FACES OF ROOT POLYTOPES

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Abstract. For every directed acyclic graph $G$, we characterize the faces of the root polytope $\tilde{Q}_G = \text{conv}\{0, e_i - e_j : (i, j) \in E(G)\}$ combinatorially. Our results specialize to state of the art results in a straightforward way.

1. Introduction

Let $A_n^+ = \{e_i - e_j : 1 \leq i < j \leq n + 1\} \subset \mathbb{R}^{n+1}$ denote the positive roots of type $A_n$. Subsets of $A_n^+$ can be encoded using a directed acyclic graph $G$ on $n + 1$ vertices with edges $(i, j) \in E(G)$ oriented so that $i < j$. Given such a graph $G$, one can consider the root polytopes

$$Q_G \overset{\text{def}}{=} \text{conv}\{e_i - e_j : (i, j) \in E(G)\} \subset \mathbb{R}^{n+1}$$

and

$$\tilde{Q}_G \overset{\text{def}}{=} \text{conv}\{0, e_i - e_j : (i, j) \in E(G)\} \subset \mathbb{R}^{n+1}.$$ 

The purpose of this paper is to completely characterize the faces of the root polytope $\tilde{Q}_G$ for every $G$. This is accomplished in Theorems 3.4 and 3.18.

Root polytopes were first studied systematically in [Pos09], where it was shown that the simplices in a triangulation of a root polytope count lattice points of a generalized permutahedron. The class of root polytopes also includes products of simplices, the triangulations of which are known to have very rich combinatorics (see e.g. [HRS00, San05, GNP18]). Triangulations and subdivision algebras of root polytopes were studied in [M´es11,M´es16], and have been used to solve a variety of other combinatorial problems, e.g. in [EM16,EM18].

Much attention has been devoted to studying the face structure of the convex hull of the entire type $A_n$ root system, and more generally to that of other root systems $\Phi$. The faces of the polytope $P_{A_n} = \text{conv}\{e_i : i \in [n+1]\}$ were characterized combinatorially already in [Cho99]; computing the $f$-vector of $P_{A_n}$ is an easy corollary of the characterization. The $f$-vectors of pulling triangulations of the boundary of $P_{A_n}$ were computed in [Het09], and the $f$-vectors of unimodular triangulations of the boundary of $P_{\Phi} = \text{conv}\{v : v \in \Phi\}$, $\Phi = A_n, C_n, D_n$, were given in [ABH+11]. The orbit classes (under an action of the Weyl group) of the faces of $P_{\Phi}$ were algebraically characterized in [CM15].

In contrast, to our knowledge the faces of convex hulls of (subsets of) positive roots have been studied only for $\Phi^+ = A_n^+$. Gelfand, Graev, and Postnikov studied faces of $Q_{K_n}$ not containing the origin in [GGP97, Prop. 8.1], but their result contains a mistake. Cho salvaged this result for facets of $Q_{K_n}$ in [Cho99, Prop. 13]. Postnikov generalized Cho’s result to facets of $\tilde{Q}_G$ for transitively closed graphs $G$ (Definition 5.4) in [Pos09, Prop. 13.3]. To our knowledge, Postnikov’s characterization [Pos09, Prop. 13.3] has been the state of the art in this direction. Our results specialize to those of Postnikov straightforwardly (spelled out in Corollary 5.8), and correct the mistake in [GGP97, Prop. 8.1] in full generality (Corollary 5.10; see also Remark 5.11).

When $G$ is an alternating graph (Definition 2.2), the faces of the affine cone generated by $\{e_i - e_j : (i, j) \in E(G)\}$ has algebrogeometric significance; it is related to the deformation theory of a certain toric variety associated to $G$. The faces of this cone, i.e. the faces of $\tilde{Q}_G$ containing the origin, were combinatorially characterized in the recent paper [Por19, Thm. 3.17], building on the work in [VV06]. We highlight and reprove their characterization in Corollaries 5.2 and 5.3.

The faces of $\tilde{Q}_G$ are again root polytopes, i.e. equal to $\tilde{Q}_H \subseteq \tilde{Q}_G$ or $Q_H \subseteq \tilde{Q}_G$ for certain subgraphs $H \subseteq G$ (Proposition 2.1). We characterize the subgraphs $H$ for which $\tilde{Q}_H \subseteq \tilde{Q}_G$ is a face in Theorem 3.4, and separately characterize the subgraphs $H$ for which $Q_H \subseteq \tilde{Q}_G$ is a face in Theorem 3.18. For $G = K_n$, the characterizations of Theorem 3.4 and 3.18 are particularly nice, and are highlighted in Corollary 5.9 and Corollary 5.10 respectively.
2. Background

**Conventions.** Unless stated otherwise, $G$ will denote a directed acyclic graph with $V(G) = [n]$. Without loss of generality, we may assume its edges $e = (i, j) \in E(G)$ are directed so that $i < j$. (The adjective acyclic will only describe directed graphs, and means that there is no directed cycle.) We use the notation $H \subseteq G$ to denote a subgraph $H$ of $G$ with $V(H) = V(G)$ and $E(H) \subseteq E(G)$. We also use the notation $G^\text{un}$ to denote the underlying undirected graph of $G$. We reserve boldface mathematical notation to denote vectors; in particular, $e_i$ is the $i$-th basis vector of $\mathbb{R}^n$.

**Root polytopes.** In [Pos09, Sec. 12], Postnikov defined the root polytopes

$$Q_G \overset{\text{def}}{=} \text{conv}\{e_i - e_j : (i, j) \in E(G)\} \subset \mathbb{R}^n$$

and

$$\tilde{Q}_G \overset{\text{def}}{=} \text{conv}\{0, e_i - e_j : (i, j) \in E(G)\} \subset \mathbb{R}^n.$$ 

It is well known that faces of root polytopes are again root polytopes:

**Proposition 2.1.** For every subgraph $H \subseteq G$, the root polytope $Q_H$ is a subpolytope of $\tilde{Q}_H$, which in turn is a subpolytope of $\tilde{Q}_G$. Every subpolytope (in particular, every face) of $\tilde{Q}_G$ is the root polytope $Q_H$ or the root polytope $\tilde{Q}_H$ for some $H \subseteq G$.

**Proof.** The inclusion of edge sets $E(H) \subseteq E(G)$ implies the inclusion of polytopes $Q_H \subseteq \tilde{Q}_H \subseteq \tilde{Q}_G$.

Conversely, every subpolytope $P$ of $\tilde{Q}_G$ is the convex hull of the vertices of $\tilde{Q}_G$ which live in $P$ (see e.g. [Zie07, Prop. 2.3]). The non-origin vertices correspond to edges of $G$, so the collection of such vertices forms a subgraph $H$ of $G$. If $P$ contains (resp. doesn’t contain) the origin, then $P = Q_H$ (resp. $P = \tilde{Q}_H$). □

**Definition 2.2.** A graph $G$ is alternating if there is no vertex $j \in [n] = V(G)$ so that $(i, j), (j, k) \in E(G)$. △

We remark that alternating graphs are nothing more than (appropriately oriented) bipartite graphs:

**Lemma 2.3.** Let $G$ be an alternating graph and suppose $G^\text{un}$ is connected. Then there is a partition of $V(G) = L \sqcup R$ into two parts so that every edge $(i, j) \in E(G)$ connects a vertex $i \in L$ to a vertex $j \in R$.

**Proof.** If $G$ has no edges, the lemma is vacuous. Otherwise, we may set

$$L \overset{\text{def}}{=} \{v \in V(G) : \text{every edge of } G \text{ incident to } v \text{ has } v \text{ as its source}\},$$

$$R \overset{\text{def}}{=} \{v \in V(G) : \text{every edge of } G \text{ incident to } v \text{ has } v \text{ as its sink}\}.$$

Every vertex of the alternating graph $G$ has an edge incident to it, so $L$ and $R$ are disjoint. If a vertex $j \in [n]$ is not in $L$, then there is an edge $(i, j) \in E(G)$ with $j$ as its sink; similarly if $j$ is not in $R$, then there is an edge $(j, k) \in E(G)$ with $j$ as its source. Since $G$ is alternating, these cannot simultaneously happen, so $j \in L \sqcup R$. We conclude $L \sqcup R = [n]$.

From the definitions of $L$ and $R$, we see that every edge of $G$ connects a vertex in $L$ to a vertex in $R$. □

The following result can be derived from [Pos09]. Here we include a full proof for completeness.

**Proposition 2.4** (cf. [Pos09, Lem. 13.2, Lem. 12.5]). Suppose $G^\text{un}$ has $r$ connected components. Then $\tilde{Q}_G$ is $(n - r)$-dimensional. If $G^\text{un}$ has $r$ connected components and $G$ is alternating, then $Q_G$ is $(n - r - 1)$-dimensional.

**Proof.** Take a spanning forest $T^\text{un} \subseteq G^\text{un}$ and let $T \subseteq G$ be its overlying directed graph. The $n - r + 1$ vertices of $\tilde{Q}_T \subseteq \tilde{Q}_G$ are affinely independent and hence form an $(n - r)$-dimensional simplex. On the other hand, $\tilde{Q}_G$ is contained in the $(n - r)$-dimensional subspace

$$W = \left\{ x \in \mathbb{R}^n : \sum_{i \in G^\text{un}} x_i = 0 \text{ for all connected components } G_j^\text{un} \text{ of } G^\text{un} \right\} \subset \mathbb{R}^n.$$ 

It follows that $\tilde{Q}_G$ is $(n - r)$-dimensional.

