Geometry as an object of experience: 
Kant and the missed debate between 
Poincaré and Einstein

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Abstract

Poincaré held the view that geometry is a convention and cannot 
be tested experimentally. This position was apparently refuted by 
the general theory of relativity and the successful confirmation of its 
predictions; unfortunately, Poincaré did not live to defend his thesis. 
In this paper, I argue that: 1) Contrary to what many authors have 
claimed, non-euclidean geometries do not rule out Kant’s thesis that 
space is a form of intuition given \textit{a priori}; on the contrary, Euclidean 
geometry is the condition for the possibility of any more general geo-
metry. 2) The conception of space-time as a Riemannian manifold 
is an extremely ingenious way to describe the gravitational field, but, 
as shown by Utiyama in 1956, general relativity is actually the gauge 
theory associated to the Lorentz group. Utiyama’s approach does not 
rely on the assumption that space-time is curved, though the equa-
tions of the gauge theory are identical to those of general relativity. 
Thus, following Poincaré, it can be claimed that it is only a mat-
ter of convention to describe the gravitational field as a Riemannian 
manifold or as a gauge field in Euclidean space.
Many scientists and philosophers have argued that the possibility of non-Euclidean geometries contradicts Kant’s thesis that geometric axioms are synthetic judgments \textit{a priori} (Kant 1929, 1986). Accordingly, it became a common place to assert that the general theory of relativity, by revealing the non-Euclidean nature of physical space, refuted Kant’s doctrine on the transcendental nature of space. However, Poincaré (1952) sustained that geometry is a convention and cannot be the object of any experience, but his point of view fall into oblivion due to the commonly held belief that the Riemannian nature of space-time is testable.

The plan of the present paper is as follows: In Section 1, a brief historical introduction to the problem is given and then it is shown that the existence of Riemannian geometry not only does not refute, but confirms Kant’s doctrine. In Section 2, the geometric convention-alism of Poincaré is reviewed. Section 3 is devoted to the alternative formulation of general relativity, as a gauge theory, given by Utiyama (1956). Finally, Section 4 compares the ideas of Poincaré and Einstein on the physical nature of space following a text written by the latter in 1949.

1 How can Riemannian geometry be possible?

The argument that non-Euclidean geometries contradict Kant’s doctrine on the nature of space apparently goes back to Helmholtz (1995) and was retaken by several philosophers of science such as Reichenbach (1958) who devoted much work to this subject.

In a essay written in 1870, Helmholtz (1995) argued that the axioms of geometry are not \textit{a priori} synthetic judgments (in the sense given by Kant), since they can be subjected to experiments. Given that Euclidian geometry is not the only possible geometry, as was believed in Kant’s time, it should be possible to determine by means of measurements whether, for instance, the sum of the three angles of a triangle is 180 degrees or whether two straight parallel lines always keep the same distance among them. If it were not the case, then
it would have been demonstrated experimentally that space is not Euclidean. Thus the possibility of verifying the axioms of geometry would prove that they are empirical and not given \textit{a priori}.

Helmholtz developed his own version of a non-Euclidean geometry on the basis of what he believed to be the fundamental condition for all geometries: “the possibility of figures moving without change of form or size”; without this possibility, it would be impossible to define what a measurement is. According to Helmholtz (1995, p. 244): “the axioms of geometry are not concerned with space-relations only but also at the same time with the mechanical deportment of solidest bodies in motion.” Nevertheless, he was aware that a strict Kantian might argue that the rigidity of bodies is an \textit{a priori} property, but “then we should have to maintain that the axioms of geometry are not synthetic propositions... they would merely define what qualities and deportment a body must have to be recognized as rigid”.

At this point, it is worth noticing that Helmholtz’s formulation of geometry is a rudimentary version of what was later developed as the theory of Lie groups, which I will mention in Section 3. As for the transport of rigid bodies, it is well known nowadays that rigid motion cannot be defined in the framework of the theory of relativity: since there is no absolute simultaneity of events, it is impossible to move all parts of a material body in a coordinated and simultaneous way. What is defined as the length of a body depends on the reference frame from where it is observed. Thus, it is meaningless to invoke the rigidity of bodies as the basis of a geometry that pretend to describe the real world; it is only in the mathematical realm that the rigid displacement of a figure can be defined in terms of what mathematicians call a \textit{congruence}.

