Coulomb and quantum oscillator problems in conical spaces with arbitrary dimensions

J. L. A. Coelho and R. L. P. G. Amaral
Instituto de Física - Universidade Federal Fluminense
Av. Litorânea, S/N, Boa Viagem, Niterói, CEP.24210-340
Rio de Janeiro - Brasil
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Abstract

The Schrödinger equations for the Coulomb and the Harmonic oscillator potentials are solved in the cosmic-string conical space-time. The spherical harmonics with angular deficit are introduced. The algebraic construction of the harmonic oscillator eigenfunctions is performed through the introduction of non-local ladder operators. By exploiting the hidden symmetry of the two-dimensional harmonic oscillator the eigenvalues for the angular momentum operators in three dimensions are reproduced. A generalization for N-dimensions is performed for both Coulomb and harmonic oscillator problems in angular deficit space-times. It is thus established the connection among the states and energies of both problems in these topologically non-trivial space-times.

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I. INTRODUCTION

There has been a growing interest for space-times with nontrivial topology and how this can affect some aspects in classical or quantum cosmological models as well as in quantum mechanics. This nontrivial topology is generated by topological defects like monopoles, strings, domain walls and branes. Their formation are associated with phase transitions in the early universe where the vacuum is degenerated [1]. Nevertheless, stable domain walls and monopoles are disastrous for cosmological models [2]. However, strings cause no harm and can be a good candidate to produce several phenomena observed in the last decades like gravitational lenses [3],[4], particle production [5] and microwave sky anisotropy [6]. The most interesting topic for our study is the metric structure of the cosmic string space-time. The metric leads to a conic space-time. Its locally flat geometry affects non-relativistic systems only through the non-trivial topology. Thus, a non-relativistic particle placed in the surroundings of a straight, infinite and static string will not suffer attraction by the cosmic string gravitational field [8].
For a cosmic string space-time the metric tensor in cylindrical coordinates \((\rho, z, \phi)\) is
\[
g_{\mu\nu} = \begin{pmatrix} 1 & & & \\
 & -1 & & \\
 & & -\rho^2 & \\
 & & & -\rho^2 \sin^2 \theta \end{pmatrix}
\]
where \(0 \leq \rho \leq \infty, -\infty \leq z \leq \infty, -\pi \alpha \leq \phi \leq \pi \alpha\) and \(\alpha = 1 - 4G\mu\), with \(\mu\) being the linear density of the cosmic string. The Laplace-Beltrami operator in these coordinates
\[
\Box \psi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) \psi
\]
take the same form as in the flat space-time. When one takes the non-relativistic limit of a system with the dynamics described by a relativistic quantum equation such as the Dirac or Klein-Gordon equations one will be led to the usual Schrödinger equation but with non-trivial boundary conditions
\[
\Phi(\phi_0) = \Phi(\phi_0 + 2\pi \alpha).
\]
The cosmic deficit angle \(\delta = 2\pi(1 - \alpha)\) shall affect any quantum wave solution that significantly encircles the string.

It has been noted that the cosmic string space-time affects the quantum solutions of central potentials. The Coulomb potential has been considered by [7] in the context of a two-dimensional potential as generated by the cosmic string itself.

Here we consider a general radial problem and define the spherical harmonics taking into account the angular deficit. We apply the results to the \((3+1)\)-D Coulomb and harmonic oscillator problems obtaining the spectra for these potentials. These spectra have been independently obtained in [9]. We consider also a generalization of these problems to a \((N + 1)\)-dimensional space-time with conic topological structure. These can be originated by a \((N - 2)\)-brane of cosmic character. The central potential is added with the origin attached to a point of the brane. The well known relationship between oscillator and coulomb problems [8] is then generalized to hyper-conic space-times.

The structure of the paper is as follows. In section II the solution of the potential problems in \((3+1)\)-Dimensions is addressed. The spherical harmonics with angular deficit are constructed. The complete eigenfunctions for the Coulomb and oscillator problems are presented and the ladder operators for the latter potential introduced. The hidden rotational symmetry discussed. In section III the \((N+1)\) dimensional generalization is performed. The dependency of the quadratic angular momentum Casimir operator eigenvalue on the angular deficit is obtained. The spectra of the Coulomb and oscillator problems are obtained and related. The section IV presents final comments.

