Isoperimetric inequalities for Bergman analytic content

Stephen J. Gardiner, Marius Ghergu and Tomas Sjödin

Abstract

The Bergman $p$-analytic content ($1 \leq p < \infty$) of a planar domain $\Omega$ measures the $L^p(\Omega)$-distance between $\overline{\Omega}$ and the Bergman space $A^p(\Omega)$ of holomorphic functions. It has a natural analogue in all dimensions which is formulated in terms of harmonic vector fields. This paper investigates isoperimetric inequalities for Bergman $p$-analytic content in terms of the St Venant functional for torsional rigidity, and addresses the cases of equality with the upper and lower bounds.

1 Introduction

The Bergman $p$-analytic content ($1 \leq p < \infty$) of a bounded planar domain $\Omega$ was introduced by Guadarrama and Khavinson [15]. It is defined by the formula

$$\lambda_{A^p}(\Omega) = \inf_{f \in A^p(\Omega)} \| \overline{\Omega} - f \|_p,$$

where $\| \cdot \|_p$ is the usual $L^p(\Omega)$-norm and $A^p(\Omega)$ is the Bergman space of $L^p(\Omega)$-integrable holomorphic functions $f$ on $\Omega$. In the case where $p = 2$, Fleeman and Khavinson [8] showed that, for any simply connected domain $\Omega$ with piecewise smooth boundary,

$$\sqrt{\rho(\Omega)} \leq \lambda_{A^2}(\Omega) \leq \frac{m(\Omega)}{\sqrt{2\pi}},$$

where $\rho(\Omega)$ denotes the torsional rigidity of $\Omega$ and $m$ is Lebesgue measure. Subsequently, Fleeman and Lundberg [9] showed that the left hand inequality is actually an equality for any bounded simply connected domain, and this relationship has been further exploited by Fleeman and Simanek [10]. Bell, Ferguson and Lundberg [3] established related inequalities concerning torsional rigidity and the norm of the self-commutator of a Toeplitz operator. The limiting case of Bergman $p$-analytic content where $p = \infty$ is the notion of analytic content, which has been studied for many years: see, for example, [11], [4], [1] for the case of the plane, and [16], [12] for its extension to higher dimensions.

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Rewriting $\lambda_{A^p}(\Omega)$ as $\inf_{\phi \in A^p(\Omega)} \| z - \phi \|_p$, we see that a natural generalization to bounded domains $\Omega$ in Euclidean space $\mathbb{R}^N (N \geq 2)$ is given by

$$
\lambda_{A^p}(\Omega) = \inf \{ \| x - f \|_{L^p} : f \in A^p(\Omega) \} \quad (1 \leq p < \infty),
$$

where $A^p(\Omega)$ denotes the space of harmonic vector fields $f = (f_1, ..., f_N)$ in $L^p \cap C^1(\Omega)$,

$$
L_p = L^p(\Omega) = (L^p(\Omega))^N, \quad \| f \|_{L^p} = \left( \int_{\Omega} \| f \|^p \, dm \right)^{1/p}
$$

and $\| \cdot \|$ is the usual Euclidean norm on $\mathbb{R}^N$. Thus $f$ satisfies $\text{div} \, f = 0$ and $\text{curl} \, f = 0$, where the latter condition means that

$$
\frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} = 0 \quad \text{for all} \quad j, k \in \{1, ..., N\} \quad \text{on} \quad \Omega.
$$

The gradient of any harmonic function is a harmonic vector field, and the converse assertion is also true when $\Omega$ is simply connected. We will assume from now on that $\Omega$ is smoothly bounded.

The purpose of this paper is to investigate isoperimetric inequalities for $\lambda_{A^p}(\Omega) (1 \leq p < \infty)$ in all dimensions, and to examine the cases of equality with the upper and lower bounds (cf. Problem 3.4 of [5]). We denote by $q$ the dual exponent of $p$, whence $1/p + 1/q = 1$ (or $q = \infty$ if $p = 1$), and note that the dual space $L^*_p$ can be identified with $L^q$. When $q < \infty$ we denote by $W^{1,q}_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ in the Sobolev space $W^{1,q}(\Omega)$; these are the functions in $W^{1,q}(\Omega)$ that have trace zero on $\partial \Omega$ (see Section 5.5 of [7]).

Since any function in $W^{1,\infty}_0(\Omega)$ has a Lipschitz representative, it is natural to denote by $W^{1,\infty}_0(\Omega)$ the subset of $W^{1,\infty}(\Omega)$ comprising those functions which vanish on $\partial \Omega$. We define

$$
Q_q(\Omega) = \sup_{u \in W^{1,q}_0(\Omega) \setminus \{0\}} \frac{N}{\| \nabla u \|_{L^q}} \int_{\Omega} u \, dm \quad (1 < q \leq \infty). \quad (1)
$$

When $q < \infty$, the quantity $(Q_q(\Omega))^q$ is known as the St Venant $q$-functional of $\Omega$. Its relationship with the torsional rigidity $\rho(\Omega)$ will be discussed in Section 4.

We begin with the case $p = 2$, where we can add the following to the results of [8] and [9].

**Theorem 1** If $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain, then $\lambda_{A^2}(\Omega) = Q_2(\Omega)$. Further, $\lambda_{A^2}(\Omega) = \sqrt{\rho(\Omega)}$ if and only if $\mathbb{R}^N \setminus \Omega$ is connected.

Next, we establish a lower bound for $\lambda_{A^p}(\Omega)$ for all $p$. 

2
Theorem 2. If $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain and $p \in [1, \infty)$, then
\[ Q_q(\Omega) \leq \lambda_{A_p}(\Omega). \] (2)

Further, equality holds if and only if either
(a) $p = 2$, or
(b) $\Omega$ is a ball or an annular region.

The case of equality above when $p \neq 2$ is a counterpart of a recent result of Abanov, Bénétteau, Khavinson and Teodorescu [1] concerning analytic content in the plane (that is, where $p = \infty$ and $N = 2$).

It remains to establish an upper bound for $\lambda_{A_p}(\Omega)$. Let $B(r)$ denote the open ball in $\mathbb{R}^N$ of centre 0 and radius $r$, and let $B = B(1)$. Further, let $r_{\Omega} > 0$ be chosen so that $m(B(r_{\Omega})) = m(\Omega)$. Then, by the generalized Faber-Krahn inequality (cf. [6]), we have $Q_q(\Omega) \leq Q_q(B(r_{\Omega}))$. The result below is new in all dimensions.