Arguments similar to those of Helmholtz were given by Reichenbach (1958) in his intent to refute Kant’s doctrine on the nature of space and time. Essentially, the argument boils down to the following: Kant assumed that the axioms of geometry are given \textit{a priori} and he only had classical geometry in mind, Einstein demonstrated that space is not Euclidean and that this could be verified empirically, \textit{ergo} Kant was wrong.

However, Kant did not state that space must be Euclidean; instead, he argued that it is a pure form of intuition. As such, space has no \textit{physical} reality of its own, and therefore it is meaningless to ascribe physical properties to it. Actually, Kant never mentioned Euclid directly in his work, but he did refer many times to the physics
of Newton, which is based on classical geometry. Kant had in mind the axioms of this geometry which is a most powerful tool of Newtonian mechanics. Actually, he did not even exclude the possibility of other geometries, as can be seen in his early speculations on the dimensionality of space (Kant 1986).

The important point missed by Reichenbach is that Riemannian geometry is necessarily based on Euclidean geometry. More precisely, a Riemannian space must be considered as locally Euclidean in order to be able to define basic concepts such as distance and parallel transport; this is achieved by defining a flat tangent space at every point, and then extending all properties of this flat space to the globally curved space (see, e.g., Eisenhart, 1959). To begin with, the structure of a Riemannian space is given by its metric tensor $g_{\mu\nu}$ from which the (differential) length is defined as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$; but this is nothing less than a generalization of the usual Pythagoras theorem in Euclidean space. As for the fundamental concept of parallel transport, it is taken directly from its analogue in Euclidean space: it refers to the transport of abstract (not material, as Helmholtz believed) figures in such a space. Thus Riemann’s geometry cannot be free of synthetic a priori propositions because it is entirely based upon concepts such as length and congruence taken from Euclid. We may conclude that Euclid’s geometry is the condition of possibility for a more general geometry, such as Riemann’s, simply because it is the natural geometry adapted to our understanding; Kant would say that it is our form of grasping space intuitively. The possibility of constructing abstract spaces does not refute Kant’s thesis; on the contrary, it reinforces it.

But then, can the axioms of geometry be verified experimentally? Let us see what Poincaré had to say on that matter.

2 Can geometry be an object of experience?

This is the fundamental question put forward by Henri Poincaré, to which his answer was that something as a “geometric experiment” cannot be performed (Poincaré, 1952):

Think of a material circle, measure its radius and circumference, and see if the ratio of the two lengths is equal to $\pi$. What have we done? We have made an experiment on
the properties of the matter with which this *roundness* has been realized, and of which the measure we used is made.

Something similar would happen with astronomical observations. For instance, Lobachevsky had suggested that it should be possible to determine the curvature of the space we live in by measuring the parallaxes of distant stars (Jammer, 1993, p. 149). But then Poincaré pointed out:

> What we call a straight line in astronomy is simply the path of a ray of light. If, therefore, we were to discover negative parallaxes, or to prove that all parallaxes are higher than a certain limit, we should have a choice between two conclusions: we could give up Euclidean geometry, or modify the laws of optics, and suppose that light is not rigorously propagated in a straight line.

He then concluded: “It is needless to add that every one would look upon this [second] solution as the more advantageous”. Poincaré wrote his essay in 1898, when the utility of non-Euclidean geometries in physics was still unknown. Evidently he was mistaken on this point: fifteen years later Einstein showed that the geometry of Riemann is more advantageous for the description of gravity.

