Convoluted Generalized White Noise, Schwinger Functions and Their Analytic Continuation to Wightman Functions

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October 22, 2018

Abstract

We construct Euclidean random fields $X$ over $\mathbb{R}^d$, by convoluting generalized white noise $F$ with some integral kernels $G$, as $X = G * F$. We study properties of Schwinger (or moment) functions of $X$. In particular, we give a general equivalent formulation of the cluster property in terms of truncated Schwinger functions which we then apply to the above fields. We present a partial negative result on the reflection positivity of convoluted generalized white noise.

Furthermore, by representing the kernels $G_\alpha$ of the pseudo–differential operators $(-\Delta + m_0^2)^{-\alpha}$ for $\alpha \in (0,1)$ and $m_0 > 0$ as Laplace transforms we perform the analytic continuation of the (truncated) Schwinger functions of $X = G_\alpha * F$, obtaining the corresponding (truncated) Wightman distributions on Minkowski space which satisfy the relativistic postulates on invariance, spectral property, locality and cluster property.

Finally we give some remarks on scattering theory for these models.
Since the work of Nelson in the early 70's the problem of the mathematical construction of models of interacting local relativistic quantum fields has been related to the one of the construction of Markovian Euclidean generalized random fields.

In models for scalar fields in space-time dimension two Markovian can be understood in the strict "global Markov sense", as proven in [4], [35], [71], [12](see also [20] and references therein). This is also true for a class of vector models in space-time dimensions $d = 2, 4$(and 8), see [11], [13], [14], [15], [55] [64]. The latter models are of the gauge-type and the construction of an associated Hilbert space, in the cases $d = 4$ and 8, presents difficulties(these
do not exist for \( d = 2 \), see [15]; for a partial result for \( d = 4 \) see [14]). For results on a model of a scalar field for \( d = 3 \)(with the Markovian property replaced by reflection positivity in the sense of [37]) see references in [37] and [21], for partial(weak and rather negative) results for scalar fields for \( d = 4 \) see [1], [37], [33],[32], [60], [62]. For partial results on conformal fields on other types of \( d = 4 \) space-times see [65].

A program of constructing Euclidean random fields of Markovian type by solving pseudo-stochastic partial differential equations of the form \( LX = F \) with \( F \) a Euclidean noise and \( L \) a suitable invariant pseudodifferential operator was started in [68], [5], [6], [7], see also [11], [14], [15], [26] in the vector case, and in [18] for the scalar case (in a recent note J. Klauder, see e.g. [49] and [50], and references therein, also advocated the use of non-Gaussian noises in stochastic PDEs resp. functional integrals to circumvent triviality for scalar fields, his suggestion is however different from ours).

In the present case we continue the work initiated in [18], extending the study of random fields of the form \( X = G * F \)(of which the above case \( G = L^{-1} \) is a special one), to more general \( G \) than in [18], in particular covering \( G_\alpha = (-\Delta + m_0^2)^{-\alpha} \) for \( \alpha \in (0,1) \) and \( m_0 \geq 0 \)(only the case \( m_0 = 0, \alpha = 1 \) was treated in [18]). For \( \alpha = \frac{1}{2}, F \) Gaussian white noise, \( X \) is Nelson’s Euclidean free field over \( \mathbb{R}^d[54] \); for \( \alpha = \frac{1}{4}, X \) is the time zero free field (over a space–time of dimension \( d + 1 \)). The idea of extending \( F \) to be a general, not necessary Gaussian Euclidean noise (“generalized white noise”) (i.e., a generalized random field ”independent at every point”, in the terminology of [34], or a ”completed scattered random measure”, which is homogeneous with respect to the Euclidean group over \( \mathbb{R}^d \) can be motivated from different points of view. Let us mention three of them (see also e.g. [5],[6],[7], and [11]):

1. From a general Euclidean noise \( F \) one can recover a Gaussian Euclidean (Gaussian white) noise \( F^g \) as weak limit, see Remark 1.4 at the end of Section 1. Thus one can look at the fields \( X = G * F \) constructed from \( F \) as perturbations of the free fields \( X = G * F^g \) constructed from \( F^g \). This perturbation is of another type than the usual perturbations of (Euclidean) quantum field theory (given by additive Feynman–Kac type functionals, see e.g. [1],[20],[66], [37]).

2. As explained in details in Remark 5.12 below, the Schwinger functions \( S_\mu \) of our model, suitably scaled by a factor \( \lambda^{-\frac{n}{2}} \), \( \lambda > 0 \), can be written in terms of the free field Schwinger functions plus a polynomial of finite order in the ”coupling constant” \( \lambda^{-1} \) without a constant term, the
coefficients being products of truncated Schwinger functions of order \(2 \leq l \leq n\). In this sense we have a parameter \(\lambda^{-1}\) which measures "the amount of Poisson component" which can also be seen as "the amount of interaction" present in the given Schwinger functions.

3. One can give a discrete approximation or "lattice approximation" of the models, by replacing \(\mathbb{R}^d\) by a lattice \(\delta\mathbb{Z}^d\), and correspondingly \(L\) and \(F\) by discrete analogues, see e.g. for special cases, [17], [18], and [19]. One can then interpret the probabilistic law of \(X\) in a bounded region as given by a Euclidean action with a non quadratic kinetic energy part (depending essentially on the Lévy measure characterizing the distribution of \(F\)).

Let us explain this a little further, starting from be the lattice approximation \(F^\Lambda_\delta\) of a generalized white noise \(F\), where the superscript \(\Lambda\) indicates a cutoff outside a bounded region \(\Lambda \subset \mathbb{R}^d\). We assume that \(F\) is determined by a Lévy characteristic \(\psi\) (cf. Section 1) such that the convolution semigroup \((\mu_t)_{t>0}\) generated by \(\psi\) [27] is absolutely continuous w.r.t. Lebesgue measure on \(\mathbb{R}\). Let \(\varrho_t\), \(t > 0\), be the corresponding densities. Then the probability distribution of \(F^\Lambda_\delta\) in every lattice point \(\delta n \in \Lambda_\delta := \delta\mathbb{Z}^d \cap \Lambda\) is given by \(\mu_{\delta}(\delta n) = \varrho_{\delta}(\delta^d x)\). We denote by \(L^\Lambda_\delta\) the lattice discretization of a partial (pseudo-) differential operator \(L\) over the lattice \(\Lambda_\delta\). Furthermore, we assume that \(L^\Lambda_\delta\) as a \(|\Lambda_\delta| \times |\Lambda_\delta|\)-matrix is invertible. Then the solvable discrete stochastic equation \(L^\Lambda_\delta X^\Lambda_\delta = F^\Lambda_\delta\) is the lattice analogue of \(LX = F\). We set

\[
W_\delta(x) := -\delta^{-d} \log \varrho_{\delta}(\delta^d x), \quad x \in \mathbb{R}.
\]

The lattice measure \(P_{X^\Lambda_\delta}\) is defined as the measure with respect to which \(X^\Lambda_\delta\) is the coordinate process. Then we have (see [18] and [19])

\[
P_{X^\Lambda_\delta}(X^\Lambda_\delta \in A) = Z^{-1} \int_A e^{-\sum_{\delta n \in \Lambda_\delta} \delta^d W_\delta((L^\Lambda_\delta X^\Lambda_\delta)(\delta n))} \prod_{\delta n \in \Lambda_\delta} dX^\Lambda_\delta(\delta n),
\]

for Borel measurable subsets \(A \subset \mathbb{R}^{\Lambda_\delta}\). Here \(\prod_{\delta n \in \Lambda_\delta} dX^\Lambda_\delta(\delta n)\) denotes the flat lattice measure and \(Z\) is a normalization constant which depends on \(\delta, \Lambda\) and \(L\). For \((\mu_t)_{t>0}\) the Gaussian semigroup (of mean zero and variance \(t\)) and \(L^\Lambda_\delta\) the discretization of \((-\Delta + m_0^2)\frac{1}{2}\) we get the usual lattice approximation of the Nelson’s free field with mass \(m_0 > 0\). If \((\mu_t)_{t>0}\) is not the Gaussian semigroup, the action \(W_\delta\) is no longer quadratic and therefore contains terms which can be identified with some kind of interaction.
In this sense the models can be looked upon as quantized versions of nonlinear field models (with some analogy with models like the Einstein–Infeld field model, see [5],[6], [7], and [11]).

In this paper we construct $X$, as given in general by $G \ast F$, study its regularity properties and the properties of the associated moment functions (Schwinger functions), proving invariance and cluster property. We also perform their analytic continuation to relativistic Wightman functions, which are shown to satisfy all Wightman axioms (possibly except for positivity), including the cluster property (a point not discussed in [18] for the special case considered there). We also provide a counterexample to the reflection positivity condition, for a particular choice of noise $F$ (with a ”sufficiently strong” Poisson component). We also indicate that despite the possible absence of reflection positivity in non Gaussian cases, one can associate scattering states which partly express a ”particle structure” of the models. We also make several comments concerning the interplay of properties of $G$ with the Markov property respectively the reflection positivity of $X$.

One main method used in our analysis is the study of truncated Schwinger functions. This, as most of the results of the present work, is based on [38]. Some of our results have been announced in [2].

Here are some details on the single sections in this paper. In section 1 we introduce the basic (white) noises $F$ used in this work. In section 2 we describe the kernels $G$ and the random fields given by $X = G \ast F$. In section 3 we discuss the basic invariance properties of the moment functions $S_n$ (Schwinger functions) of $X$. In section 4 we discuss the cluster property of the $S_n$. Section 5 is devoted to a discussion of the reflection positivity property and to an example showing that it does not hold for general $F$. In sections 6 and 7 the analytic continuation of the Schwinger functions to Wightman functions is discussed, first (section 6) for the two-point function and then for the general n-point Schwinger functions (section 7). Section 7 also contains the discussion of ”positivity properties” in a ”scattering region”.

1 The Generalized White Noise

In this section, we shall present some basic concepts as background for our discussions in later sections.

As is well–known in probability theory, an infinite divisible probability distribution $P$ is a probability distribution having the property that for
each $n \in \mathbb{N}$ there exists a probability distribution $P_n$ such that the $n$–fold convolution of $P_n$ with itself is $P$, i.e., $P = P_n \ast \ldots \ast P_n$ (n times). By Lévy–Khinchine theorem (see e.g. Lukacs [51]) we know that the Fourier transform (or characteristic function) of $P$, denoted by $C_P$, satisfies the following formula

$$C_P(t) := \int_{\mathbb{R}} e^{ist} dP(s) = e^{\psi(t)}, \quad t \in \mathbb{R},$$

where $\psi : \mathbb{R} \rightarrow \mathbb{C}$ is a continuous function, called the Lévy characteristic of $P$, which is uniquely represented as follows

$$\psi(t) = iat - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left( e^{ist} - 1 - \frac{ist}{1 + s^2} \right) dM(s), \quad t \in \mathbb{R},$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and the function $M$ satisfies the following condition

$$\int_{\mathbb{R}\setminus\{0\}} \min(1, s^2) dM(s) < \infty.$$  

On the other hand, given a triple $(a, \sigma, M)$ with $a \in \mathbb{R}, \sigma \geq 0$ and a measure $M$ on $\mathbb{R} \setminus \{0\}$ which fulfils (3), there exists a unique infinitely divisible probability distribution $P$ such that the Lévy characteristic of $P$ is given by (2).

Let $d \in \mathbb{N}$ be a fixed space time dimension. Let $S(\mathbb{R}^d)$ (resp. $S_0(\mathbb{R}^d)$) be the Schwartz space of all rapidly decreasing real – (resp. complex–) valued $C^\infty$–functions on $\mathbb{R}^d$ with the Schwartz topology. Let $S'(\mathbb{R}^d)$ (resp. $S_0'(\mathbb{R}^d)$) be the topological dual of $S(\mathbb{R}^d)$ (resp. $S_0(\mathbb{R}^d)$). We denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $S(\mathbb{R}^d)$ (resp. $S_0(\mathbb{R}^d)$) and $S'(\mathbb{R}^d)$ (resp. $S_0'(\mathbb{R}^d)$). Let $\mathcal{B}$ be the $\sigma$–algebra generated by cylinder sets of $S'(\mathbb{R}^d)$. Then $(S'(\mathbb{R}^d), \mathcal{B})$ is a measurable space.

By a characteristic functional on $S(\mathbb{R}^d)$, we mean a functional $C : S(\mathbb{R}^d) \rightarrow \mathbb{C}$ with the following properties

1. $C$ is continuous on $S(\mathbb{R}^d)$;
2. $C$ is positive–definite;
3. $C(0) = 1$.

By the well-known Bochner-Minlos theorem (see e.g. [34]) there exists a one to one correspondence between characteristic functionals $C$ and probability measures $P$ on $(S'(\mathbb{R}^d), \mathcal{B})$ given by the following relation

$$C(f) = \int_{S'(\mathbb{R}^d)} e^{i \langle f, \xi \rangle} dP(\xi), \quad f \in S(\mathbb{R}^d).$$  

(4)
We have the following result

**Theorem 1.1** Let \( \psi \) be a Lévy characteristic defined by (1). Then there exists a unique probability measure \( P_\psi \) on \( (S'(\mathbb{R}^d), \mathcal{B}) \) such that the Fourier transform of \( P_\psi \) satisfies

\[
\int_{S'(\mathbb{R}^d)} e^{i<f,\xi>} dP_\psi(\xi) = \exp \left\{ \int_{\mathbb{R}^d} \psi(f(x)) dx \right\}, \quad f \in S(\mathbb{R}^d). \tag{5}
\]

**Proof.** It suffices to show that the right hand side of (5) is a characteristic functional on \( S(\mathbb{R}^d) \). This is true, e.g., by Theorem 6 on p. 283 of [34].

**Definition 1.2** We call \( P_\psi \) in Theorem 1.1 a generalized white noise measure with Lévy characteristic \( \psi \) and \( (S'(\mathbb{R}^d), \mathcal{B}, P_\psi) \) the generalized white noise space associated with \( \psi \). The associated coordinate process

\[
F : S(\mathbb{R}^d) \times (S'(\mathbb{R}^d), \mathcal{B}, P_\psi) \to \mathbb{R}
\]

defined by

\[
F(f, \xi) = \langle f, \xi \rangle, \quad f \in S(\mathbb{R}^d), \quad \xi \in S'(\mathbb{R}^d)
\]

is called **generalized white noise**.

**Remark 1.3** In the terminology of [34], \( F \) is called a generalized random process with independent values at every point, i.e., the random variables \( \langle f_1, \cdot \rangle \) and \( \langle f_2, \cdot \rangle \) are independent whenever \( f_1(x)f_2(x) = 0 \) for \( f_1, f_2 \in S(\mathbb{R}^d) \).

Combining (2) and (5), we get, for \( f \in S(\mathbb{R}^d) \), that

\[
\int_{S'(\mathbb{R}^d)} e^{i<f,\xi>} dP_\psi(\xi) = \exp \left\{ ia \int_{\mathbb{R}^d} f(x) dx - \frac{\sigma^2}{2} \int_{\mathbb{R}^d} |f(x)|^2 dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{isf(x)} - 1 - \frac{isf(x)}{1+s^2} \right) dM(s) dx \right\}. \tag{7}
\]

From (7), we see that a generalized white noise \( F \) is composed by three independent parts, namely, we can give an equivalent (in law) realization of \( F \) as the following direct sum

\[
F(f, \cdot) = F_a(f, \cdot) \oplus F_\sigma(f, \cdot) \oplus F_M(f, \cdot) \tag{8}
\]
for \( f \in \mathcal{S}({\mathbb{R}^d}) \) with \( F_a, F_\sigma \) and \( F_M \) the coordinate processes on the probability spaces \((\mathcal{S}'({\mathbb{R}^d}), \mathcal{B}, \mathcal{P}_a)\), \((\mathcal{S}'({\mathbb{R}^d}), \mathcal{B}, \mathcal{P}_\sigma)\) and \((\mathcal{S}'({\mathbb{R}^d}), \mathcal{B}, \mathcal{P}_M)\), respectively, where \( \mathcal{P}_a, \mathcal{P}_\sigma \) and \( \mathcal{P}_M \) are defined, by Theorem 1.1, by the following relations

\[
\int_{\mathcal{S}'({\mathbb{R}^d})} e^{i<f,\xi>} d\mathcal{P}_a(\xi) = \exp \left\{ ia \int_{\mathbb{R}^d} f(x) dx \right\}
\]

\[
\int_{\mathcal{S}'({\mathbb{R}^d})} e^{i<f,\xi>} d\mathcal{P}_\sigma(\xi) = \exp \left\{ \frac{-\sigma^2}{2} \int_{\mathbb{R}^d} |f(x)|^2 dx \right\}
\]

\[
\int_{\mathcal{S}'({\mathbb{R}^d})} e^{i<f,\xi>} d\mathcal{P}_M(\xi)
=
\exp \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}\setminus\{0\}} \left( e^{isf(x)} - 1 - \frac{isf(x)}{1 + s^2} \right) dM(s) dx \right\}
\]

for all \( f \in \mathcal{S}({\mathbb{R}^d}) \). We call \( F_a, F_\sigma \) and \( F_M \) in order as degenerate (or constant), Gaussian and Poisson (with jumps given by \( M \)) noises, respectively. The first two terms \( F_a \) and \( F_\sigma \) can be clearly understood. Let us discuss further the Poisson noise. Its characteristic functional is given by the following formula

\[
\int_{\mathcal{S}'({\mathbb{R}^d})} e^{i<f,\xi>} d\mathcal{P}_M(\xi)
=
\exp \left\{ \int_{\mathbb{R}^d} \int_{\mathbb{R}\setminus\{0\}} \left( e^{isf(x)} - 1 - \frac{isf(x)}{1 + s^2} \right) dM(s) dx \right\}
\]

(9)

for \( f \in \mathcal{S}({\mathbb{R}^d}) \). The existence and uniqueness of the Poisson noise measure \( \mathcal{P}_M \) is assured by Theorem 1.1.

In what follows, we will give a representation of Poisson noise in terms of a corresponding Poisson distribution. To do this, we first introduce some notions. Let \( \mathcal{D}({\mathbb{R}^d}) \) denote the Schwartz space of all (real–valued) \( \mathcal{C}^\infty \)–functions on \( \mathbb{R}^d \) with compact support and \( \mathcal{D}'({\mathbb{R}^d}) \) its topological dual space. Clearly, \( \mathcal{D}({\mathbb{R}^d}) \subset \mathcal{S}({\mathbb{R}^d}) \). As was pointed out e.g. in [34] and [45], the Bochner Minlos theorem (and therefore our Theorem 1.1) also holds on \( \mathcal{D}({\mathbb{R}^d}) \). Especially, (9) holds for \( \mathcal{P}_M \) on \( \mathcal{D}'({\mathbb{R}^d}) \) and \( f \in \mathcal{D}({\mathbb{R}^d}) \). Namely, there exists a unique \( \mathcal{P}_M \) such that

\[
\int_{\mathcal{D}'({\mathbb{R}^d})} e^{i<f,\xi>} d\mathcal{P}_M(\xi)
=
\exp \left\{ \int_{\Lambda(f)} \int_{\mathbb{R}\setminus\{0\}} \left( e^{isf(x)} - 1 - \frac{isf(x)}{1 + s^2} \right) dM(s) dx \right\}
\]

(10)
for \( f \in \mathcal{D}(\mathbb{R}^d) \), where \( \Lambda(f) := \text{supp}(f \subset \mathbb{R}^d) \) is the support of \( f \). We assume henceforth that the first moment of \( M \) exists. In this case we can drop the third term in the exponential of the right hand side of (10) and Theorem 1.1 assures that there exists a unique measure \( \tilde{P}_M \) such that for \( f \in \mathcal{D}(\mathbb{R}^d) \)

\[
\int_{\mathcal{D}'(\mathbb{R}^d)} e^{i<f,\xi>} d\tilde{P}_M(\xi) = \exp \left\{ \int_{\Lambda(f)} \int_{\mathbb{R}\setminus\{0\}} \left( e^{isf(x)} - 1 \right) dM(s)dx \right\}. \tag{11}
\]

Set \( \kappa_f = \int_{\Lambda(f)} \int_{\mathbb{R}\setminus\{0\}} dM(s)dx \), which is a finite and strictly positive number. Then by Taylor series expansion of the exponential and dominated convergence, we have from (11) that

\[
\int_{\mathcal{D}'(\mathbb{R}^d)} e^{i<f,\xi>} d\tilde{P}_M(\xi) = e^{-\kappa_f} \sum_{n \geq 0} \frac{1}{n!} \left[ \int_{\Lambda(f)} \int_{\mathbb{R}\setminus\{0\}} e^{isf(x)} dM(s)dx \right]^n
\]

\[
= e^{-\kappa_f} \sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda(f)} \int_{\mathbb{R}\setminus\{0\}} \cdots \int_{\Lambda(f)} \int_{\mathbb{R}\setminus\{0\}} e^{i \sum_{j=1}^n s_j f(x_j)} \prod_{j=1}^n dM(s_j)dx_j \cdot \tag{12}
\]

Formula (12) might be interpreted as a representation of Poisson chaos and from it we get the following equivalent (in law) representation of Poisson noise

\[
<f, \xi> = <f, \sum_{j=1}^{N_f} \lambda_j \delta_{X_j}>, \quad f \in \mathcal{D}(\mathbb{R}^d), \tag{13}
\]

where \( \delta_x \) is the Dirac-distribution concentrated in \( x \in \mathbb{R} \) and \( N_f \) is a compound Poisson distribution (with intensity \( \kappa_f \)) given as follows

\[
\text{Pr}\{N_f = n\} = \frac{e^{-\kappa_f}(\kappa_f)^n}{n!}, \quad n = 0, 1, 2, \ldots \tag{14}
\]

where \( \{(X_j, \lambda_j)\}_{1 \leq j \leq N_f} \) is a family of independent, identically distributed random variables distributed according to the probability measure \( \kappa_f^{-1}dx \times dM(s) \) on \( \Lambda(f) \times (\mathbb{R} \setminus \{0\}) \).

