TWO DIMENSIONAL SUBSONIC AND SUBSONIC-SONIC SPIRAL FLOWS OUTSIDE A POROUS BODY

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Abstract In this paper, we investigate two dimensional subsonic and subsonic-sonic spiral flows outside a porous body. The existence and uniqueness of the subsonic spiral flow are obtained via variational formulation, which tends to a given radially symmetric subsonic spiral flow at far field. The optimal decay rate at far field is also derived by Kelvin’s transformation and some elliptic estimates. By extracting spiral subsonic solutions as the approximate sequences, we obtain the spiral subsonic-sonic limit solution by utilizing the compensated compactness. The main ingredients of our analysis are methods of calculus of variations, the theory of second-order quasilinear equations and the compensated compactness framework.

Key words Subsonic spiral flows; Euler equations; subsonic-sonic limit; a porous body

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1 Introduction and Main Results

In this paper, we are concerned with two-dimensional subsonic and subsonic-sonic spiral flows outside a porous body $D(\Gamma)$ which are governed by the following two-dimensional Euler system:

\[
\begin{align*}
\rho u_1 + \rho u_2 &= 0, \\
\rho u_1 u_2 + \rho u_2^2 + \rho u_2 &= 0, \\
\rho u_1 u_2 + \rho u_2^2 + \rho u_2 &= 0.
\end{align*}
\]

(1.1)

Here $u = (u_1, u_2)$ is the velocity field, $\rho$ is the density, and $p$ is pressure. We only consider the polytropic gas, and therefore $p = A\rho^\gamma$, where $A$ is a positive constant and $\gamma$ is the adiabatic constant with $\gamma > 1$.

Suppose that the flow is also irrotational, i.e.

\[
\partial x_2 u_1 - \partial x_1 u_2 = 0.
\]

(1.2)

Then it follows from (1.1) and (1.2) that the flow satisfies Bernoulli’s law:

\[
\frac{q^2}{2} + h(\rho) = C_0,
\]

(1.3)

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where \( h(\rho) \) is the enthalpy satisfying \( h'(\rho) = \frac{\rho'(\rho)}{\rho} \), \( q = \sqrt{u_1^2 + u_2^2} \) is the flow speed, and \( C_0 \) is a constant depending on the flow. We normalize the flow as in [7] such that \( p(\rho) = \rho^{\gamma}/\gamma \) is the pressure for the polytropic gas, and Bernoulli’s law (1.3) reduces to

\[
\frac{q^2}{2} + \int_0^\rho \frac{\rho'(\rho)}{\rho} \, d\rho = \frac{\gamma + 1}{2(\gamma - 1)}.
\]

Thus (1.4) yields a representation of the density

\[
\rho = g(q^2) = \left( \frac{\gamma + 1 - (\gamma - 1)q^2}{2} \right)^{1/(\gamma-1)}.
\]

The sound speed \( c \) is defined as \( c^2 = p'(\rho) \). At the sonic point \( q = c \), (1.5) implies that \( q^2 = 1 \). We define the critical speed \( q_{cr} \) as \( q_{cr} = 1 \). Thus the flow is subsonic when \( q < 1 \), sonic when \( q = 1 \) and supersonic when \( q > 1 \).

By the density equation in (1.1), one can introduce the stream function \( \psi \) as follows:

\[
\partial_{x_1} \psi = -\rho u_2, \quad \partial_{x_2} \psi = \rho u_1.
\]

Obviously, \( |\nabla \psi| = \rho q \). Then it follows from the normalized Bernoulli’s law (1.4) that \( \rho q \) is a nonnegative function of \( q \), which is increasing for \( q \in (0, 1) \) and decreasing for \( q \geq 1 \), and which vanishes at \( q = 0 \). Therefore \( \rho q \) attains its maximum at \( q = 1 \), and \( \rho \) is a two-valued function of \( |\nabla \psi|^2 \). Subsonic flows correspond to the branch where \( \rho > 1 \) if \( |\nabla \psi|^2 \in [0, 1) \). Set

\[
\rho = H(|\nabla \psi|^2)
\]

such that \( \rho > 1 \) if \( |\nabla \psi|^2 \in [0, 1) \). We then have that \( H \) is a positive decreasing function defined on \([0, 1)\), twice differentiable on \([0, 1)\), and satisfies \( H(1) = 1 \). Then (1.1) reduces to the single equation

\[
\text{div} \left( \frac{\nabla \psi}{H(|\nabla \psi|^2)} \right) = 0.
\]
Then the flow is described by radially symmetric smooth functions of the form $u_0(x) = U_1(r)e_r + U_2(r)e_\theta$, $\rho(x) = \rho_0(r)$ and $p(x) = p_b(r)$, which solves the system
\[
\begin{cases}
(p_b U_1)' + \frac{1}{r}p_b U_1 = 0, & r > 1, \\
U_1 U_1' + \frac{1}{\rho_b} p_b' - \frac{U_2^2}{r} = 0, & r > 1, \\
U_1 U_2' + \frac{U_1 U_2}{r} = 0, & r > 1,
\end{cases}
\]
with the boundary condition
\[\rho_b(1) = \rho_0 > 0, \quad U_1(1) = \kappa_1 > 0, \quad U_2(1) = \kappa_2.\]

Here $\rho_0$, $\kappa_1$ and $\kappa_2$ are chosen to satisfy (1.4), that is,
\[
\frac{1}{2}(\kappa_1^2 + \kappa_2^2) + \frac{\rho_0^{\gamma-1}}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}.
\]

Hence it is easy to see that
\[
\rho_b U_1 = \frac{\rho_0 \kappa_1}{r}, \quad U_2 = \frac{\kappa_2}{r}.
\]

Therefore,
\[
u_0(x) = \left(\frac{\rho_0^{-1} \rho_0 \kappa_1 x_1 - \kappa_2 x_2}{r^2}, \frac{\rho_0^{-1} \rho_0 \kappa_1 x_2 + \kappa_2 x_1}{r^2}\right).
\]

Furthermore, by (1.6), the corresponding stream functions $\psi_0$, $\psi_{10}$ and $\psi_{20}$ are
\[
\psi_{10}(\theta) = \rho_0 \kappa_1 \theta, \quad \psi_{20}(r) = -\kappa_2 \int_1^r \rho_b(s) \frac{1}{s} ds,
\[
\psi_0 = \psi_{10} + \psi_{20} = \rho_0 \kappa_1 \theta - \kappa_2 \int_1^r \rho_b(s) \frac{1}{s} ds.
\]

Denote that $M_1^2 = \frac{U_1^2}{r^2}$, $M_2^2 = \frac{U_2^2}{r^2}$ and $M^2 = M_1^2 + M_2^2$. Then, by simple calculations, we have that
\[
\begin{aligned}
\frac{d}{dr}(M_1^2) &= -\frac{(2(1 + M_1^2) + (\gamma-1)M^2)M_1^2}{r(1 - M_1^2)}, \\
\frac{d}{dr}(M_2^2) &= -\frac{(2(1 - M_1^2) + (\gamma-1)M^2)M_2^2}{r(1 - M_1^2)}, \\
\frac{d}{dr}(M^2) &= -\frac{[(\gamma-1)M^2 + 2]M^2}{r(1 - M_1^2)}.
\end{aligned}
\]

Assume that $\kappa_1^2 + \kappa_2^2 > 1$ and $\kappa_1^2 < 1$. Then there exists a smooth transonic spiral flows for all $r > 1$. The structural stability of this special transonic flows was investigated by Weng-Xin-Yuan in [24]. Assume that $\kappa_1^2 + \kappa_2^2 < 1$. Then it follows from (1.15) that $M^2 < 1$ holds for all $r > 1$, which means that the flow is uniformly subsonic.