II. COULOMB AND QUANTUM OSCILLATOR PROBLEMS IN (3+1)-D CONICAL SPACE-TIME

Let us consider spherical coordinates \((r, \theta, \phi)\) in which the cosmic string metric tensor reads
\[
g_{\mu\nu} = \begin{pmatrix} 1 & & & \\
 & -1 & & \\
 & & -r^2 & \\
 & & & -r^2 \sin^2 \theta \end{pmatrix}
\]
and the Schrödinger equation is written in standard form
\[
\left[ \nabla_r^2 + \frac{2\mu}{\hbar^2} V(r) + \frac{2\mu E}{\hbar^2} \right] \psi(r) - \frac{L^2}{\hbar^2 r^2} \psi(r) = 0,
\]
where \(L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{d}{d \theta} (\sin \theta \frac{d}{d \theta}) + \frac{1}{\sin^2 \theta} \frac{d^2}{d \phi^2} \right] \) is the angular momentum operator.
A. Eigenfunctions and energy spectrum

Let us perform the separation of variables expressing
\[ \psi(\vec{r}) = R(r)Y(\theta, \phi). \] (4)

Substituting in equation (3) we obtain the set of equations
\[ r^2 \frac{d^2}{dr^2} r - \frac{2\mu}{\hbar^2} V(r) + \frac{2\mu}{\hbar^2} r^2 = \ell(\ell + 1) \]
\[ \mathbf{L}^2 Y(\theta, \phi) = \hbar^2 \ell(\ell + 1) Y(\theta, \phi), \] (5)

where \( \ell(\ell + 1) \) is to be specified later. Separating the second equation with \( Y(\theta, \phi) = \Theta(\theta)\Phi(\phi) \) leads to
\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \frac{d}{d\theta} \Theta(\theta) = \lambda^2 \Theta(\theta) \]
\[ \frac{d^2}{d\phi^2} \Phi(\phi) = -\lambda^2 \Phi(\phi), \] (6)

where \( \lambda \) is another parameter to be determined.

1. Spherical harmonics

The presence of the string is displayed in the azimuthal equation which is subjected to the periodic boundary conditions (5). So, we have
\[ \Phi^m_{\alpha}(\phi) = \frac{1}{\sqrt{2\pi \alpha}} e^{im \alpha \phi}, \] (7)

where \( m = 0, \pm 1, \pm 2..., \alpha \in \mathbb{R}^+ \) and we identified \( \lambda = \frac{|m|}{\alpha} \). Requiring regular solutions in \( \theta = 0 \) we obtain \( \ell = k + \frac{|m|}{\alpha} \) and the polar solutions
\[ \Theta^{|m|}_{k+ \alpha}(\theta) = \frac{\Gamma(k+1)}{2\Gamma(k+2\frac{|m|}{\alpha}+1)} \sin \frac{|m|}{\alpha} \theta T^{|m|}_{k+ \alpha}(\cos \theta), \] (8)

where \( k = 0, 1, 2..., u = \cos \theta \) and \( T^{|m|}_{k+ \alpha}(u) \) are the Gegenbauer polynomials. Therefore, the generalization of the spherical harmonics required by the cosmic string space-time are
\[ Y^m_{\ell \alpha}(\theta, \phi) = \frac{(2\ell + 1)\Gamma(\ell + |m| + 1)}{4\pi \alpha \Gamma(\ell + |m| + 1)} \sin \frac{|m|}{\alpha} \theta T^{|m|}_{\ell - |m| \alpha}(\cos \theta) e^{i \frac{|m|}{\alpha} \phi}. \] (9)

For the eigenvalues of quadratic Casimir operator we observe a dependence on two integers and on the angular deficit \( \alpha \)
\[ \ell(\ell + 1) = \left( k + \frac{|m|}{\alpha} \right) \left( k + \frac{|m|}{\alpha} + 1 \right). \] (10)

This dependency of the \( l \) value on each of the states within a specific angular momentum multiplet can be understood since the operators \( \mathbf{L}_x \pm \mathbf{L}_y \) are not operators that act within the Hilbert space functions. They would create states with wrong periodicity conditions. The algebraic construction of the angular momentum states is spoiled from the beggining. We will argue latter that an attempt to redefine these operators to act in the Hilbert space as \((\mathbf{L}_x)^\frac{1}{2}\) will not work.
2. Radial equations

