On the roles of Vorob’ev cyclicities and Berry’s phase in the EPR paradox and Bell-tests

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INTRODUCTION

Many recent books on quantum theory and quantum informatics dedicate considerable sections to the Einstein-Podolsky-Rosen (EPR) \textsuperscript{1} paradox and Bell’s inequalities \textsuperscript{2,3}, which appear to represent the ultimate roadblock for an intuitive explanation of quantum phenomena. They seem to form the mathematical embodiment of Bohr’s enunciation that the atomic world cannot be explained by using the physical concepts of our macroscopic experiences and the corresponding mathematical language. In turn, removing this roadblock might lead to an interpretation of the quantum formalism without forfeiting the fundamental physical principles that lay behind our views of the macroscopic world. This fact becomes particularly obvious in reports of science writers, who commonly present Bell’s inequality as a consequence of straightforward algebra or logic and then struggle with the problem that quantum theory supposedly contradicts that logic. Their way out of the conundrum is the introduction of instantaneous influences at a distance (that Einstein called “spooky”) and/or abandoning the notion of physical reality that we ordinarily acknowledge in our macroscopic world.

The non-sequitur of such radical measures has been pointed out in several thoughtful works e.g. in \textsuperscript{4}. However, Wigner’s \textsuperscript{5} set theoretical reasoning appeared difficult to overcome even to these authors and indeed may be overcome only by the detailed mathematical physics given in the bulk of this paper.

Bell-type inequalities constrain the correlations that may appear in probabilistic models that fulfill certain generic features. Quantum phenomena appear to violate such constraints and, thus, cannot be described or understood by any of these generic models. This fact is interpreted as an experimentally verifiable proof that quantum phenomena - and, in particular, quantum entanglement - cannot be understood in terms of Kolmogorov’s classical probability concepts and a fundamental physical principle as Einstein’s notion of causality.

The so called Bell tests involve a number of variations of EPR-type experiments and are thought to present the quintessential demonstrations of whether or not quantum systems may be described in terms of physical concepts taken from the macroscopic world. In such Bell-type experiments, a source emanates pairs of entangled particles, which propagate toward two distant detection systems that test their polarizations. Each detector may be positioned in one of two available settings defined with respect to local lab frames. Upon detection each detector produces a binary response - either $-1$ or $+1$, so that the correlation between the outcomes at the two measurement stations is given by

\begin{equation}
E(\Delta) = -\cos(n \cdot \Delta),
\end{equation}

where $\Delta$ is the relative angle between the orientations of the two detectors and $n \in \mathbb{N}$ is an integer. Eq.\textsuperscript{(1)} represents the results of both quantum theory and many experiments. However, this result is considered to be inconsistent and impossible to obtain with Einstein’s causality added by Kolmogorov’s probability theory.

This supposed impossibility is very suspicious, because the correlation [\textsuperscript{1}] can be accounted for on the ground of very simple symmetry arguments and smoothness constraints, following a variational principle. Because the pair of entangled particles is invariant under rotations, the correlation between the outcomes of the two detectors (polarizers) may only depend on the relative angle $\Delta$ between them. This dependence arises from the simple fact that we compare and statistically collect 'equal' and 'not-equal' experimental outcomes when evaluating the experiments; a procedure that naturally involves the statistics of the measurement results in both wings. More detailed explanations will be given in the bulk of the paper.

The probabilities for 'equal' and 'not-equal' outcomes at the two detection systems may be written without any loss of generality as:

\begin{equation}
\begin{align*}
 p(\text{’EQUAL’}) &= \sin^2(\chi(\Delta)) \equiv p_1(\chi(\Delta)), \\
 p(\text{’NOT-EQUAL’}) &= \cos^2(\chi(\Delta)) \equiv p_2(\chi(\Delta)),
\end{align*}
\end{equation}

so that

\begin{equation}
E(\Delta) = \sin^2(\chi(\Delta)) - \cos^2(\chi(\Delta)) = -\cos(2 \cdot \chi(\Delta)).
\end{equation}

Since the correlation functions fulfill the symmetry constraints

\begin{equation}
E(\Delta + \frac{\pi}{n}) = -E(\Delta) = -E(-\Delta),
\end{equation}

