MULTI-LEVEL RANDOM WALK

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Abstract. We consider examples of multi-level random walk on the integers. We start with two levels: a particle can move at each level and jump from one level to another. At each level also absorption can happen in any state. In the next sections we study random walks on an infinite number of levels. In this case also absorption can happen at any level in any state. Using the combination of inverse z-transforms and the residue theorem we obtain absorption probabilities.

1. Introduction

In part I a particle can move on two levels, the ground level (level 0) and the excited level (level 1) and jump between the two levels. On each level the particle can also move forward or backward and there can be absorption in the current state. In part II we have an infinite number of levels and the particle can move on any level and jump only to a higher level. In this case also absorption can happen at any level in any state. In Part III and Part IV we have an infinite number of levels and we can jump both to a higher and a lower level. We obtain absorption probabilities. Applications of this theory can be found in physics and operations research.

2. Part I: Random walk on two levels

2.1. Introduction. Our discrete random walk acts on two levels, each consisting of the integers: level 0 (ground level) and level 1 (excited level). On each level we have a random walk with one-step forward probability $p$ and one-step backward probability $q$. The probability to jump from one level to another (in the same position) is $r$. In each point of each level we have probability of absorption equal to $t$. We demand: $p + q + r + t = 1$; $p q r t > 0$.

For a discrete random walk we define the expected number of visits to state $j$ when starting in state $i$ by:

$$x_j = x_{i,j} = \sum_{k=0}^{\infty} p_{i,j}^{(k)}$$

We start at ground level in state 0. Let $f_n$ be the expected number of visits to state $n$ at level 0 and $g_n$ is the expected number of visits to state $n$ at
level 1 where \( n \in \mathbb{Z} \). We have:

1. \( f_n = \delta(n, 0) + pf_{n-1} + qf_{n+1} + rg_n \)
2. \( g_n = pg_{n-1} + qg_{n+1} + rf_n \)

We define generating functions

\[
F(u) = \sum_{n=-\infty}^{\infty} f_n u^n \quad (0 < u \leq 1)
\]

\[
G(u) = \sum_{n=-\infty}^{\infty} g_n u^n \quad (0 < u \leq 1)
\]

2.2. Absorption probability in a state. Let \( \mu_1 \) and \( \mu_2 \) (\( \mu_1 > 1 > \mu_2 > 0 \)) be solutions of \( q\mu^2 - (1 + r)\mu + p = 0 \) and \( \mu_3 \) and \( \mu_4 \) (\( \mu_3 > 1 > \mu_4 > 0 \)) are solutions of \( q\mu^2 - (1 - r)\mu + p = 0 \). Define \( \zeta_1 = \left[\left((1 + r)^2 - 4pq\right)\right]^{-\frac{1}{2}} \) and \( \zeta_2 = \left[\left((1 - r)^2 - 4pq\right)\right]^{-\frac{1}{2}} \).

**Theorem 1.**

\[
F(u) = \frac{(1 - pu - \frac{q}{u})}{(1 - pu - \frac{4}{u})^2 - r^2}
\]

(3)

\[
G(u) = \frac{r}{(1 - pu - \frac{4}{u})^2 - r^2}
\]

(4)

\[
f_n = \begin{cases} \frac{1}{2}(\zeta_1\mu_1^n + \zeta_2\mu_3^n) & (n \leq 0) \\ \frac{1}{2}(\zeta_1\mu_2^n + \zeta_2\mu_4^n) & (n \geq 0) \end{cases}
\]

(5)

\[
g_n = \begin{cases} \frac{1}{2}(\zeta_1\mu_1^n + \zeta_2\mu_3^n) & (n \leq 0) \\ \frac{1}{2}(\zeta_1\mu_2^n + \zeta_2\mu_4^n) & (n \geq 0) \end{cases}
\]

(6)

**Proof.** Layer 0: \( f_n = pf_{n-1} + qf_{n+1} + rg_n + \delta(n, 0) \) gives \( F(u) = puF(u) + \frac{q}{u}F(u) + rG(u) + 1 \), so \( 1 - pu + \frac{q}{u} \) \( F(u) = 1 + rG(u) \).

