QUASIPRIMITIVE GROUPS AND BLOW-UP DECOMPOSITIONS

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Abstract. The blow-up construction by L. G. Kovács has been a very useful tool to study embeddings of finite primitive permutation groups into wreath products in product action. In the present paper we extend the concept of a blow-up to finite quasiprimitive permutation groups, and use it to study embeddings of finite quasiprimitive groups into wreath products.

1. Introduction

It is an important problem in the study of permutation groups to describe, in as much detail as possible, the inclusions among different classes of groups. For primitive groups this problem was solved by the second author in [Pra90]. The special case of describing the possible inclusions of primitive groups into wreath products in product action relied on the concept of a blow-up defined by L. G. Kovács in his seminal paper [Kov89a].

A finite permutation group is said to be quasiprimitive if all its non-trivial normal subgroups are transitive. In our research into quasiprimitive permutation groups and their actions on combinatorial objects we found it necessary to extend the results of [Pra90] and to study the class of inclusions of quasiprimitive groups into wreath products in product action. It turned out that the class of these inclusions is much richer than that of the primitive groups. In order to give a detailed description in our case, we generalised in [BPS04] the concept of a system of product imprimitivity introduced by Kovács in [Kov89b] and defined the concept of a Cartesian decomposition. The stabiliser, in a finite symmetric group, of a homogeneous Cartesian decomposition of the underlying set is a wreath product, in product action, of smaller symmetric groups. Thus the problem of finding the set of such wreath products in product action that contain a given permutation group \( G \) is equivalent to finding all \( G \)-invariant homogeneous Cartesian decompositions of the underlying set. The details of this work can be found in [BPS04, BPS05, BPSxx, PSxx]. Our efforts made it possible in [BadP03] to describe satisfactorily the inclusions of quasiprimitive groups into wreath products in product action. The philosophy behind this description is the same as in [Pra90]: define several classes of natural inclusions, and give a complete description of the exceptional inclusions.

The natural inclusions in the case of primitive groups correspond to blow-up decompositions as defined by Kovács [Kov89a]. We found it necessary to define a similar concept for quasiprimitive groups. However, we also want this new concept to fit into our more general combinatorial framework. Hence, we first introduce the class of normal Cartesian decompositions. Informally speaking, if \( \mathcal{E} = \{ \Gamma_1, \ldots, \Gamma_\ell \} \) is a normal Cartesian decomposition for a permutation group \( G \), then \( G \) has a transitive normal subgroup \( M \) which can be written as a direct product \( M = M_1 \times \cdots \times M_\ell \) in such a way that \( M_1 \times \cdots \times M_\ell \) acts in product action on \( \Gamma_1 \times \cdots \times \Gamma_\ell \). We define a blow-up decomposition for a permutation group as a special type of normal Cartesian decomposition. It is not hard to see that the blow-up decomposition given by Kovács in [Kov89a] is equivalent to our blow-up decomposition defined in Section 3 for the class of primitive groups whose socle is non-regular; see also Theorem 5.4.
The aim of this paper is to study blow-up decompositions of quasiprimitive groups. In particular we investigate the extent to which a quasiprimitive permutation group can be recovered from knowledge of its components under a blow-up decomposition. We also prove an important theorem that was stated without proof, and used, in [BadP03] (see Theorem 6.1).

The structure of the paper is as follows. In Section 2 we review some well-known facts about wreath products and quasi-primitive groups. In Section 3 we give the definition of a Cartesian decomposition, and we define normal decompositions and blow-up decompositions. In Section 4 we investigate the relationship between a quasi-primitive group and its components under a blow-up decomposition. In Section 5 we study normal decompositions that are not blow-ups. Finally in Section 6 we state and prove Theorem 6.1 which was used in the study [BadP03] of inclusions of quasi-primitive groups.

In this paper we use the following notation. Permutations act on the right: if \( \pi \) is a permutation and \( \omega \) is a point then the image of \( \omega \) under \( \pi \) is denoted \( \omega \pi \). If \( G \) is a group acting on a set \( \Omega \) and \( \Gamma \) is a subset of \( \Omega \), then \( G_\Gamma \) and \( G_{(\Gamma)} \) denote respectively the setwise and the pointwise stabiliser in \( G \) of \( \Gamma \). All groups that appear in this paper are finite.

2. Primitive and Quasi-primitive Groups

First in this section we review wreath products and their product actions, as they play an important part in our research. Let \( \Gamma \) be a finite set, \( L \leq \text{Sym} \Gamma, \ell \geq 2 \) an integer, and \( H \leq S_\ell \). The \textit{wreath product} \( L \wr H \) is the semidirect product \( L^\ell \rtimes H \), where, for \( (x_1, \ldots , x_\ell) \in L^\ell \) and \( \sigma \in H \), \( (x_1, \ldots , x_\ell)\sigma^{-1} = (x_{1\sigma}, \ldots , x_{\ell\sigma}) \). The \textit{product action} of \( L \wr H \) is the action of \( L \wr H \) on \( \Gamma^\ell \) defined by

\[
(\gamma_1, \ldots , \gamma_\ell)(x_1, \ldots , x_\ell) = (\gamma_1x_1, \ldots , \gamma_\ell x_\ell) \quad \text{and} \quad (\gamma_1, \ldots , \gamma_\ell)\sigma^{-1} = (\gamma_1\sigma, \ldots , \gamma_\ell\sigma)
\]

for all \((\gamma_1, \ldots , \gamma_\ell) \in \Gamma^\ell, x_1, \ldots , x_\ell \in L, \) and \( \sigma \in H \). The important properties of wreath products can be found in most textbooks on permutation group theory, see for instance [DM96].

