Fermion interactions, cosmological constant and space-time dimensionality in a unified approach based on affine geometry

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One of the main features of unified models, based on affine geometries, is that all possible interactions and fields naturally arise under the same standard. Here, we consider, from the effective Lagrangian of the theory, the torsion induced 4-fermion interaction. In particular, how this interaction affects the cosmological term, supposing that a condensation occurs for quark fields during the quark-gluon/hadron phase transition in the early universe. We explicitly show that there is no parity-violating pseudo-scalar density, dual to the curvature tensor (Holst term) and the spinor-bilinear scalar density has no mixed couplings of A-V form. On the other hand, the space-time dimensionality cannot be constrained from multidimensional phenomenological models admitting torsion.

Keywords: alternative theories of gravity; torsion; fermion interactions; cosmological constant.

I. INTRODUCTION

As it is well known, the Standard Model (SM) of particle physics has proven to be a very successful theory, but still presents some shortcomings. Among these, one is the Hierarchy Problem, which the current literature claims that might be solved by considering extra dimensions. In higher dimensional spacetimes, the gravitational theory by Einstein can be generalized in several ways: the Kaluza-Klein approach (or slight modifications of it) is currently used by a great percentage of the higher dimensional models treating the hierarchy trouble (see, for example, [1]).

Another important generalization of the General Relativity (GR) is the gravity with torsion that considers the possibility of a general non-symmetric connection. This point of view is also assumed in several Extended Theories of Gravity like $f(R)$-gravity [2–4] that have recently gained interest for dark energy and dark matter issues [5, 6]. The first direct geometrical generalization in this direction is due to Cartan [7]. The approach taken into account by theoretical physicists in the last decades is that, assuming as the starting point Cartan’s generalization of GR, torsion can be coupled to fermion matter in a straightforward way. The trick in such approaches is, since the field equation for the spin connection is a constraint related to the contortion, it can be used to get rid of the torsion in the original action. Then the torsion field is a non-dynamical one and the fermionic matter is added by hand. Consequently, the new “artificial” action contains standard GR and matter fields with an additional contact 4-fermion interaction [8] where the Dirac equation for the fermions is not derived from the geometrical structure of space-time.

Because of the effective 4-fermion interaction term has a coupling constant proportional to Newton’s gravitational constant, at first approximation, this interaction is highly suppressed. Nevertheless, it is currently claimed that extra dimensions could explain the hierarchy problem, and thus the (higher dimensional) fundamental gravity scale might be roughly $M^* \sim O(1)$ TeV [9–12]. The limits to the size of extra dimensions have been set up by direct searches for quantum black holes [13] and the exchange of virtual gravitons on di-lepton events [14]. On the other hand, the ATLAS collaboration has presented experimental limits for the coupling constant of 4-fermion contact interaction [14, 13]. These results are currently used for imposing bounds on the value of the fundamental gravity scale, $M^*$, and, by extension, in order to find limits on the dimensionality of the space-time. However, as we will show here, these claims could have shortcomings from the theoretical and phenomenological viewpoints.

On the other hand, there exists the cosmological constant problem, that is repeatedly faced not only from the Quantum Field Theory (QFT) point of view but also from the Quantum Gravity and early cosmology viewpoints. There are many mechanisms and scenarios trying to explain consistently the problem, some of them against the physical intuition. For example, recently Brodsky et al. have argued that quark and gluon condensates are spatially restricted to the interiors of hadrons and do not extend throughout all of space [16, 17]. Such argument seems to have some problem. Consequently, alternative possibilities need to be studied and developed. The other problem that arises here is how to generate the 4-fermion interaction from first principles and how to get its contributions to the cosmological constant. An approach where cosmological constant comes out from fermion condensations is discussed in [18].

The question that immediately arises is if there exist other mechanisms to explain faithfully the 4-fermion interaction without the drawbacks inherent to the standard Einstein-Cartan theory. Some affirmative answers are possible to this question, as we will discuss below.
Our argument is based on a gravity theory where a pure affine geometry is adopted with the gravitational Lagrangian given by

$$L_g = \sqrt{\det(R^0_\mu R_{\alpha\nu})}$$

where the specific Ricci curvature tensor is determined by

$$R^a_\mu = \lambda (e^a_\mu + f^a_\mu) + R^a_\mu,$$

that corresponds to the breaking of the $SU(2, 2)$ symmetry of a group manifold in higher dimensions with original Riemann curvature: $R^{AB}_{\mu\nu} = \partial_\mu \omega^A_\nu^B - \partial_\nu \omega^A_\mu^B + \omega^A_\mu^C \omega^B_\nu^C - \omega^A_\nu^C \omega^B_\mu^C$ (see Ref. [19, 20] for details) to the $SO(2, 2)$ group. The absolute value of the determinant in (1) is assumed. However, imposing (anti) self-duality conditions over the generalized curvature $\mathcal{R}$, an Euclidean condition is obtained in Eq. (2), $e^a_\mu$ is the tetrad field

