First-order methods for problems with $O(1)$ functional constraints can have almost the same convergence rate as for unconstrained problems

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inquired only by the matrix-vector multiplication $A(\cdot)$ and $A^\top(\cdot)$. Notice $\{x : Ax = b\} = \{x : Ax \leq b, -Ax \leq -b\}$. In addition, if $\nabla f(x) + A^\top y = 0$, then $\nabla f(x) + A^\top y^+ - A^\top y^- = 0$, where $y^+ \geq 0$ and $y^- \geq 0$ denote the positive and negative parts of $y$. Hence, if the linear-equality constrained problem has a KKT point, then so does the equivalent linear-inequality constrained problem. Therefore, the lower bound in [32] also applies to the inequality constrained problem (1), if $g$ can be accessed only through its function value and derivative. However, for the special case of $g \equiv 0$ or $m = 0$, an accelerated proximal gradient method [23, 31] can achieve a complexity result $O\left(\sqrt{\kappa} \log \frac{1}{\varepsilon}\right)$ to produce an $\varepsilon$-optimal solution of (1), when $f$ is strongly convex. Here, $\kappa$ denotes the condition number.

The worst-case instance constructed in [32] relies on the condition that $m$ is in the same or higher order of $\frac{1}{\sqrt{\varepsilon}}$. For the case with $m = o\left(\frac{1}{\sqrt{\varepsilon}}\right)$, the lower bound $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ may not hold any more. Examples of (1) with small $m$ include the Neyman-Pearson classification problem [33], fairness-constrained classification [42], and the risk-constrained portfolio optimization [11]. Therefore, we pose the following question while solving a strongly-convex problem in the form of (1):

Given $\varepsilon > 0$, can an FOM achieve a better complexity result than $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ to produce an $\varepsilon$-optimal solution of (1) when $m = o\left(\frac{1}{\sqrt{\varepsilon}}\right)$, or even achieve $\tilde{O}\left(\sqrt{\kappa}\right)$ when $m = O(1)$?

Here, an FOM for (1) only uses the function value and derivative information of $f$ and $g$ and also the proximal mapping of $h$ and its multiples, and $\tilde{O}$ suppresses a polynomial of $|\log \varepsilon|$. We will give an affirmative answer to the above question.

1.1 Algorithmic framework

The FOM that we will design and analyze is based on the inexact augmented Lagrangian method (iALM). The classic AL function of (1) is:

$$L_\beta(x, z) = F(x) + \frac{\beta}{2} \left\|g(x) + \frac{z}{\beta}\right\|^2 - \frac{\|z\|^2}{2\beta^2},$$

(2)

where $z$ is the multiplier vector, and $|a|_+$ takes the component-wise positive part of a vector $a$. The pseudocode of a first-order iALM is shown in Algorithm 1. Notice that $L_\beta$ is strongly convex about $x$ and concave about $z$. Hence, we can directly apply the accelerated proximal gradients in [23, 31] to solve each $x$-subproblem. However, that way can only give a complexity result of $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ as shown in [37], regardless of the value of $m$. To have a better overall complexity, we will design a new cutting-plane based FOM to solve each $x$-subproblem by utilizing the condition $m = O(1)$ or $m = o\left(\frac{1}{\sqrt{\varepsilon}}\right)$.

1.2 Related works

We briefly mention some existing works that also study the complexity of FOMs for solving functional constrained problems.

By using the ordinary Lagrangian function, [28, 29] analyze a dual subgradient method for general convex problems. The method needs $O(\varepsilon^{-2})$ subgradient evaluations to produce an $\varepsilon$-optimal solution (see the definition in Eq. (6) below). For a smooth problem, [27] studies the complexity of an inexact dual gradient (IDG) method. Suppose that an optimal FOM is applied to each outer-subproblem of IDG. Then to produce
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**Algorithm 1:** First-order inexact augmented Lagrangian method for (1)

```plaintext
1 Initialization: choose \( x^0, z^0 \), and \( \beta_0 > 0 \)
2 for \( k = 0, 1, \ldots \) do
3     Apply a first-order method to find \( x^{k+1} \) as an approximate solution of \( \min_{x} L_{\beta_k}(x, z^k) \).
4     Update \( z \) by \( z^{k+1} = [z^k + \beta_k g(x^{k+1})]_+ \).
5     Choose \( \beta_{k+1} \geq \beta_k \).
6     if a stopping condition is satisfied then
7         Output \((x^{k+1}, z^{k+1})\) and stop
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an \( \varepsilon \)-optimal solution, IDG needs \( O(\varepsilon^{-\frac{1}{2}}) \) gradient evaluations when the problem is convex, and the result can be improved to \( O(\varepsilon^{-\frac{1}{2}} \log \varepsilon) \) when the problem is strongly convex. For convex problems, the primal-dual FOM proposed in [41] achieves an \( O(\varepsilon^{-1}) \) complexity result to produce an \( \varepsilon \)-optimal solution, and the same-order complexity result has also been established in [38]. Based on a previous work [15] for affinely constrained problems, [24] gives a modified first-order iALM for solving convex cone programs. The overall complexity of the modified method is \( O(\varepsilon^{-1} \log \varepsilon) \) to produce an \( \varepsilon \)-KKT point (see Definition 1 below). A similar result has also been shown in [3] for convex conic programs. A proximal iALM is analyzed in [17]. By a linearly-convergent first-order subroutine for primal subproblems, [17] shows that \( O(\varepsilon^{-1}) \) calls to the subroutine are needed for convex problems and \( O(\varepsilon^{-\frac{1}{2}}) \) for strongly convex problems, to achieve either an \( \varepsilon \)-optimal or an \( \varepsilon \)-KKT point. In terms of function value and derivative evaluations, the complexity result is \( O(\varepsilon^{-1} \log \varepsilon) \) for the convex case and \( O(\varepsilon^{-\frac{1}{2}} \log \varepsilon) \) for the strongly-convex case. Complexity results of FOMs for nonconvex problems with functional constraints have also been established, e.g., [7, 8, 14, 18-20, 25, 35].

To produce an \( \varepsilon \)-KKT point, the best-known result is \( \tilde{O}(\varepsilon^{-\frac{1}{2}}) \) when the constraints are convex [18, 20] and \( \tilde{O}(\varepsilon^{-3}) \) when the constraints are nonconvex and satisfy a certain regularity condition [20].

On solving general nonlinear constrained problems, FOMs have also been proposed under the framework of the level-set method [1, 21, 22]. For convex problems, the level-set based FOMs can also achieve an \( O(\varepsilon^{-1}) \) complexity result to produce an \( \varepsilon \)-optimal solution. However, to obtain \( \tilde{O}(\varepsilon^{-\frac{1}{2}}) \), they require strong convexity of both the objective and the constraint functions.

Under the condition of strong duality, (1) can be equivalently formulated as a non-bilinear saddle-point (SP) problem. In this case, one can apply any FOM that is designed for solving non-bilinear SP problems. The work [12] generalizes the primal-dual method proposed in [9] from the bilinear SP case to the non-bilinear case. If the underlying SP problem is convex-concave, [12] establishes an \( O(\varepsilon^{-1}) \) complexity result to guarantee \( \varepsilon \)-duality gap. When the problem is strongly-convex-linear, the result can be improved to \( O(\varepsilon^{-\frac{2}{3}}) \). Notice that both results apply to the equivalent ordinary-Lagrangian-based SP problem of (1). By the smoothing technique, [13] gives an FOM (with both deterministic and stochastic versions) for solving non-bilinear SP problems. To ensure an \( \varepsilon \)-duality gap of a strongly-convex-concave problem, the method requires \( \tilde{O}(\varepsilon^{-\frac{1}{4}}) \) primal first-order oracles and \( \tilde{O}(\varepsilon^{-1}) \) dual first-order oracles. While applied to the functional constrained problem (1), the method in [13] can obtain an \( \varepsilon \)-optimal solution by \( O(\varepsilon^{-\frac{1}{2}} \log \varepsilon) \) evaluations on \( f, \nabla f, g, \) and \( J_g \). FOMs for solving the more general variational inequality (VI) problem can also be applied to (1), such as the mirror-prox method in [30], the hybrid extragradient method in [26], and the accelerated method in [10]. All of the three methods can have an \( O(\varepsilon^{-1}) \) complexity result by assuming smoothness and/or monotonicity of the involved operator.
1.3 Contributions

On solving a functional constrained strongly-convex problem, none of the existing works about FOMs (such as those we mentioned previously) could obtain a complexity result better than $O(\varepsilon^{-\frac{1}{2}})$. Without specifying the regime of $m$, the task is impossible. We show that when $m = O(1)$ in (1), an FOM can achieve almost the same-order complexity result (with a difference of at most a polynomial of $|\log \varepsilon|$) as for solving an unconstrained problem. When $m = o(\varepsilon^{-\frac{1}{2}})$, we show that a complexity result better than $O(\varepsilon^{-\frac{1}{2}})$ can be obtained. The key step in the design of our algorithm is to formulate each primal subproblem into an equivalent SP problem. The SP formulation is strongly concave about the dual variable, and the strong concavity enables the generation of a cutting plane while searching for an approximate dual solution of the SP problem. Since there are $m$ dual variables, we can apply a cutting-plane method to efficiently find an approximate dual solution when $m = O(1)$ or $m = o(\varepsilon^{-\frac{1}{2}})$. In addition, we extend the idea of a cutting-plane based FOM to the convex and nonconvex cases. For these two cases, we show that an FOM for problems with $O(1)$ functional constraints can also achieve almost the same-order complexity result as for solving unconstrained problems.

1.4 Assumptions and notation

Throughout our analysis for strongly-convex problems, we make the following assumptions.

**Assumption 1 (smoothness)** $f$ is $L_f$-smooth, i.e., $\nabla f$ is $L_f$-Lipschitz continuous. In addition, each $g_i$ is smooth, and the Jacobian matrix $J_g = [\nabla g_1^T; \ldots; \nabla g_m^T]$ is $L_g$-Lipschitz continuous.

**Assumption 2 (bounded domain and convexity)** The domain of $h$ is bounded with a diameter $D_h = \max_{x,y \in \text{dom}(h)} \|x - y\| < \infty$. The functions $h$ and $\{g_i\}$ are all convex.

The above two assumptions imply the boundedness of $g$ and $J_g$ on $\text{dom}(h)$, we use $G$ and $B_g$ respectively for their bounds, namely,

$$ G = \max_{x \in \text{dom}(h)} \|g(x)\|, \quad B_g = \max_{x \in \text{dom}(h)} \|J_g(x)\|. \quad (3) $$

**Assumption 3 (strong convexity)** The smooth function $f$ is $\mu$-strongly convex with $\mu > 0$.

**Assumption 4 (strong duality)** There is a primal-dual solution $(x^*, z^*)$ satisfying the KKT conditions of (1), i.e., $0 \in \partial F(x^*) + J_g(x^*)^T z^*$, $z^* \succeq 0$, $g(x^*) \preceq 0$, $g(x^*)^T z^* = 0$.

When Assumption 4 holds, it is easy to have (cf. [39, Eqn. 2.4])

$$ F(x) - F(x^*) + \langle z^*, g(x) \rangle \geq 0, \forall x \in \text{dom}(h). \quad (4) $$

**Notation.** For a real number $a$, we use $\lfloor a \rfloor$ to denote the smallest integer that is no less than $a$ and $\lceil a \rceil$ the smallest nonnegative integer that is no less than $a$. $B_\delta(x)$ denotes a ball with radius $\delta$ and center $x$. If $x = 0$, we simply use $B_\delta$. We define $B_\delta^+$ as the intersection of $B_\delta$ with the nonnegative orthant, so in the $n$-dimensional space, $B_\delta^+ = B_\delta \cap \mathbb{R}_+^n$. We use $V_m(\delta)$ for the volume of $B_\delta$ in the $m$-dimensional space. $[n]$ denotes the set $\{1, \ldots, n\}$. Given a closed convex set $X \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, we define $\text{dist}(x, X) = \min_{y \in X} \|y - x\|$. We use $O$, $\Theta$, and $o$ with standard meanings, while in the complexity result statement, $\tilde{O}$ has a similar meaning as $O$ but suppresses a polynomial of $|\log \varepsilon|$ for a given error tolerance $\varepsilon > 0$. 


Definition 1 (ε-KKT point) Given ε > 0, a point $\bar{x} \in \text{dom}(h)$ is called an ε-KKT point of (1) if there is $\bar{z} \geq 0$ such that
\[
\text{dist}(0, \partial_x L_0(\bar{x}, \bar{z})) \leq \varepsilon, \quad \| [g(\bar{x})]_+ \| \leq \varepsilon, \quad \sum_{i=1}^m |\bar{z}_i g_i(\bar{x})| \leq \varepsilon,
\]
where $L_0(x, z) = F(x) + z^\top g(x)$ is the ordinary Lagrangian function of (1).

By the convexity of $F$ and each $g_i$, and also Assumption 4, one can easily show that an ε-KKT point of (1) must be an $O(\varepsilon)$-optimal solution, where we call a point $\bar{x} \in \text{dom}(h)$ as an ε-optimal solution of (1) if
\[
|F(\bar{x}) - F(x^*)| \leq \varepsilon, \quad \|[g(\bar{x})]_+\| \leq \varepsilon.
\]

1.5 Outline

The rest of the paper is organized as follows. In section 2, we review an adaptive accelerated proximal gradient method (APG) and give the convergence rate of the iALM. In section 3, we design new FOMs (that are better than directly applying the APG) for solving primal subproblems in the iALM. Overall complexity results are shown in section 4. Extensions to convex and nonconvex cases are given in section 5. Numerical experiments are conducted in section 6 to demonstrate our theory, and section 7 concludes the paper.

2 An adaptive optimal FOM and convergence rate of iALM

In this section, we give an adaptive optimal FOM that will be used as a subroutine in our algorithm. Also, we establish the convergence rate of the iALM to produce an approximate KKT point.

2.1 An adaptive optimal FOM for strongly-convex composite problems

Consider the problem
\[
\min_{x \in \mathbb{R}^n} P(x) := \psi(x) + r(x),
\]
where $\psi$ is a differentiable $\mu_\psi$-strongly convex function with $L_\psi$-Lipschitz continuous gradient, and $r$ is a closed convex function. Several optimal FOMs have been given in the literature for solving (7), e.g., in [23,31]. In this paper, we choose the adaptive APG in [23], and we rewrite it in Algorithm 2 with a few modified steps for our purpose to produce near-stationary points.

The results in the next theorem are from Theorem 1 of [23].

Theorem 1 The generated sequence $\{x^k\}_{k \geq 0}$ by Algorithm 2 satisfies
\[
P(x^{k+1}) - P(x^*) \leq \left(1 - \sqrt{\frac{\mu_\psi}{\gamma_1 L_\psi}}\right)^{k+1} \left(P(x^0) - P(x^*) + \frac{\mu_\psi}{2} \|x^0 - x^*\|^2\right), \forall k \geq 0,
\]
where $x^*$ is the optimal solution of (7).

By the above theorem, we can easily bound the distance of $\bar{x}_k$ to stationarity for each $k$. 
Algorithm 2: An adaptive optimal first-order method for (7): \( \bar{x} = \text{APG}(\psi, l, L_{\text{min}}, \bar{\varepsilon}, \gamma_1, \gamma_2) \)

1. **Input:** minimum Lipschitz \( L_{\text{min}} > 0 \), increase rate \( \gamma_1 > 1 \), decrease rate \( \gamma_2 \geq 1 \), and error tolerance \( \bar{\varepsilon} > 0 \).
