The Properties of Cosmic Velocity Fields

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ABSTRACT
Understanding the velocity field is very important for modern cosmology: it gives insights to structure formation in general, and also its properties are crucial ingredients in modelling redshift-space distortions and in interpreting measurements of the kinetic Sunyaev-Zeldovich effect. Unfortunately, characterising the velocity field in cosmological \textit{N}-body simulations is inherently complicated by two facts: i) The velocity field becomes manifestly multi-valued after shell-crossing and has discontinuities at caustics. This is due to the collisionless nature of dark matter. ii) \textit{N}-body simulations sample the velocity field only at a set of discrete locations, with poor resolution in low-density regions. In this paper, we discuss how the associated problems can be circumvented by using a phase-space interpolation technique. This method provides extremely accurate estimates of the cosmic velocity fields and its derivatives, which can be properly defined without the need of the arbitrary “coarse-graining” procedure commonly used. We explore in detail the configuration-space properties of the cosmic velocity field on very large scales and in the highly nonlinear regime. In particular, we characterise the divergence and curl of the velocity field, present their one-point statistics, analyse the Fourier-space properties and provide fitting formulae for the velocity divergence bias relative to the non-linear matter power spectrum. We furthermore contrast some of the interesting differences in the velocity fields of warm and cold dark matter models. We anticipate that the high-precision measurements carried out here will help to understand in detail the dynamics of dark matter and the structures it forms.

Key words: cosmology: theory, dark matter, large-scale structure of Universe – galaxies: formation – methods: \textit{N}-body, numerical

1 INTRODUCTION
\textit{N}-body simulations in computational cosmology (e.g. Melott 1982; Efstathiou et al. 1985; Peebles et al. 1989; Springel 2005; Angulo et al. 2012) are an invaluable tool to study the flow of cosmic dark matter over time. Statistics of the resulting velocity field are important for cosmological measurements of redshift-space distortions from galaxy surveys (see e.g. Dekel 1994, for a classic review), of the kinetic Sunyaev-Zeldovich effect, as well as for more theoretically motivated questions regarding the formation and dynamics of the cosmic large-scale structure (e.g. Bertschinger & Dekel 1989; Bernardade & van de Weygaert 1996; Pichon & Bernardade 1999; Pueblas & Scoccimarro 2009; Kitaura et al. 2012).

Unfortunately, there is a fundamental problem in the determination of the dark matter velocity field from \textit{N}-body simulations: while the velocity field is defined everywhere in space, in simulations it is sampled only at the (mass-weighted) particles position. A recent discussion by Jennings et al. (2011) highlights the potentially large uncertainties in deciding how to measure the velocity power spectrum from cosmological \textit{N}-body simulations. Using a standard cloud-in-cell (CIC Hockney & Eastwood 1981) deposit to grid up simulation results leads to unacceptable levels of noise and biases to reliably measure this quantity.

There are various approaches in the literature to reconstruct a continuous velocity field and deal with this sampling issue. One is to perform a large-scale smoothing with a given fixed or adaptive kernel (e.g. Bertschinger & Dekel 1989; Melott & Shandarin 1993), which has advantages when applied to galaxy catalogs and the interest is mostly limited to the linear regime. Another approach is based on using inverse SPH-smoothed velocity fields (Colombi et al. 2007). A third class of approaches employs tessellations of the spatial distribution of sampling points. The latter offers a chance to define unique volumes around particles (by Voronoi tessellations) or by connecting the particles (Delaunay triangulations: e.g. Icke & van de Weygaert 1987; Bernardade & van de Weygaert 1996; Pueblas & Scocci-
marro 2009), and they have been particularly popular in recent analyses of N-body simulations (e.g. Pandey et al. 2013). As shown by Bernardarde & van de Weygaert (1996) and Pueblas & Scoccimarro (2009), Delaunay tessellations of the particle distribution can be used to better control noise and measurement errors of velocity power spectra from N-body simulations. In addition, Jennings (2012) showed that the DTFE estimator (Schaap & van de Weygaert 2000; Pelupessy et al. 2003; Schaap 2007), as implemented by Cautun & van de Weygaert (2011), gives much more reliable spectral properties compared to fixed kernel smoothings (confirming a similar finding by Pueblas & Scoccimarro 2009).

Despite the progress, while trying to solve the two issues described above, the methods sacrificed a direct measurement of the velocity field by that of a smoothed field and/or introduced high levels of noise. This is particularly important: due to the collisionless nature of dark matter, gravitational collapse leads to a multi-valued velocity field in multi-stream regions (quite in contrast to the behaviour of an ideal fluid), and discontinuities in the field appear. Therefore, as we will discuss throughout this paper, the properties of the volume-averaged (i.e. "coarse-grained") velocity field are not identical to the properties of the bulk velocity field. For instance, a non-zero vorticity appears in the coarse-grained velocity field even in regions with particles whose orbits have not yet crossed. Additionally, the properties of the coarse-grained field strongly depend on the (arbitrary) scale on which the volume average is performed.

In this paper, we will show how to accurately compute the velocity field and its differential properties (specifically, the vorticity and divergence of the field) without the problems outlined above. In particular, we will propose a scheme that provides an accurate (and to our knowledge the only) way to determine the bulk velocity field without coarse-graining. This is a direct consequence from the ability to define an explicit projection from phase-space into configuration space based on a reconstruction of the fine-grained distribution function. Therefore, and in contrast to coarse graining, there is no arbitrary scale in this direct projection.

Our scheme is based on the method to analyse cosmological N-body simulations proposed by Abel et al. (2012) (AHK12 hereafter). The key idea rests on the fact that structure formation in cold dark matter starts out on a very thin sheet in configuration space with an almost infinitesimally small extent in velocity space. Consequently, one can think of the modelled particles as the moving (massless) vertices of a tessellation of this initial phase space sheet. Shandarin et al. (2012) independently developed the same idea with only minor differences in the implementation. The estimates of densities, velocities and velocity dispersion therefore incorporate the information of neighbouring particles in phase-space. Such a Lagrangian tessellation is able to accurately represent anisotropic deformations of the density field which allows it to be free from artificial clumping as seen in adaptive kernel smoothing (Hahn et al. 2013; Angulo et al. 2013).

The features described above make this method unique from all the other methods such as adaptive kernel smoothing (as e.g. in SPH Monaghan 1992), CIC deposits or DTFE. We also note that our method is distinct from the phase space estimation techniques of Sharma & Steinmetz (2006) or Ascasibar (2010), which treat N-body data results as Monte Carlo sampled realisations of the micro-physical phase space structure of the dark matter distribution. Consequently all these methods are subject to noise even in the case of the heavily smoothed SPH method (see the detailed discussion in Hahn et al. 2013).

Using our explicit phase-space projection procedure applied to dark matter N-body simulations, we are able to (i) show how the velocity field switches from convergent to divergent flow in multi-stream regions, with a remnant convergent core in the centres of high-density structures such as haloes and filaments; (ii) demonstrate that vorticity is a multi-stream phenomenon, which peaks at caustics; (iii) present the 1-point statistics of the velocity divergence and vorticity, and (iv) provide high-resolution Fourier-space properties of the velocity field, in terms of velocity divergence and vorticity power spectra, and density-divergence cross-spectra. We also provide fitting formulae for the velocity divergence bias relative to the non-linear matter power spectrum. We furthermore contrast some of the interesting differences in the velocity fields of warm and cold dark matter models. All our results are presented in Sections 4 and 5.