Layer 1: \( g_n = pg_{n-1} + qg_{n+1} + rf_n \) gives \( G(u) = puG(u) + \frac{q}{u}G(u) + rF(u) \), so \( 1 - pu + \frac{q}{u} \) \( G(u) = rF(u) \).

We use the inverse z-transform: \( f_n = \frac{1}{2\pi i} \oint H(z)z^{-n}dz \), where the integration is anticlockwise along the circle \(|z| = 1\) and

\[
H(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n} = F\left(\frac{1}{z}\right) = \frac{-z(qz^2 - z + p)}{(qz^2 - (1 + r)z + p)(qz^2 - (1 - r)z + p)}
\]

So,

\[
f_n = \frac{1}{2\pi i} \oint \frac{-z^n(qz^2 - z + p)dz}{q^2(z - \mu_1)(z - \mu_2)(z - \mu_3)(z - \mu_4)}
\]
After applying the residue theorem we obtain (5). In the same way we find (6).

Absorption probabilities are $tf_n$ and $tg_n \ (n \in \mathbb{N})$. Absorption will always occur: $\sum (tf_n + tg_n) = 1$.

2.3. **Probability of absorption in a layer.** Absorption probabilities are given by $tf_n$ at ground level and $tg_n$ at excited level. We define $\pi(0)$ and $\pi(1)$ as the sum of probabilities of absorption at level 0 respectively level 1. We have $\pi(0) = \sum_{n=-\infty}^{\infty} tf_n$ and $\pi(1) = \sum_{n=-\infty}^{\infty} tg_n$

**Theorem 2.** $\pi(0) = \frac{r+t}{2r+t}$, $\pi(1) = \frac{r}{2r+t}$

**Proof.** $\sum f_n = F(1) = \frac{(1-p-q)}{(1-p-q)^2 - rt} = \frac{r+t}{(2r+t)}$ and $\sum g_n = G(1) = \frac{r}{r(2r+t)}$.

3. **Part II: Random walk on infinite set of levels with upward jumps**

3.1. **Introduction.** In this section we have an infinite number of levels $m \ (m \in \mathbb{N})$ and the particle can move on any level and jump only to a higher level. In this case also absorption can happen at any level in any state. Let $p$ and $q$ be the forward and backward probabilities, $t$ is the absorption probability in any state at each level, $r$ is the probability to jump one level upwards. We have $p + q + r + t = 1$ and we demand $pqr > 0$. We have:

$$
\begin{cases}
  f_{n,0} = pf_{n-1,0} + qf_{n+1,0} + \delta(n,0) & (\text{layer } m = 0, \ n \in \mathbb{Z}) \\
  f_{n,m} = pf_{n-1,m} + qf_{n+1,m} + rf_{n,m-1} & (\text{layer } m \geq 1, \ n \in \mathbb{Z})
\end{cases}
$$

3.2. **Generating functions.** Let $f_{n,m}$ be the expected number of visits to state $n$ at level $m \ (n \in \mathbb{Z}, m \in \mathbb{N})$. We define generating functions

$$
F(u, v) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} f_{n,m}u^n v^m \quad (0 < u \leq 1, 0 < v \leq 1)
$$

$$
G_m(u) = \sum_{n=-\infty}^{\infty} f_{n,m}u^n \quad (0 < u \leq 1, m \in \mathbb{N})
$$

**Theorem 3.**

(7) $F(u, v) = (1 - pu - qv - rv)^{-1}$

(8) $G_m(u) = \frac{1}{r} \left( \frac{r}{1 - pu - rv} \right)^{m+1} \quad (m \in \mathbb{N})$
Proof. Layer $m = 0$: $f_{n,0} = pf_{n-1,0} + qf_{n+1,0} + \delta(n,0)$ gives $G_0(u) = puG_0(u) + \frac{u}{a}G_0(u) + 1$, so $G_0(u) = \left(1 - pu - \frac{u}{a}\right)^{-1}$.

