Kinematic and Weyl singularities

W C Lim, A A Coley and S Hervik
Department of Mathematics & Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada
E-mail: wclim@mathstat.dal.ca, aac@mathstat.dal.ca and herviks@mathstat.dal.ca

Received 30 August 2006, in final form 28 November 2006
Published 22 December 2006
Online at stacks.iop.org/CQG/24/595

Abstract
We study the properties of a future singularity encountered by a perfect fluid observer in tilting spatially homogeneous Bianchi cosmologies. We derive the boost formulae for the Weyl tensor to establish that, for two observers that are asymptotically null with respect to each other, their respective Weyl parameters generally both tend to zero, constant, or infinity together. We examine three classes of typical examples and one exceptional class. Given the behaviour of the Weyl parameter, we can predict that the singularity encountered is a Weyl singularity or a kinematic singularity. The analysis suggests that the kinematic variables are also useful in indicating a singularity in these models.

PACS numbers: 98.80.Jk, 04.20.

1. Introduction
A cosmological model $(\mathcal{M}, g, u)$ is defined by specifying the spacetime geometry, determined by a Lorentzian metric $g$ defined on the manifold $\mathcal{M}$, and a family of fundamental observers, whose congruence$^1$ of worldlines is represented by the 4-velocity field $u$. The covariant derivative $u_{a;b}$ of the 4-velocity field is decomposed into kinematic variables according to

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + H (g_{ab} + u_a u_b) - \dot{u}_a u_b,$$

where $\sigma_{ab}$ is the rate of shear tensor, $\omega_{ab}$ is the vorticity tensor, $H$ is the Hubble scalar, and $\dot{u}_a$ is the acceleration vector. Sometimes there is another preferred family of fundamental observers, $\hat{u}$, whose covariant derivative defines a corresponding set of kinematic variables $(\hat{\sigma}_{ab}, \hat{\omega}_{ab}, \hat{H}, \dot{\hat{u}}_a)$, which are different from the first set.

The study of spatially homogeneous (SH) Bianchi cosmologies forms part of the effort to understand the underlying dynamics of anisotropic and inhomogeneous universes. In recent papers [1, 2] we examined the future asymptotic dynamics as experienced by the perfect fluid observer in Bianchi cosmologies with a tilted perfect fluid with linear equation

$^1$ A congruence is a family of curves such that through each point there passes precisely one curve in this family.
of state $p = (\gamma - 1)\mu$, focusing on physical properties of the models as seen by the fluid observer. We found that given a Bianchi type, for a large enough value of the equation-of-state parameter, $\gamma$, the perfect fluid observer can encounter a singularity, in spite of the fact that the SH observers, moving geodesically and orthogonally to the spatially homogeneous hypersurfaces, do not encounter any singularity into the future [3]. Although there is no universally accepted definition of a singularity [4, section 9.1], we shall make use of the congruence in defining a singularity within the context of a cosmological model: a singularity is encountered by a congruence if the proper time is finite and inextendible. This definition is consistent with the general definition adopted in the early work on Bianchi cosmologies by Ellis and King, which states that a singularity is recognized by the existence of an inextendible curve which has a finite length as measured by a generalized affine parameter [5, page 121].2 The encounter of a singularity is accompanied by an extreme tilt limit (i.e., the fluid observer become asymptotically null with respect to the SH observer). The singularity encountered by the fluid observer has the following properties3:

- the proper time needed to reach the singularity is finite;
- the Hubble scalar $H$ and some other kinematic quantities diverge;
- the matter density tends to zero (i.e., the singularity is not a matter singularity)4.

All components of the Weyl curvature tensor converge for some cases and at least one component diverges to infinity for other cases. In this paper we shall further examine the behaviour of the components of the Weyl tensor (according to the fluid observer), with the aim to predict whether they converge or diverge. In order to do so, we recall the following concept regarding the Weyl curvature tensor.

