Method for Solving the Bloch Equation from the Connection with Time-Dependent Oscillator

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Abstract

We introduce a novel method to find exact density operators for a spin-1/2 particle in time-dependent magnetic fields by using the one-mode bosonic representation of $su(2)$ and the connection with a time-dependent oscillator. As illustrative examples, we apply the method to find the density operators for constant and/or oscillating magnetic fields, which turn out to be time-dependent in general.

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I. INTRODUCTION

Recently two-level atoms have been investigated under the influence of pulsed or oscillating fields \[1,2\]. However, a system of spin-1/2 particles under external magnetic fields has been a subject of interest since the advent of quantum mechanics \[3\]. Also, two-level systems of electric dipoles under external electric fields have been studied for a long time \[4\], and the statistics of a spin-1/2 particle has been found in terms of density matrices \[3,5\]. All these systems can be described by the Lie algebra $su(2)$ whose generators satisfy the commutation relations for angular momentum (in units of $\hbar = 1$)

$$[S_a, S_b] = i\epsilon_{abc}S_c,$$ (1.1)

where $\epsilon_{abc}$ is the Levi-Civita tensor, taking $+1(-1)$ for even (odd) permutations of 1, 2, 3 and vanishing otherwise. The Lie algebra $su(2)$ can be used to find the evolution operator for the spin-1/2 particle system \[6\].

On the other hand, a harmonic oscillator has the Lie algebra $su(1,1)$, which is isomorphic to $so(2,1)$ \[7\]. The generators of $su(1,1)$ produce the spectrum of a number of quantum systems. These generators satisfy the commutation relations

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2.$$ (1.2)

We may introduce the position representation of $su(1,1)$ as

$$K_1 = \frac{1}{4} \left( \frac{\partial^2}{\partial q^2} + q^2 \right), \quad K_2 = -\frac{i}{2} \left( q \frac{\partial}{\partial q} + \frac{1}{2} \right), \quad K_3 = \frac{1}{4} \left( \frac{\partial^2}{\partial q^2} - q^2 \right).$$ (1.3)

In the above definition, we scaled $y = 2q$ as in Ref. \[7\]. The Lie algebra $su(1,1)$ has also been used to find the quantum states of an oscillator \[8\].

The compact Lie algebra $su(2)$ is isomorphic to $so(3)$, $(su(2) \approx so(3))$, whereas the noncompact Lie algebra $su(1,1)$ is isomorphic to $so(2,1)$, $(su(1,1) \approx so(2,1))$ \[7\]. In fact, the compact group $SO(3)$ generated by $so(3)$ holds $x_1^2 + x_2^2 + x_3^2$ invariant whereas the noncompact group $SO(2,1)$ generated by $so(2,1)$ holds $x_1^2 + x_2^2 - x_3^2$ invariant. Though the spin generators of $su(2)$ satisfy the commutation relations for an angular momentum operator, many years ago Schwinger introduced the two-mode bosonic representation for $su(2)$ \[9\]. It is, thus, tempting to exploit the connection between the spin particle and the oscillator. As $su(2)$ is compact but $su(1,1)$ is noncompact, the connection between these two
algebras requires a complexification of generators. This can be achieved by introducing a one-mode bosonic representation of $su(2)$, $S_x = -K_3$, $S_y = -iK_1$, and $S_z = iK_2$. The generators $S_y$ and $S_z$ are anti-Hermitian because $K_1$ and $K_2$ are Hermitian operators. Nevertheless, it is likely that the oscillator may be used to find the evolution of the spin particle just as the evolution operator of an oscillator can be found from a spin-1/2 system.

The density operator of a physical system is an important tool for understanding the statistical properties of a system. The density operator, $\rho(t)$, provides an ensemble average, $\text{Tr}[\rho(t)A]$, for every physical observable $A$. In this paper, we introduce a novel method to solve the homogeneous Bloch equation from the connection with the time-dependent oscillator that uses the one-mode bosonic representation of $su(2)$ and find a class of density operators for a spin-1/2 particle in time-dependent magnetic fields. A time-dependent oscillator is known to possess invariant operators, or density operators, that satisfy the Liouville equation. We use some known invariant operators for the oscillator to find the density operators for the spin particle, for instance, in constant and/or oscillating magnetic fields.

