Positive Planar Satisfiability Problems under 3-Connectivity Constraints

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Abstract

A 3-SAT problem is called positive and planar if all the literals are positive and the clause-variable incidence graph (i.e., SAT graph) is planar. The NAE 3-SAT and 1-in-3-SAT are two variants of 3-SAT that remain NP-complete even when they are positive. The positive 1-in-3-SAT problem remains NP-complete under planarity constraint, but planar NAE 3-SAT is solvable in $O(n^{1.5} \log n)$ time. In this paper we prove that a positive planar NAE 3-SAT is always satisfiable when the underlying SAT graph is 3-connected, and a satisfiable assignment can be obtained in linear time. We also show that without 3-connectivity constraint, existence of a linear-time algorithm for positive planar NAE 3-SAT problem is unlikely as it would imply a linear-time algorithm for finding a spanning 2-matching in a planar subcubic graph. We then prove that positive planar 1-in-3-SAT remains NP-complete under the 3-connectivity constraint, even when each variable appears in at most 4 clauses. However, we show that the 3-connected planar 1-in-3-SAT is always satisfiable when each variable appears in an even number of clauses.

1 Introduction

Boolean algebra is widely used in digital logic design to represent and simplify the Boolean operations. Possible values of the variables are true or 1, and false or 0. The negation or NOT operation is denoted by $\neg$. A SAT is a Boolean formula consisting of conjunction of clauses, e.g., \( \Phi = (x_1 \vee x_2 \vee \neg x_3) \land (x_2 \vee \neg x_5) \). An assignment for a Boolean formula is a mapping of values to its variables. With such a mapping the formula can be evaluated according to the respective rules. If the formula is satisfied, i.e. evaluates to 1, then the assignment is called satisfying and otherwise, unsatisfying. Cook [3] showed that the satisfiability problem for Boolean formulas, SAT, is NP-complete. From then on SAT has been reduced to many other NP problems to prove them as NP-complete. We refer the reader to [6] for more details on NP-completeness.

A 3-SAT problem is a SAT problem where every clause contains at most 3-literals. A SAT graph $G(\Phi)$ of a 3-SAT instance $\Phi$ consists of a vertex for each clause and a vertex for each variable, where there exists an edge between a clause vertex and a variable vertex if and only if the variable or its negation appears in that clause (e.g., see Figure 1(a)). A 3-SAT problem is called planar if its SAT graph is planar. Lichtenstein [12] showed that the planar 3-SAT problem is NP-complete.

A rich body of research investigates variants of 3-SAT problems [20, 5], and also under various restrictions on the SAT graph, e.g., when the SAT graph is planar, or 3-connected, or of bounded degree [10]. Not-All-Equal (NAE) 3-SAT and 1-in-3-SAT are two well-studied variants for 3-SAT. In a NAE 3-SAT problem, the goal is to find a truth assignment to the variables such that each clause contains
at least one true Boolean value and at least one false Boolean value. In a 1-in-3-SAT problem, the goal is to find a truth assignment to the variables such that each clause contains exactly one true Boolean value.

Both NAE 3-SAT and 1-in-3-SAT remains NP-complete even when restricted to positive SAT, i.e., when all the literals are positive (e.g., see Figure 1(b)). Planar NAE 3-SAT is known to be polynomial-time solvable. Moret [15] showed that planar NAE 3-SAT is in P, but no tight worst case time-complexity was calculated in the paper [15]. Moret’s idea was based on finding a min-cut in a planar graph, and thus the planar NAE 3-SAT can be solved in $O(n^{1.5} \log n)$ time [19]. The positive 1-in-3-SAT problem remains NP-complete even under stringent conditions, i.e., when the SAT graph is planar and cubic [14]. A cubic (resp., subcubic) graph is a graph where the degree of each vertex is exactly (resp., at most) three.

A natural question in this context is to ask whether there are nontrivial variants of the positive planar NAE 3-SAT or positive planar 1-in-3-SAT that can be solved faster. In this paper we consider the 3-connectivity constraints on the SAT graph. A rich body of research examines NP-complete graph problems under various connectivity constraints [1, 4, 7]. The planar 3-SAT problem remains NP-hard even when the SAT graph is 3-connected and each variable appears in at most 4 clauses [10]. However, 3-SAT is trivially satisfiable when the SAT graph is 3-connected and cubic (i.e., with only degree 3 vertices).

Contributions: In this paper we examine positive planar satisfiability problems under 3-connectivity constraints. Our contributions are as follows.

1. We prove that positive planar NAE 3-SAT is always satisfiable when the SAT graph is 3-connected, and a satisfiable assignment can be obtained in linear time.

2. We show that without 3-connectivity constraint, the positive planar NAE 3-SAT problem is as hard as finding a spanning 2-matching in a planar cubic graph (i.e., a planar graph with only degree 3 vertices). A spanning 2-matching of a graph is a spanning subgraph with maximum degree 2. Since no linear-time algorithm is known for finding a spanning 2-matching in a planar cubic graph, finding a linear time algorithm for NAE 3-SAT appears to be challenging.

3. We prove that positive planar 1-in-3-SAT remains NP-complete even under 3-connectivity constraint and when every variable appears in at most 4 clauses.

4. In contrast, we show that positive planar 1-in-3-SAT is always satisfiable when every variable appears in an even number of clauses, and a satisfiable assignment can be obtained in quadratic time.

