On the Gap Between Decentralized and Centralized Coded Caching Schemes

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Abstract

Caching is a promising solution to satisfy the ongoing explosive demands for multi-media traffics. Recently, Maddah-Ali and Niesen proposed both centralized and decentralized coded caching schemes, which is able to attain significant performance gains over the uncoded caching schemes. Their work indicates that there exists performance gap between the decentralized coded caching scheme and the centralized coded caching scheme. In this paper, we focus on the gap between the performances of the decentralized and centralized coded caching schemes. Most notably, we strictly prove that the multiplicative gap \(r\), the ratio of their performances, is between 1 and 1.5. The upper bound tightens the original one of 12 by Maddah Ali and Niesen, while the lower bound verifies the intuition that the centralized coded caching scheme always outperforms its decentralized counterpart. In particular, both the two bounds are achievable in some cases. Furthermore, we prove that the gap can be arbitrarily close to 1 if the number of the users is large enough, which suggests the great potential in practical applications to use the less optimal but more practical decentralized coded caching scheme.

Index Terms

Centralized, Decentralized, Coded Caching, Gap, Content Delivery Networks

I. INTRODUCTION

With the dramatic increasing demands for video streams in many applications such as Youtube and Netflix, a huge burden is placed on the underlying networks that deliver the streaming data to users. One promising technique to mitigate this burden is caching [3], whereby relatively popular contents are prefetched at the local cache memories that are accessible by the end users. Based on the cached contents, the server sends a signal when the related files are requested by the users and each user decodes its requested file from the signal as well as the cached content in its cache. With the help of the content in the caches, the amount of the transmission is reduced.

Usually, the caching network operates in two distinct phases: In the placement phase, some fractions of the content are disseminated to all the users’ caches in off peak times. It should be noted that the operations in placement phase do not depend on the actual user requests, since the server does not know the requests at this moment. In the delivery phase that happens in peak time, the user requests are uncovered. The server responds by sending a signal over the shared link to satisfy the user requests. In conventional caching design, the key idea is to deliver part of the content to caches close to the end users, so that the requested content can be served locally. Coding in the cached content or/and the transmitted signal was typically not considered. Recently, in their seminal work [1], [2], Maddah-Ali and Niesen showed that, by exploiting the cached content in the users, multicasting opportunities can be created and the central server is able to send distinct content to different users by encoding the cached content or/and the delivered signal. This type of technique is termed coded caching.

The proposed coded caching schemes in [2] and [1] are essentially centralized and decentralized respectively. In the centralized coded caching scheme [2], the placement phase should be coordinated by a central sever carefully so that the content in different caches overlap to create coded-multicasting opportunities among users with different demands. In the delivery phase, once receiving the requests of users, the server sends a predesignated signal that fully exploits the coded-multicasting opportunities. Whereas, in the decentralized coded caching scheme [1], the placement phase can be accomplished...
with an unknown number of users situated in isolated networks. Accordingly, the cached contents of the distinct users are independent with each other. Later in the delivery phase, after acquiring the information about the set of users, their cached contents and requests, the server sends a signal that efficiently exploits the coded-multicasting opportunities existed in the contents of the end users.

Obviously, an advantage of the decentralized coded caching scheme is that it extends the application of coded caching. In real networks, the centralized coordination may not be available in many cases. For example, the server may not have the knowledge of the identity or even just the number of active users in advance, or the users may be in distinct networks during the placement phases. Whereas, the decentralized coded caching scheme can be applied to such circumstances, as the scenarios with users’ nonuniform demands [6], online scenarios [5], hierarchical structured networks [4] due to its flexibility. Notably, Maddah-Ali and Niesen proved in [1] that, the performance of decentralized coded caching scheme is within a constant multiplicative gap of its centralized counterpart, i.e., the ratio of their performances does not exceed 12. Moreover, they claimed that the gap can be tighten to 1.6 numerically for any system parameters.

In this paper, we investigate the aforementioned performance gap. Specifically, we prove that the gap is between 1 and 1.5 for any system parameter setup theoretically. The upper bound tightens the proved bound 12 and justifies the claimed numerical bound 1.6 by Maddah-Ali and Niesen. The lower bound verifies that the decentralized scheme is always inferior to its centralized counterpart. Though the later conclusion seems intuitively correct, the theoretical justification has not been found to the best knowledge of the authors. Furthermore, we show that in case of large number of users, the gap always approaches 1. This indicates that in large systems, the decentralized coded caching scheme can be utilized in terms of both the achieved performance and realization requirement.

The remainder of this paper is organized as follows: In Section II, we review the network model in [1], [2] and the corresponding schemes and performances. In Section III, we present the main result. Section IV gives the proof of the main result. Finally, we conclude this paper in Section V.

II. NETWORK MODEL AND CODED CACHING SCHEMES

We review the model studied by Ali and Niesen in [1], [2] firstly, and then present their schemes and results in this section.

A. Network Model

Consider a system with a server caching $N$ popular files $W_1, W_2, \cdots, W_N$, each of which is of size $F$ bits. The server is connected to $K$ ($K \geq 2$) users $\mathcal{K}$ through an error-free shared (bottleneck) link, where $\mathcal{K} = \{1, 2, \cdots, K\}$ is the set of users. Each user $k$ is provisioned with an isolated cache $Z_k$ with size $MF$ bits, where $M \in \{0, N\}$ \footnote{For notational simplicity, we consider $M \in \{0, N\}$ and $K \geq 2$. It’s trivial to include the case $M = 0$ or/and $K = 1$ in the result.}. The system is illustrated in Fig. I.