The organization of this paper is as follows: In Sec. II, we briefly discuss the density operator for a spin-1/2 particle in a time-dependent magnetic field. The density operator satisfying the Liouville equation is determined by a solution of the Bloch or Landau-Lifshitz equation. In Sec. III, we introduce a one-mode bosonic representation of $su(2)$ and find the oscillator that corresponds to the spin-1/2 particle. Further, using an invariant operator for the oscillator, we find a density operator of the spin particle through a connection of Lie algebras $su(1,1)$ and $su(2)$. In Sec. IV, we compare our method with the evolution operator and the complex parameter method. In Sec. V, we apply the method to find density operators in constant and/or oscillating magnetic fields and compare them with the well-known results.

II. SPIN-1/2 PARTICLE IN A MAGNETIC FIELD

We consider a spin-1/2 particle under the influence of an external magnetic field. For the sake of simplicity, we set $\gamma = g\mu_B = 1$, absorbing it into the magnetic field, where $g$ and $\mu_B$ are the Landé g-factor and the Bohr magneton, respectively, of the spin particle. Then, the
spin particle has the Hamiltonian
\[ H(t) = - \sum_{a=x,y,z} B_a(t) S_a, \] (2.1)
where \( S_a \) is the \( a \)th spin component. The spin particle obeys the time-dependent Schrödinger equation for a spinor state:
\[ i \frac{\partial}{\partial t} |\Psi, t\rangle = H(t) |\Psi, t\rangle. \] (2.2)

It is well understood that any instantaneous eigenstate of a time-dependent Hamiltonian is not, in general, an exact quantum state. Nor does the evolution operator follow \( U(t) = \exp[-i \int H(t) dt] \) because \([H(t'), H(t)] \neq 0\) for \( t' \neq t \). We may directly solve either the Heisenberg or Schrödinger equation. However, instead of solving the Heisenberg equation, in this paper, we use the invariant operator method introduced by Lewis and Riesenfeld [10]. In fact, an invariant or density operator satisfying the Liouville equation
\[ i \frac{\partial \mathcal{I}(t)}{\partial t} + [\mathcal{I}(t), H(t)] = 0 \] (2.3)
leads to a general quantum state of the form
\[ |\Psi, t\rangle = \sum_n c_n e^{i \int \langle n, t | \frac{\partial}{\partial t} - H(n, t) | n, t \rangle}, \] (2.4)
where
\[ \mathcal{I}(t)|n, t\rangle = \lambda_n |n, t\rangle. \] (2.5)

For the spin 1/2-particle with \( su(2) \), one may find a density operator of the form [11]
\[ \mathcal{I}(t) = - \sum_{a=x,y,z} M_a(t) S_a, \] (2.6)
where the vector \( \mathbf{M} \) with components \( M_a \) satisfies the Bloch or Landau-Lifshitz equation
\[ \frac{d\mathbf{M}}{dt} + \mathbf{M} \times \mathbf{B} = 0. \] (2.7)
Note that \( \mathbf{M}^2 \) is a constant of the motion. There have been some attempts to find the evolution operator or density operator for a spin particle in magnetic fields [11, 12, 13].
III. CONNECTION WITH AN OSCILLATOR

We now introduce a one-mode bosonic representation for \( su(2) \):

\[
S_x = \frac{1}{4}(p^2 + q^2), \quad S_y = \frac{i}{4}(p^2 - q^2), \quad S_z = \frac{i}{4}(pq + qp),
\]  
(3.1)

where the standard commutation relation \([q, p] = i\) holds. Note that our one-mode bosonic representation of \( su(2) \) consists of \( S_x = -K_3 \), \( S_y = -iK_1 \), and \( S_z = iK_2 \). Therefore, the generators \( S_y \) and \( S_z \) are anti-Hermitian because \( K_1 \) and \( K_2 \) are Hermitian operators whereas \( S_x \) and \( K_3 \) are both Hermitian. The non-Hermitian nature of the one-mode representation follows from the fact that \( su(2) \) is compact whereas \( su(1,1) \) is noncompact. Though the bosonic representation in Eq. (3.1) indeed satisfies the spin angular momentum relation in Eq. (1.1), the relation \( S^2_a = \frac{I}{4} \) however, does not hold for the bosonic representation because the oscillator has an infinite number of states. The oscillator Hamiltonian corresponding to the spin Hamiltonian in Eq. (2.1) takes the form

