Nonperturbative path integral of 2d dilaton gravity and two-loop effects from scalar matter

W. Kummer\(^1\ast\), H. Liebl\(^1\dagger\) and D.V. Vassilevich\(^1,2\)\‡

\(^1\)Institut für Theoretische Physik
Technische Universität Wien
Wiedner Hauptstr. 8-10, A-1040 Wien
Austria

\(^2\)Department of Theoretical Physics
St. Petersburg University
198904 St. Petersburg
Russia

Abstract

Performing an nonperturbative path integral for the geometric part of a large class of 2d theories without kinetic term for the dilaton field, the quantum effects from scalar matter fields are treated as a perturbation. When integrated out to two-loops they yield a correction to the Polyakov term which is still exact in the geometric part. Interestingly enough the effective action only experiences a renormalization of the dilaton potential.

\ast e-mail: wkummer@tph.tuwien.ac.at
\dagger e-mail: liebl@tph16.tuwien.ac.at
\‡ e-mail: vasilev@snoopy.niif.spb.su
1 Introduction

Gravity in four dimensions still resists successful quantization. Also many quantum properties of the black hole and its interactions with quantized matter \[1\] have not found a generally accepted final clarification. In recent years however, remarkable progress has been made in the simplified setting of two dimensional models, including models with a dilaton field beside the metric \[2\], but also models with torsion \[3\]. In fact, by local transformations of the fields a theory with torsion may be transformed into a dilaton theory \[4\]. Then also the dilaton field may be finally transformed away \[2\]. However, it is not correct that as a consequence of this line of argument a quantization of any theory in \(d = 2\) may be achieved by quantizing theories depending on the metric alone by elimination of the dilaton field \[5\]. Profound changes of the (classical) global structure tied to the transition of such theories with respect to a conformally related dilaton theory \[6\] indicate that equally drastic changes must be expected in the respective quantum theories.

At present three basic approaches for the quantum theory of 2d gravity models are known. The first one uses the particular structure of the theory of constraints to find the explicit canonical solution yielding — in the absence of matter-fields — a finite dimensional phase space \[5, 7–13\]. However, an extension of that result so as to include interactions of the geometric variables of matter seems difficult. A second one emerged as a by-product of string theory. The corresponding dilaton theory containing black hole solutions can be solved classically in the presence of matter \[14\], but the hope that conformal field theory techniques suffice for a complete quantization could not be fulfilled. Thus also for that model only semiclassical studies were possible.

The prejudice that path-integral quantization is necessarily related to perturbation theory seems to have prevented its nonperturbative application to quantize 2d covariant theories until recently. This third approach is based upon an earlier observation that the path integral for the model of \[3\] could be performed exactly in a suitable gauge \[15\]. However, the present authors also realised that this remains even true for all matterless theories, whether dilaton fields are present or not \[16\]. It would have been difficult to find this simple result from the involved perturbative calculations in the background field gauge \[17\]. Thus full agreement with the first approach was confirmed in the sense that no local quantum effects appear.
Quantum effects in 1+1 gravity without matter are restricted to global fluctuations, e.g. zero modes of compactified directions [13]. These we exclude by assuming that the path integral is evaluated on spacetimes with simply connected topology. However, when additional matter fields are added also the method of [16] failed, except for the particular case of the Jackiw-Teitelboim model [18]. What can be done, though, is to consider perturbation theory in the matter field, treating the geometrical part still exactly by a nonperturbative path integral. Our approach thus differs fundamentally from the conventional ‘semiclassical’ one [14] in which (mostly only one loop) effects of matter are added and the resulting effective action subsequently is solved classically. It also clearly aims to go beyond the consideration of the backreaction between in- and outgoing matter [19] in a fixed background.

In the present work we compute the generating functional for connected Green functions as well as the effective action for a wide class of 2d gravity models minimally coupled to scalar matter to two loop order (Section 2). The gravitational part of the theory is treated exactly in the sense that no loops consisting only of gravitational propagators appear at all. We demonstrate the remarkable fact that the two loop contribution of the effective action vanishes. More precisely, the only effect of two–loop corrections is a renormalization of the dilaton potential. It follows that any one loop result of 2d gravity contained in our class of models is automatically true to two loops. As in the previous paper [16], our analysis is local. This means that we suppose asymptotic fall-off conditions for quantum fields and neglect all surface terms which may appear when integrating by parts.

Recently an attempt to analyse two–loop corrections in dilatonic models has been made by Mikovic and Radovanovic [20] who considered the CGHS model minimally coupled to matter. Strictly speaking, their model does not fall into the class considered in the present paper. Some implications of the calculations of that reference are discussed in the conclusion (Section 3).