Suppose now that $G^\text{un}$ has $r$ connected components and $G$ is alternating. In this case, there is a subset $L \subseteq [n] = V(G)$ so that every edge $e \in E(G)$ has source in $L$ and target not in $L$ (the set $L$ can be thought of as “source vertices” of the graph $G$).
As before, take a spanning forest $T^{\text{un}} \subseteq G^{\text{un}}$ and let $T \subseteq G$ be its overlying directed graph. The $n - r$ vertices of $Q_T \subseteq Q_G$ are affinely independent and hence form an $(n - r - 1)$-dimensional simplex. On the other hand, $Q_G$ is contained in the $(n - r)$-dimensional subspace $W$ and also in the subspace

$$\{ x \in \mathbb{R}^n : \sum_{i \in L} x_i = 1 \} \subset \mathbb{R}^n$$

intersecting $W$ transversely. Thus $Q_G$ is contained in a $(n - r - 1)$-dimensional subspace of $\mathbb{R}^n$, and $Q_G$ is $(n - r - 1)$-dimensional. \( \square \)

**Polytopes.** We refer to [Zie07] for background on polytopes in general. In what follows, let

$$\ell : (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n c_i x_i$$

denote a linear form. Recall that a face $F$ of a polytope $P \subseteq \mathbb{R}^n$ is a subset of the form

$$F = P \cap \{ x : \ell(x) = c \}$$

for some $c \in \mathbb{R}$ such that (affine) hyperplane $\{ \ell(x) = c \}$ is a supporting hyperplane (for $F$), i.e. such that

$$P \subseteq \{ x : \ell(x) \geq c \}$$

holds. A facet of a polytope is a face of codimension 1.

We will later use the following lemma.

**Lemma 2.5.** Let $F$ be a face of a polytope $P$ of codimension $d$. Then $F$ is the intersection of some $d$ facets of $P$.

**Proof.** First recall that every face $F$ of a polytope is the intersection of the facets containing it (see [Grü03, Thm 3.1.7] or [Zie07, Thm 2.7]).

Let $G$ be a face of $P$ of codimension $d - 1$ with $G \supseteq F$. By induction, we may find $d - 1$ facets $G_1, \ldots, G_{d-1}$ whose intersection is $G$. It suffices to find a facet $G_* \supseteq F$ not containing $G$, as $F = G \cap G_*$ for any such facet $G_*$. Such a facet $G_*$ must exist; otherwise, the intersection of all facets containing $F$ would contain $G$. \( \square \)

### 3. Faces of $\tilde{Q}_G$

This section contains the main results of the paper: Theorem 3.4 characterizes faces $\tilde{Q}_H \subseteq \tilde{Q}_G$, while Theorem 3.18 characterizes faces $Q_H \subseteq Q_G$. The latter theorem requires significantly more work than the former, but technicalities are summarized by Lemma 3.19. Both Theorems 3.4 and 3.18 are proven by analyzing supporting hyperplanes of the relevant subpolytopes (see Lemmas 3.1 and 3.20), then finding necessary and sufficient combinatorial conditions on $H \subseteq G$ for which a supporting hyperplane exists.

We begin with the following useful lemma:

**Lemma 3.1.** Let $H \subseteq G$ be a subgraph, so $\tilde{Q}_H \subseteq \tilde{Q}_G$. The hyperplane

$$S = \left\{ x : \sum_{i=1}^n c_i x_i = c \right\}$$

is a supporting hyperplane for $\tilde{Q}_H$ if and only if:

(a) $c = 0$
(b) $c_i \geq c_j$ for all $(i, j) \in E(G)$
(c) If $(i, j) \in E(G)$, then $c_i = c_j$ if and only if $(i, j) \in E(H)$.

**Proof.** Suppose $S$ is a supporting hyperplane for $\tilde{Q}_H$, and set

$$S_\geq \overset{\text{def}}{=} \left\{ x : \sum_{i=1}^n c_i x_i \geq c \right\}$$

Since $0 \in \tilde{Q}_H$ must be in $S$, condition (a) follows. Conditions (b) and (c) respectively follow from the conditions

(1) $\tilde{Q}_G \subseteq S_\geq$ and $\tilde{Q}_H = \tilde{Q}_G \cap S$
applied to vertices of $\tilde{Q}_G$. Conversely, if all three conditions (a), (b), and (c) hold, then
\[
\{0, e_i - e_j : (i, j) \in E(G)\} \subseteq S \quad \text{and} \quad \{0, e_i - e_j : (i, j) \in E(H)\} = \{0, e_i - e_j : (i, j) \in E(G)\} \cap S.
\]
Taking convex hulls, we deduce that (1) holds. Thus, $S$ is a supporting hyperplane for $Q_H$. □

**Definition 3.2.** Let $H \subseteq G$ be a subgraph, and let $H^{\text{un}}_1, \ldots, H^{\text{un}}_m$ be the connected components of the underlying undirected graph $H^{\text{un}}$ of $H$. The directed multigraph $H^{\text{comp}}$ is the graph with vertex set
\[
V(H^{\text{comp}}) = \{H^{\text{un}}_i : i \in [m]\}
\]
and edge multiset
\[
E(H^{\text{comp}}) = \{\{(H^{\text{un}}_i, H^{\text{un}}_j) : \text{for each edge } (v_i, v_j) \in E(G) \setminus E(H) \text{ where } v_i \in V(H^{\text{un}}_i), v_j \in V(H^{\text{un}}_j)\}\}. \quad \triangle
\]

**Example 3.3.** The multigraph $H^{\text{comp}}$ may have multiple edges, self-loops, or directed cycles. For example, let $H \subseteq G$ be as in Figure 1 below.

![Figure 1. The graphs H and G in Example 3.3.](image1)

The graph $H^{\text{un}}$ has two connected components (with vertex sets $V(H^{\text{un}}_1) = \{1, 3, 4, 5\}$ and $V(H^{\text{un}}_2) = \{2\}$), and $H^{\text{comp}}$ is as in Figure 3.3.

![Figure 2. The graph Hcomp for H and G in Example 3.3. The edges E(Hcomp) are labelled by their corresponding edges in G.](image2)

**Theorem 3.4.** Let $H \subseteq G$ be a subgraph. The subpolytope $\tilde{Q}_H \subseteq \tilde{Q}_G$ is a face of $\tilde{Q}_G$ if and only if $H^{\text{comp}}$ is loopless and acyclic.

*Proof.* Suppose $\tilde{Q}_H$ is a face of $\tilde{Q}_G$, and take a supporting hyperplane $S = \{\ell(x) = c\}$ for $\tilde{Q}_H$. By condition (c) of Lemma 3.1, the numbers $\{c_i\}_{i \in [n]}$ are constant on connected components of $H$. In particular, if $i$ and $j$ are in the same connected component of $H$, and $(i, j) \in E(G)$, then $(i, j) \in E(H)$; in other words, $H^{\text{comp}}$ is loopless. By condition (b) and (c) of Lemma 3.1, if $(H^{\text{un}}_i, H^{\text{un}}_j) \in E(H^{\text{comp}})$, then $c_{v_i} > c_{v_j}$, where $v_i \in V(H^{\text{un}}_i)$ and $v_j \in V(H^{\text{un}}_j)$. It follows that $H^{\text{comp}}$ is acyclic.

Suppose now that $H^{\text{comp}}$ is loopless and acyclic. We will define numbers $\{c_i\}_{i \in [n]}$ satisfying conditions (b) and (c) of Lemma 3.1, so that
\[
S = \left\{x : \sum_{i=1}^n c_ix_i = 0\right\}
\]
is a supporting hyperplane for $\tilde{Q}_H \subseteq \tilde{Q}_G$. Since $H_{\text{comp}}$ is loopless and acyclic, we may take a linear extension, i.e. a function

$$f : V(H_{\text{comp}}) \to \{1, \ldots, |V(H_{\text{comp}})|\}$$

so that if $(H^m_i, H^m_j) \in E(H_{\text{comp}})$, then $f(H^m_i) > f(H^m_j)$. Each vertex $v_i \in [n]$ is in some connected component $H^m$, the assignment

$$c_{v_i} = f(H^m_i)$$

works.

We pause to highlight an alternative condition equivalent to looplessness of $H_{\text{comp}}$.

**Proposition 3.5.** Let $H \subseteq G$ be a subgraph. Then $H_{\text{comp}}$ is loopless if and only if $H$ is the disjoint union of induced subgraphs $\{G|_{P_i} \}_{P_i \in \mathcal{P}}$, where $\mathcal{P} = \{P_i\}$ is a partition of $[n]$.