The system operates in two phases:

1. In the placement phase, the users are able to access the whole files $W_1, W_2, \cdots, W_N$ at the server. Then each user $k$ fills its cache $Z_k$ with some content relevant to the files. The only constraint is that, the size of the content should not be more than its cache size $MF$ bits.

2. In the delivery phase, each user $k$ requests $W_{d_k}$ from the server where $k \in \mathcal{K}$ and $d_k \in \{1, 2, \cdots, N\}$. Then the server responds by sending a signal of $R^{(d_1, \cdots, d_K)} F$ bits to the users through the shared link. Then each user $k$ decodes its requested file $W_{d_k}$ from the received signal as well as the pre-stored content in its cache $Z_k$. Specifically, $R^{(d_1, \cdots, d_K)}$ is called the rate of the system under the request $(d_1, \cdots, d_K)$.

Generally speaking, given the parameters $M, N, K$ and the request $(d_1, \cdots, d_K)$, $R^{(d_1, \cdots, d_K)}$ should be as small as possible. Let

$$R \triangleq \max_{d_1, \cdots, d_K} R^{(d_1, \cdots, d_K)}$$
Fig. 1: A central server with $N$ popular files connected to $K$ users through an error-free shared link. Each file is of $F$ bits and each user is equipped with a cache of size $MF$ bits. In the figure, $N = K = 3$, $M = 1$.

be the worst-case normalized rate for a caching scheme. Therefore, the objective is to design schemes to minimize $R$ in the literature.

B. Ali and Niesen’s Coded Caching Schemes

Ali and Niesen first investigated the above model and proposed a centralized and decentralized coded caching scheme in [2] and [1], respectively. Both schemes are proved to outperform the conventional uncoded scheme, whose rate is given by [2]

$$R_U(M) = K \left(1 - \frac{M}{N}\right) \cdot \min \left\{ \frac{1}{1 + KM/N}, \frac{N}{K} \right\}$$

(1)

In the centralized scheme, the central server is able to jointly design the content placement in the caches $Z_1, Z_2, \cdots, Z_K$. While in a decentralized coded caching scheme, the content in caches $Z_1, Z_2, \cdots, Z_K$ should be independent of each other. As mentioned in Section I, since the decentralized scheme is easier to be implemented compared to the centralized scheme, it is worth clarifying the performance difference between them.

Theorem 1 elaborates the performance of the centralized scheme in [2].

Theorem 1. (Centralized Coded Caching [2]) For $N \in \mathbb{N}^+$ files and $K \in \mathbb{N}^+$ users each with cache of size $M = s \cdot N/K$, $s \in \{0, 1, \cdots, K\}$, the centralized coded caching scheme achieves a rate

$$R_C(M) \triangleq K \left(1 - \frac{M}{N}\right) \cdot \min \left\{ \frac{1}{1 + KM/N}, \frac{N}{K} \right\}$$

$$= (K - s) \cdot \min \left\{ \frac{1}{1 + s}, \frac{N}{K} \right\}$$

(2)

For general $0 < M \leq N$, the lower convex envelop of these points is achievable, i.e.,

$$R_C(M) = \theta R_C((s - 1)N/K) + (1 - \theta)R_C(sN/K)$$

(3)

where $s \in \{1, 2, \cdots, K\}$ and $\theta \in [0, 1)$ satisfy

$$M = \theta \cdot (s - 1)\frac{N}{K} + (1 - \theta) \cdot s \frac{N}{K}$$

For $M \in \{0, N/K, 2N/K, \cdots, N\}$, Algorithm [1] illustrates the coded caching scheme proposed in [2], where $\oplus$ in Line 14 is bit-wise XOR operation. It was shown in [2] that Algorithm [1] achieves the following rate

$$K \left(1 - \frac{M}{N}\right) \cdot \frac{1}{1 + KM/N}$$

(4)

To simplify the notations, the symbols $N, K$ do not appear in notations of achievable rates in [1], [2] and [3].
Comparing (1) and (4) with (2), we see that the rate in (2) is a result of combining Algorithm 1 and the conventional uncoded caching scheme (i.e., the server always chooses the approach that leads to smaller rate between Algorithm 1 and the conventional uncoded scheme). For more general \( M \in [0, N] \), any point on the lower convex envelop given by (3) is also achievable by employing so-called memory sharing technique (2).

Algorithm 1 Centralized Coded Caching Scheme

1: procedure \( \text{Placement}(W_1, \cdots, W_N) \)
2: \( t \leftarrow \frac{KM}{N} \)
3: \( \mathcal{T} \leftarrow \{ T \subset \mathcal{K} : |T| = t \} \)
4: for \( n \in \{1, \cdots, N\} \) do
5: \( \text{Split } W_n \text{ into } \{W_n, T : T \in \mathcal{T}\} \text{ of equal size} \)
6: end for
7: for \( k \in \mathcal{K} \) do
8: \( Z_k \leftarrow \{W_n, T : n \in \{1, \cdots, N\}, T \in \mathcal{T}, k \in T\} \)
9: end for
10: end procedure

11: procedure \( \text{Delivery}(W_1, \cdots, W_N, d_1, \cdots, d_K) \)
12: \( t \leftarrow \frac{KM}{N} \)
13: \( \mathcal{S} \leftarrow \{S \subset \mathcal{K} : |S| = t + 1\} \)
14: Server sends \( \{\oplus_{k \in S} W_{d_k, S\setminus\{k\}} : S \in \mathcal{S}\} \)
15: end procedure

Theorem 2 depicts the performance of the decentralized coded caching scheme in (1).