Let $\mathcal{D}(\Gamma)$ be a porous body in $\mathbb{R}^2$, which contains the domain $B_1(0)$. $\Gamma$ is a bounded and connected $C^\infty$ smooth curve describing the boundary of $\mathcal{D}$. Suppose that $\Omega$ is the exterior domain of $\mathcal{D}(\Gamma)$, i.e., $\Omega := \mathbb{R}^2 \setminus \mathcal{D}$, which is connected and filled with compressible and inviscid fluid.

In this paper, we aim to construct a smooth subsonic spiral flow outside $\mathcal{D}(\Gamma)$ which tends to the radially symmetric subsonic spiral flow described above. $\mathcal{D}(\Gamma)$ is a porous body, which
means that on the boundary, we impose that
\[ \rho \mathbf{u} \cdot \mathbf{n} = \rho_0(r)U_1(r)\mathbf{e}_r \cdot \mathbf{n}, \quad (1.16) \]
where \( \mathbf{n} \) stands for the unit outward normal of domain \( \Omega \). It follows from (1.16) that \( \psi = \psi_{10} \) on \( \Gamma \).

Moreover, the radially symmetric subsonic spiral flow \( \mathbf{u}_0(x) \) decays like \( O(|x|^{-1}) \), so we may have the decay rate \( |\mathbf{u}(x) - \mathbf{u}_0(x)| = O(|x|^{-2}) \) as \( |x| \to \infty \). In terms of the stream function, we expect that
\[ |\nabla(\psi - \psi_0)| = O(|x|^{-2}). \quad (1.17) \]

In this paper, we will consider the following problem:

**Problem** \((\kappa_1, \kappa_2)\): Find function \( \psi \) which solves the following system:
\[
\begin{cases}
\text{div} \left( \frac{\nabla \psi}{H(|\nabla \psi|^2)} \right) = 0, & \text{in } \Omega, \\
\psi = \psi_{10}, & \text{on } \Gamma, \\
|\nabla(\psi - \psi_0)| = O(|x|^{-2}).
\end{cases} \quad (1.18)
\]

The main results of this paper can be stated as follows:

**Theorem 1.1** For any fixed \( \kappa_1 \in (0, 1) \), there exists a positive number \( \kappa_2 \) such that, for \( |\kappa_2| \in [0, \kappa_2] \), there exists a unique uniformly subsonic spiral flows to Problem \((\kappa_1, \kappa_2)\). More precisely, there exists a unique smooth solution \( \psi \in C^\infty(\Omega) \) to (1.18) such that
\[ \sup_{x \in \Omega} |\nabla \psi| < 1. \quad (1.19) \]

**Theorem 1.2** Let \( \kappa_2 \to \kappa_2 \) as \( \epsilon \to 0 \) with \( \kappa_2 < \kappa_2 \). Denote by \((u_1^\epsilon, u_2^\epsilon)\) the uniformly subsonic spiral flow corresponding to Problem \((\kappa_1, \kappa_2^\epsilon)\). Then there exists a subsequence, still labeled by \((u_1^\epsilon, u_2^\epsilon)\), such that
\[ u_1^\epsilon \to \bar{u}_1, \quad u_2^\epsilon \to \bar{u}_2, \quad (1.20) \]
\[ g((q^\epsilon)^2)u_1^\epsilon \to g(q^2)\bar{u}_1, \quad g((q^\epsilon)^2)u_2^\epsilon \to g(q^2)\bar{u}_2, \quad (1.21) \]
where \( (q^\epsilon)^2 = (u_1^\epsilon)^2 + (u_2^\epsilon)^2 \), \( q^2 = \bar{u}_1^2 + \bar{u}_2^2 \), and where \( g(q^2) \) is the function defined by (1.5). All the above convergences are almost everywhere convergent. Moreover, this limit yields a subsonic-sonic spiral flow \((\hat{\rho}, \hat{u}_1, \hat{u}_2)\), where \( \hat{\rho} = g(q^2) \), which is a weak solution of Problem \((\kappa_1, \kappa_2)\). The limit solution \((\hat{\rho}, \hat{u}_1, \hat{u}_2)\) satisfies (1.1)–(1.2) in the sense of distribution and the boundary condition (1.16) as the normal trace of the divergence-measure field \((\hat{\rho}\hat{u}_1, \hat{\rho}\hat{u}_2)\) on the boundary.

**Theorem 1.3** For any fixed \( \kappa_2 \in (-1, 1) \), there exists a positive number \( \kappa_1 \) such that for \( \kappa_1 \in (0, \kappa_1) \), there exists a unique uniformly subsonic spiral flows to Problem \((\kappa_1, \kappa_2)\). More precisely, there exists a unique smooth solution \( \psi \in C^\infty(\Omega) \) to (1.18) such that \( \sup_{x \in \Omega} |\nabla \psi| < 1 \).

**Theorem 1.4** Let \( \kappa_1^\epsilon \to \kappa_1 \) as \( \epsilon \to 0 \) with \( \kappa_1^\epsilon < \kappa_1 \). Denote by \((u_1^\epsilon, u_2^\epsilon)\) the uniformly subsonic spiral flow corresponding to Problem \((\kappa_1^\epsilon, \kappa_2^\epsilon)\). Then there exists a subsequence, still labeled by \((u_1^\epsilon, u_2^\epsilon)\) such that
\[ u_1^\epsilon \to \bar{u}_1, \quad u_2^\epsilon \to \bar{u}_2, \quad (1.22) \]
\[ g((q^\epsilon)^2)\bar{u}_1 \to g(q^2)\bar{u}_1, \quad g((q^\epsilon)^2)\bar{u}_2 \to g(q^2)\bar{u}_2, \quad (1.23) \]
where \((q')^2 = (u_1')^2 + (u_2')^2\), \(\tilde{q}^2 = \tilde{u}_1^2 + \tilde{u}_2^2\). All the above convergences are almost everywhere convergent. Moreover, this limit yields a subsonic-sonic spiral flow \((\tilde{\rho}, \tilde{u}_1, \tilde{u}_2)\), where \(\tilde{\rho} = g(\tilde{q}^2)\), which is a weak solution of Problem \((\kappa_1, \kappa_2)\). The limit solution \((\tilde{\rho}, \tilde{u}_1, \tilde{u}_2)\) satisfies the boundary condition (1.16) as the normal trace of the divergence-measure field \((\tilde{\rho}\tilde{u}_1, \tilde{\rho}\tilde{u}_2)\) on the boundary.

**Remark 1.5** The decay rate \(|\nabla (\psi - \psi_0)| = O(|x|^{-2})\) is optimal as was observed by the results in [3], but here we remove the small perturbation conditions in [3]. We employe the Kevin’s transformation and some elliptic estimates to derive it.

**Remark 1.6** We prescribe the boundary condition (1.16) on the boundary \(\Gamma\), not the one that \(\rho u \cdot \vec{n} = (\rho_b(r)U_1(r)e_r + \rho_b(r)U_2(r)e_\theta) \cdot \vec{n}\). Otherwise, the solution will be the background radially symmetric solution. One may pose \(\rho u \cdot \vec{n} = \nabla \perp \Psi \cdot \vec{n}\) on \(\Gamma\) for any given smooth function \(\Psi\); our method still works in this case.

The research on compressible inviscid flows has a long history, and has provided many significant and challenging problems. The flow past a body, through a nozzle, and past a wall are typical flows patterns, which have physical significances and physical effects. The first theoretical result on the problem for irrotational flows past a body was obtained by Frankl and Keldysh in [14]. Important progress for two dimensional subsonic irrotational flows past a smooth body with a small free stream Mach number was made by Shiffman [21]. Later on, Bers [1] proved the existence of two dimensional subsonic irrotational flows around a profile with a sharp trailing edge, and also showed that the maximum of Mach numbers approaches one as the free stream Mach number approaches the critical value. The uniqueness and asymptotic behavior of subsonic irrotational plane flows were studied by Finn and Gilbarg in [12]. The well-posedness theory for two-dimensional subsonic flows past a wall or a symmetric body was established by Chen, Du, Xie and Xin in [2]. For the three-(or higher-) dimensional cases, the existence and uniqueness of three dimensional subsonic irrotational flows around a smooth body were proved by Finn and Gilbarg in [13]. Dong and Ou [9] extended the results of Shiffman to higher dimensions by the direct method of the calculus of variations and the standard Hilbert space method. The respective incompressible case is considered in [20]. For the subsonic flow problem in nozzles, one may refer to [8, 10, 11, 16, 19, 22, 25, 26].

