Category theoretic foundation of single-photon-based decision making

Makoto Naruse¹, Song-Ju Kim², Masashi Aono³,⁴, Martin Berthel⁵,⁶, Aurélien Drezet⁵,⁶, Serge Huant⁵,⁶, and Hirokazu Hori⁷

¹ Network System Research Institute, National Institute of Information and Communications Technology, 4-2-1 Nukui-kita, Koganei, Tokyo 184-8795, Japan
² WPI Center for Materials Nanoarchitectonics, National Institute for Materials Science, 1-1 Namiki, Tsukuba, Ibaraki 305-0044, Japan
³ Earth-Life Science Institute, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8550, Japan
⁴ PRESTO, Japan Science and Technology Agency, 4-1-8 Honcho, Kawaguchi-shi, Saitama 332-0012, Japan
⁵ CNRS, Inst. NEEL, F-38042 Grenoble, France
⁶ Université Grenoble Alpes, Inst. NEEL, F-38000 Grenoble, France
⁷ Interdisciplinary Graduate School of Medicine and Engineering, University of Yamanashi, Takeda, Kofu, Yamanashi 400-8511, Japan

Email: naruse@nict.go.jp, KIM.Songju@nims.go.jp, masashi.aono@elsi.jp, martin.berthel@neel.cnrs.fr, aurelien.drezet@neel.cnrs.fr, serge.huant@neel.cnrs.fr and hirohori@yamanashi.ac.jp
Abstract

Decision making is a vital function in the age of machine learning and artificial intelligence; however, its physical realizations and their theoretical fundamentals are not yet known. In our former study, we demonstrated that single photons can be used to make decisions in uncertain, dynamically changing environments. The multi-armed bandit problem was successfully solved using the dual probabilistic and particle attributes of single photons. Herein, we revolutionize how decision making is comprehended via a category theoretic viewpoint; we present the category theoretic foundation of the single-photon-based decision making, including quantitative analysis that agrees well with the experimental results. The category theoretic model unveils complex interdependencies of the entities of the subject matter in the most simplified manner, including a dynamically changing environment. In particular, the octahedral structure and the braid structure in triangulated categories provide clear understandings and quantitative metrics of the underlying mechanisms for the single-photon decision maker. This is the first demonstration of a category theoretic interpretation of decision making, and provides a solid understanding and a design fundamental for machine learning and artificial intelligence.

Keywords: decision making, category theory, single photon, machine learning, multi-armed bandit problem, system modeling

PACS numbers: 02.50.Le, 07.05.Mh, 89.75.-k
1. Introduction

To maximize the total reward sum from multiple slot machines, one needs to explore a machine that provides the highest reward probability. However, much exploration may result in excessive loss; the best machine may change over time. At the same time, a very quick decision or insufficient exploration might result in missing the best machine. This is called the exploration–exploitation dilemma tradeoff, formulated as the multi-armed bandit problem (MAB) in the literature of machine learning [1,2,3]. MAB is important for various practical applications, such as information network management [4,5], web advertisements [6], Monte Carlo tree search [7] and clinical trials [8,9]. Meanwhile, decision theories have been intensively investigated with the quantum mechanical notions for modeling human decision making [10,11].

It is indeed important to develop high-performance MAB algorithms that can be executed on conventional electronic computing platforms for vital applications; examples of such algorithms include softmax [1,2], upper-confidence bound [12] and tug-of-war [13,14]. Meanwhile, we have been investigating physical principles and technologies for realizing artificially constructed physical decision making mechanisms. One motivation behind this is to deepen the understanding of intelligent abilities inherent in natural phenomena [15], and to extract and utilize these to pave the way for developing novel intelligent devices using state-of-the-art materials and/or photonics technologies [16,17]. In our former study [3], we proposed an architecture in which quantum attributes of a single photon are utilized for decision making and experimentally demonstrated accurate and adaptive decision making using the nitrogen-vacancy (NV) center in a nanodiamond as a single-photon source. Thanks to the quantum nature of light, single-photon detection is immediately and directly associated with decision making, which is a
decisive step toward achieving an autonomous intelligent machine based on a purely physical mechanism.

In this study, we investigate the theoretical background for clarifying the underlying mechanisms of a single-photon decision maker. In Ref. [3], the polarizations of single photons were adaptively configured such that a higher-reward-probability slot machine was selected. Using the geometry-based modeling described in this study, we can determine the dynamic change of polarizations. Based on this study, we discuss an abstracted model of the single-photon decision maker by the notions of \textit{category theory} [18-23]. Category theory is a branch of mathematics that formalizes mathematical structure into collections of objects and morphisms. Category theory extracts the essence of all mathematical subjects, including a dynamically changing environment, to reveal and formalize simple yet powerful patterns of thinking; in this paper, we revolutionize how decision making is conducted via a category theoretic picture. The complex interdependencies involved in decision making problems are elucidated by category theory. In particular, we show that the octahedral structure and the braid structure in triangulated categories provide clear qualitative understandings and quantitative metrics of the underlying mechanisms for the single-photon decision maker.

The paper is organized as follows. In section 2, we review single-photon decision maker experiments [3]. Section 3 discusses a geometry-based modeling and analysis, including experimental data analysis. Section 4 presents a category theoretical approach for the single-photon decision maker. Numerical characterizations are also discussed to investigate the correspondence between category theory modeling and actual physical system. Section 5 concludes the study.
2. Single-photon-based decision making

For the simplest case that preserves the essence of the MAB problem, we consider a player who selects one of two slot machines (slot machine 1 or slot machine 2) with the goal of maximizing a reward. A polarizing beam splitter (PBS) is prepared as shown in figure 1. Initially, the linear polarization of the input single photon is orientated at $\pi / 4$ with respect to the horizontal, enabling the photon to be detected by Channel 1 (Ch.1) or Channel 2 (Ch.2) with 50:50 probability. However, the total probability of photon detection by either Ch.1 or Ch.2 is unity. Here, photon detection by Ch.1 or Ch.2 is immediately associated with the decision to select slot machine 1 or 2, respectively. This is a notable aspect of the single-photon decision maker in the sense that the dual probabilistic (wave) and particle attributes of a single photon are utilized. A single photon polarized at an orientation of $\pi / 4$ with respect to the horizontal indicates that the system is making a thorough search for a better machine. From this initial condition, polarization is reconfigured using the following strategy:

| Polarization updating strategy in the single-photon decision maker |
|---------------------------------------------------------------|
| If the selected machine successfully dispenses a reward, polarization is shifted towards the selected machine. If no reward is dispensed from the selected machine, polarization is moved in the direction of a non-selected machine. By iteratively this process, the system guides us to a decision in which we select the correct solution, meaning that the higher-reward-probability machine is selected. The orientation of polarization is quantified by a polarization adjuster value described in detail in [3]. |

