I. INTRODUCTION

More than ten years after the supernova observations \[1,2\] led to the general acceptance that the expansion of the Universe is accelerating, we are still far from a consensus on what is responsible for the acceleration. Although there is a name for the phenomenon, dark energy, there is as of yet no convincing physical explanation.

For this reason it is imperative to learn as much as possible from observations. The fundamental observable considered so far is the equation of state parameter \( w \) which connects the average energy density \( \rho(t) \) and the average pressure \( p(t) \) through \( p(t) = w(t) \rho(t) \). This parameter characterizes the background expansion rate and the distances. While some models make specific predictions, like \( w = -1 \) for a cosmological constant, there are whole families of models that can lead to any desired evolution of \( w(t) \) (possibly with some weak constraints like \( w \geq -1 \)). Typical examples include scalar-field models like Quintessence or K-essence, or generalized gravity theories like scalar-tensor and \( f(R) \) theories. These theories cannot be ruled out based on a measurement of \( w \) alone. However, in general their perturbations evolve differently\[3,4,5,6,7,8\]. The dark energy perturbations are therefore like a fingerprint of these models and, if measured, will allow to discriminate much more precisely between competing theories, and hopefully will allow to shed some light on the physical nature of whatever accelerates the expansion of the Universe.

For many models, the behavior of perturbations at the linear level can be described in terms of those of a fluid with a certain sound speed. This is the case for Quintessence (canonical scalar-field) models for which the sound speed is \( c_s^2 = 1 \), and for many K-essence models where the sound speed is arbitrary. In this paper we concentrate on this class of models, and assume in addition that both \( c_s^2 \) and \( w \) are constant. This latter assumption is usually violated, but the results should nonetheless allow insight into the behavior of the perturbations. For models where the quantities vary only slowly with time, we expect the results to still hold in an averaged sense, due to the indirect nature of most observations. The big advantage of making these assumptions is that it allows us to solve the perturbations analytically under the additional condition of matter domination, leading to surprisingly simple results. In addition, when expressed in terms of the change of the gravitational potential relative to the case without dark energy perturbations then the simple formulae turn out to be a surprisingly good approximation until today. Our results should be seen in this context, as it is of course always possible to solve the perturbation equations numerically. However, analytical results allow a much better insight and also an easy way to see how the behavior changes as a function of the parameters.

Experiments tell us that the dark energy is at most very weakly coupled to the things that we can observe directly, like galaxies and the cosmic microwave background (CMB). It is therefore not only necessary to determine the dark energy perturbations but also to connect them to actual observables. We use our simple analytical results and make a small step in that direction, trying to establish the impact of the dark energy perturbations onto the matter power spectrum and the integrated Sachs-Wolfe (ISW) effect in the CMB. Our treatment here is far from complete and only an initial attempt to deal with the observational impact of the dark energy perturbations. In a follow-up publication \[10\] we will pay particular attention to the question whether it
is possible to observe the class of perturbations that we study here, using probes of lensing and of the matter power spectrum. In detail, the paper is organized as follows. We begin with a short discussion of the perturbation equations to set the scene and to define our variables, and remind the reader of the solution for the matter perturbations during matter domination. We then derive simplified solutions for the dark energy perturbations during matter domination. Those expressions are a good fit in their respective domain of validity, but once the dark energy starts to dominate they deviate from the numerical solution. We then show that the function $Q(k, t)$ used in [3] is well described by our solutions, even after matter domination ends. This is the main result of the paper. We finally describe our solutions, even after matter domination. We then derive simplified solutions for the dark energy perturbations during matter domination. We consider only spatially flat universes, and observables are independent of the dark energy perturbations affects the ISW effect most strongly, before concluding. The appendices give more details on how we compared our analytical formulae to numerical results and on the evolution of the decaying modes that were neglected.

II. FIRST ORDER PERTURBATIONS IN MATTER AND DARK ENERGY

Throughout this paper, we will use overdots to denote derivatives with respect to conformal time $\tau$, related to cosmic time by $dt = ad\tau$. We will denote the physical Hubble parameter with $H$ and with $\mathcal{H}$ the conformal Hubble parameter. We consider only spatially flat universes, and our metric convention is defined by the line element,

$$ds^2 = a^2 \left[ -(1 + 2\psi) d\tau^2 + (1 - 2\phi) dx_i dx^i \right].$$ (1)

We are therefore working in the Newtonian or longitudinal gauge, which influences the resulting perturbations especially on scales larger than the Hubble horizon, $k \lesssim aH$. On much smaller scales the choice of gauge is less important, and observables are independent of the gauge choice.

The perturbation equations for a fluid with equation of state parameter $w = p/\rho$ are [11] [12]

$$\delta' = 3(1 + w)\phi' - \frac{V}{Ha^2} - 3 \frac{1}{a} \left( \frac{\delta p}{\rho} - w\delta \right)$$ (2)

$$V' = - (1 - 3w) \frac{V}{a} + \frac{k^2 \delta p}{Ha^2 \rho} + (1 + w) \frac{k^2}{Ha^2} \psi$$ (3)

where $\delta = \delta p/\rho$ is the density contrast, $V = ik_j T^j_i/\rho$ is the scalar velocity perturbation (see also [13]) and the prime means the derivative with respect to the scale factor $a$. In this paper we will look only at fluids with vanishing anisotropic stress, $\sigma = 0$, so that there is a single gravitational potential,

$$k^2 \phi = -4\pi G a^2 \sum_j \rho_j \left( \delta_j + \frac{3aH}{k^2} V_j \right)$$ (4)

and $\psi = \phi$. The sum on the right hand side runs over all fluids. For our purposes, we will assume the presence of a matter fluid with $w = \delta p = 0$ as well as a dark energy fluid, parametrised by a constant $w$ and a sound speed $c_s^2$ which determines the pressure perturbation through

$$\delta p = c_s^2 \rho \delta + \frac{3aH \left( c_s^2 - c_a^2 \right)}{k^2} \rho V.$$ (5)

