Tightness results for infinite-slit limits of the chordal Loewner equation

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July 30, 2018

Abstract

In this note we consider a multi-slit Loewner equation with constant coefficients that describes the growth of multiple SLE curves connecting $N$ points on $\mathbb{R}$ to infinity within the upper half-plane. For every $N \in \mathbb{N}$, this equation provides a measure valued process $t \mapsto \{\alpha_{N,t}\}$, and we are interested in the limit behaviour as $N \to \infty$. We prove tightness of the sequence $\{\alpha_{N,t}\}_{N \in \mathbb{N}}$ under certain assumptions and address some further problems.

Keywords: chordal Loewner equation, stochastic Loewner evolution, multiple SLE, complex Burgers equation, tightness, quadratic differentials

2010 Mathematics Subject Classification: 60J67, 37L05.

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1 Introduction

In [dMS16], the second and third author noted that the conformal mappings for a certain multiple SLE (Schrödinger-Loewner evolution) process for $N$ simple curves in the upper half-plane $\mathbb{H}$ converges as $N \to \infty$. The deterministic limit has a simple description: The conformal mappings $f_t : \mathbb{H} \to \mathbb{H}$ satisfy the Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -\frac{\partial f_t(z)}{\partial z} \cdot M_t(z), \quad f_0(z) = z \in \mathbb{H},$$

where $M_t$ satisfies the complex Burgers equation

$$\frac{\partial M_t(z)}{\partial t} = -2\frac{\partial M_t(z)}{\partial z} \cdot M_t(z),$$

see Section 2.5 for more details. In several situations, partial differential equations of this type appear to describe the limit of $N$-particle systems; see [Cha92] [RS93] [CL97].

∗Supported by the JSPS KAKENHI Grant no. 26800053.
†Supported by the ERC grant “HEVO - Holomorphic Evolution Equations” no. 277691.
In Section 2 we consider again the same multiple SLE measure for $N$ curves connecting $N$ points on $\mathbb{R}$ with $\infty$. We describe the growth of these curves by a Loewner equation with weights that correspond to the speed for these curves in the growth process, and we obtain an abstract differential equation for limit points as $N \to \infty$ (Corollary 2.7).

Furthermore, in Section 3 we see that an equation of a similar type also appears in the limit behaviour of a Loewner equation describing the growth of trajectories of a certain quadratic differential.

2 Tightness of a multiple SLE process

2.1 Geometry and Loewner Theory

In this section we briefly recall the general background of hulls in the upper half-plane and the chordal Loewner equation.

A domain $D \subseteq \hat{\mathbb{C}}$ is said to be a Jordan domain if $\partial D$ is homeomorphically equivalent to the unit circle $\mathbb{T} = \partial \mathbb{D}$. Let $\Gamma$ be a subset of $\overline{D}$ such that there exist some $T > 0$ and a homeomorphism $\gamma : [0, T) \to \Gamma$ with $\gamma(0, T) \subseteq D$ and $\gamma(0) \in \partial D$. Then, if $\gamma(T) \in D$, the set $\Gamma \cap D = \Gamma \setminus \gamma(0)$ is said to be a slit in $D$, and if $\gamma(T) \in \partial D$ as well, $\Gamma$ is referred to as a chord (in $D$).

Since by the Riemann Mapping Theorem (see, e.g., [Pom75, Section 1.1]) $D$ is conformally equivalent to the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$, it suffices to consider the case $D = \mathbb{H}$ and $\gamma(0) \in \mathbb{R}$. In this setting, in particular, one may also introduce the more general notion of hull, i.e. a subset $A \subset \mathbb{H} \cup \{ \infty \}$ such that $\overline{A} \cap \mathbb{H} = A$ and $\mathbb{H} \setminus A$ is simply connected.

It is known that if $A \subset \mathbb{H}$ is a bounded hull, then there exists a unique conformal mapping $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ with hydrodynamic normalization (see [Law05, Proposition 3.34]), meaning that

$$g_A(z) = z + \frac{b}{z} + \tilde{g}(z) \quad \text{as } z \to \infty$$

for a holomorphic function $\tilde{g}$ with $\angle \lim_{z \to \infty} z \cdot \tilde{g}(z) = 0$. The quantity $b = \mathrm{hcap}(A) \geq 0$ is called the half-plane capacity of $A$.

The mapping $g_A$ can be embedded into the solution of a Loewner equation as follows. Let $T > 0$ be defined by $2T = \mathrm{hcap}(A)$. Then there exists a family $\{ \mu_t \}_{t \in [0, T]}$ of probability measures on $\mathbb{R}$, with the property that $t \mapsto \int_{\mathbb{R}} \frac{1}{t-u} \mu_t(du)$ is measurable for every $z \in \mathbb{H}$, such that the solution $\{ g_t \}_{t \in [0, T]}$ of the chordal Loewner equation

$$\begin{cases}
\frac{dg_t(z)}{dt} = \int_{\mathbb{R}} \frac{2}{g_t(z) - u} \mu_t(du) & \text{for almost every } t \in [0, T] \\
g_0(z) = z \in \mathbb{H}
\end{cases} \quad (2.1)$$

satisfies $g_A = g_T$. This follows from [GB92, Theorem 5] and considering the time-reversed flow and the inverse mapping $g_A^{-1}$.

Conversely, one can always solve (2.1) and obtain conformal mappings with hydrodynamic normalization; see [GB92, Theorem 4] or [Law05, Theorem 4.5].

For $z \in \mathbb{H}$ fixed, the solution $t \mapsto g_t(z)$ of (2.1) may have a finite lifetime $T(z) > 0$, namely $g_t(z) \in \mathbb{H}$ for all $t < T(z)$ and $\mathrm{Im}(g_t(z)) \to 0$ as $t \uparrow T(z)$.

If we fix a time $t > 0$ and let $K_t = \{ z \in \mathbb{H} \mid |T(z)| \leq t \}$, then $K_t$ is a (not necessarily bounded) hull and the mapping $z \mapsto g_t(z)$ is the conformal mapping from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ with hydrodynamic normalization. Furthermore, the hulls $K_t$ are strictly growing, i.e. $K_s \subset K_t$ whenever $s < t$, and $\mathrm{hcap}(K_t) = 2t$.

When the hull $A$ is a slit $\Gamma$, equation (2.1) necessarily has the form

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - U(t)}; \quad g_0(z) = z \in \mathbb{H}, \quad (2.2)$$

with a unique, continuous driving function $U : [0, T] \to \mathbb{R}$ (see [dMG16], and the references therein, for more details). In this case, we obtain a parametrization $\gamma$ of $\Gamma$ by setting $\gamma(0, t) = K_t$, which is equivalent to requiring $\mathrm{hcap}(\gamma(0, t)) = 2t$. We call $\gamma$ the parametrization by half-plane capacity of $\Gamma$. 
More generally, if $A$ is the union of $n$ slits $\Gamma_1, \ldots, \Gamma_n$ with pairwise disjoint closures, i.e. $\Gamma_j \cap \Gamma_k = \emptyset$ whenever $j \neq k$, then (2.1) must have the form

$$\frac{dg_k(z)}{dt} = \sum_{j=1}^n \frac{2\lambda_j(t)}{g_k(z) - U_j(t)}, \quad g_0(z) = z \in \mathbb{H},$$

(2.3)

where $U_j : [0, T] \to \mathbb{R}$ are continuous and $\lambda_j : [0, T] \to [0, 1]$ are measurable functions with $\sum_{j=1}^n \lambda_j(t) = 1$ for every $t$; see [Boe15] Theorem 2.54. In this way, we obtain parametrizations $\gamma_1, \ldots, \gamma_n$ of $\Gamma_1, \ldots, \Gamma_n$ by requiring $K_t = \cup_{j=1}^n \gamma_j(0,t)$. It is worth noting that, for $n > 1$, a representation of $A$ by (2.3) is not unique. For example, we could first generate slit $\Gamma_k$ only, i.e. $\lambda_k(t) = 1 = 1 - \lambda_j(t)$ for $j \neq k$ and $t$ small enough.

