Analysis of stability for uniform rotations of a dumbbell system in an elliptic orbit

Denilson Paulo Souza dos Santos¹, Jorge Kennety Silva Formiga²

¹Sao Paulo State University (UNESP) - São João da Boa Vista - São Paulo, Brazil
²Sao Paulo State University (UNESP), Institute of Science and Technology, São José dos Campos - São Paulo, Brazil.

Received: 29 Nov 2020;
Received in revised form: 15 Jan 2021;
Accepted: 01 Feb 2021;
Available online: 13 Feb 2021

©2021 The Author(s). Published by AI Publication. This is an open access article under the CC BY license (https://creativecommons.org/licenses/by/4.0/).

Keywords—Tether systems, Control, mathematical modeling.

I. INTRODUCTION

Tethered systems have many areas of application and has been studied in several published articles (¹¹–⁹). Beletsky and Levin⁴, begins by setting the scene for tethers in space summarizing possible applications and also discussing fact and fiction, analyzing clearly the main parameters and applications for Tethers Systems, as the density of the material the effective forces, orbital dynamics, mechanics models, attitude and possible disturbances for a flexible tethers with end masses, massless and massive variations.

The motion of tethers considering dumbbell oscillations, bodies in the central field of vibrating forces (Burov et al. (¹⁰–¹⁴)), showing stability solutions for angles on the chaotic dynamics in elliptical orbit, analyzing the problem in another aspect of Moon-tethered pendulum, to considering the uniform rotations of a two-body tethered system in planar motion and the control the length of the tether (¹⁵). In studies (¹⁵–²¹) were suggested methods of controlling the geometric configuration and Moon-tethered system with variable tether length in restricted three-body problem (¹⁶) and (²²) were studied and demonstrated important results for tethers applications in systems.

II. DYNAMICS OF THE PROBLEM

Consider planar motion of a dumbbell body in the central Newtonian gravitational field (Fig.1). The tether length (l) is small compared to the orbit the system center of mass (c.m.ₘₐₖ), moves along an elliptic keplerian orbit.
The body consists of two-point masses \( m_1 \) and \( m_2 \) connected by a light tether
\[
\rho = \frac{p}{1 + e \cos(v)}
\]
where \( p \) is the focal parameter, \( e \) the eccentricity and \( v \) the true anomaly of the orbit. The system coordinates are
\[
\begin{align*}
  x_0 &= \rho \cos(v) \\
  y_0 &= \rho \sin(v) \\
  x_1 &= x_0 + l_1 \cos(v + \varphi) \\
  y_1 &= y_0 + l_1 \sin(v + \varphi) \\
  x_2 &= x_0 - l_2 \cos(v + \varphi) \\
  y_2 &= y_0 - l_2 \sin(v + \varphi)
\end{align*}
\]

### 1.1. Potential and Kinetic Energy
The potential energy of the system can be obtained by the following expression,
\[
V = -\frac{\mu_0 m_1}{r_1} - \frac{\mu_0 m_2}{r_2}
\]
where \( m_1 \) represents the mass of the point; \( r_1 \) the position of the point mass with respect to center of the Earth; \( \mu_0 = GM \); \( G \) is the universal gravitational constant and \( M \) the mass of the Earth. Introduced the parameters, \( \mu \) and \( m \), given by
\[
\mu = \frac{m_1}{m} \quad m = m_1 + m_2
\]
Substitute on Eq. (3) leads to
\[
V = -\mu_0 \mu \left( \frac{m}{\sqrt{x_1^2 + y_1^2}} + \frac{1 - m}{\sqrt{x_2^2 + y_2^2}} \right)
\]

The Eq. (7) is cumbersome and can be simplified, introducing a new parameter \( \lambda = \frac{l}{p} \); \( \lambda \ll 1 \) the tether length is much smaller than the focal parameter \( p \). The Taylor Series expansion up to the 2nd order of \( \lambda \) for the potential energy is
\[
V = \frac{\mu_0 m (1 + e \cos(v))}{p} + \frac{\mu_0 (1 + e \cos(v))^3 (1 + 3 \cos(2\varphi)) \lambda^2}{4p}
\]
The kinetic energy of the system can be written as
\[
T = \frac{m_1 \dot{v}_1^2}{2} + \frac{m_2 \dot{v}_2^2}{2}
\]