2 Two Loop Quantization

Our starting point is the action for 1+1 gravity with a spacetime integral over the Lagrangian

\[ \mathcal{L} = \mathcal{L}^g + \mathcal{L}^m + \mathcal{L}^s , \]  

(1)
which is a sum of the gravitational, the matter and a source contribution. We will work with a gravitational term in its first order form \[ L_{(1)} = X^+ D e^- + X^- D e^+ + X d \omega + \epsilon(X^+ X^- U(X) + V(X)) , \] (2)

where \( D e^a = d e^a + (\omega \wedge e)^a \) is the torsion two form, the scalar curvature \( R \) is related to the spin connection \( \omega \) by \(-\frac{R}{2} = * d \omega \) and \( \epsilon \) denotes the volume two form \( \epsilon = \frac{1}{2} \epsilon_{ab} e^a \wedge e^b = d^2 x \det e^a_{\mu} = d^2 x (e) \). Our conventions are determined by \( \eta = \text{diag}(1, -1) \) and \( \epsilon_{ab} \) by \( \epsilon^{01} = -\epsilon^{10} = 1 \). We also have to stress that even with Greek indices \( \varepsilon^{\mu \nu} \) is always understood to be the antisymmetric Levi-Civita symbol and never the corresponding tensor. In [16] it was shown that (2) even at the quantum level is equivalent to the more familiar second order form

\[ L_{(2)} = \sqrt{-g} \left( -X^+ X^- U(X) + V(X)(\nabla X)^2 \right) . \] (3)

In many studies an exponential parametrization for the dilaton field \( X = e^{-2\phi} \) is chosen which restricts \( X \) to \( \mathbb{R}_+ \) and from the global point of view represents a further drastic assumption [4]. Our matter contribution is a minimally coupled scalar field whose Lagrangian \( L^m \) is given by

\[ L^m = \frac{1}{2} \sqrt{-g} g^{\mu \nu} \partial_\mu S \partial_\nu S = -\frac{1}{2} \epsilon^{\alpha \mu} \varepsilon_{\beta \nu} \eta_{ab} e_a^\alpha \partial_\mu S \partial_\nu S . \] (4)

The most general dilaton gravity action (3) contains the term \( U(X)(\nabla X)^2 \). This term can be removed by a dilaton dependent conformal redefinition of the metric. The matter action (4) is invariant under such a redefinition. However, the quantum theory is changed: The source terms in eq. (2) below acquire field dependent (conformal) factors, destroying straightforward quantum integrability. In addition the path integral measure for the scalar field is changed. Thus, in this section, in a first step we restrict ourselves to a subclass of 2d models with \( U(X) = 0 \) to avoid further technical complexities. Of course, in this way realistic models like spherically symmetric 4d general relativity [23] \( U(X) \propto X^{-1} \) are eliminated.

The same class of models with \( U(X) = 0 \) in (3) has also been studied in [2,7]. These theories still contain a vast variety of models allowing de facto an arbitrary topological structure which may be designed at will following a simple set of rules [22, 24]. Recently a particular example of a black hole
solution determined by \( V(X) \propto \exp(X) \) has been advocated by Cruz et al. [25] which reflects the same classical features as the CGHS model [14].

The Lagrangian \( L_s \) containing the source terms for our fields is given by

\[
L_s = j^+ e_1^- + j^- e_1^+ + j_0 + J^+ X + J^- X + J X + QS .
\]

Using an Eddington-Finkelstein gauge for the metric defined by a temporal (Weyl type) gauge for the Cartan variables

\[
e_0^+ = \omega_0 = 0 , \quad e_0^- = 1
\]

yields the trivial Faddeev-Popov determinant [16]

\[
F = (\det \partial_0)^3 .
\]

In this gauge the actions (2) and (4) are

\[
L^g_{gf} = X^+ \partial_0 e_1^- + X^- \partial_0 e_1^+ + X \partial_0 \omega_1 + X^+ \omega_1 - e_1^+ V(X) \quad (8)
\]

\[
L^m_{gf} = (e_1^- (\partial_0 S)^2 - (\partial_0 S)(\partial_1 S)) . \quad (9)
\]

Contrary to the situation in conformal gauge the matter action therefore still contains a coupling to a zweibein component which is the price to pay for a simple \( L^g \). The generating functional of Green functions derived carefully from an integral in (extended) phase space for the geometric variables [16] after integrating the ghosts becomes

\[
W = \int (\mathcal{D} \sqrt{e_1^+ S})(\mathcal{D} X)(\mathcal{D} X^-)(\mathcal{D} e_1^+)(\mathcal{D} e_1^-)(\mathcal{D} \omega_1) F \times \exp \left[ \frac{i}{\hbar} \int_x L_{gf} \right] .
\]

