Growth in Chevalley groups relatively to parabolic subgroups and some applications

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Annotation.

Given a Chevalley group $G(q)$ and a parabolic subgroup $P \subset G(q)$, we prove that for any set $A$ there is a certain growth of $A$ relatively to $P$, namely, either $AP$ or $PA$ is much larger than $A$. Also, we study a question about intersection of $A^n$ with parabolic subgroups $P$ for large $n$. We apply our method to obtain some results on a modular form of Zaremba’s conjecture from the theory of continued fractions and make the first step towards Hensley’s conjecture about some Cantor sets with Hausdorff dimension greater than $1/2$.

1 Introduction

In this paper we study some aspects of growth in Chevalley groups. Developing the ideas from [17] it was proved in [6], [34] that any finite simple group of Lie type has growth in the following sense.

**Theorem 1** Let $G$ be a finite simple group of Lie type with rank $r$ and $A$ be a generating subset of $G$. Then either $A^3 = G$ or

$$|A^3| > |A|^{1+c},$$

where $c > 0$ depends only on $r$.

In particular, there is $n \ll (\log |G|/\log |A|)^{C(r)}$ such that $A^n = G$.

Theorem above gives an affirmative answer to the well-known Babai’s conjecture [3] for finite simple groups $G$ having bounded rank. In this paper we consider two variants of this problem for Chevalley groups $G(q)$ defined over the field $\mathbb{F}_q$. The motivation both of our problems goes back to a question from Number Theory, see [30] and Section 6. Let us describe the first problem. Let $P \subseteq G(q)$ be any parabolic subgroup of $G(q)$. First of all, what can we say about size of the product of an arbitrary set $A \subseteq G(q)$ by $P$? Of course, $A$ can be a family of cosets of $P$, say, $x_1P, \ldots, x_kP$ and thus $AP$ does not grow. Similarly, if $A = \bigsqcup_j Py_j$, then $PA = A$. Nevertheless, we show that $A$ must grow either after left multiplication or after right multiplication. It reminds the sum–product phenomenon, see, e.g., [40] and indeed our new

*This work is supported by the Russian Science Foundation under grant 19–11–00001.
application to continued fractions (see Section 6 below) is connected with this area, see the discussion of the main results in [29].

Let us formulate our first theorem in the simplified form (actually, the restriction $A \cap P = \emptyset$ can be relaxed hugely, see Theorem 12 from Section 5). Our regime throughout this paper: $q$ tends to infinity and rank is fixed.

**Theorem 2** Let $G(q)$ be a Chevalley group and $P \subset G(q)$ be a parabolic subgroup. Then for any set $A \subseteq G(q)$ with $A \cap P = \emptyset$ one has

$$
\max\{ |AP|, |PA| \} \geq \frac{\sqrt{|A||P|q}}{2}.
$$

(1)

For example, if $|A| \leq |P|$, then $\max\{ |AP|, |PA| \} \gg |A|\sqrt{q}$ and this is larger than $|A|$.

Theorem above helps us to study the second problem. Let $A$ be an arbitrary subset of a group $G$ and $\Gamma$ be a subgroup of $G$. Can we guarantee that for a certain reasonable $n$ (say, $n$ depends on $\log |G|/\log |A|$ only) one has $A^n \cap \Gamma \neq \emptyset$? The representation theory (see [36], [10], [25] or Theorem 14 below) allows to show that any set $A \subset G(q)$ of size at least $G(q)q^{-r+\delta}$, where $r$ is rank of $G(q)$ and $\delta > 0$ is an arbitrary real number effectively generates the whole group $G(q)$. In particular, $A^n \cap \Gamma \neq \emptyset$ for $n \ll_{r, \delta^{-1}}$ (see Section 3) and this bound is essentially sharp. We show that if one wants to find a non-trivial intersection with any parabolic subgroup of $G(q)$, then it is possible to break this barrier.

**Theorem 3** Let $q$ be an odd number, $G(q)$ be a Chevalley group, $P \subset G(q)$ be a parabolic subgroup and $P_*$ be a proper parabolic subgroup of the maximal size. Suppose that $|A| \geq |P_*|q^{-1+\delta}$, where $\delta > 0$ is a real number. Then there is $n$, $n \ll_{r, \delta^{-1}}$ such that $A^n \cap P \neq \emptyset$.

It turns out that the method of the proof of Theorems 2, 3 has some applications to the theory of continued fractions, namely, to Zaremba’s conjecture. Let us recall the formulation. Let $a$ and $q$ be two positive coprime integers, $0 < a < q$. By the Euclidean algorithm, a rational $a/q$ can be uniquely represented as a regular continued fraction

$$
\frac{a}{q} = [0; b_1, \ldots, b_s] = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{\ddots + \frac{1}{b_s}}}}, \quad b_s \geq 2.
$$

(2)

Zaremba’s famous conjecture [42] posits that there is an absolute constant $\xi$ with the following property: for any positive integer $q$ there exists $a$ coprime to $q$ such that in the continued fraction expansion (2) all partial quotients are bounded:

$$
b_j(a) \leq \xi, \quad 1 \leq j \leq s = s(a).
$$

In fact, Zaremba conjectured that $\xi = 5$. For large prime $q$, even $\xi = 2$ should be enough, as conjectured by Hensley [15], [16]. This theme is rather popular especially at the last time, see,
e.g., [14], [23] or short surveys about this area in [29], [30]. We just mention a result of Korobov [24] who proved that one can always take growing $k$, namely, $k = O(\log q)$ for prime $q$ (such result is also true for composite $q$).

In [30] we have proved a ”modular” version of Zaremba’s conjecture.

**Theorem 4** There is an absolute constant $k$ such that for any prime number $p$ there exist some positive integers $q = O(p^{30})$, $q \equiv 0 \pmod{p}$ and $a$, a coprime with $q$ having the property that the ratio $a/q$ has partial quotients bounded by $k$.

The first theorem in this direction was proved by Hensley in [15] and after that in [26], [27]. Now using results similar to Theorems 2, 3 above and, of course, growth results in $SL_2(F_p)$ of Helfgott [17], we improve Theorem 4.

**Theorem 5** Let $\epsilon \in (0, 1]$ be any real number. There is a constant $k = k(\epsilon)$ such that for any prime number $p$ there exist some positive integers $q = O(p^{1+\epsilon})$, $q \equiv 0 \pmod{p}$ and $a$, a coprime with $q$ having the property that the ratio $a/q$ has partial quotients bounded by $k$.

Clearly, Theorem 5 is the best possible up to $\epsilon$ and it is the limit of our method.

Another result on continued fractions (see Theorem 20 from Section 6) is even more interesting than Theorem 5 because its generality and because it is the first (weak) confirmation of Hensley’s hypothesis [16, Conjecture 3]. Namely, let now the partial quotients $b_j$ belong to a finite set $A \subset \mathbb{N}$, $|A| \geq 2$ and suppose that the Hausdorff dimension of the correspondent Cantor set is strictly greater than $1/2$ (all the definitions are contained in Section 6). Then we show that a full analogue of Theorem 5 takes place (with other constants, of course).

We finish the Introduction posing a weak version of Babai’s conjecture. Even for sufficiently large subgroups $\Gamma$ the answer to our question is non–obvious.

**Problem.** Let $G$ be a finite simple non–abelian group, $\Gamma \subset G$ be a subgroup and $A \subset G$ be an arbitrary (generating) set. Is it true that $A^n \cap \Gamma \neq \emptyset$ with $n \ll (\log |G|/ \log |A|)^C$, where $C > 0$ is an absolute constant?

If $A = A^{-1}$, then the set $AA = A A^{-1}$ obviously contains the unit element and hence the answer to the problem is trivially affirmative (moreover if $|A||\Gamma| > |G|$, then the Dirichlet principle shows that $|AA^{-1} \cap \Gamma| > 1$ and hence we can find a non–trivial element in $AA^{-1}$). Thus we cannot assume that $A = A^{-1}$ and, actually, this restriction is very important for some applications as for our modular version of Zaremba’s conjecture.

We thank Nikolai Vavilov, Misha Rudnev for useful discussions and Nikolay Moshchevitin for valuable discussions and encouragement.

## 2 Definitions

Let $G$ be a group with the identity $1$. Given two sets $A, B \subset G$, define the product set of $A$ and $B$ as

$$AB := \{ab : a \in A, b \in B\}.$$
In a similar way we define the higher product sets, e.g., $A^3$ is $AAA$. Let $A^{-1} := \{a^{-1} : a \in A\}$. As usual, having two subsets $A, B$ of a group $G$, denote by

$$E(A, B) = |\{(a, a_1, b, b_1) \in A^2 \times B^2 : a^{-1}b = a_1^{-1}b_1\}|$$

the common energy of $A$ and $B$. Clearly, $E(A, B) = E(B, A)$ and by the Cauchy–Schwarz inequality

$$E(A, B)|A^{-1}B| \geq |A|^2|B|^2. \quad (3)$$

We use representation function notations like $r_{AB}(x)$ or $r_{AB^{-1}}(x)$, which counts the number of ways $x \in G$ can be expressed as a product $ab$ or $ab^{-1}$ with $a \in A$, $b \in B$, respectively. For example, $|A| = r_{AA^{-1}}(1)$ and $E(A, B) = r_{AA^{-1}BB^{-1}}(1) = \sum x r_{AB}(x)$. In this paper we use the same letter to denote a set $A \subseteq G$ and its characteristic function $A : G \to \{0, 1\}$. We write $\mathbb{F}_q^*$ for $\mathbb{F}_q \setminus \{0\}$, where $q = p^s$, $p$ is a prime number, and $(a_1, \ldots, a_l)$ for the greatest common divisor of some given positive integers $a_1, \ldots, a_l$. If $m$ divides $n$, then we write $m|n$.

Let $g \in G$ and let $A \subseteq G$ be any set. Then put $A^g = gAg^{-1}$ and, similarly, let $x^g := gxg^{-1}$, where $x \in G$. We write $N(A)$ for the normalizer of a set $A$, that is, $N(A) = \{g \in G : A^g = A\}$. If $H \subseteq G$ is a subgroup, then we use the notation $H \leq G$.

In the paper we consider the group $\mathrm{SL}_2(\mathbb{F}_q)$ of matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ab|cd), \quad a, b, c, d \in \mathbb{F}_q, \quad ad - bc = 1,$$

as well as other classical groups as $\mathrm{PSL}_n(q)$, $\mathrm{SU}_n(q)$, $\mathrm{Sp}_n(q)$, $\Omega_n^e(q)$ and so on. Also, we use the usual Lie notation $A_n(q)$, $B_n(q)$ and so on. The signs $\ll$ and $\gg$ are the usual Vinogradov symbols. All logarithms are to base 2.

