Square-Root Finding Problem In Graphs, A Complete Dichotomy Theorem.

Babak Farzad\(^1\) and Majid Karimi\(^2\)

Department of Mathematics
Brock University, St. Catharines, ON., Canada
bfarzad@brocku.ca\(^1\) mkarimi@brocku.ca\(^2\)

September, 2012

Keywords. Graph roots, Graph powers, NP-completeness.

Abstract

Graph \(G\) is the square of graph \(H\) if two vertices \(x, y\) have an edge in \(G\) if and only if \(x, y\) are of distance at most two in \(H\). Given \(H\) it is easy to compute its square \(H^2\). Determining if a given graph \(G\) is the square of some graph is not easy in general. Motwani and Sudan [11] proved that it is NP-complete to determine if a given graph \(G\) is the square of some graph. The graph introduced in their reduction is a graph that contains many triangles and is relatively dense. Farzad et al. [5] proved the NP-completeness for finding a square root for girth 4 while they gave a polynomial time algorithm for computing a square root of girth at least six. Adamaszek and Adamaszek [1] proved that if a graph has a square root of girth six then this square root is unique up to isomorphism. In this paper we consider the characterization and recognition problem of graphs that are square of graphs of girth at least five. We introduce a family of graphs with exponentially many non-isomorphic square roots, and as the main result of this paper we prove that the square root finding problem is NP-complete for square roots of girth five. This proof is providing the complete dichotomy theorem for square root problem in terms of the girth of the square roots.
1 Introduction

Graph $G$ is called the $r^{th}$ power of $H$ and $H$ is called an $r^{th}$ root of $G$, if $v$ is adjacent to $u$ in $G$ if and only if $d_H(v, u) \leq r$, where $d_H(v, u)$ is the distance between $u$ and $v$ in graph $H$. We are interested in the characterization and recognition of square graphs. Root and root finding are concepts familiar to most branches of mathematics. Root and root finding for graph also is a basic operation in graph theory. The complexity problem of root finding for graphs is an extensively studied problem in algorithmic graph theory.

The main motivation for studying the complexity of checking if a given graph is a certain power (square specifically) of another graph comes from distributed computing. In a model introduced by Linial [10], the $r^{th}$ power of graph $H$ may represents the possible flow of information in $r$ round of communication in a distributed network of processors organized according to $H$. He introduced a question about the characterization of this problem which is solved by Motvani and Sudan [11].

Mukhopadhyay [12] showed that a graph $G$ is the square of some graph if and only if there exists a complete induced subgraph $G_i$ corresponding to each vertex $v_i$ such that

- $v_i \in G_i$;
- $v_i \in G_j$ if and only if $v_j \in G_i$;
- $\bigcup G_i = G$.

However Mukhopadhyay’s theorem contains different aspects of the maximum clique problem which is an NP-hard problem. Hence it does not benefit the study from a complexity point of view.

Ross and Harary [13] characterized squares of trees and showed that tree square roots, when they exist, are unique up to isomorphism. Motwani and Sudan [11] proved that it is NP-complete to determine if a given graph has a square. The graph introduced in their reduction is a graph that contains many triangles and is relatively dense. On the other hand, there are polynomial time algorithms to compute the tree square root [9, 6, 7, 8, 4], a bipartite square root [7], and a proper interval square root [8]. Farzad et al. [5] provided an almost dichotomy theorem for the complexity of the recognition problem in terms of the girth of the square roots. They provided a polynomial time characterization of square of graphs with girth at least 6. They proved that the square root (if it exists) is unique up to isomorphism when the girth of square root is at least 7. They also proved the NP-completeness of the
problem for square roots of girth 4. Adamaszek and Adamaszek [1] proved that the square root of a graph is unique up to isomorphism when the girth of square root is at least 6 if it exits.

A summary of the the study for square root finding problem in terms of the girth of the square root is presented in Table 1 (in this table $H$ is indicating a square root graph).

| Girth $g(H)$ | Complexity Class | Unique up to isomorphism |
|-------------|-----------------|--------------------------|
| $g(H) = \infty$ | $O(|V| + |E|)$ [4] | Yes                      |
| $g(H) \geq 7$ | $O(|V| \times |E|)$ [5] | Yes [5]                  |
| $g(H) \geq 6$ | $O(|V| \times |E|)$ [5] | Yes [1]                  |
| $g(H) \geq 5$ | ?               | No                       |
| $g(H) \geq 4$ | NP-complete [5] | No                       |
| $g(H) \geq 3$ | NP-complete [11] | No                       |

Table 1: Complexity of square root problem in terms of girth of square roots.

The recognition problem has been open for square roots of girth 5. In Section 3 we show that this problem is NP-complete. The result is providing a complete dichotomy complexity theorem for square root problem. We also generalize the graph introduced in [1] to construct a family of graphs with exponential number of non-isomorphic square roots.

Definitions and notations: All graphs considered are finite, undirected and simple. Let $G = (V_G, E_G)$ be a graph with vertex set in $V_G$ and edge set $E_G$. We denote the adjacency of two vertices $u$ and $v$ in graph $G$, by $u \sim v$.