Now the question is: why does Euclid’s geometry seem so natural? Kant assumed that the basic axioms of this geometry (the only geometry known in his time) were given *a priori*, while Poincaré saw the answer in the natural selection, thanks to which “our spirit adapted to the conditions of the outer world, adopted the most advantageous geometry for the species; in other words, the most comfortable”. There is no doubt that this geometry is most comfortable since it was accepted as the natural one over two millennia, but Poincaré never explained why this was so. A Kantian, however, would argue that it is most comfortable because it is given *a priori*. In any case, Poincaré’s conclusion is clear: “geometry is not true, it is advantageous”. Accordingly, it is nowadays evident that Euclidean geometry is most comfortable to use, except for the description of the gravitational field, for which the Riemannian geometry is more advantageous. But is Riemann’s geometry indispensable for this task? It is not, as I will show next.

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1 Underlined by H. P.
3 General relativity as a gauge theory

With the general theory of relativity, Einstein successfully unified physics with geometry. Space, combined with time in a single space of four dimensions, was no longer a simple scenario for natural phenomena, but acquired a fully dynamical role. Space-time, in Einstein’s theory, is a curved space and its curvature is equivalent to a gravitational field. Thus non-Euclidean geometry found an important application in physics. Furthermore, general relativity made definite predictions, and it therefore appeared that the non-Euclidean nature of space could be an empirical fact liable to be verified.

What would Poincaré have said on this point? Unfortunately, his untimely death did not let him hear about the new theory of Einstein. In order to defend his own point of view, he would surely have insisted that the general theory of relativity simply implies that light rays do not propagate along “straight lines” but in a more complicated way. From the empirical point of view, it is perfectly equivalent to saying that light moves along “null geodesics” in a curved space, as postulated in the theory of relativity, or along a curved path in Euclidean space: there is no way to distinguish by experiments between the two possibilities, and therefore general relativity does not contradict the conventionalism of geometry. In fact, every prediction of general relativity can also be interpreted as a phenomenon in a Euclidean space, since all the classical tests can be ascribed to alterations of Newtonian physics. The perihelion shift of Mercury could be due to a correction in Newton’s law of gravity; the deflection of light rays may be a real deflection produced by gravity; and the gravitational red-shift of light is due to an actual loss of energy. Even the formation of black holes does not rely on the geometry of space: it is perfectly consistent to assume that light is strongly bent around a massive compact object, to the extent that it cannot escape from it.

As for cosmology, the expansion of the Universe is predicted by the dynamical equations of Friedmann that follow from general relativity. However, it is known from the work of McCrea and Milne (1934) that these same equations can also be obtained within the framework of Newtonian mechanics; it is only a matter of interpreting the Newtonian variables in terms of their relativistic counterparts (see Bondi, 1960, Chap. IX, for a detailed discussion). Moreover, it is a noteworthy fact that present day observations are compatible with a Universe that is spatially flat on the average; that is to say, light rays do not
diverge or converge, as would be the case in a space with negative or positive curvature, but they move in “straight” paths (actually, this supports Poincaré’s thesis even further: there is no “sufficient reason”, in a Leibnizian sense, for light rays to either diverge or converge in a perfectly homogeneous and isotropic universe).

In order to do justice to Poincaré, let us imagine for a moment what would have been of physics if Einstein had not lived. Without Einstein, special relativity would have been formulated anyhow, since all the basic ingredients of this theory were known at the beginning of the last century; it would have taken only an ingenious mind to paste all the pieces together. However, it is undeniable that general relativity is the sole achievement of Einstein, since it is a totally original theory with no more antecedents than the classic physics of Newton and the geometry developed in the XIX century.

