Universal lifting theorem and quasi-Poisson groupoids

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Abstract

We prove the universal lifting theorem: for an $\alpha$-simply connected and $\alpha$-connected Lie groupoid $\Gamma$ with Lie algebroid $A$, the graded Lie algebra of multi-differentials on $A$ is isomorphic to that of multiplicative multi-vector fields on $\Gamma$. As a consequence, we obtain the integration theorem for a quasi-Lie bialgebroid, which generalizes various integration theorems in the literature in special cases.

The second goal of the paper is the study of basic properties of quasi-Poisson groupoids. In particular, we prove that a group pair $(D, G)$ associated to a Manin quasi-triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ induces a quasi-Poisson groupoid on the transformation groupoid $G \times D/G \rightrightarrows D/G$. Its momentum map corresponds exactly with the $D/G$-momentum map of Alekseev and Kosmann-Schwarzbach.

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1 Introduction

The notion of quasi-Lie bialgebroids, first introduced by Roytenberg [23], is a natural generalization of Lie bialgebroids [19]. It also generalizes Drinfeld’s quasi-Lie bialgebras [8], the classical limit of quasi-Hopf algebras. Roytenberg defines quasi-Lie bialgebroids using Lie algebroid analogues of Manin quasi-triples. Equivalently, one may define a quasi-Lie bialgebroid as follows. Given a Lie algebroid \( A \), it is known that the anchor map together with the bracket on \( \Gamma(A) \) extends to a graded Lie bracket on \( \oplus k \Gamma(\wedge^k A) \), which makes it into a Gerstenhaber algebra \( (\oplus k \Gamma(\wedge^k A), [\cdot, \cdot], \wedge) \) [29]. Then a quasi-Lie bialgebroid is a Lie algebroid \( A \) equipped with a degree 1-derivation \( \delta : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A) \) with respect to the Gerstenhaber algebra structure on \( \oplus k \Gamma(\wedge^k A) \) such that \( \delta^2 = [\Omega, \cdot] \) for some \( \Omega \in \Gamma(\wedge^3 A) \) satisfying \( \delta \Omega = 0 \). When \( \Omega = 0 \), one obtains a differential Gerstenhaber algebra on \( \oplus k \Gamma(\wedge^k A) \), which is known to be equivalent to the usual definition of Lie bialgebroids [13, 29].

Quasi-Lie bialgebroids arise naturally in the study of the so-called twisted Poisson structures. Recall [5, 24] that a twisted Poisson structure consists of a pair \((\pi, \phi)\), where \( \pi \) is a bivector field on \( M \) and \( \phi \) is a closed 3-form, which satisfies the equation

\[
\frac{1}{2} [\pi, \pi] = (\wedge^3 \pi^\sharp)(\phi),
\]

where \( \pi^\sharp \) is the vector bundle homomorphism \( T^* M \to TM \) induced by \( \pi \) (i.e., \( \pi^\sharp(x)(\sigma) := \pi(x)(\sigma, \cdot) \), with \( x \in M \), \( \sigma \in T^*_x M \)). As explained in [24], a twisted Poisson structure induces a Lie algebroid structure on \( T^* M \) with anchor map \( \pi^\sharp \) and Lie bracket of sections \( \sigma \) and \( \tau \) defined by

\[
[\sigma, \tau] := \mathcal{L}_{\pi^\sharp(\sigma)} \tau - \mathcal{L}_{\pi^\sharp(\tau)} \sigma - d\pi(\sigma, \tau) + \phi(\pi^\sharp(\sigma), \pi^\sharp(\tau), \cdot).
\]

We will denote this Lie algebroid by \( T^* M_{(\pi, \phi)} \). Sections of its exterior algebra are ordinary differential forms. There is a derivation \( \delta : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M) \) deforming the de Rham differential \( d \) by \( \phi \), which is defined as follows. For \( f \in C^\infty(M) \), \( df = df \), and \( \delta \sigma = df - \pi^\sharp(\sigma) \mathcal{L}\sigma \), if \( \sigma \in \Omega^1(M) \). It turns out that \( \delta[\sigma, \tau] = [\delta \sigma, \tau] + [\sigma, \delta \tau], \forall \sigma, \tau \in \Omega^1(M) \), and that \( \delta^2 = [\phi, \cdot] \). Thus, one obtains a quasi-Lie bialgebroid \( (T^* M_{(\pi, \phi)}, \delta) \).

As it is known [21], a Lie bialgebroid integrates to a Poisson groupoid. It is natural to expect that a quasi-Lie bialgebroid should integrate to a quasi-Poisson groupoid. For the quasi-Lie bialgebroid corresponding to a twisted Poisson manifold, this is true as shown by Cattaneo-Xu [5] (it was proved in [5] that it integrates to a twisted symplectic groupoid, which is indeed a special example of quasi-Poisson groupoids). However, the proof in [5] relies on a twisted version of Poisson sigma-model, which does not apply for a general quasi-Lie bialgebroid. The method used by Mackenzie-Xu [21] to integrate a Lie bialgebroid, on the other hand, does not admit a generalization to the quasi case either. Therefore, we must seek some new method to tackle this integration problem.
For this purpose, we introduce the notion of multi-differentials on a Lie algebroid. By a \( k \)-differential on a Lie algebroid \( A \), we mean a linear operator \( \delta : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A) \) satisfying
\[
\delta(P \wedge Q) = (\delta P) \wedge Q + (-1)^{p(k-1)} P \wedge \delta Q,
\]
\[
\delta[P, Q] = [\delta P, Q] + (-1)^{(p-1)(k-1)} [P, \delta Q],
\]
for all \( P \in \Gamma(\wedge^p A) \) and \( Q \in \Gamma(\wedge^q A) \) (See Definition 2.23 for an equivalent definition). The space of all multi-differentials \( \mathcal{A} = \oplus_k \mathcal{A}_k \) becomes a graded Lie algebra under the graded commutator. Thus, a quasi-Lie bialgebroid \( \mathcal{A}_3 \) exactly corresponds to a 2-differential whose square is a coboundary, i.e., \( \delta : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A) \) such that \( \delta^2 = [\Omega, \cdot] \) for some \( \Omega \in \Gamma(\wedge^3 A) \) satisfying \( \delta \Omega = 0 \).

On the other hand, using the generalized coisotropic calculus of Weinstein [25], we introduce the notion of multiplicative multivector fields on a Lie groupoid. The space of multiplicative \( k \)-vector fields on \( \Gamma \) is denoted by \( \mathfrak{X}^k_{\text{mult}}(\Gamma) \). One proves that \( \oplus_k \mathfrak{X}^k_{\text{mult}}(\Gamma) \) is closed under the Schouten bracket, and therefore is a Gerstenhaber subalgebra of \( \oplus_k \mathfrak{X}^k(\Gamma) \).

The main theorem is the following

**Universal lifting theorem** Assume that \( \Gamma \Rightarrow M \) is an \( \alpha \)-simply connected and \( \alpha \)-connected Lie groupoid with Lie algebroid \( A \). Then \( \oplus_k \mathfrak{A}_k \) is isomorphic to \( \oplus_k \mathfrak{X}^k_{\text{mult}}(\Gamma) \) as graded Lie algebras.

As an immediate consequence, one concludes that there is a bijection between quasi-Lie bialgebroids \((A, \delta, \Omega)\) and quasi-Poisson groupoids \((\Gamma, \pi, \Omega)\), where \( \Gamma \Rightarrow M \) is an \( \alpha \)-simply connected and \( \alpha \)-connected Lie groupoid integrating the Lie algebroid \( A \). In particular, when \( \Omega = 0 \), one obtains a simpler proof of the Lie bialgebroid integration theorem of Mackenzie-Xu [21]. When \((A, \delta, \Omega)\) is the quasi-Lie bialgebroid corresponding to a twisted Poisson manifold, one recovers the integration theorem of Cattaneo-Xu [5].

To prove this theorem, one direction is pretty simple. Given a multiplicative \( k \)-vector field \( \Pi \in \mathfrak{X}^k_{\text{mult}}(\Gamma) \), for any \( P \in \Gamma(\wedge^p A) \), one proves that \([\Pi, \overrightarrow{P}]\) is right invariant, where \( \overrightarrow{P} \) denotes the right invariant \( p \)-vector field on \( \Gamma \) corresponding to \( P \in \Gamma(\wedge^p A) \). Therefore, there exists \( \delta_\Pi P \in \Gamma(\wedge^{k+p-1} A) \) such that for any \( P \in \Gamma(\wedge^p A) \),
\[
\overrightarrow{\delta_\Pi P} = [\Pi, \overrightarrow{P}].
\]

Thus, \( \delta_\Pi \) is indeed a \( k \)-differential. When \( \Gamma \) is a Lie group, the above construction is called the **inner derivative** [15]. To prove the other direction, we realize the groupoid \( \Gamma \) as the moduli space of the space of all \( A \)-paths \( P(A) \) moduli the gauge transformations. Such a characterization was first obtained by Cattaneo-Felder motivated by the Poisson sigma model when the Lie algebroid is the cotangent Lie algebroid associated to a Poisson manifold [2]. The general case was due to Crainic-Fernandes [7]. Heuristically, the idea can be described as follows. Let \( \delta \) be a \( k \)-differential. Then \( \delta \) naturally induces a \( k \)-vector field \( \pi_\delta \) on \( A \), which is linear along the fibers. It in turns gives rise to a \( k \)-vector field \( \overline{\pi}_\delta \) on the path space \( \overline{P}(A) \). We then prove that \( \overline{\pi}_\delta \) induces a \( k \)-vector field on the moduli space of the space of all \( A \)-paths \( P(A) \).

The second part of the paper is devoted to the study of basic properties of quasi-Poisson groupoids. Similar to Poisson groupoids, a quasi-Poisson groupoid \( \Gamma \) also defines a momentum theory via the so-called Hamiltonian \( \Gamma \)-spaces. Important properties of such spaces are also studied.
A fundamental example, which is also a driving force for our study, is the quasi-Poisson groupoid induced by a Manin quasi-triple \((\mathfrak{d}, \mathfrak{g}, \mathfrak{h})\). Given such a quasi-triple \((\mathfrak{d}, \mathfrak{g}, \mathfrak{h})\), there is an associated quasi-Lie bialgebra \((\mathfrak{g}, \delta, \phi)\), where \(\delta : \wedge^* \mathfrak{g} \to \wedge^{*+1} \mathfrak{g}\) is a derivation of the Gerstenhaber algebra \(\wedge \mathfrak{g}\) such that \(\delta^2 = [\phi, \cdot]\). It is simple to see that \(\delta\) extends to a 2-differential of the transformation Lie algebroid \(\mathfrak{g} \times D/G \to D/G\), where \(D\) and \(G\) are connected and simply connected Lie groups with Lie algebras \(\mathfrak{d}\) and \(\mathfrak{g}\) respectively, and \(\mathfrak{g}\) acts on \(D/G\) as the infinitesimal action of the left \(G\)-multiplication on \(D/G\). We explicitly describe the corresponding quasi-Poisson groupoid \(G \times D/G \Rightarrow D/G\), and prove that in this case the corresponding Hamiltonian \(T\)-spaces are equivalent to the quasi-Poisson spaces with \(D/G\)-momentum maps in the sense of Alekseev-Kosmann-Schwarzbach \([1]\). However, our approach does not require \(\mathfrak{h}\) to be admissible.

When \((\mathfrak{d}, \mathfrak{g}, \mathfrak{h})\) is a Manin triple and \(G\) is a complete Poisson group, we recover the Poisson groupoid \(G \times G^* \Rightarrow G^*[17]\) (which is symplectic in this case). When \(G\) is not necessary complete, the Poisson groupoid \(G \times D/G \Rightarrow D/G\), which is in fact a symplectic groupoid integrating the Poisson structure on \(D/G\), can be considered as a replacement of Lu-Weinstein’s symplectic groupoid \(G \times G^* \Rightarrow G^*[17]\).

A particularly interesting case is the Manin quasi-triple \((\mathfrak{g} \oplus \mathfrak{g}, \Delta(\mathfrak{g}), \frac{1}{2}\Delta^-(\mathfrak{g}))\) corresponding to a Lie algebra \(\mathfrak{g}\) equipped with a non-degenerate symmetric pairing. In this case, one obtains a quasi-Poisson groupoid structure on \(G \times G \Rightarrow G\), where \(G\) acts on \(G\) by conjugation. The discussion on this part is done in Section 4.

Finally, some remarks are in order about notations. For a Lie groupoid \(\Gamma \Rightarrow M\), by \(\alpha, \beta : \Gamma \to M\), we denote the source and target maps. Two elements \(g, h \in \Gamma\) are composable if \(\beta(g) = \alpha(h)\). We denote by \(\Gamma^{(2)} \subset \Gamma \times \Gamma\) the subset of composable pairs in \(\Gamma \times \Gamma\). By \(i : \Gamma \to \Gamma\), \(g \mapsto i(g) = g^{-1}\), we denote the inversion, and \(\epsilon : M \to \Gamma\), \(x \mapsto \epsilon(x) = \tilde{x}\), the unit map.

The cotangent Lie groupoid is denoted by \(T^*\Gamma \Rightarrow A^*, A^*\) being the dual of the Lie algebroid \(A\), and the structural functions of this groupoid are \(\tilde{\alpha}, \tilde{\beta}\) for the projections, \(\cdot\) for the multiplication, \(\tilde{i}\) for the inversion and \(\tilde{\epsilon}\) for the unit map.

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2 Multiplicative \(k\)-vector fields on Lie groupoids

2.1 Coisotropic submanifolds

In this section, we generalize Weinstein’s coisotropic calculus \([25]\) to multivector fields.

**Definition 2.1** Let \(V\) be a vector space and \(\Pi \in \wedge^k V\). We say that a subspace \(W\) of \(V\) is coisotropic with respect to \(\Pi\) if

\[
\Pi(\xi^1, \ldots, \xi^k) = 0
\]

for all \(\xi^1, \ldots, \xi^k \in W^\circ\), where \(W^\circ\) is the annihilator of \(W\), that is \(W^\circ = \{\xi \in V^* | \xi|_W = 0\}\).
A slight modification of coisotropic calculus for bivector fields developed in [25] will allow us to show several properties in the sequel. We give now the following important lemma.

**Lemma 2.2** Let $V_1, V_2$ be vector spaces and $\Pi_i \in \wedge^k V_i$ for $i = 1, 2$. If $R \subseteq V_1 \times V_2$ is coisotropic with respect to $\Pi_1 \oplus \Pi_2$ and $C \subseteq V_2$ is coisotropic with respect to $\Pi_2$ then

$$R(C) = \{ u \in V_1 \| \exists v \in C \text{ with } (u, v) \in R \}$$

is coisotropic with respect to $\Pi_1$.

**Proof.** Let $R^* \subseteq V_1^* \times V_2^*$ be the subspace given by

$$R^* = \{ (\xi^1, \xi^2) \in V_1^* \times V_2^* \| \langle \xi^1, v_1 \rangle = \langle \xi^2, v_2 \rangle, \forall (v_1, v_2) \in R \}.$$ 

Then we have $R(C)^o = R^*(C^o)$.

Now, let $\xi^1, \ldots, \xi^k \in R(C)^o = R^*(C^o)$. Then, there exists $\varphi^1, \ldots, \varphi^k \in C^o$ such that $(\xi^1, \varphi^1) \in R^*$, i.e., $(\xi^1, -\varphi^1) \in R^*$. Thus,

$$0 = (\Pi_1 \oplus \Pi_2)((\xi^1, -\varphi^1), \ldots, (\xi^k, -\varphi^k)) = \Pi_1(\xi^1, \ldots, \xi^k) + (-1)^k \Pi_2(\varphi^1, \ldots, \varphi^k)$$

$$= \Pi_1(\xi^1, \ldots, \xi^k).$$

That is, $R(C)$ is coisotropic. $\Box$

A generalization of the notion of coisotropy to manifolds is the following.

**Definition 2.3** Let $M$ be an arbitrary manifold and $\Pi \in \mathfrak{X}^k(M)$. A submanifold $S$ of $M$ is said to be coisotropic with respect to $\Pi$ if $T_xS$ is coisotropic with respect to $\Pi(x)$ for all $x \in S$.

**Remark 2.4** It is simple to see that $S$ is coisotropic with respect to the multivector field $\Pi$ if and only if $\Pi(df^1, \ldots, df^k)|_S = 0$ for any $f^1, \ldots, f^k \in C^\infty(M)$ such that $f_j|_S = 0$ (see [25] for the case of bivector fields).

**Proposition 2.5** If $S$ is coisotropic with respect to the multivector fields $\Pi$ and $\Pi'$, so is with respect to the multivector field $[\Pi, \Pi']$.

**Proof.** If $A$ is any subset of $\{1, 2, \ldots, (k + k' - 1)\}$, let $A'$ denote its complement and $|A|$ the number of elements in $A$. If $|A| = l$ and the elements in $A$ are $\{i_1, \ldots, i_l\}$ in increasing order, let us write $f_A$ for the ordered $k$-tuple $(f^{i_1}, \ldots, f^{i_l})$. Furthermore, we write $\varepsilon_A$ for the sign of the permutation which rearranges the elements of the ordered $(k + k' - 1)$-tuple $(A', A)$, in the original order. Then, the Schouten bracket of $\Pi \in \mathfrak{X}^k(M)$ and $\Pi' \in \mathfrak{X}^{k'}(M)$ is given by

$$[\Pi, \Pi'](df^{i_1}, \ldots, df^{i_1+k'-1}) = (-1)^{k+1} \sum_{|A|=k'} \varepsilon_A \Pi(d(\Pi(f_A)), df_A')$$

$$+ (-1)^{kk'} \sum_{|B|=k} \varepsilon_B \Pi'(d(\Pi(f_B)), df_B'). \hspace{1cm} (4)$$

Our result thus follows from the characterization of a coisotropic submanifold as described in Remark 2.4. $\Box$
2.2 Definition and examples

In this section, we will introduce the notion of multiplicative multivector fields and show that it generalizes several concepts which have previously appeared in the literature.

Throughout this section, we fix a Lie groupoid $\Gamma \rightrightarrows M$ and denote its Lie algebroid by $A$. Moreover, we denote by $\Lambda$ the graph of the groupoid multiplication, that is,

$$\Lambda = \{(g, h, gh) \mid \beta(g) = \alpha(h)\}.$$

**Definition 2.6** Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\Pi \in \mathfrak{X}^k(\Gamma)$ a $k$-vector field on $\Gamma$. We say that $\Pi$ is multiplicative, denoted as $\Pi \in \mathfrak{X}^k_{\text{mult}}(\Gamma)$, if $\Lambda$ is coisotropic with respect to $\Pi \oplus \Pi \oplus (-1)^{k+1}\Pi$.

An interesting characterization of multiplicative multivector fields is the following:

**Proposition 2.7** Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\Pi \in \mathfrak{X}^k(\Gamma)$ a $k$-vector field on $\Gamma$. Then, the following are equivalent:

i) $\Pi$ is multiplicative;

ii) for any $\mu_g, \nu_h \in T^*\Gamma$, such that $\tilde{\beta}(\mu_g) = \tilde{\alpha}(\nu_h)$

$$\Pi(gh)(\mu_g \cdot \nu_h, \ldots, \mu_g^k \cdot \nu_h^k) = \Pi(g)(\mu_g^1, \ldots, \mu_g^k) + \Pi(h)(\nu_h^1, \ldots, \nu_h^k);$$

iii) the linear skew-symmetric function $F_\Pi$ on $T^*\Gamma \times_{\Gamma} \mathfrak{X}^k(\Gamma)$ induced by $\Pi$,

$$F_\Pi(\mu^1, \ldots, \mu^k) = \Pi(\mu^1, \ldots, \mu^k),$$

is a 1-cocycle with respect to the Lie groupoid $T^*\Gamma \times_{\Gamma} \mathfrak{X}^k(\Gamma) \rightrightarrows A^* \times_M \mathfrak{X}^k(\Gamma)$.

**Proof.** It follows from the fact that $(\mu_g, \nu_h, \gamma_{gh}) \in N^*_{(g,h,gh)}A$ if and only if $\gamma_{gh} = -\mu_g \cdot \nu_h$, for $\mu_g, \nu_h \in T^*\Gamma$. □

**Remark 2.8** From Proposition 2.7 one can deduce that a bivector field $\Pi$ on a Lie groupoid $\Gamma$ is multiplicative if and only if $\Pi^2 : T^*\Gamma \to TT\Gamma$ is a Lie groupoid morphism, i.e.,

$$\Pi^2(\mu_g \cdot \nu_h) = \Pi^2(\mu_g) \cdot \Pi^2(\nu_h),$$

for $\mu_g, \nu_h \in T^*\Gamma$ such that $\tilde{\beta}(\mu_g) = \tilde{\alpha}(\nu_h)$.

A direct consequence of Proposition 2.5 and Definition 2.6 is the following.