## 3 Simple facts from the representation theory

First of all, we recall some notions and simple facts from the representation theory, see, e.g., [33] or [37]. For a finite group $G$ let $\hat{G}$ be the set of all irreducible unitary representations of $G$. It is well–known that size of $\hat{G}$ coincides with the number of all conjugate classes of $G$. For $\rho \in \hat{G}$ denote by $d_\rho$ the dimension of this representation. By $d_{\min}(G)$ denote the quantity $\min_{\rho \neq 1} d_\rho$. We write $(\cdot, \cdot)$ for the corresponding Hilbert–Schmidt scalar product $\langle A, B \rangle = \langle A, B \rangle_{HS} := \operatorname{tr}(AB^*)$, where $A, B$ are any two matrices of the same sizes. Put $\|A\| = \sqrt{\langle A, A \rangle}$. Clearly, $\langle \rho(g)A, \rho(g)B \rangle = \langle A, B \rangle$ and $\langle AX, Y \rangle = \langle X, A^*Y \rangle$. Also, we have $\sum_{\rho \in \hat{G}} d_\rho^2 = |G|$. For any function $f : G \to \mathbb{C}$ and $\rho \in \hat{G}$ define the matrix $\hat{f}(\rho)$, which is called the Fourier transform of $f$ at $\rho$ by the formula

$$\hat{f}(\rho) = \sum_{g \in G} f(g)\rho(g). \quad (4)$$

Then the inverse formula takes place

$$f(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \langle \hat{f}(\rho), \rho(g^{-1}) \rangle, \quad (5)$$

where $\rho$ is any given irreducible representation of $G$.
and the Parseval identity is
\[ \sum_{g \in G} |f(g)|^2 = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \|\hat{f}(\rho)\|^2. \] (6)

The main property of the Fourier transform is the convolution formula
\[ \hat{f} \ast g(\rho) = \hat{f}(\rho)\hat{g}(\rho), \] (7)
where the convolution of two functions \( f, g : G \to \mathbb{C} \) is defined as
\[ (f \ast g)(x) = \sum_{y \in G} f(y)g(y^{-1}x). \]

Finally, it is easy to check that for any matrices \( A, B \) one has \( \|AB\| \leq \|A\| \circ \|B\| \) and \( \|A\| \circ \|A\| \leq \|A\| \), where the operator \( l^2 \)-norm \( \| \cdot \| \circ \) is just the absolute value of the maximal singular value of \( A \). In particular, it shows that \( \| \cdot \| \) is indeed a matrix norm.

For any function \( f : G \to \mathbb{C} \) consider the Wiener norm of \( f \) defined as
\[ \|f\|_W := \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \|\hat{f}(\rho)\|. \] (8)

**Lemma 6** Let \( \Gamma \leq G \). Then \( \|\Gamma\|_W \leq 1. \)

**Proof.** Since \( \Gamma \) is a subgroup, we see using (6) twice that
\[ |\Gamma|^2 = |\{ \gamma_1\gamma_2 = \gamma_3 : \gamma_1, \gamma_2, \gamma_3 \in \Gamma\}| \leq \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho (\hat{\Gamma}^2(\rho), \hat{\Gamma}(\rho)) \leq \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho (\hat{\Gamma}(\rho), \hat{\Gamma}(\rho)) \|\hat{\Gamma}(\rho)\|_o \leq \frac{|\Gamma|}{|G|} \sum_{\rho \in \hat{G}} d_\rho (\hat{\Gamma}(\rho), \hat{\Gamma}(\rho)) = |\Gamma|^2, \]
because, clearly, \( \|\hat{\Gamma}(\rho)\|_o \leq |\Gamma| \). It means that for any representation \( \rho \) either \( \|\hat{\Gamma}(\rho)\| = 0 \) (and hence \( \|\hat{\Gamma}(\rho)\|_o = 0 \)) or \( \|\hat{\Gamma}(\rho)\| \geq |\Gamma| \). (alternatively, one can use the usual calculations, namely, \( \sum_{\gamma \in \Gamma} \rho(\gamma \gamma_*) = \sum_{\gamma \in \Gamma} \rho(\gamma) \cdot \rho(\gamma_*) \) for any \( \gamma_* \in \Gamma \) but then one needs to be careful with divisors of zero). Another application of (6) gives us
\[ |\Gamma| = \frac{1}{|G|} \sum_{\rho} d_\rho \|\hat{\Gamma}(\rho)\|^2 \geq |\Gamma| \cdot \frac{1}{|G|} \sum_{\rho} d_\rho \|\hat{\Gamma}(\rho)\| = |\Gamma| \|\Gamma\|_W. \] (9)
Hence \( \|\Gamma\|_W \leq 1 \) as required. \( \square \)

Lemma 6 implies a result on growth in the affine group relatively to some subgroups. Namely, the following Corollary 7 can be considered as a "baby"–version of our main results on intersections of \( A^n \) with parabolic subgroups. Clearly, the standard Borel subgroup
Let $A \subseteq \text{Aff}(\mathbb{F}_q)$ be a set, and let $\Gamma \subseteq \text{Aff}(\mathbb{F}_q)$ be a subgroup such that for any non-trivial multiplicative character $\chi$ there is $\gamma = (a,b) \in \Gamma$ such that $\chi(a) \neq 1$. Also, let $z \in \text{Aff}(\mathbb{F}_q)$ be an arbitrary element, $n \geq 1$ be a positive integer and $|A^n|\Gamma|^2 > q^{n+2}(q-1)^2$. Then $A^n \cap z\Gamma \neq \emptyset$ and $A^n \cap \Gamma z \neq \emptyset$.

**Proof.** The representation theory of $\text{Aff}(\mathbb{F}_q)$ is well-known; see, e.g., [8]. Namely, there are $(q - 1)$ one-dimensional representations $\rho_\chi$, which are given by multiplicative characters $\chi$, where $\rho_\chi((ab|01)) := \chi(a)$ and a certain $(q - 1)$-dimensional representation $\pi$. Using formula (6) with $f = A$, we have

$$\|A(\pi)\|_o < \left(\frac{|A||\text{Aff}(\mathbb{F}_q)|}{q-1}\right)^{1/2} = (|A| q)^{1/2}. \quad (10)$$

Further by the assumption for any non-trivial multiplicative character $\chi$ there is $\gamma = (a,b) \in \Gamma$ such that $\chi(a) \neq 1$. It means that for any such $\chi$ one has $\rho_\chi(\Gamma) = 0$. Applying bound (10), Lemma 6 and using formula (6) again, we obtain

$$|A^n \cap z\Gamma| = \frac{|A^n|\Gamma|}{|\text{Aff}(\mathbb{F}_q)|} + \frac{q-1}{|\text{Aff}(\mathbb{F}_q)|} (\hat{A}^n(\pi), \hat{\Gamma}(\pi)) \geq \frac{|A^n|\Gamma|}{|\text{Aff}(\mathbb{F}_q)|} - (|A|q)^{n/2} > 0,$$

provided $|A^n|\Gamma|^2 > q^{n+2}(q-1)^2$. This completes the proof. \hfill $\square$

The condition $|A^n|\Gamma|^2 > q^{n+2}(q-1)^2$ effectively works if, roughly, $|A| \gg q^{1+\varepsilon}$, where $\varepsilon > 0$ is a certain number. Further, an example of subgroup $\Gamma$ from Corollary 7 is a torus $(A0|0\lambda^{-1})$, where $\lambda$ runs over $\mathbb{F}_q^*$. In contrary, if, say, $\Gamma$ is the unipotent subgroup $U \subseteq \text{Aff}(\mathbb{F}_q)$, then one can easily construct a set $A$, $|A| \gg q^{2}/n$ such that $A^n \cap U = \emptyset$.

### 4 Some facts about Chevalley groups

We recall quickly some properties of Chevalley groups. The detailed description of such groups can be found in many books and papers, see, e.g., classical book [28] and paper [7].

Let $p$ be a prime number, $q = p^s$ and $\mathbb{F}_q$ be the finite field of size $q$. Also, let $\Phi$ be a root system, $\Pi$ its fundamental subsystem, $\Pi \subseteq \Phi^+$, $\Phi^+ \sqcup (\Phi^+)$.

Everybelow depends on the root system $\Phi$ (and hence on $\Pi, \Phi^+, -\Phi^+$ and so on) but we do not emphasis on this. Let $B$ be a Borel subgroup of $G = G(q)$, $U = O_p(B)$, $B = U H$ (the product is direct and $U$ is normal in $B$), $N = N(H)$ with $H$ an abelian $p'$-group (Cartan subgroup). The unipotent subgroup $U$ is the direct product of subgroups $\prod_{\gamma \in \Phi^+} U_{\gamma}$ and each $U_{\gamma}$ isomorphic to the field $\mathbb{F}_q$. The Weyl group $W = N/H$ is a group generated by fundamental reflections $w_{r_1}, \ldots, w_{r_l}$, $l = |\Pi|$ and $W$ acts on the root system $\Phi$. When there is no problem with coset representatives we will consider $s \in W$ as an element of $G(q)$. For $w \in W$ let $l(w)$ be the length of $w$, that is, the minimal $n$ such
that \( w = w_{r_1} \ldots w_{r_n} \) with \( r_j \in \Pi \). Another description of \( l(w) \) is \( l(w) = |\Phi^+ \cap w^{-1}(-\Phi^+)| \) and it is known that \( l(w) = 0 \) iff \( w = 1 \) (and iff \( w(\Pi) = \Pi \) and iff \( w(\Phi^+) = \Phi^+ \)). For any \( \emptyset \neq J \subseteq \Pi \) let \( W_J \) be a subgroup of \( W \) generated by \( w_r \), where \( r \in J \). It is well–known that for any Chevalley group the Bruhat decomposition takes place, namely,

\[
G = \bigsqcup_{w \in W} BwB ,
\]

where the union in \((11)\) is disjoint. It follows from the fact that for any fundamental root \( r \) and an arbitrary \( w \in W \) one has

\[
w_r Bw \subseteq BwB \cup Bw_rB .
\]