$$g_{\mu\nu} \equiv e^a_\mu e_{\nu a}, \quad \eta_{ab} \equiv e^a_\mu e^b_\mu$$

and $f^a_\mu$ is antisymmetric with respect to the index permutation

$$e_{ab} f^a_\nu = f_{\mu\nu} = -f_{\nu\mu},$$

which is associated with a central tensorial part of the original $SU(2, 2)$ group. Both $e^a_\mu$ and $f^a_\mu$ can be taken as fundamental fields from which the Palatini variational principle is applied $R_{\mu\nu}$ is the Ricci curvature tensor in a manifold with torsion, $M$ (e.g. $U_4$) and $\lambda = (1-d)$ with $d$ being the spacetime dimension. Notice, that the Ricci tensor has symmetric and antisymmetric parts corresponding to the Christoffel and torsion contributions to the connection. Here and below, we consider Greek letters $\mu, \nu$ as coordinates indices and Latin letters $a, b$ as tetrad indices. With this formalism, we have $M^a_\mu \equiv e^{\mu\nu} M_{\nu\mu}$. By using Eq. (2), the Lagrangian becomes

$$L_g = \sqrt{\det \left[ \lambda^2 \left(g_{\mu\nu} + f^a_\mu f_{\nu a} \right) + 2\lambda R_{\mu\nu} + 2\lambda f^a_\mu R_{\nu a} + R^a_\mu R_{\nu a} \right]},$$

where the Ricci tensor can be split in its symmetric and anti-symmetric part: $R_{\mu\nu} = R_{\mu\nu} + R_{\nu\mu}$ (see reference [19, 20] for details). The basis of the considered approach is a hypercomplex construction of the (metric compatible) space-time manifold $M$ [19, 21], where for each point of $M$ there exists a local affine space $A$. The connection over $A$, $\Gamma$, defines a generalized affine connection $\Gamma$ on $M$ specified by $\nabla$ and $K$, where $K$ is an invertible $(1, 1)$ tensor over $M$. Connection is compatible and rectilinear, i.e.

$$\nabla_\mu K_{\rho\sigma} = K_{\rho\sigma} T^a_{\mu\rho\sigma}, \quad \nabla_\mu g_{\mu\nu} = 0,$$

where $T^a_{\rho\sigma}$ is the torsion tensor and $g_{\mu\nu}$ is the metric tensor preserved under parallel transport. This compatibility condition ensures that the affine connection $\Gamma$ maps auto-parallel curves of $\Gamma$ on $M$ in straight lines over the affine space $A$ (locally). The first equation is the condition determining the connection $\Gamma$ in terms of the fundamental tensor $K$.

As it is well known, the Palatini variational principle determines the connection required for the space-time symmetry as well as the field equations. From here we assume a four dimensional space-time. Consequently and by construction, the action (1) yields the G-invariant conditions (namely, the intersection of the 4-dimensional Lorentz group $L_4$, the symplectic $Sp(4)$ and the almost complex group $K(4)$), without prior assumptions. As a consequence, the gravitational, Dirac and Maxwell equations arise from the Lagrangian $L_g$ as a causally connected closed system. The self-consistency is given by

$$f_{\mu\nu} \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \varphi^{\rho\sigma} = * \varphi_{\mu\nu},$$

where $\varphi_{\nu\lambda}$ is related to the torsion by $\frac{1}{6} \left( \partial_\mu \varphi_{\nu\lambda} + \partial_\nu \varphi_{\mu\lambda} + \partial_\lambda \varphi_{\mu\nu} \right) = T^a_{\nu\mu} \varphi_{\rho\lambda} + f_{\mu\nu}$ plays the role of electromagnetic field.

As it was shown in Ref. [19, 20] for this model of gravity (see Ref. [21], for astrophysical neutrino applications), the Dirac equation is derived from the same space-time manifold and acquires a coupling modification of the form

$$\gamma^\alpha j \left( \frac{1-d}{d} \right) \gamma_5 h_{\alpha},$$

where $\varphi_{\nu\lambda}$ is related to the torsion by $\frac{1}{6} \left( \partial_\mu \varphi_{\nu\lambda} + \partial_\nu \varphi_{\mu\lambda} + \partial_\lambda \varphi_{\mu\nu} \right) = T^a_{\nu\mu} \varphi_{\rho\lambda} + f_{\mu\nu}$ plays the role of electromagnetic field.
where \( h_\alpha = \varepsilon^\nu_\alpha T^\nu_\rho \) is the torsion vector defined by the duality operation in 4-dimensions and \( j \) is a parameter of pure geometrical nature. Here, the torsion described by \( h_\alpha \) is a dynamical field and the theory is Lorentz invariant by construction. This dynamical torsion vector is responsible for generation of the 4-fermion interaction, as it will be shown below.