**Proof.** If $H_{\text{comp}}$ is loopless, the partition $\mathcal{P} = \{V(H^m)\}$ works: every edge of $H$ must be contained in some $G|_{V(H^m)}$, so

$$(2) \quad H \subseteq \bigsqcup_i G|_{V(H^m)};$$

on the other hand, an edge of $G|_{V(H^m)}$ that is not in $H$ becomes a loop in $H_{\text{comp}}$, so equality holds in (2). Conversely, suppose $H$ is the disjoint union of induced subgraphs $\{G|_{P_i} \}_{P_i \in \mathcal{P}}$: if an edge $(i, j) \in E(G)$ connects two vertices $i, j$ in the same connected component of $H^m$, then $i$ and $j$ are in the same part $P_i \in \mathcal{P}$, hence must be in $E(H)$. In other words, $H_{\text{comp}}$ is loopless.

It remains to characterize faces $Q_H \subset \tilde{Q}_G$ (Theorem 3.18). To illustrate the difference between faces $\tilde{Q}_H \subseteq \tilde{Q}_G$ and faces $Q_H \subset \tilde{Q}_G$, consider the following example:

**Example 3.6.** When $H = G = K_3$, the polytope

$$Q_{K_3} = \text{conv}\{e_1 - e_2, e_1 - e_3, e_2 - e_3\}$$

is not a face of

$$\tilde{Q}_{K_3} = \text{conv}\{0, e_1 - e_2, e_1 - e_3, e_2 - e_3\}.$$  

(It turns out that $Q_{K_3}$ is a triangle and $\tilde{Q}_{K_3}$ is a rhombus, as Figure 3 below shows.) One explanation for this, which turns out to generalize, goes as follows: Suppose that a supporting hyperplane $\{\ell(x) = c\}$ for $Q_{K_3}$ exists. Since $0 \notin Q_{K_3}$, we must have $0 = \ell(0) > c$; up to scaling, we may assume $c = -1$. On one hand, $\ell(e_1 - e_2) = -1$ and $\ell(e_2 - e_3) = -1$. On the other hand, $\ell(e_1 - e_3) = -1$. This is a contradiction.  

**Figure 3.** The root polytopes $Q_{K_3}$ and $\tilde{Q}_{K_3}$. (The hyperplane $\{x_1 + x_2 + x_3 = 0\} \subset \mathbb{R}^3$ is identified with $\mathbb{R}^2$ via the projection $(x_1, x_2, x_3) \mapsto (x_1 - x_2, x_1 - x_3)$; coordinate directions in $\mathbb{R}^2$ are shown in red.)

**Definition 3.7.** A directed acyclic graph $H$ on vertex set $V(H) = [n]$ is **path consistent** if, for any pair $i, j \in [n]$ and any two undirected paths $p_{ij}^m$ and $q_{ij}^m$ in $H^m$ connecting $i$ to $j$, we have

$$(3) \quad \# \{(a, b) \in p_{ij} : a < b\} - \# \{(a, b) \in p_{ij} : a > b\} = \# \{(a, b) \in q_{ij} : a < b\} - \# \{(a, b) \in q_{ij} : a > b\}.$$  

(Here, $p_{ij}$ and $q_{ij}$ are the subsets of $E(H)$ whose underlying undirected graph are the paths $p_{ij}^m$ and $q_{ij}^m$. The sets $p_{ij}^m$ and $q_{ij}^m$ are not necessarily directed paths.) In other words, the difference between the number
of “correctly” oriented edges and the number of “incorrectly” oriented edges in any path depends only on $i$ and $j$. △

Example 3.8. The complete graph $K_3$ is not path consistent, since the paths $((1, 3))$ and $((1, 2), (2, 3))$ connecting vertices 1 and 3 have one and two correctly oriented edges respectively (cf. Example 3.6). △

Example 3.9. Any alternating graph $G$ is path consistent. Explicitly, we may apply Lemma 2.3 to each connected component of $G$ and obtain a partition $V(G) = \{n\}$ into two parts $\{n\} = L \sqcup R$ such that every vertex $i \in L$ is the source of every edge incident to it, and every vertex $j \in R$ is the sink of every edge incident to it. Then, if $p_{ij}$ is a path connecting $i$ to $j$ in $G$, we have

$$\#\{(a, b) \in p_{ij}: a < b\} - \#\{(a, b) \in p_{ij}: a > b\} = \begin{cases} 1 & \text{if } i \in L, j \in R \\ 0 & \text{if } i, j \in L \\ 0 & \text{if } i, j \in R \\ -1 & \text{if } i \in R, j \in L \end{cases}$$

so Equation (3) is satisfied. △

While path consistency turns out to be a necessary condition, it is not sufficient, as the next example shows. (The necessity will be the easier half of Theorem 3.18.)

Example 3.10. Let $H \subseteq G$ be as in Figure 4.

![Figure 4](image)

The root polytope $Q_G$ is a square with affine hull

$$\{(x_1, x_2, x_3, x_4): x_1 + x_2 = 1, x_3 + x_4 = -1\} \subseteq \mathbb{R}^4,$$

so $\tilde{Q}_G$ is a square pyramid with apex 0 (see Figure 5). The subpolytope $Q_H = \text{conv}\{e_1 - e_3, e_2 - e_4\}$ is a diagonal of the square face $Q_G$ of $\tilde{Q}_G$; hence $Q_H$ is not a face of $\tilde{Q}_G$.

![Figure 5](image)
Let us explain why \( Q_H \) is not a face of \( \tilde{Q}_G \) in a way that will generalize. Suppose that a supporting hyperplane \( \{ \ell(x) = c \} \) for \( Q_H \) exists. Since \( 0 \notin Q_H \), we must have \( 0 = \ell(0) > c \); up to scaling, we may assume \( c = -1 \). Writing

\[
\ell(x) = \sum_{i=1}^{n} c_i x_i,
\]

we have the four conditions

\[
\begin{align*}
(1, 3) \in E(H) & \implies c_1 = c_3 - 1, \\
(2, 3) \in E(G) \setminus E(H) & \implies c_2 > c_3 - 1, \\
(2, 4) \in E(H) & \implies c_2 = c_4 - 1, \\
(1, 4) \in E(G) \setminus E(H) & \implies c_1 > c_4 - 1
\end{align*}
\]

on the \( c_i \): the first two say \( c_2 > c_1 \), whereas the last two say \( c_1 > c_2 \).

\triangleq

We want to introduce a key notion used to generalize Example 3.10. We begin with:

**Definition 3.11.** Let \( H \) be a path consistent graph and assume \( H^{un} \) is path connected. For any two vertices \( u, v \in V(H) \), pick any undirected path \( p^{un} \) connecting \( u \) to \( v \) and set

\[
\ell_{uv}^{\text{def}} = \#\{(a, b) \in p: a < b\} - \#\{(a, b) \in p: a > b\}.
\]

(This quantity is well-defined because \( H \) is path consistent.) We call \( u_s \in V(H) \) a weight source if there is a vertex \( v_s \in V(H) \) so that

\[
\ell_{u_s,v_s} = \max_{u,v} \ell_{uv}.
\]

Note that a weight source always exists, but is not necessarily unique.

\triangleq

Although Definition 3.12 requires a choice of a weight source \( u_s \), we will show in Proposition 3.13 that this choice does not matter.

**Definition 3.12.** Let \( H \) be a path consistent graph and assume that \( H^{un} \) is path connected. Let \( u_s \) be a weight source. The weight function (with respect to \( u_s \)) of \( H \) is the function \( w_{u_s} : V(H) \to \mathbb{Z} \) given by

\[
w_{u_s}(i) \overset{\text{def}}{=} \ell_{u_s,i}.
\]

\triangleq

**Proposition 3.13.** Let \( H \) be a path consistent graph so that \( H^{un} \) is path connected. Let \( w_{u_s} \) denote the weight function of \( H^{un} \) with respect to \( u_s \). Then:

1. \( w_{u_s}(i) + 1 = w_{u_s}(j) \) for every edge \((i, j) \in E(G)\).
2. \( w_{u_s}(i) \geq 0 \) for all \( i \in V(H) \), and equality holds if and only if \( i \) is a weight source.
3. If \( u_s' \) is another weight source, then \( w_{u_s} = w_{u_s'} \). Thus the weight function of \( H \) is well-defined, independent of weight source.

**Definition 3.14.** Let \( H \) be a path consistent graph (with \( H^{un} \) possibly disconnected). The weight function of \( H \) is the function \( w : V(H) \to \mathbb{Z} \) obtained by gluing together weight functions \( w_j : V(H_j^{un}) \to \mathbb{Z} \).

\triangleq

**Proof of Proposition 3.13.** Item (1) is a consequence of the fact that concatenating \((i, j) \in E(G)\) to any path connecting \( u_s \) to \( i \) gives a path connecting \( u_s \) to \( j \).

More generally, concatenation of paths gives the equality

\[
\ell_{uv} + \ell_{vw} = \ell_{uw}.
\]

Let \( u_s \) be a weight source, and let \( v_s \in V(H) \) satisfy \( \ell_{u_s,v_s} = \max_{u,v} \ell_{uv} \). The equality

\[
\ell_{u_s,i} = \ell_{u_s,v_s} - \ell_{v_s,i}
\]

and the maximality of \( \ell_{u_s,v_s} \) guarantee that \( \ell_{u_s,i} \geq 0 \). Furthermore equality holds if and only if \( \ell_{iv_s} = \ell_{u_s,v_s} \), which in turn holds if and only if \( i \) is a weight source. This proves item (2).