Decomposition \((11)\) can be refined further. For \( w \in W \) put

\[
U'_w = \langle \{ U_r : r \in \Phi^+ , w(r) \in \Phi^+ \} \rangle \quad \text{and} \quad U''_w = \langle \{ U_r : r \in \Phi^+ , w(r) \in -\Phi^+ \} \rangle .
\]

Then, clearly, \( U = U'_wU''_w, B = HU'_wU''_w \) and \( wU'_ww^{-1} \subseteq U \). Thus \((11)\) can be transformed as

\[
G = \bigsqcup_{w \in W} BwU''_w ,
\]

and any element of \( G \) can be written in form \((13)\) uniquely. In particular,

\[
|G| = |B| \sum_{w \in W} |U''_w| = |H||U| \sum_{w \in W} |U''_w| = (q - 1)^{|\Pi|}q^{|\Phi^+|} \sum_{w \in W} q^l(w) .
\]

From the Bruhat decomposition and the properties of Chevalley groups, it follows that all subgroups containing \( B \) are \( 2^d \) subgroups of the form \( P_J := BW_JB \) and they are called parabolic subgroups. It is known that \( N(P_J) = P_J \), and

\[
P_J = \langle B, \{ w_j \}_{j \in J} \rangle = \langle B, \prod_{j=1}^{J} w_j \rangle = \langle B, (\prod_{j=1}^{J} w_j)B(\prod_{j=1}^{J} w_j)^{-1} \rangle .
\]

Put \( W^J = \{ w \in W : \ w(r) \in \Phi^+ \ \text{for all} \ r \in J \} \). One can check that any \( w \in W \) can be decomposed uniquely as \( w = w^Jw_J \), where \( w^J \in W^J \) and \( w_J \in W_J \) and, moreover, \( l(w) = l(w^J) + l(w_J) \). Any \( W_J \) (and \( W \) in particular) contains the unique longest element and this element is an involution. Formula \((14)\) says that, basically, the length of this longest element determines size of \( P_J \).

In paper \(25\) it was proved that Chevalley groups are quasi–random in the sense of Gowers \(10\) (also, see the first paper \(36\) where this conception was used). Namely, we have by \(25\) (a similar result takes place for any simple algebraic group \( G \)) that

\[
d_{\min}(G) \gg d q^r ,
\]

where rank \( r \) is the dimension of its maximal tori of \( G \) and \( d \) is dimension of \( G \).

Let \( \Pi_1(G(q)) \geq \Pi_2(G(q)) \geq \ldots \) be sizes of maximal proper parabolic subgroups of \( G(q) \). Consider the quantity

\[
P(G(q)) := \min \{ t : \forall H \leq G, |H| > t \implies H \text{ is parabolic} \} .
\]
In other words, \( P(G(q)) \) coincides with size of the largest (by cardinality) non–parabolic subgroup. The quantity depends on the concrete Chevalley group \( G(q) \) (e.g., \( PQ_8^+ (q) \) contains the largest (by cardinality) parabolic subgroup \( P \) and also two large non–parabolic subgroups \( \Omega_7(q) \), \( Sp_6(q) \), depending on the parity of \( q \).

Thus again \( |\Pi_1(\Omega_7(q))| \approx |Sp_6(q)| \approx q^{-1}[\Pi_1(PQ_8^+ (q))] \) see [1] Table 6). Nevertheless, we give a simple upper bound for \( P(G(q)) \). Our proof is hugely based on book [21] (which in turn uses the famous Aschbacher Theorem [2], see a good survey [20]) and follows paper [1], where the authors give a list of all maximal subgroups \( H \) of Chevalley groups, having large size, namely, \(|H| \geq |G(q)|^{1/3}\). It is easy to see that usually maximal parabolic subgroups of \( G(q) \) are even larger (clearly, \(|B| \geq (|G(q)||H|)^{1/2}\)) and hence it is enough to check all "large" subgroups from [1].

**Lemma 8** Let \( q \) be a sufficiently large number. Then we have \( P(PSL_2(q)) \leq 2(q + 1), P(PSL_3(q)) \leq q^3, \) and for \( n \geq 4 \) the following holds \( P(PSL_n(q)) \leq q^{\frac{n(n+1)}{2}}, \) provided \( q \) is a non–square.

Further, we consider \( n \geq 3 \) for \( SU_n(q), n \geq 4 \) for \( PSp_n(q), n \geq 7 \) and \( q \) is odd for \( \Omega^e_n(q), \) where \( \varepsilon = \pm \). In all cases above with an odd \( q \) and for all simple exceptional groups one has

\[
qP(G(q)) \leq \Pi_1(G(q)) = \max\{|H| : H \leq G(q), H \neq G(q)\}. \tag{16}
\]

**Proof.** We use Tables 3.5A–3.5F from [21] to determine sizes of maximal subgroups of \( G(q) \), calculations from paper [1], as well as the Aschbacher classification Theorem, which says that every maximal subgroup of a classical group belong to one of the geometric classes \( C_1-C_8 \) and an additional exceptional class \( S \). For exceptional groups we consult book [41]. Due to the existence of isomorphisms between low–dimensional classical groups (see [21] Proposition 2.9.1, for example), we may assume without losing of the generality that \( n \) satisfies the stated lower bounds.

Let \( d = (n, q - 1) \), \( \alpha = (2, q - 1) \) and let us begin with \( PSL_n(q) \). For small \( n \) it follows from the classification of subgroups of \( PSL_2(q) \) (see, e.g., [39], we use the assumption that \( q \) is a non–square to avoid the subgroup \( PGL_2(\sqrt{q}) \subset PSL_2(q) \), say), further for \( PSL_3(q) \) (we apply the assumption that \( q \) is a non–square to avoid the subgroup \( PSU_3(q) \), say) see [28], for \( PSU_3(q), PSp_4(q) \) with odd \( q \), again, see [28] and, finally, for \( PSL_4(q) \) with even \( q \), see [32] (here we appeal to the fact that \( PSL_4(q) \) contains \( PSp_4(q) \) having size less than \( q^{4(4+1)/2} \). Now let \( n \geq 4 \) and let us do not consider subgroups of the class \( S \) at the beginning. In this case the only subgroups belonging to Aschbacher’s class \( C_1 \) are maximal parabolic subgroups \( \Pi_m \) with

\[
|\Pi_m| = d^{-1}q^{m(n-m)}(q-1)|SL_m(q)||SL_{n-m}(q)| \approx q^{n^2-2nm+m^2-1} > q^{\frac{n^2}{4}-1} > q^{\frac{n(n+1)}{2}}. \tag{17}
\]

For \( H \in C_2 \), we have with \( t \geq 2 \) that \( |H| = \frac{(q-1)^t}{q} |GL_{n/t}(q)|^t \ll q^{n^2/t-1} \) and this is smaller than \( q^{\frac{n(n+1)}{2}} \). For \( H \in C_3 \), one has \( |H| = \frac{k}{d(q-1)} |GL_n/k(q^k)| \), where \( k|n \), and \( k \) is a prime number. Thus again \( |H| \leq q^{n^2/k-1} \). If \( H \in C_4 \), then \( |H| = d^{-1}|SL_a(q)||SL_{n-a}(q)|(q-1, a, n/a) \), where \( 2 \leq a < n/2 \). In other words, \( |H| \ll q^{n^2/a^2+a^2-2} \ll q^{n^2/4+2} \ll q^n(n+1)/2 \). Further, for \( H \in C_5 \), we have \( |H| = (q_0 - 1)^{-1}(q_0 - 1, (q_0^k - 1)(q_0^k - 1/2)|SL_n(q_0)| \) with \( q = q_0^k \) and \( k \) is a prime number.
Hence $|H| \ll q^{n^2-2} \leq q^{(n^2-2)/k} \leq q^{n(n+1)/2}$. If $H \in C_6$, then $|H| \leq r^{2m}|Sp_{2m}(q)|$, where $n = r^m$, $r|(q-1)$ and $r$ is an odd prime number. It follows that $|H| \leq n^2q^{m(2m+1)}$ and this quantity is very small. For $H \in C_7$, one has $|H| < |SL_n(q)|/t!, n = a^4$, $a \geq 3$, $t \geq 2$ and again this is very small. Finally, if $H \in C_8$, then either $H = PSp_n(q)$ (and we have $|PSp_n(q)| \leq q^{n(n+1)/2}$) or $|H| = |SO_n^-(q)| \leq 2aq^{n(n-1)/2}$ or $H = U_n(q_0)$, $q = q_0^2$ and $n \geq 3$. In view of (17) we see that $P(PSL_n(q)) \leq q^{n(n+1)/2}$ provided $n \geq 4$.

To finish the proof of our result in the case of $PSL_n(q)$ it remains to consider subgroups of the class $S$. We have $q^{n(n+1)/2} \geq q^{n^2-1}$ and $|\Pi_1| = q^{n^2-n} \geq q^{n(n+1)/2}$. In view of [1] Theorem 4, Table 6] for large $q$ (in the case of all groups $PSL_n(q)$, $SU_n(q)$, $PSp_n(q)$, $Ω_6^-(q)$) just three subgroups survive, namely, $PΩ_5^-(q)$ (it contains $Ω_7(q)$, $Sp_6(q)$ and smaller subgroups), $Ω_7(q)$, and $PSp_6(q)$ (the last two contain $G_2(q)$, $|G_2(q)| = q^6(q^2 - 1)(q^4 - 1) \leq q^{14}$). The group $G_2(q)$ in $Ω_7(q)$, $PSp_6(q)$ is too small because it is easy to see that $|Π_1(Ω_7(q))| \sim q^{16} \sim |Π_1(PSp_6(q))|$ (or consult estimate (19), (20) below). Similarly, for $PΩ_8^+(q)$ sizes of $Ω_7(q)$, $Sp_6(q)$ do not exceed $q^{-1}Π_1(Ω_8^+(q))$. Thus indeed $P(PSL_n(q)) \leq q^{n(n+1)/2}$ for $n \geq 4$ and we have proved (16) in the case $G(q) = PSL_n(q)$.