Through theoretical modeling and analysis, this study aims to clarify the underlying mechanisms to explain why the single-photon decision maker (described above) can derive the correct decision.
Here, we briefly review the experimental system and results of single-photon-based decision making; an analysis of experimental data is conducted in the subsequent discussion. An individual photon is supplied by a single NV color center [24] in a surface-purified 80-nm nanodiamond [25]. It passes through a polarizer and a zero-order half-wave plate and impinges on the PBS. Two avalanche photodiodes used for photon detection are connected to a time-correlated single-photon-counting system. Photon detection is associated with the decision to select a slot machine. Based on the selected slot machine’s pay-off results, linear polarization was configured towards the vertical or horizontal by rotating a half-wave plate mounted on a rotary positioner (figure 1).

Let the initial reward probabilities of slot machines 1 and 2 be given by $P_1 = 0.8$ and $P_2 = 0.2$, respectively, which means that selecting slot machine 1 is the correct decision. The reward probability is inverted every 150 cycles to represent the ability to adapt to environmental changes. The decision maker repeated 600 consecutive plays 10 times. The solid curve in figure 2 shows the evolution of the correct selection rate given by calculating the number of correct decisions divided by the number of repeat cycles. This gradually increases towards unity with time. Due to the swapping of the reward probability, the correct selection rate drops every 150 cycles but quickly recovers. The dotted curve in figure 2 shows the case in which initial reward probabilities are $P_1 = 0.6$ and $P_2 = 0.4$. As the difference between these probabilities is less than that in the previous case, making accurate decisions becomes more difficult. Although the performance is degraded relative to the previous situation, a gradual increase in the correct selection rate and an adaptation to environmental change are still observed.
3. Geometry-based modeling

Several probabilistic and unobservable processes are involved in the decision making problem, which makes their comprehension complicated. Firstly, machine selection is probabilistic; the polarization of a single photon strongly affects the resulting decision, but each individual decision is probabilistic if polarization is not completely horizontal or vertical. Secondly, we cannot be completely sure if the selected machine has a relatively higher reward probability; we only observe the dispensed reward. To clearly represent the problem, we term slot machines with relatively larger and smaller reward probabilities as ‘GOOD’ and ‘BAD’ machines, respectively. Note that GOOD/BAD information is a relation hidden from the decision maker. Finally, the result of a single play of a slot machine is probabilistic even if the selected machine is the GOOD machine. We denote the situation when the selected machine dispenses a reward or does not as ‘WIN’ or ‘LOSE’, respectively. Thus, there are eight combinations in total with regard to {Machine selection 1/2}, {GOOD/BAD machines} and {WIN/LOSE}, schematically represented as follows:

Note that each pair of the alternatives in equation (1), i.e., {Machine selection 1/2}, {GOOD/BAD machines} and {WIN/LOSE}, forms an orthogonal basis with respect to a certain reference; however, they are transformed into a mixture in the subsequent pair. That is, for example, the state ‘WIN’ is orthogonal to ‘LOSE’ but is composed of the two cases of ‘GOOD machine Wins’ and ‘Bad machine Wins’. In turn, the orthogonal states ‘Machine selection 1’ and ‘Machine selection 2’ are injected into ‘GOOD machine’ and ‘BAD machine’ based on hidden variables set by the casino from which a player is able to observe only the alternative results {WIN/LOSE}. This
viewpoint is an entry to the geometry-based analysis that follows and the category theoretic paradigm.

3.1. DECISION circle, WIN/LOSE circle and their dynamics

We quantify and analyze the problem by introducing two circular diagrams, each having a radius of unity. One of these is what we call the ‘DECISION circle’, which is concerned with the machine selection probability specified by the polarization of the single-photon decision maker. As shown in figure 3(a), $\phi$ is the angle of deviation from $\pi/4$ (ranging from $-\pi/4$ to $+\pi/4$). We regard the projection to the vertical or horizontal axis to be the probabilities of selecting slot machine 1 or 2, respectively. Rigorously speaking, the square of the projection should be considered to be the probability to satisfy the condition that the sum of the two projections is unity; however, in this study, linear correspondence is assumed for simplicity.

The other circle denotes the WIN/LOSE probability and is referred to as the ‘WIN/LOSE circle’. Here, the GOOD machine is represented by a point on the circle that deviates from the $\pi/4$ direction by $\theta$, schematically shown in figure 3(b). The vertical and horizontal projections indicate the win and lose probabilities, respectively. The BAD machine is characterized as a point on the circle that deviates from the $\pi/4$ direction by $-\theta$. By assuming $\theta > 0$, the winning probability of the GOOD machine is always higher than that of the BAD machine, which is consistent with the definition of GOOD/BAD machines.

Here, we consider an initial condition of the single-photon decision maker in which the polarization of a single photon is given by $\pi/4$, i.e., $\phi=0$ (figure 3(c)). In this situation, the probabilities of selecting slot machine 1 or 2, denoted by the projections $P^{(s)}_1$ or $P^{(s)}_2$ to the vertical or horizontal axis, respectively, are $1/\sqrt{2}$ based on the DECISION circle. For explanation, assume that the selected machine is slot machine 1 and is also the GOOD machine. The probability
of winning should be the product of the reward probability of the GOOD machine and its selection probability (in this case, the selection probability is $P_{i}^{(s)}$). By geometrically describing this situation using the WIN/LOSE circle, the winning probability is obtained as a projection of a vector with magnitude $P_{i}^{(s)}$ on the GOOD machine, schematically shown in figure 3(d).

