Relativistic Diffusion in Gödel’s Universe

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Abstract

K. Gödel [G1] discovered his celebrated solution to Einstein equations in 1949. Additional contributions were made by Kundt [K] and Hawking-Ellis ([H-E], 5.7). On the other hand, a general Lorentz invariant operator, associated to the so-called “relativistic diffusion”, and making sense in any Lorentz manifold, was introduced by Franchi-Le Jan in [F-LJ]. Here is proposed a first study of the relativistic diffusion in the framework of Gödel’s universe, which contains matter.

1 Introduction

K. Gödel [G1] published his celebrated exact solution to Einstein equations in 1949. The most striking feature of this cosmological model was to be non-causal (though locally and geodesically causal), containing closed timelike curves. For this reason, it is generally considered as rather unphysical. Possessing a series of interesting properties, it aroused however a great interest among physicists. For example, it contains rotating matter, but no singularity. Moreover, the explicit exact solutions to Einstein equations are not so many.

W. Kundt [K] studied its geodesics, and S. Hawking and G. Ellis ([H-E], section 5.7) stress on coordinates (defined by Gödel himself) showing up its rotational symmetry (about any point), to draw a nice picture of its dynamics. D. Malament ([M1],[M2]) calculated the minimal energy of a closed timelike curve. K. Gödel [G2] discussed other rotating universes, which are spatially homogeneous, finite, and expanding, and he showed in particular that there exist a lot of strongly causal such cosmological models.

The purpose of the present work is to study, in the framework of Gödel’s universe, the behaviour, and mainly the asymptotic behaviour, of the so-called relativistic diffusion.

The relativistic diffusion was introduced by J. Franchi and Y. Le Jan in [F-LJ], in the framework of general relativity, on an arbitrary Lorentz manifold, as the only diffusion which is invariant under Lorentz isometries. In this sense, it is the Lorentzian analogue of the Brownian motion on a Riemannian manifold. It lives in fact on the pseudo-unit tangent bundle of the considered Lorentz manifold, and is roughly the integral of Brownian motion of the unit pseudo-sphere of the tangent space. It can also be seen as a random perturbation of the timelike geodesic flow.
This article begins with a detailed study of timelike and lightlike geodesics, in a different and more complete way as Kundt [K] did. As a conclusion of the study of lightlike geodesics, a definition (Definition 2) of light ray (or boundary point, as an equivalence class of lightlike geodesics, without use of causality) and of convergence to a light ray is given, which appears to be rather natural in this non-causal universe (it can be reinforced to a certain extent: see Remark 10). Thus the set of light rays has a natural structure of 3-dimensional boundary, on which the isometry group of Gödel’s universe does operate.

Then the relativistic diffusion of Gödel’s universe is introduced. In order to study such a 7-dimensional diffusion, some sub-diffusions are considered, of dimensions 1, 2, and 4. A leading concern is here to bring out all asymptotic variables of the relativistic diffusion, or in other words, the tail \(\sigma\)-field of its natural filtration (which is in turn closely related to the Poisson boundary of the relativistic diffusion). The clue in this direction is the idea, suggested and partially worked out in [F-LJ] in the case of Schwarzschild solution, and very recently established in [B] in the flat case of Minkowski space, that convergence to a light ray should eventually occur. The following statement, progressively established below, shows that this general guess stands out as reinforced, also in the case of a non-empty (and even non causal) universe.

The main results of the present article are summarised in the following.

**THEOREM**

(i) The relativistic diffusion is irreducible (on its 7-dimensional phase-space).

(ii) Almost surely, the relativistic diffusion path possesses a 3-dimensional asymptotic random variable, and converges to a light ray (in the sense of Definition 2 and Remark 10).

(iii) The support of possible light rays the relativistic diffusion can converge to, is the whole 3-dimensional boundary space of light rays.

As a consequence of this theorem, and on the basis of some secondary results and considerations, the following conjecture appears to hold likely: by the showing up of the 3-dimensional asymptotic random variable evoked in (ii) above, the whole tail \(\sigma\)-field of the relativistic diffusion of Gödel’s universe, and then its whole Poisson boundary, has been brought out.

The only relativistic case in which the analogous statement has been proved, up to now, is Minkowski space, in [B], by two different methods: Doob’s \(h\)-process conditioning and then couplings, making use of an explicit expression of the laws of already found asymptotic variables; or alternatively: study of the random walk associated with a lifted relativistic diffusion on some (Poincaré) fixed locally compact group. The use of both methods does not seem to be easy in the present curved case (likely as in any other curved case), since neither are explicit the laws of the asymptotic variables, nor appears any Poincaré-like group.
2 Gödel’s pseudo-metric

**Definition 1** The Gödel’s universe is the manifold $\mathbb{R}^4$, endowed with coordinates $\xi := (t, x, y, z)$, and with the pseudo-metric (having signature $(+, -, -, -)$) defined by:

$$ds^2 := dt^2 - dx^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + 2 e^{\sqrt{2}\omega x} dt dy - dz^2,$$

for some strictly positive constant $\omega$.

The inverse matrix of this pseudo-metric $(g_{ij})$ is as follows:

$$
(\begin{pmatrix}
-1 & 0 & 2 e^{-\sqrt{2}\omega x} & 0 \\
0 & -1 & 0 & 0 \\
2 e^{-\sqrt{2}\omega x} & 0 & -2 e^{-2\sqrt{2}\omega x} & 0 \\
0 & 0 & 0 & -1
\end{pmatrix})
$$

The unit pseudo-norm relation, defining proper time $s$, is:

$$1 + \dot{t}_s^2 + \dot{x}_s^2 + \dot{z}_s^2 = \frac{1}{2} \left[e^{\sqrt{2}\omega x} \dot{y}_s + 2 \dot{t}_s\right]^2.$$

The isometry group of Gödel’s universe is the five-dimensional Lie group generated by:
1) the translations $(t, x, y, z) \mapsto (t + t_0, x, y + y_0, z + z_0)$ of the linear $(t, y, z)$ 3-subspace;
2) the hyperbolic dilatations $(t, x, y, z) \mapsto (t, x + x_0, y e^{-\sqrt{2}\omega x_0}, z)$;
3) the rotational symmetries $(u, r, \phi, z) \mapsto (u, r + \phi + \phi_0, z)$, in the new coordinates system $(u, r, \phi, z) \in \mathbb{R} \times \mathbb{R}_+ \times (\mathbb{R}/\frac{2\pi}{\omega}\mathbb{Z}) \times \mathbb{R}$ defined by $|t - u| < \pi/\omega$ and:

$$e^{2\sqrt{2}\omega x} = \text{ch}(2r) + \text{sh}(2r) \cos(\omega \phi); e^{\sqrt{2}\omega x} y = \text{sh}(2r) \sin(\omega \phi); \text{tg} \left[ \frac{\omega}{2} (\phi + t - u) \right] = e^{-2r} \text{tg} \left[ \frac{\omega \phi}{2} \right];$$

we have indeed $ds^2 = [du + 2 \text{sh}^2 r \, d\phi]^2 - 2\omega^{-2} dr^2 - \frac{1}{2} \text{sh}^2 (2r) d\phi^2 - dz^2$.

Gödel ([G1], section 4) proved that these three types of isometries generate indeed the full isometry group. As the action of this group is clearly transitive on $\mathbb{R}^4$, Gödel’s universe is an homogeneous space-time.

Letting $\omega$ go to 0, we recover Minkowski space as limit of Gödel’s universe.

### 2.1 Timelike geodesics

Geodesics are associated with the Lagrangian $L(\dot{\xi}, \xi)$, given by:

$$2 L(\dot{\xi}_s, \xi_s) = \dot{t}_s^2 - \dot{x}_s^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} \dot{y}_s^2 + 2 e^{\sqrt{2}\omega x} \dot{t}_s \dot{y}_s - \dot{z}_s^2.$$

The equation of geodesics $\frac{\partial}{\partial s} \left( \frac{\partial L(\dot{\xi}_s, \xi_s)}{\partial \dot{\xi}_s^i} \right) = \frac{\partial L(\dot{\xi}_s, \xi_s)}{\partial \xi_s^i}$ reads here:

1) $\dot{t}_s + e^{\sqrt{2}\omega x} \dot{y}_s = a$;
2) $e^{2\sqrt{2}\omega x} \dot{y}_s + 2 e^{\sqrt{2}\omega x} \dot{t}_s = b$;
3) $\dot{z}_s = c$;
\[ (4) \quad \ddot{x}_s + \left(\frac{\omega}{\sqrt{2}}\right) e^{2\sqrt{2}\omega x_s} \dot{y}_s^2 + \sqrt{2} \omega e^{2\sqrt{2}\omega x_s} \dot{t}_s \dot{y}_s = 0; \]

for constant \( a, b, c \).

Equations (1) and (2) jointly are equivalent to:

\[ (1') \quad \dot{t}_s = b e^{-\sqrt{2}\omega x_s} - a; \quad (2') \quad \dot{y}_s = 2 a e^{-\sqrt{2}\omega x_s} - b e^{-2\sqrt{2}\omega x_s}; \]

and then using Equations (1'), (2'), (3), we see that Equation (0) is equivalent to:

\[ 1 + \left[b e^{-\sqrt{2}\omega x_s} - a\right]^2 + \dot{x}_s^2 + c^2 = \frac{1}{2} b^2 e^{-2\sqrt{2}\omega x_s}, \]

or equivalently:

\[ (0') \quad \dot{x}_s^2 + \frac{1}{2} \left[2 a - b e^{-\sqrt{2}\omega x_s}\right]^2 = a^2 - c^2 - 1. \]

Note that necessarily \( a^2 \geq 1 + c^2 \), and \( ab > 0 \).

Then, owing to Equation (2), Equation (4) is equivalent to:

\[ (4') \quad \frac{\sqrt{2}}{\omega b} \dot{x} + y = Y, \quad \text{for some constant } Y. \]

Note that Equations (1) and (2) imply:

\[ (1'') \quad \ddot{t}_s + \sqrt{2} \omega e^{\sqrt{2}\omega x_s} \dot{x}_s \dot{y}_s + 2 \sqrt{2} \omega \dot{t}_s \dot{x}_s = 0; \quad (2'') \quad \ddot{y}_s - 2 \sqrt{2} \omega e^{-\sqrt{2}\omega x_s} \dot{t}_s \dot{x}_s = 0, \]

so that, using the derivative of unit pseudo-norm Relation (0), we see that Equation (4) is implied by Equations (0), (1), (2), (3), unless \( \dot{x}_s \equiv 0 \).

Setting \( R := a \sqrt{1 - (1 + c^2)/a^2} \) and \( k := R/\sqrt{2a} \in [0, 1 \sqrt{a}] \), we must have by (0'):

\[ \dot{x}_s = R \cos(\omega \varphi_s), \quad b e^{-\sqrt{2}\omega x_s} = 2 a - \sqrt{2} R \sin(\omega \varphi_s), \]

for some angular component \( \varphi_s \), whence:

\[ \dot{\varphi}_s = b e^{-\sqrt{2}\omega x_s} = 2 a - \sqrt{2} R \sin(\omega \varphi_s), \]

and then:

\[ 2 a \omega (s - s_0) = \int_{\varphi_s}^{\varphi_s} \frac{\omega d\varphi}{1 - k \sin(\omega \varphi)} = \frac{2}{\sqrt{1 - k^2}} \arctg \left[ \frac{\tan(\omega \varphi_s/2) - k}{\sqrt{1 - k^2}} \right]. \]

Therefore

\[ \tan(\omega \varphi_s/2) = \sqrt{1 - k^2} \tan(a \sqrt{1 - k^2} (s - s_0)) + k, \]

and

\[ (5) \quad e^{-\sqrt{2}\omega x_s} = \frac{2a}{b} \times \left( 1 - \frac{2k \left[ \tan(a \sqrt{1 - k^2} (s - s_0)) + k \right]}{\sqrt{1 - k^2} \tan(a \sqrt{1 - k^2} (s - s_0)) + k} \right)^2, \]
or equivalently:

\[
(5') \quad x_s = \frac{-1}{\sqrt{2} \omega} \log \left[ \frac{2 a}{b} \times \frac{(1 - k^2) \left( 1 + \tan^2\left[ a \sqrt{1-k^2} \omega (s-s_0) \right] \right)}{1 + \left( \sqrt{1-k^2} \tan\left[ a \sqrt{1-k^2} \omega (s-s_0) \right] + k \right)^2} \right].
\]

Moreover by \((1')\) we have:

\[
i_s = a - \sqrt{2} R \sin(\omega \varphi_s) = \dot{\varphi}_s - a,
\]

whence

\[
(6) \quad t_s = T_0 - a(s-s_0) + \frac{2}{\omega} \arctan \left( \sqrt{1-k^2} \tan\left[ a \sqrt{1-k^2} \omega (s-s_0) \right] + k \right).
\]

In this formula \((6)\), the successive determinations of \(\arctan\), at the successive values \(s \in s_0 + \frac{n}{a \sqrt{1-k^2} \omega} \mathbb{Z}\), are understood to be chosen conveniently, in order that the absolute time coordinate \((t_s)\) be continuous, as it must be. Observe that \((t_s)\) is strictly monotonic if and only if \(k \leq \frac{1}{\omega}\), or equivalently, if and only if \((1 + c^2) \leq a^2 \leq 2(1 + c^2)\).

Finally, by \((4')\) and \((5')\) we have:

\[
(7) \quad y_s = Y + \frac{2 a k}{b \omega} - \frac{4 a k / (b \omega)}{1 + \left( \sqrt{1-k^2} \tan\left[ a \sqrt{1-k^2} \omega (s-s_0) \right] + k \right)^2},
\]

which is consistent with \((2')\):

\[
\dot{y}_s = 4 \frac{a^2 k}{b} \cdot \left[ 1 - k \sin(\omega \varphi_s) \right] \sin(\omega \varphi_s).
\]

Observe from Equations \((5)\) and \((7)\) that every timelike geodesic has a bounded, periodic projection in the \((x, y)\)-plane, and moreover that it obeys the following relation:

\[
(8) \quad \left[ \frac{b}{2 a} e^{-\sqrt{2} \omega x_s} - 1 \right]^2 + \left[ \frac{\omega b}{2 a} (y_s - Y) \right]^2 = k^2.
\]

**Remark 1** The case \(k = 0\) is particular. It implies (using Equations \((5), (0'), (1'), (2')\)):

\(\dot{i}^2 = 1 + \dot{z}^2\) and \(\dot{x} = \dot{y} = 0\), and then:

\((x_s, y_s)\) constant and \(t_s = t_0 + a s, \ z_s = z_0 + c s, \) with \(a^2 = 1 + c^2\).

Reciprocally, if \(\dot{x}_0 = \dot{y}_0 = 0\), then (by Equations \((2'), (0')\)) the corresponding geodesic must satisfy also \(k = 0\), and then be included in the phase subspace defined by:

\[\mathcal{E}_0 := \{ \dot{x} = \dot{y} = 0 \} = \{ \dot{i}^2 = 1 + \dot{z}^2 ; \ \dot{x} = \dot{y} = 0 \}.\]

Therefore, the case \(k = 0\) corresponds to the geodesically stable phase subspace \(\mathcal{E}_0\).

