Tensor Factor Model Estimation by Iterative Projection

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Abstract

Tensor time series, which is a time series consisting of tensorial observations, has become ubiquitous. It typically exhibits high dimensionality. One approach for dimension reduction is to use a factor model structure, in a form similar to Tucker tensor decomposition, except that the time dimension is treated as a dynamic process with a time dependent structure. In this paper we introduce two approaches to estimate such a tensor factor model by using iterative orthogonal projections of the original tensor time series. The approaches extend the existing estimation procedures and our theoretical investigation shows that they improve the estimation accuracy and convergence rate significantly. The developed approaches are similar to higher order orthogonal projection methods for tensor decomposition, but with significant differences and theoretical properties. Simulation study is conducted to further illustrate the statistical properties of these estimators.

1 Introduction

Motivated by a diverse range of modern scientific applications, analysis of tensors, or multi-dimensional arrays, has emerged as one of the most important and active research areas in statistics, computer science, and machine learning. Large tensors are encountered in genomics (Alter and Golub, 2005, Omberg et al., 2007), neuroimaging analysis (Sun and Li, 2017, Zhou et al., 2013), recommender systems (Bi et al., 2018), computer vision (Liu et al., 2012), community detection (Anandkumar et al., 2014), among others. High-order tensors often bring about high dimensionality and impose significant computational challenges. For example, functional MRI produces a time series of 3-dimensional brain images, typically consisting of hundreds of thousands of voxels observed over time. Previous work has developed various tensor-based methods for independent and identically distributed (i.i.d.) tensor data or tensor data with i.i.d. noise. However, as far as we know, the statistical framework for general tensor time series data was not well studied in the literature.

Factor analysis is one of the most useful tools for understanding common dependence among multi-dimensional outputs. Over the past decades, vector factor models have been extensively studied in the statistics and economics communities. For instance, Chamberlain and Rothschild (1983), Bai and Ng (2002), Bai (2003) and Stock and Watson (2002) developed the static factor model using principal component analysis (PCA). They assumed that the common factors must have impact on most of the time series, and weak serial dependence is allowed for the idiosyncratic

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noise process. Fan et al. (2011, 2013, 2018) established large covariance matrix estimation based on the static factor model. The static factor model has been further extended to the dynamic factor model in Forni et al. (2000). The latent factors are assumed to follow a time series process, which is commonly taken to be a vector autoregressive process. Fan et al. (2016) studied semi-parametric factor models through projected principal component analysis. Pan and Yao (2008), Lam et al. (2011) and Lam and Yao (2012) adopted another type of factor model. They assumed that the latent factors should capture all dynamics of the observed process, and thus the idiosyncratic noise process has no serial dependence.

Although there have been significant efforts in developing methodologies and theories for vector factor models, there is paucity of literature on matrix- or tensor-valued time series. Wang et al. (2019) proposed a matrix factor model for matrix-valued time series, which explores the matrix structure. Chen et al. (2019a) established a general framework for incorporating domain and prior knowledge in the matrix factor model through linear constraints. Chen and Chen (2019) applied the matrix factor model to the dynamic transport network. Chen et al. (2020) developed an inferential theory of the matrix factor model under a different setting from that in Wang et al. (2019).

Recently, Chen et al. (2019b) introduced a factor approach for analyzing high dimensional dynamic tensor time series in the form

\[ \mathcal{X}_t = M_t + E_t, \]  

where \( \mathcal{X}_1, \ldots, \mathcal{X}_T \in \mathbb{R}^{d_1 \times \cdots \times d_K} \) are the observed tensor time series, \( M_t \) and \( E_t \) are the corresponding signal and noise components of \( \mathcal{X}_t \), respectively. The goal is to estimate the unknown signal tensor \( M_t \) from the tensor time series data. Following Lam and Yao (2012), it is assumed that the signal part accommodates all dynamics, making the idiosyncratic noise \( E_t \) uncorrelated (white) across time. It is further assumed that \( M_t \) is in a lower dimensional space and has certain multilinear decomposition. Specifically, we assume that \( M_t \) satisfies the following Tucker-type decomposition.

Then model (1) can be rewritten as

\[ \mathcal{X}_t = F_t \times_1 A_1 \times_2 \ldots \times_K A_K + E_t, \]  

where \( A_k \) is the deterministic loading matrix of size \( d_k \times r_k \) and \( r_k \ll d_k \), and the core tensor \( F_t \) itself is a latent tensor factor process of dimension \( r_1 \times \ldots \times r_K \). Here the \( k \)-mode product of \( \mathcal{X} \in \mathbb{R}^{d_1 \times d_2 \times \cdots \times d_K} \) with a matrix \( U \in \mathbb{R}^{d_k \times d_k} \), denoted as \( \mathcal{X} \times_k U \), is an order \( K \)-tensor of size \( d_1 \times \cdots \times d_{k-1} \times d_k^2 \times d_{k+1} \times \cdots \times d_K \) such that

\[ (\mathcal{X} \times_k U)_{i_1, \ldots, i_{k-1}, j, i_{k+1}, \ldots, i_K} = \sum_{i_k=1}^{d_k} \mathcal{X}_{i_1, i_2, \ldots, i_K} U_{j, i_k}. \]

The core tensor \( F_t \) is usually much smaller than \( \mathcal{X}_t \) in dimension. This structure provides an effective dimension reduction, as all the comovements of individual time series in \( \mathcal{X}_t \) are driven by \( F_t \). Without loss of generality, assume that \( A_k \) is of rank \( r_k \ll d_k \). It should be noted that vector and matrix factor models can be viewed as special cases of our model since a vector time series is a tensor time series composed of a single fiber and a matrix times series is one composed of a single slice.

Chen et al. (2019b) proposed two estimation procedures (TOPUP and TIPUP) for estimating the column space spanned by the loading matrix \( A_k \), for \( k = 1, \ldots, K \). Their procedures are based on two matrix unfolding operations of the auto-cross-moment of the original tensors \( \mathcal{X}_t \) and
$\mathcal{X}_{t-h}$, $h > 0$, and utilizing the assumption that the noise $\mathcal{E}_t$ and $\mathcal{E}_{t-h}$, $h > 0$ are uncorrelated. The convergence rates of their estimators critically depend on $d = d_1d_2\ldots d_K$, a potentially very large number as $d_k$, $k = 1, \ldots, K$, are large. Often they require a large $T$, the length of the time series, for accurate estimation of the loading spaces.

In this paper we propose two extensions of the estimation approaches taken by Chen et al. (2019b), motivated by the following observation. Suppose that the loading matrices $A_k$ are orthonormal with $A_k^T A_k = I$, and we are given $A_2, \ldots, A_K$. Let

$$\mathcal{Z}_t = \mathcal{X}_t \times_2 A_2^\top \times_3 \ldots \times_K A_K^\top; \quad \text{and} \quad \mathcal{E}_t^* = \mathcal{E}_t \times_2 A_2^\top \times_3 \ldots \times_K A_K^\top;$$

Then (2) leads to

$$\mathcal{Z}_t = \mathcal{F}_t \times_1 A_1 + \mathcal{E}_t^*$$

where $\mathcal{Z}_t$ is a $d_1 \times d_2 \times \ldots \times d_k$ tensor. Since $r_k \ll d_k$, $\mathcal{Z}_t$ is a much smaller tensor than $\mathcal{X}_t$. Under proper (weak) conditions on the combined noise tensor $\mathcal{E}_t^*$, estimation of the loading space of $A_1$ based on $\mathcal{Z}_t$ can be made significantly more accurate, as the convergence rate now depends on $d_1d_2\ldots d_k$ rather than $d_1d_2\ldots d_k$.

Of course, in practice we do not know $A_2, \ldots, A_K$. Similar to backfitting algorithms, we propose an iterative algorithm. With a proper initial value, we iteratively estimate the loading space of $A_k$ at iteration $j$ based on

$$\mathcal{Z}_{t,k}^{(j)} = \mathcal{X}_t \times_1 \hat{A}_1^{(j)} \times_2 \ldots \times_k \hat{A}_{k-1}^{(j)} \times_{k+1} \hat{A}_{k+1}^{(j-1)} \times_{k+2} \ldots \times_K \hat{A}_K^{(j-1)}\top,$$

using the estimate $\hat{A}_k^{(j-1)}$, $k < k' \leq K$ obtained in the previous iteration and the estimate $\hat{A}_k^{(j)}$, $1 \leq k' < k$ obtained in the current iteration. Our theoretical investigation shows that the iterative procedures for estimating $A_1$ can achieve the convergence rate as if all $A_2, \ldots, A_K$ are known and we indeed observe $\mathcal{Z}_t$ following model (3). We call the procedure iTOPUP and iTIPUP, based on the matrix unfolding mechanism used, corresponding to TOPUP and TIPUP procedures of Chen et al. (2019b). To be more specific, our algorithms have two steps: (i) We first use the estimated column space of factor loading matrices of TOPUP (resp. TIPUP) in Chen et al. (2019b) to construct the initial estimate of factor loading spaces; (ii) We then iteratively perform matrix unfolding of the auto-cross-moments of much smaller tensors $\mathcal{Z}_{t,k}^{(j)}$ to obtain the final estimators.

We note that the iterative procedure is similar to higher order orthogonal iteration (HOOI) that has been widely studied in the literature; see, e.g., De Lathauwer et al. (2000), Sheehan and Saad (2007), Liu et al. (2014), Zhang and Xia (2018), among others. However, most of the existing works are not designed for tensor time series. They do not consider the special role of the time mode nor the covariance structure in the time direction. Often the signal part are treated as fixed or deterministic. In this paper we focus on the setting that the core tensor $\mathcal{F}_t$ in (2) is dynamic which requires special treatment. Although our iterative procedures in each iteration also consist of power up and orthogonal projection operations, similar to HOOI, the matrix unfolding operation is on the auto-cross-moments that pays special attention to the time mode. Although iTOPUP proposed here can be reformulated as a twist of HOOI on the auto-covariance tensor, iTIPUP is different and cannot be recast equivalently as a HOOI. More importantly, the theoretical investigation and theoretical properties of the estimators are fundamentally different from those of HOOI, due to the dynamic structure of tensor time series and the use of auto-cross-moments, instead of covariance matrices.
In this paper, we establish upper bounds on the estimation errors for both the iTOPUP and the iTIPUP, which are much sharper than the respective theoretical guarantees for TOPUP and TIPUP in Chen et al. (2019b), demonstrating the benefits of iterative projection. It is also shown that our algorithms converge within a logarithmic number of iterations. We mainly focus on the cases where the tensor dimensions are large and of similar order. We also cover the cases where the dimensions and the ranks of the tensor factor increase with the dimensions of the tensor time series. Alternative methods to the iTOPUP and iTIPUP are discussed.

Chen et al. (2019b) showed that the TIPUP has a faster rate than the TOPUP, under a mild condition on the level of signal cancellation. In contrast, the theoretically guaranteed rate of convergence for the iTOPUP in this paper is of the same order or even faster than that for the iTIPUP under certain regularity conditions. Our results also suggest an interesting phenomenon. Using the iterative procedures, we find that both the increase in dimension and in sample size can improve the estimation of the factor loading space of the tensor factor model with the tensor order $K \geq 2$. We believe that such a super convergence rate is new in the literature. Specifically, under the strong factor conditions, the convergence rate of the iterative procedures for estimating the space of $A_k$ is $O_p(T^{-1/2}d_{-k}^{-1/2})$, where $d_{-k} = \prod_{j \neq k} d_j$, while the traditional rate of the non-iterative procedures is $O_p(T^{-1/2})$ for univariate factor model (Lam et al., 2011), matrix/tensor factor model (Chen et al., 2019b, Wang et al., 2019). While the increase in the dimensions $d_k$ ($k = 1, \ldots, K$) will not improve the performance of the non-iterative estimators, it significantly improves that of the proposed iterative estimators.

The paper is organized as follows. Section 2.1 introduces basic notation and preliminaries of tensor analysis. We present the tensor factor model and the procedures for the iTOPUP and iTIPUP in Section 2.2 and 2.3. Theoretical properties of the iTOPUP and iTIPUP are investigated in Section 3. Numerical comparison of our iterative procedures and other methods is given in Section 4. Section 5 provides a brief summary. All technical details are relegated to Appendix.

2 Tensor Factor Model by Orthogonal Iteration

2.1 Notation and preliminaries for tensor analysis

Throughout this paper, for a vector $x = (x_1, \ldots, x_p)^\top$, define $\|x\|_q = (x_1^q + \ldots + x_p^q)^{1/q}$, $q \geq 1$. For a matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, write the SVD as $A = U\Sigma V^\top$, where $\Sigma = \text{diag}(\sigma_1(A), \sigma_2(A), \ldots, \sigma_{\min(m,n)}(A))$, with the singular values $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_{\min(m,n)}(A) \geq 0$ in descending order. The matrix Frobenius norm can be denoted as $\|A\|_F = (\sum_{i,j} a_{ij}^2)^{1/2} = (\sum_{i=1}^{\min(m,n)} \sigma_i^2(A))^{1/2}$. Define the spectral norm

$$\|A\|_s = \max_{\|x\|_2 = 1, \|y\|_2 = 1} \|x^\top A y\|_2 = \sigma_1(A).$$

Let $\sigma_{\min}(A)$ (resp. $\sigma_{\max}(A)$) be the smallest (resp. largest) nontrivial singular value of $A$. Denote the projection operator onto the column space of $A$ as $P_A = A(A^\top A)^\top A^\top$, where $(\cdot)^\top$ is the Moore-Penrose pseudo-inverse. Based on the SVD $A = U\Sigma V^\top$ with $\Sigma$ nonsingular, $P_A$ can be equivalently written as $P_A = U U^\top$. For any two matrices $A \in \mathbb{R}^{m_1 \times r_1}$, $B \in \mathbb{R}^{m_2 \times r_2}$, denote the Kronecker product $\otimes$ as $A \otimes B \in \mathbb{R}^{m_1 m_2 \times r_1 r_2}$. Let $\xi = (\xi_1, \ldots, \xi_p)^\top$ be a random vector. For two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, write $a_n = O(b_n)$ (resp. $a_n \asymp b_n$) if there exists a constant $C$ such that $|a_n| \leq C|b_n|$ (resp. $1/C \leq a_n/b_n \leq C$) holds for all sufficiently large $n$, and write $a_n = o(b_n)$ if $\lim_{n \to \infty} a_n/b_n = 0$. Write $a_n \preceq b_n$ (resp. $a_n \succeq b_n$) if there exist a constant $C$ such that $a_n \leq Cb_n$.
Denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. We use $C, C_1, c, c_1, \ldots$ to denote generic constants, whose actual values may vary from line to line.

For any two matrices with orthonormal columns, say, $U$ and $\hat{U}$, suppose the singular values of $U^T \hat{U}$ are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$. A natural measure of distance between the column spaces of $U$ and $\hat{U}$ is then

$$\|\hat{U} - U\|_S = \sqrt{1 - \sigma_r^2},$$

(4)

which equals the sine of the largest principle angle between the column spaces of $U$ and $\hat{U}$.

For any two tensors $A \in \mathbb{R}^{m_1 \times m_2 \times \cdots \times m_K}$, $B \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_N}$, denote the tensor product $\otimes$ as $A \otimes B \in \mathbb{R}^{m_1 \times \cdots \times m_K \times n_1 \times \cdots \times n_N}$, such that

$$(A \otimes B)_{i_1, \ldots, i_K, j_1, \ldots, j_N} = (A)_{i_1, \ldots, i_K} (B)_{j_1, \ldots, j_N}.$$

Let vec($\cdot$) be the vectorization of matrices and tensors. The mode-$k$ unfolding (or matricization) is defined as $\text{mat}_k(A)$, which maps a tensor $A$ to a matrix $\text{mat}_k(A) \in \mathbb{R}^{m_k \times \prod_{i \neq k} m_i}$. For example, if $A \in \mathbb{R}^{m_1 \times m_2 \times m_3}$, then

$$(\text{mat}_1(A))_{i,j+m_2(k-1)} = (\text{mat}_2(A))_{j,(k+m_3(i-1))} = (\text{mat}_3(A))_{k,(i+m_1(j-1))} = A_{ijk}.$$

The tensor Hilbert Schmidt norm for a tensor $A \in \mathbb{R}^{m_1 \times m_2 \times \cdots \times m_K}$ is defined as

$$\|A\|_{\text{HS}} = \left(\sum_{i_1=1}^{m_1} \cdots \sum_{i_K=1}^{m_K} \|A_{i_1,\ldots,i_K}\|^2\right)^{\frac{1}{2}}.$$

Define the tensor operator norm for an order-4 tensor $A \in \mathbb{R}^{m_1 \times m_2 \times m_3 \times m_4}$,

$$\|A\|_{\text{op}} = \max \left\{ \sum_{i_1,i_2,i_3,i_4} u_{i_1,i_2} \cdot u_{i_3,i_4} \cdot \|A_{i_1,i_2,i_3,i_4}\| : \|U_1\|_F = \|U_2\|_F = 1 \right\},$$

where $U_1 = (u_{i_1,i_2}) \in \mathbb{R}^{m_1 \times m_2}$ and $U_2 = (u_{i_3,i_4}) \in \mathbb{R}^{m_3 \times m_4}$.

2.2 Tensor factor model

Again, we consider

$$X_t = F_t \times_1 A_1 \times_2 \cdots \times_K A_K + \varepsilon_t.$$

Without loss of generality, assume that $A_k$ is of rank $r_k$. As discussed in Chen et al. (2019b), $A_k$ is not necessarily orthonormal, which is different from the classical Tucker decomposition (Tucker (1966)). Model (2) is unchanged if we replace $(A_1, \ldots, A_K, F_t)$ by $(A_1 H_1, \ldots, A_K H_K, F_t \times_{k=1}^{K} H_k^{-1})$ for any invertible $r_k \times r_k$ matrix $H_k$. We can always rotate an estimated factor loading matrix whenever appropriate. Although $(A_1, \ldots, A_K, F_t)$ are not uniquely determined, the factor loading space, that is, the linear space spanned by the columns of $A_k$, is uniquely defined. Denote the orthogonal projection to the column space of $A_k$ as

$$P_k = P_{A_k} = A_k(A_k^T A_k)^{-1} A_k^T.$$

(5)

We use $P_k$ to represent the factor loading space of $A_k$. If $A_k$ has SVD $A_k = U_k \Lambda_k V_k^T$, then $P_k = U_k U_k^T$. The canonical representation of the tensor times series (2) is written as

$$X_t = F_t^{(\text{cano})} \times_{k=1}^{K} U_k + \varepsilon_t,$$
where \( U_k \) is the left singular matrix of \( A_k \) and the core tensor \( \mathcal{F}_t^{cmpo} = \mathcal{F}_t \times_{k=1}^K (A_k V_k^\top) \) absorbs the diagonal and right singular matrices of \( A_k \). In this canonical form, the loading matrices \( U_k \) are identifiable up to a rotation in general and up to a permutation and sign changes of the columns of \( U_k \) when the singular values are all distinct in the population version of the TOPUP or TIPUP methods, as we describe in Section 2.3 below. In what follows, we may identify the tensor time series as its canonical form, i.e. \( A_k = U_k \), without explicit declaration.

In this paper, we consider more powerful estimation procedures to achieve sharper convergence rates than the ones in Chen et al. (2019b). It is worth emphasizing that we do not impose any specific structure for the dynamics of the core tensor factor process \( \mathcal{F}_t \in \mathbb{R}^{r_1 \times \cdots \times r_K} \). The estimation procedures we use also do not require any additional structure on the noise process \( \mathcal{E}_t \). Once we have obtained an estimator of \( A_k \), say \( \hat{A}_k \), which is actually \( \hat{U}_k \) in the canonical representation, a natural estimator for the core factor process is

\[
\hat{\mathcal{F}}_t = \mathcal{X}_t \times_1 \hat{A}_1^\top \times_2 \cdots \times_K \hat{A}_K^\top.
\]

The resulting residuals are

\[
\hat{\mathcal{E}}_t = \mathcal{X}_t - \mathcal{X}_t \times_1 \hat{P}_1^\top \times_2 \cdots \times_K \hat{P}_K^\top,
\]

where

\[
\hat{P}_k = \hat{A}_k (\hat{A}_k^\top \hat{A}_k)^{-1} \hat{A}_k^\top
\]

is the estimated projection to the column space of \( A_k \). A parsimonious fitting for \( \hat{\mathcal{F}}_t \) may be obtained by appropriately rotating \( \hat{\mathcal{F}}_t \) (see, e.g., Tiao and Tsay (1989)). Such a rotation is equivalent to replacing each \( \hat{A}_k \) by \( \hat{A}_k H_k \) with an appropriate orthogonal matrix \( H_k \).

