THE HOMOGENEITY CONJECTURE FOR SUPERGRAVITY BACKGROUNDS

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Abstract. These notes record three lectures given at the workshop “Higher symmetries in Physics”, held at the Universidad Complutense de Madrid in November 2008. In them we explain how to construct a Lie (super)algebra associated to a spin manifold, perhaps with extra geometric data, and a notion of privileged spinors. The typical examples are supersymmetric supergravity backgrounds; although there are more classical instances of this construction. We focus on two results: the geometric constructions of compact real forms of the simple Lie algebras of type $B_4$, $F_4$ and $E_8$ from $S^7$, $S^8$ and $S^{15}$, respectively; and the construction of the Killing superalgebra of eleven-dimensional supergravity backgrounds. As an application of this latter construction we show that supersymmetric supergravity backgrounds with enough supersymmetry are necessarily locally homogeneous.

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1. Geometric construction of exceptional Lie algebras

The Killing–Cartan classification of complex simple Lie algebras consists of four infinite series of classical Lie algebras: $A_{n \geq 1}$, $B_{n \geq 2}$, $C_{n \geq 3}$ and $D_{n \geq 4}$, and a small number of exceptional Lie algebras: $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$, where the subscripts denote the ranks and in the classical case they have been chosen so as to avoid low rank isomorphisms. The classical Lie algebras are well-understood: they are matrix algebras and their compact real forms are the Lie algebras of

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(special) unitary matrices over $\mathbb{R}$ ($B$ and $D$), $\mathbb{C}$ ($A$) and $\mathbb{H}$ ($C$). In contrast, the exceptional Lie algebras result from “baroque” constructions involving octonions or else from constructions involving spinors, as explained by Adams in his posthumous lecture notes [1] and, for the case of $E_8$, also in [2]. It is this latter construction which we will geometrise in today’s lecture, using a device well-known in supergravity and which will be subject of the next two lectures: the so-called Killing superalgebra.

The basic idea of these lectures is to assign to a spin manifold a 2-graded algebra. In this first lecture we will consider the particular example of the exceptional Hopf fibration

$$S^7 \longrightarrow S^{15} \longrightarrow S^8,$$

where $S^n$ stands for the unit $n$-sphere in $\mathbb{R}^{n+1}$. If we think of $S^{15} \subset \mathbb{O} \oplus \mathbb{O}$ and $S^7 \subset \mathbb{O}$, then $S^8 \cong O\mathbb{P}^1$ is the octonionic projective plane. Applying the Killing superalgebra construction to the spaces in the above fibration we will obtain compact (or split) real forms of the simple Clifford algebra $C\ell$ corresponding to the relation between these Lie algebras which is suggested by the Hopf fibration relating the corresponding spaces. This first lecture is based on [3].

1.1. Clifford algebras, spin group and spinor representations. We start with a flash review of Clifford algebras, the spin group and the spinor representations. For more details, see the books [4] or [5].

Let $E, \langle -, - \rangle$ be a euclidean vector space. For instance, we could take $E = \mathbb{R}^n$ with the standard “dot” product. We define the Clifford algebra $C\ell(E)$ to be the associative algebra obtained by quotienting the tensor algebra $T(E)$ by the two-sided ideal generated by elements of the form $x \otimes x + \langle x, x \rangle 1$; in symbols,

$$C\ell(E) = T(E)/ \{ x \otimes x + \langle x, x \rangle 1 \}.$$  

Since the ideal is not homogeneous, the Clifford algebra is not graded, but only filtered. The associated graded algebra is the exterior algebra $\Lambda E$, whence we may think of the Clifford algebra as a quantisation of the exterior algebra. Since the ideal has even parity, the Clifford algebra inherits a 2-grading which agrees under the “classical limit” with the 2-grading of $\Lambda E$ into odd and even forms. When $E = \mathbb{R}^n$ relative to the usual dot product, we call the corresponding Clifford algebra $C\ell(n)$. Up to isomorphism we have the following table of euclidean Clifford algebras:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $k + 8$ |
|-----|---|---|---|---|---|---|---|---|--------|
| $C\ell(n)$ | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{H}(2)$ | $\mathbb{C}(4)$ | $\mathbb{R}(8)$ | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | $C\ell(k) \otimes_{\mathbb{R}} \mathbb{R}(16)$ |

where we use the notation $\mathbb{K}(m)$ for the algebra of $m \times m$ matrices with entries in $\mathbb{K}$, and where the last column goes by the name of Bott periodicity.

The subspace $\Lambda^2 E \subset C\ell(E)$ is a Lie subalgebra under the Clifford commutator isomorphic to $\mathfrak{so}(E)$. Exponentiating inside the (associative) Clifford algebra, gives (for $n > 2$) a simply-connected Lie group $\text{Spin}(E) \subset C\ell(E)^{\text{even}}$, called the spin group of $E$. Conjugating with $\text{Spin}(E)$ preserves $E \subset C\ell(E)$ and defines a two-to-one group homomorphism

$$\text{Spin}(E) \rightarrow \text{SO}(E).$$

When $E = \mathbb{R}^n$ with the standard dot product, we denote the spin group by $\text{Spin}(n)$.

The Clifford algebra $C\ell(n)$ is isomorphic either to a matrix algebra or to two copies of a matrix algebra, and as such has either one or two inequivalent irreducible representations. This follows from the fact that $\mathbb{R}(n)$ and $\mathbb{H}(n)$ have up to isomorphism a unique irreducible representation isomorphic to $\mathbb{R}^n$ and $\mathbb{H}^n$, respectively, whereas $\mathbb{C}(n)$ has two non-isomorphic irreducible representations: $\mathbb{C}^n$ and its complex conjugate representation. Similarly, $\mathbb{R}(n) \oplus$
\( \mathbb{R}(n) \) and \( \mathbb{H}(n) \oplus \mathbb{H}(n) \) have two inequivalent irreducible representations, isomorphic to \( \mathbb{R}^n \) or \( \mathbb{H}^n \), respectively. These are called the \textbf{pinor} representations of \( Cl(E) \). Restricting them to \( Spin(E) \) one obtains (perhaps reducible) representations called \textbf{spinor} representations and denoted \( S(E) \). The type \( (\mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}) \) of \( S(E) \) follows from the fact that \( Spin(E) \subset Cl(E)^{\text{even}} \) and that \( Cl(n)^{\text{even}} \cong Cl(n-1) \). For example, the spinor representations of \( Spin(n) \) for small \( n \), have types \( \mathbb{H} \) for \( n = 3, 4, 5 \), \( \mathbb{C} \) for \( n = 2, 6 \) and \( \mathbb{R} \) for \( n = 7, 8, 9 \). This is consistent with the low-dimensional isomorphisms \( Spin(2) \cong U(1), Spin(3) = Sp(1), Spin(4) = Sp(1) \times Sp(1), \) \( Spin(5) = Sp(2), Spin(6) = SU(4) \).

We will be using the fact that \( S(E) \) admits a \( Cl(E) \)-invariant inner product \((-,-)\) obeying
\[
(x \cdot \psi_1, \psi_2) = - (\psi_1, x \cdot \psi_2) ,
\]
for all \( \psi_1 \in S(E) \) and \( x \in E \subset Cl(E) \). This means that \((-,-)\) is also \( Spin(E) \)-invariant. For \( E \) euclidean, as we have been assuming, the spinor inner product is also positive-definite. However if we allow for \( E \) to have arbitrary signature, then all seven types of elementary inner products appear among the spinor inner products.