For the scalar field a nontrivial measure has been introduced in order to retain invariance under general coordinate transformations [26]. We explicitly display \( \hbar \) to keep track of loop orders in a transparent manner. To compute (10) we first integrate over \( e_1^- , X^- \) and \( \omega_1 \) to obtain

\[
W = \int (\mathcal{D} \sqrt{e_1^+ S})(\mathcal{D} X)(\mathcal{D} X^+)(\mathcal{D} e_1^+)(\mathcal{D} e_1^-) \delta(e_1^-) \delta(X^-) \delta(\omega_1) F \times \exp \left[ \frac{i}{\hbar} \int_x (L^m_{gf} - e_1^+ V(X) + \mathcal{L}^s) \right] .
\]
where the delta functions
\[
\delta_{(e_1)} = \delta \left( -\partial_0 X^+ + j^+ + (\partial_0 S)^2 \right),
\]
\[
\delta_{(X^-)} = \delta \left( \partial_0 e_1^+ + J^+ \right),
\]
\[
\delta_{(\omega_1)} = \delta \left( X^+ - \partial_0 X + j \right)
\]
are immediately used to integrate out the remaining variables \(X^+, e_1^+\) and \(X\). This reduces (11) to
\[
W = \int (\mathcal{D}\sqrt{e_1^+ S}) e^{\frac{i}{\hbar} \int d^2 x (J^- X^+ + J^- e_1^+ + J X - e_1^+ V(X) - (\partial_0 S)(\partial_1 S))},
\]
where \(X^+, e_1^+\) and \(X\) thus are expressed as
\[
X^+ = \frac{1}{\partial_0} j^+ + \frac{1}{\partial_0} (\partial_0 S)^2 = X_0^+ + \frac{1}{\partial_0} (\partial_0 S)^2
\]
\[
e_1^+ = -\frac{1}{\partial_0} J^+
\]
\[
X = \frac{1}{\partial_0} (X^+ + j) = X_0 + \frac{1}{\partial_0} (\partial_0 S)^2.
\]
\(X_0\) and \(X_0^+\) represent \(X\) and \(X^+\) in the absence of matter fields (zero loop order). The (ambiguous) nonlocal expressions for the Green functions \(\partial_0^{-1}\) and \(\partial_0^{-2}\) are defined properly in the Appendix. These integrations produce a factor \(\det(\partial_0)^{-3}\) which cancels exactly the Faddeev–Popov determinant (7).

We recall that this cancellation also occurs in the presence of the kinetic dilaton term in (2) [16]. In the present case this ghost cancellation is not important because (7) is field independent anyhow. To obtain the same result in conformal gauge several physical arguments such as the vanishing of the ghost contribution to Hawking radiation had to be invoked by Giddings and Strominger [27]. It should be stressed, though, that the triviality of this ghost contribution only holds in the gauge employed in our present paper.

Expanding \(V(X)\) around \(X_0\)
\[
\begin{align*}
V^0(X) & = V_0 + V_1 + \Delta V, \\
V_0 & = V(X_0), \\
V_1 & = V'(X_0) \frac{1}{\partial_0^2} (\partial_0 S)^2, \\
\Delta V & = \sum_{n=2}^{\infty} \frac{V^{[n]}(X_0)}{n!} \left( \frac{1}{\partial_0^2} (\partial_0 S)^2 \right)^n,
\end{align*}
\]
the matter field integration to arbitrary orders is contained in the factor $W_S$ of
\[ W = W_S e^{\frac{i}{\hbar} \int d^2 x (J^- X_0^+ + j^- e_1^+ + J X_0 - e_1^+ V(0))} , \tag{21} \]
i.e.
\[ W_S = \int (\mathcal{D} \sqrt{e_1^+ S}) e^{\frac{i}{\hbar} \int d^2 x (-e_1^+ \Delta V + (E_1^- (\partial_0 S)^2 - (\partial_0 S)(\partial_1 S)) - QS)} . \tag{22} \]
We introduced the abbreviation
\[ E_1^- = \frac{1}{\partial_0^2} J - \frac{1}{\partial_0} J^- - \frac{1}{\partial_0^2} (e_1^+ V'(X_0)) \tag{23} \]
in order to subsume $V(1)$ into the propagator term. $E_1^-$ clearly differs from $e_1^-$ but will formally play a similar role. Before proceeding with the remaining matter field integration we for once look back to the matterless case. There the generating functional for connected Green functions reduces to
\[ Z = J^+ \frac{1}{\partial_0^2} j^+ + J \frac{1}{\partial_0} j^+ + J \frac{1}{\partial_0} j - J^+ \frac{1}{\partial_0} \left( V \left( \frac{1}{\partial_0^2} j^+ + \frac{1}{\partial_0} j \right) + j^- \right) . \tag{24} \]
It is clear from (2) that the only possible vertex is produced by $e_1^+ V(X)$. It consists of one incoming $e_1^+$ line and multiple $X$ lines depending on the form of the potential $V(X)$. However, as we see from (24) $X$ propagates only to (the sources of) $e_1^-$ or $\omega_1$. With these propagators it is impossible to construct any loops for the matterless case and the absence of quantum corrections follows almost trivially. This (for $U(X) = 0$ simple) perturbative argument had been replaced by the nonperturbative path integral in [16].