Finally, let \( u_s' \) be another weight source. By part (2), \( w_{u_s}(u_s') = 0 \) and hence

\[
\ell_{u_s,i} = \ell_{u_s,u_s'} + \ell_{u_s',i} = 0,
\]

which proves item (3).
so that \( w_{u_1} = w_{u_1'} \). This proves item (3). \( \square \)

**Definition 3.15.** Let \( H \subseteq G \) be a subgraph, and assume \( H \) is path consistent. Let \( w \) be the weight function of \( H \). Each edge \( e = (H_i^u, H_j^u) \in E(H_{\text{comp}}) \) of the multigraph \( H_{\text{comp}} \) corresponds to a unique edge \((v_i, v_j) \in E(G) \setminus E(H)\). We define the weight decrease of \( e \) to be the quantity

\[
\text{wd}(e) \overset{\text{def}}{=} w(v_i) - w(v_j).
\]

While path consistency will be analogous to looplessness of \( H_{\text{comp}} \), the following condition will be analogous to acyclicity of \( H_{\text{comp}} \).

**Definition 3.16.** A subgraph \( H \subseteq G \) is admissible (with respect to \( G \)) if, for every directed cycle \( C \) in \( H_{\text{comp}} \), the condition

\[
\sum_{e \in C} \text{wd}(e) > -|C|
\]

holds, where the sum in (5) runs over the edges \( e \) forming the directed cycle \( C \). \( \triangle \)

**Example 3.17** (cf. Example 3.10). Returning to Example 3.10, we let \( H \subseteq G \) be as in Figure 4. The graph \( H \) is path consistent; the graph \( H^u \) has two connected components \( H_1^u \) and \( H_2^u \) consisting of vertices \( \{1, 3\} \) and \( \{2, 4\} \) respectively.

The weight function \( w : V(H) \to \mathbb{N} \) sends \( w(1) = w(2) = 1 \) and \( w(3) = w(4) = 2 \). The graph \( H_{\text{comp}} \) consists of a single cycle of length 2: there is an edge \( e = (H_1^u, H_2^u) \in E(H_{\text{comp}}) \) corresponding to the edge \((1, 4) \in E(G) \setminus E(H)\) and its weight decrease is \( \text{wd}(e) = -1 \); there is also an edge \( e' = (H_2^u, H_1^u) \in E(H_{\text{comp}}) \) corresponding to the edge \((2, 3) \in E(G) \setminus E(H)\) and its weight decrease is \( \text{wd}(e') = -1 \).

The graph \( H_{\text{comp}} \) has a single directed cycle \( C = \{e, e'\} \), and the condition (5)

\[
\text{wd}(e) + \text{wd}(e') > -2
\]

fails to hold. Thus \( H \subseteq G \) is not admissible. \( \triangle \)

We now have enough language to state our characterization of subgraphs \( H \subseteq G \) for which \( Q_H \subset Q_G \) is a face:

**Theorem 3.18.** Let \( H \subseteq G \) be a subgraph of \( G \). The subpolytope \( Q_H \subset Q_G \) is a face of \( Q_G \) if and only if \( H \) is path consistent and admissible.

To prove Theorem 3.18, we will use the following technical lemma; Section 4 is dedicated to its proof, which we feel is unenlightening in the context of this paper.

**Lemma 3.19.** Let \( H \subseteq G \) be an admissible subgraph of \( G \). There is a vector \( d = (d_e)_{e \in V(H_{\text{comp}}) \subset \mathbb{R}^{V(H_{\text{comp}})} \text{ so that}}

\[
\text{wd}(e) + d_{s(e)} - d_{t(e)} > -1
\]

for every edge \( e \in E(H_{\text{comp}}) \).

(Here, \( s(e) \) denotes the source of the edge \( e \), and \( t(e) \) denotes the target of the edge \( e \).)

We now prove the following analogue of Lemma 3.1 for faces \( Q_H \subset Q_G \):

**Lemma 3.20.** Let \( H \subseteq G \) be a subgraph, so \( Q_H \subset Q_G \). The hyperplane

\[
S = \left\{ x : \sum_{i=1}^{n} c_i x_i = c \right\}
\]

is a supporting hyperplane for \( Q_H \) if and only if:

(a) \( c < 0 \)

(b) \( c_i \geq c_j + c \) for all \( (i, j) \in E(G) \)

(c) \( \text{If } (i, j) \in E(G) \), then \( c_i = c_j + c \) if and only if \( (i, j) \in E(H) \).

**Proof.** Suppose \( S \) is a supporting hyperplane for \( Q_H \), and set

\[
S_{\geq} \overset{\text{def}}{=} \left\{ x : \sum_{i=1}^{n} c_i x_i \geq c \right\}
\]

for all \( c < 0 \). This proves item (3). \( \square \)
Since $0 \not\in Q_H$ must be in $S \geq S$, condition (a) follows. Conditions (b) and (c) respectively follow from the conditions

\[ \hat{Q}_G \subset S \geq \quad \text{and} \quad Q_H = \hat{Q}_G \cap S \]

applied to vertices of $\hat{Q}_G$. Conversely, if all three conditions (a), (b), and (c) hold, then

\[ \{0, e_i - e_j : (i, j) \in E(G)\} \subset S \geq \quad \text{and} \quad \{e_i - e_j : (i, j) \in E(H)\} = \{0, e_i - e_j : (i, j) \in E(G)\} \cap S. \]

Taking convex hulls, we deduce that (6) holds. Thus, $S$ is a supporting hyperplane for $Q_H$. \hfill \Box

**Proof of Theorem 3.18.** Let $Q_H \subset \hat{Q}_G$ be a face of $\hat{Q}_G$, and take a supporting hyperplane

\[ S = \{x : \sum_{i=1}^{n} c_i x_i = c\} \]

of $Q_H$.

Applying condition (a) of Lemma 3.20, we may assume up to scaling $c = -1$. Then, if $p_{ij}^{\text{un}}$ is an undirected path in $H^{\text{un}}$ connecting $i$ to $j$, and $p_{ij}$ is the overlying directed subgraph of $H$, we have

\[ \# \{(a, b) \in p_{ij} : a < b\} - \# \{(a, b) \in p_{ij} : a > b\} = c_j - c_i \]

by repeatedly applying condition (c) of Lemma 3.20 to the edges $e \in p_{ij} \subseteq E(H)$. This holds for any such path of $H^{\text{un}}$, so Equation (3) is satisfied, and $H$ is path consistent. Importantly, we emphasize that when $i, j \in [n]$ are in the same connected component of $H^{\text{un}}$, then

\[ c_j - c_i = w(j) - w(i), \]

where $w$ is the weight function of $H$.

Furthermore, let $C$ be a directed cycle of $H^{\text{comp}}$, consisting of edges $\{e_1^{\text{comp}}, \ldots, e_{|C|}^{\text{comp}}\}$ corresponding to edges $\{e_1, \ldots, e_{|C|}\} \subseteq E(G) \setminus E(H)$. Denote by $s_i, t_i \in [n] = V(G)$ the source and target of the edge $e_i$ respectively. Since $e_i, e_i \not\in Q_H$, condition (c) of 3.20 says

\[ c_{s_i} - c_{t_i} > -1 \]

so

\[ \sum_{i=1}^{\lfloor |C| / 2 \rfloor} w(e_{s_i}^{\text{comp}}) = |C| \sum_{i=1}^{\lfloor |C| / 2 \rfloor} (w(s_i) - w(t_i)) = |C| \sum_{i=1}^{\lfloor |C| / 2 \rfloor} (w(s_{i+1}) - w(t_i)), \]

with $s_{|C|+1} \overset{\text{def}}{=} s_1$. Since $C$ forms a cycle in $H^{\text{comp}}$, the target of $e_{s_i}^{\text{comp}} \in E(H^{\text{comp}})$ is equal to the source of $e_{s_{i+1}}^{\text{comp}} \in E(H^{\text{comp}})$. Thus, the vertices $t_i, s_{i+1} \in V(H)$ are in the same connected component of $H^{\text{un}}$. Then Equations (7) and (8) say

\[ \sum_{i=1}^{\lfloor |C| / 2 \rfloor} (w(s_{i+1}) - w(t_i)) = \sum_{i=1}^{\lfloor |C| / 2 \rfloor} (c_{s_{i+1}} - c_{t_i}) = \sum_{i=1}^{\lfloor |C| / 2 \rfloor} (c_{s_i} - c_{t_i}) > -|C|. \]

Thus we have verified Equation (5) holds for every cycle $C$, and $H$ is admissible.