**Remark 2.1.** The coefficients $\lambda_j(t)$ can be thought of as the speed of growth of the slit $\Gamma_j$ at time $t$. More precisely, we have the following relation:

Fix $j$ and $t_0 \geq 0$, assume that $g_k$ is differentiable at $t = t_0$ and consider the curve $\tilde{\gamma}(h) = g_k(\gamma_h(t_0) + h)$. Let $b(h) := \text{hcap}(\tilde{\gamma}(0,h)| = h$ be the half-plane capacity of the slit $\tilde{\gamma}(0,h)$. Then $b(h)$ is differentiable at $h = 0$ with $b(0) = \lambda_j(t_0)$, see [Boe15] Theorem 2.36.

### 2.2 Single and Multiple SLE

In what follows, $\kappa \in (0, 4]$ is a fixed parameter and $D \subseteq \mathbb{C}$ is a Jordan domain.

Fix two points $x, y \in \partial D$ and assume that $\partial D$ is analytic in neighbourhoods of $x$ and $y$.

The chordal Schramm-Loewner evolution (SLE) of a random curve $\Gamma \subset D$ for the data $D, x, y, \kappa$ can be viewed as a certain probability measure $\mu_{D,\kappa}(x, y)$ on the space of all chords connecting the points $x$ and $y$ within $D$. As one property of SLE is conformal invariance, it suffices to describe the SLE when $D = \mathbb{H}$, $x = 0$, and $y = \infty$. In this setting, the evolution of $\Gamma$ can be described efficiently as follows. Let $\gamma$ be a parametrization of $\Gamma$ with $\gamma(0) = 0$ and assume that $\gamma(0, T]$ is parametrized by half-plane capacity for every $T > 0$. The random conformal mapping $g_\gamma := g_{\gamma(0, \cdot)}$ then satisfies the Loewner equation

$$\frac{dg_\gamma(z)}{dt} = \frac{2}{g_\gamma(z) - \sqrt{k_B t}}, \quad g_\gamma(0) = z,$$

(2.4)

where $B_t$ is a standard one-dimensional Brownian motion.

Notice that one may also consider SLE for $\kappa > 4$. But then the measure is no longer supported on simple curves, and we are not interested in such a case here. For further information and a thorough treatment of SLE we refer to [Law05].

Next, we describe multiple SLE as it was introduced in [KL07].

Let $N \in \mathbb{N}$ and fix $2N$ pairwise distinct points $p_1, \ldots, p_{2N} \in \partial D$ in counter-clockwise order. Assume that $\partial D$ is analytic in a neighbourhood of $p_k$, $k = 1, \ldots, 2N$.

We call the pair $(x, y)$ of two tuples $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N)$ a configuration for these points if

a) $\{x_1, \ldots, x_N, y_1, \ldots, y_N\} = \{p_1, \ldots, p_{2N}\}$,

b) there exist $N$ pairwise disjoint chords $\gamma_k$ connecting $x_k$ to $y_k$ within $D$, $k = 1, \ldots, N$,

c) $x_1 = p_1$ and $x_1, x_2, \ldots, x_N$, as well as $x_1, x_k, y_k$, for every $k \geq 2$, are in counter-clockwise order.

The points in $x$ can be thought of as the starting points of these chords. Then $y$ represents the end points and the assumption in c) just prevents us from getting a new configuration by exchanging a starting point of one curve with its endpoint. A simple combinatorial calculation gives that there exist

$$C_N = \frac{(2N)!}{(N + 1)!N!}$$

many configurations for $2N$ points.

Fix now a configuration $(x, y)$. The **configurational multiple SLE** $Q_{D,\kappa}(x, y)$ is a positive finite measure on the space of all $N$-tuples $(\gamma_1, \ldots, \gamma_N)$, where $\gamma_k$ is a chord in $D$ connecting $x_k$ to $y_k$ and $\gamma_j \cap \gamma_j = \emptyset$ whenever $j \neq k$. One may contruct the $Q_{D,\kappa}(x, y)$ by means of the Brownian loop measure (see [KL07] for details).

If we let $H_{D,\kappa}(x, y)$ be the mass of $Q_{D,\kappa}(x, y)$, then we can write

$$Q_{D,\kappa}(x, y) = H_{D,\kappa}(x, y) \cdot \mu_{D,\kappa}(x, y)$$

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for some probability measure $\mu_{D,\kappa}(x,y)$.

Thus, one may view $Q_{D,\kappa}(x,y)$ as a probability measure for the underlying configuration with weight $H_{D,\kappa}(x,y)$. Then we may use such weights as partition functions to combine multiple SLE for different configurations. Namely, if $p = (p_1, \ldots, p_{2N})$ and $S(p)$ is the set of all configurations, then the probability for $(x,y) \in S(p)$ will be given by

$$p(x,y) = \frac{H_{D,\kappa}(x,y)}{\sum_{(v,w) \in S(p)} H_{D,\kappa}(v,w)}. \quad (2.5)$$

**Example 2.2.** Consider the case $N = 2$ and $\kappa = 3$. Then there are two possible configurations $C_1$ and $C_2$, and $\mu_{D,\kappa}(\{C_1, C_2\})$ describes the scaling limit for the Ising model with corresponding boundary conditions (see [Koz09]). The probability $p$ for obtaining configuration $C_1$ is given by

$$p = \frac{H_{D,\kappa}(C_1)}{H_{D,\kappa}(C_1) + H_{D,\kappa}(C_2)}.$$

On the other hand, $H_{D,\kappa}(x,y)$ may also be used to write down a Loewner equation that governs the growth of multiple SLE curves, see [KL07] Section 4.

Again, because of conformal invariance, it suffices just to consider the case $D = \mathbb{D}$, where $p_1, \ldots, p_{2N} \in \mathbb{R} \cup \{\infty\}$. In this setting, the number $H_{D,\kappa}(x,y)$ is known explicitly only for some special cases:

(i) for $N = 1$ and $(x,y) = (0,\infty)$, one simply takes $H_{\mathbb{D},\kappa}(0,\infty) = 1$ as a definition, which would then yield $Q_{D,\kappa} = \mu_{\mathbb{D},\kappa}$, i.e. the chordal SLE probability measure as described in [2.4];

(ii) if $N = 1$ and $x,y \in \mathbb{R}$, then $H_{\mathbb{D},\kappa}(x,y) = |y-x|^{-2\kappa}$, $b = \frac{6-\kappa}{2\kappa}$;

(iii) a special case for $\kappa = 2$ is given in [KL07] (see Remark after Proposition 3.3);

(iv) for $N = 2$, $H_{\mathbb{D},\kappa}((x_1,x_2), (y_1,y_2))$ can be expressed by a formula involving hypergeometric functions (see [KL07] Proposition 3.4).

We point out that multiple SLE can also be approached by requiring certain properties for the multi-slit Loewner equation, which leads to local properties of $H_{D,\kappa}(x,y)$ as a partition function. A framework for describing $H_{D,\kappa}(x,y)$ as the solution to certain differential equations is discussed in the recent works [FK15a, FK15b, FK15d, FK15f, KP15]. We also refer to the articles [Car03, BBK05, Gra07, Dub07].

**Remark 2.3.** Notice that one may consider $Q_{\mathbb{D},\kappa}(x,y)$ also for a configuration where $y_j = y_k$ (or $x_j = x_k$, or both) for certain $j \neq k$. This is done by considering the disjoint case $y_j \neq y_k$ first and then taking a scaled limit.

In particular, if $(x_1, \ldots, x_N) = (\infty, \ldots, \infty)$, then one has

$$H_{\mathbb{D},\kappa}((x_1, \ldots, x_N), \infty) := H_{\mathbb{D},\kappa}((x_1, \ldots, x_N), (\infty, \ldots, \infty)) := \prod_{1 \leq j < k \leq N} (x_k - x_j)^{2/\kappa}. \quad (2.6)$$

See [BBK05] Section 4.6, and the references therein, for more details.

### 2.3 The chordal Loewner equation for $H_{\mathbb{D},\kappa}((x_1, \ldots, x_N), \infty)$

Let $N \in \mathbb{N}$ and $x_{N,1} < \ldots < x_{N,N}$ be $N$ points on $\mathbb{R}$. The growth of $N$ random curves from $\mu_{\mathbb{D},\kappa}((x_{N,1}, \ldots, x_{N,N}), \infty)$ can be described by a Loewner equation as follows:

First, choose $\lambda_{N,1}, \ldots, \lambda_{N,N} \in (0,1)$ such that $\sum_{k=1}^{N} \lambda_{N,k} = 1$.

