Spin Hall Effect in Noncommutative Coordinates

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Abstract

A semiclassical constrained Hamiltonian system which was established to study dynamical systems of matrix valued non–Abelian gauge fields is employed to formulate spin Hall effect in noncommuting coordinates at the first order in the constant noncommutativity parameter $\theta$. The method is first illustrated by studying the Hall effect on the noncommutative plane in a gauge independent fashion. Then, the Drude model type and the Hall effect type formulations of spin Hall effect are considered in noncommuting coordinates and $\theta$ deformed spin Hall conductivities which they provide are acquired. It is shown that by adjusting $\theta$ different formulations of spin Hall conductivity are accomplished. Hence, the noncommutative theory can be envisaged as an effective theory which unifies different approaches to similar physical phenomena.

1 Introduction

Deformation quantization which is also known as Weyl–Wigner–Groenewold–Moyal method of quantization\textsuperscript{1}, although developed as an alternative to operator quantization of quantum mechanics became one of the main approaches of incorporating noncommutative coordinates into quantum mechanics. In this scheme one introduces star product of coordinates in terms of the constant, antisymmetric noncommutativity parameter $\theta_{\mu\nu}$ and then ordinary multiplication is replaced with star product in energy eigenvalue problems\textsuperscript{2}. In the quantum phase space given by $(\hat{x}_\mu, \hat{p}_\mu)$ this scheme is equivalent to shift the coordinates in the related Hamiltonian by

$$\hat{x}_\mu \rightarrow \hat{x}_\mu - \frac{1}{\hbar} \theta_{\mu\nu} \hat{p}_\nu,$$

as far as the first order terms in $\theta$ are retained. The $\theta$ deformed Hamiltonian can also be employed to define the related path integrals (see \textsuperscript{3} and the references given therein). When gauge fields are present in the original theory this procedure is very sensitive to the explicit realization of the gauge field. Moreover, it is not suitable to envisage how spin dependent particles would behave in noncommutative coordinates. Therefore, it is desirable to establish a systematic method of

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formulating dynamics in noncommutative coordinates which does not refer to a particular gauge choice as well as embraces spin dependent systems. Recently spin dependent dynamics was studied as a semiclassical constrained Hamiltonian system\cite{4}. We will show that this constrained system is suitable to formulate either spinless or spin dependent dynamical systems in noncommuting coordinates. Moreover, this approach is gauge independent as far as the related canonical Hamiltonian is chosen appropriately. Some spinless dynamical systems in noncommuting coordinates have already been considered as constrained Hamiltonian systems\cite{5}. However, the basic achievement reported here is to offer a systematic method of studying matrix valued observables in noncommuting coordinates.

When the noncommutativity of coordinates is introduced without an apparent physical motivation, there is a subtle issue: How the noncommutativity of coordinates in dynamical systems should be interpreted? Obviously, it can be taken literally claiming that its effects have not been observed yet due to the smallness of the noncommutativity parameter $\theta$. However, we prefer another interpretation which sounds more realistic: The dynamical system in noncommuting coordinates effectively links different manifestations of the same theory. It either relates non–interacting theory to the interacting ones or connects different kinds of interactions which explain similar phenomena\cite{3,6}. We would like to apply this point of view to obtain an effective theory of spin Hall effect which has been studied recently in terms of different methods\cite{7}-\cite{10} but an interrelation is missing (for a review see \cite{11}). We will consider two simple formulations of spin Hall effect. The first is an extension of the Drude model\cite{6} and the other is obtained as a generalization of the Hall effect employing the Rashba spin–orbit coupling\cite{4}. Studying these formulations in noncommutative space leads to deformed spin Hall conductivities which reproduce the non–deformed ones when the noncommutativity is turned off by setting $\theta = 0$. In this respect it differs from the deformed spin Hall conductivity obtained in \cite{13} which vanishes in the $\theta = 0$ limit. In \cite{13} two dimensional harmonic oscillator in noncommutative coordinates was considered adopting a similar interpretation of the noncommutativity.

We will introduce noncommutativity of coordinates within the semiclassical constrained Hamiltonian system which is briefly reviewed in Section 2. To illustrate the method we first apply it to “ordinary” Hall effect in Section 3 in a gauge invariant fashion. We also show that when gauge fields are incorporated into the Hamiltonian, manifestations of the noncommutativity depend on the choice of gauge. In Section 4 we study the spin Hall effect formulations of \cite{12} and \cite{4} in noncommutating coordinates and obtain $\theta$ deformed spin Hall conductivities which they provide. Then, we discuss how to obtain the spin Hall conductivities furnished with some different approaches from the deformed spin Hall conductivity by fixing the noncommutativity parameter $\theta$. We only deal with the deformations which are at the first order in $\theta$.