The \textit{holomorph} of an abstract group \( M \) is the semidirect product \( M \rtimes \text{Aut} M \). If \( M \) is a regular, characteristically simple permutation group acting on a set \( \Omega \), then \( \Omega \) can be identified with the underlying set of \( M \), and \( \text{Hol} M \) can also be viewed as a subgroup of \( \text{Sym} \Omega \). It is well-known that in this case \( \text{Hol} M \). Following [BadP03, Section 3], we distinguish between 8 classes of finite primitive groups, namely HA, HS, HC, SD, CD, PA, AS, TW, and 8 classes of finite quasi-primitive groups, namely HA, HS, HC, SD, CD, PA, AS, TW. The type of a primitive or quasi-primitive group \( G \) can be recognized from the structure and the permutation action of its socle, denoted \( \text{Soc} G \). Let \( G \leq \text{Sym} \Omega \) be a quasi-primitive permutation group, let \( M \) be a minimal normal subgroup of \( G \), and let \( \omega \in \Omega \). Note that \( M \) is a characteristically simple group, and, if \( M \) is non-abelian, then a subdirect subgroup of \( M \) is meant to be subdirect with respect to the unique finest direct decomposition of \( M \). The main characteristics of \( G \) and \( M \) in each primitive and quasi-primitive type are as follows.

- **HA**: \( M \) is abelian, \( C_G(M) = M \) and \( G \leq \text{Hol} M \). The group \( G \) is always primitive.
- **HS**: \( M \) is non-abelian, simple, and regular; \( \text{Soc} G = M \times C_G(M) \) and \( G \leq \text{Hol} M \). The group \( G \) is always primitive.
- **HC**: \( M \) is non-abelian, non-simple, and regular; \( \text{Soc} G = M \times C_G(M) \) and \( G \leq \text{Hol} M \). The group \( G \) is always primitive.
- **SD**: \( M \) is non-abelian and non-simple; \( M_{\omega} \) is a simple subdirect subgroup of \( M \) and \( C_G(M) = 1 \). If, in addition, \( G \) is primitive then the type of \( G \) is SD.
- **CD**: \( M \) is non-abelian and non-simple; \( M_{\omega} \) is a non-simple subdirect subgroup of \( M \) and \( C_G(M) = 1 \). If, in addition, \( G \) is primitive then the type of \( G \) is CD.
- **PA**: \( M \) is non-abelian and non-simple; \( M_{\omega} \) is a not a subdirect subgroup of \( M \) and \( M_{\omega} \neq 1; C_G(M) = 1 \). If, in addition, \( G \) is primitive then the type of \( G \) is PA.
As: $M$ is non-abelian and simple; $C_G(M) = 1$. If, in addition, $G$ is primitive then the type of $G$ is AS.

Tw: $M$ is non-abelian and non-simple; $M_G = 1$; $C_G(M) = 1$. If, in addition, $G$ is primitive then the type of $G$ is TW.

It is not hard to prove that if $G$ is a permutation group with at least two transitive minimal normal subgroups then $G$ is primitive of type HS or HC.

3. Cartesian decompositions preserved by quasiprimitive groups

A Cartesian decomposition of a set $\Omega$ is a set $\{\Gamma_1, \ldots, \Gamma_\ell\}$ of proper partitions of $\Omega$ such that

$$|\gamma_1 \cap \cdots \cap \gamma_\ell| = 1 \quad \text{for all} \quad \gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell.$$

The number $\ell$ is called the index of the Cartesian decomposition $\{\Gamma_1, \ldots, \Gamma_\ell\}$. A Cartesian decomposition is said to be homogeneous if its partitions have the same size.

If $G$ is a permutation group acting on $\Omega$, then a Cartesian decomposition $\mathcal{E}$ of $\Omega$ is said to be $G$-invariant, if the partitions in $\mathcal{E}$ are permuted by $G$. In this case, for $\Gamma \in \mathcal{E}$, the permutation group induced by $G_\Gamma$ on $\Gamma$ is denoted $G_\Gamma$ and is referred to as a component of $G$. If $G$ acts transitively on $\mathcal{E}$, then $\mathcal{E}$ is said to be a transitive $G$-invariant Cartesian decomposition.

Let $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ be a Cartesian decomposition of a set $\Omega$. It follows from the last displayed equation that the following map is a well-defined bijection between $\Omega$ and $\Gamma_1 \times \cdots \times \Gamma_\ell$:

$$\vartheta : \omega \mapsto (\gamma_1, \ldots, \gamma_\ell) \text{ where for } i = 1, \ldots, \ell, \gamma_i \in \Gamma_i \text{ is chosen so that } \omega \in \gamma_i.$$

Now suppose that $G$ is a permutation group on $\Omega$ and that $\mathcal{E}$ is $G$-invariant. Then there is a faithful action of $G$ on $\Gamma_1 \times \cdots \times \Gamma_\ell$ given by

$$(\gamma_1, \ldots, \gamma_\ell)g = (\delta_1, \ldots, \delta_\ell) \quad \text{for all} \quad \gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell \quad \text{and} \quad g \in G$$

where $\delta_1 \in \Gamma_1, \ldots, \delta_\ell \in \Gamma_\ell$ are defined by $\{\delta_1, \ldots, \delta_\ell\} = \{\gamma_1g, \ldots, \gamma_\ellg\}$. (Note that the definition of a Cartesian decomposition ensures that the sets $\Gamma_1, \ldots, \Gamma_\ell$ are disjoint.) We observe that $(\vartheta, \iota)$, where $\iota : G \to G$ is the identity map on $G$, is a permutational isomorphism from $G$ on $\Omega$ to $G$ on $\Gamma_1 \times \cdots \times \Gamma_\ell$, that is

$$(\omega \iota)g = (\omega g)\vartheta \quad \text{for all} \quad \omega \in \Omega \quad \text{and} \quad g \in G.$$ 