Next, we define $N$ random processes $V_{N,1}, \ldots, V_{N,N}$ on $\mathbb{R}$ as the solution of the SDE system

$$dV_{N,k}(t) = \sum_{j \neq k} \frac{2(\lambda_{N,k} + \lambda_{N,j})}{V_{N,k}(t) - V_{N,j}(t)} dt + \sqrt{\kappa \lambda_{N,k}} dB_{N,k}(t), \quad V_{N,k}(0) = x_{N,k}. \quad (2.7)$$

where $B_{N,1}, \ldots, B_{N,N}$ are $N$ independent standard Brownian motions and $\kappa \in [0,4]$. Although multiple SLE was only defined for $\kappa \in (0,4]$, in this particular case one may also consider the deterministic case $\kappa = 0$.

The corresponding $N$-slit Loewner equation

$$\frac{d}{dt} g_{N,z}(z) = \sum_{k=1}^{N} \frac{2\lambda_{N,k}}{g_{N,z}(z) - V_{N,k}(t)}, \quad g_{N,0}(z) = z \in \mathbb{H}, \quad (2.8)$$
describes the growth of $N$ multiple SLE curves growing from $x_{N,1},...,x_{N,N}$ to $\infty$; see [BBK05, p. 1130] (where the function $Z$ is the partition function $Z$, see equation (4) on p. 1138). The function $z \mapsto g_{N,t}(z)$ is the conformal mapping $g_{\gamma_{1,0},...\cup \gamma_{N,N}[0,t]}$ for $N$ random simple curves $\gamma_{N,k} : [0,\infty) \to \mathbb{H}$, which are non-intersecting and $\gamma_{N,k}(0) = x_{N,k}$.

We are interested in the limit $N \to \infty$ of the growing curves, i.e. the convergence of $\gamma_{N,1}[0,t] \cup ... \cup \gamma_{N,N}[0,t]$ to a hull $K_t$. To be more precise, we would like to answer the following question once that some $t>0$ has been fixed: under which conditions does the sequence $H(\gamma_{N,1}[0,t] \cup ... \cup \gamma_{N,N}[0,t])$ of domains converge to a (simply connected) domain $H \setminus K_t$ with respect to kernel convergence (check Figure 1)?

According to Carathéodory’s Kernel Theorem (Theorem 1.8 in [Pom75]), the above question is equivalent to asking for locally uniform convergence of the mappings $g_{N,t}$ to a conformal mapping $g_t : H \setminus K_t \to \mathbb{H}$. Also, we would like to be able to describe $g_t$ again by a Loewner equation.

Let $\delta_x$ be the point measure in $x$ with mass 1 and define

$$\alpha_{N,t} = \sum_{k=1}^{N} \lambda_{N,k} \delta_{V_{N,k}(t)}. \quad (2.9)$$

Then equation (2.8) can be written as

$$\frac{d}{dt} g_{N,t} = \int_{\mathbb{R}} \frac{2}{g_{N,t} - u} d\alpha_{N,t}(u). \quad (2.10)$$

Assume now that

$$\alpha_{N,0} \xrightarrow{w} \alpha \quad \text{as} \quad N \to \infty, \quad (2.11)$$

where we denoted with "$\xrightarrow{w}$" the weak convergence and where $\alpha$ is again a probability measure. We wish to know whether the sequence $\{\alpha_{N,t}\}_{\mathbb{N}}$ of stochastic measure-valued processes converges. In what follows, we show that, under certain assumptions for $x_{N,k}$ and $\lambda_{N,k}$, this sequence is tight and that each limit process satisfies the same differential equation.

Figure 1: Kernel convergence of the complement of slits.

**Definition 2.4.** Fix $T > 0$ and let $\mathcal{P}(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ endowed with the topology of weak convergence (which is a metric space due to the well-known Lévy-Prokhorov metric). We denote by $\mathcal{M}(T) = C([0,T], \mathcal{P}(\mathbb{R}))$ the space of all continuous measure-valued processes on $[0,T]$ endowed with the topology of uniform convergence.

For every $N \in \mathbb{N}$, $\alpha_{N,t}$ can be regarded as a random element from $\mathcal{M}(T)$.

**2.4 Tightness**

We call a sequence $\{\mu_N\}_{\mathbb{N}}$ of random elements from $C([0,T],\mathbb{R})$ (or $\mathcal{M}(T)$) tight if there exists a subsequence which converges in distribution. By Prohorov’s Theorem ([Bil99, Section 5]), this coincides with the usual definition of tightness.

We are now going to list certain conditions that guarantee tightness of the sequence $\{\alpha_{N,t}\}_{\mathbb{N}}$ defined in (2.9).

First of all, we make the following assumption:

there exists $C > 0$ such that for every $N \in \mathbb{N}$ it holds $\max_{k \in \{1,...,N\}} \lambda_{N,k} \leq \frac{C}{N}. \quad (a)$
Now, we introduce the "empirical distribution"
\[
\mu_{N,t} = \sum_{k=1}^{N} \frac{1}{N} \delta_{V_{N,k}(t)}
\]
and we let \( L_N : [0,1] \to [0,1] \) be defined as \( L_N(k/N) = \sum_{j=1}^{k} \lambda_{N,j} \) for \( k = 0, \ldots, N \). Next, we extend \( L_N \) to the entire unit interval \([0,1]\) by linear interpolation. Then the family \( \{L_N\}_{N \in \mathbb{N}} \) is uniformly bounded by 1 and equicontinuous by \( [a] \). The Ascoli–Arzelà Theorem implies that it is precompact. We will hence assume that the limit exists:
\[
L_N(x) \to L(x) \text{ uniformly on } [0,1] \text{ as } N \to \infty. \tag{b}
\]
Notice that if \( F_{N,t}(x) = \alpha_{N,t}(-\infty, x] \) and \( G_{N,t}(x) = \mu_{N,t}(-\infty, x] \) are the cumulative distribution functions, we have that
\[
F_{N,t}(x) = L_N(G_{N,t}(x)). \tag{2.12}
\]
Finally, the last assumption is rather a technical condition. Namely, we assume that \( \mu_{N,0} \) converges weakly to a probability measure \( \mu \) in such a way that there exists a \( C^2 \)-function \( \varphi : \mathbb{R} \to [1, \infty) \), with \( \varphi', \varphi'' \) bounded and \( \varphi(x) \to \infty \) for \( x \to \pm \infty \) such that
\[
\sup_{N \in \mathbb{N}} \int_{\mathbb{R}} \varphi(x) d\mu_{N,0}(x) < +\infty. \tag{c}
\]
Let \( C^2_b(\mathbb{R}, \mathbb{C}) \) be the space of all twice continuously differentiable functions \( f : \mathbb{R} \to \mathbb{C} \) such that \( f', f'' \) are bounded.

**Theorem 2.5.** Let \( T > 0 \). Then, under the assumptions \( [a] \), \( [b] \) and \( [c] \), the sequences \( \{\mu_{N,t}\}_{N} \) and \( \{\alpha_{N,t}\}_{N} \) are tight with respect to \( \mathcal{M}(T) \).
Moreover, if \( \mu_{N,t} \) is a converging subsequence of \( \{\mu_{N,t}\}_{N} \) with limit \( \mu_t \), then

1. \( \alpha_{N,k,t} \) converges to the process \( \alpha_t \), and for every \( t \in [0, T] \) the cumulative distribution function \( F_t \) of \( \alpha_t \) is given by
\[
F_t(x) = L \circ G_t(x) \tag{2.13}
\]
where \( G_t \) is the cumulative distribution function of \( \mu_t \);

2. \( \mu_t \) satisfies the (distributional) differential equation
\[
\begin{align*}
\frac{d}{dt} \left( \int_{\mathbb{R}} f(x) d\mu_t(x) \right) &= 2 \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} d\mu_t(x) d\alpha_t(y) \\
\mu_0 &= \mu
\end{align*}
\tag{2.14}
\]
for all \( f \in C^2_b(\mathbb{R}, \mathbb{C}) \).

**Remark 2.6.** The conditions \( [b] \) and \( [c] \) are natural in the sense that we should assume convergence of the initial conditions \( x_{N,k} \) and the coefficients \( \lambda_{N,k} \), which are encoded in the functions \( L_N \). If some \( \lambda_{N,k} \) do not converge to 0 as \( N \to \infty \), then some part of the measure \( \alpha_{N,t} \) may escape to infinity as \( N \to \infty \), see Example 2.16.

