A note on a structural definition of social-ecological network

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Abstract

A social-ecological network is a formal representation of a corresponding social-ecological system. Conventionally, such networks have been defined as encoding and representing pairwise interactions among the fundamental units of the system. We propose a combinatorial definition of social-ecological network based on its structure as a simplicial complex that serves as a formal representation of social-ecological systems that admit even higher-order interactions. We see our definition as complementary to the much successful graph theoretic paradigm of representing essentially the 1-skeletons of most of the social-ecological systems using the conventional definition of social-ecological network.

Introduction

A social-ecological system (SES) is a complex adaptive system [1, 2, 3]. For every such system, there exists a corresponding social-ecological network (SEN) that encodes the interactions of the system, giving rise to a formal representation of the SES. SENs as network theoretic representations have offered a successful framework for obtaining insight into the inherent complexity of the modelled SES, along with an understanding of its various behaviour and properties, and over the past decades has evolved as an epitome of formal modelling of SESs [4 – 12].

Most of the SENs proposed in the scholarship are based on an assumption that the SES represented by the SEN is fully described by considering the binary, or pairwise interactions among the units of the system. Such a graph theory-based SEN formalism for a SES completely models the represented system, provided the system admits only binary, that is, pairwise interactions among its fundamental units. However, for SESs that may admit not only binary but also non-pairwise higher-order interactions, such SEN representations face a twin mathematical challenge:(i) due to the definition of graph as a combinatorial object, the SEN model is necessarily limited to only capture the binary relations existing in the system, and hence are successful in capturing the system interactions only partially, and therefore, (ii) there exists a possibility of masking those complex behaviours of the modelled system that are expressed through phenomena such as self-organisation and emergence due to higher-order interactions in the system. These two limitations often result in an oversimplified model which fails to be a faithful (in the sense of being ‘isomorphic’) representation of the modelled SES.

This note proposes a combinatorial definition of an SEN that is structurally generalised (in the sense of orders of interactions) and theoretically addresses the above problems in modelling.
and describing an SES. While the proposed definition of SEN serves as a generic definition for all networks – random, regular or complex - it takes into consideration not only the existence of pairwise interactions among the system units as conventional networks do, but also the usually more abundant higher-order interactions within the SES under study, which often remain excluded in the graph theory-based SEN that represents the system as a formal model.

The proposed definition of SEN focuses on the most foundational object in an SES, namely, its structure. In order to make the note self-contained, the present section provides the preliminaries essential for constructing this definition. We begin with the definition of structure as a mathematical object, essentially following the detailed discussions presented in texts such as [13, 14]. Next, we employ this definition to formalise the concept of an SES through different orders of interactions among its constituent fundamental units with the objective to arrive at a definition of an SEN representing it.

**Preliminaries**

**Structure**

**Definition:** A structure is a tuple \( \theta = (X, R, F, C) \) where:

(i) \( X \) is a non-empty set, the universe or the domain or the carrier set of the structure;
(ii) \( R = \{R_1, \ldots, R_m \} \) is the set of relations on \( X \);
(iii) \( F = \{F_1, \ldots, F_n \} \) is the set of mappings on \( X \);
(iv) \( C = \{c_i \mid i \in I \} \), \( c_i \in X \ \forall i \in I \) is the set of constants, where \( I \) is an index set.

The *signature* of the structure is a sequence \( \{f_i, \ldots, f_n, r_1, \ldots, r_m, \kappa\} \), where \( F_i : X^{f_i} \to X \) is an \( n \)-ary map for each \( i \); \( R_j \subseteq X^{r_j} \), is an \( m \)-ary relation for each \( j \); \( \kappa = \{|c_i | i \in I \} \), cardinality of \( I \), on \( X \) where \( n, m \in \mathbb{N} \).

Often, it is the universe \( X \) that forms the focus of study, as all the mappings, relations and constants are defined on \( X \). As such, the structure may also be written as \( \theta_X = (X, F, R, C) \), or more commonly as \( X = (X, F, R, C) \), thus identifying the structure with the universe. Further, a structure in which the set \( F = \emptyset \) is called a relational structure, and would form the central component of what follows in this work. Sometimes, however, different symbols may be used when there is a need to distinguish between and emphasise on the name of the object defined by the structure and the universe of the object. We shall adopt the above notational conventions interchangeably and as per need in what follows, as we apply the definition to formally define a system.