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we postulate that the function $\chi(\Delta)$ fulfills
\[ \chi(-\Delta) = -\chi(\Delta), \quad (5) \]
\[ \chi\left(\Delta + \frac{\pi}{n}\right) = \left(\frac{\pi}{2} \pm \chi(\Delta)\right) \mod [0, \pi]. \quad (6) \]

We will show in a section below that the plausible linear relation:
\[ \chi(\Delta) = n \cdot \frac{\Delta}{2}, \quad n \in \mathbb{N}. \quad (7) \]

may actually be derived from the above symmetry constraints by using standard tools of Informatics. Thus, the quantum correlations $[1]$ are obtained as a variation of the Malus law as suggested in $[6]$.

It is the purpose of this paper to show that Eq. (1) is in no contradiction to Kolmogorov’s set theoretic probability theory nor to Einstein’s causality principle. The basis of our findings to be presented is the following. Bell-type inequalities have actually been known to mathematicians in one form or another since the early work on probability theory by Boole $[7]$, and found their most general formulation in the work of Vorob’ev $[8]$. Thus, it has been known that the constraints demanded by Bell type inequalities are a consequence of certain cyclicities of random variables on a Kolmogorov probability space. The violation of the inequalities implies that the joint probabilities associated to the random variables cannot always be defined on a single probability space, unless the cyclicities may be somehow avoided. We shall show below that the cyclicities involved in the Bell and CHSH $[9]$ theorems may be removed by well known yet subtle physics involving gauge symmetry considerations, geometric phases and other factors $[10-13]$.

**BELL-TYPE INEQUALITIES AND VOROB’EV CYCLICITIES**

Bell-functions are function-pairs related to two measurement stations with certain instrument settings usually denoted by $j, j' = a, b, c, d$. One function, $A(j, \lambda)$ describes the measurement outcomes in station 1 and the function $B(j', \lambda)$ describes those in station 2. Here, $j$ and $j'$ are variables representing the instrument settings. The variable $\lambda$ is according to Bell an element of physical reality corresponding to the pair of entangled quantum entities that are sent to the respective instruments from a common source and, therefore $\lambda$ appears equal in both functions $A, B$. The variable $\lambda$ thus describes the information shared by the two stations through the pair of entangled electrons or photons.

The purpose of the function-pairs $A, B$ is to describe in mathematical form the Einstein-Podolsky-Rosen (EPR) $[11]$ Gedanken-experiment and its variations and it was Bell’s declared intention to link the domain and co-domain of his functions to actual, performable, experiments. It was, therefore, realized in many later publications that Bell’s functions needed to depend on all possible elements of physical reality that Einstein’s theory of relativity has provided, which includes measurements with rigid rods and with clocks to describe dynamical effects. The interpretation of Bell’s $\lambda$ has, therefore, a long and checkered history. However, for the following discussions, it is sufficient to regard $\lambda$ as the mathematical symbol that describes the elements of reality emanating from the source, some of them possibly ‘hidden’ to our current knowledge. Dynamical effects related to these elements of reality may then be “absorbed” in the method of their evaluation by measurement equipment and gauge considerations.

Furthermore, Bell wished to connect his mathematical considerations to quantum physics and to actual experiments and, therefore, needed to involve probabilities. He thus assumed that some of his variables may be random variables. In Kolmogorov’s probability framework these are functions on a probability space. Mathematical work on probability themes often starts with the words: “given a probability space $\Omega$” and takes it for granted that such a probability space exists. So did Bell and all the authors following his work. They concluded that the joint probabilities predicted by quantum mechanics for Bell experiments cannot be described on a single probability space.

