NEW PERIODIC ORBITS IN THE PLANAR EQUAL-MASS THREE-BODY PROBLEM

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Abstract. It is known that there exist two sets of nontrivial periodic orbits in the planar equal-mass three-body problem: retrograde orbit and prograde orbit. By introducing topological constraints to a two-point free boundary value problem, we show that there exists a new set of periodic orbits for a small interval of rotation angle θ.

1. Introduction. The planar three-body problem studies the motion of three masses \( m_1, m_2, m_3 \) moving in a fixed plane satisfying Newton’s law of gravitation:

\[
m_i \ddot{q}_i = \frac{\partial}{\partial q_i} U(q), \quad i = 1, 2, 3,
\]

where \( q_i = (q_{ix}, q_{iy}) \), a row vector in the \( xy \) plane, is the position of \( m_i, q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \) is a \( 3 \times 2 \) matrix and the kinetic energy \( K \) and the potential energy \( U \) are given as follows:

\[
K = K(\dot{q}) = \frac{1}{2} \sum_{i=1}^{3} m_i |\dot{q}_i|^2,
\]

\[
U = U(q) = \frac{m_1 m_2}{|q_1 - q_2|} + \frac{m_2 m_3}{|q_2 - q_3|} + \frac{m_1 m_3}{|q_1 - q_3|}.
\]

It is known that in the planar equal-mass three-body problem, there exist two families of orbits [1, 7]: retrograde orbit and prograde orbit. “A retrograde orbit of the planar three-body problem is a relative periodic solution of the Newtonian equation (1) with two adjacent masses revolving around each other in one direction while their mass center revolves around the third mass in the other direction. The orbit is said to be prograde if both revolutions follow the same direction.” [3] Actually, Chen [2, 3] introduced a level estimate argument and showed the existence of such two families of orbits for a general mass set \( M = [m_1, m_2, m_3] = [1, 1, m] \).

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The purpose of this paper is to show the existence of a new set of orbits in the planar equal-mass three-body problem. A sample picture of these orbits is given in Fig. 2. We first introduce some notations before stating our main results.

Let the masses be \( m_1 = m_2 = m_3 = 1 \). We consider a two-point free boundary value problem with topological constraints. Let

\[
Q_s = \begin{bmatrix}
    a_1 - a_2 & 0 \\
    -2a_1 - a_2 & 0 \\
    a_1 + 2a_2 & 0
\end{bmatrix},
Q_e = \begin{bmatrix}
    2b_2 & 0 \\
    -b_2 & b_1 \\
    -b_2 & -b_1
\end{bmatrix} R(\theta),
\]

where \( a_1 \geq 0, a_2 \geq 0, b_1 \geq 0, b_2 \in \mathbb{R} \), and \( R(\theta) = \begin{bmatrix}
    \cos(\theta) & \sin(\theta) \\
    -\sin(\theta) & \cos(\theta)
\end{bmatrix} \). The two configuration sets are defined as follows:

\[
Q_S = \left\{ Q_s \mid a_1 \geq 0, a_2 \geq 0 \right\}, \quad Q_E = \left\{ Q_e \mid b_1 \geq 0, b_2 \in \mathbb{R} \right\},
\]

where \( Q_s \) and \( Q_e \) are defined by the matrices in (2). Geometrically, the configuration \( Q_S \) is on a horizontal line with order \( q_2 x(0) \leq q_1 x(0) \leq q_3 x(0) \). The configuration \( Q_E \) is an isosceles triangle with \( q_1 \) as its vertex, and the symmetry axis of each \( Q_e \) in (3) is a counterclockwise \( \theta \) rotation of the \( x \)-axis. Furthermore, in the configuration \( Q_e \cdot R(-\theta) \), body 2 is always above the \( x \)-axis. Pictures of the two configurations \( Q_S \) and \( Q_E \) are shown in Fig. 1 respectively.

**Figure 1.** The configurations \( Q_S \) and \( Q_E \) are shown, where blue dots represent \( q_1 \), red dots represent \( q_2 \) and black dots represent \( q_3 \). In \( Q_S \), three masses are on the \( x \)-axis with an order \( q_2 x \leq q_1 x \leq q_3 x \). In \( Q_E \), three masses form an isosceles triangle, whose symmetry axis is a counterclockwise \( \theta \) rotation of the \( x \)-axis. \( q_1 \) is on the symmetry axis and \( q_2 \) is above the symmetry axis in \( Q_E \).

**Remark 1.** In the equal-mass three-body problem, there is a remarkable periodic orbit, figure-eight orbit, which has been shown to exist [4] by Chenciner and Montgomery in 2000. Indeed, one twelfth of the figure-eight orbit can be characterized as an action minimizer connecting two specific boundary sets: a horizontal Euler configuration set

\[
Q_{S_1} = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ -a_1 & 0 \end{bmatrix} \mid a_1 \in \mathbb{R} \right\}
\]

and an isosceles configuration set

\[
Q_{E_1} = \left\{ \begin{bmatrix} 2b_2 & 0 \\ -b_2 & b_1 \\ -b_2 & -b_1 \end{bmatrix} R(\theta) \mid b_1, b_2, \theta \in \mathbb{R} \right\}.
\]
However, the two boundaries $Q_S$ and $Q_E$, do NOT satisfy our boundary conditions: $Q_S$ and $Q_E$ in (3). Actually in the figure-eight orbit [4], the vertex of the isosceles triangle in $Q_E$ corresponds to the first body of the Euler configuration set $Q_{S_1}$, which lies either on the left or on the right end of a horizontal line in $Q_{S_1}$. While in our case, the vertex of the isosceles triangle in $Q_E$ corresponds to the second body of the collinear configuration set $Q_S$ (as in Fig. 1), which lies in the middle of a horizontal line in $Q_S$. Furthermore, the angular momentum of the figure-eight orbit is 0, but the angular momenta in our case turn out to be nonzero. Therefore, the orbits in this paper are different from the well-known figure-eight orbit.

Without loss of generality, we assume the center of mass to be at the origin. That is, $q \in \chi$, where

$$\chi = \left\{ q \in \mathbb{R}^{3 \times 2} \mid \sum_{i=1}^{3} m_i q_i = 0 \right\}. \quad (4)$$

Let $\vec{a} = (a_1, a_2, b_1, b_2)$. We define the set $\Lambda$ to be

$$\Lambda = [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times \mathbb{R}. \quad (5)$$

Given $\theta = \pi/10$ and $\vec{a} \in \Lambda$, the position matrices $Q_s$ and $Q_e$ in (2) are fixed. We set $P(Q_s, Q_e)$ to be the path set in the Soblev space $H^1([0, 1], \chi) = W^{1,2}([0, 1], \chi)$ connecting the two fixed ends $Q_s$ and $Q_e$:

$$P(Q_s, Q_e) := \left\{ q \in H^1([0, 1], \chi) \mid q(0) = Q_s, q(1) = Q_e \right\}. \quad (5)$$

It is known that there exists an action minimizer $\mathcal{P}$ connecting the two fixed ends, which satisfies

$$\mathcal{A}(\mathcal{P}) = \inf_{\{q \in P(Q_s, Q_e)\}} \mathcal{A}, \quad (6)$$

where $\mathcal{A} = \int_0^1 (K + U) \, dt$. In general, the minimizer $\mathcal{P}$ is not a part of a periodic solution. In order to find a periodic or quasi-periodic solution, we consider the following free boundary value problem:

$$\inf_{\{\vec{a} \in \Lambda\}} \inf_{\{q \in P(Q_s, Q_e)\}} \mathcal{A}. \quad (6)$$

For $\theta = \pi/10$, standard results imply the existence of an action minimizer $\mathcal{P}_0 \in H^1([0, 1], \chi)$, which minimizes $\mathcal{A}$ in (6). However, $\mathcal{P}_0$ may not be a solution of the Newtonian equation (1) since it could have collision singularities. The main difficulty in this paper is to exclude possible collisions in $\mathcal{P}_0$.

By Marchal [9] and Chenciner [5], $\mathcal{P}_0$ is collision-free in $(0, 1)$. To prove $\mathcal{P}_0$ is a solution of the Newtonian equation, we only need to exclude possible boundary collisions in $\mathcal{P}_0$. Besides that, we also need to show that $\mathcal{P}_0$ is nontrivial, which means that it does not coincide with a relative equilibrium. By applying Chen's level estimate argument [2, 3, 6] and introducing a new test path, we can show that

**Theorem 1.1.** For $\theta = \pi/10$, there exists a nontrivial and collision-free minimizing path $\mathcal{P}_0 \equiv \mathcal{P}_0(t \in [0, 1])$, which satisfies

$$\mathcal{A}(\mathcal{P}_0) = \inf_{\{\vec{a} \in \Lambda\}} \inf_{\{q \in P(Q_s, Q_e)\}} \mathcal{A}. \quad (7)$$

The minimizer $\mathcal{P}_0$ can be extended to a periodic orbit.