Theorem 2. (Decentralized Coded Caching (1)) For \( N \in \mathbb{N}^+ \) files and \( K \in \mathbb{N}^+ \) users each with cache of size \( M \in (0, N] \), the decentralized coded caching scheme in (1) achieves the following rate

\[
R_D(M) \triangleq K \cdot \left(1 - \frac{M}{N}\right) \cdot \min \left\{ \frac{N}{KM} \left(1 - \left(1 - \frac{M}{N}\right)^K\right), \frac{N}{K}\right\}
\]  
(5)

The rate in (5) can be achieved by employing Algorithm 2 where \( V_{k, S} \) in Line 9 corresponds to the bits of file \( W_{d_k} \) that are present in the cache of every user in \( S \) and absent in the cache of the users outside \( S \), and all the elements \( V_{k, S\setminus\{k\}} \) are assumed to be zero padded to the length of the longest element. Algorithm 2 composes of two delivery procedures, where Delivery1 and Delivery2 correspond to the first and second item in the braces of (5) respectively. In fact, it prefers to the procedure with smaller rate.

In (1), the authors proved that for any \( K, N \in \mathbb{N}^+, M \in (0, N] \),

\[
\frac{R_D(M)}{R_C(M)} \leq 12
\]

Specifically, the authors also showed via numerical experiment that

\[
\frac{R_D(M)}{R_C(M)} \leq 1.6
\]

III. MAIN RESULTS

In this section, we first introduce the primary result of this paper in Theorem 3.

Theorem 3. For \( N \in \mathbb{N}^+ \) files and \( K \geq 2 \in \mathbb{N}^+ \) users each with a cache of size \( M \in (0, N] \), the ratio between the rates achieved by the decentralized coded caching scheme in (1) and the centralized coded caching scheme in (2) can be bounded
Algorithm 2 Decentralized Coded Caching Scheme

1: procedure PLACEMENT($W_1, W_2, \cdots, W_N$)
2:   for $k \in K, n \in \{1, \cdots, N\}$ do
3:     User $k$ independently caches a subset of $MF/N$ bits of file $W_n$, chosen uniformly at random
4:   end for
5: end procedure

6: procedure DELIVERY1($W_1, \cdots, W_N, d_1, \cdots, d_K$)
7:   for $s = K, K-1, \cdots, 1$ do
8:     for $S \subseteq K : |S| = s$ do
9:       Server sends $\oplus_{k \in S} V_{k,S \setminus \{k\}}$
10:   end for
11: end for
12: end procedure

13: procedure DELIVERY2($W_1, \cdots, W_N, d_1, \cdots, d_K$)
14:   for $n \in \{1, \cdots, N\}$ do
15:     Server sends enough random linear combinations of bits in file $W_n$ for all users requesting it to decode
16:   end for
17: end procedure

by:

$$1 \leq \frac{R_D(M)}{R_C(M)} \leq 1.5$$  \hspace{1cm} (6)

Moreover, for any fixed $N \in \mathbb{N}^+$ files and cache size $M \in (0, N]$, 

$$\lim_{K \to \infty} \frac{R_D(M)}{R_C(M)} = 1$$  \hspace{1cm} (7)

Firstly, the lower bound in (6) indicates that for any $K, N \in \mathbb{N}^+, M \in (0, N]$, 

$$R_D(M) \geq R_C(M)$$

Naturally, since all users’ caches are coordinated by the central server jointly, the centralized coded caching scheme can always outperform the decentralized coded caching scheme. Though this fact is intuitively correct, the theoretical justification is not trivial.

Secondly, the upper bound of 1.5 in (6) provides a tighter upper estimate of 12 in [1]. Actually, the upper bound is even tighter than the numerical experimental bound of 1.6 in [1]. Indeed, this result indicates that the performance of decentralized scheme is very close to that of the centralized scheme for any system parameters. Furthermore, it is shown in Section IV that

- The lower bound is achieved whenever $\lceil KM/N \rceil < K/N - 1$ (see (24));
- The upper bound is achieved when $K = 2, N \geq 2$ and $M = N/2$ (see (25));

Thus, both the lower bound and the upper bound in (6) are tight.

Thirdly, (7) depicts the case of large number of users. It is an essentially surprising result, since it says that the rate of the decentralized coded caching scheme approaches the rate of the centralized coded caching scheme, for any fixed cache size $M$ and number of files $N$. In other words, for a system with large number of users, decentralized coded caching scheme is able to achieve almost the same rate with the centralized coded caching. Thus in cases where the centralized coded caching scheme can not be implemented, it is reasonable to be replaced by the decentralized counterpart.
Finally, let us demonstrate some numerical values to gain intuition about $R_D(M)$ and $R_C(M)$. Fig. 2 presents some typical curves of $R_C(M)$, $R_D(M)$ as well as the rate $R_U(M)$ of conventional scheme as a function of $M$ for different $K, N$. In Fig. 2(a), $K = 10, N = 20$, satisfying $K \leq N$. In this case, for any $M \in \{0, N/K, \cdots, N\}$, $R_C(M)$ (or $R_D(M)$) is evaluated by the first item in the braces of (2) (or (5)). For general $0 \leq M \leq N$, $R_C(M)$ is a piecewise linear function by connecting the points

$$(M, R_C(M)) = \left(\frac{s}{N} M - s, 1 + s\right), \ s \in \{0, 1, \cdots, K\}$$

whose curve is below the curve of $R_D(M)$ as follows

$$\frac{1 - M/N}{M/N} \left(1 - \left(1 - \frac{M}{N}\right)^K\right)$$

In Fig. 2(b), $K = 15, N = 10$, satisfying $N < K < 2N$, in this case, the second item in (2) is only chosen when $s = 0$ (i.e., $M = 0$). Therefore, $R_C(M)$ is given by connecting the point $(0, N)$ and the points in (8) with $s \in \{1, 2, \cdots, K\}$. While $R_D(M)$ is given by the curve in (9) truncated by the curve (corresponding to the second item in the braces of (5))

$$N - M$$

at some point $M \in (0, 1)$.

