Decomposition-space slices are toposes

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Abstract

We show that the category of decomposition spaces and CULF maps is locally a topos. Precisely, the slice category over any decomposition space $\mathcal{D}$ is a presheaf topos, namely $\text{Decomp}_{/\mathcal{D}} \cong \text{Psh}(\text{tw}(\mathcal{D}))$.

1 Introduction

Decomposition spaces were introduced by Gálvez, Kock, and Tonks [6, 7, 8] for purposes in combinatorics, and by Dyckerhoff and Kapranov [4]—who call them unital 2-Segal spaces—for purposes in homological algebra, representation theory, and geometry. They are simplicial sets (or simplicial $\infty$-groupoids) with a property that expresses the ability to (co-associatively) decompose, just as in categories one can (associatively) compose. In particular, decomposition spaces induce coalgebras. The most nicely-behaved class of morphisms between decomposition spaces is that of CULF maps. These preserve decompositions in an appropriate way so as to induce coalgebra homomorphisms.

Apart from the coalgebraic aspect, not so much is known about the category $\text{Decomp}$ of decomposition spaces and CULF maps, and it may appear a bit peculiar. For example, the product of two decomposition spaces as simplicial sets is not the categorical product in $\text{Decomp}$—the projections generally fail to be CULF. Similarly, the terminal simplicial set (which is a decomposition space) is not terminal in $\text{Decomp}$. The simplicial-set product should rather be considered as a tensor product for decomposition spaces.

The present contribution advances the categorical study of decomposition spaces by establishing that $\text{Decomp}$ is locally a topos, meaning that all its slices are toposes—even presheaf toposes. More precisely we show:

Theorem 1.1 (Main Theorem). For $\mathcal{D}$ a decomposition space, there is a natural equivalence of categories

$$\text{Decomp}_{/\mathcal{D}} \cong \text{Psh}(\text{tw}(\mathcal{D})).$$

Here $\text{tw}(\mathcal{D})$ is the twisted arrow category of $\mathcal{D}$, obtained by edgewise subdivision—this is readily seen to be (the nerve of) a category, cf. Lemma 2.1 below.

∗ Kock was supported by grants MTM2016-80439-P (AEI/FEDER, UE) of Spain and 2017-SGR-1725 of Catalonia.

† Spivak was supported by AFOSR grants FA9550-14-1-0031 and FA9550-17-1-0058.
The functor in the direction displayed in eq. (1) is simply applying the twisted arrow category construction. The functor going in the other direction is the surprise. We construct it by exploiting an interesting natural transformation between the nerve of the category of elements and the twisted arrow category of a decomposition space:

\[
\begin{array}{c}
\text{Decomp} \\
\downarrow \lambda \\
\text{Cat.}
\end{array}
\]

\(\text{el} \circ N \Downarrow \lambda \text{tw}\)

Its component \(\lambda_D : \text{el}(N \mathcal{D}) \to \text{tw}(\mathcal{D})\) at a decomposition space \(\mathcal{D}\) sends an \(n\)-simplex of \(\mathcal{D}\) to its long edge. (In the category case, this map goes back to Thomason’s notebooks [17, p.152]; it was exploited by Gálvez–Neumann–Tonks [9] to exhibit Baues–Wirsching cohomology as a special case of Gabriel–Zisman cohomology.) The crucial property of \(\lambda\) is that it is a cartesian natural transformation. Our proof of theorem 1.1 describes the inverse to \(\text{tw}_D\) as being essentially—modulo some technical translations involving elements, presheaves, and nerves—given by \(\lambda^*\).

The importance of this result resides in making a huge body of work in topos theory available to study decomposition spaces, such as for example the ability to define new decomposition spaces from old by using the internal language. It also opens up interesting questions such as how the subobject classifier [14], isotropy group [5], etc. of the topos \(\text{Decomp}_{/\mathcal{D}}\) relates to the combinatorial structure of the decomposition space \(\mathcal{D}\).

Theorem 1.1 can be considered surprising in view of the well-known failure of such a result for categories. Lamarche (1996) had suggested that categories CULF over a fixed base category \(\mathcal{C}\) form a topos. Bunge and Niefield announced a proof, exploiting presheaves on the twisted arrow category, but Johnstone [10] found a gap in the proof and corrected the statement by identifying the precise—though quite restrictive—conditions that \(\mathcal{C}\) must satisfy. Alternative proofs were provided by Bunge–Niefield [3] and Bunge–Fiore [2], who were motivated by CULF functors as a notion of duration of processes, as already considered by Lawvere [12]. The present work also grew out of interest in dynamical systems [16]. Our main theorem can be seen as an different realization of Lamarche’s insight, allowing more general domains, thus indicating a role of decomposition spaces in category theory. From this perspective, the point is that the natural setting for CULF functors are decomposition spaces rather than categories: a simplicial set CULF over a category is a decomposition space, but not always a category.

