A $q, r$-analogue of poly-Stirling numbers of second kind with combinatorial applications

Takao Komatsu  
Department of Mathematical Sciences, School of Science  
Zhejiang Sci-Tech University  
Hangzhou 310018 China  
kamatsu@zstu.edu.cn

Eli Bagno  
Jerusalem College of Technology  
21 HaVaad HaLeumi St.  
Jerusalem, Israel  
bagnoe@g.jct.ac.il

David Garber  
Holon Institute of Technology  
52 Golomb St., P.O.Box 305  
5810201 Holon, Israel  
garber@hit.ac.il

Abstract

This paper deals with several generalizations of Stirling number of the second kind, in both analytical and combinatorial directions. Moreover, we present some analytical results regarding generalizations of the Stirling number of the first kind as well.

In the analytical part, we generalize the Comtet and Lancaster theorems, which present conditions that are equivalent to the definition of ordinary Stirling numbers of both kinds, to the case of the $q, r$-poly Stirling numbers (which are $q$-analogues of the restricted Stirling numbers defined by Broder and having a polynomial values appearing in their defining recursion).

In the combinatorial part, we generalize the approach of Cai-Readdy using restricted growth words in order to represent Stirling numbers of the second kind of Coxeter type $B$, and define a new parameter on restricted growth words of type $B$ that enables us to combinatorially realize some of the identities proven in the analytical part.
Keywords: \( q,r\)-poly-Stirling numbers, \( q\)-calculus, \( r\)-Stirling numbers, restricted growth words, Comtet Theorem, Lancaster Theorem, Coxeter groups of type \( B \), set partitions of type \( B \)

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1 Introduction

The Stirling number of the second kind, denoted \( S_{n,k} \), counts the number of partitions of the set \([n] := \{1, \ldots, n\}\) into \( k \) non-empty subsets (see [26, page 81]). Stirling numbers of the second kind arise in a variety of problems in enumerative combinatorics; they have many combinatorial interpretations, and have been generalized in various contexts and in different ways.

Stirling numbers also have a vast algebraic background and the connection between the algebraic and the combinatorial aspects is via the well-known recursion of these numbers:

\[
S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}.
\]

1.1 The Comtet and Lancaster approaches to Stirling numbers

Comtet [7] showed that the Stirling numbers of the second kind can be presented in several equivalent algebraic ways. We provide here the content of Comtet's theorem, following the presentation of Wagner [29, Section 7.2, Theorem 7.2.1]:

Theorem 1.1. Let \((b_n)_{n \geq 0}\) be a sequence of complex numbers. The following are equivalent characterizations for the array of numbers \((A_{n,k})_{n,k \geq 0}\):

1. Defining equation: For each \( n \geq 0 \):
   \[
x^n = \sum_{k \geq 0} A_{n,k} \cdot (x - b_0)(x - b_1) \cdots (x - b_{k-1}).
   \]

2. Recursion: For each \( n \geq k > 0 \):
   \[
   A_{n,k} = A_{n-1,k-1} + b_k A_{n-1,k}
   \]
   with the boundary conditions: \( A_{n,0} = b_0^n \) and \( A_{0,k} = \delta_{0k} \) for each \( n \geq k \geq 0 \).

3. Complete recursion: For \( n \geq k > 0 \):
   \[
   A_{n,k} = \sum_{j=k}^{n} A_{j-1,k-1} b_k^{n-j},
   \]
   subject to the same boundary conditions as in Condition (2).
(4) **Ordinary generating function:** For each $k \geq 0$:

$$
\sum_{n \geq 0} A_{n,k} x^n = \frac{x^k}{(1 - b_0 x) \cdots (1 - b_k x)}.
$$

(5) **Explicit formula:** For each $n \geq k \geq 0$:

$$
A_{n,k} = \sum_{d_0 + d_1 + \cdots + d_k = n - k} b_0^{d_0} \cdots b_k^{d_k}.
$$

Note that if $b_k = 1$ for all $k$, the numbers $A_{n,k}$ are the binomial coefficients $\binom{n}{k}$, and if $b_k = k$ for all $k$, then the numbers $A_{n,k}$ are the Stirling numbers of the second kind.

The *(unsigned) Stirling number of the first kind*, denoted $c_{n,k}$, counts the number of permutations of the set $[n]$ having $k$ cycles (see [26], page 32).

The recursion satisfied by these numbers is:

$$
c_{n,k} = c_{n-1,k-1} + (n-1)c_{n-1,k}.
$$

(2)

The *(signed) Stirling number of the first kind*, denoted $s_{n,k}$, is defined by the recursion:

$$
s_{n,k} = s_{n-1,k-1} - (n-1)s_{n-1,k}.
$$

The signed Stirling numbers of the first kind satisfy some orthogonality relations with the Stirling numbers of the second kind (see e.g. [12], p. 264).

Back to the unsigned Stirling numbers of the first kind, an analogue of Comtet’s theorem for these numbers was given by Lancaster [15], see also [16] and Wagner’s book [29, Section 7.2] (we fix $b_1 = 0$ in the original formulation of Lancaster):

**Theorem 1.2** (Lancaster). Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers. The following are equivalent characterizations for $(c_{n,k})_{n,k \geq 0}$:

(1) **Defining equation/generating function:**

$$(x + a_0)(x + a_1) \cdots (x + a_{n-1}) = \sum_{k=0}^{n} c_{n,k} \cdot x^k.$$

(2) **Recursion:** For each $n \geq k \geq 0$:

$$c_{n,k} = c_{n-1,k-1} + a_{n-1}c_{n-1,k}$$

with the boundary conditions: $c_{n,0} = a_0 a_1 \cdots a_{n-1}$ and $c_{0,k} = \delta_{0k}$. 

3
(3) Complete recursion: For \( n \geq k \geq 0 \):

\[
c_{n,k} = \sum_{j=k}^{n} c_{j-1,k-1} \prod_{i=j}^{n-1} a_i,
\]

subject to the same boundary conditions as in Condition (2).

1.2 Broder’s restricted Stirling numbers

Broder [4] defined an \( r \)-version to both kinds of Stirling numbers, which counts set partitions such that the first \( r \) elements are placed in \( r \) distinguished parts of the partition in the case of the second kind, and the permutations of \([n]\) which are decomposed in \( k \) cycles such that the elements \( 1, \ldots, r \) are in distinguished cycles in the case of the first kind.

The ordinary Stirling numbers can also be interpreted as the number of elements of a constant rank in the intersection lattice of hyperplane arrangements of Coxeter type \( A \). Dolgachev-Lunts [8, p. 755] and Reiner [23, Section 2] generalized this idea to hyperplane arrangements of Coxeter type \( B \) (in [8], the partitions of type \( B \) are counted by \( \tilde{S}(n,k)_0 \) in their notation). However, the concept of set partitions of type \( B \) has already appeared implicitly in Dowling [9] and also in Zaslavsky [32] in the form of signed graphs. The Stirling numbers of the second kind of type \( B \) enumerate the set partitions of type \( B \); the exact definitions will be recalled in Section 3.

The Stirling numbers were further generalized in various ways, see for example [1, 3, 6, 11, 17, 18, 20] and many more.

1.3 Poly-Stirling numbers

One type of a generalization of the Stirling numbers is obtained by exchanging \( k \) in Equation (1) or \( n-1 \) in Equation (2) for a value \( p(k) \) or \( p(n-1) \) (respectively) of a given polynomial \( p(x) \in \mathbb{Z}[x] \). In this way, we get the poly-Stirling numbers (actually one might define the polynomial \( p(x) \) also over \( \mathbb{C} \) at the expense of losing the nice combinatorial meaning).

We cite here the definitions of poly-Stirling numbers, as they appear in Miceli [20]:

Definition 1.3. Given any nonzero polynomial \( p(x) \in \mathbb{Z}[x] \), define the (unsigned) poly-Stirling numbers with respect to \( p(x) \) of the first kind by the recursion

\[
c_{n,k}^{p(x)} = c_{n-1,k-1}^{p(x)} + p(n)c_{n-1,k}^{p(x)},
\]
where \( c_{0,0}^{p(x)} = 1 \) and \( c_{n,k}^{p(x)} = 0 \) if either \( k > n \) or \( k < 0 \).

In a similar manner, the poly-Stirling numbers with respect to \( p(x) \) of the second kind are defined by the recursion

\[
S_{n,k}^{p(x)} = S_{n-1,k-1}^{p(x)} + p(k)S_{n-1,k-1}^{p(x)},
\]

where \( S_{0,0}^{p(x)} = 1 \) and \( S_{n,k}^{p(x)} = 0 \) if either \( k > n \) or \( k < 0 \).

In the current paper, we follow the work of Miceli [20], who defined two natural types of \( q \)-analogues of poly-Stirling numbers, namely taking the \( q \)-analogue of \( p(x) \) to be either \( [p(x)]_q \) (type II \( q \)-poly-Stirling) or \( p([x]_q) \) (type I \( q \)-poly-Stirling), where

\[
[n]_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}
\]

is the \( q \)-analogue of the number \( n \).

We further generalize both Miceli’s types \( q \)-poly-Stirling numbers to \( q, r \)-poly Stirling numbers, where the additional \( r \) stands for the Broder generalization [4] mentioned above. Actually, we provide a comprehensive analysis of these numbers by generalizing the theorems of Comtet and Lancaster to the \( q, r \)-poly-Stirling numbers of both kinds (see Theorem 2.4 and Theorem 6.3, respectively).

Furthermore, we present orthogonality relations between the first and the second kinds \( q, r \)-poly-Stirling numbers of both types I and II.

We also provide some identities pertaining to sum of powers:

\[
\sum_{j=r}^{n} \left( [p(j)]_q \right)^k \quad \text{and} \quad \sum_{j=r}^{n} \left( p([j]_q) \right)^k
\]

(see Theorem 7.3 and Theorem 8.4 below), which generalize several results regarding the ordinary sum of powers \( \sum_{j=1}^{n} j^k \), see \([13, 18, 25, 31]\).

### 1.4 A combinatorial realization

The second part of this paper (Sections 3-5) provides a combinatorial view for some of the expressions dealt with in the analytical part involving the \( q, r \)-poly-Stirling numbers of the second kind. A parallel work on the combinatorics of the first kind counterpart is in progress.

The Stirling numbers of the second kind are known to be in bijection with restricted growth words, see \([14, 21]\). We define a Coxeter type-\( B \) version of these words for the set partitions of type \( B \), and use them to provide a \( q \)-version of the \( r \)-variant Stirling numbers of type \( B \), \( S_{n,k,r}^{B}(q) \). Our work is motivated by the results of Cai and Readdy [5], who introduced the \( q \)-analogue using the restricted growth words for Coxeter type-\( A \).
Explicitly, we prove the following recursion for our new $q$-analogue in its $r$-variant, using restricted growth words of Coxeter type-B (see Theorem 3.11):

\[ S_{n,k,r}^{B}(q) = S_{n-1,k-1,r}^{B}(q) + [2k + 1]q \cdot S_{n-1,k,r}^{B}(q), \]

for each $1 \leq k < n$, with some boundary conditions (see Proposition 3.13 for more details). The $q$-analogue for the case $r = 0$ was also defined independently by Sagan and Swanson [24]; see also Swanson and Wallach [28, Sect. 1.10, p. 10]. Moreover, we provide a combinatorial proof an ordinary generating function, for $S_{n,k,r}^{B}(q)$. We also calculate an exponential generating function for $S_{n,k,r}^{B}(q)$, based on a combinatorial identity and we use it to present these numbers as connection constants between two bases of $\mathbb{R}[x]$.

1.5 Organization of the paper

The paper is organized as follows. Section 2 deals with the generalization of Comtet’s theorem. In Section 3 we introduce restricted growth words for Coxeter type-B and use them to provide a combinatorial $q, r$-Stirling numbers of the second kind of type $B$. In Section 4 we provide an ordinary generating function for $S_{n,k,r}^{B}(q)$ with a combinatorial proof. Section 5 deals with an exponential generating function for $S_{n,k,r}^{B}$ and we use it to present these numbers as connection constants between two bases of $\mathbb{R}[x]$.

