IMPRESSED BOUNDS FOR SOME ASYMPTOTIC FORMULAS FOR COUNTING WORDS IN SHIFT SPACES

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ABSTRACT. This paper studies a version of the counting problem in dynamical systems that is of interest, especially in conformal dynamical systems where the functions of the systems are angle preserving. Recently, M. Pollicott and M. Urbański published a result in this context for D-generic systems where the complex transfer operator behaves nicely on the critical line of the Poincaré series. Their result contains an asymptotic formula for the Apollonian circle packing. We lift the D-generic condition and conformality of the functions system in this paper to see how their asymptotic formula changes. We use some recent Tauberian theorem to show that the formula gets a form whose limit infimum and limit supremum bounds can be obtained in the sharpest sense. Further, we observed an asymptotic of length closely related to this counting problem. In fact, not only the number of words is subject to some formula, but also their length as well.

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1. INTRODUCTION

The counting problem in math has a long history dating as far back as the Gauss circle problem. Gauss tried to obtain an asymptotic formula for the number of points in the plane with integer coordinates inside a circle of radius \( T \) as \( T \) grows. Later on, Sierpinski, Walfisz, Iwaniec & Mozzochi [IM88], Huxley [Hux90], Hardy [Ivi85, p. 372], Landau and Hafner [Haf81] contributed to problems closely related to the Gauss circle problem to obtain better estimates.

The analogous problem in the context of hyperbolic spaces as well gained a lot of attention starting in 1942 with (unnoticed) work of Delsarte, where he considered the hyperbolic plane \( \mathbb{H}^2 \) and instead of \( \mathbb{Z}^2 \) he considered orbit of a point \( z \in \mathbb{H}^2 \) under the action of a Fuchsian group \( G \subseteq \text{PSL}(2, \mathbb{R}) \). He obtained an asymptotic formula for the number of \( g \in G \) that moves \( z \) at most by \( T \) as \( T \) grows. Here the distance is measured by a hyperbolic metric of constant negative curvature [Del42]. Independently, Huber published his result on this problem in 1956. His approach uses spectral decomposition of the Laplacian operator, where \( G \) doesn’t contain parabolic elements because he assumes the fundamental domain is compact [Hub56]. In the same year, Selberg extended this decomposition for the case \( G \) contains parabolic elements where the fundamental domain has a finite area. He used the celebrated trace formula for this [Sel56, p. 77]. This helped Patterson to approach the problem in generality providing some error terms as well [Pat75]. Along with these works, Margulis answered a similar question in higher dimensional hyperbolic space in 1969 [Mar04, p. 48]. Several others have contributed to this problem in different contexts including Sarnak [Sar81], Lax & Phillips [LP82], Parry & Pollicott [PP83], Lalley [Lal87], Mirzakhani [Mir08] and etc.

Recently, Pollicott & Urbański jointly obtained an asymptotic formula in the context of conformal dynamical systems, see corollary 2 or [PU21, p. 39]. For this, it is enough to have graph directed Markov system in which our functions in the system are contractions and satisfy certain properties. The most important one is the conformal property which is angle preserving orientation preserving or orientation reversing. They use the infinite theory of graph directed Markov system developed by Mauldin & Urbański [MU03] and complex transfer operator developed by Pollicott [Pol84] to obtain an asymptotic formula for counting finite words in...
the shift space for which the corresponding composition function of the system has derivative at least $e^{-T}$ as $T$ grows. Further, they introduce a slightly different system in which finitely many parabolic elements are allowed and they apply the aforementioned asymptotic formula for this system. These two kinds of conformal systems have many applications one of which is an asymptotic formula for the planer Apollonian circle packing problem. The circle packing was studied in the 1970s by Boyd [Boy73] and estimates of the number of circles of radius at 1/T were obtained by him in the 1980s [Boy82]. This estimate had major improvement due to Kontorovich & Oh in 2011 [KO11] and Oh & Shah in 2012 [OS12]. The former article focuses on two cases: (a) the number of circles of radius at least 1/T inside the biggest circle tangent to the three circles that generate the gasket, and (b) the number of circles of radius at least 1/T between two parallel lines generating the gasket up to a period of the gasket. One year later, the latter article obtains a similar formula for case (c) number of circles of radius at least 1/T bounded in a curvilinear triangle whose sides are parts of three circles tangent to each other. The method for Kontorovich-Oh-Shah is equidistribution of expanding closed horospheres on hyperbolic 3-manifolds $G\backslash \mathbb{H}^3$ where $G$ is a geometrically finite torsion-free discrete subgroup of PSL(2,$\mathbb{C}$). Further, they use Patterson-Sullivan theory of conformal density (measure) in which the Laplacian operator has simple isolated eigenvalue $-\delta_G(2 - \delta_G)$ where $\delta_G$ is the Hausdorff dimension of the limit set under the assumption $\delta_G > 1$ [Sul79, p. 195], [Pat76, p. 272].

In Pollicott & Urbański’s work the spectral theory is analyzed for the transfer operator instead, where they assume their system has D-generic property which prevents the situation that the transfer operator admitting 1 as the spectral value on the critical line Re$(s) = \delta$ of the Poincaré series except at the exponent itself $s = \delta$. The other condition they impose on the system is strong regularity which can be perceived to be analogous to the assumption $\delta_G > 1$ mentioned above.

In this paper, we relax the D-generic assumption to see how Pollicott & Urbański’s result changes, see theorem 5. We noticed that in this situation, we no longer obtain only one asymptotic formula. We may obtain continuum many relations. More precisely we can see that the ratio can converge to a full range of a closed interval rather than just a point in Pollicott & Urbański’s result, see example 4. However, we can obtain a lower bound for the infimum and an upper bound for the supremum. These bounds are shown to be sharp by an example, see example 4. We should mention that we only assume we are given a real-valued summable Hölder-type function on the shift space. We don’t assume necessarily the function is induced by a conformal system. The main result (theorem 5) involves spectral analysis of the transfer operator which we adapt from Pollicott & Urbański and a Tauberian theorem 4 due to Graham & Vaaler. Further, we investigate an asymptotic of the length for which the counting function is related to. Given $T > 0$ the maximum length contributing to the counting function is itself subject to an asymptotic formula, see proposition 19.

We use a similar approach to that of Pollicott & Urbański but we include much more details. We mention proof of important facts used by Pollicott & Urbański in section 3. Not only, do we bring the proofs but also we adapt it for our own setting lifting conformality. As well, we reprove some of the inequalities in section 5 concerning counting finite words just for the purpose of giving a clearer and shorter proof. Furthermore, we include many preliminaries that some experts can skip over. One major reason we decided to write this detailed paper is mostly for the purpose of having a self-contained article. The other reason is for the paper following this paper which targets the counting problem in the context of random dynamical systems. This way we can simply refer to any desired (deterministic) preliminaries here.

About the structure of the article, we start with some preliminaries from dynamical systems over symbolic space like pressure, Gibbs, and equilibrium state. This further includes topics like real or complex-valued summable functions, Hölder functions. Moreover, we define properties like strong regularity and D-generic property which are the main assumptions in Pollicott & Urbański’s formula. We introduce the transfer operator and talk a bit about the perturbation theory of analytic operators. This requires the concept of essential spectrum. Later, we bring notions of graph directed Markov systems and conformal graph directed Markov systems. Finally, we briefly mention two Tauberian theorems before we finish the section with some examples.

The next section is devoted to applying the perturbation theory to the transfer operator to obtain a spectral representation of the transfer operator over its maximal eigenvalues. We show first eigenvalues are simple beforehand though. This requires introducing a weighted operator involving the transfer operator. The fourth section is talking about the relation between a complex function (Poincaré series), and some counting functions. The idea is by taking the Riemann-Stieltjes integral against our target counting function, we acquire a Poincaré series. Further, we bring some estimates to find some upper and lower bounds for counting periodic words in terms of our ordinary counting function.

In the fifth section, we use the spectral representation from the third section to argue how Graham & Vaaler’s Tauberian theorem is applicable to imply the main theorem. We use this theorem to obtain Pollicott & Urbański’s
One can as well introduce a metric by
\[ d(\rho, \rho') = e^{-|\rho \wedge \rho'|}. \]

Further, we set
\[ d_a = d^a, \quad 0 < a < 1, \]
i.e. we have
\[ d_a(\rho, \rho') = e^{-|\rho \wedge \rho'|}. \]

Therefore we equip \( E^N \) with a metric space, which is called symbolic space. Note that the topology on \( E^N \) induced by this metric is the same as the Tychonoff topology where each \( E \) is equipped with ordinary discrete topology. This means for any \( a \) and \( \beta \) the topologies of \( d_a \) and \( d_\beta \) are the same, however, the metrics are not equivalent for different \( a \) and \( \beta \).

One can then see that the shift map \( \sigma : E^N \to E^N \) given by
\[ \sigma(e_1e_2...) = e_2e_3... \]
is a continuous map.

Further, we want to restrict ourselves to sequences where certain words are not appearing. We first introduce a map \( A : E \times E \to \{0, 1\} \) (sometimes called incidence or transition matrix). We use \( A_{e,e'} \) notation instead of

\[ \text{RELIMINARIES} \]

2. Preliminaries

We would like to mention that throughout this paper we try to stay loyal to the following conventions:
- \( \omega, \tau, \gamma : \) finite words
- \( \rho, \rho' : \) infinite words
- \( T : \) positive real
- \( s = x + iy : \) complex number
- \( \sigma : \) shift map
- \( m, \mu : \) measures
- \( f, g, h : \) real or complex functions on \( E^N \)
- \( 1_B : \) indicator function of set \( B \)
- \( C^{0,\alpha} : \) space of Hölder functions of exponent \( \alpha \)
- \( K, Q, c, c_1, C_1, \epsilon_\beta : \) constants
- \( \mathcal{L}, \mathcal{P}, Q, D, F, \mathcal{E} : \) operators
- \( \lambda : \) eigenvalue
- \( \mathbb{F}^+ : \) some right half-plane
- \( \eta_\rho(B, s) : \) complex function in \( s \)
- \( \mathcal{N}_\rho(B, T) : \) counting function in \( T \)

Let \( E \) be a countable (finite or infinite) set calling each of its elements a symbol, a letter or an alphabet. By \( E^N \) we mean the set of all infinite sequences of the form
\[ e_1e_2e_3...e_n... \]
where each \( e_i \) belongs to \( E \). We usually represent the first \( n \) symbols of such a sequence, also called (finite) word or block, by \( \omega \) throughout this work, i.e.
\[ \omega = e_1e_2...e_n \]
where we sometimes tend to identify \( \omega_i \) with \( e_i \) and just have
\[ \omega = \omega_0\omega_1\omega_2...\omega_n. \]

When we write \(|\omega| = n\) we just mean the word \( \omega \) has \( n \) letters. By \( E^n \) we represent all the words of length \( n \) and by \( E^\infty \) we represent \( \bigcup_{n=1}^\infty E^n \). As well we use the notation \( | \cdot, \wedge, | \) to represent the number of common initial symbols in two sequences, i.e. for \( \rho = e_1e_2... \) and \( \rho' = e'_1e'_2... \) we have
\[ |\rho \wedge \rho'| = m \Leftrightarrow e_1 = e'_1, e_2 = e'_2, ..., e_m = e'_m, e_{m+1} \neq e'_{m+1}. \]

One can as well introduce a metric by
\[ d(\rho, \rho') = e^{-|\rho \wedge \rho'|}. \]

Further, we set
\[ d_a = d^a, \quad 0 < a < 1, \]
i.e. we have
\[ d_a(\rho, \rho') = e^{-a|\rho \wedge \rho'|}. \]

The sixth section investigates an asymptotic formula for the length of the words that contribute to the counting function. We provide two asymptotic formulas for length and we propose a conjecture about the growth of the counting function relative to another counting function with some specified length. We finish with three examples to see how our estimates of bounds are sharp in the last section.
A subshift of finite type consists of the sequences \( e_1 e_2 e_3 \ldots \) in \( E_N \) such that
\[
A_{e_1 e_2} = 1, \ A_{e_2 e_3} = 1, \ \ldots, \ A_{e_n e_{n+1}} = 1, \ \ldots
\]

Of course, if \( A \) only assumes value 1, represented by \( A = 1 \), then this is just the space introduced earlier, that is why we sometimes call \((E_N, \sigma)\) full shift space.

Further, when \( A_{e_1 e_2} = 1 \) we say \( e_1 e_2 \) is \( A \)-admissible or just admissible. As well, by \( E_A^* \) we mean all admissible finite words of all lengths, by \( E_A^n \) we mean all \( \omega \in E_A^n \) such that \( \omega \sigma \) is an admissible sequence, by \( E_A^\perp \) we mean all \( \omega \in E_A^n \) such that \( \omega \sigma \) is admissible and we say \( \omega \) is periodic word, by \( \omega \) we mean the sequence \( \omega \omega \omega \omega \ldots \) and by \( E_A^\perp \) we mean all admissible words of length \( n \). Finally, for each finite word \( \omega \) of length \( n \) we define the cylinder
\[
[\omega] := \{ \rho \in E_A^N : \rho_1 \ldots \rho_n = \omega \}.
\]

**Proposition 1.** For the subshift of finite type \( E_A^N \) the followings hold:

a. All the cylinders form a countable clopen basis.

b. Every open set can be written as a countable union of mutually disjoint cylinders.

c. It is a Polish space.

**Proof.** (a). It is clear that for each positive integer \( n \), we have countably many finite words of length \( n \), therefore there are only countably many cylinders. Next, we show each cylinder is a neighborhood in \( E_A^N \). Let \( \omega \) be a finite word of length \( n \), choose any fixed \( \rho \in [\omega] \), we show \([\omega] = N(\rho, e^{-a(n-1)})\). Note that \( \rho' \) is in \([\omega]\) iff \( d_\omega(\rho, \rho') < e^{-a(n-1)} \) iff \( |p \land \rho'| > n - 1 \) iff \( |p \land \rho'| \geq n \) iff \( \rho' \in [\omega] \). To see \([\omega]\) is closed, consider a sequence \( \{\rho_j\} \in [\omega] \) converging to \( \rho \). This means \( |\rho_j \land \rho| \to \infty \) which clearly implies \( \rho \in [\omega] \). Now for every open set \( V \) and every \( \rho \in V \), note that there is \( \epsilon > 0 \) such that \( \rho \in N(\rho, \epsilon) \subset V \). We choose \( n \) large enough such that \( e^{-a(n-1)} < \epsilon \), then obviously \( [\rho_1 \rho_2 \ldots \rho_n] = N(\rho, e^{-a(n-1)}) \subset N(\rho, \epsilon) \subset V \).

(b). The fact that an open \( V \) can be written as a countable union of cylinders is clear from part a. Then part b follows from the fact that for any two cylinders \([\omega]\) and \([\tau]\) that meet each other, we have either \([\omega] \subset [\tau]\) or \([\tau] \subset [\omega]\). To show this, assume \( \rho \) belongs to both of the cylinders \([\omega]\) and \([\tau]\). Further, assume \( |\omega| \leq |\tau| \). Since \( \rho \in [\tau] \), we should have \( \rho = \tau \rho' \) for some \( \rho' \in E_A^n \), similarly \( \rho \in [\omega] \) implies that \( \rho = \omega \rho'' \) for some \( \rho'' \in E_A^n \). Thus \( \tau \rho' = \rho = \omega \rho'' \) and since \(|\omega| \leq |\tau| \) so \( \tau = \omega \omega' \) for some finite word \( \omega' \). This implies \([\tau] \subset [\omega]\).

(c). Note that a countable product of separable spaces is separable and a countable product of complete metrizable spaces is complete metrizable.

We would like to mention that we only work with probability measures over Borel sets all through this work.

**Definition 1.** For a measurable transformation \( T : X \to X \) on a measure space \((X, \mathcal{B})\) we say a measure \( \mu \) is \( T \)-invariant if for every \( A \in \mathcal{B} \):
\[
\mu(T^{-1}(A)) = \mu(A).
\]
Further we say \( \mu \) is ergodic if \( \mu \) is \( T \)-invariant measure such that if \( T^{-1}(A) = A \) then either \( \mu(A) = 0 \) or \( \mu(A) = 1 \).

**Definition 2.** We call a subshift finitely irreducible if there exists a finite set \( \Omega \) containing words such that for all \( e, e' \in E \) there is \( \omega \in \Omega \) such that \( eoe' \) is admissible. As well subshift is called finitely primitive if it is finitely irreducible and all words in \( \Omega \) are of fixed length.

Throughout this paper, we restrict ourselves to work with finitely irreducible subshifts.

**Remark 1.** Note that this notion is just a generalization of an irreducible matrix when \( E \) is countable. In fact, a finitely irreducible condition guarantees that the shift map is topologically mixing, and a finitely primitive guarantees that the shift map is topologically exact. Additionally, it is clear that if the shift space is finitely irreducible then the backward orbit of every element is dense, i.e.
\[
\bigcup_{n=0}^\infty \sigma^{-n}(\rho) = E_A^N.
\]

**Proposition 2.** If \( E \) is finite,
\[
\log r(A) = \lim_{n \to \infty} \frac{1}{n} \log \#E_A^n,
\]
where \( r(A) \) is spectral radius of matrix \( A \).

**Proof.** We refer to theorem 3.2.22 [URM21].
Next, we want to talk about the Hölder continuous maps. In Analysis textbooks [GT01, p. 52] we have different notions of Hölder continuity of exponent $\alpha$ for real or complex-valued functions on a Euclidean space $D$:

- Hölder at a point $x_0$: $\sup_{x \in U} \{|f(x) - f(x_0)|/|x - x_0|^\alpha\}$ is finite, where $U$ is a neighborhood of $x_0$ in $D$.
- Hölder: $\sup_{x \in D} \{|f(x) - f(y)|/|x - y|^\alpha\}$ is finite.
- Locally Hölder: $\sup_{x,y \in K} \{|f(x) - f(y)|/|x - y|^\alpha\}$ is finite for every compact $K \subseteq D$.