Suppose now that $H$ is path consistent and admissible. It suffices to provide numbers $c_i, i \in [n] = V(H)$, so that conditions (b) and (c) of 3.20 hold for $c = -1$, i.e.

\[ c_i - c_j > -1 \text{ for } (i, j) \in E(G) \setminus E(H) \quad \text{and} \quad c_i - c_j = -1 \text{ for } (i, j) \in E(H). \]

By Lemma 3.19, there exist numbers $d_i, i \in V(H^{\text{comp}})$, so that

\[ \text{wd}(e) + d_{s(e)} - d_{t(e)} > -1. \]

Now let $v \in [n] = V(H)$ be a vertex of $H$ and suppose $v \in V(H_v^{\text{comp}})$ is in the $(i^{\text{comp}})$-th connected component of $H^{\text{un}}$. Then

\[ c_v \overset{\text{def}}{=} w(v) + d_{i^{\text{comp}}}, \]

where $w$ is the weight function of $H$, satisfies Equation (9): if $e = (i, j) \in E(G) \setminus E(H)$ corresponds to $e^{\text{comp}} = (i^{\text{comp}}, j^{\text{comp}}) \in E(H^{\text{comp}})$ then

\[ c_i - c_j = \text{wd}(e) + d_{i^{\text{comp}}} - d_{j^{\text{comp}}} > -1. \]
while if \((i, j) \in E(H)\) then (as in Equation (7))
\[ c_i - c_j = w(i) - w(j) = -1. \]

\[ \square \]

4. Proof of Lemma 3.19

This section contains a proof of Lemma 3.19. We feel that Lemma 4.6 might be of independent interest, although it is largely irrelevant in the context of this paper. **In this section only,** we temporarily allow \(G\) to be a directed multigraph.

In what follows, we will treat signed multisets \(S\) of edges of \(G\) as formal sums
\[ S = \sum_{e \in E(G)} m_e(S) \cdot e \]
of edges, where \(m_e(S)\) is the signed multiplicity of \(e\) in \(S\). We identify the set of formal \(\mathbb{Z}\)-linear combinations of edges of \(G\) with \(\mathbb{Z}^{E(G)}\).

We treat simple directed cycles \(C\) and directed paths \(p\) as sums
\[ C = \sum_{e \in C} e \quad \text{and} \quad p = \sum_{e \in p} e. \]

**Definition 4.1.** We let \(\mathcal{M}_G\) denote the abelian group of \(\mathbb{Z}\)-linear combinations of simple directed cycles. The elements of \(\mathcal{M}_G\) are called **formal cycles**, and \(\mathcal{M}_G\) is a \(\mathbb{Z}\)-submodule of \(\mathbb{Z}^{E(G)}\). \(\triangle\)

**Example 4.2.** Simple directed cycles may satisfy relations in \(\mathcal{M}_G\). For example, consider \(G\) as in Figure 6 below.

![Figure 6. The graph \(G\) in Example 4.2.](image)

Let \(C_1 = (1, 2) + (2, 4) + (4, 1)\) and \(C_2 = (1, 4) + (4, 3) + (3, 1)\). Also let \(C_3 = (1, 2) + (2, 4) + (4, 3) + (3, 1)\) and \(C_4 = (1, 4) + (4, 1)\). These are all simple directed cycles, and in \(\mathcal{M}_G\) the relation \(C_1 + C_2 = C_3 + C_4\) holds. \(\triangle\)

**Definition 4.3.** For a formal sum of edges \(S \in \mathbb{Z}^{E(G)}\) and an edge \(e \in E(G)\), we let \(m_e(S)\) denote the coefficient of \(e\) in \(S\). The **support** of \(S\) is the set \(\{e \in E(G) : m_e(S) \neq 0\}\) and is denoted \(\text{supp}(S)\). We also set
\[ |S| \overset{\text{def}}{=} \sum_{e \in E(G)} |m_e(S)|. \]

**Lemma 4.4.** For any directed multigraph \(G\), the abelian group \(\mathcal{M}_G\) is equal to the set of formal sums \(S \in \mathbb{Z}^{E(G)}\) satisfying
\[ \sum_{e \in E(G)} m_e(S) \cdot e = \sum_{e \in E(G)} m_e(S) \cdot e \quad \text{for all } v \in V(G) \]
and
\[ \text{supp}(S) \subseteq \bigcup_{C} \text{supp}(C), \]
where the union runs over all simple directed cycles \(C\) of \(G\).

(As in Section 3, the notation \(s(e)\) and \(t(e)\) stands for the source and target of the edge \(e\) respectively.)
**Remark 4.5.** We will use Lemma 4.4 in the following way: If $H \subseteq G$ is a subgraph which is obtained as a union of directed cycles, and the formal cycle $C \in \mathcal{M}_G$ has support in $H$, then $C \in \mathcal{M}_H$. (That is to say, although $C$ comes as a $\mathbb{Z}$-linear combination of directed cycles of $G$, it may be replaced by a $\mathbb{Z}$-linear combination of directed cycles of $H$.)

For example, with notation as in Example 4.2, the formal cycle $C_4 \overset{\text{def}}{=} C_1 + C_2 - C_3 \in \mathcal{M}_G$ has support in the subgraph $H \subseteq G$ where $E(H) = \{(1, 4), (4, 1)\}$. As expected, $C_4$ is a $\mathbb{Z}$-linear combination of directed cycles of $H$, since $C_4 \neq C_4$. 

**Proof of Lemma 4.4.** Any simple directed cycle satisfies conditions (10) and (11); it follows that any formal cycle satisfies conditions (10) and (11) as well.

Conversely, suppose $S$ is a formal sum

$$S = \sum_{e \in E(G)} m_e(S) \cdot e$$

satisfying (10) and (11); our goal is to show that $S$ is a formal cycle. Adding directed cycles to $S$ if necessary, we may assume $m_e(S) \geq 0$ for every $e \in E(G)$.

Thus, it suffices to show that nonnegative formal sums of edges satisfying (10) and (11) are formal cycles. The remainder of the proof is by induction on $|S|$. Specifically, we argue that there exists a simple directed cycle $C$ of $G$ whose support is contained in $\text{supp}(S)$; because $S - C$ is again a nonnegative formal sum of edges satisfying (10) and (11), the inductive hypothesis guarantees that $S - C$ is a formal cycle.

Indeed, pick any edge $e = (s, t) \in \text{supp}(S)$; since conditions (10) and (11) holds for the vertex $t \in V(G)$, there is another edge $e' \in \text{supp}(S)$ whose source is the vertex $t$. By repeating this process, we obtain edges whose concatenation forms a directed path; this path eventually intersects itself and thus contains a simple directed cycle.

**Lemma 4.6.** Let $G$ be a directed multigraph, and let $c : \mathcal{M}_G \to \mathbb{R}$ be an additive map such that $c(C) > -|C|$ for any directed cycle $C \in \mathcal{M}_G$. Then $c$ can be extended to an additive map $c : \mathbb{Z}^E(G) \to \mathbb{R}$ so that $c(e) > -1$ for all $e \in E(G)$.

**Proof.** The proof is by induction on $|E(G)|$. Note that if $G$ has no directed cycles, the lemma is vacuous, so we may assume $G$ has at least one directed cycle. We enumerate the simple directed cycles of $G$ by $C_1, \ldots, C_r$.

Set

$$W \overset{\text{def}}{=} \min_{i \in [r]} \left\{ \frac{c(C_i)}{|C_i|} \right\} > -1; \quad I \overset{\text{def}}{=} \left\{ i \in [r] : \frac{c(C_i)}{|C_i|} = W \right\}; \quad E_I \overset{\text{def}}{=} \{ e \in E(G) : e \in C_i \text{ for some } i \in I \}.$$

Let us define an additive map $c_1 : \mathbb{Z}^{E_I} \to \mathbb{R}$ by setting $c_1(e) = W$ for all $e \in E_I$.

Treating $E_I \subseteq G$ as a subgraph, observe that Lemma 4.4 implies any formal cycle $C \in \mathcal{M}_G$ with support in $E_I$ is in fact a formal cycle in $\mathcal{M}_{E_I}$ (cf. Remark 4.5). In particular, any simple directed cycle $\{C_i : i \in I\}$ of $E_I$ satisfies $c(C_i) = W|C_i|$, so additivity of $c$ implies any formal cycle $C$ of $G$ with support in $E_I$ also satisfies $c(C) = W|C|$.

Again treating $E_I \subseteq G$ as a subgraph, we may form the multigraph $(E_I)_{\text{comp}}$. We will argue that formal cycles $D$ of $(E_I)_{\text{comp}}$ can be described as follows: its corresponding linear combination of edges $D' \subseteq G \setminus E_I$ is the restriction of some formal cycle $C$ of $G$ to $E(G) \setminus E_I$, i.e.

$$D' = \sum_{e \in E(G) \setminus E_I} m_e(C) \cdot e$$

for some $C \in \mathcal{M}_G$.

First note that $E_I$ is a union of directed cycles and hence its weak components are strongly connected; if $D$ is a directed cycle of $(E_I)_{\text{comp}}$ then the corresponding $D' \subseteq G \setminus E_I$ can be completed to a directed cycle $C$ of $G$ by appending directed paths in $E_I$. Such a directed cycle $C$ satisfies (12). When $D$ is a formal sum of cycles of $(E_I)_{\text{comp}}$, the corresponding formal sum of cycles of $G$ also satisfies (12).