\[
H(t) = B_+(t)\frac{p^2}{2} + B^*_+(t)\frac{q^2}{2} + \frac{iB_z(t)}{2}(pq + qp),
\]  
(3.2)

with

\[
B_+(t) = \frac{1}{2}(B_x + iB_y), \quad B_- = B^*_+.
\]  
(3.3)

A passing remark is that the Hamiltonian in Eq. (3.2) is not a Hermitian operator because the \( S_a \) are constructed as non-Hermitian. This means that the spin directions of \( S_a \) do not pertain to a real physical space, but to directions in a complex space. Nevertheless, the most important point is that the algebraic relations hold whatever representation one may use. Any physical interpretation should given to real operators that are obtained from algebraic relations in the end. Though invariant under the parity operator, \( q \to -q \) and \( p \to -p \), the Hamiltonian does not respect time-reversal symmetry, \( p \to -p \) and \( i \to -i \), and thus cannot be an example for \( \mathcal{PT} \)-invariant quantum mechanics [14].

From now on, we shall work on the non-Hermitian oscillator in Eq. (3.2). The spin particle or oscillator with \( B_+ = 0 \) is trivial, so this case will not be considered anymore. In the case of \( B_+ \neq 0 \), the oscillator in Eq. (3.2) may have an invariant operator of the form

\[
a(t) = -i\left[ u(t)p - \frac{1}{B_+}\left( \dot{u}(t) - i\frac{B_z}{2}u(t) \right)q \right].
\]  
(3.4)
In fact, \( a(t) \) satisfies the Liouville equation

\[
i \frac{\partial a(t)}{\partial t} + [a(t), H(t)] = 0 \tag{3.5}
\]

when \( u \) is a solution to the auxiliary equation

\[
\frac{d}{dt} \left( \frac{\ddot{u}}{B_+} \right) + \left[ \frac{B^2}{4} - B_+ \frac{d}{dt} \left( \frac{iB_z/2}{B_+} \right) \right] \left( \frac{u}{B_+} \right) = 0, \tag{3.6}
\]

or written as \( u = B_+^{1/2} v, \) \( v \) satisfies the equation in canonical form

\[
\ddot{v} + \left[ \frac{B^2}{4} - B_+ \frac{d}{dt} \left( \frac{iB_z/2}{B_+} \right) - \frac{3}{4} \left( \frac{\dot{B}_+}{B_+} \right)^2 + \frac{1}{2} \left( \frac{\dddot{B}_+}{B_+} \right) \right] v = 0. \tag{3.7}
\]

A second independent solution to Eq. (3.6) may lead to another invariant operator. However, the complex conjugate \( u^* \) cannot, in general, be a solution to Eq. (3.6) due to the complex coefficient \( B_+ \). Thus, \( a^\dagger \) is not another invariant operator, in contrast with the Hermitian oscillator case.

Our stratagem to find the density operators for the spin particle is first to find invariant operators which are quadratic in \( q \) and \( p \) and then to use the inverse relations in Eq. (9). For instance, \( a^2 \) may lead, using Eq. (9), to the density operator

\[
\mathcal{I}(t) = -2 \left\{ u^2 + \left( \frac{\ddot{u} - \frac{iB_z}{2} u}{B_+} \right)^2 \right\} S_x - i \left\{ u^2 - \left( \frac{\ddot{u} - \frac{iB_z}{2} u}{B_+} \right)^2 \right\} S_y + 2iu \left( \frac{\dddot{u} - \frac{iB_z}{2} u}{B_+} \right) S_z. \tag{3.8}
\]

