, one can define [9, 27] a \((f, w)\)-extremal toric Kähler metric as a solution of the equation

\[
- f^{w+1} \sum_{i,j=1}^{m} \left( \frac{1}{f^{w-1} H_{ij}} \right)_{ij} = \zeta_{(\Delta, L, f, w)},
\]

for \( u \in S(\Delta, L) \), \( f \) a positive affine linear function on \( \Delta \), and \( \zeta_{(\Delta, L, f, w)} \) an affine linear function determined by \((\Delta, L, f, w)\).

**Theorem 4.2 ([9, Theorem 3]).** Any two solutions \( u_1, u_2 \in S(\Delta, L) \) of (4.7) differ by an affine linear function. In particular, on a compact toric Kähler manifold \((M, \omega, J, \mathcal{T})\), for any fixed positive affine linear function in momenta \( f = f_{K, a, g} \), there exists at most one, up to a \( \mathcal{T} \)-equivariant isometry, \( \omega \)-compatible \( \mathcal{T} \)-invariant Kähler metric \( g \) for which \( g_{K, a} = f^{-2}g \) is a conformally Kähler, Einstein-Maxwell metric.

Similarly to the extremal toric case studied in [17], there exists an obstruction to find a solution to (4.7) which is called \((f, w)\)-K-stability of \((\Delta, L, f)\), which we now explain following [9, 28].

**Definition 4.3.** The \((f, w)\)-Donaldson-Futaki invariant \( F_{\Delta, L, f, w} \) of a labelled compact simple convex polytope \((\Delta, L)\) and a given positive affine linear function \( f \) on \( \Delta \) is defined by

\[
F_{\Delta, L, f, w}(\phi) = 2 \int_{\Delta} \frac{\phi}{f^{w-1}} d\sigma - \int_{\Delta} \frac{\phi}{f^{w+1}} \zeta_{(\Delta, L, f, w)} dx,
\]

where \( dx \) is an euclidean measure on \( \Delta \) and \( d\sigma \) is a measure on any facet \( F_i \subset \Delta \) defined by \( dL_i \wedge d\sigma = -dx \). In the above formula, the affine linear function \( \zeta_{(\Delta, L, f, w)} \) is the unique affine linear function such that \( F_{\Delta, L, f, w}(\phi) = 0 \) for all affine linear functions \( \phi \) on \( \Delta \).

**Definition 4.4.** A labelled polytope \((\Delta, L)\) is \((f, w)\)-K-stable if the associated \((f, w)\)-Donaldson-Futaki invariant \( F_{\Delta, L, f, w} \) is non-negative on any convex piecewise affine linear function \( \phi \) on \( \Delta \), and vanishes if and only if \( \phi \) is affine linear.

**Remark 4.5.** Note that if we take \( f \equiv 1 \) in Definition 4.3 then we recover the usual (relative) Donaldson-Futaki invariant introduced in [17, 35]. Also, the \((f, 2m)\)-Donaldson-Futaki invariant, hereafter denoted by \( F_{\Delta, L, f} \), is equal to \((2\pi)^{-m}\) times the Futaki invariant defined by (3.5), when restricted to functions \( \phi \) which are affine linear in momenta.

**Theorem 4.6 ([9]).** If \( \zeta_{(\Delta, L, f)} = c \) is constant and (4.5) admits a solution \( u \in S(\Delta, L) \) then \((\Delta, L)\) is \((f, 2m)\)-K-stable.
To summarize, the existence of \( g \in K_T^2 \) which is conformal to an Einstein-Maxwell hermitian metric is equivalent to the existence of \( u \in S(\Delta, L) \) and a positive affine linear function \( f \) on \( \Delta \), satisfying (4.6). Moreover, if a solution exists then

\[
\begin{align*}
(a) & \quad \xi(\Delta, L, f) = c \text{ is constant;} \\
(b) & \quad (\Delta, L, f) \text{ is } (f, 2m)-K\text{-stable;}
\end{align*}
\]

The constant \( c \) in \( (a) \) is prescribed by \( (\Delta, L, f) \), via the formula \((9, \text{Theorem } 2])\):

\[
c = c(\Delta, L, f) := 2 \frac{\int_{\partial \Delta} \frac{1}{f^{2m}} d \sigma}{\int_{\Delta} \frac{1}{f^{2m+1}} d \mu}
\]

In particular, it is always positive. It is not known at present whether or not \( (a) \) and \( (b) \) are sufficient in general, but a positive answer is given in the special case when \( (\Delta, L) \) is a labelled quadrilateral.

**Theorem 4.7** \((9, \text{Theorem } 5])\). Let \((M, \omega, \mathbb{T})\) be a compact symplectic toric 4-orbifold whose rational Delzant polytope is a labelled quadrilateral \( (\Delta, L) \) and \( f \) a positive affine linear function on \( \Delta \) which satisfies \( (a) \). Then \( (b) \) is equivalent to the existence of a \( \mathbb{T} \)-invariant Kähler metric \( g \) such that \( \tilde{g} = f^{-2}g \) is a conformally Kähler, Einstein-Maxwell metric on \( M \).