The aim of this work is twofold: first, we discuss the possibility to explain the nature, magnitude, bounds and contributions to the cosmological constant due to the 4-fermion interaction from the point of view of unified theories with torsion based on affine geometries. Secondly, we compare our approach with other attempts coming from the context of Riemann-Cartan theory.

The layout of the paper is the following. In Sec.II, we discuss the fermion interaction and vector torsion in view of the Hodge - de Rham decomposition. Sec.III is devoted to the derivation of the gravitational field and Dirac equations, while in Sec.IV, we discuss the fermionic structure. The effective action and the 4-fermionic interaction, together with the energy-momentum tensor, are considered in Sec. V. Conclusions are drawn in Sec.VI.

II. GENERALIZED HODGE-DE RHAM DECOMPOSITION, THE VECTOR TORSION \( h \) AND THE FERMION INTERACTION

As pointed out in references [19–24], the torsion vector \( h = h_\alpha dx^\alpha \) (the 4-dimensional dual of the torsion field \( T_{\beta\gamma\delta} \)) plays multiple roles and can be constrained in several different physical situations. Mathematically, it is defined by the Hodge-de Rham decomposition given by the 4-dimensional Helmholtz theorem which states:

If \( h = h_\alpha dx^\alpha \notin F'(M) \) (set of derivative of functions on \( M \)) is a 1-form on \( M \), then there exist a zero-form \( \Omega \), a 2-form \( \alpha = A_{[\mu\nu]} dx^\mu \wedge dx^\nu \) and a harmonic 1-form \( q = q_\alpha dx^\alpha \) on \( M \) that

\[
h = d\Omega + \delta a + q \to h_\alpha = \nabla_\alpha \Omega + \varepsilon^{\beta\gamma\delta}_\alpha \nabla_\beta A_{\gamma\delta} + q_\alpha.
\]

Notice that even if \( q_\alpha \) is not harmonic, and assuming that \( q_\alpha = (P_\alpha - eA_\alpha) \) is a vector, an axial vector can be added such that the above expression takes the form

\[
h_\alpha = \nabla_\alpha \Omega + \varepsilon^{\beta\gamma\delta}_\alpha \nabla_\beta A_{\gamma\delta} + \varepsilon^{\beta\gamma\delta}_\alpha M_{\beta\gamma\delta} + (P_\alpha - eA_\alpha)
= \nabla_\alpha \Omega + \varepsilon^{\beta\gamma\delta}_\alpha \nabla_\beta A_{\gamma\delta} + b_\alpha + (P_\alpha - eA_\alpha),
\]

where \( M_{\beta\gamma\delta} \) is a completely antisymmetric tensor. In such a way, \( \varepsilon^{\beta\gamma\delta}_\alpha M_{\beta\gamma\delta} \equiv b_\alpha \) (axial vector).

One can immediately see that, due to the theorem given above, one of the roles of \( h_\alpha \) is precisely to be a generalized energy-momentum vector, avoiding the addition "by hand" of a matter Lagrangian in the action (4). As it is well known, the addition of the matter Lagrangian leads, in general, to non-minimally coupled terms into the equations of motion of the physical fields. Consequently, avoiding the addition of energy-momentum tensor, the fields and their interactions are effectively restricted thanks to the same geometrical structure in the space-time itself.

III. GRAVITATIONAL FIELD AND DIRAC EQUATIONS

It is possible to show [19–24] that to derive the Dirac equation, one needs, as starting point, the symmetric part of the gravitational field equations derived from \( \delta g L_g = 0 \), that is

\[
\mathring{R}_{\mu\nu} = -2\lambda g_{\mu\nu} + T^j_{\mu\rho} \alpha T_{\alpha\nu}^\rho = -2\lambda g_{\mu\nu} - 2w \left( g_{\mu\nu} h_\alpha h^\alpha - h_\mu h_\nu \right) = -2\lambda g_{\mu\nu} - 2 \left( g_{\mu\nu} \Pi_\alpha \Pi^\alpha - \Pi_\mu \Pi_\nu \right).
\]

