Abstract. If $P$ is a dg-operad acting on a dg-algebra $A$ via algebra homomorphisms, then $P$ acts on the Hochschild complex of $A$. In the more general case when $P$ is a dg-prop, we show that $P$ still acts on the Hochschild complex, but only up to coherent homotopy. We moreover give a functorial dg-replacement of $P$ that strictifies the action. As an application, we obtain an explicit strictification of the homotopy coherent commutative Hopf algebra structure on the Hochschild complex of a commutative Hopf algebra.

1. Introduction

A dg-prop is a symmetric monoidal dg-category $P$ whose monoid of objects is isomorphic to $(\mathbb{N},+)$. If $P$ is a dg-prop and $C$ a symmetric monoidal dg-category, then a $P$-algebra in $C$ is a symmetric monoidal functor $P \to C$. A morphism of dg-props is a symmetric monoidal dg-functor which induces an isomorphism of object monoids, and such a morphism is called a quasi-equivalence if it induces quasi-isomorphisms on Hom-complexes. Let dgprop be the subcategory of dgCat generated by the dg-props and morphisms of dg-props. Examples of dg-props arise from dg-operads $O$ by the formula

$$P(n,m) = \bigoplus_{n_1 + \ldots + n_m = n} O(n_1) \otimes \ldots \otimes O(n_m)$$

and composition defined using the composition product of $O$. With this definition, an algebra over an operad is precisely an algebra over the dg-prop it generates. In the following, we will therefore not distinguish between a dg-operad and the dg-prop it generates.

The Hochschild complex is a functor from $A_\infty$-algebras to chain complexes. If $P$ is a prop equipped with a morphism of props $A_\infty \to P$, the Hochschild complex restricts to a functor from $P$-algebras. It is an open problem to compute the operations on the Hochschild complex of algebras over such props $P$. Partial results have been obtained in many cases, see e.g. [6, 7]. One such case is the following.

If $P$ is a dg-operad (or a dg-prop) the tensor product $\text{Ass} \otimes P$ is the dg-prop characterized by the equivalence

$$\text{Fun}^\otimes(\text{Ass} \otimes P, C) \simeq \text{Fun}^\otimes(P, \text{Alg}(C))$$

for any symmetric monoidal dg-category $C$. Evaluating the right hand side at $1 \in P$, we obtain a functor from $\text{Ass} \otimes P$-algebras to $\text{Ass}$-algebras. The Hochschild complex of an $\text{Ass} \otimes P$-algebra is by definition the Hochschild complex of the associated $\text{Ass}$-algebra. The Hochschild complex functor is lax monoidal. The structure morphisms may be used to prove that for a dg-operad $P$ and an algebra $A$ over the tensor product $\text{Ass} \otimes P$, the Hochschild complex of $A$ admits a $P$-algebra structure (see [11] and [6 Section 6.9]). On the other hand,
this fails if $P$ is a more general dg-prop. This is due to the failure of the Dold-Kan equivalence to be a symmetric monoidal equivalence. It is however true up to coherent homotopy, as the Dold-Kan equivalence is an $\mathbb{E}_\infty$-monoidal equivalence [5, Section 5]. In this paper, we give an explicit functorial strictification of the natural homotopy coherent $P$-algebra structure on the Hochschild complex of a $(P \otimes \text{Ass})$-algebra. Formally this is encoded in the following result.

**Theorem A.** Let $k$ be a commutative ring and let $\text{dgprop}$ be the category of dg-props over $k$. There is a functor

$$\widetilde{(-)} : \text{dgprop} \to \text{dgprop}$$

equipped with a natural quasi-equivalence $\widetilde{(-)} \to \text{id}$ and a natural transformation

$$\widetilde{\alpha} : \text{Fun}^\circ(\text{Ass} \otimes -, \text{Ch}_k) \to \text{Fun}^\circ(\widetilde{(-)}, \text{Ch}_k)$$

of functors $\text{dgprop}^{\text{op}} \to \text{Cat}$ such that for a dg-prop $P$ and an $\text{Ass} \otimes P$-algebra

$$\Phi : \text{Ass} \otimes P \to \text{Ch}_k$$

the value $\widetilde{\alpha}_P(\Phi)(1)$ is equal to the Hochschild complex of $\Phi(1)$.

We use explicit generators and relations to construct the functor $\widetilde{(-)}$, fattening the input prop with the structure maps of the Dold-Kan equivalence. The functor $\widetilde{(-)}$ also admits the structure of a non-unital monad.

**Example.** (Example 3.12) If $\Phi : \text{CHopf} \to \text{dgAlg}_k$ is a commutative Hopf algebra over any ring $k$, then

$$\widetilde{\alpha}_{\text{CHopf}}(\Phi) : \widetilde{\text{CHopf}} \to \text{Ch}_k$$

gives an explicit strict model for the coherent commutative Hopf algebra structure of the Hochschild complex of $\Phi(1)$.

Given a dg-prop $P$, one may ask whether $\widetilde{P}$ is cofibrant in a model structure on dg-props. In [3], Fresse constructs a model structure on the category of props over a field of characteristic zero, and a semi-model structure on certain sub-families of props in positive characteristic. However, for example Hopf algebras in positive characteristic cannot be treated in his framework. Additionally, in characteristic zero, our replacement $\widetilde{P}$ will not be cofibrant. For example, our replacement of the commutative prop still has a strictly commutative multiplication.

**Further Questions.** Theorem A displays $\widetilde{P}$ as a sub-prop of the prop of natural operations on the Hochschild complex. On the other hand, it leaves open the interaction of the $P$-action with Connes’ $B$-operator. The determination of the total prop of natural operations on $C(A)$ is still an interesting open problem with a view toward operations on cyclic homology.

The structure of the paper is as follows. In Section 2 we define the Dold-Kan structure maps and their action on Hochschild complexes, and establish necessary properties. In Section 3 we define the fattening functor $\widetilde{(-)}$ for dg-props and prove the main theorem.
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2. The cyclic bar construction and the Dold-Kan equivalence

In this section we will build a dg-category \( \tilde{N}^\Sigma \) from the structure maps of the Dold-Kan correspondence and establish the action of \( \tilde{N}^\Sigma \) on Hochschild complexes of dg-algebras. This dg-category is a key ingredient for the fattening functor we will construct in Section 3.

Convention 2.1. Throughout, we fix a commutative base ring \( k \). All algebras are assumed to be algebras over \( k \). We employ the Kozul sign convention for chain complexes. In particular, our convention for bicomplexes are that the differentials anti-commute.

We begin by recalling some basic notions from homological algebra.

We will work with the categories \( \mathbf{sMod}_k \) of simplicial \( k \)-modules and \( \mathbf{Ch}_k \) of non-negatively graded chain complexes over \( k \) with \( k \)-linear graded maps, where we consider \( \mathbf{Ch}_k \) as a category enriched in itself. \( \mathbf{sMod}_k \) is a symmetric monoidal simplicial category with tensor product given by the degreewise tensor product of \( k \)-modules. We denote this tensor product by \( \otimes \). Similarly, \( \mathbf{Ch}_k \) is a symmetric monoidal category with tensor product denoted by \( \otimes \) and given by \( (A \otimes B)_* = \oplus_{p+q=s} A_p \otimes B_q \) and differential \( d_{A \otimes B}(a \otimes b) = d_A(a) \otimes b + (-1)^{|a|} a \otimes d_B(b) \).

The category of monoids in \( \mathbf{sMod}_k \) is denoted by \( \mathbf{sAlg}_k \), and is a symmetric monoidal category with the levelwise tensor product.

For a simplicial chain complex \( A = A_{\bullet, \bullet} \) over \( k \), call \( * \) the differential degree and \( \bullet \) the simplicial degree. Write \( d_{a,b}^* : A_{a,b} \to A_{a-1,b} \) for the differential and \( d_i^{a,b} : A_{a,b} \to A_{a,b-1} \) for the simplicial face maps. Write \( \mathbf{sCh}_k \) for the category of simplicial chain complexes over \( k \).

The Dold-Kan equivalence

\[ \tilde{N} : \mathbf{sMod}_k \rightleftarrows \mathbf{Ch}_k : \Gamma \]

gives an equivalence of categories between simplicial \( k \)-modules and connective chain complexes over \( k \). The functor \( \tilde{N} : \mathbf{sMod}_k \to \mathbf{Ch}_k \), is called the normalized Moore complex functor, and takes a simplicial \( k \)-module \( M_{\bullet} \) to the chain complex \( \tilde{N}M_{\bullet} \) with \( \tilde{N}M_p = M_p / sM_{p-1} \), the quotient of \( M_p \) by the degenerate simplices, and \( d : \tilde{N}M_p \to \tilde{N}M_{p-1} \) given by the alternating sum \( d = \sum_{i=0}^p (-1)^i d_i \). The inverse functor \( \Gamma : \mathbf{Ch}_k \to \mathbf{sMod}_k \) is called the Dold-Kan construction. We can also apply \( \tilde{N} \) degreewise to a simplicial chain complex as follows:

**Definition 2.2.**

1. For \( A \in \mathbf{sCh}_k \), the bicomplex associated to \( A_{\bullet, \bullet} \) is denoted by \( \tilde{N}(A_{\bullet, \bullet})_* \), and is obtained by applying the Moore complex functor levelwise and shifting the differentials by the differential degree of \( A \). Writing this out, we have \( \tilde{N}(A_{\bullet, \bullet})_{a,b} = A_{a,b} \), the horizontal differential is \( d_b = d_A \), and the vertical differential is

\[ d_{a,b}^v = (-1)^a \sum_{i=0}^b (-1)^i d_i^{a,b} \]

We write

\[ \tilde{N}_b(A) := \text{Tot} (\tilde{N}(A_{\bullet, \bullet})_*). \]
(2) Let $A$ and $B$ be simplicial chain complexes over $k$ and denote by $A \hat{\otimes} B$ the simplicial chain complex which in simplicial degree $p$ is given by $A_{s,p} \otimes B_{s,p}$. The differential of $A \hat{\otimes} B$ is given by
\[
d^p_{A \hat{\otimes} B}(a \otimes b) = d^{|a|}_A a \otimes b + (-1)^{|a| + p} a \otimes d^{|b|}_B b.
\]

**Definition 2.3.**

1. The cyclic bar construction is the functor $B^c_y : \text{dgAlg}_k \to \text{sCh}_k$ given in simplicial degree $p$ by $B^c_y(A) = A \hat{\otimes} B$. The face maps $d_i : B^c_y(A) \to B^c_y(A)$ are given by
\[d_i : a_0 \otimes \ldots \otimes a_p \mapsto \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_p, & i = 0, \ldots, p-1 \\ (-1)^{|a_i|(|a_0|+\ldots+|a_{i-1}|)} a_i a_0 \otimes a_1 \otimes \ldots \otimes a_{p-1}, & i = p \end{cases}
\]
and the degeneracies $s_i : B^c_y(A) \to B^c_{y+1}(A)$ are given by
\[s_i : a_0 \otimes \ldots \otimes a_p \mapsto a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_p
\]
making $B^c_y(A)$ into a simplicial chain complex, called the cyclic bar construction of $A$.

