ON SOME RESULTS FOR MEROMORPHIC UNIVALENT FUNCTIONS HAVING QUASICONFORMAL EXTENSION

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Abstract. We consider the class \( \Sigma(p) \) of univalent meromorphic functions \( f \) on \( D \) having simple pole at \( z = p \in [0, 1) \) with residue 1. Let \( \Sigma_k(p) \) be the class of functions in \( \Sigma(p) \) which have \( k \)-quasiconformal extension to the extended complex plane \( \hat{C} \) where \( 0 \leq k < 1 \). We first give a representation formula for functions in this class and using this formula we derive an asymptotic estimate of the Laurent coefficients for the functions in the class \( \Sigma_k(p) \). Thereafter we give a sufficient condition for functions in \( \Sigma(p) \) to belong in the class \( \Sigma_k(p) \). Finally we obtain a sharp distortion result for functions in \( \Sigma(p) \) and as a consequence, we get a distortion estimate for functions in \( \Sigma_k(p) \).

1. Introduction and Preliminary Results

Let \( \mathbb{C} \) be the complex plane and \( \hat{\mathbb{C}} \) be the extended complex plane \( \mathbb{C} \cup \{\infty\} \). We shall use the following notations throughout the discussion of this article: \( D = \{z \in \mathbb{C} : |z| < 1\} \), \( \partial D = \{z \in \mathbb{C} : |z| = 1\} \), \( \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\} \), \( \mathbb{D}^* = \{z \in \mathbb{C} : |z| > 1\} \), \( \mathbb{D}^* = \{z \in \mathbb{C} : |z| \geq 1\} \).

The univalent analytic mappings defined in \( \mathbb{D} \) having quasiconformal extension to the whole complex plane play a vital role in Teichmüller spaces. There are number of results for such functions obtained by O. Lehto, R. Kühnau and various other mathematicians starting from the work of L. Ahlfors (see [1]) in the year 1960 till date. We refer to the following articles [11]–[16] for various other results on such mappings.

In this paper our main concern is the univalent meromorphic mappings defined in \( \mathbb{D} \) with pole at \( z = p \in [0, 1) \) having quasiconformal extension to the whole complex plane. O. Lehto extensively studied coefficient problems, growth estimate for meromorphic functions with pole at the origin \( (p = 0) \) having quasiconformal extension to the whole plane. We refer to the articles [5] and [6] for further details. In the present article, we mainly consider the class \( \Sigma_k(p) \) of meromorphic univalent functions with pole at \( z = p \in [0, 1) \) having quasiconformal extension to the whole complex plane. This newly defined class of functions has been introduced and studied in a recent article (compare [8]).

Let \( f \) be a function with \( L^1 \)-derivatives (see [3] I §4.3) in the whole complex plane \( \mathbb{C} \) and \( \Omega \) be a Jordan domain in \( \mathbb{C} \) such that \( \Omega \cup \partial \Omega =: \overline{\Omega} \subset \mathbb{C} \) with a rectifiable
boundary curve $\partial \Omega$. We also denote $\partial f := \partial f/\partial \overline{z}$ and $\partial f := \partial f/\partial z$. Now applying ‘Cauchy-Pompeiu’ (see [2, III §7]) formula for such $f$, we get

\[
(1.1) \quad f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \int_{\Omega} \frac{\overline{\partial f}(\zeta)}{\zeta - z} d\zeta d\eta,
\]

where $z \in \Omega$ and $\zeta = \xi + i\eta$.

If $f(z) \to 0$ as $z \to \infty$, then taking $\Omega = \{ \zeta \in \mathbb{C} : |\zeta| < R \}$, for some $R > 0$ and letting $R \to \infty$, the first term of (1.1) vanishes and we get,

\[
(1.2) \quad f(z) = T[\overline{\partial f}](z),
\]

where

\[
T[a](z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{a(\zeta)}{\zeta - z} d\zeta d\eta.
\]

We assume that the function $\omega$ in above expression belongs to the class $C^\infty_{0}$ - the class of infinitely many times differentiable functions with compact support in the complex plane. We then have

\[
\partial T[a](z) = H[a](z),
\]

where $H$ is the Hilbert transformation defined by

\[
H[a](z) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{a(\zeta)}{(\zeta - z)^2} d\zeta d\eta.
\]

Let $\Sigma$ be the class of univalent meromorphic functions $f$ on $\mathbb{D}$ having simple pole at the origin with residue 1. Let each $f \in \Sigma$ has the following expansion

\[
(1.3) \quad f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}.
\]

In this article our main focus will be the class of function which have pole no more at origin but at a nonzero point. We consider the class $\Sigma(p)$ of univalent meromorphic functions $f$ on $\mathbb{D}$ having simple pole at $z = p \in [0, 1)$ with residue 1. Therefore, each $f \in \Sigma(p)$ has the following expansion

\[
(1.4) \quad f(z) = \frac{1}{z - p} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}.
\]

