A Finite Horizon Stochastic Impulse Control Problem with Elephant Memory under Partial Information∗

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Abstract

We consider a stochastic impulse control problem where the terminal reward and the intervention costs depend on the entire control history. We consider the setting where the operator only has partial information of the evolution of the system and show existence of an optimal control by applying a probabilistic technique based on the concept of Snell envelopes. As a motivating example we consider the co-ordinated control of a system of hydro-power plants with hydrological coupling when the stochastic inflows are observed through noisy measurements.

1 Introduction

The standard stochastic impulse control problem is an optimal control problem that arises when an operator controls a dynamical system by intervening on the system at a discrete set of stopping times. Generally, an intervention can be represented by an element in the control set which we assume to be a compact subset of .

In impulse control the control-law, thus, takes the form where is a sequence of times when the operator intervenes on the system and is the impulse that the operator affects the system with at time .

The standard impulse control problem in finite horizon can be formulated as finding a control that maximizes

where is a controlled stochastic process that jumps at interventions (e.g. by setting ), the deterministic functions and give the running and terminal reward, respectively, and represents a cost incurred by applying the impulse .

As impulse control problems appear in a vast number of real-world applications (see e.g. for applications in finance and for applications in energy) a lot of attention has been given to various types of problems where the control is of impulse type. In the standard Markovian setting where solves a stochastic differential equation (SDE) driven by a Lévy process on the relation to quasi-variational inequalities has frequently been exploited to find optimal controls (see the seminal work in or turn to for a more recent textbook). In the non-Markovian framework an impulse control problem was solved in by utilizing the link between optimal stopping and reflected BSDEs (originally discovered in) while considering the reward functional

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where $\phi : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}$ is now a random (and not necessarily Markovian) field and the controlled process $L^u$ takes the particular form $L^u_t := L_t + \sum_{j=1}^N \mathbf{1}_{[\tau_j \leq t]} \beta_j$, with $L$ an (exogenous) non-controlled process and assuming that $U$ is a finite set. Relevant is also the treatment of multi-modes optimal switching problems in a non-Markovian setting in [13].

Almost all production systems are subject to delays in the sense that some time is required to start up the production units. In this regard a lot of effort has been directed at impulse controls where the effect of the interventions are delayed by a fixed lag. In the Markovian setting, the novel paper [2] proposes an explicit solution to an inventory problem with uniform delivery lag by taking the current stock plus pending orders as one of the states. Similar approaches are taken in [1] where explicit optimal solutions of impulse control problems with uniform delivery lags are derived for a large set of different problems and [6] that propose an iterative algorithm. Øksendal and Sulem [25] propose a solution to general Markovian impulse control problems with execution delays, by defining an operator that circumvents the delay period.

Also in the non-Markovian setting problems with delivery lag have been considered. In [18] the original work of [11] was extended to incorporate delivery lag by setting $L^u_t := L_t + \sum_{j=1}^N \mathbf{1}_{[\tau_j + \Delta \leq t]} \beta_j$ while requiring that $\tau_{j+1} \geq \tau_j + \Delta$ for a fixed $\Delta > 0$. Recently this work was extended by considering the infinite horizon setting in [12].

In the present work we aim at extending the results for the non-Markovian setting by considering a terminal reward that depends on the entire history of the control. In this regard we will generalize the recent work on optimal switching presented in [27] to impulse control. Specifically, we are interested in the problem of maximizing the reward functional

$$J(u) := \mathbb{E}_G \left[ \Psi(\tau_1, \ldots, \tau_N; \beta_1, \ldots, \beta_N) \right], \quad (1.1)$$

where $\Psi$ maps controls to values of the real line and is measurable with respect to the $\sigma$-field $G$. We consider the partial information setting and assume that we observe the system through a filtration $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}$ of sub-$\sigma$-fields of $G$ and thus restrict our attention to $\mathbb{F}$-adapted controls.

The main contribution of the present work is showing that the problem of maximizing $J$ can be solved under certain assumptions on $\Psi$ by finding an optimal control in terms of a family of interconnected value processes, that we refer to as a verification family.

The remainder of the article is organized as follows. In the next section we state the problem, set the notation used throughout the article and detail the set of assumptions that are made. Furthermore, we formulate a motivating example from renewable energy production, namely the optimization of electricity production in a system of hydrologically interconnected hydro-power stations. Then, in Section 3 a verification theorem is derived. This verification theorem is an extension of the verification theorem for the multi-modes optimal switching problem with memory developed in [27] and presumes the existence of a verification family. In Section 4 we show that, under the assumptions made, there exists a verification family, thus proving existence of an optimal control for the impulse control problem with the cost functional $J$ defined in (1.1).

2 Preliminaries

We consider a finite horizon problem and thus assume that the terminal time $T$ is fixed with $T < \infty$.

We let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space, and $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ a filtration of sub-$\sigma$-fields of $\mathcal{G}$ satisfying the usual conditions in addition to being quasi-left continuous and define $\mathcal{F} := \mathcal{F}_T$.

Remark 2.1. Recall here the concept of quasi-left continuity: A càdlàg process $(X_t : 0 \leq t \leq T)$ is quasi-left continuous if for each predictable stopping time $\gamma$ and every announcing sequence of stopping
times \( \gamma_k \sim \gamma \) we have \( X_{\gamma_k} = X_{\gamma}, \ \mathbb{P}\text{-a.s.} \)
A filtration is quasi-left continuous if \( \mathcal{F}_\gamma = \mathcal{F}_{\gamma^-} \) for every predictable stopping time \( \gamma \).

Throughout we will use the following notation:

- \( \mathcal{P}_F \) is the \( \sigma \)-algebra of \( \mathbb{F} \)-progressively measurable subsets of \([0, T] \times \Omega \).
- For \( p \geq 1 \), we let \( \mathcal{S}^p \) be the set of all \( \mathbb{R} \)-valued, \( \mathcal{P}_F \)-measurable, càdlàg processes \((Z_t : 0 \leq t \leq T)\) such that \( \mathbb{E} \left[ \sup_{t \in [0,T]} |Z_t|^p \right] < \infty \) and let \( \mathcal{S}_q^p \) be the subset of processes that are quasi-left upper semi-continuous (i.e. \( \lim_{j \to \infty} Z_{\gamma_j} \leq Z_{\gamma}, \mathbb{P}\text{-a.s.} \)).
- We let \( \mathcal{T} \) be the set of all \( \mathbb{F} \)-stopping times \( \tau \) with \( \tau \leq T, \ \mathbb{P}\text{-a.s.} \), and for each \( \gamma \in \mathcal{T} \) we let \( \mathcal{C}_\gamma \) be the corresponding subsets of stopping times \( \tau \) such that \( \tau \geq \gamma, \ \mathbb{P}\text{-a.s.} \).
- We let \( \mathcal{U} \) be the set of all \( u = (\tau_1, \ldots, \tau_N; \beta_1, \ldots, \beta_N) \), where \((\tau_j)_{j=1}^N \) is a non-decreasing sequence of \( \mathbb{F} \)-stopping times and \( \beta_j \) is a \( \mathcal{F}_\gamma \)-measurable random variable (r.v.) taking values in \( U \).
- We let \( \mathcal{U}^f \) denote the subset of \( u \in \mathcal{U} \) for which \( N \) is finite \( \mathbb{P}\text{-a.s.} \), i.e.
\[
\mathcal{U}^f := \{u \in \mathcal{U} \mid \mathbb{P}\{\omega \in \Omega : N(\omega) > k, \ \forall k \geq 0\} = 0\} \text{ and for all } k \geq 0 \text{ we let } \mathcal{U}^k := \{u \in \mathcal{U} \mid N \leq k, \mathbb{P}\text{-a.s.}\}. \text{ For } \gamma \in \mathcal{T} \text{ we let } \mathcal{U}_\gamma \text{ (and } \mathcal{U}^f_\gamma \text{ resp. } \mathcal{U}^k_\gamma \text{) be the subset of } \mathcal{U} \text{ (and } \mathcal{U}^f \text{ resp. } \mathcal{U}^k \text{) with } \tau_1 \in \mathcal{T}_\gamma. \]
- We define the set \( \mathcal{D} := \{(t_1, \ldots, b_1, \ldots) : 0 \leq t_1 \leq t_2 \leq \cdots \leq T, b_j \in U\} \) and let \( \mathcal{D}^f \) be the corresponding subset of all finite sequences. Furthermore, for \( k \geq 0 \) we let \( \mathcal{D}^k \) be the subset of sequences of length \( 2k \).
- For \( l \geq 0 \), we let \( \Pi^T_l := \{0, 2T^{-l}, 2T^{-l}, \ldots, T\} \).

### 2.1 Problem formulation

In the above notation, our problem is characterized by the complete probability space \((\Omega, \mathcal{G}, \mathbb{P})\) and the following objects:

- The filtration \( \mathbb{F} \).
- A \( \mathcal{G} \otimes \mathcal{B}(\mathcal{D}) \)-measurable map \( \Psi : \mathcal{D} \to \mathbb{R} \).

We will make the following assumptions on the terminal reward:

**Assumption 2.2.**  (i) The function \( \Psi \) can be decomposed as:

\[
\Psi(t_1, \ldots, t_n; b_1, \ldots, b_n) = \varphi(t_1, \ldots, t_n; b_1, \ldots, b_n) - \sum_{j=1}^n c(t_1, \ldots, t_j; b_1, \ldots, b_j)
\]

where \(1\):

a) \( \sup_{u \in \mathcal{U}^f} \mathbb{E}[|\varphi(u)|^2] < \infty \).

b) For all \( v \in \mathcal{U}^f \) we have \( \sup_{u \in \mathcal{U}^f} \mathbb{E}[\sup_{s \in [0,T]} \sup_{b \in U} |\varphi(v \circ (s, b) \circ u)|^2] < \infty \).

c) The intervention costs \( c \) are \( \mathcal{F} \otimes \mathcal{B}(\mathcal{D}) \)-measurable maps such that \( \mathbb{E}[c(v \circ (s, b))] \in \mathcal{S}^2 \) for all \( v \in \mathcal{U}^f \) and inf \( v \in \mathcal{D}^f \sup_{u \in \mathcal{U}^f} c(v) \geq f_k \), where \((f_k)_{k \geq 0} \subset \mathbb{R}_+ \) is a deterministic (uniformly bounded) sequence of non-negative numbers such that \( f_k := \sum_{j=0}^k f_j \to \infty \) as \( k \to \infty \).

1 We introduce the composition of \( v = (t_1, \ldots, t_n; b_1, \ldots, b_n) \) and \( v' = (t'_1, \ldots, t'_{n'}; b'_1, \ldots, b'_{n'}) \) in \( \mathcal{D}^f \) defined as \( v \circ v' := (t_1, \ldots, t_n, t'_1 \vee t_n, \ldots, t'_{n'} \vee t_n; b_1, \ldots, b_n, b'_1, \ldots, b'_{n'}) \).