Remark 1.2. It was conjectured in [8] that there exists a universal decomposition space \(\mathcal{U}\) (whose \(1\)-simplices are intervals), in the sense that every decomposition space \(\mathcal{D}\) should admit an essentially unique CULF map to \(\mathcal{U}\), given by (the decomposition-space version of) Lawvere’s interval construction [12]. The status of this conjecture is that the interval construction \(I : \mathcal{D} \to \mathcal{U}\) exists, and that every other CULF functor \(\mathcal{D} \to \mathcal{U}\) is naturally equivalent to \(I\) (but uniqueness has not been established). Size issues prevent \(\mathcal{U}\) from being a terminal object, though: the universal \(\mathcal{U}\) for \(\kappa\)-small decomposition spaces is not
itself $\kappa$-small. However, for decomposition spaces that are Möbius in the sense of [7], the corresponding universal decomposition space constructed in [8] is in fact essentially small. If the conjecture is true in this case, the universal decomposition space of Möbius intervals is a genuine terminal object in the category of Möbius decomposition spaces. Since every decomposition space CULF over a Möbius decomposition space is again Möbius, it would follow that the category of Möbius decomposition spaces is in fact a topos.

2 Preliminaries

Simplicial sets. Although decomposition spaces naturally pertain to the realm of $\infty$-categories and simplicial $\infty$-groupoids, we work in the present note with 1-categories and simplicial sets, both for simplicity and in order to situate decomposition spaces (actually just “decomposition sets”) in the setting of classical category theory and topos theory. All the results should generalize to $\infty$-categories (in the form of Segal spaces) and general decomposition spaces, and the proof ideas should also scale to this context, although the precise form of the proofs do not: where presently we exploit objects-and-arrows arguments, more uniform simplicial arguments are required in the $\infty$-case. We leave that generalization open.

Thus our setting is the category $sSet = \mathbf{Psh} (\Delta)$ of simplicial sets, i.e. functors $\Delta^{\text{op}} \to \mathbf{Set}$ and their natural transformations. Given a simplicial set $X$ and a morphism $f : [m] \to [n]$ in $\Delta$, we denote the induced function by $X_f : X_n \to X_m$. Small categories fully faithfully embed as simplicial sets via the nerve functor $N : \mathbf{Cat} \to sSet$, and decomposition spaces are defined as certain more general simplicial sets, as we now recall.

Active and inert maps. The category $\Delta$ has an active-inert factorization system: the active maps, written $g : [m] \to [n]$, are those that preserve end-points, $g(0) = 0$ and $g(m) = n$; the inert maps, written $f : [m] \to [n]$, are those that are distance preserving, $f(i+1) = f(i) + 1$ for $0 \leq i \leq m - 1$. The active maps are generated by the codegeneracy maps and the inner coface maps; the inert maps are generated by the outer coface maps $d^k$ and $d^T$. (This orthogonal factorization system is an instance of the important general notion of generic-free factorization system of Weber [19] who referred to the two classes as generic and free. The active-inert terminology is due to Lurie [13].)

Decomposition spaces. Active and inert maps in $\Delta$ admit pushouts along each other, and the resulting maps are again active and inert. A decomposition space [6] is a simplicial set (or more generally a simplicial groupoid or $\infty$-groupoid) $X : \Delta^{\text{op}} \to \mathbf{Set}$ that takes
all such active-inert pushouts to pullbacks:

\[
\begin{array}{c}
\begin{bmatrix}
[n'] \\ [m']
\end{bmatrix}
\end{array}
\begin{array}{c}
\begin{bmatrix}
[n] \\ [m]
\end{bmatrix}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\uparrow
\end{array}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X_n \\
X_m
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X_{n'} \\
X_{m'}
\end{array}
\end{array}\end{array}
\]

\[=\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\uparrow
\end{array}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X_n \\
X_m
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X_{n'} \\
X_{m'}
\end{array}
\end{array}\end{array}
\]

**CULF maps.** A simplicial map \( F : Y \to X \) between simplicial sets is called CULF [6] when it is cartesian on active maps (i.e. the naturality squares are pullbacks), or, equivalently, is right-orthogonal to all active maps \( \Delta[m] \to \Delta[n] \). The CULF maps between (nerves of) categories are precisely the discrete Conduché fibrations (see [10]). If \( \mathcal{D} \) is a decomposition space (e.g. a category) and \( F : \mathcal{E} \to \mathcal{D} \) is CULF, then also \( \mathcal{E} \) is a decomposition space (but not in general a category).

We denote by \( \text{Decomp} \) the category of decomposition spaces and CULF maps.

**Right fibrations and presheaves.** A simplicial map is called a right fibration if it is cartesian on bottom coface maps (or equivalently, right-orthogonal to the class of last-vertex inclusions \([0] \to [n])\). (This in fact implies that it is cartesian on all codegeneracy and coface maps except the top coface maps. In particular, a right fibration is CULF.) When restricted to categories, this notion coincides with that of discrete fibration. We denote by \( \text{RFib} \) the category of small categories and right fibrations. Note that for any category \( \mathcal{C} \), the inclusion functor \( \text{RFib}_{/\mathcal{C}} \to \text{Cat}_{/\mathcal{C}} \) is full.

