A Convergent Reformulation of QCD Perturbation Theory

C.J. Maxwell

Centre for Particle Theory, University of Durham
South Road, Durham, DH1 3LE, England

Abstract

We propose a generalization of Grunberg’s method of effective charges in which, starting with the effective charge for some dimensionless QCD observable dependent on the single energy scale $Q$, $\mathcal{R}(Q)$, we introduce an infinite set of auxiliary effective charges, each one describing the sub-asymptotic $Q$-evolution of the immediately preceding effective charge. The corresponding infinite set of coupled integrated effective charge beta-function equations may be truncated. The resulting approximations for $\mathcal{R}(Q)$ are the convergents of a continued function. They are manifestly RS-invariant and converge to a limit equal to the Borel sum of the standard asymptotic perturbation series in $\alpha_s(\mu^2)$, with remaining ambiguities due to infra-red renormalons. There are close connections with Padé approximation.
QCD perturbation theory in its conventional formulation of a power series expansion in the renormalization group (RG) improved coupling, $\alpha_s(\mu^2)$, suffers from a number of defects. The series has a zero radius of convergence with coefficients exhibiting factorial growth, and consequently its sum can only be reconstructed by treating it as asymptotic to some function of the coupling. Customarily Borel summation is used for this reconstruction but of course this is not the only possible choice, and the method to be used is not uniquely specified by the physical theory itself [1]. Whilst ultra-violet (UV) renormalon singularities in the Borel plane cause no problems in the definition of the Borel sum, there are ambiguities resulting from infra-red (IR) renormalons which, it has been argued, are closely connected with, and can compensate, corresponding ambiguities in the operator product expansion (OPE) [2].

In addition to these problems arising from the large-order behaviour of conventional perturbation theory, fixed-order perturbation theory is afflicted with the problem of renormalization scheme (RS) dependence. Thus fixed-order truncations of the series depend on the renormalization convention used to define $\alpha_s(\mu^2)$. A number of proposals for controlling or avoiding this RS-dependence problem have been advanced [3-6], but no consensus on the issue has been reached.

In this paper we suggest a reformulation of QCD perturbation theory which avoids many of these problems. In particular it is convergent, notwithstanding remaining ambiguities due to IR renormalons, and is manifestly RS-invariant. It can be regarded as a generalization of Grunberg’s effective charge approach [4] and turns out to have a close relation to Padé approximation [7, 8].

We begin by considering a generic dimensionless QCD observable $R(Q)$, which depends on the single energy scale $Q$ (e.g. the $e^+e^-$ R-ratio with $Q$ the c.m. energy). By raising to a power and scaling we can always arrange that the formal perturbation series for $R(Q)$ assumes the form,

$$R(Q) = a + r_1a^2 + r_2a^3 + \ldots + r_na^{n+1} + \ldots,$$  

where $a \equiv \alpha_s(\mu^2)/\pi$ is the RG-improved coupling. We shall refer to observables defined in this way as effective charges [4, 8].

Such effective charges satisfy several important properties. We shall consider SU($N$) QCD with $N_f$ flavours of massless quark, and assume that the
first beta-function coefficient $b = (11N - 2N_f)/6$ is positive. The first familiar property is Asymptotic Freedom (AF), which is equivalent to the statement that for any effective charge $R(Q)$,

$$\lim_{Q \to \infty} R(Q) = 0.$$  

(2)

The second property, which we shall refer to as “Asymptotic Scaling” (AS), is perhaps less familiar, and will be central to the reformulation of perturbation theory we are proposing. The first step is to define a universal QCD scaling function,

$$\mathcal{F}(x) \equiv e^{-1/bx}(1 + 1/cx)^{c/b},$$

(3)

where $c$ is the second (universal) beta-function coefficient,

$$c = \left[ -\frac{7}{8} C_A^2 - \frac{11}{8} C_A C_F + \frac{5}{4} C_A + \frac{3}{4} C_F \right],$$