The structure of this paper is as follows: We first discuss how bulk velocity fields are defined through a projection operation and give special attention to derivatives of the projected velocity field that are non-trivial to compute due to the discontinuous nature of the field in Section 2. Next, in Section 3, we introduce the N-body simulations used in this work. In Section 4, we present an analysis of the real-space properties of the velocity field. In Section 5, we focus on the spectral properties of velocity fields computed with our method and provide fits for the Fourier space k-dependent bias of the velocity divergence relative to the cosmic density field. We summarize our results in Section 6.

2 PROJECTIONS ONTO CONFIGURATION SPACE, DERIVATIVE OPERATORS AND DIFFERENTIALS OF VECTOR FIELDS

In this section, we discuss how to obtain bulk velocity fields from N-body simulations using the cold dark matter sheet by projecting the fine-grained distribution function from phase space onto configuration space. We then discuss the differential properties of such projected fields. We will argue that these differentials cannot in general be approximated by finite differences and derive the correct expressions for the divergence and the curl of the projected velocity field. Finally, we discuss how a piecewise linear approximation to the fine-grained distribution function based on N-body particles, as introduced in AHK12, can be used to compute the velocity field and its differentials from simulations.

2.1 The Vlasov-Poisson system and the distribution function of cold fluids

We are concerned here with the bulk velocity field of cold dark matter, a cold collisionless self-gravitating fluid. The evolution of such a fluid is fully described by the phase-space distribution function \( f(x,v,t) \) governed by the Vlasov-Poisson system of equations (see e.g. Peebles 1980; HEN 1982, the latter for a historical discussion)

\[
\frac{df(x,v,t)}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f, \tag{1}
\]

\[
\nabla^2_x \phi = \frac{4\pi G}{a^2} \int d^3 v \left( f - \bar{\rho} \right), \tag{2}
\]

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The bulk velocity field is thus given by a density-weighted average of all streams, where
\[ \bar{v} = \frac{\int \rho \mathbf{v} \, d^3v}{\int \rho \, d^3v} = \frac{\sum_{s \in S} \mathbf{v}_s(x) \rho_s(x)}{\sum_{s \in S} \rho_s(x)}, \tag{3} \]
where \( S \) is the set of all streams \( s \) that contain point \( x \), a subscript \( s \) indicates the value of a field on a given sheet, and the second equality holds for cold fluids, as discussed above. The bulk velocity field is thus given by a density-weighted average over the multi-stream velocity field (cf. Fig. 1). We want to remark explicitly that our projection operator \( \langle \cdot \rangle \) does not involve the convolution with a smoothing kernel as is often done. We note also that this projection operator, unlike kernel smoothing, is explicitly idempotent, i.e. \( \langle \cdot \rangle \circ \langle \cdot \rangle = \langle \cdot \rangle \).

We note that vorticity does not appear at the single stream level if the velocity field is not sourced by a vector potential and vorticity is initially zero. In the case of Newtonian gravity, the acceleration is given by the gradient of the gravitational potential \( \phi \). Hence, the single stream motion of a Lagrangian fluid element \((x_0, v_0)\) is curl-free non-perturbatively at all times since at fixed \( x_0 \), \[ \frac{dv_0}{dt} = -\nabla \times \nabla \phi |_{x_0} = 0 \] if it is irrotational initially (cf. also Bernardeau et al. 2002). We note that this does not imply that the mapping between Lagrangian and Eulerian space is also irrotational (cf. also Buchert 1992). The rotational nature of this latter mapping can be easily seen from the volume inversion of Lagrangian elements at a caustic, i.e. \( \partial x_i/\partial q_i \neq \partial x_j/\partial q_j \), which, in turn, rather obviously, leads to the emergence of vorticity in multi-stream regions.

### 2.3 Derivatives of the projected field

We note in this subsection that differentials of a projected variable are singular at the location of caustics where the number of streams changes and the projected field has a discontinuity. This can be easily seen directly from the definition of a derivative applied to a projected field \( \langle g \rangle \):

\[ \frac{d}{dx} \langle g(x) \rangle = \lim_{h \to 0} \frac{\langle g(x + h) \rangle - \langle g(x) \rangle}{h}. \tag{4} \]

The right-hand-side can be written as

\[ \lim_{h \to 0} \frac{1}{h} \sum_{i \in S_1} \sum_{j \in S_2} \rho_s(x + h) \rho_l(x) \left( g_s(x + h) - g_l(x) \right), \tag{5} \]

where we note that \( S_1 \) (containing points \( x + h \)) and \( S_2 \) (containing points \( x \)) can in general be sums over different numbers of streams so that the change in the number of streams has to be taken into account when performing the limit (cf. Fig. 2 and our discussion below); \( g_s \) and \( g_l \) indicate the values of \( g \) on the various intersection points of the sheet with points \( x + h \) and \( x \) respectively. In what follows, we first discuss the properties of derivatives across such discontinuities before we turn to the properties of derivatives away from discontinuities which is almost everywhere (in the mathematical sense) in configuration space.

We wish to remark that this division of space into regions where the derivative is non-singular (i.e. away from caustics) and regions of singular derivatives of measure zero is only possible with the proposed explicit phase-space projection method. Any method that uses (implicit) coarse-graining will necessarily include the singularities integrated over the coarse-graining scale in configuration space (see below) although (without coarse-graining) these regions should not contribute to any volume averages due to their zero volume measure when no coarse-graining is performed.

### Derivatives across discontinuities

As we are concerned here with projections into configuration space of the fine-grained distribution function, discontinuities in the projected fields are real discontinuities of infinitesimal extent. Consequently, the derivative across a discontinuity is a Dirac-\( \delta \) function, i.e. for a velocity field with left-sided limit \( v_l \) and right-sided limit \( v_r \) at location \( x_0 \) – given generically by \( v(x) = (v_l - v_r) \Theta(x - x_0) + v_r \), where \( \Theta \) is the Heaviside \( \Theta \)-function – the derivative near \( x_0 \) is given by

\[ \frac{dv}{dx} = (v_l - v_r) \delta_D(x - x_0). \tag{6} \]

For the three dimensional cold distribution functions we are concerned with here, these singular derivatives of finite measure...
2.4 The case of 1+1 dimensional phase-space

In the case of one spatial and one velocity dimension, the fine-grained distribution function can be approximated by connecting particles that are neighbours in the initial conditions through straight lines. The mass \( m \) of one particle can then be thought of as being distributed uniformly along the line element so that the single-stream density at every particle location is just \( \rho = 2m/l \), where \( l = \Delta x_{i-1} + \Delta x_{i+1} \) is the sum of the distances to the neighbouring particles in Lagrangian space. Single-stream velocities and densities can then be linearly interpolated in between particles. The density and velocity projected into configuration space at an arbitrary location \( x \) are readily calculated from the single-stream values by determining which line-elements intersect \( x \) and then calculate the weighted averages as discussed above.

In Fig. 1, we show the bulk velocity obtained in this way for the late stages of plane-wave collapse (red line). We compare it to the estimate that a Delaunay tessellation approach would give (blue). In this approach, the phase-space connectivity is not respected and only configuration space information is used. Particles that are closest in configuration space are connected by linear elements and velocities are linearly interpolated along these elements. Quite obviously, in multi-stream regions this leads to a noisy and incorrect projected velocity field if no coarse-graining is performed.