The above procedure is valid for any problem with a radial potential \( V(r) \). Now we choose a particular potential to solve the radial equation

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{2 \mu}{\hbar^2} V(r) + \frac{2 \mu E}{\hbar^2} \right] R(r) = \left( k + \frac{|m|}{\alpha} \right) \left( k + \frac{|m|}{\alpha} + 1 \right) R(r). \tag{11}
\]

Coulomb potential

The Coulomb potential is expressed by \(-\frac{e^2}{r}\). Substituting this in Ed. (13) we have

\[
\left[ \frac{d^2}{d\rho^2} + \frac{2}{\rho} \frac{d}{d\rho} - \frac{\ell(\ell + 1)}{\rho^2} + \frac{\beta}{\rho} - \frac{1}{4} \right] R(\rho) = 0, \tag{12}
\]

where \( \beta \) is an arbitrary parameter, \( \rho = \frac{r}{\beta r_0} \), \( \ell = k + \frac{|m|}{\alpha} \), \( E = -\frac{\epsilon_0}{\beta^2} \); with \( r_0 = \frac{\hbar^2}{2\mu e} \) (Bohr radius divided by two) and \( \epsilon_0 = \frac{\mu e^4}{2\hbar^2} \) (ionization energy). With the ansatz

\[
R(\rho) = C e^{(-\frac{\rho}{\beta})} \rho^{\ell} g(\rho), \tag{13}
\]

we are led to the radial equation for \( g(\rho) \)

\[
\rho \frac{d^2}{d\rho^2} g(\rho) + (2\ell + 2 - \rho) \frac{d}{d\rho} g(\rho) + (\beta - \ell - 1) g(\rho) = 0. \tag{14}
\]

This is the associated Laguerre equation whose normalizable solutions are the associated Laguerre polynomials \( L_{2\ell+1}^{\ell+1}(\rho) = \frac{1}{\Gamma(j+1)} \frac{\Gamma(2\ell+j+2)}{\rho^{\ell+1}} e^\rho \rho^{\ell} \frac{d}{d\rho} \rho^{\ell+2} e^{-\rho} \), with \( j = 0, 1, 2... \). Therefore, the solution for the radial differential equation is

\[
R(r) = C_{j,k,m}^\alpha \left( \frac{1}{\beta r_0} \right)^{k+\frac{|m|}{\alpha}} e^{-\frac{1}{\beta r_0}} L_{2\ell+1}^{\ell+1} \left( \frac{1}{\beta r_0} \right), \tag{15}
\]

where \( \beta \) is given by \( j + k + \frac{|m|}{\alpha} \) and \( C_{j,n,m}^\alpha \) is a normalization constant obtained as

\[
C_{j,k,m}^\alpha = \frac{1}{(j + k + \frac{|m|}{\alpha})} \left[ \frac{\Gamma(j)}{2r_0^3 \left( \Gamma(j + 2k + 2\frac{|m|}{\alpha} + 1) \right)^3} \right]. \tag{16}
\]

For the energy spectrum we obtain

\[
E_{j,k,m}^\alpha = -\frac{\epsilon_0}{(j + k + \frac{|m|}{\alpha})^2}. \tag{17}
\]

Clearly, the essential degeneracy is broken, but there is still an accidental degeneracy associated with a full symmetry of the potential [10]. It is important to point out that the energy levels depend explicitly on the angular deficit \( \alpha \) which characterizes the global structure of the metric.
Quantum harmonic oscillator

Proceeding as done before to the hydrogen atom, we can solve the radial equation (12) for $V(r) = \frac{1}{2} \mu \omega^2 r^2$ given by

$$
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\mu^2 \omega^2 r^2 + 2\mu E}{\hbar^2} \right] R(r) = \left( k + \frac{|m|}{\alpha} \right) \left( k + \frac{|m|}{\alpha} + 1 \right) R(r).
$$

(18)

Making the ansatz

$$
R(x) = C x^\ell e^{-\frac{1}{2}x^2} h(x),
$$

(19)

where $x = \left( \frac{r}{r_0} \right)^2$ and $r_0 = \sqrt{\frac{\hbar}{\mu \omega}}$. And using eq (19) in (18) we have

$$
\frac{d^2}{dx^2} h(x) + \left( \frac{2\ell + 2}{x} - 2x \right) \frac{d}{dx} h(x) + \left( \frac{2E}{\hbar \omega} - 2\ell - 3 \right) h(x) = 0.
$$

