On refined enumerations
of some symmetry classes of ASMs

A. V. Razumov, Yu. G. Stroganov

Institute for High Energy Physics
142280 Protvino, Moscow region, Russia

Abstract

Using determinant representations for partition functions of the corresponding square ice models and the method proposed recently by one of the authors, we investigate refined enumerations of vertically symmetric alternating-sign matrices, off-diagonally symmetric alternating-sign matrices and alternating-sign matrices with U-turn boundary. For all these cases the explicit formulas for refined enumerations are found. It particular, Kutin–Yuen conjecture is proved.

1 Introduction

An alternating-sign matrix (ASM) is a matrix with entries 1, 0, −1, such that 1’s and −1’s alternate in each column and each row, and such that the first and last non-zero entry in each row and column is 1. During last decade many enumeration and equinumeration results related to the ASMs were conjectured and proved. Nevertheless, there is a lot of problems to be solved. Besides of their importance for pure combinatorics enumeration results on ASMs will undoubtedly find numerous applications to problems of mathematical physics, see in this respect, for example, papers [17, 1, 18, 19, 16, 20, 5, 2, 6, 27, 4].

In the present paper, using the method proposed by one of the authors [23], we investigate refined enumerations of some symmetry classes of the ASMs. In section 2 we reproduce the necessary results of paper [23]. In section 3 the refined enumerations for ASMs with U-turn boundary and vertically symmetric ASMs are found, see Corollary 21. In section 4 we prove that the refined enumerations of vertically symmetric ASMs and off-diagonally symmetric ASMs coincide. This is the conjecture by Kutin and Yuen presented in a message [12] sent by Kuperberg to the Domino forum. In the next two paragraphs we actually quote that message.

The conjecture concerns the vertically symmetric ASMs (VSASMs) and the off-diagonally symmetric ASMs (OSASMs). These symmetry classes of ASMs are defined in the following way. An ASM is said to be vertically symmetric if it remains unchanged after the reflection with respect to the vertical line which divides the matrix into two equal parts. Note that an ASMs of odd order only can be vertically symmetric. An ASM is said to be off-diagonally symmetric if it remains unchanged after the reflection with respect to its diagonal, and whose diagonal is null. In paper [11] G. Kuperberg proved that there are the same number of $2n \times 2n$ OSASMs as $(2n + 1) \times (2n + 1)$ VSASMs. To check this equinumeration Kutin and Yuen did a computer experiment, and found that there are still the same number if one fixes the position of the 1 in the right-most column. Note that any ASM has only one 1 in the right-most column. Here one has to remind that a VSASM cannot have the right-most 1 at the top or bottom, and
OSASMs cannot have the right-most 1 at the bottom, so for both classes there are \(2n - 1\) available positions.

For example, compare the VSASMs of order 5, Figure 1, with the OSASMs of order 4,

\[
\begin{pmatrix}
0 & 0 & + & 0 & 0 \\
+ & 0 & - & 0 & + \\
0 & 0 & + & 0 & 0 \\
0 & + & - & + & 0 \\
0 & 0 & + & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & + & 0 & 0 \\
+ & - & + & 0 & + \\
0 & 0 & + & 0 & 0 \\
0 & - & 0 & + & 0 \\
0 & 0 & + & 0 & 0
\end{pmatrix},
\]

Figure 1: VSASMs of order 5

In both cases the Kutin–Yuen numbers are 1, 1, 1.

\[
\begin{pmatrix}
0 & 0 & 0 & + \\
0 & 0 & + & 0 \\
0 & + & 0 & 0 \\
+ & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & + & 0 \\
0 & 0 & 0 & + \\
+ & 0 & 0 & 0 \\
0 & + & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & + & 0 & 0 \\
+ & 0 & 0 & 0 \\
0 & 0 & 0 & + \\
0 & 0 & + & 0
\end{pmatrix}
\]

Figure 2: OSASMs of order 4

Our consideration is based on the bijection between the states of the square ice model with appropriate boundary conditions and ASMs. Consider a subset of vertices and edges of a square grid, such that each vertex is either tetravalent or univalent. A state of a corresponding square ice model is an orientation of the edges, such that two edges enter and leave every tetravalent vertex. If orientations of the edges belonging to univalent vertices is fixed we said that a boundary condition for the square ice model is given. The case of importance for studying the ASMs is the square ice model with the domain wall boundary conditions. A pattern for a state is given in Figure 3. The meaning of the labels \(x_i\) and \(y_i\) will be explained later. Such boundary conditions were first considered by Korepin [8].

\[1\text{As is now customary for alternating-sign matrices, we write } + \text{ instead of 1 and } - \text{ instead of } -1.\]
If we replace each tetravalent vertex of a state of the square ice model with the domain wall boundary condition by a number according Figure 4 we will obtain a matrix. An example is

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Figure 4: The correspondence between the square ice vertices and the entries of alternating-sign matrices
given in Figure 5. It is not difficult to check that in this way we come to the bijection between

\[
\begin{pmatrix}
0 & + & 0 & 0 \\
+ & - & + & 0 \\
0 & 0 & 0 & + \\
0 & + & 0 & 0 \\
\end{pmatrix}
\]

Figure 5: An example of the correspondence between the square ice states and alternating-sign matrices

the states and the ASMs [3].