We can see from Fig. 1 that the bulk velocity field (red line) has discontinuities at the caustics and, at that particular time, also has a positive slope everywhere except in the very centre. We argued in the previous subsection that these discontinuities (due to a change in the number of streams) lead to errors when finite differences are used and the finite difference interval brackets a caustic. In Fig. 3, we show the phase-space (top panels) together with the velocity divergence (bottom panels) for the collapse of a plane wave before (left panels) and after (right panels) shell crossing. Before shell-crossing, the velocity field is single-valued and has negative velocity divergence in the central region \( 0.4 \lesssim x \lesssim 0.6 \). After shell-crossing, the bulk velocity field is given by the red line (top right panel) with the properties we have just discussed. If the velocity divergence is computed by applying finite differences to the projected field on a grid (blue line, bottom right panel), the singular derivative (see discussion above) at the caustic locations is effectively integrated over a finite volume so that the finite velocity jump divided by the scale is recovered. If eq. (7) is used to compute the velocity divergence (red line, bottom right panel), i.e. by computing single stream finite differences on the phase space line elements followed by projection onto configuration space, the singular derivative, which exists only at a finite number of points need not be included. As a result, we see that the velocity field is expansive almost everywhere with discontinuous jumps at the caustics.

We observe that the divergence calculated directly from the sheet in the described way is somewhat noisy. This is a consequence of the low-order interpolations we are using.
particles span every relevant simplex (tetrahedron) of the tessellation reduces to decomposing one unit cube into six tetrahedra of equal size (the Delaunay triangulation of the unit cube). Consequently, one can then think of the data output from a simulation as the information that describes the evolution of the vertices of the $N_{\text{tet}} = 6N_p$ tetrahedra. Each of them carries $m_{\text{tet}} = M_{\text{box}}/N_{\text{tet}}$ of the mass $M_{\text{box}}$ in the simulation box and represents a piecewise linear interpolation between four vertices on the fine-grained distribution function. In fact, to reduce the possible impact of anisotropies due to the choice of one of the six equivalent Delaunay triangulations of the unit cube, we average over all those six possibilities. Similar methods employing the advection of a tessellation of test particles can be used in simulations of incompressible flows, allowing the calculation of Lagrangian flow properties (e.g. Pumir et al. 2013).

2.5.1 Interpolating and differentiating on tetrahedra

For a tetrahedron with vertices $x_k = (x_k, y_k, z_k)$, $k = 1 \ldots 4$, any point $x = (x, y, z)$ is $x_i$, i.e. $F_k = F(x_k)$, the linear interpolation to an arbitrary point $x$ can then be written as

$$F(x) = \sum_{k=1}^{4} F_k \zeta_k(x).$$

(10)

This implies that differentials of $F$ can be obtained through the chain-rule by simply computing

$$\frac{\partial F}{\partial x_i} = \sum_{k=1}^{4} \frac{\partial F}{\partial \zeta_k} \frac{\partial \zeta_k}{\partial x_i} = \sum_{k=1}^{4} J_{i,k}^{-1} F_k,$$

(11)

where the index $i$ refers to a Cartesian coordinate, and not to vertex $i$. Obviously, a derivative computed in this way is only accurate at first order (like the simple backward/forward finite difference operators). Gradient operators that are accurate at higher order are of course possible and would include information from more than one tetrahedron or involve quadratic tetrahedra (defined using 10 instead of 4 vertices for the linear tetrahedron). We leave this aspect for future work as first order gradients are accurate enough for the purposes of this article. We note however that, for example, a curl-free non-linear field which values of the bulk field (blue) shows a dominant negative velocity divergence at caustics but is slightly less noisy. The negative divergence reflects the finite velocity jump (that is in fact occurring over an infinitesimal volume and is thus not present in the direct projection). The correlation between velocity divergence correlation and the overdensity field changes its sign except in the innermost region and in the single-stream region.

Higher order schemes are expected to improve this but are beyond the scope of this article.

Another interesting observation from Fig. 3 is that, before shell crossing, the velocity divergence is anti-correlated with the overdensity. After shell-crossing (right column), the bulk velocity field (red, top right) is different from the fine-grained multi-stream velocity field (black). The velocity divergence (bottom right) estimated directly on the sheet (following eq. 7) is everywhere positive except in the very centre. In contrast, a finite difference estimate computed from gridded values of the bulk field (blue) shows a dominant negative velocity divergence at caustics but is slightly less noisy. The negative divergence reflects the finite velocity jump (that is in fact occurring over an infinitesimal volume and is thus not present in the direct projection). The correlation between velocity divergence correlation and the overdensity field changes its sign except in the innermost region and in the single-stream region.

2.5 Projections of phase space in N-body simulations and the dark matter sheet

In our discussion above, we have argued that knowledge of the distribution function is necessary to accurately perform its projection onto configuration space and thus determine the bulk velocity field without performing a coarse-graining operation. As we have discussed in AHK12, the fine-grained distribution function of cold dark matter can be reconstructed from $N$-body simulations using the phase-space sheet tessellation method described there. In this method, a tessellation of the Lagrangian particle coordinates $q$ is performed, decomposing the entire particle distribution into a collection of tetrahedral phase space elements.

Since, due to Liouville’s theorem, the tessellation is preserved in phase space, it can be reconstructed at any later time if the connectivity is known. In the case of an initial cubical lattice of particles, this is particularly simple since the three-dimensional Lagrangian coordinate can be encoded in the particle IDs, and the connectivity can be determined on-the-fly. In fact, the entire procedure to know which four

\[ \sum_{k=1}^{4} \frac{\partial F}{\partial \zeta_k} \frac{\partial \zeta_k}{\partial x_i} = \sum_{k=1}^{4} J_{i,k}^{-1} F_k, \]

\[ \frac{\partial F}{\partial x_i} = \sum_{k=1}^{4} \frac{\partial F}{\partial \zeta_k} \frac{\partial \zeta_k}{\partial x_i} = \sum_{k=1}^{4} J_{i,k}^{-1} F_k, \]
volume $V_{\text{tet}}$ defined above. As we have discussed in Section 2.3 above, the derivatives of the projected velocity field contain also derivatives of the fine-grained density. Hence, a piecewise constant fine-grained density will not be sufficient. We therefore compute a density estimate at each vertex from the mean volume of all $N_{\text{tet}}$ tetrahedra that share a given vertex $\mathbf{q}_i$. Specifically, the density at a given vertex $\mathbf{q}_i$ is given by $\rho(\mathbf{q}_i) = m_{\text{tet}} N_{\text{tet}} \left[ \sum_{j=1}^{N_{\text{tet}}} \frac{V_j}{V} \right]^{-1}$, i.e. the reciprocal of the sum over the volumes $V_j$ of all tetrahedra that share the vertex. We note that it is not possible to use the dual Voronoi mesh to determine the projection of a field $g$ to a given vertex $\mathbf{q}_i$.

