An Accurate and Efficient Time-domain Model for Simulating Wave Propagation in Half-space

Mi Zhao 1, Xiaojing Wang 1, Piguang Wang 2*, Jingbo Liu 2 and Yiming Huang 1

1 Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China
2 Department of civil engineering, Tsinghua University, Beijing 100084, China
*Corresponding author’s e-mail: wangpiguang1985@126.com

Abstract. The wave propagation in the unbounded domain is a hottest topic in engineering research. Based on the scaled boundary finite element method, an efficient and accuracy artificial boundary condition is proposed for the wave propagation in half space, and the shape of the artificial boundary can be arbitrary, such as ellipse and rectangle. A new continued fraction for solving the dynamic stiffness is derived, and by introducing auxiliary variables, it can be transformed into time domain conveniently. The dynamic stiffness of the artificial boundary condition in frequency domain in good agreement with the exact solution in high frequencies. The circle and ellipse artificial boundary condition is identical with the reference solution in time domain.

1. Introduction
The wave propagation in the unbounded medium is ubiquitous in many scientific and engineering fields. To solve this problem, using the numerical method such as finite element method in time domain, the artificial boundary is introduced to divide the unbounded medium into a finite domain and an infinite domain. A time-domain absorbing boundary condition is imposed on the artificial boundary of the finite domain to simulate the radiation damping effect of the truncated infinite domain. The absorbing boundary condition is also called as the radiation, non-reflecting, or transmitting boundary condition. Excellent literature reviews on absorbing boundary conditions can be found in papers and monographs [1-6].

The early absorbing boundary conditions are some approximate simplified methods in time domain, such as the viscous boundary [7], the viscous-spring boundary [8, 9], the extrapolation boundary [10, 11] and so on. The efficiency of themselves is high due to their low storage and computation costs, but the artificial boundary must be relatively far from the radiation or scattering source for obtaining the responses with the acceptable accuracy. This leads to the high costs due to a large finite element model. The infinite element method [12], the boundary element method [13-15] and the perfectly matched layer [16] are developed and studied up to now.

The Dirichlet-to-Neumann (DtN) absorbing boundary conditions [4, 5] are obtained by the analytical method to solve the problem defined in the infinite domain. It is a frequency-domain relationship between the Neumann data (stress) and the Dirichlet data (displacement) on the artificial boundary. Several accurate and efficient frequency-to-time transform schemes have been developed: the temporal locality of the spherical DtN kernel in nature [17], the Hagstrom-Hariharan operators acting on the solutions to the modal wave equations [18], the operator splitting method applying to the modal wave
equations [19,20], the rational function approximation to dynamic stiffness coefficient and its auxiliary variable realization [21,22], and the continued fraction expansion of dynamic stiffness coefficient and its auxiliary variable realization [23]. The rational function approximation has been applied to the scalar-value dynamic stiffness coefficient for the three-dimensional homogenous reservoir [24] and to the given dynamic stiffness matrix for the foundation vibration analysis [25].

The Scaled Boundary Finite Element Method (SBFEM) [26, 27] is a promising semi-analytical and semi-numerical method similar to the TLM, which can solve both the finite and infinite domain problems. A novel scaled boundary coordinatig with a scaling centre is introduced to make the SBFEM applicable to the convex artificial boundary of general geometry in the half or full space infinite domain. The dynamic-stiffness matrix, which derived from SBFEM, can be a power series expansion [3] or a numerical solution of starting from its high-frequency asymptotic expansion [28]. Using the numerical convolution computation based on the discretized displacement unit-impulse response matrix [29], the rational function approximation of the dynamic stiffness matrix [30], and the extension of the continued fraction expansion from scalar to matrix [31], the SBFEM can be transformed to time domain.

In this paper, the dynamic stiffness matrix equation is derived based on the SBFEM. Then a new continued fraction is proposed for solving the dynamic stiffness matrix, which transforms the problem into time domain. The truncated artificial boundary can be arbitrary shape, such as the rectangle, ellipse artificial boundary conditions are used to analyze the wave propagation in infinite domain. This study is efficiently decreased the storage and computation cost in engineering such as deep foundation or pile foundation.

