On the representation dimension of rank 2 group algebras and related algebras

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Abstract

The representation dimension was defined by M. Auslander in 1970 and is, due to spectacular recent progress, one of the most interesting homological invariants in representation theory. The precise value is not known in general, and is very hard to compute even for small examples. For group algebras, it is known in the case of cyclic Sylow subgroups, due to Auslander’s fundamental work. For some group algebras (in characteristic 2) of rank at least 3 the precise value of the representation dimension follows from recent work of R. Rouquier. There is a gap for group algebras of rank 2; here the deep geometric methods do not work. In this paper we show that for all \( n \geq 0 \) and any field \( k \) the commutative algebras \( k[x, y]/(x^2, y^{2+n}) \) have representation dimension 3. For the proof, we give an explicit inductive construction of a suitable generator-cogenerator. As a consequence, we obtain that the group algebras in characteristic 2 of the groups \( C_2 \times C_2^m \) have representation dimension 3. Note that for \( m \geq 3 \) these group algebras have wild representation type.

MSC Classification: 16G10 (primary); 13D05, 16S50, 20C05 (secondary).

1 Introduction

The representation dimension of an Artin algebra was defined by M. Auslander \[1\] as a way of measuring homologically how far an algebra is from being of finite representation type. (For the precise definition see Section 2 below.) In fact, Auslander showed that the representation dimension of an algebra is at most 2 if and only if the algebra has finite representation type, i.e. has only finitely many indecomposable modules (up to isomorphism).

For more than two decades it remained unclear whether Auslander’s philosophy and hopes were justified, one of the main problems being that the precise value of the representation dimension of an algebra is very hard to determine. For instance, until the late 1990’s there were still only very few examples of
algebras known with representation dimension 3, and not a single algebra with representation dimension > 3.

This situation changed dramatically over the last few years, with spectacular recent progress which we briefly mention here. O. Iyama proved that the representation dimension is always finite [6]. This integer attached to any algebra is actually an invariant under various notions and levels of equivalences of algebras. As it is defined, the representation dimension is invariant under Morita equivalences. In modern representation theory the notions of stable equivalences and especially derived equivalences became increasingly important. It was recently shown by X. Guo [4] that the representation dimension is invariant under stable equivalences. (For stable equivalences of Morita type this was already shown by C. Xi [10].) In particular, for selfinjective algebras the representation dimension is invariant under derived equivalences, thus providing one of the few explicit invariants of derived module categories known so far.

A breakthrough was obtained by R. Rouquier [8] (but see also the new revised version [9]). As a consequence of a deep general theory of dimensions of triangulated categories he obtained the first examples of algebras with representation dimension > 3, solving one of the most fundamental open questions about this invariant. Much more than that, Rouquier obtained that the representation dimension is unbounded, by showing that for any $n \geq 3$ the exterior algebra of an $n$-dimensional vector space has representation dimension $n+1$. Actually, the methods used there indicate a very geometric nature of the representation dimension in these cases, and from this approach one can expect many more explicit examples in the near future.

These recent results finally make it clear that Auslander’s philosophy of measuring how far an algebra is from being of finite representation type is working; we get a new division of algebras according to the size of their representation dimension. Note that this new division certainly does not run along the lines of the classical division into finite, tame and wild representation type. For instance, already Auslander showed that all finite-dimensional hereditary algebras have representation dimension at most 3, not depending on their representation type. There is something completely new to be discovered! So far, it remains mysterious in general what structural properties of an algebra actually determine the representation dimension.

In this paper, we will provide more examples of algebras of wild representation type having representation dimension 3. Our main motivation comes from group algebras. It is well known that the representation type of a modular group algebra $kG$ of a finite group over a field $k$ of characteristic $p$ is determined by a Sylow $p$-subgroup $P$ of $G$: $kG$ has finite representation type if and only if $P$ is cyclic; $kG$ has tame representation type precisely when $p = 2$ and $P$ is dihedral, semidihedral or quaternion. In all other cases $kG$ has wild representation type. Hence, in the cyclic case, the representation dimension of $kG$ is at most 2. It is also known that all group algebras of tame representation type (and related algebras) have representation dimension 3. (Unfortunately, there is no complete proof in the literature, but see [5], and [7], 4.4.2.)

From Rouquier’s work one gets that for any $n \geq 3$ the mod 2 group algebra
of an elementary abelian group $C_2 \times \ldots \times C_2$ of order $2^n$ has representation dimension $n + 1$. One could hope that the geometric methods used could be adapted and generalized to deal with elementary abelian $p$-groups in odd characteristic as well. However, these geometric methods do not work for elementary abelian groups of rank 2.

The main aim of this paper is to give some partial results on the rank 2 case, thus providing the first examples to fill this 'rank-2-gap'. Note that the mod $p$ group algebra of $C_{p^n} \times C_{p^n}$ is isomorphic to the commutative ring $k[x, y]/(x^{p^n}, y^{p^n})$. So the natural class of algebras to study are the commutative rings $A_{n,m} := k[x, y]/(x^m, y^n)$, for natural numbers $n, m \geq 2$.