Suppose, in addition, that $\mathcal{E}$ is homogeneous, and set $\Gamma = \Gamma_1$. Then for $i = 1, \ldots, \ell$ there exists a bijection $\alpha_i : \Gamma_i \to \Gamma$, whence we have a bijection $\vartheta' : \Omega \to \Gamma^\ell$ given by

$$(1) \quad \vartheta' : \omega \mapsto (\gamma_1\alpha_1, \ldots, \gamma_\ell\alpha_\ell) \quad \text{for all} \quad \omega \in \Omega,$$

where $\omega \vartheta = (\gamma_1, \ldots, \gamma_\ell)$. Let $\chi$ be the isomorphism $\text{Sym}(\Omega) \to \text{Sym}(\Gamma^\ell)$ induced by $\vartheta'$. Then $(\vartheta', \chi)$ restricts to a permutational isomorphism from $G$ on $\Omega$ to $G_\chi$ on $\Gamma^\ell$. It is clear that $G_\chi$ is contained in the wreath product $\text{Sym}(\Gamma) \wr S_\ell$ in its product action on $\Gamma^\ell$. Moreover the image of $\Gamma_i$ under $\vartheta'$ is the partition of $\Gamma^\ell$ with the parts indexed by $\Gamma$, such that the $\gamma$-part is the set of all $\ell$-tuples of $\Gamma^\ell$ with $i$-th entry $\gamma$. The set $\mathcal{E}\vartheta' = \{\Gamma_1\vartheta', \ldots, \Gamma_\ell\vartheta'\}$ is a $G_\chi$-invariant Cartesian decomposition of $\Gamma^\ell$.

In addition to the assumptions in the previous paragraph, suppose now $G$ is transitive on $\mathcal{E}$, and set $\Gamma = \Gamma_1$. Then for all $i = 1, \ldots, \ell$, there exists an element $g_i \in G$ such that $\Gamma_ig_i = \Gamma$. Define the bijection $\alpha_i : \Gamma_i \to \Gamma$ by $\alpha_i : \gamma \mapsto \gamma g_i$, for all $\gamma \in \Gamma_i$. Let $\vartheta'$ be given by (1), and, as before, let $\chi : \text{Sym}(\Omega) \to \text{Sym}(\Gamma^\ell)$ be the isomorphism induced by $\vartheta'$. Direct calculation shows that $G_\chi$ is contained in $G_\chi \wr S_\ell$. In this paper, we will often identify $G$ with $G_\chi$.

Let $M$ be a transitive permutation group on a finite set $\Omega$ and let $\mathcal{E}$ be an $M$-invariant Cartesian decomposition of $\Omega$. Suppose further that $M_{\{\Gamma\}} = M$ and, for $\Gamma \in \mathcal{E}$, $M$ can be written as $M = M_1 \times M_2$ where $M_1$ is a normal subgroup of $M$ and $M_2$ is the kernel of the $M$-action on $\Gamma$. It follows that the $M_1$-action on $\Gamma$ must be faithful, and so we may naturally identify $M_1$ with $M_\Gamma$. In this situation we say that the Cartesian decomposition $\mathcal{E}$ is $M$-normal if $M = \prod_{\Gamma \in \mathcal{E}} M_\Gamma$. If $\mathcal{E}$ is a $G$-invariant Cartesian
decomposition of the underlying set of some permutation group $G$ then $E$ is said to be normal if $E$ is $M$-normal for some transitive normal subgroup $M$ of $G$. In this case $M = \prod_{\Gamma \in \mathcal{E}} M^\Gamma$, and, for $\Gamma \in \mathcal{E}$, we denote $\prod_{\Gamma \neq \Gamma_0} M_{\Gamma_0}$ by $\overline{M^\Gamma}$.

In the next lemma we summarise some basic properties of normal Cartesian decompositions.

**Lemma 3.1.** Suppose that $M$ is a transitive permutation group acting on $\Omega$, $E$ is an $M$-normal Cartesian decomposition of $\Omega$, and let $\Gamma \in \mathcal{E}$. Then the following all hold.

(a) The group $M^\Gamma$ is transitive on $\Gamma$.
(b) For $\gamma \in \Gamma$ and $\omega \in \gamma$, we have $(M^\Gamma)_{\gamma} = M_\omega \cap M^\Gamma$.
(c) If $\omega \in \Omega$, then $M_\omega = \prod_{\Gamma \in \mathcal{E}} \left( M_\omega \cap M^\Gamma \right)$.
(d) The partition $\Gamma$ is the set of $\overline{M^\Gamma}$-orbits and $\overline{M^\Gamma} = M_{\{\Gamma\}}$.

**Proof.** Let $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$. Without loss of generality we will prove parts (a), (b), and (d) in the case $\Gamma = \Gamma_1$.

(a) Suppose that $\gamma_1, \gamma_2 \in \Gamma_1$ and let $\omega_1$ and $\omega_2$ be arbitrary elements of $\Omega$ such that $\omega_1 \in \gamma_1$ and $\omega_2 \in \gamma_2$. Then there is some $m \in M$ such that $\omega_1 m = \omega_2$. As $\Gamma_1$ is a system of imprimitivity for $M$, it follows that $\gamma_1 m = \gamma_2$. Suppose that $m = m_1 \cdots m_\ell$ such that $m_i \in M^\Gamma_i$. Then $\gamma_1 m = \gamma_1 m_1 = \gamma_2$. Thus $M^\Gamma_i$ is transitive on $\Gamma_i$.

(b) Let $\omega \in \Omega$ and $\gamma \in \Gamma_1$ such that $\omega \in \gamma$. As the partition $\Gamma_1$ is a system of imprimitivity in $\Omega$ for the $M$-action, we obtain that $M_\omega \cap M^\Gamma_1 \leq (M^\Gamma_1)_{\gamma}$. Suppose that $m \in (M^\Gamma_1)_{\gamma}$ and that $\{\omega\} = \gamma_1 \cap \gamma_2 \cdots \cap \gamma_\ell$ for some $\gamma_i \in \Gamma_2, \ldots, \Gamma_\ell$. Since $m \in M^\Gamma_1$, the element $m$ stabilises $\gamma_2, \ldots, \gamma_\ell$, and, by assumption, $m$ also stabilises $\gamma_1$. Hence $m$ stabilises $\omega$, and so $(M^\Gamma_1)_{\gamma_1} \leq M_\omega \cap M^\Gamma_1$. Therefore $(M^\Gamma_1)_{\gamma_1} = M_\omega \cap M^\Gamma_1$.

