CENTRAL CHARGES IN REGULAR MECHANICS

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Abstract

We consider the algebra associated to a group of transformations which are symmetries of a regular mechanical system (i.e. system free of constraints). For time dependent coordinate transformations we show that a central extension may appear at the classical level which is coordinate and momentum independent. A cochain formalism naturally arises in the argument and extends the usual configuration space cochain concepts to phase space.

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1 Introduction

The concepts of cochains and cocycles have demonstrated to be of relevance for the discussion of anomalous behavior of symmetries in QFT [1]. The applicability of these concepts has spread to classical and quantum mechanics providing a mathematical framework in which symmetry and symmetry breaking can be analyzed [2].

In a previous work [3], we have proposed a way to obtain a quantum mechanical geometrical phase for a classical system which has the characteristic that the action, but not the Lagrangian, is invariant (i.e. when the variation of the Lagrangian is a total time derivative) under contact coordinate transformation. We have encountered difficulties in applying this formalism to problems such as SUSY quantum mechanics [4] or scale invariance in two dimensional quantum mechanics [5]. The problem is that the cochain structure appearing in these problems are velocity dependent while the conventional approach to cochains is done in configuration space. The aim of this paper is two fold. First, we discuss the variation of the phase space Lagrangian (i.e. the Lagrangian written in terms of canonical variables) under finite transformations. This enable us to apply the cochain formalism in phase space. The second objective is to consider physical systems possessing a group of symmetry which is considered in order to analyze the possibility that the Poisson brackets of Noether’s charges acquire a central extension.

Noether’s theorem provides a systematic way of analyzing the conserved quantities associated to a physical system. The conventional approach consists in showing the invariance of the action under transformations forming a continuous group $G$ of dimension $\omega$. Noether’s theorem then assure the existence of $\omega$ conserved charges $Q_r$. In the Hamiltonian formalism, the Poisson brackets of the conserved (Noether’s) charges define an algebra which is isomorphic to the algebra of the global symmetry group from which the charges were obtained [6]. As a consequence, the charges $Q_r$ generate, through their Poisson brackets, the corresponding global symmetry transformations of phase space variables. There exist however the possibility that, the algebra of the charges $Q_r$ is an extension of that of the global symmetry group. At the quantum level the same statement applies to the commutators of the
charges, however due to ordering of composite operators, new terms that vanish in the classical limit ($\hbar \to 0$) may appear in the associated commutation relations, indicating the existence of an anomaly, i.e. the breaking of the classical symmetry by quantum effects [7]. A similar phenomenon may occur at the classical level, indeed as a consequence from the passage from configuration to phase space, new terms -as compared to the algebra of the original group of transformation $G$- may appear in the Poisson brackets of the $Q_r$ charges. According to our results, a necessary condition for this to happen is that the action but not the Lagrangian be invariant under the symmetry transformation. The Galilei group and the magnetic translation group provide examples where such conditions are met and a classical central extensions appear.

We tried to make the text as self contained as possible, to this end we have included in section 2 a short summary of Noether’s theorem and the corresponding expression in phase space. Section 3 and 4 are devoted to the analysis of a possible central extension of the algebra of Noether’s charges. In particular, in section 3 we consider coordinate transformations for which the variation of the Lagrangian in phase space leads to a cochain structure that implies the existence of conserved quantities. We show that such a constant of motion is related to the central extension. Section 4 is devoted to generalize the previous results, in this case however, the cochain structure does not enter the derivation of the central extension.

In order to see the ideas underlying our approach in a concrete setting, we consider the following three physical systems: $i$) Motion of a particle in two dimensions under the influence of a scale invariant potential. In this case, both the Lagrangian and the action are invariant under the transformations and there is no central extension of the algebra of Noether’s charges. $ii$) Group of magnetic translations. This system concerns the movement in two dimensions of a charged particle in a homogeneous magnetic field. The symmetry transformations to consider are translations. The central point is the incorporation of the vector potential, which lead both to the non-invariance of the Lagrangian and the modification of the translation generator. The central charge is a consequence of the non-vanishing Poisson bracket of the momentum and the vector potential. $iii$) Galilei invariance. This is a well known example [8] where a central extension of the algebra appears. We
work out details of the calculation and show the relation between the central extension and the non invariance of the Lagrangian (due to the surface term) under Galilei transformations.

2 Noether Theorem

We consider a system with \( n \) degrees of freedom, given as functions \( q_j(t) \) \((j = 1, 2, \ldots n)\) of the “time” variable \( t \). We assume that the dynamics of the system is described by the action functional:

\[
S[q_j] = \int_{t_i}^{t_f} dt L(q_j, \dot{q}_j).
\]  

(1)

Here, \( L(q_j, \dot{q}_j) \) is the Lagrange function depending on the generalized coordinates \( q_j \) and their corresponding velocities \( \dot{q}_j \), but not on time. The time evolution of the system is described by a set of \( n \) second order differential equations (the Euler-Lagrange equations of motion), which are linearly independent if the Hessian of the Lagrangian has a non-vanishing determinant,

\[
det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0.
\]

When this condition is met, the system is said to be regular. In the opposite case, when the Hessian has zero modes, the system is said to be singular, and is characterized by the existence of constraints. In this paper we restrict our selves to the study of regular systems.