Our main result in this paper gives the precise value of the representation dimension for the algebras $A_{2,m}$. Note that this result is independent of the ground field.

**Theorem 1.1** Let $k$ be a field and let $\Lambda_n = k[x, y]/(x^2, y^{2+n})$ for every non-negative integer $n$. Then the representation dimension of $\Lambda_n$ is 3.

As a consequence we get the following partial answer on the 'rank-2-gap' for group algebras.

**Corollary 1.2** Let $k$ be a field of characteristic 2. Then for any $n \geq 1$ the group algebra $k(C_2 \times C_{2^n})$ has representation dimension 3.

The paper is organized as follows. In Section 2 we recall the necessary definitions and background on representation dimensions. In particular, we outline the method for determining the precise value of the representation dimension. The final Section 3 then contains the proof of the main result.

## 2 Computing representation dimensions

The representation dimension has been defined by M. Auslander for arbitrary Artin algebras. However, for the purpose of this paper, by an algebra we mean a finite-dimensional algebra over a fixed field $k$. (It will turn out that our results will not depend on the ground field.)

A finitely generated $A$-module $M$ is called a generator-cogenerator for $A$ if all projective indecomposable modules and all injective indecomposable $A$-modules occur as direct summands of $M$.

Auslander’s basic idea was to study homological properties of endomorphism rings of modules (instead of studying modules directly). For any algebra $A$ the representation dimension is defined to be

$$\text{repdim}(A) := \inf \{ \text{gl.dim} (\text{End}_A(M)) \mid M \text{ generator-cogenerator} \}.$$ 

Note that this invariant a priori has values in $\mathbb{N}_0 \cup \{ \infty \}$. However, it was shown by O. Iyama that $\text{repdim}(A)$ is always finite.
Auslander’s fundamental result states that \( \text{repdim}(A) \leq 2 \) if and only if \( A \) has finite representation type. More precisely, \( \text{repdim}(A) = 0 \) if and only if \( A \) is semisimple, and \( \text{repdim}(A) \neq 1 \) for all algebras \( A \).

The definition indicates how to determine upper bounds for \( \text{repdim}(A) \); in fact, if one can show for some generator-cogenerator \( M \) that \( \text{gl.dim}(\text{End}_A(M)) \leq m \) then \( \text{repdim}(A) \leq m \). In particular, if one can find a generator-cogenerator \( M \) with \( \text{gl.dim}(\text{End}_A(M)) \leq 1 \), and if \( A \) is not of finite representation type, then one has shown that \( \text{repdim}(A) = 3 \). Of course, the notoriously difficult problem is to find a suitable module \( M \).

However, this is exactly the strategy we will pursue successfully in the proof of our main theorem. We will explain in Section 3 below how to construct inductively generator-cogenerators for the algebras \( k[x, y]/(x^2, y^{2+n}) \) with endomorphism rings of global dimension 3.

We now describe the general method used later for how to determine the global dimension of the endomorphism ring of a generator-cogenerator \( M \).

Let \( M \) be a finitely generated module over a finite-dimensional algebra \( A \). The identity of \( \text{End}_A(M) \) is the sum of the “identity maps” on the indecomposable direct summands of \( M \). Hence we have primitive idempotents of \( \text{End}_A(M) \) corresponding to the summands of \( M \). For any indecomposable summand \( T \) of \( M \) we denote the corresponding simple \( \text{End}_A(M) \)-module by \( E_T \). The corresponding indecomposable projective \( \text{End}_A(M) \)-module \( Q_T \) is given by all homomorphisms from \( M \) to \( T \), for abbreviation denoted by \( Q_T = \text{Hom}_A(M, T) =: (M, T) \). To prove that \( \text{gl.dim}(\text{End}_A(M)) \leq 3 \), we explicitly construct a projective resolution with length \( \leq 3 \) for every simple module \( E_T \). The general method is as follows. Recall that for a finitely generated \( A \)-module \( M \), \( \text{add} M \) denotes the full subcategory of \( \text{mod} A \) consisting of direct sums of direct summands of \( M \). For any indecomposable summand \( T \) of \( M \), we first construct a suitable exact sequence \( 0 \to K \to N_1 \to T \) with \( N_1 \in \text{add} M \) with the following property:

\( \text{(*)} \) Every homomorphism from an indecomposable summand of \( M \) to \( T \), except the multiples of the identity on \( T \), factors through \( N_1 \).

Applying the functor \( (M, -) \), we get another short exact sequence \( 0 \to (M, K) \to (M, N_1) \to (M, T) \).

If the cokernel of \( (M, N_1) \to (M, T) \) is 1-dimensional then the cokernel is \( E_T \) (because \( (M, T) \) is projective), i.e. we get the initial part of a projective resolution of the simple \( \text{End}_A(M) \)-module \( E_T \).