2. For a dg-algebra $A$, the complex
\[C(A) := N_\delta B^c_y(A)
\]
is called the Hochschild complex of $A$.

**Lemma 2.4.** The cyclic bar construction $B^c_y : \text{dgAlg}_k \to \text{sCh}_k$ is symmetric monoidal.

**Proof.** Let $A$ and $B$ be dg-algebras over $k$. Define the natural transformation $B^c_y(A) \hat{\otimes} B^c_y(B) \to B^c_y(A \otimes B)$ is given in simplicial degree $p$ by permuting tensor factors:
\[A \hat{\otimes} B \hat{\otimes} p \sigma_p \to (A \otimes B)^{\otimes p+1}
\]
\[a_0 \otimes \ldots \otimes a_p \otimes b_0 \otimes \ldots \otimes b_p \mapsto (-1)^{\text{sgn}(a,b,\sigma)} a_0 \otimes b_0 \otimes \ldots \otimes a_p \otimes b_p
\]
where $\text{sgn}(a,b,\sigma) \in \mathbb{Z}/2$ is the sign of $\sigma$ weighted by the elements $a_i, b_j$, which can be computed as
\[\text{sgn}(a,b,\sigma) \equiv \sum_{i=0}^{p-1} |b_i| \left( \sum_{j=i+1}^p |a_j| \right) \pmod{2}
\]
To check that this defines a chain map in simplicial degree $p$, we must verify that there are no sign issues. It is sufficient to consider each summand of the differential separately. For the differential acting on $a_k$ for $1 \leq k \leq p+1$, the sign we get by permuting first (i.e. the sign associated to $d^{A \hat{\otimes} B \circ \sigma}$) is
\[\text{sgn}(a,b,\sigma) + \left( \sum_{i=0}^{k-1} |a_i| + |b_i| \right)
\]
where the second term comes from the placement of $a_k$ after permuting. The sign we get by permuting second is
\[\left( \sum_{i=0}^{k-1} |a_i| \right) + \text{sgn}(a,b,\sigma) + \sum_{j=0}^{k-1} |b_j|
\]
where the third term is the correction to \(\text{sgn}_{a,b}\sigma\) when the degree of \(a_k\) is decreased by one. We see that the two are equal. For the differential acting on \(b_k\), the sign we get by permuting first is

\[
\text{sgn}(a, b, \sigma) + \left(\sum_{i=0}^{k-1} |a_i| + |b_i|\right) + |a_k|
\]

and the sign we get by permuting second is

\[
\left(\sum_{i=0}^{p} |a_i|\right) + \left(\sum_{j=0}^{k-1} |b_j|\right) + \text{sgn}(a, b, \sigma) + \sum_{j=k+1}^{p} |a_j|
\]

where the fourth term is the correction to \(\text{sgn}_{a,b}\sigma\) when the degree of \(b_k\) is decreased by one. Again we see that the two expressions are equal mod 2, hence we have a chain map.

We now verify that \(\sigma\) is a symmetric monoidal transformation. Let \(\tau\) be the symmetric monoidal twist map of \(\text{Ch}_k\), given by \(A \otimes B \to B \otimes A, a \otimes b \mapsto (-1)^{|a||b|} b \otimes a\). We check that \(\sigma \circ \tau_{p+1,p+1} = (\tau \otimes p+1) \circ \sigma\). The left hand side has sign

\[
\text{sgn}_L = \left(\sum_{i=0}^{p} |a_i|\right) \left(\sum_{j=0}^{p} |b_j|\right) + \left(\sum_{i=0}^{p-1} |a_i|\right) \left(\sum_{j=i+1}^{p} |b_j|\right) = \sum_{i=0}^{p} |a_i| \left(\sum_{j=0}^{i} |b_j|\right)
\]

and the right hand side has sign

\[
\text{sgn}_R = \sum_{i=0}^{p-1} |b_i| \left(\sum_{j=i+1}^{p} |a_j|\right) + \left(\sum_{i=0}^{p} |a_i||b_i|\right) = \sum_{i=0}^{p} |b_i| \left(\sum_{j=0}^{i} |a_j|\right)
\]

Rearranging the order of summation shows that \(\text{sgn}_L = \text{sgn}_R\), and we conclude that \(B^{cy}\) is symmetric monoidal as claimed. \(\square\)

Before discussing the monoidality properties of the cyclic bar construction and the Hochschild complex, we recall some of the monoidality properties of the Dold-Kan equivalence.

**Definition 2.5.** We will write \(\Sigma_{(p,q)} \subseteq \Sigma_{p+q}\) for the subset of \((p,q)\)-shuffles, i.e. those permutations \(\sigma\) of \((1,...,p+q)\) for which

\[
\sigma(1) < \sigma(2) < ... < \sigma(p)
\]

and

\[
\sigma(p+1) < \sigma(p+2) < ... < \sigma(p+q).
\]

We similarly denote the subgroups of \((p,q,r)\)-shuffles in \(\Sigma_{p+q+r}\) by \(\Sigma_{(p,q,r)}\).

**Lemma 2.6.** Let \(\Sigma_{p,(q,r)}\) denote the subgroup of \(\Sigma_{(p,q,r)}\) fixing the first \(p\) elements, and similarly let \(\Sigma_{(p,q),r}\) denote the subgroup of \(\Sigma_{(p,q,r)}\) fixing the last \(r\) elements. The sets of shuffles satisfy the relation

\[
\Sigma_{(p+q,r)} \Sigma_{(p,q)} = \Sigma_{(p,q,r)} = \Sigma_{(p,q+r)} \Sigma_{(p,q)}
\]

In other words, every \((p,q,r)\)-shuffle can be written uniquely as the composition of a \((p,q)\)-shuffle with a \((p+q,r)\)-shuffle and as the composition of a \((q,r)\)-shuffle with a \((p,q+r)\)-shuffle.
Proof. Let \( \theta \) be a \((p,q,r)\)-shuffle. We produce a \((p,q)\)-shuffle \( \sigma \) by removing the entries \( \theta(p+q+1),\ldots,\theta(p+q+r) \) from the list \((\theta(1),\theta(2),\ldots,\theta(p+q+r))\). We produce a \((p+q,r)\)-shuffle \( \tau \) by replacing the entries \( \theta(1),\theta(2),\ldots,\theta(p+q) \) by \((1,2,\ldots,p+q)\). It is clear that \( \tau \sigma = \theta \). Now assume \( \tau \sigma = \tau' \sigma' \) for another choice of \( \tau' \in \Sigma_{(p+q,r)} \) and \( \sigma' \in \Sigma_{(p,q)} \). Then \( \tau' = \tau \sigma (\sigma')^{-1} \), and evaluating on the entries \( i = p+q+1,\ldots,p+q+r \), we see that \( \tau(i) = \tau'(i) \) in this range. Since a \((p,q,r)\)-shuffle is determined by its values on \( p+q+1,\ldots,p+q+r \), we have \( \tau = \tau' \). Then \( \sigma' = \tau'^{-1} \tau \sigma = \sigma \), hence this decomposition is unique. The other decomposition follows similarly.

\[\textbf{Definition 2.7.}\] Let \( A \) and \( B \) be simplicial \( k \)-modules. The \textit{shuffle map} (also called the Eilenberg-Zilber map):

\[\text{sh}_{A,B} : N(A) \otimes N(B) \to N(A \hat{\otimes} B)\]

is defined on elementary tensors \( a \otimes b \in A_p \otimes B_q \) as

\[\text{sh}_{A,B}(a \otimes b) = \sum_{\sigma \in \Sigma_{(p,q)}} \text{sgn}(\sigma)s_{\sigma(p+q)}\ldots s_{\sigma(p+1)}a \otimes s_{\sigma(p)}\ldots s_{\sigma(1)}b\]

When there is no risk of confusion, we will omit \( A \) and \( B \) from the notation and simply write \( \text{sh} \) for the shuffle map.

\[\textbf{Definition 2.8.}\] Let \( A \) and \( B \) be simplicial \( k \)-modules. The \textit{Alexander-Whitney map}

\[\text{AW}_{A,B} : N(A \hat{\otimes} B) \to N(A) \otimes N(B)\]

is defined on elementary tensors \( a \otimes b \in A_n \otimes B_n \) as

\[\text{AW}_{A,B} : (a \otimes b) \mapsto \sum_{i=0}^{n} d_{i+1} \ldots d_{n-1} d_n a \otimes (d_0)^i b\]

As with the shuffle map, we omit \( A, B \) from the notation \( \text{AW}_{A,B} \) when there is no risk of confusion.

\[\textbf{Lemma 2.9.}\] The Alexander-Whitney map is associative, i.e. for \( A, B \) and \( C \) simplicial \( k \)-modules, the morphisms \( (\text{id} \otimes \text{AW}_{B,C}) \circ \text{AW}_{A,B \hat{\otimes} C} \) and \( (\text{AW}_{A,B} \otimes \text{id}) \circ \text{AW}_{A \hat{\otimes} B,C} \) from \( N(A \hat{\otimes} B \hat{\otimes} C) \) to \( N(A) \otimes N(B) \otimes N(C) \) are equal.

\[\textit{Proof.}\] Let \( a \otimes b \otimes c \in A_n \otimes B_n \otimes C_n \). For brevity, we write \( \tilde{d}_i^n = d_{i+1} \ldots d_{n-1} d_n \). Then the two compositions

\[N(A \hat{\otimes} B \hat{\otimes} C) \to N(A) \otimes N(B) \otimes N(C)\]

are

\[a \otimes b \otimes c \xrightarrow{\text{AW}_{A,B \hat{\otimes} C}} \sum_{p=0}^{n} \tilde{d}_p^n a \otimes \tilde{d}_p^n b \otimes d_0^n c\]

\[\xrightarrow{\text{id} \otimes \text{AW}_{B,C}} \sum_{p=0}^{n} \sum_{a=0}^{n-p} \tilde{d}_p^n a \otimes \tilde{d}_a^{n-p} d_0^p b \otimes \tilde{d}_0^{n+s} c\]

and

\[a \otimes b \otimes c \xrightarrow{\text{AW}_{A \hat{\otimes} B,C}} \sum_{q=0}^{n} \tilde{d}_q^n a \otimes \tilde{d}_q^n b \otimes d_0^n c\]

\[\xrightarrow{\text{AW}_{B,C} \otimes \text{id}} \sum_{q=0}^{n} \sum_{t=0}^{q} \tilde{d}_t^q \tilde{d}_q^n a \otimes \tilde{d}_t^q d_0^t \tilde{d}_q^n b \otimes \tilde{d}_0^n c.\]
Note that \( \tilde{d}_q^i \tilde{d}_q^n = \tilde{d}_q^n \). Using the simplicial identity \( d_i d_j = d_{j-1} d_i \) when \( i < j \), observe that \( d_0^n \tilde{d}_q^n = \tilde{d}_q^{n-1} d_0^n \). Writing \( (q, t) = (p + s, p) \), we now see that the two expressions are equal. \( \Box \)

The shuffle and Alexander-Whitney maps are mutually inverse quasi-isomorphisms. In particular, \( AW \circ sh = id \) and \( sh \circ AW \simeq id \). The shuffle map is a lax symmetric monoidal transformation witnessing that the Moore complex functor \( N : sMod_k \to Ch_k \), and hence also the Hochschild chains functor \( C : sAlg_k \to Ch_k \) is lax symmetric monoidal. The Alexander-Whitney map is an oplax monoidal transformation witnessing that \( N \), and hence \( C \) is oplax monoidal. However, the Alexander-Whitney map is not symmetric. Still, it is \( E_\infty \) in the following sense (see Lemma 2.13).