2. **Prestep:** choose any \( \bar{y} = y^0 \in \text{dom}(r) \) and let \( \bar{L} = L_{\text{min}}/\gamma_1 \)
3. **repeat**
   4. \( \bar{L} \leftarrow \gamma_1 \bar{L} \) and let \( \bar{x} = \text{arg min}_x (\nabla \psi(\bar{y}), x) + \frac{\bar{L}}{2} \|x - \bar{y}\|^2 + r(x) \)
   5. **until** \( \psi(\bar{x}) \leq \psi(\bar{y}) + (\nabla \psi(\bar{y}), \bar{x} - \bar{y}) + \frac{\bar{L}}{2} \|\bar{x} - \bar{y}\|^2 \)

6. **Initialization:** let \( x^{-1} = x^0 = \bar{x} \), \( L_0 = \text{max}\{L_{\text{min}}, \bar{L}/\gamma_2\} \), and \( \alpha_{-1} = 1 \)
7. **for** \( k = 0, 1, \ldots \) **do**
   8. \( \bar{L} \leftarrow L_k/\gamma_1 \)
   9. **repeat**
   10. \( \bar{L} \leftarrow \gamma_1 \bar{L} \), \( \alpha_k \leftarrow \sqrt{\mu_\psi/L} \), and \( \bar{y} \leftarrow x_k + \frac{\alpha_k(1 - \alpha_k + 1)}{\alpha_k + 1} (x_k - x_{k-1}) \)
   11. let \( \bar{x} = \text{arg min}_x (\nabla \psi(\bar{y}), x) + \frac{\bar{L}}{2} \|x - \bar{y}\|^2 + r(x) \)
   12. **until** \( \psi(\bar{x}) \leq \psi(\bar{y}) + (\nabla \psi(\bar{y}), \bar{x} - \bar{y}) + \frac{\bar{L}}{2} \|\bar{x} - \bar{y}\|^2 \)
   13. \( \hat{L} \leftarrow \bar{L}/\gamma_1 \)
   14. **repeat**
   15. increase \( L \leftarrow \gamma_2 \bar{L} \)
   16. let \( \hat{x} = \text{arg min}_x (\nabla \psi(\bar{x}), x) + \frac{\bar{L}}{2} \|x - \hat{x}\|^2 + r(x) \); \( > \) modified step to guarantee near-stationarity at \( \hat{x} \)
   17. **until** \( \psi(\hat{x}) \leq \psi(\bar{x}) + (\nabla \psi(\bar{x}), \hat{x} - \bar{x}) + \frac{\bar{L}}{2} \|\hat{x} - \bar{x}\|^2 \)
   18. set \( x^{k+1} = \bar{x}, \hat{x}^{k+1} = \hat{x} \), and \( L_{k+1} = \text{max}\{L_{\text{min}}, \hat{L}/\gamma_2\} \);
   19. **if** dist \((0, \partial P(\hat{x})) \leq \bar{\varepsilon} \) **then**
   20. return \( \hat{x} \) and stop.

**Theorem 2** The generated sequence \( \{\hat{x}^k\}_{k \geq 0} \) satisfies

\[
\text{dist}(0, \partial P(\hat{x}^{k+1})) \leq \left( \sqrt{\gamma_1 L_\psi} + \frac{L_\psi}{\sqrt{L_{\text{min}}}} \right) \sqrt{2(P(x^0) - P(x^*)) + \mu_\psi \|x^0 - x^*\|^2 \left( 1 - \frac{\mu_\psi}{\gamma_1 L_\psi} \right)^{k+1/2}}, \quad \forall k \geq 0.
\]

**Proof.** First notice that if \( \hat{L} \geq L_\psi \), it must hold \( \psi(\hat{x}) \leq \psi(\bar{x}) + (\nabla \psi(\bar{x}), \hat{x} - \bar{x}) + \frac{\hat{L}}{2} \|\hat{x} - \bar{x}\|^2 \), and when this inequality holds, we have (cf. [40, Lemma 2.1]) \( P(\bar{x}) - P(\hat{x}) \geq \frac{\hat{L}}{2} \|\bar{x} - \bar{x}\|^2 \). Since \( P(\bar{x}) - P(\hat{x}) \leq P(\bar{x}) - P(x^*) \), we have \( \frac{\hat{L}}{2} \|\bar{x} - \bar{x}\|^2 \leq P(\bar{x}) - P(x^*) \), which together with the fact \( \hat{L} \geq L_{\text{min}} \) implies

\[
\frac{\hat{L}^2}{2} \|\bar{x} - \bar{x}\|^2 \leq \hat{L}(P(\bar{x}) - P(x^*)), \quad \|\bar{x} - \bar{x}\|^2 \leq \frac{2}{L_{\text{min}}}(P(\bar{x}) - P(x^*)).
\]

In addition, from the optimality condition of \( \bar{x} \), it follows \( 0 \in \nabla \psi(\bar{x}) + \hat{L}(\bar{x} - \bar{x}) + \partial r(\bar{x}) \), and thus

\[
\text{dist}(0, \partial P(\hat{x})) \leq \|\nabla \psi(\bar{x}) - \nabla \psi(\bar{x})\| + \hat{L} \|\bar{x} - \bar{x}\| \leq (L_\psi + \hat{L}) \|\bar{x} - \bar{x}\|.
\]

By (9) and (10), we have

\[
\text{dist}(0, \partial P(\hat{x})) \leq (L_\psi + \hat{L}) \|\hat{x} - \bar{x}\| \leq \sqrt{2(P(\bar{x}) - P(x^*))} \left( \sqrt{L} + \frac{L_\psi}{\sqrt{L_{\text{min}}}} \right).
\]

Therefore, the desired result follows from (8), the fact \( \hat{L} \leq \gamma_1 L_\psi \), and the above inequality with \( \bar{x} = \hat{x}^{k+1} \) and \( \bar{x} = x^{k+1} \). \( \square \)
From [4, Theorem 3.1], we have

\[ P(x^0) - P(x^*) \leq \frac{\gamma_1 L_\psi \| y^0 - x^* \|}{2}. \]  

(11)

Hence, we can obtain the following complexity result by Theorem 2 together with (11).

**Corollary 1** Assume that \( \text{dom}(r) \) is bounded with a diameter \( D_r = \max_{x_1, x_2 \in \text{dom}(r)} \| x_1 - x_2 \| \). Given \( \varepsilon > 0, \gamma_1 > 1, \gamma_2 \geq 1 \) and \( L_{\text{min}} > 0 \), Algorithm 2 needs at most \( T \) evaluations on the objective value of \( \psi \) and the gradient \( \nabla \psi \) to produce \( \tilde{x} \) such that \( \text{dist}(0, \partial P(\tilde{x})) \leq \varepsilon \), where

\[ T = \left( 1 + \lceil \log_{\gamma_1} \frac{L_\psi}{L_{\text{min}}} \rceil + 1 \right) \left( 1 + 2 \left[ 2 \sqrt{\frac{\gamma_1 L_\psi}{L_{\text{min}}}} \log \left( \frac{L_\psi}{L_{\text{min}}} \sqrt{2 \gamma_1 L_\psi + \mu_\psi} \right) \right] \right). \]

**Proof.** Since \( \text{dom}(r) \) has a diameter \( D_r \), we have from Theorem 2 and (11) that

\[ \text{dist}(0, \partial P(\tilde{x}^{k+1})) \leq D_r \left( \sqrt{\frac{\gamma_1 L_\psi}{L_{\text{min}}}} + \frac{L_\psi}{L_{\text{min}}} \right) \sqrt{2 \gamma_1 L_\psi + \mu_\psi} \left( 1 - \sqrt{\frac{\mu_\psi}{\gamma_1 L_\psi}} \right)^{k+1}, \forall k \geq 0. \]

Hence, if \( k + 1 \geq K \), then \( \text{dist}(0, \partial P(\tilde{x}^{k+1})) \leq \varepsilon \), where

\[ K = \left[ 2 \log \left( \frac{L_\psi}{L_{\text{min}}} \sqrt{\frac{\gamma_1 L_\psi}{L_{\text{min}}} + \mu_\psi} \right) \right] + \log(1 - \frac{\mu_\psi}{\gamma_1 L_\psi} - 1), \]

namely, after at most \( K \) iterations, the algorithm will produce a point \( \tilde{x} \) satisfying \( \text{dist}(0, \partial P(\tilde{x})) \leq \varepsilon \).

Notice that the conditions in Lines 5, 11, and 17 of Algorithm 2 will hold if \( L \geq L_\psi \) and \( \tilde{L} \geq L_\psi \). Hence, every iteration will evaluate the objective value of \( \psi \) and the gradient \( \nabla \psi \) at most \( 2(1 + \lceil \log_{\gamma_1} \frac{L_\psi}{L_{\text{min}}} \rceil) \) times. Now using the fact \( \log(1 - a)^{-1} \geq a, \forall 0 < a < 1 \), we obtain the desired result by also counting the objective and gradient evaluations to obtain \( x^0 \).

\[ \square \]

### 2.2 Convergence rate of iALM

The next lemma is from Eq. (3.20) and the proof of Lemma 7 of [37].

**Lemma 1** Let \( \{(x^k, z^k)\} \) be generated from Algorithm 1 with \( z^0 = 0 \). Suppose

\[ L_{\beta_k}(x^{k+1}, z^k) \leq \min_{x} L_{\beta_k}(x, z^k) + e_k, \forall k = 0, 1, \ldots, \]

(12)

for an error sequence \( \{e_k\} \). Then

\[ \|z^k\|^2 \leq 4\|z^*\|^2 + 4 \sum_{t=0}^{k-1} \beta_t e_t, \text{ and } \|z^k\| \leq 2\|z^*\| + \sqrt{2 \sum_{t=0}^{k-1} \beta_t e_t}, \forall k \geq 1. \]

(13)

By this lemma and also the strong convexity of \( F \), we can show the following result.

**Lemma 2** Let \( \{(x^k, z^k)\} \) be generated from Algorithm 1 with \( z^0 = 0 \). If \( \text{dist}(0, \partial_x L_{\beta_k}(x^{k+1}, z^k)) \leq \varepsilon_k, \forall k \geq 0 \) for a sequence \( \{e_k\} \), then

\[ \|z^k\|^2 \leq 4\|z^*\|^2 + 4 \sum_{t=0}^{k-1} \beta_t \frac{e_t^2}{p}, \text{ and } \|z^k\| \leq 2\|z^*\| + \sqrt{2 \sum_{t=0}^{k-1} \beta_t \frac{e_t^2}{p}}, \forall k \geq 1. \]

(14)
Proof. Let $x^{k+1}$ be the minimizer of $L_{\beta_k}(x, z^k)$ about $x$. Then $0 \in \partial_x L_{\beta_k}(x^{k+1}, z^k)$. Also, it follows from dist$(0, \partial_x L_{\beta_k}(x^{k+1}, z^k)) \leq \varepsilon_k$ that there is $v \in \partial_x L_{\beta_k}(x^{k+1}, z^k)$ and $\|v\| \leq \varepsilon_k$. Since $F$ is $\mu$-strongly convex, $L_{\beta_k}(x, z^k)$ is also $\mu$-strongly convex about $x$. Then we have $\langle v, x^{k+1} - x^{k+1} \rangle \geq \mu \|x^{k+1} - x^{k+1}\|^2$, which together with the Cauchy-Schwarz inequality gives $\|x^{k+1} - x^{k+1}\| \leq \frac{\varepsilon_k^2}{\mu}$. Now by the convexity of $L_{\beta_k}(\cdot, z^k)$, it holds

$$L_{\beta_k}(x^{k+1}, z^k) - L_{\beta_k}(x^{k+1}, z^k) \leq \langle v, x^{k+1} - x^{k+1} \rangle \leq \frac{\varepsilon_k^2}{\mu},$$

and thus we have that (12) holds with $\epsilon_k = \frac{\varepsilon_k^2}{\mu}$. Therefore, (14) follows from (13). \qed

**Theorem 3 (convergence rate of IALM)** Let $(\{x^k, z^k\})$ be generated from Algorithm 1 with $z^0 = 0$. Suppose $\beta_k = \beta_0\sigma^k$, $\forall k \geq 0$ for some $\sigma > 1$ and $\beta_0 > 0$, and dist$(0, \partial_x L_{\beta_k}(x^{k+1}, z^k)) \leq \bar{\varepsilon}$, $\forall k \geq 0$ for a positive number $\bar{\varepsilon}$. Then

$$\|\|g(x^{k+1})\|\| \leq \frac{4\|x^k\|}{\beta_0\sigma^k} + \frac{\beta_0\sigma^k}{\sqrt{\mu\sigma^k}},$$

(15)

$$\sum_{i=1}^m |z_i^{k+1} g_i(x^{k+1})| \leq \frac{\bar{\varepsilon}^2(8\sigma+1)}{2\sqrt{\mu}} + \frac{\bar{\varepsilon}^2(8\sigma+1)}{4\mu(\sigma-1)}.$$  

(16)

Proof. From the update of $z$, it follows that $g_i(x^{k+1}) \leq \frac{x_i^{k+1} - x_i^k}{\beta_k}$ for each $i \in [m]$, and thus by (14), we have

$$\|\|g(x^{k+1})\|\| \leq \frac{\|x^{k+1} - x^k\|}{\beta_k} \leq \frac{4\|x^k\|^2 + \sqrt{2\sum_{i=1}^{k-1} \beta_i \frac{\|x_i^{k+1}\|}{\beta_i}} + \sqrt{2\sum_{i=1}^{k} \beta_i \frac{\|x_i^k\|^2}{\beta_i}}}{\beta_k}.$$  

Plugging into the above inequality $\varepsilon_i = \bar{\varepsilon}$, $\forall t \geq 0$ and $\beta_k = \beta_0\sigma^k$, we obtain the inequality in (15).

Furthermore, for each $i \in [m]$, we have

$$|z_i^{k+1} g_i(x^{k+1})| \leq \frac{1}{\beta_k} |z_i^{k+1} (z_i^{k+1} - z_i^k)| \leq \frac{1}{\beta_k} \left((z_i^{k+1})^2 + \frac{\|z_i^k\|^2}{8}\right),$$

and thus $\sum_{i=1}^m |z_i^{k+1} g_i(x^{k+1})| \leq \frac{1}{\beta_k} \left(\|z^k\|^2 + \frac{\|x^k\|^2}{8}\right)$. Now we obtain the result in (16) by plugging the first inequality in (14). \qed

We make a few remarks here. Given $\varepsilon > 0$, choose $\bar{\varepsilon} > 0$ such that $\frac{\bar{\varepsilon}^2(8\sigma+1)}{2\sqrt{\mu}} < \varepsilon$ in Theorem 3. Notice that $\partial_x L_{\beta_k}(x^{k}, z^k) = \partial_x L_{\beta_k}(x^{k+1}, z^k)$. Hence, from (15) and (16), it follows that to ensure $x^{k+1}$ to be an $\varepsilon$-KKT point, we need $\beta_0\sigma^k = \Theta(\frac{1}{\varepsilon})$ and solve $k = \Theta\left(\log_{\sigma} \frac{1}{\varepsilon}\right)$ $\times$-subproblems. Since the smooth part of $L_{\beta_k}(\cdot, z^k)$ has $\Theta(\beta_k)$-Lipschitz continuous gradient, it needs $O(\sqrt{\frac{1}{\varepsilon}})$ proximal gradient steps if we directly apply Algorithm 2. This way, we can guarantee an $\varepsilon$-KKT point with a total complexity $O(\sqrt{\frac{1}{\varepsilon}}) \log \varepsilon$), where $\kappa$ denotes the condition number in some sense. This complexity result has been established in a few existing works, e.g., [17, 24]. It is worse by an order of $\sqrt{\frac{1}{\varepsilon}}$ than the complexity result in Corollary 1 for the unconstrained case. Generally, we cannot improve it any more because the result matches with the lower bound given in [32].

