Implications of the $\nu_\mu \rightarrow \nu_s$ solution to the atmospheric neutrino anomaly for early Universe cosmology

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Abstract

By numerically solving the quantum kinetic equations we compute the range of parameters where the $\nu_\mu \rightarrow \nu_s$ oscillation solution to the atmospheric neutrino anomaly is consistent with a stringent big bang nucleosynthesis (BBN) bound of $N_{eff}^{BBN} \lesssim 3.6$. We show that this requires tau neutrino masses in the range $m_{\nu_\tau} \gtrsim 4$ eV (for $|\delta m_{atm}^2| = 10^{-2.5}$ $eV^2$). We discuss the implications of this scenario for hot+cold dark matter, BBN, and the anisotropy of the cosmic microwave background.

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I Introduction

Neutrino physics continues to provide the most promising window on physics beyond the standard model. There are numerous indications for neutrino oscillations including the atmospheric neutrino anomaly, solar neutrino problem, LSND experiment, structure formation in the early Universe, indications of $N_{eff}^{BBN} < 3$ etc. For a recent review, see e.g. Ref. [1].

The atmospheric neutrino anomaly [2] has been confirmed by the superKamiokande experiment [3]. The observed up-down asymmetries of the detected muons indicate [4] that the simplest solution to this anomaly is either $\nu_\mu \rightarrow \nu_\tau$ [5] or $\nu_\mu \rightarrow \nu_s$ [6–10] oscillations (although significant additional mixing with $\nu_e$ cannot currently be excluded if $\delta m^2 < 2 \times 10^{-3} \text{eV}^2$ [11]). In each case the oscillations are maximal or nearly maximal ($0.80 \approx \sin^2 2\theta \approx 1$) and

$$4 \times 10^{-4}(10^{-3}) \approx |\delta m_{\text{atmos}}^2|/\text{eV}^2 \lesssim 10^{-2},$$

for $\nu_\mu \rightarrow \nu_\tau$ ($\nu_\mu \rightarrow \nu_s$) oscillations [4]. Although experimentally similar, the $\nu_\mu \rightarrow \nu_\tau$ and $\nu_\mu \rightarrow \nu_s$ oscillation solutions to the atmospheric neutrino anomaly can be experimentally distinguished [12]. These two solutions will also have quite different implications for early Universe cosmology. Naively the $\nu_\mu \rightarrow \nu_s$ oscillation solution to the atmospheric neutrino anomaly would appear to lead to an effective number of four neutrinos [14] in the early Universe, which would make it difficult to reconcile big bang nucleosynthesis (BBN) with the estimated primordial light element abundances (for a recent discussion, see Ref. [16]). It should be stressed that the ordinary-sterile neutrino oscillations can only populate the sterile state provided that the lepton number of the Universe is very small. Since the origin of the baryon and lepton asymmetries of the Universe are unknown it is possible that a lepton asymmetry was created at some early time $T \gg 100 \text{MeV}$. Furthermore, if the lepton to photon ratio is larger than about $10^{-5}$ then the BBN bounds will be evaded [17]. Clearly this is one possibility. However another possibility is that the neutrino oscillations themselves generate the lepton number. In fact, a careful study of the ordinary-sterile neutrino oscillations in the early Universe reveal that the oscillations themselves typically generate a much larger asymmetry for a large range of parameters [18–21]. As already discussed in Ref. [20], there is a concrete mechanism whereby the large neutrino oscillation generated asymmetry can prevent the (near) maximal $\nu_\mu \rightarrow \nu_s$ oscillations from populating the sterile neutrino in the early Universe. This scenario requires the tau neutrino to be in the eV mass

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1 There is an astrophysical/cosmological argument which favours $\nu_\mu \rightarrow \nu_s$ over $\nu_\mu \rightarrow \nu_\tau$. Ref. [13] points out that the decaying neutrino theory [14] for the ionisation of hydrogen in the interstellar medium, in conjunction with the assumption that the cosmological constant is zero favours $\nu_\mu \rightarrow \nu_s$ over $\nu_\mu \rightarrow \nu_\tau$.

2 It turns out that this asymmetry is typically generated at temperatures less than 100 MeV so that the quantitative calculations of Ref. [17] are not generally applicable to the case where neutrino oscillations generate the asymmetry.
range and to mix slightly with the sterile neutrino. In this case the $\nu_\tau \to \nu_s$ oscillations generate a large tau neutrino asymmetry in the early Universe which suppresses $\nu_\mu \to \nu_s$ oscillations (the tau neutrino asymmetry can suppress $\nu_\mu \to \nu_s$ oscillations because the matter term for $\nu_\mu \to \nu_s$ oscillations also contains a part which is proportional to the tau neutrino asymmetry of the background plasma). The calculation is somewhat non-trivial because the $\nu_\mu \to \nu_s$ oscillations like to produce a large muon neutrino asymmetry which can compensate for the effect of the large tau neutrino asymmetry in the matter term for $\nu_\mu \to \nu_s$ oscillations. The computation of Ref. [20] utilised an approximate solution to the quantum kinetic equations (which was called the ‘static approximation’ in Ref. [20]) and is valid provided that the system was smooth enough. This is a very useful approximation because it saves considerable CPU time as well as giving more insight than the more complicated quantum kinetic equations. However as pointed out in Ref. [20] this approximation is not always valid for the entire range of $\sin^2 2\theta_0$, $\delta m^2$ parameter space of interest. In particular, if $\sin^2 2\theta_0$ is large enough then this approximation is generally not valid because the lepton asymmetry is created so rapidly. Thus, the main purpose of this paper is to do a more exact computation (by using the quantum kinetic equations rather than the simpler static approximation) of the region of parameter space where the $\nu_\mu \to \nu_s$ solution to the atmospheric neutrino anomaly is consistent with BBN (assuming $N_{\text{BBN}}^{\text{eff}} \approx 3.6$). Also we will also include the entire range Eq.(1) of $\delta m^2_{\text{atmos}}$. If the sterile neutrinos are not populated by $\nu_\mu \to \nu_s$ oscillations then during the low temperature evolution of the neutrino asymmetries some electron neutrino asymmetry will be transferred from the tau neutrino asymmetry (due to $\nu_\tau \to \nu_e$ oscillations), which significantly affects BBN. This observation was first made in Ref. [21] and we include a discussion of this effect here for completeness. We will also discuss the implications for the hot+cold dark matter model and the anisotropy of the cosmic microwave background (CMB).

For the purposes of this paper, we assume that there is only one light sterile neutrino. Thus, we are considering a four neutrino model. Probably the simplest four neutrino model which can explain the atmospheric neutrino anomaly through $\nu_\mu \to \nu_s$ oscillations, and also explain the solar neutrino problem is the model of Ref. [9]. In this model the solar neutrino problem is explained by the oscillations of $\nu_e \to \nu_\mu, \nu_s$ with parameters consistent with the small angle MSW solution of the solar neutrino problem. Our results (as well as the previous work [20,21]) is applicable to this model. However this model does not seem to be compatible with the LSND anomaly [22]. Actually, as we will discuss in more detail in section VIII, our results indicate that $N_{\text{eff}}^{\text{BBN}} \approx 4$ does not seem to be consistent with any four neutrino model which explains all three experimental anomalies. Hence, if experimental data indicate that $\nu_\mu \to \nu_s$ oscillations are required to explain the atmospheric neutrino data, and if the solar and LSND anomalies have been correctly interpreted in terms of neutrino oscillations, then $N_{\text{eff}}^{\text{BBN}} < 4$ actually suggests the need for more than four neutrinos.

Admittedly, the theory of neutrino oscillations in the early Universe is quite a complicated subject. The readers who are primarily interested in our results can skip directly to Figures 2,3,4. These figures summarise the main results of this paper.

The outline of this paper is as follows: In section II we briefly review the phenomenon of neutrino oscillation generated neutrino asymmetry in the early Universe. In section III we explicitly write down the quantum kinetic equations which we will need latter on, and we
numerically solve them for some illustrative examples. In section IV we obtain the region of parameter space where the \( \nu_\mu \rightarrow \nu_s \) oscillation solution of the atmospheric neutrino anomaly is consistent with a BBN bound of \( N_{\text{eff}}^{\text{BBN}} \lesssim 3.6 \). In section V, VI, VII we discuss the detailed implications of this four neutrino scheme for BBN, hot+cold dark matter model, and the anisotropy of the CMB. In section VIII we conclude.

II Oscillation generated neutrino asymmetry in the early Universe

Our notation/convention for ordinary-sterile neutrino two state mixing is as follows, the weak eigenstates \( (\nu_\alpha, \nu_s) \), with \( \alpha = e, \mu, \tau \), are linear combinations of two mass eigenstates \( (\nu_a, \nu_b) \):

\[
\nu_\alpha = \cos \theta_0 \nu_a + \sin \theta_0 \nu_b, \quad \nu_s = -\sin \theta_0 \nu_a + \cos \theta_0 \nu_b.
\] (2)

Note that we define \( \theta_0 \) such that \( \cos 2\theta_0 > 0 \) and we take the convention that \( \delta m^2 \equiv m_b^2 - m_a^2 \).

The neutrino asymmetries are defined by,

\[
L_{\nu_\alpha} \equiv \frac{n_{\nu_\alpha} - n_{\bar{\nu}_\alpha}}{n_\gamma},
\] (3)

with \( n_i \) being the number density of species \( i \). Finally, note that when we refer to a neutrino, sometimes we will mean neutrinos and anti-neutrinos. We hope the correct meaning will be clear from context.

It was shown in Ref. [18] that for \( \nu_\alpha \rightarrow \nu_s \) (\( \alpha = e, \mu, \tau \)) oscillations (with \( |\delta m^2| \gtrsim 10^{-4}\ eV^2 \)) the evolution of lepton number has the form (for small \( L_{\nu_\alpha} \))

\[
\frac{dL_{\nu_\alpha}}{dt} \simeq C \left( L_{\nu_\alpha} + \frac{\eta}{2} \right),
\] (4)

where \( \eta \) is set by the relic nucleon number densities (and is expected to be small, \( \eta/2 \sim 10^{-10} \)) and \( C \) is a function of time, \( t \) (or equivalently temperature, \( T \)). At high temperature \( C \) is negative, so that \( (L_{\nu_\alpha} + \eta/2) \simeq 0 \) is an approximate fixed point. However if \( \delta m^2 < 0 \), then \( C \) changes sign at a particular temperature \( T = T_c \). At this temperature rapid exponential growth of neutrino asymmetry occurs (unless \( \sin^2 2\theta_0 \) is very tiny, see Eq.(8) below). The temperature where \( C \) changes sign was calculated to be [18]

\[
T_c \approx 16 \left( \frac{-\delta m^2 \cos 2\theta_0}{eV^2} \right)^{\frac{1}{6}} \ MeV.
\] (5)

The generation of neutrino asymmetry occurs because the \( \nu_\alpha \rightarrow \nu_s \) oscillation probability is different to the \( \bar{\nu}_\alpha \rightarrow \bar{\nu}_s \) oscillation probability due to the matter effects in a CP asymmetric background. As the asymmetry is created, the background becomes more CP asymmetric because the neutrino asymmetries contribute to the CP asymmetry of the background.

The phenomenon of neutrino oscillation generated neutrino asymmetry was studied in more detail along with some applications in Refs. [19, 21, 23, 25]. Ref. [19] calculated the region of parameter space where the neutrino asymmetries were created under the approximation that all the neutrinos have a common momentum \( (p \sim 3.15T) \). In Ref. [20], the
neutrino momentum spread was taken into account by developing an approximate solution to the quantum kinetic equations. Further studies of this approximation, along with an alternative derivation appears in Ref. [25]. Ref. [20] also contains a detailed study of the temperature region $T \sim T_c$ where the initial exponential growth of the neutrino asymmetry occurs. In Ref. [21] the low temperature $T \sim T_c$ evolution of the neutrino asymmetry was studied. After the initial exponential growth of the neutrino asymmetry occurs, the collisions and eventually MSW transitions combine to keep the asymmetry growing. This low temperature evolution of the asymmetry is approximately independent of $\sin^2 2\theta_0$ (assuming that $\sin^2 2\theta_0 \ll 1$). The ‘final’ value of the asymmetry arises at the temperature [21]

$$T_f^\nu \simeq 0.5(|\delta m^2|/eV^2)^{1/4} \text{ MeV.}$$

(6)

The magnitude of the final value was calculated to be [21],

$$L_f^\nu/h \simeq 0.29 \text{ for } |\delta m^2|/eV^2 \gtrsim 1000,$$

$$L_f^\nu/h \simeq 0.23 \text{ for } 3 \lesssim |\delta m^2|/eV^2 \lesssim 1000,$$

$$L_f^\nu/h \simeq 0.35 \text{ for } 10^{-4} \lesssim |\delta m^2|/eV^2 \lesssim 3,$$

(7)

where $h \equiv (T_0^3/T_f^3)$. Strictly these results are valid for $\nu_\alpha \to \nu_s$ oscillations in isolation (i.e. in the idealized case where there are only the two flavour oscillations $\nu_\alpha \to \nu_s$ occurring). For the realistic case of three ordinary and one sterile neutrino considered in this paper, these results hold approximately for the $\nu_\tau \to \nu_s$ oscillations (assuming that $m_{\nu_\tau}^2 \gg m_{\nu_\mu}^2, m_{\nu_e}^2$) [20, 19]. This large neutrino asymmetry occurs for a wide range of $\sin^2 2\theta_0$, $\delta m^2$ [20, 19],

$$5 \times 10^{-10} \left[\frac{eV^2}{|\delta m^2|}\right]^{1/2} \lesssim \sin^2 2\theta_0 \lesssim 4(2) \times 10^{-5} \left[\frac{eV^2}{|\delta m^2|}\right]^{1/2}, |\delta m^2| \gtrsim 10^{-4} eV^2,$$

(8)

with $\delta m^2 < 0$ for $\nu_\mu, \tau \to \nu_s$ (nu \tau \to nu_s) oscillations. Note that the upper bound on $\sin^2 2\theta_0$ in Eq.(8) assumes that the energy density of sterile neutrinos, which arise in the period before the exponential growth of neutrino asymmetry occurs (i.e. $T \sim T_c$) is less than 0.6 of a standard neutrino species. That is $\delta N_{eff}^{^BBN} \lesssim 0.6$ is assumed in Eq.(8). Because of the present observational uncertainties in the primordial light element abundances this bound is not rigorous. It is nevertheless interesting to suppose that the bound is rigorous and to explore the consequences. In any case it is clear that the generation of large neutrino asymmetries is quite a general phenomenon which occurs for a large region of parameter space if light sterile neutrinos exist [4].