The transpose of the Clifford action \( E \otimes S(E) \rightarrow S(E) \) defines a real bilinear map \( S(E) \otimes S(E) \rightarrow E \) which in the cases of interest in this lecture will be skewsymmetric, whence defines a map \( \Lambda^2 S(E) \rightarrow E \). This will form part of a Lie bracket on a 2-graded algebra whose odd subspace will be isomorphic to \((\text{a subspace of})\ S(E)\).

1.2. \textbf{Globalisation: spin geometry}. Let \((M^n, g)\) be a riemannian manifold. At every point \( x \in M \) we can consider the orthonormal frames for the tangent space \( T_xM \). This is the fibre at \( x \) of a principal fibre bundle \( O(M) \) called, unsurprisingly, the bundle of orthonormal frames. If \( M \) is oriented, and restricting to \textit{oriented} frames, we obtain a subbundle \( SO(M) \). The obstruction to the existence of \( SO(M) \) is the triviality of \( \det(TM) \) which is captured by the first Stiefel–Whitney class \( w_1(TM) \in H^1(M; \mathbb{Z}_2) \). Hence roughly speaking half the manifolds are orientable. Assuming that \( M \) is oriented, a spin structure on \( M \) is a lift \( Spin(M) \rightarrow SO(M) \) restricting fibrewise to the two-to-one homomorphism \( Spin(n) \rightarrow SO(n) \). The obstruction is now captured by the second Stiefel–Whitney class \( w_2(TM) \in H^2(M; \mathbb{Z}_2) \), whence roughly speaking one quarter of all manifolds are spin. If \( w_2(TM) = 0 \), the set of inequivalent spin structures are in one-to-one correspondence with \( H^1(M; \mathbb{Z}_2) \cong \text{Hom}(\pi_1(M), \mathbb{Z}_2) \), so roughly speaking to the assignment of a sign to every noncontractible loop.

For example, if \( S^n \subset \mathbb{R}^{n+1} \) is the unit sphere, then \( T_xS^n \) is the perpendicular complement of the line in \( \mathbb{R}^{n+1} \) through the origin and \( x \). An oriented orthonormal basis for \( T_xS^n \) is then an oriented orthonormal frame for \( x^\perp \cong \mathbb{R}^n \) and hence in one-to-one correspondence with the points in \( SO(n) \). Adding \( x \) itself we obtain an oriented orthonormal frame for \( \mathbb{R}^{n+1} \) and hence an element of \( SO(n+1) \): the element which takes the standard orthonormal basis to that one. Conversely, we have a map \( SO(n+1) \rightarrow S^n \) sending the matrix \( g \in SO(n+1) \) to its first column, say \( x \), which is a unit vector in \( \mathbb{R}^{n+1} \). The fibre of this map consists of the remaining \( n \) columns, which form an oriented frame in the \( n \)-dimensional subspace perpendicular to \( x \). In other words, \( SO(S^n) = SO(n+1) \). The spin cover \( Spin(S^n) \) is precisely the spin group \( Spin(n+1) \) and since \( \pi_1(S^n) = \{1\} \) (for \( n > 1 \)), there is a unique such spin structure. For \( n = 1 \) there are two spin structures, which physicists like to call Neveu–Schwarz and Ramond [2].

Let \( \rho : Spin(n) \rightarrow GL(S) \) be a spinor representation of \( Spin(n) \) and define the associated vector bundle
\[
S = Spin(M) \times_\rho S ,
\]
called a bundle of spinors. Its sections are called spinor fields. (Had the lectures been given in Spanish, the spinor bundle would have been called \( \Xi! \))
The tangent space \((TxM, g_x)\) to \(M\) at \(x\) is a euclidean vector space and gives rise to a Clifford algebra \(Cl(T_xM)\). As \(x\) varies, this globalises to a bundle \(Cl(TM)\) of Clifford algebras. As a vector bundle, we have a natural isomorphism \(Cl(TM) \cong \Lambda T^* M\). We will always think of spinor representations as the restriction to the spin group of a pinor representation of the Clifford algebra. Globalising, this means that the spinor bundle \(\mathcal{S}\) will always admit an action of the Clifford bundle \(Cl(TM)\), making it into a bundle of Clifford modules. In this way, differential forms, which are sections of \(\Lambda T^* M\), will be able to act on spinor fields via the natural isomorphism \(\Lambda T^* M \cong Cl(TM)\) and the action of \(Cl(TM)\) on \(\mathcal{S}\).

The Levi-Civita connection on \(SO(M)\) lifts to a connection on \(Spin(M)\) and hence defines a connection on any associated vector bundle. In particular we have a covariant derivative \(\nabla\) on sections of \(\mathcal{S}\): for all vector fields \(X \in \mathfrak{X}(M)\) and spinor fields \(\psi \in \Gamma(\mathcal{S})\), \(\nabla_X \psi \in \Gamma(\mathcal{S})\). The covariant derivative \(\nabla_X\) along \(X\) is linear and obeys the Leibniz rule
\[
\nabla_X (f \psi) = (Xf)\psi + f \nabla_X \psi ,
\]
for all functions \(f \in C^\infty(M)\).

1.3. Killing spinors and the cone construction. The Levi-Civita connection may be used to write down natural equations on spinors, whose solutions define privileged notions of spinor fields:

- **parallel spinors**: \(\nabla \psi = 0\). By the holonomy principle, the holonomy group of \(\nabla\) must be included in the stabilizer of a spinor. The determination of which manifolds admit parallel spinors was thus solved by Wang [6] using Berger’s holonomy classification. The irreducible holonomy groups of manifolds admitting parallel spinors are \(SU(n)\), \(Sp(n)\), \(G_2\) and \(Spin(7)\).

- **Killing spinors**: \(\nabla_X \psi = \lambda X \cdot \psi\) for all \(X \in \mathfrak{X}(M)\) for some nonzero constant \(\lambda \in \mathbb{C}\), called the **Killing constant**. Iterating the definition of a Killing spinor, we find that \(M\) is Einstein with scalar curvature proportional to \(\lambda^2\), which means that \(\lambda^2 \in \mathbb{R}\) and hence \(\lambda \in \mathbb{R} \cup i\mathbb{R}\). This gives rise to two separate notions of **real** or **imaginary** Killing spinors, according to whether \(\lambda\) is real or imaginary, respectively.

In this lecture we will concentrate on the case of real Killing spinors. Moreover by rescaling the metric, if necessary, we may always take \(\lambda = \pm \frac{1}{2}\). Therefore such a manifold \(M\) is Einstein with positive scalar curvature and, if complete, is compact by the Bonnet–Myers theorem. The question of which complete spin manifolds admit real Killing spinors was solved by Bär [7] via his celebrated cone construction by mapping the problem to the problem of determining which manifolds admit parallel spinors.

Indeed, given a spin manifold \((M, g)\) we define its **metric cone** \(C(M) = \mathbb{R}^+ \times M\), with metric
\[
g_C = dr^2 + r^2 g ,
\]
where \(r > 0\) is the parameter of the \(\mathbb{R}^+\). For example, if \(M = S^n\), then \(C(M) = \mathbb{R}^{n+1} \setminus \{0\}\). In this case, and in this case alone, the metric extends smoothly to the origin and it is the flat metric on \(\mathbb{R}^{n+1}\) written in spherical polar coordinates. In all other cases, the metric has a conical singularity at \(r = 0\). Bär’s penetrating observation was that \(\nabla_X \psi = \pm \frac{1}{2} X \cdot \psi\) on \(M\) becomes the condition \(\nabla \tilde{\psi} = 0\) on the cone, where the tilded objects live on the cone. The sign in the Killing spinor equation is reflected either in the chirality of the parallel spinor in the case of \(n\) odd, or in the embedding \(Cl(n) \subset Cl(n+1)\) if \(n\) is even. This says that the existence of real Killing spinors is again a holonomy problem, albeit in an auxiliary manifold one dimension higher. A theorem of Gallot [8] says that if \(M\) is complete, then its metric cone is either flat — so that \(M\) is the round sphere — or irreducible. In this latter case, we may use
Wang’s classification of holonomy groups leaving a spinor invariant. In this way one arrives at Table 1 of (types of) complete manifolds admitting real Killing spinors.