To show that $v$ is adjacent to every element of a set $A \subseteq V(G)$, we use $v \sim A$. The neighbourhood of a vertex $v$ in graph $G$ denoted by $N_G(v)$ is the set all vertices in $G$ adjacent to $v$. The closed neighbourhood of $v$ in $G$ denoted by $N_G[v]$, is its neighbourhood containing $v$ as well, i.e. $N_G[v] = N_G(v) \cup \{v\}$. The cardinality of the set $N_G(v)$ is called the degree of $v$ in $G$. The minimum degree of a graph $G$ is shown by $\delta_G$.

Let $d_G(x, y)$ be the length i.e., number of edges of a shortest path in $G$ between $x$ and $y$. Let $G^k = (V_G, E^k)$ with $xy \in E^k$ if and only if $1 \leq d_G(x, y) \leq k$, denote the $k$-th power of $G$. If $G = H^k$ then $G$ is the $k$-th power of the graph $H$ and $H$ is a $k$-th root of $G$. Since the power of a graph $H$ is the union of the powers of the connected components of $H$, we may assume that all graphs considered are connected.
A set of vertices \( Q \subseteq V_G \) is called a **clique** in \( G \) if every two distinct vertices in \( Q \) are adjacent; a **maximal clique** is a clique that is not properly contained in another clique. Given a set of vertices \( X \subseteq V_G \), the subgraph induced by \( X \) is denoted by \( G[X] \) and \( G - X \) stands for \( G[V(G) \setminus X] \). If \( X = \{a, b, c, \ldots\} \), we write \( G[a, b, c, \ldots] \) for \( G[X] \). Also, we often identify a subset of vertices with the subgraph induced by that subset, and vice versa.

The **girth** of \( G \), \( g(G) \), is the smallest length of a cycle in \( G \); in case \( G \) has no cycles, we set \( g(G) = \infty \). In other words, \( G \) has girth \( k \) if and only if \( G \) contains a cycle of length \( k \) but does not contain any cycle of length \( \ell = 3, \ldots, k - 1 \).

A complete graph is one in which every two distinct vertices are adjacent; a complete graph on \( n \) vertices is also denoted by \( K_n \). A **star** is a graph with at least two vertices that has a vertex adjacent to all vertices and the other vertices are pairwise non-adjacent. A star on \( n + 1 \) vertices is denoted by \( K_{1,n} \).

Two graphs \( G_1 \) and \( G_2 \) are called **isomorphic** when there is a bijection \( f \) from \( V_{G_1} \) to \( V_{G_2} \) such that for all vertices \( u, v \in V_{G_1} \): \( u \overset{G_1}{\sim} v \) if and only if \( f(u) \overset{G_2}{\sim} f(v) \).

## 2 Graphs with Many Non-Isomorphic Square Roots of Girth Five

For a given graph \( G \) if there exists \( H \) where \( G = H^2 \) and \( g(H) \geq 6 \), then \( H \) is unique up to isomorphism [1]. However this is not true when the girth of \( H \) is at least 5. For \( G = K_5 \), two graphs \( K_{1,4} \) and \( C_5 \) are non-isomorphic square roots of \( G \). These two graphs can be used to introduce a family of non-isomorphic pairs of graphs with the same square, see Figure 1.

Notice that graphs in this family contain vertices of degree 1. Such vertices were a main source of technicalities in the past studies.

In [1] there is also an example of a graph with two non-isomorphic square root of girth five, see Figure 2. These two graphs are more interesting as, unlike graphs shown in Figure [1] they contain no vertex of degree 1. These two graphs are also the smallest non-isomorphic graphs with girth five, minimum degree 2 and identical squares. In this paper, we call these two graphs \( G_1 \) and \( G_2 \).
Figure 1: A family of non-isomorphic graphs with identical square graph.

Figure 2: Non-isomorphic graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with no vertex of degree 1 and identical squares.

It is also an interesting question (from a complexity point of view) to ask if there exists a graph with many non-isomorphic square roots. We
show that $\mathcal{G}_1$ and $\mathcal{G}_2$ can be used to construct a family of graphs with many non-isomorphic square roots. With current labelling of $\mathcal{G}_1$ and $\mathcal{G}_2$, we have three vertices 1, 12 and 14, that their neighbourhoods in both $\mathcal{G}_1$ and $\mathcal{G}_2$ are identical. So we may identify two graphs on one of these three vertices to construct a new graph with more than one square roots. For example, we can identify vertex 1 in both $\mathcal{G}_1$ and $\mathcal{G}_2$ as shown in Figure 3.

![Figure 3: Connecting $\mathcal{G}_1$ and $\mathcal{G}_2$ by identifying vertex 1.](image)

**Observation 1.** The square of the graph shown in Figure 3 has three non-isomorphic square roots.

**Proof.** In Figure 3, by replacing the copy of $\mathcal{G}_1$ on the vertices \{1, 2, \ldots, 16\} with a copy of $\mathcal{G}_2$, we would get a different graph with the same square. Hence switching copies of $\mathcal{G}_1$ and $\mathcal{G}_2$ constructs three non-isomorphic graphs with identical squares. 