In the general case it is sufficient to have dealt with subgroups of the classes $C_1$–$C_8$ and we begin with parabolic subgroups. For such subgroups we have analogues of formula (17), namely, (see [21] Propositions 4.1.18–4.1.20)

$$|Π_m(SU_n(q))| \sim q^{2nm-3m^2+2}|L_m(q^2)||U_{n-2m}(q)| \sim q^{n^2-2nm+3m^2-1},$$  \hspace{1cm} (18)$$

$$|Π_m(PSp_n(q))| = q^{nm+m/2-3m^2/2}(q-1)|PGL_m(q)||PSp_{n-2m}(q)| \sim q^{n^2-2nm+n+3m^2-m},$$  \hspace{1cm} (19)$$

and for $m \leq n/2$ (we do not consider smaller parabolic subgroups) one has

$$|Π_m(Ω_6^-(q))| \sim q^{nm-m/2-3m^2/2}|GL_m(q)||Ω_6^-(q)| \sim q^{n^2-2nm-n+3m^2+m},$$  \hspace{1cm} (20)$$

Analysing Tables 3.5A–3.5F from [21] and using [21] Propositions 4.1.3, 4.1.4, 4.1.6, one can easily see that others subgroups of the class $C_1$ are smaller for these parabolic groups and have form $GU_m(q) \perp GU_{n-m}(q)$, $PSp_m(q) \perp PSp_{n-m}(q)$, $O_0^+(q) \perp O_0^{-}(q)$, correspondingly, with some additional restrictions on $n, m, ε$ (e.g., $2 \leq m < n/2$, $m$ is even for $PSp_m(q) \perp PSp_{n-m}(q)$. In the case of the orthogonal group we use the assumption that $q$ is odd. More precisely, using (18)–(20), we check that (16) holds at least for all subgroups of the class $C_1$.

After that we apply the results from [1] (notice that $q^{-1}Π_1(G(q)) \geq |G(q)|^{1/3}$ and thus it requires to use the list of the subgroups from this paper) to show that almost all other subgroups of the classes $C_2$–$C_8$ are obviously small. In the case of $SU_n(q)$ it remains to check $Sp_n(q) \in C_5$ with $|Sp_n(q)| \leq q^{n(n+1)/2} \leq q^{-1}Π_1(SU_n(q))$. If $G(q) = PSp_n(q)$, then all subgroups are smaller than $q^{-1}Π_1(G(q))$. Finally, in the case of $Ω_6^-(q)$ it remains check $C_2$–subgroup $H$ of size $l||Ω_6^-(q)|| \leq q^{n(n-1)/2l}$, $t \geq 2$ and $H = GL_{n/2}(q)$ and both of these subgroups are less than $q^{-1}Π_1(G(q))$ because $n \geq 6$. The class $C_8$ exists only for $PSp_n(q)$ and it coincides with the only subgroup $O_0^+(q)$, $q$ is even with $|O_0^+(q)| \leq q^{-1}Π_1(PSp_n(q))$.

It remains to consider the exceptional groups. In this case we use [1] Theorem 5, Table 2], which says that any maximal subgroup $H$ of $G(q)$ of size $|H| \geq |G(q)|^{1/3}$ is either a maximal...
parabolic subgroup or belongs to a certain list, see [1] Table 2] (again it is easy to check that the
condition $q^{-3} \Pi_1(G(q)) \geq |G(q)|^{1/3}$ takes place). Analysing this Table, one can see that sizes of all
non–parabolic subgroups of the exceptional groups do not exceed $|B|$ with four exceptions: $F_4(q)$
(the largest subgroups are $B_4(q), C_4(q)$), further, $E_6^\circ(q)$ (the largest subgroup is $F_4(q)$), $E_7(q)$
(the largest subgroup is $(q - \varepsilon)E_6^\circ(q)$) and, finally, $E_8(q)$ with the largest subgroup $A_1(q)E_7(q)$.
For $F_4(q)$ consult [11] Section 4.5.9] to see that there is a parabolic subgroup $H \leq F_4(q)$ such that
\[ |H| = q^{15}(q - 1)|Sp_6(q)| \sim q^{37} \sim q|B_4(q)| \sim q|C_4(q)|, \]
further, for $E_6^\circ(q)$ see [11] Section 4.6.4] and [22], where it was proved that there exists a parabolic
subgroup of size $q^{20}(q - 1)|L_2(q)||L_5(q)| \sim q^{53} \sim q|F_4(q)|$. Finally, if we consider $E_8(q)$, then by
[11] Section 4.7.2] this group contains a subgroup of size $\gg q^{38}|E_7(q)|$ and this is much larger
than $q|A_1(q)||E_7(q)|$, if we take $E_7(q)$, then, similarly, by [11] Section 4.7.3] we see that $q^2|E_6^\circ(q)|$
is small. One can use another way to prove that the maximal (by size) maximal parabolic
subgroup is large: just analyse the Dynkin diagrams for $F_4(q)$, $E_6^\circ(q)$, $E_7(q)$ and $E_8(q)$. This
completes the proof.

We need a simple general lemma (a similar result can be found in [11]).

**Lemma 9** Let $G$ be a group and $\Gamma_1, \Gamma_2 \leq G$. Then
\[ \max_{x, y \in G} |x\Gamma_1 \cap \Gamma_2 y| = \max_{x \in G} |x\Gamma_1 \cap \Gamma_2 x|, \]
and $|\Gamma_1 \cap \Gamma_2| \geq |\Gamma_1||\Gamma_2|/|G|$. 

**Proof.** If the intersection $x\Gamma_1 \cap \Gamma_2 y$ is empty, then there is nothing to prove. Otherwise for any $c \in x\Gamma_1 \cap \Gamma_2 y$ one has
\[ x\Gamma_1 \cap \Gamma_2 y = ((x\Gamma_1 x^{-1}) \cap \Gamma_2)c = c(y^{-1}\Gamma_2 y \cap \Gamma_1) \]
as required.

Now from the Dirichlet principle there is $x \in G$ such that $A := x\Gamma_1 \cap \Gamma_2$ has size at least
$|\Gamma_1||\Gamma_2|/|G|$. But $A \subseteq \Gamma_2$ and hence $A^{-1}A \subseteq \Gamma_1 \cap \Gamma_2$. It remains to notice that $|A^{-1}A| \geq |A| \geq
|\Gamma_1||\Gamma_2|/|G|$. An alternative way of the proof is just use the formula $|\Gamma_1 \cap \Gamma_2| = |\Gamma_1||\Gamma_2|/|\Gamma_1\Gamma_2| \geq
|\Gamma_1||\Gamma_2|/|G|$. This completes the proof. \qed

Now we are ready to prove a result on an upper bound for $|P \cap P^g|$ for parabolic subgroups
$P$ of $G(q)$.

**Lemma 10** Let $G(q)$ be a Chevalley group and $P \subset G(q)$ be a parabolic subgroup. Then for
any $g \notin P$ one has
\[ r_{P^g P}(x) \leq \frac{2|P|}{q} \quad \text{for all} \quad x \in G(q). \quad (21) \]
Proof. In view of Lemma 9 it is enough to estimate $|P \cap P^g|$. Let $P = P_J$. From Bruhat decomposition (11) we can assume that $g \in W$ and moreover in view of (12) and Lemma 9 we can assume that $g \in W^J$, $g \notin W_J$.

First of all, let us obtain (21) for the Borel subgroup $B$ (in this case $g$ just from $W$). The equation $Bg = gB$ can be rewrittten as $Bg = HU'_g gU''_g$ and hence by (13) it has $|HU'_g| = |B|/|U''_g| = |B|q^{-l(g)}$ solutions. Clearly, $l(g) \geq 1$ and the result follows (in this case we do not even need the constant two in inequality (21) and this is absolutely sharp, take, e.g., $G(q) = SL_2(\mathbb{F}_q)$). It is easy to see that estimate (21) is, actually, equality in this case.

Now let $P$ be an arbitrary parabolic subgroup. Using the Bruhat decomposition and the arguments as in the case of the Borel subgroup, we obtain

$$|P \cap P^g| = \sum_{v_1, v_2 \in W_J} |gBv_1 U''_{v_1} \cap Bv_2 U''_{v_2} g| \leq 2 \sum_{v_1, v_2 \in W_J, l(v_2) \leq l(v_1)} \sum_{v \in W_J} q^{l(v_1) + l(v_2)} |gBv_1 B \cap Bv_2 Bg|.$$  \hfill (22)

Now using (12), we see that for any $v \in W_J$ one has $gBv \subseteq BvB \cup BgvB$. Since $g \notin P$, we get $gBv \subseteq BgvB$ and hence any element $gb v_j$, $b \in B$, $j = 1, 2$ can be written as $b_1 g v_j b_2$, $b_1, b_2 \in B$. The same is true for $vBg$, of course. Whence recalling (22), we get

$$|P \cap P^g| \leq 2 \sum_{v_1, v_2 \in W_J, l(v_2) \leq l(v_1)} q^{l(v_1) + l(v_2)} |gv_1 B \cap Bv_2 g|.$$  

Again, applying the Bruhat decomposition and transforming $gv_1 B$ as $HU'_{gv_1} gv_1 U''_{gv_1}$, we derive

$$|P \cap P^g| \leq 2 |B| \sum_{v_1, v_2 \in W_J, l(v_2) \leq l(v_1), gv_1 = v_2 g} q^{l(v_1) + l(v_2) - l(gv_1)} \leq 2 |B| \sum_{v_1, v_2 \in W_J, l(v_2) \leq l(v_1), gv_1 = v_2 g} q^{2l(v_1) - l(gv_1)}.$$  

But $g \in W^J$ and hence $l(gv_1) = l(g) + l(v_1)$. Clearly, $l(g) \geq 1$ because otherwise $g \in W_J$. In view of (14) it gives us

$$|P \cap P^g| \leq 2 |B| q^{-1} \sum_{v_1, v_2 \in W_J, l(v_2) \leq l(v_1), gv_1 = v_2 g} q^{l(v_1)} \leq 2 |B| q^{-1} \sum_{v \in W_J} q^{l(v)} = 2 |P| q^{-1}$$  

as required. \hfill \Box

It is easy to see that estimate (21) is tight up to constants (consider parabolic subgroups of $SL_n(\mathbb{F}_q)$, say).

We finish this Section by a lemma in the spirit of the well-known result of Frobenius [9] on the representation of $SL_n(\mathbb{F}_q)$.