**Proof.** To begin with, we notice that proving tightness of \( \{\mu_{N,t}\}_{N} \) can be reduced to proving tightness of stochastic real-valued processes (see [RS93, Section 3] and also [Gür88, Section 1.3]). Thus, the sequence \( \{\mu_{N,t}\}_{N} \) is tight if
\[
\left\{ \int_{\mathbb{R}} \varphi(x) d\mu_{N,t}(x) \right\}_N \text{ and } \left\{ \int_{\mathbb{R}} f(x) d\mu_{N,t}(x) \right\}_N
\]
are tight sequences (with respect to the space \( C([0, T], \mathbb{R}) \) with uniform convergence) for all \( f \in C^2_b(\mathbb{R}, \mathbb{C}) \).
Now, let \( f \in C^2_b(\mathbb{R}, \mathbb{C}) \); Itô’s formula gives
by the Stieltjes-Perron inversion formula (see \cite[Theorem F.2]{Sch12}). Denote its distribution times the Cauchy transform (or Stieltjes transform) of \( (2.16) \)

First, let \( \mu_N, t \) converge provided the boundedness of both \( \varphi' \) and \( \varphi'' \), thanks to assumption (b), the same reasoning also implies tightness of the sequence \( \{ \int f(x) d\mu_N, t(x) \} \). Hence, \( \mu_N, t \) is tight and each limit process satisfies equation (2.14).

Finally, it follows from (2.12) and assumption (b) that the convergence of \( \alpha_N, t \) converges provided the convergence of \( \mu_N, t \). In particular, it follows that relation (2.13) holds for the limit processes. \( \square \)

Now we can easily show that if \( \mu_N, t \) is a converging subsequence, then \( g_N, t \) converges as well.

First, let \( C \) be the set of all \( M(z) = \int_{\mathbb{R}} \frac{2}{z-u} d\beta(u) \), where \( \beta \) is a probability measure. So \( M \) is 2 times the Cauchy transform (or Stieltjes transform) of \( \beta \). The measure \( \beta \) can be recovered from \( M \) by the Stieltjes-Perron inversion formula (see \cite[Theorem F.2]{Sch12}). Denote its distribution function by \( F(x) \). Then \( L \circ F(x) \) is also a distribution function, which corresponds to a measure \( \beta \). In this way, we obtain a map \( L : C \to C \) defined as

\[
\int_{\mathbb{R}} \frac{2}{z-u} d\beta(u) \mapsto \int_{\mathbb{R}} \frac{2}{z-u} d\beta(u).
\]

The limit of the Loewner equation can now be described as follows.

**Corollary 2.7.** Let \( \mu_N, t \) be a converging subsequence with limit \( \mu_t \). Then \( g_N, t \) converges in distribution with respect to locally uniform convergence to \( g_t \), the solution of the Loewner equation

\[
\frac{d}{dt} g_t = (L \circ M_t)(g_t), \tag{2.15}
\]

where \( M_t = \int_{\mathbb{R}} \frac{2}{z-u} d\mu_t(u) \) solves the (abstract) differential equation

\[
\begin{aligned}
\frac{\partial}{\partial t} M_t &= -\frac{\partial}{\partial z} M_t \cdot (L \circ M_t) - M_t \cdot \frac{\partial}{\partial z} (L \circ M_t), \\
M_0(z) &= \int_{\mathbb{R}} \frac{2}{z-u} d\mu(u).
\end{aligned} \tag{2.16}
\]

**Remark 2.8.** The convergence of \( \alpha_N, t \) and \( g_N, t \) would follow immediately if we knew that equation (2.16) (or, equivalently, (2.14)) had a unique solution. If \( \lambda_N, k = \frac{1}{N} \), then (2.16) is a usual PDE and uniqueness can be shown easily (see Section 2.5).
In order to prove the above corollary, we will need the following control-theoretic result.

**Theorem 2.9.** Fix some $t > 0$. Let $\lambda$ be the Lebesgue measure on $[0, t]$ and let $\mathcal{N}(t)$ be the space of all finite measures on $\mathbb{R} \times [0, t]$ endowed with the topology of weak convergence. Let $\{\beta_{N,s}\}_{N \in \mathbb{N}}$ be a sequence of processes from $\mathcal{M}(t)$ and assume $\beta_{N,s} \times \lambda \in \mathcal{N}(t)$ converges weakly to $\beta_s \times \lambda \in \mathcal{N}(t)$ as $N \to \infty$. Denote with $h_{N,s}$, $s \in [0, t]$, the solution to the Loewner equation

$$
\frac{d}{ds} h_{N,s}(z) = \int_{\mathbb{R}} \frac{1}{h_{N,s}(u) - z} \, \beta_{N,s}(u), \quad h_{N,0}(z) = z.
$$

Then $h_{N,s}$ converges locally uniformly to $h_t$, where $h_s$, $s \in [0, t]$, is the solution to

$$
\frac{d}{ds} h_s(z) = \int_{\mathbb{R}} \frac{2}{h_s(u) - z} \, \beta_s(u), \quad h_0(z) = z.
$$

A proof of the above theorem can be found in [JVST12, Proposition 1] or [MS13, Theorem 1.1]. Notice that even though both results consider the radial Loewner equation, the proofs can be easily adapted to the chordal case.

**Proof of Corollary 2.7.** For $z \in \mathbb{H}$, let $f(x) = \frac{2}{2 + x}$. Then $f \in C^2_\text{c}(\mathbb{R}, \mathbb{C})$. Define now $M_t(z) = \int_{\mathbb{R}} f(x) \, d\alpha_t(x)$; then $(\mathcal{L} \circ M_t)(z) = \int_{\mathbb{R}} f(x) \, d\alpha_t(x)$, where $\alpha_t$ is the limit of $\alpha_{N_k,t}$, and Theorem 2.5 implies

$$
\frac{\partial}{\partial t} M_t(z) = 4 \int_{\mathbb{R}^2} \frac{1}{(z - x)^2 - (z - y)^2} \, d\alpha_t(x) d\alpha_t(y) = 4 \int_{\mathbb{R}^2} \frac{2z - x - y}{(z - x)^2(z - y)^2} \, d\mu_t(x) d\alpha_t(y)
$$

$$
= \int_{\mathbb{R}^2} \frac{2}{(z - x)^2(z - y)^2} \, d\mu_t(x) d\alpha_t(y) + \frac{2}{(z - x)(z - y)^2} \, d\mu_t(x) d\alpha_t(y)
$$

$$
= -\frac{\partial}{\partial z} M_t \cdot (\mathcal{L} \circ M_t) - M_t \cdot \frac{\partial}{\partial z} (\mathcal{L} \circ M_t).
$$

Furthermore, let $g_t$ be the solution to

$$
\frac{d}{dt} g_t = (\mathcal{L} \circ M_t)(g_t), \quad g_0(z) = z.
$$

Fix some $t > 0$. The canonical mapping $\mathcal{M}(t) \ni \alpha_s \mapsto \alpha_s \times \lambda \in \mathcal{N}(t)$ is continuous. It follows from the Continuous Mapping Theorem (see [Bil99], p. 20) that $\alpha_{N_k,t}$ converges in distribution with respect to weak convergence to $\alpha_{t} \times \lambda$.

Hence, Theorem 2.9 and again the Continuous Mapping Theorem imply that $g_{N_k,t}$, which is the solution to (2.10), converges in distribution to $g_t$ with respect to locally uniform convergence.  

### 2.5 The simultaneous case

In the case $\lambda_{N,k} = \frac{1}{N}$ for all $k$, which we call the *simultaneous* case, equation (2.7) becomes

$$
\frac{dV_{N,k}}{dt} = \sum_{j \neq k} \frac{4}{N} \frac{1}{V_{N,k} - V_{N,j}} \, dt + \frac{\sqrt{\kappa}}{\sqrt{N}} dB_{N,k}, \tag{2.17}
$$

a process that is quite similar to a Dyson Brownian motion.