Let us consider transformations of the form:

\[
q_j(t) \rightarrow q'_j(t') = f_j(q(t), t, \alpha) \\
t \rightarrow t' = f_0(t, \alpha).
\]  

(2)

In Eq. (2) \( q_i(t) \) are the coordinates in a time slice \( t \) in configuration space and \( q'_i(t') \) is the image point of \( q_i(t) \) at the time slice \( t' \), \( \alpha \) stands for the set
of \( w \) parameters specifying the transformations and \( q(t) \) is used to denote collectively the \( n \) coordinates. The parametrization is chosen in such a way that:

\[
\begin{align*}
    f_j(q(t), t, 0) &= q_j(t), \\
    f_0(t, 0) &= t
\end{align*}
\]  

(3a)

and

\[
\begin{align*}
    f_j(f(q(t), t, \alpha), f_0(t, \alpha), -\alpha) &= q_j(t) \\
    f_0(f_0(t, \alpha), -\alpha) &= t,
\end{align*}
\]  

(3b)

that is, for \( \alpha = 0 \) Eq. (2) reduces to the identity and the inverse transformation is obtained by reversing the sign of the \( \alpha \) parameters.

We assume that these transformations define a continuous group \( G \) of dimension \( w \). Before establishing the relation of Eq. (2) with Noether’s charges, we introduce the structure constants associated to \( G \). For infinitesimal transformations with parameters \( \delta \alpha_r \) we write \( \delta \alpha_r = \frac{\alpha_r}{N} \) with \( N \) arbitrary large, and unless otherwise stated, here and thereafter sum over repeated indices is assumed:

\[
\begin{align*}
    q_j'(t') &= q_j(t) + \delta q_j = q_j(t) + (\delta \alpha_r T_r) q_j, \\
    t' &= t + \delta t = t + (\delta \alpha_r S_r) t,
\end{align*}
\]  

(4)

where \( T_r, S_r, r = 1, 2, \ldots, \omega \) are the group “generators” (appropriated algebraic or differential operators). A comment about Eq. (4) is necessary. Notice that Eq. (2) implies that the coordinate and the time transformations depend upon the same set of parameters \( (\alpha_1, \alpha_2, \ldots, \alpha_\omega) \). Furthermore, the time transformations depends only on time and the \( \alpha_r \) parameters, therefore such a set of transformations must form a group by itself. That is the reason to include \( \omega \) generators \( S_r \) in (4). It may happen however that the time transformations involve only some of the \( \delta \alpha_r \). The following examples may be useful in clarifying these points.
• Consider scale transformation, defined by

\[ q_i \rightarrow q'_i = \frac{1}{\sqrt{1 + \alpha_1}} q_i \]
\[ t \rightarrow t' = \frac{1}{(1 + \alpha_1)} t. \]  
(5)

If we perform a second transformation on the coordinates we obtain:
\[ q'' = \frac{1}{\sqrt{1 + \alpha_2}} \frac{1}{\sqrt{1 + \alpha_1}} q = \frac{1}{\sqrt{1 + \alpha_1 + \alpha_2 + \alpha_1 \alpha_2}} q. \]
Which is of the type (5) and consequently the coordinate transformations form a group if the composition law \( C(\alpha_1, \alpha_2) \equiv \alpha_1 + \alpha_2 + \alpha_1 \alpha_2 \) is assumed. The point to emphasize is that the same reasoning holds for the time transformation.

For infinitesimal \( \alpha_1 \) we obtain

\[ \delta_1 q_i = -\frac{\alpha_1}{2} q_i, \quad (i = 1, 2) \]
\[ \delta_1 t = -\alpha_1 t \]

therefore, the corresponding coordinate \((T_i)\) and time \((S_i)\) generators are given by

\[ T_1 = -\frac{q_j}{2} \frac{\partial}{\partial q_j}, \quad S_1 = -t \frac{\partial}{\partial t} \]  
(6)

• As a second example we consider Galilei transformations:

\[ q'_1 = q_1 + vt + a, \quad q'_2 = q_2, \]
\[ t' = t + b. \]

To simplify our discussion we take \( a = b = 0 \). In this case, there is no variation of the time \( \delta_2 t = 0 \), whereas \( \delta_2 q_1 = vt \). The corresponding generators are
\[ T_2 = \frac{\partial}{\partial q_1}, \quad S_2 = 0 \]

A further step will be to consider both scale and Galilei transformation. It is easy to check that the \( T_i \) generators \( (i = 1, 2) \) close and that the structure constants obtained are the same as those entering in the commutation relations for the \( S_i (i = 1, 2) \) generators.

