The Kondo Temperature of SU(4) Symmetric Quantum Dots

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A path integral approach is used to derive a closed analytical expression for the Kondo temperature of the SU(4) symmetrical Anderson model. In contrast to the SU(2) case, the prefactor of the Kondo temperature is found to display a peculiar orbital energy (gate voltage) dependence, reflecting the presence of various SU(4) mixed valence fixed points. Our analytical expressions are tested against and confirmed by numerical renormalization group computations.

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Introduction - Much of our perception of strongly interacting fermion systems such as heavy fermions [1], molecules and quantum dots attached to electrodes [2], correlated states of interacting cold atoms [3], or phenomena such as the Mott transition [4] relies on the detailed study of magnetic impurities and the Kondo effect. The latter consists of the dynamical screening of a local spin entity, interacting antiferromagnetically with surrounding itinerant electrons [5], and provides the simplest example of asymptotic freedom. The fundamental energy scale, at which the Kondo screening develops is called the Kondo temperature, $T_K$. Our understanding of this curious effect has been strengthened by several recent experiments on quantum dot (QD) systems and nanotubes, which not only allow an accurate control of the model parameters, but also enable one to study out-of-equilibrium phenomena [2, 5, 6] and to realize exotic correlated states such as the two-channel Kondo state [7] or the SU(4) Kondo effect [6, 8, 9] in a controlled manner.

One of the most fascinating features of the Kondo effect is that of universality: apart from a few dimensionless numbers characterizing the asymmetry of the leads and electron-hole symmetry, at low temperatures or voltages, every physical quantity depends on the microscopic model parameters solely through $T_K$. While determining $T_K$ is therefore clearly of crucial importance, estimating it precisely is an utmost delicate problem. Its first estimate is due to Kondo [10], who constructed and studied a model of a local spin $S$ (the spin on the QD or the molecule) coupled to the (conduction) electrons’ spin density through an exchange coupling, $H_{\text{int}} = J_0 S \cdot s$. Kondo’s exchange model contains, however, logarithmic singularities, which must be regularized by a bandwidth cut-off, $D_0$. The Kondo temperature is found to depend explicitly on this unknown parameter, and is expressed as $T_{K,SU(2)} \approx D_0 \exp(-1/v_0 J_0)$ for SU(2) spins [5], with $v_0$ denoting the electrons’ density of states. Unfortunately, $D_0$ and $J_0$ being both somewhat arbitrary, the predictive power of this expression is limited.

A better estimate can be obtained through the analysis of Anderson’s more elaborate impurity model [5], as first done by Haldane [11, 12]. In Anderson’s model, some local fermionic degrees of freedom $d$, of energy $\varepsilon_d$ interact with each other by a local interaction of strength $U$ and hybridize with the leads (conduction electrons) as described by the Hamiltonian,

$$H_{\text{Au}} = \varepsilon_d \sum_{\tau} d_{\tau}^\dagger d_{\tau} + U \sum_{\tau<\tau'} n_{\tau} n_{\tau'} + \frac{i}{4} \sum_{k\tau} \left( c_{k\tau}^\dagger c_{k\tau} + \text{h.c.} \right) + \sum_{k,\tau} \varepsilon_k c_{k\tau}^\dagger c_{k\tau}.$$  

Here the operators $c_{k\tau}^\dagger$ create conduction electrons of energy $\varepsilon_k$ and “spin” $\tau$, and the strength of hybridization is characterized by the decay rate $\Delta = \pi \nu_0 e^2$. In contrast to the Kondo model, the Anderson model is not ultraviolet divergent, and has a well-defined Kondo scale in terms of its three parameters, $\varepsilon_d$, $U$, and $\Delta$, which Haldane determined accurately through a renormalization group analysis for the spin $1/2$ case, $\tau = \uparrow, \downarrow$. A closer analysis of Haldane’s work reveals, however, that, to provide a precise estimate of the Kondo temperature, one also needs to account for charge (valence) fluctuations. In this work, we study the SU(4) symmetrical Anderson model (with $\tau = 1, ..., 4$), and determine its Kondo temperature accurately. The study of the SU(4) case is motivated by its particular experimental relevance: signs of an emergent SU(4) symmetrical Kondo state were observed in carbon nanotube [13] and vertical quantum dots [9], single atom transistors [14], double QD devices [6, 15], and other exotic SU(4) Kondo regimes are also expected in fourfold degenerate QD systems where both spin and orbital degrees of freedom are conserved in tunneling [15]. Here we develop a path integral formalism introduced in Ref. [16] (see also Ref. [17] for a similar approach), to show that, in contrast to the SU(2)
FIG. 1. Sketch of the ‘phase diagram’ of the SU(4) Kondo model. In the mixed valence region MV \(_1\) (MV\(_2\)) the fourfold degenerate \(q = 1\) state competes with the \(q = 0\) empty state (the sixfold degenerate \(q = 2\) state). In the local moment region LM\(_1\) the \(q = 1\) SU(4) spin produces Kondo effect and is screened below the Kondo (Fermi liquid) scale, \(T_K\) (\(T_{\text{FL}}\), while in region LM\(_2\) a sixfold degenerate SU(4) spin is screened.

The corresponding mixed valence regions and the ‘phase diagram’ of the SU(4) Kondo model are sketched in Fig. 1 [19]. In the mixed valence regions MV\(_1\) and MV\(_2\) the fourfold degenerate charge \(q = 1\) state of the QD (d-level) competes with the \(q = 0\) empty and the sixfold degenerate \(q = 2\) states, respectively. The SU(4) spins are formed at temperatures below the energy of the QD’s charging excitations \(\Delta E_q\) in both models. Using then the well-known two-loop result for the Kondo temperature, \(T_{K,\text{SU}(N)} = D \sqrt{\frac{N\nu_0}{2}e^{-\frac{\pi^2}{\nu_0}}},\) we obtain the Kondo temperature of the SU(4) Anderson model as

\[ T_{K,\text{SU}(4)} = D_{\text{eff}} \sqrt{\frac{J^2}{4 J_0^2}} e^{-\frac{\pi^2}{\nu_0}}, \]  

with \(J_0 = -2t^2 U/[E_q(E_q + U)]\) the usual leading order expression for the Kondo coupling, expressed in terms of the shifted energies \(E_q = \epsilon_d + (q-1)U\) [20]. Eq. (3) differs from the SU(2) formula derived by Haldane in that the prefactor in Eq. (3) incorporates the effects of valence fluctuations, which have a determining impact on the functions \(f_q\).

To determine \(T_K\), we shall focus on the Kondo regime of the QD, with \(q = 1, 2\) and 3 electrons trapped on the fourfold degenerate level, and establish a mapping between the SU(4) Anderson model and the SU(4) Kondo model [20], described by the exchange interaction, \(H_{\text{exch}}^{\text{SU}(4)} = \frac{J}{2} \sum_{kk'} \tau^z \langle \sigma^z_k \sigma^z_{k'} \rangle \). We determine first \(J\) and the cut-off \(D\) by computing the logarithmic corrections to the amplitude of the exchange processes at energy \(\omega\)

\[ J(\omega) = J - \frac{N\nu_0}{2} J^2 \ln \frac{\omega}{D} + \ldots \]  

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To verify the analytical predictions, we also performed NRG computations using the Budapest Flexible DM-NRG code [22], and compared the numerically obtained Kondo temperature to the analytical result, Eq. (4). As shown in Fig. 2, an excellent agreement is found for small values of \(\nu_0 J_0\) [23]. Clearly, our results contradict the naive expectation that \(D\) should roughly correspond to the minimum charging energy of the quantum dot, in which case the edges of the functions \(f_q\) would exhibit
FIG. 3. Temperature dependence of impurity contribution to the entropy.

a linear behavior. Rather, as discussed below, for small $\Delta$’s the value of $D_{\text{eff}}$ is strongly renormalized by valence fluctuations. While for $f_2$, the discrepancy with linearity close to $x = -2$ and $-1$ is not very strong, it is much more pronounced for $f_1$ where, in addition, a strong asymmetry is observed. These differences are the results of the competition of different charge configurations in the vicinity of the mixed valences regions $MV_1$ and $MV_2$ (see Fig. 1), as we discuss next.

Valence fluctuations - The importance of charge fluctuations is evident from the numerical study of the SU(4) Anderson model. Fig. 3 displays the temperature dependence of the impurity entropy for two values of $\epsilon_d$ close to the $q = 0 \rightarrow 1$ and $q = 1 \rightarrow 2$ charge transitions, as determined by our DM-NRG computations. The impurity entropy $S_{\text{imp}}(T)$ displays steps at each temperature corresponding to a ‘phase boundary’ in Fig. 1. For $\epsilon_d \approx -0.001U$, e.g., the impurity entropy takes a value of $S_{\text{imp}} = k_B \ln 5$ at high temperatures, corresponding to the approximate (1+4)-fold degeneracy of the $q = 0$ and $q = 1$ states (region $MV_1$), and by decreasing the temperature further below $T_{FL}$, one enters directly the Fermi liquid regime with $S_{\text{imp}} = 0$. For $\epsilon_d \approx -0.999U$, on the other hand, the entropy takes a value of $k_B \ln 10$ at high temperatures, reflecting the (6+4)-fold degeneracy of the almost degenerate $q = 2$ and $q = 1$ charging states (region $MV_2$), and a reduction series of $\ln 10 \rightarrow \ln 4 \rightarrow 0$ is observed, corresponding to the $MV_2 \rightarrow LM_1$ and $LM_1 \rightarrow \text{‘Fermi liquid’ transitions}$.

