Kovalenko’s Full-Rank Limit and Overhead as Lower Bounds for Error-Performances of LDPC and LT Codes over Binary Erasure Channels

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Abstract

We present Kovalenko’s full-rank limit as a tight lower bound for decoding error probability of LDPC codes and LT codes over BEC. From the limit, we derive a full-rank overhead as a lower bound for stable overheads for successful maximum-likelihood decoding of the codes.

I. INTRODUCTION AND BACKGROUND

Binary Erasure Channels (BEC) based Low-Density Parity-Check (LDPC) codes [4], [5] and Luby Transform (LT) codes [6], [7] became quite popular for a variety of applications over packet networks such as the Internet. The popularity of LDPC and LT codes are due in part to (a) the low-complexity of the popular set of decoding algorithms that fall under the umbrella of the Message Passing Algorithm (MPA) (otherwise called Belief Propagation Algorithm for BEC) [4], [5], (b) good error performance of MPA for codes of large block lengths, and (c) the flexibility in choosing the block lengths of these codes, which make them usable for a variety of applications.
In BEC, without loss of generality, the task of both LT and LDPC decoders is to recover the unique solution of a consistent linear system

\[ HX^T = \beta^T, \quad \beta = (\beta_1, \ldots, \beta_m) \in (\mathbb{F}_2^s)^m, \]  

(I.1)

where \( H \) is an \( m \times n \) matrix over \( \mathbb{F}_2 \). This can be explained shortly as follows. In case of LT codes, to communicate an information symbol vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{F}_2^s)^n \), a sender constantly generates and transmits a syndrome symbol \( \beta_i = H_i \alpha^T \) over BEC, where \( H_i \in \mathbb{F}_2^n \) is generated uniformly at random on the fly by using the Robust Soliton Distribution \( \mu(x) = \sum \mu_d x^d \) (see [6]). A receiver then acquires a set of pairs \( \{(H_i, \beta_i)\}_{i=1}^m \) and interprets it as System (I.1). Hence, the variable vector \( X = (x_1, \ldots, x_n) \in (\mathbb{F}_2^s)^n \) in the system represents the information symbol vector \( \alpha \). In case of LDPC codes, contrastingly, a sender transmits a codeword vector \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \text{Ker}(M) = \{\alpha \in (\mathbb{F}_2^s)^N | M \cdot \alpha^T = 0\} \), where \( M \) is an \( m \times N \) binary check matrix. Due to erasures, some of symbols of \( \alpha \) may be lost and a receiver acquires a part of \( \alpha \), denote it as \( \bar{\alpha} \). Then by the rearrangements \( \alpha \equiv (\bar{\alpha}, X) \) and \( M \equiv [\bar{H}; H] \), where \( \bar{H} \) and \( H \) consist of columns of \( M \) that associate symbols of \( \bar{\alpha} \) and \( X \), respectively, the receiver interprets the kernel space constraint \( M \cdot \alpha^T = 0 \) as System (I.1), where \( \beta^T = \bar{H} \bar{\alpha}^T \). Hence in LDPC codes, \( X \) represents a lost symbol vector of \( \alpha \).

In LT codes, the column-dimension \( n \) of \( H \) is fixed, the row-dimension \( m \) of \( H \) is a variable, and a reception overhead \( \gamma = \frac{m-n}{n} \) is the key parameter for measuring error-performance of codes. In LPDC codes, however, the row-dimension \( m \) is fixed in general, the column-dimension \( n = pN \) is a variable, and an erasure rate (or loss rate) \( p = \frac{n}{N} \) is the key parameter for measuring error-performance of codes. Let \( R = 1 - \frac{m}{N} \), a code-rate of an LDPC code. By using \( m = (1+\gamma)n \), \( n = pN \), and \( R = 1 - \frac{m}{N} \), \( p \) and \( \gamma \) are expressed as

\[ p = \frac{1 - R}{1 + \gamma} \quad \text{and} \quad \gamma = \frac{m-n}{n} = \frac{1 - (R + p)}{p}. \]  

(I.2)

Like LT codes, thus, the error-performance of LDPC codes can be also measured in terms of \( \gamma \).

