Radiation induced interaction potential of two qubits strongly coupled with a quantized electromagnetic field

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We investigate the interaction of two two-level qubits with a single mode quantum field in a cavity without rotating wave approximation and considering that qubits can be located at an arbitrary distance from each other. We demonstrate that there exists a radiation induced interaction potential between atoms. We studied the properties of the system numerically and in addition constructed a simple analytical approximation. It is shown that the observable characteristics are substantially dependent on the distance between the qubits in the strong coupling regime. This allows one to perform the quantum control of the qubits, which can be exploited for the recording and transmission of quantum information.

II. CONSTRUCTION OF A MODEL HAMILTONIAN

The Hamiltonian of two identical two-level atoms (qubits) with the mass $M$, located at positions $R_1$ and $R_2$ in the dipole approximation for the interaction of resonant quantum field were recently realized experimentally. Moreover, it was possible to control the positions of qubits in a broad range with the use of tightly focused optical tweezers [16, 17]. Similar experiments were recently performed for systems of Rydberg atoms located inside a cavity [18, 19]. As a result, in our work we theoretically investigate the spectrum and dynamics of the system in the USC regime beyond the RWA and as a function of the distance $\rho$ between qubits. We demonstrate that the dependence on $\rho$ becomes important in the USC regime where the RWA in not applicable.

In our work we employ a dipole approximation for an individual qubit interacting with a quantum field. This approximation is consistent with the assumption that $\rho \sim \lambda$ due to the fact that for optical frequencies the resonance field the distance $\rho$ between qubits is much larger than the characteristic qubit size $a_0$, therefore $\rho \gg a_0$ allowing us to work in the dipole approximation for each qubit. In addition, the large qubit mass allows us to employ the adiabatic approximation in an analogy with the Born-Oppenheimer approximation of molecular physics. As a result, we treat the operator of kinetic energy of qubits as a perturbation. We show that the energy levels of the systems form potential surfaces as a function of the distance $\rho$, which define the radiation induced potential of the interaction between qubits. The form of this potential is defined by the coupling constant $f$. Moreover, the distance between qubits can be considered as an additional parameter to be used to control the system’s characteristics for the recording and the transmission of quantum information.
atoms with the field, written in natural units (ħ = c = 1) reads [20, 21]

\[
\hat{H} = -\frac{1}{2M} (\Delta R_1 + \Delta R_2) + \frac{\epsilon}{2} (\sigma^1_1 + \sigma^2_2) \\
+ \omega f \left[ (\hat{a}_1 e^{i k R_1} + \hat{a}_1^\dagger e^{-i k R_1}) \sigma^1_1 \\
+ (\hat{a}_2 e^{i k R_2} + \hat{a}_2^\dagger e^{-i k R_2}) \sigma^2_2 \right] \\
+ \omega \hat{a}^\dagger \hat{a} + V_a (R_1 - R_2),
\]

\[f = \epsilon_0 \omega \Delta d \sqrt{\frac{4\pi}{\omega^2 V}}. \tag{1} \]

Here \( V \) is the volume of the cavity, \( f \) is the dimensionless coupling constant of an atom-field interaction, \( \epsilon \) is the resonance transition energy between two qubit states \( \chi_1, \chi_2 \), \( d \) is the dipole transition matrix element; \( \epsilon_0, m_0 \) are the electron charge and mass respectively; \( \hat{a}_1, \hat{a}_2 \) are the creation and annihilation operators of the resonant quantum field with the frequency \( \omega \) and the wave vector \( k \), \( V_a (R_1 - R_2) \) is the atom-atom interaction potential due to the exchange and dipole-dipole interactions, \( \Delta \) is the Laplace operator and \( \sigma^1, \sigma^2 \) are Pauli matrices for qubits one and two respectively. The limit of \( M \to \infty \) corresponds to the situation of two immovable qubits.