Here we use the obvious duality relation between \( T \) and \( h \) and define the generalized momentum vector as: \( \sqrt{w} h_\mu = \Pi_\mu \) with \( w \) some arbitrary constant that will be conveniently fixed. Then, a mass-like shell condition is immediately obtained

\[
\Pi^2 = m^2 \Rightarrow m = \pm \sqrt{\frac{\mathring{R}}{2(1 - d)}} + d.
\]

where the definition mass (where the mass-like shell hold true) is connected with the spacetime structure, due the unified character of the theory. Notice that there exists a link between the dimension of the spacetime and the scalar "Einstenian" curvature \( \mathring{R} \). Moreover, the curvature and the mass are constrained to take definite values in order that
\(d \in \mathbb{N}\), the natural number characteristic of the dimension. On the other hand, knowing that \(\lambda = 1 - d\) and assuming that the parameter \(m \in \mathbb{R}\), the limiting condition on the physical values for the mass is \(\frac{\tilde{R}}{2(1-\sigma)} + d \geq 0\).

Admitting \(\Pi_{\mu} \rightarrow \hat{P}_{\mu} - e \hat{A}_{\mu} + \gamma^5 b_{\mu}\) (with \(b_{\mu} \equiv e_{\mu}^{\rho\sigma} M_{\nu\rho\sigma}\) an axial vector) together with the quantum condition where the classical equation is converted to an operator: \(\Pi_{\mu} \rightarrow \hat{\Pi}_{\mu}\), we have

\[
\left\{ [\gamma^\mu (\hat{P}_{\mu} - e \hat{A}_{\mu} + c_1 \gamma^5 b_{\mu}) + m] [\gamma^\nu (\hat{P}_{\nu} - e \hat{A}_{\nu} + c_1 \gamma^5 b_{\nu}) - m] \right\} \Psi = 0, \tag{14}
\]

(where \(\Psi = u + iv\) is a complex function) which leads to the Dirac equation

\[
[\gamma^\mu (\hat{P}_{\mu} - e \hat{A}_{\mu} + c_1 \gamma^5 b_{\mu}) - m] \Psi = 0, \tag{15}
\]

with \(m\) given by (13). Notice that this condition, in the Dirac case, is not obtained only passing from classical variables to quantum operators, but in the case that the action does not contain explicitly \(\Psi\), \(h_{\mu}\) remains without specification due the the gauge freedom in the momentum. Notice that the unified character of the theory make the number of equations and the field transformations above self consistent, as will be clear in Section V (see the effective action). From the second order version of (14), it is not difficult to show that for \(u^\lambda\) (remind \(\Psi = u + iv\)) :

\[
\left\{ \left( \hat{P}_{\mu} - e \hat{A}_{\mu} + c_1 \gamma^5 b_{\mu} \right)^2 - m^2 - \frac{1}{2} \sigma^{\mu\nu} \left( e \left( \nabla_{\mu} \hat{A}_{\nu} - \nabla_{\nu} \hat{A}_{\mu} \right) - c_1 \gamma^5 \left( \nabla_{\mu} b_{\nu} - \nabla_{\nu} b_{\mu} \right) \right) \right\} u^\lambda = 0 \tag{16}
\]

\(\) (the same, obviously, for \(v^\lambda\)). It is interesting to see that eq. (16) differs from that obtained by the standard expression derived in [26] due to the appearance of the last two terms: the term involving the curvature tensor is due to the spin interaction with the gravitational field (due to the torsion term in \(R^\lambda_{\mu\nu\alpha}\)) and the last term is the spin interaction with the electromagnetic and mechanical momenta. The important point here is that the spin-gravity interaction term is derived since the spinors are represented as space-time vectors whose covariant derivatives are defined in terms of the G-(affine) connection (see also [27] for the classification of torsion tensor). Other important point to remark is that, in order for Dirac equation to be global (or being covariant with respect to spin transformations if working locally) global topological conditions on M are needed. Through this work, we are working locally and global issues can be taken into account as in standard references (e.g. [37] or references quoted therein and [38], [39] for the relation between spin structures and Dirac equations).

IV. FERMIONIC STRUCTURE, ELECTROMAGNETIC FIELD AND ANOMALOUS GYROMAGNETIC FACTOR

If we introduce an expression corresponding to the antisymmetric part of the gravitational field, namely \(\nabla_{\alpha} T^{\alpha\beta\gamma} = -2 \lambda f^{\beta\gamma}\), in (16) then

\[
\left[ \left( \hat{P}_{\mu} - e \hat{A}_{\mu} + c_1 \gamma^5 b_{\mu} \right)^2 - m^2 - \frac{1}{2} \sigma^{\mu\nu} \left( e F_{\mu\nu} - c_1 \gamma^5 S_{\mu\nu} \right) \right] u^\lambda - \frac{\lambda}{d} \sigma^{\mu\nu} f_{\mu\nu} u^\lambda \tag{17}
\]

