Local dimension theory of tensor products of algebras over a ring

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Abstract

Our main goal in this paper is to set the general frame for studying the dimension theory of tensor products of algebras over an arbitrary ring $R$. Actually, we translate the theory initiated by A. Grothendieck and R. Sharp and subsequently developed by A. Wadsworth on Krull dimension of tensor products of algebras over a field $k$ into the general setting of algebras over an arbitrary ring $R$. For this sake, we introduce and study the notion of a fibred AF-ring over a ring $R$. This concept extends naturally the notion of AF-ring over a field introduced by A. Wadsworth in [14] to algebras over arbitrary rings. We prove that Wadsworth theorems express local properties related to the fibre rings of tensor products of algebras over a ring. Also, given a triplet of rings $(R, A, B)$ consisting of two $R$-algebras $A$ and $B$ such that $A \otimes_R B \neq \{0\}$, we introduce the inherent notion to $(R, A, B)$ of a $B$-fibred AF-ring which allows to compute the Krull dimension of all fiber rings of the considered tensor product $A \otimes_R B$. As an application, we provide a formula for the Krull dimension of $A \otimes_R B$ when $A$ and $B$ are $R$-algebras with $A$ is zero-dimensional as well as for the Krull dimension of $A \otimes_\mathbb{Z} B$ when $A$ is a fibred AF-ring over the ring of integers $\mathbb{Z}$ with nonzero characteristic and $B$ is an arbitrary ring. This enables us to answer a question of Jorge Matinez on evaluating the Krull dimension of $A \otimes_\mathbb{Z} B$ when $A$ is a Boolean ring. Actually, we prove that if $A$ and $B$ are rings such that $A \otimes_\mathbb{Z} B$ is not trivial and $A$ is a Boolean ring, then $\dim(A \otimes_\mathbb{Z} B) = \dim\left(\frac{B}{2B}\right)$.

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1 Introduction

All rings considered in this paper are commutative with identity element and all ring homomorphisms are unital. Here and subsequently, $R$ stands for an arbitrary ring and $k$ stands for a field. Let $A$ be a ring and $p$ be a prime ideal of $A$. Then $\text{Spec}(A)$ denotes the set of all prime ideals of $A$ and $k_A(p)$ denotes the quotient field of $\frac{A}{p}$. Also, if $n \geq 0$ is a positive integer, $A[n]$ denotes the polynomial ring in $n$ indeterminates $A[X_1, X_2, \cdots, X_n]$ and $ht(p[n])$ stands for the height of the extended ideal $p[n] := p[X_1, X_2, \cdots, X_n]$ of $p$. Further, if $A$ is an algebra over a field $k$, then $t.d.(A : k)$ denotes the transcendence degree of $A$ over $k$. Any unreferenced material is standard as in [11], [7] and [15].

It is a paper of R. Sharp on Krull dimension of tensor products of two field extensions of a field $k$ which gave the initial impetus to study the Krull dimension of tensor products. Actually, in [12], Sharp proved that, for any two extension fields $K$ and $L$ of $k$, $\dim(K \otimes_k L) = \min(t.d.(K : k), t.d.(L : k))$ (actually, this result appeared ten years earlier in Grothendieck’s EGA [9, Remarque 4.2.1.4, p. 349]). This formula is rather surprising since, as one may expect, the structure of the tensor product should reflect the way the two components interact and not only the structure of each component. This fact affords motivation to Wadsworth to work on this subject in [14]. He aimed at seeking geometric properties of primes of $A \otimes_k B$ and to widen the scope of $k$-algebras $A$ and $B$ whose tensor product Krull dimension, $\dim(A \otimes_k B)$, shows exclusive dependence on individual characteristics of $A$ and $B$. The algebras which proved to be tractable for Krull dimension computations turned out to be those rings $A$ which satisfy the altitude formula over $k$ (AF-rings for short), that is,

$$ht(p) + t.d.(\frac{A}{p} : k) = t.d.(A_p : k)$$

for all prime ideals $p$ of $A$. The class of AF-rings contains the most basic rings of algebraic geometry, including finitely generated $k$-algebras. Wadsworth proved through [14, Theorem 3.7] that if $A$ is an AF-domain and $B$ is any $k$-algebra, then

$$\dim(A \otimes_k B) = \max \left\{ ht(q[t.d.(A : k)]) + \min \left( t.d.(A : k), ht(p) + t.d.(\frac{B}{q} : k) \right) : p \in \text{Spec}(A) \text{ and } q \in \text{Spec}(B) \right\}.$$ 

As a consequence of this, [14, Theorem 3.8] states that if $A_1$ and $A_2$ are both AF-domains, then

$$\dim(A_1 \otimes_k A_2) = \min \left( \dim(A_1) + t.d.(A_2 : k), \ t.d.(A_1 : k) + \dim(A_2) \right).$$
Further, he gave a result which yields a classification of the prime ideals of $A \otimes_k B$ according to their contractions to $A$ and $B$ (cf. [14] Proposition 2.3). In [1], we continued the work of Wadsworth and transferred all his theorems in [14] on AF-domains to AF-rings. This passage from domains to rings with zero-divisors is well reflected in new formulas for the Krull dimension of tensor products involving AF-rings. As it turns out from the present work, it is these formulas that are relevant in our treatment of the general setting of tensor products over a ring $R$. We refer the reader to [1, 2, 3, 4, 5, 6, 12, 13, 14] for basics and recent investigations on the dimension theory of tensor products of algebras over a field.

The main goal of this paper is to set the general frame to study the dimension theory of tensor products of algebras over an arbitrary ring $R$. Actually, we translate the theory initiated by A. Grothendieck and R. Sharp and subsequently developed by A. Wadsworth on Krull dimension of tensor products of algebras over a field $k$ into the general setting of algebras over an arbitrary ring $R$. It turns out that Wadsworth theorem express local properties related to the fibre rings of the tensor products over an arbitrary ring $R$. For this sake, we introduce and study the notion of a fibred AF-ring over a ring $R$. Actually, we say that an $R$-algebra $A$ is a fibred AF-ring over $R$ if the fibre ring $k_R(p) \otimes_R A$ is an AF-ring over $k_R(p)$ for any prime ideal $p$ of $R$ such that $k_R(p) \otimes_R A \neq \{0\}$. When restricted to tensor products over a field the notion of a fibred AF-ring boils down to the classical one of an AF-ring. It is notable that all finitely generated algebras over $R$ proved to be fibred AF-rings as well as all zero-dimensional rings which are $R$-algebras. We prove that the fibred AF-rings inherit all properties of Wadsworth introduced AF-rings. Moreover, given a triplet of rings $(R, A, B)$ consisting of $R$-algebras $A$ and $B$ such that $A \otimes_R B \neq \{0\}$, we introduce and study the notion of a $B$-fibred AF-ring over $R$ which is a somewhat inherent concept to the given triplet $(R, A, B)$. So, when $A$ is a $B$-fibred AF-ring over $R$, we can explicit the Krull dimension of all fiber rings of $A \otimes_R B$ and in various cases this enables us to determine $\dim(A \otimes_R B)$. As an application, we compute the Krull dimension of $A \otimes_R B$ when $A$ and $B$ are $R$-algebras with $A$ is zero-dimensional. Also, we provide a formula for the Krull dimension of $A \otimes_R B$ when $A$ and $B$ are $R$-algebras with $A$ is zero-dimensional as well as for the Krull dimension of $A \otimes_Z B$ when $A$ is a fibred AF-ring over the ring of integers $Z$ with nonzero characteristic and $B$ is an arbitrary ring. This allows us to answer a question of Jorge Martinez on evaluating the Krull dimension of $A \otimes_Z B$ when $A$ is a Boolean ring. Actually, we prove that if $A$ and $B$ are rings such that $A \otimes_Z B$ is not trivial and $A$ is a Boolean ring, then $\dim(A \otimes_Z B) = \dim\left(\frac{B}{2B}\right)$.

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2 Local spectrum and effective spectrum

This section introduces the effective spectrum of a ring $R$ with respect to an $R$-algebra as well as local notions of well known concepts of dimension theory of rings such that the height of a prime ideal, the Krull dimension and the spectrum of a ring.

Let $R$ be an arbitrary ring and let $A$ be an $R$-algebra. Denote by $f_A : R \to A$, with $f_A(r) = r.1_A$ for any $r \in R$ and where $1_A$ is the unit element of $A$, the ring homomorphism defining the structure of algebra of $A$ over $R$. Let $K := \text{Ker}(f_A)$. It is easily seen that for each prime ideal $P$ of $A$, $K \subseteq p := f_A^{-1}(P)$ and that the induced homomorphisms $\overline{f}_A : \frac{R}{K} \to A$ and $\overline{f}_A : \frac{R}{p} \to \frac{A}{P}$, defined by $\overline{f}_A(\overline{r}) = f_A(r)$ and $\overline{f}_A(\overline{r}) = \overline{f}_A(r)$ for each $r \in R$, are injective. Let $S$ be a multiplicative subset of $R$. Recall that the localization of $A$ by $S$ is the $S^{-1}R$-algebra $S^{-1}A := S^{-1}R \otimes_R A$. Our first result proves that $S^{-1}A$ is isomorphic to a localization of $A$ by a multiplicative subset $\overline{S}$ of $A$. By virtue of this lemma, we deduce that, for any multiplicative subset $S$ of $R$, $S^{-1}A$ enjoys all properties satisfied by the well known localization by a multiplicative subset of $A$.

**Lemma 2.1.** Let $R$ be a ring and $A$ an $R$-algebra. Let $S$ be a multiplicative subset of $R$. Let $\overline{S} := f_A(S)$ denote the the corresponding multiplicative subset of $A$. Then the natural map $\varphi : S^{-1}A \to \overline{S}^{-1}A$ such that $\varphi\left(\sum_{i \in \Lambda} \frac{r_i \otimes_R a_i}{s_i}\right) = \sum_{i \in \Lambda} \frac{\overline{r}_i a_i}{\overline{f}_A(s_i)}$, for any finite set $\Lambda$, any $\{r_i : i \in \Lambda\} \subseteq R$, $\{s_i : i \in \Lambda\} \subseteq S$ and $\{a_i : i \in \Lambda\} \subseteq A$, is an isomorphism of $R$-algebras.