### 1.2. Lagrange Equations of Motion
For the subsequent analysis, the generalized coordinates \( \varphi \) and \( l \) are used and the system is assumed to be subject to the gravity-gradient forces.
\[
\begin{align*}
  \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} &= 0 \\
  \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{l}} \right) - \frac{\partial L}{\partial l} &= 0
\end{align*}
\]
\[
l \left( 3\mu_0 \sin(2\varphi)(1 + e \cos(v))^3 + 2p^3(\ddot{v} + \dot{\varphi}) \right) + 4p^3(\ddot{v} + \dot{\varphi}) = 0
\]
The equation is rewritten as a function of the true anomaly \( v \) (Eq. (11)–(12))
\[
( \dot{v} )' = \frac{d}{dv} \quad ( \dot{\varphi} )' = \frac{d}{dv}
\]
\[
\frac{d}{dv} = \omega_0 (1 + e \cos(v))^2 \frac{d}{dv}
\]
with
\[
\omega_0 = \frac{\mu_0}{p^3}
\]
The equations of motion of the spacecraft can be written as
\[
(1 + e \cos(v))\varphi'' + 2 \left( \frac{l}{1 + e \cos(v)} \right) \varphi' + 1 + e \sin(v) (\varphi' + 1) + 3 \cos(v) \sin(v) = 0
\]
considering a movement with uniform rotations,
\[
\varphi = \omega v + \varphi_0
\]
\[
l(v) = \eta(v) \frac{l_0}{1 + e \cos(v)}
\]
Suppose the following relation for the tether performance (Eq. ref{equa16}), the Eq. ref{equa14} with respect to the true anomaly \( v \) takes the form:
\[
\frac{\eta'(v)}{\eta(v)} = \frac{3 \sin(2(\omega v + \varphi_0))}{4(\omega + 1)(1 + e \cos(v))}
\]
The analytical integration function is difficult but substituting values for the variable \( \omega \) and solving the...
equation with respect to the variable \( \eta \), it is possible to obtain the following solutions (Table 1 and Figures ref{fig2} - ref{fig5}). Analytical solutions were found for this \( \omega \) values previously chosen. The values for \( \omega = \left( \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{5}{4}, \pm \frac{7}{4}, \pm \frac{9}{4}, \pm \frac{11}{4}, \pm \frac{13}{4}, \pm \frac{15}{4} \right) \) closed-form no solution. For \( \omega = 0 \) was examined in previous studies [12] where the relative equilibrium was studied.

1.3. Uniform Rotations: \( \varphi = \omega \nu + \varphi_0 \)

Substitute \( (\varphi = \omega \nu + \varphi_0) \) to Eq. (14)) and considering Eq. (17), it is possible to obtain the nonlinear differential equation that describes the motion of the tether.

A general analytical integration for all values of \( \omega \) cannot be found but for some values \( \omega \) the closed form solutions obtain. A numerical integration has been performed, where \( \omega \) and \( \varphi_0 \) were substituted previously, with the objective of analyzing the behavior of the cable system. The logarithmic plots of \( \ln \left( \frac{\eta}{\eta_0} \right) \) for a number of fractional angular velocities (Figures 6 - 9) was made. For the particular case when the value of \( \omega \) is zero (\( \omega = 0 \)) and \( \varphi_0 \) is constant, there are a uniform for \( \varphi_0 = \frac{(k)n}{2}, k = \{0,1,2,3,4,\ldots\} \).

The control laws are periodic in true anomaly (\( \nu \)). In Fig.2to Figure 4 describe the change in tether length depending for different eccentricities, \( e = (0.04, 0.2, 0.3, 0.5, 0.8) \). The formulation also guarantees uniform rotation for fractional values of \( \omega \), as seen in Figures 6-9.
The control law for the tether length for $\omega = 0$ and $\eta_0 = 0$.

An approximate solution for small eccentricities can be obtained using Taylor series in Eq. (17) and apply the series of order 3, in the variable $e$ obtain:

$$
\eta'(v) = \frac{3e^3\cos^2(v)\sin(2\omega)}{4(\omega + 1)} - \frac{3e^2\cos^2(v)\sin(2\omega)}{4(\omega + 1)} + \frac{3e\cos(v)\sin(2\omega)}{4(\omega + 1)} - \frac{3\sin(2\omega)}{4(\omega + 1)}
$$

One obtains the expression that has analytical integration, however this procedure introduces impossibilities in the system and restricts the solutions to the variable $\Omega$ some attenuation, which after integration generates singularities for values: $\omega = \left(\frac{-3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}\right)$. Increasing the expansion terms for order 5, the singularities also increase, $\omega = \left(\frac{-5}{2}, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}\right)$.