1.1. The Weyl parameter

The Weyl tensor is defined as the trace-free part of the Riemann curvature tensor and, consequently, is not directly involved in the Einstein field equations. In some sense, the Weyl tensor describes the ‘free gravitational field’ and therefore is of particular interest. Penrose [6] suggested that the Weyl tensor is related to a measure of a ‘gravitational entropy’ and therefore could be used to shed light on the initial state of the Universe. In this context, the Weyl tensor and its Weyl scalars have been used to characterise different behaviours of non-tilted Bianchi models [7–9].

Wainwright, Hancock and Uggla [10] studied the late-time dynamics of Bianchi type VII$_0$ cosmologies with a non-tilted perfect fluid. They introduced a quantity $\mathcal{W}$ called the ‘Weyl parameter’ (see [10], p 2580):

$$\mathcal{W} = \frac{W}{H^2},$$

where

$$W^2 = \frac{1}{6} (E_{ab}E^{ab} + H_{ab}H^{ab}).$$

They coined the term5 ‘Weyl curvature dominance’ to describe the phenomenon in which $\mathcal{W} \to \infty$. It means the ratio of some components of the Weyl curvature tensor and the square of the Hubble scalar tend to infinity6. This phenomenon also occurs in Bianchi type VI$_0$ and VIII cosmologies with a tilted perfect fluid [11, 12].

---

2 Many also use the stricter definition of a singularity as geodesic incompleteness. However, in our case this is not satisfactory since the fluid does not follow geodesics; thus we will not require the congruence to be geodesic.

3 Here, we refer to the properties of variables associated with the orthonormal frame of the fluid observer.

4 Since the source is a perfect fluid with a linear equation of state, this means that all components of the Ricci tensor also tend to zero.

5 In [9] the term ‘extreme Weyl dominance’ was used for the case $\mathcal{W} \to \infty$ while ‘Weyl dominance’ was used for the case when the Weyl curvature invariant dominated the Ricci invariant but $W$ was bounded.

6 It is worth noting that both $W$ and $H$ tend to zero in [10].
In [10–12], the Weyl parameter is defined with respect to the SH observer. However, the Weyl parameter can also be defined with respect to different observers, and particularly the fluid observer in tilted Bianchi cosmologies.\footnote{The Weyl parameter, unlike the invariant Weyl scalars, is an observer-dependent quantity.}

Although the limits of the Weyl parameter along two congruences of worldlines are in principle different, we conjecture that in general they both tend to zero, constant, or infinity together. We shall refer to this as ‘the Weyl parameters have the same convergence/divergence property’. The argument is considered in two cases: whether or not the two congruences are asymptotically null with respect to each other \((v^2 \to 1)\).

If the two congruences are not asymptotically null with respect to each other \((v^2 \not\to 1)\), then \(\Gamma = (1 - v^2)^{-1/2}\) is bounded. Then, from the boost formulae (A.5)–(A.6) for the Weyl components, in general the Weyl components \(\hat{C}_{abcd}\) and \(C_{abcd}\) relative to two different congruences have the same convergence/divergence property. Also recall from the boost formula for the Hubble scalar \(H\) ([1], equation (B.15), reproduced below)

\[
\hat{H} = \frac{1}{3} \left[ e_\mu (\Gamma v^\mu) - \frac{\Gamma}{\Gamma + 1} v^\mu e_\mu (\Gamma) + 3 \Gamma H - 2 \Gamma a_\mu v^\mu + \Gamma \dot{a}_\mu v^\mu \right] \tag{3}
\]

that \(\hat{H}\) and \(H\) also have the same convergence/divergence property. It then follows from (2) that generally the Weyl parameters \(\hat{V}\) and \(V\) have the same convergence/divergence property.

In the more interesting case for this analysis, when the two congruences are asymptotically null with respect to each other \((v^2 \to 1)\), then \(\Gamma \to \infty\) and, in general, \(\hat{C}_{abcd}\) becomes of order \(\Gamma^2 C_{abcd}\). From (3) we see that \(\hat{H}\) is of order \(\Gamma H\) as \(\Gamma \to \infty\). As a result, generally we have

\[
\frac{\hat{C}_{abcd}}{\hat{H}^2} \text{ is of order } \frac{C_{abcd}}{H^2}; \tag{4}
\]

i.e., the Weyl parameter \(\hat{W}\) of a second congruence is generally of the same order as the Weyl parameter \(W\) of the first congruence, and they therefore have the same convergence/divergence property. Exceptions are possible when cancellation occurs in the leading order terms in all the Weyl components, as we will show occurs in the LRS Bianchi type V cosmologies. Indeed, we will show that, when the argument works, we can use it to predict the type of singularity encountered by the fluid observer.