**Proof.** It is easy to see that the mapping $f : S^{-1}R \times A \to \overline{S}^{-1}A$ such that $f\left(\frac{r}{s}, a\right) = \frac{ra}{f_A(s)}$ is well defined and is $R$-biadditive. This yields the existence of the assigned homomorphism $\varphi$ of $R$-algebras. Also, it is routine to check that the map $g : \overline{S}^{-1}A \to S^{-1}R \otimes_R A$ defined by $g\left(\frac{a}{f_A(s)}\right) := g\left(\frac{a}{s,1_A}\right) = \frac{1}{s} \otimes_R a$ for each $a \in A$ and each $s \in S$ is an homomorphism of $R$-algebras. Then observe that $\varphi \circ g = \text{id}_{\overline{S}^{-1}A}$ and $g \circ \varphi = \text{id}_{S^{-1}A}$. Hence $\varphi$ is an isomorphism of $R$-algebras. \qed

The above discussion allows us to announce the following lemma which collects certain properties and facts about fibre rings. These properties were stated in [10]
Lemma 2.2. Let $R$ be a ring and let $A$ be an $R$-algebra. Let $p$ be a prime ideal of $R$. Let $S_p := \frac{R}{p} \setminus \{0\}$. Then

1) $k_R(p) \otimes_R A \cong S_p^{-1} \frac{A}{pA} := S_p^{-1} \frac{A}{f_A(p)A}$ and $k_R(p) \otimes_R P \cong S_p^{-1} \frac{P}{pA}$ for each prime ideal $P$ of $A$ such that $f_A^{-1}(P) = p$.

2) $\text{Spec}(k_R(p) \otimes_R A) = \{k_R(p) \otimes_R P : P \in \text{Spec}(A) \text{ such that } f_A^{-1}(P) = p\}$.

3) There exists an order-preserving bijective correspondence between the spectrum of $k_R(p) \otimes_R A$ and the set of prime ideals of $A$ which contract to $p$ over $R$.

4) Let $P$ be a prime ideal of $A$ and let $p := f_A^{-1}(P)$. Then

\[(k_R(p) \otimes_R A)_{k_R(p) \otimes_R P} \cong k_R(p) \otimes_R A_P.\]

5) Let $P$ be a prime ideal of $A$ and let $p := f_A^{-1}(P)$. Then

\[\frac{k_R(p) \otimes_R A}{k_R(p) \otimes_R P} \cong k_R(p) \otimes_R \frac{A}{P}.\]

Proof. In view of [10, p. 84], it remains to give a proof of (4) and (5).

4) Let $P$ be a prime ideal of $A$ and $p := f_A^{-1}(P)$. Consider the multiplicative subset $T := R \setminus p$ of $R$. Then, by (1),

\[(k_R(p) \otimes_R A)_{k_R(p) \otimes_R P} \cong \left( S_p^{-1} \frac{A}{pA} \right)_{S_p^{-1} \frac{P}{pA}} \]

\[\cong \left( \frac{A}{pA} \right)_{\frac{P}{pA}} \]

\[\cong \frac{A_P}{\frac{P}{pA}} \]

\[\cong \frac{R}{p} \otimes_R A_P.\]

Also, notice that, on the one hand,

\[T^{-1}R \otimes_R \frac{R}{p} \otimes_R A_P \cong T^{-1} \frac{R}{p} \otimes_R A_P \]

\[= S_p^{-1} \frac{R}{p} \otimes_R A_P \]

\[= k_R(p) \otimes_R A_P.\]
and, on the other,
\[ T^{-1}R \otimes_R \frac{R}{p} \otimes_R A_P \cong \frac{R}{p} \otimes_R (T^{-1}R \otimes_R A_P) \]
\[ \cong \frac{R}{p} \otimes_R T^{-1}A_P \]
\[ = \frac{R}{p} \otimes_R A_P \text{ as } f_A(T) \subseteq A \setminus P \text{ and thus each element of } \]
\[ f_A(T) \text{ is invertible in } A_P. \]

It follows that \((k_R(p) \otimes_R A)_{k_R(p) \otimes_R P} \cong k_R(p) \otimes_R A_P\), as desired.

5) Let \(P\) be a prime ideal of \(A\) and \(p := f_A^{-1}(P)\). Then, by (1),
\[ \frac{k_R(p) \otimes_R A}{k_R(p) \otimes_R P} \cong \frac{S^{-1}_p(A/pA)}{S^{-1}_p(P/pA)} \]
\[ \cong \frac{S^{-1}A}{P} \]
\[ \cong k_R(p) \otimes_R \frac{A}{P} \text{ as } p \frac{A}{P} = (0). \]

This completes the proof. \(\square\)

Given an \(R\)-algebra \(A\), it is to be noted that not all the prime ideals of \(R\) are essential in capturing the prime ideal structure of \(A\) over \(R\). Actually, there are prime ideals and chains of prime ideals of \(R\) that have no effect on the structure of the spectrum of \(A\) (see Example 2.5). That is the reason why we introduce in what follows the notion of effective spectrum of \(R\) with respect to \(A\) and effective Krull dimension of \(R\) with respect to \(A\).

**Definition 2.3.** Let \(A\) be an \(R\)-algebra.
1) A prime ideal \(p\) of \(R\) is said to be an effective prime ideal of \(R\) with respect to \(A\) if the fibre ring \(k_R(p) \otimes_R A \neq \{0\}\).
2) We define the effective spectrum of \(R\) with respect to \(A\) to be the set denoted by \(\text{Spec}^A_e(R)\) consisting of all effective prime ideals \(p\) of \(R\) with respect to \(A\), namely,
\[ \text{Spec}^A_e(R) = \{p \in \text{Spec}(R) : k_R(p) \otimes_R A \neq \{0\}\}. \]

Also, we denote by \(\text{Max}^A_e(R)\) the subset of maximal elements of \(\text{Spec}^A_e(R)\), that is, the set of maximal effective prime ideals of \(R\) with respect to \(A\).
3) Let \(p \in \text{Spec}^A_e(R)\). We define the effective height of \(p\) with respect to \(A\), denoted by \(\text{ht}^A_e(p)\), to be the supremum of lengths of chains of effective prime ideals \(p_0 \subseteq p_1 \subseteq \cdots \subseteq p_n = p\) of \(R\) terminating at \(p\).
4) We define the effective Krull dimension of \(R\) with respect to \(A\) to be the invariant
denoted by \( \dim_e^A(R) \) which is the supremum of effective heights of effective prime ideals of \( R \) with respect to \( A \), that is,

\[
\dim_e^A(R) = \sup \{ht_e^A(p) : p \in \text{Spec}_e^A(R) \}.
\]

The following result determines the effective spectrum of a ring \( R \) with respect to various constructions. Its proof is easy and thus omitted.

**Proposition 2.4.** 1) Let \( A \) be an \( R \)-algebra. Then \( \text{Spec}_e^A(R) = f_{A}^{-1}(\text{Spec}(A)) \)

2) Let \( X_1, X_2, \cdots, X_n \) be indeterminates over \( R \). Then \( \text{Spec}_e^{R[X_1, X_2, \cdots, X_n]}(R) = \text{Spec}(R) \).

3) Let \( R \rightarrow A \) be an integral extension of rings. Then \( \text{Spec}_e^A(R) = \text{Spec}(R) \).

4) Let \( S \) be a multiplicative subset of \( R \). Then \( \text{Spec}_{e^{-1}R}(R) = \{ p \in \text{Spec}(R) : p \cap S = \emptyset \} \).

5) Let \( A \) be an \( R \)-algebra and \( S \) be a multiplicative subset of \( A \). Then

\[
\text{Spec}_{e^{-1}A}(R) = f_{A}^{-1}\left( \{ P \in \text{Spec}(A) : P \cap S = \emptyset \} \right) = f_{A}^{-1}(\text{Spec}_{e^{-1}A}(A)) \subseteq \text{Spec}_e^A(R).
\]

6) Let \( I \) be an ideal of \( R \). Then \( \text{Spec}_{e^I}(R) = \{ p \in \text{Spec}(R) : I \subseteq p \} \).

Next, for any positive integer \( n \geq 2 \), we exhibit an example of a ring \( R \) and an \( R \)-algebra \( A \) such that there exists a chain of distinct prime ideals \( p_0 \subset p_1 \subset \cdots \subset p_n \) in \( R \) with both ends \( p_0, p_n \in \text{Spec}_e^A(R) \) while the intermediate elements \( p_2, \cdots, p_{n-1} \notin \text{Spec}_e^A(R) \). It would be interesting to afford a ring \( T \) issued from \( R \) and \( A \) such that \( \text{Spec}(T) = \text{Spec}_e^A(R) \). This task is not easy and it turns out from the next example that \( T \) is neither a localization of \( R \) nor a factor ring of \( R \).

**Example 2.5.** Let \( k \) be a field and \( t \) be an indeterminate over \( k \). It is known, by [10] Lemma 1, that there exists an infinite number of formal power series \( g_1(t), g_2(t), \cdots, g_m(t), \cdots \) of \( k[[t]] \) which are algebraically independent over \( k \). In fact, we can choose the \( g_i \) in the maximal ideal \( t k[[t]] \). Actually, assume that \( g_1(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_m t^m + \cdots \in k[[t]] \setminus t k[[t]] \). Then, observe that \( h_1(t) := g_1(t) - 1 \in t k[[t]] \) and, for each integer \( i \geq 2 \), \( h_i(t) := g_i(t) - g_i(0) g_1(t) \in t k[[t]] \), and the formal power series \( h_1(t), h_2(t), \cdots, h_m(t), \cdots \) are algebraically independent over \( k \) as \( k[h_1(t), h_{i_1}(t), h_{i_2}(t), \cdots, h_{i_n}(t)] = k[g_1(t), g_{i_1}(t), g_{i_2}(t), \cdots, g_{i_n}(t)] \) is a polynomial ring in \( n + 1 \) indeterminates for any finite subset \( \{ i_1, i_2, \cdots, i_n \} \) of \( N \setminus \{ 0 \} \). Therefore let \( g_1(t), g_2(t), \cdots, g_m(t), \cdots \in t k[[t]] \) be algebraically independent elements over \( k \). Let \( n \geq 2 \) be an integer and \( X_1, X_2, \cdots, X_n \) be
indeterminates over $k$. Let $R := k[X_1, X_2, \ldots, X_n]$ and $A := k[[t]]$ be the formal power series ring over $k$ which is a rank one discrete valuation ring and thus $\text{Spec}(A) = \{(0), tk[[t]]\}$. We endow $A$ with the $R$-algebra structure induced by the ring homomorphism $f_A : R \to A$ such that $f_A(X_i) = g_i(t)$ for each $i = 1, 2, \ldots, n$. Therefore $f_A^{-1}(tk[[t]]) = (X_1, X_2, \ldots, X_n)$ is a maximal ideal of $R$ of height $n$. Also, as $g_1, g_2, \ldots, g_n$ are algebraically independent over $k$, it is readily checked that $f_A$ is injective. Hence $f_A^{-1}((0)) = (0)$. It follows that $\text{Spec}_e(A) = \{(0), (X_1, X_2, \ldots, X_n)\}$ and, in particular, any prime ideal of $R$ properly between $(0)$ and the maximal ideal $(X_1, X_2, \ldots, X_n)$ is not effective with respect to $A$, as desired.