where $c_a^2 = \dot{p}/\rho$ is called the adiabatic sound speed of the fluid. This is not always a good parametrization, for example for models that cross the phantom barrier $w = -1$ [13] or when mimicking modified gravity models [7] (in which case also generically $\sigma \neq 0$). However, it covers a wide class of models, for example canonical scalar fields (Quintessence, in which case $c_s^2 = 1$) and other scalar-field models (like K-essence, which allows for $c_s^2 \neq 1$). As we only consider models with constant $w$, we have that $c_a^2 = w - \frac{\psi}{3H(1 + w)} = w$. The perturbation equations (2) and (4) become in this case

$$\delta' = - \frac{V}{Ha^2} \left( 1 + \frac{9a^2H^2 \left( c_s^2 - w \right)}{k^2} \right)$$

$$- \frac{3}{a} (c_s^2 - w) \delta + 3 (1 + w) \phi'$$ (6)

$$V' = - (1 - 3c_s^2) \frac{V}{a} + \frac{k^2 c_s^2}{Ha^2} \delta + (1 + w) \frac{k^2}{Ha^2} \phi$$ (7)

In general it is difficult to solve these equations. To simplify the problem, we will assume that the universe is matter dominated. This means that the Hubble expansion rate is given by

$$H^2 = H_0^2 \Omega_m a^{-3} = \frac{8\pi G}{3} \rho_m$$ (8)

and that only the matter perturbations contribute to the gravitational potential in Eq. [14]. We can therefore first solve for the matter perturbations alone, without considering the dark energy, and then use the resulting gravitational potential as an external source in the equations for the dark energy perturbations. Of course this approximation, and even more the assumption of a matter dominated background evolution, will change the results, and we will need to study what happens as the assumptions break down.

The perturbation equations for the matter perturba-
tions then are
\begin{align}
  \delta_m' &= \frac{V_m}{H a^2} + 3\phi' \\
  V_m' &= \frac{V_m}{a} + k^2 \phi \\
  k^2 \phi &= \frac{3\Omega_m}{2a} \left( \delta_m + \frac{3aH}{k^2} V_m \right)
\end{align}
(9) (10) (11)
As is well known, a solution to this set of equations is
\begin{align}
  \delta_m &= \delta_0 \left( a + 3 \frac{H_0^2 \Omega_m}{k^2} \right) = \delta_0 a \left( 1 + 3 \frac{H_0^2 a^2}{k^2} \right) \\
  V_m &= -\delta_0 H_0 \sqrt{\Omega_m} a^{1/2} \\
  k^2 \phi &= \frac{3}{2} \delta_0 H_0^2 \Omega_m
\end{align}
(12) (13) (14)
where the constant \( \delta_0 \) sets the overall scale (since the equations are linear). This can be verified simply by inserting them into the differential equations. The value of \( \delta_0 \) is set by the initial conditions and is in general a function of \( k \). Additionally the growth of the matter perturbations is suppressed during radiation domination, so that \( \delta_0(k) \) is smaller than expected for the scales that enter the horizon during radiation domination. However, the same happens for the dark energy perturbations so that the effects cancel out when comparing the perturbations in matter and dark energy. For more details see Appendix A which discusses how we compare the analytical results to CAMB [14].

We see that the gravitational potential is constant as a function of time on all scales. The matter perturbations grow linearly with \( a \) on small scales, and they are constant on super-horizon scales (but their behavior on those large scales depends on the gauge choice).

III. SOLUTIONS FOR THE DARK ENERGY PERTURBATIONS DURING MATTER DOMINATION

We will now use the constant \( k^2 \phi \) of the last section to look for solutions to the general perturbation equations. We will study them in different limits, and then compare the results with full numerical solutions. See also [15] for results using different approximations.

Generically, we expect at least three regimes with different behavior of the perturbations:

- Perturbations larger than the causal horizon
- Perturbations smaller than the causal horizon, but larger than the sound horizon
- Perturbations smaller than the sound horizon

We start by looking at perturbations larger than the sound horizon, \( k \ll aH/c_s \). In this case, we neglect all terms containing the sound speed in Eq. (7), effectively setting \( c_s^2 = 0 \). The solution for the velocity perturbation is (neglecting a decaying solution \( \propto 1/a \))
\[ V = -\delta_0 (1 + w) H_0 \sqrt{\Omega_m} a^{1/2}. \] (15)
Up to the prefactor \( (1 + w) \) this is the same as for the matter velocity perturbations. We find that this expression is valid on scales larger than the sound horizon even if the sound speed is non-zero.

We can now insert this solution for the dark energy velocity perturbation into Eq. (11). Again setting \( c_s^2 = 0 \) we find the solution
\[ \delta = \delta_0 (1 + w) \left( \frac{a}{1 - 3w} + \frac{3H_0^2 \Omega_m}{k^2} \right) \] (16)
where we neglected a term proportional to \( a^3 w \) which is decaying as long as \( w \) is negative. Not surprisingly, also this solution becomes equal to the one for matter perturbations for \( w \to 0 \). Relative to the matter perturbations the dark energy perturbations are suppressed by the factor \( (1 + w) \). This factor is necessarily always there, as the gravitational potential terms contain it. It can be thought of as modulating the strength of the coupling of the dark energy perturbations to the perturbations in the metric. For \( w = -1 \) the dark energy perturbations are completely decoupled (in the sense that they do not feel metric perturbations – but they can still produce them if the dark energy perturbations are not zero).

Just like Eq. (12) the solution is composed of two terms, where the second one dominates on scales larger

![FIG. 1: The figure shows the behavior of the variable \( \delta_{DE} \). The black dot-dashed line is the numerical solution with \( c_s^2 = 0.1 \) and \( w = -0.8 \) for the mode \( k = 200H_0 \). The red solid line is the approximation on scales above the sound horizon, Eq. (16) and the blue dashed line is the approximation to the scales below the sound horizon, Eq. (17). The two vertical lines give the scale factor at which the mode enters the Hubble horizon (left line) and the sound horizon (right line). The numerical solution shows how the perturbations decay at late times when matter domination ends, but radiation was omitted from the numerical calculation to allow for a longer dynamic range in \( a \) to illustrate the different regimes.](image-url)
than the horizon \((k < aH)\) and the first on smaller scales (but still larger than the sound horizon). The super-horizon part of the solution is the same as for matter apart from the overall \((1 + w)\) factor, while the sub-horizon solution is additionally suppressed by a factor of \(1/(1 - 3w)\) relative to the matter perturbations.

