Exactly solvable discrete time Birth and Death processes

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Abstract

We present 15 explicit examples of discrete time Birth and Death processes which are exactly solvable. They are related to the hypergeometric orthogonal polynomials of Askey scheme having discrete orthogonality measures. Namely, they are the Krawtchouk, three different kinds of $q$-Krawtchouk, (dual, $q$)-Hahn, ($q$)-Racah, Al-Salam-Carlitz II, $q$-Meixner, $q$-Charlier, dual big $q$-Jacobi and dual big $q$-Laguerre polynomials. The birth and death rates are determined by the difference equations governing the polynomials. The stationary distributions are the normalised orthogonality measures of the polynomials. The transition probabilities are neatly expressed by the normalised polynomials and the corresponding eigenvalues. This paper is simply the discrete time versions of the known solutions of the continuous time birth and death processes.

1 Introduction

It is known [1], [2]§6 that all hypergeometric orthogonal polynomials of a discrete variable belonging to Askey scheme [4]–[7] provide exactly solvable continuous time Birth and Death (BD) processes in one dimension. A good part of these polynomials also supply exactly solvable discrete time BD processes as will be shown in this paper. The selection criterion is the boundedness of the birth plus death rate $B(x) + D(x) < \infty$, in which birth rate $B(x)$ and death rate $D(x)$ are identified as the coefficients of the difference equations governing these polynomials.

The birth and death processes, continuous and discrete time, are typical examples of stationary Markov processes and chains with a wide range of applications [8]; demography, queueing theory, inventory models, infections and chemical dynamics, etc. In this paper, however, only the mathematical sides of the BD processes, the problem setting and solution procedures, etc are expanded. The main point of the logic is that the matrices of transition
probabilities of continuous and discrete time BD, the difference equations governing the polynomial, a real symmetric non-negative tri-diagonal matrix equation determining the polynomial \[9\] are all connected by similarity transformations in terms of a diagonal matrix determined by the *stationary probability distribution*, i.e. the orthogonality measure of the polynomial. In most literature on BD processes \[5, 10, 11, 12\], however, the implementation of orthogonal polynomials depended on the three term recurrence relations rather than the difference equations. This is why the present simple solution method have not been noticed so long time.

The present paper is prepared in a plain style so that non-experts can easily understand. It is organised as follows. An elementary introduction of Markov chains and the problem setting of discrete time BD is given in section two. The solution procedures of discrete time BD are developed in parallel with those of continuous time BD in section three. This is the main part of the paper. To an arbitrary continuous time BD with bounded \(B(x) + D(x) < \infty\), a discrete time BD is associated with a free parameter \(t_S\) representing the time scale. As shown in Theorem 3.1 they share common eigenvectors and the eigenvalues are linearly related (3.6), (3.7). A tri-diagonal matrix \(\tilde{\mathcal{H}}\) (3.8) is introduced by \(B(x)\) and \(D(x)\). A positive diagonal matrix \(\Phi\) is constructed by the ratios of \(B(x)\) and \(D(x + 1)\), (3.13), (3.14). A similarity transformation of \(\tilde{\mathcal{H}}\) in terms of \(\Phi\) produces a symmetric and positive-semidefinite tri-diagonal matrix \(\mathcal{H} = \Phi \tilde{\mathcal{H}} \Phi^{-1}\) (3.17). The complete set of eigenvalues and eigenvectors of \(\mathcal{H}\) provide the complete solutions of the continuous and discrete time BD, as their transition probability matrices \(L_{BD}\) and \(L\) are also obtained from \(\mathcal{H}\) by a similarity transformation, in the opposite direction. The solutions of the initial value problem and the transition matrix after time \(t\) (step \(\ell\)) are provided in Theorem 3.2, 3.3. The spectral representations of the transition matrices are presented in Theorem 3.4. When the birth and death rates \(B(x)\) and \(D(x)\) are chosen as the coefficients of the difference equations governing the polynomial (3.40), the corresponding BD processes, continuous and discrete, are exactly solvable as demonstrated in Theorem 3.5, 3.6, 3.7. Various data for exactly solvable BD’s, \(B(x)\), \(D(x)\), the eigenvalues, eigenvectors, etc are presented in section four and five. Section five deals with exactly solvable semi-infinite cases in which the complete eigenvectors consist of two sets of mutually orthogonal polynomials. Section six discusses two exactly solvable finite cases, which are mirror symmetric at the mid point. Section seven is for comments.
2 Markov chain

The subject of the present paper belongs to the simplest category of stationary Markov chain on a one-dimensional integer lattice $\mathcal{X}$, either finite or semi-infinite:

\[ \mathcal{X} = \{0, 1, \ldots, N\} : \text{finite}, \quad \mathcal{X} = \mathbb{Z}_{\geq 0} : \text{semi-infinite}. \]

For analytic treatment we use $x, y, \ldots$ as representing the lattice points in $\mathcal{X}$, $x, y \in \mathcal{X}$. The general problem setting is as follows. Suppose a non-negative matrix $L$ of transition probability is given. Its element

\[ L_{xy} \geq 0, \quad \sum_{x \in \mathcal{X}} L_{xy} = 1, \quad (2.1) \]

is the transition probability from $y$ to $x$. When the system has the probability distribution

\[ \mathcal{P}(x; \ell) \geq 0, \quad \sum_{x \in \mathcal{X}} \mathcal{P}(x; \ell) = 1, \]

at $\ell$-th step, the next step distribution is given by

\[ \mathcal{P}(x; \ell + 1) = \sum_{y \in \mathcal{X}} L_{xy} \mathcal{P}(y; \ell). \quad (2.2) \]

The above condition (2.1) ensures the conservation of probability,

\[ \sum_{x \in \mathcal{X}} \mathcal{P}(x; \ell + 1) = \sum_{y \in \mathcal{X}} \sum_{x \in \mathcal{X}} L_{xy} \mathcal{P}(y; \ell) = \sum_{y \in \mathcal{X}} \mathcal{P}(y; \ell) = 1. \]

Another immediate consequence of (2.1) and Perron-Frobenius theorem applied to $L$ is that its spectrum is bounded by 1 and $-1$:

\[ -1 \leq \text{Eigenvalues}(L) \leq 1. \quad (2.3) \]

This can be easily seen by considering the normalised eigenvector $v_M$ of $L$ corresponding to the maximal eigenvalue $\kappa_M$:

\[ \sum_{y \in \mathcal{X}} L_{xy} v_M(y) = \kappa_M v_M(x), \quad \sum_{y \in \mathcal{X}} |v_M(y)| = 1. \]

Since we can always choose $v_M(y) \geq 0$ by Perron-Frobenius theorem, $v_M(y)$ is a probability distribution, $\sum_{y \in \mathcal{X}} v_M(y) = 1$. This means that $\kappa_M = 1$ since $\kappa_M v_M(x)$ is also a probability
distribution, as shown above. The lower bound $-1$ in (2.3) is derived by applying Perron-Frobenius theorem to $L^2$. Other eigenvectors of $L$, having sign changes, can never constitute a probability distribution on their own.

Among many problems, the following three are most basic.

• **Initial value problem.** Given an initial probability distribution

$$
\mathcal{P}(x; 0) \geq 0, \quad \sum_{x \in \mathcal{X}} \mathcal{P}(x; 0) = 1,
$$

calculate the distribution after $\ell$ steps,

$$
\mathcal{P}(x; \ell) = \sum_{y \in \mathcal{X}} (L^\ell)_{xy} \mathcal{P}(y; 0).
$$

• **$\ell$ step transition probability.** Starting from the initial distribution concentrated at $y$, $\mathcal{P}(x; 0) = \delta_{xy}$, derive the explicit form of $\ell$ step transition probability from $y$ to $x$

$$
\mathcal{P}(x, y; \ell) = (L^\ell)_{xy}.
$$

• **Spectral representation of $L$ in terms of the eigenvalues and the eigenvectors.**

Hereafter let us restrict the matrix $L$ to be *tri-diagonal*,

$$
L_{xy} = 0 \quad \text{if } |x - y| \geq 2,
$$

a *non-negative Jacobi matrix*. The tri-diagonal restriction means that, at each step, transitions are restricted within the nearest neighbour lattice points. Such Markov chains are usually called generalised random walks. Let us specify the tri-diagonal transition matrix $L$ by using the language of random walk. This is achieved by choosing two positive functions of $x$. A walker at point $x$ advances to $x + 1$ with probability $\tilde{B}(x)$ and he retreats to $x - 1$ with probability $\tilde{D}(x)$ and stays at $x$ with probability $1 - \tilde{B}(x) - \tilde{D}(x)$, i.e.

$$
L_{x+1x} = \tilde{B}(x), \quad L_{x-1x} = \tilde{D}(x), \quad L_{xx} = 1 - \tilde{B}(x) - \tilde{D}(x),
$$

$$
0 < \tilde{B}(x) < 1, \quad 0 < \tilde{D}(x) < 1, \quad 0 < \tilde{B}(x) + \tilde{D}(x) < 1,
$$

together with the boundary condition(s)

$$
\tilde{D}(0) = 0, \quad \tilde{B}(N) = 0 : \text{ (only for finite cases)}.
$$
The matrix $L$ looks as follows:

$$
L = \begin{pmatrix}
1 - \bar{B}(0) & \bar{D}(1) & 0 & \cdots & \cdots & 0 \\
\bar{B}(0) & 1 - \bar{B}(1) - \bar{D}(1) & \bar{D}(2) & 0 & \cdots & \cdots \\
0 & \bar{B}(1) & 1 - \bar{B}(2) - \bar{D}(2) & \bar{D}(3) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\
0 & \cdots & \cdots & \bar{B}(N-2) & 1 - \bar{B}(N-1) - \bar{D}(N-1) & \bar{D}(N) \\
0 & \cdots & \cdots & 0 & \bar{B}(N-1) & 1 - \bar{D}(N) \\
\end{pmatrix}.
$$