**Remark 2.6.** 1) Let $A$ be an $R$-algebra. Then, for any $A$-algebra $B$, the natural map $f_A^{-1} : \text{Spec}_e^B(A) \to \text{Spec}_e^B(R)$ is surjective while, in general, $f_A^{-1} : \text{Spec}(A) \to \text{Spec}(R)$ is not so.

2) Let $A$ be an $R$-algebra. Then $\dim^A(R) \leq \dim(R)$ and this inequality might be strict as proved by Example 2.5 which shows that for any positive integer $n \geq 2$ there exists a ring $R$ and an $R$-algebra $A$ such that $\dim(R) = n$ while $\dim^A(R) = 1$, as desired.

To get prepared for the general setting of tensor products, we next introduce local notions of well known concepts of the dimension theory of rings, namely the height of a prime ideal of a ring $A$, the spectrum of $A$ and the Krull dimension of $A$.

**Definition 2.7.** Let $R$ be a ring and $A$ be an $R$-algebra.

1) Let $p \in \text{Spec}_e^A(R)$. Then

a) $\text{Spec}_p(A) := \{I \in \text{Spec}(A) : f_A^{-1}(I) = p\}$ denotes the set of all prime ideals of $A$ which contract to $p$ over $R$.

b) If $I \in \text{Spec}_p(A)$, then the height of $I$ at $p$, denoted by $\text{ht}_p(I)$, is the maximum of lengths of chains $I_0 \subset I_1 \subset \cdots \subset I_n = I$ of prime ideals of $A$ which contract to $p$ over $R$.

c) The Krull dimension of $A$ at $p$ is the invariant

$$\dim_p(A) := \max\{\text{ht}_p(I) : I \in \text{Spec}_p(A)\}.$$  

2) We define the fibre Krull dimension of $A$ with respect to $R$ to be the maximal length of chains of prime ideals of $A$ lying over a common (effective) prime ideal of $R$ (with respect to $A$), that is the invariant

$$f-\dim_R(A) = \sup\{\dim_p(A) : p \in \text{Spec}_e^A(R)\}.$$  

By virtue of Lemma 2.2, we get the following result which connects the above local data of $A$ with those relative to fibre rings issued from $A$. 
Corollary 2.8. Let \( R \) be a ring and \( A \) be an \( R \)-algebra. Let \( p \in \text{Spec}(R) \). Then

1) There exists an order-preserving bijective correspondence between \( \text{Spec}(k_R(p) \otimes_R A) \) and \( \text{Spec}_e^A(p) \).
2) If \( I \in \text{Spec}_e^p(A) \), then \( \text{ht}_p(I) = \text{ht}(k_R(p) \otimes_R I) \).
3) \( \text{dim}_p(A) = \text{dim}(k_R(p) \otimes_R A) \).
4) \( \text{f-dim}_R(A) = \sup\{ \text{dim}(k_A(p) \otimes_R A) : p \in \text{Spec}_e^A(R) \} \).

Next, given an \( R \)-algebra \( A \), we give lower and upper bounds of the Krull dimension of \( A \) in terms of the Krull dimension of its fibre rings and the effective Krull dimension of \( R \) with respect to \( A \). Observe that the formulas given in the following theorem are reminiscent of Seidenberg’s inequalities for the Krull dimension of polynomial rings.

Recall that a ring homomorphism \( f : A \to B \) is said to satisfy the Going-Down property (GD for short) if for any prime ideals \( p \leq q \) of \( A \) such that there exists \( Q \in \text{Spec}(B) \) with \( Q \cap A = q \), then there exists \( P \in \text{Spec}(B) \) such that \( P \cap A = p \) and \( P \subset Q \). It is then easy to see that if a ring homomorphism \( f : R \to A \) satisfies GD and if \( p \in \text{Spec}_e^A(R) \), then any prime ideal \( q \) of \( R \) such that \( q \subset p \) is an effective prime ideal of \( R \) with respect to \( A \), and thus \( \text{ht}_e^A(p) = \text{ht}(p) \).

Theorem 2.9. Let \( R \) be a ring and let \( A \) be an \( R \)-algebra. Then

1) \( \text{f-dim}_R(A) \leq \text{dim}(A) \leq \text{f-dim}_R(A) + (1 + \text{f-dim}_R(A)) \text{dim}_e^A(R) \).
2) If the homomorphism \( f_A : R \to A \) satisfies the Going-down property, then,

\[
\sup\{ \text{ht}(p) + \text{dim}_p(A) : p \in \text{Spec}_e^A(R) \} \leq \text{dim}(A) \leq \text{f-dim}_R(A) + (1 + \text{f-dim}_R(A)) \text{dim}_e^A(R). 
\]

Proof. 1) It suffices to prove the second inequality. If either \( \text{dim}_e^A(R) = +\infty \) or \( \text{f-dim}_R(A) = +\infty \), then we are done. Assume that \( \text{dim}_e^A(R) < +\infty \) and \( \text{f-dim}_R(A) < +\infty \). Let \( P_0 \subset P_1 \subset \cdots \subset P_n \) be a chain of distinct prime ideals of \( A \). Then the corresponding chain of contractions \( p_0 \subset p_1 \subset \cdots \subset p_r \) is composed of effective prime ideals of \( R \) with respect to \( A \). Observe that the number of the \( P_i \)'s lying over a fixed prime \( p_j \) is inferior than or equal to the Krull dimension of the fibre ring \( k_R(p_j) \otimes_R A \) plus one, that is, \( \text{dim}_p(A) + 1 \), and \( \text{dim}_p(A) \leq \text{f-dim}_R(A) \). Further, as \( p_0 \subset p_1 \subset \cdots \subset p_r \) is a chain composed of \( 1 + r \) effective prime ideals of \( R \) with respect to \( A \) and as \( r \leq \text{dim}_e^A(R) \), we get

\[
n \leq r(\text{f-dim}_R(A) + 1) + \text{f-dim}_R(A) \\
\leq \text{f-dim}_R(A) + (\text{f-dim}_R(A) + 1) \text{dim}_e^A(R). 
\]

This yields the desired inequality.

2) Assume that \( f_A \) satisfies GD. Let \( n := \text{ht}_e^A(p) \) and \( p_0 \subset p_1 \subset \cdots \subset p_n = p \) be a chain of distinct effective primes of \( R \) with respect to \( A \). Fixing \( P \in \text{Spec}_e^p(A) \) and applying the Going-down property yields the existence of a chain \( P_0 \subset P_1 \subset \cdots \subset P_n \) composed of effective prime ideals of \( R \) with respect to \( A \). Observe that the number of the \( P_i \)'s lying over a fixed prime \( p_j \) is inferior than or equal to the Krull dimension of the fibre ring \( k_R(p_j) \otimes_R A \) plus one, that is, \( \text{dim}_p(A) + 1 \), and \( \text{dim}_p(A) \leq \text{f-dim}_R(A) \). Further, as \( p_0 \subset p_1 \subset \cdots \subset p_r \) is a chain composed of \( 1 + r \) effective prime ideals of \( R \) with respect to \( A \) and as \( r \leq \text{dim}_e^A(R) \), we get

\[
n \leq r(\text{f-dim}_R(A) + 1) + \text{f-dim}_R(A) \\
\leq \text{f-dim}_R(A) + (\text{f-dim}_R(A) + 1) \text{dim}_e^A(R). 
\]

This yields the desired inequality.
for each prime ideal $p$ for each prime ideal $p$ for each effective prime ideal $p$

Next, we list various applications and consequences of Theorem 2.9. The first result gives a condition for coincidence of the Krull dimension of $A$ and its fiber Krull dimension with respect to $R$.

**Corollary 2.10.** Let $A$ be an $R$-algebra. If $\dim^A_e(R) = 0$, then

$$\dim(A) = f\dim R(A).$$

We aim via the following corollaries to recover Seidenberg’s inequalities for polynomial rings. The next result might be termed Seidenberg’s inequalities for algebras over an arbitrary ring.

**Corollary 2.11.** Let $A$ be an $R$-algebra such that $f_A$ satisfies GD. Assume that $f\dim R(A) = \dim^a_m(A)$ for each $m \in \max^A_e(R)$. Then

$$f\dim R(A) + \dim^A_e(R) \leq \dim(A) \leq f\dim R(A) + (1 + f\dim R(A))\dim^A_e(R).$$

**Proof.** Observe, by Theorem 2.9, that, in particular, $\sup\{\ht^A_e(m) + \dim^a_m(A) : m \in \max^A_e(R)\} \leq \dim(A)$. Then, as $f\dim R(A) = \dim^a_m(A)$ for each $m \in \max^A_e(R)$, $\sup\{\ht^A_e(m) : m \in \max^A_e(R)\} + f\dim R(A) \leq \dim(A)$. Therefore

$$\dim^A_e(R) + f\dim R(A) \leq \dim(A)$$

establishing the desired inequalities.

Next, we recover Seidenberg’s inequalities for polynomial rings.

**Corollary 2.12.** Let $R$ be a ring and let $X_1, X_2, \ldots, X_n$ be indeterminates over $R$. Then

$$n + \dim(R) \leq \dim(R[X_1, X_2, \ldots, X_n]) \leq n + (n + 1)\dim(R).$$

**Proof.** Observe that the homomorphism $R \rightarrow R[X_1, X_2, \ldots, X_n]$ satisfies GD and $\spec R[X_1, \ldots, X_n](R) = \spec(R)$, thus $\dim^a_{R[X_1, \ldots, X_n]}(R) = \dim(R)$. Also,

$$\dim_p(R[X_1, X_2, \ldots, X_n]) = \dim(k_R(p) \otimes R[X_1, X_2, \ldots, X_n])$$

$$= \dim(k_R(p)[X_1, X_2, \ldots, X_n]) = n$$

for each prime ideal $p$ of $R$. Then

$$f\dim(R[X_1, X_2, \ldots, X_n]) = \dim_p(R[X_1, X_2, \ldots, X_n]) = n$$

for each prime ideal $p$ of $R$. Now, Corollary 2.11 completes the proof.  

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Local dimension theory of algebras

10

\[ \cdots \subset P_n = P \text{ such that each } P_i \in \spec_{p_i}(A) \text{ for } i = 0, 1, \ldots, n - 1. \text{ Then } h^A_e(p) = 0 \leq h^1(P) \text{ for each } P \in \spec_p(A). \text{ Let } Q_0 \subset Q_1 \subset \cdots \subset Q_r \text{ be a chain of } \spec_p(A) \text{ such that } r := \dim(k_R(p) \otimes R A) = \dim_p(A). \text{ Therefore, as by the first step } h^A_e(p) = 0 \leq h(Q_0), \text{ we get} \]

$$h^A_e(p) + \dim^a_p(A) \leq h(Q_0) + r \leq h(Q_r) \leq \dim(A)$$

for each effective prime ideal $p$ of $R$. Hence, by (1), as $h^A_e(p) = h(P)$, we get the desired inequalities completing the proof.  