Coming back to our general discussion, finite transformations can be obtained in terms of the \( T_r, S_r \) generators by exponentiating (4)

\[
q_i'(t') = e^{\alpha_r T_r} q_i(t) \equiv g_c(\alpha) q_i(t),
\]

\[ t' = e^{\alpha_r S_r} t \equiv g_r(\alpha) t. \tag{7} \]

The group property of the transformations (2), expressed either for \( g_c \) or \( g_r \) as

\[ g(\alpha_i)g(\alpha_j) = g(c(\alpha_i, \alpha_j)), \tag{8} \]

can be used to bring out the algebra of the generators. This is achieved by considering the commutator of two infinitesimal transformations:

\[ g(\alpha_r)g(\alpha_s) - g(\alpha_s)g(\alpha_r) = g(c(\alpha_r, \alpha_s)) - g(c(\alpha_s, \alpha_r)), \]

the Taylor expansion of these expressions leads to:

\[ [T_r, T_s] = C_{rs}^u T_u, \quad [S_r, S_s] = C_{rs}^u S_u \tag{9} \]

where the structure constants \( C_{rs}^u \) are defined as:

\[ C_{rs}^u = \frac{\partial c^u(\alpha_r, \alpha_s)}{\partial \alpha_r \partial \alpha_s} \bigg|_{\alpha_r = \alpha_s = 0} - \frac{\partial c^u(\alpha_s, \alpha_r)}{\partial \alpha_r \partial \alpha_s} \bigg|_{\alpha_r = \alpha_s = 0} \tag{10} \]
For latter use, it is convenient to express this property in terms of the time and coordinate variations

\[ \delta q_j = q_j'(t') - q_j(t) = \delta \alpha_r \frac{\partial f_j(q, t, \alpha)}{\partial \alpha_r} \bigg|_{\alpha=0} \equiv \delta \alpha_r \delta^r q_j \]

\[ \delta t = \delta \alpha_r \frac{\partial f_0(t, \alpha)}{\partial \alpha_r} \bigg|_{\alpha=0} \equiv \delta \alpha_r \delta^r t. \]

Given the group property of the coordinate transformation

\[ q^{r,s}_j = f_j(f_j(q, t, \alpha_r), f_0(t, \alpha_r), \alpha_s) = f_j(q, t, \alpha_{rs}), \]

we calculate

\[ q^{r,s}_i - q^{s,r}_i = f_j(q, t, \alpha_{rs}) - f_j(q, t, \alpha_{sr}), \]

which for infinitesimal transformations results in

\[ q^{r,s}_i - q^{s,r}_i = \left( \delta^r q_j \frac{\partial}{\partial q_j} \delta^s q_i - \delta^s q_j \frac{\partial}{\partial q_j} \delta^r q_i + \delta^r t \frac{\partial}{\partial t} \delta^s q_i - \delta^s t \frac{\partial}{\partial t} \delta^r q_i \right) = C_{rs}^u \delta^u q_i, \tag{10a} \]

From Eqs. (10, 10a) we see that the structure constants \( C_{rs}^u \) are defined through the composition law of the group elements. On the other hand, we already pointed out beneath Eq. (4) that the coordinate and time transformation must have the same composition rule, therefore the commutator among the \( S_r \) generators is \( [S_r, S_s] = C_{rs}^u S_u \).

Let us consider now the relation between the transformation (2) and the physical system. At the classical level the system described by (1) is said to possess a symmetry or to be invariant if, up to surface terms, the action is form invariant under the transformations (2). In terms of the Lagrangian, this property is equivalent to the requirement

\[ \left( \frac{dt'}{dt} \right) \mathcal{L}(q', \left( \frac{dt}{dt'} \right) \frac{dq'}{dt}) = \mathcal{L}(q, \dot{q}) + \frac{d\Lambda(q)}{dt}. \tag{11} \]
When $\frac{d\alpha}{dt} = 0$ the Lagrangian is said to be invariant under (4). For infinitesimal variations and to first order, Eq. (11) reduces to the identity [6]:

$$\frac{d}{dt}(\tilde{Q}_r \delta \alpha_r) = \sum_j (\delta q_j - \delta t \dot{q}_j)(\frac{\delta L}{\delta q} - \frac{d}{dt} \frac{\delta L}{\delta \dot{q}}),$$

where

$$\tilde{Q}_r(q, \dot{q}, t) \delta \alpha_r = \frac{\partial L}{\partial \dot{q}_j} \delta q_j - \left(\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L\right) \delta t - \Lambda. \quad (12)$$

This is Noether’s theorem, which implies that for any classical solution to the equation of motion there are $w$ constants of motion, or conservation laws.