Although these factors can suppress the dark energy perturbations significantly compared to the dark matter perturbations, especially if \(w\) is close to \(-1\), the existence of a sound horizon is even more important. Inside the causal horizon, the dark matter perturbations grow linearly with \(a\) (until the perturbations become non-linear). The dark energy perturbations on the other hand will eventually encounter their sound horizon if \(c_s^2 > 0\). Once inside the sound horizon, they will stop growing. This means that the dark energy perturbation spectrum is cut off on small scales.

To get a solution on small scales, \(k \gg aH/c_s\), we start again with the equation for the velocity perturbation. However, we expect the two terms with \(k^2\) to cancel to a high degree to avoid large velocity perturbations, or in other words

\[
\delta = - \frac{(1 + w) \phi}{c_s^2} = \frac{3}{2} (1 + w) \frac{H_0^2 \Omega_m}{c_s^2 k^2} \delta_0. \quad (17)
\]

We find that the dark energy perturbations stop growing and become constant inside the sound horizon.

The velocity perturbations are now given simply by using Eq. (6) and inserting Eq. (17):

\[
V = -3Ha(c_s^2 - w) \delta = -\frac{9}{2} (1 + w)(c_s^2 - w) H_0^2 \Omega_m \frac{3}{c_s^2 k^2} a^{-1/2}. \quad (18)
\]

The extra term in brackets in Eq. (6) is not important for the scales of interest here.

Finally, we would like to remind the reader that these results have been obtained under the assumption of a time-independent \(w\) and \(c_s^2\). On the other hand, a \(k\) dependence of \(c_s^2\) is allowed.

As the horizons grow over time, a fixed wave number \(k\) will correspond to a scale that is larger than the causal horizon, \(k < aH\), at early times, and eventually it will enter the causal horizon and later the sound horizon. This makes it possible to illustrate the behavior of the perturbations in the different regimes in a single figure: In Fig. 1 we plot the numerical solution for the dark energy density contrast for \(k = 200H_0\) as well as the expressions (16) and (17). It is easy to see how the perturbations start to grow inside the causal horizon but how the growth stops when the sound horizon is encountered and pressure support counteracts the gravitational collapse.

### IV. DARK ENERGY DOMINATION AND \(Q\)

Matter domination was a crucial ingredient to compute the behavior of the dark energy perturbations, since the gravitational potential \(\phi\) is constant while matter dominates the expansion rate and the total perturbations. However, dark energy comes to dominate eventually, and then the potential starts to decay and the perturbations grow more slowly or start to decrease. This is also visible in Fig. 1 at very late times where the numerical solution for \(\delta\) starts to decrease.

It is difficult to capture this behavior accurately. But in [9] we introduced the variable \(Q(k, a)\) to describe the change of the gravitational potential due to the dark energy perturbations. \(Q\) is defined through

\[
k^2 \phi = -4\pi G a^2 Q \rho_m \left( \delta_m + \frac{3aH}{k^2} V_m \right) \quad (19)
\]

If the dark energy or modification of gravity does not contribute to the gravitational potential (for example if the dark energy is a cosmological constant) then \(Q = 1\). Otherwise \(Q\) will deviate from unity, and in general it is a function of both scale and time.

Introducing the comoving density perturbation \(\Delta \equiv \delta + 3aH/k^2\), and looking at Eq. (11) we see that we can compute \(Q\) with

\[
Q - 1 = \frac{\rho_{DE} \Delta_{DE}}{\rho_m \Delta_m}. \quad (20)
\]

Just using the results during matter domination, we find that the resulting expression for \(Q\) is surprisingly accurate even at late times (see Fig. 2). The reason is that both fluids, dark energy and matter, respond similarly to the change in the expansion rate so that most of the deviations cancel. We find that the sub-soundhorizon expression below is accurate at the percent level, while on larger scales there are deviations of about 10 to 20% by today (depending on \(w\)). The latter can be corrected “by hand” in order to obtain a more precise formula, but the expressions are sufficiently accurate for our purposes and we keep them as they are.

For matter, and during matter domination, the comoving density perturbation is extremely simple, \(\Delta_m = \delta_m a\). For the dark energy we find

\[
\Delta_{DE} = \delta_0 \frac{1 + w}{1 - 3w} a = \frac{1 + w}{1 - 3w} \Delta_m \quad (21)
\]

on scales larger than the sound horizon. This means that the relative strength of the comoving density perturbations in the dark energy and the dark matter is constant on large scales, with those in the dark energy being sub-dominant for \(w < 0\) — for \(w\) close to \(-1\) the prefactor is approximately \((1 + w)/4\). Using the scaling of the energy density in matter and dark energy, we can derive that

\[
Q - 1 = \left( \frac{1 - \Omega_m}{\Omega_m} \right) \left( \frac{1 + w}{1 - 3w} \right) a^{-3w} \equiv Q_0 a^{-3w}. \quad (22)
\]

Here we defined a constant \(Q_0\) since this expression will appear frequently later on. For \(\Omega_m = 0.25\) and \(a = 1\) \(Q_0\) interpolates smoothly between 0 for \(w = -1\) and 3 for
$w = 0$. For $w = -0.8$ we have $Q(a = 1) - 1 = Q_0 \approx 0.18$, that means that we do not expect more than about a 20% deviation of $Q$ from 1 even on large scales, given current observational limits on $w$. We will keep using $w = -0.8$ to illustrate what we can maximally expect to observe.