Let us switch to the center of mass reference system in Eq. (1)

\[R = \frac{R_1 + R_2}{2}, \quad \rho = R_1 - R_2, \tag{3}\]

in which the Hamiltonian (1) transforms into

\[
\hat{H} = -\frac{1}{4M} \Delta R - \frac{1}{M} \Delta \rho + \frac{\epsilon}{2} (\sigma^1_1 + \sigma^2_2) \\
+ \omega f \left[ (\hat{a} e^{i k (R+\rho/2)} + \hat{a}^\dagger e^{-i k (R+\rho/2)}) \sigma^1_1 \\
+ (\hat{a} e^{i k (R-\rho/2)} + \hat{a}^\dagger e^{-i k (R-\rho/2)}) \sigma^2_2 \right] \\
+ \omega \hat{a}^\dagger \hat{a} + V_a (\rho). \tag{4}\]

The system possesses translational invariance with respect to the center of mass coordinate \( \hat{R} \). Therefore, in analogy with the polaron problem we perform the Lee-Low-Pines transformation [22]

\[
\hat{H}' = \hat{L}^{-1} \hat{H} \hat{L}, \quad \hat{L} = e^{i \hat{P} \cdot (\hat{\kappa} \hat{a}^\dagger + \hat{\kappa})}, \tag{5}\]

where \( \hat{P} \) is the total momentum of the system, which in this case is an integral of motion. As a result, we find the following expression for the Hamiltonian of the system

\[
\hat{H}' = -\frac{1}{4M} (\hat{P} - \kappa \hat{a}^\dagger)^2 - \frac{1}{M} \Delta \rho + \frac{\epsilon}{2} (\sigma^1_1 + \sigma^2_2) \\
+ \omega f \left[ (\hat{a} e^{i k \rho/2} + \hat{a}^\dagger e^{-i k \rho/2}) \sigma^1_1 \\
+ (\hat{a} e^{-i k \rho/2} + \hat{a}^\dagger e^{i k \rho/2}) \sigma^2_2 \right] \\
+ \omega \hat{a}^\dagger \hat{a} + V_a (\rho). \tag{6}\]

The characteristic scale with respect to the coordinate of a relative motion \( \rho \) is defined by the wavelength of the radiation \( \lambda = 2\pi/k \) and for the optical frequencies substantially exceeds the size of the atom \( a_0 \), such that \( a_0/\lambda \sim 10^{-3} \). At these distances the contribution from
the exchange interaction into the potential $V_{\text{ex}}(\rho)$ is exponentially suppressed, while the Van der Walls potential of an atom-atom interaction in atomic units is proportional to $(a_B/\rho)^6 \sim (a_B/\lambda)^6$, with $a_B$ being the Bohr radius. It allows one to neglect the potential $V_{\text{ex}}(\rho)$ in the Hamiltonian $\hat{H}'$ (6). In addition, we can also neglect a recoil in the operator of kinetic energy since we are working in the nonrelativistic limit. Consequently, one can write
\[
\frac{1}{2M} \langle \hat{P} \hat{k} \hat{a}^\dagger \hat{a} \rangle \approx \frac{P}{2M} \omega \ll \omega.
\]

As a result, we arrive to the final expression for the Hamiltonian, which describes the interaction of two two-level atoms within the above described approximations
\[
\hat{H}' = \frac{P^2}{4M} - \frac{1}{M} \Delta \rho + \frac{\epsilon}{2} (\sigma_3^1 + \sigma_3^2) + \omega \hat{a}^\dagger \hat{a} + \omega f \left[ \left( \hat{a} e^{i \kappa \rho/2} + \hat{a}^\dagger e^{-i \kappa \rho/2} \right) \sigma_1^1 + \left( \hat{a} e^{-i \kappa \rho/2} + \hat{a}^\dagger e^{i \kappa \rho/2} \right) \sigma_1^2 \right].
\]