The lagged cross-product of the tensor version of the autocovariance is denoted as \( \Sigma_h \), which can be viewed as an order-2\( K \) tensor,

\[
\Sigma_h = \mathbb{E} \left( \sum_{t=h+1}^T \frac{\mathcal{X}_{t-h} \otimes \mathcal{X}_t}{T-h} \right) = \mathbb{E} \left( \sum_{t=h+1}^T \frac{\mathcal{M}_{t-h} \otimes \mathcal{M}_t}{T-h} \right) \in \mathbb{R}^{d_1 \times \cdots \times d_K \times d_1 \times \cdots \times d_K},
\]

for \( h = 1, \ldots, h_0 \). As \( \mathcal{M}_t = \mathcal{M}_t \times_{k=1}^K P_k \) for all \( t \),

\[
\Sigma_h = \Sigma_h \times_{k=1}^{2K} P_k = \mathbb{E} \left( \sum_{t=h+1}^T \frac{\mathcal{F}_{t-h} \otimes \mathcal{F}_t}{T-h} \right) \times_{k=1}^{2K} P_k A_k,
\]

with the notation \( A_k = A_{k-K} \) and \( P_k = P_{k-K} \) for all \( k > K \). Let \( A_k = U_k \Lambda_k V_k^\top \) be the SVD of \( A_k \), where the columns of \( U_k \) and \( V_k \) are the left singular vectors and right singular vectors, respectively. Then \( P_k = U_k U_k^\top \). The following estimating procedures are based the sample version of \( \Sigma_h \) (\( h = 1, \ldots, h_0 \)),

\[
\hat{\Sigma}_h = \sum_{t=h+1}^T \frac{\mathcal{X}_{t-h} \otimes \mathcal{X}_t}{T-h}, \quad h = 1, \ldots, h_0.
\]

Here, the term with \( h = 0 \) should be excluded, as the contemporary covariance structure of \( \mathcal{E}_t \), \( \mathbb{E}(\mathcal{E}_t \otimes \mathcal{E}_t) \neq 0 \), is involved in \( \Sigma_0 \).
2.3 Estimating procedures

We start with a quick description of the TOPUP and TIPUP procedures of Chen et al. (2019b). They serve as the starting point of our proposed iTOPUP and iTIPUP procedures.

(i). Time series Outer-Product Unfolding Procedure (TOPUP):

Let $X_{1:T} = (X_1, \ldots, X_T)$. Define an order-5 tensor as

$$\text{TOPUP}_k(X_{1:T}) = \left( \sum_{t=h+1}^{T} \frac{\text{mat}_k(X_{t-h}) \otimes \text{mat}_k(X_t)}{T-h}, \ h = 1, \ldots, h_0 \right),$$

where $h_0$ is a predetermined positive integer. Here we note that TOPUP$_k(\cdot)$ is a function mapping a tensor time series to an order-5 tensor.

Let $d_{-k} = d/d_k$ with $d = \prod_{k=1}^{K} d_k$. Then TOPUP$_k(X_{1:T})$ is a tensor of dimension $d_k \times d_{-k} \times d_k \times d_{-k} \times h_0$, and mat$_1(\text{TOPUP}(X_{1:T}))$ is a matrix of dimension $d_k \times (d^2 h_0/d_k)$. In the definition of TOPUP$_k(X_{1:T})$, the information from different time lags is accumulated, which is useful especially when the sample size $T$ is small. A relatively small $h_0$ is typically used, since the autocorrelation is often at its strongest with small time lags. Larger $h_0$ strengthens the signal, but also adds more noise in the calculation of (7); see, e.g., Wang et al. (2019).

The TOPUP method in Chen et al. (2019b) performs SVD of the mode-1 matrix unfolding of (7) to obtain the truncated left singular matrices

$$\hat{U}_{k,m}^{\text{TOPUP}}(X_{1:T}) = \text{LSVD}_m(\text{mat}_1(\text{TOPUP}_k(X_{1:T}))),$$

where LSVD$_m$ stands for the left singular matrix composed of the first $m$ left singular vectors corresponding to the largest $m$ singular values. Because of the equivalence of SVD and eigen decomposition, we can take an alternative approach by computing the eigen decomposition of mat$_1(\text{TOPUP}_k(X_{1:T}))$ mat$^T_1(\text{TOPUP}_k(X_{1:T}))$) to obtain $\hat{U}_{k,m}^{\text{TOPUP}}(X_{1:T})$.

For simplicity, we write

$$\text{UTOPUP}_k(X_{1:T}, r_k) = \hat{U}_{k,r_k}^{\text{TOPUP}}(X_{1:T}),$$

where $r_k$ is the mode-$k$ rank. Again, we emphasize that UTOPUP$_k(\cdot)$ takes input of a tensor time series of length $T$ with the target mode-$k$ having dimension $d_k$ and rank $r_k$, and produces an output matrix of size $d_k \times r_k$ as the estimate of the mode-$k$ loading matrix.

As the expectation satisfies

$$\mathbb{E} [\text{mat}_1(\text{TOPUP}_k(X_{1:T}))] = A_k \text{mat}_k \left( \sum_{t=h+1}^{T} \mathbb{E} \left( \frac{\mathcal{F}_{t-h} \otimes \mathcal{F}_t}{T-h} \right) \times \frac{1}{T-h} \right),$$

where

$$\mathbb{E} \left( \frac{\mathcal{F}_{t-h} \otimes \mathcal{F}_t}{T-h} \right) \times \frac{1}{T-h} = \left( A_{t} \times \frac{1}{T-h} \right)_l = A_{t} \times \frac{1}{T-h} A_{t}, \ h = 1, \ldots, h_0,$$

the TOPUP is expected to be consistent in estimating the column space of $A_k$.

(ii). Time series Inner-Product Unfolding Procedure (TIPUP):

Similar to (7), define a $d_k \times (d_k h_0)$ matrix as

$$\text{TIPUP}_k(X_{1:T}) = \text{mat}_1 \left( \sum_{t=h+1}^{T} \frac{\text{mat}_k(X_{t-h}) \text{mat}^T_k(X_{t})}{T-h}, \ h = 1, \ldots, h_0 \right),$$

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which replaces the tensor product by the inner product in (7). The TIPUP method performs SVD on TIPUP\(_k(X_{1:T})\):

\[
\hat{U}_{k,m}^{\text{TIPUP}}(X_{1:T}) = \text{LSVD}_{m}(\text{TIPUP}_k(X_{1:T}))
\]

for \(k = 1, ..., K\). Again, for simplicity, we write

\[
\text{UTIPUP}_k(X_{1:T}, r_k) = \hat{U}_{k,r_k}^{\text{TIPUP}}(X_{1:T}).
\]

where \(r_k\) is the mode-\(k\) rank.

Note that

\[
\mathbb{E}[\text{TIPUP}_k(X_{1:T})]
= \text{mat}_1 \left( \langle \Sigma_h, \mathcal{I}_{k,k+K} \rangle, h = 1, ..., h_0 \right)
= A_k \text{mat}_1 \left( \left\langle \sum_{t=h+1}^{T} \mathbb{E} \left( \frac{F_{t-h} \otimes F_t}{T-h} \right) \times \delta_{k,k,l} \ell_I, 1 \leq l \leq 2K A_t, \mathcal{I}_{k,k+K} \right\rangle, h = 1, ..., h_0 \right),
\]

where \(\mathcal{I}_{k,k+K}\) is an order-2\(K\) tensor with elements \((\mathcal{I}_{k,k+K})_{i,j} = I(i_{-k} = j_{-k}), \ i = (i_1, ..., i_K), \ j = (j_1, ..., j_K)\), \(i_{-k} = (i_1, ..., i_{k-1}, i_{k+1}, ..., i_K), \ j_{-k} = (j_1, ..., j_{k-1}, j_{k+1}, ..., j_K)\). Here, \(\langle \cdot, \cdot \rangle_{\{k,k+K\}}\) is defined as an inner product summation over all indices other than \(\{k, k+K\}\).

The pseudo-code for a generic iterative procedure, under the motivation described in Section 1, is provided in Algorithm 1. It incorporates two estimators/operators UINIT and UITER that map a tensor time series to an estimate of the loading matrix. UTOPUP and UTIPUP operators in (9) and (12) are examples of the operators.

When we use the TOPUP operator (9) for both UINIT and UITER in Algorithm 1, it will be called iTOPUP procedure. Similarly, iTIPUP uses TOPUP operator (12) for both UINIT and UITER. Besides these two versions, we may also use TIPUP for UINIT and TOPUP for UITER, named as TIPUP-iTOPUP. Similarly, TOPUP-iTIPUP denotes the procedure with TOPUP as UINIT and TIPUP as UITER. These variants are sometimes useful, because TOPUP and TIPUP have different theoretical properties as the initializer or for iteration, as discussed in Section 3. Other estimators of the loading spaces based on the tensor time series can also be used in place of UINIT and UITER, including those briefly mentioned in Chen et al. (2019b), such as the conventional high order SVD for tensor decomposition, which we refer to as Unfolding Procedure (UP), that simply performs SVD of the matricization along the appropriate mode of the \((K+1)\) order tensor \((X_1, ..., X_T)\) with time dimension as the additional \((K+1)\)-th mode.

**Remark 1.** As the outer product is taken with TOPUP\(_k\) in (7), the iTOPUP can be carried out by applying essentially HOOI to the order \(2K+1\) auto-cross-moment tensor \(\hat{\Sigma}_h\) in (6) with the following iteration: With \(U_{K+\ell}^{(j)} = U_{\ell}^{(j)}\) for all \(j \geq 0\),

\[
\hat{U}_{k}^{(j)} = \text{LSVD}_{r_k} \left( \text{mat}_{k}\left( \hat{\Sigma}_h \times \ell \subseteq L_k^- (\hat{U}_{\ell}^{(j)})^\top \times \ell \subseteq L_k^+ (\hat{U}_{\ell}^{(j-1)})^\top, h = 1, ..., h_0 \right) \right)
\]

where \(L_k^- = \{(1 : (k-1)) \cup (K+1) : (K+k-1)\}\) and \(L_k^+ = \{(k+1) : K\} \cup ((K+k+1) : (2K))\).

However, for iTIPUP, we need to first apply the projections to the data \(X_\ell\) before computing the auto-covariance tensor in the reduced space. Hence iTIPUP cannot be reduced to a HOOI procedure.
Algorithm 1 A generic iterative algorithm
1: Input: $X_t \in \mathbb{R}^{d_1 \times \cdots \times K d_k}$ for $t = 1, \ldots, T$, $r_k$ for all $k = 1, \ldots, K$, the tolerance parameter $\epsilon > 0$, the maximum number of iterations $J$, and the UNIT and UITER operators.
2: Let $j = 0$, initiate via applying UNIT on $\{X_{1:T}\}$, for $k = 1, \ldots, K$, to obtain
   $$\hat{U}_{k}^{(0)} = \text{UNIT}_k(X_{1:T}, r_k).$$
3: repeat
   4: Let $j = j + 1$. At the $j$-th iteration, for $k = 1, \ldots, K$, given previous estimates $(\hat{U}_{k+1}^{(j-1)}, \ldots, \hat{U}_{K}^{(j-1)})$ and $(\hat{U}_{1}^{(j)}, \ldots, \hat{U}_{k-1}^{(j)})$, sequentially calculate,
   $$Z_{t,k}^{(j)} = X_t \times_1 (\hat{U}_{1}^{(j)})^\top \times_2 \cdots \times_{k-1} (\hat{U}_{k-1}^{(j)})^\top \times_{k+1} (\hat{U}_{k+1}^{(j-1)})^\top \times_{k+2} \cdots \times K (\hat{U}_{K}^{(j-1)})^\top,$$
   for $t = 1, \ldots, T$. Perform UITER on the new tensor time series $Z_{1:T,k}^{(j)} = (Z_{1,k}^{(j)}, \ldots, Z_{T,k}^{(j)})$.
   $$\hat{U}_{k}^{(j)} = \text{UITER}_k(Z_{1:T,k}^{(j)}, r_k).$$
5: until $j = J$ or
   $$\max_{1 \leq k \leq K} \|\hat{U}_{k}^{(j)} (\hat{U}_{k}^{(j)})^\top - \hat{U}_{k}^{(j-1)} (\hat{U}_{k}^{(j-1)})^\top\|_S \leq \epsilon,$$
6: Estimate and output:
   $$\hat{U}_{k}^{\text{iF}} = \hat{U}_{k}^{(j)}, \quad k = 1, \ldots, K,$$
   $$\hat{P}_{k}^{\text{iF}} = \hat{U}_{k}^{\text{iF}} (\hat{U}_{k}^{\text{iF}})^\top, \quad k = 1, \ldots, K,$$
   $$\hat{X}_{t}^{\text{iF}} = X_t \times_1 K (\hat{U}_{k}^{\text{iF}})^\top, \quad t = 1, \ldots, T.$$

3 Theoretical Properties

3.1 Assumptions

In this section, we shall investigate the statistical properties of the proposed iTOPUP and iTIPUP described in the last section. Our theories provide theoretical guarantees for consistency and present convergence rates in the estimation of the principle space of the factor loading matrices $A_k$, under proper regularity conditions.

We introduce some notations first. Let $d = \prod_{k=1}^{K} d_k$, $d_{-k} = d / d_k$, $r = \prod_{k=1}^{K} r_k$ and $r_{-k} = r / r_k$. 

9
Define order-4 tensors

\[
\Theta_{k,h} = \sum_{t=h+1}^{T} \frac{\text{mat}_k(\mathcal{M}_{t-h}) \otimes \text{mat}_k(\mathcal{M}_{t})}{T-h} \in \mathbb{R}^{d_k \times d_{-k} \times d_k \times d_{-k}},
\]

(14)

\[
\Phi_{k,h} = \sum_{t=h+1}^{T} \frac{\text{mat}_k(\mathcal{F}_{t-h}) \otimes \text{mat}_k(\mathcal{F}_{t})}{T-h} \in \mathbb{R}^{r_k \times r_{-k} \times r_k \times r_{-k}},
\]

(15)

\[
\Phi_{k,h}^{(\text{cano})} = \sum_{t=h+1}^{T} \frac{\text{mat}_k(\mathcal{M}_{t-h} \times K_{k=1}^{K} U_k^T) \otimes \text{mat}_k(\mathcal{M}_{t-h} \times K_{k=1}^{K} U_k^T)}{T-h} \in \mathbb{R}^{r_k \times r_{-k} \times r_k \times r_{-k}},
\]

(16)

with \( U_k \) from the SVD \( A_k = U_k \Lambda_k V_k^T \). We view \( \Phi_{k,h}^{(\text{cano})} \) as the canonical version of the autocovariance of the factor process. Let \( \mathbb{E}(\cdot) = \mathbb{E}(\cdot | \{\mathcal{F}_1, \ldots, \mathcal{F}_T\}) \). The noiseless version of the order-5 TOPUP tensor (7) is

\[
\Theta_{k,1:h_0} = (\Theta_{k,h}, h = 1, \ldots, h_0) = \mathbb{E}[\text{TOPUP}_k(\mathcal{X}_{1:T})] \in \mathbb{R}^{d_k \times d_{-k} \times d_k \times d_{-k} \times h_0},
\]

(17)

and its factor version is \( \Phi_{k,1:h_0} = (\Phi_{k,h}, h = 1, \ldots, h_0) \in \mathbb{R}^{r_k \times r_{-k} \times r_k \times r_{-k} \times h_0} \). Similarly define

\[
\Theta_{k,h}^* = \sum_{t=h+1}^{T} \frac{\text{mat}_k(\mathcal{M}_{t-h}) \text{mat}_k^T(\mathcal{M}_{t})}{T-h} \in \mathbb{R}^{d_k \times d_k},
\]

(18)

\[
\Phi_{k,h}^* = \sum_{t=h+1}^{T} \frac{\text{mat}_k(\mathcal{F}_{t-h}) \text{mat}_k^T(\mathcal{F}_{t})}{T-h} \in \mathbb{R}^{r_k \times r_k},
\]

\[
\Phi_{k,h}^{*(\text{cano})} = U_k^T \Theta_{k,h}^* U_k = \sum_{t=h+1}^{T} \frac{\text{mat}_k(\mathcal{M}_{t-h} \times K_{k=1}^{K} U_k^T) \text{mat}_k^T(\mathcal{M}_{t} \times K_{k=1}^{K} U_k^T)}{T-h} \in \mathbb{R}^{r_k \times r_k}.
\]

The noiseless version of (11) without matricization is

\[
\Theta_{k,1:h_0}^* = (\Theta_{k,h}^*, h = 1, \ldots, h_0) = \mathbb{E}[\text{TOPUP}_k(\mathcal{X}_{1:T})] \in \mathbb{R}^{d_k \times d_k \times h_0},
\]

(19)

and its factor version is \( \Phi_{k,1:h_0}^* = (\Phi_{k,h}^*, h = 1, \ldots, h_0) \in \mathbb{R}^{r_k \times r_k \times h_0} \). Let \( \tau_{k,m} \) be the \( m \)-th singular value of the noiseless version of the mode-\( k \) TOPUP matrix unfolding,

\[
\tau_{k,m} = \sigma_{m}(\mathbb{E}\text{mat}_1(\text{TOPUP}_k(\mathcal{X}_{1:T}))) = \sigma_{m}(\text{mat}_1(\Theta_{k,1:h_0}^*)) = \sigma_{m}(\text{mat}_1(\Phi_{k,1:h_0}^{*(\text{cano})})).
\]

Then the signal strength for iTOPUP can be characterized as

\[
\lambda_k = \sqrt{h_0^{-1/2} \tau_{k,r_k}}.
\]

(19)

Similarly, let

\[
\tau_{k,m}^* = \sigma_{m}(\mathbb{E}(\text{TIPUP}_k(\mathcal{X}_{1:T}))) = \sigma_{m}(\text{mat}_1(\Theta_{k,1:h_0}^*)) = \sigma_{m}(\text{mat}_1(\Phi_{k,1:h_0}^{*(\text{cano})})).
\]

Then the signal strength for iTIPUP can be characterized as

\[
\lambda_k^* = \sqrt{h_0^{-1/2} \tau_{k,r_k}^*}.
\]
We note that \( \lambda_{k}^{2} \leq \| \Theta_{t,0} \|_{F}/(1 - h_{0}/T) \) by (16) and Cauchy-Schwarz inequality.

To present theoretical properties of the proposed procedures, we establish the following conditions. These assumptions are used in Chen et al. (2019b) for the theoretical properties of TOPUP and TIPUP procedures. As TOPUP and TIPUP are basic operations for the proposed iterative algorithms in this paper, the assumptions are also used here, though we present them slightly differently for better presentation.

**Assumption 1.** The error process \( \mathcal{E}_{t} \) are independent Gaussian tensors, condition on the factor process \( \{ F_{t}, t \in \mathbb{Z} \} \). In addition, there exists some constant \( \sigma > 0 \), such that

\[
\mathbb{E}(u^\top \text{vec}(\mathcal{E}_{t}))^2 \leq \sigma^2\|u\|_{2}^2, \quad u \in \mathbb{R}^{d}.
\]

This assumption is standard in most factor modelling literature. See the discussion in Chen et al. (2019b). Under this assumption the magnitude of the noise can be measured by the dimension \( d_{k} \) before the projection and by the ranks \( r_{k} \) after the projection. The main theorems (Theorems 1 and 2 in Sections 3.2 and 3.3) are based on this assumption on the noise alone, and covers all possible settings for the signal \( \mathcal{M}_{t} \). To make our discussion more clear and to provide further interpretation and simplified version of the main theorems, we provide more detailed conditions and convergence rates of the iTOPUP and iTIPUP algorithms under the following two different assumptions on the signal process \( \mathcal{M}_{t} \).

**Assumption 2.** Assume \( r_{1}, \ldots, r_{K} \) are fixed. The factor process \( F_{t} \) is weakly stationary, and

\[
\frac{1}{T - h} \sum_{t = h + 1}^{T} F_{t-h} \otimes F_{t} \longrightarrow \mathbb{E}(F_{t-h} \otimes F_{t}) \quad \text{in probability},
\]

where the elements of \( \mathbb{E}(F_{t-h} \otimes F_{t}) \) are all finite. In addition, the condition numbers of \( A_{k}^\top A_{k} \) \((k = 1, \ldots, K)\) are bounded. Furthermore, assume that \( h_{0} \) is fixed, and

(i) (TOPUP related): \( \mathbb{E}[\text{mat}_{1}(\Phi_{k,1;h_{0}})] \) is of rank \( r_{k} \) for \( 1 \leq k \leq K \).