5 \text{ The } f\text{-twist of a labelled polytope}

In this section we follow \[3\], where the authors introduce the \( f\text{-twist} \) transform of a labelled polytope. The interested reader can consult the original paper for more details. A special case of the correspondence was first seen in \[29\] (see Proposition 3) where a bijection between \textit{ambitoric Einstein-Maxwell metrics} and \textit{ambitoric extremal metrics} of positive scalar curvature was found. In \[3\], the authors introduce the \( f\text{-twist} \) transform more generally in terms of a pair of Kähler metrics arising as transversal Kähler structures of Sasaki metrics compatible with the same CR structure and having commuting Sasaki-Reeb vector fields. This leads to an interesting general equivalence between cKEM and extremal Kähler metrics in real dimension 4, which is the case we are most interest in.

**Definition 5.1.** Let \( \Delta \) be a polytope in \( \mathbb{R}^m \) containing the origin and \( f \) a positive affine linear function on \( \Delta \). We define the \( f\text{-twist} \) transform \( \tilde{\Delta} \) of \( \Delta \) to be the image of \( \Delta \) under the change of variables \( T(x) = \tilde{x} := \frac{x}{f(x)} \) where \( x = (x_1, \ldots, x_m) \) are the euclidean coordinates of \( \mathbb{R}^m \), i.e. \( \tilde{x}_i = \frac{x_i}{f(x)} \) for \( i = 1, \ldots, m \). We also define the \( f\text{-twist} \) transform of a function \( \phi \) to be the function \( \tilde{\phi}(\tilde{x}) := \frac{\phi(x)}{f(x)} \).

**Lemma 5.2.** Let \( \phi(x) \) be an affine linear function in the coordinates \( x = (x_1, \ldots, x_m) \) in \( \mathbb{R}^m \), and \( \tilde{\phi}(\tilde{x}) = \frac{\phi(x)}{f(x)} \) its \( f\text{-twist} \) transform. Then \( \tilde{\phi}(\tilde{x}) \) is an affine linear function in the coordinates \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m) \) in \( \mathbb{R}^m \). In particular, if \( (\Delta, L) \) is a labelled polytope containing the origin and \( f(x) \) is a positive affine linear function on \( \Delta \), then the \( f\text{-twist} \) transform of \( (\Delta, L) \), denoted by \( (\tilde{\Delta}, \tilde{L}) \), is a labelled polytope with respect to the labeling \( \tilde{L} := \frac{L}{f(x)} \) which contains the origin, i.e. \( \tilde{\Delta} = T(\Delta) \) is defined by \( \{\tilde{L}_i(\tilde{x}) \geq 0; i = 1, \ldots, d\} \) where \( \tilde{L}_i(\tilde{x}) := \frac{L_i(x)}{f(x)} \).
Remark 5.3. When $(\Delta, L) \subset \mathbb{R}^m$ is a rational Delzant polytope associated to a compact toric orbifold, we can assume without loss that the origin is inside $\Delta$. The last claim of Lemma 5.2 then follows from [32] and the geometric interpretation of the $f$-twist transform given in [3, Theorem 1 and Lemma 5].

Proof. Let $\phi(x) = b_0 + b_1x_1 + \cdots + b_mx_m$ be an affine linear function in the coordinates $(x_1, \ldots, x_m)$. We observe that
\begin{equation}
\tilde{\phi}(\tilde{x}) = \frac{b_0}{f(x)} + \sum_{i=1}^{n} \left( \frac{b_i}{f(x)} \tilde{x}_i \right) = \frac{b_0}{f(x)} + \sum_{i=1}^{n} b_i \tilde{x}_i.
\end{equation}

Also, for $f(x) = a_0 + \sum_{i=1}^{n} a_i x_i$, we have
\begin{equation}
\frac{1}{f(x)} = \frac{1}{a_0} (1 - a_1 \tilde{x}_1 - \cdots - a_n \tilde{x}_m).
\end{equation}

It follows from equations (5.1) and (5.2) that
\begin{equation}
\tilde{\phi}(\tilde{x}) = \frac{1}{a_0} \left( b_0 + \sum_{i=1}^{n} (b_i - b_0a_i) \tilde{x}_i \right),
\end{equation}

establishing the first part of the Lemma.

For the second part we observe that given the labelled Delzant polytope $(\Delta, L)$ then $\Delta = \{ x \in \mathbb{R}^m : L_i(x) \geq 0, i = 1, \ldots, d \}$. So $x \in \Delta$ if and only if $\tilde{x} \in \tilde{\Delta}$ or equivalently $\tilde{\Delta} := \{ \tilde{x} \in \mathbb{R}^m : \tilde{L}_i(\tilde{x}) \geq 0, i = 1, \ldots, d \} = T(\Delta)$. □

Remark 5.4.