If \( K \in \text{add} M \), then also \( (M, K) \) is projective and we have constructed a projective resolution of \( E_T \) of length 2. Otherwise, we construct another suitable short exact sequence \( 0 \to K' \to N_2 \to K \) with \( N_2 \in \text{add} M \) with the following property:

\( \text{(**)} \) Every map from an indecomposable summand of \( M \) to \( K \) factors through \( N_2 \).
Applying the functor $(M, -)$ to this short exact sequence, we get a short exact sequence

$$0 \rightarrow (M, K') \rightarrow (M, N_2) \rightarrow (M, K) \rightarrow 0.$$  

If it happens that $K' \in \text{add } M$, we have $(M, N_1), (M, N_2), (M, K')$ and $(M, T)$ are all projective $\text{End}_A(M)$-modules. Hence we get a projective resolution of the simple module $E_T$:

$$0 \rightarrow Q_{K'} \rightarrow Q_{N_2} \rightarrow Q_{N_1} \rightarrow Q_T \rightarrow E_T \rightarrow 0$$

and $\text{proj. dim } E_T \leq 3$.

The crucial aspect in this strategy is to come up with a generator-cogenerator $M$ for which this process really stops at this stage, i.e. for which $K' \in \text{add } M$ in the second step above.

### 3 Proof of the main theorem

Before proving our main theorem, we introduce some notations. For any non-negative integer $n$, we use $\Lambda_n$ to denote the (commutative) algebra $k[x, y]/(x^2, y^{n+2})$. The quotient algebra $\Lambda_n/\text{soc } (\Lambda_n)$ will be denoted by $A_n$.

Actually, in order to prove Theorem 1.1 it suffices to show that $\text{rep.dim } (A_n) = 3$, due to the following general result. Recall that an algebra $A$ is called basic if all simple $A$-modules are of dimension 1. Note that our algebras $k[x, y]/(x^2, y^{2+n})$ are basic since the trivial module $k$ is the only simple module.

**Lemma 3.1** ([3], Proposition 1.2) *Let $\Lambda$ be a basic algebra, and let $P$ be an indecomposable projective-injective $\Lambda$-module. Define $A = \Lambda/\text{soc } (P)$. If $\text{rep.dim } (A) \leq 3$, then $\text{rep.dim } (A) \leq 3$.***

The structure of the projective-indecomposable $A_n$-module can be conveniently described diagrammatically as follows.

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Here, the vertices correspond to basis vectors of the module and the edges describe the action of the generators $x$ and $y$ of $A_n$.

Now, we use a table to introduce and illustrate the notations for several classes of $A_n$-modules which will be used throughout the paper.
Notation | Shape of the $A_n$-module
--- | ---
$U_i$, $0 \leq i \leq n + 1$ | $i$ sections

Note that the projective $A_n$-module will be denoted by $A^0_n$. By Lemma 3.1, $\text{rep.dim} (A_n) \leq 3$ will imply that $\text{rep.dim} (\Lambda_n) \leq 3$. So we consider $A_n$ instead of $\Lambda_n$. For any $n \geq 0$, we will prove that $\text{rep.dim} (A_n) \leq 3$ by constructing a generator-cogenerator $M_n$ for $A_n$ such that $\text{gl.dim} (\text{End}_{A_n}(M_n)) \leq 3$.

Now we give our construction of the desired generator-cogenerators $M_n$ of $A_n$ as follows.

For $n = 0$ we put

$$M_0 = A^0_0 \oplus DA^0_0 \oplus U_0 \oplus U_1 \oplus X.$$  

(Actually, for the algebra $A_0 = k[x, y]/(x^2, y^2, xy)$ it is known that $\text{repdim} (A_0) = 3$; in fact, $A_0$ is special biserial and we can apply Corollary 1.3 from [3]. However, we give the explicit generator-cogenerator to indicate the “induction base”.)

If $n$ is positive, we put

$$M_n = \bigoplus_{i,j \geq 0, i+j \leq n} DA^i_j \oplus \bigoplus_{i \geq 0, j > 0, i+j \leq n} A^i_j \oplus A^0_n \oplus \bigoplus_{0 \leq i \leq n+1} U_i \oplus X$$

**Remark 3.2** Let us point out more clearly why this is an inductive construction, i.e. that one obtains $M_n$ from $M_{n-1}$ by a simple “combinatorial” procedure. Note that in $M_n$ we still have all summands of $M_{n-1}$, except the projective module $A^0_{n-1}$. The new summands in $M_n$ are $DA^i_{n-1}$ and $A^i_{n-1}$ (for $0 \leq i \leq n$) and the uniserial module $U_{n+1}$. With the exception of $A^0_n$ and its dual, they can be obtained by extending from summands of $M_{n-1}$ as follows. For all summands of $M_{n-1}$ we add an additional vertex on top (i.e. an additional basis vector of the module) whenever possible; but we only allow this when no new squares are created in the shape of the module. All modules obtained in this way are added as summands of $M_n$ (but only once, to avoid multiplicities).
For instance, going from $M_0$ to $M_1$, the extension of $DA_0^0$ to the four-dimensional module having the shape of a square is not allowed; on the other hand, the extension of $DA_0^0$ to the four-dimensional module $DA_1^0$ is allowed and gives a new summand. With this procedure, for $i = 0, \ldots, n-1$ the new modules $DA_i^{n-i}$ are obtained from the summands $DA_i^{n-i-1}$ of $M_{n-1}$, the new summands $A_i^{n-i}$ are obtained from $A_i^{n-i-1}$ and $U_{n+1}$ is obtained from $U_n$. Note that the new projective module $A_i^0$ and the new injective module $DA_i^0$ can not be obtained with this method from summands of $M_{n-1}$, so they have to be added separately.