(20)

Making a new change in variable $x \rightarrow x' = x^\frac{1}{2}$, we obtain the same equation as in hydrogen atom (14) with solution $h(x^\frac{1}{2}) = L_{\ell+\frac{1}{2}}(x)$. Therefore, the solution for the radial function $R(r)$ is given by

$$
R(r) = C_{j,k,m}^\alpha \left( \frac{r}{r_0} \right)^{k+\frac{|m|}{\alpha}} e^{-\frac{1}{2} \left( \frac{r}{r_0} \right)^2} L_j^{k+\frac{|m|}{\alpha}+\frac{1}{2}} \left( \frac{r^2}{r_0^2} \right),
$$

(21)

and the normalization constant is

$$
C_{j,k,m}^\alpha = \left( \frac{2}{r_0^3} \frac{\Gamma(j+1)}{\Gamma(j+k+\frac{|m|}{\alpha}+\frac{3}{2})} \right)^{\frac{3}{2}}.
$$

(22)

For the energy spectrum we obtain

$$
E_{j,k,m}^\alpha = \hbar \omega (2j + k + \frac{|m|}{\alpha} + \frac{3}{2} + 1)
$$

(23)

Again, there is a dependence on deficit angle and the degeneracy attributed to symmetry rotations (essential degeneracy) is broken. But we see clearly a persistence of the accidental degeneracy related with even $k$ states for $m = 0$.

Creation and Destruction operators

The last section results show that the eigenvalues for the Harmonic Oscillator increase in intervals of $\hbar \omega$ or of $\hbar \omega \alpha$. This suggests to investigate the construction of ladder operators for the Harmonic Oscillator which shall produce these changes. Since the Hilbert space can be constructed as the tensor product $E_{\rho,\phi} \otimes E_z$, it is sufficient to consider the 2-D quantum harmonic oscillator with angular deficit.

The Hamiltonian for 2-D quantum harmonic oscillator $V(\rho) = \frac{1}{2} \mu \omega^2 \rho^2$ without angular deficit can be written in terms of creation and destruction operators of right and left “circular quantum”
\[ H = \hbar \omega (a_g^\dagger a_d + a_g^\dagger a_g + 1), \]  

(24)

these operators are defined in terms of usual \( a_x \) and \( a_y \) operators by

\[
\begin{align*}
a_d &= \frac{1}{\sqrt{2}} (a_x - ia_y) \\
a_g &= \frac{1}{\sqrt{2}} (a_x + ia_y),
\end{align*}
\]

(25)

The non-zero commutators between the four operators \( a_d, a_g, a_d^\dagger, \) and \( a_g^\dagger \) being

\[
[ a_d, a_d^\dagger ] = [ a_g, a_g^\dagger ] = 1.
\]

These relations lead to

\[
\begin{align*}
[ H, (a_d^\dagger)^n a_d(a_g) ] &= -n (a_d^\dagger)^n \hbar \omega \\
[ H, (a_g^\dagger)^n a_g(a_d) ] &= n (a_d^\dagger)^n \hbar \omega.
\end{align*}
\]

(26)

In the case of angular deficits neither the \( a_x(y) \) nor the \( a_d(a_g) \) operators can be defined as operators acting on the Hilbert space. The reason is that they do not respect the periodicity of space, leading states that respect the periodicity, belonging to the Hilbert space, to states that do not respect, outside the Hilbert space. The products between them appearing in (24) are nevertheless well defined operators acting on the Hilbert space since these products do not change the periodicity in angular variables. The decomposition of the Hamiltonian in eq. (26) is allowed in the conic space-time once one considers the space of functions that represents the Hilbert space as embedded in a large space of functions of arbitrary periodicity. In order to properly define ladder operators acting within the Hilbert space it is necessary to define fractionary powers of the usual creation and annihilation operators. These highly non-local operators can be constructed for instance as the infinite series

\[
(a_d^\dagger)^\frac{\alpha}{n} = \lim_{\epsilon \to 0} (\epsilon + a_d^\dagger)^\frac{\alpha}{n} = \lim_{\epsilon \to 0} \sum_{i=0}^{\infty} C_i, \frac{\alpha}{n} (a_d^\dagger)^i
\]