**Remark 2** Every timelike geodesic is defined for all proper times \(s\), unbounded and causal. Moreover, it never accumulates near its past. Indeed, if for proper times \(s < s'\) we had \(x_{s'} = x_s\), then by Equation \((5)\) we should have:

\[
\tan \left[ a \sqrt{1-k^2} \omega (s' - s_0) \right] = \tan \left[ a \sqrt{1-k^2} \omega (s - s_0) \right], \quad \text{then} \quad s' = s + \frac{\pi n}{|a| \sqrt{1-k^2}} \quad \text{with} \quad n \in \mathbb{N}^*,
\]
whence by Equation (6): \( t' - t = \frac{n \pi}{\omega} \left( \frac{2 \pi n}{\omega} - \frac{\pi n}{\omega \sqrt{1 - k^2}} \right) \), and then
\[
|t' - t| = \frac{n \pi}{\omega} \left( 2 - (1 - k^2)^{-1/2} \right) \geq \frac{\pi}{\omega}.
\]

The following statement, which will be used later to ensure the irreducibility of the relativistic diffusion, shows up the non-causal structure of Gödel’s universe, despite the preceding remark 2: the causal past of any point of Gödel’s universe is the whole Gödel’s universe. In particular, the causal boundary, in the sense of Penrose (or Geroch-Kronheimer-Penrose, see ([H-E], section 6.8)), reduces to a single point.

**Proposition 1** The Gödel’s universe \((\mathbb{R}^4, ((g_{ij})))\) is geodesically transitive: any two points of it can be linked by a piece-wise geodesic timelike continuous path.

**Proof** Observe from Remark 1 that (taking \( k = c = 0 \)) there are timelike geodesics moving at will the coordinate \( t \), without changing any other coordinate, and that (taking \( k = 0, c \neq 0 \)) there are timelike geodesics moving at will the coordinate \( z \), without changing the coordinates \((x, y)\).

Hence it will be sufficient to move piece-wise geodesically the coordinates \((x, y)\), forgetting henceforth the coordinates \((t, z)\).

Observe that any given geodesic can move the coordinate \( x \) only by a uniformly bounded value, since by Equation (5) we have:
\[
e^{- \sqrt{2} \omega (x_s - x_s')} = \frac{\left[ 1 + \tan^2(\omega \varphi_s) \right] \left( 1 + \sqrt{1 - k^2} \tan(\omega \varphi_s') + k \right)^2}{\left( 1 + \sqrt{1 - k^2} \tan(\omega \varphi_s) + k \right)^2 \left[ 1 + \tan^2(\omega \varphi_s') \right]}
\],
so that several geodesic arcs are needed. Fix coordinates \((x, y)\), which we want to move to other fixed coordinates say \((x', y')\). By a finite number of geodesic moves, according to the equation displayed just above, we can move \((x, y)\) to \((x', y'')\), for some \( y'' \).

Then, the quotient in the equation displayed just above equals 1 as soon as
\[
\varphi_s + \varphi_s' = \omega^{-1} \arctan \left( \frac{2k \sqrt{1 - k^2}}{(1 + k^2 - \sqrt{1 - k^2})} \right),
\]
so that we can choose a geodesic arc, independently of the value of the parameter \( b \), such that this holds and such that \( \sqrt{1 - k^2} \tan(\omega \varphi_s') + k \neq \sqrt{1 - k^2} \tan(\omega \varphi_s) + k \). Finally, on such geodesic, by Equation (7) we have:
\[
y_s - y_s' = \frac{4a k}{b \omega} \times \left[ \frac{1}{1 + \sqrt{1 - k^2} \tan(\omega \varphi_s') + k} - \frac{1}{1 + \sqrt{1 - k^2} \tan(\omega \varphi_s) + k} \right],
\]
proving that, choosing conveniently the parameter \( b \), we can move \((x', y'')\) to \((x', y')\). \( \diamond \)
2.2 Lightlike geodesics

Equations (1), (2), (3), (4) remain the same, while the pseudo-norm equation (0) is replaced by:

\[ (0'') \quad \frac{\dot{t}_s^2 + \dot{x}_s^2 + \dot{z}_s^2}{\omega b} = \frac{1}{\omega} \left[ e^{\sqrt{2}\omega x_s} \dot{y}_s + 2 \dot{t}_s \right]^2. \]

As previously, knowing Equations (1) and (2), Equation (4) is equivalent to

\[ (4') \quad \frac{\sqrt{2}}{\omega b} \dot{x} + \dot{y} = Y, \quad \text{for some constant } Y, \]

and is implied by Equations \((0''), (1), (2), (3)\). Thus lightlike geodesics are the solutions to the system:

\[ (1') \quad \dot{t}_s = b e^{-\sqrt{2}\omega x_s} - a; \quad (2') \quad \dot{y}_s = 2a e^{-\sqrt{2}\omega x_s} - b e^{-2\omega x_s}; \quad (3) \quad \dot{z}_s = c; \]

and

\[ (0''') \quad \dot{x}_s^2 + \frac{1}{2} \left[ 2a - b e^{-\sqrt{2}\omega x_s} \right]^2 = a^2 - c^2. \]

Hence we must have again \(a^2 \geq c^2\), and \(ab > 0\) (if not, the trajectory must be constant), and, setting now

\[ \kappa := \sqrt{\frac{1}{2}(1-c^2/a^2)} \in [0, \frac{1}{\sqrt{2}}], \]

we get the same equations \((5), (6), (7), (8)\) as above, merely with \(\kappa\) replacing \(k\).

As the parameter \(s\) cannot any longer have here a meaning like proper time, but stands only for an affine parameter, determined up to a change \(s \mapsto u + v\), the constant \((a, b, c)\) is now irrelevant. Note that on the contrary, the constant \(Y\) of Equation \((4')\) is consistent.

The only meaningful a priori parameter for a lightlike geodesic is the “impact” parameter:

\[ (9) \quad B = (\ell, \varrho, Y) := \left( \frac{c}{a}, \frac{b}{a}, Y \right) \in B := [-1, 1] \times \mathbb{R}^* \times \mathbb{R}. \]

Eliminating \(s\), we see indeed that a lightlike geodesic having impact parameter \(B\) solves:

\[ \ell \, dt = (\varrho e^{-\sqrt{2}\omega x} - 1) \, dz; \quad (2 - \varrho e^{-\sqrt{2}\omega x}) e^{-\sqrt{2}\omega x} \, dt = (\varrho e^{-\sqrt{2}\omega x} - 1) \, dy; \]

\[ (\varrho e^{-\sqrt{2}\omega x} - 1) \, dx = \pm \sqrt{1 - \ell^2 - \frac{1}{2}(2 - \varrho e^{-\sqrt{2}\omega x})^2} \, dt. \]

Let us sum up the description of lightlike geodesics in the following statement.

**Proposition 2** Any lightlike geodesic \((x_\tau, y_\tau, z_\tau, t_\tau)\) having impact parameter \(B = (\ell, \varrho, Y) \in B\) satisfies, for an additional parameter \((Z_0, T_0) \in \mathbb{R}^2\) and for any real \(\tau:\)

\[ e^{-\sqrt{2}\omega x_\tau} = \frac{2}{\varrho} \times \left( 1 - \frac{2 \sqrt{1 - \ell^2} (\sqrt{1 + \ell^2} \, \text{tg} \, \tau + \sqrt{1 - \ell^2})}{2 + \left( \sqrt{1 + \ell^2} \, \text{tg} \, \tau + \sqrt{1 - \ell^2} \right)^2} \right); \]
\[ y_\tau = Y + \frac{\sqrt{2(1 - \ell^2)}}{\omega \vartheta} \left(1 - \frac{4}{2 + \left(\sqrt{1 + \ell^2} \tan \tau + \sqrt{1 - \ell^2}\right)^2}\right); \]

\[ z_\tau = Z_0 + \frac{\ell \tau}{\omega \sqrt{1 + \ell^2} / 2}; \]

\[ t_\tau = T_0 - \frac{\tau}{\omega \sqrt{1 + \ell^2} / 2} + \frac{2}{\omega} \arctg \left(\frac{\sqrt{1 + \ell^2} / 2 \tan \tau + \sqrt{1 - \ell^2} / 2}{\sqrt{1 + \ell^2} / 2}\right); \]

\[ C_B : \quad \left[\frac{\varrho}{2} e^{-\sqrt{2} \omega x_\tau} - 1\right]^2 + \left[\frac{\omega \varrho}{2} (y_\tau - Y)\right]^2 = \frac{1 - \ell^2}{2}. \]

**Remark 3** The last equation in Proposition 2 shows that to any given lightlike geodesic is associated a cylinder \( C_B \), parallel to the \((t, z)\)-coordinate plane. Reciprocally, by Proposition 2 again, any lightlike geodesic which is drawn on the cylinder \( C_B \) has a prescribed projection on the \((x, y)\)-coordinate plane (up to changing affine parameter \( \tau \)). The equations displayed in Proposition 2 define a lightlike geodesic associated to any given \( B \in \mathcal{B} \).

Considering then any continuous angular parameter \( \varphi = \varphi_\tau \) (determined modulo \( 2\pi/\omega \)) such that:

\[ \tan \left(\frac{\omega \varphi_\tau}{2}\right) = \sqrt{\frac{1 + \ell^2}{2}} \tan \tau + \sqrt{\frac{1 - \ell^2}{2}}, \]

then by Proposition 2 we have:

\[ \varrho e^{-\sqrt{2} \omega x_\tau} = 2 - \sqrt{2(1 - \ell^2)} \sin(\omega \varphi_\tau) \quad \text{and} \quad \omega \varrho y_\tau = \omega \varrho Y - \sqrt{2(1 - \ell^2)} \cos(\omega \varphi_\tau), \]

together with:

\[ t_\tau = T_0 - \frac{\tau}{\omega \vartheta \sqrt{1 + \ell^2} / 2} + \varphi_\tau, \quad z_\tau + \ell t_\tau = Z_0 + \ell T_0 + \ell \varphi_\tau. \]

Since the function \( \tau \mapsto \omega \varphi_\tau - 2 \tau \) is \( \pi \)-periodic, the functions

\[ \tau \mapsto \omega t_\tau - 2 \left(1 - [2(1 + \ell^2)]^{-1/2}\right) \tau \quad \text{and} \quad \tau \mapsto z_\tau - \frac{\ell t_\tau}{\sqrt{2(1 + \ell^2)}} = Z_0 - \frac{\ell(T_0 + \varphi_\tau - 2\tau/\omega)}{\sqrt{2(1 + \ell^2)} - 1} \]

are \( \pi \)-periodic too. This implies that \( t_\tau \) wanders out to infinity, nearly linearly, and that the projection on the \((t, z)\)-coordinate plane of any lightlike geodesic has an asymptotic direction:

\[ \lim_{\tau \to \pm \infty} \frac{z_\tau}{t_\tau} = \frac{\ell}{\sqrt{2(1 + \ell^2)} - 1}. \]

This prescribes geometrically the sign of parameter \( \ell \), which was not determined by the cylinder \( C_B \) alone, which however determines \((|\ell|, \varrho, Y)\). Note that the impact parameter \( B = (\ell, \varrho, Y) \in \mathcal{B} \) has thus indeed a clear geometrical meaning.

Note finally that the additional parameter \((Z_0, T_0)\) depends on a translation on the parameter \((\tau, \varphi_\tau)\), and then, contrary to \( B = (\ell, \varrho, Y) \), is geometrically irrelevant.
Recall that in a strongly causal space-time, it seems natural to use the causal boundary, in the sense of Penrose, to classify lightlike geodesics by gathering in an equivalence class, called a light ray, all geodesics which converge to a given causal boundary point (having asymptotically the same past, see ([H-E], section 6.8)). On the contrary, in the present setting (recall Proposition 11), such classification is totally inoperative. It seems that no alternative classification has been proposed so far, which be relevant in a non-causal setting.

However, owing to the above remark 3, we are led to adopt here the following alternative classification of lightlike geodesics into light rays, and then also, to see the 3-dimensional space of light rays as an alternative notion of (non-causal) boundary, as follows.

**Definition 2** Let us call light ray, or boundary point, of Gödel’s universe, any equivalence class of lightlike geodesics, identifying those which have the same impact parameter \( B = (\ell, \varrho, Y) \in \mathcal{B} \). Thus \( \mathcal{B} = [-1, 1] \times \mathbb{R}_+ \times \mathbb{R} \) is the boundary of Gödel’s universe.

Let us say that a path \( s \mapsto \xi_s = (t_s, x_s, y_s, z_s) \) of class \( C^1 \) in Gödel’s universe

converges to the light ray \( B = (\ell, \varrho, Y) \) if, setting:

\[
\begin{align*}
a_s &= t_s + e^{\sqrt{2} \omega x_s} \dot{y}_s, & b_s &= e^{\sqrt{2} \omega x_s} (2 \dot{t}_s + e^{\sqrt{2} \omega x_s} \dot{y}_s), & Y_s &= \frac{\sqrt{2} \dot{x}_s}{\omega b_s} + y_s,
\end{align*}
\]

the following convergences hold, as \( s \to +\infty \):

\[
\frac{\dot{z}_s}{a_s} \to \ell, \quad \frac{b_s}{a_s} \to \varrho, \quad Y_s \to Y, \quad \left[ \frac{\varrho}{2} e^{\sqrt{2} \omega x_s} - 1 \right]^2 + \left[ \frac{\omega}{2} (y_s - Y) \right]^2 \to \frac{1 - \ell^2}{2}.
\]

This notion of convergence to the boundary \( \mathcal{B} \) can be reinforced to a certain extent: see Remark 10, concluding this article. We saw above that any lightlike geodesic belonging to a light ray \( B \) converges to it. On the contrary, a timelike geodesic does not converge to any light ray: by Section 2.1, we get indeed \( \mathcal{B} = (\frac{\ell}{\varrho}, \frac{\varrho}{\ell}, Y) \in \mathcal{B} \), but the cylinder \( \mathcal{C}_B \) has to small a “radius”, since by Equation (8) we have \( k^2 = [1 - \ell^2 - a^{-2}] / 2 < [1 - \ell^2] / 2 \).

**Proposition 3** The isometry group of Gödel’s universe (recall Section 2) does operate on the boundary \( \mathcal{B} \), so that the above definition is consistent. Precisely, it works as follows.