(ii) (TIPUP related): \( \mathbb{E}[\text{mat}_{1}(\Phi_{k,1;h_{0}}^{(\text{cano})})] \) is of rank \( r_{k} \) for \( 1 \leq k \leq K \).

Under this assumption, the factor process has a fixed expected auto-cross-moment tensor with fixed dimensions. The assumption that the condition numbers of \( A_{k}^\top A_{k} \) \((k = 1, \ldots, K)\) are bounded corresponds to the pervasive condition (e.g., Stock and Watson (2002), Bai (2003)). It ensures that all the singular values of \( A_{k} \) are of the same order. Such conditions are commonly imposed in factor analysis.

As our methods are based on auto-cross-moment at nonzero lags, we do not need to assume any specific model for the latent process \( F_{t} \), except some rank conditions in Assumption 2(i) and (ii). In order to provide a more concrete understanding of Assumption 2(ii) and (ii), consider the case of \( k = 1 \) and \( K = 2 \). We write the factor process \( F_{t} = (f_{i,j,t})_{d_{1} \times d_{2}} \), and the stationary auto-cross-moments \( \phi_{1,1;1,2;2;h} = \mathbb{E}(f_{1,j,1,t-h}f_{2,j,2;t}) \). Hence \( \mathbb{E}[\text{mat}_{1}(\Phi_{k,1;h_{0}})] \) is a \( r_{k} \times (r_{k}r_{k}r_{k}r_{k}h_{0}) \) matrix, with columns being \( \phi_{1,1;1,2;2,h} \). Since \( \mathbb{E}[\text{mat}_{1}(\Phi_{k,1;h_{0}})]\mathbb{E}[\text{mat}_{1}(\Phi_{k,1;h_{0}})]^{\top} \) is a sum of many semi-positive definite \( r_{k} \times r_{k} \) matrices, if any one of these matrices is full rank, then \( \mathbb{E}[\text{mat}_{1}(\Phi_{k,1;h_{0}})] \) is of rank \( r_{k} \). Hence Assumption 2(i) is relatively easy to fulfill. On the other hand, Assumption 2(ii) is quite different. First, the condition is in the canonical form of the model as the inner product in TIPUP related procedures behaves differently. Let \( F_{t}^{(\text{cano})} = U_{t}^{\top}M_{t}U_{t} = (f_{i,j,t})_{d_{1} \times d_{2}} \), and \( \phi_{1,1;1,2;2;h}^{(\text{cano})} = \mathbb{E}(f_{1,j,1,t-h}f_{2,j,2;t})^{(\text{cano})} \). Then \( \| \Phi_{1,1;h_{0}}^{(\text{cano})} \|_{\text{HS}}^{2} = \sum_{h = 1}^{h_{0}} \sum_{i_{1},i_{2}} \sum_{j_{1},j_{2}} \phi_{1,1;1,2;2;h}^{(\text{cano})} \phi_{1,1;1,2;2;h}^{(\text{cano})} \). As \( \phi_{1,1;1,2;2;h}^{(\text{cano})} \)
may be positive or negative for different $i_1, i_2, j, h$, the summation $\sum_{j=1}^{T} \phi_{i_1:i_2,j:h}^{(c ano)}$ is subject to potential signal cancellation for $h > 0$. Assumption 2(ii) ensures that there is no complete signal cancellation that makes the rank of $E[\text{mat}_1(\Phi_{k,1:h0})]$ less than $r_k$. Note that the conditions depend on the choice of $h_0$. See the discussion in Chen et al. (2019b).

In general, the dimensions of the core factor $r_k$ ($k = 1, \ldots, K$) may diverge as the dimensions of signal tensor $d_k$ ($k = 1, \ldots, K$) grow to infinity. The following assumption provides a more concrete set of conditions that can be used to provide some insights of the properties of the iTOPUP and iTIPUP algorithms.

**Assumption 3.** For a certain $\delta_0 \in [0, 1]$, $\|\Theta_{k,0}\|_{op} \asymp \sigma^2 d^{1-\delta_0} r^{-\frac{1}{2}}$ and $\|\Theta_{k,0}\|_S \asymp \sigma^2 d^{1-\delta_0} r^{-\frac{1}{2}}$ with probability approaching one (as $T \to \infty$). For the singular values, two scenarios are considered.

(i) (TOPUP related): There exist some constants $\delta_1 \in [\delta_0, 1]$ and $c_1 > 0$ such that with probability approaching one (as $T \to \infty$), $\lambda_k \geq c_1 \sigma^2 d^{1-\delta_1} r_k^{-\frac{1}{2}}$, for all $k = 1, \ldots, K$.

(ii) (TIPUP related): There exist some constants $\delta_2 \in [\delta_0, 1]$, $c_2 > 0$ and $\delta_2 \geq 0$ such that with probability approaching one (as $T \to \infty$), $\lambda_k \geq c_2 \sigma^2 d^{1-\delta_2} r_k^{-\frac{1}{2}}$, for all $k = 1, \ldots, K$.

Assumption 3 is similar to the signal strength condition of Lam and Yao (2012), and the pervasive condition on the factor loadings (e.g., Stock and Watson (2002) and Bai (2003)). This assumption describes the signal process $M_t$ in terms of factor strength. It is more general than Assumption 2 in the sense that it allows $r_1, \ldots, r_K$ to diverge and the latent process $F_t$ does not have to be weakly stationary.

We take $\delta_0$, $\delta_1$ as measures of the strength of factors. They roughly indicate how much information is contained in the signals compared with the amount of noise, with respect to the dimensions and ranks, $d, r$ and $r_k$. In this sense, they reflect the signal to noise ratio. When $\delta_0 = \delta_1 = 0$, the factors are called strong factors; otherwise, the factors are called weak factors.

**Remark 2** (Signal Strength $\delta_0$). Following Lam and Yao (2012), we may assume $\|\Theta_{k,0}\|_{HS} \asymp d^2 (\sigma^2 d^{-\delta_0})^2$. This assumption can be seen from the following. Note that $\|\Theta_{k,0}\|_{HS}^2$ is the sum of total $d^2$ terms, each is the square of the time average of the cross-product of the entries in $M_t$. When $(d^2)^{1-\delta_0}$ of the $d^2$ terms are roughly constant (and comparable with $\sigma^2$), and the rest are almost zero, then the total sum would be $\asymp d^2 (\sigma^2 d^{-\delta_0})^2$. In this case $1 - \delta_0$ controls how sparse the signal tensor $M_t$ is. If $\delta_0 = 0$, the signal tensor $M_t$ is dense with entries comparable with the noise level $\sigma$, hence with strong factors. If the proportion of the nonzero entries in $M_t$ increases at the rate of $(d^2)^{1-\delta_0}$, or the overall average of the $d^2$ terms decreases at the rate $(d^2)^{-\delta_0}$, the signal strength is $\delta_0 > 0$. If in addition all the nonzero eigenvalues are of the same order for each of the nonnegative-definite mappings $\Theta_{k,0} : \mathbb{R}^d \to \mathbb{R}^d$ and $\Theta_{k,0}^* : \mathbb{R}^{d_k} \to \mathbb{R}^{d_k}$, with respective ranks $r = \prod_{k=1}^{K} r_k$ and $r_k$, we would have $\|\Theta_{k,0}\|_{op} \asymp \sigma^2 d^{1-\delta_0} r^{-\frac{1}{2}}$ and $\|\Theta_{k,0}^*\|_S \asymp \sigma^2 d^{1-\delta_0} r_k^{-\frac{1}{2}}$ due to $\text{tr}(\Theta_{k,0}) = \text{tr}(\Theta_{k,0}^*)$. The assumptions on the ranks of $\Theta_{k,0}$ and $\Theta_{k,0}^*$ hold when $\Phi_{k,0}$ and $\Phi_{k,0}^*$ are of full rank and rank$(A_k) = r_k$, which is a typical condition in the factor model setting.

**Remark 3** (Assumption 3(i) and the role of $\delta_1$). In fact, for TOPUP, Assumption 3(i) holds when (a) $\|E[\text{mat}_1(\text{TOPUP}_k)]\|_{HS}^2 = \sum_{h=1}^{h_0} \|\Theta_{1,h}\|_{HS}^2 \asymp h_0 \sigma^4 d^{2(1-\delta_1)}$ and (b) all the nonzero singular values of $E[\text{mat}_1(\text{TOPUP}_k)]$ are of the same order. Again, consider the case of $k = 1$ and $K = 2$. We write the factor process in the canonical form as $F_t = U_1^\top M_1 U_2 = (f_{1,j,t})_{d_1 \times d_2}$, and $\phi_{k,1:j,1:j,k}^{(c ano)} = \sum_{l=-h+1}^{T} f_{1,j,t-l,h} f_{2,j,2,h}/(T-h)$ as the time average cross product between fibers $f_{1,j,1:1:T}$ and $f_{2,j,2,1:T}$ of the factor process. The first condition (a) means $\sum_{h=1}^{h_0} \|\Theta_{1,h}\|_{HS}^2 = \sum_{h=1}^{h_0} \|\Phi_{1,h}^{(c ano)}\|_{HS}^2$
by projecting the data in other modes of the tensor time series from $R$.

Let us first study the behavior of iTOPUP procedure. By Chen et al. (2019b), the risk $\delta_1 \geq \delta_0$ in Assumption 3(i).

Remark 4 (Assumption 3(ii) and the role of $\delta_2$). However, the same rationale in Remark 3 may not be directly applicable to TIPUP. Note that $\|\Theta_{1,h}\|_{HS}^2 = \|\Phi_{1,h}^{(cano)}\|_{HS}^2 = \sum_{i_1,i_2} (\sum_{j=1}^{r_2} \phi_{i_1,j,i_2,j,h}^{(cano)})^2$. As the discussion of Assumption 2(ii), the summation $\sum_{j=1}^{r_2} \phi_{i_1,j,i_2,j,h}^{(cano)}$ is subject to signal cancellation for $h > 0$. To take into account this signal cancellation, we may assume

$$\sum_{i_1,i_2,h} (\sum_{j=1}^{r_2} \phi_{i_1,j,i_2,j,h}^{(cano)})^2 \propto r_2^{-2\delta_2} \sum_{i_1,i_2,j,h} (\phi_{i_1,j,i_2,j,h}^{(cano)})^2 \propto r_2^{-2\delta_2} \sum_{i_1,j_1,j_2,h} (\phi_{i_1,j_1,i_2,j_2,h}^{(cano)})^2$$

by counting the number of entries and measuring signal cancellation by $\delta_2$ as a rate of $r_2$. This would lead to $\|\mathbb{E}\text{mat}_1(\text{TIPUP}_k)\|_{HS}^2 \approx r_2 r_1^{-2\delta_2} \|\mathbb{E}\text{mat}_1(\text{TOPUP}_k)\|_{HS}^2$ and then $\lambda^{op}_k \geq c_2 d_0^{-\delta_2} r_2^{-\delta_2}$ with the flexibility of taking $c_2 \neq c_1$. The additional parameter $\delta_2$ in Assumption 3(ii) measures the severity of signal cancellation in the TIPUP related procedures. $\delta_2 = 0$ indicates that the signal cancellation in the TIPUP related operations has no impact on the magnitude of signal strength.

Remark 5 (Additional remarks on signal cancellation for TIPUP-related procedures). Following the above discussion on $\delta_2$ for TIPUP-related procedures, it would be reasonable to take $\delta_2 = 0$ when the majority of $\phi_{i_1,j,i_2,j,h}^{(cano)}$ are of the same sign for most of $(i_1,i_2,h)$, $\delta_2 = 1/2$ when $\phi_{i_1,j,i_2,j,h}^{(cano)}$ behave like independent zero mean variables, and $\delta_2 = \infty$ when all the signals cancel out by summation $\phi_{i_1,j_1,j_2,j,h}^{(cano)}$ over $j$. When $h_0 = 1$, $\sum_j \phi_{i_1,j,i_2,j,h}^{(cano)}$ is prone to canceling out. Increasing $h_0$ often helps to reduce the problem of signal cancellation. Of course, if the dynamics (e.g. autocovariance) of the factor process is relatively strong, then $\phi_{i_1,j,i_2,j,h}^{(cano)}$ would be small for large $h$. In this case, using large $h_0$ does not increase the signal level, but brings in more noise terms in the estimation. In practice, we may choose $h_0 \geq K$ in iTOPUP to avoid the possibility of extremely serious level of signal cancellation. In the case of fixed $r_k$, the convergence rate depends on whether $\delta_2 = \infty$ (severe signal cancellation) or $\delta_2$ being finite.

### 3.2 Theoretical properties of iTOPUP

Let us first study the behavior of iTOPUP procedure. By Chen et al. (2019b), the risk $\mathbb{E}\left\|\hat{U}_k^{(0)} - U_k^T\right\|_S^2$ of the TOPUP estimator for $U_k$, the initialization of iTOPUP, is no larger than

$$R_k^{(0)} = \lambda_k^{-2} \sigma T^{-1/2} \left\{ \sqrt{d_k d_{-k} r_{-k}} \|\Theta_{k,0}\|_S^{1/2} + \left( \sqrt{d_k + d_{-k} r_{-k}} \right) \|\Theta_{k,0}\|_{op}^{1/2} + \sigma \sqrt{d_k d_{-k} + \sigma d_k} \sqrt{d_{-k} T^{-1/2}} \right\},$$

(20)

where $d_{-k} = \prod_{j \neq k} d_j$ and $r_{-k} = \prod_{j \neq k} r_j$. The aim of iTOPUP is to achieve dimension reduction by projecting the data in other modes of the tensor time series from $\mathbb{R}^{d_j}$ to $\mathbb{R}^{r_j}$, $j \neq k$. Ideally this would reduce the above rate to

$$R_k^{(\text{ideal})} = \lambda_k^{-2} \sigma T^{-1/2} \left\{ \sqrt{d_k r_{-k}} \|\Theta_{k,0}\|_S^{1/2} + \left( \sqrt{d_k + r_{-k} T^{-1/2}} \right) \|\Theta_{k,0}\|_{op}^{1/2} + \sigma \sqrt{d_k r_{-k} + \sigma d_k} \sqrt{r_{-k} T^{-1/2}} \right\}.$$  

(21)

The following theorem provides conditions under which this ideal rate is indeed achieved.
Theorem 1. Suppose Assumption 1 holds. Let \( h_0 < T/4 \) and \( P_k, \Theta_{k,0}, \Theta_{k,0}^{*} \) and \( \lambda_k \) be as in (5), (14), (16) and (18) respectively. Let \( R(0) = \max_{1 \leq k \leq K} R_k^{(0)} \) with \( R_k^{(0)} \) in (20), Suppose that for certain constants \( C_0 \) and \( C^{(\text{iter})}_0 \),

\[
C_0 R(0) \leq 1, \tag{22}
\]

and with the \( R_k^{(\text{ideal})} \) in (21) and \( d^{*}_k = \sum_{j \neq k} d_{j} r_{j} \),

\[
C^{(\text{iter})}_0 R_k^{(\text{ideal})} + C^{(\text{iter})}_0 \lambda_k^{-2} \sigma T^{-1/2} \left( \| \Theta_{k,0}^{*} \|_S^{1/2} + \| \Theta_{k,0} \|_{\text{op}}^{1/2} + \sigma + \sigma T^{-1/2} (\sqrt{d^{*}_k} + \sqrt{d_k r_{-k}}) \right) \sqrt{d^{*}_k} \leq 1. \tag{23}
\]

Then, there exist a numerical constant \( C^{(\text{TOPUP})}_1 \) and a constant \( C^{(\text{iter})}_1 \) depending on \( K \) only such that when \( C_0 \geq 4(1 - \rho)^{-1} C^{(\text{TOPUP})}_1 \) and \( C^{(\text{iter})}_0 \geq C^{(\text{iter})}_1 / \rho, 0 < \rho < 1, \)

\[
\| \hat{P}_k^{(m)} - P_k \|_S \leq 2 C^{(\text{TOPUP})}_1 \left( (1 - \rho^m)(1 - \rho)^{-1} R_k^{(\text{ideal})} + (\rho^m / 2) R(0) \right) \tag{24}
\]

simultaneously for all \( 1 \leq k \leq K \) and \( m \geq 0 \) in an event with probability at least \( 1 - \sum_{k=1}^{K} e^{-d_k} \),

where \( R_k^{(\text{ideal})} = \max_{1 \leq k \leq K} R_k^{(\text{ideal})} \) and \( \hat{P}_k^{(m)} = \hat{U}_k^{(m)} \hat{U}_k^{(m)\top} \) with the \( m \)-step estimator \( \hat{U}_k^{(m)} \) in iTOPUP algorithm. In particular, after at most \( J = \left[ \log(\max_k d_{-k}/ \min_k r_{-k}) / \log(1/\rho) \right] \) iterations,

\[
\mathbb{E} \left[ \max_{1 \leq k \leq K} \| \hat{P}_k^{(J)} - P_k \|_S \right] \leq 3 C^{(\text{TOPUP})}_1 R_k^{(\text{ideal})} + \sum_{k=1}^{K} e^{-d_k}. \tag{25}
\]

It follows from Theorem 1 of Chen et al. (2019b) that with high probability \( \| \hat{U}_k^{(0)} \hat{U}_k^{(0)\top} - P_k \|_S \) is small when condition (22) holds with sufficiently large \( C_0 \), i.e. \( R_k^{(0)} \) are sufficiently small, so that the initialization in Step 1 of iTOPUP retains a large portion of the signal. Condition (23) has two terms on the left. The first term is the same as (22) with \( d_{-k} \) replaced by \( r_{-k} \), representing the estimation error after dimension reduction with the true projection with \( \times_{j \neq k} U_j \), and the second term represents the additional error in the estimation of \( \times_{j \neq k} U_j \) in the iteration. The essence of our analysis of iTOPUP is that under condition (23), each iteration is a contraction of the additional error in the estimation of \( \times_{j \neq k} U_j \) in a small neighborhood of it. The upper bound (24) for the error of the \( m \)-step estimator is comprised of two terms, which correspond to the error upper bound for the final estimator and the bound for the contracted error of the initial estimator, respectively.

Remark 6. The consistency of the non-iterative TOPUP estimator requires \( R^{(0)} \to 0 \) (Chen et al., 2019b). However, here we do not require the TOPUP estimator serving for initialization to be consistent. For (25) to hold, the TOPUP estimator is only required to be sufficiently close to the ground truth as in (22). As for the extra condition (23) for the iterative method to achieve (25), it is easy to verify that condition (22) implies condition (23) under many circumstances, such as when \( d_k \) are of the same order, \( r_k \) are of the same order, and \( r_k \leq d_k^{K-1} + \frac{K-1}{K} \).

Remark 7. Note that as \( \lambda_k \) typically grows with \( d_1, \ldots, d_K \) under quite general weak conditions. Under these conditions, Theorem 1 allows \( T \leq d_k^2 d_{-k}^2 \) for the consistency of the iTOPUP estimator of the loading spaces. As shown below, in many cases, \( T \) is actually allowed to be a constant for \( K > 1 \). This is in sharp contrast of the results of traditional factor analysis which requires \( T \to \infty \).
to consistently estimate the loading spaces. The main reason is that the other tensor modes provide additional information and in certain sense serve as additional samples. Roughly speaking, we have total $d_k(d_kT)$ observations in the tensor time series to estimate the $d_kr_k$ parameters in the loading space $A_k$, where $r_k \ll d_kT$ in most of the cases, depending on additional conditions on $\lambda_k$ and $\lambda^*_k$.

For a better illustration of the statistical properties of iTOPUP, we provide the following corollary as a simplified version of Theorem 1 for fixed ranks $r_k$ under some additional side conditions.

**Corollary 1.** Suppose Assumptions 1 and 2(i) hold. Let $\lambda = \prod_{k=1}^K \|A_k\|_S$ and $d_{\min} = \min\{d_1, \ldots, d_K\}$. Let $h_0 \leq T/4$ and $\sigma$ fixed. Then, there exist constants $C_{0,K}$ and $C_{1,K}$ depending on $K$ only such that when

$$\lambda^2 \geq C_{0,K}\sigma^2 \left( \frac{d}{T} + \frac{d}{\sqrt{T}d_{\min}} \right),$$

the 1-step iTOPUP estimator satisfies

$$\mathbb{E}\|\hat{P}_k^{(1)} - P_k\|_S \leq C_{1,K} \left( \frac{\sigma\sqrt{d_k}}{\lambda\sqrt{T}} + \frac{\sigma^2\sqrt{d_k}}{\lambda^2\sqrt{T}} \right) + \sum_{k=1}^K e^{-d_k}.$$  

Corollary 1 implies that, in order to recover the factor loading space for $A_k$, the signal to noise ratio needs to satisfy $\lambda/\sigma \geq C_0(\sigma^{1/2}d^{-1/2} + \sigma d^{-1/4}T^{-1/4})$. Under the assumption, the theorem indicates that the proposed iTOPUP just needs one-iteration to achieve the ideal rate of convergence. The rate is much sharper than that of the non-iterative TOPUP procedure.