Concerning with Poisson noise \( F_M \) on \( \mathcal{S}'(\mathbb{R}^d) \) determined by formula (9), we note that \( \mathcal{D}(\mathbb{R}^d) \) is dense in \( \mathcal{S}(\mathbb{R}^d) \) with respect to the topology of \( \mathcal{S}(\mathbb{R}^d) \). Therefore, by the continuity of the right hand side of (9), the chaos decomposition (12) determines the law of the coordinate process \( F_M \).
Remark 1.4 It is interesting to note, in the relation with the short discussion given in the introduction, that one can recover Gaussian fields from Poisson ones in a limit. In fact, let \( \{M_n\}_{n \in \mathbb{N}} \) be a certain sequence of functions satisfying (3) and \( \{P^M_n\}_{n \in \mathbb{N}} \) be the sequence of Poisson noise measures determined by (10), then

\[
\int_{\mathcal{D}'(\mathbb{R}^d)} e^{if,\xi} dP^n_M(\xi) \rightarrow e^{ia} \int_{\mathbb{R}^d} f(x) dx - \frac{\sigma^2}{2} \int_{\mathbb{R}^d} [f(x)]^2 dx
\]

as \( n \rightarrow \infty \), which is the characteristic functional of a Gaussian law on \( \mathcal{D}'(\mathbb{R}^d) \) with mean

\[
\mathbb{E}[\langle f, \cdot \rangle] = a \int_{\mathbb{R}^d} f(x) dx, \ f \in \mathcal{D}(\mathbb{R}^d)
\]

and covariance

\[
\mathbb{E}[\langle f, \cdot \rangle \langle g, \cdot \rangle] = \sigma^2 \int_{\mathbb{R}^d} f(x) g(x) dx, \ f, g \in \mathcal{D}(\mathbb{R}^d)
\]

iff

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{isf(x)} - 1 - \frac{isf(x)}{1 + s^2} \right) dM_n(s) dx \rightarrow ia \int_{\mathbb{R}^d} f(x) dx - \frac{\sigma^2}{2} \int_{\mathbb{R}^d} [f(x)]^2 dx
\]

as \( n \rightarrow \infty \). An example is given by Poisson laws where the left hand side is

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} (e^{isf(x)} - 1) dM_n(s) dx
\]

which converges to \( \frac{\sigma^2}{2} \int_{\mathbb{R}^d} [f(x)]^2 dx \) (i.e. to the right hand side for \( a = 0 \)) if, e.g., \( dM_n(s) = n^2 \sigma^2 \delta_\frac{1}{n}(s) ds \).

2 Euclidean Random Fields as Convoluted Generalized White Noise

Let us first give the notion of random fields.

**Definition 2.1** Let \( (\Omega, \mathcal{E}, P) \) be a probability space. By a (generalized) random field \( X \) on \( (\Omega, \mathcal{E}, P) \) with parameter space \( S(\mathbb{R}^d) \), we mean a system \( \{X(f, \omega), \omega \in \Omega\} \) of random variables on \( (\Omega, \mathcal{E}, P) \) having the following properties.
1. \( P\{\omega \in \Omega : X(c_1f_1 + c_2f_2, \omega) = c_1X(f_1, \omega) + c_2X(f_2, \omega)\} = 1, \) \( c_1, c_2 \in \mathbb{R}, \) \( f_1, f_2 \in S(\mathbb{R}^d); \)

2. \( f_n \to f \) in \( S(\mathbb{R}^d) \) implies that \( X(f_n, \cdot) \to X(f, \cdot) \) in law.

The coordinate process \( F \) in Definition 2.1 is a random field on the generalized white noise space \((S'(\mathbb{R}^d), \mathcal{B}, P_\psi)\), which follows immediately from the facts that the above property 1 is fulfilled pointwise and the property 2 is implied by the pointwise convergence \( F(f_n, \omega) \to F(f, \omega) \) as \( n \to \infty \) for all \( \omega \in S'(\mathbb{R}^d) \) and \( f_n \to f \) in \( S(\mathbb{R}^d) \) since the latter is slightly stronger than convergence in law.

Let \( G : S(\mathbb{R}^d) \to S(\mathbb{R}^d) \) be a linear and continuous mapping. Then by the known Schwartz theorem, there exists a distribution \( K \in S'(\mathbb{R}^{2d}) \), hereafter called the kernel of \( G \), such that

\[
(Gf)(x) = \int_{\mathbb{R}^d} K(x, y)f(y)dy, \quad f \in S(\mathbb{R}^d). \tag{15}
\]

It is clear that the conjugate operator \( \tilde{G} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) is a measurable transformation from \((S'(\mathbb{R}^d), \mathcal{B})\) into itself.

**Example 2.2** Let \( \Delta \) be the Laplace operator on \( \mathbb{R}^d \). Let \( G_\alpha \) be the Green function (i.e., fundamental solution) of the pseudo-differential operator \( (-\Delta + m_0^2)^{\alpha/2} \) for some arbitrary (but fixed) \( m_0 > 0 \) and \( 0 < \alpha \). Take \( K(x, y) = G_\alpha(x - y), \) \( x, y \in \mathbb{R}^d \). Then \( G = (-\Delta + m_0^2)^{-\alpha} \) is a linear continuous mapping from \( S(\mathbb{R}^d) \) to \( S(\mathbb{R}^d) \).

To see this, let \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier and inverse Fourier transforms, respectively. Namely,

\[
(\mathcal{F}f)(y) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ixy}f(x)dx
\]

\[
(\mathcal{F}^{-1}f)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ixy}f(y)dy
\]

for \( f \in S(\mathbb{R}^d) \). Then we have

\[
(Gf)(x) = \int_{\mathbb{R}^d} G_\alpha(x - y)f(y)dy, \quad f \in S(\mathbb{R}^d)
\]

and

\[
(\mathcal{F}G_\alpha)(k) = \frac{1}{(2\pi)^{\frac{d}{2}}(|k|^2 + m_0^2)^{\alpha}}, \quad k \in \mathbb{R}^d. \tag{16}
\]
Thus

\[
(F(Gf))(k) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ixk} \int_{\mathbb{R}^d} G_\alpha(x-y) f(y) dy dx
\]

\[
= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-iyk} G_\alpha(y) dy \int_{\mathbb{R}^d} e^{-ixk} f(x) dx
\]

\[
= (2\pi)^{-\frac{d}{2}} (FG_\alpha)(k) \cdot (Ff)(k)
\]

\[
= \frac{1}{(|k|^2 + m_0^2)^\alpha} \cdot (Ff)(k).
\]

Therefore

\[
Gf = (F^{-1} \left( \frac{1}{(|k|^2 + m_0^2)^\alpha} \right) \cdot Ff)(k), \quad f \in \mathcal{S}(\mathbb{R}^d).
\]

We notice that \((-\Delta + m_0^2)^{-\alpha}\) maps real test functions to real test functions. Furthermore, by Theorem IX.4 of [61], \(F\) and \(F^{-1}\) are linear continuous from \(\mathcal{S}'(\mathbb{R}^d)\) to \(\mathcal{S}'(\mathbb{R}^d)\). Hence it suffices to verify that the multiplicative operator defined by \(f(\cdot) \to \frac{1}{(|k|^2 + m_0^2)^\alpha} \cdot f(\cdot)\) is linear and continuous from \(\mathcal{S}'(\mathbb{R}^d)\) to \(\mathcal{S}'(\mathbb{R}^d)\). The linearity is obvious. The continuity is derived from the fact that \(M_h f := hf\) defines a continuous multiplicative operator from \(\mathcal{S}'(\mathbb{R}^d)\) to \(\mathcal{S}'(\mathbb{R}^d)\) if \(h\) is \(C^\infty\)-differentiable and \(h\) itself with all its derivatives are of at most polynomial increasing. This is true because in our case

\[
h(k) = \frac{1}{(|k|^2 + m_0^2)^\alpha}.
\]

In section 1, we had already defined the generalized white noise measure \(P_\psi\) on \((\mathcal{S}'(\mathbb{R}^d), \mathcal{B})\) associated with a Lévy characteristic \(\psi\). Now let \(P_K\) denote the image (probability) measure of \(P_\psi\) under \(\tilde{\mathcal{G}}\), i.e., \(P_K\) is a measure on \((\mathcal{S}'(\mathbb{R}^d), \mathcal{B})\) defined by

\[
P_K(A) := P_\psi(\tilde{\mathcal{G}}^{-1}A), \quad A \in \mathcal{B}.
\]

(17)

Then we have the following result

**Proposition 2.3** The Fourier transform of \(P_K\) is given by

\[
\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i<f,\xi>} dP_K(\xi)
\]

\[
= \exp \left\{ \int_{\mathbb{R}^d} \psi \left( \int_{\mathbb{R}^d} K(x,y) f(y) dy \right) dx \right\}, \quad f \in \mathcal{S}(\mathbb{R}^d).
\]

(18)
Conversely, given a linear and continuous mapping \( \mathcal{G} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d) \) and a Lévy characteristic \( \psi \), there exists a unique probability measure \( P_K \) such that (18) is valid.

**Proof.** For \( f \in \mathcal{S}(\mathbb{R}^d) \), by (17) and Theorem 1.1, we derive that

\[
\int_{\mathcal{S}'(\mathbb{R}^d)} e^{i<f,\xi>} dP_K(\xi) = \int_{\mathcal{S}'(\mathbb{R}^d)} e^{i<\mathcal{G}f,\xi>} dP_\psi(\xi) = \exp \left\{ \int_{\mathbb{R}^d} \psi \left( \int_{\mathbb{R}^d} K(x,y)f(y)dy \right) dx \right\}.
\]

The converse statement is derived analogously to the proof of Theorem 1.1 since the operator \( \mathcal{G} \) is continuous from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}(\mathbb{R}^d) \) and thus the RHS of (18) defines a characteristic functional.

Clearly, from Proposition 2.3, the associated coordinate process \( X : \mathcal{S}(\mathbb{R}^d) \times (\mathcal{S}'(\mathbb{R}^d),\mathcal{B},P_K) \to \mathbb{R} \) given by

\[
X(f,\xi) = <f,\xi> , \ f \in \mathcal{S}(\mathbb{R}^d) , \ \xi \in \mathcal{S}'(\mathbb{R}^d)
\]

is a random field on \( (\mathcal{S}'(\mathbb{R}^d),\mathcal{B},P_K) \). In fact, \( X \) is nothing but \( \mathcal{G}F \) which is defined by

\[
(\mathcal{G}F)(f,\xi) := F(\mathcal{G}f,\xi) , \ f \in \mathcal{S}(\mathbb{R}^d) , \ \xi \in \mathcal{S}'(\mathbb{R}^d).
\]

**Remark 2.4** Concerning Example 2.2, the above construction does not always work if \( m_0 = 0 \), because \((\Delta)^{-\alpha}\) does not map \( \mathcal{S}(\mathbb{R}^d) \) onto \( \mathcal{S}(\mathbb{R}^d) \) for \( \alpha > 0 \). In this case, if \( F \) is a Gaussian white noise, then we can obtain a random field \( X = (\Delta)^{-\alpha}F \) for \( \alpha < \frac{d}{4} \) by the following argument. Take \( \alpha < \frac{d}{4} \), then the scalar product

\[
(f_1,f_2)_\alpha := \int_{\mathbb{R}^d} \frac{(\mathcal{F}f_1)(k) \cdot (\mathcal{F}f_2)(k)}{|k|^{4\alpha}} dk , \ f_1,f_2 \in \mathcal{S}(\mathbb{R}^d)
\]

is continuous with respect to the (Schwartz) topology of \( \mathcal{S}(\mathbb{R}^d) \), where \( \mathcal{F} \) denotes the Fourier transform as introduced previously. Thus \( f \in \mathcal{S}(\mathbb{R}^d) \mapsto \exp\{-\frac{\alpha^2}{2} ||f||_\alpha^2 \} \in [0,\infty) \) is a characteristic functional on \( \mathcal{S}(\mathbb{R}^d) \). By Bochner-Minlos theorem, we get a unique measure \( P_K \) satisfying (18) and hence the associated coordinate process is precisely \( X = (\Delta)^{-\alpha}F \). However, if
one wants to follow a corresponding procedure with a generalized white noise, one needs an explicit calculation to show the continuity of the functional

\[ f \in \mathcal{S}(\mathbb{R}^d) \to \exp \left\{ \int_{\mathbb{R}^d} \psi((-\Delta)^{-\alpha} f)(x) \, dx \right\} \in \mathcal{C}, \]

where \( \psi \) is a Lévy characteristic. Then, by Bochner–Minlos theorem, one can directly construct the measure \( P_K \).

In order to derive a suitable condition for the continuity of the above functional, we note that the characteristic functional of generalized white noise (5) extends continuously from \( \mathcal{S}(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^d, dx) \), provided the corresponding generalized white noise \( F \) has mean zero and finite moments of second order (see Prop. 4.3 of [25]). Furthermore, by the above considerations for the Gaussian case and the fact that \( F \) is unitary on \( L^2(\mathbb{R}^d, dx) \), we get that \((\Delta)^{-\alpha} : \mathcal{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d, dx)\) is continuous, if \( 0 < \alpha < \frac{d}{4} \). Thus, we can construct "mass zero" random fields \( X = (\Delta)^{-\alpha} F \) for \( 0 < \alpha < \frac{d}{4} \) and \( F \) as characterized above.

In what follows, we will always assume that the continuity of \( G : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d) \) holds.

Let us now turn to discuss the invariance of the random field \( X \) under Euclidean transformations. We need to introduce some notations at first. A proper Euclidean transformation of \( \mathbb{R}^d \) is, by definition, an element of the proper Euclidean group \( E_0(\mathbb{R}^d) \) over \( \mathbb{R}^d \). In fact, the proper Euclidean group \( E_0(\mathbb{R}^d) \) is generated by

1. all translations \( T_a : x \in \mathbb{R}^d \to T_a x := x - a \in \mathbb{R}^d, a \in \mathbb{R}^d \);
2. all rotations \( R : x \in \mathbb{R}^d \to Rx \in \mathbb{R}^d \).

The (full) Euclidean group \( E(\mathbb{R}^d) \) over \( \mathbb{R}^d \) is generated by all transformations in \( E_0(\mathbb{R}^d) \) and by all reflections. The group \( E(\mathbb{R}^d) \) is an inhomogeneous orthogonal group, that is, \( E(\mathbb{R}^d) \) is the group of all nonsingular inhomogeneous linear transforms which preserve the Euclidean inner product. It is easy to see that among the reflections it is enough to consider the "time reflection" \( \theta \), defined by writing \( x \in \mathbb{R}^d \) as \( x := (x_0, \vec{x}), x_0 \in \mathbb{R}, \vec{x} \in \mathbb{R}^{d-1} \), calling \( x_0 \) "time coordinate" and setting

\[ \theta x := (-x_0, \vec{x}) \quad , \quad x = (x_0, \vec{x}) \in \mathbb{R} \times \mathbb{R}^{d-1} . \]

If \( T \) is a transformation in the Euclidean group \( E(\mathbb{R}^d) \), the corresponding transformation on a test function \( f \in \mathcal{S}(\mathbb{R}^d) \) is defined by

\[ (T f)(x) := f(T^{-1} x) \quad , \quad x \in \mathbb{R}^d ; \]
and on \( S'(\mathbb{R}^d) \) is defined by duality as follows

\[
<f, T\xi > := <T^{-1}f, \xi > , \quad f \in S(\mathbb{R}^d) , \quad \xi \in S'(\mathbb{R}^d) .
\]

The corresponding transformation on the random field \( X \) is defined by

\[
(TX)(f, \xi) := X(T^{-1}f, \xi) , \quad f \in S(\mathbb{R}^d) , \quad \xi \in S'(\mathbb{R}^d) .
\]

By the invariance of the dualization under Euclidean transformations (which follows from the invariance of Lebesgue measure), we have

\[
(TX)(Tf, \xi) = X(f, \xi) , \quad f \in S(\mathbb{R}^d) , \quad \xi \in S'(\mathbb{R}^d) .
\]

Concerning Euclidean invariance of random fields, we have the following

**Definition 2.5** By Euclidean invariance of the random field \( X \) we mean that the laws of \( X \) and \( TX \) are the same, for each \( T \in E(\mathbb{R}^d) \), i.e., the probability distributions of \( \{X(f, \cdot) : f \in S(\mathbb{R}^d)\} \) and \( \{(TX)(f, \cdot) : f \in S(\mathbb{R}^d)\} \) coincide for each \( T \in E(\mathbb{R}^d) \). In particular, if the laws of \( X \) and \( \theta X \), where \( \theta \) is the ”time–reflection” defined above, are the same, we say that the random field \( X \) is (time–)reflection invariant.

From Bochner–Minlos Theorem, the probability distribution of \( \{X(f, \cdot) : f \in S(\mathbb{R}^d)\} \) is uniquely determined by the characteristic functional \( C_X(f) \), \( f \in S(\mathbb{R}^d) \), and vice versa. Thus, the property of Euclidean invariance of random fields is also determined by means of characteristic functionals.

We say that \( G \) is \( T \)-invariant, for some \( T \in E(\mathbb{R}^d) \), if \( GT = TG \). \( G \) is called Euclidean invariant if \( G \) is invariant under all \( T \in E_0(\mathbb{R}^d) \). In case \( G \) is translation invariant its kernel \( K \) has the form \( K(x,y) = G(x-y)(\text{cf. p.39 of } [67]) \). If the kernel \( G \) of \( \mathcal{G} \) is also invariant under orthogonal transformations, then \( \mathcal{G} \) is invariant under all \( T \in E(\mathbb{R}^d) \). In this case we also say for simplicity that \( G \) is the Euclidean invariant kernel of \( \mathcal{G} \). The action of \( \mathcal{G} \) on test function in \( S(\mathbb{R}^d) \) (and by duality on \( S'(\mathbb{R}^d) \) as well as on random fields) in the translation invariant case is by convolution

\[
(\mathcal{G}f)(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy = \int_{\mathbb{R}^d} G(x-y)f(y)dy \quad x \in \mathbb{R}^d .
\]

We then also write \( \mathcal{G}f \) as \( G * f \).

**Remark 2.6** The kernel \( G_\alpha \) determined by formula (16) in Example 2.2 is given by

\[
G_\alpha(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ikx}}{(|k|^2 + m^2)^\alpha} dk , \quad x \in \mathbb{R}^d ,
\]
where the integral has to be understood in the sense of a Fourier–transform of a tempered distribution. It is invariant under all orthogonal transformations. This can be verified directly by changing integral variables in the above formula since orthogonal transforms leave $|k|$ and $dk$ invariant.

Moreover, we have following result

**Proposition 2.7** Assume that the mapping $G : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is Euclidean–invariant, then the random field $X = GF$ is Euclidean–invariant.

**Proof.** By Bochner-Minlos Theorem, it is sufficient to show that

$$C_X(f) = C_{TX}(f), \ f \in S(\mathbb{R}^d)$$

for every $T \in E(\mathbb{R}^d)$.

In fact, we have

$$C_{TX}(f) = E \left[ e^{iT X(f, \cdot)} \right]$$

$$= E \left[ e^{iX(T^{-1}f, \cdot)} \right]$$

$$= \exp \left\{ \int_{\mathbb{R}^d} \psi(G(T^{-1}f)(x)) dx \right\}, \ f \in S(\mathbb{R}^d),$$

where the last equality follows from by (18). So we need only to verify that

$$\int_{\mathbb{R}^d} \psi((G(T f))(x)) dx = \int_{\mathbb{R}^d} \psi((G f)(x)) dx .$$

This holds by using the invariance of Lebesgue measure under the transformation $x \to T^{-1}x$ and the fact that

$$(T^{-1}(G f))(x) = (G(T^{-1}f))(x) .$$

Hereafter, we only deal with Euclidean invariant kernels, the derived random fields are then also Euclidean invariant. We call such random fields Euclidean random fields. From Remark 2.6, the integral kernel $G_{\alpha}$ defined in Example 2.2 is Euclidean invariant. Moreover, since the translation invariance implies that the integral kernels are of convolution type, the corresponding Euclidean random fields are convoluted generalized white noise. We simply denote the convoluted generalized white noise $X$ by $X := G_{\alpha} * F$. 

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3 The Schwinger Functions of the Model and their Basic Properties

In 1973, E. Nelson [53] showed how to construct a relativistic quantum field theory (QFT) from an Euclidean Markov field. Inspired by this, in [56] and [57] Osterwalder and Schrader (see also [36], [42], [72]) gave a set of axioms, where Schwinger functions \( \{ S_n \}_{n \in \mathbb{N}_0} \) defined on the Euclidean space–time \( E_d \) can be analytically continued to Wightman distributions, i.e. to the vacuum expectation values of a relativistic QFT. (Here and in the following we use the "sans-serif" \( S_n \) for Schwinger functions in general, whereas the Schwinger functions of our concrete model are denoted by "italic" \( S_n \).)

Apart from existence of an analytic continuation, these axioms are \((E0)\) Temperedness, \((E1)\) Euclidean invariance, \((E2)\) Reflection positivity, \((E3)\) Symmetry and \((E4)\) cluster property. In the case of Euclidean Markov fields and also in the more general case of Euclidean reflection positive fields [33], Schwinger functions fulfilling \((E0)–(E4)\) are obtained as the moments of the Euclidean field.

In this Section we will calculate the "Schwinger functions" \( S_n \) of \( X \), which are by definition the moment functions of the convoluted generalized white noise \( X \). We will verify \((E0), (E1)\) and \((E3)\). A proof of \((E4)\) is given in Section 4. In section 5 a partial negative result on \((E2)\) is derived for the case of convoluted generalized white noises with a non–zero Poisson part.

We now fix a Lévy characteristic \( \psi \), such that the Lévy measure \( M \) has moments of all orders. With \( F \) we denote the generalized white noise determined by \( \psi \).

**Lemma 3.1** Let \( C^T_F \) denote the functional which maps \( \varphi \in S(\mathbb{R}^d) \) to \( \int_{\mathbb{R}^d} \psi(\varphi(x))dx \). Then partial derivatives of all orders of \( C^T_F \) exist everywhere on \( S(\mathbb{R}^d) \). For \( \varphi_1 \ldots \varphi_n \in S(\mathbb{R}^d) \) we get that

\[
\frac{1}{i^n} \frac{\partial^n}{\partial \varphi_1 \ldots \partial \varphi_n} C^T_F \big|_0 = c_n \int_{\mathbb{R}^d} \varphi_1 \ldots \varphi_n dx. \tag{19}
\]

Here we introduced the constants \( c_n \) defined as

\[
c_1 := a + \int_{\mathbb{R} \setminus \{0\}} \frac{s^3}{1 + s^2} dM(s)

c_2 := \sigma^2 + \int_{\mathbb{R} \setminus \{0\}} s^2 dM(s)

c_n := \int_{\mathbb{R} \setminus \{0\}} s^n dM(s), n \geq 3.
\]
As it will be explained shortly, throughout this paper the superscript "T" stands for the operation of "truncation" (cf. [41]).