For any small category \( \mathcal{C} \) there is an adjunction

\[
\partial : \text{Cat}_{/\mathcal{C}} \rightleftarrows \text{Psh}(\mathcal{C}) : \} 
\]

with \( \} \vdash \partial \). The notation is chosen because \( \partial \circ \} = \text{id} \). For any \( P \in \text{Psh}(\mathcal{C}) \), we denote \( \} P \) by \( \pi_P : \text{el}(P) \to \mathcal{C} \), and refer to \( \text{el}(P) \) as the category of elements. The resulting functor \( \pi_P \) is always a right fibration, and \( \partial \) restricts to an equivalence of categories \( \text{RFib}_{/\mathcal{C}} \simeq \text{Psh}(\mathcal{C}) \). If \( P' \to P \) is a map of presheaves, \( \text{el}(P') \to \text{el}(P) \) is a right fibration. Thus we have a functor \( \text{el} : \text{Psh}(\mathcal{C}) \to \text{RFib} \). We will make particular use of this for the case \( \mathcal{C} = \Delta \):

\[
\text{el} : \text{sSet} \to \text{RFib}.
\]

**Twisted arrow categories.** For \( \mathcal{C} \) a small category, the twisted arrow category \( \text{tw}(\mathcal{C}) \) (cf. [11]) is the category of elements of the Hom functor \( \mathcal{C}^\text{op} \times \mathcal{C} \to \text{Set} \). It thus has the arrows of \( \mathcal{C} \) as objects, and trapezoidal commutative diagrams

\[
\begin{array}{c}
\xymatrix{ f' & f \\
& f \ar[ul]
\end{array}
\]

as morphisms from \( f' \) to \( f \).
The twisted arrow category is a special case of edgewise subdivision of a simplicial set [15], as we now recall. Consider the functor
\[ Q : \Delta \to \Delta \]
\[ [n] \mapsto [n]^{\text{op}} \star [n] = [2n+1]. \]

With the following special notation for the elements of the ordinal \([n]^{\text{op}} \star [n] = [2n+1],\)
\[ 0 \to 1 \to \cdots \to n \]
\[ 0' \leftarrow 1' \leftarrow \cdots \leftarrow n', \]
the functor \(Q\) is described on arrows by sending a coface map \(d^i : [n-1] \to [n]\) to the monotone map that omits the elements \(i\) and \(i'\), and by sending a codegeneracy map \(s^i : [n] \to [n-1]\) to the monotone map that repeats both \(i\) and \(i'\).

Defining \(\text{sd} := Q^* : \text{sSet} \to \text{sSet}\), we have the commutative diagram
\[
\begin{array}{ccc}
\text{Cat} & \xrightarrow{\text{tw}} & \text{Cat} \\
N \downarrow & & \downarrow N \\
\text{sSet} & \xrightarrow{\text{sd}} & \text{sSet}.
\end{array}
\]

Note that there is a natural transformation \(L : \text{id}_\Delta \Rightarrow Q\), whose component at \([n]\) is the last-segment inclusion \([n] \subseteq [n]^{\text{op}} \star [n]\). It induces a natural map
\[ \text{cod}_X := L^* : \text{sd}(X) \to X \]
for any simplicial set \(X\), and similarly \(\text{tw}(\mathcal{C}) \to \mathcal{C}\) for any category \(\mathcal{C}\).

**Lemma 2.1.** The functor \(\text{sd} : \text{sSet} \to \text{sSet}\) sends decomposition spaces to categories and CULF maps to right fibrations. That is, there is a unique functor \(\text{tw}\) making the following diagram commute:\(^1\)
\[
\begin{array}{ccc}
\text{Decomp} & \xrightarrow{\text{tw}} & \text{RFib} \\
N \downarrow & & \downarrow N \\
\text{sSet} & \xrightarrow{\text{sd}} & \text{sSet}.
\end{array}
\]

**Proof.** Suppose \(\mathcal{D}\) is a decomposition space; we need to check that in \(\text{sd}(N\mathcal{D})\), the inert face maps \(d_T\) and \(d_\perp\) form pullbacks against each other (the Segal condition). But each top face map in \(\text{sd}(N\mathcal{D})\) is given by the composite of two outer face maps in \(N\mathcal{D}\) (removing \(n\) and \(n'\) in eq. (2)), and each bottom face map in \(\text{sd}(N\mathcal{D})\) is given by the composite of two inner face maps in \(N\mathcal{D}\) (removing \(0\) and \(0'\)). Hence the Segal condition on \(\text{sd}(N\mathcal{D})\) follows from the decomposition-space condition on \(N\mathcal{D}\).

---

\(^1\)Note that the “nerve functor” \(N : \text{Decomp} \to \text{sSet}\) on the left of the diagram is just the inclusion, but it will be convenient to have it named, so as to stress that we regard decomposition spaces as structures generalizing categories.
If \( E \rightarrow D \) is CULF, its naturality square along any active map is cartesian. The bottom face maps of \( \text{tw}(E) \) are given by (composites of) inner—hence active—maps in \( E \), and similarly for \( D \), so the naturality square along any bottom face of \( \text{tw}(E) \rightarrow \text{tw}(D) \) is again cartesian, as required for it to be a right fibration.