III. THE INTERACTION POTENTIAL OF TWO QUBITS

Let us investigate the Schrödinger equation with the Hamiltonian (8). For this purpose we choose a coordinate system in which the $x$-axis is directed along the $\hat{k}$ and seek a solution in the form $\Lambda(r) = \exp(-i(p_y y + p_z z)) \Psi(x)$
\[
\left\{ \begin{array}{l}
\frac{P^2}{4M} + \frac{(p_y^2 + p_z^2)}{M} - \frac{1}{M} \frac{d^2}{dx^2} + \frac{\epsilon}{2} (\sigma_3^1 + \sigma_3^2) + \omega \hat{a}^\dagger \hat{a} + \omega f \left[ \left( \hat{a} e^{i \phi} + \hat{a}^\dagger e^{-i \phi} \right) \sigma_1^1 + \left( \hat{a} e^{-i \phi} + \hat{a}^\dagger e^{i \phi} \right) \sigma_1^2 \right] \\
\end{array} \right\} \Psi(x) = E \Psi(x),
\]
where $\phi = \pi x/\lambda$.

The case of $x = 0$ corresponds to the situation that atoms are unified and form a system with different parameters. Therefore, we assume that in the operator (9) the coordinate $x$ is varying in the range $|x| > x_0$, where the quantity $x_0 \sim a_B \ll \lambda$. In the general case the vector $\rho$ can be directed under an arbitrary angle with respect to the $\hat{k}$. However, we consider that the following conditions $y \approx z \approx x_0$ are fulfilled for the projections of $\rho$ on the direction perpendicular to the $\hat{k}$ vector.

As it was explained in the introduction, the operator of kinetic energy of a qubit is a small correction in comparison with the operator of the interaction of a qubit and a resonance field. Therefore, we employ an adiabatic approximation and separate variables in Eq. (9)
\[
\left\{ \begin{array}{l}
\frac{\epsilon}{2} (\sigma_3^1 + \sigma_3^2) + \omega \hat{a}^\dagger \hat{a} + \omega f \left[ \left( \hat{a} e^{i \phi} + \hat{a}^\dagger e^{-i \phi} \right) \sigma_1^1 + \left( \hat{a} e^{-i \phi} + \hat{a}^\dagger e^{i \phi} \right) \sigma_1^2 \right] \\
\end{array} \right\} \psi_\nu(x) = U_\nu(x) \psi_\nu(x),
\]
\[
\left\{ \begin{array}{l}
\frac{P^2}{4M} + \frac{(p_y^2 + p_z^2)}{M} \left[ \frac{d^2}{dx^2} + U_\nu(x) \right] \Phi(x) = E \Phi(x).
\end{array} \right\}
\]

In these equations the index $\nu$ denotes a set of quantum numbers of qubit-photon system and the terms $U_\nu(x)$ define the radiation induced interaction potential of two qubits in a full analogy with the molecular physics. The solution of the Schrödinger equation (12) determines the relative motion of qubits induced by the potential $U_\nu(x)$.

In order to calculate the potential function $U_\nu(x)$ approximate methods can be employed [17]. In our work we have developed the analytical approximation for energy levels and in addition performed an exact numerical solution by diagonalizing the Hamiltonian matrix written in the basis of eigenstates of $\sigma_3$ and Fock states of the field. The details of the calculations are given in appendices A and B.

From the structure of Eq. (11) it follows that the Hamiltonian of the system is a periodic function with a period $2\lambda$.

It is convenient to express the energies in the units of
the photon frequency
\[
U_\nu(x) = \omega u_\nu(x),
\]
\[
\left\{\delta \left(\sigma_3^1 + \sigma_3^2\right) + \hat{a}^\dagger \hat{a} + f \left[\hat{a}e^{i\phi} + \hat{a}^\dagger e^{-i\phi}\right] \sigma_1^1 + \hat{a}^\dagger e^{i\phi} + \hat{a} e^{-i\phi}\right] \sigma_2^2\right\} \psi_\nu = u_\nu(x) \psi_\nu, \tag{14}
\]

where we introduced \(\delta = \epsilon / \omega\).

