Nonstochastic Bandits with Infinitely Many Experts

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Abstract
We study the problem of nonstochastic bandits with infinitely many experts: A learner aims to maximize the total reward by taking actions sequentially based on bandit feedback while benchmarking against a countably infinite set of experts. We propose a variant of Exp4.P that, for finitely many experts, enables inference of correct expert rankings while preserving the order of the regret upper bound. We then incorporate the variant into a meta-algorithm that works on infinitely many experts. We prove a high-probability upper bound of $\tilde{O}(i^* K + \sqrt{K T})$ on the regret, up to polylog factors, where $i^*$ is the unknown position of the best expert, $K$ is the number of actions, and $T$ is the time horizon. We also provide an example of structured experts and discuss how to expedite learning in such cases. Our meta-learning algorithm achieves the tightest regret upper bound for the setting considered when $i^* = \tilde{O}(\sqrt{T/K})$. If a prior distribution is assumed to exist for $i^*$, the probability of satisfying a tight regret bound increases with $T$, the rate of which can be fast.

1. Introduction
Early work on the multi-armed bandit problem commonly studied settings where the rewards of each arm are stochastically generated from some unknown distribution (Robbins 1952; Lai & Robbins 1985; Auer et al. 2002a). In general, such statistical assumptions are difficult to validate or inappropriate for some applications such as packet transmission in communication networks (Auer et al. 1995; 2002b; McMahan & Streeter, 2009). The problem of nonstochastic bandits, first investigated in (Auer et al. 1995; 2002b), makes no statistical assumptions about how the rewards are generated.

A setting of the nonstochastic bandit problem allows for incorporating expert advice. The learner interacts with an adversary over time horizon $T$ as follows. At each time, the adversary sets the rewards for the $K$ actions and keeps them secret. The learner then gets every expert’s advice on the probability of choosing each action. The learner subsequently combines the experts’ advice and samples an action. Finally, the learner observes only the reward of the action chosen, and the game repeats. The learner’s goal is to minimize regret, which is the gap between the total reward gained and the expected total reward of the best expert $i^*$ who is unknown a priori.

The framework described is a general one. First, there is no assumption about the generation of rewards except that the adversary is oblivious. In other words, the adversary’s choices are independent of the learner’s strategy. Equivalently, all rewards can be assigned before the game starts, and the learner only observes the rewards of chosen actions sequentially. Second, we do not restrict or assume knowledge of how the experts come up with their advice. Third, experts can give deterministic advice.

The problem of bandits with expert advice is not only a natural model for numerous real-world applications, such as selecting and pricing online advertisements (McMahan & Streeter 2009) but also important from a theoretical perspective. Contextual bandits can be framed as a bandits with expert advice problem by introducing policies that map a context to a probability distribution over actions (McMahan & Streeter 2009; Agarwal et al. 2017). Bandits with expert advice are also closely related to online model selection where experts correspond to model classes (Cesa-Bianchi & Lugosi 2006; Foster et al. 2017; 2019).

Prior work on nonstochastic bandits with expert advice typically assumes the number of experts to be finite (Auer et al. 1995; 2002b; McMahan & Streeter 2009; Beygelzimer et al. 2011; Neu 2015). The exponential-weight algorithm for exploration and exploitation using expert advice (Exp4), introduced by (Auer et al. 1995; 2002b), has a regret upper bound of $O(\sqrt{KT \ln N})$ in expectation, where $N$ is the number of experts. This upper bound almost matches the lower bound $\Omega(\sqrt{KT \ln N} / \ln K)$ derived by (Agarwal et al. 2012) for the expected regret when $\ln N \leq T \ln K$. However, Exp4 does not satisfy a similar regret guarantee with high probability due to the large variance of its estimates. Algorithms with high-probability guarantees are preferred for domains that need reliable methods, but such
algorithms require delicate analysis \cite{Beygelzimer2011, Neu2015}. The Exp4.P algorithm, a variant of Exp4 proposed by \cite{Beygelzimer2011}, has a regret bounded from above by \(\tilde{O}(\sqrt{KT\ln(N/\delta)})\) with probability at least \(1 - \delta\). This bound can be improved by a constant factor with the key idea of avoiding explicit exploration \cite{Neu2015}.

We study the problem of nonstochastic bandits with infinitely many experts. Our main question is: can the learner perform almost as well as the globally best expert \(i^*\) of a countably infinite set while only querying a finite number of experts? This question is motivated by challenges encountered in practical situations where it is unsatisfactory to seek advice from all experts all the time \cite{Seldin2013}. For search engine advertising, a company may need to choose among a multitude of schemes some of which also involve hyperparameter tuning \cite{McMahan2009}. As another example, there are often many features that can be used for online recommendation systems. Some features tend to be more informative than others, but their relevance is normally unknown. We can transform this problem into bandits with expert advice where each expert corresponds to a model class in a certain feature space. The number of experts can be extremely large due to the combinatorial nature. In contrast to the large number of experts available, it is desirable to query only some of them each time in consideration of computational constraints.

Our Contributions For the general case without any assumption about the experts, we propose an algorithm called Best Expert Search (BEES) and provide theoretical guarantees on its performance. BEES runs a subroutine called Exp4.R, an algorithm that we obtain by modifying Exp4.P. The “R” denotes a feature of Exp4.R: it enables inference of correct expert rankings with high probability in addition to satisfying a regret upper bound of the same order as that proved for Exp4.P. Our main result establishes a high-probability upper bound of \(\tilde{O}(i^{*1/\alpha}K + \sqrt{\alpha KT})\) on the regret of BEES, hiding only polylog factors, which depends on the position of the unknown best expert \(i^*\) and a positive integer-valued parameter \(\alpha\). This upper bound illustrates the trade-off, controlled by \(\alpha\), between exploration and exploitation for the problem of nonstochastic bandits with infinitely many experts. On the one hand, it is desirable to include a large number of experts per epoch so as to approach \(i^*\) at a fast rate. On the other hand, consulting too many experts simultaneously necessitates long epochs, which reduces the rate at which more experts are included. Although tuning \(\alpha\) needs the unknown index \(i^*\), we can simply set \(\alpha = 1\). To the best of our knowledge, our high-probability regret upper bound is the tightest for the setting considered when \(i^* = \tilde{O}(\sqrt{T/K})\). This regime is less restricted than it seems at first sight. If we assume a prior distribution on \(i^*\), then \(i^* = \tilde{O}(\sqrt{T/K})\) holds with a probability that increases with \(T\), the rate of which can be fast. Inspired by the problem of finite-time model selection for reinforcement learning (RL), we also present an example of structured experts, which simulates the trade-off between approximation and estimation. We discuss how the expert ranking property of Exp4.R can be used to expedite learning in such cases.

Related Work A natural way to consider experts as arms and use methods for infinitely many-armed bandits such as \cite{Berry1997, Kleinberg2008, Rusmevichientong2010, Carpenter2013}. However, such work relies on statistical assumptions, whereas our setting is a nonstochastic bandit problem. Our question is also related to bandits with limited advice, first posed by \cite{Seldin2013} and subsequently solved by \cite{Kale2014}, but their setting considers a finite number of experts of whom only a subset can be queried at each time.