The above correspondence was first used to solve ASMs enumeration problems by Kuperberg [10]. His strategy was as follows. Consider the partition function of the square ice model with the domain wall boundary conditions. This is the sum of the weights of all possible states. The weight of a state is the product of the weights of all tetravalent vertices which are defined in the following way. Associate spectral parameters \( x_i \) with the vertical lines of the grid and spectral parameters \( y_i \) with the horizontal ones, see Figure 3. A vertex at the intersection of the line with the spectral parameters \( x_i \) and \( y_j \) is supplied with the spectral parameter \( x_i/y_j \). The weights of the vertices depend on the value of the spectral parameter and are given in Figure 6, where \( a \) is a parameter common for all vertices and we use the convenient abbreviations

\[
\sigma(a^2), \sigma(a^2), \sigma(ax), \sigma(ax), \sigma(a\bar{x}), \sigma(a\bar{x})
\]

Figure 6: The weights of the vertices with the spectral parameter \( x \)

\footnote{In contrast with paper [23] we use multiplicative parameters.}
\[ \bar{x} = x^{-1}, \]
\[ \sigma(x) = x - \bar{x} \]

introduced by Kuperberg.

Denote the partition function of the square ice model with the domain wall boundary condition by \( Z(n; x, y) \). Here \( n \) is the size of the square ice, \( x \) and \( y \) are row-vectors constructed from the spectral parameters,

\[ x = (x_1, \ldots, x_n), \quad y = (x_1, \ldots, x_n). \]

Given a number \( x \), denote by \( A(n; x) \) the total weight of the \( n \times n \) ASMs, where the weight of an individual ASM is \( x^k \) if it has \( k \) matrix elements equal to \(-1\). Consider the partition function \( Z(n; x, y) \) for \( x = 1 \) and \( y = 1 \), where

\[ 1 = (1, 1, \ldots, 1). \]

If the number of \(-1\)’s in an \( n \times n \) alternating-sign matrix is equal to \( k \), then the number of \( 1 \)’s is equal to \( n + k \) and the number of \( 0 \)’s is \( n^2 - n - 2k \). It is clear that the weight of the corresponding state of the square ice is

\[ \sigma(a^2)^k \sigma(a^2)^{n+k} \sigma(a)^{n^2-n-2k} = \sigma(a)^{n^2-n} \sigma(a^2)^n \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^{2k}. \]

Hence we have the equality

\[ A(n; x) = \frac{1}{\sigma(a)^{n^2-n} \sigma(a^2)^n} Z(n; 1, 1), \tag{1} \]

where

\[ x = \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^2 = (a + \bar{a})^2. \]

In particular, when \( a = \exp(i \pi/3) \) we have \( x = 1 \), and relation (1) gives the total number of \( n \times n \) ASMs

\[ A(n) = \frac{1}{\sigma(a)^{n^2}} Z(n; 1, 1) \bigg|_{a = \exp(i \pi/3)}. \]

Thus, the enumeration problem for ASMs is reduced to the problem of studying the partition function \( Z(n; x, y) \) for \( x = 1 \) and \( y = 1 \).

In paper [10] Kuperberg used for \( Z(n; x, y) \) the representation via the Izergin–Korepin determinant [7, 9] and proved the formula for \( A(n) \) conjectured by Mills, Robbins and Rumsey [13, 14] and first proved by Zeilberger [25]. In paper [11] Kuperberg generalised the Izergin–Korepin determinant and the Tsuchiya determinant [24] to address enumeration problems related to numerous symmetry classes of alternating-sign matrices.

In paper [26] Zeilberger used again the representation of the partition function \( Z(n, x, y) \) via the Izergin–Korepin determinant to prove the so-called refined ASM conjecture by Mills, Robbins and Rumsey [13, 14]. Let us explain what this enumeration means. Note that the first column of an ASM contains only one 1, all other entries are 0’s. Therefore, we can try to enumerate the ASMs for which the unique 1 is at the \( r \)th position in the first column. The corresponding numbers \( A(n, r) \) give the refined enumeration of the ASMs. Similarly to the case of the usual enumeration one can assume that the weight of an individual ASM is \( x^k \) if it
has \( k \) matrix elements equal to \(-1\). This gives the polynomials \( A(n, r; x) \) which describe the weighted refined enumeration of the ASMs.

To relate the polynomials \( A(n, r; x) \) to the partition function \( Z(n; x, y) \) consider the case where \( x = 1 \) and \( y = (u, 1, \ldots, 1) \). In this case the label of a vertex which belongs to the first column is \( \bar{u} \), otherwise it is \( 1 \). Consider a state of the square ice for which the unique 1 in the first column of the corresponding ASM belongs to the \( r \)th row. Here in the first column we have one vertex of the first type, \( r - 1 \) vertices of the third type and \( n - r \) vertices of the sixth type, where type is defined in accordance with the position of the vertex in Figure 6. Hence, the contribution of the first column to the weight of the state is

\[
\sigma(a^2)^{n-1} \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^{2k} \left[ \frac{\sigma(a \bar{u})}{\sigma(a u)} \right]^{r-1},
\]

and we have the equality

\[
\sum_{r=1}^{n} A(n, r; x) t^{r-1} = \frac{Z(n; 1, (u, 1, \ldots, 1))}{\sigma(a)^{n^2-2n+1} \sigma(a^2)^n \sigma(a u)^{n-1}},
\]

where

\[
x = \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^2, \quad t = \frac{\sigma(a \bar{u})}{\sigma(a u)}.
\]

Thus, the refined enumeration problem for ASMs is reduced to the problem of studying the partition function \( Z(n; x, y) \) for \( x = 1 \) and \( y = (u, 1, \ldots, 1) \).