The process of connecting $\mathcal{G}_1$s and $\mathcal{G}_2$s by identifying one of those three vertices can form a family of graphs with girth five, minimum degree of 2 and exponentially many non-isomorphic square roots. See Figure 4 for an illustration of 16 non-isomorphic graphs with identical square.

This process is introducing a family of graphs with exponentially many non-isomorphic square roots. This family indicates that even with the
restriction $\delta_H \geq 2$ knowledge of any local neighbourhood is not sufficient to reconstruct the rest of the square root graph.

When a square root graph has no short cycle (girth of at least 6) square root finding problem is solvable by an efficient algorithm [5]. The main idea of this algorithm (and almost all attempts to find an efficient algorithm for square root finding problem) is to use a known neighbourhood of the square root graph and reconstruct the whole square root graph by only using informations from the square graph. Indeed if we know an arbitrary neighbourhood of graph $H$ of girth at least six, where $H^2 = G$, then we can recognize second neighbours (vertices of distance two) of that vertex. In this way the whole graph $H$ can be uniquely reconstructed with only using information of $G$. The family of graphs we introduced using $G_1$ and $G_2$ indicates that by knowing an arbitrary neighbourhood of the square root graph we can never decide the rest of the graph, as there are always options (to decide a second neighbourhood of a vertex) that results different (non-isomorphic) graphs. Hence knowing a constant number of neighbourhoods in the square root graph can not help to find a square root for a given graph (or to decide if there exists a square root graph).

We also use $G_1$ and $G_2$ graphs as part of our reduction in Section 3. We
need to show that the graph $G = G_1^2 = G_2^2$ has only two non-isomorphic square roots which are $G_1$ and $G_2$. For the rest of this paper we use $G$ as the square of $G_1$ (or $G_2$).

**Theorem 1.** Let $G = H^2$ for $g(H) = 5$, then $H$ is either isomorphic to $G_1$ or to $G_2$.

A proof of this theorem can be found in Appendix-A.

## 3 Square of graphs with girth five

In this section we show that the following problem is NP-complete.

**Square of Graphs With Girth Five**

*Instance:* A graph $G$.

*Question:* Does there exists a graph $H$ with girth at least 5 such that $G = H^2$?

It is an easy observation that **Square of Graphs With Girth Five** is in $NP$. We will reduce a variation of the “positive 1-in-3 SAT” problem (which is an NP-complete problem [14]) to **Square of Graphs With Girth Five**. Positive 1-in-3 SAT is a variant of the 3-satisfiability problem (3SAT). Like 3SAT, the input instance is a collection of clauses, where each clause is the disjunction of exactly three literals, and each literal is just a variable (there are no negations, which is why it is called positive). The positive 1-in-3 3SAT problem is to determine whether there exists a truth assignment to the variables so that each clause has exactly one true variable (and thus exactly two false variables). In this paper we are interested in another variation of the positive 1-in-3 SAT, which we call it **Positive and Minimum Interesting 1-in-3 SAT**.

**Positive and Minimum Interesting 1-in-3 SAT.**

*Instance:* A collection of clauses, where each clause is the disjunction of exactly three variables and two different clauses are sharing at most one variable.

*Question:* Does there exists a truth assignment to the variables so that each clause has exactly one true variable?

**Theorem 2.** **Positive and Minimum Interesting 1-in-3 SAT** is NP-complete.
Proof. It is trivial that this problem is in NP. We reduce an instance of a Positive 1-in-3 SAT to a Positive and Minimum Interesting 1-in-3 SAT. Let $\phi$ be a given collection of clauses as an instance of the positive 1-in-3 SAT.

For each pair of clauses $c : (x \lor y \lor z)$ and $d : (x \lor y \lor u)$ in $\phi$, that are sharing two variables $x$ and $y$, we know $u$ and $v$ must have the same truth value. So we may identify the two variables and thus replace $v$ with $u$ and remove the clause $d$. We construct $\phi'$ from $\phi$ by removing one of clauses in each pair of clauses that are sharing two variable. Therefore $\phi'$ is an instance of Positive and Minimum Interesting 1-in-3 SAT. This reduction shows that Positive and Minimum Interesting 1-in-3 SAT is NP-complete.

In this section we reduce the Positive and Minimum Interesting 1-in-3 SAT to Square of Graphs With Girth Five.

3.1 The Reduction

Before introducing the reduction in all details, we present three main ideas of the graph construction that we will explain below. For convenience, we represent $\forall a \in A : v \sim a$ by $v \sim A$, and also $\{x_a, y_a, z_a, \ldots\}$ by $\{x, y, z, \ldots\}_a$.