To the best of our knowledge, no work achieves high-probability regret bound of \(\tilde{O}(\sqrt{KT})\) for the setting considered. When configured correctly, Exp4 has a regret upper bound of \(\tilde{O}(\sqrt{KT\ln i^*})\) in expectation \cite{Foster2019}. However, the algorithm is computationally infeasible as it needs to handle infinitely many experts at every time step. One method of making Exp4 computationally tractable is to truncate the sequence of experts to a subset of size \(O(e^{\sqrt{KT}})\) as any larger set would make the expected regret superlinear in \(K\) or \(T\). Running Exp4 with correct configurations on this subset of experts has a regret upper bound of \(\tilde{O}((KT)^{3/4} + T\Delta)\) in expectation where \(\Delta\) is the infimum upper bound on the suboptimality gaps of the experts considered. For stochastic contextual bandits, Exp4.P can be used as a subroutine to achieve a high-probability regret bound of \(\tilde{O}(\sqrt{dT\ln T})\) with an infinite set of experts that has a finite Vapnik–Chervonenkis dimension \(d\) \cite{Beygelzimer2011}. Since the regret analysis of Exp4.P relies on the union bound, the algorithm does not apply to infinitely many experts in the nonstochastic setting. If we run Exp4.P on a finite subset of experts of size \(O(\delta \exp(\sqrt{T/(16K)})\), the regret is then bounded from above by \(\tilde{O}(K^{1/4}T^{3/4} + T\Delta)\) with probability at least \(1 - \delta\).

Outline Section 2 formally defines the problem of nonstochastic bandits with infinitely many experts. In Section 3, we introduce Exp4.R for the setting of finitely many experts and prove that it enables inference of correct expert rankings with high probability. Section 4 investigates the case of infinitely many experts and presents a meta-algorithm that runs Exp4.R as a subroutine. We prove a high-probability regret upper bound and give an example to illustrate how to expedite learning when working with structured experts. Finally, we conclude in Section 5.
2. Problem Formulation

Let \( Z_+ \) be the set of strictly positive integers. For \( N \in Z_+ \), we define \( [N] \triangleq \{1, 2, \ldots , N\} \). Let \( T \in Z_+ \) be the time horizon. Let \( \mathcal{A} \) be a set of actions where \( |\mathcal{A}| = K < \infty \).

At each time \( t \in [T] \), the adversary first sets a reward vector \( r(t) \in [0,1]^K \) where \( r_a(t) \) is the reward of action \( a \). Each expert \( i \in Z_+ \) then gives their advice \( \xi_i(t) \), which is a probability vector over \( \mathcal{A} \). After querying a finite subset of the experts’ advice but not the rewards, the learner then samples an action \( a(t) \). Finally, the learner receives the reward \( r_a(t) \) and no other information. The game proceeds to time \( t+1 \) and finishes after \( T \) time steps. The learner’s goal is to combine the experts’ advice such that the total reward is close to a benchmark, which we will define shortly.

Let \( y_i(t) \triangleq \sum_{a\in\mathcal{A}} \xi_i(t)r_a(t) \) be the expected reward of expert \( i \) at time \( t \). For any time interval \( T \subseteq Z_+ \) such that \( |T| < \infty \), we denote the expected total reward of expert \( i \) during \( T \) as \( R_i(T) \triangleq \sum_{t\in T} y_i(t) \). We define the best expert \( i^*(T;T) \) of a subset \( I \subseteq Z_+ \) during \( T \) as the one with the lowest index that has the highest total reward in expectation, namely, \( i^*(T;T) \triangleq \min \{ \arg \max_{i\in I} R_i(T) \} \).

The learner’s regret with respect to \( i^*(T;T) \) is

\[
\text{Regret}(T;I) \triangleq R_{i^*(T;T)}(T) - \sum_{t\in T} r_{a(t)}(t).
\]

For simplicity of notation, let \( \text{Regret}(T) \triangleq \text{Regret}([T];Z_+) \) and \( i^* \triangleq i^*(Z_+;[T]) \). The learner’s goal is to minimize \( \text{Regret}(T) \), the regret with respect to the globally best expert \( i^* \) for the time horizon considered.

3. Nonstochastic Bandits with a Finite Number of Experts

We start with a simplified problem where the number of experts is finite. Section 3.1 presents Exp4.R (Algorithm 1) and provides some intuition for its design. In Section 3.2, we show that Exp4.R not only preserves the regret upper bound of Exp4.P in terms of order but also enables inference of correct expert rankings with high probability.

3.1. The Algorithm

Exp4.R (Algorithm 1) is a slight variant of Exp4.P proposed by [Beygelzimer et al., 2011]. The major distinction is that Exp4.R calculates a threshold vector \( \epsilon \) which enables inference of correct expert rankings with high probability. Exp4.R takes four inputs, namely, an error rate \( \delta \in (0, 1] \), a time horizon \( T \in Z_+ \), the minimum probability \( \rho \in (0, 1/K] \) of exploration, and a finite set of experts \( I \subset Z_+ \). Without loss of generality, we suppose that \( |I| = N \).

\[
\text{Algorithm 1 Exp4.R}
\]

\[
\begin{align*}
\text{Input:} & \quad \delta \in (0, 1], T \in Z_+, \rho \in (0, 1/K], I \subset Z_+ \\
\text{Output:} & \quad w(T+1), \epsilon \\
\end{align*}
\]

\[
\begin{align*}
\beta & \leftarrow \sqrt{\ln(2N/\delta)/(KT)}. \\
w_i(1) & \leftarrow 1 \text{ for } i \in I. \\
\text{for } t = 1, \ldots , T \text{ do} & \\
& \quad \text{Get } \xi_i(t) \text{ for } i \in I. \\
& \quad q_i(t) \leftarrow w_i(t)/\sum_{i'\in I} w_{i'}(t) \text{ for } i \in I. \\
& \quad p_a(t) \leftarrow (1 - K\rho) \sum_{i\in I} q_i(t)\xi_{ai}(t) + \rho \text{ for } a \in \mathcal{A}. \\
& \quad \text{Sample action } a(t) \text{ from } p(t). \\
& \quad \text{Take action } a(t) \text{ and receive reward } r_{a(t)}(t). \\
& \quad \text{for } i \in I \text{ do} \\
& \quad \quad \hat{y}_i(t) \leftarrow \frac{\xi_i(t)r_{a(t)}(t)}{p_{a(t)}(t)}, \\
& \quad \quad \hat{v}_i(t) \leftarrow \sum_{a\in\mathcal{A}} \xi_{ai}(t) / p_a(t), \\
& \quad \quad w_i(t+1) \leftarrow w_i(t) \exp \left( \beta \frac{\hat{y}_i(t) + \beta\hat{v}_i(t)}{2} \right). \\
& \quad \text{end for} \\
& \quad \epsilon_t \leftarrow 1 + \frac{1}{KT} \sum_{t=1}^T \hat{v}_i(t) \ln \left( \frac{2N}{\delta} \right). \\
& \text{end for} \\
\end{align*}
\]

