Electromagnetic $N + \gamma^* \rightarrow R$ transition form factors of nucleons from the hard-wall AdS/QCD model

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Abstract

The electromagnetic transition form factors for the $N + \gamma^* \rightarrow R$ transition between the ground and excited states of nucleon are studied in the framework of the hard-wall model of AdS/QCD. The profile functions of the spinor and vector fields in the bulk of AdS space are presented. Using AdS/CFT correspondence between the generating functions in the bulk and boundary theories an expression for the transition form factors is obtained from the bulk action for the interaction between the photon and nucleon fields. We consider the $R(1440,1535,1710) \rightarrow N$ transitions and plot the Dirac, Pauli and electric, magnetic form factors dependencies on momentum transfer. Also, plots for the helicity amplitudes, have been presented for these transitions and compared to experimental data.

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I. INTRODUCTION

AdS/CFT correspondence, also known as gauge-gravity duality is one of the most exciting discoveries in modern theoretical physics. This correspondence adapted to describe the low-energy dynamics of QCD in the simplest models known as AdS/QCD models. These models are useful for understanding properties of QCD at low energies such as chiral symmetry breaking, hidden local symmetry, vector meson dominance at large $N$ and high energy regimes. There are two approaches in AdS/QCD correspondence called as top-down and bottom-up approaches [1]. The bottom-up approach describes various low-energy phenomenology of QCD. The hard and soft-wall models are the typical ones in the bottom-up approaches. Mass spectra, decay and coupling constants, form factors are calculated in the framework of these models gives the results in a good agreement with the data. Holographic QCD models are useful as well as for the study of internal structure of hadrons, including their excitations by means of study of the transition form factors. The last decade has been observed a progress in the experimental and theoretical studies of nucleon resonances (the radial/orbital nucleon excitations with $J^P = 1/2^+, \ J^P = 3/2^+$). Electro- and photoproduction of the resonance states helps us to study their internal structure, wave functions and interactions between constituents. The nucleon resonance $N(1440)$ with $J^P = 1/2^+$ (the Roper resonance $P_{11}$ or simply $R$) is the lowest-lying excitation of the nucleon and its structure issue is a longstanding question. Studies in the constituent quark model (CQM) framework showed that the inner structure of the Roper is possibly more complicated than the structure of the other lightest baryons. It was found that the observed mass of the Roper resonance is low and the decay width is large in comparison with the values predicted by the CQM [2]. The Roper state is the first ($2S$) radial excitation of the nucleon ground state $s^3[3]_X$. But this description fails to explain either the large decay width $\Gamma_R \simeq 300\ MeV$ or the branching ratios for the $\pi N$ (55% – 75%) and $\sigma N$ (5% – 20%) decay channels [2, 3].

The elementary theory of strong interactions QCD provides a framework, which is directly usable only at high momentum transfers. Nevertheless, the discussed data [2–4] span the momentum transfer range $0 \leq Q^2 \lesssim 4 - 5\ GeV^2$ (up to $\sim 12\ GeV^2$ for the JLab upgrade). A major challenge for theory is that a quantitative description of the transition amplitudes must include soft nonperturbative contributions as well. Holographic models are good in that sense they have no limitation on $Q^2$.