Layer $m$ $(m \geq 1)$: $f_{n,m} = pf_{n-1,m} + qf_{n+1,m} + rf_{n,m-1}$ gives $G_m(u) = r(1 - pu - \frac{u}{a})^{-1}G_{m-1}(u)$ $(m = 1, 2, \ldots)$

$$F(u, v) = \sum_{n} f_{n,0}u^n + \sum_{n} \sum_{m=1}^{\infty} f_{n,m}u^mv^m =$$

$$G_0(u) + \sum_{m=1}^{\infty} \sum_{n} \{pf_{n-1,m} + qf_{n+1,m} + rf_{n,m-1}\}u^mv^m =$$

$$G_0(u) + \sum_{m=0}^{\infty} \sum_{n} \{pf_{n-1,m} + qf_{n+1,m}\}u^mv^m +$$

$$\sum_{m=1}^{\infty} \sum_{n} rf_{n,m-1}u^mv^m - \sum_{n} \{pf_{n-1,0} + qf_{n+1,0}\}u^n =$$

$$G_0(u) + (pu + \frac{u}{a})F(u, v) + rvF(u, v) - (pu + \frac{u}{a})G_0(u), \text{ so:}$$

$$(1 - pu - \frac{u}{a} - rv)F(u, v) = (1 - pu - \frac{u}{a})G_0(u) = 1 \quad \blacksquare$$

3.3. Probability of absorption in a layer. Let $\alpha_m = \sum_{n} f_{n,m} (m \in \mathbb{N})$.

The probability of absorption in layer $m$ is $\pi_m = t\alpha_m$ $(m \in \mathbb{N})$.

Theorem 4. $\pi_m$ has a geometric distribution:

$$\pi_m = \frac{t}{r + t}(1 - \frac{t}{r + t})^m$$

Proof.

$$F(1, v) = \sum_{m} \{\sum_{n} f_{n,m}\}v^m = \sum_{m} \alpha_m v^m = [1 - (p + q + rv)]^{-1} = \sum_{k=0}^{\infty} (p + q + rv)^k$$

$\alpha_m$ is the coefficient of $v^m$ in $\sum_{k=m}^{\infty} (p + q + rv)^k$.

We get:

$$\alpha_m = r^m \sum_{j=0}^{\infty} \binom{m+j}{j} (p + q)^j = r^m (1 - p - q)^m = r^m (r + t)^{-m-1} \quad \blacksquare$$

3.4. Absorption probability in a state. Let the solutions of $q\xi^2 - \xi + p = 0$ be $\xi_1$ and $\xi_2$ where $\xi_1 > 1 > \xi_2 > 0$. Define $\zeta = (1 - 4pq)^{-1/4}$.

Theorem 5.

$$f_{n,m} = \begin{cases} (-1)^m r^m \zeta^{m+1} \xi_1^n \sum_{k=0}^{m} (-1)^k \binom{n+m}{m-k} \binom{m+k}{m} \left(\frac{\xi_1+1}{2}\right)^k & (n \leq 0, \ m \in \mathbb{N}) \\
^{m} \zeta^{m+1} \xi_2^n \sum_{k=0}^{m} \binom{n+m}{m-k} \binom{m+k}{m} \left(\frac{\xi_2-1}{2}\right)^k & (n \geq 0, \ m \in \mathbb{N}) \end{cases}$$
Proof. We use the inverse z-transform: \( f_{n,m} = \frac{1}{2\pi i} \oint H_m(z)z^{-1}dz \), where the integration is along the circle \(|z|=1\) and anticlockwise.