Next, we recover Seidenberg’s inequalities for polynomial rings. The first result gives a condition for coincidence of the Krull dimension of $A$ and its fiber Krull dimension with respect to $R$.

**Corollary 2.10.** Let $A$ be an $R$-algebra. If $\dim^A_e(R) = 0$, then

$$\dim(A) = f\dim R(A).$$

We aim via the following corollaries to recover Seidenberg’s inequalities for polynomial rings. The next result might be termed Seidenberg’s inequalities for algebras over an arbitrary ring.

**Corollary 2.11.** Let $A$ be an $R$-algebra such that $f_A$ satisfies GD. Assume that $f\dim R(A) = \dim^a_m(A)$ for each $m \in \max^A_e(R)$. Then

$$f\dim R(A) + \dim^A_e(R) \leq \dim(A) \leq f\dim R(A) + (1 + f\dim R(A))\dim^A_e(R).$$

**Proof.** Observe, by Theorem 2.9, that, in particular, $\sup\{\ht^A_e(m) + \dim^a_m(A) : m \in \max^A_e(R)\} \leq \dim(A)$. Then, as $f\dim R(A) = \dim^a_m(A)$ for each $m \in \max^A_e(R)$, $\sup\{\ht^A_e(m) : m \in \max^A_e(R)\} + f\dim R(A) \leq \dim(A)$. Therefore

$$\dim^A_e(R) + f\dim R(A) \leq \dim(A)$$

establishing the desired inequalities.

Next, we recover Seidenberg’s inequalities for polynomial rings.

**Corollary 2.12.** Let $R$ be a ring and let $X_1, X_2, \ldots, X_n$ be indeterminates over $R$. Then

$$n + \dim(R) \leq \dim(R[X_1, X_2, \ldots, X_n]) \leq n + (n + 1)\dim(R).$$

**Proof.** Observe that the homomorphism $R \rightarrow R[X_1, X_2, \ldots, X_n]$ satisfies GD and $\spec R[X_1, \ldots, X_n](R) = \spec(R)$, thus $\dim^a_{R[X_1, \ldots, X_n]}(R) = \dim(R)$. Also,

$$\dim_p(R[X_1, X_2, \ldots, X_n]) = \dim(k_R(p) \otimes R[X_1, X_2, \ldots, X_n])$$

$$= \dim(k_R(p)[X_1, X_2, \ldots, X_n]) = n$$

for each prime ideal $p$ of $R$. Then

$$f\dim(R[X_1, X_2, \ldots, X_n]) = \dim_p(R[X_1, X_2, \ldots, X_n]) = n$$

for each prime ideal $p$ of $R$. Now, Corollary 2.11 completes the proof.  

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3 Local and global Transcendence degree over an arbitrary ring

In this section we introduce the local and global transcendence degree of algebras over an arbitrary ring.

Recall that if \( k \) is a field, then it is customary to denote by

\[
\text{t.d.}(A : k) := \sup \{ \text{t.d.}(\frac{A}{p} : k) : p \in \text{Spec}(A) \}
\]

the transcendence degree of a \( k \)-algebra \( A \) over \( k \). This section aims at giving a definition of the notion of the transcendence degree of an \( R \)-algebra \( A \) over \( R \) in accordance with the field case.

In the same spirit of Definition 2.7 and in order to prepare the ground for the general case of tensor products over an arbitrary ring, we next introduce a local notion of the transcendence degree of an \( R \)-algebra \( A \) over \( R \) as well as the “general” transcendence degree of \( A \) over \( R \) which turns out to be in total accordance with the well known notion of the transcendence degree over a field.

**Definition 3.1.** Let \( R \) be a ring and \( A \) an \( R \)-algebra.

1) Let \( p \in \text{Spec}^A(R) \). We define the transcendence degree at \( p \) of \( A \) over \( R \) to be the transcendence degree of the fibre ring \( k_R(p) \otimes_R A \) (over \( p \)) over the field \( k_R(p) \), that is,

\[
\text{t.d.}_p(A : R) := \text{t.d.}(k_R(p) \otimes_R A : k_R(p)).
\]

2) We define the transcendence degree of \( A \) over \( R \) to be the supremum of the transcendence degrees of \( A \) over \( R \) at effective prime ideals \( p \) of \( R \) with respect to \( A \), that is,

\[
\text{t.d.}(A : R) := \sup \{ \text{t.d.}_p(A : R) : p \in \text{Spec}^A(R) \}
\]

We will next prove that the transcendence degree of an \( R \)-algebra \( A \) over \( R \) depends on the endowing structure of algebra of \( A \) over \( R \), namely on the ring homomorphism \( f_A \).

The following proposition shows that the transcendence degree over a ring \( R \) at an effective prime ideal of \( R \) shares all known properties of the transcendence degree over a field \( k \). Notice that, when \( P \) is a prime ideal of an \( R \)-algebra \( A \) and \( p := f_A^{-1}(P) \), then \( f_A \) induces an isomorphism between \( \frac{R}{p} \) and a subring of \( \frac{A}{P} \) which means that \( \frac{R}{p} \) might be identified with a subring of \( \frac{A}{P} \).
**Proposition 3.2.** Let $A$ be an $R$-algebra.

1) If $P$ is a prime ideal of $A$ and $p := f_A^{-1}(P)$, then

\[ \text{t.d.}_p\left( \frac{A}{P} : R \right) = \text{t.d.}\left( \frac{A}{P} : \frac{R}{p} \right) = \text{t.d.} \left( k_A(P) : k_R(p) \right). \]

2) Let $p \in \text{Spec}^A_e(R)$. Then,

\[ \text{t.d.}_p\left( A : R \right) = \sup \left\{ \text{t.d.}_p\left( \frac{A}{P} : R \right) : P \in \text{Spec}_p(A) \right\}. \]

3) If $P$ is a prime ideal of $A$ with $p := f_A^{-1}(P)$, then

\[ \text{t.d.}_p\left( A_P : R \right) = \sup \left\{ \text{t.d.}_p\left( \frac{A}{Q} : R \right) : Q \in \text{Spec}_p(A) \text{ such that } Q \subseteq P \right\} \]

\[ = \sup \left\{ \text{t.d.} \left( k_A(Q) : k_R(p) \right) : Q \in \text{Spec}_p(A) \text{ with } Q \subseteq P \right\} \]

4) $\text{t.d.} \left( A : R \right) = \sup \left\{ \text{t.d.}_p\left( \frac{A}{P} : R \right) : p \in \text{Spec}(R) \text{ and } P \in \text{Spec}_p(A) \right\}$. 

**Proof.** Let $S_p = \frac{R}{p} \setminus \{0\}$ for each prime ideal $p$ of $R$.

1) Let $P$ be a prime ideal of $A$ and let $p := f_A^{-1}(P)$. Then,

\[ \text{t.d.}_p\left( \frac{A}{P} : R \right) = \text{t.d.} \left( k_R(p) \otimes_R \frac{A}{P} : k_R(p) \right) \]

\[ = \text{t.d.} \left( S_p^{-1} \frac{A}{P} : k_R(p) \right) \quad (\text{Lemma } 2.2(1)) \]

\[ = \text{t.d.} \left( S_p^{-1} \frac{A}{P} : k_R(p) \right) \quad \text{as } p \frac{A}{P} = f_A(p) \frac{A}{P} = \{0\} \]

\[ = \text{t.d.} \left( \frac{A}{P} : R \right), \]

as desired.

2) Let $p \in \text{Spec}^A_e(R)$. Then, by (1) and Lemma 2.2((2) and (5)),

\[ \text{t.d.}_p\left( A : R \right) := \text{t.d.} \left( k_R(p) \otimes_R A : k_R(p) \right) \]

\[ = \sup \left\{ \text{t.d.} \left( k_R(p) \otimes_R \frac{A}{P} : k_R(p) \right) : P \in \text{Spec}_p(A) \right\} \]

\[ = \sup \left\{ \text{t.d.} \left( k_R(p) \otimes_R \frac{A}{P} : k_R(p) \right) : P \in \text{Spec}_p(A) \right\} \]

\[ = \sup \left\{ \text{t.d.}_p\left( \frac{A}{P} : R \right) : P \in \text{Spec}_p(A) \right\} \]

\[ = \sup \left\{ \text{t.d.}_p\left( \frac{A}{P} : \frac{R}{p} \right) : P \in \text{Spec}_p(A) \right\}. \]
Local dimension theory of algebras

3) Let $P$ be a prime ideal of $A$ and $p := f_A^{-1}(P)$. Then, by (1) and (2),
\[
\text{t.d.}_p(A_P : R) = \sup \{ \text{t.d.}(\frac{A_P}{Q} : \frac{R}{P}) : Q \in \text{Spec}_p(A) \text{ such that } Q \subseteq P \}
\]
\[
= \sup \{ \text{t.d.}(\frac{A}{Q} : \frac{R}{P}) : Q \in \text{Spec}_p(A) \text{ such that } Q \subseteq P \}
\]
\[
= \sup \{ \text{t.d.}_p(\frac{A}{Q} : R) : Q \in \text{Spec}_p(A) \text{ such that } Q \subseteq P \}.
\]

4) It follows easily from (1) and (2) completing the proof.

It is notable that the introduced notion of transcendence degree of $A$ over $R$ depends on the structure of $R$-algebra over $A$, namely, on the ring homomorphism $f_A$, as shown by the following simple example.

Example 3.3. Let $k$ be a field and $X$ an indeterminate over $k$. Let $A = k[X]$ and $B = k[X]$ and let $A = k[X]$. Consider the following ring homomorphisms $f_1, f_2 : R \rightarrow A$ such that $f_1(g(X), \alpha) = g(X)$ and $f_2(g(X), \alpha) = \alpha$. These two homomorphisms define two different $R$-algebra structures over $A$. Moreover, observe that $f_1^{-1}((0)) = (0) \times k$ and $f_2^{-1}((0)) = k[X] \times (0)$. Then $\text{Spec}^{(A,f_1)}_e(R) = \{(0) \times k\}$ and $\text{Spec}^{(A,f_2)}_e(R) = \{k[X] \times (0)\}$. Hence, by Proposition 3.2(4),
\[
\text{t.d.}(A : f_1 R) = \text{t.d.}(k(X) : \frac{k[X] \times k}{(0) \times k})
\]
\[
= \text{t.d.}(k(X) : k) = 0
\]
while
\[
\text{t.d.}(A : f_2 R) = \text{t.d.}(k(X) : \frac{k[X] \times k}{k[X] \times (0)})
\]
\[
= \text{t.d.}(k(X) : k) = 1.
\]

4 Effective spectrum with respect to tensor products

Let $R$ be a ring and let $A$ and $B$ be $R$-algebras. First, it is worth to note that the tensor product $A \otimes_R B$ over $R$ might be trivial even if $A$ and $B$ are not so. Of course, the interesting case is when $A \otimes_R B \neq \{0\}$ which makes it legitimate to introduce the notion of a triplet of rings $(R, A, B)$ consisting of a given ring $R$ and two $R$-algebras $A$ and $B$ such that $A \otimes_R B \neq \{0\}$.

