Time-dependent behaviour of Lyman $\alpha$ photon transfer in a high-redshift optically thick medium

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ABSTRACT

Using the Monte Carlo simulation method, we investigate the time-dependent behaviour of Ly$\alpha$ photon transfer in an optically thick medium of the concordance $\Lambda$CDM universe. At high redshifts, the Ly$\alpha$ photon escaping from the optically thick medium has a time-scale as long as the age of the luminous object, or even comparable to the age of the Universe. In this case, time-independent or stationary solutions of the Ly$\alpha$ photon transfer with resonant scattering will overlook important features of the escaped Ly$\alpha$ photons in physical and frequency spaces. More importantly, the expansion of the Universe means that the time-independent solutions of the Ly$\alpha$ photon transfer might not exist. We show that time-dependent solutions are sometimes essential for understanding Ly$\alpha$ emission and absorption at high redshifts. For Ly$\alpha$ photons from sources at redshift $1 + z = 10$, which are surrounded by the neutral hydrogen intergalactic medium (IGM) of the $\Lambda$CDM universe, the escape coefficient is found to be always less, or much less, than one, regardless of the age or lifetime of the sources. In such an environment, we also find that even when the Ly$\alpha$ photon luminosity of the sources is stable, the mean surface brightness gradually increases in the first $10^6$ yr, and then decreases with a power law of time. However, it never approaches a stable, time-independent state. That is, all $1 + z = 10$ sources in a neutral Hubble expanding IGM with Ly$\alpha$ luminosity $L$ have their maximum of mean surface brightness $\sim 10^{-21}(L/10^{43}$ erg s$^{-1})$ erg s$^{-1}$ cm$^{-2}$ arcsec$^{-2}$ at the age of about $10^6$ yr. We also address the time-dependent effects on the red damping wing profile.

Key words: radiative transfer – scattering – intergalactic medium – cosmology: theory.

1 INTRODUCTION

Ly$\alpha$ photons have been widely used to study the physics of the Universe at redshifts from 2 to 8. Redshifted Ly$\alpha$ photons carry information about the photon source, the halo surrounding the source and the intergalactic medium (IGM) of the early Universe. The optical afterglow of gamma-ray bursts (GRBs) has been modelled as the red wing of Ly$\alpha$ photon absorption. It has been used to estimate the column number density of neutral hydrogen of the IGM at high redshifts (Totani et al. 2006; Salvaterra et al. 2009). The transmitted flux of quasi-stellar object absorption spectrum at redshift $z > 5$ consists of complete absorption troughs separated by spikes (e.g. Becker et al. 2001; Fan et al. 2006). The spikes have been explained as Ly$\alpha$ photons leaking at low-density areas (e.g. Liu et al. 2007; Feng et al. 2008). They have been used to probe the turbulent behaviour of the IGM at high redshifts (Lu et al. 2010; Zhu, Feng & Fang 2010, 2011). Last but not least, the search for redshifted Ly$\alpha$ photons from star-forming galaxies at high redshifts is believed to be a basic tool that can be used to explore the epoch of reionization (e.g. Hayes 2010; Lehner et al. 2010). Therefore, it is crucial to have a complete understanding of the radiative transfer of Ly$\alpha$ photons, which is caused by their resonant scattering with neutral hydrogen atoms.

The radiative transfer of Ly$\alpha$ photons in a medium consisting of neutral hydrogen atoms has been extensively studied both analytically and numerically. Yet there are very few solutions on the time-dependent behaviour of Ly$\alpha$ photon transfer (Field 1958; Rybicki & Dell’Antonio 1994; Higgins & Meiksin 2009). All other analytical solutions are time-independent, based on the Fokker–Planck equation (Harrington 1973; Neufeld 1990; Dijkstra, Haiman & Spaans 2006). Time-independent solutions are important, but they can only be used to describe the ‘limiting asymptotic behaviours’ of radiative transfer (Adams 1975; Bonilha et al. 1979). They tell us nothing about the time-scales of the radiative transfer of Ly$\alpha$ photons. These time-independent solutions cannot describe the Wouthuysen–Field (W–F) effect (Wouthuysen 1952; Field 1958, 1959), which is essential for the 21-cm emission and absorption of neutral hydrogen.
at high redshifts (e.g. Roy et al. 2009b). This is because the Fokker–Planck approximation would miss the detailed balance relation of resonant scattering, which is necessary to keep the W–F local thermalization (Rybicki 2006).

A numerical method based on the Monte Carlo (MC) simulation is also popular for solving the transfer of resonant photons (e.g. Loeb & Rybicki 1999; Zheng & Miralda-Escude 2002, hereafter ZM02; Tasić et al. 2006; Verhamme, Schaerer & Maselli 2006; Laursen & Sommer-Larsen 2007; Dijkstra & Loeb 2008; Pierleoni, Maselli & Ciardi 2009; Xu & Wu 2010). However, very few works have dealt with time-dependent problems. For instance, the time-scale of the W–F local thermalization is still absent in these studies.

The time-dependent behaviour of the Lyα photon transfer is especially important for understanding observations of Lyα photons at high redshifts. First, either the lifetime or the age of photon sources at high redshift is generally short. Secondly, the optical depth of the IGM or the halo cloud around the sources is generally large. If the time-scale of the transfer of Lyα photons is comparable to the lifetime or the age of the photon source, the ‘limiting asymptotic state’ will never be approached. One more problem is that caused by the cosmic expansion, the time-scale of which is short at high redshifts. When the time-scale of the cosmic expansion is comparable to that of the resonant photon transfer in an optically thick medium, time-independent solutions probably do not exist.

Recently, a state-of-the-art numerical method based on the weighted essentially non-oscillatory (WENO) scheme has been introduced to solve the integro-differential equation of the radiative transfer of resonant photons (Roy et al. 2009a,b; Roy, Shu & Fang 2010). It reveals many interesting features of the transfer of Lyα photons in an optically thick medium, which cannot be seen with the time-independent solutions of the Fokker–Planck approximation. For instance, it shows that the time-scale of the formation of the W–F local thermal equilibrium is actually short, only about a few hundred times that of the resonant scattering. The double-peaked frequency profile of Lyα photons cannot be described by time-independent analytical solutions unless the optical depth of v0 photons is as large as about $10^6$. This result directly indicates the need for time-dependent solutions.

The WENO algorithm of the integro-differential equation of the radiative transfer is fine, but like other high-order schemes with a fixed grid without artificial viscosity, the computation time is much longer than that of the MC method. Therefore, it would be not easy to use the WENO method to deal with cases of a medium with a very high optical depth. The goal of this paper is twofold. First, we show that the MC simulation method can properly match the results of the WENO method on the time-dependent features of a moderate optical depth. Secondly, we study the time-dependent solutions of Lyα photons escaped from an optically thick medium. We do not work on specific objects, but we focus on the general time-dependent features that affect the observability of the Lyα sources embedded in or behind an optically thick medium.