Note that in such a case $\mu_{N,t} = \alpha_{N,t}$ and $\mathcal{L}$ is the identity map. If $\alpha_t$ is the limit of a converging subsequence of $\{\alpha_{N,t}\}_N$ and $M_t(z) = \int_{\mathbb{R}} \frac{2}{2 + u} \, d\alpha_t(u)$, then $M_t$ satisfies the complex Burgers equation

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} M_t = -2M_t \cdot \frac{\partial}{\partial z} M_t(z) \\
M_0(z) = \int_{\mathbb{R}} \frac{2}{z - u} \, d\alpha_0(u)
\end{array} \right., \tag{2.18}
$$

and the the limit of $g_{N_k,t}$ satisfies

$$
\frac{d}{dt} g_t = M_t(g_t), \quad g_0(z) = z. \tag{2.19}
$$

If we put $f_t = g_t^{-1}$, we obtain the Loewner PDE mentioned in Section 1

$$
\frac{\partial f_t(z)}{\partial t} = -\frac{\partial f_t(z)}{\partial z} \cdot M_t(z).
$$
Theorem 2.10. Under the assumptions of Theorem [2.5] with \( \lambda_{N,k} = \frac{k}{N} \), the sequences \( \alpha_{N,t} \) and \( g_{N,\alpha} \) converge in distribution as \( N \to \infty \).

As already mentioned, this follows as soon as we know that equation (2.18) has a unique solution, which is shown, e.g., in [RS93 Section 4] or [CL97 Section 5]. We give here another short proof.

Proof. Let \( M_t \) be a solution of (2.18). As \( M_t \) has no zeros in \( \mathbb{H} \) we can consider \( F_t := 1/M_t \) which satisfies \( \frac{d}{dt} F_t = -2F_t^{-1} \cdot \frac{d}{dt} F_t \). Next we use the fact that every \( F_t \) is univalent in a region \( \Gamma_{\alpha(t),\beta(t)} \), where

\[
\Gamma_{\alpha,\beta} := \{ z \in \mathbb{H} \mid \text{Im}(z) > \beta, \text{Im}(z) > \alpha|\text{Re}(z)| \}, \quad \alpha, \beta > 0,
\]

see [BV93] Proposition 5.4. For \( t \in [0,T] \) we find \( \alpha_0 \) and \( \beta_0 \) such that \( F_t \) is univalent in \( \Gamma_{\alpha_0,\beta_0} \) for all \( t \in [0,T] \).

Thus we can define \( V_t(z) = F_t^{-1}(z) \) for \( z \in \Gamma_{\alpha_0,\beta_0} \) and a simple calculation gives

\[
\frac{\partial}{\partial t} V_t(z) = \frac{2}{z}, \quad V_0(z) = (1/M_0)^{-1}(z).
\]

Obviously, \( V_t \) and hence also \( M_t \), is uniquely determined.

\[ \square \]

Remark 2.11. Transforms like \( \mu_t \to V_t(z) \) appear in free probability theory, which was introduced by D. Voiculescu in the 1980’s (in [AEPA09, p. 3059], \( V_t(z) - z \) is called Voiculescu transform).

We notice that Wigner’s semicircle law appears here as follows: for \( \alpha_0 = \delta_0 \), the solution of (2.18) is given by \( M_t(z) = \frac{4}{z + \sqrt{z^2 - 16t}} \), which is 2 times the Cauchy transform of the centred semicircle law with variance \( 4t \).

For relations between the chordal (and radial) Loewner equation to non-commutative probability theory, we refer to [Bau04, Sch10].

Remark 2.12. In [dMST10], the authors prove some geometric properties of the solution \( g_t \) of (2.19), under the assumption that the support of \( \alpha_0 \) is bounded. We mention one property of this case, which will be needed later on.

The measures \( \nu_t \) “grow” continuously in the following sense: \( \text{supp} \alpha_t \subset \text{supp} \alpha_s \) for all \( s \leq t \) and for each \( x \in \mathbb{R} \setminus \text{supp} \alpha_s \) there exists \( T > s \) such that \( x \notin \text{supp} \alpha_t \) for all \( t \leq T \). This is actually a consequence of the theory of the real Burgers equation (see [dMST10 Section 3.2]).

Remark 2.13. Let \( M_t \) be a solution of (2.18) and \( c > 0 \). Define \( G_t(z) := c \cdot M_{t,c}(c \cdot z) \). Then \( G_t \) also satisfies (2.18) with initial value \( G_0(z) = c \cdot M_0(c \cdot z) \). Fix some \( T > 0 \). As \( G_0(z) \to \frac{2}{z} \) when \( c \to \infty \), we obtain together with Remark 2.11 the long time behaviour

\[
\lim_{c \to \infty} c \cdot M_{t,c,T}(c \cdot z) = \frac{4}{z + \sqrt{z^2 - 16t}} \quad \text{or} \quad M_t(z) \sim \frac{4}{z + \sqrt{z^2 - 16t}} \quad \text{as} \quad t \to \infty.
\]

2.6 Examples

In the following we consider three examples. In all three cases we assume that \( \kappa = 0 \), i.e. we look at the deterministic case to make the differential equations somewhat simpler.

The proof of Theorem 2.5 shows that the sequence \( \frac{d}{dt} \left( \int f(x) d\mu_{N,t}(x) \right) \), as a sequence of functions on \([0,T]\), is uniformly bounded. In general, this is not true for \( \alpha_{N,t} \).

Example 2.14. Let \( S_N = 1 + \frac{N+1}{2N} \). We choose

\[
x_{N,k} = \frac{k}{N^2} \quad \text{and} \quad \lambda_{N,k} = \frac{1}{S_N} \left( 1 + \frac{k}{N} \right), \quad \frac{1}{N}.
\]

Obviously,

\[
\alpha_{N,0} \cdot \mu_{N,0} \overset{w}{\to} \delta_0,
\]

and \( \lambda_{N,k} \leq C/N \) for some \( C > 0 \) as \( \lambda_{N,k} \leq \lambda_{N,N} \sim \frac{2}{N^2} \) as \( N \to \infty \). Furthermore, as \( x_{N,k} \in [0,1] \) for all \( k, N \), it is easy to see that condition (1) is satisfied.

Finally, \( L_N(k/N) \) is given by \( L_N(k/N) = \sum_{j=1}^{N} \lambda_{N,j} = \frac{1}{S_N} (\frac{k}{N} + \frac{k}{N} \cdot \frac{N+1}{2N}) \), which shows that \( L_N \) converges uniformly to \( L(x) = \frac{2}{x} (x + \frac{x^2}{2}) \). Consequently, all the assumptions of Theorem 2.3 are satisfied.

Proposition 2.15. Under the assumptions of Example 2.14, there exists \( f \in C^0_b(\mathbb{R}, \mathbb{C}) \) such that \( \frac{d}{dt} \left( \int f(x) d\alpha_{N,t}(x) \right) |_{t=0} \) is unbounded.
Proof. Note that
\[
\frac{\lambda_{N,k} - \lambda_{N,j}}{x_{N,k} - x_{N,j}} = 1/S_N \frac{k/N^2 - j/N^2}{k/N^2 - j/N^2} = 1/S_N. \tag{*}
\]

Let \( f \in C^2_b(\mathbb{R}, \mathbb{C}) \). Then we obtain
\[
\frac{d}{dt} \left( \int_{\mathbb{R}} f(x) \, d\alpha_{N,t}(x) \right) = \frac{d}{dt} \left( \sum_{k=1}^{N} \lambda_{N,k} f(V_{N,k}(t)) \right) = \sum_{k=1}^{N} \lambda_{N,k} f'(V_{N,k}(t)) \sum_{j \neq k} \frac{2(\lambda_{N,k} + \lambda_{N,j})}{V_{N,k}(t) - V_{N,j}(t)}
\]
\[
= \sum_{k=1}^{N} \lambda_{N,k} f'(V_{N,k}(t)) \sum_{j \neq k} \frac{2\lambda_{N,j}}{V_{N,k}(t) - V_{N,j}(t)} + \sum_{k=1}^{N} \lambda_{N,k}^2 f''(V_{N,k}(t)) \sum_{j \neq k} \frac{2}{V_{N,k}(t) - V_{N,j}(t)}
\]
\[
= \int_{\mathbb{R}^2} \frac{2 f'(x)}{x-y} \, d\alpha_{N,t}(x) d\alpha_{N,t}(y) + 2 \sum_{j \neq k} \frac{\lambda_{N,k}^2 f''(V_{N,k}(t))}{V_{N,k}(t) - V_{N,j}(t)}
\]
\[
+ \sum_{j \neq k} \frac{\lambda_{N,k}^2 f'(V_{N,k}(t)) - \lambda_{N,j}^2 f'(V_{N,j}(t))}{V_{N,k}(t) - V_{N,j}(t)}.
\]

Now assume that \( f'(x) = 1 \) for all \( x \in [0, 1] \). It is easy to see that the first two terms are uniformly bounded. However,
\[
T_N(0) = \sum_{j \neq k} \frac{\lambda_{N,k}^2 - \lambda_{N,j}^2}{x_{N,k} - x_{N,j}} = \sum_{j \neq k} (\lambda_{N,k} + \lambda_{N,j})/S_N \leq \sum_{j \neq k} \lambda_{N,1}/S_N
\]
\[
= (N^2 - N) \left( \frac{1}{N} + \frac{1}{N^2} \right) \to \infty \quad \text{as} \quad N \to \infty.
\]

Next we have a look at two examples where condition \( (\text{ii}) \) is not satisfied, as for every \( N \) there is one coefficient \( \lambda_{N,k} = \frac{1}{2} \).