Theorem 3. If $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain and $p \in [1, 2]$, then
\[ \lambda_{A_p}(\Omega) \leq Q_q(B(r_{\Omega})). \] (3)

Further, equality holds if and only if $\Omega$ is a ball.

We will see later, in Proposition [5] that the upper bound in (3) is given explicitly by
\[ Q_q(B(r)) = \left( \frac{N}{N + p} m(B) \right)^{1/p} r^{1 + N/p} \quad (1 \leq p < \infty). \]

Recent work of the authors [12] shows that there is a harmonic function $h$ on $\Omega$ satisfying $\sup_{\Omega} \|x - \nabla h\| \leq r_{\Omega}$, whence $\lambda_{A_p}(\Omega) \leq (m(B))^{1/p} r_{\Omega}^{1 + N/p}$ for general $p$. We conjecture that balls are always the extremal domains for (3), that is, the sharper estimate of Theorem 3,
\[ \lambda_{A_p}(\Omega) \leq \left( \frac{N}{N + p} m(B) \right)^{1/p} r_{\Omega}^{1 + N/p}, \]
remains valid for all $p \in [1, \infty)$.

Theorems 2 and 3 together yield the following isoperimetric inequality for Bergman $p$-analytic content.

Corollary 4. If $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain and $p \in [1, 2]$, then
\[ Q_q(\Omega) \leq \lambda_{A_p}(\Omega) \leq Q_q(B(r_{\Omega})). \]

The remainder of the paper is devoted to proving the above results.
2 Existence and uniqueness of extremal functions

In the course of proving our results concerning \( \lambda_{A_p}(\Omega) \) and \( Q_q(\Omega) \), we are led to consider the related domain constants

\[
\lambda_{B_p}(\Omega) = \inf \{ \| x - f \|_{L^p} : f \in B_p(\Omega) \},
\]
\[
\lambda_{D_p}(\Omega) = \inf \{ \| x - f \|_{L^p} : f \in D_p(\Omega) \},
\]

where

\[
B_p(\Omega) = \{ \nabla h : h \in W^{1,p}(\Omega) \cap C^2(\Omega) \text{ and } \Delta h = 0 \text{ on } \Omega \},
\]
\[
D_p(\Omega) = \{ f \in L^p : \text{div } f = 0 \text{ on } \Omega \text{ in the sense of distributions} \}.
\]

Since \( B_p(\Omega) \subset A_p(\Omega) \subset D_p(\Omega) \), we see that

\[
\lambda_{D_p}(\Omega) \leq \lambda_{A_p}(\Omega) \leq \lambda_{B_p}(\Omega).
\]

In this section we will prove existence and uniqueness results concerning the extremal functions for \( Q_q(\Omega) \), \( \lambda_{D_p}(\Omega) \), \( \lambda_{B_p}(\Omega) \) and \( \lambda_{A_p}(\Omega) \).

Let \( \Delta_q \) denote the \( q \)-Laplacian, given by \( \Delta_q u = \nabla \cdot \left( \| \nabla u \|^{q-2} \nabla u \right) \), where \( 1 < q < \infty \). We define the \( q \)-torsion function \( w_q \) on \( \Omega \) to be the weak solution of

\[
\begin{cases}
-\Delta_q w_q = 1 & \text{in } \Omega \\
w_q = 0 & \text{on } \partial \Omega,
\end{cases}
\]

and note from [19] that \( w_q \in C^1(\overline{\Omega}) \). Further, we define \( w_\infty(x) = \text{dist}(x, \partial \Omega) \).

**Proposition 5** Let \( \Omega \subset \mathbb{R}^N \) be a smoothly bounded domain and \( p \in [1, \infty) \).

(i) There exists \( u \in W^{1,q}_0(\Omega) \) such that

\[
Q_q(\Omega) = \frac{N}{\| \nabla u \|_{L^q}^q} \int_\Omega u \, dm.
\]

(ii) The functions \( u \in W^{1,q}_0(\Omega) \) which satisfy (6) are precisely the positive multiples of \( w_q \).

(iii) \( Q_q(\Omega) = N \left( \int_\Omega w_q dm \right)^{1/p} \). Further, \( Q_q(\Omega) = N\| \nabla w_q \|_{L_q}^{q-1} \) if \( p > 1 \).

(iv) \( Q_q(B(r)) = \left( \frac{N}{N+p}m(B) \right)^{1/p} r^{1+N/p} \).

**Proof.** (i) We choose a maximizing sequence \( (u_j) \) for (1) such that \( \| u_j \|_{W^{1,q}_0(\Omega)} = 1 \) for all \( j \). (The quotient in (1) is unaffected when \( u \) is multiplied by a positive constant.)

Firstly, we suppose that \( p > 1 \). In view of the Banach-Alaoglu theorem we can arrange, by taking a subsequence, that \( (u_j) \) converges weakly to some non-zero function \( u \in W^{1,q}_0(\Omega) \). Further, by the Rellich-Kondrachov
theorem (see, for example, Section 5.7 in [7]), we can arrange that \( u_j \to u \) strongly in \( L^1(\Omega) \). Clearly \( \int_\Omega u \, dm > 0 \). By the weak lower semicontinuity of the \( L^q \)-norm,

\[
Q_q(\Omega) = \lim_{j \to \infty} \frac{N}{\| \nabla u_j \| \mathcal{L}_q} \int_\Omega u_j \, dm
\]

\[
= \lim_{j \to \infty} \frac{N \int_\Omega u_j \, dm}{\| \nabla u_j \| \mathcal{L}_q} \leq \frac{N \int_\Omega u \, dm}{\| \nabla u \| \mathcal{L}_q} \leq Q_q(\Omega),
\]

and so equality holds throughout.

If \( p = 1 \), whence \( q = \infty \), then we instead appeal to the Arzelà-Ascoli theorem to see that there is a subsequence of \((u_j)\) that converges uniformly on \( \Omega \), and make use of the fact that each \( u_j \) can be represented by a Lipschitz function.

(ii) Suppose firstly that \( p > 1 \). For any \( \phi \in C_0^\infty(\Omega) \) we define

\[
f(t) = (Q_q(\Omega))^q \int_\Omega \| \nabla (u + t\phi) \|^q \, dm - \left( N \int_\Omega (u + t\phi) \, dm \right)^q \quad (t \in \mathbb{R}).
\]

Since \( u \) is a maximizer for \( Q_q(\Omega) \), we see that \( \int_\Omega u \, dm > 0 \) and \( f'(0) = 0 \), whence

\[
(Q_q(\Omega))^q \int_\Omega \| \nabla u \|^{q-2} \nabla u \cdot \nabla \phi \, dm - N^q \left( \int_\Omega u \, dm \right)^{q-1} \int_\Omega \phi \, dm = 0 \quad (\phi \in C_0^\infty(\Omega)),
\]

and so

\[
\int_\Omega \left( (Q_q(\Omega))^q \Delta_q u + N^q \left( \int_\Omega u \, dm \right)^{q-1} \right) \phi \, dm = 0 \quad (\phi \in C_0^\infty(\Omega)).
\]

Thus \( \Delta_q u \) is a negative constant in \( \Omega \), and so \( u \) is a positive multiple of \( w_q \).