On the other hand, the subsonic-sonic limit solution can be constructed by the compactness method. The first compactness framework on sonic-subsonic irrotational flows in two dimension was introduced by [4] and [25] independently. In [4], Chen, Dafermos, Slemrod and Wang introduced a general compactness framework and proved the existence of two-dimensional subsonic-sonic irrotational flows. Xie and Xin [25] established the subsonic-sonic limit of the two-dimensional irrotational flows through infinitely long nozzles. Later on, they extended the result to the three-dimensional axisymmetric flow through an axisymmetric nozzle in [26]. Furthermore, Huang, Wang and Wang [18] established a compactness framework for the general multidimensional irrotational case. Chen, Huang and Wang [6] established the compactness framework for nonhomentropic and rotational flows and proved the existence of multidimensional subsonic-sonic full Euler flows through infinitely long nozzles. Recently, by the compactness framework in [6], the existence of subsonic-sonic flows with general conservative forces in an exterior domain was established by Gu and Wang in [17].
The rest of this article is organized as follows: in Section 2, we introduce the subsonic truncation to reformulate the problem into a second-order uniformly elliptic equation, and then establish the existence and uniqueness of the modified spiral flow by a variational method. Then we remove the truncation and complete the proof of Theorems 1.1 and 1.3. In Section 3, the compensated compactness framework for steady irrotational flows is employed to establish the existence of weak subsonic-sonic spiral flows.

2 Subsonic Spiral Flows

This section is mainly devoted to the proof of Theorems 1.1 and 1.3. The proof can be divided into 5 subsections.

2.1 Subsonic truncation

By direct calculation, it is easy to see that the derivative of function $H(s)$ goes to infinity as $s \to 1$. To control the ellipticity and avoid the singularity of $H'$, one may truncate $H$ as

$$
\tilde{H}(s) = \begin{cases} 
H(s), & \text{if } 0 \leq s \leq 1 - 2\varepsilon, \\
\text{smooth and decreasing}, & \text{if } 1 - 2\varepsilon \leq s \leq 1 - \varepsilon, \\
H(1 - \varepsilon), & \text{if } s \geq 1 - \varepsilon,
\end{cases}
$$

(2.1)

where $\varepsilon$ is a small positive constant and $\tilde{H}$ is a smooth decreasing function.

Then we consider the modified equation

$$
\begin{cases}
\text{div} \left( \frac{\nabla \psi}{H(|\nabla \psi|^2)} \right) = 0, & \text{in } \Omega, \\
\psi = \psi_{10}, & \text{on } \Gamma, \\
|\nabla (\psi - \psi_0)| = O(|x|^{-2}).
\end{cases}
$$

(2.2)

After straightforward computation, (2.2) can be rewritten as

$$
\sum_{i,j=1}^{2} a_{ij} \partial_{ij} \psi = 0,
$$

(2.3)

where

$$
a_{ij} = \frac{\tilde{H} \delta_{ij} - 2 \tilde{H}' \partial_i \psi \partial_j \psi}{\tilde{H}^2}.
$$

Then it is easy to verify that there exist two constants, $\lambda$ and $\Lambda$, such that

$$
\lambda |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \text{for } \xi \in \mathbb{R}^2.
$$

(2.4)

Here $\lambda$ and $\Lambda$ depend on $\gamma$ and $\varepsilon$.

Next, we solve (2.2) by a variational method. We first give the definition of a weak solution to be used in next subsection.

**Definition 2.1** A function $\psi \in H^1_{\text{loc}}(\Omega)$ is a weak solution of (2.2) if

$$
\int_{\Omega} (\tilde{H}(|\nabla \psi|^2))^{-1} \nabla \psi \cdot \nabla v dx = 0
$$

holds for any $v \in C^\infty_c(\Omega)$. 

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2.2 Existence of a weak solution to problem (2.2)

Define the space
\[ V = \{ \phi \in L^2_{\text{loc}}(\Omega), \nabla \phi \in L^2(\Omega), \phi|_{\Gamma} = 0 \}. \] (2.5)

\( V \) is a Hilbert space under the norm
\[ \| \phi \|_V = \left( \int_{\Omega} |\nabla \phi|^2 \, dx \right)^{\frac{1}{2}}. \]

One should look for the solution \( \psi \) to problem (2.2) with the form \( \psi = \phi_0 + \psi_0 \), where \( \phi_0 \in L^2_{\text{loc}}(\Omega) \) and \( \nabla \phi_0 \in L^2(\Omega) \). It follows from the boundary condition in (2.2) that \( \phi_0 = -\psi_0 \) on \( \Gamma \). To homogenize the boundary data on \( \Gamma \), we introduce another function \( \varphi_0 \) such that \( \phi = \phi_0 - \varphi_0 \in V \).

Let \( \zeta(x) \) be a cut-off function such that \( \zeta \equiv 1 \) near \( \Gamma \), and \( \zeta \equiv 0 \) outside a circle \( B_{R_0} \) with \( R_0 \) large enough. Then set \( \varphi_0 = -\zeta \psi_0 \). By simple calculations, \( -\zeta \psi_0 \) satisfies all the requirements that we imposed.

Define
\[ \psi = \phi + \varphi_0 + \psi_0, \quad F(s) = \int_0^s (\bar{H}(t))^{-1} \, dt, \]
and
\[ I[\phi; \varphi_0, \psi_0] = \int_{\Omega} [F(|\nabla \phi + \nabla \varphi_0 + \nabla \psi_0|^2) - F(|\nabla \psi_0|^2) - 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot (\nabla \phi + \nabla \varphi_0)] \, dx. \] (2.6)

We consider the following variational problem:
\[ l = \min_{\phi \in V} I[\phi; \varphi_0, \psi_0]. \] (2.7)

For our variational problem, we have the following theorem:

**Theorem 2.2** The functional \( I \) has uniquely one minimizer \( \phi \), i.e. \( I[\phi; \varphi_0, \psi_0] = l \). Then \( \psi = \phi + \varphi_0 + \psi_0 \) satisfies
\[ \begin{cases} \text{div} \left( \frac{\nabla \psi}{H(|\nabla \psi|^2)} \right) = 0, & \text{in } \Omega, \\ \psi = \psi_{10}, & \text{on } \Gamma, \end{cases} \] (2.8)
in the weak sense. In addition, there is a constant \( C \) such that
\[ \int_{\Omega} |\nabla \phi|^2 \, dx \leq C. \] (2.9)

The constant \( C \) here, and in the rest of this paper, depends on \( \Omega, \gamma, \varepsilon, \kappa_1 \) and \( \kappa_2 \). We will not repeat the dependence every time.