Example 2.16. For \( N \geq 2 \), let
\[
x_{N,k} = \frac{k}{N} \quad \text{and} \quad \lambda_{N,k} = \frac{1}{2(N-1)} \quad \text{for all} \quad k \neq N, \quad \text{and} \quad x_{N,N} = 2, \quad \lambda_{N,N} = \frac{1}{2}.
\]

Proposition 2.17. Let \( T > 0 \). Under the assumptions of Example 2.16 the sequence \( \{\alpha_{N,t}\}_N \) is not tight with respect to the topology of \( \mathcal{M}(T) \).

Proof. We show that \( V_{N,N}(t) \to +\infty \) as \( N \to \infty \) for every \( t > 0 \). As \( V_{N,N} \) carries the mass 1/2, this proves that \( \{\alpha_{N,t}\}_N \) is not tight.

First, we need an upper bound for \( V_{N,N-1} \). For \( k \in \{1, \ldots, N-1\} \) we have
\[
dV_{N,k}(t) \leq \sum_{j \neq k, N} \frac{2}{V_{N,k}(t) - V_{N,j}(t)} \, dt.
\]

Let \( W_{N,1}, \ldots, W_{N,N-1} \) be the of solution the system
\[
dW_{N,k}(t) = \sum_{j \neq k, N} \frac{2}{W_{N,k}(t) - W_{N,j}(t)} \, dt, \quad W_{N,k}(0) = x_{N,k}.
\]

As the function
\[
(x_1, \ldots, x_{N-1}) \mapsto \left( \sum_{j \neq 1, N} \frac{2}{x_1 - x_j}, \ldots, \sum_{j \neq N-1, N} \frac{2}{x_{N-1} - x_j} \right)
\]
\[
\text{(10)}
\]
is quasimonotone, it follows that $V_{N,k}(t) \leq W_{N,k}(t)$ for all $t \geq 0$ (Theorem 4.2 in [LL80]). Note that $W_{N,1}, \ldots, W_{N,N-1}$ is a simultaneous multiple SLE process for $N-1$ curves, each growing with “speed” $\frac{1}{2(N-1)}$. From Remark 2.12 we conclude that there exists $T_0 > 0$ and a bound $B_1 \in (1,2)$ such that $W_{N,N-1}(t) \leq B_1$ for all $t \in [0,T_0]$ and $N \geq 2$. Hence, $V_{N,k}(t) \leq B_1 < 2$ for all $t \in [0,T_0]$.

This upper bound now gives us also a lower bound as follows: As $\frac{3}{4} V_{N,N}(t) \geq 0$ and $V_{N,N}(0) = 2$ we have $V_{N,N}(t) \geq 2$ for all $t \geq 0$. Thus, for $k \in \{1, \ldots, N-1\}$ we have

$$
\sum_{j \neq k,N} \frac{2}{N-1} \left( V_{N,k}(t) - V_{N,j}(t) \right) \ dt + \frac{2}{V_{N,k}(t) - V_{N,N}(t)} \ dt \\
\geq \sum_{j \neq k,N} \frac{2}{N-1} \left( V_{N,k}(t) - V_{N,j}(t) \right) \ dt + \frac{2}{B_1 - 2} \ dt.
$$

Let $Y_{N,1}, \ldots, Y_{N,N-1}$ be the of solution the system

$$
dY_{N,k}(t) = \sum_{j \neq k,N} \frac{2}{N-1} \left( Y_{N,k}(t) - Y_{N,j}(t) \right) \ dt + \frac{2}{B_1 - 2} \ dt, \quad Y_{N,k}(0) = x_{N,k}.
$$

From [CL97], Theorem 5.1, it follows that the sequence $w_{N,t} = \sum_{k=1}^{N-1} \frac{1}{N-1} \delta_{Y_{N,k}(t)}$ of measure-valued processes converges as $N \to \infty$. This does not imply that $Y_{N,1}(t)$ is bounded from below, but we can conclude that, for example, $Y_{N,[N/2]}(t)$ is bounded from below, i.e. there exists $B_2 < 1$ such that $Y_{N,[N/2]}(t) \geq B_2$ for all $t \in [0,T_0]$.

Now we look at $V_{N,N}$, which satisfies

$$
dV_{N,N} = \sum_{j \neq N} \frac{2}{N-1} \left( V_{N,N}(t) - V_{N,j}(t) \right) \ dt + \frac{B_2}{V_{N,N}(t) - B_2} \ dt
$$

for $t \in [0,T_0]$, which implies

$$
V_{N,N}(t) \geq B_2 + \sqrt{4 - 4B_2 + B_2^2 - 2t + 2[N/2]t}.
$$

Hence, $V_{N,N}(t) \to \infty$ for every $t \in (0,T_0]$ as $N \to \infty$. As $t \mapsto V_{N,N}(t)$ is increasing, we conclude that $V_{N,N}(t) \to \infty$ for every $t > 0$.

Even though $\{\alpha_{N,t}\}_N$ is not tight in this example, it is easy to see that $g_{N,t}$ converges as $N \to \infty$. If we decompose $\alpha_{N,t} = \beta_{N,t} + \gamma_{N,t}$, then it can easily be shown that $\beta_{N,t}$ converges to a process $\beta$ and that $P_1(\alpha) = \int_{\mathbb{R}} \frac{2}{1-u} \ d\beta(u)$ satisfies a Burgers equation.

**Example 2.18.** Assume that $N = 2K + 1$, $K \in \mathbb{N}$, and let $x_{N,k} \in [-2,1]$ and $x_{N,2K+2-k} = -x_{N,k}$ for all $k \leq K$.

Assume that $x_{N,K+1} = 0$. The coefficients $\lambda_{N,k}$ are chosen as $\lambda_{N,K+1} = 1/2$, $\lambda_{N,k} = \frac{1}{4K}$, $k \neq K + 1$.

As $N \to \infty$, the sequence $L_N$ converges pointwise, but not uniformly, to $L(x) = 1/2x$, $x \in [0,1/2)$, $L(x) = 1/2x + 1/2$, $x \in [1/2,1]$.

**Proposition 2.19.** Under the assumptions of example 2.18 there exists $T_0 > 0$ such that the sequence $\{\alpha_{N,t}\}_N$ is tight with respect to the topology of $\mathcal{M}(T_0)$.

**Proof.** By symmetry, we have $V_{N,K+1}(t) = 0$ for every $K \in \mathbb{N}$ and $t \geq 0$ and we can decompose the measure $\alpha_{N,t}$ as $\alpha_{N,t} = \beta_{N,t} + \gamma_{N,t}$, where the support of $\beta_{N,t}$ is contained in $(-\infty,0)$ and $\gamma_{N,t}(A) = \beta_{N,t}(-A)$ for every Borel set $A$.

Just as in the proof of Proposition 2.17, we obtain that there exist $T_0 > 0$ and $B \in (-1,0)$ such that

$$
V_{N,K}(t) \leq B \quad \text{for all } K \in \mathbb{N} \text{ and } t \in [0,T_0].
$$

Now let $f \in C^2_0(\mathbb{R}, \mathbb{C})$. Then we have
It seems that the last two examples behave in the same way when simulations of the driving functions conclude that $t \in [0, T_0]$. As already mentioned in Remark 2.8, the convergence of $\{\alpha_{N,t}\}_N$ in distributional sense at $t = 0$ is shown to be tight. We conclude that $\{\beta_{N,t}\}_N$ and thus $\{\alpha_{N,t}\}_N$ is tight w.r.t. $\mathcal{M}(T_0)$. 