2. Terminology

We first review some terminologies about singularities and the asymptotic dynamics of the Weyl tensor.

There are several classifications of singularities in the literature. We are concerned with one particular classification regarding the convergence/divergence of the Weyl curvature tensor at the singularity.

2.1. Weyl singularity

Collins and Ellis [13, p 88] used the terminology ‘Weyl singularity’ to describe the following: at least one component of the Weyl curvature tensor (with respect to the orthonormal frame of the observer who encounters the singularity) diverges.\footnote{Collins and Ellis actually used the terminology ‘conformal singularity’, but we feel ‘Weyl singularity’ is more appropriate and avoids confusion with the ‘conformal singularity’ in the context of isotropic singularities [14, 15].}

On the other hand, the matter density and all components of the Weyl tensor can converge as the singularity is approached, while some kinematic quantities (most importantly, the
Hubble scalar $H$) diverge. We shall call such a singularity a ‘kinematic singularity’ when it is not a Weyl singularity or a matter singularity. In other words, relative to a congruence, a kinematic singularity is characterized by the blow-up of one or more kinematic variables in finite proper time, while all components of the Weyl and Ricci tensors remain bounded. This terminology is new. A prototypical example of a kinematic singularity is the initial singularity encountered by the fundamental observers of the Milne universe [16].

Later, in section 4, we shall discuss this classification in the literature.

2.2. Weyl blow-up

The concept of Weyl singularity combines two phenomena – the blow-up of Weyl and the occurrence of a singularity. To separate them, we introduce the terminology ‘Weyl blow-up’ to describe the phenomenon when some components of the Weyl tensor diverge asymptotically with respect to a particular observer, regardless of the occurrence of singularity. Weyl blow-up can be simply stated as

$$W \to \infty,$$

where $W$ is defined in (2). The phenomena $W \to \text{const}$ or $W \to 0$ will be called ‘Weyl convergence’. This terminology is new. We have not encountered examples in tilted SH cosmologies in which $W$ is bounded but whose limit does not exist.

The Weyl scalars may or may not converge asymptotically when Weyl blow-up occurs for one observer. There are three scenarios. In the trivial case where the Weyl tensor is identically zero, we have Weyl convergence for all observers. In the second case, if at least one of the four Weyl scalars diverges, then we have Weyl blow-up for all observers. In the third case, if all four Weyl scalars converge, then one observer may experience Weyl blow-up while another does not. The third case is the case of interest here.

2.3. Future singularity and Weyl blow-up

In the next section we shall examine tilted Bianchi cosmologies that have an extreme-tilt sink. On approach to the sink, the fluid observer may or may not encounter a future singularity, and may or may not experience Weyl blow-up. We summarize the four possible scenarios with a Venn diagram (see figure 1).

---

9 It is possible for Weyl blow-up to take an infinite proper time to occur, in which case there is no singularity.
We shall use the Weyl parameter to heuristically predict whether Weyl blow-up occurs if the fluid observer encounters a future singularity; i.e., whether the singularity is a Weyl singularity. We will also give a special example in which the prediction breaks down.

3. Typical examples

In this section we first examine three typical examples of tilted Bianchi cosmologies that have an extreme-tilt asymptotic regime at late times. For more details of the asymptotic dynamics of the tilt variable, the fluid cosmological time, and the fluid Hubble scalar, see [1, sections 2.1, 3.2.1, 3.2.2 and 5]. The three examples illustrate different limits of the Weyl parameter of the fluid observer:

\[ \hat{\gamma} \rightarrow 0, \quad \hat{\mathcal{W}} \rightarrow \infty, \quad \text{and} \quad \hat{\mathcal{W}} \rightarrow \text{const.} \]

respectively.