In the Hamiltonian formalism, the charges $\tilde{Q}_r$ generate, through their Poisson brackets, the global symmetry transformations on phase space. In order to analyze this property and possible generalizations, we are naturally lead to the study of the conserved charges in phase space.

The Hamiltonian $H$ is given by

$$H(q, p) = p_i \dot{q}_i - L(q, \dot{q}), \quad (13)$$

with

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}. \quad (14)$$

Since we are studying regular systems, then the conserved charges can be expressed in terms of the canonical variables

$$\tilde{Q}_r(q, \dot{q}, t) = Q_r(q, p, t) = \tilde{Q}_r(q, \frac{\partial H}{\partial p}, t), \quad (15)$$

thus the charges take the phase space form:

$$Q_r(q, p, t) \delta \alpha_r = p_i \delta q_i - H(q, p) \delta t - \Lambda. \quad (16)$$
Charge conservation in phase space is expressed as:

\[ 0 = \frac{dQ_r(q, p, t)}{dt} = \frac{\partial Q_r}{\partial q} \dot{q} + \frac{\partial Q_r}{\partial p} \dot{p} + \frac{\partial Q_r}{\partial t}. \]  

(17)

Assuming that through any point of phase space can pass a solution (there are no constraints), it follows that charge conservation is expressed in phase space in terms of the Poisson brackets:

\[ \{Q_r, H\} + \frac{\partial Q_r}{\partial t} = 0. \]  

(18)

We will refer to the linear operator acting on \( Q_r \) in Eq. (18) as the time Lie derivative. Associated to each infinitesimal transformation in configuration space \( g(\delta \alpha_r) \), we have an infinitesimal canonical transformation

\[ q^i_g(q, p) = q_i - \delta \alpha^r \{Q_r, q_i\} = q_i + \delta_c q_i, \]  

(19)

\[ p^i_g(q, p) = p_i - \delta \alpha^r \{Q_r, p_i\} = p_i + \delta_c p_i. \]  

(20)

where the subindex \( c \) indicates that these are increments due to canonical transformations generated by the \( Q \) which are related to \( \delta q \) of Eq. (16) by \( \delta_c q = \delta q - \frac{\partial H}{\partial p} \delta t \). If we restrict our attention to classical configurations corresponding to solution to the equations of motion we have:

\[ \{q^i_g(q, p), H(q^q, p^q)\} = \delta \alpha^r \left[ \left\{ -\{Q_r, q_i\}, H(q, p) \right\} + \left\{ q_i, -\{Q_r, H(q, p)\} \right\} \right] + \frac{dq_i}{dt} \]  

= \frac{dq^i_g(q, p)}{dt}. \]  

(21)

In a similar way it follows that the transformed momentum \( p^q(q, p) \) satisfies the original canonical equation. Then, the mappings (19) and (20) are symmetries of the Hamiltonian system and the charges \( Q_r \), obtained from the Lagrangian conserved charges (12), generate symmetry transformations in phase space.

Finite canonical transformation are built in terms of the \( Q_r \) charges by using the exponentiation of the generators in (19) and (20).

10
\[ U(g(\alpha)) = e^{-\{\alpha^r Q_r, \}}, \quad (22) \]

where the symbol \( \{\alpha^r Q_r, \} \) in the exponential, means a Poisson bracket understood as a linear operator acting on functions of the phase space points. It proves convenient to parametrize the transformation (22) in terms of a real arbitrary parameter \( \sigma \) and a unit vector \( (s_1, s_2, \ldots, s_w) \)

\[ U(g(\alpha)) = U(s, \sigma) = e^{\{\sigma Q_r(q,p,t)s_r, \}}. \]

Thus, finite transformation of the coordinates and momenta are given by

\[
\begin{align*}
q_i^g(q, p) &= U(s, \sigma)q_i, \\
p_i^g(q, p) &= U(s, \sigma)p_i. \\
\end{align*}
\quad (23)
\]

3 Phase Space Cochains and Central Charges

So far we have summarized infinitesimal symmetry transformations both in configuration and phase space, including Noether’s theorem and the associated conserved charges, which serve as generators of the transformations in the Hamiltonian formalism. In this section we prove that the variation of the Lagrangian under finite transformations is given by a time Lie derivative. This results allow us to introduce the cochain formalism in phase space and conclude that, under very specific conditions, a coordinate and momentum independent central extension of the algebra arises.