The rest of the paper is organized as follows. Section 3 shows that positive planar NAE 3-SAT is always satisfiable and provides a linear-time algorithm to compute such a satisfiable assignment. Section 4 reduces spanning 2-matching to positive planar NAE 3-SAT. Section 5 proves the NP-hardness of positive planar 1-in-3-SAT even when the SAT graph is 3-connected and every variable appears in at most 4 clauses. Section 6 proves that positive planar 1-in-3-SAT is always satisfiable when every variable appears in an even number of clauses. Finally, Section 7 concludes the paper suggesting directions for future research.
2 Preliminaries

In this section we give some definitions that will be used throughout the paper and present some preliminary results.

A plane graph is a planar graph with a fixed planar embedding in the plane. A planar graph may have an exponential number of embeddings. A plane graph splits the plane into connected regions called faces. The unbounded region is called the outer face and the other regions are called inner faces. The vertices that lie on the unbounded face are called outer vertices and the remaining vertices are called inner vertices.

A graph is bipartite if and only if it is bichromatic, i.e. the graph’s vertices can be colored with at most two colors such that no two adjacent vertices get the same color. The connectivity \( \kappa(G) \) of a graph \( G \) is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph. By Menger’s theorem, every pair of vertices \( u, v \) in a \( k \)-connected graph has at least \( k \) vertex-disjoint paths (except for the common vertices \( u, v \)) connecting \( u \) and \( v \). A plane graph is internally \( k \)-connected if for every inner vertex \( w \), there are \( k \) vertex-disjoint paths (except for the common vertex \( w \)) that start at \( w \) and end at an outer vertex. We refer the reader to \[18\] for basic terminologies on graphs.

**Lemma 1.** Let \( G \) be a 3-connected plane graph with a vertex \( v \) of degree \( d \geq 3 \), where \( w_1, \ldots, w_d \) are the neighbors of \( v \). Let \( H \) be a 2-connected and internally 3-connected plane graph with at most \( d \) outer vertices of degree two. Let \( G' \) be a graph obtained by replacing \( v \) with \( H \) and connecting \( w_1, \ldots, w_d \) to at least three outer vertices of \( H \) such that the graph remains planar and every degree-two outer vertex of \( H \) obtains a new edge. Then \( G' \) is 3-connected.

**Proof.** Assume for a contradiction that \( G' \) is not 3-connected and let \( u, v \) be a pair of vertices such that deleting them generates a disconnected graph. We now show that such a pair cannot exist in \( G' \).

Let \( S \) be the set of vertices in \( G' \) that correspond to the inner vertices of \( H \). Let \( G'' \) be the graph obtained by removing the vertices in \( S \) from \( G' \). Since \( H \) is 2-connected, \( G'' \) can be seen as a graph obtained from \( G \) by replacing \( v \) with a cycle where the neighbors of \( v \) are connected to at least 3 distinct neighbors on the cycle. Such graphs are known to be 3-connected \[13\]. Therefore, either both \( u, v \) lie in \( S \), or exactly one of them must lie in \( S \).

First consider the case when both \( u, v \) lie in \( S \). Since deleting \( u, v \) generates a disconnected graph, there must be a connected component \( C \) that belongs to \( H \). Let \( w \) be a vertex in \( C \). Then there cannot exist 3 vertex disjoint paths from \( w \) to the outer face of \( H \), which contradicts that \( H \) is internally 3-connected.

Consider now that exactly one of \( u \) and \( v \) lies in \( S \). Without loss of generality assume that \( u \) belongs to \( S \). Since \( G'' \) is 3-connected, deleting \( v \) does not disconnect \( G'' \). Since \( H \) is 2-connected, deleting \( u \) does not disconnect \( H \). By the construction there are three disjoint edges connecting the neighbors of \( v \) and \( H \). Hence deleting \( u \) and \( v \) cannot disconnect \( G' \).

Let \( \Phi \) be a positive planar 3-SAT and let \( G \) be its corresponding SAT graph. Let \( \Gamma \) be any arbitrary planar embedding of \( G \). We call \( \Gamma \) a quadrangulated SAT graph if every face of \( \Gamma \) has exactly four vertices, where two of them are clause vertices and two are variable vertices. A clause graph of a quadrangulated SAT graph \( \Gamma \) is obtained by adding for every face, an edge between its clause vertices, and finally, removing the variable vertices.

A perfect matching of a graph is a collection of edges \( M \) such that every vertex is incident to exactly one edge in \( M \), as illustrated in Figure 2(c). A cubic (resp. subcubic) graph where every vertex is of degree exactly 3 (resp., at most 3). By Petersen’s theorem \[15\], every bridgeless cubic graph has a perfect matching, and such a matching can be found efficiently under planarity constraint.

**Lemma 2** (Biedl et al. \[2\]). A perfect matching in a planar bridgeless cubic graph with \( n \) vertices can be found in \( O(n) \) time.

A 2-matching of a graph \( G \) is a subgraph \( H \) with maximum degree two. A 2-matching \( H \) is called spanning if every vertex of \( G \) is incident to at least one edge of \( H \), as shown in Figure 2(d). A maximum cardinality 2-matching is a 2-matching that can be computed in \( O(n^{1.5}) \) time \[3\].