The equation (2.2) connecting the $\ell$-th step distribution and $\ell + 1$-th step distribution reads for the inner points $1 \leq x \leq N - 1$,

$$
P(x; \ell + 1) = (1 - \bar{B}(x) - \bar{D}(x))P(x; \ell) + \bar{B}(x-1)P(x-1; \ell) + \bar{D}(x+1)P(x+1; \ell), \quad (2.10)
$$

and for the endpoint(s)

$$
P(0; \ell + 1) = (1 - \bar{B}(0))P(0; \ell) + \bar{D}(1)P(1; \ell), \quad (2.11)
$$

$$
P(N; \ell + 1) = \bar{B}(N-1)P(N-1; \ell) + (1 - \bar{D}(N))P(N; \ell). \quad (2.12)
$$

### 3 Discrete time Birth and Death process

Probably it is now clear that the above generalised random walk (2.7)–(2.12) can also be called discrete time Birth and Death (BD) process. They are very closely related to the continuous time Birth and Death process. For comparison, let us review it here. Let $P(x; t)$ be the probability distribution over $\mathcal{X}$ at time $t$. Let us denote the birth rate at population $x$ by $B(x) > 0$ and the death rate by $D(x) > 0$. The time evolution of the probability distribution is governed by the following differential equation:

$$
\frac{\partial}{\partial t}P(x; t) = (L_{BD}P)(x; t) = \sum_{y \in \mathcal{X}} L_{BD_{xy}}P(y; t), \quad P(x; t) \geq 0, \quad \sum_{x \in \mathcal{X}} P(x; t) = 1, \quad (3.1)
$$

$$
= -(B(x) + D(x))P(x; t) + B(x-1)P(x-1; t) + D(x+1)P(x+1; t), \quad (3.2)
$$

with the boundary condition(s)

$$
D(0) = 0, \quad B(N) = 0: \quad \text{(only for a finite case)}, \quad (3.3)
$$
which is called reflecting boundary condition. Here the matrix $L_{BD}$ is also tri-diagonal

$$
L_{BDx+1} = B(x), \quad L_{BDx-1} = D(x), \quad L_{BDx} = -B(x) - D(x),
$$

$$
L_{BDx, y} = 0, \quad |x - y| \geq 2, \tag{3.4}
$$

$$
L_{BD} =
\begin{pmatrix}
-B(0) & D(1) & 0 & \cdots & \cdots & 0 \\
B(0) & -B(1) - D(1) & D(2) & 0 & \cdots & \\
0 & B(1) & -B(2) - D(2) & D(3) & \cdots & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \\
\vdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & B(N-2) & -B(N-1) - D(N-1) & D(N) \\
0 & \cdots & \cdots & 0 & B(N-1) & -D(N)
\end{pmatrix}
$$

satisfying the condition

$$
\sum_{x \in \mathcal{X}} L_{BDx, y} = 0. \tag{3.5}
$$

This ensures the conservation of probability, that is, the condition $\sum_{x \in \mathcal{X}} p(x; t) = 1$ is preserved by the time evolution [3.1].

By introducing a parameter $t_S$ specifying the time spacing, a discrete time BD process is obtained from a continuous time BD process as shown in the following

**Theorem 3.1** For each continuous BD process with bounded $B(x) + D(x)$, a discrete time BD process is defined with one free positive parameter $t_S$ as follows,

$$
\bar{B}(x) \overset{\text{def}}{=} t_S B(x), \quad \bar{D}(x) \overset{\text{def}}{=} t_S D(x), \quad t_S \cdot \max(B(x) + D(x)) < 1, \tag{3.6}
$$

$$
\Rightarrow \quad L = I_d + t_S \cdot L_{BD}, \quad I_d: \text{Identity matrix}. \tag{3.7}
$$

If the continuous time BD is solved, the corresponding discrete time BD is solved, and vice versa, as the eigenvalues are related as

$$
\text{Eigenvalue of } L = 1 + t_S \cdot (\text{Eigenvalue of } (L_{BD})),
$$

and the corresponding eigenvectors are common.

The bigger $t_S$, the bigger is the time interval of the corresponding discrete time BD process. Obviously, $t_S$ has an upper limit given by (3.6). Later it will be shown that the spectrum of $L_{BD}$ is negative semi-definite [3.29].
Let us proceed to solve the continuous time BD (3.1)–(3.3) in the general setting, i.e. \(B(x)\) and \(D(x)\) are arbitrary positive functions restricted only by the boundary condition(s) (3.3). The special cases related to the hypergeometric orthogonal polynomials [1] will be discussed later. Let us introduce a tri-diagonal matrix \(\tilde{\mathcal{H}}\) on \(X\) in terms of the birth and death rates \(B(x)\) and \(D(x)\),

\[
\tilde{\mathcal{H}} = \begin{pmatrix}
B(0) & -B(0) & 0 & \cdots & \cdots & 0 \\
-D(1) & B(1) + D(1) & -B(1) & \cdots & \cdots & \vdots \\
0 & -D(2) & B(2) + D(2) & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & -B(N-1) + D(N-1) \\
0 & \cdots & \cdots & \cdots & -D(N) & D(N)
\end{pmatrix},
\]

and consider its eigenvalue problem

\[
(\tilde{\mathcal{H}}\tilde{\mathcal{P}}_n(x)) = \sum_{y \in X} \tilde{\mathcal{H}}_{xy}\tilde{\mathcal{P}}_n(y) = \mathcal{E}(n)\tilde{\mathcal{P}}_n(x), \ x \in \mathcal{X}, \ n \in \mathcal{X}. \quad (3.9)
\]

The equation reads explicitly as

\[
B(x) \left( \tilde{P}_n(x) - \tilde{P}_n(x + 1) \right) + D(x) \left( \tilde{P}_n(x) - \tilde{P}_n(x - 1) \right) = \mathcal{E}(n)\tilde{P}_n(x), \ n \in \mathcal{X}. \quad (3.10)
\]

Here \(\tilde{P}_n(x)\) and \(\mathcal{E}(n)\) are to be determined as an eigenvector and the corresponding eigenvalue of \(\tilde{\mathcal{H}}\). It is well known that the top component of an eigenvector of a tri-diagonal matrix is non-vanishing. We adopt the following universal normalisation of the eigenvectors \(\{\tilde{P}_n(x)\}\),

\[
\tilde{P}_n(0) = 1, \ n \in \mathcal{X}. \quad (3.11)
\]

Then it is obvious

\[
\sum_{y \in \mathcal{X}} \tilde{\mathcal{H}}_{xy} = 0, \ \tilde{P}_0(x) \overset{\text{def}}{=} 1, \ (\forall x \in \mathcal{X}), \ \implies \sum_{y \in \mathcal{X}} \tilde{\mathcal{H}}_{xy}\tilde{P}_0(y) = 0, \quad (3.12)
\]

namely, a constant vector of identical components is the eigenvector of \(\tilde{\mathcal{H}}\) of vanishing eigenvalue \(\mathcal{E}(0) = 0\). This is also obvious from the difference equation (3.10). The matrix \(\tilde{\mathcal{H}}\) is related to a real symmetric tri-diagonal matrix \(\mathcal{H}\) by a similarity transformation. Let us
introduce a positive function $\phi_0(x)$ on $X$ and a diagonal matrix $\Phi$ consisting of $\phi_0(x)$ by the ratios of $B(x)$ and $D(x + 1)$,

$$\phi_0(0) \overset{\text{def}}{=} 1, \quad \phi_0(x) \overset{\text{def}}{=} \sqrt{\frac{\prod_{y=0}^{x-1} B(y)}{D(y+1)}} \iff \frac{\phi_0(x+1)}{\phi_0(x)} = \frac{\sqrt{B(x)}}{\sqrt{D(x+1)}}, \quad x \in X. \quad (3.13)$$

$$\Phi_{xx} = \phi_0(x), \quad \Phi_{xy} = 0, \quad x \neq y. \quad (3.14)$$

For semi-infinite cases, $B(x)$ and $D(x)$ must be restricted so that $\phi_0(x)$ is square summable,

$$\sum_{x \in X} \phi_0(x)^2 < \infty. \quad (3.15)$$

Let us define

$$\phi_n(x) \overset{\text{def}}{=} \phi_0(x) \tilde{P}_n(x), \quad n \in X, \quad (3.16)$$

which constitutes the eigenvector of the real symmetric matrix $H$ defines as follows,

$$H \overset{\text{def}}{=} \Phi \tilde{H} \Phi^{-1} \iff \tilde{H} = \Phi^{-1} H \Phi \iff H_{xy} = \phi_0(x) \tilde{H}_{xy} \phi_0(y)^{-1}, \quad (3.17)$$

$$(H\phi_n)(x) = \sum_{y \in X} H_{xy} \phi_n(y) = \sum_{y \in X} \phi_0(x) \tilde{H}_{xy} \tilde{P}_n(y) = \mathcal{E}(n) \phi_n(x), \quad n \in X, \quad (3.18)$$

$$H_{xx+1} = -\sqrt{B(x)D(x+1)}, \quad H_{x-1x} = -\sqrt{B(x-1)D(x)}, \quad H_{xx} = B(x) + D(x),$$

$$H_{xy} = 0, \quad |x - y| \geq 2. \quad (3.19)$$

$$H = \begin{pmatrix}
B(0) & -\sqrt{B(0)D(1)} & 0 & \cdots & \cdots & \cdots & 0 \\
-\sqrt{B(0)D(1)} & B(1) + D(1) & -\sqrt{B(1)D(2)} & \cdots & \cdots & \cdots & \vdots \\
0 & -\sqrt{B(1)D(2)} & B(2) + D(2) & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & B(N-1) + D(N-1) & -\sqrt{B(N-1)D(N)} & D(N) \\
0 & \cdots & \cdots & \cdots & 0 & -\sqrt{B(N-1)D(N)} & D(N)
\end{pmatrix}. \quad (3.20)$$