Section 6 deals with the generalization of Lancaster’s theorem for the unsigned $q, r$-poly-Stirling numbers of the first kind. In Section 7 we present without proof the generalization of Lancaster’s theorem for the signed $q, r$-poly-Stirling numbers of the first kind, and we prove two applications of this generalization: sum of powers and orthogonality relations. Section 8 deals with some mixed relations between the two types of $q, r$-poly-Stirling numbers.

2 Type II $q, r$-poly-Stirling numbers of the second kind

In this section, we present a generalization of Comtet’s theorem (Theorem 1.1 above) to the case of type II $q, r$-poly-Stirling numbers of the second kind, mentioned in the introduction (based on Miceli’s type II $q$-poly-Stirling number of the second kind [20]). We start with its explicit definition, continue with some lemmata regarding these numbers, and finally we prove the generalization of Comtet’s theorem.

**Definition 2.1.** Let $p(x) \in \mathbb{Z}[x]$. The type II $q, r$-poly-Stirling numbers of the second kind are defined by the recurrence relation:

\[ S_{n,k,r}^{p(x)}(q) = S_{n-1,k-1,r}^{p(x)}(q) + [p(k)]q S_{n-1,k,r}^{p(x)}(q) \quad (r \leq k \leq n, \ n \geq 1) \]
with \( S_{n,r}^{(s)}(q) = 1 \) and \( S_{n,k,r}^{(s)}(q) = 0 \) for \( k < r, k > n \) or \( n < r \).

**Lemma 2.2.**

1. \( S_{n,n,r}^{(s)}(q) = 1 \)

2. \( S_{n,n-1,r}^{(s)}(q) = \sum_{i=r}^{n-1} \lfloor p(i) \rfloor_q \)

3. \( S_{n,r,r}^{(s)}(q) = (\lfloor p(r) \rfloor_q)^n \)

Hence, in general, if \( p(r) \neq 0 \), then \( S_{n,r,r}^{(s)}(q) \neq 0 \) even if \( r = 0 \).

**Proof.** By the recurrence relation (5), we get:

1. \( S_{n,n,r}^{(s)}(q) = S_{n-1,n-1,r}^{(s)}(q) + \lfloor p(n) \rfloor_q S_{n-1,n,n}^{(s)}(q) = S_{n-1,n-1,r}^{(s)}(q) = \cdots = S_{r,r,r}^{(s)}(q) = 1 \)

2. \( S_{n,n-1,r}^{(s)}(q) = S_{n-1,n-2,r}^{(s)}(q) + \lfloor p(n-1) \rfloor_q \cdot S_{n-1,n-1,r}^{(s)}(q) = \cdots = S_{r,r-1,r}^{(s)}(q) + \lfloor p(r) \rfloor_q + \cdots + \lfloor p(n-2) \rfloor_q + \lfloor p(n-1) \rfloor_q = \sum_{i=r}^{n-1} \lfloor p(i) \rfloor_q \)

3. \( S_{n,r,r}^{(s)}(q) = S_{n-1,r-1,r}^{(s)}(q) + \lfloor p(r) \rfloor_q S_{n-1,r,r}^{(s)}(q) = \cdots = (\lfloor p(r) \rfloor_q)^{n-r} S_{r,r,r}^{(s)}(q) = (\lfloor p(r) \rfloor_q)^{n-r} \)

\[ \square \]

**Lemma 2.3.** For \( n \geq r + 2 \):

\[
S_{n,r+1,r}^{(s)}(q) = \sum_{\substack{i_r + i_{r+1} + \cdots = n-r \\ i_r + i_{r+1} \geq 0}} (\lfloor p(r) \rfloor_q)^{i_r} (\lfloor p(r+1) \rfloor_q)^{i_{r+1}}
\]
Proof. We prove this lemma by induction on $n$. For the base case, substitute $n = r + 2$ in Lemma 2.2: \[ S^{p(x)}_{n,r+2,1,r}(q) = [p(r)]_q + [p(r+1)]_q. \]

Assume correctness for $n - 1$ and we prove for $n > r + 2$ using the recurrence relation in Equation (5): \[ S^{p(x)}_{n,r+1,r}(q) = \begin{cases} S^{p(x)}_{n-1,r,r}(q) + [p(r+1)]_q S^{p(x)}_{n-1,r+1,r}(q) = & \text{}\text{Assumption (2,2) + Assumption (3.1)} \\
 & \begin{cases} ([p(r)]_q)^{n-1} + [p(r+1)]_q \sum \limits_{\substack{i_r+i_{r+1}=n-r-2 \\i_r,i_{r+1} \geq 0}} ([p(r)]_q)^i ([p(r+1)]_q)^{i+1} = & \\
= \sum \limits_{\substack{i_r+i_{r+1}=n-r-1 \\i_r,i_{r+1} \geq 0}} ([p(r)]_q)^i ([p(r+1)]_q)^{i+1} & \end{cases} \end{cases} \]

When $r = 0$, the recurrence relation (5) is reduced to the one appearing in Miceli (20) Equation (37). Moreover, the boundary conditions for $r = 0$, namely $S^{p(x)}_{0,0,0}(q) = 1$ and $S^{p(x)}_{n,k,0}(q) = 0$ (for $k < 0$, $k > n$ or $n < 0$), are compatible with those of Miceli.

We are now ready for presenting and proving the generalization of Comtet’s theorem for type II $q$, $r$-poly-Stirling numbers of the second kind:

**Theorem 2.4** (Generalization of Comtet’s theorem for type II). Let $p(n)$ be a polynomial with non-negative integer coefficients and let $\varphi_k(x)$, $k \geq r$, be defined by $\varphi_r(x) = 1$ and $\varphi_k(x) = (x - [p(r)]_q)(x - [p(r+1)]_q) \cdots (x - [p(k-1)]_q)$ for $k > r$. The following are equivalent characterizations for $(S^{p(x)}_{n,k,r}(q))_{n,k,r\geq0}$ (where for all other values of the triple $(r, k, n)$, we assume $S^{p(x)}_{n,k,r}(q) = 0$):

1. **Defining equation/Change of bases:** For each $n \geq r$:
   \[ x^{n-r} = \sum_{k=r}^{n} S^{p(x)}_{n,k,r}(q) \cdot \varphi_k(x). \]

2. **Recursion:** For each $n \geq k > r$:
   \[ S^{p(x)}_{n,k,r}(q) = S^{p(x)}_{n-1,k-1,r}(q) + [p(k)]_q S^{p(x)}_{n-1,k,r}(q) \]
   with the boundary conditions: $S^{p(x)}_{r,r,r}(q) = 1$ and $S^{p(x)}_{n,k,r}(q) = 0$ for $k < r$, $k > n$ or $n < r$. 

\]
(3) **Complete recursion:** For \( n \geq k > r \):

\[
S_{n,k,r}^{(x)}(q) = \sum_{j=k}^{n} S_{j-1,k-1,r}^{(x)}(q) \left( [p(k)]_q \right)^{n-j},
\]

subject to the same boundary conditions as in Condition (2).

(4) **Ordinary generating function:** For each \( k \geq r \):

\[
\sum_{n=k}^{\infty} S_{n,k,r}^{(x)}(q)x^n = \frac{x^k}{(1 - [p(r)]_q x) \cdots (1 - [p(k)]_q x)}.
\]

(5) **Explicit formula:** For \( r \leq k \leq n \):

\[
S_{n,k,r}^{(x)}(q) = \sum_{i_r+i_{r+1}+\ldots+i_{k-r}=n-k \atop i_r \geq 0} ([p(r)]_q)^{i_r} \cdots ([p(k)]_q)^{i_k}.
\]

Before proving the theorem, we need one more lemma:

**Lemma 2.5.** Using the notations of Theorem 2.4 we have for \( r \leq n \):

\[
x \sum_{k=r}^{n-1} S_{n-1,k,r}^{(x)}(q) \cdot \varphi_k(x) = \sum_{k=r}^{n} \left( S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-1,k,r}^{(x)}(q) \right) \varphi_k(x).
\]

**Proof.** It is easy to check that for \( n = r \) both sides are 0. We compute for \( n > r \):

\[
\begin{align*}
x \sum_{k=r}^{n-1} S_{n-1,k,r}^{(x)}(q) \cdot \varphi_k(x) &= \sum_{k=r}^{n-1} \left( x - [p(k)]_q \right) S_{n-1,k,r}^{(x)}(q) \cdot \varphi_k(x) = \\
&= \sum_{k=r}^{n-1} S_{n-1,k,r}^{(x)}(q) \cdot \left( x - [p(k)]_q \right) \varphi_k(x) + \sum_{k=r}^{n-1} [p(k)]_q S_{n-1,k,r}^{(x)}(q) \cdot \varphi_k(x) = \\
&= \sum_{k=r}^{n-1} S_{n-1,k,r}^{(x)}(q) \cdot \varphi_{k+1}(x) + \sum_{k=r}^{n-1} [p(k)]_q S_{n-1,k,r}^{(x)}(q) \cdot \varphi_k(x) = \\
&= \sum_{k=r}^{n} S_{n-1,k-1,r}^{(x)}(q) \cdot \varphi_k(x) + \sum_{k=r}^{n-1} [p(k)]_q S_{n-1,k,r}^{(x)}(q) \cdot \varphi_k(x) = \\
&= \sum_{k=r}^{n} \left( S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-1,k,r}^{(x)}(q) \right) \varphi_k(x) - S_{n-1,r-1,r}^{(x)}(q) \varphi_r(x) - \underbrace{S_{n-1,n,r}^{(x)}(q)[p(n)]_q \varphi_n(x)}_{=0} = 0.
\end{align*}
\]
Lemma 2.5

Comparing the coefficients of the basis elements \( \{ \varphi_k(x) \} \) in the right-hand-side of the equation in the lemma, in order to get:

By Lemma 2.5, we can replace the right-hand-side of the above equation by the left-hand-side of the equation in the lemma, in order to get:

Comparing the coefficients of the basis elements \( \{ \varphi_k(x) \} \) on both sides, we obtain:

[(1) \implies (2)]: We first prove the boundary condition:

\[
1 = x^{n-r} \sum_{k=r}^{n} S^{p(x)}_{n,k,r} (q) \varphi_k(x) = \sum_{k=r}^{n-1} S^{p(x)}_{n-1,k-1,r} (q) + [p(k)]_q S^{p(x)}_{n-1,k,r} (q) \varphi_k(x) = S^{p(x)}_{r,r,r} (q),
\]

so we get: \( S^{p(x)}_{r,r,r} (q) = 1 \) as needed.