First is the idea of using graph $G$ to represent each copy of a variable. As we proved in Appendix A, a square root of $G$ is a graph which is isomorphic to either $G_1$ or $G_2$. We set $G_1$ to represent the FALSE value and $G_2$ to represent the TRUE value. If the square root of the subgraph that is representing a copy of a variable $x$ is isomorphic to $G_1$ we conclude that $x$ is FALSE. Otherwise, that is if it is isomorphic to $G_2$, we conclude that $x$ is TRUE.

The second idea is to represent a clause $c_i = x_i \lor y_i \lor z_i$ in such a way that exactly one of $x_i, y_i$ and $z_i$ is true (i.e., exactly one of the subgraphs that are representing the three variables is isomorphic to $G_2$ and the other two are isomorphic to $G_1$). For this, for each clause $c_i$ we introduce four new vertices $y_{i1}, \ldots, y_{i4}$ to construct a Petersen graph in the square root (that is a $K_{10}$ in the square graph) using vertices 5 and 13 in the three subgraphs representing the copies of variables in $c_i$. This construction is illustrated in Figure 5.

Lemma 1. The square of the graph shown in Figure 5 has three different (up to labelling) square roots. The other two square roots can be obtained by switching $G_1$s with $G_2$s. However, it has a unique square root of girth 5 up to isomorphism.
Proof. Let $\mathcal{C}$ be the square of the graph shown in Figure 5 on $X \cup Y \cup Z \cup I$ where $X = \{1, 2, \ldots, 16\}_x, Y = \{1, 2, \ldots, 16\}_y, Z = \{1, 2, \ldots, 16\}_z$, and $I = \{y_i^1, \ldots, y_i^4\}$. Also let $D$ to be a square root of $\mathcal{C}$. Graphs constructed by switching $G_1$s with $G_2$s. The isomorphism of these three graphs can be obtained by a permutation on $I$.

For example, assume that $D[X] \cong D[Y] \cong G_1$ and $D[Z] \cong G_2$. Then the graph obtained by the permutation $y_i^1 \leftrightarrow y_i^3$ and $y_i^2 \leftrightarrow y_i^4$ has the same square as the graph shown in Figure 5.

By Theorem 1, the square root of the subgraph induced by $X, Y$ or $Z$ is either $G_1$ or $G_2$. Now consider the neighbourhoods of vertices 5 and 13. We have $N_{G_1}[5] = N_{G_2}[13] = \{5, 13, 11, 15\}$ and $N_{G_1}[13] = N_{G_2}[5] = \{5, 13, 6, 7, 16\}$. It can be seen that if none or more than one of the square roots of the subgraph induced by $X, Y$ or $Z$ is isomorphic to $G_2$, then there would be no permutation on $I$, that form the same square as the graph shown in Figure 5.

The third idea is to make sure that different copies of the same variable have the same truth value. Again we use the fact that $N_{G_1}[5] = N_{G_2}[13] = \{5, 13, 11, 15\}$ and $N_{G_1}[13] = N_{G_2}[5] = \{5, 13, 6, 7, 16\}$. Let $x_i$ and $x_j$ be two copies of the same variable in two different clauses $c_i$ and $c_j$. We introduce
two new vertices called $v_{x_i,x_j}$ and $w_{x_i,x_j}$ which form a $C_6$ in the square root graph together with the vertices 5 and 13 in the subgraphs corresponding to $x_i$ and $x_j$. If both $x_i$ and $x_j$ are TRUE then $v_{x_i,x_j} \sim \{13_{x_i},13_{x_j}\}$ and $w_{x_i,x_j} \sim \{5_{x_i},5_{x_j}\}$, otherwise $w_{x_i,x_j} \sim \{13_{x_i},13_{x_j}\}$ and $v_{x_i,x_j} \sim \{5_{x_i},5_{x_j}\}$.

This construction is shown in Figure 6. Moreover we have the following Lemma.

**Lemma 2.** Let $X$ be the square of the graph shown in Figure 6 on the vertex set of $X_i \cup X_j \cup \{v_{x_i,x_j},w_{x_i,x_j}\}$ where $X_i = \{1,2,\ldots,16\}_{x_i}$ and $X_j = \{1,2,\ldots,16\}_{x_j}$. Let $X^2 = X$. If $X[X^i] \cong G_1$ (or $G_2$) then $X[X^j] \cong G_1$ (or $G_2$).

**Proof.** Assume otherwise and let (without loss of generality) $X[X^i] \cong G_1$ while $X[X^j] \cong G_2$, hence $v_{x_i,x_j}$ must be adjacent to $5_{x_i}$ and $13_{x_j}$ which means $5_{x_i} \not\sim 13_{x_j}$, and this is a contradiction as $5_{x_i} \not\sim 13_{x_j}$. \hfill \Box

**Reduction Graph:** Let $\phi : (c_1 \land c_2 \land \cdots \land c_n)$ be an instance of **Positive and Minimum Intersecting 1-in-3 SAT** such that $c_i = x_i \lor y_i \lor z_i$. As a convention we use $x_i$ and $x_j$ to represents two copies of variable $x$ in distinct clauses $c_i$ and $c_j$.