Several literatures showed the existence of capacity approaching LDPC codes [9] and optimal LT codes [6], [7], whose minimal overheads for successful decoding by the MPA in high probability tends to zero as block lengths (\( n \) for LT and \( N \) for LDPC codes) increase to infinity. For codes of short block lengths, however, their minimum overheads (for the successful decoding
by the MPA in high probability) are not close to zero. Furthermore, even for a nontrivial $\gamma > 0$, the full-rank probability $\Pr(\text{Rank}(H) = n)$ is not very close to 1.

System (I.1) has its unique solution, iff, $\text{Rank}(H) = n$ the full rank of $H$. In case of the full-rank, the unique solution can be recovered by using a Maximum-Likelihood Decoding Algorithm (MLDA) such as the ones in [5], [10]–[12]. These algorithms are an efficient Gaussian Elimination (GE) that fully utilize an approximate lower triangulation of $H$, which is obtainable by using the diagonal extension process with various greedy algorithms [4], [5]. Under those GE based MLDAs, thus, the probability of decoding success is precisely the $\Pr(\text{Rank}(H) = n)$.

Let us define the Decoding Error Probability (DEP) of a code as the rank-deficient probability

$$P_{\text{ML}}^{\text{err}}(1 + \gamma, n) = 1 - \Pr(\text{Rank}(H) = n),$$

(I.3) where $H$ is an $m \times n$ decoder matrix of system (I.1) with $\gamma = \frac{m-n}{n}$. Assume that $P_{\text{ML}}^{\text{err}}(1 + \gamma, n)$ is a decreasing function with respect to $\gamma$. Then for a given error-bound (or deficiency bound) $0 \leq \delta \leq 1$, define

$$\gamma_*(\delta, n) = \min_{\gamma \geq 0} \{\gamma \mid P_{\text{ML}}^{\text{err}}(1 + \gamma, n) < \delta\},$$

(I.4) and refer to as the Minimum Stable Overhead (MSO) of a code within the error-bound $\delta$. Since $P_{\text{ML}}^{\text{err}}(1 + \gamma, n)$ is decreasing, we may expect that $P_{\text{ML}}^{\text{err}}(1 + \gamma, n) \leq \delta$ for any $\gamma \geq \gamma_*(\delta, n)$. Thus, the key part of designing codes is to identify lower bounds of DEP and MSO then to obtain the codes whose DEP and MSO are close to the bounds.

In this paper, as the main contribution of this paper, we define Kovalenko’s Full-Rank Limit (KFRL), denote as $K(1 + \gamma, n)$, from Kovalenko’s rank-distribution of binary random matrices [1]–[3], and show that it is a probabilistic lower bound for $P_{\text{ML}}^{\text{err}}(1 + \gamma, n)$, i.e., $K(1 + \gamma, n) \leq P_{\text{ML}}^{\text{err}}(1 + \gamma, n)$ for any $\gamma$ and $n$. We then derive Kovalenko’s Full-Rank Overhead (KFRO) from KFRL, denote as $\gamma_K(\delta, n)$, as a lower-bound for MSO, i.e., $\gamma_K(\delta, n) \leq \gamma_*(\delta, n)$ for any $\delta$ and $n$, and show that the overhead $\gamma_K(\delta, n)$ tells the least number of symbols that a receiver should acquire to achieve $P_{\text{ML}}^{\text{err}}(1 + \gamma, n) \leq \delta$. We also provide experimental evidences which show the viability that, given a destined error-bound $\delta_0$, both LT and LDPC codes may be designed to achieve their error-performances in $P_{\text{ML}}^{\text{err}}(1 + \gamma, n)$ and $\gamma^*(\delta, n)$ that are close to $K(1 + \gamma, n)$ and $\gamma_K(\delta, n)$ for $\delta \geq \delta_0$, respectively, by supplementing enough number of dense rows to $H$ of system (I.1).
The remainder of this paper is composed of as follows. In Section II, we define KFRL and KFRO and verify them as lower bounds for DEP and MSO of LDPC and LT codes. In Section III, we present experimental results of the performances of codes in terms of DEP and overhead. We summarize the paper in Section IV.