Let $A$ and $B$ be $R$-algebras. We denote by $\mu_A : A \rightarrow A \otimes_R B$ and $\mu_B : B \rightarrow A \otimes_R B$ the canonical algebra homomorphisms over $A$ and $B$, respectively, such that $\mu_A(a) = a \otimes_R 1$ and $\mu_B(b) = 1 \otimes_R b$ for each $a \in A$ and each $b \in B$. Observe that the following diagram (D) is commutative:
From this section onward, given ideals $I, J, H$ of $A, B$ and $A \otimes_R B$, respectively, we adopt the following notation for easiness: $I \cap R := f_A^{-1}(I)$, $J \cap R := f_B^{-1}(J)$ and $H \cap A := \mu_A^{-1}(H)$, $H \cap B := \mu_B^{-1}(H)$.

We begin by recording the following isomorphisms related to the fiber rings of the tensor products over an arbitrary ring $R$.

**Lemma 4.1.** Let $R$ be a ring. Let $A$ and $B$ be algebras over $R$. Then

$$k_R(p) \otimes_R (A \otimes_R B) \cong (k_R(p) \otimes_R A) \otimes_{k_R(p)} (k_R(p) \otimes_R B) \cong S_p^{-1} \left( \frac{A}{pA} \right) \otimes (B/pB)$$

where $S_p := R \setminus \{0\}$.

**Proof.** It is direct by Lemma 2.2(1). □

**Remark.** Let $R$ be a ring and $A, B$ be two $R$-algebras. Let $I$ be a prime ideal of $A$. Notice that, by considering the ring homomorphism $\mu_A : A \to A \otimes_R B$, $k_A(I) \otimes_R B \cong k_A(I) \otimes_A (A \otimes_R B)$ stands for the fibre ring of $A \otimes_R B$ over $I$. Thus $f\text{-}dim_A(A \otimes_R B) = \sup \{\dim(k_A(I) \otimes_R B) : I \in \text{Spec}_{e}^{A \otimes_R B}(A)\}$.

The next theorem examines the effective spectrum of a ring with respect to tensor products.

**Theorem 4.2.** Let $R$ be a ring. Let $A$ and $B$ be two $R$-algebras. Let $p \in \text{Spec}(R)$, $I \in \text{Spec}(A)$ and $J \in \text{Spec}(B)$. Then

1) $\text{Spec}_{e}^{A \otimes_R B}(R) = \text{Spec}_{e}^{A}(R) \cap \text{Spec}_{e}^{B}(R)$.

2) There exists a prime ideal $P$ of $A \otimes_R B$ such that $P \cap A = I$ and $P \cap B = J$ if and only if $I \cap R = J \cap R$.

3) $\text{Spec}_{e}^{A \otimes_R B}(A) = \{I \in \text{Spec}(A) : I \cap R \in \text{Spec}_{e}^{B}(R)\}$
   $= \{I \in \text{Spec}(A) : \exists J \in \text{Spec}(B) \text{ such that } I \cap R = J \cap R\}$. 
Local dimension theory of algebras

Proof. 1) Let \( p \in \text{Spec}^{A \otimes_R B}(R) \). Then there exists a prime ideal \( P \) of \( A \otimes_R B \) such that \( p = P \cap R \). Let \( I = P \cap A \) and \( J = P \cap B \). Then, by the above commutative diagram (D), \( I \) and \( J \) are prime ideals of \( A \) and \( B \), respectively, such that \( I \cap R = J \cap R = P \cap R = p \), that is, \( p \in \text{Spec}^A(R) \cap \text{Spec}^B(R) \). Conversely, assume that \( p \in \text{Spec}^A(R) \cap \text{Spec}^B(R) \). Hence, by Lemma 4.1, the fibre ring

\[
k_R(p) \otimes_R (A \otimes_R B) \cong (k_R(p) \otimes_R A) \otimes_{k_R(p)} (k_R(p) \otimes_R B) \neq \{0\}
\]

as the fibre rings \( k_R(p) \otimes_R A \) and \( k_R(p) \otimes_R B \) are not trivial. It follows that \( p \in \text{Spec}^{A \otimes_R B}(R) \), as desired.

2) See [N Corollaire 3.2.7.1.(i)].

3) First, let \( I \in \text{Spec}(A) \) such that \( I \cap R \in \text{Spec}^B(R) \). Then, there exists \( J \in \text{Spec}(B) \) such that \( I \cap R = J \cap R \) so that by (2), there exists \( P \in \text{Spec}(A \otimes_R B) \) such that \( P \cap A = I \) (and \( P \cap B = J \)). Thus \( I \in \text{Spec}^{A \otimes_R B}(A) \). Conversely, let \( I \in \text{Spec}^{A \otimes_R B}(A) \). Then there exists \( P \in \text{Spec}(A \otimes_R B) \) such that \( P \cap A = I \), so that, using the above commutative diagram (D), \( I \cap R = P \cap R = (P \cap B) \cap R \in \text{Spec}^B(R) \) completing the proof.

The following corollary totally characterizes when two algebras \( A \) and \( B \) over a ring \( R \) constitute a triplet \((R, A, B)\) of rings.

Corollary 4.3. Let \( R \) be a ring and \( A, B \) be two \( R \)-algebras. Then the following assertions are equivalent:

1) \((R, A, B)\) is a triplet of rings;
2) \(\text{Spec}^A(R) \cap \text{Spec}^B(R) \neq \emptyset\);
3) There exists a prime ideal \( I \) of \( A \) and a prime ideal \( J \) of \( B \) such that \( I \cap R = J \cap R \).

Proof. 1) \(\Leftrightarrow\) 2) It suffices to observe that, by Proposition 2.4(1) and Theorem 4.2,

\[
f_{A \otimes_R B}^{-1}(\text{Spec}(A \otimes_R B)) = \text{Spec}^{A \otimes_R B}(R) = \text{Spec}^A(R) \cap \text{Spec}^B(R).
\]

2) \(\Leftrightarrow\) 3) It is direct.

It is easy to provide examples of nontrivial algebras over a ring \( R \) such that \( A \otimes_R B = \{0\} \) is trivial. But Corollary 4.3 characterizes when this tensor product \( A \otimes_R B \) is trivial by checking connections between the spectrum of the three components of this construction, namely \( \text{Spec}(R) \), \( \text{Spec}(A) \) and \( \text{Spec}(B) \). For instance, given a ring \( R \) and two distinct prime ideals \( p \) and \( q \) of \( R \), applying Corollary 4.3, note that \( k_R(p) \otimes_R k_R(q) = \{0\} \) since \( \text{Spec}^{k_R(p)}(R) = \{p\} \) while \( \text{Spec}^{k_R(q)}(R) = \{q\} \) and thus \( \text{Spec}^{k_R(p)}(R) \cap \text{Spec}^{k_R(q)}(R) = \emptyset \).
Corollary 4.4. Let \((R, A, B)\) be a triplet of rings. Then
\[
\dim_c^{A \otimes_R B}(R) \leq \min(\dim_c^A(R), \dim_c^B(R)).
\]
In particular, if either \(\dim_c^A(R) = 0\) or \(\dim_c^B(R) = 0\), then \(\dim_c^{A \otimes_R B}(R) = 0\).

Proof. It is straightforward from Theorem 4.2 as \(\text{Spec}_c^{A \otimes_R B}(R) = \text{Spec}_c^A(R) \cap \text{Spec}_c^B(R)\).

The next proposition allows us to give lower and upper bounds of the Krull dimension of tensor products of algebras over a ring in terms of the Krull dimension of its fibre rings and the effective Krull dimension of its components.

Proposition 4.5. Let \((R, A, B)\) be a triplet of rings. Then
1) \(f\dim_R(A \otimes_R B) \leq \dim(A \otimes_R B) \leq f\dim_R(A \otimes_R B) + \left(1 + f\dim_R(A \otimes_R B)\right) \dim_c^{A \otimes_R B}(R)\).
2) \(f\dim_A(A \otimes_R B) \leq \dim(A \otimes_R B) \leq f\dim_A(A \otimes_R B) + \left(1 + f\dim_A(A \otimes_R B)\right) \dim_c^{A \otimes_R B}(A)\).

Proof. It follows from Theorem 2.9(1).

In light of Proposition 4.5, when the effective Krull dimension of a component of a tensor product with respect to this construction is zero, the Krull dimension of the tensor product turns out to be its own fibre Krull dimension. This result will allow us to explicit the Krull dimension of the tensor products involving the ahead introduced notion of fibred AF-rings.

Corollary 4.6. Let \((R, A, B)\) be a triplet of rings.
1) If \(\dim_c^{A \otimes_R B}(R) = 0\), then
\[
\dim(A \otimes_R B) = f\dim_R(A \otimes_R B).
\]
2) If \(\dim_c^{A \otimes_R B}(A) = 0\), then
\[
\dim(A \otimes_R B) = f\dim_A(A \otimes_R B).
\]

Proof. It follows easily from Proposition 4.5.

5 Fibred AF-rings

This section introduces and studies the notion of fibred AF-rings over an arbitrary ring \(R\). This new concept extends that of AF-ring over a field introduced by A. Wadsworth in [14].
Definition 5.1. Let $R$ be a ring.
1) Let $A$ be an $R$-algebra and $p \in \text{Spec}_e^A(R)$. $A$ is said to be a fibred AF-ring at $p$ if its fiber ring $k_R(p) \otimes_R A$ is an AF-ring over $k_R(p)$.
2) An $R$-algebra $A$ is said to be a fibred AF-ring over $R$ if it is a fibred AF-ring at each effective prime ideal $p$ of $R$ with respect to $A$, that is, if each nontrivial fibre ring $k_R(p) \otimes_R A$ is an AF-ring over $k_R(p)$.
3) Let $(R, A, B)$ be a triplet of rings.
   a) If $p \in \text{Spec}_e^A(R) \cap \text{Spec}_e^B(R)$ and $A$ is a fibred AF-ring at $p$, then $A$ is said to be a $B$-fibred AF-ring at $p$.
   b) $A$ is said to be a $B$-fibred AF-ring over $R$ if $A$ is a $B$-fibred AF-ring at each common effective prime ideal $p$ of $R$ with respect to $A$ and $B$.