**System**

**Definition:** A system \( \Sigma \) is an object given by the structure \( \theta_\Sigma = (V, E, C) \), where \( V \), the vertex set is a non-empty set, the universe of the system, whose elements are the combinatorial objects called vertices and represent the fundamental units of \( \Sigma \). \( E \) is the set of relations, whose elements encode the relationships among the vertices, and are called the edges. These elements are denoted by \( e \) and are the members of the power set of \( V, \mathcal{P}(V) \), the set of all subsets of \( V \) including the empty set \( \emptyset \) and \( V \) itself. Thus, \( \forall e \in E, e \in \mathcal{P}(V) \). This implies that \( E \subseteq \mathcal{P}(V) \).
The set $C$ comprises the constants of the system, and may be considered as the set of all 0-ary or nullary maps on $\theta_\Sigma$. If there is no possibility of confusion and no particular need, we may not explicitly mention the set $C$ henceforth.

For our entire discussion in what follows, we shall assume the sets $V$ and $E$ to be finite, and write $|V| = m, m \in \mathbb{N}$, etc.

From the foregoing definitions, it is clear that the set $V$ of the system is its universe and contains the fundamental entities of the system. As already mentioned, the elements of the set $E$ are the members of $\mathcal{P}(V)$, the power set of $V$ (that is, the elements of $E$ are finite subsets of $V$ of arbitrary cardinalities), and $E$ thus comprises the set of relations of arbitrary arity on $V$. This formalism allows modelling all existing relations in a system, including as also going beyond the pair-wise or the binary relationships that may exist in the system.

Realisations of the above definition of system is instantiated by a myriad of both abstract as well as real-world examples. An example of a mathematical system would be that of the natural numbers, written as $\theta_\mathbb{N} = (\mathbb{N}, <, S, +, \cdot, 0)$, where $\mathbb{N}$ is the non-empty set of elements (symbols) called numerals, $<$ is a binary relation symbol, $S$ is a unary mapping symbol, while $+$ and $\cdot$ are binary mapping symbols, often referred to as addition and multiplication, respectively, and $0$ is a constant symbol in $\mathbb{N}$. An ecosystem serves as an example of a physical system, and can be written as $\theta_E = (\mathcal{V}_E, \mathcal{E}_E, C_E)$, where $\mathcal{V}_E$ is the non-empty set of all the components of a given ecosystem, $\mathcal{E}_E$ is the set of all relations that operate among the components within the ecosystem, and $C_E$ comprises the ecological or environmental constants in the set $\mathcal{V}_E$.

Henceforth, we shall denote a system as $\Sigma$ instead of $\theta_\Sigma$, unless stated otherwise, and when there is no possibility of confusion between the structure of the system and the system itself. Next, we formalise the concept of interaction and illustrate some useful geometrical interpretations that could be inferred from the definition.

**Interaction**

*Definition:* Let $\Sigma = (V, E, C)$ be a system. The members of the set $E$ are the *interactions* in $\Sigma$. Thus, an interaction $e = \{v_0, ..., v_{k-1}\}$ containing $k$ number of vertices, $k \in \mathbb{N}$, $k \leq |V|$ is an element of the power set of $V$: $e \in \mathcal{P}(V)$. The *order* of an interaction $e$ with $|e| = k$ is defined to be $k - 1$, and $e$ is called a $k - 1$ interaction.

The following table, for example, illustrates the inferences that could be drawn from the above definition through geometrical interpretation, with $n \in \mathbb{N}$.

| Interaction $e$ | Geometrical interpretation | Order of interaction $n$ |
|----------------|----------------------------|-------------------------|
| $e = \{v_i\}$ | A single vertex            | 0 (0-interaction)       |
| $e = \{v_i, v_j\}$ | Two vertices, interacting with each other | 1 (1-interaction)       |
| $e = \{v_i, v_j, v_k\}$ | Three vertices interacting among themselves | 2 (2-interaction)       |
| ...            |                            |                         |
| $e = \{v_i, ..., v_k\}$ | $k$ vertices interacting among themselves | $k - 1$ ($k - 1$-interaction) |
Interactions are classified as higher-order if \( n \geq 2 \) that is, if \(|e| = k \geq 3\), and as lower-order if \( n \leq 1 \), that is, if \(|e| = k \leq 2\).