**Remark 2.2.** The object part of lemma 2.1 has been observed also by Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer [1], who furthermore establish a partial converse: they show (in the more general setting of simplicial objects \( X \) in a combinatorial model category) that \( \text{sd}(X) \) is Segal if and only if \( X \) is 2-Segal [4] (i.e. a “non-unital decomposition space”).

### 3 From category of elements to twisted arrow category

In this section we describe the natural transformation from categories of elements to twisted arrow categories. First we shall need a few basic facts about \( \mathcal{Q} \) and the “last-vertex map.”

**Lemma 3.1.** The functor \( \mathcal{Q} \) sends top-preserving maps to active maps.

**Proof.** If \( f : [m] \rightarrow [n] \) preserves the top element (that is, \( f(m) = n \)), then \( \mathcal{Q}(f) \) is the map \( f^{op} \circ f : [m]^{op} \star [m] \rightarrow [n]^{op} \star [n] \), which clearly preserves both the bottom element \( m' \) and the top element \( m \).

The “last-vertex map” of \([18]\) is a natural transformation \( \text{last} : \mathcal{Q} \circ \text{el} \Rightarrow \text{id}_{\text{Set}} \), which sends a simplex \( \sigma \) in \((\text{el} X)_0\) to its last vertex. Its value in higher simplicial degree is given by the following lemma.

**Lemma 3.2.** For any \( k \), let \( \varphi \in (N \Delta)_k \) denote a sequence of maps \( [n_0] \xrightarrow{f_1} [n_1] \xrightarrow{f_2} \cdots \xrightarrow{f_k} [n_k] \in \Delta \). Then there is a unique commutative diagram \( B(\varphi) \) of the form

\[
\begin{array}{cccccc}
[0] & \xrightarrow{d^\top} & [1] & \xrightarrow{d^\top} & \cdots & \xrightarrow{d^\top} & [k] \\
\beta(n_0) & & \beta(f_1) & & \beta(f_k) & \\
[n_0] & \xrightarrow{f_1} & [n_1] & \xrightarrow{f_2} & \cdots & \xrightarrow{f_k} & [n_k]
\end{array}
\]

for which all the vertical maps are top preserving, and all the maps in the top row are \( d^\top \).

The proof is straightforward. For a hint, see the proof provided for the next lemma, where we give an two-sided refinement of this construction.

**Lemma 3.3.** For any \( k \), let \( \varphi \in (N \Delta)_k \) denote a sequence of maps \( [n_0] \xrightarrow{f_1} [n_1] \xrightarrow{f_2} \cdots \xrightarrow{f_k} [n_k] \in \Delta \). Then there is a unique commutative diagram \( A(\varphi) \) of the form

\[
\begin{array}{cccccc}
\mathcal{Q}[0] & \xrightarrow{Q(d^\top)} & \mathcal{Q}[1] & \xrightarrow{Q(d^\top)} & \cdots & \xrightarrow{Q(d^\top)} & \mathcal{Q}[k] \\
\alpha(n_0) & & \alpha(f_1) & & \alpha(f_k) & \\
[n_0] & \xrightarrow{f_1} & [n_1] & \xrightarrow{f_2} & \cdots & \xrightarrow{f_k} & [n_k]
\end{array}
\]
i.e. for which all the vertical maps are active and all the top maps are of the form \( Q(a^T) \).

**Proof.** When \( k = 0 \) it is clear: there is a unique active map \( Q[0] = [1] \to [n_0] \); call it \( a(n_0) \). The result now follows from the obvious fact that for any map \( f : [m] \to [n] \) in \( \Delta \), the following solid arrow diagram admits a unique active extension as shown:

\[
\begin{array}{c}
Q[m] \\
\downarrow f
\end{array}
\Rightarrow
\begin{array}{c}
Q[m+1] \\
\downarrow a
\end{array}
\Rightarrow
\begin{array}{c}
[n]
\end{array}
\]

**Remark 3.4.** In lemma 3.3, if \( f_i \) itself is active for any \( 1 \leq i \leq k \), then the map \( a(f_i) \) is degenerate; it factors through \( Q(s^T) \) as in the following diagram:

\[
\begin{array}{c}
Q[i-1] \\
\downarrow a(f_{i-1})
\end{array}
\Rightarrow
\begin{array}{c}
Q[i] \\
\downarrow a(f_i)
\end{array}
\Rightarrow
\begin{array}{c}
[n_{i-1}] \\
\downarrow f_i
\end{array}
\Rightarrow
\begin{array}{c}
[n_i]
\end{array}
\]

In particular, \( f_i \circ a(f_{i-1}) \) and \( a(f_i) \) have the same long edge in \([n_i]\).

**Lemma 3.5.** There is a natural transformation \( \Lambda : N \circ \text{el} \Rightarrow \text{sd} \) of functors \( \text{sSet} \to \text{sSet} \). The degree-0 component \((N \circ \text{el})_0 \to (\text{sd})_0 \) sends any simplex in \( X \) to its long edge.