We expand the recurrence relation appearing in Condition (2) and use Lemma 2.5 \( n - r \) times. Explicitly:

\[
\sum_{k=r}^{n} S^{p(x)}_{n,k,r} (q) \cdot \varphi_k(x) = \sum_{k=r}^{n-1} \left( S^{p(x)}_{n-1,k-1,r} (q) + [p(k)]_q S^{p(x)}_{n-1,k,r} (q) \right) \cdot \varphi_k(x) = \]

\[
\sum_{k=r}^{n-1} S^{p(x)}_{n-1,k,r} (q) \cdot \varphi_k(x) = x \sum_{k=r}^{n-1} S^{p(x)}_{n-1,k,r} (q) \cdot \varphi_k(x) = \]

\[
\sum_{k=r}^{n-2} S^{p(x)}_{n-2,k,r} (q) \cdot \varphi_k(x) = \cdots = \sum_{k=r}^{2} S^{p(x)}_{2,k,r} (q) \cdot \varphi_k(x) = \sum_{k=r}^{1} S^{p(x)}_{1,k,r} (q) \cdot \varphi_k(x) = \sum_{k=r}^{0} S^{p(x)}_{0,k,r} (q) \cdot \varphi_k(x) = \]

\[
x^{n-r} \sum_{k=r}^{r} S^{p(x)}_{r,k,r} (q) \cdot \varphi_k(x) =
\]
Hence, while the recursion in Condition (2) achieves the same value:

\[ (3) \]

\[
S_{n,k,r}^{(x)}(q) \cdot \varphi_r(x) = x^{n-r}.
\]

\[ (2) \rightarrow (3) \]: Expand the recurrence relation appearing in Condition (2) \( n-k \) times. Explicitly:

\[
S_{n,k,r}^{(x)}(q) = S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-1,k,r}^{(x)}(q) = \\
= S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q (S_{n-2,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-2,k,r}^{(x)}(q)) = \\
= S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-2,k-1,r}^{(x)}(q) + \\
+ ( [p(k)]_q )^2 (S_{n-3,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-3,k,r}^{(x)}(q) ) = \cdots = \\
= S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-2,k-1,r}^{(x)}(q) + ( [p(k)]_q )^2 S_{n-3,k-1,r}^{(x)}(q) + \\
+ \cdots + ( [p(k)]_q )^{n-k} S_{k-1,k-1,r}^{(x)}(q) = \sum_{j=k}^{n} S_{j-1,k-1,r}^{(x)}(q) ([p(k)]_q)^{n-j}.
\]

\[ (3) \rightarrow (2) \]: We check for \( n = r+1 \) separately: in this case, the only possible value of \( k \) is \( r+1 \). By Condition (3), we have:

\[
S_{r+1,r+1,r}^{(x)}(q) = \sum_{j=r+1}^{r+1} S_{j-1,r,r}^{(x)}(q) ([p(k)]_q)^{r+1-j} = S_{r,r,r}^{(x)}(q) = 1,
\]

while the recursion in Condition (2), achieves the same value:

\[
S_{r+1,r+1,r}^{(x)}(q) = \underbrace{S_{r,r,r}^{(x)}(q) + [p(r)]_q S_{r+1,r}^{(x)}(q)}_{=1} = 1.
\]

For \( n > r+1 \), apply Condition (3) for both \( n, n-1 > r \):

\[
S_{n,k,r}^{(x)}(q) = \sum_{j=k}^{n} S_{j-1,k-1,r}^{(x)}(q) ([p(k)]_q)^{n-j},
\]

(6)

\[
S_{n-1,k,r}^{(x)}(q) = \sum_{j=k}^{n-1} S_{j-1,k-1,r}^{(x)}(q) ([p(k)]_q)^{n-j-1}.
\]

(7)

Hence,

\[
S_{n,k,r}^{(x)}(q) = \sum_{j=k}^{n} S_{j-1,k-1,r}^{(x)}(q) ([p(k)]_q)^{n-j} =
\]
\[
\begin{align*}
S_{n-1,k-1,r}(q) + [p(k)]_q \sum_{j=k}^{n-1} S_{j-1,k-1,r}(q)\left([p(k)]_q\right)^{n-j-1} & = \\
\stackrel{(7)}{=} S_{n-1,k-1,r}(q) + [p(k)]_q S_{n-1,k,r}(q),
\end{align*}
\]

which is the recurrence relation in Condition (2).

[(2)⇒(4)]: We assume that the recurrence relation in Condition (2) holds. Denote
\[
f_k := \sum_{n=k}^{\infty} S_{n,k,r}(q)x^n \text{ for each } k \geq r.\]
Then, for \(k > r\), we have:
\[
f_k = \sum_{n=k}^{\infty} S_{n,k,r}(q)x^n = \\
\text{Cond. (2)} \Rightarrow \sum_{n=k}^{\infty} \left( S_{n-1,k-1,r}(q) + [p(k)]_q S_{n-1,k,r}(q) \right)x^n = \\
= \sum_{n=k}^{\infty} S_{n-1,k-1,r}(q)x^n + \sum_{n=k}^{\infty} [p(k)]_q S_{n-1,k,r}(q)x^n = \\
= x \sum_{n=k-1}^{\infty} S_{n,k-1,r}(q)x^n + [p(k)]_q \cdot x \sum_{n=k-1}^{\infty} S_{n,k,r}(q)x^n = \\
\left(S_{k-1,k,r}(q)=0\right) \Rightarrow \quad x \sum_{n=k-1}^{\infty} S_{n,k-1,r}(q)x^n + [p(k)]_q \cdot x \sum_{n=k}^{\infty} S_{n,k,r}(q)x^n = \\
\quad = x f_{k-1} + [p(k)]_q \cdot x f_k.
\]

This implies:
\[
f_k = \frac{x}{1 - [p(k)]_q \cdot x} \cdot f_{k-1}. \tag{8}
\]

For \(k = r\), we have:
\[
f_r = \sum_{n=r}^{\infty} S_{n,r,r}(q)x^n \overset{\text{Lem. 2.7.3}}{=} \sum_{n=r}^{\infty} \left([p(r)]_q\right)^{n-r}x^n = x^r \sum_{n=r}^{\infty} \left([p(r)]_q \cdot x\right)^{n-r} = \frac{x^r}{1 - [p(r)]_q \cdot x}.
\]

In summary, using Equation (8), we get for \(k \geq r\):
\[
\sum_{n=k}^{\infty} S_{n,k,r}(q)x^n = f_k = \frac{x^k}{\left(1 - [p(r)]_q \cdot x\right) \left(1 - [p(r+1)]_q \cdot x\right) \cdots \left(1 - [p(k)]_q \cdot x\right)}.
\]

[(4)⇒(2)]: Denote for \(k \geq r\):
\[
g_k := \frac{x^k}{\left(1 - [p(r)]_q \cdot x\right) \left(1 - [p(r+1)]_q \cdot x\right) \cdots \left(1 - [p(k)]_q \cdot x\right)} = \sum_{n=k}^{\infty} S_{n,k,r}(q)x^n.
\]
As \( g_r = \frac{x^r}{1-[p(r)]_q} \), we get: \( g_k = \frac{x^k}{1-[p(k)]_q} \cdot g_{k-1} \). Thus for \( k \geq r \):

\[
\sum_{n=k}^{\infty} S_{n,k,r}^{(x)}(q)x^n = g_k = xg_{k-1} + [p(k)]_q xg_k = \\
x \sum_{n=k}^{\infty} S_{n,k,r}^{(x)}(q)x^n + [p(k)]_q x \sum_{n=k}^{\infty} S_{n,k,r}^{(x)}(q)x^n = \\
S_{k-1,k,r}^{(x)}(q)=0 \sum_{n=k-1}^{\infty} S_{n,k-1,r}^{(x)}(q)x^{n+1} + [p(k)]_q \sum_{n=k-1}^{\infty} S_{n,k,r}^{(x)}(q)x^{n+1} = \\
n+1-n \sum_{n=k}^{\infty} (S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-1,k,r}^{(x)}(q)) x^n.
\]

Comparing the coefficients on both sides, we obtain:

\[ S_{n,k,r}^{(x)}(q) = S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-1,k,r}^{(x)}(q). \]

\[ (2) \implies (5) \]: Before proving the general case, we have to verify some specific cases. For \( r \leq k = n \), we have by the boundary condition in Condition (2) and Lemma 2.2(1):

\[ S_{n,n,r}^{(x)}(q) = 1 = \sum_{i_r+i_{r+1}+\ldots+i_k=0}^{i_k} ([p(r)]_q)^{i_r} \cdots ([p(k)]_q)^{i_k} \]

as required.

For \( r = k < n \), we have by Lemma 2.2(3):

\[ S_{n,r,r}^{(x)}(q) = ([p(r)]_q)^{n-r} \]

again as required.

Now, we pass to the general case, where \( r < k < n \). We start by induction on \( n \). The base case is \( n = r + 2 \). Since \( r < k < n \), it means that \( k = r + 1 \), and this is the case \( n = r + 2 \) of Lemma 2.3.

Now assume its correctness for \( n - 1 \geq r + 2 \) and we prove it for \( n \). This will be done by a proof for all values of \( k \) in the range \( \{r + 1, \ldots, n - 1\} \). The first case is \( k = r + 1 \), which was proved in Lemma 2.3. For \( k > r + 1 \), we prove it using the recurrence in Condition (2):

\[
S_{n,k,r}^{(x)}(q) \overset{\text{Cond. (2)}}{=} S_{n-1,k-1,r}^{(x)}(q) + [p(k)]_q S_{n-1,k,r}^{(x)}(q) = \\
\overset{\text{Assumption}}{=} \sum_{i_r+i_{r+1}+\ldots+i_k=n-k}^{i_k} ([p(r)]_q)^{i_r} ([p(r+1)]_q)^{i_{r+1}} \cdots ([p(k-1)]_q)^{i_{k-1}} +
\]

13
Then it is immediate that

\[ ([5]) \]

\[
\sum_{i_r + i_{r+1} + \ldots + i_k - 1 = -k}
\sum_{i_r + \ldots + i_k = 0}

\]

Condition (5) of Theorem 2.4 can be written in an equivalent form, as follows:

Assume that Condition (5) is satisfied:

\[ S_{n,k,r}^{(x)}(q) = \sum_{i_r + i_{r+1} + \ldots + i_k = -k}
\sum_{i_r + \ldots + i_k = 0}

\]

Then it is immediate that \( S_{r,r}^{(x)}(q) = 1 \) and \( S_{n,k,r}^{(x)}(q) = 0 \) for \( k < r, k > n \) or \( n < r \), and therefore the boundary conditions are satisfied.

Now we have to show the recurrence relation in Condition (2):

\[ S_{n,k,r}^{(x)}(q) \overset{\text{Cond. (5)}}{=} \sum_{i_r + i_{r+1} + \ldots + i_k = -k}
\sum_{i_r + \ldots + i_k = 0}

\]

Condition (5) of Theorem 2.4 can be written in an equivalent form, as follows:

**Corollary 2.6.** For \( r \leq k \leq n \), we have:

\[ S_{n,k,r}^{(x)}(q) = \sum_{j_{k-1}=0}^{n-k}
\sum_{j_{k-2}=0}^{j_{k-1}}

\]

\[ \sum_{j_{r+2}=0}^{j_{r+1}}
\sum_{j_r=0}^{j_{r+1}}

\]

which is the requested recurrence relation.
Proof. The right hand side is equivalent to the right hand side of the equation in Condition (5) of Theorem 2.4 by the following substitutions: \( j_r = i_r \) and \( j_m = i_r + i_{r+1} + \cdots + i_m \) for \( r < m < k \).

In the next theorem, we present a new way to write \( S_{n,k,r}^{\mu(x)}(q) \) based on its generating function. We start with a lemma:

**Lemma 2.7.** For \( n \geq k \) and \( r \geq 2 \), we have:

\[
\frac{d^n}{dx^n} \left[ \frac{x^k}{(1 - a_1 x) \cdots (1 - a_r x)} \right] = \sum_{j=1}^r \frac{a_j^{n-k-1}}{\prod_{i=1, i \neq j}^r (a_i - a_j)} (1 - a_j x)^{n+1}.
\]

Proof. We prove this identity by induction on \( n \). The base case is \( n = 1 \), in which one has to show that:

\[
\frac{d}{dx} \left[ \frac{x^k}{(1 - a_1 x) \cdots (1 - a_r x)} \right] = \sum_{j=1}^r \frac{a_j^{k-1}}{\prod_{i=1, i \neq j}^r (a_i - a_j)} (1 - a_j x)^2.
\]

This equality can be achieved by performing a second induction on \( r \) and using partial fractions.

The induction step on \( n \) follows by a simple derivation argument.

**Theorem 2.8.** For \( r \leq k \leq n \), we have

\[
S_{n,k,r}^{\mu(x)}(q) = \sum_{j=r}^k \frac{([p(j)]_q)^{n-r}}{\prod_{i=r, i \neq j}^k ([p(i)]_q - [p(j)]_q)}.
\]

Here, the empty product equals 1 as usual.