It seems that the last two examples behave in the same way when $\kappa > 0$. Figures 2 and 3 show simulations of the driving functions $V_{N,1}, \ldots, V_{N,N}$ for these two cases on the time interval $[0, 1]$ for $N = 51$ and $\kappa = 1$. The driving function with mass $\frac{1}{2}$ is coloured red.

Figure 2: Mass $\frac{1}{2}$ in $x_{N,N}$. 

Figure 3: Mass $\frac{1}{2}$ in $x_{N,(N-1)/2}$.

2.7 Problems and Remarks

1. As already mentioned in Remark 2.8, the convergence of $g_{N,t}$ from Corollary 2.7 follows as soon as we know that equation (2.16) has only one solution.

2. Example 2.14 suggests that the process $\alpha_t$ might not in general be differentiable (in the distributional sense) at $t = 0$.

Question: Is it always differentiable for $t > 0$?

Also, we notice that in [BBC99] it is shown that, for a special case, $\alpha_t$ has a density with respect to the Lebesgue measure for $t > 0$.

Question: Is this always true for $\alpha_t$ under the assumptions made in Theorem 2.5?

3. Fix a parameter $\kappa \in (0, 4]$. For each $N \in \mathbb{N}$, we consider $2N$ boundary points $0 < p_{N,1} < p_{N,2} < \ldots < p_{N,2N} = 1$ for multiple SLE on $\mathbb{H}$. We set $p_N := (p_{N,1}, \ldots, p_{N,2N})$. Recall that $\mathcal{S}(p_N)$ is the set of all $C_N$ configurations for these points, endowed with the probabilities given by formula (2.4).

Now we can ask for the limit of $\mathcal{S}(p_N)$ as $N \to \infty$ by using an idea from combinatorics, to encode configurations into Dyck paths.

An $N$–Dyck path is a continuous function $d : [0, 2N] \to [0, \infty)$ defined as follows:
• $d(0) = 0$ and $d(2N) = 0$,
• $d(k) - d(k + 1) \in \{-1, +1\}$ for all $k \in \{0, ..., 2N - 1\}$,
• for all other points $x \in [0, 2N] \setminus \{0, 1, ..., 2N\}$, $d(x)$ is defined by linear interpolation.

The set of all $N$--Dyck paths corresponds to the set $S(p_N)$ in the following way. An $N$--Dyck path can be completely described by $2N$ numbers $L_1, ..., L_{2N} \in \{-1, +1\}$ representing the slopes of the $2N$ line segments. These numbers are determined by a configuration for $p_N$ as follows (see the figures below for an example):

(i) $L_k = +1$ and $L_{k+1} = -1$ if and only if $p_k$ and $p_{k+1}$ are connected by a simple curve;
(ii) $L_k = L_{k+1} = +1$ if and only if the curve connecting $p_{k+1}$ is “contained” in the curve connecting $p_k$;
(iii) $L_k = L_{k+1} = -1$ if and only if the curve connecting $p_k$ is “contained” in the curve connecting $p_{k+1}$.

![Figure 4: A Dyck path for $N = 5$.](image)

![Figure 5: The configuration corresponding to Figure 1.](image)

Define also $p_{N,0} := 0$ and fix some $\gamma \in (0, 1]$. Normalize now such a Dyck path $d$ to define a normalized Dyck path as a continuous function $e_N : [0, 1] \to [0, \infty)$ with $e_N(p_{N,0}) = 0$ and

$$e_N(t) = e_N(p_{N,k}) + t \cdot \frac{d(k + 1) - d(k)}{(p_{N,k+1} - p_{N,k})^\gamma} \quad (2.21)$$

for $t \in [p_{N,k}, p_{N,k+1}]$, $k = 0, ..., 2N - 1$. Then the set of all normalized Dyck paths is a subset of the space $C([0, 1], \mathbb{R})$ endowed with the topology of uniform convergence. It becomes a probability space by taking the corresponding probabilities from the set $S(p_N)$. Let $E_N(t)$ be a random path from this set.

**Question:** Does $E_N(t)$ converge in distribution as $N \to \infty$?

**Remark 2.20.** Take $p_k = \frac{k}{N}$ and $\gamma = \frac{1}{2}$. If all the probabilities are equally distributed, i.e. the probability for each normalized Dyck path is $\frac{1}{2^N}$, then the corresponding random path $E_N(t)$ converges in distribution to a Brownian excursion process of duration 1 (see [Ric09, Section 1.2] and [MM03]).

Furthermore, we note that the probabilities for configurations are also considered for $\kappa > 4$, e.g. in [KP15].

**Question:** What can be said about the limit of the probabilities for the set $S(p_N)$ as $\kappa \to 0$?

4. The above questions can be extended to different settings like radial multiple SLE or multiple SLE in multiply connected domains (refer to [Law11]). For instance, in [Car03], the author describes the Loewner equation for radial SLE where $N$ simple curves grow from the boundary of the unit disc $\mathbb{D}$ within $\mathbb{D}$ towards the interior point 0. The radial analogue of Theorem 2.10, i.e. the coefficients in the Loewner equation are $\frac{1}{N}$, can be obtained simply by using the main result of [CL01].

### 3 Trajectories of a certain quadratic differential

Finally, we take a look at a Loewner equation that describes the growth of $N$ trajectories of a certain quadratic differential. By using the methods from the previous section, we obtain again an abstract differential equation for the limit case $N \to \infty$, which reduces to the Burgers equation in a special case.
Moreover, let a finite (signed) measure \( \mu \) and \( \alpha_{N,j} \in \mathbb{Z} \).

Remark 3.2. If \( \alpha_{N,j} \) such that \( \sum_{k=1}^{N} \alpha_{N,j} = 1 \). The Loewner equation

\[
\frac{d}{dt} V_{N,k}(t) = \sum_{j \neq k} \frac{2 \lambda_{N,j}}{V_{N,k}(t) - V_{N,j}(t)} + \sum_{j=1}^{M_N} \frac{\alpha_{N,j} \lambda_{N,k}}{V_{N,k}(t) - S_{N,j}(t)} + \sum_{j=1}^{M_N} \frac{\alpha_{N,j} \lambda_{N,k}}{V_{N,k}(t) - S_{N,j}(t)}
\]

where \( S_{N,j}(t) = g_{N,t}(\sigma_{N,j}) \).

Remark 3.1. This follows from [Tsa09, Theorem 5.1], where all the degrees \( \mu_k^\pm \) are equal to 0 in our case, as the trajectories form a 90°-angle with the real axis, which is also a trajectory of \( Q(z)dz^2 \) (check p. 564 in [Tsa09]).

Next, define the probability measure \( \mu_{N,t} = \sum_{k=1}^{N} \lambda_{N,k} \delta_{V_{N,k}(t)} \).

Remark 3.2. If \( M_N = 0 \) for all \( N \in \mathbb{N} \) and \( \lambda_{N,k} \leq C/N \) for all \( k,N \) and some \( C > 0 \), then, by the proof of Theorem 3.3, it is easy to that the following holds:

If \( \mu_{N,0} \to \mu \) as \( N \to \infty \) such that \( [c] \) is satisfied, then the limit \( \mu_t \) of \( \mu_{N,t} \) exists, and the transform \( M_t(z) = \int_{\mathbb{R}} \frac{2}{z - u} d\mu_t(u) \) satisfies the Burgers equation

\[
\frac{\partial}{\partial t} M_t = -M_t \cdot \frac{\partial}{\partial z} M_t(z), \quad M_0(z) = \int_{\mathbb{R}} \frac{2}{z - u} d\mu_t(u).
\]

Note that this is equation [2.18] with the 2 replaced by 1. The limit \( g_t \) of \( g_{N,t} \) satisfies \( \frac{\partial}{\partial t} g_t = M_t(g_t) \) and a simple calculation shows that \( \frac{\partial}{\partial t} M_t(g_t(z)) = 0 \), which implies that \( t \mapsto g_t(z_0) \), for \( z_0 \in \mathbb{H} \) fixed, describes a straight line, and that \( M_t(g_t(z_0)) = M_0(z) \).