**Definition 2.10.** ([4, p.552]) A functor \( F : C \to D \) between symmetric monoidal categories is \( E_\infty \)-**monoidal** if there is an \( E_\infty \) operad \( O \) in \( D \) and maps

\[
\mu_n : O(n) \otimes (F(A_1) \otimes ... \otimes F(A_n)) \to F(A_1 \otimes ... \otimes A_n)
\]

such that

1. the action is unital, i.e. if \( I \) denotes the monoidal unit of \( D \) and \( \eta : I \to O(1) \) is the unit of the operad, then the following diagram commutes:

\[
\begin{array}{ccc}
I \otimes F(A) & \xrightarrow{\eta \otimes id} & O(1) \otimes F(A) \\
\downarrow{\simeq} & & \downarrow{\mu_1} \\
F(A) & & F(A)
\end{array}
\]

2. The action is equivariant: for each \( \sigma \in \Sigma_n \), the action \( \mu_n \) is compatible with the action of \( \Sigma_n \) on \( O(n) \) and by permuting indices of the \( A_i \). I.e. the following diagram commutes:

\[
\begin{array}{ccc}
O(n) \otimes F(A_1) \otimes ... \otimes F(A_n) & \xrightarrow{\mu_n} & F(A_1 \otimes ... \otimes A_n) \\
\downarrow{\sigma \otimes \sigma} & & \downarrow{F(\sigma)} \\
O(n) \otimes F(A_{\sigma^{-1}(1)}) \otimes ... \otimes F(A_{\sigma^{-1}(n)}) & \rightarrow & F(A_{\sigma^{-1}(1)} \otimes ... \otimes A_{\sigma^{-1}(n)})
\end{array}
\]

3. The action is associative, i.e. is compatible with the operad multiplication. \( E_\infty \)-**comonoidal** functors are similarly defined by using structure maps

\[
\nu_n : O(n) \otimes F(A_1 \otimes ... \otimes A_n) \to F(A_1) \otimes ... \otimes F(A_n).
\]

We will now define chain complexes which assemble into a dg-operad (and later a symmetric monoidal dg-category) witnessing that \( AW \) is an \( E_\infty \)-**comonoidal** transformation.

**Definition 2.11.** Define the functors

\[
N^{\hat{o}}_n, N^{\otimes n} : sMod_k^{\times n} \to Ch_k
\]

given by

\[
N^{\hat{o}}_n(A_1, ..., A_n) = N(A_1 \hat{\otimes} ... \hat{\otimes} A_n)
\]

\[
N^{\otimes n}(A_1, ..., A_n) = N(A_1) \otimes ... \otimes N(A_n)
\]
and let
\[ \mathcal{O}(n) := \text{Nat}_{s\mathsf{Mod}_k^n}(N^{\otimes n}, N^{\otimes n}) \]

**Notation 2.12.** Since the elements of \( \mathcal{O}(n) \), and of the complex \( \tilde{N}^\Sigma((n),(n)) \) which we define below, are natural transformations, we can in particular view them as 2-morphisms in the 2-category of categories, and so 2-categorical constructions, like horizontal composition, can be applied to them. For a 4-tuple of morphisms \( f, f': a \to b \) and \( g, g': b \to c \) and a pair of 2-morphisms \( \alpha : f \to f' \) and \( \beta : g \to g' \), we write \( \beta \circ \alpha \) for their horizontal composition \( \beta \circ \alpha : gf \to g'f' \).

**Lemma 2.13.** The complexes \( \mathcal{O}(n) \) assemble into an \( \mathbb{E}_\infty \)-operad witnessing that \( N^{\otimes n} \) and \( N^{\otimes n} \) are \( \mathbb{E}_\infty \)-comonoidal functors and that \( AW : N^{\otimes 2} \to N^{\otimes 2} \) is an \( \mathbb{E}_\infty \)-comonoidal transformation.

**Proof.** Let \( n_1 + ... + n_i = n \) be natural numbers. The operad structure on \( \mathcal{O} \) is given by the maps
\[ \mathcal{O}(i) \otimes (\mathcal{O}(n_1) \otimes ... \otimes \mathcal{O}(n_i)) \to \mathcal{O}(n) \]
given by \( (\phi, \gamma_1, ..., \gamma_i) \mapsto \phi \circ (\gamma_1 \ast ... \ast \gamma_i) \). The \( \Sigma_n \)-action is given by conjugation, i.e. for \( \chi \in \Sigma_n \) and \( \psi \in \mathcal{O}(n) \) we have \( \chi \cdot \psi = \chi \circ \psi \circ \chi^{-1} \). It is known (see [2, Satz 1.6]) that the complex of natural transformations
\[ \mathcal{O}(n) = \text{Nat}_{s\mathsf{Mod}_n^k}(N^{\otimes n}, N^{\otimes n}) \]
is acyclic with zero-th homology \( k \). It follows (see [4, Section 7] and [5, Section 5]) that the functors \( N \) is an \( \mathbb{E}_\infty \)-comonoidal functor and that \( AW : N^{\otimes 2} \to N^{\otimes 2} \) is an \( \mathbb{E}_\infty \)-comonoidal transformation. \( \square \)

We will look at the complex
\[ \tilde{N}((n),(n)) := \text{Nat}_{s\mathsf{Mod}_n^k}(N^{\hat{\otimes} n}, N^{\hat{\otimes} n}) \]
which is homotopy equivalent to \( \text{Nat}_{s\mathsf{Mod}_n^k}(N^{\hat{\otimes} n}, N^{\otimes n}) \), seen by post-composing with shuffle and Alexander-Whitney maps, but with the difference that maps in \( \tilde{N}((n),(n)) \) may be composed, giving rise to an algebra structure. In the rest of this section, we will construct a dg-category with morphism complexes built from \( \text{Nat}_{s\mathsf{Mod}_n^k}(N^{\hat{\otimes} n}, N^{\hat{\otimes} n}) \), and the notation is chosen with this in mind.

**Definition 2.14.** The symmetric group \( \Sigma_n \) acts on \( s\mathsf{Mod}_n^k \) by \( \chi(A_1, ..., A_n) = (A_{\chi^{-1}(1)}, ..., A_{\chi^{-1}(n)}) \). Let \( \tilde{N}^\Sigma((n),(n)) \) be the complex
\[ \tilde{N}^\Sigma((n),(n)) = \bigoplus_{\chi \in \Sigma_n} \text{Nat}_{s\mathsf{Mod}_n^k}(N^{\hat{\otimes} n}, N^{\hat{\otimes} n} \circ \chi) =: \bigoplus_{\chi \in \Sigma_n} \tilde{N}^\Sigma_\chi((n),(n)) \]

**Lemma 2.15.** The chain complex \( \tilde{N}^\Sigma((n),(n)) \) admits a \( \Sigma_n \)-graded algebra structure and contracts to \( k\Sigma_0 \) in degree 0.

**Proof.** Let \( f \in \tilde{N}^\Sigma_\chi((n),(n)) \) and \( g \in \tilde{N}^\Sigma((n),(n)) \). We treat \( f \) and \( g \) as 2-morphisms in the 2-category of dg-categories as in Remark 2.12. The product of \( g \) and \( f \) is given by \( (g \circ \text{id}_\chi) \circ f : N^{\hat{\otimes} n} \to N^{\hat{\otimes} n} \circ (\chi' \chi) \), which may also be visualized by the pasting diagram
This gives the graded algebra structure. As for the contraction, the components $\tilde{N}_\chi^\Sigma((n), (n))$ are isomorphic to $\tilde{N}((n), (n))$ by pre-composition by $\chi$ and $\chi^{-1}$. As $\tilde{N}((n), (n))$ contracts onto $\id_{\tilde{N}^\otimes_n}$, $\tilde{N}_\chi^\Sigma((n), (n))$ contracts similarly to $\chi$. □

**Definition 2.16.** For $A_1, ..., A_k$ simplicial $k$-modules, we introduce the shorthand

$\mathcal{N}^{(k_1, ..., k_n)}(A_1, ..., A_k) = N(A_1 \otimes ... \otimes A_{k_1} \otimes ... \otimes N(A_{k_1+...+k_{n-1}+1} \otimes ... \otimes A_k)$

where $k = k_1 + ... + k_n$. Let $m_1 + ... + m_l = k$. Writing $\vec{k} = (k_1, ..., k_n)$ and similarly for $\vec{m}$, define the complex

$\tilde{N}(\vec{k}, \vec{m}) := \text{Nat}_{sMod^k}^\times(N^\vec{k}, N^\vec{m})$

Its symmetrized version $\tilde{N}_\Sigma^\Sigma(\vec{k}, \vec{m})$ is defined as before by

$$\tilde{N}_\Sigma^\Sigma(\vec{k}, \vec{m}) = \bigoplus_{\chi \in \Sigma_k} \text{Nat}_{sMod^k}^\times(\chi^\Sigma N^\vec{k}, \chi^\Sigma N^\vec{m} \circ \chi) =: \bigoplus_{\chi \in \Sigma_k} \tilde{N}_\chi^\Sigma(\vec{k}, \vec{m})$$

- We will refer to a finite sequence of integers $\vec{k} = (k_1, ..., k_n)$ as a **vector**. The sum of the entries of a vector is called its **length** and denoted $|\vec{k}| := k_1 + ... + k_n$.
- We write $\tilde{N}_\Sigma^\Sigma$ for the dg-category whose objects are vectors $\vec{k}$, and whose morphism complexes are given by the $\tilde{N}_\Sigma^\Sigma(\vec{k}, \vec{m})$ defined above.

**Notation 2.17.** For any $\vec{k} = (k_1, ..., k_n)$ with $|\vec{k}| = k$, by Lemma 2.6 composing shuffle maps gives rise to a well-defined shuffle map which we write $\text{sh}_{\vec{k}} : N^\vec{k} \to N^{(k)} = N^{\otimes k}$. Similarly, the Alexander-Whitney map is associative by Lemma 2.9, so composing AW-maps gives rise to a well-defined map $AW_{\vec{k}} : N^{(k)} \to N^\vec{k}$. Note that $AW_{\vec{k}} \circ \text{sh}_{\vec{k}} \simeq \id_{N^\vec{k}}$ and $\text{sh}_{\vec{k}} \circ AW_{\vec{k}} \simeq \id_{N^{(k)}}$.

**Lemma 2.18.** For every pair $\vec{k} = (k_1, ..., k_n)$, $\vec{m} = (m_1, ..., m_l)$, the assignment

$$\phi : f \mapsto AW_{\vec{m}} \circ f \circ \text{sh}_{\vec{k}}$$

defines a homotopy equivalence $\tilde{N}_\Sigma^\Sigma((n), (n)) \to \tilde{N}_\Sigma^\Sigma(\vec{k}, \vec{m})$ with homotopy inverse

$$\psi : g \mapsto \text{sh}_{\vec{m}} \circ g \circ AW_{\vec{k}}$$

In particular, $\tilde{N}_\Sigma^\Sigma(\vec{k}, \vec{m})$ contracts onto the degree zero subcomplex of elements of the form $AW_{\vec{m}} \circ \chi \circ \text{sh}_{\vec{k}}$ for some $\chi \in \Sigma_k$.