Proof. Since the differentiability of \( \varphi \rightarrow ia \int_{\mathbb{R}^d} \varphi dx \) and \( \varphi \rightarrow \frac{\sigma^2}{2} \int_{\mathbb{R}^d} \varphi^2 dx \) is immediate, we only have to deal with the Poisson part of \( \psi \). We remark that \( |e^{iy} - 1| \leq |y| \) for all \( y \in \mathbb{R} \). Therefore

\[
\frac{1}{t_1} |e^{is[\varphi(x)+t_1\varphi_1(x)]} - e^{is\varphi(x)}| \leq |s\varphi_1(x)|,
\]

for all \( t_1 > 0, s \in \mathbb{R} \). This shows that the LHS is uniformly bounded (in \( t_1 \)) by a function in \( L^1(\mathbb{R} \setminus \{0\} \times \mathbb{R}^d, dM \otimes dx) \)-function. Analogously, for all \( t_n > 0, \)

\[
\frac{1}{t_n} |s^{n-1}\varphi_1(x) \cdots \varphi_{n-1}(x)(e^{is[\varphi(x)+t_n\varphi_n(x)]} - e^{is\varphi(x)})| \leq |s^n\varphi_1(x) \cdots \varphi_n(x)|,
\]

and again the RHS is an uniform \( L^1(\mathbb{R} \setminus \{0\} \times \mathbb{R}^d, dM \otimes dx) \)-bound. Thus we may interchange partial derivatives and integration by the dominated convergence theorem:

\[
\frac{1}{i^n} \frac{\partial^n}{\partial \varphi_1 \cdots \partial \varphi_n} \int_{\mathbb{R}^d} \int_{\mathbb{R} \setminus \{0\}} \left( e^{is\varphi} - 1 - \frac{is\varphi(x)}{1 + s^2} \right) dM(s) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R} \setminus \{0\}} e^{is\varphi_1(x) \cdots \varphi_n(x)} s^n dM(s) dx
\]

if \( n \geq 2 \) and

\[
\cdots = \int_{\mathbb{R}^d} \int_{\mathbb{R} \setminus \{0\}} e^{is\varphi(x)} \varphi_1(x) s^3 \frac{1}{1 + s^2} dM(s) dx
\]

if \( n = 1 \). Setting \( \varphi = 0 \) and taking into account the linear and Gaussian part, we get (19).

We have \( C_F = \exp C_T^F \). Consequently \( C_F \) has partial derivatives of any order, and it follows, that all moments of \( F \) — i.e. the expectation values of \( \langle \varphi_1, F \rangle \cdots \langle \varphi_n, F \rangle \), exist and are equal \( i^{-n} \) times the \( n \)-th order partial derivative of \( C_F \) w.r.t. \( \varphi_1 \cdots \varphi_n \) at the point \( \varphi = 0 \), cf. [51].

Definition 3.2 Let \( \varphi_1 \cdots \varphi_n \in \mathcal{S}(\mathbb{R}^d) \). We define \( M^F_n \), the \( n \)-th moment function of \( F \), by

\[
M^F_n(\varphi_1 \otimes \cdots \otimes \varphi_n) = \int_{\mathcal{S}'(\mathbb{R}^d)} \langle \varphi_1, \omega \rangle \cdots \langle \varphi_n, \omega \rangle dP_\psi(\omega)
\]
By the above remark this equals

\[ \frac{\partial^n}{\partial \varphi_1 \ldots \partial \varphi_n} C_F |_0 \]

(21)

In order to calculate partial derivatives of \( C_F \) of any order we need a generalized chain rule:

**Lemma 3.3** Let \( V \) be a vector space. Let \( g : V \rightarrow \mathbb{C} \) be infinitely often partial differentiable and let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be analytic. Then for \( v_1 \ldots v_n \in V \)

\[ \frac{\partial^n}{\partial v_1 \ldots \partial v_n} f \circ g = \sum_{k=1}^{n} f^{(k)} \circ g \sum_{I \in P_k(n)} \prod_{\{j_1 \ldots j_l\} \in I} \frac{\partial^l g}{\partial v_{j_1} \ldots \partial v_{j_l}} \]

(22)

holds on \( V \). Here \( P_k(n) \) is the collection of partitions of \( \{1 \ldots n\} \) into exactly \( k \) disjoint subsets, \( f^{(k)}(x) = \left( \frac{d^k}{dx^k} f \right)(x) \).

**Proof.** We proceed by induction over \( n \). The statement for \( n = 1 \) is Leibniz’ chain rule. Observe that

\[ \frac{\partial^{n+1}}{\partial v_{n+1} \partial v_1 \ldots \partial v_n} g \circ f \]

\[ = \frac{\partial}{\partial v_{n+1}} \left( \sum_{k=1}^{n} f^{(k)} \circ g \sum_{I \in P_k(n)} \prod_{\{j_1 \ldots j_l\} \in I} \frac{\partial^l g}{\partial v_{j_1} \ldots \partial v_{j_l}} \right) \]

\[ = \sum_{k=1}^{n} \left\{ f^{(k+1)} \circ g \sum_{I \in P_k(n)} \prod_{\{j_1 \ldots j_l\} \in I} \frac{\partial^l g}{\partial v_{j_1} \ldots \partial v_{j_l}} \right\} \frac{\partial g}{\partial v_{n+1}} \]

\[ + f^{(k)} \circ g \sum_{I \in P_k(n)} \prod_{m=1}^{k} \frac{\partial^l g}{\partial v_{j_1} \ldots \partial v_{j_l}} \prod_{i=1}^{k} \frac{\partial^l+1 g}{\partial v_{j_{i+1}} \ldots \partial v_{j_m} \partial v_{n+1}} \}

We denote \( I \in P_k(n) \) by \( I = \{I_1, \ldots, I_k\} \) and set \( I_i = \{j_{i_1}, \ldots, j_{l_i}\} \) (where, of course, \( l \) depends on \( i \)). Using collections of partitions \( P_k(n+1) \), we can “reindex” the sums in the latter expression and get:

\[ = \sum_{k=1}^{n} \left\{ f^{(k+1)} \circ g \sum_{I \in P_{k+1}^{(n+1)}} \prod_{\{j_1 \ldots j_l\} \in I} \frac{\partial^l g}{\partial v_{j_1} \ldots \partial v_{j_l}} \right\} \]

\[ + f^{(k)} \circ g \sum_{I \in P_{k+1}^{(n+1)}} \prod_{i \in I} \frac{\partial^l g}{\partial v_{j_1} \ldots \partial v_{j_l}} \}

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\[
\sum_{k=1}^{n+1} \left\{ f^{(k)} \circ g \sum_{I \in P_k^{(n+1)}, \{n+1\} \in I, \{j_1, \ldots, j_l\} \in I} \prod_{j_1, \ldots, j_l} \frac{\partial^l g}{\partial v_{j_1} \cdots \partial v_{j_l}} \right. \\
+ f^{(k)} \circ g \sum_{I \in P_k^{(n+1)}, \{n+1\} \notin I, \{j_1, \ldots, j_l\} \in I} \prod_{j_1, \ldots, j_l} \frac{\partial^l g}{\partial v_{j_1} \cdots \partial v_{j_l}} \right\}
\]

On the RHS we have also reindexed \( k + 1 \) in the first sum in the braces to \( k \) with the sum ranging from 2 to \( n + 1 \). Since the only partition in \( P_1^{(n+1)} \), \( \{1, \ldots, n+1\} \), does not contain \( \{n+1\} \), we may sum from 1 to \( n + 1 \). Similarly in the second sum we may extend the sum from \( k = 1 \) to \( k = 1 \ldots n + 1 \), since \( P_{n+1}^{(n+1)} \) contains only \( \{1\}, \ldots, \{n+1\} \) and thus gives no contributions to this sum. But the last expression obtained obviously equals (22) with \( n \) replaced by \((n + 1)\).

We remark that for the case of only one variable \( v = v_1 = \ldots = v_n \) we get Faa di Brunos expansion formula \([46]\).

**Corollary 3.4** (Cumulant formula) Let \( V, g \) as in Lemma 3.3 and let \( f \) be the exponential function. Furthermore assume \( g(0) = 0 \). Then \( f^{(k)} \circ g(0) = 1 \) for all \( k \in \mathbb{N} \). Let \( P^{(n)} \) stand for the collection of all partitions \( I \) of \( \{1 \ldots n\} \) into disjoint subsets. It follows that for \( v_1, \ldots, v_n \in V \) we get:

\[
\frac{\partial^n}{\partial v_1 \cdots \partial v_n} \exp g \big|_0 = \sum_{I \in P^{(n)}} \prod_{\{j_1, \ldots, j_l\} \in I} \frac{\partial^l g}{\partial v_{j_1} \cdots \partial v_{j_l}} g \big|_0 .
\]

**Corollary 3.5** (Wick’s Theorem) Let \( f \) be the exponential function and \( g \) a quadratic function on a vector space \( V \), i.e. \( g(tv) = t^2 g(v) \) for all \( v \in V \). Then by Corollary 3.4

\[
\frac{\partial^n}{\partial v_1 \cdots \partial v_n} \exp g \big|_0 = \begin{cases} 
\sum_{I \in \text{pairings} \{j_1, j_2\} \in I} \prod_{\{j_1, j_2\} \in I} \frac{\partial^2 g}{\partial v_{j_1} \partial v_{j_2}} g \big|_0 & \text{for } n \text{ even} \\
0 & \text{for } n \text{ odd}
\end{cases}
\]

Here the pairings are those partitions \( I \) of \( \{1 \ldots n\} \), where all subsets in \( I \) do contain exactly two elements. This follows from the fact that all partial derivatives of \( g \) are zero, except for partial derivatives of order 2.

**Proposition 3.6** Set \( \varphi_1 \ldots \varphi_n \in \mathcal{S}(\mathbb{R}^d) \). Then

\[
M^F_n (\varphi_1 \otimes \ldots \otimes \varphi_n) = \sum_{I \in P^{(n)}} \prod_{\{j_1, \ldots, j_l\} \in I} c_I \int_{\mathbb{R}^d} \varphi_{j_1} \cdots \varphi_{j_l} dx
\]

(25)
Proof. (25) follows directly from (21), $C_F = \exp C_F^T$, Lemma 3.1 and Corollary 3.4.

Remark 3.7 Choosing $a$ in (20) such that $c_1 = 0$ implies that $F$ has mean zero. If furthermore $M$ is a symmetric measure w.r.t. reflections at zero, then all $c_n$ vanish for $n$ odd. For such $n$ also $M_n^F$ is zero, since in this case at least one $c_l$ with $l$ odd appears in every summand on the RHS of (25).

We now fix, as in section 2, a linear continuous map $G : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ and denote its dual by $\tilde{G}$. Furthermore, we assume that $G$ is Euclidean invariant. Then there exists a convolution kernel $G$ which is invariant under orthogonal transformations, such that $G \varphi = G * \varphi$ for all $\varphi \in S(\mathbb{R}^d)$ (cf. Section 2). Define, as explained in section 2, $P_G = P_\psi \circ (\tilde{G})^{-1}$ and let $X$ be the coordinate process w.r.t. $P_G$.

Definition 3.8 ("Schwinger functions of $X"$) Set $\varphi_1 \ldots \varphi_n \in S(\mathbb{R}^d)$. We define the $n$-th Schwinger function $S_n$ as the $n$-th moment of $X$, i.e.

$$S_n(\varphi_1 \otimes \ldots \otimes \varphi_n) = \int_{S'(\mathbb{R}^d)} <\varphi_1, \omega> \ldots <\varphi_n, \omega> dP_G(\omega) \quad n \in \mathbb{N}_0 \quad (26)$$

Proposition 3.9 The Schwinger functions $S_n$ defined above are symmetric and Euclidean invariant tempered distributions, i.e. $S_n \in S'(\mathbb{R}^{dn})$ for $n \geq 1$. Furthermore for $\varphi_1 \ldots \varphi_n \in S(\mathbb{R}^d)$ we have

$$S_n(\varphi_1 \otimes \ldots \otimes \varphi_n) = \sum_{I \in P(n)} \prod_{\{j_1 \ldots j_l\} \in I} c_l \int_{\mathbb{R}^d} G * \varphi_{j_1} \ldots G * \varphi_{j_l} d\omega \quad (27)$$

Proof. The symmetry follows directly from Definition 3.8. Euclidean invariance of the moment functions follows from the Euclidean invariance in law of the random field $X$.

Now we first prove (27). By the transformation formula, the RHS of (26) is equal to

$$\int_{S'(\mathbb{R}^d)} <\varphi_1, \tilde{G}\omega> \ldots <\varphi_n, \tilde{G}\omega> dP_\psi(\omega) = \int_{S'(\mathbb{R}^d)} <G * \varphi_1, \omega> \ldots <G * \varphi_n, \omega> dP_\psi(\omega)$$

This together with Proposition 3.6 now implies (27).

Concerning temperedness we remark that by (27) $S_n$ is a sum of tensor products of linear functionals, say $S_l$, which map $\varphi_1 \otimes \ldots \otimes \varphi_l$ into
\[ c_l \int_{\mathbb{R}^d} G * \varphi_1 \ldots G * \varphi_l \, dx. \]

Fix \( \varphi_1 \ldots \varphi_{j-1} \varphi_{j+1} \ldots \varphi_l \in S(\mathbb{R}^d) \). Then \( \varphi_j \mapsto G * \varphi_1 \mapsto G * \varphi_j \prod_{m=1, m \neq j} G * \varphi_m \mapsto c_l \int_{\mathbb{R}^d} G * \varphi_1 \ldots G * \varphi_l \, dx \) is a map composed of \( S(\mathbb{R}^d) \)–continuous mappings and therefore is continuous in \( \varphi_j \) alone, provided the other \( \varphi_i \)'s are fixed. A use of Schwartz nuclear theorem yields the temperedness of the \( S^T \) and a second application of the nuclear theorem then implies \( S_n \in S'(\mathbb{R}^d) \).

4 Truncated Schwinger Functions and the Cluster Property

In this section, let \( \{S_n\}_{n \in \mathbb{N}_0} \) be a sequence of distributions, where \( S_0 = 1 \) and \( S_n \in S'(\mathbb{R}^{dn}) \) for \( n \geq 1 \). We define truncated distributions \( S^T_n \) \( n \geq 1 \) in the sense of Haag [41]. The \( \{S_n\}_{n \in \mathbb{N}_0} \) determine the corresponding sequence of truncated distributions uniquely and vice versa, and we can translate many properties of one sequence into properties of the other. This is quite obvious e.g. for \( (E0) \), \( (E1) \) and \( (E3) \).

Making use of arguments in the classical papers on the so–called asymptotic condition in axiomatic QFT ([41],[22],[23],[48]) we obtain the equivalence of the cluster property \( (E4) \) of translation invariant \( \{S_n\}_{n \in \mathbb{N}_0} \) and the cluster property of the truncated Schwinger functions \( \{S^T_n\}_{n \in \mathbb{N}} \).

As an immediate consequence of formula (27) we have explicit formulae for the truncated Schwinger functions of our model and we can easily check their "truncated" cluster property in order to verify \( (E4) \).

Definition 4.1 Let \( \{S_n\}_{n \in \mathbb{N}_0} \) be a sequence of distributions with \( S_0 = 1 \) and \( S_n \in S'(\mathbb{R}^{dn}) \) for \( n \geq 1 \). Let \( \varphi_1 \ldots \varphi_n \in S(\mathbb{R}^d) \). By the relation

\[ S_n(\varphi_1 \otimes \ldots \otimes \varphi_n) = \sum_{I \in \mathcal{P}(n)} \prod_{\{j_1, \ldots, j_l\} \in I} S^T_{l}(\varphi_{j_1} \otimes \ldots \otimes \varphi_{j_l}) \quad n \geq 1 \quad (28) \]

we recursively define the \( n \)–th truncated distribution \( S^T_n \). Here, for \( \{j_1, \ldots, j_l\} \in I \) we assume \( j_1 < j_2 < \cdots < j_l \).

Remark 4.2

1. By the Schwartz nuclear theorem the sequence \( \{S_n\}_{n \in \mathbb{N}_0} \) uniquely determines the sequence \( \{S^T_n\}_{n \in \mathbb{N}} \) and vice versa.

2. All \( S^T_n \) are Euclidean (translation) invariant if and only if all \( S_n \) are euclidean (translation) invariant. The same equivalence holds for temperedness and symmetry (see e.g. [22] for the symmetry).
From now on we assume at least translation invariance for \( \{S_n\}_{n \in \mathbb{N}_0} \) and \( \{S^T_n\}_{n \in \mathbb{N}} \) respectively. And we will call these distributions (truncated) Schwinger functions, even though at this level they might have little to do with QFT.

**Definition 4.3** Let \( a \in \mathbb{R}^d, a \neq 0 \) and \( \lambda \in \mathbb{R} \). Let \( T_{\lambda a} \) denote the representation of the translation by \( \lambda a \) on \( S(\mathbb{R}^{dn}) \), \( n \in \mathbb{N} \). Take \( m, n \geq 1, \varphi_1 \ldots \varphi_{m+n} \in S(\mathbb{R}^d) \).

1. **cluster property** \( (E4) \) A sequence of Schwinger functions \( \{S_n\}_{n \in \mathbb{N}_0} \) has the cluster property if for all \( n, m \geq 1 \)

\[
\lim_{\lambda \to \infty} \left\{ S_{m+n}(\varphi_1 \otimes \ldots \otimes \varphi_m \otimes T_{\lambda a}(\varphi_{m+1} \otimes \ldots \otimes \varphi_{m+n})) - S_m(\varphi_1 \otimes \ldots \otimes \varphi_m)S_n(\varphi_{m+1} \otimes \ldots \otimes \varphi_{m+n}) \right\} = 0 . \tag{29}
\]

2. **cluster property of truncated Schwinger functions** \( (E4T) \) A sequence of truncated Schwinger functions \( \{S^T_n\}_{n \in \mathbb{N}} \) has the cluster property of truncated Schwinger functions, if for all \( n, m \geq 1 \)

\[
\lim_{\lambda \to \infty} S^T_{m+n}(\varphi_1 \otimes \ldots \otimes \varphi_m \otimes T_{\lambda a}(\varphi_{m+1} \otimes \ldots \otimes \varphi_{m+n})) = 0 . \tag{30}
\]

**Remark 4.4** We could also replace \( \lim_{\lambda \to \infty} (\cdot) \) in (29) and (30) by \( \lim_{\lambda \to \infty} |\lambda|^N (\cdot) \) for \( N \) arbitrary. Such conditions should be seen as cluster-properties for short-range interactions. Indeed they would exclude the mass zero cases. Take e.g. the free Markov field of mass zero \( X = (-\Delta)^{-\frac{1}{2}}F \) in \( d \) dimensions, \( d \geq 3 \), where \( F \) is a Gaussian white noise. Then (29) and (30) tend to zero only as \( \lambda^{-d+2} \), as \( \lambda \to \infty \). Nevertheless, by the same proofs as in Theorem 4.5 and Corollary 4.7 below we can also show that the Schwinger functions of our model have the "short-range" cluster property. This already indicates the existence of a "mass-gap" in the corresponding relativistic theory, obtained by the analytic continuation of the (truncated) Schwinger functions in Sections 6 and 7.

The following result is the Euclidean analogue to one proven by Araki [22], [23] for the case of (truncated) Wightman functions.
Theorem 4.5 Let \( \{ S_n \}_{n \in \mathbb{N}}, \{ S^T_n \}_{n \in \mathbb{N}} \) be as in Definition 4.1 and let the \( \{ S_n \}_{n \in \mathbb{N}_0}, \{ S^T_n \}_{n \in \mathbb{N}} \) be translation invariant. Then \( \{ S_n \}_{n \in \mathbb{N}_0} \) has the cluster property, if and only if \( \{ S^T_n \}_{n \in \mathbb{N}} \) has the cluster property of the truncated Schwinger functions.

**Proof** \((E4) \Rightarrow (E4T)\) Assume there exist \( m, n \geq 1 \), such that \( \varphi_1 \ldots \varphi_{n+m} \in S(\mathbb{R}^d) \) and a \( a \in \mathbb{R}^d \setminus \{0\} \), such that the limit in (28) is not zero or does not exist. Let furthermore \( n + m \) be minimal w.r.t. this property.

Define \( \varphi^\lambda_k = \varphi_k \) for \( k = 1 \ldots m \) and \( = T_\lambda a \varphi_k \) for \( k = m + 1 \ldots m + n \). By translation invariance the LHS of (29) is then equal to

\[
\lim_{\lambda \to \infty} \sum_{I \in \mathcal{P}_{m,n}^{(m+n)}} \prod_{(j_1 \ldots j_l) \in I} S^T_I (\varphi^\lambda_{j_1} \otimes \ldots \otimes \varphi^\lambda_{j_l})
\]

The symbol \( \mathcal{P}_{m,n}^{(m+n)} \) stands for all partitions \( I \) of \( \{1 \ldots n + m\} \) into disjoint subsets, such that in each partition \( I \) there is at least one subset \( \{j_1, \ldots, j_l\} \in I \) such that \( \{j_1, \ldots, j_l\} \cap \{1 \ldots m\} \neq \emptyset, \{j_1, \ldots, j_l\} \cap \{m + 1 \ldots m + n\} \neq \emptyset \). By the assumption \((E4)\), the above expression equals zero.

Furthermore, every summand except for the one indexed by \( I = \{\{1 \ldots n + m\}\} \) tends to zero, since in each such summand at least one truncated Schwinger function in the product is evaluated on some \( \varphi^\lambda_k \)'s \( 1 \leq k \leq m \) and some \( \varphi^\lambda_k \)'s \( m + 1 \leq k \leq m + n \), at the same time. By the minimality of \( n + m \) this factor tends to zero as \( \lambda \to \infty \). The other factors in the product either tend to zero (by minimality of \( n + m \)) as \( \lambda \to \infty \) or are constant, either by the definition of the \( \varphi^\lambda_k \) for \( k = 1, \ldots, m \) or by the translation invariance of the \( S^T_I \).

Consequently also the summand indexed by \( I = \{\{1, \ldots, n + m\}\} \) has to converge to zero as \( \lambda \to \infty \), which is in contradiction with the above assumptions on \( n, m \).

\((E4T) \Rightarrow (E4)\) Fix \( \varphi_1 \ldots \varphi_{n+m} \) and define \( \varphi^\lambda_k \) as above. Then

\[
\lim_{\lambda \to \infty} S_{m+n} (\varphi^\lambda_1 \otimes \ldots \otimes \varphi^\lambda_{m+n})
= \lim_{\lambda \to \infty} \left\{ \sum_{I \in \mathcal{P}_{m,n}^{(m+n)}} \prod_{(j_1 \ldots j_l) \in I} S^T_I (\varphi^\lambda_{j_1} \otimes \ldots \otimes \varphi^\lambda_{j_l})
+ \sum_{I \in \mathcal{P}_{m,n}^{(m+n)}} \prod_{(j_1 \ldots j_l) \in I} S^T_I (\varphi^\lambda_{j_1} \otimes \ldots \otimes \varphi^\lambda_{j_l}) \right\},
\]
where \((P_{m,n}^{(m+n)})^c := P_{m,n}^{(m+n)} \setminus P_{m,n}^{(m+n)}\). In the second term all products contain at least one factor that tends to zero as \(\lambda \to \infty\) by \((E4T)\). The other factors are constant (either by the definition of the \(\phi^\lambda_k\) or by the translation invariance of the \(\mathcal{S}_T^\lambda\)) or tend to zero by \((E4T)\), again. Thus, the whole second sum vanishes for \(\lambda \to \infty\).