We call each of the above suprema the Hölder coefficient. Of course, $D$ can be replaced with the metric space $E^N$ to obtain similar notions on the shift space. We denote the set of complex-valued Hölder continuous functions of Hölder exponent $\alpha$ on $E^N$ by $C^{0,\alpha}(E^N, \mathbb{C})$ or simply $C^{0,\alpha}$. We remind that the usual Hölder coefficient is defined by:

$$|g|_\alpha = \sup_{\rho,\rho' \in E^N} \left\{ \frac{|g(\rho) - g(\rho')|}{d(\rho,\rho')} \right\}.$$ 

We would like to define another Hölder coefficient that is justified later. We set:

$$V_{a,n}(f) := \sup \{|f(\rho_1) - f(\rho_2)|e^{\alpha(n-1)} : |\rho_1 \wedge \rho_2| \geq n \geq 1\},$$

and

$$V_a(f) := \sup_{n \geq 1} V_{a,n}(f).$$

There is another notion of Hölder continuity useful for our purposes.

**Definition 3.** A complex-valued function $f$ on $E^N$ is called Hölder-type continuous with exponent $\alpha > 0$ if $V_a(f) < \infty$.

We define a norm on $C^{0,\alpha}(E^N, \mathbb{C})$ by

$$(1) \quad \|g\|_\alpha := \|g\|_{\infty} + V_a(g).$$

We are ready to find relations between these different notions of Hölder continuity.

**Proposition 3.** The followings hold:

(a) On $E^N$ every complex-valued function is Hölder continuous iff it is Hölder-type continuous and bounded.

(b) The norm given above in (1) is equivalent to usual $\|\cdot\|_{C^{0,\alpha}} = \|\cdot\|_{\infty} + \|\cdot\|_\alpha$ norm over $C^{0,\alpha}(E^N, \mathbb{C})$.

(c) $(C^{0,\alpha}(E^N, \mathbb{C}), \|\cdot\|_\alpha)$ is Banach space.

(d) A Hölder-type continuous function is locally Hölder continuous and Hölder continuous at every point.

**Proof.**

a) Assume $f$ is Hölder continuous function, then there is $M$ such that

$$|f(\rho_1) - f(\rho_2)| \leq M d(\rho_1, \rho_2)^\alpha = M e^{-\alpha|\rho_1 \wedge \rho_2|},$$

for every $\rho_1$ and $\rho_2$. Therefore

$$|f(\rho_1)| \leq |f(\rho_1) - f(\rho_2)| + |f(\rho_2)| \leq M + |f(\rho_2)|.$$ 

This gives boundedness of $f$. For Hölder-type, assuming $|\rho_1 \wedge \rho_2| \geq n$, it follows

$$|f(\rho_1) - f(\rho_2)|e^{\alpha(n-1)} \leq Me^{-\alpha},$$

i.e. $V_a(f) \leq Me^{-\alpha}$.

For the converse, assuming that $|f| \leq K$ for some constant $K$, and $|\rho_1 \wedge \rho_2| = n \geq 1$ we have

$$|f(\rho_1) - f(\rho_2)|e^{\alpha(n-1)} \leq V_a(f).$$

Therefore

$$|f(\rho_1) - f(\rho_2)| \leq V_a(f)e^{-\alpha(n-1)} = V_a(f)e^\alpha d(\rho_1, \rho_2)^\alpha.$$ 

In case $|\rho_1 \wedge \rho_2| = 0$, we use boundedness of $f$ to get

$$|f(\rho_1) - f(\rho_2)| \leq 2K = 2Kd(\rho_1, \rho_2)^\alpha.$$ 

Thus

$$|f(\rho_1) - f(\rho_2)| \leq \max\{2K, V_a(f)e^\alpha\}d(\rho_1, \rho_2)^\alpha,$$

for every $\rho_1$ and $\rho_2$.

b) From the proof above we realize that $V_a(f) \leq \|f\| e^{-\alpha}$ which leaves

$$\|f\|_\alpha \leq \|f\|_{\infty} + \|f\| e^{-\alpha} \leq \|f\|_{\infty} + \|f\|_\alpha.$$
Furthermore \( |f|_a \leq \max\{2K, V_a(f)e^a\} \) gives us
\[
\|f\|_\infty + |f|_a \leq 3\|f\|_\infty + V_a(f)e^a \leq (3 + e^a)\|f\|_a.
\]
c) This is a well-known fact, see for example [GT01, p. 73] for a Euclidean space.

d) This is easy to show. ■

**Remark 2.** We want to justify why we used the terminology Hölder-type:

- The Hölder-type continuous functions subject of study in this paper in the case of infinite alphabets are summable. This makes them unbounded and so they are not Hölder.
- Let \( E = \mathbb{N} \). One can see that \( f : E^\mathbb{N} \to \mathbb{R} \) defined by \( f(kn_1n_2n_3...) = \ln 1/n_k^2 \) is Hölder continuous at each point (consider \( [kn_2n_3...n_k] \)) and locally Hölder continuous but is not Hölder-type continuous.
- Note that locally Hölder continuous on \( E^\mathbb{N} \) wouldn’t imply continuity necessarily, however Hölder continuity at a point clearly implies continuity.
- Regarding Hölder continuity at a point even if we were able to find a uniform bound for Hölder coefficients that worked for all the points it still doesn’t imply Hölder-type continuity necessarily.
- Over shift space with finite alphabets Hölder continuity and Hölder-type continuity coincide.

Below we need to use a sequence of finite words in the lemma. For that we use the notation \( \omega_{i(i)} \), to denote that it is not the \( i^{th} \) coordinate of \( \omega \) which we represent by \( \omega_i \).

**Lemma 1.** Let \( \{\omega_{i(i)}\}_{i \in I} \) be any collection of finite words with bounded length, i.e. there exists a positive integer \( k \) such that \( |\omega_{i(i)}| \leq k \) for each \( i \). If the cylinders \( \{[\omega_{i(i)}]\}_{i \in I} \) are mutually disjoint, then the indicator function of \( H := \bigcup_{i \in I} [\omega_{i(i)}] \) is Hölder continuous, i.e. \( 1_H \in C^{0,\alpha}(E^\mathbb{N}_A, \mathbb{C}) \).

**Proof.** We want to show there exists \( M > 0 \) such that
\[
\|1_H(\rho) - 1_H(\rho')\| \leq M d(\rho, \rho'),
\]
for every \( \rho, \rho' \in E^\mathbb{N}_A \). If \( \rho, \rho' \in H \), there is nothing to prove as the left-hand side is 0. Similarly if \( \rho, \rho' \notin H \). If \( \rho \in H \) and \( \rho' \notin H \), then there is \( i \) such that \( \rho \in [\omega_{i(i)}] \). But \( |\rho \land \rho'| \leq |\omega_{i(i)}| \), otherwise \( \rho' \in [\omega_{i(i)}] \). Therefore
\[
e^{-k} \leq e^{-|\omega_{i(i)}|} \leq e^{-|\rho \land \rho'|} = d(\rho, \rho').
\]
Thus if we just pick \( M = e^k \), then for each \( \rho, \rho' \) we have
\[
\|1_H(\rho) - 1_H(\rho')\| \leq M d(\rho, \rho').
\]

**Lemma 2.** If \( f : E^\mathbb{N}_A \to \mathbb{C} \) is Hölder-type continuous with \( V_a(f) < \infty \) then there exists \( K_f > 0 \) such that for any \( \omega \in E^\mathbb{N}_A \) and any \( \rho, \rho' \in E^\mathbb{N}_A \) where \( \omega \rho, \omega \rho' \) are admissible we have
\[
|S_n f(\omega \rho) - S_n f(\omega \rho')| \leq K_f d(\rho, \rho').
\]

**Proof.** We refer to [MU03, p. 26]. ■

A sequence \( \{a_n\} \) of real numbers is called **subadditive** if for every positive integer \( m, n \):
\[
a_{m+n} \leq a_m + a_n.
\]

**Lemma 3** (Fekete’s Lemma). For every subadditive sequence \( \{a_n\} \), the limit of the sequence \( \{a_n/n\} \) exists and it is equal to \( \inf_n \{a_n/n\} \).

**Proof.** We refer to [MU03, p. 5]. ■

**Lemma 4.** Let \( f_i(T) \) be a collection of non-negative functions defined on \( T > 0 \). Then
\[
\sum_i \liminf_{T \to \infty} f_i(T) \leq \liminf_{T \to \infty} \sum_i f_i(T)
\]

**Proof.** Of course, if the collection is finite, this is clear. We show it for an infinite countable collection. As each \( f_i \) is non-negative so for each \( n \)
\[
\sum_{i=1}^n f_i(T) \leq \sum_i f_i(T).
\]
Taking liminf from both sides
\[ \sum_{i=1}^{n} \liminf_{T \to -\infty} f_i(T) \leq \liminf_{T \to -\infty} \sum_{i} f_i(T). \]
This holds for each \( n \), therefore we get the inequality. ■

Unfortunately, analogous inequality for limsup doesn’t hold even if \( \sum_{i} f_i(T) \) is uniformly bounded above. Alternatively, we mention the following inequality.

**Lemma 5.** For any two non-negative functions \( f(T), g(T) \) defined on \( T > 0 \), we have
\[ \liminf_{T \to -\infty} (f(T) + g(T)) \leq \liminf_{T \to -\infty} f(T) + \limsup_{T \to -\infty} g(T) \leq \limsup_{T \to -\infty} (f(T) + g(T)). \]

**Proof.** Let \( \underline{L} = \liminf_{T \to -\infty} (f(T) + g(T)) \), and \( \overline{L} = \limsup_{T \to -\infty} g(T) \). For \( \varepsilon > 0 \) there is \( T_0 \) such that for \( T > T_0 \) we have
\[ \underline{L} - \varepsilon \leq f(T) + g(T) \leq f(T) + \overline{g} + \varepsilon, \]
\[ \underline{L} - \overline{L} - 2\varepsilon \leq f(T), \]
which establishes the left inequality. A similar argument gives the right inequality. ■

A real-valued function \( f \) on \( E_A^{(N)} \) is called **summable** if
\[ \sum_{e \in \omega} \exp(\sup_{e} f) < \infty. \]

One purpose of this definition is to define an operator on the space of bounded complex-valued continuous functions on \( E_A^{(N)} \). Therefore we can extend this definition to complex-valued functions.

**Definition 4.** A complex-valued function \( f \) on \( E_A^{(N)} \) is called **summable** if
\[ \sum_{e \in \omega} \exp(\sup_{e} \text{Re}(f)) < \infty. \]

**Definition 5.** For a complex-valued Hölder-type summable function \( f \) we introduce **Ruelle-Perron-Frobenius** operator, also known as **transfer** operator
\[ \mathcal{L}_f : C_b(E_A^{(N)}, \mathbb{C}) \to C_b(E_A^{(N)}, \mathbb{C}) \]
\[ \mathcal{L}_f(g)(\rho) = \sum_{e \in \omega} \exp(f(e\rho)) g(e\rho), \]
where the sum is taken over all \( e \in \omega \) that \( e\rho \) is admissible, i.e. \( A_{e\rho_1} = 1 \).

**Remark 3.** Here we would like to mention:
- If this \( f \) over shift with infinite letters is summable, then definition 4 yields that \( \text{Re}(f) \) should go to \( -\infty \), i.e. \( f \) is unbounded. Therefore it is not Hölder continuous, see proposition 3.
- As well it is clear that when \( E \) is finite then every real-valued \( f \) is summable.
- Further one can see that this operator preserves \( C^{0,0}(E_A^{(N)}, \mathbb{C}) \).

Next, we want to consider the adjoint operator \( \mathcal{L}_f^* \) acting on \( C_b(E_A^{(N)}, \mathbb{C})^* \) which is the space of all regular bounded additive set functions [DSS58, p. 262] (by an additive set function we mean a complex-valued function \( g \) defined on the algebra, not necessarily \( \sigma \)-algebra, generated by the closed sets such that \( g \) is finitely additive, not necessarily countably additive). Below we mention a result which for case \( E \) finite is due to Ruelle [Rue76] and for \( E \) infinite is due to Mauldin-Urbański [MU03, p. 50].

**Theorem 1.** If \( f : E_A^{(N)} \to \mathbb{R} \) is real-valued summable and Hölder-type continuous function, then the adjoint operator \( \mathcal{L}_f^* \) admits an eigenmeasure \( m \) with eigenvalue \( \exp(P(f)) \).

This \( P(f) \) is introduced below in definition 7.

**Definition 6.** A **Gibbs state** for a real-valued function \( f \) on \( E_A^{(N)} \) is a probability measure \( m \) on \( E_A^{(N)} \) for which there is \( Q > 1 \) and \( P \in \mathbb{R} \) such that:
\[ Q^{-1} \leq \frac{m(\omega)}{\exp(S_{\omega}f(\omega\rho) - Pn)} \leq Q, \quad \forall \omega \in E_A^{(N)}, \forall \omega\rho \text{ admissible.} \]
It is clear that a Gibbs state has full support, i.e.
\[
\text{supp}(m) = E^\text{Ni}_n.
\]
Another important fact is that once we get an eigenmeasure from theorem 1 it follows that it is actually a Gibbs state for \( f \) [MU03, p. 28]. Using this Gibbs state an invariant ergodic Gibbs state \( \mu \) for \( f \) can be constructed as well [MU03, p. 14]. Furthermore, it is clear that if \( f \) is Hölder-type so is any constant multiple of \( f \). However, the summable property of \( f \) doesn’t necessarily carry on to any constant multiple of \( f \). We set
\[
\Gamma := \{ x \in \mathbb{R} : xf \text{ summable} \}.
\]
Clearly, if \( E \) is finite then \( \Gamma = \mathbb{R} \) and if \( E \) is infinite then definition 4 tells us \( x_1 \in \Gamma \) implies \( x_2 \in \Gamma \) for any \( x_2 > x_1 \), i.e. \( \Gamma \) is half line. Therefore using the above explanation we obtain Gibbs state for \( xf (x \in \Gamma) \) as well:
\[
Q^{-1}_x \leq \frac{m_\omega (|\omega|)}{\exp \left( xS_n f(\omega) - P(x)n \right)} \leq Q_x, \quad \forall \omega \in E^n_A, \quad \forall \omega \text{ admissible.}
\]
\[ (2) \]

**Definition 7.** *The topological pressure* of a real-valued function \( f \) on \( E^N_A \) is defined by
\[
P(f) = \lim_{n \to \infty} \frac{1}{n} \ln \left( \sum_{\omega \in E^n_A} \exp(\sup_{[\omega]} S_n f) \right).
\]
This limit exists by Fekete’s lemma 3.

**Definition 8.** A invariant ergodic measure \( \mu \) is called *equilibrium state* for a real-valued function \( f \) on \( E^N_A \) if it is a Gibbs state for \( f \) and it established the following equation:
\[
P(f) = h_\mu (\sigma) + \int fd\mu,
\]
where \( h_\mu \) is Kolmogorov entropy of the shift map \( \sigma \). Note that in general under a much weaker assumption for \( f \) we have the following equation known as variational principle:
\[
P(f) = \sup \{ h_\mu (\sigma) + \int fd\mu \},
\]
where the supremum is taken over invariant ergodic measures \( \mu \). Furthermore, we set
\[
\chi_\mu = - \int fd\mu,
\]
and call it *Lyapunov exponent*.

One can see that \( P \) in definition 6 is actually the same as the topological pressure of \( f \) [MU03, p. 13]. This means
\[
P(x) = P(xf), \quad x \in \Gamma.
\]
We can actually show this function is strictly decreasing on \( \Gamma \) assuming some weak condition. This is a well-known fact for function systems, but here we don’t assume \( f \) is induced by a function system and so we prove it. First, we need the following lemma.

**Lemma 6.** If \( \mu \) is an invariant ergodic Gibbs measure then
\[
\lim_{n} \sup_{\omega \in F^n_A} \mu(|\omega|) = 0.
\]

**Proof.** Let \( b_n = \sup_{\omega \in F^n_A} \mu(|\omega|) \). Note that this supremum is attained so \( b_n \) is decreasing, therefore \( b_n \) is convergent to some \( b \). Fix \( 0 < \epsilon < b \) and for each \( n \) define
\[
F_n := \{ \omega \in E^n_A : \epsilon \leq \mu(|\omega|) \}.
\]
Clearly, \( F_n \) is finite. If \( \omega e \in F_{n+1} \) then \( \epsilon \leq \mu(|\omega e|) \leq \mu(|\omega|) \) which implies \( \omega e \in F_n \), i.e. each \( F_{n+1} \) extends some of \( F_n \). If this extension process stops at moment \( m \) or in other words, \( F_m = \emptyset \) then \( \mu(|\omega|) < \epsilon \) for all \( \omega \in E^m_A \), i.e. \( b \leq b_m \leq \epsilon \). Therefore this process cannot stop and so we get at least one element \( \rho = e_1 e_2 e_3 \cdots \in E^N_A \) such that \( \epsilon \leq \mu(|e_1 \cdots e_n|) \) for each \( n \). This means \( \epsilon \leq \mu(|\rho|) \). We will show \( \rho \) is periodic and the periodic orbit of \( \rho \) \( O_\rho (\rho) \) has full measure which is a contradiction.

Let \( A := \cup_{n \geq 0} \sigma^{-n}(|\rho|) \). Clearly either \( \sigma^{-1}(A) = A \) or \( \sigma^{-1}(A) \cup \{ \rho \} = A \). In the latter case \( \mu(\sigma^{-1}(A) + \mu(\{ \rho \}) = \mu(A) \) which yields \( \mu(\{ \rho \}) = 0 \) using invariant property of \( \mu \). In the former case, \( \rho \) must be periodic with some
Proposition 4. If $P(x_0) \leq 0$ for some $x_0$ then $P(x)$ is strictly decreasing on $\Gamma$.