(c) Assume that $\{\omega\} = \gamma_1 \cap \cdots \cap \gamma_\ell$ as in part (b). Then $(M^\Gamma_1)_{\gamma_1} \times \cdots \times (M^\Gamma_\ell)_{\gamma_\ell} \leq M_\omega$. On the other hand, by part (a),

$$|M : (M^\Gamma_1)_{\gamma_1} \times \cdots \times (M^\Gamma_\ell)_{\gamma_\ell}| = |M^\Gamma_1 : (M^\Gamma_1)_{\gamma_1}| \times \cdots \times |M^\Gamma_\ell : (M^\Gamma_\ell)_{\gamma_\ell}|$$

and so

$$M_\omega = (M^\Gamma_1)_{\gamma_1} \times \cdots \times (M^\Gamma_\ell)_{\gamma_\ell} = (M_\omega \cap M^\Gamma_1) \times \cdots \times (M_\omega \cap M^\Gamma_\ell).$$

(d) Suppose, as above, that $\{\omega\} = \gamma_1 \cap \cdots \cap \gamma_\ell$ for some $\omega \in \Omega$ and $\gamma_i \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell$. As $\Gamma_1$ is a block-system for the action of $M$ on $\Omega$, the block $\gamma_1$ is stabilised by

$$K_1 = (M_\omega \cap M^\Gamma_1) \times \overline{M^\Gamma_1} = (M^\Gamma_1)_{\gamma_1} \times \overline{M^\Gamma_1}.$$
Theorem 3.2. Let $G$ be a permutation group on $\Omega$, $M$ a transitive, non-abelian, minimal normal subgroup of $G$, and let $\mathcal{E}$ be a normal $G$-invariant Cartesian decomposition of $\Omega$. Then $\mathcal{E}$ is $M$-normal and $G$ is transitive on $\mathcal{E}$.

Proof. Let $N$ be a normal subgroup of $G$, such that $\mathcal{E}$ is $N$-normal, and let $N_i = N^{\Gamma_i}$, where $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$. By the definition of $N$-normal, $N = N_1 \times \cdots \times N_\ell$. Suppose first that $M \leq N$. We claim that

$$M = (N_1 \cap M) \times \cdots \times (N_\ell \cap M).$$

Note that $N = N_1 \times \overline{N_1}$ and $M \leq N$. Since $M$ is a non-abelian minimal normal subgroup of $G$, $M$ is a direct product of isomorphic non-abelian simple groups. Let $T$ be a simple direct factor of $M$. For $i \in \{1, \ldots, \ell\}$ let $\sigma_{N_i}$ be the projection map $N \to N_i$. As $T$ is a non-trivial subgroup of $N$, there exists $i$ such that $\sigma_{N_i}(T)$ is non-trivial, whence $\sigma_{N_i}(T) \cong T$ as $T$ is a non-abelian simple group. We claim that $T \leq N_i$. Choose $x \in \sigma_{N_i}(T)$ with $x \neq 1$. As $M$ is a normal subgroup of $G$, the subgroup $T^x$ is also a minimal normal subgroup of $M$, and we see that either $T = T^x$ or $[T, T^x]$ is trivial. If the latter, then

$$1 = \sigma_{N_i}([T, T^x]) = [\sigma_{N_i}(T), (\sigma_{N_i}(T))^x] = [\sigma_{N_i}(T), \sigma_{N_i}(T)];$$

but the last term is non-trivial as $\sigma_{N_i}(T)$ is a non-abelian simple group. Hence the former holds, that is $T = T^x$. As $\sigma_{N_i}(T) \cong T$, if $t, t' \in T$ then $t^x = t'$ if and only if $\sigma_{N_i}(t)^x = \sigma_{N_i}(t')$. As $\sigma_{N_i}(T)$ is non-abelian simple, conjugation by $x$ induces a non-trivial automorphism of $\sigma_{N_i}(T)$. Thus it must induce a non-trivial automorphism of $T$ as well. However if $\sigma_{N_i}(T)$ is non-trivial for some $j \neq i$, then $x$ centralises $\sigma_{N_j}(T) \cong T$ and so, as shown by a similar argument, conjugation by $x$ induces a trivial automorphism of $T$. Thus $\sigma_{N_i}(T)$ is trivial for all $j \neq i$, and $T \leq N_i$ as claimed. Thus each simple direct factor of $M$ is contained in some $N_i$ and it follows that (2) holds. In this case, therefore, $\mathcal{E}$ is $M$-normal, so $M = \prod_{\Gamma_i \in \mathcal{E}} M^{\Gamma_i}$. Since $G$ is transitive on the simple direct factors of $M$, it follows that $G$ is transitive on $\mathcal{E}$.

Thus we may assume that $M \not\leq N$, so $M \cap N = 1$, by minimality of $M$. As both $M$ and $N$ are transitive, we have that they are both regular and, for a fixed $\omega \in \Omega$, the map $\vartheta : N \to M$ given by

$$x\vartheta = y^{-1} \text{ if and only if } \omega x = \omega y$$

is an isomorphism between $N$ and $M$. For each $i$ set $M_i = N_i \vartheta$ and $\overline{M_i} = N_i \vartheta$. Since $\vartheta$ is an isomorphism, $M = M_1 \times \cdots \times M_\ell$. Also by the definition of $\vartheta$, the $\overline{M_i}$-orbits are the same as the $N_i$-orbits, and by Lemma 3.1(d) the $\overline{M_i}$-orbits form the partition $\Gamma_i$. Thus $M^{\Gamma_i} = (M_i)^{\Gamma_i}$. If $K_i$ is the kernel of the action of $M_i$ on $\Gamma_i$, then $K_i \vartheta^{-1}$ (by the definition of $\vartheta$) also acts trivially on $\Gamma_i$, and hence lies in $N_i^{\Gamma_i} = N_i$. However $K_i \vartheta^{-1} \leq N_i$, so $K_i \vartheta^{-1} \leq N_i \cap \overline{N_i} = 1$, whence also $K_i = 1$. Hence $M^{\Gamma_i} = M_i$, and $M = \prod_{i=1}^\ell M^{\Gamma_i}$, so $\mathcal{E}$ is $M$-normal. As $M$ is a minimal normal subgroup of $G$, the conjugation action by $G$ on the $M^{\Gamma_i}$ is transitive, and hence $G$ is transitive on $\mathcal{E}$. □

Motivated partly by Theorem 3.2, we now introduce the class of innately transitive permutation groups.