**Proof Step 1** \( I[\phi; \varphi_0, \psi_0] \) is coercive on \( V \). Let \( p = (p_1, p_2) \), \( F(p) = F(|p|^2) \). It follows from (2.4) that
\[ \forall \xi \in \mathbb{R}^2, \quad \lambda |\xi|^2 \leq \partial^2_{p_i p_j} F(p) \xi_i \xi_j \leq \Lambda |\xi|^2. \] (2.10)

Then, by direct calculation, we have
\[
\begin{align*}
F(|\nabla \phi + \nabla \varphi_0 + \nabla \psi_0|^2) - F(|\nabla \psi_0|^2) - 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot (\nabla \phi + \nabla \varphi_0) \\
= F(\nabla \psi) - F(\nabla \psi_0) - \partial_p F(\nabla \psi_0) \cdot (\nabla \phi + \nabla \varphi_0) \\
= \int_0^1 t \partial^2_{p_i p_j} F(\nabla \psi_0 + (1-t)(\nabla \phi + \nabla \varphi_0)) \partial_i (\phi + \varphi_0) \partial_j (\phi + \varphi_0) \, dt.
\end{align*}
\]
This implies that
\[ \Lambda \int_\Omega |\nabla \phi + \nabla \varphi_0|^2 \, dx \geq I \geq \lambda \int_\Omega |\nabla \phi + \nabla \varphi_0|^2 \, dx \geq \frac{\lambda}{2} \|\phi\|_V^2 - \lambda \int_\Omega |\nabla \varphi_0|^2 \, dx. \]  
(2.11)

Therefore we obtain
\[ I \geq \frac{\lambda}{2} \|\phi\|_V^2 - C. \]  
(2.12)

From (2.12), one can conclude that the energy functional is coercive on \( \mathcal{V} \).

**Step 2** The existence of the minimizer \( \phi \in \mathcal{V} \). By the coerciveness of the functional \( I(\cdot; \varphi_0, \psi_0) \) on \( \mathcal{V} \), we know that \( I = \inf_{\phi \in \mathcal{V}} I(\phi; \varphi_0, \psi_0) \) exists. Since \( I(0; \varphi_0, \psi_0) \leq \Lambda \|\varphi_0\|_V^2 \leq C \), we have \( I \leq C \). By definition, there is a sequence \( \{\phi_k\}_{k=1}^\infty \subset \mathcal{V} \) such that \( \lim_{k\to\infty} I(\phi_k; \varphi_0, \psi_0) = I \).

For sufficiently large \( k \),
\[ C \geq I(\phi_k; \varphi_0, \psi_0) \geq \frac{\lambda}{2} \|\phi_k\|_V^2 - C, \]
hence \( \|\phi_k\|_V^2 \leq \frac{4C}{\lambda} \). Since \( \mathcal{V} \) is a Hilbert space, there exists a subsequence of \( \{\phi_k\} \) (still denoted by \( \{\phi_k\} \)) that converges weakly to some \( \phi \in \mathcal{V} \).

Next, we need to show that the functional \( I(\cdot; \varphi_0, \psi_0) \) is lower semi-continuous with respect to the weak convergence of \( \mathcal{V} \), that is
\[ I(\phi; \varphi_0, \psi_0) \leq \liminf_{k\to\infty} I(\phi_k; \varphi_0, \psi_0), \text{ if } \phi_k \rightharpoonup \phi \text{ in } \mathcal{V}. \]
(2.13)

It follows from \( \phi_k \rightharpoonup \phi \) in \( \mathcal{V} \) that \( \nabla \phi_k \rightharpoonup \nabla \phi \) in \( L^2_{\text{loc}}(\Omega) \). Furthermore, one has
\[ 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \in L^2_{\text{loc}}(\Omega). \]

Denote \( \Omega_{R_j} = \Omega \cap B_{R_j}(0) \), where \( \{R_j\}_{j=1}^\infty \subset \mathbb{R} \) is an increasing sequence and \( R_j \to \infty \) as \( j \to \infty \). Then
\[ \lim_{k\to\infty} \int_{\Omega_{R_j}} 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot (\nabla \phi_k + \nabla \varphi_0) \, dx = \int_{\Omega_{R_j}} 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot (\nabla \phi + \nabla \varphi_0) \, dx. \]  
(2.14)

For \( \psi_k = \phi_k + \varphi_0 + \psi_0 \), a simple calculation yields that
\[ F(|\nabla \psi_k|^2) - F(|\nabla \psi|^2) - 2F'(|\nabla \psi|^2) \nabla \psi \cdot (\nabla \phi_k - \nabla \phi) \]
\[ = \int_0^1 t \partial^2_{\psi^2} F(\nabla \psi + (1 - t)(\nabla \phi_k - \phi)) \partial_t (\nabla \phi_k - \phi) \partial_j (\phi_k - \phi) \, dt \geq 0. \]  
(2.15)

Similarly to (2.14), we have
\[ \lim_{k\to\infty} \int_{\Omega_{R_j}} 2F'(|\nabla \psi|^2) \nabla \psi \cdot (\nabla \phi_k - \nabla \phi) = 0. \]

So integrating both sides of (2.15) over \( \Omega_{R_j} \) leads to
\[ \liminf_{k\to\infty} \int_{\Omega_{R_j}} F(|\nabla \psi_k|^2) - F(|\nabla \psi|^2) \, dx \geq 0. \]  
(2.16)

Collecting (2.14)-(2.15) and (2.16) together gives
\[ \liminf_{k\to\infty} \int_{\Omega_{R_j}} [F(|\nabla \psi_k|^2) - F(|\nabla \psi|^2) - 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot (\nabla \phi_k + \nabla \varphi_0)] \, dx \]
\[ \geq \int_{\Omega_{R_j}} [F(|\nabla \psi|^2) - F(|\nabla \psi_0|^2) - 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot (\nabla \phi + \nabla \varphi_0)] \, dx. \]  
(2.17)
It follows from (2.17) that we deduce
\[
\liminf_{k \to \infty} I[\phi_k; \varphi_0, \psi_0] \geq \int_{\Omega_{R_j}} F(|\nabla \psi|^2) - F(|\nabla \psi_0|^2) - 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot (\nabla \phi + \nabla \phi_0) \, dx. \tag{2.18}
\]
This inequality holds for each $R_j$. Let $R_j \to \infty$, and use the monotone convergence theorem to conclude
\[
\liminf_{k \to \infty} I[\phi_k; \varphi_0, \psi_0] \geq I[\varphi; \varphi_0, \psi_0].
\]
Therefore we proved (2.13). $\phi \in V$ is actually a minimizer.

**Step 3** The uniqueness of the minimizer $\phi$. For any $\phi_1, \phi_2 \in V$, we derive that
\[
I[\phi_1; \varphi_0, \psi_0] + I[\phi_2; \varphi_0, \psi_0] - 2I[\frac{\phi_1 + \phi_2}{2}; \varphi_0, \psi_0] = \int_{\Omega} F(\nabla \psi_1) - F(\nabla \psi_2) - 2F\left(\frac{\nabla \psi_1 + \nabla \psi_2}{2}\right) \, dx \geq \frac{\lambda}{2} \|\psi_1 - \psi_2\|_{V}^2.
\]
For the uniqueness, suppose that $\phi_1$ and $\phi_2$ are minimizers. Then $\frac{\phi_1 + \phi_2}{2} \in V$, and
\[
I[\phi_1; \varphi_0, \psi_0] + I[\phi_2; \varphi_0, \psi_0] \leq 2I\left[\frac{\phi_1 + \phi_2}{2}; \varphi_0, \psi_0\right]. \tag{2.20}
\]
Using (2.20), one has that
\[
0 \geq I[\phi_1; \varphi_0, \psi_0] + I[\phi_2; \varphi_0, \psi_0] - 2I\left[\frac{\phi_1 + \phi_2}{2}; \varphi_0, \psi_0\right] \geq \frac{\lambda}{2} \|\phi_1 - \phi_2\|_{V}^2, \tag{2.21}
\]
which implies that the minimizer is unique in $V$.