Proof. We start with the following estimate and we use 2 for it:

$$\exp \left( x_0 \sup_{[\omega]} S_n f - nP(x_0) \right) \leq Qm([\omega]).$$

Next we use the above lemma to find $N$ such that for every $n \geq N$ and every $\omega \in E^n_A$:

$$\exp \left( x_0 \sup_{[\omega]} S_n f - nP(x_0) \right) \leq Q \mu([\omega]) \leq Q \sup_{\omega \in E^n_A} \mu([\omega]) \leq e^{-1}.$$

Then for all $k > 0$ and $\omega' \in E^k_A$:

$$\exp \left( x_0 \sup_{[\omega']} S_{kn} f - knP(x_0) \right) \leq \exp \left( x_0 k \sup_{[\omega']} S_n f - NkP(x_0) \right) \leq e^{-k}.$$

Consider $x_1 < x_2$ in $\Gamma$, we use the above estimate to find

$$\sum_{\omega' \in E^n_A} \exp(x_2 \sup_{[\omega']} S_{nk} f) = \sum_{\omega' \in E^n_A} \exp(x_1 \sup_{[\omega']} S_{nk} f) \exp \left( x_2 - x_1 \sup_{[\omega']} S_{nk} f \right)$$

$$= \sum_{\omega' \in E^n_A} \exp(x_1 \sup_{[\omega']} S_{nk} f) \exp \left( \frac{x_2 - x_1}{x_0} \left( x_0 \sup_{[\omega']} S_{nk} f - nP(x_0) \right) \right) \exp \left( \frac{x_2 - x_1}{x_0} knP(x_0) \right)$$

$$\leq \sum_{\omega' \in E^n_A} \exp(x_1 \sup_{[\omega']} S_{nk} f) \exp(-k\frac{x_2 - x_1}{x_0}) = \exp \left( -k\frac{x_2 - x_1}{x_0} \right) \sum_{\omega' \in E^n_A} \exp(x_1 \sup_{[\omega']} S_{nk} f)$$

Now if we take log, divide by $kn$ and let $k \to \infty$, we obtain

$$P(x_2) \leq -\frac{x_2 - x_1}{Nx_0} + P(x_1) < P(x_1).$$

\[ \square \]

Definition 9. A real-valued function $f : E^N_A \to \mathbb{R}$ is called regular if $P(x) = 0$ for some $x > 0$ and is called strongly regular if it is regular and $0 < P(x) < \infty$ for some $x > 0$.

Remark 4. It is worth mentioning

- If $P(x)$ is strictly decreasing, it can have only one root say $\delta$. Further, strong regularity means
  $$\inf \Gamma < \delta.$$
- The above proposition can be proved under weaker assumption: $\inf_{x \in \Gamma} P(x) \leq 0$.

Proposition 5. If $f$ is strongly regular, the first derivative of $P$ at $\delta$ is $P'(\delta) = -\lambda \mu_\delta$.

Proof. We refer to proposition 2.6.13 in [MU03, p. 47].

\[ \square \]
Next, we want to consider a spectral theory of the transfer operator. We start with considering a family of functions \( \{ s f \} \) where \( s \) is usually a complex number in the right half plane \( \Gamma^+ = \Gamma \times \mathbb{R} \). Then definitions 4 and 3 are applicable for such functions, however, definitions 7 is not applicable as \( s f \) is not real anymore unless for real \( s \). It is clear that when \( s \in \Gamma^+ \) then the series
\[
\sum_{e \in E} \sup_{|e|} |\exp(sf)| = \sum_{e \in E} \exp(\text{Re}(s) \sup_{|e|} f)
\]
converges. Thus having a Hölder-type summable function, spectral theory of transfer operator on the right half plane \( \Gamma \) makes sense. Note that
- \( \mathcal{L}_s := \mathcal{L}_{sf} \) is an operator on \( C^{0,\alpha}(E^N, \mathbb{C}) \) for any \( s \in \Gamma^+ \).
- The property function \( P \) is defined on \( \Gamma \).

Another important property of the transfer operator to be discussed is the D-generic property. This property prohibits the possibility of admitting specific eigenvalue. We adopt its definition from [PU21, p. 32]. Before mentioning the definition, we need an equivalency.

**Proposition 6.** The following conditions are equivalent:

1. The multiplicative group \( G_{a}(y) := \{ \exp(-|o|a) \exp(y S_{[o]} f(\omega)) : \omega \in E^*_\text{per} \} \) is generated by an integer power of \( e^{2\pi} \).
2. \( \exp(P(x) + ia) \) is an eigenvalue of \( \mathcal{L}_{x+iy} : C^{0,\alpha}(E^N_A, \mathbb{C}) \to C^{0,\alpha}(E^N_A, \mathbb{C}) \), for some \( x \in \Gamma \).
3. There exists \( u \in C_b(E^N_A, \mathbb{C}) \) such that \( yf - a + uo \sigma - u \in C_b(E^N_A, 2\pi \mathbb{Z}) \).

**Proof.** We refer to Proposition 2 in [Pol84, p. 138] and Proposition 2.3.5 in [PU21, p. 32].

**Definition 10.** We say a potential \( f \) is **D-generic** if one of the statements of the above proposition fails for all non-zero \( y \) and \( a = 0 \). In other words, \( \mathcal{L}_{x+iy} : C^{0,\alpha}(E^N_A, \mathbb{C}) \to C^{0,\alpha}(E^N_A, \mathbb{C}) \) doesn’t admit \( \exp(P(x)) \) as eigenvalue if \( y \neq 0 \).

Further, we say the potential \( f \) is **strongly D-generic**, if either of the above statements (i) or (ii) from the above proposition fails for all non-zero \( y \) and all real \( a \). In other words, \( \mathcal{L}_{x+iy} : C^{0,\alpha}(E^N_A, \mathbb{C}) \to C^{0,\alpha}(E^N_A, \mathbb{C}) \) doesn’t admit any eigenvalue of magnitude \( \exp(P(x)) \) for any \( y \neq 0 \).

One can obtain an alternative statement for D-generic and strongly D-generic properties.

**Proposition 7.** A potential \( f \) is D-generic iff the additive subgroup generated by the following set is not cyclic:
\[
\{ S_{[o]} f(\omega) : \omega \in E^*_\text{per} \}.
\]
And it is strongly D-generic iff the additive subgroup generated by the following set is not cyclic for any real \( \beta \).
\[
\{ S_{[o]} f(\omega) - n\beta : \omega \in E^*_\text{per} \}.
\]

**Proof.** We only prove the first claim. The other one can be proved in a similar manner. Assuming \( f \) is not D-generic, then for some non-zero \( y \) and \( a = 0 \), statement (i) holds. Then for any \( \omega \in E^*_\text{per} \), there exists integer \( k_o \) such that \( yS_{[o]} f(\omega) = 2\pi k_o \). Therefore \( < S_{[o]} f(\omega) : \omega \in E^*_\text{per} \supset \mathbb{Z} < 2\pi k/y : k \in \mathbb{Z} >. \) Since the latter group is cyclic, so is the first one. For the converse, we assume that \( < S_{[o]} f(\omega) : \omega \in E^*_\text{per} \supset \mathbb{Z} >. \) Then there exists a non-zero \( y \) such that \( S_{[o]} f(\omega) = yk_o \) for any \( \omega \in E^*_\text{per} \), where \( k_o \in \mathbb{Z} \). Therefore, \( G_{0}(2\pi/y) \) is generated by \( e^{2\pi} \).

Next, we would like to bring some facts from spectral theory. We mostly refer to [DS58], [Kat76], [Bro61] or [Bau85]. Assume \( \mathcal{B} \) is a Banach space, \( \mathcal{L} \) a bounded operator on \( \mathcal{B} \).

The **spectrum** of the bounded operator \( \mathcal{L} \), denoted by \( \text{Sp}(\mathcal{L}) \), is defined to be all the complex numbers \( \zeta \) such that the operator \( (\mathcal{L} - \zeta I) \) is not injective. Further the **spectral radius** of \( \mathcal{L} \) is defined to be
\[
r(\mathcal{L}) := \sup \{|\zeta| : \zeta \in \text{Sp}(\mathcal{L})\}
\]
There is an alternative expression of spectral radius known as Gelfand’s formula.
\[
r(\mathcal{L}) = \lim \|\mathcal{L}^n\|^\frac{1}{n}.
\]

Next, we mention the essential spectrum definition. We indicate that there are several other definitions of this concept in the math community, however, the radius of the essential spectrum (defined below) remains the same for all the definitions. We adapt the following definition from [Bro61, p. 107].
Definition 11. The complex number $\zeta$ belongs to the essential spectrum of the operator $\mathcal{L}$, denoted by $\text{Sp}_{\text{ess}}(\mathcal{L})$, if at least one of the following condition holds:

(i) the operator $(\mathcal{L} - \zeta I)$ has a range which is not closed in $\mathcal{B}$.
(ii) $\bigcup_{\varepsilon > 0} \ker (\mathcal{L} - \zeta I)^{\varepsilon}$ is infinite dimensional.
(iii) the point $\zeta$ is a limit point of the spectrum of $\mathcal{L}$.

Furthermore, the essential spectral radius is

$$r_{\text{ess}}(\mathcal{L}) := \sup \{|\zeta| : \zeta \in \text{Sp}_{\text{ess}}(\mathcal{L})\}$$

Nussbaum showed the essential spectral radius follows a Gelfand’s type formula. Before bringing his formula we need to introduce a semi-norm. Consider, $\mathcal{B}$ the ideal of all bounded compact operators on $\mathcal{B}$, then

$$\|\mathcal{L}\|_\mathcal{B} := \inf_{C \in \mathcal{B}} \|\mathcal{L} + C\|,$$

defines a semi-norm on the space of bounded linear operators on $\mathcal{B}$ [Nus70, p. 474].

Proposition 8. $r_{\text{ess}}(\mathcal{L}) = \lim_{n \to \infty} \|\mathcal{L}_n\|^{1/n}_{\mathcal{B}}$

**Proof.** We refer to [Nus70, p. 477]. ■

Next, we briefly talk about the perturbation theory of linear operators. Our main sources are [Kat76], [DS58] and [Bau85].

It is now clear from definition 11 that for every $r$ where $r_{\text{ess}}(\mathcal{L}) < r \leq \rho(\mathcal{L})$ we should have only finitely many $\zeta \in \text{Sp}(\mathcal{L})$ with $|\zeta| \geq r$, each of which is isolated eigenvalue with finite algebraic multiplicity. Kato calls these finite $\zeta$’s, finite system of eigenvalues [Kat76, p. 181] or [Bau85, p. 363]. This concept shows up in [DS58, p. 572] as spectral set. According to Schwartz-Dunford, spectral set is any clopen subset of the spectrum. The purpose is to obtain a perturbation theorem for a holomorphic family of operators $\mathcal{L}_s$ in complex variable $s$. The original idea of the perturbation theory of self-adjoint operators over Hilbert space goes back to Schrödinger. The first major math result in this area was obtained by Rellich. Later on Sz. Nagy and Kato independently worked on this topic to generalize Rellich’s result to a general closed operator over Banach space [Kat52]. Many of these results can be found in [DS58, VII.6] or [Kat76, Ch. VII] or [Bau85, Ch. 10]. We first want to define a holomorphic family of operators. Note that there are several definitions for this but all in the context of bounded operator-valued over a fixed Banach space coincide [Bau85, 10.1], [Bau85, 10.3].

Definition 12. Let $(X, \|\cdot\|)$ be Banach space, $\mathcal{B}(X)$ be the space of all bounded linear operator on $X$, $G$ a region in the complex plane and $s \mapsto \mathcal{L}_s$ a function from $G$ into $\mathcal{B}(X)$. We say $\mathcal{L}_s$ is holomorphic in $G$ if there exists an operator-valued function $s \mapsto \mathcal{L}_s'$ such that

$$\left\| \frac{\mathcal{L}_{s+h} - \mathcal{L}_s}{h} - \mathcal{L}_s' \right\| \to 0,$$

for all $s \in G$ and $h \to 0$.

We are ready to express one major result in the perturbation theory of a holomorphic family of bounded operators.

Theorem 2. Let $\mathcal{L}_s$ be a holomorphic family of bounded operators from a region $G$ into $\mathcal{B}(X)$. Let $s_0 \in G$ and $\lambda_1, ..., \lambda_n$ be finite system of eigenvalues of $\mathcal{L}_{s_0}$, each of which with algebraic multiplicity 1. Then there is small enough neighborhood of $s_0$ such that $\mathcal{L}_s$ has the spectral representation

$$\mathcal{L}_s = \sum_{i=1}^{n} \lambda_i(s) P_{i,s} + D(s),$$

where each $\lambda_i(s)$ is holomorphic function, $P_i(s)$ is holomorphic operator-valued function and a projection, $D(s)$ holomorphic operator-valued function and further

$$\lambda_i(s_0) = \lambda_i,$$

for each $i = 1, ..., n$.

In general, if the multiplicity of an eigenvalue is higher than 1 the eigenvalues may have algebraic singularities at $s_0$. The idea of the proof is first reducing it to the case where $X$ is finite-dimensional and then one can apply perturbation theory of holomorphic operators in finite dimension. For a detailed proof, first see theorem 1 in [Bau85, p. 367], then theorem 1 in [Bau85, p. 243], [Bau85, p. 129] and [Bau85, p. 131]. Another source of proof for the general form of the result is theorem 9 in [DS58, p. 587]. As well theorem 1.8 in [Kat76, p.
Proposition 9. Suppose \( \mathfrak{B}(X) \) is a Banach algebra of operators on the Banach space \( X \). Let \( \mathcal{L} \in \mathfrak{B}(X) \). Further, assume the spectrum of \( \mathcal{L} \) can be written as
\[
\text{Sp}(\mathcal{L}) = F_1 \cup F_2,
\]
for disjoint nonempty closed sets \( F_1, F_2 \). Then there is a nontrivial idempotent \( E \in \mathfrak{B}(X) \) such that
- if \( \mathcal{L} \subseteq \mathcal{L} E \), then \( BE = E \mathcal{L} \).
- if \( \mathcal{L}_1 = \mathcal{L} E \) and \( \mathcal{L}_2 = \mathcal{L}(1 - E) \), then \( \mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}_1 \mathcal{L}_2 = \mathcal{L}_2 \mathcal{L}_1 = 0 \).
- \( \text{Sp}(\mathcal{L}_1) = F_1 \cup \{0\} \), \( \text{Sp}(\mathcal{L}_2) = F_2 \cup \{0\} \).

Definition 13. We first consider directed multi-graph \( (V, E, i, t) \) and an incidence matrix \( A : E \times E \to \{0, 1\} \), where \( V \) is the finite set of vertices, \( E \) is the countable (finite or infinite) set of directed edges and \( i, t \) (initial and tail) are functions
\[
i, t : E \to V,
\]
such that
\[
A_{ab} = 1 \Rightarrow t(a) = i(b).
\]
In addition, we have a finite family of Euclidean compact metric spaces \( \{X_v\}_v \subseteq V \) and countable family of contractions \( \{\phi_v\}_v \subseteq E \) and \( \kappa \in (0, 1) \) such that
\[
|\phi_v(x) - \phi_v(y)| \leq \kappa|x - y|,
\]
for all \( e \in E \) and \( x, y \in X_{t(e)} \). Then
\[
S = \{\phi_e : X_{t(e)} \to X_{t(e)}\}_{e \in E}
\]
is called attracting graph directed Markov system.

We extend the functions \( i, t : E \to V \) in a natural way to \( E^n \) as follows:
\[
t(\omega) := t(\omega_n), \quad i(\omega) := i(\omega_1).
\]
If \( \omega \in E^n \), we define:
\[
\phi_\omega = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X_{t(\omega)} \to X_{t(\omega)}.
\]
Now for any \( \rho \in E^n \), the sets \( \{\phi_{\rho_1, \rho_2, \ldots, \rho_n}(X_{t(\rho)})\}_{n \geq 1} \) form a descending sequence of non-empty compact sets and therefore \( \cap_{n \geq 1} \phi_{\rho_1, \rho_2, \ldots, \rho_n}(X_{t(\rho)}) \) is non-empty. Further since
\[
\text{diam}(\phi_{\rho_1, \rho_2, \ldots, \rho_n}(X_{t(\rho)})) \leq \kappa^n \text{diam}(X_{t(\rho)}),
\]
we find that this intersection is actually a singleton and we denote it by \( \pi(\rho) \), in this way we have defined a map
\[
\pi : E^N \to \bigsqcup_{v \in V} X_v,
\]
where \( \bigsqcup_{v \in V} X_v \) is the disjoint union of the compact spaces \( \{X_v\}_v \).

Definition 14. The set
\[
J = \pi(E^N)
\]
is called the limit set of system \( S \).

Definition 15. We call a graph directed Markov system conformal if the following conditions are satisfied for some \( d \in \mathbb{N} \):
(a) For every \( v \in V \), \( X_v \) is compact connected subset of \( \mathbb{R}^d \) and \( X_v = \text{Int}(X_v) \).
(b) (Open Set Condition) For all different \( e, e' \in E \),
\[
\phi_e(\text{Int}(X_{t(e)})) \cap \phi_{e'}(\text{Int}(X_{t(e')})) = \emptyset.
\]
(c) (Conformality) For every \( v \in V \) there is an open connected \( W_v \) containing \( X_v \). Further for each \( e \in E \), \( \phi_e \) extends to a \( C^1 \) conformal diffeomorphism from \( W_{t(e)} \) into \( W_{t(e)} \) with Lipschitz constant bounded by \( \kappa \).
(d) (Bounded Distortion Property) There are two constants $L \geq 1$ and $\alpha > 0$ such that for every $e \in E$ and every $x, y \in X_{\tau(e)}$

$$\frac{|\phi'_e(y)|}{|\phi'_e(x)|} - 1 \leq L \|y - x\|^\alpha,$$

where $|\phi'_e(x)|$ denotes the scaling factor of the derivative of $\phi'_e$ at $x$.

To a conformal graph directed Markov system we assign a real-valued function by

$$f : E^N_A \to \mathbb{R}, \quad f(\rho) = \log |\phi'_e(\pi(\sigma\rho))|.$$  

Also known as potential.

Before finishing the section, we mention two main Tauberian theorems needed later on. First Ikehara & Wiener’s theorem [Wie33, p. 127] and then Graham & Vaaler’s theorem [GV81, p. 294] which is just a refinement of the Ikehara-Wiener theorem. The motivation for the Ikehara-Wiener theorem was to provide a simpler proof of the prime number theorem. We know that PNT was proved in the late 19th century. However, Ikehara & Wiener used a theorem of Wiener to obtain the following result in the early 1930s that implies PNT [Kor04, p. 127].