In all of the examples we have used the time parameter \( \tau \) to be the dynamical time of the SH observer; i.e., \( \tau \) is defined as

\[ \frac{d\tau}{dt} = H, \]

where \( t \) and \( H \) are the cosmological time and the Hubble parameter of the SH observer, respectively. We shall be using \( \tau \) in the asymptotic expansions of both SH and fluid quantities. For those who would prefer to express the expansions in terms of the fluid cosmological time \( t_{\text{fluid}} \), simply use the relation (valid along each fluid worldline but not across the fluid congruence)

\[ dt_{\text{fluid}} = \frac{\sqrt{1 - v^2}}{H} d\tau. \]

For further details and discussion, see [1, page 3575]. In [1] we claimed that the limits of the kinematic variables along the SH congruence (or the normal congruence in the inhomogeneous example) and those along the fluid congruence are the same. We take this opportunity to justify that claim. The examples in [1] are special in that they have spatially homogeneous limits along the SH or normal congruence as \( \tau \) tends to infinity. The fluid congruence is also special in that it must pass by all events passed by the SH or normal congruence. Thus, moving along a fluid worldline, the limit of a kinematic variable must be the same as the limit along an SH or normal worldline.

3.1. Irrotational Bianchi type V

The dynamics of irrotational tilted Bianchi type V cosmologies was studied by Hewitt and Wainwright [17]. As the solutions approach the extreme-tilt Milne solution \( M^- \) (which is a sink for \( \gamma \) satisfying \( \frac{5}{2} < \gamma < 2 \)) at late times, the (orthogonal) SH observer sees

\[ H \approx H_0 e^{-\tau}, \quad W \approx H_0^2 |\Sigma_{-0}| e^{-4\tau}, \quad \mathcal{W} \approx |\Sigma_{-0}| e^{-2\tau}, \]

as \( \tau \rightarrow \infty \).

As a result

\[ H \rightarrow 0, \quad W \rightarrow 0, \quad \mathcal{W} \rightarrow 0, \]

for all \( \frac{5}{2} < \gamma < 2 \).

\[ \text{The asymptotic expansions for kinematic and matter variables are obtained by linearizing equations (2.10) and (2.11) in [17], and solving them. These are then substituted into equations (14) and (15) in [18, appendix 3] to obtain the asymptotic expansions for the Weyl components.} \]
According to the (tilted) fluid observer, however, the asymptotic expansions are (from the boost formulae (3) and (A.5)–(A.6))

\[
\Gamma \approx \Gamma_0 \exp \left( \frac{5\gamma - 6}{2 - \gamma} \tau \right), \quad \hat{H} \approx \frac{4}{3(2 - \gamma)} \Gamma_0 H_0 \exp \left( \frac{2(3\gamma - 4)}{2 - \gamma} \tau \right),
\]

(8)

\[
\hat{W} \approx 2\sqrt{2} \xi_0 H_0 |\Sigma_{-0}| \exp \left( \frac{2(7\gamma - 10)}{2 - \gamma} \tau \right), \quad \hat{V} \approx \frac{9\sqrt{2}}{8} (2 - \gamma)^2 |\Sigma_{-0}| \exp^{-2\tau},
\]

(9)

as \( \tau \to \infty \), so

\[
\hat{H} \to \infty \quad \text{for} \quad \frac{4}{3} < \gamma < 2,
\]

\[
\hat{W} \to \infty \quad \text{for} \quad \frac{10}{7} < \gamma < 2,
\]

\[
\hat{V} \to 0 \quad \text{for all} \quad \frac{6}{5} < \gamma < 2.
\]

We note that the convergence/divergence property of \( H \) and \( W \) is observer dependent, while \( W \) tends to zero for both observers. For the fluid observer, the \( \gamma \) threshold for encountering a singularity is lower than that for Weyl blow-up. A singularity is not necessarily a Weyl singularity in this case (i.e., it could be a kinematic singularity).