Lemma 11 Let $G(q)$ be a Chevalley group and $P \subseteq G(q)$ be a parabolic subgroup. Suppose that $\rho$ is an arbitrary non-trivial irreducible representation of $P$ such that $H(\rho) \neq 0$. Then $d_\rho \geq \frac{q-1}{2}$. 

proof. At the beginning let $P$ be a Borel subgroup $B$ and suppose that $\rho(h) = 1$ for any $h \in H$. We know that $B = UH$ and thus there is $r \in \Phi^+$ such that $\rho(U_\tau) \neq 1$. Since there is a canonical homomorphism from $\text{SL}_2(\mathbb{F}_q)$ onto $\langle U_\tau, U_{-\tau} \rangle$, where $r \in \Phi$ is an arbitrary and
\[
\left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \lambda^{-1} & 0 \\ 0 & \lambda \end{array} \right) = \left( \begin{array}{cc} 1 & \lambda^2 t \\ 0 & 1 \end{array} \right)
\]
we see that, say, $g := (1|101)$ is conjugated with $g^m$, where $m$ runs over all quadratic residues of $\mathbb{F}_q$. In other words, the operation $x \to x^m$ permutes all eigenvalues of $\rho(g)$ and hence the dimension $d_\rho$ is at least $\frac{q-1}{2}$ (strictly speaking, the arguments above hold for $\mathbb{F}_p$ but it is easy to show that for $\mathbb{F}_q$ a similar method works, see, e.g., [5, Proposition 8.10]).

Now we can assume that $\rho(u) = 1$ for any $u \in U$ (because otherwise we can apply the arguments above) but there is $h_0 \in H$ such that $\rho(h_0) \neq 1$. As we know $H$ is an abelian group equals the product of $l = |\Pi|$ cyclic subgroups which are isomorphic to $\mathbb{F}_q^*$. Clearly, for any $h \in H$ one has $\tilde{H}(\rho) = \rho(h)\tilde{H}(\rho) = \tilde{H}(\rho)\rho(h)$ in particular, $\tilde{H}(\rho) = \rho(h_0)\tilde{H}(\rho) = \tilde{H}(\rho)\rho(h_0)$. Thus the matrix $\rho(h_0)$ has a non–trivial invariant subspace $\mathcal{L}$, corresponding to the eigenfunctions with the eigenvalue equal one because otherwise $\tilde{H}(\rho) = 0$. Obviously, matrices $\rho(h)$, $h \in H$ commute with $\rho(h_0)$ and hence $\rho(h)\mathcal{L} \subseteq \mathcal{L}$. Since $B =HU$, it follows that $\rho(b)\mathcal{L} \subseteq \mathcal{L}$ for any $b \in B$. But then we find an invariant subspace of $\rho$, contradicting our assumption.

Finally, let $P = P_J$ be an arbitrary parabolic subgroup. Then we have $\rho$ is 1 on $B$ because otherwise we can use the previous arguments. We know that $P_J = \langle B, \{w_j\}_{j \in J} \rangle = \langle B, \prod_{j=1}^J w_j \rangle = \langle B, (\prod_{j=1}^J w_j)B(\prod_{j=1}^J w_j)^{-1} \rangle$ and hence $\rho$ is 1 on $P$. This completes the proof. \qed

5 Growth relatively to parabolic subgroups

Now let us obtain a result on growth of subsets from $G(q)$ under left/right multiplications by parabolic subgroups.

For any sets $A,B,C$ put $\sigma_A(B,C) := \sum_{x \in A} r_{BC}(x)$. Bounds in Theorem 12 below depend on the quantities $\sigma_p(A^{-1}, A)$, $\sigma_p(A, A^{-1})$, where $A$ is an arbitrary subset of $G(q)$ and $P$ is a parabolic subgroup. The sense of these expressions is rather obvious, namely, $\sigma_p(A^{-1}, A)$ and $\sigma_p(A, A^{-1})$ are small if the intersection of $A$ with left/right cosets of $P$ is small in average.

**Theorem 12** Let $G(q)$ be a Chevalley group and $P \subset G(q)$ be a parabolic subgroup. Then for any set $A \subseteq G(q)$ one has either

\[ |AP||A \cap P| \geq 2^{-1}|A|^2 \]

or

\[ |AP||PA| \geq 2^{-2}|A||P|q. \]  \hspace{1cm} (23)

In particular,

\[ \max\{|AP|, |PA|\} \geq 2^{-1} \min\{|A|^2|A \cap P|^{-1}, (|A||P|q)^{1/2}\}. \]  \hspace{1cm} (24)

Similarly,

\[ |APA| \geq |P|/4 \cdot \min\{q, |A|^4\sigma_p^{-1}(A^{-1}, A)\sigma_p^{-1}(A, A^{-1})\}, \]

\[ \text{(25)} \]

\[ \text{where } \sigma_p^{-1}(A, B) := \sum_{x \in A} r_{AB}(x), \]

\[ \text{and } \sigma_p^{-1}(A^{-1}, B) := \sum_{y \in A^{-1}} r_{BA}(y). \]
and if \( A \subseteq P \), then
\[
|PAB| \geq q|P|.
\] (26)

**Proof.** Let \( g \notin P \) and put \( A_g = A \cap gP \). Also, let \( \Delta = \max_{g \notin P} |A_g| \). We have
\[
E(A^{-1}, P) = \sum_x r_{AP}^2(x) = \sum_{x \in P} r_{AP}^2(x) + \sum_{x \notin P} r_{AP}^2(x) \leq |P| \sum_{x \in P} r_{AP}(x) + \Delta |P||A| = |P|^2 |A \cap P| + \Delta |P||A|.
\] (27)

In view of (3), we get
\[
|AP| \geq 2^{-1} \min\{ |A||P|\Delta^{-1}, |A|^2 |A \cap P|^{-1} \}.
\] (28)

On the other hand, using Lemma 10, we derive
\[
E(P, A_g) = \sum_x r_{AP_g}^2(x) \leq \sum_x r_{PA_g}(x) r_{PgP}(x) \leq 2|P|^2 |A_g|q^{-1},
\] (29)

and hence by the Cauchy–Schwarz inequality, we get
\[
|PA| \geq |PA_g| \geq \frac{|P|^2 |A_g|^2}{E(P, A_g)} \geq 2^{-1} q|A_g| = 2^{-1} q\Delta,
\] (30)

where we choose \( g \) such that \( |A_g| = \Delta \). Combining (28) and (30), we arrive to (24).

Similarly, let us obtain (25). In view of Lemma 9 and Lemma 10, we have
\[
\sigma := \sum_x r_{AP}^2(x) = \sum_{z, z'} r_{A^{-1}A}(z) r_{A^{-1}A}(z') |zP \cap Pz'| = \sum_{z, z' \in P} r_{A^{-1}A}(z) r_{A^{-1}A}(z') |zP \cap Pz'| + \sum_{z, z' \notin P} r_{A^{-1}A}(z) r_{A^{-1}A}(z') |zP \cap Pz'| \leq |P| \sigma_P(A^{-1}, A) \sigma_P(A, A^{-1}) + 2|P|q^{-1}|A|^4.
\] (31)

By the Cauchy–Schwarz inequality, we know that \( \sigma |APA| \geq |A|^4 |P|^2 \) and combining this with (31), we obtain the required result.

It remains to obtain (26). Since \( A \) does not belong to \( P = P_J \), it follows that there are \( w_J \in W_J, 1 \neq w \in W^J \), \( b_1, b_2 \in B \) such that the product \( b_1 w_J w b_2 \) is an element from \( A \). It easily follows from the Bruhat decomposition. Then \( Pw_J w B \subseteq PAB \) and in view of (12), we have \( Pw_J = P \). Thus we see that \( PAB \) contains disjoint sets \( Bw_J B \) for any \( v \in W_J \) and hence by (14)
\[
|PAB| \geq \sum_{v \in W_J} |Bw_J B| \geq |B| \sum_{v \in W_J} q(l(v)) = |B| q(l(w^J)) \sum_{v \in W_J} q(l(v)) \geq q|B| \sum_{v \in W_J} q(l(v)) = q|P_J|.
\]

This completes the proof. \( \square \)
Remark 13 It is easy to see that bound (21) is tight. Indeed, let \( P = B \) be a Borel subgroup and \( A = B \setminus Bw_rB \), where \( w_r \) is a fundamental reflection. In particular, \( l(w_r) = 1 \) and \( A \) is a parabolic subgroup. Then \( AB = BA = A \) but by (14), we have \( |A| \sim q|B| \sim \sqrt{|A||B|}q \).

Now we are ready to obtain a result on intersections of powers of \( A \) with parabolic subgroups. We use quasi–random technique from [10], [36].

**Theorem 14** Let \( G(q) \) be a Chevalley group and \( P \subset G(q) \) be a parabolic subgroup. Also, let \( n \geq 1 \) be a positive integer and \( X, Y_1, \ldots, Y_n \subset G(q) \) be nonempty sets such that \( X \cap P = \emptyset \) and

\[
q|X||P|^3 d_{\min}^{n+2} \prod_{j=1}^n |Y_j| \geq 4|G|^{n+4}.
\]  

Then \( XY_1 \ldots Y_n X \cap P \neq \emptyset \).

**Proof.** First of all, let us obtain a general upper bound for \( \|A(\rho)\|_\diamond \), where \( A \) is any subset of \( G = G(q) \) and \( \rho \) is an arbitrary non–trivial representation of \( G \). Using formula (10) with \( f = A \), we have

\[
\|\hat{A}(\rho)\|_\diamond < \left(\frac{|A| |G|}{d_{\min}}\right)^{1/2}.
\]  

Now if \( XY_1 \ldots Y_n X \cap P = \emptyset \), then \( (PX)Y_1 \ldots Y_n (XP) \cap P = \emptyset \). In terms of the representation theory it can be rewritten as

\[
0 = \frac{|PX||Y_1| \ldots |Y_n||XP||P|}{|G|} + \frac{1}{|G|} \sum_{\rho \in G, \rho \neq 1} \langle \widehat{PX}(\rho) \hat{Y}_1(\rho) \ldots \hat{Y}_n(\rho) \hat{XP}(\rho), \hat{P}(\rho) \rangle
\]

Since \( X \cap P = \emptyset \), we know by estimate (23) of Theorem 12 that \( |PX||XP| \geq 2^{-2} |X||P|q \). Using this fact and applying Lemma 6 combining with bound (33) for the sets \( Y_j \), we obtain

\[
\frac{|PX||Y_1| \ldots |Y_n||XP||P|}{|G|} < \|P\|_W \left(\frac{|PX||G|}{d_{\min}}\right)^{1/2} \left(\frac{|XP||G|}{d_{\min}}\right)^{1/2} \prod_{j=1}^n \left(\frac{|Y_j||G|}{d_{\min}}\right)^{1/2} \leq \left(\frac{|G|}{d_{\min}}\right)^{(n+2)/2} \left(\frac{|PX||XP|}{\prod_{j=1}^n |Y_j|}\right)^{1/2}
\]

or, in other words,

\[
q|X||P|^3 d_{\min}^{n+2} \prod_{j=1}^n |Y_j| < 4|G|^{n+4}.
\]

This completes the proof. \( \square \)

Let \( P \) be a parabolic subgroup of size at least \( |G(q)|/d_{\min} \) and let \( A \cap P = \emptyset \). Then Theorem 14 says us that \( A^{n+2} \cap P \neq \emptyset \), provided

\[
|A| \gg \frac{|G(q)|}{d_{\min}} \cdot \left(\frac{d_{\min}^2}{q}\right)^{1/(n+1)}.
\]  

(34)
In other words, if we want to generate $G(q)$ by powers of $A$, then we have a natural barrier $|A| \gg (\frac{|G(q)|}{d_{min}})^{1+\epsilon}$. Our next aim is to relax the last condition.