**Proof.** A \( k \)-simplex in \( N \circ \text{el}(X) \) is a sequence \([n_0] \to \cdots \to [n_k]\) together with an \( n_k \)-simplex \( \sigma \in X_{n_k} \). By lemma 3.3, there is an induced map \( Q[k] \to [n_k] \), and hence \( \Delta[k] \to \text{sd}(X) \). Naturality follows from the uniqueness of the diagram in (3).

**Remark 3.6.** The two arguments in lemmas 3.2 and 3.3 can be compared by means of the last-segment inclusion \( L_m : [m] \to Q[m] \), by which \( \text{cod} \) was defined. One finds that \( \Lambda \) mediates between the natural transformations last and \( \text{cod} \) as follows:

\[
\begin{array}{c}
N \circ \text{el} \\
\downarrow \text{last}
\end{array}
\Rightarrow
\begin{array}{c}
\Lambda
\end{array}
\Rightarrow
\begin{array}{c}
\text{sd} \\
\downarrow \text{cod}
\end{array}
\Rightarrow
\begin{array}{c}
\text{id}
\end{array}
\]

**Lemma 3.7.** For any sequence \( \varphi \in (N \Delta)_k \) as in lemmas 3.2 and 3.3, we have

\[ Q(B(\varphi)) = A(Q(\varphi)) \]

**Proof.** Just observe that \( Q(B(\varphi)) \) is a diagram with bottom row \( Q(\varphi) \), and it satisfies the condition of lemma 3.3, as a consequence of lemma 3.2 and lemma 3.1. Therefore, by uniqueness in lemma 3.3, the diagram must be \( A(Q(\varphi)) \).
For $X$ a simplicial set with corresponding right fibration $p: \text{el}(X) \to \Delta$, there is a functor $\omega_X: \text{el}(\text{sd} X) \Rightarrow \text{el}(X)$ induced by pullback along $Q$:

$$
\begin{align*}
\text{el}(\text{sd} X) @>\omega_X>> \text{el}(X) \\
\text{sd} p @>\downarrow p>> \downarrow p \\
\Delta @>Q>> \Delta
\end{align*}
$$

(4)

It sends an $n$-simplex in $\text{sd}(X)$ to the corresponding $(2n+1)$-simplex in $X$. These functors assemble into a natural transformation

$$
\omega: \text{el} \circ \text{sd} \Rightarrow \text{el}.
$$

**Lemma 3.8.** The following diagram in the category of endofunctors on $\text{sSet}$ commutes:

$$
\begin{array}{ccc}
N \circ \text{el} \circ \text{sd} & \xleftarrow{N \omega} & N \circ \text{el} \\
\downarrow \Lambda & & \downarrow \Lambda \\
N \circ \text{el} & \xrightarrow{\text{last}_\text{sd} \Lambda} & \text{sd}.
\end{array}
$$

**Proof.** Given an $k$-simplex $x \in N \text{el}(\text{sd} X)_k$, i.e. a sequence $[n_0] \to \cdots \to [n_k] \to \text{sd}(X)$, applying $\text{last}_\text{sd}$ returns the dotted arrow as shown (see lemma 3.2):

$$
\begin{array}{cccc}
[0] & \xrightarrow{d^*} & [1] & \xrightarrow{d^*} \cdots \xrightarrow{d^*} & [k] \\
\downarrow & & \downarrow & & \downarrow \\
[n_0] & \xrightarrow{} & [n_1] & \xrightarrow{} \cdots & [n_k] \xrightarrow{} \text{sd}(X).
\end{array}
$$

Here we have written $Q^* X$ instead of the usual $\text{sd}(X)$ because we will make use of its adjoint $Q$, which restricts to $Q$ on representables, i.e. $Q[n] = [2n+1]$. Applying instead $\Lambda \circ N \omega$ returns the dotted arrow as shown:

$$
\begin{array}{cccc}
Q_1[0] & \xrightarrow{Q_1(d^*)} & Q_1[1] & \xrightarrow{Q_1(d^*)} \cdots \xrightarrow{Q_1(d^*)} & Q_1[k] \\
\downarrow & & \downarrow & & \downarrow \\
Q_1[n_0] & \xrightarrow{} & Q_1[n_1] & \xrightarrow{} \cdots & Q_1[n_k] \xrightarrow{} X.
\end{array}
$$

To see that these two dotted maps represent the same $k$-simplex of $\text{sd}(X)$ via the $Q_1 \dashv Q^*$ adjunction, we invoke lemma 3.7 and the uniqueness from lemma 3.3.

**Proposition 3.9.** On decomposition spaces, the map $\Lambda$ restricts to a cartesian natural transformation $\lambda: \text{el} \circ N \Rightarrow \text{tw}$:

$$
\begin{array}{ccc}
\text{Decomp} & \xrightarrow{\text{el} \circ N} & \text{Cat} \\
\downarrow \text{N} & \xRightarrow{\text{tw}} & \downarrow \text{N} \\
\text{sSet} & \xleftarrow{N \circ \text{el}} & \text{sSet}
\end{array}
$$

8
Proof. If \( \mathcal{D} \) is a decomposition space, \( \text{tw}(\mathcal{D}) \) is a category by lemma 2.1, and so is \( \text{el}(\mathcal{N}\mathcal{D}) \). Because \( N : \text{Cat} \to \text{sSet} \) is fully faithful, the natural transformation \( \Lambda \) lifts uniquely to a natural transformation \( \lambda \) as shown.