**Step 4** $\psi = \phi + \varphi_0 + \psi_0$ satisfies (2.8) in the weak sense. It follows from Steps 1–3 that the functional $I$ has a unique minimizer $\phi$. For any $v \in C^\infty_c(\Omega)$, define $I[\phi + \tau v; \varphi_0, \psi_0]$ for $\tau > 0$. Then
\[
I(\tau) - I(0) = \int_{\Omega} [F(\nabla \phi + \tau \nabla v + \nabla \varphi_0 + \nabla \psi_0) - F(\nabla \phi + \nabla \varphi_0 + \nabla \psi_0)] \, dx - \tau \int_{\Omega} 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot \nabla v \, dx = V_1 + \tau V_2.
\]
Since $\text{div}(F'(|\nabla \psi_0|^2) \nabla \psi_0) = 0, V_2 = 0$ for any $v \in C^\infty_c(\Omega)$. Hence we deduce that
\[
\frac{I(\tau) - I(0)}{\tau} = \frac{1}{\tau} \int_{\Omega} [F(\nabla \phi + \tau \nabla v + \nabla \varphi_0 + \nabla \psi_0) - F(\nabla \phi + \nabla \varphi_0 + \nabla \psi_0)] \, dx - \int_{\Omega} 2F'(|\nabla \psi_0|^2) \nabla \psi_0 \cdot \nabla v \, dx = W_1 + \tau W_2 + \tau W_3.
\]
Due to truncation, $F'$ is bounded. Then $W_2 \leq \tau \Lambda \|v\|_{V}^2$, so $\lim_{\tau \to 0} W_2 = 0$. By the Lebesgue convergence theorem, we get $\lim_{\tau \to 0} W_1 = 0$. Since $I'(0) = 0$, this gives $W_3 = 0$. Therefore $\psi = \phi + \varphi_0 + \psi_0$ is a weak solution to (2.8). \hfill \square
2.3 Regularity of weak solution

In this section, we verify that the weak solution obtained from the variation in subsection 2.2 is a classical solution.

Lemma 2.3 Suppose that \( \psi \in H^1_{\text{loc}}(\Omega) \) is a weak solution to problem (2.2). Then there is a constant \( \alpha \in (0, 1) \) such that, for any subregion \( \Omega_1 \subset (\Gamma \cup \Omega) \), it holds that

\[
\sup_{x \in \Omega_1} |\nabla \psi| + \sup_{x_1, x_2 \in \Omega_1} \frac{|\nabla \psi(x_1) - \nabla \psi(x_2)|}{|x_1 - x_2|^\alpha} \leq C,
\]

where \( C \) depends on \( \Omega_1, \Omega, \gamma, \epsilon, \rho_0, \kappa_1, \kappa_2 \).

Proof Denote \( \bar{\psi}_k = \partial_k \psi \) for \( k = 1, 2 \). It is easy to verify that

\[
\partial_i (a_{ij} \partial_j \bar{\psi}_k) = 0,
\]

where \( a_{ij} = F'(|\nabla \psi|^2) \delta_{ij} + 2F''(|\nabla \psi|^2) \partial_i \psi \partial_j \psi \). \( [a_{ij}] \) has uniformly positive eigenvalues.

For the interior estimate, let \( U \) be a bounded interior subregion of \( \Omega \). It follows from Theorem 8.24 in [15] that \( ||\bar{\psi}_k||_{C^\alpha(U)} \leq C||\bar{\psi}_k||_{L^2(\Omega)} \). According to (2.9), we know that \( \|\bar{\psi}_k\|_{L^2(\Omega)} \) is uniformly bounded. From the standard elliptic estimate, we obtain (2.22).

Next, for the boundary estimate near \( \Gamma \), one can apply Theorem 8.29 in [15] and follow the arguments above to show that (2.22) holds.

Then a standard bootstrap argument implies that \( \psi \) actually belongs to \( C^\infty(\Omega) \) and that \( \psi \) is a classical solution.

Now, we use an \( L^\infty \) bound of the gradient to show that \( |\nabla (\psi - \psi_0)| \) tends to zero as \( |x| \to \infty \) with a decay rate \( |x|^{-2} \) in \( \Omega \). \( \square \)

Lemma 2.4 We have the following estimate of the continuity of \( \nabla \psi \) at infinity:

\[
|\nabla \psi - \nabla \psi_0| \leq \frac{C}{(1 + |x|)^2},
\]

Proof For a fixed \( R \gg 1 \) with \( N = \{|x| \geq R\} \subset \Omega \), set \( \tilde{\psi}_1 = \partial_1 \psi = -\rho u_2 \) and \( \tilde{\psi}_2 = \partial_2 \psi = \rho u_1 \). Then we have the elliptic system

\[
-a_{22}(x) \partial_{x_2} \tilde{\psi}_2 = a_{11}(x) \partial_{x_1} \tilde{\psi}_1 + 2a_{12}(x) \partial_{x_1} \tilde{\psi}_1, \quad \partial_{x_1} \tilde{\psi}_2 = \partial_{x_2} \tilde{\psi}_1,
\]

where

\[
\begin{align*}
a_{22}(x) &= F'(|\nabla \psi|^2) + 2F''(|\nabla \psi|^2) \tilde{\psi}_2^2, \\
a_{12}(x) &= 2F'(|\nabla \psi|^2) + 2F''(|\nabla \psi|^2) \tilde{\psi}_1 \tilde{\psi}_2, \\
a_{11}(x) &= F'(|\nabla \psi|^2) + 2F''(|\nabla \psi|^2) \tilde{\psi}_1^2.
\end{align*}
\]

Applying Theorem 3 in [12], we know that \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) can be represented in the form

\[
\begin{align*}
\begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2
\end{pmatrix}
&= \begin{pmatrix}
\tilde{\psi}_{10} \\
\tilde{\psi}_{20}
\end{pmatrix} + \frac{1}{c_0 x_1^2 + b_0 x_1 x_2 + a_0 x_2^2} \begin{pmatrix}
\beta_1 x_1 + \beta_2 x_2 \\
\gamma_1 x_1 + \gamma_2 x_2
\end{pmatrix} \left( O(|x|^{-1-\alpha}) \right) + O(|x|^{-1-\alpha}) \), as \( |x| \to \infty \),
\end{align*}
\]

\( \square \) Springer
where $0 < \alpha < 1$ and
\[
\begin{align*}
\lim_{|x| \to \infty} \psi_1 &= \psi_{10}, \\ \lim_{|x| \to \infty} \psi_2 &= \psi_{20},
\end{align*}
\]
\[
c_0 = F'(\psi_{10} + \psi_{20}) + 2F''((\psi_{10} + \psi_{20})^2)\psi_{20},
\]
\[
b_0 = 2F''((\psi_{10} + \psi_{20})^2)\psi_{10}\psi_{20},
\]
\[
a_0 = F'((\psi_{10} + \psi_{20})^2) + 2F''((\psi_{10} + \psi_{20})^2)\psi_{10}^2,
\]
\[
c_0(\gamma_1 + \beta_2) + 2b_0\beta_1 = 0, \quad c_0\gamma_2 = a_0\beta_1.
\]
Since $\nabla \psi - \nabla \psi_0 \in L^2(\Omega)$, it follows from (2.26) that $\psi_{10} = \psi_{20} = 0$. Thus, $a_0 = c_0 = F'(0)$, $b_0 = 0$, $\gamma_1 + \beta_2 = 0$ and $\gamma_2 = \beta_1$. Hence (2.26) can be rewritten as
\[
\nabla \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{F'(0)(x_1^2 + x_2^2)} \begin{pmatrix} \beta_1(x_1 \ x_2) + \beta_2 \left( \begin{array}{c} x_2 \\ -x_1 \end{array} \right) \end{pmatrix} + \begin{pmatrix} O(|x|^{-1-\alpha}) \\ O(|x|^{-1-\alpha}) \end{pmatrix}, \quad \text{as } |x| \to \infty.
\]
(2.27)

Then $\nabla \psi - \nabla \psi_0 \in L^2(\Omega)$ implies that
\[
\nabla \psi_0 = \frac{1}{F'(0)(x_1^2 + x_2^2)} \begin{pmatrix} \beta_1(x_1 \ x_2) + \beta_2 \left( \begin{array}{c} x_2 \\ -x_1 \end{array} \right) \end{pmatrix}, \quad \text{as } |x| \to \infty.
\]
(2.28)