Remark. 1) From this definition, it is clear that the notion of a fibred AF-ring extends that of an AF-ring introduced by Wadsworth as any algebra $A$ over a field $k$ possesses only one fiber ring over $k$ which is $A$ itself.
2) Note that the notion of $B$-fibred AF-ring is inherent to the considered triplet of rings $(R, A, B)$ and it is easy to see that the following assertions are equivalent:
   a) $A$ is a fibred AF-ring over $R$;
   c) $A$ is a $B$-fibred AF-ring over $R$ for any $R$-algebra $B$ such that $(R, A, B)$ is a triplet;
   c) $A$ is an $R$-fibred AF-ring over $R$.

Let $k$ be a field and $A$ be a $k$-algebra. Recall that, for each prime ideal $P$ of $A$, $ht(P) + \text{t.d.} \left( \frac{A}{P} : k \right) \leq \text{t.d.}(A_P : k)$ and $A$ is said to be an AF-ring if this inequality turns out to be an equality, that is, $ht(P) + \text{t.d.} \left( \frac{A}{P} : k \right) = \text{t.d.}(A_P : k)$ for each prime ideal $P$ of $A$. Next, we show that these properties translate into local data for algebras over an arbitrary ring $R$.

Proposition 5.2. Let $R$ be a ring and let $p$ be a prime ideal of $R$. Let $A$ be an $R$-algebra.
1) If $P$ is a prime ideal of $A$ such that $P \cap R = p$, then
   $$ht_p(P) + \text{t.d.}_p \left( \frac{A}{P} : R \right) \leq \text{t.d.}_p(A_P : R).$$
2) Let $B$ be an $R$-algebra such that $(R, A, B)$ is a triplet of rings. Assume that $p \in \text{Spec}_e^A(R)$ (resp., $p \in \text{Spec}_e^A(R) \cap \text{Spec}_e^B(R)$). Then $A$ is a fibred AF-ring (resp., $B$-fibred AF-ring) at $p$ if and only if
   $$ht_p(P) + \text{t.d.}_p \left( \frac{A}{P} : R \right) = \text{t.d.}_p(A_P : R)$$
for each prime ideal $P$ of $A$ such that $p = P \cap R$. 

Proof. 1) Let $P$ be a prime ideal of $A$ such that $P \cap R = p$. Then, by Lemma 2.2,

$$ht_p(P) + t.d_p\left(\frac{A}{P} : R\right) = ht(k_R(p) \otimes_R P) + t.d.\left(\frac{k_R(p) \otimes_R A}{k_R(p) \otimes_R P} : k_R(p)\right)$$

$$= ht(k_R(p) \otimes_R P) + t.d.\left(\frac{k_R(p) \otimes_R A}{k_R(p) \otimes_R P} : k_R(p)\right)$$

$$\leq t.d.\left((k_R(p) \otimes_R A)_{k_R(p) \otimes_R P} : k_R(p)\right)$$

$$= t.d.\left(k_R(p) \otimes_R A : k_R(p)\right)$$

$$= t.d.\left(A_P : R\right),$$

as desired.

2) It is direct from Definition 5.1 taking into account the following equalities which figure in Lemma 2.2: $t.d.\left(A_P : R\right) = t.d.\left(\frac{k_R(p) \otimes_R A}{k_R(p) \otimes_R P} : k_R(p)\right)$ and $t.d.\left(k_R(p) \otimes_R A : k_R(p)\right)$ for each prime ideal $P$ of $A$ with $p := P \cap R$.

The following result exhibits various classes of fibred AF-rings.

**Proposition 5.3.** 1) If $k$ is a field, then the fibred AF-rings over $k$ are exactly the AF-rings over $k$.

2) Any finitely generated $R$-algebra $R[x_1, x_2, ..., x_n]$ is a fibred AF-ring over $R$.

3) Let $(R, A, B)$ be a triplet of rings. If $\dim_{A \otimes_R B}(A) = 0$, then $A$ is a $B$-fibred AF-ring over $R$.

4) Any zero-dimensional ring $A$ which is an $R$-algebra is a fibred AF-ring over $R$.

**Proof.** 1) It is straightforward.

2) Let $A = R[X_1, X_2, ..., X_n]$ be a polynomial ring in $n$-variables over $R$. Let $p \in \text{Spec}(R)$. Then $k_R(p) \otimes_R A \cong k_R(p)[X_1, X_2, ..., X_n]$ is clearly an AF-ring over $k_R(p)$ [14] Corollary 3.2]. Hence $A$ is a fibred AF-ring over $R$. Now, let $A = R[x_1, x_2, ..., x_n]$ be any finitely generated $R$-algebra and let $p \in \text{Spec}(R)$. Then $A \cong R[x_1, X_2, ..., X_n]_I$ for some ideal $I$ of $R[X_1, X_2, ..., X_n]$ and thus, by Lemma
2.2, 
\[ k_R(p) \otimes_R A \cong k_R(p) \otimes_R \frac{R[X_1, X_2, \ldots, X_n]}{I} \]
\[ \cong (k_R(p) \otimes_R R[X_1, X_2, \ldots, X_n]) \otimes_R \frac{R[X_1, X_2, \ldots, X_n]}{I} \]
\[ \cong k_R(p)[X_1, X_2, \ldots, X_n] \subsetneq k_R(p)[y_1, y_2, \ldots, y_n], \text{ where } y_i := X_i \]
is a finitely generated \( k_R(p) \)-algebra which is an AF-ring over \( k_R(p) \) by \[14\] page 395. Hence \( A \) is a fibred AF-ring over \( R \) proving (2).

3) Let \((R, A, B)\) be a triplet of rings such that \( \dim (A \otimes_R B) = 0 \). Let \( p \in \text{Spec}_e^A(R) \cap \text{Spec}_e^B(R) = \text{Spec}_e^{A \otimes_R B}(R) \). Let \( P \) be a prime ideal of \( A \) such that \( P \cap R = p \). Then, by Proposition 3.2,
\[ \text{t.d.}_p(A_P : R) = \text{sup} \left\{ \text{t.d.}\left(\frac{A}{Q} : \frac{R}{p}\right) : Q \in \text{Spec}_p(A) \text{ with } Q \subseteq P \right\} \]
Let \( Q \subseteq P \) be a prime ideal of \( A \) with \( Q \cap R = p \). Then, as \( p \in \text{Spec}_e^{A \otimes_R B}(R) \), by Theorem 4.2(3), \( Q \in \text{Spec}_e^{A \otimes_R B}(A) \). Now, since \( \dim (A \otimes_R B) = 0 \), we get \( Q = P \). Therefore, by Proposition 3.2,
\[ \text{t.d.}_p(A_P : R) = \text{t.d.}\left(\frac{A}{P} : \frac{R}{p}\right) = \text{t.d.}_p\left(\frac{A}{P} : R\right) \]
Also, as \( \dim (A \otimes_R B) = 0 \), \( \text{ht}\left(\frac{P}{pA}\right) = 0 \) since any prime ideal \( Q \) such that \( pA \subseteq Q \subseteq P \) is an effective prime ideal of \( A \) with respect to \( A \otimes_R B \) (see Theorem 4.2(3)).
Then, by Corollary 2.8, \( \text{ht}_p(P) = \text{ht}\left(\frac{P}{pA}\right) = 0 \). It follows that
\[ \text{ht}_p(P) + \text{t.d.}_p\left(\frac{A}{P} : R\right) = \text{t.d.}_p(A_P : R) \]
Then, by Proposition 5.2, \( A \) is a \( B \)-fibred AF-ring over \( R \), as desired.

4) Note that if \( \dim(A) = 0 \), then, in particular, \( \dim_e^{A \otimes_R B}(A) = 0 \) for any triplet \((R, A, B)\) of rings. Hence, by (3), \( A \) is a \( B \)-fibred AF-ring over \( R \) for any \( R \)-algebra \( B \) such that \((R, A, B)\) is a triplet of rings. In particular for \( B = R \), \( A \) is an \( R \)-fibred AF-ring over \( R \) which means that \( A \) is a fibred AF-ring over \( R \), as desired. \[ \square \]

We next establish the stability of the fibred AF-ring notion under various type of constructions.
Proposition 5.4. Let $R$ be a ring and let $A$ be an $R$-algebra. Let $p$ be a prime ideal of $R$.
1) If $A$ is a fibred AF-ring at $p$ and $S$ is a multiplicative subset of $A$ such that $p \in \text{Spec}_{e}S^{-1}A(R)$, then the localization $S^{-1}A$ is a fibred AF-ring at $p$.
2) Let $A_1, A_2, \ldots, A_n$ be fibred AF-rings at $p$. Then $A_1 \otimes_R A_2 \otimes_R \cdots \otimes_R A_n$ is a fibred AF-ring at $p$.
3) If $A$ is a fibred AF-ring at $p$, then the polynomial ring $A[X_1, X_2, \ldots, X_n]$ is a fibred AF-ring at $p$.

Proof. 1) Let $A$ be a fibred AF-ring at $p$ and $S$ be a multiplicative subset of $A$ such that $k_R(p) \otimes_R S^{-1}A \neq \{0\}$. Then, by Proposition 5.4, there exists $P \in \text{Spec}(A)$ such that $P \cap S = \emptyset$ and $p = P \cap R$. Let $\overline{S}$ be the image of $S$ via the homomorphism $A \to \frac{A}{pA}$. Observe that $pA \cap S = \emptyset$ and $\frac{A}{pA} \cap \overline{S} = \emptyset$. Then, by Lemma 2.2,

$$
\text{ht}_p(S^{-1}P) + \text{t.d.}_p\left(\frac{S^{-1}A}{S^{-1}P} : R\right) = \text{ht}\left(\frac{S^{-1}P}{pS^{-1}A}\right) + \text{t.d.}\left(\frac{S^{-1}A}{S^{-1}P} : R \mid p\right)
$$

$$
= \text{ht}\left(\frac{S^{-1}P}{pA}\right) + \text{t.d.}\left(\frac{A}{pA} : R \mid p\right)
$$

$$
= \text{ht}_p(P) + \text{t.d.}_p\left(\frac{A}{pA} : R \mid p\right)
$$

as $A$ is a fibred AF-ring at $p$. Moreover, note that $\text{t.d.}_p((S^{-1}A)_{S^{-1}P} : R) = \text{t.d.}_p(A_P : R)$. It follows that

$$
\text{ht}_p(S^{-1}P) + \text{t.d.}_p\left(\frac{S^{-1}A}{S^{-1}P} : R\right) = \text{t.d.}_p((S^{-1}A)_{S^{-1}P} : R).
$$

Hence $S^{-1}A$ is a fibred AF-ring at $p$.