(i) Equation (5.2) in the proof of the Lemma 5.2 defines a distinguished affine linear function in the new coordinates $\tilde{x}$, which hereafter we will denote by
\begin{equation}
\tilde{f}(\tilde{x}) := \frac{1}{f(x)} = \frac{1}{a_0} (1 - a_1 \tilde{x}_1 - \cdots - a_n \tilde{x}_m);
\end{equation}

(ii) For a given affine linear function $\phi$ defined on $\Delta$, we have
\begin{equation}
\phi(x) = (T^* \phi)(\tilde{x}) = \frac{\tilde{\phi}(\tilde{x})}{\tilde{f}(\tilde{x})}.
\end{equation}

For a symplectic potential $u \in \mathcal{S}(\Delta, L)$ we consider the $f$-twist transform of $u$ defined by
\begin{equation}
\tilde{u}(\tilde{x}) := \frac{u(x)}{f(x)}.
\end{equation}

Then we have

Lemma 5.5. If $u \in \mathcal{S}(\Delta, L)$, then $\tilde{u} \in \mathcal{S}(\tilde{\Delta}, \tilde{L})$.

In the case when $(\Delta, L)$ is rational, this result compared with [3, Theorem 1] and Lemma 5.2 yields the claim in Lemma 5.5 Here we give a general argument for the sake of completeness. In order to prove Lemma 5.5, we first recall a result from [2] (see also [8, Lemma 3]).
Theorem 5.6 ([2] Theorem 2]). Let \((M, \omega, \mathbb{T})\) be the toric symplectic manifold associated to a labelled Delzant polytope \((\Delta, L)\), and \(J\) any \(\omega\)-compatible toric complex structure. Then \(J\) is determined in moment-angle coordinates \((x, t) \in M^0 \cong \Delta^0 \times \mathbb{T}^n\) by

\[
\begin{align*}
J dt &= - \langle G^u(x), dx \rangle \\
J dx &= - \langle H^u(x), dt \rangle
\end{align*}
\]

in terms of a symplectic potential \(u \in S(\Delta, L)\) of the form

\[
u = u_\Delta + h,
\]

where \(u_\Delta = \frac{1}{2} \sum L_r \log(L_r)\) is the so-called Guillemin potential, \(h\) is a smooth function on the whole of \(\Delta\), the matrix \(H^u = (G^u)^{-1}\) with \(G^u = \text{Hess}(u)\) positive definite on \(\Delta^0\) and having determinant of the form

\[
\text{det}(G) = \delta(x)/\Pi_r L_r(x),
\]

where \(\delta\) is a smooth and strictly positive function on the whole \(\Delta\).

Conversely any such \(u\) determines a compatible toric complex structure on \((M, \omega)\), which in suitable \((x, t)\) coordinates of \(\Delta^0 \times \mathbb{T}^n\) has the form

\[
\begin{align*}
J dt &= - \langle G^u(x), dx \rangle \\
J dx &= - \langle H^u(x), dt \rangle
\end{align*}
\]

Remark 5.7. The arguments in [4, Proposition 1] show that, more generally, (5.5) and (5.6) are equivalent with the defining smoothness, positivity, and boundary conditions (see (4.2) above) of \(S(\Delta, L)\), independent of the integrality of \((\Delta, L)\).

We will also need the following

Lemma 5.8. [3] Let \(u \in S(\Delta, L)\) and consider \(f(x) = a_0 + \sum_{i=1}^d a_i x_i\) an affine linear function which is positive on \(\Delta\) containing the origin. If \(\tilde{a}(\tilde{x}) = \frac{u(\tilde{x})}{f(\tilde{x})}\), then \(G = \text{Hess}_x(u)\) and \(\tilde{G} = \text{Hess}_{\tilde{x}}(\tilde{a})\) are related by

\[
\text{det}(\tilde{G}) = \frac{f^{m+2}(x)}{a_0^2} \text{det}(G).
\]

Proof. This follows from the definition of the projective Hessian in [3 pp. 10–12]. For the sake of completeness, we present here a direct argument in the case \(m = 2\) (which we shall use to prove Theorem 6.3).

Let \(f(x) = a_0 + a_1 x_1 + a_2 x_2\). We have \(x_i = \frac{\tilde{x}_i}{f(\tilde{x})}\) where \(\tilde{f}(\tilde{x}) = \frac{1}{f(x)}\). Then we obtain:

\[
\begin{align*}
\delta_k (x_j) &= \frac{\partial x_j}{\partial \tilde{x}_k} = \frac{\partial}{\partial \tilde{x}_k} \left( \frac{x_j}{f(\tilde{x})} \right) \\
&= \frac{\delta_{kj} \tilde{f}(\tilde{x}) - \delta_k \tilde{x}_j}{\tilde{f}^2(\tilde{x})} \\
&= f(x) \left( \delta_{kj} f(x) + \frac{a_k}{a_0} x_j \right),
\end{align*}
\]

(5.7)
where \(k, j = 1, 2\). Also, we observe that:

\[
\delta_{ij} = \frac{\partial^2}{\partial \bar{x}_i \partial x_j} u
\]

\[
\frac{\partial}{\partial \bar{x}_i} \left( \frac{\partial x_1}{\partial \bar{x}_j} \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \bar{x}_j} \frac{\partial}{\partial x_2} \right)
\]

\[
= \left( \frac{\partial x_1}{\partial \bar{x}_j} \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \bar{x}_j} \frac{\partial}{\partial x_2} \right) \circ \left( \frac{\partial x_1}{\partial \bar{x}_i} \frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \bar{x}_i} \frac{\partial}{\partial x_2} \right).
\]