Since each element of \( E \) is an interaction in the system, therefore in accordance with the structure schemata, the \( n \)-ary relations that comprise \( E \) encode (that is, the data collected for the system are formulated as a representation using the combinatorial formalism) all interactions in \( \Sigma \). Owing to this fact, we shall use the term relation interchangeably with the interaction that this relation encodes, in the rest of this article.

It may be remarked here that our definitions of system and interaction makes interaction a central object to the existence of a system.

Having given a formal and combinatorial description of a system comprising \( n \)-ary interactions, we next proceed to represent such a system mathematically. For doing so, we need to introduce some objects from algebraic topology in the following section.

**Simplicial Complex**

In this section, we define and describe a combinatorial object known as simplicial complex in classical algebraic topology, and argue that a simplicial complex is equipped with an appropriate structure to capture interactions of all possible (finite) orders between the fundamental units of a given system. A rich body of literature has been devoted to the study of simplicial complexes, and the definitions that follow this paragraph to make the paper self-contained are essentially found in the standard texts [15, 16, 17]. In this work we are interested to develop a combinatorial description of social-ecological systems, and hence shall remain primarily concerned with the concept of an abstract simplicial complex, while making only a passing reference to its geometric counterpart.

**Definition:** Let \( V \) be a non-empty, finite set with \(|V| = n + 1, n \in \mathbb{N}\), whose elements are the vertices, and are denoted by \( v_i, \ i = 0, ..., n \). Any member of the power set of \( V, \mathcal{P}(V) \), is a combinatorial object called a simplex over \( V \) (or, a simplex). In the rest of our discussion, we shall assume that all simplices (also, simplexes) are non-empty. Thus, by a simplex we shall understand a non-empty, finite subset of \( V \).

A simplex \( \sigma \) containing \( k+1 \) vertices ((\( k < n \)) is defined to have a dimension \( k \), written as \( \text{dim}(\sigma) = k \). Often, \( \sigma \) is then called a \( k \)-simplex. Any subset of a simplex \( \sigma \) is called a face of \( \sigma \). If \( \tau \) is a face of \( \sigma \), then it is written as \( \tau \subseteq \sigma \). A 0-face (0-dimensional subset of a simplex) is called a vertex and written as \( \{v_i\} \), and a 1-face is called an edge, written as \( \{v_i, v_j\} \), with the indices belonging to some index set.

**Definition:** Given a set \( V \) as in the above definition, a family \( \Delta \) of simplices is called a simplicial complex if it is closed under inclusion, that is, under taking of (finite, non-empty) subsets. Often, this object is also referred to as an abstract simplicial complex.

Though the collection \( \Delta \) may either be finite or infinite, we shall assume it to be finite for our discussion.

The above definition means that the following condition holds for \( \Delta \):

\[
\forall \sigma \in \Delta, (\tau \subseteq \sigma \Rightarrow \tau \in \Delta). \quad (1)
\]
This condition further implies that two arbitrary simplices in the simplicial complex $\Delta$ are either disjoint, or they intersect as a face of $\Delta$. That is, $\forall \sigma, \tau \in \Delta \Rightarrow (i) \sigma \cap \tau = \emptyset$, or $\sigma \cap \tau \in \Delta$.

Let $V = \{v_0, ..., v_n\}$. Let $P(V)$ be written as $(v_0, ..., v_n)$. Thus according to the above definition, if $\sigma$ is an $n$-simplex over the vertices of $V$, then the simplicial complex $\Delta$ composed of $\sigma$ itself and the rest of its $d$-dimensional faces ($d < n$) is written as $\Delta = \langle v_0, ..., v_n \rangle$.

For example, given a set $V = \{v_i, v_j, v_k\}$, the simplicial complex $\Delta$ comprising the 2-simplex $\{v_i, v_j, v_k\}$ and all its 0- and 1-dimensional faces is written as $\Delta = \{\{v_i\}, \{v_j\}, \{v_k\}, \{v_i, v_j\}, \{v_i, v_k\}, \{v_j, v_k\}, \{v_i, v_j, v_k\}\} = \langle v_i, v_j, v_k \rangle$. It may be noted that this simplicial complex does not include the empty simplex $\emptyset$ as per our earlier assumption, while some authors include the empty simplex too while describing the simplicial complex [18].