Therefore, we obtain the density operator in Eq. (2.6) with the complex components

\[
\mathcal{M}_x = -2 \left\{ u^2 + \left( \frac{\ddot{u} - \frac{iB_z}{2} u}{B_+} \right)^2 \right\}, \\
\mathcal{M}_y = 2i \left\{ u^2 - \left( \frac{\dddot{u} - \frac{iB_z}{2} u}{B_+} \right)^2 \right\}, \\
\mathcal{M}_z = -4iu \left( \frac{\dddot{u} - \frac{iB_z}{2} u}{B_+} \right). \tag{3.9}
\]

A direct calculation shows that the components in Eq. (3.9) satisfy Eq. (2.7) for the density operator. Furthermore, the complex \( \mathcal{M} \) satisfies the constraint equation

\[
\mathcal{M}_x^2 + \mathcal{M}_y^2 + \mathcal{M}_z^2 = 0. \tag{3.10}
\]

Note that Eq. (3.10) is a consistent condition because \( \mathcal{M}^2 \) is a constant of the motion for Eq. (2.7), so the eigenvalue of the invariant operator, here \( \lambda^2 = \mathcal{M}^2 \), should be constant.
The real part $M_r$ and the imaginary part $M_i$ of $M$ individually satisfy Eq. (2.7), a linear equation. We, thus, have two real density operators:

$$\mathcal{I}_r = - \sum_{a=x,y,z} M_r a S_a, \quad \mathcal{I}_i = - \sum_{a=x,y,z} M_i a S_a.$$  

(3.11)

Note that the real and the imaginary parts have the same magnitude, but they are orthogonal to each other

$$M_r^2 = M_i^2, \quad M_r \cdot M_i = 0.$$  

(3.12)

IV. COMPARISON WITH OTHER METHODS

We shall compare the method in Sec. III with other methods: the evolution operator and the method recently introduced by Kobayashi [16, 17]. The evolution operator is determined, for instance, by Eq. (6.23a) of Ref. [18]. The Schrödinger equation leads the evolution operator $U$,

$$U = \begin{pmatrix} v & i w \\ iw^* & v^* \end{pmatrix}, \quad v^* v + w^* w = 1,$$

(4.1)

to satisfy the equation

$$\frac{d}{dt} \left( \frac{x}{B_-} \right) + \left[ \frac{B^2}{4} + B_- \frac{d}{dt} \left( \frac{i B_z/2}{B_-} \right) \right] \left( \frac{x}{B_-} \right) = 0, \quad (x = v, w).$$  

(4.2)

Note that Eq. (4.2) is the complex conjugate of Eq. (3.6), thus implying $x = u^*$. In this sense, Eq. (3.6) will determine not only the evolution operator but also the density matrix, as was shown in Sec. III.

Kobayashi recently introduced two methods for solving the Bloch equation. In his first method, the Bloch equation is transformed to the rotating reference frame [16]. This idea of a rotating frame is in essence similar to using the time-dependent creation and annihilation operators in the Fock space of time-dependent harmonic oscillators [15, 19]. These operators are chosen to satisfy the Liouville equation, Eq. (3.5) or Eq. (2.3), so the eigenvalue of the number operator $\hat{N}(t) = \hat{a}^\dagger(t)\hat{a}(t)$ is a constant of the motion, and the exact quantum state in Eq. (2.4) is an instantaneous eigenstate of the number operator up to a time-dependent phase factor.

On the other hand, in his second method Kobayashi introduced complex parameters for the magnetization [17]. As the magnitude $|M|$ of the magnetization vector $M$ is a constant...
of the motion, the magnetization vector can be normalized:

$$m = \frac{M}{|M|}, \quad m_x^2 + m_y^2 + m_z^2 = 1. \quad (4.3)$$

He then introduced two complex parameters:

$$\xi = \frac{m_x + im_y}{1 - m_z} = \frac{1 + m_z}{m_x - im_y},$$
$$\eta = -\frac{1 - m_z}{m_x - im_y} = -\frac{m_x + im_y}{1 + m_z}. \quad (4.4)$$

The Bloch equation, Eq. (2.7), in his notation reading as

$$\frac{dm}{dt} + \gamma B \times m = 0, \quad (4.5)$$

leads to the Riccati equation

$$\dot{\xi} = \frac{\gamma}{2}(B_y + iB_z)\xi^2 - i\gamma B_z\xi + \frac{\gamma}{2}(B_y - iB_z). \quad (4.6)$$