Exp4.R first initializes a weight \( w_i(1) = 1 \) for each expert \( i \in I \). At time \( t \in [T] \), normalizing \( w(t) \) gives a probability distribution \( q(t) \) over \( I \). After getting advice \( \xi_i(t) \) from each expert \( i \), Exp4.R constructs a probability distribution \( p(t) \) over \( \mathcal{A} \) by weighting all advice according to \( q(t) \) and mixing in uniform exploration so that \( p_a(t) \geq \rho \) for all \( a \in \mathcal{A} \). Specifically, for all \( a \), let

\[
p_a(t) = (1 - K\rho) \sum_{i\in I} q_i(t)\xi_{ai}(t) + \rho. \tag{1}
\]

Exp4.R subsequently takes action \( a(t) \) sampled according to \( p(t) \) and receives the reward \( r_{a(t)}(t) \). Time \( t \) concludes with weight updates as specified below. For \( i \in I \), Exp4.R estimates \( \hat{y}_i(t) \) by \( \hat{y}_i(t) \) and calculates an upper bound on the variance of \( \hat{y}_i(t) \) conditional on history until time \( t - 1 \) as given by

\[
\hat{y}_i(t) = \frac{\xi_i(t)r_{a(t)}(t)}{p_{a(t)}(t)}, \quad \hat{v}_i(t) = \sum_{a\in\mathcal{A}} \frac{\xi_{ai}(t)}{p_a(t)}. \tag{2}
\]

Exp4.R updates each expert’s weight \( w_i(t) \) using

\[
w_i(t+1) = w_i(t) \exp \left( \epsilon \frac{\hat{y}_i(t) + \beta\hat{v}_i(t)}{2} \right), \tag{3}
\]

where \( \beta = \sqrt{\ln(2N/\delta)/(KT)} \). The game ends in \( T \) time steps and gives two outputs, namely, the final weight vector...
We establish in Proposition 1 that, with high probability, the following conditions hold: (i) Assumption 1. For simplicity of notation, we denote \( R_i(\tau) = \sum_{t=1}^{\tau} \hat{v}_i(t) \) as \( R_i(T) \). Updating weights using (3) allows us to construct a confidence bound for each \( R_i(T) \). For \( i \in \mathcal{I} \), let \( \hat{R}_i(T) \equiv \sum_{t=1}^{T} \hat{v}_i(t) \). For any \( \delta \in (0, 1] \), let \( \mathcal{E}(\delta) \) be an event defined by

\[
\forall i \in \mathcal{I}, \quad -\ln \left( \frac{2N}{\delta} \right) + \sqrt{\frac{KT}{\ln N}} \hat{v}_i(T) \leq R_i(T) - \hat{R}_i(T) \leq \sqrt{\ln \left( \frac{2N}{\delta} \right)} \left( \hat{V}_i(T) + \sqrt{KT} \right).
\]

Lemma 1 shows that the estimates \( \hat{R}_i(T) \) are concentrated around the true values \( R_i(T) \). The proof relies on a Freedman-style inequality for martingales from (Beygelzimer et al., 2011), which we defer to the appendix.

Lemma 2 establishes an upper bound on the regret of Exp4.R. Since Lemma 2 is a slight variant of Theorem 2 in (Beygelzimer et al., 2011), the proof is very similar to the original one and hence omitted here. We note that Theorem 2 in (Beygelzimer et al., 2011) holds for a smaller regime than stated in the original paper. To be specific, the condition \( T = \Omega(K \ln N) \) is essential for \( \rho = \sqrt{\ln N/(KT)} \leq 1/K \) to be true. We make the correction in Lemma 2.

Lemma 3 validates the correctness of the inferred expert rankings when the concentration event \( \mathcal{E}(\delta) \) holds. Corollary 1 shows that the uncertainty gap for ranking any pair of experts is the sum of their thresholds given by Exp4.R. We can prove Corollary 1 by first taking the contrapositive of the statement in Lemma 3 and then switching \( i \) and \( i' \).

Finally, we combine the lemmas to prove Proposition 1.

Same as Exp4.P, the computational complexity of Exp4.R is \( \mathcal{O}(KN) \) for space and \( \mathcal{O}(KN^2) \) for runtime.

Assumption 1. The following conditions hold: (i) \( \max \{ 4K \ln N, \ln(2N/\delta)/[(\epsilon - 2)K] \} \leq T \), (ii) and there exists a uniform expert \( i \in \mathcal{I} \) such that \( \xi_a(t) = 1/K \) for all \( a \in \mathcal{A} \) and \( t \in \mathbb{Z}_+ \).
Event $\mathcal{E}(\delta)$ implies that
\[
R_i(T) - \hat{R}_i(T) + \hat{R}_{i'}(T) - R_{i'}(T) \\
\geq -\ln \left( \frac{2N}{\delta} \right) \sqrt{\frac{KT}{\ln N}} - \sqrt{\ln N} \frac{K}{KT} \hat{V}_i(T) \\
- \sqrt{\ln \left( \frac{2N}{\delta} \right) \left( \frac{\hat{V}_i(T)}{\sqrt{KT}} + \sqrt{KT} \right)}.
\]
Adding (5) and (6) and then simplifying the algebra give
\[
R_i(T) - R_{i'}(T) > 0.
\]

**Corollary 1.** Under the conditions of Lemma 3 for all $i, i' \in I$, it holds that

(i) if $\ln w_i(T + 1) - \ln w_{i'}(T + 1) > \epsilon_i$, then $R_i(T) > R_{i'}(T)$;

(ii) if $R_i(T) \geq R_{i'}(T)$, then
\[
\ln w_i(T + 1) - \ln w_{i'}(T + 1) \leq -\epsilon_i.
\]

**Proposition 1.** Under Assumption 1 for any $\delta \in (0, 1]$, with probability at least $1 - \delta$, Exp4.R configured with $\rho = \sqrt{\ln N / (KT)}$ satisfies that

(i) Regret$(T; I) \leq 7\sqrt{KT \ln (2N/\delta)}$;

(ii) for all $i, i' \in I$, if $\ln w_i(T + 1) - \ln w_{i'}(T + 1) > \epsilon_i$, then $R_i(T) > R_{i'}(T)$.

**Proof.** Proposition 1 follows directly from Lemmas 1, 3.

4. Selection Among Infinitely Many Experts

In this section, we study the problem of nonstochastic bandits with a countably infinite set of experts. We make no assumptions about the experts or how they are indexed. For this general case, we propose a meta-algorithm called Best Expert Search (BEES, Algorithm 2) that runs Exp4.R as a subroutine and provide a high-probability upper bound on regret. Section 4.1 provides an example of structured experts and discusses how the expert ranking property of Exp4.R can be used to expedite learning in such case.