(4)

with $C_A = N, C_F = (N^2 - 1)/2N$, SU($N$) Casimirs. Then AS corresponds to the statement that for any effective charge $R(Q)$,

$$\lim_{Q \to \infty} Q \mathcal{F}(R(Q)) = \Lambda_R,$$

(5)

where $\Lambda_R$ is a scaling constant with dimensions of energy which depends on the observable. Given sufficiently large values of $Q$ this property can evidently serve as a test of QCD, but since $\Lambda_R$ is not universal it cannot usefully be applied at fixed values of $Q$. However if the next-to-leading order (NLO) perturbative coefficient $r \equiv r_{1\,\overline{MS}}(\mu = Q)$ has been calculated (in the $\overline{MS}$ scheme with scale $\mu^2 = Q^2$ for instance), then $\Lambda_R$ can be converted into a universal scaling constant $\Lambda_{\overline{MS}}$ via the exact Celmaster-Gonsalves relation,

$$\Lambda_R e^{-r/b} = \Lambda_{\overline{MS}}.$$  

(6)

Use of different subtraction procedures results in different constants which, however, are still universal. We stress that $r$ is independent of $Q$. Thus we arrive at the property of “Universal Asymptotic Scaling” (UAS),

$$\lim_{Q \to \infty} Q \mathcal{F}(R(Q)) e^{-r/b} = \Lambda_{\overline{MS}}.$$  

(7)
This property can now be used to test QCD at fixed $Q$ by looking at the scatter in $Q \mathcal{F}(\mathcal{R}(Q)) e^{-r/b}$ for various observables \cite{3}. At finite $Q$ UAS will be violated and one will have

$$Q \mathcal{F}(\mathcal{R}(Q)) e^{-r/b} = \Lambda_{\overline{MS}}(1 + \eta_{\mathcal{R}}(Q)),$$

where $\eta_{\mathcal{R}}(Q)$ represents the non-scaling sub-asymptotic effects which vanish as $Q \to \infty$. The crucial observation is that $\eta_{\mathcal{R}}(Q)$ has the formal perturbation series

$$\eta_{\mathcal{R}}(Q) = \frac{\rho_2}{b} (a + O(a^2)),$$

where $\rho_2$ is the next-NLO (NNLO) effective charge (EC) beta-function coefficient \cite{4,6}, given by the RS-invariant combination

$$\rho_2 = c_2 + r_2 - r_1 c - r_1^2,$$

with $c_2$ the three-loop beta-function coefficient. Thus $\eta_{\mathcal{R}}$ is proportional to another effective charge, which will itself satisfy the UAS property, and will have its own non-scaling contribution proportional to yet another effective charge, etc. This self-similar construction based on the exact property of UAS provides the reformulation of QCD perturbation theory which we are proposing.

The construction can usefully be regarded as a generalization of Grunberg’s effective charge approach \cite{4,6}, and it will help to illuminate the discussion so far if we briefly review that formalism. We refer the reader to reference \cite{6} for full details.

The key ingredient in the EC approach is the EC beta-function $\rho(\mathcal{R}(Q))$ which determines the $Q$-evolution of the observable $\mathcal{R}(Q)$,

$$\frac{d\mathcal{R}(Q)}{d\ln Q} \equiv -b \rho(\mathcal{R}(Q))$$

$$= -b(\mathcal{R}^2 + c\mathcal{R}^3 + \rho_2 \mathcal{R}^4 + \ldots + \rho_n \mathcal{R}^{n+2} + \ldots),$$

where the $\rho_n$ are $Q$-independent RS-invariant combinations of the $r_i$, $c_i$ ($i \leq n$) (see eq.(10) for $\rho_2$). $\rho(\mathcal{R}(Q))$ can be regarded as an observable to be reconstructed from the measured running of $\mathcal{R}(Q)$ \cite{6}. Integrating up eq.(11) and imposing Asymptotic Freedom as a boundary condition yields,
\[ F(\mathcal{R}(Q)) = b \ln \frac{Q}{\Lambda_{\mathcal{R}}} - \int_{0}^{\mathcal{R}(Q)} dx \left[ -\frac{1}{\rho(x)} + \frac{1}{x^2(1+cx)} \right] \]
\[ \equiv b \ln \frac{Q}{\Lambda_{\mathcal{R}}} - \Delta \rho_0(Q) , \quad (12) \]