The proof of Theorem 1.1 can be found in Theorem 4 and Theorem 5.1.
Remark 2. The periodic orbit $q(t) = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}(t)$ extended by $\mathcal{P}_0$ satisfies

$$q(t + 2) = q(t)R(\pi/5), \quad q(2 - t) = q(t)BR(\pi/5),$$

where $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $R(\pi/5)$ is the rotation matrix defined in (2). It implies that this orbit has a symmetry group $D_{10}$.

However, $\mathcal{P}_0$ is not the action minimizer under the symmetry constraint $D_{10}$ since it is different from the retrograde orbit. In fact, $\mathcal{P}_0$ can be seen as a local action minimizer under the $D_{10}$ symmetry group constraint. A picture in Fig. 2 shows $\mathcal{P}_0$ and its periodic extension.

![Figure 2](image_url)

**Figure 2.** A picture of the periodic orbit extended by the action minimizer $\mathcal{P}_0$ with $\theta = \pi/10$. The three dots represent the starting configuration $Q_S$, and the three crosses represent the ending configuration $Q_E$ of $\mathcal{P}_0$. In the graph, the red curve is the trajectory of body 2, the blue curve is for body 1 and the black curve is for body 3.

Actually, the argument in Theorem 1.1 can be extended to the case when $\theta$ is close to $\pi/10$. Recall that the two boundary sets $Q_S$ and $Q_E$ in (2) are

$$Q_S = \left\{ Q_s \mid a_1 \geq 0, a_2 \geq 0 \right\}, \quad Q_E = \left\{ Q_e \mid b_1 \geq 0, b_2 \in \mathbb{R} \right\},$$

where

$$Q_s = \begin{bmatrix} a_1 - a_2 & 0 \\ -2a_1 - a_2 & 0 \\ a_1 + 2a_2 & 0 \end{bmatrix}, \quad Q_e = \begin{bmatrix} 2b_2 & 0 \\ -b_2 & b_1 \\ -b_2 & -b_1 \end{bmatrix} R(\theta),$$

with $R(\theta)$ defined in (2).
By the definition of $Q_S$ and $Q_E$ in (8), the action minimizer corresponding to a given $\theta$ could be a part of an Euler orbit, which is a relative equilibrium. Besides the difficulty of excluding collisions in the action minimizer, another challenge is to show that this minimizer is not a part of a relative equilibrium. By carefully choosing the test paths, the following theorem holds, while its proof is partially computer assisted and can be found in Theorem 6.1. Rough numerical simulations indicate that these solutions are dynamically unstable. It will be very interesting if one can understand their stabilities mathematically.

**Theorem 1.2.** For each $\theta \in [0.084\pi, 0.183\pi]$, there exists a nontrivial and collision-free minimizing path $P_{0,\theta} = P_{0,\theta}(t \in [0,1])$ connecting the two configuration sets $Q_S$ and $Q_E$ in (3), and it can be extended to a periodic or quasi-periodic orbit.

The paper is organized as follows. Section 2 shows the coercivity of the action functional under our topological constraints. Section 3 defines the test path for $\theta = \pi/10$, while $\theta$ represents the angle between the symmetry axis of $Q_E$ and the $x$–axis as shown in Fig. 1. In Section 4, a lower bound of the action $A(P_0)$ is given if the minimizer $P_0$ has some collision singularities. Section 5 extends the minimizer $P_0$ to a periodic orbit for $\theta = \pi/10$. In the end, Section 6 applies the argument in Sections 3 and 4 to an interval of $\theta$ and proves the existence of periodic or quasi-periodic orbits for $\theta \in [0.084\pi, 0.183\pi]$.

2. **Topological constraint and coercivity.** In this section, we introduce a general coercivity result and apply it to our case, which implies the existence of a minimizer $P_0$ satisfying

$$A(P_0) = \inf_{\{d \in \Lambda\}} \inf_{\{q \in P(Q_s, Q_e)\}} A.$$

Furthermore, if one of the boundaries of $P_0$ has no collision, we show that it can be extended.

Let $\chi = \left\{ q \in \mathbb{R}^{N \times d} \mid \sum_{i=1}^{N} m_i q_i = 0 \right\}$. We set

$$Q_s = \begin{bmatrix} q_1(a_1, \ldots, a_k) \\ \vdots \\ q_N(a_1, \ldots, a_k) \end{bmatrix}, \quad Q_e = \begin{bmatrix} q_1(b_1, \ldots, b_s) \\ \vdots \\ q_N(b_1, \ldots, b_s) \end{bmatrix},$$

(10)

where $q_i \in \mathbb{R}^d$ $(i = 1, 2, \ldots, N, d = 1, 2, \text{or } 3)$ are row vectors, and $Q_s, Q_e \in \chi$. The variational argument is a two-step minimizing procedure. First, we consider a fixed-end boundary value problem, which is also known as the Bolza problem. For given values of $a_1, \ldots, a_k$ and $b_1, \ldots, b_s$, the two matrices $Q_s$ and $Q_e$ are fixed. There exists an action minimizer $P$, which satisfies

$$A(P) = \inf_{\{q \in P(Q_s, Q_e)\}} A = \inf_{\{q \in P(Q_s, Q_e)\}} \int_{0}^{1} [K(\dot{q}(t)) + U(q(t))] dt,$$

where $K(\dot{q}(t))$ is the standard kinetic energy, $U(q(t))$ is the standard potential energy, and $P(Q_s, Q_e)$ is defined as follows:

$$P(Q_s, Q_e) := \{ q \in H^1([0,1], \chi) \mid q(0) = Q_s, q(1) = Q_e \}.$$

If one wants $P$ to be a part of a periodic solution, the boundaries must be special. Hence, we introduce another minimizing procedure. Instead of fixing the boundaries, we free several parameters on the boundaries $q(0) = Q_s$ and $q(1) = Q_e$. The Lagrangian action functional is then minimized over these parameters. The
resulting minimizing path may be extended to a periodic or quasi-periodic solution. A general coercivity theorem [11] is introduced here to show the existence of the minimizer in a subset of $H^1([0, 1], \chi)$. This coercivity result is standard, while its proof basically follows by Arzelà-Ascoli theorem and can be found in [8, 11]. Similar coercivity results can also be found in [2].

**Theorem 2.1.** $Q_s, Q_e \in \chi$ are defined by (10), where $q_i \in \mathbb{R}^d, (i = 1, \ldots, N)$ and the variables $a_1, \ldots, a_k, b_1, \ldots, b_s$ are independent. Let $\vec{\alpha} = (a_1, \ldots, a_k) \in S_1, \vec{\beta} = (b_1, \ldots, b_s) \in S_2$, where $S_1 \subset \mathbb{R}^k$ and $S_2 \subset \mathbb{R}^s$ are closed subsets. $S_1 \cup S_2 = S$. Assume that both $\{Q_s|\vec{\alpha} \in \mathbb{R}^k\}$ and $\{Q_e|\vec{\beta} \in \mathbb{R}^s\}$ are linear spaces and their intersection satisfies:

$$\{Q_s|\vec{\alpha} \in \mathbb{R}^k\} \cap \{Q_e|\vec{\beta} \in \mathbb{R}^s\} = \{0\}.$$ 

Then there exist a path sequence $\{P_{n_i}\}$ and a minimizer $P_0$ in $H^1([0, 1], \chi)$, such that for each $n_i$,

$$A(P_{n_i}) = \inf_{\{q \in P(Q_{s_{n_i}}, Q_{e_{n_i}})\}} A, \quad A(P_0) = \inf_{\{q \in P(Q_{s}, Q_{e})\}} A = \inf_{\{q \in P(Q_{s_{0}}, Q_{e_{0}})\}} A,$$

where $P(Q_{s_{n_i}}, Q_{e_{n_i}}) = \{q \in H^1([0, 1], \chi) \mid q(0) = Q_s(\vec{\alpha}_{n_i}), q(1) = Q_e(\vec{\beta}_{n_i})\}$ and $P(Q_{s_0}, Q_{e_0}) = \{q \in H^1([0, 1], \chi) \mid q(0) = Q_s(\vec{\alpha}_0), q(1) = Q_e(\vec{\beta}_0)\}$ with $\vec{\alpha}_{n_i} = (a_{1n_i}, \ldots, a_{kn_i})$ and $\vec{\beta}_{n_i} = (b_{1n_i}, \ldots, b_{sn_i})$. For $t \in [0, 1]$, $P_{n_i}(t)$ converges to $P_0(t)$ uniformly. In particular,

$$\lim_{n_i \to \infty} \vec{\alpha}_{n_i} = \vec{\alpha}_0, \quad \lim_{n_i \to \infty} \vec{\beta}_{n_i} = \vec{\beta}_0.$$ 