In the first term all \(\mathcal{S}_T^\lambda\) are evaluated on \(\varphi^\lambda_{j_1} \ldots \varphi^\lambda_{j_l}\) such that either \(\{j_1 \ldots j_l\} \subset \{1 \ldots m\}\) or \(\subset \{m+1 \ldots m+n\}\). It follows from the definition of the \(\phi^\lambda_k\) or the translation invariance of the \(\mathcal{S}_T^\lambda\), that all factors do not depend on \(\lambda\). In the first term, we may omit the \(\lambda\)'s therefore. The first term by this argument equals

\[
\left( \sum_{I \in P^{(m)}} \prod_{\{j_1 \ldots j_l\} \in I} \mathcal{S}_T^\lambda(\varphi_{j_1} \otimes \ldots \otimes \varphi_{j_l}) \right) 
\times \left( \sum_{I \in P^{(n)}} \prod_{\{j_1 \ldots j_l\} \in I} \mathcal{S}_T^\lambda(\varphi_{j_1+m} \otimes \ldots \otimes \varphi_{j_l+m}) \right) 
= \mathcal{S}_m(\varphi_1 \otimes \ldots \otimes \varphi_m) \mathcal{S}_n(\varphi_{m+1} \otimes \ldots \otimes \varphi_{m+n}),
\]

from which we get (29).

\textbf{Remark 4.6} Let \(\{S_n\}_{n \in \mathbb{N}_0}\) be defined according to Definition 3.8. Let \(\{\mathcal{S}_T^\lambda_n\}_{n \in \mathbb{N}}\) denote the sequence of truncated Schwinger functions. Then, of course, by comparison of (28) and (27) we get

\[
\mathcal{S}_T^\lambda_n(\varphi_1 \otimes \ldots \otimes \varphi_n) = c_n \int_{\mathbb{R}^d} G^* \varphi_1 \ldots G^* \varphi_n dx \quad (31)
\]

for \(\varphi_1 \ldots \varphi_n \in \mathcal{S}(\mathbb{R}^d)\).

\textbf{Corollary 4.7} Let \(\{S_n\}_{n \in \mathbb{N}_0}\) be as in Definition 3.8. Then \(\{S_n\}_{n \in \mathbb{N}_0}\) has

the cluster property \((E4)\).

\textbf{Proof.} By Theorem 4.5 it suffices to show \((E4T)\) for \(\{\mathcal{S}_T^\lambda_n\}_{n \in \mathbb{N}}\). Fix \(a \neq 0\) in \(\mathbb{R}^d\) and let \(\lambda \in \mathbb{R}^d\), \(\varphi_1 \ldots \varphi_{n+m} \in \mathcal{S}(\mathbb{R}^d)\), \(m, n \geq 1\). We have

\[
\lim_{\lambda \to \infty} \mathcal{S}_T^{m+n}(\varphi_1 \otimes \ldots \otimes \varphi_m T_{\lambda a}(\varphi_{m+1} \otimes \ldots \otimes \varphi_{m+n}))
= \lim_{\lambda \to \infty} c_{m+n} \int_{\mathbb{R}^d} G^* \varphi_1 \ldots G^* \varphi_m G \ast T_{\lambda a} \varphi_{m+1} \ldots G \ast T_{\lambda a} \varphi_{m+n} dx
= \lim_{\lambda \to \infty} c_{m+n} \int_{\mathbb{R}^d} G^* \varphi_1 \ldots G^* \varphi_m T_{\lambda a}(G^* \varphi_{m+1} \ldots G^* \varphi_{m+n}) dx
\]
On the RHS we shift away a fast falling function $G \ast \varphi_{m+1} \ldots G \ast \varphi_{m+n}$ from the fixed fast falling function $G \ast \varphi_1 \ldots G \ast \varphi_m$. The RHS therefore approaches zero faster than any negative power of $\lambda$ falls to zero.

5 On Reflection Positivity

In Section 4 we saw that many properties of Schwinger functions can be directly translated into related properties of truncated Schwinger functions. How about reflection positivity then?

Let us first discuss a simple related case. If $C_\mu$ denotes the Fourier–transform of a probability measure $\mu$ on the real line, and $C_\mu$ is analytic in a neighborhood of zero, then the derivatives of $C_\mu$ fulfill the positivity condition

$$\left(\frac{a_n}{i^n} \frac{d^n}{dx^n} + \ldots + \frac{a_1}{i} \frac{d}{dx} + a_0\right)^2 C_\mu |_{0 \geq 0} \geq 0 \quad (32)$$

for all $a_0 \ldots a_n \in \mathbb{R}$. This is related to $(E2)$. Now suppose, that $C_\mu = \exp(C_\mu^T)$ and consider $C_\mu^\lambda = \exp(\lambda C_\mu^T)$. Schönbergs theorem (see [27]) says that $C_\mu^\lambda$ is positive definite for all $\lambda \in \mathbb{R}_+$ if and only if $C_\mu^T$ is a Lévy characteristic cf. section 1). One can show, that the derivatives of $C_\mu^T$ at zero (the cumulants or "truncated moments of $\mu$") fulfill the same positivity condition (32) if we set $a_0 = 0$, since a Lévy characteristic can be approximated by a sequence $b_n - C_n(0) + C_n$, where for $n \in \mathbb{N}$ $C_n$ is a positive definite function and $b_n \geq 0$ (cf. [27]). On the other hand, if we can disprove one of the latter positivity conditions, we will, as a consequence of Schönbergs theorem, find some $\lambda \in \mathbb{R}_+$ such that $C_\mu^\lambda$ is not positive definite and therefore (32) does not hold for some $\lambda$, $n$, $a_l \neq 0$, $l = 0, \ldots, n$.

Along these lines we now construct some counter examples of convoluted generalized white noises $X = G \ast F$ with nonzero Poisson part, which do not have the property of reflection positivity. It is interesting that such $X$ do exist even in such cases where the corresponding convoluted Gaussian white noise is reflection positive. Roughly speaking, the Schwinger functions $\{S_n\}_{n \in \mathbb{N}_0}$ belonging to $X$ do not have the property of reflection positivity, if the terms in the $S_n$ emerging from the "interaction" are large in comparison with the "free" terms. It remains an open question, whether reflection positivity holds or does not hold in other cases.

Let us start with some definitions borrowed from the theory of infinitely divisible random distributions.
Definition 5.1 Let $\{S_n\}_{n \in \mathbb{N}_0}, S_n \in S'(\mathbb{R}^{d_n})$ for $n \geq 1$, $S_0 = 1$ be a sequence of distributions and $\{S^n_T\}_{n \in \mathbb{N}}$ the corresponding truncated sequence. By $\mathbb{R}^d_+$ we denote the set of $x \in \mathbb{R}^d : x = (x^0, \vec{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}, x^0 > 0$. Let $S((\mathbb{R}^d_+)^n)$ be the Schwartz–functions on $\mathbb{R}^{dn}$ with support in $(\mathbb{R}^d_+)^n$. $\theta$ is the "time–reflection", i.e. $\theta(x^0, \vec{x}) = (-x^0, \vec{x})$.

1. We call $\{S_n\}_{n \in \mathbb{N}}$ reflection infinitely divisible if for all $\lambda \in \mathbb{R}_+$ the sequence of Schwinger functions $\{S^n_\lambda\}_{n \in \mathbb{N}_0}$ determined by the truncated sequence $\{\lambda S^n_T\}_{n \in \mathbb{N}}$ is reflection positive.

2. We say, a sequence of truncated Schwinger functions $\{S^n_T\}_{n \in \mathbb{N}}$ is conditional reflection positive, if for test functions $\varphi_k \in S((\mathbb{R}^d_+)^k), k = 1, \ldots, n$ the inequality

$$\sum_{k,j=1}^n S^n_{k+l} (\theta \varphi_k \otimes \varphi_l) \geq 0 \quad (33)$$

holds.

Observe that the sum in (33) is over $k, l = 1 \ldots n$, which distinguishes conditional reflection positivity from reflection positivity (E2), where the sum is over $k, l = 0 \ldots n$ and $\varphi_0 \in \mathbb{R}$.

Remark 5.2 The positivity–condition introduced here is a little more strict than the original positivity–condition in [56]. Here we demand ”positivity” of the Schwinger functions $\{S_n\}_{n \in \mathbb{N}_0}$ for test functions $\varphi_k \in S((\mathbb{R}^d_+)^k), k = 1 \ldots n, \varphi_0 \in \mathbb{R}$ instead of $\varphi_k \in S_{+<}((\mathbb{R}^d)^k), k = 1 \ldots n$, which is the original condition from [56]. Here $S_{+<}((\mathbb{R}^d)^k)$ is the space of all test functions with support in $(\mathbb{R}^d_+)^k = (x_1 \ldots x_k) \in \mathbb{R}^{dk}, x_j = (x^0_j, \vec{x}_j), j = 1 \ldots 2, 0 < x^0_j < \ldots < x^0_k$. Since the latter space is contained in the former, our condition is more strict.

But in the present case, they are equivalent: The reason is that for a large class of convolution kernels $G$ the Schwinger functions $S_n$ belonging to $X = G \ast F$ (cf. section 3), are more regular than tempered distributions namely they are locally integrable functions (cf. Lemma 7.7). One can therefore evaluate such $S_n$ on test functions with "jumps". Thus, by application of symmetry, we may calculate for $\varphi_k \in S((\mathbb{R}^d_+)^k), k = 1 \ldots n, \varphi_0 \in \mathbb{R}$:

$$\sum_{k,j=0}^n S_{k+j} (\theta \varphi_k \otimes \varphi_k)$$

27
\[
\sum_{k,j=0}^{n} \sum_{\pi \in \text{Perm}(k)} \sum_{\pi' \in \text{Perm}(j)} S_{k+j}(\theta(1_{x_1^0 < \ldots < x_{\pi(k)}^0}) \varphi_k) \\
\otimes (1_{x_1^0 < \ldots < x_{\pi'(j)}^0}) \varphi_j) \\
= \sum_{k,j=0}^{n} S_{k+j}(\theta \tilde{\varphi}_k \otimes \tilde{\varphi}_j),
\]

where \( \text{Perm}(k) \) is the group of permutations of \( \{1, \ldots, k\} \) and \( \tilde{\varphi}_k \) is defined as

\[
\tilde{\varphi}_k(x_1, \ldots, x_k) := 1_{x_1 < \ldots < x_k}(x_1, \ldots, x_k) \sum_{\pi \in \text{Perm}(k)} \varphi(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(k)}).
\]

Now the functions \( \tilde{\varphi}_k \) can be approximated by functions from \( S_{+}(<IR^{dk}) \) and by application of the dominated convergence theorem we can derive the sharpened reflection positivity condition \((E4)\) from the original one of [56].

**Lemma 5.3** Let \( \{S_n\}_{n \in \mathbb{N}_0} \) and \( \{S_n^T\}_{n \in \mathbb{N}} \) be as in Definition 5.1. If \( \{S_n\}_{n \in \mathbb{N}_0} \) is reflection infinitely divisible, then \( \{S_n^T\}_{n \in \mathbb{N}} \) is conditional reflection positive.

More precisely: If \( \{S_n^T\}_{n \in \mathbb{N}} \) is not conditionally reflection positive, then there exists a \( \lambda_0 > 0 \), such that for all \( \lambda, 0 < \lambda < \lambda_0 \), the sequence of Schwinger functions \( \{S_n^\lambda\}_{n \in \mathbb{N}_0} \) is not reflection positive.

**Proof.** Suppose, \( \{S_n^T\}_{n \in \mathbb{N}} \) is not conditionally reflection positive. Then there are test functions \( \varphi_1 \ldots \varphi_n \) as in definition 5.1 such that the LHS of (33) is negative. Since

\[
\lim_{\lambda \to +0} \left[ \frac{1}{\lambda} \sum_{k,l=1}^{n} S_{k+l}^\lambda(\theta \varphi_k \otimes \varphi_l) \right] = \sum_{k,l=1}^{n} S_{k+l}^T(\theta \tilde{\varphi}_k \otimes \tilde{\varphi}_l) < 0
\]

there exists a \( \lambda_0 > 0 \) such that for all \( \lambda, 0 < \lambda < \lambda_0 \), also the LHS of the above equation is negative. This implies the statement of the lemma.

**Remark 5.4** Provided a quite weak growth condition in \( n \) is fulfilled by the \( \{S_n^T\}_{n \in \mathbb{N}} \) and the \( S_n, S_n^T \) are symmetric, we also have: If \( \{S_n^T\}_{n \in \mathbb{N}} \) is conditional reflection positive then \( \{S_n\}_{n \in \mathbb{N}_0} \) is reflection infinitely divisible. (see [38]).

From now on we want to impose some restrictions on the kernel \( G \). These restrictions are typical for the Green’s functions of a large class of (pseudo) differential operators.
Condition 5.5 We want to consider kernels $G$ that are continuous, real functions on $\mathbb{R}^d \setminus \{0\}$, and have a singularity at the origin, i.e. $\lim_{|x| \to 0} G(x) = \pm \infty$ and fall to zero as $|x| \to \infty$. Furthermore assume that the mapping $f \mapsto G \ast f$ is well defined and continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. Finally, $G$ is assumed to be invariant under orthogonal transformations.

Remark 5.6 Of course, the Green’s functions of the pseudo differential operators $(-\Delta + m^2)^\alpha$, $\alpha \in (0, 1)$, fulfill Condition 5.5 (see e.g. the representation we give in Section 6).

Definition 5.7 Let $\phi \in \mathcal{S}(\mathbb{R}^d_+)$. For $x \in \mathbb{R}^d$ we write $(x_0, \vec{x})$. Define $\phi_s(x) = \frac{1}{2} \phi(x)$ for $x_0 > 0$ and $\phi_a(x) = \frac{1}{2} \phi(\theta x)$ for $x_0 < 0$. Let $\phi^a = \text{sign}(x^0) \phi^a$. By $q(\phi)$ we denote the function

$$q(\phi) = G \ast \theta \phi G \ast \phi = (G \ast \phi^a)^2 - (G \ast \phi^a)^2.$$  \hspace{1cm} (34)

Clearly $\phi^a, \phi^a$ and $q(\phi)$ are fast falling functions. For $\varphi_1, \ldots, \varphi_n \in \mathcal{S}(\mathbb{R}^d_+)$ we get by the $\theta$-invariance of $G$ and the definition of $q$:}

$$S_{2n} \left( \theta(\varphi_1 \otimes \ldots \otimes \varphi_n) \otimes \varphi_1 \ldots \otimes \varphi_n \right) = 2c_{2n} \int_{\mathbb{R}^d_+} q(\varphi_1) \ldots q(\varphi_n) dx.$$  \hspace{1cm} (35)

We concentrate on

$$S_{4n+2} \left( \theta(\varphi_1^{\otimes 2n} \otimes \varphi_2) \otimes \varphi_1^{\otimes 2n} \otimes \varphi_2 \right) = 2c_{4n+2} \int_{\mathbb{R}^d_+} (q(\varphi_1))^{2n} q(\varphi_2) dx$$  \hspace{1cm} (35)

and try to choose $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ such that the RHS of (35) is smaller than zero, provided $c_{4n+2} > 0$. To this aim, we at first need to proof some technical lemmas.

Here and in the following $B_{\varepsilon}(x), x \in \mathbb{R}^d$ shall denote the open ball of radius $\varepsilon$ around $x$.

Lemma 5.8 Fix $x^0 > 0$ and let $x = (x^0, 0) \in \mathbb{R}^d$. Then there exists a function $\varphi_2 \in \mathcal{S}(\mathbb{R}^d_+)$ and an $\varepsilon_2 > 0$ such that $q(\varphi_2) < 0$ on $B_{\varepsilon_2}(x)$.

Proof. $q(\varphi)(y)$ is continuous in $y$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d_+)$. Therefore it suffices to pick a $\varphi_2$ s.t. $q(\varphi_2)(x) < 0$. The existence of an $\varepsilon_2$ follows.

Define continuous linear functionals $F^\pm$ on $\mathcal{S}(\mathbb{R}^d_+)$ by

$$F^\pm : f \mapsto \int_{\mathbb{R}^d_+} G(x \mp y) f(y) dy.$$
Let $x \in \mathbb{R}^d_+, \, x = (x^0, 0)$ such that $\mathcal{G}((2x^0, 0)) \neq 0$. For every $\epsilon_2 > 0$ there exist a function $\varphi_1 \in \mathcal{S}(\mathbb{R}^d_+)$ and an $\epsilon_1, \epsilon_2 > \epsilon_1 > 0$, in such a way, that $|q(\varphi_1)| > 2$ on $B_{\epsilon_1}(x)$ and $|q(\varphi_1)| < \frac{1}{2}$ on $\mathbb{R}^d_+ \setminus B_{\epsilon_2}(x)$.

**Proof.** First we investigate what happens, if we take $\delta_x$, the Dirac measure with mass one in $x$, as a "test function":

$$q(\delta_x)(y) = \mathcal{G}(y - \Theta x) \mathcal{G}(y - x).$$

$G$ is continuous on $\mathbb{R}^d_+ \setminus \{0\}$ and $\mathcal{G}((2x^0, 0)) \neq 0$. It follows, that $q(\delta_x)$ on $\mathbb{R}^d_+ \setminus \{0\}$ has a unique singularity in $y = x$, i.e. $\lim_{|x - y| \to 0} |q(\delta_x)(y)| = \infty$, and is continuous on $\mathbb{R}^d_+ \setminus \{y\}$. Now let $\delta_{\rho,x} \in \mathcal{S}(\mathbb{R}^d_+)$ be an approximation of $\delta_x$, i.e. $\delta_{\rho,x} \to \delta_x$ for $\rho \to 0$. Then $|q(\delta_{\rho,x})|$ takes arbitrarily large values in a neighborhood of $x$ for $\rho \to 0$. At the same time, on $\mathbb{R}^d_+ \setminus B_{\epsilon_1}(x)$ $|q(\delta_{\rho,x})|$ is bounded uniformly in $\rho$ by a positive constant, say $\frac{D}{2}$. Choose $\epsilon_1 < \epsilon_1 < \epsilon_2$ small enough, so that $|q(\delta_{\rho,x})| > 2D$ on $B_{\epsilon_1}(x)$ for some small $\rho$. Fix such a $\rho$ and let $\varphi_1 = D^{-\frac{1}{2}} \delta_{\rho,x}$. Then $\varphi_1$ and $\epsilon_1$ fulfill the conditions of the lemma.

We are now able to construct very sharp maxima for the functions $|q(\varphi_1)|^{2n}$ on arbitrarily small neighborhoods $B_{\epsilon_1}(x)$ of certain points $x$. Moreover, we can also achieve that $|q(\varphi_1)|^{2n}$ takes very small values outside a ball $B_{\epsilon_2}(x)$. Of course, now we want to enforce the "negative spot" of $q(\varphi_2)$, detected in Lemma 5.8 through multiplication by an adequately chosen $|q(\varphi_1)|^{2n}$.

**Lemma 5.10** It is possible to choose $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d_+)$ and $n \in \mathbb{N}$, such that the integral on the RHS of (35) takes a negative value.

**Proof.** Fix $x = (x^0, \bar{0})$ as in Lemma 5.9. According to Lemma 5.8 there exists a function $\varphi_2 \in \mathcal{S}(\mathbb{R}^d_+)$ and an $\epsilon_2 > 0$ such that $q(\varphi_2) < 0$ on $B_{\epsilon_2}(x)$. Furthermore choose $\varphi_1 \in \mathcal{S}(\mathbb{R}^d_+)$ fulfilling the conditions of Lemma 5.9 with the above fixed $\epsilon_2, x$. Now
It is clear, that for $n$ large enough, the RHS of this inequality becomes negative.

The derivation of this section’s main result is now easy.

**Proposition 5.11** Let $G$ fulfill Condition 5.5, and $\psi$ be a Lévy–characteristic with non–zero Poisson part and a Lévy–measure $M$, such that all moments of $M$ exist. Let $F^\lambda$ denote the noise determined by $\lambda \psi$, $\lambda > 0$, and let $X^\lambda = G * F^\lambda$ in the sense of Section 2. Then there exists a $\lambda_0 > 0$ s.t. for all $\lambda$, $0 < \lambda < \lambda_0$, the Schwinger functions of $X^\lambda$ given by Definition 3.8 are not reflection positive.

**Proof.** Since $c_{4n+2} > 0$ for a $\psi$ with $M \neq 0$, equation (35) together with Lemma 5.10 imply that conditional reflection does not hold for the truncated Schwinger functions of $X = G * F^1$. Therefore, Proposition 5.11 follows from Lemma 5.3.

**Remark 5.12** We assume — and in the next section we will present some examples — that at least the 2–point function $S_2 = S^2_T$ of a convoluted generalized white noise with mean zero is reflection positive. Furthermore we may choose the Lévy measure $M$ of $\psi$ symmetric w.r.t the reflection at 0. In this case all Schwinger functions $S_n$, $n$ odd, vanish (cf. Remark 3.7).

We choose $\lambda > 0$. By scaling the test functions in the reflection positivity condition $\varphi_0 \in \mathbb{R} \mapsto \varphi_0 \in \mathbb{R}, \varphi_k \in S(\mathbb{R}^d_+) \mapsto \lambda^{k/2} \varphi_k \in S(\mathbb{R}^d_+)\}$ for $k = 1, \ldots, n$, we get that the sequence of Schwinger functions $\{S^\lambda_n\}_{n \in \mathbb{N}_0}$ of the Lévy–characteristic $\psi$ is reflection positive (fulfills the Osterwalder–Schrader axioms) if and only if the sequence of Schwinger functions $\{\tilde{S}^\lambda_n\}_{n \in \mathbb{N}_0}$ defined by $\tilde{S}^\lambda_n := \lambda^{-n/2} S^\lambda_n$ is reflection positive (fulfills the Osterwalder–Schrader axioms) [56].

Take $\varphi_1 \ldots \varphi_{2n} \in S(\mathbb{R}^d)$ and write

$$\tilde{S}^\lambda_{2n}(\varphi_1 \otimes \ldots \otimes \varphi_{2n})$$
\[ \begin{align*}
&= \sum_{I \in \text{pairings}} \prod_{\{j_1,j_2\} \in I} S^T_2(\varphi_{j_1} \otimes \varphi_{j_2}) \\
&\quad + \sum_{k=1}^{n-1} \lambda^{k-n} \sum_{I \in P_{k(2n)}} \prod_{\{j_1,\ldots,j_l\} \in I} S^T_l(\varphi_{j_1} \otimes \ldots \otimes \varphi_{j_l}).
\end{align*} \]

For \( \lambda \) large we may interpret \( \tilde{S}^\lambda_{2n} \) as the \( 2n \)-point Schwinger function of a "perturbed" Gaussian reflection–positive random field with covariance function \( S^T_2 \). The "perturbation" is a polynomial without a constant term of degree \( n - 1 \) in the "coupling constant" \( \lambda^{-1} \). In Proposition 5.11, we have shown that the reflection positivity breaks down if the "coupling constant" \( \lambda^{-1} \) is larger than a certain threshold \( \lambda_0^{-1} \). It remains an open question what happens if \( \lambda^{-1} \) is small. We will not study this problem here.