**Table 1.** Manifolds admitting real Killing spinors

| $n$ | $\text{hol } \nabla$ | $C(M)$ | $M$ |
|-----|-----------------|---------|-----|
| $2m-1$ | SU($m$) | Calabi–Yau | Sasaki–Einstein |
| $4m-1$ | Sp($m$) | hyperkähler | 3-Sasaki |
| 6 | $G_2$ | nearly Kähler (non-Kähler) | |
| 7 | Spin(7) | weak $G_2$ holonomy | |

Such manifolds play an important rôle in the AdS/CFT correspondence, as pointed out originally in [9].

1.4. **The Killing superalgebra.** Let $(M, g)$ be a spin manifold and $\mathcal{S} \to M$ a bundle of $C^\infty(TM)$-modules. We will assume that $M$ admits real Killing spinors and, without loss of generality (i.e., rescaling the metric and reversing orientation, if necessary), assume that the Killing constant $\lambda = \frac{1}{2}$. We define a 2-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is the vector space of Killing vector fields on $M$ and

$$
\mathfrak{g}_1 = \left\{ \psi \in \Gamma(\mathcal{S}) \mid \nabla_X \psi = \frac{1}{2} X \cdot \psi \quad \forall X \in \mathfrak{X}(M) \right\}
$$

(9) is the vector space of Killing spinors. Remember we have a real bilinear map $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{X}(M)$ by transposing the Clifford action. Explicitly, given $\psi_1, \psi_2 \in \mathfrak{g}_1$, we let $[\psi_1, \psi_2] \in \mathfrak{X}(M)$ be defined by

$$
g([\psi_1, \psi_2], X) = (\psi_1, X \cdot \psi_2)
$$

(10) for all $X \in \mathfrak{X}(M)$. The following result explains the terminology.

**Lemma 1.** $[\psi_1, \psi_2]$ is a Killing vector field.

**Proof.** This is a simple calculation. For all $X, Y \in \mathfrak{X}(M)$, we have

$$
g(\nabla_X [\psi_1, \psi_2], Y) = X \cdot g([\psi_1, \psi_2], Y) - g([\psi_1, \psi_2], \nabla_X Y)
$$

\[= X \cdot (\psi_1, Y \cdot \psi_2) - (\psi_1, \nabla_X Y \cdot \psi_2)\]

\[= (\nabla_X \psi_1, Y \cdot \psi_2) + (\psi_1, Y \cdot \nabla_X \psi_2)\]

\[= \frac{1}{2} (X \cdot \psi_1, Y \cdot \psi_2) + \frac{1}{2} (\psi_1, Y \cdot X \cdot \psi_2)\]

\[= \frac{1}{2} (\psi_1, (Y \cdot X - X \cdot Y) \cdot \psi_2),\]

whence

$$
g(\nabla_X [\psi_1, \psi_2], Y) + g(\nabla_Y [\psi_1, \psi_2], X) = 0 .
$$

(11) \hfill \square

Let $K \in \mathfrak{g}_0$ be a Killing vector field. Then $A_K : TM \to TM$, defined by $A_K(Y) = -\nabla_Y K$, is a skewsymmetric endomorphism of $TM$. Define the following **Lie derivative** on spinor fields:

$$
\mathcal{L}_K = \nabla_K + \rho(A_K),
$$

(12) where $\rho : \mathfrak{so}(TM) \to \mathfrak{g}(\mathcal{S})$ is the spinor representation at the level of the Lie algebra.

Taking for $\rho$ any other representation of $\mathfrak{so}(TM)$ defines equally well a Lie derivative on sections of the corresponding associated vector bundle. For example, if we take $\rho$ to be the defining representation on $TM$, we have

$$
\mathcal{L}_K X = \nabla_K X + A_K X = \nabla_K X - \nabla_X K = [K, X],
$$

(13)
which is the standard Lie derivative of vector fields.

**Proposition 2.** The Lie derivative $\mathcal{L}_K$ on $\Gamma(\mathbb{S})$ obeys a number of properties:

1. $[\mathcal{L}_{K_1}, \mathcal{L}_{K_2}]\psi = \mathcal{L}_{[K_1,K_2]}\psi$,
2. $\mathcal{L}_K(f\psi) = (Kf)\psi + f\mathcal{L}_K\psi$,
3. $\mathcal{L}_K(X \cdot \psi) = [K,X] \cdot \psi + X \cdot \mathcal{L}_K\psi$, and
4. $\mathcal{L}_K \nabla_X \psi = \nabla_X \mathcal{L}_K\psi + \nabla_{[K,X]}\psi$,

for all Killing vector fields $K, K_1, K_2$ and all $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\psi \in \Gamma(\mathbb{S})$.

We remark that properties (1) and (2) justify calling $\mathcal{L}_K$ a Lie derivative, whereas (3) and (4) say that $\mathcal{L}_X$ leaves invariant the Clifford action and $\nabla$, respectively. The definition of $\mathcal{L}_K$ goes back to Lichnerowicz and appears in the thesis of Kosmann-Schwarzbach [10]. It appeared also more recently, in a supergravity context, in [11].

The last two properties in Proposition 2 imply that if $\psi \in \mathfrak{g}_1$ is a Killing spinor, then so is $\mathcal{L}_K\psi$ for all Killing vector fields $K$. Indeed,

$\nabla_X \mathcal{L}_K\psi = \mathcal{L}_K \nabla_X \psi - \nabla_{[K,X]}\psi$ \hspace{1cm} by (4) in Proposition 2

$= \frac{1}{2} \mathcal{L}_K(X \cdot \psi) - \frac{1}{2} [K,X] \cdot \psi$ \hspace{1cm} since $\psi \in \mathfrak{g}_1$

$= \frac{1}{2} X \cdot \mathcal{L}_K\psi$ \hspace{1cm} by (3) in Proposition 2

whence $\mathcal{L}_K\psi \in \mathfrak{g}_1$. This defines a real bilinear map $\mathfrak{g}_0 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$, denoted $(K, \psi) \mapsto [K,\psi] := \mathcal{L}_K\psi$.

We now have defined a 2-graded multiplication on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, denoted by a bracket $[-,-]$ anticipating the fact that in some cases it will be a Lie (super)algebra. To wit, we have a map $\Lambda^2 \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$, given by the Lie bracket of vector fields, the above-defined map $\mathfrak{g}_0 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ given by the spinorial Lie derivative, and the map $\mathfrak{g}_1 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$ given by transposing the Clifford action. This last map is either symmetric or skew-symmetric depending on dimension. This suggests that it may define a Lie (super)algebra structure on $\mathfrak{g}$. This requires satisfying the relevant Jacobi identity. Being a 2-graded algebra, the jacobator, the element in $\text{Hom}(\mathfrak{g}^{\otimes3}, \mathfrak{g})$ whose vanishing implies the Jacobi identity, breaks up into four components depending on whether we have three, two, one or no elements in $\mathfrak{g}_0$:

- all elements in $\mathfrak{g}_0$: this is simply the Jacobi identity of the Lie bracket of vector fields;
- two elements in $\mathfrak{g}_0$ and one in $\mathfrak{g}_1$:

  $[K_1, [K_2, \psi]] - [K_2, [K_1, \psi]] = \mathcal{L}_{K_1} \mathcal{L}_{K_2}\psi - \mathcal{L}_{K_2} \mathcal{L}_{K_1}\psi$