**Theorem 3 (Ikehara-Wiener).** Let $a(T)$ be a monotone increasing function continuous from right such that

$$\eta(s) = \int_0^\infty e^{-sT} d\alpha(T)$$

converges for $\text{Re}(s) > \delta > 0$. If

$$\eta(s) - \frac{A}{s - \delta} = g(s)$$

has continuous extension to $\text{Re}(s) = \delta$, then

$$e^{-\delta T} a(T) \to \frac{A}{\delta}, \quad \text{as} \quad T \to \infty.$$  

In the 1980s, Graham & Vaaler on their way to study extremal (minorant and majorant) functions in Fourier analysis for some special classes of functions, obtained a refinement of the Ikehara-Wiener theorem as a corollary. One may want to know that the early work in the construction of extremal functions was done by Beurling and later on by Selberg (unpublished). For the proof of the following result see [GV81, p. 294].

**Theorem 4 (Graham-Vaaler).** Let $a$ be a Borel measure on $[0, \infty)$ and that the Laplace-Stieltjes transform

$$\eta(s) = \int_0^\infty e^{-sT} d\alpha(T), \quad s = x + 2\pi iy,$$

exists for $\text{Re}(s) > \delta$. Suppose that for some number $y_0 > 0$, there is a constant $A > 0$ such that the analytic function $\eta(s) - A/(s - \delta)$ extends to a continuous function on the set $\{\delta + 2\pi iy : |y| < y_0\}$. Then

$$A y_0^{-1} \{\exp(\delta y_0^{-1}) - 1\}^{-1} \leq \liminf_{T \to \infty} e^{-\delta T} a[0, T]$$

$$\leq \limsup_{T \to \infty} e^{-\delta T} a[0, T]$$

$$\leq A y_0^{-1} \{\exp(\delta y_0^{-1}) - 1\}^{-1} \exp(\delta y_0^{-1})$$

**Remark 5.** It is worth noting that

- If $\eta(s) - A/(s - \delta)$ has continuous extension to the whole line $x = \delta$ then we may let $y_0 \to \infty$, this implies Ikehara-Wiener theorem 3.
- Graham & Vaaler or Korevaar [Kor04, p. 30] assumed that $A$ should be positive or non-negative. But since $\eta$ is real non-negative on the real line, this assumption can be relaxed, i.e. $A$ can be any complex number. Then one can see it has to be real non-negative. Furthermore, it is clear that for us the measure $\mu$ (possibly infinite measure) is just taken to be the Borel measure generated by the right continuous, increasing function $N_\mu(B, T)$, see 16 and [Fol99, Thm 1.16]. Moreover, Graham & Vaaler provide an example to show their bounds are both sharp.

**Example 1.** Consider the iterated function system where in the multi-graph $(V, E, i, t)$, $V$ is Singleton $\{v\}$, $E$ is finite $i = t$ are maps from $E$ to the only element of $V$ and the mapping $A : E \times E \to \{0, 1\}$ is just constant 1. This is an iterated function system. Then for the conformal graph directed Markov systems, we consider $X_v = [0, 1]$ and

$$\phi_e(t) = \alpha t + \beta_e,$$
where $\alpha_e, \beta_e$ are chosen appropriately enough from $(0, 1)$ so that we have all conditions for conformal graph directed Markov system satisfied, see definition 15. Then we know from below the definition 15 the potential is

$$f(\rho) = \log |(\phi_{\rho_i})' (\pi(\sigma \rho))| = \log \alpha_{\rho_i}$$

and

$$S_n f(\rho) = \sum_{e \in E} n_e \log \alpha_e,$$

where $n_e$ is just number of letter $e$ appearing in the word $\rho_1...\rho_n$. We can find the pressure:

$$P(x) = \lim_{n} \frac{1}{n} \log \sum_{|\omega| = n} \|\phi_{\omega}^t\|^x = \log \left( \sum_{e \in E} \alpha_e^x \right)$$

As well we know the following should hold

$$m_x([e_0 \omega_1...\omega_n]) = \exp(-P(x)) \int_{[\omega_1...\omega_n]} |(\phi_{e_0})' (\pi(r))|^x dm_x(r),$$

for $e \in E$, Gibbs state $m_x$ and pressure $P(x)$. This actually leaves

$$\frac{m_x([e_0 \omega_1...\omega_n])}{m_x([\omega_1...\omega_n])} = \frac{\alpha_e^x}{\sum_{e \in E} \alpha_e^x}$$

Example 2. Consider an iterated function system containing conformal maps

$$\phi_e(t) = at + \beta_e, \quad \alpha, \beta_e \in (0, 1),$$

where $\beta_e$ are appropriate enough for conformal conditions, see Definition 15, $E = \{0, 1,..., k - 1\}$, with some irreducible incidence matrix $A$. Then we know for this system we should have the potential

$$f(\alpha) = \log |(\phi_{\omega_1})' (\pi(\sigma \omega))| = \log \alpha$$

and a Gibbs state has the form

$$m_x([e_0 \omega_1...\omega_n]) = \exp(-P(x)) \int_{[\omega_1...\omega_n]} |(\phi_{e_0})' (\pi(r))|^x dm_x(r),$$

for appropriate $e \in E$, Gibbs state $m_x$ and pressure $P(x)$. This actually yields

$$P(x) = \log \left( \frac{m_x([\omega_1...\omega_n])}{m_x([e_0 \omega_1...\omega_n])} \right) + x \log \alpha = \log r(A) + x \log \alpha.$$ 

Note that the first term of the above sum does not depend on $t$ and it is actually equal to $\lim_n \log \#E_n^n / n = \log r(A)$, see Proposition 2, where $r(A)$ is the spectral radius of the incidence matrix $A$, we can see this simply by the pressure formula:

$$P(x) = \lim_{n} \frac{1}{n} \log \sum_{\omega \in E_n} \|\phi_{\omega}^t\|^x = \lim_{n} \frac{\log \#E_n^n}{n} + x \log \alpha = \log r(A) + x \log \alpha.$$ 

3. Spectral Analysis of Transfer Operator

In this section we assume we have a summable strongly regular H"older-type function (potential) $f : E_A^n \to \mathbb{R}$ with $P(f) = P(1) = 0$. We recall $C^{0,\alpha}(E_A^n, \mathbb{C})$ is Banach space of H"older continuous complex-valued functions over $E_A^n$, and $\mathfrak{B} := \mathfrak{B} \left( C^{0,\alpha}(E_A^n, \mathbb{C}) \right)$ is Banach space of all bounded linear operators over $C^{0,\alpha}(E_A^n, \mathbb{C})$. For every $s \in \Gamma^+$ it was stated in the previous section that $\mathcal{L}_s$ belongs to $\mathfrak{B}$. One major step is to establish holomorphy of the operator $\mathcal{L}_s$.

Lemma 7. For every $n \in \mathbb{N}$, the operator-valued function $s \mapsto \mathcal{L}_s^n$ is holomorphic on $\Gamma^+$.

Proof. For each $\omega \in E_A^n$ one can consider the (idempotent) function $i_{[\omega]} \in C^{0,\alpha}(E_A^n, \mathbb{C})$ where it is defined to be $1$ on $\xi$ such that $\omega \xi$ is admissible and $0$ otherwise. Then for each $s$ in the right half plane $\Gamma^+$ and $g \in C^{0,\alpha}(E_A^n, \mathbb{C})$ we define $F_{\omega,s} g$:

$$F_{\omega,s} g(\rho) : = i_{[\omega]}(\rho) \exp(s S_n f(\omega \rho)) g(\omega \rho).$$

We want to show $F_{\omega,s}$ is an operator on $C^{0,\alpha}(E_A^n, \mathbb{C})$. First note that

$$\|F_{\omega,s} g(\rho)\|_{\infty} \leq \exp(\text{Re}(s) \sup_{[\omega]} S_n f) \|g\|_{\infty}.$$
To find Hölder coefficient of $F_{\alpha,s}$ we let $|\rho \wedge \rho'| \geq k \geq 1$:

$$|F_{\alpha,s}(\rho) - F_{\alpha,s}(\rho')| \leq |\exp(s S_{n}(f(\omega \rho)))g(\omega \rho) - \exp(s S_{n}(f(\omega \rho')))g(\omega \rho')|$$

$$= |(\exp(s S_{n}(f(\omega \rho))) - \exp(s S_{n}(f(\omega \rho')))) g(\omega \rho) + \exp(s S_{n}(f(\omega \rho'))) (g(\omega \rho) - g(\omega \rho')) |$$

$$\leq \exp(\text{Re}(s) \sup_{[0]} S_{n,f}) |s| |S_{n,f}(\omega \rho) - S_{n,f}(\omega \rho')| \|g\|_{\infty}$$

$$+ \exp(\text{Re}(s) \sup_{[0]} S_{n,f}) |g(\omega \rho) - g(\omega \rho')|.$$  

By Lemma 2, we get

$$|F_{\alpha,s}(\rho) - F_{\alpha,s}(\rho')| \exp(\alpha k) \leq |F_{\alpha,s}(\rho) - F_{\alpha,s}(\rho')| \exp(\alpha |\rho \wedge \rho'|)$$

$$\leq \exp(\text{Re}(s) \sup_{[0]} S_{n,f}) |s| K \|g\|_{\infty} + \exp(\text{Re}(s) \sup_{[0]} S_{n,f}) V_{a}(g)$$

$$\leq \exp(\text{Re}(s) \sup_{[0]} S_{n,f}) |g\|_{a}(1 + |s| K),$$

where $K$ depends only on $f$. Therefore we can write:

$$\|F_{\alpha,s}\|_{a} = \|F_{\alpha,s}g\|_{\infty} + V_{a}(F_{\alpha,s}g)$$

$$\leq \exp(\text{Re}(s) \sup_{[0]} S_{n,f}) |g\|_{\infty} + \exp(\text{Re}(s) \sup_{[0]} S_{n,f}) |g\|_{a}(1 + |s| K)$$

$$\leq \exp(\text{Re}(s) \sup_{[0]} S_{n,f}) |g\|_{a}(2 + |s| K).$$

so

$$\|F_{\alpha,s}\|_{a} \leq \exp(\text{Re}(s) \sup_{[0]} S_{n,f})(2 + |s| K).$$

Next, we want to show the map $s \mapsto F_{\alpha,s}$ is holomorphic on $\Gamma^{+}$. As expected derivative is

$$F'_{\alpha,s}g(\rho) = i_{[a]}(\rho). \exp(s S_{n,f}(\omega \rho)). S_{n,f}(\omega \rho). g(\omega \rho),$$

we first need to show this defines an operator on $C^{0,\alpha}(E_{\mathbb{A}}, \mathbb{C})$ and then to check it is actually bounded. Note that $|S_{n,f}|$ is bounded on $[a]$ by some $C$, see definition 6. If we review all the inequalities above and replace all the $g(\omega \rho)$ with $S_{n}(\psi)(\omega \rho)g(\omega \rho)$ we get:

$$\|F'_{\alpha,s}\|_{a} = \|F'_{\alpha,s}g\|_{\infty} + V_{a}(F'_{\alpha,s}g)$$

$$\leq \exp(\text{Re}(s) \sup_{[a]} S_{n,f}) |g\|_{\infty} C + \exp(\text{Re}(s) \sup_{[a]} S_{n,f}) |g\|_{a}(C + |s| KC + K)$$

$$\leq \exp(\text{Re}(s) \sup_{[a]} S_{n,f}) |g\|_{a}(2C + |s| KC + K).$$

Fix $s_{0}$ in $\Gamma^{+}$, we write:

$$F_{\alpha,s} - F_{\alpha,s_{0}} - (s - s_{0}) F'_{\alpha,s_{0}} g(\rho)$$

$$= i_{[a]}(\rho). (\exp(s S_{n,f}(\omega \rho)) - \exp(s_{0} S_{n,f}(\omega \rho))$$

$$- (s - s_{0}) \exp(s_{0} S_{n,f}(\omega \rho)). S_{n,f}(\omega \rho). g(\omega \rho),$$

therefore

$$s \mapsto F_{\alpha,s} \in \mathcal{B}$$

is holomorphic iff

$$s \mapsto i_{[a]}(\rho) (\exp(s S_{n,f}(\omega \rho)) \in C^{0,\alpha}(E_{\mathbb{A}}, \mathbb{C})$$

is holomorphic. But

$$i_{[a]}(\rho) \exp(s S_{n,f}(\omega \rho)) = i_{[a]}(\rho) \exp(s i_{[a]}(\rho) S_{n,f}(\omega \rho))$$

and $i_{[a]}(\rho) S_{n,f}(\omega \rho) \in C^{0,\alpha}(E_{\mathbb{A}}, \mathbb{C})$, thus since $i_{[a]}(\rho)$ is a constant function of $s$ problem boils down to holomorphy of the function $s \mapsto \exp(s \mathcal{T})$ for $\mathcal{T} \in C^{0,\alpha}(E_{0}^{\alpha}, \mathbb{C})$, and this is clearly holomorphic. Thus the map $s \mapsto F_{\alpha,s}$ defines a holomorphic $\mathfrak{B}$-valued function on the right half plane $\Gamma^{+}$. Now because for $s \in \Gamma^{+}$, $\text{Re}(s)f$ admits Gibbs state, $\mathfrak{B}$-valued function

$$s \mapsto \mathcal{L}_{s} = \sum_{a \in E_{\mathbb{A}}} F_{\alpha,s}$$

converges and so is holomorphic on $\Gamma^{+}$. 

\[ \square \]

**Proposition 10.** The spectral radius of $\mathcal{L}_{s}$ is at most $e^{\rho(s)}$ and $r_{e^{s}}(\mathcal{L}_{s}) < e^{\rho(s)}$. 

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Proof. For the case $E$ is finite we just refer to [Pol84, p. 140]. Assuming $E$ is infinite, the former part is a straightforward consequence of Doeblin inequality (also known as Ionescu Tulcea-Marinescu inequality or Lasota-Yorke inequality) shown in [MU03, p. 32]:

\begin{equation}
\| L^n g \|_a \leq e^{nP(x)}(Q e^{-an} \| g \|_a + C \| g \|_\infty),
\end{equation}

which leaves

\begin{equation}
\| L^n \|_a \leq e^{nP(x)}(Q + C).
\end{equation}

First for every $\omega \in E^n_A$ choose $\hat{\omega} \in [\omega]$ arbitrarily. Then for every $n \geq 1$ consider the operator $E_n$ on $C^{0,a}(E^n_A, \mathbb{C})$ defined by:

\[ E_n(g) := \sum_{\omega \in E^n_A} g(\hat{\omega}) \|_{[\omega]}. \]

Therefore $E_n g$ is constant on each cylinder $[\omega]$. It is clear that $\| E_n g \|_\infty \leq \| g \|_\infty$. We want to show $V_a(E_n g) \leq V_a(g)$. Remembering definition 3 if $m \geq n$ then clearly $V_{n,m} = 0$, in case $1 \leq m < n$ and $|\rho_1 \wedge \rho_2| \geq m$ there should be $\omega_1, \omega_2 \in E^n_A$ such that $\rho_1 \in [\omega_1]$ and $\rho_2 \in [\omega_2]$, therefore $|\hat{\omega_1} \wedge \hat{\omega_2}| \geq m$ and

\[ |E_n g(\rho_1) - E_n g(\rho_2)| e^{a(m-1)} = |g(\hat{\omega_1}) - g(\hat{\omega_2})| e^{a(m-1)} \leq V_{a,m}(g). \]

Thus we have:

\begin{equation}
\| E_n g \|_a \leq \| g \|_a.
\end{equation}

Next without loss of generality assume $E = \mathbb{N}$ and for each $N \geq 1$ define

\[ E^n_A(N) := \{ \omega \in E^n_A : \omega_1, \omega_2, \ldots, \omega_n \leq N \} \]

\[ E^n_A(N+) := E^n_A \setminus E^n_A(N) \]

\[ E_{n,N} g := \sum_{\omega \in E^n_A(N)} g(\hat{\omega}) \|_{[\omega]}. \]

Note that $n$ and $N$ are independent. Moreover notice that this time since we have finite sum the operator $E_{n,N}$ on $C^{0,a}(E^n_A, \mathbb{C})$ is of finite rank and so compact. We use triangle inequality to write:

\begin{equation}
\| L^n - L^n e_{n,N} \|_a \leq \| (L^n - L^n e_n) + (L^n e_n - L^n e_{n,N}) \|_a \\
\leq \| L^n (I - E_n) \|_a + \| L^n (E_n - E_{n,N}) \|_a,
\end{equation}

where $I$ is just the identity operator. Note that 11 implies $\| g - E_n g \|_a \leq 2 \| g \|_a$. Furthermore, for any $\rho \in E^n_A$ if set $\omega = \rho_1 \ldots \rho_n$ then we have $|\rho \wedge \hat{\omega}| \geq n$ and

\[ |g(\rho) - E_n g(\rho)| e^{a(n-1)} = |g(\rho) - g(\hat{\omega})| e^{a(n-1)} \leq V_{a,n}(g) \leq V_a(g). \]

Since $\rho$ is arbitrary, we obtain

\[ \| g - E_n g \|_\infty \leq V_a(g) e^{a} e^{-an} \leq \| g \|_a e^{a} e^{-an}. \]

Thus using two recent inequalities and 9 we find

\begin{equation}
\| L^n (I - E_n) g \|_a \leq e^{nP(x)}(Q e^{-an} 2 \| g \|_a + C \| g \|_a e^{a} e^{-an}) \\
\leq C_1 e^{nP(x)} \| g \|_a e^{-an},
\end{equation}

for some constant $C_1 > 0$. Recalling $F_{\omega,s}$ from the proof of previous lemma, we can write

\[ F_{\omega,s}(E_n g - E_{n,N} g) = \sum_{|\omega'|=a} g(\hat{\omega}) F_{\omega',s}(1_{[\omega]}) - \sum_{\omega \in E^n_A(N)} g(\hat{\omega}) F_{\omega',s}(1_{[\omega]}) \]

\[ = \sum_{\omega \in E^n_A(N+)} g(\hat{\omega}) F_{\omega',s}(1_{[\omega]}) = g(\hat{\omega}') F_{\omega',s}(1_{[\omega]}) \]

depending on $\omega' \in E^n_A(N+)$ or not, so

\[ L^n (E_n g - E_{n,N} g) = \sum_{\omega' \in E^n_A} F_{\omega',s}(E_n g - E_{n,N} g) = \sum_{\omega \in E^n_A(N+)} g(\hat{\omega}) F_{\omega',s}(1_{[\omega]}). \]