On the other hand, on small scales the growth of dark energy perturbations is stopped by pressure support. The dark energy perturbations stop growing once they are inside the sound horizon, while the matter perturbations (for which there is no sound horizon) continue to grow. This leads to

$$\Delta_{DE} \approx \frac{3}{2}(1 + w) \left( \frac{Ha}{c_s k} \right)^2 \Delta_m / a$$

(23)

on scales below the sound horizon. Here we neglected the dark energy velocity perturbations, since their contribution to $\Delta_{DE}$ is suppressed by a factor proportional to $(Ha/k)^2$ relative to $\delta_{DE}$. From this expression, we then find that

$$Q - 1 = \left( \frac{1 - \Omega_m}{\Omega_m} \right) \frac{3}{2}(1 + w) \left( \frac{Ha}{c_s k} \right)^2 a^{-3w}$$

(24)

$$= \left( 1 - \Omega_m \right) \frac{3}{2}(1 + w) \frac{H_0^2}{c_s^2 k^2} a^{-3w}$$

(25)

where the result is more accurate when using the matter $Ha$ as done above. Also on small scales, deviation of $Q$ from 1 is maximal at late times for any dark energy that leads to acceleration, $w < -1/3$. But in general it will be suppressed relative to the large scales by the lack of growth of the dark energy perturbations once they are inside the sound horizon.

For the variable $Q$ we can construct a unified formula which accounts both for modes below and above the sound horizon,

$$Q_{tot} - 1 = \frac{1 - \Omega_m}{\Omega_m} (1 + w) \frac{a^{-3w}}{1 - 3w + \frac{2k^2c_s^2 a}{3H_0^2 \Omega_m}}$$

(26)

In Fig. 2 we compare $Q_{tot}$ with the numerical solution from CAMB for different values of the dark energy sound speed. The formula for $Q_{tot}$ interpolates between the two asymptotic regions. Close to the sound horizon it is not very accurate, but it is sufficient to use in e.g. Fisher-matrix codes to estimate the size of effects due to the presence of dark energy perturbations.

V. IMPACT ON SOME OBSERVATIONAL QUANTITIES

A. Growth and shape of the matter power spectrum

What impact does the change in $\phi$ from $Q \neq 1$ have on the matter perturbations? Notice that the effect of the change in $H$ from the onset of dark energy domination at late times is expected to be larger than the effect from the presence or absence of dark energy perturbations, but here we want to quantify only the latter. We can write $\phi = \phi_m + \phi_{DE}$ where $\phi_m$ is the solution given by Eq. 14 and $\phi_{DE}$ is $(Q - 1)\phi_m$. Since the differential equation is linear, the solution with the total source is just the sum of the solution for each source (plus, as before, a decaying solution $\propto 1/a$). The expressions we have found for $Q$ are always power-laws in $a$, which guarantees a simple form for the velocity perturbation.

Even though the dark matter does not have a sound horizon, the solutions will now depend on whether the $k$ mode in question is larger or smaller than the sound horizon of the dark energy, simply because the deviation of $\phi$ from $\phi_m$ depends on this. For $k$-modes larger than the dark energy sound horizon, we find for the matter velocity perturbation

$$V_m = -\delta_0 H_0 \sqrt{\Omega_m} \sqrt{a} \left( 1 + \left( \frac{Q_0}{1 - 3w} \right) a^{-3w} \right)$$

(27)

where $w$ is the equation of state parameter of the dark energy fluid, and the constant $Q_0$ was defined in Eq. 22. At late times $V_m$ will deviate from this formula because of the change in the expansion rate, but we are again interested in the impact of the perturbations, corresponding to the factor in curly brackets. The factor is less sensitive that the velocity itself, and it tracks the numerical result closely until dark energy domination sets in, giving at least the right order of magnitude even today. For the optimistic case with $w = -0.8$ we find a deviation of about 6.5% at late times due to the presence of the dark energy.
energy perturbations. The fully numerical integration for this case gives about 4.5% change. The agreement is not as good as for the dark energy perturbations themselves, but still acceptable, especially since we are dealing with an indirect effect.

This solution we can insert in the one for the density perturbation. In this case, $\phi' \neq 0$, but since we are again dealing with a linear equation, we end up with a sum of three solutions, one for each source (the two parts of $V_m$ and $\phi'$) as well as the constant peculiar solution:

$$
\delta_m = \delta_0 \left\{ \frac{3H_0^2\Omega_{m}}{k^2} \left(1 + \frac{3}{2}Q_0a^{-3w}\right) + a \left(1 + \frac{Q_0a^{-3w}}{1 - 5w + 6w^2}\right) \right\}
$$

On large scales, larger than the causal horizon, the matter perturbations are therefore enhanced by a factor $1 + (3Q_0)/2a^{-3w}$, which is of the same order as $Q$. On smaller scales, but still larger than the sound horizon, the factor is $1 + Q_0a^{-3w}/(1 - 5w + 6w^2)$ which is smaller by a factor of 10 or so (depending on $w$). Inside the sound horizon, the growth of the dark energy perturbations is suppressed.

It is customary to consider two different aspects of the matter perturbations: the power spectrum today and the growth rate of the perturbations. For the power spectrum today we can for example look at the ratio of $\delta_m^2$ with dark energy perturbations to $\delta_m^2$ without the perturbations. For our usual optimistic benchmark with $w = -0.8$ we expect a 4% enhancement of $P(k)$ on scales larger than the sound horizon. A numerical calculation shows that it is closer to 2%, see Fig. 3.

Observationally there are two difficulties to detect the effect: first, it is unclear what happens with the dark energy perturbations on scales where the matter perturbations become non-linear, especially if those scales are larger than the dark energy sound horizon. Second, for a dark energy model with a very large sound speed like Quintessence, for which $c_s^2 = 1$, the matter power spectrum will only be affected on the largest scales which have not been observed. It is obviously much easier to measure the sound speed if it is low. However, the feature visible in Fig. 3 would provide a clear signature for the presence (and size) of a sound-horizon, especially if the corresponding change in the matter velocity field could be detected as well.