3.2. Bianchi type VII\( _0 \)

The dynamics of tilted Bianchi type VII\( _0 \) cosmologies was studied by Hervik et al \[11\]. As the solutions approach the extreme-tilt limit \( \hat{P}_4 \) (which is an attractor for \( \gamma \) satisfying \( \frac{4}{3} < \gamma < 2 \)) at late times, the SH observer sees

\[
H \sim e^{-2\tau}, \quad W \sim e^{-3\tau}, \quad \mathcal{W} \sim e^{\tau},
\]

(11)

as \( \tau \to \infty \). As a result,

\[
H \to 0, \quad W \to 0, \quad \mathcal{W} \to \infty,
\]

(12)

for all \( \frac{4}{3} < \gamma < 2 \).

According to the fluid observer, however, the asymptotic rates are

\[
\Gamma \sim \exp \left( \frac{3\gamma - 4}{2 - \gamma} \tau \right), \quad \hat{H} \sim \exp \left( \frac{5\gamma - 8}{2 - \gamma} \tau \right),
\]

(13)

\[
\hat{W} \sim \exp \left( \frac{9\gamma - 14}{2 - \gamma} \tau \right), \quad \hat{V} \sim e^{\tau},
\]

(14)

as \( \tau \to \infty \), so

\[
\hat{H} \to \infty \quad \text{for} \quad \frac{8}{5} < \gamma < 2,
\]

\[
\hat{W} \to \infty \quad \text{for} \quad \frac{14}{5} < \gamma < 2,
\]

(15)

\[
\hat{V} \to 0 \quad \text{for all} \quad \frac{5}{2} < \gamma < 2.
\]

Again, we note that the convergence/divergence property of \( H \) and \( W \) is observer dependent, while \( W \) tends to infinity (i.e., Weyl dominance) for both observers. For the fluid observer, the \( \gamma \) threshold for encountering a singularity is higher than that for Weyl blow-up. A singularity is necessarily a Weyl singularity in this case\(^{11}\).

\(^{11}\) Tilted Bianchi type VIII cosmologies also exhibit the same qualitative behaviour (see \[12, 19\]).
3.3. Bianchi-type VII\textsubscript{h}

The dynamics of tilted Bianchi type VII\textsubscript{h} cosmologies was studied in [20, 21]. As the solutions approach the extreme-tilt vacuum plane $\tilde{\mathcal{L}}_{\text{VII}_h}$ (which is a sink for $\gamma$ satisfying \(\frac{6}{\Sigma_2} < \gamma < 2, -\frac{1}{4} < \Sigma_3 < 0\) at late times, the SH observer sees

$$H \approx H_0 e^{(1-2\Sigma_3)\tau}, \quad W \sim H^2,$$

as $\tau \to \infty$. As a result

$$H \to 0, \quad W \to 0, \quad \mathcal{W} \to \text{const.},$$

for all $\frac{6}{\Sigma_2} < \gamma < 2$.

According to the fluid observer, however, the asymptotic rates are

$$\Gamma \approx G_0 \exp\left(\frac{(5 + 2 \Sigma_2)\gamma - 6}{2 - \gamma} \tau\right),$$

$$\dot{H} \sim \exp\left(\frac{2(3\gamma - 2(1 - \Sigma_3))}{2 - \gamma} \tau\right), \quad \dot{W} \sim \dot{H}^2,$$

as $\tau \to \infty$, so

$$\dot{H} \to \infty \quad \text{for} \quad \frac{2}{3}(2 - \Sigma_3) < \gamma < 2,$$

$$\dot{W} \to \infty \quad \text{for} \quad \frac{2}{3}(2 - \Sigma_3) < \gamma < 2,$$

$$\mathcal{W} \to \text{const.} \quad \text{for all} \quad \frac{6}{\Sigma_2} < \gamma < 2.$$

Again, we notice that the convergence/divergence property of $H$ and $W$ is observer-dependent, while $\mathcal{W}$ tends to a constant for both observers. For the fluid observer, the $\gamma$ threshold for encountering a singularity is equal to that for Weyl blow-up. A singularity is necessarily a Weyl singularity in this case.

From the three typical examples above, we have formulated the following conjecture.