The ground state, $J^P = (1/2)^+$ with radial quantum number is $n = 0$ and angular momentum $l = 0$; first excited state, $J^P = (1/2)^-$ with $(n, l) = (0, 1)$; second excited state, $J^P = (1/2)^+$.
Recently the electromagnetic nucleon-Roper transition has been studied in the framework of soft-wall AdS/QCD model \[5,7\]. \( N + \gamma^* \rightarrow R(1440) \) transition was studied in Refs. \[8,9\] within the soft-wall model at the finite and zero temperature cases. Electromagnetic transition form factors of nucleons were studied in the light-front holographic QCD in Refs. \[10,12\]. In Ref. \[8\] the Roper electroproduction was considered in a soft-wall AdS/QCD model \[6,8,9,13,14\] with inclusion of the leading three-quark (3q) state and higher Fock components, as well. AdS/QCD models have an advantage is that it contains the correct power scaling description of form factors and helicity amplitudes at large \( Q^2 \) \[15\]. These models also provide agreement with data at low and intermediate values of the momentum transfer \( Q^2 \) for these functions. The nucleon-Roper electromagnetic transition within the light front holographic QCD was studied in Ref. \[5\], where the Dirac form factor was considered. In Ref. \[7\] authors extended the soft-wall model of AdS/QCD to the description of all nucleon resonances with adjustable quantum numbers and considered form factors, helicity amplitudes and charge radii for the transition. In Ref. \[16\] authors showed that the in the framework of the extended version of the effective action the electromagnetic form factors of the nucleon and of the Roper resonance are sufficiently improved. This was achieved by including non-minimal interaction terms into the Lagrangian consistent with gauge invariance, which contributes to the momentum dependence of the form factors and helicity amplitudes. Moreover, in Ref. \[7\] it was presented a description of electromagnetic properties of the nucleon and the Roper at small finite temperatures using the formalism developed in Ref. \[17\]. In the \[17\] the authors present the study of the Roper resonance state \( N(1535) \) which has a negative-parity. In Ref. \[18\] the \( N(1520) \) and \( N(1535) \) wave functions are defined without any adjustable parameters and are used to make predictions for the valence quark contributions to the transition form factors in semirelativistic approximation and the results are compared to data particularly for high momentum transfer. The authors study the structure of the \( N(1710) \) resonance and calculate the \( N + \gamma^* \rightarrow R(1710) \) electromagnetic form factors, helicity amplitudes, and predict that they are almost identical to those of the \( N + \gamma^* \rightarrow R(1440) \) transition in the high momentum transfer region in covariant spectator quark model in Ref. \[19\]. In Ref. \[20\] has been presented a comprehensive review of the electromagnetic transition form factors of baryons where the nucleon excitations \( N^* (\gamma^*N \rightarrow N^*) \) in the first, second and third nucleon resonance states and also discussed the limitations of some parametrizations of the data of the helicity amplitudes and multipole form factors. Despite some good results for the abovementioned transitions within the soft-wall model, to check holographic models for a form factors calculation, it is reasonable to
consider transition form factors in the hard-wall model framework as well because, in several other form factor cases, the results of these models agree one with another \[21, 22\] and, in some cases, they differ \[23\].

Here, we aim to consider the $N + \gamma^* \rightarrow R(1440, 1710)$ transitions with positive parity Roper states and $N + \gamma^* \rightarrow R(1535)$ transition with negative parity Roper state and study the Dirac-Pauli electromagnetic and electric, magnetic Sachs form factors, helicity amplitudes in the framework of hard-wall model.

This paper is structured as follows: In Secs. II and III we introduce the profile function for the nucleons and bulk-to-boundary propagator for the electromagnetic field. The electromagnetic current of the Roper nucleon transition is defined in Sec. IV and in Sec. V, we discuss the form factors, charge and magnetic radii and helicity amplitudes of the $N + \gamma^* \rightarrow R$ transition. In Sec. VI, we present our numerical analysis and results and comment these results in VII section.

II. NUCLEONS IN HARD-WALL MODEL

Electromagnetic transition form factors are an important probe for the study of the internal structure of nucleons. These form factors were studied in the framework of certain models. We apply the hard-wall model of the holographic QCD for the calculation of Roper-nucleon transition form factors. The metric for this model is the five-dimensional AdS metric in Poincare patch:

$$ds^2 = \frac{1}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2). \quad (2.1)$$

The fifth coordinate $z$ extends from 0 to $z_m$. These boundaries of the AdS space are called the ultraviolet (UV) and infrared (IR) ones, respectively. $\eta_{\mu\nu}$ is the metric of the Minkowski space ($\eta_{\mu\nu} = diag(1, -1, -1, -1); \mu, \nu = 0, 1, 2, 3$).

In order to describe nucleons in the AdS/CFT correspondence framework, it is necessary to introduce two independent fermion fields in the bulk of AdS space, which describe to chiral components of the nucleons as is known from \[24\]. The minimal bulk action for the spinor field is written in the following:

$$S = \int dzd^4x \sqrt{g}(i\bar{N}_1e_A^M\Gamma^A D_MN_1 - m_5\bar{N}_1N_1), \quad (2.2)$$

where $g$ is the determinant of the AdS metric, $e_A^M$ is the vierbein for the metric \(2.1\) and $\Gamma^A = (\gamma^\mu, -i\gamma^5)$ are the Dirac matrices. $m_5 = \pm \frac{5}{2}$ for $\Psi_{1,2}$ correspondingly \[25\]. The Lorentz and gauge-covariant derivative is defined:

$$D_M = \partial_M - \frac{i}{4}\omega_M^{AB}\Sigma_{AB}, \quad (2.3)$$
where $\omega^{AB}_M$ is the spin connection and $\Sigma_{AB}$ is defined as: $\Sigma_{AB} = \frac{1}{2!} [\Gamma^A, \Gamma^B]$. Nonvanishing components of the spin connections for the metric (2.1) are the following ones $\omega^{5A}_\mu = -\omega^{A5}_\mu = \frac{1}{z} \delta^A_\mu$.