A potentially more promising place to look for the impact of the dark energy perturbations is in way the matter perturbations grow over time. Often the impact of the dark energy on the growth rate of the matter perturbations is parametrised in terms of a growth index $\gamma$ \cite{10,17}, defined through

$$
\frac{d\log(\Delta_m)}{d\log(a)} = \Omega(a)\gamma
$$

We will now try to connect our expression \cite{28} to $\gamma$ through some rather crude approximations which should nonetheless be good enough to result in an order of magnitude estimate of the change of $\gamma$ due to the presence of the extra perturbations. On small scales (but always larger than the dark energy sound horizon) the differences between $\delta_m$ and $\Delta_m$ are suppressed by $(Ha/k)^2$ and we have

$$
\Delta_m(k \gg Ha) \approx \delta_0a \left(1 + Q_0a^{-3w}_{\Delta m}(1 - 5w + 6w^2)\right)
$$

Using $\Delta_m^{(0)} = \delta_0a$ to model partially the late-time change of the expansion rate, and assuming that this term obeys the form of Eq. (29) with unperturbed growth index $\gamma_0$ we find by performing the derivative

$$
\frac{d\log(\Delta_m)}{d\log(a)} = \frac{\Delta_m^{(0)}a}{\Delta_m^{(0)}} - \frac{3wQ_0a^{-3w}}{Q_0a^{-3w} + (1 - 5w + 6w^2)} = \Omega(a)^\gamma + \frac{3(Q - 1)}{5 - 6w - Q/w}
$$

where we used Eq. (22) to reintroduce $Q - 1$. In general this cannot be cast in the form of $\Omega_m(a)^\gamma$ since the dark energy perturbations change the matter growth rate even during matter domination where $\Omega_m(a) = 1$. We can however connect to that form at least in the limit when $\Omega_m(a)$ just starts to deviate from unity.

We would like to end up with $\Omega(a)^{\gamma_1}$ with $\gamma_1 = \gamma_0 + \epsilon$. If $\epsilon$ is small enough then we can use that $\Omega_m(a)^\epsilon \approx 1 + \epsilon \log(\Omega_m(a))$ and if we are close to $\Omega(a) = 1$ then additionally $\log(\Omega_m(a)) \approx \Omega_m(a) - 1$, giving finally

$$
\epsilon(\Omega_m(a) - 1) \approx \frac{3(Q - 1)}{5 - 6w - Q/w}
$$
According to \cite{18} (see also \cite{19}), the growth index depends on \( Q \) through the combination
\[
\gamma = \frac{3(1 - w - A(Q))}{5 - 6w}
\]
\[
A(Q) = \frac{Q - 1}{1 - \Omega_m(a)}.
\]
so that in terms of our \( \epsilon \) above
\[
\epsilon = \frac{-3(Q - 1)}{(1 - \Omega_m(a))(5 - 6w)}
\]
which is sufficiently close to Eq. (32), given the crudeness of the approximations used. Numerically the expression above is very close to Eq. (31) — closer than to the full numerical solution which remains more constant at late times. Both predict the correct deviation at early times, though.

The denominator of the function \( A \) is
\[
1 - \Omega_m(a) = \frac{(1 - \Omega_m)a^{-3w}}{\Omega_m + (1 - \Omega_m)a^{-3w}}.
\]
Not surprisingly, this is \( 1 - \Omega_m \) today, so that \( A \approx (1 + w) \), while at early times the second term in the denominator is suppressed so that for \( a \ll 1 \) \( A \) becomes \( \Delta_{DE}/\Delta_m \approx (1 + w)/4 \) on large scales. This has to be compared to \( 1 - w \approx 2 \). The dark energy perturbations thus change the growth index by a few percent or about \( 0.02 \) for \( w = -0.8 \) on scales larger than the sound horizon. This will be challenging to measure even by full-sky surveys like the proposed Euclid satellite mission which expects to achieve an error on \( \gamma \) of less than this \( 0 \) but only if enough modes can be measured, i.e. if the sound speed is close to zero. Once the perturbations enter the sound horizon, they stop to grow and so their impact on the matter perturbations decreases and becomes rapidly negligible and thus impossible to detect. However, notice that this is purely the change of the growth rate due to the perturbations in the dark energy fluid! We defer a more detailed investigation of the detectability of the dark energy perturbations in the dark matter power spectrum to a later publication \cite{10}.

\section{The integrated Sachs-Wolfe effect}

After the matter power spectrum, we focus on the integrated Sachs-Wolfe (ISW) effect for cosmological models with non-zero contribution from the dark energy perturbations. The motivation is that the ISW part of the spectrum is the most affected by the dark energy, see \cite{20,21,22,23}.

The ISW effect results from the late-time decay of gravitational potentials. The total blueshifting or redshifting of the CMB photons caused by the change in the potential during the passage of the photons induces temperature fluctuations \cite{24}:
\[
\zeta = \frac{\Delta T (\hat{n})}{T_0} = 2 \int \frac{\partial \phi}{\partial \tau} d\tau = -2 \int_0^{\chi_H} a^2 H \frac{\partial \phi}{\partial a} d\chi,
\]
where \( \tau \) denotes conformal time. In the last step, we have replaced the integration variable by the comoving distance \( \chi \) which is related to the conformal time by \( d\chi = -cd\tau = -cdt/a \); here we assume a zero anisotropic stress component for all the species in the Universe.

In Fourier space, the derivative of the gravitational potential with respect to the scale factor \( a \) can be expressed as:
\[
\phi' = \frac{3 H^2 \Omega_m}{2 a k^2} \left\{ Q (a, k) \Delta'_m (a, k) + Q' (a, k) \Delta_m (a, k) - \frac{1}{a} Q (a, k) \Delta_m (a, k) \right\}.
\]
where the prime denotes the derivative with respect the scale factor. There are three main ways in which the dark energy perturbations change the ISW effect: They change \( \Delta_m \) itself, as discussed in the last section, both the shape (a very small effect) and the growth. They change \( \phi' \) additionally through the presence of \( Q \) in the last and first term of Eq. (38) and their variation enters as well through \( Q' \).