The real symmetry and the positive semi-definiteness of $H$ can be seen clearly by the following factorisation in terms of an upper triangular matrix $A$,

$$H = tA A \Rightarrow H = t^* H, \quad A_{xx} = \sqrt{B(x)}, \quad A_{xx+1} = -\sqrt{D(x+1)}, \quad A_{xy} = 0, \quad \text{otherwise}. \quad (3.21)$$
in which $\mathcal{A}$ is the transposed matrix of $\mathcal{A}$. This guarantees the reality and non-negativeness of the eigenvalues

$$\mathcal{E}(n) \geq 0, \quad n \in \mathcal{X}, \quad (3.22)$$

and the orthogonality of the eigenvectors $\{\phi_n(x)\}$ of $\mathcal{H}$, since the simpleness of the eigenvalues is due to its tri-diagonality. It should be stressed that $\phi_0(x)$ is the zero mode (eigenvector) of $\mathcal{A}$ and $\mathcal{H}$,

$$0 = (\mathcal{A}\phi_0)(x) = \sqrt{B(x)}\phi_0(x) - \sqrt{D(x+1)}\phi_0(x+1) \Rightarrow (\mathcal{H}\phi_0)(x) = 0, \quad (3.23)$$

and $\phi_0(x)^2$ provides the orthogonality measure of the eigenvectors $\{\tilde{P}_n(x)\}$ of $\tilde{\mathcal{H}}$,

$$(\phi_n, \phi_m) \overset{\text{def}}{=} \sum_{x \in \mathcal{X}} \phi_n(x)\phi_m(x) = \sum_{x \in \mathcal{X}} \phi_0(x)^2\tilde{P}_n(x)\tilde{P}_m(x) = \frac{1}{d_n^2}\delta_{nm}, \quad n, m \in \mathcal{X}, \quad (3.24)$$

in which the normalisation constants $\{d_n > 0\}$ are calculated after all the eigenvectors $\{\tilde{P}_n(x)\}$ are known. It should be stressed that $\phi_0(x)$ and $\tilde{P}_n(x)$ are uniquely specified by the normalisation condition (3.11), (3.13)

$$\phi_0(0) = 1 = \tilde{P}_n(0), \quad n \in \mathcal{X}.$$  

Let us define orthonormal vectors $\{\hat{\phi}_n(x)\}$

$$\hat{\phi}_n(x) \overset{\text{def}}{=} d_n\phi_n(x) = d_n\phi_0(x)\tilde{P}_n(x), \quad (\hat{\phi}_n, \hat{\phi}_m) = \delta_{nm}, \quad n, m \in \mathcal{X}, \quad (3.25)$$

and the square of the normalised zero mode

$$\pi(x) \overset{\text{def}}{=} \hat{\phi}_0(x)^2 = d_0^2\phi_0(x)^2 = d_0^2 \prod_{y=0}^{x-1} \frac{B(y)}{D(y+1)}, \quad (3.26)$$

$$\sum_{x \in \mathcal{X}} \pi(x) = 1 \Leftarrow \frac{1}{d_0^2} = \sum_{x \in \mathcal{X}} \prod_{y=0}^{x-1} \frac{B(y)}{D(y+1)} = \sum_{x \in \mathcal{X}} \prod_{y=0}^{x-1} \frac{\tilde{B}(y)}{\tilde{D}(y+1)}, \quad (3.27)$$

which will turn out to be the stationary distribution.

With these preparations, let us return to $L_{BD}$ (3.4). It is now clear that $L_{BD}$ is also related to $\mathcal{H}$ by a similarity transformation in terms of $\Phi$, but in the opposite direction to $\tilde{\mathcal{H}}$ together with a negative sign,

$$L_{BD} = -\Phi \mathcal{H} \Phi^{-1}, \quad (3.28)$$
\[ L_{BD_{x+1}} = -H(x) - D(x), \]
\[ L_{BD_{x-1}} = -\phi_0(x + 1)H_{x+1}\phi_0(x)^{-1} = \phi_0(x + 1)\sqrt{B(x)D(x+1)}\phi_0(x)^{-1} = B(x), \]
\[ L_{BD_{x-1}} = -\phi_0(x - 1)H_{x-1}\phi_0(x)^{-1} = \phi_0(x - 1)\sqrt{B(x-1)D(x)}\phi_0(x)^{-1} = D(x), \]
\[ (L_{BD}\hat{\phi}_n(x)) = -E(n)\hat{\phi}_0(x)\hat{\phi}_n(x), \quad n \in \mathcal{X}. \tag{3.29} \]

Thus we arrive at the solutions of the general continuous time BD by the following

**Theorem 3.2** If we obtain the complete set of eigensystem of the matrix \( \tilde{H} \) (3.8)–(3.9), the solution of the initial value problem of the continuous time BD (3.1)–(3.4) is given by

\[ P(x; t) = \hat{\phi}_0(x) \sum_{n \in \mathcal{X}} c_n e^{-E(n)t} \hat{\phi}_n(x), \tag{3.30} \]

in which \( \{c_n\} \) are determined as the expansion coefficients of the initial distribution \( P(x; 0) \),

\[ P(x; 0) = \hat{\phi}_0(x) \sum_{n \in \mathcal{X}} c_n \hat{\phi}_n(x) \Rightarrow c_0 = 1, \quad c_n = \sum_{x \in \mathcal{X}} \hat{\phi}_n(x)\hat{\phi}_0(x)^{-1}P(x; 0), \quad n = 1, \ldots. \tag{3.31} \]

The transition matrix from \( y \) to \( x \) after time \( t \) is

\[ P(x, y; t) = \hat{\phi}_0(x)\hat{\phi}_0(y)^{-1} \sum_{n \in \mathcal{X}} e^{-E(n)t} \hat{\phi}_n(x)\hat{\phi}_n(y). \tag{3.32} \]

The approach to the stationary distribution \( \pi(x) \) (3.26) is guaranteed by the positivity of \( E(n) > 0 \) for the non-zero modes,

\[ \lim_{t \to \infty} P(x; t) = \pi(x), \quad \lim_{t \to \infty} P(x, y; t) = \pi(x). \tag{3.33} \]

Based on the relationship between the discrete and continuous BD’s **Theorem 3.1** (3.6)–(3.7) we arrive at the solutions of the general discrete time BD by the following

**Theorem 3.3** The complete set of eigensystem of the matrix \( \tilde{H} \) (3.9) provides the complete set of eigensystem of the matrix \( L \) (2.7) for the discrete time BD with the identification (3.6)

\[ (L\hat{\phi}_n)(x) = \kappa(n)\hat{\phi}_0(x)\hat{\phi}_n(x), \quad \kappa(n) = 1 - t_s \cdot E(n), \quad n \in \mathcal{X}. \tag{3.34} \]

The solution of the initial value problem of the discrete time BD (2.7)–(2.9) after \( \ell \) steps is given by

\[ P(x; \ell) = \hat{\phi}_0(x) \sum_{n \in \mathcal{X}} c_n \kappa(n)^\ell \hat{\phi}_n(x), \tag{3.35} \]
in which \( \{c_n\} \) are given in (3.31). The \( \ell \) step transition matrix from \( y \) to \( x \) is
\[
P(x, y; \ell) = \hat{\phi}_0(x)\hat{\phi}_0(y)^{-1} \sum_{n \in \mathcal{X}} \kappa(n)^\ell \hat{\phi}_n(x)\hat{\phi}_n(y).
\] (3.36)

The approach to the stationary distribution is about the same as (3.33). It should be stressed that for both continuous and discrete time BD’s, the stationary distribution \( \pi(x) \) is the same and it is determined by the input \( B(x), D(x) \) and \( \tilde{B}(x), \tilde{D}(x) \) only without solving the eigenvalue problem of \( \tilde{H} \) (3.9).

**Theorem 3.4** The complete set of eigensystem of the matrix \( \tilde{H} \) (3.9) provides the spectral representation of the symmetric matrix \( H \) (3.17),
\[
H_{xy} = \sum_{n \in \mathcal{X}} \mathcal{E}(n)\hat{\phi}_n(x)\hat{\phi}_n(y).
\] (3.37)

This in turn supplies the spectral representations of \( L_{BD} \) and \( L \) through (3.28) and (3.7),
\[
L_{BD_{xy}} = -\hat{\phi}_0(x) \sum_{n \in \mathcal{X}} \mathcal{E}(n)\hat{\phi}_n(x)\hat{\phi}_n(y)\hat{\phi}_0(y)^{-1} = -\pi(x) \sum_{n \in \mathcal{X}} \mathcal{E}(n)(d_n^2/d_0^2)\tilde{P}_n(x)\tilde{P}_n(y),
\] (3.38)
\[
L_{xy} = \hat{\phi}_0(x) \sum_{n \in \mathcal{X}} \kappa(n)\hat{\phi}_n(x)\hat{\phi}_n(y)\hat{\phi}_0(y)^{-1} = \pi(x) \sum_{n \in \mathcal{X}} \kappa(n)(d_n^2/d_0^2)\tilde{P}_n(x)\tilde{P}_n(y),
\] (3.39)
in which \( \pi(x) \) (3.26) is the stationary distribution.