What would have been, then, of general relativity without Einstein? The answer can be found in the wider context of what is presently known as a gauge theory, a concept that appeared in the fifties and turned out to be extremely important in theoretical physics. With the formalism of gauge theory, it is possible to deduce the equations that describe an interaction on the sole basis of their symmetries; in other words, symmetry determines dynamical laws! Gauge theories are based on two fundamental mathematical concepts developed throughout the XIX century: abstract mathematical spaces and the theory of continuous groups of transformations. Mathematicians discovered that it is possible to define abstract spaces without resorting to any coordinate system, and that these more general spaces have a much more complex and interesting structures than the usual one of three dimensions. Curved spaces of many dimensions, such as those of Riemann, are only particular cases. In quantum mechanics, for example, the state of an atomic system is described as a vector in Hilbert spaces: a Hilbert space can have any number of dimensions, even infinite, and the “coordinates” are complex numbers. On the other hand, the theory of continuous groups of transformations was developed mainly by Sophus Lie. Continuous transformations in abstract spaces generalize the concepts of rotations and translations of material bodies according to rules of motion in usual space. The crucial point is that a given generalized motion corresponds to a particular symmetry invariance and can be described by a non commutative algebra, forming a Lie group, which can be classified in well defined categories.

The theory of Lie groups has crucial applications in physics, partic-
ularly in the description of fundamental interactions between atomic particles. The basic idea is that the various states of an atomic system are described by vectors in an abstract space, and that these vectors can be transformed without altering the physics of the system, just as the translation of a solid body does not alter its properties. The essential point is that these transformations can be interpreted as symmetry properties and the mathematical formalism permits one to deduce all the equations that describe the interactions. Such a formalism was first used by C. N. Yang and R. L. Mills (1954) to obtain the equations that describe nuclear interactions.

The idea put forward by Yang and Mills was generalized a year later by Ryoyu Utiyama, in an article that had a great impact in theoretical physics. Utiyama (1955) showed that if the symmetry properties of a fundamental interaction are known in a given point of an abstract space, then the dynamical equations of that interaction, valid everywhere in the same space, could be deduced precisely. Utiyama’s formalism is based on the theory of continuous Lie groups. Essentially, if the group of transformations that does not alter the form of the interaction is known \textit{locally}, then it is possible to deduce the equations that are valid \textit{globally}. More specifically, Utiyama stated the problem in the following form:

Let us consider a system of fields $Q^A(x)$ which is invariant under some transformation group $G$ depending on parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$. Suppose that the aforementioned parameter-group is replaced by a wider group $G'$ derived by replacing the parameters $\epsilon'$ by a set of arbitrary functions $\epsilon'(x)$, and that the system considered is invariant under the wider group $G'$.

Under these conditions, he showed that a new field $A(x)$ can be introduced and the new Lagrangian $L'(Q, A)$ can be deduced from the original one $L(Q)$, together with the field equations. The idea is that, given a Lie group with operators $\hat{T}_a$ that satisfy the commutation relations $[\hat{T}_a, \hat{T}_b] = f_{abc} \hat{T}_c$, the derivative of a field $Q^A$ must be generalized to $\partial_\mu Q^A \rightarrow \partial_\mu Q^A - T^A_{\alpha B} Q^B A_\alpha^\mu$, with a new field $A_\alpha^\mu$ that in turn defines a gauge field

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - \frac{1}{2} f_{abc} (A_b^B A_c^\mu - A_b^\mu A_c^B),$$

(3.1)
in terms of the structure constants of the Lie group, $f_{abc}$. 

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Gauge field theory has found its paramount application in the
Standard Model of elementary particles, a model that has been most
successful in describing nuclear and electromagnetic interactions. Ac-
tually, the model is the gauge theory associated to a combination of
three simple groups of transformations in an abstract space: SU(3),
SU(2) and U(1).

As for the general theory of relativity, Utiyama showed in his ar-
ticle of 1956 that it is actually the gauge theory associated to the
group of Lorentz transformations in Minkowski space. In fact, it is
clearly seen from the formulation of Utiyama that there is an equiv-
alence between the electromagnetic and the gravitational fields: the
electromagnetic field is a spin-1 field described by a tensor of rank 2,
and the gravitation field is a spin-2 field described by a tensor of rank
4. The gravitational field tensor, by a lucky coincidence, has precisely
the same algebraic structure as the Riemann tensor that characterizes
a Riemannian space of four dimensions.