\[
H_m(z) = \sum_{n=-\infty}^{\infty} f_{n,m}z^{-n} = G_m(\frac{1}{z}) = \frac{(-1)^{m+1}r^m z^{m+1}}{q^{m+1}(z-\xi_1)^{m+1}(z-\xi_2)^{m+1}}
\]

We apply the residue theorem, using the notation \( n^k = \frac{n!}{(n-k)!} \).

\[
\text{Res}_{z=\xi_1} [H_m(z)] = \frac{(-1)^{m+1}r^m}{q^{m+1}m!} \lim_{z \to \xi_1} \frac{d^m}{dz^m} [z^{m+n}(z-\xi_2)^{-m-1}] = \frac{(-1)^{m+1}r^m}{q^{m+1}m!} \sum_{k=0}^{m} \binom{m}{k} (m+n)^{m-k} \xi_1^{n+k}(-1)^k(m+k)\xi_1^{-m-k} = \frac{(-1)^{m+1}r^m \xi_1^{m+1} \xi_2^n}{(m+1)} \sum_{k=0}^{m} (-1)^k \binom{n+m}{m} \binom{m+k}{m} \left(\frac{\xi_1+1}{2}\right)^k
\]

where we used \( \xi_1 - \xi_2 = (q\xi)^{-1} \) and \( q\xi_1 = \frac{\xi_1+1}{2} \). Because of \( \xi_1 > 1 \) we have to integrate clockwise, introducing a factor \(-1\). Proceeding along the same lines we find the residu in \( \xi_2 \). \( \square \)

Probability of absorption in state \( n \) of layer \( m \) is given by \( tf_{n,m} \). We work out solutions for the first three layers.

\[
f_{n,0} = \begin{cases} \zeta_1^n & (n \leq 0) \\ \zeta_2^n & (n \geq 0) \end{cases}
\]

\[
f_{n,1} = \begin{cases} (-n+\zeta)r^2\zeta_1^{n} & (n \leq 0) \\ (n+\zeta)r^2\zeta_2^{n} & (n \geq 0) \end{cases}
\]

\[
f_{n,2} = \begin{cases} \frac{1}{2}(n^2 - 3\zeta n + 3\zeta^2 - 1)r^2\zeta_1^{n} & (n \leq 0) \\ \frac{1}{2}(n^2 + 3\zeta n + 3\zeta^2 - 1)r^2\zeta_2^{n} & (n \geq 0) \end{cases}
\]

**Theorem 6.** Solutions of

\[
\begin{cases} f_{n,0} = pf_{n-1,0} + qf_{n+1,0} + \delta(n, 0) & (\text{layer } m = 0, \ n \in \mathbb{Z}) \\ f_{n,m} = pf_{n-1,m} + qf_{n+1,m} + rf_{n,m-1} & (\text{layer } m \geq 1, \ n \in \mathbb{Z}) \end{cases}
\]

are of the form

\[
f_{n,m} = \begin{cases} r^m \zeta^{m+1} \xi_1^n \sum_{k=0}^{m} (-1)^k c_{k,m} n^k & (n \leq 0, \ m \in \mathbb{N}) \\ r^m \zeta^{m+1} \xi_2^n \sum_{k=0}^{m} c_{k,m} n^k & (n \geq 0, \ m \in \mathbb{N}) \end{cases}
\]

**Proof.** \( f_{n,m} = r^m \zeta^{m+1} \xi_2^n \sum_{k=0}^{m} c_{k,m} n^k \) \((n \geq 0)\) follows from Theorem 5.

Consider the dual RW where roles of \( p \) and \( q \) are interchanged (reflected RW). \( \zeta \) is invariant under this transformation \( T \). The formula for \( n \geq 0 \) is in
the dual walk valid for $n \leq 0$. The transformation $T$ maps $\xi_2 = \frac{1-\sqrt{(1-4pq)}}{2q}$ to $\frac{1-\sqrt{(1-4pq)}}{2p} = \xi_1^{-1}$, so we have:

$$r^m \xi^{m+1} \xi_2^m \sum_{k=0}^{\infty} c_{k,m} n^k T \rightarrow r^m \xi^{m+1} \left(\xi_1^{-1}\right)^{-n} \sum_{k=0}^{\infty} c_{k,m} (-n)^k$$

\[\square\]