As its application, we show that the two configurations $Q_s$ and $Q_e$ defined in (2) satisfy the assumptions in Theorem 2.1. Let $\vec{a} = (a_1, a_2, b_1, b_2)$ and $\Lambda = [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times \mathbb{R}$. When $\theta \in (0, \pi/2)$, it is easy to check that

$$\{Q_s \mid \vec{\alpha} \in \mathbb{R}^k\} \cap \{Q_e \mid \vec{\alpha} \in \mathbb{R}^k\} = \{0\}.$$ 

By Theorem 2.1, there exists an action minimizer $P_0 \in H^1([0, 1], \chi)$ and a vector $\vec{a}_0 \in \Lambda$, such that

$$A(P_0) = \inf_{\{q \in P(Q_s, Q_e)\}} A = \inf_{\{q \in P(Q_s, Q_e), \vec{a}=\vec{a}_0\}} A.$$ 

Let $q(t) (t \in [0, 1])$ be the position matrix path of $P_0$. We then show that if one of the ends $q(0)$ or $q(1)$ of $P_0$ is collision-free, the path $P_0 = P_0([0, 1])$ can be extended.

**Lemma 2.2.** Let $q(t) (t \in [0, 1])$ be the position matrix path of $P_0$. If $q(0) = Q_s(\vec{a}_0)$ has no collision, then $q(t) (t \in [0, 1])$ can be smoothly extended to $t \in [-1, 1]$. If $q(1) = Q_e(\vec{a}_0)$ has no collision, then $q(t) (t \in [0, 1])$ can be smoothly extended to $t \in [0, 2]$.

**Proof.** The proof mainly follows by the first variation formulas. If $q(0) = Q_s(\vec{a}_0)$ has no collision, by the first variation formulas, the velocities $q_i(0) (i = 1, 2, 3)$ must satisfy

$$\dot{q}_{1x}(0) = \dot{q}_{2x}(0) = \dot{q}_{3x}(0) = 0.$$
Then the extension of \( q(t) \) \((t \in [0, 1])\) can be defined as follows

\[
q(t) = \begin{cases} 
(q_1^T(t), q_2^T(t), q_3^T(t))^T, & t \in [0, 1], \\
(q_1^T(-t), q_2^T(-t), q_3^T(-t))TB, & t \in [-1, 0],
\end{cases}
\] (11)

where \( B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). It is easy to check that \( q(t) \) in (11) is smoothly connected at \( t = 0 \). Hence, by the uniqueness of solution of initial value problem in an ODE system, \( q(t) \) \((t \in [-1, 1])\) in (11) is a smooth extension of \( q(t) \) \((t \in [0, 1])\).

If \( q(1) = Q_e(a_0) \) has no collision, by the first variation formulas, the velocities \( q_i(1) (i = 1, 2, 3) \) satisfy

\[
\dot{q}_1(1) = -\dot{q}_1(1)BR(2\theta), \quad \dot{q}_3(1) = -\dot{q}_2(1)BR(2\theta), \quad \dot{q}_2(1) = -\dot{q}_3(1)BR(2\theta),
\]

where \( B \) is as in (11) and \( R(2\theta) \) is as in (2). Then the extension of \( q(t) \) \((t \in [0, 1])\) can be defined as follows

\[
q(t) = \begin{cases} 
(q_1^T(t), q_2^T(t), q_3^T(t))^T, & t \in [0, 1], \\
(q_1^T(2-t), q_2^T(2-t), q_3^T(2-t))TB(2\theta), & t \in [1, 2].
\end{cases}
\] (12)

It is clear that \( q(t) \) in (12) is smooth at \( t = 1 \). It follows that \( q(t) \) \((t \in [0, 2])\) in (11) is a smooth extension of \( q(t) \) \((t \in [0, 1])\). The proof is complete. \(\square\)

3. An upper bound of \( A(P_0) \). Let

\[
P(Q_S, Q_E) = \{ q \in H^1([0, 1], \chi) \mid q(0) \in Q_S, q(1) \in Q_E \},
\]

where \( Q_S \) and \( Q_E \) are given by (8). By choosing a test path in \( P(Q_S, Q_E) \) and calculating its action, we can find an upper bound of \( A(P_0) \) in the case when \( \theta = \pi/10 \). This test path is basically a continuous and piecewise linear function, which can be seen as a linear approximation of the minimizing path \( P_0 \).

Let \( q(t) = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \) be the position matrix of the minimizer \( P_0 = P_{0, \pi/10} \). Approximated values of \( q_{i,k} = q_i(k/10) (i = 1, 2, 3; k = 0, 1, 2, \ldots, 10) \) are given as follows:

\[
q_{1,0} = (0.2680, 0), \quad q_{2,0} = (-0.9999, 0), \\
q_{1,1} = (0.2870, 0.0971), \quad q_{2,1} = (-0.9951, 0.0689), \\
q_{1,2} = (0.3339, 0.1742), \quad q_{2,2} = (-0.9809, 0.1374), \\
q_{1,3} = (0.3919, 0.2266), \quad q_{2,3} = (-0.9576, 0.2051), \\
q_{1,4} = (0.4504, 0.2582), \quad q_{2,4} = (-0.9254, 0.2716), \\
q_{1,5} = (0.5044, 0.2735), \quad q_{2,5} = (-0.8846, 0.3360), \\
q_{1,6} = (0.5515, 0.2766), \quad q_{2,6} = (-0.8353, 0.3978), \\
q_{1,7} = (0.5907, 0.2702), \quad q_{2,7} = (-0.7780, 0.4563), \\
q_{1,8} = (0.6216, 0.2565), \quad q_{2,8} = (-0.7127, 0.5108), \\
q_{1,9} = (0.6436, 0.2371), \quad q_{2,9} = (-0.6398, 0.5605), \\
q_{1,10} = (0.6904, 0)R(\pi/10), \quad q_{2,10} = (-0.3452, 0.7480)R(\pi/10),
\]
and \( q_{3,k} = -q_{1,k} - q_{2,k} \), \((k = 0, 1, 2, \ldots, 10)\). It is clear that the two boundary positions satisfy
\[
\begin{bmatrix}
q_{1,0} \\
q_{2,0} \\
q_{3,0}
\end{bmatrix} \in Q_S = \left\{ Q_s \mid a_1 \geq 0, a_2 \geq 0 \right\},
\]
and
\[
\begin{bmatrix}
q_{1,10} \\
q_{2,10} \\
q_{3,10}
\end{bmatrix} \in Q_E = \left\{ Q_e \mid b_1 \geq 0, b_2 \in \mathbb{R} \right\}.
\]

Let \( \tilde{q}(t) = \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \end{bmatrix} \) be the position matrix of the test path \( P_{test} = P_{test}([0,1]) \). This path is defined by connecting the adjacent points \( q_{i,k} \) and \( q_{i,(k+1)} \), \((i = 1, 2, 3, k = 0, 1, \ldots, 9)\) with straight lines, where the positions \( \tilde{q}_i(t) \) \((i = 1, 2, 3)\) satisfy
\[
\tilde{q}_i(t) = q_{i,k} + 10 \left( t - \frac{k}{10} \right) \left[ q_{i,(k+1)} - q_{i,k} \right], \quad t \in \left[ \frac{k}{10}, \frac{k + 1}{10} \right].
\]

It follows that the test path \( P_{test} \) is a continuous and piecewise smooth function. Actually, \( P_{test} \in H^1([0,1], \chi) \). Hence,
\[
P_{test} \in P(Q_S, Q_E).
\]

It implies that the action of \( P_{test} \) is an upper bound of the minimum action \( A(P_0) \), that is
\[
A(P_0) \leq A(P_{test}). \tag{13}
\]

The following lemma shows that the minimum action value \( A(P_0) \) is less than 3.964.