II. KOVALEKO’S FULL-RANK LIMIT AND OVERHEADS

Let us first clarify terms and notations for the remainder of this section. Let $|H_i|$ denote the number of nonzero entries of a row $H_i$ of $H$ and refer to as the degree of $H_i$. Given an overhead $\gamma$, we shall assume that $\gamma n = k$ for some integer $k \geq 0$. Let $\hat{H}$ denote an $m \times n$ random binary matrix over $\mathbb{F}_2$ that consists of random rows $\hat{H}_i = (\hat{h}_{i1}, \ldots, \hat{h}_{in})$ for $1 \leq i \leq m$, such that $\Pr(\hat{h}_{ij} = 1) = \frac{1}{2}$ for $1 \leq j \leq n$. Finally, let $\xi_k(n-s) = \Pr(\text{Rank}(\hat{H}) = n-s)$ the probability that $\text{Rank}(\hat{H}) = n-s$, where $k = m-n$ (or $k = \gamma n$).

Let us introduce Kovaleko’s rank-distribution of $\hat{H}$. It is shown in [1]–[3] by Kovalenko that, for any fixed integers $k$ and $s$ with $l = k+s \geq 0$,

$$
\xi_k(n-s) = \frac{S(n-s,l)}{2^{ls}} \prod_{i=s+1}^{n} \left(1 - \frac{1}{2^i}\right),
$$

where

$$
S(n-s,l) = \sum_{i_1=0}^{n-s} \sum_{i_2=i_1}^{n-s} \cdots \sum_{i_l=i_{l-1}}^{n-s} 2^{-i_1} \cdots 2^{-i_l}.
$$

Since $\lim_{n \to \infty} S(n-s,l) = \prod_{i=1}^{k+s} \frac{1}{1 - \frac{1}{2^i}}$, it holds that

$$
\lim_{n \to \infty} \xi_k(n-s) = \frac{1}{2^{s(k+s)}} \frac{\prod_{i=s+1}^{\infty} (1 - \frac{1}{2^i})}{\prod_{i=1}^{k+s} (1 - \frac{1}{2^i})}.
$$

In fact, the limit distribution above still holds when entries of $\hat{H}$ meet the density constraint

$$
\frac{\ln(n) + x}{n} \leq \Pr(\hat{h}_{ij} \neq 0) \leq 1 - \frac{\ln(n) + x}{n},
$$

where $x \to \infty$ arbitrarily slowly. The limit distribution, however, is not directly applicable to $H$ in System (I.1), because entries of $H$ may not follow the constraint (II.4).

In the following, we define KFRL and verify it as a lower bound for $1-\xi_k(n) = \Pr(\text{Rank}(\hat{H}) < n)$. We then define KFRO from KFRL and verify it as a lower bound for MSO. Foremost, notice
that the sequence \( \{S(n-s,l)\}_{n=s}^{\infty} \) is in fact increasing, therefore,

\[
S(n-s,l) \leq \lim_{n \to \infty} S(n-s,l) = \prod_{i=1}^{k+s} \left(1 - \frac{1}{2^i}\right)^{-1}.
\] (II.5)

By Plugging in \( s = 0 \) into (II.1) and (II.5), we have

\[
1 - \prod_{i=k+1}^{n} \left(1 - \frac{1}{2^i}\right) \leq 1 - \xi_k(n).
\] (II.6)

With the left-hand side above, where \( k = \gamma n \), define

\[
K(1 + \gamma, n) = 1 - \prod_{i=k+1}^{n} \left(1 - \frac{1}{2^i}\right),
\] (II.7)

and refer to as KFRL. For a given error-bound \( \delta \) now, define

\[
\gamma_K(\delta, n) = \min_{\gamma \geq 0} \{\gamma \mid K(1 + \gamma, n) \leq \delta\},
\] (II.8)

and refer to as the KFRO with \( \delta \). Notice that KFRL is decreasing with respect to \( \gamma \), and thus, \( K(1 + \gamma, n) \leq \delta \) for any \( \gamma \geq \gamma_K(\delta, n) \). Observe from (II.7) that \( g(k+1, n) = \left(1 - \frac{1}{2^{k+1}}\right) g(k, n) \).

