EVEN (\(\bar{s}, \bar{t}\))-CORE PARTITIONS AND SELF-ASSOCIATE CHARACTERS OF \(\tilde{S}_n\).

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Abstract. A partition is a \(\bar{s}\)-core if it is the result of removing all of the \(s\)-bars from a partition. We extend a method of Olsson and Bessenrodt to determine the number of even partitions that are simultaneously \(\bar{s}\)-core and \(\bar{t}\)-core. When \(p\) and \(q\) are distinct primes, this also determines the number of self-associate characters of \(\tilde{S}_n\) that are simultaneously defect 0 for \(p\) and \(q\).

1. Introduction

Navarro and Willems [5] were the first to investigate the question of when a block was simultaneously a \(p\)-block and a \(q\)-block for a group \(G\) and odd primes \(p\) and \(q\). In particular, they conjectured that if a \(p\)-block and \(q\)-block coincided, then that block consists of a single character. While Bessenrodt disproved this conjecture, Olsson and Stanton [6] showed that it was true in the case where \(G\) was the symmetric group \(S_n\). Olsson and Bessenrodt [2] then showed that Navarro’s conjecture was true in the case where \(G\) was the spin symmetric group \(\tilde{S}_n\). There has been considerable work enumerating characters with defect 0 in \(p\) and \(q\), as well as enumerating the subclass of self-associate characters. Anderson [1] showed that the number of characters of \(S_n\) with defect 0 in \(p\) and \(q\) is \(\frac{1}{p+q} \left(\frac{p+q}{p}\right)\), while Ford, Mai, and Sze [4] showed that the number of self-associate characters of \(S_n\) with defect 0 in \(p\) and \(q\) is \(\left(\frac{p}{s} + \frac{q}{t}\right)\). On the other hand, Bessenrodt and Olsson [2] showed that the number of spin characters in \(\tilde{S}_n\) with defect 0 for \(p\) and \(q\) was also \(\left(\frac{p}{s} + \frac{q}{t}\right)\). However, they left the problem of enumerating the number of self-associate spin characters of \(\tilde{S}_n\) with defect 0 for \(p\) and \(q\) unresolved.

Much of this work has concentrated on the partitions that correspond combinatorially to \(p\)-blocks for \(S_n\) and \(\tilde{S}_n\). In the case of \(S_n\), they are the \(p\)-core partitions; in the case of \(\tilde{S}_n\), they are the \(\tilde{p}\)-core partitions. Furthermore, a character of \(S_n\) is self-associate if and only if the corresponding partition is self-conjugate, while a spin character of \(\tilde{S}_n\) is self-associate if and only if the corresponding partition is even. In this paper, we will enumerate the self-associate spin characters in \(\tilde{S}_n\) with defect 0 for \(p\) and \(q\) by counting the number of even \((\bar{s}, \bar{t})\)-cores for relatively prime odd integers \(s\) and \(t\) greater than 1.

Theorem 1.1. Suppose \(s, t > 1\) are relatively prime odd positive integers.

- If \(s, t \equiv 3 \pmod{4}\), the number of even \((\bar{s}, \bar{t})\)-core partitions is

\[
\frac{1}{2} \left(\frac{s-1}{2} + \frac{t-1}{2}\right) - \frac{1}{2}.
\]
• Otherwise, the number of even \((\bar{s}, \bar{t})\)-core partitions is
\[
\frac{1}{2} \left( \left( \frac{\bar{s}-1}{2} + \frac{\bar{t}-1}{2} \right) + (-1)^{(s-1)(t-1)/8} \left( \frac{8}{7} \right) \left( \left\lfloor \frac{s}{4} \right\rfloor + \left\lfloor \frac{t}{4} \right\rfloor \right) \right).
\]

where \((\frac{7}{s})\) denotes the Jacobi symbol.

Our proof will use the bijection between spin characters of defect 0 for \(p\) and \(q\) in \(\tilde{S}_n\) and monotone \((\frac{p-1}{2}, \frac{q-1}{2})\) paths. In particular, self-associate spin characters will correspond to paths of a certain “parity”.