In linear perturbation theory all \( k \) modes evolve independently, so that we can decompose the dark matter density contrast as
\[
\Delta_m (a, k) = a G (a, k) \Delta_m (k).
\]
Here \( \Delta_m (k) \equiv \Delta_m (a = 1, k) \). The so-called growth factor \( G \) is usually independent of scale: during matter domination \( G \) is constant and the late-time change in the expansion rate affects all scales equally. However, the contribution from the dark energy perturbations induces a scale dependence because of the existence of the dark energy sound horizon, see for instance \cite{25}. For the growth rate, we will use the \( \gamma (a, k) \) from Eqs. (33) and (34), which provides a sufficiently good approximation.

We can then write Eq. (38) as:
\[
\phi' = -\frac{3 H^2 \Omega_m}{2 k^2} \frac{\partial}{\partial a} \left\{ G (a, k) Q (a, k) \right\} \Delta_m (k).
\]
The line of sight integral for the ISW-temperature perturbation \( \zeta \) can now be written as
\[
\zeta = \int_0^{\chi_H} d\chi W_\zeta (\chi) \Delta_m (k)
\]
where we introduced the weight function:
\[
W_\zeta (\chi) = \frac{3 H^2 \Omega_m}{c^3} a^2 H \frac{\partial}{\partial a} \left\{ G (a, k) Q (a, k) \right\}
\]
which allows the expressions for the ISW-auto spectrum \( C_{\zeta \zeta} (\ell) \) to be written in a compact notation, applying the Limber-projection \cite{26} in the flat-sky approximation, for simplicity:
\[
C_{\zeta \zeta} (\ell) = \int_0^{\chi_H} d\chi \left( \frac{W_\zeta^2 (\chi)}{\chi^2} \right) \tilde{P}_{\Delta \Delta} (k = \ell/\chi).
\]
\( \bar{P}_{\Delta\Delta}(k) \) is the linear matter power spectrum today, which can be written as:

\[
\frac{k^3 \bar{P}_{\Delta\Delta}(k)}{2\pi^2} = \delta_H^2 \left( \frac{k}{H_0} \right)^{n+3} T^2(k). \tag{44}
\]

Here \( \delta_H \) is the amplitude of the present-day density fluctuations at the Hubble scale and \( T(k) \) is the transfer function for CDM. We adopt the fit by Eisenstein & Hu \cite{27}. We have neglected the additional impact from the dark energy perturbations onto the dark matter power spectrum as it is only of the order of a few percent (see the discussion in the last section). The Eisenstein & Hu fit agrees with the CAMB output to a precision of about 4\% for both low and high values of \( c_s^2 \), more than sufficient for the purposes of this section.

In Fig. 4 we plot \( \ell(\ell + 1) C_\ell/(2\pi) \) using Eq. \eqref{eq:43} for two different values of the dark energy sound speed. The differences between the two curves come from the term \((\partial \bar{Q}G)/\partial a)^2\). Since the ISW power spectrum depends on the derivatives of the product of the growth factor \( G(a,k) \) and \( Q(a,k) \) we need to look at

\[
(GQ)' = G'Q + GQ' \tag{45}
\]

The deviation of \( Q \) from 1 is never enough to explain the differences between the two curves in Fig. 4. Indeed, taking \( Q = 1 \) while keeping \( Q' \) barely changes the results. The relative size between the first and second term depends mainly on \( Q'G/G' \). From Eq. \eqref{eq:26} we see that \( aQ' \propto (Q - 1) \) with a proportionality factor of about 2 to 3. This is a small number, but it is boosted by \( G' \ll G \), since \( aG'/G = \Omega_m(a)^7 - 1 \). The dark energy slows down the growth of the dark matter perturbations so that \( G' < 0 \). However, \( Q' > 0 \) because of the relative increase of the dark energy density enhancing the importance of the dark energy fluctuations. The two contributions will partially cancel and so decrease the result, unless one term dominates strongly, e.g. for modes inside the sound horizon. This is the reason why the ISW contribution to the CMB power spectrum decreases as the sound speed increases.

\[
\frac{d^2P_{\Delta\Delta}(k)}{da^2} = \frac{dP_{\Delta\Delta}(k)}{da} \left( \frac{\partial G(a)}{\partial a} \right)^2 \tag{46}
\]

If we assume that the dark energy does not cluster (i.e. \( Q = 1 \)), Eq. \eqref{eq:46} reads:

\[
P_{\phi'\phi'} = \left( \frac{3H_0^2\Omega_m}{2k^2} \right)^2 \left\{ \frac{\partial}{\partial a} G(a,k)Q(a,k) \right\}^2 \bar{P}_{\Delta\Delta}(k). \tag{47}
\]

and the growth factor becomes again a function of time only.

Comparing Eq. \eqref{eq:10} and Eq. \eqref{eq:47} we can define a magnification parameter for the ISW power spectrum:

\[
A^2 = \left\{ \frac{d(G(a,k)Q(a,k))/da}{dG(a)/da} \right\}^2. \tag{48}
\]

This is precisely the expression discussed above, \( A = Q + Q'G/G' \).

In Fig. 4 we plot the magnification factor \( A^2 \) for \( k = 200H_0 \) and \( w = -0.8 \) for two different values of the sound speed: \( c_s^2 = 10^{-4} \) (red solid line) and \( c_s^2 = 1 \) (blue dashed line). The vertical line at \( a = 0.62 \) shows when the \( k \)-mode enters the sound horizon for \( c_s^2 = 10^{-4} \) (for the high sound speed the mode enters at \( a = 0.62 \times 10^{-4} \)).