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3 The alternative case where $\nu_\mu$ and $\nu_\tau$ are approximately maximal mixtures of two mass eigenstates is discussed in Ref. [24].

4 In the region of parameter space where $|\delta m^2| \ll 10^{-4} eV^2$, the evolution of lepton number is dominated by oscillations between collisions and the lepton number tends to be oscillatory [26, 27, 14]. Note however that, if say $\nu_\tau \to \nu_s$ oscillations have $|\delta m^2| \gtrsim 10^{-4} eV^2$, then the final value of $L_{\nu_\tau}$ is so large that oscillations of say $\nu_e \to \nu_s$ with $|\delta m^2| \ll 10^{-4} eV^2$ are heavily suppressed by the large matter effects (caused by the large $L_{\nu_e}$) for $T \sim 0.5$ MeV, and thus such oscillations cannot have any effect for BBN.
Oscillation generated large neutrino asymmetries can have several important implications for cosmology, including,

(1) The BBN bounds on ordinary - sterile neutrino oscillations can be evaded \[20,23\].

(2) If electron asymmetry is generated, then big bang nucleosynthesis can be directly affected though the modification of nuclear reaction rates (such as \(\nu_e + N \leftrightarrow P + e^-\), \(\bar{\nu}_e + P \leftrightarrow N + e^+\)) \[21,24\].

(3) The modification of the neutrino number densities will have implications for structure formation and will affect the anisotropy of the cosmic microwave background.

In this paper we will illustrate all three of these implications within the framework of the four neutrino model which solves the atmospheric neutrino anomaly by \(\nu_\mu \to \nu_s\) oscillations.

### III The quantum kinetic equations for ordinary - sterile neutrino oscillations in the early Universe

The density matrix \[28–30\] for an ordinary neutrino, \(\nu_\alpha\) (\(\alpha = e, \mu, \tau\)), of momentum with magnitude \(p\) oscillating with a sterile neutrino in the early Universe can be parameterised as follows,

\[
\rho_{\nu_\alpha}(p) = \frac{1}{2} \tilde{P}_0(p)[I + \mathbf{P}(p)\sigma], \quad \rho_{\bar{\nu}_\alpha}(p) = \frac{1}{2} \bar{\tilde{P}}_0(p)[I + \bar{\mathbf{P}}(p)\sigma],
\]

where \(I\) is the \(2 \times 2\) identity matrix and \(\mathbf{P}(p) = P_x(p)\hat{x} + P_y(p)\hat{y} + P_z(p)\hat{z}\), \(\sigma = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}\) (the \(\sigma_i\) are the Pauli matrices). It will be understood that the density matrices and the quantities \(P_i\) also depend on time \(t\) or, equivalently, temperature \(T\) (the time temperature relation for \(m_e \lesssim T \lesssim m_\mu\) is \(dt/dT \simeq -M_P/5.5T^3\), where \(M_P \simeq 1.22 \times 10^{22}\) MeV is the Planck mass).

We will normalise the density matrix such that the momentum distributions of \(\nu_\alpha\), \(\nu_s\) are given by

\[
N_{\nu_\alpha} = \frac{1}{2} P_0(p)[I + P_z(p)]N^0(p), \quad N_{\nu_s} = \frac{1}{2} P_0(p)[I - P_z(p)]N^0(p),
\]

where

\[
N^0(p) = \frac{1}{2\pi^2} \frac{p^2}{1 + exp\left(\frac{p}{T}\right)}.
\]

Note that there are analogous equations for the anti-neutrinos (with \(\mathbf{P}(p) \to \bar{\mathbf{P}}(p)\) and \(P_0 \to \tilde{P}_0\)). The evolution of \(P_0(p)\) and \(\mathbf{P}(p)\) [or \(\tilde{P}_0(p)\), \(\bar{\mathbf{P}}(p)\)] are governed by the equations \[29,28\],

\[
\frac{\partial \mathbf{P}(p)}{\partial t} = \mathbf{V}(p) \times \mathbf{P}(p) + [1 - P_z(p)][\frac{\partial}{\partial t} ln P_0(p)]\hat{z}
\]
\[
-D(p) + \frac{\partial}{\partial t} \ln P_0(p)[P_x(p)\dot{x} + P_y(p)\dot{y}],
\]

\[
\frac{\partial P_0(p)}{\partial t} \approx \Gamma(p) \left[ K(p) - \frac{1}{2} P_0(p)(1 + P_z(p)) \right],
\]

where \( D(p) = \Gamma(p)/2 \) and \( \Gamma(p) \) is the total collision rate of the weak eigenstate neutrino of momentum \( p \) with the background plasma and \( K(p) \equiv N^{eq}(p)/N^0(p) \), with \( N^{eq}(p) \) being the expected number of neutrinos in thermal equilibrium, i.e.

\[
N^{eq}(p) \equiv \frac{1}{2\pi^2} \frac{p^2}{1 + \exp \left( \frac{m_\alpha - \mu}{T} \right)},
\]

For anti-neutrinos, \( \mu_{\nu_\alpha} \rightarrow \mu_{\bar{\nu}_\alpha} \) in the above equation. The chemical potentials \( \mu_{\nu_\alpha}, \mu_{\bar{\nu}_\alpha} \), depend on the lepton asymmetry. In general, for a distribution in thermal equilibrium

\[
L_{\nu_\alpha} = \frac{1}{4\zeta(3)} \int_0^\infty \frac{x^2 dx}{1 + e^{x + \mu_\alpha}} - \frac{1}{4\zeta(3)} \int_0^\infty \frac{x^2 dx}{1 + e^{x + \tilde{\mu}_\alpha}},
\]

where \( \tilde{\mu}_\alpha \equiv \mu_{\nu_\alpha}/T \) and \( \tilde{\mu}_\alpha \equiv \mu_{\bar{\nu}_\alpha}/T \). Expanding out the above equation,

\[
L_{\nu_\alpha} \simeq -\frac{1}{24\zeta(3)} \left[ \pi^2 (\mu_\alpha - \tilde{\mu}_\alpha) - 6(\tilde{\mu}_\alpha - \bar{\mu}_\alpha)\ln2 + (\tilde{\mu}_\alpha - \bar{\mu}_\alpha)^3 \right],
\]

which is an exact equation for \( \tilde{\mu}_\alpha = -\bar{\mu}_\alpha \), otherwise it holds to a good approximation provided that \( \tilde{\mu}_{\alpha,\bar{\alpha}} \leq 1 \). For \( T \gtrsim T_{dec}^\alpha \) (where \( T_{dec}^\alpha \approx 2.5 \text{ MeV} \) and \( T_{dec}^{\alpha,\bar{\alpha}} \approx 3.5 \text{ MeV} \) are the chemical decoupling temperatures), \( \mu_{\nu_\alpha} \simeq -\mu_{\bar{\nu}_\alpha} \) because processes such as \( \nu_\alpha + \bar{\nu}_\alpha \leftrightarrow e^+ + e^- \) are rapid enough to make \( \tilde{\mu}_\alpha + \tilde{\mu}_\alpha \simeq \tilde{\mu}_{e^+} + \tilde{\mu}_{e^-} \simeq 0 \). However, for \( 1 \text{ MeV} \lesssim T \lesssim T_{dec}^\alpha \), weak interactions are rapid enough to approximately thermalise the neutrino momentum distributions, but not rapid enough to keep the neutrinos in chemical equilibrium. In this case, the value of \( \tilde{\mu}_\alpha \) is approximately frozen at \( T \simeq T_{dec}^\alpha \) (taking for definiteness \( L_{\nu_\alpha} > 0 \)), while the anti-neutrino chemical potential \( \tilde{\mu}_\alpha \) continues increasing until \( T \simeq 1 \text{ MeV} \).

The form of the evolution equation, Eq. (12) is very easy to understand. The \( \mathbf{V} \times \mathbf{P} \) term is simply the quantum mechanical coherent evolution of the states (this is derived in many textbooks, see e.g. Ref. [28]). The damping term \( \langle D[P_x\dot{x} + P_y\dot{y}] \rangle \) is just the destruction of the coherence of the ensemble due to collisions. The only surprising thing is that \( D(p) \) is half of the collision frequency i.e. \( D(p) = \Gamma(p)/2 \) [rather than \( \Gamma(p) \)] [24]. The rate of change of \( P_0 \) is due only to the repopulation of ordinary neutrinos due to weak interactions (since, from Eq. (11), \( P_0(p) = (N_{\nu_\alpha} + N_{\bar{\nu}_\alpha})/N^0 \), which is obviously unchanged by the \( \nu_\alpha \rightarrow \nu_\alpha \) oscillations). Note

5 From Ref. [31, 25] it is given by \( \Gamma(p) = yG_F^2 T^5(p/3.15T) \) where \( y \simeq 4.0 \) for \( \nu = \nu_e \) and \( y \simeq 2.9 \) for \( \nu = \nu_\mu, \nu_\tau \) (for \( m_e \simeq T \lesssim m_\mu \)).

6 The chemical and thermal decoupling temperatures are so different because the inelastic collision rates are much less than the elastic collision rates. See e.g. Ref. [31] for a list of the collision rates.
that Eq. (13) is approximate and holds provided that the distributions of the neutrinos and background particles are in near thermal equilibrium (see Ref. 25 for a detailed derivation of this formula). Finally the $\partial \ln P_0 / \partial t$ terms arise because the repopulation does not populate sterile states or mixtures of states. [For example, because the repopulation does not affect the number of sterile states, it follows that

$$\partial P_z / \partial t |_{\text{repopulation}} = (1 - P_z) \partial \ln P_0 / \partial t,$$

since $\frac{1}{2} P_0 (1 - P_z)$ is unchanged by the re-population].

The quantity $V(p)$ is given by [24, 30]

$$V(p) = \beta(p) \hat{x} + \lambda(p) \hat{z},$$

(17)

where $\beta(p)$ and $\lambda(p)$ are

$$\beta(p) = \frac{\delta m^2}{2p} \sin 2\theta_0, \quad \lambda(p) = -\frac{\delta m^2}{2p} \left[ \cos 2\theta_0 - b(p) \pm a(p) \right],$$

(18)

in which the $+(-)$ sign corresponds to neutrino (anti-neutrino) oscillations. The dimensionless variables $a(p)$ and $b(p)$ contain the matter effects [33] (more precisely they are the matter potential divided by $\delta m^2 / 2p$). For $\nu_\alpha \to \nu_s$ oscillations $a(p), b(p)$ are given by [34]

$$a(p) = -4 \zeta(3) \sqrt{2} G_F T^3 L^{(\alpha)} p / \pi^2 \delta m^2, \quad b(p) = -4 \zeta(3) \sqrt{2} G_F T^4 A_\alpha p^2 / \pi^2 \delta m^2 M_W^2,$$

(19)

where $\zeta(3) \approx 1.202$ is the Riemann zeta function of 3, $G_F$ is the Fermi constant, $M_W$ is the $W$-boson mass, $A_e \approx 17$ and $A_{\mu,\tau} \approx 4.9$ (for $m_e \lesssim T \lesssim m_\mu$). The quantity $L^{(\alpha)}$ is given by

$$L^{(\alpha)} = L_{\nu_\alpha} + L_{\nu_e} + L_{\nu_\mu} + L_{\nu_\tau} + \eta,$$

(20)

where $\eta$ is a small term due to the asymmetry of the electrons and nucleons and is expected to be very small, $\eta \sim 5 \times 10^{-10}$. Recall that the neutrino asymmetry is defined in Eq. (3).