When the dimensions of the core factor $r_k$ ($k = 1, \ldots, K$) diverge as the dimensions of signal tensor $d_k$ ($k = 1, \ldots, K$) grow to infinity, we characterize the convergence rate of iTOPUP in terms of $d_k$, $r_k$ and $T$ under Assumption 3(i).

**Corollary 2.** Suppose Assumptions 1 and 3(i) hold. Let $h_0 \leq T/4$, $d^* = \sum_{j \neq k} d_j r_j$ and $r = \prod_{k=1}^K r_k$. Suppose that for a sufficiently large $C_0$ not depending on $\{d_k, r_k, k \ll K\}$,

$$T \geq C_0 \max_k \left( \frac{d^{2\delta_1} d_0^{3/2}}{r_k} + \frac{d^{2\delta_1} r_k^{3/2}}{d_k^{1/2}} + \frac{d_k^{1/2} r_k^{1/2}}{d^{1/2 - \delta_1}} + \frac{d_0^{1/2} r_k^{1/2}}{d^{1+\delta_0 - 2\delta_1}} + \frac{d_k^{1/2} (d^*_k)^{1/2}}{d^{1-\delta_1}} \right).$$

Then, after $J = O(\log d)$ iterations, we have the following upper bounds for iTOPUP,

$$\|\hat{P}_k^{(J)} - P_k\|_S = O\left( \frac{d_k^{1/2} r_k^{5/4}}{T^{1/2} d^{1/2 + \delta_0/2 - \delta_1} r_k} + \frac{d_k^{1/2} r_k^{5/4}}{T^{1/2} d^{1+\delta_0 - 2\delta_1} r_k} \right).$$

Moreover, (29) holds after at most $J = O(\log r)$ iterations, where $r = \prod_{k=1}^K r_k$, if any one of the following three conditions holds in addition to (28): (i) $d_k$ ($k = 1, \ldots, K$) are of the same order, (ii) $\lambda_k$ ($k = 1, \ldots, K$) are of the same order, (iii) $(\lambda^*_k)^{-2} \sqrt{T_k}$ ($k = 1, \ldots, K$) are of the same order.

Note that the second part of Corollary 2 says that when the condition is right, iTOPUP algorithm only needs a small number of iterations to converge, as $O(\log r)$ is typically very small. The noise level $\sigma$ does not appear directly in the rate since it is incorporated in the signal to noise ratio in the tensor form in Assumption 3. In Corollary 2, we show that as long as the sample size $T$ satisfies (28), the iTOPUP achieves consistent estimation under regularity conditions. To digest
(28), consider that the growth rate of \( r_k \) is much slower than \( d_k \) and the factors are strong with \( \delta_0 = \delta_1 = 0 \). Then (28) becomes \( T \geq C_0 \max_{k} (r^{3/2} r_k^{-1}) \).

The advantage of using index \( \delta_0, \delta_1 \) is to link the convergence rates of the estimated factor loading space explicitly to the strength of factors. It is clear that the stronger the factors are, the faster the convergence rate is. Moreover, the stronger the factors are, the smaller the sample size is required.

### 3.3 Theoretical properties of iTIPUP

Now, let us consider the statistical performance of iTIPUP procedure. Again, by Chen et al. (2019b) the TIPUP risk in the estimation of \( P_k \) is bounded by

\[
\mathbb{E}[\| \hat{P}_k^{(\text{TIPUP})} - P_k \|_S] \leq R_k^{(0)} = (\lambda_k^*)^{-2} \sigma T^{-1/2} \sqrt{d_k} \left( \| \Theta_{k,0}^* \|_S^{1/2} + \sigma \sqrt{d_{-k}} \right)
\]  

(30)

with \( d_{-k} = \prod_{j \neq k} d_j \), and the aim of iTIPUP is to achieve the ideal rate

\[
R_k^{*\text{(ideal)}} = (\lambda_k^*)^{-2} \sigma T^{-1/2} \sqrt{d_k} \left( \| \Theta_{k,0}^* \|_S^{1/2} + \sigma \sqrt{r_{-k}} \right)
\]  

(31)

through dimension reduction, where \( r_{-k} = \prod_{j \neq k} r_j \). The following theorem, which allows the ranks \( r_k \) to grow to infinity as well as \( d_k \) when \( T \to \infty \), provides sufficient conditions to guarantee this ideal convergence rate for iTIPUP.

**Theorem 2.** Suppose Assumption 1 holds. Let \( P_k, \Theta_{k,0}^* \) and \( \lambda_k^* \) be as in (5), (16) and (19) respectively. Let \( h_0 < T/4 \), and

\[
R_k^{*0} = \max_{1 \leq k \leq K} R_k^{*0}, \quad R_k^{*\text{(ideal)}} = \max_{1 \leq k \leq K} R_k^{*\text{(ideal)}}.
\]

with \( R_k^{*0} \) in (30) and \( R_k^{*\text{(ideal)}} \) in (31). Suppose that for certain constants \( C_0 \) and \( C_0^{\text{(iter)}} \),

\[
C_0 R_k^{*0} \leq \min_{1 \leq k \leq K} \lambda_k^*/\| \Theta_{k,0}^* \|_S,
\]

(32)

\[
C_0^{\text{(iter)}} R_k^{*\text{(ideal)}} + C_0^{\text{(iter)}} \max_{1 \leq k \leq K} \sqrt{d_{-k}/d_k} R_k^{*\text{(ideal)}} \leq 1,
\]

(33)

where \( d_{-k} = \sum_{j \neq k} d_j r_j \), Then, there exist a numerical constant \( C_1^{\text{TIPUP}} \) and a constant \( C_1^{\text{iter}} \) depending on \( K \) only such that when \( C_0 \geq 8(1-\rho)^{-1} C_1^{\text{TIPUP}} \) and \( C_0^{\text{(iter)}}/(1 + 1/C_0^{\text{(iter)}}) \geq C_1^{\text{iter}} / \rho \), 0 < \( \rho < 1 \),

\[
\| \hat{P}_k^{(m)} - P_k \|_S \leq 2 C_1^{\text{TIPUP}} \left( (1 - \rho m)(1 - \rho)^{-1} R_k^{*\text{(ideal)}} + (\rho^m/2) R_k^{*0} \right)
\]

(34)

simultaneously for all \( 1 \leq k \leq K \) and \( m \geq 0 \) in an event with probability at least \( 1 - \sum_{k=1}^K e^{-d_k} \), where \( \hat{P}_k^{(m)} = \hat{U}_k^{(m)} \hat{U}_k^{(m)\top} \) with the m-step estimator \( \hat{U}_k^{(m)} \) in iTIPUP algorithm, i.e. In particular, after at most \( J = [\log(\max_k d_{-k} \min_k r_{-k}) / \log(1/\rho)] \) iterations,

\[
\mathbb{E} \left[ \max_{1 \leq k \leq K} \| \hat{P}_k^{(J)} - P_k \|_S \right] \leq 3 C_1^{\text{TIPUP}} \frac{1}{1 - \rho} R_k^{*\text{(ideal)}} + \sum_{k=1}^K e^{-d_k}.
\]

(35)
By (16), (19) and the Cauchy-Schwarz inequality, $(1 - h_0/T)\lambda_k^2 \leq \|\Theta_{1,0}^*\|s$, so that (32) guarantees a sufficiently small $R^{*}(0)$, which implies a sufficiently small error in the initialization of iTIPUP by (30). Condition (33) again has two terms respectively reflecting the ideal rate after dimension reduction by the true $U_{-k} = \bigcap_{j \neq k} U_j$ in the estimation of $U_k$ and the extra cost of estimating $U_{-k}$. The upper bound (34) for the error of the $m$-step estimator is also comprised of two terms, the bound for the final estimator and the bound for the contracted error of the initial estimator.

**Remark 8.** Again, (32) and (33) do not demand a specific relationship between $T$ and the dimensions, as $\lambda_k^*$ are allowed to depend on the dimensions and ranks, as discussed in Remark 7.

We present more explicit error bound in Corollary 3 under the additional Assumption 2(ii), which assumes there is no severe signal cancellation in iTIPUP.

**Corollary 3.** Suppose Assumptions 1 and 2(ii) hold. Let $\lambda = \prod_{k=1}^{K} \|A_k\|s$ and $d_{\max} = \max\{d_1, ..., d_K\}$. Let $h_0 \leq T/4$ and $\sigma$ fixed. Then, there exist constants $C_{0,K}$ and $C_{1,K}$ depending on $K$ only such that when

$$\lambda^2 \geq C_{0,K} \sigma^2 \left( \frac{d_{\max}}{T} + \frac{\sqrt{d}}{T} \right),$$

the 1-step iTIPUP estimator satisfies

$$\mathbb{E}\|\hat{P}_k(1) - P_k\|s \leq C_{1,K} \left( \frac{\sigma \sqrt{d_k}}{\lambda \sqrt{T}} + \frac{\sigma^2 \sqrt{d_k}}{\lambda^2 \sqrt{T}} \right) + \sum_{k=1}^{K} e^{-d_k}. \quad (37)$$

The signal to noise ratio assumption $\lambda/\sigma \geq C_{0,K}(\sqrt{d_{\max}/T} + (d/T)^{1/4})$ is required here to guarantee the performance of iTIPUP algorithm. Again, under the conditions, iTIPUP requires one iteration to achieve the much sharper ideal convergence rate, comparing to that of the non-iterative TIPUP algorithm.

When the ranks $r_k$ ($k = 1, ..., K$) also diverge and there is no severe signal cancellation in iTIPUP, we have the following convergence rate for iTIPUP under Assumption 3(ii).

**Corollary 4.** Suppose Assumptions 1 and 3(ii) hold. Let $h_0 \leq T/4$ and $d_{\max} = \sum_{k \neq k} d_j r_j$. Suppose that for a sufficiently large $C_0$ not depending on $\{\sigma, d_k, r_k, k \leq K\}$,

$$T \geq \max_k \left( \frac{d_k r_k^{3/2+4\delta_2}}{d_1^{1+3\delta_0-4\delta_1} r_k^{4\delta_2+1}} + \frac{r_k^{1+4\delta_2}}{d_1^{1+2\delta_0-4\delta_1} r_k^{4\delta_2}} + \frac{d_{\max} r_k^{1/2+2\delta_2}}{d_1^{1+\delta_0-2\delta_1} r_k^{2\delta_2}} + \frac{d_{\max} r_k^{1+2\delta_2}}{d_1^{2-\delta_1} r_k^{2\delta_2}} \right). \quad (38)$$

Then, after at most $J = O(\log d)$ iterations, the iTIPUP estimator satisfies

$$\|\hat{P}_k(J) - P_k\|s = O\left( \frac{d_k^{1/2} r_k^{1/4+\delta_2}}{T^{1/2} d_1^{1+\delta_0-2\delta_1} r_k^{\delta_2}} + \frac{d_k^{1/2} r_k^{1/2+\delta_2}}{T^{1/2} d_1^{1-\delta_1} r_k^{\delta_2}} \right). \quad (39)$$

Moreover, (39) holds after at most $J = O(\log r)$ iterations, if any one of the following three conditions holds in addition to condition (38), (i) $d_k$ ($k = 1, ..., K$) are of the same order, (ii) $\lambda_k$ ($k = 1, ..., K$) are of the same order, (iii) $(\lambda_k^*)^{-2} \sqrt{d_k}$ ($k = 1, ..., K$) are of the same order.
Corollary 4 provides the sample complexity of the algorithm in (38). In the case that the growth rate of \( r_k \) is much slower than \( d_k \) and the factors are strong with \( \delta_0 = \delta_1 = 0 \), the required sample size reduces to \( T \geq C_0 \max_k \left( r_{-k}^{4\delta_2 + 1} r_{-k}^{-1/2} d_{-k}^{-1} + r_{-k}^{2\delta_2} d_{-k}^* r_{-k}^{1/2} d_{-k}^{-1} \right) \), where \( r_{-k} = r/r_k \) and \( d_{-k} = d/d_k \). By comparing with Corollary 2, the sample complexity for iTIPUP is smaller, if \( \delta_2 \) is a small constant. As expected, the convergence rate is slower in the presence of weak factors. When the factors are strong (\( \delta_0 = \delta_1 = 0 \)) and all \( r_k \) and \( \delta_2 \) are fixed finite constants, the rate becomes \( O_p(T^{-1/2} d_{-k}^{-1/2}) \), the same as that for iTOPUP.

### 3.4 Mixed Procedures

As discussed in Section 2.3, we can mixed TOPUP and TIPUP operations for the initiation and iterative operations in Algorithm 1. Based on Theorems 1 and 2, for the mixed TIPUP-iTOPUP procedure, (24) (with \( R^{(0)} \) replaced by \( R^{(0)*} \)) and (25) still hold if condition (22) is replaced by

\[
C_0 R^{(0)*} \leq 1, \tag{40}
\]

where \( R^{(0)*} = \max_{1 \leq k \leq K} R_k^{(0)*} \) for \( R_k^{(0)*} \) in (30). On the other hand, for TIPUP-iTOPUP, conclusions (34) (with \( R^{(0)*} \) replaced by \( R^{(0)} \)) and (35) still hold when condition (32) is replaced by

\[
C_0 R^{(0)} \leq \min_{1 \leq k \leq K} \lambda_k^{*2} / \| \Theta_{1,0}^* \|_S, \tag{41}
\]

where \( R^{(0)} = \max_{1 \leq k \leq K} R_k^{(0)} \) with \( R_k^{(0)} \) in (20).

### 3.5 Comparisons

**Comparison between the non-iterative procedures and iterative procedures**

Theorems 1 and 2 show that the convergence rates of the non-iterative estimators TOPUP and TIPUP can be improved by the iterative procedure. Particularly, when the dimensions \( r_k \) for the factor process are fixed and the respective signal strength conditions are fulfilled, the proposed iTOPUP and iTIPUP just need one-iteration to achieve the much sharper ideal rate \( R^{(\text{ideal})} \) in (21) and \( R^{(\text{ideal})} \) (31), comparing to the rate (20) of TOPUP and (30) of TIPUP derived in Chen et al. (2019b), respectively. The improvement is achieved, as motivated shown in Section 1, through replacing the much larger \( d_{-k} \) by \( r_{-k} \), via orthogonal projection. When the factors are strong with \( \delta_0 = \delta_1 = 0 \) and the factor dimensions are fixed, the non-iterative TOPUP-based estimators of Lam et al. (2011) for the vector factor model, Wang et al. (2019) for the matrix factor and Chen et al. (2019b) for tensor factor models all have the same \( O_p(T^{-1/2}) \) convergence rate for estimating the loading space. In comparison, the convergence rate \( O_p(T^{-1/2} d_{-k}^{-1/2}) \) of both iterative estimators, iTOPUP and iTIPUP (when there is no severe signal cancellation, with bounded \( \delta_2 \)), is much sharper. Intuitively, when the signal is strong, the orthogonal projection operation helps to consolidate signals while potentially averaging out the noises, when the projection reduces the mode-\( k \) unfolded matrix from a \( d_k \times d_{-k} \) matrix for the tensor \( X_t \) into a \( d_k \times r_{-k} \) matrix for the projected tensor \( Z_t \), as the motivation we provided in Section 1. As a result, the increase in the dimension \( d_{-k} \) improves the iterative estimator.

When \( r_k \) are allowed to diverge, the iTOTUP and iTIPUP algorithms converge after at most \( O(\log(d)) \) iterations to achieve the ideal rate according to Theorems 1 and 2. The number of iterations needed can be as few as \( O(\log(r)) \) when the condition is right. The power iterations are
necessary in order to refine the reliable initial estimates to achieve sharper guaranteed convergence rate. Moreover, stronger signal \( \lambda \) will result in faster exponential contraction with smaller \( \rho \) in (24) and (34), in other words, converging more quickly to the final estimators.

**Comparison between iTIPUP and iTOPUP:** The inner product operation in (11) for TIPUP-related procedures enjoys significant amount of noise cancellation comparing to the outer product operation in (7) for TOPUP-related procedures. Compared with iTOPUP, the benefit of noise cancellation of the iTIPUP procedure is still visible through the reduction of \( r_{\delta_k} \) in (21) to \( \sqrt{r_{\delta_k}} \) in (31) in the ideal rates. However, this benefit is much less pronounced compared with the reduction of \( d_{\delta_k} \) in (20) for TOPUP to \( \sqrt{d_{\delta_k}} \) in (30) for TIPUP in the non-iterative rates. Meanwhile, the potential damage of signal cancellation in the TIPUP related schemes persists as \( \lambda_{\delta_k}^* \) and \( \lambda_k \) are unchanged between the initial and ideal rates. When \( r_{\delta_k} \) are allowed to diverge to infinity, and \( \delta_2 > 1 \), then \( r_{\delta_k}^{-1} \rightarrow \infty \). In this case, the iTOPUP has a faster rate than the iTIPUP in view of Corollary 2 and 4. Otherwise, if \( r_{\delta_k}^{-1} \rightarrow 0 \), the rate of iTIPUP is faster.

If all the assumptions in Corollary 1 and 3 are satisfied, iTOPUP and iTIPUP will have exactly the same convergence rate. There are two considerations. First is the quality of the initial estimator. In theory, as long as the initial estimates are reasonably good, both iTOPUP and iTIPUP achieve consistent estimation. In practice, one would prefer to use TIPUP for initialization when there is no severe signal cancellation. The second is the signal strength. In general iTIPUP has much weaker signal strength \( \lambda_{\delta_k}^* \) than iTOPUP \( \lambda_k \), depending on the severity of signal cancellation, though it also requires smaller signal to noise ratio. Under extremely serious level of signal cancellation (\( \delta_2 = \infty \)), the convergence rate of iTIPUP will be much slower.

In practice, when the rank \( r_k \) is small, iTOPUP procedures are recommended, as it has the same convergence rate as the iTIPUP, but safeguards the signal cancellation cases. If there is a concern of possible signal cancellation, initiation should also use TOPUP, though TIPUP typically provides more accurate initial estimators when there is no signal cancellation.

**Comparison of signal to noise ratio requirement with that of HOOI:** The signal to noise ratio condition mainly depends on the existence of an initial estimator with sufficiently small estimation error. In the typical factor model setting, we have condition number of \( A_k^T A_k \) bounded and ranks \( r_k \) fixed. By comparing (26) in Corollary 1 with (36) in Corollary 3, iTIPUP is able to handle tensor factor model estimation under much weaker signal to noise ratio. In the traditional tensor power iteration method (e.g., Zhang and Xia (2018)), the observations are assumed to be composed by deterministic signal and i.i.d. noise. Then, averaging the observations before orthogonal projection, HOOI requires signal to noise ratio \( \lambda/\sigma \geq C_0d^{1/4}/T \). However, in our tensor factor models, signal part is random and serial correlated. The time mode should be viewed as an additional mode of the signal part. Working with the average of cross-products of the observations leads to \( T^{-1/2} \) in the rate of iTOPUP and iTIPUP, instead of \( T^{-1} \) in HOOI. Then, this phenomenon results in \( \lambda/\sigma \geq C_{0,K}d^{1/4}/T^{1/2} \) in (36). Thus, if there is no severe signal cancellation, iTIPUP and TIPIP-iTOPUP have comparable signal to noise ratio condition with HOOI.

### 4 Simulation Study

In this section, we compare the empirical performance of different procedures of estimating the loading matrices of a tensor factor model, under various simulation setups. Specifically, we consider the following procedures: the non-iterative and iterative methods, and the intermediate output from
the iterative procedures when the number of iteration is 1 after initialization. If TIPUP is used as UINIT and UITER, the one step procedure will be denoted as 1TIPUP. Similar for 1UP and 1TOPUP. We consider the following combinations of UINIT and UITER.

