SECOND MAIN THEOREM WITH TROPICAL HYPERSURFACES AND DEFECT RELATION

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Abstract. The tropical Nevanlinna theory is Nevanlinna theory for tropical functions or maps over the max-plus semiring by using the approach of complex analysis. The main purpose of this paper is to study the second main theorem with tropical hypersurfaces into tropical projective spaces and give a defect relation which can be regarded as a tropical version of the Shiffman’s conjecture. On the one hand, our second main theorem improves and extends the tropical Cartan’s second main theorem due to Korhonen and Tohge [Advances Math. 298(2016), 693-725]. The growth of tropical holomorphic curve is also improved to \( \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0 \) (rather than just hyperorder strictly less than one) by obtaining an improvement of tropical logarithmic derivative lemma. On the other hand, we obtain a new version of tropical Nevanlinna’s second main theorem which is different from the tropical Nevanlinna’s second main theorem obtained by Laine and Tohge [Proc. London Math. Soc. 102(2011), 883-922]. The new version of the tropical Nevanlinna’s second main theorem implies an interesting defect relation that \( \delta_f(a) = 0 \) holds for a non-constant tropical meromorphic function \( f \) with \( \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0 \) and any \( a \in \mathbb{R} \) such that \( f \oplus a \not\equiv a \).

1. Introduction

Recently the so-called tropical approach to mathematics has attracted much attention from researchers in several fields including combinatorics, optimization, mathematical physics and algebraic. Surprisingly, many results from classical algebraic geometry have tropical analogues. The tropical geometry (see [20] or a recent book [19]) is a piecewise linear version of complex algebraic geometry, which is geometry over the tropical semiring (that is, the max-plus semi-ring first arose in Kleenes 1956 paper on nerve sets and automata [13]).

The tropical Nevanlinna theory can be seen as Nevanlinna theory over the tropical semiring by using the approach of complex analysis. Functions or maps are naturally defined on tropical semiring. In [9], Halburd and Southall first proved the tropical versions of Poisson-Jensen theorem, Nevanlinna first main theorem by introducing tropical versions of Nevanlinna characteristic function, proximity
function, counting function, logarithmic derivative lemma for tropical meromorphic functions with finite order similarly as in the classical Nevanlinna theory (see for examples [10, 3, 24]). Using this theory, they investigated the existence of finite-order max-plus meromorphic solutions which can be considered to be an ultra-discrete analogue of the Painlevé property, in which they proposed a tropical version of Clunie theorem. Laine and Tohge [17] considered tropical Laurent series with arbitrary real value powers and extended the definition of tropical meromorphic functions, then in this case the multiplicities of poles (respectively, zeros) may be arbitrary real numbers instead of being integers (respectively, rationals) which is in certain respects fundamentally different from the counterparts in the classical meromorphic functions. After modifying the Halburd-Southall’s version of Tropical Poisson-Jensen theorem and first main theorem, Laine and Tohge gave the tropical Nevanlinna second main theorem for tropical meromorphic functions with hyperorder strictly less than one. Laine, Liu and Tohge [16] presented tropical counterparts of some classical complex results related to Fermat type equations, Hayman conjecture and Brück conjecture. Recently, Korhonen and Tohge [15] extended the tropical Nevanlinna theory to tropical holomorphic curves in a finite dimensional tropical projective space, and obtained a tropical version of Cartan second main theorem for tropical holomorphic curves with hyperorder strictly less than one.

The main purpose of this paper is to study the second main theorem for tropical holomorphic curves intersecting tropical hypersurfaces (Theorem 4.8) and then obtain a defect relation (Theorem 4.10) which can be regarded as a tropical version of the Shiffman’s conjecture in Classical Nevanlinna theorem completely proved by Ru [23]. The growth condition for tropical holomorphic curves is improved to the case \( \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0 \) (rather than just hyperorder strictly less than one), by obtaining an improvement of the tropical logarithmic derivative lemma (Theorem 3.1) due to [9, 17]. As shown in our main result (Theorem 4.8), we obtain an equality form of second main theorem under the assumption \( \text{ddg}(\{P_{M+2} \circ f, \ldots, P_q \circ f\}) = 0 \) so that the Korhonen and Tohge’s version of tropical Cartan second main theorem [15] is improved. Moreover, from the improvement of tropical Cartan second main theorem, we obtain new versions of tropical Nevanlinna second main theorem (Theorem 6.3 and Theorem 6.5) which are different from the Laine and Tohge’s version of tropical second main theorem, and thus propose an interesting defect relation: \( \delta_f(a) = 0 \) holds for a nonconstant tropical meromorphic function \( f \) with \( \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0 \) and any \( a \in \mathbb{R} \) such that \( f \oplus a \neq a \). It is very interesting that the assumption \( f \oplus a_j \neq a_j \) in our results are clearly better than the assumptions in Laine and Tohge’s version of tropical Nevanlinna second main theorem. During the study of the tropical Nevanlinna second main theorem, we find that there exists one gap in the proof of Korhonen and Tohge’s improvement of tropical Nevanlinna second main theorem due to Laine and Tohge.

This paper is organized as follows. In the next section, we will briefly introduce some definitions and notations of the tropical semiring, tropical linear algebra, Nevanlinna theory for tropical meromorphic functions, and tropical holomorphic curves into tropical projective spaces. After proving a strong lemma for a nondecreasing convex continuous function, we give an improved tropical version of logarithmic derivative lemma (Theorem 3.1) in Section 3. In Section 4, we firstly show that the identity of the two definitions of tropical algebraically nondegenerated and
algebraically independently in the Gondran-Minoux sense, and then propose mainly
the second main theorem with tropical hypersurfaces (Theorem 4.8). This main re-
sult improves the tropical Cartan second main theorem (Corollary 4.12). After
establishing the tropical first main theorem for tropical holomorphic curve inter-
secting tropical hypersurfaces (Theorem 4.7) similarly as in Classical Nevanlinna
theory, we then obtain the defect relation (Theorem 4.10) according to Theorem
4.8. The proof of our main result (Theorem 4.8) is individually proved in Sec-
tion 5. The new versions of tropical Nevanlinna second main theorem (Theorem
6.3 and Theorem 6.5) will be discussed in Section 6, from which we also obtained
an interesting result on defect relation for tropical meromorphic functions (Theorem
6.6).

2. Preliminaries

According to [22] the term "tropical" appeared in computer science in honor
of Brazil and, more specifically, after Imre Simon (who is a Brazilian com-
puter scientist) by Dominique Perrin. In computer science the term is usua-
lly applied to (min, +)−semiring. Here we use the tropical (max-plus) semiring ($\mathbb{R} \cup \{-\infty\}$, $\oplus$, $\odot$)
endowing $\mathbb{R} \cup \{-\infty\}$ with (tropical) addition
$$x \oplus y := \max(x, y)$$
and (tropical) multiplication
$$x \odot y := x + y.$$ But the two semirings are isomorphic each other by $x \mapsto -x$. We also use notations
$x \odot y := \frac{x}{y} = x - y$ and $x \odot a := ax$ for $a \in \mathbb{R}$. The identity elements for the
tropical operations are $0_o = -\infty$ for addition and $1_o = 0$ for multiplication (see for
examples, [20, 9, 17]).

**Definition 2.1.** [14] A tropical entire function $f : \mathbb{R} \to \mathbb{R}$ is a finite or infinite
linear combination of tropical monomials:
$$f(x) := \sum_{k=0}^{+\infty} c_k \odot x^{\odot s_k} = \max_{k=0}^{+\infty} (s_k x + c_k)$$
where $c_1, c_2, \ldots$ and $s_1, s_2, \ldots$ are real numbers such that $c_1 > c_2 > \ldots (\rightarrow -\infty)$
and $s_0 < s_1 < \cdots$. The tropical polynomial $P : \mathbb{R} \to \mathbb{R}$ defined as a finite linear
combination of tropical monomials.

Remark that a tropical entire function $f : \mathbb{R} \to \mathbb{R}$ has the following three
important properties:
(i) $f$ is continuous,
(ii) $f$ is piecewise-linear, where the number of pieces is finite or infinite,
(iii) $f$ is convex, that is $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in \mathbb{R}$.

**Definition 2.2.** [14] A continuous piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ is called to
be tropical meromorphic. A point $x \in \mathbb{R}$ of derivative discontinuity of $f$ is a pole of
$f$ with multiplicity $-\omega_f(x)$ whenever
$$\omega_f(x) := \lim_{\varepsilon \to 0^+} \{f'(x + \varepsilon) - f'(x - \varepsilon)\} < 0,$$ and a zero (or root) with multiplicity $\omega_f(x)$ if $\omega_f(x) > 0$. 
Note that the multiplicities are positive numbers, not necessarily integers as it is the case in classical complex analysis. From the viewpoint of geometry, we see that a zero (or pole) of \( f \) is the point \( x \in \mathbb{R} \) at which the graph of \( f \) is nonlinear and convex (or concave). Korhonen and Tohge [15, Proposition 3.3] proved that for any tropical meromorphic function \( f \), there exist two tropical entire functions \( g \) and \( h \) such that \( f = h \circ g \), where \( g \) and \( h \) do not have any common zeros.

The tropical proximity function for tropical meromorphic functions \( f \) in one real variable is defined by

\[
m(r, f) := \frac{1}{2} \sum_{\sigma = \pm 1} f^+(\sigma r) = \frac{1}{2} \{f^+(r) + f^+(-r)\}
\]

where \( f^+(x) = \max\{f(x), 0\} \). Denote by \( n(r, f) \) the number of poles of \( f \), counted with multiplicities, in the interval \((-r, r)\), i.e.,

\[
n(r, f) = \sum_{|b_r| < r} \tau_f(b_r)
\]

and define the tropical counting function

\[
N(r, f) := \frac{1}{2} \int_0^r n(t, f) dt = \frac{1}{2} \sum_{|b_r| < r} \tau_f(b_r)(r - |b_r|)
\]

where all \( b_r \) are poles of \( f \). The tropical Nevanlinna characteristic function is given by

\[
T(r, f) := m(r, f) + N(r, f)
\]

which is then an increasing convex function of \( r \) by the tropical Cartan identity [14, Theorem 3.8]. For example, a nonconstant tropical rational function satisfies \( T(r, f) = O(r) \) (see [14, Theorem 1.14]). The tropical Poisson-Jensen formula [9, 17] (see also [14, Theorem 3.1]) implies the tropical Jensen formula

\[
N(r, 1_0 \odot f) - N(r, f) = \frac{1}{2} \sum_{\sigma = \pm 1} f(\sigma r) - f(0),
\]

which is very useful and will be used many times throughout this paper. Furthermore, there exists the tropical first main theorem (see [14, Theorem 3.5])

\[
T(r, 1_a \odot (f \odot a)) = T(r, f) + O(1)
\]

provided that \(-\infty < a < L_f := \inf\{f(b) : \omega_f(b) < 0\}\).

Since the tropical linear space \( L \) in \( \mathbb{R}^{n+1} \) is closed under tropical scalar multiplication, \( L = L + \mathbb{R}(1, 1, \ldots, 1) \). Therefore the tropical projective space is defined as \( \mathbb{T}^n = \mathbb{R}^{n+1}/\mathbb{R}(1, 1, \ldots, 1) \). That is, \( \mathbb{T}^n = \mathbb{R}^{n+1}_{\text{max}}/\{0_n\} \sim (a_0, a_1, \ldots, a_n) \sim (b_0, b_1, \ldots, b_n) \) if and only if

\[
(a_0, a_1, \ldots, a_n) = \lambda \odot (b_0, b_1, \ldots, b_n) = (\lambda \odot b_0, \lambda \odot b_1, \ldots, \lambda \odot b_n)
\]

for some \( \lambda \in \mathbb{R} \). Denote by \([a_0 : a_1 : \cdots : a_n] \) the equivalence class of \((a_0, a_1, \ldots, a_n)\). For instance, \( \mathbb{T}^1 \) is just the completed max-plus semiring \( \mathbb{R}_{\text{max}} \cup \{+\infty\} = \mathbb{R} \cup \{\pm \infty\} \).

Let \( f := [f_0 : f_1 : \cdots : f_n] : \mathbb{R} \rightarrow \mathbb{T}^n \) be a tropical holomorphic map where \( f_0, f_1, \ldots, f_n \) are tropical entire functions and do not have any zeros which are common to all of them. Denote \( f = (f_0, f_1, \ldots, f_n) : \mathbb{R} \rightarrow \mathbb{R}^{n+1} \). Then the map \( f \)
that is, independent) in the Gondran-Minoux sense \[4, 5\] if there exist (respectively there do not exist) two disjoint subsets

\{\}
where the sum is taken over all permutations \(\\{\\)

\text{where the sum is taken over all permutations } \{n \}

is called a reduced representation of the tropical holomorphic curve \(f \in \mathbb{T}P^n\). The tropical Cartan characteristic function of \(f\) is defined by

\[
T_f(r) := \frac{1}{2} \sum_{\sigma=\pm 1} \|f(\sigma r)\| - \|f(0)\| = \frac{1}{2} \left[\|f(r)\| + \|f(-r)\|\right] - \|f(0)\|
\]

where \(\|f(x)\| = \max\{f_0(x), \ldots, f_n(x)\}\) is defined according to canonical coordinates of \(\mathbb{T}P^n\). It is well defined, that is, \(T_f(r)\) is independent of the reduced representation of \(f\) \[15\] Proposition 4.3. The order and hyperorder of \(f\) are given by

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log T_f(r)}{\log r},
\]

and

\[
\rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T_f(r)}{\log r},
\]

respectively. If \(f\) is a tropical meromorphic function, then \(T_f(r) = T(r, f) + O(1)\) (see \[15\] Proposition 4.4).