1. The translation \( (t, x, y, z) \mapsto (t + t_0, x, y + y_0, z + z_0) \) changes \((\ell, \varrho, Y)\) into \((\ell, \varrho, Y + y_0)\).
2. The hyperbolic dilatation \( (t, x, y, z) \mapsto (t, x + x_0, y e^{-\sqrt{2} \omega x_0}, z) \) changes \((\ell, \varrho, Y)\) into

\[
(\ell, \varrho e^{-\sqrt{2} \omega x_0}, Y e^{-\sqrt{2} \omega x_0}).
\]
3. The rotational symmetry \( (u, r, \phi, z) \mapsto (u, r, \phi + \phi_0, z) \) changes \((\ell, \varrho, Y)\) into

\[
(\ell, \alpha + [\varrho - \alpha] \cos(\omega \phi_0) - \omega \varrho Y \sin(\omega \phi_0)), \quad \frac{\omega \varrho \cos(\omega \phi_0) + [\varrho - \alpha] \sin(\omega \phi_0)}{\omega [\alpha + [\varrho - \alpha] \cos(\omega \phi_0) - \omega \varrho \sin(\omega \phi_0)]},
\]

where \( \alpha := \frac{2 \arctan(2r)}{u + 2 \sqrt{r^2 - 1}} \) is constant under \( \phi \mapsto \phi + \phi_0 \), and on each geodesic. We have indeed: \( \alpha = \frac{\varrho}{2} (1 + \omega^2 Y^2) + \frac{1 + \ell^2}{e} \), or equivalently: \( \varrho - \alpha = \frac{\varrho}{2} (1 - \omega^2 Y^2) - \frac{1 + \ell^2}{e} \).
The two first items are straightforward. On the other hand, the action of the rotational symmetry \((u, r, \phi, z) \mapsto (u, r, \phi + \phi_0, z)\) is not so obvious. However, a computation shows that we have in coordinates \((u, r, \phi, z)\): \(a = \dot{u} + 2(\text{sh} r)^2 \dot{\phi}\), and:

\[
b = A + \Psi \cos(\omega \phi) - 2\omega^{-1}\dot{r} \sin(\omega \phi), \quad Z := \omega b Y = \Psi \sin(\omega \phi) + 2\omega^{-1}\dot{r} \cos(\omega \phi),
\]

with

\[
A := 2a \text{ch}(2r) - \text{sh}^2(2r) \dot{\phi} \quad \text{and} \quad \Psi := [2a - \text{ch}(2r) \dot{\phi}] \text{sh}(2r).
\]

Note that \(A = 2a + 4[a - \text{ch}^2 r \dot{\phi}] \text{sh}^2 r = 2a + 2 \frac{\partial L}{\partial \dot{r}}\) is seen to be constant on each geodesic, by looking at the expression of the Lagrangian \(L\) in coordinates \((u, r, \phi, z)\). Alternatively, a computation yields: \(2A = b + \omega^2 b (2Y y - y^2) + (4a - b e^{-\sqrt{2} \omega x}) e^{-\sqrt{2} \omega x}\), whence by using Proposition 2:

\[
2\alpha - \varrho = \frac{2A}{a} - \varrho = \omega^2 \varrho \left[ Y^2 - \frac{2(1 - \ell^2)}{\omega^2} \left[ 1 - \frac{4}{2 + \text{tg}^2(\frac{\omega \varphi}{2})} \right]^2 \right] + \frac{1}{\varrho} \left[ 4 - \frac{4\sqrt{1 - \ell^2} \text{tg}^2(\frac{\omega \varphi}{2})}{2 + \text{tg}^2(\frac{\omega \varphi}{2})} \right]^2,
\]

\[
= \omega^2 \varrho Y^2 + \frac{4}{\varrho} - 2 \frac{(1 - \ell^2)}{\varrho}, \quad \text{whence} \quad \alpha = \frac{\varrho}{2} (1 + \omega^2 Y^2) + \frac{1 + \ell^2}{\varrho}.
\]

Now we have at once: under \(\phi \mapsto \phi + \phi_0\), \((a, b, Z)\) is changed into

\[
(a, A + [b - A] \cos(\omega \phi_0) - Z \sin(\omega \phi_0), Z \cos(\omega \phi_0) + [b - A] \sin(\omega \phi_0)),
\]

so that \((\ell, \varrho, Y)\) is changed into (recall from the above that \(\alpha = A/a\)):

\[
\left( \ell, \frac{A}{a} + [\varrho - \frac{A}{a}] \cos(\omega \phi_0) - \varrho Y \sin(\omega \phi_0), \frac{\omega \varrho \cos(\omega \phi_0) + [\varrho - \frac{A}{a}] \sin(\omega \phi_0)}{\omega \left[ \frac{A}{a} + [\varrho - \frac{A}{a}] \cos(\omega \phi_0) - \varrho \sin(\omega \phi_0) \right]} \right).
\]

### 2.3 Ricci curvature and energy tensor

Recall that the Christoffel symbols are computed by:

\[
\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left( \frac{\partial g_{ij}}{\partial \xi^\ell} + \frac{\partial g_{ij}}{\partial \xi^\ell} - \frac{\partial g_{ij}}{\partial \xi^\ell} \right),
\]

and that the Ricci tensor \((R_{ij})\) is computed by:

\[
R_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial \xi^k} - \frac{\partial \Gamma^k_{ik}}{\partial \xi^j} + \Gamma^k_{ik} \Gamma^l_{lj} - \Gamma^k_{il} \Gamma^l_{jk}.
\]

From Equations (1'), (2'), (3), we find all non-vanishing Christoffel coefficients:

\[
\Gamma^t_{xy} = \Gamma^y_{tx} = (\omega/\sqrt{2}) e^{\sqrt{2} \omega x}, \quad \Gamma^t_{tx} = \sqrt{2} \omega, \quad \Gamma^y_{yy} = (\omega/\sqrt{2}) e^{2\sqrt{2} \omega x}, \quad \Gamma^y_{tx} = -\sqrt{2} \omega e^{-\sqrt{2} \omega x}.
\]
Therefore we get all non-vanishing Ricci coefficients:

\[ R_{tt} = 2\omega^2; \quad R_{ty} = R_{yt} = 2\omega^2 e^{\sqrt{2}\omega x}; \quad R_{yy} = 2\omega^2 e^{2\sqrt{2}\omega x}. \]

Hence, the scalar curvature is \( R = g^{ij} R_{ij} = 2\omega^2. \)

Einstein equations

\[ R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = T_{ij} \]

are satisfied, with cosmological constant \( \Lambda = \omega^2 \) representing a positive pressure, and energy tensor \( (T_{ij}) = (R_{ij}) = (u_i u_j) \), where \( u := (\sqrt{2}\omega, 0, \sqrt{2}\omega e^{\sqrt{2}\omega x}, 0) \) represents the four-velocity of matter, which rotates with constant velocity \( \omega \). The energy is thus:

\[ E(\xi, \dot{\xi}) := T_{ij}(\xi) \dot{\xi}^i \dot{\xi}^j = 2\omega^2 \left[ \dot{t} + e^{\sqrt{2}\omega x} \dot{y} \right]^2 = 2\omega^2 a(\xi, \dot{\xi})^2. \]

3 Relativistic diffusion

Recall from [F-LJ] that the general expression of the relativistic operator \( \mathcal{L} \) is:

\[ \mathcal{L} = \dot{\xi}^k \frac{\partial}{\partial \xi^k} + \left( \frac{3\sigma^2}{2} \dot{\xi}^k - \ddot{\xi}^i \dot{\xi}^j \Gamma^k_{ij}(\xi) \right) \frac{\partial}{\partial \xi^k} + \frac{\sigma^2}{2} (\dot{\xi}^k \dot{\xi}^l - g^{kl}(\xi)) \frac{\partial^2}{\partial \xi^k \partial \xi^l}, \]

\( \sigma \) being an arbitrary fixed positive (speed or heat) parameter.

Equivalently in the present setting, the relativistic diffusion \( (\xi_s, \dot{\xi}_s) \), in coordinates \( \xi = (t, x, y, z) \), solves the following system of stochastic differential equations:

\[
\begin{align*}
    dt_s &= \dot{t}_s \, ds; \quad dx_s = \dot{x}_s \, ds; \quad dy_s = \dot{y}_s \, ds; \quad dz_s = \dot{z}_s \, ds; \\
    dt_s &= -2\sqrt{2} \omega t_s \dot{x}_s \, ds + \sqrt{2} \omega e^{\sqrt{2}\omega x} \dot{x}_s \dot{y}_s \, ds + \frac{3\sigma^2}{2} \dot{t}_s \, ds + \sigma dM^t_s; \\
    d\dot{x}_s &= -\sqrt{2} \omega e^{\sqrt{2}\omega x} \dot{t}_s \dot{y}_s \, ds - (\omega/\sqrt{2}) e^{2\sqrt{2}\omega x} \dot{x}_s \dot{y}_s \, ds + \frac{3\sigma^2}{2} \dot{\dot{x}}_s \, ds + \sigma dM^x_s; \\
    d\dot{y}_s &= 2\sqrt{2} \omega e^{-\sqrt{2}\omega x} \dot{t}_s \, ds + \frac{3\sigma^2}{2} \dot{\dot{y}}_s \, ds + \sigma dM^y_s; \\
    d\dot{z}_s &= \frac{3\sigma^2}{2} \dot{\dot{z}}_s \, ds + \sigma dM^z_s;
\end{align*}
\]

where the \( \mathbb{R}^4 \)-valued martingale \( M_s := (M^t_s, M^x_s, M^y_s, M^z_s) \) has (rank 3) quadratic covariation matrix:

\[
((K_s^{ij})) := \frac{\langle dM^i_s, dM^j_s \rangle}{ds} = \begin{pmatrix}
    \dot{t}_s^2 + 1 & \dot{t}_s \dot{x}_s & \dot{t}_s \dot{y}_s - 2 e^{-\sqrt{2}\omega x} & \dot{t}_s \dot{z}_s \\
    \dot{t}_s \dot{x}_s & \dot{x}_s^2 + 1 & \dot{x}_s \dot{y}_s & \dot{x}_s \dot{z}_s \\
    \dot{t}_s \dot{y}_s - 2 e^{-\sqrt{2}\omega x} & \dot{x}_s \dot{y}_s & \dot{y}_s^2 + 2 e^{-2\sqrt{2}\omega x} & \dot{y}_s \dot{z}_s \\
    \dot{t}_s \dot{z}_s & \dot{x}_s \dot{z}_s & \dot{y}_s \dot{z}_s & \dot{z}_s^2 + 1
\end{pmatrix}.
\]

Recall that the unit pseudo-norm relation reads:

\[
0 \quad 1 + \dot{t}_s^2 + \dot{x}_s^2 + \dot{z}_s^2 = \frac{1}{2} \left[ e^{\sqrt{2}\omega x} \dot{y}_s + 2 \dot{t}_s \right]^2.
\]
Thus, the relativistic diffusion $\left(\xi_s, \dot{\xi}_s\right)$ is 7-dimensional, having phase space:

$$\mathcal{E} := \{(t, x, y, z, t, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^8 \mid 1 + t^2 + x^2 + z^2 = \frac{1}{2} \left[e^{\sqrt{2} \omega x} \dot{y} + 2 \dot{t} \right]^2\},$$

or equivalently:

$$\mathcal{E} = \{(t, x, y, z, \dot{t}, \dot{x}, \dot{y}, \dot{z}) \in \mathbb{R}^8 \mid 1 + \dot{x}^2 + \dot{z}^2 + \frac{1}{2} e^{2\sqrt{2} \omega x} \dot{y}^2 = \left[t + e^{\sqrt{2} \omega x} \dot{y}\right]^2\}.$$  

Note that the particular phase subspace distinguished in Remark 1:

$$\mathcal{E}_0 := \mathcal{E} \cap \{ \dot{x} = \dot{y} = 0 \} = \mathcal{E} \cap \{ \dot{t}^2 = 1 + \dot{z}^2 ; \dot{x} = \dot{y} = 0 \}$$

is clearly not stable under the relativistic diffusion $\left(\xi_s, \dot{\xi}_s\right)$, contrary to the geodesic flow, and even instantly unstable: starting from any point in $\mathcal{E}_0$, its exit time from $\mathcal{E}_0$ is null.

### 3.1 Reduction of the dimension

The study of geodesics induces to consider the following quantities (which, as $\dot{z}_s$, are constant along each geodesic), setting (as in Definition 2):

$$\tag{10} a_s := \dot{t}_s + e^{\sqrt{2} \omega x} \dot{y}_s \quad \text{and} \quad b_s := e^{\sqrt{2} \omega x} (2 \dot{t}_s + e^{\sqrt{2} \omega x} \dot{y}_s).$$

Then we have:

$$da_s = \frac{3\sigma^2}{2} a_s \, ds + \sigma \, dM^a_s = \frac{3\sigma^2}{2} a_s \, ds + \sigma \left( dM^t_s + e^{\sqrt{2} \omega x} dM^y_s \right) ;$$

and

$$db_s = \frac{3\sigma^2}{2} b_s \, ds + \sigma \, dM^b_s = \frac{3\sigma^2}{2} b_s \, ds + \sigma e^{\sqrt{2} \omega x} \left( 2 dM^t_s + e^{\sqrt{2} \omega x} dM^y_s \right).$$

Moreover we have:

$$d\dot{x}_s = (\omega/\sqrt{2}) e^{-2\sqrt{2} \omega x} b_s^2 \, ds - \sqrt{2} \omega e^{-\sqrt{2} \omega x} a_s b_s \, ds + \frac{3\sigma^2}{2} \dot{x}_s \, ds + \sigma \, dM^x_s,$$

and the $\mathbb{R}^4$-valued martingale $\tilde{M}_s := (M^a_s, M^b_s, M^x_s, M^z_s)$ has (rank 3) quadratic covariation matrix:

$$\left((\tilde{K}_s^{ij})\right) = \begin{pmatrix}
  a_s^2 - 1 & a_s b_s - 2 e^{\sqrt{2} \omega x} & a_s \dot{x}_s & a_s \dot{z}_s \\
  a_s b_s - 2 e^{\sqrt{2} \omega x} & b_s^2 - 2 e^{2\sqrt{2} \omega x} & b_s \dot{x}_s & b_s \dot{z}_s \\
  a_s \dot{x}_s & b_s \dot{x}_s & \dot{x}_s^2 + 1 & \dot{x}_s \dot{z}_s \\
  a_s \dot{z}_s & b_s \dot{z}_s & \dot{x}_s \dot{z}_s & \dot{z}_s^2 + 1
\end{pmatrix}.$$  

From this, we deduce the following.

**Corollary 1** The (7-dimensional) relativistic diffusion $\left(\xi_s, \dot{\xi}_s\right)$ admits the following sub-diffusions: $\left(a_s\right); \quad (\dot{z}_s); \quad (z_s, \dot{z}_s); \quad (a_s, \dot{z}_s); \quad (a_s, z_s, \dot{z}_s); \quad (x_s, \dot{x}_s, a_s, b_s).$
The unit pseudo-norm relation can be written:

\[(00) \quad 1 + \dot{x}_s^2 + \dot{z}_s^2 + (a_s - e^{-\sqrt{2}\omega x} b_s)^2 = \frac{1}{2} e^{-2\sqrt{2}\omega x} b_s^2, \]

or equivalently:

\[(00') \quad 1 + \dot{x}_s^2 + \dot{z}_s^2 + \frac{1}{2} (2 a_s - e^{-\sqrt{2}\omega x} b_s)^2 = a_s^2. \]

Hence the phase space \(\mathcal{E}\) of the relativistic diffusion \((\xi_s, \dot{\xi}_s)\) can be written equivalently:

\[\mathcal{E} = \left\{ (t, x, y, z, a, b, \dot{x}, \dot{z}) \in \mathbb{R}^8 \mid 1 + \dot{x}^2 + \dot{z}^2 + \frac{1}{2} (2 a - e^{-\sqrt{2}\omega x} b)^2 = a^2 \right\}. \]

And the particular phase subspace \(\mathcal{E}_0\) distinguished in Remark 4 can be written:

\[\mathcal{E}_0 = \mathcal{E} \cap \left\{ a^2 = 1 + \dot{z}^2 ; 2 a = e^{-\sqrt{2}\omega x} b ; \dot{x} = 0 \right\}. \]

Remark 4 We see in particular that \(a_s^2 \geq 1\) and that \(b_s^2 \geq 2 e^{2\sqrt{2}\omega x_s}\), for any proper time \(s \geq 0\). Therefore, \((a_s)\) and \((b_s)\) almost surely never vanish. Moreover, they must have the same sign, since \((00')\) implies \(\left| e^{-\sqrt{2}\omega x_s} \frac{b_s}{a_s} - 2 \right| \leq \sqrt{2}\) and then \(e^{-\sqrt{2}\omega x_s} \frac{b_s}{a_s} \geq 2 - \sqrt{2}\).

This implies also \(\left| e^{\sqrt{2}\omega x_s} \frac{a_s}{b_s} - 1 \right| \leq \frac{1}{\sqrt{2}}\).

### 3.2 Study of the one-dimensional sub-diffusions \((a_s)\) and \((\dot{z}_s)\)

These two one-dimensional sub-diffusions are easily handled.