The dimension of a simplicial complex $\Delta$, denoted by $dim(\Delta)$, is defined to be $r \geq 0$ where $r$ is the largest natural number such that $\Delta$ contains an $r$-simplex. If $dim(\Delta) = d$, then every face of dimension $d$ is called a cell, while that of dimension $d-1$ is called a facet.

**Definition:** Let $\Delta$ be a simplicial complex. The boundary of $\Delta$, denoted by $\partial \Delta$, is the set of all the faces of $\Delta$. $\partial \Delta := \{\tau | \tau \subseteq \sigma \in K\}$, where $\sigma$ is a unique $k$-simplex of $\Delta$.

The above definition implies that for a given $k$-simplex in $\Delta$, every face that is a ($k$-1)-simplex is a boundary face or a facet. Further,

**Definition:** Let $\Delta$ be a simplicial complex on a set $V$. Let $p \in \mathbb{N}$. The simplicial complex given by $\Omega := \{\sigma \in \Delta | dim(\sigma) \leq p\}$ is defined to be the $p$-skeleton of $\Delta$.

$\Omega$ is therefore the collection of all faces of $\Delta$, that have a dimension at most $p$.

For example, a simple graph $G = (V, E)$ with a finite, non-empty vertex set $V$ and an edge set $E$ is a 1-skeleton comprising all 0-simplices (vertices) and 1-simplices (links) of a simplicial complex $\Delta$ with $dim(\Delta) \geq 1$. Therefore, all simple graphs are simplicial complexes of dimension at most 1.

As has been remarked earlier, a simplicial complex is a combinatorial object: it is a collection of simplices satisfying condition (1). Every simplicial complex $\Delta$, however, corresponds to a geometric object which is a subspace of the $m$-dimensional Euclidean space $\mathbb{R}^m$ via a mapping $\varphi: V \rightarrow \mathbb{R}^m$, where $V$ is the vertex set on which $\Delta$ is defined. The geometric realisation of $\Delta$ with respect to the mapping $\varphi$ is defined to be the set $|\Delta|_\varphi = \bigcup_{\sigma \in \Delta} |\sigma|_\varphi$. The object on the right hand side of the equality sign is the geometric simplex, and is a subspace of $\mathbb{R}^m$. This subspace has the set of $\varphi$ images of the vertices of the simplex $\sigma \in \Delta$ as a basis, and is thus the convex hull of points in the vertex set of $|\sigma|_\varphi$. The left-hand side object therefore becomes a subspace of $\mathbb{R}^m$, and is a topological space by virtue of the definition, called the polyhedron. Thus the topological object $|\Delta|_\varphi$, often written as $|\Delta|$ when the mapping is understood, is the geometric (and topological) counterpart of the combinatorial object $\Delta$. Note that if the mapping $\varphi$ is an affine embedding, then the topology of $|\Delta|_\varphi$ is independent of $\varphi$. However, as our interest is in modelling the system purely through its combinatorial information, we shall not discuss the geometric simplicial complex anymore, and shall remain focussed on the abstract simplicial
complex $\Delta$ for our further discussion in this note. For details on the geometric realisation, the reader may refer to [16, 17]. Though even for the purpose of visualisation, the geometric simplicial complex is highly useful, we shall not have much opportunity to refer to it in this note.

**Methodology**

In order to apply the foregoing definitions and discussions to establish a formal representation of an SES as an SEN, it is necessary to formulate the SES as a system in accordance with the structural perspective discussed so far. Once this is achieved, the interactions of all orders – both lower as well as higher - can be encoded and represented as simplices and simplicial complexes of appropriate dimensions, and the SES can be represented by an appropriate SEN that encodes the interactions in the SES. To obtain such a formulation, we propose a definition of SES from a structural perspective in the present section, followed by a definition of network that includes all possible orders of interactions in its structure.