Hence, loosely speaking, the simple recipe for constructing $M_n$ from $M_{n-1}$ is as follows.

(i) For all summands of $M_{n-1}$ construct all possible modules obtained by adding a vertex on top, but without creating new squares in the shapes of the modules. Add these modules as new summands (if they are not already summands).

(ii) Remove the old projective module $A_{n-1}^0$. (But keep all other summands of $M_{n-1}$.)

(iii) Add the new projective $A_n^0$ and its dual $DA_n^0$ as summands.

This inductive construction of a generator-cogenerator for $A_n$ is actually the crucial step in the proof of our main result. We will show below that the endomorphism ring of $M_n$ has global dimension 3, which then implies that $A_n$ has representation dimension 3, as claimed.

We are now in the position to give the proof our main result.

**Proof of Theorem 1.1.** We have to show that $\text{gl.dim} \ (\text{End}_{A_n}(M_n)) \leq 3$ for any $n \geq 0$.

1. **The case** $n = 0$. We have $M_0 = A_0^0 \oplus DA_0^0 \oplus U_0 \oplus U_1 \oplus X$.

   (1) The projective module $A_0^0$. Since the radical of the projective module $A_0^0$ is $U_0 \oplus U_0 \in \text{add } M_0$ and the exact sequence $0 \to \text{rad } A_0^0 \to A_0^0$ clearly has the property (*), we can easily get a projective resolution of the simple $\text{End}_{A_0}(M_0)$-module $E_{A_0^0}$:

   $$0 \to Q_{U_0} \oplus Q_{U_0} \to Q_{A_0^0} \to E_{A_0^0} \to 0.$$

   by applying the functor $(M_0, -)$ to $0 \to \text{rad } A_0^0 \to A_0^0$. Hence proj.dim $E_{A_0^0} = 1$.

   (2) The injective module $DA_0^0$. There is an exact sequence $0 \to U_0 \to U_1 \oplus X \to DA_0^0 \to 0$ with the property (*). (Note that there is no epimorphism from an indecomposable summand of $M_0$ to $DA_0^0$, except the multiples of the identity on $DA_0^0$). Since $U_0$ and the middle term are in $\text{add } M_0$, by applying the functor $(M_0, -)$, we get a projective resolution of the simple $\text{End}_{A_0}(M_0)$-module $E_{DA_0^0}$:

   $$0 \to Q_{U_0} \to Q_{U_1} \oplus Q_X \to Q_{DA_0^0} \to E_{DA_0^0} \to 0.$$

   Hence proj.dim $E_{DA_0^0} \leq 2$. 

7
(3) The module $X$. There is a short exact sequence $0 \to U_0 \to A_0^0 \to X \to 0$ with the property (*) (Clearly, epimorphisms only occur from $A_0^0$ to $X$ and they factor through the middle term. Those non-epimorphisms from an indecomposable summand of $M_0$ to $X$ must have image in $\text{rad } X$ and clearly they also factor through the middle term.) Applying $(M_0, -)$ to the exact sequence, we get a projective resolution of the simple $\text{End}_{A_0}(M_0)$-module $E_X$:

$$0 \to Q_{U_0} \to Q_{A_0^0} \to Q_X \to E_X \to 0$$

and hence $\text{proj.dim } E_X \leq 2$.

(4) The modules $U_i$ where $i = 0, 1$. By symmetry, it suffices to show that $\text{proj.dim } E_{U_0} \leq 2$. For $U_0$, there is a short exact sequence $0 \to K \to DA_0^0 \oplus DA_0^0 \to U_0 \to 0$ with the property (*) and the kernel $K$ has the following shape:

Since $K$ is not a summand of $M_0$, we need to find a short exact sequence ending at $K$ with the property (**). There is a short exact sequence $0 \to U_0 \oplus U_0 \to U_1 \oplus A_0^0 \oplus X \to K \to 0$ with the property (**). Applying the functor $(M_0, -)$ to the above two short exact sequences and putting the resulting exact sequences together, we get a project resolution of $E_{U_0}$:

$$0 \to Q_{U_0} \oplus Q_{U_0} \to Q_{U_1} \oplus Q_{A_0^0} \oplus Q_X \to Q_{DA_0^0} \oplus Q_{DA_0^0} \to Q_{U_0} \to E_{U_0} \to 0$$

Hence $\text{proj.dim } E_{U_0} \leq 3$.

From the above, we have that $\text{gldim } (\text{End } M_0) \leq 3$, and as a consequence $\text{repdim } (A_0) \leq 3$. Since $A_0$ is not of finite representation type we can actually deduce that $\text{repdim } (A_0) = 3$.