2 The semiclassical approach

We would like to recall the main ingredients of the semiclassical approach established in \cite{4} for studying spin dependent dynamical systems. It is based on an extension of the deformation quantization\cite{1}: Let us deal with matrix observables which may depend on $\hbar$, although they are functions of the classical phase space variables $(\pi_\mu, x_\mu)$. In terms of the star product

$$ \star = \exp \left[ \frac{i\hbar}{2} \left( \frac{\overleftarrow{\partial}}{\partial x_\mu} \frac{\overrightarrow{\partial}}{\partial \pi_\mu} - \frac{\overleftarrow{\partial}}{\partial \pi_\mu} \frac{\overrightarrow{\partial}}{\partial x_\mu} \right) \right], $$
one introduces the Moyal bracket of the matrix observables $K_{ab}(\pi, x)$ and $N_{ab}(\pi, x)$ as
\[
([K(\pi, x), N(\pi, x)]_\star)_{ab} = K_{ac}(\pi, x) \star N_{cb}(\pi, x) - N_{ac}(\pi, x) \star K_{cb}(\pi, x).
\] (2)

The semiclassical approximation is introduced by the bracket
\[
\{K(\pi, x), N(\pi, x)\}_C \equiv -i\hbar [K, N] + \frac{1}{2}\{K(\pi, x), N(\pi, x)\} - \frac{1}{2}\{N(\pi, x), K(\pi, x)\},
\] (3)

obtained from the Moyal bracket (2) by retaining the lowest two terms in $\hbar$. The first term on the right hand side is the ordinary commutator of matrices and the others are Poisson brackets defined as
\[
\{K(\pi, x), N(\pi, x)\} \equiv \frac{\partial K}{\partial x^\nu} \frac{\partial N}{\partial \pi^\nu} - \frac{\partial K}{\partial \pi^\nu} \frac{\partial N}{\partial x^\nu}.
\]

It is a semiclassical approximation because in (3), where the observables $K$ and $N$ may depend on $\hbar$, only the two lowest order terms in $\hbar$ are detained.

Semiclassical dynamical equations are given by employing the semiclassical bracket (3) to define the time evolution of a semiclassical observable $K(\pi, x)$ as
\[
\dot{K}(\pi, x) = \{K(\pi, x), H(\pi, x)\}_C,
\]
in terms of the semiclassical Hamiltonian $H(\pi, x)$. As usual the dot over the observables indicates the time derivative.

### 2.1 A semiclassical constrained Hamiltonian system

Semiclassical Hamiltonian dynamics is developed emulating the rules of the ordinary Hamiltonian formulation by replacing Poisson brackets with the semiclassical brackets (3). It is a systematic method of introducing and studying noncommutativity of phase space variables. Let us consider the gauge fields $A_\alpha, B_\alpha; \alpha = 1, \cdots, n$; which are in general $N \times N$ matrices and the underlying first order matrix Hamiltonian
\[
\mathcal{L} = \dot{r}^\alpha \left( \frac{1}{2} I p_\alpha + \rho A_\alpha(r, p) \right) - \dot{p}^\alpha \left( \frac{1}{2} I r_\alpha - \xi B_\alpha(r, p) \right) - \mathcal{H}_0(r, p).
\] (4)

$\rho, \xi$ are the coupling constants and $I$ denotes the unit matrix. Canonical momenta are defined as usual:
\[
\pi^\alpha_r = \frac{\partial \mathcal{L}}{\partial \dot{r}^\alpha}, \quad \pi^\alpha_p = \frac{\partial \mathcal{L}}{\partial \dot{p}^\alpha}.
\]

Being a first order Hamiltonian (4) leads to the primary constraints
\[
\psi^{1\alpha} \equiv (\pi^\alpha_r - \frac{1}{2} p^\alpha) I - \rho A^\alpha,
\] (5)
\[
\psi^{2\alpha} \equiv (\pi^\alpha_p + \frac{1}{2} r^\alpha) I - \xi B^\alpha.
\] (6)

In terms of the Lagrange multipliers $\lambda^z_\alpha; z = 1, 2$, and the canonical Hamiltonian $\mathcal{H}_0$ one introduces the extended Hamiltonian
\[
\mathcal{H}_e = \mathcal{H}_0 + \lambda^z_\alpha \psi^z_\alpha.
\] (7)
To employ the semiclassical approach we set \( \pi^\mu = (\pi_p^\alpha, \pi_r^\alpha) \) and \( x_\mu = (p_\alpha, r_\alpha) \). Now, one can observe that the constraints obey the semiclassical brackets

\[
\{ \psi_1^1, \psi_1^1 \}_C = \rho F_{\alpha \beta}, \\
\{ \psi_2^1, \psi_2^1 \}_C = \xi G_{\alpha \beta}, \\
\{ \psi_1^1, \psi_2^1 \}_C = -\delta_{\alpha \beta} - M_{\alpha \beta},
\]

where \( \delta_{\alpha \beta} \) is the Kronecker delta and the field strengths are defined by

\[
F_{\alpha \beta} = \frac{\partial A_\beta}{\partial r^\alpha} - \frac{\partial A_\alpha}{\partial p^\beta} - \frac{i\rho}{\hbar} [A_\alpha, A_\beta], \\
G_{\alpha \beta} = \frac{\partial B_\beta}{\partial p^\alpha} - \frac{\partial B_\alpha}{\partial r^\beta} - \frac{i\xi}{\hbar} [B_\alpha, B_\beta], \\
M_{\alpha \beta} = \xi \frac{\partial B_\beta}{\partial r^\alpha} - \rho \frac{\partial A_\alpha}{\partial p^\beta} - \frac{i\xi \rho}{\hbar} [A_\alpha, B_\beta].
\]