(28)

where the regularization parameter \( \epsilon \) is to be removed after summing the series. In this way it is straightforward to extend the commutation relations (27) for

\[
\begin{align*}
[ H, (a_d(a_g))^\frac{n}{\alpha} ] &= -\frac{n}{\alpha} (a_d(a_g))^\frac{n}{\alpha} \hbar \omega \\
[ H, (a_g^\dagger(a_d))^\frac{n}{\alpha} ] &= \frac{n}{\alpha} (a_g^\dagger(a_d))^\frac{n}{\alpha} \hbar \omega.
\end{align*}
\]

(29)

what allows for an interpretation in terms of fractionary quantum creation and destruction operators. The operator product \((a_g^\dagger a_d^\dagger)^\frac{\alpha}{n}\) also does not change the periodicity condition and can be in principle defined in the Hilbert space. Indeed the axial symmetric states not depending on the angular variable are eigenstates in both the usual and in the conic space cases, being insensitive to the topological defect of the space-time. These states are created by applying this operator product on the fundamental state. We are thus led to construct the general basis state vector as

\[
| n, n', \alpha \rangle \equiv | n, n', \alpha \rangle_{g(d)} = (a_g^\dagger a_d)^n(a_g^\dagger)^{n'}(a_d^\dagger)^{\alpha}(0, 0).
\]

(30)

Here we avoided the use of the fractionary power operators \((a_g^\dagger)^{\frac{\alpha}{n}}\) and \((a_d^\dagger)^{\frac{\alpha}{n}}\) at the same time. Indeed inspection shows that the action of their product on the fundamental state leads to non-normalizable states. Therefore, all states of the model are given by linear combinations of
The energy of the basis states can be calculated, through relations (27) and (29) as

$$E = \hbar \omega (2n + \frac{n'}{\alpha} + 1).$$

(32)

It is also straightforward, using eq. (28), to see that

$$\langle \rho, \phi | 0, n' \rangle_{g(d)}^{\alpha} = \langle \rho, \phi \mid (a_{g(d)}^{\dagger})^{n'} | 0, 0 \rangle$$

(33)

gives the eigenfunctions obtained by the direct solution of the differential equations. Further applying the differential operator \((a_{g(d)}^{\dagger})^{n}\) reproduces all the basis eigenfunctions.

Let us now discuss the Hidden symmetry of the oscillator. It is well known that angular momentum algebra describes the degeneracy of the 2-D oscillator. The “angular momentum” operators are defined as

$$J_\pm = a_{d(g)}^{\dagger} a_{g(d)}$$

(34)

and

$$J_z = \frac{1}{2}(a_{d}^{\dagger} a_{g} - a_{g}^{\dagger} a_{d}).$$

(35)

The Casimir operator \(J^2\) is

$$J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2 = \frac{1}{2}(N_g + N_d)(\frac{1}{2}(N_g + N_d) + 1),$$

(36)

where

$$N_g + N_d = a_{g}^{\dagger} a_{g} + a_{d}^{\dagger} a_{d} = \frac{H}{\hbar \omega} - 1.$$  

(37)

This leads to the identification of the \(J\) quantum number associated to the square of the angular momentum with the eigenvalue of \(\frac{N_g + N_d}{2}\)

$$J = \langle \frac{N_g + N_d}{2} \rangle.$$  

(38)

In the case of angular deficit neither the \(J_+\) nor the \(J_-\) operators are defined as operators acting in the Hilbert space of the 2-D harmonic oscillator spoiling the hidden SU(2) symmetry. These operators should be exchanged by \(J_{\pm}^2\) to act in the Hilbert space. Nevertheless the composite operators \(J^2, J_z, N_g\) and \(N_d\) are bona-fide operators. The relationship expressed by e’s. (36-38) are extended to the deficit angular space case. Let us consider then the action of \(N_g + N_d\) on the basis states eq. (30). Taking \(n' = 2m\) in that equation it is straightforward to see that

$$\frac{N_g + N_d}{2} | n, 2m \rangle_{g(d)}^{\alpha} = (n + \frac{|m|}{\alpha}) | n, 2m \rangle_{g(d)}^{\alpha}. $$

(39)
In other words these states have quantum numbers

\[ j = n + \frac{|m|}{\alpha} \]  

This reproduces exactly the form obtained in the section (2.A) by the resolution of the angular differential equations.