One immediate consequence is the following corollary:

Corollary 1.2. Suppose \(p\) and \(q\) are distinct odd primes.

• If \(p, q \equiv 3 \pmod{4}\), the number of self-associate spin characters of \(\tilde{S}_n\) which are simultaneously of defect 0 for \(p\) and \(q\) is
\[
\frac{1}{2} \left( \left( \frac{p-1}{2} + \frac{q-1}{2} \right) \right).
\]

• Otherwise, the number of self-associate spin characters of \(\tilde{S}_n\) which are simultaneously of defect 0 for \(p\) and \(q\) is
\[
\frac{1}{2} \left( \left( \frac{p-1}{2} + \frac{q-1}{2} \right) + (-1)^{(p-1)(q-1)/8} \left( \frac{p}{q} \right) \left( \left\lfloor \frac{p}{4} \right\rfloor + \left\lfloor \frac{q}{4} \right\rfloor \right) \right).
\]

In addition, spin characters of \(\tilde{S}_n\) split on restriction to \(\tilde{A}_n\) if and only if the spin character is self-associate. Thus we also immediately get the number of spin characters of \(\tilde{A}_n\) that are of defect 0 for \(p\) and \(q\).

Corollary 1.3. Suppose \(p\) and \(q\) are distinct odd primes.

• If \(p, q \equiv 3 \pmod{4}\), the number of spin characters of \(\tilde{A}_n\) which are simultaneously of defect 0 for \(p\) and \(q\) is
\[
\frac{3}{2} \left( \left( \frac{p-1}{2} + \frac{q-1}{2} \right) \right).
\]

• Otherwise, the number of spin characters of \(\tilde{A}_n\) which are simultaneously of defect 0 for \(p\) and \(q\) is
\[
\frac{3}{2} \left( \left( \frac{p-1}{2} + \frac{q-1}{2} \right) \right) + \frac{1}{2} \left( -1 \right)^{(p-1)(q-1)/8} \left( \frac{p}{q} \right) \left( \left\lfloor \frac{p}{4} \right\rfloor + \left\lfloor \frac{q}{4} \right\rfloor \right).
\]

Proof. By the result of Bessenrodt and Olsson, there are \(\left( \frac{p-1}{2} + \frac{q-1}{2} \right)\) spin characters of \(\tilde{S}_n\) that are defect 0 for \(p\) and \(q\). The corollary then follows immediately from the fact that the number of spin characters of \(\tilde{A}_n\) (with defect 0 for \(p\) and \(q\)) is equal to the number of spin characters of \(\tilde{S}_n\) (with defect 0 for \(p\) and \(q\)) plus the number of spin characters of \(\tilde{S}_n\) (with defect 0 for \(p\) and \(q\)) that split upon restriction to \(\tilde{A}_n\). \qed
Recall the definition of an $\bar{s}$-core partition and a $(\bar{s}, \bar{t})$-core partition.

**Definition** A bar partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition satisfying $\lambda_1 > \cdots > \lambda_k$.

Note that since the constituents of $\lambda$ are distinct, we can talk about positive integers $a$ either being in $\lambda$ or not being in $\lambda$, much like a set.

The definition of a $\bar{s}$-core partition is a bit involved, and for the technical details, see [3]. However, we will use the following (equivalent) combinatorial specification.

**Proposition 2.1.** A bar partition $\lambda$ is a $\bar{s}$-core partition if and only if the following three constraints hold:

- No part in $\lambda$ is divisible by $s$.
- For all $a$ in $\lambda$ with $a > s$, $a - s \notin \lambda$.
- For all $a$ in $\lambda$ with $1 \leq a \leq s - 1$, $s - a \notin \lambda$.

**Definition** A partition $\lambda$ is a $(\bar{s}, \bar{t})$-core partition if it is both a $\bar{s}$-core partition and a $\bar{t}$-core partition.