There is a similar Riccati equation for $\eta$. In our notation, it is written as

$$\dot{\xi} = i\gamma B_-\xi^2 - i\gamma B_z\xi - i\gamma B_+. \quad (4.7)$$

We linearize the Ricatti equation in Eq. (4.7) by introducing a new variable

$$\xi = \frac{i}{\gamma B_-} (\frac{\dot{z}}{z} - i\frac{\gamma}{2} B_z). \quad (4.8)$$

Finally, we can obtain

$$\frac{d}{dt} \left( \frac{\dot{z}}{\gamma B_-} \right) + \left[ \gamma^2 \frac{B^2}{4} - \gamma B_- \frac{d}{dt} \left( \frac{i\gamma B_z/2}{\gamma B_-} \right) \right] \left( \frac{z}{\gamma B_-} \right) = 0. \quad (4.9)$$

Note that Eq. (4.9) is the same as Eq. (4.2) for the evolution operator because we scaled $\gamma$ as $\gamma = 1$ in Eq. (2.7) and $B \leftrightarrow -B$. This means that our method based on the connection between a spin-1/2 particle in time-dependent magnetic fields and a time-dependent oscillator with time-dependent mass and/or frequency is equivalent to the complex parameter method of Kobayashi. Hence, our method provides a group theoretical foundation for Kobayashi’s method.
V. EXACT DENSITY OPERATORS IN CONSTANT AND/OR OSCILLATING MAGNETIC FIELDS

We now apply our method to find the class of density operators in Eq. (3.8) for a spin particle in magnetic fields. To show the nontrivial nature of the density operator, we find the density operator in the trivial case of constant and/or oscillating magnetic fields and compare it with the known result.

A. Constant Field

First, we directly solve Eq. (2.7) for the density operator. In the case of a constant field, \( \mathbf{M} \parallel \mathbf{B}_0 \) satisfies Eq. (2.7). The density operator with the components

\[
M_x = M_0 B_{0x}, \quad M_y = M_0 B_{0y}, \quad M_z = M_0 B_{0z},
\]

(5.1)

where \( M_0 \) is an arbitrary constant, is proportional to the Hamiltonian itself. In the coordinate system with the \( z \)-direction along \( \mathbf{B}_0 \), a second solution may be found:

\[
M_x = M_{0\perp} \cos(B_0 t + \varphi), \quad M_y = -M_{0\perp} \sin(B_0 t + \varphi), \quad M_z = M_{0z},
\]

(5.2)

where \( M_{0\perp} \), \( M_{0z} \), and \( \varphi \) are constants that are determined by the initial conditions. The vector \( \mathbf{M} \) rotates around the magnetic field due to the torque \( \tau = \mathbf{M} \times \mathbf{B}_0 \).

Second, using the connection with the oscillator, we may find the density operator given in Eq. (3.9). We look for a solution to Eq. (3.6) of the form

\[
u_+(t) = B_{0+} M_0^{1/2} e^{i(B_0 t + \varphi)/2}
\]

(5.3)

for a positive constant \( M_0 \) and a phase constant \( \varphi \). The complex components of the density operator are given by

\[
\mathcal{M}_x = -\frac{1}{2} M_0 \left[ (B_{0x} + iB_{0y})^2 - (B_0 - B_{0z})^2 \right] e^{i(B_0 t + \varphi)},
\]

\[
\mathcal{M}_y = \frac{i}{2} M_0 \left[ (B_{0x} + iB_{0y})^2 + (B_0 - B_{0z})^2 \right] e^{i(B_0 t + \varphi)},
\]

\[
\mathcal{M}_z = M_0 (B_{0x} + iB_{0y})(B_0 - B_{0z}) e^{i(B_0 t + \varphi)}.
\]

(5.4)

Finally, we obtain a real density operator

\[
\mathcal{I}_r(t) = -\sum_{a=1}^{3} M_{(r)a} S_a.
\]