\[-\frac{1}{2} e \sigma^{\mu\nu} \left( \hat{A}_{\mu} \hat{P}_{\nu} - \hat{A}_{\nu} \hat{P}_{\mu} \right) u^\lambda = 0 \]

as a consequence, we have

\[
\left[ \left( \hat{P}_{\mu} - e \hat{A}_{\mu} + c_1 \gamma^5 b_{\mu} \right)^2 - m^2 - \frac{1}{2} \sigma^{\mu\nu} \left( e F_{\mu\nu} - c_1 \gamma^5 S_{\mu\nu} + \frac{\lambda}{d} f_{\mu\nu} + e \left( \hat{A}_{\mu} \hat{P}_{\nu} - \hat{A}_{\nu} \hat{P}_{\mu} \right) \right) \right] u^\lambda = 0 \tag{18}
\]

where clearly appear the contributions to the \((g-2)\) factor due the axial vector \(b_{\mu}\) and the geometry through the commutation relation between the covariant derivatives \(\nabla\). We can go ahead and see that if \(\omega_2 F_{\mu\nu} = S_{\mu\nu}\) and
\( \omega_1 F_{\mu\nu} = \sigma'_{\mu\nu} \) the last expression assumes the suggestive form

\[
\left\{ \left( \tilde{P}_\mu - e \tilde{A}_\mu + c_1 \gamma^5 \tilde{b}_\mu \right)^2 - m^2 - \frac{1}{2} \sigma^{\mu\nu} \left[ \left( e - c_1 \omega_2 \gamma^5 - \omega_3 \frac{\lambda}{d} \right) F_{\mu\nu} + \omega_1 \frac{\lambda}{d} \sigma_{\mu\nu} \right] \right\} u^\lambda - \frac{e}{2} \sigma^{\mu\nu} \left( \tilde{A}_\mu \tilde{P}_\nu - \tilde{A}_\nu \tilde{P}_\mu \right) u^\lambda = 0
\]

with the result that the gyromagnetic factor results modified accordingly, and a 4-fermion coupling is introduced constructively, thanks to \( f_{\mu\nu} \). Although the anomalous term is clearly determined from the above equations, it is extremely useful in order to compare the present scheme to other theoretical approaches.

With these considerations in mind, it is important to derive the anomalous momentum for the electron. Specifically, from the last expression, one gets the correction to the lepton anomalous momentum in the form

\[
\Delta a_e = \frac{\omega_1 \lambda}{e d} = \frac{\omega_1}{e} \left( 1 - \frac{1}{d} \right).
\]

The experimental precision in measurement of this quantity is \( \Delta a_e^{\exp} = 0.28 \times 10^{-12} \)

and then the upper bound for the universal geometric parameter \( \omega_1 \) is

\[
\omega_1 < \frac{e}{d} \left( \frac{d-1}{d} \right) 0.28 \times 10^{-12}
\]

then, in 4-dimensions, we have \( \omega_1 < 0.28 \times 10^{-12} \). This result is useful in order to give constraints to the theory.

Another important consideration is related to the anomalous magnetic momentum. As it is well known from the quantum point of view, in the lowest-order diagram, the anomalous magnetic momentum term is given by

\[
\Delta \Gamma_\mu(p, p' = p + q) = -2iA^2 \int \frac{d^4k}{(2\pi)^4} \Gamma_a S(k' = k + q) \gamma_\mu S(k) \Gamma_a
\]

where

\[
S(k) = \frac{1}{\tilde{k} - M(k)}
\]

is the formal propagator and \( \Gamma_a = \{ I, \gamma_\mu, \gamma_\mu \gamma_5 \} \) for \( a = S, P, V, A, T \) (scalar, pseudoscalar, vector, axial vector and tensor bilinears). The Fierz transformation for the integrand, necessary to reduce and rewrite the matrix quantities in a convenient computational form, has a general form as

\[
\Gamma_a \left( \tilde{k}' - M(k') \right) \gamma_\mu \left( \tilde{k} - M(k) \right) \Gamma_a = \sum_{a=s, p, v, a, t} C_a \text{Tr} \left[ \left( \tilde{k}' + M(k') \right) \gamma_\mu \left( \tilde{k} + M(k) \right) \Omega_a \right] \Omega_a,
\]

where \( C_a \) are the coefficients of the Fierz transformation. This makes the link between experimental data and theory, as masses, phase space and matrix elements from cross sections. See also [32] for details.