Now, using (5.7) and (5.8) we obtain the following formulas for \(\tilde{\delta}_{ij} = \tilde{\delta}_{ij}(\tilde{x})\):

\[
\tilde{\delta}_{11}(\tilde{x}) = \frac{f(x)}{a_0} \left( (a_1 x_1 + a_0)^2 \frac{\partial^2 u(x)}{\partial x_1^2} + 2a_1x_2 \left( a_1 x_2 + a_0 \right) \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} + \frac{a_1}{2} x_2^2 \frac{\partial^2 u(x)}{\partial x_2^2} \right)
\]

\[
\tilde{\delta}_{12}(\tilde{x}) = \frac{f(x)}{a_0} \left( a_0 f(x) + 2a_1 a_2 x_1 x_2 \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} + \frac{a_1}{2} a_2 x_1^2 \frac{\partial^2 u(x)}{\partial x_1^2} + \frac{a_1}{2} a_2 x_2^2 \frac{\partial^2 u(x)}{\partial x_2^2} \right)
\]

\[
\tilde{\delta}_{22}(\tilde{x}) = \frac{f(x)}{a_0} \left( (a_2 x_2 + a_0)^2 \frac{\partial^2 u(x)}{\partial x_2^2} + 2a_2 x_1 \left( a_2 x_1 + a_0 \right) \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} + \frac{a_2}{2} x_1^2 \frac{\partial^2 u(x)}{\partial x_1^2} \right)
\]

Finally, straightforward computation of \(\det(\tilde{G}) = \det(\text{Hess}(\tilde{u}))\) using (5.9) yields

\[
\det(\tilde{G}) = \frac{f^4(x)}{a_0^4} \det G.
\]

\[\square\]

Proof of Lemma 5.5 To prove Lemma 5.5, we shall check the equivalent conditions for \(\tilde{u} \in \mathcal{S}(\tilde{\Delta}, \tilde{L})\) given by Theorem 5.6. In order to use Theorem 5.6, the first step is to check that \(\tilde{u}(\tilde{x}) = \tilde{u}_\lambda(\tilde{x}) + \tilde{\phi}(\tilde{x})\), where \(\tilde{u}_\lambda\) is the Guillemin potential of \((\tilde{\Delta}, \tilde{L})\) and \(\tilde{\phi}\) is a smooth function on \(\tilde{\Delta}\). Since \(u \in \mathcal{S}(\Delta, L)\), we have \(u(x) = u_\lambda(x) + h(x)\) where \(u_\lambda\) is the Guillemin potential of \((\Delta, L)\) and \(h\) is a smooth function on the whole of \(\Delta\). Now, using that \(\tilde{u}(\tilde{x}) = \frac{u(x)}{f(x)}\) we can write

\[
\tilde{u}(\tilde{x}) = \tilde{u}_\lambda(\tilde{x}) + \tilde{\phi}(\tilde{x}),
\]

where \(\tilde{u}_\lambda\) is the Guillemin potential of \((\tilde{\Delta}, \tilde{L})\) and \(\tilde{\phi}(\tilde{x}) = h \left( \frac{\tilde{\lambda}}{f} \right) \tilde{f} - \log(\tilde{f}) \left( \sum_r L_r \right)\). The smoothness of \(\tilde{\phi}\) on \(\tilde{\Delta}\) follows from the smoothness of \(h\) on \(\Delta\) and the positivity of \(\tilde{f}\) on \(\tilde{\Delta}\).

The second step is to check the positivity of \(\tilde{G}\) in \(\tilde{\Delta}^0\) and its behaviour on \(\partial \tilde{\Delta}\). The positivity of \(\tilde{G}\) on \(\tilde{\Delta}^0\) follows from [3] Theorem 1 and [3] Lemma 5 (which identifies \(\tilde{u}\) with the simplectic potential of a Kähler metric over \(\tilde{\Delta}^0 \times \mathbb{T}^m\)).

To check the behaviour of \(\det(\tilde{G})\) on \(\partial \tilde{\Delta}\) we need to show that

\[
\det(\tilde{G}) = \frac{\tilde{\delta}(x)}{\Pi r L_r(x)},
\]
with \( \delta \) being a smooth and strictly positive function on the whole \( \tilde{\Delta} \). This follows from Lemma 5.8. Indeed, since \( u \in S(\Delta, L) \) according to Theorem 5.6

\[
\text{det}(G) = \frac{\delta(x)}{\prod_{r=1}^{d} L_r(x)}
\]

Using (5.10) we obtain,

\[
\text{det} \tilde{G} = \frac{(f(x))^{m+2} \delta(x)}{a_0 \prod_{r=1}^{d} L_r(x)}
\]