**Lemma 3.1.**
\[
A(P_0) \leq 3.964.
\]

**Proof.** The proof is basically a calculation of \( A(P_{test}) \). We show that
\[
A(P_{test}) \leq 3.964.
\]
Let \( A_k \) be the action of the path for \( t \in [\frac{k}{10}, \frac{k + 1}{10}] \), \((k = 0, 1, \ldots, 9)\). At \( t = k/10 \), the position matrix of \( P_{test} = P_{test}([0,1]) \) is
\[
\begin{bmatrix}
\hat{q}_{1,k} \\
\hat{q}_{2,k} \\
\hat{q}_{3,k}
\end{bmatrix} = \begin{bmatrix}
q_{1,k} \\
q_{2,k} \\
q_{3,k}
\end{bmatrix} + \frac{10}{9} \begin{bmatrix}
\tilde{q}_1(k/10) \\
\tilde{q}_2(k/10) \\
\tilde{q}_3(k/10)
\end{bmatrix}. \tag{14}
\]

That for each \( i = 1, 2, 3 \) and for \( t \in [\frac{k}{10}, \frac{k + 1}{10}] \),
\[
\tilde{q}_i(t) = q_{i,k} + 10(q_{i,(k+1)} - q_{i,k})(t - \frac{k}{10}), \quad t \in \left[ \frac{k}{10}, \frac{k + 1}{10} \right].
\]
It follows that
\[
A_k = \int_{\frac{k}{10}}^{\frac{k + 1}{10}} \frac{1}{2} \sum_{i=1}^{3} |\dot{\tilde{q}}_i(t)|^2 + U(\tilde{q}) \, dt = \frac{5}{2} \sum_{i=1}^{3} |q_{i,(k+1)} - q_{i,k}|^2 + \sum_{1 \leq i < j \leq 3} \int_{\frac{k}{10}}^{\frac{k + 1}{10}} \frac{1}{|\tilde{q}_i(t) - \tilde{q}_j(t)|} \, dt.
\]
In fact, by the definition of \( \tilde{q}_i(t) \) (\( t \in \left[ \frac{k}{10}, \frac{k+1}{10} \right] \)), \( k = 0, 1, \ldots, 9 \) in (14), the integral
\[
\int_0^{k+1} \frac{1}{|q_i(t) - \tilde{q}_i(t)|} \, dt
\]
can be simplified to the following form
\[
\int_0^{k+1} \frac{1}{\sqrt{(a + bt)^2 + (c + dt)^2}} \, dt,
\]
where \( a, b, c, d \) are constants. A direct calculation shows that
\[
\int_0^{k+1} \frac{1}{\sqrt{(a + bt)^2 + (c + dt)^2}} \, dt
= \frac{1}{\sqrt{b^2 + d^2}} \ln \left( \frac{ab + cd}{b^2 + d^2} + t + \sqrt{\frac{(a + bt)^2 + (c + dt)^2}{b^2 + d^2}} \right) \bigg|_0^{k+1}.
\]
Therefore, the values of \( A_k \) \( k = 0, 1, \ldots, 9 \) can be calculated by using the coordinates of \( q_{i,k} \) \( i = 1, 2, 3, \quad k = 0, 1, \ldots, 10 \). It turns out that
\[
A(P_{\text{test}}) = \sum_{k=0}^{9} A_k \approx 3.96390 \leq 3.964.
\]
Hence,
\[
A(P_0) \leq A(P_{\text{test}}) \leq 3.964.
\]
The proof is complete.

**4. Exclusion of collisions.** In this section, we show that

**Theorem 4.1.** The minimizing path \( P_0 = P_0([0, 1]) \) has no collision singularity.

The proof of this theorem follows by Lemma 4.3, Corollary 1 and Lemma 4.5. Note that, by the works of Marchal and Chenciner, the action minimizer \( P_0 \) has no singularity in \( (0, 1) \). The upper bound 3.964 of \( A(P_0) \) in Section 3 will be used to exclude the possible collisions on the two boundaries \( q(0) \) and \( q(1) \) in \( P_0 = P_0([0, 1]) \).

We assume that in the minimizing path \( P_0 \), \( q(0) \) and \( q(1) \) are
\[
q(0) = \begin{bmatrix} a_{10} - a_{20} & 0 \\ -2a_{10} - a_{20} & 0 \\ a_{10} + 2a_{20} & 0 \end{bmatrix}, \quad q(1) = \begin{bmatrix} 2b_{20} & 0 \\ -b_{20} & b_{10} \\ -b_{20} & -b_{10} \end{bmatrix} R(\theta),
\]
where \( \theta = \pi/10, \quad a_{10} \geq 0, \quad a_{20} \geq 0, \quad b_{10} \geq 0 \) and \( b_{20} \in \mathbb{R} \).

The lower bound estimate of collision paths is based on the following result, which is introduced by Chen [2, 3]. Give any \( \theta \in (0, \pi), \quad T > 0 \), consider the following path spaces:
\[
\Gamma_{T, \theta} := \{ \vec{r} \in H^1([0, 1], \mathbb{R}^2) : < \vec{r}(0), \vec{r}(T) > = |\vec{r}(0)||\vec{r}(T)| \cos \theta \},
\]
\[
\Gamma^*_{T, \theta} := \{ \vec{r} \in \Gamma_{T, \theta} : \vec{r}(t) = 0 \text{ for some } t \in [0, T] \}.
\]
The symbol \( < \cdot, \cdot > \) stands for the standard scalar product in \( \mathbb{R}^2 \) and \( | \cdot | \) represents the standard norm in \( \mathbb{R}^2 \). Define the Keplerian action functional \( I_{\mu, \alpha,T} : H^1([0, T], \mathbb{R}^2) \to \mathbb{R} \cup \{ +\infty \} \) by
\[
I_{\mu, \alpha,T}(\vec{r}) := \int_0^T \frac{\mu}{2} |\vec{r}|^2 + \frac{\alpha}{|\vec{r}|} \, dt.
\]
Lemma 4.2 ([2, 3]). Let $\theta \in (0, \pi]$, $T > 0$, $\mu > 0$, $\alpha > 0$ be constants. Then

$$\inf_{\vec{r} \in \Gamma_{T, \theta}} I_{\mu, \alpha, T}(\vec{r}) = \frac{3}{2} (\mu \alpha^2 \theta^2 T)^{\frac{1}{3}},$$

$$\inf_{\vec{r} \in \Gamma_{T, \theta}} I_{\mu, \alpha, T}(\vec{r}) = \frac{3}{2} (\mu \alpha^2 \pi^2 T)^{\frac{1}{3}}.$$

Note that the center of mass is assumed to be at the origin, it follows that the action $A$ can be rewritten as

$$A = \int_0^1 (K + U) dt$$

$$= \int_0^1 \sum_{i=1}^3 \frac{1}{2} |\dot{q}_i|^2 + \frac{1}{3} \sum_{1 \leq j < k \leq 3} |q_j - q_k| dt$$

$$= \frac{1}{3} \sum_{1 \leq i < j \leq 3} \int_0^1 \frac{1}{2} |\dot{q}_i - \dot{q}_j|^2 + \frac{3}{|q_i - q_j|} dt. \tag{19}$$

Let $A_{ij} := \int_0^1 \frac{1}{2} |q_i - q_j|^2 + \frac{3}{|q_i - q_j|} dt$, it follows that

$$A = \frac{1}{3} [A_{12} + A_{23} + A_{13}]. \tag{20}$$

4.1. Exclusion of triple collisions. The following result implies that whenever there is a triple collision in $P_0$, its action must be greater than 3.964.

Lemma 4.3. If $P_0$ has a triple collision, then

$$A(P_0) \geq 6.6927 > 3.964.$$

Therefore, $P_0$ has no triple collision.

Proof. If $P_0$ has a triple collision, by Lemma 4.2, we have

$$A = \frac{1}{3} [A_{12} + A_{23} + A_{13}] \geq \frac{3}{2} (9\pi^2)^{\frac{1}{3}} \geq 6.6927.$$

Hence, the lower bound of action for paths with triple collision is about 6.6927, which is greater than 3.964. By Lemma 3.1, $P_0$ must have no triple collision. The proof is complete. \hfill \square

4.2. Exclusion of binary collisions at $t = 1$. Note that the definition of $Q_e$ is

$$Q_e = \begin{bmatrix} 2b_2 & 0 \\ -b_2 & b_1 \\ -b_2 & -b_1 \end{bmatrix} R(\theta), \tag{21}$$

where $b_1 \geq 0, b_2 \in \mathbb{R}$, and $R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$. Let $\theta = \pi/10$. To show $P_0$ has no collision at $t = 1$, we only need to exclude the binary collision between body 2 and body 3. We will give a lower bound for the action of orbits with binary collision between body 2 and body 3 at $t = 1$: $q_2(1) = q_3(1)$.