Returning to (21) we still have to perform the $S$ integration. As usual in perturbative quantum field theory the terms of higher than quadratic order in a Green function are replaced by the functional derivatives with respect to the sources $Q$, viz. $i\hbar \delta / \delta Q$.

The integration of the term quadratic in $S$ and thus comprising the full propagator in the (exact!) geometric background yields
\[ \int (\mathcal{D} \sqrt{e_1^+ S}) e^{\frac{i}{\hbar} \int d^2 x E_1^- (\partial_0 S)^2 - (\partial_0 S)(\partial_1 S) - QS} = e^{i \int d^2 x P_1^-(e_1^+) e^{\frac{i}{\hbar} \int Q \Theta^{-1} Q}} . \tag{25} \]
where $\Theta^{-1}$ is defined as the inverse of the differential operator

$$\Theta = \partial_0 \partial_1 - \partial_0 E^- \partial_0 = \partial_0 \vartheta .$$  \hfill (26)

In terms of the regularized $\partial^{-1}_\mu$ of the Appendix we define the corresponding inverse $\vartheta^{-1} = (\partial_1 - E^- \partial_0)^{-1}$. $S_P$ denotes the Polyakov-Liouville action, formally written as

$$S_P = \sqrt{-g} R \frac{1}{\Box} R ,$$  \hfill (27)

where, however, $R$ and $\Box$ have to be expressed in terms of $E^-$. Thus in a dilaton theory with $V \neq 0$ according to (23) more types of external zweibein (or metric) lines are possible. From (22) to second loop order in $\Delta V$

$$\exp \left( \frac{i}{\hbar} \int (-e^+ \Delta V) \exp \frac{-i}{4\hbar} \int Q \Theta^{-1} Q \right)$$

$$= \left( 1 - \frac{i}{\hbar} \int (e^+ \frac{V''(X_0)}{2} \left( \partial_0^{-2} (\partial_0 \frac{\delta}{\delta Q})^2 \right)^2 + \ldots ) \right) e^{\frac{\pi i}{4} \int Q \Theta^{-1} Q |_{Q=0}}$$

$$= 1 + \int_x i \hbar e^+ \frac{V''(X_0)}{8} \gamma(x) + O(\hbar^2)$$  \hfill (28)

follows, where we introduced the abbreviation $\gamma = \gamma(x), x = (x^0, x^1)$, in the last line which in a suggestive notation for the nonlocal expressions $\partial_0^{-2}$ is given by

$$\gamma(z) = \int dydz \langle x | \partial_0^{-2} | y \rangle \langle x | \partial_0^{-2} | z \rangle$$

$$\left( \partial_0 \vartheta^{-1}(y, y) \vartheta^{-1}(z, z) + 2 \left( \partial_0 \vartheta^{-1}(y, z) \right)^2 \right)$$  \hfill (29)

The expression $\partial_0 \vartheta^{-1}(z, z)$ has to be understood as $\lim_{y \to z} \partial_0 \vartheta^{-1}(y, z)$, i.e. the differential only refers to the second argument. $\langle z | O | z \rangle$ generically denotes the corresponding operator $O$ in a $z$ representation. Thus the two-loop contribution consists of two parts. The first one represents two tadpole like single loops attached to the point $z$ by something resembling a free propagator, the second term in (29) is a genuine two loop one (Fig. 1).

### 2.1 Two Loop Field Independence

Before we continue the computation of the two loop results for the generating functional for connected Green functions we demonstrate that the
\[ e_1^+ V''(x) \]
\[ (\partial_0^{-2})_{xz} \quad (\partial_0^{-2})_{xy} \]
\[ \partial_0 \vartheta^{-1}(z, z) \quad \partial_0 \vartheta^{-1}(y, y) \]
\[ e_1^+ V''(x) \quad (\partial_0^{-2})_{xy} \quad (\partial_0^{-2})_{xz} \]
\[ [\partial_0^z \vartheta^{-1}(y, z)]^2 \]

Figure 1: The two diagrams corresponding to (29).