Next, consider a case in which $\phi$ is slightly increased but is still smaller than $\theta$ ($\theta > \phi > 0$) (figure 3(e)). As in the initial condition, the winning probability of the GOOD machine is given by the projection of the vector along the GOOD machine with the probability of machine selection. That is, the WIN probability when the selected machine is GOOD is

$$P_{GW} = \sin\left(\frac{\pi + \phi}{4} + \frac{\theta}{2}\right) \sin\left(\frac{\pi + \theta}{4}\right),$$

(2)

whereas the LOSE probability with the GOOD machine selected is

$$P_{GL} = \sin\left(\frac{\pi + \phi}{4} + \frac{\theta}{2}\right) \cos\left(\frac{\pi + \theta}{4}\right),$$

(3)

where ‘GW’ and ‘GL’ are abbreviations of ‘Good machine Wins’ and ‘Good machine Loses’, respectively. $P_{GW}$ and $P_{GL}$ are plotted in the vertical axis of the WIN/LOSE circle. Similarly, when the selected machine is BAD, the WIN and LOSE probabilities are given by

$$P_{BW} = \cos\left(\frac{\pi - \phi}{4} + \frac{\theta}{2}\right) \sin\left(\frac{\pi - \theta}{4}\right),$$

(4)

and

$$P_{BL} = \cos\left(\frac{\phi - \pi}{4} + \frac{\theta}{2}\right) \cos\left(\frac{\pi - \theta}{4}\right),$$

(5)

respectively. The difference between equation (2) and equation (4) and the difference between equation (5) and equation (3) are respectively simplified/given by
\[ P_{GW} - P_{BW} = \sin\left(\frac{\phi + \theta}{2}\right), \quad P_{BL} - P_{GL} = \sin\left(\frac{\theta - \phi}{2}\right). \]  

Let \( \theta \) be a positive fixed value, then the derivatives of equation (6) with respect to \( \phi \) are given by

\[
\frac{d(P_{GW} - P_{BW})}{d\phi} \propto \cos\left(\frac{\phi + \theta}{2}\right), \quad \frac{d(P_{BL} - P_{GL})}{d\phi} \propto -\cos\left(\frac{\theta - \phi}{2}\right),
\]

indicating that the former and latter terms give positive and negative values, respectively, based on the condition that \( \theta > \phi > 0 \). In other words, the probability difference \( P_{GW} - P_{BW} \) is increasing while \( P_{BL} - P_{GL} \) is decreasing. This means that by increasing \( \phi \), the probability of selecting the GOOD machine is increasing (figure 3(f)). Based on the repetition strategy of the single-photon decision maker described earlier, an increase in \( \phi \) is implemented.

When \( \phi = \theta \), both \( P_{GL} \) and \( P_{BL} \) are projected onto the same point in the WIN/LOSE circle, diminishing their difference, which is confirmed by the second equation in equation (6).

Next, consider the case where \( \phi > \theta \) (figure 3(g)). In this case, the difference along the LOSE axis is modified from the previous case (the second term in equation (6)) and is given by

\[ P_{GL} - P_{BL} = \sin\left(\frac{\phi - \theta}{2}\right). \]  

The derivative of equation (8) with regard to \( \phi \) is given by \( 1/2 \sin[(\phi - \theta)/2] \), which is always positive as \( \phi > \theta \). This indicates that \( P_{GL} - P_{BL} \) is increasing (figure 3(h)).

Finally, \( \phi \) ultimately reaches the vertical axis, i.e., \( \phi / 2 = \pi / 4 \) (figure 3(i)). Here, the probability of selecting machine 1 is unity in the DECISION circle. The machine selection vector is projected directly onto the axis of the GOOD machine in the WIN/LOSE circle, whereas the
projection along the axis of the BAD machine completely disappears. Therefore, the WIN/LOSE probability stems only from the GOOD/BAD machines’ property specified by $\theta$, schematically shown in figure 3(j). Using the abovementioned mechanism, the polarization $\theta/2$ of the single-photon decision maker is autonomously directed towards the GOOD machine.

We conducted simulations to examine the mechanism discussed above. We first considered the case in which the reward probabilities of slot machines 1 and 2 are given by 0.6 and 0.4, respectively. In the WIN/LOSE circle, this corresponds to the case where the GOOD machine is characterized by the value $\theta/2$ for which the vector (0.4, 0.6) intersects the unit circle. Specifically, $\theta/2$ corresponds to 0.20 rad. The BAD machine is represented in the circle as a deviation from $\pi/4$ by $-\theta/2$. Here, 500 consecutive plays were repeated 500 times. We evaluated the following conditional probabilities to examine the dynamics behind the decision making compatible with the theoretical modeling:

(i) The probability of winning by selecting the GOOD machine if the result is win $[P(\text{GOOD WIN} \mid \text{WIN})]$;
(ii) The probability of winning by selecting the BAD machine if the result is win $[P(\text{BAD WIN} \mid \text{WIN})]$;
(iii) The probability of losing by selecting the GOOD machine if the result is lose $[P(\text{GOOD LOSE} \mid \text{LOSE})]$;
(iv) The probability of losing by selecting the BAD machine if the result is lose $[P(\text{BAD LOSE} \mid \text{LOSE})]$.

These are depicted respectively by the solid red, dashed green, dotted blue and dash dot magenta curves in figure 4(a). Here, the probabilities are calculated as ensemble averages over the samples.
The difference between (i) and (ii) increases over time throughout the playing cycles, which is closely consistent with the increase in magnitude of the arrow projected onto the WIN axis in the WIN/LOSE circle. On the other hand, the difference between (iii) and (iv) decreases until a given playing cycle (around cycle number 9). This is consistent with the decrease in magnitude of the arrow projected onto the LOSE axis in the WIN/LOSE cycle in which $\phi < \theta$ (figure 3(e)). After the cycle, the difference between (iii) and (iv) increases, which is consistent with the theoretical modeling in the case where $\phi > \theta$ (figure 3(g)).

Figure 4(b) shows the same analysis when the reward probabilities of slot machines 1 and 2 are given by 0.8 and 0.2, respectively. As the difference between the GOOD and BAD machines has become larger ($\theta / 2$ corresponds to 0.54 rad) than that in the previous case, the decision maker can find the GOOD machine more easily. The dynamics of the conditional probability follow similar trajectories to the previous case and flip over time from $\phi < \theta$ to $\phi > \theta$, which corresponds to the intersection between $P(\text{GOOD LOSE} | \text{LOSE})$ and $P(\text{BAD LOSE} | \text{LOSE})$ appearing in cycle 11.

3.2. Experimental analysis

We analyzed experimental data generated by single-photon-based decision making based on the formula described above. As reviewed in the beginning, the correct selection rate approaches unity as time elapses. We focused on the initial 150 cycles, starting from the initial condition that polarization is $\pi / 4$ or equivalently $\phi / 2 = 0$.