The paper is organized as follows. In Section 2, we present the basic models that we study. In Section 3, we give the MC simulation method and its tests for time-dependent problems of Lyα photon transfer in an optically thick medium. We give the major results of the time-dependent solutions of Lyα photon emission in the damped Lyα (DLA) and IGM models in Sections 4 and 5, respectively. We present the problems of absorption by an optically thick medium in Section 6. We give a discussion and conclusions in Section 7. All the relevant equations of the radiative transfer of Lyα photons and the details of MC simulation are presented in the appendices.

2 Problem

We study the time-dependent transfer of Lyα photons in two typical models of the neutral hydrogen H I medium. The first is the so-called DLA system halo model, in which a source is surrounded by a static spherical halo of physical radius $r_p$, consisting of homogeneously distributed neutral hydrogen with number density $n_{HI}$ and temperature $T$. The second model is a source at redshift $(1 + z) = 10$ located in the homogeneously Hubble expanding IGM, for which the density and temperature are given by the parameters of the concordance $\Lambda$CDM universe. We call this the IGM model. The radiative transfer equations of the two models are given in Appendix A.

In order to compare with the WENO solutions, for the DLA halo model we use dimensionless time and radial coordinates, defined as $\eta = c n_{HI} \sigma T$ and $r = n_{HI} \sigma_T r_p$, respectively. Here, $c$ and $r_p$ are the physical variables of the time and radial coordinates, and $\sigma_T/\sqrt{\pi}$ is the cross-section of scattering at the resonant frequency $v_0 = 2.46 \times 10^{15}$ s$^{-1}$. Therefore, $\eta$ and $r$ are the time and length in units of the mean free flight time and the mean free path of photon $v_0$, respectively. The value of $r$ is actually equal to the optical depth of the spherical halo from $r = 0$ to $r$ at frequency $v_0$. For a signal propagating in a radial direction with the speed of light, we have $r = \eta + \text{const}$.

As usual, in frequency space, we use variable $x \equiv (\nu - \nu_0)/\Delta \nu_D$, where

$$\Delta \nu_D = v_0 \frac{v_T}{c} = \nu_0 \sqrt{\frac{2 \nu_0 T}{m_H c^2}} = 1.06 \times 10^{11} \left( \frac{T}{10^4} \right)^{1/2} \text{Hz}$$

is the Doppler broadening at frequency $v_0$ by the thermal motion $v_T$ of the gas with temperature $T$. The variable $x$ is then the deviation of frequency $\nu$ from $\nu_0$ in units of the Doppler broadening.

With the dimensionless variables, the specific number intensity of photons is $I(\eta, r, x, \mu)$, where $\mu = \cos \theta$ is the direction relative to the radial vector $r$. Thus, none of the solutions of $I(\eta, r, x, \mu)$ refers to a specific density $n_{HI}$ and size $r_p$ (see Appendix A). This helps us to see the common features of the DLA halo model.

The optical depth of a halo or cloud with column density $N_{HI}$ at frequency $x$ is

$$\tau(x) = N_{HI} \sigma(x) = \tau_0 \phi(x, a). \tag{1}$$

Here, $\sigma(x)$ is the scattering cross-section, $\tau_0 = N_{HI} \sigma_0$ and $\phi(x, a)$ is the normalized Voigt profile given by

$$\phi(x, a) = \frac{a}{\pi a^2} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(x - y)^2 + a^2} dy. \tag{2}$$

Here, $a$ is the ratio of natural to Doppler broadening. For the Lyα line, $a = 4.7 \times 10^{-4} (T/10^4)^{-1/2}$. The profile equation (2) describes the joint effect of the Gaussian distribution of the velocity of the neutral hydrogen atom and the Lorentz profile of the cross-section in the rest frame of the atom. For an expanding (or collapsing) halo or turbulent gas cloud, the bulk velocity of the gas might be larger than the thermal velocity $v_T$. Even in this case, the Doppler or thermal broadening is still important, as it is the key factor leading to the local thermal equilibrium of Lyα photons (the W–F effect).

1 Because of the different normalization scheme of the Voigt function of equation (2), our definition of $\sigma_0$ is different from that sometimes used in the literature by a factor of $\pi^{1/2}$. Consequently, the expressions for mean flight time, mean free path and optical depth might be different by a factor of $\pi^{1/2}$. 
In an optically thick spherical cloud, most time-dependent behaviours can be described by two time-dependent distribution functions:

(i) the angularly averaged, $r^2$-rescaled, specific number intensity

$$j(\eta, r, x) = \frac{r^2}{2} \int_{-1}^{+1} I(\eta, r, x, \mu) d\mu;$$

(ii) the $r^2$-rescaled number flux describing the photons escaped from a spherical halo of radius $r$

$$f(\eta, r, x) = \frac{r^2}{2} \int_{-1}^{+1} \mu I(\eta, r, x, \mu) d\mu.$$

The equations of $j$ and $f$ are given in equations (A1) and (A2). For the IGM model, we use the physical variables $t$ and $r_p$ because of the specific cosmological model we have adopted. The temperature of the IGM is taken as $T = 100$ K, and $y = 1/\tau_{\text{esc}}$ (see equation A3), which describes the Hubble expansion, is equal to $1.4 \times 10^{-6}$ in the concordance $\Lambda$CDM universe.

3 METHOD AND TEST

3.1 Time-dependent Monte Carlo simulations

We use the MC method to simulate $j(\eta, r, x)$ and $f(\eta, r, x)$ of Section 2. Most MC codes that simulate the time-independent solutions of the radiative equation can easily be modified to deal with time-dependent problems. If the feedback of photon transfer on the parameters of neutral hydrogen is negligible, the radiative transfer equations are linear with respect to $I$, and thus to $j$ and $f$. Thus, the linear superposition of the solutions of the sources is valid. The time-dependent solutions can simply be given by a weighted summation over the results of a single flash. The weight of the summation is proportional to the time-dependent flux of the light source.

We employ the same MC algorithm as used in Xu & Wu (2010), for which some of the details are given in Appendices B and C. The major modification from the earlier methods (e.g. ZM02) is to record the time at each collision of the photon, and to take snapshots of the photon distribution in spatial and frequency space based on these time stamps. Thus, the time-dependent solutions $j$ and $f$ of an arbitrary source can be given by a synthesis of the fluxes at different epochs from a single flash source.

3.2 Test with $x$-profile of flux

As a test of the MC method, we calculate the flux $f(\eta, r, x)$ at the outer boundary $r_D = 10^2$ of a DLA halo. This result is shown in Fig. 1, in which the curves are the MC results with $\eta = 500$, 2000 and 3000. They show a typical double-peaked profile. The curves of $\eta = 2000$ and 3000 are actually the same. The overlapping indicates that $f(\eta, r, x)$ is already at the limiting asymptotic state, or saturated for time $\eta \geq 2000$. In Fig. 1, the data points are given by the WENO numerical solutions of Roy et al. (2010). Therefore, the MC method can match the time-dependent WENO solutions.