In Fig. 2(c), $K = 20, N = 10$, satisfying $K \geq 2N$. In this case, the second item in (2) is chosen when $M \in \{0, N/K, \cdots, s^*N/K\}$, where $s^* = \lfloor K/N - 1 \rfloor$. Thus, $R_C(M)$ is given by connecting the point $(0, N)$, $(M, N - M) (0 \leq M \leq s^*N/K)$ and the points in (8) with $s \in \{s^* + 1, \cdots, K\}$. While $R_D(M)$ is similar as illustrated in Fig. 2(b). Note that, $R_C(M)$ coincides with (10) for $M \in (0, s^*N/K)$ in this case.

**IV. PROOFS OF THEOREM 3**

**A. Preliminaries**

In this subsection, we present two useful lemmas, whose proofs will be given in Appendix.

**Lemma 1.** Let $x$ be a fixed positive real number, and $m$ be a fixed nonnegative integer, then for $n \in \mathbb{N}^+$, s.t., $n \geq x$, the series

$$(1 - \frac{x}{n})^{n+m}$$

increases with $n$ and the limit is given by

$$\lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^{n+m} = e^{-x}$$

In order to derive our proof, it is more convenient to set

$$s = \lfloor KM/N \rfloor \in \{1, 2, \cdots, K\}$$

rewrite

$$\frac{M}{N} = \theta \cdot \frac{s - 1}{K} + (1 - \theta) \cdot \frac{s}{K} = \frac{s - \theta}{K}$$

and

$$\theta = \lfloor KM/N \rfloor - KM/N \in [0, 1)$$

and re-express $R_C(M)$ given by (3) as the following piecewise function

$$R_C(M) = \begin{cases} \theta \cdot \frac{K - s + 1}{s} + (1 - \theta) \cdot \frac{K - s}{s + 1}, & \text{if } K/N \leq s \leq K \\ \theta \cdot \frac{K - s + 1}{K} + (1 - \theta) \cdot \frac{K - s}{s + 1}, & \text{if } K/N - 1 \leq s < K/N \\ N - M, & \text{if } 1 \leq s < K/N - 1 \end{cases}$$

(11) (12) (13)
Further, to simplify the proof, we define three functions as
\[
\begin{align*}
    r_C(q, K) &\triangleq \frac{K(1-q)}{1+Kq} \\
    \tilde{r}_C(q, K) &\triangleq \theta \cdot r_C \left( \frac{s-1}{K}, K \right) + (1-\theta) \cdot r_C \left( \frac{s}{K}, K \right) \\
    r_D(q, K) &\triangleq \frac{1-q}{q} \left( 1 - (1-q)^K \right)
\end{align*}
\]
where \( q \in (0, 1], K \in \mathbb{N}^+, s = \lceil Kq \rceil, \) and \( \theta = s - Kq. \) Obviously, if setting \( q = \frac{M}{N} \)
then we get the following relations between \( R_C(M) \) and \( \tilde{r}_C(q, K) \), as well as \( R_D(M) \) and \( r_D(q, K) \):

1) If \( K/N \leq s \leq K \), then
\[
R_C(M) = \theta \cdot \frac{K-s+1}{s} + (1-\theta) \cdot \frac{K-s}{s+1} = \tilde{r}_C(q, K)
\]

2) If \( K/N - 1 \leq s < K/N \), then
\[
R_C(M) = \theta \cdot N \cdot \frac{K-s+1}{K} + (1-\theta) \cdot \frac{K-s}{s+1} < \tilde{r}_C(q, K)
\]
3) \[
R_D(M) = \min\{r_D(q,K), N - M\} \tag{19}
\]
\[
= \min\left\{ \frac{K - s + \theta}{s - \theta} \cdot \left(1 - \left(\frac{K - s + \theta}{K}\right)^K\right), N \cdot \frac{K - s + \theta}{K} \right\} \tag{20}
\]

Finally, we elaborate the relations between \(r_C(q,K)\), \(\tilde{r}_C(q,K)\), and \(r_D(q,K)\) as follows, which associated with (17)-(19), will play a key role in our proof.

**Lemma 2.** For \(\forall q \in (0, 1]\), \(\forall K \in \mathbb{N}^+\),
\[
r_C(q,K) \leq \tilde{r}_C(q,K) \leq r_D(q,K) \tag{21}
\]
and moreover, if \(K \geq 3\), then
\[
\frac{r_D(q,K)}{r_C(q,K)} < 1.5 \tag{22}
\]
Furthermore, for any fixed \(q \in (0, 1]\),
\[
\lim_{K \to \infty} \frac{r_D(q,K)}{r_C(q,K)} = 1 \tag{23}
\]

**B. Proof of The Lower Bound in Theorem 3**

The proof is divided into 2 cases according to the expressions of \(R_C(M)\) in (11)-(13).