**Proof.** Fix homotopies $\alpha_{\vec{k}} : AW_{\vec{k}} \circ \text{sh}_{\vec{k}} \to \id$ and $\beta_{\vec{k}} : \text{sh}_{\vec{k}} \circ AW_{\vec{k}} \to \id$. Then we get homotopies

$$\beta_{\vec{m}} \ast \id \ast \beta_{\vec{k}} : \psi \circ \phi \to \id$$

$$\alpha_{\vec{m}} \ast \id \ast \alpha_{\vec{k}} : \phi \circ \psi \to \id$$
so that φ and ψ are mutually inverse homotopy equivalences. Now the composition
\[ k\Sigma_n \hookrightarrow \tilde{N}^\Sigma((n), (n)) \to \tilde{N}^\Sigma(k, \tilde{m}) \]
takes χ to \( AW_{\tilde{m}} \circ \chi \circ \text{sh}_{\tilde{k}} \) and is a homotopy equivalence since \( k\Sigma_n \hookrightarrow \tilde{N}^\Sigma((n), (n)) \) is by Lemma \ref{lemma:totalization-equivalence}. The inverse
\[ \tilde{N}^\Sigma(k, \tilde{m}) \to \tilde{N}^\Sigma((n), (n)) \to k\Sigma_n \]
sends \( AW_{\tilde{m}} \circ \chi \circ \text{sh}_{\tilde{k}} \) to χ, so \( \tilde{N}^\Sigma(k, \tilde{m}) \) contracts as claimed. \( \square \)

We now turn to establishing the action of \( \tilde{N}^\Sigma \) on Hochschild complexes of dg-algebras. We begin by constructing a way of differentially extending functors between additive categories.

**Construction 2.19.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be additive categories and let \( \text{Fun}^\text{pt}(\mathcal{A}, \mathcal{B}) \) be the category of pointed functors between them, that is, functors \( F : \mathcal{A} \to \mathcal{B} \) such that \( F(0) \simeq 0 \). Note that \( \text{Fun}^\text{pt}(\mathcal{A}, \mathcal{B}) \) is itself an additive category. Denote by \( m\text{-Ch}(\mathcal{B}) \) the additive category of \( m \)-fold chain complexes in \( \mathcal{B} \). We will produce an additive functor
\[ (\_)_\varepsilon : \text{Fun}^\text{pt}(\mathcal{A}^{\times n}, m\text{-Ch}(\mathcal{B})) \to \text{Fun}^\text{pt}(\text{Ch}(\mathcal{A})^{\times n}, (n + m)\text{-Ch}(\mathcal{B})). \]

Let \( F : \mathcal{A}^{\times n} \to m\text{-Ch}(\mathcal{B}) \) be a pointed functor. Then \( F_\varepsilon \) sends an \( n \)-tuple of chain complexes \( (A_1, \ldots, A^n) \) in \( \mathcal{A} \) to the \((n + m)\)-fold chain complex in \( \mathcal{B} \) given in multidegree \((p_1, \ldots, p_n, q_1, \ldots, q_m)\) by \( F(A^1_{p_1}, \ldots, A^n_{p_n})_{q_1, \ldots, q_m} \). In the same multidegree, the differentials are given by
\[ d_i = \begin{cases} (-1)^{p_1 + \ldots + p_i - 1}d^{A_i}, & 1 \leq i \leq n \\ (-1)^{p_1 + \ldots + p_n + q_1 + \ldots + q_{i-1} - 1}d^{F(A^1_{p_1}, \ldots, A^n_{p_n})}, & n + 1 \leq i \leq n + m \end{cases} \]
Since \( F \) is a pointed functor, this does indeed define a \((n + m)\)-fold chain complex in \( \mathcal{B} \). Similarly, \( F_\varepsilon \) sends an \( n \)-tuple of morphisms \((f^i : A^i \to B^i)_{1 \leq i \leq n}\) to the morphism given on the first \( n \) multidegrees \((p_1, \ldots, p_n)\) by the morphism
\[ F(f^1_{p_1}, \ldots, f^n_{p_n}) : F(A^1_{p_1}, \ldots, A^n_{p_n}) \to F(B^1_{p_1}, \ldots, B^n_{p_n}) \]
Hence \( F_\varepsilon \) is indeed a functor.

Now let \( F, G : \mathcal{A}^{\times n} \to m\text{-Ch}(\mathcal{B}) \) be pointed functors and let \( \alpha : F \to G \) be a natural transformation. Then \( \alpha_\varepsilon \) is the natural transformation given by applying \( \alpha \) levelwise, i.e. for an \( n \)-tuple of chain complexes \( (A_1, \ldots, A^n) \) in \( \mathcal{A} \), the morphism
\[ (\alpha_\varepsilon)_{(A^1, \ldots, A^n)} : F_\varepsilon(A^1, \ldots, A^n) \to G_\varepsilon(A^1, \ldots, A^n) \]
is given in the first \( n \) multidegrees \((p_1, \ldots, p_n)\) by the morphism
\[ \alpha(A^1_{p_1}, \ldots, A^n_{p_n}) : F(A^1_{p_1}, \ldots, A^n_{p_n}) \to G(A^1_{p_1}, \ldots, A^n_{p_n}). \]
With this definition it is clear that \((\_)_\varepsilon\) is an additive functor.

**Definition 2.20.** Building on Construction \ref{construction:pointed-functors} we define the functor
\[ (\_)_\delta : \text{Fun}^\text{pt}(\mathcal{A}^{\times n}, m\text{-Ch}(\mathcal{B})) \to \text{Fun}^\text{pt}(\text{Ch}(\mathcal{A})^{\times n}, \text{Ch}(\mathcal{B})) \]
as the composition of \((\_)_\varepsilon\) with the totalization functor \((n + m)\text{-Ch}(\mathcal{B}) \to \text{Ch}(\mathcal{B})\).

Before considering the monoidality properties of the functor \((\_)_\delta\), we need an observation about totalizations of \( n \)-fold chain complexes.
Observation 2.21. Let $\mathcal{A}$ be an additive category. The symmetric group on $n$ letters acts on the category of $n$-fold chain complexes in $\mathcal{A}$ by reordering the differentials. Specifically, if $A$ is an $n$-fold chain complex in $\mathcal{A}$ and $\chi \in \Sigma_n$, we have

$$(\chi \cdot A)_{p_1,\ldots,p_n} = A_{p_{\chi^{-1}(1)},\ldots,p_{\chi^{-1}(n)}}$$

and the differentials are similarly reordered. Let $\chi(p_1,\ldots,p_n)$ be the image of $\chi$ under the blow-up homomorphism $\Sigma_n \to \Sigma_{p_1+\ldots+p_n}$. There is a natural transformation $g_\chi : \text{Tot} \to \text{Tot} \circ \chi$ given in multidegree $(p_1,\ldots,p_n)$ by the sign of $\chi(p_1,\ldots,p_n)$. Composition of these transformations has the same effect as applying the sign associated to the composite permutation, such that $(g_{\chi'} \circ \text{id}_A) \circ g_\chi = g_{\chi' \chi}$ for any pair $\chi, \chi' \in \Sigma_n$.

Lemma 2.22. Let the pointed functor $F : \mathcal{A}^\times \to \text{Ch}(\mathcal{B})$ be given by $F(A^1,\ldots,A^n) = F'(A^1,\ldots,A^i) \otimes F''(A^{i+1},\ldots,A^n)$ for a pair of pointed functors $F' : \mathcal{A}^{\times i} \to \text{Ch}(\mathcal{B})$ and $F'' : \mathcal{A}^{\times n-i} \to \text{Ch}(\mathcal{B})$. Then there is a natural isomorphism

$$F_{\delta}(A^1,\ldots,A^n) \simeq F_{\delta}'(A^1,\ldots,A^i) \otimes F_{\delta}''(A^{i+1},\ldots,A^n).$$

Furthermore, this isomorphism is associative.

Proof. The tensor product $F'(A^1,\ldots,A^i) \otimes F''(A^{i+1},\ldots,A^n)$ is the totalization of a bicomplex, so we can lift $F_{\delta}$ to a functor $F_\epsilon : \text{Ch}_k^{\times n} \to (n+2)-\text{Ch}_k$ such that $\text{Tot} \circ F_\epsilon = F_{\delta}$. As before, the differentials in multidegree $(p_1,\ldots,p_n,q_1,q_2)$ are given by

$$d_i = \begin{cases} 
(1)p_1+\ldots+p_{i-1}dA^i, & 1 \leq i \leq n \vspace{1ex} \\
(1)p_1+\ldots+p_n dF'(A^1,\ldots,A^i), & i = n+1 \\
(1)p_1+\ldots+p_n+q_1 dF''(A^{i+1},\ldots,A^n), & i = n+2
\end{cases}$$

Now the tensor product $F_\epsilon'(A^1,\ldots,A^i) \otimes F_\epsilon''(A^{i+1},\ldots,A^n)$ is obtained by reordering the differentials and totalizing. Specifically, we must pass $d_{n+1}$ past $d_{i+1},\ldots,d_n$, which in multidegree $(p_1,\ldots,p_n,q_1,q_2)$ incurs a sign $(-1)^{q_1(p_{i+1}+\ldots+p_n)}$. The natural isomorphism in the statement of the lemma is thus obtained as the transformation $g_\chi_1$ of Observation 2.21 where $\chi_1$ is the cycle $(i+1,\ldots,n,n+1) \in \Sigma_{n+2}$. The above recipe generalizes readily to a version with more than two tensor factors by replacing $\chi_1$ with the permutation $\chi_{i_1,\ldots,i_m} \in \Sigma_{n+m}$ given by the composition of cycles

$$\chi_{i_1,\ldots,i_m} = (i_1 + \ldots + i_m + 1,\ldots,n+m) \circ \ldots \circ (i_1 + 1,\ldots,n+1)$$

To see that this is associative, it is sufficient to look at the case of three factors:

$$F(A_1,\ldots,A_n) = F^1(A_1,\ldots,A_i) \otimes F^2(A_{i+1},\ldots,A_{i+j}) \otimes F^3(A_{i+j+1},\ldots,A_n)$$

Associativity of the natural isomorphism above now follows from the identity

$$(i+j+2,\ldots,n+1,n+2) \circ (i+1,\ldots,n,n+1) = (i+1,\ldots,n+1,n+2) \circ (i+j+1,\ldots,n+1,n+2)$$

in $\Sigma_{n+2}$. \hfill $\square$

Definition 2.23. Write $\zeta_n$ for the natural isomorphism

$$\zeta_n : (F^1 \otimes \ldots \otimes F^n)_{\delta} \xrightarrow{\sim} F^1_{\delta} \otimes \ldots \otimes F^n_{\delta}$$

of functors $\mathcal{A}^{\times i} \to \text{Ch}(\mathcal{B})$ given by Lemma 2.22.
We therefore get a morphism

\[ A \mapsto C^\vec{k}(A) = C(A^{\otimes k_1}) \otimes \ldots \otimes C(A^{\otimes k_n}) \]

i.e. \( C^\vec{k}(A) = N_\delta B^{cy}(A^{\otimes k_1}) \otimes \ldots \otimes N_\delta B^{cy}(A^{\otimes k_n}) \).

**Definition 2.25.** Recall that the cyclic bar construction \( B^{cy} : \text{dgAlg}_k \to \text{sCh}_k \) is a symmetric monoidal functor. We denote the natural structure isomorphism by

\[ \theta : B^{cy}(A_1) \otimes \ldots \otimes B^{cy}(A_n) \to B^{cy}(A_1 \otimes \ldots \otimes A_n) \]

The isomorphism is given in simplicial degree \( k - 1 \) (in which we have \( nk \) tensor factors) by the permutation \( \chi_{nk} \in \Sigma_{nk} \) sending \( i + dk \) to \( d + 1 + (i - 1)n \) for \( 0 < i \leq k \) and \( 0 \leq d < n \), with a sign like that in the proof of Lemma 2.4.