Using the representation of the partition function via the Izergin–Korepin determinant, Kuperberg and Zeilberger had to overcome its singularity at desired values of the spectral parameters. One of the authors of the present paper proposed a new way to deal with the Izergin–Korepin determinant valid at \( a = \exp(i \pi/3) \). It allowed to give more simple proof of the refined ASM conjecture. In the present paper we use the method of paper [23] to obtain formulas for the refined enumeration of some symmetry classes of the ASMs and prove the Kutin–Yuen conjecture.

### 2 The case of general ASMs

In this section we describe the method to investigate refined enumerations proposed in paper [23]. We obtain at first a new determinant representation for the partition function \( Z(n; x, y) \) valid for \( a = \exp(i \pi/3) \). Actually this representation is not connected directly with enumeration problems, but reveals a useful symmetry of the partition function.

Let us start with the representation of the partition function \( Z(n; x, y) \) via the Izergin–Korepin determinant. Recall that the proof of the validity of that representation is based on the following three lemmas.

**Lemma 1** The partition function \( Z(n; x, y) \) is symmetric in the coordinates of \( x \) and in the coordinates of \( y \).
Lemma 2  The partition function $Z(n; x, y)$ satisfies the recurrent relation
\[
\frac{Z(n; x_1, \ldots, x_n, y_1, \ldots, y_n, a x_n)}{Z(n-1; x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1})} = \sigma(a^2) \prod_{k=1}^{n-1} \sigma(a \bar{x}_k y_k) \prod_{k=1}^{n-1} \sigma(a \bar{x}_k y_n).
\]

Lemma 3  For each $i = 1, \ldots, n$ the product $x_i^{n-1} Z(n; x, y)$ is a polynomial in $x_i$ of degree $n - 1$. For each $i = 1, \ldots, n$ the product $y_i^{n-1} Z(n; x, y)$ is a polynomial in $y_i$ of degree $n - 1$.

It is clear that these three lemmas, together with the initial condition $Z(1, x, y) = \sigma(a^2)$, inductively define $Z(n; x, y)$. The representation of $Z(n; x, y)$ via the Izergin–Korepin determinant is the content of the following theorem.

Theorem 4  The partition function $Z(n; x, y)$ can be represented as
\[
Z(n; x, y) = \frac{\sigma(a^2)^n \prod_{i,j} \alpha(x_i, y_j)}{\prod_{i<j} \sigma(x_i x_j) \sigma(y_i y_j)} \det M(n; x, y),
\]
where $M(n; x, y)$ is an $n \times n$ matrix defined by the equality
\[
M(n; x, y)_{ij} = \frac{1}{\alpha(x_i y_j)}.
\]
and the function $\alpha$ is defined by
\[
\alpha(x) = \sigma(ax) \sigma(a \bar{x}).
\]

One can prove this theorem directly checking that the right side of equality (3) satisfies Lemmas 1, 2, and 3.

Consider the case of $a = \exp(i \pi/3)$. Note that in this case one has the equalities
\[
1 - a + a^2 = 0, \quad 1 - \frac{1}{a} + \frac{1}{a^2} = 0,
\]
which can be used to prove that
\[
\sigma(a^2) = \sigma(a)
\]
and that
\[
\sigma(x) \sigma(ax) \sigma(a^2 x) = -\sigma(x^3).
\]
The last equality implies
\[
\alpha(x) = -\frac{\sigma(x^3)}{\sigma(x)}.
\]
Using this formula and denoting denote $u_{2i-1} = x_i, u_{2i} = y_i$, we rewrite equality (3) as
\[
Z(n; u) = \frac{(-1)^n \sigma(a)^n \prod_{i,j} \sigma(u_{2i-1}^3 u_{2j}^3)}{\prod_{i<j} \sigma(u_{2i-1} u_{2j-1})} \prod_{i,j} \sigma(u_{2i-1} u_{2j}) \det M(n; u).
\]
Introduce now the function

\[ F(n; \mathbf{u}) = \prod_{\mu < \nu} \sigma(u_\mu \bar{u}_\nu) Z(n; \mathbf{u}). \tag{6} \]

Here and below we assume that Greek indices run from 1 to 2n. Using the determinant representation for \( Z(n; \mathbf{u}) \), we can also write

\[ F(n; \mathbf{u}) = (-1)^n \sigma(a)^n \prod_{i,j} \sigma(u_3^i \bar{u}_3^j) \det M(n; \mathbf{u}). \tag{7} \]

The following simple lemma is very important for our consideration.