Proof. The generating function appearing in Condition (4) of Theorem 2.4 is:

\[
\sum_{n=k}^{\infty} S_{n,k,r}^{\mu(x)}(q)x^n = \frac{x^k}{(1 - [p(r)]_q x) \cdots (1 - [p(k)]_q x)}.
\]

Then, we get by Lemma 2.7 for \( r^\prime = k - r + 1 \) and substituting \([p(i)]_q \) for \( a_i \):

\[
S_{n,k,r}^{\mu(x)}(q) = \frac{1}{n!} \cdot \frac{d^n}{dx^n} \left[ \frac{x^k}{\prod_{j=r}^k (1 - [p(j)]_q \cdot x)} \right]_{x=0} = \sum_{j=r}^k \frac{([p(j)]_q)^{n-r}}{\prod_{i=r, i \neq j}^k ([p(i)]_q - [p(j)]_q)}.
\]
3 Restricted growth words for type $B$ and set partitions of type $B$

In this section, we supply a combinatorial realization for a specific case of the analytical results that preceded.

3.1 Restricted growth words for type $A$

We start by recalling the definition of restricted growth words for type $A$ (introduced by Hutchinson [14], and first called as such by Milne [21]; see also [10, Sec. 1.7].

**Definition 3.1.** The word $x_1 \cdots x_n$ over the alphabet $\{1, \ldots, n \}$ is called a restricted growth (RG)-word if $x_1 = 1$ and for each $2 \leq t \leq n$ one has:

$$x_t \leq \max \{x_1, \ldots, x_{t-1}\} + 1.$$

For example, $\omega = 122123$ is an RG-word, but $\omega = 14213$ is not.

There is a well-known bijection (see [21]) between restricted growth words and set partitions of the set $\{1, \ldots, n\}$, which we describe here. Let $P = \{B_1, \ldots, B_k\}$ be a set partition of $\{1, \ldots, n\}$, whose blocks are ordered in such a way that the set of minimum elements of the blocks is increasing.

For such a set partition, we associate an RG-word $x_1 \cdots x_n$ as follows: $x_j$ is the number of the block where $j$ is located. For instance, the RG-word $\omega = 122123$ is matched with the set partition $\{\{1, 4\}, \{2, 3, 5\}, \{6\}\}$ of the set $\{1, \ldots, 6\}$.

3.2 Set partitions for type $B$

We now recall the definition of set partitions for type $B$ (see Dolgachev-Lunts [8, p. 755] and Reiner [23, Section 2]; implicitly in Dowling [9] and Zaslavsky [32] in the form of signed graphs):

**Definition 3.2.** A set partition of $[\pm n] = \{\pm 1, \ldots, \pm n\}$ of type $B$ is a set partition of the set $[\pm n]$ such that the following conditions are satisfied:

- If $B$ appears as a block in the partition, then $-B$ (which is obtained by negating all the elements of $B$) also appears in that partition.
- There exists at most one block satisfying $-B = B$. This block is called the zero block (if it exists, it is a subset of $[\pm n]$ of the form $\{\pm i \mid i \in C\}$ for some $C \subseteq [n]$).
For example, the following is a set partition of $[\pm 6]$ of type $B$:
\[
\{ \{1, -1, 4, -4\}, \{2, 3, -5\}, \{-2, -3, 5\}, \{6\}, \{-6\}\}.
\]

Note that every non-zero block $B$ has a corresponding block $-B$ attached to it. For the sake of convenience, we write for the pair of blocks $B, -B$, only the representative block containing the minimal positive number appearing in $B \cup -B$. For example, the pair of blocks $B = \{-2, -3, 5\}, -B = \{2, 3, -5\}$ will be represented by the single block $\{2, 3, -5\}$.

**Definition 3.3.** Let $S^B_{n,k}$ be the number of set partitions of type $B$ having $k$ representative non-zero blocks. This is known as the Stirling number of the second kind of type $B$ (see sequence A085483 in OEIS [22]).

It is easy to see that $S^B_{n,n} = S^B_{n,0} = 1$ for each $n \geq 0$. The following recursion for $S^B_{n,k}$ is well-known ([9] Theorem 7; see the Erratum], [2] Corllary 3, for $m = 2$, and [30] Equation (1)), for $m = 2, c = 1$:

**Proposition 3.4.** For each $1 \leq k < n$,
\[
S^B_{n,k} = S^B_{n-1,k-1} + (2k + 1)S^B_{n-1,k}.
\]

Note that $S^B_{n,k}$ is a special case of the $q, r$-poly-Stirling number of the second kind, for $q = 1, r = 0$ and $p(x) = 2x + 1$.

Proposition 3.4 is a special case of Proposition 5.11 which is proven in the next subsection.

We present here a new generalization of the Stirling number of the second kind of type $B$, based on the work of Broder [4] for type $A$. Define the $r$-Stirling number of the second kind of type $B$ as follows:

**Definition 3.5.** Let $S^B_{n,k,r}$ be the number of set partitions of type $B$ of the set $[\pm n]$ into $k$ non-zero blocks such that the numbers $1, \ldots, r$ are in distinct non-zero blocks. The numbers $\{1, \ldots, r\}$ will be called special elements.

For example, the following is an element of $S^B_{7,3,2}$:
\[
\{ \{4, -4\}, \{1, 3, -5\}, \{-1, -3, 5\}, \{2, 6\}, \{-2, -6\}, \{7\}, \{-7\}\}.
\]

Note that the case $r = 0$ brings us back to the definition of the Stirling number of type $B$ given in Definition 3.3.

The recursion for the $r$-Stirling numbers of type $B$ is identical to the one given in Proposition 3.4 where the only differences are the initial conditions, as the following proposition claims:
Proposition 3.6. If $n < r$, then $S_{n,k,r}^B = 0$. If $n = r$, then $S_{n,k,r}^B = \delta_{k,r}$. If $n > r$, then:

$$S_{n,k,r}^B = S_{n-1,k-1,r}^B + (2k + 1)S_{n-1,k,r}^B.$$ 

Note that $S_{n,k,r}^B$ is also a special case of the $q, r$-poly-Stirling number of the second kind, for $q = 1$ and $p(x) = 2x + 1$.

Proposition 3.6 is also a special case of Proposition 3.11 which is proven in the next subsection.

3.3 Restricted growth words for type $B$ and their weights

We are actually going to prove a $q$-analogue of Propositions 3.4 and 3.6. For this end, we modify Definition 3.1, to produce an extended version of restricted growth words, which are in bijection with the set partitions of type $B$.

Definition 3.7. Let $\Sigma^B$ be the alphabet $\{0, \pm 1, \pm 2, \ldots, \pm n\}$ and define the following order on $\Sigma^B$:

$$0 < -1 < 1 < -2 < 2 < \cdots < -n < n.$$ 

A restricted growth (RG-)word of type $B$ is a word $\omega = \omega_1 \cdots \omega_n$ in the alphabet $\Sigma^B$ which satisfies the following conditions:

1. We have $\omega_1 = 0$ or $\omega_1 = 1$.

2. For each $2 \leq t \leq n$, the following inequality holds:

$$\omega_t \leq \max \left\{ |\omega_1|, \ldots, |\omega_{t-1}| \right\} + 1,$$

with respect to the order defined above.

In the case that $\omega_t = \max \left\{ |\omega_1|, \ldots, |\omega_{t-1}| \right\} + 1$, we demand: $\omega_t > 0$.

Denote by $R^B(n,k)$ the set of RG-words of type $B$ over the alphabet $\Sigma^B$ such that the absolute value of the maximal element is $k$.

Now, let $P = \{ B_0, B_1, \ldots, B_k \}$ be a set partition of $[\pm n]$ of type $B$, such that its non-zero blocks are ordered in such a way that the set of minimum positive elements of the non-zero blocks is increasing (where if the zero block $B_0$ appears, it appears as the first block).

For such a set partition, we associate an RG-word $\omega = \omega_1 \cdots \omega_n$ of type $B$ as follows: for each $1 \leq j \leq n$, $\omega_j$ is the number of the representative block where $j$ or $-j$ appears. If the element $j$ appears in the representative block, then $\omega_j$ is the
number of the block containing \( j \); otherwise \( \omega_j \) will be the number of this block, with a negative sign. Note that if \( j \) is the smallest element in its block (in absolute value), then by the definition of a representative block, it should appear in it, so that we demand that \( \omega_j > 0 \) (as Equation (10) requires).

**Example 3.8.** Let

\[
P = \{ \{2, 5, -2, -5\}, \{4, -6, -3\}, \{-4, 6, 3\}, \{-1, 7\}, \{1, -7\}\},
\]

be a set partition of \([\pm 7]\) of type \( B \).

The representative blocks in increasing order, together with the zero block, are:

\[
\{ B_0 = \{2, 5, -2, -5\}, B_1 = \{1, -7\}, B_2 = \{3, -4, 6\}\}.
\]

Then, its associated RG-word of type \( B \) is:

\[
\omega = (1, 0, 2, -2, 0, 2, -1).
\]

Note, on the other hand, that the following word:

\[
\omega' = (1, 0, -2, 2, 0, -2, -1)
\]

is not an RG-word of type \( B \), since the first appearance of 2 is negative.

It is easy to see that this forms a bijection between the set partitions of \([n]\) of type \( B \), having \( k \) non-zero representative blocks and the set \( R^B(n, k) \).

Motivated by Cai and Readdy [5], who dealt with the Stirling number of the second kind (for set partitions of type \( A \)), we equip each RG-word of type \( B \) (or, equivalently, each set partition of \([\pm n]\) of type \( B \)) with a weight, which generates the \( q \)-Stirling numbers of the second kind of type \( B \). Note that this is the first appearance of \( q \)-Stirling number of the second kind of type \( B \), except for the work of Sagan and Swanson [24], who reached an equivalent definition independently, using set partitions of type \( B \).

**Definition 3.9.** Let \( \omega = \omega_1 \cdots \omega_n \in R^B(n, k) \). Define the weight of \( \omega \) by

\[
wt(\omega) = \prod_{i=1}^{n} wt_i(\omega),
\]

where \( wt_1(\omega) = 1 \) and for \( 2 \leq i \leq n \),

\[
wt_i(\omega) = \begin{cases} 1 & \text{if } \omega_i > \max\{\omega_1, \ldots, \omega_{i-1}\} \text{ or } \omega_i = 0 \\ q^{2|\omega_i|} & \text{if } |\omega_i| \leq \max\{|\omega_1|, \ldots, |\omega_{i-1}|\} \text{ and } \omega_i < 0 \\ q^{2|\omega_i|} & \text{if } |\omega_i| \leq \max\{|\omega_1|, \ldots, |\omega_{i-1}|\} \text{ and } \omega_i > 0 \end{cases}
\]

19
We also define the $q$-Stirling number $S_{n,k}^B(q)$ of the second kind of type $B$ as follows:

$$S_{n,k}^B(q) := \sum_{\omega \in R^B(n,k)} \text{wt}(\omega).$$

Note that for each $i \in \{\pm 1, \ldots, \pm k\}$, the first occurrence of $|i|$ has no contribution to the weight, but each of its next occurrences contributes $q^{2|i|-1}$ for $i < 0$ and $q^{2|i|}$ for $i > 0$. Moreover, the elements in the zero block have no contribution to the weight.

By the bijection between the set $R^B(n,k)$ and the set of set partitions of $[\pm n]$ of type $B$, we have for $q = 1$: $S_{n,k}^B(1) = S_{n,k}^B$.

**Example 3.10.** Given the set partition of $[\pm 6]$ of type $B$:

$$\{ B_0 = \{2, -2\}, B_1 = \{1, -3\}, B_2 = \{4, -5, 6\} \},$$

its associated RG-word is: $\omega = (1, 0, -1, 2, -2, 2)$, so we have:

$$\text{wt}(\omega) = \begin{array}{ccccccc}
1 & 1 & q & 1 & q^3 & q^4 \\
\text{wt}_1 & \text{wt}_2 & \text{wt}_3 & \text{wt}_4 & \text{wt}_5 & \text{wt}_6
\end{array} = q^8.$$

We can now prove combinatorially a Stirling-type recursion for the $q$-Stirling number of the second kind of type $B$:

**Proposition 3.11.** For each $1 \leq k < n$,

$$S_{n,k}^B(q) = S_{n-1,k-1}^B(q) + [2k + 1]_q \cdot S_{n-1,k}^B(q),$$

with the boundary conditions: $S_{n,0}^B(q) = S_{n,n}^B(q) = 1$.