**Definition** A partition $\lambda$ is said to be even if an even number of its parts are even.

Note that a partition $\lambda$ being even is equivalent to $|\lambda| - \ell_\lambda$ being even, where $|\lambda|$ and $\ell_\lambda$ denote the size and length of $\lambda$, respectively.

**Proposition 2.2.** Associate classes of spin characters of $\tilde{S}_n$ correspond to bar partitions of $n$. In addition,

- Associate classes classes with defect 0 for $p$ correspond to $\bar{p}$-core partitions.
- Self-associate spin characters of $\tilde{S}_n$ correspond to even partitions.

For a proof (and more background on the theory of characters of $\tilde{S}_n$), see Hoffman, §10. [3]

### 3. Yin-Yang Diagrams

From now on we assume that $s, t > 1$ are relatively prime odd integers. Set $m = [s/2], n = [t/2], a = [s/4], b = [t/4]$.

The Yin-Yang diagram of $(s, t)$ can be represented as an $m \times n$ grid of integers. If the lower left corner of the grid is $(0, 0)$ and upper right corner is $(n, m)$, then for each $1 \leq x \leq n$ and $1 \leq y \leq m$, we place the value $|sx - ty|$ in the square whose upper right corner is $(x, y)$. The Yin half of the diagram corresponds to the region of the Yin-Yang diagram corresponding to ordered pairs $(x, y)$ where $sx - ty < 0$; the Yang half corresponds to the portion of the diagram where $sx - ty > 0$.
For a path $P$ from $(0, 0)$ to $(n, m)$, let $L_P$ be the region of the Yin-Yang diagram bounded above by $P$. Then there is a map from paths to bar partitions, given by

$$P \mapsto L_P \triangle L_{P_0},$$

where $P_0$ denotes the path from $(0, 0)$ to $(n, m)$ separating the Yin and Yang regions and $\Delta(S, T) = (S \cup T) \setminus (S \cap T)$ denotes the symmetric difference of $S$ and $T$. (Strictly speaking, this gives a map into sets, but a bar partition $\lambda$ can be represented uniquely as a set because all of its parts are distinct.)

**Lemma 3.1.** The map given in (1) maps into $(\bar{s}, \bar{t})$-core partitions, and gives a bijection between $(\bar{s}, \bar{t})$-cores and monotonic paths in the Yin-Yang diagram.

**Proof.** See [2].

**Definition.** For $S$ a subset of the positive integers, let $E(S) = S \cap 2\mathbb{Z}$ (i.e. the subset consisting of all of the even integers of $S$). Define the parity of a path $P$ to be the parity of $E(L_P)$ (i.e. a path is even if $E(L_P)$ is even and odd if $E(L_P)$ is odd.)

**Lemma 3.2.** Even $(\bar{s}, \bar{t})$-core partitions correspond exactly to paths $P$ that have the same parity as $P_0$.

**Proof.** We have

$$|E(L_P \triangle L_{P_0})| = |E(L_P) \triangle E(L_{P_0})| = |E(L_P)| + |E(L_{P_0})| - 2|E(L_P) \cap E(L_{P_0})|$$

so $|E(L_P \triangle L_{P_0})|$ is even if and only if $|E(L_P)| \equiv |E(L_{P_0})| \pmod{2}$, as desired.

**Lemma 3.3.** Suppose $s \equiv 1 \pmod{4}$. Then $(-1)^{|E(L_{P_0})|} = \left(\frac{s}{2}\right)$, where $\left(\frac{s}{2}\right)$ denotes the Jacobi symbol.