We begin with a brief reminder of the cochain and coboundary concepts [1]. Consider a transformation \( g \) which belongs to a group of transformations. Suppose \( g \) acts on abstract space variables according to a definite rule,

\[ x \xrightarrow{g} x^g \]
and the group composition law is
\[ g_1 g_2 = g_{12}. \]

The application of two successive transformations yields\(^3\)
\[ x \xrightarrow{s} x^{g_1} \xrightarrow{g_2} (x^{g_1})^{g_2} = x^{g_{12}}. \]

Quantities that depend on \(x\) and \(n\) group elements are called \(n\)-cochains \(\omega_n(x, g_1, g_2, \ldots g_n)\). The coboundary operation \(\Delta\) is defined as:
\[
\Delta \omega_n \equiv \omega_n(x^{g_1}; g_2, \ldots, g_{n+1}) - \omega_n(x; g_{12}, g_3, \ldots, g_{n+1}) + (-)^m \\
\omega_n(x; g_1, \ldots, g_{mm+1}, \ldots, g_{n+1}) + (-)^{n+1} \omega_n(x; g_1, \ldots, g_2). \tag{24}
\]

The coboundary has the important property that \(\Delta^2 = 0\). Further details about cochains, cocycles and coboundaries can be found in [1] and references there in.

The Lagrangian can be expressed in terms of canonical variables as follows
\[ \mathcal{L}(q, p) = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}(q, p). \tag{25} \]

A finite transformation of the Lagrangian is obtained in the following way:
\[
\mathcal{L}(q^g, p^g) = \left( \mathcal{U} \left( s, \frac{\sigma}{N} \right) \right)^N \mathcal{L}(q, p) \\
= \left[ e^{-\left( \frac{\sigma}{N} + \Sigma r \right)} \right]^N \mathcal{L}(q, p). \tag{26}
\]

The finite transformation has been expressed as the product of a large number \(N\) of identical infinitesimal mappings.

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\(^3\)Given a symmetry group of a classical system, this property holds in configuration space. However, once we go over phase space, this need not be the case.
Using the Jacobi identity and the conservation law (18), the Poisson bracket is rewritten as:

\[ e^{-\frac{\sigma s_r}{N}\{Q_r, \}} \mathcal{L}(q, p) \cong \mathcal{L}(q, p) - \frac{\sigma s_r}{N}\{Q_r, \mathcal{L}(q, p)\} + \ldots \]  

(27)

This can be expressed in terms of Noether’s conserved charge (see Eqs. 16 and 18):

\[ -\frac{\sigma s_r}{N}\{Q_r, \mathcal{L}(q, p)\} = \frac{\sigma s_r}{N} \left[ -\{q_i, Q_r\}\{\mathcal{H}, p_i\} - \{\mathcal{H}, \{q_i, Q_r\}\}p_i + \{q_i, \frac{\partial q_r}{\partial t}\}p_i \right] \]

\[ = \left[ -\{\mathcal{H}, \} + \frac{\partial}{\partial t} \right] p_i \{q_i, Q_r\} \frac{\sigma s_r}{N} \]

(28)

\[ = \left[ -\{\mathcal{H}, \} + \frac{\partial}{\partial t} \right] p_i \delta \epsilon q_i \]

The basic assumptions we will make in this section are the following (in the next section we will prove the validity of the two last assumptions):

- Only regular systems are considered.
- In phase space, the transformation satisfies the condition \((q^{g_1})^{g_2} = q^{g_{12}}\) and \((p^{g_1})^{g_2} = p^{g_{12}}\).
- The central extension \(L_{rs}\) is momentum independent. (See eq. (33), below).

In order to show the appearance of a central extension, we begin calculating the variation of the Lagrangian under a finite transformation. The finite transformation are built starting from (29), and (26). In terms of the intermediary variables

\[ \]
\[ q_i^m(q, p) = \exp\left[ -\frac{m\sigma s_r}{N} \{Q_r, \ } \right] q_i, \]
\[ p_i^m(q, p) = \exp\left[ -\frac{m\sigma s_r}{N} \{Q_r, \ } \right] p_i, \]

the variation of the Lagrangian (25) is given by:
\[
L(q^{g(\sigma, s)}, p^{g(\sigma, s)}) - L(q, p) = \left\{ -\{H, \} + \frac{\partial}{\partial t} \right\} \sum_{m=1}^{N-1} \Lambda^T(q^m, p^m, g(\frac{\sigma}{N}, s)) + O\left(\frac{1}{N}\right),
\]

where \( \Lambda^T \) has been defined by

\[
\Lambda^T(q, p, g(\frac{\sigma}{N}, s)) = (\Lambda^r(q) - L(q, p)\delta_r t)s_r \sigma/N.
\]

We are interested in the \( N \to \infty \) limit, for which the sum will approach an integral. This is neatly seen writing \( \Lambda^T = \Lambda^T r \delta \alpha_r \), where \( \delta \alpha_r \) stands for the infinitesimal parameter associated to the transformations (19) and (20).