Having a density estimate defined at each vertex, we can now linearly interpolate easily to any position $\mathbf{x}$ inside a given tetrahedron using eq. (10). This achieves a piecewise linear field with a piecewise constant derivative across the tessellation. Similarly, for each tetrahedron, the fine-grained velocity $\mathbf{v}_{1-4}$ is known at the four vertices $\mathbf{q}_i$, and can be linearly interpolated to position $\mathbf{x}$ using eq. (10), and eq. (11) can be used to compute derivatives on each tetrahedron. In this way, the gradient of the fine-grained (i.e. per stream) density field $\nabla \rho$ as well as the gradient tensor of the fine-grained velocity field $\nabla \otimes \mathbf{v}$ can be easily calculated from the values of the density as well as the velocity at the four vertices of each tetrahedron.

### 2.5.3 Projected fields using the dark matter sheet

To determine the projection of a field $g(\mathbf{q})$, defined on the dark matter sheet, at an arbitrary point $\mathbf{x}$, we first determine all the tetrahedra which contain $\mathbf{x}$. The velocity and density due to each stream is computed by linearly interpolating from the four vertices of each tetrahedron to the point. Having now a linearly interpolated quantity $g(\mathbf{x})$ for tetrahedron $i$, we next average over all the tetrahedra $i$ that contain that point and weight each contribution by the density of the tetrahedron, i.e. make the approximation

$$g(\mathbf{x}) \approx \sum \frac{N_i}{\sum_i N_i} \rho_i(\mathbf{x}) g_i(\mathbf{x}),$$

where $N_i$ is the number of tetrahedra that intersect point $\mathbf{x}$. Note that $g_i$ is only defined inside the corresponding tetrahedron and is equal to zero outside. This is done most efficiently by looping over all tetrahedra and adding their contributions to all the cells in the uniform grid whose cell centres are contained inside the tetrahedron. In fact, we subsample each cell 8 times, i.e. we compute the value for each cell by averaging over the values at 8 points inside the cell to arrive at a value closer to an actual volume average for each cell. We perform this operation for all points of a cubical lattice to obtain a three-dimensional data cube of velocity information.

### 3 SIMULATIONS

We have performed a series of cosmological simulations covering a considerable range in box sizes and mass resolutions to ensure convergence of our results. In all cases, we generated initial conditions using MUSIC (Hahn & Abel 2011) adopting the parametrisation of the transfer function of Eisenstein & Hu (1999) for the cold-dark matter (CDM) simulations as well as a truncated transfer function for the warm-dark matter (WDM) simulations that we detail below. We use cosmological parameters consistent with the WMAP7 data release (Komatsu et al. 2011). Specifically, we use the density parameters $\Omega_m = 0.276$, $\Omega_L = 0.724$ and $\Omega_b = 0.045$, a Hubble parameter of $h = 0.703$, power spectrum normalisation $\sigma_8 = 0.811$ and spectral index $n_s = 0.96$.

The initial conditions are generated by perturbing a regular lattice of particles using the Zel’dovich approximation at $z = 100$. The details on all the simulations that we use in this work are summarised in Table 1. All our analysis is performed at $z = 0$.

#### 3.1 CDM simulations

For the CDM simulations, our four box sizes range between $3 \h^{-1}$Gpc and $100 \h^{-1}$Mpc, the mass resolution is $512^3$ particles in all cases, the $1 \h^{-1}$Gpc box has been run at a higher resolution of $1024^3$ particles. The gravitational evolution between $z = 100$ and $z = 0$ for these simulations has been performed using the tree-PM code L-GADGET 3 (Angulo et al. 2012) using a 1024$^3$ PM mesh whenever for the simulations with $512^3$ particles and a 2048$^3$ mesh for the simulations with $1024^3$ particles. The softening adopted for the tree force is given in Table 1 for all cases.

#### 3.2 WDM simulations

We consider also simulations starting from a perturbation spectrum with small-scale suppression in this work. This has the advantage that, unlike in CDM, the simulations are able to capture the full dynamic range of perturbations with enough resolution. We consider the same simulations as in Hahn et al. (2013), i.e., specifically, we adopt the parametrisation of the WDM transfer function from Bode et al. (2001) to modify the CDM transfer function

$$T_{\text{WDM}}(k) = T_{\text{CDM}}(k) \left[ 1 + (\alpha k)^2 \right]^{-5.0},$$

and

$$\alpha = \frac{0.05}{h^{-1} \text{Mpc}} \left( \frac{\Omega_m}{0.4} \right)^{0.15} \left( \frac{h}{0.65} \right)^{1.3} \left( \frac{m_{\text{dm}}}{1 \text{keV}} \right)^{-1.15},$$

which for the WDM particle mass of 300eV equals a cut-off scale $\alpha = 0.21 \h^{-1}$Mpc. For these simulations, we employed a modified version of GADGET-2 (Springel 2005) that uses only the particle-mesh force, evaluated on a $512^3$ mesh (Note that we employ only the standard $N$-body simulations from Hahn et al. 2013 here). We simulated the gravitational evolution of one $40 \h^{-1}$Mpc box using $128^3$, $256^3$ as well as $512^3$ particles. The cut-off scale $\alpha$ is resolved in all three runs.

| Simulation Name | $L_{\text{box}}$ | $N_p$ | $m_p$ | $\epsilon$ |
|-----------------|-----------------|-------|-------|-----------|
| L3000N512       | 3000            | 512$^3$ | $1.5 \times 10^{13}$ | 200       |
| L1000N512       | 1000            | 512$^3$ | $5.7 \times 10^{11}$ | 65        |
| L300N512        | 300             | 512$^3$ | $1.5 \times 10^{10}$ | 20        |
| L1000N1024      | 100             | 512$^3$ | $5.7 \times 10^{8}$  | 6.5       |
| WDM512          | 40              | 512$^3$ | $3.7 \times 10^{7}$  | 78        |
| WDM256          | 40              | 256$^3$ | $2.9 \times 10^{8}$  | 78        |
| WDM128          | 40              | 128$^3$ | $2.3 \times 10^{9}$  | 78        |

Table 1. Labels and specifics of the simulations used in this work.
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Figure 4. Slice of the cosmic velocity field from the L300N512 simulation obtained with the phase space tessellation method discussed in this work (top) and the DTFE method (bottom). Shown is the $x$-component of the velocity. The insets show enlargements of a $50 \times 50 h^{-1}$Mpc$^2$ region highlighting differences between the two methods in multi-stream regions. Equivalently to Fig. 1, noise in the lower panel inset arises due to interpolation between unrelated particles in the absence of phase-space information. Obtaining the correct point-wise bulk-velocity field in multi-stream regions is only possible through a projection of the reconstructed fine-grained distribution function.

4 REAL-SPACE PROPERTIES OF THE COSMIC VELOCITY FIELD

In this section, we present the results of our analysis of velocity fields in real space. First, we show slices of the velocity divergence and vorticity in cosmological simulations of CDM and WDM structure formation. We then present their 1-point statistical properties and give a preliminary discussion of the velocity fields inside dark matter haloes.

4.1 Validation

We compare our results for velocity fields with corresponding results obtained with a Delaunay tessellation in Eulerian space, i.e. based on the particle positions only, which has been employed commonly in the literature (e.g. Icke & van de Weygaert 1987; Bernardeau & van de Weygaert 1996; Pueblas & Scoccimarro 2009; Jennings et al. 2011). Such a method does not involve a projection of the actual distribution function, but interpolates between the particles on the fine-grained distribution in configuration space (i.e. performs the tessellation after the projection) which necessarily leads to inconsistencies in multi-stream regions when the mesh on which the velocities are evaluated is of higher resolution than the Delaunay tessellation. Quite obviously, this approach will perform better when the mesh cells are large and an effective coarse-graining is performed. As in our approach, velocities at an arbitrary point $x$ can be determined by linearly interpolating the velocity...
of the four vertices (particles) of the Delaunay tetrahedron containing $\mathbf{x}$ to $\mathbf{x}$. To perform this calculation, we used the publicly available version 1.1.1 of DTFE\(^1\) (Cautun & van de Weygaert 2011).