From the previous analysis it is clear that, in the vicinity of the charging transitions, the mixed valence fixed points govern the physics over a large energy window, and have a substantial impact on quantum fluctuations - determining the precise Kondo scale. In fact, the basic structure of the functions $f_q$ can be understood by means of a relatively simple renormalization group procedure constructed by Haldane [11], accounting also for charge fluctuations. Before proceeding with the full derivation of the functions $f_q$, let us present this simple and instructive derivation. We first consider the case $\Delta \ll -\epsilon_d \ll U$ where the $q = 0$ and $q = 1$ states compete in region $MV_1$. In a standard renormalization group (RG) picture, $\epsilon_d$ is first renormalized by charge fluctuations in the region $MV_1$, and replaced by

$$
\epsilon_d' = \epsilon_d + \frac{3\Delta}{\alpha} \ln \frac{U}{|\epsilon_d|} \tag{5}
$$

in the region $LM_1$, with $\alpha$ a constant of order one. The factor 3 in Eq. (5) accounts for the balance between the empty state and the fourfold degenerate singly occupied state. In region $LM_1$ (below a cutoff energy $D \approx \alpha \epsilon_d$), the QD can only be singly occupied, and a standard SU(4) Kondo effect takes place. The exchange constant $J$ can be estimated here by the Schrieffer-Wolff approach as $J = -2\Delta/(\pi \epsilon_d)$. Combining $D$ and $J$ within the two-loop expression of the Kondo temperature, we recover Eqs. (3) with $D_{\text{eff}} = \sqrt{\alpha |\epsilon_d| U^{3/4}}$, yielding $f_1(x)_{x \rightarrow 0} \propto \sqrt{x}$ in agreement with the analytical expression of $f_1(x)$ (with $\alpha^{-1} = \gamma_c$ identified as Euler’s constant). The reason for the emerging non-integer power is the logarithmic correction to $\epsilon_d$, Eq. (5), which - after exponentiation in the Kondo temperature’s expression - rescales the effective cutoff $D_{\text{eff}}$ to be of a non-trivial power law form.

The same reasoning can be extended to the proximity of the region $MV_2$. Note that there are six different states with double occupancy. Each of them is coupled to two singly occupied states by the tunneling term, while each singly occupied state is coupled to three states with double occupancy. As a result we find $D_{\text{eff}}/U = \alpha^{1/4}(|\epsilon_d|/U + 1)^{1/4}$ for $\epsilon_d > -U$ and $D_{\text{eff}}/U = (\alpha |\epsilon_d|/U + 1)^{1/4}$ for $\epsilon_d < -U$ in agreement with the general expressions obtained for $f_1(x)$ and $f_2(x)$ (with again $\alpha = 1/\gamma_c$). The three limiting expressions derived so far by simple scaling arguments are sufficient to qualitatively understand the shape of the effective cutoff $D_{\text{eff}}$ shown in Fig. 2, in particular the strong asymmetry in $f_1$.

Path integral approach - To determine the functions $f_q$ precisely, we must go beyond the previous simple analysis. The Schrieffer-Wolff (SW) transformation, applied to Eq. (1), gives the Kondo coupling $J_0$ of Eq. (3), but only a rough estimate of the cutoff $D_{\text{eff}}$. Moreover, any subleading correction to $J_0$ renormalizes $D_{\text{eff}}$. A precise determination of $D_{\text{eff}}$ thus requires a second order calculation in $\Delta$, or fourth order in the tunneling $t$, beyond SW. We follow the path integral approach devised in Ref. [16], where a systematic expansion in $\Delta$ is performed, and extend it to the SU(4) symmetry. Bosonic (fermionic) slave fields, associated with each even (odd) charged state of the QD, are introduced. They give a quadratic Anderson Hamiltonian at $t = \Delta = 0$ around
which a diagrammatic perturbation theory in $\Delta$ is used. Below, we focus on the case $q = 1$, but a similar discussion applies equally for the cases $q = 2$ and $q = 3$.

In the path integral formalism, the high energy slave-fields, associated with QD states with $q \neq 1$, are integrated out, and the Anderson model is mapped onto a SU(4) Kondo-like action

$$S_K = \sum_{\tau' = 1}^4 \text{Tr} \left[ \frac{J}{2} \left( f^\dagger_{\tau} f_{\tau'} - \frac{1}{4} \right) \epsilon_{\tau}^c \epsilon_{\tau'}^c \right].$$  \hspace{1cm} (6)

The trace in Eq. (6) refers to summations over all wavevectors and Matsubara frequencies. In contrast to the genuine Kondo model, $J$ in Eq. (6) is a frequency-dependent coupling [16]. This dependence is a remainder of the integrated charge fluctuations in the Anderson model in addition to the Kondo-like spin fluctuations. To leading order in $\Delta$ and neglecting its frequency dependence, $J$ reduces to the SW coupling $J_0$. For energies $\omega$ below the charging energy $U$, dot excitations out of the $q = 1$ subspace are frozen and the Anderson model maps (for $-\epsilon_d \gg \Delta$ and $\epsilon_d + U \gg \Delta$) onto the Kondo model. The Kondo temperature $T_K$ becomes the only relevant energy scale, which controls the Kondo crossover. As a result, the expansion of the spin-exchange process $\mathcal{J}(\omega)$ in the Anderson model recovers exactly the same expansion as Eq. (2), derived within the Kondo model (see Eq. (7a)). We can thus identify $J$ and $\mathcal{D}$ from this weak coupling expansion, and determine $T_K$.

Equipped with the action Eq. (6), we calculate the spin-exchange process $\mathcal{J}(\omega)$, given by the series of diagrams represented in Fig. 4. To be consistent with the second order expansion in $\Delta$, self-energy vertex corrections must also be considered. They lead to a renormalization of the orbital energy, analogous to Eq. (5), $\epsilon_d \to \epsilon_d = \epsilon_d + \nu_0^2 \ln(-\epsilon_d/\Xi + 3\ln(\epsilon_d + U)/\Xi)$, with $\Xi$ denoting an UV regularization cutoff (to be distinguished from $\mathcal{D}$), eventually sent to infinity. The self-energy corrections also give rise to a quasi-particle weight $Z = 1 + \nu_0^2 [1/\epsilon_d - 3/(\epsilon_d + U)]$ for the slave-fermions $f_\tau$, which renormalizes the spin-exchange process. The complete calculation leads to the renormalized exchange amplitude,

$$\mathcal{J}(\omega) = J_0 + 2\nu_0 \nu_0^2 g_1 \left( \frac{\epsilon_d}{U} \right) - 2\nu_0 J^2_0 \ln \frac{\omega}{\Xi},$$  \hspace{1cm} (7a)

$$g_1(x) = \frac{3x - 2}{4} x + 2 - \frac{x^3}{2} \left( x^2 + 3x + 3 \right) \ln \frac{2x + 3}{x + 1},$$  \hspace{1cm} (7b)

with $\mathcal{D} = \sqrt{-\epsilon_d(\epsilon_d + U)}.$ Comparison with Eq. (2) for $N = 4$ yields then the exchange coupling $J = J_0 + 2\nu_0 \nu_0^2 g_1 (\epsilon_d/U)$. Finally, computing the Kondo temperature $T_K$, we obtain Eq. (3) with

$$f_1(x) \equiv \sqrt{-x(x + 1)} \exp \left[ g_1(x) \right].$$  \hspace{1cm} (8)

The function $f_2(x)$ can be derived by a similar procedure. Details of its rather lengthy derivation as well as the analytical form of $f_2$ are given in the Supplemental Material [24].

Conclusions - In this work, we determined the Kondo temperature of the SU(4) Kondo model. In contrast to the SU(2) case, the prefactor of $T_K$ was found to depend strongly on $\epsilon_d$ through universal functions of $\epsilon_d/U$, which we determined analytically and numerically. As demonstrated through an analysis similar to that of Haldane, charge fluctuations play a decisive role in determining the afore-mentioned prefactors, and lead to a power law suppression of the Kondo scale close to the SU(4) mixed valence fixed points. In addition to being of fundamental interest, the anomalous dependence of the Kondo temperature found here should be of importance when characterizing SU(4) Kondo systems.

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[21] This result is exact in the scaling limit ($U, |\varepsilon_d| \to \infty$, $T_K = \text{finite}$) up to an overall coefficient which depends on the definition of the Kondo temperature. Here we define $T_K$, in Eq. (3), such as to resum all leading and next to leading logarithmic singularities [5] in the exchange coupling expansion Eq. (2). Notice that the same $T_K$ appears in the magnetization at large magnetic field in the SU(2) case.

[22] We used the open-access Budapest Flexible DM-NRG code, http://www.phy.bme.hu/~dmnrg/; O. Legeza, C. P. Moca, A. I. Tóth, I. Weymann, G. Zarárd, arXiv:0809.3143 (2008) (unpublished).

[23] Numerically, we identified the Kondo temperature as the half-width of the d-level’s spectral function. We changed $\Delta$ and $\varepsilon_d$ simultaneously while keeping the coupling $\nu_0 J_0$ constant, extracted the $D_{\text{eff}}/U$ from the numerics, and compared that with the analytical expressions we obtained the functions $f_q(x)$.

[24] See Supplemental Material for the detailed discussion about analytical and numerical calculations.
S-I. PATH INTEGRAL AND SLAVE BOSON REPRESENTATION OF THE SU(4) ANDERSON MODEL

Here we provide detailed discussion of the techniques developed and the calculations carried out to derive the analytical results in the main text. To describe the Hilbert space of the dot we adopt a slave boson approach. In Fig. S-1, we provide a graphical representation of the quantum dot Hilbert space together with the associated energies.