We construct an instance $G = G(\phi)$ and we show that there exists a square root $H$ of of girth 5 of graph $G$ corresponds to a satisfying assignment of $\phi$.

The vertex set of graph $G(\phi)$ consists of:
For every copy $x_i$ of variable $x$, $V_{x_i} = V_{G_1} = \{1, 2, \ldots, 16\} = \{1, 2, \ldots, 16\} = \{1, 2, \ldots, 16\}$, representing 16 vertices of a graph $\phi$.

For each clause $c_i$, $V_i = \{y^1_i, y^2_i, y^3_i, y^4_i\}$.

$W_{x_i,x_j} = \{v_{x_i,x_j}, w_{x_i,x_j}\}$, corresponding to two copies $x_i$ and $x_j$ of the same variable $x$, in two distinct clauses $c_i$ and $c_j$.

The edge set of $G(\phi)$ consists of:

- Variable edges: for each $x_i$, $G[V_{x_i}] = \emptyset$.

- Clause edges: For each clause $c_i = x_i \lor y_i \lor z_i$ where $i \notin \{j, k, m\}$:
  - $y^1_i \sim \{v_{x_i,x_j}, w_{x_i,x_j}, v_{y_i,y_k}, v_{z_i,z_m}\}$,
  - $y^2_i \sim \{w_{x_i,x_j}, v_{y_i,y_k}, v_{z_i,z_m}\}$,
  - $y^3_i \sim \{6_{x_i}, 16_{x_i}, 6_{y_i}, 16_{y_i}, 6_{z_i}, 16_{z_i}\}$,
  - $y^4_i \sim \{11_{x_i}, 15_{x_i}, 6_{y_i}, 16_{y_i}, 6_{z_i}, 16_{z_i}\}$, see Figure 7.

- Intra clause edges: for each clause $c_i = x_i \lor y_i \lor z_i$ where $i \notin \{j, k, m\}$:
  - $y^1_i \sim \{v_{x_i,x_j}, w_{y_i,y_k}, v_{z_i,z_m}\}$,
  - $y^2_i \sim \{w_{x_i,x_j}, v_{y_i,y_k}, v_{z_i,z_m}\}$,
Figure 8: A subgraph of $G(\phi)$ corresponding to the clause $c_i = x_i \lor y_i \lor z_i$.

Notice that we may have only a subset of these edges depending on the existence of $x_j$ (the copy of variable $x$ in $c_j$), $y_k$ (the copy of variable $y$ in $c_k$) and $z_m$ (the copy of variable $z$ in $c_m$).

- Edges for different copies of a variable: for each arbitrary pair $x_i$ and $x_j$ which are different copies of the same variable,
  
  $v_{x_i, x_j} \sim \{13_{x_i}, 13_{x_j}, 5_{x_i}, 5_{x_j}\}$,
  
  $v_{x_i, x_j} \sim \{11_{x_i}, 15_{x_i}, 11_{x_j}, 15_{x_j}\}$,
  
  $w_{x_i, x_j} \sim \{13_{x_i}, 13_{x_j}, 5_{x_i}, 5_{x_j}\}$,
  
  $w_{x_i, x_j} \sim \{6_{x_i}, 7_{x_i}, 16_{x_i}, 6_{x_j}, 7_{x_j}, 16_{x_j}\}$, see Figure 8.

- Edges of variable copies:
  
  for an arbitrary variable $x$ and all $i \neq j$ and $k \neq l$, we have $w_{x_i, x_j} \sim w_{x_k, x_l}$ and $v_{x_i, x_j} \sim v_{x_k, x_l}$.

It is an easy observation to see that $G(\phi)$ can be be constructed from $\phi$ in polynomial time.
Lemma 3. There exists a truth assignment to variables in instance $\phi$ of POSITIVE AND MINIMUM INTERSECTING 1-in-3 SAT that satisfies the formula if and only if there exists a graph $H$ of girth five such that $G(\phi) = H^2$.

Proof. 

- **Satisfiability to squareness:**
  
  - $H$ construction: we construct the graph $H$ by using a satisfying assignment of $\phi$ as follows:

    * For all $i$ such that there exists a clause $c_i$ where $x_i \in c_i$, $H[\{1, 2, \ldots, 16\}_{x_i}] = G_2$ if $x$ is true and $H[\{1, 2, \ldots, 16\}_{x_i}] = G_1$ if $x$ is false.

    * For each pair of $x_i$ and $x_j$ where $i \neq j$ if $x$ is true then $v_{x_i,x_j}^H \sim \{13z_i, 13z_j\}$ and $w_{x_i,x_j}^H \sim \{5x_i, 5x_j\}$. Otherwise, that is if $x$ is false, $w_{x_i,x_j}^H \sim \{13z_i, 13z_j\}$ and $v_{x_i,x_j}^H \sim \{5x_i, 5x_j\}$.