In the WENO method, the solutions of the angularly averaged specific intensity $j(\eta, r, x)$ and the flux $f(\eta, r, x)$ at boundary $r_D$ should satisfy the condition $j(\eta, r_D, x) = 2f(\eta, r_D, x)$ (Unno 1955). This is the result of the Eddington approximation and the assumption that there are no incoming photons at the boundary. Our MC simulation results of $j(\eta, r_D, x)$ at time $\eta = 500$ are shown at various radii in Fig. 2. At the surface where $r = 100$, we see $j \approx 2f$ at the centre frequencies when compared with Fig. 1. This shows that the MC simulation can pass the test of Unno’s boundary condition. We also find that the $j = 2f$ relation is only valid at the boundary at centre frequencies. Slightly beneath the surface, at radii $r = 99$ and 98, we find that $j \approx 4f$ and $5f$, respectively. The enhanced photon intensity is a result of backward-scattered photons near the boundary.

Besides the curve of $r = r_D = 100$, Fig. 2 also plots $j(\eta, r, x)$ at time $\eta = 500$, but for $r = 80$, 90, 95, 98 and 99. We see that all the curves of $r < 100$ are almost flat in the range of $|x| < 2$. This means that the frequency distribution of photons is thermalized near the resonant frequency $\nu_0$. That is, the frequency distribution is of Boltzmann shape $\exp\{-(h\Delta\nu_0/kT)x\}$, where $T$ is the kinetic temperature of the neutral hydrogen gas in the halo. This is the $W$–$F$ local thermalization (Wouthuysen 1952; Field 1958, 1959). Fig. 2 shows that the $W$–$F$ local thermalization is achieved by resonant scattering.
even at $r = 99$. Yet photons at the outermost layer $r = r_0$ have not yet been thermalized, as the optical thin layer does not have a large enough number of scatterings. This result is similar to the WENO solution (Roy et al. 2009b, 2010).

### 3.3 Test with $|x_\pm|-\tau_0$ relation

The second test is on the $|x_\pm|\sim\tau_0$ relation, where $x_\pm$ are the frequencies of the two peaks of the double-peaked profile, as shown in Fig. 1, and $\tau_0$ is the optical depth defined in equation (1). This relation has been studied by many time-independent solutions, based on the Fokker–Planck equation. The major conclusion is the so-called $(\tau_0)^{1/3}$ law, that is, $x_\pm = \pm A(\tau_0)^{1/3}$, where $A$ is a constant of order 1 (Adams 1972, 1975; Harrington 1973; Neufeld 1990; Dijkstra et al. 2006). It is well known that the $(\tau_0)^{1/3}$ law is available only when the optical depth $\tau_0$ is very large. However, the $x_\pm-\tau_0$ relation, which is available for various $\tau_0$, has not been calculated until very recently. This might be because of the absence of a proper numerical solver of the integro-differential equation of resonant photon transfer in an optical thick medium. The WENO solver provided the first $x_\pm-\tau_0$ relation of a DLA halo with moderate and high optical depth.

The $x_\pm-\tau_0$ relation given by the MC simulation is presented in Fig. 3, which shows that the $(\tau_0)^{1/3}$ law deviates significantly from the MC results when $\tau_0 \leq 10^2$, where the optical depth is smaller than $\tau_0 = 2 \times 10^2$ at $a = 5 \times 10^{-3}$. This result is the same as the WENO solution. In Fig. 3, the range of $|x_\pm|$ is larger than that of the WENO solution (see fig. 4 of Roy et al. 2010).

In the range $10^{-2} < \tau_0 < 10^2$, the $|x_\pm|\sim\tau_0$ relation is almost flat with $|x_\pm| \simeq 2$. This is because the double-peaked profile is from photons stored in the frequency range of $|x| < 2$ and in the local thermal equilibrium state. The positions of the two peaks, $x_\pm$, are actually about the same as the frequency range of the local thermalization. The frequency width $|x| \leq 2$ of the local thermal equilibrium state is determined by the Doppler broadening, and is very weakly dependent on $\tau_0$. Thus, once the photons in the local thermal equilibrium state are dominant, we always have $x_\pm \simeq \pm 2$. This point can also be seen with Figs 1 and 2, in which the positions of the two peaks are kept at $|x_\pm| \simeq 2$ despite the fact that the intensity of the flux increases with time significantly.

When $\tau_0 < 10^{-2}$, the curves of $|x_\pm|$ are no longer determined by one variable $\tau_0$ but are instead determined by variables $a$ and $\tau_0$ separately. The value of $|x_\pm|$ shows a quick drop to zero at $\tau_0 \sim 10^{-2}$ for $a = 5 \times 10^{-3}$, and $\tau_0 \sim 10^{-3}$ for $a = 5 \times 10^{-4}$. The W–F thermal equilibrium cannot be established in haloes with small optical depth $\tau_0 \ll 10^2$, and therefore photons from these haloes do not have a double-peaked profile.

Because thermalization will erase all frequency features within the range $|x| \sim 2$, the double-peaked structure does not retain information about the photon frequency distribution within $|x| \leq 2$ at the source. It is impossible to probe the frequency profile for $|x| < 2$ Ly$\alpha$ photons of the source from the escaped Ly$\alpha$ photons. This property can also be used as a test of the simulation code. That is, the simulation results should be independent of the profile of Ly$\alpha$ emission from the sources, only if the profile is non-zero within the range $|x| < 2$ (i.e. it should not matter whether the source is monochromatic or has a finite width around $v_0$).

### 4 Ly$\alpha$ Photon Emission: DLA Model

#### 4.1 Time-dependent Ly$\alpha$ escape

It is well known that the spatial transfer of Ly$\alpha$ photons in an optically thick halo is not simply a Brownian random walk. The time-scale of the Ly$\alpha$ photons escape from an optically thick halo is much shorter than that of Brownian diffusion. This is because the spatial transfer depends on the diffusion in frequency space. This is the so-called ‘single longest excursions’ process (Adams 1972). However, earlier estimates of the escaping time-scale based on the ‘single longest excursion’ cannot describe the details of the time-dependent behaviour of photons escaping from an optically thick medium.