Hence by \( g(0, n) := 0.288788095066 \) for \( n \geq 50 \), \( K(1 + \gamma, n) \) can be computed explicitly by (II.8), and consequently, \( \gamma_K(\delta, n) \) is obtainable from the graph of \( K(1 + \gamma, n) \).

The following proposition shall be conveniently used for upper bounds for \( K(1 + \gamma, n) \) and \( \gamma_K(\delta, n) \), and for the proof of Lemma II.1.

**Proposition II.1.** Let \( V = (v_1, \ldots, v_n) \in \mathbb{F}_2^n \) be given with \( |V| = k > 0 \), and let \( W = (w_1, \ldots, w_n) \in \mathbb{F}_2^n \) be a random vector such that \( \Pr(w_i = 1) = \frac{d}{n} \) for \( 1 \leq i \leq n \). Then

\[
\Pr(W \cdot V^T = 0) = \frac{1 + (1 - \frac{2d}{n})^k}{2},
\] (II.9)

where \( W \cdot V^T = \sum_{i=1}^{n} w_i v_i \) over \( \mathbb{F}_2 \).

**Proof:** From binomial expansions, we have

\[
\sum_{s \text{ even}} \binom{k}{s} a^s b^{k-s} = \frac{(a + b)^k + (-a + b)^k}{2}.
\] (II.10)

Let \( p_i = \Pr(w_i = 1) \) for \( 1 \leq i \leq n \). Since \( |V| = k \), assume without loss of generality that \( v_i = 1 \)
for $1 \leq i \leq k$ and $v_i = 0$ for $k + 1 \leq i \leq n$, so that $\Pr(W \cdot V^T = 0) = \Pr\left(\sum_{i=1}^{k} w_i = 0\right)$. Then since $\sum_{i=1}^{k} w_i = 0$ iff. $w_i = 1$ for even number of $i$’s,

$$\Pr\left(\sum_{i=1}^{k} w_i = 0\right) = \sum_{s \text{ even}}^{k} \binom{k}{s} \prod_{i \in I_s} p_i \prod_{i \notin I_s} \left(1 - p_i\right)$$

(II.11)

where $I_s \subset \{1, 2, \ldots, k\}$ with $|I_s| = s$. Hence by $p_i = \frac{d}{n}$ for $1 \leq i \leq n$, we have

$$\Pr(W \cdot V^T = 0) = \sum_{s \text{ even}}^{k} \binom{k}{s} \left(\frac{d}{n}\right)^s \left(1 - \frac{d}{n}\right)^{k-s}.$$  

(II.12)

Taking $a = \frac{d}{n}$ and $b = 1 - \frac{d}{n}$ into (II.10) verifies (II.9).

**Theorem II.1 (Upper-Bound for $\gamma_K(\delta, n)$).** For a given error-bound $\delta$, let $k_\delta > 0$ be an integer such that

$$\log_2\left(\frac{1}{\delta}\right) \leq \left(\frac{k_\delta}{n}\right) \leq \frac{1 + \log_2\left(\frac{1}{\delta}\right)}{n},$$

(II.13)

i.e., $k_\delta = \min\{k \in \mathbb{Z} \mid 2^{-k} \leq \delta\}$. It then follows that

$$\gamma_K(\delta, n) \leq \frac{1 + \log_2(1/\delta)}{n}.$$  

(II.14)

**Proof:** Let $\hat{H}$ be an $m \times n$ binary random matrix with $m = n + k_\delta$ such that, for each row $\hat{H}_i = (\hat{h}_i, \ldots, \hat{h}_m)$, $\Pr(\hat{h}_{ij} = 1) = \frac{1}{2}$ for $1 \leq j \leq n$. By Proposition [II.1], $\Pr(\hat{H}_i \cdot V^T = 0) = \frac{1}{2}$ for $1 \leq i \leq n$ and $V \neq 0$. Then since each $\hat{H}_i$ is independent of all other rows,

$$\Pr(V \in \text{Ker}(\hat{H})) = \prod_{i=1}^{m} \Pr(\hat{H}_i \cdot V^T = 0) = \frac{1}{2^m}.$$  