Then 8 leaves:

\[ \| L^n (E_n g - E_{n,N} g) \|_a \leq \| g \|_\infty \sum_{\omega \in E^n_A(N+)} \| F_{\omega,s}(1_{[\omega]}) \|_a. \]
\[\|g\|_\infty (2 + |s|K) \sum_{\omega \in E^n_A(N+)} \exp(x \sup_{[\omega]} S_n f).\]

Now since \( A \) is finitely irreducible, there exists a finite set \( \Omega \subseteq E^n_A = \bigcup_{\rho \in E^n_A} \) such that for every \( e \in E \) and \( \rho \in E^n_A \), there is \( \omega \in \Omega \) with \( e \omega \rho \) being admissible. Thus there exists a finite set \( F \subseteq E^n_A \) such that for every \( e \in E \), there is \( \tau \in F \) with \( e \tau \) being admissible. For every \( \omega \in E_A^n \) choose \( \tau_\omega \in F \) with \( \omega \tau_\omega \) admissible. Therefore using 2 we can continue

\[\|g\|_\infty (2 + |s|K)Q^2 \sum_{\omega \in E^n_A(N+)} \exp(xS_n f(\omega \tau_\omega)).\]

Moreover, if we consider

\[c_N := \sup_{j \geq N} \exp(\sup f[j]),\]

then the fact that \( f \) is summable implies that \( c_N \rightarrow 0 \). Now for each \( \omega \in E^n_A(N+) \) there is \( \omega_i > N \) so

\[\exp(S_n f(\omega \tau_\omega)) = \exp(S_{i-1} f(\omega \tau_\omega)) + f(\omega_i \ldots \omega_n \tau_\omega) + S_{n-i} f(\omega_i+1 \ldots \omega_n \tau_\omega))\]

\[\leq Q c_N Q = Q^2 c_N.\]

Therefore for small enough \( \epsilon > 0 \) we have

\[\| \mathcal{L}^n_s (E_n g - E_n \mathcal{E}_n g) \|_a\]

\[\leq \|g\|_\infty (2 + |s|K)Q^2 \sum_{\omega \in E^n_A(N+)} \exp(c_n f(\omega \tau_\omega)) \exp((x - \epsilon) S_n f(\omega \tau_\omega))\]

\[\leq \|g\|_\infty (2 + |s|K)Q^2 Q^2 c_n^\epsilon \sum_{\omega \in E^n_A(N+)} \exp((x - \epsilon) S_n f(\omega \tau_\omega))\]

\[\leq \|g\|_\infty (2 + |s|K)Q^4 c_n^\epsilon \sum_{\tau \in F} \mathcal{L}_x \mathcal{E}_n(1)(\tau) \leq \|g\|_\infty (2 + |s|K)Q^4 c_n^\epsilon \# F \| \mathcal{L}^n_{x-\epsilon} \|_a.\]

This together with 10 yields

\[\| \mathcal{L}^n_s(E_n - E_n \mathcal{E}_n) \|_a \leq (2 + |s|K)Q^4 c_n^\epsilon \# F \| \mathcal{L}^n_{x-\epsilon} \|_a\]

\[\leq (2 + |s|K)Q^4 c_n^\epsilon \# F (Q + C)e^{nP(x-\epsilon)}.\]

For large enough \( N \) we get

\[\| \mathcal{L}^n_s(E_n - E_n \mathcal{E}_n) \|_a \leq e^{nP(x-\epsilon)} e^{-an}.\]

Thus since \( P \) is strictly decreasing, the above inequality combined with 12 and 13 implies

\[\| \mathcal{L}^n_s - \mathcal{L}^n_s \mathcal{E}_n \mathcal{E}_n \|_a \leq C_1 e^{nP(x) - an} + e^{nP(x-\epsilon)} e^{-an} \leq C_2 e^{nP(x-\epsilon)} e^{-an}.\]

Therefore, we can estimate the essential spectral radius by proposition 8:

\[r_{ess} (\mathcal{L}_s) = \lim_{a} \| \mathcal{L}^n_s \|_a^{1/n} \leq \lim_{a} \| \mathcal{L}^n_s - \mathcal{L}^n_s \mathcal{E}_n \mathcal{E}_n \|_a^{1/n} \leq e^{P(x-\epsilon)} e^{-a}.\]

Since \( \epsilon \) was chosen small enough, this completes the proof. \( \blacksquare \)

We want to introduce two operators closely related to the transfer operator. The first operator is \( \mathcal{L}_0 \). There is \( s \) hidden in the definition but we don’t write that. It is defined by:

\[\mathcal{L}_0 := e^{-P(s)} \mathcal{L}_s,\]

and another operator is the weighted operator defined by:

\[\tilde{\mathcal{L}}_s g := e^{-P(s)} \frac{1}{h_x} \mathcal{L}_s (g h_x),\]

where \( h_x \) is a fixed point of \( \mathcal{L}_0 \) obtained in [MU03, p. 34] as the (compactly) convergent point of the sequence \( \left\{ \frac{1}{n} \sum_{i=0}^{n-1} e^{-P(s)} \mathcal{L}_s^i(1) \right\} \). In other words, \( h_x \) is actually an eigenfunction of \( \mathcal{L}_s \) corresponding to the eigenvalue \( e^{-P(s)} \). Moreover, it is clear that \( \int h_x dm_x = 1 \).

**Lemma 8.** There is \( c > 0 \) such that \( h_x > c \).
Proof. We use theorem 2.3.5 from [MU03, p. 29] to show this. Let $n_k - 1 = (M + 1)\ell_k + r_k$ where $0 \leq r_k \leq M$ then
\[
\frac{1}{n_k} \sum_{i=0}^{n_k-1} L^i_0(1) \geq \frac{1}{n_k} \sum_{i=1}^{(M+1)\ell_k} L^i_0(1) \geq \frac{1}{n_k} \ell_k R
\]
which leaves $h_x \geq \frac{R}{M+1}$.

**Lemma 9.** If $g \in C^{0,\alpha}(E^N_A, \mathbb{C})$ is non-negative then \( \left\{ \frac{1}{n} \sum_{j=1}^{n} \tilde{L}^j xg \right\} \) has a converging subsequence with limit \( \int gd\mu_x \), where $\mu_x$ is the equilibrium state of $xf$.

**Proof.** Observe that $\tilde{L}_x(1) = 1$ and so $\tilde{L}_x^j(1) = 1$ for each $j \geq 1$. Then one can start with $\|L^j_x g\|_{\infty} \leq \|g\|_{\infty}$ and follow the same proof of theorem 2.4.3 [MU03, p. 34] to find that \( \left\{ \frac{1}{n} \sum_{j=1}^{n} \tilde{L}^j xg \right\} \) has a converging subsequence with limit $g_1 \in C^{0,\alpha}(E^N_A, \mathbb{C})$, where $\tilde{L}_x g_1 = g_1$. This leaves $g_1 h_x$ as a fixed point of $L_0$. Since $g$ is non-negative so is $g_1$ and $g_1 h_x$. Now theorem 2.4.7 [MU03, p. 39] tells us that
\[
\left( \frac{g_1}{d} h_x m_\alpha \right) \circ \sigma^{-1} = \frac{g_1}{d} h_x m_\alpha, \quad d = \int g_1 h_x dm_\alpha,
\]
where $m_\alpha$ is eigenmeasure of $L_\alpha$. Therefore if one defines a measure by $\mu_1(A) = \frac{1}{d} \int_A g_1 h_x dm_\alpha$, we find that
\[
\mu_1(\sigma^{-1}(A)) = \frac{1}{d} \int_{\sigma^{-1}(A)} g_1 h_x dm_\alpha
\]
\[
= \frac{1}{d} \int_A g_1 \circ \sigma^{-1} h_x \circ \sigma^{-1} d(m \circ \sigma^{-1}) = \frac{1}{d} \int_A g_1 h_x dm_\alpha = \mu_1(A).
\]
That leaves an invariant absolutely continuous measure with respect to $m_\alpha$. Then theorem 10.4.2 [URM21] implies that $\mu_1$ must be $\mu_x$, therefore the Randon-Nikodym derivative of $\mu_1$ with respect to $m$ is the same as that of $\mu$ with respect to $m$ a.e. which means $g_1 = d$ a.e. and since $g_1$ is continuous so $g_1 = d = \int g_1 h_x dm_\alpha = \int g_1 d\mu_x$ everywhere. Furthermore, it is not hard to see that $(\tilde{L}_x)^n(\mu_x) = \mu_x$ see theorem 2.4.4 [MU03, p. 36]. Since we had
\[
\frac{1}{n} \sum_{j=1}^{n} \tilde{L}^j xg \to g_1
\]
on a sub-sequence, then
\[
\int gd\mu_x = \int \frac{1}{n} \sum_{j=1}^{n} \tilde{L}^j xg d\mu_x \to \int g_1 d\mu_x,
\]
i.e. $\int gd\mu_x = \int g_1 d\mu_x$.

**Proposition 11.** The transfer operator $L_x$ has at most finitely many eigenvalues of modulus $e^{P(x)}$ all of which with multiplicity one.

**Proof.** The previous proposition implies there are at most finitely many spectral values of $L_x$ with modulus $e^{P(x)}$ all are isolated eigenvalues with finite (algebraic) multiplicity, see definition 11. We would like to show first for each eigenvalues $\lambda$ with $|\lambda| = e^{P(x)}$ the transfer operator $L_x$ acts on $X := P_{\lambda}(C^{0,\alpha}(E^N_A, \mathbb{C})) = \cup_{m \geq 1} \ker(L_x - \lambda)^m$ diagonally. To see this we consider the Jordan normal form of $L := L_x$ on finite-dimensional space $X$, so there is an invertible transformation $P$, such that $PLP^{-1}$ is the Jordan normal form of $L$. Consider a $k \times k$ Jordan block in matrix representation that has $1$ above the diagonal. The $n^{th}$ power of the block looks like
\[
\begin{bmatrix}
\lambda^n & (n)\lambda^{n-1} \\
\lambda^n & (n)\lambda^{n-2} & \cdots & (n)\lambda^{n-k+1} \\
\lambda^n & (n)\lambda^{n-1} & \cdots & (n)\lambda^{n-k+2} \\
& \vdots \\
& \vdots \\
& \vdots \\
& \lambda^n
\end{bmatrix}
\]
Then for $e = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}^T$ we have
\[
\begin{bmatrix}
\lambda^n & \binom{n}{1} \lambda^{n-1} & \binom{n}{2} \lambda^{n-2} & \ldots & \binom{n}{k} \lambda^{n-k+1} \\
\lambda^n & \binom{n}{1} \lambda^{n-1} & \binom{n}{2} \lambda^{n-2} & \ldots & \binom{n}{k} \lambda^{n-k+1} \\
\vdots & \vdots & & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \vdots \\
\lambda^n & \binom{n}{1} \lambda^{n-1} & \binom{n}{2} \lambda^{n-2} & \ldots & \binom{n}{k} \lambda^{n-k+1}
\end{bmatrix} e = \begin{bmatrix}
(k_n) \lambda^{n-k+1} \\
(k_{n-1}) \lambda^{n-k+2} \\
(k_{n-2}) \lambda^{n-k+3} \\
\vdots \\
(k_1) \lambda^{n-k+1}
\end{bmatrix}.
\]
Notice that $\binom{n}{k} \lambda^{n-k+1}$ is the $(k-1)$th coordinate of this vector. If we equip $X$ with the norm
\[||x|| = |x_1| + \ldots + |x_t|, \quad t = \dim X,
\]
and if we view $e$ and the above vector in $X$, we will have:
\[
\left(\binom{n}{k} \lambda^{n-k+1}\right) \leq ||P^n P^{-1} e|| \leq ||P^n|| ||P^{-1}|| \leq C_0 ||\lambda^n||
\]
for some constant $C_0$, where the last inequality holds by proposition 10 and because on finite-dimensional space all the norms are equivalent. This is clearly a contradiction. Therefore there is no non-trivial Jordan block, i.e. $L$ is diagonalizable. This implies
\[X = \ker(\mathcal{L}_s - \lambda).
\]
It is clear that if $g$ is in $\ker(\mathcal{L}_s - \lambda)$ then $g / h_s$ is in $\ker(\mathcal{L}_s - e^{-P(x)} \lambda)$. Therefore to show each $\ker(\mathcal{L}_s - e^{-P(x)} \lambda)$ is one dimensional, it is enough to show $\ker(\mathcal{L}_s - e^{-P(x)} \lambda)$ is one dimensional. Let $g \in \ker(\mathcal{L}_s - e^{-P(x)} \lambda)$, for each $n$
\[|\vec{g}| = |e^{-P(x)} \lambda g| = |\vec{\mathcal{L}}^n g| \leq \vec{\mathcal{L}}^n |g|.
\]
Therefore if we apply the above lemma to the function $|g|$ we obtain
\[|g| \leq \int \left| g \right| d\mu_\lambda.
\]
Continuity of $g$ and the fact that $\text{supp}(\mu_\lambda) = \mathcal{E}_\lambda^\prime$ (see explanation below the definition 6) makes this inequality into equality, i.e. every eigenvector has constant modulus. It is not hard to see that
\[
\vec{\mathcal{L}}^n g(\rho) = e^{-nP(x)} \sum_{\omega \in F_\lambda} \exp(\theta S_\omega f(\omega \rho)) \frac{1}{h_\lambda(\sigma^{-1} \omega \rho) h_\lambda(\sigma^{-2} \omega \rho)} \ldots \frac{1}{h_\lambda(\sigma \omega) h_\lambda(\omega \rho)} g(\omega \rho).
\]
Moreover since $\vec{\mathcal{L}}_\lambda(1) = 1$ we get:
\[
1 = \vec{\mathcal{L}}^n_\lambda(1)(\rho) = \sum_{\omega \in F_\lambda} e^{-nP(x)} \frac{1}{h_\lambda(\rho)} \exp(\theta S_\omega f(\omega \rho)) \frac{1}{h_\lambda(\sigma^{-1} \omega \rho) h_\lambda(\sigma^{-2} \omega \rho)} \ldots \frac{1}{h_\lambda(\sigma \omega) h_\lambda(\omega \rho)} g(\omega \rho).
\]
Note that every term in this sum, say $u_\omega$, is positive. Eventually, we find:
\[
e^{-nP(x)} \lambda^n g(\rho) = \vec{\mathcal{L}}^n g(\rho) = \sum_{\omega \in F_\lambda} u_\omega \exp(iy S_\omega f(\omega \rho)) g(\omega \rho).
\]
Now note that $|\sum_j a_j| = \sum_j |a_j|$ implies all $a_j$ are co-linear, this along with the fact that $g$ has constant modulus we get
\[g(\omega \rho) = e^{-nP(x)} \lambda^n \exp(-iy S_\omega f(\omega \rho)) g(\rho).
\]
This means values of $g$ on the dense set $\cup_\lambda \sigma^{-n}(\rho)$ (see remark below the definition 2) is determined by $g(\rho)$, so $g$ spans $\ker(\mathcal{L}_s - e^{-P(x)} \lambda)$ as long as $g$ has at least one non-zero point. This shows $\lambda$ is a simple eigenvalue and it finishes the proof.

Thus everything is ready to obtain spectral representation of $\mathcal{L}_s$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$ of modulus $e^{P(x)}$. We use the above proposition to see that for each $s = x + iy \in \Gamma^+$, $\mathcal{L}_s$ has only finitely many eigenvalues $\lambda_1(s), \ldots, \lambda_p(s)$ of modulus $e^{P(x)}$ each of which isolated in the spectrum and actually they are all simple eigenvalues. Therefore we may use theorem 2 to obtain the following spectral representation of the transfer operator:
\[\mathcal{L}_s = \lambda_1(s) P_{1,s} + \lambda_2(s) P_{2,s} + \ldots + \lambda_p(s) P_{p,s} + D_s,
\]
where each $P_{i,s}$ is projection. Note that in this equation the operators are analytic operators and eigenvalues are analytic functions. Further, the composition of every two different operators on the right-hand side vanishes by proposition 9. This yields

$$L^s = \lambda_1(s)^{\hat{0}} P_{1,s} + \lambda_2(s)^{\hat{0}} P_{2,s} + \ldots + \lambda_n(s)^{\hat{0}} P_{n,s} + D^s.$$ 

Finally proposition 9 implies:

$$\text{Sp}(L_s) \cup \{0\} = \{\lambda_1(s)\} \cup \{\lambda_2(s)\} \cup \ldots \cup \{\lambda_n(s)\} \cup \text{Sp}(D_s) \cup \{0\}.$$ 

We finish this section with the following lemma.

**Lemma 10.** For every $s_0$ on the line $x = 1$, there is a neighborhood $U$ of $s_0$, $0 < \beta < 1$ and constant $C > 0$ such that for every positive integer $m$

$$\|D_s^{\beta m}\|_a \leq C \beta^m, \quad s \in U.$$ 

**Proof.** The above spectral decomposition implies the spectral radius of $D_s$ to be strictly less than that of $L_s$. Furthermore, proposition 10 implies $r(L_{s_0}) \leq e^{P(1)} = 1$ so for $s_0$ there is $0 < \beta < 1$ such that $r(D_{s_0}) < \beta$. Thus there is constant $C_1$ and natural number $q$ such that

$$\|D_s^q\|_a \leq C_1 \beta^q \leq \frac{\beta}{2}.$$ 

Additionally, using continuity on a small enough ball $U$ at $s_0$ we have

$$\|D_s^q - D_{s_0}^q\|_a < \frac{\beta}{2}.$$ 

Combining these two recent inequalities yields $\|D_s^q\|_a \leq \beta$ on $U$. Furthermore, there is constant $C_2$ such that for each integer $r$ with $0 \leq r < q$, we have $\|D_s^r\|_a \leq C_2$ on $U$. Since for each positive integer $m$ we can write $m = lq + r$, we eventually get for some $C > 0$:

$$\|D_s^m\|_a \leq C(\beta^{1/q})^m$$ 

on $U$. 

4. **Counting Function and Poincaré Series**

Given $\rho \in E^n_A$ and $B \subseteq E^n_A$, for every $T > 0$ we define several **counting functions**.