We plot in Fig. 1 the potential surface as a function of the coupling constant \(f\) for the ground state of the system when \(\delta = 1\). The depth of the potential wells is varying by two orders of magnitude when the coupling constant is changing in the range \(\epsilon \in [0,1]\). When the system is in the USC regime the depth of the wells is significant enough. As a result, we will be mainly investigating this most interesting regime.

We pay attention to the fact that the period of oscillations of the potential equal to \(\lambda / 4\), which is different from the periodicity of the Hamiltonian of the system. This symmetry arises also in the analytical approximation derived in the appendix A (See also Fig. 7).

We also note here, that the dependence of the energy of the ground state on \(x\) is caused mainly by the counter rotating terms in the Hamiltonian (8). This follows from the fact that in the RWA the exact ground state of the system is given by \(\psi_\nu^{\text{RWA}} = \chi_1^1 \chi_2^2 |0\rangle\) and by acting with \(\hat{H}'\) on \(\psi_\nu^{\text{RWA}}\) one finds that \(u_\nu^{\text{RWA}}(x) = -\delta\) and is independent of \(x\).

**IV. OBSERVABLE CHARACTERISTICS OF THE SYSTEM**

One of the observable consequences of the dependence of the energy of the system on the distance between qubits is the variation of the transition frequencies with the variation of \(x\), i.e.

\[\Omega_{\nu_1 \nu_2}(x) = \omega(u_{\nu_2}(x) - u_{\nu_1}(x)).\]  \tag{15}

In Fig. 2 we plot this example. As a result, for different \(x\) the period of the Rabi oscillations and the time evolution of the system is modified that is demonstrated in Figs. 3 - 4. In these Figs. instead of the time evolution of the ground state in the time domain \(P_{-1}(t)\) we plot its Fourier transform \(P_{-1}(\omega)\), which is an even function of the frequency. As a result, we plotted only the region of positive frequencies \(\omega\). This quantity demonstrates the changes in the dynamics of the system with the change of the dimensionless coupling constant \(f\). As such, for small values of the coupling constant (see Fig. 3) the spectrum possesses a single maximum, to which coincides a period of oscillations with only a single Rabi frequency. When the coupling constant increases we start to observe the doubling of frequencies that signifies that the system transitions in the chaotic regime [23]. In the strong coupling regime (Fig. 4) the system possesses a spectrum with a broad range of frequencies. However, we pay attention to the fact that the spectrum changes significantly with the distance between qubits.

In addition, to the spectrum and dynamics of the system we also investigate the correlation properties such as the average number of photons inside the cavity and the entanglement of the qubit states. The dependence of these quantities on the distance \(x\) between qubits allows one to perform their control. For this purpose let
us investigate these quantities for the ground state of the system.

We represent the wave function of the system in the basis of Fock states $|n\rangle$ and $\chi_{s_1}^{1}, \chi_{s_2}^{2}$ the spin states of qubits

$$\psi_0 = \sum_{n,s_1,s_2} C_{n,s_1,s_2}^0 |n\rangle \chi_{s_1}^{1} \chi_{s_2}^{2},$$

(16)

The coefficients $C_{n,s_1,s_2}^0$ are computed numerically by diagonalizing the Hamiltonian matrix in this basis. Consequently, with the help of this expansion we can compute the average number of photons inside the cavity as

$$\langle n(x,f) \rangle = \langle \hat{n}^\dagger \hat{n} \rangle = \sum_{n,s_1,s_2} n |C_{n,s_1,s_2}^0|^2,$$

(17)

which we plot in Fig. 5. This figure demonstrates that in the USC regime even when the system is in the ground state the interaction between qubits and the quantum field leads to the excitation of three photons. However, this number significantly depends on the distance between qubits.