Using the \( \sigma, s \) parametrization (see discussion beneath Eq. (22)) we can write \( \Lambda^T = \Lambda^T r s, \frac{s}{N} \to \Lambda^T r s d\beta \) and therefore

\[
L(q^{g(\sigma, s)}, p^{g(\sigma, s)}) - L(q, p) = \left\{ -\{H, \} + \frac{\partial}{\partial t} \right\} \int_0^\sigma d\beta \Lambda^T_r(q^{g(\beta, s)}, p^{g(\beta, s)}) s_r = -\{H, \Lambda^T_f \} + \frac{\partial \Lambda^T_f}{\partial t},
\]

where

\[
\Lambda^T_f = \Lambda^T_f(q, p, g(\sigma, s)) = \int_0^\sigma d\beta \Lambda^T_r(q^{g(\beta, s)}, p^{g(\beta, s)}) s_r.
\]

Thus, the variation of the “phase-space” Lagrangian turns out to be given by a time Lie derivative of the “surface” term \( \Lambda^T_f \).
The central point of this section relies on the observation that the variation of the Lagrangian under finite transformations defines a coboundary in phase space (Eq. 24 for \( n = 0 \)).

\[
\Delta \mathcal{L}(q, p) = \mathcal{L}(q^g, p^g) - \mathcal{L}(q, p).
\]

Applying the coboundary operation to (30), and using the property \( \Delta^2 = 0 \), we obtain:

\[
\Delta \Delta \mathcal{L}(q, p) = 0 = \Delta \left[ -\{H, \Lambda_f^T \} \right] + \frac{\partial \Lambda_f^T}{\partial t}.
\]

The last equality can be verified by considering the explicit definition of the coboundary operation. Eq. (31) tell us that \( \omega_2(q, p, g_1, q_2) \equiv \Delta \Lambda_f^T(q, p, g_1) \) is conserved in time.

The coboundary \( \omega_2(q, p, g_1, g_2) \) depends on two group elements, \( q \) and \( p \). For infinitesimal \( g_1 \) and \( g_2 \) we will parametrize the difference of two such coboundaries as

\[
D \omega_2(q, p, g_1, g_2) = \omega_2(q, p, g_1, g_2) - \omega_2(q, p, g_2, g_1) = L_{rs} \alpha_1 \alpha_2,
\]

moreover, using Eq. (24) with \( n = 1 \)

\[
\begin{align*}
\omega_2(q, p, g_1, g_2) - \omega_2(q, p, g_2, g_1) &= (\Lambda_f^T(q^{g_1}, p^{g_1}, g_2) - \Lambda_f^T(q, q, g_2)) \\
&+ (\Lambda_f^T(q, p, g_1) - \Lambda_f^T(q^{g_2}, p^{g_2}, g_1)) \\
&- \Lambda_f^T(q, p, g_{12}) + \Lambda_f^T(q, p, g_{21}).
\end{align*}
\]

In terms of Noether’s conserved charges
\[-\Lambda_f^T(q, p, g_{12}) + \Lambda_f^T(q, p, g_{21}) = Q_r\alpha_{12r} - Q_r\alpha_{21r} - p\delta^g_{12}q + p\delta^g_{21}q,\]

\[\Lambda_f^T(q^{g_1}, p^{g_1}, g_2) - \Lambda_f^T(q, p, g_2) = -\{\alpha_1, Q_r, \Lambda_f^T(q, p, g_2)\} = -\{\alpha_1, Q_r, \alpha_{2s}\Lambda_s^T\}.

Thus, we obtain:

\[\omega_2(q, p, g_1, g_2) - \omega_2(q, p, g_2, g_1) = -\alpha_1, \alpha_2s\{Q_r, \Lambda_s^T\} + \alpha_2, \alpha_1s\{Q_r, \Lambda_s^T\} + Q_r(\alpha_{12r} - \alpha_{21r}) + p(\delta^{g_21}q - \delta^{g_{12}}q).

The Poisson brackets in the expression are evaluated by expressing \(\Lambda_s^T\) in terms of the charges.

\[\omega_2(q, p, g_1, g_2) - \omega_2(q, p, g_2, g_1) = \alpha_1, \alpha_2s\{Q_r, Q_s\} + Q_t(\alpha_{12t} - \alpha_{21t}).\]

Using Eqs. (8,9) it is not difficult to show that \((\alpha_{12} - \alpha_{21})_t = \alpha_1, \alpha_2sC^t_{rs}.\) Comparing with (32), we finally conclude

\[\{Q_r, Q_s\} - C^t_{rs}Q_t = L_{rs} \quad (33)\]

The conservation of \(w_2\) leads to

\[\frac{\partial}{\partial t}D\omega_2 + \{D\omega_2, H\} = \frac{\partial}{\partial q_i}D\omega_2 \frac{\partial H}{\partial p_i} + \frac{\partial}{\partial t}D\omega_2\]

Using the assumption that \(D\omega_2\) is momentum independent, taking the derivative of this expression respect to \(p_i\) we conclude that

\[\left(\frac{\partial}{\partial q_i}D\omega_2(q, g_1, g_2)\right)\frac{\partial^2 H}{\partial p_j \partial p_i} = 0.\]

Since we are considering regular systems, the Hessian \(\frac{\partial^2 H}{\partial p_j \partial p_i}\) has no zero modes, which requires

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\[
\frac{\partial}{\partial q_i} L_{rs} = 0.
\]

Therefore the \( L_{rs} \) are coordinate and momentum independent. (From the conservation of \( \omega_2 \) it also follows \( \frac{dL}{dt} = 0 \)).

