Holographic Holes in Higher Dimensions

Sotaro Sugishita (Osaka Univ.)

with

Robert C. Myers & Junjie Rao

arXiv:1403.3416 (JHEP 1406 (2014) 044)

Osaka, April 12, 2016
Introduction

- Black hole has an entropy proportional to the area of the event horizon.

**Bekenstein-Hawking entropy**

\[ S = \frac{A}{4G_N} \]
**Introduction**

- Black hole has an entropy proportional to the area of the event horizon.

**Bekenstein-Hawking entropy**

\[
S = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}
\]

This formula seems to represent a fundamental property of quantum gravity.
Bekenstein-Hawking entropy

• The BH entropy formula can be applied to any Killing horizon. (Rindler, de Sitter, ...)

  ➢ How about other surfaces?
  ➢ Is area related to entropy?

• It is expected that BH entropy is entanglement entropy between the inside and outside of the horizon.

[Bombelli, Koul, Lee & Sorkin (1986), Srednicki (1993), Frolov & Novikov (1993), Solodukhin (2011) ...]
Entanglement Entropy

- a measure of how a system is entangled

Hilbert space \( \mathcal{H}_{\text{tot}} = \mathcal{H}_a \otimes \mathcal{H}_b \)

reduced density matrix \( \rho_a = \text{tr}_b \rho_{\text{tot}} \)

\[
S_a = -\text{tr}_a \rho_a \log \rho_a
\]

Ex: two spins

\[
|\psi\rangle = |\uparrow\rangle_a |\uparrow\rangle_b \quad \rightarrow \quad S_a = 0
\]

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_a |\uparrow\rangle_b + |\downarrow\rangle_a |\downarrow\rangle_b) \quad \rightarrow \quad S_a = \log 2
\]

entangled
Entanglement Entropy for QFT

✓ Divide a space into two regions.
✓ Trace over the d.o.f. in region b.

\[ \rho_a = \text{tr}_b \rho_{\text{tot}} \]
\[ S_a = -\text{tr}_a \rho_a \log \rho_a \quad \text{(geometric entropy)} \]

cf. momentum-space entanglement

• EE of the ground state for local QFT follows the area law.

\[ S_a \sim \frac{A(\partial a)}{\epsilon^{d-2}} \]

• Difficult to compute EE for general QFTs.
Holographic Entanglement Entropy

In the context of the AdS/CFT correspondence, a simple formula to compute EE is proposed.

\[ S(a) = \text{ext} \frac{A(c)}{4G_N} \]

The EE for a region in the boundary can be evaluated by the area of the minimal surface in the bulk.

Ryu & Takayanagi (2006)
The extremal surface is generally not a horizon.

\[ \text{the BH entropy of the extremal surface} = \text{entanglement entropy} \]

The table shows the correspondence between bulk and boundary quantities:

| Bulk                              | Boundary           |
|-----------------------------------|--------------------|
| BH entropy of black hole          | thermal entropy    |
| BH entropy of extremal surface    | entanglement entropy|
| BH entropy of general surfaces    | ???                |
“Hole-ographic” construction

Balasubramanian, Chowdhury, Czech, de Boer, Heller (2013)

It was shown that the surface area of a general hole in AdS$_3$ can be obtained by a combination of EE of 2-dim CFT using HEE formula.

\[
\frac{\text{(length)}}{4G_N}
\]
“Hole-ographic” construction

Balasubramanian, Chowdhury, Czech, de Boer, Heller (2013)

It was shown that the surface area of a general hole in AdS$_3$ can be obtained by a combination of EE of 2-dim CFT using HEE formula.

\[
\frac{(\text{length})}{4G_N} = \sum_{k}^{\infty} \left[ S(I_k) - S(I_k \cap I_{k+1}) \right]
\]

We call this combination of EE the **differential entropy**. This is a field theoretic quantity.
It was shown that the surface area of a general hole in AdS$_3$ can be obtained by a combination of EE of 2-dim CFT using HEE formula.

We generalized this construction to higher dimensional and more general backgrounds.  

Bekenstein-Hawking entropy of co-dim 2 surface = differential entropy
Holography

• How do we reconstruct a bulk spacetime from the boundary data?