**Proposition 2.26.** The dg-category \( \bar{N}_E \) acts on Hochschild complexes of dg-algebras. That is, we have natural transformations

\[ \bar{N}_E(\vec{k}, \vec{m}) \otimes C^\vec{k} \to C^{\vec{m}} \]

of functors \( \text{dgAlg}_k \to \text{Ch}_k \) compatible with composition. This action exhibits \( C : \text{dgAlg}_k \to \text{Ch}_k \) as a symmetric monoidal, \( \mathbb{E}_\infty \)-comonoidal functor.

**Proof.** For \( \vec{k} = (k_1, \ldots, k_n) \) denoting

\[ N_\delta \theta_{\vec{k}} = (N_\delta(\theta_{k_1}) \otimes \ldots \otimes N_\delta(\theta_{k_n})) \circ \zeta \]

\[ N_\delta(\Delta_k B^{cy}(A)) \to N_\delta B^{cy}(A^{\otimes k_1}) \otimes \ldots \otimes N_\delta B^{cy}(A^{\otimes k_n}) = C^\vec{k}(A) \]

where \( \Delta_k : \text{sCh} \to \text{sCh}^{\times k} \) is the diagonal functor. If \( f \in \bar{N}_E \), we write \( (f_\delta)_{(A_1, \ldots, A_n)} \) for the component of \( f_\delta \) at the \( n \)-tuple \( (A_1, \ldots, A_n) \) of simplicial chain complexes over \( k \). We now have a composite morphism

\[ C^\vec{k}(A) \xrightarrow{N_\delta(\theta_{\vec{k}})^{-1}} N_\delta^\vec{k}(B^{cy}(A), \ldots, B^{cy}(A)) \]

\[ \xrightarrow{(f_\delta)_{(B^{cy}(A), \ldots, B^{cy}(A))}} N_\delta^\vec{m}(B^{cy}(A), \ldots, B^{cy}(A)) \]

\[ C^{\vec{m}}(A) \xleftarrow{N_\delta(\theta_{\vec{m}})} N_\delta^\vec{m}(B^{cy}(A), \ldots, B^{cy}(A)) \]

We therefore get a morphism

\[ i_{\vec{k}, \vec{m}} : \bar{N}_E(\vec{k}, \vec{m}) \to \text{Nat}(C^\vec{k}(A), C^{\vec{m}}(A)) \]

\[ f \mapsto N_\delta \theta_{\vec{m}} \circ (f_\delta \ast \text{id}_{\Delta_k \circ B^{cy}}) \circ (N_\delta \theta_{\vec{k}})^{-1} \]

whose adjoint is the morphism in the statement of the proposition. The contractibility of \( \bar{N} \) (see Lemma 2.18) now implies that \( C : \text{dgAlg}_k \to \text{Ch}_k \) is \( \mathbb{E}_\infty \)-monoidal and comonoidal. However, since the shuffle maps are strictly symmetric, it is in fact symmetric monoidal as claimed.

**Lemma 2.27.** The images of \( \bar{N}_E(\chi((n), (n))) \) and \( \bar{N}_E(\chi'(n), (n))) \) in \( \text{End}_{\text{Ch}_k}(C((\cdot)^{\otimes n})) \) are disjoint for \( \chi \neq \chi' \).
This proves the claim.

\[H_0(C((-)\otimes n)) = HH_0((-)\otimes n) = (-)^{\otimes n}\]

so the induced action of \(f \in \tilde{N}_X((n),(n))\) on \(H_0(C((-)\otimes n))\) is given by permuting tensor factors. Namely, \(\tilde{N}_X((n),(n)) = \tilde{N}((n),(n))\) acts as the identity since each \(f \in \tilde{N}((n),(n))\) is homotopic to the identity map. Now each \(\tilde{N}_X((n),(n))\) is isomorphic to \(\tilde{N}((n),(n))\), the map given by postcomposition by \(\chi_*\), and it follows that each \(f \in \tilde{N}_X((n),(n))\) is homotopic to \(\chi_*\). In particular, for \(A = k[x]\) we see that \(\chi\) and \(\chi'\) act differently on \(k[x]^{\otimes n} \simeq k[x_1,\ldots,x_n]\). This proves the claim.

3. DG-fattening of props

In this section we will build the fattening functor for dg-props and prove Theorem A. The fattening functor will associate to a dg-prop \(P\) a certain full subcategory of the free symmetric monoidal dg-category on \(P\) and \(\tilde{N}_X\), modulo relations expressing that the Dold-Kan morphisms are natural with respect to the morphisms of \(P\).

**Remark 3.1.** To spell out what Theorem A means, to each dg-prop \(P\), there is a natural homotopy-coherent \(P\)-action on the Hochschild complex of \(\text{Ass} \otimes P\)-algebras. The homotopies that make up the coherencies are encoded in a replacement dg-prop \(\hat{P}\) which strictify the homotopy-coherent \(P\)-action. This strictification is moreover functorial in the prop.

In order to produce the functor \(\widehat{(-)}\), we first construct an auxiliary functor \(Q : \text{dgprop} \to \text{dgCat}^{\otimes}\) landing in symmetric monoidal dg-categories. \(Q\) is constructed using a natural family of generators and relations and will contain \(\widehat{(-)}\) as a full subfunctor, i.e. there will be a natural transformation \(\widehat{(-)} \to Q\) whose components are inclusions of full subcategories.

The following definition describes a way of functorially arranging the entries of an \(n\)-tuple of integers \(\vec{k}\) according to the entries of a vector \(\vec{a}\) of length \(n\), which we use to define the functor \(Q\). Informally one should think of \(\text{Par}_{\vec{k}}(\vec{a})\) as given by arranging the entries of \(\vec{k}\) according to the entries of \(\vec{a}\). Similarly, for a morphism \(\gamma : \vec{a} \to \vec{b}\), one may think of \(\text{Par}_{\vec{k}}(\gamma)\) as the natural transformation whose \((A_1,\ldots,A_k)\)-component equals the \((A_1 \otimes \cdots \otimes A_{k_1},\ldots,A_{k_1+\ldots+k_{i-1}+1} \otimes \cdots \otimes A_k)\)-component of \(\gamma\).

**Definition 3.2.** For each \(n\)-tuple \(\vec{k} = (k_1,\ldots,k_n), k = |\vec{k}|,\) let \(\iota_{\vec{k}} : \text{Mod}^{\times k} \to \text{Mod}^{\times n}\) be the functor taking a \(k\)-tuple \((A_1,\ldots,A_k)\) to the \(n\)-tuple \((B_1,\ldots,B_n)\) where

\[B_i = A_{k_1+\ldots+k_{i-1}+1} \otimes \cdots \otimes A_{k_1+\ldots+k_i}\]

Writing \(N_n\) for the full subcategory of \(\tilde{N}\) on the objects \(\vec{a}\) with \(|\vec{a}| = n\), let

\[\text{Par}_{\vec{k}} : N_n \to N_k\]

be the functor taking \(\vec{a} = (a_1,\ldots,a_i)\) to

\[\text{Par}_{\vec{k}}(\vec{a}) := (k_1 + \cdots + k_{a_1},k_{a_1+1} + \cdots + k_{a_1+a_2},\ldots,k_{a_1+\ldots+a_{i-1}+1} + \cdots + k_n)\]

i.e. the unique vector such that \(N^\vec{a} \circ \iota_{\vec{k}} = N^{\text{Par}_{\vec{k}}(\vec{a})}\). For a morphism \(\gamma : \vec{a} \to \vec{b}\), \(\text{Par}_{\vec{k}}(\gamma)\) is given by \(\gamma \ast \text{id}_{\vec{k}}\). In particular, the following diagram commutes.
Recollection 3.3. We recall some ideas from enriched category theory. Let $\mathcal{V}$ be a co-complete symmetric monoidal category. Then there is a free-forgetful adjunction between $\mathcal{V}$-enriched categories and $\mathcal{V}$-enriched graphs [8, Thm 2.13]. If $\Gamma$ is a $\mathcal{V}$-enriched graph, the free $\mathcal{V}$-enriched category on $\Gamma$ has morphism objects

$$F \Gamma(a, a') = \bigoplus_{n \geq 0} \Gamma(a_n, a') \otimes \Gamma(a_{n-1}, a_n) \otimes \ldots \otimes \Gamma(a, a_1)$$

We call the objects $\Gamma(a_i, a_{i+1})$ a family of generators of a $\mathcal{V}$-enriched category $\mathcal{C}$ if $\mathcal{C}$ is a quotient of $F \Gamma$. In the following, we give a definition of $Q(P)$ in terms of generators and relations.

Before giving the definition of the functor $Q$, we will clarify some technicalities. Using the above free-forgetful adjunction and the monad associated to the commutative prop, we can for a $\mathcal{V}$-graph $\Gamma$ make the free symmetric monoidal $\mathcal{V}$-category $F^{\otimes} \Gamma$, whose objects are given by the free commutative monoid on Ob $\Gamma$. If (Ob $\Gamma, +$) is a commutative monoid, we may quotient $F^{\otimes} \Gamma$ by the relation $a \otimes b \sim (a + b)$ for $a, b \in$ Ob $\Gamma$. We will write $F^{\otimes}_{\text{Ob}} \Gamma$ for this quotient.

The situation we are interested in is the following. Let $\Gamma$ be a $\mathcal{V}$-graph whose objects are equipped with a commutative monoid structure, and $\Gamma(a, b) = C(a, b) \otimes D(a, b)$, where $C$ is a $\mathcal{V}$-category and $D$ is a symmetric monoidal $\mathcal{V}$-category such that there is an inclusion Ob $C \hookrightarrow$ Ob $D$ (we write $C(a, b) = 0$ unless both $a$ and $b$ are in Ob $C$). Then $F^{\otimes}_{\text{Ob}} \Gamma$ contains $FC$ as a subcategory and $F^{\otimes}_{\text{Ob}} D$ as a symmetric monoidal subcategory.

Definition 3.4. Let $C$ and $D$ be as above. By the free symmetric monoidal $\mathcal{V}$-category generated by $C$ and $D$ we mean the quotient of $F^{\otimes}_{\text{Ob}} (\Gamma)$ by the relations defined by the structure maps

$$FC \to C$$

$$F^{\otimes}_{\text{Ob}} D \to D$$

For a symmetric monoidal $\mathcal{V}$-category $\mathcal{E}$ to be generated by $C$ and $D$ we mean that $\mathcal{E}$ is a quotient of the free symmetric monoidal $\mathcal{V}$-category generated by $C$ and $D$.

Definition 3.5. For a dg-prop $P$, let $Q(P)$ be the symmetric monoidal category enriched in bicomplexes whose monoid of objects is the monoid of vectors $\vec{k}$ under concatenation, and is generated as a symmetric monoidal category enriched in bicomplexes by $P$ and $\tilde{N}^{\Sigma}$. Here $(f) \in P(n, m)$ has bidegree $(|f|, 0)$ and $\psi \in \tilde{N}^{\Sigma}(\vec{n}, \vec{m})$ has bidegree $(0, |\psi|)$. The horizontal and vertical differentials act on $P$ and $\tilde{N}^{\Sigma}$ respectively. We write $+$ for the symmetric
monoidal structure of $P$. For each $\vec{a} = (a_1, ..., a_l)$ in $N_n$ and each $n$-tuple $f = (f_1, ..., f_n): \vec{k} \to \vec{m}$ of morphisms in $P$, we have morphisms

$$(k_{a_1+...a_i+1} + ... + k_{a_1+...a_{i+1}}) \xrightarrow{(f_{a_1+...a_i+1}+...+f_{a_1+...a_{i+1}})} (m_{a_1+...a_i+1} + ... + m_{a_1+...a_{i+1}})$$

in $P$, and we write $\text{Par}_f(\vec{a}): \text{Par}_k(\vec{a}) \to \text{Par}_m(\vec{a})$ for the tensor product of these morphisms for $1 \leq i \leq l$.