2 To retain adaptedness we abuse notation set \( c(v \circ (s, b)) := \mathbb{E}[c(u \circ (s, b))|\mathcal{F}_s] \) for \( s \in [0, \tau_N] \).
Proposition 2.5. Suppose that there is a control problem. Hence, \( \hat{u} \) for all \( k \in \mathbb{N} \) the limit \( \lim_{t' \to t, b' \in U^k} \chi(t', b') = 0 \),

\[
\lim_{b' \to b} \chi(t, b') \leq \chi(t, b).
\]

Remark 2.3. We remark that one way (and maybe the most natural way) of interpreting the functions \( \varphi \) and \( c \) have the following regularity:

a) \( \varphi \) (resp. \( c \)) is \( \mathbb{P} \)-a.s. right-continuous with left limits in the intervention times uniformly in the interventions and upper semi-continuous (resp. lower semi-continuous) in the interventions. That is, (for \( \chi = \varphi, -c \)) for each \( k \geq 0 \), there is a \( \mathbb{P} \)-null set \( N \) such that for all \( \omega \in \Omega \setminus N \) and all \( (t, b) \in D^k \) the limit \( \lim_{(t', b') \to (t, b)} \chi(t', b') \) exists and

\[
\lim_{t' \to t, b' \in U^k} |\chi(t', b') - \chi(t, b')| = 0,
\]

b) For any \( v, u \in U^I \) we have that for any predictable stopping time \( \gamma \in T \) and any announcing sequence \( \gamma_j \uparrow \gamma \) with \( \gamma_j \in T \) (again for \( \chi = \varphi, -c \))

\[
\lim_{j \to \infty} \sup_{b} \chi(v \circ (\gamma_j, b) \circ u) \leq \chi(v \circ (\gamma, b) \circ u),
\]

\( \mathbb{P} \)-a.s. for all \( b \in U \) (where the exception set is independent of \( b \)).

(iii) For each \( v \in U^I \) we have that, outside of a \( \mathbb{P} \)-null set, \( \Psi(v) > \Psi(v \circ (T, b)) \) for all \( b \in U \)

The above assumptions are mainly standard assumptions for impulse control problems translated to our setting. Assumption \([i.a]\) implies that the expected maximal reward is finite. Assumption \([iii]\) implies that it is never optimal to intervene at the terminal time. We show below that the fact that \( F_k \to \infty \) as \( k \to \infty \) together with \([i.a]\) implies that, with probability one, the optimal control (whenever it exists) can only make a finite number of switches.

Remark 2.3. We remark that one way (and maybe the most natural way) of interpreting the functions \( \varphi \) and \( c \) are in the sense of functionals (resp. functions) of some impulsively controlled stochastic process \( X^u \), for example, by setting

\[
\varphi(t; b) = \int_0^T \phi(s, X^t;b)ds + \psi(X^t;b)
\]

and

\[
c(t; b) = \hat{c}(t_n, b_n, X^t;b).
\]

Remark 2.4. Note that we may hide part of the intervention cost within the function \( \varphi \) which implies that we can handle problems with negative intervention costs as in [23].

We consider the following problem:

Problem 1. Find \( u^* \in U \), such that

\[
J(u^*) = \sup_{u \in U} J(u).
\]

As a step in solving Problem 1 we need the following proposition which is a standard result for impulse control problems.

Proposition 2.5. Suppose that there is a \( u^* \in U \) such that \( J(u^*) \geq J(u) \) for all \( u \in U \). Then \( u^* \in U^I \).

Proof. Pick \( \hat{u} := (\hat{t}_1, \ldots, \hat{t}_N; \hat{\beta}_1, \ldots, \hat{\beta}_N) \in U \setminus U^I \) and let \( B := \{ \omega \in \Omega : \hat{N}(\omega) > k, \forall k > 0 \} \), then \( \mathbb{P}[B] > 0 \). Furthermore \( ^3 \)

\[
J(\hat{u}) \leq \sup_{u \in U} \mathbb{E}[|\varphi(\tau_1, \ldots, \beta_1, \ldots)|] + F_k \mathbb{P}[B] \leq C - F_k \mathbb{P}[B],
\]

for all \( k \geq 0 \), by Assumption \([2.2] [i.a]\). However, again by Assumption \([2.2] [i.a]\) we have \( J(\emptyset) \geq -C \). Hence, \( \hat{u} \) is dominated by the strategy of doing nothing and the assertion follows. \( ^\Box \)

\(^3\)Throughout \( C \) will denote a generic positive constant that may change value from line to line.
2.2 A motivating example

We consider a system of $m$ hydro-power plants located in the same river-system (see Fig. 1 for a possible setup when $m = 4$) and introduce the following quantities:

- $M_t^i$: amount of water in reservoir $i$ (where $M_i$ is the capacity and $\bar{M}_i$ is a minimum level) at time $t \in [0, T]$.
- $V_t^i$: exogenous inflow to reservoir $i$ at time $t \in [0, T]$.
- $\xi_t^i$: volumetric flow-rate through power plant $i$ at time $t \in [0, T]$, taking values in the compact subset $U_i$ of $\mathbb{R}_+$, with $0 \in U_i$.
- $\delta_{i,j}$: flow time from the outlet of plant $i$ to reservoir $j$.

for $i, j \in \{1, \ldots, m\}$ with $j \neq i$. We assume that the electricity price ($R_t : 0 \leq t \leq T$) and the vector of inflows ($V_t : 0 \leq t \leq T$) are càdlàg processes adapted to a complete filtration $\mathcal{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ with $\mathcal{G}_T = \mathcal{G}$ and set $U := U_1 \times \cdots \times U_m$.

As the level of water in each reservoir cannot exceed the capacity of the reservoir, water may have to be spilled and we find that the amount of water in reservoir $i$ follows the relation

$$M_t^i = m_0^i + \int_0^t (V_s^i - \xi_s^i)ds - K_t^i$$

$$+ \sum_{j: A_{j,i}=1} \left\{ \int_0^t \xi_s^{j,i} ds + \bar{K}_{t-\delta_{j,i}}^{j,i} \right\}$$

where $m_0 \in [\bar{M}_1, M_1] \times \cdots \times [\bar{M}_m, M_m]$ is the vector of initial water levels in the reservoirs, $A_{j,i} = 1$ if reservoir $i$ is located directly downstream from plant $j$ and 0 otherwise and $\bar{K}_t$ is the minimal spill in $[0, t]$ necessary to have $M_t^i \leq \bar{M}_i$ for all times $t \in [0, T]$. The process $\bar{K}$ is, thus, non-decreasing with $\bar{K}_0 = 0$ if $A_{i,j} = 1$ and $\bar{K}_0 = 0$ if $A_{i,j} = 0$ for all $j \in \{1, \ldots, m\}$. Furthermore, $\bar{K}^i$ satisfies the Skorokhod condition,

$$\int_0^T (\bar{M}_i - M_t^i)^+ d\bar{K}_t^i = 0.$$

In particular we find that

$$\frac{d}{dt} \bar{K}_t^i = \mathbb{1}_{[M_t^i = \bar{M}_i]} (V_t^i - \xi_t^i) + \sum_{j: A_{j,i}=1} (\xi_{t-\delta_{j,i}}^j + \frac{d}{dt} \bar{K}_{t-\delta_{j,i}}^j).$$

On the other side, to assure that $M_t^i \geq \bar{M}_i$ the natural thing is to force a shut-down of unit $i$ whenever $M_t^i = \bar{M}_i$. Given the control law $u = (\tau_1, \ldots, \tau_N; \beta_1, \ldots, \beta_N) \in \mathcal{U}$ we then let $\xi$ follow the relation

$$\xi_t := \zeta_0(t) \mathbb{1}_{[0, \tau_1)}(t) + \sum_{j=1}^N \zeta_j(t) \mathbb{1}_{[\tau_j, \tau_{j+1})}(t),$$

figure 1: A system of four hydro-power units.

4Where we for this specific example require that $\tau_{j+1} > \tau_j$. 


where the processes \( \zeta_j \) are defined as

\[
\zeta_j(t) := \beta_j \mathbb{1}_{[0, \eta_{j,1})}(t) + \sum_{k \geq 1} \kappa_{j,k} \mathbb{1}_{[\eta_{j,k}, \eta_{j,k+1})}(t)
\]

with

\[
\eta_{j,1} := \inf \{ t \geq \tau_j : \exists i, [\beta_j]_i > 0, M^i_t = \dot{M}_i \}
\]

and

\[
\kappa_{j,1} := \beta_j - [\beta_j]_{e_{j,1}}, \text{ with } t_{j,1} \text{ a measurable selection of }
\]

\[
t_{j,1} \in \{ i \in \{1, \ldots, m \} : [\beta_j]_i > 0, M^i_{\eta_{j,1}} = \dot{M}_i \}
\]

and then recursively we let

\[
\eta_{j,k} := \inf \{ t \geq \eta_{j,k-1} : \exists i, [\kappa_{j,k-1}]_i > 0, M^i_t = \dot{M}_i \}
\]

and define

\[
\kappa_{j,k} := \kappa_{j,k} - [\kappa_{j,k}]_{t_{j,k}} e_{j,k}, \text{ with } t_{j,k} \text{ a measurable selection of }
\]

\[
t_{j,k} \in \{ i \in \{1, \ldots, m \} : [\kappa_{j,k-1}]_i > 0, M^i_{\eta_{j,k}} = \dot{M}_i \}
\]

for \( k > 1 \) (with the convention that \( \inf \emptyset := \infty \)).

We note that this implies that \( (\xi_t : 0 \leq t \leq T) \) are the flow-rate levels given the control \( u \) considering the fact that a unit is turned off if the water level in the corresponding reservoir reaches the minimal allowed level for that particular reservoir.

We also assume that a change in flow-rate from \( x \in U \) to \( x' \in U \) incurs a cost \( c_{x,x'} \), due to, for example, wear and tear on equipment and lower efficiency during production-shifts and that the power production in unit \( i, p_i \), depends continuously on the flow-rate, \( \xi_i \), and the drop-height (i.e. \( M^i \)).

The total revenue during an operation period \([0, T]\) can then be written

\[
J(u) := \mathbb{E} \left[ \int_0^T R_t \sum_{i=1}^m p_i(M^i_t, \xi^i_t) dt + q(\dot{M}) - \sum_{t: \Delta \xi_t \neq 0} c_{\xi_t - \xi_{t-}} \right],
\]

where \( \Delta \xi_t := \xi_t - \xi_{t-} \) is the jump that \( \xi \) makes at time \( t \), \( q : \mathbb{R}^m \to \mathbb{R} \) represents the value of stored water at time \( T \) as a function of the amount of water in the system and

\[
\dot{M}_i := M^i_T + \sum_{j: A_{j,i} = 1} \int_{T - \delta_{j,i}}^T \xi^j_s ds.
\]

Now, as the amount of water in the reservoirs and the inflows are in general only available to the operator through noisy measurements, we assume that we observe the process \( O_t := (R_t, M^O_t, V^O_t) \), where \( M^O_t := M_t + \varepsilon_t \) and \( V^O_t := V_t + \varepsilon'_t \) with \( (\varepsilon_t : 0 \leq t \leq T) \) and \( (\varepsilon'_t : 0 \leq t \leq T) \) two \( \mathbb{G} \)-adapted processes representing measurement noise and let \( \mathbb{F} \) be the augmented natural filtration generated by \( O \).