4. Part III: Random walk on infinite number of levels using bivariate generating functions

4.1. Introduction. In this section we have an infinite number of levels and the particle can move on any level and jump to both an lower or a higher level. In this case also absorption can happen at any level in any state. Let $p$ and $q$ be the forward and backward probabilities, $t$ is the absorption probability in any state at each level, $r$ and $s$ are the probabilities to jump one level upwards and downwards. We have $p + q + r + s + t = 1$ and we demand $pqrst > 0$. We start in state 0 of layer 0. We have:

$$f_{n,m} = pf_{n-1,m} + qf_{n+1,m} + rf_{n,m-1} + sf_{n,m+1} + \delta(n,0)\delta(m,0) \quad (m,n \in \mathbb{Z})$$

We define a generating function

$$F(u,v) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_{n,m} u^n v^m \quad (0 < u \leq 1, 0 < v \leq 1)$$

Let $\alpha_m = \sum_n f_{n,m}$. Let $\pi_m$ be the probability of absorption in layer $m$ $(m \in \mathbb{Z})$. $\pi_m = t\alpha_m$.

Let $\tau = 1 - p - q = r + s + t$.

Let $v_1$ and $v_2$ be the solutions of $rv_1^2 - \tau v_1 + s = 0$ ($v_1 > 1$, $0 < v_2 < 1$).

Define $\rho = (\tau^2 - 4rs)^{-\frac{1}{2}}$.

Theorem 7.

$$F(u,v) = (1 - pu - \frac{q}{u} - rv - \frac{s}{v})^{-1}$$

$$\alpha_m = \begin{cases} \rho v_1^m & (m < 0) \\ \rho v_2^m & (m \geq 0) \end{cases}$$

Proof.

$$F(u,v) = puF(u,v) + \frac{q}{u}F(u,v) + rvF(u,v) + \frac{s}{v}F(u,v) + 1$$

$$F(1,v) = \sum_n \sum_m f_{n,m} v^m = \sum_m \{\sum_n f_{n,m}\} v^m = \sum_m \alpha_m v^m$$

We also have

$$F(1,v) = (1 - p - q - rv - \frac{s}{v})^{-1} = \frac{-v}{rv^2 - \tau v + s}$$
We find the desired result by applying the inverse \( v \)-transform and using the residue theorem.

We found probabilities for absorption in a layer, but not for each point. In the next section we adapt the random walk (more symmetric) and find closed forms for individual absorption probabilities.

5. Part IV: Semi-symmetric random walk on infinite number of levels using difference equations

5.1. Introduction. In this section we have an infinite number of levels and the particle can move on any level and jump to both a lower or a higher level. In this case also absorption can happen at any level in any state. Let \( p \) be the forward and backward probability, \( t \) is the absorption probability in any state at each level, \( r \) is the probability to jump one level upwards or downwards. We have \( 2p + 2r + t = 1 \) and we demand \(prt > 0\). We use the same procedure as in McCrea & Whipple [1]. We first construct a solution in the rectangular domain to obtain solutions for semi-infinite strips, infinite strips, half-plane and our goal: the plane.