For consistency the constraints (5) and (6) should be constant in time:

\[
\{ \psi_z^z, H^e \}_C \approx 0, \quad (8)
\]

where \( \approx \) denotes that the equality is satisfied on the hypersurface where constraints vanish. The conditions (8) are solved by fixing the Lagrange multipliers as

\[
\lambda_\alpha^z = -\{ \psi_{z'}^z, H_0 \}_C C_{z z'}^{-1} \delta_{\alpha z'}, \quad (9)
\]

where we utilized the definitions

\[
C_{\alpha \beta}^{zz'} = \{ \psi_\alpha^z, \psi_{\beta'}^{z'} \}_C; \quad C_{\alpha \gamma}^{zz''} C_{\gamma \beta}^{-1} = \delta_{\alpha \beta}. 
\]

To impose vanishing of the second class constraints (5), (6) effectively, one introduces the semiclassical Dirac bracket

\[
\{ K, N \}_CD \equiv \{ K, N \}_C - \{ \psi_z^z, N \}_C C_{z z'}^{-1} \{ \psi_{z'}^z, N \}_C. \quad (10)
\]

Extending the usual Dirac procedure we replace the semiclassical bracket of observables (3) with the semiclassical Dirac bracket (10) in dynamical equations. Hence, the phase space variables satisfy the relations

\[
\{ \rho^\alpha, r^\beta \}_CD = C_{11}^{-1} \delta_{\alpha \beta} + \xi (MG)^{\alpha \beta} + \xi (MG)^{\beta \alpha} - \rho \xi^2 (GF^2)^{\alpha \beta} + \cdots, \\
\{ p^\alpha, p^\beta \}_CD = C_{22}^{-1} \delta_{\alpha \beta} + \rho (MF)^{\alpha \beta} + \rho (MF)^{\beta \alpha} - \rho^2 \xi (GF)^{\alpha \beta} + \cdots, \quad (11)
\]

\[
\{ r^\alpha, p^\beta \}_CD = C_{12}^{-1} \delta_{\alpha \beta} + \delta_{\alpha \beta} + M^{\beta \alpha} - \rho \xi (GF)^{\alpha \beta} - (MM)^{\alpha \beta} + \cdots. \quad (12)
\]

Observe that \( r_\alpha \) and \( p_\alpha \) should be considered as coordinates and the corresponding momenta, respectively, after effectively eliminating the constraints (5), (6).

Consistent with the usual procedure

\[
\hat{O}(r, p) = \{ O(r, p), H_e \}_C
\]
defines the equation of motion of the observable $O(r,p)$. $H_ε$ is defined in (7) with the Lagrange multipliers given as in (9). Therefore, one can find that

$$\dot{r}_\alpha = \xi \left( \frac{\partial H_0}{\partial \dot{r}_\beta} - \frac{i \rho}{\hbar} [\mathcal{A}_\beta, H_0] \right) \left( G^{\alpha \beta} - (MG)^{\alpha \beta} + (MG)^{\beta \alpha} - \rho \xi (GFG)^{\alpha \beta} + \cdots \right)$$

$$+ \left( \frac{\partial H_0}{\partial \dot{r}_\beta} - \frac{i \xi}{\hbar} [\mathcal{B}_\beta, H_0] \right) \left( \delta^{\alpha \beta} + M^{\alpha \beta} - \rho \xi (GF)^{\alpha \beta} - (MM)^{\alpha \beta} + \cdots \right),$$

(14)

$$\dot{p}_\alpha = \left( \frac{\partial H_0}{\partial r_\beta} - \frac{i \rho}{\hbar} \mathcal{A}_\beta, H_0 \right) \left( \delta^{\alpha \beta} - M^{\alpha \beta} + \rho \xi (FG)^{\alpha \beta} + (MM)^{\alpha \beta} + \cdots \right)$$

$$+ \rho \left( \frac{\partial H_0}{\partial p_\beta} - \frac{i \xi}{\hbar} \mathcal{B}_\beta, H_0 \right) \left( F^{\alpha \beta} - (MF)^{\alpha \beta} + (MF)^{\beta \alpha} - \rho \xi (FGF)^{\alpha \beta} + \cdots \right),$$

(15)

are the equations of motion of the phase space variables. Although this formalism is valid in any dimensions, in the following we will consider physical systems which are effectively 2–dimensional.

3 Hall Effect in Noncommutative Coordinates

Electrons moving in a thin slab in the presence of uniform external magnetic field perpendicular to the plane will experience the Lorentz force. Hence they will be pushed on a side of the slab producing a potential difference between the sides. This is known as Hall effect. If one applies an external electric field balancing the potential difference, electrons will move without deflection\[14]. This approach gives a simple formalism of deriving the Hall conductivity\[4]. We would like to study this system in noncommuting coordinates. Although we still use the terminology of Section 2, as far as spinless systems are considered semiclassical brackets coincide with their classical counterparts.