Let us paraphrase Utiyama’s formulation in a more modern no-
tation using the language of differential geometry. The form ulation
runs along the following lines. Given the Minkowski metric $\eta_{ab} =
(-1,1,1,1)$ in flat space with Cartesian coordinates, define a tetrad
(also known as Cartan repère mobile or vierbein) $e^a_\alpha$ such that, at a
given point, the differential line element is

$$ds^2 = \eta_{ab} e^a e^b = g_{\alpha\beta} dx^\alpha dx^\beta,$$

where $g_{\alpha\beta}$ is the metric tensor and $e^a = e^a_\alpha dx^\alpha$ is a one-form (see, e.g.,
Flanders 1963). Under an infinitesimal Lorentz group transformation,
the tetrad $e^a$ transforms as $e^a \rightarrow e^a + \epsilon^{ab} e^b$, where $\epsilon_{ab} = -\epsilon_{ba}$. Next,
let $Q^A$ be a tensor with $A$ being an index referring to a particular in-
reducible representation of the Lorentz group. Then, for a Lagrangian
$L(Q^A, \partial_{\mu} Q^A)$ to be invariant under a “generalized Lorentz transfor-
mation” with $e_{ab}(x)$, the partial derivatives must be substituted by

$$\partial_{\mu} Q^A \rightarrow \partial_{\mu} Q^A - T_{ab} A^B Q^B A^a_{\mu},$$

where $T_{ab}$ are the generators of the Lorentz group,

$$[T_{ab}, T_{cd}] = \frac{1}{2} f_{abcd} T_{mn}.$$  

\footnote{Utiyama used the notation $h^a_\alpha$ for $e^a_\alpha$, but he did not recognize this as a tetrad.}
Then, according to (3.1), a new field is defined as

\[ R^{ab\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu - \frac{1}{2} f_{kl}^{~~ab} \, mn (A_{kl}^{\mu} A_{mn}^{\nu} - A_{mn}^{\mu} A_{kl}^{\nu}). \] (3.5)

Thus the field is described by the tensor \( R^{ab\mu\nu} \) from which the scalar \( R = e^a_{\alpha} e^b_{\beta} R^{ab\mu\nu} \) can be constructed. The next step is to take \( \det (e^a_{\alpha}) R \) as the Lagrangian of the field, add to it the Lagrangian of the matter-source, and, as explained in any text book of general relativity, obtain the usual Einstein equations by varying with respect to \( g_{\alpha\beta} \).

Now, in the language of differential geometry, if \( e^a_{\alpha} \) is identified with a tetrad, then \( A^{ab}_\alpha \) are the Ricci rotation coefficients (related to the Christoffel symbols) and \( R^{\alpha\beta\mu\nu} = e^\alpha_{a} e^\beta_{b} R^{ab\mu\nu} \) is the Riemann tensor (see, e.g., Flanders 1963).

Just for the sake of comparison, recall that the electromagnetic field \( F_{\alpha\beta} \) can be deduced as the gauge field of the Abelian group \( U(1) \):

\[ F_{\alpha\beta} = \partial_\alpha A_{\beta} - \partial_\beta A_{\alpha}, \]

where the electromagnetic potential \( A_{\alpha} \) plays the same role as the Ricci rotation coefficients for the gravitational field. The Lagrangian is taken as \( L = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \), which is quadratic in the field; unlike the case with the Riemann tensor, it is not possible to construct a scalar from \( F_{\alpha\beta} \) that is linear on the field; this is an important difference between the two fields.

As for the equations of motion for a test particle, these can be obtained, in the case of an electromagnetic field, from the action \( \int A_{\mu} U^{\mu} d\tau \) where \( U^{\mu} \) is the four-velocity of the particle and \( \tau \) its proper time: the Lorentz force equation follows. For the gravitational field, the action is of the form \( \int \sqrt{g_{\mu\nu}} U^{\mu} U^{\nu} d\tau \), and its variation gives rise to the well known geodesic equations if \( g_{\mu\nu} \) is interpreted as the metric of a Riemannian space.