We can illustrate the effect by looking at the gravito-power spectrum,

\[
P_{\phi'\phi'} = \left( \frac{3H_0^2\Omega_m}{2k^2} \right)^2 \left\{ \frac{\partial}{\partial a} G(a,k) \right\}^2 \bar{P}_{\Delta\Delta}(k). \tag{46}
\]

for \( Q = 1 \). From Eq. \eqref{eq:10} we see that the dark energy density grows relative to the dark matter density. As expected, they do not affect the ISW effect significantly in this case. For small sound speeds the dark energy perturbations partially cancel the contribution from \( G' \) and decrease \( A \) by about 30\% for scales above the dark energy sound horizon. Their slower growth inside the sound horizon leads to a smaller cancellation, as can be seen in Fig. 4. Based on these observations, we expect that a low sound speed decreases the ISW effect (which goes like \( A^2 \)) by about 50\%, consistent with Fig. 4. Overall, it seems that \( Q' \) can provide a more sensitive probe of the dark energy perturbations than \( Q \).
VI. CONCLUSIONS

In this paper we have derived simple analytical expressions for the behavior of the perturbations in fluid dark energy models with vanishing anisotropic stress, equations (15) to (18). These expressions are valid for models with constant $w \leq 0$ and $c_s^2 \geq 0$. They were derived under the assumption of matter domination, but they allow to compute the function $Q(k, a; w, c_s^2)$ which describes the deviation of the Poisson equation from the case without dark energy perturbations and which is relatively insensitive to the late-time deviations from matter domination. The expressions for $Q$ are given in Eq. (22) for scales larger than the sound horizon and Eq. (25) for scales smaller than the sound horizon. We also give a single interpolating equation in Eq. (26) which is useful for Fisher-matrix calculations that include dark energy perturbations. In models without anisotropic stress $Q$ completely characterizes the dark energy perturbations and represents a fingerprint that allows to differentiate between different models with the same background expansion rate but a different evolution of the perturbations.

We expect our results to hold generically for scalar-field like models. Large changes can appear because of rapidly varying (especially oscillating) $w$ leading to resonance-like behavior (see e.g. the rather contrived "phaxion" model in [13]), for models with non-zero effective anisotropic stresses (like DGP [7, 28]) or couplings between the dark energy and the dark matter [24, 51, 81], so such models will lead to different results.

We find that the dark energy perturbations are always smaller than the perturbations in the dark matter, at least by a factor $(1 + w)$, but on scales larger than the dark energy sound horizon they are only suppressed by an additional factor of order unity. The dark energy perturbations do not grow on scales smaller than the sound horizon, so that these perturbations are much more suppressed relative to the dark matter. The impact of the dark energy perturbations on the gravitational potential is additionally influenced by the relative energy density, so that measurable deviations tend to appear only at late times.

To demonstrate the usefulness of the equations we then used the formula for $Q$ to investigate the change in the dark matter power spectrum and the ISW effect if $w = -0.8$. The changes in the matter power spectrum at late times are of the order $(Q(a, k) - 1)/5$ which corresponds to a few percent on scales larger than the sound horizon of the dark energy. The growth index $\gamma$ is changed by about 0.02 on the same scales. We further find that the impact on the ISW effect is due to the growth of the dark energy perturbations, $Q'$ having a larger effect on $\phi'$ than naively expected from the size of $Q$. It is much larger than the impact of the dark energy perturbations on the matter power spectrum, but because of cosmic variance it is more difficult to constrain observationally.

The dark energy perturbations of the class of models investigated here vanish as $w \rightarrow -1$. However, if $w$ is different from $-1$ and especially if the sound speed of the dark energy is less than the speed of light, then there is hope that the effects from the dark energy perturbations could be seen with cosmological observations. Although very challenging, it is nonetheless worth the effort since the perturbations provide a much more precise "fingerprint" of the dark energy than the equation of state parameter $w$.

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APPENDIX A: COMPAREING WITH CAMB

Since we are dealing with linear perturbation theory, all $k$-modes evolve independently and for each $k$-mode the perturbations depend linearly on the normalization given by the constant $\delta_0$. As we can choose this constant arbitrarily for each $k$-mode, we should really think of it as $\delta_0(k)$. In the standard cosmology, its value is set in the very early universe by inflation. This scenario then provides the initial conditions for the differential equations. However, when we compare our results with numerical solutions from e.g. CAMB, we have to take into account as well that the Universe was radiation dominated at early times. Since both the expansion rate of the Universe and the dominant contribution to the gravitational potentials are different during radiation domination, we expect to find a different behavior for the matter and dark energy perturbations. It turns out that the change in the expansion rate strongly suppresses the growth rate of the perturbations, leading to an only logarithmically growing solution [32].

This is not directly relevant to our solutions as we are content to limit our expressions to the matter dominated and later epochs. One nuisance is that we have to disregard radiation domination in some of the figures where we prefer a larger dynamical range to show the (formal) evolution of the perturbations in the different regimes. We note this in the figure captions where applicable. Another issue concerns the normalization for comparison with numerical codes: as the perturbation growth of sub-horizon modes is delayed by the radiation dominated epoch, the $k$ modes which enter the horizon during that period end up with a lower normalization than expected if they are normalised at early times. The pragmatic solution here is to normalize the perturbations instead in the late universe, after the onset of matter dominance,

\[ \delta_m(a = a_1) = \delta_m = \delta_0 \left( a_1 + \frac{3H_0^2\Omega_{m0}}{k^2} \right) \]  

which fixes $\delta_0(k)$ in terms of $\delta_m(k)$ at a given scale factor $a_1$. We also note that radiation pushes the growth of both
dark matter and dark energy perturbations to later times so that the ratio (which is relevant for \( Q \)) is basically unchanged.

If a more detailed treatment is desirable, then this can be obtained by following the discussion in standard texts like e.g. [32]. We prefer to avoid these additional complications here since they are not required for the main points of our work.