(II.15)

Note that $\text{Rank}(\hat{H}) < n$ iff. $\hat{H} \cdot V^T = 0$ for some $V \neq 0$, and there are of total $2^n - 1$ nonzero vectors in $\mathbb{F}_2^n$. Therefore,

$$1 - \xi_{k_\delta}(n) \leq \sum_{V \neq 0} \Pr\left(V \in \text{Ker}(\hat{H})\right) \leq \frac{2^n - 1}{2^m} \frac{1}{2^{k_\delta}} < \delta.$$  

(II.16)

Hence by (II.6), $K(1 + \gamma, n) < \delta$, and by the definition of $\gamma_K(\delta, n)$, $\gamma_K(\delta, n) \leq \gamma_\delta$. The inequality (II.14) is then clear by (II.13).

Although the authors of the paper are not able to provide any mathematical proofs, experiments exhibited that $K(1 + \gamma, n)$ and $2^{-n}$ are almost identical as $\delta$ decreases. Hence $\gamma_K(\delta, n)$ is in
fact very close to $\gamma_n = \frac{k_n}{n}$. Notice that, since $\lim_{n \to \infty} \frac{1 + \log_2 (1/\delta)}{n} = 0$ as long as $\delta \geq 2^{-c}n^c$ for $c < 1$, $\lim_{n \to \infty} \gamma_K(\delta, n) = 0$ for such $\delta$ by Theorem II.1.

In the following lemma, we show that $K(1 + \gamma, n) \leq P_{\text{err}}^{\text{ML}}(1 + \gamma, n)$. As a consequence of the lemma, we show in Theorem II.2 that $\gamma_K(\delta, n) \leq \gamma_*(\delta, n)$.

**Lemma II.1** (KFRL as a lower-bound for DEP). Let $H$ be an $m \times n$ matrix of System (I.1), where $m = (1 + \gamma)n$ with $\gamma \geq 0$. Then

$$K(1 + \gamma, n) \leq P_{\text{err}}^{\text{ML}}(1 + \gamma, n). \tag{II.17}$$

**Proof:** Let $k = \gamma n$, $m = (1 + \gamma)n$, and $\hat{H}$ an $m \times n$ binary random matrix such that $\Pr(h_{ij} = 1) = \frac{1}{2}$. We first show that

$$\Pr(\text{Rank}(H) = n) \leq \Pr(\text{Rank}(\hat{H}) = n). \tag{II.18}$$

In LT codes, each row $H_i$ of $H$ in system (I.1) follows the uniform probability $\Pr(h_{ij} = 1) = \frac{d}{n}$ with $d \leq \frac{n}{2}$, where $d = |H_i|$ with probability $\mu_d$ of the RSD $\mu(x) = \sum \mu_d x^d$. In LDPC codes, $H$ of system (I.1) is formed by randomly chosen $n = pN$ columns of the check matrix $M$. In both LT and LDPC codes, thus, $\Pr(h_{ij} = 1) \leq \frac{1}{2}$ for $1 \leq j \leq n$. Then by Proposition II.1, $\Pr(\hat{H}_i \cdot V^T = 0) \leq \Pr(H_i \cdot V^T = 0)$ for $V \in \mathbb{F}_2^n$, and this is true for every $1 \leq i \leq m$. Therefore, $\Pr(\hat{H} \cdot V^T = 0) \leq \Pr(H \cdot V^T = 0)$, and in expectation sense, $|\text{Ker} (\hat{H})| \leq |\text{Ker} (H)|$, and hence, the inequality (II.18) is verified. The inequality (II.17) is then clear by the lower bound in (II.6).