These generators are subject to the following relations: for each morphism $\gamma: \vec{a} \to \vec{b}$ in $N_n$ and $n$-tuple $f = (f_1, ..., f_n): \vec{k} \to \vec{m}$ of morphisms in $P$, the following diagram commutes:

$$\begin{array}{ccc}
\text{Par}_k(\vec{a}) & \xrightarrow{\text{Par}_f(\vec{a})} & \text{Par}_m(\vec{a}) \\
\downarrow \text{Par}_k(\gamma) & & \downarrow \text{Par}_m(\gamma) \\
\text{Par}_k(\vec{b}) & \xrightarrow{\text{Par}_f(\vec{b})} & \text{Par}_m(\vec{b})
\end{array}$$

For a morphism of dg-props $g: P \to P'$, let $Q(g)$ be the symmetric monoidal functor $Q(P) \to Q(P')$ which is the identity on objects and on generators in $\tilde{N}^{\Sigma}$, and acts by $g$ on the generators in $P$. Note that this assignment preserves the relations.

**Remark 3.6.** The relations imply that if $\vec{k} = (k_1, ..., k_n)$ with $|\vec{k}| = k$, $\vec{m} = (k'_1, ..., k'_n)$ with $|\vec{k}'| = k'$ and $\vec{f} = (f_1, ..., f_n): \vec{k} \to \vec{k}'$ is an $n$-tuple of morphisms in $P$ and $f = f_1 \otimes ... \otimes f_n \in P(k, k')$, the following squares commute.

$$\begin{array}{ccc}
\vec{k} & \xrightarrow{\vec{f}} & \vec{k}' \\
\downarrow \text{sh}_n & & \downarrow \text{sh}_n \\
(k) & \xrightarrow{(f)} & (k')
\end{array} \quad \begin{array}{ccc}
(k) & \xrightarrow{(f)} & (k') \\
\downarrow \text{AW}_n & & \downarrow \text{AW}_n \\
\vec{k} & \xrightarrow{\vec{f}} & \vec{k}'
\end{array}$$

**Definition 3.7.** For $\mathcal{C}$ a category enriched in bicomplexes, let $\text{Tot}(\mathcal{C})$ be the dg-category whose morphism complexes are the $\oplus$-totalization of the morphism bicomplexes in $\mathcal{C}$.

**Lemma 3.8.** There is a natural symmetric monoidal functor $F: \text{Tot}(Q(P)) \to P$ defined on objects by taking $\vec{k}$ to $|\vec{k}|$, and on morphisms by taking $f: (k) \to (m)$ in $P(k, m)$ to $f: k \to m$ and $\gamma: \vec{k} \to \vec{m}$ in $\tilde{N}(\vec{k}, \vec{m})_i$ (if $|\vec{k}| = |\vec{m}|$) to $\text{id}_{|k|}$ if $i = 0$ and 0 otherwise.

**Proof.** It is clear that the assignment is natural in $P$ if it is well-defined, which we now verify. Given a morphism $\gamma: \vec{a} \to \vec{b}$ in $N_n$ and $f = (f_1, ..., f_n): \vec{k} \to \vec{m}$ in $P^{\times n}$, we must verify that the diagram
remains commutative after applying $F$. It is sufficient to assume that $\gamma \in \tilde{N}(\tilde{a}, \tilde{b})_0$. But $F$ takes $\text{Par}_k(\gamma)$ to the identity and $F(\text{Par}_f(\tilde{a})) = F(\text{Par}_f((n))) = F(\text{Par}_f(\tilde{b}))$, so $F$ is well defined. To see that $F$ preserves the differentials on each morphism complex, note that for a general morphism

$$g = \gamma^n \circ f^n \circ ... \circ \gamma^0 \circ f^0$$

the differential is given by

$$dg = (d^0 \gamma^n) \circ f^n \circ ... \circ \gamma^0 \circ f^0 + (-1)^{\gamma^n} \gamma^n \circ (d^f f^n) \circ ... \circ \gamma^0 \circ f^0 + ...$$

$$+ (-1)^{\gamma^n + |f^n| + ... + |\gamma^n|} \gamma^n \circ f^n \circ ... \circ \gamma^0 \circ d^h f^0$$

In the case that $Fg$ is non-zero (i.e. each $|\gamma^n| = 0$) this differential is identical to the differential in $P$. \qed

**Notation 3.9.** We write $(1)^n$ for the vector $(1, ..., 1)$ of length $n$. Note that for any $n$-tuple $f = (f_1, ..., f_n)$ of morphisms in $P$, we have $\text{Par}_f((1)^n) = f$ and $\text{Par}_f(n) = (F(f))$.

**Lemma 3.10.** Let $P$ be a dg-prop and let $\tilde{k}, \tilde{m} \in \text{Ob } Q(P)$. The map

$$\text{Hom}_{\text{Tot}(Q(P))}(\tilde{k}, \tilde{m}) \to \text{Hom}_{P}(k, m)$$

induced by $F$ is a quasi-isomorphism.

**Proof.** Denote by $c_v \text{Hom}_P(k, m)$ the bicomplex which has $\text{Hom}_P(k, m)$ concentrated in vertical degree 0, and consider the map of bicomplexes

$$A : c_v \text{Hom}_P(k, m) \simeq \tilde{N}((m), (m)) \otimes \text{Hom}_P(k, m) \to \text{Hom}_{Q(P)}(\tilde{k}, \tilde{m})$$

taking $\gamma \otimes f$ to $AW_{\tilde{m}} \circ \gamma \circ f \circ \text{sh}_{\tilde{m}}$. We will show that the totalization of $A$ is a quasi-isomorphism and a quasi-inverse to the map induced by $F$ on Hom-complexes. Let $f : \tilde{k} \to \tilde{m}$ with $|f| = (d, d')$ in $Q(P)$ be a composition of generators of $Q(P)$. If for any such $f$, the homology class of $f$ is represented by a composition $AW_{\tilde{m}} \circ \tilde{\gamma} \circ (F(f)) \circ \text{sh}_{\tilde{m}}$, where $\tilde{\gamma} \in \tilde{N}((m), (m))_{d'}$, then the map $A$ above is a quasi-isomorphism after totalizing. Indeed, assume that $f$ is a cycle with respect to the vertical differential. $f$ is given by a sum

$$f = f_1 + ... + f_n \in \text{Hom}_{Q(P)}(\tilde{k}, \tilde{m})_{d,n}$$

where each $f_i$ is a composition of generators of $Q(P)$. We may assume that each $f_i$ has the form $AW_{\tilde{m}} \circ \gamma_i \circ (F(f_i)) \circ \text{sh}_{\tilde{m}}$. Using the contractibility of $\tilde{N}^\Sigma$, we may in fact assume that the $\gamma_i$ are identical, such that $f$ represents the same homology class as

$$AW_{\tilde{m}} \circ \gamma \circ (F(f)) \circ \text{sh}_{\tilde{m}}$$

in vertical homology for some $\gamma \in \tilde{N}^\Sigma((m), (m))$. Now the vertical differential acts only on $\gamma$, which must be a cycle, hence a boundary in $\tilde{N}^\Sigma((m), (m))$ unless $d = 0$, hence this cycle
represents a trivial homology class if \( d > 0 \). In the case \( d = 0 \), \( \gamma = \text{id} \) is a cycle which is not a boundary. It follows that on homology,

\[
H_*(\text{Hom}_{Q(P)}(\tilde{k}, \tilde{m}); d_v) \simeq \text{Hom}_P(k, m).
\]

We get an isomorphism of \( E_1 \)-pages of the spectral sequence for a double complex:

\[
H_*(\tilde{N}^\Sigma((m), (m)) \otimes \text{Hom}_P(k, m); d_v) \xrightarrow{\sim} H_*(\text{Hom}_{Q(P)}(\tilde{k}, \tilde{m}); d_v)
\]

hence \( A \) is a quasi-isomorphism after totalizing. Now, for any \( f \in \text{Hom}_P(k, m) \) we have

\[
F \circ \text{Tot}(A)(f) = F(AW_{\tilde{m}} \circ \gamma \circ (f) \circ \text{sh}_{\tilde{k}}) = f
\]
such that \( F \circ \text{Tot}(A) \) is the identity. The result now follows from the 2-out-of-3 property for quasi-isomorphisms.

In the following, for \( a, b \) elements of a bicomplex \( C \) with \( |a| = |b| = (d, d') \), a vertical homotopy \( h : a \simeq b \) means an element \( h \) of \( C \) with \( |h| = (d, d' + 1) \) such that \( d_v h = b - a \).

We are left to show that each element admits a homotopy to the desired form. This is \textit{a priori} by a sequence of generators

\[
\tilde{k}_{0,0} \xrightarrow{\phi^0} \tilde{k}_{0,1} \xrightarrow{\gamma^0} \tilde{k}_{1,0} \xrightarrow{\phi^1} \ldots \xrightarrow{\gamma^{m-1}} \tilde{k}_{m,0}
\]

where \( \phi^i \in P^{\times n_i} \) and \( \gamma^i \in \tilde{N}(\tilde{k}_{i,1}, \tilde{k}_{i+1,0})_0 \). To begin, we may fix for each \( \gamma^i \) a vertical homotopy \( c(\gamma^i) : \gamma^i \simeq AW_{\tilde{k}_{i+1,0}} \circ \text{sh}_{\tilde{k}_{i,1}} \). Applying the \( c(\gamma^i) \) we obtain a new morphism

\[
g = \tilde{k}_{0,0} \xrightarrow{\phi^0} \tilde{k}_{0,1} \xrightarrow{\text{AW}_{\tilde{k}_{1,0}} \circ \text{sh}_{\tilde{k}_{0,1}}} \tilde{k}_{1,0} \xrightarrow{\phi^1} \ldots \xrightarrow{\text{AW}_{\tilde{k}_{m,0}} \circ \text{sh}_{\tilde{k}_{m-1,1}}} \tilde{k}_{m,0}
\]
equipped with a vertical homotopy \( f \simeq g \). Now repeated application of the relations

\[
\begin{align*}
\tilde{k}_{i,0} & \xrightarrow{\phi^i} \tilde{k}_{i,1} & (k_{i,0}) & \xrightarrow{\text{Par}_{\phi^i}(n_i)} (k_{i,1}) \\
\text{sh}_{\tilde{k}_{i,1}} & & & \\
(k_{i,0}) & \xrightarrow{\text{Par}_{\phi^i}(n_i)} (k_{i,1})
\end{align*}
\]

allows us to rewrite \( g \) as the composition

\[
g = \text{AW}_{\tilde{k}_{m,0}} \circ \text{Par}_{\gamma^{m-1}}(((n_{m-1})) \circ (\text{AW}_{\tilde{k}_{m-1,0}} \circ \text{sh}_{\tilde{k}_{m-1,0}}) \circ \ldots \\
\ldots \circ \text{Par}_{\phi^i}(n_i)) \circ (\text{AW}_{\tilde{k}_{i,0}} \circ \text{sh}_{\tilde{k}_{i,0}}) \circ \text{Par}_{\phi^0}(n_0) \circ \text{sh}_{\tilde{k}_{0,0}}
\]

Now choose vertical homotopies \( \beta_{\tilde{k}_{i,0}} : (\text{AW}_{\tilde{k}_{i,0}} \circ \text{sh}_{\tilde{k}_{i,0}}) \rightarrow \text{id}_{(k_{i,0})} \), giving us a composition of the desired form. This completes the base case.