    * For each clause $c_i = x_i \lor y_i \lor z_i$:

      - if $x_i$ is true then $y_i^1 \sim \{13x_i, 13y_i, 5z_i\}$, $y_i^2 \sim \{5x_i, 5y_i, 5z_i\}$, $y_i^3 \sim \{5x_i, 13y_i, 13z_i\}$, $y_i^4 \sim \{13x_i, 5y_i, 13z_i\}$.

      - if $y_i$ is true then $y_i^1 \sim \{5x_i, 5y_i, 5z_i\}$, $y_i^2 \sim \{13x_i, 13y_i, 5z_i\}$, $y_i^3 \sim \{13x_i, 5y_i, 13z_i\}$, $y_i^4 \sim \{5x_i, 13y_i, 13z_i\}$.

      - if $z_i$ is true then $y_i^1 \sim \{5x_i, 13y_i, 13z_i\}$, $y_i^2 \sim \{13x_i, 5y_i, 13z_i\}$, $y_i^3 \sim \{13x_i, 13y_i, 5z_i\}$, $y_i^4 \sim \{5x_i, 5y_i, 5z_i\}$.

      Recall that in all cases 10 vertices $y_i^1, y_i^2, y_i^3, y_i^4, 5x_i, 5y_i, 5z_i, 13x_i, 13y_i, 13z_i$ form a Petersen graph in $H$.

  - $H^2 = G(\phi)$: trivial.

- **Squareness to satisfiability:**

  Let $H$ be a square root of $G(\phi)$. By Theorem [1] graph $H[V_{x_i}]$ (for each copy of an arbitrary $x$) is isomorphic either to $G_1$ or $G_2$. We set $x$ to be true when $H[V_{x_i}] \cong G_2$ and false otherwise. By Lemma [2] all other copies of $x$ would also have the same truth value. By Lemma [1]
this assignment is a truth assignment to $\phi$ since exactly one variable in each clause is evaluated as true.

As an example let $\phi : c_1 \land c_2 \land c_3$ and $c_1 = x_1 \lor y_1 \lor z_1$, $c_2 = x_2 \lor u_2 \lor v_2$ and $c_3 = y_3 \lor a_3 \lor b_3$, where $x = b = TRUE$ and $y = z = u = a = v = FALSE$. The graph shown in Figure 9 is the square root of $G(\phi)$.

**Theorem 3.** SQUARE OF GRAPHS WITH GIRTH FIVE is NP-complete.

**Theorem 4** (The Complete Dichotomy Theorem). SQUARE OF GRAPHS WITH GIRTH $g$ is NP-complete if and only if $g \leq 5$.

### 4 Conclusions

We have disproved the conjecture in [5] by showing that SQUARE OF GRAPHS WITH GIRTH FIVE is NP-complete. Together with results provided by Motwani and Sudan [11] and Farzad et al. [5], we presented Theorem 4 as a complete dichotomy theorem for square root finding problem.

The problem of square root finding for graphs can be restated for higher roots.
$k^{th}$ Power of a Graph With Girth $r$

Instance: A graph $G$.

Question: Does there exists a graph $H$ with girth $r$ such that $G = H^k$.

The problem of root finding for higher root is an open problem in terms of the $r^{th}$-root of the power graph. Results provided by Adamaszek and Adamaszek [2] is the closest result to a complete girth-parametrized complexity dichotomy. They proved that the recognition problem of $k^{th}$ Power of a Graph With Girth $r$ is NP-complete when $r = k$ while there is a polynomial time algorithm to find all $k^{th}$-roots of girth $2k + 3$ for a given graph.

The problem of finding a complete girth-parametrized complexity dichotomy for $k^{th}$ Power of a Graph With Girth $r$ is open, and we conjectured the following:

**Conjecture 1.** $k^{th}$ Power of a Graph With Girth $r$ for $r = 2k + 1$ is NP-complete.

References

[1] A. Adamaszek, M. Adamaszek, Uniqueness of Graph Square Roots of Girth Six, *Electr. J. Comb.* 18 (1) (2011).

[2] A. Adamaszek, M. Adamaszek, Large-girth roots of graphs, *SIAM J. Discrete Math.* 24 (4) (2010) 1501–1514.

[3] A. Brandstädt, V. B. Le, and R. Sritharan, Structure and linear time recognition of 4-leaf powers, *ACM Transactions on Algorithms*, 15 (1) (2008).

[4] M. Chang, M. Ko, and Hsueh-I Lu, Linear time algorithms for tree root problems, *Lecture Notes in Computer Science*, 4059 (2006) 411–422.

[5] B. Farzad, L. C. Lau, V. B. Le, N. N. Tuy, Computing Graph Roots Without Short Cycles *Proc. 26th STACS* (2009) 397–408.

[6] P. E. Kearney, D. G. Corneil, Tree powers, *J. Algorithms* 29 (1998) 111–131.

[7] L. C. Lau, Bipartite roots of graphs, *ACM Transactions on Algorithms* 2 (2006) 178–208.
[8] L. C. Lau, D. G. Corneil, Recognizing powers of proper interval, split and chordal graphs, SIAM J. Discrete Math. 18 (2004) 83–102.