It remains to show that for any CULF map \( F : \mathcal{E} \to \mathcal{D} \), the diagram

\[
\begin{align*}
\text{el}(\mathcal{N}\mathcal{E}) & \xrightarrow{\Lambda_{\mathcal{E}}} \text{tw}(\mathcal{E}) \\
\text{el}(\mathcal{N}F) & \downarrow \quad \downarrow \text{tw}(F) \\
\text{el}(\mathcal{N}\mathcal{D}) & \xrightarrow{\lambda_{\mathcal{D}}} \text{tw}(\mathcal{D})
\end{align*}
\]

is cartesian. On objects and morphisms respectively, this amounts to showing that a unique lift exists for any solid-arrow squares (arbitrary \( \tau, \sigma, f \)) as follows:

\[
\begin{align*}
\Delta[1] & \xrightarrow{\tau} \mathcal{E} \\
\Delta[n] & \xrightarrow{\sigma} \mathcal{D} \\
\Delta[0] & \xrightarrow{a(\sigma)} \mathcal{D} \\
\Delta[3] & \xrightarrow{\tau} \mathcal{E} \\
\Delta[n] & \xrightarrow{\sigma} \mathcal{D} \\
\Delta[0] & \xrightarrow{a(\sigma)} \mathcal{D} \\
\Delta[1] & \xrightarrow{\tau} \mathcal{E} \\
\Delta[n] & \xrightarrow{\sigma} \mathcal{D} \\
\Delta[0] & \xrightarrow{a(\sigma)} \mathcal{D}
\end{align*}
\]

Here the \( a \)'s denote the unique active maps, as in lemma 3.3. These two lifts do indeed exist uniquely because \( F \) is CULF.

\[\square\]

4 Proof of main theorem

Lemma 4.1. For every decomposition space \( \mathcal{D} \), the following diagram commutes (up to isomorphism):

\[
\begin{array}{cccc}
\text{Decomp}_{\mathcal{D}} & \xrightarrow{N_{\mathcal{D}}} & \text{sSet}_{/\mathcal{N}\mathcal{D}} & \xrightarrow{\text{el}_{\mathcal{N}\mathcal{D}}} & \text{RFib}_{/\text{el}(\mathcal{N}\mathcal{D})} & \xrightarrow{Q^*_{\text{el}(\mathcal{N}\mathcal{D})}} & \text{RFib}_{/\text{el}(\text{sd}(\mathcal{N}\mathcal{D}))} \\
\text{tw}_{\mathcal{D}} & \text{RFib}_{/\text{tw}(\mathcal{D})} & \text{sd}_{\mathcal{N}\mathcal{D}} & \text{sSet}_{/\text{sd}(\mathcal{N}\mathcal{D})} & \text{el}_{\text{sd}(\mathcal{N}\mathcal{D})} & \text{RFib}_{/\text{el}(\text{sd}(\mathcal{N}\mathcal{D}))}
\end{array}
\]

where \( Q^* : \text{RFib}_{/\Delta} \to \text{RFib}_{/\Delta} \) is pullback along \( Q \).

Proof. We already have the following (up-to-isom) commutative diagram:

\[
\begin{array}{cccc}
\text{Decomp} & \xrightarrow{N} & \text{sSet} & \xrightarrow{\text{el}} & \text{RFib}_{/\Delta} & \xrightarrow{Q^*} & \text{RFib}_{/\Delta} \\
\text{tw} & \text{RFib} & \text{sd} & \text{sSet} & \text{el} & \text{RFib}_{/\Delta}
\end{array}
\]

Indeed, the left side is lemma 2.1, and the right side is just the translation between simplicial sets and right fibrations over \( \Delta \). Now slice it over \( \mathcal{D} \). (As always when slicing (up-to-isom) commutative diagrams, the result involves the identifications already expressed by the diagrams. In the present case we use \( \text{sd}(\mathcal{N}\mathcal{D}) \cong N \text{ tw}(\mathcal{D}) \) and \( Q^*(\text{el}(\mathcal{N}\mathcal{D})) \cong \text{el}(\text{sd}(\mathcal{N}\mathcal{D})) \cong \text{el}(N \text{ tw}(\mathcal{D})) \).)

\[\square\]
A key ingredient in the proof of the main theorem is to see that $\lambda^*_D$ provides a sort of splitting of the double rhombus diagram from the previous lemma:

**Lemma 4.2.** For any $D \in \text{Decomp}$, the following diagram commutes up to natural isomorphism.

$$
\begin{array}{cccc}
\text{Decomp}_D / N_D & \xrightarrow{N_D} & \text{sSet} / N_D & \xrightarrow{\text{el}_N D} & \text{RFib} / \text{el}(N_D) \\
& & \xrightarrow{\lambda^*_D} & & \xrightarrow{Q^*_\text{el}(N_D)} \\
\text{RFib} / \text{tw}(D) & & \xrightarrow{\text{el}_{\text{tw}(D)}} & & \text{sSet} / \text{sd}(N_D) & \xrightarrow{\text{el}_{\text{sd}(N_D)}} & \text{RFib} / \text{el}(\text{sd} N_D)
\end{array}
$$

(*7*)

**Proof.** The commutativity of the left square follows immediately from the fact that $\lambda$ is cartesian, cf. proposition 3.9; see in particular the pullback square in eq. (5).