The choice of the signs is fixed to preserve the fermionic anticommutation relations \( \{d_0^\dagger, d_{\tau'}\} = \delta_{\tau,\tau'} \) and \( \{d_0^\dagger, d_{\tau}\} = 0 \) between the original operators. In the new basis, the action associated to the Anderson Hamiltonian Eq. (1) becomes quadratic for \( \delta = 0 \)

\[
S_0 = -\int_0^\beta d\tau \left\{ \sum_{k\nu} c^\dagger_{k\nu}(\tau) G^{-1}_k c_{k\nu}(\tau) + b^\dagger_0(\tau) F_0^{-1} b_0(\tau) + \sum_{\nu} f^\dagger_{1,\nu}(\tau) F^{-1}_{1,1} f_{1,\nu}(\tau) + \sum_{\nu < \nu'} b^\dagger_{\nu,\nu'}(\tau) F_2^{-1} b_{\nu,\nu'}(\tau) + \sum_{\nu} f^\dagger_{3,\nu}(\tau) F^{-1}_{3,3} f_{3,\nu}(\tau) + b^\dagger_{1}(\tau) F_4^{-1} b_{1}(\tau) \right\},
\]

with \( G_k = (-\partial_\tau - \varepsilon_k)^{-1} \) the lead electron free propagator and \( F_q = (-\partial_\tau - E_q)^{-1} \) the slave-field free propagator, with \( E_q = q\varepsilon_d + q(q - 1)U/2 + \lambda \). Note that the para-energy \( \lambda \) must be introduced in free slave-state propagators to perform Abrikosov’s projection onto the physical sector \( [1] \), defined by the constraint

\[
N_{\text{slave}} = b^\dagger_0 b_0 + \sum_\tau f^\dagger_{1,\tau} f_{1,\tau} + \sum_\tau < \tau' b^\dagger_{\tau,\tau'} b_{\tau,\tau'} + \sum_\tau f^\dagger_{3,\tau} f_{3,\tau} + b^\dagger_{1} b_{1} = 1.
\]

As \( [N_{\text{slave}}, H] = 0 \), the Anderson Hamiltonian is diagonal in sectors with integer \( N_{\text{slave}} \). \( \text{Tr}_n \) is the trace carried out in the subspace defined by \( N_{\text{slave}} = n \), \( n = 1 \) being the physical one. Denoting \( \langle O \rangle_\lambda = \text{Tr}[O e^{-\beta H - \beta N_{\text{slave}}}] / \text{Tr}[e^{-\beta H - \beta N_{\text{slave}}}] \) the expectation value of the operator \( O \) in the whole Hilbert space spanned by the action Eq. (S-2), the physical average \( \langle O \rangle = \text{Tr}[O e^{-\beta H}] / \text{Tr}[e^{-\beta H}] \) is recovered with the following prescription \[2\]

\[
Z = \lim_{\lambda \to \infty} \frac{\partial}{\partial e^{-\beta \lambda}} Z_\lambda, \quad \langle O \rangle = \lim_{\lambda \to \infty} \left[ \langle O \rangle_\lambda + Z_\lambda \frac{\partial}{Z} e^{-\beta \lambda} \langle O \rangle_\lambda \right].
\]

Supplementary material for the paper “The Kondo Temperature of SU(4) Symmetric Quantum Dots”

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FIG. S-1: Representation of the SU(4) quantum dot Hilbert space with slave-boson and para-fermions. In the case of a SU(4) symmetric carbon nanotube, the label $\tau = 1, \ldots, 4$ merges the fourfold valley and spin degeneracy. This leads to 12 possible states, which are here represented with their energy and total charge occupation. To every configuration a slave-field is associated: bosons for even occupancy and fermions for odd occupancy of the quantum dot.

$Z = \text{Tr}_1 \left[ e^{-\beta H} \right]$ is the physical partition function of the Anderson Hamiltonian. The tunneling term reads

$$S_T = t \int_0^\beta dt \sum_{k\nu} \left( c_{k\nu}^\dagger(\tau)d_{\nu}(\tau) + d_{\nu}^\dagger(\tau)c_{k\nu}(\tau) \right), \quad (S-5)$$

in which the transformation Eq. (S-1) has to be operated.

The path integral representation permits to map the SU(4) Anderson model onto the SU(4) Kondo action Eq. (6) and carry out the diagrammatic calculations of the exchange processes $J(\omega)$ in the three charge sectors $q = 1, 2, 3$. The procedure can be summarized in the following way. The slave-fields associated to the high energy charge sectors $Z$ is associated: bosons for even occupancy and fermions for odd occupancy of the quantum dot.

$\Sigma = \text{Tr}_1$ for the fields with energies $\epsilon_k$ larger than the energy scale $\omega$. Their propagators are replaced by the full propagators $G^\omega_k$ and $F^\omega_k$ with Dyson equations, e.g. $F^\omega_k(\omega_n) = [F^1_k(\omega_n)^{-1} - \Sigma^\omega_k(\omega_n)]^{-1}$, $\Sigma^\omega$ being the self-energy. The superscript $\omega$ indicates that all internal lines in diagrams contributing to $\Sigma^\omega$ are integrated for momenta $k$ such that $\omega \leq \epsilon_k \leq \Xi$. $\Xi$ is an intermediate cutoff introduced to regularize UV divergences, and safely sent to infinity at the end of calculation. The interaction term in Eq. (6) is replaced in the

### Energy

| $4\epsilon_d + 6U$ | Charge |
|-------------------|--------|
| $3\epsilon_d + 3U$ | 4      |
| $2\epsilon_d + U$ | 3      |
| $\epsilon_d$     | 2      |
| 0                 | 1      |

$T$ and $t$ are the vectors collecting the 15 matrices forming the fundamental representation of the SU(4) group [3], providing a compact notation of the SU(4) exchange interaction

$$H_{\text{exch}}^{SU(4)} = \frac{J}{2} \sum_{k,k',\tau\tau'} \left( f_{\tau\tau'}^\dagger f_{\tau} - \frac{q}{4} \right) c_{k\nu}^\dagger c_{k'\nu'}^\dagger = J \sum_{\nu\nu'\tau\tau'} T_{\tau\tau'} \cdot t_{\nu\nu'} c_{\nu'}^\dagger f_{\tau'}^\dagger f_{\tau}, \quad (S-7)$$
FIG. S-2: Virtual paths in the SU(4) Anderson model to fourth order in the tunnel coupling $t$. We consider different Coulomb blockade regimes with $q$ charges in the quantum dot. The circles signal the departure charge state to which one has to come back in perturbation theory. We emphasize graphically the symmetry between the sectors $q = 1$ and $3$. Notice that, for $q = 1$ and $3$, the charge sectors $4$ and $0$ can be respectively neglected to fourth order in $t$.

course of renormalization by the four-point vertex function $V^\omega$ [7]

$$V^\omega_{\nu\nu',\tau\tau'}(k, i\omega, i\omega_m; k', i\omega_n, i\omega_o) = -\beta \frac{\langle c_{k\nu} (i\omega) g_{\tau} (i\omega_m) g_{\nu'}^\dagger (i\omega_o) c_{k'\nu'}^\dagger (i\omega_n) \rangle}{G_k (i\omega) G_{k'} (i\omega_m) F_{\nu} (i\omega_m) F_{\nu'} (i\omega_o)}.$$  \hspace{1cm} (S-8)

Renormalization also generates higher order vertices, which are irrelevant and discarded from the effective action. Before writing the resulting effective action, let us proceed with a simple rescaling of fields. The slave-fermion propagator $F^\omega_k$ is expanded close to its renormalized pole $\delta_d = \varepsilon_d + \lambda + \Sigma^\nu (\varepsilon_d)$ so we can write $F^\omega_k (i\omega_n) = Z / (i\omega_n - \delta_d)$. $Z = (1 - \partial_{\omega_n} \Sigma(i\omega_n)|_{\omega \rightarrow \delta_d})$ is the quasi-particle weight of the slave-fermions, corresponding to the renormalization of their wavefunction. Rescaling the fields $f^{(1)}_\tau \rightarrow f^{(1)}_\tau = f^{(1)}_\tau / \sqrt{Z}$ and expanding the vertex function $V^\omega$ close to the propagator poles $\varepsilon_F = 0$ (the Fermi energy) and $\delta_d$, the effective action of the system is derived

$$S' = -\sum_{k,\omega,\tau} \langle \sum_{\omega_m}\tilde{f}_1 \tilde{F}_1^{-1} \tilde{f}_1 + \sum_{k'\omega\tau'\nu\nu'} V^R_{\tau\tau'\nu\nu'} c_{k\nu}^\dagger c_{k'\nu'} c_{k'\nu'}^\dagger f_\tau f_{\nu'} + \ldots \rangle \triangleq T \mathbf{F} + \mathcal{W} \mathbf{1}. \hspace{1cm} (S-9)$$

This quantity involves the spin-exchange process $J(\omega)$, which recovers the universal expression Eq. (2) in the main text, valid for a pure SU(4) Kondo model. The couple $(J, D)$ is then set and the Kondo temperature determined through $T_{K,SU(N)} = D \sqrt{N_U J/2} \ e^{-\frac{\pi}{\sqrt{\pi}}}$. We neglect any frequency dependent correction to the action Eq. (S-9) as they are irrelevant in the renormalization flow. The renormalized vertex reads [9–12]

$$V^R = J(\omega) \mathbf{T} \cdot \mathbf{T} + \mathcal{W} \mathbf{1}. \hspace{1cm} (S-10)$$