In fact, Vorob’ev had shown previously that certain expectation values and corresponding probabilities for function-pair-outcomes cannot consistently exist on a single common probability space $\Omega$, if a combinatorial-topological cyclicity is involved in the concatenation of random variables (functions on a probability space). Vorob’ev’s generality of argument makes it necessary to involve combinatorial topology. The essence and principle of his reasoning can be made clear from the graphical representation for the special case of Bell’s functions shown in Fig. $[1]$

Following quantum theory and experimental results, Bell used the important constraint $A(j, \lambda) = -B(j, \lambda)$ for ‘equal’ settings $j$, which defines the equipment-settings of complete anti-correlation corresponding to $\Delta = 0$. (Incomplete anti-correlations do not introduce any changes of the following presented principles) This constraint is essential for the derivation of Bell-type inequalities. We may, therefore, replace throughout the functions $B$ by their negative equivalent $A$ for the same equipment setting $j$ and $\lambda$. The Bell inequality deals, thus, only with the three function-pairs:

\[ A(a, \lambda) A(b, \lambda); A(a, \lambda) A(c, \lambda); A(b, \lambda) A(c, \lambda), \quad (8) \]

Here we have used, with Bell, identical $\lambda$-s for all of the three pairs, which is equivalent to the assumption that all variables are defined on one common probability space. The Vorob’ev cyclicity that corresponds to these three function-pairs is that of the triangle shown in Fig. $[1]$ Vorob’ev has emphasized that
the arbitrary prescription of joint pair probability-distributions to the first two pairs does not permit complete freedom to choose the joint distribution of the last pair. This fact puts a constraint on the possible pair expectation values in form of Bell-type inequalities. The form of the cyclicity determines the form of the inequalities.

Similar considerations apply to the CHSH inequalities \[9\] as shown in Fig.2. The corresponding cyclicity is represented for four pairs of Bell-type functions:

\[ A(a, \lambda) A(b, \lambda); A(a, \lambda) A(c, \lambda); A(d, \lambda) A(b, \lambda) \quad (9) \]

and

\[ A(d, \lambda) A(c, \lambda). \]

The cyclicity imposes again constraints if we restrict ourselves to one common probability space.

Bell’s theorem is widely understood as an experimentally testable statement that QM joint probabilities for the separate pair-wise measurement outcomes in a Bell-type experiment cannot be defined on a single probability space and, hence, there cannot exist an underlying more fundamental description of quantum phenomena. We want to show here that these joint probabilities can be properly defined on a single probability space through the use of the gauge symmetries of the problem. It is our declared purpose to show that Vorob’ev’s cyclicities that are inherent in Bell’s tests can be eliminated by a careful consideration of the involved gauge symmetries, geometric phases and other factors, so that Bell-type constraints can be completely avoided.

In order to show how to eliminate the cyclicities it is important to note that in Bell’s formulation the variables \( j, j' \) that describe the settings of the instruments, as well as the variables \( \lambda \) that describe the elements of reality of the entangled pairs, are defined with respect to local lab frames. However, while the settings \( j, j' \) can be defined with respect to local lab frames, it is not always true that the variables \( \lambda \) can be so defined when there are cyclicities involved. Neither is it necessarily true that the response functions of the instruments can be defined in terms of variables defined with respect to lab frames as assumed in Bell’s formulation. In general, the variables \( \lambda \) may be properly defined only with respect to the setting of each instrument, and the response of the instrument would then be a function only of the variable \( \lambda \) defined with respect to the instrument setting.

To this regard, we must note that the original EPR argument involves only instruments in parallel settings, for which their outcomes are fully (anti)correlated. Hence, the original EPR argument does not need to independently define the setting of each one of the instruments. The introduction of independently defined settings for each one of the instruments was done by Bell, but not by Einstein and his collaborators.

\[ E(\Delta) = \sin^2(\chi(\Delta)) - \cos^2(\chi(\Delta)) = -\cos(2 \cdot \chi(\Delta)). \]
In order to link this equation to the results of quantum theory and experiments, we still have to show the linearity of the function \( \chi(\Delta) \). This may be accomplished as follows: Because the Shannon entropy for the random game discussed in the introduction is given by [13] [15]

\[
S[\chi(\Delta)] = - \sum_{i=\pm1} p_i(\chi(\Delta)) \cdot \log(p_i(\chi(\Delta))),
\]

we define the average entropy, with the help of some symmetry considerations, as

\[
\mathcal{D}[\chi(\Delta)] = \frac{n}{\pi} \int_0^{\pi/n} d\Delta \ S[\chi(\Delta)],
\]

and look for its extrema,

\[
\frac{\delta \mathcal{D}[\chi(\Delta)]}{\delta \chi(\Delta)} = \frac{n}{\pi} \int_0^{\pi/n} d\Delta \ \frac{\delta S[\chi(\Delta)]}{\delta \chi(\Delta)} = 0.
\]