\( \gamma \)-contribution is field independent. Both terms on the r.h.s. of (29) contain the expression \( \langle x | \vartheta^{-1} \partial_0 | y \rangle \). Realizing that \( \partial_0^{-2} \) is local in the \( x_1 \) coordinate means that with our definition of \( \vartheta^{-1} \) the building blocks of our diagrams are

\[ \langle x^0, x^1 | \vartheta^{-1} \partial_0 | y^0, x^1 \rangle = \langle x^0, x^1 | \frac{1}{\partial_1 - E_1^{-1}} \partial_0 | y^0, x^1 \rangle \]  

(30)

where \( x_0 \) and \( x_1 \) are "time" and "space" coordinates. The operator on the r.h.s. may be represented as

\[ (P^{-1} \partial_1 P)^{-1} = P^{-1} \partial_1^{-1} P \]

\[ P(x^1) = \mathcal{P} \exp \left( - \int_{x^1}^{y^1} dz^1 E_1^{-1}(z^1) \partial_0 \right) \]  

(31)

where \( \mathcal{P} \exp \) is the path ordered exponential satisfying the operator identity \( \partial_1 P(x^1) = P(x^1)(-E_1^{-1}(x^1) \partial_0) \). Only \( \partial_1^{-1} \) is non-local in \( x_1 \). Hence we obtain

\[ \langle x^0, x^1 | P^{-1} (\partial_1^{-1}) P \partial_0 | y^0, x^1 \rangle = c \partial_0 \delta(x^0 - y^0) \]  

(32)

where \( c \) denotes the limit \( x^1 \rightarrow y^1 \) of the regularized \( \langle x^1 | (\partial_1^{-1}) | y^1 \rangle = \mp \theta(x^1 - y^1) e^{\pm i \mu (x^1 - y^1)} \) the sign depending on the regularization \( \partial_1 \rightarrow \nabla_1 \) or \( \tilde{\nabla}_1 \). This expression, as well as \( \gamma \) itself, is therefore independent of sources and thus may be absorbed in a renormalization of the potential \( V \). Actually the first term of (29) after taking the limit \( x^0 \rightarrow y^0 \) vanishes because of (32).
In the generating functional for connected Green functions

\[ Z = \frac{\hbar}{i} \log W \]
\[ = J X_0^+ + j e_1^+ + J X_0 - e_1^+ V_R(X_0) + \hbar S_P(E_1^-, e_1^+) + O(\hbar^3) \quad (33) \]
\[ V_R = V - \hbar^2 \gamma V'' \quad (34) \]
everything is expressed in terms of the sources (cf. (16) and (23)).

For some time it has been argued [17] that a generalized dilaton theory in \( d = 1+1 \) can be considered ‘renormalized’ if the ‘potential’ \( V(X) \) by quantum corrections is changed into \( V_R \) in (34). In those previous studies such a ‘renormalization’ was supposed to occur even in the absence of matter which we have found to be an artefact of their approach by working in a more suitable gauge [15]. However, in (34) this renormalization phenomenon — as induced by quantum corrections from matter — is a genuine one.

Of course, only our present two-loop effect produces as a correction a second derivative of \( V \). But let us assume that this remains true for higher orders (cf. eq. (20)) leading to corresponding higher derivatives, and that \( V(X) \) may be decomposed in a power series in \( X \)

\[ V(X) = \sum \frac{v_n}{n!} X^n. \quad (35) \]

If the sum in (35) extends to infinity, each term will acquire quantum corrections and thus a separate normalization condition. Keeping in mind that the global properties of the classical solution depend crucially on \( V \), such a renormalization essentially means to fix the global properties order by order, something not acceptable in a ‘renormalizable’ theory of the usual sense.

The situation becomes even worse when singularities are present (poles of \( V \) or noninteger powers). This would be the case in physically more interesting models. Then differentiation inescapably produces completely new terms in each new loop order.