If the central source of a DLA halo is assumed to be a photon flash, the time dependence of the luminosity of the photon source is proportional to $\delta(\eta)$. Without scattering, the luminosity at the boundary of the halo with size $r$ should still be a delta function as $\propto \delta(\eta - r)$ (i.e. it is also a flash, but with a retarded time $r$, which is the time needed for a freely streaming photon to go from the centre to the edge of the halo at the speed of light). Considering the effect of resonant scattering, the luminosity of the escaped photons will no longer be a flash. The light curve of the luminosity from such a source for a halo with size $r = 2 \times 10^2$ and $a = 5 \times 10^{-4}$ is shown in Fig. 4. Here, $F(\eta) = \int f(\eta, x) \, dx$ is the flux integrated over all frequencies of escaped photons.

Fig. 4 shows that the light curve lasts from time $\eta \sim 10^8$ to $5 \times 10^8$. The peak of the light curve $F_{\text{max}}$ is at $\eta_{\text{max}} \sim 2 \times 10^8$, and the time duration $\Delta \eta$ of $F(\eta) > F_{\text{max}}/2$ is also about $2 \times 10^8$. Because the source of the photons does not contain any time-scale, both the numbers $\eta_{\text{max}}$ and $\Delta \eta$ are from the size or optical depth of the halo. The time-scale $\eta_{\text{max}}$ means that the retarded time for the escape of photons from the halo is as large as $\sim 10r$ or $10r_0$. The amount $\Delta \eta$ means that the time distribution of the escaped photons is significantly spread out from a Delta function $\delta(\eta - \Delta \eta \sim 10r_0$).

Without resonant scattering, photons emitted from the source will escape from the halo at time $\eta = r = r_0$. With resonant scattering, the majority of photons emitted from the source will not escape from the halo until time $\eta = \eta_{\text{max}} \sim 10^8$. Therefore, a huge number of resonant photons are stored in the halo. The resonant nature of Ly$\alpha$ photon scattering allows the photons to remain in the halo with a time-scale equal to 10 times the optical depth $\tau_0$ of the halo.

In our model, the destruction processes of Ly$\alpha$ photons, such as the two-photon process, are not considered, and dust absorption...
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Figure 4. The light curve of a flash of Ly$\alpha$ photons in a DLA halo model of $a = 0.0005$ with optical depth $\tau_0 = r_0 = 2 \times 10^7$. One photon is supposed to be emitted from a central source at time $\eta = 0$ (i.e. the integral of the light curve is equal to 1).

Figure 5. Time-dependent solutions of the escape coefficient $f_{\text{esc}}(\eta)$ for photon sources located in the centre of a DLA halo of $T = 10^5$ K with optical depth $\tau_0 = 2 \times 10^7$. The lifetime of the source $\eta_{\text{life}}$ is taken to be $\eta_{\text{life}} = 10^8$, $10^9$, $10^{10}$ and $10^{10}$, respectively, from the bottom up (lines are coloured orange, violet, cyan, green and black, respectively).

is also ignored. The number of photons is conserved. Thus, we have $\int F(\eta) d\eta = 1$. Therefore, the light curve $F(\eta)$ of Fig. 4 can be understood as the probability distribution of the time for Ly$\alpha$ photons to escape from a halo of $r = 2 \times 10^7$. With dimensionless variables, the curve of Fig. 4 is also the probability distribution of the total length of the path of a photon transferring from $r = 0$ to $r = 2 \times 10^7$. In this context, $\eta_{\text{max}}$ and $\Delta \eta$ can be used as the most probable path length of the path and the variance of the distribution, respectively. Considering that the most probable path length $\eta_{\text{max}}$ and the variance $\Delta \eta$ have the same order, the spatial transfer of Ly$\alpha$ photons in an optically thick halo is essentially still a random process of diffusion.

The light curve of a flash source (Fig. 4) can easily be generalized to arbitrary sources. Considering that the total flux is linearly dependent on sources, the total flux $L(\eta)$ is given by

$$L(\eta) = \int_0^\eta F(\eta - \eta') s(\eta') d\eta'.$$

Here, $F(\eta)$ is the curve of Fig. 4 and $s(\eta)$ is the time-dependent Ly$\alpha$ photon flux of the sources.

4.2 Escape coefficient

For a source with a stable luminosity of Ly$\alpha$ photons, the escape coefficient of Ly$\alpha$ photons emerging from a halo with size $r$ can be calculated by

$$f_{\text{esc}}(\eta, r) \equiv F(\eta)/F_0 = \int f(\eta, r, x) dx,$$

if the luminosity of sources is normalized. Because the number of Ly$\alpha$ photons is conserved, the total number of escaped photons should be equal to the total number of photons emitted by the source when a stable state is reached. Thus, the escape coefficient of a stable source should reach 1 when $\eta$ is large enough. However, before the system approaches a stable state, the escape coefficient $f_{\text{esc}}(\eta, r)$ can be much less than 1.

We have calculated the time-dependent solution of the escape coefficient $f_{\text{esc}}(\eta)$ for a stable photon source with an unlimited lifetime, located at the centre of a halo with an optical depth $\tau_0 = 2 \times 10^7$. At time $\eta_{\text{max}} = 2 \times 10^8$, the escape coefficient $f_{\text{esc}}(\eta) \sim 0.37$. If we define the time-scale $\eta_{\text{sat}}$ of reaching a saturated or stable state as $f_{\text{esc}}(\eta_{\text{sat}}) = 0.95$, this yields $\eta_{\text{sat}} = 6.1 \times 10^8$ or $\sim 30\tau_0$. Therefore, $f_{\text{esc}}$ is always significantly less than 1, when the age is less than about $30\tau_0$.

Fig. 5 presents the time-dependent solutions of the escape coefficient $f_{\text{esc}}(\eta)$ for photon sources in a $T = 10^5$ K neutral medium with a limited lifetime of $\eta_{\text{life}} = 10^8$, $10^9$, $10^{10}$ and $10^{10}$, respectively, from the bottom to the top. This shows that the escape coefficient is always less than 1 if $\eta_{\text{life}} < 10^7$. In this case, the time-scale of the escape of Ly$\alpha$ photons in a $r_0 = 2 \times 10^7$ halo is much larger than the lifetime of the source. Therefore, photons emitted within the time duration $\eta = 0$ to $\eta = \eta_{\text{life}}$ are fully spread over a time-scale $\Delta \eta \gg \eta_{\text{life}}$. Thus, the escape coefficient of a source with a short lifetime is always much less than 1 when the DLA halo is optically thick. This mechanism could be important for understanding the observable features of high-redshift objects, such as GRBs or first stars. Even when dust absorption is negligible, the escape coefficient can be small, even very small, when the age of the object is small.

It is interesting to see that although the different curves of Fig. 5 correspond to very different $\eta_{\text{life}}$, all the curves with $\eta_{\text{life}} < 10^7$ approach their maximum at about the same time (i.e. $\eta \sim 2 \times 10^9$). This time-scale is actually the $\eta_{\text{max}}$ shown in Fig. 4. The stored photons yield a delayed emission with a time-scale of about $\eta_{\text{max}} \sim 2 \times 10^7$. The delay time is independent of the lifetime $\eta_{\text{life}}$ of the source, but dependent only on the optical depth of the halo.