Lemma 4.4. Assume that bodies 2 and 3 have a binary collision at $t = 1$ in the minimizing path $P_0 = P_0([0,1])$, its action satisfies

$$A(P_0) \geq 4.7911.$$
Proof. By assumption, bodies 2 and 3 collide at \( t = 1 \). By Lemma 4.2,
\[
A_{23} \geq \frac{3}{2} (9\pi^2)^{1/3}.
\]
(22)
In the minimizing path \( \mathcal{P}_0 \), the position matrix is given by
\[
q(1) = Q_e = \begin{bmatrix}
2b_{20} & 0 \\
-b_{20} & b_{10} \\
-b_{20} & -b_{10}
\end{bmatrix} R(\pi/10)
\]
with \( b_{10} \geq 0 \). Note that we assume bodies 2 and 3 collide at \( t = 1 \), it implies that \( b_{10} = 0 \). If meanwhile \( b_{20} = 0 \), it is a total collision. By Lemma 4.3, it is not possible for \( \mathcal{P}_0 \). It follows that \( b_{10} = 0 \) and \( b_{20} \neq 0 \).

Next, we find a lower bound of \( A_{12} + A_{13} \) for \( \mathcal{P}_0 \) with a binary collision between bodies 2 and 3 at \( t = 1 \). If in \( \mathcal{P}_0 \), \( q(0) = \begin{bmatrix}
a_{10} - a_{20} & 0 \\
-2a_{10} - a_{20} & 0 \\
a_{10} + 2a_{20} & 0
\end{bmatrix} \) has no collision, we have the following two cases. First, if \( b_{20} > 0 \), then the vector \( q_2q_1 = q_1 - q_2 \) rotates at least \( \frac{\pi}{10} \) and the vector \( q_3q_1 = q_1 - q_3 \) rotates at least \( \frac{9\pi}{10} \). Second, if \( b_{20} < 0 \), the vector \( q_2q_1 = q_1 - q_2 \) rotates at least \( \frac{9\pi}{10} \) and the vector \( q_3q_1 = q_1 - q_3 \) rotates at least \( \frac{\pi}{10} \). Therefore, by Lemma 4.2,
\[
A_{12} + A_{13} \geq \frac{3}{2} (9\pi^2)^{1/3} \left[ \left( \frac{1}{10} \right)^{2/3} + \left( \frac{9}{10} \right)^{2/3} \right].
\]
(23)
If in \( \mathcal{P}_0 \), \( q(0) = \begin{bmatrix}
a_{10} - a_{20} & 0 \\
-2a_{10} - a_{20} & 0 \\
a_{10} + 2a_{20} & 0
\end{bmatrix} \) has a binary collision, it can only happen in one of the two pairs: bodies 1 and 2, bodies 1 and 3. Then by Lemma 4.2, it follows that
\[
A_{12} + A_{13} \geq \frac{3}{2} (9\pi^2)^{1/3} \left[ 1 + \left( \frac{1}{10} \right)^{2/3} \right].
\]
(24)
By inequalities (22), (23) and (24),
\[
A(\mathcal{P}_0) = \frac{1}{3} [A_{12} + A_{13} + A_{23}] \geq 4.7911.
\]
The proof is complete.

Corollary 1. In the minimizing path \( \mathcal{P}_0 \), there is no collision singularity at \( t = 1 \).

Proof. The proof follows directly by Lemma 3.1, Lemma 4.3 and Lemma 4.4. The proof is complete.

4.3. Exclusion of binary collisions at \( t = 0 \). In \( \mathcal{P}_0 \),
\[
q(0) = Q_s = \begin{bmatrix}
a_{10} - a_{20} & 0 \\
-2a_{10} - a_{20} & 0 \\
a_{10} + 2a_{20} & 0
\end{bmatrix}
\]
with \( a_{10} \geq 0 \) and \( a_{20} \geq 0 \). By Lemma 4.3 in Section 4.1, there is no triple collision in \( q(0) \) of \( \mathcal{P}_0 \). Then \( a_{10} \) and \( a_{20} \) can not be 0 at the same time. We are left to exclude two possible binary collisions: collision between bodies 1 and 2, collision between bodies 1 and 3.

Lemma 4.5. The minimizing path \( \mathcal{P}_0 \) has no collision at \( t = 0 \).
Lemma 4.2, \[ A_{23} \geq \frac{3}{2} (9\pi^2)^{1/3} \left( \frac{2}{5} \right)^{2/3}. \] (25)

In order to find a good lower bound for the collision path, we will apply the extension formulas in Lemma 2.2 of Section 2. By Corollary 1, there is no singularity on \( q(1) \) of \( P_0 \). It follows that the orbit can be extended to the time period \( t \in [0, 2] \). And the action value for \( t \in [0, 2] \) is \( 2A(P_0) \). If at \( t = 0 \), bodies 1 and 2 collide, that is \( a_{10} = 0 \). By the extension formula (12) in Lemma 2.2, it implies that at \( t = 2 \), bodies 1 and 3 collide. Similarly, the binary collision between bodies 1 and 3 at \( t = 0 \) implies a binary collision between bodies 1 and 2 at \( t = 2 \). Therefore, by Lemma 4.2,

\[ 2A(P_0) \geq \frac{1}{3} \left[ 2 \cdot \frac{3}{2} (9\pi^2)^{1/3} \left( \frac{2}{5} \right)^{2/3} + 2 \cdot \frac{3}{2} (18\pi^2)^{1/3} \right] \geq 8.0438. \] (26)

It follows that

\[ A(P_0) \geq 4.0219 > 3.964. \]

By Lemma 3.1, \( P_0 \) has no collision singularity at \( t = 0 \). The proof is complete. \( \square \)

4.4. \( P_0 \) is nontrivial. An orbit is called nontrivial if it does not coincide with a relative equilibrium in the N-body problem. At the end of Section 4, we show that \( P_0 \) is not a part of a relative equilibrium. By the definition of \( Q_S \) and \( Q_E \) in (8) and (9), the only possible relative equilibrium is an Euler orbit. In this case, the isosceles configuration in \( Q_E \) degenerates to an Euler configuration. And from a straight line in \( Q_S \) to an Euler configuration in \( Q_E \), this Euler orbit rotates \( 2\pi/5 \). It follows that the corresponding action of this part of the Euler orbit is

\[ A(P_0) = A_{Euler} = 2 \cdot \frac{3}{2} \left( \frac{5}{4} \right)^{2/3} \left( \frac{2}{5} \right)^{2/3} \approx 4.0539 > 3.964. \] (27)

By Lemma 3.1, \( P_0 \) is nontrivial.

5. Extension to a periodic orbit. In Section 4, we show that the minimizing path \( P_0 = P_0([0, 1]) \) is collision-free. The extension of \( P_0 \) is shown by the first variation formulas and the uniqueness of solution of initial value problem in an ODE system.

Theorem 5.1. When \( \theta = \pi/10 \), the minimizing path \( P_0 = P_0([0, 1]) \) can be extended to a nontrivial periodic orbit.

Proof. By Lemma 2.2, \( P_0 = P_0([0, 1]) \) can be extended to \( t \in [0, 2] \). The extension is defined as follows

\[ q(t) = \begin{cases} (q^T_1(t), q^T_2(t), q^T_3(t))^T, & t \in [0, 1], \\
(q^T_1(2-t), q^T_2(2-t), q^T_3(2-t))^T BR(2\theta), & t \in [1, 2], \end{cases} \] (28)

where \( B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) and \( R(2\theta) \) is the rotation matrix defined in (2). It is clear that for \( t \in [0, 2] \), the path \( P_0([0, 2]) \) in (28) is smooth. Note that by the first variation formulas, at \( t = 0 \), the velocity \( \dot{q}_i(0) \) of the minimizing path \( P_0 \) satisfy

\[ \dot{q}_{1x}(0) = \dot{q}_{2x}(0) = \dot{q}_{3x}(0) = 0. \]
Then one can extend the path $P_0([0, 2])$ to $t \in [0, 4]$:

$$q(t) = \begin{cases} 
(q_1^T(t), q_2^T(t), q_3^T(t))^T, & t \in [0, 2], \\
(q_1^T(t - 2), q_2^T(t - 2), q_3^T(t - 2))^T R(2\theta), & t \in [2, 4].
\end{cases} \tag{29}$$

It is easy to check that the extension in (29) is smooth. Furthermore, at $t = 4$,

$$q_i(4) = q_i(0)R(4\theta), \quad \dot{q}_i(4) = \dot{q}_i(0)R(4\theta), \quad (i = 1, 2, 3).$$

By the uniqueness of solution of initial value problem in an ODE system, the position matrix $q(t)$ of path $P_0([0, 4])$ can be extended smoothly to any $t \in \mathbb{R}$:

$$q(t) = q(t - 4n)R(4n\theta), \quad t \in [4n, 4(n+1)], \quad n \in \mathbb{Z}. \quad \tag{30}$$

Note that $\theta = \pi/10$. By taking $n = 5$, we have $q(t + 20) = q(t)$. Hence, $q(t)$ is periodic. The proof is complete. \hfill \Box

Remark 3. The periodic orbit (Fig. 2) extended by $P_0 \equiv P_{0, \pi/10}$ has a nonzero angular momentum $J$:

$$J \approx -1.66365.$$

The initial condition of this orbit is

$$\begin{bmatrix} q_1 & \dot{q}_1 \\
q_2 & \dot{q}_2 \\
q_3 & \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0.268018677656945 & 0 & 0 & 1.013345278728911 \\
-0.999909017383165 & 0 & 0 & 0.689219989691781 \\
0.731890339726020 & 0 & 0 & -1.702565268420692 \end{bmatrix}.$$

By running the above initial condition in the ode solver ODE45 of Matlab, this orbit looks unstable numerically since it breaks the periodic shape when $t = 40$. However, we have not investigated its stability rigorously.