**Conjecture:** In general (except in a zero-measure set of special solutions where the leading order term of each Weyl component vanishes), the convergence/divergence property of the Weyl parameter $\mathcal{W}$ is observer independent. In the extreme-tilt limit, according to the fluid observer,

- If $\dot{\mathcal{W}} \to 0$, then the $\gamma$ threshold for encountering a singularity is lower than that for Weyl blow-up.
- If $\dot{\mathcal{W}} \to \infty$, then the $\gamma$ threshold for encountering a singularity is higher than that for Weyl blow-up.
- If $\dot{\mathcal{W}} \to \text{const.}$, then the $\gamma$ threshold for encountering a singularity is equal to that for Weyl blow-up.

This conjecture works for typical examples, but may fail due to simplification in special solutions. Let us present one such example:

3.4. Exceptional example: LRS Bianchi type V

The LRS Bianchi type V cosmologies are a special case of the irrotational tilted Bianchi type V cosmologies. They were studied by Collins and Ellis [13].

The shear component $\Sigma_{\ldots}$ is zero in this class, and the SH observer does not see the asymptotic rates (6), but rather

$$H \approx H_0 e^{-\tau}, \quad W \approx \frac{5}{2} H_0^2 |\Sigma_{40}| e^{-6\tau}, \quad \mathcal{W} \approx \frac{5}{2} |\Sigma_{40}| e^{-2\tau},$$

as $\tau \to \infty$. As a result

$$H \to 0, \quad W \to 0, \quad \mathcal{W} \to \text{const.},$$

for all $\frac{6}{\Sigma_2} < \gamma < 2$. 

Again, we notice that the convergence/divergence property of $H$ and $W$ is observer-dependent, while $\mathcal{W}$ tends to a constant for both observers. For the fluid observer, the $\gamma$ threshold for encountering a singularity is equal to that for Weyl blow-up. A singularity is necessarily a Weyl singularity in this case.

From the three typical examples above, we have formulated the following conjecture.

**Conjecture:** In general (except in a zero-measure set of special solutions where the leading order term of each Weyl component vanishes), the convergence/divergence property of the Weyl parameter $\mathcal{W}$ is observer independent. In the extreme-tilt limit, according to the fluid observer,

- If $\dot{\mathcal{W}} \to 0$, then the $\gamma$ threshold for encountering a singularity is lower than that for Weyl blow-up.
- If $\dot{\mathcal{W}} \to \infty$, then the $\gamma$ threshold for encountering a singularity is higher than that for Weyl blow-up.
- If $\dot{\mathcal{W}} \to \text{const.}$, then the $\gamma$ threshold for encountering a singularity is equal to that for Weyl blow-up.

This conjecture works for typical examples, but may fail due to simplification in special solutions. Let us present one such example:
as $\tau \to \infty$. Nonetheless, (7) still holds:

$$H \to 0, \quad W \to 0, \quad \mathcal{V} \to 0,$$

for all $\frac{5}{6} < \gamma < 2$.

It turns out that there is only one independent component of the Weyl tensor $C_{abcd}$, namely

$$E_+ = (H + \sigma)\sigma + \frac{1}{4} \frac{\gamma}{G_+} V^2.$$  

(23)

The boost formulae (A.5)–(A.6) then imply that $\hat{E}_+$ is the only nonzero component in any other frame, and that

$$\hat{E}_+ = E_+;$$

(24)
i.e., the leading order terms $\Gamma^2 C_{abcd}$ in $\hat{C}_{abcd}$ cancel. As a result, equation (4) fails, and this is a special example when general behaviour predicted by the conjecture fails.

Indeed, the fluid observer sees

$$\Gamma \approx \Gamma_0 \exp \left(\frac{5\gamma - 6}{2 - \gamma} \tau\right),$$

(25)

$$\hat{H} \approx \frac{4}{3(2 - \gamma)} \Gamma_0 H_0 \exp \left(\frac{2(3\gamma - 4)}{2 - \gamma} \tau\right),$$

(26)

$$\hat{W} \approx W \approx \frac{5}{2} H_0^2 |\Sigma_{\phi0}| \exp \left(-\frac{2(3\gamma - 2)}{2 - \gamma} \tau\right),$$