- UP based: (i) UP, (ii) 1UP and (iii) iUP
- TIPUP based: (iv) TIPUP, (v) 1TIPUP and (vi) iTIPUP
- TOPUP based: (vii) TOPUP, (viii) 1TOPUP and (ix) iTOPUP
- mixed iterative: (x) TIPUP-1TOPUP, and (xi) TIPUP-iTOPUP
- mixed iterative: (xii) TOPUP-1TIPUP, and (xiii) TOPUP-iTIPUP

Our empirical results show that (xii) and (xiii) are always inferior than (v) and (vi) respectively. Hence results used (xii) and (xiii) are not shown here.

We demonstrate the performance of all procedures under the setting of a matrix factor model,

\[ X_t = A_1 F_t A_2^T + E_t = \lambda U_1 F_t U_2^T + E_t. \]  

(42)

Here, \( E_t \) is white noise with no autocorrelation, \( E_t \perp E_{t+h}, h > 0 \), and generated according to \( E_t = \Psi_1^{1/2} Z_t \Psi_2^{1/2} \), where \( \Psi_1, \Psi_2 \) are the column and row covariance matrices with the diagonal elements being 1 and all off diagonal elements being 0.2. All of the elements in the \( d_1 \times d_2 \) matrix \( Z_t \) are i.i.d \( N(0,1) \). The elements of the loading matrices \( U_1 \) (of size \( d_1 \times r_1 \) ) and \( U_2 \) (of size \( d_2 \times r_2 \) ) are first generated from i.i.d. \( N(0,1) \), and then orthonormalized through QR decomposition. We set different \( \lambda \) values for different signal-to-noise ratio. All of the entries \( f_{ijt} \) in the factor matrix \( F_t \) follow independent univariate AR(1) model \( f_{ijt} = \phi_{ij} f_{ij(t-1)} + \epsilon_{ijt} \) with standard \( N(0,1) \) innovation. We fix dimensions \( d_1 = d_2 = 16 \) here and consider different choices of sample size \( T \). We use the estimation error for \( A_1 \) as criterion: \( \| \hat{P}_1 - P_1 \|_s \).

We considered three experimental configurations:

I. We set \( r_1 = r_2 = 1 \) and the univariate \( f_t \) follows AR(1) with AR coefficient \( \phi = 0.8 \) to see the effect of sample size \( T \) and signal strength \( \lambda \);

II. We fix the signal strength and consider the rank setup \( r_1 = 1 \) and \( r_2 = 2 \), in which case \( F_t \) is a \( 1 \times 2 \) matrix and the two univariate time series \( f_{it} \) follow AR(1) \( f_{it} = \phi_i f_{i(t-1)} + \epsilon_{it} \) independently with two AR coefficients \( \phi_1 = 0.8 \) and \( \phi_2 = 0.6 \) respectively;

III. We consider the same setting as II, except that \( \phi_2 \) is negative instead, \( \phi_2 = -0.8 \). We repeat all the experiments 100 times.

Settings I and II satisfy Assumption 2 since the rank \( r_1 \) and \( r_2 \) are fixed and the factor process is stationary. The \( \lambda \) in (42) is \( \lambda = \prod_{k=1}^2 \| A_k \|_S \) in Corollaries 1 and 3. The signal to noise ratio is \( \lambda/\sigma = \lambda \). Setting III satisfy Assumption 2(i). When \( h_0 = 1 \) it does not satisfy Assumption 2(ii) as there is a severe signal cancellation. However, using \( h_0 = 2 \) significantly reduces signal cancellation as lag 2 auto-cross-covariance does not cancel each other.

Under Setting I, Figure 1 shows the boxplot of the logarithm of the estimation errors with three choices of signal strengths and two choices of sample sizes for methods (i)-(ix). The performance of the mixed algorithms (x)-(xi) is not shown because they are identical to the corresponding methods (v) and (vi), under the rank one setting when the same initialization is used. We use \( h_0 = 1 \) and
\( \hat{r}_2 = 1 \) in the process of the estimation. It can be seen easily from Figure 1 that UP, 1UP, and iUP are always the worst, showing the advantage of the methods that accommodate time series features and the disadvantage of neglecting temporal correlation. We will exclude UP, 1UP, and iUP from comparison under Settings II and III. When the sample size is small and the signal is weak \((T = 256 \text{ and } \lambda = 1)\), none of the methods work well, though procedures using TIPUP work sometimes. When the sample size is not too small or the signal strength is not too weak (shown in all panels except for the top left one), one-step methods (1TIPUP and 1TOPUP) are better than the noniterative methods (TIPUP and TOPUP), and iterative methods (iTIPUP and iTOPUP) are in turn better than the one-step methods. When the sample size and signal strength increase, all methods perform better, but meanwhile the advantage of iterative methods over one-step methods and the advantage of one-step methods over initialization methods become smaller. When the sample size is large and signal is strong, the one-step methods are similar to the iterative methods after convergence.

It is somewhat surprising to observe, from the top left panel in Figure 1 \((T = 256 \text{ and } \lambda = 1)\), that in the small sample size and low signal strength case, the median error of iTIPUP is larger than that of 1TIPUP, which in turn is larger than TIPUP, whereas the order is reversed under stronger signal to noise ratio or with larger sample size shown in the other panels. Furthermore, the top right panel in Figure 1 shows that, with weak signal to noise ratio, the TIPUP based methods perform better than the TOPUP based methods. This observation coincides with the results in Corollaries 1 and 3. Figure 2 produces some deeper insight, where the trajectories of the iterative methods (including initial estimations, estimations after one iteration, and the estimations after final convergence) of the 100 repetitions are connected, for the \( T = 256 \) case. The top two panels show that when signal is weak and the sample size is small, the initial estimates may be poor, and the iterative methods may need certain accuracy in the initial estimates to produce further improvement. This reemphasizes the condition on the initial estimate in the theorems. The bottom two panels show that when signal is stronger, the relatively more accurate initial estimates enable the iterative methods to improve the estimates. Again, TOPUP initial estimates are not as accurate as the TIPUP estimates.

Under Setting II, Figure 3 shows the boxplot of the logarithm of the estimation errors of 8 methods including (iv)-(ix) and mixed (x)-(xi) with TIPUP initiation and TOPUP iteration. Again, the performance of the mixed (xii)-(xiii) procedures with iTIPUP iteration is not as good as that of iTIPUP hence not shown. Here we use different sample sizes, with the signal strength fixed at \( \lambda = 1 \) and two \( h_0 \) values: \( h_0 = 1 \) and \( h_0 = 2 \). The theoretical \( \lambda_1 \) defined in (18) and \( \lambda^*_1 \) in (19) under the stationary auto-cross-moments of the factor process are given in the figure. Note that they are different for different \( h_0 \). It shows that the mixed TIPUP-1TOPUP method can slightly improve 1TOPUP because of the better initialization. With larger sample size \( T = 1024 \), TIPUP-1TOPUP also slightly outperforms 1TIPUP. In this case, using the larger \( h_0 = 2 \) provides slightly poorer performance than \( h_0 = 1 \), as the lag-2 autocorrelation is significantly smaller than that of lag 1 for the underlying AR(1) process with \( \phi_2 = 0.6 \). The extra term adds limited signal, shown by the small differences in \( \lambda_1 \) and \( \lambda^*_1 \), but incorporates extra noise terms in the estimators. To see more clearly the impact of \( h_0 \), we show the boxplots of the estimated \( \lambda^*_1 \) and \( \lambda_1 \) using iTIPUP and iTOPUP, respectively, for \( h_0 = 1, 2 \) and 3, under different sample sizes in Figure 4. The theoretical values are marked with a diamond. It is seen that the estimated values are relatively close to the theoretical values. More importantly, they decrease as \( h_0 \) increases in this no-signal cancellation case.
Figure 1: Boxplot of the logarithm of the estimation error of $A_1$ under Setting I. 9 methods are considered in total. Three rows correspond to three signal-to-noise strengths $\lambda = 1, 2, 4$. Two columns correspond to two sample sizes $T = 256, 1024$. 
Figure 2: Trajectory of the logarithm of the estimation error of $A_1$ under Setting I with fixed sample size $T = 256$. Two rows correspond to two signal-to-noise strengths $\lambda = 1, 2$. Two columns correspond to TIPUP-based and TOPUP-based methods respectively.
$\phi_2 = 0.6$

Figure 3: Boxplot of the logarithm of the estimation error of $A_1$ under Setting II. 8 methods are considered in total. Two rows correspond to two sample sizes $T = 512, 1024$. Two columns correspond to two choices of $h_0$. The population signal strengths $\lambda_2^1$ (18) and $\lambda_2^2$ (19) for different $h_0$ are provided on the top.
Under Setting III, when $\phi_1 = 0.8$ and $\phi_2 = -0.8$, we can readily check that $E(F_t F_{t-1}^T) = (\phi_1 + \phi_2)\sigma^2 = 0$. Therefore, in the TIPUP-related procedure for estimating $A_1$ with $h_0 = 1$, the signal completely cancels out. Since the ranks $r_1$ and $r_2$ are fixed, we have $\delta_2 = \infty$ for $h_0 = 1$, and the corresponding $\lambda_1^* = 0$. Figure 5 shows the boxplot of the logarithm of the estimation error of $A_1$ for 8 methods including (iv)-(ix) and mixed (x)-(xi) with two choices of $h_0 = 1$ and $h_0 = 2$. We fix the signal strength to be $\lambda = 1$ to isolate the effect of $h_0$. When $h_0 = 1$, both initialization TIPUP and TOPUP do not perform well. But 1TOPUP and iTOPUP improve the performance of TOPUP significantly with TOPUP iteration while 1TIPUP and iTIPUP cannot improve TIPUP. This is because signal cancellation has significant impact on TIPUP based procedures while having no impact on TOPUP based procedures. To our pleasant surprise, when $h_0 = 1$, the mixed TIPUP-1TOPUP is better than both 1TIPUP and 1TOPUP, and the mixed TIPUP-iTOPUP is similar to iTOPUP and much better than iTIPUP. When using $h_0 = 2$, the noise cancellation is mild and $(\lambda_1^*)^2 = 1.78$. Since $r_k$ are fixed, we have $\delta_2 < \infty$. Note that in this case the signal using TIPUP only comes from lag-2 cross product and is weaker than that using TOPUP related procedures. The difference does not have impact on the convergence rate, but on the signal to noise ratio. Comparing the left two subfigures with the right ones of Figure 5, it is seen that using $h_0 = 2$ always boosts the performance of TIPUP-related methods significantly. Meanwhile, the TOPUP based methods are not sensitive to the choice of $h_0$. When $h_0 = 2$, the non-iterative TIPUP performs better than TOPUP, 1TIPUP performs better than 1TOPUP, but after convergence, iTOPUP performs better than iTIPUP. Because the initialization TIPUP is better than TOPUP for $h_0 = 2$, it is of no surprise to see that TIPUP-1TOPUP behaves better than 1TIPUP and 1TOPUP, and TIPUP-iTOPUP is similar as iTOPUP and slightly better than iTIPUP.
Figure 5: Boxplot of the logarithm of the estimation error of $A_1$ under Setting III. 8 methods are considered in total. Two rows correspond to two sample sizes $T = 512, 1024$. Two columns correspond to two choices of $h_0$. The population signal strengths $\lambda_1^2$ (18) and $\lambda_1^2$ (19) for different $h_0$ are provided on the top.
Figure 6: Boxplot of the sample estimates of the signal strengths $\lambda_1^2$ (18) and $\lambda_{1*}^2$ (19) over 100 replications for iTIPUP and iTOPUP with three choices of $h_0$ under Setting III. Two panels correspond to two sample sizes $T = 512, 1024$. The superimposed red diamonds are the population version of the signal strengths.

Again, to see more clearly the impact of $h_0$ in this case with noise cancellation, we show the boxplots of the estimated $\lambda_1^*$ and $\lambda_1$ using iTIPUP and iTOPUP, respectively, for $h_0 = 1, 2$ and 3 in Figure 6. It is seen that the iTOPUP procedure remains robust in estimating $\lambda_1$ under the noise-cancellation case. And $\lambda_1$ decreases as $h_0$ increases. However, iTIPUP is very different. Although when using $h_0 = 1$ the estimated $\lambda_1^*$ significantly overestimates the theoretical value $\lambda_1^* = 0$, they are still much less than those from using $h_0 = 2$ and 3. The reversed order of the magnitude of $\lambda_1^*$ as $h_0$ increases can be potentially used to detect signal cancellation in practice, though the theoretical property of the estimators of $\lambda_1^*$ (e.g. standard deviation) is technically challenging to obtain. In practice, when one observes such a reversed order, it is recommended to use iTOPUP as a conservative estimator. Of course, the behaviors of $\lambda_k$ and $\lambda_{k*}$ depend on the auto-cross-moment structure of the underlying factor process. For example, if the factor process follows a MA(2) model with zero lag-1 autocorrelation ($f_t = e_t + \theta_2 e_{t-2}$), then $\lambda_1$ and $\lambda_1^*$ under $h_0 = 2$ would be larger than those under both $h_0 = 1$ and $h_0 = 3$. But we expect that the pattern of $\lambda_1$ under different $h_0$ would be similar to that of $\lambda_{1*}$ under different $h_0$, if there is no severe signal cancellation. Severe signal cancellation would make the patterns different.

5 Summary

In this paper we propose new estimation procedures for tensor factor model via iterative projection, and focus on two procedures: iTOPUP and iTIPUP. Theoretical analysis shows the asymptotic properties of the estimators. Simulation study is used to illustrate the finite sample properties of
the estimators. While theoretical results are obtained under very general conditions, specific cases are considered. Specifically, under the typical factor model setting where the condition numbers of $A_k^TA_k$ are bounded and the ranks $r_k$ are fixed, the proposed iterative procedures, iTOPUP method and iTIPUP method (with no severe signal cancellation) lead to a convergence rate $O_p((Td_{-k})^{-1/2})$ under strong factors settings due to information pooling of the orthogonal projection of the other $d_{-k}$ dimensions. This rate is much sharper than the classical rate $O_p(T^{-1/2})$ in non-iterative estimators for vector, matrix and tensor factor models. It implies that the accuracy can be improved by increasing the dimensions, and consistent estimation of the loading spaces can be achieved even with a fixed finite sample size $T$, different from the requirement of traditional factor model analysis. The proposed iterative estimation methods not only preserve the tensor structure, but also result in sharper convergence rate in the estimation of factor loading space.

The iterative procedure requires two operators, one for initialization and one for iteration. Under certain conditions of the signal to noise ratio, we only need the initial estimator to have sufficiently small estimation errors but not the consistency of the initial estimator. Often, one iteration is sufficient. Under more complicated cases, at most $O(\log(d))$ iterations are needed to achieve ideal rate of convergence. Based on the theoretical results and empirical evidence, we suggest to use iTOPUP for iteration when the ranks $r_k$ are small. In terms of initiation, we suggest to use TIPUP if no danger of signal cancellation and TOPUP otherwise. Using a slightly large $h_0$ often solve the noise cancellation problem, as the empirical results show. By examination of the patterns of estimated singular values under different lag values $h_0$, using iTOPUP and iTIPUP, it is possible to detect noise cancellation, which has significant impact on iTIPUP estimators.

The proposed iterative procedure is similar to HOOI algorithms in spirit, but the detailed operations and the theoretical challenges are significantly different.

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6 Appendix

6.1 Proofs of Theorem 2

We focus on the case of $K = 2$ as the iTIPUP begins with mode-$k$ matrix unfolding. In particular, we sometimes give explicit expressions only in the case of $k = 1$ and $K = 2$. For $K = 2$, we observe a matrix time series with $X_t = A_1 F_t A_2^T + E_t \in \mathbb{R}^{d_1 \times d_2}$. Under the conditional expectation $\mathbb{E}_n F_1, \ldots, F_T$ are fixed. Let $U_1$, $U_2$ be the left singular matrices of $A_1$ and $A_2$ respectively with $r_k = \text{rank}(U_k) = \text{rank}(A_k)$.

We outline the proof as follows. Let $L_k^{(m)}$ be the loss (4) for $\hat{P}_k^{(m)}$ or equivalently the spectral norm error for $\hat{P}_k^{(m)} = \hat{U}_k^{(m)} \hat{U}_k^{(m)\top}$, $k = 1, \ldots, K$, and $L^{(m)}$ their maximum,

$$L_k^{(m)} = \|\hat{P}_k^{(m)} - P_k\|_S, \quad L^{(m)} = \max_{k = 1, 2, \ldots, K} L_k^{(m)}. \quad (43)$$

From Chen et al. (2019b), $\mathbb{E}[L_k^{(0)}] \leq R_k^{* (0)}$ as we mentioned in (30). By applying the Gaussian concentration inequality for Lipschitz functions and Lemma 2 in their analysis, we have

$$L^{(0)} \leq C_1^{(\text{TIPUP})} R_k^{* (0)} \quad \text{with} \quad R_k^{* (0)} = \max_{1 \leq k \leq K} R_k^{* (0)} \quad (44)$$

in an event $\Omega_0$ with $\mathbb{P}(\Omega_0) \geq 1 - 5^{-1} \sum_{k=1}^{K} e^{-d_k}$. This is similar to (51) below.

After the initialization with $\hat{U}_k^{(0)}$, the algorithm iteratively produces estimates $\hat{U}_k^{(m)}$ from $m = 1$ to $m = J$. For any $\hat{U}_2 \in \mathbb{R}^{d_2 \times r_2}$, let

$$V_{1, h}^*(\hat{U}_2) = \sum_{t=h+1}^{T} \frac{X_{t-h} \hat{U}_2 \hat{U}_2^\top X_{t}}{T - h} \in \mathbb{R}^{d_1 \times d_1}, \quad \text{TIPUP}_1(\hat{U}_2) = \text{mat}_1(V_{1, h_0}^*(\hat{U}_2)) \in \mathbb{R}^{d_1 \times (d_1 h_0)},$$

as matrix-valued functions of matrix-valued variable $\hat{U}_2$, where $V_{1, h_0}^*(\hat{U}_2) = (V_{1, h}^*(\hat{U}_2), h = 1, \ldots, h_0) \in \mathbb{R}^{d_1 \times d_1 \times h_0}$. Given $\hat{U}_2^{(m)}$, the $(m + 1)$-th iteration produces estimates

$$\hat{U}_1^{(m+1)} = \text{LSVD}_{r_1}(\text{TIPUP}_1(\hat{U}_2^{(m)})), \quad \hat{P}_1^{(m+1)} = \hat{U}_1^{(m+1)} \hat{U}_1^{(m+1)\top}.$$  

The “noiseless” version of this update is given by

$$\Theta_{1, h}^*(\hat{U}_2) = \sum_{t=h+1}^{T} \frac{A_1 F_{t-h} A_2^\top \hat{U}_2 \hat{U}_2^\top A_2 F_t A_1^\top}{T - h}, \quad \mathbb{E}[\text{TIPUP}_1](\hat{U}_2) = \text{mat}_1(\Theta_{1, h_0}^*(\hat{U}_2)) \quad (45)$$

with $\Theta_{1, h_0}^*(\hat{U}_2) = (\Theta_{1, h}^*(\hat{U}_2), h = 1, \ldots, h_0)$ as in (17), giving error free “estimates”,

$$U_1 = \text{LSVD}_{r_1}(\mathbb{E}[\text{TIPUP}_1](\hat{U}_2^{(m)})), \quad P_1 = U_1 U_1^\top,$$

when $\mathbb{E}[\text{TIPUP}_1](\hat{U}_2^{(m)})$ is of rank $r_1$. Thus, by Wedin’s theorem (Wedin (1972)),

$$L_1^{(m+1)} = \|\hat{P}_1^{(m+1)} - P_1\|_S \leq \frac{2\|\text{TIPUP}_1(\hat{U}_2^{(m)}) - \mathbb{E}[\text{TIPUP}_1](\hat{U}_2^{(m)})\|_S}{\sigma_{r_1}(\mathbb{E}[\text{TIPUP}_1](\hat{U}_2^{(m)}))}. \quad (46)$$

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We note that \( \mathbb{E}[\text{TIPUP}_1(\tilde{U}_2) \mid \tilde{U}_2] \) is treated as a matrix-valued function of matrix-valued variable \( \tilde{U}_2 \) and \( \tilde{U}_2^{(m)} \) is plugged-in after the conditional expectation in \( \mathbb{E}[\text{TIPUP}_1(\tilde{U}_2^{(m)})] \). For general \( 1 \leq k \leq K \), we define \( \text{TIPUP}_k(U_{-k}) \) and \( \mathbb{E}[\text{TIPUP}_k](U_{-k}) \) as matrix-valued functions of \( U_{-k} = \bigcirc_{j \neq k} U_j \).