(5.11)

\[
= \frac{\delta(x)}{a_0^d (f(x))^{d-(m+2)} \prod_{r=1}^{d} L_r(x)}
\]

where \( \tilde{\delta}(\tilde{x}) = \frac{1}{a_0^d} \delta \left( \frac{\tilde{x}}{f(x)} \right) (\tilde{f}(\tilde{x}))^{d-(m+2)} \) is a positive function on \( \tilde{\Delta} \).

\[\square\]

**Definition 5.9** (see [3] p.9 and Lemma 5). For a toric Kähler metric \( g \) over \( M^0 = \Delta^0 \times \mathbb{T}^n \) given in moment-angle coordinates by (4.1), and an affine linear function \( f(x) = a_0 + \sum_{i=1}^{n} a_i x_i \) positive on \( \Delta \) with \( a_0 > 0 \), we define the f-twist transform of \( g \) to be the toric Kähler metric \( \tilde{g} \) over \( \Delta \times \mathbb{T}^n \) given by

\[
\tilde{g} = \langle d\tilde{x}, \tilde{G}(\tilde{x}), d\tilde{x} \rangle + \langle d\tilde{t}, \tilde{H}(\tilde{x}), d\tilde{t} \rangle,
\]

\[
\tilde{\omega} = \langle d\tilde{x} \wedge d\tilde{t} \rangle,
\]

(5.12)

\[
\tilde{f} \tilde{t} = -\langle \tilde{G}(\tilde{x}), d\tilde{x} \rangle,
\]

\[
\tilde{f} d\tilde{x} = -\langle \tilde{H}(\tilde{x}), d\tilde{t} \rangle,
\]

with

\[
\tilde{t}_j = t_j - \frac{a_j}{a_0} t_0, \quad j = 1, \ldots, n, \quad \text{and} \quad \tilde{u}(\tilde{x}) = \frac{u(x)}{f(x)}.
\]

**Theorem 5.10** ([3] Theorem 1, Lemma 5]). \((\tilde{g}, \tilde{f})\) is extremal if and only if \((g, f, I, f)\) is \((f, m+2)\)-extremal.

We complete the above observation with the following

**Proposition 5.11.** Let \((\Delta, L)\) be a simple compact convex labelled polytope in \( \mathbb{R}^n \) which contains the origin, and \( f(x) \) an affine linear function which is positive on \( \Delta \). Consider \((\Delta, L)\) to be the f-twist transform of \((\Delta, L)\). Then,

\[
F_{\Delta, L, f, m+2}(\phi) = \frac{1}{f(0)} F_{\Delta, L}(\tilde{\phi}),
\]

where \( \zeta_{(\Delta, L)}(x) = \zeta_{(\Delta, L)}(\tilde{x}) \) and \( \tilde{\phi}(\tilde{x}) = \frac{\phi(x)}{f(x)} \).

**Proof.** Let \( T: \Delta \to \tilde{\Delta} \) be the diffeomorphism given by \( \tilde{x} := T(x) = \frac{x}{f(x)} \). We consider the Lebesgue measure \( dx = dx_1 \wedge \ldots \wedge dx_m \) on \( \Delta \) and the induced measures \( d\sigma \) on each facet \( F_i \subset \partial \Delta \) defined by letting \( dL_i \wedge d\sigma = -dx \). In the same way we define \( d\tilde{x} \) on \( \Delta \) and \( d\tilde{\sigma} \) on \( \tilde{F}_i \subset \tilde{\Delta} \), respectively.

We observe that:

\[
T^*(dx) = \frac{f(0)}{(f(x))^{m+1}} d\tilde{x}
\]

(5.13)

\[
T^*(d\sigma) = \frac{f(0)}{(f(x))^m} d\tilde{\sigma}.
\]
where characterizes the possible positive affine linear function $s_T$ for (6.1) and labelling satisfy condition Theorem 6.1 manifold classified by Corollary 5.12.

Let $f$ (6.2)

\[
\text{Denote by } E(\alpha, \beta) = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} \text{ and for } k \geq 1, \text{and by } \Delta_{p,k} \text{ the Delzant polytope of the } k\text{-th Hirzebruch surface } \mathbb{F}_k \text{ endowed with a } T^2\text{-invariant Kähler metric in the Kähler class } \Omega_p = \mathcal{L} - (1 - p)\mathcal{E}, \text{ where } \mathcal{L} \text{ and } \mathcal{E} \text{ are respectively the Poincaré duals of a projective line and the infinity section of } \mathbb{F}_k \text{ (see [23]). It can be shown that the corresponding Delzant polytope } \Delta_{p,k} \text{ is the convex hull of } (0,0), (p,0), (p,(1-p)k), (0,k), (0 < p < 1), \text{ and labelling } L_{p,k} = \{ e_1, e_2, \ldots, x_k \} \text{ where } e_1 = (1,0), e_2 = (0,1), e_3 = -e_1, e_4 = -ke_1 + e_2), e_5 = x_1, e_6 = (e_2, x) = x_2, \text{ and } L_1(x) = \langle e_1, x \rangle = x_1, L_2(x) = \langle e_2, x \rangle = x_2, L_3(x) = \langle e_3, x \rangle + p = -x_1 + p, L_4(x) = \langle e_4, x \rangle + k = k(1 - x_1) - x_2.