As for the solutions of the above continuous time Birth and Death process (3.1)–(3.5), we have the following

**Theorem 3.5** The above continuous time Birth and Death process (3.1)–(3.5) is exactly solvable when \( B(x) \) and \( D(x) \) are chosen to be the coefficient functions of certain difference equations [7]
\[
B(x) \left( \tilde{P}_n(x) - \tilde{P}_n(x + 1) \right) + D(x) \left( \tilde{P}_n(x) - \tilde{P}_n(x - 1) \right) = \mathcal{E}(n)\tilde{P}_n(x), \quad n \in \mathcal{X},
\] (3.40)
\[
\iff \sum_{y \in \mathcal{X}} \tilde{H}_{x,y} \tilde{P}_n(y) = \mathcal{E}(n)\tilde{P}_n(x), \quad n \in \mathcal{X},
\]

which determine hypergeometric orthogonal polynomials \( \{\tilde{P}_n(x)\} \) with discrete orthogonality measures belonging to Askey scheme. In this case all the eigenvalues of \( L_{BD} \) and the corresponding eigenvectors are explicitly known. The eigenvalues of \( L_{BD} \) are the same as those in (3.40) with a minus sign \( \{-\mathcal{E}(n)\} n = 0, 1 \ldots \), and the eigenvectors are proportional
to \{\tilde{P}_n(x)\}$ of (3.40), as explicitly given in (3.29). \{\tilde{P}_n(x)\} are hypergeometric orthogonal polynomial

\[
\tilde{P}_n(x) = P_n(\eta(x)), \quad n \in \mathcal{X},
\]

in a certain sinusoidal coordinate \(\eta(x)\), which takes the following five types [9]. They are linear or quadratic in \(x\) and linear or ‘quadratic’ in \(q^{\pm x}\), with \(0 < q < 1\),

\[
\eta(x) : \quad x, \quad x(x + d), \quad 1 - q^x, \quad q^{-x} - 1, \quad (q^{-x} - 1)(1 - dq^x); \quad \eta(0) = 0.
\]

It should be stressed that, except for the cases of \(\eta(x) = x\), \(\tilde{P}_n(x)\) is not a degree \(n\) polynomial in \(x\).

**Theorem 3.6** The discrete time BD (2.7)–(2.9) is exactly solvable if \(\tilde{B}(x)\) and \(\tilde{D}(x)\) are related to \(B(x)\) and \(D(x)\) of an exactly solvable continuous time BD by the relation (3.6). The general forms of the solutions of the initial value problem and the transition matrix will be presented in Theorem 3.7 for both continuous and discrete time BD processes.

**Remark** Theorem 3.5 was proven in [1] for the continuous time BD processes related to 16 different polynomials. Some of them have unbounded \(B(x) + D(x)\). The continuous time BD processes related to the polynomials having Jackson integral type measures are defined on a direct sum of two semi-infinite integer lattices, \(\mathcal{X} = \mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0}\), and they require a different formalism. The big \(q\)-Jacobi is the typical example. The solutions of those BD processes are given in [2]. Those polynomials have no corresponding discrete time BD processes as they all have unbounded \(B(x) + D(x)\). However, the dual polynomials of some of them, e.g. dual big \(q\)-Jacobi and dual big \(q\)-Laguerre have the above continuous time BD processes (3.1)–(3.4), whose solutions are also provided in [2]. The solutions of the discrete time BD processes corresponding to dual big \(q\)-Jacobi and dual big \(q\)-Laguerre are given in §5.3 and §5.4. The solutions of the continuous time BD processes related with \(q\)-Meixner and \(q\)-Charlier reported in [1] were flawed due to the lack of completeness of those polynomials listed in the literature [6, 7]. The complete solutions are given in [2]. The solutions of the discrete time BD processes corresponding to \(q\)-Meixner and \(q\)-Charlier are listed in §5.1 and §5.2.

**Theorem 3.7** For the exactly solvable continuous and discrete time BD’s stated in Theorem 3.5, 3.6, the formulas of the solutions for the initial value problem, the transition
matrix and the spectral representations are the same as those given in Theorem 3.2, 3.3, 3.4, so far as the polynomials determined by the difference equations (3.40) form a complete set.

Remark The examples presented in the next section §4 belong to this category. All quantities appearing in the formulas in Theorem 3.2, 3.3, 3.4 are explicitly known and reported in the subsections having the names of the corresponding polynomials. In contrast, the solutions of the exactly solvable BD’s presented in section 5 require another set of polynomials on top of those determined by $B(x)$ and $D(x)$ (3.40) for completeness. The extra set of polynomials is also associated with another BD process with its birth and death probabilities $B^(-(x))$ and $D^-(x)$, which are related to the original $B(x)$ and $D(x)$ by certain parameter transformations (involutions). The explicit formulas for these two discrete time BD’s are given in Theorem 5.1, 5.2, 5.3, 5.4 in the subsections having the names of the polynomials. The formulas for the corresponding continuous time BD’s are not listed since they are already reported in [2] §6.A.

4 Explicit Examples I

Here we present the data for exactly solvable discrete time BD processes, for which $B(x) + D(x)$ is bounded and the corresponding polynomials are complete. The ranges of parameters in the birth and death rates are restricted by the positivity of $B(x)$ and $D(x)$. We list a representative one only. For more general information of the polynomials, we refer to [9] and [6]. The format for the normalisation constant $d^2_n$ consists of two parts separated by a $\times$ symbol: $d^2_n = (d^2_n/d^2_0) \times d^2_0$. The second part $d^2_0$ satisfies the relation $\sum_x \phi_0(x)^2 = 1/d^2_0$.

Throughout sections 4 and 5 the parameter $q$ is $0 < q < 1$.

4.1 Krawtchouk

The case of linear birth and death rates is a very well-known example (the Ehrenfest model) of an exactly solvable birth and death processes [8, 11]:

\begin{align*}
B(x) &= p(N-x), \quad D(x) = (1-p)x, \quad 0 < p < 1, \\
\mathcal{E}(n) &= n, \quad \eta(x) = x, \\
\phi_0(x)^2 &= \frac{N!}{x!(N-x)!} \left( \frac{p}{1-p} \right)^x, \quad d^2_n = \frac{N!}{n!(N-n)!} \left( \frac{p}{1-p} \right)^n \times (1-p)^N.
\end{align*}

\[ (4.1) \quad (4.2) \quad (4.3) \]
\[ \tilde{P}_n(x) = P_n(\eta(x)) = 2F_1\left(\begin{array}{c} -n, -x \\ -N \end{array} \right) \mid p^{-1}. \] (4.4)

The stationary probability \( \pi(x) = \phi_0(x)^2 d_0^2 = \binom{N}{x} p^x (1 - p)^{N-x} \) is the binomial distribution.

### 4.2 Hahn

This is a well-known example of quadratic (in \( x \)) birth and death rates,

\[ B(x) = (x + a)(N - x), \quad D(x) = x(b + N - x), \quad a > 0, \quad b > 0. \] (4.5)

It has a quadratic energy spectrum

\[ \mathcal{E}(n) = n(n + a + b - 1), \quad \eta(x) = x, \quad \phi_0(x)^2 = \frac{N!}{x!(N-x)!} \frac{(a)_n (b)(N-n)}{(b)_n (n + a + b - 1)_n}, \] (4.6)

\[ d_n^2 = \frac{N!}{n!(N-n)!} \frac{(a)_n (2n + a + b - 1)(a + b)_N}{(b)_n (n + a + b - 1)_n} \times \frac{(b)_N}{(a + b)_N}, \] (4.7)

\[ \tilde{P}_n(x) = P_n(\eta(x)) = {}_3F_2\left(\begin{array}{c} -n, n + a + b - 1, -x \\ a, -N \end{array} \right) \mid 1. \] (4.8)

### 4.3 dual Hahn

The birth and death rates are rational functions of \( x \), with \( a > 0, \quad b > 0, \)

\[ B(x) = \frac{(x + a)(x + a + b - 1)(N - x)}{(2x - 1 + a + b)(2x + a + b)}, \quad D(x) = \frac{x(x + b - 1)(x + a + b + N - 1)}{(2x - 2 + a + b)(2x - 1 + a + b)}, \] (4.9)

\[ \mathcal{E}(n) = n, \quad \eta(x) = x(x + a + b - 1), \quad \phi_0(x)^2 = \frac{N!}{x!(N-x)!} \frac{(a)_n (2x + a + b - 1)(a + b)_N}{(b)_n (x + a + b - 1)_n}, \] (4.10)

\[ d_n^2 = \frac{N!}{n!(N-n)!} \frac{(a)_n (b)_n}{(b)_n} \times \frac{(b)_N}{(a + b)_N}, \] (4.11)

\[ \tilde{P}_n(x) = P_n(\eta(x)) = {}_3F_2\left(\begin{array}{c} -n, x + a + b - 1, -x \\ a, -N \end{array} \right) \mid 1. \] (4.12)

### 4.4 Racah

The function \( B(x) \) and \( D(x) \) depend on four real parameters \( a, b, c \) and \( d \), with one of them, say \( c \), being related to \( N, c \equiv -N \):

\[ B(x) = -\frac{(x + a)(x + b)(x + c)(x + d)}{(2x + d)(2x + 1 + d)}, \quad D(x) = -\frac{(x + d - a)(x + d - b)(x + d - c)x}{(2x - 1 + d)(2x + d)}, \] (4.13)

\[ a > b, \quad d > 0, \quad a > N + d, \quad 0 < b < 1 + d, \] (4.14)
\[ \mathcal{E}(n) = n(n + \tilde{d}), \quad \eta(x) = x(x + d), \quad \tilde{d} \overset{\text{def}}{=} a + b + c - d - 1, \quad (4.15) \]
\[ \phi_0(x)^2 = \frac{(a, b, c, d)_x}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_x} \frac{2x + d}{d}, \quad (4.16) \]
\[ d_n^2 = \frac{(a, b, c, d)_n}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_n} \frac{2n + \tilde{d}}{d} \times \frac{(-1)^N(1 + d - a, 1 + d - b, 1 + d - c)_N}{(d + 1)_N(d + 1)_{2N}}, \quad (4.17) \]
\[ \tilde{P}_n(x) = P_n(\eta(x)) = _4F_3\left(\begin{array}{c} -n, n + \tilde{d}, -x, x + d \\ a, b, c \end{array} \bigg| 1 \right). \quad (4.18) \]