On the first look, Proposition 5.11 may be discouraging. The lack of reflection positivity in the general case leads to some difficulties in the physical interpretation of the model. In the "state space" of the reconstruction theorem in [56] in general we may find some "states" with negative norm. A straightforward probabilistic interpretation is therefore difficult in this case, since "negative probabilities" would occur.

Nevertheless, even for the case of a large "coupling constant" \( \lambda^{-1} \), some of the difficulties can be overcome, at least that is what we hope at the moment: Let us note, that in the situation of Proposition 5.11 the obstructions to reflection positivity come from the higher order truncated Schwinger functions. In Section 6 and Section 7 we analytically continue the truncated Schwinger functions "by hand". The truncated Wightman distributions obtained by this procedure fulfill the spectral condition of QFTs with a "mass-gap", Poincaré invariance and locality. From Haag–Ruelle theory (see e.g. p.317 of [61] Vol. III), we know, that such truncated Wightman distributions of order \( n \geq 3 \) do not contribute to the norm of a state approaching the asymptotic resp. scattering region \( x^0 \to \pm \infty \), because of the short range of the forces involved (in the case of a QFT with a "mass-gap"). Therefore, if stable one particle states exist (take e.g. our model \( X = G_\alpha * F \), where \( \alpha = \frac{1}{2} \)) the pseudonorm of the states approaching the scattering region should get positive. (For a precised discussion in a special case, see Subsection 7.6).
6 Analytic Continuation I: Laplace–Representation for the Kernel of \((-\Delta + m_0^2)^{-\alpha}\), \(\alpha \in (0, 1)\)

In this section, we will give a representation of the kernel \((-\Delta + m_0^2)^{-\alpha}\), for \(m_0 > 0\) and \(\alpha \in (0, 1)\), in terms of a Laplace transform (which is specified later on). In [18], an analytic continuation of the kernel associated with the pseudo differential operator \((-\Delta)^{-1}\) (corresponding to the case that \(m_0 = 0\) and \(\alpha = 1\)) was obtained, the starting point of which was a representation of the kernel of \((\Delta)^{-1}\) as a Laplace transform. In [13], based on the same representation of \((\Delta)^{-1}\), an analytic continuation of the (vector) kernel of \(\partial^{-1}\) was derived, where \(\partial\) is the quaternionic Cauchy–Riemann operator.

This section should be regarded as an introduction to the next section, where we extend the methods concerning analytic continuation of Schwinger functions of random fields in [13] and [18]. It is also interesting to extend our approach here to the case of vector kernels including the case of mass \(m_0 > 0\). We intend to investigate this problem in forthcoming papers.

We notice that the kernel of \((-\Delta + m_0^2)^{-\alpha}\) is given by

\[
G_\alpha(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{ikx}}{|k|^2 + m_0^2} e^{\alpha} \, dk, \quad x \in \mathbb{R}^d
\]

(36)

which is the Fourier transform of a tempered distribution (see Example 2.2). The idea we will realize here is that we represent the integral (36), which is over the conjugate variable \(k^0\) on the real time axis (i.e., the \(k^0\)-axis in \(k = (k^0, \vec{k}) \in \mathbb{R} \times \mathbb{R}^{d-1}\)), by an integral over (a part of) the upper half part of the imaginary axis \(ik^0\) (thus the \(k^0\)-axis being replaced by an imaginary axis) so that the above Fourier transform goes over to a Laplace transform.

For simplicity of exposition, we assume first that \(d = 1\). We want to evaluate the integral

\[
\int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2 + m_0^2)^{\alpha}} \, dk, \quad x \in \mathbb{R}
\]

(37)

by some complex integral. This can be done by a contour integral around a branch point as follows.

We denote by \(\log\) the main branch of the complex logarithm which is holomorphic on \(\mathbb{C} \setminus (-\infty, 0]\). Set

\[
(iz + m_0)^{-\alpha} = f_1(z) = \exp\{-\alpha \log(iz + m_0)\}, \quad z \in \mathbb{C} \setminus i[m_0, \infty);
\]

\[
(-iz + m_0)^{-\alpha} = f_2(z) = \exp\{-\alpha \log(-iz + m_0)\}, \quad z \in \mathbb{C} \setminus i(-\infty, -m_0].
\]
Clearly, $f_1$ and $f_2$ are holomorphic functions on the indicated domains, respectively. Therefore, for arbitrarily fixed $x \in \mathbb{R}$

$$h(z) := e^{ix} f_1(z)f_2(z)$$

is a holomorphic extension of the function $k \in \mathbb{R} \rightarrow \frac{e^{ikx}}{(k^2 + m_0^2)^\alpha} \in \mathcal{C}$, which is defined on the real line, to the domain $\mathcal{C} \ \{ iy : y \in \mathbb{R}, |y| > m_0 \}$. Take $C \subset \mathcal{C} \ \{ iy : y \in \mathbb{R}, |y| > m_0 \}$ as indicated in Fig. 1. By the well-known Cauchy integral theorem, we get

$$\int_C h(z)dz = 0 .$$

On the other hand

$$\int_C h(z)dz = \int_{-t}^{t} \frac{e^{ikx}}{(k^2 + m_0^2)^\alpha}dk + \sum_{j=1}^{5} \int_{C_j} h(z)dz ,$$

the curves $C_j$ being as in Fig.1.

Thus we have derived

$$\int_{-t}^{t} \frac{e^{ikx}}{(k^2 + m_0^2)^\alpha}dk = - \sum_{j=1}^{5} \int_{C_j} h(z)dz . \quad (38)$$

Moreover, we have the following result
Lemma 6.1 For every $\alpha \in (0, 1)$ and $x > 0$,
\[
\int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2 + m_0^2)^\alpha} dk = 2 \sin(\pi \alpha) \int_{m_0}^{\infty} \frac{e^{-rx}}{(r^2 - m_0^2)^\alpha} dr .
\]

Proof. We remark at first that for any fixed $x \in \mathbb{R}$
\[
\int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2 + m_0^2)^\alpha} dk = \lim_{t \to \infty} \int_{-t}^{t} \frac{e^{ikx}}{(k^2 + m_0^2)^\alpha} dk ,
\]
where by Leibniz criterion the right hand side converges for every $\alpha > 0$. Thus we should analyse each curve integral on the right hand side of (38). We shall use polar coordinates for each integral.

1. Using the polar coordinate representation $z = re^{i\beta}$, $r > 0$, $-\frac{3\pi}{2} < \beta < \frac{\pi}{2}$, $C_1 = \{(r, \beta) : r = t, 0 \leq \beta \leq \beta_1 \}$, we have the following derivation for $x \geq 0$ and $\beta_1 \in (0, \frac{\pi}{2})$
\[
\left| \int_{C_1} h(z)dz \right| = \left| \int_{0}^{\beta_1} tie^{i\beta} e^{ixe^{i\beta}} f_1(t e^{i\beta}) f_2(t e^{i\beta}) d\beta \right|
\]
\[
\leq \int_{0}^{\beta_1} t e^{-xt \sin \beta} \left| e^{2i\beta} \frac{m_0^2}{t^2} \right|^\alpha d\beta
\]
\[
= t^{1-2\alpha} \int_{0}^{\beta_1} \frac{e^{-xt \sin \beta}}{1 + 2 \left( \frac{m_0}{t} \right)^2 \cos 2\beta + \left( \frac{m_0}{t} \right)^4} \frac{d\beta}{2\beta}
\]
\[
\leq t^{1-2\alpha} \left[ 1 - \left( \frac{m_0}{t} \right)^2 \right]^{-\alpha} \int_{0}^{\beta_1} e^{-xt \sin \beta} d\beta
\]
\[
\leq t^{1-2\alpha} \left[ 1 - \left( \frac{m_0}{t} \right)^2 \right]^{-\alpha} \frac{1}{\cos \beta_1} \int_{0}^{\sin \beta_1} e^{-xt \sin \beta} d\beta
\]
\[
= \frac{xt^{2\alpha}}{1 - \left( \frac{m_0}{t} \right)^2} \frac{1}{\cos \beta_1}
\]
\[
\to 0 \quad \text{as } t \to \infty.
\]

2. Analogously, using the representation $z = re^{i\beta}$, $r > 0$, $-\frac{3\pi}{2} < \beta < \frac{\pi}{2}$, $C_5 = \{(r, \beta) : r = t, -\pi - \beta_1 \leq \beta \leq -\pi \}$. we have for any arbitrary fixed $\beta_1 \in (0, \frac{\pi}{2})$
\[ \left| \int_{C_5} h(z)dz \right| \to 0 \]
as \( t \to \infty \).

3. Using the polar coordinates representation \( z = im_0 + ire^{i\beta} \), \( r > 0 \), \( 0 \leq \beta < 2\pi \), \( C_3 = \{(r, \beta) : r = s, \beta_2 \leq \beta \leq 2\pi - \beta_2\} \), we have for any fixed \( \beta_2 \in (0, \frac{\pi}{2}) \) and \( s < m_0 \) that

\[
\left| \int_{C_3} h(z)dz \right| = \left| - \int_{\beta_2}^{2\pi - \beta_2} e^{i(m_0 + ise^{i\theta})x} \left( \frac{1}{(im_0 + ise^{i\beta})^2 + m_0^2} \right) isie^{i\beta} d\beta \right| \\
\leq \int_{\beta_2}^{2\pi - \beta_2} \frac{s^{1-\alpha} e^{-(m + s \cos \beta)x}}{(4m_0^2 + 4ms \cos \beta + s^2)^{\frac{\alpha}{2}}} d\beta \\
\leq s^{1-\alpha} \int_{\beta_2}^{2\pi - \beta_2} \frac{e^{-(m-s)x}}{(2m - s)^{\frac{\alpha}{2}}} d\beta \\
= \frac{s^{1-\alpha}(2\pi - 2\beta_2)e^{-(m-s)x}}{2(m - s)^{\frac{\alpha}{2}}} \\
\to 0
\]
as \( s \to 0 \).

4. Using \( z = im_0 + ire^{i\beta} \), \( r > 0 \), \( 0 \leq \beta < 2\pi \), \( C_4 = \{(r, \beta) : s \leq r \leq t_1, \beta = \beta_2\} \) with \( t_1 = (t^2 + m_0^2 - 2tm \sin \beta_1)^{\frac{1}{2}} \), we have, for \( \beta_2 \in (0, \frac{\pi}{2}) \) and \( s < 1 < t \), the following derivation

\[
\int_{C_4} h(z)dz \\
= \int_{s}^{t_1} e^{i(m_0 + ire^{i\beta_2})x} f_1(im_0 + ire^{i\beta_2})f_2(im_0 + ire^{i\beta_2})ie^{i\beta_2} dr \\
= \int_{s}^{t_1} e^{i(m_0 + ire^{i\beta_2})x}ie^{i\beta_2} \\
= ie^{i\beta_2} \int_{s}^{t_1} e^{-(m_0 + re^{i\beta_2})x} dr \\
= ie^{i\beta_2} \int_{s}^{t_1} e^{-(m_0 + re^{i\beta_2})x} dr \\
= ie^{i\beta_2} \int_{s}^{t_1} e^{-(m_0 + re^{i\beta_2})x} dr
\]
\begin{align*}
&= i e^{i \pi \alpha} e^{i \beta_2 (1 - \alpha)} \int_s^{t_1} \frac{e^{-(m_0 + r e^{i \beta_2}) x}}{r^\alpha (2m_0 + r e^{i \beta_2})^\alpha} dr \\
&= i e^{i \pi \alpha} e^{i \beta_2 (1 - \alpha)} \int_s^{t_1} \frac{e^{-(m_0 + r e^{i \beta_2}) x}}{r^\alpha (2m_0 + r e^{i \beta_2})^\alpha} dr ,
\end{align*}

\text{where we had used the representation } -1 = e^{-i \pi} \text{ in the fourth equality.}

\text{Now we want to consider the limit of (40) as } s \to 0 \text{ and } t_1 \to \infty. \text{ In order to do that, by an application of Lebesgue theorem, we estimate the integrand in (40) as follows:}

\begin{align*}
\left| \int_s^{t_1} \frac{e^{-(m_0 + r e^{i \beta_2}) x}}{r^\alpha (2m_0 + r e^{i \beta_2})^\alpha} dr \right| &\leq \int_s^{t_1} \frac{e^{-(m_0 + r \cos \beta_2) x}}{r^\alpha |2m_0 + r e^{i \beta_2}|^\alpha} dr \\
&= \int_s^{t_1} \frac{e^{-(m_0 + r \cos \beta_2) x}}{r^\alpha (r^2 + 4m_0 r \cos \beta_2 + 4m_0^2)^\alpha} dr \\
&= \left( \int_s^1 + \int_1^{t_1} \right) \frac{e^{-(m_0 + r \cos \beta_2) x}}{r^\alpha (r^2 + 4m_0 r \cos \beta_2 + 4m_0^2)^\alpha} dr ,
\end{align*}

\text{and}

\begin{align*}
\int_s^1 \frac{e^{-(m_0 + r \cos \beta_2) x}}{r^\alpha (r^2 + 4m_0 r \cos \beta_2 + 4m_0^2)^\alpha} dr &\leq \int_s^1 \frac{e^{-m_0 x}}{r^\alpha (r^2 + 4m_0^2)^\alpha} dr \\
&\leq \frac{e^{-m_0 x}}{(2m_0)^\alpha} \int_s^1 \frac{dr}{r^\alpha} \\
&= \frac{e^{-m_0 x}}{(2m_0)^\alpha} \frac{1 - s^{1 - \alpha}}{1 - \alpha} \\
&\to \frac{e^{-m_0 x}}{(2m_0)^\alpha (1 - \alpha)} \quad (42)
\end{align*}

\text{as } s \to 0. \text{ Moreover}

\begin{align*}
\int_1^{t_1} \frac{e^{-(m_0 + r \cos \beta_2) x}}{r^\alpha (r^2 + 4m_0 r \cos \beta_2 + 4m_0^2)^\alpha} dr
\end{align*}
$$\leq \frac{e^{-m_0 x}}{(1 + 4m_0 \cos \beta_2 + 4m_0^2)^{\frac{\alpha}{2}}} \int_1^{t_1} e^{-r x \cos \beta_2} dr$$

$$= \frac{e^{-m_0 x}}{e^{-x \cos \beta_2} - e^{-x t_1 \cos \beta_2}} \frac{x \cos \beta_2}{e^{-(m_0 + \cos \beta_2)x}}$$

$$\rightarrow \frac{e^{-(m_0 + \cos \beta_2)x}}{(1 + 4m_0 \cos \beta_2 + 4m_0^2)^{\frac{\alpha}{2}}} x \cos \beta_2$$

(43)

as $t_1 \to \infty$ (or, equivalently, $t \to \infty$). Using the above facts (40)–(43), we see that the following limit exists and is given by

$$\lim_{s \to 0, t \to \infty} \int_{C_4} h(z) dz = i e^{i \pi \alpha} e^{i(1-\alpha)\beta_2} \int_0^\infty \frac{e^{-(m_0 + r \cos \beta_2)x} - i r x \sin \beta_2}{r^\alpha (2m_0 + r e^{i \beta_2})^\alpha} dr .$$

Now setting $\beta_2 \to 0$, by Lebesgue theorem again, we get

$$\lim_{\beta_2 \to 0} \lim_{s \to 0, t \to \infty} \int_{C_4} h(z) dz = i e^{i \pi \alpha} \int_0^\infty \frac{e^{-(m_0 + r)x}}{r^\alpha (2m_0 + r)^\alpha} dr$$

$$= i e^{\alpha \pi i} \int_{m_0}^\infty \frac{e^{-r x}}{(r^2 - m_0^2)^\alpha} dr .$$

5. Similarly, we get

$$\lim_{\beta_2 \to 0} \lim_{s \to 0, t \to \infty} \int_{C_2} h(z) dz = -i e^{-\alpha \pi i} \int_m^\infty \frac{e^{-r x}}{(r^2 - m_0^2)^\alpha} dr .$$

Finally, combining the above steps 1 to 5, we derive from (38) that

$$\int_{-\infty}^\infty \frac{e^{ikx}}{(k^2 + m_0^2)^\alpha} dk = -\left[ i e^{\alpha \pi i} - i e^{-\alpha \pi i} \right] \int_{m_0}^\infty \frac{e^{-r x}}{(r^2 - m_0^2)^\alpha} dr$$

$$= 2 \sin(\pi \alpha) \int_{m_0}^\infty \frac{e^{-r x}}{(r^2 - m_0^2)^\alpha} dr .$$

Hence we obtain (39).

Now we give the notion of Laplace transform in one variable. The definition the of Laplace transform in the multi variable case will be given in Section 7.
Definition 6.2 Let \( M_d \) denote the \( d \)-dimensional Minkowski space–time with Minkowski inner product \(< , >_M\). Choose \( e_0 \in M_d \) such that \(< e_0, e_0 >_M = 1 \) and let \( \{ e_0, \ldots, e_{d-1} \} \) be an orthonormal frame in \( M_d \). Let \( V_0^+ \) be the forward light cone, namely,

\[
V_0^+ := \{ k \in M_d : k^2 > 0, < k, e_0 >_M < 0 \},
\]

and \( V^*_+ \) its closure. We recall that \( \mathbb{R}^d_+ := \{ x = (x^0, \vec{x}) \in \mathbb{R} \times \mathbb{R}^{d-1} : x^0 > 0 \} \). Notice that for \( x = (x^0, \vec{x}) \in \mathbb{R}^d_+ \), the function \( e(x, k) := e^{-x^0 k^0 + i \vec{x} \cdot \vec{k}} \) on the forward light cone \( V^*_+ \) behaves as a fast falling function, i.e., there exists \( h_x \in \mathcal{S}'(\mathbb{R}^d) \) such that \( h_x(k) = e(k, x) \) for \( k \in V_0^+ \). Therefore, for a tempered distribution \( f \in \mathcal{S}'(\mathbb{R}^d) \) with \( \text{supp} f \subset V_0^+ \), we can define

\[
(Lf)(x) := < e(\cdot, x), f > := < h_x, f > ,
\]

which is well defined since by the fact that \( \text{supp} f \subset V_0^+ \) there is no ambiguity arising from the choice of \( h_x \). We call \( Lf \) the Laplace transform of \( f \in \mathcal{S}'(\mathbb{R}^d) \) with \( \text{supp} f \subset V_0^+ \).

Clearly, the Laplace transform of a test function \( f \in \mathcal{S}(\mathbb{R}^d_+) \) is given by the following formula

\[
(Lf)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d_+} e^{-x^0 k^0 + i \vec{x} \cdot \vec{k}} f(k) dk , \quad x \in \mathbb{R}^d_+ .
\]

Sometimes \( L \) (e.g. in [13]) is called the Fourier–Laplace transform. Our definition here is close to the one given in [67].

The following proposition is the main result of this section, which gives a representation of \( G_\alpha \) in terms of a Laplace transform.

Proposition 6.3 For \( \alpha \in (0, 1) \) and \( x \in \mathbb{R}^d \setminus \{ 0 \} \), we have the following formula

\[
G_\alpha(x) = 2(2\pi)^{-d} \sin(\pi \alpha) \int_{\mathbb{R}^d_+} e^{-|x^0| k^0 + i \vec{x} \cdot \vec{k}} \frac{1_{\{ k^0 > |\vec{k}|^2 + m_0^2 \}^\frac{1}{2}}(k)}{(k^0^2 - |\vec{k}|^2 - m_0^2)^\alpha} dk ,
\]

where \( 1_{\{ k^0 > |\vec{k}|^2 + m_0^2 \}^\frac{1}{2}}(k) \) is the indicator of the subset \( \{ k = (k^0, \vec{k}) \in \mathbb{R} \times \mathbb{R}^{d-1} : k^0 > (|\vec{k}|^2 + m_0^2)^{\frac{1}{2}} \} \subset \mathbb{R}^d_+ .
\]
Proof. By (36) and (39), for \( x \in \mathbb{R}^d_+ \), we obtain (45) by Fubini theorem (cf. [38] for details on using Fubini theorem) and the following derivation

\[
G_\alpha(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{ikx}}{\sqrt{|k|^2 + m_0^2}} \alpha \, dk
\]

By changing variables in (45) as \((k^0, \vec{k}) \to ((|\vec{k}|^2 + m_0^2)^{\frac{1}{2}}, \vec{k})\), we get a "Källen–Lehmann representation" for \( G_\alpha \) (we refer the reader to Theorem IX.33 of [61] or Theorem II.4 of [66] for the Källen–Lehmann representation).

Corollary 6.4 For \( \alpha \in (0,1) \) and \( x \in \mathbb{R}^d \setminus \{0\} \), the kernel \( G_\alpha \) of \((-\Delta + m_0^2)^{-\alpha}\) has the following representation

\[
G_\alpha(x) = \int_0^\infty C_m(x) \rho_\alpha(dm_0^2) , \tag{46}
\]

where

\[
C_m(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-x^0 k^0 + i\vec{k} \cdot \vec{x}} 1_{\{k^0 > 0\}}(k) \delta(k^0 - |\vec{k}|^2 - m^2) \, dk , \tag{47}
\]

\[
\rho_\alpha(dm_0^2) = 2 \sin(\pi \alpha) 1_{\{m^2 > m_0^2\}} \frac{dm_0^2}{(m^2 - m_0^2)^\alpha} , \tag{48}
\]

Remark 6.5 In general, the Källen–Lehmann representation characterizes the 2-point Schwinger functions of a Lorentz–invariant field–theory. Comparing our formula (47) with formula (6.2.6) of [37], we see that \( C_m \) is a representation of the 2-point Schwinger function (i.e. the free covariance) of the relativistic free field of mass \( m \).
Corollary 6.6

For \( p \leq 40 \) of \([67]\) and translation invariance of \( s \), we have

\[
C_m(x) = (2\pi)^{\frac{d}{2}} \frac{m}{|x|^{d/2}} K_{d-2}^\frac{1}{2}(m|x|) \quad , \quad x \in \mathbb{R}^d \setminus \{0\} , \tag{49}
\]

where \( K_{d-2} > 0 \), for \( x \in \mathbb{R}^d \setminus \{0\} \), is the modified Bessel function. Thus \( C_m \) is a positive function on \( \mathbb{R}^d \setminus \{0\} \) with a singularity at the origin and exponential decay as \( m|x| \to \infty \). This implies that \( G_\alpha \) is singular at the origin, as was assumed in Section 5. The consequence that \( \rho \) has a "mass gap", i.e., \( \text{supp} \rho_\alpha \subset [m_0^2, \infty) \). (For \( d = 1 \), the singularity of \( G_\alpha \) is only due to \( \rho_\alpha \) being an infinite measure).