Let us deal with the Hamiltonian

$$H_0 = \frac{1}{2m} \left( p_1^2 + p_2^2 \right) + V(r),$$

(16)

in terms of the scalar potential

$$V(r) = -eE_i r_i$$

(17)

yielding the uniform external electric field $E_i; i = 1, 2$. We consider an electron moving on the $r_1 r_2$ plane but we choose the coupling constant to be $\rho = e/c$ and the $\mathcal{A}_i$ field such that there is a uniform magnetic field $\mathbf{B} = B\hat{r}_3$ perpendicular to the plane of motion:

$$F_{ij} = B \epsilon_{ij}.$$  

(18)

By inspecting the relation (11) it is evident that to introduce noncommutating coordinates we need to choose

$$G_{ij} = \epsilon_{ij}$$

(19)

and identify the related coupling constant with the noncommutativity parameter: $\xi = \theta$. The gauge field yielding (19) can be defined for example by $\mathcal{B}_i = -\epsilon_{ij} p_j / 2$, though its specific form is
not needed. Employing (18)–(19) in (11)–(13) and retaining the terms at the first order in $\theta$ and $eB/c$, one can show that

$$\{r_i, r_j\}_{CD} = \theta \epsilon_{ij}, \quad (20)$$
$$\{p_i, p_j\}_{CD} = \frac{eB}{c} \epsilon_{ij}, \quad (21)$$
$$\{r_i, p_j\}_{CD} = \left(1 + \frac{eB\theta}{c}\right) \delta_{ij}. \quad (22)$$

In this formalism $p_i$ act as kinematic momenta (21) and due to this fact the semiclassical bracket (22) possesses a term depending on the noncommutativity parameter $\theta$, in contrary to the formalisms where canonical momenta are adopted (see (28)–(30)).

Keeping the terms at the first order in $\theta$ and $eB/c$, (14) and (15) yield the following equations of motion of the phase space variables

$$\dot{r}_i = -e\theta \epsilon_{ij} E_j + \left(1 + \frac{eB\theta}{c}\right) \frac{p_i}{m}, \quad (23)$$
$$\dot{p}_i = \left(1 + \frac{eB\theta}{c}\right) e E_i + \frac{eB}{mc} \epsilon_{ij} p_j. \quad (24)$$

Expressing $p_i$ in terms of the velocity $v_i \equiv \dot{r}_i$, the force acting on the electron follows from (23) and (24) as

$$\mathcal{F}_i = m \ddot{r}_i = \left(1 + 2 \frac{eB\theta}{c}\right) e E_i + \frac{eB}{mc} \epsilon_{ij} (v_j + e\theta \epsilon_{jk} E_k).$$

For not being deflected the force acting on the electron should vanish:

$$\mathcal{F}_i = 0,$$

which can be solved for the velocity as

$$v_i = \frac{c}{B} \left(1 + \frac{eB\theta}{c}\right) \epsilon_{ij} E_j. \quad (25)$$

We would like to associate the above formulation with a system of electrons. To this aim let us introduce the density of electrons $\kappa$ and define the electric current as

$$j_i = e\kappa v_i. \quad (26)$$

Employing the velocity obtained demanding that the electrons move without deflection (25) in (26) yields the electric current

$$j_i = -\sigma_H(\theta) \epsilon_{ij} E_j,$$

where the deformed Hall conductivity is

$$\sigma_H(\theta) = -\left(1 + \frac{eB\theta}{c}\right) \frac{eck}{B}. \quad (27)$$

(27) is gauge invariant in the sense that it does not depend on how the vector potential $A_i$ is realized explicitly. (27) depends on the field strength (18). Deformation of Hall conductivity
in noncommuting coordinates has already been addressed in [3]. However, there is a discrepancy between the deformed Hall conductivities reported there and the one obtained here (27). The reason resides in the fact that the deformation of Hall conductivity obtained in [3] by the custom method (1), depends on how the gauge field is realized. In [3] vector potential was in symmetric gauge. We would like to show that within our approach, by choosing the Hamiltonian suitable to the custom method of deformation one obtains the same factors of deformation in symmetric gauge but other choices lead to a non–deformed Hall conductivity. Deal with the Hamiltonian

\[ H_0 = \frac{1}{2m} (p_i - (e/c)A_i)^2 + V(r), \]

where the scalar potential is still as in (17) and the noncommutativity furnished through (19). There is no other gauge field. Hence, \( p_i \) behave as canonical momenta and the deformed canonical relations are now given by

\[
\begin{align*}
\{r_i, r_j\}_{CD} &= \theta \epsilon_{ij}, \\
\{p_i, p_j\}_{CD} &= 0, \\
\{r_i, p_j\}_{CD} &= \delta_{ij}.
\end{align*}
\]

(28) (29) (30)