In the rest of the paper, we show that in some special cases, a better complexity can be obtained. When $m = O(1)$, we show that we can achieve a complexity result $O(\sqrt{\kappa}) \log \varepsilon^{\frac{3}{2}}$, which is in almost the same order as the optimal result for the unconstrained case. For a general $m$, we can achieve $O(m\sqrt{\kappa}) \log \varepsilon^{\frac{1}{2}}$, which is better than $O(\sqrt{\frac{1}{\varepsilon}} \log \varepsilon^{\frac{3}{2}})$ in the regime of $m = o(\sqrt{\frac{1}{\varepsilon}})$. 


3 Better first-order methods for x-subproblems

When \( m \) is small in (1), we do not directly apply Algorithm 2 to solve the x-subproblem \( \min_x \mathcal{L}_{\beta_k}(x, z^k) \) in Algorithm 1. Instead, we design new and better FOMs that use Algorithm 2 as a subroutine in the framework of a cutting-plane method. Our key idea is to reformulate the x-subproblem into a strongly-convex-strongly-concave saddle-point problem, which has a unique primal-dual solution. For the saddle-point formulation, we first find a sufficient-accurate dual solution by a cutting-plane based FOM. Then we find a sufficient-accurate primal solution based on the obtained approximate dual solution.

Below, we give more precise description on how to design better FOMs. Given \( z \geq 0 \), let
\[
\theta(x) = g(x) + z.
\]

From (3) and the Mean-Value Theorem, it follows that \( \theta \) is \( B_g \)-Lipschitz continuous, namely,
\[
\|\theta(x_1) - \theta(x_2)\| \leq B_g \|x_1 - x_2\|, \forall x_1, x_2.
\]

With \( \theta \), we can rewrite the problem \( \min_x \mathcal{L}_{\beta}(x, z) \) into
\[
\min_{x \in \mathbb{R}^n} \phi(x) := F(x) + \frac{\beta}{2} \|\theta(x)\|_+^2.
\]

Notice that \( \frac{1}{2}\|\theta(x)\|_+^2 = \max_{y \geq 0} \{ y^T \theta(x) - \frac{1}{2} \|y\|^2 \} \) and \( y = [\theta(x)]_+ \) reaches the maximum. We re-write (18) into
\[
\min_{x \in \mathbb{R}^n} \max_{y \geq 0} \phi(x, y) := F(x) + \beta (y^T \theta(x) - \frac{1}{4} \|y\|^2).
\]

Define
\[
d(y) = \min_{x \in \mathbb{R}^n} \phi(x, y), \quad \text{and} \quad \bar{y} = \arg \max_{y \geq 0} d(y).
\]

Notice that \( d \) is \( \beta \)-strongly concave, so \( \bar{y} \) is the unique maximizer of \( d \). Also, for a given \( y \geq 0 \), define \( x(y) \) as the unique minimizer of \( \phi(\cdot, y) \), i.e.,
\[
x(y) = \arg \min_{x} \phi(x, y).
\]

In our algorithm design, we first find an approximate solution \( \bar{y} \) of \( \max_{y \geq 0} d(y) \) and then find an approximate solution \( \bar{x} \) of \( \min_x \phi(x, \bar{y}) \). By controlling the approximation errors, we can guarantee \( \bar{x} \) to be a near-stationary point of \( \phi \). On finding \( \bar{y} \), we use a cutting-plane method. Since \( d \) is strongly concave, a cutting plane can be generated at a query point \( y \geq 0 \), though we can only have an estimate of \( \nabla d(y) \) by approximately solving \( \min_x \phi(x, y) \). It is unclear whether the same idea works if we directly play with the augmented (or ordinary) Lagrangian dual function because it is not strongly concave.

3.1 Preparatory lemmas

We first establish a few lemmas. The next lemma indicates that the complexity of solving \( \min_x \phi(x, y) \) by the APG can be independent of \( \beta \), if \( \|y\| \) is in the same order of \( \|\bar{y}\| \). This fact is the key for us to design a better FOM for solving ALM subproblems.
Lemma 3 Suppose \( \bar{x} \) is the minimizer of \( \phi \) in (18). Then \( \bar{y} = [\theta(\bar{x})]_+ \) is the solution of max_{\gamma \geq 0} \( d(y) \), and \((\bar{x}, \bar{y})\) is the saddle point of \( \Phi \). In addition, let \((x^*, z^*)\) be the point in Assumption 4. Then

\[
\|y\| = \|\theta(\bar{x})\| \leq \frac{2\|x^*\| + \|z^*\|}{\beta}.
\]

Proof. It is easy to see that \( \bar{y} = [\theta(\bar{x})]_+ \) is the solution of max_{\gamma \geq 0} \( d(y) \) and \((\bar{x}, \bar{y})\) is a saddle point of \( \Phi \); cf. [34, Corollary 37.3.2]. We only need to show (22). Since \( \bar{x} \) is the minimizer of \( \phi \), it holds

\[
F(\bar{x}) + \frac{\beta}{2} \|\theta(\bar{x})\| + \|z^*\| + \langle \theta(\bar{x}), y \rangle \leq \frac{\|x\|^2}{2\beta} + (z^*, \theta(\bar{x})) \leq \frac{\|x\|^2}{2\beta}
\]

which implies the inequality in (22).

Lemma 4 For any \( y \geq 0 \), it holds that

\[
\nabla d(y) = \beta(\theta(x(y)) - y),
\]

where \( x(y) \) is defined in (21). In addition,

\[
\beta(y_1 - y_2, \theta(x(y_1)) - \theta(x(y_2))) \leq -\mu\|x(y_1) - x(y_2)\|^2, \quad \forall y_1, y_2 \geq 0,
\]

and

\[
\|x(y_1) - x(y_2)\| \leq \frac{\beta_B}{\mu} \|y_1 - y_2\|, \quad \forall y_1, y_2 \geq 0.
\]

Proof. The result in (23) follows from the Danskin Theorem (cf. [5]). We only need to show (24) and (25).

For \( i = 1, 2 \), denote \( x_i = x(y_i) \). From the definition of \( x(y) \) and the \( \mu \)-strong convexity of \( F \), it holds

\[
\begin{align*}
F(x_1) + \beta y_1^\top \theta(x_1) &\leq F(x_2) + \beta y_1^\top \theta(x_2) - \frac{\mu}{2} \|x_1 - x_2\|^2, \\
F(x_2) + \beta y_2^\top \theta(x_2) &\leq F(x_1) + \beta y_2^\top \theta(x_1) - \frac{\mu}{2} \|x_1 - x_2\|^2.
\end{align*}
\]

Adding the above two inequalities gives the result in (24). Now using the \( B_\gamma \)-Lipschitz continuity of \( \theta \), we have (25) from (24) and complete the proof.

Lemma 5 (approximate dual gradient) Given \( \hat{y} \geq 0 \) and \( \delta \geq 0 \), let \( \hat{x} \) be an approximate minimizer of \( \Phi(\cdot, \hat{y}) \) such that \( \text{dist}(0, \partial_x \Phi(\hat{x}, \hat{y})) \leq \delta \). Then

\[
\|\theta(\hat{x}) - \theta(x(\hat{y}))\| \leq B_\gamma \frac{\delta}{\mu}, \quad \|\beta(\theta(\hat{x}) - \hat{y}) - \nabla d(\hat{y})\| \leq \beta B_\gamma \frac{\delta}{\mu}.
\]

Proof. From the \( \mu \)-strong convexity of \( F \), it follows that for each \( y \geq 0 \), \( \Phi(\cdot, y) \) is \( \mu \)-strongly convex, and thus \( \mu \|\hat{x} - x(\hat{y})\| \leq \text{dist}(0, \partial_x \Phi(\hat{x}, \hat{y})) \leq \delta \), which gives \( \|\hat{x} - x(\hat{y})\| \leq \frac{\delta}{\mu} \). Hence, by the \( B_\gamma \)-Lipschitz continuity of \( \theta \), we have \( \|\theta(\hat{x}) - \theta(x(\hat{y}))\| \leq B_\gamma \frac{\delta}{\mu} \), and thus from (23),

\[
\|\beta(\theta(\hat{x}) - \hat{y}) - \nabla d(\hat{y})\| = \beta\|\theta(\hat{x}) - \theta(x(\hat{y}))\| \leq \beta B_\gamma \frac{\delta}{\mu}.
\]

This completes the proof.
Lemma 6 Given $\hat{y} \geq 0$, it holds
\[
\text{dist}(0, \partial \phi(\hat{x})) \leq \text{dist}(0, \partial x \Phi(\hat{x}, \hat{y})) + \beta \|J_\theta(\hat{x})\| \cdot \|\theta(\hat{x})\| - \hat{y} \|, \forall \hat{x} \in \text{dom}(h).
\]
Proof. It is easy to have $\partial \phi(\hat{x}) = \partial_x \Phi(\hat{x}, \hat{y}) + \beta J_\theta(\hat{x})(\theta(\hat{x}) - \hat{y})$. The desired result now follows from the triangle inequality and the Cauchy-Schwarz inequality.

Lemma 7 Given $\bar{e} > 0$, if $\bar{y} \geq 0$ is an approximate solution of $\max_{y \geq 0} d(y)$ such that $\|\theta(x(\bar{y}))\| + - \bar{y} \| \leq \frac{\bar{e}}{3B_y}$, and $\hat{x}$ is an approximate minimizer of $\Phi(\cdot, \bar{y})$ such that $\text{dist}(0, \partial_x \Phi(\hat{x}, \bar{y})) \leq \frac{\bar{e}}{3B_y}$, then $\text{dist}(0, \partial \phi(\hat{x})) \leq \bar{e}$.

Proof. Since $\text{dist}(0, \partial_x \Phi(\hat{x}, \bar{y})) \leq \frac{\bar{e}}{3B_y}$, we use Lemma 5 with $\delta = \frac{\bar{e}}{3B_y}$ to have $\|\theta(\hat{x}) - \theta(x(\bar{y}))\| \leq \frac{\bar{e}}{3B_y}$. In addition, from the nonexpansiveness of $[\cdot]_+$, it follows that $\|\theta(x(\bar{y}))\| + - \bar{y} \| \leq \frac{\bar{e}}{3B_y}$. Because $\|\theta(x(\bar{y}))\| + - \bar{y} \| \leq \frac{\bar{e}}{3B_y}$, we have from the triangle inequality that $\|\theta(\hat{x})\| + - \bar{y} \| \leq \frac{2\bar{e}}{3B_y}$. The desired result now follows from Lemma 6 and $\|J_\theta(x)\| \leq B_y$, $\forall x \in \text{dom}(h)$.

3.2 the case with a single constraint

For simplicity, we start with the case of $m = 1$, so the bold letters $y, \theta$ are actually scalars in this subsection.

We show the complexity to produce a point $\hat{x}$ satisfying $\text{dist}(0, \partial \phi(\hat{x})) \leq e$ for a specified error tolerance $e > 0$. By Lemma 7, we can first find a $\bar{y} \geq 0$ such that $\|\theta(x(\bar{y}))\| + - \bar{y} \| \leq \frac{\bar{e}}{3B_y}$ and then approximately solve $\min_x \Phi(x, \bar{y})$ to obtain $\hat{x}$.

Our idea of finding a desired approximate solution $\hat{y}$ is to first obtain an interval that contains the solution $\bar{y} = \arg \max_{y \geq 0} d(y)$ and then to apply a bisection method. The following lemma shows that for a given $\bar{y} \geq 0$, we can either check if it is a desired approximate solution or obtain the sign of $\nabla d(\bar{y})$ so that we know the search direction to have a desired solution.

Lemma 8 Given $\delta > 0$ and $\bar{y} \geq 0$, let $\hat{x} \in \text{dom}(h)$ be a point satisfying $\text{dist}(0, \partial_x \Phi(\hat{x}, \bar{y})) \leq \frac{\delta \beta}{B_y}$. If $\|\theta(\hat{x})\| + - \bar{y} \| \leq \frac{\delta \beta}{B_y}$, then $\|\theta(x(\bar{y}))\| + - \bar{y} \| \leq \delta$. Otherwise, $\|\theta(x(\bar{y}))\| + - \bar{y} \| > \frac{\delta \beta}{B_y}$ and $\nabla d(\bar{y})(\theta(\hat{x}) - \bar{y}) > 0$.

Proof. From Lemma 5 and the condition on $\hat{x}$, it follows that
\[
\|\theta(\hat{x}) - \theta(x(\bar{y}))\| \leq \frac{\delta \beta}{B_y}, \quad \text{and} \quad \|\beta(\theta(\hat{x}) - \bar{y}) - \nabla d(\bar{y})\| \leq \frac{\delta \beta}{B_y}.
\]
(26)

Hence, by the nonexpansiveness of $[\cdot]_+$, it holds $\|\theta(x(\bar{y}))\| + - \|\theta(\hat{x})\| + \|\theta(x(\bar{y}))\|\| + - \bar{y} \| \leq \delta$ if $\|\theta(\hat{x})\| + - \bar{y} \| \leq \frac{\delta \beta}{B_y}$ and $\|\theta(x(\bar{y}))\| + - \bar{y} \| > \frac{\delta \beta}{B_y}$ otherwise.

When $\|\theta(\hat{x})\| + - \bar{y} \| > \frac{\delta \beta}{B_y}$, it must hold $\|\theta(\hat{x}) - \bar{y} \| > \frac{\delta \beta}{B_y}$ because $\bar{y} \geq 0$, and thus $\|\beta(\theta(\hat{x}) - \bar{y}) - \nabla d(\bar{y})\| \geq \|\beta(\theta(\hat{x}) - \bar{y}) - \bar{y} \|$.

Therefore, from the second inequality in (26), we conclude that $\nabla d(\bar{y})$ must have the same sign as $\theta(\hat{x}) - \bar{y}$, because otherwise $\|\beta(\theta(\hat{x}) - \bar{y}) - \nabla d(\bar{y})\| \geq \|\beta(\theta(\hat{x}) - \bar{y})\| > \frac{3\delta \beta}{B_y}$. This completes the proof.

By this lemma, we design an interval search algorithm that can either return a point $\bar{y} \geq 0$ such that $\|\theta(x(\bar{y}))\| + - \bar{y} \| \leq \delta$ or return an interval $Y = [a, b] \subseteq [0, \infty)$ that contains the solution $\bar{y}$. The pseudocode is shown in Algorithm 3.

Once the stopping condition in Line 4 or 10 is satisfied, then by Lemma 8, we immediately obtain a desired $\bar{y}$ such that $\|\theta(x(\bar{y}))\| + - \bar{y} \| \leq \delta$. The next lemma shows that the algorithm must exist the while loop within a finitely many iterations.
Algorithm 3: Interval search: \( Y = \text{IntV}(\beta, \mathbf{z}, \delta, L_{\min}, \gamma_1, \gamma_2) \)

1. **Input**: multiplier vector \( \mathbf{z} \geq 0 \), penalty \( \beta > 0 \), target accuracy \( \delta > 0 \), \( L_{\min} > 0 \), and \( \gamma_1 > 1, \gamma_2 \geq 1 \)
2. **Overhead**: define \( \theta(x) = g(x) + \frac{\beta}{\beta} \cdot \Phi(x, y) \) as in (19), and \( \bar{\epsilon} = \frac{\mu \delta}{4B g} \).