where

\[ F(x) \equiv -b \ln \mathcal{F}(x) = \frac{1}{x} + c \ln \left( \frac{cx}{1+cx} \right) , \quad (13) \]

and with \( \Lambda_{\mathcal{R}} \) a constant of integration. Assuming \( AF \) the AS property in eq.(5) then follows directly on rearranging eq.(12) and taking the \( Q \to \infty \) limit, in which \( \Delta \rho_0(Q) \to 0 \). Further, as demonstrated in ref.[6], one can then identify \( b \ln(Q/\Lambda_{\mathcal{R}}) \) with the NLO RS-invariant \( \rho_0(Q) \) \[3, 4\].

\[ b \ln \frac{Q}{\Lambda_{\mathcal{R}}} = \rho_0(Q) \equiv b \ln \frac{Q}{\Lambda_{\overline{\text{MS}}}} - r , \quad (14) \]

from which the Celmaster-Gonsalves relation in eq.(6) follows. The sub-asymptotic \( \eta_{\mathcal{R}}(Q) \) contribution in eq.(8) is equivalent to \( (\exp(\Delta \rho_0/b) - 1) \), with \( \Delta \rho_0(Q) \) as defined in eq.(12).

In this notation the violation of UAS is controlled by \( \Delta \rho_0(Q) \). Expanding the integrand in eq.(12) as a power series in \( x \) and integrating term-by-term we obtain

\[ \Delta \rho_0(Q) = \rho_2 \mathcal{R} + (\rho_3 - 2\rho_2 c) \frac{\mathcal{R}^2}{2} + (\rho_4 - 2 c \rho_3 + 2 c^2 \rho_2 + 2 c^2 - \rho_2^2) \frac{\mathcal{R}^3}{3} + \ldots \quad (15) \]

We can then write \( \Delta \rho_0(Q) \equiv \rho_2 \mathcal{R}^{(1)}(Q) \), where \( \mathcal{R}^{(1)}(Q) \) is another effective charge whose perturbative expansion follows on substituting \( \mathcal{R} = a + r_1 a^2 + \ldots \) in eq.(15),

\[ \mathcal{R}^{(1)}(Q) = a + (r_1 + \frac{\rho_3}{2\rho_2} - c) a^2 + \ldots , \quad (16) \]

from which the coefficients \( r_1^{(1)}, r_2^{(1)} , \ldots \), can be obtained. \( \mathcal{R}^{(1)}(Q) \) will satisfy eq.(12) with \( \rho_0(Q), \Delta \rho_0(Q) \) replaced by \( \rho_0^{(1)}(Q) \) and \( \Delta \rho_0^{(1)}(Q) \) where
\[
\rho_0^{(1)}(Q) = b \ln \frac{Q}{\Lambda_{\overline{MS}}} - r^{(1)\overline{MS}} = b \ln \frac{Q}{\Lambda_{\overline{MS}}} - r - \frac{\rho_3}{2\rho_2} + c. \tag{17}
\]

Writing \(\Delta \rho_0^{(1)}(Q) \equiv \rho_2^{(1)} \mathcal{R}^{(2)}(Q)\) with \(\mathcal{R}^{(2)}(Q)\) yet another effective charge the self-similar construction continues.

Thus perturbation theory is reformulated as an infinite set of effective charges \(\mathcal{R}(Q), \mathcal{R}^{(1)}(Q), \ldots, \mathcal{R}^{(n)}(Q), \ldots\), satisfying the set of coupled equations,

\[
\begin{align*}
F(\mathcal{R}(Q)) &= \rho_0(Q) - \rho_2 \mathcal{R}^{(1)}(Q) \\
F(\mathcal{R}^{(1)}(Q)) &= \rho_0^{(1)}(Q) - \rho_2^{(1)} \mathcal{R}^{(2)}(Q) \\
F(\mathcal{R}^{(2)}(Q)) &= \rho_0^{(2)}(Q) - \rho_2^{(2)} \mathcal{R}^{(3)}(Q) \\
\vdots & \quad \vdots \\
F(\mathcal{R}^{(n)}(Q)) &= \rho_0^{(n)}(Q) - \rho_2^{(n)} \mathcal{R}^{(n+1)}(Q) \\
\vdots & \quad \vdots 
\end{align*}
\tag{18}
\]