In the case when the reward probabilities of slot machines 1 and 2 were 0.8 and 0.2, the factorized evolution of conditional probabilities is summarized in figure 4(c). Due to the number of repeated samples being limited to 10, which is not large, the curves occasionally fluctuate. However, similar dynamics are successfully observed in figure 4(b), as predicted by theory.
Figure 4(d) summarizes conditional probabilities when the reward probabilities of slot machines 1 and 2 are given by 0.6 and 0.4, respectively. The blue dotted curve \([P(\text{GOOD LOSE} | \text{LOSE})]\) and dash dot magenta curve \([P(\text{BAD LOSE} | \text{LOSE})]\) intersect each other, which is consistent with the theory (figure 4(a)). On the other hand, the difference between \([P(\text{GOOD WIN} | \text{WIN})]\) and \([P(\text{BAD WIN} | \text{WIN})]\) decreases in the initial 24 cycles, which apparently does not agree with the abovementioned theoretical analysis. Note that the initial condition \(\phi / 2 = 0\) indicates that the single-photon decision maker is exploring both selections. Therefore, \(\phi / 2\) may occasionally take negative values (figure 4(e)). In such cases, the probability of selecting slot machine 2, which is on the horizontal axis of the DECISION circle, becomes larger, and machine simulation gets projected onto the BAD machine in the WIN/LOSE circle in the theory. Hence, in such situations, \(P_{\text{GW}} - P_{\text{BW}}\) decreases.

### 4. Category theoretic modeling and analysis

In the previous section, we discussed the underlying structure inherent in the decision-making problem via simple geometry-based modeling and analysis. However, the complex interdependencies involved in the subject matter have not uncovered yet; for example, the polarization updating strategy tells us that the polarizer setup depends both on the current decision and the betting result whereas the former simple model does not represent such interrelation. Furthermore, the effects of environmental conditions have not been revealed yet; such perspectives become further important in considering adaptive decision making, or autonomous intelligence, in dynamically changing environments. More generally speaking, this issue is closely related to the problem of describing or formalizing non-equilibrium open systems in physics. Based on such
motivation, in this section, we further generalize single-photon-based decision making by employing the notion of category theory.

As mentioned in the introduction, category theory [18-23] is a branch of mathematics that formalizes mathematical structure into collections of objects and morphisms (or arrows) that connect them. Category theory extracts the essence of all mathematical subjects to reveal and formalize stunningly simple yet extremely powerful patterns of thinking, which has revolutionized how mathematics is done [22]. Mathematically, a category is defined as follows [18,19]:

**Definition 1** (Category)

1. Objects: $A, B, C, \cdots$
2. Morphisms: $f, g, h, \cdots$
3. For each morphism $f$, there are given objects $\text{dom}(f)$ and $\text{cod}(f)$ called the domain and codomain of $f$, respectively. We write $f : A \to B$ to indicate that $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
4. Given morphisms $f : A \to B$ and $g : B \to C$, there is a given morphism $g \circ f : A \to C$ called the composite of $f$ and $g$.
5. For each object $A$, there is a given morphism $1_A : A \to A$ called the identity morphism of $A$.
6. Associability $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B$, $g : B \to C$, $h : C \to D$.
7. Unit $f \circ 1_A = f = 1_B \circ f$ for all $f : A \to B$.

A category can be anything that satisfies the abovementioned definition. One of the significant features of category theory is that objects and morphisms are determined by the role they play in a category via their relations to other objects and morphisms, i.e., by their position in a structure and **not** by **what they are** or **what they are made of** [19]. We consider the possibility
that these properties of category theory can be highly revealing when it comes to understanding
the decision-making problem; the category theoretic viewpoint may divulge the underlying
structure. In this study, we do not examine exact mathematical formulae and proofs; instead, our
goal is to obtain physical insight into the single-photon decision maker via category theoretic
viewpoints.

We start with a rudimentary picture of the decision-making problem, schematically shown
in figure 5(a), in which two objects are connected by a morphism. One object represents ‘Machine
selection’ or equivalently ‘Decision’, and the other object represents ‘Result’; these are denoted
by P and Q, respectively. In other words, ‘Machine selection’ and ‘Result’ are ‘Initial’ and ‘Final’
states of an apparent playing process, respectively. Indeed, the single-photon-based decision
maker and probabilistically behaving slot machines are absent in this simple diagram.

4.1. Product and coproduct

As the first step, we include the notions of ‘product’ and ‘coproduct’, which are obtained
from a basic category theoretic perspective [18].

**Definition 2** (product and coproduct)

In any category, a *product diagram* for the objects A and B consists of an object S and
morphisms \( A \leftarrow S \rightarrow B \) satisfying the following. Given any diagram of the form
\( A \leftarrow Z \rightarrow B \), there exists a unique \( u : Z \rightarrow S \), making the diagram

\[
\begin{array}{c}
Z \\
\downarrow z_1 \\
A \quad S \quad B \\
\downarrow z_2 \\
p_1 & u & p_2
\end{array}
\]

commute, i.e., \( z_1 = p_1 \circ u \) and \( z_2 = p_2 \circ u \). S is written as \( A \otimes B \).
A diagram $A \xrightarrow{p} T \xleftarrow{p} B$ is a coproduct of $A$ and $B$, represented by $T = A \oplus B$, if for any $Z$ and $A \xrightarrow{q} Z \xleftarrow{q} B$, there is a unique $u : T \rightarrow Z$ with $u \circ q_1 = z_1$ and $u \circ q_2 = z_2$ as indicated in

$$Z \xrightarrow{u} T \xleftarrow{q_1} A \xleftarrow{q_2} B.$$  

(10)

We introduce the notions of product and coproduct into the diagram shown in figure 5(a) and obtain the diagram shown in figure 5(b). Here, we have chosen one representative description of the slot playing process to demonstrate that it corresponds to obtaining either ‘WIN’ or ‘LOSE’ based on whether ‘Machine 1’ or ‘Machine 2’ is selected. Indeed, the practical slot playing processes as a whole involve complicated environmental substances, but the essence is describable based on the structure shown in figure 5(b) by taking all the environmental conditions preceding the slot play into the product, and the succeeding environmental conditions into the coproduct. Here, we provide the physical interpretation of each object and morphism. The product $P \otimes Q$, denoted by $X$ in figure 5(b), indicates the ‘Casino Setting’, including the entire environmental condition of the slot playing. The morphism from the Casino Setting ($X$) to Result ($Q$) indicates that the Casino Setting obviously generates the betting Result (WIN or LOSE) and all the environmental conditions are injected into ‘0’ in $Q$, i.e., the equivalence class to the Result. Therefore, all environmental conditions are included in the “kernel” of the morphism $X \rightarrow Q$. According to category theory, the kernel can be regarded as a set consisting of the preceding object and morphism in a short exact sequence; hence the ‘all environmental conditions’ of the slot play, also referred to as ‘Machine operation environment’, is placed as object $M$ shown in figure 5(c). [21]. At the same time, the Casino Setting affects the Decision, which is manifested by the
morphism from $X$ to $P$. Here the angle $\theta/2$, which is introduced in the geometrical analysis discussed earlier, is marked in the vicinity of $X$ because the **Casino Setting** involves the *hidden* adjustment concerned with the GOOD/BAD machine.