S-II. CALCULATION OF THE SU(4) SPIN EXCHANGE PROCESSES FOR $q = 1$

1. Integration of the high energy slave-fields

The quantum dot hosts a single charge if $-U < \varepsilon_d < 0$. In Fig. S-2 we show that virtual processes to fourth order in the tunneling $t$ do not involve the charge sector occupied by four charges. The associated slave-field $b_4$ can then be neglected. The action of the system reduces to $S = S_0 + S_{12} + S_{23}$, where $S_0$ is the quadratic action Eq. (S-2) without the $b_4$ field and $S_{nm}$ hybridizes the slave-fields associated to $n$ and $m$ charges on the quantum dot. For instance, $S_{01}$ reads

$$S_{01} = t \text{ Tr } \sum_{k\tau} \langle b_0^\dagger c_{k\tau}^\dagger f_{1,\tau} + \text{ h.c.} \rangle . \hspace{1cm} (S-11)$$
The trace refers equally to the summation over the imaginary-time or Matsubara frequency dependence of the fields, e.g.

\[
\text{Tr} \left[ c_k^\dagger G^{-1}_k c_k \right] = \int_0^\beta d\tau c_k^\dagger(\tau) (-\partial_\tau - \varepsilon_k) c_k(\tau) = \sum_n c_k^\dagger(i\omega_n) (i\omega_n - \varepsilon_k) c_k(i\omega_n). \tag{S-12}
\]

The integration over the \( b_0 \) slave-field leads to an effective interaction of the form Eq. (S-6)

\[
S'_{01} = -\frac{t^2}{\beta} \sum_{n,m,l} F_0(i\nu_n) c_{k'n}(i\omega_m) c_{k''l}(i\omega_l) f_{1,\tau}(i\nu_n + i\omega_l) f_{1,\tau}(i\nu_n + i\omega_m) \left[ 2T_{\tau\tau'} \cdot t_{\nu\nu'} - \frac{1}{4} \delta_{\tau\tau'} \delta_{\nu\nu'} \right], \tag{S-13}
\]

in which we make use of Eq. (S-7) with \( q = 1 \). The \( f_{3,\tau} \) slave-fields are then integrated. The action \( S_{23} \) has the algebraic form

\[
S_{23} = t \text{Tr} \left[ F^\dagger \cdot C \cdot b \right] + \text{h.c.} \tag{S-14}
\]

\[
F = \begin{pmatrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} \\ f_{1,2} & f_{1,3} & f_{1,4} & f_{1,3} \\ f_{1,3} & f_{1,4} & f_{1,3} & f_{1,4} \\ f_{1,4} & f_{1,3} & f_{1,4} & f_{1,3} \end{pmatrix}, \quad b = \begin{pmatrix} b_{12} \\ b_{13} \\ b_{14} \\ b_{23} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & c_4 & c_4 \\ 0 & -c_4 & c_3 & 0 & 0 \\ c_4 & 0 & -c_2 & 0 & c_1 \\ -c_3 & c_2 & 0 & -c_1 & 0 \end{pmatrix}. \tag{S-15}
\]

To keep simple notations, we omit the momentum index \( k \) for lead electrons. The integration of the para-fermions \( f_{3,\tau} \) leads to the effective interaction

\[
S'_{23} = t^2 \text{Tr} \left[ b^\dagger \cdot \mathbf{A}_2 \cdot b \right], \tag{S-16}
\]

with \( \mathbf{A}_2 = C^\dagger \cdot F_3 \mathbb{I}_4 \cdot C \) and \( \mathbb{I}_4 \) the 4 \times 4 identity matrix. This term renormalizes the action involving the \( b_{\tau\tau'} \) slave-bosons

\[
S_{12} + S'_{23} = -\text{Tr} \left[ b^\dagger \cdot \Phi^{-1}_2 \cdot b \right] + t \text{Tr} \left[ b^\dagger \cdot w + w^\dagger \cdot b \right], \tag{S-17}
\]

with

\[
\Phi^{-1}_2 = F_2^{-1} \mathbb{I}_6 - t^2 \mathbf{A}_2 \quad \text{and} \quad w = \begin{pmatrix} f_{1,2} c_1 - f_{1,1} c_2 \\ f_{1,3} c_1 - f_{1,1} c_3 \\ f_{1,4} c_1 - f_{1,1} c_4 \\ f_{1,3} c_2 - f_{1,2} c_3 \\ f_{1,4} c_2 - f_{1,2} c_4 \\ f_{1,3} c_3 - f_{1,2} c_4 \end{pmatrix}. \tag{S-18}
\]

Eq. (S-17) is quadratic in \( b \) and its integration is also Gaussian, leading to

\[
S'_{12} = t^2 \text{Tr} \left[ w^\dagger \cdot \Phi_2 \cdot w \right] = t^2 \text{Tr} \left[ w^\dagger \cdot F_2 \mathbb{I}_6 \cdot w \right] + t^4 \text{Tr} \left[ w^\dagger \cdot F_2 \cdot \mathbf{A}_2 \cdot F_2 \cdot w \right]. \tag{S-19}
\]

In the last equality, we have expanded the operator \( \Phi_2 \) in \( t \). The mapping of the first term of Eq. (S-19) onto the Kondo-like interaction Eq. (S-6) possible by considering the identity \( w^\dagger \cdot w = \sum_{\nu\nu'\tau\tau'} c_{\nu\nu'} f_{1,\tau}^\dagger f_{1,\tau'} \left[ \frac{3}{4} \delta_{\nu\nu'} \delta_{\tau\tau'} - 2T_{\tau\tau'} \cdot t_{\nu\nu'} \right] \), a consequence of the relation \( \sum_{\nu} f_{1,\tau}^\dagger f_{1,\tau} = 1 \). This is a reformulation of the constraint Eq. (S-3), once all the slave-states participating to it, with the exception of the para-fermions \( f_{1,\tau} \), have been integrated out. The effective action is cast into the form

\[
S'_{1} = S'_{0} + S_{SU(4)} + t^4 \text{Tr} \left[ w^\dagger \cdot F_2 \cdot \mathbf{A}_2 \cdot F_2 \cdot w \right], \tag{S-20}
\]

with \( S'_{0} \) given by the two first terms of Eq. (S-2). \( S_{SU(4)} \) has the structure of Eq. (S-6), with \( \tilde{J} = -2t^2 [F_2 + F_0] \) and \( W = (t^2/4) [3F_2 - F_0] \). Their frequency dependence is explicit in the Matsubara representation of the fields. Adopting the same notations as in Eq. (S-6)

\[
\tilde{J}_{(i\nu_n, i\omega_m, i\omega_l)} = -2t^2 [F_0(i\nu_n) + F_2(i\nu_n + i\omega_m + i\omega_l)], \quad \tilde{W}_{(i\nu_n, i\omega_m, i\omega_l)} = -\frac{t^2}{4} [F_0(i\nu_n) - 3F_2(i\nu_n + i\omega_m + i\omega_l)]. \tag{S-21}
\]
These two expressions are in agreement with the Schrieffer-Wolff transformation for \( q = 1 \) \([13, 14]\). This can be seen by substituting the frequencies of Eq. (S-21) with the poles of \( G_k \) and \( F_1 \): \( \nu_n \rightarrow \varepsilon_d + \lambda \) and \( i\omega_{m,t} \rightarrow 0 \). The last interaction term in Eq. (S-20) reads

\[
\frac{t^4}{\beta} \text{Tr} \left[ w^1 \cdot F_2 \cdot A_2 \cdot F_2 \cdot w \right] = \frac{t^4}{\beta} \sum_{\beta} F_2(i\omega_l + i\omega_m - i\omega_n + i\omega_o - i\omega_p) F_3(i\omega_l + i\omega_m + i\omega_n) \times \]

\[
F_2(i\omega_l + i\omega_n) c^{\dagger}_{\alpha \beta}(i\Omega) c_{\kappa \nu \gamma}(i\omega_n) c_{\kappa' \nu' \delta}(i\omega_l) f_{1,\tau}(i\omega_l) f_{1,\tau}(i\omega_n) .
\]

The sum runs over all frequencies and momenta. \( i\Omega = i\omega_l + i\omega_m - i\omega_n + i\omega_o - i\omega_p \) ensures energy conservation. The sum over the orbital labels is set by the specific algebraic form of \( A_2 \) and \( w \) in Eqs. (S-6) and (S-18) respectively. A mean-field treatment of the four point interaction \((c^\dagger c)^2\) close to the free propagator poles permits to map also this interaction onto Eq. (S-6). All correction to the mean-field and frequency expansion is irrelevant in the renormalization flow towards low energies and can be neglected. Lead electron operators in Eq. (S-22) are contracted by pairs and replaced by their free propagator. For instance

\[
c^{\dagger}_{\alpha \beta}(i\Omega) c_{\kappa \nu \gamma}(i\omega_n) c_{\kappa' \nu' \delta}(i\omega_l) f_{1,\tau}(i\omega_l) f_{1,\tau}(i\omega_n) = -G_k(i\omega_n) c^{\dagger}_{\kappa' \nu' \delta}(i\omega_l + i\omega_o - i\omega_p) c_{\kappa \nu \gamma}(i\omega_l) f_{1,\tau}(i\omega_p) f_{1,\tau}(i\omega_n) .
\]