While considering a system on the one hand and a simplicial complex on the other, it is immediate to note the one-to-one correspondence due to definition (though we do not define it here) and consequently an identification of the universe of the system with the vertex set of the simplicial complex, and the interactions of various orders in the system with simplexes of various dimensions belonging to this simplicial complex. Based on the foregoing, we may state that in a given system, any interaction of order $k \in \mathbb{N}, k \leq n$ would be represented by the corresponding $k$-simplex in entirety. The boundary of the system is represented by the boundary of the corresponding simplicial complex, which is a collection of faces of the simplicial complex. It may be noted that all binary relations present in the system are represented by interactions of order 1, that is, pairwise interactions, and are encoded (that is, the data collected for the system are formulated as a representation using the combinatorial formalism) as 1-simplexes and 0-simplexes in the complex, thus yielding a graph. However, the system may comprise higher-order interactions too, each encoded as a $k$-simplex ($k \geq 2$) in the complex, which the pairwise interaction (that is lower order interaction) formalism fails to capture and represent owing to its structure, as the graph representing such pairwise, binary interactions would form only the 1-skeleton of the $n$-dimensional simplicial complex.

In order to arrive at a structural definition for SES, we premise that the fundamental ecological units of interest are the different ecological entities comprising an ecosystem and constitute its universal set $V_E$, while the fundamental social units are the social entities comprising the set $V_S$, the universe of a social system such that $V_S \cap V_E = \emptyset$. The last condition asserts that the social and the ecological units are separate entities, and no ecological unit is a social unit and vice-versa. We further assume that the social and ecological systems are autonomous systems and exist independently of one another, with their structures given by $\theta_S = (V_S, E_S, C_S)$ and $\theta_E = (V_E, E_E, C_E)$, respectively. The sets $E_S$ and $E_E$ comprise the social and the ecological interactions respectively defined on the sets $V_S$ and $V_E$, with $E_S \cap E_E = \emptyset$. The last condition expresses the fact that the social interactions (recall, interactions encode relations) are distinct and different from the ecological interactions, and vice-versa.

**Definition:** A social-ecological system (SES) is an object with its structure given by $\theta_{SES} = (V, E, C)$, where the universe is the vertex set $V \neq \emptyset$ and comprises all the social-ecological units with $V = V_S \cup V_E$, the disjoint union of the sets $V_S \neq \emptyset$ and $V_E \neq \emptyset$. The set $E$ comprises the interactions of $\theta_{SES}$, and each interaction corresponds to an element of the power set of $V$, ...
The set \( C \) comprises the constants in the given SES, which are the non-negotiables of the system under study.

According to the foregoing definition, each element \( e \in \mathcal{P}(V) \) of arbitrary cardinality, that is, of arbitrary order comprising the collection \( E \) is encoded and represented by a simplex of an appropriate dimension. Together with condition (1), \( E \) constitutes a simplicial complex of dimension \( d > 0 \) over \( V \).

Let the environment be defined by the structure \( \theta_E = (X) \) where the universe \( X \) comprises all biotic and abiotic components of the environment. It may be noted that by the definition of environment, no interactions between its biotic and abiotic components are included, and hence the structure comprises only the non-empty set \( X \), while the other components of the structure are empty sets, and hence are omitted. Also, since both the social as well as ecological entities are members of the environment, therefore \( V_S \subseteq X, V_E \subseteq X \), with \( X = V_S \cup V_E \). It is important to note that \( E_S \cup E_E \subseteq E \), which captures the fact that the set \( E \) of relations in the SES contains all relations of all arity and hence interactions of all orders that exist at various scales and levels of integration within the social and the ecological components individually. These interactions could be purely social-social and ecological-ecological types, but may also encode relations at various scales and at various levels of integration that exist between the social and the ecological components, that is, the social-ecological interactions [6, 4, 12].

Finally, since \( V = V_S \cup V_E \subseteq X \), there exists an inclusion map \( f : V \hookrightarrow X \) defined by \( v \mapsto f(v) = v \in X \). This completes the inclusion of the universe \( V \) of the SES into the universe of the environment. The inclusion map embeds the focal SES within the environment. Further, we have \( X = V \cup V^c \), where \( V^c = \{ x | x \in X, x \notin V \} \) is the complement of the vertex set \( V \) and comprises all those elements of the environment that are not included in the SES under study, and hence are external to the system.