A finite permutation group is said to be innately transitive if it has a transitive minimal normal subgroup (see [BamP04] for a comprehensive study of these groups). In particular, primitive and quasiprimitive groups are innately transitive. In [BPS05] we introduced six disjoint classes of transitive Cartesian decompositions that may be preserved by an innately transitive group. Theorem 3.2 has the following consequence: if $\mathcal{E}$ is a transitive $G$-invariant normal Cartesian decomposition, for an innately transitive group $G$, then $\mathcal{E}$ belongs to one of only two of the six classes in [BPS05], namely $\text{CD}_S(G)$ or $\text{CD}_1(G)$ (see [BPS05] for the notation). Moreover, the Cartesian decompositions in these two families are normal.

This simple observation will be refined somewhat in Theorem 6.1(d).

Suppose that $\mathcal{E}$ is a $G$-invariant Cartesian decomposition for some permutation group $G$. We say that $\mathcal{E}$ is a blow-up decomposition for $G$ if $\mathcal{E}$ is transitive and it is $M$-normal for some transitive normal subgroup $M$ of $G$ such that, for all $\Gamma \in \mathcal{E}$, we have $M^{\Gamma} = \text{Soc}(G^{\Gamma})$. 


The concept of a ‘blow-up’ is due to Kovács [Kov89a]. In the above we have simply translated his definition to the current context. The terminology ‘blow-up’ is intended to stress the intuitive idea that a permutation group $G$ on a set $\Omega$ with a blow-up decomposition $E$ is simply a ‘blown-up’ version of the smaller permutation group $G^\Gamma$ for $\Gamma \in E$; thus we talk of $G$ as being a ‘blow-up’ of its components. Likewise, if $E$ is a Cartesian decomposition, then there is a sense in which $G$ can be thought of as being constructed from its components (which are necessarily groups smaller than $G$), although in general the relationship between $G$ and its components is not as strong as in the blow-up case.

4. Components and blow-up decompositions in quasiprimitive groups

The aim of this section is to study the relationship between a quasiprimitive permutation group and its components under a blow-up decomposition.

**Lemma 4.1.** Let $G$ be a permutation group on a set $\Omega$ and suppose that $E$ is a blow-up decomposition for $G$ such that, for $\Gamma \in E$, $G^\Gamma$ is quasiprimitive on $\Gamma$. Suppose further that either $\text{Soc}(G^\Gamma)$ is non-abelian or $G$ is quasiprimitive. Then the rule $K \mapsto K^\Gamma$ defines a bijection from the set of minimal normal subgroups of $G$ to the set of minimal normal subgroups of $G^\Gamma$.

**Proof.** To prove this we adapt the argument preceding [Kov89a, (2.1)]. Set $H = G^\Gamma$, $M = \text{Soc}H$, and $\ell = |E|$. As explained in Section 3, we may assume without loss of generality that $G$ is a subgroup of the wreath product $W = H \wr S_\ell$ in its product action on $\Gamma^\ell$, and (since $E$ is a blow-up decomposition) that $G$ contains $M^\ell = (\text{Soc}H)^\ell$. Since $G^\Gamma$ is quasiprimitive, $\mathcal{C}_H(M) \leq M$. It follows easily that $\mathcal{C}_W(M^\ell) \leq M^\ell$. Thus each minimal normal subgroup of $G$ must lie in $M^\ell$, and hence must lie in $G_{(E)}$.

If $M$ is non-abelian, then $M$, and also $M^\ell$, is a direct product of non-abelian simple groups. Given that $M^\ell \leq G$, we deduce that each minimal normal subgroup of $G$ is a direct product of the $G$-conjugates of some simple direct factor of $M^\ell$. Since $G$ is transitive on $E$, the projection of $G$ onto the top group $S_\ell$ of the wreath product $W$ is a transitive subgroup of $S_\ell$. Thus each minimal normal subgroup $K$ of $G$ is of the form $K_0^\ell$, where $K_0$ is a characteristically simple normal subgroup of $H$. Moreover, the minimality of $K$ implies that $H = G^\Gamma$ is transitive on the simple direct factors of $K_0$, and so $K_0 = K^\Gamma$ is a minimal normal subgroup of $H$. Conversely for each minimal normal subgroup $K_0$ of $H$, $K_0^\ell$ is a minimal normal subgroup of $G$.

If $M$ is abelian then, by assumption, in this case, $G$ is quasiprimitive. Moreover $G$ has a minimal normal subgroup contained in $M^\ell$ that is abelian, and hence $G$ is quasiprimitive of type HA. This implies that $G$ is primitive. We now apply [Kov89a, (2.1)] directly to deduce that $\text{Soc}G = M^\ell$. As both $G$ and $H$ are now quasiprimitive with an abelian socle, they both have a unique minimal normal subgroup, namely $\text{Soc}G$ and $M$ respectively. This gives the required result. \[\square\]

**Corollary 4.2.** Suppose that $G$ is a permutation group on $\Omega$ and $E$ is a blow-up decomposition for $G$. Let $\Gamma_0 \in E$.

(i) If $G$ is quasiprimitive then the component $G^{\Gamma_0}$ is quasiprimitive and $\text{Soc}G = \prod_{\Gamma \in E} \text{Soc}(G^\Gamma)$.

(ii) If the component $G^{\Gamma_0}$ is quasiprimitive and $\text{Soc}(G^{\Gamma_0})$ is non-abelian, then $G$ is quasiprimitive.