Looking the matter from a modern perspective, it can be ascertained now that Einstein used an extremely ingenious logical reasoning in order to develop a theory based on the formal analogy between the equations of Riemannian geometry and those of a relativistic gravitational field. Thus Einstein was four decades ahead of his time in formulating what would be known as a gauge theory. If Einstein had not had this insight, it would certainly correspond to Utiyama the merit of formulating a fully relativistic theory of gravity.

The formulation of a gravitational theory by Utiyama does not rely on a curved space, but rather on an abstract space that has the same mathematical structure as a Riemannian space of four dimensions. In this way he arrived to a basic set of equations that are iden-
tical to those obtained by Einstein. Thus, Poincaré would be perfectly right in claiming that it is a mere question of convenience whether to use one formulation or the other: there is no difference, from a purely formal point of view, to use Riemannian geometry or gauge field theory, although the elegant formulation of Einstein is more comfortable and easier to visualize because it is based on a beautiful geometric analogy.

4 An imaginary dialogue

Poincaré and Einstein met only once, during the 1911 Solvay Congress at Brussels, a year before the untimely death of the great French mathematician. Poincaré already knew about the special theory of relativity, but he was not entirely convinced of it, although he had a great esteem for his young creator. As for Einstein, who was usually reluctant to acknowledge the contributions of his predecessors, it is only on his later years that he mentioned Poincaré as he deserved.

In 1949, Paul Schilpp edited a volume dedicated to Einstein with several essays written by distinguished scientists and philosophers of science (Schilpp 1949). In the final chapter, Einstein commented each essay and, in particular, he took the opportunity to imagine a dialogue between Poincaré and Reichenbach on the geometric conventionalism that the latter author had criticized in his essay. Einstein summarized the antagonist positions in a simple and clear way: “Is a geometry — looked at from the physical point of view — verifiable (viz., falsifiable) or not? Reichenbach, together with Helmholtz, says: Yes, provided that the empirically given solid body realizes the concept of ‘distance’. Poincaré says no and consequently is condemned by Reichenbach.”

Next, Einstein imagined the following dialogue:

Poincaré: The empirically given bodies are not rigid, and consequently cannot be used for the embodiment of geometric intervals. Therefore, the theorems of geometry are not verifiable.

Reichenbach: I admit that there are no bodies which can be credited immediately adduced for the “real definition” of the interval. Nevertheless, this real definition can be achieved by taking the thermal volume-dependence, elasticity, electro- and magnetostriction, etc., into consideration. That this is really and without contradiction possible,
classical physics has surely demonstrated.

Poincaré: In gaining the real definition improved by yourself you have made use of physical laws, the formulation of which presupposes (in this case) Euclidean geometry. The verifications, of which you have spoken, refer, therefore, not merely to geometry but to the entire system of physical laws which constitute its foundation.

At the end of the imaginary conversation, Einstein manifested his agreement with Kant in so far as “there are concepts (as, for example, that of causal connection), which play a dominating role in our thinking, and which, nevertheless, cannot be deduced by means of a logical process from the empirically given.” According to Einstein, this was Kant’s most important contribution and not the belief that “Euclidean geometry is necessary to thinking and offers assured (i.e., not dependent upon sensory experience) knowledge concerning the objects of ‘external’ perception.”

It is thus evident that Einstein was still convinced of the non-Euclidean nature of space, but he was unable to give a convincing argument against Poincaré and only paraphrased him. There is no doubt that Einstein believed that the curvature of space must be an object of experience; nevertheless, it can be seen from the above dialogue he imagined, that he was quite aware that the problem is considerably more difficult than what Helmholtz or Reichenbach thought. Surely, his conviction was sustained on the fact that, during his lifetime, the only known form of unifying gravity and relativity was using the mathematical tool of Riemannian geometry. The development of gauge theories and, particularly, the independent formulation of Utiyama of the same relativistic theory of gravity (which Einstein did not live to see), made it evident that Riemannian geometry is very advantageous for the theory describing this fundamental interaction, but, as Poincaré would have said, it is a very convenient convention, but a convention anyway.

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