**Lemma 5** For every \( \mu = 1, \ldots, 2n \) one has

\[ F(n; (u_1, \ldots, u_\mu, \ldots, u_{2n})) + F(n; (u_1, \ldots, a^2u_\mu, \ldots, u_{2n})) + F(n; (u_1, \ldots, a^4u_\mu, \ldots, u_{2n})) = 0. \]

**Proof.** For \( a = \exp(i \pi/3) \) one has

\[ M(n, \mathbf{x}, \mathbf{y})_{ij} = -\frac{\sigma(x_i \bar{y}_j)}{\sigma(x_3^i \bar{y}_3^j)}. \]

Using equalities (4), we obtain

\[ \sigma(x) + \sigma(a^2x) + \sigma(a^4x) = 0 \tag{8} \]

that gives for the first column

\[ M(n; (x_1, \ldots, x_n), \mathbf{y})_{1j} + M(n; (a^2x_1, \ldots, x_n), \mathbf{y})_{1j} + M(n; (a^4x_1, \ldots, x_n), \mathbf{y})_{1j} = 0. \]

Since \( \det M(n; \mathbf{x}, \mathbf{y}) \) for every \( j \) linearly depends on \( M(n; \mathbf{x}, \mathbf{y})_{1j} \), the above equality implies that

\[ \det M(n; (x_1, \ldots, x_n), \mathbf{y}) + \det M(n; (a^2x_1, \ldots, x_n), \mathbf{y}) + \det M(n; (a^4x_1, \ldots, x_n), \mathbf{y}) = 0. \]

Taking into account equality (7), we conclude that the statement of the lemma is valid for \( \mu = 1 \). For all other values of \( \mu \) one can apply the same reasonings. \( \square \)

Lemma 5 and relation (6) imply the following lemma.

**Lemma 6** For every \( \mu = 1, \ldots, 2n \) the function \( u_\mu^{3n-2} F(n, \mathbf{u}) \) is a polynomial of degree \( 3n - 2 \) in \( u_\mu^2 \).

We also have the following evident lemma.

**Lemma 7** The function \( F(n, \mathbf{u}) \) turns to zero if \( u_\mu^2 = u_\nu^2 \) for some \( \mu \neq \nu \).

Let us show now that Lemmas 5, 6, and 7 determine the function \( F(n, \mathbf{u}) \) uniquely up to a constant factor.
Lemma 8 The function $F(n, \mathbf{u})$ satisfying Lemmas 5, 6 and 7 is proportional to the determinant of the matrix

$$P(n; \mathbf{u}) = \begin{pmatrix}
  u_1^{3n-2} & u_2^{3n-2} & u_3^{3n-2} & \cdots & u_2^{3n-2} \\
  u_1^{3n-4} & u_2^{3n-4} & u_3^{3n-4} & \cdots & u_2^{3n-4} \\
  u_1^{3n-8} & u_2^{3n-8} & u_3^{3n-8} & \cdots & u_2^{3n-8} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  u_1^{-3n+2} & u_2^{-3n+2} & u_3^{-3n+2} & \cdots & u_2^{-3n+2}
\end{pmatrix}.$$

Proof. Taking into account Lemma 6 for some fixed value of $\mu$ we write

$$F(n; \mathbf{u}) = \sum_{k=1}^{3n-1} a_k^{(\mu)} (u_1, \ldots, \hat{u}_\mu, \ldots, u_{2n}) u_\mu^{3n-2k},$$

where hat means that the corresponding argument is omitted. From Lemma 5 it follows that $a_k^{(\mu)} = 0$ if $k$ is divisible by 3. Therefore, one has

$$F(n; \mathbf{u}) = \sum_{\substack{k=1 \atop k \neq 0 \mod 3}}^{3n-1} a_k^{(\mu)} (u_1, \ldots, \hat{u}_\mu, \ldots, u_{2n}) u_\mu^{3n-2k}.$$

Hence, we have only $2n$ unknown functions $a_k^{(\mu)}$. Lemma 7 implies that for any $\mu \neq \nu$ one has

$$\sum_{\substack{k=1 \atop k \neq 0 \mod 3}}^{3n-1} a_k^{(\mu)} (u_1, \ldots, \hat{u}_\mu, \ldots, u_{2n}) u_\mu^{3n-2k} = 0.$$

Thus, we have a system of $2n - 1$ linear equations for $2n$ functions $a_k^{(\mu)}$. In general position the rank of this system is equal to $2n - 1$. Hence, it determines the functions $a_k^{(\mu)}$ uniquely up to a common factor which cannot depend on $u_\mu$. Considering all values of $\mu$, we conclude that Lemmas 5, 6 and 7 determine the function $F(n, \mathbf{u})$ uniquely up to a constant factor. It is easy to see that the determinant of the matrix $P(n; \mathbf{u})$ satisfies Lemmas 5, 6 and 7. Hence, the function $F(n; \mathbf{u})$ must be proportional to this determinant.

Lemma 9 The determinant of the matrix $P(n; \mathbf{u})$ satisfies the recurrent relation

$$\frac{\det P(n; (u_1, \ldots, u_{2n-2}, u_{2n-1}, a_{2n-1}))}{\det P(n-1; (u_1, \ldots, u_{2n-2}))} = (-1)^n \sigma (a) \prod_{\mu=1}^{2n-2} \sigma (u_\mu^{3} \bar{u}_{2n-1}^{3}).$$