**Proposition 2.9** The Schouten bracket of multiplicative multivector fields is still multiplicative. That is, $(\oplus_k \mathfrak{X}^k_{\text{mult}}(\Gamma), [\cdot, \cdot])$ is a graded Lie subalgebra of $(\oplus_k \mathfrak{X}^k(\Gamma), [\cdot, \cdot])$. 

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Below we list some well-known examples.

**Example 2.10** Let $G$ be a Lie group and $\Pi \in \mathfrak{x}^k(G)$ a $k$-vector field. From $\tilde{\beta}(\mu^i_g) = \tilde{\alpha}(\nu^i_h)$ we deduce that

\[
\mu^i_g = (L_g^{-1})^*(R_h)^*\nu^i_h.
\]

Moreover, we have that

\[
\mu^i_g \cdot \nu^i_h = (L_g^{-1})^*\nu^i_h.
\]

Therefore, we see that Eq. (5) is equivalent to

\[
\left( (L_{g^{-1}})^*\Pi(g)h - (L_{g^{-1}})^*(R_h)^*\Pi(g) - \Pi(h) \right)(\nu^1_h, \ldots, \nu^k_h) = 0.
\]

That is, $\Pi(g)h = (R_h)^*\Pi(g) + (L_g)^*\Pi(h)$. The converse is obvious. Therefore, our definition of multiplicative $k$-vector fields is indeed a generalization of the usual notion for Lie groups (see [16]).

**Example 2.11** We say that a function $\sigma$ on $\Gamma$ is multiplicative if $\sigma(gh) = \sigma(g) + \sigma(h)$ for $(g, h) \in \Gamma^{(2)}$. Therefore, multiplicative functions are multiplicative $0$-vector fields.

**Example 2.12** A vector field $X \in \mathfrak{x}(\Gamma)$ is said to be multiplicative if it is a Lie groupoid morphism $X : \Gamma \to T\Gamma$ from a Lie groupoid $\Gamma \Rightarrow M$ to its corresponding tangent Lie groupoid $T\Gamma \Rightarrow TM$ (see [20]). Therefore, we have $X(gh) = X(g) \cdot X(h)$ for $\beta(g) = \alpha(h)$. Using this fact and that

\[
(\mu_g \cdot \nu_h)(u_g \cdot v_h) = \mu_g(u_g) + \nu_h(v_h), \text{ for all } (u_g, v_h) \in T\Gamma^{(2)},
\]

we deduce that $X$ is a multiplicative $1$-vector field in the sense of Definition 2.6.

**Example 2.13** From Definition 2.6, we see that the Poisson tensor on a Poisson groupoid is a multiplicative bivector field [25].

**Example 2.14** Given a Lie groupoid $\Gamma$ with Lie algebroid $A$, if $P \in \Gamma(\wedge^k A)$, then $\tilde{\Omega} - \tilde{P}$ is a multiplicative $k$-vector field on $\Gamma \Rightarrow M$.

### 2.3 Multiplicative and affine multivector fields

Let $\Omega$ be the submanifold of $\Gamma \times \Gamma \times \Gamma \times \Gamma$ consisting of elements $(l, h, g, w)$ such that $w = hl^{-1}g$. $\Omega$ is called the *affinoid diagram* corresponding to the groupoid $\Gamma$ by Weinstein [25]. A characterization of multiplicative $k$-vector fields in terms of the affinoid diagram is the following:

**Proposition 2.15** Let $\Pi$ be a $k$-vector field on a Lie groupoid. Then $\Pi$ is multiplicative if and only if $\Omega$ is coisotropic with respect to $\Pi \oplus (-1)^{k+1}\Pi \oplus (-1)^{k+1}\Pi \oplus \Pi$ and $M$ is coisotropic with respect to $\Pi$.

**Proof.** Suppose that $\Pi$ is a multiplicative $k$-vector field on $\Gamma$. Using Eq. 5 and the identity $\overline{\epsilon}(\phi_m^1) \cdot \overline{\epsilon}(\phi_m^k) = \overline{\epsilon}(\phi_m^k)$ for all $\phi_m \in A^*_m$, we have

\[
\Pi(\epsilon(m))(\overline{\epsilon}(\phi_m^1), \ldots, \overline{\epsilon}(\phi_m^k)) = 2\Pi(\epsilon(m))(\overline{\epsilon}(\phi_m^1), \ldots, \overline{\epsilon}(\phi_m^k)),
\]

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∀φ^j_m, . . . , φ^k_m ∈ A^*_m, m ∈ M. Since we have e(A^*) = N^*M, we deduce that M is coisotropic with respect to Π.

Now we proceed as in [26]. In the product Γ × ˜Γ × ˜Γ × Γ × ˜Γ (where ˜Γ denotes Γ endowed with the k-vector field (−1)^k+1Π), we consider the coisotropic submanifold \( R = \{(g, h, l, x, y, z) \mid g y = l \text{ and } h z = x \} \). On the other hand, it is simple to see that the diagonal \( ∆ \subset Γ × ˜Γ \) is a coisotropic submanifold. Therefore, using Lemma 2.2, we have that \( R(∆) \) is coisotropic submanifold of \( Γ × ˜Γ × ˜Γ × Γ \). It is simple to see that \( R(∆) = Ω \). Our result thus follows.

Conversely, let Π be a k-vector field on a Lie groupoid such that Ω is coisotropic with respect to Π ⊕ (−1)^k+1Π ⊕ (−1)^k+1Π ⊕ Π and M is coisotropic with respect to Π. Applying Lemma 2.2 to \( R = Ω \) and \( C = M \), we obtain that Λ is coisotropic with respect to Π ⊕ Π ⊕ (−1)^k+1Π. That is, Π is multiplicative. □

Next we recall the definition of an affine multivector field on a Lie groupoid and show that any multiplicative multivector field is affine.

**Definition 2.16** A multivector field Π on Γ is affine if for any \( g, h ∈ Γ \) such that \( β(g) = α(h) = m \) and any bisections \( X, Y \) through the points \( g, h \), we have

\[
Π(gh) = (R_Y)_*Π(g) + (L_X)_*Π(h) - (R_Y*L_X)_*Π(e(m)). \tag{7}
\]

A useful characterization of affine multivector fields is the following

**Proposition 2.17** [21] Let Π be a k-vector field on a Lie groupoid Γ ⇒ M. Then Π is affine if and only if \([X, Π]\) is right-invariant for all \( X ∈ Γ(A) \).

**Proposition 2.18** If Π is a multiplicative k-vector field on a Lie groupoid Γ, then Π is affine.

**Proof.** If Π is multiplicative, according to Proposition 2.15, we know that Ω is coisotropic with respect to Π ⊕ (−1)^k+1Π ⊕ (−1)^k+1Π ⊕ Π, and M is coisotropic with respect to Π. For any \( μ ∈ T_{gh}Γ \), it follows from Lemma 2.6 in [26] that \(-μ, L^*_Xμ, R^*_Yμ, -L^*_X^tR^*_Y^tμ\) is conormal to Ω. Therefore, for any \( μ^1, . . . , μ^k ∈ T_{gh}Γ \), we have

\[-Π(gh)(μ^1, . . . , μ^k) + Π(h)(L^*_Xμ^1, . . . , L^*_X^tμ^k) + Π(g)(R^*_Yμ^1, . . . , R^*_Y^tμ^k) -Π(e(m))(L^*_X^tR^*_Y^tμ^1, . . . , L^*_X^tR^*_Y^tμ^k) = 0.\]

Thus Eq. (7) follows immediately. □

Similar to the case of multiplicative bivector fields, we can give another useful characterization of multiplicative k-vector fields.

**Theorem 2.19** Let Γ ⇒ M be a Lie groupoid and Π ∈ \( X^k(Γ) \) a k-vector field on Γ. Then Π is multiplicative if and only if the following conditions hold:

i) Π is affine, i.e., Eq. (7) holds;

ii) M is a coisotropic submanifold of Γ;

iii) both \( α_\ast Π(g) \) and \( β_\ast Π(g) \) only depend on \( α(g) \) and \( β(g) \), respectively.
\( \forall \eta^1, \eta^2 \in \Omega^1(M), \text{ we have } (\alpha^* \eta^1 \wedge \beta^* \eta^2) \lhd \Pi = 0; \)

\( \forall \theta \in \Omega^p(M), 1 \leq p < k, \text{ then } (\beta^* \theta) \lhd \Pi \text{ is a left-invariant } (k - p)\text{-vector field on } \Gamma. \)

**Proof.** Let \( \Pi \) be a multiplicative \( k \)-vector field on \( \Gamma \). From Propositions \[2.15\] and \[2.18\], we obtain i) and ii).

Next, since

\[
0_g \cdot (\beta^* \eta)_h = (\beta^* \eta)_{gh}
\]

for any \( \eta \in \Omega^1(M) \), from Eq. \[5\], it follows that

\[
\Pi((\beta^* \eta^1)_{gh}, \ldots, (\beta^* \eta^k)_{gh}) = \Pi(0_g \cdot (\beta^* \eta^1)_h, \ldots, 0_g \cdot (\beta^* \eta^k)_h) = \Pi(\beta^* \eta^1)_h, \ldots, (\beta^* \eta^k)_h)
\]

\( \forall \eta^1, \ldots, \eta^k \in \Omega^1(M) \). Hence \( \beta_* \Pi_g \) only depends on \( \beta(g) \). Similarly, from the equation

\[
(\alpha^* \eta)_g \cdot 0_h = (\alpha^* \eta)_{gh}
\]

\( \forall \eta \in \Omega^1(M) \), we also deduce that \( \alpha_* \Pi_g \) only depends on \( \alpha(g) \). Hence, iii) holds.

Moreover, from Eqs. \[5\], \[8\] and \[9\], we have iv).

Finally, let us prove v). Let \( \theta \in \Omega^p(M) \) be a \( p \)-form on \( M \), \( 1 \leq p < k \). Then, using Eqs. \[5\], \[8\] and \[9\], we see that \( (\beta^* \theta) \lhd \Pi \) is tangent to \( \alpha \)-fibers. Therefore, it suffices to prove that

\[
(L_X)_*((\beta^* \theta) \lhd \Pi)_{(g)_h}) = ((\beta^* \theta) \lhd \Pi)_{(gh)_v}, \quad \forall(g, h, gh) \in \Lambda,
\]

where \( X \) is an arbitrary bisection through \( g \). According to \[28\], for any \( \mu \in T_g \Gamma \), there exists \( \nu \in T_g \Gamma \), which is characterized by the equation:

\[
\langle \nu, v_g \rangle = \langle \mu, (R_h)_*(v_g - (L_X)_*\beta_*v_g) \rangle, \quad \forall v_g \in T_g \Gamma,
\]

and \( \nu \cdot (L_X)_* \mu = \mu \). Using this fact and Eq. \[9\], it follows that

\[
((\beta^* \theta) \lhd \Pi)_{(h)}((L_X)_* \mu^1, \ldots, (L_X)_* \mu^{p-k}) = ((\beta^* \theta) \lhd \Pi)_{(gh)}(\mu^1, \ldots, \mu^{p-k})
\]

\( \forall \mu^1, \ldots, \mu^{p-k} \in \Omega^1(\Gamma) \). Thus, v) follows.

To prove the converse, we first note that the following three types of vectors span the whole conormal space of \( \Omega \) at a point \( (g, h, l, w) \): \( (-\mu, L_X \mu, R_y \mu, -L_X R_y \mu) \) for any \( \mu \in T_l \Gamma \), \( (0, \beta^* \eta, \beta^* \eta, 0, 0) \) for any \( \eta \in T^*_{\alpha(k)} \Gamma \), and \( (-\alpha^* \zeta, 0, \alpha^* \zeta, 0) \) for any \( \zeta \in T^*_{\beta(z)} \Gamma \), where \( X \) and \( Y \) are any bisections through the points \( g \) and \( h \), respectively (see \[28\]). From i), iii), iv) and v) we deduce that \( \Omega \) is coisotropic with respect to \( \Pi \oplus (-1)^k \Pi \oplus \Pi \) (see Theorem 2.8 in \[28\]). Using this fact, ii) and Proposition \[2.15\], the conclusion follows. \( \square \)

**Corollary 2.20** If \( \Pi \) is a multiplicative \( k \)-vector field on \( \Gamma \) then for all \( \theta \in \Omega^p(M) \), \( 1 \leq p < k \), then \( (\alpha^* \theta) \lhd \Pi \) is a right-invariant \( (k - p) \)-vector field on \( \Gamma \).

Finally, let us show an interesting property that generalizes the one obtained for multiplicative bivector fields in \[25\].
Proposition 2.21 Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\Pi \in \mathfrak{X}^k(\Gamma)$ be a multiplicative $k$-vector field on $\Gamma$. Then there exists a unique $k$-vector field $\pi$ on $M$ such that

$$\alpha_* \Pi = \pi, \quad \beta_* \Pi = (-1)^{k+1} \pi.$$ 

Proof. Since $\Pi$ is multiplicative, using property $iii)$ in Theorem 2.19, we can define a $k$-vector field $\pi$ on $M$ by setting $\pi = \alpha_* \Pi$. Now, let us investigate the relation between $\Pi$, $\pi$ and the map $\beta$.

First, we show that if $i : \Gamma \rightarrow \Gamma$ is the groupoid inversion then

$$i_* \Pi = (-1)^{k+1} \Pi. \quad (10)$$

This is an immediate consequence of property $ii)$ in Theorem 2.19 and the fact that the inverse $(\mu_g)^{-1}$ of $\mu_g \in T^*_g \Gamma$ is given by $(\mu_g)^{-1} = -i^* (\mu_g)$. In fact,

$$0 = \Pi(\tilde{\epsilon}(\tilde{\beta}(\mu^1_g)), \ldots, \tilde{\epsilon}(\tilde{\beta}(\mu^k_g)))$$

$$= \Pi(\mu^1_g, (\mu^1_g)^{-1}, \ldots, \mu^k_g, (\mu^k_g)^{-1})$$

$$= \Pi(\mu^1_g, \ldots, \mu^k_g) + (-1)^k (i_\ast \Pi)(\mu^1_g, \ldots, \mu^k_g)$$

for any $\mu^1_g, \ldots, \mu^k_g \in T^*_g \Gamma$. Finally, using Eq. (10) and the relation $\alpha \circ i = \beta$, we conclude that $\beta_* \Pi = (-1)^{k+1} \pi$. \qed

Example 2.22 From Proposition 2.21 and property $iv)$ in Theorem 2.19 we obtain that the map $(\alpha, \beta) : \Gamma \rightarrow M \times M$ satisfies

$$(\alpha, \beta)_* \Pi = \pi \oplus (-1)^{k+1} \pi.$$ 

In particular, when $\Gamma$ is a pair groupoid $M \times M \rightrightarrows M$, since $(\alpha, \beta) : M \times M \rightarrow M \times M$ is a diffeomorphism, we conclude that the only multiplicative $k$-vector fields are of the form $\pi \oplus (-1)^{k+1} \pi, \pi \in \mathfrak{X}^k(M)$ (see [20] and [25] for the case $k = 1$ and $k = 2$, respectively).

2.4 $k$-differentials on Lie algebroids

We now turn to the study of the Lie algebroid counterpart of multiplicative $k$-vector fields, namely, $k$-differentials.

Definition 2.23 Let $(A, [\cdot, \cdot], \rho)$ be a Lie algebroid over $M$. An almost $k$-differential is a pair of linear maps $\delta : C^\infty(M) \rightarrow \Gamma(\wedge^{k-1} A)$ and $\delta : \Gamma(A) \rightarrow \Gamma(\wedge^k A)$ satisfying

i) $\delta(fg) = g(\delta f) + f(\delta g)$, for all $f, g \in C^\infty(M)$.

ii) $\delta(fX) = (\delta f) \wedge X + f \delta X$, for all $f \in C^\infty(M)$ and $X \in \Gamma(A)$.

An almost $k$-differential is said to be a $k$-differential if it satisfies the compatibility condition

$$\delta[X,Y] = [\delta X,Y] + [X,\delta Y], \quad (11)$$

for all $X, Y \in \Gamma(A)$. 
Example 2.25 Let $\mathfrak{g}$ be a Lie algebra. A $k$-differential on $\mathfrak{g}$ is just a linear map $\delta : \mathfrak{g} \to \wedge^k \mathfrak{g}$ such that it is a 1-cocycle with respect to the adjoint representation of $\mathfrak{g}$ on $\wedge^k \mathfrak{g}$ [16].

Example 2.26 Let $\delta$ be a 0-differential, that is, $\delta f = 0$ for $f \in C^\infty(M)$, and $\delta X \in C^\infty(M)$ for $X \in \Gamma(A)$. From $ii)$ in Definition 2.23 we deduce that $\delta(fX) = f\delta X$. Therefore, there exists $\phi \in \Gamma(A^*)$ such that

$$\delta X = \phi(X) \quad \text{for} \quad X \in \Gamma(A).$$

Moreover, using Eq. (11), we obtain that $\phi$ is a 1-cocycle in the Lie algebroid cohomology of $A$. Thus, we can conclude that 0-differentials are just Lie algebroid 1-cocycles with trivial coefficients.

Example 2.27 Let $\delta$ be an almost 1-differential, that is, $\delta f \in C^\infty(M)$ for $f \in C^\infty(M)$, and $\delta X \in \Gamma(A)$ for $X \in \Gamma(A)$. From $i)$ in Definition 2.23 we deduce that there exists $X_0 \in \mathfrak{X}(M)$ such that

$$\delta f = X_0(f) \quad \text{for} \quad f \in C^\infty(M).$$

Moreover, using $ii)$, we obtain that

$$\delta(fX) = f\delta X + X_0(f)X,$$

that is, $\delta$ is a covariant differential operator on $A$, with anchor $X_0$ (see [18, 20]). If, moreover, $\delta$ is a 1-differential, from Eq. (11) we see that $\delta \in \Gamma(\text{CDO}(A))$ is a derivation of the bracket on $\Gamma(A)$.

Example 2.28 It is well-known that a Lie algebroid structure on a vector bundle $A \to M$ is equivalent to an almost 2-differential $\delta : \Gamma(\wedge^1 A^*) \to \Gamma(\wedge^2 A^*)$ of square 0 (see, for instance, [13, 29]). Thus, we see that a Lie bialgebroid (a notion first introduced in [19]) corresponds to a 2-differential of square 0 on a Lie algebroid $A$. 

For a given Lie algebroid $A$, it is known that the anchor map together with the bracket on $\Gamma(A)$ extends to a graded Lie bracket on $\oplus_k \Gamma(\wedge^k A)$, which makes it into a Gerstenhaber algebra ($\oplus_k \Gamma(\wedge^k A), [[\cdot, \cdot], \cdot]$ [29].

A $k$-differential $\delta$ extends naturally to sections of $\wedge A$ as follows:

$$\delta(X_1 \wedge \cdots \wedge X_s) = \sum_{i=1}^s (-1)^{(i+1)(k+1)} X_1 \wedge \cdots \wedge (\delta X_i) \wedge \cdots \wedge X_s (12)$$

$\forall X_1, \ldots, X_s \in \Gamma(A)$. In this way, we obtain a linear operator $\delta : \Gamma(\wedge^1 A) \to \Gamma(\wedge^{1+k-1} A)$. This following proposition can be directly verified.

Proposition 2.24 A $k$-differential on a given Lie algebroid $A$ is equivalent to a derivation of the associated Gerstenhaber algebra ($\oplus_k \Gamma(\wedge^k A), [[\cdot, \cdot], \cdot]$), i.e., a linear operator $\delta : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A)$ satisfying

$$\delta(P \wedge Q) = (\delta P) \wedge Q + (-1)^{p(k-1)} P \wedge \delta Q,$$

$$\delta[P, Q] = [\delta P, Q] + (-1)^{(p-1)(k-1)} [P, \delta Q],$$

for all $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$.

As we see below, $k$-differentials reduce to various well-known notions in special cases.
Example 2.29 If $P \in \Gamma(\wedge^k A)$, then $ad(P) = [P, \cdot]$ is clearly a $k$-differential, which is called the coboundary $k$-differential associated to $P$.

The space of almost differentials can be endowed with a graded Lie algebra structure as shown in the following:

**Proposition 2.30** Let $\hat{A}_k$ denote the space of almost $k$-differentials and $\hat{A} = \oplus_k \hat{A}_k$. If we define

$$[\delta_1, \delta_2] = \delta_1 \delta_2 - (-1)^{(k+1)(l+1)} \delta_2 \delta_1$$

(16)

for $\delta_1 \in \hat{A}_k$ and $\delta_2 \in \hat{A}_l$, then

i) $[\delta_1, \delta_2] \in \hat{A}_{(k+l-1)}$;

ii) $(\hat{A}, [\cdot, \cdot])$ is a graded Lie algebra.

Moreover, the space $A$ of $k$-differentials, is a graded Lie subalgebra.