4.3 Surface brightness and the size of a DLA halo

For a stable source, the time-independent surface brightness (SB) of a stable source located in a DLA halo is simply proportional to $r_{\text{DLA}}^{-2}$, where $r_{\text{DLA}}$ is the radius of the halo. When the lifetime of the source is short, the time-dependent behaviour of the surface brightness, $SB(t)$, is not simply proportional to $r_{\text{DLA}}^{-2}$. To demonstrate this point, we calculate the surface brightness of a source with lifetime $10^6$ yr and Ly$\alpha$ luminosity $L = 10^{42}$ erg s$^{-1}$ in a medium of $a = 0.0005$. The results are presented in Fig. 6, in which the sizes of the haloes are taken to be $r_{\text{DLA}} = 1$ kpc, 10 kpc, 100 kpc and 1 Mpc. In Fig. 6, the optical depth of the DLA halo surrounding the source is assumed to be $\tau_0 = 2 \times 10^7$, which corresponds to a typical DLA halo with a column density $N_{\text{HI}} = 2 \times 10^{20}$ cm$^{-2}$ (equation 1).

First, we see from Fig. 6 that the curves do not appear to be proportional to $r^{-2}_{\text{DLA}}$. The maximum surface brightness of a halo...
with $r_{\text{DLA}} = 1 \text{kpc}$ is about $10^{-17} \text{erg s}^{-1} \text{cm}^{-2} \text{arcsec}^{-2}$, while the maximum surface brightness of $r_{\text{DLA}} = 1 \text{Mpc}$ is only $\sim 10^{-25} \text{erg s}^{-1} \text{cm}^{-2} \text{arcsec}^{-2}$. That is, when $r_{\text{DLA}}$ decreases by a factor of $10^6$, the maximum surface brightness decreases by a factor of $10^9$.

Secondly, the shapes of the curves of Fig. 6 for different $r_{\text{DLA}}$ are very different. The curve of $r_{\text{DLA}} = 1 \text{kpc}$ shows saturation at $t > 3 \times 10^4 \text{yr}$, while none of the others has a saturated phase.

These time-dependent behaviours are also a result of the time-scale of Ly$\alpha$ photon transfer. For a halo with a given column density $N_{\text{HI}},$ or $r_0$, the time-scale of the Ly$\alpha$ photon transfer is about $\eta_{\text{max}} \sim 10r_0$, which is independent of the size $r_{\text{DLA}}$ of the halo. However, $\eta_{\text{max}} \sim 10r_0$ corresponds to a physical time $t_{\text{max}} = \eta_{\text{max}}r_{\text{DLA}}/c$, which is $r_{\text{DLA}}$-dependent. For a DLA halo with $r_{\text{DLA}} = 1 \text{kpc}$, the time-scale $t_{\text{max}} = \eta_{\text{max}}r_{\text{DLA}}/c \sim 3 \times 10^4 \text{yr}$. This is much less than $10^6 \text{yr}$, and therefore the surface brightness is saturated when $t > 3 \times 10^4 \text{yr}$. For a DLA halo with $r_{\text{DLA}} = 100 \text{kpc}$, the time-scale $t_{\text{max}} = \eta_{\text{max}}r_{\text{DLA}}/c \sim 3 \times 10^4 \text{yr}$. This is larger than the lifetime of the source, and therefore the surface brightness cannot approach a saturated state. For a DLA halo with $r_{\text{DLA}} > 100 \text{kpc}$, the time-scale $t_{\text{max}} = \eta_{\text{max}}r_{\text{DLA}}/c$ would be much larger than $10^6 \text{yr}$. The source is more like a single flash and its radiative transfer is highly time-dependent (Fig. 4).

Thus, we can conclude that for DLA haloes with a given column density, the surface brightness will be proportional to $r_{\text{DLA}}^{-2}$ and $\kappa > 2$. This result could be useful to estimate the size of a DLA halo with observed surface brightness and a model of the source.

5 LY$\alpha$ PHOTON EMISSION: IGM MODEL

5.1 Ly$\alpha$ escape

The mechanism of the escape of Ly$\alpha$ photons from an expanding opaque IGM at high redshift is different from that of DLA haloes. For the former, besides the diffusion in frequency space caused by the resonant scattering, we should also consider the frequency redshift caused by cosmic expansion. Photons will escape from the Gunn–Peterson (GP) trough once their frequency is redshifted enough.

As in Section 4, we calculate the escape coefficient using the number of escaped photons. Fig. 7 presents the time-dependent solutions of the escape coefficient $f_{\text{esc}}(t)$ for sources at redshift $1+z = 10$ surrounded by the IGM of $T = 100 \text{K}$ of the expanding Universe. The time duration of the emission of the source is taken to be $t_{\text{life}} = 10^3, 10^4, 10^5, 10^6$ and $10^7 \text{yr}$, respectively, from the bottom up (lines are coloured blue, cyan, green, yellow, orange and black, respectively). $t$ is the delayed time of an escaped Ly$\alpha$ photon with respect to the first escaped free-streaming photon from the source, measured at the redshift of the source.

The curves of Fig. 7 look very similar to those of Fig. 5. Therefore, we can explain Fig. 7 in the same way as Fig. 5, replacing the optical depth of the DLA halo with the GP optical depth (equation A5) of the expanding IGM at $(1+z) = 10$. In Fig. 7, all the short lifetime curves with $t_{\text{life}} < 10^5 \text{yr}$ reach their maximum at about the same time, $t_{\text{max}} \sim 10^6 \text{yr}$, which is the time-scale of the Ly$\alpha$ photon escape from the GP trough in an expanding IGM at $1+z = 10$. Thus, we can conclude that the escape coefficient of sources with lifetimes less than $t_{\text{max}} \sim 10^6 \text{yr}$ should always be less than 1, even without dust absorption.

The curve of the shortest lifetime ($t_{\text{life}} = 10^5$) of Fig. 7 can be thought to represent the light curve of a flash source in an IGM model, like that of Fig. 4 for the DLA model. The curve of $t_{\text{life}} = 10^3$ of Fig. 7 has a long tail. The long tail is basically a power law of $t$, and is a joint result of radiative transfer and the Hubble expansion velocity field. As mentioned in Section 4.1, for the light curve of Fig. 4, the maximum $\eta_{\text{max}}$ and variance (or the width of the light curve) $\Delta t$ are about the same. Yet, the power-law long tail means that the maximum $t_{\text{max}}$ of the curve of $t_{\text{life}} = 10^3$ of Fig. 7 is much less than the width of the curve, because the suppression of the escape coefficient is mainly a result of the width of the light curve. Therefore, the power-law long tail of Fig. 7 implies that the stable state of $f_{\text{esc}} = 1$ takes a much longer time to approach, or can never be approached within, the age of the Universe.