We now argue, for any formal cycle $D \in \mathcal{M}_{(E_I)_{\text{comp}}}$, that the quantity

$$c_2(D) \overset{\text{def}}{=} c(C) - W \sum_{e \in E_I} m_e(C)$$

...
is well-defined, independent of the choice of formal cycle $C$ satisfying (12). Let $C_1, C_2$ be formal cycles satisfying (12), and consider the formal sums
\[ C_1 \cap E_I \overset{\text{def}}{=} \sum_{e \in E_I} m_e(C_1) \cdot e \quad \text{and} \quad C_2 \cap E_I \overset{\text{def}}{=} \sum_{e \in E_I} m_e(C_2) \cdot e. \]

By definition, we may decompose $C_1 \cap E_I$ and $C_2 \cap E_I$ into a $\mathbb{Z}$-linear combination of directed paths of $E_I$, each of which connects endpoints of edges of $D' \subseteq G \setminus E_I$. Treating paths as sums of edges, we may write
\[ C_1 \cap E_I = \sum_i a_i \cdot p_{i,1} \quad \text{and} \quad C_2 \cap E_I = \sum_i b_i \cdot p_{i,2}, \]
where $a_i, b_i \in \mathbb{Z}$. For a directed path $p$, let $s(p)$ and $t(p)$ denote the source and target respectively. For $v \in V(E_I)$ and $j \in \{1, 2\}$ let
\[ S(v; j) \overset{\text{def}}{=} \{ i : s(p_{i,j}) = v \} \quad \text{and} \quad T(v; j) \overset{\text{def}}{=} \{ i : t(p_{i,j}) = v \}. \]

Since $C_1$ and $C_2$ satisfy (12) for the same formal cycle $D' \in \mathcal{M}_{E_I}^{\text{comp}}$, we have
\[ \sum_{i \in S(v; 1)} a_i = \sum_{i \in S(v; 2)} b_i \quad \text{and} \quad \sum_{i \in T(v; 1)} a_i = \sum_{i \in T(v; 2)} b_i \]
for all $v \in V(E_I)$.

Thus, $C_1 \cap E_I$ and $C_2 \cap E_I$ can be simultaneously completed to a formal cycle of $E_I$, i.e. there exist formal cycles $(C_1)^I$ and $(C_2)^I$ of $E_I$ so that
\[
\begin{align*}
\{ m_e((C_1)^I) = m_e(C_1) \quad &\text{for all } e \in E_I \cap \text{supp}(C_1) \\
\{ m_e((C_2)^I) = m_e(C_2) \quad &\text{for all } e \in E_I \cap \text{supp}(C_2) \\
m_e((C_1)^I) = m_e((C_2)^I) \quad &\text{for all other } e \in E_I.
\end{align*}
\]

Note that $C_1 + (C_2)^I = (C_1)^I + C_2$ in $\mathcal{M}_G$, and hence
\[ c(C_1) + W|(C_2)^I| = c(C_2) + W|(C_1)^I|. \]
Rearranging terms, we obtain
\[ c(C_1) - W|C_1 \cap E_I| = c(C_2) - W|C_2 \cap E_I|. \]
We conclude $c_2 : \mathcal{M}_{E_I}^{\text{comp}} \rightarrow \mathbb{R}$ is well-defined.

The function $c_2 : \mathcal{M}_{E_I}^{\text{comp}} \rightarrow \mathbb{R}$ is additive, since restriction commutes with summation: if $C_1$ and $C_2$ are formal cycles of $G$ whose restrictions to $G \setminus E_I$ are $(D_1)^I$ and $(D_2)^I$ respectively, then the restriction of $C_1 + C_2$ to $G \setminus E_I$ is $(D_1)^I + (D_2)^I$.

Furthermore, if $D$ is a directed cycle of $(E_I)^{\text{comp}}$, then minimality of $W$ implies
\[
\frac{c_2(D)}{|D|} = \frac{c(C) - W|C \cap E_I|}{|D|} > \frac{c(C) - c(E_I)}{|E_I|} \frac{|C \cap E_I|}{|D|} = \frac{c(C)|D|}{|C|} > -1.
\]
Since $|E((E_I)^{\text{comp}})| < |E(G)|$, the inductive hypothesis asserts that the function $c_2$ extends to an additive map $c_2 : \mathbb{Z}^{E((E_I)^{\text{comp}})} \rightarrow \mathbb{R}$ so that $c_2(e) > -1$ for all $e \in E((E_I)^{\text{comp}})$; identifying $E((E_I)^{\text{comp}})$ with $E(G) \setminus E_I$ we obtain an additive map $c_2 : \mathbb{Z}^{E(G) \setminus E_I} \rightarrow \mathbb{R}$.

The functions $c_1 : \mathbb{Z}^{E(I)} \rightarrow \mathbb{R}$ and $c_2 : \mathbb{Z}^{E(G) \setminus E_I} \rightarrow \mathbb{R}$ glue to a function $\mathbb{Z}^{E(G)} \rightarrow \mathbb{R}$ which we claim extends $c : \mathcal{M}_G \rightarrow \mathbb{R}$.

To verify this claim, we must check that if $C$ is a simple directed cycle of $G$, then
\[ \sum_{e \in C \cap E_I} c_1(e) + \sum_{e \in C \cap (G \setminus E_I)} c_2(e) = c(C). \]

By the definitions of $c_1$ and $c_2$, we have
\[ \sum_{e \in C \cap E_I} c_1(e) = W|C \cap E_I| \quad \text{and} \quad \sum_{e \in C \cap (G \setminus E_I)} c_2(e) = c(C) - W|C \cap E_I|, \]
so Equation (13) is satisfied.

We can now prove Lemma 3.19, restated here for convenience:
Lemma 3.19. Let $H \subseteq G$ be an admissible subgraph of $G$. There is a vector $d = (d_v)_{v \in V(H)} \in \mathbb{R}^V(H)$ so that
\[ w_d(e) + d_{s(e)} - d_{t(e)} > -1 \]
for every edge $e \in E(H)$. 

Proof of Lemma 3.19. Let $M^T$ denote the transpose of the incidence matrix of $H$, i.e. the matrix corresponding to the linear transformation
\[ M^T : \mathbb{R}^V(H) \to \mathbb{R}^E(H) \]
-defined by
\[ e_i \mapsto \sum_{e \in E(H)} e_c - \sum_{e \in E(H)} e_c. \]

Let $w_d(H) \subseteq \mathbb{R}^E(H)$ denote the vector whose component indexed by $e \in E(H)$ is $w_d(e)$. Then Lemma 3.19 asks for a vector $d \in \mathbb{R}^V(H)$ so that
\[ w_d(H) + M^T d > -1, \]
where $1 \in \mathbb{R}^E(H)$ is the vector whose components are all equal to 1.

The image of $M^T$ is equal to the cut space of $H$, i.e. the space
\[ W = \left\{ x \in \mathbb{R}^E(H) : \sum_{e \in \mathcal{C}} x_e = 0 \text{ for all directed cycles } \mathcal{C} \text{ of } H \right\}; \]
see e.g. [Bol98, Thm. II.3.9, Ex. II.4.39].

Because $H$ is admissible, the additive function
\[ c : \mathcal{M}_H \to \mathbb{R} \]
\[ \mathcal{C} \mapsto \sum_{e \in \text{supp}(\mathcal{C})} m_e(\mathcal{C}) \cdot w_d(e) \]
satisfies $c(\mathcal{C}) > -|\mathcal{C}|$ for every directed cycle $\mathcal{C}$ of $\mathcal{M}_H$, so Lemma 4.6 guarantees that $c$ can be extended to an additive function $c : \mathbb{R}^E(H) \to \mathbb{R}$ with $c(e) > -1$. Let $c \in \mathbb{R}^E(H)$ denote the vector whose component indexed by $e \in E(H)$ is $c(e)$; by definition, $c > -1$. The condition that
\[ \sum_{e \in \mathcal{C}} c(e) = \sum_{e \in \mathcal{C}} w_d(e) \]
for every directed cycle $\mathcal{C}$ of $H$ is exactly the condition
\[ w_d(H) - c \in W; \]
so $w_d(H) - c = M^T v$ for some $v \in \mathbb{R}^V(H)$. Rearranging,
\[ w_d(H) + M^T (-v) = c > -1, \]
so $d := -v$ satisfies Equation (15). \hfill \Box

5. Consequences of Theorems 3.4 and 3.18; relations to previous results

In this section, we explore consequences of Theorems 3.4 and 3.18. In Corollaries 5.2 and 5.3 we highlight a result of Portakal characterizing faces of the form $Q_H \subseteq Q_G$ for alternating graphs $G$. In Corollary 5.8, we show that Theorem 3.18 specializes to a result of Postnikov characterizing facets of the form $Q_H \subseteq Q_G$ for transitively closed graphs $G$ (Definition 5.4). We also highlight the special case $G = K_n$ in Corollaries 5.9 and 5.10; the latter corollary corrects a result of Gelfand, Graev, and Postnikov (see Remark 5.11).

We will use the following notation.

**Definition 5.1.** Let $G$ be a directed graph and let $A \subseteq V(G)$. The set of **neighbors** of $A$, denoted $N(A)$, is the set
\[ N(A) \equiv \{ v \in V(G) : (v, a) \in E(G) \text{ for some } a \in A \} \cup \{ v \in V(G) : (a, v) \in E(G) \text{ for some } a \in A \}. \]
We say $A \subseteq V(G)$ is **independent** if it is disjoint from $N(A)$. \hfill \triangle
Recall (see Lemma 2.3) that the vertex set of an alternating graph may be partitioned into disjoint sets $L$ and $R$ consisting of source and sink vertices respectively. In this setting, Theorem 3.4 may be recast as follows.