(5.5)
where
\[
M_{(r)x} = -\frac{1}{2}M_0\left[B_{0x}^2 + B_{0y}^2 - (B_0 - B_{0z})^2\right]\cos(B_0t + \varphi) + M_0B_{0x}B_{0y}\sin(B_0t + \varphi),
\]
\[
M_{(r)y} = -\frac{1}{2}M_0\left[B_{0x}^2 + B_{0y}^2 + (B_0 - B_{0z})^2\right]\sin(B_0t + \varphi) - M_0B_{0x}B_{0y}\cos(B_0t + \varphi),
\]
\[
M_{(r)z} = M_0(B_0 - B_{0z})\left[B_{0x}\cos(B_0t + \varphi) - B_{0y}\sin(B_0t + \varphi)\right].
\] (5.6)

A few remarks are in order. Note that the components in Eq. (5.6) satisfy Eq. (2.7) or the Liouville equation. The two eigenvalues of the density operator in Eq. (5.5) are
\[
\lambda_{\pm} = \pm\frac{1}{2}M_{(r)} = \pm\frac{1}{2}M_0B_0(B_0 - B_{0z}).
\] (5.7)

The imaginary components simply given by
\[
\mathbf{M}_{(i)} = -\frac{d}{dt}\mathbf{M}_{(r)}
\] (5.8)
also satisfy Eq. (2.7) because the time derivative of a solution is another solution. Another solution to Eq. (3.6),
\[
u_-(t) = B_0 + M_0^{1/2}e^{-i(B_0t+\varphi)/2},
\] (5.9)
leads to the components in Eq. (5.6) for another density operator now with $B_0$ replaced by $-B_0$ and $(B_0t + \varphi)$ by $-(B_0t + \varphi)$.

\section*{B. Oscillating Field}

We turn to a spin particle in constant and oscillating magnetic fields
\[
B_+ = B_{0+}e^{-i\omega t}, \quad B_z = B_{0z},
\] (5.10)
where $B_x = 2B_{0+}\cos(\omega t)$ and $B_y = -2B_{0+}\sin(\omega t)$. The oscillating magnetic field provides a resonance of the spin particle in the constant field $B_{0z}$. By directly solving Eq. (2.7), we may find the density operator with the components
\[
M_x = \frac{2B_{0+}M_{0z}}{\omega + B_{0z}}\cos(\omega t), \quad M_y = -\frac{2B_{0+}M_{0z}}{\omega + B_{0z}}\sin(\omega t), \quad M_z = M_{0z},
\] (5.11)
where $M_{0z}$ is a constant.

We now use the connection with the oscillator to find the density operator. Even the general case of $B_x = B_{0x}\cos(\omega t) + B_{0y}\sin(\omega t)$ and $B_y = -B_{0x}\sin(\omega t) + B_{0y}\cos(\omega t)$ has a solution to Eq. (3.6) of the form
\[
u_+(t) = B_{0+}M_0^{1/2}e^{i[(\Omega-\omega)t+\varphi]/2}, \quad \Omega = \sqrt{B_0^2 + 2\omega B_{0z} + \omega^2}.
\] (5.12)
We then have the complex components

\[ M_x = -\frac{1}{2} M_0 \left[ (B_{0x} + iB_{0y})^2 e^{-i\omega t} - (\Omega - \omega - B_{0z})^2 e^{i\omega t} \right] e^{i(\Omega t + \varphi)}, \]

\[ M_y = \frac{i}{2} M_0 \left[ (B_{0x} + iB_{0y})^2 e^{-i\omega t} + (\Omega - \omega - B_{0z})^2 e^{i\omega t} \right] e^{i(\Omega t + \varphi)}, \]

\[ M_z = M_0 (B_{0x} + iB_{0y}) (\Omega - \omega - B_{0z}) e^{i(\Omega t + \varphi)}. \]  

(5.13)