Quantum Information is a key concept.
Contents

1. Introduction

2. Strong subadditivity and geometric inequality

3. Diff. entropy = BH entropy in AdS$_3$

4. Higher-dimensional and more general backgrounds

5. Discussion and summary
Contents

1. Introduction

2. Strong subadditivity and geometric inequality

3. Diff. entropy = BH entropy in AdS$_3$

4. Higher-dimensional and more general backgrounds

5. Discussion and summary
Strong Subadditivity

\[ S(I_1 \cup I_2) + S(I_1 \cap I_2) \leq S(I_1) + S(I_2) \]

This inequality is a fundamental property of EE. It can be proved by using properties of the von Neumann entropy. e.g. Nielsen & Chuang. As far as I know, the proof is not simple.
A holographic proof of strong subadditivity

If we use the HEE formula, we can give a simple geometrical proof of strong subadditivity.

$$S(I_1 \cup I_2) + S(I_1 \cap I_2) \leq S(I_1) + S(I_2)$$
A holographic proof of strong subadditivity

The following inequalities hold, since the left hand sides correspond to the minimal surfaces.
We have the following inequalities:

\[
S(I_1 \cup I_2) \leq \hat{S}(I_1, I_2) \leq S(I_1) + S(I_2) - S(I_1 \cap I_2)
\]

\[
\hat{S}(I_1, I_2) = \text{(area of outer envelope)}/4G
\]
Geometric inequality

\[ \hat{S}(I_1, I_2) \leq S(I_1) + S(I_2) - S(I_1 \cap I_2) \]

This inequality is stronger than strong subadditivity.

\[ S(I_1 \cup I_2) \leq \hat{S}(I_1, I_2) \]

global coordinates of AdS3

outer envelope
We consider the case where the intervals are chosen to cover the entire boundary.
We consider the case where the intervals are chosen to cover the entire boundary.

geometric inequality:

\[
\hat{S}(\{I_k\}) \leq \sum_k [S(I_k) - S(I_k \cap I_{k+1})]
\]

(length of outer envelope)/4G

When the bulk is dual to any pure state in the bdry, \( S(\bigcup I_k) = 0 \).

In contrast, the length of the outer envelope is finite, \( \hat{S}(\{I_k\}) \neq 0 \).
We take a continuum limit, i.e. take the number of intervals to infinity. In this limit, the outer envelope becomes a smooth curve.
We take a continuum limit, i.e. take the number of intervals to infinity. In this limit, the outer envelope becomes a smooth curve.

We will show that the geometric inequality is saturated in this limit.

\[
\frac{\text{(area)}}{4G_N} = \sum_k^\infty \left[ S(I_k) - S(I_k \cap I_{k+1}) \right]
\]

The differential entropy reconstructs the Bekenstein-Hawking entropy.
Contents

1. Introduction

2. Strong subadditivity and geometric inequality

3. Diff. entropy = BH entropy in AdS$_3$

4. Higher-dimensional and more general backgrounds

5. Discussion and summary
Holographic entanglement entropy in AdS$_3$

- evaluate the holographic EE of an interval of width $\Delta x$

EE in 2-dim CFT

- Poincaré coordinates of AdS$_3$

$$\text{d}s^2 = \frac{L^2}{z^2} \left( \text{d}z^2 - \text{d}t^2 + \text{d}x^2 \right)$$

(L : AdS radius)

boundary: $z=0$

- the extremal surface is given by a semi-circle:

$$z^2 + x^2 = \left( \frac{\Delta x}{2} \right)^2$$
Holographic entanglement entropy in $\text{AdS}_3$

- evaluating the length of the semi-circle

$$S(\Delta x) = \frac{1}{4G_N} \text{(length of the semi-circle)}$$

$$= \frac{L}{2G_N} \log \frac{\Delta x}{\delta} + O(\delta)$$

If we use the relation, $c = \frac{3L}{2G_N}$ Brown & Henneaux (1986)

this reproduces the famous result of 2-dim CFT,

$$S(\Delta x) = \frac{c}{3} \log \frac{\Delta x}{\delta} + \text{(non univerasal terms)}$$