**Lemma 1** There exist two standard real Brownian motions \((w_s)\) and \((w'_s)\), and two almost surely converging processes \((\eta_s)\) and \((\eta'_s)\), such that we have:

\[|a_s| = \exp \left[ \sigma^2 s + \sigma w_s + \eta_s \right] \quad \text{for any proper time } s \geq 0, \]

and

\[|\dot{z}_s| = \exp \left[ \sigma^2 s + \sigma w'_s + \eta'_s \right] \quad \text{for any sufficiently large proper time } s. \]

**Proof** The stochastic differential equations satisfied by \((a_s)\) and \((\dot{z}_s)\) are respectively:

\[da_s = \frac{3\sigma^2}{2} a_s ds + \sigma \sqrt{a_s^2 - 1} dw_s, \quad \text{and} \quad d\dot{z}_s = \frac{3\sigma^2}{2} \dot{z}_s ds + \sigma \sqrt{\dot{z}_s^2 + 1} dw'_s, \]

for two standard real Brownian motions \((w_s)\) and \((w'_s)\).

These equations are solved as follows; we have real Brownian motions \((W_u)\) and \((W'_u)\) such that:

\[a_s = F\left( W\left[ \inf \left\{ u \mid \int_0^u (W_v^2 - 1)^{-2} dv > \sigma s \right\} \right] \right). \]
and
\[ \dot{z}_s = G\left(W' \left| \inf \left\{ u \left| \int_0^u (1 - |W''_v|^2)^{-2} dv > \sigma^2 s \right\} \right. \right), \]
with \( F(W) := \frac{W}{\sqrt{1 - |W'|^2}} \) and \( G(W') := \frac{W'}{\sqrt{1 - |W'|^2}} \).

Clearly, as \( u \) increases to the hitting time of 1 by \( W \), then \( \int_0^u (W_u^2 - 1)^{-2} dv \) increases to infinity, \( W[u] \) goes to 1, and \( F(W[u]) \) goes to infinity, showing that \( a_s \) goes almost surely to infinity with \( s \). The same reasoning holds for \( (\dot{z}_s) \) (except that \( |W'_0| \) must be smaller than 1, while \( |W_0| \) must be larger than 1), so that, in the same way, \( |\dot{z}_s| \) goes almost surely to infinity with \( s \). The invariant measure of the diffusion \( (a_s) \) is \( \frac{1}{\{a > 1\}} \sqrt{a^2 - 1} da \), and the invariant measure of the diffusion \( (\dot{z}_s) \) is \( \sqrt{z^2 + 1} dz \).

Then we have almost surely (since \( |\dot{z}_s| \to \infty \)), for any sufficiently large proper time \( s \):
\[ d \log |a_s| = (1 + \frac{1}{2a^2}) \sigma^2 ds + \sigma \sqrt{1 - a^{-2}} dw_s, \]
and \( d \log |\dot{z}_s| = (1 - \frac{1}{2a^2}) \sigma^2 ds + \sigma \sqrt{1 + \dot{z}_s^{-2}} dw'_s \).

Whence, for real Brownian motions \( \tilde{w}, \tilde{w}' \) and for sufficiently large proper times \( s_0, s \):
\[ \log |a_s| = \log |a_{s_0}| + \sigma^2 s + \sigma w_s + \int_{s_0}^s \frac{\sigma^2 du}{2a^2} - \tilde{w} \left[ \int_{s_0}^s \frac{\sigma^2 a^{-2}_u du}{1 + \sqrt{1 - a^{-2}_u}} \right] = \sigma^2 s + o(s) > \sigma^2 s/2, \]
and similarly:
\[ \log |\dot{z}_s| = \sigma^2 (s - s_0) + \sigma (w'_s - w'_{s_0}) - \int_{s_0}^s \frac{\sigma^2 du}{2\dot{z}_u^2} + \tilde{w}' \left[ \int_{s_0}^s \frac{\sigma^2 \dot{z}_u^{-2} du}{1 + \sqrt{1 + \dot{z}_u^{-2}}} \right] = \sigma^2 s + o(s) > \sigma^2 s/2. \]
This implies the convergence of the integrals in the above formulas, and then the result. 

\[ \textbf{Corollary 2} \quad \text{For any sufficiently large proper time} \ s \ \text{we have} : \ |z_s| = e^{\sigma^2 s + o(s^{1/2})}. \]

We have also the following lower control, which we shall use later.

\[ \textbf{Lemma 2} \quad \text{For any} \ A > \sqrt{3}, \ \text{we have} \ \mathbb{P} \left( \exists \ s > 0 \ |\dot{z}_s| \leq A e^{\sigma^2 s/2} / |\dot{z}_0| \geq A^2 \right) < 1/\sqrt{A}, \]
and \( \mathbb{P} \left( \exists \ s > 0 \ |a_s| \leq A e^{\sigma^2 s/2} / |a_0| \geq A^2 \right) < 1/A. \)

\[ \textbf{Proof} \quad \text{Fix} \ A > \sqrt{3} \ \text{and} \ |\dot{z}_0| \geq A^2. \ \text{The stochastic differential equation satisfied by} \ (\log |\dot{z}_s|), \ \text{already written in the proof of Lemma} \text{ is equivalent to:} \]
\[ d \log |\dot{z}_s e^{-\sigma^2 s/2}| = \frac{\sigma^2}{2} \left( 1 - \dot{z}_s^{-2} \right) ds + \sigma \sqrt{1 + \dot{z}_s^{-2}} dw'_s. \]

Let us apply the comparison theorem (see for example ([I-W], Theorem 4.1)): setting
\[ T_A := \inf \{ s \ | \ |\dot{z}_s| = A e^{\sigma^2 s/2} \} \quad \text{and} \quad \log r_s := \log A^2 + \frac{\sigma^2}{2} \left( 1 - A^{-2} \right) s + \sigma \sqrt{1 + A^{-2}} w'_s, \]
we have: \[
\inf_{0 \leq s \leq T_A} | \dot{z}_s e^{-\sigma^2 s/2} | \geq \inf_{0 \leq s \leq T_A} r_s , \quad \text{whence}
\]
\[
\mathbb{P}[T^z_A < \infty] \leq \mathbb{P} [\log r_s \text{ hits } \log A] = \mathbb{P} [w' - \frac{1}{1+A}-2 s/2 \text{ hits } \log A] = A^{-\frac{1}{1+A}-2} < 1/\sqrt{A} . \]
Similarly, for \(|a_0| \geq A^2\) and \(T^a_A := \inf\{u \mid |a_u| = A e^{\sigma^2 s/2}\}\), since
\[
d|a_s e^{-\sigma^2 s/2}| = \frac{\sigma^2}{2} a_s^{-2} \, ds + \sigma \sqrt{1 - a_s^{-2}} \, dw_s ,
\]
we get:
\[
\mathbb{P}[T^a_A < \infty] \leq \mathbb{P} [w_s - s/2 \text{ hits } \log A] = 1/A . \diamond
\]

### 3.3 Study of the two-dimensional sub-diffusion \((a_s, \dot{z}_s)\)

Recall from Section 3.1 that we have:
\[
a_s^2 - 1 \geq \dot{z}_s^2 ,
\]
due to the \(\mathbb{R}^2\)-valued martingale \(\dot{M}_s := (M^a_s, M^z_s)\) has quadratic covariation matrix:
\[
((K^{ij}_s)) = \begin{pmatrix} a_s^2 - 1 & a_s \dot{z}_s \\ a_s \dot{z}_s & \dot{z}_s^2 + 1 \end{pmatrix} .
\]

We get easily an asymptotic variable of the sub-diffusion \((a_s, \dot{z}_s)\).

**Proposition 4** The process \((\dot{z}_s/a_s)\) converges almost surely, toward some random limit \(\ell\) such that \(0 < |\ell| \leq 1\).

**Proof** Using Itô’s Formula and the above expressions for \(da_s\) and \(d\dot{z}_s\), we get:
\[
d\left[\frac{\dot{z}_s}{a_s}\right] = \frac{d\dot{z}_s}{a_s} - \frac{\dot{z}_s}{a_s^2} \, da_s + \frac{\dot{z}_s}{a_s^2} \, (da_s) + \frac{\langle da_s, d\dot{z}_s\rangle}{a_s^2} = \frac{\sigma}{a_s} \left[ dM^z_s - \frac{\dot{z}_s}{a_s} \, dM^a_s \right] - \frac{\sigma^2}{a_s^3} \, \dot{z}_s \, ds ,
\]
with
\[
\langle a_s^{-1} \left[ dM^z_s - \frac{\dot{z}_s}{a_s} \, dM^a_s \right] \rangle = \frac{\ln |a_s|}{a_s^2} \, ds .
\]
Hence, we have some real Brownian motion \(\dot{W}\) such that almost surely, for any \(s \geq 0\):
\[
\frac{\dot{z}_s}{a_s} = \frac{\dot{z}_0}{a_0} - \sigma^2 \int_0^s \frac{\dot{z}_u}{a_u^2} \, du + \sigma \dot{W} \left[ \int_0^s \frac{1 - |\dot{z}_u/a_u|^2}{a_u^2} \, du \right] ,
\]
which almost surely converges toward some random limit \(\ell \in \mathbb{R}\), by Lemma [\(\Box\)] The unit pseudo-norm relation \((00')\) implies \(a_s^2 - 1 \geq \dot{z}_s^2\), and then \(\ell^2 \leq 1\).

Similarly, using Lemma [\(\Box\)] again, for any large enough proper time \(s\) we have:
\[
d\left[\frac{a_s}{\dot{z}_s}\right] = \frac{da_s}{\dot{z}_s} - \frac{a_s \, d\dot{z}_s}{\dot{z}_s^2} + \frac{a_s \, (d\dot{z}_s)}{\dot{z}_s^3} - \frac{\langle da_s, d\dot{z}_s\rangle}{\dot{z}_s^2} = \frac{\sigma}{\dot{z}_s} \left[ dM^a_s - \frac{a_s}{\dot{z}_s} \, dM^z_s \right] + \frac{\sigma^2}{\dot{z}_s^3} a_s \, ds ,
\]
\[15\]
with
\[ \langle \hat{z}_s^{-1} \left[ dM_s^a - \frac{a_s}{\hat{z}_s} dM_s^s \right] \rangle = \frac{\left| a_s/\hat{z}_s \right|^2 - 1}{\hat{z}_s^2} \, ds , \]
whence the almost sure convergence of \((a_s/\hat{z}_s)\), which proves that \(\ell \neq 0\) almost surely. \(\diamond\)

The following statement ensures that the range of possible limits \(\ell\) in Proposition 4 is the whole \([-1,0[\cup)0,1]\). This provides a continuum of non-trivial bounded harmonic functions for the relativistic operator \(L\).

**Proposition 5** For any real \(\ell_0\) such that \(0 < |\ell_0| \leq 1\), and for any \(\varepsilon > 0\), we have
\[ \mathbb{P}[\ell_0 - \varepsilon < \ell = \lim_{s \to \infty} \frac{\hat{z}_s}{a_s} < \ell_0 + \varepsilon] > 1 - \varepsilon, \]
provided \(\hat{z}_0/a_0\) is close enough from \(\ell_0\) and \(|a_0|\) is large enough.

**Proof** Fix \(A > 9\), \(|a_0| > A^2\) and \(\hat{z}_0 \geq A^2\), such that \(\hat{z}_0/a_0\) is close to \(\ell_0\) (precisely, we demand \(|\log(\hat{z}_0/a_0)\| < A^{-2}\), and consider the event:
\[ \mathcal{A} := \left\{ a_s^2 > 1 + A^2 e^{\sigma^2 s} \quad \text{and} \quad \hat{z}_s^2 > A^2 e^{\sigma^2 s} \quad \text{for all} \ s \geq 0 \right\} . \]

By Lemma 2 we have: \(\mathbb{P}(\mathcal{A}) > 1 - 2/\sqrt{A}\). Now, on \(\mathcal{A}\) we have:
\[ \int_0^\infty \frac{du}{a_u^2 - 1} + \int_0^\infty \frac{du}{\hat{z}_u^2} \leq 2 \sigma^{-2} A^{-2} \quad \text{and} \quad \int_0^\infty \frac{du}{\hat{z}_u^2} \leq \sigma^{-2} A^{-2} . \]

Hence, we see from the expression giving \(\log \left[ \frac{|\hat{z}_s|}{\sqrt{a_s^2 - 1}} \right]\), displayed in the proof of Proposition 4, that we have on \(\mathcal{A}\):
\[ |\log(\ell/\ell_0)| \leq 2 A^{-2} + \sigma \max\{|\hat{W}_s| | 0 \leq s \leq \sigma^{-2} A^{-2}\} . \]

Finally, as
\[ \mathbb{P}[\sigma \max\{|\hat{W}_s| | 0 \leq s \leq \sigma^{-2} A^{-2}\} > A^{-1/2}] \leq 2 \mathbb{P}[\max\{\hat{W}_s| | 0 \leq s \leq A^{-2}\} > A^{-1/2}] \]
\[ = 2 \mathbb{P}[|\hat{W}_{A^{-2}}| > A^{-1/2}] = 4 \mathbb{P}[\hat{W}_1 > \sqrt{A}] < e^{-A/2}, \]
we obtain:
\[ \mathbb{P}[|\log(\ell/\ell_0)| \leq 2 A^{-2} + A^{-1/2}] > 1 - 2/\sqrt{A} - e^{-A/2} . \ \diamond \]

**Proposition 6** The process \(\lambda_s := \argsh(\sqrt{a_s^2 - \hat{z}_s^2 - 1})\) solves the following stochastic differential equation:
\[ d\lambda_s = \sigma d\tilde{w}_s + \sigma^2 \coth(2\lambda_s) ds , \]
for some standard real Brownian motion \(\tilde{w}\), and then is a real diffusion which goes almost surely to infinity as \(s \to \infty\). Moreover, the process \(\tilde{\eta}_s := \lambda_s - \sigma \tilde{w}_s - \sigma^2 s\) converges almost surely (in \(\mathbb{R}\)) as \(s \to \infty\).
Proof. Using Itô’s Formula and the expressions for $da$ and $dz$, we get:

$$d\lambda_s = \frac{2a_s}{\text{sh}(2\lambda_s)} \left[ \frac{3\sigma^2}{2} a_s ds + \sigma dM^a_s \right] - \frac{2\dot{z}_s}{\text{sh}(2\lambda_s)} \left[ \frac{3\sigma^2}{2} \dot{z}_s ds + \sigma dM^z_s \right] + \frac{a_s^2 \dot{z}_s^2 \text{ch}(2\lambda_s)}{\text{ch}^3 \lambda_s \text{sh}^3 \lambda_s} \sigma^2 ds$$

$$- \frac{\sigma^2}{\text{sh}(2\lambda_s)} \left[ \frac{a_s^2}{\text{ch}^2 \lambda_s} + \frac{\dot{z}_s^2 + 1}{\text{sh}^2 \lambda_s} \right] (a_s^2 - 1) ds - \frac{\sigma^2}{\text{sh}(2\lambda_s)} \left[ \frac{a_s^2}{\text{ch}^2 \lambda_s} + \frac{a_s^2 - 1}{\text{sh}^2 \lambda_s} \right] (\dot{z}_s^2 + 1) ds$$

$$= \frac{2\sigma}{\text{sh}(2\lambda_s)} \left[ a_s dM^a_s - \dot{z}_s dM^z_s \right] + \frac{3\sigma^2}{\text{sh}(2\lambda_s)} \frac{\text{ch}^2 \lambda_s}{\text{sh}(2\lambda_s)} ds + \frac{a_s^2 \dot{z}_s^2 \text{ch}(2\lambda_s)}{\text{ch}^3 \lambda_s \text{sh}^3 \lambda_s} \sigma^2 ds$$

$$- \frac{\sigma^2}{\text{sh}(2\lambda_s)} \left[ \frac{a_s^4 + \dot{z}_s^4 - \text{ch}^2 \lambda_s}{\text{ch}^2 \lambda_s} + 2 \frac{a_s^2 \dot{z}_s^2 + \text{sh}^2 \lambda_s}{\text{sh}^2 \lambda_s} \right] ds$$

$$= \frac{2\sigma}{\text{sh}(2\lambda_s)} \left( (a_s^2 - \dot{z}_s^2) (a_s^2 - \dot{z}_s^2 - 1) \right)^{1/2} d\tilde{w}_s + \frac{\sigma^2}{\text{sh}(2\lambda_s)} \left[ 3 \frac{\text{ch}^2 \lambda_s}{\text{sh}(2\lambda_s)} + 2 \frac{a_s^4 \dot{z}_s^2 - a_s^4 + \dot{z}_s^4 - 1}{\text{ch}^2 \lambda_s} \right] ds$$

$$= \sigma d\tilde{w}_s + \sigma^2 \text{coth}(2\lambda_s) ds.$$

Since $\text{coth}(2\lambda_s) > 1$, the comparison theorem ensures that we have almost surely:

$$\lambda_s \geq \lambda_0 + \sigma \tilde{w}_s + \sigma^2 s \rightarrow +\infty.$$