**Remark 2.31** From Eqs. (13) and (16), we deduce that

$$[\delta, ad(P)] = ad(\delta P)$$

(17)

for any $k$-differential $\delta \in A$ and any multisection $P \in \Gamma(\wedge^\bullet A)$.

To end this section, we note that one can introduce a graded Lie algebra structure on $\hat{A} \times \Gamma(\wedge A)$, where $\Gamma(\wedge A) = \oplus_k \Gamma(\wedge^k A)$. This is defined by

$$[(\delta_1, P_1), (\delta_2, P_2)] = ([\delta_1, \delta_2], \delta_1(P_2) - \delta_2(P_1)),$$

for any $(\delta_1, P_1), (\delta_2, P_2) \in \hat{A} \times \Gamma(\wedge A)$.

Note that this is the semi-direct product Lie bracket when we consider the natural representation of $\hat{A}$ on $\Gamma(\wedge A)$.

**Lemma 2.32** If $\delta$ is a $k$-differential on a Lie algebroid $A$, then there exists a $k$-vector field $\pi_M$ on $M$ given by

$$\pi_M(df_1, \ldots, df_k) = (-1)^{k+1} \langle \rho(\delta f_1), df_2 \wedge \ldots \wedge df_k \rangle,$$

for all $f_1, \ldots, f_k \in C^\infty(M)$, where $\rho : \Gamma(\wedge^k A) \to \mathfrak{X}_k(M)$ is the natural extension of the anchor $\rho : \Gamma(A) \to \mathfrak{X}(M)$.

**Proof.** Since $\delta f \in \Gamma(\wedge^{k-1} A)$, we have for any $i \geq 2$

$$\{f_1, f_2, \ldots, f_i, f_{i+1}, \ldots, f_k\} = (-1)^{k+1} \langle \rho(\delta f_1), df_2 \wedge \ldots \wedge df_i \wedge df_{i+1} \wedge \ldots \wedge df_k \rangle$$

$$= (-1)^{k+1} \langle \rho(\delta f_1), df_2 \wedge \ldots \wedge df_{i+1} \wedge df_i \wedge \ldots \wedge df_k \rangle$$

$$= -\{f_1, f_2, \ldots, f_{i+1}, f_i, \ldots, f_k\}.$$

On the other hand, since $\delta$ is a $k$-differential, using Eq. (13) we get that, $\forall f, g \in C^\infty(M)$

$$0 = \delta[f, g] = [\delta f, g] + (-1)^{k+1}[f, \delta g]$$

$$= (-1)^k d_A g \Delta f + (-1)^k d_A f \Delta g.$$
where $d_A$ is the differential of the Lie algebroid $A$. Thus we can deduce that for any $f_1, \ldots, f_k \in C^\infty(M)$,

$$0 = (-1)^{k+1} \langle df_2 \mathcal{J} \rho(\delta f_1), df_3 \wedge \ldots \wedge df_k \rangle + (-1)^{k+1} \langle df_1 \mathcal{J} \rho(\delta f_2), df_3 \wedge \ldots \wedge df_k \rangle \equiv \{f_1, f_2, \ldots, f_k\} + \{f_2, f_1, \ldots, f_k\}.$$

Therefore $\{\ldots, \ldots\}$ is indeed skew-symmetric. Moreover, from the fact that both $\delta$ and $d$ are derivations, we can deduce that $\{\ldots, \ldots\}$ is a derivation with respect to each argument. That is, $\{\ldots, \ldots\}$ induces a $k$-vector field $\pi_M \in \mathfrak{X}^k(M)$. □

**Example 2.33** If $P \in \Gamma(\wedge^k A)$ and $ad(P)$ is the coboundary $k$-differential associated to $P$, then the corresponding $k$-vector field on $M$ is just $\rho(P)$.

### 2.5 From multiplicative $k$-vector fields to $k$-differentials

Assume that $\Pi$ is a multiplicative $k$-vector field on $\Gamma$. For any $f \in C^\infty(M)$ and $X \in \Gamma(A)$, it is known from Propositions 2.17, 2.18 and Corollary 2.20 that $[\Pi, \alpha^* f]$ and $[\Pi, \vec{X}]$ are right invariant, where $\vec{X}$ denotes the right invariant vector field on $\Gamma$ corresponding to $X \in \Gamma(A)$. Therefore there exists $\delta_{\Pi} f \in \Gamma(\wedge^{k-1} A)$ and $\delta_{\Pi} X \in \Gamma(\wedge^k A)$ such that

$$\delta_{\Pi} f = [\Pi, \alpha^* f], \forall f \in C^\infty(M),$$

$$\delta_{\Pi} X = [\Pi, \vec{X}], \forall X \in \Gamma(A).$$

(18)

It is simple to see that $\delta_{\Pi}$ is indeed a $k$-differential. We are now ready to state the main theorem of the paper.

**Theorem 2.34** Assume that $\Gamma \Rightarrow M$ is an $\alpha$-simply connected and $\alpha$-connected Lie groupoid with Lie algebroid $A$. Then the map

$$\delta : \oplus_k \mathfrak{X}^k_{\text{mult}}(\Gamma) \rightarrow \oplus_k A_k$$

$$\Pi \mapsto \delta_{\Pi}$$

is a graded Lie algebra isomorphism.

We divide the proof into several steps. The surjectivity of $\delta$ will be postponed to Section 3. Here we prove the following result.

**Proposition 2.35** Under the same hypothesis as in Theorem 2.34, $\delta$ is an injective graded Lie algebra homomorphism.

**Proof.** Using the graded Jacobi identity of the Schouten brackets and Eq. (18), we deduce that if $\Pi \in \mathfrak{X}^k_{\text{mult}}(\Gamma)$ and $\Pi' \in \mathfrak{X}^l_{\text{mult}}(\Gamma)$ then

$$\delta_{[\Pi, \Pi']} = [\delta_{\Pi}, \delta_{\Pi'}].$$

Therefore, $\delta$ is a graded Lie algebra homomorphism.

Next, let us prove that $\delta$ is injective. We will use the following lemma (see Theorem 2.6 in [21]):

**Lemma 2.36** If $\Pi$ is an affine multivector field on an $\alpha$-connected Lie groupoid $\Gamma \Rightarrow M$ then $\Pi = 0$ if and only if $\delta_{\Pi} X = 0$, $\forall X \in \Gamma(A)$, and $\Pi$ vanishes on the unit space $M$. 

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Suppose that $\Pi$ is multiplicative $k$-vector field on $\Gamma$ such that $\delta_\Pi = 0$. It remains to show that $\pi|_M = 0$. We know that $T^*_\epsilon(m)$ is spanned by the differential of functions of the type $\alpha^* f$, with $f \in C^\infty(M)$, and the differential of functions $F$ which are constant along $M$, i.e., such that $dF \in N^*M$. Since $M$ is coisotropic, we get that

$$\Pi(\epsilon(m))(dF_1, \ldots, dF_k) = 0.$$  

Moreover,

$$i_{\alpha^* f} \Pi = (-1)^{k+1}[\Pi, \alpha^* f] = (-1)^{k+1}\delta_\Pi(f) = 0,$$

which implies that

$$\Pi(\epsilon(m))(\alpha^* f_1, \ldots, \alpha^* f_j, dF_1, \ldots, dF_l) = 0,$$

for $j + l = k$ and $j \geq 1$. Therefore, $\Pi|_M = 0$.

From Lemma 2.36, it follows that $\Pi = 0$. Thus $\delta$ is injective. □

Following [16], we also call $\delta_\Pi$ the inner derivative of $\Pi$.

Theorem 2.34 has many interesting corollaries. Below is a list of well known results which are in the literature.

**Example 2.37** Let $G$ be a simply connected and connected Lie group with Lie algebra $\mathfrak{g}$ and $\Pi \in \mathfrak{X}^k_{\text{mult}}(G)$. Then $\delta_\Pi: \mathfrak{g} \to \wedge^k \mathfrak{g}$ is the 1-cocycle obtained by taking the inner derivative of $\Pi$ [16]. Thus we have a one-to-one correspondence between 1-cocycles $\mathfrak{g} \to \wedge^* \mathfrak{g}$ and multiplicative multivector fields on $G$ (see [16]).

**Example 2.38** If $\sigma$ is a multiplicative function on $\Gamma$, i.e., $\sigma: \Gamma \to \mathbb{R}$ is a groupoid 1-cocycle, then $\delta_\sigma \in \Gamma(A^*)$ is exactly its corresponding Lie algebroid 1-cocycle. Thus we conclude that for an $\alpha$-connected and $\alpha$-simply connected Lie groupoid, there is a one-to-one correspondence between groupoid 1-cocycles $\sigma: \Gamma \to \mathbb{R}$ and Lie algebroid 1-cocycles $\delta \in \Gamma(A^*)$. This result was first proved in [27].

**Example 2.39** Multiplicative vector fields on a Lie groupoid are exactly infinitesimals of Lie groupoid automorphisms [20]. One-differentials on a Lie algebroid, on the other hand, are covariant differential operators on $A$ which are derivations with respect to the bracket [20]. These are exactly infinitesimals of the Lie algebroid automorphisms. Thus for an $\alpha$-connected and $\alpha$-simply connected Lie groupoid, we have a one-to-one correspondence between infinitesimals of the Lie groupoid automorphisms and infinitesimals of the corresponding Lie algebroid automorphisms [20].

**Example 2.40** Let $P \in \Gamma(\wedge^k A)$ be a $k$-section of $A$, and $\Pi = \overrightarrow{P} - \overleftarrow{P}$ the corresponding multiplicative $k$-vector field on $\Gamma$. From the definition of $\delta_\Pi$, we see that $\delta_\Pi$ is just the coboundary $k$-differential $\text{ad}(P) = [P, \cdot]$. Thus we conclude that for an $\alpha$-connected and $\alpha$-simply connected Lie groupoid, there is a one-to-one correspondence between coboundary multiplicative multivector fields on the Lie groupoid and coboundary $k$-differentials on its Lie algebroid.

**Example 2.41** Let $(\Gamma \rightrightarrows M, \Pi)$ be a Poisson groupoid, i.e., $\Pi \in \mathfrak{X}^2_{\text{mult}}(\Gamma)$ such that $[\Pi, \Pi] = 0$. From Theorem 2.34 there exists a 2-differential $\delta_\Pi$ on $A$. Moreover,

$$\delta_\Pi^2 = \delta_\Pi \circ \delta_\Pi = \frac{1}{2}[\delta_\Pi, \delta_\Pi] = \frac{1}{2}[\delta_\Pi, \Pi] = 0.$$
Thus, $\delta_{\Pi}$ defines a Lie algebroid structure on $A^*$. Moreover, since $\delta_{\Pi}[X, Y] = [\delta_{\Pi}X, Y] + [X, \delta_{\Pi}Y]$ for all $X, Y \in \Gamma(A)$, we deduce that $(A, A^*)$ is a Lie bialgebroid. As a consequence, we obtain the integration theorem of Mackenzie-Xu [19]: there is a one-to-one correspondence between $\alpha$-connected and $\alpha$-simply connected Poisson groupoids and Lie bialgebroids.

### 3 Lifting of $k$-differentials

This section is devoted to the proof of the surjectivity of $\delta$ in Theorem 2.34.

#### 3.1 $A$-paths

From now on, we use the notation $I = [0, 1]$. Let $(A, [\cdot, \cdot], \rho)$ be the Lie algebroid of an $\alpha$-connected and $\alpha$-simply connected Lie groupoid $\Gamma$. Following [7], by $\tilde{P}(A)$ we denote the Banach manifold of all $C^1$-paths in $A$. A $C^1$-path $a : I \to A$ is said to be an $A$-path if

$$\rho(a(t)) = \frac{d\gamma(t)}{dt}, \quad (19)$$

where $\gamma(t) = (p \circ a)(t)$ is the base path ($p : A \to M$ is the bundle projection). The set of $A$-paths, denoted by $P(A)$, is a Banach submanifold of $\tilde{P}(A)$.

It is well-known that integrating along $A$-paths yields a $\Gamma$-path. We recall this construction in order to be self-contained. Following [7], we call a $\Gamma$-path a $C^2$-path $r(t)$ on the groupoid $\Gamma$ such that $r(0) \in M$ and $\alpha(r(t)) = r(0)$ for all $t \in I$.

There is a diffeomorphism $\mathfrak{I}$ from the space of $A$-paths to the space of $\Gamma$-paths. For any $a \in P(A)$, we define $\mathfrak{I}(a)$ to be the solution $r(t)$ of the initial value problem

$$\begin{cases}
\frac{dr(t)}{dt} = \overrightarrow{a(t)}r(t) \\
r(0) = \gamma(0)
\end{cases} \quad (20)$$

where $\gamma = p \circ a$ is the base path. The inverse of $\mathfrak{I}$ is, for any $\Gamma$-paths $r(t)$, given by $\mathfrak{I}^{-1}(r(t)) = L_{r^{-1}(t)} \cdot \frac{dr(t)}{dt}$.

The purpose of this section is to study properties of the smooth map $\tau$ from $P(A)$ to $\Gamma$ given by

$$\tau(a) = r(1), \quad (21)$$

where $r = \mathfrak{I}(a)$.

First, we study the covariance of $\tau$. We recall that the (pseudo-) group of (local) bisections of $\Gamma \rightrightarrows M$ acts on $\Gamma$ by

$$r \to \tilde{g}^{-1} \cdot r \cdot \tilde{g}$$

for any bisection $\tilde{g}$ and $r \in \Gamma$ (this last expression makes sense provided that the local bisection is chosen so that both products in the above equation are defined). Differentiating this action with respect to $r$, one obtains an automorphism of the Lie algebroid $A \to M$, which is denoted by $Ad_{\tilde{g}} : A_m \to A_{\tilde{g} \cdot m}$. Differentiating this action with respect to $\tilde{g}$, and using the fact that the Lie algebra of the (pseudo-) group of bisections is the Lie algebra of (local) sections of $A$, one constructs, for all $\xi \in \Gamma(A) \text{ and } b \in A$, an element of $T_bA$, which we denote by $ad_\xi b \in T_bA$. 

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The reader should not confuse $b \to ad_\xi b$, which is a tangent vector on $A$, with the adjoint action of the Lie algebra $\Gamma(A)$ on itself.

For any $\Gamma$-path $r(t)$ and any $C^2$-path $g(t)$ in $\Gamma$, such that $r(t)$ and $g(t)$ are composable for all $t \in I$ (i.e. $\beta(r(t)) = \alpha(g(t))$), the path $g^{-1}(0) \cdot r(t) \cdot g(t)$ is a $\Gamma$-path again. It is simple to check that $\mathcal{J}^{-1}(g(0)^{-1} \cdot r(t) \cdot g(t))$ is equal to $Ad_{\tilde{g}_t} \mathcal{J}^{-1}(r(t)) + \left( R_{\tilde{g}_t^{-1}} \cdot \frac{d\tilde{g}_t}{dt} \right) |_{\tilde{g}_t \gamma(t)}$, where $\tilde{g}_t$ is, for all $t \in I$, a (local) bisection of $\Gamma \Rightarrow M$ through $g(t)$ (defined in a neighborhood of $g(t)$). In short, for any $a \in P(A)$, and any time-dependent (local) bisection $\tilde{g}_t$ through $g(t)$, with a $C^2$-dependence in $t$,

$$\tau \left( Ad_{\tilde{g}_t} a(t) + \left( R_{\tilde{g}_t^{-1}} \cdot \frac{d\tilde{g}_t}{dt} \right) |_{\tilde{g}_t \gamma(t)} \right) = g^{-1}(0) \cdot \tau(a) \cdot g(1).$$

(22)

For any $A$-paths $a_1(t)$ and $a_2(t)$, if $\tau(a_1) = \tau(a_2)$, then $\mathcal{J}(a_1) = \mathcal{J}(a_2) \cdot g(t)$ for some path $g(t)$ on $\Gamma$ such that $g(0), g(1) \in M$. Hence, it follows from Eq. (22) that $\tau(a_1) = \tau(a_2)$ if and only if there exists a time-dependent bisection $\tilde{g}_t$ such that

$$a_2(t) = Ad_{\tilde{g}_t} a_1(t) + R_{\tilde{g}_t^{-1}} \cdot \frac{d\tilde{g}_t}{dt} |_{\tilde{g}_t \gamma(t)}$$

and $\tilde{g}_0$ and $\tilde{g}_1$ are unital elements of the pseudo-group of local bisections.

Since $\Gamma$ is $\alpha$-simply connected, the pseudo-group of time-dependent local bisections $\tilde{g}_t$ is connected. Therefore, $\tau(a_1) = \tau(a_2)$ if and only if $a_1$ and $a_2$ can be linked by a differentiable path in $P(A)$ which is tangent to the differential of the action described by Eq. (22). But vectors in $T_a P(A)$ tangent to this action are precisely the vectors of the form, at a given $A$-path $a(t)$ with base path $\gamma(t)$,

$$G_{\xi} |a : t \to ad_\xi(t) a(t) + \left. \frac{d\xi(t)}{dt} \right|_{\gamma(t)},$$

(23)

where $\xi(t)$ is a $C^2$-time-dependent section of $A \to M$ with $\xi(0) = \xi(1) = 0$ and $\left. \frac{d\xi(t)}{dt} \right|_{\gamma(t)}$, an element of $A_{\gamma(t)}$, is considered as an element of $T_{a(t)} A$.

For any $C^2$-time dependent section $\xi(t)$ of $A \to M$ with $\xi(0) = \xi(1) = 0$, the vector field $G_{\xi}$ on $P(A)$ given by Eq. (23), is called a gauge vector field. A smooth function $f : P(A) \to \mathbb{R}$ is said to be invariant under the gauge transformation if and only if $G_{\xi}(f) = 0$ for any $G_{\xi}$ of the form described by Eq. (23) with $\xi(0) = \xi(1) = 0$. The following proposition summarizes the above discussion.

**Proposition 3.1** Assume that $\Gamma$ is an $\alpha$-connected and $\alpha$-simply connected Lie groupoid. Then the map $\tau : P(A) \to \Gamma$ induces an isomorphism between $C^\infty(\Gamma)$ and the algebra of smooth functions on $P(A)$ invariant under the gauge transformation.

Note that it follows from Eqs. (22) (23) that for any time-dependent section $\xi(t)$ of $\Gamma(A)$, we have

$$\tau_* \left( ad_\xi(t) a(t) + \left. \frac{d\xi(t)}{dt} \right|_{\gamma(t)} \right) = \xi(1) - \xi(0).$$

(24)

This relation will be useful later on.

Next we need to introduce regular extensions to $\tilde{P}(A)$ of the 1-form $d\tau^* f$ for a smooth function $f$ on $\Gamma$. First, we give some definitions related to the cotangent spaces of $P(A)$ and $\tilde{P}(A)$.
For any $a \in \tilde{P}(A)$ (resp. $P(A)$), the cotangent space of $\tilde{P}(A)$ is denoted by $T^*_a \tilde{P}(A)$ (resp. $T^*_a P(A)$). For any real-valued function $f$ on $\tilde{P}(A)$ (resp. on $P(A)$), the differential (if it exists) at the point $a \in \tilde{P}(A)$ (resp. $a \in P(A)$) is an element of $T^*_a \tilde{P}(A)$ (resp. $T^*_a P(A)$), and is denoted by $df_a$.

For any $a \in \tilde{P}(A)$, denote by $P_a(T^*A)$, the space of $C^1$-maps $\eta : I \to T^*A$ such that $\forall t \in I$ the identity $\pi _\eta (t) = a(t)$ holds, where $\pi$ denotes the projection $T^*A \to A$. The space $P_a(T^*A)$ can be considered as a sub-space of $T^*_a \tilde{P}(A)$, by associating to any $\eta(t) \in P_a(T^*A)$ the linear map

$$T^*_a \tilde{P}(A) \rightarrow \mathbb{R}$$
$$X(t) \rightarrow \int_0^1 \langle \eta(t), X(t) \rangle dt,$$

considered as an element of $T^*_a \tilde{P}(A)$, where $\langle , \rangle$ denotes the pairing between the cotangent and tangent vectors. Throughout this section, we will always consider $P_a(T^*A)$ as a subspace of $T^*_a \tilde{P}(A)$, and, therefore, the vector bundle $P(T^*A) \to \tilde{P}(A)$ as a vector sub-bundle of $T^* \tilde{P}(A) \to \tilde{P}(A)$. We denote by $P(T^*A)_{|\tilde{P}(A)} \to P(A)$ and $T^* \tilde{P}(A)_{|\tilde{P}(A)} \to P(A)$ the restrictions of these vector bundles to $P(A)$.

**Definition 3.2** Given a 1-form $\omega$ on the Banach submanifold $P(A)$ of $A$-paths, by an extension (resp. regular extension), we mean a smooth section $\Phi_\omega$ of $T^* \tilde{P}(A)_{|\tilde{P}(A)}$ such that $\Phi_\omega|_a$, is for any $a \in P(A)$, an extension of $\omega|_a$ (i.e., the restriction of $\Phi_\omega|_a$ to $T_a P(A)$ is $\omega|_a$ for any $a \in P(A)$).