2. Problem Statement

The scalar wave propagation problem in the two-dimensional space is shown in figure 1, such as the dynamic structure-foundation interaction problem. To numerically solve the wave propagation problem in time domain by using the finite-domain-based method such as finite element method, an artificial boundary is introduced to enclose the domain of interest (called as finite domain) and to truncate the resting domain (called as infinite domain). The finite domain includes the structure and wave source to lead to the non-homogeneity and nonlinearity and is simulated by the finite element method. The infinite domain is defined as homogeneous and linear. Its wave radiation effect is analysed by the so-called radiation boundary condition imposing on the artificial boundary of finite domain.

The control equation of the infinite domain outside of the artificial boundary is the scalar wave equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]  

(1)

where \( u(x, y, t) \) is the unknown wave filed function (called as displacement); \( c \) is the wave propagation velocity; the dot over variable denotes the derivative to time \( t \). The initial condition of the infinite domain is at rest.

The boundary condition of the infinite domain is

\[
\left. \frac{\partial u}{\partial \alpha} \right|_{\alpha=0} = 0
\]  

(2)

The radiation condition is satisfied at infinity.
3. The Scaled Boundary Finite Element

The scaled boundary finite element equation for elastodynamics is detailed derived by Song and Wolf in [27]. In this paper, the derivation will not give unnecessary details; only a brief summary of the equations necessary for the completeness is presented in this section.

The scaling center is chosen as the Cartesian origin $O$. Moreover, the transform of the scaled boundary coordinate is shown in figure 2. Only the boundary $S$ visible from the scaling center is discretized, as shown in figure 2(a). The geometry of the isoparametric element is interpolated using the shape functions $N\eta$ formulated in the local coordinates $\eta$ of an element on boundary as

$$x = \xi N(\eta) X, \; y = \xi N(\eta) Y$$

(3)

where $\xi$, $\eta$ are the scaled boundary coordinates; $x$, $y$ are arbitrary node coordinates in Cartesian; $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are the coordinate of the nodes of an element in Cartesian coordinate system.

The same shape functions are used to interpolated in the nodal displacement functions $u$

$$u = N(\eta) u(\xi)$$

(4)

where $u(\xi) = \{u_1, u_2\}$ are the displacement of the nodes of an element, $u$ is the displacement function of an arbitrary node.

The scaled boundary finite element equation can be derived by utilizing the Galerkin’s weighted residual technique to the governing equation, and it can be expressed in frequency domain as
\[ \ddot{\xi}^2 E_0 \mathbf{u}(\xi)_{\dot{\xi}} + \left( E_0 - E_1 + E_1^T \right) \ddot{\xi}^2 \mathbf{u}(\xi) + E_1 \mathbf{u}(\xi)_{\dot{\xi}} + \omega^2 \ddot{\xi}^2 M_0 \mathbf{u}(\xi) = \mathbf{0} \]  

(5)

where \( \omega \) is the circle frequency; and \( E_0^e, E_1^e, E_2^e \) and \( M_0^e \) are the coefficient matrices of an element, are expressed as

\[ E_0^e = \int_{-1}^{1} B_1^T B_1 |J| d\eta = \frac{\Delta_2^2 + \Delta_3^2}{12 |J|} \left[ \begin{array}{c} 2 \\ 1 \\ 2 \end{array} \right] \]  

(6a)

\[ E_1^e = \int_{-1}^{1} B_2^T B_1 |J| d\eta = \frac{\Delta_2^2 + \Delta_3^2}{24 |J|} \left[ \begin{array}{c} -1 \\ 1 \\ -1 \end{array} \right] - \frac{\Delta_3^2 \left( y_1 + y_2 \right) + \Delta_2 \left( x_1 + x_2 \right)}{8 |J|} \left[ \begin{array}{c} -1 \\ 1 \\ 1 \end{array} \right] \]  

(6b)

\[ E_2^e = \int_{-1}^{1} B_2^T B_2 |J| d\eta = \frac{3 \left( y_1 + y_2 \right)^2 + 3 \left( x_1 + x_2 \right)^2 + \Delta_2 + \Delta_3}{24 |J|} \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \]  