In this case \( \mathcal{F}_t \) is a sub-\( \sigma \)-field of \( \mathcal{G}_t \) (and, thus also of \( \mathcal{G} \)) and we note that, under rather mild conditions on the involved parameters, this problem fits into the setting described above.

In the remainder of this section we will recall some well known results that will be useful in showing that an optimal control for Problem 1 exists, starting with the concept of Snell envelopes.\footnote{With \( e_i \) the \( i \)-th coordinate vector in \( \mathbb{R}^m \).}
2.3 The Snell envelope

In this section we gather the main results concerning the Snell envelope that will be useful later on. Recall that a progressively measurable process $X$ is of class $[D]$ if the set of random variables $\{X_\tau : \tau \in T\}$ is uniformly integrable.

**Theorem 2.6** (The Snell envelope). Let $X = (X_t)_{0 \leq t \leq T}$ be an $\mathcal{F}$-adapted, $\mathbb{R}$-valued, càdlàg process of class $[D]$. Then there exists a unique (up to indistinguishability), $\mathbb{R}$-valued càdlàg process $Z = (Z_t)_{0 \leq t \leq T}$ called the Snell envelope of $X$, such that $Z$ is the smallest supermartingale that dominates $X$. Moreover, the following holds (with $\Delta X_t := X_t - X_{t^-}$):

(i) For any stopping time $\gamma$,

$$Z_\gamma = \text{ess sup}_{\tau \in \mathcal{T}_\gamma} \mathbb{E} \left[ X_\tau | \mathcal{F}_{\gamma} \right].$$

(ii) The Doob-Meyer decomposition of the supermartingale $Z$ implies the existence of a triple $(M, K^c, K^d)$ where $(M_t : 0 \leq t \leq T)$ is a uniformly integrable right-continuous martingale, $(K^c_t : 0 \leq t \leq T)$ is a non-decreasing, predictable, continuous process with $K^c_0 = 0$ and $(K^d_t : 0 \leq t \leq T)$ is non-decreasing purely discontinuous predictable with $K^d_0 = 0$, such that

$$Z_t = M_t - K^c_t - K^d_t.$$  

Furthermore, $\{\Delta K^d_t > 0\} \subset \{\Delta X_t < 0\} \cap \{Z_- = X_{t-}\}$ for all $t \in [0, T]$.

(iii) Let $\theta \in \mathcal{T}$ be given and assume that for any predictable $\gamma \in \mathcal{T}_\theta$ and any increasing sequence $\{\gamma_j\}_{j \geq 0}$ with $\gamma_j \in \mathcal{T}_\theta$ and $\lim_{j \to \infty} \gamma_j = \gamma$, $\mathbb{P}$-a.s., we have $\lim \sup_{j \to \infty} X_{\gamma_j} \leq X_\gamma$, $\mathbb{P}$-a.s. Then, the stopping time $\tau^*_\theta$ defined by $\tau^*_\theta := \inf\{s \geq \theta : Z_s = X_s\} \wedge T$ is optimal after $\theta$, i.e.

$$Z_\theta = \mathbb{E} \left[ X_{\tau^*_\theta} | \mathcal{F}_\theta \right].$$

Furthermore, in this setting the Snell envelope, $Z$, is quasi-left continuous, i.e. $K^d \equiv 0$.

(iv) Let $X^k$ be a sequence of càdlàg processes converging increasingly and pointwisely to the càdlàg process $X$ and let $Z^k$ be the Snell envelope of $X^k$. Then the sequence $Z^k$ converges increasingly and pointwisely to a process $Z$ and $Z$ is the Snell envelope of $X$.

In the above theorem (i)-(iii) are standard. Proofs can be found in [14] (see [22] for an English version), Appendix D in [19], [16] and in the appendix of [8]. Statement (iv) was proved in [13].

The Snell envelope will be the main tool in showing that Problem 1 has a solution.

2.4 The section and projection theorems

In this section we recall two fundamental results from the general theory of stochastic processes, namely the measurable selection and the optional projection theorems.

For any space $E$ we define the projection of a set $A \subset \Omega \times E$ onto $\Omega$ as $\pi_\Omega(A) := \{\omega \in \Omega : \exists x \in E, (\omega, x) \in A\}$. 

**Theorem 2.7** (Measurable projection). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $E$ a Polish space. For every $A \in \mathcal{F} \otimes \mathcal{B}(E)$ the set $\pi_\Omega(A)$ is $\mathcal{F}$-measurable.

A proof can be found in e.g. [9] Capter III. In particular we need the following corollary result:
**Corollary 2.8.** Consider a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \(h(\omega, x)\) be a real valued, measurable function defined on the product space \((\Omega \times \mathbb{R}^m, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m))\). Then for all \(A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m)\), the function

\[
g(\omega) := \sup_{x \in \mathbb{R}^m} \{ h(\omega, x) : (\omega, x) \in A \}
\]

(with the convention \(\sup \emptyset = -\infty\)) is \(\mathcal{F}\)-measurable.

**Proof.** For each \(K \in \mathbb{R}\) we have \(\{g(\omega) > K\} = \pi_{\Omega}(A \cap h^{-1}((K, \infty)))\). Now, since \(h\) is measurable, the set \(A \cap h^{-1}((K, \infty))\) is in \(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m)\) and the result follows by the measurable projection theorem. \(\square\)

**Theorem 2.9** (Measurable selection). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space and let \(E\) be a Borel space with \(E := \mathcal{B}(E)\). For every \(A \in \mathcal{F} \otimes \mathcal{B}(E)\) there is a \(\mathcal{F}\)-measurable r.v. \(\beta\) taking values in \(E := E \cup \{\partial\}\) (with \(\partial\) a cemetery point) such that

\[
\{(\omega, \beta(\omega)) \in \Omega \times E\} \subset A \quad \text{and} \quad \{\omega \in \Omega : \beta(\omega) \in E\} = \pi_{\Omega}(A).
\]

This is a standard result and proofs can be found in e.g. Chapter 7 in [5] or [20]. In particular we need the following corollary result:

**Corollary 2.10.** Consider a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \(h(\omega, x)\) be a measurable function defined on the product space \((\Omega \times \mathbb{R}^m, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^m))\), such that for \(\mathbb{P}\)-almost every \(\omega\) the map \(x \mapsto h(\omega, x)\) is upper semi-continuous. Then, with \(U\) a compact subset of \(\mathbb{R}^m\), there exists a \(\mathcal{F}\)-measurable r.v. \(\beta\) such that

\[
h(\omega, \beta(\omega)) = \sup_{x \in \mathbb{R}^m} \{ h(\omega, x) : (\omega, x) \in \Omega \times U \},
\]

\(\mathbb{P}\)-a.s.

**Proof.** Since \(A := \Omega \times U \in \mathcal{F} \otimes \mathcal{B}(E)\) (where now \(E = \mathbb{R}^m\)) the function \(g(\omega) = \sup_{x \in \mathbb{R}^m} \{ h(\omega, x) : (\omega, x) \in A \}\) is \(\mathcal{F}\)-measurable. Furthermore, as \(h\) is \(\mathcal{F} \otimes \mathcal{B}(E)\)-measurable, the set \(B := \{(\omega, x) \in \Omega \times U : h(\omega, x) = g(\omega)\}\) is in \(\mathcal{F} \otimes \mathcal{B}(E)\). Now, by Theorem 2.9 there is a \(\mathcal{F}\)-measurable \(E\)-valued r.v. \(\beta\) such that \(\{(\omega, \beta(\omega)) \in \Omega \times E\} \subset B\) and \(\{\omega \in \Omega : \beta(\omega) \in E\} = \pi_{\Omega}(B)\). As \(U\) is compact and \(b \mapsto h(\omega, b)\) is u.s.c. on \(\Omega \setminus \mathcal{N}\) with \(\mathbb{P}(\mathcal{N}) = 0\), we have that \(B^\omega := \{b \in U : (\omega, b) \in B\} = \{b \in U : h(\omega, b) = g(\omega)\} \neq \emptyset\) for all \(\omega \in \Omega \setminus \mathcal{N}\) and, hence, \(\mathbb{P}(\pi_{\Omega}(B)) = 1\). \(\square\)

The last result that we need is the optional projection theorem.

**Theorem 2.11** (Optional projection). Assume that \((X_t : 0 \leq t \leq T)\) is a measurable process (not necessarily adapted to the filtration \(\mathcal{F}\)) with \(\mathbb{E}[|X_T|] < \infty\) for all stopping times \(\tau \in \mathcal{T}\), then there exists a unique optional process \((\mathring{X}_t : 0 \leq t \leq T)\) such that

\[
\mathring{X}_\tau := \mathbb{E}[X_\tau | \mathcal{F}_\tau],
\]

for all stopping times \(\tau \in \mathcal{T}\). If, furthermore, \(X\) is càdlàg then \(\mathring{X}\) is also càdlàg.

A proof of Theorem 2.11 can be found in [10].

### 3 A verification theorem

Our approach to finding a solution to Problem 1 is based on deriving an optimal control under the assumption that a specific family of processes exists, and then showing that the family indeed does exist. We will refer to any such family of processes as a verification family.
**Definition 3.1.** We define a verifying family to be a family of càdlàg supermartingales \((Y^v_s)_{0 \leq s \leq T}: v \in \mathcal{U}^f\) such that for each \(v := (\eta_1, \ldots, \eta_M; \alpha_1, \ldots, \alpha_M) \in \mathcal{U}^f\):

a) The family satisfies the recursion

\[
Y^v_s = \text{ess sup}_{\tau \in T_s} \mathbb{E}\left[1_{[\tau = T]} \varphi(v) + 1_{[\tau < T]} \sup_{b \in U} \{-c(v \circ (\tau, b)) + Y^v_{\tau^b}(\tau, b)\}\right] F_s. \tag{3.1}
\]

b) The family is uniformly bounded in the sense that \(\sup_{u \in \mathcal{U}} \mathbb{E}[\sup_{s \in [0, T]} |Y^u_s|^2] < \infty\).

c) For every \(\tau \in T\), the map \(b \mapsto Y^v_{\tau^b}\) is u.s.c. outside of a \(\mathbb{P}\)-null set.

d) The process \((\sup_{b \in U} Y^v_{\tau^b}(s, b): 0 \leq s \leq T)\) has a version that is in \(S^2_q\). Furthermore, the version in \(S^2_q\) agrees with \((\sup_{b \in U} Y^v_{\tau^b}(s, b): 0 \leq s \leq T)\), outside of a \(\mathbb{P}\)-null set, at each stopping time \(\tau \in T\).