Moreover, we have almost surely for large enough $s_0$ and for $s \geq s_0$:

$$\lambda_s \geq \sigma^2 s/2.$$

Hence, we deduce that $\tilde{\eta}_s = \tilde{\eta}_{s_0} + 2\sigma^2 \int_{s_0}^s \frac{du}{e^{\lambda_u} - 1}$ converges almost surely. ♦

**Remark 5.** The equation satisfied by $(\lambda_s)$ can be precisely solved as follows, provided $\lambda_0 > 0$, using some real Brownian motion $\beta$ started from $\beta_0 = \frac{1}{2} \log[\text{coth} \lambda_0]$:

$$\lambda_s = \frac{1}{2} \log \left[ \text{coth} \left( \beta \left[ \inf \left\{ u \mid \int_0^u \text{sh}^{-2}(2\beta_v) \, dv = \sigma^2 s \right\} \right] \right) \right].$$

This implies that we have almost surely: $\lambda_s > 0$ for any $s > 0$: the state subspace $E_0$ of Remark 1 is polar for the relativistic diffusion. Recall from Section 3 (before Section 3.1) that it is also instantly unstable. Hence we can always restrict the state space $E$ of the relativistic diffusion to $E \setminus E_0$.

By the unit pseudo-norm relation (00') and by Remark 5, we have almost surely for any $s > 0$: $a_s^2 - 1 > \dot{z}_s^2$. Therefore there exist two independent standard real Brownian motions $(w_s)$ and $(\tilde{w}_s)$ such that:

$$da_s = \frac{3\sigma^2}{2} a_s ds + \sigma \sqrt{a_s^2 - 1} dw_s \quad \text{and} \quad dz_s = \frac{3\sigma^2}{2} \dot{z}_s ds + \sigma \frac{a_s \dot{z}_s}{\sqrt{a_s^2 - 1}} dw_s + \sigma \sqrt{1 - \frac{\dot{z}_s^2}{a_s^2 - 1}} d\tilde{w}_s.$$
We shall need to know that in fact \(|\ell| < 1\) almost surely. Let us set:

\[
A_s := \sqrt{1 - \frac{1 + \dot{z}_s^2}{a_s^2}} = \sqrt{\left[\frac{\dot{x}_s}{a_s}\right]^2 + \frac{1}{2} \left[2 - e^{-\sqrt{2} a_s \beta_s} \frac{b_s}{a_s}\right]^2} = \frac{\text{sh} \lambda_s}{a_s}.
\]

Note that the phase subspace \(E_0\) (of Remarks 1 and 5) is precisely:

\[
E_0 = E \cap \{A = 0\}.
\]

**Proposition 7** The random limit \(\ell = \lim_{s \to \infty} (\dot{z}_s/a_s)\) of Proposition 4 satisfies almost surely: \(0 < |\ell| < 1\).

**Proof** By Proposition 4 and Notation (11), we have almost surely:

\[
\log(1 - \ell^2) = 2 \log A_\infty.
\]

On the other hand, using Itô’s Formula and Proposition 6, we get:

\[
d(\log A_s) = d(\log[\text{sh} \lambda_s]) - d(\log|a_s|) = \coth \lambda_s d\lambda_s - \frac{d(\lambda_s)}{2 \text{sh}^2 \lambda_s} - \sigma^2(1 + \frac{1}{2} a_s^{-2}) ds - \frac{\sigma}{a_s} dM_s^a
\]

\[
= \sigma \left[\left(\frac{a_s}{\text{sh}^2 \lambda_s} - \frac{1}{a_s}\right) dM_s^a - \frac{\dot{z}_s}{\text{sh}^2 \lambda_s} dM_s^a\right] + \frac{\sigma^2}{2} \left[2 \coth \lambda_s \coth (2 \lambda_s) - \frac{1}{\text{sh}^2 \lambda_s} - 2 - \frac{1}{a_s^2}\right] ds
\]

\[
= \frac{\sigma}{a_s \text{sh}^2 \lambda_s} \left[(\dot{z}_s^2 + 1) dM_s^a - a_s \dot{z}_s dM_s^a\right] - \frac{\sigma^2}{2 a_s^2} ds,
\]

whence, for some real Brownian motion \(\tilde{B}\):

\[
\log(1 - \ell^2) = 2 \log A_0 + 2 \sigma \tilde{B} \left[\int_0^\infty \frac{\dot{z}_s^2 + 1}{a_s^2 (a_s^2 - \dot{z}_s^2 - 1)} ds\right] - \sigma^2 \int_0^\infty \frac{ds}{a_s^2},
\]

which converges (in \(\mathbb{R}\)) almost surely, by Lemma 1, Proposition 4 and Proposition 6, showing that indeed \(\ell^2 < 1\) almost surely. 

We have furthermore the following.

**Proposition 8** The law of the random limit \(\ell = \lim_{s \to \infty} (\dot{z}_s/a_s)\) has no atom.

**Proof** Fix any \(\ell_0 \in ]-1,1[^\prime\), and set \(\delta_s := \dot{z}_s - \ell_0 a_s\). The stochastic differential equation satisfied by \((\delta_s)\) is easily seen to be:

\[
d\delta_s = \frac{3 \sigma^2}{2} \delta_s ds + \sigma \sqrt{\delta_s^2 + 1 - \ell_0^2} d\beta_s,
\]

for some standard real Brownian motion \((\beta_s)\). This diffusion equation can be solved as follows: we have a real Brownian motion \((W_u)\) (started from \(W_0 \in ]\frac{1}{1-\ell_0}, \frac{1}{1-\ell_0}[\)) such that:

\[
\delta_s = F\left(W\left[\inf\left\{u \mid \int_0^u \frac{(1 - \ell_0^2) dv}{1 - (1 - \ell_0^2)^2 W_0^2} > \sigma s\right\}\right]\right),
\]

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with \( F(W) := \frac{(1 - \ell_0^2)^{3/2} W}{\sqrt{1 - (1 - \ell_0^2)^2 W^2}} \). As \( u \) increases to the hitting time of \( \pm (1 - \ell_0^2)^{-1} \) by \( W \), then 
\[
\int_0^u \frac{(1 - \ell_0^2) dv}{1 - (1 - \ell_0^2)^2 W^2} \to \infty,
\]
increases to infinity, \( W_u \) goes to \( \pm (1 - \ell_0^2)^{-1} \), and \( F(W_u) \) goes to \( \pm \infty \), showing that \( |\delta_s| \) goes almost surely to infinity with \( s \). (The invariant measure of the diffusion \( (\delta_s) \) is \( \sqrt{\delta^2 + 1 - \ell_0^2 d\delta} \).

Then we have almost surely, for any sufficiently large proper time \( s \):
\[
d\log |\delta_s| = (1 - \frac{1 - \ell_0^2}{2\delta_s^2}) \sigma^2 ds \pm \sigma \sqrt{1 + \frac{1 - \ell_0^2}{\delta_s^2}} d\beta_s.
\]
Whence, for real Brownian motions \( w, \bar{w} \) and for sufficiently large proper times \( s_0, s \):
\[
\log |\delta_s| = \sigma^2(s - s_0) + \sigma(w, w - w_{s_0}) - \frac{\sigma^2(1 - \ell_0^2)}{2} \int_{s_0}^s \frac{du}{\delta_u^2} + \sigma^2(1 - \ell_0^2) \bar{w} \left[ \int_{s_0}^s \frac{du}{\delta_u^2} \right] = \sigma^2 s + o(s) > \sigma^2 s/2.
\]
This implies the convergence of the integrals in the above formula, and then the existence of a standard real Brownian motion \((w_{s_0}^\delta, s)\) and of an almost surely converging process \((\eta_s^\delta)\), such that almost surely, for any sufficiently large proper time \( s \) we have:
\[
|\delta_s| = |\dot{z}_s - \ell_0 a_s| = \exp \left[ \sigma^2 s + \sigma w_{s_0}^\delta + \eta_s^\delta \right] = \exp \left[ \sigma^2 s + o(s^{5/9}) \right].
\]

For the same \( \ell_0 \) and \( (\delta_s) \) as above, we get as in the proof of Proposition 4:
\[
\log \frac{|\delta_s/a_s|}{|\delta_{s_0}/a_{s_0}|} = -\sigma^2 \int_{s_0}^s \frac{\dot{z}_u du}{a_u^2 \delta_u} - \frac{\sigma^2}{2} \int_{s_0}^s \frac{1 - (\dot{z}_u/a_u)^2 du}{\delta_u^2} + \sigma W \left[ \int_{s_0}^s \frac{1 - (\dot{z}_u/a_u)^2 du}{\delta_u^2} \right],
\]
almost surely for any sufficiently large \( s_0, s \). Using the above, this shows the almost sure convergence of \( \log \left| \frac{\dot{z}_s}{a_s} - \ell_0 \right| \), hence by Proposition 4 that indeed \( \mathbb{P}[\ell = \ell_0] = 0 \).

The following statement implies that the asymptotic \( \sigma \)-algebra of \((a_s, \dot{z}_s)\) is generated by the only variable \( \ell \).

**Proposition 9** The process \((\dot{z}_s - \ell a_s)\) converges in law, toward some smooth law, but not in probability. Furthermore, we have \((\dot{z}_s - \ell a_s) = o(|a_s|^{2/9})\) almost surely.

**Proof** Since
\[
\left\langle \frac{dM_s^a}{a_s} \right\rangle = (1 - a_s^{-2})ds,
\]
\[
\left\langle dM_s^a - \frac{\dot{z}_s}{a_s} dM_s^a \right\rangle = (1 - \left| \frac{\dot{z}_s}{a_s} \right|^2)ds, \quad \text{and} \quad \left\langle \frac{dM_s^a}{a_s}, dM_s^a - \frac{\dot{z}_s}{a_s} dM_s^a \right\rangle = \frac{\dot{z}_s}{a_s^2} ds,
\]
there exist two independent standard real Brownian motions \( w, w' \) such that:
\[
\frac{dM_s^a}{a_s} = \sqrt{1 - a_s^{-2}} dw_s, \quad \text{and} \quad dM_s^a - \frac{\dot{z}_s}{a_s} dM_s^a = \frac{\dot{z}_s}{a_s^2 \sqrt{1 - a_s^{-2}}} dw_s + \sqrt{\frac{a_s^2 - 1 - \dot{z}_s^2}{a_s^2 - 1}} dw'.
\]
Hence, the expression for $d \log a_s = \sigma^2(1 + \frac{1}{2a_s^2})ds + \sigma \frac{\sigma}{a_s} dM^a_s$, we have for any $0 \leq s \leq u$ (recall Lemma 1):

$$\frac{a_s}{a_{s+u}} = \exp \left[ -\sigma^2 s - \frac{\sigma^2}{2} \int_s^{s+u} \frac{dv}{a_v^2} - \sigma \int_s^{s+u} \sqrt{1 - a_v^{-2}} \, dw_v \right] = e^{-\sigma^2 s - \frac{\sigma}{a_s} (w_{s+u} - w_s) + o(1)}.$$

Hence, the expression for $d[\dot{z}_s/a_s]$ used for Proposition 1 implies that almost surely:

$$\dot{z}_s - \ell a_s = \sigma^2 a_s \int_s^{\infty} \frac{\dot{z}_u}{a_u^3} du - \sigma a_s \int_s^{\infty} a_u^{-1} \left[ dM_u^z - \frac{\dot{z}_u}{a_u} dM_u^a \right]$$

$$= \sigma^2 a_s \int_s^{\infty} \mathcal{O}(a_u^{-2}) \, du - \sigma a_s \int_s^{\infty} \frac{\dot{z}_u dw_u}{a_u^3 \sqrt{1 - a_u^{-2}}} - \sigma a_s \int_s^{\infty} \left[ 1 - \frac{\dot{z}_u}{a_u} a_u \right] \, dw_u.$$

Now, there exist Brownian Motions $\tilde{W}, \tilde{W}'$ (independent of $s$) such that:

$$\int_s^{\infty} \frac{\dot{z}_u dw_u}{a_u^3 \sqrt{1 - a_u^{-2}}} = \tilde{W} \int_s^{\infty} \frac{\dot{z}_u^2 du}{a_u^6 (1 - a_u^{-2})} = o \left[ \int_s^{\infty} \frac{\dot{z}_u^2 du}{a_u^6 (1 - a_u^{-2})} \right]^{4/9} = o \left[ \int_s^{\infty} a_u^{-4} \, du \right]^{4/9}$$

$$= o \left( |a_s|^{-16/9} \right) \times \left[ \int_0^s \frac{a_u}{a_{s+u}} \, du \right]^{4/9} = o \left( |a_s|^{-16/9} \right) \times \left( \int_0^\infty e^{-4\sigma^2 u - 4\sigma(w_{s+u} - w_s)} \, du \right)^{4/9}$$

$$= o \left( |a_s|^{-1/3} \right) \times \left( \int_0^\infty \exp \left[ -7 u + (s+u) \left( 1 + \frac{8 w_{s+u}}{\sigma(s+u)} \right) + s \left( 1 - \frac{6 w_s}{\sigma s} \right) \right] \sigma^2/2 du \right)^{4/9} = o \left( |a_s|^{-1/3} \right),$$

and similarly:

$$\int_s^{\infty} \left[ 1 - \frac{\dot{z}_u^2}{a_u^2} a_u \right] du = \tilde{W}' \int_s^{\infty} \left[ 1 - \frac{\dot{z}_u^2}{a_u^2} a_u \right] du = o \left[ \int_s^{\infty} a_u^{-2} \, du \right]^{4/9}$$

$$= o \left( |a_s|^{-8/9} \right) \times \left[ \int_0^s \frac{a_u}{a_{s+u}} \, du \right]^{4/9} = o \left( |a_s|^{-8/9} \right) \times \left( \int_0^\infty e^{-2\sigma^2 u - 2\sigma(w_{s+u} - w_s)} \, du \right)^{4/9}$$

$$= o \left( |a_s|^{-2/3} \right) \times \left( \int_0^\infty \exp \left[ -7 u + (s+u) \left( 1 + \frac{8 w_{s+u}}{\sigma(s+u)} \right) + s \left( 1 - \frac{6 w_s}{\sigma s} \right) \right] \sigma^2/4 du \right)^{4/9} = o \left( |a_s|^{-7/9} \right).$$

This shows also that $\int_s^{\infty} \mathcal{O}(a_u^{-2}) \, du = o \left( |a_s|^{-7/4} \right)$.