Now let \( p = 1 \), so that \( q = \infty \). In the formula (1) we can normalize to consider only those functions \( u \) such that \( \| \nabla u \| \mathcal{L}_\infty = 1 \), whence \( u \) is majorized by the Lipschitz function \( w_\infty \). Further, the supremum can only be attained among functions \( u \) satisfying \( \| \nabla u \| \mathcal{L}_\infty = 1 \) by the function \( w_\infty \). More generally, the supremum can only be attained by a positive multiple of \( w_\infty \).

(iii) If \( p > 1 \), then we see from (1) that

\[
\int_\Omega w_q \, dm = \int_\Omega w_q(-\Delta_q w_q) \, dm = \int_\Omega \| \nabla w_q \|^{q-2} \nabla w_q \cdot \nabla w_q \, dm = \| \nabla w_q \|^{q}_\mathcal{L}_q.
\]

Hence, by parts (i) and (ii), \( Q_q(\Omega) = N \| \nabla w_q \|^{q-1}_\mathcal{L}_q = N \left( \int_\Omega w_q \, dm \right)^{1/p} \).

If \( p = 1 \), then it is immediate that \( Q_\infty(\Omega) = N \int_\Omega w_\infty \, dm \).
(iv) If \( \Omega = B(r) \), then \( w_q(x) \) is clearly a multiple of \( x \mapsto r^p - \|x\|^p \).

Letting \( u(x) = (r^p - \|x\|^p) / p \), we have \( \|\nabla u\| = \|x\|^{p-1} \) and

\[
\|\nabla u\| \mathcal{L}_q = \left( \frac{Nm(B)}{N+p} r^{N+p} \right)^{1/q} , \quad \int_\Omega u \ dm = \frac{m(B)}{N+p} r^{N+p}.
\]

Thus, by parts (i) and (ii),

\[
Q_q(B(r)) = \frac{N}{\|\nabla u\| \mathcal{L}_q} \int_\Omega u \ dm = \left( \frac{N}{N+p} m(B) \right)^{1/p} r^{1+N/p}.
\]

As usual, we define

\[
D_p(\Omega) = \left\{ g \in \mathcal{L}_q(\Omega) : \int_\Omega f \cdot g \ dm = 0 \text{ for all } f \in D_p(\Omega) \right\}.
\]

\( B_p(\Omega) \) and \( A_p(\Omega) \) are defined analogously.

**Proposition 6** Let \( \Omega \subset \mathbb{R}^N \) be a smoothly bounded domain and \( p \in (1, \infty) \).

(i) There exists \( f_0 \in D_p(\Omega) \) such that \( \lambda_{D_p}(\Omega) = \|x - f_0\| \mathcal{L}_p \).

(ii) This function \( f_0 \) satisfies \( \|x - f_0\|^{p-2} (x - f_0) \in D_p(\Omega)^\perp \).

(iii) There exists \( u_0 \in W^{1,q}_0(\Omega) \) such that \( \nabla u_0 = -\|x - f_0\|^{p-2} (x - f_0) \).

(iv) The function \( u_0 \) is a positive multiple of \( w_q \), and

\[
\lambda_{D_p}(\Omega) = Q_q(\Omega) = \frac{-1}{\|\nabla w_q\| \mathcal{L}_q} \int_\Omega (x - f) \cdot \nabla w_q \ dm \quad (f \in D_p(\Omega)). \quad (7)
\]

**Proof.**

(i) We choose a sequence \((f_j)\) in \( D_p(\Omega) \) such that \( \|x - f_j\| \mathcal{L}_p \to \lambda_{D_p}(\Omega) \). By weak compactness we can arrange, by choosing a suitable subsequence, that \((f_j)\) is weakly convergent to some \( f_0 \) in \( \mathcal{L}_p \). Further, \( \text{div} f_0 = 0 \) on \( \Omega \) in the sense of distributions, so \( f_0 \in D_p(\Omega) \). Finally, \( \lambda_{A_p}(\Omega) = \|x - f_0\| \mathcal{L}_p \) by the weak lower semicontinuity of the norm.

(ii) For any \( g \in D_p(\Omega) \) we can differentiate the function \( t \mapsto \int_\Omega \|x - f_0 - tg\|^{p} \ dm \) and then set \( t = 0 \) to see that

\[
\int_\Omega \|x - f_0\|^{p-2} (x - f_0) \cdot g \ dm = 0.
\]

(iii) If \( f \in \mathcal{L}_p \), then by definition,

\[
f \in D_p(\Omega) \Leftrightarrow \int_\Omega f \cdot \nabla \phi \ dm = 0 \quad (\phi \in C_c^\infty(\Omega)). \quad (8)
\]

Hence

\[
D_p(\Omega)^\perp = \{ \nabla \phi : \phi \in C_c^\infty(\Omega) \} ^{\mathcal{L}_q(\Omega)} \quad (9)
\]
since, if $g \in L^q(\Omega)$ does not belong to the above closure, the Hahn-Banach theorem would yield the existence of $f \in L^q(\Omega) \equiv L^p(\Omega)$ such that
\[
\int_{\Omega} f \cdot g \, dm = 1, \quad \int_{\Omega} f \cdot \nabla \phi \, dm = 0 \quad (\phi \in C_\infty^c(\Omega)),
\]
whence $f \in D^p(\Omega)$ and so $g \not\in D^p(\Omega)\perp$.

We claim next that\[D^p(\Omega) \perp = \{\nabla u : u \in W^{1,q}_0(\Omega)\}\] (10)
Clearly the right hand side of (10) is contained in the right hand side of (9).
To see the reverse inclusion, let \((\phi_k)\) be a sequence in $C_\infty^c(\Omega)$ such that \((\nabla \phi_k)\) converges in $L^q(\Omega)$. Then \((\phi_k)\) is Cauchy in $L^q(\Omega)$, by Poincaré’s inequality for $W^{1,q}_0(\Omega)$. It follows that \((\phi_k)\) converges in $W^{1,q}_0(\Omega)$ to some function $u$ and $\lim_{k \to \infty} \nabla \phi_k = \nabla u$. Hence (10) holds, and the desired conclusion now follows from part (ii).