BEES takes five inputs including an error rate $\delta \in (0, 1]$, the number of epochs $L \in \mathbb{Z}_+$, and three constants $\alpha, c, C \in \mathbb{Z}_+$ that control the exponential growth of the epoch length and the number of experts consulted in each epoch. At a high level, BEES supplies Exp4.R with an increasing (but still finite) number of experts over epochs, prioritizing those with lower indices. This scheme can be considered as putting a prior on the experts implicitly where the experts that are believed to perform well are given low indices. Since we make no assumptions about the experts, they can be ordered using domain knowledge before being given to BEES as inputs. Growing the epoch length and the number of experts at exponential rates allows us to derive a regret upper bound of the same order as that of Exp4.R when the best expert $i^*$ has a relatively low index. This idea is similar to, though not the same as, the doubling trick (Besson & Kaufmann, 2018) as the latter only deals with the epoch length. For our setting, we need to increase the number of experts at an appropriate rate relative to the epoch length.

**Algorithm 2 Best Expert Search (BEES)**

1: \textbf{Input}: $\delta \in (0, 1]$, $\alpha \in \mathbb{Z}_+$, $L \in \mathbb{Z}_+$, $c \in \mathbb{Z}_+$, $C \in \mathbb{Z}_+$
2: \textbf{for} epoch $l = 1, \ldots, L$ \textbf{do}
3: \hspace{1em} $N_l \leftarrow c2^{n_l}$, $l_l \leftarrow c2^l$.
4: \hspace{1em} $\rho_l \leftarrow \sqrt{\ln N_l / (KT_l)}$.
5: \hspace{1em} $I_l \leftarrow [N_l]$.
6: \hspace{1em} Exp4.R$(\delta/L, l, l_l, \rho_l, I_l)$.
7: \textbf{end for}

Theorem 1 provides sample complexity guarantees for BEES by establishing a high-probability regret upper bound. Corollary 2 simplifies this bound for specific parameter values. Corollary 2 shows that BEES, when tuned right, satisfies Regret$(T) = \tilde{O} \left( (i^*)^{1/\alpha} K + \sqrt{\alpha KT} \right)$ with high probability, where $\tilde{O} (\cdot)$ omits only polylog factors. This upper bound illustrates the trade-off between exploration and exploitation for the problem of bandits with infinitely many experts. On the one hand, we want to include a large number of experts in each epoch so as to approach $i^*$ at a fast rate. On the other hand, querying too many experts simultaneously necessitates long epochs, which reduces the rate at which more experts are included. This trade-off is controlled by the parameter $\alpha \in \mathbb{Z}_+$. The term $\tilde{O} \left( (i^*)^{1/\alpha} K \right)$ in the upper bound corresponds to regret from not considering $i^*$ sooner. The other term $\tilde{O} \left( \sqrt{\alpha KT} \right)$ is the regret that benchmarks against the best expert in each epoch. Another consideration for not using an arbitrarily large value of $\alpha$ is that the minimum time horizon required by BEES which is $T = \Omega(C(\alpha, c, K, \delta))$ increases with $\alpha$. Although tuning $\alpha$ needs the unknown index of the best expert $i^*$, we can simply set $\alpha = 1$. BEES has space complexity $\tilde{O} \left( K(1 + T/K)^n \right)$ and time complexity $\tilde{O} \left( (K^2(1 + T/K)^{n+1}) \right)$.

To the best of our knowledge, Theorem 1 establishes the tightest high-probability regret bound for the setting considered when $i^* = \tilde{O} \left( \sqrt{T/K} \right)$. This regime is less restricted than it seems at first sight. Assuming a prior distribution...
on \(i^*\) shows that the condition on \(i^*\) is satisfied with a probability that increases with \(T\), the rate of which can be fast. For simplicity, let \(\alpha = 1\) and \(c = 1\). In order for \(\text{Regret}(T) = \breve{O}(\sqrt{KT})\) to hold with high probability, we need \(i^* = \breve{O}(\sqrt{T/K})\). We denote the complement of this event as \(\mathcal{B}\). If we suppose that \(F(i) = P(i^* > i)\) for \(i \in \mathbb{Z}_+\) and some function \(F: \mathbb{Z}_+ \to [0, 1]\), then \(P(\mathcal{B})\) decreases with \(T\). For example, if \(F(i) \propto e^{-s}i^{-s}\) for some \(s > 0\), then \(P(\mathcal{B})\) is roughly proportional to \(e^{-s\sqrt{T/K}}\). If \(F(i) \propto e^{-si}\) for some \(s > 0\), then \(P(\mathcal{B})\) is roughly proportional to \(e^{-s\sqrt{T/K}}\).

Before stating Theorem 1, we provide some intuition for the proof. Lemma 1 implies that \(\sum_{i \in T_i} \hat{y}_i(t) \approx K_i(T_i)\) for each expert \(i\) and every epoch \(T\) with high probability. For this reason, we can prove a lower bound on the regret with respect to the best expert in each epoch, namely, \(\sum_{i = 1}^{L} R_i(T_i) - \sum_{i = 1}^{T} r_{a(i)}(t) = \breve{O}(\sqrt{\alpha KT})\). We then derive an upper bound on the gap between the globally best expert and the best expert in each epoch, which is given by \(R_{i^*}(T_i) - \sum_{i = 1}^{L} R_i(T_i) = \breve{O}(i^*^{1/\alpha}K)\). Adding the upper bounds, we get \(\text{Regret}(T) = \breve{O}(i^*^{1/\alpha}K + \sqrt{\alpha KT})\).

For simplicity of notation, we suppose that the total number of epochs \(L = \log_2\left[1 + T/(2C)\right]\) so that \(T = \sum_{i = 1}^{L} T_i\) where \(T_i = C2^l\) for \(l \in [L]\). We use \([\cdot]\) to denote the floor and ceiling functions, respectively. For the general case of \(T \geq 2C\), let \(L = \left[\log_2\left[1 + T/(2C)\right]\right], T_i = C2^l\) for \(l \in [L - 1]\), and \(T_L = T - \sum_{i = 1}^{L - 1} T_i\).

**Theorem 1.** If a uniform expert is available in each epoch, then there exist absolute constants \(\alpha \in \mathbb{Z}_+\) and \(c \in \mathbb{Z}_+\) such that, for some \(C(\alpha, c, K, \delta) \in \mathbb{Z}_+\), BEES satisfies that, for any \(\delta \in (0, 1)\) with probability at least \(1 - \delta\), we have

\[
\text{Regret}(T) < 20\sqrt{\alpha K(T + 2C) \ln \left(\frac{cL(2 + T/C)}{\delta}\right)} + 2L \left(\frac{i^*}{c}\right) ^ {\frac{1}{\alpha}}.
\]

**Corollary 2.** Under the conditions of Theorem 1 running BEES with \(\alpha \in \mathbb{Z}_+, c \in \mathbb{Z}_+,\) and \(C = \lceil \alpha K \ln(16C^4/\delta) \rceil\) satisfies that, for any \(\delta \in (0, 1)\), with probability at least \(1 - \delta\), \(\text{Regret}(T) = \breve{O}(\sqrt{\alpha KT})\).