3. **Initial step**: call Alg. 2: \( \hat{x} = \text{APG}(\psi, h, \mu, L_{\min}, \bar{\epsilon}, \gamma_1, \gamma_2) \) with \( \psi = \Phi(\cdot) - h. \) \( \triangleright \) so \( \text{dist}(0, \partial_x \Phi(\hat{x}, 0)) \leq \frac{\mu \delta}{4B g} \)
4. if \( [\theta(\hat{x})]_+ - b \leq \frac{3\delta}{4} \) then
5. \( \text{Return } Y = \{0\} \) and stop. \( \triangleright \) otherwise, \( \nabla d(0) \) is positive
6. Let \( a = 0, b = \frac{1}{\beta} \) and call Alg. 2: \( \hat{x} = \text{APG}(\psi, h, \mu, L_{\min}, \bar{\epsilon}, \gamma_1, \gamma_2) \) with \( \psi = \Phi(\cdot) - b. \) \( \triangleright \) set \( b = O(\frac{1}{\beta}) \)
7. while \( [\theta(\hat{x})]_+ - b \geq \frac{3\delta}{4} \) and \( \theta(\hat{x}) - b > 0 \) do
8. let \( a \leftarrow b, \) and increase \( b \leftarrow 2b. \)
9. call Alg. 2: \( \hat{x} = \text{APG}(\psi, h, \mu, L_{\min}, \bar{\epsilon}, \gamma_1, \gamma_2) \) with \( \psi = \Phi(\cdot) - b. \)
10. if \( [\theta(\hat{x})]_+ - b \leq \frac{3\delta}{4} \) then
11. \( \text{Return } Y = \{b\} \) and stop. \( \triangleright \) found \( \bar{y} = b \) such that \( [\theta(x(\bar{y}))]_+ - \bar{y} \leq \delta \)
12. else
13. Return \( Y = [a, b] \) and stop. \( \triangleright \) found an interval containing \( \bar{y} \)

Lemma 9 Given \( \delta > 0 \), if \( b \geq \frac{2\|x^*\|\|z\|}{\beta} \) and \( \text{dist}(0, \partial_x \Phi(\hat{x}, b)) \leq \frac{\mu \delta}{4B g} \), then either \( [\theta(\hat{x})]_+ - b \leq \frac{3\delta}{4} \) or \( \theta(\hat{x}) - b < 0 \).

Proof. From Lemma 3, it follows that \( \bar{y} = [\theta(x(\bar{y}))]_+ \leq \frac{2\|x^*\|\|z\|}{\beta}. \) The result in (24) indicates the decreasing monotonicity of \( \theta(x(y)) \) with respect to \( y. \) Hence, if \( b \geq \frac{2\|x^*\|\|z\|}{\beta} \), then \( \theta(x(b)) \leq \theta(x(\bar{y})) \leq \frac{2\|x^*\|\|z\|}{\beta} \leq b, \) and thus \( \theta(x(b)) - b \leq 0. \) Now if \( [\theta(\hat{x})]_+ - b > \frac{3\delta}{4} \), we know from Lemma 8 that \( \nabla d(b)(\theta(\hat{x}) - b) > 0, \) and thus \( \theta(\hat{x}) - b < 0 \) since \( \nabla d(b) = \beta(\theta(x(b)) - b) \leq 0. \) This completes the proof. \( \square \)

When Algorithm 3 exits the while loop, it can output a single point or an interval. The lemma below shows that if an interval is returned, then it will contain the solution \( \bar{y} \).

Lemma 10 Given \( \delta > 0 \), let \( Y \) be the return from Algorithm 3. If \( Y \) contains a single point \( \bar{y} \), then \( [\theta(x(\bar{y}))]_+ - \bar{y} \leq \delta. \) Otherwise, \( Y \) is an interval \([a, b]\), and it holds that \( \nabla d(a) > 0, \nabla d(b) < 0, \) and \( \bar{y} \in [a, b]. \)

Proof. If \( Y \) contains a single point \( \bar{y} \), then the condition in either Line 4 or 10 of Algorithm 3 is satisfied, and we immediately have \( [\theta(x(\bar{y}))]_+ - \bar{y} \leq \delta \) from Lemma 8.

Now suppose that \( Y \) is an interval \([a, b]\). From Lemma 8 and the setting in Line 8 of Algorithm 3, we always have \( \nabla d(a) > 0. \) When the algorithm exits the while loop and returns an interval, we have \( [\theta(\hat{x})]_+ - b \geq \frac{3\delta}{4} \) but \( \theta(\hat{x}) - b \leq 0. \) Then it follows from Lemma 8 that \( \nabla d(b) < 0. \) Therefore, the unique solution \( \bar{y} \) must lie in \([a, b]\) by the Mean-Value Theorem and the strong concavity of \( d. \) \( \square \)

Remark 1 Suppose Algorithm 3 returns an interval \([a, b]\). Then Lemma 9 indicates that \( b \leq \frac{1}{\beta} \max \{1, 4\|z^*\| + 2\|z\|\} \), and in addition, at most \( T + 2 \) calls are made to Alg. 2, where \( T \) is the smallest non-negative integer such that \( 2^T \geq 2\|z^*\| + \|z\| \).

Suppose Algorithm 3 returns an interval \([a, b]\). We can then use the bisection method to obtain a desired point \( \bar{y}. \) The pseudocode is given in Algorithm 4.

By Lemma 8 and the lemma below, it holds that the returned point \( \bar{y} \) from Algorithm 4 must satisfy \( [\theta(x(\bar{y}))]_+ - \bar{y} \leq \delta. \)
Algorithm 4: Bisection method for $\max_{y \geq 0} d(y)$: $(\bar{x}, \bar{y}) = \text{BiSec}(\beta, z, \delta, L_{\min}, \gamma_1, \gamma_2)$

1. **Input:** multiplier vector $z \geq 0$, penalty $\beta > 0$, target accuracy $\delta > 0$, $L_{\min} > 0$, and $\gamma_1 > 1, \gamma_2 \geq 1$

2. **Overhead:** define $\theta(x) = g(x) + \frac{\beta}{2} \cdot f(x, y)$ as in (19), and $\varepsilon = \frac{\mu^3}{2B_y^2}$.

3. Call Alg. 3: $Y = \text{IntV}(\beta, z, \delta, L_{\min}, \gamma_1, \gamma_2)$ and denote it as $[a, b]$. $\triangleright$ If $Y$ is a singleton, then $a = b$

4. while $b - a > \frac{\mu^3}{\mu + \beta B_y^2}$ do

5. let $c = \frac{a + b}{2}$ and call Alg. 2: $\bar{x} = \text{APG}(\psi, h, \mu, L_{\min}, \varepsilon, \gamma_1, \gamma_2)$ with $\psi = \Phi(-c - h)

6. if $||\theta(\bar{x})||_+ - c \leq \frac{\mu^3}{2}$ then

7. Let $\bar{y} = c$, return $(\bar{x}, \bar{y})$, and stop

8. else if $||\theta(\bar{x})|c > 0$ then

9. let $a \leftarrow c$

10. else

11. let $b \leftarrow c$

12. Let $\tilde{y} = \frac{a + b}{2}$ and $\bar{x} = \text{APG}(\psi, h, \mu, L_{\min}, \varepsilon, \gamma_1, \gamma_2)$ with $\psi = \Phi(\cdot, \bar{y}) - h$, return $(\bar{x}, \tilde{y})$, and stop.

Lemma 11 Let $Y = [a, b] \subseteq (0, \infty)$. If $\nabla d(a) > 0$, $\nabla d(b) < 0$, and $b - a \leq \frac{\mu^3}{\mu + \beta B_y^2}$ for a positive $\delta$, then $||\theta(x(\tilde{y}))||_+ - \tilde{y} | \leq \delta$ for any $\tilde{y} \in [a, b]$.

Proof. Recall from Lemma 3 that $\tilde{y} = [\theta(x(\tilde{y})))_+$. Hence, for any $\tilde{y} \in [a, b]$, we have

$$
||\theta(x(\tilde{y}))||_+ - \tilde{y} || = ||\theta(x(\tilde{y}))||_+ - \tilde{y} - [\theta(x(\tilde{y}))]|_+ + \tilde{y} ||
\leq ||[\theta(x(\tilde{y}))]|_+ - [\theta(x(\tilde{y}))]|_+|| + ||\tilde{y} - \tilde{y}||
\leq B_y ||x(\tilde{y}) - x(\tilde{y})|| + ||\tilde{y} - \tilde{y}||
\leq \frac{\mu^3}{\mu + \beta B_y^2} ||\tilde{y} - \tilde{y}|| + ||\tilde{y} - \tilde{y}||,
$$

where we have used the non-expansiveness of $|[.|_+|$. In the second inequality, the third inequality follows from (17), and the last inequality holds because of (25). Now since $\tilde{y} \in [a, b]$, we have $||\tilde{y} - \tilde{y}|| \leq b - a \leq \frac{\mu^3}{\mu + \beta B_y^2}$, and thus the desired result follows.

Remark 2 Since the bisection method halves the interval every time, it takes at most $\left[\log_2 \frac{(b - a)(\mu + \beta B_y^2)}{\mu^3}\right]_+$ halves to reduce an initial interval $[a, b]$ to one with length no larger than $\frac{\mu^3}{\mu + \beta B_y^2}$. Notice $a \geq 0$ and $b \leq \frac{1}{\mu} \max\{1, 4\|x\| + 2\|z\|\}$ from Remark 1. Hence, after $Y$ is obtained, Algorithm 4 will call Algorithm 2 at most

$$
\left[\log_2 \max\left\{1, 4\|x\| + 2\|z\|\right\} \left(\frac{\mu + \beta B_y^2}{\mu^3}\right)\right]_+ + 1 \text{ times}.\n$$

Below we establish the complexity result of Algorithm 4 to return $\tilde{y}$.

Theorem 4 (Iteration complexity of BiSec) Under Assumptions 1–4, Algorithm 4 needs at most $T$ evaluations on $f$, $\theta$, $\nabla f$, and $J_\theta$ to output $\bar{x}$ and $\bar{y} \geq 0$ that satisfy $\text{dist}(0, \partial_x \Phi(\bar{x}, \bar{y})) \leq \varepsilon$ and $||\theta(x(\bar{y}))||_+ - \bar{y} | \leq \delta$, where $\varepsilon = \frac{\mu^3}{2B_y^2}$, and

$$
T = K \left(1 + \left[\log_{\gamma_1 L_{\min}}\right]_+\right) \left(1 + 2 \cdot \frac{\sqrt{\gamma_1 L_{\min}}}{\mu} \log \left(\frac{\beta}{\mu} \left(\sqrt{\gamma_1 L_{\min}} + \frac{L_{\min}}{\sqrt{\gamma_1 L_{\min}}} \sqrt{2\gamma_1 L_{\min} + \mu}\right)\right)_+\right),
$$

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with $L_z = L_f + L_g \max \{1, 4\|z\| + 2\|z\|\}$ and

$$K = 3 + \left\lceil \log_2(2\|z\| + \|z\|) \right\rceil + \left\lceil \log_2 \frac{\max \{1, 4\|z\| + 2\|z\|\} (\mu + \beta B^2)}{\beta \mu \delta} \right\rceil.$$

(28)

**Proof.** By Remarks 1 and 2, Algorithm 4 calls Algorithm 2 at most $K$ times, where $K$ is given in (28). Notice that the gradient of $\psi = \psi(\cdot, b) - h$ is Lipschitz continuous with constant $L_f + \beta b L_g$. Since $b \leq \frac{1}{\mu} \max \{1, 4\|z\| + 2\|z\|\}$ from Remark 1, we apply Corollary 1 to obtain the desired result. 

3.3 the case with multiple constraints

In this subsection, we consider the case of $m > 1$. Similar to the case of $m = 1$, we use a cutting-plane method to approximately solve $\max_{y \geq \theta} d(y)$. The next lemma is the key. It provides the foundation to generate a cutting plane if a query point is not sufficiently close to the solution $\bar{y}$.

**Lemma 12** Let $b > 0$, and suppose $\|\bar{y}\| \leq b$. Given $\delta > 0$ and $\bar{y} \geq 0$, let $\bar{x} \in \text{dom}(h)$ be a point satisfying $\text{dist}(0, \partial_x \Phi(\bar{x}, \bar{y})) \leq \min \left\{ \frac{\mu}{\beta}, \frac{\mu \delta}{\beta B^2_{\beta B^2}} \right\}$. If $\|\theta(\bar{x})\| + \|\bar{y}\| \leq \frac{3\delta}{4}$, then $\|\theta(\bar{x}\bar{y})\| + \|\bar{y}\| \leq \delta$. Otherwise, $\|\theta(\bar{x}\bar{y})\| + \|\bar{y}\| > \frac{3\delta}{4}$, and also $\theta(\bar{x} - \bar{y}, y - \hat{y}) \geq 0$ for any $y \in B_\eta(\bar{y}) \cap B^+_\eta$, where $\eta = \min \{b, \eta_+\}$, and $\eta_+$ is the positive root of the equation

$$\frac{\mu + \beta B^2}{\mu} \left( \eta + \sqrt{2\eta B_d} \right) = \frac{\delta}{4}, \quad \text{with} \quad B_d = \max_{y \in B^+_\eta} \nabla d(y).$$

(29)

**Proof.** By the same arguments in the proof of Lemma 8, we can show that $\|\theta(\bar{x}\bar{y})\| + \|\bar{y}\| \leq \delta$ if $\|\theta(\bar{x})\| + \|\bar{y}\| \leq \frac{3\delta}{4}$, and $\|\theta(\bar{x}\bar{y})\| + \|\bar{y}\| > \frac{3\delta}{4}$ otherwise. Hence, we only need to show $\theta(\bar{x} - \bar{y}, y - \hat{y}) \geq 0$ for any $y \in B_\eta(\bar{y}) \cap B^+_\eta$ in the latter case, and we prove this by contradiction.

Suppose $\|\theta(\bar{x})\| + \|\bar{y}\| > \frac{3\delta}{4}$ and the following condition holds

$$\theta(\bar{x} - \bar{y}, y - \hat{y}) < 0, \quad \text{for some} \ y \in B_\eta(\bar{y}) \cap B^+_\eta.$$

(30)

By the $\beta$-strong concavity of $d$, it holds

$$d(y) \leq d(\bar{y}) + \langle \nabla d(\bar{y}), y - \bar{y} \rangle - \frac{\beta}{2} \|y - \bar{y}\|^2.$$

(31)

From the Mean-Value Theorem, it follows that there is $\bar{y}$ between $y$ and $\bar{y}$ such that $d(y) - d(\bar{y}) = \langle \nabla d(\bar{y}), y - \bar{y} \rangle \geq -\eta B_d$, where the inequality holds because $y \in B_\eta(\bar{y})$ and $\bar{y}$ must fall in $B^+_\eta$. Since $d(\bar{y}) \geq d(\bar{y})$, we have $d(\bar{y}) - d(y) \leq d(\bar{y}) - d(\bar{y}) - \eta B_d$. Hence, (30) and (31) imply

$$\frac{\beta}{2} \|y - \bar{y}\|^2 \leq \eta B_d + \langle \beta(\bar{x} - \bar{y}) - \nabla d(\bar{y}), \hat{y} - y \rangle.$$  

(32)

From Lemma 5 and the condition $\text{dist}(0, \partial_x \Phi(\bar{x}, \bar{y})) \leq \frac{\mu \delta}{\beta B^2_{\beta B^2}}$, it follows $\|\beta(\bar{x} - \bar{y}) - \nabla d(\bar{y})\| \leq \frac{\beta \mu \delta}{\beta B^2_{\beta B^2}}$, which together with (32) and the Cauchy-Schwartz inequality gives

$$\frac{\beta}{2} \|y - \bar{y}\|^2 \leq \eta B_d + \frac{\beta \mu \delta}{\beta B^2_{\beta B^2}} \|\bar{y} - y\|.$$
Solving the above inequality, we have \(|y - \bar{y}| \leq \sqrt{\frac{2\eta B_2}{\beta}} + \frac{\mu \delta}{4(\mu + \beta B_2)}\), and since \(|y| \leq \eta\), it holds \(|\bar{y} - \bar{y}| \leq \eta + \sqrt{\frac{2\eta B_2}{\beta}} + \frac{\mu \delta}{4(\mu + \beta B_2)}\). Now using (27), we have

\[
\|\theta(x(\bar{y}))\| + |\bar{y}| \leq \frac{\mu + \beta B_2}{\mu} \left(\eta + \sqrt{\frac{2\eta B_2}{\beta}} + \frac{\mu \delta}{4(\mu + \beta B_2)}\right) + \frac{\delta}{4} \leq \frac{\delta}{2},
\]

where the last inequality follows from the choice of \(\eta\).