The purpose of the present section is to reduce the solution of Problem 1 to showing existence of a verifying family. This is done in the following verification theorem:

**Theorem 3.2.** Assume that there exists a verifying family \((Y^v_s)_{0 \leq s \leq T}: v \in \mathcal{U}^f\). Then the family is unique (i.e. there is at most one verifying family, up to indistinguishability for each \(Y^v\)) and:

(i) Satisfies \(Y_0 = \sup_{u \in \mathcal{U}} J(u)\) (where \(Y := Y^0\)).

(ii) Defines the optimal control, \(u^* = (\tau^*_1, \ldots, \tau^*_{N^*}; \beta^*_1, \ldots, \beta^*_N)\), for Problem 1, where \((\tau^*_j)_{1 \leq j \leq N^*}\) is a sequence of \(\mathbb{F}\)-stopping times given by

\[
\tau^*_j := \inf \{ s \geq \tau^*_{j-1} : Y^v_{s \tau^*_{j-1}, \ldots, \tau^*_{j-1}; \beta^*_{j-1}} = \sup_{b \in U} \{-c(\tau^*_{1}, \ldots, \tau^*_{j-1}, s; \beta^*_{1}, \ldots, \beta^*_{j-1}, b) + Y^v_{\tau^*_{1}, \ldots, \tau^*_{j-1}, s; \beta^*_{1}, \ldots, \beta^*_{j-1}, b}\} \land T,
\]



(iii) \(N^* = \max\{ j : \tau^*_j < T \}\), with \(\tau^*_0 := 0\).

**Proof.** The proof is divided into three steps where we first, in steps 1 and 2, show that for any \(0 \leq j \leq N^*\), we have

\[
Y^v_{\tau^*_1, \ldots, \tau^*_j; \beta^*_1, \ldots, \beta^*_j} = \text{ess sup}_{\tau \in T_{\tau^*}} \mathbb{E}\left[1_{[\tau = T]} \varphi(\tau^*_1, \ldots, \tau^*_j; \beta^*_1, \ldots, \beta^*_j)\right] + 1_{[\tau < T]} \sup_{b \in U} \{-c(\tau^*_1, \ldots, \tau^*_j, \tau; \beta^*_1, \ldots, \beta^*_j, b) + Y^v_{\tau^*_1, \ldots, \tau^*_j, \tau; \beta^*_1, \ldots, \beta^*_j, b}\}\right] F_s \tag{3.2}
\]

\(\mathbb{P}\)-a.s. for \(s \in [\tau^*_j, \tau^*_{j+1}]\). Then in Step 3 we show that \(u^*\) is the optimal control, establishing (i) and (ii). A straightforward generalization to arbitrary initial conditions \(v \in \mathcal{U}^f\) then gives that