**Proof of Theorem 1.** We can show that, for all \(\delta \in (0, 1)\), \(\alpha \in \mathbb{Z}_+, c \in \mathbb{Z}_+,\) there exists \(C(\alpha, c, K, \delta) \in \mathbb{Z}_+\) such that \(4K \ln(c2^\alpha) \leq C2^l\) and \(\ln(c2^\alpha + 1/\delta) \leq (e - 2)CK2^l\) for all \(l \in \mathbb{Z}_+\). For example, we can set \(C = \lceil \alpha K \ln(16C^4/\delta) \rceil\). Together with the definitions of \(N_l\) and \(T_l\) in Algorithm 2, we have that, for all \(\alpha \in \mathbb{Z}_+, c \in \mathbb{Z}_+,\) there exists \(C \in \mathbb{Z}_+\) such that \(4K \ln N_l \leq T_l\) and \(\ln(2N_l/\delta) \leq (e - 2)KT_l\) for all \(l \in \mathbb{Z}_+.\) We fix such integers \(\alpha, c, C \in \mathbb{Z}_+\) for the rest of the proof.

For simplicity of notation, we first consider running Exp4.R(\(\delta, T_i, \rho_i, \mathcal{I}_i\)) in each epoch \(l\) for any \(\delta \in (0, 1/L)\) and then apply a change of variables at the end of the proof. We suppose that a uniform expert is available in each epoch. Assumption 1 is then satisfied for all epochs. For now, we assume that event \(\mathcal{E}(l)\) holds for all epochs, the probability of which will be discussed at the end of the proof. For simplicity of notation, let \(i_l^* \triangleq i^*(T_l; \mathcal{I})\) for \(l \in [L]\).

Let \(U_l \triangleq \alpha l + \log_2(2c/\delta)\) for \(l \in [L]\). Recall that \(T_i\) is the time interval of epoch \(l\) where \([T_i] = T_i\). By Lemma 2,

\[
\sum_{l = 1}^{L} R_{i_l^*}(T_l) - \sum_{l = 1}^{L} r_{a(l)}(t) \leq \sum_{l = 1}^{L} 7\sqrt{KTC \ln \left(\frac{2N_l}{\delta}\right)} \leq 7\sqrt{KCU_L \ln \left(\frac{2L}{\delta}\right)} \leq 20\sqrt{KCU_L \left(2L^2/2 - 1\right)}.
\]

Since \(L = \log_2\left[1 + T/(2C)\right]\), we have

\[
\sum_{l = 1}^{L} R_{i_l^*}(T_l) - \sum_{l = 1}^{L} r_{a(l)}(t) < 20\sqrt{KCU_L \left(\sqrt{1 + \frac{T}{2C}} - 1\right)} \leq 20\sqrt{K} \left[\alpha L + 2 \ln \left(\frac{2c}{\delta}\right)\right] \left(C + \frac{T}{2}\right).
\]

We first discuss the case where \(i^* \notin \mathcal{I}_1\). Let \(L'\) be the last epoch such that \(i^*\) is not considered in Algorithm 2. Since \(|\mathcal{I}_l| = N_l\), we have \(L' = \min \left(L, \left[\alpha^{-1} \log_2(i^*/c)\right] - 1\right)\). Since \(i^* \notin \mathcal{I}_1\), we get \(L' \geq 1\). By the definition of \(i^*_l\), we have \(R_{i_l^*}(T_l) \geq R_{i_l^*}(T_l)\) for all \(l > L'\). Thus,

\[
R_{i_l^*}(T_l) = \sum_{l = 1}^{L} R_{i_l^*}(T_l) - \sum_{l = 1}^{L'} (R_{i_l^*}(T_l) - R_{i_l^*}(T_l)) \leq \sum_{l = 1}^{L'} (R_{i_l^*}(T_l) - R_{i_l^*}(T_l)) \leq \sum_{l = 1}^{L'} T_l \leq C2^{L'+1} < 2C \left(\frac{i^*}{c}\right) ^ {\frac{1}{\alpha}}.
\]

We now consider the case where \(i^* \in \mathcal{I}_l\). It follows from Algorithm 2 that \(i^* \in \mathcal{I}_l\) for all \(l\). Thus, the definition
of \( i^* \) implies that \( R_{i^*}(T_l) \geq R_{i^*}(T_l) \) for all \( l \). We define
\[ D \triangleq R_{i^*}([T]) - \sum_{l=1}^{L} R_{i^*}(T_l). \]
We then have \( D \leq 0 \). However, the definition of \( i^* \) implies that \( D \geq 0 \). Therefore, \( D = 0 \) and (8) is satisfied.

Adding (7) and (8) gives
\[
\text{Regret}(T) \leq 20 \sqrt{K \left[ \alpha L + 2 \ln \left( \frac{2C}{\delta} \right) \right] \left( C + \frac{T}{2} \right)} + 2C \left( \frac{i^*}{c} \right)^{\frac{1}{2}}.
\]

Using Lemma 1 and the union bound over all \( L \) epochs, we conclude that (9) holds with probability at least \( 1 - L\delta \). A change of variables gives that, for any \( \delta \in (0, 1] \), with probability at least \( 1 - \delta \), we have
\[
\text{Regret}(T) \leq 20 \sqrt{K \left[ \alpha L + 2 \ln \left( \frac{2CL}{\delta} \right) \right] \left( C + \frac{T}{2} \right)} + 2C \left( \frac{i^*}{c} \right)^{\frac{1}{2}}.
\]

4.1. Structured Experts

In this section, we present an example of structured experts that is inspired by the problem of finite-time model selection for RL and discuss how the expert ranking property of Exp4.R can be used to expedite learning in such cases.

As RL becomes increasingly integrated into autonomous systems such as agile robots (Hwangbo et al. 2019), self-driving vehicles (Kuderer et al. 2015), customized fertilizer formulation (Binas et al. 2019), and personalized medication dosing (Nemati et al. 2016), it is crucial that the techniques are robust (Matni et al. 2019). An aspect of robustness is the capability to detect and adjust for model errors. For RL, this entails both model selection and parameter estimation. How to achieve both objectives simultaneously while maintaining provably good performance is an active area of research (Ni & Wang 2019; Abbasi-Yadkori et al. 2020). The crux of the problem of online model selection for RL is to balance approximation and estimation errors in a time-dependent manner. As an example, we suppose that there is an infinite sequence of nested model classes. This structure arises naturally when an RL algorithm incorporates increasingly many features over time. Some new features may also just become obtainable while an RL algorithm is running. In fact, it is unknown a priori for many applications what is a minimal feature space that contains an optimal policy. Given an infinite sequence of model classes, the best class to use depends on the horizon or, equivalently, the amount of trajectory data that will become available. Although a larger model class has a smaller approximation error, it tends to have a higher estimation error for a fixed finite horizon. Moreover, if several classes have the same approximation power, the simplest one is typically preferred in consideration of time and space complexity.