So far, we have shown that almost surely, as $s \to \infty$:

$$\dot{z}_s - \ell a_s = o \left( |a_s|^{-1/3} \right) - \sigma a_s \int_s^{\infty} \left[ 1 - \frac{\dot{z}_u^2}{a_u^2} a_u \right] \, du = o \left( |a_s|^{2/9} \right).$$
Considering then a standard real Brownian motion \( w'' \), independent from \( (a_s, \hat{z}_s) \), we have:

\[
\left\langle \int_s^\infty \frac{a_s}{a_u} \sqrt{1 - \frac{\sigma^2}{a_u^2} - \sqrt{1 - \ell^2}} \, dw_u'' \right\rangle
= \int_0^\infty e^{-2\sigma^2 u - 2\sigma(w_u + w_s) + o_s(1)} \left[ \sqrt{1 - \frac{\sigma^2}{a_u^2} - \sqrt{1 - \ell^2}} \right]^2 \, du
\leq \sqrt{\int_0^\infty e^{-2\sigma^2 u - 2\sigma(w_u + w_s) + o_s(1)} \, du},
\]

which goes to 0 in probability as \( s \to \infty \). Hence, \((\hat{z}_s - \ell a_s)\) behaves in probability as:

\[-\sigma \sqrt{1 - \ell^2} \int_s^\infty e^{-\sigma^2(u-s) - \sigma(w_u - w_s)} \, dw_u'',\]

and then converges in law, toward the law of:

\[\sqrt{1 - \ell^2} \int_0^\infty e^{-u-w_u} \, dw_u'' \equiv \sqrt{(1 - \ell^2) \int_0^\infty e^{-2\sigma^2 u - 2\sigma w_u} \, du} \times N,\]

\(N\) denoting a \( \mathcal{N}(0,1) \) Gaussian variable, independent from \((\ell, w)\).

On the other hand, the above proves also that, as \( t \geq s \to \infty \), \((\hat{z}_t - \ell a_t) - (\hat{z}_s - \ell a_s)\) behaves in probability as:

\[-\sigma \sqrt{1 - \ell^2} \left[ \int_s^\infty e^{-\sigma^2(u-s) - \sigma(w_u - w_s)} \, dw_u'' - \int_t^\infty e^{-\sigma^2(u-t) - \sigma(w_u - w_t)} \, dw_u'' \right],\]

which diverges in probability: for small enough \( \varepsilon > 0 \) and large enough \((t - s)\), we have

\[\mathbb{P}\left[ \int_t^\infty e^{-\sigma^2(u-s) - \sigma(w_u - w_s)} \, dw_u'' < \varepsilon \right] > 1 - \varepsilon ,\]

and by independence:

\[\mathbb{P}\left[ \left| \int_s^t e^{-\sigma^2(u-s) - \sigma(w_u - w_s)} \, dw_u'' - \int_t^\infty e^{-\sigma^2(u-t) - \sigma(w_u - w_t)} \, dw_u'' \right| > 2\varepsilon \right] > 1 - \varepsilon .
\]

Propositions \( \ref{prop:1} \), \( \ref{prop:2} \) and Notation \( (11) \) imply at once the following.

**Corollary 3** We have almost surely: \( A_s = \sqrt{1 - \ell^2} + o(|a_s|^{-7/9}) \).

### 3.4 Study of the four-dimensional sub-diffusion \((x_s, \hat{x}_s, a_s, b_s)\)

Recall from Section \( \ref{sec:3.1} \) that we have \( db_s = \frac{3\sigma^2}{2} b_s \, ds + \sigma dM^b_s \), and

\[d\hat{x}_s = \frac{\omega}{\sqrt{2}} \left( e^{-\sqrt{2} \omega x_s} b_s a_s - 2 \right) e^{-\sqrt{2} \omega x_s} a_s b_s \, ds + \frac{3\sigma^2}{2} \hat{x}_s \, ds + \sigma dM^x_s,
\]

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Recalling from Remark 4 that for any proper time $b$, any vanishing of $\log$ the $R$ shows that there exists another real Brownian motion $\tilde{w}$, so that there exists another real Brownian motion $\tilde{w}$ such that:

$$
\left((\tilde{K}^{ij})\right) = \begin{pmatrix}
    a_s^2 - 1 & a_s b_s - 2 e^{b_s} & a_s \dot{x}_s \\
    a_s b_s - 2 e^{b_s} & b_s^2 - 2 e^{2b_s} & b_s \dot{x}_s \\
    a_s \dot{x}_s & b_s \dot{x}_s & \dot{x}_s^2 + 1
\end{pmatrix}.
$$

The process $(\dot{s})$ alone is easily handled, analogously to Lemma 2

**Lemma 3**  There exists a real standard real Brownian motion $(w''_s)$, and an almost surely converging process $(\eta''_s)$, such that we have:

$$
|s| = \exp \left[ \sigma^2 s + \sigma \omega''_s + \eta''_s \right] \quad \text{for any proper time } s.
$$

**Proof**  We already noticed in Remark 4 that the unit pseudo-norm relation $(00')$ forbids any vanishing of $b_s$. Thus there exists a real standard real Brownian motion $(w''_s)$ such that for any proper time $s$ we have:

$$
d \log |s| = \sigma^2 ds + \sigma^2 e^{2\omega''_s} b_s^{-2} ds + \sigma \sqrt{1 - 2 e^{2\omega''_s} b_s^{-2} du''_s},
$$

so that there exists another real Brownian motion $\tilde{w''}$ such that:

$$
\log |s| = \log |b_0| + \sigma^2 s + \sigma^2 \int_0^s e^{2\omega''_u} du b_u^{-2} + \sigma \omega''_s + \sqrt{2 \sigma} \int_0^s \left( \frac{e^{2\omega''_u}}{b_u^2 + 2 e^{2\omega''_u}} \right) du.
$$

Now using Remark 4 and Lemma 1 we get:

$$
\int_0^\infty e^{2\omega''_u} du \frac{du}{b_u^2} = \int_0^\infty \left[ e^{\omega''_u} \frac{a_u^{-2}}{b_u} \right] \frac{a_u^{-2} du}{b_u} < \infty,
$$

which shows that $\eta''_s := \log |s| - \sigma^2 s - \sigma \omega''_s$ almost surely converges as $s \to \infty$. \checkmark

Then we get easily a new asymptotic variable of the relativistic diffusion.

**Lemma 4**  The process $\log(b_s/a_s)$ converges almost surely as $s \to \infty$.

**Proof**  Recalling from Remark 4 that $b_s/a_s > 0$, we have for any proper time $s \geq 0$:

$$
d \log \left( \frac{b_s}{a_s} \right) = \sigma^2 e^{2\omega''_s} b_s^{-2} ds - \frac{1}{2} \sigma^2 a_s^{-2} ds + \sigma \left( b_s^{-1} dM^b_s - a_s^{-1} dM^a_s \right),
$$

or equivalently, for some real Brownian motion $W$:

$$
\log \left( \frac{b_s}{a_s} \right) - \log \left( \frac{b_0}{a_0} \right) = \sigma^2 \int_0^s e^{2\omega''_u} du \frac{du}{b_u^2} - \sigma^2 \int_0^s \frac{du}{2 a_u^2} + \sigma W \left( \int_0^s \left[ \frac{4 e^{\omega''_u}}{a_u b_u} - \frac{e^{2\omega''_u}}{b_u^2} - \frac{1}{a_u^2} \right] du \right).
$$

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Now, as we already noticed in the proof of Lemma 3 for \( \int_0^s e^{2\sqrt{2} \omega x_u} \frac{du}{b_u^2} \), by Remark 4 and Lemma 1 we get:

\[
\int_0^\infty e^{\sqrt{2} \omega x_u} \frac{du}{a_u b_u} = \int_0^\infty e^{\sqrt{2} \omega x_u} \frac{a_u}{b_u} \frac{a_u^{-2}}{du} < \infty,
\]

and then the almost sure convergence of \( \log(b_s/a_s) \) as \( s \to \infty \).

By Remark 4 again, we deduce at once the following.

**Corollary 4** The process \((x_s)\) is almost surely bounded. Setting \( \varrho := \lim_{s \to \infty} \frac{b_s}{a_s} \), we have precisely:

\[
0 < (1 - \frac{1}{\sqrt{2}}) \varrho \leq \liminf_{s \to \infty} e^{\sqrt{2} \omega x_s} \leq \limsup_{s \to \infty} e^{\sqrt{2} \omega x_s} \leq (1 + \frac{1}{\sqrt{2}}) \varrho < \infty.
\]

The following statement, analogous to Proposition 5, ensures that the range of possible limits \( \varrho \) in Corollary 4 (and Lemma 4), is the whole \([0, \infty)\). This provides another continuum of non-trivial bounded harmonic functions for the relativistic operator \( \mathcal{L} \).

**Proposition 10** For any real \( \varrho_0 \) such that \( 0 < \varrho_0 < \infty \), and for any \( \varepsilon > 0 \), we have

\[
\mathbb{P}[\varrho_0 - \varepsilon < \varrho = \lim_{s \to \infty} \frac{b_s}{a_s} < \varrho_0 + \varepsilon] > 1 - \varepsilon,
\]

provided \( b_0/a_0 \) is close enough from \( \varrho_0 \) and \( |a_0| \) is large enough.

**Proof** Fix \( A > 1 \), \( |a_0| > A^2 \), and \( y_0 \) such that \( Y_0 \) is close to \( \varrho_0 \), and use the expression displayed for \( \log(b_s/a_s) \) in the proof of Lemma 4, Remark 4, and Lemma 2, to get on an event of probability \( 1 - 1/A \):

\[
|\log \varrho - \log \left[ \frac{b_0}{a_0} \right]| \leq \sigma^2 (1 + \frac{1}{\sqrt{2}})^2 \int_0^\infty \frac{du}{a_u} + \sigma \max \left\{ |W_u| : 0 \leq u \leq 4(1 + \frac{1}{\sqrt{2}}) \int_0^\infty \frac{du}{a_u^2} \right\}
\]

\[
\leq 3 A^{-2} + \sigma \max \{|W_u| : 0 \leq u \leq 7 \sigma^{-2} A^{-2}\},
\]

so that

\[
\mathbb{P}\left( |\log \varrho - \log \varrho_0| \leq \left| \log \left[ \frac{b_0}{a_0 \varrho_0} \right] \right| + 3 A^{-2} + A^{-1/2} \right) > 1 - 1/A - e^{-A/14}.
\]

Recall that we set (in Section 3.3):

\[
A_s = \sqrt{\left( \frac{\dot{x}_s}{a_s} \right)^2 + \frac{1}{2} \left[ 2 - e^{-\sqrt{2} \omega x_s} \frac{b_s}{a_s} \right]^2} = \sqrt{1 - \frac{1 + z_s^2}{a_s^2}} = \left| \frac{\operatorname{sh} \lambda_s}{a_s} \right|.
\]

Let us set also:

\[
\dot{x}_s = a_s A_s \cos \gamma_s; \quad 2 - e^{-\sqrt{2} \omega x_s} \frac{b_s}{a_s} = \sqrt{2} A_s \sin \gamma_s.
\]
Remark 6 By Corollary 3 we have almost surely:
\[ e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} = 2 - \sqrt{2(1 - \ell^2)} \sin \gamma_s + o(|a_s|^{-7/9}) \] and \[ \dot{x}_s = \sqrt{1 - \ell^2} a_s \cos \gamma_s + o(|a_s|^{2/9}). \]

Proposition 11 There exists a standard real Brownian motion \( W \) such that (provided \( A_0 > 0 \)) we have almost surely, for any \( s \geq 0 \):
\[ \gamma_s = \gamma_0 + \omega \int_0^s e^{-\sqrt{\omega \cdot x_u}} b_u du + \sigma W \left[ \int_0^s \frac{du}{A_u^2 a_u^2} \right]. \]

Proof Let us differentiate the relation
\[ \cot \gamma_s = \frac{\dot{x}_s}{a_s} \left[ 2 - e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \right]^{-1}, \]
which makes sense almost surely at least for a dense subset of positive \( s \). We get:
\[ (1 + \cot^2 \gamma_s)(-d\gamma_s + \cot \gamma_s \langle d\gamma_s \rangle) = \sqrt{2} \left[ 2 - e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \right]^{-1} \frac{d[\dot{x}_s]}{a_s}, \]
with
\[ d[\dot{x}_s] = \frac{d\dot{x}_s}{a_s} - \frac{\ddot{x}_s}{a_s} ds + \frac{\langle d\dot{x}_s, d\dot{x}_s \rangle}{a_s^2} + \frac{\dot{x}_s}{a_s^3} \sigma^2 ds + \frac{\sigma}{a_s} \left[ dM_s - \frac{\dot{x}_s}{a_s} dM_s^a \right], \]
and
\[ d\left[ e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \right] = \left[ 2 - e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \right] \left[ \frac{2\sigma^2}{a_s^2} - e^{-\sqrt{\omega \cdot x_s}} \frac{a^2 b_s}{a_s^3} - \sqrt{2} \omega e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \ddot{x}_s \right] ds + \frac{\sigma}{a_s} \left[ dM_s^b - \frac{b_s}{a_s} dM_s^a \right]. \]

Hence we get
\[ \sqrt{2} A_s^2 (d\gamma_s - \cot \gamma_s \langle d\gamma_s \rangle) = \left[ e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} - 2 \right] \left[ e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} - 2 \right] e^{-\sqrt{\omega \cdot x_s}} \frac{\omega b_s}{\sqrt{2}} - \sigma^2 \frac{\dot{x}_s}{a_s^3} ds \]
\[ + \left[ e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} - 2 \right] \frac{\sigma}{a_s} \left[ dM_s^b - \frac{b_s}{a_s} dM_s^a \right] + \left[ 2 - e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \right] \frac{\sigma^2}{a_s^2} \left( -e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \ddot{x}_s \right) ds \]
\[ - \frac{\sigma}{a_s^2} \left[ dM_s^b - \frac{b_s}{a_s} dM_s^a \right] - \frac{2\sigma^2}{a_s^2} e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \ddot{x}_s ds \]
\[ - \left[ 2 - e^{-\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} \right]^{-1} \frac{\sigma^2}{a_s^2} e^{-2\sqrt{\omega \cdot x_s}} \left[ 4 e^{\sqrt{\omega \cdot x_s}} \frac{b_s}{a_s} - \frac{b_s}{a_s^2} - 2 e^{2\sqrt{\omega \cdot x_s}} \right] ds. \]
\[
\frac{\sigma}{a_s} \left[ e^{-\sqrt{2} \omega x_s} \frac{b_s}{a_s} - 2 \right] \frac{dM^x_s - \frac{x_s}{a_s} e^{-\sqrt{2} \omega x_s} dM^b_s + 2 \frac{x_s}{a_s} dM^a_s}{\sqrt{2} \omega e^{-\sqrt{2} \omega x_s} b_s A_s^2 ds}
\]

whence for some standard real Brownian motion \( \tilde{W} \):

\[
d\gamma_s = \frac{\sigma}{A_s a_s} d\tilde{W}_s + \omega e^{-\sqrt{2} \omega x_s} b_s ds + \left[ e^{-\sqrt{2} \omega x_s} \frac{b_s}{a_s} - 2 \right] \frac{\sigma^2 \dot{x}_s}{\sqrt{2} A_s^2 a_s^3} ds
\]

This implies

\[
2 A_s^4 \langle d\gamma_s \rangle = \frac{\sigma^2}{a_s^2} \left[ e^{-\sqrt{2} \omega x_s} \frac{b_s}{a_s} - 2 \right] \frac{dM^x_s - \frac{x_s}{a_s} e^{-\sqrt{2} \omega x_s} dM^b_s + 2 \frac{x_s}{a_s} dM^a_s}{\sqrt{2} \omega e^{-\sqrt{2} \omega x_s} b_s A_s^2 ds}
\]

By Proposition \(6\) and by the definition (11) of \( A_s \), we have almost surely:

\[
A_u^2 a_u^2 = s^2 \lambda_u = s^2 [\sigma^2 u + \sigma \dot{w}_u + \ddot{\eta}_u],
\]

which proves the almost sure convergence of \( \int_0^{\infty} \frac{du}{A_u^2 a_u^2} \). Hence we deduce the following.