6. Properties of the action minimizer for $\theta \in [0.084\pi, 0.183\pi]$. In this last section, we will follow the idea in Section 4 and show that the action minimizer $P_0 = P_{0, \theta}$ is nontrivial and collision-free for $\theta \in [0.084\pi, 0.183\pi]$.

Theorem 6.1. For each $\theta \in [0.084\pi, 0.183\pi]$, there exists a nontrivial and collision-free minimizing path $P_{0, \theta} \equiv P_{0, \theta}(t \in [0, 1])$ connecting the two configuration sets $Q_S$ and $Q_E$ in (3), and it can be extended to a periodic or quasi-periodic orbit.

Proof. By Theorem 2.1, for $\theta \in [0.084\pi, 0.183\pi]$, there exists an action minimizer $P_{0, \theta} \equiv P_{0, \theta}(t \in [0, 1])$, such that

$$\mathcal{A}(P_{0, \theta}) = \inf_{\{\delta \in \Lambda\}} \inf_{(q \in P(Q_t, Q_c))} \mathcal{A}.$$

The main challenge in the theorem is to exclude the boundary collisions in $P_{0, \theta} \equiv P_{0, \theta}(t \in [0, 1])$ and show that this minimizer is not a part of an Euler orbit. The extension of the minimizer $P_{0, \theta} \equiv P_{0, \theta}(t \in [0, 1])$ then follows similarly by Theorem 5.1.
Let
\[
\begin{align*}
f_1(\theta) & = \frac{1}{2} (9)^{1/3} \left[ \pi^{2/3} + \theta^{2/3} + (\pi - \theta)^{2/3} \right], \\
f_2(\theta) & = \frac{1}{2} (9)^{1/3} \left[ (2\pi^2)^{1/3} + \left( \frac{\pi}{2} - \theta \right)^{2/3} \right], \\
g(\theta) & = 3 \left( \frac{5}{4} \right)^{2/3} \left( \frac{\pi}{2} - \theta \right)^{2/3}.
\end{align*}
\] (31)

When \( \theta \in [0.084\pi, 0.183\pi] \),
\[
\begin{align*}
f_1(\theta) - f_2(\theta) & = \frac{1}{2} (9)^{1/3} \left[ \pi^{2/3} + \theta^{2/3} + (\pi - \theta)^{2/3} - (2\pi^2)^{1/3} - \left( \frac{\pi}{2} - \theta \right)^{2/3} \right] \\
& \geq \frac{1}{2} (9\pi^2)^{1/3} \left[ 1 + (0.084)^{2/3} + (1 - 0.183)^{2/3} - 2^{1/3} - (0.5 - 0.084)^{2/3} \right] \\
& \approx 0.5545 > 0.
\end{align*}
\]

Note that by the analysis in Section 4.1, if \( \mathcal{P}_{0, \theta} = \mathcal{P}_{0, \theta}(t \in [0, 1]) \) has some collision singularities, its action \( \mathcal{A}_{\text{col}} \) satisfies

\( \mathcal{A}_{\text{col}} \geq \min \{ f_1(\theta), f_2(\theta) \} = f_2(\theta), \quad \text{when } \theta \in [0.084\pi, 0.183\pi]. \) (32)

We denote the action of an Euler orbit connecting \( Q_s \) and \( Q_e \) (in (2)) by \( \mathcal{A}_{\text{Euler}} \). It is clear that
\[ \mathcal{A}_{\text{Euler}} = g(\theta) = 3 \left( \frac{5}{4} \right)^{2/3} \left( \frac{\pi}{2} - \theta \right)^{2/3}. \]

For each \( \theta \in [0.084\pi, 0.183\pi] \), we need to define a test path \( \mathcal{P}_{\text{test}} = \mathcal{P}_{\text{test}, \theta} \), such that its action \( \mathcal{A}_{\text{test}} = \mathcal{A}(\mathcal{P}_{\text{test}, \theta}) \) satisfies
\[ \mathcal{A}_{\text{test}} < \min \{ f_2(\theta), g(\theta) \} \leq \min \{ \mathcal{A}_{\text{col}}, \mathcal{A}_{\text{Euler}} \}. \]

Note that the difference \( f_2(\theta) - g(\theta) \) is
\[
\begin{align*}
f_2(\theta) - g(\theta) & = \frac{1}{2} (9)^{1/3} \left[ (2\pi^2)^{1/3} + \left( \frac{\pi}{2} - \theta \right)^{2/3} \right] - 3 \left( \frac{5}{4} \right)^{2/3} \left( \frac{\pi}{2} - \theta \right)^{2/3} \\
& = \frac{1}{2} (18\pi^2)^{1/3} + \left( \frac{\pi}{2} - \theta \right)^{2/3} \left[ \frac{1}{2} (9)^{1/3} - 3 \left( \frac{5}{4} \right)^{2/3} \right].
\end{align*}
\]

It turns out that when \( \theta \in [0.084\pi, 0.106722\pi] \),
\[ \min \{ f_2(\theta), g(\theta) \} = f_2(\theta). \] (33)

When \( \theta \in [0.106723\pi, 0.183\pi] \),
\[ \min \{ f_2(\theta), g(\theta) \} = g(\theta). \] (34)

The test path \( \mathcal{P}_{\text{test}, \theta} \) is defined as follows. We first choose 7 different \( \theta_0 \):
\[
\theta_0 = 0.182\pi, \ 0.18\pi, \ 0.173\pi, \ 0.16\pi, \ 0.132\pi, \ 0.1\pi, \ 0.085\pi.
\]
For each given $\theta_0$, let $q(t) = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} (t)$ be the position matrix path of the minimizer $P_{0, \theta_0}$. Let $\tilde{q}(t) = \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \end{bmatrix} (t)$ be the position matrix path of $P_{\text{test}, \theta}$. We can then define a test path $P_{\text{test}, \theta}$ by connecting the following 11 points:

$$\tilde{q} \left( \frac{i}{10} \right) = q \left( \frac{i}{10} \right), \ (i = 0, 1, \ldots, 9), \quad \tilde{q} (1) = q (1) R (\theta_0) R (\theta).$$

In fact, $\tilde{q}(t)$ satisfies

$$\tilde{q}(t) = \tilde{q} \left( \frac{i}{10} \right) + 10 \left( t - \frac{i}{10} \right) \left[ q \left( \frac{i + 1}{10} \right) - \tilde{q} \left( \frac{i}{10} \right) \right], \quad t \in \left[ \frac{i}{10}, \frac{i + 1}{10} \right], \quad (35)$$

where $i = 0, 1, \ldots, 9$. It is easy to check that $\tilde{q}(0) \in Q_S$ and $\tilde{q}(1) \in Q_E$, where $Q_S$ and $Q_E$ are the boundary configuration sets defined in (3). Once the values of $q \left( \frac{i}{10} \right) (i = 0, 1, \ldots, 10)$ in $P_{0, \theta_0}$ are given, the action of the test path $A_{\text{test}} = A (P_{\text{test}, \theta})$ can be calculated accurately as in the proof of Lemma 3.1. For readers’ convenience, the data of the 7 test paths and the corresponding figures of action values are given in Appendix A.

Note that for a given set of 11 interpolation points, formula (16) implies that the action of the test path $A_{\text{test}} = A (P_{\text{test}, \theta})$ is a smooth function with respect to $\theta$. In the last step, we compare the value of the two smooth functions: $A_{\text{test}} = A (P_{\text{test}, \theta})$ and $\min \{ f_2(\theta), g(\theta) \}$ in different intervals of $\theta$ as in Fig. 3. To do so, we calculate the value of the two functions for $\theta \in [0.084\pi, 0.183\pi]$ with a step $\pi \times 10^{-6}$.