Inspired by the problem of finite-time model selection for RL, we propose to consider experts structured in a way that simulates the trade-off between approximation and estimation. In particular, we suppose that the experts are ranked in ascending order of complexity. Assumption 2 stipulates that the total reward is weakly unimodal in expectation with respect to the expert index during any period of time. In addition, the index of the globally best expert is a nondecreasing function of the time horizon. See Figure 1 for an illustration. The proposed time-dependent unimodal structure is fundamentally related to oracle inequalities in empirical risk minimization (Wainwright 2019). Although the experts’ performance may fluctuate around the proposed structure in practice, solutions to the stylized setting are of theoretical interest.

Assumption 1. For all \( T \subset [T] \), if \( i \leq i^*(Z_i^*; T) \), then \( R_{i-1}(T) \leq R_i(T) \). Otherwise, \( R_{i^*}(T) \geq R_{i+1}(T) \). Moreover, \( i^*(Z_i^*; T) \leq i^*(Z_i^*; T') \) if \( T' \subset [T] \) and \( |T'| > |T| \).

\[ \text{Expected total reward} \quad R_i(T) \]

\[ i^*(T) \quad i^*(2T) \]

\[ \text{Expert} \]

Figure 1. An illustration of Assumption 2

Under Assumption 2 the outputs of Exp4.R give a threshold rule that allows us to find a lower bound for \( i^* \), which can accelerate the rate of approaching \( i^* \). We modify BEES to incorporate lower bound estimation (BEES.LB. Algorithm 3). BEES.LB runs Exp4.R and Probabilistic Thresholding Search (PTS. Algorithm 4) as subroutines. In each epoch, BEES.LB eliminates experts identified as suboptimal. Lemma 4 shows that the estimated lower bound is correct if the concentration event \( E(\delta) \) holds. Theorem 2 provides sample complexity guarantees for BEES.LB by establishing a high-probability regret upper bound. The
proof of Theorem 2 is similar to that of Theorem 1 hence deferred to the appendix. PTS has space complexity $O(N)$ and time complexity $O(N^2)$. PTS can be efficiently implemented by first sorting the input $w$. BEES.LB takes the same space $O(K(1 + T/K)^\alpha)$ as BEES. The time complexity of BEES.LB is $O(K^2(1 + T/K)^{\alpha+1} + (1 + T/K)^{2\alpha})$, which reduces to the runtime of BEES for sufficiently small $\alpha \in \mathbb{Z}_+$. The upper bound given in Theorem 2 is the same as that for the general case of unstructured experts because the lower bound from PTS can stay at 1 in the worst case. A trivial example is that all experts are the same. For cases where the experts’ performance differs by sufficient margins, the actual improvement of BEES.LB over BEES should become obvious.

If the globally best expert $i^*$ is fixed over time, then we can modify BEES.LB to additionally estimate an upper bound on $i^*$, initialized to $\infty$. We can show that the confidence interval for $i^*$ contracts over epochs. While the epoch length always grows exponentially, the set of experts considered in each epoch is data-dependent. If no upper bound on $i^*$ has been identified, then the number of experts considered will increase by a factor of $2^\alpha$ in the next epoch. Otherwise, only the experts in the non-expanding confidence interval will be considered from now on.

### 5. Discussion

In this paper, we have proposed an algorithm for the problem of nonstochastic bandits with infinitely many experts under the constraint of having access to only a finite subset of experts. We have established a high-probability upper bound on the regret of our meta-algorithm BEES, which matches the lower bound up to polylog factors if the globally best expert has a relatively low index. If we assume that there exists a prior distribution on the best expert, then the probability that our regret upper bound is tight will increase with the time horizon, the rate of which can be fast. The expert ranking property of the subroutine Exp4.R enables learning acceleration if the structure of the experts is known.

We have demonstrated this point with an example that is inspired by the problem of finite-time model selection for RL. One interesting direction for future work is to obtain instance-dependent upper bounds in terms of the experts’ suboptimality gaps. Such instance-dependent bounds can be used to prove the learning acceleration enabled by Exp4.R. A simple implementation of our algorithms inherits the computational complexity of Exp4.P. It is worthwhile to design efficient implementation for specific applications.
Acknowledgments

The authors would like to thank John N. Tsitsiklis, Dylan J. Foster, Caroline Uhler, Devavrat Shah, and Thibaut Horel for helpful discussions. This work was supported by the OCP Group.

References

Abbasi-Yadkori, Y., Pacchiano, A., and Phan, M. Regret balancing for bandit and RL model selection. arXiv preprint arXiv:2006.05491, 2020.

Agarwal, A., Dudik, M., Kale, S., Langford, J., and Schapire, R. E. Contextual bandit learning with predictable rewards. In International Conference on Artificial Intelligence and Statistics, pp. 19–26, 2012.

Agarwal, A., Luo, H., Neyshabur, B., and Schapire, R. E. Corralling a band of bandit algorithms. In Conference on Learning Theory, pp. 12–38, 2017.

Auer, P., Cesa-Bianchi, N., Freund, Y., and Schapire, R. E. Gambling in a rigged casino: The adversarial multi-armed bandit problem. In IEEE 36th Annual Foundations of Computer Science, pp. 322–331, 1995.

Auer, P., Cesa-Bianchi, N., and Fischer, P. Finite-time analysis of the multiarmed bandit problem. Machine Learning, 47(2):235–256, 2002a.

Auer, P., Cesa-Bianchi, N., Freund, Y., and Schapire, R. E. The nonstochastic multiarmed bandit problem. SIAM J. Comput., 32(1):48–77, 2002b.

Berry, D. A., Chen, R. W., Zame, A., Heath, D. C., and Shepp, L. A. Bandit problems with infinitely many arms. Ann. Statist., 25(5):2103–2116, 1997.

Besson, L. and Kaufmann, E. What doubling tricks can and can’t do for multi-armed bandits. arXiv preprint arXiv:1803.06971, 2018.

Beygelzimer, A., Langford, J., Li, L., Reyzin, L., and Schapire, R. E. Contextual bandit algorithms with supervised learning guarantees. In International Conference on Artificial Intelligence and Statistics, pp. 19–26, 2011.

Binas, J., Luginbuehl, L., and Bengio, Y. Reinforcement learning for sustainable agriculture. CCAI Workshop at the 36th International Conference on Machine Learning, 2019.

Carpentier, A. and Valko, M. Simple regret for infinitely many armed bandits. In International Conference on Machine Learning, pp. 1133–1141, 2015.

Cesa-Bianchi, N. and Lugosi, G. Prediction, Learning, and Games. Cambridge University Press, 2006.

Foster, D. J., Kale, S., Mohri, M., and Sridharan, K. Parameter-free online learning via model selection. In Advances in Neural Information Processing Systems, pp. 6020–6030, 2017.

Foster, D. J., Krishnamurthy, A., and Luo, H. Model selection for contextual bandits. In Advances in Neural Information Processing Systems, pp. 14741–14752, 2019.