The operations of tropical addition \(\oplus\) and tropical multiplication \(\odot\) for the \((n + 1) \times (n + 1)\) matrices \(A = (a_{ij})\) and \(B = (b_{ij})\) are defined by

\[
A \oplus B = (a_{ij} \oplus b_{ij})
\]

and \(A \odot B = (\bigoplus_{k=0}^n a_{ik} \odot b_{kj})\), respectively. If an \((n + 1) \times (n + 1)\) matrix \(A\) contains at least one element different from 0 in each row, then \(A\) is called regular.

The tropical determinant \(|A|_o\) of \(A\) is defined by

\[
|A|_o = \bigoplus a_{0(\pi(0))} \odot a_{1(\pi(1))} \odot \cdots \odot a_{n(\pi(n))},
\]

where the sum is taken over all permutations \{\(\pi(0), \pi(1), \ldots, \pi(n)\)\} of \{0, 1, \ldots, n\}. Note that an \((n + 1) \times (n + 1)\) matrix \(A\) is regular if and only if \(|A|_o \neq 0_o\).

Choose \(c \in \mathbb{R} \setminus \{0\}\). Let \(f : \mathbb{R} \to \mathbb{T}P^n\) be a tropical holomorphic map with reduced representation \((f_0, f_1, \ldots, f_n)\). We use short notations

\[
\overline{f}_j^{[0]} := f_j(x), \quad \overline{f}_j^{[1]} := f_j(x + c), \quad \overline{f}_j^{[k]} := f_j(x + kc) = f_j(x \odot c^\otimes k)
\]

for all \(j, k \in \{0, 1, \ldots, n\}\). The tropical Casorati determinant, or tropical Casorati, of \(f\) is defined by

\[
C_o(f) := C_o(f_0, f_1, \ldots, f_n) = \bigoplus f_0^{[\pi(0)]} \odot f_1^{[\pi(1)]} \odot \cdots \odot f_n^{[\pi(n)]}
\]

where the sum is taken over all permutations \{\(\pi(0), \ldots, \pi(n)\)\} of \{0, \ldots, n\}.

Tropical meromorphic functions \(g_0, \ldots, g_n\) are linearly dependent (respectively independent) in the Gondran-Minoux sense \[4, 5\] if there exist (respectively there do not exist) two disjoint subsets \(I\) and \(J\) of \(K := \{0, \ldots, n\}\) such that \(I \cup J = K\) and

\[
\bigoplus_{i \in I} a_i \odot g_i = \bigoplus_{j \in J} a_j \odot g_j,
\]

that is,

\[
\max\{a_i + g_i\} = \max\{a_j + g_j\},
\]
where the constants \( a_0, a_1, \ldots, a_n \in \mathbb{R}_{\text{max}} \) are not all equal to 0. If \( a_0, \ldots, a_n \in \mathbb{R}_{\text{max}} \) and \( f_0, \ldots, f_n \) are tropical entire functions, then

\[
F = \bigoplus_{\nu=0}^{n} a_{\nu} \odot f_{\nu} = \bigoplus_{i=1}^{j} a_{k_i} \odot f_{k_i}
\]
is called a tropical linear combination of \( f_0, f_1, \ldots, f_n \) over \( \mathbb{R}_{\text{max}} \), where the index set \( \{k_1, \ldots, k_j\} \subset \{0, \ldots, n\} \) is such that \( a_{k_i} \in \mathbb{R} \) for all \( i \in \{1, \ldots, j\} \), while \( a_{\nu} = 0 \) if \( \nu \notin \{k_1, \ldots, k_j\} \). Note that if \( f_0, \ldots, f_n \) are linearly independent in the sense of Gondran and Minoux, then the expression of \( \nu \) cannot be rewritten by means of any other index set which is different from the set \( \{k_1, \ldots, k_j\} \).

Let \( G = \{f_0, \ldots, f_n\}(\neq \{o_o\}) \) be a set of tropical entire functions, linearly independent in the Gondran-Minoux sense, and denote

\[
\mathcal{L}_G = \text{span} < f_0, \ldots, f_n > = \left\{ \bigoplus_{k=0}^{n} a_k \odot f_k : (a_0, \ldots, a_n) \in \mathbb{R}_{\text{max}}^{n+1} \right\}
\]
to be their linear span. The collection \( G \) is called the spanning basis of \( \mathcal{L}_G \). The dimension of \( \mathcal{L}_G \) is defined by

\[
\dim(\mathcal{L}_G) = \max\{\ell(F) : F \in \mathcal{L}_G \setminus \{0_o\}\},
\]
where \( \ell(F) \) is the shortest length of the representation of \( F \in \mathcal{L}_G \setminus \{0_o\} \) defined by

\[
\ell(F) = \min\{j \in \{1, \ldots, n+1\} : F = \bigoplus_{i=1}^{j} a_{k_i} \odot f_{k_i}\}
\]
where \( a_{k_i} \in \mathbb{R} \) with integers \( 0 \leq k_1 < k_2 < \cdots < k_j \leq n \). Note that usually the dimension of the tropical linear span space of \( G \) may not be \( n+1 \), which is different from the classical linear algebraic. If \( \ell(F) = n+1 \) for a tropical linear combination \( F \) of \( f_0, \ldots, f_n \), then \( F \) is said to be complete, that is, the coefficients \( a_k \) in any expression of \( F \) of the form \( F = \bigoplus_{k=0}^{n} a_k \odot f_k \) must satisfy \( a_k \in \mathbb{R} \) for all \( k \in \{0, \ldots, n\} \) and in this case, \( \mathcal{L}_G = n+1 \).

Let \( G = \{f_0, \ldots, f_n\} \) be a set of tropical entire functions, linearly independent in the Gondran-Minoux sense, and let \( Q \subset \mathcal{L}_G \) be a collection of tropical linear combinations of \( G \) over \( \mathbb{R}_{\text{max}} \). The degree of degeneracy of \( Q \) is defined to be

\[
\text{ddg}(Q) := \text{card}\{F \in Q : \ell(F) < n+1\}.
\]
If \( \text{ddg}(Q) = 0 \), then we say \( Q \) is non-degenerate. This means that the degree of degeneracy of a set of tropical linear combinations is the number of its non-complete elements. In this way the number of complete elements of \( Q \) is the ‘actual dimension’ of the subspace spanned by \( Q \), and thus the \( \text{ddg}(Q) \) is the ‘codimension’ of the subspace spanned by \( Q \) (see [14, Page 120-121]).

3. Tropical version of logarithmic derivative lemma

In the classical Nevanlinna theory, the logarithmic derivative lemma plays a key role in significant expression of the Nevanlinna second main theorem for meromorphic functions and Cartan second main theorem for holomorphic curve intersecting hyperplanes. The tropical analogue of the lemma on the logarithmic derivative for tropical meromorphic functions with finite order was obtained by Halburd and Southall [9], and was extended to the case of hyper-order strictly less than one by

\[
\text{ddg}(Q) := \text{card}\{F \in Q : \ell(F) < n+1\}.
\]
In this section, we will improve the tropical logarithmic derivative lemma, and extend the condition of growth of meromorphic functions to the case not exceeding to the hyperorder 1 minimal type, i.e., \( \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0 \) (rather than just hyperorder strictly less than one). This result will be used to prove our tropical second main theorem with tropical hypersurfaces in next section.

**Theorem 3.1.** (Tropical version of logarithmic derivative lemma) Let \( c \in \mathbb{R} \setminus \{0\} \).

If \( f \) is a tropical meromorphic function on \( \mathbb{R} \) with

\[
\limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0,
\]

then

\[
m(r, f(x + c) \ominus f(x)) = o(T_f(r))
\]

where \( r \) runs to infinity outside of a set of zero upper density measure \( E \), i.e.,

\[
dens E = \limsup_{r \to \infty} \frac{1}{r} \int_{E \cap [1, r]} dt = 0.
\]

**Remark 3.2.** (i). We note that the condition (3) implies that \( \rho_2(f) \leq 1 \) and the equality can possibly take happened. In fact, assume that (3) holds, then there exists \( r_0 > 0 \) such that for any \( r > r_0 \), we have \( \log T_f(r) < r \) and thus \( \rho_2(f) \leq 1 \). Moreover, whenever \( f \) is taken to satisfy, for example \( \log T_f(r) = r (\log r)^m \) where \( m > 0 \), one can easily get both (3) and \( \rho_2(f) = 1 \). Hence, Theorem 3.1 is an improvement of the tropical logarithmic derivative lemma obtained in \([9, 17]\) for hyperorder strictly less than one.

(ii). By the improved version of tropical logarithmic derivative lemma, the tropical Clunie and Mohon’ko type theorems \([14, \text{Corollaries 4.4, 4.11, 4.12 and 7.15}]\) (see also \([9, 17, 18]\)) can be also improved to the case \( \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0 \).

Before giving the proof, we show the following lemma which is an improvement of a result on growth properties of nondecreasing continuous real functions \([7, \text{Lemma 2.1}]\) and \([9, 17, 18]\). Here we establish a strong lemma of real convex functions for improving the tropical logarithmic derivative lemma, an analogue result for a logarithmic convex function is obtained in \([27]\).

**Lemma 3.3.** Let \( T(r) \) be a nondecreasing positive, convex, continuous function on \([1, +\infty)\) with

\[
\liminf_{r \to \infty} \frac{\log T(r)}{r} = 0.
\]

Then for the function

\[
\phi(r) := \max_{1 \leq t \leq r} \left\{ \left( \frac{t}{\log T(t)} \right)^\delta \right\}, \quad \delta \in (0, \frac{1}{2}),
\]

we have

\[
T(r) \leq T(r + \phi(r)) \leq (1 + \varepsilon(r))T(r),
\]

where \( \varepsilon(r) \to 0 \) as \( r \) tends to infinity outside of a set of zero lower density measure \( E \), i.e.,

\[
dens E = \liminf_{r \to \infty} \frac{1}{r} \int_{E \cap [1, r]} dt = 0;
\]

Especially, for any fixed positive real value \( c(\neq 0) \),

\[
T(r) \leq T(r + c) \leq (1 + \varepsilon(r))T(r), \quad r \notin E \to \infty.
\]
Furthermore, if the growth assumption is changed into
\[
\limsup_{r \to \infty} \frac{\log T(r)}{r} = 0,
\]
then the exceptional set \(E\) is a set with zero upper density measure, i.e.,
\[
\text{dens} E = \limsup_{r \to \infty} \frac{1}{r} \int_{E \cap [1, r]} dt = 0.
\]

Proof. Since \(T(r)\) is a nondecreasing positive, convex, continuous function on \([1, +\infty)\), it follows that for any \(r \in [1, \infty)\) and \(R = r + \phi(r)\), we have
\[
T(R) \leq T(r) + \frac{dT(R)}{dR} (R - r) \\
\leq T(r) + \frac{T'(r + \phi(r))}{T(r + \phi(r))} \phi(r) T(r + \phi(r)).
\]

Define
\[
\hat{\tau}(r) = \sqrt{\frac{\log T(r + \phi(r)) - T(1 + \phi(1))}{r}}, \quad r \in [1, +\infty).
\]

Since
\[
\liminf_{r \to \infty} \frac{\log T(r)}{r} = 0,
\]
there exists one sequence \(r_n\) with \(\lim_{n \to \infty} r_n = \infty\) such that
\[
\hat{\tau}(r_n) = \min_{1 \leq t \leq r_n} \hat{\tau}(t).
\]

This gives
\[
0 < \hat{\tau}(r_{n+1}) \leq \hat{\tau}(r_n) \to 0
\]
as \(n \to \infty\). Define \(\tau(r) := \hat{\tau}(r_n)\) for \(r \in [r_n, r_{n+1}]\) and so \(\tau(r)\) tends to zero as \(r\) tends to \(\infty\). Now consider the set
\[
E := \{ r \in [1, +\infty) : \frac{T'(r + \phi(r))}{T(r + \phi(r))} \geq \tau(r) \}.
\]

Then by the Riemann-Stieljies integral, it follows that
\[
\log T(r + \phi(r)) - \log T(1 + \phi(1)) \\
\geq \int_1^r d(\log T(t + \phi(t))) \\
= \int_1^r \frac{T'(t + \phi(t))}{T(t + \phi(t))} d(t + \phi(t)) \\
= \int_1^r \frac{T'(t + \phi(t))}{T(t + \phi(t))} dt + \int_1^r \frac{T'(t + \phi(t))}{T(t + \phi(t))} d\phi(t) \\
\geq \int_1^r \frac{T'(t + \phi(t))}{T(t + \phi(t))} dt \\
\geq \int_{E \cap [1, r]} \tau(t) dt \\
\geq \tau(r) \int_{E \cap [1, r]} dt.
\]
Hence for \( r = r_n \),
\[
\frac{1}{r} \int_{E \cap [1, r]} dt \leq \frac{1}{r} \frac{\log T(r_n + \phi(r_n)) - \log T(1 + \phi(1))}{r_n} \\
\leq \sqrt{\frac{\log T(r_n + \phi(r_n)) - \log T(1 + \phi(1))}{r_n}}
\]
and thus
\[
dens E = \liminf_{r \to \infty} \frac{1}{r} \int_{E \cap [1, r]} dt = 0.
\]

Therefore, for all \( r \notin E \), we have
\[
T(r + \phi(r)) \leq T(r) + \tau(r)\phi(r)T(r + \phi(r))
\]
which implies
\[
T(r + \phi(r)) \leq (1 + \varepsilon(r))T(r)
\]
where \( \varepsilon(r) := \frac{\tau(r)\phi(r)}{1 - \tau(r)\phi(r)} \). Since \( \delta \in (0, \frac{1}{2}) \), we get that \( \tau(r)\phi(r) \to 0 \), and thus \( \varepsilon(r) \to 0 \), as \( r \to \infty \).