Let $\Sigma_k$ be the class of functions in $\Sigma$ that have $k$-quasiconformal extension ($0 \leq k < 1$) to the whole plane $\mathbb{C}$ and let $\Sigma_k(p)$ be the class of functions in $\Sigma(p)$ that have $k$-quasiconformal extension to the whole plane $\mathbb{C}$. Here, a mapping $f : \mathbb{C} \to \mathbb{C}$ is called $k$-quasiconformal if $f$ is a homeomorphism and has locally $L^2$-derivatives on $\mathbb{C} \setminus \{ f^{-1}(\infty) \}$ (in the sense of distribution) satisfying $|\partial f| \leq k|\overline{\partial f}|$ a.e. Thus the complex dilatation $\mu_f(z)$ of $f$ satisfies $|\mu_f(z)| \leq k$, for $z \in \overline{\mathbb{D}}^*$ and vanishes on $\mathbb{D}$. Let $\Sigma_k^0(p)$ be the class of functions in $\Sigma_k(p)$ such that $b_0 = 0$. Therefore, each $f$ in $\Sigma_k^0(p)$ has the expansion of the following form.
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\[ f(z) = \frac{1}{z-p} + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}. \]

Let \( f \in \Sigma^0_k(p) \) and we make a change of variable \( \psi(z) = f(1/z) \), so that \( \psi \) has the following expansion

\[ \psi(z) = z/(1-pz) + \sum_{n=1}^{\infty} b_n z^{-n}, \quad z \in \mathbb{D}^*. \]

As \( \psi \) is obtained by composing a Möbius transformation with a \( k \)-quasiconformal mapping, therefore it is also a \( k \)-quasiconformal map in \( \hat{\mathbb{C}} \). Hence the complex dilatation of \( \psi \) satisfies \(|\mu_\psi(z)| \leq k\) for \( z \in \hat{\mathbb{D}} \) and \( \mu_\psi(z) \) vanishes outside \( \mathbb{D} \), i.e. \( \mu_\psi(z) \) has bounded support. We see that \(|\mu_f| = |\mu_\psi| = |\mu|\). Now, if \( f \in \Sigma^0_k(p) \), then \( \psi(z) - z/(1-pz) \to 0 \) as \( z \to \infty \). Thus from (1.2), we have

\[ \psi(z) - z/(1-pz) = T[\mathbf{D}(\psi(z) - z/(1-pz))] = T[\mathbf{D}\psi](z). \]

Taking partial derivative of both sides w.r.t. \( z \) and using \( \partial T[\omega] = H[\omega] \), we get

\[ \partial \psi(z) = 1/(1-pz)^2 + H[\mathbf{D}\psi](z). \]

As \( \mathbf{D}\psi = \mu \partial \psi \), the above equation takes the form

\[ \overline{\partial}\psi(z) = \mu/(1-pz)^2 + \mu H[\mathbf{D}\psi](z). \]

Now we wish to solve this equation. We will see that it is solvable in \( L^2 \) but if we assume that \((|\mu|_{\infty}\|H\|_q) < 1\), for \( q \geq 2 \) then it will be solvable in \( L^q \). From ‘Calderon-Zygmund’ inequality (see [3, I p. 26]), we know that \( \|H\omega\|_q \leq A_q\|\omega\|_q \), where \( A_q \) is a constant, i.e. \( \|H\|_q \) is bounded in \( L^q \). In particular we have \( \|H\|_2 = 1 \).

We first define inductively

\[ \phi_1 = \mu/(1-pz)^2, \quad \text{and} \quad \phi_n = \mu H[\phi_{n-1}], \quad n = 2, 3, \ldots. \]

Now, since \( \mu(z) = 0 \) for \( z \in \mathbb{D}^* \), then for \( i = 2, 3, \ldots \), we have

\[ \|\phi_i\|_q = \|\mu H[\phi_{i-1}]\|_q \]
\[ = |\mu|\|H[\phi_{i-1}]\|_q \]
\[ \leq |\mu|_{\infty}\|H\|_q\|\phi_{i-1}\|_q \]
\[ \leq \cdots \]
\[ \leq (|\mu|_{\infty}\|H\|_q)^{i-1}\|\phi_1\|_q. \]