For the right square, let $p : \mathcal{F} \to \text{tw}(D)$ be a right fibration. We must establish a map $\text{el}(N \mathcal{F}) \to Q^*_\text{el}(N_D)(\lambda^*_D \mathcal{F})$ and show it is an isomorphism over $\text{el}(\text{sd} N D)$. By eq. (4), the functor $Q^*_\text{el}(N_D)$ is pullback along $\omega_{N D}$ in the following diagram of categories:

$$
\begin{array}{cccc}
\mathcal{F}' & \xrightarrow{\lambda^*_D} & \mathcal{F} & \\
\text{el}(\text{sd} N D) & \xrightarrow{\omega_{N D}} & \text{el}(N D) & \xrightarrow{\lambda^*_D} & \text{tw}(D) & \\
\Lambda & \xrightarrow{Q} & \Lambda
\end{array}
$$

Thus we have $\mathcal{F}' \cong Q^*_\text{el}(N_D)(\lambda^*_D \mathcal{F})$. The nerve of the middle composite,

$$
N \text{el}(\text{sd} N D) \xrightarrow{N \omega_{N D}} N \text{el}(N D) \xrightarrow{\Lambda N} \text{sd}(N D),
$$

is identified with $\text{last}_{\text{sd}(N D)}$ by lemma 3.8 and proposition 3.9, which in particular gives $N \lambda = \Lambda N$. The naturality square for the last-vertex map

$$
\begin{array}{ccc}
N \text{el}(N \mathcal{F}) & \xrightarrow{\text{last}_{N \mathcal{F}}} & N \mathcal{F} \\
N \text{el}(N \rho) & \downarrow & \downarrow N \rho \\
N \text{el}(\text{sd} N D) & \xrightarrow{\text{last}_{\text{sd}(N D)}} & \text{sd}(N D)
\end{array}
$$

induces a morphism $N \text{el}(N \mathcal{F}) \to N \mathcal{F}'$ by the universality of $\mathcal{F}'$ as a pullback, and the fact that $N : \text{Cat} \to \text{sSet}$ preserves limits. Since $N$ is fully faithful, we obtain our desired comparison map $u : \text{el}(N \mathcal{F}) \to \mathcal{F}'$ of discrete fibrations over $\text{el}(\text{sd} N D)$. It is enough to check that the restriction of $u$ to each fiber is a bijection; to do so we describe these fibers and the map $u$ between them in concrete terms.

An object in $\text{el}(\text{sd} N D)$ is a pair $([n], \sigma)$ where $\sigma : [n] \to \text{sd}(N D)$. As an $n$-simplex in $\text{tw}(D)$, it is sent by $\omega_{N D}$ to the same pair $([n], \sigma)$, but where now $\sigma$ is considered a $(2n+1)$-simplex in $D$. Applying $\lambda_D$ returns the long edge of the simplex, $[1] \to [2n+1] \to D$,
which we denote \( \ell \in \text{Ob}\ tw(D) \). The fiber of \( \mathcal{F} \) over \( \sigma \) is the \( p \)-fiber over \( \ell \), that is the discrete set \( \{ z \in \text{Ob}\mathcal{F} \mid p(z) = \ell \} \). In other words, we can identify an object in this fiber with a commutative diagram

\[
\begin{array}{ccc}
[0] & \xrightarrow{z} & \mathcal{F} \\
\downarrow{\mu} & & \downarrow{p} \\
[n] & \xrightarrow{\sigma} & \text{sd}(N\mathcal{D}).
\end{array}
\]

(8)

On the other hand, an object in \( \text{el}(N\mathcal{F}) \) over \( \sigma \) can be identified with an \( n \)-chain of arrows \( z_0 \to \cdots \to z_n \) in \( \mathcal{F} \), lying over \( \sigma \). The comparison functor \( u : \text{el}(N\mathcal{F}) \to \mathcal{F} \) sends such an \( n \)-chain to its last element, \( z_n \). Thus it suffices to show that there is a unique lift in eq. (8). But this is exactly the condition that \( p \) is a right fibration. □

**Corollary 4.3.** With notation as in eqs. (6) and (7), we have natural isomorphisms

\[
\partial_{\text{sd}(N\mathcal{D})} \circ Q^*_\text{el}(N\mathcal{D}) \cong \text{sd}_{N\mathcal{D}} \circ \partial_{N\mathcal{D}} \quad \text{and} \quad N_{\text{tw}(\mathcal{D})} \cong \partial_{\text{sd}(N\mathcal{D})} \circ Q^*_\text{el}(N\mathcal{D}) \circ \lambda^*_D.
\]