In order to compute entanglement $\mathcal{E}$ we use the definition [24]

$$\mathcal{E}(x,f) = -\sum_{s_1} p_{s_1} \log_2 p_{s_1},$$

(18)

$$p_{s_1} = \sum_{n,s_2} |C_{n,s_1,s_2}^0|^2,$$

(19)

which is demonstrated in Fig. 6. As was expected, for large values of the coupling constant $f$ the states of both qubits are entangled and the degree of entanglement depends on the distance between qubits. This fact can be used for encoding of quantum information in the system. In addition, an analytical approximation (A6) demonstrates that the spin part of the system is entangled with the field part.

V. CONCLUSION

In our work we have investigated a system of two qubits, which interact with the quantum electromagnetic field inside a cavity. The qubits are located on the distances of the order of the wavelength of the radiation that is sufficiently larger than the corresponding qubit size. This allowed us to describe the individual qubit within the dipole approximation. In addition, we have employed the adiabatic approximation and considered the operator of the kinetic energy of the qubits as a perturbation. As a consequence, we have approximately separated the variables in the center of mass reference system of qubits. In a full analogy with molecular physics we have computed the radiation induced potential (terms) of the system of two qubits. As a result, the observable characteristics of the system became functions of the distance between qubits and the coupling constant $f$ of the qubits-field interaction.

We have identified the most interesting range of the coupling constant (USC regime) when the interaction between qubits and the field is the strongest. We have calculated the observable characteristics numerically and constructed an analytical approximation. Moreover, we have demonstrated that by varying the distance between qubits the observable characteristics are strongly changing. In addition, by changing the qubits positions one can perform a quantum control of the system, which is useful for the recording and the transmission of quantum information.
A. ANALYTICAL APPROXIMATION FOR ENERGY LEVELS

In this appendix we will derive an analytical approximation for eigenvalues and eigenvectors of the system of two two-level atoms interacting with a single mode quantum field.

Let us first consider the case when $\delta$ equals zero in the Hamiltonian of the system Eq. (14). In this case the problem becomes exactly solvable, since both spins are diagonalized by one of the following wave functions $\chi_1^\uparrow \chi_1^\downarrow$, $\chi_2^\uparrow \chi_2^\downarrow$, $\chi_3^\downarrow \chi_3^\downarrow$, where $\chi_i^\uparrow$, $\chi_i^\downarrow$ are eigenvectors of $\sigma_i$. The field part of the Hamiltonian contains only the first powers of operators. As is well known, in this case the field Hamiltonian is diagonalized by displacing the classical component from the field operators or in other words by performing a unitary transformation of the Hamiltonian with the operator of the coherent state $\hat{D}(u) = \exp(u\hat{a}^\dagger - u^*\hat{a})$ that transforms the field operators as

$$\hat{D}^\dagger(u)\hat{a}\hat{D}(u) = \hat{a} + u,$$

$$\hat{D}^\dagger(u)\hat{a}^\dagger\hat{D}(u) = \hat{a}^\dagger + u^*.$$  \hfill (A1)

The parameter $u$ is then chosen from the condition that the first power of the operators vanishes. By performing the unitary transformation of the field part of the Hamiltonian we can find out the following expressions for each of the four spin wave functions listed above

$$\hat{H}_{1f} = \hat{a}^\dagger\hat{a} + u_1(\hat{a}^\dagger + \hat{a}) + u_1^2 + 2f(\hat{a}^\dagger + \hat{a})\cos \phi + 4u_1f\cos \phi,$$

$$\hat{H}_{2f} = \hat{a}^\dagger\hat{a} - iu_2(\hat{a}^\dagger - \hat{a}) + u_2^2 + 2i\hat{f}(\hat{a}^\dagger - \hat{a})\sin \phi - 4u_2f\sin \phi,$$

$$\hat{H}_{3f} = \hat{a}^\dagger\hat{a} - iu_3(\hat{a}^\dagger - \hat{a}) + u_3^2 - 2i\hat{f}(\hat{a}^\dagger - \hat{a})\sin \phi + 4u_3f\sin \phi,$$

$$\hat{H}_{4f} = \hat{a}^\dagger\hat{a} + u_4(\hat{a}^\dagger + \hat{a}) + u_4^2 - 2f(\hat{a}^\dagger + \hat{a})\cos \phi - 4u_4f\cos \phi.$$  \hfill (A2)

Here we considered that parameters of the coherent states for the cases 1, 4 are pure real and for the cases 2, 3 are pure imaginary numbers. From here we can conclude that $u_{1,4} = \pm 2f\cos \phi$ and $u_{2,3} = \pm 2if\sin \phi$.