The last equation can be written as:

\[
n \int_0^{\pi/n} d\Delta \ \sin(2\chi(\Delta)) \log(|\tan(\chi(\Delta))|) = 0,
\]

whose solutions are:

\[
\chi(\Delta) = n \cdot \Delta/2, \quad n \in \mathbb{N}.
\]

which corresponds to the general quantum correlation.

This whole procedure goes, of course against the grain of anyone who has followed the work of Bell-CHSH, because \( \Delta \) contains the instrument settings of both experimental wings, which apparently spells some kind of non-locality. However, as explained above, the computation of the correlation uses judgements for equal and not-equal measurement outcomes i.e. judgements relative to the other experimental wing. Such judgements are naturally based on global facts as opposed to only local facts within the measurement stations. In fact, Bell-type constraints do not even rule out correlations of the form \( E(\Delta) = -1 + 2|\Delta(\text{mod}[-\pi, \pi])|/\pi \), which can be easily obtained in random games with macroscopic carriers, but only rule out correlations depending on \( \Delta \) in the form predicted by quantum mechanics.

**A perspective on Vorob’ev’s cyclicities in terms of gauge symmetries**

The authors of this present work have more recently proposed that relativity and gauge symmetry considerations permit and actually demand to take steps that remove the Vorob’ev cyclicity for a correct theoretical analysis of actual EPR experiments [10][13]. These considerations follow the observation that the absolute direction of the polarizers or magnets in a Bell experiment is a redundant gauge variable, while the relative orientation between the two completely defines the setting of the measurement instruments. A most important consequence of this observation is that we may remove the cyclicity in the Bell-type inequalities as shown in Fig.3. Here we have fixed the instrument setting in one wing as a reference direction and just chosen the instrument settings in the other wing to obtain the correct Bell-CHSH angle-differences. As one can see, the cyclicity is removed and so is the constraint by Bell-CHSH inequalities and the equivalent inequalities given by Wigner’s procedure. The probability distribution of the elements of reality emanating from the source is not changed by this procedure as it must not be. However, the evaluation of the relative outcomes by the measurement instruments may change and give different numbers for the equal and not-equal outcomes depending on the relative instrument settings. Under certain circumstances that we shall now detail, this procedure is essential in order to make it possible to describe all the involved pair-wise random games on a single probability space.

We assume that the space of random events available at every repetition of the experiment form an unbiased sample within the whole space of events, so that we can consider for the sake of simplicity that the latter is always available.

Thus, let \( (\Omega, \Sigma, \mu) \) be a probability space, where \( \Omega \) is a non-empty sample space, \( \Sigma \) is the \( \sigma \)-algebra of all its subsets and \( \mu \) is a (probability) measure defined on it, and let \( \xi : \Omega \to [-1, +1] \subset \mathbb{R} \) be a random variable defined on it that takes values on the real interval \([-1, +1]\).

Furthermore, let \( \{\mathcal{F}_\Delta\}_{\Delta \in \mathbb{Z}} \) be a (continuous or discrete) group of isomorphic parameterizations of the probability space labelled over an additive group \( \mathbb{Z} \). That is, \( \mathcal{F}_\Delta : \Omega \to \Omega \) are bijective applications from the sample space onto the sample space that preserve the probability measure:

\[
(\forall S \subseteq \Omega)(\mu(\mathcal{F}_\Delta[S]) = \mu(S)).
\]

In particular, \( \mathcal{F}_0 = \mathbb{I} : \Omega \to \Omega \) denotes the identity transformation.