(1.10)
Next we estimate
\[
\|\phi_i\|_q = \left( \iint_{|z|\leq 1} \frac{|\mu(z)|^q}{|1-pz|^{2q}} \, dxdy \right)^{1/q}
\leq \|\mu\|_\infty \left( \iint_{0}^{1} \frac{r}{(1-pr)^{2q}} \, dr \, d\theta \right)^{1/q}
\]
(1.11)
where \(C(p, q)\) is a constant depending on \(p\) and \(q\), and after a little calculation we find it as
\[
C(p, q) = \left( \frac{2\pi}{p^2} \left[ \frac{(1-p)^{(2-2q)}}{2-2q} - \frac{(1-p)^{(1-2q)}}{1-2q} + \frac{1}{(1-2q)(2-2q)} \right] \right)^{1/q}.
\]
(1.12)
Thus from (1.10), we get
\[
\|\phi_i\|_q \leq C(p, q) \|H\|^{i-1}_q \|\mu\|_\infty^i, \quad i = 2, 3, \ldots
\]
(1.13)
Now we define \(\omega_n = \sum_{i=1}^{n} \phi_i\) and wish to show that \(\{\omega_n\}_{n \geq 1}\) is a Cauchy sequence in \(L^q\). Since \(L^q\) is complete, \(\{\omega_n\}\) will converge in \(L^q\). For \(n > m\),
\[
\|\omega_n - \omega_m\|_q \leq \sum_{i=m+1}^{n} \|\phi_i\|_q
\leq C(p, q) \sum_{i=m+1}^{n} \|H\|^{i-1}_q \|\mu\|_\infty^i
\]
\[
= \frac{C(p, q)}{\|H\|_q} \sum_{i=m+1}^{n} (\|H\|_q \|\mu\|_\infty)^i
\]
\[
\leq \frac{C(p, q)}{\|H\|_q} \sum_{i=m+1}^{\infty} (\|H\|_q \|\mu\|_\infty)^i
= \frac{C(p, q)}{\|H\|_q} \left( \frac{M^{m+1}}{1-M} \right) \to 0 \quad \text{as} \quad n > m \to \infty,
\]
where \(M = \|H\|_q \|\mu\|_\infty < 1\). Hence the sequence \(\{\omega_n\}\) is convergent so that \(\lim_{n \to \infty} \omega_n = \sum_{i=1}^{\infty} \phi_i =: \omega \in L^q\). We now have \(H[\omega_n] = \sum_{i=1}^{n} H[\phi_i]\), as \(H\) is a linear operator. Hence,
\[
\mu H[\omega_n] = \sum_{i=1}^{n} \mu H[\phi_i] = \sum_{i=1}^{n} \phi_{i+1} = \sum_{i=2}^{n+1} \phi_i,
\]
which imply
\[
\frac{\mu}{(1 - pz)^2} + \mu H[\omega_n] = \sum_{i=1}^{n+1} \phi_i = \omega_{n+1}.
\]
Taking limit both sides of above equation as \(n \to \infty\), we have
\[
\frac{\mu}{(1 - pz)^2} + \mu H[\omega] = \omega.
\]
So, from above equation it follows that \(\omega = \overline{\partial \psi}\) satisfies equation (1.8). Using this result we provide a representation theorem for functions in \(f \in \Sigma_{k}(p)\). We follow the idea due to Lehto [3, I §4.3]. This is one of the main contents in the next section.

In 1976, J. G. Krzyż [7], gave a sufficient condition for functions to belong in the class \(\Sigma_{k}(p)\).

**Theorem A.** Let \(f \in \Sigma\) have the expansion of the form (1.3) in \(D\). If there exists \(k\), \(0 \leq k < 1\), such that
\[
|z^2 f'(z) + 1| \leq k |z|^2, \quad \text{for all } z \in D.
\]
Then \(f \in \Sigma_{k}\).

We also provide a sufficient condition for functions to belong in the class \(\Sigma_{k}(p)\) in the following section. Next, we state a theorem proved by K. Löwner [9] for the class \(\Sigma\).

**Theorem B.** Let \(f \in \Sigma\) have the expansion of the form (1.3). Then
\[
|z^2 f'(z)| \leq \frac{1}{1 - |z|^2}, \quad z \in D,
\]
where equality holds at a point \(z = z_0 \in D\) if and only if
\[
f(z) = \frac{1}{z} + b_0 - \frac{(z_0^{-1} - \overline{z}_0)z_0 z}{1 - \overline{z}_0 z}, \quad z \in D,
\]
where \(b_0\) is a constant.

The above theorem was slightly improved by T. Sugawa [10, Theorem 1] as follows:

**Theorem C.** For \(f \in \Sigma\) with the expansion of the form (1.3), the inequality
\[
|z^2 f'(z) + 1| \leq \frac{|z|^2}{1 - |z|^2}
\]
holds for each \(z \in D\). Moreover, equality holds at a point \(z = z_0 \in D\) if and only if
\[
f(z) = \frac{1}{z} + b_0 - \frac{(z_0^{-1} - \overline{z}_0)z_0 z}{1 - \overline{z}_0 z}, \quad z \in D,
\]
for a constant \(b_0\).
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Using the Area theorem for the class $\Sigma_k$ (see [5, §3]) and the above theorem we get that, if $f \in \Sigma_k$ with the expansion of the form (1.3) in $D$, then

$$|z^2 f'(z) + 1| \leq \frac{k|z|^2}{1 - |z|^2}, \quad z \in D.$$ 

In the next section we generalize Theorem C for functions in the class $\Sigma(p)$ and as a consequence we obtain a distortion result for functions in $\Sigma_k(p)$.

2. Main Results

We start the Section with the following result which we will use to find a representation formula for functions in $\Sigma_k^0(p)$.

**Theorem 1.** If $f \in \Sigma_k^0(p)$, then $f(z) = 1/(z - p) + \sum_{i=1}^{\infty} T[\phi_i](1/z)$ for $z \in \mathbb{C}$, where $\phi_i$’s are defined in (1.9).