By noting that \( \phi(r) \to \infty \) as \( r \to \infty \), we have, for a fixed positive real number \( c \), \( T(r + c) \leq T(r + \phi(r)) \). Therefore we obtain the desired result.

Obviously, if the growth assumption is, instead, \( \limsup_{r \to \infty} \frac{\log T(f)}{r} = 0 \), then the above sequence \( \{r_n\} \) can be arbitrarily chosen, and thus the exceptional set \( E \) is a set with zero upper density measure.

\( \square \)

Proof of Theorem 3.1 By [13] Theorem 3.24, for all \( \alpha(> 1) \) and all \( r > 0 \), we have
\[
m(r, f(x + c) \odot f(x)) \leq \frac{16|c|}{r + |c|} \frac{1}{\alpha - 1} T_f(\alpha(r + |c|)) + \frac{|f(0)|}{2}.
\]
Let
\[
\alpha := 1 + \frac{(r + |c|)^{\delta - 1}}{(\log T_f(r + |c|))^{\delta}}, \quad \delta(0, \frac{1}{2}).
\]
Under the assumption (3), we get that
\[
\frac{1}{(\alpha - 1)(r + |c|)} = \left( \frac{\log T_f(r + |c|)}{r + |c|} \right)^{\delta} = o(1)(r \to \infty).
\]
Note that
\[
\alpha(r + |c|) = (r + |c|) + \left( \frac{r + |c|}{\log T_f(r + |c|)} \right)^{\delta}.
\]
Take \( \phi(r) = \max_{1 \leq \ell \leq \tau} \left( \frac{f}{T_f(r)} \right)^{\delta} \) in Lemma 3.3 and so \( \left( \frac{r + |c|}{T_f(r + |c|)} \right)^{\delta} \leq \phi(r + |c|) \). We get that
\[
T_f(\alpha(r + |c|)) \leq (1 + \varepsilon(r + |c|))T_f(r + |c|) \leq (1 + \varepsilon(r + |c|))(1 + \varepsilon(r))T_f(r)
\]
holds for all \( r \) with \( r \notin E \) where \( dens E = 0 \). Therefore, we have
\[
m(r, f(x + c) \odot f(x)) \leq 16|c|o(1)(1 + \varepsilon(r + |c|))(1 + \varepsilon(r))T_f(r) + \frac{|f(0)|}{2} \text{ holds for all } r \notin E \text{ where } dens E = 0.
\]
\( \square \)
In terms of the Hinkkanen’s Borel type growth lemma, in another way, we establish the following

**Theorem 3.4.** Let $c \in \mathbb{R} \setminus \{0\}$. If $f$ is tropical meromorphic function on $\mathbb{R}$ with

$$
\limsup_{r \to \infty} \frac{\log T_f(r)(\log r)^\varepsilon}{r} = 0
$$

holds for any $\varepsilon > 0$, then

$$
m(r, f(x + c) \odot f(x)) = o(T_f(r))
$$

where $r$ runs to infinity outside of a set $E$ satisfying

$$
\int_E \frac{dt}{t \log t} < +\infty.
$$

The next lemma is the Hinkkanen’s Borel type growth lemma (or see also a similar lemma [3, Lemma 3.3.1].

**Lemma 3.5.** [11, Lemma 4] Let $p(r)$ and $h(r) = \varphi(r)/r$ be positive nondecreasing functions defined for $r \geq sp > 0$ and $r \geq \tau > 0$, respectively, such that $\int_e^\infty \frac{dr}{p(r)} = \infty$ and $\int_r^\infty \frac{dr}{\varphi(r)} < \infty$. Let $u(r)$ be a positive nondecreasing function defined for $r \geq r_0 \geq sp > 0$ such that $u(r) \to \infty$ as $r \to \infty$. Then if $C$ is real with $C > 1$, we have

$$
u \left( r + \frac{p(r)}{h(u(r))} \right) < Cu(r)
$$

whenever $r \geq r_0$, $u(r) > \tau$, and $r \notin E$ where

$$
\int_E \frac{dr}{p(r)} \leq \frac{1}{h(w)} + \frac{C}{C - 1} \int_w^\infty \frac{dr}{\varphi(r)} < \infty
$$

and $w = \max\{\tau, u(r_0)\}$.

Now we give the proof of Theorem 3.4.

**Proof of Theorem 3.4.** In Lemma 3.5 take

$$
u(r) = T_f(r), \quad p(r) = r \log r, \quad h(r) = \frac{\varphi(r)}{r},
$$

where

$$
\varphi(r) = r \log r (\log \log r)^{1+\varepsilon}
$$

with the $\varepsilon > 0$ given by the assumption of theorem. Then it is obvious that $\int_e^\infty \frac{dr}{p(r)} = \infty$ and $\int_r^\infty \frac{dr}{\varphi(r)} < \infty$ for $r \geq sp > 0$ and $r \geq \tau > 0$. Let

$$
\alpha := 1 + \frac{p(r + |c|)}{(r + |c|)h(T_f(r + |c|))} = 1 + \frac{\log(r + |c|)}{\log T_f(r + |c|)(\log \log T_f(r + |c|))^{1+\varepsilon}}.
$$

Note that

$$
T_f(\alpha(r + |c|)) = T_f \left( r + |c| + \frac{(r + |c|) \log(r + |c|)}{\log T_f(r + |c|)(\log \log T_f(r + |c|))^{1+\varepsilon}} \right).
$$

Applying Lemma 3.5 we have

$$
T_f(\alpha(r + |c|)) \leq CT_f(r + |c|)
$$
for all \( r \) possibly outside a set \( E \) satisfying
\[
\int_E \frac{dt}{p(t)} = \int_E \frac{dt}{t \log t} \leq \frac{1}{\log w(\log \log w)^{1+\varepsilon}} + \frac{C}{C-1} \int_w^{\infty} \frac{dt}{t \log t (\log \log t)^{1+\varepsilon}} < +\infty.
\]

By [14, Theorem 3.24], for all \( \alpha(>1) \) and all \( r > 0 \), we have
\[
m(r, f(x+c) \odot f(x)) \leq \frac{16|c|}{r+|c|} \frac{1}{\alpha-1} T_f(\alpha(r+|c|)) + \frac{|f(0)|}{2}.
\]

Thus we get from (5) that for the above defined \( \alpha \),
\[
m(r, f(x+c) \odot f(x))(6) \leq \frac{16|c|}{r+|c|} \frac{1}{\alpha-1} T_f(\alpha(r+|c|)) \leq \frac{16|c|}{r+|c|} T_f(r+|c|)
\]
\[
+ \frac{|f(0)|}{2}
\]
holds for \( r \notin E \).

Under the condition (4) (it implies \( \rho_2(f) \leq 1 \) according to Remark 3.2(i)), we get
\[
\frac{\log T_f(r+|c|)(\log(r+|c|))^\varepsilon}{r+|c|} = o(1) \ (r \to \infty),
\]
and
\[
\frac{\log \log T_f(r+|c|)}{\log(r+|c|)} = O(1) \ (r \to \infty).
\]

Furthermore, by [15], we also have
\[
T_f \left( r + \frac{r \log r}{\log T_f(r)(\log \log T_f(r))^{1+\varepsilon}} \right) \leq C T_f(r)
\]
for all \( r \notin E_c = \{ x : x - |c| \in E \} \). It follows also from the condition (4) that for all sufficiently large \( r \),
\[
\frac{\log T_f(r)(\log r)^\varepsilon}{r} < 2^{-1-\varepsilon}|c|^{-1}, \quad \frac{\log \log T_f(r)}{\log r} \leq 2.
\]

Hence, for all sufficiently large \( r \),
\[
\frac{r \log r}{\log T_f(r)(\log \log T_f(r))^{1+\varepsilon}} = \frac{r}{\log T_f(r)(\log r)^\varepsilon} \left( \frac{\log r}{\log \log T_f(r)} \right)^{1+\varepsilon} > |c|.
\]

Hence, we have
\[
T_f(r+|c|) \leq T_f \left( r + \frac{r \log r}{\log T_f(r)(\log \log T_f(r))^{1+\varepsilon}} \right) \leq C T_f(r)
\]
Let \( q \) and \( n \) be positive integers with \( q > n \), and let \( \epsilon > 0 \). Given \( n + 1 \) tropical entire functions \( g_0, \ldots, g_n \) without common roots, and linearly independent in Gondran-Minoux sense, let \( q + 1 \) tropical linear combinations \( f_0, \ldots, f_q \) of the \( g_j \) over the semiring \( \mathbb{R}_{\text{max}} \) be defined by

\[
f_k(x) = a_{0k} \odot g_0(x) \oplus a_{1k} \odot g_1(x) \oplus \cdots \oplus a_{nk} \odot g_n(x), 0 \leq k \leq q.
\]

Let \( \lambda = \text{ddg}((f_{n+1}, \ldots, f_q)) \) and

\[
L = \frac{f_0 \odot f_1 \odot \cdots \odot f_n \odot f_{n+1} \odot \cdots \odot f_q}{C_0(f_0, f_1, \ldots, f_n)} \oplus L.
\]

If the tropical holomorphic curve \( g \) of \( \mathbb{R} \) into \( \mathbb{T} \mathbb{P}^n \) with reduced representation \( g = (g_0, \ldots, g_n) \) is of hyper-order \( \rho_2(g) = \rho_2 < 1 \), then

\[
(q - n - \lambda) T_g(r) \leq N(r, 1 \odot L) - N(r, L) + o \left( \frac{T_g(r)}{r^{1 - \rho_2 - \epsilon}} \right),
\]

where \( r \) approaches infinity outside an exceptional set of finite logarithmic measure.

Consider a homogeneous tropical polynomial with degree \( d \) in \( n \) dimensional tropical projective space \( \mathbb{T} \mathbb{P}^n \) of the form

\[
P(x) = \bigoplus_{I_i \in J_d} c_{I_i} \odot x^{I_i}
\]

\[
= \bigoplus_{i_0 + i_1 + \cdots + i_n = d} c_{i_0, i_1, \ldots, i_n} \odot x_0^{\odot i_0} \odot x_1^{\odot i_1} \cdots \odot x_n^{\odot i_n},
\]

where \( J_d \) is the set of all \( I_i = (i_0, i_1, \ldots, i_n) \in \mathbb{N}^{n+1} \) with \( \#I_i = i_0 + i_1 + \cdots + i_n = d \). The (homogeneous) tropical hypersurface \( V_P \) in \( \mathbb{T} \mathbb{P}^n \) is the set of zeros (roots) \( x = (x_0, x_1, \ldots, x_n) \) of \( P(x) \), that is, the graph of \( P \) is nonlinear at these points (corner locus). In particular, \( V_P \) is called a tropical hyperplane whenever \( d = 1 \). For more general definition of tropical hypersurfaces associated to a tropical Laurent
polynomial, please see [20] Definition 3.6]. It is shown that $V_P$ is the set of points where more than one monomial of $P$ reaches its maximal value [20] Proposition 3.3).

Set $M := \binom{n+d}{d} - 1$. For any $I_i = (i_0, \ldots, i_n) \in J_d$, $i \in \{0, 1, \ldots, M\}$, denote $f^I_i := f_0^{\circ i_0} \circ \cdots \circ f_n^{\circ i_n}$. Then one can see that the composition function

$$P(f) := P \circ f = \bigoplus_{i_0+i_1+\cdots+i_n = d} = \bigoplus_{i=0}^M c_{i_0,i_1,\ldots,i_n} \circ f_0^{\circ i_0} \circ f_1^{\circ i_1} \cdots f_n^{\circ i_n}$$

for a tropical holomorphic curve $f := [f_0, \ldots, f_n] : \mathbb{R} \to \mathbb{P}^n$ and tropical hypersurface $V_P$ is a tropical algebraical combination of $f_0, \ldots, f_n$ in the Gondran-Minoux sense. From which, we may also regard $P \circ f$ as a tropical linear combination of $f^{I_0}, f^{I_1}, \ldots, f^{I_M}$ in the Gondran-Minoux sense. From this viewpoint, we introduce some definitions similarly as in Section 2.