Proof. Recalling that we consider the case of $a = \exp(i \pi/3)$, one can easily see that the determinant of the matrix $P(n; \mathbf{u})$ for $u_{2n} = a u_{2n-1}$ is

$$(-1)^{n+1} \begin{vmatrix}
  u_1^{3n-2} & u_2^{3n-2} & \cdots & u_2^{3n-2} \\
  u_1^{3n-4} & u_2^{3n-4} & \cdots & u_2^{3n-4} \\
  u_1^{3n-8} & u_2^{3n-8} & \cdots & u_2^{3n-8} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_1^{-3n+8} & u_2^{-3n+8} & \cdots & u_2^{-3n+8} \\
  u_1^{-3n+4} & u_2^{-3n+4} & \cdots & u_2^{-3n+4} \\
  u_1^{-3n+2} & u_2^{-3n+2} & \cdots & u_2^{-3n+2}
\end{vmatrix}.$$
Let us subtract from the first row of the above determinant its third row multiplied by \( u_{2n-1}^6 \), subtract from the second row the forth row again multiplied by \( u_{2n-1}^6 \), and so on whenever is possible. As the result we have the following expression for \( \det P(n; \mathbf{u}) \):

\[
(-1)^{n+1} \begin{vmatrix}
(u_1^6 - u_{2n-1}^6) u_1^{3n-8} & (u_2^6 - u_{2n-1}^6) u_2^{3n-8} & \cdots & 0 \\
(u_1^6 - u_{2n-1}^6) u_1^{3n-10} & (u_2^6 - u_{2n-1}^6) u_2^{3n-10} & \cdots & 0 \\
(u_1^6 - u_{2n-1}^6) u_1^{3n-14} & (u_2^6 - u_{2n-1}^6) u_2^{3n-14} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(u_1^6 - u_{2n-1}^6) u_1^{3n+2} & (u_2^6 - u_{2n-1}^6) u_2^{3n+2} & \cdots & 0 \\
\end{vmatrix}.
\]

Now the statement of the Lemma is evident. \( \square \)

**Theorem 10** For \( a = \exp(i \pi / 3) \) the partition function \( Z(n; \mathbf{u}) \) has the following determinant representation:

\[
Z(n; \mathbf{u}) = (-1)^{\frac{n(n+1)}{2}} \prod_{\mu < \nu} \frac{\sigma(a)^n}{\sigma(u_\mu \bar{u}_\nu)} \det P(n; \mathbf{u}). \tag{9}
\]

**Proof.** One can show that in our case the recurrent relation of Lemma 2 takes the form

\[
\frac{Z(n; (u_1, \ldots, u_{2n-2}, u_{2n-1}, a u_{2n-1}))}{Z(n-1; (u_1, \ldots, u_{2n-2}))} = \sigma(a) \prod_{\mu=1}^{2n-2} \sigma(a u_\mu \bar{u}_{2n-1}).
\]

Taking this fact and Lemma 2 into account, we easily see that the right hand side of the relation (9) satisfies Lemmas 1 and 3 Therefore, since the equality (9) is valid for \( n = 1 \), it is valid for any \( n \).

**Corollary 11** For \( a = \exp(i \pi / 3) \) the partition function \( Z(n, \mathbf{x}, \mathbf{y}) \) is symmetric in the union of the coordinates of \( \mathbf{x} \) and \( \mathbf{y} \).

Let us return to enumeration problems. As was noted in section 1, to solve them we should consider the partition function \( Z(n; \mathbf{x}, \mathbf{y}) \) for \( \mathbf{x} = 1, \mathbf{y} = 1 \), or for \( \mathbf{x} = 1, \mathbf{y} = (u, 1, \ldots, 1) \). Unfortunately, both determinants, \( \det M(n, \mathbf{u}) \) and \( \det P(n, \mathbf{u}) \), become singular under such specialisation of the spectral parameters. Kuperberg 10 proposed to find values of the Izerging–Korepin determinant along a curve that includes a desired values of the spectral parameters. The same method was used by Zeibelberg 26 to prove the refined ASM conjecture. The method used in paper 23 is different. To follow this method let us define for a fixed value of \( \mu \) the function

\[
F^{(\mu)}(n, \mathbf{u}) = \prod_{\nu \neq \mu} \sigma(u_\mu \bar{u}_\nu) Z(n, \mathbf{u}),
\]

and specialise then the values of the spectral parameters in the following way

\[
f(n, \mathbf{u}) = F^{(\mu)}(n, (u_{\mu-1} \ldots 1, u_{2n-\mu} 1)).
\]
Lemma 13 A function \( f(n; u) \) has the following properties.

(a) The function \( u^{3n-2} f(n; u) \) is a polynomial of degree \( 3n - 2 \) in \( u^2 \).

(b) The function \( f(n; u) \) satisfies the relation
\[
f(n; \bar{u}) = -f(n; u).
\]

(c) The function \( f(n; u) \) obeys the equality
\[
f(n; u) + f(n; a^2 u) + f(n; a^4 u) = 0.
\]

(d) The Laurent polynomial \( f(n; u) \) is divisible by \( \sigma(u)^{2n-1} \).

Probably, only the property (b) has to be discussed. Actually it follows from the equality
\[
Z(n; (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{2n})) = Z(n; (u_1, u_2, \ldots, u_n))
\]
which is a consequence of the left-right and top-bottom symmetries of the square ice with the domain wall boundary.

The next important lemma was proved by one of the authors in paper [22].