**Proof.** Let $c_j$ be the number of even integers in the $j$th column of the Yang half of the diagram. Once again, we will associate each square in the Yin-Yang diagram with the coordinates of its upper-right corner. Then for each $j$, the squares in the Yang half of the diagram with $x = j$ correspond to those satisfying $1 \leq y \leq \left\lfloor \frac{sx}{2} \right\rfloor$. If $j$ is odd, then the even squares correspond to squares whose $y$ coordinate is odd, so

$$c_j = \left\lfloor \frac{\frac{s}{2} + 1}{2} \right\rfloor = \left\lfloor \frac{s}{2} + 1 \right\rfloor.$$

On the other hand, if $j$ is even, then the even squares in the $j$th column correspond to squares whose $y$ coordinate is even, so in this case,

$$c_j = \left\lfloor \frac{\frac{s}{2}}{2} \right\rfloor = \left\lfloor \frac{s}{4} \right\rfloor.$$

In either case, the number of even numbers in the $j$th column is

$$c_j = \left\lfloor \frac{\frac{s}{2} + j}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor = \left\lfloor \frac{s}{2} + 1 \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor.$$

Thus

$$|E(L_{P_0})| = \sum_{j=1}^{\frac{s}{4}} c_j = \sum_{j=1}^{\frac{s}{4}} \left\lfloor \frac{s}{2} + 1 \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor.$$
Since \( t \) is odd, we have

\[
\left\lfloor \frac{j}{2} \right\rfloor + \left\lfloor \frac{t-j}{2} \right\rfloor = \frac{t-1}{2},
\]

and for all \( j \) not divisibly by \( t \), we have.

\[
\left\lfloor \frac{(\frac{s}{t} + 1)j}{2} \right\rfloor + \left\lfloor \frac{(\frac{s}{t} + 1)(t-j)}{2} \right\rfloor = \frac{s+t}{2} - 1.
\]

Thus

\[
\left\lfloor \frac{(\frac{s}{t} + 1)j}{2} \right\rfloor - \left\lfloor \frac{j}{2} \right\rfloor + \left\lfloor \frac{(\frac{s}{t} + 1)(t-j)}{2} \right\rfloor - \left\lfloor \frac{t-j}{2} \right\rfloor = \frac{s-1}{2}.
\]

Since \( s \equiv 1 \pmod{4} \), this means that the right hand side of (5) is even, which means that in (2), we can replace the \( j = 1, 2, \ldots, \frac{t-1}{2} \) with \( j = 2, 4, \ldots, t-1 \). Indeed, for every \( 1 \leq j \leq \frac{t-1}{2} \), we can replace \( j \) with the unique even residue in \( \{j, t-j\} \) without changing the parity of the summation. Substituting this back into (2) gives

\[
E(s, t) = \sum_{k=1}^{\frac{t-1}{2}} \left\lfloor \frac{(\frac{s}{t} + 1)(2k)}{2} \right\rfloor - \left\lfloor \frac{(2k)}{2} \right\rfloor = \sum_{k=1}^{\frac{t-1}{2}} \left\lfloor \frac{sk}{t} \right\rfloor \pmod{2}.
\]

However, \( [7] \) showed that for odd integers \( s, t \),

\[
\left(\frac{s}{t}\right) = (-1)^{\sum_{i=1}^{(t-1)/2} \left\lfloor \frac{is}{t} \right\rfloor}.
\]

Thus \((-1)^{E(s, t)} = \left(\frac{s}{t}\right)\), as desired. \( \Box \)

4. COUNTING EVEN AND ODD PATHS

In this section, suppose \( x, y \) are arbitrary positive integers. For each unit square in the plane, color the square with upper right corner \((i, j)\) red if \( i + j \) is even; otherwise, color the square blue. In this case, define a path from \((0, 0)\) to \((x, y)\) to be \textit{even} if the number of red squares between the path and the \( x \)-axis is even and \textit{odd} otherwise. Note that in the case of the Yin-Yang diagram, red squares correspond to squares that contain an even integer, so the definitions agree.

**Lemma 4.1.** Let \( D(x, y) \) be the number of even paths from \((0, 0)\) to \((x, y)\) minus the number of odd paths from \((0, 0)\) to \((x, y)\).

- If \( x = 2k, y = 2l \), then \( D(x, y) = \binom{k+l}{k} \).
- If \( x = 2k, y = 2l + 1 \), then \( D(x, y) = \binom{k+l}{k} \).
- If \( x = 2k + 1, y = 2l \), then \( D(x, y) = (-1)^{k}(\binom{k+l}{k}) \).
- If \( x = 2k + 1, y = 2l + 1 \), then \( D(x, y) = 0 \).