Since $\psi = \phi_0 + \psi_0$, we have that
\[
\nabla \phi_0 = \begin{pmatrix} O(|x|^{-1-\alpha}) \\ O(|x|^{-1-\alpha}) \end{pmatrix}, \quad \text{as } |x| \to \infty.
\]
(2.29)

Furthermore, it follows from (2.25) that we can derive
\[
\sum_{i,j=1}^2 a_{ij}(x)\partial_{x_i x_j} \tilde{\psi}_1 + \sum_{i=1}^2 c_i(x)\partial_{x_i} \tilde{\psi}_1 = 0,
\]
(2.30)

where
\[
c_1(x) = \partial_{x_1} a_{11}(x) - \frac{a_{11}(x)}{a_{22}(x)} \partial_{x_1} a_{22}(x),
\]
\[
c_2(x) = 2\partial_{x_1} a_{12}(x) - \frac{2a_{12}(x)}{a_{22}} \partial_{x_1} a_{22}(x).
\]

We set $\tilde{\phi}_0 = \partial_1 \phi_0$ and $\tilde{\psi}_0 = \partial_1 \psi_0$. Then $\tilde{\phi}_0$ satisfies that
\[
\sum_{i,j=1}^2 a_{ij}(x)\partial_{x_i x_j} \tilde{\phi}_0 + \sum_{i=1}^2 c_i(x)\partial_{x_i} \tilde{\phi}_0 + \sum_{i,j=1}^2 a_{ij}(x)\partial_{x_i x_j} \tilde{\psi}_0 + \sum_{i=1}^2 c_i(x)\partial_{x_i} \tilde{\psi}_0 = 0.
\]
(2.31)

Introduce Kelvin’s transform
\[
y_1 = \frac{R^2}{|x|^2} x_1 =: \Phi^1(x), \quad y_2 = \frac{R^2}{|x|^2} x_2 =: \Phi^2(x),
\]
(2.32)

and write
\[
y = \Phi(x), \quad x = \Psi(y),
\]
where $\Psi = \Phi^{-1}$.

Set $\phi_0^\pm(y) = \tilde{\phi}_0(\Psi(y))$ and $\psi_0^\pm(y) = \tilde{\psi}_0(\Psi(y))$. Then it follows from (2.29) and (2.32) that $\phi_0^\pm(y) = O(|y|^{1+\alpha})$. Thus,
\[
\phi_0^\pm(0) = 0, \quad \nabla \phi_0^\pm(0) = 0.
\]
(2.33)
Meanwhile, under this transformation, (2.31) can be written as

\[
\sum_{i,j=1}^{2} a_{ij}^2(y) \partial_{y_i y_j} \phi_{0}^2 + \sum_{i=1}^{2} c^2_i(y) \partial_{y_i} \phi_{0}^2 = f^2(y),
\]  

(2.34)

where

\[
\begin{align*}
& a_{11}^2(y) = a_{11}(\Psi(y))(\Phi_{x_i}^1)^2 + 2a_{12}(\Psi(y))\Phi_{x_i}^1 \Phi_{x_2}^1 + a_{22}(\Psi(y))(\Phi_{x_2}^1)^2, \\
& a_{12}^2(y) = a_{21}(\Psi(y)) = a_{11}(\Psi(y))(\Phi_{x_1}^1)^2 + a_{12}(\Psi(y))(\Phi_{x_1}^1 \Phi_{x_2}^1 + \Phi_{x_2}^1 \Phi_{x_1}^1) + a_{22}(\Psi(y))(\Phi_{x_2}^1)^2, \\
& a_{22}^2(y) = a_{11}(\Psi(y))(\Phi_{x_1}^2)^2 + 2a_{12}(\Psi(y))\Phi_{x_1}^2 \Phi_{x_2}^2 + a_{22}(\Psi(y))(\Phi_{x_2}^2)^2, \\
& c^2_1(y) = a_{11}(\Psi(y))\Phi_{x_1 x_1}^1 + 2a_{12}(\Psi(y))\Phi_{x_1 x_2}^1 + a_{22}(\Psi(y))\Phi_{x_2 x_2}^1, \\
& c^2_2(y) = a_{11}(\Psi(y))\Phi_{x_1 x_1}^2 + 2a_{12}(\Psi(y))\Phi_{x_1 x_2}^2 + a_{22}(\Psi(y))\Phi_{x_2 x_2}^2,
\end{align*}
\]

\[f^2(y) = -\sum_{i,j=1}^{2} a_{ij}^2(y) \partial_{y_i y_j} \psi_{0}^2 - \sum_{i=1}^{2} c^2_i(y) \partial_{y_i} \psi_{0}^2.\]

Applying Schauder estimates in [15], we deduce that

\[|\phi_{0}^2|_{2,\alpha;B_{R/2}} \leq C(|\phi_{0}^2|_{0;B_{R}} + |f^2|_{0;\alpha;B_{R}}).\]

(2.35)

Since \(\partial_i(a_{ij} \partial_j \phi_{0}) + \partial_i(a_{ij} \partial_j \psi_{0}) = 0\), we can apply Theorem 8.15 in [15] to derive that \(|\phi_{0}|_{0;N} \leq C\). By substituting \(|\phi_{0}^2|_{0;B_{R}} = |\tilde{\phi}_{0}|_{0;N} \leq C\) into (2.35), we infer that

\[|\phi_{0}^2|_{2,\alpha;B_{R/2}} \leq C.\]

(2.36)

Combining that with (2.33), we obtain that

\[|\phi_{0}^2| \leq C|y|^2, \quad \text{in } B_{R/2}.\]

(2.37)

Hence,

\[|\tilde{\phi}_{0}| \leq \frac{C}{|x|^2}, \quad \text{in } \{|x| \geq 2R\}.\]

(2.38)

Thus (2.24) can be obtained from (2.38).

\[\square\]

2.4 Modified spiral flows

For the modified equation (2.2), we have the following theorem:

**Theorem 2.5** For every \(\psi_{0}, \ (2.2)\) has a unique classical spiral solution \(\psi\) such that \(\psi = \phi + \varphi_{0} + \psi_{0}\) with \(\phi \in V\). Furthermore, for each fixed \(\kappa_{1}, \nabla \psi\) depends on \(\kappa_{2}\) continuously. In particular, \(\max_{\Omega} |\nabla \psi|\) is a continuous function of \(|\kappa_{2}|\). For each fixed \(\kappa_{2}\), we have the similar conclusion.

\[\square\] Springer
Proof The existence follows from Theorem 2.2, and the regularity estimate follows from Lemmas 2.3 and 2.4. To prove the uniqueness, we assume that classical solutions
\[ \psi_i = \phi_i + \varphi_0 + \psi_0, \quad i = 1, 2 \quad \text{with } \phi_i \in \mathcal{V} \]
would be both critical points of \( I[\phi; \varphi_0, \psi_0] \). Let \( I_\phi' \) be the Fréchet derivative. Then we have
\[
0 = (I_\phi'(\phi_1; \varphi_0, \psi_0) - I_\phi'(\phi_2; \varphi_0, \psi_0), \phi_1 - \phi_2)
= \int_{\Omega} \left[ \partial_p F(\nabla \phi_1 + \varphi_0 + \psi_0) - \partial_p F(\nabla \phi_2 + \varphi_0 + \nabla \psi_0) \right]|(\nabla \phi_1 - \nabla \phi_2) dx
= \int_{\Omega} \int_0^1 \partial_{p}^2 F(t(\nabla \phi_1 + (1-t)\nabla \phi_2 + \nabla \varphi_0 + \nabla \psi_0) \partial_t (\phi_1 - \phi_2) dtdx
\geq \lambda \| \phi_1 - \phi_2 \|^2.
\]
Hence, \( \phi_1 = \phi_2 \).