By a regular extension of a smooth function $g \in C^\infty(P(A))$, we mean a regular extension of its differential. A regular extension of the zero function on $P(A)$ will be called a regular extension of zero. Also, we use the following notation: for any regular extension $\Phi_\omega$, we denote by $\Phi_\omega(t)$ the corresponding path in $P_a T^*A$.

Given a vector field $X$ on $\tilde{P}(A)$ tangent to $P(A)$, and a 1-form $\omega$ on $P(A)$, the Lie derivative of a regular extension $\Phi_\omega$ of $\omega$ is defined by

$$(\mathcal{L}_X \Phi_\omega)(Y) = \Phi_\omega([Y, X]) + X(\Phi_\omega(Y)),$$

where $Y$ is the restriction to $P(A)$ of a vector field on $\tilde{P}(A)$. This definition needs to be justified. It is clear that the right hand side of Eq. (25) is $C^\infty$-linear with respect to $Y$ and depends only on its restriction to $P(A)$. It therefore defines a section of $T^* \tilde{P}(A)_{|\tilde{P}(A)} \to P(A)$. It follows from Eq. (26) that if $Y$ itself is tangent to $P(A)$, then $(\mathcal{L}_X \Phi_\omega)(Y) = (\mathcal{L}_X \omega)(Y)$. Therefore, $\mathcal{L}_X \Phi_\omega$ is an extension of $\mathcal{L}_X \omega$.

In the following subsections, we need to investigate regular extensions $\Phi_{df} \tau^* f$ of $df \tau^* f$ for a smooth function $f \in C^\infty(\Gamma)$. The following technical lemma will be very useful.

**Lemma 3.3** $\forall a \in P(A),$

i) for any smooth function $f : \Gamma \to \mathbb{R}$, the pull-back function $\tau^* f : P(A) \to \mathbb{R}$ admits a regular extension $\Phi_{df^\tau} f$;

ii) for any smooth function $f : \tilde{P}(A) \to \mathbb{R}$ whose restriction to $P(A)$ vanishes, there exists $g_t : I \to C^\infty(M)$ with $g_0 = g_1 = 0$ such that $df|_a = df g|_a$. Here for any time dependent 1-form $\omega : t \to \omega_t$ on $M$ and any $a \in \tilde{P}(A)$ with base path $\gamma(t)$,

$$\mathcal{F}_\omega(a) = \int \langle \omega_t|_{\gamma(t)}, \frac{d\gamma(t)}{dt} - \rho(a) \rangle dt.$$
PROOF. i) We divide the proof of i) into four steps. In what follows, $\gamma$ always denotes the base path of $a \in P(A)$. We say that a path $e(t)$ in a vector bundle $p : E \to M$ is over a base path $\gamma$ if $p \circ e = \gamma$.

Step 1. We first describe the differential $\tau_*$ of the map $\tau$ defined in Eq. (21).

The map $(r, a) \to \tilde{\alpha}$ is a map from the fibered product $\Gamma \times_{a,M,p} A$ to $TT$ that maps a pair $(r, a)$ (with $r \in R, a \in A$) to an element of $T_T_R$. Considering the fibered product $\Gamma \times_{a,M,p} A$ as a submanifold of $\Gamma \times A$, one can extend this map to a smooth map $F(a, a)$ from $\Gamma \times A$ to $TT$ satisfying the condition $F(r, a) \in T_{r} \Gamma$ for all $r \in \Gamma$. This allows us to rewrite $\tau$ more conveniently. Consider the map $\tilde{\tau} : \tilde{P}(A) \times M \to \Gamma$ given by

$$\tilde{\tau}(a, m) = r(1),$$

where $r(t) : I \to \Gamma$ is the solution of the initial value problem

$$\begin{align*}
\frac{dr(t)}{dt} &= F(r(t), a(t)) \\
\tau(0) &= m
\end{align*}$$

By definition, we have $\tau(a) = \tilde{\tau}(a, (p \circ a)(0))$.

Linearizing the previous equation, one obtains that the differential of the map $\tilde{\tau}$ can be written under the form

$$\tilde{\tau}_*(\delta a, \delta m) = L(\delta m(0)) + \int_{t \in I} L_t(\delta a(t)) dt,$$

where $\delta a, \delta m$ are elements of $T_a \tilde{P}(A)$ and $T_m M$ respectively, and $L_t, L$ are linear maps from $T_{\tau(t)} \Gamma$ and $T_m M$ to $T_{\tilde{\tau}(a, m)} \Gamma$ respectively. Thus the differential of $\tau : P(A) \to \Gamma$ can be expressed under the form

$$\tau_*(\delta a) = \int_{t \in I} L_t(\delta a(t)) dt + L(p_*(\delta a(0)))$$

where $\delta a \in T_a P(A)$.

At this point, we need to explain why some technical difficulties arise in the construction of a regular extension of $\tilde{\tau}^* f$ and what remains to be done in order to avoid it. It follows from Eq. (20) that the differential $d\tilde{\tau}^* f = df \circ \tau_*$ is the sum of two 1-forms, namely $\omega_1 : \delta a \to \int_{t \in I} L_t(\delta a(t)) dt$ and $\omega_2 : \delta a \to L(p_*(\delta a(0)))$. It is easy to find a regular extension of $\omega_1$: we can just choose, for any $\delta a \in T_a P(A)$,

$$\Phi_{\omega_1}(\delta a) = \int_{t \in I} L_t(\delta a(t)) dt.$$}

Unfortunately, it is not so easy to find a regular extension of $\omega_2$, since this 1-form is “concentrated” in $0$.

Step 2. We describe explicitly the tangent space $T_a P(A)$ of $P(A)$.

Let us choose a connection $\nabla^A$ on the vector bundle $A \to M$. This connection allows us to decompose the tangent space of $A$ as a direct sum $T_b A = T_{p(b)} M \oplus A_{p(b)}$ for any $b \in A$. With this convention, for any $a \in \tilde{P}(A)$, an element $\delta a$ of $T_a \tilde{P}(A)$ becomes a pair $\delta a = (\epsilon, b)$ where $\epsilon : I \to TM$ and $b : I \to A$ are $C^1$-maps over the base path $\gamma = p \circ a$. 

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We choose now a connection $\nabla^M$ on the tangent bundle $TM \to M$. By differentiating the relation $\frac{d}{dt} = \rho(a)$, we obtain that $\delta a = (\epsilon, \beta) \in T_a \tilde{P}(A)$ is an element of the tangent space of $T_a P(A)$ if and only if:

$$\nabla^M_a \epsilon = (\nabla_{\epsilon} \rho)(a) + \rho(\beta),$$

where $(\nabla_{\epsilon} \rho)(a) = \nabla^M_a \rho(a) - \rho(\nabla^A_a a)$ is a path in $TM$ (over $\gamma$ again).

**Step 3.** For any $A$-path $a$, we want to construct a linear map $\Pi_a$ from $T_{a} \tilde{P}(A)$ to $T_{\gamma(0)} M$, where $\gamma = p \circ a$ is the base path, depending smoothly on $a \in P(A)$, whose restriction to $T_a P(A)$ is simply the differential $a \to \gamma(0)$, and which is “given by an integral”. That is, the differential is of the form

$$\Pi_a(\delta a) = \int_{s \in I} M(t)(\delta a(t)) dt,$$

where $M_t$ is, for all $t \in I$, a linear map from $T_{a}(t) A$ to $T_{\gamma(t)} M$. We proceed as follows. Set $\delta a = (\epsilon, \beta) \in T_{\tilde{P}(A)}$ as in Step 2.

First, for any $s \in I$, we define $\eta_s(t) : I \to TM$ to be the unique solution of the initial value problem:

$$\begin{cases}
\nabla^M_{\frac{d}{dt}} \eta_s(t) = (\nabla_{\eta_s(t)} \rho)(a) + \rho(\beta(t)) \\
\eta_s(s) = \epsilon(s)
\end{cases}$$

(this equation is a linear equation of order 1, which guarantees the existence and uniqueness of the solution). Then we define $\Pi_a(\epsilon, \beta)$ by

$$\Pi_a(\epsilon, \beta) = \int_{s \in I} \eta_s(0) ds.$$

It follows from a classical result of ordinary linear differential equations (see, for instance, [11]) that

$$\eta_s(0) = L(s)(\epsilon(s)) + \int_{s}^{0} M(s, u)(\beta(u)) du$$

for some smooth function $M(s, u)$ from $A_{\gamma(u)}$ to $T_{\gamma(u)} M$. It is simple to check that for any $\delta a = (\epsilon, \beta)$

$$\Pi_a(\delta a) = \int_{s \in I} L(s)(\epsilon(s)) ds - \int_{s \in I} \int_{u=0}^{s} M(s, u)(\beta(u)) duds.$$

The right-hand side of this equation is of the form given by Eq. (31). Now it remains to check that the restriction of $\Pi_a$ to $TP(A)$ is equal to the map $\delta a \to p_\ast(\delta a(0))$. For any $\delta = (\epsilon, \beta)$ tangent to $P(A)$, by the uniqueness of the solution of an initial value problem, we have $\eta_s = \epsilon$ for all $s \in I$. Therefore, the restriction of $\Pi_a$ to $T_a P(A)$ is equal to

$$\Pi_a(\delta a) = \int_{I} \epsilon(0) dt = \epsilon(0) = p_\ast(\delta a(0)).$$

**Step 4.** We now can define for any $f \in C^\infty(M)$

$$\Phi_{d^r f}(\delta a) := \tilde{\tau}_s(\delta a, \Pi_a(\delta a)).$$
It follows from Eqs. (27, 31) that \( \Phi_{d\tau^*f} \) is a regular extension of \( \tau^*f \). This achieves the proof of \( i) \).

\( ii) \) We identify an element \( \delta a \in \tilde{P}(A) \) with a pair of paths \( \epsilon(t) \) in \( TM \), and \( \beta(t) \) in \( A \) over a base path \( \gamma(t) \) as previously.

Since \( \Phi_{d\tau^*f} \) is a regular extension, we have

\[
\Phi_{d\tau^*f}(\epsilon, \beta) = \int_I \langle M(t), \epsilon(t) \rangle dt + \int_I \langle A(t), \beta(t) \rangle dt
\]

for some \( C^1 \)-maps \( M(t), A(t) \) from \( I \) to \( T^*M \) and \( A^* \) respectively, which are over the base path \( \gamma(t) \).

Let \( \omega(t) : t \rightarrow T^*_\gamma(t)M \) be a path over \( \gamma \) which is a solution of the initial value problem

\[
\begin{cases}
\nabla_{\frac{d\tau(t)}{dt}} M \omega(t) + ((\nabla \rho)a(t))^*(\omega(t)) = M(t) \\
\omega(1) = 0
\end{cases}
\]

where, for any fixed \( t \in I \), \( ((\nabla \rho)a(t))^* \in \text{End}(T^*_\gamma(t)M) \) is the dual of the endomorphism \( v \rightarrow (\nabla \rho)(a(t)) \) of \( T^*_\gamma(t)M \).

Using integration by part, we obtain

\[
\int_I \langle M(t), \epsilon(t) \rangle dt = -\int_I \langle \omega(t), \nabla_{\frac{d\tau(t)}{dt}} M \epsilon(t) \rangle dt + \int_I \langle \omega(t), (\nabla_{\epsilon(t)} \rho)(a(t)) \rangle dt + \langle \omega(0), \epsilon(0) \rangle.
\]

Since \( \Phi_{d\tau^*f}(\epsilon, \beta) \) vanishes as long as the conditions

\[
\begin{cases}
\beta(t) = 0 \\
\nabla_{\frac{d\tau(t)}{dt}} M \epsilon(t) - (\nabla_{\epsilon(t)} \rho)a(t) = 0
\end{cases}
\]

are satisfied, we must have \( \omega(0) = 0 \).

Now, since

\[
\Phi_{d\tau^*f}(\epsilon, \beta) = \int_I \langle \omega(t), \nabla_{\frac{d\tau(t)}{dt}} M \epsilon(t) + (\nabla_{\epsilon(t)} \rho)a(t) \rangle dt + \int_I \langle A(t), \beta(t) \rangle dt.
\]

must vanish whenever \( \epsilon, \beta \) satisfy Eq. (30), we must have \( A(t) = \rho^* \omega(t) \). As a consequence, we have

\[
\Phi_{df}(\delta a) = \int_I \langle \omega(t), \nabla_{\frac{d\tau(t)}{dt}} M \epsilon(t) - (\nabla_{\epsilon(t)} \rho)a(t) + \rho(\beta(t)) \rangle dt.
\]

According to Lemma 3.4 \( i) \), there exists a family of time dependent functions \( g_t \) from \( I \) to \( C^\infty(M) \) vanishing in \( t = 0, 1 \) such that \( dg_t|_{\gamma(t)} = \omega(t) \). The result now follows from Lemma 3.4 \( ii) \). \( \square \)

**Lemma 3.4**  
\( i) \) For any \( C^1 \)-path \( \omega(t) : I \rightarrow T^*M \), there exists a time-dependent function \( g : t \rightarrow g_t \) vanishing at \( t = 0, 1 \) such that \( dg_t|_{\gamma(t)} = \omega(t) \) for any \( t \in ]0, 1[ \).

\( ii) \) The function \( a \rightarrow dF_{dg|_a} \) is a regular extension of zero whose differential is of the form

\[
dF_{dg|_a}(\epsilon, \beta) = \int_I \langle \omega(t), \nabla_{\frac{d\tau(t)}{dt}} M \epsilon(t) - (\nabla_{\epsilon(t)} \rho)a(t) + \rho(\beta(t)) \rangle dt.
\]

where \( \epsilon, \beta, \nabla M, \nabla \) are as in Eq. (30).
PROOF. i) We denote by \( \exp_m : T_m M \to M \) the exponential map associated to the connection \( \nabla^M \). Recall that \( \exp_m \) is a local diffeomorphism from a neighborhood of \( 0 \in T_m M \) to a neighborhood of \( m \). Let \( f(t) \) be a real-valued smooth function that vanishes at \( t = 0 \) and \( t = 1 \).

Define \( g_t(m) = f(t)\psi(t,m)(\omega(t),\exp_{\gamma(t)}^{-1}(m)) \), where \( \psi(t,m) \) is a smooth function on \( I \times M \) satisfying two conditions: (1) \( \psi(t,m) \) is identically equal to 1 in a neighborhood of the curve \( (t,\gamma(t)) \) for all \( t \in I \) (2) \( \psi(t,m) \) vanishes outside an open set on which \( \exp_{\gamma(t)}^{-1} \) is well-defined. We leave it to the reader to check that these conditions can be satisfied and that the function \( g_t \) has the requested properties.

ii) The function \( F_{dg} \) is identically zero on \( P(A) \) by definition. One checks that its differential is of the form \( [\mathcal{K}] \) by a direct computation. □

3.2 Almost differentials and linear multivector fields

In this subsection, we study a particular case of multivector fields on a vector bundle.

Let \( p : A \to M \) be a vector bundle. It is clear that there exists a bijection between the space \( \Gamma(A^*) \) of sections of the dual bundle \( p^*: A^* \to M \) and the set \( C^\infty_{lin}(A) \) of functions which are linear on each fiber. In fact, for any section \( \phi \in \Gamma(A^*) \) the corresponding linear function \( \ell_\phi \) is given by \( \ell_\phi(X_m) = (\phi(m),X_m) \forall X_m \in A_m \). On the other hand, by basic functions, we mean functions on \( A \) which are the pull-back functions from \( M \).

Definition 3.5 Let \( p : A \to M \) be a vector bundle and \( \pi \in \mathfrak{X}^k(A) \) a \( k \)-vector field on \( A \). We say that \( \pi \) is linear if \( \pi(df_1,\ldots,df_k) \) is a linear function whenever all \( f_1,\ldots,f_k \in C^\infty(A) \) are linear functions.

Linear multivector fields were called homogeneous in [9]. They satisfy the following properties.

Proposition 3.6 Let \( p : A \to M \) be a vector bundle and \( \pi \in \mathfrak{X}^k(A) \) a linear \( k \)-vector field on \( A \) with \( k \geq 2 \). Then \( \pi(df_{\phi_1},\ldots,df_{\phi_{k-1}},d(p^*f_1)) \) is a basic function on \( A \) and

\[
(d(p^*f_1) \wedge d(p^*f_2)) \cdot \pi = 0 \quad \forall \phi_1,\ldots,\phi_{k-1} \in \Gamma(A^*) \quad \text{and} \quad f_1, f_2 \in C^\infty(M).
\]

Proof. Proceed as in [6], using Leibniz rule and the fact that \( (p^*f)\ell_\phi = \ell_{f\phi} \) for \( f \in C^\infty(M) \) and \( \phi \in \Gamma(A^*) \). □

Let \( \pi \in \mathfrak{X}^k_{lin}(A) \) be a linear \( k \)-vector field on \( A \). Proposition 3.6 enables us to introduce a pair of operations

\[
\rho_* : \Gamma(A^*) \times \left( \binom{k-1}{1} \right) \times \Gamma(A^*) \to \mathfrak{X}(M) \quad \text{and}
\]

\[
[,] : \Gamma(A^*) \times \left( \binom{k}{1} \right) \times \Gamma(A^*) \to \Gamma(A^*)
\]

by

\[
\rho_*(\phi_1,\ldots,\phi_{k-1})(f)p = \pi(d\ell_{\phi_1},\ldots,d\ell_{\phi_{k-1}},d(f\cdot p)),
\]

\[
\ell_{[\phi_1,\ldots,\phi_k]} = \pi(d\ell_{\phi_1},\ldots,d\ell_{\phi_k}),
\]

\( \forall \phi_1,\ldots,\phi_k \in \Gamma(A^*) \) and \( f \in C^\infty(M) \).  

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It is simple to see that the following identities are satisfied:

$$\rho_*(\phi_1, \ldots, f \phi_i, \ldots, \phi_{k-1}) = f \rho_*(\phi_1, \ldots, \phi_i, \ldots, \phi_{k-1}),$$

$$[\phi_1, \ldots, f \phi_i, \ldots, \phi_k]_* = (-1)^{i+1} \rho_*(\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_{k-1})(f)\phi_i$$  \hspace{1cm} (35)

$$+f[\phi_1, \ldots, \phi_i, \ldots, \phi_k]_*$$

for any $\phi_1, \ldots, \phi_k \in \Gamma(A^*)$ and $f \in C^\infty(M)$. In particular, we see that $\rho_*$ induces a bundle map $\wedge^{k-1}A^* \to TM$.

Conversely, if we have a pair $(\rho_*, [\cdot, \ldots, \cdot])_*$ satisfying Eq. \hspace{1cm} (35), we can define a linear k-vector field on $A$ using Eq. \hspace{1cm} (34).

Now, assume that $\delta$ is an almost k-differential. We construct a pair $([\cdot, \ldots, \cdot], \rho_*)$ satisfying Eq. \hspace{1cm} (35) as follows. Let

$$\rho_*(\phi_1, \ldots, \phi_{k-1})(f) = \{\delta f, \phi_1 \wedge \ldots \wedge \phi_{k-1}\}, \text{ and}$$

$$\langle [\phi_1, \ldots, \phi_k], X \rangle = \sum_{i=1}^{k} (-1)^{i+k} \rho_*(\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_k)(\phi_i(X))$$

$$-\langle \delta X, \phi_1 \wedge \ldots \wedge \phi_k \rangle,$$

for all $\phi_1, \ldots, \phi_k \in \Gamma(A^*)$, $X \in \Gamma(A)$ and $f \in C^\infty(M)$. Conversely, using these equations, one can construct an almost k-differential from $([\cdot, \ldots, \cdot], \rho_*)$.

A combination of the above discussion leads to the following

**Proposition 3.7** For a given Lie algebroid $A$, there is a one-to-one correspondence between linear multivector fields and almost differentials on $A$.

For an almost differential $\delta$ on $A$, we denote its corresponding linear multivector field by $\pi_\delta$. In local coordinates, the correspondence between almost k-differentials and k-vector fields $\pi_\delta$ on $A$ can be described as follows. Let $(x_1, \ldots, x_n)$ be local coordinates on $M$ and $\{e_1, \ldots, e_s\}$ a local basis of $\Gamma(A)$. Assume that

$$\delta x_i = \sum a_i^{i_1 \ldots i_{k-1}}(x)e_{i_1} \wedge \ldots \wedge e_{i_{k-1}},$$

$$\delta e_i = \sum c_i^{i_1 \ldots i_k}(x)e_{i_1} \wedge \ldots \wedge e_{i_k},$$

Then

$$\pi_\delta = \sum a_i^{i_1 \ldots i_{k-1}}(x)\frac{\partial}{\partial v_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial v_{i_{k-1}}} \wedge \frac{\partial}{\partial x_i} - c_i^{i_1 \ldots i_k}(x)v_i \frac{\partial}{\partial v_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial v_{i_k}},$$

where $(v_1, \ldots, v_s)$ are the corresponding linear coordinates on the fibers.