(iv) By the divergence theorem, (10) and Hölder’s inequality,
\[
\frac{N}{\|\nabla u\|_{L^q}} \int_{\Omega} u \, dm = \frac{-1}{\|\nabla u\|_{L^q}} \int_{\Omega} x \cdot \nabla u \, dm
\]
\[= \frac{-1}{\|\nabla u\|_{L^q}} \int_{\Omega} (x - f_0) \cdot \nabla u \, dm\]
\[\leq \|x - f_0\|_{L^p} \quad (u \in W^{1,q}_0(\Omega) \setminus \{0\}),
\]
with equality precisely when $\nabla u$ is a negative multiple of $\|x - f_0\|^{p-2} (x - f_0)$. It now follows from (1) and Proposition 5(ii) that $u_0$ is a positive multiple of $w_q$, and from part (i) and (10) that (7) holds.

The next result shows that (7) also holds when $p = 1$. Inequality (2) will follow from (11) in view of (4).

**Proposition 7** If $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain and $p \in [1, \infty)$, then
\[
\lambda_{D^p}(\Omega) = Q_q(\Omega) = \frac{-1}{\|\nabla w_q\|_{L^q}} \int_{\Omega} (x - f) \cdot \nabla w_q \, dm \quad (f \in D^p(\Omega)).
\] (11)

**Proof.** We know from Theorem 1 of [17] that $w_q \to w_\infty$ uniformly on $\Omega$ as $q \to \infty$. Since the function $p \mapsto (m(\Omega))^{-1/p} \lambda_{D^p}(\Omega)$ is increasing, we see from Propositions 6(iv) and 5(iii) that
\[
\lambda_{D^1}(\Omega) \leq (m(\Omega))^{1-1/p} \lambda_{D^p}(\Omega) = N (m(\Omega))^{1-1/p} \left(\int_{\Omega} w_q \, dm\right)^{1/p}
\]
\[\to N \int_{\Omega} w_\infty \, dm \quad (p \to 1)
\]
\[= Q_\infty(\Omega). \quad (12)
\]
For large $k \in \mathbb{N}$ let $v_k$ be a mollification of $(w_\infty - k^{-1})^+$ that belongs to $C^\infty_c(\Omega)$. Since $\nabla v_k \in D_1(\Omega)^\perp$ by (8), and $(\nabla v_k)$ is boundedly convergent almost everywhere to $\nabla w_\infty$, we see that $\nabla w_\infty \in D_1(\Omega)^\perp$. Thus, by the divergence theorem,

$$Q_\infty(\Omega) = N \int \nabla w_\infty \cdot dm = - \int x \cdot \nabla w_\infty \cdot dm = - \int (x-f) \cdot \nabla w_\infty \cdot dm \leq \|x-f\|_L^1 \quad (f \in D_1(\Omega)).$$

Hence $Q_\infty(\Omega) \leq \lambda_{D_1}(\Omega)$, and (11) follows in view of (12). \hfill \blacksquare

We note that

$$\lambda_{B_p}(\Omega) = \inf\{\|\nabla u\|_{L_p} : u \in W^{1,p}(\Omega) \text{ and } \Delta u = Nm \text{ in } \Omega\}. \quad (13)$$

**Proposition 8** Let $p \in [1, \infty)$.

(i) There exists $f \in B_p(\Omega)$ such that $\lambda_{B_p}(\Omega) = \|x-f\|_{L_p}$; equivalently, there exists $u_0 \in W^{1,p}(\Omega)$ such that $\Delta u_0 = Nm$ in $\Omega$ and $\lambda_{B_p}(\Omega) = \|\nabla u_0\|_{L_p}$.

(ii) The function $u_0$ satisfies $\|\nabla u_0\|^{p-2} \nabla u_0 \in B_p(\Omega)^\perp$.

(iii) The function $f_0 = \|\nabla u_0\|^{p-2} \nabla u_0 \in L_q$ satisfies $\lambda_{B_p}(\Omega) = \left(\int _\Omega f_0 \cdot x \cdot dm\right) / \|f_0\|_{L_q}$.

(iv) The function $u_0$ is uniquely determined by the properties

$$\left\{ \begin{array}{l} \|\nabla u_0\|^{p-2} \nabla u_0 \in B_p(\Omega)^\perp \\quad \Delta u_0 = Nm \text{ in } \Omega \end{array} \right.. \quad (14)$$

**Proof.** (i) We can choose a minimizing sequence $(u_j)$ for (13), where $\int _\Omega u_j \cdot dm = 0$ for each $j$. By Poincaré’s inequality $(u_j)$ is bounded in $W^{1,p}(\Omega)$, and by the Rellich-Kondrachov theorem we can arrange that $(u_j)$ converges strongly in $L^1(\Omega)$ to a function $u_0$. Since the functions $\{u_j : j \geq 0\}$ all have distributional Laplacian equal to $Nm$, we can choose smooth representatives of these functions and arrange that $u_j \to u_0$ and $\partial u_j / \partial x_i \to \partial u_0 / \partial x_i$ locally uniformly on $\Omega$ for each $i$. Now

$$\left| \int _\Omega \nabla u_j \cdot \phi \cdot dm \right| \leq \|\nabla u_j\|_{L_p} \|\phi\|_{L_q} \quad (\phi \in (C^\infty_c(\Omega))^N; j \geq 1),$$

so we can let $j \to \infty$ and use the density of $C^\infty_c(\Omega)$ in $L^q(\Omega)$ to see that $\|\nabla u_0\|_{L_p} \leq \lambda_{B_p}(\Omega)$. (When $p = 1$ and so $q = \infty$, we instead use the fact that, for any $g \in (L^\infty(\Omega))^N$, there is a sequence $(\phi_n)$ in $(C^\infty_c(\Omega))^N$ that converges pointwise almost everywhere to $g$ on $\Omega$ and satisfies $\sup_\Omega \|\phi_n\| \leq \text{ess sup}_\Omega \|g\|$ for all $n$.) Similarly, $u_0 \in L^p(\Omega)$, so $u_0 \in W^{1,p}(\Omega)$ and $u_0$ is a minimizer for (13).
(ii) Given any $h \in W^{1,p}(\Omega) \cap C^2(\Omega)$ such that $\Delta h = 0$ on $\Omega$, we differentiate $\|\nabla (u_0 + th)\|^p_{L^p}$ with respect to $t$ and then put $t = 0$ to see that

$$\int_\Omega \|\nabla u_0\|^{p-2} \nabla u_0 \cdot \nabla h \, dm = 0. \quad (15)$$

(When $p = 1$, we know that $m(\{\|\nabla u_0\| = 0\}) = 0$, and the above equation still follows by dominated convergence, since $\|\nabla (u_0 + th)\| - \|\nabla u_0\| / t \leq \|\nabla h\|$.) Thus $\|\nabla u_0\|^{p-2} \nabla u_0 \in B_p(\Omega)^\perp$.