Equation of motion obtained from the action (2.2) has an explicit form:

$$(z \gamma^5 \partial_z + iz \not \partial - 2 \gamma^5) N_1 - m_5 N_1 = 0. \tag{2.4}$$

It is favorable to solve this equation in terms of Fourier components:

$$N_{1,2} = \frac{1}{2\pi} \int \left[ F_{1,2L}(p, z)\psi_{1,2L}(p) + F_{1,2R}(p, z)\psi_{1,2R}(p) \right] e^{-ipx} d^4p, \tag{2.5}$$

where the 4D spinors satisfy the free Dirac equation:

$$ip\Psi(p) = |p|\Psi(p). \tag{2.6}$$

Here $p$ denotes $p = |p| = \sqrt{p^2}$ for a time-like four-momentum $p$. Then the Dirac equation (2.4) will be written as equations for the profile functions $F_{1,2L,R}$ [25]:

$$\left( \partial_z^2 - \frac{4}{z} \partial_z + \frac{6 + m_5 - m_5^2}{z^2} \right) F_{1,2L}(p, z) = -p^2 F_{1,2L}(p, z),$$

$$\left( \partial_z^2 - \frac{4}{z} \partial_z + \frac{6 + m_5 - m_5^2}{z^2} \right) F_{1,2R}(p, z) = -p^2 F_{1,2R}(p, z). \tag{2.7}$$

Extra components of these spinors are eliminated by the UV and IR boundary conditions [25] imposed on the $N_{1,2}$ spinors. The mass eigenvalues corresponding to the Kaluza-Klein modes are determined by the IR boundary $F_R(z_m) = 0$. Solutions to the equations (2.7) are expressed in terms of Bessel functions $J_{2,3}$ [25] [26]:

$$F_{1L}^n(p, z) = -c_1^n z^\frac{5}{2} J_2(p, z), F_{1R}^n(p, z) = c_1^n z^\frac{5}{2} J_3(p, z),$$

$$F_{2L}^n(p, z) = -c_2^n z^\frac{5}{2} J_3(p, z), F_{2R}^n(p, z) = c_2^n z^\frac{5}{2} J_2(p, z), \tag{2.8}$$

where $c_{1,2}^n$ are constants were found from the normalization conditions and are equal [26]:

$$|c_{1,2}^n| = \frac{\sqrt{2}}{z_m J_2(M_n z_m)}. \tag{2.9}$$

Here, $M_n$ is the mass spectrum of the Kaluza-Klein modes corresponding to the ground and excited states of the boundary nucleons. The spectrum $M_n$ is expressed in terms of zeros $\alpha_n^{(3)}$ of the Bessel function $J_3$:

$$M_n = \frac{\alpha_n^{(3)}}{z_m}. \tag{2.10}$$
III. VECTOR FIELD

Action for the bulk vector field $V_M$ corresponding to the photon field in the boundary theory is written in the form:

$$S_V = -\frac{1}{2g^2} \int d^5x \sqrt{g} Tr V_{MN}^2,$$

(3.1)

where $V_{MN} = \partial_M V_N - \partial_N V_M$ has been defined up to motion quadratic order in the action. The usual gauge fixing of the $V_z$ component is the axial gauge $V_z = 0$. Fourier transform of the vector field $V_\mu$ will be written as [16]:

$$V_\mu(x, z) = \int \frac{d^4q}{(2\pi)^4} e^{-iqx} V_\mu(q)V(q, z),$$

(3.2)

where the bulk-to-boundary propagator $V(q, z)$ obeys the equation of motion

$$\partial_z (\frac{1}{z} \partial_z V(q, z)) + \frac{q^2}{z} V(q, z) = 0,$$

(3.3)

which is obtained from the action (3.1) and imposed the boundary conditions $V(q, \epsilon) = 1$ and $\partial_z V(q, z_m) = 0$. $V_\mu(q)$ is the UV boundary value of the vector field. The solution to the equation (3.3) is written in terms of the Bessel function [27, 28]:

$$V(q, z) = \frac{\pi}{2} zq \left( \frac{Y_0(qz_m)}{J_0(qz_m)} J_1(qz) + Y_1(qz) \right).$$

(3.4)