5.2. Rectangular region. To obtain a solution in two dimensional plane, we start with a rectangular region \( R = \{(x, y)|0 \leq x \leq c + 1, 0 \leq y \leq d + 1\} \). We define an interior of \( R \): \( I = \{(x, y)|1 \leq x \leq c, 1 \leq y \leq d\} \). The boundary of \( R \) is \( B \), which consist of absorbing barriers. We define \( f^{(a,b)}_{x,y} \) as the expected number of departures from \((x, y)\) when starting in the interior source \((a, b)\) on a lattice. We shall often use the abbreviation \( f_{x,y} \). We have for \( I \):

\[
(13) \quad f_{x,y} = \delta_{a,x} \delta_{b,y} + \sigma \{f_{x+1,y} + f_{x-1,y}\} + \eta \{f_{x+1,y+1}f_{x,y-1}\} 
\]

and for \( B \):

\[
(14) \quad f_{x,y} = 0 
\]

The homogeneous part of the difference equation (13) has solutions \( f_{x,y} = Ae^{x\alpha + iy\beta} \), where \( 2\sigma \cosh \alpha + 2\eta \cos \beta = 1 \), so \( f_{x,y} = C' \sinh \alpha x \sin \beta y \).

We construct solutions of (13) and (14):

\[
f_{x,y} = \sum_{r=1}^{d} C(r) \sin \frac{yr\pi}{d+1} \sinh x\alpha_r \sinh[(c + 1 - a)\alpha_r] \quad (x \leq a) 
\]

\[
f_{x,y} = \sum_{r=1}^{d} C(r) \sin \frac{yr\pi}{d+1} \sinh a\alpha_r \sinh[(c + 1 - x)\alpha_r] \quad (x \geq a) 
\]

where

\[
(15) \quad 2\eta \cos \frac{r\pi}{d+1} + 2\sigma \cosh \alpha_r = 1
\]
We substitute these solutions in (13) with \( x = a \) and get after some calculations, using (15):

\[
\sigma \sum_{r=1}^{d} C(r) \sin \frac{yr \pi}{d+1} \sinh \alpha_r \sinh[(c+1)\alpha_r] = \delta_{b,y}
\]

Using

\[
\frac{2}{d+1} \sum_{r=1}^{d} \sin \frac{br \pi}{d+1} \sin \frac{yr \pi}{d+1} = \delta_{b,y}
\]

we get:

\[
f_{x,y} = \frac{2}{(d+1)\sigma} \sum_{r=1}^{d} \sin \frac{br \pi}{d+1} \sin \frac{yr \pi}{d+1} \sinh \alpha_r \sinh[(c+1-a)\alpha_r] \quad (x \leq a)
\]

\[
f_{x,y} = \frac{2}{(d+1)\sigma} \sum_{r=1}^{d} \sin \frac{br \pi}{d+1} \sin \frac{yr \pi}{d+1} \sinh \alpha_r \sinh[(c+1-x)\alpha_r] \quad (x \geq a)
\]

where

\[
2\eta \cos \frac{\pi \lambda}{d+1} + 2\sigma \cosh \alpha_r = 1
\]

5.3. Plane. In four steps we get a solution for the plane.

Step 1: Semi-infinite strips.
Taking \( d \to \infty \) in the rectangular solution:

\[
f_{x,m} = \frac{2}{\pi \sigma} \int_{0}^{\pi} \sin \frac{b \lambda}{d+1} \sin \frac{y \lambda}{d+1} \sinh \alpha \sinh[(c+1-a)\alpha] \exp(-a\mu) \sinh \mu \, d\lambda \quad (x \leq a)
\]

where

\[
2\eta \cos \lambda + 2\sigma \cosh \mu = 1
\]

Step 2: Infinite strips.
In the semi-infinite strip solution we take \( b, y \to \infty \) and let \( m = y - b, m \) finite.