The movement in two dimensions of a particle in a homogeneous magnetic field provides an example where the approach so far developed can be applied. The system under consideration is described by the Lagrangian:

\[
\mathcal{L}(q, \dot{q}) = \frac{M}{2} \sum_i \dot{q}_i^2 + \frac{e}{c} A_i(q) \dot{q}_i, \quad i = 1, 2
\]

where

\[
A_i(q) = \frac{B}{2} \epsilon_{ij} q_j; \quad \partial_i A_i = 0, \quad \text{and} \quad \epsilon_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The symmetry involved in this problem, is the translation group, defined by the transformations:

\[
q'(t) = f_i(q, t, \alpha) = q_i(t) + \alpha_i. \quad (34)
\]

The variation of the Lagrangian under (34) is

\[
\delta\mathcal{L} = \frac{d}{dt} \left( -\frac{e}{c} A_i(q) \alpha_i \right).
\]

The \( \sigma, s \) parametrization is achieved by introducing

\[
s_i = \frac{\alpha_i}{\sqrt{\alpha_i \alpha_i}}, \quad \sigma = \sqrt{\alpha_i \alpha_i}, \quad i = 1, 2,
\]

and

\[
q^{(\beta, s)} = q_i + \beta s_i
\]

The finite cochain is given by:
Given $\Lambda_f(q, g_1)$, it is straightforward to calculate

$$\omega_2(g_1, g_2) - \omega_2(g_2, g_1) = B\epsilon_{ij}\alpha_1\alpha_2.$$ 

Comparing with (32), we get the central charge $L_{ij} = B\epsilon_{ij}$.

This result is easily verified. In configuration space, translations in orthogonal directions commute. On the other hand, in phase space, the Poisson brackets of Noether’s charges results in

$${\{Q_i, Q_j\}} = B\epsilon_{ij}.$$ 

In this section we present an alternative derivation of the central extension of the algebra, which is not based on the cochain structure, and furthermore has the advantage of showing that the central extension $L_{rs}$ depends only on the coordinates.

Consider the Poisson bracket of Noether’s charges (16):

$$\{Q_r, Q_s\} = \{p_i\delta^r q_i, p_j\delta^s q_j\} - \{Q_r, H\delta^s t\} - \{H\delta^r t, Q_s\} - \{\Lambda_r, p_j\delta^s q_j\} - \{p_j\delta^r q_j, \Lambda_s\},$$

this expression is obtained taking into account that the time variation is $q$-independent and therefore $\{Q_r, \delta t\} = 0$. Using charge conservation (18) we obtain
\{Q_r, Q_s\} = \{p_i \delta^r q_i, p_j \delta^s q_j\} + p_i \left(\delta^s t \frac{\partial}{\partial t} \delta^r q_i - \delta^r t \frac{\partial}{\partial t} \delta^s q_i\right) \\
- H \left(\delta^s t \frac{\partial}{\partial t} \delta^r t - \delta^r t \frac{\partial}{\partial t} \delta^s t\right) - \{\Lambda_r, p_j \delta^s q_j\} - \{p_j \delta^r q_j, \Lambda_s\}.

This result can be written in terms of the structure constants introduced in (10a).

\{Q_r, Q_s\} = C^u_{rs} Q_u + L_{rs}, \quad \text{(35)}

where

\[ L_{rs} = \{\Lambda_s, p_i \delta^r q_i\} - \{\Lambda_r, p_i \delta^s q_i\} - C^u_{rs} \Lambda_u \]

Notice that $L_{rs}$ will not depend on the momenta and that, as it should be, it is antisymmetric in the $r-s$ indices. Explicit evaluation of the Poisson bracket taking into account that $\delta^r q$ are $p$ independent leads to:

\[ L_{rs} = \left(\frac{\partial \Lambda_s}{\partial q_j}\right) \delta^r q_j - \left(\frac{\partial \Lambda_r}{\partial q_j}\right) \delta^s q_j - C^u_{rs} \Lambda_u. \quad \text{(36)} \]

In fact, if Noether charge is conserved, then (35) implies that $L_{rs}$ is also conserved. Indeed, the time Lie derivative of $L_{rs}$ is given by:

\[-\{H, L_{rs}\} + \frac{\partial L_{rs}}{\partial t} = -\{H, \{Q_r, Q_s\}\} + \frac{\partial}{\partial t} \{Q_r, Q_s\} + C^u_{rs} \left[\{H, Q_u\} - \frac{\partial Q_u}{\partial t}\right].\]