Note that the vector field function $V_\mu(q)$ is the source for the nucleon current $J_\mu^a = N \gamma_\mu t^aN$. In the limit $Q^2 \rightarrow 0$ the bulk-to-boundary propagator of the vector field can be defined as [29]

$$V(q, z) = 1 - \frac{Q^2z^2}{4} \left( 1 - 2ln \left( \frac{z}{z_m} \right) \right),$$

(3.5)

and in this limit

$$\partial_z V(q, z) = Q^2 z ln \left( \frac{z}{z_m} \right).$$

(3.6)

IV. THE ELECTROMAGNETIC CURRENT

Electromagnetic transition form factors for the nucleon to $R$ transition have been studied in several theoretical approaches. These form factors are defined due to Lorentz and gauge invariance of the interaction by the following matrix element [16, 30]:

$$M^\mu(p_1, \lambda_1, p_2, \lambda_2) = \bar{u}(p_1, \lambda_1) [\gamma_\mu F_1(Q^2) + i\sigma^{\mu\nu} \frac{q_\nu}{M} F_2(Q^2)] u(p_2, \lambda_2),$$

(4.1)
where \( \uppi (p_1, \lambda_1) \) and \( u (p_2, \lambda_2) \) are the spinors describing the \( R \) resonance and the nucleon, respectively. \( M \) is the sum of the nucleon and Roper nucleon masses \( M = M_N + M_R \); \( \gamma^\mu_1 = \gamma^\mu - q^\mu \frac{4}{\sigma} \); \( q = p_1 - p_2 \). The final and initial nucleons helicities (\( \lambda, \lambda_1 \)) are related with the \( \lambda_2 = \lambda_1 - \lambda \) photon helicity.

The four-momenta of Roper, nucleon and photon and the polarization vector of a photon are specified as [31, 32]:

\[
p_1 = (M_1, 0), \quad p_2^2 = (E_2, 0, 0, -|P|), \quad q^\mu = (q^0, 0, 0, |p|)
\]

\[
\epsilon^\mu(\pm) = (0, \epsilon^\pm), \quad \epsilon^\pm = \frac{1}{\sqrt{2}}(\pm 1, i, 0),
\]

\[
\epsilon^\mu(0) = \frac{1}{\sqrt{Q^2}}(|p|, 0, 0, q^0),
\]

where \( E_2 = \frac{Q_+}{2M_R} - M_N, \quad Q_\pm = M^2_\pm + Q^2, \quad Q^2 = -q^2, \quad M_\pm = M_R \pm M_N \) and \( |p| = \sqrt{Q^2 + Q^2} \) is the value of the three-momentum of the nucleon or the photon. For the positive parity Roper state the electromagnetic current of the Roper-nucleon transitions is defined as below [24, 33–35]:

\[
J^\mu = \pi_f(P_f)|\gamma^\mu T F_i^f(Q^2) + \frac{1}{m_{f_i}}\sigma_{\mu\nu}Q_{\nu}F_{2i}^f(Q^2)|u_i(P_i).
\]

Here, \( P_{i,f} \) are four-momenta of the incoming/outgoing nucleons: \( P_{2i}^2 = m_{i,f}^2; \quad Q = P_f - P_i; \quad m_{f,i} = m_f + m_i \). \( F_{1,2} \) are called Dirac and Pauli form factors, respectively.

V. DIRAC AND PAULI FORM FACTORS. HELICITY AMPLITUDES IN ADS/QCD

A. Form factors

The action for the bulk interaction between the fields is defined as:

\[
S_{int} = \int d^4x dz \sqrt{g} L_{int}(x, z),
\]

where \( L_{int}(x, z) \) is the Lagrangian of the interaction between the spinor and vector fields [6, 16]:

\[
L_{int}(x, z) = \sum c^R c^N \Psi^R_{i,r}(x, z) \hat{V}_i(x, z) \Psi^N_{i,r}(x, z).
\]

Here \( c_r \) are the mixing parameters that contribute of the AdS fermion fields with different twist dimension [16]. \( \nabla_\pm (x, z) \) denotes:

\[
\nabla_\pm (x, z) = \tau_3 \Gamma^M V_M(x, z) \pm \frac{i}{4} \eta_{\nu} [\Gamma^M, \Gamma^N] V_{MN}(x, z) \pm g_\nu \tau_3 \Gamma^M \Gamma^z V_M(x, z).
\]
Here $\tau_3$ is the isospin Pauli matrix, $\eta_V$ is the matrix of the $\eta_p, \eta_n$ coupling constants:

$$\eta_V = \text{diag}(\eta_p, \eta_n). \quad (5.4)$$

After taking into account the definitions (5.2), (5.3) in (5.1), $S_{\text{int}}$ gets following explicit form:

$$S_{\text{int}} = \int \frac{dz}{z^5} d^4p d^4p' V(q, z) V(q) \left[ c \frac{e}{2} [ F_{L,L}^{\tau}(z) F_{R,L}(z) + F_{R,R}^{\tau}(z) F_{R,L}(z) + F_{R,L}^{\tau}(z) F_{R,L}(z) - F_{R,L}^{\tau}(z) F_{R,L}(z)] \eta_V z^3 q^\nu [F_{R,R}(z) F_{R,L}(z) - F_{R,L}(z) F_{R,L}(z)]\pi(p' \gamma^\nu u(p) - \eta_V z^2 q^\nu [F_{R,R}(z) F_{R,L}(z) - F_{R,L}(z) F_{R,L}(z)]\pi(p' \gamma^\nu u(p)] \right]. \quad (5.5)$$

Here $|p| = M_{\text{nuc}}, |p'| = M_{\text{Rop}}$ and $q^\mu = p - p'$. The profile functions $F_{L,R}^\tau(|p'|, z)$ and $F_{L,R}(|p|, z)$ describe the Roper state and nucleon, respectively. According to AdS/CFT correspondence of the bulk and boundary theories the generating functionals of the gauge and AdS gravity theories are equivalent:

$$Z_{\text{gauge}} = Z_{\text{AdS}}. \quad (5.6)$$

The vector current of nucleons, which is defined in the QCD theory, can be calculated from the $Z_{\text{AdS}}$ using the correspondence equality (5.6). The vacuum expectation value of the nucleon’s vector current $J_\mu$ in the boundary QCD theory will be found applying the holographic formula:

$$\langle J_\mu \rangle = -i \frac{\delta Z_{\text{QCD}}}{\delta V_\mu(q)}|_{V_\mu=0} = -i \frac{\delta e^{i S_{\text{int}}}}{\delta V_\mu(q)}|_{V_\mu=0}. \quad (5.7)$$

From the comparison the electromagnetic currents of the Roper-nucleon transitions (4.5) with the nucleon vector current (5.7) for the $Q^2 = -q^2 > 0$ momentum transfer region the holographic expressions of the $G_i(Q^2)$ form factors will be written in terms of integrals over the $z$ variable:

$$G_1(Q^2) = \frac{1}{2} \int_0^{z_m} dz V(Q, z) \sum_\tau c_\tau^{R_N} (F_{\tau,0}^{L}(z) F_{\tau,1}^{L}(z) + F_{\tau,0}^{R}(z) F_{\tau,1}^{R}(z)), \quad (5.8)$$

$$G_2(Q^2) = \frac{1}{2} \int_0^{z_m} dz V(Q, z) \sum_\tau c_\tau^{R_N} (F_{\tau,0}^{R}(z) F_{\tau,1}^{L}(z) - F_{\tau,0}^{L}(z) F_{\tau,1}^{R}(z)), \quad (5.9)$$

$$G_3(Q^2) = \frac{1}{2} \int_0^{z_m} dz \partial_z V(Q, z) \sum_\tau c_\tau^{R_N} (F_{\tau,0}^{L}(z) F_{\tau,1}^{L}(z) - F_{\tau,0}^{R}(z) F_{\tau,1}^{R}(z)), \quad (5.10)$$

$$G_4(Q^2) = \frac{M}{2} \int_0^{z_m} dz V(Q, z) \sum_\tau c_\tau^{R_N} (F_{\tau,0}^{L}(z) F_{\tau,1}^{R}(z) + F_{\tau,1}^{L}(z) F_{\tau,0}^{R}(z)). \quad (5.11)$$

The Dirac and Pauli form factors $F_1(Q^2)$ and $F_2(Q^2)$ are defined by means of the $G_i(Q^2)$ form factors [9, 36]:

$$F_1(Q^2) = G_1(Q^2) + g_V G_2(Q^2) + \eta_V G_3(Q^2), \quad (5.12)$$

$$F_2(Q^2) = G_2(Q^2) + g_V G_1(Q^2) + \eta_V G_4(Q^2). \quad (5.13)$$
\[ F_2(Q^2) = \eta V G_4(Q^2). \quad (5.13) \]

The \( F_{1,2}(Q^2) \) form factors are normalized to the electric charge \( e_N \) and anomalous magnetic moment \( \mu_a \) of the nucleon: \( F_1(0) = e_N \) and \( F_2(0) = \mu_a = g(e/2M) = 1.79 \mu_B \). A point particle of charge \( e \) and total magnetic moment \( (g+1) \mu_B \) is a particle for which \( F_1(Q^2) = e \) and \( F_2(Q^2) = g \mu_B \) for all values of \( Q^2 \). The functions \( F_1(Q^2) \) and \( F_2(Q^2) \) are called the charge and moment form factors of the nucleon, respectively.