However, we know that when \(|\theta(x(\bar{y}))| + |\bar{y}| \geq \frac{3\delta}{2}\), it holds \(|\theta(x(\bar{y}))| + |\bar{y}| \geq \frac{\delta}{2}\), and (33) contradicts to this fact. Therefore, the assumption in (30) cannot hold. This completes the proof. \(\square\)

Suppose \(|\bar{y}| \leq b\) for some \(b > 0\). For a given \(\bar{y} \geq 0\), let \(\bar{x}\) satisfy the condition required in Lemma 12. Then if \(|\theta(x(\bar{y}))| + |\bar{y}| \geq \frac{3\delta}{2}\), we find a half-space containing the set \(B_\eta(\bar{y}) \cap B_\delta^+\), whose volume is at least \(4^{-m} V_\eta(\eta)\) if \(\eta \leq b\). Therefore, we can apply a cutting-plane method to find a near-optimal \(\bar{y}\). For simplicity, we use the ellipsoid method. The pseudocode is shown in Algorithm 5. In general, the ellipsoid method is numerically inefficient for high-dimensional problems. However, it can converge fast for solving the low-dimensional dual problem min\(_{y \geq 0} \alpha d(y)\), as we will show in the numerical experiments.

---

**Algorithm 5:** Ellipsoid Method for max\(_{\gamma \geq 0} \alpha d(y)\): \((\bar{x}, \bar{y}, \text{FLAG}) = \text{Ellipsoid}(\beta, z, \delta, B, L_{\text{min}}, \gamma_1, \gamma_2)\)

1. **Input:** multiplier vector \(z \geq 0\), penalty \(\beta > 0\), target accuracy \(\delta > 0\), \(b > 0\), \(L_{\text{min}} > 0\), and \(\gamma_1 > 1, \gamma_2 \geq 1\)
2. **Overhead:** define \(\theta(x) = z(x) + \frac{\bar{z}}{\beta}, \phi(z, y)\) as in (19), \(\varepsilon = \min\{\frac{\mu \delta}{4B_\eta}, \frac{\mu^2 \delta}{8B_\beta(\mu + \beta B_2)}\}\), and FLAG = 0.
3. Let \(\eta_\varepsilon\) be the positive root of (29) and \(\eta \leftarrow \min\{b, \eta_\varepsilon\}\), and set \(k = 0\).
4. Set \(\varepsilon_k = \{y \in \mathbb{R}^m : (y - \bar{y})^T B^{-1}(y - \bar{y}) \leq 1\}\) with \(B = \beta^2 I\) and \(y = \bar{y} = 0\) \(\triangleright \) initial ellipsoid
5. **while** the volume of \(\varepsilon_k \geq 4^{-m} V_\eta(\eta)\) **do**
6.  **if** \(\bar{y} \geq 0\) **then**
7.  Let \(a = -e_0\) where \(e_0 = \arg\min_{y \in [0,\varepsilon]} \bar{y}\)
8.  \(\triangleright \) add a cutting plane \(y_0 \geq \bar{y}_0\)
9.  Set \(\varepsilon_{k+1} = \{y \in \mathbb{R}^m : (y - \bar{y})^T B^{-1}(y - \bar{y}) \leq 1\}\) with updated \(B\) and \(\bar{y}\) by
10. \(B \leftarrow \frac{m^2}{m^2 - 1} \left(\frac{B}{m + 1} a^T Ba \frac{Ba}{a^T Ba}\right)\) \(\triangleright \) add a cutting plane \((\bar{y}, y - \bar{y}) \leq 0\)
11. **else if** \(|\bar{y}| > b\) **then**
12.  Let \(a = \bar{y}\)
13.  \(\triangleright \) add a cutting plane \((\bar{y}, y - \bar{y}) \leq 0\)
14.  Set \(\varepsilon_{k+1} = \{y \in \mathbb{R}^m : (y - \bar{y})^T B^{-1}(y - \bar{y}) \leq 1\}\) with updated \(B\) and \(\bar{y}\)
15. **else**
16.  Call Alg. 2; \(\bar{x} = \text{APG}(\psi, h, \mu, L_{\text{min}}, \varepsilon, \gamma_1, \gamma_2)\) with \(\psi = \phi(y, \bar{y}) - h\)
17.  **if** \(|\theta(x)\| + |\bar{y}| \geq \frac{3\delta}{2}\) **then**
18.  FLAG = 1, return \((\bar{x}, \bar{y}, \text{FLAG})\), and stop \(\triangleright \) found \(\bar{y}\) such that \(|\theta(x(\bar{y}))\| + |\bar{y}| \leq \delta\)
19.  **else**
20.  Let \(a = \bar{y} - \theta(x)\)
21.  \(\triangleright \) add a cutting plane \((\bar{y} - \theta(x), y - \bar{y}) \leq 0\)
22.  Set \(\varepsilon_{k+1} = \{y \in \mathbb{R}^m : (y - \bar{y})^T B^{-1}(y - \bar{y}) \leq 1\}\) with updated \(B\) and \(\bar{y}\)
23. **increase** \(k \leftarrow k + 1\).

From Lemma 12 and the property of the ellipsoid method (cf. [6]), we can show the finite convergence of Algorithm 5, and furthermore, we can estimate its total complexity by Corollary 1 if \(|\bar{y}| \leq b\).
Theorem 5. Under Assumptions 1–4, Algorithm 5 will stop within at most \(2m(m+1)\log \frac{4B}{\eta} \) iterations, where \(\eta\) is defined in Line 3 of the algorithm. If \(\|\hat{y}\|_b^2 \leq b\), it must return \(\text{FLAG} = 1\) and a vector \(\hat{y} \succeq 0\) satisfying \(\|\theta(x(\hat{y})))_+ - \hat{y}\|_b \leq \delta\) with at most \(T\) evaluations of \(f, \nabla f, \theta, \) and \(J_\theta\), where

\[
T = K \left(1 + \left[\log \gamma_1 L_{\min} \right]_+ \right) \left(1 + 2 \sqrt{2} \sqrt{\frac{\gamma_1 L_{\max}}{\mu} \log \frac{D_\eta}{\varepsilon}} \left(\sqrt{2} \gamma_1 L_{\max} + \frac{L_{\phi}}{\sqrt{L_{\min}}} \right) \sqrt{\gamma_2 L_{\max} + \mu} \right)_+, \tag{35}
\]

with \(K = \left[2m(m+1) \log \frac{4B}{\eta} \right]\), \(L_{\psi} := L_f + \beta b L_g\), and \(\varepsilon = \min \left\{ \frac{\mu^2 \delta}{4B^2}, \frac{\gamma_1 L_{\max} + \mu}{8(\mu^2 + \beta B^2)} \right\}\).

Proof. By the property of the ellipsoid method, we have (cf. [6, Eq. 2.11])

\[
\text{vol}(\mathcal{E}_k) \leq e^{-\frac{k}{2m+1}} \text{vol}(\mathcal{E}_{k-1}) \leq e^{-\frac{k}{2m+1}} \text{vol}(\mathcal{E}_0), \quad \forall k \geq 1.
\]

Hence, to satisfy the stopping condition \(\text{vol}(\mathcal{E}_k) \leq 4^{-m}V_m(\eta)\), it suffices to have \(e^{-\frac{k}{2m+1}} \text{vol}(\mathcal{E}_0) \leq 4^{-m}V_m(\eta)\). Since \(\mathcal{E}_0\) is a ball of radius \(b\), this requirement is equivalent to \(e^{-\frac{k}{2m+1}} \leq \left(\frac{4}{b}\right)^m\), which holds if \(k \geq \left[2m(m+1) \log \frac{4B}{\eta} \right]\). We below estimate the number of evaluations of the function value and gradient.

Notice that when Algorithm 2 is called, \(\|\hat{y}\|_b^2 \leq b\), and thus the smooth function \(\psi\) has \((L_f + \beta b L_g)\)-Lipschitz continuous gradient. Since Algorithm 2 is called at most \(2m(m+1) \log \frac{4B}{\eta}\) times, we have from Corollary 1 that the total number of function and gradient evaluations is \(T\) given in (35).

By Theorem 5, we can guarantee to find a desired approximate solution \(\hat{y}\) by gradually increasing the search radius \(b\). The algorithm is shown below.

\textbf{Algorithm 6: Search by the Ellipsoid Method for max}_{\bf{y}} \geq 0 d(y): (x, \hat{y}) = \text{StEM}(\beta, z, \delta, L_{\min}, \gamma_1, \gamma_2)

\begin{enumerate}
  \item \textbf{Input:} multiplier vector \(z \geq 0\), penalty \(\beta > 0\), target accuracy \(\delta > 0\), \(L_{\min} > 0\), and \(\gamma_1 > 1, \gamma_2 \geq 1\)
  \item \textbf{Overhead:} define \(\theta(x) = g(x) + \frac{\mu}{2} \cdot \Phi(x, y)\) as in (19), and set \(k = 0, b_0 = \frac{1}{4} \mu\) and \(\text{FLAG} = 0\).
  \item \textbf{while} \(\text{FLAG} = 0\) \textbf{do}
    \begin{enumerate}
      \item Call Alg. 5: \((x, \hat{y}, \text{FLAG}) = \text{Ellipsoid}(\beta, z, \delta, b_k, L_{\min}, \gamma_1, \gamma_2)\).
      \item Let \(b_{k+1} \leftarrow 2b_k\) and increase \(k \leftarrow k + 1\).
    \end{enumerate}
  \item \textbf{Output} \((x, \hat{y})\).
\end{enumerate}

Theorem 6. Under Assumptions 1–4, if \(\delta \leq \frac{8(\mu + \beta B^2)}{\beta \mu}\), then the output \((\hat{x}, \hat{y})\) of Algorithm 6 must satisfy \(\text{dist}(0, \partial_\theta \Phi(x(\hat{y}))) \leq \varepsilon, \hat{y} \succeq 0\) and \(\|\theta(x(\hat{y})))_+ - \hat{y}\|_b \leq \delta\), where \(\varepsilon = \min \left\{ \frac{\mu^2 \delta}{4B^2}, \frac{\gamma_1 L_{\max} + \mu}{8(\mu^2 + \beta B^2)} \right\}\). In addition, it needs at most \(T\) evaluations of \(f, \nabla f, \theta, \) and \(J_\theta\) to give the output, where

\[
T \leq 3CK + 4C \sqrt{\gamma_1} \log \left(\frac{D_\eta}{\varepsilon} \left(\sqrt{\gamma_1 L_{\max} + \frac{L_{\max}}{L_{\min}}} \right) \sqrt{2} \gamma_1 L_{\max} + \mu \right) \left(2 \gamma_1 L_{\max} + \mu \right) \left(1 \right),
\]

with the constants defined as

\[
L_{\max} = L_f + L_g(4\|z\|_b + 2\|z\|), \quad C = 2 \left[2m(m+1) \log R \right] \cdot \left(1 + \|z\|_b \frac{L_{\max}}{L_{\min}} \right)_+, R = \frac{64(2\|z\|_b + \varepsilon)}{\beta \mu} \left(\frac{4(\beta G + 4\|z\|_b + 2\|z\|_b)(\mu + \beta B^2)^2}{\beta \mu^2} \right) + \mu + \beta B^2.
\]
Proof. By the quadratic formula, we can easily have the positive root of (29) to be

\[
\eta_+ = \frac{\sqrt{2\beta G} + \sqrt{2\beta G + \frac{\mu^2}{\mu^2 + \beta B^2}}}{\frac{\sqrt{2\beta G}}{\beta} + \frac{\mu^2}{\mu^2 + \beta B^2}} \geq \frac{(\frac{\sqrt{2\beta G}}{\beta} + \frac{\mu^2}{\mu^2 + \beta B^2})^2}{8(\frac{\sqrt{2\beta G}}{\beta} + \frac{\mu^2}{\mu^2 + \beta B^2})^2}.
\]

Hence, it holds that

\[
\frac{b}{\eta_+} \leq \frac{8b(\frac{\sqrt{2\beta G}}{\beta} + \frac{\mu^2}{\mu^2 + \beta B^2})}{(\frac{\sqrt{2\beta G}}{\beta} + \frac{\mu^2}{\mu^2 + \beta B^2})^2} = 8b \left( \frac{4B_d(\mu + \beta B^2)^2}{\beta(\mu b)^2} + \frac{\mu + \beta B^2}{\mu b} \right).
\]

When \( b \geq \frac{\eta}{\eta_+} \), the right hand side of the above inequality is greater than one by the assumption \( \delta \leq \frac{8(\mu + \beta B^2)}{\beta \mu} \), and since \( \eta = \min\{\eta_+, b\} \) in Algorithm 5, we have

\[
\frac{b}{\eta} = \max\{\frac{b}{\eta_+}, 1\} \leq 8b \left( \frac{4B_d(\mu + \beta B^2)^2}{\beta(\mu b)^2} + \frac{\mu + \beta B^2}{\mu b} \right) \leq 8b \left( \frac{4(\beta G + \|x\| + \beta b)(\mu + \beta B^2)^2}{\beta(\mu b)^2} + \frac{\mu + \beta B^2}{\mu b} \right),
\]

(37)

where we have used \( \nabla d(y) = \beta(g(x(y)) + \frac{\beta}{\beta} - y) \) in (23) and thus the bound of \( \nabla d(y) \) over \( B^+_0 \) satisfies \( B_d \leq \beta G + \|x\| + \beta b \) with \( G \) defined in (3). Furthermore, by Lemma 3 and Theorem 5, Algorithm 5 must return FLAG = 1 and a vector \( \tilde{y} \) satisfying \( \|\theta(x(\tilde{y}))\| - \tilde{y} \leq \delta \) when \( b \geq 2\|x^*\| + \|z\| \). Since \( b_0 = \frac{\eta}{\eta_+} \) and \( b_{k+1} = 2b_k \), Algorithm 6 must stop after making at most \( K \) calls to Algorithm 5, where \( K \) is the smallest positive integer such that \( 2^{K-1} \geq 2\|x^*\| + \|z\| \), i.e.,

\[
K = \lceil \log_2(2\|x^*\| + \|z\|) \rceil + 1.
\]

In addition, from \( b_{k+1} = 2b_k \), it holds

\[
b_k = \frac{\eta}{\eta_+} \leq \max\{1, 4\|x^*\| + 2\|z\|\}, \quad \text{for each } 0 \leq k \leq K - 1.