**Proof.** Let $\psi(z) = f(1/z)$, $z \in \mathbb{D}^*$. Therefore from (1.6) we have, $\psi(z) - z/(1 - pz) \to 0$ as $z \to \infty$, so by (1.7) we get

$$
\psi(z) = \frac{z}{1 - pz} + T[\overline{\partial} \psi](z) = \frac{z}{1 - pz} + T \left[ \sum_{i=1}^{\infty} \phi_i \right](z).
$$

We have to show that $T \left[ \sum_{i=1}^{\infty} \phi_i \right](z) = \sum_{i=1}^{\infty} T[\phi_i](z)$. As $T$ is linear it seems to be obvious but we have to only show that the last series in the above expression is convergent. Using Hölder’s inequality we now have, $(\zeta = \xi + i\eta)$

$$
|T[\phi_i](z)| = \frac{1}{\pi} \left| \int_{|\zeta| \leq 1} \frac{\phi_i(\zeta)}{\zeta - z} \, d\xi \, d\eta \right| 
\leq \frac{1}{\pi} \int_{|\zeta| \leq 1} \left| \frac{\phi_i(\zeta)}{\zeta - z} \right| \, d\xi \, d\eta 
\leq \frac{1}{\pi} \left( \int_{|\zeta| \leq 1} |\phi_i(\zeta)|^q \, d\xi \, d\eta \right)^{1/q} \left( \int_{|\zeta| \leq 1} \frac{1}{|\zeta - z|^s} \, d\xi \, d\eta \right)^{1/s} (1/q + 1/s = 1) 
= \frac{1}{\pi} \|\phi_i\|_q \left( \int_{|\zeta| \leq 1} \frac{1}{|\zeta - z|^s} \, d\xi \, d\eta \right)^{1/s}.
$$

We note that $z$ lies within the whole plane but the integral in the last equation has to be understood by the Cauchy-Principal value. Now using the estimate derived in (1.13), we have

$$
(2.1) \quad |T[\phi_i](z)| \leq C'(p, q)(\|H\|_q \|\mu\|_\infty)^t,
$$

where $C'(p, q)$ is a constant depending on $p$, $q$, and $t$.
where \( \|H\|_q \) and \( C'(p,q) \) are constants. So applying the Weierstrass-M test, we conclude that the series \( \sum_{i=1}^{\infty} T[\phi_i](z) \) is absolutely and uniformly convergent in \( \mathbb{C} \), and hence we can write

\[
\psi(z) = \frac{z}{1-pz} + \sum_{i=1}^{\infty} T[\phi_i](z), \quad z \in \mathbb{C}.
\]

Thus we have the following desired representation formula:

\[
f(z) = \frac{1}{z-p} + \sum_{i=1}^{\infty} T[\phi_i](1/z), \quad z \in \mathbb{C}.
\]

This ends the proof of the Theorem. \( \square \)

Next, in order to establish asymptotic estimates for \(|f(z) - 1/(z - p)|\) and \(|b_n|\) for functions in the class \( \Sigma^0_k(p) \) having expansion (1.5), we first establish another representation formula using previous theorem for functions in the class \( \Sigma^0_k(p) \).

**Theorem 2.** Let \( f \in \Sigma^0_k(p) \) and \( k < k_0 < 1 \). As \( k \to 0 \), we have

\[
f(z) = \frac{1}{z-p} - \frac{1}{\pi} \int_D \frac{z \mu(\zeta) \, d\xi \, d\eta}{(1-p\zeta)^2(z \zeta - 1)} + O(k^2), \quad (\zeta = \xi + i\eta)
\]

in the whole plane \( \mathbb{C} \), where \(|O(k^2)| \leq ck^2 \), and the constant \( c \) depends only on \( k_0 \).

**Proof.** Since \( f \in \Sigma^0_k(p) \) then from (2.2), we have

\[
\psi(z) = z/(1-pz) + T[\phi_1](z) + \sum_{i=2}^{\infty} T[\phi_i](z).
\]

If \( q \geq 2 \) and \( k_0 \|H\|_q < 1 \), then \( \|\mu\|_\infty \|H\|_q \leq k\|H\|_q < k_0\|H\|_q < 1 \). Using (2.1) we have

\[
\sum_{i=2}^{\infty} |T[\phi_i](z)| \leq C'(p,q) \sum_{i=2}^{\infty} (k\|H\|_q)^i \leq C'(p,q) k^2 \|H\|_q^2 \sum_{i=0}^{\infty} (k_0\|H\|_q)^i
\]

\[
= k^2 \left( \frac{C'(p,q)\|H\|_q^2}{1-k_0\|H\|_q} \right) = ck^2, \quad \text{where } c \text{ depends only on } k_0.
\]

Hence by the definition of \( T \) and \( \phi_1 \) we have

\[
\psi(z) = \frac{z}{1-pz} - \frac{1}{\pi} \int_D \frac{\mu(\zeta) \, d\xi \, d\eta}{(1-p\zeta)^2(\zeta - z)} + O(k^2), \quad z \in \mathbb{C}.
\]
Hence,
\[
f(z) = \frac{1}{z-p} - \frac{1}{\pi} \int_\mathbb{D} \frac{z \mu(\zeta) \, d\xi \, d\eta}{(1-p\zeta)^2(z\zeta-1)} + O(k^2), \quad z \in \mathbb{C},
\]
where \(|O(k^2)| \leq ck^2\).