(27)

$$\mathcal{V} \approx \frac{45}{32} (2 - \gamma)^2 \Gamma_0^{-2} |\Sigma_{\phi0}| \exp \left(-\frac{2(3\gamma - 2)}{2 - \gamma} \tau\right),$$

(28)

as $\tau \to \infty$. As a result,

$$\hat{H} \to \infty \quad \text{for} \quad \frac{4}{7} < \gamma < 2,$$

$$\hat{W} \to 0 \quad \text{for all} \quad \frac{5}{6} < \gamma < 2,$$

$$\mathcal{V} \to 0 \quad \text{for all} \quad \frac{5}{6} < \gamma < 2.$$  

(29)

Comparing with equation (10), the quantity $\hat{W}$ never tends to infinity.

4. Discussion

The future asymptotic dynamics in tilting SH Bianchi cosmologies has recently been studied, with an emphasis on the physical properties of the models as experienced by the fluid observer [1, 2]. In this paper we have been primarily interested in the properties of the singularity that the perfect fluid observer can encounter, with particular emphasis on the behaviour of the Weyl tensor (according to the fluid observer). As we have seen in the examples, Weyl blow-up is not a good indicator of a singularity. A better measure is the blow-up of kinematic variables.

We now discuss the work by Collins and Ellis [13] and Ellis and King [5] on Weyl blow-up at the singularity. In Collins and Ellis [13], it is claimed that the future singularity in the LRS Bianchi type V example is a Weyl singularity (i.e., a ‘conformal singularity’, see page 97), in that some components of Weyl tensor diverge. This is not the case, for the only component of the Weyl tensor is $E_+$, and it tends to zero. All components of the Riemann tensor tend to zero at the singularity. Some kinematic variables diverge, and are responsible for the singular behaviour.
Ellis and King [5] classify singularities as ‘curvature singularities’, ‘locally extendible singularities’ or ‘intermediate singularities’. Curvature singularities correspond to our Weyl singularities with the blow-up of at least one of the Weyl scalars; locally extendible singularities correspond to our kinematic singularities; and intermediate singularities correspond to our Weyl singularities with bounded Weyl scalars. According to this classification, all of our examples of Weyl singularities are intermediate singularities. We prefer the terminology ‘kinematic singularity’ over ‘locally extendible singularity’ since in our examples the fluid worldlines are certainly not extendible at the singularity, due to the blow up of the kinematic variables; indeed, our examples are not ‘locally extendible’ in the sense of Clarke [22].

5. Conclusion

In discussions regarding singularities, usually attention is focussed on the Weyl tensor components and the matter density. We have argued that in a cosmological context more emphasis should be put on the kinematic variables.

In this paper we have derived the boost formulae for the components of the Weyl tensor. We have applied the boost formulae to establish that, for two observers that are asymptotically null with respect to each other, their respective Weyl parameter generally both tend to zero, constant, or infinity together. We examined three classes of typical examples and one exceptional class. Given the behaviour of the Weyl parameter, we can predict that the singularity encountered is a Weyl singularity or a kinematic singularity. The examples suggest that the kinematic variables are more useful than the Weyl tensor components in indicating a singularity.

Acknowledgments

This work was supported by an AARMS Postdoctoral Fellowship (SH) and the Natural Sciences and Engineering Research Council of Canada (AC).

Appendix. Boost formulae for the Weyl tensor components

In [1] the boost formulae for the kinematic and the matter variables are derived. Here we extend the formulae to include the Weyl components.

Recall from [1] that the boost formulae for the orthonormal frame vector fields are

\[ \mathbf{e}_0 = \Gamma \mathbf{e}_0 + \Gamma v^\mu \mathbf{e}_\mu, \]  
\[ \mathbf{e}_\alpha = \Gamma v_\alpha \mathbf{e}_0 + B_\alpha^\mu \mathbf{e}_\mu, \]  

where

\[ B_\alpha^\mu = \left[ \delta_\alpha^\mu + \frac{\Gamma^2}{\Gamma + 1} v_\alpha v^\mu \right], \quad \Gamma = \frac{1}{\sqrt{1 - v^2}}, \quad v^2 = v_\mu v^\mu. \]