**Corollary 5.2 ([Por19, Thm. 3.12]).** Let $G$ be an alternating graph and suppose $G^\text{un}$ is connected. The subgraph $H \subseteq G$ defines a facet $\tilde{Q}_H \subseteq \tilde{Q}_G$ if and only if $H^\text{un}$ has two connected components and

$$H = G|_{\bar{A} \cup N(A)} \cup G|_{[n] \setminus (\bar{A} \cup N(A))}$$

for some set $A \subseteq R$ of sink vertices.

**Proof.** Suppose first that $\tilde{Q}_H$ is a facet of $\tilde{Q}_G$. Since $G$ is connected, we have $\dim(\tilde{Q}_G) = n - 1$ and hence $\dim(\tilde{Q}_H) = n - 2$. Proposition 2.4 implies $H^\text{un}$ must have two connected components which we denote by $H^\text{un}_1$ and $H^\text{un}_2$. Theorem 3.4 asserts that the two-vertex graph $H^\text{comp}_1$ is loopless and acyclic; because $G$ is connected the graph $H^\text{comp}_2$ must have an edge which, without loss of generality, sends $H^\text{comp}_1 \in V(H^\text{comp}_2)$ to $H^\text{comp}_2 \in V(H^\text{comp}_2)$.

Proposition 3.5 implies that $H$ is a disjoint union of induced subgraphs of $G$: specifically, we may write

$$H = G|_{V(H^\text{un}_1)} \cup G|_{V(H^\text{un}_2)}.$$ 

Set

$$A \overset{\text{def}}{=} V(H^\text{un}_1) \cap R;$$

observe that $H_1 \subseteq G|_{\overline{A} \cup N(A)}$; every edge of $H_1$ has target in $A$. Furthermore, because $H_1$ is a source vertex in $H^\text{comp}_1$, every edge $e = (v, a) \in E(G)$ incident to a vertex in $A$ must be in $H_1$. It follows that $H_1 = G|_{\overline{A} \cup N(A)}$. Since $V(H^\text{un}) = [n] \setminus V(H^\text{un}_1)$, we conclude that $H$ has the form (16).

Suppose now that $H^\text{un}$ has two connected components $H^\text{un}_1$ and $H^\text{un}_2$ and has the form (16). Proposition 3.5 implies $H^\text{comp}$ is loopless.

Observe that $(G|_{\overline{A} \cup N(A)})^\text{un}$ and $(G|_{[n] \setminus (\overline{A} \cup N(A))})^\text{un}$ are both connected: if either had (at least) two connected components, then the other would be empty and $H = G$. We conclude that the two vertices of $H^\text{comp}$ correspond to $H_1 := G|_{\overline{A} \cup N(A)}$ and $H_2 := G|_{[n] \setminus (\overline{A} \cup N(A))}$. Note that no edge of $E(G) \setminus E(H)$ can have target in $A$. Furthermore, since $A \subseteq R$ we have $N(A) \subseteq L$; hence no edge of $E(G) \setminus E(H)$ can have target in $N(A)$.

Hence $H^\text{comp}$ has no edges whose target is $H^\text{comp}_1 \in V(H^\text{comp}_2)$. In total, we have shown $H^\text{comp}$ is loopless and acyclic, so Theorem 3.4 implies $\tilde{Q}_H \subseteq \tilde{Q}_G$ is a facet. Since $H$ has two connected components, $\dim(\tilde{Q}_H) = n - 2$ and $\tilde{Q}_H$ is a facet. □

**Corollary 5.3 ([Por19, Thm. 3.17]).** Let $G$ be an alternating graph and suppose $G^\text{un}$ is connected. The subgraph $H \subseteq G$ defines a face $\tilde{Q}_H \subseteq \tilde{Q}_G$ of codimension $d$ if and only if $H^\text{un}$ has $d + 1$ connected components and can be written as the intersection $H = H_1 \cap \cdots \cap H_d$ of $d$ many graphs for which $\tilde{Q}_H$ is a facet of $\tilde{Q}_G$.

**Proof.** Suppose $\tilde{Q}_H \subseteq \tilde{Q}_G$ is a face of codimension $d$. By Lemma 2.5, it is the intersection of some $d$ facets $F_1, \ldots, F_d$ of $\tilde{Q}_G$. These facets must contain the origin, so $F_i = \tilde{Q}_H$, for some graphs $H_i$. Furthermore, since $\tilde{Q}_H$ has codimension $d$, Proposition 2.4 asserts $H^\text{un}$ must have $d + 1$ connected components.

Now suppose $H^\text{un}$ has $d + 1$ connected components and assume $H = H_1 \cap \cdots \cap H_d$ where $\tilde{Q}_H$ is a facet of $\tilde{Q}_G$. Observe that the vertices of the polytope $\tilde{Q}_H$ are precisely the common vertices of the polytopes $\tilde{Q}_H_i$ for $i \in [d]$. It follows that the polytope $\tilde{Q}_H$ is the intersection of the polytopes $\tilde{Q}_H_i$, for $i \in [d]$. Hence $\tilde{Q}_H \subseteq \tilde{Q}_G$ is a face; because $H^\text{un}$ has $d + 1$ connected components, Proposition 2.4 asserts $H^\text{un}$ must have codimension $d$. □

To state Postnikov’s result we begin with the following definition:

**Definition 5.4.** A graph $G$ is called **transitively closed** if whenever $(i, j), (j, k) \in E(G)$ are edges of $G$, then $(i, k) \in E(G)$ is also an edge of $G$. △

**Definition 5.5.** Let $L, R \subseteq [n]$ be disjoint subsets of $[n] = V(G)$. The subgraph $G_{L,R} \subseteq G$ is the (alternating) graph whose edge set is

$$E(G_{L,R}) = \{(i, j) \in E(G): i \in L, j \in R\}.$$

We call such graphs **alternating-induced** subgraphs of $G$. △

**Example 5.6.** Let $G = K_5$, $L = \{1, 3\}$, and $R = \{2, 5\}$. Then $G_{L,R}$ is the graph in Figure 7. △
Let than necessary for the purposes of this paper, the proof is essentially the same, so we include it here.

Suppose there exist vertices $i, j, k \in [n]$ such that $(i, j), (j, k) \in E(H)$, then $(i, k) \in E(G)$ because $G$ is transitively closed. If $(i, k) \in E(H)$, then $H$ is not path consistent, since $p_{ik} = ((i, j), (j, k))$ and $q_{ik} = ((i, k))$ violate Equation (3), and if $(i, k) \notin E(H)$, then $H$ is not admissible, since the edge $e = (i, k) \in E(G) \setminus E(H)$ corresponds to a self-loop $e_{\text{comp}} \in E(H_{\text{comp}})$ with $\text{wd}(e_{\text{comp}}) = -2$, violating Equation (5).

Now let $H_i$ denote the overlying directed graph of the connected component $H_i^{un}$ of $H^{un}$. If $H_i$ is not an isolated vertex, every vertex $v \in V(H_i)$ is either the source of an edge or the target of an edge in $E(H_i)$. Define the (disjoint) subsets

$$L_i \overset{\text{def}}{=} \{v \in V(H_i) : v \text{ is the source of some } e \in E(H_i)\},$$

$$R_i \overset{\text{def}}{=} \{v \in V(H_i) : v \text{ is the target of some } e \in E(H_i)\}.$$

Observe that $H_i \subseteq (G|_{V(H_i)})_{L_i, R_i}$. An edge $e \in E((G|_{V(H_i)})_{L_i, R_i}) \setminus E(H_i)$ corresponds to a loop $e_{\text{comp}} \in (H_i^{un}, H_i^{un}) \subseteq E(H_{\text{comp}})$ with $\text{wd}(e_{\text{comp}}) = -1$, violating Equation (5). It follows that $H_i = (G|_{V(H_i)})_{L_i, R_i}$. 

We use Proposition 5.7 to deduce Postnikov’s result from Theorem 3.18. For $G = K_n$, this result appeared in the earlier work of [Cho99, Prop. 13].

**Corollary 5.8** ([Pos09, Prop. 13.3]). Let $G$ be transitively closed and suppose $G^{un}$ is connected. The subgraph $H \subseteq G$ defines a facet $Q_H \subseteq \hat{Q}_G$ of $\hat{Q}_G$ not containing the origin if and only if $H^{un}$ is connected and $H = G_{L,R}$ is alternating-induced by some partition $L \sqcup R = [n]$.

**Proof.** Let $Q_H \subseteq \hat{Q}_G$ be a facet. By Theorem 3.18, $H \subseteq G$ is path consistent and admissible. By Proposition 5.7, $H$ has the form (17).

Since $G^{un}$ is connected, Proposition 2.4 says $\hat{Q}_G$ is $(n - 1)$-dimensional, so the facet $Q_H \subseteq \hat{Q}_G$ is $(n - 2)$-dimensional. Since $H$ is alternating, Proposition 2.4 implies that $H^{un}$ has one connected component. It follows that the partition $P$ appearing in (17) can only contain one part, i.e., $H = G_{L,R}$ for some disjoint $L, R \subseteq [n]$. If $H$ is to contain no isolated vertices, we further obtain $L \sqcup R = [n]$.