We are now in a position to present the following asymptotic estimate:

**Corollary 1.** Each \(f \in \Sigma_k^0(p)\) satisfies the following asymptotic bound

\[
|f(z) - \frac{1}{z-p}| \leq \frac{k}{\pi} \int_\mathbb{D} \frac{|z| \, d\xi \, d\eta}{|1-p\zeta|^2|z\zeta-1|} + ck^2, \quad (\zeta = \xi + i\eta), \quad z \in \mathbb{C},
\]

where \(|O(k^2)| \leq ck^2\).

**Proof.** It follows from (2.3) that

\[
\left|\psi(z) - \frac{z}{1-pz}\right| \leq \frac{k}{\pi} \int_\mathbb{D} \frac{d\xi \, d\eta}{|1-p\zeta|^2|\zeta-z|} + |O(k^2)|, \quad z \in \mathbb{C}.
\]

Now the inequality (2.4) follows by applying a change of variable \(f(z) = \psi(1/z)\) in the above inequality. Here we note that in (2.3) if

\[
\mu(\zeta) = ke^{i\theta} \frac{\zeta-z}{|\zeta-z|} \left(\frac{1-p\zeta}{|1-p\zeta|}\right)^2, \quad \text{a.e. in } \mathbb{D},
\]

then equality will hold in (2.5) and consequently in (2.4). We choose here \(\theta \in (0, 2\pi]\) such that the second and the third term of the right hand side of (2.3) have the same argument so that equality holds in (2.5).

**Remark.** We note here that whenever \(p \to 0\) in (2.5), we obtain the estimate proved by O. Lehto for functions in \(\Sigma_k\) (see f.i. [3, Cor. 3.2]).

Using the above representation formula, we have the following asymptotic coefficient estimates for functions in \(\Sigma_k^0(p)\).

**Theorem 3.** Let \(f(z) \in \Sigma_k^0(p), \; (0 < p < 1)\) with the expansion as given in (1.5). Then

\[
|b_n| \leq 2k \sum_{m=0}^{\infty} \frac{p^{2m}}{n+2m+1} + Ck^2, \quad n \geq 1.
\]

Here the constant \(C\) is given by

\[
C = \frac{C(p)}{(n\pi)^{1/2}(1-k)}, \quad \text{where} \quad C(p) = \left[\frac{\pi(3-p)}{3(1-p)^3}\right]^{1/2}.
\]

Now the equality
\[ |b_n| = 2k \sum_{m=0}^{\infty} \frac{\mu^{2m}}{n + 2m + 1}. \]

holds for those functions in \( \Sigma_k^0(p) \) whose complex dilatation is given by
\[ \mu(z) = k \left( \frac{z}{z} \right)^{\frac{n-3}{2}} \left( \frac{z - p}{z - \frac{p}{z}} \right), \quad z \in \mathbb{D}^*. \]

**Proof.** First we note that for \( \zeta \in \mathbb{D} \) and \( z \in \mathbb{D}^* \), we have
\[
\sum_{n=1}^{\infty} \frac{\zeta^{n-1}}{z^n} = \frac{-1}{\zeta - z}.
\]

Therefore, for \( z \in \mathbb{D}^* \), it follows that
\[
T[\phi_i](z) = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{\phi_i(\zeta)}{\zeta - z} \, d\xi \, d\eta
\]
\[= \frac{1}{\pi} \int_{\mathbb{D}} \sum_{n=1}^{\infty} (\phi_i(\zeta) \zeta^{n-1} z^{-n}) \, d\xi \, d\eta
\]
\[= \frac{1}{\pi} \sum_{n=1}^{\infty} \left( \int_{\mathbb{D}} \phi_i(\zeta) \zeta^{n-1} d\xi d\eta \right) z^{-n}.
\]

Now using the representation in (2.2), we get
\[\psi(z) - z/(1 - pz) = \sum_{i=1}^{\infty} T[\phi_i](z)
\]
\[= \sum_{n=1}^{\infty} \left( \frac{1}{\pi} \sum_{i=1}^{\infty} \int_{\mathbb{D}} \phi_i(\zeta) \zeta^{n-1} d\xi d\eta \right) z^{-n}.
\]

Thus comparing the above representation of \( \psi \) with the expansion in (1.6), we have the coefficients of \( \psi \) as
\[b_n = \frac{1}{\pi} \sum_{i=1}^{\infty} \int_{\mathbb{D}} \phi_i(\zeta) \zeta^{n-1} d\xi d\eta.
\]

From the inequality (1.13) one can easily obtain
\[\|\phi_i\|_2 \leq C(p, 2) \|H\|_{i-1} \|\mu\|_i \leq C(p) k^i,
\]
by virtue of the fact that \( \|H\|_2 = 1 \) and \( \|\mu\|_\infty \leq k \) and denoting \( C(p, 2) =: C(p) \).