It can be also understood why the operators \((J_\pm)^{1/\alpha}\) do not generate all states of a multiplet. Since \(a^\dagger g(d)\) and \(a g(d)\) appear simultaneously with fractional powers in \((J_\pm)^{1/\alpha}\) they generate non-normalizable functions when applied to the basis states of eq.(30). This is necessary to allow for the change in \(j\) value within the multiplet.

III. (N+1)-DIMENSIONAL GENERALIZATION

In order to construct a N-dimensional generalization for Coulomb and Quantum oscillator problems we consider the metric

\[ ds^2 = dt - (d\rho^2 + \frac{1}{\rho^2} d\phi^2 + dx_3^2 + ... + dx_N^2). \]  

(41)

The variable \(\phi\) is assumed to present an angular deficit, \(\phi \to \phi\alpha\). This space-time generalizes the cosmic string space-time and the \((N-2)\)-brane is considered in \(x_1 = x_2 = 0\). We have taken out one generator of the angular momentum algebra and assumed non-trivial boundary conditions for the orbit it generates in real space breaking thus the \(SO(N)\) symmetry.

In hyper-spherical coordinates the metrics reads:

\[ g^{\mu\nu} = diag(1, -1, r, rsin\theta, rsin\theta sin\phi_1, ..., rsin\theta sin\phi_1...sin\phi_{N-1}) \]  

(42)

where

\[ 0 \leq r \leq \infty \]
\[ 0 \leq \theta \leq \pi \]
\[ -\pi \leq \phi_1 \leq \pi \]
\[ \vdots \]
\[ -\pi\alpha \leq \phi_{N-1} \leq \pi\alpha \]

(43)

In this way the N-dimensional Schrödinger equation can be written by

\[ \left[ \nabla^2_r - \frac{L^2}{\hbar^2 r^2} - \frac{\mu V(r)}{\hbar^2} + \frac{2\mu E}{\hbar^2} \right] \Psi(r) = 0 \]  

(44)

where

\[ \nabla^2_r = \frac{1}{r^{N-1} dr} \left( r^{N-1} \frac{d}{dr} \right) \]  

(45)

and
\[ L^2 = \frac{1}{\sin^{N-2} \theta} \frac{d}{d \theta} \left( \frac{\sin^{N-2} \theta}{\sin \theta} \frac{d}{d \theta} \right) + \frac{1}{\sin^2 \theta \sin^{N-3} \phi_1} \frac{d}{d \phi_1} \left( \frac{\sin^{N-3} \phi_1}{\sin \phi_1} \frac{d}{d \phi_1} \right) + \cdots \]
\[ \cdots + \frac{1}{\sin^2 \theta \sin^{N-3} \phi_1 \cdots \sin^2 \phi_{N-4}} \frac{1}{\sin \phi_{N-3}} \frac{d}{d \phi_{N-3}} \left( \frac{\sin \phi_{N-3}}{\sin \phi_{N-3}} \frac{d}{d \phi_{N-3}} \right) + \frac{1}{\sin^2 \theta \sin^{N-3} \phi_1 \cdots \sin^2 \phi_{N-3}} \frac{d^2}{d \phi_{N-2}^2} \]  

(46)

For \( V(r) \) strictly radial we can perform the separation of variables method \( N \) times and obtain the angular equation

\[ L^2 Y_{n_0, \ldots, n_{N-3}}(\theta, \phi_1, \ldots, \phi_{N-2}) = \ell(\ell + N - 2)Y_{n_0, \ldots, n_{N-3}}(\theta, \phi_1, \ldots, \phi_{N-2}), \]  

(47)

where \( N \) is the number of dimensions. Introducing

\[ k = \sum_{i=0}^{N-3} n_i \]

where the integers \( n_i \) are separation constants the non-trivial boundary condition affects the quantum number \( \ell \) in the same form as in the three-dimensional case

\[ \ell = k + \frac{|m|}{\alpha} \]  

(48)

The radial equation will be

\[ \left[ r^2 \frac{d^2}{dr^2} + r(N - 1) \frac{d}{dr} + r^2 \left( \frac{2\mu E}{\hbar^2} - \frac{\mu}{\hbar^2} V(r) \right) \right] R(r) = \ell(\ell + N - 2)R(r). \]  