### 4.5 affine q-Krawtchouk

The birth and death rates are quadratic in \( q^x \):

\[ B(x) = (q^{x-N} - 1)(1 - pq^{x+1}), \quad D(x) = pq^{x-N}(1 - q^x), \quad 0 < p < q^{-1}, \quad (4.19) \]
\[ \mathcal{E}(n) = q^{-n} - 1, \quad \eta(x) = q^{-x} - 1, \quad (4.20) \]
\[ \phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x(q; q)_{N-x}} \frac{(pq; q)_x}{(pq)_x}, \quad d_n^2 = \frac{(q; q)_N}{(q; q)_n(q; q)_{N-n}} \frac{(pq; q)_n}{(pq)_n} \times (pq)^N, \quad (4.21) \]
\[ \tilde{P}_n(x) = P_n(\eta(x)) = _3\phi_2\left(\begin{array}{c} q^{-n}, q^{-x}, 0 \\ pq, q^{-N} \end{array} \bigg| q; q \right). \quad (4.22) \]

### 4.6 q-Krawtchouk

The birth and death rates are linear in \( q^x \):

\[ B(x) = q^{x-N} - 1, \quad D(x) = p(1 - q^x), \quad p > 0, \quad (4.23) \]
\[ \mathcal{E}(n) = (q^{-n} - 1)(1 + pq^n), \quad \eta(x) = q^{-x} - 1, \quad (4.24) \]
\[ \phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x(q; q)_{N-x}} \frac{pq^x q^{2x-1-xN}}{p^{-x}}, \quad (4.25) \]
\[ d_n^2 = \frac{(q; q)_N}{(q; q)_n(q; q)_{N-n}} \frac{(-p; q)_n}{(-pq^{N+1}; q)_n p^n q^{2n(N+1)}} \frac{1 + pq^{2n}}{1 + p} \times \frac{p^N q^{2N(N+1)}}{(-pq; q)_N}, \quad (4.26) \]
\[ \tilde{P}_n(x) = P_n(\eta(x)) = _3\phi_2\left(\begin{array}{c} q^{-n}, q^{-x}, -pq^n \\ pq, q^{-N}, 0 \end{array} \bigg| q; q \right). \quad (4.27) \]

### 4.7 quantum q-Krawtchouk

The birth and death rates are quadratic in \( q^x \):

\[ B(x) = p^{-1}q^x(q^{x-N} - 1), \quad D(x) = (1 - q^x)(1 - p^{-1}q^{x-N-1}), \quad (4.28) \]
\( \mathcal{E}(n) = 1 - q^n, \quad \eta(x) = q^{-x} - 1, \quad p > q^{-N}, \)

\( \phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}} \frac{p^{-x} q^{x(x-1-N)}}{(p^{-1} q^{-N}; q)_x}, \)

\( d_n^2 = \frac{(q; q)_N}{(q; q)_n (q; q)_{N-n}} \frac{p^{-n} q^{-Nn}}{(p^{-1} q^{-n}; q)_n} \times (p^{-1} q^{-N}; q)_N, \)

\( \tilde{P}_n(x) = P_n(\eta(x)) = 2 \phi_1\left( \frac{q^{-n}, q^{-x}}{q^{-N}}, \frac{q}{pq^{n+1}} \right). \)

### 4.8 q-Hahn

The birth and death rates are quadratic polynomials in \( q^x \):

\( B(x) = (1 - a q^x)(q^{x-N} - 1), \quad D(x) = a q^{-1}(1 - q^x)(q^{x-N} - b), \quad 0 < a, b < 1, \)

\( \mathcal{E}(n) = (q^{-n} - 1)(1 - abq^{-n}), \quad \eta(x) = q^{-x} - 1, \)

\( \phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}} \frac{(a; q)_x (b; q)_N}{(b; q)_N a^x}, \)

\( d_n^2 = \frac{(q; q)_N}{(q; q)_n (q; q)_{N-n}} \frac{(a, abq^{-1}; q)_n}{(ab q^N, b; q)_n a^n} \frac{1 - abq^{-n-1}}{1 - ab^{-1}} \times \frac{(b; q)_N a^n}{(ab; q)_N}, \)

\( \tilde{P}_n(x) = P_n(\eta(x)) = 3 \phi_2\left( \frac{q^{-n}, abq^{-n-1}, q^{-x}}{a, q^{-N}}, q; \right). \)

### 4.9 dual q-Hahn

The birth and death rates are rational functions of \( q^x \):

\( B(x) = \frac{(q^{x-N} - 1)(1 - a q^x)(1 - abq^{-x-1})}{(1 - abq^{2x-1})(1 - abq^{2x})}, \quad 0 < a, b < 1, \)

\( D(x) = a q^{x-N-1} \frac{(1 - q^x)(1 - abq^{x+N-1})(1 - bq^{-x-1})}{(1 - abq^{2x-2})(1 - abq^{2x-1})}, \)

\( \mathcal{E}(n) = q^{-n} - 1, \quad \eta(x) = (q^{-x} - 1)(1 - abq^{-x-1}), \)

\( \phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}} \frac{(a, abq^{-1}; q)_x}{(abq^N, b; q)_x a^x} \frac{1 - abq^{2x-1}}{1 - ab^{-1}}, \)

\( d_n^2 = \frac{(q; q)_N}{(q; q)_n (q; q)_{N-n}} \frac{(a; q)_n (b; q)_{N-n}}{(b; q)_N a^n} \times \frac{(b; q)_N a^n}{(ab; q)_N}, \)

\( \tilde{P}_n(x) = P_n(\eta(x)) = 3 \phi_2\left( \frac{q^{-n}, abq^{-n-1}, q^{-x}}{a, q^{-N}}, q; \right). \)
4.10 \( q \)-Racah

The birth and death rates are rational functions of \( q^x \),

\[
B(x) = -\frac{(1-aq^x)(1-bq^x)(1-cq^x)(1-dq^x)}{(1-dq^{2x})(1-dq^{2x+1})},
\]

\[
D(x) = -\tilde{d}\frac{(1-a^{-1}dq^x)(1-b^{-1}dq^x)(1-c^{-1}dq^x)(1-q^x)}{(1-dq^{2x})(1-dq^{2x+1})},
\]

\[
c = q^{-N}, \quad a \leq b, \quad 0 < d < 1, \quad 0 < a < q^N d, \quad q d < b < 1,
\]

\[
E(n) = (q^{-n} - 1)(1 - \tilde{d}q^n), \quad \eta(x) = (q^{-x} - 1)(1 - dq^x), \quad \tilde{d} \overset{\text{def}}{=} abcd^{-1}q^{-1},
\]

\[
\phi_0(x)^2 = \frac{(a, b, c, d; q)_x}{(a^{-1}dq, b^{-1}dq, c^{-1}dq, q; q)_x} \frac{1 - dq^{2x}}{1 - d},
\]

\[
d^2_n = \frac{(a, b, c, \tilde{d}; q)_n}{(a^{-1}dq, b^{-1}dq, c^{-1}dq, q; q)_n} \frac{1 - \tilde{d}q^m}{1 - d} \times \frac{(-1)^N (a^{-1}dq, b^{-1}dq, c^{-1}dq; q)_N \tilde{d}N q^{1/2}N(N+1)}{(dq; q)_N(dq; q)_2N},
\]

\[
\check{P}_n(x) = P_n(q(x)) = _4\phi_3\left( q^{-n}; \tilde{d}q^n, q^{-x}, dq^x \bigg| a, b, c \right) \frac{q}{q}.
\]

The BD processes corresponding to the 10 polynomials from Krawtchouk (§4.1) to \( q \)-Racah (§4.10) are on finite lattices, \( \mathcal{X} = \{0, 1, \ldots, N\} \).

4.11 Al-Salam-Carlitz II

The BD processes corresponding to this polynomial are on a semi-infinite lattice \( \mathcal{X} = \mathbb{Z}_{\geq 0} \). The birth and death rates are quadratic in \( q^x \), and thus bounded,

\[
B(x) = aq^{2x+1}, \quad D(x) = (1 - q^x)(1 - aq^x), \quad 0 < a < q^{-1},
\]

\[
E(n) = 1 - q^n, \quad \eta(x) = q^{-x} - 1, \quad \tilde{d} = \frac{abcd}{q},
\]

\[
\phi_0(x)^2 = \frac{a x q^{x+2}}{(q, aq; q)_x}, \quad d^2_n = \frac{(aq)_n}{(q; q)_n} \times (aq; q)_{\infty},
\]

\[
\check{P}_n(x) = P_n(\eta(x)) = 2\phi_0\left( q^{-n}; \frac{q^{-x}}{a^{-1}q^n} \bigg| q; a^{-1}q^n \right).
\]

5 Explicit Examples II

The solutions of the discrete time BD processes described in this section take different forms from those presented in Theorems 3.2, 3.3 (3.30)–(3.36), although the structure of the problem setting itself (2.6)–(2.9) is common. The eigenvectors of \( L (2.7) \) consist of two sets
of polynomials \( \{ \phi_n(x) \} \) and \( \{ \phi_n^{(-)}(x) \} \), having the same name. They are mutually orthogonal and constitute a complete set of basis of the corresponding Hilbert space. These discrete time BD processes are all Markov chains on a semi-infinite lattice \( \mathcal{X} = \mathbb{Z}_{\geq 0} \).