To bound the numerator on the right-hand side of (46), we write

\[
\text{TIPUP}_1(\tilde{U}_2) - \mathbb{E}[\text{TIPUP}_1](\tilde{U}_2) = \sum_{j=1}^{3} \left( \Delta_{j,1,h}^* (\tilde{U}_2 \tilde{U}_2^T), h = 1, \ldots, h_0 \right) \in \mathbb{R}^{d_1 \times (d_1 h_0)}
\]

as both \( \text{TIPUP}_1(\tilde{U}_2) \) and \( \mathbb{E}[\text{TIPUP}_1](\tilde{U}_2) \) are linear in \( \tilde{U}_2 \), where for any \( d_2 \times d_2 \) matrix \( \hat{M}_2 \)

\[
\Delta_{1,1,h}^* (\hat{M}_2) = \Delta_{1,1,h}^* (\hat{M}_2) = \sum_{t=h+1}^{T} A_1 F_{t-h} A_2^T \hat{M}_2 E_t^T / (T - h),
\]

\[
\Delta_{2,1,h}^* (\hat{M}_2) = \Delta_{2,1,h}^* (\hat{M}_2) = \sum_{t=h+1}^{T} E_{t-h} \hat{M}_2 A_2 F_t^T / (T - h),
\]

\[
\Delta_{3,1,h}^* (\hat{M}_2) = \Delta_{3,1,h}^* (\hat{M}_2) = \sum_{t=h+1}^{T} E_{t-h} \hat{M}_2 E_t / (T - h).
\]

As \( \Delta_{j,1,h}^* (\hat{M}_2) \) is linear in \( \hat{M}_2 \), the numerator on the right-hand of (46) can be bounded by

\[
\left\| \text{TIPUP}_1(\tilde{U}_2^{(m)}) - \mathbb{E}[\text{TIPUP}_1](\tilde{U}_2^{(m)}) \right\|_S \leq \left\| \text{TIPUP}_1(\tilde{U}_2) - \mathbb{E}[\text{TIPUP}_1](\tilde{U}_2) \right\|_S + L^{(m)}(2K - 2) \sum_{j=1}^{3} h_0^{1/2} \max_{h \leq h_0} \left\| \Delta_{j,1,h}^* \right\|_{1,S,S}
\]

with an application of Cauchy-Schwarz inequality for the sum over \( h = 1, \ldots, h_0 \), where \( \left\| \Delta_{j,1,h}^* \right\|_{1,S,S} \) are norms of the \( \mathbb{R}^{d_2 \times d_2} \rightarrow \mathbb{R}^{d_1 \times d_1} \) linear mappings \( \Delta_{j,1,h}^* \) defined as

\[
\left\| \Delta_{j,1,h}^* \right\|_{1,S,S} = \max_{\hat{M}_2} \left\| \Delta_{j,1,h}^* (\hat{M}_2) \right\|_S.
\]

For general \( 1 \leq k \leq K \), \( \Delta_{j,k,h}^* \) is an \( \mathbb{R}^{d_{-k} \times d_{-k}} \) to \( \mathbb{R}^{d_1 \times d_k} \) mapping. Because it applies to \( \hat{M}_{-k} = \bigcirc_{j \neq k} (\tilde{U}_j \tilde{U}_j^T) - \bigcirc_{j \neq k} (U_j U_j^T) \), (70) of Lemma 1 (iii) gives the general version of (48) with

\[
\left\| \Delta_{j,k,h}^* \right\|_{k,S,S} = \max_{\hat{M}_j} \left\| \Delta_{j,k,h}^* (\bigcirc_{j \neq k} \hat{M}_j) \right\|_S.
\]

We claim that in certain events \( \Omega_j, j = 1, 2, 3 \), with \( \mathbb{P}(\Omega_j) \geq 1 - 5^{-1} \sum_{k=1}^{K} e^{-d_k} \),

\[
\left\| \Delta_{j,k,h}^* \right\|_{k,S,S} \leq \rho \lambda_{k}^{*2} / (24(K - 1)), \quad \forall 1 \leq k \leq K.
\]

For simplicity, we will only prove this inequality for \( k = 1 \) and \( K = 2 \) in the form of

\[
\mathbb{P} \left\{ \left\| \Delta_{j,k,h}^* \right\|_{1,S,S} = \max_{\hat{M}_2} \left\| \Delta_{j,1,h}^* (\hat{M}_2) \right\|_S \geq \rho \lambda_{1}^{*2} / 24 \right\} \leq 5^{-1} e^{-d_2}
\]

as the proof of its counter part for general \( 1 \leq k \leq K \) is similar.

Define the ideal version of the ratio in (46) for general \( 1 \leq k \leq K \) as

\[
L_k^{(\text{ideal})} = \frac{2 \left\| \text{TIPUP}_k(U_{-k}) - \mathbb{E}[\text{TIPUP}_k](U_{-k}) \right\|_S}{\sigma_{r_k} \left( \mathbb{E}[\text{TIPUP}_k](U_{-k}) \right)}, \quad L_k^{(\text{ideal})} = \max_{1 \leq k \leq K} L_k^{(\text{ideal})}.
\]
As $U_{-k} = \bigcap_{j\neq k} U_j$ is true and deterministic, the proof of (44) also implies
\[
L^{(\text{ideal})} \leq C_1^{(\text{TIPUP})} R^{(\text{ideal})} \quad \text{with} \quad R^{* (\text{ideal})} = \max_{1 \leq k \leq K} R^*_k
\]  
(51)
in an event $\Omega_4$ with $\mathbb{P}(\Omega_4) \leq 5^{-1} \sum_{k=1}^K e^{-d_k}$, where $R^*_k$ is as in (31).

We aim to prove that in the event $\cap_{j=0}^4 \Omega_j$
\[
L^{(m+1)} \leq 2 L^{(m)} + 24(K-1) L^{(m)} \max_{j,k,h} \| \Delta^*_{j,k,h} \|_{k,S,S} / \lambda^*_k \leq 2 L^{(m)} + \rho L^{(m)} ,
\]  
(52)
which would imply by induction
\[
L^{(m+1)} \leq 2(1 + \ldots + \rho^m) L^{(\text{ideal})} + \rho^{m+1} L^{(0)} ,
\]  
(53)
and then the conclusions would follow from (44) and (51). To this end, we notice that by (43), (46), (48) and (49), (52) holds provided that $\sigma_{r_k}(\mathbb{E}[\text{TIPUP}_k](\hat{L}^{(m)}))$, the numerator in (46), is no smaller than a half of its ideal version as in (19), e.g.
\[
2 \sigma_{r_h}(\mathbb{E}[\text{TIPUP}_1](\hat{L}^{(m)}_2)) \geq \sigma_{r_1}(\mathbb{E}[\text{TIPUP}_1](U_2)) = h_0^{1/2} \lambda^*_2.
\]  
(54)
in the case of $k = 1$ and $K = 2$. We divide the rest of the proof into 4 steps to prove (50) for $j = 1, 2, 3$ and (54).

**Step 1.** We prove (50) for the $\Delta^*_1(\hat{M}_2)$ in (47). By Lemma 1(ii), there exist $\hat{M}^{(1)} \in \mathbb{R}^{d_2 \times d_2}$ of the form $W_\ell Q_\ell^\top$ with $W_\ell \in \mathbb{R}^{d_2 \times 2r_2}$, $Q_\ell \in \mathbb{R}^{d_2 \times 2r_2}$, $1 \leq \ell, \ell' \leq N_{d_2 r_2,1/8} = 17^{2 d_2 r_2}$, such that $\| \hat{M}^{(1)} \|_S \leq 1$, $\text{rank}(\hat{M}^{(1)}) \leq 2r_2$ and
\[
\| \Delta^*_{1,1,h} \|_{1,S,S} \leq 2 \max_{1 \leq \ell, \ell' \leq N_{d_2 r_2,1/8}} \left\| \frac{T}{T - h} \sum_{t=h+1}^T A_1 F_{t-h} A_2 \hat{M}^{(1)} E_t^\top \right\|_S.
\]

To bound $\| \Delta^*_{1,k,h} \|_{1,S,S}$ for general $k \leq K$, we just need to replace $\hat{M}^{(1)}$ by $\bigcap_{j \neq k} \hat{M}^{(1)}_j$ and $N_{d_2 r_2,1/8}$ by $N_{d_2 r_2,1/(8K-8)}$ with $d^*_j = \sum_{j \neq k} d_j$ as in Lemma 1 (iii). We apply the Gaussian concentration inequality to the right-hand side above. Elementary calculation shows that
\[
\left\| \frac{T}{T - h} \sum_{t=h+1}^T A_1 F_{t-h} A_2 \hat{M}^{(1)} E_t^\top \right\|_S \leq \left\| (A_1 F_1 A_2^\top, \ldots, A_1 F_T A_2^\top) \left( \hat{M}^{(1)}(E_{h+1}^\top - E_{h+1}^\top) \right) \right\|_S
\]
\[
\leq \left\| (A_1 F_1 A_2^\top, \ldots, A_1 F_T A_2^\top) \right\|_S^{1/2} \left\| \text{diag}(\hat{M}^{(1)})(E_{h+1}^\top - E_{h+1}^\top) \right\|_S
\]
\[
\leq \sqrt{T} \Theta_{1,0}^{1/2} \left\| \left( \begin{array}{c} E_{h+1}^\top - E_{h+1}^\top \\ \vdots \\ E_T^\top - E_T^\top \end{array} \right) \right\|_F.
\]
That is, $\left\| \sum_{t=h+1}^{T} A_{1} F_{t-h} A_{2}^{\top} \tilde{M}^{(t)} E_{t}^{\top} \right\|_{\mathcal{S}}$ is a $\sqrt{T} \left\| \Theta_{1,0}^{*} \right\|_{\mathcal{S}}^{1/2}$ Lipschitz function in $(E_{1}, \ldots, E_{T})$. Employing similar arguments in the proof of Theorem 2 in Chen et al. (2019b), we have

$$
\mathbb{E} \left\| \sum_{t=h+1}^{T} A_{1} F_{t-h} A_{2}^{\top} \tilde{M}^{(t)} E_{t}^{\top} \right\|_{\mathcal{S}} \leq \frac{\sigma (8 T d_{1})^{1/2}}{T-h} \left\| \Theta_{1,0}^{*} \right\|_{\mathcal{S}}^{1/2}.
$$

Then, by Gaussian concentration inequalities for Lipschitz functions,

$$
P \left( \left\| \sum_{t=h+1}^{T} A_{1} F_{t-h} A_{2}^{\top} \tilde{M}^{(t)} E_{t}^{\top} \right\|_{\mathcal{S}} - \frac{\sigma (8 T d_{1})^{1/2}}{T-h} \left\| \Theta_{1,0}^{*} \right\|_{\mathcal{S}}^{1/2} \right) \leq 2 e^{-\frac{\sigma^{2}}{2}}.
$$

Hence,

$$
P \left( \left\| \Delta_{1}^{*} \right\|_{1,S,S/2} \leq \frac{\sigma (8 T d_{1})^{1/2}}{T-h} \left\| \Theta_{1,0}^{*} \right\|_{\mathcal{S}}^{1/2} + \frac{\sigma \sqrt{T}}{T-h} \left\| \Theta_{1,0}^{*} \right\|_{\mathcal{S}}^{1/2} \right) \leq 2 N_{2 d_{2} r_{2},1/8} e^{-\frac{\sigma^{2}}{2}}.
$$

As $T \geq 4 h_{0}$ and $K = 2$, this implies with $x = \sqrt{d_{2} r_{2}}$ that in an event with at least probability $1 - e^{-d_{2}/5}$,

$$
\left\| \Delta_{1}^{*} \right\|_{1,S,S} \leq \frac{C_{1,K}^{(iter)} \sigma T^{-1/2} \left\| \Theta_{1,0}^{*} \right\|_{\mathcal{S}}^{1/2} (\sqrt{d_{1}} + \sqrt{d_{2} r_{2}}) + C_{1,K}^{(iter)} (1 + \sqrt{d_{2} r_{2}/d_{1}}) \lambda_{1}^{2} R_{1}^{(ideal)}}{24 (K-1)}
$$

with the $R_{k}^{(ideal)}$ in (31) and a constant $C_{1,K}^{(iter)}$ depending on $K$ only. In this event, (33) gives

$$
24 (\lambda_{k}^{2})^{-2} \left\| \Delta_{1,k,h}^{*} \right\|_{k,s,S} \leq \frac{C_{1,K}^{(iter)} / C_{0}^{(iter)}}{24} \leq \rho.
$$

Thus, (50) holds for $\Delta_{1}^{*} (\tilde{M}_{2})$.

**Step 2.** Inequality (50) for $\Delta_{2}^{*} (\tilde{M}_{2})$ follow from the same argument as the above step.

**Step 3.** Here we prove (50) for the $\Delta_{3}^{*} (\tilde{M}_{2})$ in (47). By Lemma 1 (ii), we can find $U_{2}^{(\ell)} \in \mathbb{R}^{d_{2} \times r_{2},1, \ell, \ell', \ell, \ell', \ell, \ell'} \leq N_{d_{2} r_{2},1/8}$ such that $\left\| U_{2}^{(\ell)} \right\|_{S} \leq 1$, $\left\| U_{2}^{(\ell')} \right\|_{S} \leq 1$ and

$$
\left\| \Delta_{3}^{*} \right\|_{1,S,S} = \left\| \sum_{t=h+1}^{T} E_{t-h} U_{2}^{(\ell)} U_{2}^{(\ell')} E_{t}^{\top} \right\|_{S} \leq \frac{\max_{1 \leq \ell, \ell' \leq \ell, \ell'} \left\| \sum_{t=h+1}^{T} E_{t-h} U_{2}^{(\ell)} U_{2}^{(\ell')} E_{t}^{\top} \right\|_{S}}{T-h}.
$$

We split the sum into two terms over the index sets, $S_{1} = (h, 2 h] \cup (3 h, 4 h] \cup \cdots \cup (h, T]$ and its complement $S_{2}$ in $(h, T]$, so that $\left\{ E_{t-h}, t \in S_{1} \right\}$ is independent of $\left\{ E_{t}, t \in S_{2} \right\}$ for each $a = 1, 2$. Let $n_{a} = \left| S_{a} \right| r_{2}$. Define $G_{a} = \left( E_{t-h} U_{2}^{(t)}, t \in S_{a} \right)$ in $\mathbb{R}^{d_{1} \times n_{a}}$ and $H_{a} = \left( E_{t} U_{2}^{(t)}, t \in S_{a} \right)$ in $\mathbb{R}^{d_{1} \times n_{a}}$. Then, $G_{a}, H_{a}$ are two independent Gaussian matrices. Note that

$$
\left\| \sum_{t \in S_{a}} E_{t-h} U_{2}^{(t)} U_{2}^{(t')} E_{t}^{\top} \right\|_{S} = \left\| G_{a} H_{a}^{\top} \right\|_{S}.
$$

Moreover, by Assumption 1, $\text{Var} (u^{\top} \text{vec} (G_{a})) \leq \sigma^{2}$ and $\text{Var} (u^{\top} \text{vec} (H_{a})) \leq \sigma^{2}$ for all unit vectors $u \in \mathbb{R}^{d_{1} n_{a}}$, so that by Lemme 2 (i)

$$
P \left\{ \left\| G_{a} H_{a}^{\top} \right\|_{S} / \sigma^{2} \geq d_{1} + 2 \sqrt{d_{1} n_{a}} + x (x + 2 \sqrt{n_{a} + 2 \sqrt{d_{1}}}) \right\} \leq 2 e^{-x^{2}/2}, \quad x > 0.
$$

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As $\sum_{a=1}^{2} n_a = r_2(T - h)$, it follows from (55), (56) and the above inequality that
\[
\mathbb{P} \left\{ \frac{\| \Delta_{3,1,h}^{*} \|_{1,S}}{4\sigma^2} \geq \frac{(\sqrt{d_1} + x)^2}{T - h} + \frac{\sqrt{2r_2}(\sqrt{d_1} + x)}{\sqrt{T - h}} \right\} \leq 4N_{d_2r_2,1/8}^2e^{-x^2/2}.
\]
Thus, with $h_0 \leq T/4$, $x = \sqrt{d_1} + \sqrt{d_{-1}}$ and some constant $C_{1,K}^{(\text{iter})}$ depending on $K$ only,
\[
\| \Delta_{3,1,h}^{*} \|_{1,S} \leq \frac{C_{1,K}^{(\text{iter})}(1 - h_0/T)^2\sigma^2}{24(K - 1)} \left( \frac{(\sqrt{d_1} + \sqrt{d_{-1}})(\sqrt{d_{-1}} - 1)}{T^{1/2}} + \frac{(\sqrt{d_1} + \sqrt{d_{-1}})^2}{T - h_0} \right)
\]
with at least probability $1 - e^{-d_1}/5$. As $\lambda_2^* \leq \| \Theta_{1,0}^* \|_S^{1/2}/(1 - h_0/T)^{1/2}$ by (19) and (16),
\[
R_1^{(\text{ideal})} \geq (\lambda_2^*)^{-1}(1 - h_0/T)\sigma(T - h_0)^{-1/2}\sqrt{d_1} + (\lambda_1^*)^{-2}\sigma T^{-1/2}\sigma \sqrt{d_{-1}r_{-1}}
\]
by (31). Thus, in the event (57) and for $k = 1$ and $K = 2$,
\[
\| \Delta_{3,1,h}^{*} \|_{1,S} \leq \frac{C_{1,K}^{(\text{iter})}}{24} \left( 1 + \sqrt{\frac{d_{-1}}{d_1}} \right) \lambda_1^* \lambda_2^* R_1^{(\text{ideal})} + \left( 1 + \sqrt{\frac{d_{-1}}{d_1}} \right)^2 \lambda_2^* (R_1^{(\text{ideal})})^2
\]
\[
\leq \frac{\lambda_2^* C_{1,K}^{(\text{iter})}(1 + 1/C_0^{(\text{iter})})}{24 C_0^{(\text{iter})}},
\]
which is no greater than $\lambda_1^* \rho/24$ by the condition on $C_0^{(\text{iter})}$. This yields (50) for $\Delta_3^{*}(\hat{M}_2)$.

**Step 4.** Next, we prove (54) in the event $\gamma^2_2 = 0$. Note that,
\[
\| \Theta_{1,h}^{*}(-\hat{U}_2^{(m)}) - \Theta_{1,h}^{*}(U_2) \|_S = \left\| \sum_{r=h+1}^{T} \frac{A_2 F_{r-h} A_2^{\top} (\hat{U}_2^{(m)} - U_2^{(m)})(\hat{U}_2^{(m)} - U_2^{(m)})}{T - h} A_2 F_{r-h} A_2^{\top} \right\|_S
\]
\[
= \frac{1}{T - h} \left\| (A_1 F_1 A_2^{\top}, ..., A_1 F_{T-h} A_2^{\top}) \begin{pmatrix} (\hat{U}_2^{(m)} - U_2^{(m)}) A_2 F_{T-h} A_2^{\top} \\ \vdots \\ (\hat{U}_2^{(m)} - U_2^{(m)}) A_2 F_{T-h} A_2^{\top} \end{pmatrix} \right\|\]
\[
\leq \frac{1}{T - h} \left\| (A_1 F_1 A_2^{\top}, ..., A_1 F_{T-h} A_2^{\top}) \text{diag}(\hat{U}_2^{(m)} - U_2^{(m)}) A_2 F_{T-h} A_2^{\top} \right\|_S
\]
\[
\leq L^{(m)} \| \Theta_{1,0}^* \|_S/(1 - h_0/T).
\]
Hence, by the Cauchy-Schwarz inequality and (45),
\[
\| \mathbb{E}[	ext{TIPUP}_1(-\hat{U}_2^{(m)})] - \mathbb{E}[	ext{TIPUP}_1(U_2)] \|_S \leq \sqrt{h_0} L^{(m)} \| \Theta_{1,0}^* \|_S/(1 - h_0/T).
\]
By (19), $\lambda_1^2 h_0^{1/2} = \sigma_r(\text{mat}_1(\Theta_{1,1,h}^*)) = \sigma_r(\mathbb{E}[	ext{TIPUP}_1](U_2))$. Thus, by Weyl’s inequality
\[
\sigma_r(\mathbb{E}[	ext{TIPUP}_1](\hat{U}_2^{(m)})) \geq \lambda_1^2 h_0^{1/2} - 2\sqrt{h_0} L^{(m)} \| \Theta_{1,0}^* \|_S \geq \sigma_r(\mathbb{E}[	ext{TIPUP}_1](U_2))/2 = \lambda_1^2 h_0^{1/2}/2.
\]
when \( m_k \leq k\lambda_k^2 \geq 4L^{(m)}|\Theta_{1,0}^*|s \). We prove this condition by induction in the event \( \cap_{j=0}^{1} \Omega_j \).