In the symmetric gauge

\[ A_i = -B \frac{\epsilon_{ij}r_j}{2}, \]

from (11)–(13) we obtain the equations of motion

\[
\begin{align*}
\dot{r}_i &= -e\theta \epsilon_{ij}E_j + \left( 1 + \frac{eB\theta}{2c} \right) \left( \frac{p_i}{m} - \frac{e}{mc}A_i \right), \\
\dot{p}_i &= eE_i + \frac{eB}{2c} \epsilon_{ij} \left( \frac{p_j}{m} - \frac{e}{mc}A_j \right).
\end{align*}
\]

Now, following the procedure given above one can solve the condition \( m\ddot{r}_i = 0 \) for the velocity \( \dot{r}_i \) and plug into the current (26). This yields the following deformed Hall conductivity

\[
\sigma^S_{\text{H}}(\theta) = -\left( 1 - \frac{eB\theta}{4c} \right) \frac{\epsilon c \kappa B}{\hbar},
\]

which is the result obtained in [3] up to a rescaling of the noncommutativity parameter with \( \hbar \). The rescaling is needed due to the fact that the noncommutativity parameter of [3] is obtained like (11) with another \( \hbar^{-1} \) factor. This result is gauge dependent. Indeed, if one adopts other gauge choices like \( A_i = (-Br_2, 0) \) or \( A_i = (0, Br_1) \), following the same procedure one can show that there will be no \( \theta \) dependent contribution to Hall conductivity though coordinates are noncommuting.

We would like to elucidate the idea of interpreting a non interacting dynamical system in noncommutating coordinates as an effectively interacting one [3],[6] considering the fractional quantum Hall effect whose conductivity is

\[
\sigma^F_{\text{H}} = \nu \frac{e^2}{\hbar},
\]

where \( \nu = 1/3, 2/3, 1/5, \ldots \). The idea is to fix the noncommutativity parameter \( \theta \) as

\[
\theta_F = -\frac{\nu}{\kappa \hbar} - \frac{e}{eB},
\]
so that (27) yields
\[ \sigma_H(\theta)|_{\theta=\theta_F} = \sigma_{H}^{F}. \]

Hence, the fractional quantum Hall effect where electrons are interacting can be obtained from the Hall effect in noncommutative space which is a non–interacting theory. Thus, one avoids to deal with a complicated interacting theory favoring an effective theory which is simpler.

4 Spin Hall Effect in Noncommutative Coordinates

To understand spin Hall effect, which basically occurs due to spin currents produced by spin–orbit coupling terms in the presence of electric field, diverse models were developed\[7\]–\[10\] (for a complete list see\[11\]). However, there are two simple semiclassical formulations which are suitable to investigate spin Hall effect in noncommuting coordinates: i) The extension of the Drude model given in\[12\]. ii) Generalization of the Hall effect formulation proposed in\[4\].

4.1 Deformation of the Drude model type formulation

The extension of the Drude model discussed in\[12\] can be obtained within the semiclassical approach of Section 2. The appropriate gauge field is
\[ A_{a} = \frac{\epsilon_{abc} \sigma_{b}}{4mc^{2}} \frac{\partial V}{\partial r_{c}}, \]
where \( a = 1, 2, 3 \); \( \sigma_{a} \) are the Pauli spin matrices and we set the coupling constant to be \( \rho = -\hbar \). Without ignoring the \( 1/c^{4} \) terms, the related field strength becomes
\[ F_{ab} = \frac{\epsilon_{bcd} \sigma_{c}}{4mc^{2}} \frac{\partial^{2} V}{\partial r_{a} \partial r_{d}} - \frac{\epsilon_{acd} \sigma_{c}}{4mc^{2}} \frac{\partial^{2} V}{\partial r_{b} \partial r_{d}} - \frac{\epsilon_{abc} \sigma}{8m^{2}c^{4}} \nabla \frac{\partial V}{\partial r_{c}}. \]

We deal with an electron moving on the noncommutative \( r_{1}r_{2} \) plane, so that like the previous section we consider the Hamiltonian\[16\] and set \( \xi = \theta \) and \( G_{ij} = \epsilon_{ij} \). Hence, ignoring the terms at the order of \( \hbar^{2} \) one can show from\[11\]–\[13\] that the phase space variables satisfy
\[ \{ r_{i}, r_{j} \}_{CD} = \theta \epsilon_{ij}, \]
\[ \{ p_{i}, p_{j} \}_{CD} = -\hbar F_{ij}, \]
\[ \{ r_{i}, p_{j} \}_{CD} = \delta_{ij} + \hbar \theta \epsilon_{ik} F_{kj}. \]

As before \( p_{i} \) are kinematic momenta thus the \( \theta \) dependent term appears in\[35\].