\(\rho_0^{(n)}\) involves \(b \ln(Q/\Lambda_{\overline{MS}})\), \(r\), and the \(\rho_k\) EC RS invariants for \(k \leq 2n + 1\) (we define \(\rho_1 \equiv c\)). Thus it is determined given a complete \(N^{2n+1}\text{LO}\) perturbative calculation for \(\mathcal{R}(Q)\). \(\rho_2^{(n)}\) involves the \(\rho_k\) for \(k \leq 2n + 2\) and so is determined given a \(N^{2n+2}\text{LO}\) calculation for \(\mathcal{R}(Q)\).

The effective charge \(\mathcal{R}^{(1)}(Q)\) describes the violation of UAS of \(\mathcal{R}(Q)\), \(\mathcal{R}^{(2)}(Q)\) describes the scaling violation of \(\mathcal{R}^{(1)}(Q)\) etc. If we assume that \(\mathcal{R}^{(n)}(Q)\) scales and set \(\mathcal{R}^{(n+1)}(Q) = 0\), then the truncated set of equations in eqs.(18) can be solved for \(\mathcal{R}(Q)\), given \(\ln(Q/\Lambda_{\overline{MS}})\), the NLO \(\overline{MS}\) coefficient \(r\), and the \(\rho_0^{(i)}, \rho_2^{(i)}\), that is given a complete \(N^{2n+1}\text{LO}\) perturbative calculation for \(\mathcal{R}(Q)\).

Equivalently we can iteratively eliminate \(\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \ldots\) from eqs.(18). If we define \(G(x)\) to be the inverse function of \(F(x)\) in eq.(13), that is \(G(F(x)) = x\), then we obtain the continued function representation for \(\mathcal{R}(Q)\),

\[
\mathcal{R}(Q) = G(\rho_0 - \rho_2 G(\rho_0^{(1)} - \rho_2^{(1)} G(\rho_0^{(2)} - \rho_2^{(2)} G(\ldots))) \ldots). \tag{19}
\]

Equation (19) is our main result. The first NLO approximation \(G(\rho_0)\) is precisely the conventional NLO perturbative approximation in the EC
scheme. Subsequent approximations differ from conventional fixed-order perturbation theory. An interesting feature of the approach is that each extra level in the construction of eqs.(18) adds two further orders in perturbation theory. One can interpolate by using \( G(\rho_0 - \rho_2 G(\rho_0)) \) as the NNLO approximation, and then the N^3LO approximation is \( G(\rho_0 - \rho_2 G(\rho_0^{(1)})) \), \( G(\rho_0 - \rho_2 G(\rho_0^{(2)}) - \rho_2^{(1)} G(\rho_0^{(1)}))) \) interpolates the N^4LO, etc.

Our claim is that if only ultra-violet (UV) renormalon singularities are present then this sequence of approximations converges, and the limit is precisely the Borel sum of the divergent asymptotic conventional perturbation series in \( \alpha_s(\mu^2) \). If infra-red (IR) renormalons are present in addition then strict convergence is prevented \[8\]. Since by construction eqs.(18) are a set of exact equations this indicates that these equations are insufficient to determine \( \mathcal{R}(Q) \) and correspondingly extra information must be added, presumably in the form of further OPE terms.

We shall now motivate this convergence claim by considering the simplified case of QCD with \( c = 0 \). In this limit \( F(x) = G(x) = 1/x \) and the continued function of eq.(19) becomes a continued fraction.

\[
\mathcal{R}(Q) = \frac{1}{|\rho_0| - \frac{\rho_2}{|\rho_0^{(1)}|} - \frac{\rho_2^{(1)}}{|\rho_0^{(2)}} - \ldots}.
\]

(20)

As we shall now show the successive convergents of this continued fraction are the diagonal \([n/n]\) Padé approximants of the original perturbation series for \( \mathcal{R}(Q) \) in eq.(1), in the RS where \( c_2 = c_3 = \ldots = c_k = \ldots = 0 \), the so-called 't Hooft scheme \[10\]. This is in accord with the recent observation by Gardi that in the “large- \( \beta_0 \)” limit diagonal Padé approximants are RS-invariant \[7\].