Hwangbo, J., Lee, J., Dosovitskiy, A., Bellicoso, D., Tsou, V., Koltun, V., and Hutter, M. Learning agile and dynamic motor skills for legged robots. Science Robotics, 4(26), 2019.

Kale, S. Multiarmed bandits with limited expert advice. In Conference on Learning Theory, pp. 107–122, 2014.

Kleinberg, R., Slivkins, A., and Upfal, E. Multi-armed bandits in metric spaces. In ACM Symposium on Theory of Computing, pp. 681–690, 2008.

Kuderer, M., Gulati, S., and Burgard, W. Learning driving styles for autonomous vehicles from demonstration. In IEEE International Conference on Robotics and Automation, pp. 2641–2646, 2015.

Lai, T. L. and Robbins, H. Asymptotically efficient adaptive allocation rules. Adv. Appl. Math, 6(1):4–22, 1985.

Matni, N., Proutiere, A., Rantzer, A., and Tu, S. From self-tuning regulators to reinforcement learning and back again. In IEEE Conference on Decision and Control, pp. 3724–3740, 2019.

McMahan, H. B. and Streeter, M. Tighter bounds for multiarmed bandits with expert advice. In Conference on Learning Theory, 2009.

Nemati, S., Ghassemi, M. M., and Clifford, G. D. Optimal medication dosing from suboptimal clinical examples: A deep reinforcement learning approach. In Annual International Conference of the IEEE Engineering in Medicine and Biology Society, pp. 2978–2981, 2016.

Neu, G. Explore no more: Improved high-probability regret bounds for non-stochastic bandits. In Advances in Neural Information Processing Systems, pp. 3168–3176, 2015.

Ni, C. and Wang, M. Maximum likelihood tensor decomposition of Markov decision process. In IEEE International Symposium on Information Theory, pp. 3062–3066, 2019.

Robbins, H. Some aspects of the sequential design of experiments. Bull. Amer. Math. Soc., 58(5):527–535, 1952.

Rusmevichientong, P. and Tsitsiklis, J. N. Linearly parameterized bandits. Mathematics of Operations Research, 35 (2):395–411, 2010.
Seldin, Y., Crammer, K., and Bartlett, P. Open problem: Adversarial multiarmed bandits with limited advice. In Conference on Learning Theory, pp. 1067–1072, 2013.

Wainwright, M. J. High-Dimensional Statistics: A Non-Asymptotic Viewpoint. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2019.

A. Proof of Lemma 1

Let $E_t[\cdot]$ denote the conditional expectation given history until time $t-1$. We can show that $\hat{y}_i(t)$ is a conditionally unbiased estimator for $y_i(t)$. In other words, $E_t[\hat{y}_i(t)] = y_i(t)$ for all $i$ and $t$. Lemma 5 shows that $\hat{v}_i(t)$ is an upper bound on the conditional variance of $\hat{y}_i(t)$. Lemma 6 is a Friedman-style inequality for martingales from Beygelzimer et al. (2011). The proof of Lemma 1 relies on Lemmas 5 and 6.

Lemma 5 (From proof of Lemma 3 in Beygelzimer et al. (2011)). For all $t \in \mathbb{Z}_+$ and $i \in I$, we have $E_t[(y_i(t) - \hat{y}_i(t))^2] \leq \hat{v}_i(t)$.

Lemma 6 (Beygelzimer et al. (2011), Theorem 1). Let $X_1, \ldots, X_T$ be a sequence of real-valued random variables. For any real-valued random variable $Y$, we define $E_t[Y] \triangleq E[Y | X_1, \ldots, X_{t-1}]$. We assume that, $X_t \leq B$ and $E_t[X_i] = 0$ for all $t$. We define the random variables

$$S \triangleq \sum_{i=1}^T X_i, \quad V \triangleq \sum_{i=1}^T E_t[X_i^2].$$

For any fixed estimate $V' > 0$ of $V$, and for any $\delta \in (0, 1]$, with probability at least $1 - \delta$, we have

$$S \leq \begin{cases} \sqrt{(e-2) \ln \frac{t}{\delta}} \left( \frac{V}{\sqrt{V'}} + \sqrt{V'} \right), & \text{if } V' \geq \frac{B^2 \ln(1/\delta)}{e-2}, \\ B \ln(1/\delta) + (e-2) \frac{V}{\sqrt{V'}}, & \text{otherwise}. \end{cases}$$

Proof of Lemma 7 We now fix any $i \in I$ and $t \in \mathbb{Z}_+$. By definition, we have $y_i(t) \in [0, 1]$. Using (1) and the assumption that $\rho \in [0, 1/K]$, we get $p_\alpha(t) \geq \rho$ for all $\alpha \in A$. Thus, (2) implies that $\hat{y}_i(t) \in [0, 1/\rho]$ almost surely. Let $X_t = y_i(t) - \hat{y}_i(t)$. We then have $-1/\rho \leq X_t \leq 1$ almost surely. We can show that $E_t[\hat{y}_i(t)] = y_i(t)$ and hence $E_t[X_t] = 0$. We recall that $R_i(T) = \sum_{t=1}^T y_i(t)$. Applying Lemma 6 to $(X_t)$ and $(-X_t)$, respectively and then taking a union bound, we conclude that, for any $\delta \in (0, 1]$, with probability at least $1 - \delta/N$, the inequality $-B_1 \leq R_i(T) - \hat{R}_i(T) \leq B_2$ holds, where

$$B_1 = \begin{cases} \sqrt{(e-2) \ln \frac{2N}{\delta}} \left( \frac{V}{\sqrt{V'}} + \sqrt{V'} \right), & \text{if } V' \geq \frac{\ln(2N/\delta)}{(e-2)/\rho^2}, \\ \frac{\ln(2N/\delta)}{(e-2)/\rho^2}, & \text{otherwise}, \end{cases}$$

$$B_2 = \begin{cases} \sqrt{(e-2) \ln \frac{2N}{\delta}} \left( \frac{V}{\sqrt{V'}} + \sqrt{V'} \right), & \text{if } V' \geq \frac{\ln(2N/\delta)}{(e-2)/\rho^2}, \\ \frac{\ln(2N/\delta)}{(e-2)/\rho^2}, & \text{otherwise}. \end{cases}$$

We now fix an arbitrary $\delta \in (0, 1]$. Assumption (1) implies that $\ln(2N/\delta) \leq (e-2)KT$. Taking $\rho = \sqrt{\ln N/(KT)}$ and $V' = KT$, we have

$$\frac{\ln(2N/\delta)}{(e-2)/\rho^2} \leq V' < \frac{\ln(2N/\delta)}{(e-2)/\rho^2}.$$

Lemma 5 implies that $V \leq \hat{V}_i(T)$. Therefore, with probability at least $1 - \delta/N$, we have

$$\begin{align*}
- \ln \left( \frac{2N}{\delta} \right) &\sqrt{\frac{KT}{\ln N}} - \sqrt{\frac{\ln N}{KT}} \hat{V}_i(T) \\
&\leq R_i(T) - \hat{R}_i(T) \\
&\leq \sqrt{\ln \left( \frac{2N}{\delta} \right) \left( \frac{\hat{V}_i(T)}{\sqrt{KT}} + \sqrt{KT} \right)}.
\end{align*}$$

Applying the union bound over $i \in I$, we conclude that $\mathbb{P}(E(\delta)) \geq 1 - \delta$.