**Definition 4.2.** Tropical meromorphic functions $f_0, \ldots, f_n$ are algebraically dependent (respectively independent) in the Gondran-Minoux sense if and only if $f^{I_0}, \ldots, f^{I_M}$ are linearly dependent (respectively independent) in the Gondran-Minoux sense.

**Definition 4.3.** Let $G = \{f_0, \ldots, f_n\}(\neq \{0\})$ be a set of tropical entire functions, algebraically independent in the Gondran-Minoux sense, and denote

$$\hat{\mathcal{L}}_G = \text{span} < f^{I_0}, \ldots, f^{I_M} > := \left\{ \bigoplus_{k=0}^M a_k \circ f^{I_k} : (a_0, \ldots, a_M) \in \mathbb{R}^{M+1} \right\}$$

to be their algebraic span. The collection $G$ is called the algebraic spanning basis of $\hat{\mathcal{L}}_G$. The dimension of $\hat{\mathcal{L}}_G$ is defined by

$$\dim(\hat{\mathcal{L}}_G) = \max\{\hat{\ell}(F) : F \in \hat{\mathcal{L}}_G \setminus \{0\}\}$$

where $\hat{\ell}(F)$ is the shortest length of the representation of $F \in \hat{\mathcal{L}}_G \setminus \{0\}$ defined by

$$\hat{\ell}(F) = \min\{j \in \{1, \ldots, M+1\} : F = \bigoplus_{i=1}^j a_{k_i} \circ f^{I_{k_i}} \}$$

where $a_{k_i} \in \mathbb{R}$ with integers $0 \leq k_1 < k_2 < \cdots < k_j \leq M$.

Note that usually the dimension of the tropical algebraic span space of $G$ may not be $M+1$, which is different from the classical linear algebraic. If $\hat{\ell}(F) = M + 1$ for a tropical algebraic combination $F$ of $f_0, \ldots, f_n$, then $F$ is said to be complete, that is, the coefficients $a_k$ in any expression of $F$ of the form $F = \bigoplus_{k=0}^M a_k \circ f_k$ must satisfy $a_k \in \mathbb{R}$ for all $k \in \{0, \ldots, M\}$ such that $\hat{\mathcal{L}}_G = M + 1$.

Furthermore, the tropical Casorati determinant $\hat{C}(f) = C(f^{I_0}, \ldots, f^{I_M})$ is given as

$$\hat{C}_\alpha(f) = C_\alpha(f^{I_0}, \ldots, f^{I_M}) = \bigoplus_{\pi(0)} f^{I_0}_{\pi(0)} \circ f^{I_1}_{\pi(1)} \circ \cdots \circ f^{I_M}_{\pi(M)}$$

where the sum is taken over all permutations $\{\pi(0), \ldots, \pi(M)\}$ of $\{0, 1, \ldots, M\}$. Clearly, when $d = 1$, we have $\hat{C}_\alpha(f) = C_\alpha(f)$.

Recall that in the classical algebraical geometry, if the image of a holomorphic curve $f = [f_0 : f_1 : \ldots : f_n] : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ cannot be contained in any hypersurface
(respectively hyperplane) in \( \mathbb{P}^n(\mathbb{C}) \), then say that \( f \) is algebraically (respectively linearly) nondegenerated that means that the entire functions \( f_0, f_1, \ldots, f_n \) are algebraically (respectively linearly) independently. We give a similar definition for tropical holomorphic curves.

**Definition 4.4.** Let \( f = [f_0 : f_1 : \ldots : f_n] : \mathbb{R} \to \mathbb{T}\mathbb{P}^n \) be a tropical holomorphic curve. If for any tropical hypersurface (respectively hyperplane) \( V_P \) in \( \mathbb{T}\mathbb{P}^n \) defined by a homogeneous tropical polynomial \( P \) in \( \mathbb{R}^{n+1} \), \( f(\mathbb{R}) \) is not a subset of \( V_P \), then we say that \( f \) is tropical algebraically (respectively linearly) nondegenerated.

Then we give a relationship between tropical algebraically nondegenerated and algebraically independently in Gondran-Minoux sense.

**Proposition 4.5.** A tropical holomorphic curve \( f : \mathbb{R} \to \mathbb{T}\mathbb{P}^n \) with reduced representation \( f = (f_0, \ldots, f_n) \) is tropical algebraically (respectively linearly) nondegenerated if and only if \( f_0, \ldots, f_n \) are algebraically (respectively linearly) independently in the Gondran-Minoux sense.

**Proof.** Assume that \( f = (f_0, \ldots, f_n) \) is tropical algebraically (respectively linearly) nondegenerated, this means that for any hypersurface (respectively hyperplane) \( V_P \) in \( \mathbb{T}\mathbb{P}^n \) defined by a homogeneous tropical polynomial \( P \) in \( \mathbb{R}^{n+1} \), \( f(\mathbb{R}) \not\subset V_P \). Now if \( f_0, \ldots, f_n \) are algebraically (respectively linearly) independently in the Gondran-Minoux sense, then there exist two nonempty disjoint subsets \( I \) and \( J \) of \( K := \{0, \ldots, M\} \) such that \( \bigcup \{a_i \circ f_0^{i_0} \circ \cdots \circ f_n^{i_n} \}_{i \in I} = \bigcup \{a_j \circ f_0^{j_0} \circ \cdots \circ f_n^{j_n} \}_{j \in J} \)

where \( M = \binom{n+d}{d} - 1 \), all \( a_i, a_j \in \mathbb{R}_{\text{max}} \) and \( i_0 + \ldots + i_n = j_0 + \ldots + j_n \in \mathcal{J}_d \). Hence, it gives a homogeneous tropical polynomial \( \tilde{P}(x) = \bigoplus_{k=0}^{M} a_k \circ x_0^{k_0} \circ \cdots \circ x_n^{k_n} \) with degree \( d = k_0 + \ldots + k_n \in \mathcal{J}_d \) such that

\[
\tilde{P}(f) = \bigoplus_{k=0}^{M} a_k \circ f_0^{k_0} \circ \cdots \circ f_n^{k_n} = \bigoplus_{i \in I} a_i \circ f_0^{i_0} \circ \cdots \circ f_n^{i_n} = \bigoplus_{j \in J} a_j \circ f_0^{j_0} \circ \cdots \circ f_n^{j_n}.
\]

This implies that \( (f_0(x), \ldots, f_n(x)) \) are points of \( V_P \) for all \( x \in \mathbb{R} \). We obtain a contradiction. Hence \( f_0, \ldots, f_n \) must be algebraically (respectively linearly) independently in the Gondran-Minoux sense.

Now assume that \( f_0, \ldots, f_n \) are algebraically (respectively linearly) independently in the Gondran-Minoux sense. If \( f \) is tropical algebraically (respectively linearly) degenerated, then there exists one hypersurface (respectively hyperplane) \( V_P \) in \( \mathbb{T}\mathbb{P}^n \) defined by a homogeneous tropical polynomial \( P(x) = \bigoplus_{k=0}^{M} a_k \circ x_0^{k_0} \circ \cdots \circ x_n^{k_n} \) with degree \( d = k_0 + \ldots + k_n \in \mathcal{J}_d \) in \( \mathbb{R}^{n+1} \), such that \( f(\mathbb{R}) \subset V_P \). This means that for all \( x \in \mathbb{R} \), \( (f_0(x), \ldots, f_n(x)) \) are roots of \( P(x) \). Since \( V_P \) consists of some lines (including segment lines and half lines) or located points at which the maximum is attained by two or more of the tropical monomials in \( P \). According to the continuity of the tropical curve \( f \), all image points \( (f_0(x), \ldots, f_n(x)) \) are located
in continuous piecewise lines or located points in $V_P$ on which the maximum can be taken at least twice in

$$P \circ f = \bigoplus_{k=0}^{M} a_k \odot f_0^{ \odot k_0} \odot \cdots \odot f_n^{ \odot k_n}.$$ 

Denote by $\{L_j\}_{j=1}^{m}$ the set of all the lines or located points of $f(\mathbb{R}) \subset V_P$. Then we know that for each $L_j$ ($j \in \{1, \ldots, m\}$), the maximum can be attained by two tropical monomials of $P \circ f$, say that, given by

$$a_{j^1} \odot f_0^{ \odot j_0^1} \odot \cdots \odot f_n^{ \odot j_n^1} = a_{j^2} \odot f_0^{ \odot j_0^2} \odot \cdots \odot f_n^{ \odot j_n^2}$$

where $\{j^1, j^2\} \subset \{0, 1, \ldots, M\}$, $j_0^1 + \ldots + j_n^1 = j_0^2 + \ldots + j_n^2 \in J_d$, both $a_{j^1}$ and $a_{j^2}$ are not equal to $0_o$. Hence We get that for all $x \in \mathbb{R}$,

$$\bigoplus_{i \in I} b_i \odot f_0^{ \odot i_0} \odot \cdots \odot f_n^{ \odot i_n} = \bigoplus_{j \in J} b_j \odot f_0^{ \odot j_0} \odot \cdots \odot f_n^{ \odot j_n}$$

where $I \cup J = \{0, 1, \ldots, M\}$, $I \cap J = \emptyset$ and

$$b_k = \begin{cases} a_{j^1}, & k = j^1 \in I; \\ a_{j^2}, & k = j^2 \in J; \\ 0_o, & \text{others.} \end{cases}$$

This contradicts the assumption that $f_0, \ldots, f_n$ are algebraically (respectively linearly) independently in the Gondran-Minoux sense. Hence $f$ is tropical algebraically (respectively linearly) nondegenerated. \hfill \Box

For giving a new tropical first main theorem, we introduce the tropical Weil function, tropical proximity function for a tropical holomorphic map $f$ with respect to the tropical hypersurface $V_P$ with degree $d$, similarly as in Classical Nevanlinna theory.

**Definition 4.6.** Let $f : \mathbb{R} \to \mathbb{T}P^n$ be a tropical holomorphic map, let $V_P$ be a tropical hypersurface with degree $d$ defined by a homogeneous polynomial $P$ of degree $d$ and let $a$ be the vector defined by the polynomial $P$. The tropical proximity function $m_f(r, V_P)$ of $f$ with respect to $V_P$ is defined as

$$m_f(r, V_P) := \frac{1}{2} \sum_{\sigma = \pm 1} \lambda_{V_P}(f(\sigma r)),$$

where $\lambda_{V_P}(f(\sigma r))$ means the tropical Weil function defined by

$$\lambda_{V_P}(f(x)) := \frac{\|f(x)\|^{\odot d} \odot \|a\|^{\odot d}}{P(f)(x)} \odot.$$

Note that $P(f)$ is a tropical holomorphic function on $\mathbb{R}$ which thus doesn’t have any pole. Hence by the tropical Jensen formula, we have

$$N(r, 1_o \odot P(f)) = \frac{1}{2} \sum_{\sigma = \pm 1} P(f)(\sigma r) - P(f)(0).$$

Now one can easily deduce the following first main theorem for tropical hypersurfaces by the definitions of tropical characteristic function, counting function and approximation function.
**Theorem 4.7. (First Main Theorem)** If \( f(\mathbb{R}) \not\subset V_P \), then

\[
m_f(r, V_P) + N(r, 1_o \odot P(f)) = dT_f(r) + O(1)
\]

In Section 6 we will discuss the difference of the counting function \( N(r, 1_o \odot P(f)) \) from the counting function \( N(r, \frac{1}{P(f)}) \) for a tropical meromorphic function intersecting a value \( a \) in \( \mathbb{T}^p \). Hence the new tropical first main theorem (Theorem 4.7) we obtained is different from the classical Nevanlinna theory. It is surprising that whenever \( \lambda = 0 \), all inequalities become equalities. This is very different from the classical Nevanlinna theory.

**Theorem 4.8. (Second Main Theorem with tropical hypersurfaces)** Let \( q \) and \( n \) be positive integers with \( q \geq n \). Let the tropical holomorphic curve \( f : \mathbb{R} \to \mathbb{T}^p \) be tropical algebraically nondegenerated. Assume that tropical hypersurfaces \( V_P \) are defined by homogeneous tropical polynomials \( P_j \) \( (j = 1, \ldots, q) \) with degree \( d_j \), respectively, and \( d \) are the least common number of \( d_1, \ldots, d_q \). Let \( M = \binom{n+d}{d} - 1 \). If \( \lambda = d \log(\{P_{M+2} \circ f, \ldots, P_q \circ f\}) \) and

\[
\limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0,
\]

then

\[
(q - M - 1 - \lambda)T_f(r) \\
\leq \sum_{j=1}^{q} \frac{1}{d_j} N(r, \frac{1_o}{P_j \circ f} \odot) - d N(r, \frac{\frac{1_o}{P_{M+1} \circ f}}{C_o(P_1 \circ f, \ldots, P_{M+1} \circ f)} \odot) + o(T_f(r))
\]

\[
= \sum_{j=M+2}^{q} \frac{1}{d_j} N(r, \frac{1_o}{P_j \circ f} \odot) + o(T_f(r))
\]

\[
\leq (q - M - 1)T_f(r) + o(T_f(r))
\]

where \( r \) approaches infinity outside an exceptional set of zero upper density measure.