For example, the number density of $\nu_\alpha$ is

$$n_{\nu_\alpha} = \int_0^\infty N_{\nu_\alpha} dp = \int_0^\infty \frac{1}{2} P_0 (1 + P_z) N^0 dp.$$  

(21)

Note that in the interests of notational simplicity, in the above equation and in the following discussion, the functional dependence of $P_i, N^0, \beta, \lambda, D$ on the neutrino momentum will not always be made explicit. The rate of change of lepton number is given by

$$\frac{dL_{\nu_\alpha}}{dt} = \frac{d}{dt} \left[ \frac{n_{\nu_\alpha} - n_{\bar{\nu}_\alpha}}{n_\gamma} \right] = -\frac{d}{dt} \left[ \frac{n_{\nu_e} - n_{\bar{\nu}_s}}{n_\gamma} \right].$$

(22)

Thus, using Eq. (11),

$$\frac{dL_{\nu_\alpha}}{dt} = \frac{-1}{2n_\gamma} \int \left[ \frac{\partial P_0}{\partial t} (1 - P_z) - P_0 \frac{\partial P_z}{\partial t} + \frac{\partial P_0}{\partial t} (1 - P_z) + P_0 \frac{\partial P_z}{\partial t} \right] N^0 dp.$$  

(23)

It is possible to numerically integrate the system of coupled differential equations, Eq. (12,13,23), although it is (CPU) time consuming. Unfortunately we do not have a parallel supercomputer handy. We therefore employ the useful time saving approximation of
integrating the oscillation equations, Eq.(12), in the region around the MSW resonance (we will define this region precisely in a moment). Actually away from the resonance the oscillations are typically suppressed by the matter effects or by $\sin^2 2\theta_0$ (or both). Thus, this should be a good approximation, which we can check by taking larger slices of momentum space around the resonance.

In the adiabatic approximation, the oscillation probability for $\nu_\alpha \rightarrow \nu_s$ is given by the formula,

$$P = \sin^2 2\theta_m \langle \sin^2 \frac{\tau}{2L_m} \rangle.$$  \hspace{1cm} (24)

Of course this formula is only valid provided that $\sin^2 2\theta_m$ is approximately constant on the interaction time scale $(1/D)$. In Eq.(24), the brackets $\langle ... \rangle$ denotes the average over the interaction times $\tau$, and $\sin^2 2\theta_m, L_m$ are the matter mixing angle and oscillation length respectively. In terms of $\beta, \lambda, D$,

$$\sin^2 2\theta_m = \frac{\beta^2}{\beta^2 + \lambda^2}, \hspace{0.5cm} L_m^2 = \frac{1}{\beta^2 + \lambda^2},$$

$$\langle \sin^2 \frac{\tau}{2L_m} \rangle = D \int_0^\infty e^{-\tau D} \sin^2 \frac{\tau}{2L_m} d\tau.$$  \hspace{1cm} (25)

It is straightforward to show that \[21\]

$$P = \frac{1}{2} \frac{\beta^2}{\beta^2 + D^2 + \lambda^2}.$$  \hspace{1cm} (26)

Thus the oscillation probability has the MSW resonance when $\lambda(p) = 0$ (note that for notational convenience we call this the ‘MSW’ resonance even if $D^2 \gg \beta^2$). Solving the equation, $\lambda(p = p_{res}) = 0$ we find,

$$p_{res} = \frac{X_2}{2X_1} + \sqrt{\left( \frac{X_2}{2X_1} \right)^2 + \frac{\cos 2\theta_0}{X_1}},$$  \hspace{1cm} (27)

where $X_1 \equiv b(p)/p^2$ and $X_2 \equiv a(p)/p$ (note that $X_i$ are independent of $p$). The resonance width in momentum space, $\Delta$, can be obtained from $\lambda(p = p_{res} \pm \frac{1}{2}\Delta) \approx \beta^2 + D^2$,

$$\Delta \approx \frac{4p_{res}}{\delta m^2} \frac{\sqrt{\beta^2 + D^2}}{(2p_{res}X_1 - X_2)}.$$  \hspace{1cm} (28)

where $\beta, D$ are evaluated at $p = p_{res}$. For anti-neutrinos, $X_2 \rightarrow -X_2$ in Eqs.(27,28).

For the numerical work the continuous variable $p/T$ is replaced by a finite set of momenta $p_n/T$ (with $n = 1, 2, ..., N$) on a logarithmically spaced mesh. The variables $P(p)$ and $P_0(p)$ are replaced by the set of $N$ variables $P(x_n)$ and $P_0(x_n)$. The evolution of each of these variables is governed by Eqs.(12,13), where for each value of $n$, the variables $V(p)$ and $D(p)$ are replaced by $V(x_n)$ and $D(x_n)$. The anti-neutrinos are similarly treated. Thus the oscillations of the neutrinos and anti-neutrinos can be described by $8N$ simultaneous differential equations. For each time step the lepton number is computed from Eq.(23) and
the chemical potentials are obtained from Eq. (16). In order to integrate first order differential equations initial conditions must be specified. At very high temperatures the oscillations are heavily suppressed by the collisions so we take $P_{x,y}(p) = 0, \bar{P}_{x,y}(p) = 0$ initially [to see this observe that at high temperature, $D \gg \beta$ so that $P \to 0$ from Eq. (26)]. Also, we assume that there is initially a negligible number of sterile neutrinos which means that we take $P_{z}(p) = \bar{P}_{z}(p) = 0$ initially. The value of our initial temperature must be significantly higher than the temperature where the exponential growth of lepton number occurs [we typically used $T_{\text{initial}} \sim 4T_{c}$, where $T_{c}$ is given in Eq. (5)]. In our numerical work we integrate Eq. (12) around the region

$$p_{\text{res}} - f \frac{\Delta}{2} < p < p_{\text{res}} + f \frac{\Delta}{2},$$

(29)

with $f = 7$. Note that repopulation equation Eq. (13) needs to be integrated over the entire region in momentum space, which we typically approximated to be $0 < p/T < 20$. We also take into account the damping in the momentum region away from resonance. That is for the momentum region,

$$p < p_{\text{res}} - f \frac{\Delta}{2} \text{ and } p > p_{\text{res}} + f \frac{\Delta}{2},$$

(30)

we neglect the oscillations and Eqs. (12,13) are truncated to,

$$\frac{\partial P_{x,y}(p)}{\partial t} = - \left( D(p) + \frac{\partial}{\partial t} \ln P_{0}(p) \right) P_{x,y}(p),$$

$$\frac{\partial P_{z}(p)}{\partial t} = (1 - P_{z}(p)) \frac{\partial}{\partial t} \ln P_{0}(p),$$

$$\frac{\partial P_{0}(p)}{\partial t} \approx \Gamma(p) \left[ K(p) - \frac{1}{2} P_{0}(p)(1 + P_{z}(p)) \right],$$

(31)

and similarly for $P_{x,y,z}(p), P_{0}(p)$. We checked the stability of our results by integrating Eq. (12) over larger slices of momentum space [i.e. taking $f > 7$].

We have numerically solved these equations for three illustrative examples. In Figure 1 we have plotted $|L_{\nu_{e}}|/h$ versus $T$ (recall that $h \equiv T_{3}^{3}/T_{3}^{3}$) for $\nu_{\tau} \rightarrow \nu_{s}$ oscillations. We have chosen $\delta m^{2}/eV^{2} = -0.5, -50, -5000$ and $\sin^{2}2\theta_{0} = 10^{-8}$ with initial condition $L_{\nu_{e}} = 0$ at $T_{\text{initial}} = 300$ MeV. In these examples the rapid exponential growth of lepton number occurs when $T_{c} \simeq 16, 37, 80$ MeV. In the region $T > T_{c}, L^{(\alpha)} \rightarrow 0$ is an approximate fixed point and this explains why $L_{\nu_{e}} \rightarrow -\eta/2 \simeq -2.5 \times 10^{-10}$. In fact this behaviour occurs provided that the initial value of $L_{\nu_{e}}$ is less than $10^{-5}$. In the lower temperature region $T < T_{c}$, the oscillations increase the size of the asymmetry and the final value of the asymmetry is also given approximately by Eq. (7) (assuming of course that the magnitude of the initial asymmetry was less than $L_{\nu_{e}}^{0}$ to begin with).

\footnote{For initial values of $L_{\nu_{e}} \gtrsim 10^{-5}$ the oscillations are not able to drive $L^{(\alpha)} \rightarrow 0$. In this case the oscillations do not significantly destroy the initial asymmetry in the region $T > T_{c}$. In the lower temperature region $T < T_{c}$ the oscillations increase the size of the asymmetry and the final value of the asymmetry is also given approximately by Eq. (7) (assuming of course that the magnitude of the initial asymmetry was less than $L_{\nu_{e}}^{0}$ to begin with).}
The behaviour illustrated in Figure 1 is well understood \cite{20,21} and quite general provided that \( \delta m^2 < 0, |\delta m^2| \gtrsim 10^{-4} \) eV\(^2\) and \( \sin^2 2\theta_0 \) is within the wide range, Eq.\((5)\). In the region before the exponential growth of lepton number occurs \( (T \gtrsim T_c) \), \( L^{(a)} \approx 0 \) (which means that \( X_2 \approx 0 \)) so that the MSW oscillation resonance for neutrinos \( p^\nu_{\text{res}} \) has approximately the same value as the MSW oscillation resonance for anti-neutrinos \( (p^{\bar{\nu}}_{\text{res}}) \). As the neutrino asymmetry is created (taking \( L_{\nu_0} > 0 \) for definiteness), \( p^\nu_{\text{res}}/T \gg 1 \) (and typically \( \gtrsim 20 \)) and \( p^{\bar{\nu}}_{\text{res}}/T < 1 \) (and typically \( \lesssim 0.2 \)). In the region after the exponential growth \( (T \approx T_c) \) the collisions keep \( p^\nu_{\text{res}}/T < 1 \). For lower temperatures \( T \approx T_c/2 \), the \( b(p) \) term \([\text{in Eq.}(13)]\) can be approximately neglected (relative to the \( a(p) \) term) and

\[
\frac{p^{\nu}_{\text{res}}}{T} \approx \frac{1}{TX_2} \alpha \frac{1}{T^4 L^{(\tau)}}.
\]

(32)

As \( T \) increases, \( p^\nu_{\text{res}}/T \) increases and MSW transitions convert \( \bar{\nu}_\alpha \rightarrow \bar{\nu}_s \) at the momentum \( p = p^\nu_{\text{res}} \). The effect of this is to keep \( L_{\nu_0} \) growing until all of the \( \bar{\nu} \) have passed through the MSW resonance\(^6\). If there were no re-population effects (which is approximately true for the case of low \( |\delta m^2| \lesssim 3 \) eV\(^2\), where the asymmetry doesn’t become large until low temperatures, \( T \approx 1 \) MeV) then the expected asymmetry is \( L_{\nu_0} \approx n_{\nu_s}/n_{\nu} \approx 0.375 \) (since all of the \( \nu_0 \) have been converted into \( \nu_s \)). However in the case where \( L_{\nu_0} \) becomes very large in the region above about 1 MeV, repopulation effects must be taken into account, and the final value of the asymmetry is typically reduced somewhat \([\text{see Eq.}(5)]\). Thus, this explains why the final asymmetry is so large and approximately independent of \( \sin^2 2\theta_0 \) so long as it is small. Note that in the case of relatively large \( \sin^2 2\theta_0 \), significant number of \( \nu_s, \bar{\nu}_s \) are populated in the region \( T \approx T_c/2 \). In this case, the MSW transitions at low temperature are less effective at creating \( L_{\nu_0} \). Indeed, in the limit where the \( \nu_s, \bar{\nu}_s \) are fully populated, MSW transitions would simply interchange equal numbers of \( \nu_\alpha \) with \( \bar{\nu}_s \) and thus there would be no significant generation of neutrino asymmetry in this case.

Note that for the examples in Figure 1, the sign of \( L_{\nu_0} \) changed at the temperature \( T \approx T_c/2 \). This behaviour is expected (see Ref.\[20\] for a discussion of this point). Actually our numerical integration of the quantum kinetic equations reveals that for very large \( \sin^2 2\theta_0 \), oscillations of \( L_{\nu_0} \) occur. In this case, it would presumably be impossible to predict the sign of \( L_{\nu_0} \) (see also the discussion in Ref.\[19\]). Finally note that this result was not found in Ref.\[20\] because the static approximation developed there is not valid for \( \sin^2 2\theta_0 \) sufficiently large (due to the extremely rapid exponential growth of lepton number).

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\(^8\) See Ref.\[21\] for a figure illustrating the evolution of \( p^{\nu}_{\text{res}} \) for an example.