1.4. Stability Conditions

We study stability using the Floquet Theory, let’s linearize the equation of motion in the vicinity of solution (Eq. (16)), it is possible to check the stability analyzing system behavior around small oscillations. Considering Eq. (19) and Eq. (20) analyzing the neighborhood of the point $\eta_0 = 0$, it follows:

$$
\varphi = \omega v + \delta \varphi
$$

Substituting Eq. (17) in Eq. (15) one obtains:

$$
4(\omega + 1 + \delta \varphi')(1 + e \cos(v)) \eta'(v) / \eta(v) + 3 \sin(2(\omega v + \delta \varphi)) + 2(1 + e \cos(v)) \delta \varphi'' = 0
$$

Replacing Eq. (18), concerning the $\eta(v)$ variable, $\delta \varphi''(1 + e \cos(v)) + 3 \cos(2\omega v + \delta \varphi) \sin(\delta \varphi) - \frac{3 \sin(\delta \varphi')}{2(\omega + 1)} = 0 \quad (22)$

This is the nonlinear equation of the perturbed motion. The linearized equation is:

$$
(1 + e \cos(v)) \delta \varphi'' + \frac{3}{2} \left(2 \cos(2\omega v) \delta \varphi - \sin(2\omega v) \delta \varphi'\right) / \omega + 1 = 0
$$

Applying the Floquet theory find the monodromy matrix $A$ for this system, one is obtained numerically by integrating a periodic orbit ($2\pi$) and the variation equations (Eq. (23)).

A good numerical method to refine the closure of the orbit is essential to obtain an accurate monodromy matrix $(A)$, which is obtained to analyze to stability with respect to small perturbations $(\delta \varphi)$ of the orientation angle $\varphi$. It is possible to analyze the stability using the Floquet theory, since this equation is a differential equation of the second order. The linearized equation constrains this analysis to small variations of $\varphi$. The stability monodromy matrix $(A)$ has some important properties, which are (13) and (19):

1. $\det(A) = 1$;
2. $\{\lambda_1 = \lambda_2 = \lambda^{-1}\}$ eigenvalues;
3. $Tr(A) = 2 + \sum \lambda_i$

The indicator of stability $2 - |Tr(A)|$ is given by $Tr(A)$ between $0$ and $2$ ($0 < 2 - |Tr(A)| \leq 2$) cite[ref13], where $Tr(A)$ means Trace of the matrix $A$. Positive values between the described boundaries correspond to stable solutions (linear approximation), negative values correspond to instability and zero values correspond to critical cases. The conditions of stability for $\omega$ are demonstrated in Figures 8 -- 18, where the positive values of $\omega$ correspond to direct rotations (direction of the orbital motion).

**Table 1 - Intervals of stability.**

| $\omega$ | $e$ |
|----------|-----|
| -4       | [0, 0.2956) U [0.9063, 0.9334] |
| -3.75    | [0.7821, 0.8333] |
| -3.5     | [0, 0.0966) U [0.6824, 0.7690] |
| -3.25    | [0.6328, 0.7542] |
| -3       | [0.7895] |
| -2.75    | [0, 0.6370] |
| -2.5     | [0, 0.4906] |
The regions of stability with solutions (Figures 8–18 and Table 1) have been shown for a number of $\omega \in [-4,4]$. When $\omega = -1$ there is no solution and when $\omega = 1$ there is a small stable range (Table 1). The Table 1 shows the complete set of stable solutions for the monodromy matrix, as a function of the eccentricity and of the true anomaly.