(II) **The case $n > 0$.** We have

$$M_n = \bigoplus_{i,j \geq 0, i+j \leq n} DA_i^j \oplus \bigoplus_{i \geq 0, j > 0, i+j \leq n} A_i^j \oplus A_0^0 \oplus \bigoplus_{0 \leq i \leq n+1} U_i \oplus X.$$ 

Again, for any indecomposable summand $N$ of $M_n$, we show that $\text{proj.dim } E_N \leq 3$ by explicitly constructing a projective resolution for the corresponding simple $\text{End}_{A_n}(M_n)$-module $E_N$. For the convenience of the reader, we list the indecomposable summands of $M_n$ as follows.

$$
\begin{array}{cccccccc}
A_n^0 & A_{n-1}^1 & \cdots & A_1^1 & X & A_0^0 & A_0^1 & A_0^2 \\
A_n^2 & A_{n-2}^1 & \cdots & A_2^1 & & A_1^0 & & \\
& \vdots & \vdots & \vdots & & & \vdots & \\
& A_1^{n-1} & & A_0^{n-1} & & & & A_0^n \\
\end{array}
$$
(1) The projective module $A^0_n$. Clearly, the exact sequence $0 \to \text{rad} A^0_n \to A^0_n \to 0$ has the property (*). Since $\text{rad} A^0_n = A^1_{n-1} \in \text{add} M_n$, applying the functor $(M_n, -)$ to the exact sequence gives us a projective resolution of the simple $\text{End}_{A_n}(M_n)$ module $E_{A^0_n}$ of the form

$$0 \to Q_{A^1_{n-1}} \to Q_{A^0_n} \to E_{A^0_n} \to 0.$$ 

Hence $\text{proj.dim } E_{A^0_n} = 1$.

(2) The module $X$. There is a short exact sequence $0 \to A^1_{n-1} \to U_0 \oplus A^0_n \to X \to 0$ with the property (*) (Actually, the only epimorphisms, apart from the multiples of the identity on $X$, are from $A^0_n$, so clearly they factor through the middle term. All other maps factor through the radical $\text{rad} X = U_0$). Since $A^1_{n-1}$ and the middle term are in $\text{add} M_n$, applying the functor $(M_n, -)$, we get a projective resolution of the simple $\text{End}_{A_n}(M_n)$-module $E_X$:

$$0 \to Q_{A^1_{n-1}} \to Q_{U_0} \oplus Q_{A^0_n} \to Q_X \to E_X \to 0$$ 

Hence $\text{proj.dim } E_X \leq 2$.

(3) The modules $A^1_i$. If $i = 0$, then there is a short exact sequence $0 \to D A^0_0 \to U_1 \oplus U_0 \oplus DA^0_1 \to A^0_0 \to 0$ with the property (*) (Except for the multiples of the identity on $A^0_1$, the epimorphisms are from $A^1_1$ ($s > 0$) and $DA^0_1$. They all factor through $DA^1_1$). Since all terms of the exact sequence are in $\text{add} M_n$, by applying the functor $(M_n, -)$, we get a projective resolution of the simple $\text{End}_{A_n}(M_n)$-module $E_{A^1_0}$:

$$0 \to Q_{DA^1_0} \to Q_{U_1} \oplus Q_{U_0} \oplus Q_{DA^0_1} \to Q_{A^0_1} \to E_{A^1_0} \to 0$$ 

Hence $\text{proj.dim } E_{A^1_0} \leq 2$.

If $i > 0$, then there is a short exact sequence $0 \to DA^1_i \to A^2_{i-2} \oplus DA^0_{i+1} \to A^1_i$ with the property (*) (Apart from the multiples of the identity on $A^1_i$, the epimorphisms to $A^1_i$ are from $A^1_s$ ($s > i$) and $DA^1_t$ ($t > i$). They all factor through $DA^1_i$. The non-epimorphisms all factor through the middle term by the definition of the middle term). Note that all the terms of the short exact sequence are in $\text{add} M_n$. Applying the functor $(M_n, -)$, we get a projective resolution of the simple $\text{End}_{A_n}(M_n)$-module $E_{A^1_i}$:

$$0 \to Q_{DA^1_i} \to Q_{A^2_{i-1}} \oplus Q_{DA^0_{i+1}} \to Q_{A^1_i} \to E_{A^1_i} \to 0$$

9
(4) The modules $A^j_i (j > 1, i > 0)$. For every such module, there is a short exact sequence $0 \to K \to A^{j+1}_{i-1} \oplus A^{j-1}_i \oplus DA^{j-1}_{i+1} \to A^i_j \to 0$ with the property (*) (In fact, the only epimorphisms from summands other than $A^j_i$ are from $A^s_i (s > i)$ and $DA^{j-1}_{i+1} (t > i)$. They all factor through $DA^{j-1}_{i+1}$. Other maps factor through $A^{j+1}_{i-1} \oplus A^{i-1}_i$). The kernel $K$ has the following shape:

$$\begin{array}{c}
\text{j sections} \\
123 \\
\text{DA} \\
123 \\
\text{i + 1 sections}
\end{array}$$