(iii) If we take $h = u_0 - \|x\|^2/2$ in $(15)$, then we find that

$$\int_\Omega f_0 \cdot x \, dm = \int_\Omega \|\nabla u_0\|^p \, dm.$$

Since $\|f_0\|_{L^q} = \|\nabla u_0\|^{p/q}_{L^p}$, we obtain the desired equality.

(iv) In view of parts (i) and (ii) it only remains to check that $(14)$ uniquely determines $u_0$ up to a constant. (When $p > 1$, the uniqueness of the gradient $\nabla u_0$ also follows from the strict convexity of the $L_p$-norm.) To see this, let $v$ be another such function and consider the harmonic function $v - u_0$. It follows from $(15)$ that

$$\int_\Omega \|\nabla u_0\|^p \, dm = \int_\Omega \|\nabla v\|^{p-2} \nabla u_0 \cdot \nabla v \, dm$$

and

$$\int_\Omega \|\nabla v\|^p \, dm = \int_\Omega \|\nabla v\|^{p-2} \nabla v \cdot \nabla u_0 \, dm.$$

Hölder’s inequality now shows that $\|\nabla u_0\|_{L^p} = \|\nabla v\|_{L^p}$, and we deduce that $\nabla u_0 \equiv \nabla v$. (If $p = 1$, then Hölder’s inequality is unnecessary.)

**Proposition 9** Let $p \in [1, \infty)$.

(i) There exists $f \in A_p(\Omega)$ such that $\lambda_{A_p}(\Omega) = \|x - f\|_{L^p}$.

(ii) The function $f$ satisfies $\|x - f\|^{p-2}(x - f) \in A_p(\Omega)^\perp$.

(iii) The function $f_0 = \|x - f\|^{p-2}(x - f) \in L_q$ satisfies $\lambda_{A_p}(\Omega) = (\int_\Omega f_0 \cdot x \, dm) / \|f_0\|_{L^q}$.

(iv) The function $f$ is uniquely determined by the properties

$$\begin{cases} \|x - f\|^{p-2}(x - f) \in A_p(\Omega)^\perp \\
\text{div } f = 0 \text{ and } \text{curl } f = 0 \text{ in } \Omega \end{cases}.$$

**Proof.** (i) We choose a sequence $(f^{(j)})$ in $A_p(\Omega)$ such that $\|x - f^{(j)}\|_{L^p} \to \lambda_{A_p}(\Omega)$. Since $(\|f^{(j)}\|_{L^p})$ is bounded and the functions $\|f^{(j)}\|$ are subharmonic (by Theorem 3.4.5 of [2]), the harmonic co-ordinate functions $f^{(j)}_i$ ($i = 1, \ldots, N$) are locally uniformly bounded. Thus, by taking a subsequence,
we can arrange that \((f^{(j)})\) converges locally uniformly to some function \(f\) satisfying \(\text{div} \ f = 0\) and \(\text{curl} \ f = 0\) on \(\Omega\). Since
\[
\left| \int (x - f^{(j)}) \cdot \phi \ dm \right| \leq \left\| x - f^{(j)} \right\|_{L^p} \left\| \phi \right\|_{L^q} \quad (\phi \in (C_c^\infty(\Omega))^N),
\]
we can let \(j \to \infty\) and use the density of \(C_c^\infty(\Omega)\) in \(L^q(\Omega)\) to see that \(\left\| x - f^{(j)} \right\|_{L^p} \leq \lambda_{A_p}(\Omega)\). (When \(p = 1\) we make the same adjustments to this argument as in the proof of Proposition 8(i).) The reverse inequality is trivial.

(ii) - (iv) The arguments are analogous to those given for the previous proposition.

3 Proofs of Theorems 2 and 3

As noted previously, inequality (2) follows from (11) and (4). In this section we will complete the proofs of Theorem 2 (except where \(p = 2\)) and Theorem 3. In view of (4) and Proposition 7, Theorem 3 is a consequence of the result below.

**Theorem 10** If \(\Omega \subset \mathbb{R}^N\) is a smoothly bounded domain and \(p \in [1, 2]\), then
\[
\lambda_{B_p}(\Omega) \leq Q_q(B(r\Omega)).
\]
Further, equality holds if and only if \(\Omega\) is a ball.

**Proof.** Let \(u\) be the Green potential satisfying \(\Delta u = N\) on \(\Omega\) and \(u = 0\) on \(\partial\Omega\). Next, let \(w(x) = (\|x\|^2 - r_{\Omega}^2)/2\), so that \(\Delta w = N\) in \(B(r\Omega)\) and \(w = 0\) on \(\partial B(r\Omega)\). We make use of a result of Talenti [23] concerning spherical rearrangements. Theorem 1(v) of that paper tells us that, provided \(p \leq 2\), we have \(\left\| \nabla u \right\|_{L^p(\Omega)} \leq \left\| \nabla w \right\|_{L^p(B(r\Omega))}\). Hence
\[
\lambda_{B_p}(\Omega) \leq \left\| \nabla u \right\|_{L^p} \leq \left\{ \int_{B(r\Omega)} \|x\|^p dm \right\}^{1/p} = \left\{ \frac{N}{N + p} m(B) r_{\Omega}^{p+N} \right\}^{1/p} = Q_q(B(r\Omega)),
\]
by (13) and then Proposition 5(iv).

Finally, if \(\left\| \nabla u \right\|_{L^p(\Omega)} = \left\| \nabla w \right\|_{L^p(B(r\Omega))}\), then Propositions 3.2.1 and 3.2.2 of Kesavan [18] tell us that \(\Omega\) must be a ball. ■

**Lemma 11** Let \(p \in [1, \infty)\). If \(\Omega\) is either a ball or an annular region, then
\[
\lambda_{B_p}(\Omega) = \lambda_{A_p}(\Omega) = \lambda_{D_p}(\Omega) = Q_q(\Omega).
\]

10
Proof. In view of (4) and Proposition 7 it is enough to show that \( \lambda_{B_p}(\Omega) \leq Q_q(\Omega) \) when \( \Omega \) is either a ball or an annular region. If \( \Omega = B(r) \), then (cf. (17))

\[
\lambda_{B_p}(B(r)) \leq \|x\|_{L_p} = Q_q(B(r)).
\]

Thus it remains to consider the case where \( \Omega = B(R) \setminus B(r) \) and \( 0 < r < R \).