The correspondence between almost differentials and linear multivector fields preserves the graded Lie algebra structure, as shown in the following

**Proposition 3.8** Given a Lie algebroid $A$, the map $\delta \mapsto \pi_\delta$ is a graded Lie algebra isomorphism from almost differentials on $A$ to linear multivector fields on $A$. I.e., for any almost differentials $\delta_1$ and $\delta_2$, we have,

$$[\pi_{\delta_1}, \pi_{\delta_2}] = \pi_{[\delta_1, \delta_2]}.$$
PROOF. Note that if $P \in \Gamma(\wedge^p A)$ is given locally by $\sum_{j_1,\ldots,j_p} P^{j_1,\ldots,j_p}(x)e_{j_1} \wedge \cdots \wedge e_{j_p}$ under a given local basis $\{e_1,\ldots,e_s\}$ of $\Gamma(A)$, then

$$\delta P = \sum_i \frac{\partial P^{j_1,\ldots,j_p}(x)}{\partial x_i} a_i \delta e_{j_1} \cdots \delta e_{j_p} + \sum_i (-1)^{(i+1)(k+1)} P^{j_1,\ldots,j_p}(x)e_{j_1} \wedge \cdots \wedge \delta e_{j_i} \wedge \cdots \wedge e_{j_p}.$$  

The assertion follows from a tedious computation and is left to the reader. □

Example 3.9 If $\delta$ is an almost 0-differential, we know that $\delta$ is just the contraction operator by an element $\phi \in \Gamma(A^*)$. In this case, one shows that $\pi_\delta \in \mathfrak{X}_{lin}^0(A) = C_{lin}^\infty(A)$ is just $-\ell_\phi$.

Example 3.10 When $k = 1$, as a consequence of Proposition 3.8 one obtains a Lie algebra isomorphism between $\Gamma(CDO(A))$, the space of covariant differential operators on $A$, and $\Gamma(T^{LN}(A))$, the space of linear vector fields on $A$ (see [20]).

Example 3.11 Let $\delta$ be a 2-differential of square zero. We know that $\delta$ induces a Lie algebroid structure on $A^*$ (see [13, 29]). On the other hand, from Proposition 3.8 it follows that $[\pi_\delta, \pi_\delta] = 0$, and therefore $\pi_\delta$ defines a Poisson structure on $A$. Such a correspondence between Lie algebroids structures on $A^*$ and linear Poisson structures on the dual bundle is standard (see [6] for instance). A generalization to arbitrary linear 2-vector fields and pre-Lie algebroids was considered in [10].

We end this subsection by recalling two kinds of liftings from $\Gamma(\wedge^* A)$ to multivector fields on $A$, the complete and vertical lifts. Let $P \in \Gamma(\wedge^p A)$, and $ad(P)$ its corresponding coboundary differential. Then the corresponding linear $p$-vector field $\pi_{ad(P)}$ is the so-called complete lift of $P$, which is denoted by $P^c$ (see [4]).

On the other hand, for any multisection $P \in \Gamma(\wedge^p A)$, there exists another kind of lift, called the vertical lift, denoted by $P^v$. It is obtained via the natural identification $A_m \simeq T^*_{a\vert Vert}(A), \forall a \in A$. In local coordinates, if $P = P^{j_1,\ldots,j_p}(x)e_{j_1} \wedge \cdots \wedge e_{j_p}$, for a given local basis $\{e_1,\ldots,e_s\}$, then

$$P^v = P^{j_1,\ldots,j_p}(x) \frac{\partial}{\partial v_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial v_{j_p}}.$$  

Complete and vertical lifts satisfy the following properties [9]:

$$f^c = p^* f; \quad (36)$$
$$f^c = \ell_{da} f; \quad (37)$$
$$(P \wedge Q)^c = P^c \wedge Q^c + P^v \wedge Q^v; \quad (38)$$
$$(P \wedge Q)^v = P^v \wedge Q^v; \quad (39)$$
$$[P^c, Q^c] = [P, Q]^c; \quad (40)$$
$$[P^c, Q^v] = [P, Q]^v; \quad (41)$$
$$[P^v, Q^v] = 0; \quad (42)$$

$\forall f \in C^\infty(M)$ and $P, Q \in \Gamma(\wedge^* A)$. 

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Proposition 3.12 Let \( \delta \) be a \( k \)-differential and \( P \in \Gamma(\wedge^k A) \). Then
\[
[\pi_\delta, P^c] = (\delta P)^c, \\
[\pi_\delta, P^v] = (\delta P)^v.
\]

Proof. Using Eq. (17) and Proposition 3.8 we have
\[
[\pi_\delta, P^c] = [\pi_\delta, \pi_{ad}(P)] = \pi_{ad(\delta P)} = (\delta P)^c.
\]
The second identity can be checked directly using local coordinates. □

Let \( P \in \Gamma(\wedge^k A) = \{ I \ni t \mapsto P(t) \in \Gamma(\wedge^k A) : P(t) \text{ is of class } C^2 \text{ in } t \} \) and
\[ P_0 \Gamma(\wedge^k A) = \{ I \ni t \mapsto P(t) \in \Gamma(\wedge^k A) : P(0) = P(1) = 0, P(t) \text{ is of class } C^2 \text{ in } t \}. \]
If \( P \in P_0 \Gamma(\wedge^k A) \), we define \( G(P) \) as the time-dependent multivector field on \( A \) given by:
\[
G(P) = P^c + \left( \frac{dP}{dt} \right)^v.
\]

The multivector fields \( G(P) \) satisfy the following properties.

Proposition 3.13 If \( P, Q \in P \Gamma(\wedge^k A) \) and \( \delta \) is a multi-differential on \( A \), then
\[
[G(P), G(Q)] = G([P, Q]), \\
G(P \wedge Q) = G(P) \wedge Q^v + P^v \wedge G(Q), \\
[\pi_\delta, G(P)] = G(\delta(P)).
\]

In particular, Eq. (45) implies that
\[
G(f(t)P) = f(t)G(P) + \frac{df(t)}{dt} P^v.
\]

Proof. Using Eqs. (10), (11) and (12) and that \( \frac{d}{dt} \) is a derivation with respect to the Schouten bracket, we deduce that Eq. (11) holds.

Eq. (10) is a direct consequence of Proposition 3.12 and the fact that \( \delta \) commutes with \( \frac{d}{dt} \). □

3.3 From \( k \)-vector fields on \( A \) to \( k \)-vector fields on \( \widetilde{P}(A) \)

We start this subsection with a construction that must be thought, at least heuristically, as a lifting of a time-dependent multivector fields on \( A \) to a multivector field on \( \widetilde{P}(A) \).

Let \( \pi(t) \) be a time-dependent \( k \)-vector field on \( A \) with \( k \geq 1 \). For given \( k \) one-forms \( \eta_1, \ldots, \eta_k \in \Omega^1(P(A)) \) and their regular extensions \( \Phi_{\eta_1}, \ldots, \Phi_{\eta_k} \), define a smooth function \( \bar{\pi}(\Phi_{\eta_1}, \ldots, \Phi_{\eta_k}) \) on \( P(A) \) as follows. For any \( a \in P(A) \),
\[
\bar{\pi}(\Phi_{\eta_1}, \ldots, \Phi_{\eta_k})(a) := \int_I \pi(t)|_{a(t)}(\Phi_{\eta_1}(a(t)), \ldots, \Phi_{\eta_k}(a(t))) dt
\]
Remark 3.14 Eq. (48) still makes sense when one of the extension \( \Phi_{\eta_i} \) is not necessary regular, with the following modification of its definition. Assume that \( \Phi_{\eta_i} \) is not necessary regular, and that \( (\Phi_{\eta_2}, \ldots, \Phi_{\eta_k}) \) are. Then \( (\pi_t(\Phi_{\eta_2}(a(t)), \ldots, \Phi_{\eta_k}(a(t)))) \) is an element of \( TP(A) \) and we can therefore define

\[
\tilde{\pi}(\Phi_{\eta_1}, \ldots, \Phi_{\eta_k})(a) = \Phi_{\eta_1}\left(\pi_t(\Phi_{\eta_2}(a(t)), \ldots, \Phi_{\eta_k}(a(t)))\right).
\]

We recover of course the previous definition of \( \tilde{\pi} \) if \( \Phi_{\eta_1} \) is regular.

For instance, consider the case \( k = 0 \). If \( f_t \) is a time-dependent function on \( A \), i.e., \( f \in C^\infty(I \times A) \), then Eq. (48) gives

\[
\tilde{f}(a) = \int_I f(t, a(t)) \, dt.
\]

Eq. (49) still makes sense for any \( a \in \tilde{P}(A) \) and hence defines a function on \( \tilde{P}(A) \) that we denote by \( \tilde{f} \) again. Note also that for any time-dependent vector field \( X : t \to X_t \) on \( A \), \( \tilde{X} \) is a vector field on \( \tilde{P}(A) \), which, at any \( a(t) \in \tilde{P}(A) \), is given by \( t \to X_{t(a(t))} \).

For any time-dependent vector field \( X : t \mapsto X_t \) on \( \tilde{P}(A) \), tangent to \( P(A) \), we define the Lie derivative \( L_X \tilde{\pi} \) of \( \tilde{\pi} \) as follows:

\[
(L_X \tilde{\pi})(\Phi_{\eta_1}, \ldots, \Phi_{\eta_k}) = X(\tilde{\pi}(\Phi_{\eta_1}, \ldots, \Phi_{\eta_k})) - \sum_{i=1}^k \tilde{\pi}(\Phi_{\eta_1}, \ldots, L_{X_i} \Phi_{\eta_i}, \ldots, \Phi_{\eta_k}).
\]

(50)

This definition needs to be justified. Of course, the Lie derivative of a regular extension is not necessary a regular extension, but it is an extension and by Remark 3.14 the left hand side of Eq. (50) makes sense.

Lemma 3.15 i) For any time-dependent function \( g : t \to g_t \) from \( I \) to \( C^\infty(M) \) with \( g_0 = g_1 = 0 \) (i.e., \( g \in P\Gamma(A^0) \)), we have \( \tilde{G}(g) = \mathcal{F}_{dg} \).

ii) If \( \xi \in P_0 \Gamma(A) \), then \( \tilde{G}(\xi) \) is the gauge vector field given by Eq. (21).

iii) If \( \xi \in P\Gamma(A) \), then \( \tau_*(\tilde{G}(\xi)) = \tilde{\xi}(0) - \tilde{\xi}(1) \), where \( \tau : P(A) \to \Gamma \) is the map defined by Eq. (20).

iv) For any multi-differential \( \delta \) on \( A \), we have

\[
L_{\tilde{G}(\xi)} \tilde{\pi}_{\delta} = \tilde{G}(\delta(\xi)).
\]

v) For any multi-differential \( \delta \) on \( A \) and time-dependent function \( g : t \to g_t \) on \( M \), we have

\[
\tilde{\pi}_{\delta}(\tilde{G}(g), \Phi_{\eta_2}, \ldots, \Phi_{\eta_k}) = \tilde{G}(\delta(g))(\Phi_{\eta_2}, \ldots, \Phi_{\eta_k}).
\]

Proof. i) For any time-dependent function \( g : t \to g_t \) from \( I \) to \( C^\infty(M) \), it follows from Eq. (27) and (13) that

\[
g^\circ = \frac{1}{0} (dg_t, \rho(a(t))) \, dt.
\]

(51)
Now using Eqs. (36) and (39) we have

\[
\left(\frac{dg}{dt}\right)^\nu = \int_0^1 \frac{dg_t}{dt}(p\cdot a(t)) dt = -\int_0^1 \langle dg_t, \frac{dp\cdot a(t)}{dt}\rangle dt,
\]

where the last equality is obtained by integration by parts using the boundary condition \(g_0 = g_1 = 0\). Thus \(i)\) follows.

\(ii)\) follows from the fact that for any \(\xi \in \Gamma(A)\), \(ad_\xi\) is equal to \(\xi^c\), a fact that can be easily checked in local coordinates.

\(iii)\) is easily deduced from \(ii)\) and Eq. (24).

\(iv)\) From the definition of Lie derivatives given by Eqs. (26) and (50), it follows that

\[
L_{\xi^c}(g) = \xi^c(g) = 0.
\]

By Eq. (46), we have

\[
\frac{d}{dt}\left(\int_0^1 \langle df_{t|a(t)}(X(t)), X(t)\rangle dt\right) = \int_0^1 \langle df_{t|a(t)}(X(t)), X(t)\rangle dt
\]

for any \(X \in T_{a\tilde{P}}(A)\), where \(df_{t|a(t)}\) is the differential of the smooth function \(f_t\) at the point \(a(t)\).

By the definition of \(\tilde{\pi}_t\), we have

\[
\tilde{\pi}_t(d\tilde{G}(g), \Phi_{\eta_2}, \ldots, \Phi_{\eta_k})|_a = \int_I \pi_t(dG(g)|_{a(t)}, \Phi_{\eta_2}(t), \ldots, \Phi_{\eta_k}(t)) dt
\]

for any A-path \(a(t)\). By Eq. (26), we have

\[
\pi_t(dG(g), \Phi_{\eta_2}, \ldots, \Phi_{\eta_k})|_a = \int_I G(\delta(g)|_{a(t)}(\Phi_{\eta_2}(t), \ldots, \Phi_{\eta_k}(t)) dt = G(\delta(g))(\Phi_{\eta_2}, \ldots, \Phi_{\eta_k}).
\]

This proves \(v)\). \(\Box\)

**Proposition 3.16** For any \(P \in P_\Gamma(\Lambda^k A)\), any functions \(f_1, \ldots, f_k \in C^\infty(\Gamma)\), and any regular extensions \(\Phi_{dr^*f_1}, \ldots, \Phi_{dr^*f_k}\) of \(\tau^*f_1, \ldots, \tau^*f_k\), we have

\[
\overline{G}(P)(\Phi_{dr^*f_1}, \ldots, \Phi_{dr^*f_k}) = \overline{P}(\overline{1})(df_1, \ldots, df_k) - \overline{P}(0)(df_1, \ldots, df_k)
\]

First we need the following

**Lemma 3.17** For any \(\xi \in P_\Gamma(A)\), \(a \in P(A)\), any covector \(\eta \in T^*_a P(A)\) conormal to the gauge orbit, and any its regular extension \(\Phi_{\eta}\), the following identity holds

\[
\frac{d}{dt}(\Phi_{\eta}(a(t)), \xi^c(t)) = \langle \Phi_{\eta}(t), G(\xi)(t)\rangle,
\]

\(55)
In particular, for any smooth function \( f : \Gamma \to \mathbb{R} \) and any regular extension \( \Phi_{d^* f} \) of \( \tau^* f \), the following identity holds
\[
\frac{d}{dt}(\Phi_{d^* f}(t), \xi^v(t)) = (\Phi_{d^* f}(t), G(\xi)(t)).
\] (56)

**Proof.** Since \( \eta \) is conormal to the gauge orbits, for any smooth map \( \chi(t) : I \to \Gamma(A) \) with \( \chi(0) = \chi(1) = 0 \), we have
\[
\langle \eta, G(\chi) \rangle_a = \langle \Phi_\eta, G(\chi) \rangle_a = \int_I (\Phi_\eta(t), G(\chi(t)) \rangle_{a(t)}) dt = 0.
\]
Applying this equality to \( \chi(t) := \psi(t)\xi(t) \), where \( \psi(t) \) is any smooth function with \( \psi(0) = \psi(1) = 0 \) and using Eq. (57) we obtain
\[
0 = \int_0^1 \psi(t) \langle \Phi_\eta(t), G(\xi(t)) \rangle dt + \int_0^1 \frac{d\psi(t)}{dt} \langle \Phi_\eta(t), \xi^v(t) \rangle dt.
\] (57)
The result follows by integration by part. \( \square \)

Now we are ready to prove Proposition 3.16.

**Proof.** According to Lemma 3.15 we have
\[
\overline{G(\xi)}(\tau^* f) = \xi^v(1)(f) - \xi^v(0)(f), \forall f \in C^\infty(\Gamma).
\]
On the other hand, we have, by definition,
\[
\overline{G(\xi)(\tau^* f)} = \int_0^1 \langle \Phi_{d^* f}(t), G(\xi(t)) \rangle dt.
\]
Therefore, we have
\[
\int_0^1 \langle \Phi_{d^* f}(t), G(\xi(t)) \rangle dt = \xi^v(1)(f) - \xi^v(0)(f).
\]
By Lemma 3.17 it follows that
\[
\int_0^1 \frac{d}{dt}(\langle \Phi_{d^* f}(t), \xi^v(t) \rangle) dt = \xi^v(1)(f) - \xi^v(0)(f).
\]
Hence
\[
\langle \Phi_{d^* f}(1), \xi^v(1) \rangle - \langle \Phi_{d^* f}(0), \xi^v(0) \rangle = \xi^v(1)(f) - \xi^v(0)(f).
\]
Since this identity holds for any time-dependent section \( \xi \), we have
\[
\langle \Phi_{d^* f}(1), \xi^v(1) \rangle = \xi^v(1)(f) \quad \text{and} \quad \langle \Phi_{d^* f}(0), \xi^v(0) \rangle = \xi^v(0)(f).
\]
This implies that for any \( P \in P\Gamma(\wedge^k A) \), we have
\[
\begin{align*}
P^v(1)(\Phi_{d^* f_1}(1), \ldots, \Phi_{d^* f_k}(1)) &= \overline{P(1)}(f_1, \ldots, f_k) \\
P^v(0)(\Phi_{d^* f_1}(0), \ldots, \Phi_{d^* f_k}(0)) &= \overline{P(0)}(f_1, \ldots, f_k).
\end{align*}
\] (58)

Now assume that \( P(t) = \xi_1(t) \wedge \ldots \wedge \xi_k(t) \) for some time-dependent sections \( \xi_1, \ldots, \xi_k \) of \( \Gamma(A) \). Then, according to Eq. (55),
\[
G(P)(t) = \sum_{i=1}^k \xi_i^v(t) \wedge \cdots \wedge G(\xi_i(t)) \wedge \cdots \wedge \xi_k^v(t).
\]
Therefore, for any \( a(t) \in P(A) \),
\[
\widetilde{G}(P)(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k})(a(t))
\]
\[
= \int_0^1 G(P)(t)(\Phi_{\tau^*f_1}(t), \ldots, \Phi_{\tau^*f_k}(t))dt
\]
\[
= \sum_{i=1}^k \int_0^1 (\xi_i^v(t) \wedge \ldots \wedge G(\xi_i(t))|_{a(t)} \wedge \ldots \wedge \xi_k^v(t))(\Phi_{\tau^*f_1}(t), \ldots, \Phi_{\tau^*f_k}(t))dt
\]
\[
= \sum_{\sigma \in S_k} \sum_{i=1}^k (-1)^{[\sigma]} \int_0^1 \langle \Phi_{\tau^*f_1}(t), \xi_{\sigma(1)}^v(t) \rangle \ldots \langle \Phi_{\tau^*f_1}(t), G(\xi_{\sigma(i)}(t)) \rangle \ldots \langle \Phi_{\tau^*f_k}(t), \xi_{\sigma(k)}^v(t) \rangle dt
\]
(By Lemma 3.17)
\[
= \sum_{\sigma \in S_k} \sum_{i=1}^k (-1)^{[\sigma]} \int_0^1 \frac{d}{dt} \langle \Phi_{\tau^*f_1}(t), \xi_{\sigma(1)}^v(t) \rangle \ldots \langle \Phi_{\tau^*f_k}(t), \xi_{\sigma(k)}^v(t) \rangle dt
\]
\[
= \sum_{\sigma \in S_k} (-1)^{\sigma} \int_0^1 \frac{d}{dt} \langle \Phi_{\tau^*f_1}(t), \xi_{\sigma(1)}^v(t) \rangle \ldots \langle \Phi_{\tau^*f_k}(t), \xi_{\sigma(k)}^v(t) \rangle dt
\]
\[
= P^v(t)(\Phi_{\tau^*f_1}(t), \ldots, \Phi_{\tau^*f_k}(t))|_0^1
\]
The result now follows from Eq. (58).  □

### 3.4 From \( k \)-vector fields on \( \widetilde{P}(A) \) to \( k \)-vector fields on \( \Gamma \)

Assume that \( \Gamma \) is an \( \alpha \)-simply connected and \( \alpha \)-connected Lie groupoid with Lie algebroid \( A \). Let \( \delta \) be a \( k \)-differential on \( A \) and \( \tilde{\delta} \) the corresponding \( k \)-vector field on \( \widetilde{P}(A) \). The goal of this section is to construct a \( k \)-vector field \( \Pi_{\delta} \) on \( \Gamma \) from \( \tilde{\delta} \).