Now let us consider the general situation of $\delta \neq 0$. In this case the spins and the field are entangled and the approximate wave function of the system is expressed as a linear combination of the previously found four possibilities, namely

$$|\Psi\rangle = A_1\chi_1^\uparrow \chi_1^\downarrow |n, 2f\cos \phi\rangle + A_2\chi_1^\uparrow \chi_1^\downarrow |n, -2if\sin \phi\rangle + A_3\chi_2^\uparrow \chi_2^\downarrow |n, 2if\sin \phi\rangle + A_4\chi_1^\downarrow \chi_3^\downarrow |n, -2f\cos \phi\rangle,$$  \hfill (A6)

where we introduced the notation $|n, u\rangle = \hat{D}(u)|n\rangle$. As a result, the energy of the system is given as the solution of the eigenvalue problem $\hat{H}|\Psi\rangle = E|\Psi\rangle$, with the Hamiltonian $\hat{H}$ defined by Eq. (14) in the finite basis, consistent of four wave functions. The solution of this eigenvalue problem leads to the desired coefficients $A_i$, $i = 1..4$ and the eigenvalues. For example, for the first four states one finds the following expressions for the Hamiltonian matrix (here we employed the notation for the coherent states $|0, u\rangle = |u\rangle$)

$$H_{ij} = \begin{pmatrix}
    u_1^2 + 4fu_1\cos \phi & \frac{\delta}{2}\langle u_1 |i u_2\rangle & 0 & \frac{\delta}{2}\langle u_1 |i u_3\rangle \\
    \frac{\delta}{2}\langle i u_2 |u_1\rangle & u_2^2 - 4fu_2\sin \phi & 0 & 0 \\
    0 & 0 & u_3^2 + 4fu_3\sin \phi & \frac{\delta}{2}\langle i u_4 |u_3\rangle \\
    \frac{\delta}{2}\langle u_4 |i u_3\rangle & \frac{\delta}{2}\langle i u_3 |u_4\rangle & 0 & u_4^2 - 4fu_4\cos \phi
\end{pmatrix},$$  \hfill (A7)

where the overlapping integrals between different coherent states are defined as

$$\langle u|u\rangle = \langle 0|e^{v^*\hat{a} - v\hat{a}^\dagger}e^{u\hat{a}^\dagger - u^*\hat{a}}|0\rangle = \langle 0|e^{(u-v)\hat{a}^\dagger - (u^* - v^*)\hat{a}}|0\rangle e^{1/2(u^* - vu^*)} = e^{-1/2|u-v|^2}e^{1/2(u^* - vu^*)}.$$
the first four states

\[
E_1 = -2f^2 - \sqrt{4f^4 \cos^2 2\phi + \delta^2 e^{-4f^2}},
E_2 = -2f^2 \sin^2 \phi,
E_3 = -2f^2 \cos^2 \phi,
E_4 = -2f^2 + \sqrt{4f^4 \cos^2 2\phi + \delta^2 e^{-4f^2}}.
\]

Analogous formulas can be obtained for other states of the system.

Finally, we pay attention to the fact that the period of oscillations is equal to \(\pi/2\), despite that fact that the Hamiltonian of the system has periodicity \(2\pi\) in the dimensionless distance \(\phi\) between qubits.