The propagators appearing in Eq. (S-22) are expanded close to the fixed point frequencies \( i\omega \rightarrow E_F = 0 \) for lead electrons and \( i\omega \rightarrow \varepsilon_d \approx \varepsilon_d + \lambda \) for dot para-fermions, leading to contributions of the form

\[
V = A_0 \times \frac{t^4}{\beta} \sum_{\nu,\nu'} c^{\dagger}_{\kappa \nu \gamma}(i\omega_l + i\omega_o - i\omega_p) c_{\kappa' \nu' \delta}(i\omega_l) f_{1,\tau}(i\omega_p) f_{1,\tau}(i\omega_n) ,
\]

in which different contractions lead to

\[
A_0 = \frac{F_2(i\omega_l + i\omega_o)}{\beta} \sum_{\nu,\nu'} G_p(i\omega_n) F_3(i\omega_n + i\omega) \frac{\nu_0}{(\varepsilon_d + U)^2} \ln \frac{2\varepsilon_d + 3U}{\Xi} ,
\]

\[
A_1 = \frac{F_2(i\omega_l + i\omega_o)}{\beta} \sum_{\nu,\nu'} G_p(i\omega_n) F_3(i\omega_n + i\omega) \frac{\nu_0}{(\varepsilon_d + U)(\varepsilon_d + 2U)} \ln \frac{2\varepsilon_d + 3U}{\varepsilon_d + U} ,
\]

\[
A_2 = \frac{1}{\beta} \sum_{\nu,\nu'} G_p(i\omega_n) F_3(i\omega_n + i\omega) \frac{-\nu_0}{(\varepsilon_d + U)(\varepsilon_d + 2U)} + \frac{\nu_0}{(\varepsilon_d + 2U)^2} \ln \frac{2\varepsilon_d + 3U}{\varepsilon_d + U} .
\]

The Matsubara sums are carried out first and the analytical continuations \( i\omega \rightarrow 0 \) and \( i\omega \rightarrow \varepsilon_d + \lambda \) are done later. The cutoff \( \Xi \) must be introduced to prevent UV divergences and can be safely sent to infinity at the end of calculations. Applying the constraint \( \sum_{\tau} f_{i,\tau} = 1 \) and Eq. (S-7), Eq. (S-22) maps onto Eq. (S-6), namely

\[
\frac{t^4}{\beta} \sum_{kk'} \text{Tr} \left[ (8A_1 - 4A_0 - 4A_2) T_{k\tau' \tau} \cdot T_{k'\nu' \nu} + \left( \frac{3}{2} A_0 + \frac{3}{2} A_2 - 3A_1 \right) \delta_{\nu\nu'} \delta_{\tau\tau'} \right] c^{\dagger}_{k\nu \gamma} c_{k'\nu' \delta} f_{1,\tau}(i\omega_l) f_{1,\tau}(i\omega_n) .
\]

The effective interaction of the SU(4) Anderson Hamiltonian recovers then Eq. (S-6) with couplings

\[
J_{n,m,t} = J_{n,m,t}^0 + t^4 (8A_1 - 4A_0 - 4A_2) , \quad \bar{W}_{n,m,t} = \bar{W}_{n,m,t}^0 + t^4 \left( \frac{3}{2} A_0 + \frac{3}{2} A_2 - 3A_1 \right) .
\]

2. Calculation of the slave-field propagator

To leading order in \( t \), the self-energy \( \Sigma(i\omega_n) \) is given by the diagram pictured in Fig. S-3

\[
\Sigma_1(i\omega_l) = \frac{4}{\beta} \sum_{\nu} G_p(i\omega_l) \bar{W}_{i\omega_l,\nu}\bar{W}_{i\omega_l,\nu} = \nu_0 t^2 \left[ \ln \frac{\lambda - i\omega}{\Xi} + 3 \ln \frac{2\varepsilon_d + U + \lambda - i\omega}{\Xi} \right] .
\]

Notice that \( \omega \) can be safely sent to zero at this stage. The anti-symmetric properties of the SU(N) matrices \( T \) and \( t \), imply that only \( \bar{W}^1 \) in Eq. (S-21), and not \( \bar{J}^1 \), contributes to the self energy. The full propagator reads then

\[
F_1(i\omega_l) = \frac{2}{i\omega_l - \varepsilon_d} ,
\]
involving the renormalized pole \( \tilde{\varepsilon}_d = \varepsilon_d + \lambda + \Sigma_d(\tilde{\varepsilon}_d) \) and the quasi-particle weight \( Z = 1/[1 - \Sigma'(\tilde{\varepsilon}_d)] \)

\[
Z = 1 + \nu_0 t^2 \left[ \frac{1}{\tilde{\varepsilon}_d} - \frac{3}{\varepsilon_d + U} \right], \quad \tilde{\varepsilon}_d = \varepsilon_d + \lambda + \nu_0 t^2 \left( \ln \frac{-\varepsilon_d}{\Xi} + 3 \ln \frac{\varepsilon_d + U}{\Xi} \right). \tag{S-30}
\]

3. Calculation of the four-point vertex function

To fourth order in \( t \), the calculation of the vertex Eq. (S-8) involves the diagrams in Fig. S-4. These are calculated using the \( S_{SU(4)} \) interaction Eq. (S-6) with the couplings Eq. (S-27). The first contribution to the renormalized vertex \( \mathcal{V}^R \) is given by the first diagram of the series in Fig. S-4 multiplied by the quasi-particle weight \( Z \) Eq. (S-30) and performing the analytical extension \( i\omega \rightarrow \tilde{\varepsilon}_d \), given in Eq. (S-30). This contribution reads

\[
\mathcal{V}_1 = Z \left[ J_1 \mathbf{T} \right] \left[ W_1 \mathbf{T} \right] \left( \ln \frac{-\varepsilon_d}{\Xi} + 3 \ln \frac{-\varepsilon_d + U}{\Xi} \right) \left( \frac{1}{\varepsilon_d} + \frac{1}{(\varepsilon_d + U)^2} \right) +
\]

\[
\left[ W_1 Z + \frac{\nu_0 t^4}{4} \left( \ln \frac{-\varepsilon_d}{\Xi} + 3 \ln \frac{-\varepsilon_d + U}{\Xi} \right) \left( \frac{1}{\varepsilon_d} - \frac{3}{(\varepsilon_d + U)^2} \right) \right], \tag{S-31}
\]

with \( J_1 = -2t^2 U/\varepsilon_d(\varepsilon_d + U) \) and \( W_1 = -t^2(4\varepsilon_d + U)/4\varepsilon_d(\varepsilon_d + U) \), the couplings obtained by the Schrieffer-Wolff transformation. The leading diagram in Fig. S-4 also carries a subleading contribution from Eqs. (S-26) and (S-27). This reads

\[
\mathcal{V}_1' = t^2 \left( 8A_1 - 4A_0 - 4A_2 \right) \mathbf{T} \cdot \mathbf{t} + t^4 \left( \frac{3}{2} A_0 + \frac{3}{2} A_2 - 3A_1 \right) \mathbf{1}. \tag{S-32}
\]

The diagrams of order \( t^4 \) in Fig. S-4 are calculated and cast in the form

\[
\mathcal{V}_a = -\frac{1}{2} \sum_{p_\omega \alpha \beta} G_p(i\omega + i\tilde{\omega} - i\omega_n) F_1(i\omega_n) \left[ \left( -\frac{1}{4} J_a^2 + 2J_a W_a \right) \mathbf{T} \cdot \mathbf{t} + \left( \frac{15}{64} J_a^2 + W_a^2 \right) \mathbf{1} \right], \tag{S-33}
\]

\[
\mathcal{V}_b = -\frac{1}{2} \sum_{p_\omega \alpha \beta} G_p(i\omega + i\omega_n - i\tilde{\omega}) F_1(i\omega_n) \left[ \left( \frac{7}{4} J_b^2 + 2J_b W_b \right) \mathbf{T} \cdot \mathbf{t} + \left( \frac{15}{64} J_b^2 + W_b^2 \right) \mathbf{1} \right],
\]

with

\[
J_a = -2t^2 \left[ F_0(i\omega_n - i\omega) + F_2(i\omega - i\tilde{\omega}) \right], \quad W_a = -\frac{t^4}{4} \left[ F_0(i\omega_n - i\omega) - 3F_2(i\omega - i\tilde{\omega}) \right],
\]

\[
J_b = -2t^2 \left[ F_0(i\omega - i\tilde{\omega}) + F_2(i\omega + i\omega_n) \right], \quad W_b = -\frac{t^4}{4} \left[ F_0(i\tilde{\omega} - i\omega) - 3F_2(i\omega + i\omega_n) \right]. \tag{S-34}
\]
FIG. S-4: Diagrammatic series for the vertex function Eq. (S-8). This is a more detailed version of Fig. 3 in the main text. The arrows indicate the frequency dependence of the couplings in Eq. (S-21).