To do this in a particular case of $SL_2(F_q)$ we need a result on growth in $SL_2(F_q)$, which provides us some concrete bounds for growth, see [30] Theorem 14 (which in turn develops the ideas of [17, 35]). In the general case we apply Lemma 8.

**Theorem 15** Let $q \geq 5$, $A \subseteq SL_2(F_q)$ be a generating set, $q^{2-\epsilon} \ll |A| \ll q^{\frac{72}{35}}$, $\epsilon < \frac{2}{25}$. Then $|AAA| \gg |A|^\frac{72}{35}$.

Now we are ready to prove a result, which breaks the limit from (34). The absolute constants in 2), 3) can be easily computed but we do not specify them.

**Theorem 16** Let $B$ be a Borel subgroup of $SL_2(F_q)$ and $A \subseteq SL_2(F_q)$ be an arbitrary set. Then the following holds

1) If $|A| \geq q^{2-c}$, $c < \frac{2}{25}$, then there is $n \leq \lceil \frac{24(1+c)}{2-25c} \rceil$ such that $A^{3n+2} \cap B \neq \emptyset$.

2) If $|A| \geq q^1+\delta$, then there is $n \ll 1/\delta$ with $A^n \cap B \neq \emptyset$.

3) In general, let $q$ an odd number, $G(q)$ be a Chevalley group and $P \subset G(q)$ be a parabolic subgroup. Suppose that $|A| \geq \Pi_1(G(q))q^{-1-\delta}$. Then there is $n, n \ll 1/\delta$ such that $A^n \cap P \neq \emptyset$.

**Proof.** We can assume that $A \cap B = \emptyset$ because otherwise there is nothing to prove. Let $U := \{(1u|01): u \in F_q\}$. If $A$ generates $SL_2(F_q)$, then by Theorem 15 either $|A| \geq q^{\frac{72}{35}}$ or $|AAA| \gg |A|^\frac{72}{35} > q^{2+\frac{25c}{2}}$. Applying Theorem 14 with $P = B$, $X = A$, $Y_j = AAA$ and $d_{min} = \frac{q+1}{2}$ we see that $A^{3n+2} \cap B \neq \emptyset$ provided $n \geq \lceil \frac{24(1+c)}{2-25c} \rceil$. If $|A| \geq q^{\frac{72}{35}}$, then Theorem 14 with $P = B$, $X = A$, $Y_j = A$ gives us even better upper bound for $n$.

Now suppose that $A$ does not generate $SL_2(F_q)$. By the well–known subgroups structure of $SL_2(F_q)$ see, e.g., [39], we have that $A$ is a subset of a Borel subgroup and conjugating we can assume that $A$ is a subset of the standard Borel subgroup $B_\ast$ of the upper–triangular matrices. Also, we have $B = g^{-1}B_\ast g$ for a certain $g \in SL_2(F_q)$. We can assume that $g \notin B_\ast$ because otherwise $B_\ast = B$ and hence $A = B_\ast \cap A = B \cap A \neq \emptyset$. One can carefully use inequalities (25, 26) of Theorem 12 and prove that $A^{-1}BA^{-1}$ has size at least $|SL_2(F_q)| - (1 + o(1))|B|$. It is not enough for our purposes and we consider $A^n$ directly. By the Bruhat decomposition the element $g$ can be written as $bwu$, where $b \in B$, $u \in U$ and $w = (01|(-1)0)$. Then any element of $B = g^{-1}B_\ast g$ has the form

$$
\begin{pmatrix}
1 & -v \\
n & 1
\end{pmatrix}
\begin{pmatrix}
\lambda & 0 \\
u & \lambda^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & v \\
n0 & 1
\end{pmatrix} =
\begin{pmatrix}
\lambda - vu & v(\lambda - uv) - v\lambda^{-1} \\
u & uv + \lambda^{-1}
\end{pmatrix},
$$

where the variables $\lambda, u$ run over $F_q^\ast, F_q$, correspondingly, and $v$ is a fixed element. Since $A^n \subseteq B_\ast$, it follows that it is enough to find an element $(\lambda(v\lambda - v\lambda^{-1})(0\lambda^{-1})) \in B_\ast \cap B$ in $A^n$. The intersection $T := B_\ast \cap B$ is, clearly, is subgroup of size $q - 1$ and $T$ is, actually, a torus. Applying Corollary 7 (here we use the representation theory for $B$ not $Aff(F_q)$), we obtain that $A^3 \cap T \neq \emptyset$, provided $|A| \gg q^{5/3}$.

Now let us prove that the condition $|A| \geq q^{1+\delta}$ implies that there is $n \ll 1/\delta$ such that $A^n \cap B \neq \emptyset$. Again, if $A$ generates $SL_2(F_q)$, then we consequently apply Theorem 15 (also, see...
and hence $\Gamma$ is a parabolic subgroup, and hence it is enough to have
$$|\Gamma| \geq |A| \geq \Pi_1(G(q)) q^{-1+\delta} > \Pi_1(G(q)) q^{-1} \geq P(G(q))$$
and hence $\Gamma$ is a parabolic subgroup, $|\Gamma| \leq \Pi_1(G(q))$.

Recall that the intersection of two Borel subgroups contains a maximal torus of $G(q)$. Indeed, by the Bruhat decomposition we have $B := gB_s g^{-1} = u w B_s w^{-1} u^{-1}$, where $u \in U$, $w \in W$ and hence $u H w^{-1} \subseteq B_s \cap B$ because $w^{-1} H w = H \subseteq B_s$. In particular, the subgroup $P \cap \Gamma$ contains a torus $T$. Applying the arguments from the proof of Corollary 7 for the group $\Gamma$, as well as Lemma 11 we see that $A^n \cap T \neq \emptyset$, if
$$|A| > \frac{\Pi_1(G(q))}{q} \cdot \left( \frac{\Pi_1(G(q))}{|T|} \right)^{2/n} \geq \frac{|\Gamma|}{q} \cdot \left( \frac{|\Gamma|}{|T|} \right)^{2/n}.$$

By the assumption $|A| \geq \Pi_1(G(q)) q^{-1+\delta}$ and hence it is enough to have $n \gg I \delta^{-1}$. This completes the proof. \hfill \Box

**Example.** Let $B^+, B^-$ be the standard Borel subgroups of the upper/lower–triangular matrices from $\text{SL}_2(F_p)$ and $p \equiv -1 \pmod{4}$. Let also $A \subseteq B^+ \setminus B^-$ such that all elements of matrices from $A$ are quadratic residues. Then one can see that $A \cap B^- \text{ and } A^2 \cap B^-$ are empty. Also, we have $|A| \gg p^2$. It means that in Theorem 16 we need at least three multiplications even for sets $A$ with $|A| \gg p^2$.

### 6 Two applications to Zaremba’s conjecture

Using inequality (23) of Theorem 12 combining with Theorem 14 and applying the method from [30] one can decrease the constant 30 in Theorem 4 to 24. We go further, using the specific of our problem and obtain Theorem 5 from the Introduction.

Denote by $F_M(Q)$ the set of all *rational* numbers $\frac{u}{v}, (u, v) = 1$ from $[0, 1]$ with all partial quotients in $2$ not exceeding $M$ and with $v \leq Q$:
$$F_M(Q) = \left\{ \frac{u}{v} = [0; b_1, \ldots, b_s] : (u, v) = 1, 0 \leq u \leq v \leq Q, b_1, \ldots, b_s \leq M \right\}.$$

By $F_M$ denote the set of all *irrational* numbers from $[0, 1]$ with partial quotients less than or equal to $M$. From [14] we know that the Hausdorff dimension $w_M := \text{HD}(F_M)$ of the set $F_M$ satisfies
$$w_M = 1 - \frac{6}{\pi^2} \frac{1}{M} - \frac{72 \log M}{\pi^4 M^2} + O \left( \frac{1}{M^2} \right), \quad M \to \infty,$$  
(36)
however here it is enough for us to have a simpler result from \[12\], which states that

$$1 - w_M \asymp \frac{1}{M}$$  \hspace{1cm} (37)

with some absolute constants in the sign \(\asymp\). Explicit estimates for dimensions of \(F_M\) for certain values of \(M\) can be found in \[18\], \[19\] and in other papers. For example, see \[19\]

$$w_2 = 0.5312805062772051416244686... > \frac{1}{2}$$  \hspace{1cm} (38)

In papers \[12\] \[13\] Hensley gives the bound

$$|F_M(Q)| \asymp M^{2w_M}.$$  \hspace{1cm} (39)

More generally (see \[16\]), let \(A \subset \mathbb{N}\) be a finite set with at least two points and let \(F_A\) be the set of all irrational numbers such that \(b_j \in A\) (previously, \(A = \{1, \ldots, M\}\)). Then it is known \[12\], \[16\] that for the correspondent discrete set \(F_A(Q)\) formula (39) takes place (the constants there depend on \(A\) of course). The Hausdorff dimension \(\text{HD}(F_A)\) of the set \(F_A\) it is known to exist and satisfies \(0 < \text{HD}(F_A) < 1\).

We associate a set of matrices from \(G = \text{SL}_2(\mathbb{F}_p)\) with the continued fractions. One has

$$\begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_s \end{pmatrix} = \begin{pmatrix} p_{s-1} & p_s \\ q_{s-1} & q_s \end{pmatrix},$$  \hspace{1cm} (40)

where \(p_s/q_s = [0; b_1, \ldots, b_s]\) and \(p_{s-1}/q_{s-1} = [0; b_1, \ldots, b_{s-1}]\). Clearly, \(p_{s-1}q_s - p_s q_{s-1} = (-1)^s\).