### 5.1 \( q \)-Meixner

The birth and death rates are quadratic in \( q^x \),

\[
B(x) = c q^x (1 - bq^{x+1}), \quad D(x) = (1 - q^x)(1 + bcq^x), \quad 0 < b < q^{-1}, \quad c > 0, \quad (5.1)
\]

\[
\mathcal{E}(n) = 1 - q^n, \quad \eta(x) = q^{-x} - 1, \quad (5.2)
\]

\[
\phi_0(x)^2 = \frac{(bq:q)_x}{(q,-bcq;q)_x} c q^x, \quad d_n^2 = \frac{q^n(bq:q)_n}{(q,-c^{-1}q;q)_n} \times \frac{(-bcq;q)_\infty}{(-c;q)_\infty}, \quad (5.3)
\]

\[
\tilde{P}_n(x) = P_n(\eta(x)) = 2 \phi_1 \left( q^{-n}, q^{-x} \Bigg| cq \right), \quad (5.4)
\]

It turned out that the above \( q \)-Meixner polynomials did not form a complete set [13]. It can be seen clearly by (3.15) of [2]. For the completeness another set of orthogonal polynomials is necessary. They are obtained from the original set by the parameter change (involution)

\[
(b, c) \rightarrow (-bc, c^{-1}),
\]

\[
B^{(-)}(x) = c^{-1} q^x (1 + bcq^{x+1}), \quad D^{(-)}(x) = (1 - q^x)(1 - bq^x), \quad (5.5)
\]

\[
\tilde{P}_n^{(-)}(x) = P_n^{(-)}(\eta(x)) = 2 \phi_1 \left( q^{-n}, q^{-x} \Bigg| -bcq \right), \quad \mathcal{E}'(n) \overset{\text{def}}{=} 1 + cq^n, \quad \eta(x) = q^{-x} - 1, \quad (5.6)
\]

\[
\phi_0^{(-)}(x)^2 = \frac{(-bcq;q)_x}{(q,bq;q)_x} c^{-x} q^x, \quad \phi_0^{(-)}(x) = (1)^2 \prod_{y=0}^{x-1} - \frac{B^{(-)}(y)}{D^{(-)}(y+1)} \left( -1 \right)^x \phi_0^{(-)}(x) > 0. \quad (5.7)
\]

The orthogonality relations are \( (n, m = 0, 1, \ldots) \)

\[
(\phi_n, \phi_m) = \sum_{x=0}^{\infty} \phi_n(x) \phi_m(x) = \frac{\delta_{n,m}}{d_n^2}, \quad \hat{\phi}_n(x) \overset{\text{def}}{=} \phi_n(x) d_n, \quad (5.8)
\]

\[
(\phi_n^{(-)}, \phi_m^{(-)}) = \sum_{x=0}^{\infty} \phi_n^{(-)}(x) \phi_m^{(-)}(x) = \frac{\delta_{n,m}}{d_n^{(-)}}, \quad d_n^{(-)} \overset{\text{def}}{=} d_n \bigg|_{(b,c) \rightarrow (-bc,c^{-1})}, \quad (5.9)
\]

The formulas for \textit{continuous time} BD involving two sets of orthogonal polynomials are slightly different from the previous case as given in the following
**Theorem 5.1** The solution of the initial value problem is

\[ \mathcal{P}(x; t) = \hat{\phi}_0(x) \sum_{n=0}^{\infty} \left( c_n e^{-\mathcal{E}(n)t} \hat{\phi}_n(x) + c_n^{(-)} e^{-\mathcal{E}'(n)t} \hat{\phi}_n^{(-)}(x) \right), \]  

(5.10)

in which

\[ c_n \overset{\text{def}}{=} \sum_{x=0}^{\infty} \hat{\phi}_n(x) \hat{\phi}_0(x)^{-1} \mathcal{P}(x; 0), \quad c_n^{(-)} \overset{\text{def}}{=} \sum_{x=0}^{\infty} \hat{\phi}_n^{(-)}(x) \hat{\phi}_0(x)^{-1} \mathcal{P}(x; 0). \]  

(5.11)

The transition probability from site \( y \) at time \( t = 0 \) to site \( x \) at a later time \( t \) in the continuous time BD is

\[ \mathcal{P}(x, y; t) = \hat{\phi}_0(x) \sum_{n=0}^{\infty} \left( e^{-\mathcal{E}(n)t} \hat{\phi}_n(x) \hat{\phi}_n(y) + e^{-\mathcal{E}'(n)t} \hat{\phi}_n^{(-)}(x) \hat{\phi}_n^{(-)}(y) \right) \hat{\phi}_0(y)^{-1} \quad (t > 0), \]  

(5.12)

These formulas are reported in [2] §6.A (6.20).

The formulas for discrete time BD involving two sets of orthogonal polynomials are shown in the following

**Theorem 5.2** The solution of the initial value problem and the transition matrix after \( \ell \) step are

\[ \mathcal{P}(x; \ell) = \hat{\phi}_0(x) \sum_{n=0}^{\infty} \left( c_n \kappa(n)^\ell \hat{\phi}_n(x) + c_n^{(-)} \kappa^{(-)}(n)^\ell \hat{\phi}_n^{(-)}(x) \right), \]  

(5.13)

\[ \mathcal{P}(x, y; \ell) = \hat{\phi}_0(x) \sum_{n=0}^{\infty} \left( \kappa(n)^\ell \hat{\phi}_n(x) \hat{\phi}_n(y) + \kappa^{(-)}(n)^\ell \hat{\phi}_n^{(-)}(x) \hat{\phi}_n^{(-)}(y) \right) \hat{\phi}_0(y)^{-1} \quad \ell = 1, 2, \ldots, \]  

(5.14)

in which \( c_n \) and \( c_n^{(-)} \) are given in (5.11) and

\[ \kappa(n) = 1 - t_S \cdot \mathcal{E}(n), \quad \kappa^{(-)}(n) = 1 - t_S \cdot \mathcal{E}'(n). \]  

(5.15)

**Theorem 5.3** The spectral representation of the symmetric matrix \( \mathcal{H} \) (3.17) for the present case reads

\[ \mathcal{H}_{xy} = \sum_{n=0}^{\infty} \mathcal{E}(n) \hat{\phi}_n(x) \hat{\phi}_n(y) + \sum_{n=0}^{\infty} \mathcal{E}'(n) \hat{\phi}_n^{(-)}(x) \hat{\phi}_n^{(-)}(y). \]  

(5.16)

This in turn supplies the spectral representations of \( L_{BD} \) and \( L \) through (3.28) and (3.7),

\[ L_{BDxy} = -\hat{\phi}_0(x) \left( \sum_{n=0}^{\infty} \mathcal{E}(n) \hat{\phi}_n(x) \hat{\phi}_n(y) + \sum_{n=0}^{\infty} \mathcal{E}'(n) \hat{\phi}_n^{(-)}(x) \hat{\phi}_n^{(-)}(y) \right) \hat{\phi}_0(y)^{-1} \]  

(5.17)
in which \( \pi(x) \) is the stationary distribution.

The \( q \)-Meixner polynomials provide another exactly solvable BD with birth rate \( B(x) \) and death rate \( D(x) \). Discrete time BD is also possible as shown in the following

**Theorem 5.4** The formulas for the initial value problem and the transition matrix for the discrete BD defined by the second set of polynomials are

\[
\mathcal{P}(x; \ell) = \hat{\phi}_0^{(-)}(x) \sum_{n=0}^{\infty} \left( \bar{c}_n \kappa(n) \ell \hat{\phi}_n^{(-)}(x) + c_n^{(+)} \kappa^{(+)}(n) \ell \hat{\phi}_n(x) \right),
\]

\[
\bar{c}_n \overset{\text{def}}{=} \sum_{x=0}^{\infty} \hat{\phi}_n^{(-)}(x) \hat{\phi}_0^{(-)}(x)^{-1} \mathcal{P}(x; 0), \quad c_n^{(+)} \overset{\text{def}}{=} \sum_{x=0}^{\infty} \hat{\phi}_n(x) \hat{\phi}_0^{(-)}(x)^{-1} \mathcal{P}(x; 0).
\]

\[
\mathcal{P}(x, y; \ell) = \hat{\phi}_0^{(-)}(x) \sum_{n=0}^{\infty} \left( \kappa(n) \ell \hat{\phi}_n^{(-)}(x) \hat{\phi}_n^{(-)}(y) + \kappa^{(+)}(n) \ell \hat{\phi}_n(x) \hat{\phi}_n(y) \right) \hat{\phi}_0^{(-)}(y)^{-1} \quad \ell = 1, 2, \ldots,
\]

in which

\[
\kappa(n) = 1 - t_s \cdot \mathcal{E}(n), \quad \kappa^{(+)}(n) = 1 - t_s \cdot \mathcal{E}^{(+)}(n), \quad \mathcal{E}^{(+)}(n) \overset{\text{def}}{=} 1 + c^{-1} q^n.
\]

The corresponding formulas for the continuous time BD are obtained from those in **Theorem 5.1** by changing \( \hat{\phi}_n \leftrightarrow \hat{\phi}_n^{(-)}, \quad c_n \leftrightarrow c_n^{(-)}, \) etc.
5.2 \(q\)-Charlier

The \(q\)-Charlier polynomials are obtained from those of \(q\)-Meixner by setting \(b = 0\) and \(c = a > 0\). The birth and death rates and the complete set of orthogonal vectors involving the \(q\)-Charlier polynomials are

\[
B(x) = aq^x, \quad D(x) = 1 - q^x, \quad B^{(-)}(x) = a^{-1}q^x, \quad D^{(-)}(x) = 1 - q^x,
\]

\[
P_n(x) = P_n(\eta(x)) = 2\phi_1(q^{-n}, q^{-x}; 0; q; -a^{-1}q^{n+1}), \quad \phi_0(x) = \sqrt{\frac{a^xq^{x(x-1)}}{(q; q)_x}},
\]

\[
P^{(-)}_n(x) = P^{(-)}_n(\eta(x)) = 2\phi_1(q^{-n}, q^{-x}; 0; q; -aq^{n+1}), \quad \phi^{(-)}_0(x) = (-1)^x \sqrt{\frac{a^{-x}q^{x(x-1)}}{(q; q)_x}},
\]

\[
\mathcal{E}(n) = 1 - q^n, \quad \mathcal{E}'(n) \overset{\text{def}}{=} 1 + aq^n, \quad \mathcal{E}^{(+)}(n) \overset{\text{def}}{=} 1 + a^{-1}q^n, \quad \eta(x) = q^x - 1,
\]

\[
d_n^2 = \frac{q^n}{(q, -a^{-1}q; q)_n} \times \frac{1}{(-a; q)_\infty}, \quad d_n^{(-)} \overset{\text{def}}{=} d_n|_{a \to a^{-1}}.
\]

The orthogonality relations have the same forms as (5.8)–(5.9). The formulas in Theorem 5.1–5.4 apply for the \(q\)-Charlier.