\[
Y^v_s = \text{ess sup}_{u \in \mathcal{U}_s} \mathbb{E}\left[\varphi(v \circ u) - \sum_{j=M}^{M+N} c([v \circ u]_{1,j})\right] F_s, \tag{3.3}
\]

```
with \([v]_{1:j} := (t_1, \ldots, t_j; b_1, \ldots, b_j)\), by which uniqueness follows.

**Step 1** We start by showing that for each \(v \in \mathcal{U}^I\) the recursion (3.1) can be written in terms of a \(\mathbb{F}\)-stopping time and that the inner supremum is attained, \(\mathbb{P}\)-a.s. From (3.1) we note that, by definition, \(Y^v\) is the smallest supermartingale that dominates
\[
X^v := \left(1_{[s=T]} E[\varphi(v)|\mathcal{F}_T] + 1_{[s<T]} \sup_{b \in U} \{-c(v \circ (s, b)) + Y^{v_o(s,b)}_s\} : 0 \leq s \leq T \right).
\] (3.4)

Now, by property [d] in the definition of a verification family (Definition 3.1 and Assumption 2.2 [ii]) we note that \(X^v\) is a càdlàg process of class [D] that is quasi-left upper semi-continuous on \([0, T]\). Furthermore, by Assumption 2.2 [iii], property [d] and quasi-left continuity of the filtration we get that for any sequence \((\gamma_k)_{k \geq 0} \subset T\) such that \(\gamma_k \nearrow T\), \(\mathbb{P}\)-a.s. we have \(\lim_{k \to \infty} X^v_{\gamma_k} \leq X^v_T\), \(\mathbb{P}\)-a.s. By Theorem 2.6 [iii] it thus follows that for any \(\theta \in \mathcal{T}_0\), there is a stopping time \(\tau^\theta \in \mathcal{T}_0\) such that:
\[
Y^v_\theta = E\left[1_{[\tau^\theta=T]}\varphi(v) + 1_{[\tau^\theta<T]} \sup_{b \in U} \{-c(v \circ (\tau^\theta, b)) + Y^{v_o(\tau^\theta,b)}_{\tau^\theta}\}|\mathcal{F}_\theta\right].
\]

Now, by property [c] and Assumption 2.2 [iii] the map \(b \mapsto -c(v \circ (\tau^\theta, b)) + Y^{v_o(\tau^\theta,b)}_{\tau^\theta}\) is u.s.c. outside of a \(\mathbb{P}\)-null set and Corollary 2.10 implies that there is a \(\mathcal{F}_{\tau^\theta}\)-measurable r.v. \(\beta^\theta\) such that
\[
Y^v_\theta = E\left[1_{[\tau^\theta=T]}\varphi(v) + 1_{[\tau^\theta<T]} \{-c(v \circ (\tau^\theta, \beta^\theta)) + Y^{v_o(\tau^\theta,\beta^\theta)}_{\tau^\theta}\}|\mathcal{F}_\theta\right].
\]

**Step 2** We now show that \(Y_0 = J(u^*)\). We start by noting that \(Y\) is the Snell envelope of
\[
\left(1_{[s=T]} E[\varphi(v)|\mathcal{F}_T] + 1_{[s<T]} \sup_{b \in U} \{-c(s, b) + Y^s_{b}\} : 0 \leq s \leq T \right),
\]
where \(\varphi_0 := \varphi(\emptyset)\), and by step 1 we thus have
\[
Y_0 = \sup_{\tau \in \mathcal{T}} E\left[1_{[\tau=T]}\varphi_0 + 1_{[\tau<T]} \sup_{b \in U} \{-c(\tau, b) + Y^\tau_{b}\}\right]
= E\left[1_{[\tau_1^*=T]}\varphi_0 + 1_{[\tau_1^*<T]} \sup_{b \in U} \{-c(\tau_1^*, b) + Y^{\tau_1^*}_{\beta_1^*}\}\right]
= E\left[1_{[\tau_1^*=T]}\varphi_0 + 1_{[\tau_1^*<T]} \{-c(\tau_1^*, \beta_1^*) + Y^{\tau_1^*}_{\beta_1^*}\}\right].
\]

Moving on we pick \(j \in \{1, \ldots, N^*\}\) and note that \((\tau_1^*, \ldots, \tau_j^*; \beta_1^*, \ldots, \beta_j^*) \in \mathcal{U}^I\). But then, by Step 1, we have that
\[
Y^{\tau_1^*,\ldots,\tau_j^*;\beta_1^*,\ldots,\beta_j^*}_{\tau_j^*} = E\left[1_{[\tau_{j+1}=T]}\varphi(\tau_1^*, \ldots, \tau_j^*; \beta_1^*, \ldots, \beta_j^*)
+ 1_{[\tau_{j+1}<T]} \{-c(\tau_1^*, \ldots, \tau_j^*; \beta_1^*, \ldots, \beta_{j+1}^*) + Y^{\tau_1^*,\ldots,\tau_j^*,\beta_1^*,\ldots,\beta_{j+1}^*}_{\tau_{j+1}}\}|\mathcal{F}_{\tau_j^*}\right].
\]

By induction we get that for each \(K \geq 0\),
\[
Y_0 = E\left[1_{[N^* \leq K]}\varphi(\tau_1^*, \ldots, \tau_K^*; \beta_1^*, \ldots, \beta_K^*)
- \sum_{j=1}^{N^* \wedge K} c(\tau_1^*, \ldots, \tau_j^*; \beta_1^*, \ldots, \beta_{j+1}^*)
+ 1_{[N^* > K]} \{-c(\tau_1^*, \ldots, \tau_{K+1}^*; \beta_1^*, \ldots, \beta_{K+1}^*) + Y^{\tau_1^*,\ldots,\tau_{K+1}^*;\beta_1^*,\ldots,\beta_{K+1}^*}_{\tau_{K+1}^*}\}\right].
\]

Now, arguing as in the proof of Proposition 2.5 and using property [b] we find that \(u^* \in \mathcal{U}^I\). Letting \(K \to \infty\) and using dominated convergence we, thus, conclude that \(Y_0 = J(u^*)\).
Step 3 It remains to show that the strategy \( u^* \) is optimal. To do this we pick any other strategy \( \hat{u} := (\hat{\tau}_1, \ldots, \hat{\tau}_N; \hat{\beta}_1, \ldots, \hat{\beta}_N) \in \mathcal{U}^f \). By the definition of \( Y_0 \) in (3.1) we have
\[
Y_0 \geq \mathbb{E} \left[ \mathbb{1}_{[\hat{\tau}_1 \geq T]} \varphi_0 + \mathbb{1}_{[\hat{\tau}_1 < T]} \sup_{b \in \mathcal{U}} \{-c(\hat{\tau}_1, b) + Y^\hat{\tau}_1;\hat{b}\} \right] \\
\geq \mathbb{E} \left[ \mathbb{1}_{[\hat{\tau}_1 \geq T]} \varphi_0 + \mathbb{1}_{[\hat{\tau}_1 < T]} \{-c(\hat{\tau}_1; \hat{\beta}_1) + Y^\hat{\tau}_1;\hat{\beta}_1\} \right]
\]
(with \( \hat{\tau}_{N+1} := \infty \)) but in the same way
\[
Y^\hat{\tau}_1;\hat{\beta}_1 \geq \mathbb{E} \left[ \mathbb{1}_{[\hat{\tau}_2 \geq T]} \varphi(\hat{\tau}_1, \hat{\beta}_1) + \mathbb{1}_{[\hat{\tau}_2 < T]} \{-c(\hat{\tau}_1, \hat{\tau}_2; \hat{\beta}_1, \hat{\beta}_2) + Y^\hat{\tau}_1,\hat{\tau}_2;\hat{\beta}_1,\hat{\beta}_2\} \bigg| \mathcal{F}_{\hat{\tau}_1} \right],
\]
P-a.s. By repeating this argument and using the dominated convergence theorem we find that \( J(u^*) \geq J(\hat{u}) \) which proves that \( u^* \) is in fact optimal. Repeating steps 2 and 3 with \( v \in \mathcal{U}^f \) as initial condition (3.3) follows.

Remark 3.3. To realize why we are required to take the supremum and not the essential supremum in the interior maximization over U in (3.1), consider the case when \( T = 2 \) and \( \mathbb{F} \) is the augmented natural filtration generated by a Brownian motion \( (B_t)_{0 \leq t \leq 2} \) and set
\[
\varphi(v) := (3 - t_1) \mathbb{1}_{[t_1 \geq 1]} \mathbb{1}_{[t_1 = B_1]}
\]
and \( c = 1 \). Then, \( \Psi \) satisfies the requirements in Assumption 2.2 and the optimal control is \( u^* = (1, B_1) \), with \( J(u^*) = 1 \). However, \( Y_1^{(1,0)} = \mathbb{1}_{[b = B_1]} \) and we have
\[
\text{ess sup}_{b \in \mathcal{U}} Y_1^{(1,0)} = 0.
\]

4 Existence

Theorem 3.2 presumes existence of the verification family \( ((Y_s^v)_{0 \leq s \leq T} : v \in \mathcal{U}^f) \). To obtain a satisfactory solution to Problem 1, we thus need to establish that a verification family exists. This is the topic of the present section. We will follow the standard existence proof which goes by applying a Picard iteration (see [7, 13, 17]). We, therefore, define the sequence of families of processes \( ((Y_s^{v,k})_{0 \leq s \leq T} : v \in \mathcal{U}^f)_{k \geq 0} \) as
\[
Y_s^{v,0} := \mathbb{E} \left[ \varphi(v) \bigg| \mathcal{F}_s \right], \quad (4.1)
\]
and
\[
Y_s^{v,k} := \text{ess sup}_{\tau \in \mathcal{T}_s} \mathbb{E} \left[ \mathbb{1}_{[\tau = T]} \varphi(v) + \mathbb{1}_{[\tau < T]} \sup_{b \in \mathcal{U}} \{-c(v \circ (\tau, b)) + Y_{\tau}^{v_o(\tau, b),k-1}\} \bigg| \mathcal{F}_s \right], \quad (4.2)
\]
for \( k \geq 1 \).

Proposition 4.1. Assume that, for each \( v \in \mathcal{U}^f \), \( Y_s^{v,k} \) is well defined for all \( k \geq 0 \). Then, the sequence \( ((Y_s^{v,k})_{0 \leq s \leq T} : v \in \mathcal{U}^f)_{k \geq 0} \) is uniformly bounded in the sense that there is a \( K > 0 \) such that,
\[
\sup_{u \in \mathcal{U}} \mathbb{E} \left[ \sup_{s \in [0,T]} |Y_s^{u,k}|^2 \right] \leq K,
\]
and for each \( v \in \mathcal{D}^f \)
\[
\mathbb{E} \left[ \sup_{s \in [0,T]} \sup_{b \in \mathcal{U}} |Y_s^{v_o(s,b),k}|^2 \right] \leq K,
\]
for all \( k \geq 0 \).
Proof. By the definition of $Y_{v,k}$ we have that for any $v \in \mathcal{U}^f$, 
\[ \mathbb{E}\left[ \varphi(v) \mid \mathcal{F}_s \right] \leq Y_{v,k}^s \leq \text{ess sup}_{u \in \mathcal{U}} \mathbb{E}\left[ \varphi(u) \mid \mathcal{F}_s \right]. \]

Doob’s maximal inequality gives that for any $u \in \mathcal{U}$
\[ \mathbb{E}\left[ \sup_{s \in [0,T]} \mathbb{E}\left[ |\varphi(u)| \mid \mathcal{F}_s \right]^2 \right] \leq C \mathbb{E}\left[ |\varphi(u)|^2 \right]. \]

Taking the supremum over all $u \in \mathcal{U}$ on both sides and using that the right hand side is uniformly bounded by Assumption 2.2 (i.a) the first bound follows.

Concerning the second claim, note that
\[ \mathbb{E}\left[ \sup_{s \in [0,T]} \sup_{b \in \mathcal{U}} |Y_{v_0(s,b)}^s| \right] \leq \sup_{u \in \mathcal{U}} \mathbb{E}\left[ \sup_{s \in [0,T]} \sup_{r \in [0,T]} |\varphi(v \circ (r,b) \circ u)| \right]. \]

Now, arguing as above we find that
\[ \mathbb{E}\left[ \sup_{s \in [0,T]} \sup_{b \in \mathcal{U}} |Y_{v_0(s,b)}^s| \right] \leq C \sup_{u \in \mathcal{U}} \mathbb{E}\left[ \sup_{s \in [0,T]} |\varphi(v \circ (r,b) \circ u)| \right]. \]

where the right hand side is bounded by Assumption 2.2 (b).