[9] Y. Lin, S. S. Skiena, Algorithms for square roots of graphs, SIAM J. Discrete Math. 8 (1995) 99–118.

[10] N. Linial, Locality in distributed graph algorithms, SIAM J. on Computing 21 (1992) 193–201.

[11] R. Motwani, M. Sudan, Computing roots of graphs is hard, Discrete Appl. Math. 54 (1994) 81–88.

[12] A. Mukhopadhyay, The square root of a graph, J. Combin. Theory 2 (1967) 290–295.

[13] I.C. Ross, F. Harary, The square of a tree, Bell System Tech. J. 39 (1960) 641–647.

[14] T. J. SCHAEFER, The complexity of satisfiability problems, In Proceedings of the 10th Annual ACM Symposium on Theory of Computing (New York), ACM, New York (1978) 126–226.
Appendix

A: Proof of Theorem 1

Unique pair of square roots for $\mathfrak{G}$: In this appendix we show that the graph $\mathfrak{G} = \mathfrak{G}_1^2 = \mathfrak{G}_2^2$ has only two non-isomorphic square roots which are $\mathfrak{G}_1$ and $\mathfrak{G}_2$. For the rest of this subsection we denote $\mathfrak{G}$ for the square of $\mathfrak{G}_1$ (or $\mathfrak{G}_2$).

Proof.

Lemma 4. Let $\mathfrak{G} = H^2$ for $g(H) = 5$, then $N_H(1) = \{7, 11, 12\}$.

Proof. We show this in the following four steps:

I $1 \overset{H}{\sim} 12$: Assume otherwise and let $1 \overset{H}{\sim} 12$, now since $1 \overset{\mathfrak{G}}{\sim} 12$ then $N_H(1) \subseteq (N_\mathfrak{G}(12) \cap N_\mathfrak{G}(1)) = \{11, 7, 2, 3\}$. In other hand we have $11 \overset{\mathfrak{G}}{\sim} 7, 2 \overset{\mathfrak{G}}{\sim} 3$ but non of $11$ or $7$ is not adjacent to any of $2$ and $3$, therefore either $N_H(1) \subseteq \{2, 3\}$ or $N_H(1) \subseteq \{7, 11\}$. If $N_H(1) \subseteq \{2, 3\}$ then we have a contradiction with $11 \in N_\mathfrak{G}(1)$, and if $N_H(1) \subseteq \{7, 11\}$ we again have a contradiction with $2 \in N_\mathfrak{G}(1)$, this implies $1 \overset{H}{\sim} 12$.

II $1 \overset{H}{\sim} 11$: Assume otherwise and let $1 \overset{H}{\sim} 11$, now since $1 \overset{\mathfrak{G}}{\sim} 11$ then $N_H(1) \subseteq (N_\mathfrak{G}(11) \cap N_\mathfrak{G}(1)) = \{7, 5, 12, 13, 14\}$. But according to part $I$, we know that $1 \overset{H}{\sim} 12$, therefore $N_H(1) \subseteq \{7, 12\}$, and this is a contradiction because non of $12$ and $7$ are not adjacent to $14$ in $\mathfrak{G}$, so it implies $1 \overset{H}{\sim} 11$.

III $1 \overset{H}{\sim} 7$: Assume otherwise and let $1 \overset{H}{\sim} 7$, now since $1 \overset{\mathfrak{G}}{\sim} 7$ then $N_H(1) \subseteq (N_\mathfrak{G}(7) \cap N_\mathfrak{G}(1)) = \{5, 11, 12, 13\}$. Again according to part $I$ and $II, 1 \overset{H}{\sim} 11, 12$, therefore $N_H(1) = \{11, 12\}$. Here we have two possibilities, either $7 \overset{H}{\sim} 11$ or $7 \overset{H}{\sim} 12$. If $7 \overset{H}{\sim} 11$, since $14 \overset{\mathfrak{G}}{\sim} 12$ then $14 \overset{H}{\sim} 11$ and this is a contradiction since $7 \overset{\mathfrak{G}}{\sim} 14$. If $7 \overset{H}{\sim} 12$, since $2 \overset{\mathfrak{G}}{\sim} 11$ then $2 \overset{H}{\sim} 12$ and this is a contradiction since $7 \overset{\mathfrak{G}}{\sim} 2$. So it implies $1 \overset{H}{\sim} 7$.

IV $N_H(1) = \{7, 11, 12\}$: Since $\{7, 11, 12\}$ is a maximal clique in $\mathfrak{G}$, and $\{7, 11, 12\} \subseteq N_H(1)$ therefore $N_H(1) = \{7, 11, 12\}$.

For more convenient we use the following notation. For $v \in V_\mathfrak{G}$ let $x \in N_\mathfrak{G}(v) - N_H(v)$, we define $L_v(x)$ as follows:
\[L_v(x) = \{u \in N_H(v) \mid ux \in E_G\} = N_\emptyset(x) \cap N_H(v)\]

Since girth of \(H\) is 5 for all vertices \(v \) and \(x\) where \(x \in N_\emptyset(v) - N_H(v)\), there is a unique \(u \in L_v(x)\) such that \(u \sim x_H\).