2) Let $p$ be a prime ideal of $R$. Then

$$
k_R(p) \otimes_R (A_1 \otimes_R \cdots \otimes_R A_n) \cong (k_R(p) \otimes_R A_1) \otimes_{k_R(p)} (k_R(p) \otimes_R (A_2 \otimes_R \cdots \otimes_R A_n))
$$

$$
\cong (k_R(p) \otimes_R A_1) \otimes_{k_R(p)} \cdots \otimes_{k_R(p)} (k_R(p) \otimes_R A_n).
$$

First, as each $k_R(p) \otimes_R A_i \neq \{0\}$, $k_R(p) \otimes_R (A_1 \otimes_R A_2 \otimes_R \cdots \otimes_R A_n) \neq \{0\}$. Hence, since each $k_R(p) \otimes_R A_i$ is an AF-ring over $k_R(p)$, we get, by [14] Proposition 3.1, $(k_R(p) \otimes_R A_1) \otimes_{k_R(p)} \cdots \otimes_{k_R(p)} (k_R(p) \otimes_R A_n)$ is an AF-ring over $k_R(p)$ so that $k_R(p) \otimes_R (A_1 \otimes_R \cdots \otimes_R A_n)$ is an AF-ring over $k_R(p)$. It follows that $A_1 \otimes_R \cdots \otimes_R A_n$ is a fibred AF-ring at $p$.

3) It follows easily from (2) as $A[X_1, X_2, \ldots, X_n] \cong R[X_1, X_2, \ldots, X_n] \otimes_R A$ and, by Proposition 5.3(2), $R[X_1, X_2, \ldots, X_n]$ is a fibred AF-ring at $p$. □
Corollary 5.5. Let $R$ be a ring.
1) If $A$ is a fibred AF-ring over $R$ and $S$ is a multiplicative subset of $A$, then the localization $S^{-1}A$ is a fibred AF-ring over $R$.
2) Let $A_1, A_2, ..., A_n$ be fibred AF-rings over $R$. Then $A_1 \otimes_R A_2 \otimes_R \cdots \otimes_R A_n$ is a fibred AF-ring over $R$.
3) If $A$ is a fibred AF-ring over $R$, then the polynomial ring $A[X_1, X_2, ..., X_n]$ is a fibred AF-ring over $R$.

The following two corollaries give the $B$-fibred AF-ring versions of the above Proposition 5.4 and Corollary 5.5. Their proofs are straightforward.

Corollary 5.6. Let $(R, A, B)$ be a triplet of rings. Let $p$ be a prime ideal of $R$.
1) If $A$ is a $B$-fibred AF-ring at $p$ and $S$ is a multiplicative subset of $A$ such that $p \in \text{Spec} S^{-1}A(R)$, then the localization $S^{-1}A$ is a $B$-fibred AF-ring at $p$.
2) Let $A_1, A_2, ..., A_n$ be $B$-fibred AF-rings at $p$. Then $A_1 \otimes_R A_2 \otimes_R \cdots \otimes_R A_n$ is a $B$-fibred AF-ring at $p$.
3) If $A$ is a $B$-fibred AF-ring at $p$, then the polynomial ring $A[X_1, X_2, ..., X_n]$ is a $B$-fibred AF-ring at $p$.

Corollary 5.7. Let $(R, A, B)$ be a triplet of rings.
1) If $A$ is a $B$-fibred AF-ring over $R$ and $S$ is a multiplicative subset of $A$, then the localization $S^{-1}A$ is a $B$-fibred AF-ring over $R$.
2) Let $A_1, A_2, ..., A_n$ be $B$-fibred AF-rings over $R$. Then $A_1 \otimes_R A_2 \otimes_R \cdots \otimes_R A_n$ is a $B$-fibred AF-ring over $R$.
3) If $A$ is a $B$-fibred AF-ring over $R$, then the polynomial ring $A[X_1, X_2, ..., X_n]$ is a $B$-fibred AF-ring over $R$.

6 Krull dimension of tensor products involving fibred AF-rings

The goal of this section is to discuss and compute the Krull dimension of the tensor product of algebras over $R$ involving fibred AF-rings in various settings.

The following theorem allows to compute the Krull dimension of all fibre rings of the tensor product of algebras $A$ and $B$ over a ring $R$ in the case when $A$ is a fibred AF-ring over $R$. This result translates Wadsworth theorem [14, Theorem 3.7] into the general setting of tensor products over an arbitrary ring $R$. We give the next more general version of a triplet $(R, A, B)$ of rings such that $A$ is a $B$-fibred AF-ring over an effective prime ideal $p$ of $R$.
Notation. 1) Let $A$ be a ring and $P$ be a prime ideal of $A$. Let $n \geq 1$ be a positive integer. Then, for easiness of notation, we denote by $A[n]$ the polynomial ring in $n$ indeterminates $A[X_1, X_2, \cdots, X_n]$ and by $P[n]$ the extended prime ideal $P[X_1, X_2, \cdots, X_n]$ of $A[X_1, X_2, \cdots, X_n]$.

2) Let $A$ be an algebra over a field $k$. Let $0 \leq d \leq s$ be positive integers. Then, in [14], Wadsworth adopted the following notation:

$$D(s, d, A) := \sup \left\{ \text{ht}(P[s]) + \min \left( s, d + \text{t.d.} \left( \frac{A}{P} : k \right) \right) : P \in \Spec(A) \right\}.$$  

3) Let $A$ be an algebra over a ring $R$ and $p$ be a prime ideal of $R$. Let $0 \leq d \leq s$ be positive integers. Then, we adopt the following notation for a local invariant of the above $D(s, d, A)$:

$$D_p(s, d, A) := D(s, d, k_R(p) \otimes_R A).$$

We begin by expliciting the local invariant $D_p(s, d, A)$ in terms of the local invariants of the height and transcendence degree.

**Lemma 6.1.** Let $R$ be a ring. Let $A$ be an algebra over $R$ and $p$ be a prime ideal of $R$. Let $0 \leq d \leq s$ be positive integers. Then

$$D_p(s, d, A) = \sup \left\{ \text{ht}_p(P[s]) + \min \left( s, d + \text{t.d.}_p \left( \frac{A}{P} : R \right) \right) : P \in \Spec_p(A) \right\}.$$  

It is worth noting that if $A$ and $B$ are algebras over a ring $R$ and $X_1, X_2, \cdots, X_n$ are indeterminates, then

$$(A \otimes_R B)[X_1, X_2, \cdots, X_n] \cong A \otimes_R B \otimes_R R[X_1, X_2, \cdots, X_n] \cong A[X_1, X_2, \cdots, X_n] \otimes_R B \cong A \otimes_R B[X_1, X_2, \cdots, X_n].$$

**Proof.** Observe that, using Lemma 2.2 and Corollary 2.8,

$$D_p(s, d, A) = D(s, d, k_R(p) \otimes_R A)$$

$$= \sup \left\{ \text{ht}(k_R(p) \otimes_R P)[s] + \min \left( s, d + \text{t.d.} \left( \frac{k_R(p) \otimes_R A}{k_R(p) \otimes_R P} : k_R(p) \right) \right) : P \in \Spec_p(A) \right\}$$

$$= \sup \left\{ \text{ht}(k_R(p) \otimes_R (P[s])) + \min \left( s, d + \text{t.d.} \left( k_R(p) \otimes_R \frac{A}{P} : k_R(p) \right) \right) : P \in \Spec_p(A) \right\}$$

$$= \sup \left\{ \text{ht}_p(P[s]) + \min \left( s, d + \text{t.d.}_p \left( \frac{A}{P} : R \right) \right) : P \in \Spec_p(A) \right\},$$

as desired. \qed
Recall that Wadsworth proved in \cite{14} that, given a field $k$, if $A$ is an AF-domain and $B$ is any $k$-algebra, then
\[ \dim(A \otimes_k B) = D(t.d.(A), \dim(A), B) \] \cite{14}, Theorem 3.7.\]

We generalized this result in \cite{1} to AF-rings by proving that if $A$ is an AF-ring and $B$ is any $k$-algebra, then,
\[ \dim(A \otimes_k B) = \sup \left\{ D\left( t.d.(A_P : k), \dim(A_P), B \right) : P \in \text{Spec}(A) \right\} \] \cite{1}, Theorem 1.4.\]

Our first main result gives a new version of the above-cited Wadsworth theorem in the general setting of tensor products of algebras over an arbitrary ring $R$.

**Theorem 6.2.** Let $(R, A, B)$ be a triplet of rings and let $p \in \text{Spec}^A(R) \cap \text{Spec}^B(R)$. Assume that $A$ is a $B$-fibred AF-ring at $p$. Then
\[ \dim_p(A \otimes_R B) = \sup \left\{ D_p\left( t.d.(A_I : R), ht_p(I), B \right) : I \in \text{Spec}_p(A) \right\}. \]

**Proof.** Observe that
\[ k_R(p) \otimes_R (A \otimes_R B) \cong (k_R(p) \otimes_R A) \otimes_{k_R(p)} (k_R(p) \otimes_R B) \]
and that $k_R(p) \otimes_R A$ is an AF-ring over the field $k_R(p)$. Then, using \cite{1} Theorem 1.4, we get, by Lemma 2.2(4),
\[ \dim(k_R(p) \otimes_R (A \otimes_R B)) = \dim \left( (k_R(p) \otimes_R A) \otimes_{k_R(p)} (k_R(p) \otimes_R B) \right) \]
\[ = \sup \left\{ D\left( t.d.\left( (k_R(p) \otimes_R A) \otimes_{k_R(p)} (k_R(p) \otimes_R B) : I \in \text{Spec}_p(A) \right) \right) \]
\[ = \sup \left\{ D\left( t.d.(A_I : R), ht_p(I), k_R(p) \otimes_R B \right) : I \in \text{Spec}_p(A) \right\} \]
\[ = \sup \left\{ D_p\left( t.d.(A_I : R), ht_p(I), B \right) : I \in \text{Spec}_p(A) \right\}, \]
as desired.\]

The following corollaries compute the Krull dimension of tensor products involving algebras whose (effective) Krull dimension is zero. It is clear that if $\dim(A) = 0$, then for any nontrivial $A$-algebra $C$, $\dim^e_C(A) = 0$. Also, in Example 6.6, we record the existence of various cases of triplets of rings $(R, A, B)$ such that either $\dim^e_{A \otimes_R B}(R) = 0$ or $\dim^e_{A \otimes_R B}(A) = 0$.\]
Corollary 6.3. Let \((R, A, B)\) be a triplet of rings such that \(A\) is a \(B\)-fibred AF-ring and \(\dim_{e}^{A \otimes R B}(R) = 0\) (in particular, \(\dim(R) = 0\)). Then,
\[
\dim(A \otimes R B) = \sup\left\{ D_p\left(t.d_p(A_I : R), ht_p(I), B\right) : p \in \Spec_e^A(R) \cap \Spec_e^B(R) \text{ and } I \in \Spec_p(A) \right\}.
\]
\[\text{Proof.} \text{ As } \dim_{e}^{A \otimes R B}(R) = 0, \text{ by Corollary 4.6(1),}
\[
\dim(A \otimes R B) = f\dim_{R}(A \otimes R B)
\]
\[= \sup\{\dim_p(A \otimes R B) : p \in \Spec_{e}^{A \otimes R B}(R)\}.\]

Then, Theorem 6.2 completes the proof. \(\Box\)

Corollary 6.4. Let \((R, A, B)\) be a triplet of rings such that \(A\) is a \(B\)-fibred AF-ring and either \(\dim_{e}^{A}(R) = 0\) or \(\dim_{e}^{B}(R) = 0\). Then,
\[
\dim(A \otimes R B) = \sup\left\{ D_p\left(t.d_p(A_I : R), ht_p(I), B\right) : p \in \Spec_e^A(R) \cap \Spec_e^B(R) \text{ and } I \in \Spec_p(A) \right\}.
\]
\[\text{Proof.} \text{ It is straightforward by Corollary 4.4 and Corollary 6.3.} \Box\]

We devote the following theorem to the case where one component of a tensor product is a zero-dimensional ring.