The objective in the remainder of this section is to show that the limit family that we get when letting $k \to \infty$ in $((Y_{s,v,k})_{0 \leq s \leq T} : v \in \mathcal{U}^f)_{k \geq 0}$ is a verification family. To obtain this result we will make use of the following induction hypothesis (where $k$ is a non-negative integer):

**H.k.** There is a sequence of families of processes $((Y_{s,v,k'})_{0 \leq s \leq T} : v \in \mathcal{U}^f)_{0 \leq k' \leq k}$ such that for $k' = 0, \ldots, k$ and $v \in \mathcal{U}^f$:

i) The relation (4.1) holds for $k' = 0$ and (4.2) holds for $k' > 0$.

ii) For every $\tau \in \mathcal{T}$, the map $b \mapsto Y_{\tau}^{v_0(r,b),k'}$ is u.s.c., outside of a $\mathbb{P}$-null set.

iii) The process $(\sup_{b \in \mathcal{U}} \{-c(v \circ (t,b)) + Y_{s,v_0(s,b),k'}^s \} : 0 \leq s \leq T)$ has a version which is in in $\mathcal{S}_q^2$.

iv) At each stopping time $\tau \in \mathcal{T}$, the version of $(\sup_{b \in \mathcal{U}} \{-c(v \circ (t,b)) + Y_{s,v_0(s,b),k'}^s \} : 0 \leq s \leq T)$ in $\mathcal{S}_q^2$ agrees with $(\sup_{b \in \mathcal{U}} Y_{s,v_0(s,b)}^s : 0 \leq s \leq T)$, outside of a $\mathbb{P}$-null set.

Remark 4.2. Note that for all $v \in \mathcal{U}^f$, (4.1) only defines the process $Y_{t,v,0}^s$ up to a $\mathbb{P}$-null set for each $t \in [0,T]$ and what we want, as made explicit in (iii), is for $(\sup_{b \in \mathcal{U}} Y_{s,v_0(s,b),0}^s : 0 \leq s \leq T)$ to have a càdlàg version and that the map $b \mapsto Y_{t,v_0(r,b),0}^s$ is u.s.c. outside of a $\mathbb{P}$-null set. On the other hand, whenever (iii) holds for $k - 1$, (4.2) defines $Y_{v,k}$ up to indistinguishability as the Snell envelope of a càdlàg process. However, this still means that $Y_{v_0(s,b),k}^s$ is only defined up to a $\mathbb{P}$-null set for each $s \in [0,T]$ and we conclude that, for our purposes, we have the same type of flexibility for $k > 0$.

Before proceeding to show that the induction hypothesis holds, we extend the set of admissible controls to non-causal sequences by allowing the $\beta_j$ to be $\mathcal{G}$-measurable rather than merely $\mathcal{F}_{\tau_j}$-measurable. We let
\( \mathcal{U}^l \) denote \( \mathcal{U}^l \) after this relaxation. We note that for any \( v \in \mathcal{U}^l \) the processes \( \mathbb{E}[\varphi(v)|\mathcal{F}_s] : 0 \leq s \leq T \) and

\[
\mathbb{E}[ - c(v \circ (s, \beta)) + \varphi(v \circ (s, \beta))|\mathcal{F}_s] : 0 \leq s \leq T),
\]

with \( \beta \) a \( \mathcal{G} \)-measurable r.v., are well defined. Furthermore, under Hypothesis H.k. we can apply an induction argument to show that \( Y^{r,k+1} \) and

\[
\mathbb{E}[ - c(v \circ (s, \beta))|\mathcal{F}_s] + Y^{v_0(s, \beta),k+1} : 0 \leq s \leq T),
\]

are well defined.

**Proposition 4.3.** Hypothesis H.0. (i) and (iii) hold.

**Proof.** To simplify notation we define for all \( \mathcal{G} \)-measurable random variables \( \beta \) taking values in \( U \), the process

\[
\Gamma^t,\beta_s := \mathbb{E}[ - c(v \circ (t, \beta)) + \varphi(v \circ (t, \beta))|\mathcal{F}_s].
\]

We note that \( (\Gamma^s,\beta_s : 0 \leq s \leq T) \) has a version which is \( \mathbb{P} \)-optionally measurable and càdlàg by Assumption 2.2.(ii) and Theorem 2.11 and, furthermore, for each \( \tau \in \mathcal{T} \), the map \( b \mapsto \Gamma^r,b \) is \( \mathbb{P} \)-a.s. u.s.c. by Assumption 2.2(ii).

Let \( (s^l_0, \ldots, s^l_{2^l+1}) \) be an ordering of the elements of \( \Pi^l \) and note that for \( j = 0, \ldots, 2^l \), there is a \( \mathcal{F}^l \)-measurable r.v. \( \beta^l_{j+1} \) such that

\[
\sup_{b \in U} \Gamma^l_{s^j_{j+1},b} = \Gamma^l_{s^j_{j+1},\beta^l_{j+1}}
\]

\( \mathbb{P} \)-a.s. Define the sequence of càdlàg processes \( (\hat{\Gamma}^l)_{l \geq 0} \) as

\[
\hat{\Gamma}^l_r := \sum_{j=0}^{2^l} 1_{[s^l_j, s^l_{j+1})}(r) K^j_\tau + 1_{[r = \tau]} \sup_{b \in U} \Gamma^T,b
\]

where \( (K^j_\tau : 0 \leq s \leq T) \) is a càdlàg version of \( (\Gamma^s,\beta^l_s : 0 \leq s \leq T) \).

Now, we let \( \Gamma^{0,0} := \hat{\Gamma}^0 \) and then recursively define \( \Gamma^{0,l} := \Gamma^{0,l-1} \vee \hat{\Gamma}^l \) for \( l \geq 1 \). Then, \( (\Gamma^{0,l})_{l \geq 0} \) is a non-decreasing sequence of càdlàg processes and

\[
\sup_{b \in U} \Gamma^r,b \geq \Gamma^{0,l},
\]

\( \mathbb{P} \)-a.s., for all \( r \in [0, T] \) and all \( l \geq 0 \). Now, we have

\[
\sup_{b \in U} \Gamma^r,b = \sup_{b \in U} \Gamma^r,b + (\Gamma^r,\beta^l_{j+1} - \Gamma^r,\beta^l_j) + (\Gamma^r,\beta^l_{j+1} - \Gamma^r,\beta^l_{j+1})
\]

\[
= \Gamma^r,\beta^l_{j+1} + \sup_{b \in U} \Gamma^r,b - \Gamma^r,\beta^l_{j+1} + (\Gamma^r,\beta^l_{j+1} - \Gamma^r,\beta^l_{j+1}).
\]

Furthermore, as

\[
\sup_{b \in U} \Gamma^r,b - \Gamma^r,\beta^l_{j+1} \leq \sup_{b \in U} \{\Gamma^r,b - \Gamma^r,\beta^l_{j+1}\} + (\Gamma^r,\beta^l_{j+1} - \Gamma^r,\beta^l_{j+1})
\]

\[
\leq \sup_{b \in U} \{\Gamma^r,b - \Gamma^r,\beta^l_{j+1}\} + \sup_{b \in U} \{\Gamma^r,\beta^l_{j+1} - \Gamma^r,\beta^l_{j+1}\} + (\Gamma^r,\beta^l_{j+1} - \Gamma^r,\beta^l_{j+1})
\]

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we conclude that
\[
\sup_{b \in U} \Gamma^{r,b}_{r_j} \leq \Gamma^{r,b}_{r_j+1} + \sup_{b \in U} \{ \Gamma^{r,b}_{r_j - 1} - \Gamma^{r,b}_{r_j} \} + \sup_{b \in U} \{ \Gamma^{r,b}_{r_j+1} - \Gamma^{r,b}_{r_j+1} \}
\]
\[
+ (\Gamma^{r,b}_{r_j+1} - \Gamma^{r,b}_{r_j+1}) + (\Gamma^{r,b}_{r_j} - \Gamma^{r,b}_{r_j}).
\]
This implies that if we let \( \bar{\Gamma}^{0,l} := \bar{\Gamma}^{0,l} + Z_l \) where
\[
Z_l := \sum_{j=0}^{2^l} 1_{[s_j, s_{j+1})}(r) \left( 2 \mathbb{E} \left[ \sup_{b \in U} |c(v \circ (s_{j+1}^l, b)) - c(v \circ (r, b))| + \sup_{b \in U} |\varphi(v \circ (r, b)) - \varphi(v \circ (s_{j+1}^l, b))| | F_r \right] + \sup_{b \in U} \{ \Gamma^{r,b}_{s_j} - \Gamma^{r,b}_{s_j} \} + (\Gamma^{r,b}_{s_j+1} - \Gamma^{r,b}_{s_j+1}) \right),
\]
then
\[
\sup_{b \in U} \Gamma^{s,b}_s \leq \bar{\Gamma}^{0,l}_s,
\]
\( \mathbb{P} \)-a.s., for all \( s \in [0, T] \) and all \( l \geq 0 \). Furthermore, by once again applying the optional projection theorem in combination with Assumption 2.2 (ii.a) we find that \( \bar{\Gamma}^{0,l} \) has a càdlàg version.

Now, for each \( r \in (0, T) \) the sequence \( \sum_{j=0}^{2^l} 1_{[s_j, s_{j+1})}(r) \) approaches \( r \) from above as \( l \to \infty \). We, thus find that (for \( \chi = \varphi, -c \))
\[
\sum_{j=0}^{2^l} 1_{[s_j, s_{j+1})}(r) \sup_{b \in U} |\chi(v \circ (r, b)) - \chi(v \circ (s_{j+1}^l, b))| \to 0,
\]
\( \mathbb{P} \)-a.s. by Assumption 2.2 (ii.b) and since \( \sum_{j=0}^{2^l} 1_{[s_j, s_{j+1})}(r) \) approaches \( r \) from below, uniform integrability combined with quasi-left continuity of the filtration implies that \( \bar{Z}_l \to 0 \), \( \mathbb{P} \)-a.s. as \( l \to \infty \). In particular we get that for any sequence \( r_j \searrow r \)
\[
\liminf_{r_j \searrow r} \sup_{b \in U} \Gamma^{r,b}_{r_j} \geq \liminf_{r_j \searrow r} \bar{\Gamma}^{0,l}_{r_j} = \bar{\Gamma}^{0,l}_{r},
\]
and
\[
\limsup_{r_j \searrow r} \sup_{b \in U} \Gamma^{r,b}_{r_j} \leq \limsup_{r_j \searrow r} \bar{\Gamma}^{0,l}_{r_j} = \bar{\Gamma}^{0,l}_{r}.
\]
Letting \( l \) tend to infinity we find that
\[
\liminf_{r_j \searrow r} \sup_{b \in U} \Gamma^{v_0(r_j,b),0}_{r_j} = \limsup_{r_j \searrow r} \sup_{b \in U} \Gamma^{v_0(t_i,b),0}_{r_j} = \sup_{b \in U} \Gamma^{v_0(r,b),0}_{r}
\]
which implies that \( \sup_{b \in U} \{ -c(v \circ (s, b)) + Y_{s}^{v_0(s,b),0} : 0 \leq s \leq T \} \) has a càdlàg version that in addition is \( \mathcal{F}_r \)-measurable.

Now, let \( \tau \in \mathcal{T} \) be predictable and suppose that \( (\tau_j)_{j \geq 0} \subset \mathcal{T} \) is an announcing sequence for \( \tau \), so that \( \tau_j \nearrow \tau \), \( \mathbb{P} \)-a.s., and let \( \beta_{\tau} \) be the \( \mathcal{F}_{\tau_j} \)-measurable maximizer for \( b \mapsto \Gamma^{\tau,b}_{\tau_j} \) and denote by \( \beta \) the maximizer of \( b \mapsto \Gamma^{\tau,b}_{\tau} \), then
\[
\limsup_{j \to \infty} \sup_{b \in U} \Gamma^{\tau_j,b}_{\tau_j} = \lim_{j \to \infty} \Gamma^{\tau_j,b}_{\tau_j} = \lim_{j \to \infty} \{ -c(v \circ (\tau_j, \beta_j)) + \mathbb{E}[\varphi(v \circ (\tau_j, \beta_j)) | \mathcal{F}_{\tau_j}] \}
\]
\[
\leq -c(v \circ (\tau, \beta)) + \mathbb{E}[\varphi(v \circ (\tau, \beta)) | \mathcal{F}_{\tau}] = \sup_{b \in U} \Gamma^{\tau,b}_{\tau},
\]
by quasi-left continuity of the filtration, uniform integrability and Assumption 2.2.(ii.b). We conclude that (i)-(iii) hold for \( k = 0 \).

We note that Proposition 4.3 implies the existence of a càdlàg version of \( \left( \sup_{b \in U} \{ -c(v \circ (s, b)) + Y^\varphi_{s,b,0} \} : 0 \leq s \leq T \right) \). That this càdlàg version is useful is shown in the following proposition:

**Proposition 4.4.** For each \( \tau \in \mathcal{T} \) the càdlàg version of \( \left( \sup_{b \in U} \{ -c(v \circ (s, b)) + Y^\varphi_{s,b,0} \} : 0 \leq s \leq T \right) \) satisfies\(^6\):

\[
\sup_{b \in U} \{ -c(v \circ (\tau, b)) + Y^\varphi_{\tau,b,0} \} = \sup_{b \in U} \{ -c(v \circ (\tau, b)) + \mathbb{E}[\varphi(v \circ (\tau, b))|\mathcal{F}_\tau] \}, \tag{4.4}
\]

\( \mathbb{P} \)-a.s., i.e. hypothesis H.0.\(^{[iii]} \) holds.

**Proof.** Let \((\tau_l)_{l \geq 0}\) be a non-increasing sequence of stopping times in \( \mathcal{T}^\Pi \) (the subset of \( \mathcal{T} \) of stopping times taking values in the countable set \( \Pi \)) such that \( \tau_l \downarrow \tau \). We may, for example, set \( \tau_l := \inf \{ s \in \Pi^T : s \geq \tau \} \). Since \( \Pi \) is countable we have

\[
\sup_{b \in U} \{ -c(v \circ (\tau, b)) + Y^\varphi_{\tau,b,0} \} = \sup_{b \in U} \{ -c(v \circ (\tau, b)) + \mathbb{E}[\varphi(v \circ (\tau, b))|\mathcal{F}_{\tau_l}] \}, \tag{4.5}
\]

\( \mathbb{P} \)-a.s. Now, by right-continuity we get that

\[
\lim_{l \to \infty} \sup_{b \in U} \{ -c(v \circ (\tau, b)) + Y^\varphi_{\tau,b,0} \} = \sup_{b \in U} \{ -c(v \circ (\tau, b)) + Y^\varphi_{\tau,b,0} \}.
\]

Furthermore, letting \((\beta_l)_{l \geq 0}\) be a sequence of maximizers for the right-hand side of (4.5) at times \((\tau_l)_{l \geq 0}\) we get

\[
\lim_{l \to \infty} \sup_{b \in U} \{ -c(v \circ (\tau_l, b)) + \mathbb{E}[\varphi(v \circ (\tau_l, b))|\mathcal{F}_{\tau_l}] \} = \lim_{l \to \infty} \{ -c(v \circ (\tau, \beta_l)) + \mathbb{E}[\varphi(v \circ (\tau, \beta_l))|\mathcal{F}_{\tau}] \}
\]

\( \mathbb{P} \)-a.s. where \( \beta \) is the maximizer corresponding to time \( \tau \). On the other hand, we also have

\[
\lim_{l \to \infty} \sup_{b \in U} \{ -c(v \circ (\tau, b)) + \mathbb{E}[\varphi(v \circ (\tau_l, b))|\mathcal{F}_{\tau_l}] \} = \lim_{l \to \infty} \{ -c(v \circ (\tau, \beta_l)) + \mathbb{E}[\varphi(v \circ (\tau_l, \beta_l))|\mathcal{F}_{\tau_l}] \}
\]

which establishes (4.5).

**Proposition 4.5.** Hypothesis H.k holds for all \( k \geq 0 \).

**Proof.** Assume that Hypothesis H.k–1 holds. Applying a reasoning similar to that in the proof of Theorem 3.2 we find that

\[
Y^u_{s,k} = \text{ess sup}_{u \in U^k} \mathbb{E}[\varphi(v \circ u) - \sum_{j=M}^{M+N} c([v \circ u]_{1:j})|\mathcal{F}_s]. \tag{4.6}
\]

\(^{[iii]}\) We abuse notation and let \( \left( \sup_{b \in U} \{ -c(v \circ (s, b)) + Y^\varphi_{s,b,0} \} : 0 \leq s \leq T \right) \) denote its càdlàg version. 15
For each $t \in [0, T]$ and every $u \in U$, we note that the map $U \to L^2(\Omega, \mathcal{F}_t, \mathbb{P})$:
\[
b \mapsto \mathbb{E}\left[\varphi(v \circ (t, b) \circ u) \big| \mathcal{F}_t\right]
\]
is $\mathbb{P}$-a.s. u.s.c. and we conclude that the map $b \mapsto Y_t^{v_0(t,b),k}$ is u.s.c. outside of a $\mathbb{P}$-null set. In particular we have that for each $t \in [0, T]$, there is a $\mathcal{F}_t$-measurable r.v. $\beta_t$ such that
\[
Y_t^{v_0(t,\beta_t),k} = \sup_{b \in U} Y_t^{v_0(t,b),k}.
\]

For $s \in [0, T]$ we note that the process $(Y_r^{v_0(s,\beta_t),k} : 0 \leq r \leq T)$ given by
\[
Y_r^{v_0(s,\beta_t),k} = \operatorname{ess sup}_{\tau \in T_r} \mathbb{E}\left[\mathbbm{1}_{\{|r-T|\}} \varphi(v \circ (s, \beta_t)) + \mathbbm{1}_{\{|r-T|\}} \sup_{\beta_t \in U} \left\{-c(v \circ (s, \beta_t) \circ (\tau, \beta_t')) + Y_r^{v_0(s,\beta_t \circ (\tau,\beta_t'),k-1}\right\} \big| \mathcal{F}_r\right],
\]
is the Snell envelope of a càdlàg process of class $[D]$ and thus itself a càdlàg process by Theorem 2.6. Again we let $(s_1^l, \ldots, s_{2^l+1})$ be an ordering of the elements of $\Pi_T^l$ and define
\[
\tilde{Y}_r^l = \sum_{j=0}^{2^l} \mathbbm{1}_{[s_j^l, s_{j+1}^l)}(r) Y_r^{v_0(s_{j+1}^l, \beta_t),k} + \mathbbm{1}_{\{|r-T|\}} Y_T^{v_0(T,\beta_t),k}.
\]
Then, $\tilde{Y}_r^l$ is a sequence of càdlàg processes and
\[
|Y_r^{v_0(r,\beta_t),k} - \tilde{Y}_r^l| \leq \sum_{j=0}^{2^l} \mathbbm{1}_{[s_j^l, s_{j+1}^l)}(r) \operatorname{ess sup}_{u \in U_k^l} \mathbb{E}\left[|\varphi(v \circ (r, \beta_t) \circ u) - \varphi(v \circ (s_{j+1}^l, \beta_t) \circ u)|\right] + \sum_{j=M}^{M+N+1} |c([v \circ (r, \beta_t) \circ u]_{1:j}) - c([v \circ (s_{j+1}^l, \beta_t) \circ u]_{1:j})||\mathcal{F}_r|.
\]
Letting $\check{Z}_r^l$ denote a càdlàg version of the left hand side, we find that $(\check{Z}_r^l)_{l \geq 0}$ is a sequence of càdlàg processes such that $\check{Z}_r^l \to 0$, $\mathbb{P}$-a.s. as $l \to \infty$. This implies that $(Y_r^0 - \check{Z}_r^0 \vee \cdots \vee \check{Y}_r^l - \check{Z}_r^l)_{l \geq 0}$ and $(\check{Y}_r^0 - \check{Z}_r^0 \wedge \cdots \wedge \check{Y}_r^l - \check{Z}_r^l)_{l \geq 0}$ is a non-decreasing (resp. non-increasing) sequence of càdlàg processes that is dominated (resp. dominates) $(Y_r^{v_0(r,\beta_t),k} : 0 \leq r \leq T)$. Arguing as in the proof of Proposition 4.3 it thus follows that the process $(Y_r^{v_0(r,\beta_t),k} : 0 \leq r \leq T)$ has a càdlàg version.

We now proceed as in the proof of Proposition 4.3 and define
\[
\Gamma_{s,\beta,k}^l := \mathbb{E}\left[-c(v \circ (t, \beta))|\mathcal{F}_s\right] + Y_s^{v_0(t,\beta),k-1}
\]
for all $G$-measurable random variables $\beta$. We then let
\[
\hat{\Gamma}_r^l := \sum_{j=0}^{2^l} \mathbbm{1}_{[s_j^l, s_{j+1}^l)}(r) K_{r}^{j+1,l,k} + \mathbbm{1}_{\{|r-T|\}} \sup_{b \in U} \Gamma_{T,\beta,k}^l,
\]
where $(K_{s}^{j,l,k} : 0 \leq s \leq T)$ is a càdlàg version of $(\Gamma_{s}^{j,l,k} : 0 \leq s \leq T)$ and $\beta_{l+1}^l$ is a $\mathcal{F}_{l+1}^j$-measurable r.v. such that
\[
\sup_{b \in U} \Gamma_{s}^{j+1,l,b,k} = \Gamma_{s}^{j+1,\beta_{l+1}^l,k}.
\]
\[ \mathbb{P}\text{-a.s. for } j = 0, \ldots, 2^l. \] Again, let \( \bar{\Gamma}^l_0 := \Gamma^l_0 \) and then recursively define \( \bar{\Gamma}^l_0 \) as \( \bar{\Gamma}^l_1 := \bar{\Gamma}^l_0 \lor \Gamma^l_1 \) for \( l \geq 1 \). We find that \( (\bar{\Gamma}^l_0)_{l \geq 0} \) is an increasing sequence of càdlàg processes and

\[ \sup_{b \in U} \Gamma^l_{s,b,k} > \Gamma^l_{s} \text{,} \]

\( \mathbb{P}\text{-a.s., for all } s \in [0, T] \). Furthermore, letting

\[ \bar{\bar{\Gamma}}^l_0 := \Gamma^l_0 + Z^l, \]

we find that \( (\bar{\bar{\Gamma}}^l_0)_{l \geq 0} \) is a sequence of càdlàg processes and \( \bar{\bar{\Gamma}}^l_0 \to \bar{\bar{\Gamma}}^l_1 \) \( \mathbb{P}\text{-a.s.} \) for all \( r \in [0, T] \), as \( l \to \infty \). Again we can use the fact that \( (\sup_{b \in U} \bar{\bar{\Gamma}}^l_0 : 0 \leq s \leq T) \) is trapped between two sequences of càdlàg processes whose difference tends to 0, \( \mathbb{P}\text{-a.s.} \) to conclude that \( (\sup_{b \in U} \{ -c(v \circ (s, \beta)) + \bar{\bar{\Gamma}}^l_0(v, s, \beta, k) : 0 \leq s \leq T \}) \) is càdlàg version of \( \bar{\bar{\Gamma}}^l_0 \) is u.s.c., outside of a \( \mathbb{P}\text{-null set.} \)

Furthermore, as \( (\bar{\bar{\Gamma}}^l_0(v, s, \beta, k) : 0 \leq s \leq T) \) is only defined up to indistinguishability (see the discussion in Remark 4.2) for each \( v \in U \) we can chose \( \bar{\bar{\Gamma}}^l_0(v, s, \beta, k) \) in such a way that for all \( \tau \in \mathcal{T} \), the map \( b \mapsto \Gamma^l_{\tau, b,k} \)

\( \mathbb{P}\text{-a.s.} \) and

\[ \lim_{j \to \infty} \mathbb{E} \left[ \varphi(v \circ (\tau_j, \beta_j) \circ u) | \mathcal{F}_{\tau_j} \right] \leq \mathbb{E} \left[ \varphi(v \circ (\tau, \beta) \circ u) | \mathcal{F}_{\tau} \right], \]

for any \( u \in U \), by quasi-left continuity of the filtration, uniform integrability and Assumption 2.2. Taking the supremum over \( u \) it follows that \( \lim_{j \to \infty} \Gamma^l_{\tau, b,k, j} \leq \Gamma^l_{\tau, b,k} \).

Finally, repeating the argument in the proof of Proposition 4.4 and using (4.6) we find that for each \( \tau \in \mathcal{T} \), the càdlàg version of \( (\sup_{b \in U} \{ -c(v \circ (s, b)) + Y^\tau_{s,v}(v, s, b, k) : 0 \leq s \leq T \}) \) satisfies\(^7\)

\[ \sup_{b \in U} \{ -c(v \circ (\tau, b)) + Y^\tau_{s,v}(\tau, b) \} = \sup_{b \in U} \{ -c(v \circ (\tau, b)) + \mathbb{E} \left[ \varphi(v \circ (\tau, b) \circ u) \right] \}

\[ - \sum_{j=0}^{M+N+1} c([v \circ (\tau, b) \circ u]_{1,j}) | \mathcal{F}_{r_j} \}, \quad \text{(4.8)} \]

\( \mathbb{P}\text{-as} \).

We conclude that \( \Box \) hold for \( k \) as well. Applying the induction argument this extends to all \( k \geq 0 \).

\(^7\)We again abuse notation and let \( (\sup_{b \in U} \{ -c(v \circ (s, b)) + Y^\tau_{s,v}(v, s, b, k) : 0 \leq s \leq T \}) \) denote its càdlàg version.