**Proof.** We start by checking the boundary conditions. Note that $R^B(n,0)$ consists of the single RG-word $(0, 0, \ldots, 0)$ of length $n$, which corresponds to the set partition $\{ B_0 = \{1, -1, 2, -2, \ldots, n, -n\} \}$. Therefore:

$$S_{n,0}^B(q) = \sum_{\omega \in R^B(n,0)} \text{wt}(\omega) = \text{wt}((0, 0, \ldots, 0)) = 1.$$ 

Note that $R^B(n,n)$ consists of the single RG-word $(1, 2, \ldots, n)$ of length $n$, which corresponds to the set partition $\{ B_1 = \{1\}, B_2 = \{2\}, \ldots, B_n = \{n\} \}$. Therefore:

$$S_{n,n}^B(q) = \sum_{\omega \in R^B(n,n)} \text{wt}(\omega) = \text{wt}((1, 2, \ldots n)) = 1.$$
We now prove the recurrence relation. Let
\[ f : R^B(n, k) \to R^B(n - 1, k) \cup R^B(n - 1, k - 1) \]
be the function defined by removing the last element, i.e.
\[ f(\omega_1 \cdots \omega_n) = \omega_1 \cdots \omega_{n-1}. \]

For \( \omega \in R^B(n, k) \), we have exactly two possibilities:

- If the maximal element \( k \) of \( \omega \) appears only once, at position \( n \) (i.e. \( \omega_n = k \) and \( \omega_j < k \) for all \( j < n \)), then \( f(\omega) \in R^B(n - 1, k - 1) \). Actually, the RG-words satisfying this condition in \( R^B(n, k) \) are in bijection with \( R^B(n - 1, k - 1) \). In this case, we have \( \text{wt}(f(\omega)) = \text{wt}(\omega) \).

- Otherwise, the maximal element \( k \) appears in \( \omega \) before position \( n \), i.e. \( \omega_j = k \) for some \( j < n \) (note that each element \( 1 \leq i \leq k \) also appears before \( \omega_n \)). Then: \( f(\omega) \in R^B(n - 1, k) \). For each \( \omega = \omega_1 \cdots \omega_{n-1} \in R^B(n - 1, k) \), there are \( 2k + 1 \) possibilities for choosing the value for \( \omega_n \), i.e. \( \omega_n \in \{0, \pm 1, \ldots, \pm k\} \). Each such possibility contributes \( 1, q, q^2, \ldots, q^{2k} \) to the weight, respectively, giving in total \( 2k + 1 \).

Summing up both possibilities yields the requested recursion. \( \square \)

In order to establish a \( q \)-analogue for the \( r \)-variant \( S^B_{n,k,r} \), we define the following subset of \( R^B(n, k) \):

**Definition 3.12.** Let \( R^B_r(n, k) \) be the subset of \( R^B(n, k) \) consisting of all RG-words of type \( B \) such that the first \( r \) entries are \( 1, 2, \ldots, r \) in increasing order.

It is easy to see that the bijection defined above from set partitions to RG-words of type \( B \) restricts to a bijection between set partitions of \([n]\) of type \( B \) in \( k \) blocks such that the first \( r \) elements are in distinct blocks to the set \( R^B_r(n, k) \). Therefore, we define the \( q, r \)-Stirling number \( S^B_{n,k,r}(q) \) of the second kind of type \( B \) as follows:
\[
S^B_{n,k,r}(q) := \sum_{\omega \in R^B_r(n, k)} \text{wt}(\omega).
\]

Then, we have the following recurrence relation:

**Proposition 3.13.** For each \( r \leq k < n \),
\[
S^B_{n,k,r}(q) = S^B_{n-1,k-1,r}(q) + [2k + 1]_q \cdot S^B_{n-1,k,r}(q),
\]
with the boundary conditions:
\[
S^B_{r,r,r}(q) = 1 \quad \text{and} \quad S^B_{n,k,r}(q) = 0 \quad \text{for} \quad k < r, k > n \text{ or } n < r.
\]
Proof. The proof of the recurrence is identical to the proof of the recurrence in Proposition 3.11, so we check here only the boundary conditions. Note that \( R^B_r(r, r) \) consists of the single RG-word \((1, 2, \ldots, r)\), which corresponds to the set partition \( B_1 = \{1\}, B_2 = \{2\}, \ldots, B_r = \{r\} \). Therefore:

\[
S^B_{r,r,r}(q) = \sum_{\omega \in R^B(r, r)} \text{wt}(\omega) = \text{wt}((1, 2, \ldots, r)) = 1.
\]

If \( k < r, k > n \) or \( n < r \), we have \( R_r(n, k) = \emptyset \), and therefore the sum vanishes. \( \square \)

4 Ordinary generating functions of \( S^B_{n,k}(q) \) and \( S^B_{n,k,r}(q) \)

The ordinary generating function of \( S^B_{n,k}(q) \) can now be easily computed using the RG-words of type \( B \), defined in Definitions 3.7 and 3.12 above. Note that the expression for this ordinary generating function is also a consequence of the generalization of Comtet’s Theorem (Theorem 2.4) for \( p(x) = 2x + 1 \), but we offer here a combinatorial proof.

We start with the case \( r = 0 \):

**Proposition 4.1.** Let \( S^B_{n,k}(q) \) be the \( q \)-analogue of the Stirling number of second kind of type \( B \). Then for all \( k \in \mathbb{N} \), the ordinary generating function of \( S^B_{n,k}(q) \) is:

\[
\sum_{n=0}^{\infty} S^B_{n,k}(q) t^n = \frac{t^k}{(1-t)(1-|3_1q|t) \cdots (1-[2k+1]q)t)}.
\]

**Proof.** We start by proving the equality for \( q = 1 \). We claim that:

\[
\sum_{n=0}^{\infty} S^B_{n,k}(1) t^n = \sum_{n=0}^{\infty} S^B_{n,k}(1) \cdot t^n = \frac{t^k}{(1-t)(1-3t) \cdots (1-(2k+1)t)}.
\]

Every RG-word \( \omega \) of type \( B \) can be divided in \( k + 1 \) sub-words as follows: the first sub-word (if exists), which will be denoted \( W_0 \), contains all the first 0 digits of \( \omega \) up to the first appearance of 1 (not included). The second sub-word, which will be denoted \( W_1 \), contains the next digits of \( \omega \) until the first occurrence of the digit 2 (not included), and so on.

Note that by the rules of RG-words of type \( B \), the sub-word \( W_0 \) may contain any non-negative number of 0’s. Hence, its generating function is: \( 1 + t + t^2 + \cdots = \frac{1}{1-t} \).

The sub-word \( W_i \) for \( i > 0 \) may contain any number of elements from the alphabet \( \{0, \pm1, \ldots, \pm i\} \), but the first element must be \( i \) (so the sub-word \( W_i \) is not
empty) and so the number of possibilities to construct \( W_i \) is \((2i + 1)^{m_i - 1}\), where \( m_i \) is the length of \( W_i \). Hence, the generating function of \( W_i \) is:

\[
t(1 + (2i + 1)t + ((2i + 1)t)^2 + \cdots) = \frac{t}{1 - (2i + 1)t}.
\]

Combining the \( k + 1 \) sub-words together, we proved the case \( q = 1 \).

For the general case of an arbitrary \( q \), by a similar argument, the weight each sub-word \( W_i \) contributes is:

\[
t(1 + [2i + 1]_q t + ([2i + 1]_q t)^2 + \cdots) = \frac{t}{1 - [2i + 1]_q t},
\]

and the general result follows.

Miceli’s proof for this result [20, Theorem 18] (where \( p(x) = 2x + 1 \)) is based on a different combinatorial argument. Moreover, Sagan and Swan son [24] proved this proposition using symmetric functions.

Note that Proposition 4.1 is a special case of the next corollary for \( r = 0 \):

**Corollary 4.2.** Let \( S_{n,k,r}^B(q) \) be the \( q \)-analogue of the \( r \)-Stirling number of second kind of type \( B \). Then for all \( k \in \mathbb{N} \), the ordinary generating function of \( S_{n,k,r}^B(q) \) is:

\[
\sum_{n=0}^{\infty} S_{n,k,r}^B(q)t^n = \frac{t^k}{(1 - [2r + 1]_q t)(1 - [2r + 3]_q t) \cdots (1 - [2k + 1]_q t)}.
\]

**Proof.** By the definition of the set \( R_r^B(n,k) \), the first \( r \) elements of any word \( \omega \in R_r^B(n,k) \) are \( 1, \ldots, r \), so the proof is identical to the proof of the \( q \)-analogue in Proposition 4.1 except for the fact that we start with the \( (r + 1) \)th sub-word.

## 5 Exponential generating functions of \( S_{n,k}^B \) and \( S_{n,k,r}^B \) and an application to connection constants between bases of \( \mathbb{R}[x] \)

In this section, we deal with the exponential generating functions of \( S_{n,k}^B \) and \( S_{n,k,r}^B \) (Sections 5.1 and 5.2 respectively). In Section 5.3, we introduce an application of the exponential generating function of \( S_{n,k,r}^B \) to connection constants between bases of \( \mathbb{R}[x] \).
5.1 An exponential generating function of \( S_{n,k}^B \)

**Proposition 5.1.** For \( k \geq 0 \),

\[
\sum_{n=k}^{\infty} S_{n,k}^B \frac{t^n}{n!} = \frac{1}{k!2^k} e^t (e^{2t} - 1)^k. \tag{11}
\]

Here we propose an analytic proof of Theorem 5.1 while a generalization of this result will be proved in a combinatorial way in Theorem 5.3.

**Proof of Proposition 5.1.** Denote \( S_{k}^B(t) := \sum_{n=k}^{\infty} S_{n,k}^B \frac{t^n}{n!} \) for all \( k \geq 0 \). By the recursion (9), we have:

\[
\frac{d}{dt} S_{k}^B(t) = \frac{d}{dt} \left[ \sum_{n=k}^{\infty} S_{n,k}^B \frac{t^n}{n!} \right] =
\]

\[
= \frac{d}{dt} \left[ \sum_{n=k}^{\infty} S_{n-1,k-1}^B \frac{t^{n-1}}{n-1} \right] + (2k + 1) \frac{d}{dt} \left[ \sum_{n=k}^{\infty} S_{n-1,k}^B \frac{t^{n-1}}{n-1} \right] =
\]

\[
= \sum_{n=k}^{\infty} S_{n-1,k-1}^B \frac{t^{n-1}}{n-1} + (2k + 1) \sum_{n=k}^{\infty} S_{n-1,k}^B \frac{t^{n-1}}{n-1} =
\]

\[
= -\sum_{n=k-2}^{\infty} S_{k-2,k-1}^B \frac{t^{n-1}}{n-1} + \sum_{n=k-1}^{\infty} S_{n,k-1}^B \frac{t^{n-1}}{n-1} + (2k + 1) \sum_{n=k}^{\infty} S_{n,k}^B \frac{t^{n-1}}{n-1} =
\]

\[
= \sum_{n=k-1}^{\infty} S_{n,k-1}^B \frac{t^{n}}{n!} - \sum_{n=k-1}^{\infty} S_{k-2,k-1}^B \frac{t^{n}}{n!} + (2k + 1) \sum_{n=k}^{\infty} S_{n,k}^B \frac{t^{n}}{n!} - \sum_{n=k}^{\infty} S_{k-1,k}^B \frac{t^{n}}{n!} =
\]

\[
= S_{k-1}^B(t) + (2k + 1) S_{k}^B(t),
\]

so we get:

\[
\frac{d}{dt} S_{k}^B(t) = S_{k-1}^B(t) + (2k + 1) S_{k}^B(t). \tag{12}
\]