Let \(Q = p - 1\) and consider the set \(F_M(Q)\). Any \(u/v \in F_M(Q)\) corresponds to a matrix from (40) such that \(b_j \leq M\). The set \(F_M(Q)\) splits into ratios with even and with odd \(s\),换句话说 \(F_M(Q) = F_M^{\text{even}}(Q) \sqcup F_M^{\text{odd}}(Q)\). Let \(A \subseteq \text{SL}_2(\mathbb{F}_p)\) be the set of matrices of the form above with even \(s\). It is easy to see from (39), multiplying if it is needed the set \(F_M^{\text{odd}}(Q)\) by \(\begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix}\),\(1 \leq b \leq M\) that \(|F_M^{\text{even}}(Q)| > M \cdot |F_M(Q)| > M \cdot Q^{2w_M}\). Let \(B\) be the standard Borel subgroup of \(\text{SL}_2(\mathbb{F}_p)\), i.e., the set of all upper–triangular matrices. It is easy to check that if for a certain \(n\) one has \(A^n \cap B \neq \emptyset\), then \(q_{s-1}\) equals zero modulo \(p\) and hence there is \(u/v \in F_M((2p)^n)\) such that \(v \equiv 0 \pmod{p}\). Actually, if we find any number from \(p_s, q_s, p_{s-1}, q_{s-1}\) equals zero modulo \(p\), then we can do the same, see \[15\] (but we do not need this fact).

**Lemma 17** We have

$$\sigma_B(A, A^{-1}) \leq p|A| \quad \text{and} \quad \sigma_B(A^{-1}, A) \leq M^2 p|A|.$$  \hspace{1cm} (41)

Moreover,

$$\max_{g \in \text{SL}_2(\mathbb{F}_p)} \{|A \cap gB|, |A \cap Bg|\} \leq Mp,$$  \hspace{1cm} (42)

$$\max_{g, h \in \text{SL}_2(\mathbb{F}_p)} |A \cap gBh| \ll_M |A| \cdot p^{-2w_M - \frac{1}{4}}.$$  \hspace{1cm} (43)
do not exceed \( p \pmod{p} \). Bound (42) can be obtained exactly in the same way. We can assume that \( \alpha, c \neq 0 \).

**Proof.** Let us begin with the estimation of \( \sigma_B(A, A^{-1}) \). We see that the product

\[
\left( \begin{array}{cc}
p_{s-1} & p_s \\
q_{s-1} & q_s
\end{array} \right) \left( \begin{array}{cc}
q'_t & -p'_t \\
-q'_{t-1} & p'_{t-1}
\end{array} \right) \in B,
\]

iff \( q'_t q_{s-1} \equiv q_s q'_{t-1} \pmod{p} \). It is well-known that \( \frac{q_s}{q_{s-1}} = [b_s; b_{s-1}, \ldots, b_1] \) and hence the number of pairs \( (q_{s-1}, q_s) \) is at most \(|A|\). Further, fixing \( q'_{t-1} \) as well as a pair \( (q_{s-1}, q_s) \), we find \( q'_t \) uniquely modulo \( p \) and hence we find \( q'_t \) because \( q'_t \leq p - 1 \). Thus \( \sigma_B(A, A^{-1}) \leq p|A| \) because all variables do not exceed \( p - 1 \). The argument showing that \( \sigma_B(A^{-1}, A) \leq M p|A| \) is even simpler because in this case we have the equation \( p'_{t-1} q_{s-1} = p_{s-1} q'_{t-1} \pmod{p} \) and any pair \( (p_{s-1}, q_{s-1}, q'_{t-1}) \) determines \( p'_{t-1} \). It remains to notice that we can reconstruct \( (p_s, q_s) \) from \( (p_{s-1}, q_{s-1}) \) in at most \( M \) ways. Bound (42) can be obtained exactly in the same way.

Finally, to get (43) we see that the inclusion

\[
\left( \begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array} \right) \left( \begin{array}{cc}
p_{s-1} & p_s \\
q_{s-1} & q_s
\end{array} \right) \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right) \in B
\]

(45)
gives us

\[
a(\gamma p_{s-1} + \delta q_{s-1}) \equiv -c(\gamma p_s + \delta q_s) \pmod{p}.
\]

(46)

We can assume that \( a, c \neq 0 \) because this case was considered above and the same situation for \( \gamma = 0 \). If \( \delta = 0 \), then \( ap_{s-1} \equiv -cp_s \pmod{p} \) and fixing \( p_s \) we find \( p_{s-1} \) uniquely. But \( \frac{p_{s-1}}{p_s} = [b_s; b_{s-1}, \ldots, b_2] \) and we determine the whole matrix, choosing \( b_1 \) in at most \( M \) ways. Thus suppose that all coefficients in (46) do not vanish. In view of the Bruhat decomposition (i.e. one can put \( d = \alpha = 0, \beta = b = 1, \gamma = c = -1 \)) equation (46) can be rewritten as

\[
a(\delta q_{s-1} - p_{s-1}) \equiv \delta q_s - p_s \pmod{p}
\]

(47)
or, in other words,

\[
\delta (q_s + \omega q_{s-1}) \equiv p_s + \omega p_{s-1} \pmod{p},
\]

(48)

where \( \omega = -a \). Equation (48) can be interpreted easily: any Borel subgroup fixes a point (the standard Borel subgroup fixes \( \infty \)) and hence inclusion (45) says that our set \( A \) transfers \( \omega \) to \( \delta \). In other terms, identity (48) says that the tuples \( (q_s, q_{s-1}, p_s, p_{s-1}) \) belongs to a hyperspace with the normal vector \( (\delta, \delta \omega, -1, -\omega) \) and hence for some other solutions of (48), we get

\[
\left| \begin{array}{cccc}
p_s & p_{s-1} & q_s & q_{s-1} \\
p'_s & p'_s & q'_s & q'_{s-1} \\
 p''_s & p''_{s-1} & q''_s & q''_{s-1} \\
p'''_s & p'''_{s-1} & q'''_s & q'''_{s-1}
\end{array} \right| \equiv 0 \pmod{p}.
\]

(49)

Now consider the set \( \tilde{A} \subset A \) which is constructing in an analogous way from \( F_M(2^{-5}Q^{1/k}) \), \( k = 4 \) but not from \( F_M(Q) \). Our first task is to prove

\[
\max_{g, h \in \text{SL}_2(\mathbb{F}_p)} |\tilde{A} \cap gBh| \leq Mp^{1/k}.
\]

(50)
Clearly, $|\tilde{A}| \sim |A|^{1/k} \sim p^{2w_M/k}$ and hence (50) would give us an almost square–root saving as $M$ tends to $\infty$. If we solve equation (49) with elements from $\tilde{A}$, then we arrive to an equation

$$Xq_s + Yq_{s-1} + Zp_s + Wp_{s-1} \equiv 0 \pmod{\tilde{p}},$$

where $|X|, |Y|, |Z|, |W| < 2^{-2}p^{3/k}$ which is, actually, an equation in $\mathbb{Z}$. We can assume that not all integer coefficients $X, Y, Z, W$ (which itself are some determinants of matrix from (49)) vanish because otherwise we obtain a similar equation with a smaller number of variables. Without losing of the generality, assume that $X \neq 0$ and substitute $q_s$ into the identity $q_sp_{s-1} - q_sq_{s-1} = (-1)^s = 1$. We derive

$$q_{s-1}p_sX = -p_{s-1}(Yq_{s-1} + Zp_s + Wp_{s-1}) - 1$$

or, in other words,

$$(Xq_{s-1} + Zp_{s-1})(Xp_s + Yp_{s-1}) = YZp_{s-1}^2 - X(Wp_{s-1}^2 + 1) := f(p_{s-1}). \quad (51)$$

Fix $p_{s-1} < 2^{-5}p^{1/k}$ and suppose that $f(p_{s-1}) \neq 0$. Then the number of the solutions to equation (51) can be estimated in terms of the divisor function as $p^{o(1)}$. Further if we know $(q_{s-1}, p_s, p_{s-1})$, then we determine the matrix from $A$ in at most $M$ ways. Now in the case $f(p_{s-1}) = 0$, we see that there are at most two variants for $p_{s-1}$ and fixing $q_s \leq 2^{-5}p^{1/k}$ in $q_p + p_{s-1} = 1$, we find the remaining variables in at most $p^{o(1)}$ ways (or just use formula (51)). Thus we have obtained (50).

To derive (43) from (50) notice that $A \subseteq \tilde{A}X$, where $X$ is constructing in an analogous way from $F_M(2^5(M + 1)Q^{1-1/k})$. Then, using (50), we get

$$\max_{g,h \in SL_2(\mathbb{F}_p)} |A \cap gBh| \leq \sum_{x \in X} |\tilde{A}x \cap gBh| \leq MP^{1/k}|X| \ll M p^{2w_M + \frac{(1-2w_M)}{4}} \sim |A| \cdot p^{1-2w_M}. \quad (51)$$

This completes the proof of the lemma. \hfill $\Box$

Assume that $|A| \sim p^{2w_M} \gg p^{3/2}$. Using formula (25) of Theorem 12, as well as Lemma 17, we obtain an optimal lower bound for $|A^{-1}BA^{-1}|$.

**Corollary 18** Let $w_M > 3/4$. Then

$$|ABA|, \; |A^{-1}BA^{-1}| \gg p^3.$$

Now we are ready to prove Theorem 5. First of all we obtain the result with the constant equals five and with the exact bounds (on $M$, say) and then subsequently refine the constant, using some additional arguments (which give worse dependence on $M$). The method of obtaining the constant five is more general and can be generalized further, see Theorem 20 below and remarks after it. One more time, decreasing $C$ in the condition $q = O(p^C)$, we increasing the constant $\xi$.

Take $n \geq 1$ and consider the equation $ay_1 \ldots y_n a' = b$, where $y_j \in Y$, $a, a' \in A$, $b \in B$ and we will choose the set $Y$ later. If this equation has no solutions, then the equation $sy_1 \ldots y_n s' = b$,
we can estimate the energies $E$.