**Proof.** We induct on \( \min(x, y) \). Clearly \( D(x, y) = 1 \) if \( x = 0 \) or \( y = 0 \), and this agrees with the formulas above.

Otherwise, we have \( D(x, y) = D(x, y-1) + (-1)^{c_{x}(y)}D(x-1, y) \), where \( c_{x}(y) \) is the number of red squares in the rectangle \([x-1, x] \times [0, y] \). As before, we have that \( c_{x}(y) = \left\lfloor \frac{xy}{2} \right\rfloor - \left\lfloor \frac{x}{2} \right\rfloor \). Note that \( c_{x}(y) = \left\lfloor \frac{y}{2} \right\rfloor \) unless \( x \) and \( y \) are both odd, in which case it is equal to \( \left\lfloor \frac{y}{2} \right\rfloor + 1 \). There are four cases for the parities of \( x \) and \( y \):
\begin{itemize}
  \item $x = 2k, y = 2l$.
  \[
  D(x, y) = D(2k, 2l - 1) + (-1)^l D(2k - 1, 2l)
  = \binom{k + l - 1}{l - 1} + (-1)^l \cdot \binom{(k - 1) + l}{l}
  = \binom{k + l}{l}
  \]

  \item $x = 2k, y = 2l + 1$.
  \[
  D(x, y) = D(2k, 2l) + (-1)^l D(2k - 1, 2l + 1)
  = \binom{k + l}{l} + 0
  = \binom{k + 1}{l}
  \]

  \item $x = 2k + 1, y = 2l$.
  \[
  D(x, y) = D(2k + 1, 2l - 1) + (-1)^l D(2k, 2l)
  = 0 + (-1)^l \binom{k + l}{l}
  = (-1)^l \binom{k + 1}{l}
  \]

  \item $x = 2k + 1, y = 2l + 1$.
  \[
  D(x, y) = D(2k + 1, 2l) + (-1)^{k+l+1-k} D(2k, 2l + 1)
  = (-1)^l \binom{k + l}{l} + (-1)^{l+1} \binom{k + l}{l}
  = 0
  \]
\end{itemize}

In particular, we have that if $y$ is even, then $D(x, y) = (-1)^{xy/2} \binom{k + l}{l}$.

5. **Even $(\tilde{s}, \tilde{t})$-cores**

We are now ready to enumerate the even $(\tilde{s}, \tilde{t})$-core partitions.

**Theorem 5.1.** Let $s, t > 1$ be relatively prime odd integers, and set $m = \lfloor s/2 \rfloor, n = \lfloor t/2 \rfloor, a = \lfloor s/4 \rfloor, b = \lfloor t/4 \rfloor$. In addition, let $\left( \frac{s}{n} \right)$ denote the Jacobi symbol. Then the number of even $(\tilde{s}, \tilde{t})$-core partitions is

\begin{itemize}
  \item $\frac{1}{2} \binom{m+n}{n}$ if $s, t \equiv 3(4)$,
  \item $\frac{1}{2} \left( \binom{m+n}{n} + (-1)^{mn/2} \left( \frac{\tilde{s}}{\tilde{t}} \right) \left( \frac{\tilde{a}+\tilde{b}}{\tilde{b}} \right) \right)$ otherwise.
\end{itemize}

When $s$ and $t$ are primes, this gives the number of self-associate spin characters in $\tilde{S}_n$ that are defect 0 for both $s$ and $t$.