The electric and magnetic Sachs form factors of the nucleon \( G_E(Q^2) \), \( G_M(Q^2) \) are alternative Lorentz invariant quantities, and are defined in terms of Dirac and Pauli form factors within the relations:

\[ G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4m_N^2} F_2(Q^2), \quad (5.14) \]
\[ G_M(Q^2) = F_1(Q^2) + F_2(Q^2). \quad (5.15) \]

The slopes of the form factors in the limit \( Q^2 \to 0 \) are defined as the electric and magnetic charge radii of the nucleon:

\[ \langle r_E^2 \rangle = -6 \frac{dG_E(Q^2)}{dQ^2} |_{Q^2=0}, \quad (5.16) \]
\[ \langle r_M^2 \rangle = -6G_M(0) \frac{dG_M(Q^2)}{dQ^2} |_{Q^2=0}. \quad (5.17) \]

**B. Helicity amplitudes**

In addition to the electromagnetic transition form factors the helicity amplitudes \( A_{1/2}(Q^2) \) and \( S_{1/2}(Q^2) \) are defined through the transition matrix element of the transverse electromagnetic interaction between the nucleon and the Roper resonance states:

\[ A_{1/2}(Q^2) = \sqrt{\frac{2\pi\alpha}{K}} \left\langle R, S'_z = +\frac{1}{2} |\varepsilon_+ J| N, S_z = -\frac{1}{2} \right\rangle, \quad (5.18) \]
\[ S_{1/2}(Q^2) = \sqrt{\frac{2\pi\alpha}{K}} \left\langle R, S'_z = +\frac{1}{2} |\varepsilon_0 J| N, S_z = -\frac{1}{2} \right\rangle \frac{|q|}{Q}, \quad (5.19) \]

where \( q \) is the photon three-momentum in the rest frame of nucleons, \( \varepsilon_+^\lambda(\lambda = 0, \pm 1) \) is the photon polarization vector, \( \alpha \simeq 1/137 \) is the fine-structure constant, \( K = |q| = \frac{M_R^2 - M_N^2}{2M_R} \). The magnitude of the photon three-momentum is defined:

\[ |q| = \frac{\sqrt{Q_+^2 Q_-^2}}{2M_R}, \quad (5.20) \]
where $Q_\pm = (M_R \pm M_N)^2 + Q^2$. Note that when $Q^2 = 0$, one has $K = |q| = \frac{M_R - M_N}{2M_R}$. For the $N + \gamma^* \rightarrow R(1440, 1710)$ transition, where Roper state is the $J^P = +1/2$ state, the $A_{1/2}(Q^2)$, $S_{1/2}(Q^2)$ helicity amplitudes are written in terms of the Dirac and Pauli form factors as following \cite{37, 38}:

\begin{align}
A_{1/2}(Q^2) &= R[F_1(Q^2) + \frac{M_R - M_N}{M_R + M_N}F_2(Q^2)], \tag{5.21} \\
S_{1/2}(Q^2) &= \frac{R}{\sqrt{2}} |q| \frac{M_R + M_N}{Q^2} (F_1(Q^2) - \tau F_2(Q^2)), \tag{5.22}
\end{align}

where $\tau$ and $R$ denote: $\tau = \frac{Q^2}{(M_R + M_N)^2}$, $R = \sqrt{\frac{4\pi Q^2}{M_R M_N}}$. For the $N + \gamma^* \rightarrow R(1535)$ transition, where Roper state is the $J^P = -1/2$ state the $A_{1/2}(Q^2)$, $S_{1/2}(Q^2)$ helicity amplitudes can be defined in terms of the $F_{1,2}(Q^2)$ form factors as below \cite{11}:

\begin{align}
A_{1/2}(Q^2) &= 2A_R[F_1(Q^2) + \frac{M_R - M_N}{M_R + M_N}F_2(Q^2)], \tag{5.23} \\
S_{1/2}(Q^2) &= -\sqrt{2}A_R(M_R + M_N) \frac{|q|}{Q^2} \left[ \frac{M_R - M_N}{M_R + M_N} (F_1(Q^2) - \tau F_2(Q^2)) \right]. \tag{5.24}
\end{align}

**VI. NUMERICAL ANALYSIS**

We perform the numerical calculations for of the $F_1(Q^2)$, $G_{E,M}(Q^2)$ form factors and $A_{1/2}(Q^2)$, $S_{1/2}(Q^2)$ helicity amplitudes according to the (5.12), (5.13), (5.14), (5.15) and (5.21), (5.22), (5.23), (5.24) formulas. The parameters and constants in these formulas are fixed as $c_3^{RN} = 0.72$, $\eta_p = 0.453$ \cite{6}, $\gamma^{-1} = 0.205 \text{ GeV}^{-1}$ \cite{23}.