In [20], the authors computed the critical points of the volume functional which characterizes the possible positive affine linear functions $f$ on $(\Delta_{p,k}, L_{p,k})$ which satisfy condition (a) in (4.9).

**Theorem 6.1** ([20,31]). Let $M = \mathbb{F}_k$ be the $k$-th Hirzebruch surface considered as a toric manifold classified by $(\Delta_{p,k}, L_{p,k})$. Let $0 < r_k < s_k < 1$ be the real roots of

\[
F_k(p) = 4(1 - p)^2k^2 - 4(p - 1)(p - 2)pk + p^4.
\]

(i) For any $k$ and $0 < p < 1$, the affine linear function

\[
f_p = \frac{p + 2\sqrt{1 - p} - 2}{2p^2} x_1 - \frac{\sqrt{1 - p} - 1}{2p}
\]

is positive on $\Delta_{p,k}$ and $(\Delta_{p,k}, L_{p,k}, f_p)$ satisfy the conditions (a) and (b) in (4.9).

(ii) For $k = 1$ and for $\frac{8}{9} < p < 1$, the two affine linear functions

\[
f_{p}^{\pm} = \frac{p \mp \sqrt{9p^2 - 8p}}{4p^2} x_1 + \frac{3 \mp \sqrt{9p^2 - 8p}}{8p},
\]
are positive on $\Delta_{p,k}$ and $(\Delta_{p,k}, L_{p,k}, f_{p,k}^\pm)$ satisfy the conditions (a) and (b) in (4.9).

(iii) For $k = 1, 2, 3, 4$ and $0 < p < r_k$, let

$$a_{p,k}^\pm = \pm \frac{\sqrt{F_k(p) + 2(p-1)k - p(p-2)}}{2((2(p-1)(p-2)k - p^3))},$$

$$b_{p,k}^\pm = \pm \frac{\sqrt{F_k(p)}}{k(2(p-1)(p-2)k - p^3)},$$

$$c_{p,k}^\pm = \frac{1}{4}(1 + (p-2)kb_{p,k}^\pm - 2pa_{p,k}^\pm).$$

and consider the two affine linear functions (6.3)

$$f_{p,k}^\pm := a_{p,k}^\pm x_1 + b_{p,k}^\pm x_2 + c_{p,k}^\pm.$$

The $f_{p,k}^\pm$ defined are positive on $\Delta_{p,k}$ and satisfy the condition (a).

**Remark 6.2.** In [20] the authors showed that the families (6.1) and (6.2) of affine linear functions satisfying condition (a) correspond to the Killing potentials of the cKEM metrics constructed in [31] Theorem D] and [31] Theorem B] respectively. Combined with Theorem 4.6 above, it follows that these families also satisfy the condition (b).

In view of Theorem 6.1 and Remark 6.2, the following question arises:

**Question 1** ([20]). **Does the affine linear function given by** (6.3) **in Theorem 6.1 define a Killing potential for a toric cKEM metric?**

In the case of $f_{p,1}^\pm$ ($k = 1$ in (6.3)), numerical evidence towards a positive answer appears in [21].

In view of Theorem 4.7 Question 1 reduces to verifying whether or not $(\Delta_{p,k}, L_{p,k}, f)$ is $(f, 4)$-K-stable, i.e. whether or not the condition (b) holds true for $f$ given by (6.3). A conclusive answer follows from the following

**Proposition 6.3.** Let $(\Delta, L, f)$ be a triple described by (i), (ii) or (iii) of Theorem 6.1. Then $(\Delta, L, f)$ verifies the condition (b) in (4.9), i.e. $(\Delta, L, f)$ is $(f, 4)$-K-stable.

Combining Proposition 6.3 and Theorem 4.6 we obtain

**Corollary 6.4.** For $f$ given by (6.3), there exist a conformally Einstein-Maxwell, toric Kähler metric on $\mathbb{F}_k$ compatible in $\Omega_p$. Furthermore, in this case the Kähler metric is regular ambitoric of hyperbolic type.

**Remark 6.5.** For the last claim of Corollary 6.4 see the proof of [9] Theorem 5] and [6] Sec. 5.4].

In order to prove Proposition 6.3, we shall use Corollary 5.12 which states that we need to show $(\Delta, L)$ is (relatively) K-stable.

Now, recall the following definition introduced in [33].

**Definition 6.6.** Let $\Delta$ be a quadrilateral with vertices $s^1, \ldots, s^4$, such that $s^1$ is not consecutive to $s^3$. We say that a function $f$ is equipoised on $\Delta$ if

$$\sum_{i=1}^{4} (-1)^i f(s^i) = 0.$$

A labelled polytope $(\Delta, L)$ is called equipoised if its extremal affine function $\zeta_{(\Delta, L)}$, introduced by (4.4), is equipoised on $\Delta$. 
**Theorem 6.7** ([33]). If $(\Delta, L)$ is an equipoised labelled compact convex quadrilateral then it is K-stable and the Abreu equation ([4]) admits a solution $u \in S(\Delta, L)$. Furthermore, the extremal Kähler metric corresponding to $u$ is either a product, or Calabi-type or an orthotoric metric.