**Corollary 5** There exists a converging process \( \tilde{\eta} \) such that we have almost surely, for any \( s \geq 0 \):

\[
\gamma_s = \omega \int_0^s e^{-\sqrt{2} \omega x_u} b_u du + \tilde{\eta}_s.
\]

Using Corollary \(5\), Corollary \(4\), Lemma \(3\) and Remark \(6\) we deduce also easily the following.

**Corollary 6** We have almost surely, for large \( s \) : \( \gamma_s = \text{sign}(b_0) e^{\sigma^2 s + o(s^{5/9})} \),

\[
e^{-\sqrt{2} \omega x_s} = \frac{2}{\sqrt{2}} \frac{\sqrt{2(1-e^{-2})}}{\text{sign}(b_0) \sqrt{2}} \sin \left( e^{\sigma^2 s + o(s^{5/9})} \right) + O(e^{-\frac{2}{5} \sigma^2 s}), \quad \frac{x_s}{a_s} = \sqrt{1 - \ell^2} \cos \left( e^{\sigma^2 s + o(s^{5/9})} \right) + O(e^{-\frac{2}{5} \sigma^2 s}).
\]

In the same vein as Proposition \(9\) we have the following.
Proposition 12  We have \((b_s - \varrho a_s) = o(|a_s|^{2/9}),\) almost surely as \(s \to \infty.\)

Proof  We have:  
\[
d\frac{b_s}{a_s} = 2\sigma^2 \frac{e^{\sqrt{2}\omega x_s}}{a_s^2} ds - \sigma^2 \frac{b_s}{a_s^3} ds + \frac{\sigma}{a_s} \left[ dM_s^b - \frac{b_s}{a_s} dM_s^a \right],
\]
and
\[
\left\langle dM_s^b - \frac{b_s}{a_s} dM_s^a \right\rangle = \left( 4 e^{\sqrt{2}\omega x_s} \frac{b_s}{a_s} - 2 e^{2\sqrt{2}\omega x_s} \frac{b_s^2}{a_s^2} \right) ds,
\]
and
\[
\left\langle dM_s^a, dM_s^b - \frac{b_s}{a_s} dM_s^a \right\rangle = \left( \frac{b_s}{a_s} - 2 e^{\sqrt{2}\omega x_s} \right) ds.
\]
Hence there exist two independent standard real Brownian motions \(w, w'\) such that:
\[
dM_s^b - \frac{b_s}{a_s} dM_s^a = \frac{b_s}{a_s} - 2 a_s e^{\sqrt{2}\omega x_s} \frac{a^2}{\sqrt{1 - a_s^{-2}}} dw_s + \sqrt{\frac{2 e^{2\sqrt{2}\omega x_s} (\dot{z}_s^2 + \dot{z}_s^2)}{a_s^{-2} - 1}} dw'_s.
\]
and as in the proof of Proposition 9,
\[
\left\langle dM_s^a, dM_s^b - \frac{b_s}{a_s} dM_s^a \right\rangle = \frac{b_s}{a_s} - 2 e^{\sqrt{2}\omega x_s} ds.
\]
Therefore, we get almost surely:
\[
\varrho - \frac{b_s}{a_s} = 2\sigma^2 \int_s^\infty \frac{e^{\sqrt{2}\omega x_u}}{a_u^2} du - \sigma^2 \int_s^\infty \frac{b_u}{a_u^3} du + \sigma \int_s^\infty \frac{b_u}{a_u^3} \frac{2 (\dot{z}_u^2 + \dot{z}_u^2)}{a_u^2 - 1} du + \sigma \int_s^\infty \frac{b_u}{a_u^3} \frac{2 (\dot{z}_u^2 + \dot{z}_u^2)}{a_u^2 - 1} du'.
\]
Now, as in the proof of Proposition 9 we have
\[
\int_s^\infty \frac{e^{\sqrt{2}\omega x_u}}{a_u^2} du + \int_s^\infty \frac{b_u}{a_u^3} du = o\left(|a_s|^{-7/4}\right), \text{ and } \int_s^\infty \frac{b_u}{a_u^3} \frac{2 (\dot{z}_u^2 + \dot{z}_u^2)}{a_u^2 - 1} du = o\left(|a_s|^{-4/3}\right),
\]
and
\[
\int_s^\infty \frac{e^{\sqrt{2}\omega x_u}}{a_u^2} \frac{2 (\dot{z}_u^2 + \dot{z}_u^2)}{a_u^2 - 1} du' = o\left(|a_s|^{-7/9}\right).
\]
So far, we have shown that almost surely, as \(s \to \infty:\)
\[
b_s - \varrho a_s = o\left(|a_s|^{-1/3}\right) - \sigma a_s \int_s^\infty \frac{e^{\sqrt{2}\omega x_u}}{a_u^3} \frac{2 (\dot{z}_u^2 + \dot{z}_u^2)}{a_u^2 - 1} du' = o\left(|a_s|^{2/9}\right).
\]
Remark 7  Almost the same proof as above shows that we have in fact \((b_s - \varrho a_s) = o(|a_s|^\varepsilon)\) almost surely, and similarly, \((\dot{z}_s - \ell a_s) = o(|a_s|^\varepsilon)\) almost surely, for any \(\varepsilon > 0\). But Propositions 9 [12] (as Corollaries 2, 3, 6) as stated are sufficient for our purpose.

It is possible to complete Proposition 12 in a way somewhat similar to Proposition 9: starting from the last formula above (in the proof of Proposition 12, and proceeding as in the proof of Proposition 9) we can see that the process \((b_s - \varrho a_s)\) asymptotically behaves as

\[
\sigma a_s \int_s^\infty \frac{e^{\sqrt{2} \omega u}}{a_u} \sqrt{2 \left[ \frac{\dot{z}_u^2 + \dot{\chi}_u^2}{\sigma_u^2} \right]} \, du,
\]

for a standard real Brownian motion \(w''\), independent from the filtration of \((a_s, b_s, \dot{x}_s, \dot{z}_s)\), and then, using the pseudo-norm relation (00′) together with Proposition 12 as

\[
\sigma a_s \int_s^\infty \sqrt{4 \varrho e^{\sqrt{2} \omega u} - 2 e^{2 \sqrt{2} \omega u} - \varrho^2} \frac{du}{a_u} = W'' \left[ \int_0^\infty \left[ 4 \varrho e^{\sqrt{2} \omega u} - 2 e^{2 \sqrt{2} \omega u} - \varrho^2 \right] \frac{\sigma^2 a_s^2}{a_{s+u}^2} \, du \right].
\]

Note that to complete the Brownian representation of the martingale \((\bar{M}_s)\) in the proof of Proposition 12 we can find a Brownian motion \(w''\) in the filtration of \((a_s, b_s, \dot{x}_s, \dot{z}_s)\), independent from \(w, w'\), such that

\[
dM_s^\varrho = \frac{a_s \dot{x}_s}{\sqrt{a_s^2 - 1}} \, dw_s - \frac{(b_s - 2 \, a_s \, e^{\sqrt{2} \omega x_s}) \dot{x}_s}{\sqrt{a_s^2 - 1}} \, dw' + \frac{\dot{z}_s}{\sqrt{\dot{z}_s^2 + \dot{\chi}_s^2}} \, dw''.
\]

Now

\[
\left( \varrho, \int_0^\infty e^{\sqrt{2} \omega (x_{s+u} - x_s)} \frac{a_s^2}{a_{s+u}^2} \, du, \int_0^\infty e^{\sqrt{2} \omega (x_{s+u} - x_s)} \frac{a_s^2}{a_{s+u}^2} \, du, \int_0^\infty \frac{a_s^2}{a_{s+u}^2} \, du, e^{\sqrt{2} \omega x_s} \right)
\]

converges in law to

\[
\left( \varrho, \int_0^\infty e^{\sqrt{2} \omega x_u - 2 \sigma^2 u - 2 \sigma w_u} \, du, \int_0^\infty e^{2 \sqrt{2} \omega x_u - 2 \sigma^2 u - 2 \sigma w_u} \, du, \int_0^\infty e^{-2 \sigma^2 u - 2 \sigma w_u} \, du, V \right),
\]

where \(V^{-1} := \frac{2}{\varrho} - \frac{\sqrt{2(1 - \varepsilon^2)}}{\varrho} \sin U\), \(U\) denoting an independent variable, uniform on the circle. We get thus the convergence in law, but not in probability, of \((b_s - \varrho a_s)\), to the smooth:

\[
W'' \left[ \sigma^2 \int_0^\infty \left[ 4 \varrho V e^{\sqrt{2} \omega x_u} - 2 V^2 e^{2 \sqrt{2} \omega x_u} - \varrho^2 \right] e^{-2 \sigma^2 u - 2 \sigma w_u} \, du \right].
\]

3.5 Irreducibility

Proposition 13  (i) The relativistic diffusion is irreducible: from any starting point, it hits any non-empty open subset of the phase-space \(E \setminus E_0\) with a strictly positive probability.

(ii) For any starting point (in \(E\), the law of the asymptotic variable \((\ell, \varrho)\) charges any non-empty open subset of the range \([-1, 0[ \cup [0, 1[^*] \times]0, \infty[^*].\)
Proof (i) We know from Proposition [1] (in Section [2.1]) that there are piece-wise geodesic timelike continuous paths, and then trajectories in the support of the relativistic diffusion \((\xi, \dot{\xi})\), moving at will the coordinates \((t, x, y, z)\).

Owing to the the quadratic covariation (rank 3) matrix of the \(\mathbb{R}^3\)-valued martingale \((M^a_s, M^b_s, M^z_s)\) (recall Section [3.1]) and to the unit pseudo-norm relation (00), we can find three independent standard real Brownian motions \((w^1, w^2, w^3)\) such that:

\[
d\dot{z}_s = \frac{3\sigma^2}{2} \dot{z}_s ds + \sigma \sqrt{\dot{z}^2_s + 1} \, dw^1_s;
\]

\[
da_s = \frac{3\sigma^2}{2} a_s ds + \sigma - \frac{a_s \dot{z}_s}{\sqrt{\dot{z}^2_s + 1}} \, dw^1_s + \sigma \sqrt{\frac{a^2_s - \dot{z}^2_s - 1}{\dot{z}^2_s + 1}} \, dw^2_s;
\]

\[
db_s = \frac{3\sigma^2}{2} b_s ds + \sigma \frac{b_s \dot{z}_s}{\sqrt{\dot{z}^2_s + 1}} \, dw^1_s + \sigma - \frac{a_s b_s - 2 e^{\sqrt{2} \omega x_s} (\dot{z}_s^2 + 1)}{\sqrt{(\dot{z}^2_s + 1)(a^2_s - \dot{z}^2_s - 1)}} \, dw^2_s + \sigma \frac{\sqrt{2} e^{\sqrt{2} \omega x_s} \dot{x}_s}{\sqrt{a^2_s - \dot{z}^2_s - 1}} \, dw^3_s.
\]

Let us use the support theorem of Stroock and Varadhan (see for example ([I-W], Theorem VI.8.1)). We see thus from the above stochastic differential system, that the following trajectories belong to the support of \((\xi, \dot{\xi})\) \(= (t, x, y, z, \dot{z}, a, b, \dot{x})\):

- trajectories moving at will the coordinate \(\dot{z}\), without changing the coordinates \((t, x, y, z)\);
- trajectories moving at will the coordinate \(a\), without changing the coordinates \((t, x, y, z)\);
- trajectories moving at will the coordinate \(b\), provided \(\dot{z} \neq 0\), without changing the coordinates \((t, x, y, z)\).

So far, it has become clear that it is possible, within the support of the relativistic diffusion, to move any point of the phase space \(E\) having given first coordinates \((t, x, y, z, \dot{z}, a) \in \mathbb{R}^6\), onto some point of the phase space \(E\) having prescribed first coordinates \((t', x', y', z', \dot{z}', a') \in \mathbb{R}^6\).

It remains only to consider the last two coordinates \((b, \dot{x})\). They are of course constrained by the unit pseudo-norm relation (00'), which tells precisely that they run some ellipse of this plane of coordinates, which is centred on the axis \(\{\dot{x} = 0\}\). The last type of trajectory mentioned above allows now to move \((b, \dot{x})\) arbitrarily on the upper half and on the lower half this ellipse, without changing the other coordinates, within the support of the relativistic diffusion.

Finally, we must make clear that we can cross the axis \(\{\dot{x} = 0\}\), within the support of the relativistic diffusion, at the cost of an arbitrarily small move of all coordinates. Now, owing to the the quadratic covariation (rank 3) matrix of the \(\mathbb{R}^3\)-valued martingale \((M^a_s, M^b_s, M^z_s)\) (recall Section [3.1]) and to the unit pseudo-norm relation (00), we can find three independent standard real Brownian motions \((\tilde{w}^1, \tilde{w}^2, \tilde{w}^3)\) such that:

\[
d\dot{x}_s = (\omega/\sqrt{2}) e^{-\sqrt{2} \omega x_s} b^2_s ds - \sqrt{2} \omega e^{-\sqrt{2} \omega x_s} a_s b_s ds + \frac{3\sigma^2}{2} \dot{x}_s ds + \sigma \sqrt{\dot{x}^2_s + 1} \, d\tilde{w}^1_s;
\]
Recall from Formulas (10) that we have
\[ \dot{a}_s = \frac{3a^2}{2} a_s ds + \sigma \frac{a_s \dot{x}_s}{\sqrt{\dot{x}_s^2 + 1}} d\bar{w}_s^1 + \sigma \sqrt{\frac{a_s^2 - \dot{x}_s^2 - 1}{\dot{x}_s^2 + 1}} d\bar{w}_s^2; \]
\[ \dot{b}_s = \frac{3b^2}{2} b_s ds + \sigma \frac{b_s \dot{x}_s}{\sqrt{\dot{x}_s^2 + 1}} d\bar{w}_s^1 + \sigma \frac{a_s b_s - 2 e^{\sqrt{\bar{2}} \omega x_s} (\dot{x}_s^2 + 1)}{(\dot{x}_s^2 + 1)(a_s^2 - \dot{x}_s^2 - 1)} d\bar{w}_s^2 + \sigma \frac{\sqrt{2} e^{\sqrt{\bar{2}} \omega x_s} \dot{z}_s}{\sqrt{a_s^2 - \dot{x}_s^2 - 1}} d\bar{w}_s^3. \]
This shows that, arrived in an arbitrarily thin \( \delta \)-neighbourhood of the axis \( \{ \dot{x} = 0 \} \), we can cross this axis (within the support of the relativistic diffusion, acting on the Brownian component \( \bar{w}_s^1 \)) without changing the coordinates \( (t, x, y, z) \), and perturbing the coordinates \( (a, b, \dot{x}, \dot{z}) \) only by some move of order \( \delta \). This ends the proof of irreducibility.