We have already discussed the difference between employing only configuration space information (such as DTFE) and using full phase space information in Section 2.4. Similar differences become readily visible when we inspect slices of the velocity field in one of the cosmological simulations. In the top panel of Fig. 4, we show the $x$-component of the velocity field for the L300N512 run. In the bottom panel, we show the respective velocity field obtained using DTFE. In both cases, we have resampled to a cube of $512^3$ cells. For the Delaunay tessellation, the small scale jitter due to linear interpolation between vertices that are not close in phase space and the lack of averaging in multi-stream regions is clearly apparent in the insets that zoom into the velocity field.

As discussed in Section 2.3, it is non-trivial to compute derivatives of the bulk velocity field since a commutator between the derivative and the projection operator appears. We will illustrate this now by looking at slices of the velocity divergence field computed from the WDM512 simulation in Fig. 5 using various derivative estimators. Specifically, we compare the correct expression based on the fine-grained distribution function from eq. (7) (top left panel), with a divergence computed on the gridded velocity field using a 4th order finite difference divergence operator (top right) and with the Fourier-space divergence operator (bottom left) that we will later employ when computing the spectral properties of velocity fields. The latter is given by computing $\text{div} \mathbf{v} = \mathcal{F}^{-1} [-i \mathbf{k} \cdot \mathcal{F} [\mathbf{v}]]$ using the fast Fourier transform (FFT) on the mesh on which we have computed the velocity field. Finally, we also show results when computing the velocity divergence for the DTFE estimate using again the 4th order finite difference operator. The differences for the velocity divergence fields computed using the various methods are readily visible. As expected, computing derivatives on the noisier DTFE field leads to a noisy divergence field. The spectral derivative performs badly by producing Gibbs ringing due to the discontinuous field. Finally, we also clearly see that the finite difference estimate performs an implicit coarse-graining by differentiating on a finite scale across caustics leading to strongly compressive features on the infall side of every caustic, just as in the one-dimensional case that we have discussed above in Section 2.2. In fact, these compressive features, while depending on the scale on which the derivative is taken, are clearly the highest magnitudes of divergence compared to what is estimated away from caustics with eq. (7).

By using phase-space information to average over multi-stream regions, we thus expect that our method will greatly improve estimates of the velocity field and its derivatives on non-linear scales. We explore this in more detail in the remainder of this paper.

4.2 Divergence and vorticity in CDM cosmic velocity fields

As we have discussed above and shown already in Fig. 5, pronounced differences in the real-space properties exist between divergence and vorticity estimated directly from the sheet and estimates based on finite differencing of the discontinuous velocity field.

In Fig. 6, we show slices through the CDM simulation L100N512, our simulation resolving the smallest scales. Specifically, we show the overdensity field (left panel), the velocity divergence (middle panels) as well as the vorticity field (right panels). Note that the colour map of the velocity divergence has been inverted with respect to that of the vorticity field (also in the remainder of the paper), to allow for a more easy comparison in terms of the anti-correlation between the two fields in linear theory. Again, we show two versions of velocity derivative estimation, the sheet-based estimate following eqs. (7) and (8) in the top panels as well as a finite difference estimate in the bottom panels. All fields were sampled onto a cubical lattice of $512^3$ cells and each slice corresponds to one slice of the data cube.

We recover for these CDM simulations the same qualitative differences in the velocity divergence fields as already discussed above for the plane-wave as well as the WDM case: finite differencing leads to filaments surrounded by envelopes of strong compression which are a result of the differentiation across the caustic on a finite scale (see Section 2.3). In the sheet-based estimates, the velocity field is almost everywhere expansive, aside from small filaments that have not shell-crossed yet. This results is not easily discernible using finite-difference estimates for the derivatives.

Similar differences resulting from the derivative estimation can be seen for the vorticity field. Applying eq. (8) leads to a vorticity that is explicitly zero in single-stream regions. The finite difference approach rather obviously fails to recover this aspect. In addition, due to differentiation across caustics, a similar envelope of high vorticity is visible at caustics as the maxima of compression discussed above. We observe that, as discussed by Pichon & Bernardeau (1999), a jump in vorticity occurs at the same locations as the caustics and with a similar magnitude in velocity change.

4.3 The velocity field in haloes

We continue our qualitative discussion of velocity fields by investigating in more detail the most massive halo of mass $M_h \sim 1.4 \times 10^{14} h^{-1} \text{M}_\odot$ (which corresponds to ~ 4 million particles) in our WDM512 simulation. We perform this analysis for the WDM case since the truncation of the perturbation spectrum allows us to resolve the collapse of those perturbations that are present with enough resolution. This cannot be achieved in the CDM case since perturbations up to the resolution limit exist.

In Fig. 7, we show slices of the density field (top left), velocity divergence (bottom left), vorticity (top right) as well as the $y$-component of the vorticity (bottom right) through the centre of this most massive halo. We now clearly see how the velocity divergence is positive in both void regions as well as predominantly positive in regions that have shell crossed. The only regions of convergence are given by regions of moderate overdensity that are not yet shell-crossed (but will do so in the future and are visible in red colour in the divergence map and not visible in the vorticity maps in Fig. 7). This can be clearly seen from the vorticity slice, where the convergent regions are not visible, indicating that they have not yet shell-crossed. Vorticity increases with every caustic, being highest in the outer parts of the virialised regions of the halo. In the very

\(^1\) http://www.astro.rug.nl/~voronoi/DTFE/dtfe.html
Figure 6. Slice through the overdensity (left panel), velocity divergence (middle panels) and vorticity (right panels) fields of the L100N512 simulation. Divergence and vorticity in the top row have been calculated directly from the dark matter sheet using eqs. (7) and (8), while those in the bottom row have been calculated from velocity fields computed from the dark matter sheet for which divergence and vorticity were calculated using finite differences. Note that the vorticity is plotted logarithmically, while the velocity divergence (due to its changing sign) is not.

centre of the halo, the velocity field is convergent and the vorticity decreases again.

We show these aspects in more details in Fig. 8, where we plot spherically averaged profiles of the density, vorticity, velocity divergence and the three tidal field eigenvalues. The density profile has not been calculated with the dark matter sheet, instead we simply plot the overdensity of $N$-body particles per shell to avoid the centrally biased profiles we have observed for the sheet in AHK12. The drop in vorticity and, more importantly, the sign reversal of the velocity divergence in the central region are now more clearly visible than in Fig. 7.

We observe a behaviour of the vorticity and divergence profiles that is consistent with our visual inspection of the slices. Velocity divergence rises as we enter the virial radius, is highest just inside the virial radius, then declines and becomes negative for $r \lesssim 0.2h^{-1}\text{Mpc}$. This qualitative behaviour is present for both the divergence of the bulk velocity field, as well as the averaged single stream divergence. We note that, at least for the halo we inspect, the radius where the divergence flips sign is close to the point where $\frac{d \log \delta}{d \log r} \sim 2$. In Appendix A, we show that while the details of the velocity divergence profiles converge rather slowly with increasing resolution, the qualitative features – i.e. the shape of the profile as well as the peak position and, most importantly, the radius at which the profile switches sign – are remarkably stable across resolutions.