**V. EFFECTIVE ACTION, \( \Theta \) TERM AND 4-FERMION INTERACTION**

Now, let us analyze the theory from a different point of view. In some gravitational models, a link between torsion and CP violating terms certainly appears. An illustrative example is given by Ashtekar that has rewritten Einstein’s theory, in its Hamiltonian formulation, as a set of differential equations obeyed by an SO(3) connection and its canonically conjugate momenta corresponding to the SO(3) gauge [29]. Bengtsson and Peldan [30] have shown that if one performs a particular canonical transformation involving Ashtekar’s variables and the corresponding SO(3) gauge fields, the expression for the Hamiltonian constraint changes when other constraints remain unaffected. This corresponds precisely to the addition of a "CP-violating"-term to the corresponding Lagrangian. Mullick and Bandyopadhyay have shown that this CP-violating-term is responsible for nonzero torsion [31]. This \( \theta \)-term effectively corresponds to the chiral anomaly when a fermion chiral current interacts with a gauge field. Here, the contrary statement is found from first principles: the theory with torsion leads, at effective level, a \( \theta \)-term directly related to the space-time dimensions through the "cosmological" constant \( \lambda = 1 - d \) as we will see soon.
A. Deriving the effective Lagrangian

As in the case of massive vector particle with spin 1, let us derive the effective Lagrangian. The procedure (see for example [26]) has to be performed in 2 steps in order to avoid several subsidiary conditions: all the information for dynamics must be obtained from the same variational procedure. The starting effective Lagrangian is

\[
L_{\text{eff}} = \theta f^* f^{\mu
u} + \frac{\theta}{2\lambda} f^{\mu
u} (\nabla_{\mu} h_{\nu} - \nabla_{\nu} h_{\mu}) + \frac{\theta}{2\lambda} f^{\rho\mu\nu}\nabla_{\mu} T_{\nu}^{\rho} + A\bar{\Psi} \left( [\rho^{\mu} h_{\mu} + m] \Psi + B h_{\mu} h^{\mu} \right)
\]

where

\[
\rho_{\mu} = (a_{c} + b_{c} \gamma^{5}) \gamma_{\mu}^{c} + \epsilon_{\mu}^{\alpha \beta \gamma} (c_{c} \gamma_{\alpha}^{c} \sigma_{\beta \gamma}) \quad \text{(21)}
\]

that corresponds to the decomposition of a general vector element of the Lie algebra of SU(2, 2). Let us notice that if \(b_{c} + c_{c} = 0\), the pseudo vectorial part of \(\rho_{\mu}\) is eliminated. This fact is directly related, due to the variational procedure of the effective Lagrangian (20), with the generalized Hodge-de Rham decomposition that we have considered before, that is

\[
A\bar{\Psi} \rho^{\mu} \Psi = \bar{\Psi} (a^{c} \gamma^{\mu}) \Psi + \bar{\Psi} \left( (b^{c} + c^{c}) \gamma^{5} \gamma^{\mu} \right) \Psi
\]

Then, following the standard procedure (Berestetsky et al. [26]) after deriving the equations of motion and the related constraints, the effective Lagrangian takes the form

\[
L_{\text{eff}} = L_{1} + L_{2} + L_{3}
\]

being

\[
L_{1} = (\theta + \lambda) f^{*} f^{\mu \nu} \rightarrow (\text{theta term}) \quad \text{(24)}
\]

\[
L_{2} = A\bar{\Psi} \left( [\rho^{\mu} (P^{\mu} - \epsilon A_{\mu} + \gamma^{5} b_{\mu} + \nabla_{\mu} \Omega + \epsilon_{\mu}^{\beta \gamma} \nabla_{\beta} A_{\gamma \delta}) + m] \right) \Psi \rightarrow (\text{Dirac-like term}) \quad \text{(25)}
\]

\[
L_{3} = A^{2}\bar{\Psi} \rho_{\mu} \bar{\Psi} \rho^{\mu} \Psi \rightarrow (4 - \text{fermion term}) \quad \text{(26)}
\]

It is important to note also that all dependence on coefficient values are charged on the respective parameters in order to avoid the unboundedness problem for the Lagrangian (eg; \(\theta, A, \text{etc.}\)).

B. Energy-momentum tensor and cosmological term

It is worth noticing that the mass term in the Dirac equation (15) contains the GR curvature scalar plus the cosmological term \(\lambda = (1 - d)\). In the analysis already made in [32], the mass is a constant, then it is naturally included into the Dirac equation and then into the energy-momentum tensor. Also here, the gravitational part of the Lagrangian (containing the curvature) has been avoided. We can write the effective energy-momentum tensor derived from the effective Lagrangian density \(L_{\text{eff}}\), as

\[
T_{\rho \sigma} \propto 4 (\theta + \lambda) \left[ f^{*}_{\alpha \rho} f^{\alpha}_{\sigma} - g_{\rho \sigma} \frac{f^{*}_{\nu} f^{\mu \nu}}{4} \right] - A\bar{\Psi} \left( g_{\rho \sigma} \rho^{\mu} h_{\mu} - \left( \rho^{\mu} h_{\mu} + h_{\mu} \rho_{\mu} \right) \pm \frac{\rho^{\sigma} R^{\mu} + \rho^{\mu} R^{\sigma}}{\sqrt{R^{\mu}} \sqrt{R^{\sigma}}} \right) \Psi
\]

\[- 2 A^{2} \left[ \frac{g_{\rho \sigma}}{2} \bar{\Psi} \rho_{\mu} \bar{\Psi} \rho^{\mu} \Psi - \bar{\Psi} \rho_{\rho} \bar{\Psi} \rho_{\rho} \Psi \right].
\]