From Eq. (5.13), we finally obtain the density operator with real components

\[ M_{(r)x} = -\frac{1}{2} M_0 \left[ \{B_{0x}^2 - B_{0y}^2 - (\Omega - \omega - B_{0z})^2\} \cos(\omega t) + 2B_{0x}B_{0y} \sin(\omega t) \right] \cos(\Omega t + \varphi) \]

\[ -\frac{1}{2} M_0 \left[ \{B_{0x}^2 - B_{0y}^2 + (\Omega - \omega - B_{0z})^2\} \sin(\omega t) - 2B_{0x}B_{0y} \cos(\omega t) \right] \sin(\Omega t + \varphi), \]

\[ M_{(r)y} = -\frac{1}{2} M_0 \left[ \{B_{0x}^2 - B_{0y}^2 + (\Omega - \omega - B_{0z})^2\} \cos(\omega t) + 2B_{0x}B_{0y} \sin(\omega t) \right] \sin(\Omega t + \varphi) \]

\[ +\frac{1}{2} M_0 \left[ \{B_{0x}^2 - B_{0y}^2 + (\Omega - \omega - B_{0z})^2\} \sin(\omega t) - 2B_{0x}B_{0y} \cos(\omega t) \right] \cos(\Omega t + \varphi), \]

\[ M_{(r)z} = M_0 (\Omega - \omega - B_{0z}) \left[ B_{0x} \cos(\Omega t + \varphi) - B_{0y} \sin(\Omega t + \varphi) \right]. \]  

(5.14)

We, thus, find the general time-dependent density operator. Note that in the limit of \( \omega = 0 \) we recover the result in Eq. (5.6) for a constant field. Using an independent solution

\[ u_-(t) = B_{0z} M_0^{1/2} e^{-i[(\Omega + \omega) t + \varphi]/2}, \]  

(5.15)

we can find another density operator obtained by replacing \( \Omega \) by \( -\Omega \) and \( (\Omega t + \varphi) \) by \( -(\Omega t + \varphi) \).

It would be interesting to compare the solutions Eqs. (5.6) and (5.14) with those in Sec. III of Ref. [16]. However, our procedure to get solutions is simpler than that in Refs. [16, 17].

VI. CONCLUSION

A spin-1/2 particle has a compact Lie algebra \( su(2) \), which can be connected through complexification with the oscillator algebra \( su(1,1) \). We introduced a non-Hermitian one-mode bosonic representation (3.1) for \( su(2) \). Using the bosonic representation, we found the oscillator in Eq. (3.2) corresponding to the spin particle in an external magnetic field. The spin particle in the time-dependent magnetic field corresponds to a time-dependent oscillator. It is well known that the time-dependent oscillator has invariant operators which
provide exact quantum states up to time-dependent phase factors. A caveat is that the bosonic representation for $su(2)$ is non-Hermitian and, as a consequence, the corresponding oscillator has, in general, a non-Hermitian Hamiltonian. However, this connection still provides us with a novel method for finding the nontrivial density operators for the spin-1/2 particle, even for the well-known case of constant and/or oscillating magnetic fields.

We have made use of the connection between the spin-1/2 particle and an oscillator to develop a novel method for finding the density operator in Eq. (3.9) for the spin particle. The solution of the auxiliary equation, Eq. (3.6), leading to the density operator is found to be the complex conjugate of the solution of Eq. (4.2) for the evolution operator, which in turn is the same as the solution of Eq. (4.9) for the complex parameters of the magnetization. As illustrative examples, we applied the method to the spin particle in constant and/or oscillating magnetic fields. The density operators in Eqs. (5.6) and (5.14) have a complicated time-dependence. In the case of the constant field, the density operator in Eq. (5.1) or (5.2) is widely used whereas the density operator in Eq. (5.6) is the most general time-dependent one, which can also be obtained by using Kobayashi’s method. Similarly, the density operator in Eq. (5.14) differs from that in Eq. (5.11) and seems to be the most general one for the oscillating field. Thus, the connection with an oscillator provides an effective method for finding the general density operator for a spin-1/2 particle.

The oscillator with a constant mass and frequency is known to possess not only time-independent invariant operators but also time-dependent ones, which lead to squeezed states of time-independent states [19]. The physical meanings and applications of the time-dependent density operator for constant field and/or oscillating magnetic fields will be addressed in a future work. The method of this paper can be easily generalized to spin chain systems. Each spin particle corresponds to an oscillator; thus, a spin system is equivalent to a system of coupled oscillators. The invariant operator for the oscillator chain with time-dependent masses, frequencies, and couplings may be found in a similar manner [20]. It would be interesting to study the spin chain system in connection with the oscillator chain system.
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