  $= [\mathcal{L}_{K_1}, \mathcal{L}_{K_2}]\psi$

  $= \mathcal{L}_{[K_1,K_2]}\psi$ \hspace{1cm} by (1) in Proposition 2

  $= [[K_1, K_2], \psi]$ ;

- one element in $\mathfrak{g}_0$ and two in $\mathfrak{g}_1$: this is property (3) in Proposition 2 and

- all elements in $\mathfrak{g}_1$: this does not follow from the formalism and has to be checked case by case. For Lie algebras, it lives in $(\Lambda^3 \mathfrak{g}_1 \otimes \mathfrak{g}_1)^{\mathfrak{g}_0}$, whereas for Lie superalgebras it lives in $(\mathfrak{s}^{3} \mathfrak{g}_1^1 \otimes \mathfrak{g}_1)^{\mathfrak{g}_0}$. In some cases, representation theory shows that such spaces are 0, and hence this last component of the jacobator vanishes. In other cases, such spaces are not 0, but the jacobator vanishes all the same. In most cases, however, this last component of the jacobator will not vanish. Hence the generic situation is a 2-graded $\frac{3}{4}$-Lie (super)algebra.
1.5. Some examples of 2-graded Lie algebras. Consider now the unit spheres $S^7 \subset \mathbb{R}^8$, $S^9 \subset \mathbb{R}^9$ and $S^{15} \subset \mathbb{R}^{16}$, thought of as riemannian manifolds with the canonical spin structure given by the spin groups. In all these cases, the spinor inner product is real symmetric and positive-definite. Since Clifford action is skewsymmetric, so is its transpose, whence the odd-odd bracket is similarly skewsymmetric, defining a map $\Lambda^2 g_1 \to g_0$. In other words, $g$ is a 2-graded (possibly) Lie algebra. Notice that $[g_1, g_1]$ is an ideal of $g_0$: this does not use the vanishing of the last component of the jacobator. Since $g_0$, the isometry Lie algebra of the above spheres, is simple, we see that $[g_1, g_1] = g_0$ in this case.

To determine $g_1$ as a $g_0$-module we use the cone construction and the fact, proved in [11], that this construction is equivariant under the action of $g_0$, which is naturally a Lie subalgebra of the isometries of the cone. This means that $[K, \psi] = \mathcal{L}_K \psi$ can be lifted and calculated on the cone:

$$L_{\tilde{K}} \tilde{\psi} = \tilde{\nabla}_{\tilde{K}} \tilde{\psi} + \tilde{\rho}(A_{\tilde{K}}) \tilde{\psi},$$

(14)

but $\tilde{\psi}$ is parallel and (relative to flat coordinates on the cone) $\tilde{\psi}$ and $A_{\tilde{K}}$ are constant because $\tilde{K}$ is a linear vector field. Therefore this is the standard action of $g_0 = \mathfrak{so}(n+1)$ $(n = 7, 8, 15)$ on (positive-chirality, when applicable) spinors: $S(8)_+, S(9)$ and $S(15)_+$, all of which are irreducible representations.

In all cases, a roots-and-weights calculation (made less painful by using LiE [12]) shows that

$$\left(\Lambda^3 g_1^* \otimes g_1\right)^{g_0} = 0,$$

(15)

whence the Jacobi identity is satisfied and $g$ becomes a 2-graded Lie algebra. To identify the Lie algebras in question we simply observe that $g_0$ being simple and $g_1$ being irreducible, implies that $g$ is simple. The dimensions are easy to compute and the Lie algebras are thus easy to recognise from the Killing–Cartan classification. The results are summarised in Table 2.

| $M$ | $g_0$ | $\dim g_0$ | $g_1$ | $\dim g_1$ | $\dim g$ | $g$ |
|-----|-------|-------------|-------|------------|----------|-----|
| $S^7$ | $\mathfrak{so}(8)$ | 28 | $S(8)_+$ | 8 | 36 | $B_4$ |
| $S^8$ | $\mathfrak{so}(9)$ | 36 | $S(9)$ | 16 | 52 | $F_4$ |
| $S^{15}$ | $\mathfrak{so}(16)$ | 120 | $S(16)_+$ | 128 | 248 | $E_8$ |

Since the inner products on $g_0$ and $g_1$ are invariant and positive-definite, $g$ is a compact real form of the corresponding complex simple Lie algebra. By the usual device of taking $i g_1$ instead of $g_1$, we may obtain the maximally split real forms.

2. Supergravity backgrounds

Supergravity is an extension of Einstein (or Einstein–Maxwell) theory. At the level of its solutions, it is given by some geometric data $(g, F, \ldots)$, where $g$ is a local lorentzian metric and $F, \ldots$ stand for extra fields, all subject to partial differential equations of the form

- **Einstein**
  $$\text{Ric}(g) - \frac{1}{2} Rg = T(F, \ldots)$$

- **“Maxwell”**
  $$dF = 0 \quad \text{and} \quad d \star F = \ldots$$

The details depend on the supergravity theory in question and have hence kept purposefully vague in the above description. We will consider here only so-called Poincaré supergravities. There are other supergravities: massive, gauged,\ldots Supergravity theories are dictated by the
representation theory of the Poincaré superalgebras. There are (physically interesting) supergravity theories in dimension \( d \leq 11 \) and Lorentzian signature, meaning that the local metric \( g \) is Lorentzian. Supergravity theories are among the jewels of twentieth century theoretical physics and a good review of the structure of supergravity theories from the representation theory point of view can be found in \([13]\).

2.1. **Eleven-dimensional supergravity.** My favourite, and to some extent the simplest yet nontrivial, supergravity theory is the unique eleven-dimensional supergravity theory. Its existence was conjectured by Nahm \([14]\), whereas it was constructed by Cremmer, Julia and Scherk \([15]\). Its field content is a Lorentzian eleven-dimensional metric \( g \) and a closed 4-form \( F \). We can motivate this as follows.

Supergravity is a theory invariant under local supersymmetry, hence the spectrum should carry a representation of the corresponding supersymmetry algebra. In the case of eleven-dimensional supergravity this is the eleven-dimensional Poincaré superalgebra \((\mathfrak{so}(1,10) \oplus \mathbb{R}^{1,10}) \oplus S(1,10)\), where \( S(1,10) \) is the spinor representation of \( \text{Spin}(1,10) \). It is not hard to show, using Bott periodicity, that \( \text{Cl}(1,10) = \mathbb{R}(32) \oplus \mathbb{R}(32) \) and hence \( S(1,10) \cong \mathbb{R}^{32} \).

The supertranslation ideal generated by \( \mathbb{R}^{1,10} \oplus S(1,10) \) has as nonzero brackets the projection \( S^2 S(1,10) \to \mathbb{R}^{1,10} \) of the symmetric square of the spinor representation \( S^2 S(1,10) \cong \mathbb{R}^{1,10} \oplus \Lambda^2 \mathbb{R}^{1,10} \oplus \Lambda^5 \mathbb{R}^{1,10} \) onto the vector representation of \( \text{Spin}(1,10) \).