The diagonal \([n/n]\) Padé approximants are defined by

\[
\mathcal{R}[n/n] \equiv \frac{a + A_2a^2 + \ldots + A_na^n}{1 + B_1a + \ldots + B_n a^n} = a + r_1a^2 + r_2a^3 + \ldots + r_{2n-1}a^{2n} + O(a^{2n+1}),
\]

(21)

where the \( A_i \) and \( B_i \) coefficients are uniquely fixed by demanding that the \( r_i, i \leq 2n - 1 \) are reproduced on expanding the quotient.

Corresponding to the original perturbation series in eq.(1) one can define the standard Stieltjes \[11\] continued fraction for \( \mathcal{R}(Q) \),
\[ \mathcal{R}(Q) = \frac{K_1a}{|1|} + \frac{K_2a}{|1|} + \frac{K_3a}{|1|} + \ldots, \quad (22) \]

where \( K_1 = 1, K_2 = -r_1, K_3 = r_1 - r_2/r_1, \ldots \) The successive convergents of eq.(22) are respectively the diagonal \([n/n]\) and off-diagonal \([n/n + 1]\) Padé approximants to \( \mathcal{R}(Q) \). We can also define the so-called “associated” or Jacobi form of the continued fraction \([\]\).

\[ \mathcal{R}(Q) = \frac{K_1a}{|(1 + K_2a)|} - \frac{K_2K_3a^2}{|1 + (K_3 + K_4)a|} - \frac{K_4K_5a^2}{|1 + (K_5 + K_6)a|} - \ldots. \quad (23) \]

The convergents of eq.(23) are the diagonal \([n/n]\) Padé approximants of \( \mathcal{R} \). If we scale the partial quotients in eq.(23) by \( a \) we obtain

\[ \mathcal{R}(Q) = \frac{K_1a}{|(1/a) + K_2|} - \frac{K_2K_3a}{|(1/a) + K_3 + K_4|} - \frac{K_4K_5a^2}{|(1/a) + K_5 + K_6|} - \ldots. \quad (24) \]

In the ‘t Hooft scheme \((c_2 = c_3 = \ldots = 0)\), and with \( c = 0 \), \( \rho_0^{(n)} = (1/a) - r_1^{(n)} \), and so on comparing eq.(24) with eq.(20) we can identify \( \rho_0^{(n)} = (1/a) + K_{2n+1} + K_{2n+2} \) and \( \rho_2^{(n)} = K_{2n+2}K_{2n+3} \), and we see that indeed the successive (manifestly RS-invariant) convergents of eq.(20) are precisely the successive convergents of the Jacobi form of the continued fraction- the diagonal \( \mathcal{R}[n/n] \) Padé approximants.

The convergence of the diagonal Padé approximants is assured if the perturbation series for \( \mathcal{R}(Q) \) is asymptotic to a “Stieltjes function” \([\]\) \([\]\), and if this is the case this function is identical to the Borel sum of the series, if it exists \([\]\).

A Stieltjes function is of the form \([\]\)

\[ \mathcal{R}(a) = a \int_0^\infty dt \frac{W(t)}{(1 + ta)}, \quad (25) \]

where \( W(t) \) is nonnegative throughout the range of integration. For such a function the continued fraction coefficients \( K_n \) are all nonnegative \([\]\).