Holzhey, Larsen, Wilczek (1994)
Calabrese, Cardy (2004)
We will confirm that the geometric inequality is saturated for general curves in AdS$_3$. Myers, SS, Rao (2014)

\[ ds^2 = \frac{L^2}{z^2} (dz^2 - dt^2 + dx^2) \]

- impose periodic boundary condition in x-direction. (period $\ell_1$)
- arbitrary closed curve: $z(x)$
- Bekenstein-Hawking entropy for this curve:

\[ \frac{A}{4G_N} = \frac{L}{4G_N} \int_0^{\ell_1} dx \frac{\sqrt{1 + z'^2}}{z} \]
• constant profile $z(x) = z_*$

• consider a series of $n$ equally spaced intervals with a width $\Delta x = 2z_*$

• the length of overlap: $o = \Delta x - \ell_1/n$

$$\sum_{k}^{n} [S(I_k) - S(I_k \cup I_{k+1})]$$

$$= \frac{nL}{2G_N} \left[ \log \left( \frac{\Delta x}{\delta} \right) - \log \left( \frac{o}{\delta} \right) \right]$$

$$= -\frac{nL}{2G_N} \log \left( 1 - \frac{\ell_1}{2nz_*} \right)$$

no UV divergence

• take the “continuum” limit ($n \to \infty$)

$$\frac{L \ell_1}{4G_N z_*} = \frac{A}{4G_N}$$

geom. inequality is saturated
arbitrary profile \( z(x) \)

\[
a(x) = z(x)z'(x) : \text{shift from the midpt. of interval to the tangent pt.}
\]

\[
\Delta x(x) = 2z(x)\sqrt{1 + z'(x)^2}
\]

\[
o_+ = \frac{1}{2}(\Delta x(x) + \Delta x(x + \,dx)) + (a(x) - a(x + \,dx)) - \,dx
\]

\[
= 2z\sqrt{1 + z'^2} - (1 + z'^2 + zz'')\,dx + \frac{\Delta x'}{2}\,dx
\]
arbitrary profile $z(x)$

\[
S(I_k) - \frac{1}{2}S(I_k \cap I_{k+1}) - \frac{1}{2}S(I_{k-1} \cap I_k) = \frac{L}{4G_N} \frac{1 + z'^2 + zz''}{z \sqrt{1 + z'^2}} \, dx + \mathcal{O}(dx^2)
\]

$a(x) = z(x)z'(x)$: shift from the midpt. of interval to the tangent pt.

$\Delta x(x) = 2z(x)\sqrt{1 + z'(x)^2}$

$o_+ = \frac{1}{2}(\Delta x(x) + \Delta x(x + dx)) + (a(x) - a(x + dx)) - dx$

\[
= 2z\sqrt{1 + z'^2} - (1 + z'^2 + zz'')dx + \frac{\Delta x'}{2}dx
\]
The differential entropy reproduces the BH entropy.

$$S(I_k) - \frac{1}{2} S(I_k \cap I_{k+1}) - \frac{1}{2} S(I_{k-1} \cap I_k)$$

$$= \frac{L}{4G_N} \frac{1 + z'^2 + zz''}{z \sqrt{1 + z'^2}} \, dx + O(dx^2)$$

$$\sum_{k=1}^{n} [S(I_k) - \frac{1}{2} S(I_k \cap I_{k+1}) - \frac{1}{2} S(I_{k-1} \cap I_k)]$$

$$= \frac{L}{4G_N} \int_0^{l_1} \frac{1 + z'^2 + zz''}{z \sqrt{1 + z'^2}} \, dx$$

$$= \frac{L}{4G_N} \int_0^{l_1} \left[ \frac{\sqrt{1 + z'^2}}{z} + \frac{z''}{\sqrt{1 + z'^2}} \right] \, dx = \frac{A}{4G_N}$$
Contents

1. Introduction

2. Strong subadditivity and geometric inequality

3. Diff. entropy = BH entropy in AdS₃

4. Higher-dimensional and more general backgrounds

5. Discussion and summary
Higher dimensions

We consider the following metric in (d+1)-dimensional background:

\[ ds^2 = -g_0(z)dt^2 + \sum_{i=1}^{d-1} g_i(z)(dx^i)^2 + g_1(z)f(z)dz^2 \]

bdry: \( z = 0 \)

e.g. planar AdS black hole

\[ g_0(z) = \frac{L^2}{z^2} \left( 1 - \frac{z^d}{z_h^d} \right), \quad g_i(z) = \frac{L^2}{z^2}, \quad f(z) = \left( 1 - \frac{z^d}{z_h^d} \right)^{-1} \]

assume that \( x^i \)-directions are periodic with periods \( \ell^i \)
We only consider the case where const. time slice in the bdry is partitioned by a set of overlapping strips as the following fig.