The following integrals are computed

\[
I_{00} = \frac{1}{\beta} \sum_{p,n} G_p(i\omega + i\tilde{\omega} - i\omega_n) F_1(i\omega_n) F_0^2(i\omega_n - i\omega) = \frac{\nu_0}{\varepsilon_d} \ln \frac{-\varepsilon_d}{\omega} - 1,
\]

\[
I_{02} = \frac{1}{\beta} \sum_{p,n} G_p(i\omega + i\tilde{\omega} - i\omega_n) F_1(i\omega_n) F_0(i\omega_n - i\omega) F_2(i\omega + i\tilde{\omega}) = -\frac{\nu_0}{\varepsilon_d} \ln \frac{\omega}{\varepsilon_d},
\]

\[
I_{22} = \frac{1}{\beta} \sum_{p,n} G_p(i\omega + i\tilde{\omega} - i\omega_n) F_1(i\omega_n) F_0^2(i\omega + i\tilde{\omega}) = -\frac{\nu_0}{\varepsilon_d(\varepsilon_d + U)} \ln \frac{\omega}{\varepsilon_d},
\]

\[
L_{00} = \frac{1}{\beta} \sum_{p,n} G_p(i\omega + i\omega_n - i\tilde{\omega}) F_1(i\omega_n) F_0^2(i\tilde{\omega} - i\omega) = \frac{\nu_0}{\varepsilon_d} \ln \frac{\varepsilon_d + U}{\omega},
\]

\[
L_{02} = \frac{1}{\beta} \sum_{p,n} G_p(i\omega + i\omega_n - i\tilde{\omega}) F_1(i\omega_n) F_0(i\omega_n - i\omega) F_2(i\omega + i\tilde{\omega}) = \frac{\nu_0}{\varepsilon_d(\varepsilon_d + U)} \ln \frac{\varepsilon_d + U}{\omega},
\]

\[
L_{22} = \frac{1}{\beta} \sum_{p,n} G_p(i\omega + i\omega_n - i\tilde{\omega}) F_1(i\omega_n) F_0^2(i\omega + i\tilde{\omega}) = \frac{\nu_0}{\varepsilon_d(\varepsilon_d + U)^2} \left[1 - \ln \frac{\varepsilon_d + U}{\omega}\right].
\]

In this case, one has to take care of the infra-red cutoff \(\omega\) for sums over momenta. Only the integrals \(I_{22}\) and \(L_{00}\) depend on the high energy cut-off \(\Xi\). The contributions to the vertex function read then

\[
\mathcal{V}^a = t^4 \left(4I_{02} + 4I_{22}\right) \mathbf{T} \cdot \mathbf{t} - t^4 \left(I_{00} + \frac{3}{2} I_{22} + \frac{3}{2} I_{02}\right) \mathbf{1},
\]

\[
\mathcal{V}^b = -t^4 \left(8L_{00} + 12L_{02} + 4L_{22}\right) \mathbf{T} \cdot \mathbf{t} - t^4 \left(L_{00} + \frac{3}{2} L_{22} + \frac{3}{2} L_{02}\right) \mathbf{1}.
\]

The summation of Eq. (S-31), (S-32) and (S-36) leads to the final expression of the renormalized vertex in the form Eq. (S-10). The \(\Xi \to \infty\) limit can be safely performed. The spin-exchange part \(\mathcal{J}(\omega)\) recovers Eq. (7a) in the main text. For completeness, we also report the result for the potential scattering term

\[
\mathcal{W}_1 = W_1 + \frac{1}{\nu_0} \left(\frac{\Delta}{\pi U}\right)^2 \eta_1 \left(\frac{\varepsilon_d}{U}\right),
\]

with

\[
\eta_1(x) = \frac{3}{4} \left(\frac{1}{x^2} + \frac{1}{(x + 1)^2}\right) + \frac{1}{x^2} \ln \sqrt[4]{-x(x + 1)} - \frac{1}{(x + 1)^2} \ln \sqrt[4]{x(x + 1)} + \frac{3}{2} \ln \frac{x + 3}{x + 1} + \frac{3}{2} \ln \frac{x}{x + 1} + \frac{3}{2} \ln \frac{2x + 3}{x + 1} - \frac{1}{x + 1}(x + 1)^2 \left(\frac{1}{2} + \ln \frac{2x + 3}{x + 1}\right).
\]

Notice that even if \(W_1 = -t^2(4\varepsilon_d + U)/(4\varepsilon_d(\varepsilon_d + U))\) vanishes in the middle of the Coulomb valley for \(\varepsilon_d = -U/2\), \(\eta_1\) does not. This is explained by the fact that the model is effectively particle-hole symmetric at \(\varepsilon_d = -U/2\) only to second order in \(t\).
S-III. CALCULATION OF THE SU(4) SPIN EXCHANGE PROCESSES FOR $q = 2$

4. Integration of the high energy slave-fields

For $-2U < \varepsilon_d < -U$, two charges are frozen on the dot to low energy. To fourth order in the tunneling $t$, all the other occupancies of the quantum dot contribute virtually to the effective action, see Fig. S-2. The highest energy charge modes are $b_0$ and $b_4$. They are integrated first and the effective action reads

$$ S_2 = -\text{Tr} \left[ b^\dagger \cdot F_2^{-1} b \right] - \text{Tr} \left[ f^\dagger \cdot \Phi_1^{-1} \cdot f \right] - \text{Tr} \left[ F^\dagger \cdot \Phi_3^{-1} \cdot F \right] + t \text{Tr} \left[ D^\dagger \cdot f \right] + \text{Tr} \left[ B_1^\dagger \cdot F \right] + \text{Tr} \left[ F^\dagger \cdot B_3 \right]. $$

(S-39)

The vectors $F$ and $b$ are the same as in Eq. (S-15) and we introduce

$$ \Phi_1^{-1} = F_1^{-1} \Phi_4 - i^2 A_1, \quad \Phi_3^{-1} = F_3^{-1} \Phi_4 - i^2 A_1, $$

(S-40)

The integration of the remaining para-fermions collected in the vectors $f$ and $F$ is also Gaussian

$$ S'_2 = -\text{Tr} \left[ b^\dagger \cdot F_2^{-1} \cdot b \right] + \text{Tr} \left[ J^2 \left( b^\dagger \cdot T \cdot b \right) \cdot (e^\dagger \cdot t \cdot c) + \tilde{W}^2 \left( b^\dagger \cdot 4 \cdot b \right) \cdot (e^\dagger \cdot 4 \cdot c) \right] + t^4 \text{Tr} \left[ B_1^\dagger \cdot F_1 \cdot A_1 \cdot B_1 \right] + t^4 \text{Tr} \left[ F_3 \cdot A_3 \cdot F_1 \cdot B_3 \right], $$

(S-41)

with this time $J^2 = -2t^2 [F_1 + F_3]$ and $\tilde{W}^2 = t^2/2 [F_3 - F_1]$. In the Matsubara representation

$$ \tilde{J}^2_{iv_n,iv_{\omega_m},i\omega_l} = -2t^2 \left[ F_1 (iv_n - i\omega_l) + F_3 (iv_n + i\omega_m) \right], \quad \tilde{W}^2_{iv_n,iv_{\omega_m},i\omega_l} = -t^2/2 \left[ F_1 (iv_n - i\omega_l) - F_3 (iv_n + i\omega_m) \right]. $$

(S-42)

Taking the analytical continuations $iv_n \rightarrow \varepsilon_d$ and $t^4 \rightarrow 0$, also these expressions recover the Schrieffer-Wolff couplings $J_2 = -2t^2 U/(\varepsilon_d + U)\varepsilon_d$ and $W_2 = -t^2 (2\varepsilon_d + 3U)/A(\varepsilon_d + U)$, valid for $q = 2$. The SU(4) representation of the interaction $(b^\dagger \cdot T \cdot b) \cdot (c^\dagger \cdot t \cdot c)$ deserves a specific discussion. The vector $b$ has six components. This implies that the matrices composing the invariant SU(4) vector $T$ are not those of the fundamental representation [3] and present in Eq. (S-7). The explicit expression of the product in Eq. (S-41) reads

$$ 2 \left( b^\dagger \cdot T \cdot b \right) \cdot (e^\dagger \cdot t \cdot c) = c_1 c_2 \left( b^\dagger_{14} b_{14} + b^\dagger_{23} b_{23} + b^\dagger_{13} b_{13} + b^\dagger_{24} b_{24} - b^\dagger_{13} b_{13} - b^\dagger_{24} b_{24} \right) + c_1 c_3 \left( b^\dagger_{24} b_{14} - b^\dagger_{23} b_{12} + b^\dagger_{14} b_{24} - b^\dagger_{32} b_{14} \right) + c_2 c_3 \left( b^\dagger_{23} b_{24} + b^\dagger_{13} b_{12} + b^\dagger_{24} b_{24} - b^\dagger_{13} b_{12} - b^\dagger_{23} b_{23} + b^\dagger_{34} b_{13} \right) + \text{h.c.}. $$

(S-43)

This is derived by applying the constraint $\sum_{\tau \tau'} b^\dagger_{\tau \tau'} b_{\tau \tau'} = 1$, which is the corrected version of Eq. (S-3), once all the other slave-fields have been integrated. The mapping onto the fermionic representation of the SU(4) spin-exchange interaction Eq. (S-6) is obtained refermionizing the slave-bosons $b_{\tau \tau'}$. These fields are associated to a two-electron(fermion) state $b^\dagger_{\tau \tau'} = f^\dagger_{\tau} f^\dagger_{\tau'}$. The $f_{\tau}$ are para-fermion fields obeying the constraint $\sum_{\tau} f^\dagger_{\tau} f_{\tau} = 2$. Substituting in Eq. (S-43), Eq. (S-7) for $q = 2$ is recovered, according to the effective SU(4) symmetric interaction Eq. (S-6).
The subleading interaction in Eq. (S-41) is simplified with a mean-field treatment, analog to that carried out in Section S-II 1, leading to

\[ S_{st} = t^4 \sum_{k k' \nu' \tau' \rho' \rho} \text{Tr} \left\{ 2(\mathcal{V}_0 - \mathcal{V}_4) \mathbf{T}_{\tau \tau', \rho \rho'} \cdot \mathbf{t}_{\nu \nu'} + \frac{\mathcal{V}_0 + \mathcal{V}_4}{2} \delta_{\tau \tau', \rho \rho'} \delta_{\nu \nu'} \right\} \left( c^\dagger_{k \nu} c_{k' \nu'} b_{\tau \rho'}^\dagger b_{\tau' \rho} \right) \]  