In Fig.1, we present our numerical results for the $N + \gamma^* \rightarrow R(1440)$ electromagnetic transition in the $0 \leq Q^2 \leq 5 \text{ GeV}^2$ interval. We plot the Dirac and Pauli form factors and compare our results to the MAID \cite{29}, CLAS \cite{36} experimental data and valence quark contributions model \cite{37}. As is seen from the Fig.1a the Dirac form factor for this transition is close to the data from CLAS and MAID experiments in the $1 \leq Q^2 \leq 5 \text{ GeV}^2$ interval. Plot for the Pauli form factor is presented in the Fig.1b and agrees with experimental data starting from $Q^2 = 0.9 \text{ GeV}^2$ value of momentum transfer. However, at low values of $Q^2$ ($Q^2 < 1$) $\text{GeV}^2$ the hard-wall model results don’t describe data for this transition in good agreement.

In Fig.2, the helicity amplitude $S_{1/2}(Q^2)$ presented for this transition. Though helicity amplitudes have same shape of dependence the hard-wall model results describe the experimental data with less accuracy than the valence quark contribution result.
In Figs.[3,4] we present plots for the form factors and helicity amplitudes for the negative parity $R(1535)$ state transition. For this transition the plot for $F_1(Q^2)$ form factor (Fig.3 left) is close to experimental data in the region $2 \leq Q^2 \leq 4 \text{ GeV}^2$ and again at the low energy limit hard-wall model fails in description of transition. For the $F_2(Q^2)$ form factor (Fig.3 right) the hard-wall results is better than semirelativistic approximation ones and close to experimental data in the $2 \leq Q^2 \leq 4 \text{ GeV}^2$ region. The helicity amplitudes for the $R(1535)$ transition $A_{1/2}(Q^2)$ and $S_{1/2}(Q^2)$ for the $R(1535)$ transition are close to the data in the $Q^2 \geq 1 \text{ GeV}^2$ region (Fig.4).

Graphs for the $R(1710)$ transition are presented in Figs.5 and 6. The form factors $F_1(Q^2)$ and $F_2(Q^2)$ agrees with data in the region $1 \leq Q^2 \leq 5 \text{ GeV}^2$. As seen from Fig.6, neither the hard-wall model nor the non-relativistic quark one correctly describes the helicity amplitudes $A_{1/2}(Q^2)$ and $S_{1/2}(Q^2)$ in the $R(1710)$ transition state case.

Electric and magnetic form factors $G_E(Q^2)$ and $G_M(Q^2)$ for the negative parity $R(1535)$ state transition are given in Fig.7. The hard-wall results for the $G_E(Q^2)$ form factor is close to MAID experimental data in the region $1 \leq Q^2 \leq 6 \text{ GeV}^2$. The $G_M(Q^2)$ graph is close to CLAS experimental data in the interval $1 \leq Q^2 \leq 6 \text{ GeV}^2$. Also, both graphs for $G_E(Q^2)$, $G_M(Q^2)$ are close semirelativistic approximation graphs in this interval. It is worth to notice that for this transition, the hard-wall model results for the $G_E(Q^2)$ and $G_M(Q^2)$ are in a good agreement with experimental data. In Fig.8 plots of the $G_E(Q^2)$ and $G_M(Q^2)$ form factors were presented for the $R(1440)$ and $R(1710)$ transitions which are in good agreement with experimental and light front holography data.

It is seen from the Table1 the hard-wall results for the protons are in a good agreement with data.