**Proof.** (i) Set $H = G^{\Gamma_0}$ and $\ell = |E|$. We may assume that $G$ is a quasiprimitive subgroup of the wreath product $H \wr S_\ell$ in its product action on $(\Gamma_0)^\ell$. Since, $E$ is a blow-up decomposition, $G$ contains $(\text{Soc}H)^\ell$. If $H$ is not quasiprimitive, then there exists a minimal normal subgroup $K$ of $H$ with $K$ intransitive on $\Gamma_0$. Then $K^\ell \leq (\text{Soc}H)^\ell$ and $K^\ell$ is a normal subgroup of $H \wr S_\ell$ contained in $G$ and is intransitive on $(\Gamma_0)^\ell$. This contradicts the quasiprimitivity of $G$. Hence $H$ is quasiprimitive. Then by Lemma 4.1, for a minimal normal subgroup $K$ of $G$, $K^{\Gamma_0}$ is a minimal normal subgroup of $G^{\Gamma_0}$ which implies the assertion.
(ii) By Lemma 4.1, a minimal normal subgroup of $G$ is of the form $\prod_{\Gamma \in \mathcal{E}} K^\Gamma$ where, for each $\Gamma$, $K^\Gamma$ is a minimal normal subgroup of $G^\Gamma$. Since, for $\Gamma \in \mathcal{E}$, the group $G^\Gamma$ is quasiprimitive, $K^\Gamma$ is transitive on $\Gamma$, and hence $\prod_{\Gamma \in \mathcal{E}} K^\Gamma$ is transitive on $\Omega$. $\square$

We interpret the previous results as saying that, given a group $G$ with a blow-up decomposition, the structure of the socle of $G$ is strongly related to the structure of the socles of the components of $G$. Also there is a strong link between possible quasiprimity of $G$ and of its components. The following example shows the necessity of the restriction that $\text{Soc}(G^{T_0})$ cannot be abelian in Corollary 4.2(ii).

**Example 4.3.** Suppose that $H = \langle (1, 2, 3) \rangle$ acting on $\Gamma = \{1, 2, 3\}$ and let $G = H \downarrow D_8$ where $D_8$ is the dihedral group acting on the set $\{1, 2, 3, 4\}$ preserving the block system $\{\{1, 2\}, \{3, 4\}\}$. We consider $G$ as a permutation group acting in product action on $\Gamma^4$. Then $G^\Gamma = H$, and so $G^\Gamma$ is quasiprimitive with a unique minimal normal subgroup (namely itself). On the other hand $G$ has three minimal normal subgroups

$$
M_1 = \{(x, x, x, x) \mid x \in H\}; \\
M_2 = \{(x, x, x^2, x^2) \mid x \in H\}; \\
M_3 = \{(x, y, y^2, x) \mid x, y \in H\}.
$$

and $(M_1)^\Gamma = (M_2)^\Gamma = (M_3)^\Gamma = H$. So $G$ is not quasiprimitive, and hence the condition in Corollary 4.2(ii) that $\text{Soc}(G^{T_0})$ is non-abelian is necessary. This example also shows that the correspondence $M \mapsto M^\Gamma$ in Lemma 4.1 is not always one-to-one if $\text{Soc}(G^\Gamma)$ is abelian but $G$ is not quasiprimitive.

The statement of the primitive analogue of Corollary 4.2, can be obtained by replacing all occurrences of “quasiprimitive” by “primitive”, and the single occurrence of “non-abelian” by “non-regular” in the statements of these theorems. The validity of these analogues follows from [Kov89a, Theorem 1 and (2.1)]. Given that for primitive groups the restriction to non-regular socle is a stronger condition than the restriction to non-abelian socle, we see that the concept of a blow-up in fact behaves better with respect to quasiprimity than it does to primitivity.

**5. Normal decompositions and blow-up decompositions**

The observant reader may ask whether, for a permutation group $G$, it is possible that a transitive, $G$-invariant, normal decomposition with quasiprimitive components is not a blow-up. This question is answered by the results of this section.

Suppose that $G \leq \text{Sym}\Omega$ is a finite permutation group with a non-abelian, non-simple, regular, minimal normal subgroup $M_1$. The centraliser $C_{\text{Sym}\Omega}(M_1)$ is isomorphic to $M_1$, and so it is isomorphic to $T^k$ where $T$ is a non-abelian finite simple group, and $k \geq 2$. If $C_G(M_1)$ is a proper subdirect subgroup of $C_{\text{Sym}\Omega}(M_1)$, then, using the terminology of [BamP04], $G$ is said to be an *innately transitive group of diagonal quotient type*. In this case $G$ has two minimal normal subgroups $M_1$ and $N_1$ where $M_1 \cong T^k$ and $N_1 \cong T^{k/m}$ for some divisor $m$ of $k$ such that $1 < m < k$. Further, $N_1$ is semiregular and intransitive, and, in particular, $G$ is not quasiprimitive.

**Theorem 5.1.** Let $G$ be a permutation group on a set $\Omega$ and suppose that $\mathcal{E}$ is a transitive, normal, $G$-invariant Cartesian decomposition of $\Omega$, and that, for $\Gamma \in \mathcal{E}$, the component $G^\Gamma$ is quasiprimitive. Then exactly one of the following possibilities holds.

1. $\mathcal{E}$ is a blow-up decomposition.
2. $G$ is quasiprimitive of type $\Gamma W$ and $G^\Gamma$ is primitive of type HS or HC.
3. $G$ is innately transitive with diagonal quotient type and $G^\Gamma$ is primitive of type HC or HS.

**Proof.** Set $H = G^\Gamma$ and $\ell = |\mathcal{E}|$. We may assume that $G$ is a subgroup of $H \downarrow S_\ell$ in its product action on $\Gamma^\ell$. As $\mathcal{E}$ is a normal Cartesian decomposition, there exists a transitive normal subgroup $M$ of $G$ such
that \( M \leq G(E) \leq H^\ell \) and \( M = (M^\Gamma)^\ell \). If \( \text{Soc} \, H \leq M^\Gamma \), then by the transitivity of \( G \) on \( E \), we have \( (\text{Soc} \, H)^\ell \leq G \). Hence \( E \) is a blow-up decomposition and part (i) holds. Assume now that \( \text{Soc} \, H \not\leq M^\Gamma \).