(6c)

\[ M_0^e = \int_{-1}^{1} N^T N |J| d\eta = \frac{|J|}{3} \left[ \begin{array}{c} 2 \\ 1 \\ 1 \end{array} \right] \]  

(6d)

where \( B_1 \) and \( B_2 \) represent the strain-nodal displacement relationship and \( J \) is the determinant of the Jacobian matrix on the boundary. It is obvious that the coefficient matrices \( E_0^e \) and \( M_0^e \) are positive definite, \( E_2^e \) is symmetric. To model the total domain, an assemblage as in the conventional finite element is performed

\[ \ddot{\xi}^2 E_0 \mathbf{u}(\xi)_{\dot{\xi}} + \left( E_0 - E_1 + E_1^T \right) \ddot{\xi}^2 \mathbf{u}(\xi) + E_1 \mathbf{u}(\xi)_{\dot{\xi}} + \omega^2 \ddot{\xi}^2 M_0 \mathbf{u}(\xi) = \mathbf{0} \]  

(7)

where \( E_0, E_1, E_2 \) and \( M_0 \) are the assembled coefficient matrices.

The internal nodal forces, which can be derived by integrating the surface traction over element, on a surface with a constant \( \xi \) can be expressed as

\[ \mathbf{q}(\xi) = E_0 \ddot{\xi}^2 \mathbf{u} \left( \xi \right)_{\dot{\xi}} + E_1^T \mathbf{u} \left( \xi \right) \]  

(8)

The nodal forces \( \mathbf{R} \) are related to the internal forces on the boundary, which is expressed as for the unbounded domain

\[ \mathbf{R} = -\mathbf{q}(\xi = 1) = - \left( E_0 \ddot{\xi}^2 \mathbf{u}(\xi)_{\dot{\xi}} + E_1^T \mathbf{u}(\xi) \right) \]  

(9)

Defined the dynamic stiffness matrix of unbounded domain as

\[ \mathbf{R} = \mathbf{S} \mathbf{u} \]  

(10)

where \( \mathbf{R} \) and \( \mathbf{u} \) are the amplitude value of the nodal force and displacement on the boundary \( \xi = 1 \) in frequency domain. The dynamic stiffness equation for the SBFEM can be derived from Equations (7), (8), (9) and (10) [27]

\[ (\mathbf{S} + E_0) E_0^{-1} \left( \mathbf{S} + E_1^T \right) - \omega^2 \mathbf{S} = -\mathbf{E}_2 + \omega^2 \mathbf{M}_0 = \mathbf{0} \]  

(11)

The Equation (11) is non-linear first order ordinary differential equations in the independent variable \( \omega \). It can be solved by numerical solution of continued fraction [29, 30]. A continued fraction has much larger convergence range and higher convergence rather than the corresponding power series.
4. A High-order Local Transmitting Boundary Condition

4.1. The continued fraction solution

In this section, a continued fraction for the dynamic stiffness matrix is determined directly from the dynamic stiffness by scaled boundary finite element equation. The continued fraction of the dynamic stiffness matrix is defined as

\[ S = g_0 + i\omega h_0 - i\omega S_1^{-1} \]  
\[ S_j = g_j + i\omega h_j - S_{j+1}^{-1} \]

where \( g_0 \), \( h_0 \), \( g_j \) and \( h_j \) are the undetermined coefficient matrix.