There are only two exceptions: Either a polynomial behaviour of \( V \), where only a finite number of coefficients need to be redefined, or an exponential behaviour \( V(X) = \alpha \exp(\beta X) \) where the renormalized potential is an exponential again and only one parameter \( \alpha \) needs to be renormalized. Note that this potential gives black hole solutions [25], sharing however with the dilaton black hole of [14] the null-completeness at the singularity [6].
2.2 Effective Action

In semiclassical calculations the effective action $\Gamma$ including quantum corrections [14] is used extensively. Starting from (33) the ‘mean fields’ to order $\hbar^2$ are

$$e^+ = \frac{\delta Z}{\delta j} = e^+_1,$$  \hspace{1cm} (36)

$$\omega_1 = \frac{\delta Z}{\delta j} = \frac{1}{\partial_0} J + \frac{1}{\partial_0} e^+_1 V_R' + \hbar \frac{1}{\partial_0} \left( S'_P e^+_1 \frac{1}{\partial_0^2} V_R'' \right) + O(\hbar^3),$$  \hspace{1cm} (37)

$$e^- = \frac{\delta Z}{\delta j} = -\frac{1}{\partial_0} J - \frac{1}{\partial_0} J' - \frac{1}{\partial_0^2} \left( e^{-}_1 V_R' + \hbar S'_P e^+_1 \frac{1}{\partial_0^2} V_R'' \right) + O(\hbar^3),$$  \hspace{1cm} (38)

$$X^- = \frac{\delta Z}{\delta j} = X_0 + \frac{1}{\partial_0} S'_P + O(\hbar^3),$$  \hspace{1cm} (39)

$$X^+ = \frac{\delta Z}{\delta j} = X_0^+ + \frac{1}{\partial_0} S'_P + O(\hbar^3).$$  \hspace{1cm} (40)

$X^-$ is not needed in the following. $e^+_1$, $X_0$, $X_0^+$ are functions of sources as in (16) and $V_R$ is a function of $X_0$ only.

Using these relations extensive cancellations occur and we are left with

$$\Gamma = Z - JX - J^+X^+ - j\omega_1 - j^+e^+_1$$

$$= S_{cl}(\overline{X}, e, \omega) + hS_P(E^+_1, e^+_1) - h^2 S'_P e^+_1 \frac{1}{\partial_0} V_R''(X_0) \frac{1}{\partial_0^2} S'_P + O(\hbar^3).$$  \hspace{1cm} (41)

Using (38) and (39),

$$e^-_1 = E^-_1 - \hbar \frac{1}{\partial_0} \left( S'_P e^+_1 \frac{1}{\partial_0^2} V_R''(X_0) \right) + O(\hbar^2)$$

$$\overline{X} = X_0 + O(\hbar)$$

yields for the loop expansion for the generalized Polyakov term (27) with (23)

$$S_P(E^+_1, e^+_1) = S_P(e^-_1, e^-_1) + hS'_P e^+_1 \frac{1}{\partial_0} V_R''(X_0) \frac{1}{\partial_0^2} S'_P + O(\hbar^2).$$  \hspace{1cm} (44)

The final result is

$$\Gamma = S_{cl}(\overline{e^+_1}, \overline{\omega}, \overline{X}) + hS_P(\overline{e^-_1}, \overline{e^-_1}) + O(\hbar^3)$$  \hspace{1cm} (45)
where the potential $V$ is to be replaced by $V_R$ everywhere. Hence also in that expression the whole effect of two-loop corrections is a renormalization of the dilaton potential. It should be noted that in the effective action to two-loops the corrected Polyakov term still is expressed purely in terms of the mean field $\bar{e}_1$ whereas in (33) its two-loop corrections required the presence of $E_1^{-}$. 

3 Conclusions

The crucial ingredient of our approach is the use of the Eddington-Finkelstein gauge (or a Weyl-type gauge for the Cartan-variable) and a first order formulation for the action. In our previous work [16] it helped us to calculate the effective action for pure dilaton gravity to all orders of perturbation theory. Here we were able to calculate the two-loop effective action in dilaton gravity with matter treating the latter as a perturbation. Thanks to a field-independence of the correction term in 2-loops — which does not seem likely to persist in higher loop orders, though — the result allows the interpretation as a renormalization of the potential. A two loop generalization of the (generalized) Polyakov term could be derived as well. Interestingly enough the next loop order in the effective action does not depend on all the complicated further terms found for the Polyakov corrections.

The attentive reader will have noticed that throughout the present paper the technique of generating functionals has been used relying heavily on $J_i$, the sources for the $X_i$ which are nothing else but the sources of the canonical momenta in phase space. In fact, in an expression like (33) the geometric part vanishes when those sources are eliminated by naively putting $J_i = 0$. As we will show in forthcoming work a well defined procedure for eliminating $J_i$ exists yielding a generating functional with sources for the canonical coordinates only. The results obtained in this way are fully consistent with those of the second reference in [15], where canonical momenta were integrated out first and no sources for them were introduced.

To compare with related recent results of other authors we note that at one-loop order the authors of [20] obtained the BPP action [28] instead of the usual Polyakov term. Originally the BPP action was introduced ”by hand" to achieve solvability of the effective equations of motion. In one-loop calculations it can appear only in the case of non-minimal coupling of the scalar field [29]. A possible origin of such a term might be that the authors [20] assumed that the classical condition $\exp \rho = \exp \phi$, connecting the dilaton
and the scale factor of the metric in conformal gauge, can be preserved by loop corrections.