The interacting units of the social and the ecological systems individually and independently of each other form networks of interactions between their respective fundamental units, as stated in the definitions above. The SEN as a formal representation of the corresponding SES comprises a collection of such networks, encoding the inter-system interactions between the social units on the one hand and the ecological units on the other. As we have already noted, these (usually complex) networks do not necessarily encode only the dyadic interactions and therefore necessarily be represented by graphs, but may encode higher-order interactions present in the systems, represented by simplicial complexes of appropriate dimensions, particularly so when group interactions with group size larger than two are being represented. A SEN, therefore, must be understood as an empirical relational structure that comprises a simplicial complex representing the different orders of interactions that encode the \( n \)-ary relations that exist in the system, with \( n \geq 1 \). The mathematical object traditionally referred to as a SEN in graph-theoretic sense is the 1-skeleton of this simplicial complex. This 1-skeleton, which is a graph and is a 1-simplex, therefore only represents the pair-wise (thus, lower-order, at the most of order 1) interactions that encode at most only the binary relations that exist in the SES. Finally, based on the foregoing we define an SEN as a relational structure in the following paragraph.

**Definition:** A social-ecological network (SEN) is a combinatorial object whose structure is given by \( \Lambda_{SES} = (\Delta, \Lambda) \) along with an algorithm \( A \) such that for \( \Lambda \neq \emptyset, i \in \Lambda \subset \mathbb{N}, \Delta \) is a simplicial complex over \( V \), the universe of an SES, with \( \dim(\Delta) \geq 1 \) given by the algorithm \( A(i) \). A network is called static if the temporal component \( \Lambda \) consist of a single element \( i \), otherwise the network is a dynamic network.
The above definition implies that in the instance of an SES admitting interactions at most of order 1 (that is, only lower-order, at most binary interactions), the network $N_{\text{SES}}$ at every step indexed by $i$ can be identified with a 1-skeleton of $\Delta$. Such a network is often called the underlying graph of $\Delta$, and is given by the structure $G = (V, E)$, consisting of the vertex set $V$ a non-empty set of finite abstract combinatorial objects called vertices, and a family $E$ of two-element subsets of $V$: $E \subseteq \mathcal{P}(V)$ such that $\forall e \in E, |e| = 2$, that is, each $e$ is a 1-simplex called an edge of $G$, as mentioned earlier.

The foregoing implies that whenever a system admits interactions beyond the pairwise ones, the SEN representing this system should be structurally generalised, as suggested here, to include the non-pairwise, higher-order interactions to obtain a more ‘isomorphic’ representation and thus offer a deeper insight into the behaviour of the SES that the SEN represents. Our definition generalises the concept of an SEN structurally from a graph to a simplicial complex, and now it is able to account for and formally represent all orders of interactions that may involve groups or collections of vertices of the focal SES, even when the collection has more than two vertices.

We also distinguish between a graph and a network, in general, and specifically refer to the simplicial object defined as SEN in the foregoing as higher-order network, when non-pairwise interactions too are included in the representation of the corresponding SES. Only those SESs that admit solely lower-order interactions at most up to order 1, that is, admit at most pairwise interactions among its fundamental units, are the ones that are represented entirely by the graph theoretic (conventional) SENs. Such instantiations are the special cases of our definition of SEN, where by setting $\dim(\Delta) = 1$ we recover the conventional, graph theoretic network representing lower-order interactions in the SES. Representing any other SES by a graph would effectively amount to loss of information about the non-binary interactions existing in the system, and hence the structure of the SES that is being represented. Evidently, such SEN representations would result into an erroneous and incomplete insight into the structure and behaviour of the SES modelled.

Conclusion

The formal representation of the complex interactions in SESs via a combinatorial definition of an SEN as a simplicial structure as proposed, is expected to be significantly more ‘isomorphic’ or structurally more closer to the SES being modelled, compared to that using the conventional graph theoretic definition of SEN. Based on this definition, the modelling framework of formally representing an SES as an SEN holds the promise of facilitating a significantly detailed insight into the structure, and hence the interactions responsible for the inherent complexity and the complex behaviour of the SES being studied.

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