Here we get the value of \( C(p) \) by putting \( q = 2 \) in (1.12) as
\[C(p) = \left[ \frac{\pi(3 - p)}{3(1 - p)^3} \right]^{1/2}.
\]
Now applying Cauchy-Schwartz inequality in $L^2$, we obtain for each $i \geq 2$,

$$\left| \int_\mathbb{D} \phi_i(\zeta) \zeta^{n-1} d\zeta d\eta \right| \leq \left( \int_\mathbb{D} |\phi_i(\zeta)| |\zeta|^{n-1} d\zeta d\eta \right)^{1/2} \leq \|\phi_i\|_2 \left( \int_\mathbb{D} \zeta^{2(n-1)} d\zeta d\eta \right)^{1/2} = \|\phi_i\|_2 (\pi/n)^{1/2} \leq C(p)k^i(\pi/n)^{1/2}. \quad (2.7)$$

Now we can write

$$b_n = \frac{1}{\pi} \int_\mathbb{D} \frac{\mu(\zeta)\zeta^{n-1}}{(1-p\zeta)^2} d\zeta d\eta + \frac{1}{\pi} \sum_{i=2}^\infty \int_\mathbb{D} \phi_i(\zeta)\zeta^{n-1} d\zeta d\eta.$$

Hence using (2.7) we get,

$$\left| \frac{1}{\pi} \sum_{i=2}^\infty \int_\mathbb{D} \phi_i(\zeta)\zeta^{n-1} d\zeta d\eta \right| \leq \frac{C(p)}{\pi} \frac{(\pi/n)^{1/2}}{\sum_{i=2}^\infty k^i} = \frac{C(p)}{(n\pi)^{1/2}} \left( \frac{k^2}{1-k} \right) = Ck^2, \quad \text{where } C = \frac{C(p)}{(n\pi)^{1/2}(1-k)}. \quad (2.8)$$

Consequently, we have the following asymptotic representation of the coefficients of the functions in the class $\Sigma^0_k(p)$:

$$b_n = \frac{1}{\pi} \int_\mathbb{D} \frac{\mu(\zeta)\zeta^{n-1}}{(1-p\zeta)^2} d\zeta d\eta + O(k^2), \quad n = 1, 2, \cdots$$

where $|O(k^2)| \leq Ck^2$. Hence we have,

$$|b_n| \leq \frac{k}{\pi} \int_\mathbb{D} \frac{|\zeta|^{n-1}}{|1-p\zeta|^2} d\zeta d\eta + Ck^2 = \frac{k}{\pi} \int_0^{2\pi} \int_0^1 \frac{r^n}{1-2pr\cos \theta + p^2r^2} dr d\theta + Ck^2.$$

To find the value of the last integral we use the fact that
\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos \theta + r^2} d\theta = 1, \quad \text{for } r < 1.
\]

Using this formula and noting that \( pr < 1 \), we get from above

\[
|b_n| \leq \frac{k}{\pi} \int_0^1 \frac{r^n}{1 - pr^2} \left( \int_0^{2\pi} \frac{1 - p^2 r^2}{1 - 2pr \cos \theta + p^2 r^2} d\theta \right) dr + Ck^2
\]

\[
= 2k \int_0^1 \frac{r^n}{1 - pr^2} dr + Ck^2
\]

\[
= 2k \int_0^1 r^n (1 + p^2 r^2 + p^4 r^4 + p^6 r^6 + \cdots) dr + Ck^2
\]

\[
= 2k \sum_{m=0}^{\infty} \frac{p^{2m}}{n + 2m + 1} + Ck^2.
\]

Now the following equality

\[
|b_n| = 2k \sum_{m=0}^{\infty} \frac{p^{2m}}{n + 2m + 1}
\]

holds whenever

\[
|b_n| = \frac{k}{\pi} \int_D \int |\zeta|^{n-1} |1 - p\zeta|^2 d\xi d\eta.
\]

Therefore, it is clear that the above equality holds for the functions whose complex dilatation is given by

\[
\mu(\zeta) = k \left( \frac{\zeta}{\zeta} \right)^{\frac{n-1}{2}} \left( \frac{1 - p\zeta}{1 - p\zeta} \right)^2 \quad \text{a.e. in } D,
\]

i.e.,

\[
\mu(z) = k \left( \frac{z}{\bar{z}} \right)^{\frac{n-3}{2}} \left( \frac{z - p}{\bar{z} - p} \right) \quad \text{for } z \in D^*.
\]

Remark. For the case \( p = 0 \) i.e. if \( f \in \Sigma_k \) with the expansion as given by (1.3) in \( D \), the estimate given by (2.6) becomes

\[
|b_n| \leq \frac{2k}{n + 1} + Ck^2.
\]

In this case the constant in (1.11) is given by \( C(q) = \pi^{1/q} \) so that the constant in (2.7) is replaced by \( C(p) = \pi^{1/2} \). Hence the constant in (2.8) finally coming as \( C = \frac{n-1/2}{1-k} \). This result was proved by O. Lehto (see [3, II p.74]).
Next, we prove a sufficient condition for functions to belong in the class $\Sigma_k(p)$.