The Weyl curvature tensor \( C_{\alpha \beta \gamma \delta} \) can be decomposed into ‘electric’ and ‘magnetic’ components as follows:

\[ C_{\alpha \beta} = E_{\alpha \beta}, \quad C_{a \beta \gamma \delta} = -\epsilon^{\mu \nu} E_{\gamma \delta}, \quad C_{\alpha \beta \gamma \delta} = \epsilon^{\mu \nu} H_{\gamma \delta}. \]

Boosting the orthonormal frame results in the following boost formulae for the Weyl components:
\[ \dot{E}_{\alpha\beta} = (2\Gamma^2 - 1) E_{\alpha\beta} - \frac{2\Gamma^2(2\Gamma + 1)}{\Gamma + 1} v^\mu E_{\mu(\alpha} v_{\beta)} + \frac{\Gamma^4}{(\Gamma + 1)^2} E_{\mu\nu} v^\mu v^\nu v_{(\mu \nu)} v_{(\alpha} v_{\beta)} + 2\Gamma^2 v_\mu e^{\mu
u}_{(\alpha} H_{\beta)\nu} \]
\[ - 2\Gamma^2 v_\mu e^{\mu
u}_{(\alpha} E_{\beta)\nu} + 2\Gamma^3 v_\mu e^{\mu
u}_{(\alpha} v_{\beta)} E_{\nu\rho} v^\rho. \] (A.5)

\[ \dot{H}_{\alpha\beta} = (2\Gamma^2 - 1) H_{\alpha\beta} - \frac{2\Gamma^2(2\Gamma + 1)}{\Gamma + 1} v^\mu H_{\mu(\alpha} v_{\beta)} + \frac{\Gamma^4}{(\Gamma + 1)^2} H_{\mu\nu} v^\mu v^\nu v_{(\mu \nu)} v_{(\alpha} v_{\beta)} - 2\Gamma^2 v_\mu e^{\mu
u}_{(\alpha} H_{\beta)\nu} + 2\Gamma^3 v_\mu e^{\mu
u}_{(\alpha} v_{\beta)} E_{\nu\rho} v^\rho. \] (A.6)

References

[1] Coley A A, Hervik S and Lim W C 2006 Class. Quantum Grav. 23 3573
[2] Coley A A, Hervik S and Lim W C 2006 Phys. Lett. B 638 310
[3] Rendall A D 1995 Math. Proc. Camb. Phil. Soc. 118 511
[4] Wald R M 1984 General Relativity (Chicago, IL: The University of Chicago Press)
[5] Ellis G F R and King A R 1974 Commun. Math. Phys. 38 119
[6] Penrose R 1979 General Relativity, an Einstein Centenary Survey ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
[7] Wainwright J and Anderson P J 1984 Gen. Rel. Grav. 16 609
[8] Goode S W and Wainwright J 1984 Class. Quantum Grav. 2 99
[9] Barrow J D and Hervik S 2002 Class. Quantum Grav. 19 5173
[10] Wainwright J, Hancock M J and Uggla C 1999 Class. Quantum Grav. 16 2577
[11] Hervik S, van den Hoogen R J, Lim W C and Coley A A 2006 Class. Quantum Grav. 23 845
[12] Hervik S and Lim W C 2006 Class. Quantum Grav. 23 3017
[13] Collins C B and Ellis G F R 1979 Phys. Rep. 56 65
[14] Anguige K and Tod K P 1999 Ann. Phys., NY 276 257
[15] Goode S W, Coley A A and Wainwright J 1992 Class. Quantum Grav. 9 445
[16] Rindler W 2000 Phys. Lett. A 276 52
[17] Hewitt C G and Wainwright J 1992 Phys. Rev. D 46 4242
[18] Uggla C, van Elst H, Wainwright J and Ellis G F R 2004 Phys. Rev. D 68 103502
[19] Coley A A, Hervik S and Lim W C 2006 Preprint gr-qc/0605089
[20] Coley A A and Hervik S 2005 Class. Quantum Grav. 22 2579
[21] Hervik S, van den Hoogen R J and Coley A A 2005 Class. Quantum Grav. 22 607
[22] Clarke C J S 1973 Commun. Math. Phys. 32 205