**Theorem II.2** (KFRO as a lower-bound for MSO). To solve system (I.1) uniquely with a destined bound $P_{\text{err}}^{\text{ML}}(1 + \gamma, n) \leq \delta$, it should hold that

$$\gamma_*(\delta, n) \geq \gamma_K(\delta, n). \tag{II.19}$$

To achieve $P_{\text{err}}^{\text{ML}}(1 + \gamma, n) \leq \delta$, therefore, the numbers of symbols that receivers should acquire is at least $(1 + \gamma K(\delta, n))n$ for LT codes, and $\frac{R + \gamma K(\delta, n)}{1 + \gamma K(\delta, n)} N$ for LDPC codes.

**Proof:** The inequality (II.19) is clear by Lemma II.1 and by the definitions of $\gamma_*(\delta, n)$ and $\gamma_K(\delta, n)$ in (I.4) and (II.8), respectively. To achieve $P_{\text{err}}^{\text{ML}}(1 + \gamma, n) \leq \delta$ with LT codes, the inequality (II.19) implies that the number of symbols of $\beta$, equivalently, the row-dimension $m$
of $H$ in System (I.1), should be at least $(1 + \gamma_K(\delta, n))n$. In case of LDPC codes, recall that $m = (1 - R)N$ and $n = pN$. To achieve $P_{err}^{\text{ML}}(1 + \gamma, n) < \delta$ with LDPC codes, hence, (II.19) implies that $m \geq (1 + \gamma_K(\delta, n))pN$. In other words, the number of lost symbols, equivalently the column-dimension of $H$ in System (I.1) that is $n = pN$, should be at most $(1 - p)N$ where $(1 - R)N = m$. Therefore, the number of acquired symbols by receivers, i.e., $(1 - p)N$, should be at least $\frac{\gamma_K(\delta, n) + R}{1 + \gamma_K(\delta, n)}N$.

Example II.1. Red curves in Fig. 1 represent the KFRL $K(1 + \gamma, n)$, where $n = 100$ for LT codes (top) and $n = p200$ for LDPC codes (bottom) with $0 \leq p \leq \frac{1}{2}$. When $\delta = 10^{-4}$, for an example, $1 + \gamma_K(10^{-4}, n) \approx 1.14$ in both LT and LDPC codes. To verify 1.14 with LDPC codes, use the conversions in (I.2) with $p_K \approx 0.43$ in the bottom figure. This implies that by Lemma II.1 since $K(1 + \gamma, n) \geq 10^{-4}$ for $1 + \gamma \leq 1.14$, the DEP of both LT and LDPC codes can not be better than $10^{-4}$, i.e., $P_{err}^{\text{ML}}(1 + \gamma, n) \geq 10^{-4}$ for $\gamma \leq 0.14$. Again by Theorem II.2 to achieve $P_{err}^{\text{ML}}(1 + \gamma, 100) \leq 10^{-4}$ with LT codes, the minimum overhead $\gamma_s(10^{-4}, 100)$ should be larger than 0.14, i.e., $\gamma_s(10^{-4}, n) \geq 0.14$. Analogously, to achieve $P_{err}^{\text{ML}}(1 + \gamma, n) \leq 10^{-4}$ with the LDPC codes, where $n = p200$, the maximum tolerable loss rate $p^* = \frac{0.5}{1 + \gamma_s(10^{-4}, n)}$ (use the conversion in (I.2)), should be less than $p_K = \frac{0.5}{1 + \gamma_{100}(10^{-4}, n)} \approx 0.43$, i.e., $p^* \leq 0.43$.

Another thing should be noticed is that, as mentioned earlier, the two curves $K(1 + \gamma, n)$ and $2^{-\gamma n}$ in the top figure are almost identical as $\delta$ decreases. In this respect, $\gamma_K(10^{-4}, 100) \approx \frac{k_\delta}{100}$, where $k_\delta$ is the smallest integer $k$ such that $2^{-k} \leq 10^{-4}$. It is not hard to see by direct computation that $k_\delta = 14$ for $\delta = 10^{-4}$ and $\gamma_\delta \approx \frac{14}{100} = 0.14$, that is precisely the $\gamma_K(10^{-4}, 100)$.