Let \( \Gamma \) be a planar embedding of a set of disjoint cycles. Then a genealogical tree \( T \) of \( \Gamma \) is defined as follows:
Figure 2: (a) A quadrangulated positive planar 3-SAT graph $\Gamma$, where the clause vertices are shown in black square and the variable vertices are shown in circles. (b) The corresponding clause graph $C$ of $\Gamma$. (c) A perfect matching and (d) a spanning 2-matching, where the edges in the matching and 2-matching are shown in bold.

- Each vertex $v$ in $T$ corresponds to a face $f_v$ in $\Gamma$.
- The root of $T$ corresponds to the outerface of $\Gamma$.
- There exists an edge from a parent node $v$ to a child node $w$ if and only if the face $f_v$ encloses the face $f_w$, and $f_v$ and $f_w$ share a common cycle on their boundaries.

3 Positive Planar 3-Connected NAE 3-SAT

In this section we show that a positive planar 3-connected NAE 3-SAT is always satisfiable and a satisfying assignment can be obtained in $O(n + m)$ time, where $n$ and $m$ are the number of variables and clauses, respectively.

**Theorem 3.** Let $R$ be an arbitrary positive planar 3-connected NAE 3-SAT expression. Then $R$ is always satisfiable and a satisfiable assignment of $R$ can be computed in linear time.

Since the SAT graph $G$ is 3-connected, it has a unique plane embedding (upto the choice of the outerface) and hence the clause graph is the same for every choice of the outerface. Before we prove Theorem 3, we consider a simpler case when the SAT graph $G$ is planar, 3-connected and quadrangulated, as stated below.

**Lemma 4.** Let $R$ be any arbitrary positive planar 3-connected NAE 3-SAT instance and let $G$ be the corresponding planar SAT graph. If $G$ is quadrangulated, then $R$ is always satisfiable.

**Proof.** We first show that the clause graph corresponding to $G$ must have a perfect matching and we use that matching to find a satisfying truth assignment. Since $G$ is quadrangulated, every face in $G$ has exactly four vertices, where two are variable vertices and two are clause vertices. Let $C$ be the clause graph obtained from $G$. Since each clause has exactly three literals, the corresponding clause vertex $v$ has exactly three neighbors in $G$. Since $v$ is incident to exactly 3 faces in $G$, it must have exactly three neighbors in the clause graphs. Therefore, the clause graph $C$ is a bridgeless cubic graph. We now show that $C$ is a bridgeless cubic graph. Suppose for a contradiction that $C$ has a bridge $(v, w)$ and deleting the bridge results into two disjoint connected components $H_1$ and $H_2$ (e.g. see Figure 3(a)). Since the faces of $G$ are quadrangulated, there must be a face $p, v, q, w$ in $G$, where $p, q$ are variable vertices. Since $G$ is 3-connected, there must be three vertex disjoint paths between $v$ and $w$. Hence there exists a path $v, \ldots, w$ in $G$ that does not pass through $p$ or $q$ (e.g. see Figure 3(b)). The sequence of clause vertices in this path connects $v$ and $w$ in $C$ (e.g. see Figure 3(c)). Therefore, $(v, w)$ cannot be a bridge in $C$. By Petersen’s theorem [16], $C$ contains a perfect matching $M$.

We now show how to compute a satisfying assignment for the NAE 3-SAT instance $R$. Let $C'$ be the graph obtained from $C$ by deleting the edges of $M$, e.g. see Figures 4(a)–(b). Since $C$ is a planar cubic graph, $C'$ must be a planar disjoint collection of cycles. Let $T$ be the genealogical tree of $C'$. We compute a two coloring of $T$ with red and black colors, e.g. see Figure 4(c). For each vertex (i.e., face)
which has been colored red, we set the corresponding variable vertices (i.e., the variable vertices lying inside face) to be true.

We now prove that the resulting truth assignment is a satisfying truth assignment, i.e., for each clause at least one variable vertex must be true and at least one must be false. Consider a clause vertex $q$ in $G$. Note that $q$ is incident to three variable vertices $v_1, v_2$ and $v_3$ in $G$, and let $D$ be the cycle passing through $q$ in $C'$. Then $D$ either contains two variable vertices in its interior and the other variable vertex remains outside, or $D$ contains one variable vertex in its interior and the other two remain outside. Since the genealogical tree is two colored, at most two of these variable vertices of $q$ can be true, and the remaining ones must be false.

We now consider the case when the SAT graph $G$ is not necessarily quadrangulated. If the SAT graph is not quadrangulated, then the clause graph may not be cubic or planar. Therefore, instead of using the clause graph, we define a saturated clause graph $C_s$ as follows.

**Saturated clause graph $C_s$:** Let $\Gamma$ be a plane embedding of $G$. Let $F$ be a face $v_1, v_2, \ldots, v_p$ of $\Gamma$ with $p > 4$ vertices. Since $G$ is 3-connected and bipartite, $p$ must be even and exactly half of the vertices would be clause vertices. Let $v_2, v_4, \ldots, v_p$ be the clause vertices. We define a saturation operation that first adds a cycle $w_1, w_2, \ldots, w_{p/2}$ of $p/2$ dummy vertices interior to $F$ and then adds the edges $(w_i, v_{2i})$. Figures 5(a)–(b) illustrate the saturation operation. The saturated clause graph is obtained by adding for each quadrangular face an edge between the clause vertices, and then applying the saturation operation to all the faces of length more than four in $\Gamma$.