Substituting Equation (12a) into (11) and after some manipulations obtains

\[
(i\omega)^2 (h_0 E_0^{-1} h_0 - M_0) + (i\omega) \left[ (g_0 + E_1) E_0^{-1} h_0 + h_3 E_0^{-1} (g_0 + E_1^T) - h_0 \right] \\
+ (g_0 + E_1) E_0^{-1} (g_0 + E_1^T) - E_2 - i\omega S_1^{-1} E_0^{-1} (g_0 + E_1^T + i\omega h_0) - i\omega (g_0 + E_1 + i\omega h_0) E_0^{-1} S_1^{-1} \\
+ (i\omega)^2 S_1^{-1} E_0^{-1} S_1^{-1} - i\omega S_1^{-1} + i\omega^2 S_1^{-1} = 0
\]

Setting the quadratic and constant term coefficients of the polynomial Equation (13) with respect to \((i\omega)\) equal to zero, respectively, the equation with respect to \(h_0\) and \(g_0\) can be obtained

\[
h_0 E_0^{-1} h_0 - M_0 = 0 \quad (14a)
\]
\[
(g_0 + E_1) E_0^{-1} (g_0 + E_1^T) - E_2 = 0 \quad (14b)
\]

which can be solved by the function ‘care’ in MATLAB.

The remaining part of the Equation (13) is the equation of \(S_1^{-1}\) and eliminating \((i\omega)\), then pre- and post-multiply by \(S_1^{-1}\)

\[
S_1 V_1 S_1 + \left( V_2^1 + i\omega V_3^1 \right) S_1 + S_1 \left( V_4^1 + i\omega V_5^1 \right) + i\omega V_6^1 + S_1 - \omega S_{1,\omega} = 0
\]

with the coefficient

\[
V_1^1 = h_0 E_0^{-1} (g_0 + E_1^T) + (g_0 + E_1) E_0^{-1} h_0 - h_0 \quad (16a)
\]
\[
V_2^1 = -E_0^{-1} (g_0 + E_1^T) \quad (16b)
\]
\[
V_3^1 = -E_0^{-1} h_0 \quad (16c)
\]
\[
V_4^1 = -(g_0 + E_1) E_0^{-1} \quad (16d)
\]
\[
V_5^1 = -h_0 E_0^{-1} \quad (16e)
\]
\[
V_6^1 = E_0^{-1} \quad (16f)
\]

Substituting Equation (12b) into (15) and after some manipulations obtains
\[(i\omega)^2 \left( h_j V_j/h_j + V_3/h_j + h_j V_j' \right) \]

\[+ (i\omega) \left[ h_j V_j/g_j + g_j V_j/h_j + V_2/h_j + V_j/g_j + g_j V_j/h_j + h_j V_4/h_j + V_6/h_j + (-1)^{j+1} h_j - h_j \right] \]

\[+ g_j V_j/g_j + V_2/g_j + g_j V_4/h_j + (-1)^j g_j - S_{j+1}^{-1} \left[ V_j/\left( g_j + i\omega h_j \right) + V_4/h_j + i\omega V_j \right] \]

\[- \left[ (g_j + i\omega h_j) V_j/h_j + V_2/h_j + i\omega V_j \right] S_{j+1}^{-1} + S_{j+1}^{-1} V_j/h_j + V_6/h_j + (-1)^{j+1} h_j - h_j = 0 \]

(17)

Setting the quadratic and linear term coefficients of the polynomial Equation (17) with respect to \((i\omega)\) equal to zero, respectively, the equation with respect to \(h_j\) and \(g_j\) can be obtained

\[h_j V_j/h_j + V_3/h_j + h_j V_j' = 0 \] (18a)

\[h_j V_j/g_j + g_j V_j/h_j + V_2/h_j + V_j/g_j + g_j V_j/h_j + h_j V_4/h_j + V_6/h_j + (-1)^{j+1} h_j - h_j = 0 \] (18b)

Pre- and post-multiplying Equation (18a) with \(h_j^{-1}\) respectively yields

\[V_j' + h_j^{-1} V_j' + V_3/h_j' = 0 \] (19)

The Equation (18b) and (19) can be solved by the function ‘lyap’ in MATLAB. And the remaining terms of Equation (17) yields

\[S_{j+1} V_{j+1}^{(j+1)} S_{j+1} + \left( V_{j+1}^{(j+1)} + i\omega V_{j+1}^{(j+1)} \right) S_{j+1} + S_{j+1} \left( V_{j+1}^{(j+1)} + i\omega V_{j+1}^{(j+1)} \right) + V_{j+1}^{(j+1)} + (-1)^{j+1} S_{j+1} - \omega S_{j+1,\omega} = 0 \]