Clearly, $K$ is not in add $M_n$. There is a short exact sequence $0 \to DA^{j-1}_i \to A^{j-1}_{i-1} \oplus DA^i_i \oplus DA_i^{j-2} \to K \to 0$ with the property (**). Applying the functor $(M_n, -)$ to the above two exact sequences and putting together the resulting sequences together, we get a projective resolution of the simple $\text{End}_{A_n}(M_n)$-module $E_{A^j_i}$:

$$0 \to Q_{DA^{j-1}_i} \to Q_{A^{j-1}_{i-1}} \oplus Q_{DA^i_i} \oplus Q_{DA_i^{j-2}} \to Q_{A^{j-1}_{i-1}} \oplus Q_{A^i_i} \oplus Q_{DA_i^{j-1}} \to Q_{A^i_j} \to E_{A^i_j} \to 0$$

Hence $\text{proj.dim } E_{A^i_j} \leq 3$ for $j > 1$ and $i > 0$.

(5) The modules $A^j_0 (j > 1)$. For any $A^j_0$ with $j > 1$, there is a short exact sequence $0 \to K \to U_j \oplus A^{j-1}_0 \oplus DA^{j-1}_i \to A^0_0 \to 0$ with the property (*) (The only epimorphisms not from $A^j_0$ itself are from $A^j_i (s > 0)$ or $DA^j_i (i > 0)$. They all factor through $DA_i^{j-1}$). The kernel $K$ has the following shape:

$$\begin{array}{c}
\text{j sections} \\
123 \\
\text{U} \\
123 \\
\text{2} \\
\text{DA} \\
123 \\
\text{3}
\end{array}$$

Again, $K$ is not in add $M_n$, but there is a short exact sequence $0 \to U_j \to U_{j-1} \oplus U_{j+1} \oplus DA_i^{j-2} \to K \to 0$ with the property (**). Applying the functor $(M_n, -)$ to the above two short exact sequences and putting together the resulting sequences, we get a projective resolution of the simple $\text{End}_{A_n}(M_n)$-module $E_{A^j_0}$:

$$0 \to Q_{U_j} \to Q_{U_{j-1}} \oplus Q_{U_{j+1}} \oplus Q_{DA_i^{j-2}} \to Q_{U_j} \oplus Q_{A^{j-1}_0} \oplus Q_{DA_i^{j-1}} \to Q_{A^0_0} \to E_{A^0_0} \to 0$$

Hence $\text{proj.dim } E_{A^j_0} \leq 3$ for $j > 1$.

(6) The module $DA^0_n$. There is a short exact sequence $0 \to A^1_{n-1} \to DA^1_{n-1} \oplus A^0_n \to DA^0_n \to 0$ with the property (*) and all terms of the short exact sequence are in add $M_n$. Applying the functor $(M_n, -)$ to the short exact sequence, we get a projective resolution of the simple $\text{End}_{A_n}(M_n)$-module $E_{DA^0_n}$:

$$0 \to Q_{A^1_{n-1}} \to Q_{DA^1_{n-1}} \oplus Q_{A^0_n} \to Q_{DA^0_n} \to E_{DA^0_n} \to 0$$
Hence \( \text{proj.dim } DA_n^0 \leq 2 \).

(7) The modules \( DA_i^0 (0 < i < n) \). For any \( DA_i^0 \) with \( 0 < i < n \), there is a short exact sequence \( 0 \to A_{i-1}^0 \to DA_{i-1}^0 + A_i^1 \to DA_i^0 \to 0 \) with the property (*). (Except for the multiples of the identity on \( DA_i^0 \), the only epimorphisms to \( DA_i^0 \) are from \( DA_i^0 (s > i) \) and \( A_i^1 (t \geq i) \). They all factor through \( A_i^1 \).) Note that all terms of the short exact sequence are in \( \text{add } M_n \). Applying the functor \( (M_n, -) \), we get a projective resolution of \( E_{DA_i^0} \):

\[
0 \to Q_{A_{i-1}^0} \to Q_{DA_{i-1}^0} + Q_{A_i^1} \to Q_{DA_i^0} \to E_{DA_i^0} \to 0
\]

Hence we have \( \text{proj.dim } E_{DA_i^0} \leq 2 \) for \( 0 < i < n \).

(8) The module \( DA_i^0 \). There is a short exact sequence \( 0 \to K \to X \oplus A_i^1 \to DA_i^0 \to 0 \) with the property (*). (Note that all epimorphisms from an indecomposable summand of \( M_n \) to \( DA_i^0 \), except the multiples of the identity on \( DA_i^0 \), factor through \( A_i^1 \)). The kernel \( K \) is isomorphic to \( A_i^1 \), which is not in \( \text{add } M_n \). There is a short exact sequence \( 0 \to A_{i-1}^0 \to U_0 \oplus U_0 \oplus A_n^0 \to K \to 0 \) with the property (**). Applying the functor \( (M_n, -) \) to the above two exact sequences and putting the resulting sequences together, we get a projective resolution of the simple \( \text{End } M_n \)-module \( E - DA_i^0 \):

\[
0 \to Q_{A_{i-1}^0} \to Q_{U_0} \oplus Q_{U_0} \oplus Q_{A_n^0} \to Q_X \oplus Q_{A_i^1} \to Q_{DA_i^0} \to E_{DA_i^0} \to 0
\]

Hence we have \( \text{proj.dim } E_{DA_i^0} \leq 3 \).