**Proposition 3.18** Let \( f_1, \ldots, f_k \in C^\infty(\Gamma) \) be a family of smooth functions on \( \Gamma \) and \( \Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k} \) regular extensions of \( \tau^*f_1, \ldots, \tau^*f_k \). Then \( \tilde{\delta}(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k}) \) is a smooth function on \( P(A) \), which is

i) independent of the choice of the chosen regular extensions \( \Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k} \) of \( d\tau^*f_1, \ldots, d\tau^*f_k \), and

ii) invariant under gauge transformations.

**Proof:** i) It suffices to prove that if \( \Phi_{\tau^*f_1} \) is a regular extension of zero, the function \( \tilde{\delta}(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k}) \) vanishes. By Lemma 3.3 ii), for any \( a \in P(A) \), there exists \( g : I \rightarrow C^\infty(M) \) with \( g_0 = g_1 = 0 \) such that \( \Phi_{\tau^*f_1}|_a = dF_{dg}|_a \). Then
\[
\tilde{\delta}(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k})|_a = \tilde{\delta}(dF_{dg}, \Phi_{\tau^*f_2}, \ldots, \Phi_{\tau^*f_k}).
\]
According to Lemma 3.15 i), we have \( F_{dg} = \widetilde{G}(g) \). Lemma 3.15 v) implies that
\[
\tilde{\delta}(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k})|_a = \widetilde{G}(\tilde{\delta}(g))(\Phi_{\tau^*f_2}, \ldots, \Phi_{\tau^*f_k})|_a.
\]
The latter vanishes according to Proposition 3.16 since by assumption \( \xi(0) = \xi(1) = 0 \). This proves \( i \).

Before starting the proof of \( ii \), we have to add a comment. We have proven in \( i \) that for any regular extension \( \Phi \) of 0, and any regular extensions \( \Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k} \) of \( \tau^*f_2, \ldots, \tau^*f_n \), we have

\[
\pi_\delta(\Phi, \Phi_{\tau^*f_2}, \ldots, \Phi_{\tau^*f_k}) = 0
\]

Regular extensions are dense with respect to the induced topology of \( T_{\Phi}^*P(A) \). Thus,

\[
\pi_\delta(\Phi, \Phi_{\tau^*f_2}, \ldots, \Phi_{\tau^*f_k}) = 0,
\]

for any extension \( \Phi \) of 0.

\( ii \) Since the functions \( \tau^*f_1, \ldots, \tau^*f_k \) are invariant under the gauge transformation, \( \forall \xi \in P_0\Gamma(A) \), the Lie derivative \( \mathcal{L}_{G(\xi)}(\Phi_{\tau^*f_i}) \) is, for all \( i \in \{1, \ldots, k\} \), an extension of zero. Therefore, by Eq. 59, for all \( i \in \{1, \ldots, k\} \), \( \pi_\delta(\Phi_{\tau^*f_1}, \ldots, \mathcal{L}_{G(\xi)}(\Phi_{\tau^*f_i}), \ldots, \Phi_{\tau^*f_k}) = 0 \).

By Eq. 50, we have

\[
\pi_\delta(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k}) = (\mathcal{L}_{G(\xi)}(\Phi_{\tau^*f_1}), \ldots, \Phi_{\tau^*f_k})).
\]

By Lemma 3.15 \( v \), we have

\[
G(\xi)(\pi_\delta(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k})) = G(\pi_\delta)(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k}),
\]

which is identically equal to 0 according to Proposition 3.16. This proves \( ii \). □

By Proposition 3.1 and 3.15 \( ii \) above, the function \( \pi_\delta(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k}) \) descends to a smooth function on \( \Gamma \), which will be denoted by \( \{f_1, \ldots, f_k\} \). I.e.,

\[
\pi_\delta(\Phi_{\tau^*f_1}, \ldots, \Phi_{\tau^*f_k}) = \tau^*\{f_1, \ldots, f_k\}.
\]

It is straightforward to check that the map \( f_1, \ldots, f_k \to \{f_1, \ldots, f_k\} \) indeed defines a \( k \)-vector field \( \Pi_\delta \) on \( \Gamma \), i.e.,

\[
\{f_1, \ldots, f_k\} = \Pi_\delta(df_1, \ldots, df_k).
\]

**Proposition 3.19** \( \Pi_\delta \) is a multiplicative \( k \)-vector field on \( \Gamma \).

**Proof.** By definition, \( \Pi_\delta \) is given, for any \( \eta_1, \ldots, \eta_k \in T_\Phi^*\Gamma \) by

\[
\Pi_\delta(\eta_1, \ldots, \eta_k) = \int_I (\pi_\delta|_{\alpha(t)})(\Phi_{\tau^*\eta_1}(t), \ldots, \Phi_{\tau^*\eta_k}(t)) \, dt
\]

where \( \Phi_{\tau^*\eta_1}(t), \ldots, \Phi_{\tau^*\eta_k}(t) \) are any regular extensions of \( \tau^*\eta_1, \ldots, \tau^*\eta_k \), with a smooth dependence on the variable \( t \) and \( a(t) \) is any \( A \)-path with \( \tau(a) = g \). Since smooth functions are dense in the space of piecewise continuous functions with finitely many discontinuities, we obtain that Eq. 62 remains valid when the extensions \( \Phi_{\tau^*\eta_1}(t), \ldots, \Phi_{\tau^*\eta_k}(t) \) are just assumed to be piecewise continuous in \( t \) (with finitely many discontinuities).

It is straightforward to check that for any \( g \in \Gamma \) with \( \alpha(g) = m \) and \( \beta(g) = n \), there exists an \( A \)-path \( a(t) \) with \( \tau(a) = g \) such that \( a(t) \) is constantly equal to \( m \) in a neighborhood of \( t = 0 \) and constantly equal to \( n \) in a neighborhood of \( t = 1 \). Consider two
composable elements \(g_1, g_2 \in \Gamma\) and two composable elements \(\eta_1 \in T_{g_1}^* \Gamma\) and \(\eta_2 \in T_{g_2}^* \Gamma\). Choose now \(A\)-paths \(a_1(t)\) and \(a_2(t)\) satisfying the previous condition. Define \(a(t)\) by \(a(t) = a_1(2t)\) for \(t \in [0, \frac{1}{2}]\) and \(a(t) = a_2(2t - 1)\) for \(t \in [\frac{1}{2}, 1]\). By construction, \(a(t)\) is an \(A\)-path and \(\tau(a) = \tau_1 \gamma_2\).

Consider \(\Phi_{\tau^* \eta_1}(t)\) and \(\Phi_{\tau^* \eta_2}(t)\) two regular extensions of \(\tau^* \eta_1\) and \(\tau^* \eta_2\). Then the map defined by \(\Phi(t) = \Phi_{\tau^* \eta_1}(2t)\) for \(t \in [0, \frac{1}{2}]\) and \(\Phi(t) = \Phi_{\tau^* \eta_2}(2t - 1)\) for \(t \in [\frac{1}{2}, 1]\) is a regular extension of \(\Phi_{\eta_1 \eta_2} \in T_{g_1 g_2}^* \Gamma\), and it may have a point of discontinuity in \(t = \frac{1}{2}\).

We choose now \(k\) compatible pairs \(\eta_i^1, \eta_i^2\) for \(i = 1, \ldots, k\) of elements of \(T_{g_1}^* \Gamma\) and \(T_{g_2}^* \Gamma\). For all \(i = 1, \ldots, k\) we consider two regular extensions \(\Phi_{\tau^* \eta_i^1}\) and \(\Phi_{\tau^* \eta_i^2}\) of \(\tau^* \eta_i^1\) and \(\tau^* \eta_i^2\) respectively. And, for all \(i = 1, \ldots, k\) again, we form \(\Phi_{\eta_i^1 \eta_i^2}(t)\) as above.

Eq. (62) being valid even for piecewise regular extension, the first of the identities below is valid; the other ones are routine.

\[
\Pi_\delta(\eta_1^1 \eta_2^1, \ldots, \eta_k^1 \eta_k^2) = \int_0^1 (\pi_\delta)_{a(t)} \left( \Phi_{\eta_1^1 \eta_2^1}(t), \ldots, \Phi_{\eta_k^1 \eta_k^2}(t) \right) dt
\]

(63)

By Proposition 3.16(ii), \(\Pi_\delta\) is multiplicative. □

Finally to complete the proof of Theorem 2.34, we need to show that the map \(\delta \to \Pi_\delta\) constructed above is indeed the inverse of \(\Pi \to \delta_{\Pi}\). This is due to the following

**Proposition 3.20** For any \(k\)-differential \(\delta\), the identities

\[
[\Pi_\delta, \vec{X}] = \delta \vec{X} \quad \text{and} \\
[\Pi_\delta, \alpha^* f] = \vec{\delta} \vec{f}
\]

hold.

**PROOF.** For any functions \(f_1, \ldots, f_k \in C^\infty(\Gamma)\) and any \(X \in \Gamma(A)\), one has

\[
[\Pi_\delta, \vec{X}] (f_1, \cdots, f_k) = \vec{X} (\Pi_\delta (f_1, \cdots, f_k)) - \sum_{i=1}^k \Pi_\delta (f_1, \cdots, \vec{X} (f_i), \cdots, f_k).
\]

(64)

Let \(\xi \in P\Gamma(A)\) be an element such that \(\xi(1) = X\) and \(\xi(0) = 0\). According to Lemma 3.15 we have \(\tau_\delta(G(\xi)) = \vec{X}\). Therefore, for any regular extension \(\Phi_{\tau^* f}\) of \(\tau^* f\), where \(f \in C^\infty(\Gamma)\), \(\mathcal{L}_{G(\xi)} \Phi_{\tau^* f}\) is a regular extension of \(\tau^*(\vec{X} (f))\).

Let us choose some regular extensions \(\Phi_{\tau^* f_1}, \ldots, \Phi_{\tau^* f_k}\) of \(\tau^* f_1, \cdots, \tau^* f_k\). Applying \(\tau^*\) to both sides of Eq. (61) and using Eq. (61), we obtain

\[
\tau^* ([\Pi_\delta, \vec{X}] (f_1, \cdots, f_k)) = G(\xi) \cdot \vec{\tau}_\delta (\Phi_{\tau^* f_1}, \ldots, \Phi_{\tau^* f_k})
\]

\[
- \sum_{i=1}^k \vec{\tau}_\delta (\Phi_{\tau^* f_1}, \ldots, \mathcal{L}_{G(\xi)} \Phi_{\tau^* f_i}, \ldots, \Phi_{\tau^* f_k}).
\]

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Using the definition as given by Eq. (50), the right hand side of the equation above is
\[(\mathcal{L}_{G(\xi)}\pi_{\xi})(\Phi_{d^*f_1}, \ldots, \Phi_{d^*f_k})\]
which is again, by Lemma 3.15 (iv), equal to \(G(\xi)(\Phi_{d^*f_1}, \ldots, \Phi_{d^*f_k})\). By Proposition 3.16, the latter is equal to \(\tau^*\delta(X)(f_1, \ldots, f_k)\). This proves the first equality. The proof of the other equality is similar. □

4 Quasi-Poisson groupoids

In this section, we will introduce the notion of quasi-Poisson groupoids. We describe their infinitesimal invariants, namely quasi-Lie bialgebroids, and study the corresponding momentum map theory.

4.1 Definition and properties

Definition 4.1 A quasi-Poisson groupoid is a triple \((\Gamma \xrightarrow{\Pi} M, \Omega)\), where \(\Gamma \xrightarrow{\Pi} M\) is a Lie groupoid, \(\Pi\) is a multiplicative bivector field on \(\Gamma\) and \(\Omega \in \Gamma(\wedge^3 A)\) such that the following compatibility conditions hold

\[
\frac{1}{2}[\Pi, \Pi] = \overrightarrow{\Omega} - \overleftarrow{\Omega}, \quad \text{and} \quad [\Pi, \overrightarrow{\Omega}] = 0. \tag{65}
\]

Using Proposition 2.21 and Eqs. (65) and (66), we obtain the following

Proposition 4.2 Given a quasi-Poisson groupoid \((\Gamma \xrightarrow{\Pi} M, \Omega)\) there exists a bivector field \(\Pi_M\) on \(M\) such that

\[
\Pi_M = \alpha_\ast \Pi = -\beta_\ast \Pi.
\]

\(\Pi_M\) satisfies the relations

\[
\frac{1}{2}[\Pi_M, \Pi_M] = \Omega_M, \quad \text{and} \quad [\Pi_M, \Omega_M] = 0, \tag{67}
\]

where \(\Omega_M\) is the 3-vector field \(\Omega_M = \rho(\Omega)\), and \(\rho : \Gamma(\wedge^3 A) \to \mathfrak{X}^3(M)\) is the extension of the anchor map.

Some interesting examples of quasi-Poisson groupoids are listed below.

Example 4.3 If \(\Omega = 0\) then we just have a multiplicative Poisson structure \(\Pi\) on a Lie groupoid \(\Gamma \xrightarrow{\Pi} M\). I.e., \((\Gamma \xrightarrow{\Pi} M, \Pi)\) is a Poisson groupoid.

Example 4.4 If \(G\) is a Lie group, then we recover the notion of quasi-Poisson structures on a Lie group of Kosmann-Schwarzbach [12]. That is, a multiplicative bivector field \(\Pi\) on \(G\) and an element \(\Omega \in \wedge^3 \mathfrak{g}\) such that \(\frac{1}{2}[\Pi, \Pi] = \overrightarrow{\Omega} - \overleftarrow{\Omega}\) and \([\Pi, \overrightarrow{\Omega}] = 0\).
Proposition 4.5 Let \((\Gamma \rightrightarrows M, \Pi, \Omega)\) be a quasi-Poisson groupoid such that \(\Pi \in \mathcal{X}^2(\Gamma)\) is non-degenerate. Let \(\omega \in \Omega^2(\Gamma)\) be its corresponding non-degenerate 2-form and \(\phi \in \Omega^3(M)\) be the 3-form on \(M\) defined by \((\wedge^3 \omega^3)(\Omega) = \alpha^* \phi\). Then \((\Gamma \rightrightarrows M, \omega, \phi)\) is a non-degenerate twisted symplectic groupoid in the sense of \([5]\). That is,

1. \(d\phi = 0\);  
2. \(d\omega = \alpha^* \phi - \beta^* \phi\); and  
3. \(\omega\) is multiplicative, i.e., the 2-form \((\omega, \omega, -\omega)\) vanishes when being restricted to the graph of the groupoid multiplication \(\Lambda \subset \Gamma \times \Gamma \times \Gamma\).

Proof. Since \(\Pi\) is multiplicative, it follows from Remark 2.4 that \(\omega\) satisfies the formula \(m^* \omega = pr^*_1 \omega + pr^*_2 \omega\). That is, \(\omega\) is multiplicative.

Since \(\omega\) is multiplicative, we know (see \([5]\)) that the Lie algebroid \(A\) of \(\Gamma\) is isomorphic to \(T^* M\) as a vector bundle and the isomorphism \(\lambda : A \to T^* M\) is characterized by \(\omega^b(\vec{X}) = \alpha^* \eta\), for \(X \in \Gamma(A)\) and \(\eta \in \Omega^1(M)\). In general, we have

\[
(\wedge^k \omega^b)(\Omega^k(M)) = \alpha^* \varphi, \quad (\wedge^k \omega^b)(\Omega^k(M)) = \beta^* \varphi,
\]

\(\forall P \in \Gamma(\Lambda^k A)\) and \(\varphi \in \Omega^k(M)\).

Define \(\phi \in \Omega^3(M)\) as the 3-form on \(M\) such that \((\wedge^3 \omega^b)(\Omega) = \alpha^* \phi\) or, equivalently, \(\Omega = (\wedge^3 \Pi^2)(\alpha^* \phi)\). Using that \(\Pi^2\) is the inverse of \(\omega^b\), \((\wedge^3 \Pi^2)(d \omega) = \frac{1}{2}[\Pi, \Pi]\) and Eq. (65), we deduce that

\[
(\wedge^3 \Pi^2)(d \omega) = \frac{1}{2}[\Pi, \Pi] = \Omega - \Omega = (\wedge^3 \Pi^2)(\alpha^* \varphi - \beta^* \phi).
\]

As a consequence, we have \(d \omega = \alpha^* \phi - \beta^* \phi\).

Let \(\Pi_M = \alpha_\ast \Pi\). We will prove that

\[
\lambda(-\delta_\Pi P) = \delta(\lambda(P)), \text{ for } P \in \Gamma(\Lambda^* A),
\]

where \(\delta_\Pi\) is the 2-differential corresponding to \(\Pi\) (see Theorem 2.4.4), and \(\delta : \Omega^*(M) \to \Omega^{*+1}(M)\) is the map characterized by

\[
\delta f = df, \forall f \in C^\infty(M),
\]

\[
\delta \eta = d \eta - \Pi^2_M(\eta) \wedge \phi, \forall \eta \in \Omega^1(M).
\]

From Lemma 2.4 in \([5]\), we know that \(\delta f = df = -\lambda(\delta_\Pi f)\). If \(\eta \in \Omega^1(M)\) and \(X \in \Gamma(A)\) such that \(\lambda(X) = \eta\) (that is, \(\vec{X} = \Pi^2(\alpha^* \eta)\)), then using the relation

\[
(\wedge^2 \Pi^2)(d \gamma) = [\Pi^2(\gamma), \Pi] - \frac{1}{2}(\gamma \wedge [\Pi, \Pi]), \quad \forall \gamma \in \Omega^1(M)
\]

and Eqs. (33) and (65), we obtain

\[
(\wedge^2 \omega^b)(d \alpha^* \eta) = [\Pi^2(\alpha^* \eta), \Pi] - (\alpha^* \eta \wedge (\Omega - \Omega)) = -\delta_\Pi \vec{X} - (\alpha^* \eta \wedge (\wedge^3 \Pi^2)(\alpha^* \phi - \beta^* \phi))).
\]

Applying \(\wedge^2 \omega^b\) to both sides of this equation and using that

\[
(\wedge^2 \omega^b)(\alpha^* \eta \wedge ((\wedge^3 \Pi^2)(\alpha^* \phi - \beta^* \phi))) = -\alpha^*(\Pi^2_M(\eta) \wedge \phi),
\]

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we conclude that

\[-(\wedge^2 \omega)(\delta \Pi \chi) = \alpha^*(d\eta - \Pi_M^2(\eta) \cdot \phi).\]

I.e., \(-\lambda(\delta \Pi X) = \delta \eta\). As a consequence, by Eqs. (55), (69) and that \(\lambda(\Omega) = \phi\), we have \(\delta \phi = 0\). Thus \(d\phi = 0\). Thus, we conclude that \((\Gamma \Rightarrow M, \omega, \phi)\) is a non-degenerate twisted symplectic groupoid. □

4.2 Quasi-Poisson groupoids and quasi-Lie bialgebroids

In this subsection, we will describe the infinitesimal invariants of quasi-Poisson groupoids, i.e., quasi-Lie bialgebroids. The notion of quasi-Lie bialgebroids was first introduced by Roytenberg [23]. Here, we give an alternative definition using 2-differentials.

Definition 4.6 A quasi-Lie bialgebroid corresponds to a 2-differential whose square is a coboundary, i.e., \(\delta : \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A)\) such that \(\delta \circ \delta = [\Omega, \cdot]\) for some \(\Omega \in \Gamma(\wedge^3 A)\) satisfying \(\delta \Omega = 0\).

An interesting example of a quasi-Lie bialgebroid is the following.

Example 4.7 Recall that a twisted Poisson structure \((M, \pi, \phi)\) is a bivector field \(\pi \in \mathfrak{X}^2(M)\) and a closed 3-form \(\phi \in \Omega^3(M)\) such that

\[
\frac{1}{2} [\pi, \pi] = (\wedge^3 \pi^\sharp)(\phi).
\]

(70)

In this situation, let \(A = T^*M\) be the corresponding twisted Lie algebroid (see Eq. (2) and, for more details, [23]). Define \(C^\infty(M) \overset{\delta_{\pi,\phi}}{\to} \Omega^1(M) \overset{\delta_{\pi,\phi}}{\to} \Omega^2(M)\) as follows. On \(C^\infty(M)\), \(\delta_{\pi,\phi}\) is the usual de Rham differential, while on \(\Omega^1(M)\), \(\delta_{\pi,\phi}\eta = d\eta - \pi^\sharp(\eta) \cdot \phi\), for all \(\eta \in \Omega^1(M)\). Then \((T^*M, \delta_{\pi,\phi}, \phi)\) is a quasi-Lie bialgebroid.

A direct consequence of Lemma 2.32 is the following

Proposition 4.8 Let \((A, \delta, \Omega)\) be a quasi-Lie bialgebroid. Then, \(\delta\) induces a bivector field \(\pi_M\) on \(M\) such that

\[
\frac{1}{2} [\pi_M, \pi_M] = \Omega_M, \quad [\pi_M, \Omega_M] = 0,
\]

where \(\Omega_M\) is the 3-vector field \(\Omega_M = \rho(\Omega)\), and \(\rho : \Gamma(\wedge^3 A) \to \mathfrak{X}^3(M)\) is the extension of the anchor map.