Although Fig. 7 shows that the escape coefficient of sources with $t_{\text{life}} \geq 10^6 \text{yr}$ is larger than those of sources with $t_{\text{life}} \leq 10^5 \text{yr}$, this does not mean that the former is easier to observe than the latter.
This point can be more clearly revealed with the time-dependent solution of surface brightness.

5.2 Surface brightness

We calculate the mean surface brightness

\[ SB(t) \equiv \frac{L}{\pi \langle r_{\text{esc}}^2 \rangle}, \]

defined as the flux divided by the averaged area of the source, where \( r_{\text{esc}} \) is the projected distance to the source on the sky from which a photon escapes. Thus, \( \pi \langle r_{\text{esc}}^2 \rangle(t) \) is the mean of the area at time \( t \) on the plane perpendicular to the line of sight. The result is presented in Fig. 8, which plots \( SB(t) \) for sources with Lyα luminosity (erg s\(^{-1}\)) and lifetime (yr) paired as \((10^{46}, 10^2), (10^{45}, 10^3), (10^{44}, 10^4), (10^{43}, 10^5), (10^{42}, 10^6)\) and \((10^{41}, 10^7)\). That is, all sources emit the same amount of \( 2 \times 10^9 \) Ly\( \alpha \) photons in their whole life-span. The sources start to emit at time \( t = 0 \).

Fig. 8 has two remarkable features. First, the four curves of \((10^{46}, 10^2), (10^{45}, 10^3), (10^{44}, 10^4)\) and \((10^{43}, 10^5)\) are almost the same. The common curve has a maximum at \( t \sim 10^6 \) yr and then starts decaying with a power law \( SB(t) \propto t^{-3} \). The time-scales of the maximum and the power-law tail are about the same as that of the light curve of Fig. 7 with \( t_{\text{flx}} = 10^3 \) yr. Therefore, for sources with lifetimes less than \( t \sim 10^6 \) yr, the surface brightness is only dependent on the total number \( N \) of Ly\( \alpha \) photons emitted from the source, regardless of their lifetime. This is because the photons should wait for about \( t \sim 10^6 \) yr before escaping from the IGM. The stored photons are locally thermalized, and therefore the information about the ‘age’ of the photons emitted at time \( t < 10^6 \) will be forgotten during the thermalization. This property is actually also valid for the sources of \((10^{42}, 10^3)\) and \((10^{41}, 10^4)\). For these two cases, the total numbers of Ly\( \alpha \) photons emitted in \( t \leq 10^6 \) yr are smaller (2 \( \times \) \( 10^{46} \) and \( 2 \times 10^{45} \), respectively). Therefore, their \( SB(t) \) at \( t \sim 10^6 \) yr is less than that of \( N = 10^{46} \) by factors of 10 and 100, respectively, in Fig. 8. The curves of \( t_{\text{flx}} = 10^4 \) and \( 10^3 \) yr become scalable from \( \sim 1 \) Myr. The \( t_{\text{flx}} = 10^6 \) yr curve is just starting to move away from the flash source solutions. The relaxation time seems to be around 0.3–1 Myr when the flash source reaches its maximum. This is in agreement with the estimate from Rybicki & Dell’Antonio (1994), where the relaxation time is \( (af^2)^{1/3} \times t_1 \sim 0.3 \) Myr for our chosen redshift \( 1/z = 10 \) and temperature \( T = 100 \) K. Fig. 8 shows various scaling relations. For example, for the flash source of \( t_{\text{flx}} = 10^3 \) yr, the asymptotic slope sets in from \( \sim 1 \) Myr as a result of radiative transfer in the Hubble expansion velocity field. For the \( t_{\text{flx}} = 10^6 \) yr curve in Fig. 8, another straight part sets in from 1 to 50 Myr even before the asymptotic slope is reached, as a result of photon emission, photon transfer and Hubble expansion. These time-scales are comparable to the possible lifetime of the sources, as previously pointed out by Higgins & Meiksin (2009).

The second feature of Fig. 8 is the monotonous decrease of \( SB(t) \) when \( t \) is large, regardless of the lifetime of the source. It is very different from all the curves of Fig. 5, which approach a stable or saturated state if the lifetime of the source is long enough. For any sources in the expanding neutral IGM, the mean surface brightness \( SB(t) \) does not approach a saturated or stable state. In other words, a time-independent solution of the surface brightness does not exist in the IGM model. This is simply because the increase in the volume of the expanding IGM, which stores Ly\( \alpha \) photons, is faster than the number of Ly\( \alpha \) photons redshifted to frequency \( x < -2 \). This feature is similar to the evolution of ionized haloes around an ultraviolet photon source in the expanding Universe. The ionized radius can never approach a stable state required by a Strömgren sphere, because the increase of the ionized radius is always lower than the comoving velocity (Shapiro & Gioumou 1987). Therefore, in Fig. 8 there is no flat section for any curve. The maximum of the surface brightness of a Ly\( \alpha \) photon source with luminosity \( L \) at \( 1+z = 10 \) scales approximately as \( \sim 10^{-21} \times (L/10^{43} \text{ erg s}^{-1}) \text{ erg s}^{-1} \text{ cm}^{-2} \) arcsec. We should emphasize that the maximum can be reached only for sources with ages equal to about \( 10^6 \) yr. The surface brightness would be less than the maximum when the source age is younger or older than \( 10^6 \) yr. That is, in terms of surface brightness, a stable source will yield a time-dependent curve.

6. LY\( \alpha \) PHOTON ABSORPTION

6.1 Time-dependent red damping wing: DLA model

A typical problem of Ly\( \alpha \) absorption at high redshift is the red damping wing, or the \( \text{H} \text{i} \) damping wing of high-redshift sources. The optical afterglow of GRBs at high redshifts has shown this feature (e.g. Totani et al. 2006; Salvaterra et al. 2009). We calculate the time dependence of the red damping wing of a DLA halo with radius and optical depth \( \tau_0 \) to be \( r = \tau_0 \times 2 \times 10^7 \). The frequency spectrum of the central photon source is assumed to be flat with a flux equal to 1. If the source starts to emit photons at time \( \eta = 0 \), photons that do not undergo scattering or collisions will escape from the halo at time \( \eta_c = 2 \times 10^7 \). The profiles of the red damping wing at times later than \( \eta_c \) are plotted in Fig. 9, in which we take \( \eta = (1 + t)\eta_c \), where \( t = 0.01, 0.5, 2, 5, 10 \) and 50.