(9) The modules \( DA_i^j (i, j > 0, i + j = n) \). For each module \( DA_i^j \) with \( i, j > 0 \) and \( i + j = n \), there is a short exact sequence \( 0 \to DA_{i-1}^j \to DA_{i-1}^{j+1} + DA_{i-1}^{j-1} \to DA_i^j \to 0 \) with the property (*) and all the terms of the short exact sequence are in \( \text{add } M_n \). Applying the functor \( (M_n, -) \) to the exact sequence, we get a projective resolution of the simple \( \text{End } M_n \)-module \( E_{DA_i^j} \):

\[
0 \to Q_{DA_{i-1}^j} \to Q_{DA_{i-1}^{j+1}} \oplus Q_{DA_{i-1}^{j-1}} \to Q_{DA_i^j} \to E_{DA_i^j} \to 0
\]

Hence we have \( \text{proj.dim } E_{DA_i^j} \leq 2 \) for \( i, j > 0 \) and \( i + j = n \).

(10) The modules \( DA_i^0 (j > 0) \). First, we consider \( DA_i^0 \). There is a short exact sequence \( 0 \to U_n \to U_{n+1} \oplus DA_{i-1}^{n-j} \to DA_i^0 \to 0 \) with the property (*). Applying the functor \( (M_n, -) \) to the short exact sequence, we get a projective resolution of the simple \( \text{End } A_n(M_n) \)-module \( E_{DA_i^0} \):

\[
0 \to Q U_n \to Q U_{n+1} \oplus Q_{DA_{i-1}^{n-j}} \to Q_{DA_i^0} \to E_{DA_i^0} \to 0
\]

Hence we have \( \text{proj.dim } E_{DA_i^0} \leq 2 \). For each \( DA_i^0 \) with \( 0 < j < n \), there is a short exact sequence \( 0 \to A_0^j \to DA_{i-1}^{n-j} \oplus A_i^{j+1} \to DA_i^0 \to 0 \) with the property (*). Applying the functor \( (M_n, -) \) to the short exact sequence, we get a projective resolution of the simple \( \text{End } A_n(M_n) \)-module \( E_{DA_i^0} \):

\[
0 \to Q_{A_0^j} \to Q_{DA_{i-1}^{n-j}} \oplus Q_{A_i^{j+1}} \to Q_{DA_i^0} \to E_{DA_i^0} \to 0
\]
Hence we have \( \text{proj.dim} \ E_{DA_i^j} \leq 2 \).

(11) The modules \( DA_i^j(i, j > 0, i + j < n) \). For each module \( DA_i^j \) with \( i, j > 0 \) and \( i + j < n \), there is a short exact sequence \( 0 \rightarrow K \rightarrow DA_i^{j-1} \oplus DA_i^{j+1} \rightarrow DA_i^j \) with the property (*) (Except the multiples of the identity on \( DA_i^j \), the only epimorphisms to \( DA_i^j \) are from \( DA_s^j \) \((s > i)\) and \( A_t^{j+1} \) \((t \geq i)\). They all factor through \( A_t^{j+1} \). Other maps factor through the middle term by the definition of the middle term.) The kernel \( K \) has the following shape:

 Clearly, \( K \) is not in \( \text{add} \ M_n \). However, there is a short exact sequence \( 0 \rightarrow A_t^{j+1} \rightarrow DA_i^{j-1} \oplus A_t^{j+1} \rightarrow DA_i^j \rightarrow K \rightarrow 0 \) with the property (**). Applying the functor \( (M_n, -) \) to the above two exact sequences and putting the resulting sequences together, we get a projective resolution of the simple \( \text{End}_{A_n}(M_n) \)-module \( E_{DA_i^j} \):

\[
0 \rightarrow Q_{A_t^{j+1}} \rightarrow Q_{DA_i^{j-1}} \oplus Q_{A_t^{j+1}} \oplus Q_{A_t^{j+2}} \rightarrow Q_{DA_i^{j-1}} \oplus Q_{DA_i^{j+1}} \oplus Q_{A_t^{j+1}} \rightarrow Q_{DA_i^j} \rightarrow E_{DA_i^j} \rightarrow 0
\]

Hence we have \( \text{proj.dim} \ E_{DA_i^j} \leq 3 \) for \( i, j > 0 \) and \( i + j < n \).