In this paper we are working in the Newtonian gauge, but CAMB uses the synchronous gauge. In order to compare the CAMB output with our formulae, we need to transform the relevant quantities. The energy-momentum tensor \( T^\mu_\nu ( SYN) \) in the synchronous gauge is related to the \( T^\mu_\nu (Con) \) in the conformal Newtonian gauge by the transformation:

\[
T^\mu_\nu (SYN) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} T^\sigma_\rho (Con) \tag{A2}
\]

where \( \tilde{x}^\mu \) and \( x^\mu \) denote the synchronous and the conformal Newtonian coordinates respectively. The relevant transformations then are [11]

\[
\begin{align*}
\delta (SYN) & = \delta (Con) - \alpha \frac{\dot{\rho}}{\rho}, \quad \text{(A3)} \\
V (SYN) & = V (Con) - \alpha (1 + w) k^2, \quad \text{(A4)} \\
\delta p (SYN) & = \delta p (Con) - \alpha \dot{\rho}, \quad \text{(A5)} \\
\sigma (SYN) & = \sigma (Con), \quad \text{(A6)}
\end{align*}
\]

where \( \rho \) is the energy density at background, \( \delta \) is the density contrast, \( V \) is the velocity perturbation, \( \delta p \) is the pressure perturbation, \( \sigma \) is the anisotropic stress and the function \( \alpha = \dot{x}^0 - x^0 = (\dot{h} + \dddot{\eta})/2k^2 \). Here \( \eta \) and \( h \) are two scalar fields characterizing the scalar modes of the metric perturbation in synchronous gauge. This transformation also applies to individual species when more than one particle species contributes to the energy-momentum tensor.

**APPENDIX B: DECAYING MODES**

In the main text we were able to find simple approximate solutions to the full system of equations by neglecting certain terms, especially those linking \( \delta \) and \( V \) for the dark energy perturbations. This works very well for the parameter values of interest to us, specifically \( w \leq 0 \) and \( \delta^2 \lesssim 1 \). For \( \delta^2 < 0 \) we expect rapid growth of the density perturbations from a well-known instability. This instability, however, requires exactly the coupling which we neglected. In this appendix we want to take a closer look at the “decaying modes” and their behavior especially for \( \delta^2 < 0 \). For this purpose we need to revert to the full system. To allow for consistent simplifications we can then cast it in the form of a single second-order differential equation by combining Eqs. (6) and (7):

\[
\delta'' + \frac{3}{a} (1 - w) + \frac{H'}{H} - \frac{A'}{A} \delta' + \frac{3}{a^2} (c_s^2 - w) \left( \frac{2 - 3c_s^2}{c_s^2} + \frac{aH'}{H} - a \frac{A'}{A} \right) \delta + \frac{Ak^2c_s^2}{(a^2H)^2} \delta + \frac{(1 + w) A}{(a^2H)^2} k^2 \phi = 0 \tag{B1}
\]

where:

\[
\begin{align*}
A & = 1 + \frac{9a^2H^2 (c_s^2 - w)}{k^2}, \quad \text{(B2)} \\
A' & = \frac{9(2aH^2 + 2a^2HH') (c_s^2 - w)}{k^2}. \quad \text{(B3)}
\end{align*}
\]

We expect to see rapid perturbation growth for imaginary sound speeds on small scales. For \( k \gg 1 \) then \( A \approx 1 \) and \( A' \approx 0 \); Eq. (B1) becomes:

\[
\delta'' + \frac{3}{2a} (1 - 2w) \delta' + \frac{k^2c_s^2}{H_0^2\Omega_m a} + \frac{3}{2a^2} (c_s^2 - w) (1 - 6c_s^2) \delta + \frac{(1 + w) k^2 \phi}{(a^2H)^2} = 0 \tag{B4}
\]

At the sound horizon \( c_s^2 k^2 = (aH)^2 = H_0^2\Omega_m/a \) so that for sub-sound-horizon modes \( \nu(a)^2 \equiv k^2c_s^2/(H_0^2\Omega_m) \gg |(c_s^2 - w)(1 - 6c_s^2)| \), and we can neglect the contribution from the second term. \( \nu \) quantifies how deep the mode is inside the sound horizon, and it will become complex for \( c_s^2 < 0 \) (notice also that \( \nu \) grows as \( \sqrt{a} \)). We are then left with:

\[
\delta'' + \frac{3}{2a} (1 - 2w) \delta' + \frac{k^2c_s^2}{H_0^2\Omega_m a} + \frac{(1 + w) k^2 \phi}{(a^2H)^2} = 0 \tag{B5}
\]

The full solution to this equation contains the one found earlier, Eq. (17), as well as two additional ones,

\[
\begin{align*}
\delta_1 & \propto \nu(a)^{-n} J_n(2\nu(a)\Gamma(3/2 - 3w)) \tag{B6} \\
\delta_2 & \propto \nu(a)^{-n} J_{-n}(2\nu(a)\Gamma(1/2 + 3w)) \tag{B7}
\end{align*}
\]

where \( n = (1 - 6w)/2 \). If \( c_s^2 < 0 \) then the absolute value of the Bessel functions will grow exponentially fast since its argument is complex, as expected for the instability.

For super-horizon modes \( (k \ll 1) \) Eq. (B2) reduces to \( A = 9a^2H^2 (c_s^2 - w)/k^2 \) and Eq. (B1) becomes:

\[
\delta'' + \frac{5 - 6w}{2a} \delta' + \frac{9 (c_s^2 - w)}{2a^2} \delta + \frac{9 (1 + w) (c_s^2 - w)}{a^2k^2} k^2 \phi = 0 \tag{B8}
\]

The full solution to this equation contains the one found earlier, Eq. (16), as well as the following ones,

\[
\begin{align*}
\delta_1 & \propto a^{3/2} (-1 + 2w - \sqrt{-8c_s^2 + (1 + 2w)^2}) \tag{B9} \\
\delta_2 & \propto a^{3/2} (-1 + 2w + \sqrt{-8c_s^2 + (1 + 2w)^2}) \tag{B10}
\end{align*}
\]
The interesting term here is the square root, in order to find a growing solution it needs to be real: \(-8c_s^2 + (1 + 2w)^2 \geq 0\). This is the case for
\[
(1 + 2w)^2 \geq 8c_s^2. \tag{B11}
\]
This means that if we want to keep the sound speed positive we need to have an equation of state parameter that is either very negative or well larger than \(-1/2\). In either case the solutions are decaying for \(w < 0\).

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