\]

In the \( k \)-th call to Algorithm 5, let \( \eta_k \) denote the \( \eta \) used in Line 3 of Algorithm 5, \( L_{\psi_k} = L_f + \beta L_g b_k \) the gradient Lipschitz constant of the smooth function \( \psi \), and \( T_k \) the total number of gradient and function evaluations. Then, by (38) and the definition of \( L_{\max} \), we have \( L_{\psi_k} \leq L_{\max} \). Also, from (37), (38), and the definition of \( R \), it follows \( \frac{b_k}{\eta_k} \leq R \) for each \( 0 \leq k \leq K - 1 \). Moreover, we have from (35) that

\[
T_k \leq \left[ 2m(m + 1) \log R \right] \left( 1 + \log_{\gamma_1} \frac{L_{\psi_k}}{L_{\min}} \right) + 1 + 2 \left[ 2 \gamma_1 L_{\psi_k} \log \left( \frac{D_p}{\gamma_1 L_{\psi_k}} \right) \sqrt{2\gamma_1 L_{\psi_k}} \right] \leq C \left( 1 + 2 \left[ 2 \gamma_1 L_{\psi_k} \log \left( \frac{D_p}{\gamma_1 L_{\max}} \right) \sqrt{2\gamma_1 L_{\max}} \right] \right) \leq 3C + 4C \gamma_1 L_{\psi_k} \log \left( \frac{D_p}{\gamma_1 L_{\max}} \right) \sqrt{2\gamma_1 L_{\max}}.
\]

Notice that \( \sqrt{L_{\psi_k}} \leq \sqrt{L_f} + \sqrt{\beta L_g b_k} \) and, thus

\[
\sum_{k=0}^{K-1} \sqrt{L_{\psi_k}} \leq K \sqrt{L_f} + \sum_{k=0}^{K-1} \sqrt{\beta L_g b_k} = K \sqrt{L_f} + \sqrt{L_g} \sum_{k=0}^{K-1} \sqrt{2\gamma_1 L_{\max}} \leq K \sqrt{L_f} + \sqrt{L_g} \max \left\{ 1, \frac{2\sqrt{2\|x^*\| + \|z\|}}{\sqrt{2-1}} \right\}.
\]

Therefore, \( T \) must satisfy the condition in (36) since \( T \leq \sum_{k=0}^{K-1} T_k \).
Remark 3 In terms of the dependence on \( m \), the number \( T \) in (36) is proportional to \( m^2 \). We can improve it to the order of \( m \) if a more advanced cutting-plane method is used, such as the volumetric-center cutting-plane method in [36], and the analytic-center cutting-plane method in [2], and the faster cutting plane method in [16].

4 Overall iteration complexity of the first-order augmented Lagrangian method

In this section, we specify the implementation details in Algorithm 1. We use the method derived in section 3 as the subroutine to find each \( x^{k+1} \). In addition, we choose a geometrically increasing sequence \( \{ \beta_k \} \) and stop the algorithm once an \( \varepsilon \)-KKT point is obtained. The pseudocode is given in Algorithm 7.

Algorithm 7: Cutting-plane first-order iALM for problems in the form of (1) with \( m = O(1) \)

1. **Input:** \( \beta_0 > 0, \sigma > 1, \text{tolerance } \varepsilon > 0, L_{\text{min}} > 0, \gamma_1 > 1, \text{and } \gamma_2 \geq 1 \)
2. **Initialization:** choose \( x^0 \in \text{dom}(h) \), and set \( z^0 = 0 \)
3. for \( k = 0, 1, \ldots \) do
   4. Choose \( \varepsilon_k \leq \min \{ \varepsilon, \frac{24B_2(\mu + \beta_k B_2^2)}{\mu} \} \) and set \( \delta_k = \frac{\varepsilon_k}{\gamma_k B_2^2} \).
   5. if \( m = 1 \) then
      6. Call Alg. 4: \( (x^{k+1}, y^{k+1}) = \text{BiSec}(\beta_k, z^k, \delta_k, L_{\text{min}}, \gamma_1, \gamma_2) \)
   7. else
      8. Call Alg. 6: \( (x^{k+1}, y^{k+1}) = \text{STEM}(\beta_k, z^k, \delta_k, L_{\text{min}}, \gamma_1, \gamma_2) \)
   9. if \( m = 1 \) and \( \frac{\mu}{\delta_k B_2^2} > 1 \), or \( m > 1 \) and \( \min \{ \frac{\mu}{\delta_k B_2^2}, \frac{\sigma}{\gamma_k B_2^2} \} > 1 \) then
      10. Call Alg. 2: \( x^{k+1} = \text{APG}(\psi, h, \mu, L_{\text{min}}, \varepsilon_k/3, \gamma_1, \gamma_2) \) with \( \psi(x) = f(x) + \beta_k(y^{k+1} - g(x)) \).
11. Update \( z \) by \( z^{k+1} = [z^k + \beta_k g(x^{k+1})]_+ \).
12. Let \( \beta_{k+1} \leftarrow \sigma \beta_k \).
13. if \( (x^{k+1}, z^{k+1}) \) is an \( \varepsilon \)-KKT point of (1) then
   14. Output \((x, z) = (x^{k+1}, z^{k+1})\) and stop

The next theorem gives a bound on the number of calls to the subroutine.

**Theorem 7** Suppose that Assumptions 1 through 4 hold. Let \( (\beta_0, \sigma, \varepsilon, \gamma_1, \gamma_2) \) be the input of Algorithm 7 and \( \{(x^k, y^k, z^k)\}_{k \geq 0} \) be the generated sequence. Then dist \( 0, \partial\mathcal{L}_{\beta_k}(x^{k+1}, z^k) \) \( \leq \varepsilon_k \) for each \( k \geq 0 \). Suppose \( \bar{\varepsilon} = \min \{ \varepsilon, \frac{24B_2(\mu + \beta_k B_2^2)}{\mu} \} \), \( \forall k \geq 0 \). Let \( \varepsilon_k = \bar{\varepsilon} \) for all \( k \geq 0 \). Then after at most \( K - 1 \) iterations, Algorithm 7 will produce an \( \varepsilon \)-KKT point of (1), where

\[
K = \max \left\{ \left[ \log_{\sigma} \frac{9\|z^u\|^2}{\rho \sigma} \right]_+, \left[ \log_{\sigma} \frac{8\|z^u\|^2}{\rho \sigma} \right]_+, \left[ \log_{\sigma} \frac{4}{\rho \sigma} \right]_+ \right\} + 1. \tag{39}
\]

In addition, the output multiplier vector \( \bar{z} \) satisfies

\[
\| \bar{z} \| \leq 2\|z^u\| + \sqrt{\frac{2\sigma^2}{\rho \sigma + 1}} \max \{3\|z^u\|, 2\sqrt{2\|z^u\|}, 2\}. \tag{40}
\]
Proof. For each $k \geq 0$, define
$$
\theta_k(x) = g(x) + \frac{z^k}{\beta_k} \phi_k(x) = F(x) + \frac{\beta_k}{2} \|\theta_k(x)\|_+ , \quad \Phi_k(x, y) = F(x) + \beta_k \left( y^\top \theta_k(x) - \frac{1}{2} \|y\|^2 \right).
$$

When $m = 1$, if $(x^{k+1}, y^{k+1})$ is obtained in Line 6 of Alg. 7, then we have from Theorem 4 that
$$
\text{dist}(0, \partial_x \Phi_k(x^{k+1}, y^{k+1})) \leq \frac{\delta_k}{\beta_k}, \quad \text{and} \quad \|\theta_k(x(y^{k+1}))\|_+ - y^{k+1} \leq \delta_k,
$$
where $x(y^{k+1}) = \arg \min_x \Phi_k(x, y^{k+1})$. Furthermore, notice that if $\frac{\beta_k}{\beta_k + \beta_k} > 1$, we will do Line 10 in Alg. 7 to obtain a new $x^{k+1}$ that satisfies $\text{dist}(0, \partial_x \Phi_k(x^{k+1}, y^{k+1})) \leq \frac{\delta_k}{3}$. Now by Lemma 7 and the choice of $\delta_k = \frac{\epsilon}{\beta_k + \beta_k}$, we have $\text{dist}(0, \partial_x \Phi_k(x^{k+1}, z^k)) = \text{dist}(0, \partial_x \Phi_k(x^{k+1})) \leq \epsilon_k$.

When $m > 1$, by the choice of $\epsilon_k$ and $\delta_k$, it holds $\delta_k \leq \frac{\epsilon}{\beta_k + \beta_k}$ for each $k$. Hence, we can use Theorem 6 and Lemma 7 to show $\text{dist}(0, \partial_x \Phi_k(x^{k+1}, z^k)) \leq \epsilon_k$ by the same arguments as in the case of $m = 1$.

Therefore, for $m \geq 1$, if $\epsilon_k = \tilde{\epsilon}$ for all $k$, we have from Theorem 3 that the inequalities in (15) and (16) hold. By the choice of $\tilde{\epsilon}$, it holds $\frac{\epsilon^2}{2\mu(\sigma-1)} \leq \tilde{\epsilon}$. Since $K - 1 \geq \log \frac{\alpha}{\beta_k} \frac{\|z^k\|_2}{\beta_k}$, then $\frac{\epsilon}{\beta_k(\sigma-1)} \leq \frac{\epsilon}{\beta_k} \leq \frac{\epsilon}{2}$, and thus we have from (16) that $\sum_{i=1}^m |z^k_i| g_i(x^K) \leq \epsilon$. In addition, noticing $\sqrt{\sigma(\sigma+1)} \leq 1$ and $\tilde{\epsilon} \leq \sqrt{\frac{\epsilon}{\beta_k(\sigma-1)}}$, we have $\epsilon^3(\sqrt{\sigma} + 1) \sqrt{\frac{2}{\mu(\sigma-1)}} \leq \epsilon^3$, and thus (15) implies
$$
\|g(x^K)\|_+ \leq \frac{4\|z^k\|_2}{\beta_k(\sigma-1)} + \frac{\sqrt{\epsilon}}{\beta_k(\sigma-1)}.
$$

Now by the setting of $K$ in (39), we have that both terms on the right hand side of the above inequality are no greater than $\epsilon/2$. Hence, $\|g(x^K)\|_+ \leq \epsilon$, and thus $x^K$ must be an $\epsilon$-KKT point of (1).

To show (40), we have from the second inequality in (14) and the fact $\epsilon_k = \tilde{\epsilon} \leq \sqrt{\frac{\epsilon^3(\sigma-1)}{8\sigma+1}}$, $\forall k$ that
$$
\|z^k\| \leq 2\|z^k\| + \sqrt{\frac{2\sigma^2}{\mu} \frac{\sigma}{\sigma+1}} \leq 2\|z^k\| + \sqrt{\frac{2\sigma^2}{8\sigma+1}}, \forall k \geq 1.
$$

Hence, for each $1 \leq k \leq K$ with the $K$ given in (39), it holds
$$
\|z^k\| \leq 2\|z^k\| + \sqrt{\frac{2\sigma^2}{8\sigma+1}} \leq 2\|z^k\| + \sqrt{\frac{2\sigma^2}{8\sigma+1}} \max \left\{ \|z^k\|, 2\sqrt{2\|z^k\|} \right\}.
$$

Since the output $\tilde{z}$ must be one of $\{z^k\}_{k=1}^K$, we complete the proof. \hfill \square

By Theorem 7, we establish the overall iteration complexity of Algorithm 7 to produce an $\epsilon$-KKT point of (1). We first give the result for the case of $m = 1$.

**Theorem 8 (Iteration complexity when $m = 1$)** Suppose that Assumptions 1 through 4 hold, and $m = 1$ in (1). Let $(\beta_0, \sigma, \epsilon, \gamma_1, \gamma_2)$ be the input of Algorithm 7 and $\{B_k, y^k, z^k\}_{k=0}^\infty$ be the generated sequence. Suppose $\tilde{\epsilon} = \min \left\{ \epsilon, \frac{\epsilon^3(\sigma-1)}{8\sigma+1} \right\} \leq \epsilon, \frac{1}{2\mu}, \frac{4B_0(\mu+\beta)\beta_0^2}{\beta_k \mu}$, $\forall k \geq 0$. Let $\epsilon_k = \tilde{\epsilon}$ for all $k \geq 0$. Then Algorithm 7 needs at most $T_{\text{total}} = O \left( \frac{L_f+L_g(1+\|z^k\|)}{\mu} \log \epsilon^3 \right)$ evaluations on $f$, $\nabla f$, $g$, and $J_g$ to produce an $\epsilon$-KKT point of (1).
Proof. Let $K$ be the integer given in (39) and $L_{g^k} = L_f + L_g \max\{1, 4\|z^*\| + 2\|z^k\|\}$ for $0 \leq k \leq K - 1$. Also, let $T_k$ be the number of evaluations on $f$, $\nabla f$, $g$, and $J_g$ during the $k$-th iteration of Algorithm 7. From Theorem 4 and the setting $\delta_k = \frac{\epsilon}{\sqrt{2} \epsilon_k},$ we have that the complexity incurred by Line 6 of Algorithm 7 is $O(\sqrt{\frac{L_{g^k}}{\mu}} |\log \epsilon|^2)$. In addition, the complexity incurred by Line 10 is $O(\sqrt{\frac{L_{g^k}}{\mu}} |\log \epsilon|)$. From (14) with $\varepsilon_t = \bar{\epsilon}, \forall t$, it follows $\|z^k\| = O(\|z^*\|)$, and thus $L_{g^k} = O(L_f + L_g(1 + \|z^*\|))$ for $0 \leq k \leq K - 1$. Therefore, $T_k = O\left(\sqrt{\frac{L_f + L_g(1 + \|z^*\|)}{\mu}} |\log \epsilon|^2\right)$. Since $K = O(|\log \epsilon|)$ in (39), the total complexity $T_{\text{total}} = \sum_{k=0}^{K-1} T_k = O\left(\sqrt{\frac{L_f + L_g(1 + \|z^*\|)}{\mu}} |\log \epsilon|^3\right)$, which completes the proof.

Remark 4 If $\beta_0$ is taken in the order of $\frac{1}{\sqrt{\epsilon}}$, then $K = O(1)$ in (39). In this case, the total complexity of Algorithm 7 is $O\left(\sqrt{\frac{L_f + L_g(1 + \|z^*\|)}{\mu}} |\log \epsilon|^2\right)$ to produce an $\varepsilon$-KKT point.

Similarly, we can show the complexity result for the case of $m > 1$ by using Theorem 6.

Theorem 9 (Iteration complexity when $m > 1$) Suppose that Assumptions 1 through 4 hold, and $m > 1$ in (1). Let $(\beta_0, \sigma, \varepsilon, \gamma_1, \gamma_2)$ be the input of Algorithm 7 and $(\{x^k, y^k, z^k\})_{k \geq 0}$ be the generated sequence. Suppose $\bar{\epsilon} = \min \left\{ \varepsilon, \sqrt{\frac{\mu(\sigma - 1)}{8\sigma + 1}} \right\} \leq \left\{ \varepsilon, \frac{24B_\gamma(\mu + \beta_0^2)}{\mu} \right\}, \forall k \geq 0$. Let $\varepsilon_k = \bar{\epsilon}$ for all $k \geq 0$. Then Algorithm 7 needs at most $T_{\text{total}} = O(m^2 \sqrt{\frac{L_f + L_g(1 + \|z^*\|)}{\mu}} |\log \epsilon|^3)$ evaluations on $f$, $\nabla f$, $g$, and $J_g$ to produce an $\varepsilon$-KKT point of (1).

Remark 5 Similar to Remark 4, the total complexity can be improved to $O(m^2 \sqrt{\frac{L_f + L_g(1 + \|z^*\|)}{\mu}} |\log \epsilon|^2)$ if $\beta_0 = O(\frac{1}{\sqrt{\epsilon}})$. Ignoring the term $|\log \epsilon|$, our result is better than the best known result $O\left(\sqrt{\frac{L_f + L_g(1 + \|z^*\|)}{\mu \varepsilon}} |\log \epsilon|\right)$ if $m = O(\varepsilon^{-\frac{1}{2}})$. As we discussed in Remark 3, the dependence on $m^2$ can be improved to $m$ if a more advanced cutting plane method is used. In this case, we can obtain a result $O\left(\frac{m}{\sqrt{\mu(\sigma - 1)/8\sigma + 1}} |\log \epsilon|^2\right)$ that is better than $O\left(\sqrt{\frac{L_f + L_g(1 + \|z^*\|)}{\mu \varepsilon}} |\log \epsilon|\right)$ if $m = O(\varepsilon^{-\frac{1}{2}})$ by ignoring the logarithmic term $|\log \epsilon|$.

