Set Covering with Ordered Replacement: 
Additive and Multiplicative Gaps

Friedrich Eisenbrand\textsuperscript{1}, Naonori Kakimura\textsuperscript{2,*},
Thomas Rothvoß\textsuperscript{1,**}, and Laura Sanità\textsuperscript{1,***}

\textsuperscript{1} EPFL, Lausanne, Switzerland
\textsuperscript{2} University of Tokyo, Japan

Abstract. We consider set covering problems where the underlying set
system satisfies a particular replacement property w.r.t. a given partial
order on the elements: Whenever a set is in the set system then a set
stemming from it via the replacement of an element by a smaller element
is also in the set system.

Many variants of Bin Packing that have appeared in the literature
are such set covering problems with ordered replacement. We provide
a rigorous account on the additive and multiplicative integrality gap
and approximability of set covering with replacement. In particular we
provide a polylogarithmic upper bound on the additive integrality gap
that also yields a polynomial time additive approximation algorithm if
the linear programming relaxation can be efficiently solved.

We furthermore present an extensive list of covering problems that
fall into our framework and consequently have polylogarithmic additive
gaps as well.

1 Introduction

Set Cover is a prominent combinatorial optimization problem that is very
well understood from the viewpoint of multiplicative approximation. There ex-
ists a polynomial time factor $O(\log n)$ approximation for Set Cover [2] and
a corresponding hardness result [9]. Also the (multiplicative) integrality gap
of the standard linear programming relaxation for Set Cover is known to be
$O(\log n)$ [14].

Let $S$ be a family of subsets of $[n] = \{1, \ldots, n\}$, $w : S \to \mathbb{R}_+$ be a cost
function and let $\chi(S) \in \{0, 1\}^n$ denote characteristic vector of a set $S \in \mathcal{S}$. The
Set Cover integer program

$$\min \left\{ \sum_{S \in \mathcal{S}} w(S)x_S \mid \sum_{S \in \mathcal{S}} x_S \cdot \chi(S) \geq 1, x \geq 0, x \text{ integral} \right\}$$

\textsuperscript{*} Supported in part by Grant-in-Aid for Scientific Research and by Global COE
Program “The research and training center for new development in mathematics”,
MEXT, Japan.

\textsuperscript{**} Supported by the Alexander von Humboldt Foundation within the Feodor Lynen
program.

\textsuperscript{***} Supported by Swiss National Science Foundation within the project “Robust Net-
work Design”.

O. Günlük and G.J. Woeginger (Eds.): IPCO 2011, LNCS 6655, pp. 170–182, 2011.
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and its linear programming relaxation is also in the focus of this paper. However, we are interested in the additive gap of a certain class of set covering problems. This additive gap is the difference between the optimum value of the integer program (1) and its linear programming relaxation. While there exists an extensive amount of literature on the (multiplicative) gap and (multiplicative) approximation algorithms, the additive gap and algorithms to construct integer solutions that are within the corresponding additive range have received less attention.

Why is it interesting to study the additive integrality gap of set covering problems? Suppose, for example that we know of a certain class of set covering problems that the additive gap is polylogarithmic, \( \log n \) say. If we then, at the same time, know that the optimum solution is at least \( \sqrt{n} \), then the linear programming relaxation of (1) asymptotically approaches the optimum solution of the integer program yielding a \((1 + \log n/\sqrt{n})\)-factor approximation algorithm if an integer solution respecting the gap can be efficiently computed.

Two prominent covering problems whose additive gap has been studied are Multi-Edge Coloring \([12,16]\) and Bin Packing. \([13]\) For Bin Packing, Karmarkar and Karp \([13]\) showed that the additive gap is bounded by \( O(\log^2 n) \) and they also provide a polynomial time algorithm that constructs a solution within this range. There is an extensive amount of literature on variants of Bin Packing (see e.g. \([7,6,5,8,7,3,19,1]\)). The question whether the Set Cover linear programming relaxations of such variants also exhibit small additive gaps is in the focus of our paper.

It is easy to see that the additive gap of general Set Cover is \( \Theta(n) \). For example, the Vertex Cover problem on a disjoint union of triangles exhibits this additive gap. What makes Bin Packing so special that polylogarithmic additive gaps can be shown to hold? It turns out that it is essentially the fact that in a feasible packing of a bin, we can replace any item by a smaller item and still remain feasible. In the setting of Set Cover this is reflected by the following. There is a partial order \( \preceq \) of the elements that we term replacement order. The order is respected by \( S \) if

\[
S \in S, i \in S, j \notin S, j \preceq i \Rightarrow ((S \setminus \{i\}) \cup \{j\}) \in S
\]

We will also consider costs \( w(S) \) of sets in the family \( S \). These costs are normalized in the sense that \( w(S) \in [0, 1] \) for each \( S \in S \). The costs respect the replacement order if \( w(S) \geq w(S') \) whenever \( S' \) is obtained from \( S \) by replacing one element \( i \in S \) with an element \( j \preceq i \) and if \( w(S') \leq w(S) \) for any \( S' \subseteq S \). Given a family \( S \) and costs respecting the replacement order \( \preceq \), the Set Cover With Ordered Replacement problem is to solve the integer program (1). We denote the optimum value of (1) and its relaxation by \( OPT(S) \) and \( OPT_f(S) \), respectively. The additive gap of \( S \) is thus \( OPT(S) - OPT_f(S) \).

**Contributions.** We provide a rigorous account on additive and multiplicative integrality gaps and approximability of Set Cover With Ordered Replacement if \( \preceq \) is a total order. Our main results are as follows.

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1 Even though coined bin “packing”, it is a covering problem.