We are now ready to prove Theorem 3.

**Proof of Theorem 3** Let $R$ be any arbitrary positive planar 3-connected NAE 3-SAT expression and let $G$ be the corresponding planar 3-connected SAT graph. Let $C_s$ be the saturated clause graph of $G$. Since $G$ is 3-connected, each clause vertex in $G$ is adjacent to exactly three variable vertices. Therefore, it is straightforward to observe from the construction of saturated clause graph that $C_s$ is planar and cubic. We now show that $C_s$ is bridgeless.

Suppose for a contradiction that $(v, w)$ is a bridge in $C_s$. If $(v, w)$ is an edge inside a quadrangular face of $G$, then we can prove that there must be another path connecting $v$ and $w$ in $C_s$ in the same way as we proved the clause graph to be bridgeless in Lemma 3. If $(v, w)$ is an edge that has been added during the saturation operation on some face $F$, then both $v$ and $w$ cannot be on the added cycle. Therefore, we may assume without loss of generality that $v$ is a clause vertex (Figure 5(c)) and $w$ is a dummy vertex. Let $v'$ be another clause vertex on $F$. Since $G$ is 3-connected, there must be a path $P$ in $G$ between $v$ and $v'$ that does not contain any vertex of $F$. Hence we can construct a path in $C_s$ between $v$ and $v'$ outside of $F$, and extend it inside $F$ to form a cycle that contains $(v, w)$. Hence $(v, w)$ cannot be a bridge in $C_s$.

Since $C_s$ is planar bridgeless cubic graph, by Petersen’s theorem $C_s$ contains a perfect matching. We can now use the same argument as in the proof of Lemma 2 using this perfect matching to construct a satisfying truth assignment of $R$.

It now remains to prove that the time complexity of the whole process is linear in the number of vertices of $G$. A planar embedding $\Gamma$ of the SAT graph $G$ can be obtained in linear time. The construction of $C_s$ requires iterating through each face of $\Gamma$ and spending a time proportional to the length of each face. Hence we can compute $C_s$ in linear time. Since $C_s$ is a planar bridgeless cubic graph, by Lemma 2 one can obtain a perfect matching $M$ of $C_s$ in linear time. Given a perfect matching, one can delete the edges of $M$ from $\Gamma$ and then recursively traverse the cycles on the outer face to construct the genealogical
Figure 4: Illustration for computing a satisfying truth assignment from a perfect matching. (a) A quadrangulated SAT graph, where clause graph $C$ is shown in solid lines. A perfect matching $M$ is shown in bold. (b) The graph $C'$. (c) A genealogical tree $T$, with a two coloring where \{f_1, f_3, f_6, f_5\} are colored with the same color. (d) A satisfying truth assignment obtained from the two coloring of $T$.

tree $T$. Thus the construction of the $T$ takes linear time. Finally, coloring the tree with two colors and setting the corresponding truth values takes a linear-time traversal of the tree and a linear-time traversal of $G$. Thus the overall time complexity remains linear.

4 Positive Planar NAE 3-SAT without 3-Connectivity Constraint

In this section we consider the case of general planar SAT graphs. We show that the problem of solving a positive planar NAE 3-SAT is as hard as the problem of deciding whether a planar cubic graph contains a spanning 2-matching. Figure 6(a) illustrates a spanning 2-matching in a cubic graph. Although a rich body of literature examines 2-factor and maximum 2-matching in cubic graphs [9], to the best of our knowledge, no linear-time algorithm is known for deciding whether a planar cubic graph admits a spanning 2-matching.

**Theorem 5.** The problem of deciding whether a connected planar cubic graph admits a spanning 2-matching is linear-time reducible to positive planar NAE 3-SAT.

**Proof.** Let $G$ be a planar cubic graph. We construct a graph $G'$ by subdividing each edge of $G$ with a division vertex, i.e., each edge $(u, v)$ of $G$ is replaced by a path $u, d_{uv}, v$ in $G'$, where $d_{uv}$ is the division vertex.

We now consider $G'$ as a SAT graph where the original vertices of $G$ are the clause vertices and the division vertices are the variable vertices. In the following we show that $G$ has a spanning 2-matching if and only if the planar NAE 3-SAT $I$ corresponding to $G'$ has an affirmative not-all-equal solution.
Figure 5: Illustration for the proof of Lemma 4. (a) A face of length 12 in $G$. (b) Illustration for the saturation operation, where the clause vertices are shown in squares and the vertices of the added cycle are shown in triangles. (c) The vertices $v$ and $w$ lie on a cycle in $C_s$.

Figure 6: (a) A planar cubic graph $G$ with a spanning 2-matching (shown in red). (b) The corresponding positive planar NAE 3-SAT, and the associated affirmative solution. The literals which are true are shown in red.

First assume that $G$ has a spanning 2-matching (i.e., a spanning subgraph with maximum degree 2), and let $F$ be the set of edges in that spanning subgraph. Let $F_d$ be the division vertices in $G'$ that correspond to $F$. We now set the literals of $I$ determined by $F_d$ to be true, and the remaining literals to false. Suppose for a contradiction that such an assignment gives rise to a clause with all true or all false literals. If all three are false, then it contradicts that $F$ is spanning. If all three are true, then there must be a vertex in $G$ that is incident to three edges in $F$, contradicting that $F$ corresponds to a subgraph with maximum degree 2.