According to Lemma 4 we have:

- \(N_H(12) = \{2, 3, 1\}\), since \(L_{1}(2) = L_{1}(3) = 12\).

Also \(14 \sim 11\) (because \(L_{1}(14) = 11\)), but \(L_{1}(5) = L_{1}(13) = \{7, 11\}\), hence we have two possibilities:

I. Case 1: \(5 \sim 11\) and \(13 \sim 7\):

- \(N_H(7) = \{1, 11\}\): trivial.
- \(N_H(11) = \{1, 5, 14\}\): trivial.
- \(N_H(13) = N_G(7) - N_H(7) - \{11, 12\} = \{5, 6, 16\}\): since 13 and 1 are the only neighbours of 7.

We now consider the set \(N_\emptyset(13) - N_H(13) = \{1, 3, 8, 10, 11, 14, 15\}\), we have \(L_{13}(3) = \{16\}, L_{13}(8) = \{6, 16\}, L_{13}(10) = \{16\}, L_{13}(14) = \{5, 6, 16\}, L_{13}(15) = \{5\}\), therefore:

- \(N_H(5) = \{13, 15, 11\}\), since \(14 \not\sim 11\) (otherwise we have a cycle of length four).
- \(N_H(16) = \{13, 3, 8, 10\}\), since \(14 \not\sim 3\).
- \(N_H(6) = \{13, 14\}\), trivial.

We now consider the set \(N_G(12) - N_H(12) = \{8, 9, 15, 16\}\), we have \(L_{12}(8) = L_{12}(9) = L_{12}(15) = L_{12}(16) = \{2, 3\}\), however we know that \(16 \sim 3, 8\):

- \(N_H(2) = \{12, 15, 8\}\), since \(16 \sim 8\) (otherwise we have a cycle of length four), and \(9 \not\sim 8\).
- \(N_H(3) = \{12, 9, 16\}\), trivial.

- \(N_H(15) = \{2, 4, 5, 9\}\), considering \(N_\emptyset(5) - N_H(5)\) and refining the known neighbours.
- \(N_H(14) = \{4, 6, 8, 11\}\), similar argument to vertex 15, considering the vertex 11.
- \(N_H(4) = \{10, 14, 15\}\), similar argument to vertex 15, considering the vertex 15.
- $N_H(8) = \{2, 14, 16\}$, trivial.
- $N_H(9) = \{3, 15\}$, trivial.
- $N_H(10) = \{4, 16\}$, trivial.

It can be seen that the above graph is $G_1$.

**II Case 2: $5 \sim H 7$ and $13 \sim H 11$:**

- $N_H(7) = \{1, 5\}$: trivial.
- $N_H(11) = \{1, 13, 14\}$: trivial.
- $N_H(5) = N_G(7) - N_H(7) - \{1, 7\} = \{6, 7, 13, 16\}$: since 5 and 1 are the only neighbours of 7.

We now consider the set $N_G(5) - N_H(5) = \{1, 2, 4, 9, 11, 14, 15\}$, we have $L_5(2) = \{16\}$, $L_5(4) = \{6, 16\}$, $L_5(9) = \{16\}$, $L_5(14) = \{6, 13, 16\}$, $L_5(15) = \{13\}$, therefore:

- $N_H(13) = \{5, 11, 15\}$, since $14 \not\sim H 11$ (otherwise we have a cycle of length three).

- $N_H(16) = \{2, 4, 5, 9\}$, since $14 \not\sim H 9$, and also $4 \not\sim H 13$ but $4 \not\sim H 11$, therefore $4 \not\sim H 6$.

- $N_H(6) = \{5, 14\}$, trivial.

We now consider the set $N_G(12) - N_H(12) = \{8, 9, 15, 16\}$, we have $L_{12}(8) = L_{12}(9) = L_{12}(15) = L_{12}(16) = \{2, 3\}$, however we know that $16 \sim H 3, 8$:

- $N_H(2) = \{12, 15, 8\}$, since $16 \sim H 8$ (otherwise we have a cycle of length four), and $9 \sim H 8$.

- $N_H(3) = \{12, 9, 16\}$, trivial.

- $N_H(15) = \{3, 8, 10, 13\}$, considering $N_G(5) - N_H(5)$ and refining the known neighbours.

- $N_H(14) = \{4, 6, 8, 11\}$, similar argument to vertex 15, considering the vertex 11.

- $N_H(4) = \{10, 14, 16\}$, similar argument to vertex 15, considering the vertex 15.

- $N_H(8) = \{2, 14, 15\}$, trivial.

- $N_H(9) = \{3, 16\}$, trivial.
- \( N_H(10) = \{4, 15\} \), trivial.

It can be seen that the above graph is \( G_2 \).

So \( H \) is either isomorphic to \( G_1 \) or \( G_2 \). \( \square \)