### B. Exact Numerical Solution

We consider that both atoms in the initial moment of time occupy the down states \(\chi_1^\downarrow\) and \(\chi_2^\downarrow\) and the field is prepared in the coherent state with the amplitude \(\alpha\)

\[
|\Psi(0)\rangle = |\alpha\rangle \chi_1^\uparrow \chi_2^\uparrow.
\]

The functions \(\chi_1^\downarrow\) and \(\chi_2^\downarrow\) are the eigenfunctions of \(\sigma_3\). We also use the following convention and ordering for the extended spin space unit base vectors:

\[
\begin{align*}
|\chi_1\rangle &\equiv \chi_1^\downarrow \chi_1^\uparrow = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & |\chi_2\rangle &\equiv \chi_1^\uparrow \chi_1^\uparrow = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\
|\chi_3\rangle &\equiv \chi_2^\downarrow \chi_2^\uparrow = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, & |\chi_4\rangle &\equiv \chi_2^\uparrow \chi_2^\uparrow = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

First one needs to obtain the exact numerical solution of the eigenvalue problem. For this purpose we introduce the following matrix elements of the Hamiltonian (14) in the Fock field states and spin base vectors (B2):

\[
H_{nk} = \left( \frac{\delta}{2}(\sigma_3 \otimes I_2 + I_2 \otimes \sigma_3) + nI_4 \right) \delta_{nk}
\]

\[
+ f\left( (\sqrt{k} e^{i\phi} \delta_{n,k-1} + \sqrt{k+1} e^{-i\phi} \delta_{n,k+1}) \sigma_1 \otimes I_2
+ (\sqrt{k} e^{-i\phi} \delta_{n,k-1} + \sqrt{k+1} e^{i\phi} \delta_{n,k+1}) I_2 \otimes \sigma_1 \right),
\]

where \(I_4\) is a unit 4 \(\times\) 4 matrix, \(I_2\) is the unit 2 \(\times\) 2 matrix. By numerically solving the eigenvalue problem for (B3), we obtain the set of eigenvalues \(\{E_{\alpha}\}\) and corresponding eigenvectors \(\{\psi_{\alpha}\}\) (in the form of a list of expansion coefficients \(\{C_{nk}^\alpha\}\) for each eigenvector). As a result, one can construct the time-dependent wave vector as an expansion over the stationary states:

\[
|\Psi(t)\rangle = \sum_{\alpha} A_{\alpha} |\psi_{\alpha}\rangle e^{-iE_{\alpha} t},
\]

where

\[
|\psi_{\alpha}\rangle = \sum_{k=0}^{\infty} \sum_{q=1}^{4} C_{nkq}^\alpha |k\rangle |\chi_q\rangle = \sum_{k=0}^{\infty} \left( \begin{array}{c} C_{k1}^\alpha \n\\ C_{k2}^\alpha \n\\ C_{k3}^\alpha \n\\ C_{k4}^\alpha \n\end{array} \right) |k\rangle
\]

The coefficients \(A_{\alpha}\) can be obtained from the initial condition:

\[
A_{\alpha} = \langle \psi_{\alpha} | \Psi(0) \rangle = \sum_{k=0}^{\infty} \left( \begin{array}{c} C_{k1}^\alpha \n\\ C_{k2}^\alpha \n\\ C_{k3}^\alpha \n\\ C_{k4}^\alpha \n\end{array} \right) \frac{\alpha_k}{\sqrt{k!}} e^{-\frac{\sigma^2}{2}}
\]

The density matrix of the system is

\[
\hat{\rho} = |\Psi(t)\rangle \langle \Psi(t)|
\]

and the density matrix of the atomic subsystem can be calculated by tracing out the field degrees of freedom in Eq. (B7)

\[
\hat{\rho}_{TQ} = \sum_{n=0}^{\infty} \langle n|\Psi(t)\rangle \langle \Psi(t)|n \rangle
\]

Finally, the probability of finding both atoms in the down state can be obtained in the way as follows:

\[
P_{-1}(t) = \langle \chi_1 | \hat{\rho}_{TQ} | \chi_4 \rangle = \\
= \sum_{n} \sum_{\alpha} A_{\alpha} e^{-iE_{\alpha} t} C_{n4}^\alpha
\]
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