A Regularized Inverse Derivatives

In the preceding sections we frequently encountered inverse derivative operators. Here we shall define a proper infrared regularization scheme and list the corresponding calculation rules which where used in the main text. We introduce two regularized Green functions $\nabla^{-1}_0$ and $\tilde{\nabla}^{-1}_0$ to replace $\partial^{-1}_0$.

\[
\partial^{-1}_0 \Rightarrow \begin{cases} 
\lim_{\mu \to 0} (\partial_0 - i\mu)^{-1} = \lim_{\mu \to 0} (\nabla^{-1}_0) \\
\lim_{\mu \to 0} (\partial_0 + i\mu)^{-1} = \lim_{\mu \to 0} (\tilde{\nabla}^{-1}_0) 
\end{cases} \tag{46}
\]

where $\mu = \mu_0 - i\varepsilon$. $\mu_0 \to +0$ represents the IR regularization, proper asymptotic behavior (cf. (47),(48) below) is provided by $\varepsilon \to +0$. Note that a partial integration transforms $\nabla^{-1}_0$ into $\tilde{\nabla}^{-1}_0$ and also that $\tilde{\nabla}^{-1}_0$ is not the complex conjugate of $\nabla^{-1}_0$.

The inverse operators are defined as the Green functions $\nabla_0$ and $\tilde{\nabla}_0$ and are calculated straightforwardly

\[
\begin{align*}
(\nabla^{-1}_0)_{x,y} &= -\theta(y - x)e^{i\mu(x-y)} \\
(\tilde{\nabla}^{-1}_0)_{x,y} &= \theta(x - y)e^{-i\mu(x-y)} 
\end{align*} \tag{47,48}
\]

where $\theta$ denotes the step function. The inverse squared operators are defined as the Green functions of $(\nabla_0)^2$ and $(\tilde{\nabla}_0)^2$ and are given by

\[
\begin{align*}
(\nabla^{-2}_0)_{x,y} &= (y - x)\theta(y - x)e^{i\mu(x-y)} \\
(\tilde{\nabla}^{-2}_0)_{x,y} &= (x - y)\theta(x - y)e^{-i\mu(x-y)} 
\end{align*} \tag{49,50}
\]

Using (47) to (50) the following rules may be verified easily

\[
\begin{align*}
\nabla_0 \nabla^{-2}_0 &= \nabla^{-1}_0 \\
\nabla_0 \tilde{\nabla}^{-2}_0 &= \nabla^{-1}_0 - 2i\mu\tilde{\nabla}^{-2}_0 \\
\nabla_0 \tilde{\nabla}^{-1}_0 &= \delta(x - y) - 2i\mu\tilde{\nabla}^{-1}_0 \\
\tilde{\nabla}_0 \nabla^{-1}_0 &= \tilde{\nabla}^{-1}_0 + 2i\mu\nabla^{-2}_0 \\
\tilde{\nabla}_0 \tilde{\nabla}^{-1}_0 &= \delta(x - y) + 2i\mu\tilde{\nabla}^{-2}_0 
\end{align*} \tag{51,52,53}
\]

Note that in the main text only these types of operations appeared and therefore the limit $\mu \to 0$ does not cause any divergencies, because $\mu$ does not appear with negative powers. Therefore this regularization scheme seems much superior to the one used in [15].
Acknowledgement

This work has been supported by Fonds zur Förderung der wissenschaftlichen Forschung (FWF) Project No. P 10221–PHY. One of the authors (D.V.) thanks GRACENAS and the Russian Foundation for Fundamental Research, grant 97-01-01186, for financial support.

References

[1] S.W. Hawking, Commun. Math. Phys. 43, 199 (1975); S.M. Christensen and S.A. Fulling, Phys. Rev. D 15, (1977), 2088 .

[2] D. Louis-Martinez, J. Gegenberg and G. Kunstatter, Phys. Lett. B321, (1994), 193; Phys. Rev. D51, (1995), 1781; W.M. Seiler and R.W. Tucker, Phys. Rev. D53, (1996), 4366; A. Barvinsky and G. Kunstatter, Phys. Lett. B389, (1996), 231.

[3] M.O. Katanev and I.V. Volovich, Phys. Lett. B 175, (1986), 413.

[4] M.O. Katanaev, W. Kummer and H. Liebl, Phys. Rev. D 53, (1996), 5609.

[5] K.V. Kuchař, J.D. Romano and M. Varadarajan, Phys. Rev. D55, (1997), 795.