\((ii)\) This is a direct consequence of \((i)\) above and of Propositions \(5\) and \(10\): by \((i)\), it is indeed enough to start the relativistic diffusion so that \( \dot{z}_0/a_0 \) be close to a given \( \ell_0 \in \left[ \frac{1}{2}, 1 \right] \), \( b_0/a_0 \) be close to a given \( \rho_0 > 0 \), and \( |a_0| \) be large enough. \( \diamond \)

### 3.6 Convergence to a lightlike geodesic

From Section 2.2 (Proposition 2 and Remark 3), for a lightlike geodesic we have three geometrically meaningful real parameters \( \ell, \rho, Y \) such that:
\[ \frac{\dot{z}_\tau}{a} = \ell \in [0, 1]; \quad \frac{b}{a} = \rho \in ]0, \infty[; \quad \frac{\sqrt{2}}{\omega b} \dot{x}_\tau + y_\tau = Y \in \mathbb{R}, \]
\[ \text{and} \quad \left[ \frac{b}{2} e^{-\sqrt{2} \omega x_\tau} - 1 \right] + \left[ \frac{\omega b}{2} (y_\tau - Y) \right]^2 = \frac{1}{2} (1 - \ell^2), \]
where \( \tau \) denotes an (irrelevant) affine parameter.

Accordingly, let us exhibit now a third asymptotic random variable \( Y \) for the relativistic diffusion.

**Proposition 14** The process \( Y_s := y_s + \frac{\sqrt{2}}{\omega} \frac{\dot{x}_s}{b_s} \) converges almost surely, as \( s \to \infty \), toward some real random variable \( Y \). And we have \( Y - Y_s = o(|a_s|^{-4/9}) \) almost surely.

**Proof** Recall from Formulas (10) that we have \( \dot{y}_s = e^{-\sqrt{2} \omega x_s} (2 a_s - e^{-\sqrt{2} \omega x_s} b_s) \). We have then:
\[ d \left[ \frac{\dot{x}_s}{b_s} \right] = \frac{d\dot{x}_s}{b_s} - \frac{\dot{x}_s d\bar{w}_s^1}{b_s^2} + \frac{\dot{x}_s (d\bar{w}_s^2)}{b_s^3} - \frac{(d\bar{w}_s, d\dot{x}_s)}{b_s^2} \]
\[ = \frac{\omega}{\sqrt{2}} e^{-2 \sqrt{2} \omega x_s} b_s ds - \sqrt{2} \omega e^{-\sqrt{2} \omega x_s} a_s ds - 2 \sigma^2 e^{2 \sqrt{2} \omega x_s} \frac{\dot{x}_s}{b_s^2} ds + \frac{\sigma}{b_s} dM_s^x - \frac{\dot{x}_s}{b_s^2} dM_s^b, \]
whence
\[ \frac{\omega}{\sqrt{2}} dY_s = -2 \sigma^2 e^{2 \sqrt{2} \omega x_s} \frac{\dot{x}_s}{b_s^3} ds + \frac{\sigma}{b_s} dM_s^x - \frac{\dot{x}_s}{b_s^2} dM_s^b. \]
and for some Brownian motion $W'$:

$$\frac{\omega}{\sqrt{2}} Y_s = \frac{\omega}{\sqrt{2}} Y_0 - 2\sigma^2 \int_0^s e^{2\sqrt{2}\omega x_u} \frac{\dot{x}_u}{b_u^2} du + \sigma W' \left[ \int_0^s \left[ 1 - e^{2\sqrt{2}\omega x_u} \frac{\dot{x}_u}{b_u^2} \right] \frac{du}{b_u^2} \right].$$

By Corollary 4, Remark 6 and Proposition 12, the two above integrals, and then $Y_s$, converge almost surely, and moreover, we have (for some Brownian motion $W'$):

$$Y_s - Y = Y_s - Y_\infty = \frac{2\sqrt{2}\sigma^2}{\omega} \int_s^\infty e^{2\sqrt{2}\omega x_u} \frac{\dot{x}_u}{b_u^2} du + W \left[ \frac{2\sigma^2}{\omega^2} \int_s^\infty \left[ 1 - e^{2\sqrt{2}\omega x_u} \frac{\dot{x}_u}{b_u^2} \right] \frac{du}{b_u^2} \right]$$

$$= O(1) \int_s^\infty a_s^{-2} ds + W \left[ O(1) \int_s^\infty a_u^{-2} du \right] = o \left[ \int_s^\infty a_u^{-2} du \right]^{4/9} = o \left( a_s^{-8/9} \right) \left[ \int_s^\infty \frac{a_s}{a_s + u}^2 du \right]^{4/9}$$

$$= o \left( a_s^{-4/9} \right) \phi.$$

Proposition 14 together with Remark 6 and Corollary 4 implies easily the following.

**Corollary 7** We have almost surely, as $s \to \infty$:

$$\left[ \frac{\sigma}{2} e^{-2\sqrt{2}\omega x} - 1 \right]^2 + \left[ \frac{\omega}{2} (y_s - Y) \right]^2 \to \frac{1}{2} (1 - \ell^2).$$

The following statement, analogous to Propositions 5 and 10, ensures that the range of possible limits $Y$ in Proposition 14 is the whole $\mathbb{R}$. This provides again another continuum of non-trivial bounded harmonic functions for the relativistic operator $\mathcal{L}$.

**Proposition 15** For any real $y$ and any $\varepsilon > 0$, we have $\mathbb{P}[y - \varepsilon < Y < y + \varepsilon] > 1 - \varepsilon$, provided $Y_0$ is close enough from $y$ and $|a_0|$ is large enough.

**Proof** Recall from Lemma 2 that the event $\mathcal{A} := \{ |a_s| \geq \sqrt{\sqrt[3]{2} a_0} e^{\sigma^2 s/2} \}$ for any $s \geq 0 \}$ has (for $|a_0| > 3$) probability larger than $1 - |a_0|^{-1/2}$. The proof of Proposition 14 shows that

$$|Y - Y_0| = O(1) \int_0^\infty \frac{du}{a^2} + \max \left\{ |W_s| \left\{ 0 \leq s \leq O(1) \int_0^\infty \frac{du}{a^2} \right\} = O(\frac{1}{\max(|a_0|)}) + W^*(\frac{1}{\max(|a_0|)}) \} on \mathcal{A},$$

so that $\mathbb{P}(|Y - Y_0| \leq 2 |a_0|^{-1/3}) > 1 - 2 |a_0|^{-1/2}$, for large enough $|a_0|$. \phi

Proposition 15 improves Proposition (13(ii)). We deduce indeed at once the following.

**Corollary 8** For any starting point $(\ell, \varrho, Y)$, the law of the asymptotic variable $(\ell, \varrho, Y)$ charges any non-empty open subset of the range $\left( \left\{ 1 \right\} - 1, 0 \right] \cup \left\{ 0, 1 \right\} \times 0, \infty x \mathbb{R} \right.$.

More precisely, if the starting point of the relativistic diffusion satisfies: $z_0/a_0$ close enough to $\ell_0 \in [-1, 1]$, $b_0/a_0$ close enough to $\varrho_0 > 0$, $Y_0$ close enough to $y \in \mathbb{R}$, and $|a_0|$ large enough, then with arbitrary large probability, $(\ell, \varrho, Y)$ is arbitrary close to $(\ell_0, \varrho_0, y)$. 30
Remark 8 A rapid look at Remark 3 could let think that there could be a fourth asymptotic random variable for the relativistic diffusion, namely a possible almost sure limit for $X_s := z_s + \ell t_s - (\ell/\gamma)s$. But as a matter of fact, there is no such limit, in accordance with the last sentence of Remark 3 on the geometric irrelevance of additional parameter $(Z_0, T_0)$. Indeed, since by Equations (10) we have $\dot{t}_s = e^{-\sqrt{\sigma}x_s}b_s - a_s$, we deduce from Corollary 5 that $X_s - \int_0^s (\dot{z}_u - \ell a_u)du$ converges almost surely, and then, by (the proof of) Proposition 9 that so does also

$$X_s + \sigma \sqrt{1 - \ell^2} \int_0^s \int_u^\infty e^{-\sigma^2(v-u) - \sigma(w_v-w_u)} dw_v'' du.$$

Now, as (setting $w^{(n)} := w_n - w_n$, for any fixed $n \in \mathbb{N}$)

$$\int_n^{n+1} \int_u^\infty e^{-\sigma^2(v-u) - \sigma(w_v-w_u)} dw_v'' du = W^n \left[ \int_0^\infty \left[ \int_0^{\min\{1,v\}} e^{\sigma^2 u + \sigma w_v^{(n)}(u)} du \right]^2 \int_0^\infty e^{-2\sigma^2 v - 2\sigma w_v^{(n)}(v)} dv \right]$$

has a constant law, we see that $(X_s)$ indeed cannot converge in probability.

The theorem of the introduction (section 1) is now established. Indeed, gathering successively Remark 5 and Proposition 13, Propositions 4 and 7, Corollary 4, Proposition 9, and Corollary 8, we get the following main result (for which $\sigma > 0$ is necessary, due to the observation made after Definition 2 in Section 2.2).

**Theorem 1** (i) The relativistic diffusion is irreducible, on its phase space $\mathcal{E} \mathcal{B}\setminus \mathcal{E}_0$.

(ii) Almost surely, the relativistic diffusion path possesses a 3-dimensional asymptotic random variable $(\ell, \varrho, Y)$, and converges to the light ray $B = (\ell, \varrho, Y) \in \mathcal{B}$, in the sense of Definition 2. Indeed, we have almost surely, as proper time $s \to \infty$:

$$\dot{z}_s/a_s \longrightarrow \dot{\ell} \in [0, 1] \setminus \{0\}; \quad b_s/a_s \longrightarrow \varrho \in [0, \infty]; \quad Y_s \longrightarrow Y \in \mathbb{R};$$

$$\left[ \frac{\varrho}{2} e^{-\sqrt{\varrho}x_s} - 1 \right]^2 + \left[ \frac{\varrho}{2} (y_s - Y) \right]^2 \longrightarrow \frac{1}{2} (1 - \ell^2).$$

(iii) The asymptotic random variable $(\ell, \varrho, Y)$ can be arbitrary close to any given $(\ell_0, \varrho_0, y) \in [0, \infty \times \mathbb{R}$, with positive probability. Hence, the whole boundary (space of light rays) $\mathcal{B}$ is the support of light rays the relativistic diffusion can converge to.

**Remark 9** From the proofs of Lemma 4, Proposition 13, and Proposition 4, we have the following representation of the asymptotic variable $B = (\ell, \varrho, Y)$:

$$\varrho = \frac{b_0}{a_0} + 2\sigma^2 \int_0^\infty e^{\sqrt{\varrho}x_u} \frac{du}{a_u^2} - \sigma^2 \int_0^\infty \frac{b_u du}{a_u^2} + \sigma \int_0^\infty a_u^{-1} \left[ dM_u^b - \frac{b_u}{a_u} dM_u^a \right];$$

$$Y = Y_0 - 2\sigma^2 \int_0^\infty e^{\sqrt{\varrho}x_u} \frac{\dot{x}_u}{b_u^2} du + \frac{\sqrt{\varrho}}{\omega} \int_0^\infty b_u^{-1} \left[ dM_u^x - \frac{\dot{x}_u}{b_u} dM_u^b \right];$$

$$X_s := z_s + \ell t_s - (\ell/\gamma)s.$$
\[ \ell = \frac{\dot{z}_0}{a_0} - \sigma^2 \int_0^\infty \frac{\dot{z}_u}{a_u^3} du + \sigma \int_0^\infty a_u^{-1} \left[ dM_u^z - \frac{\dot{z}_u}{a_u} dM_u^a \right]. \]

By Proposition 8, the law of the asymptotic variable \( B \) has no atom, and by Theorem (i(iii)), it is really three-dimensional. None of \( \ell, \varrho, Y \) is a function of the two others.

Recall that the random excitement of the relativistic diffusion is a standard three-dimensional Brownian motion. Therefore, Theorem 1, reinforced by Proposition 9 and Remarks 7, 8, 9, incites to believe in the following.

**Conjecture** The tail \( \sigma \)-field and the invariant \( \sigma \)-field of the relativistic diffusion in Gödel’s universe are the \( \sigma \)-field generated by the asymptotic three-dimensional random variable \( B = (\ell, \varrho, Y) \) of Theorem 1 (exhibited by Proposition 4, Corollary 4, and Proposition 14).

**Remark 10** The convergence of Theorem (i(iii)), of the generic diffusion path \( \tilde{\xi} = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) \), in the sense of Definition 2, occurs in fact in some stronger sense. Indeed, by Remark 6, Propositions 12 and 4, and Lemma 1, we have on one hand:

\[
e^{-\sqrt{2}\omega x_s} = \frac{2}{\varrho} - \frac{\sqrt{2(1-\ell^2)}}{\varrho} \sin \gamma_s + o(e^{-2\sigma^2s/3}) \quad \text{and} \quad y_s = Y - \frac{\sqrt{2(1-\ell^2)}}{\omega \varrho} \cos \gamma_s + o(e^{-2\sigma^2s/3}),
\]

while by Remark 3 using the increasing diffeomorphism \( \varphi = (\tau \mapsto \varphi_{s}) \), we have on the other hand:

\[
e^{-\sqrt{2}\omega \varphi^{-1}(\gamma_s/\omega)} = \frac{2}{\varrho} - \frac{\sqrt{2(1-\ell^2)}}{\varrho} \sin \gamma_s \quad \text{and} \quad \varphi^{-1}(\gamma_s/\omega) = Y - \frac{\sqrt{2(1-\ell^2)}}{\omega \varrho} \cos \gamma_s.
\]

Hence, we have in the \((x, y)\)-plane a strong convergence, of the projection of the generic relativistic diffusion path to the projection of a lightlike geodesic, in the Skorohod topology:

\[
|x_s - \tilde{x}_{\varphi^{-1}(\gamma_s/\omega)}| + |y_s - \tilde{y}_{\varphi^{-1}(\gamma_s/\omega)}| = o(e^{-2\sigma^2s/3}).
\]

Otherwise, by Proposition 9 Corollaries 5 and 6 and Remark 3 we have:

\[
z_s + \ell t_s = z_s + \ell t_0 + \ell \int_0^s (e^{-\sqrt{2}\omega x_s} b_u - a_u) du = \frac{\ell}{\omega} \gamma_s + O(1) + \int_0^s (\dot{z}_u - \ell a_u) du = \frac{\ell}{\omega} \gamma_s + o(|\gamma_s|^{1/3})
\]

\[
= \tilde{z}_{\varphi^{-1}(\gamma_s/\omega)} + \ell \tilde{t}_{\varphi^{-1}(\gamma_s/\omega)} + o(|\gamma_s|^{1/3}) = \left( \tilde{z}_{\varphi^{-1}(\gamma_s/\omega)} + \ell \tilde{t}_{\varphi^{-1}(\gamma_s/\omega)} \right) [1 + o(e^{-\sigma^2s/2})] \longrightarrow \infty.
\]

Hence, again in the Skorohod topology, and in the \((z, t)\)-plane, the projection of the limiting light ray stands for an asymptotic direction for the projection of the generic relativistic diffusion path, but there is no exactly asymptotic lightlike geodesic (only a parabolic branch occurs).
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