(S-44)

\( \mathcal{V}_0 \) and \( \mathcal{V}_4 \) are the contributions to the vertex coming from the 0 and 4 charge sector respectively. They read

\[ \mathcal{V}_0 = B_0 + B_2 - 2B_1, \]
\[ \mathcal{V}_4 = C_0 + C_2 - 2C_1, \]  

(S-45)

with

\[ B_0 = \frac{1}{\beta} \sum_{p,n} G_p(-i\omega_n) F_p^2(i\nu - i\omega) F_0(i\nu - i\omega + i\omega_n) = \frac{\nu_0}{(\varepsilon_d + U)^2} \ln \left( \frac{2\varepsilon_d - U}{\varepsilon_d + U} \right), \]
\[ B_1 = \frac{1}{\beta} \sum_{p,n} G_p(-i\omega_n) F_1(i\nu + i\omega_n) F_0(i\nu - i\omega + i\omega_n) F_1(i\nu - i\omega) = \frac{\nu_0}{\varepsilon_d(\varepsilon_d + U)} \ln \left( \frac{\varepsilon_d + U}{2\varepsilon_d + 2U} \right), \]
\[ B_2 = \frac{1}{\beta} \sum_{p,n} G_p(-i\omega_n) F_2^2(i\nu + i\omega_n) F_0(i\nu - i\omega + i\omega_n) = \frac{\nu_0}{\varepsilon_d(\varepsilon_d + U)} \ln \left( \frac{\varepsilon_d + U}{2\varepsilon_d + 2U} \right), \]
\[ C_0 = \frac{1}{\beta} \sum_{p,n} G_p(i\omega_n) F_3^2(i\nu + i\omega) F_0(i\nu + i\omega + i\omega_n) = \frac{\nu_0}{(\varepsilon_d + 2U)^2} \ln \left( \frac{2\varepsilon_d + 5U}{\varepsilon_d + U} \right), \]
\[ C_1 = \frac{1}{\beta} \sum_{p,n} G_p(i\omega_n) F_1(i\nu + i\omega_n) F_0(i\nu + i\omega + i\omega_n) F_1(i\nu + i\omega) = \frac{\nu_0}{(\varepsilon_d + 3U)(\varepsilon_d + 2U)} \ln \left( \frac{2\varepsilon_d + 5U}{\varepsilon_d + 2U} \right), \]
\[ C_2 = \frac{1}{\beta} \sum_{p,n} G_p(i\omega_n) F_2^2(i\nu + i\omega_n) F_0(i\nu + i\omega + i\omega_n) = \frac{\nu_0}{(\varepsilon_d + 3U)^2} \ln \left( \frac{2\varepsilon_d + 5U}{\varepsilon_d + 2U} \right). \]  

(S-46)

The total effective interaction maps onto the Kondo-like interaction Eq. (S-6) with effective couplings

\[ \tilde{J}_{n,m,l} = J_{n,m,l}^2 + t^2 (\mathcal{V}_0 - \mathcal{V}_4); \quad \tilde{W}_{n,m,l} = \tilde{W}_{n,m,l}^2 + t^4 (\mathcal{V}_0 + \mathcal{V}_4). \]  

(S-47)

5. Calculation of the slave-field propagator

To second order in \( t \), the self-energy of the boson propagator \( \Sigma_2(i\nu_n) = -\langle b(i\nu_n) b^\dagger(i\nu_n) \rangle \) reads

\[ \Sigma_2(i\nu_n) = \frac{4}{\beta} \sum_{p,l} W (i\nu_n, i\omega_l, i\omega_l) G_p(i\omega_l) = \frac{2\Delta}{\pi} \left[ \ln \left( \frac{-i\nu_n + 3\varepsilon_d + 3U + \lambda}{\varepsilon_d + \lambda - i\nu_n} \right) \right]. \]  

(S-48)

The self-energy leads to the renormalized pole \( i\tilde{\nu} \) and quasi-particle weight \( Z_2 \)

\[ i\tilde{\nu} \rightarrow \tilde{\varepsilon}_d = 2\varepsilon_d + U + \lambda + \frac{2\Delta}{\pi} \left[ \ln \left( \frac{\varepsilon_d + 2U}{\varepsilon_d + U} \right) + \ln \left( \frac{-\varepsilon_d - U}{\varepsilon_d - U} \right) \right], \]
\[ Z_2 = 1 - \nu_0 J_2, \]  

(S-49)

with \( J_2 = -2t^2 U/(\varepsilon_d + U)(\varepsilon_d + 2U) \) given by the Schrieffer-Wolff transformation.

6. Calculation of the four-point vertex function

Although the possibility of the fermionic representation of the slave-boson fields, it is much simpler to calculate the vertex function for the slave-bosons, similarly to Eq. (S-8)

\[ \mathcal{V}_{\nu' \tau' \rho' \rho}(k, i\omega_l, i\omega_n; k', i\omega_m, i\nu_n) = -\beta \frac{\left( \left< c_{k \nu}(i\omega_l) b_{\tau \rho}(i\nu_n) b_{\tau' \rho'}^\dagger(i\nu_n) c_{k' \nu'}^\dagger(i\omega_m) \right> \right)}{G_k(i\omega_l) G_{k'}(i\omega_m) F_2(i\nu_n) F_2(i\nu_l)}. \]  

(S-50)
This takes the form Eq. (S-10), but the vector \( \mathbf{T} \) is made of the \( 6 \times 6 \) matrices appearing in Eq. (S-41), leading to Eq. (S-43). We calculate now all the contributions to the renormalized vertex \( \mathcal{V}^\alpha_2 = \tilde{Z}_2 \mathcal{V}^\alpha \). The contribution corresponding to the first diagram of the series in Fig. S-4 is readily obtained

\[
\mathcal{V}_{1-2} = \mathbf{T} \cdot \mathbf{t} \left[ J_2 - \nu_0 J_2^2 + 2\nu_0 \left( \frac{J_2^2}{4} + 4W_2^2 \right) \ln \left( \frac{-\varepsilon_d - U}{\varepsilon_d + 2U} \right) \right] + \left[ W_2 - \nu_0 J_2 W_2 + \nu_0 J_2 W_2 \ln \left( \frac{-\varepsilon_d - U}{\varepsilon_d + 2U} \right) \right]. \tag{S-51}
\]

This includes the corrections to the pole and the quasi-particle weight Eq. (S-49). Also the contributions coming from the subleading interaction Eq. (S-44) have to be taken into account

\[
\mathcal{V}'_2 = 2t^4 (\mathcal{V}_0 - \mathcal{V}_2) \mathbf{T} \cdot \mathbf{t} + \frac{t^4}{2} (\mathcal{V}_0 + \mathcal{V}_2) \mathbf{1}. \tag{S-52}
\]

Eventually, we have the contributions which are the analog of the contributions \( \mathcal{V}^a \) and \( \mathcal{V}^b \) in Eq. (S-33)

\[
\begin{align*}
\mathcal{V}^a_2 &= -\frac{1}{\beta} \sum_{p,n} G_p(i\omega + i\nu_\omega - i\nu_n) F_2(i\nu_n) \left[ (-J_a^2 + 2J_a W_a) \mathbf{T} \cdot \mathbf{t} + \left( \frac{5}{16} J_a^2 + W_a \right) \mathbf{1} \right], \\
\mathcal{V}^b_2 &= -\frac{1}{\beta} \sum_{p,n} G_p(i\omega + i\nu_n - i\nu_\omega) F_2(i\nu_n) \left[ (J_b^2 + 2J_b W_b) \mathbf{T} \cdot \mathbf{t} + \left( \frac{5}{16} J_b^2 + W_b \right) \mathbf{1} \right],
\end{align*}
\tag{S-53}
\]

with

\[
\begin{align*}
J_a &= -2t^2 \left[ F_1(i\nu_\omega - i\omega) - F_3(i\nu_\omega + i\omega) \right], \quad W_a = \frac{t^4}{\beta} \left[ F_1(i\nu_\omega - i\omega) - F_3(i\nu_\omega + i\omega) \right], \\
J_b &= -2t^2 \left[ F_1(i\nu_\omega - i\omega) - F_3(i\nu_\omega + i\nu_n) \right], \quad W_b = \frac{t^2}{\beta} \left[ F_1(i\nu_\omega - i\omega) - F_3(i\nu_\omega + i\nu_n) \right].
\end{align*}
\tag{S-54}
\]