**Case \(K/N - 1 \leq s \leq K\):** In this case, applying Lemma 2 to (17) and (18), we have
\[
R_C(M) \leq \tilde{r}_C(q,K) \leq r_D(q,K)
\]

On the other hand, we always have
\[
R_C(M) \leq \theta \cdot N \cdot \frac{K - s + 1}{K} + (1 - \theta) \cdot N \cdot \frac{K - s}{K}
\]
\[
= \frac{N}{K} (K - s + \theta)
\]
\[
= N \left(1 - \frac{s - \theta}{K}\right)
\]
\[
= N \left(1 - \frac{M}{N}\right)
\]
\[
= N - M
\]

Thus, by (19), we have
\[
R_C(M) \leq R_D(M)
\]

**Case \(1 \leq s < K/N - 1\):** By Lemma 2, we have \(r_C(q,K) \leq r_D(q,K)\), which gives
\[
\frac{1}{1 + KM/N} < \frac{N}{K M} \left(1 - \left(\frac{M}{N}\right)^K\right)
\]
for \(q = M/N \in (0, 1]\) and \(K \in \mathbb{N}^+\). Besides, note that
\[
\frac{N}{K} < \frac{1}{1 + s} \leq \frac{1}{1 + KM/N}
\]

Then, the above two inequalities imply
\[
\frac{N}{K} < \frac{N}{K M} \left(1 - \left(\frac{M}{N}\right)^K\right)
\]
which means \(R_D(M) = N - M\) by (5). Whereas \(R_C(M) = N - M\) in this case by (13), i.e.,
\[
R_C(M) = R_D(M) \tag{24}
\]

Combing these two cases, we derive the lower bound.
C. Proof of The Upper Bound in Theorem 3

Hereafter we only need to verify the upper bound for \( s \geq K/N - 1 \) in (24).

**Case \( K/N \leq s \leq K \):**

- If \( K = 2 \), then
  
  - If \( N = 1 \), then \( s = 2 \), as implied by \( K/N \leq s \leq K \). Then by (11) and (20), we have
    \[
    R_C(M) = R_D(M) = \frac{\theta}{2}
    \]
  
  - If \( N \geq 2 \), then \( s = 1, 2 \). By (11) and (20), we can derive
    \[
    R_C(M) = \frac{3\theta + (2-s)s}{s(s+1)}
    \]
    and
    \[
    R_D(M) = \min \left\{ \frac{(2-s+\theta) \cdot (4-s+\theta)}{4}, N \cdot \frac{2-s+\theta}{2} \right\}
    \]
    Thus,
    \[
    \frac{R_D(M)}{R_C(M)} = \frac{s(s+1)}{4} \cdot \frac{(2-s+\theta) \cdot (4-s+\theta)}{3\theta + s(2-s)}
    \]
    \[
    = \begin{cases} 
      (1+\theta)(3+\theta)/(6\theta+2), & \text{if } s = 1 \\
      (2+\theta)/2, & \text{if } s = 2 
    \end{cases}
    \]
    It is easy to verify that \((1+\theta)(3+\theta)/(6\theta+2)\) decreases with \( \theta \) while \((2+\theta)/2\) increases with \( \theta \) on \((0, 1)\). Therefore,
    \[
    \frac{R_D(M)}{R_C(M)} \leq \begin{cases} 
      (1+\theta)(3+\theta)/(6\theta+2)|_{\theta=0}, & \text{if } s = 1 \\
      (2+\theta)/2|_{\theta=1}, & \text{if } s = 2 
    \end{cases}
    \]
    \[
    = 1.5
    \]

- If \( K \geq 3 \), then by (17) and (19)
  \[
  R_D(M) \leq r_D(q, K) \\
  R_C(M) = \tilde{r}_C(q, K)
  \]
  Applying Lemma 2 we have
  \[
  \frac{R_D(M)}{R_C(M)} \leq \frac{r_D(q, K)}{\tilde{r}_C(q, K)} \leq \frac{r_D(q, K)}{r_C(q, K)} < 1.5
  \]

**Case \( K/N - 1 \leq s < K/N \):** In this case,

\[
R_C(M) = \theta \cdot N \cdot \frac{K-s+1}{K} + (1-\theta) \cdot \frac{K-s}{s+1}
\]
\[
\geq \theta \cdot \frac{K}{s+1} \cdot \frac{K-s+1}{K} + (1-\theta) \cdot \frac{K-s}{s+1}
\]
\[
= \frac{K-s+\theta}{s+1}
\]
by (18).

- If \( s = 1 \), then
  \[
  \frac{1}{2} \leq \frac{N}{K} < 1
  \]
We can deduce two upper bounds on \( \frac{R_D(M)}{R_C(M)} \). On one hand, by (20) and (26), we have
\[
\frac{R_D(M)}{R_C(M)} \leq \frac{N(K - 1 + \theta)/K}{\theta N + (1 - \theta)(K - 1)/2} \\
= \frac{\theta + (1 - \theta)(K - 1)/K}{\theta + (1 - \theta)(K - 1)/(2N)} \\
\leq \frac{\theta + (1 - \theta)(K - 1)/(2(K - 1))}{\theta} \\
= 2 \cdot \frac{1 - (1 - \theta)/K}{\theta} \\
= 2 \cdot \frac{1 + K - 2}{1 + \theta} \\
= \frac{\theta}{1 + \theta} \tag{29}
\]
where in (29) we use the fact \( N \leq K - 1 \) given by (28). On the other hand, with (20) and (27), we then have
\[
\frac{R_D(M)}{R_C(M)} \leq \frac{(K - 1 + \theta)/(1 - \theta) \cdot (1 - ((K - 1 + \theta)/K)^K)}{(K - 1 + \theta)/2} \\
= 2 \cdot \frac{1 - (1 - (1 - \theta)/K)^K}{1 - \theta} \\
= \frac{2}{K} \cdot \sum_{i=0}^{K-1} \left( \frac{K - 1 + \theta}{K} \right)^i \\
\tag{30}
\]
Note that, by observing (30) and (31), the first upper bound increases with \( K \) and decreases with \( \theta \). While with the result of Lemma 1, (32) and (33) indicates that the second upper bound decreases with \( K \) and increases with \( \theta \). So, we decompose this case into five subcases:

- If \( 2 \leq K \leq 3 \), with (31), we have
  \[
  \frac{R_D(M)}{R_C(M)} \leq \frac{2}{3} \cdot \left( 1 + \frac{1}{1 + \theta} \right) \\
  \leq \frac{4}{3} \\
  < 1.5 \\
  \]

- If \( 4 \leq K \leq 8 \), \( \theta > 6/25 \), still with (31),
  \[
  \frac{R_D(M)}{R_C(M)} < \frac{2}{8} \cdot \left( 1 + \frac{6}{1 + 6/25} \right) = \frac{181}{124} < 1.5 \\
  \]

- If \( 4 \leq K \leq 8 \), \( \theta \leq 6/25 \), with (33), we have
  \[
  \frac{R_D(M)}{R_C(M)} < \frac{2}{4} \cdot \frac{3 + 6/25}{4} = \approx 1.4988 < 1.5 \\
  \]

- If \( K \geq 9 \), \( \theta > 1/3 \), with (30),
  \[
  \frac{R_D(M)}{R_C(M)} < \frac{2}{1 + \theta} < \frac{2}{1 + 1/3} = 1.5 \\
  \]
- If $K \geq 9$, $\theta \leq 1/3$, with (33),
\[
\frac{R_D(M)}{R_C(M)} \leq \frac{2}{9} \sum_{i=0}^{8} \left( \frac{8 + 1/3}{9} \right)^i \\
\approx 1.4993 \\
< 1.5
\]

- If $s \geq 2$, then by (20) and (27), we have
\[
\frac{R_D(M)}{R_C(M)} \leq \frac{N \cdot (K - s + \theta)/K}{(K - s + \theta)/(s + 1)} \\
= \frac{N}{K} \cdot (s + 1) \\
< \frac{s + 1}{s} \\
\leq 1.5
\]

Collecting all the above cases, we deduce the desired upper bound.

D. Proof of The Limit in Theorem 3

In this subsection, we prove (7). For any fixed $N \in \mathbb{N}^+$, $M \in (0, N]$,

- If $M \geq 1$, since $s = \lfloor KM/N \rfloor \geq K/N$, then $R_C(M)$ is given by (17). It is easy to check that $R_D(M)$ is evaluated by the first item in (5), thus $R_D(M) = r_D(q, K)$ by (19). Therefore,
\[
\frac{R_D(M)}{R_C(M)} = \frac{r_D(q, K)}{r_C(q, K)}
\]

According to (21), we have
\[
1 \leq \frac{r_D(q, K)}{r_C(q, K)} \leq \frac{r_D(q, K)}{r_C(q, K)}
\]

The limit then follows from (23).

- If $M < 1$, we use the expressions of $R_C(M)$ and $R_D(M)$ in (2) and (5), i.e.,
\[
R_C(M) = \left( 1 - \frac{M}{N} \right) \cdot \min \left\{ \frac{K}{1 + KM/N}, N \right\} \\
R_D(M) = \left( 1 - \frac{M}{N} \right) \cdot \min \left\{ \frac{N}{M} \left( 1 - \left( 1 - \frac{M}{N} \right)^K \right)^N \right\}
\]

Note that as $K \to \infty$, the first items in the braces of the above expressions approaches $N/M > N$ for $M < 1$, we then conclude that for sufficiently large $K$,
\[
R_D(M) = R_C(M) = N \left( 1 - \frac{M}{N} \right)
\]

Therefore, the limit is proved.

V. Conclusions

In this paper, we investigated the ratio between the rates of the decentralized and centralized coded caching schemes proposed by Maddah-Ali and Niesen. We strictly proved that for any system parameters, the ratio is between 1 and 1.5. This verified two facts: Firstly, the centralized scheme always outperforms the decentralized scheme due to the central coordination of the server. Secondly, the rate of the decentralized coded caching scheme is within a constant multiplicative gap 1.5 of the rate of the centralized coded caching scheme, which is tighter than the claimed numerical upper bound 1.6 in [1]. Both the upper bound and the lower bound are tight since they can be achieved.
Furthermore, we showed that when the number of users $K$ goes to infinity, the ratio always approaches 1. This suggests that in the systems with large number of users, even without the centralized coordination from the central server, the decentralized coded caching scheme is able to achieve almost the same performance with its centralized counterpart. Therefore, the decentralized coded caching scheme is a reasonable alternative solution for the centralized coded caching scheme in scenarios that the centralized coordination is unavailable.

Finally, we have to point out that there exist other centralized coded caching schemes (for example the one in [7]) and other decentralized coded caching schemes (for example the ones in [8]). Noted that, the original centralized coded caching scheme in [2] has been shown in [7] to achieve the minimal rate $R_{C}(M)$ within the so-called regular placement delivery array framework; Some unexploited multicasting opportunities in the original decentralized coded caching scheme in [1] can be utilized to decrease the rate [8]. Thus, it would be very desirable and also very possible to design new decentralized coded caching scheme to further decrease $R_{D}(M)$. We leave this interesting topic for future research.

**APPENDIX**

**A. Proof of Lemma 1**

\[
(1 - \frac{x}{n})^{n+m} = \left(1 - \frac{x}{n}\right)^{n} \cdot (1 - \frac{x}{n})^{m} < \left(n \cdot \frac{1 - x/n + 1}{n + 1}\right)^{n+1} \cdot \left(1 - \frac{x}{n + 1}\right)^{m}
\]  
\[
= \left(1 - \frac{x}{n + 1}\right)^{n+1+m}
\]

where in (34) we use the fact that the geometric mean does not exceed arithmetic mean for nonnegative numbers. Then, the limit is given by

\[
\lim_{n \to \infty} (1 - \frac{x}{n})^{n+m} = \lim_{n \to \infty} (1 - \frac{x}{n})^{n}. \lim_{n \to \infty} (1 - \frac{x}{n})^{m} = \lim_{n \to \infty} \left((1 - \frac{1}{n/x})\right)^{n} = e^{-x}
\]

where in the last identity we used the fact $\lim_{n \to \infty} (1 - 1/n)^n = e^{-1}$.