Concerning the truncated Schwinger function \( S^T_2 \) associated with \( X_\alpha = G_\alpha \ast F \), we have the following result, which gives a representation of the integral kernel \( s \) of \( S^T_2 \) as a Laplace transform, while by formula (2–25) on p. 40 of \([67]\) and translation invariance of \( s \), we have

\[
S^T_2(f_1 \otimes f_2) = c_2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_1(x)s(x-y)f_2(y)dx dy , \quad f_1, f_2 \in \mathcal{S}(\mathbb{R}^d) .
\]

Corollary 6.6 For \( \alpha \in (0, \frac{1}{2}) \),

\[
s(x-y) = 2(2\pi)^{-\frac{d}{2}} \sin(\pi 2\alpha) \mathcal{L} \left( \frac{1}{(k^2 - |\vec{k}|^2 - m_0^2)^{2\alpha}} \right)(x-y) . \tag{50}
\]

**Proof.** By (27), we have

\[
S^T_2(f_1 \otimes f_2) = c_2 < (-\Delta + m_0^2)^{-\alpha} f_1, (-\Delta + m_0^2)^{-\alpha} f_2 >_{L^2}
\]

\[
= c_2 < f_1, (-\Delta + m_0^2)^{-2\alpha} f_2 >_{L^2} , \quad f_1, f_2 \in \mathcal{S}(\mathbb{R}^d) ,
\]

where \( c_2 \) is given by (27). Therefore by (36) we get

\[
s(x-y) = c_2 G_{2\alpha}(x-y)
\]

\[
= 2(2\pi)^{-d} \sin(2\pi \alpha) \times
\]

\[
\times \int_{\mathbb{R}^d} e^{-|x-y|} e^{i\vec{k} \cdot (x-y)} \mathcal{L} \left( \frac{1}{(k^0)^2 - |\vec{k}|^2 - m_0^2)^{2\alpha}} \right)(x-y) .
\]

\[
= 2(2\pi)^{-\frac{d}{2}} \sin(2\pi \alpha) \mathcal{L} \left( \frac{1}{(k^0)^2 - |\vec{k}|^2 - m_0^2)^{2\alpha}} \right)(x-y) .
\]

\[\blacksquare\]
Remark 6.7 We should point out that in section 7 we shall give a different derivation of $s$. By the above formula (47), one can analytically continue $s$ to the kernel, $w$ say, of the 2-point (truncated) Wightman distribution $W_2^T = w(x - y)$ with

$$w(x - y) := 2(2\pi)^{-\frac{d}{2}} c_2 \sin(2\pi \alpha) F^{-1} \left( \frac{1}{((k^0)^2 - |k|^2 - m_0^2)^{2\alpha}} \right) (x - y).$$

Remark 6.8 Concerning the Schwinger function $S_2$ and the 2-point Wightman distribution for the case $\alpha = \frac{1}{2}$, see the discussion Section II.5 of [66].

Remark 6.9 Suppose $F$ is a Gaussian white noise, then $X_\alpha = (-\Delta + m_0^2)^{-\alpha} F$ is a generalized free field for each $\alpha \in (0, \frac{1}{2})$ and $m_0 > 0$. We refer the reader to e.g. [42] and [66] for the notion of generalized free field. Our argument here is as follows. In this case, all Wightman distributions $W_n \in \mathbb{N}$, are given symbolically by

$$W_n(x_1, \ldots, x_n) = \begin{cases} 0, & n \text{ is odd} \\ \sum W_2^T(x_{j_1}, x_{l_1}) \ldots W_2^T(x_{j_n}, x_{l_n}), & n \text{ is even} \end{cases}$$

where the sum is over $\{(j_1, \ldots, j_\frac{n}{2}, l_1, \ldots, l_\frac{n}{2}) \in \mathbb{N}^{\frac{n}{2}} : 1 \leq j_1 < j_2 < \ldots < j_\frac{n}{2} < n \text{ and } j_k < l_k \text{ for } 1 \leq k < \frac{n}{2} \}$. This shows that all $W_n, n \in \mathbb{N}$, are determined by $W_2^T$. The corresponding Schwinger function $S_n, n \in \mathbb{N}$, satisfy reflection positivity since by Corollary 6.4, $W_2^T$ has a Källén–Lehmann representation, therefore the field $X_\alpha$ is a generalized free Euclidean field. Especially $X_\alpha$ is reflection positive in the sense of [37].

Remark 6.10 Finally in this section, we should point out that concerning the kernel $G_1$ of $(-\Delta + m_0^2)^{-1}$, i.e., $\alpha = 1$, the above procedure can not be performed. One can apply the residue theorem instead of Cauchy integral theorem to get a Laplace transform representation of $G_1$. We therefore have a representation of $G_1$ by using the following basic formula (see e.g. (II.3) of [40] or (II.4) of [66]):

$$\int_{-\infty}^{\infty} \frac{e^{its}}{t^2 + r^2} dt = \frac{2\pi e^{-r|s|}}{r}, \quad r > , s \in (-\infty, \infty).$$

In fact, we have the following derivation
\[ G_1(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{ikx}}{|k|^2 + m_0^2} \, dk = (2\pi)^{-d} \int_{\mathbb{R}^{d-1}} e^{i\vec{k}\vec{x}} \left( \int_{-\infty}^{\infty} \frac{e^{ik^0x^0}}{k^0^2 + |\vec{k}|^2 + m_0^2} \, dk^0 \right) \, d\vec{k} = (2\pi)^{-d+1} \int_{\mathbb{R}^{d-1}} \frac{e^{-|\vec{k}|^2 + m_0^2 \frac{1}{2}|x^0| + i\vec{k}\vec{x}}}{2(|\vec{k}|^2 + m_0^2)^{\frac{1}{2}}} \, dk \].

The above integral representation of \( G_1 \) in case \( m_0 = 0 \) was used in [18] (see the corresponding formula (6.6) in Minkowski space in [18]) as a starting point for the analytic continuation of the Schwinger functions.

7 Analytic Continuation II: Continuation of the (Truncated) Schwinger Functions

7.1 Preliminary remarks

In this section we present the analytic continuation of the Euclidean (truncated) Schwinger functions obtained from convoluted generalized white noise \( X = (\Delta + m_0^2)^{-\alpha}F \), \( m_0 > 0 \), \( \alpha \in (0, \frac{1}{2}] \), to relativistic (truncated) Wightman distributions. More generally, formal expressions are obtained for the class of random fields \( X \) given by \( X = G \ast F \), where \( G \) is the Laplace–transform of a Lorentz–invariant signed measure on the forward lightcone. But as we will illustrate for \( X = (\Delta + m_0^2)^{-\alpha}F \), \( \alpha \in (\frac{1}{2}, 1) \), the question whether such relativistic expressions really represent the analytic continuation of the corresponding (truncated) Schwinger functions, deserves a separate discussion.

We derive manifestly Poincaré–invariant formulas for the Fourier–transform of the (truncated) Wightman functions. The obtained Wightman distributions fulfill the strong spectral condition of a QFT, with a "mass- gap". Locality, Hermiticity and the cluster property of the (truncated) Wightman functions follow from the (truncated) Schwinger functions’ symmetry, \( \theta \)-invariance and cluster property respectively, as a result of the general procedures of axiomatic QFT. We continue the discussion, started at the end of section 5, for the special case \( X = (\Delta + m_0^2)^{-\frac{1}{2}}F \), \( d = 4 \).

To the reader’s convenience (and to keep the length of the paper within reasonable bounds) not every step is presented with all details (for complete proofs see [38]). The methods applied here are based on and extend those of [13], [18].
7.2 The mathematical background

Let us first clarify the notations and the mathematical background. In this subsection \( n \) is a fixed integer with \( n \geq 2 \). From Definition 6.2 recall the meaning of \( M_d, <, >_M \) and let \( \{e_0, \ldots, e_{d-1}\} \) be an orthonormal frame in \( M_d \), by which \( M_d \) is identified with \( \mathbb{R} \times \mathbb{R}^{d-1} \cong \mathbb{R}^d \). From now on we write for \( x \in M_d \) \( x = (x^0, \vec{x}) \) and \( x^2 = <x, x>_M = x^0^2 - |\vec{x}|^2 \). The forward mass cone of mass \( m_0 > 0 \) is defined in analogy with the forward light cone \( V^+_0 \) as

\[
V^+_{m_0} := \{ k \in M_d : k^2 > m_0^2, <k, e_0>_M < 0 \} \quad m_0 \geq 0 .
\] (51)

By \( V^+_m \) we denote its closure. Let \( \theta \) again denote the time–reflection.

The backward (closed) mass cone/lightcone is defined by

\[
\theta V^-_0 := \{ k \in M_d : k^2 < m_0^2, <k, e_0>_M > 0 \} \quad m_0 \geq 0 .
\]

Since \( M_d \cong \mathbb{R}^d \subset \mathbb{C}^d \) there is a complexification of the Minkowski inner product \( <,>_M \) such that it is analytic in the coordinates with respect to \( \{e_0, \ldots, e_{d-1}\} \). We denote this complexification by \( <,>_C \).

Let \( T^n \) be the tubular domain in \( \mathbb{C}^d \) with base \( V^-_0 \), i.e.

\[
T^n := \{ \vec{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^d)^n : z_j - z_{j+1} \in M_d + iV^-_0, \ 1 \leq j \leq n - 1 \}\]

(52)

\( T^n \) is called the backward tube. Finally define

\[
e(k, \vec{z}) := (2\pi)^{-\frac{d+1}{2}} \exp \left\{ i \sum_{l=1}^{n} <k_l, z_l>_M \right\} ,
\]

(53)

where \( k = (k_1, \ldots, k_n) \in (\mathbb{R}^d)^n \) and \( \vec{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^d)^n \). We now give the definition of spectral conditions that are crucial for the theory of the Laplace–transforms in \( n \) arguments as well as for the notion of causality in a relativistic QFT.

**Definition 7.1** 1. Let \( \hat{\mathcal{W}} \in S'_c(\mathbb{R}^{dn}) \). We say that \( \hat{\mathcal{W}} \) fulfils the spectral condition, if there is a \( \hat{w} \in S(\mathbb{R}^{d(n-1)}) \) such that

\[
\hat{\mathcal{W}}(k_1 \ldots k_n) = \hat{w}(k_1, k_1 + k_2, \ldots, k_1 + \ldots + k_{n-1}) \delta \left( \sum_{l=1}^{n} k_l \right) \quad (54)
\]

and \( \text{supp} \ \hat{w} \subset (V^+_0)^{n-1} \).
2. Let \( \{ \hat{W}_l \}_{l \in \mathbb{N}} \) be a sequence of truncated distributions determined by a sequence of distributions \( \{ \hat{W}_l \}_{l \in \mathbb{N}_0} \), \( \hat{W}_l \in S_C^{d} (\mathbb{R}^d) \), \( \hat{W}_0 = 1 \). We say that the \( \{ \hat{W}_l \}_{l \in \mathbb{N}} \) (the \( \{ \hat{W}_l \}_{l \in \mathbb{N}_0} \) respectively) fulfil the strong spectral condition with a mass gap \( m_0 > 0 \) if all the distributions \( \hat{W}_l \), \( l \geq 2 \), fulfil the spectral condition (54), where \( V_0^{\ast} \) is replaced by \( V_0^{\ast - m_0} \).

The following two theorems, taken from the theory of Laplace transforms, provide us with the necessary mathematical tools for the analytic continuation of Schwinger functions.

**Theorem 7.2** Assume, that \( \hat{W} \in S_C^{d} (\mathbb{R}^d) \) fulfils the spectral condition. Then

1. \( \mathcal{L}(\hat{W})(\hat{\mathbf{z}}) = < \hat{W}, e(\cdot, \hat{\mathbf{z}}) > \) is well–defined and holomorphic in the variables \( z_j - z_{j+1} \), \( j = 1 \ldots n - 1 \) on the domain \( \hat{\mathbf{z}} \in T^n \). \( \mathcal{L}(\hat{W}) \) is called the Laplace transform of \( \hat{W} \).

2. \( \mathcal{F}^{-1}(\hat{W})(\Re \mathbf{z}) \) is the boundary–value of \( \mathcal{L}(\hat{W})(\hat{\mathbf{z}}) \) for \( \Im (z_j - z_{j+1}) \to 0 \) inside \( T^n \), i.e. the relation

\[
\lim_{\Im (z_j - z_{j+1}) \to 0} \mathcal{L}(\hat{W})(\hat{\mathbf{z}}) = \mathcal{F}^{-1}(\hat{W})(\Re \mathbf{z})
\]

holds in the sense of tempered distributions in the argument \( \Re \mathbf{z} \in \mathbb{R}^d \).

Here \( \Gamma \subset V_0^{\ast} \) is a subcone of \( V_0^{\ast} \) such that \( \Gamma \cup \{0\} \) is closed in \( \mathbb{R}^d \), and the Fourier transform \( \mathcal{F} \) is taken w.r.t. the Minkowski inner product on \( M_d \cong \mathbb{R}^d \).

Well–definedness in Theorem 7.2 holds, since \( e(\cdot, \hat{\mathbf{z}}) \) on the support of \( \hat{W} \) behaves like a fast decreasing function, if \( \hat{\mathbf{z}} \in T^n \) (see also Definition 6.2). We remark, that \( < \hat{W}, e(\cdot, \hat{\mathbf{z}}) > \) does only depend on the variables \( z_j - z_{j+1} \), \( j = 1 \ldots n - 1 \) as a consequence of (54) and up to the factor \( (2\pi)^{d/2} \) equals the Laplace–transform of \( \hat{w} \), defined in the sense of [67]. (In [67] a different convention on the Fourier–transform leads to the interchange of forward and backward cones). Theorem 7.2 is essentially equal to Theorem 2.6 and Theorem 2.9 of [67].

Let us remark, that most textbooks work with other conventions on the Laplace–transform as we do here, since we here mostly use the variables \( z_j \) and not the difference variables \( \zeta_j := z_j - z_{j+1} \). Lastly, let us remark that for distributions that depend only on one variable \( k \in \mathbb{R}^d \) we keep the conventions of Section 6.
We observe that

$$(E_d)^n_< := \left\{ \bar{z} \in (M_d^n)^n : \Im(z_j^0 - z_{j+1}^0) < 0, \ j = 1 \ldots n-1, \right.\left. \Im z_j = 0, \ \Re z_j^0 = 0 \ j = 1 \ldots n \right\}$$

is included in $T^n$ and is a $dn$-dimensional real manifold. We can calculate the coefficients of a local expansion of a function holomorphic in $T^n$ around a point $o \in (E_d)^n_<$ by performing "real" differentiations inside $(E_d)^n_<$. Therefore, functions which are single valued holomorphic on $T^n$ are determined by their values on $(E_d)^n_<$. This gives rise to the following

**Theorem 7.3** Let $S \in S'_c(\mathbb{R}^{dn})$ and $\hat{W} \in S_c(\mathbb{R}^{dn})$ fulfil the spectral condition. Assume

$$S\left((\Im z_1^0, \Re \vec{z}_1), \ldots, (\Im z_n^0, \Re \vec{z}_n)\right) = \mathcal{L}(\hat{W})(\bar{z}) \quad \bar{z} \in (E_d)^n_<$$

Then $\hat{W}$ is determined uniquely by this requirement. If $S$ is invariant under Euclidean transformations, then $W = F^{-1}(\hat{W})$ is Poincaré–invariant. If $S$ is, in addition, symmetric, then $W$ is local in the sense that

$$W(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = W(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n)$$

if $x_j - x_{j+1}$ is space–like, i.e. $(x_j - x_{j+1})^2 < 0$. Furthermore, if $S$ is a real distribution, $\theta$–invariance of $S$ implies Hermiticity of $W$, i.e.

$$W(x_1, \ldots, x_n) = \overline{W(x_n, \ldots, x_1)}$$

Theorem 7.3 is part of the reconstruction theorem in[56] and [66]. We remark, that for symmetric and Euclidean invariant $S$, $\mathcal{L}(W)$ possesses a single valued holomorphic extension to the so–called permuted extended tube $T^n_{p,e}$ (see [67] for definitions and proofs). Our restriction of Theorem 7.3 to tempered distributions $S$ is motivated only by our model, where "time–coincident"Schwinger functions are well defined. In [56] Theorem 7.3 is proved for $S \in S'_{\neq,c}(\mathbb{R}^{dn})$, which is a larger class of distributions than $S'_c(\mathbb{R}^{dn})$.

**Remark 7.4** Let $y_l = (y_l^0, \vec{y}_l) = (\Im z_l^0, \Re \vec{z}_l)$. Then (56) reads

$$S(y_1, \ldots, y_n) = (2\pi)^{-\frac{dn}{2}} \int_{\mathbb{R}^{dn}} e^{-\sum_{l=1}^n k_l^0 y_l^0 + i\vec{k}_l \vec{y}_l} \hat{W}(k_1, \ldots, k_n) \bigotimes_{l=1}^n dk_l$$

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where \( \underline{y} = (y_1 \ldots y_n) \in (\mathbb{R}^d)_<^n := \{ x \in \mathbb{R}^{dn} : x_1^0 < x_2^0 < \ldots < x_n^0 \} \).

### 7.3 Truncated Schwinger functions with ”sharp masses”

Our goal in this subsection is to represent the truncated Schwinger functions \( S^T_{n,n} \geq 2 \) of a convoluted generalized white noise as the restriction to the Euclidean region \( (E_d)_<^n \) of a Laplace transform of a tempered distribution \( \tilde{W}^T_{n,n} \) having the spectral property. In [13] and [18] this is done for the truncated Schwinger functions of \( X = \Delta^{-1}F \). The kernel \( \Delta^{-1}(x) \) can be represented as the Laplace transform of the essentially (up to a multiplication with a positive constant) unique Lorentz–invariant measure on the forward mass ”hyperboloid” with mass \( m = 0 \). Given that, the crucial step in the analytic continuation is a change of variables, depending on the value of \( m \).

Since Section 6 shows that we have to deal with a continuum of masses rather than with a sharp mass, we proceed in three steps: First in this subsection we give an integral representation of \( S^T_{n,n} \) in terms of ”truncated Schwinger functions with sharp masses” \( S^T_{\underline{m},n} \), \( \underline{m} = (m_1, \ldots, m_n) \). Then the subsection 7.4 deals with the representation of \( S^T_{\underline{m},n} \) as the Laplace–transform of a tempered distribution \( \tilde{W}^T_{\underline{m},n} \) restricted to \( (E_d)_<^n \). Finally, we integrate \( \tilde{W}^T_{\underline{m},n} \) over the masses to obtain the Fourier transform of the truncated \( n \)-point Wightman function \( \hat{W}^T_n \) (subsection 7.5). In that subsection we also collect the properties of the obtained (truncated) Wightman distributions arising at the main theorem of the present section.

From now on, we restrict ourselves to convoluted generalized white noises \( X = G * F \) with a kernel \( G \) which admits a representation of the form

\[
G(x) = \int_{\mathbb{R}^+} C_m(x) \rho(dm^2), \ x \in \mathbb{R}^d \setminus \{0\}
\]

where \( \rho \) is a (possibly signed) Borel measure on \( \mathbb{R}_+^d \). Furthermore we restrict ourselves to \( \rho \)'s which fulfill the following

**Condition 7.5**

1. There exists a mass \( m_0 > 0 \), such that \( \text{supp} \rho \subset [m_0^2, \infty) \).

2. \( \int_{\mathbb{R}^+} \frac{1}{m^2} |\rho|(dm^2) < \infty \).

**Remark 7.6**

1. Since \( C_m \) is the Laplace transform of the Lorentz invariant distribution \( (2\pi)^{-\frac{d}{2}} \delta_{m_0}^+(k) := (2\pi)^{-\frac{d}{2}} 1_{\{k^0 > 0\}}(k)\delta(k^2 - m_0^2) \) in the sense of Section 6, (60) and Condition 7.5 mean that \( G \) is the
Laplace transform of a signed Lorentz invariant measure on the forward mass–cone $V_{m_0}^*$.  

2. Condition 7.5 is also a sufficient condition for $G : f \in S(\mathbb{R}^d) \mapsto G * f \in S(\mathbb{R}^d)$ to be well–defined and continuous. The proof of this statement can be verified using similar techniques as in the proof of Lemma 7.7 [38].

3. The $\rho$’s obtained in Section 6 obviously fulfil Condition 7.5.

Lemma 7.7 Let $\rho$ and $G$ be as above and $S^T_n$ as in (31). Then for $\varphi \in S(\mathbb{R}^{dn})$

$$< S^T_n, \varphi > = c_n \int (\mathbb{R}^+)^n < S^T_{m,n}, \varphi > \rho(dm^2)$$

where $m = (m_1, \ldots, m_n)$ and $\rho(dm^2) = \rho^{\otimes n}(dm_1^2 \times \ldots \times dm_n^2)$. $S^T_{m,n}$ is defined by

$$< S^T_{m,n}, \varphi_1 \otimes \ldots \otimes \varphi_n > = \int_{\mathbb{R}^d} C_{m_1} \ast \varphi_1 \ldots C_{m_n} \ast \varphi_n dx \ , \varphi_1 \ldots \varphi_n \in S(\mathbb{R}^d)$$

(62)

Proof. By the nuclear theorem, (62) well–defines $S^T_{m,n} \in S'((\mathbb{R}^d)^n)$. To prove (61) let again $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_n$, with $\varphi_1 \ldots \varphi_n \in S(\mathbb{R}^d)$. We remark that $C_m(x) = m^{d-2} C_1(mx)$ for $x \in \mathbb{R}^d \setminus \{0\}$ and $C_1 \in L^1(\mathbb{R}^d, dx)$ (cf. [37] p. 126). Therefore

$$\int (\mathbb{R}^+)^n \int_{\mathbb{R}^d} \int_{\mathbb{R}^{dn}} \prod_{l=1}^n |C_{m_l}(y_l)\varphi_l(x-y_l)| dy_1 \ldots dy_n dx |\rho|(dm^2)$$

$$= \int (\mathbb{R}^+)^n \int_{\mathbb{R}^d} \int_{\mathbb{R}^{dn}} \prod_{l=1}^n |C_1(y_l')\varphi_l(x-y_l')| dy_1' \ldots dy_n dx \prod_{l=1}^n \rho|y_l'|^2 (dm_l^2)$$

$$\leq \left[ \|C_1\|_{L^1(\mathbb{R}^d, dx)} \int_{\mathbb{R}^+} \frac{1}{m^2} |\rho|(dm^2) \right] \prod_{l=1}^{n-1} \|\varphi_l\|_{L^\infty(\mathbb{R}^d, dx)} \|\varphi_n\|_{L^1(\mathbb{R}^d, dx)}$$

$$< \infty$$

(63)

where we have applied

$$\int_{\mathbb{R}^d} \prod_{l=1}^n |\varphi_l(x-z_l)| dx \leq \prod_{l=1}^{n-1} \|\varphi_l\|_{L^\infty(\mathbb{R}^d, dx)} \|\varphi_n\|_{L^1(\mathbb{R}^d, dx)} \ , \ z_1 \ldots z_n \in \mathbb{R}^d.$$
Now (63) allows us to apply Fubini’s theorem to the LHS of (61). The LHS and the RHS of (61) are therefore equal for \( \varphi = \varphi_1 \otimes \ldots \otimes \varphi_n \). At the same time (63) shows, by the nuclear theorem, that both sides of (61) denote tempered distributions. These give, by the above argument, equal values if evaluated on test functions \( \varphi = \varphi_1 \otimes \ldots \otimes \varphi_n \). Therefore, by the nuclear theorem again, the distributions on both sides of (61) are equal.