The equations of motion following from\[14\] and\[15\] are
\[ \dot{r}_{i} = \frac{p_{i}}{m} + \frac{\hbar \theta}{m} \epsilon_{ij} F_{jk} p_{k} + \theta \epsilon_{ij} \frac{\partial V}{\partial r_{j}}, \]
\[ \dot{p}_{i} = -\frac{\partial V}{\partial r_{i}} - \hbar \theta F_{ij} \epsilon_{jk} \frac{\partial V}{\partial r_{k}} - \frac{\hbar}{m} F_{ij} p_{j}. \]
In the spirit of the Drude model by adding the drag force $-p_i/\tau$ where $\tau$ is the relaxation time and retaining the terms linear in the velocity $v_i$, (36), (37) yield the total force

$$ \mathcal{F}_i = m\ddot{r}_i - \frac{p_i}{\tau} = -\frac{\partial V}{\partial r_i} - h\theta F_{ij}\epsilon_{jk} \frac{\partial V}{\partial r_k} - hF_{ij}v_j + \theta \epsilon_{ij}v_k \frac{\partial^2 V}{\partial r_j \partial r_k} - \frac{m}{\tau} v_i + \frac{m\hbar\theta}{\tau} \epsilon_{ij} F_{jk}v_k + \frac{m\theta}{\tau} \epsilon_{ij} \frac{\partial V}{\partial r_j}. \quad (38) $$

The \( \hbar^2 \) terms are ignored and we used (36) to express the momentum \( p_i \) in terms of the velocity \( v_i \) as

$$ \frac{p_i}{m} = v_i - \theta \epsilon_{ij} \frac{\partial V}{\partial r_j} - h\theta \epsilon_{ij} F_{jk}v_k. \quad (39) $$

In the absence of the drag force and for \( \theta = 0 \) the force (38) has already been derived in [15] employing the gauge field (31) and also in [4] within another approach. According to [12] let us deal with

$$ V = V_l + V_e \quad (40) $$

where \( V_l \) is the potential collecting effects of crystal and

$$ V_e = -e E \cdot r $$

is the external scalar potential. Moreover, in the force (38) we replace potential terms with their volume averages. Considering cubic lattice the following average over the volume is given in terms of the constant \( A \) as

$$ \langle \frac{\partial^2 V}{\partial r_a \partial r_b} \rangle = -eA\delta_{ab}. $$

Because of the external electric field \( E \) we also have

$$ \langle \frac{\partial V}{\partial r_a} \rangle = -eE_a. $$

Hence, average value of the field strength (32) is

$$ \langle F_{ij} \rangle = -\left( \frac{eA}{2mc^2}\sigma_3 + \frac{e^2}{8m^2c^4}\sigma \cdot EE_3 \right) \epsilon_{ij}. \quad (41) $$

In [16] it is claimed that the constant \( A \) should vanish due to electrical neutrality of conductors. On the other hand in [17] it is argued that the original formulation [12] makes use of the localized charges leading to an estimate for \( A \) which is nonvanishing (see also [18]). Nevertheless, as far as the average (11) is nonvanishing the extension of Drude model is successful though the estimated value of spin Hall conductivity may differ.

One demands that electron moves with a constant velocity, so that the total force acting on the electron should vanish:

$$ \mathcal{F}_i = 0. \quad (42) $$

Plugging (41) into (42) yields

$$ eE_i - \frac{mv_i}{\tau} - e\theta \epsilon_{ij} \left( Av_j - \frac{mE_j}{\tau} \right) + \left( \frac{eA}{2mc^2}\sigma_3 + \frac{e^2}{8m^2c^4}\sigma \cdot EE_3 \right) \left( e\theta E_i + \epsilon_{ij}v_j + \frac{m\theta v_i}{\tau} \right) = 0. \quad (43) $$
The steady–state solution can be obtained treating the $\langle F_{ij} \rangle$ dependent contribution in (43) as perturbation. Hence let
\[ v_i = v^0_i + v^I_i \]
where the lowest order solution is
\[ v^0_i = \frac{e\tau}{m} E_i. \]  
(44)
The first order solution in the perturbation can be shown to be
\[ v^I_i = \frac{2e\tau\theta}{m} \left( \frac{e\hbar A}{2mc^2}\sigma_3 + \frac{e^2\hbar}{8m^2c^4}\sigma \cdot E E_3 \right) E_i \]
+ \left[ e\theta \left( -1 + \frac{\tau^2 e A}{m} \right) + \frac{\tau^2 e}{m} \left( \frac{e\hbar A}{2mc^2}\sigma_3 + \frac{e^2\hbar}{8m^2c^4}\sigma \cdot E E_3 \right) \right] \epsilon_{ij} E_j. \]  
(45)

Let us introduce the density matrix
\[ N = \frac{1}{2} n (1 + \xi \cdot \sigma) \]  
(46)
in terms of the spin polarization vector $\xi$ whose magnitude is
\[ \xi = \frac{n^\uparrow - n^\downarrow}{n^\uparrow + n^\downarrow}, \]
where $n^\uparrow$ and $n^\downarrow$ denote concentration of states with spins along the $\hat{\xi}$ and $-\hat{\xi}$ directions and $n = n^\uparrow + n^\downarrow$. We choose the spin polarization to point in the third direction:
\[ \xi = \xi \hat{r}_3. \]