| $\omega$ | Solutions |
|---------|-----------|
| -4      | $[0, 0.1935]$ |
| -3      | $[0, 0.5453]$ |
| -2      | No Solutions found |
| -1.75   | No Solutions found |
| -1.5    | No Solutions found |
| -1.25   | No Solutions found |
| -1      | No Solutions found |
| -0.75   | No Solutions found |
| -0.5    | No Solutions found |
| -0.25   | $[0.4521, 0.9999]$ |
| 0       | $[0, 0.9999]$ |
| 0.25    | No Solutions found |
| 0.5     | No Solutions found |
| 0.75    | $[0.0177, 0.0325]$ |
| 1       | $[0.8789, 0.8805]$ |
| 1.25    | No Solutions found |
| 1.5     | $[0.0, 0.1338]$ |
| 1.75    | $[0.0, 0.2684]$ |
| 2       | $[0.0, 0.5539] \cup [0.9778, 0.9789]$ |
| 2.25    | $[0, 0.5820]$ |
| 2.5     | $[0, 0.6451]$ |
| 2.75    | $[0.6426, 0.7665]$ |
| 3       | $[0, 0.0989] \cup [0.9666, 0.9859]$ |
| 3.25    | $[0, 0.0890] \cup [0.7768, 0.8296]$ |
| 3.5     | $[0, 0.2961] \cup [0.7656, 0.8173]$ |
| 3.75    | $[0, 0.3412] \cup [0.8506, 0.8792]$ |
| 4       | $[0, 0.4207]$ |

Fig. 8: Stability region for $\omega = \{-4; -\frac{15}{4}; -\frac{7}{2}; -\frac{13}{4}\}$.

Fig. 9: Stability curve for $\omega = \{-3; -\frac{11}{4}; -\frac{5}{2}; -\frac{9}{4}\}$.

Fig. 10: Stability curve for $\omega = \{-\frac{1}{4}\}$.

Fig. 11: Stability curve for $\omega = \{0; \frac{3}{4}\}$.
The stability analysis suggests the system's viability. Some parameters for the system are shown in monodromy matrix.

III. FORCES IN TETHERS

The force on the tethers can be calculated using the relation of the forces involved in the problem, applied to the tether,

\[ m_1 \ddot{r}_1 = -\mu_0 \frac{m_1}{|r_0^2|} r_0^2 + \ddot{\mathbf{T}} \]  

(24)

The force on the tether for a system composed of: \( m_1 = m_2 = 1000 \text{ kg} \), \( l_0 = 100\text{km} \), \( p = 7000\text{km} \), the
direction is tether length ($l$) and sense $m_1$ to $m_2$, and $T$ is magnitude of the tether force, solving the Eq.(16) with respect to $l$, it is possible to show that the cable suffers a large variation in tension and that these values are periodic and, in some cases, increase with grows to the eccentricity (Figures 19 – 32).

Fig.19: Tether Force $T$ for $\omega = 0$ and $e = 0$.

Fig.20: Tether Force $T$ for $\omega = \{1/2, 1\}$ and $e=0$.

Fig.21: Tether Force $T$ for $\omega = 0$.

Fig.22: Tether Force $T$ for $\omega = 1$.

Fig.23: Tether Force $T$ for $\omega = 2$.

Fig.24: Tether Force $T$ for $\omega = 3$.

Fig.25: Tether Force $T$ for $\omega = 1/2$. 
Fig. 26: Tether Force $T$ for $\omega = \frac{1}{4}$

Fig. 27: Tether Force $T$ for $\omega = -\frac{1}{2}$

Fig. 28: Tether Force $T$ for $\omega = -\frac{5}{2}$

Fig. 29: Tether Force $\vec{T}$ with $\omega = 0$ for several eccentricities ($e$).

Fig. 30: Tether Force $\vec{T}$ with $\omega = 1$ for several eccentricities ($e$).

Fig. 31: Tether Force $\vec{T}$ with $\omega = \frac{1}{2}$ for several eccentricities ($e$).

Fig. 32: Tether Force $\vec{T}$ with $\omega = -4$ for several eccentricities ($e$).
IV. CONCLUSION

The uniform rotations of a dumbbell and with several possibilities are considered in the present study, as well as the stability analysis and the viable control laws. In some cases are possible obtain solutions closes-form, in other cases only numerical solutions for the control of the tether systems are available. The necessary conditions of stability for uniform rotations were analyzed using the parameters $\omega$ and the eccentricity of the orbit, generating the control laws for the cable length.

ACKNOWLEDGEMENTS

This work was accomplished with the support of São Paulo Research Foundation (FAPESP) #17/04643-4 and #16/15675-1.