(a) The central counting function for us is

$$N_\rho(B, T) := \#\{\omega \in \bigcup_{n=1}^\infty E^n_A : \omega \rho \text{ admissible, } \omega \rho \in B, S_{[\omega]} f(\omega \rho) \geq -T\}.$$ 

It is not so hard to see that this is a step function of $T$, continuous from right and increasing. In order to associate a complex function to this counting function we set $N_\rho(B, T) = 0$ for $T < 0$ and we consider the Laplace–Stieltjes transform of $T \mapsto N_\rho(B, T)$ which we call it **Poincaré series**:

$$\eta_\rho(B, s) := \int_0^\infty \exp(-sT) d N_\rho(B, T).$$ 

We will talk about its convergence in the next proposition. Below we introduce other counting functions appropriate for our purposes.

(b) Let $H = \{\tau_{(i)}\}_{i \in I}$ be a countable (finite or infinite) collection of finite words of bounded length, i.e. there exists a positive integer $k$ such that $|\tau_{(i)}| \leq k$ for each $i \in I$. Further, assume the cylinders $\{\tau_{(i)}\}_{i \in I}$ are mutually disjoint. We denote

$$[H] := \bigcup_{i \in I} [\tau_{(i)}],$$

then the corresponding Poincaré series is of the form

$$\eta_\rho([H], s) = \int_0^\infty \exp(-sT) d N_\rho([H], T).$$

$$= \sum_{n=1}^\infty \exp(-sT_n) \left( N_\rho([H], T_n) - N_\rho([H], T_{n-1}) \right)$$

where $T_1 < T_2 < T_3 < \ldots$ is the increasing sequence of discontinuities of $T \mapsto N_\rho([H], T)$. Eventually this sums up to

$$\eta_\rho([H], s) = \sum_{n=1}^\infty \sum_{\omega \rho \in [H]} \exp(sS_n f(\omega \rho)) = \sum_{n=1}^\infty \mathcal{L}_n^\omega(\tau_{[H]})(\rho).$$
Lemma 11. Let $\omega \in \{\omega \in \mathbb{N} : \omega \rho \text{ admissible}, \omega \rho \in [H], S_{[\omega]}f(\omega \rho) \geq -T\}$, then its Poincaré series would be
\[
\eta_\rho([H], s) = \sum_{n=1}^{\infty} L_n^\rho([1_H])(s).
\]

Proof. It is enough to apply Lemma 2:
\[
|S_{[\tau\omega]}f(\tau\omega\gamma) - S_{[\tau\gamma]}f(\tau\gamma\omega\tau(\gamma+))| \leq K_f e^{-(k+q)\alpha},
\]
where $K$ only depends on $f$.

Proof. It is enough to apply Lemma 2:
\[
|S_{[\tau\gamma]}f(\tau\gamma\omega\tau(\gamma+))| \leq K_f d(\tau\gamma, \tau(\gamma+)) \leq K_f e^{-(k+q)\alpha}.
\]

Lemma 12. Let $q$ be a positive integer, then the following inequalities hold:

(i) \[
N_{\text{per}}([\tau], [q], T) \leq N_{\text{per}}([\tau], q, T + K),
\]

(ii) \[
\sum_{\tau \in E_A^{\text{per}}} N_{\tau(\gamma+)}(\tau \gamma, T - Ke^{-(k+q)\alpha}) \leq N_{\text{per}}(\tau, T),
\]

(iii) \[
N_{\text{per}}([\tau], T) \leq \sum_{\tau \in E_A^{\text{per}}} N_{\tau(\gamma+)}([\tau \gamma], T + Ke^{-(k+q)\alpha}),
\]

(iv) For $i \geq k + q$
\[
N_{\tau(\gamma+)}([\tau \gamma], i, T) \leq N_{\tau(\gamma+)}([\tau \gamma], i, T + K),
\]
(v) If $F$ is any finite subset of $E_A^N$ and $F' = E_A^N \setminus F$, then
\[ N_{\text{per}}((\tau), T) \leq \sum_{\gamma \in F} N_{\tau \gamma}(\gamma, T) + Ke^{-(k+q)\alpha} \]
\[ + \sum_{\gamma \in F'} N_{\tau \gamma + i}(\gamma, T + 2K) + \sum_{i=1}^{k+q-1} N_{\tau \gamma + i}(\tau, i, T + K) \]
where $K$ only depends on $f$.

**Proof.**

(i) Let $\omega$ be a finite word contributing to $N_{\text{per}}(\omega, q, T)$, then $|\omega| = q$. The fact that $\overline{\omega} \in [\tau]$ gives $\omega_1 = \tau_1$. Therefore since $\omega_1 \omega_1$ is admissible, so is $\omega \tau^+$. If $q \geq k = |\tau|$, since $\overline{\omega} \in [\tau]$ so is $\omega \tau^+\gamma$, and if $q < k$, since $\overline{\omega} \in [\tau]$, we can write $\tau$ as $m$ copies of $\omega$ and some remainders, i.e., $\tau = \omega^m \omega_1 \ldots \omega_r$. It is clear that the first $k$ letters of $\omega^{m+1} \omega_1 \ldots \omega_r$ is again $\tau$. Thus $\omega \tau^+ \in [\tau]$. It remains to show $S_{[\tau]} \psi(\omega \tau^+) \geq -T - K$. From our assumption $S_{[\tau]} \psi(\overline{\omega}) \geq -T$, we can apply Lemma 2 to see that
\[ S_{[\tau]} \psi(\overline{\omega}) \leq S_{[\tau]} \psi(\omega \tau^+) + K. \]
This finishes the proof for part (i).

(ii) Let $\gamma$ be a word of length $q$ with $\tau \gamma$ admissible. Let $\omega$ be a finite word contributing to $N_{\tau \gamma}(\gamma, T - K e^{-(k+q)\alpha})$, we want to show $\gamma \omega$ contributes to $N_{\text{per}}(\tau, T)$. It is clear that this way of contribution is injective, which proves (ii). Since $\gamma \omega \gamma \tau(\gamma) + \gamma$ is admissible, so is $\gamma \overline{\omega} \gamma \tau(\gamma)$. Moreover, we know $S_{[\gamma \omega \gamma \tau(\gamma)]} \psi(\gamma \omega \gamma \tau(\gamma)) \geq -T - K e^{-(k+q)\alpha}$. If we use the above lemma we find
\[ -T + K e^{-(k+q)\alpha} \leq S_{[\gamma \omega \gamma \tau(\gamma)]} \psi(\gamma \omega \gamma \tau(\gamma)) \leq S_{[\gamma \omega \gamma \tau(\gamma)]} \psi(\gamma \overline{\omega} \gamma \tau(\gamma)) + K e^{-(k+q)\alpha}, \]
which shows $-T \leq S_{[\gamma \omega \gamma \tau(\gamma)]} \psi(\gamma \overline{\omega} \gamma \tau(\gamma))$ as needed.

(iii) Let $\omega$ be a finite word contributing to $N_{\text{per}}([\tau], T)$ of length $n$. The fact that $\overline{\omega} \in [\tau]$ gives $\omega_1 = \tau_1$. Therefore since $\omega_1 \omega_1$ is admissible, so is $\omega \tau$. Note that $\tau = \cup' \gamma \tau$, where the union is over all $\gamma$ with length $q$ such that $\tau \gamma$ is admissible. Since $\overline{\omega} \in [\tau]$, there should be $\gamma$ such that $\overline{\omega} \in [\tau \gamma]$. Since $\omega \tau$ is admissible, so is $\omega \tau \gamma$. Next we want to show $\omega \tau \gamma(\gamma) + \omega \tau$ is admissible. If we separate into two cases where $n \geq k + q$ and $n < k + q$, then in exactly a similar manner as in part (i) we obtain this. It remains only to show $S_{[\tau]} \psi(\omega \tau \gamma(\gamma)) \geq -T - K e^{-(k+q)\alpha}$. We have already $S_{[\tau]} \psi(\overline{\omega}) \geq -T$, furthermore if we use lemma 2 we see that
\[ |S_{[\tau]} \psi(\overline{\omega}) - S_{[\tau]} \psi(\omega \tau \gamma(\gamma))| \leq K d(\overline{\omega}, \tau \gamma(\gamma)) \leq K e^{-(k+q)\alpha}, \]
where the last inequality is due to $\overline{\omega} \in [\tau \gamma]$. Thus from this inequality, we obtain
\[ -T - K e^{-(k+q)\alpha} \leq S_{[\tau]} \psi(\omega \tau \gamma(\gamma)). \]
This completes part (iii).

(iv) Take $\omega$ that contributes to $N_{\tau \gamma}(\gamma, t, T)$. Clearly, $\omega \tau^+ \gamma$ is admissible. Since $|\omega| \geq k + q$ then we have clearly $\omega \tau^+ \in [\tau \gamma]$ as well. Further, note that
\[ |S_{[\tau]} (\omega \tau \gamma(\gamma)) - S_{[\tau]} (\omega \tau^+ \gamma)| \leq K. \]
(v) Take $\omega$ such that it contributes to $N_{\text{per}}([\tau], T)$. If its length is less than $k + q$, then we use part (i). This contributes to the third sum on the right-hand side. If the length of $\omega$ is at least $k + q$, then (iii) and (iv) tell us $\omega$ contributes to either of the first two sums on the right-hand side. This finishes the proof of (v) and the lemma.

Below we want to prove the item (v) from the above lemma without $[\tau]$. Let $\rho \in E_A^N$, then due to our assumption that shift space is finitely irreducible, there exists a finite set consisting of finite words
\[ \Omega = \{ \tau(1), \ldots, \tau(r) \} \]
such that for every finite word $\omega$ there exists $\tau(j) \in \Omega$ with $\omega \tau(j) \rho$ being admissible. Below we have a summation over all $\tau(j) \rho$, while this might not be admissible for all $j = 1, \ldots, r$. Note that the sum is only taken over those $j$s where $\tau(j) \rho$ is admissible. Note, that this $\Omega$ and $r$ is independent of $\rho$. 

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Lemma 13. If $F$ is any finite subset of $E^q_A$ and $F' = E^q_A \setminus F$, for any $\rho \in E^q_A$ we have
\[
N_{\text{per}}(T) \leq \sum_{\gamma \in F} N_{\gamma^+}([\gamma], T + Ke^{-qa}) + \sum_{j=1}^r N_{\tau(j)\rho}([F'], T + K) + \sum_{j=1}^r \sum_{i=1}^{q-1} N_{\tau(j)\rho}(i, T + K),
\]
where $K$ only depends on $f$.

Proof. The proof is similar to item (v) in the above lemma. Let $\omega$ be a finite word contributing to $N_{\text{per}}(T)$, pick $\tau(j) \in \Omega$ such that $\omega \tau(j)\rho$ is admissible. If $|\omega| < q$, clearly $\omega$ is contributing to the third term on the right hand of the inequality. If $|\omega| \geq q$ and $\omega_1 \ldots \omega_q \in F$, we want to show $\omega$ contributes to $N_{\gamma^+}([\gamma], T + Ke^{-qa})$ where $\gamma = \omega_1 \ldots \omega_q$. Since $\omega$ is periodic $\omega \omega_1$ is admissible, and so is $\omega \gamma \gamma^+$. It is clear that $\omega \gamma \gamma^+ \in [\gamma]$ as well. Further, note that
\[
|S_{\omega} f(\omega \gamma \gamma^+) - S_{\omega} f(\omega)| \leq K d(\gamma \gamma^+, \omega) \leq Ke^{-qa}.
\]
Finally, in case $|\omega| \geq q$ and $\omega_1 \ldots \omega_q \in F'$ we want to show $\omega$ is contributing to the second sum on the right-hand side. This is similar to our previous case. \hfill \blacksquare

Moreover, we have the following two estimates for the eigenfunction $h$ and the equilibrium state $\mu$.

Lemma 14. Let $\omega$ be a word of length $n$ such that $\omega \rho$, $\omega \rho'$ are admissible, then we have
\[
1 - K_1 e^{-na} \leq \frac{h(\omega \rho)}{h(\omega \rho')} \leq 1 + K_1 e^{-na},
\]
where $K_1$ only depends on $h$.

Proof. We know from [MU03, p. 34] that $h$ is Hölder continuous, therefore there is a constant $K_0$, such that
\[
|h(\omega \rho) - h(\omega \rho')| \leq K_0 d(\omega \rho, \omega \rho') \leq K_0 e^{-na}.
\]
Dividing by $h(\omega \rho)$ and using lemma 8, we obtain
\[
\left|\frac{h(\omega \rho)}{h(\omega \rho')} - 1\right| \leq \frac{K_0}{h(\omega \rho)} e^{-na} \leq K_1 e^{-na},
\]
where $K_1 = K_0 \frac{M+1}{R}$. \hfill \blacksquare

Lemma 15. Let $\omega$ be a finite word of length $n$ such that $\omega \rho$ is admissible, then
\[
(1 - K_1 e^{-na})h(\omega \rho)m(|\omega|) \leq \mu(|\omega|) \leq (1 + K_1 e^{-na})h(\omega \rho)m(|\omega|),
\]
where $K_1$ is a constant depending only on $h$.

Proof. We saw in the proof of the lemma 9 that $\mu(A) = \int_A h dm$. Therefore we have
\[
\left(\inf_{|\omega|} h\right) m(|\omega|) \leq \mu(|\omega|) \leq \left(\sup_{|\omega|} h\right) m(|\omega|).
\]
Now we use the above lemma to see
\[
(1 - K_1 e^{-na})h(\omega \rho) \leq \inf_{|\omega|} h \leq \sup_{|\omega|} h \leq (1 + K_1 e^{-na})h(\omega \rho).
\]
This finishes the proof. \hfill \blacksquare

Proposition 12. The functions $\eta_\rho([H], s)$, $\eta_\rho(H, s)$ are holomorphic on $\text{Re}(s) > 1$, and the function $\eta_\rho([H], q, s)$ is holomorphic on $\Gamma^+$. 

Proof. Using the relation 18, if we show $\eta_\rho([H], s)$ is holomorphic then $\eta_\rho(H, s)$ will be holomorphic as well. In order to show $\eta_\rho([H], s)$ is holomorphic we need $|\mathcal{L}_2^r([H])|_\infty$:
\[
|\mathcal{L}_2^r([H])|_\infty \leq |\mathcal{L}_2^r(1)|_\infty \leq \sum_{\omega \in E^q_A} \exp(\text{Re}(s)) \sup_{|\omega|} S_{\omega} f.
\]
This reminds us of the pressure function. Using the fact that \( P \) is strictly decreasing on \( \Gamma \) from proposition 4, consider an arbitrary \( s_0 = x_0 + iy_0 \) with \( x_0 > 1 \), for any \( s \) with \( x \geq x_0 \) there is a negative \( r \) such that \( P(x) < r < 0 \), therefore there is \( N \) such that for \( n > N \):

\[
\frac{1}{n} \ln \left( \sum_{\omega \in E_k} \exp(x \sup_{[a]} S_n f) \right) < r,
\]

so

\[
|\mathcal{L}^n(\{1\}_H)|_\infty \leq |\mathcal{L}^n(1)|_\infty \leq \sum_{\omega \in E_k} \exp(x \sup_{[a]} S_n f) < e^n.
\]

This shows \( \eta_\rho([H], s) \) converges uniformly on compact sets, thus \( \eta_\rho([H], s) \) as a sum of holomorphic functions is holomorphic on \( \text{Re}(s) > 1 \).

The above expression of \( \eta_\rho([H], q, s) \) shows it is holomorphic on \( \Gamma^+ \).

**Proposition 13.** If \( f : E_A^\mathbb{N} \to \mathbb{R} \) has D-generic property, then each \( \eta_\rho([H], s) \) and \( \eta_\rho(H, s) \) at each point of the critical line \( \text{Re}(s) = 1 \) except \( s = 1 \) admits analytic continuation and at \( s = 1 \) admits a meromorphic extension with a simple pole and residue

\[
\text{Res}(\eta_\rho, 1) = \frac{h(\rho)}{\lambda_\mu} m([H]).
\]

If we lift the D-generic property, then there exists \( y_1 > 0 \) such that the above statement holds on the segment \( \{1 + iy : |y| < y_1 \} \) with the same residue at the simple pole \( s = 1 \). Furthermore, this \( y_1 \) doesn’t depend on \( H \) or \( \rho \).