**Theorem 4.** Let $f \in \Sigma(p)$ has an expansion of the form (1.4) in $\mathbb{D}$. If there exists $k, 0 \leq k < 1$ and $p$, $0 \leq p < 1$, such that

$$|(z - p)^2 f'(z) + 1| \leq \frac{k|z - p|^2}{(1 + p)^2} \text{ for all } z \in \mathbb{D},$$

then $f \in \Sigma_k(p)$.

**Proof.** For $z \in \mathbb{D}$, we have

$$f(z) = \frac{1}{z - p} + \sum_{n=0}^{\infty} b_n z^n.$$

Therefore we can write

$$f(z) = \frac{1}{z - p} + \omega(z),$$

where $\omega(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in $\mathbb{D}$. Hence for $z \in \mathbb{D}$,

$$f'(z) + (z - p)^{-2} = \omega'(z).$$

It follows from the given condition that $|\omega'(z)| \leq \frac{k}{(1 + p)^2}, z \in \mathbb{D}$. Hence by [8, Theorem 2], we conclude that $f \in \Sigma_k(p)$.

Our next result deals with a distortion inequality for functions in the class $\Sigma(p)$.

**Theorem 5.** Each function $f \in \Sigma(p)$ of the form (1.4) satisfies the inequality

$$f'(z) + \frac{1}{(z - p)^2} \leq \frac{1}{(1 - p^2)(1 - |z|^2)}, \text{ for } z \in \mathbb{D}.$$ (2.9)

Moreover, equality holds for a constant $b_0$ at $z = z_0 \in \mathbb{D}$, if and only if

$$f(z) = \frac{1}{z - p} + b_0 - \left( \frac{z_0 - 1 - z_0}{1 - p^2} \right) \left( \frac{z_0 - p}{z_0 - p} \right) \left( \frac{-z_0 z}{1 - z_0 z} \right).$$ (2.10)

**Proof.** Let $f \in \Sigma(p)$ with the expansion as

$$f(z) = \frac{1}{z - p} + \sum_{n=0}^{\infty} b_n z^n, \text{ for } z \in \mathbb{D}.$$

We now give a change of variable $z = 1/\zeta$ and let $f(z) = f(1/\zeta) = \psi(\zeta)$. So $\psi$ is defined in $\mathbb{D}^*$ and has Laurent’s series expansion as

$$\psi(\zeta) = \frac{\zeta}{1 - p \zeta} + \sum_{n=0}^{\infty} b_n \zeta^{-n}, \quad \zeta \in \mathbb{D}^*.$$ (2.11)

Thus by Chichra’s area theorem (see [4]), we have

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq \frac{1}{(1 - p^2)^2}.$$
From equation (2.11), taking derivative of both sides we get,

\[ \psi'(\zeta) - \frac{1}{(1 - p\zeta)^2} = \sum_{n=1}^{\infty} (-nb_n)\zeta^{-n-1}, \quad \zeta \in \mathbb{D}^*. \]

Now applying Cauchy-Schwartz inequality in above we have,

\[ \left| \psi'(\zeta) - \frac{1}{(1 - p\zeta)^2} \right| \leq \sum_{n=1}^{\infty} n|b_n||\zeta|^{-n-1} \]

\[ \leq \sqrt{\sum_{n=1}^{\infty} n|b_n|^2 \sum_{n=1}^{\infty} n|\zeta|^{-2n-2}} \]

\[ = \sqrt{\sum_{n=1}^{\infty} n|b_n|^2 \frac{1}{(|\zeta|^2 - 1)^2}}. \] \hspace{1cm} (2.12)

Now using Chichra’s area theorem in above inequality, we get

\[ \left| \psi'(\zeta) - \frac{1}{(1 - p\zeta)^2} \right| \leq \frac{1}{(1 - p)(|\zeta|^2 - 1)}, \quad \text{for} \quad \zeta \in \mathbb{D}^*. \] \hspace{1cm} (2.13)

Returning back to the original variable \( z = 1/\zeta \) and noting that \(-z^2f'(z) = \psi'(\zeta)\), the above inequality yields

\[ \left| z^2f'(z) + \frac{z^2}{(z - p)^2} \right| \leq \frac{|z|^2}{(1 - p^2)(1 - |z|^2)}, \quad \text{for} \quad z \in \mathbb{D}, \]

which gives us the desired result.