If \( p > 1 \), then it follows from spherical symmetry that there exists \( v \in C^1(\Omega) \) such that \( \nabla v = \|\nabla w_q\|^{q-2} \nabla w_q \). Writing \( w = -Nv \), we see that

\[
\Delta w = -N\Delta_q w_q = N \quad \text{and so, by (13) and Proposition 5(iii)},
\]

\[
\lambda_{B_p}(\Omega) \leq \|\nabla w\|_{L_p} = N \|\nabla w_q\|_{L_q}^{q-1} = Q_q(\Omega),
\]

as required.

Now suppose that \( p = 1 \). By Proposition 5(iii) again,

\[
Q_\infty(\Omega) = N \int_{\Omega} w_\infty(x) \, dm
\]

\[
= N \left( \int_{\{r < \|x\| < (R+r)/2\}} (\|x\| - r) \, dm(x) + \int_{\{(R+r)/2 < \|x\| < R\}} (R - \|x\|) \, dm(x) \right)
\]

\[
= \frac{Nm(B)}{N+1} \left( R^{N+1} + r^{N+1} - \frac{(R+r)^{N+1}}{2^N} \right). \tag{18}
\]

If we define

\[
u(x) = \begin{cases} \frac{\|x\|^2}{2} + \frac{1}{N-2} \left( \frac{R+r}{2} \right)^N \|x\|^{2-N} & (N \geq 3) \\ \frac{\|x\|^2}{2} - \left( \frac{R+r}{2} \right)^2 \log \|x\| & (N = 2) \end{cases},
\]

then

\[
\lambda_{B_1}(\Omega) \leq \|\nabla u\|_{L_1} = \int_{\Omega} \left( \|x\| - \left( \frac{R+r}{2} \right)^N \|x\|^{1-N} \right) \, dm
\]

\[
= \int_{\{r < \|x\| < (R+r)/2\}} \left( \frac{R+r}{2} \right)^N \|x\|^{1-N} \, dm
\]

\[
+ \int_{\{(R+r)/2 < \|x\| < R\}} \left( \|x\| - \left( \frac{R+r}{2} \right)^N \|x\|^{1-N} \right) \, dm
\]

\[
= \frac{Nm(B)}{N+1} \left( R^{N+1} + r^{N+1} - \frac{(R+r)^{N+1}}{2^N} \right) = Q_\infty(\Omega),
\]

by (18). □
Proposition 12. Let $p \in [1, \infty)$. If there exists $f \in A_p(\Omega)$ satisfying $\|x-f\|_{L_p} = Q_q(\Omega)$, then $f \in B_p(\Omega)$.

Proof. First suppose that $p > 1$, so that $q < \infty$. By (4) and Proposition 7

$$\|x-f\|_{L_p} = \frac{1}{\|\nabla w_q\|_{L_q}} \int_\Omega (x-f) \cdot \nabla w_q \, dm.$$  

Since

$$-\int_\Omega (x-f) \cdot \nabla w_q \, dm = \|x-f\|_{L_p} \|\nabla w_q\|_{L_q}, \quad (19)$$

the equality case of Hölder’s inequality implies that

$$x - f = c \|\nabla w_q\|^{q-2} \nabla w_q$$

on $\Omega$ for some constant $c$. Hence $x - f$ has a continuous extension to $\Omega$, and on $\partial \Omega$ it is normal to $\partial \Omega$. We now choose a function $v \in C^1(\mathbb{R}^N \setminus \Omega)$ such that $v = 0$ and $\nabla v = x - f$ on $\partial \Omega$. (Such a function exists by [13], for example.) Thus we obtain a continuous extension of $f$ to $\mathbb{R}^N$ by defining it to be $x - \nabla v$ on $\mathbb{R}^N \setminus \Omega$.

We claim that this extended function, which we also denote by $f$, is curl-free in the sense of distributions. By using a partition of unity it is enough to show that, for some $\delta > 0$,

$$\int \left( f_i \frac{\partial \phi}{\partial x_j} - f_j \frac{\partial \phi}{\partial x_i} \right) \, dm = 0 \quad (i \neq j) \quad (20)$$

whenever $\phi \in C^\infty(\mathbb{R}^N)$ and $\text{diam}(\text{supp}(\phi)) < \delta$. This equation trivially holds when $\text{supp}(\phi) \cap \partial \Omega = \emptyset$, so it is enough to consider the case where

$$\text{supp}(\phi) \subset K := \bigtimes_{i=1}^N (y_i - r, y_i + r)$$

for some $y \in \partial \Omega$ and $r > 0$.

Without loss of generality we may assume that

$$K \cap \partial \Omega = \{(x_1, \ldots, x_{N-1}, g(x_1, \ldots, x_{N-1})) : x_i \in (y_i - r, y_i + r) \text{ whenever } i < N\}$$

for some smooth function $g$. If $i < N$ and the co-ordinates $x_j \,(j \neq i, N)$ are fixed, then

$$\int_{y_{N-r}}^{y_{N+r}} \int_{y_{i-r}}^{y_{i+r}} \left( f_i \frac{\partial \phi}{\partial x_N} - f_N \frac{\partial \phi}{\partial x_i} \right) \, dx_i dx_N \cdot dA(x_i, x_N),$$

where $D_1, D_2$ are the components of $\{(x_i, x_N) : (x_1, \ldots, x_N) \in K \setminus \partial \Omega\}$ and $A$ denotes two-dimensional measure. Two applications of Green’s theorem, together with the fact that $\partial f_i / \partial x_N = \partial f_N / \partial x_i$ on $\mathbb{R}^N \setminus \partial \Omega$, show that this
latter integral expression reduces to self-cancelling terms along the common boundary curve of \(D_1, D_2\). Hence (20) holds when \(j = N\). If \(j \neq N\), we apply a small rotation in the \((x_j, x_N)\)-plane to see similarly that
\[
\int \left\{ f_i \left( \frac{\partial \phi}{\partial x_N} \cos \theta + \frac{\partial \phi}{\partial x_j} \sin \theta \right) - (f_N \cos \theta + f_j \sin \theta) \frac{\partial \phi}{\partial x_i} \right\} \, dm = 0,
\]
whence (20) again follows.

We now use a rotationally invariant smoothing kernel \(\psi_\varepsilon\) supported by a ball of radius \(\varepsilon\) to obtain a mollification \(f_\varepsilon\) of \(f\), which is also curl-free since \(\frac{\partial}{\partial x_j} \psi_\varepsilon(x - y) = -\frac{\partial}{\partial y_j} \psi_\varepsilon(x - y)\).