**Proof.** By reviewing equations 17 and 14, it is clear that we can write

\[
\eta_\rho([H], s) = \sum_{k=1}^\infty \mathcal{L}^k(1_{[H]})
\]

\[
= \sum_{k=1}^\infty \left( \lambda_1(s)^k P_{1,1}(1_{[H]}) + \ldots + \lambda_n(s)^k P_{n,1}(1_{[H]}) + D_s(1_{[H]}) \right).
\]

Now we use proposition 4 to see \( |\lambda_i(s)| = e^{P_i(s)} < 1 \) if \( x > 1 \). Therefore we can continue the above equation

\[
= \lambda_1(s)(1 - \lambda_1(s)^{-1} P_{1,1}(1_{[H]}) + \ldots + \lambda_n(s)(1 - \lambda_n(s)^{-1} P_{n,1}(1_{[H]}) + Q_s(1_{[H]}),
\]

where \( Q_s = \sum_{k=1}^\infty D_s^k \) converges using lemma 10. This is a valid relation for the Poincaré series \( \eta_\rho \) on \( x > 1 \). We fix \( s_0 \) on the line \( x = 1 \), it is clear that \( Q_s(1_{[H]}) \) is a holomorphic function on the neighborhood \( U \) of \( s_0 \) obtained in lemma 10. Additionally all the projections \( P_{i,1} \) and function \( \lambda_i(s) \) are analytic as discussed just above the equation 14. Therefore the right-hand side of the above equation is analytic on some neighborhood \( U_0 \) of \( s_0 \), as long as \( \lambda_i(s_0) \neq 1 \). As we know for real \( s = x + iy \), one of the eigenvalues of the transfer operator is \( e^{P_i(s)} \) by theorem 1. We let \( \lambda_i(s) \) represent this eigenvalue, it is clear that \( \lambda_i(s) \) is not constant on any neighborhood of \( s = 1 \) as \( |\lambda_i(s)| = e^{P_i(s)} \) and \( P \) is strictly decreasing by proposition 4. Since \( \lambda_i(s) \) are isolated, simple eigenvalues and further analytic functions identity theorem from complex analysis guarantee the existence of \( y_1 > 0 \) for which the equations \( \lambda_i(s) = 1 \) on \( \{1 + iy : |y| < y_1 \} \) have a solution only if \( i = 1 \) and \( s = 1 \). We deduce the right-hand side of the equation above defines an analytic function on a neighborhood of \( \{1 + iy : 0 < |y| < y_1 \} \). Note that \( \lambda_i(s) \) is simple eigenvalue, so near \( s = 1 \) we expect

\[
(1 - \lambda_i(s)) \sim s - 1.
\]

In other words, we find that \( \eta_\rho([H], s) - A/(s - 1) \) admits analytic extension to the segment \( \{1 + iy : |y| < y_1 \} \), where

\[
A = \lim_{s \to 1} \eta_\rho([H], s)(s - 1) = \lambda_1(1) P_{1,1}(1_{[H]}) \lim_{s \to 1} \frac{s - 1}{1 - \lambda_1(s)}.
\]

It is clear that using the D-generic property \( y_1 \) can be taken to be \( \infty \). Thus, it only remains to compute \( A \). It is clear that \( \lambda_1 = \lambda_1(1) = e^{P_1(1)} = 1 \). To compute \( P_{1,1}(1_{[H]}) \) first note that \( \mathcal{L}_0 P_{1,1} = \lambda_1 P_{1,1} \) for each \( i \), so

\[
\int P_{1,1}(g)dm = \int \mathcal{L}_0 P_{1,1}(g)dm = \lambda_1 \int P_{1,1}(g)dm.
\]

This gives \( \int P_{1,1}(g) = 0 \) for every \( g \in C^0_s(E_A^\mathbb{N}, C) \) and \( i \neq 1 \). Therefore with respect to the measure \( m \) for each \( k \):

\[
\int g = \int \mathcal{L}_k^g = \int P_{1,1}(g) + \lambda_2^k P_{2,1}(g) + \ldots + \lambda_n^k P_{n,1}(g) + \int D_k^g.
\]
now implementing the inequality obtained in lemma 10 would yield

$$\int g = \int P_{1,1}(g).$$

This actually determines the action of $P_{1,1}$ since if $P_{1,1}(g) = k_g h$ then $k_g = \int g$, i.e.

$$P_{1,1}(g) = h \int g \, dm.$$

And lastly

$$\lim_{s \to 1} \frac{1 - \lambda_1(s)}{s - 1} = \lim_{s \to 1} \frac{1 - e^{P(s)}}{s - 1} = -P'(1)e^{P(1)} = - \int f \, d\mu = \chi_\mu,$$

where the equality to the last follows from proposition 2.6.13 in [MU03, p. 47]. Thus we find that the residue is $\rho(m([H])/\chi_\mu)$.  

### 5. ASYMPTOTIC FORMULA FOR COUNTING

In this section, we assume $f$ is strongly regular, summable and Hölder-type continuous with $P(1) = P(f) = 0$. We keep this assumption to the end of a proposition 18 and after that, we consider general functions with $P(\delta) = P(\delta f) = 0$ for some $\delta > 0$. We want to find an asymptotic formula for the counting functions presented in the previous section. We can provide a formula for some estimate of the lower bound and upper bound of all possible values. As well in this section by $y_0$ we mean

$$y_0 = \frac{y_1}{2\pi},$$

where $y_1$ was obtained in proposition 13. As mentioned in that proposition, this $y_0$ doesn’t depend on $H$ in $\eta_\rho([H], T)$. Further, we set

$$c_1 := y_0^{-1} (\exp(y_0^{-1}) - 1)^{-1}, \quad c_2 := y_0^{-1} (\exp(y_0^{-1}) - 1)^{-1} \exp(y_0^{-1}).$$

**Proposition 14.**

$$c_1 \frac{h(\rho)}{\chi_\mu} m([H]) \leq \liminf_{T \to \infty} \frac{N_\rho(H, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_\rho(H, T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_\mu} m([H]),$$

and

$$c_1 \frac{h(\rho)}{\chi_\mu} m([H]) \leq \liminf_{T \to \infty} \frac{N_\rho([H], T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_\rho([H], T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_\mu} m([H]),$$

and for every positive integer $q$

$$\lim_{T \to \infty} \frac{N_\rho([H], q, T)}{\exp(T)} = 0.$$

**Proof.** The first two lines of inequalities follow from proposition 13 and applying Graham-Vaaler theorem 4. The last equality follows from proposition 12 and applying Ikehard-Wiener theorem 3.  

**Proposition 15.**

$$c_1 \frac{1}{\chi_\mu} \mu([\tau]) \leq \liminf_{T \to \infty} \frac{N_{per}(\tau, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{per}(\tau, T)}{\exp(T)} \leq c_2 \frac{1}{\chi_\mu} \mu([\tau]),$$

and

$$c_1 \frac{1}{\chi_\mu} \mu([\tau]) \leq \liminf_{T \to \infty} \frac{N_{per}([\tau], T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{per}([\tau], T)}{\exp(T)} \leq c_2 \frac{1}{\chi_\mu} \mu([\tau]).$$
Proof. Let $\sum'$ represent the sum over all $\gamma$ with length $q$ such that $\tau\gamma$ is admissible. Then using part (ii) of lemma 12, lemma 4 and proposition 14 we can write:

$$
\liminf_{T \to \infty} \frac{N_{\text{per}}(\tau, T)}{\exp(T)} \geq \liminf_{T \to \infty} \sum_{\tau} \frac{N_{\tau}(\tau, T - K e^{-(k+q)u})}{\exp(T)} \\
\geq \exp\left(-K e^{-(k+q)u}\right) \sum_{\tau} \liminf_{T \to \infty} \frac{N_{\tau}(\tau, T - K e^{-(k+q)u})}{\exp(T - K e^{-(k+q)u})} \\
= \exp\left(-K e^{-(k+q)u}\right) \sum_{\tau} c_i \frac{h(\tau\gamma)^+}{X_{\mu}} m([\tau\gamma]).
$$

We use lemma 15 at this step and continue:

$$
\liminf_{T \to \infty} \frac{N_{\text{per}}(\tau, T)}{\exp(T)} \geq c_1 \exp\left(-K e^{-(k+q)u}\right) \sum_{\tau} \frac{(1 + K_1 e^{-(k+q)u})^{-1} \mu([\tau\gamma])}{X_{\mu}} \\
= c_1 \frac{1}{X_{\mu}} \mu([\tau]).
$$

Since $q$ is arbitrary, by $q \to \infty$ we obtain

$$
\liminf_{T \to \infty} \frac{N_{\text{per}}(\tau, T)}{\exp(T)} \geq c_1 \frac{1}{X_{\mu}} \mu([\tau]).
$$

If we show

$$
\limsup_{T \to \infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq c_2 \frac{1}{X_{\mu}} \mu([\tau]),
$$

we are done with the proof. We use lemma 12 part (v) for this and then we apply proposition 14 several times.

$$
\limsup_{T \to \infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq \limsup_{T \to \infty} \sum_{\tau \in E_A^\infty} \frac{N_{\tau}(\tau, T + K e^{-(k+q)u})}{\exp(T)} \\
+ \limsup_{T \to \infty} \sum_{\tau \in E_A^\infty} \frac{N_{\tau}(\tau, T + 2K)}{\exp(T)} + \limsup_{T \to \infty} \sum_{i=1}^{k+q-1} \frac{N_{\tau}(\tau, i, T + K)}{\exp(T)}.
$$

Now the first limsup easily passes through the finite sum and we use proposition 14 with $H = \tau\gamma$, for the second limsup note that

$$
\sum_{\tau \in E_A^\infty \backslash \tau F^\infty} N_{\tau}(\tau, T + 2K) = N_{\tau}(\tau F^\infty, T + 2K),
$$

therefore we apply proposition 14 with $H = \tau F^\infty$ and the last limsup is clearly 0 using again proposition 14. Thus we get

$$
\limsup_{T \to \infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq \sum_{\tau \in E_A^\infty \backslash \tau F^\infty} \frac{N_{\tau}(\tau, T + K e^{-(k+q)u})}{\exp(T + K e^{-(k+q)u})} \exp\left(K e^{-(k+q)u}\right) \\
+ \limsup_{T \to \infty} \frac{N_{\tau}(\tau, T + 2K)}{\exp(T + 2K)} \exp(2K) \\
= \exp\left(K e^{-(k+q)u}\right) \sum_{\tau \in E_A^\infty \backslash \tau F^\infty} c_2 \frac{h(\tau\gamma)^+}{X_{\mu}} m([\tau\gamma]) + c_2 \frac{h(\tau\gamma)^+}{X_{\mu}} m([\tau F^\infty]) \exp(2K).
$$

Notice that since $F$ was arbitrary for $\epsilon > 0$ we choose $F$ such that

$$
c_2 \frac{h(\tau\gamma)^+}{X_{\mu}} m([\tau F^\infty]) \exp(2K) < \epsilon,
$$

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then we obtain
\[ \limsup_{T \to \infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq \exp \left( K e^{-(k+q)\alpha} \right) \sum_{\gamma \in F} c_2 \frac{h(\tau \gamma)}{\mu(\gamma)} m(\gamma) + \epsilon. \]

Now we apply left-hand side of the lemma 15:
\[ \limsup_{T \to \infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq \exp \left( K e^{-(k+q)\alpha} \right) \sum_{\gamma \in F} c_2 \frac{1}{\mu(\gamma)} + \epsilon. \]

Eventually we let \( q \to \infty \) to get
\[ \limsup_{T \to \infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq \sum_{\gamma \in F} c_2 \frac{1}{\mu(\gamma)} + \epsilon = c_2 \frac{1}{\mu(F)} + \epsilon \leq c_2 \frac{1}{\mu(\tau)} + \epsilon. \]

Since \( \epsilon \) was arbitrary we have
\[ \limsup_{T \to \infty} \frac{N_{\text{per}}([\tau], T)}{\exp(T)} \leq c_2 \frac{1}{\mu(\tau)}. \]

**Proposition 16.**
\[ \limsup_{T \to \infty} \frac{N_{\text{per}}(T)}{\exp(T)} \leq c_2 \frac{1}{\mu}. \]

**Proof.** This proof is exactly similar to the proof of the previous proposition for limsup and implementing lemma 13.

**Proposition 17.** For every open set \( V \subseteq E^n \) we have
\[ c_1 \frac{h(\rho)}{\mu} \leq \liminf_{T \to \infty} \frac{N_\rho(V, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_\rho(V, T)}{\exp(T)} \leq c_1 \frac{h(\rho)}{\mu} + y_0^{-1} h(\rho), \]
and
\[ c_1 \frac{1}{\mu} \leq \liminf_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} \leq c_1 \frac{1}{\mu} + y_0^{-1} \frac{1}{\mu}. \]

**Proof.** We know from proposition 1 that \( V \) can be written as a union of disjoint cylinders, so \( V = \cup_i \{\tau_i\} \).

Therefore using lemma 4 and proposition 14 with \( H = \tau_{i(0)} \) one can write
\[ \liminf_{T \to \infty} \frac{N_\rho(V, T)}{\exp(T)} = \liminf_{T \to \infty} \sum_i \frac{N_\rho([\tau_i), T)}{\exp(T)} \geq \sum_i \liminf_{T \to \infty} \frac{N_\rho([\tau_i), T)}{\exp(T)} \]
\[ \geq \sum_i c_1 \frac{h(\rho)}{\mu} m(\tau_i) = c_1 \frac{h(\rho)}{\mu} m(V). \]

For the limsup we use lemma 5 and the above inequality for the open set \( V' \) to find
\[ c_1 \frac{h(\rho)}{\mu} m(V') + \limsup_{T \to \infty} \frac{N_\rho(V, T)}{\exp(T)} \]
\[ \leq \liminf_{T \to \infty} \frac{N_\rho(V', T)}{\exp(T)} + \limsup_{T \to \infty} \frac{N_\rho(V, T)}{\exp(T)} \]
\[ \leq \limsup_{T \to \infty} \frac{N_\rho(V', T) + N_\rho(V, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_\rho(T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\mu}, \]
where the last inequality holds if we apply proposition 14 for \( H = E \) (all the alphabets). This yields
\[ \limsup_{T \to \infty} \frac{N_\rho(V, T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\mu} - c_1 \frac{h(\rho)}{\mu} m(V') = c_1 \frac{h(\rho)}{\mu} m(V) + y_0^{-1} h(\rho). \]
For counting periodic words, the idea is similar. Again we implement lemma 4 and this time proposition 15 to obtain:

\[
\liminf_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} = \liminf_{T \to \infty} \sum_{i} \frac{N_{\text{per}}(\tau_i, T)}{\exp(T)} \geq \sum_{i} \liminf_{T \to \infty} \frac{N_{\text{per}}(\tau_i, T)}{\exp(T)} \geq \sum_{i} c_1 \frac{1}{\chi_{\mu}} \mu(\tau_i) = c_1 \frac{1}{\chi_{\mu}} \mu(V).
\]

Applying lemma 5 and the above inequality for the open set \( \overline{V} \) gives us:

\[
c_1 \frac{1}{\chi_{\mu}} \mu(\overline{V}) + \limsup_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} \leq \liminf_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} + \limsup_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} \leq c_2 \frac{1}{\chi_{\mu}},
\]

where the last inequality is due to the above proposition. This eventually gives

\[
\limsup_{T \to \infty} \frac{N_{\text{per}}(V, T)}{\exp(T)} \leq c_2 \frac{1}{\chi_{\mu}} - c_1 \frac{1}{\chi_{\mu}} \mu(\overline{V}) = c_1 \frac{1}{\chi_{\mu}} \mu(\overline{V}) + y_0^{-1} \frac{1}{\chi_{\mu}}.
\]

**Proposition 18.** For every Borel set \( B \subseteq E_A^n \) we have

\[
c_1 \frac{h(\rho)}{\chi_{\mu}} m(B^n) \leq \liminf_{T \to \infty} \frac{N_{\rho}(B, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{\rho}(B, T)}{\exp(T)} \leq c_1 \frac{h(\rho)}{\chi_{\mu}} m(B) + y_0^{-1} h(\rho),
\]

and

\[
c_1 \frac{1}{\chi_{\mu}} \mu(B^n) \leq \liminf_{T \to \infty} \frac{N_{\text{per}}(B, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{\text{per}}(B, T)}{\exp(T)} \leq c_1 \frac{1}{\chi_{\mu}} \mu(B) + y_0^{-1} \frac{1}{\chi_{\mu}}.
\]

**Proof.** We only prove the first line of inequalities. The other one is proved in a similar manner. We apply the above proposition to open set \( B^c \):

\[
c_1 \frac{h(\rho)}{\chi_{\mu}} m(B^n) \leq \liminf_{T \to \infty} \frac{N_{\rho}(B^c, T)}{\exp(T)} \leq \liminf_{T \to \infty} \frac{N_{\rho}(B, T)}{\exp(T)}.
\]

For \( \limsup \) we use lemma 5 and this inequality for \( \overline{B}^c \):

\[
\liminf_{T \to \infty} \frac{N_{\rho}(\overline{B}, T)}{\exp(T)} + \limsup_{T \to \infty} \frac{N_{\rho}(\overline{B}, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{\rho}(\overline{B}, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{\rho}(\overline{B}, T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_{\mu}},
\]

where the last inequality holds if we apply proposition 14 for \( H = E \) (all the alphabets). Thus

\[
\limsup_{T \to \infty} \frac{N_{\rho}(B, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_{\rho}(\overline{B}, T)}{\exp(T)} \leq c_2 \frac{h(\rho)}{\chi_{\mu}} - c_1 \frac{h(\rho)}{\chi_{\mu}} m(B^n) = c_1 \frac{h(\rho)}{\chi_{\mu}} m(B) + y_0^{-1} \frac{1}{\chi_{\mu}} h(\rho).
\]

This finishes the proof.

Note that so far we focused on the systems with \( P(1) = 0 \). We want to show that this is not restrictive and we can otherwise get the corresponding counting formula as well. For a general Hölder-type function \( f : E_A^n \to \mathbb{R} \) we remember that \( x \in \Gamma \) iff \( xf \) is summable. Assuming strong regularity we know there exists \( \delta > 0 \) such that \( P(\delta) = 0 \) and \( \inf \Gamma < \delta \). Now if we consider a new function \( g = \delta f \), first it is clear that \( g \) as well is strongly regular. Secondly, since \( P(xg) = P(x\delta f) \) we have \( P_1(1) = 0 \). Therefore all the results obtained above are applicable for \( g \). Additionally, note that \( S_{\delta g}(\rho) = \delta S_g(\rho) \), so we find that

\[
N^g(\delta T) = N(T).
\]

Moreover, it is clear

\[
\mathcal{L}_{1g} = \mathcal{L}_{\delta f}.
\]
Therefore if $L_{\delta} h_\delta = h_\delta$ then $L_{1g} h_\delta = h_\delta$, similarly if $L_{\delta}^* m_\delta = m_\delta$ then $L_{1g}^* m_\delta = m_\delta$. Additionally, if $L_{sf}$ avoids $\exp\left( P(\delta f) \right) = 1$ as eigenvalue on

$$\{ \delta + iy : 0 < |y| < y_0(\delta) \},$$

then $L_{sg}$ does so on

$$\{ 1 + iy : 0 < |y| < \frac{y_0(\delta)}{\delta} \}.$$

This implies $y_0(g) = \frac{y_0(\delta)}{\delta}$ and so

$$c_1(g) = (\frac{y_0}{\delta})^{-1} \left( \exp\left( \frac{y_0}{\delta} \right) - 1 \right)^{-1}.$$

It is now enough to use proposition 18 for $g$ with $m = m_\delta$, $\mu = \mu_\delta$ and $h = h_\delta$ to estimate:

$$c_1(g) \frac{h(\rho)}{\chi_\mu} m(B^o) \leq \liminf_{T \to \infty} \frac{N_\rho(B, T)}{\exp(T)} \leq \limsup_{T \to \infty} \frac{N_\rho(B, T)}{\exp(T)} \leq c_1(g) \frac{h(\rho)}{\chi_\mu} m(B) + \frac{y_0}{\delta}^{-1} \frac{h(\rho)}{\chi_\mu}.$$

Furthermore, note that

$$\chi_\mu = - \int g \, d\mu_\delta = -\delta \int f \, d\mu_\delta = \delta \chi_{\mu_\delta}.$$

Now we replace $T$ with $\delta T$ and use 19 to obtain the following estimate for $f$:

$$y_0^{-1} \left( \exp(\delta y_0^{-1}) - 1 \right)^{-1} \frac{h_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta(B^o) \leq \liminf_{T \to \infty} \frac{N_{\rho}(B, T)}{\exp(\delta T)} \leq \limsup_{T \to \infty} \frac{N_{\rho}(B, T)}{\exp(\delta T)} \leq y_0^{-1} \left( \exp(\delta y_0^{-1}) - 1 \right)^{-1} \frac{h_\delta(\rho)}{\chi_{\mu_\delta}} m_\delta(B) + y_0^{-1} \frac{h_\delta(\rho)}{\chi_{\mu_\delta}}.$$

Similarly, we can obtain a formula for $N_{\rho_{\text{per}}}(B, T)$ which we omit its proof. We set

$$(20) \quad c_\delta := y_0^{-1} \left( \exp(\delta y_0^{-1}) - 1 \right)^{-1}$$

and capture all the aforementioned arguments in the following theorem.