The use of Jacobi’s identity and charge conservation, simplifies this expression to

\[-\{H, L_{rs}\} + \frac{\partial L_{rs}}{\partial t} = \{Q_s, \{H, Q_r\}\} + \{Q_r, \{Q_s, H\}\} + \frac{\partial}{\partial t} \{Q_r, Q_s\} = 0. \]
In order to prove the central extension character of $L_{rs}$ it will be sufficient to show that $L_{rs}$ is $q$ and $p$ independent. (From Eq. (36) it is already clear that $L_{rs}$ is $p$ independent). To this end consider the time Lie derivative of $L_{rs}$:

$$0 = \{L_{rs}, H\} + \frac{\partial L_{rs}}{\partial t} = H \frac{\partial L_{rs}}{\partial p_j} + \frac{\partial L_{rs}}{\partial q_j}.$$

Taking the derivative of this expression respect $p_i$ we obtain:

$$\frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial L_{rs}}{\partial q_j} = 0,$$

since we restrain our analysis to regular systems, the Hessian can not have zero modes, which implies

$$\frac{\partial L_{rs}}{\partial q_j} = 0.$$

Thus, we have shown that the Poisson bracket of Noether’s charges can acquire only coordinate and momentum independent central extensions. This result justify the second assumption of the previous section. In fact, the third assumption can also be validated. To this end consider the difference of two successive transformations applied in reserved order:

$$\{Q_r, \{Q_s, \} \} = \{\{Q_r, Q_s\}, \} = \{C_{rs}^t Q_t + L_{rs}^t, \} = \{C_{rs}^t Q_t, \}$$

The last equality follows from the $q$ and $p$ independence of the $L_{rs}$ central charges. Thus the central charges have no effect whatsoever on the analogous of the Baker-Campbell-Hausdorf formula, therefore $(q^{g_1})^{g_2} = q^{g_{12}}$ and $(p^{g_1})^{g_2} = p^{g_{12}}$.

As an application of this approach, let us consider a free particle and the Galilei symmetry group. It is well known that the mass of the particle is involved in the algebra of the group and it is considered as a central extension [8]. The system under consideration is described by:

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\begin{align*}
\mathcal{L} &= \frac{M}{2} \sum_{i=1}^{3} \dot{q}_i^2, \\
\mathcal{H} &= \sum_{i=1}^{3} \frac{p_i^2}{2M}.
\end{align*}

The Galilei transformations, lead to the infinitesimal variations

\[ \delta q_j = (\delta v_j) t + \delta a_j, \quad \delta \dot{q}_j = \delta v_j. \]

The \( \delta q_j \) must be considered as the combination of two independent variations. A pure boost characterized by the parameters \( (\delta v_j) \) and pure translations \( (\delta a_j) \)

\[ \delta^r q_j(\text{boost}) \equiv \frac{\delta q_j}{\delta v_r} = t \delta_{jr}, \]

\[ \delta^r q_j(\text{trans}) \equiv \frac{\delta q_j}{\delta a_r} = \delta_{jr}. \]

For infinitesimal transformations, the variation of the Lagrangian is:

\[ \delta \mathcal{L} = \frac{d}{dt} (M q_i \delta v_i). \]

Thus, in this case, \( \Lambda = M q_i \delta v_i = \Lambda^\text{boost} \delta v_r + \Lambda^\text{trans} \delta a_r \). Clearly \( \Lambda^\text{boost} = M q_r \) and \( \Lambda^\text{trans} = 0 \). Noether’s theorem leads to the independent conserved charges:

\[ Q_r = p_r t - M q_r, \quad P_r = p_r \quad r = 1, 2, 3. \]

The Poisson brackets of these charges are:

\[ \{ Q_r, Q_s \} = 0, \quad \{ P_r, P_s \} = 0, \quad \{ P_r, Q_s \} = M \delta_{rs}. \]

On the other hand, according to our discussion, the central extension -if it exist- should be given by (36). It is straightforward to show using (16)
that for this example $C^u_{rs} = 0$. Furthermore, if the indices $r$ and $s$ refer both to boost, or both to translations $L_{rs} = 0$. So, the only possibility left is:

$$L_{rs} = \left( \frac{\partial \Lambda_{\text{boost}}^s}{\partial q_j^r} \right) \delta^r_j(\text{trans}) - \left( \frac{\partial \Lambda_{\text{trans}}^r}{\partial q_j^s} \right) \delta^r_j(\text{boost})$$

$$= M \delta_{sj} \delta_{jr} = M \delta_{rs}$$

Therefore, we conclude that the mass is a central extension.
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