**Proof.** For the sake of a self-contained presentation we sketch the proof. Following [33], we recall that given $(\Delta, L)$ and $g = g_u$ defined by $u \in S(\Delta, L)$, we say that

- $g = g_u$ is of **product type** if $\Delta^0$ admits product coordinates $\xi, \eta$ such that on $M^0 = \Delta^0 \times \mathbb{T}^2$ we have

\[
g|_{\Delta^0} = \frac{d\xi^2}{A(\xi)} + \frac{d\eta^2}{B(\eta)} + A(\xi)dt_1^2 + B(\eta)dt_2^2.
\]

(6.4)

In this case, the momentum coordinates $x = (x_1, x_2)$ are given by $x_1 = \xi$, $x_2 = \eta$ and we can assume $\text{Im} \Delta^0 \xi = (a_1, a_2)$ and $\text{Im} \Delta^0 \eta = (\beta_1, \beta_2)$ with $0 < \beta_1 < \beta_2 < a_1 < a_2$, and $A \in C^\infty([a_1, a_2])$ and $B \in C^\infty([\beta_1, \beta_2])$ are positive on $(a_1, a_2)$ and $(\beta_1, \beta_2)$, respectively, satisfying the first order boundary conditions

\[
\begin{align*}
A(\alpha_i) &= 0 = B(\beta_i), \\
A'(\alpha_1) &= r_{a_1}, A'(\alpha_2) &= -r_{a_2}, \\
B'(\beta_1) &= r_{\beta_1}, B'(\beta_2) &= -r_{\beta_2},
\end{align*}
\]

(6.5)

with $r_{a_1} > 0, r_{\beta_i} > 0$ for $i = 1, 2$ prescribed by the labelling $L$.

- $g = g_u$ is of **Calabi-type** if $\Delta^0$ admits Calabi coordinates $\bar{\xi}, \bar{\eta}$ such that on $M^0 = \Delta^0 \times \mathbb{T}^2$ we have

\[
g|_{\Delta^0} = \bar{\xi} \frac{d\xi^2}{A(\xi)} + \bar{\eta} \frac{d\eta^2}{B(\eta)} + A(\bar{\xi})dt_1^2 + B(\bar{\eta})dt_2^2.
\]

(6.6)

In this case, the momentum coordinates $x = (x_1, x_2)$ are given by $x_1 = \bar{\xi}$, $x_2 = \bar{\eta} \xi$ and we can assume $\text{Im} \Delta^0 \bar{\xi} = (a_1, a_2)$ and $\text{Im} \Delta^0 \bar{\eta} = (\beta_1, \beta_2)$ with $0 < \beta_1 < \beta_2 < a_1 < a_2$, $A \in C^\infty([a_1, a_2])$ and $B \in C^\infty([\beta_1, \beta_2])$ are positive on $(a_1, a_2)$ and $(\beta_1, \beta_2)$, respectively, satisfying the first order boundary conditions (6.5) at $a_1, a_2$ and $\beta_1, \beta_2$ (see [33] Proposition 4.4).

- $g = g_u$ is **orthotoric** if $\Delta^0$ admits orthotoric coordinates $\bar{\xi}, \bar{\eta}$ such that on $M^0 = \Delta^0 \times \mathbb{T}^2$ we have

\[
g|_{\Delta^0} = \frac{(\bar{\xi} - \eta)}{\bar{\xi} - \eta} d\xi^2 + \frac{(\bar{\xi} - \eta)}{\bar{\xi} - \eta} d\eta^2 + \frac{A(\bar{\xi})}{\bar{\xi} - \eta} dt_1^2 + B(\bar{\eta})dt_2^2.
\]

(6.7)

In this case, the momentum coordinates $x = (x_1, x_2)$ are given by $x_1 = \bar{\xi} + \eta x_2 = \bar{\xi} \eta$ and we can assume $\text{Im} \Delta^0 \bar{\xi} = (a_1, a_2)$ and $\text{Im} \Delta^0 \bar{\eta} = (\beta_1, \beta_2)$ with $0 < \beta_1 < \beta_2 < a_1 < a_2$, $A \in C^\infty([a_1, a_2])$ and $B \in C^\infty([\beta_1, \beta_2])$ are positive on $(a_1, a_2)$ and $(\beta_1, \beta_2)$, respectively, satisfying the first order boundary conditions (6.5) at $a_1, a_2$ and $\beta_1, \beta_2$ (see [33] Proposition 3.1).