\(^9\) Note that the MSW transitions keep \( p^{\nu}_{\text{res}}/T \) approximately constant from \( T \approx T_c/2 \) until \( L_{\nu_0} \) is quite large \((\sim 10^{-2})\). From Eq.\((12)\), \( p^{\nu}_{\text{res}}/T \approx \text{constant} \), implies that \( L_{\nu_0} \) is proportional to \( 1/T^4 \). This explains why \( \frac{d\log L_{\nu_0}}{d\log T} \approx -4 \), in the region after the exponential growth of lepton number occurs for the examples in Figure 1.
We now turn to the main issue of this paper, that is, to accurately determine the region of parameter space where the $\nu_\mu \to \nu_s$ oscillation solution to the atmospheric neutrino anomaly is consistent with a stringent BBN bound of $N_{\text{eff}}^{\text{BBN}} \lesssim 3.6$.

Let us begin by introducing some notation. We will denote the parameters, $\sin^2 2\theta_{0\alpha s}$, $|\delta m^2_{\mu s}|$, and the matter terms $a_{\alpha s}(p)$, $b_{\alpha s}(p)$ for $\nu_\alpha \to \nu_s$ oscillations by $\sin^2 2\theta_{0\alpha s}$, $\delta m^2_{\mu s}$, and $a_{\alpha s}(p)$, $b_{\alpha s}(p)$ respectively ($\alpha = e, \mu, \tau$). We are assuming that there is only one sterile neutrino which mixes maximally or near maximally with the muon neutrino so that the atmospheric neutrino anomaly is solved, i.e.

$$\sin^2 2\theta^\mu_0 \simeq 1, \quad 10^{-3} \lesssim |\delta m^2_{\mu s}|/eV^2 \lesssim 10^{-2}. \quad (33)$$

We assume that

$$m_{\nu_e}, m_{\nu_\mu}, m_{\nu_s} < m_{\nu_\tau}, \quad (34)$$

and that the $\nu_\tau$ oscillates with the $\nu_s$ with small mixing, i.e. $\sin^2 2\theta^\tau_0 \ll 1$. In fact it turns out that the above assumptions are actually necessary if $N_{\text{eff}}^{\text{BBN}} = 4$ is to be avoided.

Of course there will be many specific particle physics models consistent with the assumptions, Eqs.(33,34). For example, Ref. [9] discusses one such model where in addition to approximately maximal $\nu_\mu \to \nu_s$ oscillations there are $\nu_e \to \nu_\mu, \nu_s$ oscillations with parameters consistent with the small angle MSW solution of the solar neutrino problem ($\sin^2 2\theta_0 \sim 10^{-2}$, $|\delta m^2| \sim 10^{-5} eV^2$). Alternatively it is also possible that $\nu_e \to \nu_\mu$ oscillate in the region of parameter space suggested by the LSND experiment [22]. Note however that models with 3 ordinary and 3 sterile neutrinos may be quantitatively different due to the additional oscillation modes possible. We intend to discuss some of these models in a forthcoming paper.

In this four neutrino scenario $\nu_\tau$ is assumed to be the heaviest neutrino so that $\nu_\tau \to \nu_s$ oscillations have $\delta m^2_{\tau s} < 0$, and create $L_{\nu_\tau}$ first at a temperature

$$T_{\nu_\tau} \approx 16 \left(\frac{|\delta m^2_{\tau s}|}{eV^2}\right)^{\frac{1}{2}} \text{MeV}. \quad (35)$$

Note that the creation of $L_{\nu_\tau}$ implies that the matter term for $\nu_\mu \to \nu_s$ oscillations is also generated (note that the $a_{\mu s}(p)$ term in Eq.(19) is proportional to $L^{(\mu)} \simeq 2L_{\nu_\mu} + L_{\nu_\tau} + L_{\nu_e}$). Thus, the creation of a large $L_{\nu_\tau}$ asymmetry also implies the creation of a large $L^{(\mu)}$ function which can potentially suppress the $\nu_\mu \to \nu_s$ oscillations. For example, for maximal vacuum mixing, the matter mixing angle for $\nu_\mu \to \nu_s$ oscillations is,

$$\sin^2 2\theta^\mu_0 = \frac{1}{1 + (a_{\mu s} + b_{\mu s})^2}. \quad (36)$$

In order to simply explain the qualitative behaviour of this oscillation system it is useful to consider the quantities $\langle a_{\alpha s} \rangle$, where $\langle a_{\alpha s} \rangle \equiv a_{\alpha s}(p = 3.15 T)$ and $p \simeq 3.15T$ is the mean
momentum of a neutrino in a Fermi Dirac distribution with zero chemical potential. [Of course in our numerical work the momentum distribution will be taken into account]. In the region \( T \sim T_c^{rs} \) where the exponential growth of \( L_{\nu_e} \) occurs, \( L_{\nu_e} \) is generated so that \( \langle a_{rs} \rangle \sim \) 10. If \( \nu_\mu \rightarrow \nu_s \) oscillations do not create significant \( L_{\nu_\mu} \), then

\[
|\langle a_{\mu s} \rangle| \simeq \frac{1}{2} \frac{|\delta m_{\mu s}^2|}{|\delta m_{\mu s}^2|} |\langle a_{\tau s} \rangle| \gg 1 \text{ if } |\delta m_{\mu s}^2| \ll |\delta m_{\mu s}^2|.
\] (37)

Thus, if \( \nu_\mu \rightarrow \nu_s \) oscillations do not have time to generate significant \( L_{\nu_\mu} \) then the large matter term \( a_{\mu s} \) will be generated which will suppress the \( \nu_\mu \rightarrow \nu_s \) oscillations. During the subsequent evolution, \( \langle a_{\tau s} \rangle \sim 1 \), so that \( \langle a_{\mu s} \rangle \) remains very large and thus the \( \nu_\mu \rightarrow \nu_s \) oscillations will always be heavily suppressed (until very low temperatures where \( T \ll T_{\mu s}^f \)). It is important to realise though, that in response to the \( \nu_\mu \rightarrow \nu_s \) oscillations can potentially create \( L_{\nu_\mu} \) such that \( \langle a_{\mu s} \rangle \rightarrow 0 \), since this is an approximate fixed point of the \( \nu_\mu \rightarrow \nu_s \) oscillation system. Note that the evolution of \( L_{\nu_\mu} \) due to \( \nu_\mu \rightarrow \nu_s \) oscillations is dominated by the oscillations in the resonance region where \( b_{\mu s} \approx a_{\mu s} \) (assuming that \( L_{\nu_e} > 0 \) for definiteness\(^{[8]} \)). The resonance momentum for maximal \( \nu_\mu \rightarrow \nu_s \) oscillations can be obtained from the condition \( \lambda(p) = 0 \). From Eqs.(18,19) it follows that \( p_{res}^{\mu s} \) is given by

\[
p_{res}^{\mu s} = \frac{M_W^2 L^{(\mu)}}{TA_\mu}.
\] (38)

Thus, \( p_{res}^{\mu s} \) is proportional to \( L^{(\mu)} \simeq 2L_\nu + L_{\nu_\tau} + L_{\nu_e} \). As \( L_{\nu_\tau} \) is created this causes \( p_{res}^{\mu s} \) to increase. The \( \nu_\mu \rightarrow \nu_s \) oscillations either create \( L_{\nu_\mu} \) fast enough so that \( p_{res}^{\mu s} \simeq \langle p \rangle \simeq 3.15T \) or they do not. If the \( \nu_\mu \rightarrow \nu_s \) oscillations create \( L_{\nu_\mu} \) sufficiently, then \( |\langle a_{\mu s} - b_{\mu s} \rangle| \simeq 10 \). Furthermore the \( \nu_\mu \rightarrow \nu_s \) oscillations will continue to keep \( |\langle a_{\mu s} - b_{\mu s} \rangle| \simeq 10 \) for lower temperatures because these oscillations become more effective at lower temperatures since they are not suppressed so much by the collisions. Thus, we need to compute the evolution of \( L_{\nu_\mu}, L_{\nu_\tau} \) due to \( \nu_\tau \rightarrow \nu_s \) and \( \nu_\mu \rightarrow \nu_s \) oscillations through the high temperature region \( (\frac{T^{rs}}{2} \approx T \approx T_{initial}) \) in order to obtain the parameter space where \( \nu_\tau \rightarrow \nu_s \) oscillations generate a large \( L^{(\mu)} \) which is not subsequently destroyed by \( \nu_\mu \rightarrow \nu_s \) oscillations. Of course the lower temperature \( (T \sim T_{\nu_\tau}^{rs}/2) \) evolution of lepton number can also affect BBN in several ways, and we will return to this issue in section V.

As in Ref. [20], we assume that the \( \nu_\tau, \nu_\mu, \nu_s \) system reduces to two two-flavour oscillations, \( \nu_\tau \rightarrow \nu_s \) and \( \nu_\mu \rightarrow \nu_s \). This simplifying assumption is discussed in some detail in Ref. [23]. Heuristically, it is expected that this simplifying assumption is justified because the MSW resonance momentum of the two oscillation modes are generally different (and different to the \( \nu_\tau \rightarrow \nu_\mu \) resonance momentum due to the matter effects). Also it should be noted that the \( \nu_\tau \rightarrow \nu_\mu, \nu_\tau \rightarrow \nu_e \) and \( \nu_\mu \rightarrow \nu_e \) oscillations can be approximately neglected at high temperatures \([\text{Here ‘high’ means } T \sim T_{\nu_\tau}^{rs} \text{ which is typically in the range} \]

\(^{10}\) Of course if \( L_{\nu_\tau} > 0 \), then the \( \nu_\mu \rightarrow \nu_s \) MSW resonance occurs for the neutrino oscillations (rather than the anti-neutrino oscillations), which create \( L_{\nu_\mu} < 0 \).
20 \sim T^{\tau_s}/\text{MeV} \sim 60], because these oscillations do not create or destroy lepton number efficiently at high temperature because there are almost equal numbers of $\nu_e, \nu_\mu, \nu_\tau$ neutrinos. Also $\nu_e \rightarrow \nu_s$ oscillations would be expected to be heavily suppressed by the large tau lepton number (assuming that $|\delta m^2_{es}| \ll |\delta m^2_{\tau\tau}|$).

Furthermore, the population of sterile neutrinos by $\nu$-oscillations and similarly for the anti-neutrinos. This obviously implies that the rate of change of lepton number (assuming that $N_{\nu_\tau} = N_{\nu_\nu} = N_{\nu_s}$) due to $\nu_s$ oscillations obviously does not directly affect the number of $\nu_\mu$ neutrinos. Thus,

$$\frac{\partial}{\partial t} N_{\nu_\mu}(p)|_{\nu_\tau \rightarrow \nu_s} = \frac{\partial}{\partial t} [Q_0(p)(1 + Q_z(p))]|_{\nu_\tau \rightarrow \nu_s} = 0. \quad (42)$$

Solving Eqs. (11,12) (and the analogous equations for the anti-neutrinos) implies that

$$N_{\nu_\tau}(p) = \frac{1}{2} P_0(p) (1 + P_z(p)) N^0(p), \quad N_{\nu_\mu}(p) = \frac{1}{2} Q_0(p) (1 + Q_z(p)) N^0(p),$$

$$N_{\nu_s}(p) = \frac{1}{2} P_0(p) (1 - P_z(p)) \frac{1}{2} Q_0(p) (1 - Q_z(p)) N^0(p). \quad (39)$$