**Theorem 5.** If $f : E_A^N \to \mathbb{R}$ is strongly regular Hölder-type function with $P(\delta f) = 0$, for every Borel set $B \subseteq E_A^N$ and $\rho \in E_A^N$ we have

$$c_\delta \frac{h(\rho)}{\chi_\mu} m_\delta(B^o) \leq \liminf_{T \to \infty} \frac{N_{\rho_{\text{per}}}(B, T)}{\exp(\delta T)} \leq \limsup_{T \to \infty} \frac{N_{\rho_{\text{per}}}(B, T)}{\exp(\delta T)} \leq c_\delta \frac{\mu(B)}{\chi_\mu} + y_0^{-1} \frac{h(\rho)}{\chi_\mu}.$$
Corollary 2 (Pollicott-Urbański). Let $S = \{\phi_e\}_{e \in E}$ be a strongly regular conformal graph directed Markov system with $D$-generic property. Let $\delta$ be the Hausdorff dimension of the limit set of $S$, then for every Borel set $B \subseteq E_A^\infty$ with boundary of measure 0 and $\rho \in E_A^\infty$ we have

$$
\lim_{T \to \infty} \frac{N_{\rho}(B,T)}{\exp(\delta T)} = \frac{h_f(\rho)}{\delta \mu_\delta(B)},
$$

and

$$
\lim_{T \to \infty} \frac{N_{\rho|\delta}(B,T)}{\exp(\delta T)} = \frac{1}{\delta \mu_\delta(B)}.
$$

Proof. It follows from the previous corollary. Note that when $S$ is $D$-generic then we are allowed to let $y_0 \to \infty$ and this gives $c_\delta \to \frac{1}{\delta}$ from 20.

6. Asymptotic Formula for Length

Before bringing some examples we would like to talk about counting with specified length. As indicated at the beginning of the previous section item (d) we had $N_\rho([H],q,T)$ which is counting the number of words $\omega$ satisfying $s_{[\omega]}f(\omega \rho) \geq -T$ of length $q$. We addressed in proposition 14 that growth of this relative to $\exp(\delta T)$ tends to 0. Therefore if we would like to obtain fairly interesting growth we have to focus on some counting where $q$ as well grows as $T$ grows. We know $N_\rho(T) \sim C \exp(\delta T)$ but if we write

$$
N_\rho(T) = \sum_{i=1}^{\infty} N_\rho(i,T),
$$

first, we should note that this sum is terminating at some point. More precisely, for $\rho$ if we set

$$
m(T) := \sup_{\omega \in E_\rho^T} |\omega| : S_{[\omega]}f(\omega \rho) \geq -T, \forall \omega' \in E_\rho^T, |\omega'| \leq |\omega|, \quad b_n := \inf_{\omega \in E_\rho^T} S_n(\omega \rho),
$$

$$
M(T) := \sup_{\omega \in E_\rho^T} \{|\omega| : S_{[\omega]}f(\omega \rho) \geq -T\}, \quad d_n := \sup_{\omega \in E_\rho^T} S_n(\omega \rho),
$$

then $N_\rho(i,T) = 0$ for $i > M(T)$, therefore

$$
N_\rho(T) = \sum_{i=1}^{M(T)} N_\rho(i,T).
$$

The question we ask is which term of the above sum on the right-hand side might have growth comparable to the left-hand side, i.e. for which $i(T)$ the growth of $N_\rho(T)/N_\rho(i(T),T)$ is not too fast?! With the tools we have, we couldn’t answer this question, however, we have some words on that. First, we prove the following.

Proposition 19. Both of the following limits exist:

$$
\lim_{T \to \infty} \frac{M(T)}{T} = r, \quad \lim_{T \to \infty} \frac{M(T)}{T} = s.
$$

Proof. First, we prove the latter. We set $M := M(T)$, let $\omega$ be a finite word making the supremum possible in the definition of $M(T)$, then for any $\tau \in E_\rho^{M+1}$ we find

$$
d_M \geq S_M f(\omega \rho) \geq -T > S_{M+1}(\tau \rho),
$$

$$
d_M \geq -T \geq d_{M+1},
$$

$$
\frac{d_M}{M} \geq \frac{-T}{M} \geq \frac{d_{M+1}}{M+1} \frac{M+1}{M}.
$$

Therefore it is enough to show that $d_M/n$ is convergent. To do so, we note that for arbitrary $\tau$, $\gamma$ with $|\tau| = m$, $|\gamma| = n$ where $\tau \gamma \rho$ is admissible, we can find $\omega \in \Omega$ such that $\tau \omega \rho$ is as well admissible by finitely irreducible definition 2. By lemma 2 we find:

$$
\delta S_{m+n}(\tau \gamma \rho) = \delta S_m(\tau \gamma \rho) + \delta S_n(\gamma \rho) \leq \delta S_m f(\tau \omega \rho) + \delta S_n f(\gamma \rho) + K_\delta f
$$

$$
= \delta S_{m+|\omega|} f(\tau \omega \rho) - \delta S_{|\omega|} f(\omega \rho) + \delta S_n f(\gamma \rho) + K_\delta f
$$

$$
= \delta S_{|\omega|} f(\tau \omega \rho) + \delta S_m f(\sigma |\omega|)(\tau \omega \rho) - \delta S_{|\omega|} f(\omega \rho) + \delta S_n f(\gamma \rho) + K_\delta f.
$$

Now by 2 we know $\delta S_{|\omega|} f \leq \log Q_\delta$ and since $\Omega$ is finite, there is $C > 0$ such that

$$
S_{m+n}(\tau \gamma \rho) \leq S_m f(\sigma |\omega|)(\tau \omega \rho) + S_n f(\gamma \rho) + C \leq d_m + d_n + C.
$$
Thus we have $d_{m+n} \leq d_m + d_n + C$ and we can use Fekete’s lemma 3 with $a_n = d_n + C$ to get convergence of $d_n/n$.

For the other one, note that if $E$ is infinite then using 2 there are infinitely many $n$ for which $b_n = -\infty$, therefore $m(T) = \sup \theta$ which we set it $-\infty$ and so $m(T)/T = -\infty$ for all $T > 0$. Let $E$ be finite, for arbitrary $\tau, \gamma$ with $|\tau| = m, |\gamma| = n$ where $\tau\rho$ and $\gamma\rho$ are admissible there is $\omega \in \Omega$ such that $\tau\gamma\rho$ is admissible as well. Therefore by lemma 2 we find:

$$\delta S_m f(\tau\rho) + \delta S_n f(\gamma\rho) \geq \delta S_m f(\tau\gamma\rho) - K_{\delta f} + \delta S_n f(\gamma\rho)$$

$$= \delta S_m f(\tau\gamma\rho) - \delta S_{|\alpha|} f(\omega\gamma\rho) - K_{\delta f} + \delta S_n f(\gamma\rho)$$

$$= \delta S_{m+|\alpha|} f(\tau\gamma\rho) - \delta S_n f(\gamma\rho) - K_{\delta f}$$

Now for large $m$ it is clear that by lemma 2 we have $\delta S_{m+|\alpha|} f(\tau\gamma\rho) \geq \delta S_{m+|\alpha|} f(\tau\rho) - K_{\delta f}$, so again we use 2 and the fact that $E$ and $\Omega$ are finite to obtain $C > 0$ such that:

$$\delta S_m f(\tau\rho) + \delta S_n f(\gamma\rho) \geq \delta S_{m+n} f(\sigma f(\tau\gamma\rho)) - C.$$  

This gives $b_m + b_n \geq b_{m+n} - C$, and once again we use Fekete’s lemma to find that $b_n/n$ is convergent. Note that similar to above we can set $m := m(T)$ and let $\omega$ be a finite word making the supremum possible in the definition of $m(T)$, so:

$$\frac{b_m}{m} \geq \frac{T}{m} \geq \frac{b_{m+1}}{m+1} \geq \frac{m+1}{m}$$

This finishes the proof.

Note that $m(T)$ is the cutoff integer where before that the counting problem is just counting $\sum_{i=1}^{m(T)} \# E_i^\omega$, while after that not all words with generic length are included in $N_{\rho}(T)$. We continue this omitting process till we reach $M(T)$ where no finite word of length bigger is counted anymore. Furthermore, it is obvious that $r \leq s$. We know equality and strict inequality are both possible, examples 4, 5 correspondingly. Our guess is the following

$$\frac{N_{\rho}(T)}{N_{\rho}(i(T), T)} = O(T) \iff \frac{i(T)}{T} \to 1 / \chi_{\mu_{\rho}}$$

where $O$ is just the big O notation and $m(T) \leq i(T) \leq M(T)$. As stated, we couldn’t show this with the tools we have. Note that this last assumption cannot be relaxed, for taking $i(T) = M(T) + 1$ in example 4 gives

$$\frac{N_{\rho}(T)}{1 + N_{\rho}(i(T), T)} = N_{\rho}(T) = O(\exp(\delta T)), \quad \frac{i(T)}{T} \to \frac{1}{\chi_{\mu_{\rho}}}.$$  

In example 4 we have only one choice $i(T) = m(T) = M(T)$ and then $N_{\rho}(T)/N_{\rho}(i(T), T) = 1$. However, computations get much harder for example 5. Our computations using an asymptotic formula for partial sum of binomials [GKP94, p. 492] suggest $N_{\rho}(T)/N_{\rho}(i(T), T) = O(T)$. In case, such a relation holds in general, it tells us that the main contributor to $N_{\rho}(T)$ is asymptotically $N_{\rho}(i(T), T)$. This is important because in some cases one needs to deal with words of specified length rather than any length when working with $N_{\rho}(T)$.

7. Examples

Example 3. Recalling example 2 from the preliminaries section, we apply theorem 5. We know that the transfer operator $L_x$ for real $x = s$ due to Ruelle’s theorem [Pol84, p. 136], has only one eigenvalue of modulus $e^{F(x)}$ and this eigenvalue is $e^{E(x)}$. Therefore by equation 6 for real $s = x$ the eigenvalue is of the form

$$\lambda(x) = \exp(\log r(A) + x \log \alpha).$$

and since eigenvalue is an analytic function using the identity theorem we obtain for every $s$

$$\lambda(s) = \exp(\log r(A) + s \log \alpha).$$

We observe that $\lambda(s) = 1$ when

$$s = \frac{\log r(A)}{-\log \alpha} + \frac{2\pi k}{-\log \alpha} i, \quad k \in \mathbb{Z}.$$  

Therefore $\delta = \log r(A)/ -\log \alpha$ and $\eta_{\rho} - 1/(s - \delta)$ has continuous extension on the segment

$$\{ s \in \mathbb{C} : s = \delta + iy, \quad |y| < \frac{2\pi}{-\log \alpha} \}.$$
of the critical line. Then theorem 5 for

\[ y_0 = \frac{y_1}{2\pi} = \frac{-1}{\log \alpha}, \quad \delta = \frac{\log r(A)}{-\log \alpha}, \quad \chi_{\mu_\delta} = - \int \log ad\mu_\delta = - \log \alpha \]

gives us the following estimate:

(21) \[ \frac{h_\delta(\rho)}{r(A) - 1} \leq \lim \inf_{T} \frac{N_\rho(T)}{\exp(\delta T)} \leq \lim \sup_{T} \frac{N_\rho(T)}{\exp(\delta T)} \leq \frac{h_\delta(\rho)}{r(A) - 1} r(A). \]

**Example 4.** In the previous example if we consider the full shift with two letters and \( \alpha = \frac{1}{3} \) with maps

\[ \phi_0(x) = \frac{1}{3} x, \quad \phi_1(x) = \frac{1}{3} x + \frac{2}{3}, \]

then the limit set of this system is the Cantor set on the unit interval. Therefore

\[ f(\rho) = \log |\phi'_1(\sigma(\rho))| = \log \frac{1}{3}, \]

\[ \mathcal{L}_\mu(\rho) = \exp(sf(0\rho)) + \exp(sf(1\rho)) = 2\left(\frac{1}{3}\right), \quad h = 1 \]

\[ \log r(A) = \lim_{n} \frac{1}{n} \log \# E^n = \lim_{n} \frac{1}{n} \log 2^n = \log 2. \]

Thus

(22) \[ 1 \leq \lim \inf_{T} \frac{N_\rho(T)}{\exp(\delta T)} \leq \lim \sup_{T} \frac{N_\rho(T)}{\exp(\delta T)} \leq 2. \]

Now we show that the left and right inequalities in the above line are actually equalities. For this, we need to explicitly compute \( N_\rho(T) \). If \( |\omega| = n \):

\[ S_n f(\omega\rho) = n \log \frac{1}{3} \]

\[ S_n f(\omega\rho) \geq -T \iff n \leq \frac{T}{\log 3}. \]

Therefore

\[ \frac{N(T)}{\exp(\delta T)} = \frac{2^{\frac{T}{\log 3} + 1} - 2}{2^{\log 3}}. \]

We dropped the notation \( \rho \) in \( N_\rho(T) \) as it is independent. For any \( A \in [0, 1] \) it is clear that we can choose a sequence \( T_n \) with \( T_n \rightarrow \infty \) such that \( \frac{T_n}{\log 3} - \left\lfloor \frac{T_n}{\log 3} \right\rfloor \rightarrow A \). Then we obtain:

\[ \frac{N(T_n)}{\exp(\delta T_n)} \rightarrow 2^{-A+1}, \]

This in particular means

\[ \lim \inf_{T \rightarrow \infty} \frac{N(T)}{\exp(\delta T)} = 1, \quad \lim \sup_{T \rightarrow \infty} \frac{N(T)}{\exp(\delta T)} = 2. \]

Note that in general computing \( N_\rho(T) \) is not so easy even for simple systems. The following example is one in that regard.

**Example 5.** Recalling example 1, consider the deterministic system with conformal maps of the unit interval

\[ \phi_0(x) = \frac{1}{2} x + \frac{1}{20}, \quad \phi_1(x) = \frac{1}{3} x + \frac{1}{30} \]

on the full shift space \( E^\infty = \{0, 1\}^\infty \). Clearly, we have

\[ f(\rho) = \log |\phi'_1(\sigma(\rho))|, \]

\[ S_n f(\omega\rho) = n_0 \log \frac{1}{2} + n_1 \log \frac{1}{3}, \]
where \( n_0 = n_0(\omega) = S_n \mathbb{1}_{\{0\}}(\alpha \omega) \) and \( n_1 = n_1(\omega) = S_n \mathbb{1}_{\{1\}}(\alpha \omega) \). Basically, \( n_0 \) is the number of 0s and \( n_1 \) is the number of 1s in \( \omega \in E^n \). The pressure is calculated to be

\[
P(\chi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|\omega| = n} \| \phi_\omega^t \| \chi = \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{|\omega| = n} \left( \frac{1}{2n_0(\omega)} \right) \frac{1}{3n_1(\omega)} \right)^n
\]

And a Gibbs state by 4 can be found first on \( [\omega_1] \), then on \( [\omega_1 \omega_2] \) and so on:

\[
m_\chi ([\omega]) = \frac{\left( \frac{1}{2} \right)^{n_0} \left( \frac{1}{3} \right)^{n_1}}{\left( \frac{1}{2} + \frac{1}{3} \right)^n}, \quad \omega \in E^n.
\]

Note that, it defines a system with D-generic property. One way to see that the system is D-generic is by Proposition 7. Note that \( E_{\text{per}}^n \) is the set of periodic words of any length which is exactly \( E^n \), since we work with the full shift. Therefore if the set

\[
\{ S_{[\omega]} f(\alpha \omega) : \omega \in E_{\text{per}}^n \} = \{ n_0 \log \frac{1}{2} + n_1 \log \frac{1}{3} : n_0 + n_1 = n \in \mathbb{N} \},
\]

generates a cyclic additive group with a generator \( \beta \), then there exist integers \( k, k' \) such that \( k \beta = \log 1/2 \) and \( k' \beta = \log 1/3 \). This yields \( k/k' = \log 2/\log 3 \) is rational. The other way to see that our system has D-generic property is by directly solving the following equation for the eigenvalue of the maximal modulus of the transfer operator:

\[
1 = \lambda(s) = \frac{1}{2^s} + \frac{1}{3^s}, \quad x = \delta
\]

so by properties of the triangle inequality, there exists \( b \geq 0 \) such that

\[
\frac{1}{2^s} b \frac{1}{3^s} \Rightarrow b = \frac{3^s}{2^s} = \frac{3^\delta}{2^\delta} \exp(iy \log 3 - iy \log 2) \Rightarrow b = \frac{3^\delta}{2^\delta}, \quad y = \frac{2k\pi}{\log(3/2)}.
\]

But,

\[
1 = \frac{1}{2^s} + \frac{1}{3^s} = \frac{b}{3^s} + \frac{1}{3^s} = \frac{b}{3^\delta} \exp(-iy \log 3) + \frac{1}{3^\delta} \exp(-iy \log 3) = \frac{2k\pi}{\log 3},
\]

i.e. \( y \) can only be 0. Now we are ready to apply corollary 2 to find

\[
\frac{N_p(T)}{\exp(\delta T)} \to \frac{1}{\delta X_{\mu}}, \quad T \to \infty.
\]

In order to keep this paper short we didn’t bring examples with infinite alphabets. One example would be the Apollonian circle packing problem for which there is a good exposition already in [PU21, ch. 6].

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