In the special case whenever \( \lambda = 0 \),

\[
(q - M - 1)T_f(r) \\
= \sum_{j=1}^{q} \frac{1}{d_j} N(r, \frac{1_o}{P_j \circ f} \odot) - d N(r, \frac{\frac{1_o}{P_{M+1} \circ f}}{C_o(P_1 \circ f, \ldots, P_{M+1} \circ f)} \odot) + o(T_f(r))
\]

\[
= \sum_{j=M+2}^{q} \frac{1}{d_j} N(r, \frac{1_o}{P_j \circ f} \odot) + o(T_f(r))
\]

where \( r \) approaches infinity outside an exceptional set of zero upper density measure.

**Definition 4.9.** The defect of a tropical holomorphic curve \( f : \mathbb{R} \to \mathbb{T}^p \) intersecting a tropical hypersurface \( V_P \) given by a tropical polynomial \( P \) with degree \( d \) on \( \mathbb{R}^{n+1} \) is defined by

\[
\delta_f(V_P) := \liminf_{r \to \infty} \frac{m_f(r, V_P)}{dT_f(r)} = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{P \circ f} \odot)}{dT_f(r)}.
\]
Then by Theorem 4.8 we obtain immediately the following defect relation.

**Theorem 4.10.** *(Defect relation)* Let q and n be positive integers with $q \geq n$. Let the tropical holomorphic curve $f : \mathbb{R} \to \mathbb{T}^n$ be tropical algebraically nondegenerated. Assume that $P_j$ $(j = 1, \ldots, q)$ are homogeneous tropical polynomials with degree $d_j$, and $d$ are the least common number of $d_1, \ldots, d_q$. Let $M = \binom{n+d}{d} - 1$. If $\lambda = ddg(\{P_{M+2} \circ f, \ldots, P_q \circ f\})$ and

$$
\limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0,
$$

then

$$
\sum_{j=1}^{q} \delta_f(V_{P_j}) \leq M + 1 + \lambda, \quad \text{and} \quad \sum_{j=M+2}^{q} \delta_f(V_{P_j}) \leq \lambda.
$$

In special case whenever $\lambda = 0$, we get that $\delta_f(V_{P_j}) = 0$ for each $j \in \{M+2, \ldots, q\}$.

This theorem can be regarded as a tropical version of the Shiffman’s conjecture [26] on defect relation in the classical Nevanlinna theory which says that $\sum_{j=1}^{q} \delta_f(Q_j) \leq n + 1$ holds for a algebraically nondegenerated holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ and hypersurfaces $\{Q_j\}_{j=1}^{q}$ with the same degree $d$ located in general position in $\mathbb{P}^n(\mathbb{C})$. The Shiffman’s conjecture was completely proved by Ru [26].

For further study, we propose also the tropical version of the Griffith’s conjecture [6] (see also [26, 23, 25, 12, 11]) (that is, $\sum_{j=1}^{q} \delta_f(Q_j) \leq \frac{n+1}{d}$) as follows, which is partially proved by Biancofiore [11] for a class of holomorphic curves, by Siu [24] with $n + 1 = 3$ using meromorphic connections, and by Hu-Yang [12] solving a weaker form for a special class of holomorphic curves, respectively.

**Conjecture 4.11.** Let q and n be positive integers with $q \geq n$. Let the tropical holomorphic curve $f : \mathbb{R} \to \mathbb{T}^n$ be tropical algebraically nondegenerated. Assume that $P_j$ $(j = 1, \ldots, q)$ are homogeneous tropical polynomials with degree $d_j$, and $d$ are the least common number of $d_1, \ldots, d_q$. Let $M = \binom{n+d}{d} - 1$. If $\lambda = ddg(\{P_{M+2} \circ f, \ldots, P_q \circ f\})$ and $\limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0$, then

$$
\sum_{j=1}^{q} \delta_f(V_{P_j}) \leq \frac{n+1+\lambda}{d}.
$$

By Theorem 4.8 we get a new tropical version of Cartan-Nochka’s second main theorem as follows. Whenever $d = d_j = 1$, we have $M = \binom{n+d}{d} - 1 = n$; and from the case (i) in the proof of Theorem 4.8 in the next section we have

$$
N(r, 1_o \circ L) - N(r, L) = \sum_{j=1}^{q} N(r, 1_o \circ P_j \circ f) - N(r, 1_o \circ C_o(P_1 \circ f, \ldots, P_{n+1} \circ f)),
$$

where

$$
L := \frac{\left(\frac{1}{C_o(P_1 \circ f, P_2 \circ f, \ldots, P_{n+1} \circ f)} \right) \circ \left(\frac{1}{C_o(P_1 \circ f, P_2 \circ f, \ldots, P_{n+1} \circ f)} \right)}{C_o(P_1 \circ f, P_2 \circ f, \ldots, P_{n+1} \circ f)}.
$$

Hence Theorem 4.11 is just a special case for tropical hyperplanes, that is, $d = d_j = 1$ for all $j = 1, 2, \ldots, q$. Furthermore, we adopt the idea from the proof of [14, Corollary 7.2] and can deal with the ramification term $N(r, 1_o \circ C_o(P_1 \circ f, \ldots, P_{n+1} \circ f)) \circ$. From this, it might be possible to consider the truncated form of tropical Cartan second main theorem in future.
Corollary 4.12. (Second main theorem with tropical hyperplanes) Let $q$ and $n$ be positive integers with $q \geq n$. Let the tropical holomorphic curve $f : \mathbb{R} \to \mathbb{T} \mathbb{P}^n$ be tropical linearly nondegenerated. Assume that $V_{P_j}$ are tropical hyperplanes in $\mathbb{T} \mathbb{P}^n$ defined by homogeneous tropical polynomials $P_j$ ($j = 1, \ldots, q$) with degree $1$, respectively. If $\lambda = \text{ddg}(\{P_{n+2} \circ f, \ldots, P_q \circ f\})$ and

$$\limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0,$$

then

$$(q - n - 1 - \lambda)T_f(r) \leq N(r, 1_o \circ L) - N(r, L) + o(T_f(r))$$

$$= \sum_{j=1}^{q} N(r, \frac{1_o}{P_j \circ f} \circ) - N(r, \frac{1_o}{C_o(P_1 \circ f, \ldots, P_{n+1} \circ f) \circ} + o(T_f(r))$$

$$\leq \sum_{j=n+2}^{q} N(r, \frac{1_o}{P_j \circ f} \circ) + o(T_f(r))$$

$$\leq (q - n - 1)T_f(r) + o(T_f(r))$$

where $r$ approaches infinity outside an exceptional set of zero upper density measure.

In special case whenever $\lambda = 0$, we have

$$(q - n - 1)T_f(r) = N(r, 1_o \circ L) - N(r, L) + o(T_f(r))$$

$$= \sum_{j=1}^{q} N(r, \frac{1_o}{P_j \circ f} \circ) - N(r, \frac{1_o}{C_o(P_1 \circ f, \ldots, P_{n+1} \circ f) \circ} + o(T_f(r))$$

$$= \sum_{j=n+2}^{q} N(r, \frac{1_o}{P_j \circ f} \circ) + o(T_f(r))$$

where $r$ approaches infinity outside an exceptional set of zero upper density measure.

5. Proof of Theorem 4.8

(i). We first assume that $d_j = d$ holds for all $j = 1, \ldots, q$ and

$$P_j(x) = \bigoplus_{i_0, \ldots, i_m = d} a_{i_0, \ldots, i_m} \circ x_1 \otimes \cdots \otimes x_{n+1}$$

$$= \bigoplus_{k=0}^{M} a_{i_k} x_1^{i_k}.$$ 

Take $f^1 = f_0^{\otimes i_0} \circ \cdots \circ f_n^{\otimes i_n}$ ($i = 0, \ldots, M$) which are still tropical entire functions on $\mathbb{R}$. Since $f = (f_0, f_1, \ldots, f_n)$ is tropical algebraically nondegenerated, it follows from Proposition 4.5 and Definition 4.2 that $f = (f_0, \ldots, f_n)$ is algebraically independent in the Gondran-Minoux sense, and thus $F = (f_0^1, f_1^1, \ldots, f_n^1)$ is linearly independent in the Gondran-Minoux sense. Denote $g_{j-1} := P_j \circ f$ for all $j = 1, 2, \ldots, q$. By the properties of tropical Casorati determinant,

$$C_0(g_0, \ldots, g_M) = g_0 \circ g_0 \circ \cdots \circ g_0^{[M]} \circ C_0(1_o, g_1 \circ g_0, \ldots, g_M \circ g_0).$$
Take
\[ \tilde{L} := \frac{g_0 \circ \overline{g_1} \circ \cdots \circ \overline{g_{M}} \circ g_{M+1} \circ \cdots \circ g_{q-1}}{C_0 (g_0, g_1, \ldots, g_M)} \]
and
\[ \psi := g_{M+1} \circ \cdots \circ g_{q-1}, \]
then
\[ \psi = \tilde{L} \circ K, \]
where
\[ K = C_0 (1, g_1 \circ g_0, \ldots, g_M \circ g_0) \circ (\overline{g_0} \circ \overline{g_1} \circ \cdots \circ (\overline{g_0} \circ \overline{g_{M}}) \circ \overline{g_{M}}). \]

We denote the \( g_\nu, \ (M + 1 \leq \nu \leq q - 1), \) to be
\[ g_\nu := \bigoplus_{j \in S_\nu} t_{j\nu} \circ f^{I_{j\nu}} (x) = \max \{ t_{j\nu} + f^{I_{j\nu}} (x) \}, \ t_{j\nu} \in \mathbb{R}, \]
for index sets \( S_\nu \subset \{0, 1, \ldots, M\} \) with cardinality \( \#S_\nu \leq M + 1 \). Then we have

\[ \sum_{\nu=M+1, \#S_\nu=M+1}^{q-1} \left( \frac{1}{2} \sum_{\sigma=\pm 1} g_\nu (\sigma r) \right) \]
\[ = \sum_{\nu=M+1, \#S_\nu=M+1}^{q-1} \frac{1}{2} \left( \max_{j \in S_\nu} \{ t_{j\nu} + f^{I_{j\nu}} (r) \} + \max_{j \in S_\nu} \{ t_{j\nu} + f^{I_{j\nu}} (-r) \} \right) \]
\[ \geq \sum_{\nu=M+1, \#S_\nu=M+1}^{q-1} \frac{1}{2} \left( \max_{j \in S_\nu} \{ f^{I_{j\nu}} (r) \} + \max_{j \in S_\nu} \{ f^{I_{j\nu}} (-r) \} \right) \]
\[ \quad + \sum_{\nu=M+1, \#S_\nu=M+1}^{q-1} \min_{j \in S_\nu} \{ t_{j\nu} \} \]
\[ = \sum_{\nu=M+1, \#S_\nu=M+1}^{q-1} \frac{1}{2} \left( \max_{j_0+\cdots+j_n=d} (j_{0\nu} f_0 (r) + \cdots + j_{n\nu} f_n (r)) \right) \]
\[ + \max_{j_0+\cdots+j_n=d} \{ j_{0\nu} f_0 (-r) + \cdots + j_{n\nu} f_n (-r) \} \]
\[ + \sum_{\nu=M+1, \#S_\nu=M+1}^{q-1} \min_{j \in S_\nu} \{ t_{j\nu} \}. \]

The condition \( \lambda = ddg(f_{M+2} \circ f, \ldots, P_q \circ f) \) which is the number of its non-complete elements means that there exist \( q - M - 1 - \lambda \) complete elements in the set \( \{ f_{M+2} \circ f, \ldots, P_q \circ f \} \). Since \( g_\nu, \ (M + 1 \leq \nu \leq q - 1) \) are piecewise linear entire functions on \( \mathbb{R} \), there exist \( \alpha_\nu, \beta_\nu \in \mathbb{R} \) and an interval \( [r_1, r_2] \subset \mathbb{R} \) containing the origin such that \( r_1 < r_2 \) and
\[ g_\nu (x) = \alpha_\nu x + \beta_\nu \]
for all $x \in [r_1, r_2]$. Then we get that
\[ g_\nu(0) = \beta_\nu = \max_{j \in S_\nu} \{ t_{j\nu} + f^{t_{j\nu}}(0) \}. \]

If define
\[ h_\nu(x) := \alpha_\nu(x) + \beta_\nu \]
for all $x \in \mathbb{R}$, then by the convexity of the graph of $g_\nu$ we get that
\[ g_\nu(x) \geq h_\nu(x) \]
for all $x \in \mathbb{R}$. Hence
\[ \frac{1}{2} \sum_{\sigma = \pm 1} g_\nu(\sigma x) \geq \frac{1}{2} \sum_{\sigma = \pm 1} h_\nu(\sigma x) = \beta_\nu. \]