Having established that Hypothesis H.k. holds we move on to investigate the limit family that we get when $k \to \infty$.

**Proposition 4.6.** For each $v \in \mathcal{U}^I$, the limit $\bar{Y}^v := \lim_{k \to \infty} Y^{v,k}$, exists as an increasing pointwise limit, $\mathbb{P}$-a.s.

**Proof.** Since $\mathcal{U}_t^k \subset \mathcal{U}_t^{k+1}$ we have that, $\mathbb{P}$-a.s.,

$$Y^{v,k}_t \leq Y^{v,k+1}_t \leq \esssup_{u \in \mathcal{U}_s} \mathbb{E}\left[|\varphi(v \circ u)|\middle|\mathcal{F}_s\right],$$

where the right hand side is bounded $\mathbb{P}$-a.s. by Proposition 4.1. Hence, the sequence $(Y^{v,k}_t : 0 \leq s \leq T)_{k \geq 0}$ is non-decreasing and $\mathbb{P}$-a.s. bounded, thus, it converges $\mathbb{P}$-a.s. for all $s \in [0,T]$.

**Proposition 4.7.** For each $v \in \mathcal{U}^I$, the convergence is uniform in the sense that (outside of a $\mathbb{P}$-null set):

i) For each $\tau \in \mathcal{T}$ we have $\sup_{b \in U} |Y^{v,(\tau,b),k}_\tau - Y^{v,(\tau,b)}_\tau| \to 0$;

ii) We have $Y^{v,k}_s \to Y^v_s$ and $\sup_{b \in U} \{ -c(v \circ (s,b)) + Y^{v,(s,b),k}_s \} \to \sup_{b \in U} \{ -c(v \circ (s,b)) + Y^{v,(s,b)}_s \}$, both uniformly in $s$,

as $k \to \infty$.

**Proof.** We note that for $p \in (1,2)$, we have

$$\sup_{s \in [0,T]} \sup_{b \in U} \sup_{\tau \in [0,T]} Y^{v,(\tau,b),k}_s \leq \sup_{s \in [0,T]} \sup_{b \in U} \sup_{r \in [0,T]} Y^{v,(r,b),k}_s$$

$$\leq \sup_{s \in [0,T]} \esssup_{u \in \mathcal{U}} \mathbb{E}\left[ \sup_{s \in [0,T]} \sup_{r \in [0,T]} |\varphi(v \circ (r,b) \circ u)|\middle|\mathcal{F}_s\right]$$

$$\leq 1 + \sup_{s \in [0,T]} \esssup_{u \in \mathcal{U}} \mathbb{E}\left[ \sup_{s \in [0,T]} \sup_{r \in [0,T]} |\varphi(v \circ (r,b) \circ u)|^p\middle|\mathcal{F}_s\right] =: K(\omega)$$

for all $k \geq 0$ (where the inequalities hold $\mathbb{P}$-a.s.). Now, arguing as in the proof of Proposition 4.1 we have

$$\mathbb{E}\left[ \sup_{s \in [0,T]} \esssup_{u \in \mathcal{U}} \mathbb{E}\left[ \sup_{s \in [0,T]} \sup_{r \in [0,T]} |\varphi(v \circ (r,\beta) \circ u)|^p\middle|\mathcal{F}_s\right]^{2/p}\right]$$

$$\leq C \sup_{u \in \mathcal{U}} \mathbb{E}\left[ \sup_{r \in [0,T]} |\varphi(v \circ (r,\beta) \circ u)|^2\right] < \infty.$$ 

We thus conclude that there is a $\mathbb{P}$-null set $\mathcal{N}$ such that for each $\omega \in \Omega \setminus \mathcal{N}$ we have $K(\omega) < \infty$.

For $\omega \in \Omega \setminus \mathcal{N}$ (in the remainder of the proof $\mathcal{N}$ denotes a generic $\mathbb{P}$-null set), we thus have for $\tau \in \mathcal{T}$ and $\beta$ a $\mathcal{F}_\tau$-measurable r.v.,

$$-K(\omega) \leq Y^{v,(\tau,\beta),k}_\tau \leq \mathbb{E}\left[|\varphi(v \circ (\tau,\beta) \circ (\tau^k,\ldots,\tau^k_{N_k};\beta^k_1\ldots,\beta^k_{N_k})) - \Delta F_M(N^k)|\middle|\mathcal{F}_\tau\right]$$

$$\leq K(\omega) - \mathbb{E}[\Delta F_M(N^k)|\mathcal{F}_\tau],$$

where $\Delta F_i(j) := \sum_{l=i+1}^{i+j} f(l)$ and $(\tau^k_1,\ldots,\tau^k_{N_k};\beta^k_1\ldots,\beta^k_{N_k}) \in \mathcal{U}^k$ is a control corresponding to the value $Y^{v,(\tau,\beta),k}_\tau$. This implies that for $k' > 0$ we have,

$$\mathbb{P}[N^k > k'|\mathcal{F}_\tau] \leq 2K(\omega)/\Delta F_M(k').$$
Now, for all $0 \leq k' \leq k$ we have,

$$
\hat{Y}_T^{v; \nu_0(\tau, \beta), k, k'} := \mathbb{E}\left[ \varphi(v \circ (\tau, \beta) \circ (\tau^k_1, \ldots, \tau^k_{N^k A_k}; \beta^k_{N^k A_k}) - \sum_{j=1}^{N^k A_k} c(v \circ (\tau, \beta) \circ (\tau^j_1, \ldots, \tau^j_{N^k A_k}; \beta^j_1, \ldots, \beta^j_{N^k A_k})) \big| \mathcal{F}_T \right]
$$

$$\leq Y_T^{v; \nu_0(\tau, \beta), k} - Y_T^{v; \nu_0(\tau, \beta), k'} \leq Y_T^{v; \nu_0(\tau, b), k},$$

where we have introduced the process $\hat{Y}_T^{v; k, k'}$ corresponding to the truncation $(\tau^k_1, \ldots, \tau^k_{N^k A_k}; \beta^k_1, \ldots, \beta^k_{N^k A_k})$ of the optimal control. As the truncation only affects the performance of the controller when $N^k > k'$ we have

$$Y_T^{v; \nu_0(\tau, \beta), k} - Y_T^{v; \nu_0(\tau, \beta), k'} = \mathbb{E}\left[ \mathbb{1}_{[N^k > k']} \left( \varphi(v \circ (\tau, \beta) \circ (\tau^k_1, \ldots, \tau^k_{N^k A_k}; \beta^k_1, \ldots, \beta^k_{N^k A_k}) - \sum_{j=1}^{N^k A_k} c(v \circ (\tau, \beta) \circ (\tau^j_1, \ldots, \tau^j_{N^k A_k}; \beta^j_1, \ldots, \beta^j_{N^k A_k})) \big| \mathcal{F}_T \right) \right]
$$

Applying Hölder’s inequality we get that for $\frac{1}{p} + \frac{1}{q} = 1$, there is thus a $\mathbb{P}$-a.s. bounded $\mathcal{F}$-measurable r.v. $C = C(\omega)$ such that

$$Y_T^{v; \nu_0(\tau, \beta), k} - Y_T^{v; \nu_0(\tau, \beta), k'} < C \Delta F_M(k')^{-1/q},$$

for all $\tau \in \mathcal{T}$ and all $\mathcal{F}_T$-measurable random variables $\beta$. Now, for each $\tau, k, k'$, and each $\epsilon > 0$, there is a $\mathcal{F}_T$-measurable r.v. $\beta^\epsilon$ such that (apply the proof of Corollary 2.10 with $g^\epsilon := g - \epsilon$ instead of $g$),

$$\sup_{b \in U} \left\{ Y_T^{v; \nu_0(\tau, \beta), k} - Y_T^{v; \nu_0(\tau, \beta^\epsilon), k'} \right\} \leq Y_T^{v; \nu_0(\tau, \beta^\epsilon), k} - Y_T^{v; \nu_0(\tau, \beta^\epsilon), k'} + \epsilon$$

and assertion $\square$ follows since $C \Delta F_M(k')^{-1/q} \to 0$, $\mathbb{P}$-a.s. as $k' \to \infty$.

Similarly, we find that for any $v \in U^f$ there is a $\mathbb{P}$-a.s. bounded $\mathcal{F}$-measurable r.v. $C = C(\omega)$ such that

$$Y_s^{v, k} - Y_s^{v, k'} < C \Delta F_M(k')^{-1/q},$$

for all $s \in [0, T]$ outside of a $\mathbb{P}$-null set which gives the first part of $\square$ by right-continuity of the processes $(Y^{v, k})_{k \geq 0}$.

To get the second part we note that for each $s \in [0, T]$,

$$\sup_{b \in U} \left\{ -c(v \circ (s, b)) + Y_s^{v; \nu_0(s, b), k} \right\} - \sup_{b \in U} \left\{ -c(v \circ (s, b)) + Y_s^{v; \nu_0(s, b), k'} \right\} \leq \sup_{b \in U} \left\{ Y_s^{v; \nu_0(s, b), k} - Y_s^{v; \nu_0(s, b), k'} \right\},$$

$\mathbb{P}$-a.s. and the result follows by the above using right-continuity, this time of $(\sup_{b \in U} Y_s^{v; \nu_0(s, b), k}: 0 \leq s \leq T)$ for all $k \geq 0$. 

$\square$
Proposition 4.8. The family \( (\bar{Y}_s^v)_{0 \leq s \leq T} : v \in \mathcal{U}^f \) is a verification family.

Proof. As \( \bar{Y}^v \) is the pointwise limit of an increasing sequence of càdlàg supermartingales it is a càdlàg supermartingale (see p. 86 in [10]). Furthermore, as the sequence \( (\sup_{b \in \mathcal{U}} Y_{s}^{v(\tau, b)} : 0 \leq s \leq T)_{k \geq 0} \) is a sequence of càdlàg functions that converges uniformly outside of a \( \mathbb{P} \)-null set (Proposition 4.7, ii) the limit is a càdlàg process.

We treat each remaining property in the definition of a verification family separately:

a) Applying the convergence result to the right hand side of (4.2) and using the fact that, by Proposition 4.6,
\[
1_{\{s \geq T\}} \varphi(v) + 1_{\{s < T\}} \sup_{b \in \mathcal{U}} \{-c(v \circ (s, b)) + \bar{Y}_s^v(s, b)\}
\]
is a càdlàg process, (iv) of Theorem 2.6 gives
\[
\bar{Y}_s^v = \text{ess sup}_{\tau \in \mathbb{T}_s} \mathbb{E}\left[1_{\{\tau \geq T\}} \varphi(v) + 1_{\{\tau < T\}} \sup_{b \in \mathcal{U}} \{-c(v \circ (\tau, b)) + \bar{Y}_\tau^{v(\tau, b)}\}\Big| \mathcal{F}_s\right].
\]

b) Uniform boundedness was shown in Proposition 4.1.

c) This is immediate from Proposition 4.5 and Proposition 4.7.

d) We know from Propositions 4.5 and 4.7 that \( (\sup_{b \in \mathcal{U}} Y_{s}^{v(\tau, b)} : 0 \leq s \leq T) \) has a càdlàg version and by Proposition 4.1 we have for \( k \geq 0 \),
\[
|\bar{Y}_s^{v(\gamma_j, \beta_j)} - \bar{Y}_s^{v(\gamma, \beta)}| \leq |Y_{s}^{v(\gamma_j, \beta_j)},k - Y_{s}^{v(\gamma, \beta)}(\gamma, \beta),k| + 2C(\omega)\Delta F_M(k)^{-1/q},
\]
\( \mathbb{P} \)-a.s. By Proposition 4.5 the first part on the right hand side tends to zero \( \mathbb{P} \)-a.s. as \( j \to \infty \). Since \( k \) was arbitrary and \( C \) is \( \mathbb{P} \)-a.s. bounded the desired result follows. The fact that
\[
\sup_{b \in \mathcal{U}} \{-c(v \circ (\tau, b)) + \bar{Y}_\tau^{v(\tau, b)}\} = \sup_{b \in \mathcal{U}} \{-c(v \circ (\tau, b)) + \text{ess sup}_{\eta \in \mathbb{T}_s} \mathbb{E}\left[1_{\{\eta = T\}} \varphi(v \circ (\tau, b)) + 1_{\{\eta < T\}} \sup_{b \in \mathcal{U}} \{-c(v \circ (\tau, b) \circ (\eta, b)) + \bar{Y}_\eta^{v(\tau, b)(\eta, b)}\}\Big| \mathcal{F}_s\right]\}
\]
follows similarly from Proposition 4.5 and Proposition 4.7. This finishes the proof. \( \Box \)

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