The evolution of the functions $P_i(p), Q_i(p)$ are given by Eqs. (12,13), where $\delta m^2, \sin^2 2\theta_0 = \delta m^2_{\tau\tau}, \sin^2 2\theta_0^{s} \ [\delta m^2_{\mu\mu}, \sin^2 2\theta_0^{\mu\mu}]$ in Eq. (14) for $P_i(p)$ and $Q_i(p)$ (of course, for the evolution equations for $P_i(p), Q_i(p)$, $\alpha = \tau, \mu$ in Eq. (19)). At each time step, the lepton numbers $L_{\nu_\tau}, L_{\nu_\mu}$ are computed using Eq. (23) [in the case of $dL_{\nu_\mu}/dt, P_i(p) \rightarrow Q_i(p)$]. The two sets of evolution equations for $P_i(p), Q_i(p)$ are coupled together because they each depend on both $L_{\nu_\tau}$ and $L_{\nu_\mu}$. They are also coupled together through the population of the sterile state. In particular the population of $\nu_s$ by $\nu_\tau \rightarrow \nu_s$ oscillations can affect $\nu_\mu \rightarrow \nu_s$ oscillations. Similarly the population of $\nu_s$ by $\nu_\mu \rightarrow \nu_s$ oscillations can affect $\nu_\tau \rightarrow \nu_s$ oscillations. This effect can only be important if the population of sterile neutrinos becomes relatively large, i.e. $n_{\nu_s} \sim 0.1$. Note that the population of $\nu_s$ due to $\nu_\mu \rightarrow \nu_s$ oscillations is negligible in the temperature region $T \sim T^{\tau_s}_{c}$. This is because $T^{\tau_s}_{c}$ is typically high enough so that the $\nu_\mu \rightarrow \nu_s$ oscillations are suppressed by the matter effects and also damped by collisions so that they cannot populate a significant number of sterile states. However, if $\sin^2 2\theta_0^{s}$ is large enough, then the $\nu_\tau \rightarrow \nu_s$ oscillations generate significant and approximately equal numbers of $\nu_s, \bar{\nu}_s$ in the region $T \sim T^{\tau_s}_{c}$. This population of sterile neutrinos will obviously affect the $\nu_\mu \rightarrow \nu_s$ oscillations. The population of $\nu_s, \bar{\nu}_s$ by $\nu_\tau \rightarrow \nu_s$ oscillations can be incorporated by noting that

$$\frac{N_{\nu_s}(p)}{N^0(p)} = \frac{1}{2} P_0(p) (1 - P_z(p)) = \frac{1}{2} Q_0(p) (1 - Q_z(p)). \quad (40)$$

and similarly for the anti-neutrinos. This obviously implies that the rate of change of $Q_i(p)$ due to $\nu_\tau \rightarrow \nu_s$ oscillations (we denote this quantity by $\frac{\partial}{\partial t} [Q_i(p)|_{\nu_\tau \rightarrow \nu_s}$) satisfies

$$\frac{\partial}{\partial t} [Q_0(p)(1 - Q_z(p))]|_{\nu_\tau \rightarrow \nu_s} \simeq \frac{\partial}{\partial t} [P_0(p)(1 - P_z(p))]. \quad (41)$$

Furthermore, the population of sterile neutrinos by $\nu_\tau \rightarrow \nu_s$ oscillations obviously does not directly affect the number of $\nu_\mu$ neutrinos. Thus,
\[ \frac{\partial}{\partial t} \tilde{Q}_0(p)|_{\nu_\tau \rightarrow \nu_s} = \frac{\partial}{\partial t} Q_0(p)|_{\nu_\tau \rightarrow \nu_s} = \frac{1}{2} \left( \frac{\partial P_0(p)}{\partial t} (1 - P_z(p)) - \frac{\partial P_z(p)}{\partial t} P_0(p) \right), \]
\[ \frac{\partial}{\partial t} \tilde{Q}_z(p)|_{\nu_\tau \rightarrow \nu_s} = \frac{\partial}{\partial t} Q_z(p)|_{\nu_\tau \rightarrow \nu_s} = -\frac{1}{2} \frac{1(1 + Q_z(p))}{Q_0(p)} \left( \frac{\partial P_0(p)}{\partial t} (1 - P_z(p)) - \frac{\partial P_z(p)}{\partial t} P_0(p) \right). \]

where here we have neglected the tiny difference in \( \nu_s, \bar{\nu}_s \) populations (which is actually necessary for self consistency). When solving the evolution equations for \( P_i(p), Q_i(p) \) we have included the contribution, Eq.(43).

Our initial conditions for the numerical integration are (as explained in section III), \( P_{x,y}(p) = \tilde{P}_{x,y}(p) = 0 \) and \( P_z(p) = \tilde{P}_z(p) = P_0(p) = \tilde{P}_0(p) = 1 \) (and similarly for the \( Q_i(p), \bar{Q}_i(p), i = x, y, z, 0 \)). We also set the initial values of all of the neutrino asymmetries to zero, and took \( T_{\text{initial}} = 4T_{\text{res}} \). Of course the results do not depend on \( T_{\text{initial}} \) so long as it is high enough (and \( T_{\text{initial}} = 4T_{\text{res}} \) is certainly high enough). As we discussed in section III (and in Ref. [20]), the results are also independent of the initial values of the neutrino asymmetries so long as they are less than about \( 10^{-5} \). We also utilise the approximation of integrating around the region of the MSW resonance [taking \( f = 7 \) in Eq.(29)] for both \( \nu_\tau \rightarrow \nu_s \) and \( \nu_\mu \rightarrow \nu_s \) oscillations. We checked the stability of our results by taking larger slices of momentum space.

Performing the necessary numerical work, we obtained the obtained the region of parameter space where the \( L^{(\mu)} \) created by \( \nu_\tau \rightarrow \nu_s \) oscillations is not destroyed by maximal \( \nu_\mu \rightarrow \nu_s \) oscillations. This region is given in Figure 2. Of course, we must also require that the sterile neutrinos do not become significantly populated by \( \nu_\tau \rightarrow \nu_s \) oscillations. Recall from Eq.(8) that \( \delta N_{\text{eff}}^{BBN} \lesssim 0.6 \) implies the constraint [20],
\[ \sin^2 2\theta_0 \lesssim 4 \times 10^{-5} \left[ \frac{eV^2}{|\delta m_{\tau s}^2|} \right]^{\frac{1}{2}}. \]

Note that \( \nu_\tau \rightarrow \nu_s \) oscillations populate the sterile neutrinos in the temperature region \( T \gtrsim T_{\text{res}}^{\nu_\tau} \) where \( a_{\tau s}(p) \) is very small. The constraint, Eq.(44) is given by the dashed-dotted line in Figure 2. There are also other contributions to \( N_{\text{eff}}^{BBN} \), which are due to the low temperature evolution of \( L_{\nu_\tau}, L_{\nu_\mu} \) and \( L_{\nu_e} \), including the population of sterile neutrinos at low temperatures. We will return to these issues in section V.

Let us now compare our new results with the earlier calculations of Ref. [20]. In Figure 7 of Ref. [20] we took \( |\delta m_{\text{atmos}}^2| = 10^{-2} \) \( eV^2 \) and computed the allowed region by numerically integrating an approximate solution of the quantum kinetic equations. This approximation (which we called the ‘static approximation’ in Ref. [20]) holds provided that the system is smooth on the interaction time scale \( 1/D(p) \). For this \( \delta m_{\text{atmos}}^2 \) our new results are in agreement with the previous calculation of Ref. [20] in the region \( \sin^2 2\theta_0 \gtrsim 3 \times 10^{-7} \). This result was anticipated in Ref. [20], where it was found that the evolution of lepton number was smooth enough for the static approximation to be acceptable in this parameter range. In the case where \( \sin^2 2\theta_0 \gtrsim 3 \times 10^{-7} \) our new results show that the allowed region is somewhat smaller when compared with the approximation of Ref. [20].
Finally recall that in the context of the standard big bang model, the age of the Universe implies an upper bound on the mass of the tau neutrino which is

\[ m_{\nu_{\tau}} \lesssim 100 \text{ eV}, \] (45)

and hence

\[ |\delta m_{rs}^2| \lesssim 10^4 \text{ eV}^2 \] (46)

Some authors (including myself) have taken a more stringent bound of \( m_{\nu_{\tau}} \lesssim 40 \text{ eV} \) (which implies \( |\delta m_{rs}^2| \lesssim 10^3 \text{ eV}^2 \)). While the latter bound certainly appears to be favoured, we feel it is prudent to be cautious, and we therefore adopt the weaker bound Eq.(46).

V Detailed implications for big bang nucleosynthesis

Let us briefly summarise the story so far. In the region above the solid line(s) in Figure 2, \( \nu_{\tau} \to \nu_s \) oscillations create a large \( L_{\nu_{\tau}} \) which suppresses \( \nu_{\mu} \to \nu_s \) oscillations so that these oscillations cannot significantly populate the sterile state (for \( T \gtrsim 0.5 \text{ MeV} \)). Let us denote the contribution to \( N_{\text{BBN}}^{\text{eff}} \) from the oscillations, \( \nu_{\mu} \to \nu_s \) (\( \nu_{\tau} \to \nu_s \) in the region \( T \gtrsim T_{\nu_{\tau}}^{\text{eq}} \)) by the notation \( \delta_1 N_{\text{eff}}^{\text{BBN}} (\delta_2 N_{\text{eff}}^{\text{BBN}}) \). With this notation, in the parameter region to the right (left) of the dashed-dotted line \( \delta_2 N_{\text{eff}}^{\text{BBN}} > 0.6 \) (\( \delta_2 N_{\text{eff}}^{\text{BBN}} < 0.6 \)). Also, in the region above (below) the solid line(s) in Figure 2, \( \delta_1 N_{\text{eff}}^{\text{BBN}} \ll 0.1 \) (\( \delta_1 N_{\text{eff}}^{\text{BBN}} + \delta_2 N_{\text{eff}}^{\text{BBN}} \simeq 1 \)).

It is important to realise that \( N_{\text{eff}}^{\text{BBN}} \) depends on the number densities of \( \nu_e, \bar{\nu}_e \) through BBN nuclear reaction rates (as well as the expansion rate). Thus, it is important to study the evolution of the four neutrino system down to low temperatures \( T \sim 0.5 \text{ MeV} \). The evolution of this four neutrino system down to low temperatures has already been studied in some detail in Ref. [21] and we include a discussion here for completeness. Since the final value of \( L_{\nu_{\tau}}, L_{\nu_{\mu}} \) is quite big there is a significant modification to the momentum distribution of the tau and anti-tau neutrinos. If the tau neutrinos also oscillate into muon and electron neutrinos, then some of the tau lepton number can be transferred to the electron and muon neutrinos. In other words small \( L_{\nu_{\mu}}, L_{\nu_{e}} \) will be created. The details are quite independent of the intergenerational mixing angles so long as they are small (here, small means \( \sin^2 2\theta \gtrsim 0.1 \)) and depend only on \( \delta m^2 \). In Ref. [21] the contribution to \( N_{\text{eff}}^{\text{BBN}} \) due to the modification of the \( \nu_e, \bar{\nu}_e \) momentum distributions (which we denote by \( \delta_3 N_{\text{eff}}^{\text{BBN}} \)) was found to be

\[
\begin{align*}
\delta_3 N_{\text{eff}}^{\text{BBN}} &\simeq 0(0), \text{ for } |\delta m^2|/eV^2 \lesssim 3, \\
\delta_3 N_{\text{eff}}^{\text{BBN}} &\simeq -0.45(0.45) \text{ for } 3 \lesssim |\delta m^2|/eV^2 \lesssim 1000, \\
\delta_3 N_{\text{eff}}^{\text{BBN}} &\simeq -0.50(0.50) \text{ for } |\delta m^2|/eV^2 \gtrsim 1000.
\end{align*}
\] (47)

11They cannot be arbitrarily small lest the oscillations become non adiabatic (typically they must be greater than about \( 3 \times 10^{-10} \)).
for $L_{\nu_e} > 0$ ($L_{\nu_e} < 0$). Note that $\delta_3 N_{eff}^{BBN}$ in the above equation is actually continuous so there is really a smooth transition region between the three $\delta m^2$ regions which we have neglected for simplicity. The creation of the lepton number can also change the expansion rate since the energy density of the Universe will be modified a bit by the modification of the number and momentum distributions that occur when the large $L_{\nu}$ neutrino asymmetry is created. The contribution to $N_{eff}^{BBN}$ due to this effect (which we denote by $\delta_4 N_{eff}^{BBN}$) was calculated to be \[ \delta_4 N_{eff}^{BBN} \simeq 0 \quad \text{for } |\delta m^2|/eV^2 \lesssim 3, \]
\[ \delta_4 N_{eff}^{BBN} \simeq -0.05 \quad \text{for } 3 \lesssim |\delta m^2|/eV^2 \lesssim 1000, \]
\[ \delta_4 N_{eff}^{BBN} \simeq 0.40 \quad \text{for } |\delta m^2|/eV^2 \gtrsim 1000. \] \hspace{1cm} (48)

Finally note that the results in Eq.(47) and Eq.(48) are only valid provided that $\delta_1 N_{eff}^{BBN} + \delta_2 N_{eff}^{BBN} \lesssim 0.1$. This is only approximately true in the region above the solid line(s) in Figure 2 and in the region where $\nu_\tau \to \nu_s$ oscillations do not significantly populate $\nu_s$ at high temperature $T \sim T_c$. The region of parameter space where $\nu_\tau \to \nu_s$ oscillations do not significantly populate $\nu_s$ at high temperature (i.e. $\delta_2 N_{eff}^{BBN} \lesssim 0.1$) is given by \[ \sin^2 2\theta_0 \lesssim 1.3 \times 10^{-5} \left[ \frac{eV^2}{|\delta m^2|} \right]^{1/2}. \] \hspace{1cm} (49)

Comparing this equation with Eq.(8) (which assumes that $\delta_2 N_{eff}^{BBN} \lesssim 0.6$) we see that it is only slightly more stringent. If there is a significant number density of $\nu_s$ coming from $\nu_\tau \to \nu_s$ oscillations at high temperature, then this will reduce the size of $L_{\nu_\tau}$. The effects given in Eq.(47) and Eq.(48) will therefore be reduced.