Thus, two observers related by a relative ’displacement’ \( \Delta \) would describe the same random event, respectively, as \( S \subseteq \Omega \) and \( \mathcal{F}_\Delta[S] \subseteq \Omega \). Hence, the correlation between their descrip-
Bell-type inequalities constrain the two-parties correlations \( \{E(\Delta)\}_{i=1,2,...,n} \) that can exist between parties for which, see Fig\textsuperscript{2}

\[
\sum_{i=1,2,...,n} \Delta_i = 0, \tag{17}
\]

assuming that this cyclicity constraint requires that

\[
F_n \circ \ldots \circ F_2 \circ F_1 = \mathcal{F}_0. \tag{18}
\]

However, we notice in this paper that whenever gauge symmetries are involved a non-zero geometric (Berry) phase \( \alpha \in \mathbb{Z} \) may appear through some finite cyclic sequences \( \{17\} \):

\[
F_n \circ \ldots \circ F_2 \circ F_1 = \mathcal{F}_\alpha \neq \mathcal{F}_0. \tag{19}
\]

In other words, we consider the case in which cyclic sequences may be associated by a re-definition of the identity of symmetric events \( \{10–13\} \), so that the parties cannot be all described within a single probability space. In such a case the same symmetry consideration can and must be used in order to remove the cyclicities, as shown in Fig\textsuperscript{2}. Obviously, this freedom must not be allowed when all parties can test the random events at once, since it would imply that the events could have a double identity for at least one of the involved parties. On the other hand, this freedom must be considered in cases in which every random event can be tested only by a strict subset of the involved parties. In this case, the freedom \( \{19\} \) is equivalent to stating that the identities of single parties cannot be properly defined, but only their relative ‘displacements’. In physical terms we shall say that the identity of the parties is a gauge (non-physical) degree of freedom. This gauge freedom is tantamount to relaxing the cyclicity constraint \( \{17\} \) and, therefore, it allows to avoid the constraints that would appear otherwise.

**Remark on the relationship to experiments**

We find it important to remark that the dissolution of the Vorob’ev cyclicity discussed above is not necessarily associated to an actual geometrical rearrangement of the sources and measuring instruments involved in the EPRB experiments. The dissolution can be done as part of the theoretical description and analysis of the experiment, by taking advantage of the gauge symmetries involved and the fact that the polarization of each particle of every pair of entangled photons or electrons can be tested along a single orientation. In fact, when a non-zero Berry phase \( \{19\} \) appears through the considered cyclic arrangement it is a must to dissolve the cyclicity in order to describe all the involved pair-wise experiments together.

Thus, our analysis applies equally well to experiments in which the detectors at the two experimental wings could be actually rotated with respect to local lab frames \( \{16\} \), or to the experiment of Giustina et al. \( \{17\} \), which uses electro optical modulators (EOMs) located at any place between source and detectors in both experimental wings, or to hypothetical experiments, not yet performed, with different sources emitting the entangled pairs and detectors arranged geometrically in a Vorob’ev cyclicity. Current optical fiber technology does appear to permit the construction of such experiments.

**CONCLUSION**

Bell-type inequalities for random games involving at least three functions defined on a single probability space have been known to mathematicians since the early works of Boole on probability theory \( \{7\} \), and found their more general formulation in the work of Vorob’ev \( \{8\} \). These inequalities constrain the correlations between pairs of random variables that can appear in this kind of games, and it is known that they are associated to certain cyclicities in the way how the variables are concatenated. The violation of the inequalities is understood to imply that the variables cannot be jointly defined on a single probability space.

In particular, the violation of this kind of inequalities by the correlations predicted by Quantum Mechanics has been understood as the ultimate proof of the impossibility to describe quantum phenomena in terms of any underlying more fundamental theory based on the same fundamental physical principles derived from our macroscopic experience, and thus it represents the ultimate roadblock towards an intuitive interpretation of the quantum formalism.

In this paper we have shown, however, how subtle physically motivated considerations related to the gauge symmetries involved in the considered games may allow to remove the cyclicities and, hence, lift the constraints derived from the inequalities, paving the way to an explanation of the quantum formalism within the framework of standard Einstein’s causality and Kolmogorov’s probability theory \( \{10–13\} \).

As a byproduct of our symmetry argumentation we have presented an elegant way of obtaining the correlations predicted by quantum mechanics for the Bell experiment \( \{2\} \) from a simple variational principle using standard tools of information theory.

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