Adopting the definition of the current given in [12] as
\[ j_i \equiv e\text{Tr} (N v_i) \]  
(47)
and making use of the steady–state solution given by (44)–(45), we obtain
\[ j_i = \sigma_C(\theta) E_i - \sigma_{SH}^D(\theta) \xi \epsilon_{ij} E_j, \]
where the $\theta$ deformed conductivity is
\[ \sigma_C(\theta) = \left( 1 + \frac{e\hbar \theta A}{mc^2} + \frac{e^2\hbar \theta E_3^2}{4m^2c^4} \right) \frac{ne^2\tau}{m} \]
and the $\theta$ deformed spin Hall conductivity is
\[ \sigma_{SH}^D(\theta) = -\frac{nh^2e^3A}{2m^3c^2} - \frac{nh^2e^4E_3^2}{8m^4c^4} + \frac{ne^2\theta}{\xi} \left( 1 + \frac{e\tau^2 A}{m} \right). \]  
(48)

Indeed, when one sets $\theta = 0$ and ignore $1/c^4$ terms the formalism of [12] follows.
4.2 Deformation of the Hall Effect type formulation

Another simple method of deriving spin Hall conductivity was developed in [4] generalizing the formulation of Hall effect presented in Section 3. In this approach one introduces the non–Abelian gauge field

\[ A_i = \epsilon_{ij} \sigma_j, \] (49)

which is consistent with the Rashba spin–orbit coupling term[19]. It leads to

\[ F_{ij} = -\frac{i\rho}{\hbar} [A_i, A_j] = \frac{2\rho}{\hbar} \sigma_3 \epsilon_{ij}, \] (50)

The main ingredients of the dynamical equations (11)–(15) are the field strengths. Explicit forms of gauge fields do not matter as far as they lead to the same field strengths and commute with the related canonical Hamiltonian. We will deal with the Hamiltonian (16) which is a scalar, so that instead of the gauge field (49) we may equivalently choose

\[ A_i = \sigma_i, \] (51)

yielding the same field strength (50) up to a minus sign. The gauge field (51) is consistent with the Dresselhaus spin–orbit coupling term[20]. Obviously, we may also let the both gauge fields (49) and (51) be present by different coupling constants.

As before noncommutativity of the \( r_1 r_2 \equiv xy \) plane is furnished by letting the coupling constant be \( \xi = \theta \) and

\[ G_{ij} = \epsilon_{ij}. \] (52)

Moreover, as it is announced the canonical Hamiltonian is given by (16) with the scalar potential (17).

By plugging (50), (52) into (11)–(13) and keeping only the first order terms in \( \theta \) and \( \rho^2 \) one obtains the relations

\[
\{r_i, r_j\}_{CD} = \theta \epsilon_{ij}, \\
\{p_i, p_j\}_{CD} = \frac{2\rho^2}{\hbar} \sigma_3 \epsilon_{ij}, \\
\{r_i, p_j\}_{CD} = \left( 1 + \frac{2\rho^2 \theta}{\hbar} \sigma_3 \right) \delta_{ij}.
\]

As it is typical to our formalism \( p_i \) act as kinematic momenta.

At the same order in \( \theta \) and \( \rho \), making use of (50), (52) in (14), (15) the equations of motion of the phase space variables are obtained as

\[
\dot{r}_i = \left( 1 + \frac{2\rho^2 \theta}{\hbar} \sigma_3 \right) \frac{p_i}{m} - e \theta \epsilon_{ij} E_j, \\
\dot{p}_i = \left( 1 + \frac{2\rho^2 \theta}{\hbar} \sigma_3 \right) e E_i + \frac{2\rho^2}{\hbar m} \sigma_3 \epsilon_{ij} p_j.
\]

In terms of the velocity \( v_i \equiv \dot{r}_i \), we can write the momentum in the first order in \( \theta \) as

\[
\frac{p_i}{m} = \left( 1 - \frac{2\rho^2 \theta}{\hbar} \sigma_3 \right) v_i + e \theta \epsilon_{ij} E_j.
\]
Hence, at the first order in $\theta$ and $\rho^2$ the force acting on the particle is

$$\mathcal{F}_i = m\ddot{r}_i = eE_i + \frac{2\rho^2}{\hbar}\sigma_3\epsilon_{ij}v_j + \frac{2e\rho^2\theta}{\hbar}\sigma_3E_i.$$  

Imitating the formulation of the Hall effect we demand that

$$\mathcal{F}_i = 0$$  \hfill (53)

in order to have a motion without deflection. The condition (53) can be solved for the velocity as

$$v_i^\uparrow = \left(\frac{e\hbar}{2\rho^2} + e\theta\right)\epsilon_{ij}E_j,$$  \hfill (54)

$$v_i^\downarrow = -\left(\frac{e\hbar}{2\rho^2} - e\theta\right)\epsilon_{ij}E_j,$$  \hfill (55)