Irreducible unitary representations of the Poincaré superalgebra are induced by representations of the supertranslation ideal generated by \( S(1,10) \oplus \mathbb{R}^{1,10} \). This is done by first fixing a character of the abelian translation ideal \( \mathbb{R}^{1,10} \); that is, a momentum \( p \in (\mathbb{R}^{1,10})^\ast \). Since we are interested in a theory of gravity, which we expect even in eleven dimensions to be a long range force, we require a massless representation, whence \( p^2 = 0 \), but \( p \neq 0 \). The little group of \( p \), which is the maximal compact subgroup of the stabiliser of \( p \) in \( \text{Spin}(1,10) \) is isomorphic to \( \text{Spin}(9) \). Once a momentum \( p \) has been fixed, the supertranslation ideal takes the form of a Clifford algebra

\[
\begin{align*}
[Q_1, Q_2] &= -2 (Q_1, p \cdot Q_2) 1 , \\
\end{align*}
\]  

where \( Q_i \in S(1,10) \) and the spinor inner product is symplectic in this signature, whence the bracket here is symmetric. The bilinear form defining the Clifford algebra, \( (Q_1, Q_2) = (Q_1, p \cdot Q_2) \) is degenerate because \( p^2 = 0 \). In fact, it has rank 16. This is shown by exhibiting a “dual” momentum \( q \) such that \( q^2 = 0 \) and \( p \cdot q = 1 \). Then \( S(1,10) = \ker p \oplus \ker q \), where the kernel refers to the Clifford action. The common stabiliser of \( p \) and \( q \) in \( \text{Spin}(1,10) \) is a Spin(9)-subgroup, which we can identify with the little group of either \( p \) or \( q \). The degenerate Clifford algebra \([18]\) becomes an honest Clifford algebra on \( \ker q \subset S(1,10) \) isomorphic to \( \text{Cl}(16) \), and in fact, as Spin(9)-module, \( \ker q \) is the spinor module \( S(9) \). There is, up to isomorphism, a unique irreducible representation of \( \text{Cl}(16) \) and it is real and of dimension 256. Indeed, by Bott periodicity,

\[
\text{Cl}(16) \cong \text{Cl}(8) \otimes_{\mathbb{R}} \mathbb{R}(16) \cong \mathbb{R}(16) \otimes_{\mathbb{R}} \mathbb{R}(16) \cong \mathbb{R}(16^2) .
\]

As a Spin(9)-module, this is nothing but \( S(S(9)) \); that is, spinors of spinors! A roots-ands weights calculation shows that as a representation of \( \text{Spin}(9) \) we have

\[
S(S(9)) \cong S_0^2(\mathbb{R}^9) \oplus \Lambda^3 \mathbb{R}^9 \oplus \text{RS}(\mathbb{R}^9) ,
\]

where \( \mathbb{R}^9 \) stands for the vector representation of \( \text{Spin}(9) \), \( S_0^2 \) denotes traceless symmetric tensors and \( \text{RS} \) stands for the Rarita–Schwinger representation, which is the subrepresentation of \( \mathbb{R}^9 \otimes S(9) \) consisting of the kernel of the Clifford action \( \mathbb{R}^9 \otimes S(9) \to S(9) \). Counting dimensions, we see that for the bosonic part of the representation

\[
\dim S_0^2(\mathbb{R}^9) + \dim \Lambda^3 \mathbb{R}^9 = 44 + 84 = 128 ,
\]
whereas for the fermionic part of the representation
\[ \dim \text{RS}(\mathbb{R}^9) = \dim \mathbb{R}^9 \otimes S(9) - \dim S(9) = 16 \times 9 - 16 = 128, \]
whence the physical degrees of freedom match, as expected. In terms of fields, \( S^2_{\mathbb{R}}(\mathbb{R}^9) \) parametrise the fluctuations of a metric tensor \( g \), whereas \( \Lambda^3 \mathbb{R}^9 \) parametrises the fluctuations of a (locally defined) 3-form potential \( A \), and \( \text{RS}(\mathbb{R}^9) \) parametrises the fluctuations of a gravitino.

The supergravity action, ignoring terms involving the gravitino, consists of three terms: an Einstein–Hilbert term, a Maxwell term and a Chern–Simons term. The lagrangian density is given by
\[ R \text{dvol}_g - \frac{1}{4} F \wedge \ast F + \frac{1}{12} F \wedge F \wedge A, \]
(23)
where \( F = dA \) locally. Although \( A \) appears explicitly in the above lagrangian, the Euler–Lagrange equations only involve \( F \). The equations are of Einstein–Maxwell type with a twist provided by the Chern–Simons term; namely, the Maxwell equation is nonlinear:
\[ d \ast F = -\frac{1}{2} F \wedge F. \]
(24)

2.2. Supersymmetric supergravity backgrounds. We define a (bosonic) eleven-dimensional supergravity background to be an eleven-dimensional lorentzian spin manifold \( (M, g, \mathbb{S}) \) and a closed 4-form \( F \in \Omega^4(M) \) subject to the Einstein–Maxwell equations derived from the lagrangian (23).

The lagrangian (23) admits a supersymmetric completion by adding extra terms involving the gravitino \( \Psi \in \Omega^1(M, \mathbb{S}) \). The variation of the gravitino under supersymmetry defines a connection \( D \) on the spinor bundle \( \mathbb{S} \):
\[ D_X \psi := \nabla_X \psi + \frac{1}{6} e^i X^i F \cdot \psi + \frac{1}{12} X^b \wedge F \cdot \psi, \]
for all \( \psi \in \Gamma(\mathbb{S}) \) and \( X \in \mathfrak{X}(M) \) and where \( X^b \in \Omega^1(M) \) is the one-form such that \( X^b(Y) = g(X,Y) \) for all \( Y \in \mathfrak{X}(M) \).

The connection \( D \) is the fundamental object in this game, as it encodes virtually all the information of the theory. For example, the Einstein–Maxwell equations can be recovered by demanding the vanishing of the Clifford-trace of its curvature. More explicitly, let \( e_i \) be a pseudo-orthonormal frame for \( M \) and let \( e^i \) denote the dual frame, defined by \( g(e^i, e_j) = \delta_j^i \).

Then, as shown in [16], the field equations defining the notion of a supergravity background are equivalent to
\[ \sum_i e^i \cdot R^D(e_i, X) = 0 \quad \forall \ X \in \mathfrak{X}(M). \]
(26)

A nonzero spinor field \( \psi \in \Gamma(\mathbb{S}) \) which is \( D \)-parallel is called a (supergravity) Killing spinor. Although this seems a priori to be a generalisation of the notion of a parallel spinor, it is in fact the original notion of a Killing spinor. The geometrical notion in the first lecture is a special case of the supergravity Killing spinor equation for a particular Ansatz for \( (M, g, F) \), known as a Freund–Rubin background [17].

Being a linear equation, Killing spinors form a vector space, which anticipating the construction of the Killing superalgebra, will be denoted \( \mathfrak{g}_1 \). Being defined by a parallel condition, a Killing spinor is determined by its value at a point, whence the dimension of \( \mathfrak{g}_1 \) is bounded above by the rank of the spinor bundle; that is, \( \dim \mathfrak{g}_1 \leq 32 \). The ratio
\[ \nu = \frac{\dim \mathfrak{g}_1}{32} \]
(27)
is called the supersymmetry fraction of the background \( (M, g, F) \). If \( \nu > 0 \), \( (M, g, F) \) is said to be supersymmetric.
2.3. **Examples.** A large number of supersymmetric backgrounds are known. Maximally supersymmetric backgrounds – those with $\nu = 1$ – have been classified in [18, 19, 20]. For such backgrounds, $D$ is flat and the equations of motions are automatically satisfied. These backgrounds are related as follows:

\[
\begin{array}{c}
\text{AdS}_4 \times S^7 \\
\text{PL} \\
\text{KG} \\
\text{PL} \\
\text{AdS}_7 \times S^4
\end{array}
\]

where $\text{AdS}_n$ is the $n$-dimensional anti de Sitter spacetime — i.e., lorentzian hyperbolic space —, KG is a special type of plane wave [21] whose geometry is described by a lorentzian symmetric space of Cahen–Wallach type [22], and $\mathbb{R}^{1,10}$ is Minkowski spacetime with $F = 0$. The arrows labelled “PL” are Penrose limits, described in this context in [23, 24], but tracing their origin to work of Güven [25] and, of course, Penrose [26]. The undecorated arrows are zero-curvature limits.