7.4 The Schwinger functions with ”sharp masses” as Laplace transforms

From now on we assume \( \underline{y} = (y_1, \ldots, y_n) \in (\mathbb{R}^d)_\prec, \underline{m} = (m_1, \ldots, m_n) \in (\mathbb{R}^+)^n \). Then

\[
S_{\underline{m},n}^T(y_1, \ldots, y_n) = \int_{\mathbb{R}^d} C_{m_1}(x - y_1) \ldots C_{m_n}(x - y_n) \, dx
\]
is well–defined as a function, since the singularities of the functions \( C_{m_i}(x - y_l) \) are all separated from each other and are therefore all integrable and, furthermore, for large \( x \) the integrand falls off exponentially to zero (cf. [37] p. 126).

Taking into account that

\[
C_{m_i}(x - y_l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-k_0^0|x^0 - y_l^0| + i\bar{k}_l(x - \bar{y}_l)} \delta_{m_i}(k_l) \, dk_l,
\]
we get by Fubini’s theorem

\[
S_{\underline{m},n}^T(y_1, \ldots, y_n) = (2\pi)^{d-1-dn} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \prod_{l=1}^n e^{-k_0^0|x^0 - y_l^0|} \, dx^0 \right] \times e^{-i \sum_{l=1}^n \bar{k}_l \bar{y}_l} \prod_{l=1}^n \delta_{m_l}(k_l) \delta \left( \sum_{l=1}^n \bar{k}_l \right) \otimes dk_l \quad (64)
\]

where we have also applied the distributional identity \( \mathcal{F} \left( \frac{1}{\sqrt{2\pi}} \right) = \delta \) in the distribution space \( \mathcal{S}'(\mathbb{R}^{d-1}) \).

The RHS of (64) has to be considered as an integral over a submanifold of \( \mathbb{R}^{dn} \) determined by the \( \delta \)–distributions. If not stated otherwise, all products of distributions which occur in the following are defined in this way.

The expression in the brackets \( [\ldots] \) in (64) equals

\[
\frac{1}{\sum_{l=1}^n k_l^0} \prod_{l=2}^n e^{-k_l^0(y_l^0 - y_l^0)} \sum_{j=1}^{n-1} \prod_{l=1}^{j-1} e^{-k_l^0(y_j^0 - y_l^0)} (y_{j+1} - y_j^0)
\]

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If we insert (65) into (64), then the RHS of (64) splits up into \( n + 1 \) summands, say

\[
S_{m,n}^T(y_1, \ldots, y_n) = I_0(y_1 \ldots y_n) + \sum_{j=1}^{n-1} (y_j^0 - y_{j+1}^0) I_j(y_1 \ldots y_n) + I_n(y_1 \ldots y_n) \tag{66}
\]

We are going to write each of these summands in the form of (59). This is easy for \( I_0 \) and \( I_n \):

\[
I_0(y_1, \ldots, y_n) = (2\pi)^{d-1-n} \int_{\mathbb{R}^{dn}} \prod_{l=2}^{n} e^{-k_l^0 y_l^0 - i\vec{k}_l \vec{y}_l} e^{-(-\sum_{l=2}^{n} k_l^0) y_1^0 - i\vec{k}_1 \vec{y}_1} \delta_{m_l}(k_l) \left( \sum_{l=1}^{n} k_l \right)^n dk_l
\]

\[
= (2\pi)^{d-1-n} \int_{\mathbb{R}^{d(n-1)} \times \mathbb{R}^{d-1}} \prod_{l=2}^{n} e^{-k_l^0 y_l^0 - i\vec{k}_l \vec{y}_l} e^{-(-\sum_{l=1}^{n} k_l^0) y_1^0 - i\vec{k}_1 \vec{y}_1} \prod_{l=1}^{n} \delta_{m_l}(k_l) \left( \sum_{l=1}^{n} k_l \right)^n \frac{dk_l}{\omega_l} \times \prod_{l=2}^{n} \frac{1}{\omega_1 + \sum_{l=2}^{n} k_l^0} \prod_{l=2}^{n} \delta_{m_l}(k_l) \left( \sum_{l=1}^{n} k_l \right)^n \frac{dk_l}{\omega_1 - k_1^0}
\]

where we have introduced \( \omega_l = \sqrt{k_l^2 + m_l^2} \), \( l = 1 \ldots n \). Thus, \( I_0 \) is the Laplace transform of the tempered distribution

\[
(2\pi)^{d-1-n} \prod_{l=2}^{n} \frac{1}{\omega_1 - k_l^0} \prod_{l=2}^{n} \delta_{m_l}(k_l) \left( \sum_{l=1}^{n} k_l \right)^n \tag{67}
\]

restricted to \( (E_d)^n \). By an analogous calculation we find that \( I_n \) is the Laplace transform of the tempered distribution
Furthermore, we integrate over $\delta_m$ be directly deduced from these formulas (cf. the proof of proposition 7.8 below).

Let us now turn to the more complicated calculations for the $I_j$'s $j = 1 \ldots n - 1$.

\[
I_j(y_1 \ldots y_n) = (2\pi)^{d-1-dn} \int_{\mathbb{R}^{d(n-2)}} e^{\sum_{l=1}^{j-1} k_l^0 y_l^0 - i\vec{k}_l \vec{y}_l} \times \int_0^1 e^\left(\sum_{l=1}^{j-1} k_l^0 s + \sum_{l=1}^{j-1} k_l^0 (1-s)\right) \times e^{-\left(\sum_{l=1}^{j-1} k_l^0 s + \sum_{l=1}^{j-1} k_l^0 (1-s)\right) - \sum_{l=1}^{n-j} k_l^0} \delta_{m_l}^+(k_l) \delta \left(\sum_{l=1}^n k_l\right) \otimes dk_l.
\]

We may interchange $ds$ and $\otimes_{l=1}^n dk_l$ integrations by Fubini's theorem. Furthermore, we integrate over $\delta_{m_j}^+(k_j) \delta_{m_{j+1}}^+(k_{j+1})$ and change coordinates $k_l^0 \mapsto -k_l^0$ for $l = 1 \ldots j - 1$, getting the RHS to be equal to

\[
(2\pi)^{d-1-dn} \int_0^1 \int_{\mathbb{R}^{d(n-2)} \times \mathbb{R}^{d(d-1)}} \prod_{l=1}^{j-1} e^{-k_l^0 y_l^0 - i\vec{k}_l \vec{y}_l} \times \prod_{l=j}^{j+1} e^{-k_l^0 y_l^0 - i\vec{k}_l \vec{y}_l} \prod_{l=j+1}^{j-1} e^{-k_l^0 y_l^0 - i\vec{k}_l \vec{y}_l} \times \prod_{l=1}^{n-j} \delta_{m_l}^+(k_l) \prod_{l=j+2}^{n-j+1} \delta_{m_l}^+(k_l) \otimes dk_l \otimes \frac{d\vec{k}_1}{2\omega_l} ds,
\]

where we have used the following notations

\[
\vec{k}_j^0 = \vec{k}_j^0(k_1^0, \ldots, k_{j-1}^0, l_j^0, k_{j+1}^0, k_{j+2}^0, \ldots, k_n^0)
\]
In these cases we may, by Fubini’s theorem again, change the order of the integrations over new variables.

\[ I_j(k_1, \ldots, k_{j+1}) := \{ (-\sum_{l=1}^{j-1} k_l^0 + \omega_j)s + (\omega_{j+1} + \sum_{l=j+2}^{n} k_l^0)(1 - s) + \sum_{l=j+2}^{n} k_l^0 \} \; ; \]

\[ \tilde{k}_{j+1}^0 = \tilde{k}_{j+1}^0(k_1^0, \ldots, k_{j-1}^0, \tilde{k}_j, \tilde{k}_{j+1}, k_{j+2}^0 \ldots k_n^0) \]

\[ := \{ (-\sum_{l=1}^{j-1} k_l^0 + \omega_j)s + (\omega_{j+1} + \sum_{l=j+2}^{n} k_l^0)(1 - s) - \sum_{l=j+2}^{n} k_l^0 \}. \]

We remark that \( \sum_{l=1}^{j-1} k_l^0 + \tilde{k}_j^0 + \tilde{k}_{j+1}^0 + \sum_{l=j+2}^{n} e \cdot k_l^0 = 0 \). Therefore we may introduce new “integrations” over new variables \( k_j^0, k_{j+1}^0 \) using the measure

\[ \delta(k_j^0 - \tilde{k}_j^0(k_j^0, \ldots, k_{j-1}^0, \tilde{k}_j, \tilde{k}_{j+1}, k_{j+2}^0 \ldots k_n^0)) \delta(\sum_{l=1}^{n} k_l^0) \] \( dk_j^0 dk_{j+1}^0 \),

where

\[ a(k_j, k_{j+1}) = -k_j + \omega_j - k_{j+1} + \omega_{j+1} \]

\[ b(k_{j+1}) = -k_{j+1} + \omega_{j+1} . \]

In this way we get

\[ I_j(y_1, \ldots, y_n) \]

\[ = (2\pi)^{d-1-dn} \int_0^1 \int_{\mathbb{R}^d} \prod_{l=1}^{n} e^{-k_l^0 y_l^0} \frac{1}{\omega_{j+1}^{j+1}} \prod_{l=1}^{j-1} \delta_m(k_l) \]

\[ \times \frac{\delta(a(k_j, k_{j+1})s - b(k_{j+1}))}{4\omega_{j+1} \omega_{j+1}} \prod_{l=j+2}^{n} \delta_m(k_l) \delta(\sum_{l=1}^{n} k_l) \bigotimes_{l=1}^{n} dk_l ds . \] (69)

For \( n \geq 3 \) or \( n = 2, m_1 \neq m_2 \) \( a(k_j, k_{j+1}) \neq 0 \) holds almost everywhere with respect to the measure \( \prod_{l=1}^{n} \delta_m(k_l) \prod_{l=j+2}^{n} \delta_m(k_l) \delta(\sum_{l=1}^{n} k_l) \bigotimes_{l=1}^{n} dk_l \).

In these cases we may, by Fubini’s theorem again, change the order of the integrations. This together with

\[ \int_0^1 \delta(as - b) ds = \frac{1}{|a|} \left[ 1_{\{0 < b < a\}}(a, b) + 1_{\{a < b < 0\}}(a, b) \right] \] for \( a \neq 0 \)

inserted into (69), allows us to conclude that \( I_j \) is the Laplace– transform of the distribution

\[ H_j(k_1 \ldots k_n) \]

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\[
(2\pi)^{d-1} - \frac{dn}{2} \sum_{l=1}^{n} \delta_{m_l}(k_l) \prod_{l=j+2}^{n} \delta_{m_l}^{+}(k_l) \delta \left( \sum_{l=1}^{n} k_l \right)
\]

restricted to \((E_d)^n\).

The temperedness of \(H_j\) can be derived from an integral representation like that in (69), where the exponential functions have to be replaced by a test function \(\varphi \in S(\mathbb{R}^d)\):

\[
|\langle H_j, \varphi \rangle| \leq (2\pi)^{d-1} - \frac{dn}{2} \sum_{l=1}^{n} m_l^{-1} \left( \int_{\mathbb{R}^{d-1}(n-1)} \frac{1}{(1 + |\vec{k}|^2)^{dn/2}} d\vec{k} \right) ||\varphi||_{0,dn} \, ,
\]

where \( ||\varphi||_{0,dn} := \sup_{\vec{k} \in \mathbb{R}^{dn}} |(1 + |\vec{k}|^2)^{dn/2} \varphi(\vec{k})| \).

Since \(I_j(y_1, \ldots, y_n)\) is the Laplace–transform of \(H_j(k_1, \ldots, k_n), \,(y_{j+1}^0 - y_j^0)I_j(y_1, \ldots, y_n)\) is the Laplace transform of the tempered distribution \((\partial_{j+1}^0 = \partial_{j}^0 l = 1 \ldots n\). Terms that depend only on \(k_j^0 + k_{j+1}^0\) give a zero contribution when derived with respect to \(\partial_{j+1}^0 - \partial_j^0\).

This applies to \(a(k_j, k_{j+1})\) and \(\delta(\sum_{l=1}^{n} k_l)\). Thus, only the derivatives of the characteristic functions in (70) contribute to \((\partial_{j+1}^0 - \partial_j^0)H_j\).

Taking into account

\[
1_{\{0 < b(k_{j+1}) - a(k_j, k_{j+1})\}} = 1_{\{k_j^0 < -\omega_j\}} 1_{\{k_{j+1}^0 < \omega_{j+1}\}} ,
\]

\[
1_{\{a(k_j, k_{j+1}) - b(k_{j+1}) < 0\}} = 1_{\{k_j^0 > -\omega_j\}} 1_{\{k_{j+1}^0 > \omega_{j+1}\}} ,
\]

\[
\frac{d}{dx} 1_{\{0 < x\}}(y) = \delta(y)\quad\text{and also} \quad (2\omega_l)^{-1} \delta(k_l^0 \mp \omega_l) = \delta_{m_l}^{+}(k_l) ,
\]

we calculate

\[
\begin{align*}
\frac{1}{4\omega_j \omega_{j+1}} & \left( (\partial_{j+1}^0 - \partial_j^0) 1_{\{0 < b(k_{j+1}) - a(k_j, k_{j+1})\}} (k_j, k_{j+1}) \right) \\
& = (2\omega_j)^{-1} 1_{\{k_j^0 < -\omega_j\}} (k_j) \delta_{m_{j+1}}^+(k_{j+1}) \\
& \quad + \delta_{m_j}^-(k_j) (2\omega_{j+1})^{-1} 1_{\{k_{j+1}^0 < -\omega_{j+1}\}} (k_{j+1}) \\
\frac{1}{4\omega_j \omega_{j+1}} & \left( (\partial_{j+1}^0 - \partial_j^0) 1_{\{a(k_j, k_{j+1}) - b(k_{j+1}) < 0\}} (k_j, k_{j+1}) \right) \\
& = (2\omega_j)^{-1} 1_{\{k_j^0 > -\omega_j\}} (k_j) \delta_{m_{j+1}}^+(k_{j+1}) \\
& \quad + \delta_{m_j}^-(k_j) (2\omega_{j+1})^{-1} 1_{\{k_{j+1}^0 < \omega_{j+1}\}} (k_{j+1})
\end{align*}
\]

(72)
Adding up both sides of (72) yields

\[
(\partial^0_{j+1} - \partial^0_j) H_j(k_1, \ldots, k_n) = (2\pi)^{d-1} \frac{4\pi}{\omega_j k_j^2 + \omega_j + k_j^2 - \omega_j + 1} \left\{ \frac{\sign(k_j^2 + \omega_j)}{2\omega_j k_j^2 + \omega_j + k_j^2 - \omega_j + 1} + \frac{\delta_{m_1}(k_j) \sign(\omega_j - k_j^2)}{2\omega_j k_j^2 + \omega_j + k_j^2 - \omega_j + 1} \right\} \\
\times \prod_{l=1}^{j-1} \delta_{m_l}(k_l) \prod_{l=j+2}^{n} \delta_{m_l}(k_l) \delta \left( \sum_{l=1}^{n} k_l \right) \\
= (2\pi)^{d-1} \frac{4\pi}{\omega_j k_j^2 + \omega_j + k_j^2 - \omega_j + 1} \left\{ \frac{\delta_{m_1}(k_{j+1})}{2\omega_j (k_{j+1}^2 + \omega_j)} + \frac{\delta_{m_1}(k_j)}{2\omega_j (k_{j+1}^2 + \omega_j - 1)} \right\} \\
\times \prod_{l=j+2}^{n} \delta_{m_l}(k_l) \delta \left( \sum_{l=1}^{n} k_l \right)
\]

(73)

A closer analysis shows, that the singularities on the RHS of (73) have to be understood in the sense of Cauchy’s principal value.

Keeping in mind that

\[
\frac{1}{2\omega(\omega + k^0)} + \frac{1}{2\omega(\omega - k^0)} = \frac{(-1)}{k^2 - m^2}, \quad \omega = \sqrt{k^2 + m^2},
\]

by adding up (67), (73) for \( j = 1 \ldots n - 1 \) and (68) (recall also (66) ) we get the following

**Proposition 7.8** For \( \underline{m} = (m_1 \ldots m_n) \in (\mathbb{R}_+)^n \), let \( \hat{W}^T_{\underline{m},n} \) denote the distribution

\[
(2\pi)^{d-1} \frac{4\pi}{\omega^2} \left\{ \sum_{j=1}^{n} \prod_{l=1}^{j-1} \delta_{m_l}(k_l) \frac{(-1)}{k_j^2 - m_j^2} \prod_{l=j+2}^{n} \delta_{m_l}(k_l) \right\} \delta \left( \sum_{l=1}^{n} k_l \right) \\
\]

(74)

for \( n \geq 3 \) or \( n = 2 \), \( m_1 \neq m_2 \) and

\[
(2\pi)^{-1} \left\{ (2\omega_1)^{-2} \delta_{m_1}(k_1) - (2\omega_1)^{-1} (\partial^0_1 \delta_{m_1})(k_1) \right\} \delta(k_1 + k_2)
\]

(75)

for \( n = 2 \), \( m_1 = m_2 \).

Then

1. \( \hat{W}^T_{\underline{m},n} \) is tempered and fulfils the strong spectral condition with the mass gap \( m_0 \leq \min\{m_l : l = 1 \ldots n\} \)
2. \( S_{m,n}^T \) is the restriction of \( \mathcal{L}(W_{m,n}^T) \) to the Euclidean region \((E_\mathbb{R}^d)^n_\prec \) in the sense of (56). This property determines \( \check{W}_{m,n}^T \) uniquely.

3. \( W_{m,n}^T = F^{-1}(\check{W}_{m,n}^T) \) is Poincaré invariant.

**Proof.** We only deal with the case \( n \geq 3 \) or \( n = 2 \) if \( m_1 \neq m_2 \). Concerning the support properties, let us concentrate on the \( j \)th summand in (74). Let \( \vec{k} = (k_1, \ldots, k_n) \) be in the support of this summand. For \( r < j \), \( \sum_{l=1}^r k_l \in V_{m_0}^{*-} \) holds, since each \( k_l, l = 1 \ldots r \), is in this cone. For \( n - 1 \geq r \geq j \) we get \( \sum_{l=1}^r k_l = -\sum_{l=r+1}^n k_l \in V_{m_0}^{*-} \), since each \( k_l, l = r + 1 \ldots n \), is in \( V_0^{*-} \) and thus \( -k_l \in V_0^{*-} \).

The facts that \( \check{W}_{m,n}^T \) is tempered, and that \( S_{m,n}^T \) is the restriction of \( \mathcal{L}(W_{m,n}^T) \) to \((E_\mathbb{R}^d)^n_\prec \) summarize the above discussion. We have worked in the notations of (59) rather than in that of (56), but, as already remarked, these two relations are equivalent. The other statements follow from Theorem 7.3.

The derivation of (75) is relatively easy. In the following we will not use this formula and therefore leave the calculation as an exercise.

Nevertheless, the formula (75) has some consequences: If we take a noise \( F \) with mean zero and \( X = (-\Delta + m_1^2)^{-1}F = (2\pi C_{m_1} * F) \), then it is easy to see that (75) gives the Fourier transform of the 2–point Wightman function of the model. Since the distribution in (75) does not admit a Källen–Lehmann representation, not all of the one-particle and free ”states” of this model have positive norm. Thus, a good physical interpretation of such models is impossible, even if \( F \) has zero Poisson part.

Two more details may be of interest:

**Remark 7.9**  
1. Let \( L^1_+ (\mathbb{R}^d) \) denote the proper orthochronous Lorentz group. For \( \Lambda \in L^1_+ (\mathbb{R}^d) \) and \( a \in \mathbb{R}^d \), we recall that the Poincaré group acts on functions \( \varphi(k_1, \ldots, k_n) \) defined on the momentum space \((\mathbb{R}^d)^n_\prec \) as follows

\[ (T_{(\Lambda,a)} \varphi)(k_1, \ldots, k_n) = \varphi((\Lambda^*)^{-1} k_1, \ldots, (\Lambda^*)^{-1} k_n) e^{i<\sum_{l=1}^n k_l, a>_M}. \]

where the adjoint \( \Lambda^* \) is w.r.t. \( <,>_M \), the Minkowski inner product. From (74) we know that \( \check{W}_{m,n}^T \) are "manifestly" Poincaré invariant.

2. Only if \( m_1 = m_2 = \ldots = m_n \) we can expect \( W_{m,n}^T \) to be a "local" distribution (cf. Theorem 7.3).
7.5 The analytic continuation of the truncated Schwinger functions

Proposition 7.8 immediately implies

Theorem 7.10 Suppose that the distribution

\[ \hat{W}_n^T := c_n \int_{(\mathbb{R}^+)^n} \hat{W}_{\underline{m},n} \rho(d\underline{m}^2), \]

i.e.

\[ <\hat{W}_n^T, \varphi> = c_n \int_{(\mathbb{R}^+)^n} <\hat{W}_{\underline{m},n}, \varphi> \rho(d\underline{m}^2) \quad \varphi \in S(\mathbb{R}^{dn}) \quad (76) \]

is well-defined for all \( \varphi \in S(\mathbb{R}^{dn}) \) and furthermore \( \hat{W}_n^T \in S'(\mathbb{R}^{dn}) \). Then

1. \( \hat{W}_n^T \) fulfils the strong spectral condition with respect to the mass gap \( m_0 \), where \( m_0 \) is as in Condition 7.5.

2. \( S_n^T \) is the restriction of the Laplace transform \( \mathcal{L}(\hat{W}_n^T) \) to the Euclidean region \( (E_d)_< \). This determines \( \hat{W}_n^T \) uniquely. Furthermore \( \hat{W}_n^T = \mathcal{F}^{-1}(\hat{W}_n^T) \) is the boundary-value of the analytic function \( \mathcal{L}(\hat{W}_n^T)(z), z \in T^n \), for \( \Im z \to 0 \) as described in Theorem 7.2. In this sense, we call the truncated \( n \)-point Wightman distribution \( W_n^T \) the analytic continuation of \( S_n^T \) to the Minkowski space-time.