REFERENCES

[1] Belestsky, Vladimir V.; Levin, Evgenii M., 1993. Dynamics of Space Tethers Systems. San Diego - California: Advances in the Astronautical Sciences, Vol. 83 - American Astronautical Society ISSN 0-87703-370-6.
[2] J. Pearson, “Anchored Lunar Satellites for Cislunar Transportation and Communication,” Journal of the Astronautical Sciences, Vol. XXVII, No. 1, 1979, pp. 39-62.
[3] Artsutanov, Y. N., “Into the Cosmos without Rockets,” Znanije-Sila 7, 25, 1969.
[4] Artsutanov, Y. N., “The Earth-to-Moon Highway,” Technics to Youth, No. 4, 21, 35, 1979 (in Russian).
[5] Bainum, Peter M., and V. K. Kumar. “Optimal control of the shuttle-tethered-subsatellite system.” Acta Astronautica 7.12 (1980): 1333-1348.
[6] Misra, A. K., Z.Amier, and V. J. Modi. “Attitude dynamics of three-body tethered systems.” Acta Astronautica 17.10 (1988): 1059-1068.
[7] Kalantzis, S., Modi, V. J., Pradhan, S., Misra, A. K. (1998). Dynamics and control of multibody tethered systems. Acta astronautica, 42(9), 503-517.
[8] Misra, A. K. "Dynamics and control of tethered satellite systems." Acta Astronautica 63.11 (2008): 1169-1177.
[9] ChebotoyV.A,MainsD.L, “Fether satellite system collision study”, Acta Astronautica, 44 (7–12) (1999), pp. 543–551.
[10] Burov A.A. and KosenkoI.L , “On relative equilibria of an orbital station in regions near the triangular libration points,” Doklady Physics, Vol. 52, Issue 9, 2007, pp.507-509.
[11] BurovA.A., PascalM., and StepanovS.Ya., “The gyroscopic stability of the triangular stationary solutions of the generalized planar three-body problem,” Journal of Applied Mathematics and Mechanics, Vol. 64,Issue 5, 2000a, pp. 729-738.
[12] Burov A.A. and TrogerH, “The relative equilibria of an orbital pendulum suspended on a tether,” Journal of

Applied Mathematics and Mechanics, Vol. 64, Issue 5, 2000b, pp. 723-728.
[13] Burov,A., KononovO. I., and GuermanA. D., “Relative equilibria of a Moon - tethered spacecraft”, Advances in the Astronautical Sciences, v. 136, 2011a, pp. 2553-2562.
[14] Burov A. and KosenkoI.L, “Plane oscillations of a body with variable mass distribution in an elliptic orbit,” Proc. of ENOC 2011, July 24–29, 2011b, Rome, Italy.
[15] BurovA.A., KosenkoI.L, and GuermanA. D., Dynamics of a moon-anchored tether with variable length. Advances in the Astronautical Sciences, 2012, Vol. 142, pp.3495-3507.
[16] Burov, A. A., Guerman, A. D., Kosenko, I. I. (2013). Equilibrium configurations and control of a moon-anchored tethered system. Advances in the Astronautical Sciences, 146, 251-266.
[17] Jin, D. P., and H. Y. Hu. "Optimal control of a tethered subsatellite of three degrees of freedom.” Nonlinear Dynamics 46.1-2 (2006): 161-178.
[18] CartmellM.P., McKenzieD.J., "A review of space tether research, Progress in Aerospace Sciences", Volume44, Issue 1, January 2008, Pages 1-21, ISSN 0376-0421, http://dx.doi.org/10.1016/j.paerosci.2007.08.002.
[19] Birkhoff, G. D. Dynamical system. American Mathematical Society, Colloquium Publications, v. 9, p. 423,1927
[20] Santos, D. P. S. dos; Ferreira, A. “Three-dimensional two-body tether system — equilibrium solutions”. Journal of Physics: Conference Series, v. 641, n. 1, p. 012009, 2015. http://dx.doi.org/10.1088/1742-6596/641/1/012009.
[21] Santos, D. P. S. dos; Morant, S. A. R. B. A. Guerman, A. D.; Burov, A. “Stability solutions of a dumbbell-like system in an elliptical orbit”. Journal of Physics: ConferenceSeries, v. 641, n. 1, p. 012004, 2015, http://dx.doi.org/10.1088/1742-6596/641/1/012004.
[22] Ferreira, A. F. S., Burov, A. A., Guerman, A. D., Prado, A. F. B. A., Nikonov, V. I.(2019). Stationary configurations of space tether anchored on smaller primary in three-body problem. Acta Astronautica, 160, 572-582. doi:10.1016/j.actaastro.2019.01.031