Fig. 9 shows that the red damping wing at time \( \eta \leq 1.01 \eta_c \) can be well described by a Voigt profile \( f(\tau) = e^{-\tau^2} \), where \( \tau(\tau) \) is given by equation (1). Therefore, it is reasonable to fit the red damping wing of a GRB’s optical afterglow with a Voigt absorption profile, because the red damping wing is measured only a few hours or a few days after the GRB explosion (i.e. the time \( \eta \) is very close to \( \eta_c \)). Even when \( \eta = 1.5 \eta_c \) (the blue line in Fig. 9), the red damping wing can still be approximately fitted by a Voigt profile. However, the fitting at \( \eta = 1.5 \eta_c \) will yield a smaller \( \tau_0 \) than that of the...
The IGM model makes the problem of the red damping wing very different from that of the DLA model. If the frequency spectrum of the source is flat, the cosmic redshift will continuously provide Ly\(\alpha\) photons from the blue side of the spectrum. Because the time-scale of the W–F thermalization is much shorter than the time-scale of the cosmic expansion, the photons that move into the frequency range \(\sim v_0\) from \(v > v_0\) are quickly thermalized. Thus, a huge number of thermalized photons are stored in the frequency range \(|x| < 2\) (Roy et al. 2009b, 2010). This process marks the difference between the IGM and the DLA models, and also the difference between our IGM model and the IGM models of some other authors (Loeb & Rybicki 1999), in which the source emits only \(v_0\) photons.

Figure 10 shows the red damping wings of a source in the IGM with \(T = 100\) K at \(1 + z = 10\) with age \(t = 0, 10^2, 10^5, 5 \times 10^5\) and \(10^7\) yr. When the time is less than \(10^5\) yr, the Ly\(\alpha\) photons have not yet been significantly stored, and the profile of the red damping wing is about the same as pure absorption; this has been addressed by Miralda-Escude (1998). Once the time is larger than \(10^5\) yr, the stored photons yield a shoulder at a peak frequency \(-x \sim 500\). Fig. 10 also shows that the profile of the red damping wing does not yet seem to approach a saturated or time-independent state at \(t = 10^7\) yr.

### 7 Discussion and Conclusions

The time-dependent behaviours in the physical and frequency spaces of the Ly\(\alpha\) photons that emerge from an optically thick medium have been extensively studied using the MC method. The first conclusion is that time-dependent solutions are essential not only for strong time-dependent sources such as GRBs, but also for stable sources, especially when cosmic expansion needs to be considered. Cosmic expansion makes the radiation transfer equation not invariant with respect to the transformation of time shift. Consequently, in principle, the radiative transfer equation does not have time-independent solutions. The time-dependent behaviour is essential.

With the model of the IGM in the Hubble expanding universe, we show that the time dependence of the brightness of a stable source at high redshifts is like that of the light curve of an ‘explosion’ (i.e. in the first phase, the surface brightness is increasing and approaching a maximum, and then, in the second phase, it decays monotonically). Although our calculation is only based on a model source at redshift \((1 + z) = 10\), we believe that for various high-redshift sources there is no time-independent solution, or saturated state, of the surface brightness by IGM scattering.

The IGM model might be too simple and ideal. Many effects have not been considered, such as the effects given by the inhomogeneity of the H\(\text{I}\) density distribution, the non-uniform ionization, the bulk velocity, the wind and the turbulence of the IGM field, and the observational aperture, etc. However, one point seems to be clear – under these circumstances, stable solutions will not exist. All these effects should be studied with time-dependent solutions of the Ly\(\alpha\) photon transfer. Because some high-redshift Ly\(\alpha\) emitters have already been observed, static solutions might not be enough to understand these observations.
For the DLA model, the saturated state, or the time-dependent state, does exist. Therefore, it is reasonable to find this state directly using a time-independent solution. However, time-independent solutions cannot give us the suppression of the escape coefficient in the time range of $\eta < 10 \tau_0$, where $\tau_0$ is the optical depth of the halo. When $\tau_0$ is large, the escape coefficient will be substantially suppressed for a time range comparable to the age of the Universe. In other words, the escape coefficient is much less than 1 all the time.

The low value of the escape coefficient is often explained by dust absorption. Dust extinction is effective only when the medium is optically thick. Therefore, once we use the model of dust absorption at high redshifts, the time-dependent behaviour must be considered. A WENO time-dependent solution of the Ly$\alpha$ photon transfer in a dusty medium will be discussed in a future paper.

We have not considered the Ly$\alpha$ photons from the recombination in the ionized halo around the central source. Simply changing the luminosity of the central light source by adding the photons from the stable recombination in the halo is a poor approximation, as the time-scale of the recombination is not small. Therefore, with the DLA model, time-dependent solutions are also essential for understanding Ly$\alpha$-related phenomena.

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APPENDIX A: RADIATIVE TRANSFER OF LY\textalpha PHOTONS

With dimensionless variables, the specific intensity $I$ of the photon number is a function of $\eta$, $r$, $x$, and $\mu$. When the optical depth is large, we can take the Eddington approximation. The radiative transfer equations are (Roy et al. 2009c)

$$
\frac{\partial j}{\partial \eta} + \frac{\partial f}{\partial r} = -\phi(x; a) j + \int R(x', \eta') dx' + \gamma \frac{\partial j}{\partial x} + r^2 S
$$

(A1)

and

$$
\frac{\partial f}{\partial \eta} + \frac{1}{3} \frac{\partial j}{\partial r} - 2 \frac{j}{r} - \gamma \frac{\partial f}{\partial x} = -\phi(x; a) f,
$$

(A2)

where $j$ and $f$ are the rescaled quantities of physical intensity,

$$
\int j(\eta, r, x) = r^2 j_0 = r^2 \frac{1}{2} \int_0^{\eta} I(\eta, r, x) d\mu
$$

is the angularly averaged specific intensity and

$$
\int f(\eta, r, x) = r^2 f_p = r^2 \frac{1}{2} \int_0^{\eta} \mu I(\eta, r, x) d\mu
$$

is the flux. The mean intensity $I(\eta, r, x)$ describes the photons trapped in the halo by the resonant scattering, while the flux $f(\eta, r, x)$ describes the photons in transit. We can calculate $\gamma = 1/\tau_{cp}$ as

$$
\tau_{cp} = 4.9 \times 10^9 h^{-1} f_{hi} \left( \frac{0.25}{\Omega_\Lambda} \right)^{1/2} \left( \frac{\Omega_b h^2}{0.022} \right)^{3/2} \left( \frac{1+z}{10} \right)^{3/2}.
$$

(A3)

The term $S$ describes a constant physical photon source. Because equations (A1) and (A2) are linear with respect to $j$ and $f$, the solutions of $j$ and $f$ for sources $S(\eta, x)$ can be given by a superposition of solutions $j(\eta, r, x; \eta_0, x_0)$ and $f(\eta, r, x; \eta_0, x_0)$, which are the solutions of the source $S = \delta(\eta - \eta_0) \delta(x - x_0)$.