**B. Proof of Lemma 2**

**Proof of (21):** We prove that for $\forall q \in (0,1], K \in \mathbb{N}^+$,

\[ r_{C}(q,K) \leq \tilde{r}_{C}(q,K) \leq r_{D}(q,K) \]

Note that $r_{C}(q,K)$ is convex on $q \in (0,1)$. Moreover, we have

\[
q = \frac{s - \theta}{K} = \frac{s - 1}{K} + (1 - \theta) \frac{s}{K}
\]

since $s = [Kq]$ and $\theta = s - Kq$. Then, the first inequality follows from the well-known Jensen’s inequality.
For the second inequality, by substituting (35) into (15) and (16) respectively, it is therefore sufficient to prove
\[
\theta \cdot \frac{K - s + 1}{s} + (1 - \theta) \cdot \frac{K - s}{s + 1} \leq \frac{1 - (s - \theta)/K}{(s - \theta)/K} \left( 1 - \left(1 - \frac{s - \theta}{K}\right)^K \right)
\]
\[
\Leftrightarrow \frac{s - \theta}{K^2} \left( \theta \frac{K + 1}{s(s + 1)} + \frac{K + 1}{s + 1} - 1 \right) \leq \left( 1 - \frac{s - \theta}{K} \right) \cdot \left( 1 - \left(1 - \frac{s - \theta}{K}\right)^K \right)
\]
for all \( K \in \mathbb{N}^+, \ s \in \{1, 2, \ldots, K\} \) and \( \theta \in [0, 1) \)

Define
\[
f(\theta, n, s) = \left( 1 - \frac{s - \theta}{n} \right) \cdot \left( 1 - \left(1 - \frac{s - \theta}{n}\right)^n \right) - \frac{s - \theta}{n} \left( \theta \frac{n + 1}{s(s + 1)} + \frac{n + 1}{s + 1} - 1 \right)
\]
\[
= \frac{n - s}{n(s + 1)} + \frac{n + 1}{ns(s + 1)} \theta^2 - \left( 1 - \frac{s - \theta}{n} \right)^{n+1}
\]
(36)

In what follows, we only need to verify \( f(\theta, n, s) \geq 0 \) for any \( \theta, n \) and \( s \), s.t., \( \theta \in [0, 1) \), \( n, s \in \mathbb{N}^+, \ n \geq s \).

- If \( s = n \in \mathbb{N}^+ \),
\[
f(\theta, n, n) = \frac{\theta^2}{n^2} - \frac{(\frac{\theta}{n})^{n+1}}{n^{n+1}}
\]
\[
= \frac{\theta^2}{n^2} \left( 1 - \left(\frac{\theta}{n}\right)^{n-1} \right)
\]
\[
\geq 0
\]

- If \( s = 1 \) and \( n > s \), then
\[
f(\theta, n, 1) = \frac{n - 1}{2n} + \frac{n + 1}{2n} \theta^2 - \left( \frac{n - 1 + \theta}{n} \right)^{n+1}
\]
\[
\frac{\partial f(\theta, n, 1)}{\partial \theta} = \frac{n + 1}{n} \left( \theta - \left( \frac{n - 1 + \theta}{n} \right)^n \right)
\]
(37)

and
\[
\frac{\partial^2 f(\theta, n, 1)}{\partial \theta^2} = \frac{n + 1}{n} \left( 1 - \left( \frac{n - 1 + \theta}{n} \right)^{n-1} \right) > 0, \ \forall \ \theta \in (0, 1)
\]
(38)

(38) tells us that (37) is increasing on \( \theta \in (0, 1) \). Therefore, it follows that
\[
\frac{\partial f(\theta, n, 1)}{\partial \theta} < \left. \frac{\partial f(\theta, n, 1)}{\partial \theta} \right|_{\theta=1} = 0, \ \forall \ \theta \in (0, 1)
\]
which indicates that \( f(\theta, n, 1) \) is decreasing on \( \theta \in [0, 1) \), thus
\[
f(\theta, n, 1) \geq f(1, n, 1) = 0, \ \forall \ \theta \in [0, 1)
\]

- If \( s \geq 2 \) and \( n > s \), we begin with a lower bound for \( f(\theta, n, s) \). Note that \((1 - x)^{n+1}\) is convex on \( x \in (0, 1) \). Thus, using Jensen’s inequality we have
\[
\left( 1 - \frac{s - \theta}{n} \right)^{n+1} = \left( 1 - \theta \frac{s - 1}{n} - (1 - \theta) \frac{s}{n} \right)^{n+1}
\]
\[
\leq \theta \left( 1 - \frac{s - 1}{n} \right)^{n+1} + (1 - \theta) \left( 1 - \frac{s}{n} \right)^{n+1}
\]