Lemma 12 A function \( f(n; u) \) satisfying Lemma [12] is proportional to the function
\[
\varphi(n, u) = \frac{(-1)^{n-1}}{\sigma(a)^{2n-2}} \sum_{k=0}^{n-1} \binom{n - \frac{4}{3}}{n - 1 - k} \binom{n - \frac{2}{3}}{k} \sigma(u^{3n-2-6k}).
\]

The function \( \varphi(n; u) \) is normalized by the condition
\[
\varphi(n; a) = 1.
\] (10)

The normalization condition (10) can be verified using the identity
\[
\sum_{k=0}^{n-1} \binom{n - \frac{4}{3}}{n - 1 - k} \binom{n - \frac{2}{3}}{k} = \binom{2n-2}{n-1}.
\]

Equality (2) for \( a = \exp(i \pi/3) \) and \( u = a \) gives
\[
Z(n, (a, 1, \ldots, 1) = A(n-1) \sigma(a)^{n^2},
\]
where we have taken into account the evident equality \( A(n, 1) = A(n-1) \). Therefore, we have
\[
f(n, u) = A(n-1) \sigma(a)^{n^2+2n-1} \varphi(n; u)
\]
that implies
\[
Z(n, (u, 1, \ldots, 1) = A(n-1) \sigma(a)^{n^2+2n-1} \frac{\varphi(n; u)}{\sigma(u)^{2n-1}}.
\]

It is instructive to rewrite equality (2) for \( a = \exp(i \pi/3) \) in terms of the function \( \varphi(n; u) \):
\[
\frac{1}{A(n-1)} \sum_{r=1}^{n} A(n, r) t^{r-1} = \frac{\sigma(a)^{3n-2} \varphi(n; u)}{\sigma(a u)^{n-1} \sigma(u)^{2n-1}}.
\] (11)

In paper [23] this relation was used to prove the refined ASM conjecture and to obtain some results on doubly refined ASM enumeration. In the next sections we will show that the method described above effectively works in other cases.
3 ASMs with U-turn boundary

The Kutin-Yuen conjecture concerns vertically symmetric and off-diagonally symmetric alternating-sign matrices. One can formulate the ice square model related to vertically symmetric ASMs, but as far as we know there is no a determinant formula for this model with arbitrary spectral parameters. However, one can show that the set of vertically symmetric matrices can be considered as a subset of the so-called U-turn alternating-sign matrices (UASMs). Let us define consider these matrices and the partition function for the corresponding square ice model.

Recall [11] that a $2n \times n$ ASM with U-turn boundary looks vertically as a usual ASM, horizontally the 1’s and $-1$’s alternate if we walk along an odd row from left to right and then along the next even row from right to left, see example in Figure 7. The states of the corresponding square ice model are constructed in accordance with pattern given in Figure 8. Here at the additional bivalent U-turn vertices one of the edges points in and the other one points out. The corresponding weights are given in Figure 9. Note that an additional parameter $b$ is introduced. It can be easily shown that we have a bijective correspondence between the states and UASMs.

The square ice model with U-turn boundary was considered first by Tsuchiya [24]. He obtained the determinant formula for the partition function. We will use its modification which was suggested by Kuperberg [11] and looks as

$$Z_U(n; x, y) = \frac{\sigma(a^2)^n \prod_i [\sigma(b \bar{y}_i) \sigma(a^2 x_i^2)] \prod_{i,j} [\alpha(x_i, \bar{y}_j) \alpha(x_i, y_j)]}{\prod_{i<j} [\sigma(\bar{x}_i x_j) \sigma(y_i \bar{y}_j)] \prod_{i<j} [\sigma(\bar{x}_i \bar{x}_j) \sigma(y_i y_j)] \det M_U(n; x, y)},$$

Figure 7: Example of UASM

Figure 8: Square ice with U-turn boundary

Figure 9: Weights of U-turn vertices
where \( n \times n \) matrix \( M_U \) is defined as

\[
M_U(n, x, y)_{i,j} = \frac{1}{\alpha(x_i, y_j)} - \frac{1}{\alpha(x_i, y_j)}.
\]

Denote by \( A_U(2n, r; x, y) \) the total weight of the UASMs whose unique 1 in the first column belongs to the \( r \)th row, each \(-1\) has the multiplicative weight \( x \) and each upward oriented U-turn has the multiplicative weight \( y \). To relate these numbers to the partition function \( Z_U(n; x, y) \) consider again the case where \( x = 1 \) and \( y = (u, 1, \ldots, 1) \). Let for a state of the square ice with U-turn boundary the corresponding UASM has 1 in the intersection of the first column and the \( r \)th row. Here again in the first column we have one vertex of the first type, \( r - 1 \) vertices of the third type and \( 2n - r \) vertices of the sixth type, where type is determined again by Figure 6. Hence, the contribution of the first column to the weight of the state is \( \sigma(a^2) \sigma(a \bar{u})^{r-1} \sigma(a u)^{n-r} \). Let \( k \) be the number of \(-1\)’s in the ASM under consideration. Then the number of 1’s in the ASM with the first column removed is equal to \( n - 1 + k \), and the number of 0’s is equal to \( 2n^2 - 3n + 1 - 2k \). Denote by \( l \) the number of upward U-turns. After all one can see that the weight of the state under consideration is

\[
\sigma(a)^{2n^2-3n+1} \sigma(a^2)^n \sigma(a u)^{2n-1} \sigma(a b)^n \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^{2k} \left[ \frac{\sigma(a \bar{u})}{\sigma(a u)} \right]^{r-1} \left[ \frac{\sigma(\bar{a} b)}{\sigma(a b)} \right]^l,
\]

and we have the equality

\[
\sum_{r=1}^{2n} A_U(2n, r; x, y) t^{r-1} = \frac{Z_U(n; 1, (u, 1, \ldots, 1))}{\sigma(a)^{2n^2-3n+1} \sigma(a^2)^n \sigma(a u)^{2n-1} \sigma(a b)^n}, \tag{12}
\]

where

\[
x = \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^2, \quad y = \frac{\sigma(\bar{a} b)}{\sigma(a b)}, \quad t = \frac{\sigma(a \bar{u})}{\sigma(a u)}.
\]