Further, since each component \(f_i\) of \(f\) is harmonic in \(\Omega\), the functions \(f_\varepsilon\) and \(f\) are equal on the set \(\{x : \text{dist}(x, \mathbb{R}^N \setminus \Omega) > \varepsilon\}\). Hence line integrals of \(f\) in \(\Omega\) are path independent, so \(f\) is of the form \(\nabla v\), where \(\Delta v = \nabla \cdot f = 0\), and thus \(f \in B_p(\Omega)\).

Finally, if \(p = 1\), then (19) still holds, and now shows that \(x - f = -\|x - f\| \nabla w_\infty\) on \(\Omega\). We can thus apply the above argument to \(\Omega_\eta = \{x : \text{dist}(x, \mathbb{R}^N \setminus \Omega) > \eta\}\) to deduce that \(f \in B_1(\Omega_\eta)\) for arbitrarily small \(\eta > 0\), and so \(f \in B_1(\Omega)\).

We now consider the overdetermined problem
\[
\begin{cases}
\Delta v = 1 & \text{in } \Omega \\
v = c_i \text{ and } \frac{\partial v}{\partial n} = a_i & \text{on } \Gamma_i,
\end{cases}
\tag{21}
\]
where \(n\) denotes the exterior unit normal, \(a_i, c_i \in \mathbb{R}\) (\(i = 0, \ldots, j\)) and \(\{\Gamma_i\}\) are the components of \(\partial \Omega\). (We use \(\Gamma_0\) for the outer boundary component.) The following theorem, which generalizes earlier work of Serrin [21], is contained in Theorem 2 of Sirakov [22].

**Theorem 13** Let \(c_0 = 0, a_0 \geq 0,\) and \(c_i < 0, a_i \leq 0\) (\(i = 1, \ldots, j\)). Then there exists \(v \in C^2(\Omega)\) satisfying (21) if and only if \(\Omega\) is a ball or an annular region. In either case, \(v\) is a radial function.

The case \(p \neq 2\) of the equality statement in Theorem 2 is established in the next result. The case where \(p = 2\) will be addressed in Section 4.

**Theorem 14** Let \(p \in [1, \infty),\) where \(p \neq 2\). Then \(\lambda_{A_p}(\Omega) = Q_q(\Omega)\) if and only if \(\Omega\) is either a ball or an annular region.

**Proof.** For the “if” part we refer to Lemma 11. For the “only if” part it is enough, given Propositions 9 and 12 to show that, if there exists \(v \in\)
$W^{1,p}(\Omega)$ such that $\Delta v = Nm$ and $\|\nabla v\|_{L^p} = Q_\eta(\Omega)$, then $\Omega$ is either a ball or an annular region.

If $p > 1$, then we see from Proposition 5 that

$$Q_\eta(\Omega) = \frac{\int_\Omega w_q \text{dm}}{\|\nabla w_q\|_{L^q}} = \frac{\int_\Omega w_q \Delta v \text{dm}}{\|\nabla w_q\|_{L^q}} = -\frac{\int_\Omega \nabla w_q \cdot \nabla v \text{dm}}{\|\nabla w_q\|_{L^q}},$$

where the last equality can be justified using the facts that $w_q \in W^{1,q}_0(\Omega)$ and that $C^{\infty}_c(\Omega)$ is dense in $W^{1,q}_0(\Omega)$. By Hölder’s inequality,

$$\left| \int_\Omega \nabla w_q \cdot \nabla v \text{dm} \right| \leq \|\nabla v\|_{L^p} \|\nabla w_q\|_{L^q},$$

where equality occurs if and only if $\nabla w_q, \nabla v$ are always parallel, $\nabla w_q \cdot \nabla v$ does not change sign, and $\|\nabla w_q\|^q = c\|\nabla v\|^p$ in $\Omega$ for some constant $c > 0$. Further, $\nabla w_q \neq 0$ on $\partial \Omega$ by Hopf’s lemma (see Theorem 5.5.1 of [20]). Thus the equality $Q_\eta(\Omega) = \|\nabla v\|_{L^p}$ implies that each component of any level surface of $v$ is also a component of a level surface of $w_q$. Hence, for each component $\Gamma_i$ of $\partial \Omega$, there is a function $g_i$ such that $v = g_i \circ w_q$ near $\Gamma_i$. Further, $\nabla v = g_i'(w_q)\nabla w_q$, so

$$\|\nabla v\| = |g_i'(w_q)| \|\nabla w_q\| = c^{1/q} |g_i'(w_q)| \|\nabla v\|^{p/q}.$$  

Since $p \neq 2$, we have $p \neq q$ and so $\|\nabla v\| = c^{1/(q-p)} |g_i'(w_q)|^{q/(q-p)}$. Thus $\|\nabla v\|$ is constant on each component of a level surface of $w_q$ (which is also a level surface of $v$).

Since $\nabla w_q \cdot n < 0$ on $\partial \Omega$ and $\nabla w_q \cdot \nabla v$ does not change sign, we can apply the divergence theorem to $\nabla v$ to see that $\nabla w_q \cdot \nabla v < 0$ near $\partial \Omega$ and hence $\nabla v \cdot n > 0$ on $\Gamma_0$. Now let $\varepsilon > 0$ be small and let $\Omega_\varepsilon$ be the component of $\{0 < w_q < \varepsilon\}$ which has $\Gamma_0$ as a boundary component. Since $w_q \in C^1(\Omega)$, $\|\nabla w_q\|^q = c\|\nabla v\|^p$ and $\|\nabla v\| = c^{1/(q-p)}|g_0'(w_q)|^{q/(q-p)}$, it follows that $|g_0'(0)| < \infty$ and certainly $|g_0(0)| < \infty$. Thus $v$ has a (finite) constant value on each component of $\partial \Omega_\varepsilon$. Since $\Delta v = Nm$ on $\Omega$ we conclude that $v \in C^2(\bar{\Omega}_\varepsilon)$ (see Theorem 6.14 of [14]). Thus we can apply Theorem 13 to $(v - g_0(0))/N$ on $\Omega_\varepsilon$ to see that $\Gamma_0$ is a sphere and $v$ is a radial function. By the analyticity of $v$, any other boundary of component of $\Omega$ must be a concentric sphere. Thus $\Omega$ is either a ball or an annular region.