This gives that
\begin{equation}
(11) \quad \frac{1}{2} \sum_{\sigma = \pm 1} \psi(\sigma r) = \sum_{\nu = M+1}^{q-1} \left( \frac{1}{2} \sum_{\sigma = \pm 1} g_\nu(\sigma r) \right)
= \sum_{\nu = M+1, \# S_\nu = M+1}^{q-1} \left( \frac{1}{2} \sum_{\sigma = \pm 1} g_\nu(\sigma r) \right) + \sum_{\nu = M+1, \# S_\nu < M+1}^{q-1} \left( \frac{1}{2} \sum_{\sigma = \pm 1} g_\nu(\sigma r) \right)
\geq \sum_{\nu = M+1, \# S_\nu = M+1}^{q-1} \left( \frac{1}{2} \sum_{\sigma = \pm 1} g_\nu(\sigma r) \right) + \sum_{\nu = M+1, \# S_\nu < M+1}^{q-1} \beta_\nu.
\end{equation}

According to the definition of tropical Cartan characteristic function,
\[ T_f(r) + \max_{j=0,1,\ldots,n} \{ f_j(0) \} = \frac{1}{2} \max \{ f_0(r), \ldots, f_n(r) \} + \frac{1}{2} \max \{ f_0(-r), \ldots, f_n(-r) \}. \]

Then we get from (10) that
\begin{equation}
(12) \quad \sum_{\nu = M+1, \# S_\nu = M+1}^{q-1} \left( \frac{1}{2} \sum_{\sigma = \pm 1} g_\nu(\sigma r) \right)
\geq \sum_{\nu = M+1, \# S_\nu = M+1}^{q-1} d \left( T_f(r) + \max_{j=0}^{n} \{ f_j(0) \} \right) + \sum_{\nu = M+1, \# S_\nu = M+1}^{q-1} \min_{j \in S_\nu} \{ t_{j\nu} \}
= (q - M - 1 - \lambda) d \left( T_f(r) + \max_{j=0}^{n} \{ f_j(0) \} \right) + \sum_{\nu = M+1, \# S_\nu = M+1}^{q-1} \min_{j \in S_\nu} \{ t_{j\nu} \}.
\end{equation}

Therefore, it follows from (11) and (12) that
\[ \frac{1}{2} \sum_{\sigma = \pm 1} \psi_\nu(\sigma r) \geq (q - M - 1 - \lambda) d \left( T_f(r) + \max_{j=0}^{n} \{ f_j(0) \} \right)
+ \sum_{\nu = M+1, \# S_\nu = M+1}^{q-1} \min_{j \in S_\nu} \{ t_{j\nu} \} + \sum_{\nu = M+1, \# S_\nu < M+1}^{q-1} \beta_\nu. \]
This gives an inequality of characteristic function $T_f(r)$ as follows

\[
(q - M - 1 - \lambda)T_f(r) \leq \frac{1}{d} \left( \frac{1}{2} \sum_{\sigma = \pm 1} \psi(\sigma r) \right) + \frac{1}{d} \sum_{\nu = M+1, \# S_\nu = M+1}^{q-1} \min_{j \in S_\nu} \{ t_{j\nu} \} \\
+ \frac{1}{d} \sum_{\nu = M+1, \# S_\nu < M+1}^{q-1} \beta_\nu + (q - M - 1 - \lambda) \max_{j=0}^{n} \{ f_j(0) \}.
\]

Next we need obtain an estimation on the first term of the right side of the above inequality. By the tropical Jensen’s theorem and the definition of $\psi$, we deduce

\[
\frac{1}{2} \sum_{\sigma = \pm 1} \psi(\sigma r) \\
= \frac{1}{2} \sum_{\sigma = \pm 1} \hat{L}(\sigma r) + \frac{1}{2} \sum_{\sigma = \pm 1} K(\sigma r) \\
= N(r, 1_o \odot \hat{L}) - N(r, \hat{L}) + \hat{L}(0) + \frac{1}{2} \sum_{\sigma = \pm 1} K^+(\sigma r) - \frac{1}{2} \sum_{\sigma = \pm 1} (-K)^+(\sigma r) \\
= N(r, 1_o \odot \hat{L}) - N(r, \hat{L}) + \hat{L}(0) + m(r, K) - m(r, 1_o \odot K) \\
\leq N(r, 1_o \odot \hat{L}) - N(r, \hat{L}) + \hat{L}(0) + m(r, K).
\]

Denote

\[
L = \frac{g_0 \odot g_1 \odot \cdots \odot g_M \odot \cdots \odot g_{q-1}}{C_o(g_0, g_1, \ldots, g_M)} \odot,
\]

which gives

\[
\hat{L} = L \odot \frac{g_1^2 \odot g_2^2 \cdots \odot g_M^M}{g_1 \odot g_2 \odot \cdots \odot g_M} \odot.
\]

Then the above inequalities give

\[
(14) \quad \frac{1}{2} \sum_{\sigma = \pm 1} \psi(\sigma r) \\
\leq N(r, 1_o \odot \hat{L}) - N(r, \hat{L}) + L(0) + \sum_{j=0}^{M} g_j(j) - \sum_{j=0}^{M} g_j(0) + m(r, K).
\]
We will estimate $m(r, K)$ below. Since $g_j \otimes g_0$ are tropical meromorphic functions all $j \in \{1, 2, \ldots, M\}$, we have

$$T_{g_j \otimes g_0}(r) = \frac{1}{2} \sum_{\sigma = \pm 1} \max \{g_j(\sigma r), g_0(\sigma r)\} - \max \{g_j(0), g_0(0)\}$$

$$= \frac{1}{2} \sum_{\sigma = \pm 1} \max \{\{c_{j_0, \ldots, j_n} + j_0 f_0(\sigma r) + \ldots + j_n f_n(\sigma r)\} - \max \{g_j(0), g_0(0)\}\}$$

$$= \frac{1}{2} \sum_{\sigma = \pm 1} \max_{j_0 + \ldots + j_n = d} \{j_0 f_0(\sigma) + \ldots + j_n f_n(\sigma)\} - \max \{g_j(0), g_0(0)\} + \max \{c_{j_0, \ldots, j_n}\}$$

$$\le \frac{1}{2} \sum_{\sigma = \pm 1} d \max \{f_0(\sigma), f_1(\sigma), \ldots, f_n(\sigma)\} - \max \{g_j(0), g_0(0)\} + \max \{c_{j_0, \ldots, j_n}\}$$

$$\le d T_f(r) + d \max \{f_0(0), f_1(0), \ldots, f_n(0)\} - \max \{g_j(0), g_0(0)\} + \max \{c_{j_0, \ldots, j_n}\}.$$

This implies that

$$\limsup_{r \to \infty} \frac{\log T_{g_j \otimes g_0}(r)}{r} \le \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0,$$

and then by Lemma 3.3 we get that for any $k \in \mathbb{N}$,

$$T_{g_j \otimes g_0}(r) = (1 + \varepsilon(r)) T_{g_j \otimes g_0}(r) = T_f(r) + o(T_f(r))$$

holds for all $r \notin E$ with $\text{dens} E = 0$ (Throughout this proof, $E$ always means having the property $\text{dens} E = 0$). Therefore, for any $k \in \mathbb{N}$

$$\limsup_{r \to \infty} \frac{\log T_{g_j \otimes g_0}(r)}{r} \le \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0.$$

Note that

$$K = C_0(1, 2, g_1 \otimes g_0, \ldots, g_M \otimes g_0) \circ (\overline{g_1} \otimes \overline{g_0}) \circ \cdots \circ (\overline{g_M} \otimes \overline{g_0})$$

$$= \bigoplus \left( \frac{(g_1 \otimes g_0)^{\pi(0)}}{g_1 \otimes g_0} \right) \circ \cdots \circ \left( \frac{(g_M \otimes g_0)^{\pi(M)}}{g_M \otimes g_0} \right)$$

where the tropical sum is taken over all permutations $\{\pi(0), \ldots, \pi(M)\}$ of the set $\{0, 1, \ldots, M\}$. Now by Theorem 5.1 we deduce that

$$m(r, K) = o(T_f(r))$$

holds for all $r \notin E$ with $\text{dens} E = 0$.

Therefore, it follows from (13), (14) and (15) that

$$\limsup_{r \to \infty} \frac{\log T_f(r)}{r} \le \limsup_{r \to \infty} \frac{\log T_{g_j \otimes g_0}(r)}{r} = 0.$$
for all $r \notin E$ with $\overline{\text{dens}E} = 0$.

The next step is to estimate $N(r, 1_o \odot \tilde{L})$ and $N(r, \tilde{L})$. Note that

$$\tilde{L} = L \odot \frac{g_1 \odot g_2 \odot \cdots \odot g_M}{g_1 \odot g_2 \odot \cdots \odot g_M}$$

and that $g_1, \ldots, g_M$ are tropical entire functions. Then by the tropical Jensen formula,

\begin{equation}
N(r, 1_o \odot \tilde{L}) - N(r, \tilde{L}) = \frac{1}{2} \sum_{\sigma = \pm 1} \tilde{L}(\sigma r) - \tilde{L}(0)
\end{equation}

\begin{align*}
&= \frac{1}{2} \sum_{\sigma = \pm 1} L(\sigma r) - L(0) + \sum_{j=1}^{M} \frac{1}{2} \left( \sum_{\sigma = \pm 1} g^{[j]}_{\sigma}(\sigma r) + g^{[j]}_{\sigma}(0) \right) \\
&\quad - \sum_{j=1}^{M} \frac{1}{2} \left( \sum_{\sigma = \pm 1} g_j(\sigma r) + g_j(0) \right) \\
&= N(r, 1_o \odot L) - N(r, L) + \sum_{j=1}^{M} \left( N(r, 1_o \odot g^{[j]}_j) - N(r, 1_o \odot g_j) \right) \\
&\leq N(r, 1_o \odot L) - N(r, L) + \sum_{j=1}^{M} \left( N(r + jc, 1_o \odot g_j) - N(r, 1_o \odot g_j) \right),
\end{align*}

where the last inequality is deduced from the geometric meaning between $N(r, 1_o \odot g^{[j]}_j)$ and $N(r + jc, 1_o \odot g_j)$. Using the tropical Jensen formula again, we deduce that

\begin{align*}
N(r, 1_o \odot g_j) &= \frac{1}{2} \sum_{\sigma = \pm 1} g_j(\sigma r) - g_j(0) \\
&= \frac{1}{2} \sum_{\sigma = \pm 1} \max_{j_0 + \cdots + j_n = 0} \{ c_{j_0, \ldots, j_n} \sigma \} + f_0(\sigma r) + \cdots + f_n(\sigma r) - g_j(0) \\
&= \frac{1}{2} \sum_{\sigma = \pm 1} \max_{j_0 + \cdots + j_n = 0} \{ c_{j_0, \ldots, j_n} \sigma \} + f_0(\sigma r) + \cdots + f_n(\sigma r) \\
&\quad - g_j(0) + \max_{j_0 + \cdots + j_n = 0} \{ c_{j_0, \ldots, j_n} \sigma \} \\
&\leq \frac{1}{2} \sum_{\sigma = \pm 1} d \max\{ f_0(\sigma r), f_1(\sigma r), \ldots, f_n(\sigma r) \} \\
&\quad - g_j(0) + \max_{j_0 + \cdots + j_n = 0} \{ c_{j_0, \ldots, j_n} \sigma \} \\
&\leq dT_f(r) + d \max\{ f_0(0), f_1(0), \ldots, f_n(0) \} - g_j(0) \\
&\quad + \max_{j_0 + \cdots + j_n = 0} \{ c_{j_0, \ldots, j_n} \sigma \},
\end{align*}

This implies

$$\limsup_{r \to \infty} \frac{\log N(r, 1_o \odot g_j)}{r} \leq \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0.$$
Hence by Lemma 3.3

\[ N(r + j, 1_o \odot g_j) - N(r, 1_o \odot g_j) = \varepsilon(r)N(r, 1_o \odot g_j) = o(T_f(r)) \]

holds for \( r \not\in E \) with \( \text{dens} E = 0 \). Therefore we get from (17) and (18) that

\[ N(r, 1_o \odot \tilde{L}) - N(r, \tilde{L}) \leq N(r, 1_o \odot L) - N(r, L) + o(T_f(r)) \]

holds for \( r \not\in E \) with \( \text{dens} E = 0 \). Combining this with (19) gives

\[ (q - M - 1 - \lambda)T_f(r) \leq \frac{1}{d}N(r, 1_o \odot L) - \frac{1}{d}N(r, L) + o(T_f(r)). \]

for all \( r \not\in E \) with \( \text{dens} E = 0 \).