**Proof.** The number of even paths in the $(s, t)$ Yin-Yang diagram minus the number of odd paths is equal to $D(n, m)$ (as given in Lemma 4.1). Thus the number of even $(\tilde{s}, \tilde{t})$-core partitions minus the number of odd $(\tilde{s}, \tilde{t})$-core partitions is equal
to \((-1)^{\left| E(L_{R_0}) \right|} D(n, m)\). Since the overall number of \((\bar{s}, \bar{t})\)-core partitions is \(\binom{m+n}{n}\), this means that the number of even \((\bar{s}, \bar{t})\)-core partitions is

\[
\frac{1}{2} \left( \binom{m+n}{n} + (-1)^{\left| E(L_{R_0}) \right|} \cdot D(n, m) \right).
\]

If \(s, t\) are both 3 (mod 4), then \(m\) and \(n\) are both odd. Thus \(D(n, m) = 0\), so the number of even \((\bar{s}, \bar{t})\)-core partitions is

\[(8) \quad \frac{1}{2} \left( \binom{m+n}{n} \right).\]

Otherwise, suppose without loss of generality that \(s \equiv 1 \pmod{4}\). Then \(m\) is even, so by Lemma 4.1

\[D(n, m) = (-1)^{mn/2} \left( \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \right) = (-1)^{mn/2} \left( \frac{a+b}{b} \right).\]

In addition, by Lemma 3.3

\[(-1)^{\left| E(L_{R_0}) \right|} = \left( \frac{s}{t} \right).\]

Thus the number of even \((\bar{s}, \bar{t})\)-core partitions in this case is

\[(9) \quad \frac{1}{2} \left( \binom{m+n}{n} + (-1)^{mn/2} \left( \frac{s}{t} \right) \left( \frac{a+b}{b} \right) \right).\]

Note that by quadratic reciprocity, this formula is symmetric in \(s\) and \(t\), so \[(9)\] does in fact give the number of even \((\bar{s}, \bar{t})\)-core partitions as long as \(s\) and \(t\) are not both 3 (mod 4).

\[\square\]

6. Future Direction

Bessenrodt and Olsson [2] showed that if \(p < q\) are odd primes, then the Yin half of the \((p, q)\) Yin-Yang diagram is in some sense the “largest” \((\bar{p}, \bar{q})\)-core partition, and thus the maximum \(n\) for which there exists an associate class of spin characters in \(\tilde{S}_n\) with defect 0 for \(p\) and \(q\) is just the sum of the numbers in the Yin diagram. More precisely, they showed that any \((\bar{p}, \bar{q})\)-core can be contained in the partition represented by the Yin half of the diagram. We can ask the same question for even \((\bar{p}, \bar{q})\)-core partitions.

**Conjecture.** For any pair of distinct odd primes \(p, q\), there exists a \((\bar{p}, \bar{q})\)-core partition \(\lambda\) such that any even \((\bar{p}, \bar{q})\)-core partition is contained in \(\lambda\).

Regardless of whether the conjecture is true, we can also ask the following question:

**Question.** What is the maximum \(n\) for which a self-associate character in \(\tilde{S}_n\) of defect 0 for \(p\) and \(q\) exists?

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References

[1] J. Anderson. Partitions which are simultaneously $t_1$- and $t_2$-core. Discrete Math., 248(13):237 – 243, 2002.

[2] C. Bessenrodt and J. B. Olsson. Spin block inclusions. J. Algebra, 306(1):3 – 16, 2006.

[3] P. N. Hoffman and J. F. Humphreys. Projective representations of the symmetric groups : $Q$-functions and shifted tableaux. Clarendon Press Oxford University Press, Oxford New York, 1992.

[4] B. Ford, H. Mai and L. Sze. Self-conjugate simultaneous $p$- and $q$-core partitions and blocks of $A_n$. J. Number Th., 129(4):858 – 865, 2009.

[5] G. Navarro and W. Willems. When is a $p$-block a $q$-block? Proc. Ame. Math. Soc., 125(6):1589–1591, 1997.

[6] J.B. Olsson and D. Stanton. Block inclusions and cores of partitions. Aequationes Math., 74(1-2):90–110, 2007.

[7] B. D. Tangedal. Eisenstein’s lemma and quadratic reciprocity for Jacobi symbols. Math. Mag., 73(2):130–134, 2000.