Assume now that $I$ has an affirmative not-all-equal solution. We construct a set $D$ of division vertices by taking for each clause vertex, the division vertices corresponding to the literals which are true. Let $F$ be the edges of $G$ corresponding to the set $D$.

We now show that that $F$ corresponds to a spanning 2-matching of $G$. Since every clause contributed to $D$, $F$ must be a spanning subgraph. Suppose for a contradiction that the graph determined by $F$ contains a vertex of degree 3. By construction of $F$, this implies that there is a clause where all its literals are assigned the value true, which contradicts that the initial assignment is a not-all-equal assignment.

5 Positive Planar 3-Connected 1-in-3-SAT

The 1-in-3-SAT problem remains NP-hard even if each variable appears in at most 4 clauses [23]. Laroche [11] proved that the positive planar 1-in-3-SAT problem is NP-complete. Moore and Robson [14] proved that the positive planar 1-in-3-SAT remains NP-hard even when the SAT graph is cubic. None of these hardness reductions ensures the 3-connectivity of the underlying SAT graph. However, the problem remains hard even in cases where additional edges can be added to the SAT graph (keeping it planar) to form a cycle containing all variable and clause vertices [17]. Although such an edge augmented graph may be 3-connected, the SAT graph itself may not be 3-connected.

In this section we show that positive planar 1-in-3-SAT remains NP-hard even when the SAT graph is 3-connected and each variable appears in at most 4 clauses.
5.1 Outline of the Reduction

We reduce planar 3-connected 3-SAT which is shown to be NP-hard even when every variable appears in at most 4 clauses \cite{10}. For convenience we will refer to the SAT graph as a 3-SAT graph or a 1-in-3-SAT graph depending on the SAT instance. Let \( G \) be a 3-SAT graph corresponding to a planar 3-connected 3-SAT instance \( I \), as illustrated in Figure 7(a). The hardness reduction is carried out in two phases.

In the first phase, we replace each variable with a variable gadget and each clause with a clause gadget. We will use the resulting planar graph \( G' \) as a 1-in-3-SAT graph of a positive planar 1-in-3-SAT instance \( I' \) and show that \( I \) is satisfiable if and only if \( I' \) is satisfiable. While constructing \( G' \), we will ensure that every variable appears in at most 4 clauses. However, \( G' \) would not be 3-connected. In the second phase, we will add additional gadgets to \( G' \) to construct a planar 3-connected graph \( G'' \) ensuring that each variable appears in at most 4 clauses.

To complete the proof we will use the resulting planar graph \( G'' \) as a 1-in-3-SAT graph of a positive planar 1-in-3-SAT instance \( I'' \) and show that \( I \) is satisfiable if and only if \( I'' \) is satisfiable.

5.2 Construction of \( G' \)

The graph \( G' \) is constructed by replacing the vertices and clauses with the vertex and clause gadgets.

5.2.1 Clause Gadgets

We first replace each variable vertex \( v \) of degree \( d \) in \( G \) with a cycle \( L_v \) of \( d \) vertices. We will refer to \( L_v \) as the lamina of \( v \). We then connect the vertices on \( L_v \) with the neighbors of \( v \) such that the resulting graph remains planar, e.g., see Figures 7(a)–(b). For each clause vertex \( c = (x \lor y \lor z) \), the clause gadget contains the clauses \( c_1 = (x \lor p \lor r) \), \( c_2 = (y \lor p \lor q) \) and \( c_3 = (z \lor s \lor q) \). Figure 7(c) illustrates a clause gadget in blue. It is known that \( c \) evaluates to true if and only if \( (c_1 \land c_2 \land c_3) \) admits a satisfiable truth assignment where each clause contains exactly one true value \cite{23}.

5.2.2 Variable Gadgets

To remove the negated variables we now replace each lamina with a variable gadget. The variable gadget consists of one inner ring, and one or more outer rings, as described below.

A \( k \)-ring is a planar 1-in-3-SAT graph of \( 3k \) clause vertices on its outerface, as illustrated in Figure 8(a). Here \( k \geq 3 \) is a positive integer. A \( k \)-ring consists of \( k \) groups, each containing 6 clause vertices, as shown in red shaded region. A \((k+1)\)-ring can be constructed by adding a group to a \( k \)-ring, as shown in Figure 8(b). Later, we will refer to a \( k \)-ring just as a ring for simplicity. We will use the following property of a ring.

Remark 1. In every satisfiable truth assignment of a ring, the truth values on the outerface appear in the following sequence \( T, F, F, \ldots, T, F, F \).

A variable gadget of length \( q \) is a planar 1-in-3-SAT graph with \( q \) blocks. Here \( q \geq 3 \) is a positive integer and each block contains 3 clause vertices. Figure 8(i) illustrates a variable gadget of length 3 where the three blocks are enclosed in three disjoint red shaded regions. A variable gadget contains two or more rings: one inner ring, and one or more outer rings. A ring is called an outer ring if it lies on the outer face. Otherwise, it is called an inner ring. The gadget in Figure 8(j) contains two rings: one inner ring and one outer ring. We first describe the details of the gadget without the outer rings, as illustrated in Figure 8(c). We will refer to the cycle passing through the blocks (shown in bold) as the variable band.