[6] M.O. Katanaev, W. Kummer and H. Liebl, Nucl. Phys. B 486, (1997), 353 .

[7] D. Louis-Martinez, J. Gegenberg and G. Kunstatter, Phys. Lett. B321, (1994), 193.

[8] D. Cangemi and R. Jackiw, Phys. Rev. D50, (1994), 3913; Phys. Lett. B337, (1994), 271.

[9] E. Benedict, Phys. Lett. B340, (1994), 43.

[10] T. Thiemann and H.A. Kastrup, Nucl. Phys. B399, (1993), 211; H.A. Kastrup and T. Thiemann, Nucl. Phys. B425, (1994), 665.

[11] K. Kuchař, Phys. Rev. D50, (1994), 3961.
[12] T. Strobl, Phys. Rev. D50, (1994), 7346.

[13] P. Schaller and T. Strobl, Class. Quant. Grav. 11, (1994), 331.

[14] G. Mandal, A. Sengupta and S.R. Wadia, Mod. Phys. Lett. A6, (1991), 1685; S. Elitzur, A. Forge and E. Rabinovici, Nucl. Phys. B 359, (1991), 581; E. Witten, Phys. Rev. D44, (1991), 314; C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D, 45, (1992), 1005; J. Russo, L. Susskind and L. Thorlacius, Phys. Lett. B 292, (1992), 13; T. Banks, A. Dabholkar, M. Douglas and M. O' Laughlin, Phys. Rev. D 45, (1992), 3607; S.P. deAlwis, Phys. Lett. B 289, (1992), 278.

[15] W. Kummer and D.J. Schwarz, Nucl. Phys. B382, (1992), 171; F. Haider and W. Kummer, Int. J. Mod. Phys. A9, (1994), 207.

[16] W. Kummer, H. Liebl and D.V. Vassilevich, Nucl. Phys. B493, (1997), 491.

[17] J.G. Russo, A.A. Tseytlin, Nucl. Phys. B382, (1992), 259; R. Kantowski and C. Marzban, Phys. Rev. D46, (1992), 5449; E. Elizalde, S. Naftulin and S.D. Odintsov Int. J. Mod. Phys. A9, (1994), 933; E. Elizalde, P. Fosalba-Vela, S. Naftulin and S. D. Odintsov, Phys. Lett. B352, (1995), 235.

[18] C. Teitelboim, Phys. Lett. 126B, (1983), 41; R. Jackiw, 1984 Quantum Theory of Gravity, ed S. Christensen (Bristol: Hilger) p.403

[19] Y. Kiem and H. Verlinde, Phys. Rev. D 52, (1995), 7053.

[20] A. Mikovic and V. Radovanovic, Nucl. Phys. B481 (1996) 719.

[21] H. Grosse, W. Kummer, P. Prešnažder and D.J. Schwarz, J. Math. Phys. 33 (11), (1992), 3892.

[22] T. Klösch, T. Strobl, Class. Quantum Grav. 13, (1996), 965; W. Kummer and W. Widerin, Phys. Rev. D 52, (1995), 6965.

[23] P. Thomi, B. Isaak and P. Hajicek, Phys. Rev. D30 (1984), 1168; P. Hajicek, Phys. Rev. D30, (1984), 1178; S.R. Lau, Class. Quant. Grav. 13, (1996), 1541.

[24] T. Klösch and T. Strobl, Class Quantum Grav. 14, (1997), 1689.
[25] J. Cruz, J. Navarro-Salas, M. Navarro and C.F. Talavera, Symmetries and Black Holes in 2D Dilaton Gravity, [hep-th/9606097].

[26] K. Fujikawa, U. Lindström, N.K. Nielsen, M. Rocek and P. van Nieuwenhuizen, Phys. Rev. D37, (1988), 391; D.J. Toms Phys. Rev. D35, (1987), 3796.

[27] S.B. Giddings and A. Strominger, Phys. Rev. D47, (1993), 2454.

[28] S. Bose, L. Parker and Y. Peleg, Phys. Rev. D52, (1995), 3512.

[29] E. Elizalde, S. Naftulin and S. Odintsov, Phys. Rev. D49, (1994), 2852; S. Nojiri and S. Odintsov, Trace anomaly and non–local effective action for 2D conformally invariant scalar interaction with dilaton, [hep-th/9706009]; R. Bousso and S.W. Hawking, Trace anomaly of dilaton coupled scalars in two dimensions, [hep-th/9705236]; W. Kummer, H. Liebl and D. Vassilevich, Hawking radiation for non-minimally coupled matter from generalized 2d black hole model, [hep-th/9707041].