For \( d' > 0 \), we may again write \( f \) as a sequence of generators

\[
\tilde{k}_{0,0} \xrightarrow{\phi^0} \tilde{k}_{0,1} \xrightarrow{\gamma^0} \tilde{k}_{1,0} \xrightarrow{\phi^1} \ldots \xrightarrow{\gamma^{m-1}} \tilde{k}_{m,0}
\]

where \( \phi^i \in P^{\times n_i} \) and now \( \gamma^i \in \tilde{N}(\tilde{k}_{i,1}, \tilde{k}_{i+1,0})_{d'} \). We now consider two cases. Assume first that \( d'_i < d' \) for each \( i \). Let \( j \) be the least \( i \) such that \( |\gamma^i| > 0 \). By our assumption on the \( d'_i \), \( j < m - 1 \). Write \( f' \) for the composition

\[
\begin{align*}
\tilde{k}_{j,1,0} & \xrightarrow{\gamma^{j+1}} \tilde{k}_{j+1,1} \xrightarrow{\gamma^{j+2}} \ldots \xrightarrow{\gamma^{m-1}} \tilde{k}_{m,0}
\end{align*}
\]
and write $f''$ for the composition
\[ \overline{k}_{0,0} \overset{\phi^0}{\longrightarrow} \overline{k}_{0,1} \overset{\gamma^0}{\longrightarrow} \overline{k}_{1,0} \overset{\phi^1}{\longrightarrow} ... \overset{\gamma^{m-1}}{\longrightarrow} \overline{k}_{j+1,0}. \]

By induction, we may rewrite $f'$ and $f''$ up to homotopy as
\[ f' \simeq AW_{m} \circ \overline{\gamma}' \circ (F(f')) \circ \text{sh}_{k_{j+1,0}} \]
where $\overline{\gamma}' \in \tilde{N}((m), (m))_{d_{j+1} + ... + d_{m-1}}$ and
\[ f'' \simeq AW_{j+1,0} \circ \overline{\gamma}'' \circ (F(f'')) \circ \text{sh}_{\overline{k}} \]
where $\overline{\gamma}'' \in \tilde{N}(([k_{j+1,0}]), (([k_{j+1,0}]))_{d_{j} + ... + d_{j}}$. Hence we get a homotopy
\[ f \simeq AW_{m} \circ \overline{\gamma}' \circ (F(f')) \circ \overline{\gamma}'' \circ (F(f'')) \circ \text{sh}_{\overline{k}} \]
We may now rewrite $(F(f')) \circ \overline{\gamma}'' \simeq \overline{\gamma}'' \circ (F(f'))$ to obtain a composition of the desired form.

Finally, assume that there is a $j$ such that $d_{j} = d'$. If $j = m - 1$, then the result follows from the base case and contractibility of $\tilde{N}((m), (m))$. If $j < m - 1$, we will provide a homotopy between $f$ and another morphism $f'$ for which $d_{j+1} = d'$. By the above, this will finish the argument. We apply a homotopy $\gamma^j \simeq AW_{k_{j+1,0}} \circ \overline{\gamma}^j \circ \text{sh}_{k_{j,0}}$ where $\overline{\gamma}^j \in \tilde{N}([k_{j+1,0}], [k_{j+1,0}])_{d_{j}}$. Note that we may assume that $\gamma^j$ is of the form $\text{Par}_{k_{j,0}}(\gamma^j)$ for a $\gamma^j \in \tilde{N}(([n_{j}]), (n_{j}))$. To name a concrete such element, one can use the (higher) homotopies witnessing $\text{sh}$ and $AW$ as mutual homotopy inverses. Now using the relation
\[ \begin{array}{c}
(k_{j,0}) \xrightarrow{\text{Par}_{\phi^j}(\gamma^j)} (k_{j,1}) \\
\hspace{1cm} \downarrow \text{Par}_{\phi^j}(\gamma^j) \\
(k_{j,0}) \xrightarrow{\text{Par}_{\phi^j}(\gamma^j)} (k_{j,1})
\end{array} \]
We see that we have a composition
\[ f \simeq f' = \overline{k}^j_{0,0} \overset{\phi^0}{\longrightarrow} \overline{k}^j_{0,1} \overset{\gamma^0}{\longrightarrow} \overline{k}^j_{1,0} \overset{\phi^1}{\longrightarrow} ... \overset{\gamma^{m-1}}{\longrightarrow} \overline{k}^j_{m,0} \]
where for $i \neq j + 1$ we have $\overline{k}^j_{i,0} = \overline{k}_{i,0}$, $\overline{k}^j_{i,0} = \overline{k}_{i,0}$, for $i \neq j$ we have $\phi^i = \phi^j$, and for $i \neq j + 1, j$ we have $\gamma^i = \gamma^j$. Finally, $\overline{k}^j_{j+1,0} = (k_{j,0})$, $\overline{k}^j_{j+1,1} = (k_{j+1,1})$, $\phi^j = \text{Par}_{\phi^j}(n_{j})$, $\gamma^j = \text{sh}_{k_{j,1}}$, and $\gamma^{j+1} = \gamma^j \circ AW_{k_{j+1,0}} \circ \text{Par}_{k_{j+1,1}}(\gamma^j)$. We see now that for the composition $f'$, $|\gamma^{j+1}| = d'$, and this finishes the argument. \[ \square \]

Recall that for a prop $P$, a $P$-algebra is a symmetric monoidal functor $\Phi : P \rightarrow Ch_k$ and a $\mathcal{A}ss \otimes P$-algebra is the same as a symmetric monoidal functor $P \rightarrow \text{dgAlg}_k$.

**Lemma 3.11.** The functor $\text{Tot}(Q(-)) : \text{dgprop} \rightarrow \text{dgCat}^\otimes$ has the property that there is a natural transformation of functors $\text{dgprop}^{op} \rightarrow \text{Cat}$
\[
\alpha : \text{Fun}^\otimes(-, \text{dgAlg}_k) \rightarrow \text{Fun}^\otimes(\text{Tot}(Q(-)), Ch_k)
\]
such that $\alpha_P(\Phi)(1) = C(\Phi(1))$. 

Proof. We divide the proof into several steps. First we construct the functors $\alpha_P(\Phi)$. Then we show functoriality in $\Phi$. Finally we will show naturality in $P$.

**Step 1:** Constructing $\alpha_P(\Phi)$.

Let $P$ be a dg-prop, and let $\Phi : P \to \text{dgAlg}_k$ be a symmetric monoidal functor. We will produce a symmetric monoidal functor $\alpha_P(\Phi) : \text{Tot}(Q(P)) \to \text{Ch}_k$ that sends $\vec{k}$ to $C^k(\Phi(1))$. Throughout this section of the proof, we write $A = \Phi(1)$ for ease of notation. We describe the functoriality of $\alpha_P(\Phi)$ in terms of the generators of $\text{Tot}(Q(P))$. If $f : n \to m$ is in $P$, then $(f)$ acts by

$$C^n(A) \simeq C(\Phi(n)) \xrightarrow{C(f)} C(\Phi(m)) \simeq C^m(A)$$

Furthermore $\tilde{N}$ acts according to Proposition 2.26. This determines how the generators of $\text{Tot}(Q(P))$ act. Now let $f = (f_1, \ldots, f_n) : \vec{k} \to \vec{m}$ be an $n$-tuple of morphisms in $P$ and let $\gamma : \vec{a} \to \vec{b}$ be a morphism in $N_n$. We are left to verify that the relations

$$\text{Par}_{\vec{k}}(\vec{a}) \text{Par}_{\vec{m}}(\vec{a})$$

are preserved by the action. Now there is an isomorphism (see Proposition 2.26)

$$\alpha_P(\Phi)(\text{Par}_{\vec{k}}(\vec{a})) = C^{\text{Par}_{\vec{k}}(\vec{a})}(A) \xrightarrow{N\theta_{\vec{a}}^{-1}} N\vec{a}(B^{cy}(A^k_1), \ldots, B^{cy}(A^k_n))$$

Write $N\vec{a}(B^{cy}(A^k))$ for the latter. Consider the following diagrams:

$$\gamma_{B^{cy}(A^k)} \downarrow \quad \gamma_{B^{cy}(A^m)} \quad N\theta_{\vec{a}}^{-1} \quad N\theta_{\vec{a}}^{-1}$$

The left diagram commutes by the definition of $\tilde{N}$, while the right diagram commutes by the naturality of the symmetric monoidal structure maps of $B^{cy}$. Finally, observe that

$$\alpha_P(\Phi)(\text{Par}_{\vec{k}}(\gamma)) = N\theta_{\vec{a}} \circ \gamma_{B^{cy}(A^k)} \circ N\theta_{\vec{a}}^{-1}$$

Together these facts imply that the relations in $\text{Tot}(Q(P))$ are preserved by the action, such that $\alpha_P(\Phi)$ is a functor. Then by the definition of $C^k$ it is clearly symmetric monoidal.

**Step 2:** Showing that $\alpha_P$ is a functor.

We will notationally identify an object in $\text{Fun}^\otimes(\text{Ass} \otimes P, \text{Ch}_k)$ with its value at 1. Let $\phi : A \to B$ be a morphism in $\text{Fun}^\otimes(P, \text{dgAlg}_k)$. We will produce a natural transformation
\( \alpha_P(A) \to \alpha_P(B) \). The component at \( \vec{l} \in Q(P) \) is given by applying \( \phi_1 : A \to B \) component-wise, i.e. \( \alpha_P(\phi)_{\vec{l}} = C^\vec{l}(\phi_1) : C^\vec{l}(A) \to C^\vec{l}(B) \). It is sufficient to check naturality against the generators of \( Q(P) \). If \( f : n \to m \) is a morphism in \( P \), then the following diagram commutes because it commutes before applying \( C(\cdot) \).

\[
\begin{array}{ccc}
C(A^{\otimes n}) & \xrightarrow{C(\phi_n)} & C(B^{\otimes n}) \\
C(f_A) & & C(f_B) \\
C(A^{\otimes m}) & \xrightarrow{C(\phi_m)} & C(B^{\otimes m})
\end{array}
\]

Let \( \gamma : \vec{k} \to \vec{m} \) be a morphism in \( \widetilde{N} \) and consider the following diagrams:

\[
\begin{array}{ccc}
C^\vec{k}(A) & \xrightarrow{N^\vec{k}_{\vec{k}}^{-1}} & N^\vec{k}(B^{cy}(A)) \\
C^\vec{k}(\phi) & & N^\vec{k}(B^{cy}(\phi)) \\
C^\vec{k}(B) & \xrightarrow{N^\vec{k}_{\vec{k}}^{-1}} & N^\vec{k}(B^{cy}(B))
\end{array}
\quad
\begin{array}{ccc}
N^\vec{k}(B^{cy}(A)) & \xrightarrow{N^\vec{k}(B^{cy}(\phi))} & N^\vec{k}(B^{cy}(\phi)) \\
N^\vec{k}(B^{cy}(\vec{m})) & \xrightarrow{N^\vec{k}(B^{cy}(\phi))} & N^\vec{k}(B^{cy}(\vec{m}))
\end{array}
\]

The left diagram commutes by naturality of the symmetric monoidal structure maps of \( B^{cy} \) and the right diagram commutes by the definition of \( \widetilde{N} \). Since \( \gamma \) acts by

\[
\alpha_P(A)(\gamma) = N\theta_m \circ N^\gamma(B^{cy}(A)) \circ N\theta_{\vec{k}}^{-1}
\]

the commutativity of these two families of diagrams implies naturality with respect to the morphisms in \( \tilde{N}^{\Sigma} \).