Using \( \lim_{b \to \infty} \int_{0}^{\pi} \cos b \lambda g(\lambda) = 0 \) we get:

\[
f_{x,m} = \frac{2}{\pi \sigma} \int_{0}^{\pi} \cos m \lambda \sinh \alpha \sinh[(c+1-a)\alpha] \exp(-a\mu) \sinh \mu \, d\lambda \quad (x \leq a)
\]

\[
f_{x,m} = \frac{2}{\pi \sigma} \int_{0}^{\pi} \cos m \lambda \sinh \alpha \sinh[(c+1-x)\alpha] \exp(-a\mu) \sinh \mu \, d\lambda \quad (x \geq a)
\]

Step 3: Half-plane.
By taking \( c \to \infty \) in solution of infinite strip, we get:

\[
f_{x,m} = \frac{1}{2\pi \sigma} \int_{0}^{\pi} \cos m \lambda \sinh \alpha \exp(-a\mu) \sinh \mu \, d\lambda \quad (x \leq a)
\]
\[ f_{x,m} = \frac{1}{2\pi \sigma} \int_0^\pi \frac{\cos m\lambda \sinh a\mu \exp \left(-x\mu\right)}{\sinh \mu} d\lambda \quad (x \geq a) \]

Step 4: Plane.

**Theorem 8.** In the plane we have a unique solution:

\[ f_{n,m} = f_{n,m}^{(0,0)} = \frac{1}{2\pi \sigma} \int_0^\pi \frac{\cos m\lambda \exp \left(-|n|\mu\right)}{\sinh \mu} d\lambda \]

where

\[ 2\eta \cos \lambda + 2\sigma \cosh \mu = 1 \]

**Proof.** By taking \( a, x \to \infty, n = x - a \) finite, in solution of infinite strip, we get (16).

First we prove that (16) & (17) is a solution of (1) in the plane:

\[ f_{n,m} - \sigma \{f_{n+1,m} + f_{n-1,m}\} - \eta \{f_{n,m+1} + f_{n,m-1}\} = \frac{1}{2\pi \sigma} \int_0^\pi \frac{T(n,m)}{\sinh \mu} d\lambda \]

where

\[ T(n,m) = \cos m\lambda e^{-|n|\mu} - \sigma \{\cos m\lambda e^{-|n+1|\mu} + \cos m\lambda e^{-|n-1|\mu}\} + \eta \{\cos (m+1)\lambda e^{-|n|\mu} + \cos (m-1)\lambda e^{-|n|\mu}\} \]

If \(|n| \geq 1\) then \( T \) vanishes. When \( n = 0 \), then \( T \) reduces to \( 2\sigma \cos m\lambda \sinh \mu \) and we have:

\[ \frac{1}{2\pi \sigma} \int_0^\pi \frac{T(m,n)}{\sinh \mu} d\lambda = \frac{1}{\pi} \int_0^\pi \cos m\lambda d\lambda = \delta_{m,0} \]

The solution is unique: see Feller [2], (p.362) \( \square \)

We combine the results of Theorems [7] and [8]:

\[ \sum_n f_{n,m} = \frac{1}{2\pi \sigma} \int_0^\pi \frac{\cos m\lambda \sum_n \exp \left(-|n|\mu\right)}{\sinh \mu} d\lambda = \frac{1}{2\pi \sigma} \int_0^\pi \frac{\cos m\lambda}{\tanh \frac{1}{2\mu} \sinh \mu} d\lambda \]

Theorem [8] gives \( \alpha_m = \rho v_2^m \) \((m \in \mathbb{Z})\) (use: \( v_1 v_2 = 1 \)). So we get:

\[ \frac{1}{2\pi} \int_0^\pi \frac{\cos m\lambda}{\tanh \frac{1}{2\mu} \sinh \mu} d\lambda = \sigma \rho v_2^m \quad (m \in \mathbb{Z}) \]

where

\[ 2\eta \cos \lambda + 2\sigma \cosh \mu = 1 \quad (\eta > 0, \sigma > 0, 2\eta + 2\sigma < 1) \]

\[ \rho = \{(1 - 2p)^2 - 4r^2\}^{-\frac{1}{2}} \quad rv^2 - (1 - 2p)v + r = 0 \]
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