The coproduct $P \oplus Q$, denoted by $Y$ in figure 5(b), is introduced based on the representative descriptions that the slot playing processes are synthesized as the combination of the **Decision** ($P$) and resultant **Result** ($Q$), so as to extract the knowledge for better decisions in subsequent trials; i.e., the information of whether the GOOD machine is associated with either ‘Machine 1’ or ‘Machine 2’ based on the betting result. The coproduct $P \oplus Q$ involves all environmental conditions after a slot play is completed. The visible part of coproduct $Y$ is physically accommodated in the polarization updating strategy based on the Result, so that $Y$ is referred to as the ‘**Polarizer Setting**’ ($\phi/2$) for decision making in which the hidden parameter of the slot machines ($\theta/2$) is inferred. Since the morphism $Q \rightarrow Y$ dominates the polarization of the single photon source which determines the quantum mechanical probability distribution in the decision maker, the environmental conditions for the optical fields correspond to ‘*co-kernel*’ of the morphism $Q \rightarrow Y$. According to category theory, the co-kernel can be regarded as a set consisting of the subsequent object and morphism in a short exact sequence; thus the optical environmental conditions (namely, the ‘**Optical environment**’) of the single-photon decision maker, also referred to as ‘**Machine operation environment**’, is placed as object $F$ shown in figure 5(c) [21]. Such dependencies are clearly supported by the reconfiguration strategy of the single-photon decision maker. It is noted that all the unobservable environmental conditions after the single slot-play are implicitly included in the **Optical environment** $F$. Such high reception of objects and morphisms reveals the expressive power of category theory.
It is noteworthy that the category theoretic picture shown in figure 5(b) indicates that the product $X = P \otimes Q$ corresponds to the ‘governor’ dominating the initial and final states of apparent slot-playing events, whereas the coproduct $Y = P \oplus Q$ is the ‘observer’ who attempts to infer the governor. In the mathematical context of category theory, the diagram shown in figure 5(b) corresponds to a representative description of the slot playing process $P \to Q$ based on the representative morphisms $X \to P$ and $Q \to Y$, indicated by the arrows with double lines, belonging to multiplicative system of morphisms, i.e. one can describe the slot playing process in a different but equivalent manner based on any set of representative morphisms belonging to the multiplicative system [21]. The morphisms $X \to Q$ and $P \to Y$ are referred to, respectively, as the right and left quotient morphisms of $P \to Q$ based on the representatives $X \to P$ and $Q \to Y$. The most important feature of the diagram shown in figure 5(b) resides in the commutative relation between $X \to P \to Y$ and $X \to Q \to Y$ which leads us to the description of the polarization updating strategy in the following.

4.2. Complex and characteristic arrow

We can naturally assume that the category theoretic graph shown in figure 5(c) indicates the relationships between objects when they have established certain stationary states, i.e., a single slot-playing process, including adjustment of the single-photon decision maker for subsequent slot-playing, is completed. Then we can proceed to the next slot-playing based on the prepared optical environment. In order to construct the category theoretic picture of the polarization updating strategy of the single-photon decision maker, we need to introduce the notion of ‘complex’:

**Definition 3** (complex)
A complex $A^*$ is a sequence of objects $\{A^i\}_{i \in \mathbb{Z}}$ and morphisms $d^i_j : A^i \to A^{i+1}$ such that $d^i_j \circ d^{i-1}_j = 0$ for all $j$, i.e., the nature of boundary operator or differential operator.

Due to this nature, the object $A^{i-1}$ in the complex is injected as the image of the morphism $d^{i-1}_j$ into the kernel of the morphism $d^i_j$. It is noted that the quotient of kernel of $d^i_j$ divided by the equivalent class of the image of $d^{i-1}_j$ is referred to as the $j$-th order homology (or co-homology) $H^j(A^*)$ of the complex $A^*$. The remarkable feature is that the homology is irrelevant to the preceding object and is transferred to the subsequent object as the equivalent class of 0 object; that is, the homology $H^j(A^*)$ represents the local feature added only to the object $A^i$ in the complex $A^*$. Therefore, a complex describes a sequential evolution of objects with a history of sequential addition of homology (co-homology).

It is useful to introduce ‘shift’ or ‘translation’ of complex; $C^* = A^*[1]$ consisting of $\{C^i\}_{i \in \mathbb{Z}} = A^{i+1}$ with $d^i_c = -1 d^i_j$ [21].

A morphism of complex, $f : A^* \to B^*$, is a set of morphism $f^i : A^i \to B^i$ which satisfies $f^{i+1} \circ d^i_j = d^{i+1}_j \circ f^i$ for all $j$. Here, one of the most remarkable features known in category theory is about the ‘chain-wise exact sequence of complex’ given by $0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{R} \to 0$, which consists of short exact sequences $0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{R} \to 0$ for all $j$. What is important is that the chain-wise exact sequence of complex induces the long exact sequence of homology [21]:

$$\cdots \to \bullet \to H^{i-1}(R) \to H^i(P) \to H^i(Q) \to H^i(R) \to H^{i+1}(P) \to \cdots.$$ (11)

Moreover, category theory tells us that, within a certain equivalent class of homotopy, one can find a ‘characteristic arrow’ also called ‘translation morphism’ $R^* \to \mathcal{P}^*[1]$ which maintains the
long exact sequence of homology. That is, the evolution of chain-wise exact sequence of complex is described by a ‘triangular’ structure, $P' \rightarrow Q' \rightarrow R' \rightarrow P'[1]$ [21].