Directly from the boundary condition, one computes:

\[
S_{0}^B(t) = \sum_{n=0}^{\infty} S_{n,0}^B \frac{t^n}{n!} = \sum_{n=0}^{\infty} 1 \cdot \frac{t^n}{n!} = e^t.
\]

24
For proving the proposition, we check that the right hand side of Equation (11), \( \frac{1}{k!2^k} e'(e^{2t} - 1)^k \), satisfies Equation (12) and the boundary condition \( S_0^B(t) = e' \). Indeed, substituting 0 for \( k \) in the right hand side yields: \( \frac{1}{0!2^0} e'(e^{2t} - 1)^0 = e' \) as needed. Now we check that it satisfies Equation (12):

\[
\frac{d}{dt} \left[ \frac{1}{k!2^k} e'(e^{2t} - 1)^k \right] = \frac{1}{k!2^k} e'(e^{2t} - 1)^k + \frac{1}{k!2^k} e' \cdot k(e^{2t} - 1)^{k-1} \cdot 2e^{2t} = \n\]

\[
= \frac{1}{k!2^k} e'(e^{2t} - 1)^k + \frac{1}{(k-1)!2^{k-1}} e' e^{2t} (e^{2t} - 1)^{k-1} = \n\]

\[
= \frac{1}{k!2^k} e'(e^{2t} - 1)^k + \frac{1}{(k-1)!2^{k-1}} e' (e^{2t} - 1)^{k-1} + \frac{1}{(k-1)!2^{k-1}} e'(e^{2t} - 1)^k = \n\]

\[
= \frac{1}{(k-1)!2^{k-1}} e'(e^{2t} - 1)^{k-1} + (2k + 1) \frac{1}{k!2^k} e'(e^{2t} - 1)^k \]

as required.

\[\square\]

Proposition [5.1] is generalized by Sagan and Swanson [24] where the Stirling numbers are replaced by complete homogeneous symmetric functions.

### 5.2 An exponential generating function of \( S_{n,k,r}^B \)

In this subsection, we present the exponential generating function for \( S_{n,k,r}^B \). We generalize an approach appearing in Mező’s book [19, Section 8.2].

We start by proving the following nice combinatorial identity:

**Lemma 5.2.** Let \( n, k, r \in \mathbb{N} \) and let \( 0 \leq p \leq r \). Then

\[
S_{n,k,r}^B = \sum_{m=0}^{n-r} \binom{n-r}{m} \cdot S_{n-p-m,k-p,r-p}^B \cdot (2p)^m. \tag{13}
\]

**Proof.** Given \( p \leq r \), we call the first \( p \) elements \( \{1, \ldots, p\} \) out of the first \( r \) special elements (see Definition [5.5] - most special elements). For each one of them, its positive value will serve as the minimal element of a distinct block. These \( p \) blocks will be called the most special blocks.

In the first step, we choose the number \( m \) of additional elements that will occupy the \( p \) most special blocks (in addition to the most special elements \( 1, \ldots, p \)). Obviously, the absolute value of each of these \( m \) elements is greater than \( r \). The number \( \binom{n-r}{m} \) counts the ways to choose \( m \) such elements.

Then, we insert the remaining \( n - p - m \) elements in \( k - p \) blocks, such that the \( r - p \) elements \( p + 1, \ldots, r \) are special, i.e. \( p + 1, \ldots, r \) are in distinct blocks,
in $S^B_{n-p,m,k-p,r-p}$ possibilities. Next, we fill the $p$ most special blocks with $p + m$ elements in $S^B_{p+m,p,p}$ ways. Note, however, that $S^B_{p+m,p,p} = (2p)^m$, since each one of the additional $m$ elements can be located in one of the $p$ most special blocks, and can be either positive or negative. This gives us the required equality.

From here, one can easily deduce an expression for the exponential generating function of $S^B_{n+r,k+r,r}$ as follows.

**Theorem 5.3.** Let $k, r \in \mathbb{N}$. Then:

$$
\sum_{n=k}^{\infty} S^B_{n+r,k+r,r} \frac{t^n}{n!} = \frac{1}{k!2^k} e^{(2r+1)y}(e^{2r} - 1)^k.
$$

**Proof.** By substituting in Equation (13) $n + r$, $k + r$ and $r$ for $n$, $k$ and $p$ respectively, we get:

$$
S^B_{n+r,k+r,r} = \sum_{m=0}^{n} \binom{n}{m} S^B_{(n+r)-(r-m,(k+r)-r-r)} \cdot (2r)^m \ rac{m-n-m}{(2r)^{n-m}} \cdot S^B_{m,k}.
$$

Since the exponential generating function of the (ordinary) Stirling numbers for type $B$ is: $\sum_{n=k}^{\infty} S^B_{n,k} \frac{t^n}{n!} = \frac{1}{k!2^k} e^{(2r-1)^k}$ (see Sagan and Swanson [24], who proved it using the theory of species), we conclude from the rule of product of exponential generating functions (see e.g. [27] Prop. 5.1.1) that

$$
\sum_{n=k}^{\infty} S^B_{n+r,k+r,r} \frac{t^n}{n!} = e^{2rt} \cdot \frac{1}{k!2^k} e^{(2r-1)^k} = \frac{1}{k!2^k} e^{(2r+1)y}(e^{2r} - 1)^k
$$
as required. \qed

Note that by substituting $r = 0$ in Theorem 5.3, we get again Proposition 5.1.

### 5.3 An application to connection constants

Given $n, r \in \mathbb{N}$, let $\{(x + 2r)^n\}_{n \in \mathbb{N}}$ and $\{(x - 1)(x - 3) \cdots (x - 2n + 1)\}_{n \in \mathbb{N}}$ be two sequences of polynomials, forming two different bases of $\mathbb{R}[x]$. The next result presents the numbers $S^B_{n+r,k+r,r}$ as connection constants between these two bases of $\mathbb{R}[x]$:

**Theorem 5.4.** Let $n, r \in \mathbb{N}$. Then we have:

$$(x + 2r)^n = \sum_{k=0}^{n} S^B_{n+r,k+r,r} \cdot (x-1)(x-3) \cdots (x-2k+1).$$
Proof. We calculate:

\[
\sum_{n=0}^{\infty} \frac{(2x + 2r)^n}{n!} = e^{2(x + r)t} = e^{2r}t e^{2(x - \frac{1}{2})t} = e^{2r}t (1 + (\sqrt{e} - 1))^{(x - \frac{1}{2})} =
\]

\[
= \sum_{k=0}^{\infty} e^{2r}t^k \frac{(x - \frac{1}{2}) \cdots (x - \frac{k+1}{2})}{2^k k!} =
\]

\[
= \sum_{k=0}^{\infty} S_{n+r,k+r,r}^B t^k \frac{(2x - 1)(2x - 3) \cdots (2x - 2k + 1)}{n!} =
\]

By comparing the coefficients of \(t^n\), we have:

\[
(2x + 2r)^n = \sum_{k=0}^{n} S_{n+r,k+r,r}^B \cdot (2x - 1)(2x - 3) \cdots (2x - 2k + 1).
\]

By substituting \(x\) for \(2x\), the result follows. \(\square\)

6 The type II \(q, r\)-poly-Stirling numbers of the first kind

In this section, we move to the Stirling numbers of the first kind, and we present a generalization of Lancaster’s theorem (Theorem 1.2 above) to the case of unsigned type II \(q, r\)-poly-Stirling numbers of the first kind, mentioned in the introduction (based on Miceli’s type II \(q\)-poly-Stirling number of the first kind \(\cite{20}\)). As in Section 2 we start with its explicit definition, continue with some lemmata regarding these numbers, and finally we prove the generalization of Lancaster’s theorem to our framework.

Definition 6.1. Let \(p(x) \in \mathbb{Z}[x]\). The (unsigned) type II \(q, r\)-poly-Stirling numbers of the first kind, denoted \(c_{n,k,r}^{(x)}(q)\), are defined by the recurrence relation:

\[
c_{n,k,r}^{(x)}(q) = \frac{c_{n-1,k-1,r}^{(x)}(q) + [p(n - 1)]_q c_{n-1,k,r}^{(x)}(q)}{r \leq k \leq n, \ n \geq 1} \quad (14)
\]

with \(c_{r,r,r}^{(x)}(q) = 1\) and \(c_{n,k,r}^{(x)}(q) = 0\) (for \(k < r, k > n\) or \(n < r\)).

When \(r = 0\), the recurrence \(\text{(14)}\) is reduced to that in Miceli \(\cite{20}\) Eqn. (39)). Moreover, the boundary conditions for \(r = 0\): \(c_{0,0,0}^{(x)}(q) = 1\) and \(c_{n,k,0}^{(x)}(q) = 0\) (for \(k < 0, k > n\) or \(n < 0\)) are compatible with those of Miceli.

Using recurrence \(\text{(14)}\), we get:
Lemma 6.2. For \( r < n \):

1. \( c^{p(x)}_{n,n,r}(q) = 1 \).

2. \( c^{p(x)}_{n,n-1,r}(q) = \sum_{i=r}^{n-1} [p(i)]_q \).

Proof. By the recurrence (14), we get:

1. \( c^{p(x)}_{n,n,r}(q) = c^{p(x)}_{n-1,n-1,r}(q) = \cdots = c^{p(x)}_{r,r,r}(q) = 1 \).

2. \( c^{p(x)}_{n,n-1,r}(q) = c^{p(x)}_{n-1,n-2,r}(q) + [p(n-1)]_q \cdot c^{p(x)}_{n-1,n-1,r}(q) = \cdots = c^{p(x)}_{r+1,r,r}(q) + [p(r+1)]_q + \cdots + [p(n-2)]_q + [p(n-1)]_q = [p(r)]_q + [p(r+1)]_q + \cdots + [p(n-2)]_q + [p(n-1)]_q \).

\( \square \)

Next, we extend and generalize Lancaster’s result to the case of type II unsigned \( q, r \)-poly-Stirling numbers of the first kind:

Theorem 6.3 (Generalized Lancaster’s theorem for \( q, r \)-poly-Stirling numbers). Let \( p(n) \) be a polynomial with non-negative integer coefficients. The following are equivalent characterizations for \( (c^{p(x)}_{n,k,r}(q))_{n,k,r\mid 0 \leq r \leq k \leq n} \) (where for all other values of the triple \((n, k, r)\), we assume \( c^{p(x)}_{n,k,r}(q) = 0 \)).

1. Defining equation/generating function:

\[
(x + [p(r)]_q)(x + [p(r+1)]_q) \cdots (x + [p(n-1)]_q) = \sum_{k=r}^{n} c^{p(x)}_{n,k,r}(q) \cdot x^{k-r}. 
\]

2. Recursion: For each \( n \geq k > r \):

\[
c^{p(x)}_{n,k,r} = c^{p(x)}_{n-1,k-1,r}(q) + [p(n-1)]_q c^{p(x)}_{n-1,k,r}(q) 
\]

with the boundary conditions: \( c^{p(x)}_{n,r,r}(q) = [p(r)]_q[p(r+1)]_q \cdots [p(n-1)]_q \) and \( c^{p(x)}_{r,k,r}(q) = \delta_{kr} \).
(3) **Complete Recursion:** For $n \geq k > r$:

$$c_{n,k,r}^{(x)}(q) = \sum_{j=k}^{n} c_{j-1,k-1,r}^{(x)}(q) \prod_{i=j}^{n-1} [p(i)]_q,$$

subject to the same boundary conditions as in Condition (2).

(4) **Explicit formula:** For $n \geq k \geq r$:

$$c_{n,k,r}^{(x)}(q) = \prod_{j=r}^{n-1} [p(j)]_q \sum_{r \leq i_{r+1} < \cdots < i_{k} \leq n-1} 1/[p(i_{r+1})]_q \cdots [p(i_{k})]_q.$$

**Proof.** The proof consists of the following six parts.

[**(1)\(\Rightarrow\)**(2)]: We first prove the boundary conditions. For proving the first one:

$$c_{n,r,r}^{(x)}(q) = [p(r)]_q[p(r + 1)]_q \cdots [p(n - 1)]_q,$$

note that its two sides are the corresponding free coefficients of the two sides of Condition (1).