Next we consider the equality case. Now equality holds for \( f(z) \) in (2.9) for some point \( z = z_0 \in \mathbb{D} \) if and only if it does hold for \( \psi(\zeta) \) in (2.13) for the point \( \zeta = \zeta_0 = 1/z_0 \in \mathbb{D}^* \). This implies that equality will occur in Cauchy-Schwartz inequality as well as in the Chichra’s area theorem. For the first case we can say that the two sequences \( \{b_n\} \) and \( \{\zeta_0^{-n-1}\} \) are proportional, which means there exists a complex constant \( a \) such that \( b_n = a\zeta_0^{-n} \) for \( n \geq 1 \). Again, if equality holds in the Chichra’s area theorem, then we have \( |a| = \frac{(R-1/R)}{1-p^2} \), where \( R = |\zeta_0| > 1 \). Hence

\[ \psi(\zeta) = \frac{\zeta}{1 - p\zeta} + b_0 + a \sum_{n=1}^{\infty} (\zeta_0)^{-n-1} \zeta^{-n} \]

\[ = \frac{\zeta}{1 - p\zeta} + b_0 + \frac{a}{\zeta_0 \zeta - 1}, \quad \text{for} \quad \zeta \in \mathbb{D}^*. \] \hspace{1cm} (2.14)

For this function clearly equality holds in (2.13) at the point \( \zeta = \zeta_0 \in \mathbb{D}^* \) and hence it does hold in (2.9) at \( z_0 = 1/\zeta_0 \in \mathbb{D} \). Next, we show that \( \psi \) in (2.14) is univalent in \( \mathbb{D}^* \) for a particular value of \( a \).
We first make a change of variable \( \eta = \zeta - \frac{1}{1 + \frac{p}{\zeta}} \). Then \( \eta \in \mathbb{D}^* \) if and only if \( \zeta \in \mathbb{D}^* \). Using this transformation and letting \( b = \zeta \), we get from (2.14)

\[
(1 - p^2)\psi \left( \frac{\eta + p}{1 + p\eta} \right) = \eta + \frac{a(1 - p^2)}{b(\eta + p)} \eta + K, \quad \text{where } K \text{ is a constant.}
\]

\[
= \eta + \frac{a \left( \frac{1 - p^2}{1 - bp} \right)}{(\frac{b - p}{1 - bp})\eta - 1} + \frac{ap(1 - p^2)}{b - p}
+ K_0, \quad K_0 \text{ is another constant.}
\]

\[
= \eta + \frac{a \left( \frac{1 - p^2}{1 - bp} \right)}{(\frac{b - p}{1 - bp})\eta - 1} + \frac{ap(1 - p^2)}{(b - p) \left( \frac{b - p}{1 - bp} \right) \eta - 1} + K_0
\]

\[
= \eta + \frac{ab(1 - p^2)^2}{(b - p)(b - p)} \left( \frac{1 - p^2}{1 - bp} \right) \eta - 1 + K_0
\]

\[
= \eta + \frac{A}{B\eta - 1} + K_0,
\]

where

\[
A = \frac{ab(1 - p^2)^2}{(1 - bp)(b - p)} \quad \text{and} \quad B = \frac{b - p}{1 - bp}.
\]

Let us denote

\[
\phi(\eta) = (1 - p^2)\psi \left( \frac{\eta + p}{1 + p\eta} \right), \quad \eta \in \mathbb{D}^*.
\]

then from the above calculation we see that \( \phi \) takes the form

\[
(2.15) \quad \phi(\eta) = \eta + \frac{A}{B\eta - 1} + K_0, \quad \eta \in \mathbb{D}^*.
\]

Hence it is clear that \( \phi \) is univalent in \( \mathbb{D}^* \) if and only if \( \psi \) is univalent in \( \mathbb{D}^* \). Now we apply [10, Lemma 2] in (2.15) to obtain that \( \phi \) is univalent in \( \mathbb{D}^* \) precisely when \( |A + B - B^{-1}| + |AB| \leq |B|^2 - 1 \). Using the relation \( |a| = |b| - |b|^{-1}/(1 - p^2) \) and performing a simple calculation, the above inequality gives

\[
a = - \left( \frac{\zeta - \overline{\zeta}}{1 - p^2} \right) \left( \frac{1 - p\overline{\zeta}}{1 - p\zeta} \right).
\]

Putting this value of \( a \) in (2.11), we get the univalent extremal function for the inequality (2.13) at the point \( \zeta = \zeta_0 \in \mathbb{D}^* \) as

\[
\psi(\zeta) = \frac{\zeta}{1 - p\zeta} + b_0 - \left( \frac{\zeta_0 - \overline{\zeta}}{1 - p^2} \right) \left( \frac{1 - p\overline{\zeta_0}}{1 - p\zeta_0} \right) \left( \frac{1}{1 - \zeta_0 \overline{\zeta}} \right), \quad \zeta \in \mathbb{D}^*.
\]

Consequently, a change of variable \( f(z) = \psi(1/\zeta) \) will yield the required extremal function (2.10) for the inequality (2.9). \( \square \)
Using the area theorem for $\Sigma_k(p)$ (see [8, Theorem 1]) in (2.12) and the above theorem, we now have the following distortion result. However, sharpness of this bound is not being established.

**Corollary 2.** Let $f \in \Sigma_k(p)$ and have the expansion (1.4) in $D$. Then

$$\left| f'(z) + \frac{1}{(z-p)^2} \right| \leq \frac{k}{(1-p^2)(1-|z|^2)}, \quad \text{for} \quad z \in D.$$ 

**Acknowledgement:** The authors thank Toshiyuki Sugawa for his suggestions and careful reading of the manuscript.

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