Assume now \( \lambda_{N,k} = \frac{1}{N} \) for all \( k \) and \( N \). First, we introduce a second measure-valued process

\[
\sigma_{N,t} = \sum_{j=1}^{M_N} \frac{\alpha_{N,j}}{N} \delta_{S_{N,j}(t)},
\]

and we assume that there exists a compact set \( K \subset \mathbb{H} \) such that

\[
\text{supp } \sigma_{N,0} \subset K \quad \text{for all } N \in \mathbb{N}.
\]

Theorem 3.3. Let \( \lambda_{N,k} = \frac{1}{N} \) for all \( k \) and \( N \). Assume that \( \mu_{N,0} \) converges weakly to the probability measure \( \mu \) such that \( [c] \) holds. Furthermore, assume that \( [d] \) holds and that \( \sigma_{N,0} \) converges weakly to a finite (signed) measure \( \sigma \) as \( N \to \infty \). Then there exists \( T > 0 \) such that \( \{\mu_{N,t}\}_{N \in \mathbb{N}} \) is tight as a sequence in \( \mathcal{M}(T) \).

Moreover, let \( \mu_{N,t} \) be a converging subsequence with limit \( \mu_t \). Then the following two statements hold:

(i) \( \sigma_{N,t} \) converges to a process \( \sigma_t \) and

\[
\frac{d}{dt} \left( \int_{\mathbb{R}} f(x) d\mu_t(x) \right) = \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} d\mu_t(x)d\mu_t(y) + 2Re \left( \int_{\mathbb{H}} \int_{\mathbb{R}} \frac{f'(x)}{x - z} d\mu_t(x)d\sigma_t(z) \right),
\]

\[
\frac{d}{dt} \left( \int_{\mathbb{H}} h(z) d\sigma_t(z) \right) = \int_{\mathbb{H}} \frac{2h'(z)}{z - y} d\sigma_t(z)d\mu_t(y),
\]

for every \( f \in C^2_b(\mathbb{R}, \mathbb{C}) \) and continuously differentiable \( h : \mathbb{H} \to \mathbb{C} \) with \( h' \) bounded.
Finally, we can write the first equation of (i) as
\[ d\frac{dt}{dt}g_t(z) = M_t(g_t), \]
\[ \frac{\partial}{\partial t} M_t(z) = \frac{\partial}{\partial z} M_t(z) \cdot M_t(z) + 2 \text{Re} \left( \int_{\mathbb{R}} \frac{M_t(z)}{(z - g_t(w))^2} \cdot \frac{M_t(g_t(w))}{(z - g_t(w))^2} \cdot \frac{\partial}{\partial z} M_t(z) \cdot d\sigma(w) \right). \]

Proof. First we note that \( \sigma_{N,t} \) is the pushforward of \( \sigma_{N,0} \) w.r.t. \( g_{N,t} \), i.e.
\[ \sigma_{N,t} = \left( g_{N,t} \right)_* (\sigma_{N,0}) \tag{3.1} \]
A normality argument plus assumption (iii) would yield the existence of \( T > 0 \) and a compact set \( K_T \subset \mathbb{H} \) such that
\[ \text{supp} \sigma_{N,t} \subset K_T \text{ for all } t \in [0, T]. \tag{3.2} \]

Now let \( f \in C^2_b(\mathbb{R}, \mathbb{C}) \). Then
\[
\frac{d}{dt} \left( \int_{\mathbb{R}} f(x) d\mu_{N,t}(x) \right) = \frac{d}{dt} \left( \sum_{k=1}^{N} \frac{1}{N} f(V_{N,k}(t)) \right)
\]
\[
= \sum_{k=1}^{N} \frac{1}{N} f'(V_{N,k}(t)) \cdot \left( \sum_{j \neq k} \frac{2/N}{V_{N,k}(t) - V_{N,j}(t)} \right) + \sum_{j=1}^{M_N} \frac{\alpha_{N,j}/N}{V_{N,k}(t) - S_{N,j}(t)} + \sum_{j=1}^{M_N} \frac{\alpha_{N,j}/N}{V_{N,k}(t) - S_{N,j}(t)}
\]
\[
= \int_{x \neq y} \frac{f'(x) - f'(y)}{x - y} d\mu_{N,t}(x) d\mu_{N,t}(y) + 2 \text{Re} \left( \int_{\mathbb{H}} \int_{\mathbb{H}} \frac{f'(x)}{x - z} d\mu_{N,t}(x) d\sigma_{N,t}(z) \right).
\]
As in the proof of Theorem 2.5, we conclude that the first term is bounded. The second one is bounded for all \( t \in [0, T] \) because of (3.2) and as \( \sigma \) is finite.
Recall that \( S_{N,j}(t) = \mu_{N,t}(s_{N,j}) \). Thus, for any continuously differentiable \( h : \mathbb{H} \to \mathbb{C} \), with \( h' \) bounded, we get
\[
\frac{d}{dt} \left( \int_{\mathbb{R}} h(z) d\mu_{N,t}(z) \right) = \frac{d}{dt} \left( \sum_{j=1}^{M_N} \frac{\alpha_{N,j}/N}{N} h(S_{N,j}(t)) \right)
\]
\[
= \sum_{j=1}^{M_N} \frac{\alpha_{N,j}/N}{N} h'(S_{N,j}(t)) \cdot \sum_{k=1}^{N} \frac{2/N}{S_{N,j}(t) - V_{N,k}(t)} = \int_{\mathbb{R}} \int_{\mathbb{H}} \frac{2h'(z)}{z - y} d\sigma_{N,t}(z) d\mu_{N,t}(y),
\]
which is also bounded for all \( t \in [0, T] \).
As in the proof of Theorem 2.5, we conclude tightness of the sequences \( \{\mu_{N,t}\}_{n \in \mathbb{N}} \) and \( \{\sigma_{N,t}\}_{n \in \mathbb{N}} \). It should be noted that we do not need a condition like (c) for the convergence of \( \sigma_{N,0} \), as we assumed that the support of \( \sigma_{N,0} \) is contained in a compact set independent of \( N \).

Now let \( \mu_t \) be the limit of a converging subsequence \( \mu_{N,t} \). Relation (3.1) implies that \( \sigma_{N,t} \) converges to \( \sigma := (g_t)_*(\sigma) \) as \( N \to \infty \), and we obtain statement (i).
As in the proof of Corollary 2.7, we conclude that \( g_{N,t} \) converges locally uniformly to \( g_t \) which satisfies
\[ \frac{d}{dt} g_t = M_t(g_t). \]

Finally, we can write the first equation of (i) as
\[
\frac{d}{dt} \left( \int_{\mathbb{R}} f(x) d\mu_t(x) \right) = \int_{\mathbb{R}} f'(x) - f'(y) d\mu_t(x) d\mu_t(y) + 2 \text{Re} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} f'(x) d\mu_t(x) d\sigma(w) \right).
\]
For \( f(x) = \frac{2}{x^2} \), \( z \in \mathbb{H} \), this becomes (and we use the calculation from the proof of Corollary 2.7)
\[
\frac{\partial}{\partial t} M_t(z) = \frac{\partial}{\partial z} M_t(z) \cdot M_t(z) + 2 \text{Re} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{2}{(z - x)(z - g_t(w))^2} - \frac{2}{(z - x)(z - g_t(w))^2} + \frac{2}{(x - z)(z - g_t(w))^2} d\mu_t(x) d\sigma(w) \right)
\]
\[
= \frac{\partial}{\partial z} M_t(z) \cdot M_t(z) + 2 \text{Re} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{M_t(z)}{(z - g_t(w))^2} - \frac{M_t(g_t(w))}{(z - g_t(w))^2} + \frac{2}{\partial z} M_t(z) \cdot d\sigma(w) \right)
\]
and we are done. \( \square \)
Figure 6 shows a stream plot of the trajectories for $Q(z) = \prod_{k=0}^{9} (z-2k/9+1)^2 \cdot (z-i)^{10} \cdot (z+i)^{10}$, and in Figure 7 $z=i$ is a zero of $Q$, i.e. $Q(z) = \prod_{k=0}^{9} (z-2k/9+1)^2 \cdot (z-i)^{10} \cdot (z+i)^{10}$.

Figure 6: Pole of order $N$ at $z=i$.

Figure 7: Zero of order $N$ at $z=i$.

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