assume that bulk surfaces have translational sym. in $x^j$-directions ($j=2,\ldots,d-1$),
i.e. the bulk profiles have the form $z = z(x^1)$
We demand that there be the extremal surface which is tangent to the given profile $z(x_1)$ at each point $x_1$.

$$z = h(\tilde{x}; x_1):$$ the extremal surfaces which are tangent to the bulk surface at $x_1$.

$$\begin{cases}
    h(\tilde{x} = x_1; x_1) = z(x_1) \\
    \frac{dh}{d\tilde{x}}(\tilde{x} = x_1; x_1) = z'(x_1)
\end{cases}$$

$h_0(x_1)$: the maximal height of $h(\tilde{x}; x_1)$
Holographic EE of a strip of width $\Delta x$

$$S(\Delta x) = \frac{\ell_2 \cdots \ell_{d-1}}{4G_N} \int_0^{\Delta x} dx \sqrt{G(h)(1 + f(h)h'^2)}$$

$$G(z) = g_1(z) \cdots g_{d-1}(z)$$

- the extremal curve $h$ satisfies $$\frac{\sqrt{G(h)}}{\sqrt{(1 + f(h)h'^2)}} = \sqrt{G(h_0)}$$

- the widths of the intersections

$$o_\pm = \Delta x - (1 + a' \mp \Delta x')dx$$

combination of EE

$$S(\Delta x) - \frac{1}{2}[S(o_+) + S(o_-)] = \frac{dS}{d\Delta x} (1 + a')dx$$

to leading order in $dx$
We can show that the difference of the integrands is a total derivative.

\[
E = \frac{A}{4G_N}
\]

The geom. inequality is saturated.
Contents

1. Introduction

2. Strong subadditivity and geometric inequality

3. Diff. entropy = BH entropy in AdS$_3$

4. Higher-dimensional and more general backgrounds

5. Discussion and summary
Discussion 1/2

• Interpretation of the differential entropy

Susskind & Witten (1998)

The Bekenstein-Hawking entropy for a cut-off surface gives a number of degrees of freedom for the boundary theory.

\[ \frac{A}{4G_N} \sim N^2 \cdot \frac{V}{\delta^3} \]

The diff entropy for a constant profile in AdS$_3$

\[ E \sim c \cdot \frac{\ell_1}{z_*} \]
Discussion 2/2

• Reconstruction of the bulk geometry from QFT

The linearized Einstein eq. was obtained from the “first law” of EE using Ryu-Takayanagi formula.

[Lashkari, McDermott, Van Raamsdonk (2013), Faulkner, Guica, Hartman, Myers, Van Raamsdonk (2013), Swingle & Van Raamsdonk (2014), Jacobson (2015)]

By considering small surfaces, we can evaluate local geometry in the bulk from the diff entropy.

It is interesting to find a property of diff entropy like the “first law” of EE.
Summary

- BH entropy of a class of surfaces can be evaluated by the differential entropy.
- this construction is extended to more general surfaces and backgrounds which also include time dependence.
  
  [Czech, Dong & Sully (2014), Headrick, Myers & Wien (2014)]

- our construction also extends to higher derivative gravity.

- we should find a direct interpretation of the differential entropy in terms of the boundary theory.
  
  [Czech, Lamprou, McCandlish, Sully (2015)]

- Our results provide the relation between geometry and entanglement, which is a key concept in understanding quantum gravity.
The spacetime entanglement conjecture proposed that in a theory of quantum gravity, the short-range entanglement entropy between the degrees of freedom describing any large region and its complement is finite and described in terms of geometry of the entangling surface and the leading contribution is given by the BH entropy formula.

\[ S = \frac{A}{4G_N} + \cdots \]

assuming that the Einstein-Hilbert action emerges as the low-energy effective gravitational action.