(20)

with the coefficient

\[V_j^{(j+1)} = g_j V_j/g_j + V_2/g_j + g_j V_4/h_j + (-1)^{j-1} g_j + V_1 \]

\[V_2^{(j+1)} = -V_j/g_j - V_4 \]

\[V_3^{(j+1)} = -V_j/h_j - V_5 \]

\[V_4^{(j+1)} = -g_j V_j/h_j - V_2 \]

\[V_5^{(j+1)} = -h_j V_j/h_j - V_3 \]

\[V_6^{(j+1)} = V_j \] (21f)

It can be seen that the coefficient matrices \(g_0, h_0, g_j\) and \(h_j\) of the continued fraction are determined by Equations (14), (18b) and (19), which are solved efficiently in MATLAB. The truncated order \(M\) of the continued fraction with the approximation \(S_{j+1}^{-1} = 0\). As the increasing of the order of the continued fraction, the coefficient matrices does not require recalculation for a lower order.

4.2. The high transmitting boundary condition in time domain

The relation of the interaction forces and displacements on the boundary can be obtained by substituting the first term of the continued fraction solution in Equation (12a) into (10)

\[R = \left( g_0 + i\omega h_0 - i\omega S_1 \right) u \]

(22)

Introducing the auxiliary variable \(u_j\) as

\[u = S_j u_1 \text{ and } u_{j-1} = S_j u_j \]

(23)
Substituting into Equation (22) and applying inverse Fourier transform, a high-order temporally local transforming boundary condition is obtained

\[ R = g_u u + h_o u - I u_j \]  

(24)

Substituting Equation (23) into (12b), the remaining terms of the continued fraction solution

\[ g_j u_j + h_o u - I u_j = 0 \]  

(25)

Equation (24) and (25) can be coupled seamlessly and straightforwardly with finite elements modelling the near field.

5. Verification and Numerical

5.1. Verification of dynamic stiffness in frequency domain

In this section, the half-circle, the half-ellipse and the half-rectangle artificial boundary conditions are accomplished using the proposed method. The velocity of waves is taken as 200m/s, the mass density is 1900kg/m3 and the uniform force on the interface as \(\varepsilon_{\text{ext}}\). The right top node is treated as the viewpoint, and the displacement calculated by this method is compared with the exacted solution, as shown in figure 4. For the circle model, the reference solution is the exact solution. For the ellipse and rectangle model, the reference solution is calculated by finite element coupled with exact half circle artificial boundary condition.

![Figure 3. Three models of wave propagation in infinite domain.](image)

![Figure 4. Amplitude displacement response of circle model in frequency domain.](image)
In figure 4, it can be seen that the proposed method is identical with the exact solution for circle model. In figure 5 and figure 6, it can be seen that the proposed method for ellipse and rectangle shape interface is more accurate at high frequencies. So that the proposed method in seismic response analysis is adoptable.

5.2. Verification of dynamic stiffness in time domain

In this section, a circle and an ellipse model are used to verify the proposed method in time domain. The exciting force is taken as an impulse, which is a finite difference approximation of Dirac delta function described by Zhao [22], as shown in figure 7. The reference solution is inverse Fourier transformed from the exact frequency domain. The displacement response of right top node is shown in figure 8. It obvious that the proposed method result is identical with the exact solution. Due to the function ‘care’ function in MATLAB, require the coefficient matrix is strict; it can not be accomplished for the arbitrary mesh size.
6. Conclusion
A high efficient boundary condition for the wave propagation in half space infinity domain is proposed in this study, which can be located on the structure surface. For the arbitrary shape artificial boundary, the dynamic stiffness equation is obtained by utilizing the SBFEM. A continued fraction is introduced to simulate the artificial boundary condition in frequency domain. By introducing the auxiliary variable, the proposed artificial boundary condition can be transformed into time domain. In frequency domain, it has been verified that the proposed artificial boundary condition with circle, ellipse and rectangle shape in good agreement with the exact solution. Then the proposed artificial boundary condition is verified in time domain, and the results is highly consistent with the reference solution.

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