One might thus speculate whether the change from convergent to divergent could be caused by a corresponding change of the tidal field. We also show the three eigenvalues of the tidal tensor in Fig. 8. The tidal field eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ are obtained by diagonalising the tidal field tensor $T_{ij} \equiv -\partial_i \partial_j \hat{\phi}$, where $\hat{\phi}$ is the gravitational potential normalised in such a way that $\text{tr} T_{ij} = \lambda_1 + \lambda_2 + \lambda_3 = -\delta$. We find that the eigenvalue signature is $(-+)$ at all radii, implying one-dimensional stretching and two-dimensional compression. It thus seems not very plausible that the change from convergent to divergent flow is of a simple tidal origin. The associated drop in vorticity might point towards an isotropised inner region, where particle velocities are well aligned with density gradients.

We only pointed out a few, rather qualitative, observations of the properties of bulk velocity fields in haloes. A more rigorous and detailed inspection is beyond the scope of this paper and will be followed up in future work.

4.4 One-point statistics

In this section, we return to the global statistics of the velocity field in cosmological volumes. In Fig. 9, we show the volume weighted probability distribution functions (PDFs) of the velocity divergence and vorticity fields. We find that the CDM vorticity PDFs that we show are reasonably well described by
a lognormal distribution

\[ p(\omega | \mu, \sigma) = \frac{1}{\sqrt{2\pi \sigma \omega}} \exp \left( -\frac{(\log \omega - \mu)^2}{2\sigma^2} \right), \]  

(15)

with best fit parameters \((\mu, \sigma)\) of \((3.55, 1.3)\), \((3.42, 1.14)\) and \((3.45, 1.10)\) for L100N512, L300N512 and L1000N1024, respectively. Quite in contrast, the WDM vorticity distribution has a distinctly different shape with strongly enhanced wings and a change in slope at large vorticity. Similar behaviour can be observed in the PDFs of the velocity divergence. While the CDM distributions have power-law tails at both positively and negatively large values, the WDM distribution has a distinctly enhanced tail.

Most remarkably, all velocity divergence PDFs have a distinct feature, a bump just below \( \text{div} \mathbf{v} \sim 70 \text{ km/s/h}^{-1} \text{Mpc} \). The origin of this feature becomes apparent when we plot the two dimensional distribution of overdensity-velocity divergence pairs. We do this, separately for the CDM simulation L100N512 and WDM512, in Fig. 10. From this figure, and rather as expected, it is readily apparent that in underdense regions, a tight correlation between velocity divergence and overdensity exists. This correlation weakens with increasing density, and completely disappears for \( \log_{10} 1 + \delta \gtrsim -0.5 \) in the case of CDM, and for \( \delta \gtrsim 0 \) for WDM. In the WDM case, we observe a very sharp upper limit to the possible velocity divergence at a given underdensity. This limit can be easily understood, since the expansion rate \( \theta_{\text{void}} \equiv H_0^{-1} \text{div} \mathbf{v} \) of a spherical void is directly related to its overdensity \( \delta \) (which, of course, is in fact an underdensity) through the nonlinear equations (cf. Bernardeau et al. 1997; van de Weygaert & Bond 2008)

\[ \theta_{\text{void}} = \frac{3}{2} \frac{\Omega_m^{0.6} - \Omega_{\text{void}}^{0.6}}{1 + \Omega_{\text{void}}^{0.6}/2}, \quad \Omega_{\text{void}} = \frac{\Omega_m (1 + \delta)}{(1 + \theta_{\text{void}}/3)^2}. \]  

(16)

We indicate the spherical void expansion rate by a dashed orange line in Fig. 10. It attains a maximum value of \( \sim 69 \text{ km/s/h}^{-1} \text{Mpc} \) for \( \delta = -1 \), which we show as the vertical line in Figure 9. In the CDM case, this does not in fact provide an upper limit but rather describes the median relation for \( \delta \lesssim -0.5 \) very well. This is plausibly simply due to the small scale noise inherent to CDM simulations.

We note that, as discussed in Section 2.3, the discontinuous velocity jumps with associated singular derivative live on a subspace of measure zero and thus do not contribute to the volume-weighted one-point statistics without coarse-graining presented above.
5 SPECTRAL PROPERTIES OF THE COSMIC VELOCITY FIELD

In this section, we present our main results regarding the spectral properties of cosmic velocity fields. We first outline fundamental differences between divergence and vorticity calculated in real and in Fourier space, before we will present power spectra of the velocity divergence and vorticity as well as density-divergence cross spectra for all our CDM and WDM simulations along with a convergence study and a comparison with results obtained with the DTFE.

5.1 Radial and transversal velocity modes vs. divergence and curl

The Fourier transform of the gradient operator $\nabla$ is given by $-ik$, so that in Fourier space the divergence and the curl of the velocity field $v$ become purely radial and transversal projections of the Fourier transformed velocity field:

$$\tilde{\theta} \equiv -ik \cdot \hat{v}$$

$$\tilde{\omega} \equiv -ik \times \hat{v},$$

where the tilde denotes a Fourier transformed field, which we will omit in what follows.

We note that do not use the divergence and curl computed using the sheet to study the spectral properties, but rather the velocity field itself and then perform the radial and transversal projections. The reason for this is that a Fourier transform of the singular derivatives at the caustics needs to be performed as well, rather than just the field away from the caustics in order to obtain the correct spectra.

One might be tempted to believe that only spectral derivatives should be used and derivatives on the tetrahedra should be avoided. While this is certainly warranted for the spectral properties of the velocity field, the inverse discrete Fourier transforms of the fields $\theta$ and $\omega$ have little to do with the divergence and curl of the bulk velocity field. This is, of course, obvious since the bulk velocity fields are discontinuous and thus slowly decaying in Fourier space so that their Nyquist limited spectral derivatives are particularly ill behaved at the caustic locations where the derivative is infinite. We remind the reader of the comparison of divergence fields obtained with a spectral derivative, a 4th order accurate finite difference operator, as well as by using 1st order accurate finite differences on the tetrahedra with explicit evaluation of the derivative of the projected field (according to eq. 7) in Fig. 5.

2 This trivial result can be easily seen from the definition of the Fourier transform $\int f(x) \exp(-ikx) \, dx = (f \ast \delta)(k)$. The dashed gray line in the bottom panel shows the reasonably good fit of a lognormal distribution to the vorticity PDF of simulation L100N512.
Their respective power spectra are given by

\[ P_\theta(k) = \langle \delta(k) \delta'\langle k'\rangle \rangle, \]

\[ P_\omega(k) = \langle \omega(k) \omega'\langle k'\rangle \rangle, \]

where \( \delta \) is the overdensity and \( \omega \) is the velocity potential. These expressions are valid in the linear regime, where the linear perturbation theory is applicable.

At late times, vorticity is generated and can reverse its sign on scales where shell-crossing has occurred, which is exactly what we will find below.