The calculation of the Matsubara sums and of the integrals over momenta leads to the following result

\[
\begin{align*}
\mathcal{V}^a_2 &= 2t^4 (I_{11} + 3I_{33} + 4I_{13}) \mathbf{T} \cdot \mathbf{t} - \frac{t^4}{\beta} \left( 3I_{11} + 3I_{33} + 4I_{13} \right) \mathbf{1}, \\
\mathcal{V}^b_2 &= -2t^4 (3I_{11} + L_{13}) \mathbf{T} \cdot \mathbf{t} - \frac{t^4}{\beta} (3I_{11} + L_{13}) \mathbf{1},
\end{align*}
\tag{S-55}
\]

with

\[
\begin{align*}
I_{11} &= \frac{1}{\beta} \sum_{p,n} F_2(i\nu_n) G_p(i\omega + i\nu_\omega - i\nu_n) F_1^2(i\nu_n - i\omega) = -\frac{\nu_0}{(\varepsilon_d + U)^2} \left[ 1 + \ln \frac{\omega}{-\varepsilon_d - U} \right], \\
I_{13} &= \frac{1}{\beta} \sum_{p,n} F_2(i\nu_n) G_p(i\omega + i\nu_n - i\nu_\omega) F_1(i\nu_n - i\omega) F_2(i\omega + i\nu_\omega) = \frac{\nu_0}{(\varepsilon_d + U)(\varepsilon_d + 2U)} \ln \frac{\omega}{-\varepsilon_d - U}, \\
I_{33} &= \frac{1}{\beta} \sum_{p,n} F_2(i\nu_n) G_p(i\omega + i\nu_\omega - i\nu_n) F_3^2(i\nu_\omega + i\omega) = \frac{\nu_0}{(\varepsilon_d + 2U)^2} \ln \frac{\omega}{\Xi}, \\
L_{11} &= \frac{1}{\beta} \sum_{p,n} F_2(i\nu_n) G_p(i\omega + i\nu_n - i\nu_\omega) F_1^2(i\nu_\omega - i\omega) = \frac{\nu_0}{(\varepsilon_d + U)^2} \ln \frac{\omega}{\Xi}, \\
L_{13} &= \frac{1}{\beta} \sum_{p,n} F_2(i\nu_n) G_p(i\omega + i\nu_n - i\nu_\omega) F_1(i\omega + i\nu_\omega) F_2(i\omega + i\nu_n) = -\frac{\nu_0}{(\varepsilon_d + U)(\varepsilon_d + 2U)} \ln \frac{\omega}{\varepsilon_d + 2U}, \\
L_{33} &= \frac{1}{\beta} \sum_{p,n} F_2(i\nu_n) G_p(i\omega + i\nu_\omega - i\nu_n) F_3^2(i\omega + i\nu_\omega) = \frac{\nu_0}{(\varepsilon_d + 2U)^2} \left[ 1 + \ln \frac{\omega}{\varepsilon_d + 2U} \right].
\end{align*}
\tag{S-56}
\]

The total expression of the renormalized vertex is obtained by summing up Eqs. (S-51), (S-55) and (S-52). The limit \( \Xi \to \infty \) is safely carried out. The renormalized vertex recovers Eq. (S-10) and the spin-exchange coupling reads

\[
J_2(\omega) = J_2 + 2\nu_0 J_2^2 \frac{\varepsilon_d}{U} - 2\nu_0 J_2^2 \ln \frac{\omega}{e^{-\frac{1}{\beta} \sqrt{(-\varepsilon_d - U)(\varepsilon_d + 2U)}}}, \tag{S-57}
\]
with

\[ g_2(x) = \frac{(x+2)^2}{4x^2} \left[ x + \ln \frac{x+1}{2x+1} \right] + \frac{(x+1)^2}{4(x+3)^2} \left[ -x - 3 + \ln \frac{x+2}{2x+5} \right]. \]  

(S-58)

Eq. (S-57) maps onto Eq.(2) in the main text if \( J \rightarrow J_2 + 2\nu_0J_2^2g_2(\frac{\epsilon_d}{U}) \) and \( D \rightarrow D_2 = e^{-\frac{1}{4}}\sqrt{(-\epsilon_d - U)(\epsilon_d + 2U)}. \) The SU(4) Kondo temperature for \( q = 2 \) is derived in the form Eq. (3), with the prefactor \( f_2(x) \)

\[ f_2(x) = \frac{\sqrt{(-1-x)(x+2)}}{\epsilon^2}e^{g_2(x)}. \]  

(S-59)

This function is plotted in the main text in Fig. 2, showing an excellent agreement with the numerical renormalization group calculations. For completeness we also give the potential scattering term

\[ W_2 = W_2 + \frac{\Delta}{\pi U} \left\{ \eta_2 \left( \frac{\epsilon_d}{U} \right) \right\}, \]  

(S-60)

with

\[ \eta_2(x) = \frac{1}{2} \frac{2x+1}{x(x+1)^2} - \frac{1}{2} \frac{2x+5}{(x+3)(x+2)^2} - \frac{2}{(x+1)(x+2)} \ln \frac{x+2}{x+1} + \frac{1}{2} \frac{1-x}{(x+1)x^2} \ln \frac{x+1}{2x+1} + \frac{1}{2} \frac{x+4}{(x+2)(x+3)^2} \ln \frac{x+2}{2x+5} + \frac{1}{2} \frac{(x+1)^2}{(x+2)^2} \ln \frac{\sqrt{-x-1}(2x-1)}{\sqrt{(x+2)(2x+5)}}. \]  

(S-61)

This function vanishes at the particle-hole symmetric point \( x = \epsilon_d/U = -3/2 \). This coincides with the middle of the Coulomb valley, in contrast with the \( q = 1 \) case, see Eq. (S-37).

S-IV. CALCULATION OF THE SU(4) SPIN EXCHANGE PROCESSES FOR \( q = 3 \)

The case of triple dot occupancy requires \(-3U < \epsilon_d < -2U\). The case \( q = 3 \) sector is the particle-hole symmetric to \( q = 1 \). All results are readily derived by making the following set of substitutions in Section S-II

\[
\begin{align*}
E_0 - E_1 &= -\epsilon_d, & E_4 - E_3 &= \epsilon_d + 2U, \\
E_2 - E_1 &= \epsilon_d + U, & E_2 - E_3 &= -\epsilon_d - 3U, \\
E_3 - E_1 &= 2\epsilon_d + 3U, & E_4 - E_3 &= -2\epsilon_U.
\end{align*}
\]  

(S-62)

The prefactor for the SU(4) Kondo temperature, Eq. (3) in the main text, reads then \( f_3(\epsilon_d/U) = f_1(-2 - \epsilon_d/U) \). The derivation of the effective SU(4) symmetric action Eq. (S-6) deserves a specific discussion. We recall that the constraint Eq. (S-3), imposing \( N_{\text{slave}} = 1 \), must be adapted upon the integration of the high energy slave-fields. In this case \( \sum_{\tau} f_{3,\tau}^* f_{3,\tau} = 1 \). In this representation, the spin-exchange interaction reads

\[ 2T \cdot t = -\sum_{\tau} \epsilon_{k}\epsilon_{k'} \left( f_{3,\tau}^* f_{3,\tau'} - \frac{1}{4} \delta_{\tau,\tau'} \right). \]  

(S-63)

This expression recovers the canonical one Eq. (S-7) with \( q = 3 \), if we substitute \( f_{3,\tau}^* f_{3,\tau'} = -f_{\tau}^* f_{\tau} \), in which these last fermionic fields obey the constraint \( \sum_{\tau} f_{\tau}^* f_{\tau} = 3 \).

S-V. NUMERICAL RENORMALIZATION GROUP APPROACH

In this section we provide some details regarding the numerical renormalization group (NRG) calculations. Our NRG computations were performed with a discretization parameter \( \Lambda = 2 \), while exploiting the full SU(4) symmetry of the Hamiltonian (with 300 kept multiplets corresponding to \( \approx 4200 \) kept states in each iteration). These calculations were checked against computations performed by using the four U(1)×U(1)×SU(2)×SU(2) symmetries with 3000 multiplets kept. [15]

The value of the Kondo temperature [16] depends on its precise definition and the quantity from which it is extracted. To determine it, we computed the spectral function of the \( d \)-level, \( \mathcal{A}(\omega) = -\text{Im} G_{d,d}^x(\omega)/\pi \), and defined
$T_{K,SU(4)}^{NRG}$ as the half-width at half maximum of $A(\omega)$. This definition takes into account that the peak of the spectral function is generally not centered at the Fermi energy (excepting the electron-hole symmetrical point, $\epsilon_d = -3U/2$) and shifts gradually as a function of $\epsilon_d$. This definition is found to agree perfectly with the analytical expression, Eq. (5), up to a prefactor of the order one, $T_{K,SU(4)}^{NRG} \approx 0.94T_{K,SU(4)}$. In the NRG calculations, we changed $\Delta$ and $\epsilon_d$ simultaneously such that the size of the coupling $|\epsilon_0J_0|$ remained constant, and extracted the $\epsilon_d$ dependent prefactor this way.

The various fixed points discussed in the main text can be identified from the flow diagram of the NRG levels [17, 18], shown in Fig. S-5 for parameters close to the mixed valence regime, $\epsilon_d \approx \{0, -U\}$. For $\epsilon_d = -U$, the ground state of the isolated dot ($\Delta = 0$) would be 10 fold degenerate. For finite but small enough $\Delta$, valence fluctuations mix these 10 states by quantum fluctuations and a valence fluctuation regime emerges. A transition to this valence fluctuation regime is expected therefore for $\epsilon_d \approx -U$ as soon as the energy drops below $U$. The interaction $U$ being relatively large, this mixed valence regime and the corresponding 10-fold ground state degeneracy is almost immediately observed in the finite size spectrum, after only a few iterations. Apart from the curious ground state degeneracy, the observed mixed valence spectrum displays a regular Fermi liquid-like structure, characteristic of the SU(4)-symmetrical mixed valence fixed point.

For small $\Delta$’s, (and $|\epsilon_d|, |\epsilon_d + U| > \Delta$) a second transition can be observed in the spectrum, occurring approximately at the charging energy, $\Delta E_0 \approx \text{min}_q \left| \epsilon_q \right|$, slightly shifted by mixed valence fluctuations. The ground state is gradually split and its degeneracy is partially removed when the energy drops below $\Delta E_0$, and the local moment regime develops.

Getting closer to the valence transition point (i.e., for $|\epsilon_d|, |\epsilon_d + U| < \Delta$), the Kondo temperature gradually increases, the local moment regime shrinks, and at some point it disappears. This happens on the right panel of Fig. S-5, e.g., where one enters the Fermi liquid state directly from the mixed valence region.

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