As for the second boundary condition, we get by Condition (1) for $n = r$:

$$1 \quad \text{Cond. (1)} \quad \sum_{k=r}^{r} c_{r,k,r}^{(x)}(q) \cdot x^{r-r} = c_{r,r,r}^{(x)}(q).$$

Moreover, if $k \neq r$, then we have by definition that $c_{r,k,r}^{(x)}(q) = 0$, and so we get $c_{r,k,r}^{(x)}(q) = \delta_{k,r}$ as needed.

Now, we prove the recurrence relation. We have:

$$\sum_{k=r}^{n} c_{n,k,r}^{(x)}(q)x^{k-r} \quad \text{Cond. (1)} \quad = (x + [p(r)]_q)(x + [p(r + 1)]_q) \cdots (x + [p(n - 1)]_q) =$$

$$\text{Cond. (1)} \quad = (x + [p(n - 1)]_q) \sum_{k=r}^{n-1} c_{n-1,k,r}^{(x)}(q) \cdot x^{k-r} =$$

$$= \sum_{k=r}^{n-1} c_{n-1,k,r}^{(x)}(q) \cdot x^{k-r+1} + \sum_{k=r}^{n-1} [p(n - 1)]_q c_{n-1,k,r}^{(x)}(q) \cdot x^{k-r} =$$

$$k+1 \to k \quad = \sum_{k=r+1}^{n} c_{n-1,k-1,r}^{(x)}(q) \cdot x^{k-r} + \sum_{k=r}^{n-1} [p(n - 1)]_q c_{n-1,k,r}^{(x)}(q) \cdot x^{k-r} =$$

29
and we prove it for $\alpha_1$ obtain:

$$\sum_{k=r}^{n} \left( c_{n-1,k-1,r}^{(x)}(q) + [p(n-1)]_q \right) \cdot x^{k-r} = \sum_{k=r}^{n} \left( c_{n-1,r-1,r}^{(x)}(q)x^0 - [p(n-1)]_q \right) c_{n-1,n,r}^{(x)}(q)x^{n-r} =$$

$$= \sum_{k=r}^{n} \left( c_{n-1,k-1,r}^{(x)}(q) + [p(n-1)]_q \right) c_{n-1,k,r}^{(x)}(q) \cdot x^{k-r}.$$

Comparing the coefficients of the basis elements $\{x^{k-r}\}_{k \in \mathbb{N}}$ on both sides, we obtain the recurrence relation:

$$c_{n,k,r}^{(x)}(q) = c_{n-1,k-1,r}^{(x)}(q) + [p(n-1)]_q \cdot c_{n-1,k,r}^{(x)}(q)$$

as needed.

$$(2) \Rightarrow (1):$$ We prove by induction on $n$. The base case $n = r$ is $c_{r,r,r}^{(x)} = 1$.

Now assume its correctness for $n - 1$:

$$\sum_{k=r}^{n-1} c_{n-1,k,r}^{(x)}(q)x^{k-r} = (x + [p(r)]_q)(x + [p(r + 1)]_q) \cdots (x + [p(n - 2)]_q),$$

and we prove it for $n$. By the recurrence relation appearing in Condition (2), we obtain:

$$\sum_{k=r}^{n} c_{n,k,r}^{(x)}(q)x^{k-r} = \sum_{k=r}^{n} \left( c_{n-1,k-1,r}^{(x)}(q) + [p(n-1)]_q \right) c_{n-1,k,r}^{(x)}(q) \cdot x^{k-r} =$$

$$= x \sum_{k=r}^{n} c_{n-1,k-1,r}^{(x)}(q)x^{k-1-r} + [p(n-1)]_q \sum_{k=r}^{n} c_{n-1,k,r}^{(x)}(q)x^{k-r} =$$

$$= x \sum_{k=r-1}^{n-1} c_{n-1,k,r}^{(x)}(q)x^{k-r} + [p(n-1)]_q \sum_{k=r}^{n-1} c_{n-1,k,r}^{(x)}(q)x^{k-r} +$$

$$+ c_{n-1,r-1,r}^{(x)}(q)x^{0} + [p(n-1)]_q \sum_{n-1,r,n,r}^{(x)}(q)x^{n-r} =$$

$$= (x + [p(n-1)]_q) \sum_{k=r}^{n-1} c_{n-1,k,r}^{(x)}(q)x^{k-r} =$$

30
Hence, we also have:

\[
(x + [p(n-1)]_q) \cdot (x + [p(r)]_q) \cdots (x + [p(n-2)]_q).
\]

**[(2)⇒(3)]:** Using \( n - k \) times the recurrence relation in Condition (2), we get:

\[
c^p_{n,k,r}(q) = c^p_{n-1,k-1,r}(q) + [p(n-1)]_q c^p_{n-1,k,r}(q) =
\]

\[
= c^p_{n-1,k-1,r}(q) + [p(n-1)]_q (c^p_{n-2,k-1,r}(q) + [p(n-2)]_q c^p_{n-2,k,r}(q)) =
\]

\[
= c^p_{n-1,k-1,r}(q) + [p(n-1)]_q c^p_{n-2,k-1,r}(q) +
\]

\[
+ ([p(n-1)]_q [p(n-2)]_q c^p_{n-3,k-1,r}(q) + [p(n-3)]_q c^p_{n-3,k,r}(q)) = \cdots =
\]

\[
= c^p_{n-1,k-1,r}(q) + [p(n-1)]_q c^p_{n-2,k-1,r}(q) + ([p(n-1)]_q [p(n-2)]_q) c^p_{n-3,k-1,r}(q) +
\]

\[
+ \cdots + ([p(n-1)]_q \cdots [p(k)]_q) c^p_{k-1,k-1,r}(q) =
\]

\[
= \sum_{j=k}^{n} c^p_{j-1,k-1,r}(q) \prod_{i=j}^{n-1} [p(i)]_q.
\]

**[(3)⇒(2)]:** We check for \( n = r + 1 \) separately: in this case, the only possible value of \( k \) is \( r + 1 \). By Condition (3), we have:

\[
c^p_{r+1,r+1,r}(q) = \sum_{j=r+1}^{r+1} c^p_{j-1,r,r}(q) \cdot 1 = c^p_{r,r,r}(q) = 1,
\]

which is exactly what we expect to obtain by the recursion in Condition (2):

\[
c^p_{r+1,r+1,r}(q) = c^p_{r,r,r}(q) + [p(r)]_q c^p_{r+1,r+1,r}(q) = 1.
\]

For \( n > r + 1 \), since the equation in Condition (3) holds for all \( n > r \):

\[
c^p_{n,k,r}(q) = \sum_{j=k}^{n} c^p_{j-1,k-1,r}(q) \prod_{i=j}^{n-1} [p(i)]_q,
\]

we also have:

\[
c^p_{n-1,k,r}(q) = \sum_{j=k}^{n-1} c^p_{j-1,k-1,r}(q) \prod_{i=j}^{n-2} [p(i)]_q.
\]

Hence,

\[
c^p_{n,k,r}(q) \overset{\text{Cond. (3)}}{=} \sum_{j=k}^{n} c^p_{j-1,k-1,r}(q) \prod_{i=j}^{n-1} [p(i)]_q
\]

31
and therefore the boundary conditions are satisfied.

then it is immediate that

which is the recurrence relation in Condition (2).

\[(2)\Rightarrow(4)\]: We prove the identity by induction on \(n\). The base case is: \(n = k = r\). Indeed,

\[ c_{r,r,r}^{p(x)}(q) = \prod_{j=r}^{r-1} [p(j)]_q = 1 \]

as required.

Assume correctness for \(n - 1\), and we prove the correctness for \(n\): by the recursion \[14\], we have:

\[ c_{n,k,r}^{p(x)}(q) = \prod_{j=r}^{n-2} [p(j)]_q + \sum_{j=k}^{n-2} \frac{1}{[p(i_{r+1})]_q \cdots [p(i_{k-1})]_q} + [p(n - 1)]_q \prod_{j=r}^{n-2} [p(j)]_q \sum_{r\leq i_{r+1} < \ldots < i_{k-1} \leq n-2} \frac{1}{[p(i_{r+1})]_q \cdots [p(i_{k-1})]_q [p(n - 1)]_q} = \]

\[ = \prod_{j=r}^{n-1} [p(j)]_q \sum_{r\leq i_{r+1} < \ldots < i_{k-1} \leq n-2} \frac{1}{[p(i_{r+1})]_q \cdots [p(i_{k-1})]_q} + \prod_{j=r}^{n-1} [p(j)]_q \sum_{r\leq i_{r+1} < \ldots < i_{k-1} \leq n-2} \frac{1}{[p(i_{r+1})]_q \cdots [p(i_{k})]_q} = \]

\[ [(4)\Rightarrow(2)]: Assume that we have Condition (4):

\[ c_{n,k,r}^{p(x)}(q) = \prod_{j=r}^{n-1} [p(j)]_q \sum_{r\leq i_{r+1} < \ldots < i_{k-1} \leq n-1} \frac{1}{[p(i_{r+1})]_q \cdots [p(i_{k})]_q}, \]

then it is immediate that \( c_{n,r,r}^{p(x)}(q) = [p(r)]_q [p(r + 1)]_q \cdots [p(n - 1)]_q \) and \( c_{r,r,r}^{p(x)}(q) = \delta_{kr} \), and therefore the boundary conditions are satisfied.
Now we show the recurrence relation in Condition (2):

\[
c_{n,k,r}^{(x)}(q) = \text{Cond. (4)} \quad \sum_{r \leq i_{r+1} < \cdots < i_k \leq n-1} \frac{1}{[(p(i_{r+1})]_q \cdots [p(i_k)]_q} = \\
= \left[ \prod_{j=r}^{n-1} [p(j)]_q \right] \sum_{r \leq i_{r+1} < \cdots < i_k \leq n-2} \frac{1}{[(p(i_{r+1})]_q \cdots [p(i_k)]_q} + \\
+ [p(n-1)]_q \left( \left[ \prod_{j=r}^{n-2} [p(j)]_q \right] \sum_{r \leq i_{r+1} < \cdots < i_k \leq n-2} \frac{1}{[p(i_{r+1})]_q \cdots [p(i_k)]_q} \right) = \\
\text{Cond. (4)} \quad c_{n-1,k-1,r}^{(x)}(q) + [p(n-1)]_q c_{n-1,k,r}^{(x)}(q),
\]

as required. \qed

7 Orthogonality relations between poly-Stirling numbers of both kinds

Until now, we focused on the unsigned version of the \( q, r \)-poly-Stirling numbers of the first kind \( c_{n,k}^{(x)}(q) \) due to their general combinatorial significance, but there also exists a signed version which we discuss now in order to present two of their applications: an identity related to a sum of powers (Section 7.1) and the orthogonality relations between the Stirling numbers of first and second kinds (Section 7.2).

Definition 7.1. Let \( p(x) \in \mathbb{Z}[x] \). The type II (signed) \( q, r \)-poly-Stirling numbers of the first kind, denoted \( s_{n,k,r}^{(x)}(q) \), are defined by the recurrence relation:

\[
s_{n,k,r}^{(x)}(q) = s_{n-1,k-1,r}^{(x)}(q) - [p(n-1)]_q s_{n-1,k,r}^{(x)}(q) \quad (r \leq k \leq n, n \geq 1)
\]

with \( s_{r,r,r}^{(x)}(q) = 1 \) and \( s_{n,k,r}^{(x)}(q) = 0 \) (for \( k < r, k > n \) or \( n < r \)).