4.3. Octahedral structure in decision making

Based on the category theoretical descriptions of the step-wise evolving relationship of objects in terms of complex and morphism discussed above, we can naturally transform the single slot playing process with the single-photon decision maker described in figure 5(c) into the diagram of complexes evolving under the polarization updating strategy as shown in figure 5(d). In this diagram, the updating processes of the entire environmental condition $M$ and the optical environment $F$ have been introduced with the characteristic arrows indicated by wiggly lines. The positions of $M$, $F$, and the characteristic arrows are determined based on the commutative relation between $X^* \rightarrow P^* \rightarrow Y^*$ and $X^* \rightarrow Q^* \rightarrow Y^*$. Since the step-wise short exact sequences $0 \rightarrow M \rightarrow X \rightarrow Q \rightarrow 0$ and $0 \rightarrow Q \rightarrow Y \rightarrow F \rightarrow 0$ fulfil certain equilibrium conditions for each slot playing process, we can derive the composite morphisms $M^* \rightarrow Q^*$ and $Q^* \rightarrow F^*$; accordingly, the category theoretic picture in figure 5(d) includes triangular structures given by $M^* \rightarrow Q^* \rightarrow Y^* \rightarrow M^*[1]$ (marked by ‘B3’) and $X^* \rightarrow Q^* \rightarrow F^* \rightarrow X^*[1]$ (‘B1’) corresponding to the polarization updating strategy. This diagram naturally coincides with a physical interpretation of subsequent machine selection that is made based on the optical environmental conditions. This is done through the single-photon decision maker, which takes into consideration all environmental conditions after a slot play in order to prepare the subsequent environmental conditions of slot play.

Since the diagram in figure 5(d) has the same complexes $M^*$ and $F^*$ at the top and bottom, we can construct an equivalent diagram while having $P^*$ and $Q^*$ placed at the top and bottom, as shown in figure 5(e). This indicates the commutativity of $Y^* \rightarrow F^* \rightarrow X^*$ and $Y^* \rightarrow M^* \rightarrow X^*$.
including characteristic arrows. We can complete the diagram by adding composite morphisms $F^* \to P^*$ and $P^* \to M^*$ as characteristic arrows as shown in figure 5(f). Here, we can find another two triangular structures given by $P^* \to Y^* \to F^* \to P^*[1]$ (denoted by ‘B4’ in Fig. 5(f)) $M^* \to X^* \to P^* \to M^*[1]$ (‘B2’) corresponding to the polarization updating strategy.

The diagrams in figure 5(f) have a compact picture of octahedral structure by directly connecting the same complexes at the top and bottom of the diagram with each other, leading to a three-dimensional diagram shown in figure 5(g). This structure corresponds to one of the most important consequences of ‘triangulated category’ or ‘derived category’ called the octahedron axiom [19]. The octahedron consists of four short exact sequences corresponding to triangular category and four triangular diagrams as indicated in figure 5(g). The triangles B1 and B2 are located in the upper half of the octahedron while those of B3 and B4 are in the lower half.

Here, one important remark must be made about the interpretation of the induced long exact sequence of homology or the triangular structure of the complex in decision making. Firstly, it is indeed natural to associate homology induced in each complex with the ‘local environment’. For example, the betting results of each play is determined based on a spontaneous symmetry break occurring in the slot machine, which is included in the homology of Machine operation environment ($M^*$). Generally, descriptions of the intention, will or preference of the decision maker and the slot machine are represented in homology. On the other hand, the short exact sequences, which involve no homology, tightly restrict the evolution of complexes in the triangulated category; hence the sequence of homology indicates the ‘history’ of evolution experienced by each object via the triangulated structure.

In the category theoretical context, the octahedral structure is known to be resolved into two Mayer-Vietoris sequences [21].
\[
X^* \rightarrow Q^* \oplus F^* \rightarrow Y^* \rightarrow X^*[1], \quad (12)
\]
\[
X^* \rightarrow Y^* \rightarrow M^*[1] \oplus F^* \rightarrow X^*[1] \quad (13)
\]
which correspond, respectively, to the two commutative diagrams shown in figures 5(d) and 5(e).

These Mayer-Vietoris sequences imply that the structure of the initially unknown structure \( X^* \) is transferred to the observer \( Y^* \); namely, correct decision making is realized. In the following section, we investigate the geometrical properties based on the braid structure of the octahedral diagram.

One remark we have is that category theory provides us a useful framework to picture complicated physical systems and their functionalities for which one could not necessarily identify exact physical entities. This is because category theory allows us to describe and analyze relationships and functionalities of conceptualized objects, and to derive phenomenal substances of the systems and their functionalities. Therefore, category theory is especially significant in describing systems that exert functionalities in non-equilibrium open systems involving a certain group or hierarchy of environmental systems. Furthermore, by using ‘functor’ in category theory, one can transfer functional structures from one category to another while preserving essential substances.

4.4. Braid structure in decision making

In order to deepen understanding of the physical and mathematical implications of the single-photon decision maker via geometrical considerations, we have simulated the evolution of the polarization updating process based on its ‘braid structure’ of octahedron shown in figure 5(f) (regarding the relation between the hidden variable \( \theta/2 \) and polarization setting \( \phi/2 \)).

As a series of decision making and polarization updating processes, the diagram shown in figure 5(f) can be extended by appending shifted diagrams of octahedral structure as shown in
figure 6(a). More specifically, the extended diagram in figure 6(a) is derived by repeating the
diagram in figure 5(f) while swapping the positions of $P^*$ and $Q^*$; consequently, the four short
exact sequences are arranged sequentially. Following the category theoretical context, this diagram
produces a ‘braid structure’ [21,26] of octahedron shown in figure 6(b), which consists of the
following four exact sequences as braids;

\[
\begin{array}{c|c}
\text{Braid}_{1} & \text{Braid}_{4} \\
\downarrow & \downarrow \\
0 \rightarrow M^* \rightarrow X^* \rightarrow P^* \rightarrow 0 & \text{Braid}_{2} \\
\downarrow & \downarrow \\
0 \rightarrow M^* \rightarrow Q^* \rightarrow Y^* \rightarrow 0 & \text{Braid}_{3} \\
\downarrow & \downarrow \\
F^* \rightarrow F^* & \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

(14)

This braid structure reveals the geometrical structure or interdependence underlying the single-
photon decision maker in a totally simplified manner. Here, we further explore the braid concept
by examining the ‘knots’ of the braids. Since our main concern in decision making is whether
Machine 1 or 2 corresponds to the GOOD machine, we can analyze the relative behaviors of the
braids with respect to the hidden variable of the Casino Setting $\theta/2$ in the following.