5 Extensions to convex or nonconvex problems

In this section, we extend the idea of the cutting-plane based FOM to constrained problems with a convex or nonconvex objective. Similar to the strongly convex case, we show that FOMs for solving problems with $O(1)$ nonlinear functional constraints can achieve a complexity result of almost the same order as for solving unconstrained problems.

5.1 Extension to the convex case

We still consider the problem in (1). Suppose that the conditions in Assumptions 1 and 2 hold. Instead of the strong convexity in Assumption 3, we assume the convexity of $f$ in this subsection.
Given a target accuracy $\varepsilon > 0$, to find an $\varepsilon$-KKT point of (1), we follow [15] and solve a perturbed strongly-convex problem:

$$\min_{x \in \mathbb{R}^n} F_\varepsilon(x) := f_\varepsilon(x) + h(x), \text{ s.t. } g(x) := [g_1(x), \ldots, g_m(x)] \leq 0,$$

where

$$f_\varepsilon(x) = f(x) + \frac{\varepsilon}{4D_h} \|x - x^0\|^2 \text{ with } x^0 \in \text{dom}(h).$$

Let $\tilde{x} \in \text{dom}(h)$ be an $\frac{\varepsilon}{7}$-KKT point of (41), i.e., there is $\bar{z} \geq 0$ such that

$$\text{dist}\left(0, \partial_x L_0(\tilde{x}, \bar{z}) + \frac{\varepsilon}{2D_h}(\tilde{x} - x^0)\right) \leq \frac{\varepsilon}{2}, \quad \|\bar{z}\|_\infty \leq \frac{\varepsilon}{2}, \quad \sum_{i=1}^m |\bar{z}_i| g_i(\tilde{x}) \leq \frac{\varepsilon}{2},$$

where $L_0$ is the Lagrange function of (1). Since $\frac{\varepsilon}{2D_h}(\tilde{x} - x^0) \leq \frac{\varepsilon}{7}$, $(\tilde{x}, \bar{z})$ must satisfy the conditions in (5), and thus $\tilde{x}$ is an $\varepsilon$-KKT point of (1). Based on this observation, we can apply Algorithm 7 to the perturbed problem (41). By Theorems 8 and 9 and noticing that $f_\varepsilon$ in (42) is $\frac{\varepsilon}{2D_h}$-strongly convex, we obtain the following complexity result.

Theorem 10 (complexity result for convex cases) Assume that the conditions in Assumptions 1 and 2 hold and that $f$ is convex. Given $\varepsilon > 0$, suppose that the problem (41) has a KKT point $x_\varepsilon^*$ with a corresponding multiplier $z_\varepsilon^*$. Apply Algorithm 7 to find an $\frac{\varepsilon}{2}$-KKT point $\bar{x}$ of (41). Then $\bar{x}$ is an $\varepsilon$-KKT point of (1), and the total number of evaluations on $f, \nabla f, g$, and $J_g$ is $O\left(m^2 \sqrt{D_h\left(L_f + L_g(1+\|z\|_2^2)\right)} \log \varepsilon^{-3}\right)$.

5.2 Extension to the nonconvex case

In this subsection, we assume Assumptions 1 and 2 but do not assume the convexity of $f$. For the nonconvex case, we follow [20] and design an FOM within the framework of the proximal-point method, namely, we solve a sequence of problems in the form of

$$\bar{x}^{k+1} \approx \arg\min_{x \in \mathbb{R}^n} \left\{ F_k(x) := f(x) + L_f \|x - \bar{x}^k\|^2 + h(x), \text{ s.t. } g(x) := [g_1(x), \ldots, g_m(x)] \leq 0 \right\},$$

Under Assumptions 1 and 2, the above problem is convex, and its objective is $L_f$-strongly convex. Hence, we can apply Algorithm 7 to find $\bar{x}^{k+1}$. Let $x_\varepsilon^{k+1}$ be the unique optimal solution to (43). To ensure the existence of a corresponding multiplier for each $k$ and also a uniform bound, we assume the Slater’s condition on the original problem (1).

Assumption 5 (Slater’s condition) There is $x_{\text{feas}} \in \text{relint}(h)$ such that $g_i(x_{\text{feas}}) < 0$ for all $i = 1, \ldots, m$.

With the Slater’s condition, the solution $x_\varepsilon^{k+1}$ to (43) must be a KKT point (cf. [34]). Let $z_\varepsilon^{k+1} \geq 0$ be a corresponding multiplier. We give a uniform bound of $z_\varepsilon^{k+1}$ below.

Lemma 13 (uniform bound of multipliers) Assume Assumptions 1, 2, and 5. Let $x^*$ be a minimizer of (1), and let $x_\varepsilon^{k+1}$ be the KKT point of (43) with a corresponding Lagrangian multiplier $z_\varepsilon^{k+1}$. Then

$$\|z_\varepsilon^{k+1}\| \leq B_z := \frac{F(x_{\text{feas}}) - F(x^*) + L_f D^2_f}{\min_i \left(-g_i(x_{\text{feas}})\right)}, \forall k \geq 0.$$
Proof. From the KKT system, we have that
\[
- \sum_{i=1}^{m} (z_i^{k+1})_i \nabla g_i(x_i^{k+1}) \in \partial F_k(x_i^{k+1}), \quad (z_i^{k+1})_i g_i(x_i^{k+1}) = 0, \forall i = 1, \ldots, m. \quad (45)
\]
Then we have
\[
\sum_{i=1}^{m} (z_i^{k+1})_i g_i(x_{\text{feas}}) \geq \sum_{i=1}^{m} (z_i^{k+1})_i \left( g_i(x_i^{k+1}) + \langle x_{\text{feas}} - x_i^{k+1}, \nabla g_i(x_i^{k+1}) \rangle \right) \\
= \langle x_{\text{feas}} - x_i^{k+1}, \sum_{i=1}^{m} (z_i^{k+1})_i \nabla g_i(x_i^{k+1}) \rangle \\
\geq F_k(x_i^{k+1}) - F_k(x_{\text{feas}}), \quad (46)
\]
where the first inequality is from the convexity of each \( g_i \) and the nonnegativity of \( z_i^{k+1} \), the equality holds because of the second equation in (45), and the last inequality follows from the convexity of \( F_k \) and the first equation in (45).

Since the diameter of \( \text{dom}(h) \) is \( D_h \), it holds that
\[
-F_k(x_i^{k+1}) + F_k(x_{\text{feas}}) = F(x_{\text{feas}}) + L_f \| x_{\text{feas}} - x_i^{k + 1} \|^2 - F(x_i^{k+1}) - L_f \| x_i^{k+1} - x_i^{k} \|^2 \\
\leq F(x_{\text{feas}}) - F(x_i^{k+1}) + L_f D_h^2. \quad (47)
\]
Notice \( F(x_i^{k+1}) \geq F(x^*) \). Hence, \( F(x_{\text{feas}}) - F(x_i^{k+1}) \leq F(x_{\text{feas}}) - F(x^*) \), and from (47), it follows that \(-F_k(x_i^{k+1}) + F_k(x_{\text{feas}}) \leq F(x_{\text{feas}}) - F(x^*) + L_f D_h^2 \). Now we have from (46) that
\[
\| z_i^{k+1} \|_1 \leq -\frac{F_k(x_i^{k+1}) + F_k(x_{\text{feas}})}{\min_i (-g_i(x_{\text{feas}}))} \leq \frac{F(x_{\text{feas}}) - F(x^*) + L_f D_h^2}{\min_i (-g_i(x_{\text{feas}}))},
\]
and we complete the proof by \( \| z_i^{k+1} \|_2 \leq \| z_i^{k+1} \|_1 \).

Similar to our discussion in section 5.1, we notice that if \( x_i^{k+1} \) is an \( \varepsilon \)-KKT point of (43) and also \( 2L_f \| x_i^{k+1} - x_i^{k} \| \leq \varepsilon \), then \( x_i^{k+1} \) is an \( \varepsilon \)-KKT point of (1). Below, we show that the sum of \( \| x_i^{k+1} - x_i^{k} \|^2 \) can be controlled if each \( x_i^{k+1} \) is obtained with sufficient accuracy, and thus a near-KKT point of (1) can be produced.

**Theorem 11 (Complexity result for nonconvex cases)** Assume Assumptions 1, 2, and 5. Let \( x^* \) be a minimizer of (1). Let \( \varepsilon > 0 \) be given and \( x^0 \in \text{dom}(h) \). Generate the sequence \( \{(x^k, z^k)\}_{k \geq 1} \) by applying Algorithm 7 to (43) with the target accuracy \( \varepsilon = \min \{ \frac{\varepsilon}{2}, \frac{\varepsilon^2}{64L_f(D_h + 2B_z)} \} \), where
\[
B_z := 2B_z + \sqrt{\frac{\varepsilon^2}{8D_h}} \max \{ 3B_z, 2\sqrt{2B_z}, 2 \}, \quad (48)
\]
with \( B_z \) defined in (44). Then after solving at most \( K \) proximal point subproblems as that in (43), we can find an \( \varepsilon \)-KKT point of (1), where
\[
K = \left[ \frac{64L_f(F(x^0) - F(x^*) + L_f D_h^2 + B_z \| g(x^0) \|)}{\varepsilon^2} \right]. \quad (49)
\]
In addition, the total number of evaluations on \( f, \nabla f, g, \) and \( J_g \) is \( O\left( \frac{n^2}{\varepsilon^3} \log \varepsilon \right) \).
Proof. Since each \( (\bar{x}^{k+1}, \bar{z}^{k+1}) \) is an output from Algorithm 7 applied to (43) and with a target accuracy \( \bar{\varepsilon} \), then \( \bar{x}^{k+1} \) is an \( \bar{\varepsilon} \)-KKT point of the problem in (43), and thus there is a subgradient \( \nabla F_k(\bar{x}^{k+1}) \in \partial F_k(\bar{x}^{k+1}) \) such that
\[
\|\nabla F_k(\bar{x}^{k+1}) + J^T_g(\bar{x}^{k+1})\bar{z}^{k+1}\| \leq \bar{\varepsilon}, \quad \|g(\bar{x}^{k+1})\| \leq \bar{\varepsilon}, \forall k \geq 0.
\] (50)
From the first inequality in (50) and recalling that the diameter of \( \text{dom}(h) \) is \( D_h \), we have
\[
\left\langle \bar{x}^{k+1} - \bar{x}_k, \nabla F_k(\bar{x}^{k+1}) + J^T_g(\bar{x}^{k+1})\bar{z}^{k+1} \right\rangle \leq D_h \bar{\varepsilon}.
\]
Hence, by the \( L_f \)-strong convexity of \( F_k \) and convexity of each \( g_i \), we have
\[
D_h \bar{\varepsilon} \geq \left\langle \bar{x}^{k+1} - \bar{x}_k, \nabla F_k(\bar{x}^{k+1}) + J^T_g(\bar{x}^{k+1})\bar{z}^{k+1} \right\rangle \\
\geq F_k(\bar{x}^{k+1}) - F_k(\bar{x}_k) + \frac{L_f}{2}\|\bar{x}^{k+1} - \bar{x}_k\|^2 + \langle \bar{z}^{k+1}, g(\bar{x}^{k+1}) - g(\bar{x}_k) \rangle \\
= F(\bar{x}^{k+1}) - F(\bar{x}_k) + L_f\|\bar{x}^{k+1} - \bar{x}_k\|^2 + \langle \bar{z}^{k+1}, g(\bar{x}^{k+1}) - g(\bar{x}_k) \rangle.
\] (51)
By (40) and (44), we have \( \|\bar{z}^{k+1}\| \leq \bar{B}_z, \forall k \geq 0 \), where \( \bar{B}_z \) is given in (48). Hence, it follows from the second inequality in (50) that \( \langle \bar{z}^{k+1}, g(\bar{x}^{k+1}) - g(\bar{x}_k) \rangle \geq -2\bar{\varepsilon}\bar{B}_z, \forall k \geq 1 \). Now summing up (51), we obtain
\[
L_f \sum_{k=0}^{K-1} \|\bar{x}^{k+1} - \bar{x}_k\|^2 \leq KD_h \bar{\varepsilon} + F(\bar{x}_0) - F(\bar{x}_K) + (2K - 1)\bar{\varepsilon}\bar{B}_z + \bar{B}_z\|g(\bar{x}_0)\|_+\|,
\] (52)
where we have used \( \langle \bar{z}^1, g(\bar{x}_0) \rangle \leq \|\bar{z}^1\| \cdot \|g(\bar{x}_0)\|_+ \leq \bar{B}_z\|g(\bar{x}_0)\|_+ \).
Because \( x^*_k \) is a KKT-point of (43) with a corresponding multiplier \( z^*_k \), we have from (4) that
\[
F_{K-1}(\bar{x}^*_K) - F_{K-1}(x^*_K) + \langle z^*_K, g(\bar{x}^*_K) \rangle \geq 0.
\]
Plugging \( F_{K-1}(\cdot) = F(\cdot) + L_f\|\cdot - \bar{x}^{K-1}\|^2 \) into the above equation gives
\[
F(\bar{x}^*_K) + L_f\|\bar{x}^*_K - \bar{x}^{K-1}\|^2 - F(x^*_K) - L_f\|x^*_K - \bar{x}^{K-1}\|^2 + \langle z^*_K, g(\bar{x}^*_K) \rangle \geq 0.
\]
Now using (44), \( \|g(\bar{x}^*_K)\| \leq \bar{\varepsilon}, \|\bar{x}^*_K - \bar{x}^{K-1}\|^2 \leq D_h^2, \) and the fact \( F(\bar{x}^*_K) \geq F(x^*_K) \), we have from the above inequality that \( -F(\bar{x}^*_K) \leq -F(x^*_K) + L_fD_h^2 + \bar{\varepsilon}\bar{B}_z \). This inequality together with (52) gives
\[
L_f \sum_{k=0}^{K-1} \|\bar{x}^{k+1} - \bar{x}_k\|^2 \leq KD_h \bar{\varepsilon} + F(\bar{x}_0) - F(x^*_K) + L_fD_h^2 + 2K\bar{\varepsilon}\bar{B}_z + \bar{B}_z\|g(\bar{x}_0)\|_+\|.
\] (53)
Multiplying \( L_f \) to both sides of the above inequality and taking square root, we have
\[
\min_{0 \leq k < K} L_f\|\bar{x}^{k+1} - \bar{x}_k\| \leq \sqrt{L_f(D_h \bar{\varepsilon} + 2\bar{B}_z \bar{\varepsilon})} + \sqrt{\frac{L_f\left(F(\bar{x}_0) - F(x^*_K) + L_fD_h^2 + 2K\bar{\varepsilon}\bar{B}_z + \bar{B}_z\|g(\bar{x}_0)\|_+\right)}{K}}.
\] (54)
Therefore, by the setting of \( \bar{\varepsilon} \) and \( K \), we have \( \min_{0 \leq k < K} L_f\|\bar{x}^{k+1} - \bar{x}_k\| \leq \frac{\xi}{4} \). Suppose \( L_f\|\bar{x}^{k0} - \bar{x}^{k0}\| \leq \frac{\xi}{4} \).