III. Experimental Results with LT and LDPC Codes

In this section, we provide experimental results which show the viability that both LT and LDPC codes may achieve the error-performances in $P_{err}^{\text{ML}}(1 + \gamma, n)$ and $\gamma_s(\delta, n)$ that are close to $K(1 + \gamma, n)$ and $\gamma_K(\delta, n)$, respectively, when enough number of dense rows or columns are supplemented to $H$ in System (I.1). Codes for experiments are arranged as following. For LDPC codes, two check matrices of block dimension $100 \times 200$ (thus $R = \frac{1}{2}$), say $M$ and $\bar{M}$, were arranged by using PEG algorithm in [8]: $M$ was generated with the column-degree distribution $\rho(x)$ in TABLE I and $\bar{M}$ was generated by supplementing 15 random rows of degree $\frac{N}{2} = 100$ to a check matrix of dimension $85 \times 200$ arranged with $\rho(x)$. For LT codes, two row-degree distributions $\mu(x)$ and $\bar{\mu}(x)$ in TABLE I were used for constructing codes of block length $n = 100$. 

January 13, 2009
| $\mu(x)$ | $\begin{cases} (\mu_d)^{5}_{d=1} = (0.012, 0.482, 0.153, 0.082, 0.047) \\ (\mu_d)^{10}_{d=6} = (0.035, 0.024, 0.023, 0.012, 0.012) \\ \mu_{25} = 0.059, \mu_{35} = 0.059 \end{cases}$ |
|---|---|
| $\bar{\mu}(x)$ | Normalization of $\mu(x) + (0.17)x^0$ |
| $\rho(x)$ | $\begin{cases} (\rho_d)^{5}_{d=2} = (0.46, 0.32, 0.021, 0.06, 0.04, 0.025) \\ \rho_9 = 0.01, \rho_{19} = 0.02, \rho_{20} = 0.05 \end{cases}$ |

**TABLE I**

The row-degree distributions $\mu(x)$ and $\bar{\mu}(x)$ for LT codes (top), and the column-degree distributions $\rho(x)$ for LDPC codes (bottom).

In Fig. 1, curves represent $K(1+\gamma, n)$’s (red ones) and $P_{ML}^{\text{err}}(1+\gamma, n)$’s of LT and LDPC codes (blue and black ones), where $n=100$ for LT and $n=p200$ for LDPC codes with $0 \leq p \leq 0.5$. At each point of the DEP curves, the value of $P_{ML}^{\text{err}}(1+\gamma, n)$ is estimated by the fraction of the number of rank-deficient cases of $m \times n$ matrices $H$ with $m = (1+\gamma)n$ (or the fraction of decoding failure cases of system (I.1)) based on more than $10^6$ random constructions of $(H, \beta)$ of system (I.1). The the Separated MLDA in [11], [12] was used to check the rank-deficiency.

It can be clearly seen from the figure that, when check matrices of codes are constructed with $\mu(x)$ and $\rho(x)$ that have no dense fractions (i.e. $\mu_{50} = \rho_{100} = 0$), their DEP (black ones) never drop to the error-bounds, $\delta = 10^{-2}$ with LT codes and $\delta = 10^{-3}$ with LDPC codes. These error-flooring phenomena are obviously due to the deficient cases of $H$, i.e., $\eta = \dim \text{Ker}(H) > 0$ that occur sporadically for large $\gamma$. Most of the deficient cases, however, $\eta$ is merely one or two for large $\gamma$ (small $p$ for LDPC codes). This small deficiency can be readily removed by supplementing a fraction of dense rows. To improve their DEP, we altered $\mu(x)$ of the LT code into $\bar{\mu}(x)$ by supplementing the dense fraction $\mu_{50} = 0.17$ (thus $\bar{\mu}_{50} \approx 0.15$), and the check matrix $M$ was redesigned to $\bar{M}$ by supplementing 15 random rows of degree 100 as stated before. Thus, $H$ in system (I.1) by $\bar{\mu}(x)$ and $\bar{M}$ can have enough number of dense rows. By doing so, the altered codes achieved their DEP curves (blue ones) and MSO $\gamma(\delta, n)$ that are close to the lower bounds KFRL and KFRO for $\delta \leq 10^{-4}$, respectively.