Now we examine the continuity of \( I[\phi; \varphi_0, \psi_0] \). Letting \( \psi_i = \phi_i + \varphi_0 + \psi_0, \quad i = 1, 2, \) one has
\[
\begin{align*}
&= \int_{\Omega} \left[ \partial_p F(\nabla \phi_1 + \varphi_0 + \nabla \psi_0) - \partial_p F(\nabla \psi_0) \right]|(\nabla \phi_1 - \nabla \phi_2) dx
\end{align*}
\]
This implies that
\[
|I[\phi_1; \varphi_0, \psi_0] - I[\phi_2; \varphi_0, \psi_0]| \leq C(1 + \| \phi_1 \|_V + \| \phi_2 \|_V)(\| \phi_1 - \phi_2 \|_V), \quad (2.39)
\]
Since \( \psi_0 = \psi_{10} + \psi_{20}, \varphi_0 = -\zeta \psi_{20} \), the continuity of \( I[\phi; \varphi_0, \psi_0] \) on \( \psi_{10} \) and \( \psi_{20} \) follows from the equality
\[
\begin{align*}
I[\phi; \varphi_0, \psi_0] &= \int_{\Omega} t^2 P, F(\nabla \psi_0 + (1-t)(\nabla \phi + \nabla \varphi_0) \partial_t (\phi + \varphi_0) \partial_j (\phi + \varphi_0) dt
\end{align*}
\]
Now, for each fixed \( \kappa_1 \), we will prove the continuity of the solution on \( \kappa_2 \). Let \( \kappa_2 \) be a convergent sequence such that \( \kappa_2^m \to \kappa_2^* \) as \( m \to \infty \). Denote that
\[
\psi^m = \phi^m + \varphi_0 + \psi_{10} + \psi_{20} = \phi^m + \psi_0 + (1 - \zeta) \psi_{20}^m,
\]
where \( \psi_{20}^m = -\kappa_2^m \int_0^1 \rho_V(s) \mathrm{d}s \). First we show that \( \phi^m \to \phi_0^* \) in \( \mathcal{V} \), by using that \( \phi^m \) is a minimizing sequence of \( I[\phi; -\zeta \psi_{20}^*, \psi_{10} + \psi_{20}^*] \). Since \( \kappa_2^m \to \kappa_2^* \) as \( m \to \infty \), \( (1 - \zeta) \psi_{20}^m \to (1 - \zeta) \psi_{20}^* \), where \( \psi_{20}^* = -\kappa_2^* \int_0^1 \rho_V(s) \mathrm{d}s \). All estimates in Lemmas 2.3 and 2.4, as well as in (2.9), can be taken uniformly. In particular, \( \int_{\Omega} |\nabla \phi^m|^2 \) and \( \max_{\Omega} |\nabla \phi^m| \) are uniformly bounded.

For any given \( \delta > 0 \) and sufficiently large \( m \), it follows from (2.39) and (2.40) that we have
\[
\begin{align*}
&\begin{cases}
|I[\phi^m; -\zeta \psi_{20}^*, \psi_{10} + \psi_{20}^*] - I[\phi^m; -\zeta \psi_{20}^*, \psi_{10} + \psi_{20}^*]| < \delta,
|I[\phi^m; -\zeta \psi_{20}^*, \psi_{10} + \psi_{20}^*] - I[\phi^m; -\zeta \psi_{20}^*, \psi_{10} + \psi_{20}^*]| < \delta.
\end{cases}
\end{align*}
\]
Combining this with the minimality of $\phi^m$ for $I[\phi; -\zeta \psi^m_{20}, \psi_{10} + \psi^m_{20}]$, we derive that

$$I[\phi^m; -\zeta \psi_{20}, \psi_{10} + \psi^m_{20}] \leq I[\phi^m; -\zeta \psi^m_{20}, \psi_{10} + \psi^m_{20}] + \delta \leq I[\phi^*_2; -\zeta \psi^*_2, \psi_{10} + \psi^*_2] + 2\delta.$$ 

Therefore, $\phi^m$ is a minimizing sequence of $I[\phi; -\zeta \psi^*_2, \psi_{10} + \psi^*_2]$. By the proof of the existence of the minimizer, we know that $\phi^m \to \phi^*_2$ in $V$. The uniform convergence of $\nabla \psi^m$ to $\nabla \psi^*_2$ follows from the uniform estimates in Lemmas 2.3 and 2.4 and an argument by contradiction. Then we conclude that $\max_{\Omega} |\nabla \psi^m| \to \max_{\Omega} |\nabla \psi^*_2|$. Hence $\max_{\Omega} |\nabla \psi|$ is a continuous function of $|\kappa_2|$.

Finally, for each fixed $\kappa_2$, we will prove the continuity of the solution on $\kappa_1$. Let $\kappa^i_2$ be a convergent sequence such that $\kappa^i_2 \to \kappa_1^*$ as $n \to \infty$. Denote that $\psi^i_n = \phi^i_n + \varphi_0 + \psi^m_{10} + \psi_{20}$. Similarly to the above proof, we have that $\max_{\Omega} |\nabla \psi^i_n| \to \max_{\Omega} |\nabla \psi^*_2|$. Hence $\max_{\Omega} |\nabla \psi|$ is a continuous function of $|\kappa_1|$.

### 2.5 Removal of the truncation

In this subsection, we remove the truncation and complete the proof of Theorems 1.1 and 1.3. We only need to prove Theorem 1.1, as the proof of Theorem 1.3 is similar.

We have thus far shown for a fixed parameter $\varepsilon$ that (2.2) has a unique classical solution $\psi$. Now we let $\{\varepsilon_i\}_{i=1}^\infty$ be a strictly decreasing sequence such that $\varepsilon_i \to 0$ as $i \to \infty$. For fixed $\kappa_1$ and $i$, there exists a maximum interval $[0, \kappa^i_2)$ such that, for $|\kappa_2| \in [0, \kappa^i_2)$, $|\nabla \psi| < \sqrt{1 - 2\varepsilon_i}$. Then $\psi$ is the solution to the original equation (1.18). From the uniqueness of (2.2), we can see that $\kappa^i_2 \leq \kappa_2$ for $i \leq j$. Thus $\{\kappa^i_2\}_{i=1}^\infty$ is an increasing sequence with the upper bound 1, which implies the convergence of the sequence. Therefore, we have

$$\dot{\kappa}_2 = \lim_{i \to \infty} \kappa^i_2.$$  \hspace{1cm} (2.41)

If $\kappa^i_2 < \kappa_2$ for any $i$, then, for any $|\kappa_2| \in [0, \kappa_2)$, there exists an index $i$ such that $|\kappa_2| \in [0, \kappa^i_2)$, so the truncation can be removed such that $\psi$ is the classical solution of (1.18) and $\max_{\Omega} |\nabla \psi| < 1$.

The uniqueness of subsonic spiral flows is easy to obtain. Suppose that there are two solutions to (1.18) such that $\psi_1 = \phi_i + \varphi_0 + \psi_0$, $i = 1, 2$ with $\phi \in V$ and $\max_{\Omega} |\nabla \psi_1| < 1$. A small $\varepsilon$ can be picked such that $\max_{\Omega} |\nabla \psi| < \sqrt{1 - 2\varepsilon}$. Both solutions will be solutions to the modified equation (2.2) with $\varepsilon$. It follows from the uniqueness of (2.2) that $\psi_1 = \psi_2$. Therefore the proof of Theorem 1.1 is complete.

Moreover, we have that $\max_{\Omega} |\nabla \psi| \to 1$ as $|\kappa_2| \to \dot{\kappa}_2$. It is expected that subsonic spiral flows will tend to some subsonic-sonic spiral flows as $|\kappa_2| \to \dot{\kappa}_2$. We will study this limiting behavior by a compensated compactness framework.

### 3 Subsonic-Sonic Spiral Flows

In this section, similarly to the subsonic case, we only need to prove Theorem 1.2. First, let us recall the compensated compactness framework for steady irrotational flows [18].