One clause vertex from each block is adjacent to the inner ring. The \( i \)th block, where \( 0 \leq i \leq (q - 1) \), is adjacent to the \((3i+1)\)th variable vertex on the outerface of the inner ring. Figure 8(c) illustrates the positions of the vertices of the inner ring where the blocks connect to (i.e., shown in numeric labels). By Remark 1, these \((3i+1)\)th variables must all have the same truth value. For example, in Figure 8(c) they are all true, and in Figure 8(d)–(e) they are all false. Hence, in a satisfiable truth assignment, the truth values on the variable band are either all false (Figure 8(d)), or appear in the following sequence \( T, F, \ldots, T, F \) (Figures 8(c) and (e)).

An outer ring enforces the truth value sequence on the variable band to be \( T, F, \ldots, T, F \). Therefore, with an outer ring, Figure 8(d) becomes infeasible. A variable gadget of length \((q+1)\) can be constructed
Figure 7: (a) The 3-SAT graph $G$. (b) Replacing each variable vertex with a lamina. (c)–(d) Construction of $G'$. The clause gadget is shown in a blue rectangle. The variable gadget is shown in a thick black circle.

by adding a new block to a variable gadget of length $q$, as shown in Figure 8(f). Hence, we can attach outer rings as necessary.

There can be one or more outer rings exterior to the variable belt. Each outer ring is adjacent to 8 consecutive variable vertices on the belt. Figure 8(g) illustrates the positions where the variable vertices on the outer ring form connections with the belt. By Remark 1, these three variable vertices on the outer ring must take a truth assignment from $\{(T,F,T), (F,T,F), (F,F,F)\}$. However, two of these assignments, i.e., $\{(T,F,T), (F,T,F)\}$ cannot be extended to a full satisfiable assignment. Figures 8(g) and (h) illustrate this by showing how these truth assignments impose both true and false values on the inner ring. The other truth assignment, i.e., $(F,F,F)$, enforces the sequence $T,F,...,T,F$ on the variable belt, as shown in Figures 8(i)–(j).

**Remark 2.** In every satisfiable truth assignment of a variable gadget, the truth values on the variable belt in the following sequence $T,F,...,T,F$.

We complete the construction of $G'$ by replacing each lamina with a variable gadget. Since every variable vertex in $G$ appears in at most 4 clauses, a lamina can have at most 4 incident edges. Figure 9 illustrates various scenarios while replacing a lamina of four variable vertices $a,b,c,d$. Since each of $a,b,c,d$ is a positive literal (type P) or a negative literal (type N) of the same variable, we can have 16 different scenarios. These cases have been grouped into four configurations. While replacing a lamina with a variable gadget, one must choose a configuration based on the literal types. Figures 9(e)–(h) illustrate the first four scenarios of Figure 9(b) in more detail. For example, if $a,b,c,d$ are of type PPNP (e.g., see Figure 9(e)), then we can choose the configuration of Figure 9(b). If $a,b,c,d$ are of type PPPN (e.g., see Figure 9(g)), then we choose the configuration of Figure 9(b) by taking a mirror reflection and relabeling...
The 1-in-3-SAT graph $G$ to a satisfying truth assignment for $I$, the truth values of the positive and negative literals have been set consistently. Since all clauses are satisfied in $I'$, the truth assignment also satisfies $I$. On the other hand, any satisfying truth assignment of $I$ determines a consistent set of truth values on the belt, and thus can be extended to a satisfying truth assignment for $I'$.

\[\Box\]

5.3 Construction of $G''$

The 1-in-3-SAT graph $G'$ is not 3-connected. We now add some more clauses and variables to $G$ to construct another 1-in-3-SAT graph $G''$ such that $G''$ is planar 3-connected with the degree of variable vertices bounded by 4.

**Lemma 6.** Let $I$ and $I'$ be the 3-SAT and 1-in-3-SAT instances corresponding to $G$ and $G'$, respectively. Then $I$ is satisfiable if and only if $I'$ is satisfiable.

**Proof.** The clause gadget ensures that a clause in the 3-SAT instance is satisfiable if and only if each clause in the clause gadget obtains exactly one true value. The variable gadgets ensure that in every satisfying truth assignment $I'$, the truth values of the positive and negative literals have been set consistently. Since all clauses are satisfied in $I'$, the truth assignment also satisfies $I$. On the other hand, any satisfying truth assignment of $I$ determines a consistent set of truth values on the belt, and thus can be extended to a satisfying truth assignment for $I'$.

Figure 8: (a) A ring with its schematic representation where the truth assignments of the variables at positions 1, 4 and 8 are shown. The feasible truth value assignments are illustrated using T and F. The clause vertices are shown in squares. (b) An addition of a new group into a ring. (c)–(f) Illustration for variable gadgets without the outer rings. Only the outer cycle of the ring (in blue) is drawn for simplicity. (g)–(h) Variable gadgets with two infeasible truth value assignments. (h)–(j) Variable gadgets with two feasible truth value assignments.