3. \( W_n^T \) is a Poincaré invariant, local, hermitian distribution, which fulfils, in addition, the cluster-property of the truncated Wightman distributions, i.e. for \( \varphi_1 \ldots \varphi_{n+m} \in S(\mathbb{R}^{dn}) \) and a spacelike \( a \in \mathbb{R}^d (a^2 < 0) \) we have

\[ W_n^T(\varphi_1 \otimes \ldots \varphi_m \otimes T_{\lambda a} (\varphi_{m+1} \otimes \ldots \otimes \varphi_{m+n})) \to 0 \quad \text{if } \lambda \to \infty, \quad (77) \]

where \( T_{\lambda a} \) is the translation by \( \lambda a \).

Proof.

1. By Proposition 7.8 all the \( \hat{W}_{\underline{m},n}^T, \underline{m} = (m_1 \ldots m_n), m_l \in \text{supp } \rho \) fulfil the strong spectral condition with the mass gap \( m_0 > 0 \) given by Condition 7.5. Since \( \hat{W}_n^T \) is a superposition of such \( \hat{W}_{\underline{m},n}^T \), the same applies to \( \hat{W}_n^T \).

2. As remarked before, \( \hat{e}(\underline{k}, y) := (2\pi)^{-d} \frac{1}{2} e^{-\sum_{l=1}^{n} k_l^0 y_l^0 + i\underline{k} \cdot \underline{y}} \) on the support of \( \hat{W}_n^T \) behaves like a fast falling function in \( \underline{k} \in \mathbb{R}^{dn} \), whenever
\[ y = (y_1, \ldots, y_n) \in (\mathbb{R}^d)^n. \] Therefore, the following equations hold:

\[
\begin{align*}
\langle \hat{W}_n^T, \tilde{e}(\cdot, y) \rangle &= c_n \int_{(\mathbb{R}^+)^m} \langle \hat{W}_m^T, \tilde{e}(\cdot, y) \rangle \rho(dm^2) \\
&= c_n \int_{(\mathbb{R}^+)^m} S_m^T(y_1, \ldots, y_n) \rho(dm^2) \\
&= S_n^T(y_1, \ldots, y_n).
\end{align*}
\]

The second equality is valid by Proposition 7.8, where the RHS makes sense dy a.e. by Lemma 7.7, which also implies the third equality. Theorem 7.2 and 7.3 now imply 2.

3. Except for the cluster–property, everything follows from 2 and Theorem 7.3. For the cluster–property, we refer to [56], Theorem 4.5 and Corollary 4.7 (see alternatively [61] Vol. III p. 324).

\[
\textbf{Corollary 7.11} \text{ Let } \{W_n\}_{n \in \mathbb{N}_0} \text{ be the Wightman distributions determined by the truncated sequence } \{W_n^T\}_{n \in \mathbb{N}} \text{ and } W_0 = 1. \text{ Then } \hat{W}_n \text{ fulfills the spectral condition, for } n \in \mathbb{N}. \text{ The statements 1, 2 and 3 of Theorem 7.10 hold, if the } W_n^T, S_n^T \text{ are replaced by } W_n, S_n, \text{ respectively and the cluster–property of the truncated Wightman function is replaced by that for the Wightman functions (see [67] or [56]).}
\]

For a proof of Corollary 7.11 we refer to similar discussions in Section 4 and to [22].

Let us now turn to the question of the temperedness of the formal expressions (76). It is e.g. not difficult to see that for \(\rho\)'s that have compact support in \(\mathbb{R}_+\), temperedness follows (c.f. (71)). Nevertheless, in these cases we cannot expect the two–point Wightman function to admit a Källen–Lehmann representation. The reader is asked to convince herself/himself that there is no such representation e.g. for the case \(\rho(dm^2) = f(m^2)dm^2, f > 0\), where \(f \in S(\mathbb{R})\) has compact support in \(\mathbb{R}_+\). In this case again, we cannot give a good physical interpretation, even not for one–particle or free states.

Therefore we restrict ourselves to the \(\rho\)'s obtained in Section 6, i.e.

\[
\rho_{\alpha}(dm^2) = 2\sin \pi \alpha 1_{\{m^2 > m_0^2\}}(m^2) \frac{dm^2}{(m^2 - m_0^2)^{\alpha}} \quad \alpha \in (0, 1) \quad (78)
\]

Again it is possible to show the temperedness of the distributions \(\hat{W}_{n, \alpha}^T\), defined by (76) with \(\rho = \rho_{\alpha}\) if \(\alpha \in (1/2, 1)\) by a direct estimate, using (71).
Let us therefore turn to the case $\alpha \in (0, \frac{1}{2}]$. We introduce the notations

\[
\begin{align*}
\mu^+_\alpha(k) & = (2\pi)^{-d/2} \sin \pi \alpha \mathbb{1}_{\{k^2 > m_0^2, k^0 > 0\}}(k) \frac{1}{(k^2 - m_0^2)^\alpha} \\
\mu^-_\alpha(k) & = (2\pi)^{-d/2} \sin \pi \alpha \mathbb{1}_{\{k^2 > m_0^2, k^0 < 0\}}(k) \frac{1}{(k^2 - m_0^2)^\alpha} \\
\mu_\alpha(k) & = (2\pi)^{-d/2} \left( \cos \pi \alpha \mathbb{1}_{\{k^2 > m_0^2\}}(k) + 1_{\{k^2 < m_0^2\}}(k) \right) \frac{1}{|k^2 - m_0^2|^\alpha}
\end{align*}
\]

and we get:

**Proposition 7.12** Let $\hat{W}^T_{n,\alpha}$ be defined as in (76) with $\rho = \rho_\alpha$, $\alpha \in (0, \frac{1}{2})$. Then $\hat{W}^T_{n,\alpha}$ is tempered and equal to

\[
c_n(2\pi)^{d-\frac{d+\alpha}{2}} \left\{ \sum_{j=1}^{n} \prod_{l=1}^{j-1} \mu^+_\alpha(k_l) \mu_\alpha(k_j) \prod_{l=j+1}^{n} \mu^+_\alpha(k_j) \right\} \delta \left( \sum_{l=1}^{n} k_l \right) n \geq 2 \quad (79)
\]

**Proof.** Despite the fact, that one cannot apply Fubini’s theorem because of the presence of the Cauchy principal values in (74), it can be shown by a regularization (of the $\rho_\alpha$ and the $\hat{W}^T_{m,n}$) and a passage to the limit, that the integration w.r.t. $\rho_\alpha(d\mathbf{m})$ and the evaluation with a test function $\varphi \in \mathcal{S}(\mathbb{R}^{dn})$ can be interchanged in (76). Therefore, we get for $\hat{W}^T_{n,\alpha}$:

\[
\begin{align*}
& c_n(2\pi)^{d-1-\frac{d+\alpha}{2}} \left\{ \sum_{j=1}^{n} \prod_{l=1}^{j-1} \sin \pi \alpha \int_{m_0^2}^{\infty} \frac{\delta^-_m(k_l)}{(m_l^2 - m_0^2)^\alpha} \right\} \\
& \times \left[ \int_{\mathbb{R}^+} \rho_\alpha(d\mathbf{m})^2 \prod_{l=j+1}^{n} \sin \pi \alpha \int_{m_0^2}^{\infty} \frac{\delta^-_m(k_l)}{(m_l^2 - m_0^2)^\alpha} \right] \delta \left( \sum_{l=1}^{n} k_l \right) \\
& = c_n(2\pi)^{d-1-\frac{d+\alpha}{2}} \left\{ \sum_{j=1}^{n} \prod_{l=1}^{j-1} \mu^-_\alpha(k_l) \left[ \int_{\mathbb{R}^+} \frac{\rho_\alpha(d\mathbf{m})^2}{m_l^2 - k_j^2} \right] \\
& \times \prod_{l=j+1}^{n} \mu^+_\alpha(k_l) \right\} \delta \left( \sum_{l=1}^{n} k_l \right)
\end{align*}
\]

But (cf. [31] p.70)

\[
\begin{align*}
& \int_{m_0^2}^{\infty} \frac{1}{(m^2 - k^2) (m^2 - m_0^2)^\alpha} dm^2 \\
& = \int_{0}^{\infty} \frac{1}{(x - (k^2 - m_0^2))x^\alpha} dx
\end{align*}
\]

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Thus, we get (79). It is not difficult to show the temperedness of (79) by application of a Cauchy–Schwarz inequality, making also use of the fact that \((\mu^+_\alpha)^2, (\mu^-\alpha)^2, (\mu_\alpha)^2\) are locally integrable functions.  

Poincaré–invariance in (79) follows from Theorem 7.10, but is also "manifest" (cf. Remark 7.9 1.). Let us make sure that (79) in the case \(n = 2\) yields the same result as Section 6. We have

\[
c_2(2\pi)^{d/2} \left\{ \mu_\alpha(k_1)\mu^+_\alpha(k_2) + \mu^-\alpha(k_1)\mu_\alpha(k_2) \right\} \delta(k_1 + k_2) = c_4 \sin \pi \alpha \cos \pi \alpha \left[ 1_{(k_1^2 > m_0^2, k_1^0 < 0)}(k_1) \frac{1}{(k_1^2 - m_0^2)^\alpha} \right]^2 \delta(k_1 + k_2) = c_2 2 \sin 2\pi \alpha 1_{(k_1^2 > m_0^2, k_1^0 < 0)}(k_1) \frac{1}{(k_1^2 - m_0^2)^{2\alpha}} \delta(k_1 + k_2) \tag{80}
\]

for \(\alpha \in (0, \frac{1}{2})\), where we have applied \(\sin \pi \alpha \cos \pi \alpha = \frac{1}{2} \sin(2\pi \alpha)\). (80) differs only by a time reflection \(\theta\) from the distribution defined in Remark 6.9, which arises from different conventions in the definition of the Laplace–transform in Section 6 and Section 7. Thus, we have the same result as in Section 6. (Note that the additional factor \((2\pi)^{-d}\) in Remark 6.7 arises from the fact, that the normalisation factor of the Fourier transform in one argument is \((2\pi)^{-d/2}\) while for the Fourier transform in 2 arguments it is \((2\pi)^{-d})\).

**Corollary 7.13** For \(n \geq 3\), Proposition 7.12 also applies to \(\hat{\tilde{W}}^T_{n,2}\). A technical calculation shows, that (79) is also tempered for \(\alpha = \frac{1}{2}\). Furthermore, the 2-point-function \(W^T_{2,\frac{1}{2}} = (2\pi)c_2 F^{-1} [\delta_{m_0}(k_1)\delta(k_1 + k_2)]\) is the well-known 2-point function of the relativistic free field. Therefore, also in the case \(\alpha = \frac{1}{2}\) Theorem 7.2 applies.

**Remark 7.14** For \(0 < \alpha < \frac{1}{2}\), \(c_1 = 0\), \(\hat{\tilde{W}}_{2,\alpha} = \hat{W}^T_{2,\alpha}\) admits a Källen–Lehmann representation. Therefore the corresponding Gaussian Euclidean field with covariance–function \(S_{2,\alpha}\) is reflection positive (but not Markov, the latter being seen from a general theorem of Pitt [59]).

For \(\alpha = \frac{1}{2}\) the corresponding Gaussian Euclidean field is the Markov free field of mass \(m_0\) ([54]). This can be also taken from the Equation (80) by the following considerations: For \(\alpha \uparrow \frac{1}{2}\), on one hand we have that the
coefficient \( \sin(2\pi \alpha) \downarrow 0 \). This implies that the Fourier transformed truncated 2-point functions \( \hat{W}_{2,\alpha} \) vanish on open sets in \( \mathbb{R}^{d_2} \) which do not intersect the mass-shell \( \{ k_1^2 = m_0^2, k_0^2 < 0, k_1 + k_2 = 0 \} \). On the other hand, the singularity of the \( W_{2,\alpha} \) on this mass-shell causes non-integrability for \( \alpha \uparrow \frac{1}{2} \).

Combining this two aspects in a quantitative calculation [38], one can show that

\[
\lim_{\alpha \uparrow \frac{1}{2}} \frac{2 \sin 2\pi \alpha_1 (k_1^2 > m_0^2, k_0^2 < 0)}{(k_1^2 - m_0^2)^{2\alpha}} = 2\pi \delta_{m_0}(k_1),
\]

where the limit is the weak limit in \( S'(\mathbb{R}^d) \).

Let us now have a look on (80) for \( 1 > \alpha > \frac{1}{2} \). First of all we note, that (80) is no more tempered, since the exponent \(-2\alpha\) is smaller than -1 and (80) is not locally integrable. Therefore, formula (80) in this case can not represent \( \hat{W}_{2,\alpha} \). Nevertheless, one may speculate that (80) still holds if \( k_1 \) stays away from the mass-shell \( \{ k_1^2 = m_0^2, k_0^2 < 0, k_1 + k_2 = 0 \} \). If this were true, an interesting observation can be made: Since the function \( \sin 2\pi \alpha \) at \( \alpha = \frac{1}{2} \) changes its sign from + to −, the corresponding truncated 2-point Schwinger function \( S_{2,\alpha} \) for \( \alpha \in (\frac{1}{2}, 1) \) would be "reflection negative", rather than "reflection positive".

Thus, it seems, as if the Markov–property in the case of Gaussian Euclidean random fields would appear at the "boundary" of reflection–positivity. Nevertheless, a proper treatment of this problem has to be left to future work.

### 7.6 Positivity in the scattering region

Let us concentrate on \( \alpha = \frac{1}{2} \) and \( d = 4 \), \( c_1 = 0 \). We thus consider the truncated Wightman functions of the convoluted generalized white noise \( X = (-\Delta + m_0^2)^{-\frac{1}{2}} F \). If \( F \) is Gaussian, \( X \) is the free Markov field of mass \( m_0 \) and the analytic continuation of its only nonzero Schwinger function \( S_{2} \) yields that \( S_{2} \) on \((E_d)^2\) is the Laplace transform of

\[
\hat{W}_{2}(k_1, k_2) = (2\pi)^{d+1} c_2 \delta_{m_0}^-(k_1) \delta(k_1 + k_2).
\]

This is the Fourier transform of the 2–point function of the relativistic free field of mass \( m_0 \) [54]. Let \( \{W_n\}_{n \in \mathbb{N}} \) be the sequence of Wightman functions composed from the truncated sequence \( \{W_n\}_{n \in \mathbb{N}} \), \( W_n = 0 \) \( n \neq 2 \), \( W_n = \mathcal{F}^{-1}(W_n) \). \( W_n \) is composed from the \( W_{2,\alpha} \) in the way of Corollary 3.5. For \( \varphi \in \mathcal{S}(\mathbb{R}^{dn}) \) define \( \varphi^* \) by \( \varphi^*(x_1 \ldots x_n) = \hat{f}(x_n \ldots x_1) \). It is well known, that in this case positivity holds for the \( \{W_n\}_{n \in \mathbb{N}_0} \), i.e. for \( \varphi^0 \in \mathcal{C} \), \( \varphi_l \in \mathcal{S}(\mathbb{R}^{dl}) \), \( l = 1 \ldots n \), we can define a seminorm for the vector \( \Psi = (\varphi^0, \varphi_1 \ldots \varphi_n, 0 \ldots) \)
in the Bochner algebra \( \bigoplus_{n=0}^{\infty} \mathcal{S}(\mathbb{R}^{dn}) =: \mathcal{S} \) as
\[
\|\Psi\|^2 := \sum_{l,m=0}^{n} W_{l+m}^T(\varphi_l^* \otimes \varphi_m) \geq 0
\] (81)

This allows us to look at equivalence classes of vectors with norm larger than zero as the "physical states" of the theory. Let now \( F \) be a generalized white noise. We define the general non–definite (cf. Section 5) squared pseudo–norm \( \| \cdot \|_2 \) on \( \mathcal{S} \) in analogy to (80), where the \( W_{l+m}^T \)'s are replaced by \( W_{l+m} \)'s obtained from the truncated Wightman distributions \( W^T_{m,\frac{1}{2}} = \mathcal{F}^{-1}(\hat{W}^T_{m,\frac{1}{2}}) \) defined in Corollary 7.13. Let us now fix \( \varepsilon > 0 \) and define a region \( U \) in the Minkowski space time \( M_d \) as \( U := \{ k \in M_d : k^2 \in [m_0^2 - \varepsilon, m_0^2 + \varepsilon], k^0 > 0 \} \). Let \( \mathcal{S}(U) \) denote the Schwartz functions on \( \mathbb{R}^{d} \) with support in \( U \). For \( \varphi \in \mathcal{S}(\mathbb{R}^{d}) \) such that \( \hat{\varphi} \in \mathcal{S}(U) \) define \( \varphi(x,t) := \mathcal{F}^{-1}(\hat{\varphi}(t))(x) \) where \( \hat{\varphi}(t) = \varphi(k)e^{i(k^0 - \omega)t} \). Let us concentrate on \( \Psi(t) \in \mathcal{S} \) such that \( \Psi(t) = (\varphi_0, \varphi_1(t), \ldots, \varphi_n(t)0 \ldots 0 \ldots) \) where \( \varphi_r(t) = \bigotimes_{s=1}^{r} \varphi_r^s(t) \), with \( \varphi_r^s(t) \) defined as above. It is well known [43] that the "wave packet" \( \varphi_r^s(t)(x) \) is concentrated near the plane \( x^0 = t \). In this sense, we say \( \Psi(t) \) approaches the asymptotic region \( x^0 \to \pm \infty \) if \( t \) goes to that limit.

Let us quote [61] Vol III p. 324 ff. and [43] for the following basic results of Haag–Ruelle–Theory, based only on locality, strong spectral condition and invariance of the \( W_n^T \) \( n \geq 3 \) and the special form of \( W_2^T \):
\[
\frac{d}{dt} W_2^T \left( \varphi_{r_1}^{s_1}(t) \otimes \varphi_{r_2}^{s_2}(t) \right) = 0 
\] (82)

and
\[
W_n^T \left( \varphi_{r_1}^{s_1}(t) \otimes \ldots \otimes \varphi_{r_n}^{s_n}(t) \right) \to 0 \quad \text{for} \quad n \geq 3 \quad \text{as} \quad t \to \pm \infty. 
\] (83)

The limit in (83) is approached as \((1 + |t|)^{\frac{3}{2}(n-2)} \) falls to zero in the general case, and faster than \((1 + |t|)^{-N} N \in \mathbb{N} \) falls to zero, if the \( \varphi_r^s(0) \)'s are non–overlapping in velocity–space, i.e. \( \omega_i^{-1}k_i^{+1} \neq \omega_j^{-1}k_j^{-1} \) for \( k_i, k_j \in \text{supp} \varphi_r^s \) \( i,j = 1 \ldots n \ i \neq j \).

Since we can expand \( \|\Psi(t)\|^2 \) into products of truncated Wightman functions of the type \( W_2^T \left( \varphi_{r_1}^{s_1}(t) \otimes \varphi_{r_2}^{s_2}(t) \right) \) and those of the type of the LHS of (83), we get

**Proposition 7.15** Let \( \Psi(t) \in \mathcal{S} \), \( \| \cdot \|^2 \) \( \| \cdot \|_g^2 \) as above. Then
\[
\|\Psi(t)\|^2 \to \|\Psi(0)\|^2_g \quad \text{as} \quad t \to \pm \infty
\] (84)
The limit here is approached as $(1 + |t|)^{-\frac{3}{2}} \to 0$ in general and faster as $(1 + |t|)^{-N} \to 0$ for any $N \in \mathbb{N}$ for a "non–overlapping" $\Psi(0)$.

**Remark 7.16**

1. Let us point out, that the existence of stable one–particle respectively of free states, i.e. the holding of (82), is special for $\alpha = \frac{1}{2}$. We do not have an analogue of these statements e.g. in the case $\alpha \in (0, \frac{1}{2})$.

2. We remark that the result of Proposition 7.15 in the non–overlapping case holds for all dimensions $d \geq 2$ [43]. Proposition 7.15 can also be generalized to the dimensions $d \geq 4$, but the classical literature only deals with the physical–space–time $d = 4$.

3. If there is at least one $\varphi_s$, such that all $\hat{\varphi}_{rs}^r = 1 \ldots s$ take nonzero values on the mass-shell of mass $m_0$, then we have $\|\Psi(0)\|^2_g > 0$ and thus also $\|\Psi(t)\|^2$ becomes positive for large $t$.

4. $\| \cdot \|_g$ may also be called the norm of a free or noninteracting state. In the very vague sense of (84) we may therefore say, that $\Psi(t)$ approaches a free state. Nevertheless, we cannot define asymptotic free states as in Haag–Ruelle theory, since at the moment we have no Hilbert topology in which the $\Psi(t)$‘s could converge.

5. For further investigations it seems therefore to be necessary to introduce a suitable auxiliary topology. A promising candidate is the Krein topology: In [3] we proved that the Wightman distributions of our model fulfill the Hilbert-structure condition of [52]. The model developed here thus fulfills the modified Wightman axioms for "fields in an indefinite metric" (see e.g. [28] for a definition of such fields). By the general results on spaces with indefinite inner product (see again[52] and references therein) we get that there exists a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ such that $\mathcal{S} \subset \mathcal{H}$ is dense. If $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{S}$ induced by the sequence of Wightman distributions $\{W_n\}_{n \in \mathbb{N}_0}$, then there is a continuous self–adjointed operator $\eta$ on $\mathcal{H}$ s.t. $\eta^2 = 1$ and $\langle \cdot, \eta \cdot \rangle = \langle \cdot, \cdot \rangle$ holds on $\mathcal{S}$.

Finally, we would like to summarize this section as follows: In Proposition 7.12 and Corollary 7.13 we give explicit formulae for the Fourier–transformed (truncated) Wightman distributions that belong to the random field $X = (-\Delta + m_0^2)^{-\alpha} F$, $\alpha \in (0, \frac{1}{2}]$. The sequence of Wightman distributions constructed from the former distributions fulfills all Wightman axioms.
(cf. [66], [67]) of relativistic QFT, except for the positivity of the square norm in the state–space, which in some cases does not hold and in others is uncertain (it is certain only for the case where $F$ is Gaussian). Nevertheless, in the case $\alpha = \frac{1}{2}$ there exists stable one resp. free states, and therefore Haag–Ruelle theory allows us to derive a positivity condition for states $\Psi(t)$ approaching the asymptotical respectively scattering regions as $t \to \pm \infty$.

Acknowledgements We thank D. Applebaum, C. Becker, P. Blanchard, R. Gielerak, Z. Haba, J. Schäfer and Yu. M. Zinoviev for stimulating discussions. The financial support of D.F.G. is gratefully acknowledged.

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