By (32) and (44), \( 4L^{(0)}|\Theta_{1,0}^*|s/\min_{k \leq K} \lambda_k^2 \leq 4C_1^{(TOPUP)}/C_0 \leq 1 \). Given the induction assumption \( 4L^{(m)}|\Theta_{1,0}^*|s/\min_{k \leq K} \lambda_k^2 \leq 1 \), (54) holds for the same \( m \), so that (53), (51) and (44),

\[
L^{(m+1)} \leq C_1^{(TOPUP)} \{2(1 + \ldots + \rho^m)R^*(\text{ideal}) + \rho^{m+1}R^*(0)\} \leq C_1^{(TOPUP)}2(1 - \rho)^{-1}R^*(0).
\]

It then follows from (32) that \( 4L^{(m+1)}|\Theta_{1,0}^*|s/\min_{k \leq K} \lambda_k^2 \leq C_1^{(TOPUP)}8(1 - \rho)^{-1}/C_0 \leq 1 \). This completes the induction and the proof of the entire theorem.

Remark 9. Alternatively, if we assume \( 2(1 - \rho)^{-1}R^*(\text{ideal}) \leq R^*(0) \), then \( 2(1 + \ldots + \rho^m)R^*(\text{ideal}) + \rho^{m+1}R^*(0) \leq R^*(0) \), so that the condition \( 4C_1^{(TOPUP)}/C_0 \leq 1 \) would be sufficient. The weakest condition is

\[
(4C_1^{(TOPUP)})^{\max\{2(1 + \ldots + \rho^m)|R^*(\text{ideal})/R^*(0) + \rho^{m+1}\}} \leq 1.
\]

6.2 Proofs of Theorem 1

Again, we focus on the case of \( K = 2 \) as the iTOPUP also begins with mode-\( k \) matrix unfolding. In particular, we sometimes give explicit expressions only in the case of \( k = 1 \) and \( K = 2 \). For \( K = 2 \), we observe a matrix time series with \( X_t = A_1 F_t A_2^\top + E_t \in \mathbb{R}^{d_1 \times d_2} \). Under the conditional expectation \( E, F_1, \ldots, F_T \) are fixed. Let \( U_1, U_2 \) be the left singular matrices of \( A_1 \) and \( A_2 \) respectively with \( r_k = \text{rank}(U_k) = \text{rank}(A_k) \). Recall \( \otimes \) is kronecker product and \( \otimes \) is tensor product.

We outline the proof as follows, which has exactly the same structure as the proofs of Theorem 2.

Recall \( L_k^{(m)} \) and \( L^{(m)} \) in (43). From Chen et al. (2019b), \( \mathbb{E}[L_k^{(0)}] \leq R_k^{(0)} \) as we mentioned in (20). By applying the Gaussian concentration inequality for Lipschitz functions and Lemma 2 in their analysis, we have

\[
L^{(0)} \leq C_1^{(TOPUP)}R^{(0)} \quad \text{with} \quad R^{(0)} = \max_{1 \leq k \leq K} R_k^{(0)}
\]

in an event \( \Omega_0 \) with \( \mathbb{P}(\Omega_0) \geq 1 - 5^{-1} \sum_{k=1}^{K} e^{-d_k} \). This is similar to (64) below.

After the initialization with \( \hat{U}_k^{(0)} \), the algorithm iteratively produces estimates \( \hat{U}_k^{(m)} \) from \( m = 1 \) to \( m = J \). For any \( \hat{U}_2 \in \mathbb{R}^{d_2 \times r_2} \), let

\[
V_{1,h}^{(m)}(\hat{U}_2) = \sum_{t=h+1}^{T} \frac{X_{t-h}^{(m)} \otimes X_t \hat{U}_2}{T-h} \in \mathbb{R}^{d_1 \times r_2 \times d_1 \times r_2}, \quad \text{mat}_1^{(\text{TOPUP})}(\hat{U}_2) = \text{mat}_1(V_{1,h}^{(m)}(\hat{U}_2) \in \mathbb{R}^{d_1 \times (d_1 r_2^2 h_0)},
\]

where \( V_{1,h}^{(m)}(\hat{U}_2) = (V_{1,h}^{(m)}(\hat{U}_2), h = 1, \ldots, h_0) \in \mathbb{R}^{d_1 \times r_2 \times d_1 \times r_2 \times h_0} \). Given \( \hat{U}_2^{(m)} \), the \((m+1)\)-th iteration produces estimates

\[
\hat{U}_2^{(m+1)} = \text{LSVD}_{r_1}(\text{mat}_1^{(\text{TOPUP})}(\hat{U}_2^{(m)})), \quad \hat{U}_2^{(m+1)} = \hat{U}_2^{(m+1)} \hat{U}_2^{(m+1)^\top}.
\]

The “noiseless” version of this update is given by

\[
\Theta_{1,h}^{(m)}(\hat{U}_2) = \sum_{t=h+1}^{T} \frac{A_1 F_{t-h} A_2^\top \hat{U}_2 \otimes A_1 F_t A_2^\top \hat{U}_2}{T-h}, \quad \mathbb{E}[\text{mat}_1^{(\text{TOPUP})}](\hat{U}_2) = \text{mat}_1(\Theta_{1,h}^{(m)}(\hat{U}_2))
\]

(59)
with \( \Theta_{1,1,h}(\tilde{U}_2) = (\Theta_{1,h}(\tilde{U}_2), h = 1, \ldots, h_0) \) as in (15), giving error free “estimates”,

\[
U_1 = \text{LSVD}_r(\mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](\tilde{U}_2^{(m)})), \quad P_1 = U_1 U_1^T.
\]

Thus, by Wedin’s theorem (Wedin (1972)),

\[
L_1^{(m+1)} = \left\| \hat{P}_1^{(m+1)} - P_1 \right\|_S \leq \frac{2\|\text{mat}_1(\text{TOPUP}_1)(\tilde{U}_2^{(m)}) - \mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](\tilde{U}_2^{(m)})\|_S}{\sigma_r(\mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](\tilde{U}_2^{(m)}))}.
\] (60)

To bound the numerator on the right-hand side of (60), for any matrix \( \tilde{M}_2 = \tilde{U}_2 \odot I_{d_1} \odot \tilde{U}_2 \), we write

\[
\Delta_1(\tilde{M}_2) = \Delta_{1,1,h}(\tilde{M}_2) = \frac{1}{T - h} \sum_{t=h+1}^{T} \text{mat}_1(A_1 F_{t-h} A_j^T \tilde{U}_2 \otimes E_t \tilde{U}_2),
\]

\[
\Delta_2(\tilde{M}_2) = \Delta_{2,1,h}(\tilde{M}_2) = \frac{1}{T - h} \sum_{t=h+1}^{T} \text{mat}_1(E_{t-h} \tilde{U}_2 \otimes A_1 F_t^T A_j^T \tilde{U}_2),
\]

\[
\Delta_3(\tilde{M}_2) = \Delta_{3,1,h}(\tilde{M}_2) = \frac{1}{T - h} \sum_{t=h+1}^{T} \text{mat}_1(E_{t-h} \tilde{U}_2 \otimes E_t \tilde{U}_2) = \frac{1}{T - h} \sum_{t=h+1}^{T} \text{mat}_1(E_{t-h} \otimes E_t) \tilde{M}_2.
\]

Let \( P_{U_2} = U_2 U_2^T \) and \( I = P_{U_2} + P_{U_2^\perp} \). Note that, for \( \tilde{M}_2 = \tilde{U}_2 \odot I_{d_1} \odot \tilde{U}_2 \) and \( M_2 = U_2 \odot I_{d_1} \odot U_2 \),

\[
\|\Delta_3(\tilde{M}_2)\|_S \leq \left\| \sum_{t=h+1}^{T} \frac{\text{mat}_1(E_{t-h} P_{U_2} \tilde{U}_2^{(m)} \otimes E_t P_{U_2} \tilde{U}_2^{(m)})}{T - h} \right\|_S + \left\| \sum_{t=h+1}^{T} \frac{\text{mat}_1(E_{t-h} P_{U_2} \tilde{U}_2^{(m)} \otimes E_t P_{U_2} \tilde{U}_2^{(m)})}{T - h} \right\|_S
\]

\[
+ \left\| \sum_{t=h+1}^{T} \frac{\text{mat}_1(E_{t-h} P_{U_2^\perp} \tilde{U}_2^{(m)} \otimes E_t P_{U_2^\perp} \tilde{U}_2^{(m)})}{T - h} \right\|_S + \left\| \sum_{t=h+1}^{T} \frac{\text{mat}_1(E_{t-h} P_{U_2^\perp} \tilde{U}_2^{(m)} \otimes E_t P_{U_2^\perp} \tilde{U}_2^{(m)})}{T - h} \right\|_S
\]

\[
\leq \|\Delta_3(M_2)\|_S + 3L^{(m)}\|\Delta_3\|_{1,s,s},
\]

where \( \|\Delta_j\|_{1,s,s} \) are defined as

\[
\|\Delta_j\|_{1,s,s} = \|\Delta_{j,1,h}\|_{1,s,s} = \max_{\|\tilde{U}_2\|_S \leq s, \text{rank}(\tilde{U}_2) = r_2, \|\tilde{U}_2\|_S \leq 1, \text{rank}(\tilde{U}_2) = r_2} \|\Delta_{j,1,h}(\tilde{M}_2)\|_S.
\]

Then, the numerator on the right-hand of (60) can be bounded by

\[
\|\text{mat}_1(\text{TOPUP}_1)(\tilde{U}_2^{(m)}) - \mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](\tilde{U}_2^{(m)})\|_S \leq \|\text{mat}_1(\text{TOPUP}_1)(U_2) - \mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](U_2)\|_S + L^{(m)}(K^2 - 1) \sum_{j=1}^{3} h_0^{1/2} \max_{h \leq h_0} \|\Delta_{j,1,h}\|_{1,s,s}.
\] (61)

For general \( 1 \leq k \leq K \), \( \Delta_{j,k,h} \), applying \( P_{\odot j \neq k} U_j \) gives the general version of (61) with

\[
\|\Delta_{j,k,h}\|_{k,s,s} = \max_{|M_j|_S \leq s} \|\Delta_{j,k,h}(\odot j \neq k \tilde{M}_j)\|_S.
\]
We claim that in certain events $\Omega_j, j = 1, 2, 3$, with $\overline{P}(\Omega_j) \geq 1 - 5^{-1} \sum_{k=1}^{K} e^{-d_k}$,

$$\|\Delta_{j,k,h}\|_{k,S,S} \leq \rho \lambda_k^2 / (12 (K^2 - 1)), \quad \forall 1 \leq k \leq K. \quad (62)$$

For simplicity, we will only prove this inequality for $k = 1$ and $K = 2$ in the form of

$$\overline{P} \left\{ \|\Delta_j\|_{1,S,S} = \max_{\|\overline{M}_2\| \leq 1} \|\Delta_{j,1,h}(\overline{M}_2)\|_S \geq \rho \lambda_1^2 / 36 \right\} \leq 5^{-1} e^{-d_2}. \quad (63)$$

Define the ideal version of the ratio in (60) for general $1 \leq k \leq K$ as

$$L_k^{(\text{ideal})} = \frac{2\|\text{mat}_1(\text{TOPUP}_k)(x_{j\neq k} U_j) - \mathbb{E}[\text{mat}_1(\text{TOPUP}_k)](x_{j\neq k} U_j)\|_S}{\sigma_{1,k}(\mathbb{E}[\text{mat}_1(\text{TOPUP}_k)](x_{j\neq k} U_j))}, \quad L^{(\text{ideal})} = \max_{1 \leq k \leq K} L_k^{(\text{ideal})}. \quad (64)$$

The proof of (58) also implies

$$L^{(\text{ideal})} \leq C_1^{(\text{TOPUP})} R^{(\text{ideal})} \quad \text{with} \quad R^{(\text{ideal})} = \max_{1 \leq k \leq K} R_k^{(\text{ideal})} \quad (64)$$

in an event $\Omega_4$ with $\overline{P}(\Omega_4) \leq 5^{-1} \sum_{k=1}^{K} e^{-d_k}$, where $R_k^{(\text{ideal})}$ is as in (21).

We aim to prove that in the event $\cap_{j=0}^4 \Omega_j$

$$L_k^{(m+1)} \leq 2 L_k^{(\text{ideal})} + 12(K^2 - 1) L_k^{(m)} \max_{j,k,h} \|\Delta_{j,k,h}\|_{k,S,S} / \lambda_k^2 \leq 2 L_k^{(\text{ideal})} + \rho L_k^{(m)}, \quad (65)$$

which would imply by induction

$$L_k^{(m+1)} \leq 2(1 + \ldots + \rho^m) L_k^{(\text{ideal})} + \rho^{m+1} L_k^{(0)}, \quad (66)$$

and then the conclusions would follow from (58) and (64). We notice that by (43), (60), (61) and (62), (65) holds provided that $\sigma_{r_k}(\mathbb{E}[\text{mat}_1(\text{TOPUP}_k)](U_j^{(m)}, j \neq k))$, the numerator in (60), is no smaller than a half of its ideal version as in (18), e.g.

$$2\sigma_{r_1}(\mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](U_2^{(m)})) \geq \sigma_{r_1}(\mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](U_2)) = h_0^{1/2} \lambda_1^2, \quad (67)$$

in the case of $k = 1$ and $K = 2$. Again, we divide the rest of the proof into 4 steps to prove (63) for $j = 1, 2, 3$ and (67).

**Step 1.** We prove (63) for the $\Delta_1(\overline{M}_2)$. By Lemma 1 (ii), there exist $U_2^{(\ell)}, U_2^{(\ell')} \in \mathbb{R}^{d_2 \times r_2}$, $1 \leq \ell, \ell' \leq N_{d_2 r_2, 1/8} = 17^{d_2 r_2}$, such that $\|U_2^{(\ell)}\|_S \leq 1$, $\|U_2^{(\ell')}\|_S \leq 1$ and

$$\|\Delta_1\|_{1,S,S} = \|\Delta_{1,1,h}\|_{1,S,S} \leq 2 \max_{\ell,\ell' \leq N_{d_2 r_2, 1/8}} \left\| \sum_{t=h+1}^{T} \frac{\text{mat}_1(A_1 F_{t-h} A_2^T U_2^{(\ell)} \otimes E_t U_2^{(\ell')})}{T-h} \right\|_S.$$
We apply the Gaussian concentration inequality to the right-hand side above. Elementary calculation shows that
\[
\begin{align*}
\left\| \sum_{t=h+1}^{T} \text{mat}_1(A_1 F_{t-h} A_2 U_2^{(f)} \otimes E_t U_2^{(e)}) - \sum_{t=h+1}^{T} \text{mat}_1(A_1 F_{t-h} A_2 U_2^{(f)} \otimes E_t^* U_2^{(e)}) \right\|_S \\
\leq \sum_{t=h+1}^{T} \text{mat}_1(A_1 F_{t-h} A_2 U_2^{(f)} \otimes (E_t - E_t^*) U_2^{(e)}) \\
\leq \left( \text{mat}_1(A_1 F_{1} A_2 \otimes I_{d_1}), \ldots, \text{mat}_1(A_1 F_{T-h} A_2 \otimes I_{d_1}) \right) \\
\leq \sqrt{T} \| \Theta_{1,0}^* \|_S^{1/2} \left\{ U_2^{(f)} \|_S U_2^{(e)} \right\}_S \left( E_{h+1} - E_{h+1}^* \right) \left( E_T - E_T^* \right) \right}_F.
\end{align*}
\]
That is, \( \left\| \sum_{t=h+1}^{T} \text{mat}_1(A_1 F_{t-h} A_2 U_2^{(f)} \otimes E_t U_2^{(e)}) \right\|_S \) is a \( \sigma \sqrt{T} \| \Theta_{1,0}^* \|_S^{1/2} \) Lipschitz function in \( (E_1, \ldots, E_T) \).

Employing similar arguments in the proof of Theorem 1 in Chen et al. (2019b), we have
\[
\mathbb{E} \left\| \sum_{t=h+1}^{T} \text{mat}_1(A_1 F_{t-h} A_2 U_2^{(f)} \otimes E_t U_2^{(e)}) \right\|_S \leq \frac{\sigma(2T)^{1/2}(\sqrt{d_1} + \sqrt{d_1 r_2^2})}{T-h} \| \Theta_{1,0}^* \|_S^{1/2} \cdot \Omega_{1,0}^{1/2}. \]

Then, by Gaussian concentration inequalities for Lipschitz functions,
\[
P \left( \left\| \sum_{t=h+1}^{T} \text{mat}_1(A_1 F_{t-h} A_2 U_2^{(f)} \otimes E_t U_2^{(e)}) \right\|_S \geq \frac{\sigma(2T)^{1/2}(\sqrt{d_1} + \sqrt{d_1 r_2^2})}{T-h} \| \Theta_{1,0}^* \|_S^{1/2} \cdot \Omega_{1,0}^{1/2} \right) \leq 2 e^{-\frac{x^2}{2}}.
\]

Hence,
\[
P \left( \| \Delta_1 \|_{1,S,S/2} \leq \frac{\sigma(2T)^{1/2}(\sqrt{d_1} + \sqrt{d_1 r_2^2})}{T-h} \| \Theta_{1,0}^* \|_S^{1/2} \cdot \Omega_{1,0}^{1/2} \right) \leq 2 N_d^2 e^{-\frac{2}{2}}.
\]

As \( T \geq 4 h_0 \), this implies with \( x \approx \sqrt{d_2 r_2} \) that in an event with at least probability \( 1 - e^{-d_2/5} \),
\[
\| \Delta_1 \|_{1,S,S} \leq \frac{C_{1,K}^{(iter)} \sigma T^{-1/2} \| \Theta_{1,0}^* \|_S^{1/2} (\sqrt{d_1 r_2} + \sqrt{d_2 r_2})}{12 (K^2 - 1)} \leq \frac{C_{1,K}^{(iter)} (\lambda_1^2 R_{1}^{(ideal)}) + \sigma T^{-1/2} \| \Theta_{1,0}^* \|_S^{1/2} \sqrt{d_2 r_2}}{36},
\]
with the \( R_k^{(ideal)} \) in (21) and a constant \( C_{1,K}^{(iter)} \) depending on \( K \) only. In this event, (23) gives
\[
36(\lambda_k)^{-2} \| \Delta_1 \|_{1,h/4,1} \leq C_{1,K}^{(iter)} / C_0^{(iter)} \leq \rho. \quad \text{Thus, (63) holds for } \Delta_1^{*} (\tilde{M}_2).
\]

**Step 2.** Note that
\[
\| \Delta_2 \|_{1,S,S} = \max_{U_2 \in \mathbb{R}^{d_2 \times r_2}} \sum_{t=h+1}^{T} \text{mat}_1(E_{t-h} U_2 \otimes U_2^T A_1 F_{t-h} U_2^T) \leq \frac{36(\lambda_k)^{-2} \| \Delta_1 \|_{1,h/4,1}}{36},
\]

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Then, inequality (63) for $\Delta_2(\tilde{M}_2)$ follow from the same argument as the above step.