(12) The module \( U_0 \). There is a short exact sequence \( 0 \rightarrow K \rightarrow DA_0^0 \oplus DA_0 \rightarrow U_0 \rightarrow 0 \) with the property (*) (Except the multiples of the identity on \( U_0 \), all maps from an indecomposable summand of \( M_n \) to \( U_0 \) factor either through \( DA_0^0 \) or \( DA_0 \)). The kernel \( K \) has the following shape:

Clearly, \( K \) is not in \( \text{add} \ M_n \). However, there is a short exact sequence \( 0 \rightarrow U_n \oplus U_0 \rightarrow U_{n+1} \oplus A_0^0 \oplus X \rightarrow K \rightarrow 0 \) with the property (**). Applying the functor \( (M_n, -) \) to the above exact sequence and putting the resulting sequences together, we get a projective resolution of the simple \( \text{End}_{A_n}(M_n) \)-module \( E_U \):

\[
0 \rightarrow Q_{U_n} \oplus Q_{U_0} \rightarrow Q_{U_{n+1}} \oplus Q_{A_0^0} \oplus Q_X \rightarrow Q_{DA_0} \oplus Q_{DA_0} \rightarrow Q_{DA_0} \rightarrow E_U \rightarrow 0
\]

Hence we have \( \text{proj.dim} \ E_{U_0} \leq 3 \).

(13) The module \( U_1 \). There is a short exact sequence \( 0 \rightarrow K \rightarrow DA_1^0 \oplus A_0^0 \rightarrow U_1 \rightarrow 0 \) with the property (*) (Except the multiples of the identity on \( U_1 \), every epimorphism from an indecomposable summand of \( M_n \) to \( U_1 \) factors through either \( DA_1^0 \) or \( A_0^0 \). The maps having image in the radical of \( U_1 \) factor through \( A_0^0 \)). The kernel \( K \) has the following shape:
There is another short exact sequence \( 0 \to A_0^n \to U_n \oplus A_1^{n-1} \oplus DA_0^0 \to K \to 0 \)
with the property (**). Applying the functor \((M_n, -)\) to the above two exact sequences and putting together the resulting sequences, we get a projective resolution of the simple \( \text{End} A_n(M_n)\)-module \( E_{U_i} \):

\[
0 \to Q_{A_0^n} \to Q_{U_n} \oplus Q_{A_1^{n-1}} \oplus Q_{DA_0^0} \to Q_{DA_0^0} \oplus Q_{A_0^n} \to Q_{U_i} \to E_{U_i} \to 0
\]

Hence we have \( \text{proj.dim} E_{U_i} \leq 3 \).

(14) The modules \( U_i \), \((1 < i < n+1)\). For each module \( U_i \) with \( 1 < i < n+1 \), there is a short exact sequence \( 0 \to K \to U_{i-1} \oplus DA_0^i \oplus A_{i-1}^{n-i+1} \to U_i \to 0 \) with the property (*) (In fact, the epimorphisms from an indecomposable summand of \( M_n \) to \( U_i \), except the multiples of the identity on \( U_i \), either factor through \( DA_0^i \) or \( A_{i-1}^{n-i+1} \). Other maps factor through the \( U_{i-1} \), which is the radical of \( U_i \)). The kernel \( K \) has the following shape:

There is a short exact sequence \( 0 \to A_{i-1}^{n-i+1} \oplus 2 \to A_{i-1}^{n-i+1} \oplus A_{i-2}^{n-i+2} \oplus Da_{i-1}^{i-1} \to K \to 0 \) with the property (**). Applying the functor \((M_n, -)\) to the above two exact sequences and putting together the resulting sequences, we get a projective resolution of the simple \( \text{End} M_n \) module \( E_{U_i} \):

\[
0 \to Q_{A_{i-1}^{n-i+1}} \to Q_{A_{i-1}^{n-i+1}} \oplus Q_{A_{i-2}^{n-i+2}} \oplus Q_{DA_{i-1}^{i-1}} \to Q_{U_{i-1}} \oplus Q_{DA_0^i} \oplus Q_{A_{i-1}^{n-i+1}} \to Q_{U_i} \to E_{U_i} \to 0
\]

Hence \( \text{proj.dim} E_{U_i} \leq 3 \) for \( 1 < i < n+1 \).

(15) The module \( U_{n+1} \). There is a short exact sequence \( 0 \to A_{n-1}^1 \to U_n \oplus A_0^n \to U_{n+1} \to 0 \) with the property (**). (In fact, except the multiples of the identity on \( U_{n+1} \), the only epimorphisms are from \( A_0^n \). They factor through \( A_0^n \). Other maps factor through the radical \( U_n \).) Since all terms of the short exact sequence are in add \( M_n \), by applying the functor \((M_n, -)\), we get a projective resolution of the simple \( \text{End} A_n(M_n)\)-module \( E_{U_{n+1}} \):

\[
0 \to Q_{A_{n-1}^1} \to Q_{U_n} \oplus Q_{A_0^n} \to Q_{U_{n+1}} \to E_{U_{n+1}} \to 0
\]

Hence we have \( \text{proj.dim} E_{U_{n+1}} \leq 2 \).

Thus we have shown that \( \text{gl.dim} \text{End} A_n(M_n) \leq 3 \) for any non-negative integer \( n \). Hence we have \( \text{rep.dim} A_n \leq 3 \) for all \( n \geq 0 \). On the other hand, the algebras \( A_n \) \((n \geq 0)\) are not representation finite, so we have \( \text{rep.dim} A_n > 2 \) for all \( n \geq 0 \). Altogether, we come to the conclusion that \( \text{rep.dim} A_n = 3 \) for all non-negative integers \( n \) and this completes the proof of Theorem 1.1.

\( \bigcirc \)
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