Using the Dirac equation and rearranging the 4-fermion term, the above tensor can be rewritten in order to identify the effective contribution to the cosmological term from the fermion sector. We obtain

\[
T_{\rho \sigma} \propto 4 (\theta + \lambda) \left[ f^{*}_{\alpha \rho} f^{\alpha}_{\sigma} - g_{\rho \sigma} \frac{f^{*}_{\nu} f^{\mu \nu}}{4} \right] - A\bar{\Psi} \left( - \left( \rho^{\mu} h_{\mu} + h_{\mu} \rho_{\mu} \right) \pm \frac{R^{\mu}_{\rho} + R^{\rho}_{\mu}}{R^{\mu} + \lambda d} \right) \Psi
\]

\[+ A^{2} g_{\rho \sigma} \bar{\Psi} \rho_{\mu} \bar{\Psi} \rho^{\mu} \Psi.
\]
As firstly pointed out by Eddington [33], the mass term is directly related to the curvature and implied by the Mach principle. Here, we want to stress that also fermion interactions can contribute to the cosmological term and then can take part to the cosmic dynamics as a sort of dark energy contribution [18]. Notice that from the above expression, the pure fermionic contribution to the cosmological constant, due to the 4-fermion interaction is

\[ \Lambda_f \equiv \kappa \rho_{\Lambda_f} = +\kappa A^2 g_{\rho\mu} \bar{\Psi} \gamma^{\rho} \gamma^{\mu} \Psi \]

(where the units of the constant are \([\kappa] = m_{Pl}^2\)). Considering the possibility of quark condensates, it was conjectured [24] that a nonzero vacuum expectation value of the 4-fermion term arises from a spontaneous breaking of the global chiral symmetry by the \((\bar{q}q)\) condensate, which sets the energy scale of the condensation to the QCD scale of the running strong-interaction coupling, \(\Lambda_{QCD}\). To see this, the Shifman-Vainshtein-Zakharov (SVZ) approximation can be effectively used [25] given the following result

\[ \langle 0 | \Lambda_f | 0 \rangle = \frac{16\kappa A^2}{9}(a_\epsilon a^\epsilon - (b_\epsilon + c_\epsilon)(b^\epsilon + c^\epsilon)) \langle 0 | \bar{\Psi} \Psi | 0 \rangle^2. \]

Here, the contribution corresponding to the axial/axial vector channel is identically zero (only \(A-A\) and \(V-V\) expectation values give contributions to the cosmological constant (see the explicit channel computations below) and the arbitrary constants \(a_\epsilon, b_\epsilon, c_\epsilon\) can be defined accordingly. It is also useful to consider that, from the above formula, all the parameters can be fixed from the corresponding experimental data. The traces for vector and axial-vector channels explicitly are

\[
\begin{align*}
\text{Tr} \left[ \left( \hat{k} + M (k') \right) \gamma_\mu \left( \hat{k} + M (k) \right) I \right] &= 4 (k'M(k) + kM(k')) \mu, \\
\text{Tr} \left[ \left( \hat{p}_2 + m \right) \gamma_\mu (\hat{p}_1 + m) \gamma_5 \right] &= 0, \\
\text{Tr} \left[ \left( \hat{p}_2 + m \right) \gamma_\mu (\hat{p}_1 + m) \gamma_\nu \gamma_5 \right] &= 4 \left[ m^2 g_{\mu\nu} + (p_2 p_1 \epsilon_{\mu\nu}) + p_{2\mu} p_{1\nu} \right], \\
\text{Tr} \left[ \left( \hat{p}_2 + m \right) \gamma_\mu (\hat{p}_1 + m) \gamma_5 \gamma_5 \right] &= 4 \epsilon_{\alpha\beta\gamma\delta} p_{2\alpha} p_{1\beta}, \\
\text{Tr} \left[ \left( \hat{p}_2 + m \right) \gamma_\mu (\hat{p}_1 + m) \sigma_{\lambda\rho} \right] &= 4
\end{align*}
\]

and then the above result \(\langle 0 | \Lambda_f | 0 \rangle\) is explicitly recovered.