**Step 3.** Now we prove (63) for the $\Delta_3(\tilde{M}_2)$. We split the sum into two terms over the index sets, $\mathcal{S}_1 = \{(h, 2h) \cup (3h, 4h) \cup \cdots \} \cap (h, T]$ and its complement $\mathcal{S}_2$ in $(h, T]$, so that $\{E_{t-h}, t \in \mathcal{S}_2\}$ is independent of $\{E_{t-h}, t \in \mathcal{S}_1\}$ for each $a = 1, 2$. Let $n_a = |\mathcal{S}_a|$

By Lemma 1 (ii), we can find $U_2^{(\ell)}, U_2^{(\ell')} \in \mathbb{R}^{d_2 \times r_2}$, $1 \leq \ell, \ell' \leq N_{d_2 r_2, 1/8}$ such that $\|U_2^{(\ell)}\|_S \leq 1$, $\|U_2^{(\ell')}\|_S \leq 1$. In this case,

$$\|\Delta_3\|_{1,S,S} = \|\Delta_{3,1,h}\|_{1,S,S} \leq 2 \max_{1 \leq \ell, \ell' \leq N_{d_2 r_2, 1/8}} \left\| \frac{\sum_{t=h+1}^T \text{mat}_1(E_{t-h}U_2^{(\ell)} \otimes E_{t}U_2^{(\ell')})}{T-h} \right\|_S. \tag{68}$$

Define $G_a = (E_{t-h}U_2^{(\ell)}, t \in \mathcal{S}_a)$ and $H_a = (E_{t}U_2^{(\ell)}, t \in \mathcal{S}_a)$. Then, $G_a, H_a$ are two independent Gaussian matrices. By Lemma 2(ii), for any $x > 0$,

$$\mathbb{P}\left( \left\| \sum_{t \in \mathcal{S}_a} \text{mat}_1(E_{t-h}U_2^{(\ell)} \otimes E_{t}U_2^{(\ell')}) \right\|_S \geq d_1 \sqrt{2} + 2r_2 \sqrt{d_1 n_a} + x^2 + \sqrt{n}x + 2 \sqrt{d_1 r_2x} \right) \leq 2e^{-x^2/2}.
$$

As in the derivation of $\|\Delta_3^*\|_{1,S,S}$ in the proof of Theorem 2, we have, with $x = \sqrt{d_2 r_2}$ and some constant $C_{1,K}^{(iter)}$ depending on $K$ only,

$$\mathbb{P}\left( \|\Delta_{3,1,h}\|_{1,S,S} \geq \frac{C_{1,K}^{(iter)} \sigma^2}{36} \left( \frac{r_2 \sqrt{d_1} + \sqrt{d_2 r_2}}{T^{1/2}} + \frac{d_1 \sqrt{r_2} + d_2 r_2 + r_2 \sqrt{d_1 d_2}}{T} \right) \right) \leq e^{-d_2/5}.
$$

This yields (63) for $\Delta_3(\tilde{M}_2)$ as in the end of Step 1 for $\Delta_1(\tilde{M}_2)$.

**Step 4.** Next, we consider the $r_1$-th largest singular value of $\sigma_{r_1}(\mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](\tilde{U}_2^{(m)}))$ in the event $\cap_{j=0}^4 \Omega_j$. By definition, the left singular subspace of $\mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](\tilde{U}_2^{(m)})$ is $U_1$. Then,

$$\sigma_{r_1}(\mathbb{E}[\text{mat}_1(\text{TOPUP}_1)](\tilde{U}_2^{(m)})) = \sigma_{r_1} \left( \text{mat}_1 \left( \sum_{t=h+1}^T \frac{A_1 F_{t-h}A_2^T \tilde{U}_2^{(m)} \otimes A_1 F_{t}A_2^T \tilde{U}_2^{(m)}}{T-h}, h = 1, ..., h_0 \right) \right)$$

$$= \sigma_{r_1} \left( \text{mat}_1 \left( \sum_{t=h+1}^T \frac{A_1 F_{t-h}A_2 A_2^T U_2 \tilde{U}_2^{(m)} \otimes A_1 F_{t}A_2 A_2^T U_2 \tilde{U}_2^{(m)}}{T-h}, h = 1, ..., h_0 \right) \right)$$

$$= \sigma_{r_1} \left( \text{mat}_1 \left( \sum_{t=h+1}^T \frac{A_1 F_{t-h}A_2 A_2^T U_2 \otimes A_1 F_{t}A_2 A_2^T U_2}{T-h}, h = 1, ..., h_0 \right) \right) \cdot \sigma_{\min} \left( U_2^T \tilde{U}_2^{(m)} \otimes I_{d_1} \otimes \tilde{U}_2^T \tilde{U}_2^{(m)} \otimes I_{h_0} \right)$$

$$\geq \sigma_{r_1} \left( \text{mat}_1 \left( \sum_{t=h+1}^T \frac{A_1 F_{t-h}A_2 A_2^T U_2 \otimes A_1 F_{t}A_2 A_2^T U_2}{T-h}, h = 1, ..., h_0 \right) \right) \cdot \sigma_{\min} \left( U_2^T \tilde{U}_2^{(m)} \right) \cdot \sigma_{\min} \left( \tilde{U}_2^{(m)} \right)$$

$$\geq \sigma_{r_1} \left( \text{mat}_1 \left( \sum_{t=h+1}^T \frac{A_1 F_{t-h}A_2 A_2^T U_2 \otimes A_1 F_{t}A_2 A_2^T U_2}{T-h}, h = 1, ..., h_0 \right) \right) \cdot \|\Delta_{3,1,h}\|_{1,S,S} \cdot \sigma_{\min}(\tilde{U}_2^{(m)})$$

$$\geq \sqrt{h_0} \lambda_1^2 (1 - L^{(m)}).$$
The last step follows from the definitions in (4) and (43). If $L^{(m)} \leq 1/2$, then
\[
\sigma_{r_1}(\mathbb{E}[\text{mat}_1(\text{TUPUP}_1)](\hat{U}_2^{(m)})) \geq \sqrt{\nu_0} \lambda_0^2/2.
\]
By (18), $\lambda_{h_0}^{2/2} = \sigma_{r_1}(\text{mat}_1(\Theta_{1,1:h_0})) = \sigma_{r_1}(\text{mat}_1(\mathbb{E}[\text{TUPUP}_1](U_2)))$. We prove this condition in the event $\cap_{j=0}^{4} \text{O}_j$. By (22) and (58), $L(0) \leq C^{\text{TUPUP}}_1 R^{(0)} \leq C^{\text{TUPUP}}_1/C_0 \leq 1/2$. By induction, given $L^{(m)} \leq 1/2$, (67) holds for the same $m$. Applying (58), (64) and (66),
\[
L^{(m+1)} \leq C^{\text{TUPUP}}_1 \{2(1 + \ldots + \rho^m)R^{(\text{ideal})} + \rho^{m+1}R^{(0)}\} \leq C^{\text{TUPUP}}_1 2(1-\rho)^{-1}R^{(0)} \\
\leq C^{\text{TUPUP}}_1 2(1-\rho)^{-1}/C_0 \leq 1/2.
\]
This completes the induction and the proof of the entire theorem.

6.3 Proofs of other Corollaries

Proposition 1. Let $\lambda = \prod_{k=1}^K \|A_k\|_S$. Assume that the condition numbers of $A_k^T A_k$ ($k = 1, \ldots, K$) are bounded. Then, for all $1 \leq k \leq K$, we have,
\[
\|\Theta_{k,0}\|_{\text{op}} = \lambda^2 \|\Phi_{k,0}\|_{\text{op}}, \quad \|\Theta_{k,0}^*\|_{S} = \lambda^2 \|\Phi_{k,0}^*\|_{S},
\]
\[
\tau_{k,r_k} = \lambda^2 \times \sigma_{r_k} (\text{mat}_1(\Phi_{k,1:h_0})),
\]
\[
\tau_{k,r_k}^* = \lambda^2 \times \sigma_{r_k} \left(\Phi_{k,1:h_0}/\lambda^2\right).
\]
Proof. If the condition numbers of $A_k^T A_k$ ($k = 1, \ldots, K$) are bounded, all the singular values of $A_k$ are at the same order. Then Proposition 1 immediately follows. □

Proofs of Corollary 1 and 3. Employing Proposition 1, under Assumption 2 and $\mathbb{E}[\text{mat}_1(\Phi_{k,1:h_0})]$ is of rank $r_k$, we can show $\lambda_k = \lambda$. When the ranks $r_k$ are fixed, condition (23) can be written as $C^{(\text{iter})}_0 R^{(\text{ideal})} \leq 1$. Thus, for $C_0 = 1/R^{(0)} = 6C^{\text{TUPUP}}_1$ and $C^{(\text{iter})}_0 = 1/R^{(\text{ideal})} = C^{(\text{iter})}_1/\rho$, we have
\[
\rho = C^{(\text{iter})}_1/C^{(\text{iter})}_0 = C_0/C^{(\text{iter})}_0.
\]
For $m = 1$, this gives the rate $R^{(\text{ideal})}$ by (24). Then Corollary 1 follows from the results of Theorem 1.

Similarly, Applying Proposition 1, under Assumption 2 and $\mathbb{E}[\Phi_{k,1:h_0}^*/\lambda^2]$ is of rank $r_k$, we can obtain $\lambda_k^* = \lambda$. Then Corollary 3 follows from the results of Theorem 2. □

Proofs of Corollary 2 and 4. Applying Assumption 3 to Theorem 1 (resp. Theorem 2), then Corollary 2 (resp. Corollary 4) follows. □

6.4 Technical Lemmas

Lemma 1. Let $d, d_j, d_s, r \leq d \wedge d_j$ be positive integers, $\epsilon > 0$ and $N_{d,\epsilon} = \lfloor (1 + 2/\epsilon)^d \rfloor$.

(i) For any norm $\| \cdot \|$ in $\mathbb{R}^d$, there exist $M_j \in \mathbb{R}^d$ with $\|M_j\| \leq 1$, $j = 1, \ldots, N_{d,\epsilon}$, such that
\[
\max_{\|M\| \leq 1} \min_{1 \leq j \leq N_{d,\epsilon}} \|M - M_j\| \leq \epsilon. 
\]
Consequently, for any linear mapping $f$ and norm $\| \cdot \|_s$,
\[
\sup_{M \in \mathbb{R}^d, \|M\| \leq 1} \|f(M)\|_s \leq 2 \max_{1 \leq j \leq N_{d,1/2}} \|f(M_j)\|_s.
\]
(ii) Given \( \epsilon > 0 \), there exist \( U_j \in \mathbb{R}^{d \times r} \) and \( V_j' \in \mathbb{R}^{d' \times r} \) with \( \|U_j\|_S \vee \|V_j'\|_S \leq 1 \) such that

\[
\max_{M \in \mathbb{R}^{d \times d'}, \|M\|_S \leq 1, \text{rank}(M) \leq r} \min_{j \in N_{d,r,1/2}, j' \in N_{d',r,1/2}} \|M - U_j V_j'^T\|_S \leq \epsilon.
\]

Consequently, for any linear mapping \( f \) and norm \( \|\cdot\|_* \) in the range of \( f \),

\[
\sup_{M, \tilde{M} \in \mathbb{R}^{d \times d'}, \|M - \tilde{M}\|_S \leq 1} \frac{\|f(M - \tilde{M})\|_*}{\epsilon 2^{r+d \cdot d'}} \leq \sup_{\|M\|_S \leq 1} \|f(M)\|_* \leq 2 \max_{1 \leq j \leq N_{d,r,1/8}} \|f(U_j V_j'^T)\|_*.
\]

(iii) Given \( \epsilon > 0 \), there exist \( U_{j,k} \in \mathbb{R}^{d_k \times r_k} \) and \( V_{j,k}' \in \mathbb{R}^{d'_k \times r_k} \) with \( \|U_{j,k}\|_S \vee \|V_{j,k}'\|_S \leq 1 \) such that

\[
\max_{M_k \in \mathbb{R}^{d_k \times d'_k}, \|M_k\|_S \leq 1} \min_{\|v\|_S \leq 1, \forall k \in K} \|\bigcirc_{k=2}^K M_k - \bigcirc_{k=2}^K (U_{j,k} V_{j,k}'^T)\|_{\text{op}} \leq \epsilon(K - 1).
\]

For any linear mapping \( f \) and norm \( \|\cdot\|_* \) in the range of \( f \),

\[
\sup_{M_k, \tilde{M}_k \in \mathbb{R}^{d_k \times d'_k}, \|M_k - \tilde{M}_k\|_S \leq 1} \|\bigcirc_{k=2}^K f(M_k - \tilde{M}_k)\|_* \leq \sup_{\|M_k\|_S \leq 1} \|\bigcirc_{k=2}^K f(M_k)\|_* \leq \epsilon(2K - 2)
\]

and

\[
\sup_{M_k \in \mathbb{R}^{d_k \times d'_k}, \|M_k\|_S \leq 1} \|\bigcirc_{k=2}^K f(M_k)\|_* \leq 2 \max_{1 \leq j \leq N_{d_k r_k,1/(8K-8)}} \|\bigcirc_{k=2}^K f(U_{j,k} V_{j,k}'^T)\|_*.
\]

Proof. (i) The covering number \( N_\epsilon \) follows from the standard volume comparison argument as the \((1 + \epsilon/2)\)-ball under \( \|\cdot\| \) and centered at the origin contains no more than \((1 + 2/\epsilon)^d\) balls centered at \( M \). The inequality follows from the “subtraction argument”.

\[
\sup_{\|M\|_S \leq 1} \|f(M)\|_* - \max_{1 \leq j \leq N_{d,1/2}} \|f(M_j)\|_* \leq \sup_{\|M - M_j\|_{L^2} \leq 1/2} \|f(M - M_j)\|_* \leq \sup_{\|M\|_S \leq 1} \|f(M)\|_*/2.
\]

(ii) The covering numbers are given by applying (i) to both \( U \) and \( V \) in the decomposition \( M = UV^T \) as Lemma 7 in Zhang and Xia (2018). The first inequality in (69) follows from the fact that for \( r < d \land d' \), \( (M - \tilde{M})/\epsilon \) is a sum of two rank-\( r \) matrices with no greater spectrum norm than 1, and the second inequality of (69) again follows from the subtraction argument although we need to split \( M - U_j V_j'^T \) into two rank \( r \) matrices to result in an extra factor of 2.

(iii) The proof is nearly identical to that of part (ii). The only difference is the factor \( K - 1 \) when \( \|\bigcirc_{k=2}^K M_k - \bigcirc_{k=2}^K \tilde{M}_k\|_{\text{op}} \leq (K - 1) \max_{2 \leq k \leq K} \|M_k - \tilde{M}_k\|_S \) is applied. \(\square\)

Lemma 2. (i) Let \( G \in \mathbb{R}^{d_1 \times n} \) and \( H \in \mathbb{R}^{d_2 \times n} \) be two centered independent Gaussian matrices such that \( \mathbb{E}(u^T \text{vec}(G))^2 \leq \sigma^2 \forall u \in \mathbb{R}^{d_1 n} \) and \( \mathbb{E}(v^T \text{vec}(H))^2 \leq \sigma^2 \forall v \in \mathbb{R}^{d_2 n} \). Then,

\[
\|G H^T\|_S \leq \sigma^2 (\sqrt{d_1 d_2} + \sqrt{d_1 n} + \sqrt{d_2 n}) + \sigma^2 x (x + 2 \sqrt{n} + \sqrt{d_1} + \sqrt{d_2}).
\]
with at least probability $1 - 2e^{-x^2/2}$ for all $x \geq 0$.

(ii) Let $G_i \in \mathbb{R}^{d_1 \times d_2}$, $H_i \in \mathbb{R}^{d_3 \times d_4}$, $i = 1, \ldots, n$, be independent centered Gaussian matrices such that $\mathbb{E}(u^\top \text{vec}(G_i))^2 \leq \sigma^2 \ \forall \ u \in \mathbb{R}^{d_1 d_2}$ and $\mathbb{E}(v^\top \text{vec}(H_i))^2 \leq \sigma^2 \ \forall \ v \in \mathbb{R}^{d_3 d_4}$. Then,

$$
\left\| \text{mat}_1 \left( \sum_{i=1}^n G_i \otimes H_i \right) \right\|_S \leq \sigma^2 (\sqrt{d_1 n} + \sqrt{d_1 d_3 d_4} + \sqrt{nd_2 d_3 d_4}) + \sigma^2 x (x + \sqrt{n} + \sqrt{d_1} + \sqrt{d_2} + \sqrt{d_3 d_4})
$$

with at least probability $1 - 2e^{-x^2/2}$ for all $x \geq 0$.

**Proof.** Assume $\sigma = 1$ without loss of generality. Let $x \geq 0$.

(i) Independent of $G$ and $H$, let $\zeta_j \in \mathbb{R}^n$, $j = 1, 2$, be independent standard Gaussian vectors. As in Chen et al. (2019b), the Sudakov-Fernique inequality provides

$$
\mathbb{E}\left[ \|GH^\top\|_S|G\right] \leq \mathbb{E}\left[ \max_{|u|_2 = 1} u^\top G \zeta_2 |G\right] + \|G\|_S \sqrt{d_2}.
$$

Thus, by the Gaussian concentration inequality

$$
\mathbb{P}\left\{ \|GH^\top\|_S \geq \mathbb{E}\left[ \max_{|u|_2 = 1} u^\top G \zeta_2 |G\right] + \|G\|_S (\sqrt{d_2} + x) |G\right\} \leq e^{-x^2/2}.
$$

Applying the Sudakov-Fernique inequality again, we have

$$
\mathbb{E}\left[ \max_{|u|_2 = 1} u^\top G \zeta_2 |G\right] + \|G\|_S (\sqrt{d_2} + x) \leq \sqrt{d_1 n} + (\sqrt{d_1} + \sqrt{n}) (\sqrt{d_2} + x).
$$

Moreover, as the Lipschitz norm of $\mathbb{E}\left[ \max_{|u|_2 = 1} u^\top G \zeta_2 |G\right] + \|G\|_S (\sqrt{d_2} + x)$ is bounded by $\sqrt{n} + \sqrt{d_2} + x$, by the Gaussian concentration inequality

$$
\mathbb{E}\left[ \max_{|u|_2 = 1} u^\top G \zeta_2 |G\right] + \|G\|_S (\sqrt{d_2} + x) \leq \sqrt{d_1 n} + (\sqrt{d_1} + \sqrt{n}) (\sqrt{d_2} + x) + x(\sqrt{n} + \sqrt{d_2} + x)
$$

holds with at least probability $1 - e^{-x^2/2}$.

(ii) We treat $G = (G_1, \ldots, G_n) \in \mathbb{R}^{d_1 \times d_2 \times n}$ and $H = (H_1, \ldots, H_n) \in \mathbb{R}^{d_3 \times d_4 \times n}$ as tensors. Let $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{d_2 \times n}$ be a standard Gaussian matrix independent of $H$. For $u \in \mathbb{R}^{d_1}$ and $V \in \mathbb{R}^{d_2 \times (d_3 d_4)}$,

$$
\mathbb{E}\left[ \left\| \text{mat}_1 \left( \sum_{i=1}^n G_i \otimes H_i \right) \right\|_S \right| H \right|
$$

$$
= \mathbb{E}\left[ \sup_{|u|_2 = 1 \setminus |V|_F = 1} u^\top \text{mat}_1(G) \text{vec}(\text{mat}_3(H)V^\top) \left| H \right| \right]
$$

$$
\leq \sqrt{d_1} \sup_{|V|_F = 1} \|\text{mat}_3(H)V^\top\|_F + \mathbb{E}\left[ \sup_{|V|_F = 1} (\text{vec}(\xi))^\top \text{vec}(\text{mat}_3(H)V^\top) \left| H \right| \right]
$$

$$
= \sqrt{d_1} \|\text{mat}_3(H)\|_S + \mathbb{E}\left[ \left( \sum_{j=1}^{d_2} \sum_{k=1}^{d_3 d_4} \left( \sum_{i=1}^n \xi_{i,j} \text{vec}(H_i)_k \right)^2 \right)^{1/2} \right| H \right]
$$
\[ \leq \sqrt{d_1} \| \text{mat}_3(H) \|_S + \sqrt{d_2} \| \text{vec}(H) \|_2 \]

By the Gaussian concentration inequality,

\[ \mathbb{P}\left\{ \| \text{mat}_1 \left( \sum_{i=1}^{n} G_i \otimes H_i \right) \|_S \geq (\sqrt{d_1} + x) \| \text{mat}_3(H) \|_S + \sqrt{d_2} \| \text{vec}(H) \|_2 \right\} H \leq e^{-x^2/2}. \]

Moreover, as \( \mathbb{E}[ (\sqrt{d_1} + x) \| \text{mat}_3(H) \|_S + \sqrt{d_2} \| \text{vec}(H) \|_2] \leq (\sqrt{d_1} + x)(\sqrt{n} + \sqrt{d_3d_4}) + \sqrt{d_2nd_3d_4} \) and the Lipschitz norm of \( (\sqrt{d_1} + x) \| \text{mat}_3(H) \|_S + \sqrt{d_2} \| \text{vec}(H) \|_2 \) is bounded by \( \sqrt{d_1} + x + \sqrt{d_2} \),

\[ (\sqrt{d_1} + x) \| \text{mat}_3(H) \|_S + \sqrt{d_2} \| \text{vec}(H) \|_2 \leq (\sqrt{d_1} + x)(\sqrt{n} + \sqrt{d_3d_4}) + \sqrt{nd_2d_3d_4} + x(\sqrt{d_1} + x + \sqrt{d_2}) \]

holds with at least probability \( 1 - e^{-x^2/2} \). \( \square \)