VII. SUMMARY

We investigate nucleon electromagnetic form factors and helicity amplitudes within the hard-wall model for the nucleon-Roper transitions $N + \gamma^* \rightarrow R(1440, 1535, 1710)$. We also check this model for the charge and magnetic radii of the nucleon in these transition cases. As is seen from the comparison of the graphs, the holographic hard-wall model gives good results for the Dirac and Pauli form factors and helicity amplitude behaviors in the $Q^2 \geq 1 \text{ GeV}^2$ values of momentum transfer. Failure of the hard-wall model in the description of nucleon transition form factors at low momentum transfer values may be related to several reasons:

1) Simplicity of the model in the sense that it doesn’t take into account more complicated
physical situation at low energies, i.e., it takes into account interaction of particles at low and high values of momentum transfer in the same way. But it may thought that transitions at high transfer momentum occur with higher probability than at low values of it;

2) It doesn’t take into account interaction with the spins or magnetic moments of particles;

3) The study here doesn’t include higher order derivative terms in the interaction Lagrangian.

Despite this, the hard-wall model gives results describing the magnetic and charge form factors for the $R(1440, 1710)$ Roper state transition case well, and it is worth carrying out the phenomenological studies within the hard-wall model as well.

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FIG. 1: $N + \gamma^* \rightarrow R(1440)$ transition Dirac and Pauli form factors (red lines) is compared with CLAS experimental data (squares with error bars) [39], MAID fit (dashed lines) [32] and valence quark contributions (solid lines) [40].

FIG. 2: $N + \gamma^* \rightarrow R(1440)$ transition $A_{1/2}(Q^2)$ and $S_{1/2}(Q^2)$ helicity amplitudes in units of $10^{-3} \text{ GeV}^{-1/2}$ (red lines) is compared with CLAS experimental data (squares with error bars) [39], MAID fit (dashed lines) [32] and valence quark contributions (green solid lines) [40] are also shown.
FIG. 3: Results for the $N + \gamma^* \rightarrow R(1535)$ transition Dirac and Pauli form factors (red lines) is compared with CLAS experimental data (full circles) [39], MAID (full squares) [41,42], JLab/Hall C (triangles) [43] and the semirelativistic approximation (green thick solid line) [11].

FIG. 4: Results for the $N + \gamma^* \rightarrow R(1535)$ transition $A_{1/2}(Q^2)$ and $S_{1/2}(Q^2)$ helicity amplitudes in units of $GeV^{-1/2}$ (red lines) is compared with CLAS experimental data (full circles) [39], MAID (full squares) [41,42], JLab/Hall C (triangles) [43], PDG (empty squares) [44] and the semirelativistic approximation (green thick solid line) [11].
FIG. 5: $N + \gamma^* \rightarrow R(1710)$ transition Dirac and Pauli form factors (red lines) compared with CLAS experimental data (squares with error bars) [39], MAID fit (dashed lines) [32] and nonrelativistic quark model (green lines) [19].

FIG. 6: $N + \gamma^* \rightarrow R(1710)$ transition $A_{1/2}(Q^2)$ and $S_{1/2}(Q^2)$ helicity amplitudes in units of $10^{-3} \, GeV^{-1/2}$ (red lines) is compared with CLAS experimental data (squares with error bars) [39], MAID fit (dashed lines) [32] and nonrelativistic quark model (green lines) [19].
FIG. 7: The hard-wall model results for the $G_E(Q^2)$ and $G_M(Q^2)$ form factors for the $N + \gamma^* \rightarrow R(1535)$ with negative parity state (red line). Data from PDG [44] (full squares) and CLAS [39, 45-46] (full circles). The thin solid line represent the fit to the MAID data [47] and the semirelativistic approximation (green thick solid line) [11].

FIG. 8: The hard-wall model results for the $G_E(Q^2)$ and $G_M(Q^2)$ form factors for the $N + \gamma^* \rightarrow R(1440)$ (blue line) and $N + \gamma^* \rightarrow R(1710)$ (green line) transitions. Data from Refs. [48-53] and light front holography (red line) [9].
TABLE I: Electromagnetic properties of the Roper nucleons

| Quantity | $N$(1440) | $N$(1535) | $N$(1710) | Soft-wall model [6] | Data [3] | Experimental Data |
|----------|-----------|-----------|-----------|---------------------|---------|-------------------|
| $r_p^E (fm)$ | 0.8207 | 0.7722 | 0.6640 | 0.840 | 0.8768 ± 0.0069 | 0.831 [54] |
| $r_M^p (fm)$ | 0.687 | 0.638 | 1.109 | 0.785 | 0.777 ± 0.013 ± 0.010 | 0.831 [55, 56] |
| $\langle r_E^2 \rangle^a (fm^2)$ | -0.566 | -0.312 | -0.132 | -0.117 | -0.1161 ± 0.0022 | -0.111 [57] |
| $r_M^a (fm)$ | 0.8207 | 0.7722 | 0.6640 | 0.792 | 0.862^{+0.009}_{-0.008} | 0.864 [58] |