(49)

A. N-dimensional solutions for hydrogen atom and quantum harmonic oscillator

Particularizing to the Coulomb problem the radial solutions are:

\[ R(r) = C \left( \frac{r}{r_0} \right) ^\ell e^{-\frac{1}{2} \left( \frac{r}{r_0} \right)} L_{\ell}^{2\ell + N - 2} \left( \frac{r}{k r_0} \right) \]  

(50)

where \( k^2 = -\frac{\epsilon_0}{\epsilon}, \ i = 1, 2 \ldots \) and \( C \) is a normalization constant. The energy spectrum is

\[ E = -\frac{\epsilon_0}{\left( i + \ell + \frac{N-3}{2} \right)^2} \]  

(51)

For the oscillator problem the radial solutions are:

\[ R(r) = C \left( \frac{r}{r_0} \right) ^\ell e^{-\frac{1}{2} \left( \frac{r}{r_0} \right)} ^2 L_{\ell}^{2\ell + N - 2} \left( \frac{r^2}{r_0^2} \right) \]  

(52)

where \( C \) is a normalization constant and and \( i = 1, 2 \ldots \). The energy spectrum is

\[ E = \hbar \omega \left[ \ell + 2i + \frac{N - 4}{2} \right] \]  

(53)

Let us now discuss the relationship between both potential solutions along the lines discussed for trivial topology by [8].
B. Relationship

With the generalization of the space-time above it is straightforward to generalize the mapping of the states of both problems:

|                         | Hydrogen atom | Harmonic oscillator |
|-------------------------|---------------|---------------------|
| radial Variable        | \(\frac{1}{\beta} \frac{r}{r_0}\) | \((\frac{r}{r_0})^2\) |
| energy                 | \((\frac{\epsilon_0}{E})^\frac{1}{2}\) | \(\frac{E'}{2\hbar}\) |
| generalized angular momentum quantum number | \(2\ell\) | \(\ell + \lambda\) |
| spatial dimension      | \(N\) | \(\frac{N'}{2} - \lambda + 1\) |
| azimuthal quantum number | \(2\frac{|m|}{\alpha}\) | \(\frac{|m'|}{\alpha'}\) |
| angular deficit        | \(\alpha\) | \(\alpha'\left(\frac{N'}{2}\right)\) |

where \(\lambda\) allows for a freedom in the mapping.

This mapping reveals that the direct relation of the even states of quantum harmonic oscillator and all states of hydrogen atom is attainable in the space-time with angular deficit. It is to be pointed out that the relation can be established with different angular deficits \(\alpha\) and \(\alpha'\) and azimuthal quantum numbers just keeping \(2|m|/\alpha = |m'|/\alpha'\).

IV. CONCLUSIONS

In this work, we studied the solutions of the Schrödinger equation in cosmic strings-like space-times with a point in the string acting as a source for a radial potential. We performed an extension of the spherical harmonics to the non-trivial space-time. We verified that the global characteristic of the space-time is present explicitly in the structure of states and energy spectrum. The deficit angle splits the degeneracies associated with rotational symmetry in energy spectrum, but the accidental degeneracies are still partially present.

The extension of the algebraic method of construction of the harmonic oscillator states through the introduction of fractionally powered ladder operators allowed the discussion of its hidden symmetry. This sheds some light on the dependency of the angular momentum Casimir operator values on the angular deficit and on the algebraic construction of the angular momentum states. In peculiar \(\alpha\) values the operators here introduced become local. For instance if \(\alpha = 1/2\) the raising operators become \((a^\dagger)^2\) and the obstruction pointed out to the simultaneous use of left and right raising operators disappears. In these cases the presence of the string affects the quantum states simply as a supperselection rule.

Moreover, the remarkable point raised by this study is that the attempt to allow for the relationship between the coulomb potential in 3D and the oscillator problem in higher dimensions, if the 3D space presents conical topology, leads naturally to the construction of the conic space-times of higher dimensions. In this sense a quantum mechanical issue is serving as a guide to relate topological space-times in different dimensions.

The somewhat artificial consideration of the source of the potential point exactly over the string restricts severally any attempt to use these results as a means to detect a real cosmic string.
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