As we have discussed in AHK12 and Hahn et al. (2013), the density field estimated from the sheet is somewhat biased high in the densest regions (inner regions of haloes) since the sheet, linearly interpolated between particles, no longer tracks the true distribution function. We thus expect the cross-spectrum to drop rapidly at the scales where vorticity is generated and to reverse its sign on scales where shell-crossing has occurred, which is exactly what we will find below.

In the analysis of the correlation (cf. Fig. 3, but also Fig. 7 for an illustration of the reversal of the density–velocity divergence correlation), the growth of vorticity further destroys the potential flow, transversal modes become important. We thus expect the cross-spectrum to drop rapidly at the scales where vorticity is generated and to reverse its sign on scales where shell-crossing has occurred, which is exactly what we will find below.

5.3 Velocity divergence and vorticity power spectra for WDM

We first investigate the velocity divergence and vorticity power spectra for our WDM simulations, which all have the same PM force resolution but have varying mass resolution. The 300 eV initial WDM power spectrum leads to a truncation of perturbations that is resolved in all simulations (see also the discussion in Angulo et al. 2013). At fixed force resolution this should mean that one can arrive at perfectly converged properties. This is in stark contrast to the CDM case that we will discuss below, where more perturbations are introduced when the mass resolution is increased since the perturbation spectrum of CDM continues to almost arbitrarily small scales (see also our discussion of this in AHK12 and Hahn et al. 2013).

For this reason, we expect to achieve actual convergence if the simulation resolves both the cut-off scale and the non-linear scale. We thus re-analyse the simulations introduced in Hahn et al. (2013) for a toy model 300eV WDM particle. We do not use the simulation outputs obtained with the new simulation technique introduced in that paper, but make use of the standard PM results in order to keep the discussion purely focused on the analysis of the simulation data rather than the simulation technique.

We show the divergence and vorticity power spectra for our three WDM runs in Fig. 11, comparing again with dTFE. It is quite remarkable that the divergence spectra determined using the sheet are converged irrespective of resolution at all \( k \lesssim 10 h/\text{Mpc} \) with very small discrepancy at larger \( k \). The vorticity spectra show a small resolution dependence in amplitude.
Cosmic Velocity Fields

Figure 11. Power spectra at $z = 0$ of the divergence (top panel) and the vorticity (bottom panel) for the WDM simulations. All simulations used the same PM force resolution of $512^3$ cells, but had varying mass resolution: $128^3$ (red), $256^3$ (yellow) and $512^3$ (blue). Results obtained using the sheet tessellation are shown with a solid line, respective results obtained with dtfe are shown with dashed lines. While the two methods yield comparable results on large scales, only the sheet method shows convergence on small scales.

increasingly so at larger $k$, but not in shape. For the divergence spectra obtained with dtfe, we observe slower convergence, in particular scales $k \gtrsim 3h$/Mpc show a very strong dependence on the mass resolution. Even more pronounced is the lack of convergence for the vorticity spectra. Here, in the case of dtfe, all scales show a resolution dependence, unlike for the sheet estimated vorticity which is perfectly converged on small scales. In addition, for dtfe, at small scales, the vorticity spectra appear to converge slowly to a power-law with positive index. This is most likely due to the lack of an actual projection of the distribution function in this approach, where the single particle velocity dispersion becomes sampled in multi-stream regions rather than the bulk velocity. These results are a particularly good example of the strength of our phase-space based estimate of cosmic velocity fields. A remarkable difference is that the phase space sheet estimate of the vorticity spectrum converges from below, while the dtfe estimate converges from above.

5.4 Results: Power spectra of the velocity and density field for CDM

In Fig. 12, we present, for all 5 CDM simulations, the velocity divergence (top panel), and vorticity (second from top panel) power spectra, as well as the density-velocity divergence cross spectrum (third panel from top) and the density power spectrum (bottom panel). Our results are largely consistent with results of Pueblas & Scoccimarro (2009). In particular, we observe as well that converged power spectra require that the non-linear scale is well resolved. Pueblas & Scoccimarro (2009) observe that a mass resolution below $10^9h^{-1}M_\odot$ is necessary at $z = 0$ for converged spectra, consistent with our results. Both, the results of these authors as well as ours indicate that this is mostly due to the slow convergence of the flow vorticity. Somewhat unexpectedly however, Pueblas & Scoccimarro (2009) do not find an obvious resolution dependence of the convergence spectrum (while they do for the vorticity spectrum).

Since vorticity is driven by small-scale non-linearities, we find that the slow convergence of vorticity is, at least to some degree, driven by a lack of resolution at the non-linear scale: when running the L1000N512 box with a force resolution of only 8 times the mean inter-particle distance we observe a significant drop of the large-scale vorticity spectrum (not shown).

We next discuss the differences we see in spectra estimated using the phase space sheet compared to a standard Delaunay tessellation approach. In Figure 13, we compare the divergence and vorticity spectra computed from the velocity field estimates based on the dark matter sheet with those obtained using
We find that the small scale resolution for the vorticity spectra, which are clearly the most challenging to estimate reliably, an essentially identical picture to what was discussed above in Section 4.2, (3) convergence at all scales is rise at the smallest scales which is due to the small scale noise of the density fields, obtained with CIC deposits, contain a shot-noise term that we have not subtracted. The level of agreement between the different simulations (with different amplitudes of shot-noise) implies however that the discrepancy between the two biases cannot be explained by shot-noise alone. The difference between the fits is thus plausibly caused by either a non-linear relation between density and velocity-divergence or another more complicated additive noise component.

5.5 Fitting formulae

In this section, we determine fitting functions for the bias of the velocity divergence relative to the matter density field for $k < 1 \, h \, \text{Mpc}^{-1}$. The divergence bias $b_\theta$ can be defined either through the density–velocity divergence cross spectrum as

$$b_\theta^{(1)} = -\frac{1}{H f} \frac{\delta \theta}{\delta \delta}$$

or through the velocity divergence power spectrum as

$$b_\theta^{(2)} = \frac{1}{H f} \sqrt{ \frac{P_{\omega \omega}}{P_{b b}}},$$

where $H f \equiv d \log D/d \log \tau$ and $D(\tau)$ is the linear growth factor. We find that both are well fit for $k \lesssim 1 \, h \, \text{Mpc}^{-1}$ by the exponential function

$$b_\theta(k) = \exp \left[ -(k/\alpha)^{\beta} \right].$$

We determine the fit parameters $\alpha$ and $\beta$ by combining the spectra of the three boxes L100N512, L300N512 and L1000N512 for $k < 1 \, h \, \text{Mpc}^{-1}$. The divergence bias calculated from eq. (23) is shown in Fig. 15 (top panel, left) for the three boxes, the one calculated from eq. (24) in the top panel, right. When fitting the bias from the cross-spectrum, we find best fitting parameters $\alpha = 0.606 \pm 0.004 \, h \, \text{Mpc}^{-1}$ and $\beta = 1.176 \pm 0.017$. When fitting from the power spectrum, the best fit parameters are $\alpha = 0.7423 \pm 0.006 \, h \, \text{Mpc}^{-1}$ and $\beta = 1.112 \pm 0.018$. The relative error with respect to the fits is shown in the bottom panels of Fig. 15. We find that for $k \lesssim 1 \, h \, \text{Mpc}^{-1}$, the fits describe the data with about 5 per cent accuracy or better on larger scales. For a linear bias in the absence of noise, the two bias determinations should yield the same answer, but we find that this is only approximately true. Particularly, the parameter $\alpha$ implies a cut-off at slightly larger scales for the bias calculated from the cross-spectrum. We note that the density fields, obtained with CIC deposits, contain a shot-noise term that we have not subtracted. The level of agreement between the different simulations (with different amplitudes of shot-noise) implies however that the discrepancy between the two biases cannot be explained by shot-noise alone. The difference between the fits is thus plausibly caused by either a non-linear relation between density and velocity-divergence or another more complicated additive noise component.