It is interesting to note that $K(1+\gamma, n)$ is very close to $2^{-\gamma n}$ for small $\delta$. In this case, $\gamma_K(\delta, n)$ can be understood as the integer $k_\delta$ such that $\log_2(1/\delta) \leq k_\delta \leq 1 + \log_2(1/\delta)$, i.e., $\gamma_K(\delta, n) := \frac{k_\delta}{n}$.

Although we do not present experimental evidences, supplementing about 15 percent of dense
rows to $H$ of system (I.1) does not degrade the computational complexity of solving system (I.1) seriously. For an example, with the LT codes generated by the $\tilde{\mu}(x)$, the number of symbol additions on $\beta$ of system (I.1) to compute the solution of the system under the Separated MLDA is within $1,100$ (that is $11n$). Similarly with the LDPC codes by $\tilde{M}$, the number of symbol addition on $\beta$ is within $1,600$ (that is $8N$).

IV. SUMMARY

We presented that Kolvalenko’s full-rank limit and its overhead are tight lower bounds for decoding error probability and minimum stable overheads, respectively, of LT and LDPC codes. We also provided experimental evidences which show the viability that, when enough number of dense rows are supplemented to check matrices, both LT and LDPC codes may achieve the code performances in decoding error probability and minimum stable overheads that are close to Kolvalenko’s full-rank limit and its overhead, respectively.

REFERENCES

[1] I. N. KOVALENKO, On the Limit Distribution of the Number of Solutions of a Random System of Linear Equations in the Class of Boolean Functions (in Russian), Theory of Probab. Appl., 12:51-61, 1967.
[2] V. F. KOLCHIN, Random Graphs, Cambridge University Press 1999.
[3] C. COOPER, On the rank of random matrices, Random Structures and Algorithms, 1999.
[4] T. RICHARDSON, R. URBANKE, Efficient Encoding of Low-Density Parity-Check Codes, IEEE Trans. Inform. Theory, 47:638-656, 2001.
[5] DAVID BURSHTEIN, GADI MILLER, An Efficient Maximum Likelihood Decoding of LDPC Codes Over the Binary Erasure Channel, IEEE Trans. Inform. Theory, 50:2837-2844, 2004.
[6] M. LUBY, LT Codes, 43$^{rd}$ Annual IEEE Symposium on Foundations of Computer Science, 2002.
[7] A. SHOKROLLAHI, Raptor codes, Digital Fountain, Inc., Tech. Rep. DF2003-06-001, June 2003.
[8] H. XIAO, A. H. BANIHASHemi, Improved Progressive Edge-Growth (PEG) Construction of Irregular LDPC Codes, IEEE Comm. Letters, Vol 8, No. 12, Dec., 2004.
[9] P. OSWALD AND A. SHOKROLLAHI, Capacity Achieving Sequences for the Erasure Channel, IEEE Trans. Inform. Theory, 48:3017-3028, 2002.
[10] A. SHOKROLLAHI, S. Lassen, R. Karp, Systems and Processes for Decoding Chain Reaction Codes Through Inactivation, US Patent 1856263, Feb. 15, 2005.
[11] KI-MOON LEE AND HAYDER RADHA, The Maximum Likelihood Decoding Algorithm of LT codes and Degree Distribution Design with Dense Fractions, Proceedings on ISIT 2007.
[12] KI-MOON LEE, HAYDER RADHA, AND HO-YOUNG CHEONG, LT Codes from an Arranged Encoder Matrix and Degree Distribution Design with Dense Rows, Allerton Conference on Communication, Control and Computing 2007.
Fig. 1. Top figure shows the error-performance of LT codes by $\mu(x)$ (black) and $\bar{\mu}(x)$ (blue) in DEP vs. overhead. Bottom figure shows the error-performance of LDPC codes by $M$ (black) and $\bar{M}$ (blue) in DEP vs. erasure rate, where $p = \frac{1}{1+\gamma}$. 