Note that a variable belt can have vertices of degree two, which are shown in large red vertices. The configurations ensure that the edges incident to the lamina are attached to these vertices. In addition, the degree of each variable vertex does not exceed 4. We now have the following lemma.
Figure 9: Replacing each lamina with a variable gadget, where the degree two vertices on the variable belt are shown in red. (a)–(d) Four configurations based on the 16 scenarios of \( a, b, c, d \). (e)–(h) Details for the first four scenarios of Figure 9(b).

Let \( r, p, q, s \) be the variables in a clause gadget of \( G' \), e.g., see Figure 7(c). To construct \( G'' \), we augment the clause gadgets with two modified vertex gadgets. Note that an outer ring in a variable gadget enforces that the variable belt must obtain an alternating sequence of true and false values. A modified vertex gadget attaches the outer ring to the variable belt at some positions, as illustrated in Figure 10(a), such that the outer ring no longer enforces the variable belt to contain both true and false values. The vertices \( r, p \) (similarly, \( q, s \)) lie on two consecutive vertices on a variable belt. We ensure that \( p \) and \( q \) are vertices of degree four (i.e., they are degree-two vertices on the variable belt).

**Lemma 7.** Let \( I' \) and \( I'' \) be the 1-in-3-SAT instances corresponding to \( G' \) and \( G'' \), respectively. Then \( I' \) is satisfiable if and only if \( I'' \) is satisfiable.

**Proof.** Consider first the case when \( I'' \) admits a satisfying truth assignment \( \phi'' \). Since the clauses of \( I' \) are included in \( I'' \), \( \phi'' \) will set exactly one variable to true in each clause of \( I' \). Therefore, \( I' \) will be satisfied.

Assume now that \( I' \) admits a satisfying truth assignment \( \phi' \). We now show how the truth assignment can be extended to satisfy all the clauses of \( I'' \). Let \((u \lor p \lor r) \land (v \lor p \lor q) \land (w \lor q \lor s)\) be three clauses in \( I' \) that belong to a clause gadget of \( G' \). We now consider the following cases depending on the truth values of \( u \) and \( w \) in \( \phi' \).

**Case 1 (\( u \) and \( w \) both obtain true values).** In this case \( r, p, q, s \) must obtain false values. Since \( I' \) is satisfied, \( v \) must be true. Since the outer ring here no longer enforces the truth value sequence on the variable band to be \( T, F, \ldots, T, F \), we can use a truth value sequence \( F, F, \ldots, F, F \) for the variable belt.

The variable belt is connected to the outer ring at the third, sixth and ninth variable vertices on the ring. By the property of the ring (Remark 4), in any satisfying truth assignment, they must all have the same truth value. Thus the outer rings obtain true values at positions 3, 6 and 9. This truth value assignment is consistent also for the clause connecting the two rings. Hence we can satisfy all clauses with exactly one true value per clause, as illustrated in Figure 10(a).

**Case 2 (exactly one of \( u \) and \( w \) obtains a true value).** Without loss of generality assume that \( u \) obtains a true value and \( w \) obtains a false value (Figure 10(b)). The other case is similar, as shown in Figure 10(c). Since \( u \) obtains a true value, \( r, p \) must obtain false values and this enforces false values on the corresponding ring. Since \( W \) obtains a false value, exactly one of \( q, s \) obtains a true value. We choose a feasible truth value assignment based on the value of \( v \) in \( \phi' \). Hence the corresponding ring obtains
false values. We can extend this assignment to also satisfy the clause connecting the two rings. Hence we can satisfy all clauses with exactly one true value per clause.

Case 3 (u and w both obtain false values). In this case one of p, r obtains a true value and the other variable obtains a false value. Similarly, one of q, s obtains a true value and the other variable obtains a false value. We choose a feasible truth value assignment based on the value of v in φ'. In this scenario, the outer rings obtain false values at positions 3, 6 and 9. This truth value assignment is consistent also for the clause connecting the two rings. Hence we can satisfy all clauses with exactly one true value per clause, as shown in Figure 10(d).

Lemma 8. The 1-in-3-SAT graph $G''$ is 3-connected with the degree of each variable vertex bounded by 4.

Proof. The input SAT graph $G$ is 3-connected, and by Lemma 1, it remains 3-connected after we replace each variable vertices with a lamina. Each variable gadget is 2-connected and internally 3-connected. Hence by Lemma 1, replacing each lamina with a variable gadget (e.g., Figure 9) keeps the graph 3-connected. Each clause gadget consists of two modified vertex gadgets, e.g., see the subgraph interior the rectangular region in Figure 10(a). Such clause gadgets are 2-connected and internally 3-connected. Hence by Lemma 1, replacing the clauses with clause gadget keeps the resulting graph $G''$ 3-connected.

Figure 10: Extension of the truth value assignments of u, v and w. (a) u and w both obtain true values. (b) u obtains a true value and w obtains a false value. (c) u obtains a false value and w obtains a true value. (d) u and w both obtain false values.
By construction, the vertices of each variable and clause gadget are of maximum degree 4. Hence the degree of the vertices of $G''$ are also bounded by 4.

**Theorem 9.** Planar 3-connected 1-in-3-SAT is NP-hard even when every variable appears in at most 4 clauses.

*Proof.* Let $I$ be a planar 3-connected 3-SAT instance and let $G$ be the corresponding SAT graph. Let $G'$ be the graph obtained by modifying $G$ using the vertex and clause gadgets, and let $G''$ be the 1-in-3SAT graph obtained from $G'$. Let $I'$ and $I''$ be the 1-in-3-SAT instances corresponding to $G'$ and $G''$, respectively. By Lemma 6 and Lemma 7, $I$ is satisfiable if and only if $I''$ is satisfiable. By Lemma 8, $G''$ is a 3-connected graph with the degree of its variable vertices bounded by four.