A relevant example, motivated by UV renormalon singularities \([\]\), is to consider a simple pole at \( z = -1 \) in the Borel plane,
\[ R(a) = \int_0^\infty dz \frac{e^{-z/a}}{(1 + z)} = a \int_0^\infty dt \frac{e^{-t}}{(1 + ta)}, \]

so that the non-negative Stieltjes weight function in eq.(25) is \( W(t) = e^{-t} \). This corresponds to perturbative coefficients \( r_n = (-1)^n n! \) which results in the (nonnegative) continued fraction coefficients \( K_1 = 1, K_{2n} = K_{2n+1} = n, n > 1 \), from which we can obtain \( \rho_0^{(n)} = (1/a) + 2n + 1 \) and \( \rho_2^{(n)} = (n + 1)^2 \) for the coefficients in eqs.(18). The continued fraction in eq.(20) then converges to the well-defined Borel sum in eq.(26). Branch point UV renormalon singularities can also be reduced to Stieltjes form by a change of variables. A subtlety is that if one changes the RS, and hence the definition of \( a \), the series may no longer be Stieltjes, equivalently the \( K_n \) are RS-dependent and will not all be nonnegative in a general RS. To prove convergence, however, it is only necessary that the series is Stieltjes in one particular RS, since the convergents are RS-invariant.

By analyzing the large-\( n \) behaviour of eqs.(18) using \( \rho_0^{(n)} \approx 2n \) and \( \rho_2^{(n)} \approx n^2 \) one can infer that for the above example \( R^{(n)}(Q) \approx 1/n \), so as \( n \to \infty \), \( R^{(n)}(Q) \to 0 \), and successive violations of UAS become ever smaller, underwriting the convergence of the truncations of eqs.(18).

To conclude, we regard the important feature of this approach to be its physical motivation based on iterating an exact property of massless QCD, which we have termed Universal Asymptotic Scaling, and which is defined in eq.(7). Thus if the infinite set of coupled equations in eqs.(18) has a solution this determines \( R(Q) \). It is incidental that this approach can reproduce the result of applying Borel summation to the original divergent asymptotic series. For instance if \( r_n = (-1)^n (2n)! \) the Borel method is inapplicable, nonetheless the set of coupled equations eqs.(18) in this case generates a continued fraction (assuming \( c = 0 \)) which converges to a unique Stieltjes function \( [12] \).

We plan to give a more complete discussion of the convergence of the approach, and the manner in which ambiguities induced by the presence of IR renormalons manifest themselves in this language, in a future work. The links with Padé methods are also interesting and should be explored further.
Acknowledgements

We thank Jan Fischer for a number of enjoyable discussions. The CERN TH Division is thanked for its hospitality during the period when this work was completed.
References

[1] For a recent review see: Jan Fischer, “On the Role of Power Expansions in Quantum Field Theory”, PRA-HEP 97/06 [hep-ph/9704351].

[2] G. Grunberg, Phys. Lett. 325 (1994) 441.

[3] P.M. Stevenson, Phys. Rev. D23 (1981) 2916.

[4] G. Grunberg, Phys. Lett. B95 (1980) 70; Phys. Rev. D29 (1984) 2315.

[5] G.P. Lepage and P.B. Mackenzie, Phys. Rev. D28 (1983) 228; S.J. Brodsky, G.P. Lepage and P.B. Mackenzie, Phys. Rev. D28 (1983) 228.

[6] D.T. Barclay, C.J. Maxwell and M.T. Reader, Phys. Rev. D49 (1994) 3480.

[7] Einan Gardi, “Why Padé approximants reduce the renormalization scale dependence in quantum field theory?” TAUP-2393-96 [hep-ph/9611453].

[8] M.A. Samuel, J. Ellis and M. Karliner, Phys. Rev. Lett. 74 (1995) 4380; J. Ellis, E. Gardi, M. Karliner and M.A. Samuel, Phys. Lett. B366 (1996) 268, and Phys. Rev. D54 (1996) 6986.

[9] W. Celmaster and R.J. Gonsalves, Phys. Rev. D20 (1979) 1420.

[10] G. ’t Hooft in: “The Whys of Subnuclear Physics”, Erice (1977), ed. A. Zichichi (Plenum, New York, 1977).

[11] George A. Baker, Jr. and Peter Graves-Morris: Padé Approximants (Second Edition); Cambridge University Press, 1996.

[12] Carl M. Bender and Steven A. Orszag: Advanced Mathematical Methods for Scientists and Engineers; McGraw-Hill Book Company, 1978, Chap.8.