Note that \( g_j \) (\( j = 0, \ldots, q-1 \)) and \( C_o(g_0, \ldots, g_M) \) are all tropical entire functions. Then according to the definition of \( L \), we can get from the tropical Jensen formula that

\[ N(r, 1_o \odot L) - N(r, L) = \frac{1}{2} \sum_{\sigma = \pm 1} L(\sigma r) - L(0) \]

\[ = \frac{1}{2} \sum_{j=0}^{q-1} \left( \frac{1}{2} \sum_{\sigma = \pm 1} g_j(\sigma r) - g_j(0) \right) \]

\[ - \left( \frac{1}{2} \sum_{\sigma = \pm 1} C_o(g_0, \ldots, g_M)(\sigma r) - C_o(g_0, \ldots, g_M)(0) \right) \]

\[ = \frac{1}{2} \sum_{j=0}^{q-1} N(r, 1_o \odot g_j) - N(r, 1_o \odot C_o(g_0, \ldots, g_M)). \]

Now combining (19) and (20), we get the form of the second main theorem that

\[ (q - M - 1 - \lambda)T_f(r) \leq \frac{1}{d} \sum_{j=0}^{q-1} N(r, 1_o \odot g_j) - \frac{1}{d}N(r, 1_o \odot C_o(g_0, \ldots, g_M)) + o(T_f(r)). \]

for all \( r \not\in E \) with \( \text{dens} E = 0 \).

Now we estimate \( N(r, 1_o \odot C_o(g_0, \ldots, g_M)) \). According to the definition of tropical Casorati determinant, we have

\[ C_o(g_0, \ldots, g_M) \]

\[ = \bigoplus \left[ g_0^{[\pi(0)]} \odot \cdots \odot g_M^{[\pi(M)]} \right] \]

\[ = \left\{ \bigoplus \left[ \left( g_0^{[\pi(0)]} \odot \cdots \odot g_M^{[\pi(M)]} \right) \odot (g_0 \odot \cdots \odot g_M) \right] \right\} + g_0 \odot \cdots \odot g_M \]

\[ = \bigoplus \left[ \left( g_0^{[\pi(0)]} \odot g_0 \right) \odot \cdots \odot \left( g_M^{[\pi(M)]} \odot g_M \right) + g_0 \odot \cdots \odot g_M, \right] \]

where the sum is taken over all permutations \( \{\pi(0), \ldots, \pi(M)\} \) of \( \{0, 1, \ldots, M\} \). If denote

\[ D = \bigoplus \left[ \left( g_0^{[\pi(0)]} \odot g_0 \right) \odot \cdots \odot \left( g_M^{[\pi(M)]} \odot g_M \right) \right], \]

then

\[ D \geq \left( g_0^{[\pi(0)]} \odot g_0 \right) \odot \cdots \odot \left( g_M^{[\pi(M)]} \odot g_M \right) \]
for any permutation \(\{\pi(0), \ldots, \pi(M)\}\) of \(\{0, 1, \ldots, M\}\). By the tropical Jensen formula and (18),

\[
\frac{1}{2} \sum_{\sigma = \pm 1} D(\sigma r) - D(0) \geq \sum_{j=0}^{M} \left( \frac{1}{2} \sum_{\sigma = \pm 1} g_j^{\pi(j)}(\sigma r) - \frac{1}{2} \sum_{\sigma = \pm 1} g_j(\sigma r) \right) - D(0)
\]

holds for \(r \not\in E\) with \(\overline{\text{dens}E} = 0\). Hence using the tropical Jensen formula again, it gives by (22) and (23) that

\[
N(r, 1_o \odot (g_0, \ldots, g_M)) = \frac{1}{2} \sum_{\sigma = \pm 1} C_o(g_0, \ldots, g_M)(\sigma r) - C_o(g_0, \ldots, g_M)(0)
\]

\[
= \frac{1}{2} \sum_{\sigma = \pm 1} D(\sigma r) - D(0) + \sum_{j=0}^{M} \left( \frac{1}{2} \sum_{\sigma = \pm 1} g_j(\sigma r) - g_j(0) \right)
\]

\[
= \frac{1}{2} \sum_{\sigma = \pm 1} D(\sigma r) - D(0) + \sum_{j=0}^{M} N(r, 1_o \odot g_j)
\]

\[
\geq o(T_f(r)) + \sum_{j=1}^{M+1} N(r, \frac{1_o}{P_j \circ f})
\]

holds for \(r \not\in E\) with \(\overline{\text{dens}E} = 0\). Submitting (24) into (21) gives that

\[
(q - M - 1 - \lambda)T_f(r) \leq \frac{1}{d} \sum_{j=0}^{q-1} N(r, 1_o \odot g_j) - \frac{1}{d} N(r, 1_o \odot C_o(g_0, \ldots, g_M)) + o(T_f(r))
\]

\[
\leq \frac{1}{d} \sum_{j=M+2}^{q} N(r, \frac{1_o}{P_j \circ f}) + o(T_f(r))
\]

where \(r\) approaches infinity outside an exceptional set of zero upper density measure.
(ii). We now consider general case whenever the degree of homogeneous polynomials $P_j$ ($j = 1, \ldots, q$) are $d_j$ respectively. Assume that

$$P_j(x) = \bigoplus_{i_{j1}, \ldots, i_{jn}=d_j} a_{i_{j1}, \ldots, i_{jn}} x_1^{i_{j1} \circ} \cdots \circ x_n^{i_{jn} \circ}$$

$$= \max_{i_{j1}, \ldots, i_{jn}=d_j} \{ a_{i_{j1}, \ldots, i_{jn}} + i_{j1}x_1 + \cdots + i_{jn}x_n+1 \}.$$

Then

$$P_j^{\circ \frac{df}{dx}}(x) = \frac{d}{d_j} P_j(x)$$

$$= \frac{d}{d_j} \bigoplus_{i_{j1}, \ldots, i_{jn}=d_j} a_{i_{j1}, \ldots, i_{jn}} x_1^{i_{j1} \circ} \cdots \circ x_n^{i_{jn} \circ}$$

$$= \max_{i_{j1}, \ldots, i_{jn}=d_j} \{ \frac{d}{d_j} a_{i_{j1}, \ldots, i_{jn}} + \frac{d}{d_j} i_{j1}x_1 + \cdots + \frac{d}{d_j} i_{jn}x_n+1 \}$$

$$= \frac{d}{d_j} \sum_{i_{j1}, \ldots, i_{jn}=d_j} \bigoplus_x \bigoplus_{o=1}^n \{ \frac{d}{d_j} a_{i_{j1}, \ldots, i_{jn}} \circ x_1^{i_{j1} \circ} \cdots \circ x_n^{i_{jn} \circ} \}.$$

Thus all $P_j^{\circ \frac{df}{dx}}(x)$ are of degree $d$. Furthermore, we can see that if $x_0$ is a root of $P_j \circ f$ with multiplicity $\omega_{P_j \circ f}(x_0) > 0$, then $x_0$ should be also a root of $P_j^{\circ \frac{df}{dx}} \circ f$ ($= \frac{d}{d_j} P_j \circ f$) with multiplicity $\omega = \frac{d}{d_j} \omega_{P_j \circ f}(x_0) > 0$. The inverse is also true. This implies that

$$N(r, \frac{1}{P_j^{\circ \frac{df}{dx}} \circ f} \circ) = \frac{d}{d_j} N(r, \frac{1}{P_j \circ f} \circ).$$

Hence by the conclusion (i), we have

$$(q - M - 1 - \lambda) T_f(r)$$

$$\leq \frac{1}{d} \sum_{j=1}^q N(r, \frac{1}{P_j^{\circ \frac{df}{dx}} \circ f} \circ) \circ - \frac{1}{d} N(r, \frac{1}{C_o(P_1^{\circ \frac{df}{dx}} \circ f, \ldots, P_{M+1}^{\circ \frac{df}{dx}} \circ f)} \circ) + o(T_f(r))$$

$$\leq \frac{1}{d} \sum_{j=M+2}^q N(r, \frac{1}{P_j^{\circ \frac{df}{dx}} \circ f} \circ) + o(T_f(r))$$

$$= \sum_{j=M+2}^q \frac{1}{d_j} N(r, \frac{1}{P_j \circ f} \circ) + o(T_f(r))$$

where $r$ approaches infinity outside an exceptional set of finite upper density measure. By the first main theorem (Theorem 4.7) we have

$$N(r, \frac{1}{P_j \circ f} \circ) \leq d_j T_f(r)$$

for all $j = 1, \ldots, q$. Therefore, the theorem is proved immediately.
6. Tropical Nevanlinna second main theorem and defect relation

In this section we mainly give new versions of tropical Nevanlinna second main theorem which are very different from before and then obtain the defect relation. Recall that the first version of second main theorem for tropical meromorphic functions was obtained by Laine and Tohge [17], which is an analogue of the Nevanlinna second main theorem for meromorphic functions on the complex plane in the Classical Nevanlinna theory [21]. Note that the equality (27) below was written as an inequality form in the original statement [17], and holds from the fact that $N(r, 1_o \odot (f \oplus a_j)) \leq T(r, f) + O(1)$ for all $j \in \{1, \ldots, q\}$ according to the first main theorem (23).

**Theorem 6.1.** [17] If $f$ is a nonconstant tropical meromorphic function of hyper-order $\rho_2(f) < 1$, if $\varepsilon > 0$, and $q (\geq 1)$ distinct values $a_1, \ldots, a_q \in \mathbb{R}$ satisfying

$$\max\{a_1, \ldots, a_q\} < \inf\{f(\alpha) : \omega_f(\alpha) < 0\},$$

and

$$\inf\{f(\beta) : \omega_f(\beta) > 0\} > -\infty.$$  \hspace{1cm} (25)

Then

$$qT(r, f) = \sum_{j=1}^{q} N(r, 1_o \odot (f \oplus a_j)) + o\left(\frac{T(r, f)}{r^{1-\rho_2(f)-\varepsilon}}\right)$$

for all $r$ outside of an exceptional set of finite logarithmic measure.

Recently, Korhonen and Tohge [15] Corollary 7.2] said that from Theorem 4.4 they improved Theorem 6.1 by dropping the assumption (26). However, we find that there exists one gap. In [15 Page 722], it was said that from the assumption (25) (that is, (7.4) in [15]) it follows that “the roots of $g_0$ are exactly the poles of $a_k \oplus f$, counting multiplicity, for all $k = 1, 2, \ldots, q$.” This is very true! But they continued to assert that by $f_k = (a_{k-1} \odot g_0) \oplus (1_o \odot g_1)$,

$$N(r, 1_o \odot f_k) = N(r, 1_o \odot (f \oplus a_{k-1}))$$

for all $k = 2, \ldots, q+1$. In fact, this is not true! Firstly, $N(r, 1_o \odot (f \oplus a_{k-1}))$ means the roots of $f \oplus a_{k-1}$ but not poles of $f \oplus a_{k-1}$; secondly, the roots of $g_0$ (which is the poles of $f = g_1 \odot g_0$ as in the statement in [15]) is not equal to the roots of $f_k = (a_{k-1} \odot g_0) \oplus (1_o \odot g_1)$, and hence it is not true of $N(r, 1_o \odot g_0) = N(r, 1_o \odot f_k)$. Therefore, it is not reasonable enough to assert that $N(r, 1_o \odot f_k) = N(r, 1_o \odot (f \oplus a_{k-1}))$. Below we will consider the relationship of the two counting functions to explain this.

Let $f = [f_0 : f_1] : \mathbb{R} \to \mathbb{T}\mathbb{P}^1$ be a tropical nonconstant meromorphic function, and let $a = [a_1 : a_0]$ be a value of $\mathbb{T}\mathbb{P}^1$ which defining a tropical polynomial $P(x) = (a_0 \odot x_0) \oplus (a_1 \odot x_1)$ on $\mathbb{R}^2$. Then the polynomial $P(x)$ gives a tropical hyperplane $V_P = \{a\}$. We can see that

$$P \circ f(x) = (a_0 \odot f_0(x)) \oplus (a_1 \odot f_1(x))$$

$$= ((a_0 - a_1) \odot (f_1(x) - f_0(x))) + (a_1 + f_0(x))$$

$$= (a \odot f(x)) \oplus (a_1 + f_0(x)).$$
In special case whenever 

\[ N(r, 1_0 \odot (P \circ f)) \leq N(r, 1_0 \odot (f \oplus a) + N(r, 1_0 \odot (a_1 + f_0)) \]

\[ = N(r, 1_0 \odot (f \oplus a)) + N(r, f). \]

If discussing some special cases, then it is easy to get the following proposition.

**Proposition 6.2.** (I). If \( f \oplus a = f \) (for example, \( a = -\infty = [1_0 : 0_o] \)), and thus \( P \circ f(x) = f_1(x) + a_1 \), then in this case we have

\[ N(r, 1_0 \odot (P \circ f)) = N(r, 1_0 \odot (f_1 + a_1)) = N(r, 1_0 \odot (f \oplus a)). \]

(II). If \( f \oplus a = a \) (for example, \( a = +\infty = [0_o : 1_o] \)), and thus \( P \circ f(x) = a + a_1 + f_0(x) \), then in this case we have \( N(r, 1_0 \odot (P \circ f)) = N(r, 1_0 \odot (a + a_1 + f_0)) = N(r, f) \) and \( N(r, 1_0 \odot (f \oplus a)) = 0 \).

By Corollary [4.12] (or Theorem [4.8]), we get directly a new tropical version of Nevanlinna’s second main theorem.