[Study 1: Knots at X] Braid 1 and 2 intersect with each other at $X$. Because $X$ represents the
Casino Setting $\theta/2$, let us assume that $\theta > 0$ means that Braid 1 stays over Braid 2, whereas
$\theta < 0$ represents the converse case. If the Casino Setting is unchanged (i.e., $\theta/2$ is constant), one
braid is always on top of the other; hence, there is no knotting of braids, schematically shown in
figure 6(c).

[Study 2: Knots at Y] Braid 3 and 4 intersect at $Y$, which physically corresponds to the
Polarizer Setting ($\phi/2$). A knot is induced when two braids are ‘wrapped’, schematically shown
in figure 6(d). The purpose of the decision maker is to unfold the knots at $Y$ such that an adequate $\phi/2$ is derived by inferring the Casino Setting $(\theta/2)$. In quantitatively analyzing the braids, we investigate the system simulated in the numerical analysis shown earlier. We let the reward probabilities of slot machines 1 and 2 be 0.6 and 0.4 and all other conditions of numerical simulation are the same as discussed before. Figure 6(e) shows an incidence histogram of the number of knots at $Y$. As time elapses, knots are unfolded, meaning that the Casino Setting $\theta/2$ is inferred by the decision maker. Figure 6(f) evaluates when the ‘last’ knot in 500 consecutive plays is induced in the system; the incidence frequency decays quickly, meaning that polarization adaptation is promptly completed.

[Study 3: Knots at $P$] Braids 2 and 4 intersect at $P$, which physically corresponds to the Decision. The machine selection is probabilistically conducted based on the polarization $(\phi/2)$ of an incident single photon. This means that the knot of Machine Selection ($P$) is not equivalent to that of the Polarizer Setting ($Y$). Actually, as shown in figure 6(g), the histogram of the total number of knots at $P$ differs from that of the Polarizer Setting (figure 6(e)); the number of knots at $P$ is larger than that at $Y$. As shown in figure 6(h), showing the last appearance of knots over the 500 plays, knots are induced even after hundreds of plays are conducted. These observations clearly manifest the fundamental architecture of the decision-making problem and the dynamics of the single-photon decision maker.

[Study 4: Knots at $F$] Braids 1 and 4 intersect at $F$, which corresponds to the polarization of a single photon. This polarization is deterministically configured by the Polarizer Setting with the properties of knots at $F$ equal to those at $Y$ (see Study 2).

[Study 5: Knots at $Q$ and $M$] Braids 1 and 3 intersect at $Q$ and Braids 2 and 3 intersect at $M$. The physical interpretation of $Q$ is the WIN/LOSE betting results, whereas that of $M$ is the
Machine operation environment. Both these entities cannot be controlled by the decision maker; hence, knots at \( Q \) and \( M \) are never unfolded. This can naturally be understood by considering the case in which even completely correct decisions cannot avoid ‘LOSE’ events if the reward probability of the GOOD slot machine is not 100%.

Furthermore, both \( Q \) and \( M \) are on the same braid, \( \text{Braid 3: } \rightarrow M' \rightarrow Q' \rightarrow Y' \rightarrow \), so that the other object on \( \text{Braid 3}, Y \), physically indicates that the polarized angle exhibits fluctuations due to the uncontrollable entities on \( \text{Braid 3} \). This is another insight provided by the category theoretic analysis.

5. Conclusion

In conclusion, we presented theoretical foundations for single-photon-based decision making using category theory. The notion of DECISION and WIN/LOSE circles through which the evolution of polarization adaptation is theoretically formulated and analyzed was introduced. The simulation and experimental results agree well with each other in accordance with the geometrical modeling. Next, we demonstrated a theoretical model of single-photon decision making by category theory. We showed that the octahedral structure in triangulated category clearly reveals the underlying mechanisms of the single-photon decision maker. The effects of environment were properly included as objects in the octahedral structure. The Mayer-Vietoris sequences also support the principle of decision making in the form transferring the initially unknown structure to the observer. The braid structure, known in triangulated category, provided various insights including quantitative decision making metrics, such as the fact that braid knots unfolding corresponds to adaptation to a given problem, and that braids that cannot be unfolded clearly represent uncontrollable entities inherent to the system. This study provides a solid foundation for
the single-photon decision maker, and paves the way for future usage of the category theoretical approach for the general understanding and design of intelligence.

Acknowledgments

The authors thank M. Agu and K. Kitahara for their strong encouragement toward the study of functionalities in a non-equilibrium open system. This work was supported in part by the Grant-in-aid in Scientific Research and the Core-to-Core Program, A. Advanced Research Networks from the Japan Society for the Promotion of Science, CREST program from by Japan Science and Technology Agency, and Agence Nationale de la Recherche, France, through the SINPHONIE and PLACORE project (Grant No. ANR-12-NANO-0019 and ANR-13-BS10-0007).
References

1. Daw N, O’Doherty J, Dayan P, Seymour B and Dolan R 2006 Cortical substrates for exploratory decisions in humans *Nature* **441** 876–879

2. Sutton R S and Barto A G 1998 *Reinforcement Learning: An Introduction* (Cambridge: The MIT Press)

3. Naruse M, Berthel M, Drezet A, Huant S, Aono M, Hori H and Kim S-J 2015 Single-photon decision maker *Sci. Rep.* **5**, 13253

4. Lai L, Gamal H, Jiang H and Poor V 2011 Cognitive Medium Access: Exploration, Exploitation, and Competition *IEEE Trans. Mob. Comput.* **10** 239–253

5. Kim S–J and Aono M 2014 Amoeba-inspired algorithm for cognitive medium access *NOLTA* **5** 198–209

6. Agarwal D, Chen B-C and Elango P 2009 Explore/exploit schemes for web content optimization *Proc. of ICDM* 1–10

7. Kocsis L and Szepesvári C 2006 Bandit based Monte Carlo planning. *Machine Learning: ECML 2006, LNCS* **4212** 282–293

8. Press W H 2009 Bandit solutions provide unified ethical models for randomized clinical trials and comparative effectiveness research *PNAS* **106** 22387–22392

9. Takahashi H 2013 Molecular neuroimaging of emotional decision-making. *Neuroscience Research* **75** 269–274.

10. Pothos EM, Busemeyer J 2009 A quantum probability explanation for violations of ‘rational’ decision theory *Proc. Royal Soc. London B: Bio. Sci. rspb-2009*