As \( M^\Gamma \) is normal in \( H \), it follows that \( H \) has at least two minimal normal subgroups. As discussed in Section 2, the group \( H \) is primitive of type HS or HC, and \( H \) has exactly two minimal normal subgroups \( M_1, N_1 \), both regular, non-abelian and transitive on \( \Gamma \). Without loss of generality we may assume that \( M_1 \leq M^\Gamma \). Then \( (M_1)^\ell \leq G \) and therefore, by definition, \( E \) is \( (M_1)^\ell \)-normal, so we may assume that \( M = (M_1)^\ell \). Thus \( M \) is a regular non-abelian minimal normal subgroup of \( G \). Let \( C = \mathbb{C}_{\text{Sym} \, \Gamma}(M_1) \) and note that \( \text{Soc} \, G \leq M \times C \). By [DM96, Theorem 4.3B], \( C \cong M \). Since \( N_1 = \mathbb{C}_{\text{Sym} \, \Gamma}(M_1) \leq H \), we have \( C = (N_1)^\ell \leq H \wr S_\ell \); therefore \( \text{Soc} \, (H \wr S_\ell) = M \times C \). We may write \( C = \prod_{x \in S} T_x \), where \( S \) is a set of size \( k\ell \), and each \( T_x \) is isomorphic to a non-abelian simple group \( T \). As \( H = G^\Gamma \) is transitive on the set of minimal normal subgroups of \( N_1 = C^\Gamma \) and \( G \) is transitive on \( E \), we have that \( G \) induces a transitive permutation group on \( S \) of degree \( k\ell \). If \( C \leq G \) then \( \text{Soc} \, G = M \times C \), and so \( G \) is quasiprimitive with two minimal normal subgroups. In this case \( \text{Soc} \, (G^\Gamma) = M_1 \times N_1 \) and \( (\text{Soc} \, G^\Gamma)^\ell = M \times C \leq G \).

Therefore \( E \) is a blow-up decomposition and part (i) holds. If \( C \cap G = 1 \) then \( G \) has a unique minimal normal subgroup, which is regular. Therefore \( G \) is quasiprimitive of type \( Tw \) and part (ii) holds.

Thus we may assume that \( 1 < C \cap G < C \). In this case \( 1 \neq (C \cap G)^\ell \leq G^\ell \) and \( (C \cap G)^\ell \leq G^\Gamma \). Since \( G^\Gamma \) is quasiprimitive, \( (C \cap G)^\Gamma \) must be transitive. Recall that \( G^\Gamma \) is regular. Thus \( (C \cap G)^\Gamma = G^\Gamma \cong T^k \), and also \( N = C \cap G = \mathbb{C}_G(M) \neq 1 \). Therefore, \( N \) is a proper subdirect subgroup of \( C \) where \( C \) viewed as a direct product of its minimal normal subgroups, and hence \( N \) is a direct product of full diagonal subgroups. Thus \( G \) is innately transitive with diagonal quotient type.

These possibilities are mutually exclusive. For if \( E \) is a blow-up and \( \text{Soc} \, G \) is non-abelian then the other two possibilities cannot occur, by Corollary 4.2. On the other hand, if \( \text{Soc} \, G \) is abelian then only possibility (i) can occur. \( \square \)

We construct an example to show that the situation described by Theorem 5.1(iii) is possible.

**Example 5.2.** Let \( T \) be a non-abelian finite simple group, let \( H \) be any subgroup of the holomorph

\[
\text{Hol}(T^k) = T^k \rtimes \text{Aut}(T^k) = T^k \rtimes (\text{Aut}(T) \wr S_\ell)
\]

such that \( H \) has two minimal normal subgroups \( M_1 \) and \( N_1 \) where \( M_1 \cong N_1 \cong T^k \). Then \( H \), considered as a permutation group on \( \Gamma = T^k \), is a primitive group of type HS if \( k = 1 \) or type HC if \( k \geq 2 \). Let \( \ell \geq 2 \), and let \( G \) be a subgroup of \( H \wr S_\ell \) in its product action on \( \Omega = \Gamma^\ell \), such that \( G \) contains \( M_1^\ell \cong (T^k)^\ell \), a regular normal subgroup, and \( \hat{G} = M_1^\ell (H^\delta \times S_\ell) \) where \( \delta \) is the diagonal embedding \( \delta : H \to H^\ell \) defined by \( h^\delta = (h, \ldots, h) \). Then \( \text{Soc} \, \hat{G} = M_1^\ell \times N_1^\ell \delta \), and, in addition, \( N_1 \delta \) is a semiregular and intransitive minimal normal subgroup of \( G \). Thus \( G \) is innately transitive with diagonal quotient type in its action on \( \Gamma^\ell \). Moreover, the component of \( G \) induced on \( \Gamma \) is \( H \), which is primitive.

Theorem 5.1 demonstrates that the quasiprimitivity of the components of a transitive normal Cartesian decomposition for \( G \) often implies that \( G \) also is quasiprimitive. More precisely, the following result is valid.

**Corollary 5.3.** Suppose that \( G \leq \text{Sym} \, \Omega \) is a permutation group and \( \mathcal{E} \) is a transitive, normal \( G \)-invariant Cartesian decomposition of \( \Omega \) such that, for \( \Gamma \in \mathcal{E} \), the component \( G^\Gamma \) is quasiprimitive but not of type HA, HS or HC. Then \( \mathcal{E} \) is a blow-up decomposition and \( G \) is a quasiprimitive group.

**Proof.** As the type of a component is not HS or HC, it follows from Theorem 5.1 that \( \mathcal{E} \) is a blow-up decomposition. Then Corollary 4.2(ii) implies that \( G \) is quasiprimitive. \( \square \)

Next in this section we present a sufficient and necessary condition to decide if a transitive Cartesian decomposition is a blow-up decomposition.
Theorem 5.4. Let $G$ be a quasiprimitive permutation group acting on a set $\Omega$ and let $E$ be a transitive $G$-invariant Cartesian decomposition of $\Omega$. Then $E$ is a blow-up decomposition if and only if $E$ is $(\text{Soc } G)$-normal and, for $\Gamma \in E$, $C_{G^\Gamma}((\text{Soc } G)^\Gamma) \leq (\text{Soc } G)^\Gamma$.