**Theorem 6.3.** Assume that \( f = [f_0 : f_1] : R \to TP^1 \) is a nonconstant tropical meromorphic function with

\[ \lim_{r \to \infty} \sup \frac{\log T_f(r)}{r} = 0, \]

and \( a_j = [a_j : a_j] \) \((j = 1, \ldots, q)\) are distinct values of \( TP^1 \) which defining tropical polynomials \( P_j(x) = a_j \circ x_o \oplus a_j \odot x_1 \) on \( R^2 \), respectively. If \( ddg(P_1, \ldots, P_q \circ f) = \lambda \), then

\[ (q - 2 - \lambda) T_f(r) \]

\[ \leq \sum_{j=1}^{q} N \left( r, \frac{1_o}{P_j \odot f} \odot \right) - N \left( r, \frac{1_o}{C_0(P_1 \circ f, P_2 \circ f) \odot} \right) + o(T_f(r)) \]

\[ \leq \sum_{j=3}^{q} N \left( r, \frac{1_o}{P_j \odot f} \odot \right) + o(T_f(r)) \]

\[ \leq (q - 2) T_f(r) + o(T_f(r)), \]

where \( r \) approaches infinity outside an exceptional set of zero upper density measure. In special case whenever \( \lambda = 0 \),

\[ (q - 2) T_f(r) \]

\[ = \sum_{j=1}^{q} N \left( r, \frac{1_o}{P_j \odot f} \odot \right) - N \left( r, \frac{1_o}{C_0(P_1 \circ f, P_2 \circ f) \odot} \right) + o(T_f(r)) \]

\[ = \sum_{j=3}^{q} N \left( r, \frac{1_o}{P_j \odot f} \odot \right) + o(T_f(r)), \]

where \( r \) approaches infinity outside an exceptional set of zero upper density measure.

**Proof.** Due to \( f \) tropical meromorphic on \( R \), we may assume \( f = f_1 \circ f_0 = [f_0 : f_1] : R \to TP^1 \). According to Corollary [4.12] we only need prove that \( f \) is linearly independent in Gondran-Minoux sense (or say, tropical linearly nondegenerated).
Proposition 6.4. Assume that $f$ is a nonconstant tropical meromorphic function with and $a_j = [a_{j1} : a_{j0}]$ (for $j = 1,\ldots,q$) are distinct values of $\mathbb{R}$ defining tropical polynomials $P_j(x) = a_{j0} \odot x_0 \oplus a_{j1} \odot x_1$ on $\mathbb{R}^2$, respectively. If $f \not\equiv (f \oplus a_j) \not\equiv a_j$ for all $j = 1, 2,\ldots,q$, then $r := \Delta g([P_1 \circ f,\ldots,P_q \circ f]) = 0$.

Proof. Assume $\lambda > 0$. Then there exists at least one of $\{P_1 \circ f,\ldots,P_q \circ f\}$, say $P_k \circ f$, satisfying $\ell(P_k \circ f) < 2$ by the definition of the degree of degeneracy. This means $P_k \circ f, f \equiv a_{k0} \odot f_0$ or $P_k \circ f \equiv a_{k1} \odot f_1$. Thus we have either $(a_{k0} + f_0) \oplus (a_{k1} + f_1) \equiv a_{k0} \oplus f_0$ or $(a_{k0} + f_0) \oplus (a_{k1} + f_1) \equiv a_{k1} \oplus f_1$, which contradicts either $f \oplus a_k \not\equiv a_k$ or $f \oplus a_k \not\equiv f$ respectively. □

Then we obtain a tropical Nevanlinna theorem which is very similar to Theorem 6.1. But the counting functions are very different from each other. Furthermore, here we only need one assumption $f \oplus a_j \not\equiv a_j$, which means that it demands only at least one point $x_0$ such that $f(x_0) > a_j$ for each $a_j$. Thus this assumption is obviously better than the condition (25) in Theorem 6.1 due to Laine and Tohge [17] which implies that the value $f(x)$ at each pole should be large strictly than $\max\{a_1,\ldots,a_q\}$.

Theorem 6.5. Assume that $f$ is a nonconstant tropical meromorphic function with

$$\limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0,$$

and $a_j$ (for $j = 1, 2,\ldots,q$) defining tropical polynomial $P_j$ are distinct values of $\mathbb{R}$ such that $f \oplus a_j \not\equiv a_j$, respectively. Then

$$qT_f(r) = \sum_{j=1}^{q} N \left( r, \frac{1}{P_j \circ f} \right) + o(T_f(r)),$$

where $r$ approaches infinity outside an exceptional set of zero upper density measure.

Proof. If $f \oplus a_j \not\equiv f$ is not satisfied for all $j = 1, 2,\ldots,q$. Suppose there exists one nonempty subset $S$ of $\{1, 2,\ldots,q\}$ such that $f \oplus a_k \equiv f$ for all $k \in S$, and $f \oplus a_i \not\equiv f$ for all $i \in \{1, 2,\ldots,q\} \setminus S$. Then for $k \in S$, we have

$$T_f(r) = T(r,f) + O(1) = T(r, 1_{a_k} \odot f) + O(1) = T(r, 1_{a_k} \odot (f \oplus a_k)) + O(1) \leq N(r, 1_{a_k} \odot (f \oplus a_k)) + \max\{-a_k, 0\} + O(1) = N(r, 1_{a_k} \odot (f \oplus a_k)) + O(1)$$
Further, by (II) of Proposition 6.2 we have
\[ N(r, 1_0 \odot (P_k \circ f)) = N(r, 1_0 \odot (f \oplus a_k)). \]
Hence
\[
(31) \quad \#ST_f(r) \leq \sum_{k \in S} N(r, 1_0 \odot (P_k \circ f)) + O(1)
\]
\[
\leq \#ST_f(r) + O(1)
\]
where the last inequality comes from the tropical first main theorem (that is, Theorem 6.7).

On the other hand, by Theorem 6.3 and Proposition 6.4, we get that for \( i \in \{1, 2, \ldots, q\} \setminus S, \)
\[
(32) \quad (q - \#S)T_f(r) = \sum_{i \in \{1, 2, \ldots, q\} \setminus S} N(r, 1_0 \odot (P_i \circ f)) + o(T_f(r)),
\]
where \( r \) approaches infinity outside an exceptional set of zero upper density measure. Therefore the corollary is obtained immediately from (31) and (32).

By Theorem 6.3 and Theorem 6.5, we have defect relation as follows. Define that a value \( a \) is called a Picard exceptional value of \( f \) if \( \delta_f(a) = 1 \). Hence by (ii) below, we get that all finite real values are the Picard exceptional values of a nonconstant tropical meromorphic function.

**Theorem 6.6.** Assume that \( f = [f_0 : f_1] : \mathbb{R} \to \mathbb{T}^1 \) is a nonconstant tropical meromorphic function with \( \limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0. \)

(i) Let \( a_j = [a_{j1} : a_{j0}] \) \( (j = 1, \ldots, q) \) be distinct values of \( \mathbb{T}^1 \) which defining tropical polynomials \( P_j(x) = a_{j0} \odot x_0 \oplus a_{j1} \odot x_1 \) on \( \mathbb{R}^2 \), respectively. If \( \text{ddg}(\{P_3 \circ f, \ldots, P_q \circ f\}) = \lambda \), then \( \sum_{j=3}^{q} \delta_f(a_j) = \lambda \) for all \( a_j \) \( (j = 3, 4, \ldots, q) \). In special case whenever \( \lambda = 0, \delta_f(a_j) = 0 \) holds for each \( a_j \) \( (j \in \{3, 4, \ldots, q\}) \).

(ii) \( \delta_f(a) = 0 \)
holds for each \( a \in \mathbb{R} \) such that \( f \oplus a \neq a \).

**Remark 6.7.** Let \( \alpha \) be a real number with \( |\alpha| > 1 \). Define a tropical meromorphic function \( e_\alpha(x) \) on \( \mathbb{R} \) by
\[
e_\alpha(x) := \alpha^{|x|}(x - [x]) + \sum_{j=-\infty}^{[x]-1} \alpha^{|x|}(x - [x]) + \frac{1}{|\alpha| - 1}.
\]
Similarly, if \( \beta \) is a real number with \( |\beta| < 1 \), the corresponding tropical function defined as
\[
e_\beta(x) := \beta^{|x|}(\frac{1}{1-\beta} - x + [x]).
\]
By [14], Proposition 1.22 and Proposition 1.24, both \( e_\alpha(x) \) and \( e_\beta(x) \) are of hyperorder one and \( \limsup_{r \to \infty} \frac{\log T_{e_\alpha}(r)}{r} = \limsup_{r \to \infty} \frac{\log T_{e_\beta}(r)}{r} = 1 \neq 0. \) By Remark 1.25, we have
\[
1 - \limsup_{r \to \infty} \frac{N(r, 1_0 \odot (e_\beta(x) \oplus a))}{T_{e_\beta}(x)(r)} \geq \frac{1}{2} > 0
\]
for all $a < 0$. Then by \((28)\) and the tropical first main theorem, we get $\delta_{e_j}(a) \geq \frac{1}{2} > 0$ for all $a < 0$. This means that the assumption $\limsup_{r \to \infty} \frac{\log T_j(r)}{r} = 0$ cannot be deleted in (ii) of Theorem 6.3.

Recall that the Classical Nevanlinna second main theorem for meromorphic functions on the complex plane $\mathbb{C}$ have a truncated form as follows

$$(q - 2)T_f(r) \leq N^{(1)}(r, \frac{1}{f - a_j}) + o(T_f(r))$$

holds for all $r$ possibly outside a set with finite linear measure, where $a_1, \ldots, a_q$ are distinct values in $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ and $N^{(1)}(r, \frac{1}{f - a_j})$ is the counting function of zeros of $f - a_j$ with multiplicities truncated by one (or say ignoring multiplicities). Now let $f$ be a tropical meromorphic function and let $a$ be a value in $\mathbb{TP}^1$ which defining a tropical linear polynomial $P$ in $\mathbb{TP}^1$. Then it might be interesting to consider the truncated counting function $N^{(1)}(r, \frac{1}{f - a_j} \otimes \circ)$ corresponding to $N(r, \frac{1}{f - a_j} \otimes \circ)$ ignoring the multiplicities of roots of $P_j \circ f$ in our results on the tropical Nevanlinna second main theorem (Theorem 6.3 and Theorem 6.5). We give a conjecture as follows.

Conjecture 6.8. Assume that $f$ is a nonconstant tropical meromorphic function with $\limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0$.

(i). Let $a_j$ ($j = 1, \ldots, q$) be distinct values of $\mathbb{TP}^1$ which defining tropical linear polynomials $P_j(x)$ on $\mathbb{R}^2$, respectively. If $\lambda = \ddg(\{P_1 \circ f, \ldots, P_q \circ f\})$, then

$$(q - 2 - \lambda)T_f(r) \leq \sum_{j=1}^{q} N^{(1)}(r, \frac{1}{P_j \circ f} \otimes \circ) + o(T_f(r)),$$

where $r$ approaches infinity outside an exceptional set of zero upper density measure.

(ii). Let $a_j$ ($j = 1, \ldots, q$) be distinct finite values in $\mathbb{R}$ which defining tropical linear polynomials $P_j(x)$ on $\mathbb{R}^2$ such that $f \otimes a_j \neq a_j$, respectively. Then

$$qT_f(r) = \sum_{j=1}^{q} N^{(1)}(r, \frac{1}{P_j \circ f} \otimes \circ) + o(T_f(r)),$$

where $r$ approaches infinity outside an exceptional set of zero upper density measure.

Moreover, we may give a conjecture on truncated form of tropical Cartan second main theorem corresponding to Corollary 4.12 as follows. Here $N^{(k)}(r, f)$ is denoted to be the counting function of poles of $f$ with multiplicities truncated by $k$ (that is, the multiplicity is only counted $k$ whenever its multiplicity is greater than $k$).

Conjecture 6.9. Let $q$ and $n$ be positive integers with $q \geq n$. Let the tropical holomorphic curve $f : \mathbb{R} \to \mathbb{TP}^n$ be tropical algebraically nondegerated. Assume that $V_{P_j}$ are defined by are homogeneous tropical polynomials $P_j$ ($j = 1, \ldots, q$) with degree 1, respectively. If $\lambda = \ddg(\{P_{n+2} \circ f, \ldots, P_q \circ f\})$ and $\limsup_{r \to \infty} \frac{\log T_f(r)}{r} = 0$, then

$$(q - n - 1 - \lambda)T_f(r) \leq \sum_{j=n+2}^{q} N^{(n)}(r, \frac{1}{P_j \circ f} \otimes \circ) + o(T_f(r))$$
where \( r \) approaches infinity outside an exceptional set of zero upper density measure. In special case whenever \( \lambda = 0 \), we have

\[
(q - n - 1)T_f(r) = \sum_{j=n+2}^{q} N^n_j (r, \frac{1}{P_j \circ f}) + o(T_f(r))
\]

where \( r \) approaches infinity outside an exceptional set of zero upper density measure.

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