Observe that in the region in Figure 2 below the solid line(s), the sterile neutrino will be populated by $\nu_\mu \to \nu_s$ oscillations at a temperature $T \sim 6 - 10$ MeV. When this occurs the growth in $L_{\nu_\tau}$ is cutoff because the number of tau and sterile neutrinos become approximately equal. A these temperatures, $|L_{\nu_\tau}| \lesssim 10^{-2}$ so that the possible modifications of $\nu_e, \bar{\nu}_e$ are negligible in this case. In other words $N_{eff}^{BBN} \simeq 4$, with $\delta_1 N_{eff}^{BBN} + \delta_2 N_{eff}^{BBN} \simeq 1$, $\delta_3 N_{eff}^{BBN}$, $\delta_4 N_{eff}^{BBN} \ll 0.1$.

To summarise, it is possible to identify 4 distinct contributions to $N_{eff}^{BBN}$. They are:

(1) $\delta_1 N_{eff}^{BBN}$. This is the contribution to $N_{eff}^{BBN}$ which arises from the change in the expansion rate of the Universe due to the population of the sterile neutrinos by $\nu_\mu \to \nu_s$ oscillations.

\[ \sin^2 2\theta_0 \lesssim 1.3 \times 10^{-5} \left[ \frac{eV^2}{|\delta m^2|} \right]^{1/2}. \] \hspace{1cm} (49)

\[ \delta_4 N_{eff}^{BBN} \simeq 0 \quad \text{for } |\delta m^2|/eV^2 \lesssim 3, \]
\[ \delta_4 N_{eff}^{BBN} \simeq -0.05 \quad \text{for } 3 \lesssim |\delta m^2|/eV^2 \lesssim 1000, \]
\[ \delta_4 N_{eff}^{BBN} \simeq 0.40 \quad \text{for } |\delta m^2|/eV^2 \gtrsim 1000. \] \hspace{1cm} (48)

12 Actually it should be noted that $\delta_2 N_{eff}^{BBN}$, $\delta_3 N_{eff}^{BBN}$ and $\delta_4 N_{eff}^{BBN}$ are all continuous above the solid line(s) in Figure 2. This means that their values change smoothly. The contribution $\delta_1 N_{eff}^{BBN}$, on the other hand is discontinuous across the solid line(s).

13 Note that there is a mistake in the corresponding equation in Ref. [21]. Eq.(19) is the correct equation.
\( \delta_2 N_{\text{eff}}^{BBN} \). This is the contribution to \( N_{\text{eff}}^{BBN} \) which arises from the change in the expansion rate due to the population of sterile neutrinos by \( \nu_\tau \rightarrow \nu_s \) oscillations for temperatures before the exponential growth of the \( L_{\nu_\tau} \) occurs (i.e. \( T \gtrsim T_c^{rs} \)).

\( \delta_3 N_{\text{eff}}^{BBN} \). This is the contribution to \( N_{\text{eff}}^{BBN} \) from the direct modification of the nuclear reaction rates (such as \( \nu_e + N \leftrightarrow P + e^- \), \( \bar{\nu}_e + P \leftrightarrow N + e^+ \)). This occurs because of the modification of the momentum distributions of \( \nu_e, \bar{\nu}_e \) due to a small \( L_{\nu_\tau} \) asymmetry which is transferred from \( L_{\nu_\tau} \) by \( \nu_\tau \rightarrow \nu_e \) oscillations.

\( \delta_4 N_{\text{eff}}^{BBN} \). This is the contribution to \( N_{\text{eff}}^{BBN} \) from the change in the expansion rate due to the modification of the energy densities of the neutrinos at low temperature \( T \sim T_f^{\tau s} \) which is caused by the large \( L_{\nu_\tau} \).

Note that both the effects (2) and (4) are essentially due to the population of \( \nu_s \) by \( \nu_\tau \rightarrow \nu_s \) oscillations. We label them distinctly because the effect (2) occurs at high temperatures before the generation of neutrino asymmetry occurs (in this region \( \nu_s, \bar{\nu}_s \) are populated approximately in equal numbers) while the effect (4) occurs because of the population of \( \bar{\nu}_s \) (taking \( L_{\nu_\tau} > 0 \) for definiteness) which necessarily occurs in the temperature region where the neutrino asymmetry nears its final value. Finally, the effective neutrino number for BBN is related to the contributions, \( \delta_i N_{\text{eff}}^{BBN} \), in the obvious way

\[
N_{\text{eff}}^{BBN} = 3 + \sum_{i=1}^{4} \delta_i N_{\text{eff}}^{BBN}.
\] (50)

The results of this section are summarised in Figures 3,4. Figure 3 (Figure 4) is for the case where \( L_{\nu_e} > 0 \) (\( L_{\nu_e} < 0 \))\(^{14} \). As these figures show, there is a significant chunk of parameter space where \( N_{\text{eff}}^{BBN} < 3 \). Thus, not only is the \( \nu_\mu \rightarrow \nu_s \) oscillation solution to the atmospheric neutrino anomaly consistent with a stringent BBN bound of \( N_{\text{eff}}^{BBN} < 3.6 \), but there is a range of parameters where this solution implies that there are less than 3 neutrinos for BBN. It is interesting that there are some recent indications that \( N_{\text{eff}}^{BBN} < 3 \) is actually favoured by the data especially if the low deuterium abundance measurements \( D/H \sim 3 \times 10^{-5} \)\(^{37} \) in quasar absorption systems are correct \(38,39\)\(^{15} \). The lesson here is that there are well motivated extensions of the standard model which give \( N_{\text{eff}}^{BBN} < 3 \). Thus, if it turns out that \( N_{\text{eff}}^{BBN} < 3 \) is required for consistency of the standard big bang

\(^{14}\) The sign of \( L_{\nu_e} \) is the same as the sign of \( L_{\nu_\tau} \), which cannot be predicted at the moment (see Ref. \( 20\) for some discussion on this point).

\(^{15}\) It has been argued that \( N_{\text{eff}}^{BBN} \) can be as large as 4.5 \(^{40} \) and still be consistent with the data. However this paper assumed that \( D/H \) could be as large as \( 2.5 \times 10^{-4} \) which seems to be disfavoured by recent studies \( 37 \) (see also Ref. \( 16 \) for a review). Nevertheless, things are not completely clear at the moment, so one should keep in mind the possibility that \( N_{\text{eff}}^{BBN} \) as large as 4.5, as suggested by Ref. \( 40 \), may be consistent with the standard big bang model.
model, then this should be taken seriously as an indication that sterile neutrinos exist\textsuperscript{16}.

Finally, the scenario we have given in this section holds provided that $\nu_e \rightarrow \nu_s$ oscillations can be neglected. In the case where $|\delta m^2_{es}| \ll 1 \text{ eV}^2$, as happens for the particular case studied in Ref. 3 (where $\nu_e \rightarrow \nu_{\mu}, \nu_s$ oscillations solve the solar neutrino problem), there is no impact for BBN from $\nu_e \rightarrow \nu_s$ oscillations. However, in the case where $|\delta m^2_{es}|/\text{eV}^2 \gtrsim 1$ (which is compatible with the LSND experiment), it turns out that the $\nu_e \rightarrow \nu_s$ oscillations cannot be neglected for BBN. In particular, consider the possibility that

$$m_{\nu_e} < m_{\nu_{\mu}} \approx m_{\nu_s} < m_{\nu_{\tau}} \quad \text{and} \quad \delta m^2_{es} \sim 1 \text{ eV}^2,$$

along with the requirement that the atmospheric neutrino anomaly is solved by $\nu_{\mu} \rightarrow \nu_s$ oscillations, Eq.(33). We will assume that $\sin^2 2\theta^e_0$ is small enough so that $\nu_e \rightarrow \nu_s$ oscillations are unable to create significant $L_{\nu_e}$ at high temperatures, $T \sim T^r_{\nu_s}$. [The situation at high temperature is qualitatively similar to the case studied in section IV]. The low temperature evolution of $L_{\nu_e}$ will be dominated by the MSW transitions. It is instructive to consider the resonance momentum of $\nu_e \rightarrow \nu_s$ oscillations, $p^e_{res}$. At low temperatures, where the $b(p)$ terms can be neglected, the resonance condition arises from the equation $\pm \delta(p) = \cos 2\theta_0$ ($\approx 1$ in this case for both $\nu_\tau \rightarrow \nu_s$ and $\nu_e \rightarrow \nu_s$ oscillations). Taking $L_{\nu_{\tau}} > 0$ for definiteness, which means that at low temperatures, the $\bar{\nu}_\tau \rightarrow \nu_s$ have a MSW resonance, and the $\nu_e \rightarrow \nu_s$ oscillations have a MSW resonance (rather than the $\bar{\nu}_e \rightarrow \nu_s$ oscillations). This opposite behaviour occurs because $\delta m^2_{es} > 0$ while $\delta m^2_{es} < 0$. Thus it is clear that the $\nu_e \rightarrow \nu_s$ oscillations generate $L_{\nu_e}$ with the opposite sign to $L_{\nu_{\tau}}$. Using Eq.(33),

$$\frac{p^e_{res}}{T} \approx \frac{\pi^2 \delta m^2_{es}}{4\zeta(3)\sqrt{2}G_F T^4 L^{(e)}} \approx 0.5 \frac{\delta m^2_{es} \text{MeV}^4}{\text{eV}^2} \frac{0.23}{T^4}.$$  \hspace{1cm} (52)

Eventually, for $T \sim T^f_{\nu_e}$ [see Eq.(3)], $L_{\nu_e} \rightarrow L^f_{\nu_e} \approx 0.23$ (for $3 \gtrsim |\delta m^2_{es}|/\text{eV}^2 \approx 1000$). For these temperatures, $L^{(e)} \approx 2L_{\nu_e} + L^f_{\nu_e}$. Thus, as $T \rightarrow 0$, Eq.(52) implies that $p^e_{res}/T \rightarrow \infty$ and there will be significant MSW conversion of $\nu_e \rightarrow \nu_s$. In this case there cannot be complete MSW conversion because of the structure of Eq.(52). In fact Eq.(52), implies that the creation of $L_{\nu_e}$ actually increases the rate at which $p^e_{res}/T$ moves through the momentum distribution. Thus, $L_{\nu_e}$ is created much more rapidly than $L_{\nu_{\tau}}$. Indeed it must be created so rapidly that the oscillations are not completely adiabatic which is why incomplete MSW conversion occurs in this case. In fact, it is possible to show that as $p^e_{res}/T \rightarrow \infty$, $L^{(e)} \rightarrow 0$ (approximately), i.e. $L_{\nu_e} \rightarrow -L^f_{\nu_e}/2 \approx -0.12$. Since significant generation of $L_{\nu_e}$ cannot occur until temperatures where $p^e_{res}/T \gtrsim 1$, it follows from Eq.(52) that for $\delta m^2_{es} \ll 1 \text{eV}^2$ the significant creation of $L_{\nu_e}$ occurs at temperatures much less than 0.5 MeV, so that there

\textsuperscript{16} Of course neutrino oscillations is not the only mechanism which can give $N_{eff}^{BBN} < 3$, however it is perhaps the most well motivated given the existing neutrino anomalies. For some other ways to get $N_{eff}^{BBN} < 3$, see for example, Ref. 11.\textsuperscript{17}

\textsuperscript{17}Note that from Eq.(33) it follows that $\nu_{\mu}$ and $\nu_s$ are maximal mixtures of two nearly degenerate mass eigenstates in this case.
is no significant impact for BBN in this case. However, for $\delta m^2_{\nu_s} \gtrsim 1 \text{ eV}^2$, the $L_{\nu_e}$ would be created during the time when the $N + \nu_e \leftrightarrow P + e^-$, $P + \bar{\nu}_e \leftrightarrow N + e^+$ reactions are still occurring. Thus, in this case there will be a significant modification to BBN. This connection between the oscillations in the LSND parameter range and BBN is extremely interesting, but we will not present any quantitative numerical results here. However, we do intend to discuss this effect more fully in the context of models with 3 ordinary and 3 sterile neutrinos in the forthcoming paper [35].