\section{VI. DISCUSSION AND CONCLUSIONS}

Let us now analyze each term of the Lagrangian (23). There is a possible screening between \(\theta\) plus \(\lambda\) Lagrangian terms. A similar relation between \(\theta\) and \(\lambda\) has been conjectured in Ref. [31].

There are no Holst term and FMT term, in contrast with other theories involving gravitation in canonical formulation (on the possibility of getting Holst-like terms in f(R) theories was analyzed in [40] and [41]). The vector-vector and the axial-axial terms are also in the FMT Lagrangian but the term that we have here is not constrained by any extra-parameter as the Barbero-Immirzi one. It is clear that the term proportional to the axial-vector coupling is not present into the model discussed here due to the fundamental geometrical structure of our construction. Here we have

\[ A \bar{\Psi} \gamma^\mu \Psi = \bar{\Psi} \left( a_\epsilon \gamma_\mu \right) \Psi + \bar{\Psi} \left( b_\epsilon + c_\epsilon \right) \gamma^5 \gamma_\mu \Psi \equiv \Psi + A \]

where the sum of axial and vector terms appears. Immediately we see that the 4-fermion interaction \(\bar{\Psi} \gamma^\mu \gamma_\mu \Psi\) geometrically only picks \(V-V\) and \(A-A\) interactions, if we suppose coefficients real (in particular \(b_\epsilon\) and \(c_\epsilon\)). This important fact has been experimentally probed as pointed out in [34]. In that paper, the effective 4-fermion interaction was focused on the case of neutrinos endowed with non-standard interactions. These are a natural outcome of many neutrino mass models [23] and can be of two types: flavour-changing (FC) and non-universal (NU). As it is well known, see-saw-type models leads to a non-trivial structure of the lepton mixing matrix characterizing the charged and neutral current weak interactions. This leads to gauge induced non standard interactions which may violate lepton flavor and CP even with massless neutrinos. Alternatively, non-standard neutrino interactions may also arise in models where neutrino masses are “calculable” from radiative corrections.

Finally, in some supersymmetric unified models, the strength of non-standard neutrino interactions may be a calculable renormalization effect. How sizable are non-standard interactions will be a model-dependent issue. In some models, non-standard interaction strengths are too small to be relevant for neutrino propagation, because they are suppressed by some large scale and/or restricted by limits on neutrino masses. However, this could not be the case, and there are interesting models where moderate strength non-standard interactions remain in the limit of light (or
even massless) neutrinos. Such a fact may occur even in the context of fully unified models like SO(10). Non-standard interactions may, in principle, affect neutrino propagation properties in matter as well as the detection cross sections. Thus their existence can modify the solar neutrino signal observed by the experiments.

There appears, at effective level, a $\theta$ (parity violating) term that is not present in other formulations as the standard Einstein-Cartan (see for example the discussion in [32]) and loop-quantum gravity inspired [30, 31]. For quarks, a non-zero vacuum expectation value of the 4-fermion term arises from a spontaneous breaking of the global chiral symmetry by the $\langle \bar{q}q \rangle$ condensate, which sets the energy scale of the condensation to the QCD scale of the running strong-interaction coupling, $\Lambda_{\text{QCD}}$. Quark condensates are associated to the color degree of freedom, and characterize the confined phase of quark matter and constitute, together with gluon condensates, the QCD vacuum. For leptons, which do not interact strongly and are not subjected to confinement, less is known about the form and scale of condensation. Is important to note that in the study of cosmological constant one should bear in mind that vector torsion gives the contribution to conformal anomaly (see, for instance, [12]) which may give qualitatively same effect to cosmology (anomaly-driven inflation or LCDM DE) as pure cosmological constant.

Solving the question about the interplay between gravity models with torsion, space-time dimensionality and 4-fermion interaction is still complicated, although some claims in the recent literature point out the contrary [35].

The basic points under discussion are: dimensional compactification in space-times with torsion, the origin of 4-fermion interaction and the specific extra-dimensional models that have to be considered. From the viewpoint of the model presented here, the main question to be addressed is, in a space-time with dimensionality $> 4$, that the torsion dual will not be a vector but a higher rank tensor field. However, the total antisymmetry of the torsion in such a case, simplifies any physical analysis. In a forthcoming paper, we will discuss the possible experimental tests of the unified scheme presented here.

VII. ACKNOWLEDGEMENTS

DJCL is very grateful to the people of the Bogoliubov Laboratory of Theoretical Physics (BLTP) and JINR Directorate by they hospitality and financial support, and also to Professors J.W.F. Valle and F.J. Escrihuela for bring me important references on the subject.

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