Proof. Set $H = G^\Gamma$, $\ell = |E|$, and $M = \text{Soc } G$. We may assume that $G$ is a subgroup of $H \wr S_\ell$ in its product action on $\Gamma^\ell$. If $E$ is a blow-up decomposition, then, by Corollary 4.2(i), $M = (\text{Soc } H)^\ell$. In particular, $E$ is $M$-normal and $M^\Gamma = \text{Soc } H$. Furthermore, Corollary 4.2(i) implies that the component $H$ is quasiprimitive whence we have $C_H(M^\Gamma) \leq M^\Gamma$ as required.

Conversely, assume that $E$ is $M$-normal, and that the centraliser $C_H(M^\Gamma)$ is contained in $M^\Gamma$. By the definition of a blow-up it is enough to show that $M^\Gamma = \text{Soc } H$. If $M$ is abelian then $G$ is of type $\text{HA}$ and $M$ is the unique minimal normal subgroup of $G$. We also know that $M$ is regular and elementary abelian, hence so is $M^\Gamma$. As $M^\Gamma \leq H$ we must have that $H \leq \text{Hol}(M^\Gamma)$, the normaliser of $M^\Gamma$ in $\text{Sym } \Gamma$. The subgroup $M^\Gamma$ must be a minimal normal subgroup of $H$, for if $N < M^\Gamma$ and $N \leq H$, then $N^\Gamma < M$ is a normal subgroup of $G$ properly contained in $M$, which is impossible by the minimality of $M$. Therefore $H$ is primitive of type $\text{HA}$ on $\Gamma$ and $M^\Gamma = \text{Soc } H$.

Thus we may assume that $M$ is non-abelian. Then $M^\Gamma$, which is a homomorphic image of $M$, is the direct product of non-abelian simple groups and so $M^\Gamma$ is necessarily a direct product of minimal normal subgroups of $H$. However, as $C_H(M^\Gamma) \leq M^\Gamma$ it follows that $M^\Gamma$ must contain all minimal normal subgroups of $H$, whence $M^\Gamma = \text{Soc } H$ as required. $\square$

We end this section with an example to show that a quasiprimitive group may have non-quasiprimitive components with respect to a normal Cartesian decomposition.

Example 5.5. Let $T$ be a non-abelian finite simple group and let $P$ be a finite group with a core-free subgroup $Q$ and a homomorphism $\varphi : Q \to \text{Aut } T$ such that $\varphi$ induces a non-trivial, proper subgroup $R$ of $\text{Inn } T$. Then the twisted wreath product $W = T \varphi \wr Q$ is a quasiprimitive permutation group acting on $\Omega = T^{P\wr Q}$; see [Bad93]. Further, the natural Cartesian decomposition of $\Omega$ is clearly $(\text{Soc } W)$-normal. However, a component of $W$ has a regular minimal normal subgroup isomorphic to $T$, and an intransitive normal subgroup isomorphic to $R$. Thus a component of $W$ is not quasiprimitive.

6. Inclusions of Quasiprimitive Groups

In [BadP03] the first two authors studied inclusions of quasiprimitive groups into primitive ones. The description of such inclusions in the case when the primitive group has type $\text{PA}$ relied on the following theorem, which was stated without proof in [BadP03, Theorem 4.7]. Here we give the first published proof. Recall that a Cartesian decomposition $E$ is homogeneous if $|\Gamma|$ is the same for all $\Gamma \in E$.

Theorem 6.1. If $G \leq \text{Sym } \Omega$ is a transitive permutation group, $E$ is a homogeneous $G$-invariant Cartesian decomposition of $\Omega$, and $\Gamma_0 \in E$, then the following all hold.

(a) If $E$ is a blow-up decomposition and $G$ is quasiprimitive on $\Omega$ then the component $G^{\Gamma_0}$ is quasiprimitive and $\text{Soc } G = \prod_{\Gamma \in E} \text{Soc } (G^\Gamma)$.

(b) If $E$ is a blow-up decomposition and the component $G^{\Gamma_0}$ is quasiprimitive on $\Gamma_0$ not of type $\text{HA}$ then $G$ is quasiprimitive on $\Omega$ and $\text{Soc } G = \prod_{\Gamma \in E} \text{Soc } (G^\Gamma)$.

(c) The group $G$ is not quasiprimitive of type $\text{SD}$.

(d) If $G$ is quasiprimitive of type $\text{CD}$ then $E$ is a blow-up decomposition and the component $G^{\Gamma_0}$ is quasiprimitive of type $\text{SD}$ or $\text{CD}$.

Proof. Parts (a) and (b) follow from Corollary 4.2. By [BPS05, Corollary 1.3], quasiprimitive groups of type $\text{SD}$ do not preserve Cartesian decompositions, and so part (c) also holds.

Let us now prove part (d). Suppose that $G$ is a quasiprimitive group of type $\text{CD}$. Let $M$ denote the socle of $G$. Then $M$ is a minimal normal subgroup of $G$. Hence $M$ is a non-abelian characteristically simple group and so $M = T^k$ where $T$ is a non-abelian finite simple group. Moreover, a point stabiliser
\( M_\omega \) is a subdirect subgroup of \( M \). It follows from [BPS05, Theorem 1.2(c)] that \( \mathcal{E} \) is \( M \)-normal and that \( G \) is transitive on \( \mathcal{E} \). Thus \( M \) may be written as \( M = \prod_{\Gamma \in \mathcal{E}} M^\Gamma \). Further, for \( \Gamma \in \mathcal{E} \), the subgroups \( M^\Gamma \) are permuted by \( G \). By Lemma 3.1, the partition \( \Gamma \) is the set of \( M^\Gamma \)-orbits of \( \Omega \), and so \( G_{\Gamma} = N_G(M^\Gamma) = N_G(M^\Gamma) \). The subgroup \( M^\Gamma \) is a transitive minimal normal subgroup of \( G^\Gamma \). As a point stabiliser in \( M^\Gamma \) is a subdirect subgroup, we obtain from [BamP04, Proposition 5.5] that \( G^\Gamma \) is quasiprimitive of type \( Sd \) or \( Cd \). Therefore Theorem 5.1 implies that \( \mathcal{E} \) is a blow-up decomposition. □

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