The argument for the case $p = 1$ is mostly similar. Since $\|\nabla w_\infty\| = 1$, we have

$$Q_{\infty}(\Omega) = N \int_\Omega w_\infty \text{dm} = -\int_\Omega \nabla w_\infty \cdot \nabla v \text{dm}$$

and

$$\left| \int_\Omega \nabla w_\infty \cdot \nabla v \text{dm} \right| \leq \|\nabla v\|_{L^1}.$$
The equality $Q_\infty(\Omega) = \|\nabla v\|_{L^1}$ implies that $\nabla w_\infty$ and $\nabla v$ are parallel. Thus, for each component $\Gamma_i$ of $\partial \Omega$, there is a function $g_i$ such that $v = g_i \circ w_\infty$ near $\Gamma_i$. We no longer claim that this equation holds on $\partial \Omega$. However, as in the proof of Proposition 12 we can work instead with $\Omega_\eta = \{ x : \text{dist}(x, \mathbb{R}^N \setminus \Omega) > \eta \}$ for small $\eta > 0$ and now argue as before to conclude that $\Omega$ is either a ball or an annular region.

4 The case where $p = 2$

It follows from Proposition 8(i) that there exist harmonic functions $h$ satisfying $\lambda_{B_2}(\Omega) = \| x - \nabla h \|_{L^2}$. We will now identify all such functions. (This was already done in [8] in the case of planar domains.)

**Theorem 15** The harmonic functions $h \in W^{1,2}(\Omega)$ which satisfy $\lambda_{B_2}(\Omega) = \| x - \nabla h \|_{L^2}$ are precisely the functions of the form $H_{\Omega} \| x \|_2^2/2 + c$, where $H_{\Omega}$ is the solution to the Dirichlet problem on $\Omega$ with boundary data $g$, and $c \in \mathbb{R}$.

**Proof.** Let $h \in W^{1,2}(\Omega)$ be a harmonic function satisfying $\lambda_{B_2}(\Omega) = \| x - \nabla h \|_{L^2}$, and let $k \in C^1(\Omega)$ be harmonic on $\Omega$. Since the function $t \mapsto \| x - \nabla (h + tk) \|_2^2$ has a minimum at $t = 0$, we see that

$$\int_{\Omega} (x - \nabla h) \cdot \nabla k \, dm = 0.$$ 

Hence, by the divergence theorem,

$$\int_{\partial \Omega} \left( \frac{\|x\|^2}{2} - h \right) \frac{\partial k}{\partial n} \, d\sigma = 0,$$ 

where $\sigma$ denotes surface area measure. Since we can solve the Neumann problem

$$\begin{cases} \Delta k = 0 & \text{in } \Omega \\ \frac{\partial k}{\partial n} = \phi & \text{on } \partial \Omega \end{cases}$$

for any smooth function $\phi$ satisfying $\int_{\partial \Omega} \phi \, d\sigma = 0$, we see from (22) that $\|x\|^2/2 - h(x)$ is constant on $\partial \Omega$. ■

The torsional rigidity of $\Omega$ is defined by

$$\rho(\Omega) = \int_{\Omega} \|\nabla v\|^2 \, dm,$$

where $v$ is the solution to the Dirichlet problem

$$\begin{cases} -\Delta v = N & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_0 \\ v = c_i & \text{on } \Gamma_i \text{ for } i = 1, \ldots, j \end{cases};$$

(23)
here \( \Gamma_0 \) is again the boundary of the unbounded component of \( \mathbb{R}^N \setminus \Omega \), while \( G_1, G_2, \ldots, G_j \) are the bounded components of \( \Omega^c \) with boundaries \( \Gamma_1, \Gamma_2, \ldots, \Gamma_j \) and the constants \( c_i \) are chosen so that
\[
\int_{\Gamma_i} \frac{\partial v}{\partial n} \, d\sigma = 2m(G_i) \quad (i = 1, 2, \ldots, j).
\] (24)

From Proposition \ref{prop:boundary_integrals} iii) we see that
\[
Q_2(\Omega) = \sqrt{\rho(\Omega)}
\] (25)
when \( \mathbb{R}^N \setminus \Omega \) has no bounded components.

Theorem \ref{thm:main} is contained in the result below.

**Theorem 16** If \( \Omega \subset \mathbb{R}^N \) is a smoothly bounded domain, then
\[
\lambda_{B^2}(\Omega) = \lambda_{A^2}(\Omega) = \lambda_{D^2}(\Omega) = Q_2(\Omega).
\] (26)

Further, these quantities are equal to \( \sqrt{\rho(\Omega)} \) if and only if \( \mathbb{R}^N \setminus \Omega \) is connected.

**Proof of Theorem 16** Let \( u(x) = H_{\Omega}^{\|x\|^2/2} - \|x\|^2/2 \). By Theorem \ref{thm:isoperimetric}
\[
(\lambda_{B^2}(\Omega))^2 = \int_{\Omega} \|x - \nabla H_{\Omega}^{\|x\|^2/2}\|^2 \, dx = \int_{\Omega} \|\nabla u\|^2 \, dx = N \int_{\Omega} u \, dx,
\]
where for the last step we applied the divergence theorem and noted that \( u = 0 \) on \( \partial \Omega \). Hence
\[
\lambda_{B^2}(\Omega) = \|\nabla u\|_{L^2} = \frac{N \int_{\Omega} u \, dm}{\|\nabla u\|_{L^2}} \leq Q_2(\Omega).
\]
Equation (26) now follows from Proposition \ref{prop:isoperimetric} and (25).

We know from (25) that \( Q_2(\Omega) = \sqrt{\rho(\Omega)} \) if \( \mathbb{R}^N \setminus \Omega \) is connected. Conversely, suppose that \( \mathbb{R}^N \setminus \Omega \) is not connected, and let \( c_k = \min\{c_1, \ldots, c_j\} \).

If \( c_k \leq 0 \), then the Hopf boundary point lemma (see Section 6.4.2 of \cite{[7]}) would tell us that \( \partial v/\partial n < 0 \) on \( \Gamma_k \), which contradicts (24). Thus \( c_i > 0 \) (\( i = 1, \ldots, j \)) in (23), so \( v \) cannot be a multiple of \( w_2 \), and it now follows from Proposition \ref{prop:isoperimetric} that \( Q_2(\Omega) > \sqrt{\rho(\Omega)} \).

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*Stephen J. Gardiner and Marius Ghergu*

School of Mathematics and Statistics
University College Dublin
Dublin 4, Ireland
stephen.gardiner@ucd.ie
marius.ghergu@ucd.ie

*Tomas Sjödin*

Department of Mathematics
Linköping University
581 83, Linköping
Sweden
tomas.sjodin@liu.se