VI Implications for the hot+cold dark matter model

In the scenario considered in this paper, where $\nu_\mu \rightarrow \nu_s$ oscillations solve the atmospheric neutrino anomaly, a BBN bound of $N_{\text{BBN}}^{\text{eff}} < 3.6$ implies $m_{\nu_e} \gtrsim 4 \text{ eV}$ for $|\delta m^2_{\text{atmos}}| \simeq 10^{-2.5} \text{ eV}^2$ (see Figure 2). Neutrino masses in the eV range have long been considered to be interesting for cosmology. This is because these neutrinos can make a significant contribution to the energy density of the Universe. In the standard big bang model, the contribution of massive neutrinos to the energy density is given by the well known formula,

$$\Omega_{\text{neutrino}} = \frac{\sum_\alpha m_{\nu_\alpha}}{h^2 92 \text{ eV}},$$

(53)

where $h$ is the usual cosmological parameter parameterising the uncertainty in the Hubble constant. Thus, neutrinos in the eV mass range are a well known and well motivated candidate for hot dark matter. Some studies have suggested that a hot+cold dark matter mixture with $\Omega_{\text{neutrinos}} \simeq 0.20 - 0.25$ can nicely explain the structure formation [42]. If only one eV neutrino is assumed, which we take to be the tau neutrino, then the neutrino mass favoured in the usual hot+cold dark matter scenario is [42]

$$3 \text{ eV} \lesssim m_{\nu_\tau} \lesssim 7 \text{ eV},$$

(54)

This situation is modified somewhat when light sterile neutrinos exist. The reason is that the $\nu_\tau \rightarrow \nu_s$ oscillations generate such a large $L_{\nu_\tau}$ that the total number of tau neutrinos is actually significantly reduced. In the region where $3 \lesssim |\delta m^2_{\tau s}|/\text{eV}^2 \lesssim 1000$, the $L_{\nu_\tau} \simeq 0.23$ [from Eq.(7)]. The large final lepton number occurs because about 70% of the anti-neutrinos have been depleted (assuming $L_{\nu_\tau} > 0$) which means that the total number of tau neutrinos (i.e. tau + anti-tau neutrinos) is approximately 0.65 of a standard neutrino species. (Note that the total number of neutrinos hasn’t changed much, the missing heavy anti-tau neutrinos have just been converted into light sterile states.) This significant depletion of tau neutrinos implies that the tau neutrino mass favoured in the hot+cold dark matter scenario (which suggests that $\Omega_{\text{neutrinos}} \simeq 0.20 - 0.25$) is actually about 50% larger than the naive expectation. Thus when $\nu_\tau$ oscillates with a light $\nu_s$, the hot+cold dark matter scenario actually favours a tau neutrino mass of

$$5 \text{ eV} \lesssim m_{\nu_\tau} \lesssim 10 \text{ eV},$$

(55)

rather than $3 - 7 \text{ eV}$. This means that $|\delta m^2_{\tau s}|$ is expected to be in the range

$$25 \text{ eV}^2 \lesssim |\delta m^2_{\tau s}| \lesssim 100 \text{ eV}^2,$$

(56)
assuming here that the sterile neutrino is much lighter than the tau neutrino. The hot+cold
dark matter region, Eq. (56) is the shaded band on Figure 2. From this figure, we see
that for $|\delta m^2_{\text{atmos}}| = 10^{-2}$ eV$^2$ there is only a relatively small range of $\sin^2 2\theta_{0s}$ where the
favoured hot+cold dark matter region is compatible with $N^\text{BBN}_{\text{eff}} \lesssim 3.6$. However, for lower
values of $|\delta m^2_{\text{atmos}}|$ which are actually favoured by the superKamiokande data, there is a
significant range for $\sin^2 2\theta_{0s}$ where the favoured hot+cold dark matter region is compatible
with $N^\text{BBN}_{\text{eff}} \lesssim 3.6$.

Finally note that detailed studies [42] of structure formation in the hot+cold dark matter
model show that they are sensitive to the number of eV neutrinos (not just $\Omega_{\text{neutrinos}}$). These
studies typically assume that the number of eV neutrinos (usually taken to be degenerate)
at the epoch of matter radiation equality (we will denote this quantity by $N^\text{dm}_{\text{eff}}$) is an integer,
i.e., $N^\text{dm}_{\text{eff}} = 1, 2, 3$. It is important to understand that this is only true provided that sterile
neutrinos do not exist. Indeed, as we have just explained above, we expect $N^\text{dm}_{\text{eff}} \simeq 0.65 in$
the case where only $\nu_\tau$ is in the eV mass range and oscillates with a light sterile neutrino[41].

VII Implications for the cosmic microwave background

During the next decade or so, high precision measurements of the anisotropy of the
cosmic microwave background (CMB) will be performed by several experiments (such as
the PLANK and MAP missions). From these experiments it may be possible to estimate
accurately the radiation content of the Universe at the epoch of photon - matter decoupling
[44]. In this context it is important to note that sterile neutrinos can leave their ‘imprint’
on the cosmic microwave background [15]. Thus it is useful to introduce the quantities
$N^\text{lu}_{\text{eff}}, N^\text{hv}_{\text{eff}}$ where $N^\text{lu}_{\text{eff}}$ ($N^\text{hv}_{\text{eff}}$) is the effective number of light neutrinos (heavy neutrinos)
at the epoch of photon decoupling. In this context, ‘light’ means much less than about
an eV (which will make them relativistic at the epoch of photon decoupling) and ‘heavy’
means more than about an eV (which will make them approximately non-relativistic). Of
course in the minimal standard model of particle physics which has 3 massless neutrinos,
$N^\text{BBN}_{\text{eff}} = N^\text{lu}_{\text{eff}} = 3, N^\text{hv}_{\text{eff}} = 0$. However in the four neutrino model case that we are discussing
in this paper, $N^\text{BBN}_{\text{eff}} \neq N^\text{lu}_{\text{eff}}$ and $N^\text{hv}_{\text{eff}} \neq 0$ in general. [Note that $N^\text{BBN}_{\text{eff}} \neq N^\text{lu}_{\text{eff}}$ not just
because of the heavy tau neutrino but also because of the contribution of $L_\nu$ to $N^\text{BBN}_{\text{eff}}$].

If we assume that $m_{\nu_\tau} \gtrsim 4$ eV (as suggested by Figure 2) and $m_{\nu_e}, m_{\nu_\mu}, m_{\nu_s} \ll 1$ eV then then

$$N^\text{hv}_{\text{eff}} \simeq \frac{n_{\nu_e}}{n_0}, \quad N^\text{lu}_{\text{eff}} = \frac{\rho_{\nu_e} + \rho_{\nu_\mu} + \rho_{\nu_s}}{\rho_0},$$

(57)

where $n_i$ ($\rho_i$) are the mass (energy) densities with $n_0$ ($\rho_0$) being the mass (energy) density of

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18 In Ref. [24], the 4 neutrino model of Ref. [43] where $\nu_\mu \rightarrow \nu_\tau$ oscillations solved the atmospheric
neutrino anomaly was studied. In that particular model $\nu_\mu, \nu_\tau$ are assumed to be approximately
degenerate and in the eV mass range. In that model the oscillations of the $\nu_\mu \rightarrow \nu_\beta$ and $\nu_\tau \rightarrow \nu_\beta$
created a large $L_\mu \simeq L_\tau \simeq 0.16$ and $N^\text{dm}_{\text{eff}} \simeq 1.5$.
a massless neutrino with Fermi-Dirac distribution with zero chemical potential. Assuming that $3 \lesssim |\delta m_{\tau s}^2|/eV^2 \lesssim 1000$, using results from Ref. [21],

$$N_{\nu}^{e f f} \simeq N_{\nu}^{e f f} \simeq 0.65, \quad N_{\nu}^{e f f} \simeq 2.2.$$  \hspace{1cm} (58)

The precise measurements of the CMB may well prove to be quite useful in distinguishing between various competing explanations of the neutrino anomalies, since each model should leave quite a distinctive imprint on the CMB.

**VIII Concluding remarks**

In this paper we have numerically integrated the quantum kinetic equations to obtain the region of parameter space where the $\nu_\mu \rightarrow \nu_s$ oscillation solution to the atmospheric neutrino anomaly is consistent with a BBN bound of 3.6 effective neutrinos. The consistency occurs because the $\nu_\tau \rightarrow \nu_s$ oscillations create a large $L_{\nu_\tau}$ asymmetry. This large asymmetry modifies the effective potential for $\nu_\mu \rightarrow \nu_s$ oscillations which become heavily suppressed for the range of parameters given in Figure 2. These dynamically induced matter effects prevent the $\nu_\mu \rightarrow \nu_s$ oscillations from significantly populating the sterile state for the entire ‘allowed region’ of Figure 2. This work confirms and improves the previous study [20] which utilised an approximate solution of the quantum kinetic equations (which we called the ‘static’ approximation in Ref. [20]). We have also discussed the detailed implications of this scenario for BBN (see figures 3,4), the hot+cold dark matter model, and also for the forthcoming precision measurements of the cosmic microwave background.

In this paper we assumed the existence of only one light sterile neutrino. Probably the simplest four neutrino scheme which can solve the atmospheric neutrino anomaly by $\nu_\mu \rightarrow \nu_s$ oscillations and also solve the solar neutrino problem is the case considered in Ref. [9] (where small angle MSW enhanced $\nu_e \rightarrow \nu_\mu, \nu_s$ oscillations solve the solar neutrino problem). Our results indicate that the tau neutrino mass should be larger than $3 - 4$ eV if consistency with a stringent BBN bound of 3.6 neutrinos is required. Note however that the scenario of Ref. [4] cannot explain the LSND anomaly [22]. The LSND experiment [22] has provided strong evidence for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ and $\nu_\mu \rightarrow \nu_e$ oscillations. The suggested parameter range is $\sin^2 2\theta_0 \sim 10^{-2}$ and

$$0.2 \lesssim |\delta m_{lsnd}^2|/eV^2 \lesssim 6.$$  \hspace{1cm} (59)

One may well wonder whether or not there exists any four neutrino scheme which can solve all three experimental anomalies (with the $\nu_\mu \rightarrow \nu_s$ oscillations solving the atmospheric neutrino anomaly), and still be consistent with a stringent BBN bound of 3.6 neutrinos? The only potential scheme which comes to mind is [10],

$$m_{\nu_e} \simeq m_{\nu_\mu} > m_{\nu_\mu}, m_{\nu_s}.$$  \hspace{1cm} (60)

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19Please excuse the sloppy notation in Eq.(60). Of course the mass eigenstates are linear combinations of weak eigenstates.
with $\nu_e \rightarrow \nu_\tau$ oscillations solving the solar neutrino problem, either through vacuum oscillations or through MSW enhanced oscillations (this means that $|\delta m^2_{\tau e}| \approx 10^{-3} \text{ eV}^2$). The LSND anomaly is explained by $\nu_\mu \rightarrow \nu_e$ oscillations with

$$\delta m^2_{\tau s} \simeq \delta m^2_{es} \simeq \delta m^2_{e\mu} \simeq \delta m^2_{\text{lsnd}} \approx 6 \text{ eV}^2.$$  \hfill (61)

In this scenario, $\nu_\tau \rightarrow \nu_s$ oscillations will generate $L_{\nu_\tau}$ and $\nu_e \rightarrow \nu_s$ oscillations generate $L_{\nu_e}$. However because the $|\delta m^2_{\text{lsnd}}|$ is so low, the results in Figure 2 indicate that, except for a very small region of parameter space ($|\delta m^2_{\mu s}| \approx 10^{-3} \text{ eV}^2, \sin^2 2\theta \sim 10^{-5}$), we expect the $\nu_\mu \rightarrow \nu_s$ oscillations to generate a $L_{\nu_\mu}$ asymmetry such that $L^{(\mu)} \rightarrow 0$. This means that we expect that the $\nu_\mu \rightarrow \nu_s$ oscillations will eventually fully populate the sterile neutrino at a temperature around $6 - 10 \text{ MeV}$. We conclude that this scheme does not seem to be compatible with a BBN bound of 3.6 neutrinos. Hence, if experimental data indicate that $\nu_\mu \rightarrow \nu_s$ oscillations are required to explain the atmospheric neutrino data, and if the solar and LSND anomalies have been correctly interpreted in terms of neutrino oscillations, then $N_{\text{BBN}}^{\text{eff}} < 4$ actually suggests the need for more than four neutrinos. Of course, if light sterile neutrinos exist then the theoretically most attractive possibility is that there are three of them, just like there are three types of ordinary neutrinos and three types of charged leptons etc. Furthermore there are well motivated extensions of the standard model which have three light sterile neutrinos. For example, parity conserving theories \cite{parity} (models with similar neutrino phenomenology but without a mirror sector have also been proposed in Ref. \cite{models}) necessarily have three light sterile neutrinos which can naturally explain all of the neutrino anomalies if neutrinos have mass (see the second paper of Ref. \cite{summary} for a summary). These schemes also have the nice property that they can explain these neutrino anomalies without assuming any large mixing between generations, and they can also have a neutrino mass heirarchy. These last two features are really expected given the situation with the quarks and the charged leptons. The implications of these models for early Universe cosmology is in progress \cite{progress}.

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\textsuperscript{20} Note that the additional oscillation mode, $\nu_e \rightarrow \nu_s$ could increase the ‘allowed region’ in Figure 2 a little, but not substantially.
REFERENCES

[1] P. Langacker, to appear in the Proceedings of the XVIII International Conference on Neutrino Physics and Astrophysics, Takayama, Japan, June 1998.