**Step 3:** Showing that \( \alpha \) is natural in \( P \).

Let \( i : P \to P' \) be a morphism of dg-props. We need to check commutativity of the diagram

\[
\begin{array}{ccc}
\text{Fun}^{\otimes}(P', \text{dgAlg}_k) & \xrightarrow{\alpha_{P'}} & \text{Fun}^{\otimes}(\text{Tot}(Q(P')), \text{Ch}_k) \\
i^* & & \text{Tot}(Q(i))^* \\
\text{Fun}^{\otimes}(P, \text{dgAlg}_k) & \xrightarrow{\alpha_P} & \text{Fun}^{\otimes}(\text{Tot}(Q(P)), \text{Ch}_k)
\end{array}
\]

Let \( \Phi : P' \to \text{dgAlg}_k \) be a symmetric monoidal functor. We first show that the functors \( \alpha_P(i^*\Phi) \) and \( \text{Tot}(Q(i))^*\alpha_{P'}(\Phi) \) are equal. Since \( i \) and \( \text{Tot}(Q(i)) \) are isomorphisms on objects, we have

\[
\alpha_P(i^*\Phi)(\vec{k}) = C^\vec{k}(i^*\Phi(1)) = C^\vec{k}(\Phi(1))
\]

and

\[
\text{Tot}(Q(i))^*\alpha_{P'}(\Phi)(\vec{k}) = \alpha_{P'}(\Phi)(\vec{k}) = C^\vec{k}(\Phi(1))
\]
so they are equal on objects. Let $\gamma : \tilde{k} \to \tilde{m}$ be a morphism in $\tilde{N}^\Sigma$. Since $\text{Tot}(Q(i))$ is the identity on $\tilde{N}^\Sigma$, we similarly have

$$\text{Tot}(Q(i))^* \alpha_{P'}(\Phi)(\gamma) = \alpha_{P'}(\Phi)(\gamma) = N\theta_m \circ N^\gamma(B^w(\Phi(1))) \circ N\theta_k^{-1}$$

and

$$\alpha_P(i^*\Phi)(\gamma) = N\theta_m \circ N^\gamma(B^w(i^*\Phi(1))) \circ N\theta_k^{-1} = N\theta_m \circ N^\gamma(B^w(\Phi(1))) \circ N\theta_k^{-1}$$

so the action of $\tilde{N}^\Sigma$ coincides as well. We now compare the action by a morphism $f : k \to m$ in $P$. We have

$$\alpha_P(i^*\Phi)(f) : C(k)(\Phi(1)) \simeq C(\Phi(k)) \overset{C(id_k)}{\longrightarrow} C(\Phi(m)) \simeq C(m)(\Phi(1)).$$

Notice that $\alpha_P(i^*\Phi)(f) = \alpha_{P'}(\Phi)(i(f))$. Now since $\text{Tot}(Q(i))(f) = i(f)$ in $\text{Tot}(Q(P'))$, we have

$$\text{Tot}(Q(i))^* \alpha_{P'}(\Phi)(f) = \alpha_{P'}(\Phi)(i(f))$$

so the two functors coincide on objects.

Before we verify that the functors also agree on morphisms, we recall a basic fact about compositions of natural transformations. If $\mathcal{C}, \mathcal{C}', \mathcal{D}$ are categories, $j : \mathcal{C} \to \mathcal{C}'$ is a functor and $\alpha : F \Rightarrow G : \mathcal{C}' \to \mathcal{D}$ is a natural transformation, then the pullback of $\alpha$ along $j$ is given componentwise by $(\alpha \ast \text{id}_{j(\cdot)})_c = \alpha_{j(c)}$.

For a morphism: $\psi : \Phi \to \Psi$, we have the natural transformations

$$\alpha_P(i^*\psi) : \alpha_P(i^*\Phi) \to \alpha_P(i^*\Psi)$$

and

$$\text{Tot}(Q(i))^* \alpha_{P'}(\psi) : \text{Tot}(Q(i))^* \alpha_{P'}(\Phi) \to \text{Tot}(Q(i))^* \alpha_{P'}(\Psi)$$

of functors $\text{Tot}(Q(P)) \to \text{Ch}_k$. It is sufficient to check that they coincide on components. Let $\tilde{k}$ be an object of $\text{Tot}(Q(P))$. Then since $i$ is an isomorphism on objects, we get

$$\alpha_P(i^*\psi)(\tilde{k}) = C_{\tilde{k}}(i^*\psi(1)) = C_{\tilde{k}}(\psi(1))$$

and

$$\text{Tot}(Q(i))^* \alpha_{P'}(\psi)(\tilde{k}) = \alpha_{P'}(\psi)(\tilde{k}) = C_{\tilde{k}}(\psi(1))$$

so they are equal. \hfill \square

**Proof of Theorem A:** Define $\overline{(-)} : \text{dgprop} \to \text{dgprop}$ to be the functor taking a dg-prop $P$ to the full subcategory of $\text{Tot}(Q(P))$ generated by the objects $\{(1)^n\}_{n \geq 0}$. To see that this defines a functor, recall from Definition 3.3 that for a morphism of dg-props $P \to P'$, the induced symmetric monoidal functor $\text{Tot}(Q(P)) \to \text{Tot}(Q(P'))$ is the identity on object monoids, hence it restricts to a prop morphism $\overline{P} \to \overline{P}'$.

The natural quasi-equivalence $\overline{(-)} \to \text{id}.$

Let $F|_{\overline{P}} : \overline{P} \to P$ be the composition

$$\overline{P} \hookrightarrow \text{Tot}(Q(P)) \xrightarrow{F} P$$

It is clear that $F|_{\overline{P}}$ induces an isomorphism on object monoids. By Lemma 3.10, $F|_{\overline{P}}$ also induces quasi-isomorphisms on Hom-complexes, hence it is a quasi-equivalence. Naturality of $F$ and the inclusion $\overline{P} \to \text{Tot}(Q(P))$ imply that $F|_{\overline{P}}$ is a natural quasi-equivalence.
The natural transformation $\tilde{\alpha}$.

To produce the natural transformation $\tilde{\alpha}$, we use the transformation $\alpha$ from Lemma 3.11. Recall that there is an equivalence of categories
\[
\text{Fun}^\otimes(\text{Ass} \otimes P, \text{Ch}_k) \simeq \text{Fun}^\otimes(P, \text{dgAlg}_k).
\]
The natural inclusion $i : (-) \to \text{Tot}(Q(-))$ gives us a natural transformation
\[
i^* : \text{Fun}^\otimes(\text{Tot}(Q(-)), \text{Ch}_k) \to \text{Fun}^\otimes((-), \text{Ch}_k)
\]
and we define $\tilde{\alpha}$ to be the composition
\[
\tilde{\alpha} = i^* \circ \alpha : \text{Fun}^\otimes(\text{Ass} \otimes -, \text{Ch}_k) \to \text{Fun}^\otimes((-), \text{Ch}_k).
\]
Because $\tilde{\alpha}$ is a restriction of $\alpha$, we have that for any prop $P$, and symmetric monoidal functor $\Phi : \text{Ass} \otimes P \to \text{Ch}_k$, there is an equality $\tilde{\alpha}_P(\Phi)(1) = \alpha_P(\Phi)(1) = C(\Phi(1))$, hence $\tilde{\alpha}$ has the stated properties.

Example 3.12. Consider the example $P = \mathcal{CH}\text{opf}$, the prop encoding a commutative Hopf algebra structure. Note that every morphism in $\mathcal{CH}\text{opf}$ is an algebra homomorphism, hence we have an equivalence $\text{Ass} \otimes \mathcal{CH}\text{opf} \simeq \mathcal{CH}\text{opf}$ and Theorem A gives a recipe for the natural coherent commutative Hopf algebra structure on Hochschild chains of commutative Hopf algebras. In particular, $\mathcal{CH}\text{opf}$ is generated in degree 0 by the morphisms
\[
(\alpha) \xrightarrow{\eta} (1)
\]
\[
(1,1) \xrightarrow{\text{sh}} (2) \xrightarrow{m} (1)
\]
\[
(1) \xrightarrow{\Delta} (1,1)
\]
\[
(1) \xrightarrow{\epsilon} (1)
\]
An example of a generator in degree 1 is the bialgebra relation, in which we need the homotopy $\theta \in \tilde{N}^\Sigma((2,2),(2,2))_1$ to interpolate between the upper and lower legs of the diagram. Here $F \in \Sigma_4$ is the transposition $(2,3)$. 
In a similar way, we need the contraction \( \alpha_{(1,1)} : AW \circ \text{sh} \simeq \text{id} \in \tilde{N}^\Sigma((1,1),(1,1)) \) for the antipode diagrams.

Note that \( \tilde{C\text{Hopf}} \) still has a strictly commutative multiplication. If \( C\mathbb{E}_n\text{Hopf} \) encodes commutative and \( \mathbb{E}_n \)-cocommutative Hopf algebras, then \( C\tilde{\mathbb{E}_n\text{Hopf}} \) will also be \( \mathbb{E}_n \) cocommutative for \( n \leq \infty \), but if \( C\text{Hopf} \) is the prop encoding a Hopf algebra structure which is both commutative and cocommutative, then \( C\tilde{\text{Hopf}} \) is strictly commutative but only \( \mathbb{E}_\infty \)-cocommutative, since \( AW \) is not a symmetric monoidal transformation.

References

[1] Morten Brun, Zbigniew Fiedorowicz, and Rainer M. Vogt, *On the multiplicative structure of topological Hochschild homology*, Algebr. Geom. Topol., 7:16331650, 2007

[2] Albrecht Dold, *Über die Steenrodschen Kohomologieoperationen*, Ann. of Math. 73, 1961, 258-294.

[3] Benoit Fresse, *Props in model categories and homotopy invariance of structures*, arXiv:0812.2738v4, 5 Dec 2008.

[4] Birgit Richter, \( \mathbb{E}_\infty \)-structure for \( Q_*(R) \), Math. Ann. 316, 547-564 (2000).

[5] Birgit Richter, *Symmetry Properties of the Dold-Kan Correspondence*, Mathematical Proceedings of the Cambridge Philosophical Society, 134(1), pp. 95102, 2003.

[6] Nathalie Wahl and Craig Westerland, *Hochschild homology of structured algebras*, Advances in Math. 288 (2016), 240-307.

[7] Nathalie Wahl, *Universal operations in Hochschild homology*, J. Reine Angew. Math., to appear, 2012.

[8] Harvey Wolff, *V-cat and V-graph*, J. Pure Appl. Algebra 4 (1974), 123135.