By the arguments as in the proof of Theorem 14, we obtain (recall that $d_{\text{min}}(\text{SL}_2(\mathbb{F}_p)) \geq \frac{p-1}{2}$))

$$|Y|^n |S||S'||B| \ll |G| \left( \frac{|G||S|}{p} \right)^{1/2} \left( \frac{|G||S'|}{p} \right)^{1/2} \left( \frac{|G||Y|}{p} \right)^{n/2}$$

or, in other words,

$$|Y|^n |A|^2 \ll p^{2n+4}. \tag{53}$$

It remains to choose $Y$. Let $K = |AAA|/|A|$ and $\tilde{K} = |AA|/|A|$. If $\tilde{K} \gg p^6/|A|^3$, then $|AA| \gg p^6/|A|^2$ and this is a contradiction with inequality (53) for $Y = AA$ and $n = 1$. Suppose that $\tilde{K} \ll p^6/|A|^3$. In inequality (30), using the Helfgott’s method [17], [35], it was proved that

$$|A|^2 p^{-1} \ll_M K \tilde{K} |A|^3 K^{2/3} |A|^{1/3},$$

provided

$$|A| \gg p^{3/2} K^{5/2} \tag{54}$$

Combining the last estimate with $\tilde{K} \ll p^6/|A|^3$, we get

$$K \gg_M \frac{|A|^{11/5}}{p^{21/5}}.$$  

It is easy to check, that if (54) has no place, then we obtain even better lower bound for $K$. Applying inequality (53) with $Y = AAA$ and $n = 1$ we arrive to a contradiction, provided

$$|A| \sim p^{2w_M} \gg p^{51/26}.$$  

In view of (57) we can satisfy the last condition taking sufficiently large $M$. Thus $A^5 \cap B \neq \emptyset$ and one can calculate the required $M$ by formula (57).

To replace the constant five in Theorem 5 to four it is enough to show (see inequality (53)) that $|AA| \gg |A|^{1+c}$, where $c > 0$ is an absolute constant and for the last in view of (3) it is enough to obtain a non–trivial upper bound for the energy of $A$ of the form $E(A, A) \ll |A|^{3-c}$. Suppose for a certain $T \geq 1$, $E(A, A) = |A|^3/T$. By the non–commutative Balog–Szemerédi–Gowers Theorem, see [31, Theorem 32] or [40, Proposition 2.43, Corollary 2.46] there is $a \in A$ and $A_s \subseteq a^{-1}A$, $|A_s| \gg_T |A|$ such that $|A_s|^3 \ll_T |A_s|$. Here the signs $\ll_T, \gg_T$ mean that all dependences on $T$ are polynomial. In view of the Helfgott’s growth result or Theorem 15 it is enough to show that $A_s$ does not belong to a coset of a Borel subgroup. But it easily follows from bound (43) of Lemma 17 (here we assume that $w_M > 1/2$) and the lower bound for size of $A$ (and hence size of $A_s$).

To replace the constant four in Theorem 5 to three notice that the Parseval identity (6) gives us $\|A\|_o^4 \ll E(A, A) \|G\|/p \ll |A|^{3-c} p^2$. Here $c > 0$ is an absolute constant, $\|A\|_o =$$
max_{p \neq 1} \| \hat{A}(\rho) \|_o \text{ and } M \text{ is taken to be large enough. Hence we find a solution to the equation } sas^t = b \text{ provided } \\
\frac{|S|^2 |A| |B|}{|G|} \gg p|A|^3 > \frac{|S| |G|}{p} |A| \| A \|_o \gg |A| p^3 (|A|^{3-c} p^2)^{1/4} \tag{55}
\]
or, in other words, \(|A| \gg p^{10/7} \). In view of (37) we can satisfy the last condition taking sufficiently large \(M\).

Finally, we replace the constant three in Theorem 5 to two and further to \(1 + \varepsilon\). Let \( \Lambda \subset A \) be a set constructing in an analogous way from \( F_M(\sqrt{Q}) \) but not from \( F_M(Q) \). Clearly, \(|\Lambda| \sim p^{3M} \sim \sqrt{|A|}\) and \( \Lambda^2 \subseteq A \).

**Lemma 19** Let \( X \subseteq B \) be an arbitrary set. We have \( E(\Lambda, X) = |\Lambda||X| \) and \( E(\Lambda^{-1}, X) \leq M^4|\Lambda||X| \). In particular, \(|BA| = |B||\Lambda| \) and \(|AB| > |B||\Lambda|/M^4 \).

**Proof.** As in the proof of Lemma 17 we see that \( \Lambda^{-1} \Lambda \in B \) iff \( q_l q_{s-1} \equiv q_s q_{l-1} \) (mod \(p\)) (we use the notation from the lemma). The set \( \Lambda \) has been constructed from \( F_M(\sqrt{Q}) \) and hence we have \( q_l q_{s-1} = q_s q_{l-1} \). Obviously, \((q_{s-1}, q_s) = (q_{l-1}, q_l) = 1\) and hence \( q_s = q_l, q_{s-1} = q_{l-1} \). After that we reconstruct both matrices and obtain \( E(\Lambda, X) = |\Lambda||X| \).

Similarly, \( \Lambda^{-1} \Lambda \in B \) iff \( p_l p_{s-1} \equiv p_s p_{l-1} \) (mod \(p\)) and whence \( p_l p_{s-1} = p_s p_{l-1} \). Again, \((p_{s-1}, p_s) = (p_{l-1}, p_l) = 1\) and hence \( p_s = p_l, p_{s-1} = p_{l-1} \). After that we reconstruct both matrices in at most \(M^2\) ways. Finally, from (40) it follows that the image \( \Lambda^{-1} \Lambda \) belongs to a set of cardinality at most \(M^2\) and whence we obtain \( E(\Lambda, X) \leq M^4|\Lambda||X| \). This completes the proof of the lemma.

After that we redefine \( S \) and \( S' \) as \( BA \), \( \Lambda B \), respectively, and use the calculations from (55). It gives
\[
\frac{|S|^2 |A| |B|}{|G|} \gg p^3 |A|^2 > \frac{|S| |G|}{p} \| A \|_o \gg |A|^{1/2} p^4 (|A|^{3-c} p^2)^{1/4} \tag{56}
\]
or, in other words, \(|A| \gg p^{6/7} \). In view of (37) we can satisfy the last condition taking sufficiently large \(M\). Thus we have obtained the integer constant two but it is easy to see that this quantity is, actually, \(2 - \tilde{c}\), where the absolute constant \(\tilde{c}\) depends on \(c\). Indeed, just replace \(\sqrt{p-1}\) in the definition of the set \( \Lambda \) to \(p^{(1-c)/2}\) for sufficiently small \(\varepsilon = \varepsilon(c) > 0\) and repeat the calculations above.

In the last step we take an integer parameter \(k \sim 1/\varepsilon\) and consider \( \Lambda_k \subset A \), constructed from \( F_M(2^{-1}Q^{1/k}) \). Let \( \hat{A} = \Lambda_k \subset A \) and we have \( |\hat{A}| \sim_k |A| \) (more precisely, \(|A| \gg |\hat{A}| \gg \eta^k |A| \), where \(\eta < 1\) is an absolute constant). In other words, \( A \) and \( \hat{A} \) have comparable sizes. In particular, Lemma 17 takes place for \( \hat{A} \), hence \( E(\Lambda) \ll_k |\hat{A}|^{3-c} \) and whence \( \| \hat{A} \|_o \ll_k |\hat{A}|^{1-c} \). The set \( \hat{A} \) is the direct product of \(k\) copies of \(\Lambda_k\) and hence the set of all eigenvalues of the Fourier transform \(\hat{A}(\rho)\) is the \(k\)th power of the set of all eigenvalues of \(\hat{A}(\rho)\). We will show a little bit later that this relation, indeed, implies a power saving for the operator norm of \(\hat{A}(\rho)\).

It means that \(\| \Lambda_k \|_o \ll_k |\Lambda_k|^{1-c}(k)\) for a certain \(c_s(k) > 0\) and calculations in (56) for the equation \(s\lambda_k s^t = b\), \(\lambda_k \in \Lambda_k\) give us
\[
p^3 |A| \| \Lambda_k \| \gg |\Lambda_k|^{1-c_s(k)} |A|^{1/2} p^4 \sim |\Lambda_k|^{1-c_s(k)} |\Lambda| p^4 \gg_k |S| \| \Lambda_k \|_o p^2 \tag{57}
\]
and this is attained for any sufficiently large $M = M(\epsilon)$, $M \gg k/c_\epsilon(k)$ because inequality (57) is equivalent to
\[ pw_M + 2c_\epsilon(k)w_M/k \gg p. \] (58)

To demonstrate the required power saving for the operator norm of $\hat{\Lambda}_k(\rho)$ we need to obtain an analogue of Lemma 17 for the set $\Lambda_k$ and $p$ replaced by $p^{1/k}$. But because of similarity of $A$ and $\Lambda_k$ the proof is the same and moreover for $k \geq 4$ we have the following uniform bound $|\Lambda_k \cap gBh| \ll_M p^{1/k}$ for all $g, h \in \text{SL}_2(\mathbb{F}_p)$ (see the arguments of the proof of Lemma 17 in particular, bound (50)). As for the intersection of $\Lambda_k$ with the dihedral groups $\Gamma$ (another class of maximal subgroups of $\text{SL}_2(\mathbb{F}_p)$) the arguments are the same again (any dihedral subgroup provides even two linear restrictions for the tuple $(p_{s-1}, q_{s-1}, p_s, q_s)$) and they give the estimate $|\Lambda_k \cap g\Gamma h| \ll_M p^{1/k}$ for all $g, h \in \text{SL}_2(\mathbb{F}_p)$ and $k \geq 4$ (the details can be found in [4] and in [29, Lemma 21]). This completes the proof of Theorem 5.

Applying the second part of Theorem 16 and the arguments of the proof of the result above (avoid using of Lemma 17 and Lemma 19 which appellate to the specific structure of the set $A$), we obtain

**Theorem 20** Let $A \subset \mathbb{N}$ be a finite set, $|A| \geq 2$ such that $\text{HD}(F_A) > 1/2 + \delta$, where $\delta > 0$. There is an integer constant $C_A(\delta)$ such that for any prime number $p$ there exist some positive integers $q = O_A(p^{C_A(\delta)})$, $q \equiv 0 \pmod{p}$ and $a, (a, q) = 1$ having the property that the ratio $a/q$ has partial quotients belonging to $A$.

Thanks to [38] we see in particular, that Theorem 20 takes place for $A = \{1, 2\}$. Previously, this fact was obtained in [30] by another approach (although one can check that now our new constant $C_A(\delta)$ is better). As the reader can see from the proof, our method is rather general and we do not even need, actually, in restrictions of the form $b_j \in A$ and it is possible to consider other (say, Markov–type) conditions for the partial quotients (of course we still need that the Hausdorff dimension of the corresponding Cantor set is greater than $1/2$).

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