In order to compare the two functions for every $\theta$, a linear interpolation method is introduced. Actually, we use straight lines to replace the real curves of the two functions in each step. The error of this linear interpolation is $\frac{1}{8} (\pi \times 10^{-6})^2 \Delta$, where $\Delta$ is the maximum of the second derivative of the corresponding function. For $\theta \in [0.084\pi, 0.183\pi]$, it turns out that $\Delta \leq \frac{20}{\pi}$ for both functions. It implies that the error is bounded by

$$\frac{1}{8} (\pi \times 10^{-6})^2 \Delta \leq \frac{1}{8} (\pi \times 10^{-6})^2 \frac{20}{\pi} \approx 7.85 \times 10^{-12}.$$  

By (33) and (34), it follows that

$$\min \{ f_2(\theta), g(\theta) \} = \begin{cases} f_2(\theta), & \text{for } 0.084\pi \leq \theta \leq 0.106722\pi; \\ g(\theta), & \text{for } 0.106723\pi \leq \theta \leq 0.183\pi. \end{cases}$$

Numerically, for $\theta \in [0.084\pi, 0.183\pi]$, the minimum value of $\min \{ f_2(\theta), g(\theta) \} - A_{\text{test}}$ in Fig. 3 is $3.08 \times 10^{-4} > 7.85 \times 10^{-12}$.

Therefore, for each given $\theta \in [0.084\pi, 0.183\pi]$, the action of the test path $A_{\text{test}}$ satisfies

$$A_{\text{test}} < \min \{ f_2(\theta), g(\theta) \}.$$  

It follows that the action minimizer $P_{0, \theta}$ is nontrivial and collision-free when $\theta \in [0.084\pi, 0.183\pi]$. The proof is complete. \hfill \square

**Remark 4.** Similar to Remark 3, we choose 6 special angles

$$\theta = 0.084\pi, \ 0.1\pi, \ 0.12\pi, \ 0.14\pi, \ 0.16\pi, \ 0.183\pi$$
to calculate the angular momentum of the minimizer \( \mathcal{P}_{0,\theta} \). It turns out that their angular momenta are

\[
J(0.084\pi) \approx -1.61, \quad J(0.1\pi) \approx -1.66, \quad J(0.12\pi) \approx -1.75,
\]

\[
J(0.14\pi) \approx -1.86, \quad J(0.16\pi) \approx -2.01, \quad J(0.183\pi) \approx -2.22.
\]

Hence, it is reasonable to believe that the angular momenta of this set of orbits are nonzero when \( \theta \in [0.084\pi, 0.183\pi] \).

In summary, this paper studies a new set of periodic orbits in the planar equal-mass three-body problem. For each given \( \theta \in [0.084\pi, 0.183\pi] \), the orbit can be characterized as a local action minimizer connecting a collinear configuration and an isosceles configuration. Topological constraints are introduced on the two boundary configurations as in Fig. 1, and the orbit generated by the corresponding minimizer is different from the retrograde orbit, the prograde orbit in [3] and the well-known figure-eight orbit [4].

The topological constraints introduced in this paper are simple and have clear geometric meanings, which can be extended to study periodic orbits in four-body or five-body problem.

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Appendix A. Data of the 7 test paths.

1. \( \theta_0 = 0.182\pi \) : a test path \( \mathcal{P}_{\text{test}, \theta} \) is defined for \( \theta \in \left[ 0.181\pi, 0.183\pi \right] \), where \( \mathcal{P}_{\text{test}, \theta} \) connects the adjacent points \( \hat{q}_i(\frac{\pi}{10}) \) \( (i=1, 2, 3; \ j=0, 1, \ldots, 10) \) in Table 1 by straight lines (as in (35)), where \( \hat{q}_3 = -\hat{q}_1 - \hat{q}_2 \).

| \( t \) | \( \hat{q}_1 \) | \( \hat{q}_2 \) |
|---|---|---|
| 0 | (0.1109, 0) | (-1.1189, 0) |
| 0.1 | (0.11378134, 0.035120592) | (-1.1144897, 0.089194621) |
| 0.2 | (0.12212758, 0.068373015) | (-1.1012708, 0.17787543) |
| 0.3 | (0.13510199, 0.098115021) | (-1.0792782, 0.26552064) |
| 0.4 | (0.15147483, 0.12309328) | (-1.0485685, 0.35159462) |
| 0.5 | (0.16980434, 0.14251378) | (-1.0092177, 0.43553762) |
| 0.6 | (0.18860929, 0.15603011) | (-0.96132044, 0.51676178) |
| 0.7 | (0.20644197, 0.16368275) | (-0.90499096, 0.59464214) |
| 0.8 | (0.22204784, 0.16582253) | (-0.84036636, 0.66850889) |
| 0.9 | (0.23430082, 0.16304077) | (-0.76761388, 0.73763832) |
| 1 | (0.28822372, 0) | (-0.144111185, 1.0455201) |

Table 1. The positions of \( \hat{q}_{i,j} = \hat{q}_i(\frac{j\pi}{10}) \) \( (i=1, 2; \ j=0, 1, \ldots, 10) \) in \( \mathcal{P}_{\text{test}, \theta} \) corresponding to \( \theta \in [0.181\pi, 0.183\pi] \).
2. $\theta_0 = 0.18\pi$: a test path $P_{\text{test}, \theta}$ is defined for $\theta \in \left[ 0.176\pi, 0.181\pi \right]$, where $P_{\text{test}, \theta}$ connects the adjacent points $\tilde{q}_i \left( \frac{j}{10} \right)$ (i=1, 2, 3; j=0, 1, ..., 10) in Table 2 by straight lines (as in (35)), where $\tilde{q}_3 = -\tilde{q}_1 - \tilde{q}_2$:

| $\theta_0 = 0.18\pi$, $\theta \in \left[ 0.176\pi, 0.181\pi \right]$ |
|---|
| $t$ | $q_1$ | $q_2$ |
| 0 | (0.1198, 0) | (-1.1166, 0) |
| 0.1 | (0.12300003, 0.038069081) | (-1.1122108, 0.087838457) |
| 0.2 | (0.13225041, 0.074032720) | (-1.0990555, 0.17518597) |
| 0.3 | (0.14657647, 0.10606411) | (-1.0771702, 0.206154328) |
| 0.4 | (0.16456742, 0.13280675) | (-1.0466117, 0.34639491) |
| 0.5 | (0.18460176, 0.15344287) | (-1.0074550, 0.42920169) |
| 0.6 | (0.20503978, 0.16765957) | (-0.95979063, 0.50939364) |
| 0.7 | (0.22435124, 0.17556129) | (-0.90372458, 0.58636242) |
| 0.8 | (0.24118113, 0.17757208) | (-0.83938055, 0.65945294) |
| 0.9 | (0.25429011, 0.17435346) | (-0.76690612, 0.72795349) |
| 1 | (0.31146657, 0) $R(\theta)$ | (-0.155733285, 1.0357709) $R(\theta)$ |

Table 2. The positions of $q_{i,j} = \tilde{q}_i \left( \frac{j}{10} \right)$ (i = 1, 2, j = 0, 1, ..., 10) in $P_{\text{test}, \theta}$ corresponding to $\theta \in [0.176\pi, 0.181\pi]$.

3. $\theta_0 = 0.173\pi$: a test path $P_{\text{test}, \theta}$ is defined for $\theta \in [0.165\pi, 0.176\pi]$, where $P_{\text{test}, \theta}$ connects the adjacent points $\tilde{q}_i \left( \frac{j}{10} \right)$ (i=1, 2, 3; j=0, 1, ..., 10) in Table 3 by straight lines (as in (35)), where $\tilde{q}_3 = -\tilde{q}_1 - \tilde{q}_2$:

| $\theta_0 = 0.173\pi$, $\theta \in [0.165\pi, 0.176\pi]$ |
|---|
| $t$ | $q_1$ | $q_2$ |
| 0 | (0.1454, 0) | (-1.1064, 0) |
| 0.1 | (0.14676577, 0.046886901) | (-1.0620407, 0.083918190) |
| 0.2 | (0.16194195, 0.090810510) | (-1.0889788, 0.16740323) |
| 0.3 | (0.18067059, 0.12832570) | (-1.0672503, 0.25001086) |
| 0.4 | (0.20377305, 0.16080003) | (-1.0389499, 0.33127610) |
| 0.5 | (0.22901187, 0.1845175) | (-0.99813266, 0.41070550) |
| 0.6 | (0.25428011, 0.20016641) | (-0.95039814, 0.48770701) |
| 0.7 | (0.27777826, 0.20832504) | (-0.89534906, 0.56190183) |
| 0.8 | (0.29786797, 0.20961764) | (-0.83155952, 0.63247939) |
| 0.9 | (0.31364961, 0.20493038) | (-0.75965878, 0.69882326) |
| 1 | (0.37739476, 0) $R(\theta)$ | (-0.18869738, 1.00216461) $R(\theta)$ |

Table 3. The positions of $q_{i,j} = \tilde{q}_i \left( \frac{j}{10} \right)$ (i = 1, 2, j = 0, 1, ..., 10) in $P_{\text{test}, \theta}$ corresponding to $\theta \in [0.165\pi, 0.176\pi]$.