We first notice that in [5], the authors show that for the metrics above to be extremal, the functions $A(\bar{\xi})$ and $B(\bar{\eta})$ must be polynomials of degree $\leq 4$ satisfying certain linear relations between their coefficients. We refer to pairs of polynomials $(A(\bar{\xi}), B(\bar{\eta}))$ satisfying these relations an extremal pair $(A, B)$. [33] Theorem 1.1]
then states that if \((\Delta, L)\) is an equipoised quadrilateral, one can associate to \((\Delta, L)\) real numbers \(0 < \beta_1 < \beta_2 < \alpha_1 < \alpha_2\) and an extremal pair \((A, B)\), verifying the first order boundary conditions (6.5), such that they define an extremal Kähler metric in \(S(\Delta, L)\), should they be positive on \((a_1, a_2)\) and \((\beta_1, \beta_2)\), respectively. Also, it is shown in [33, Theorem 1.1] that \((\Delta, L)\) is \(K\)-stable if and only the extremal pair \((A, B)\) is positive on their respective intervals of definition. We now argue that \(K\)-stability (i.e. positivity of \(A\) and \(B\)) follows automatically from the equipoised condition.

By [33], if \((\Delta, L)\) is equipoised, then the solution of the Abreu equation (4.4) (if it exists) must be given by one of the three types described above, according to whether \((\Delta, L)\) is a equipoised parallelogram, trapezoid which is not a parallelogram, or a quadrilateral which is not a trapezoid, respectively. Furthermore, it is observed [33] that equipoised parallelogram are always \(K\)-stable and admit extremal Kähler metrics of product type. This follows from the following observation: according to [33, Proposition 4.6], a metric of Calabi-type (6.6) is extremal if and only if \(A(\xi) = \sum_{i=0}^{4} a_i \xi^{4-i}\) has degree at most 4, \(B(\eta)\) has degree 2 and

\[
B''(\eta) = -2a_2 = -A''(0).
\]

We notice that the boundary conditions (6.5) impose that \(B(\eta)\) is positive on \((\beta_0, \beta_1)\) which in turn yields \(A''(0) = 2a_2 > 0\). If we suppose that \(A\) is not positive in \((\alpha_1, \alpha_2)\) this would imply that the two roots of \(A''(\xi)\) belong to the interval \((\alpha_1, \alpha_2)\) due to the boundary conditions (6.5). However, since \(0 < \alpha_1 < \alpha_2\), for \(A''(0)\) to be positive \(A''(\xi)\) would have to admit a third root in the interval \((0, \alpha_1)\) which is not possible since \(\text{deg} A'' = 2\). Then we conclude that \(A(\xi)\) must be positive on \((\alpha_1, \alpha_2)\).

The \(K\)-stability of an equipoised labelled quadrilateral which is neither a parallelogram nor a trapezoid was latter observed in [6, Example 1]. This follows from the fact that in this case, \((A, B)\) is an extremal pair if and only if \(\text{deg}(A + B) \leq 1\) [6, Proposition 3]. Then, between any maximum of \(A\) on \((a_1, a_2)\) and of \(B\) on \((\beta_1, \beta_2)\), the quadratic \(A'' = -B''\) has a unique root; the boundary conditions thus force again \(A\) and \(B\) to be positive on \((a_1, a_2)\) and \((\beta_1, \beta_2)\), respectively.

\[\square\]

**Proof of Proposition 6.3** The proof is similar in all cases, so we present only the case \(f_{p,k}^\pm\) below.

We first notice that as \(\zeta(\Lambda_{p,k}, \mathcal{L}_{f_{p,k}^\pm}) = c\) by Theorem 6.1, we have \(\zeta(\Lambda_{p,k}, \mathcal{L}) = \frac{1}{f_{p,k}^\pm}\) by Proposition 5.11. It follows that \((\tilde{\Lambda}_{p,k}, \tilde{\mathcal{L}})\) is equipoised if and only if

\[
\sum_{i=1}^{4} (-1)^i \frac{1}{f_{p,k}^\pm(s^i)} = 0.
\]
Now, using Theorem 6.7, the proof is completed by a straightforward computation verifying (6.8). For the reader’s convenience we present here the case \( k = 1 \), we also drop the index \( k \) to simplify the notation.

\[
\sum_{i=1}^{4} (-1)^{i} f_p(s^i) = \frac{1}{p^2 + 2p - 2 + \sqrt{F(p)}} + \frac{1}{-p^3 + 3p^2 - 4p + 2 - (1-p) \sqrt{F(p)}} + \frac{1}{p^3 - 3p^2 + 4p - 2 - (1-p) \sqrt{F(p)}} + \frac{1}{-p^2 - 2p + 2 + \sqrt{F(p)}}
\]

If we write \( U = p^2 + 2p - 2, W = \sqrt{F(p)} \) and \( V = p^3 - 3p^2 + 4p - 2 \), the RHS of (6.9) is given by

\[
\frac{1}{U + W} + \frac{1}{-V - (1-p)W} + \frac{1}{V - (1-p)W} + \frac{1}{-U + W} = \frac{-2W [V^2 + (1-p)(pW^2 - U^2)]}{(U + W)(-V - (1-p)W)(V - (1-p)W)(-U + W)}
\]

Now replacing \( U, V, W \), and \( F(x) = x^4 - 4x^3 + 16x^2 - 16x + 4 \), we can check that \( V^2 + (1-p)(pW^2 - U^2) = 0 \) in (6.10). We have performed similar verifications for any \( k \). \( \square \)

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