It is convenient to introduce the modified partition function

\[
Z'_U(n; x, y) = \frac{Z_U(n; x, y)}{\prod_i [\sigma(b \bar{y}_i) \sigma(a^2 x_i^2)]}.
\]

Assume that \( a = \exp(i \pi/3) \). In this case, using equality (5), we obtain

\[
Z'_U(n; u) = (-1)^n \frac{\sigma(a)^n \prod \{ \sigma(u_{2i-1}^3 \bar{u}_{2j}^3) \sigma(u_{2i-1}^3 u_{2j}^3) \}}{\prod_{\mu < \nu} \sigma(u_\mu \bar{u}_\nu) \prod_{\mu < \nu} \sigma(u_\mu u_\nu)} \det M_U(n; u),
\]

where we again use the notation \( u_{2i-1} = x_i, u_{2i} = y_i \). Consider the function

\[
F_U(n; u) = \prod_{\mu < \nu} \sigma(u_\mu \bar{u}_\nu) \prod_{\mu < \nu} \sigma(u_\mu u_\nu) Z'_U(n; u)
\]

which can be also defined as

\[
F_U(n; u) = (-1)^n \sigma(a)^n \prod_{i,j} \{ \sigma(u_{2i-1}^3 \bar{u}_{2j}^3) \sigma(u_{2i-1}^3 u_{2j}^3) \} \det M_U(n; u).
\]

We have the following analogue of Lemmas 5, 6 and 7.
Lemma 14 \textit{The function $F_U(n; u)$ has the following properties.}

(a) \textit{For every $\mu = 1, \ldots, 2n$ one has}

$$F_U(n; (u_1, \ldots, u_\mu, \ldots, u_{2n})) + F_U(n; (u_1, \ldots, a^2u_\mu, \ldots, u_{2n})) + F_U(n; (u_1, \ldots, a^4u_\mu, \ldots, u_{2n})) = 0.$$  

(b) \textit{For every $\mu = 1, \ldots, 2n$ the function $u_\mu^{6n-2}F_U(n, u)$ is a polynomial of degree $6n-2$ in $u_\mu^2$.}

(c) \textit{The function $F_U(n, u)$ turns to zero if for some $\mu \neq \nu$ either $u_\mu^2 = u_\nu^2$ or $u_\mu^2 = \bar{u}_\nu^2$, and if $u_\mu^4 = 1$ for some $\mu$.}

Lemma 15 \textit{The function $F_U(n, u)$ satisfying Lemma 14 is proportional to the determinant of the matrix}

$$P_U(n; u) = \begin{pmatrix}
\sigma(u_1^{6n-2}) & \sigma(u_2^{6n-2}) & \sigma(u_3^{6n-2}) & \ldots & \sigma(u_{2n}^{6n-2}) \\
\sigma(u_1^{6n-4}) & \sigma(u_2^{6n-4}) & \sigma(u_3^{6n-4}) & \ldots & \sigma(u_{2n}^{6n-4}) \\
\sigma(u_1^2) & \sigma(u_2^2) & \sigma(u_3^2) & \ldots & \sigma(u_{2n}^2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma(u_1^2) & \sigma(u_2^2) & \sigma(u_3^2) & \ldots & \sigma(u_{2n}^2)
\end{pmatrix}.$$  

Proof. Taking into account the statement (b) of Lemma 14 for some fixed value of $\mu$ we write

$$F_U(n; u) = \sum_{k=1}^{6n-1} a_k^{(\mu)}(u_1, \ldots, \hat{u}_\mu, \ldots, u_{2n}) u_\mu^{6n-2k}.$$  

From the statement (a) of Lemma 14 it follows that $a_k^{(\mu)} = 0$ if $k$ is divisible by $3$. Therefore, one has

$$F_U(n; u) = \sum_{\substack{k=1 \atop k \neq 0 \mod 3}}^{6n-1} a_k^{(\mu)}(u_1, \ldots, \hat{u}_\mu, \ldots, u_{2n}) u_\mu^{6n-2k}.$$  

Hence, we have only $4n$ unknown functions $a_k^{(\mu)}$. Using the statement (c) of Lemma 14 for all $\nu \neq \mu$ one has

$$\sum_{\substack{k=1 \atop k \neq 0 \mod 3}}^{6n-1} a_k^{(\mu)}(u_1, \ldots, \hat{u}_\mu, \ldots, u_{2n}) u_\nu^{6n-2k} = 0,$$

and, furthermore,

$$\sum_{\substack{k=1 \atop k \neq 0 \mod 3}}^{6n-1} a_k^{(\mu)}(u_1, \ldots, \hat{u}_\mu, \ldots, u_{2n}) \bar{u}_\nu^{6n-2k} = 0,$$

and,

$$\sum_{\substack{k=1 \atop k \neq 0 \mod 3}}^{6n-1} a_k^{(\mu)}(u_1, \ldots, \hat{u}_\mu, \ldots, u_{2n}) = 0.$$  

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Thus, we have a system of $4n - 1$ linear equations for $4n$ functions $a_k^{(\mu)}$. In general position the rank of this system is equal to $4n - 1$. Hence, it determines the functions $a_k^{(\mu)}$ uniquely up to a common factor which cannot depend on $u_\mu$. Considering all values of $\mu$, we conclude that Lemma 14 determines the function $F_U(n, u)$ uniquely up to a constant factor. It is not difficult to get convinced that the determinant of the matrix $P_U(n; u)$ satisfies Lemma 14. Hence, the function $F_U(n; u)$ must be proportional to this determinant.