5.3 dual big \(q\)-Jacobi

The big \(q\)-Jacobi polynomials are the most generic member of the family having orthogonality measures of Jackson integral type. The big \(q\)-Laguerre, Al-Salam-Carlitz I, discrete \(q\)-Hermite I,II and \(q\)-Laguerre belong to this family. They all have unbounded \(B(x)\) and \(D(x)\). This means that the discrete time BD corresponding to these polynomials cannot be constructed, although the solutions of their continuous time BD processes show quite interesting features, as reported in [2] (II§6A).

Some of the dual polynomials of this family, however, provide exactly solvable discrete time BD processes as they have bounded \(B(x)\) and \(D(x)\). Reflecting the structure of Jackson integrals, the corresponding dual orthogonal polynomials consist of two components, similar to the \(q\)-Meixner case presented in §5.1. The dual polynomials belonging to this family are extensively reported in [13]–[14].

The data for the dual big \(q\)-Jacobi polynomials are

\[
0 < a < q^{-1}, \quad 0 < b < q^{-1}, \quad c < 0, \quad \eta(x) \overset{\text{def}}{=} (q^{-x} - 1)(1 - abq^{x+1}),
\]

\[
B(x) = -cq^{x+1}(1 - aq^{x+1})(1 - abq^{x+1})(1 - abc^{-1}q^{x+1}) \frac{1}{(1 - abq^{2x+1})(1 - abq^{2x+2})},
\]
\[ D(x) = aq \frac{(1 - q^2)(1 - bq^2)(1 - cq^2)}{(1 - abq^2x)(1 - abq^{2x+1})}, \] (5.33)

\[ B(-)(x) = \frac{(1 - b - bq^x + 1)(1 - abq^{x+1})(1 - cq^{x+1})}{(1 - abq^{2x+1})(1 - abq^{2x+2})}, \] (5.34)

\[ D(-)(x) = -cq \frac{(1 - q^2)(1 - aq^x)(1 - abc^{-1}q^x)}{1 - abq^{2x}(1 - abq^{2x+1})}, \] (5.35)

\[ \hat{P}_n(x) = P_n(\eta(x)) = 3\phi_2 \left( \frac{q^{-n}, abq^{x+1}, q^{-x}}{aq, abc^{-1}q}; q \right), \] (5.36)

\[ \hat{P}_n(-)(x) = P_n(-)(\eta(x)) = 3\phi_2 \left( \frac{q^{-n}, abq^{x+1}, q^{-x}}{bq, cq}; q \right), \] (5.37)

\[ \phi_0(x)^2 = q^{\frac{x(x-1)}{2}} \frac{(ab^{-1}q; q)_x}{(cq; q)_x} \frac{1 - abq^{2x+1}}{(aq, abq^2; q)_x}, \] (5.38)

\[ \phi_0(-)(x)^2 = q^{\frac{x(x-1)}{2}} \frac{(bq; q)_x}{(aq, abc^{-1}x; q)_x} \frac{1 - abq^{2x+1}}{(aq, abq^2; q)_x}, \] (5.39)

\[ d_n^2 = q^n \frac{(aq, abc^{-1}q; q)_n}{(q, ac^{-1}q; q)_n} \frac{(bq, cq; q)_n}{(abq^2, a^{-1}c; q)_n}, \] (5.40)

\[ d_n^-2 = q^n \frac{(bq, cq; q)_n}{(aq, abc^{-1}q; q)_n} \frac{(aq, abc^{-1}q; q)_n}{(abq^2, ac^{-1}; q)_n}. \] (5.41)

The following parameter substitution (involution)

\[(a, b, c) \leftrightarrow (c, abc^{-1}, a), \] (5.42)

gives rise to the interchange of the basic and the (-) objects. The orthogonality relations for \(\{\phi_n(x)\}\) and \(\{\phi_n(-)(x)\}\) have the same form as (5.8)–(5.9).

Discrete time BD based on \(B(x), D(x) \ (5.32),(5.33)\) has the same formulas as those in Theorem 5.2 given for the \(q\)-Meixner systems \(5.13\) and \(5.14\) with the replacements

\[ E(n) = aq(1 - q^n), \quad E'(n) = q(a - cq^n). \] (5.43)

Likewise discrete time BD based on \(B(-)(x), D(-)(x) \ (5.34),(5.35)\) has the same forms as those in Theorem 5.4 \((5.21)-(5.23)\) with the replacements

\[ E(n) = -cq(1 - q^n), \quad E'(-)(n) = q(-c + aq^n). \] (5.44)

### 5.4 dual big \(q\)-Laguerre

The basic data are as follows. They are obtained from those of the dual big \(q\)-Jacobi polynomial by setting \(b \to 0\) and \(c \to b\), with \(0 < a < q^{-1}\) and \(b < 0\),

\[ B(x) = -bq^{x+1}(1 - aq^{x+1}), \quad D(x) = aq(1 - q^x)(1 - bq^x), \] (5.45)
\[ B^{-}(x) = aq^{x+1}(1 - bq^{x+1}), \quad D^{-}(x) = -bq(1 - q^{x})(1 - aq^{x}), \] (5.46)

\[ \tilde{P}_{n}(x) = P_{n}(\eta(x)) = 2\phi_{1}\left( q^{-x}; \frac{q^{-x}}{aq} \bigg| q; ab^{-1}q^{n+1} \right), \quad \eta(x) \overset{\text{def}}{=} q^{-x} - 1, \] (5.47)

\[ \hat{P}_{n}^{-}(x) = P_{n}^{-}(\eta(x)) = 2\phi_{1}\left( q^{-x}; \frac{q^{-x}}{aq} \bigg| q; a^{-1}bq^{n+1} \right), \] (5.48)

\[ \phi_{0}(x)^{2} = \frac{q^{\frac{1}{2x}(x-1)} - (aq; q)_{x}}{(-ab^{-1})^{x} (aq; q)_{x} (q; q)_{x}}, \] (5.49)

\[ \phi_{0}^{-}(x)^{2} = \frac{q^{\frac{1}{2x}(x-1)} - (aq; q)_{x}}{(-a^{-1}b) (aq; q)_{x} (q; q)_{x}}, \] (5.50)

\[ d_{n}^{2} = q^{n} \frac{(aq; q)_{n}}{(q, ab^{-1}q; q)_{n}} \times \frac{(bq; q)_{\infty}}{(a^{-1}b; q)_{\infty}}, \] (5.51)

\[ d_{n}^{-2} = q^{n} \frac{(aq; q)_{n}}{(q, a^{-1}bq; q)_{n}} \times \frac{(bq; q)_{\infty}}{(ab^{-1}; q)_{\infty}}, \] (5.52)

\[ \phi_{n}^{-}(x) \overset{\text{def}}{=} \phi_{0}(x) \tilde{P}_{n}^{-}(x), \quad \hat{\phi}_{n}^{-}(x) \overset{\text{def}}{=} \phi_{n}^{-}(x) d_{n}^{-}. \] (5.53)

The basic objects and the \((-\)) objects are interchanged by the parameter substitution (involution) \(a \leftrightarrow b\).

Discrete time BD based on \(B(x), D(x)\) (5.45) has the same formulas as those in Theorem 5.2 for the \(q\)-Meixner systems (5.13) and (5.14) with the replacements

\[ \mathcal{E}(n) = aq(1 - q^{n}), \quad \mathcal{E}'(n) = q(a - bq^{n}). \] (5.54)

Likewise discrete time BD based on \(B^{-}(x), D^{-}(x)\) (5.46) has the same forms as those in Theorem 5.4 (5.21) - (5.23) with the replacements

\[ \mathcal{E}(n) = -bq(1 - q^{n}), \quad \mathcal{E}'(n) = q(-b + aq^{n}). \] (5.55)

6 \hspace{1em} Mirror symmetric Birth and Death processes

Here we present a simple mirror symmetric discrete time BD process. Among the exactly solvable discrete time BD’s on a finite integer lattice listed in section 4, those related to two polynomials, the Krawtchouk §4.1 and Hahn §4.2 can be made mirror symmetric

Mirror symmetry: \(D(N - x) = B(x), \quad B(N - x) = D(x), \) (6.1)

\[ \implies \phi_{0}(N - x)^{2} = \phi_{0}(x)^{2}, \quad \hat{P}_{n}(N - x) = (-1)^{n} \hat{P}_{n}(x), \] (6.2)
by adjusting the parameters. These two polynomials have \( \eta(x) = x \). As for the Krawtchouk
(4.1)–(4.4) with \( p = 1/2 \), we have

Krawtchouk: \( B(x) = (N - x)/2, \ D(x) = x/2, \ \eta(x) = x, \ \mathcal{E}(n) = n, \) \( (6.3) \)
\[ \phi_0(x)^2 = \frac{N!}{x!(N - x)!}, \quad d_n^2 = \frac{N!}{n!(N - n)!} \times 2^{-N}, \]
\[ \hat{P}_n(x) = P_n(x) = \begin{pmatrix} -n,\ -x \\ -N \end{pmatrix} = (-1)^n P_n(N - x), \] \( (6.5) \)
due to Pfaff’s transformation formula (see [4]p79)
\[ 2F_1\left( \begin{array}{c} a, \ b \\ c \end{array} \bigg| \frac{x}{x-1} \right) = (1 - x)^{-a} 2F_1\left( \begin{array}{c} a - b, \ c \\ x \end{array} \bigg| \frac{x}{x-1} \right). \]