[2] T. Haines et al., Phys. Rev. Lett. 57, 1986 (1986); Kamiokande Collaboration, K. S. Hirata et al., Phys. Lett. B205, 416 (1988); ibid. B280, 146 (1992); Y. Fukuda et al., Phys. Lett. B335, 237 (1994); IMB Collaboration, D. Casper et al., Phys. Rev. Lett. 66, 2561 (1991); R. Becker-Szendy et al., Phys. Rev. D46, 3720 (1992); NUSEX Collaboration, M. Aglietta et al., Europhys. Lett. 8, 611 (1989); Frejus Collaboration, Ch. Berger et al., Phys. Lett. B227, 489 (1989); ibid. B245, 305 (1990); K. Daum et al., Z. Phys. C66, 417 (1995); Soudan 2 Collaboration, W. W. M., Allison et al., Phys. Lett. B391, 491 (1997).

[3] Super-Kamiokande Collaboration, Y. Fukuda et al., hep-ex/9803006; hep-ex/9805006; hep-ex/9807003; see also Kamiokande Collaboration, S. Hatakeyama et al., hep-ex/9806038; MACRO Collaboration, M. Ambrosio et al., hep-ex/9807005; hep-ex/9808001.

[4] R. Foot, R. R. Volkas and O. Yasuda, Phys. Rev. D58, 013006 (1998); see also O. Yasuda, to appear in the Proceedings of the XVIII International Conference on Neutrino Physics and Astrophysics, Takayama, Japan, June 1998 (hep-ph/9809206); M. C. Gonzalez-Garcia, H. Numokawa, O. L. G. Peres and J. W. F. Valle, hep-ph/9807304.

[5] J. G. Learned, S. Pakvasa and T. J. Weiler, Phys. Lett. B207, 79 (1988); V. Barger and K. Whisnant, Phys. Lett. B209, 365 (1988); K. Hidaka, M. Honda and S. Midorikawa, Phys. Rev. Lett. 61, 1537 (1988).

[6] E. Akhmedov, P. Lipari and M. Lusignoli, Phys. Lett. B300, 128 (1993).

[7] R. Foot, Mod. Phys. Lett. A9, 169 (1994); R. Foot and R. R. Volkas, Phys. Rev. D52, 6595 (1995).

[8] J. Bowes and R. R. Volkas, J. Phys. G24, 1249 (1998); A. Geiser, CERN-EP-98-056; P. Langacker, hep-ph/9805281; Y. Koide and H. Fusaoka, hep-ph/9806510; W. Krolikowski, hep-ph/9808307; see also, C. Giunti, C. W. Kim and U. W. Kim, Phys. Rev. D46, 3034 (1992); M. Kobayashi, C. S. Lim and M. M. Nojiri, Phys. Rev. Lett. 67, 1685 (1991).

[9] Q. Y. Liu and A. Yu. Smirnov, Nucl. Phys. B524, 505 (1998); M. Bando and K. Yoshioka, hep-ph/9806400.

[10] J. T. Peltoniemi, D. Tommasini and J. W. F. Valle, Phys. Lett. B298, 383 (1993); N. Okada and O. Yasuda, Int. J. Mod. Phys.A12, 3669 (1997); S. M. Bilenky, C. Giunti and W. Grimus, Eur. Phys. J. C1, 247 (1998); V. Barger, S. Pakvasa, T. J. Weiler and K. Whisnant, hep-ph/9806328; A. S. Joshipura and A. Yu. Smirnov, hep-ph/9806376.

[11] R. Foot, R. R. Volkas and O. Yasuda, Phys. Lett. B433, 82 (1998); O. Yasuda, hep-ph/9804400; G. L. Fogli, E. Lisi, A. Marrone and G. Scioscia, hep-ph/9808203; E. Kh. Akhmedov, A. Dighe, P. Lipari and A. Yu. Smirnov, hep-ph/9808270.

[12] T. Kajita, Talk at the topical workshop on neutrino physics, Institute of theoretical physics, University of Adelaide, Nov 1996; F. Vissani and A. Yu. Smirnov, Phys. Lett. B432, 376 (1998); Q. Y. Liu, S. P. Mikheyev and A. Yu. Smirnov, hep-ph/9803415; P. Lipari and M. Lusignoli, hep-ph/9803410; J. G. Learned, S. Pakvasa and J. L. Stone, hep-ph/9805343; L. J. Hall and H. Murayama, hep-ph/9806218; see also Refs. [1, 4].
[13] D. W. Sciama, SISSA preprint, 1998.
[14] D. W. Sciama, Modern cosmology and the dark matter problem, (Cambridge, CUP, 1993).
[15] P. Langacker, University of Pennsylvania Preprint, UPR 0401T, September (1989); R. Barbieri and A. Dolgov, Phys. Lett. B237, 440 (1990); Nucl. Phys. B349, 743 (1991); K. Kainulainen, Phys. Lett. B244, 191 (1990); K. Enqvist, K. Kainulainen and M. Thomson, Nucl. Phys. B 373, 498 (1992); J. Cline, Phys. Rev. Lett. 68, 3137 (1992); X. Shi, D. N. Schramm and B. D. Fields, Phys. Rev. D48, 2568 (1993); G. Raffelt, G. Sigl and L. Stodolsky, Phys. Rev. Lett. 70, 2363 (1993); C. Y. Cardall and G. M. Fuller, Phys. Rev. D54, 1260 (1996).
[16] D. N. Schramm and M. S. Turner, Rev. Mod. Phys. 70, 303 (1998).
[17] R. Foot and R. R. Volkas, Phys. Rev. Lett. 75, 4350 (1995).
[18] R. Foot, M. J. Thomson and R. R. Volkas, Phys. Rev. D53, 5349 (1996).
[19] X. Shi, Phys. Rev. D54, 2753 (1996).
[20] R. Foot and R. R. Volkas, Phys. Rev. D55, 5147 (1997).
[21] R. Foot and R. R. Volkas, Phys. Rev. D56, 6653 (1997); Erratum (to appear).
[22] LSND Collaboration, C. Athanassopoulos et al (LSND Collab), Phys. Rev. C54, 2685 (1996); Phys. Rev. Lett. 77, 3082 (1996); Phys. Rev. Lett. 81, 1774 (1998).
[23] R. Foot and R. R. Volkas, Astropart. Phys. 7, 283 (1997).
[24] N. F. Bell, R. Foot and R. R. Volkas, hep-ph/9805255 (to appear in Phys. Rev. D58).
[25] N. F. Bell, R. R. Volkas and Y.Y.Y. Wong, University of Melbourne Preprint, hep-ph/9809363.
[26] K. Enqvist, K. Kainulainen and J. Maalampi, Nucl. Phys. B349, 754 (1991).
[27] D. P. Kirilova and M. V. Chizhov, Phys. Lett. B393, 375 (1997); hep-ph/9806441.
[28] For a basic introduction to the density matrix see the book by Merzbacher, Quantum Mechanics, Wiley, second edition, 1970, Chapter 13.
[29] B. H. J. McKellar and M. J. Thomson, Phys. Rev. D49, 2710 (1994) and references there-in.
[30] The application of the density matrix to ordinary-sterile neutrino oscillations was first discussed in the papers: R. A. Harris and L. Stodolsky, Phys. Lett. B78, 313 (1978); A. Dolgov, Sov. J. Nucl. Phys. 33, 700 (1981).
[31] See the paper by Enqvist et al., in Ref. [15].
[32] L. Stodolsky, Phys. Rev. D36, 2273 (1987); M. Thomson, Phys. Rev. A45, 2243 (1991).
[33] L. Wolfenstein, Phys. Rev. D17, 2369 (1978); S. P. Mikheyev and A. Yu. Smirnov, Nuovo Cim. C9, 17 (1986); see also V. Barger et al, Phys. Rev. D22, 2718 (1980).
[34] D. Notzold and G. Raffelt, Nucl. Phys. B307, 924 (1988).
[35] R. Foot and R. R. Volkas, in preparation.
[36] See e.g. S. Sarkar, Rept. Prog. Phys. 59, 1493 (1996).
[37] D. Tytler, X-M. Fan and S. Burles, Nature 381, 207 (1996); S. Burles and D. Tytler, astro-ph/9712108 astro-ph/9712109; see also D. Tytler, S. Burles and D. Kirkman, astro-ph/9612121.
[38] N. Hata et al, Phys. Rev. Lett. 75, 3977 (1995).
[39] Y. Cardall and G. M. Fuller, astro-ph/9603071; N. Hata, G. Steigman, S. Bludman and P. Langacker, Phys. Rev. D55, 540 (1997); C. Copi, D. N. Schramm and M. S. Turner, ibid. 55, 3389 (1997).
[40] P. J. Kernan and S. Sarkar, Phys. Rev. D 54, 3681 (1996).
[41] R. Foot and H. Lew, Mod. Phys. Lett. A8, 3767 (1993); S. Dodelson, G. Gyuk and M. S. Turner, Phys. Rev. D49, 5068 (1994) and references therein; A. D. Dolgov, S. Pastor, J. C. Romão and J. W. F. Valle, Nucl. Phys. B496, 24 (1997); K. Kohri, M. Kawasaki and K. Sato, Phys. Lett. B430, 132 (1998); S. Hannestad, Phys. Rev. D57, 2213 (1998); K. Kainulainen, H. Kurki-Suonio and E. Sihvola, astro-ph/9807098.
[42] R. K. Schaefer, Q. Shafi and F. W. Stecker, Astrophys. J., 347, 575 (1989); J. Holtzman, Astrophys. J., Suppl. 74, 1 (1989); J. Primack, J. Holtzman, A. Klypin and D. O. Caldwell, Phys. Rev. Lett. 74 (1995) 2160. For a review, see e.g. J. Primack, astro-ph/9610078.
[43] D. O. Caldwell and R. N. Mohapatra, Phys. Rev. D48, 3259 (1993); J. T. Peltoniemi and J. W. F. Valle, Nucl. Phys. B406, 409 (1993).
[44] R. E. Lopez, S. Dodelson, A. Heckler and M. S. Turner, astro-ph/9803095, and references there-in.
[45] S. Hannestad and G. Raffelt, astro-ph/9805223.
Figure Captions

Figure 1. The creation of $|L_{\nu_{\tau}}|/h$ by $\nu_{\tau} \rightarrow \nu_{s}$ oscillations as a function of temperature, $T/MeV$. In these examples, $\sin^{2}2\theta_{0} = 10^{-8}$ and $\delta m^{2}/eV^{2} = -0.5, -50, -5000$ for the dashed line, solid line and dotted line respectively.

Figure 2. The ‘allowed region’ of parameter space in the $\sin^{2}2\theta_{\tau s}^{s}, -\delta m_{\tau s}^{2}/eV^{2}$ plane, where the $\nu_{\tau} \rightarrow \nu_{s}$ oscillations create tau lepton number in such a way as to prevent maximal $\nu_{\mu} \rightarrow \nu_{s}$ oscillations from populating the sterile neutrino for $T \gtrsim 0.5 MeV$. The bold solid line, upper solid line, lower solid line are the boundaries assuming $|\delta m_{\text{atmos}}^{2}|/eV^{2} = 10^{-2.5}, 10^{-2}, 10^{-3}$ respectively. The dashed-dotted line is the bound Eq.(44) which arises by requiring that the $\nu_{\tau} \rightarrow \nu_{s}$ oscillations do not populate the sterile neutrino in the temperature region before the exponential growth of $L_{\nu_{\tau}}$ occurs. The shaded region is the favoured region in hot + cold dark matter models (see the discussion in Section VI).

Figure 3. Detailed predictions for $N_{\text{BBN}}^{\text{eff}}$ in the $\sin^{2}2\theta_{\tau s}^{s}, -\delta m_{\tau s}^{2}/eV^{2}$ plane. For simplicity, $|\delta m_{\text{atmos}}^{2}| = 10^{-2.5} eV^{2}$ is used. In this figure $L_{\nu_{e}} > 0$ is assumed. Note that $N_{\text{eff}}^{\text{BBN}}$ is actually continuous across the horizontal boundary at $\delta m^{2} = -1000 eV^{2}$. The transition region is roughly $400 \lesssim -\delta m^{2}/eV^{2} \lesssim 3000$ (the transition region is not shown for simplicity).

Figure 4. Same as Figure 3, except that $L_{\nu_{e}} < 0$ is assumed.
Figure 1
Figure 2

Allowed Region

$-\delta m^2/eV^2$

$\sin^2 2\theta_0$
Figure 3

\[ N_{\text{BBN}}^{\text{eff}} = 2.9 \]

\[ 2.5 < N_{\text{BBN}}^{\text{eff}} < 3.5 \]

\[ 3.5 < N_{\text{BBN}}^{\text{eff}} < 4.0 \]

\[ 3.6 < N_{\text{BBN}}^{\text{eff}} < 4.0 \]

\[ \sin^2 2\theta_0 \]

\[ -\delta m^2 / eV^2 \]
Figure 4

$-\delta m^2/eV^2$

$3.9 < N_{BBN}^{eff} < 4.0$

$3.4 < N_{BBN}^{eff} < 4.0$

$3.7 < N_{BBN}^{eff} < 4.0$

$N_{BBN}^{eff} = 3.9$

$N_{BBN}^{eff} = 3.4$

$N_{BBN}^{eff} = 4.0$

$\sin^2 2\theta_0$