4. $\theta_0 = 0.16\pi$: a test path $P_{\text{test}, \theta}$ is defined for $\theta \in [0.146\pi, 0.165\pi]$, where $P_{\text{test}, \theta}$ connects the adjacent points $\tilde{q}_i \left( \frac{j}{10} \right)$ (i=1, 2, 3; j=0, 1, ..., 10) in Table 4 by straight lines (as in (35)), where $\tilde{q}_3 = -\tilde{q}_1 - \tilde{q}_2$.

5. $\theta_0 = 0.132\pi$: a test path $P_{\text{test}, \theta}$ is defined for $\theta \in [0.12\pi, 0.146\pi]$, where $P_{\text{test}, \theta}$ connects the adjacent points $\tilde{q}_i \left( \frac{j}{10} \right)$ (i=1, 2, 3; j=0, 1, ..., 10) in Table 5 by straight lines (as in (35)), where $\tilde{q}_3 = -\tilde{q}_1 - \tilde{q}_2$.

6. $\theta_0 = 0.1\pi$: a test path $P_{\text{test}, \theta}$ is defined for $\theta \in [0.089\pi, 0.12\pi]$, where $P_{\text{test}, \theta}$ connects the adjacent points $\tilde{q}_i \left( \frac{j}{10} \right)$ (i=1, 2, 3; j=0, 1, ..., 10) in Table 6 by straight lines (as in (35)), where $\tilde{q}_3 = -\tilde{q}_1 - \tilde{q}_2$.
| $\theta_0 = 0.16\pi$ | $\theta \in [0.146\pi, 0.165\pi]$ |
|----------------------|----------------------------------|
| $t$ | $\hat{q}_1$  | $\hat{q}_2$  |
| 0 | $(0.1803, 0)$ | $(-1.0860, 0)$ |
| 0.1 | $(0.18658246, 0.059576674)$ | $(-1.0816238, 0.07868710)$ |
| 0.2 | $(0.20429715, 0.11438966)$ | $(-1.0685224, 0.15716562)$ |
| 0.3 | $(0.23056861, 0.16992393)$ | $(-1.0467705, 0.23480016)$ |
| 0.4 | $(0.26186698, 0.19737903)$ | $(-1.0164725, 0.31125003)$ |
| 0.5 | $(0.29489886, 0.22358233)$ | $(-0.9777453, 0.38665495)$ |
| 0.6 | $(0.32695826, 0.23945199)$ | $(-0.93070180, 0.46871449)$ |
| 0.7 | $(0.35597786, 0.24655415)$ | $(-0.87545128, 0.52868620)$ |
| 0.8 | $(0.38039590, 0.24583791)$ | $(-0.81208876, 0.59538118)$ |
| 0.9 | $(0.39901814, 0.23851003)$ | $(-0.74069823, 0.65815496)$ |
| 1 | $(0.46894463, 0)R(\theta)$ | $(-0.24472315, 0.94630690)R(\theta)$ |

Table 4. The positions of $\hat{q}_{i,j} = \hat{q}_i \left( \frac{j}{10} \right)$ ($i = 1, 2, j = 0, 1, \ldots, 10$) in $\mathcal{P}_{\text{test}, \theta}$ corresponding to $\theta \in [0.146\pi, 0.165\pi]$.

| $\theta_0 = 0.132\pi$ | $\theta \in [0.12\pi, 0.146\pi]$ |
|----------------------|----------------------------------|
| $t$ | $\hat{q}_1$  | $\hat{q}_2$  |
| 0 | $(0.2306, 0)$ | $(-1.0421, 0)$ |
| 0.1 | $(0.24197875, 0.079876150)$ | $(-1.0375636, 0.072168012)$ |
| 0.2 | $(0.27249901, 0.14948693)$ | $(-1.0240177, 0.14401233)$ |
| 0.3 | $(0.31440456, 0.20354167)$ | $(-1.0016223, 0.21516044)$ |
| 0.4 | $(0.36651601, 0.24151617)$ | $(-0.97657606, 0.28517262)$ |
| 0.5 | $(0.40588817, 0.26512985)$ | $(-0.93107630, 0.35354922)$ |
| 0.6 | $(0.44781116, 0.27643491)$ | $(-0.8839444, 0.41974435)$ |
| 0.7 | $(0.48345138, 0.27815124)$ | $(-0.82742450, 0.48317696)$ |
| 0.8 | $(0.51262863, 0.27151295)$ | $(-0.7635450, 0.54323463)$ |
| 0.9 | $(0.53418979, 0.25829537)$ | $(-0.69191492, 0.59928148)$ |
| 1 | $(0.50692629, 0)R(\theta)$ | $(-0.294863145, 0.84227284)R(\theta)$ |

Table 5. The positions of $\hat{q}_{i,j} = \hat{q}_i \left( \frac{j}{10} \right)$ ($i = 1, 2, j = 0, 1, \ldots, 10$) in $\mathcal{P}_{\text{test}, \theta}$ corresponding to $\theta \in [0.12\pi, 0.146\pi]$.

| $\theta_0 = 0.1\pi$ | $\theta \in [0.089\pi, 0.12\pi]$ |
|----------------------|----------------------------------|
| $t$ | $\hat{q}_1$  | $\hat{q}_2$  |
| 0 | $(0.2680, 0)$ | $(-0.9999, 0)$ |
| 0.1 | $(0.28609887, 0.097087533)$ | $(-0.90513460, 0.06886847)$ |
| 0.2 | $(0.3390610, 0.17420817)$ | $(-0.90094971, 0.13739831)$ |
| 0.3 | $(0.39193700, 0.22664276)$ | $(-0.95760880, 0.20514818)$ |
| 0.4 | $(0.45044849, 0.25815778)$ | $(-0.92539244, 0.27156414)$ |
| 0.5 | $(0.50439276, 0.27354220)$ | $(-0.88455797, 0.33601715)$ |
| 0.6 | $(0.55509942, 0.27662332)$ | $(-0.83534366, 0.39783468)$ |
| 0.7 | $(0.59074871, 0.27023456)$ | $(-0.77798406, 0.45632275)$ |
| 0.8 | $(0.62156413, 0.25649856)$ | $(-0.71271941, 0.51078103)$ |
| 0.9 | $(0.64359767, 0.23707560)$ | $(-0.63979436, 0.56051180)$ |
| 1 | $(0.69032914, 0)R(\theta)$ | $(-0.34516457, 0.74809598)R(\theta)$ |

Table 6. The positions of $\hat{q}_{i,j} = \hat{q}_i \left( \frac{j}{10} \right)$ ($i = 1, 2, j = 0, 1, \ldots, 10$) in $\mathcal{P}_{\text{test}, \theta}$ corresponding to $\theta \in [0.089\pi, 0.12\pi]$. 
7. \( \theta_0 = 0.085\pi \): a test path \( P_{\text{test}, \theta} \) is defined for \( \theta \in [0.084\pi, 0.089\pi] \), where \( P_{\text{test}, \theta} \) connects the adjacent points \( \tilde{q}_i(\frac{j}{10}) \) (i=1, 2, 3; j=0, 1, ..., 10) in Table 7 by straight lines (as in (35)), where \( \tilde{q}_3 = -\tilde{q}_1 - \tilde{q}_2 \).

| \( \tilde{q}_i = ( \tilde{q}_{1i}, \tilde{q}_{2i}, \tilde{q}_{3i} ) \) |
|-----------------|
| \( \tilde{q}_1 \) | \( \tilde{q}_2 \) |
| \( \tilde{q}_3 \) |
| \( \tilde{q}_4 \) |
| \( \tilde{q}_5 \) |
| \( \tilde{q}_6 \) |
| \( \tilde{q}_7 \) |
| \( \tilde{q}_8 \) |
| \( \tilde{q}_9 \) |
| \( \tilde{q}_{10} \) |

Table 7. The positions of \( \tilde{q}_{i,j} = \tilde{q}_i(\frac{j}{10}) \) (i = 1, 2, 3; j = 0, 1, ..., 10) in \( P_{\text{test}, \theta} \) corresponding to \( \theta \in [0.084\pi, 0.089\pi] \).

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Figure 3. In each figure, the horizontal axis is $\theta/\pi$, and the vertical axis is the action value $A$. In each subfigure, the black curve is the graph of the test path’s action $A_{test}$; the purple curve is the graph of $A_{Euler} = g(\theta)$ in (31) and the red curve is the graph of $f_2(\theta)$ in (31), which is the lower bound of $A_{col}$. 