Taking \( a = b \) for the Hahn (4.5)–(4.8), we obtain

Hahn: \( B(x) = (x + a)(N - x), \ D(x) = x(a + N - x), \) \( (6.6) \)
\[ \eta(x) = x, \ \mathcal{E}(n) = n(n + 2a - 1), \]
\[ \phi_0(x)^2 = \frac{N!}{x!(N - x)!} \frac{(a)_x (a)_{N-x}}{(a)_N}, \quad d_n^2 = \frac{N!}{n!(N - n)!} \frac{(2n + 2a - 1)(2a)_n}{(n + 2a - 1)(n+1)_N} \times \frac{(a)_N}{(2a)_N}, \]
\[ \hat{P}_n(x) = P_n(x) = 3F_2\left( \begin{array}{c} -n,\ n + 2a - 1,\ -x \\ a,\ -N \end{array} \bigg| 1 \right) = (-1)^n P_n(N - x). \] \( (6.9) \)
due to the following transformation formula (see [4]p142)
\[ 3F_2\left( \begin{array}{c} -n,\ a,\ b \\ d,\ e \end{array} \bigg| 1 \right) \times (e - a)_n 3F_2\left( \begin{array}{c} -n,\ a - b \\ d,\ a + 1 - n - e \end{array} \bigg| 1 \right). \]

Let us consider the following transition matrix \( L_{BD}^M \)
\[ (L_{BD}^M) \mathcal{P}(x; t) = \sum_{y \in \mathcal{X}} L_{BD}^M y \mathcal{P}(y; t) \]
\[ = -(B(x) + D(x)) \mathcal{P}(N - x; t) + B(x - 1) \mathcal{P}(N - x + 1; t) \]
\[ + D(x + 1) \mathcal{P}(N - x - 1; t), \] \( (6.11) \)
\[ L_{BD}^M = \begin{pmatrix} 0 & 0 & \cdots & \cdots & D(1) & -B(0) \\ 0 & 0 & \cdots & D(2) & -B(1) - D(1) & B(0) \\ 0 & \cdots & \cdots & -B(2) - D(2) & B(1) & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & D(N - 1) & \cdots & \cdots & \cdots & 0 \\ D(N) & -B(N - 1) - D(N - 1) & \cdots & \cdots & 0 & 0 \\ -D(N) & B(N - 1) & \cdots & \cdots & 0 & 0 \end{pmatrix}. \]
which is obtained from $L_{BD}$ (3.4) by mirror reflection. That is

$$L_{BD}^M = L_{BD}J,$$

$$J \overset{\text{def}}{=} \text{anti-diagonal}\{1, 1, \ldots, 1\}, \text{ or } J_{xy} \overset{\text{def}}{=} \delta_{x,N-y}, \sum_{y \in X} J_{xy} \hat{P}_n(y) = (-1)^n \hat{P}_n(x).$$

From the general relation (3.29)

$$(L_{BD} \hat{\phi}_0 \hat{\phi}_n)(x) = -\mathcal{E}(n) \hat{\phi}_0(x) \hat{\phi}_n(x), \ n \in \mathcal{X},$$

we obtain

$$(L_{BD}^M \hat{\phi}_0 \hat{\phi}_n)(x) = -(-1)^n \mathcal{E}(n) \hat{\phi}_0(x) \hat{\phi}_n(x), \ n \in \mathcal{X}. \quad (6.14)$$

Since the spectrum of $L_{BD}^M$ is not negative semi-definite, $L_{BD}^M$ does not define a stochastic process. However, it is interesting to consider its discrete time version.

Let us introduce the mirror image of $L$ (2.7),

$$L_{x+1-N-x}^M = \bar{B}(x), \ L_{x-1-N-x}^M = \bar{D}(x), \ L_{xN-x}^M = 1 - \bar{B}(x) - \bar{D}(x),$$

$$L_{xy}^M = 0 \ \text{for} \ x + y < N - 1, \ x + y > N + 1, \quad (6.15)$$

This defines another exactly solvable stochastic process, as $L^M$ is a non-negative tri-anti-diagonal matrix with

$$L^M \overset{\text{def}}{=} LJ = \begin{pmatrix}
0 & 0 & \cdots & \cdots & \bar{D}(1) & 1 - \bar{B}(0) \\
0 & 0 & \cdots & \bar{D}(2) & 1 - \bar{B}(1) - \bar{D}(1) & \bar{B}(0) \\
0 & \cdots & \cdots & 1 - \bar{B}(2) - \bar{D}(2) & \bar{B}(1) & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
0 & \bar{D}(N-1) & \cdots & \cdots & 0 & 0 \\
\bar{D}(N) & 1 - \bar{B}(N-1) - \bar{D}(N-1) & \cdots & \cdots & 0 & 0 \\
1 - \bar{D}(N) & B(N-1) & \cdots & \cdots & 0 & 0
\end{pmatrix}. $$

This defines another exactly solvable stochastic process, as $L^M$ is a non-negative tri-anti-diagonal matrix with

$$(L^M \hat{\phi}_0 \hat{\phi}_n)(x) = \kappa_M(n) \hat{\phi}_0(x) \hat{\phi}_n(x), \ 1 \geq \kappa_M(n) \overset{\text{def}}{=} (-1)^n (1 - t_S \cdot \mathcal{E}(n)) \geq -1, \ n \in \mathcal{X}. \quad (6.16)$$

However, this is simply the mirror image of the original process. Let us introduce the process governed by the sum of $L$ and $L^M$,

$$L^S \overset{\text{def}}{=} \frac{1}{2}(L + L^M) = \frac{1}{2}L(I_d + J), \quad (6.17)$$
which is exactly solvable having interesting properties,

\[
\left( L^S \hat{\phi}_0 \hat{\phi}_n \right) (x) = \kappa_S(n) \hat{\phi}_0(x) \hat{\phi}_n(x),
\]

\[
1 \geq \kappa_S(n) \overset{\text{def}}{=} \frac{1}{2} \left( 1 + (-1)^n \right) \left( 1 - t_S \cdot \mathcal{E}(n) \right) \geq -1, \quad n \in \mathcal{X}.
\]

All odd eigenvalues vanish

\[
\kappa_S(2n + 1) = 0, \quad n = 0, 1, \ldots, \left[ \frac{N - 1}{2} \right],
\]

and the convergence to the stationary distribution is accelerated. By expanding the initial distribution as (3.31), the distribution after \( \ell \) step is described by the even eigenvectors only

\[
\mathcal{P}(x; \ell) = \hat{\phi}_0(x) \sum_{n=0}^{\left[ \frac{N}{2} \right]} c_{2n} \left( \kappa_S(2n) \right)^\ell \hat{\phi}_{2n}(x), \quad \ell = 1, 2, \ldots.
\]

Likewise, the transition matrix from \( y \) at \( \ell = 0 \) (\( \mathcal{P}(x; 0) = \delta_{x,y} \)) to \( x \) after \( \ell \) steps is

\[
\mathcal{P}(x, y; \ell) = \hat{\phi}_0(x) \sum_{n=0}^{\left[ \frac{N}{2} \right]} \left( \kappa_S(2n) \right)^\ell \hat{\phi}_{2n}(x) \hat{\phi}_{2n}(y) \hat{\phi}_0(y)^{-1}.
\]

In other words, the asymmetric part of the initial distribution \( \mathcal{P}^{AS}(x; 0) \)

\[
\mathcal{P}^{AS}(x; 0) \overset{\text{def}}{=} \frac{1}{2} \left( \mathcal{P}(x; 0) - \mathcal{P}(N - x; 0) \right) = \left( \frac{1}{2} (I_d - J) \mathcal{P} \right) (x; 0),
\]

is erased by one action of \( L^S \) (6.17)

\[
\left( L^S \mathcal{P}^{AS} \right) (x; 0) = \frac{1}{4} \left( L(I_d + J)(I_d - J) \right) \mathcal{P}(x; 0) = 0.
\]

A Markov chain having similar eigenvectors was reported in \[15\]. A very special case of

discrete time BD with \( \bar{B}(x) + \bar{D}(x) = 1 \) based on \( p = 1/2 \) Krawtchouk was reported in \[16\].

6.1 Dual systems

For finite systems, an apparently different looking exactly solvable system can be constructed

from a known exactly solvable one by the similarity transformation by the anti-diagonal

matrix \( J \) (6.13),

\[
L^d \overset{\text{def}}{=} JLJ, \quad L_{BD}^d \overset{\text{def}}{=} JL_{BD}J.
\]

That is,

\[
\bar{B}^d(x) \overset{\text{def}}{=} \bar{D}(N - x), \quad \bar{D}^d(x) \overset{\text{def}}{=} \bar{B}(N - x); \quad B^d(x) \overset{\text{def}}{=} D(N - x), \quad D^d(x) \overset{\text{def}}{=} B(N - x),
\]

26
and it is called a ‘dual system’ \(17\) §7. The original and its dual system have common eigenvalues and the eigenvectors are mapped by \(J\). For most of examples listed in §11 dual polynomials are the same as the original one with parameter change (involution).

### 7 Comments

It should be stressed that the same solution procedures apply to BD processes related to various new orthogonal polynomials \(18\). They are obtained from the classical orthogonal polynomials, e.g. the Racah and \(q\)-Racah, etc by multiple applications of the discrete analogue of the Darboux transformations or the Krein-Adler transformations. Since there are many different ways to deform the classical orthogonal polynomials, these new polynomials offer virtually infinite examples of exactly solvable BD processes.

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