Now we are ready to state the main theorem of this section: quasi-Lie bialgebroids are indeed the infinitesimal invariants of quasi-Poisson groupoids.

Theorem 4.9 If \((\Gamma \Rightarrow M, \Pi, \Omega)\) is a quasi-Poisson groupoid, then there exists a natural quasi-Lie bialgebroid structure \((\delta, \Omega)\) on the Lie algebroid \(A\) of \(\Gamma\).

Conversely, if \((A, \delta, \Omega)\) is a quasi-Lie bialgebroid, where \(A\) is the Lie algebroid of an \(\alpha\)-connected and \(\alpha\)-simply connected Lie groupoid \(\Gamma\), there exists a quasi-Poisson groupoid structure \((\Pi, \Omega)\) on \(\Gamma\) such that the corresponding quasi-Lie bialgebroid is \((\delta, \Omega)\).
Theorem 4.9 generalizes a result of Kosmann-Schwarzbach regarding quasi-Lie bialgebroids and quasi-Lie bialgebras \[12\]. On the other hand, when \( \Omega = 0 \), that is, \((A, \delta, \Omega)\) is a quasi-Lie bialgebroid.

Corollary 4.10 Let \((M, \pi, \phi)\) be a twisted Poisson structure. If the Lie algebroid \( T^* M \) can be integrated to an \( \alpha \)-simply connected and \( \alpha \)-connected Lie groupoid \( \Gamma \), then \( \Gamma \) is a non-degenerate twisted symplectic groupoid.

Proof. Let \( \lambda : T^* M \to A \) be the Lie algebroid isomorphism between the Lie algebroid \( A \) of \( \Gamma \) and \( T^* M \). Denote by \( \delta \) and \( \Omega \) the almost 2-differential and the 3-section of \( A \) respectively such that

\[
\Omega = \lambda(\phi); \\
-\lambda(\delta_{\pi,\phi}(\varphi)) = \delta(\lambda(\varphi)), \forall \varphi \in \Omega^*(M),
\]

(71)

where \( \delta_{\pi,\phi} \) is defined as in Example 4.7. \((A, \delta, \Omega)\) is clearly a quasi-Lie bialgebroid induced by the quasi-Lie bialgebroid structure \((T^* M, \delta_{\pi,\phi}, \phi)\). Therefore, according to Theorem 4.9, there exists a 2-differential, according to Theorem 2.34, there exists a bivector field \( \Pi \) such that \((\Gamma \propto M, \eta, \Omega)\) is a quasi-Poisson groupoid integrating the quasi-Lie bialgebroid \((A, \delta, \Omega)\).

Remark Theorem 4.9 generalizes a result of Kosmann-Schwarzbach regarding quasi-Poisson Lie groups and quasi-Lie bialgebras \[12\]. On the other hand, when \( \Omega = 0 \), that is, \((A, \delta)\) is a Lie bialgebroid, we recover the classical results in \[19, 21\]: there exists a one-to-one correspondence between Lie bialgebroids and Poisson groupoids.

As another consequence of Theorem 4.9, we recover the construction of the non-degenerate twisted symplectic groupoid associated with a twisted Poisson manifold \[5\].
Since any element of the cotangent space \( T^*_\vartheta \Gamma \) \((m \in M)\) can be written as \( \xi + \alpha^*df \), with \( \xi \in A^* \) and \( f \in C^\infty(M) \), and \( \lambda \) and \( \lambda^* \) are injective (\( \lambda \) is a vector bundle isomorphism), it follows that \( \Pi \) is non-degenerate along \( M \). Following Theorem 5.3 in [21], one can extend this non-degeneracy for every point in \( \Gamma \). From Proposition 4.5, we conclude that \( \Gamma \) is the twisted symplectic groupoid integrating \((M, \pi, \phi)\). □

We end this section with the following proposition, which reveals the relation between the bivector fields obtained by Propositions 4.2 and 4.8.

**Proposition 4.11** Let \((\Gamma \to M, \Pi, \Omega)\) be a quasi-Poisson groupoid, and \((A, \delta, \Omega)\) its corresponding quasi-Lie bialgebroid. By \( \Pi_M \) and \( \pi_M \) we denote the bivector fields on \( M \) induced from the quasi-Poisson groupoid structure on \( \Gamma \) and the quasi-Lie bialgebroid structure on \( A \) as in Proposition 4.2 and Proposition 4.8 respectively. Then \( \Pi_M = \pi_M \).

**Proof.** Let \( f, g \in C^\infty(M) \). Then, using Eq. (18),

\[
-\langle \rho(\delta f), dg \rangle = -\langle \alpha^*([\Pi, \alpha^* f]), dg \rangle = \Pi(\alpha^* df, \alpha^* dg) = (\alpha_* \Pi)(df, dg).
\]

The conclusion thus follows. □

### 4.3 Hamiltonian \( \Gamma \)-spaces of quasi-Poisson groupoids

Let \( \Gamma \to M \) be a Lie groupoid. Recall that a \( \Gamma \)-space is a smooth manifold \( X \) with a map \( J : X \to M \), called the momentum map, and an action

\[
\Gamma \times_M X = \{(g, x) \in \Gamma \times X \mid \beta(g) = J(x)\} \to X, \quad (g, x) \mapsto g \cdot x
\]

satisfying

1. \( J(g \cdot x) = \alpha(g) \), for \( (g, x) \in \Gamma \times_M X \);
2. \( (gh) \cdot x = g \cdot (h \cdot x) \), for \( g, h \in \Gamma \) and \( x \in X \) such that \( \beta(g) = \alpha(h) \) and \( J(x) = \beta(h) \);
3. \( \epsilon(J(x)) \cdot x = x \), for \( x \in X \).

Hamiltonian \( \Gamma \)-spaces for Poisson groupoids were studied in [14]. For quasi-Poisson groupoids, one can introduce Hamiltonian \( \Gamma \)-spaces in a similar fashion.

**Definition 4.12** Let \((\Gamma \to M, \Pi, \Omega)\) be a quasi-Poisson groupoid. A Hamiltonian \( \Gamma \)-space is a \( \Gamma \)-space \( X \) with momentum map \( J : X \to M \) and a bivector field \( \Pi_X \in \mathfrak{X}^2(X) \) such that:

1. the graph of the action \( \{(g, x, g \cdot x) \mid J(x) = \beta(g)\} \) is a coisotropic submanifold of \((\Gamma \times X \times X, \Pi \oplus \Pi_X \oplus -\Pi_X)\);
2. \( \frac{1}{2}[\Pi_X, \Pi_X] = \hat{\Omega} \), where the hat denotes the map \( \Gamma(\wedge^3 A) \to \mathfrak{X}^3(X) \), induced by the infinitesimal action of the Lie algebroid on \( X \): \( \Gamma(A) \to \mathfrak{X}(X), Y \mapsto \hat{Y} \).
**Proposition 4.13** Let \((\Gamma \rightrightarrows M, \Pi, \Omega)\) be a quasi-Poisson groupoid. If \((X, \Pi_X)\) is a Hamiltonian \(\Gamma\)-space with momentum map \(J\), then \(J\) maps \(\Pi_X\) to \(\Pi_M\), where \(\Pi_M\) is given by Proposition 4.13.

**Proof.** The result follows using the coisotropy condition and the fact that \((-\beta^*\eta, J^*\eta, 0)\), with \(\eta \in T^*M\), is conormal to the graph of the groupoid action. Indeed,

\[
0 = \Pi(-\beta^*\eta_1, -\beta^*\eta_2) + \Pi(J^*\eta_1, J^*\eta_2) = \left(\beta^*\Pi + J_*\Pi_X\right)(\eta_1, \eta_2)
\]

for any \(\eta_1, \eta_2 \in T^*M\). Thus, \(J_*\Pi_X = -\beta^*\Pi = \Pi_M\). □

The following theorem gives an equivalent description of Hamiltonian \(\Gamma\)-spaces of quasi-Poisson groupoids in terms of their infinitesimal objects.

**Theorem 4.14** Let \((\Gamma \rightrightarrows M, \Pi, \Omega)\) be a quasi-Poisson groupoid with corresponding quasi-Lie bialgebroid \((A, \delta, \Omega)\). Then \((X, \Pi_X)\) is a Hamiltonian \(\Gamma\)-space with momentum map \(J : X \to M\) if and only if \(\frac{1}{2}[\Pi_X, \Pi_X] = \hat{\Omega}\) and

\[
\begin{align*}
[\Pi_X, J^*f] &= \hat{\delta}_{\Pi f}, \forall f \in C^\infty(M), \\
[\Pi_X, \hat{Y}] &= \hat{\delta}_\Pi(\hat{Y}), \forall Y \in \Gamma(A).
\end{align*}
\] (72)

**Proof.** Using Theorem 7.1 in [14] we know that the coisotropy condition is equivalent to the following conditions:

1. For any \(f \in C^\infty(M), X_{J^*f}(x) = (r_x)_*X_{\alpha^*f}(u)\), where \(x \in X, u = J(x)\) and \(r_x\) denotes the map \(g \mapsto g \cdot x\) from \(\beta^{-1}(u)\) to \(X\).

2. For any compatible \((g, x) \in \Gamma \times_M X\),

\[
\Pi_X(g \cdot x) = (L_x)_*\Pi_X(x) + (R_y)_*\Pi(g) - (R_y)_*(L_x)_*\Pi(u),
\]

where \(u = \beta(g) = J(x)\), \(\alpha\) is any local bisection through \(g\), and \(\gamma\) is any local section of \(J\) through the point \(x\).

From the first condition and the equation \(\delta_{\Pi f} = [\Pi, \alpha^*f]\), it follows that \([\Pi_X, J^*f] = \hat{\delta}_{\Pi f}\).

On the other hand, let \(x \in X\) and \(u = J(x)\). Applying the second condition to the family of (local) bisections \(X_t = \exp tY\) associated to \(Y \in \Gamma(A)\) and \(g_t = (\exp tY)(u)\), one obtains that

\[
(L_{X_t}^{-1})_*\Pi_X(g_t \cdot x) = \Pi_X(x) + (R_Y)_*(L_{X_t}^{-1})_*\Pi(g_t) - (R_Y)_*\Pi(u),
\]

for any section \(Y\) of \(J\). Then, taking derivatives, we have

\[
[\Pi_X, \hat{Y}] = \hat{\delta}_\Pi(\hat{Y}).
\]

The other direction follows just by going backwards. □

**Remark** Note that Eq. (72) is equivalent to that \([\Pi_X, \hat{P}] = \hat{\delta}_{\Pi} \hat{P}\) for any \(P \in \Gamma(\wedge^kA), k \geq 0\). I.e., the following diagram is commutative,

\[
\begin{array}{ccc}
\Gamma(\wedge^kA) & \overset{\sim}{\longrightarrow} & \mathcal{X}^k(X) \\
\delta_{\Pi} \downarrow & & \downarrow [\Pi_X, \cdot] \\
\Gamma(\wedge^{k+1}A) & \overset{\sim}{\longrightarrow} & \mathcal{X}^{k+1}(X)
\end{array}
\]

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4.4 Twists of quasi-Lie bialgebroids

Just like quasi-Lie bialgebras, one can talk about twists of a quasi-Lie bialgebroid. Given a quasi-Lie bialgebroid $\mathbf{A}, \delta, \Omega$ and a section $t \in \Gamma(\wedge^2 \mathbf{A})$, let $\delta^t = \delta + [t, \cdot]$, and $\Omega^t = \Omega + \delta t + \frac{1}{2}[t, t]$. Using Eq. (13) and the properties of the Schouten bracket, it is simple to see that $\mathbf{A}, \delta^t, \Omega^t$ is also a quasi-Lie bialgebroid, which will be called the twist of $\mathbf{A}, \delta, \Omega$ by $t \in \Gamma(\wedge^2 \mathbf{A})$.

The following proposition describes how a twist affects the integration (see [1] for the case of quasi-Lie bialgebras).

**Proposition 4.15** Assume that $(\Gamma \to M, \Pi, \Omega)$ is a quasi-Poisson groupoid with corresponding quasi-Lie bialgebroid $\mathbf{A}, \delta, \Omega$, and $t \in \Gamma(\wedge^2 \mathbf{A})$. Then $(\Gamma \to M, \Pi^t, \Omega^t)$ is a quasi-Poisson groupoid with quasi-Lie bialgebroid $\mathbf{A}, \delta^t, \Omega^t$, where $\Pi^t = \Pi + \tau_{-t} - \tau_t$.

**Proof.** It is a direct consequence of Theorem 2.34. Note that if $\Pi$ and $\Pi'$ are multiplicative bivector fields then $\delta^{\Pi + \Pi'} = \delta^\Pi + \delta^\Pi'$. Also we have $\delta^\tau_{-t} = \text{ad}(t)$ (see Example 2.40). $\Box$

Next, we will show how a Hamiltonian $\Gamma$-space of a quasi-Poisson groupoid $(\Gamma \to M, \Pi, \Omega)$ must be modified in order to obtain a Hamiltonian $\Gamma$-space for the twisted quasi-Poisson structure $(\Pi^t, \Omega^t)$.

**Proposition 4.16** Assume that $(\Gamma \to M, \Pi, \Omega)$ is a quasi-Poisson groupoid and $t \in \Gamma(\wedge^2 \mathbf{A})$. There is a bijection between Hamiltonian $\Gamma$-spaces of $(\Gamma \to M, \Pi, \Omega)$ and those of $(\Gamma \to M, \Pi^t, \Omega^t)$.

More precisely, if $(X \to M, \Pi_X)$ is a Hamiltonian $\Gamma$-space of $(\Gamma \to M, \Pi, \Omega)$, then $(X \to M, \Pi_X + \hat{t})$ is a Hamiltonian $\Gamma$-space of $(\Gamma \to M, \Pi^t, \Omega^t)$.

**Proof.** Let $(X \to M, \Pi_X)$ be a Hamiltonian $\Gamma$-space of the quasi-Poisson groupoid $(\Gamma \to M, \Pi, \Omega)$. Using Theorem 4.14, we deduce that $[\Pi_X, \hat{P}] = \hat{\delta^\Pi P}$ for any $P \in \Gamma(\wedge^k \mathbf{A})$, $k \geq 0$.

Therefore, from the fact that $\oplus_k \Gamma(\wedge^k \mathbf{A}) \to \oplus_k \mathfrak{X}^k(X)$, $Y \mapsto \hat{Y}$ is a graded Lie algebra morphism, one gets that

$$[\Pi_X + \hat{t}, \hat{P}] = \hat{\delta^\Pi P} + [t, P] = \hat{\delta^\Pi^t P}.$$ 

On the other hand, it is trivial to see that

$$\frac{1}{2} [\Pi_X + \hat{t}, \Pi_X + \hat{t}] = \hat{\Omega^t}.$$ 

As a consequence of Theorem 4.14 $(X \to M, \Pi_X + \hat{t})$ is a Hamiltonian $\Gamma$-space of the quasi-Poisson groupoid $(\Gamma \to M, \Pi^t, \Omega^t)$. $\Box$

4.5 Quasi-Poisson groupoids associated to Manin pairs

In this subsection we will describe an example of quasi-Poisson groupoid associated to a Manin quasi-triple.

Let $(\mathfrak{d}, \mathfrak{g})$ be a Manin pair, that is, $\mathfrak{d}$ is an even dimensional Lie algebra with an invariant, nondegenerate symmetric bilinear form, and $\mathfrak{g}$ is a maximal isotropic subalgebra
of $\mathfrak{d}$. In this case, one can integrate the Manin pair $(\mathfrak{d}, \mathfrak{g})$ to the so-called *group pair* $(D, G)$, where $D$ and $G$ are connected and simply connected Lie groups with Lie algebra $\mathfrak{d}$ and $\mathfrak{g}$ respectively. Furthermore, the action of the Lie group $D$ on itself by left multiplication induces an action of $D$ on $S = D/G$, and in particular a $G$-action on $S$, which is called the *dressing action*. As in \[\Pi\], the infinitesimal dressing action is denoted by $v \mapsto v_S$ for any $v \in \mathfrak{d}$.

If $\mathfrak{h}$ is an isotropic complement of $\mathfrak{g}$ in $\mathfrak{d}$, by identifying $\mathfrak{h}$ with $\mathfrak{g}^*$, we obtain a quasi-Lie bialgebra structure on $\mathfrak{g}$, with cobracket $F : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ and $\Omega \in \wedge^3 \mathfrak{g}$. If $\{e_i\}$ is a basis of $\mathfrak{g}$ and $\{e^i\}$ the dual basis of $\mathfrak{g}^* \cong \mathfrak{h}$, then $F(e_i) = \frac{1}{2} \sum_{j,k} F_{ij}^k e_j \wedge e_k$ and $\Omega = \frac{1}{6} \sum_{i,j,k} \Omega^{ijk} e_i \wedge e_j \wedge e_k$. Moreover, the bracket on $\mathfrak{g} \cong \mathfrak{g} \oplus \mathfrak{h}$ can be written as

$$[e_i, e_j]_\mathfrak{d} = \sum_{k=1}^n c_{ij}^k e_k, \quad [e_i, e^j]_\mathfrak{d} = \sum_{k=1}^n -c_{ik}^j e^k + F_{ij}^k e_k, \quad [e^i, e^j]_\mathfrak{d} = \sum_{k=1}^n F_{jk}^i e^k + \Omega^{ijk} e_k, \quad (73)$$

where $c_{ij}^k$ are the structure constants of the Lie algebra $\mathfrak{g}$ with respect to the basis $\{e_i\}$.

**Example 4.17** Let $\mathfrak{g}$ be a Lie algebra endowed with a nondegenerate symmetric bilinear form $K$. On the direct sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ one can construct a scalar product $(\cdot | \cdot)$ by

$$(u_1, u_2)(v_1, v_2) = K(u_1, v_1) - K(u_2, v_2),$$

for $(u_1, u_2), (v_1, v_2) \in \mathfrak{d}$. Then, $(\mathfrak{d}, \Delta(\mathfrak{g}), \frac{1}{2} \Delta_-(\mathfrak{g}))$ is a Manin quasi-triple, where $\Delta(v) = (v, v)$ and $\Delta_-(v) = (v, -v)$, $\forall v \in \mathfrak{g}$ (see \[\Pi\]). In this case, as far as the corresponding quasi-Lie bialgebra is concerned, the cobracket $F$ vanishes and $\Omega$ can be identified with the trilinear form on $\mathfrak{g}$ given by $(u, v, w) \mapsto \frac{1}{2} K(w, [u, v]_\mathfrak{g})$.

Let $\lambda : T_s^* S \to \mathfrak{g}$ be the dual map of the infinitesimal dressing action $\mathfrak{g}^* \cong \mathfrak{h} \to T_s S$. That is,

$$\langle \lambda(\theta_s), \eta \rangle = \langle \theta_s, \eta S(s) \rangle, \forall \theta_s \in T_s^* S \text{ and } \eta \in \mathfrak{h}.$$ 

A direct consequence is that

$$\lambda(df) = \sum_{i=1}^n (e^i)_S(f) e_i, \text{ for } f \in C^\infty(S). \quad (74)$$

**Remark** Recall that an isotropic complement $\mathfrak{h}$ is said to be *admissible* at a point $s \in S = D/G$ if the infinitesimal dressing action restricted to $\mathfrak{h}$ defines an isomorphism from $\mathfrak{h}$ onto $T_s S$ \[\Pi\]. In this case, we can define an isomorphism from $\mathfrak{g}$ to $T_s^* S$, $\xi \mapsto \xi(h)(s)$, as follows,

$$\langle \xi(h)(s), \eta S(s) \rangle = -(\xi(\eta)), \quad \forall \eta \in \mathfrak{h},$$

where in the right-hand side $(\cdot | \cdot)$ is the bilinear form on $\mathfrak{d}$. $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ is said to be an *admissible* Manin quasi-triple if $\mathfrak{h}$ is admissible at every point of $S$. In this case, if $\lambda(\theta_s) = \xi$ then $\xi(h)(s) = -\theta_s$.

Next, we will show that on the transformation Lie algebroid $\mathfrak{g} \times S \to S$ there exists a natural quasi-Lie bialgebroid structure.
Proposition 4.18 Assume that \((\mathfrak{d}, \mathfrak{g}, \mathfrak{h})\) is a Manin quasi-triple with associated quasi-Lie bialgebra \((\mathfrak{g}, F, \Omega)\). On the transformation Lie algebroid \(\mathfrak{g} \times S \to S\) (where \(\mathfrak{g}\) acts on \(S\) by the infinitesimal dressing action) define an almost 2-differential
\[
\delta(f) = \lambda(df), \text{ for } f \in C^\infty(S),
\]
\[
\delta \xi = -F(\xi), \text{ for } \xi \in \mathfrak{g},
\]
where \(\lambda\) is defined by Eq. (74) and \(\xi \in \mathfrak{g}\) is considered as a constant section of the Lie algebroid \(\mathfrak{g} \times S \to S\), extending this operation to arbitrary section using the derivation law.