**Theorem 3.1** Let $u'(x_1, x_2) = (u'_1, u'_2)(x_1, x_2)$ be sequence of functions satisfying the following set of conditions (A):

(A.1) $q'(x_1, x_2) = |u'(x_1, x_2)| \leq 1$ a.e. in $\Omega$.
(A.2) curl \( \mathbf{u}^\epsilon \) and \( \text{div} (g(q^\epsilon)^2 \mathbf{u}^\epsilon) \) are confined in a compact set in \( H^{-1}_\text{loc} (\Omega) \).

Then there exists a subsequence (still labeled) \( \mathbf{u}^\epsilon \) that converges a.e. as \( \epsilon \to 0 \) to \( \hat{\mathbf{u}} \) satisfying that

\[
\hat{q}(x_1, x_2) = |\hat{\mathbf{u}}(x_1, x_2)| \leq 1 \quad \text{a.e.} \quad (x_1, x_2) \in \Omega.
\]

Let \( (\rho^\epsilon, u_1^\epsilon, u_2^\epsilon) \) denote the solutions obtained in Theorem 1.1 to problem \((\kappa_1, \kappa_2)\). Then we have that

\[
\begin{cases}
\partial_{x_1} ((g(q^\epsilon)^2)u_1^\epsilon) + \partial_{x_2} ((g(q^\epsilon)^2)u_2^\epsilon) = 0, \\
\partial_{x_2} u_1^\epsilon - \partial_{x_1} u_2^\epsilon = 0.
\end{cases}
\]

Thus conditions (A.1) and (A.2) in Theorem 3.1 are both satisfied. Theorem 3.1 implies that the solution sequence has a subsequence (still denoted by) \( (\hat{\rho}, \hat{u}_1, \hat{u}_2) \) that converges a.e. to a vector function \((\hat{\rho}, \hat{u}_1, \hat{u}_2)\). Since (1.1)\(_2\) and (1.1)\(_3\) hold for the sequence of subsonic solutions \((\rho^\epsilon, u_1^\epsilon, u_2^\epsilon)\), it is easy to see that \((\hat{\rho}, \hat{u}_1, \hat{u}_2)\) also satisfies (1.1)\(_2\) and (1.1)\(_3\) in the sense of distribution.

The boundary condition is satisfied in the sense of Chen and Frid [5], which implies that

\[
\int_{\Gamma} \eta(\omega)(\rho^\epsilon u_1^\epsilon)(\omega) \cdot \vec{n}(\omega) d\mathcal{H}(\omega) = \int_{\Omega} (\rho^\epsilon u_1^\epsilon)(x) \cdot \nabla \eta(x) dx + \langle \text{div} (\rho^\epsilon \mathbf{u}^\epsilon) |_{\Omega}, \eta \rangle
\]

for \( \eta \in C_0^1(\mathbb{R}^2) \). From the above, we can see \( \langle \text{div} (\rho^\epsilon \mathbf{u}^\epsilon) |_{\Omega}, \eta \rangle = 0 \). We now let \( \epsilon \to 0 \) and obtain

\[
\int_{\Gamma} \eta(\omega)(\hat{\rho} \hat{u}_1)(\omega) \cdot \vec{n}(\omega) d\mathcal{H}(\omega) = \int_{\Gamma} \eta(\omega)(\rho_0 U_1)(\omega) e_r \cdot \vec{n}(\omega) d\mathcal{H}(\omega);
\]

that is, \( \hat{\rho} \hat{u}_1 \cdot \vec{n} = \rho_0 U_1 e_r \cdot \vec{n} \) in the sense of the boundary trace.

References

[1] Bers L. Existence and uniqueness of a subsonic flow past a given profile. Comm Pure Appl Math, 1954, 7: 441–504
[2] Chen C, Du L, Xie C, et al. Two dimensional subsonic Euler flows past a wall or a symmetric body. Arch Ration Mech Anal, 2016, 221(2): 559–602
[3] Cui D C, Li J. On the existence and stability of 2-D perturbed steady subsonic circulatory flows. Sci China Math, 2011, 54(7): 1421–1436
[4] Chen G Q, Dafermos C M, Slemrod M, et al. On two-dimensional sonic-subsonic flow. Commun Math Phys, 2011, 271: 635–647
[5] Chen G Q, Frid H. Divergence-measure fields and hyperbolic conservation laws. Arch Rational Mech Anal, 1999, 147: 89–118
[6] Chen G Q, Huang F M, Wang T Y. Subsonic-sonic limit of approximate solutions to multidimensional steady Euler equations. Arch Ration Mech Anal, 2016, 219(2): 719–740
[7] Courant R, Friedrichs K O. Supersonic Flow and Shock Waves. New York: Interscience Publishers Inc, 1948
[8] Duan B, Weng S. Global smooth axisymmetric subsonic flows with nonzero swirl in an infinitely long axisymmetric nozzle. Z Angew Math Phys, 2018, 69(5): Paper No 135, 17 pp
[9] Dong G C, Ou B. Subsonic flows around a body in space. Comm Partial Differential Equations, 1993, 18: 355–379
[10] Du L, Xin Z, Yan W. Subsonic flows in a multi-dimensional nozzle. Arch Ration Mech Anal, 2011, 201: 965–1012
[11] Du L, Xie C, Xin Z. Steady subsonic ideal flows through an infinitely long nozzle with large vorticity. Commun Math Phys, 2014, 328: 327–354
[12] Finn R, Gilbarg D. Asymptotic behavior and uniqueness of plane subsonic flows. Comm Pure Appl Math, 1957, 10: 23–63
[13] Finn R, Gilbarg D. Three-dimensional subsonic flows and asymptotic estimates for elliptic partial differential equations. Acta Math, 1957, 98: 265–296
[14] Frankl F, Keldysh M. Die äussere neumann’she aufgabe für nichtlineare elliptische differentialgleichungen mit anwendung auf die theorie der flugel im kompressiblen gas. Bull Acad Sci, 1934, 12: 561–697
[15] Gilbarg D, Trudinger N. Elliptic partial differential equations of second order. Second Edition. New York: Springer-Verlag, 1983
[16] Gu X, Wang T Y. On subsonic and subsonic-sonic flows in the infinity long nozzle with general conservatives force. Acta Math Sci, 2017, 37B(3): 752–767
[17] Gu X, Wang T Y. On subsonic and subsonic-sonic flows with general conservatives force in exterior domains. arxiv:2001.09300
[18] Huang F M, Wang T Y, Wang Y. On subsonic and subsonic-sonic flows in the infinity long nozzle with general conservatives force. Acta Math Sci, 2017, 37B(3): 752–767
[19] Liu L, Yuan H. Steady subsonic potential flows through infinite multi-dimensional largely-open nozzles. Calc Var Partial Differ Equ, 2014, 49(1/2): 1–36
[20] Ou B. An irrotational and incompressible flow around a body in space. J Partial Differential Equations, 1994, 7(2): 160–170
[21] Shiffman M. On the existence of subsonic flows of a compressible fluid. J Rational Mech Anal, 1952, 1: 605–652
[22] Weng S. Subsonic irrotational flows in a two-dimensional finitely long curved nozzle. Z Angew Math Phys, 2014, 65(2): 203–220
[23] Weng S, Xin Z, Yuan H. Steady compressible radially symmetric flows with nonzero angular velocity in an annulus. J Differential Equations, 2021, 286: 433–454
[24] Weng S, Xin Z, Yuan H. On Some Smooth Symmetric Transonic Flows with Nonzero Angular Velocity and Vorticity. arXiv:2101.00450
[25] Xie C, Xin Z. Global subsonic and subsonic-sonic flows through infinitely long nozzles. Indiana Univ Math J, 2007, 56: 2991–3023
[26] Xie C, Xin Z. Global subsonic and subsonic-sonic flows through infinitely long axially symmetric nozzles. J Differential Equations 2010, 248(11): 2657–2683