6 SUMMARY AND CONCLUSIONS

In this paper, we have extended the phase-space sheet tessellation method introduced in AHK12 to estimate the properties of volume-weighted cosmic velocity fields and their differentials, explicitly, the velocity divergence and vorticity. To our knowledge, this method is the only one that is able to take into
We can summarise the advantages of our approach as follows:

(i) The phase space sheet allows a proper definition of a projection operator from phase space onto configuration space and thus properly averaged velocity fields in multi-stream regions. An estimate that does not respect the phase-space connectivity leads to small-scale jitter by interpolating between velocities that are close in configuration space but not close in phase space. Using the correct phase-space connectivity significantly reduces small-scale noise in the velocity fields.

(ii) We derived exact expressions for the divergence and curl of the bulk velocity field in multi-stream regions, eqs. (7) and (8), that explicitly do not include the discontinuities of the velocity field at caustics in multi-stream regions where derivatives are singular.

(iii) By excluding the singular derivatives (which occupy a volume of measure zero), our technique can compute the differential properties of the bulk velocity field without coarse-graining and thus the introduction of an arbitrary scale. By doing so, the differentials represent only the bulk dynamics of the dark matter fluid instead of including also the motion of the singular caustics.

Thus, in summary, we find that our new method provides significantly improved estimates of cosmic velocity fields and their differential properties: it allows a proper definition of a phase-space projection operator that significantly reduces small-scale noise.

We applied this estimator to a set of cold and warm dark matter N-body simulations. We discussed in detail the differences that arise when ordinary finite differencing schemes and spectral derivatives are used. We showed that those operators perform well in linear perturbation theory, appears as a strong correlation before shell crossing. After shell-crossing, the correlation reverses its sign. In shell-crossed regions, the outer regions of the halo have a positive divergence, while the inner regions are convergent. This tentative result certainly warrants further investigation.

Our science results can be summarised as:

(i) We demonstrated explicitly that the mapping between density and velocity divergence, i.e. $\text{div} \mathbf{v} \propto -\delta$, which holds in linear perturbation theory, appears as a strong correlation before shell crossing. After shell-crossing, the correlation reverses its sign. In shell-crossed regions, the only locations of predominantly negative divergence are the very centres of filaments and haloes.

(ii) Our results from WDM simulations indicate that, inside of haloes, the spherically averaged vorticity drops sharply and the velocity divergence changes its sign around the location where the density profile has a slope of $-2$. The outer regions of the halo have a positive divergence, while the inner regions are convergent. This tentative result certainly warrants further investigation.

(iii) We discussed the 1-point statistics of the divergence and vorticity fields and found that the divergence PDF exhibits a pronounced feature at the maximum void expansion rate. The vorticity PDF is reasonably well fit by a lognormal distribution. Furthermore, the two-dimensional histogram of overdensity vs. velocity divergence shows that for CDM the correlation between overdensity and velocity divergence is well described by the void expansion rate for $\delta \lesssim -0.7$ while for WDM, the void expansion rate

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure15}
\caption{Fit of the divergence bias $b_0$ from the divergence-density cross spectrum (left panel) and from the divergence power spectrum (right panel) at $z = 0$ with an exponential function $b_0(k) = \exp[-(k/\alpha)^{\beta}]$ with best fitting parameters $\alpha = 0.006 \pm 0.004$ $h$ Mpc$^{-1}$ and $\beta = 1.176 \pm 0.017$ when estimated from the cross spectrum (left), as well as $\alpha = 0.7423 \pm 0.006$ $h$ Mpc$^{-1}$ and $\beta = 1.112 \pm 0.018$ when estimated from the divergence power spectrum (right). The vertical grey line indicates $k = 1$ $h$ Mpc$^{-1}$, the wave number up to which our fit was performed and is valid with an accuracy of about 5 per cent. At even larger $k$, $b_0$ changes its sign (see Fig. 12).}
\end{figure}
provides a sharp upper limit to the velocity divergence at a given overdensity for $\delta \lesssim -0.7$.

(iv) Velocity spectra for CDM obtained with the phase space sheet show faster convergence behaviour with resolution, but both our method and dtfe converge to the same spectra when the non-linear scale is very well resolved. Consistent with previous results, we find that the vorticity spectra require higher resolution in order to be converged than divergence spectra.

(v) We complemented our discussion of spectral properties with an analysis of WDM simulations, where the entire perturbation spectrum can be resolved. Here, we are able to estimate spectra that are almost independent of the resolution of the underlying simulation, while the estimate on the position-space tessellation shows only slow convergence with resolution, is dominated by small scale noise and lacks convergence to the correct spectrum at small scales.

(vi) Finally, we provided fits for a bias parameter $b_b$ in terms of a function exponentially decaying in wave number $k$ that allows the CDM divergence power spectrum $P_{\delta\delta}$ as well as the divergence–overdensity cross spectrum $P_{\delta b}$ to be related to the non-linear matter power spectrum $P_{\delta\delta}$.

It will be interesting to follow up this preliminary analysis by probing the effect of cosmic large-scale velocity fields on the assembly of haloes and thus the alignment of their spins with larger scales as well as the connection between velocity fields and gravitational tidal fields (c.f. Aragón-Calvo et al. 2007; Hahn et al. 2007a,b; Hoffman et al. 2012; Libeskind et al. 2013; Laigle et al. 2013). Another application of our method is to more reliably measure the “effective field” properties of dark matter (e.g. Carrasco et al. 2012). Quite obviously, future work should also study the second moment of the projected velocity field, i.e. the anisotropic stress tensor, in addition to the properties of the bulk field.

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APPENDIX A: RESOLUTION STUDY OF THE VELOCITY FIELD IN HALOES

In this appendix, we assess the degree of convergence of the velocity fields estimated from the dark-matter sheet inside of virialized structures. We show a close-up version of the most massive halo in the WDM simulations in Fig. A1, to be compared to the respective panels of Fig. 7. We observe that the larger scale structures in the density field are remarkably stable with increasing resolution. Convergence in the velocity divergence field is however considerably slower.

We complement this rather qualitative comparison with radial velocity divergence profiles of the averaged single-stream velocity divergence in Fig. A2. Consistent with the slices discussed above, the qualitative features are stable across resolutions. While the velocity divergence profile is not yet converged, the peak location as well as the location of sign flip are consistent among the different resolutions.

Figure A1. Convergence study of slices of the dark-matter sheet estimated density field and velocity divergence through the centre of the most massive halo of the WDM simulations. The bottom row panels are a close-up version of the corresponding panels in Fig. 7.

Figure A2. Convergence study of the spherically averaged single stream velocity divergence. Profiles are shown for all WDM simulations. While the peak properties have not yet converged, the point where the velocity field changes from convergent to divergent flow is consistent among all resolutions. Shaded regions correspond to the error on the median estimated from the 16th and 84th percentile in each radial bin.