6 **Positive Planar 3-Connected 1-in-3-SAT with Even Variable Frequency**

In this section we prove that every positive planar 3-connected 1-in-3-SAT with each variable appearing in an even number of clauses is always satisfiable and a satisfying truth assignment can be computed in quadratic time.

![Figure 11](image-url)

Figure 11: (a) An instance of a positive planar 3-connected 1-in-3-SAT, and (b) the corresponding saturated clause graph shown in solid blue.

Let $R$ be an arbitrary positive planar 3-connected 1-in-3-SAT expression and let $G$ be its corresponding SAT graph. Figure 11(a) illustrates such an instance where the clause and variable vertices are shown in squares and circles, respectively.

We first construct a saturated clause graph $C_s$, as illustrated in Figure 11(b). The vertices of the added cycle are shown in triangles. We have shown in the proof of Theorem 3 that $C_s$ is a bridgeless cubic graph. However, here we need to show that $C_s$ is 3-connected. To observe this we can think of the construction of $C_s$ in three steps where each step remains the graph 3-connected, as follows.

Step 1: For each quadrangular face of the clause graph $C$, add an edge between the clause vertices. Let $C'$ be the resulting graph. Since $C$ is 3-connected and $C'$ is obtained by adding some edges to $C$, $C'$ is 3-connected.

Step 2: Apply the saturation operation to all the faces of length more than four. Let $C''$ be the resulting graph. Since $C$ is 3-connected and $C''$ is obtained by adding some edges to $C$, $C''$ is 3-connected.

Step 3: Delete the variable vertices from $C''$ to obtain $C_s$. If the neighbors of a vertex $v$ in a 3-connected graph induces a cycle, then $v$ can be deleted to obtain another 3-connected graph [22]. Using this property one can show that $C_s$ is 3-connected.
Since $C_5$ is a planar 3-connected cubic graph, the dual of $C_5^*$ is a triangulated planar graph. Since every variable appears in an even number of clauses, each vertex of $C_5^*$ has an even degree. Every triangulated planar graph where each vertex has an even degree admits a 3 vertex coloring [21]. Such a planar triangulation is known as an even triangulation. Therefore, we can color the faces of $C_5$ with 3 colors $c_1, c_2, c_3$ such that no two adjacent faces receive the same color. Figure 12(a) illustrates such a coloring of the faces. We take the faces that are colored with $c_1$ and set the variables corresponding to those faces to true. Figure 12(b) illustrates these variables in filled circles (orange). Finally, we set the remaining variables to false.

We now show that the resulting truth assignment satisfies the 1-in 3-SAT instance $R$. Since the SAT graph $G$ is 3-connected, every clause is adjacent to exactly three variables. Therefore, the three faces around a clause vertex must have 3 different colors. One of these is colored with $c_1$ and the remaining two must be colored with $c_2$ and $c_3$. Therefore, exactly one variable associated to that clause is set to true.

The construction of $C_5$ and the dual graph takes linear time. A 3-coloring of an even triangulation can be computed using a concept of ‘non-crossing Eulerian circuit’ [21], which is straightforward to compute in quadratic time. Hence in quadratic time, one can find a satisfying truth assignment for $R$.

![Figure 12: Construction of a satisfying truth assignment from a face coloring of $C_5$.](image)

The following theorem summarizes the results of this section.

**Theorem 10.** Let $R$ be an arbitrary positive planar 3-connected 1-in-3-SAT expression. If every variable appears in an even number clauses, then $R$ is always satisfiable and a satisfiable assignment of $R$ can be computed in quadratic time.

### 7 Conclusions

We have shown that positive planar 3-connected NAE 3-SAT is always satisfiable and a satisfiable assignment can be obtained in linear time. We have also shown that deciding whether a positive planar NAE 3-SAT is satisfiable or not is as hard as finding a spanning 2-matching in a cubic graph. Hence we pose the following open problems.

**Open Problem 1.** Does there exist a linear-time algorithm for finding a spanning 2-matching in a cubic (or, subcubic) planar graph?

It would be interesting to examine whether a linear-time algorithm can be obtained for the general case. In fact, we are not aware of any $o(n^{3/2})$-time algorithm for the general case.

**Open Problem 2.** Does there exist an $o(n^{3/2})$-time algorithm for solving positive planar NAE 3-SAT?

One can also try to find out other tractable versions for 3-SATs based on connectivity constraints on the SAT graph. We have shown that planar 3-connected 1-in-3-SAT remains NP-hard when every variable appears in at most 4 clauses. Thus the following is another intriguing question.
Open Problem 3. Does planar 3-connected 1-in-3-SAT remain NP-hard when every variable appears in 3 clauses?

We have shown that if every variable appears in an even number clauses, then a positive planar 3-connected 1-in-3-SAT is always satisfiable and a satisfiable assignment can be computed in quadratic time. The time complexity relies on computing a 3-coloring of an even triangulation. Hence it would be interesting to examine whether there exist faster algorithms to compute a 3-coloring for an even triangulation.

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