Conversely, suppose $H \subseteq G$ is a subgraph so that $H^{un}$ is connected and $H = G_{L,R}$ for some $L \sqcup R = [n]$. By Proposition 2.4, dim $\hat{Q}_G = n - 1$ and dim $Q_H = n - 2$, so it suffices to show that $Q_H \subseteq \hat{Q}_G$ is a facet. Since $H$ is alternating, it is automatically path consistent (as shown in Example 3.9).

Note also that $H_{\text{comp}}$ consists of a single vertex with a self loop corresponding to each edge $e = (i, j) \in E(G) \setminus E(G_{L,R})$, and $\text{wd}(e) = 0$ when $i, j \in L$ or $i, j \in R$, whereas $\text{wd}(e) = 1$ when $i \in R$ and $j \in L$. In
both cases, Equation (5) is satisfied and \( H \) is admissible. Since \( H \subseteq G \) is path consistent and admissible, Theorem 3.18 implies \( Q_H \subset \tilde{Q}_G \) is a face. \( \square \)

The case \( G = K_n \) of Theorems 3.4 and 3.18 is of special interest, and we spell them out here.

**Corollary 5.9.** The subgraph \( H \subseteq K_n \) forms a face \( \tilde{Q}_H \subseteq \tilde{Q}_{K_n} \) if and only if

\[
H = K_{[1,n_1]} \sqcup K_{[n_1+1,n_2]} \sqcup \cdots \sqcup K_{[n_k+1,n]}
\]

is a disjoint union of complete graphs on vertex sets \([n_i + 1, n_{i+1}] \defeq \{n_i + 1, n_i + 2, \ldots, n_{i+1}\}\).

**Proof.** By Theorem 3.4, it suffices to characterize subgraphs \( H \subseteq K_n \) so that \( H_{\text{comp}} \) is loopless and acyclic. By Proposition 3.5, \( H_{\text{comp}} \) is loopless if and only if \( H \) is the disjoint union of induced subgraphs \( \{(K_i)_{P_i} \}_{P_i \in \mathcal{P}} \), which are just complete graphs \( \{K_i\}_{P_i \in \mathcal{P}} \) on vertex sets \( P_i \subseteq [n] \). The acyclicity of \( H_{\text{comp}} \) implies that if \( i < j < k \) and \( i \in P_a, j \in P_b \neq P_a, \text{then } k \notin P_a \). Thus, \( P_a \) consists of consecutive numbers \( \{n_i + 1, \ldots, n_{i+1}\} \).

If the partition \( \mathcal{P} = \{P_i\} \) is of the form \( P_i = [n_i + 1, n_{i+1}] \), it is immediate that \( H_{\text{comp}} \) is acyclic. \( \square \)

**Corollary 5.10.** The subgraph \( H \subseteq K_n \) forms a face \( \tilde{Q}_H \subseteq \tilde{Q}_{K_n} \) if and only if

\[
H = (K_{[1,n_1]})_{L_1,R_1} \sqcup (K_{[n_1+1,n_2]})_{L_2,R_2} \sqcup \cdots \sqcup (K_{[n_k+1,n]})_{L_{k+1},R_{k+1}}
\]

is a disjoint union of alternating-induced subgraphs of complete graphs on vertex sets \([n_i + 1, n_{i+1}]\).

**Proof.** By Theorem 3.18, it suffices to characterize path consistent, admissible subgraphs \( H \subseteq K_n \). Let \( H \subseteq K_n \) be such a graph. Since \( K_n \) is transitively closed, Proposition 5.7 asserts that

\[
H = \bigcup_{P_i \in \mathcal{P}} (K_{P_i})_{L_i,R_i}
\]

is a disjoint union of alternating-induced subgraphs of complete graphs on vertex sets \( P_i \in \mathcal{P} \). To show that \( H \) is of the form (19), it suffices to show that if \( i, j, k \in [n] = V(H) \) with \( i < j < k \), either \( (i, j) \in E(H), (j, k) \in E(H) \), or \( j \) is an isolated vertex. (This would imply that the partition \( \mathcal{P} = \{P_i\} \) can be chosen so that \( i, k \in P_i \), implies \( j \in P_i \), i.e., so that the parts are consecutive blocks of numbers.)

With the above goal in mind, consider any triple \( i < j < k \) with \( i, k \in E(H) \), and suppose \( j \) is not in the same connected component of \( H^{\text{un}} \) as \( i \) and \( k \); we want to show that \( j \) is isolated. If there is an edge \( (j, \ell) \in E(H) \), then the edges \( e_{i\ell} = (i, \ell) \) and \( e_{jk} = (j, k) \) of \( E(K_n) \) give rise to a directed cycle \( C = \{(e_{i\ell})_{\text{comp}}, (e_{jk})_{\text{comp}}\} \) in \( H_{\text{comp}} \). Since \( \text{wd}(e_{i\ell})_{\text{comp}} = \text{wd}(e_{jk})_{\text{comp}} = -1 \), Equation (5) is violated and \( H \) is not admissible.

Similarly, if there is an edge \( (\ell, j) \in E(H) \), then the edges \( e_{jk} = (\ell, k) \) and \( e_{ij} = (i, j) \) of \( E(K_n) \) give rise to a directed cycle \( C = \{(e_{jk})_{\text{comp}}, (e_{ij})_{\text{comp}}\} \) in \( H_{\text{comp}} \). Since \( \text{wd}(e_{jk})_{\text{comp}} = \text{wd}(e_{ij})_{\text{comp}} = -1 \), Equation (5) is violated and \( H \) is not admissible.

Conversely, if \( H \) is of the form (19), then it is alternating and hence path consistent, and the directed graph \( H_{\text{comp}} \) is nothing more than the complete graph on \( V(H_{\text{comp}}) \); in particular it is acyclic and Equation (5) is satisfied, so \( H \) is admissible as well. \( \square \)

**Remark 5.11.** The faces \( Q_H \subset \tilde{Q}_{K_n} \) were studied already in [GGP97, Prop. 8.1]. Their result contains a mistake; it states that there is a bijection

\[
\rho: \{H: Q_H \subset \tilde{Q}_{K_n} \text{ is a face}\} \leftrightarrow \{\text{alternating-induced subgraphs } (K_n)_{L_i,R_i}\}
\]

such that \( H \subseteq \rho(H) \). This is false for \( n = 4 \), as for the graphs \( H_1 \) and \( H_2 \) in Figure 8, the condition \( H \subseteq \rho(H) \) forces \( \rho(H_1) = \rho(H_2) = H_2 \). Yet, \( Q_{H_1} \) is an edge of the triangular facet \( Q_{H_2} \) of \( \tilde{Q}_{K_4} \), and indeed Corollary 5.10 asserts that

\[
H_1 = (K_{[1,2]})_{\{1,2\}} \sqcup (K_{[3,4]})_{\{3,4\}} \quad \text{and} \quad H_2 = (K_4)_{\{1,3\},\{2,4\}}
\]

form distinct faces of \( Q_H \subset \tilde{Q}_{K_4} \).

Compare [GGP97, Prop. 8.1] to Corollary 5.10, which asserts that the identity map is a bijection

\[
\text{id}: \{H: Q_H \subset \tilde{Q}_{K_n} \text{ is a face}\} \leftrightarrow \{\text{disjoint unions of alternating-induced subgraphs } (K_{[n_i+1,n_{i+1}+1]})_{L_i,R_i}\}.
\]

\( \triangle \)
Remark 5.12. Corollaries 5.9 and 5.10 give rise to the tantalizing question of explicitly computing the $f$-vector of $\tilde{Q}_{K_n}$. Specifically, let us highlight that by Proposition 2.4 there are

$$\#\{\text{graphs of the form (18) with } n - d \text{ connected components}\}$$

$$+ \#\{\text{graphs of the form (19) with } n - d - 1 \text{ connected components}\}$$

faces of dimension $d$. The first term of the summand is easily shown to be

$$\#\{\text{graphs of the form (18) with } n - d \text{ connected components}\} = \binom{n-1}{n-d-1},$$

as the graph $H$ is uniquely determined by the numbers $1 \leq n_1 < \cdots < n_{n-d-1} \leq n - 1$.

We record here that a graph $H$ of the form (19) arises from a unique choice of $L_i, R_i$ satisfying the additional condition

$$\min(L_i \cup R_i) \in L_i \quad \text{and} \quad \max(L_i \cup R_i) \in R_i,$$

and that conversely any collection of disjoint sets $L_i, R_i$ satisfying condition (20) and $\max(R_i) < \min(L_{i+1})$ uniquely determines the graph $H$, since we may recover

$$E(H) = \{(a, b) : a \in L_i, b \in R_i \text{ for some } i\}.$$  

In other words, we have a bijection

$$\{ H \text{ of the form (19) } \} \longleftrightarrow \{ \text{disjoint sets } L_i, R_i \subset [n] \text{ satisfying (20) and } \max(R_i) < \min(L_{i+1}) \}.$$  

The graph $H$ corresponding to the sets $\{L_1, R_1, \ldots, L_\ell, R_\ell\}$ under this bijection is so that $H^\text{un}$ has $\ell$ connected components containing an edge, along with

$$n - \sum_{i=1}^\ell (|L_i| + |R_i|)$$

many isolated vertices. \(\triangle\)

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