The arrows $\uparrow$ and $\downarrow$ correspond to the positive and negative eigenvalue of the Pauli spin matrix $\sigma_3$. It is natural to define the spin current as

$$j_i^z = \frac{\hbar}{2} (n^\uparrow v_i^\uparrow - n^\downarrow v_i^\downarrow),$$  \hfill (56)

where $n^\uparrow$ and $n^\downarrow$ denote concentration of states with spins along the $\hat{r}_3 \equiv \hat{z}$ and $-\hat{z}$ directions. Employing (54), (55) in (56) yields

$$j_i^z = -\sigma_{SH}(\theta)\epsilon_{ij}E_j,$$

where now, the $\theta$ deformed spin Hall conductivity is given by

$$\sigma_{SH}(\theta) = -\frac{e\hbar^2n}{4\rho^2} - \frac{1}{2}e\hbar n \xi \theta.$$  \hfill (57)

Let $n = n^\uparrow + n^\downarrow$ be given as the concentration of states occupying the lower energy state of the Rashba Hamiltonian\cite{19} times a constant $l$:

$$n = \frac{\rho^2 l}{\pi \hbar^2}.$$  

Denote that the original Rashba coupling constant $\alpha$ is related to $\rho$ as $\rho = -\alpha m/\hbar[19]$. Then, the $\theta$ deformed spin Hall conductivity can be written as

$$\sigma_{SH}(\theta) = -\frac{el}{4\pi} - \frac{e\hbar \bar{\theta}}{2},$$  \hfill (58)

where we defined

$$\bar{\theta} \equiv (n^\uparrow - n^\downarrow) \theta.$$  

Observe that, the $\theta = 0$ limit of (58) agrees with the universal behavior obtained in [8] for $l = 1/2$.  

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4.3 Deformed Spin Hall Conductivity

The difference in the $\xi$ dependence of (48) and (58) is due to the fact that in the former the density matrix (46) is employed to define the current (47), but in the latter we avoided it in the definition of the spin current (56). Nevertheless, both of the formalisms which we presented here lead to deformed spin Hall conductivities which can be written symbolically as

$$\sigma_S(\theta) = \Sigma_0 + \theta \Sigma_1,$$

which yields spin Hall conductivity when the noncommutativity is switched off by setting $\theta = 0$. In [13] another $\theta$ deformed spin Hall conductivity was delivered which vanishes in the limit $\theta = 0$.

For concreteness in the following discussions we will concentrate on (58). In the spirit of interpreting the noncommutativity as a link between similar physical phenomena $\theta$ can be fixed to obtain other formulations of spin Hall effect. We will illustrate this point of view by considering spin Hall conductivities obtained by inclusion of impurities, the Rashba type spin–orbit couplings with higher order momenta and the quantum spin Hall effect.

When impurity effects included into the Rashba Hamiltonian which is linear in momenta, the universal behavior of spin Hall conductivity[8] is swept out[9]. This can be obtained by fixing the noncommutativity parameter in (58) as

$$\tilde{\theta}_0 = -\frac{l}{2\pi \hbar}$$

yielding

$$\sigma_{SH}(\theta)|_{\tilde{\theta} = \tilde{\theta}_0} = 0.$$

On the other hand when one deals with the Rashba type Hamiltonian with higher order momenta

$$H = \epsilon_k - \frac{1}{2} b_i(k) \sigma_i + V(r),$$

one finds a non–vanishing spin Hall conductivity[10]. Here $k$ is the kinematic momentum and $\epsilon_k$ is the energy dispersion in the absence of spin–orbit coupling. By defining $b_1 + i b_2 \equiv b_0(k) \exp(iN\theta)$ and

$$\tilde{N} = \frac{d\ln |b_0|}{d\ln k}, \quad 1 + \zeta = \frac{d\ln v}{d\ln k},$$

in terms of the velocity $v$, the spin Hall conductivity

$$\sigma_{\text{SH}}^{HR} = -\frac{eN}{4\pi} \left( \frac{N^2 - 1}{N^2 + 1} \right) \left( \tilde{N} - \zeta - 2 \right)$$

results. This can be achieved from (58) as

$$\sigma_{SH}(\theta)|_{\tilde{\theta} = \tilde{\theta}_{HR}} = \sigma_{\text{SH}}^{HR},$$

by setting $l = N$ and fixing the noncommutativity parameter as

$$\tilde{\theta}_{HR} = \frac{N}{2\pi \hbar} \left[ -1 + \left( \frac{N^2 - 1}{N^2 + 1} \right) \left( \tilde{N} - \zeta - 2 \right) \right].$$
Quantization of the spin Hall conductance in units of $e/2\pi$ was predicted in [21]. Hence the quantized spin Hall conductivity can be written as

$$\sigma_{SH}^Q = -\frac{e}{2\pi \mu}$$

where $\mu$ is a number depending on the physical system considered. This can be obtained from (58) by fixing the noncommutativity parameter as

$$\sigma_{SH}(\theta)|_{\tilde{\theta} = \tilde{\theta}_Q} = \sigma_{SH}^Q,$$

where

$$\tilde{\theta}_Q = \frac{1}{2\pi \hbar} (-l + 2\mu).$$

Therefore, the spin Hall effect in noncommutative coordinates can be considered as the master formulation such that fixing the noncommutativity parameter $\theta$ yields different manifestations of the same physical phenomenon.

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