The $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$ backgrounds depend on a parameter, interpreted as the scalar curvature of the eleven-dimensional geometry. The radii of curvature of the factors are in a ratio of 2 : 1, whence these backgrounds do not describe realistic compactifications as they once were thought to do. They are known as Freund–Rubin backgrounds. The Killing spinors of the Freund–Rubin backgrounds are $\otimes$ of geometric Killing spinors on the two factors: real on the riemannian factor and imaginary on the lorentzian factor. One can substitute either factor by an Einstein manifold with the same scalar curvature and admitting the relevant kind of Killing spinors. In particular one can consider $\text{AdS}_4 \times X^7$, where $X$ is a riemannian manifold admitting real Killing spinors, whence its cone has holonomy contained in Spin(7). Whenever $X$ is not a sphere, the resulting background has a smaller fraction $\nu$ of supersymmetry. They can be understood as near-horizon geometries of M2-branes, to which we now turn.

The **M2-brane** is a interesting background with $\nu = \frac{1}{2}$, discovered in [27] and interpreted as an interpolating soliton in [28]. It is described as follows:

\[
g = H^{-2/3} ds^2(\mathbb{R}^{1,2}) + H^{1/3} \left( dr^2 + r^2 ds^2(S^7) \right) \\
F = \text{dvol}(\mathbb{R}^{1,2}) \wedge dH^{-1} \\
H(r) = \alpha + \frac{\beta}{r^6} ,
\]

where $ds^2(\mathbb{R}^{1,2})$ and $\text{dvol}(\mathbb{R}^{1,2})$ are the metric and volume of 3-dimensional Minkowski spacetime, $ds^2(S^7)$ is the metric on the unit sphere in $\mathbb{R}^8$ and $H$ is a two-parameter harmonic function on $\mathbb{R}^8$. If we take $\beta \to 0$ while keeping $\alpha \neq 0$ fixed, we obtain eleven-dimensional Minkowski spacetime with $F = 0$, but taking $\alpha \to 0$ while keeping $\beta \neq 0$ fixed, one obtains $\text{AdS}_4 \times S^7$ with scalar curvature depending on $\beta$. Therefore the M2-brane interpolates between these two maximally supersymmetric backgrounds. The Killing spinors are given by

\[
\psi = H^{1/6} \psi_\infty ,
\]

where $\psi_\infty$ is a parallel spinor in the asymptotic Minkowski spacetime obeying the projection condition

\[
\text{dvol}(\mathbb{R}^{1,2}) \cdot \psi_\infty = \psi_\infty .
\]
Since $\text{dvol}(\mathbb{R}^{1,2})^2 = 1$ and the parallel spinors of $\mathbb{R}^{1,10}$ split into two half-dimensional eigenspaces of $\text{dvol}(\mathbb{R}^{1,2})$. As a result the solution has $\nu = \frac{1}{2}$.

As observed in [9], replacing $S^7$ by another seven-dimensional manifold admitting real Killing spinors — that is, weak $G_2$-holonomy, Sasaki-Einstein or 3-Sasaki manifolds — we obtain an M2-brane at a conical singularity in an 8-dimensional manifold with Spin(7), SU(4) or Sp(2) holonomy, respectively.

To this day a large class of backgrounds with various values of $\nu$ are known to exist. General local metrics with minimal supersymmetry have been written down in [16, 29]. To date, the only fraction which has been ruled out is $\nu = \frac{31}{32}$ [30, 31].

3. The Killing superalgebra of supergravity backgrounds

In the first lecture we saw that from a spin manifold admitting Killing spinors one could define (in the good cases) a 2-graded Lie algebra and in this way we recovered the compact real forms of the simple Lie algebras of types $B_4$, $F_4$ and $E_8$. At its most basic, what we have is a spin manifold with a privileged subspace of spinor fields which then generates a 2-graded algebra with the spinors being the odd-subspace.

In the second lecture we saw how supersymmetric (eleven-dimensional) supergravity backgrounds gave rise to precisely such a situation: an eleven-dimensional lorentzian spin manifold with a privileged notion of spinor: the supergravity Killing spinors, which are parallel with respect to a connection $D$ on the spinor bundle. Unlike the spin connection $\nabla$, the connection $D$ is not induced from a connection on the tangent bundle: it is genuinely a spinor connection. The spinor bundle is a real rank-32 symplectic vector bundle, but $D$ does not preserve the symplectic structure. In fact, as shown by Hull [32], the holonomy algebra of $D$ is generically contained in $\mathfrak{sl}(32, \mathbb{R})$ since only the ‘determinant’ is preserved.

In this third and last lecture we will see that to every supersymmetric background of eleven-dimensional supergravity one can assign a Lie superalgebra by the techniques in the first lecture and using it we will show that if $\nu$ is sufficiently large, the background is forced to be homogeneous. This lecture is based on [33].

3.1. The Killing superalgebra. Let $(M, g, F)$ be a supersymmetric eleven-dimensional supergravity background. Following the method in the first lecture, let us define a 2-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_0 = \{ X \in \mathfrak{X}(M) | \mathcal{L}_X g = 0 = \mathcal{L}_X F \}$$

is the Lie algebra of $F$-preserving isometries of the background, and

$$\mathfrak{g}_1 = \{ \psi \in \Gamma(\$) | D\psi = 0 \}$$

is the space of Killing spinors. Clearly $\mathfrak{g}$ is finite-dimensional, since as mentioned above $\dim \mathfrak{g}_1 \leq 32$ and $\dim \mathfrak{g}_0 \leq 66$, which is the maximum dimension of the isometry algebra of an eleven-dimensional lorentzian manifold. It is only for $\mathbb{R}^{1,10}$ with $F = 0$ that both of these upper bounds are realised.

Given $\psi \in \Gamma(\$)$, we may define $[\psi, \psi] \in \mathfrak{X}(M)$ by transposing the Clifford action. This is nonzero because this bilinear product is now symmetric, since the inner product on $\$ is symplectic. In fact, it is not difficult to show that $[\psi, \psi]$ is always causal; that is, it has non-positive minkowskian norm.

**Lemma 3.** If $\psi \in \mathfrak{g}_1$ then $[\psi, \psi] \in \mathfrak{g}_0$. 

Proof. We first show that $[\psi, \psi]$ is Killing. We first have that
\[
g(\nabla_X [\psi, \psi], Y) = X g([\psi, \psi], Y) - g([\psi, \psi], \nabla_X Y) \quad \text{since } \nabla g = 0
\]
\[
= X (\psi, Y \cdot \psi) - (\psi, \nabla_X Y \cdot \psi) \quad \text{by definition of } [\psi, \psi]
\]
\[
= (\nabla_X \psi, Y \cdot \psi) + (\psi, Y \cdot \nabla_X \psi)
\]
\[
= 2 (\nabla_X \psi, Y \cdot \psi) \quad \text{since } (\psi_1, Y \cdot \psi_2) = (\psi_2, Y \cdot \psi_1)
\]
\[
= -2 (Y \cdot \nabla_X \psi, \psi) .
\]
Now since $D\psi = 0$,
\[
\nabla_X \psi = -\frac{1}{6} \iota_X F \cdot \psi - \frac{1}{12} X^b \wedge F \cdot \psi ,
\]
whence
\[
g(\nabla_X [\psi, \psi], Y) = \frac{1}{3} (Y \cdot \iota_X F \cdot \psi, \psi) + \frac{1}{6} (Y \cdot (X^b \wedge F) \cdot \psi, \psi)
\]
\[
= \frac{1}{3} (Y^b \wedge \iota_X F - \iota_Y \iota_X F) \cdot \psi, \psi
\]
\[
+ \frac{1}{6} (Y^b \wedge X^b \wedge F + g(X, Y)F - X^b \wedge \iota_Y F) \cdot \psi, \psi
\]
\[
= -\frac{1}{3} (\iota_Y \iota_X F \cdot \psi, \psi) + \frac{1}{6} (Y^b \wedge X^b \wedge F \cdot \psi, \psi) ,
\]
where we have used that for every 4-form $\Phi \in \Omega^4(M)$,
\[
(\Phi \cdot \psi, \psi) = 0 .
\]
It follows that
\[
g(\nabla_X [\psi, \psi], Y) + g(\nabla_Y [\psi, \psi], X) = 0 ,
\]
whence $[\psi, \psi]$ is a Killing vector. One can also prove that $\mathcal{L}_{[\psi, \psi]} F = 0$. Indeed, since $dF = 0$, $\mathcal{L}_{[\psi, \psi]} F = d\iota_{[\psi, \psi]} F$ and it is just a calculation to show that
\[
\iota_{[\psi, \psi]} F = -dB ,
\]
where $B \in \Omega^2(M)$ is the 2-form in the square of $\psi$:
\[
B(X, Y) = (\psi, X^b \wedge Y^b \cdot \psi) .
\]