**Proof.** Using \( \partial \circ \text{el} = \text{id} \), these statements follow from eqs. (6) and (7), right parts. □

**Lemma 4.4.** For any decomposition space \( \mathcal{D} \), there exists a functor \( \text{untw}_\mathcal{D} : \text{RFib}_{/\text{tw}(\mathcal{D})} \to \text{Decomp}_{/\mathcal{D}} \) with

\[
\text{untw}_\mathcal{D} \circ \text{tw}_\mathcal{D} = \text{id}_{\text{Decomp}_{/\mathcal{D}}}. 
\]

Furthermore, both the left-to-right and the right-to-left squares commute:

\[
\begin{array}{ccc}
\text{Decomp}_{/\mathcal{D}} & \xrightarrow{\text{untw}_\mathcal{D}} & \text{RFib}_{/\text{tw}(\mathcal{D})} \\
\downarrow{N_{\mathcal{D}}} & & \downarrow{\lambda^*_D} \\
\text{sSet}_{/N\mathcal{D}} & \xleftarrow{\delta_{N\mathcal{D}}} & \text{RFib}_{/\text{el}(N\mathcal{D})}.
\end{array}
\]

(9)

**Proof.** We already know that the left-to-right square commutes by lemma 4.2 (left part). To define \( \text{untw}_\mathcal{D} \), making the right-to-left square commute, it suffices to show that for any right fibration \( p : \mathcal{F} \to \text{tw}(\mathcal{D}) \), the simplicial map \( \partial_{N\mathcal{D}} \lambda^*_D(p) \) is CULF, because then it lands in \( \text{Decomp} \) (see the preliminaries on decomposition spaces, p.4). Unwinding the statement, we need to show that in the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{j} & \mathcal{F} \\
\downarrow{q} & & \downarrow{p} \\
V & \xrightarrow{j} & \text{el}(N\mathcal{D}) \xrightarrow{\lambda_D} \text{tw}(\mathcal{D}) \\
\downarrow{\Delta_{\text{active}}} & & \downarrow{\Delta} \\
\Delta & \xrightarrow{\text{incl}} & \Delta
\end{array}
\]
the right fibration $q : U \to V$ corresponds to a cartesian natural transformation

$$
\Delta^\text{op}_{\text{active}} \xymatrix{ U \ar[d]_{\psi} \ar[r] & \Set. \ar[l]_{\psi} \\
V }
$$

The cartesian condition in turn can be read off directly on $q$: we need to check that for every active $a : [m] \to [n]$ the following square is a pullback of sets:

$$
\begin{array}{ccc}
U_m & \xymatrix{ a^* \ar[r] & U_n } & \\
V_m & \xymatrix{ a^* \ar[r] & V_n } & \\
\end{array}
$$

As in the previous proof, we check this by computing these $q$-fibers in term of $p$-fibers. By remark 3.4 (with $i = 1$), $\lambda_D$ sends any arrow lying over an active map to an identity. Thus the $a^*$ maps are fiberwise bijections, so it is clear the square is a pullback.

Finally the main statement follows easily: we first read off from the commutativity of eq. (9) that

$$
N_D \circ \text{untw}_D \circ \text{tw}_D = \partial_N D \circ \lambda_D^* \circ \text{tw}_D = \partial_N D \circ \text{el}_N D \circ N_D = N_D.
$$

Since the nerve functor is fully faithful, so is the slice $N_D$, and we have established $\text{untw}_D \circ \text{tw}_D = \text{id}_{\text{Decomp}_D}$. \hfill \Box

**Theorem 4.5 (Main theorem).** For any decomposition space $D$, we have natural inverse equivalences of categories

$$
\begin{array}{ccc}
\text{Decomp}_D & \xymatrix{ \text{tw}_D \ar[r] & \text{RFib}_{/ \text{tw}(D)} } & \\
\text{untw}_D \ar[u] \end{array}
$$

**Proof.** We established $\text{untw}_D \circ \text{tw}_D = \text{id}_{\text{Decomp}_D}$ in lemma 4.4. For the other direction, since nerve is fully faithful, it suffices to prove that $N_{\text{tw}(D)} \circ \text{tw}_D \circ \text{untw}_D = N_{\text{tw}(D)}$. This is the outer square in the diagram below:

The commutativity of each of the squares inside was proven earlier as indicated. \hfill \Box
Remark 4.6. All the proof ingredients are readily seen to be natural in $\mathcal{D}$. In fact the main theorem can be seen as the $\mathcal{D}$-component of a natural equivalence of $\text{Cat}$-valued functors

\[
\text{Decomp}\xrightarrow{\simeq} \text{Cat}
\]

\[
\text{Decomp}_{/\pi} \xrightarrow{\simeq} \text{RFib}_{/\pi(\mathcal{D})}
\]

Corollary 4.7. For $\mathcal{D}$ a decomposition space and $\mathcal{F} \to \text{tw}(\mathcal{D})$ a right fibration, $\mathcal{F}$ is again the twisted arrow category of a decomposition space.

Corollary 4.8. The category $\text{Decomp}$ is locally cartesian closed.

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