Then by (36),
\[
f(\theta, n, s)
\]
\[
\geq \frac{n - s}{n(s + 1)} + \frac{n + 1}{ns(s + 1)} \theta^2 - \theta \left( 1 - \frac{s - 1}{n} \right)^{n+1} - (1 - \theta) \left( 1 - \frac{s}{n} \right)^{n+1}
\]
\[
= \frac{n - s}{n(s + 1)} - \left( 1 - \frac{s}{n} \right)^{n+1} + \frac{n + 1}{ns(s + 1)} \left( \theta^2 - \theta \frac{ns(s + 1)}{n + 1} \left( 1 - \frac{s - 1}{n} \right)^{n+1} - (1 - \frac{s}{n})^{n+1} \right)
\]
When \( \theta = ns(s + 1)/(2(n + 1)) \cdot ((1 - (s - 1)/n)^{n+1} - (1 - s/n)^{n+1}) \), the righthand side achieves its minimal value, which is defined as

\[
g(n, s) \triangleq \frac{n - s}{n(s + 1)} - \left(1 - \frac{s}{n}\right)^{n+1} - \frac{1}{4} \cdot \frac{ns(s + 1)}{n + 1} \left((1 - \frac{s - 1}{n})^{n+1} - (1 - \frac{s}{n})^{n+1}\right)^2
\]

By Lemma 1 both the items \((1 - s/n)^{n+1}\) and \((1 - (s - 1)/n)^{n+1}\) strictly increases with \(n > s\) and

\[
\lim_{n \to \infty} \left(1 - \frac{s}{n}\right)^{n+1} = e^{-s}
\]

\[
\lim_{n \to \infty} \left(1 - \frac{s - 1}{n}\right)^{n+1} = e^{-2s+2}
\]

which implies \((1 - s/n)^{n+1} < e^{-s}\) and \((1 - (s - 1)/n)^{n+1} < e^{-2s+2}\). Therefore, we have

\[
g(n, s) \geq \frac{1}{s + 1} - \frac{s}{n(s + 1)} - e^{-s} - \frac{e^2}{4} \cdot \frac{s(s + 1)}{e^{2s}} \cdot (1 - (1 - \frac{1}{n - s + 1})^{n+1})^2
\]

\[
\triangleq h(n, s)
\]

Up to now, we have derived

\[
f(\theta, n, s) \geq g(n, s) \geq h(n, s) \quad \forall \theta \in [0, 1)
\]

(39)

Note that \(h(n, s)\) increases with \(n\) since the positive number \((1 - 1/(n - s + 1))^{n+1}\) increases with \(n\) by Lemma 1.

Therefore,

- If \(s \geq 4\), it is easy to check that \(e^{2s} > 4s(s + 1)^3\) and \(e^s > 2(s + 1)^2\) for \(s \geq 4\). Consequently, we have

\[
h(n, s) \geq h(s + 1, s)
\]

\[
> \frac{1}{s + 1} - \frac{s}{(s + 1)^2} - e^{-s} - \frac{e^2}{4} \cdot \frac{s(s + 1)}{e^{2s}}
\]

\[
> \frac{1}{s + 1} - \frac{s}{(s + 1)^2} - \frac{1}{2(s + 1)^2} - \frac{e^2}{16} \cdot \frac{1}{(s + 1)^2}
\]

\[
= \left(1 - \frac{e^2}{16}\right) \cdot \frac{1}{(s + 1)^2}
\]

\[
> 0
\]

(40)

where in (40) we make use of the fact \(e^2 < 8\).

- If \(s = 3\), \(n \geq 5\), we have

\[
h(n, 3) \geq h(5, 3) \approx 0.0045 > 0
\]

- If \(s = 2\), \(n \geq 8\), we have

\[
h(n, 2) \geq h(8, 2) \approx 0.0004 > 0
\]

- For other cases, i.e., \(s = 3, n = 4\) or \(s = 2, n = 3, 4, 5, 6, 7\), we compute \(g(n, s)\) directly

\[
g(4, 3) \approx 0.0593 > 0, \quad g(3, 2) \approx 0.0602 > 0
\]

\[
g(4, 2) \approx 0.0845 > 0, \quad g(5, 2) \approx 0.0953 > 0
\]

\[
g(6, 2) \approx 0.1012 > 0, \quad g(7, 2) \approx 0.1047 > 0
\]

Therefore, we conclude \(f(\theta, n, s) > 0\) for \(\theta \in [0, 1)\) and \(2 \leq s < n\), by (39).
Proof of (22): We prove that
\[
\frac{r_D(q, K)}{r_C(q, K)} < 1.5
\]
for all \( K \geq 3 \in \mathbb{N}^+ \) and \( q \in (0, 1) \).

Note from (14) and (16),
\[
\frac{r_D(q, K)}{r_C(q, K)} = \frac{1 + Kq}{Kq} \left(1 - (1 - q)^K\right) = \frac{1 + x}{x} \left(1 - \left(1 - \frac{x}{K}\right)^K\right)
\]
where \( x \overset{\Delta}{=} Kq \). Therefore, it suffices to show
\[
l_n(x) \overset{\Delta}{=} \frac{1 + x}{x} \left(1 - \left(1 - \frac{x}{n}\right)^n\right) < 1.5
\]
for any \( x \in (0, n] \), where \( n \geq 3 \) and \( n \in \mathbb{N}^+ \).

- If \( x \in (0, 3] \),
  \[
l_n(x) \leq \sup_{z \in (0, 3]} l_n(z)
  \leq \sup_{z \in (0, 3]} l_3(z)
  = \sup_{z \in (0, 3]} \frac{z^3 - 8z^2 + 18z + 27}{27}
  = \frac{1001 + 20\sqrt{10}}{729}
  \approx 1.4599
  < 1.5
\]
  where in (42) we use the fact that for any fixed \( z \in (0, 3] \), \((1 - z/n)^n\) increases with \( n \) by Lemma [1].

- If \( 3 < x \leq n \), then
  \[
l_n(x) \leq \frac{1 + x}{x} < \frac{1 + 3}{3} < 1.5
\]

Proof of (23): The limit (23) is clear from the expression (41).

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