Then, \((\mathfrak{g} \times S, \delta, \Omega)\) is a quasi-Lie bialgebroid.

Proof. We remark that it suffices to check the axioms of a quasi-Lie bialgebroid for functions on \(S\) and constant sections of \(\mathfrak{g} \times S\), since the general case follows from the derivation law.

If \(f \in C^\infty(S)\), then using Eq. (73),
\[
\begin{align*}
\delta^2 f &= \sum_{i=1}^n \delta((e^i)_S(f))e_i \\
&= \sum_{i,j=1}^n (e^i)_S((e^j)_S(f))e_j \wedge e_i + \sum_{i=1}^n (e^i)_S(f)\delta(e_i) \\
&= \sum_{i<j} ((e^i)_S((e^j)_S(f)) - (e^j)_S((e^i)_S(f)))e_i \wedge e_j + \sum_{i=1}^n (e^i)_S(f)\delta(e_i) \\
&= \sum_{i<j} ([e^i, e^j]_S(f))e_i \wedge e_j + \sum_{k=1}^n (e^k)_S(f)\delta(e_k) \\
&= \sum_{i<j} \sum_{k=1}^n (F_{ikj} (e^k)_S(f) + \Omega_{ijk} (e_k)_S(f))e_i \wedge e_j - \sum_{k=1}^n (e^k)_S(f)\sum_{i<j} F_{ikj} e_i \wedge e_j \\
&= \sum_{i<j} \sum_{k=1}^n \Omega_{ijk} (e_k)_S(f)e_i \wedge e_j \\
&= \frac{1}{2} \sum_{i,j,k=1}^n \Omega_{ijk} (e_k)_S(f)e_i \wedge e_j \\
&= [\Omega, f].
\end{align*}
\]

Moreover, since \((\mathfrak{g}, F, \Omega)\) is a quasi-Lie bialgebra, we have
\[
\delta^2 \xi = [[\Omega, \xi]], \forall \xi \in \mathfrak{g}.
\]

Next, let us show that \(\delta[[\xi, f]] = [\delta \xi, f] + [\xi, \delta f]\), for \(\xi \in \mathfrak{g}\) and \(f \in C^\infty(S)\). Taking \(\xi = e_i\)
and using Eq. (73),

\[ [\delta e_i, f] + [e_i, \delta f] - \delta [e_i, f] = \sum_{j,k=1}^{n} (e_j)_S(f) F_{ik}^j e_k + \sum_{j=1}^{n} (e_i)_S(e^j)_S(f) e_j + \sum_{j,k=1}^{n} (e^j)_S(f) c_{ik}^j e_k - \sum_{j=1}^{n} (e^i)_S(e_j)_S(f) e_j ] \]

\[ = \sum_{j=1}^{n} \left( (e_i)_S(e^j)_S(f) e_j - (e^i)_S(e_j)_S(f) e_j \right) + \sum_{j,k=1}^{n} \left( (e_j)_S(f) F_{ik}^j + (e^j)_S(f) c_{ik}^j \right) e_k \]

\[ = \sum_{j=1}^{n} \left[ (e_i)_S, (e^j)_S \right](f) e_j - \sum_{j,k=1}^{n} \left( F_{ik}^j e_k - c_{ik}^j e_k \right)_S(f) e_j \]

\[ = \sum_{j=1}^{n} \left[ (e_i)_S, (e^j)_S \right](f) e_j - \sum_{j=1}^{n} \left[ (e_i, e^j)_S \right](f) e_j \]

\[ = 0. \]

Since we also know that \( \delta [\xi_1, \xi_2] = [\delta \xi_1, \xi_2] + [\xi_1, \delta \xi_2] \) for any \( \xi_1, \xi_2 \in g \), the conclusion thus follows. \( \square \)

Now, consider the transformation groupoid \( \Gamma : G \times S \to S \) associated to the dressing action. Theorem 4.19 implies that \( \Gamma \) is a quasi-Poisson groupoid. In what follows, we will explicitly describe the multiplicative bivector field \( \Pi \) on \( \Gamma \).

The quasi-Lie bialgebra \( (g, F, \Omega) \) implies that \( G \) is a quasi-Poisson Lie group with multiplicative bivector field denoted by \( \Pi_G \). Moreover, there exists a bivector field \( \Pi_S \) on \( S \) given by \( \Pi_S = -\sum_{i=1}^{n} (e_i)_S \otimes (e^i)_S \), i.e.,

\[ \Pi_S(df, dg) = -\sum_{i=1}^{n} (e^i)_S(f)(e_i)_S(g), \forall f, g \in C^\infty(S). \] (76)

Using Proposition 4.8, Eqs. (70) and (76), we directly deduce the following

**Proposition 4.19** Let \( (\mathfrak{d}, g, \mathfrak{h}) \) be a Manin quasi-triple and \( (g \times S \to S, \delta, \Omega) \) the corresponding quasi-Lie bialgebroid. Then, the bivector field on \( S \) induced by \( \delta \) as in Proposition 4.8 coincides with \( \Pi_S = -\sum_{i=1}^{n} (e_i)_S \otimes (e^i)_S \).

**Proof.** If \( f, g \in C^\infty(S) \), then

\[ -(\delta f)_S(g) = -\sum_{i=1}^{n} (e^i)_S(f)(e_i)_S(g) = \Pi_S(df, dg). \]

\( \square \)

**Example 4.20** Let \( g \) be a Lie algebra endowed with a nondegenerate symmetric bilinear form \( K \) and consider the corresponding Manin quasi-triple \( (\mathfrak{d}, \Delta(g), 1/2 \Delta_-(g)) \) (see Example 4.17). Then, \( g \) acts on \( S = G \) by the adjoint action and the 2-differential is given by

\[ \delta(f) = \frac{1}{2} \sum_{i=1}^{n} (\bar{e}_i + \bar{e}_i)(f)e_i, \quad \forall f \in C^\infty(G), \]

\[ \delta \xi = 0, \quad \forall \xi \in g. \]
where \( \{ e_i \} \) is an orthonormal basis of \( g \). Moreover, the bivector field on \( G \) induced by \( \delta \) is

\[
\Pi_G = \frac{1}{2} \sum_{i=1}^{n} \overrightarrow{e_i} \wedge \overrightarrow{e_i},
\]

which was first obtained in \cite{1} (see also \cite{2}).

Now we are ready to describe the quasi-Poisson groupoid structure on the transformation groupoid \( G \times S \rightrightarrows S \).

**Theorem 4.21** Assume that \((\mathfrak{d}, \mathfrak{g}, \mathfrak{h})\) is a Manin quasi-triple. Define a bivector field \( \Pi \) on \( G \times S \) by

\[
\Pi\left( (\theta_g, \theta_s), (\theta'_g, \theta'_s) \right) = \Pi_G(\theta_g, \theta'_g) - \Pi_S(\theta_s, \theta'_s) + (\theta'_s, (L_g^* \theta_g)s) - (\theta_s, (L_g^* \theta'_g)s),
\]

for any \((g, s) \in G \times S, \theta_g, \theta'_g \in T^*_g G, \theta_s, \theta'_s \in T^*_s S\), where \((L_g^* \theta_g)s\) denotes the vector field on \( S \) corresponding to the dressing action of \( L_g^* \theta_g \in \mathfrak{g}^* \cong \mathfrak{h} \subset \mathfrak{d} \), similarly for \((L_g^* \theta'_g)s\).

Then \((G \times S \rightrightarrows S, \Pi, \Omega)\) is a quasi-Poisson groupoid integrating the 2-differential \( \delta \) given by Eq. (77).

**Proof.** Let \( \Pi \) be the multiplicative bivector field on \( G \times S \) integrating \( \delta \). Then, using Propositions 4.11 and 4.19 and the fact that \( \beta(g, s) = s \) for any \((g, s) \in G \times S\),

\[
\Pi((0, df), (0, dg)) = \Pi(\beta^*(df), \beta^*(dg)) = (\beta_* \Pi)(df, dg) = -\Pi_S(df, dg).
\]

Therefore,

\[
\Pi((0, \theta_s), (0, \theta'_s)) = -\Pi_S(\theta_s, \theta'_s), \forall \theta_s, \theta'_s \in T^*_s S.
\]

Next, if \( f \in C^\infty(S) \) and \( \theta_g \in T^*_g G \), then from Eqs. (13), (21) and (25), it follows that

\[
\Pi((0, df), (\theta_g, 0)) = \langle \Pi^f(\beta^*(df)), (\theta_g, 0) \rangle = -\langle L_{(g, s)}(\delta f), (\theta_g, 0) \rangle = -\langle L_g^s(\delta f), \theta_g \rangle = -\langle \delta f, (L_g^* \theta_g)s \rangle.
\]

Thus,

\[
\Pi((0, \theta_s), (\theta_g, 0)) = -(\theta_s, (L_g^* \theta_g)s), \forall \theta_s \in T^*_s S.
\]

Finally, fixing \( s \in S \), we write \( \Pi_s \) the bivector field on \( G \) defined by \( \Pi_s(g)(\theta_g, \theta'_g) = \Pi(g, s)((\theta_g, 0), (\theta'_g, 0)) \). Then, it is clear that \( \Pi_s \) is a multiplicative bivector field on \( G \).

Moreover, if \( \xi \in \mathfrak{g} \) is considered as a constant section of \( \mathfrak{g} \times S \rightrightarrows S \), then \( \overrightarrow{\xi} = (\overrightarrow{\xi}_G, 0) \), where \( \overrightarrow{\xi}_G \) is the right-invariant vector field on \( G \). Using Eqs. (13) and (25), we get

\[
\langle [\Pi_s, \overrightarrow{\xi}_G], 0 \rangle = \langle [\Pi, \overrightarrow{\xi}], 0 \rangle = \delta \xi = \langle -\overrightarrow{F(\xi)}G, 0 \rangle = \langle -\overrightarrow{\xi}_G, \Pi_G \rangle, 0 \rangle = (\langle \Pi_G, \overrightarrow{\xi} \rangle, 0).
\]

As a consequence, \( \Pi_s \) coincides with \( \Pi_G \), that is

\[
\Pi((\theta_g, 0), (\theta'_g, 0)) = \Pi_G(\theta_g, \theta'_g), \forall \theta_g, \theta'_g \in T^*_g G.
\]

Summing up, we can conclude that the multiplicative bivector field \( \Pi \) integrating \( \delta \) is the one given by Eq. (77) and that \((G \rightrightarrows S, \Pi, \Omega)\) is a quasi-Poisson groupoid. \( \square \)
Example 4.22 In the particular case when $\mathfrak{h}$ is also a Lie subalgebra of $\mathfrak{g}$, that is, $(\mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ is a Manin triple, then $\Omega = 0$ and $(G, \Pi_G)$ is a Poisson Lie group. Moreover, if the dressing action is complete, we have $D/G \cong G^*$, the dual Poisson Lie group. Thus, we recover a Poisson groupoid structure on $G \times G^* \rightrightarrows G^*$, whose Poisson bivector field is described in [15].

On the other hand, when the dressing action is not complete, we can still have a Poisson groupoid $G \times (D/G) \rightrightarrows (D/G)$, while the groupoid $G \times G^* \rightrightarrows G^*$ does not exist any more. It would be interesting to study the relation between this Poisson groupoid and the symplectic groupoid by Lu-Weinstein in [17].

From Examples 4.17 and 4.20 and Theorem 4.21, one can deduce the following

Corollary 4.23 Assume that $\mathfrak{g}$ is a Lie algebra endowed with a nondegenerate symmetric bilinear form $K$ and $G$ is its corresponding connected and simply connected Lie group. Then the transformation groupoid $G \times G \rightrightarrows G$, where $G$ acts on $G$ by conjugation, together with the multiplicative bivector field $\Pi$ on $G \times G$:

$$\Pi(g, s) = \frac{1}{2} \sum_{i=1}^{n} \overrightarrow{e_i} \wedge \overrightarrow{e_i} - \overrightarrow{e_i} \wedge \overleftarrow{e_i} - (\text{Ad}_{g^{-1}e_i})^2 \wedge \overrightarrow{e_i},$$

the bi-invariant 3-form $\Omega := \frac{1}{2} K(\cdot, \cdot, \cdot)_{\mathfrak{g}} \in \wedge^3 \mathfrak{g}^* \cong \Omega^3(G)^G$ on $G$, is a quasi-Poisson groupoid. Here $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}$ and the superscripts refer to the respective $G$-component.

Remark We remark that, under the change of coordinates $(g, s) \mapsto (a, b) = (s^{-1}g^{-1}, g)$, $\Pi$ becomes the bivector field on $G \times G$ obtained in Example 5.3 of [2], i.e.,

$$\Pi = \frac{1}{2} \sum_{i=1}^{n} \overrightarrow{e_i} \wedge \overrightarrow{e_i} + \overleftarrow{e_i} \wedge \overrightarrow{e_i}.$$

4.6 $D/G$ momentum maps

Next, we will investigate the relation between quasi-hamiltonian spaces with $D/G$-momentum map in the sense of [1] and Hamiltonian $\Gamma$-spaces, where $\Gamma$ is the quasi-Poisson groupoid $G \times S \rightrightarrows S$ associated to a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ as described in Section 4.5.

First, we recall the notion of a quasi-hamiltonian space with $D/G$-momentum map.

Definition 4.24 [1] Let $(G, \Pi_G, \Omega)$ be a connected quasi-Poisson Lie group acting on a manifold $X$ with a bivector field $\Pi_X$. The action $\Phi : G \times X \to X$ of $G$ on $X$ is said to be a quasi-Poisson action if and only if

$$\Phi_* (\Pi_G \oplus \Pi_X) = \Pi_X,$$

$$\frac{1}{2} [\Pi_X, \Pi_X] = \Omega_X$$

where $\Omega_X \in \mathfrak{X}^3(X)$ is defined using the map $\wedge^3 \mathfrak{g} \to \mathfrak{X}^3(X)$ induced by the infinitesimal action.
Now, we recall the definition of a quasi-hamiltonian action with a momentum map $J$ \cite{1}. In fact our definition does not require the assumption of $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ being admissible, so is more general.

**Definition 4.25** Let $\Phi$ be a quasi-Poisson action of a quasi-Poisson Lie group $(G, \Pi_G, \Omega)$ on $(X, \Pi_X)$. A $G$-equivariant map $J : X \to S$ is called a momentum map if

$$
\Pi_X^\sharp(J^*\theta_s) = -(\lambda(\theta_s))_X, \text{ for } \theta_s \in T_s^*S,
$$

(80)

where $G$ acts on $S$ by dressing action. The action is called quasi-hamiltonian if it admits a momentum map and $X$ is called a quasi-hamiltonian space.

**Remark** If we consider an admissible isotropic complement $\mathfrak{h}$ of $(\mathfrak{d}, \mathfrak{g})$, then our definition coincides with the one given in \cite{1}.

**Proposition 4.26** Let $(G, \Pi_G, \Omega)$ be a quasi-Poisson Lie group. If $X$ is a quasi-hamiltonian space then the momentum map is a bivector map from $(X, \Pi_X)$ to $(S, \Pi_S)$, i.e.,

$$
J_* \Pi_X = \Pi_S.
$$

(81)

**Proof.** Let $f, g \in C^\infty(S)$. Using Eqs. (74), (80) and the $G$-equivariance, we have that

$$
\Pi_X(J^*df, J^*dg) = \langle \Pi_X^\sharp(J^*df), J^*dg \rangle = -J_s(\lambda(df)_X)(g) = -(\lambda(df))_S(g) = \Pi_S(df, dg).
$$

\(\square\)

**Theorem 4.27** Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})$ be a Manin quasi-triple. If $(X, \Pi_X)$ is a quasi-hamiltonian space with momentum map $J : X \to S$, then $X$ is a Hamiltonian $\Gamma$-space, where $\Gamma$ is the quasi-Poisson groupoid $(G \times S \rightrightarrows S, \Pi, \Omega)$. Here the $\Gamma$-action is given by

$$
(g, s) \cdot x = \Phi(g, x), \forall g \in G, s \in S \text{ and } x \in X \text{ such that } J(x) = s,
$$

(82)

and $\Phi(g, x)$ denotes the $G$-action on $X$.

Conversely, if $(X, \Pi_X)$ is a Hamiltonian $\Gamma$-space with momentum map $J : X \to S$, then $(X, \Pi_X)$ is a quasi-hamiltonian space with momentum map $J : X \to S$, where the $G$-action on $X$ is given by

$$
\Phi(g, x) = (g, J(x)) \cdot x, \forall g \in G \text{ and } x \in X.
$$

**Proof.** First of all, using the fact that $J : X \to S$ is $G$-equivariant, we know that $\Phi$ can be extended to an action of $\Gamma$ on $X$ by Eq. (82). In addition, the conormal space of the graph of the $\Gamma$-action at a point $((g, x), s, \Phi(g, x))$ is spanned by vectors of the form $(-\langle (\Phi_x)^\sharp\theta, 0 \rangle, -\langle (\Phi_g)^\sharp\theta, 0 \rangle)$ for $\theta \in T_{(g, x)}^*X$ and $(0, \gamma, -J^*\gamma, 0)$ for $\gamma \in T_s^*S$. We accordingly divide our proof into three different cases:

**Case 1.** The 2 covectors are of the first type. From the definition of $\Pi$ (see Eq. (77)) we see that the coisotropy condition is equivalent to

$$
\Phi_* (\Pi_G \oplus \Pi_X) = \Pi_X.
$$
**Case 2.** The 2 covectors are of the second type. From Eq. (77), we deduce that \( \Pi \oplus \Pi_X \oplus -\Pi_X \) is coisotropic is equivalent to

\[
J_sPi_X = Pi_S.
\]

**Case 3.** One covector is of the first type and the other of the second. In this case, the coisotropy condition is just the momentum map condition, i.e.,

\[
\Pi^\chi_j(J^*\gamma) + (\lambda(\gamma))_X = 0.
\]

Finally, we note that Eq. (79) is equivalent to

\[
\frac{1}{2}[\Pi_X,\Pi_X] = \hat{\Omega}.
\]

We conclude our proof. \( \square \)

In [2], the authors study quasi-hamiltonian spaces for the particular case when the Manin pair is the one associated with a Lie algebra \( g \) endowed with a nondegenerate symmetric bilinear form \( K \) (see Example 4.17). We now recall their definitions and discuss the relation with Hamiltonian \( \Gamma \)-spaces of quasi-Poisson groupoids.

**Definition 4.28** [1, 2] A quasi-Poisson manifold is a \( G \)-manifold \( X \), equipped with an invariant bivector field \( \Pi_X \) such that

\[
\frac{1}{2}[\Pi_X,\Pi_X] = \Omega_X.
\]

Moreover, one can also introduce the following specific definition of momentum maps.

**Definition 4.29** [1, 2] An \( \text{Ad-equivariant} \) map \( J : X \to G \) is called a momentum map for the quasi-Poisson manifold \( (X, \Pi_X) \) if

\[
\Pi^\chi_j(d(J^*f)) = (J^*(\mathcal{D}f))_X, \forall f \in C^\infty(G),
\]

where \( \mathcal{D} : C^\infty(G) \to C^\infty(G, g) \) is defined by \( \mathcal{D}f = \frac{1}{2} \sum_{i=1}^n (\vec{e}_i + \vec{e}_i)(f)e_i \). The triple \( (X, \Pi_X, J) \) is then called a Hamiltonian quasi-Poisson manifold.

**Remark** Note that the operator \( \mathcal{D} \) is exactly the 2-differential \( \delta \) on the Lie algebroid \( g \times S \to S \) applying to functions (see Example 4.20).

As a consequence of Theorem 4.27 we deduce the following

**Corollary 4.30** Let \( G \) be a Lie group with Lie algebra \( g \) endowed with a nondegenerate symmetric bilinear form \( K \). If \( (X, \Pi_X) \) is a hamiltonian quasi-Poisson manifold with momentum map \( J : X \to G \), where the action is denoted by \( \Phi(g, x) \), then \( X \) is a Hamiltonian \( \Gamma \)-space, where \( \Gamma \) is the quasi-Poisson groupoid \( (G \times G \Rightarrow G, \Pi, \Omega) \) obtained in Corollary 4.23. Here the \( \Gamma \)-action is given by

\[
(g,s) \cdot x = \Phi(g, x), \forall g \in G, s \in S \text{ and } x \in X \text{ such that } J(x) = s.
\]

Conversely, if \( (X, \Pi_X) \) is a Hamiltonian \( \Gamma \)-space with momentum map \( J : X \to G \), then \( (X, \Pi_X) \) is a quasi-Poisson manifold with momentum map \( J : X \to G \), where the \( G \)-action on \( X \) is given by

\[
\Phi(g, x) = (g, J(x)) \cdot x, \text{ for } g \in G \text{ and } x \in X.
\]
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