11. Cheon T and Takahashi T 2010 Interference and inequality in quantum decision theory. *Phys. Lett. A* **375** 100–104
12. Auer P, Cesa-Bianchi N and Fischer P 2002 Finite-time analysis of the multi-armed bandit problem *Machine Learning* **47** 235–256

13. Kim S -J, Aono M and Hara M 2010 Tug-of-war model for the two-bandit problem: Nonlocally-correlated parallel exploration via resource conservation *BioSystems* **101** 29–36

14. Kim S -J, Aono M and Hara M 2010 Tug-of-war model for multi-armed bandit problem *LNCS 6079* 69–80

15. Kim S -J, Aono M and Nameda E 2015 Efficient decision-making by volume-conserving physical object *New J. Phys.* **17** 083023

16. Kim S -J, Naruse M, Aono M, Ohtsu M and Hara M 2013 Decision Maker based on Nanoscale Photo-excitation Transfer *Sci. Rep.* **3** 2370

17. Naruse M, Nomura W, Aono M, Ohtsu M, Sonnefraud Y, Drezet A, Huant S and Kim S -J 2014 Decision making based on optical excitation transfer via near-field interactions between quantum dots *J. Appl. Phys.* **116** 154303

18. Mac Lane S 1971 *Categories for the Working Mathematician* (Berlin: Springer)

19. Awodey S 2010 *Category Theory* (Oxford: Oxford University Press)

20. Simmons H 2011 *An Introduction to Category Theory* (Cambridge: Cambridge University Press)

21. Iversen B 1986 *Cohomology of sheaves* (Berlin: Springer-Verlag)

22. Spivak D I 2014 *Category Theories for the Sciences* (Cambridge: The MIT press)

23. Kashiwara M and Schapira P 2006 *Categories and Sheaves* (Berlin: Springer)

24. Beveratos A, Brouri R, Gacoin T, Poizat J –P and Grangier P 2001 Nonclassical radiation from diamond nanocrystals *Phys. Rev. A* **64** 061802
25. Rondin L, Dantelle G, Slablab A, Grosshans F, Treussart F, Bergonzo P, Perruchas S, Gacoin T, Chaigneau M, Chang H -C, Jacques V and Roch J -F 2010 Surface-induced charge state conversion of nitrogen-vacancy defects in nanodiamonds *Phys. Rev. B* **82** 115449

26. Iversen B 1986 Octahedra and braids. *Bulletin de la Société Mathématique de France* **114** 197–213
Figure legends

**Figure 1.** Architecture of single-photon decision maker. The polarization of single photons is configured such that the higher-reward-probability slot machine is selected. See reference (3) for details. This study is concerned with theoretical backgrounds of decision making. (Adapted by permission from Macmillan Publishers Ltd: Scientific Reports [3], Copyright 2015)

**Figure 2.** Experimental demonstration of single-photon decision maker. The reward probabilities of slot machines 1 and 2 are configured as \{0.8 and 0.2\} (solid curve) and \{0.6 and 0.4\} (dotted curve). These situations, including experimental results, are investigated in the theoretical modeling and analysis. (Adapted by permission from Macmillan Publishers Ltd: Scientific Reports [3], Copyright 2015)

**Figure 3.** Geometry-based modeling of single-photon decision maker. Two circles are introduced: (a) **DECISION circle** where the angle $\phi/2$ indicates the polarization of a single photon and (b) **Win/Lose circle** where the angle $\theta/2$ indicates the GOOD and BAD slot machines. The decision, given based on the Decision circle, is mapped onto this Win/Lose circle.

(c–j) Explaining the mechanism behind the single-photon decision maker autonomously shifting towards an accurate decision. (c, d) Initial situation ($\phi = 0$). (e, f) Second stage ($\phi < \theta$). (g, h) Third stage ($\phi > \theta$). (i, j) Final stage ($\phi/2 = \pi/4$). See main text for details.

**Figure 4.** Analysis of simulations and experiments from the geometry-based modeling aspect. Conditional probabilities of $P(\text{GOOD WIN} | \text{WIN})$, $P(\text{BAD WIN} | \text{WIN})$, $P(\text{GOOD LOSE} | \text{LOSE})$ and $P(\text{BAD LOSE} | \text{LOSE})$ are respectively depicted by the solid red,
dashed green, dotted blue and dash dot magenta curves. The reward probabilities are \((A, D)\) 0.6 and 0.4 \((B, D)\) 0.8 and 0.2. The simulation, experiment and theoretical predictions agree well with each other.

**Figure 5.** Category theoretic modeling. (a) Rudimentary picture of decision making. (b) Product and coproduct involving Machine selection and Result. (c–g) Synthesis of octahedron structure. (c) Addition of two important objects (complexes): Machine selection environment (or photon environment) \((F^*)\) and Machine operation environment \((M^*)\). (d) Addition of two composite morphisms \((M^* \rightarrow Q^*\) and \(Q^* \rightarrow F^*\)) and two translation morphisms \((Y^* \rightarrow M^*[1]\) and \(F^* \rightarrow X^*[1]\)). (e) Another commutative diagram \((Y^* \rightarrow F^* \rightarrow X^*\) and \(Y^* \rightarrow Q^* \rightarrow X^*\)). (f) Addition of two translation morphisms \((P^* \rightarrow M^*[1]\) and \(F^* \rightarrow P^*[1]\)) and four triangulated structures \((B1, \cdots, B4)\) are derived. (g) Octahedral structure. The triangles B1 and B2 are located in the upper half and those of B3 and B4 are in the lower half.

**Figure 6.** Braids and knots in decision making. (a) Concatenated structure of the octahedral structure. (b) Four braids are obtained from the octahedron structure: B1: \(X^* \rightarrow Q^* \rightarrow F^*\), B2: \(M^* \rightarrow X^* \rightarrow P^*\), B3: \(M^* \rightarrow Q^* \rightarrow Y^*\), B4: \(P^* \rightarrow Y^* \rightarrow F^*\). (C, D) Knots of braids. There is no knot when one braid stays on top of the other (c), whereas a knot is induced in the situation shown in (d). (e–h) Evaluate the knots at certain positions in the braid structure. Autonomous decision making corresponds to unwrapping the knots at the complexes of Polarizer Setting \((Y^*)\) and Machine selection \((P^*)\), which are demonstrated respectively in (e, f) and (g, h).
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5
Figure 6