Then by our discussion above Theorem 11, \( \bar{x}^{k0} \) is an \( \bar{\varepsilon} \)-KKT point of (1). From Theorems 8 and 9, the complexity of solving one problem as that in (43) is \( O(m^2 \log \varepsilon |z|) \), and thus the total complexity is \( O(Km^2 \log \varepsilon |z|^3) = O\left(\frac{m^2}{\varepsilon^2}\right) \log \varepsilon |z|^3 \). This completes the proof. \( \square \)
6 Experimental results

In this section, we demonstrate the established theory by performing numerical experiments on solving quadratically-constrained quadratic program (QCQP):

\[
\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q_0 x + x^T c_0, \quad \text{s.t.} \quad \frac{1}{2} x^T Q_j x + x^T c_j + d_j \leq 0, \quad j = 1, \ldots, m; \quad x_i \in [l_i, u_i], \quad i = 1, \ldots, n. \tag{55}
\]

In the experiment, \(Q_0\) is generated to be positive definite, \(Q_j\) is positive semidefinite but rank-deficient for each \(j = 1, \ldots, m\), and \(l_i = -10\) and \(u_i = 10\) for each \(i\). All \(d_j\) are negative so the Slater’s condition holds.

We compare two implementations of the iALM in Algorithm 1. One directly applies the APG method in Algorithm 2 to solve each ALM subproblem, and we call it “APG-based iALM”. The other uses the proposed cutting-plane based FOM to solve subproblems, namely, we implement Algorithm 7 to solve (55), and we call it “cutting-plane iALM”. For both implementations, we set \(\beta_k = 10^{k-1}\) for each outer iteration \(k \geq 1\) and run the iALM to 5 outer iterations. The target accuracy for a near-KKT point is set to \(\varepsilon = 10^{-4}\). In the implementation of the APG-based iALM, due to the quadratic penalty term, we apply Algorithm 2 with line search for a local smoothness constant and set the parameters to \(\gamma_1 = 1.5, \gamma_2 = 2, L_{\min} = 1\). In the implementation of the cutting-plane iALM, we use Algorithm 2 to solve problems in the form of (21), for which we can explicitly compute the global smoothness constant, and thus we simply set \(L_{\min}\) to the global smoothness constant.

We test three groups of QCQP instances, each of which has \(n = 1000\). The first group has \(m = 1\) constraint, the second has \(m = 2\), and the third has \(m = 5\). For each group, we conduct 5 independent trials. For each instance, we report the number of gradient and function evaluations, the primal residual, dual residual, and complementarity violation, which are denoted as \#grad, \#func, pres, dres, and compl, for solving each ALM subproblem. In order to demonstrate the worst-case theoretical result, we use randomly-generated initial point while solving each ALM subproblem. The performance of the iALM can be much better if the warm-start technique is adopted. The results are shown in Tables 1–3. For the cutting-plane iALM, its \#func. is zero and not shown in the tables, because we feed the APG an explicitly-computed smoothness constant and no line search is performed.

From the results, we see that as the penalty parameter increases, the APG-based iALM needs significantly more iterations to solve the subproblems, while the cutting-plane iALM does not suffer from the big penalty parameter. However, the cutting-plane iALM has worse scalability to \(m\), and this matches with our theory.

7 Concluding remarks

We have proposed a cutting-plane based first-order method (FOM) for solving strongly-convex problems with \(m\) functional constraints. If \(m = O(1)\), our method can achieve a complexity result of \(\tilde{O}(\sqrt{\kappa})\), where \(\kappa\) denotes the condition number of the underlying problem in some sense. In general, a complexity result of \(\tilde{O}(m^2 \sqrt{\kappa})\) has been established. To give an \(\varepsilon\)-KKT point, our result is better than an existing lower bound if \(m = o(\varepsilon^{-\frac{4}{3}})\). Our result can be further improved to \(\tilde{O}(m \sqrt{\kappa})\) by using a more advanced cutting-plane method as the key ingredient in our algorithm. We have also extended the idea of the cutting-plane based FOM to convex cases and nonconvex cases. Similarly, when \(m = O(1)\), we obtained almost the same-order complexity results (with a difference of a polynomial of \(|\log \varepsilon|\)) as for solving an unconstrained problem.
Table 1 Results by the APG based first-order iALM and the proposed cutting-plane based first-order iALM for solving QCQP (55) with \( m = 1 \) and \( n = 1000 \).

| trial 1 | total running time = 1592 sec. | total running time = 25 sec. |
|---------|--------------------------------|-----------------------------|
| 1       | 6281                           | 9188                        |
| 2       | 9188                           | 5.71e-02                    |
| 3       | 9188                           | 9.70e-05                    |
| 4       | 9188                           | 3.26e-03                    |
| 5       | 9188                           | 3.26e-03                    |

| trial 2 | total running time = 1609 sec. | total running time = 24 sec. |
|---------|--------------------------------|-----------------------------|
| 1       | 6307                           | 9226                        |
| 2       | 9226                           | 4.37e-02                    |
| 3       | 9226                           | 9.95e-05                    |
| 4       | 9226                           | 1.91e-03                    |
| 5       | 9226                           | 1.91e-03                    |

| trial 3 | total running time = 1699 sec. | total running time = 23 sec. |
|---------|--------------------------------|-----------------------------|
| 1       | 6704                           | 9806                        |
| 2       | 9806                           | 4.78e-02                    |
| 3       | 9806                           | 9.56e-05                    |
| 4       | 9806                           | 2.29e-03                    |
| 5       | 9806                           | 2.29e-03                    |

| trial 4 | total running time = 1679 sec. | total running time = 23 sec. |
|---------|--------------------------------|-----------------------------|
| 1       | 6649                           | 9682                        |
| 2       | 9682                           | 4.31e-02                    |
| 3       | 9682                           | 9.56e-05                    |
| 4       | 9682                           | 1.86e-03                    |
| 5       | 9682                           | 1.86e-03                    |

| trial 5 | total running time = 1659 sec. | total running time = 26 sec. |
|---------|--------------------------------|-----------------------------|
| 1       | 6541                           | 9568                        |
| 2       | 9568                           | 5.14e-02                    |
| 3       | 9568                           | 9.66e-05                    |
| 4       | 9568                           | 2.65e-03                    |
| 5       | 9568                           | 2.65e-03                    |

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Table 2 Results by the APG based first-order iALM and the proposed cutting-plane based first-order iALM for solving QCQP (55) with \( m = 2 \) and \( n = 1000 \).

|          | APG based iALM | proposed cutting-plane iALM |
|----------|-----------------|-----------------------------|
|          | out.\( \text{iter} \) | \( \beta \) | \#grad | \#func | pres | dres | compl | total running time | total running time |
| trial 1  | 1               | 6615 | 9676 | 5.40e-02 | 9.89e-05 | 2.09e-03 | 5.59e-02 | 8.97e-09 | 1.94e-03 |
|          | 2               | 21978 | 32122 | 5.96e-07 | 9.97e-05 | 2.15e-08 | 4.89e-07 | 6.98e-09 | 1.83e-08 |
|          | 3 \( 10^2 \)   | 72140 | 105404 | 3.21e-09 | 1.00e-04 | 5.68e-10 | 3.57e-09 | 3.80e-09 | 1.50e-10 |
|          | 4 \( 10^3 \)   | 235420 | 349334 | 1.95e-09 | 9.80e-05 | 8.61e-11 | 7.07e-10 | 1.63e-11 | 1.63e-11 |
|          | 5 \( 10^4 \)   | 766572 | 1198688 | 1.99e-09 | 9.96e-05 | 8.75e-12 | 5.14e-09 | 3.77e-09 | 1.73e-12 |
| trial 2  | 1               | 6615 | 9730 | 5.76e-02 | 9.72e-05 | 2.46e-03 | 5.47e-02 | 7.64e-09 | 2.16e-03 |
|          | 2               | 22145 | 32366 | 6.59e-07 | 9.99e-05 | 3.03e-08 | 5.16e-07 | 9.72e-09 | 2.07e-08 |
|          | 3 \( 10^2 \)   | 72255 | 105572 | 1.01e-08 | 9.97e-05 | 4.97e-10 | 4.14e-09 | 2.06e-10 | 2.06e-10 |
|          | 4 \( 10^3 \)   | 234871 | 343132 | 4.62e-09 | 9.97e-05 | 2.20e-10 | 1.98e-09 | 2.24e-11 | 2.24e-11 |
|          | 5 \( 10^4 \)   | 763890 | 1115950 | 2.56e-10 | 9.99e-05 | 1.41e-11 | 9.78e-09 | 3.01e-12 | 3.01e-12 |
| trial 3  | 1               | 6986 | 10218 | 6.98e-02 | 9.94e-05 | 3.53e-03 | 5.91e-02 | 9.39e-09 | 2.74e-03 |
|          | 2               | 23158 | 33846 | 1.11e-06 | 9.82e-05 | 4.94e-08 | 5.68e-07 | 5.57e-09 | 2.53e-08 |
|          | 3 \( 10^2 \)   | 75312 | 110038 | 9.48e-09 | 9.87e-05 | 4.33e-10 | 1.01e-10 | 5.59e-09 | 4.54e-10 |
|          | 4 \( 10^3 \)   | 245766 | 359048 | 1.90e-09 | 9.95e-05 | 1.70e-10 | 5.87e-10 | 6.93e-09 | 3.54e-11 |
|          | 5 \( 10^4 \)   | 796022 | 1162890 | 1.37e-10 | 9.98e-05 | 6.25e-12 | 6.97e-10 | 9.53e-12 | 9.53e-12 |
| trial 4  | 1               | 7038 | 10294 | 6.39e-02 | 9.95e-05 | 3.00e-03 | 5.56e-02 | 3.99e-10 | 2.28e-03 |
|          | 2               | 23247 | 33976 | 1.09e-06 | 9.85e-05 | 5.16e-08 | 5.25e-07 | 3.74e-11 | 2.21e-08 |
|          | 3 \( 10^2 \)   | 76117 | 111038 | 4.82e-09 | 9.87e-05 | 4.33e-10 | 1.01e-10 | 5.59e-09 | 4.54e-10 |
|          | 4 \( 10^3 \)   | 237846 | 347132 | 6.52e-09 | 9.97e-05 | 2.20e-10 | 1.18e-10 | 1.98e-09 | 2.24e-11 |
|          | 5 \( 10^4 \)   | 772485 | 1128506 | 2.56e-10 | 9.99e-05 | 1.41e-11 | 8.47e-11 | 9.78e-09 | 3.01e-12 |
| trial 5  | 1               | 6715 | 9822 | 6.15e-02 | 9.80e-05 | 3.00e-03 | 5.91e-02 | 3.99e-10 | 2.28e-03 |
|          | 2               | 22286 | 32572 | 7.34e-07 | 9.90e-05 | 3.51e-08 | 4.85e-07 | 4.42e-09 | 2.17e-08 |
|          | 3 \( 10^2 \)   | 73326 | 107046 | 5.92e-08 | 9.94e-05 | 3.11e-09 | 4.89e-09 | 5.36e-09 | 3.79e-10 |
|          | 4 \( 10^3 \)   | 237846 | 347478 | 6.31e-09 | 9.95e-05 | 4.05e-10 | 2.13e-10 | 8.53e-09 | 2.85e-11 |
|          | 5 \( 10^4 \)   | 772485 | 1128506 | 1.27e-10 | 9.99e-05 | 5.95e-12 | 6.50e-12 | 1.66e-09 | 2.25e-12 |

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Table 3 Results by the APG based first-order iALM and the proposed cutting-plane based first-order iALM for solving QCQP (55) with \( m = 5 \) and \( n = 1000 \).

| trial | \( \beta \) | APG based iALM | proposed cutting-plane iALM |
|-------|-------------|----------------|----------------------------|
|       |             | total running time = 6190 sec. | total running time = 740 sec. |
| t1    | 1           | 7075           | 10348           | 7.77e-02 | 9.3e-05 | 2.97e-03 | 32538 | 7.77e-02 | 4.21e-09 | 2.97e-03 |
|       | 2           | 23340          | 34112           | 1.72e-06 | 9.92e-05 | 7.07e-08 | 32744 | 8.60e-07 | 6.13e-09 | 3.33e-08 |
| 3     | 10 \(^2\)   | 76514          | 111794          | 5.58e-08 | 9.96e-05 | 1.71e-09 | 32946 | 8.54e-09 | 9.41e-09 | 3.76e-10 |
| 4     | 10 \(^3\)   | 249880         | 365058          | 6.85e-09 | 9.92e-05 | 2.56e-10 | 33232 | 5.83e-10 | 2.88e-09 | 2.95e-11 |
| 5     | 10 \(^4\)   | 816213         | 1192386         | 5.24e-10 | 9.99e-05 | 2.91e-11 | 33402 | 3.42e-11 | 6.35e-09 | 2.71e-12 |
|       |             | total running time = 5915 sec. | total running time = 722 sec. |
|       | 1           | 7072           | 10342           | 7.50e-02 | 9.99e-05 | 2.61e-03 | 32784 | 7.50e-02 | 9.61e-09 | 2.61e-03 |
|       | 2           | 23195          | 33900           | 8.41e-07 | 9.93e-05 | 2.99e-08 | 32950 | 7.89e-07 | 3.69e-09 | 2.78e-08 |
| 3     | 10 \(^2\)   | 76206          | 111344          | 4.94e-08 | 9.92e-05 | 2.04e-09 | 33234 | 2.30e-09 | 3.69e-09 | 1.63e-10 |
| 4     | 10 \(^3\)   | 248830         | 365224          | 4.04e-09 | 9.98e-05 | 1.42e-08 | 33458 | 3.54e-10 | 3.70e-09 | 1.15e-11 |
| 5     | 10 \(^4\)   | 812789         | 1187381         | 4.97e-11 | 9.96e-05 | 9.95e-12 | 33648 | 0.00e+00 | 5.79e-09 | 2.77e-12 |
|       |             | total running time = 5939 sec. | total running time = 699 sec. |
|       | 1           | 7120           | 10414           | 7.10e-02 | 9.58e-05 | 2.40e-03 | 33270 | 7.10e-02 | 3.61e-09 | 2.40e-03 |
|       | 2           | 23633          | 34540           | 1.31e-06 | 9.88e-05 | 4.00e-08 | 33464 | 7.38e-07 | 7.78e-09 | 2.54e-08 |
| 3     | 10 \(^2\)   | 77367          | 113040          | 3.45e-08 | 9.93e-05 | 9.96e-07 | 33318 | 6.70e-09 | 6.85e-09 | 2.47e-10 |
| 4     | 10 \(^3\)   | 253174         | 369870          | 4.98e-11 | 9.97e-05 | 9.95e-12 | 33686 | 3.42e-10 | 9.49e-09 | 1.32e-11 |
| 5     | 10 \(^4\)   | 824422         | 1204378         | 4.97e-11 | 9.96e-05 | 9.95e-12 | 34194 | 7.07e-11 | 7.66e-09 | 3.69e-12 |
|       |             | total running time = 5775 sec. | total running time = 708 sec. |
|       | 1           | 7012           | 10256           | 8.19e-02 | 9.26e-05 | 3.06e-03 | 32678 | 8.19e-02 | 4.81e-09 | 3.06e-03 |
|       | 2           | 23154          | 33840           | 1.25e-06 | 9.98e-05 | 4.71e-08 | 33126 | 8.47e-07 | 9.51e-09 | 3.20e-08 |
| 3     | 10 \(^2\)   | 76076          | 111154          | 3.45e-08 | 9.93e-05 | 9.96e-07 | 33318 | 6.70e-09 | 6.85e-09 | 2.47e-10 |
| 4     | 10 \(^3\)   | 247554         | 361660          | 3.82e-09 | 9.97e-05 | 1.54e-10 | 33538 | 5.65e-10 | 9.78e-09 | 2.30e-11 |
| 5     | 10 \(^4\)   | 803441         | 1173728         | 2.17e-10 | 9.97e-05 | 2.04e-11 | 33748 | 4.82e-11 | 3.95e-09 | 1.37e-12 |
|       |             | total running time = 5587 sec. | total running time = 727 sec. |

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