The resonant scattering is described by the redistribution function $R(x', \eta; x)$, which is the probability of a photon absorbed at the frequency $x'$, and re-emitted at the frequency $x$. This depends on the details of the scattering (Heneyy 1941; Hummer 1962, 1969). If we consider coherent scattering without recoil, the redistribution function with the Voigt profile is

$$
R(x', \eta; x) = \frac{1}{\pi^{1/2}} \int_{|x' - x|/\sigma} e^{-u^2}
$$

$$
\times \left[ \tan^{-1} \left( \frac{x_{\text{max}} + u}{a} \right) - \tan^{-1} \left( \frac{x_{\text{max}} - u}{a} \right) \right] du,
$$

(A4)

where $x_{\text{min}} = \min(x, x')$ and $x_{\text{max}} = \max(x, x')$. This redistribution function is normalized as

$$
\int_{-\infty}^{\infty} R(x', x) dx' = \phi(x, 0) = \pi^{-1/2} e^{-x^2}.
$$

Thus, the frequency of photons will be changed during the evolution governed by equations (A1) and (A2), while the total number of
photons is conserved. That is, the destruction processes of Lyα photons, such as the two-photon process (Spitzer & Greenstein 1951; Osterbrock 1962) and dust absorption, are ignored in equations (A1) and (A2).

APPENDIX B: MONTE CARLO SOLVER

The MC simulation starts by releasing a photon at the source placed along a random but isotropic direction. The frequency distribution of the new photon follows that of the source, either a Gaussian distribution with a Doppler core or a continuum. Once the photon enters the gas medium, the length of the free path is determined by calculating the optical depth variable \( \tau \) travelled during the free flight, following the distribution function \( e^{-\tau} \). It is then straightforward to convert from \( \tau \) to distance. The location of the next scattering is thus determined. If the place is outside the H I cloud in a halo model, or if the optical depth travelled is larger than the GP optical depth in an expanding IGM model, the photon is labelled as escaped.

At the site of the scattering, the velocity of the H I atom is generated by two steps. First, the velocity components \( v_x \) and \( v_z \) (normalized to Doppler velocity \( V_D \), and with \( z \) being the propagation direction of the photon) are generated following a Maxwellian distribution \( e^{-v'^2} \). Secondly, the velocity \( v_z \) is generated following the distribution

\[
\frac{e^{v'^2}}{(x - v_x)^2 + a^2},
\]

which is the joint requirement of the Gaussian distribution and the Lorentz profile for the rest-frame cross-section of resonant scattering. The direction of the resonantly scattered photons is assumed to be isotropic, but it can be easily adapted to other types of angular dependence. Once the direction is generated, the frequency of the outgoing photon can be calculated. Using \( x \) and \( x' \) to represent the laboratory frequency of the incoming and outgoing photons, respectively, we have

\[
x' = x - v \cos \eta + v \cos \eta \cos \theta + v \sin \theta \sin \eta \cos \phi - b(1 - \cos \theta),
\]

where \( b = (\hbar v_0)/(m_e V_D c) \) is the recoil parameter, \( \eta \) is the angle between the incoming and outgoing photons and \( \theta \) and \( \phi \) are the two spherical coordinates of the outgoing photon, where the coordinates are chosen such that the incoming photon is in the \( z \)-direction (we follow the scattering geometry and notations of Field 1959). With the new frequency and direction of the photon, we repeat the above procedures of calculating the next scattering and the determination on escape. Each photon is followed all the way along its path until it escapes.

Because the effectiveness of generating \( v_z \) determines crucially the speed of calculation, special algorithms have been proposed (ZM02). We basically follow the algorithm of ZM02 for medium to large \( x \) (0.6 \( \leq x \) \( \leq 17 \)). For smaller \( x \) (\( x < 0.6 \)), the method of plain rejection (not employing the algorithm of ZM02) is used. For very large \( x \) (\( x > 17 \)), our treatment for \( u > u_0 \) is similar to that of ZM02. However, for \( u < u_0 \), we switch the roles of the two functions, using the distribution function \( e^{-u^2} \) as the transformation method to generate \( v_z \). We then use \( 1/[(x - v_z)^2 + a^2] \) as the comparison function to reject.

Twenty million photons are used by the MC simulation for each model. Simulations are performed only for sources with a single flash of photons. In these simulations, a time stamp can be recorded for each photon at each step of the collision, and at its escape. For sources of arbitrary time dependence, a new random variable is used, representing the birth time of the photon. The time stamps can be generated by adding a photon’s birth time to the recorded time stamps from a single-flash simulation. For example, with 10\(^4\) trials of randomly generated birth time for each original photon in a single-flash simulation, we form a subgroup of 10\(^4\) photons, which have the same history of collisions but which occur at different epochs. In this way, we greatly improve the very low usage rate when the data of each recorded photon are coupled with only one birth time. Statistics over this enlarged group of photons give better continuity and smaller Poisson errors on the surveyed quantities, but do not add new physics.

APPENDIX C: GUNN–PETERSON OPTICAL DEPTH

The free path of a Lyα photon in a Hubble streaming IGM can be derived by using the GP optical depth for frequency \( x \) measured at source redshift \( z \), which is

\[
\tau(x, z) = \tau_{GP}(z) \int_{x_\Lambda}^x \phi(x') \, dx'.
\]

Here, the GP optical depth is given by equation (A3).

In MC simulations, the Hubble streaming always makes the frequency \( x \) lower and the GP optical depth smaller. Let \( x_\Lambda \) be the photon frequency immediately after the last scattering and let \( x_B \) be the photon frequency before the next scattering. The change of GP optical depth \( \delta \tau \) during the free flight is always negative and equals \( -1 \) on average. The particular value of each \( \delta \tau \) is determined by generating a random number \( \delta \tau \leq 0 \) following the distribution function \( e^{\delta \tau} \). With this \( \delta \tau \), we can derive the new frequency \( x_B \) by solving the equation

\[
\delta \tau = \tau_{GP}(z) \int_{x_\Lambda}^{x_B} \phi(x') \, dx'.
\]

By compiling a large data table of \( \tau(x, z) \) and using linear interpolation, \( x_B \) can be quickly solved.

The free length \( l \) before the next scattering can be found by considering that the frequency change is caused by the Doppler effect of Hubble motion:

\[
x_B - x_\Lambda = -\frac{H(z) l}{c} \frac{v_0}{\Delta V_D}.
\]

Here, the Hubble constant at redshift \( z \) is

\[
H(z) = H_0 \sqrt{\Omega_M (1 + z)^3 + \Omega_K (1 + z)^2 + \Omega_\Lambda} \approx H_0 \sqrt{\Omega_M (1 + z)^3},
\]

where all \( \Omega_M \), \( \Omega_K \) and \( \Omega_\Lambda \) refer to the present values.

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