B. Proof of Theorem 2

Proof of Theorem 2 We can show that, for all $\delta \in (0, 1]$, $\alpha \in \mathbb{Z}_+$, and $c \in \mathbb{Z}_+$, there exists $C(\alpha, c, K, \delta) \in \mathbb{Z}_+$ such that $4K \ln (c^2 \alpha^l) \leq C^2$ and $\ln (c^2 \alpha^l + 1/\delta) \leq (e-2)C^2$ for all $l \in \mathbb{Z}_+$. For example, we can set $C = \lfloor \alpha K \ln(16c^4/\delta) \rfloor$. Together with the definitions of $N_l$ and $T_l$ in Algorithm 3, we have that, for all $\alpha \in \mathbb{Z}_+$ and $c \in \mathbb{Z}_+$, there exists $C \in \mathbb{Z}_+$ such that $4K \ln N_l \leq T_l$ and $\ln(2N_l/\delta) \leq (e-2)KT_l$ for all $l \in \mathbb{Z}_+$. We fix such integers $\alpha, c, C \in \mathbb{Z}_+$ for the rest of the proof.

For simplicity of notation, we first consider running Exp$\text{4.R}((\delta, T_l, \rho_l, I_l))$ in each epoch $l$ of Algorithm 3 for any $\delta \in (0, 1/L]$ and then apply a change of variables at the end of the proof. We suppose that a uniform expert is available in each epoch. Assumption (1) is then satisfied for all epochs. For now, we assume that event $E(\delta)$ holds for all epochs, the probability of which will be discussed at the end of the proof. For simplicity of notation, let $i^*_l \triangleq i^*(I_l; T_l)$ for $l \in [L]$. Let $U_l \triangleq \log \left( 2c / \delta \right)$ for $l \in [L]$. Recall that $T_l$ is the
time interval of epoch \( l \) where \( |T_l| = T_l \). By Lemma \[2\]
\[
\sum_{l=1}^{L} R_{i_l^*}(T_l) - \sum_{t=1}^{T} r_{a(t)}(t) \leq \sum_{l=1}^{L} 7 \sqrt{KT_l \ln \left( \frac{2N_l}{\delta} \right)} \\
= \sum_{l=1}^{L} 7 \sqrt{KC2^l \ln \left( \frac{\epsilon^2L}{\delta} \right)} \\
= 7\sqrt{KC} \ln 2 \sum_{l=1}^{L} \sqrt{2U_l} \\
\leq 7\sqrt{KC}U \ln 2 \sum_{l=1}^{L} 2^{l/2} \\
< 20\sqrt{KC}U \left( 2^{L/2} - 1 \right).
\]

Since \( L = \log_2[1 + T/(2C)] \), we have
\[
\sum_{l=1}^{L} R_{i_l^*}(T_l) - \sum_{t=1}^{T} r_{a(t)}(t) \\
< 20\sqrt{KC}U \left( \sqrt{1 + \frac{T}{2C} - 1} \right) \\
< 20\sqrt{K} \left[ \alpha L + 2 \ln \left( \frac{2c}{\delta} \right) \right] \left( C + \frac{T}{2} \right). 
\tag{10}
\]

We first discuss the case where \( i^* \notin \mathcal{I}_l \). Let \( L'' \) be the last epoch such that \( i^* \) is not considered in Algorithm \[3\]. In other words, \( L'' = \max \{ l \in [L] \mid i^* \notin \mathcal{I}_l \} \). Lemma \[4\] implies that \( i^* \in \mathcal{I}_l \) for all \( l > L'' \). By the definition of \( i_l^* \), we have \( R_{i_l^*}(T_l) \geq R_{i_*}(T_l) \) for all \( l > L'' \). Thus,
\[
R_{i_*}(|T|) - \sum_{l=1}^{L} R_{i_l^*}(T_l) \leq \sum_{l'=1}^{L''} (R_{i_*}(T_l) - R_{i_l^*}(T_l)) \\
\leq \sum_{l'=1}^{L''} T_l \\
< C2^{L''+1}.
\]

We now provide an upper bound on \( L'' \). By Algorithms \[3\] and \[4\] we have \( |\mathcal{I}_l| = N_l \) and \( 1 \leq i_l \leq i_{l+1} \) for all \( l \). Let \( L' \) be the last epoch such that \( i^* \) is not considered in the worst case where \( i_{l_l} = 1 \) for all \( l \). In other words, \( L' = \min \{ L, \lceil \alpha^{-1} \log_2(i^*/c) \rceil - 1 \} \). Under the assumption that \( i^* \notin \mathcal{I}_l \), we get \( L' \geq 1 \). By the definitions of \( L' \) and \( L'' \), we have \( L'' \leq L' \) and hence
\[
R_{i_*}(|T|) - \sum_{l=1}^{L} R_{i_l^*}(T_l) < C2^{L'+1} < 2C \left( \frac{i^*}{c} \right)^{\frac{1}{\delta}}. 
\tag{11}
\]

We now consider the case where \( i^* \in \mathcal{I}_l \). It follows from Lemma \[4\] that \( i^* \in \mathcal{I}_l \) for all \( l \). Thus, the definition of \( i_l^* \) implies that \( R_{i_l^*}(T_l) \geq R_{i_*}(T_l) \) for all \( l \). We define \( D \triangleq R_{i_*}(|T|) - \sum_{l=1}^{L} R_{i_l^*}(T_l) \). We then have \( D = 0 \).

However, the definition of \( i^* \) implies that \( D \geq 0 \). Therefore, \( D = 0 \) and (11) is satisfied.

Adding (10) and (11) gives
\[
\text{Regret}(T) < 20\sqrt{K} \left[ \alpha L + 2 \ln \left( \frac{2c}{\delta} \right) \right] \left( C + \frac{T}{2} \right) \\
+ 2C \left( \frac{i^*}{c} \right)^{\frac{1}{\delta}}. 
\tag{12}
\]

Using Lemma \[1\] and the union bound over all \( L \) epochs, we conclude that (12) holds with probability at least \( 1 - L\delta \). A change of variables gives that, for any \( \delta \in (0,1] \), with probability at least \( 1 - \delta \), we have
\[
\text{Regret}(T) < 20\sqrt{K} \left[ \alpha L + 2 \ln \left( \frac{2cL}{\delta} \right) \right] \left( C + \frac{T}{2} \right) \\
+ 2C \left( \frac{i^*}{c} \right)^{\frac{1}{\delta}} \\
< 20\alpha K(T + 2C) \ln \left( \frac{cL(2 + T/C)}{\delta} \right) \\
+ 2C \left( \frac{i^*}{c} \right)^{\frac{1}{\delta}}.
\]