This result explains why $\psi$ is a called a Killing spinor, since it is the “square root” of a Killing vector. It follows by the usual polarisation trick that if $\psi_1, \psi_2 \in g_1$, then $[\psi_1, \psi_2] \in g_0$. We therefore have a symmetric bilinear map $g_1 \times g_1 \rightarrow g_0$ denoted by $(\psi_1, \psi_2) \mapsto [\psi_1, \psi_2]$.

We now define a bilinear map $g_0 \times g_1 \rightarrow g_1$ using the spinorial Lie derivative $\mathcal{L}_X$ defined in equation \([\ref{12}]\). Let $X \in g_0$. It follows from Proposition \([\ref{2}]\) and the fact that $X$ preserves $F$, that $\mathcal{L}_X$ preserves $D$; that is,
\[
[\mathcal{L}_X, D_Y] \psi = D_{[X,Y]} \psi \quad \forall \psi \in \Gamma(\mathfrak{s}) ,
\]
whence if $D\psi = 0$, also $D\mathcal{L}_X \psi = 0$. Therefore $[X, \psi] = \mathcal{L}_X \psi$ defines the desired bilinear map. Together with the Lie bracket of vector fields, under which $g_0$ becomes a Lie algebra, we have on $\mathfrak{g} = g_0 \oplus g_1$ the structure of a superalgebra.

In checking the Jacobi identity, one again sees as in the first lecture that $3/4$ of the jacobator is identically zero because of properties of the Lie derivative $\mathcal{L}_X$. The fourth component of the jacobator vanishes if and only if for all $\psi \in g_1$,
\[
[[\psi, \psi], \psi] = 0 \quad \text{or equivalently } \mathcal{L}_{[\psi, \psi]} \psi = 0 .
\]
Representation theory is not useful here, since we are interested in a general result for unspecified $g_0$ and $g_1$. An explicit calculation (made less painful with Mathematica or Maple) shows that this is indeed the case. Therefore we have a Lie superalgebra called the symmetry
The Poincaré superalgebra, whereas the Killing superalgebra is the supertranslation ideal, as explained in Section 2.1. The maximal superalgebra is obtained by taking the Poincaré superalgebra, whereas the Killing superalgebra is the supertranslation ideal, as explained in Section 2.1. The maximal superalgebra is obtained by taking $m_0 = S^2 g_1$ and declaring $m_0$ to be central. The extra elements in $m_0$ not in the Killing superalgebra can be understood in terms of brane charges, as explained, for example, in [35].

Backgrounds with $F = 0$ are said to be purely gravitational. The Killing spinors are parallel with respect to the Levi-Civita connection $\nabla$. This means that the holonomy of $\nabla$ is contained in the stabiliser of a spinor in Spin(1, 10). There are two types of spinor orbits in $S(1, 10)$ and hence two stabilisers, up to isomorphism. As shown by Bryant [36] and the author [37], the orbits are labelled by the value of a quartic polynomial $q$, whose value $q(\psi)$ at $\psi \in S(1, 10)$ is the minkowskian norm of the vector $[\psi, \psi]$, which as mentioned above is always non-negative. If $q(\psi) = 0$ we must distinguish between the $\psi = 0$ and a 25-dimensional orbit with stabiliser isomorphic to $(\text{Spin}(9) \ltimes \mathbb{R}^8) \times \mathbb{R} \subset \text{Spin}(1, 10)$, whereas if $q(\psi) < 0$, the stabiliser is isomorphic to SU(5). This dichotomy gives rise to two types of supersymmetric purely gravitational backgrounds: one generalising the M-wave [38], where $[\psi, \psi]$ is a lightlike parallel vector and thus the geometry is described by a Brinkmann metric, and another generalising the Kaluza–Klein monopole [39, 40, 41]. This latter class gives rise to reducible geometries of the form $\mathbb{R} \times N$, where $N$ is a riemannian ten-dimensional manifold with holonomy contained in SU(5). The Kaluza–Klein monopole is the case $N = \mathbb{R}^6 \ltimes K$, with $K$ a 4-dimensional hyperkähler manifold. Because $\psi$ is parallel with respect to $\nabla$, so is $[\psi, \psi]$ and the resulting Killing superalgebras are of the supertranslation type, sketchily $[Q, Q] = P_+$, where $P_+$ is the parallel null vector in the case of the waves, or else $[Q, Q] = \text{translation in the flat factor}$, for the generalised Kaluza–Klein monopoles.

For backgrounds with nonzero $F$, as in the M2-brane discussed in the second lecture, the Killing superalgebra is still of the supertranslation type, where now $[Q, Q] = \text{translations along the brane worldvolume.}$

For the maximally supersymmetric Freund–Rubin backgrounds the symmetry superalgebra is simple and isomorphic to $\mathfrak{osp}(8|4)$ in the case of $\text{AdS}_4 \times S^7$ and to $\mathfrak{osp}(6, 2|2)$ for $\text{AdS}_7 \times S^4$. Simplicity implies that the Killing superalgebra agrees with the symmetry superalgebra. In [34] we showed via an explicit geometric construction that the maximal superalgebra is isomorphic to $\mathfrak{osp}(1|32)$.

It was shown in [24] that the Penrose limit contracts the Killing superalgebra, whence for the maximally supersymmetric KG background, obtained via the Penrose limit from the above Freund–Rubin backgrounds, the Killing superalgebra is a contraction of either of the orthosymplectic superalgebras $\mathfrak{osp}(8|4)$ or $\mathfrak{osp}(6, 2|2)$. This contraction was performed explicitly in [32] obtaining the Killing superalgebra previously computed in [18].

3.3. The homogeneity conjecture. Because squaring Killing spinors one obtains Killing vectors, it stands to reason that the more supersymmetric a background is, the more isometries it has. It is therefore tempting to conjecture that there given sufficient supersymmetry — that
is, a sufficiently large value of the fraction $\nu$ — the background might be homogeneous, where we say that a supergravity background $(M, g, F)$ is homogeneous if it admits the transitive action of a Lie group via $F$-preserving isometries. So the question is whether there is some critical fraction $\nu_c$ such that if a background has a fraction $\nu > \nu_c$, then it is forced to be homogeneous. All maximally supersymmetric backgrounds discussed in the second lecture are symmetric spaces, whence in particular homogeneous. On the other hand, the M2-brane, which has $\nu = \frac{1}{2}$, has cohomogeneity one: with orbits labelled by the radial coordinate $r$ in the solution. This suggests that $\nu_c \geq \frac{1}{2}$. Furthermore, inspecting the catalogue of known solution with $\nu > \frac{1}{2}$, one sees that they are always homogeneous. This prompted Patrick Meessen to state the

**Homogeneity Conjecture.** All supergravity backgrounds with $\nu > \frac{1}{2}$ are homogeneous.

In fact, we have to be a little careful because in practice we have that $g, F$ are only locally defined in some open neighbourhood of $\mathbb{R}^{11}$, so that a more relevant notion is that of local homogeneity, which is implied by local transitivity, by which we mean that around every point there is a local frame consisting of $F$-preserving Killing vectors.