Theorem 6.5. Let \((R, A, B)\) be a triplet of rings such that \(\dim_{e}^{A \otimes R B}(A) = 0\) (in particular, \(\dim(A) = 0\)). Then
\[
\dim(A \otimes R B) = \sup\left\{ D_p\left(t.d_p(k_A(I) : R), 0, B\right) : p \in \Spec_e^A(R) \cap \Spec_e^B(R) \text{ and } I \in \Spec_p(A) \right\}.
\]
\[\text{Proof.} \text{ As } \dim_{e}^{A \otimes R B}(A) = 0, \text{ by Corollary 4.6(2),}
\[
\dim(A \otimes R B) = f\dim_{A}(A \otimes R B)
\]
\[= \sup\{\dim(k_A(I) \otimes R B) : I \in \Spec_{e}^{A \otimes R B}(A)\}.\]

Let \(I \in \Spec_{e}^{A \otimes R B}(B)\) and \(p := I \cap R\). Note that, by Proposition 5.3, \(k_A(I)\), being zero-dimensional, is a fibred AF-ring over \(R\). Also, as \(I \in \Spec_{e}^{A \otimes R B}(A)\), then \(p \in \Spec_{e}^{A \otimes R B}(R)\). Therefore, Theorem 6.2 yields
\[
\dim_p(k_A(I) \otimes R B) = D_p(t.d_p(A_I : R), ht_p(I), B).
\]
Now, since \( p \in \text{Spec}^{A \otimes_R B}(R) \), by Theorem 4.2(3), any \( J \in \text{Spec}_p(A) \) is an effective prime ideal of \( A \) with respect to \( A \otimes_R B \). Therefore, since \( \dim^{A \otimes_R B}(A) = 0 \), we get

\[
\text{ht}_p(I) = \text{ht}(\frac{I}{pA}) = 0.
\]

Furthermore, as \( \dim^{k_A(p)}(R) = 0 \), we get, by Corollary 4.4, \( \dim^{k_A(p) \otimes_R B}(R) = 0 \). It follows, by Corollary 4.6(1) and as \( k_A(I) \otimes_R B \) possesses only one fibre ring which is \( k_R(p) \otimes_R (k_A(I) \otimes_R B) \), that

\[
\dim(k_A(I) \otimes_R B) = t \cdot \dim_R(k_A(I) \otimes_R B) = \dim_p(k_A(I) \otimes_R B) = D_p(t.d.(A_I : R), \text{ht}_p(I), B) = D_p(t.d.(A_I : R), 0, B).
\]

Consequently,

\[
\dim(A \otimes_R B) = \sup \{ D_p(t.d.(A_I : R), 0, B) : p \in \text{Spec}^{A \otimes_R B}(R) \text{ and } I \in \text{Spec}_p(A) \}
\]

completing the proof as, by Theorem 3.3, \( \text{Spec}^{A \otimes_R B}(R) = \text{Spec}_p^A(R) \cap \text{Spec}_p^B(R) \).

**Example 6.6.** 1) Let \( R := \mathbb{Z} \) and \( A, B \) be rings such that \( (\mathbb{Z}, A, B) \) is a triplet of rings. Let \( \text{char}(A) = n \) and \( \text{char}(B) = m \) such that \( n \neq 0 \) and \( m \neq 0 \). Then \( \text{Spec}_e \mathbb{Z}(\otimes_A B) = \{ p\mathbb{Z} : p \text{ is a common prime divisor of } n \text{ and } m \} \) and thus \( \dim_e \mathbb{Z}(\otimes_A B) = 0 \).

2) Let \( R := \mathbb{Z} \). Let \( n \geq 1 \) be an integer. Let \( A = \mathbb{Z}_{p\mathbb{Z}} + X \mathbb{Q}[[X]] \) be a \( D + m \) construction issued from the local ring \( \mathbb{Q}[[X]] = \mathbb{Q} + X \mathbb{Q}[[X]] \). Let \( B \) be a \( \mathbb{Z}_{p\mathbb{Z}} \)-algebra and let \( p \) be a prime divisor of \( n \). Then, \( \text{Spec}(A) = \{ (0), X \mathbb{Q}[[X]], p\mathbb{Z}_{p\mathbb{Z}} + X \mathbb{Q}[[X]] \} \) and \( \text{Spec}_e \mathbb{Z}(A) = \{ p\mathbb{Z} \} \) and \( \text{Spec}_e \mathbb{Z}(B) = \{ p\mathbb{Z}_{p\mathbb{Z}} + X \mathbb{Q}[[X]] \} \). It follows that \( \dim_e A(\otimes_B \mathbb{Z}) = 0 \) and \( \dim_e B(\otimes_{A \otimes_B} \mathbb{Z}) = 0 \).

Next, we deal with tensor products over the ring of integers \( \mathbb{Z} \). This allows us to answer a question raised by Jorge Martinez on evaluating the Krull dimension of the tensor product over \( \mathbb{Z} \) of two rings one of which is a Boolean ring.

**Corollary 6.7.** Let \( (\mathbb{Z}, A, B) \) be a triplet of rings such that \( \text{char}(A) =: n \neq 0 \). Assume that \( A \) is a \( B \)-fibred AF-ring over \( \mathbb{Z} \). Let \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \) be the decomposition of \( n \) into prime factors. Then

\[
\dim(A \otimes_\mathbb{Z} B) = \sup \left\{ D_{p_i}(t.d.(A_I : \mathbb{Z}), \text{ht}_{p_i}(I), B) : i = 1, 2, \cdots, n \text{ and } I \in \text{Spec}_{p_i}(A) \right\}.
\]
**Proof.** Observe that, as \( \text{char}(A) = n \), \( \frac{\mathbb{Z}}{n\mathbb{Z}} \) is identified to a subring of \( A \) and that \( \text{Spec}\left( \frac{\mathbb{Z}}{n\mathbb{Z}} \right) = \left\{ \frac{p_1\mathbb{Z}}{n\mathbb{Z}}, \frac{p_2\mathbb{Z}}{n\mathbb{Z}}, \ldots, \frac{p_r\mathbb{Z}}{n\mathbb{Z}} \right\} \). Then, for each \( j = 1, 2, \ldots, r \), \( \frac{p_j\mathbb{Z}}{n\mathbb{Z}} \) is a minimal prime ideal of \( \frac{\mathbb{Z}}{n\mathbb{Z}} \) and thus there exists \( I_j \in \text{Spec}(A) \) such that \( I_j \cap \frac{\mathbb{Z}}{n\mathbb{Z}} = \frac{p_j\mathbb{Z}}{n\mathbb{Z}} \). Hence, for each \( j = 1, 2, \ldots, r \), there exists \( I_j \in \text{Spec}(A) \) such that \( I_j \cap \mathbb{Z} = p_j\mathbb{Z} \). Therefore \( \text{Spec}^A_\mathbb{Z}(\mathbb{Z}) = \{ p_1\mathbb{Z}, p_2\mathbb{Z}, \ldots, p_r\mathbb{Z} \} \). Thus \( \dim_A^A(\mathbb{Z}) = 0 \). Now, Corollary 6.4 completes the proof.

We close with the following corollary which presents an answer to a question of Jorge Martinez on evaluating the Krull dimension of the tensor product over the ring of integers \( \mathbb{Z} \) of two rings one of which is Boolean.

First, we record the following well known characteristics of Boolean rings.

**Lemma 6.8.** Let \( R \) be a Boolean ring. Then
1) \( R \) is commutative.
2) \( \text{char}(R) = 2 \).
3) \( \dim(R) = 0 \).
4) \( \frac{R}{p} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \) for each prime ideal \( p \) of \( R \).
5) \( \frac{R}{p} \) is an algebraic field extension of \( \frac{\mathbb{Z}}{2\mathbb{Z}} \) for each prime ideal \( p \) of \( R \).

**Corollary 6.9.** Let \((\mathbb{Z}, A, B)\) be a triplet of rings such that \( A \) is a Boolean ring. Then
\[
\dim(A \otimes \mathbb{Z} B) = \dim\left( \frac{B}{2B} \right).
\]

**Proof.** Using the proof of Corollary 6.7, we get \( \text{Spec}_\mathbb{Z}(A) = \{ 2\mathbb{Z} \} \). Also, as \( \dim(A) = 0 \), by Proposition 5.3, \( A \) is a fibred AF-ring over \( \mathbb{Z} \). Moreover,
\[
\text{t.d.}_{2\mathbb{Z}}(A : \mathbb{Z}) = \sup \left\{ \text{t.d.}\left( \frac{A}{I} : \frac{\mathbb{Z}}{2\mathbb{Z}} \right) : I \in \text{Spec}(A) \right\} = 0
\]
as \( \frac{A}{I} \) is algebraic over \( \frac{\mathbb{Z}}{2\mathbb{Z}} \), by Lemma 6.8(5). It follows, by Theorem 6.5, Lemma 6.1 and Lemma 6.8, that
\[
\dim(A \otimes \mathbb{Z} B) = \dim(\frac{B}{2B}) = \sup \left\{ \text{ht}\left( \frac{J}{2B} \right) : J \in \text{Spec}_{2\mathbb{Z}}(B) \right\} = \dim\left( \frac{B}{2B} \right)
\]
completing the proof.
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