We present here the generalization of Lancaster’s theorem for the signed case. Its proof is almost identical to the unsigned case (Theorem 6.3), except for the appropriate sign changes, so we present it without proof:

Theorem 7.2 (Generalized Lancaster’s theorem for signed type II \( q, r \)-poly-Stirling numbers of the first kind). Let \( p(x) \in \mathbb{Z}[x] \). The following are equivalent characterizations for \( (s_{n,k,r}^{(x)}(q))_{n,k,r} \) (where for all other values of the triple \( (n, k, r) \), we assume \( s_{n,k,r}^{(x)}(q) = 0 \):
(1) Defining equation/generating function:

\[(x - [p(r)]_q)(x - [p(r + 1)]_q) \cdots (x - [p(n - 1)]_q) = \sum_{k=r}^{n} s_{n,k,r}^{p(x)}(q) \cdot x^{k-r}.
\]

(2) Recursion: For each \(n \geq k \geq r\):

\[s_{n,k,r}^{p(x)}(q) = s_{n-1,k-1,r}^{p(x)}(q) - [p(n - 1)]_q \cdot s_{n-1,k,r}^{p(x)}(q),\]

with the boundary conditions:

\[s_{n,r,r}^{p(x)}(q) = (-1)^{n-r}[p(r)]_q[p(r + 1)]_q \cdots [p(n - 1)]_q \]

and \(s_{r,r,r}^{p(x)}(q) = \delta_{kr}\).

(3) Complete recursion: For \(n \geq k \geq r\):

\[s_{n,k,r}^{p(x)}(q) = \sum_{j=k}^{n} (-1)^{n-j} s_{j-1,k-1,r}^{p(x)}(q) \prod_{i=j}^{n-1} [p(i)]_q,
\]

subject to the same boundary conditions as in Condition (2).

(4) Explicit formula: For \(n \geq k \geq r\):

\[s_{n,k,r}^{p(x)}(q) = (-1)^{n-k} \left[ \prod_{j=r}^{n-1} [p(j)]_q \right] \sum_{r \leq i_1 < \cdots < i_{n-1} \leq n-1} \frac{1}{[p(i_{r+1})]_q \cdots [p(i_k)]_q}.
\]

7.1 An application to sum of powers

In this subsection, we provide an application of Theorem 7.2 to the sum of powers of the expressions \([p(j)]_q\):

**Theorem 7.3.** For integers \(n\) and \(k\) with \(n \geq r\) and \(k \geq 1\), we have:

\[
\sum_{j=r}^{n} ([p(j)]_q)^k = - \sum_{\ell = 1}^{n-r+1} \ell \cdot s_{n+1,n+1-\ell,r}^{p(x)}(q) s_{n+k-\ell,n,r}^{p(x)}(q).
\]

**Proof.** By Condition (1) of Theorem 7.2 for \(n + 1\) instead of \(n\) we have:

\[x^n \prod_{j=r}^{n} (x - [p(j)]_q) = \sum_{k=r}^{n+1} s_{n+1,k,r}^{p(x)}(q)x^k. \quad (16)
\]
Next, we substitute \( \frac{1}{t} \) for \( x \):

\[
\left( \frac{1}{t} \right)^r \prod_{j=r}^{n} \left( \frac{1}{t} - [p(j)]_q \right) = \sum_{k=r}^{n+1} S_{n+1,k,r}^{(x)}(q) \left( \frac{1}{t} \right)^k.
\]

Multiplying this by \( t^{n+1} \), yields:

\[
F_1(t) := \prod_{j=r}^{n} (1 - t[p(j)]_q) = \sum_{k=r}^{n+1} S_{n+1,k,r}^{(x)}(q) t^{n-k+1} \sum_{\varepsilon = 0}^{n-r+1} \sum_{\sigma = 0}^{n} S_{n+1,n+1-\varepsilon,r}^{(x)} t^{\sigma}.
\]

Recall from Theorem 2.4(4) that

\[
\sum_{n=k}^{\infty} S_{n,k,r}^{(x)}(q)x^n = \frac{x^k}{(1 - x[p(r)]_q) \cdots (1 - x[p(k)]_q)}.
\]

By the substitutions (in this order): \( n \to n + \nu \), \( k \to n \) and \( x \to t \), and dividing both sides by \( t^n \), we get:

\[
\sum_{\nu=0}^{\infty} S_{n+\nu,n,r}^{(x)}(q) t^{\nu} = \frac{1}{(1 - t[p(r)]_q) \cdots (1 - t[p(n)]_q)} = \prod_{j=r}^{n} (1 - t[p(j)]_q)^{-1} =: F_2(t).
\]

Note that \( F_1(t)F_2(t) = 1 \). Now, we have:

\[
\frac{d}{dt} \log F_2(t) = \frac{d}{dt} \left[ \sum_{j=r}^{n} - \log \left( 1 - t[p(j)]_q \right) \right] = \sum_{j=r}^{n} \frac{[p(j)]_q}{1 - t[p(j)]_q} = \sum_{j=r}^{n} \sum_{k=1}^{\infty} ([p(j)]_q)^k t^{k-1}.
\]

Note that:

\[
\frac{d}{dt} F_1(t) = \frac{d}{dt} \left[ \prod_{j=r}^{n} \left( 1 - t[p(j)]_q \right) \right] = \sum_{m=r}^{n} -[p(m)]_q \prod_{j=r,j \neq m}^{n} (1 - t[p(j)]_q).
\]

Therefore, we have

\[
- \left( \frac{d}{dt} F_1(t) \right) F_2(t) = \sum_{m=r}^{n} \frac{[p(m)]_q}{1 - t[p(m)]_q} \overset{\text{18}}{=} \frac{d}{dt} \log F_2(t).
\]

On the other hand, by Equation (17), we also have:

\[
\frac{d}{dt} F_1(t) = \sum_{\varepsilon = 1}^{n-r+1} \varepsilon S_{n+1,n+1-\varepsilon,r}^{(x)}(q)t^{\varepsilon-1}.
\]
Combining Equations (18) and (19), we have:

\[- \left( \sum_{\ell = 1}^{n+1} \ell^\ell S_{n+1,n+1-\ell,r}(q) t^{\ell-1} \right) \left( \sum_{v=0}^{\infty} S^p_{n+v,n,r}(q) t^v \right) = \sum_{k=1}^{\infty} \sum_{j=r}^{n} ([p(j)]_q)^k t^{k-1},\]

which can be also written as:

\[- \sum_{\ell = 1}^{n+1} \sum_{v=0}^{\infty} \ell^\ell S^p_{n+1,n+1-\ell,r}(q) S^p_{n+v,n,r}(q) t^{\ell-1+v} = \sum_{k=1}^{\infty} \sum_{j=r}^{n} ([p(j)]_q)^k t^{k-1}.\]

Comparing the coefficients of \(t^{k-1}\) in both sides (imposing that \(v = k - \ell\)), yields the requested formula.

Note that Theorem 1.1 of Merca [18] is a special case of Theorem 7.3 where \(r = 0\) and \(p(x) = x\).

### 7.2 An application to orthogonality relations

As a consequence of Theorems 2.4 and 7.2, we get the orthogonality relations between the type II \(q\)-, \(r\)-poly-Stirling numbers of the first and second kinds:

**Theorem 7.4.** For \(r \leq \ell \leq n\) and \(r < n\), we have:

\[\sum_{k=\ell}^{n} S^p_{n,k,r}(q) S^p_{n,k,\ell,r}(q) = \delta_{n\ell},\]  

\[\sum_{k=\ell}^{n} S^p_{n,k,r}(q) S^p_{n,k,\ell,r}(q) = \delta_{n\ell}.\]  

**Proof.** By Theorem 2.4(1), we have:

\[x^{n-r} = \sum_{k=r}^{n} S^p_{n,k,r}(q) \prod_{j=r+1}^{k} (x - [p(j - 1)]_q) .\]

On the other hand, by Theorem 7.2(1) we have:

\[\prod_{j=r}^{n-1} (x - [p(j)]_q) = \sum_{k=r}^{n} S^p_{n,k,r}(q) \cdot x^{k-r} .\]

By substituting either one of these equations into the other and changing the order of summations, we readily obtain the orthogonality relations. \(\square\)
8  Type I $q, r$-poly-Stirling numbers of both kinds and mixed relations between both types

In this section, we present another version of $q, r$-poly-Stirling numbers of the first and second kinds. Based on this version, we present some mixed relations between both types of these numbers.

8.1  Type I $q, r$-poly-Stirling numbers of both kinds

Here we present the generalization of Miceli’s Type I $q$-poly-Stirling numbers of the first and second kind (see [20]):

**Definition 8.1.** The type I $q, r$-poly-Stirling numbers of the second kind are defined by the recurrence:

$$S_{n,k,r}(q) = S_{n-1,k-1,r}(q) + p([k]_q)S_{n-1,k,r}(q) \quad (r \leq k \leq n, \ n \geq 1),$$

(22)

with $S_{r,r}(q) = 1$ and $S_{n,k,r}(q) = 0$ for $k < r, k > n$ or $n < r$.

**Definition 8.2.** The type I signed $q, r$-poly-Stirling numbers of the first kind are defined by the recurrence:

$$S_{n,k,r}(q) = S_{n-1,k-1,r}(q) - p([n-1]_q)S_{n-1,k,r}(q) \quad (r \leq k \leq n, \ n \geq 1)$$

(23)

with $S_{r,r}(q) = 1$ and $S_{n,k,r}(q) = 0$ for $k < 0, k > n$ or $n < r$.

**Definition 8.3.** The type I unsigned $q, r$-poly-Stirling numbers of the first kind are defined by the recurrence:

$$S_{n,k,r}(q) = S_{n-1,k-1,r}(q) + p([n-1]_q)S_{n-1,k,r}(q) \quad (r \leq k \leq n, \ n \geq 1)$$

(24)

with $S_{r,r}(q) = 1$ and $S_{n,k,r}(q) = 0$ for $k < 0, k > n$ or $n < r$.

All the results of Sections [2, 6] and [7] and their proofs can be transferred verbatim to type I as well (the only change is the location of the squared brackets); nevertheless, we provide here the formulation of the result parallel to Theorem 7.3 for our use in the next subsection:

**Theorem 8.4.** For integers $n$ and $k$ with $n \geq r$ and $k \geq 1$, we have

$$\sum_{j=r}^{n} (p([j]_q))^k = - \sum_{\ell=1}^{n-r+1} \ell S_{n+1,n+1-\ell,r}(q)S_{n+k-\ell,n,r}(q).$$
8.2 Mixed relations

We can obtain a mixed relation of both types of $q,r$-poly-Stirling numbers:

**Theorem 8.5.** Let $u \geq 1$ be a positive integer. For $n \geq r$, we have

$$
\sum_{\ell = 1}^{n-r+1} \ell^x S_{n+1,n+1-\ell,r}^x (q) S_{n+ku-\ell,n,r}^x (q) = \sum_{\ell = 1}^{n-r+1} \ell^x S_{n+1,n+1-\ell,r}^x (q) \overline{S}_{n+k-\ell,n,r}^x (q).
$$

**Proof.** If we put $p(x) = x$ in Theorem 7.3, we get:

$$
\sum_{j=r}^{n} (\lfloor j \rfloor_q)^k = - \sum_{\ell = 1}^{n-r+1} \ell^x S_{n+1,n+1-\ell,r}^x (q) S_{n+k-\ell,n,r}^x (q).
$$

Substituting $ku$ for $k$, we get:

$$
\sum_{j=r}^{n} (\lfloor j \rfloor_q)^{ku} = - \sum_{\ell = 1}^{n-r+1} \ell^x S_{n+1,n+1-\ell,r}^x (q) S_{n+k-\ell,n,r}^x (q). \tag{25}
$$

On the other hand, if we put $p(x) = x^u$ in Theorem 8.4, we get:

$$
\sum_{j=r}^{n} ((\lfloor j \rfloor_q)^u)^k = - \sum_{\ell = 1}^{n-r+1} \ell^x S_{n+1,n+1-\ell,r}^x (q) \overline{S}_{n+k-\ell,n,r}^x (q). \tag{26}
$$

Comparing Equations (25) and (26) yields the requested equality. \qed

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