In [33] we proved something weaker and at the same stronger than the homogeneity conjecture. We proved that if a background has $\nu > \frac{3}{4}$ then it is locally homogeneous, but we proved that already $[g_1, g_1]$ acts locally transitively. In other words, local homogeneity is a direct consequence of supersymmetry.

### 3.4. Status of the conjecture.

The conjecture does not just make reference to eleven-dimensional supergravity, but in fact to any Poincaré supergravity theory. Concentrating for definiteness on the ten-dimensional supergravity theories, similar results exist for these theories as well. In [43] we showed that any background of either type IIA or IIB supergravity with $\nu > \frac{3}{4}$ is locally homogeneous, whereas any background of type I/heterotic supergravity with $\nu > \frac{1}{2}$ is locally homogeneous. This latter result benefited from the classification of parallelisable backgrounds [44], which in turn was made possible by the fact that the Killing spinors are defined by the lift to the spin bundle of a metric connection with torsion. We believe that the conjecture is true as stated, but proving this for type II and eleven-dimensional supergravities will require a better understanding of the connection $D$.

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REFERENCES

[1] J. F. Adams, *Lectures on exceptional Lie groups*. The University of Chicago Press, 1996. Edited by Zafer Mahmud and Mamoru Mimura.

[2] M. Green, J. Schwarz, and E. Witten, *Superstring Theory*. Cambridge University Press, Cambridge, UK, 1987. 2 vols.

[3] J. M. Figueroa-O'Farrill, “A geometric construction of the exceptional Lie algebras $F_4$ and $E_8$,” *Comm. Math. Phys.* **283** (2008) 663–674.

[4] F. R. Harvey, *Spinors and calibrations*. Academic Press, 1990.

[5] H. Lawson and M. Michelsohn, *Spin geometry*. Princeton University Press, 1989.

[6] M. Wang, “Parallel spinors and parallel forms,” *Ann. Global Anal. Geom.* **7** (1990), no. 1, 59–68.

[7] C. Bär, “Real Killing spinors and holonomy,” *Comm. Math. Phys.* **154** (1993) 509–521.

[8] S. Gallot, “Equations différentielles caractéristiques de la sphère,” *Ann. Sci. École Norm. Sup.* **12** (1979) 235–267.

[9] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, and B. Spence, “Branes at conical singularities and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 1249–1286.

[10] Y. Kosmann, “Dérivées de Lie des spineurs,” *Annali di Mat. Pura Appl. (IV)* **91** (1972) 317–395.

[11] J. M. Figueroa-O'Farrill, “On the supersymmetries of Anti-de Sitter vacua,” *Class. Quant. Grav.* **16** (1999) 2043–2055.

[12] M. A. van Leeuwen, “LiE, a software package for Lie group computations,” *Euromath Bull.* **1** (1994), no. 2, 83–94.

[13] A. Van Proeyen, “Structure of supergravity theories,” *hep-th/0301005*

[14] W. Nahm, “Dynamics of dimensional reduction,” *Phys. Lett.* **B97** (1980) 233–235.

[15] J. P. Gauntlett and S. Pakis, “The geometry of $D = 11$ Killing spinors,” *J. High Energy Phys.* **04** (2003) 039.

[16] M. Blau, J. M. Figueroa-O'Farrill, and G. Papadopoulos, “Penrose limits and maximal supersymmetry,” *Class. Quant. Grav.* **19** (2002) L87–L95.

[17] R. G"uven, “Plane wave limits and T-duality,” *Phys. Lett.* **B482** (2000) 255–263.

[18] R. Penrose, “Any space-time has a plane wave as a limit,” in *Differential geometry and relativity*, pp. 271–275. Reidel, Dordrecht, 1976.

[19] M. Duff and K. Stelle, “Multi-membrane solutions of $D=11$ supergravity,” *Phys. Lett.* **253B** (1991) 113–118.

[20] M. Duff, G. Gibbons, and P. K. Townsend, “Macroscopic superstrings as interpolating solitons,” *Phys. Lett.* **332B** (1994) 321–328.
[29] J. P. Gauntlett, J. Gutowski, and S. Pakis, “The geometry of $D=11$ null Killing spinors,” J. High Energy Phys. 12 (2003) 049, hep-th/0311112.
[30] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest, “$N = 31, D = 11$,” J. High Energy Phys. 02 (2007) 043, hep-th/0610331.
[31] J. M. Figueroa-O’Farrill and S. Gadhia, “M-theory preons cannot arise by quotients,” J. High Energy Phys. 06 (2007) 043, hep-th/0702055.
[32] C. M. Hull, “Holonomy and symmetry in M-theory,” hep-th/0305039.
[33] J. M. Figueroa-O’Farrill, P. Meessen, and S. Philip, “Supersymmetry and homogeneity of M-theory backgrounds,” Class. Quant. Grav. 22 (2005) 207–226, hep-th/0409170.
[34] J. M. Figueroa-O’Farrill, E. Hackett-Jones, G. Moutsopoulos, and J. Simón, “On the maximal superalgebra of supergravity backgrounds,” arXiv:0809.5034 [hep-th].
[35] P. K. Townsend, “M-theory from its superalgebra,” hep-th/9712004.
[36] R. L. Bryant, “Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor,” in Global analysis and harmonic analysis (Marseille-Luminy, 1999), J. P. Bourgignon, T. Branson, and O. Hijazi, eds., vol. 4 of Sémin. Congr., pp. 53–94. Soc. Math. France, Paris, 2000. math.DG/0004073.
[37] J. M. Figueroa-O’Farrill, “Breaking the M-waves,” Class. Quant. Grav. 17 (2000) 2925–2947, hep-th/9904124.
[38] C. M. Hull, “Exact pp-wave solutions of eleven-dimensional supergravity,” Phys. Lett. 139B (1984) 39–41.
[39] R. Sorkin, “Kaluza–Klein monopole,” Phys. Rev. Lett. 51 (1983) 87–90.
[40] D. Gross and M. Perry, “Magnetic monopoles in Kaluza–Klein theories,” Nucl. Phys. B226 (1983) 29.
[41] S. Han and I. Koh, “$N=4$ remaining supersymmetry in a Kaluza–Klein monopole background in $D=11$ supergravity theory,” Phys. Rev. D31 (1985) 2503.
[42] M. Hatsuda, K. Kamimura, and M. Sakaguchi, “Super-pp-wave algebra from super-AdS $\times S$ algebras in eleven-dimensions,” Nucl. Phys. B637 (2002) 168–176, hep-th/0204002.
[43] J. M. Figueroa-O’Farrill, E. Hackett-Jones, and G. Moutsopoulos, “The Killing superalgebra of ten-dimensional supergravity backgrounds,” Class. Quant. Grav. 24 (2007) 3291–3308, hep-th/0703192.
[44] J. M. Figueroa-O’Farrill, T. Kawano, and S. Yamaguchi, “Parallelisable heterotic backgrounds,” J. High Energy Phys. 10 (2003) 012, hep-th/0308141.

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