**Lemma 16** The determinant of the matrix $P_U$ satisfies the recurrent relation

$$\frac{\det P_U(n; (u_1, \ldots, u_{2n-2}, u_{2n-1}, a u_{2n-1}))}{\det P_U(n-1; (u_1, \ldots, u_{2n-2}))} = \sigma(a) \sigma(u_{2n-1}^6) \prod_{\mu=1}^{2n-2} \left[ \sigma(u_\mu \bar{u}_{2n-1}) \sigma(u_\mu u_{2n-1}) \right].$$

**Proof.** Consider the matrix

$$P'_U(n; u) = \begin{pmatrix}
\sigma(u_1^{6n-2}) & \ldots & \sigma(u_{2n-1}^{6n-2}) & \sigma(u_2^{6n-2}) \\
\sigma(u_1^{6n-8}) & \ldots & \sigma(u_{2n-1}^{6n-8}) & \sigma(u_2^{6n-8}) \\
\vdots & \ddots & \vdots & \vdots \\
\sigma(u_1^{6n+10}) & \ldots & \sigma(u_{2n-1}^{6n+10}) & \sigma(u_2^{6n+10}) \\
\sigma(u_1^{6n+4}) & \ldots & \sigma(u_{2n-1}^{6n+4}) & \sigma(u_2^{6n+4})
\end{pmatrix}.$$  

One can get convinced that

$$\det P_U(n; u) = (-1)^n \det P'_U(n; u).$$

For $a = \exp(i \pi/3)$ the determinant of the matrix $P'_U(n; u)$ is

$$\begin{vmatrix}
\sigma(u_1^{6n-2}) & \ldots & \sigma(u_{2n-1}^{6n-2}) & \sigma(\bar{a}^2 u_{2n-1}^{6n-2}) \\
\sigma(u_1^{6n-8}) & \ldots & \sigma(u_{2n-1}^{6n-8}) & \sigma(\bar{a}^2 u_{2n-1}^{6n-8}) \\
\vdots & \ddots & \vdots & \vdots \\
\sigma(u_1^{6n+16}) & \ldots & \sigma(u_{2n-1}^{6n+16}) & \sigma(\bar{a}^2 u_{2n-1}^{6n+16}) \\
\sigma(u_1^{6n+10}) & \ldots & \sigma(u_{2n-1}^{6n+10}) & \sigma(\bar{a}^2 u_{2n-1}^{6n+10}) \\
\sigma(u_1^{6n+4}) & \ldots & \sigma(u_{2n-1}^{6n+4}) & \sigma(\bar{a}^2 u_{2n-1}^{6n+4})
\end{vmatrix}.$$ 

Let us substract the second row multiplied by $u_{2n-1}^{6} + u_{2n-1}^{-6}$ from the first row, and then add the third row. Repeat this procedure starting from the second row and so on whenever is possible. As the result we have the determinant

$$\begin{vmatrix}
\sigma(u_1^2 \bar{u}_{2n-1}^3) \sigma(u_1^3 u_{2n-1}^3) \sigma(u_1^{6n-8}) & \ldots & 0 & 0 \\
\sigma(u_1^3 \bar{u}_{2n-1}^2) \sigma(u_1^3 u_{2n-1}^3) \sigma(u_1^{6n-14}) & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\sigma(u_1^3 \bar{u}_{2n-1}^2) \sigma(u_1^3 u_{2n-1}^3) \sigma(u_1^{6n+10}) & \ldots & 0 & 0 \\
\sigma(u_1^{6n+10}) & \ldots & \sigma(u_{2n-1}^{6n+10}) & \sigma(\bar{a}^2 u_{2n-1}^{6n+10}) \\
\sigma(u_1^{6n+4}) & \ldots & \sigma(u_{2n-1}^{6n+4}) & \sigma(\bar{a}^2 u_{2n-1}^{6n+4})
\end{vmatrix}.$$ 

The last expression gives the equality

$$\frac{\det P'_U(n; (u_1, \ldots, u_{2n-2}, u_{2n-1}, a u_{2n-1}))}{\det P'_U(n-1; (u_1, \ldots, u_{2n-2}))} = -\sigma(a) \sigma(u_{2n-1}^{6}) \prod_{\mu=1}^{2n-2} \left[ \sigma(u_\mu \bar{u}_{2n-1}) \sigma(u_\mu u_{2n-1}) \right]$$

which implies the statement of the Lemma. 

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**Theorem 17** For \( a = \exp(i \pi / 3) \) the function \( Z'_U(n; u) \) has the following determinant representation:

\[
Z'_U(n; u) = \prod_{\mu < \nu} \frac{\sigma(u_{\mu} \bar{u}_{\nu}) \prod_{\mu \leq \nu} \sigma(u_{\mu} u_{\nu})}{\sigma(a)^n} \det P_U(n; u).
\]

Similarly as it was for general ASMs, we can prove the above theorem using Lemma 16 and the corresponding recurrent relation for \( Z_U(n; x, y) \) found by Kuperberg [11].

**Corollary 18** For \( a = \exp(i \pi / 3) \) the Laurent polynomial

\[
\frac{Z_U(n; x, y)}{\prod_{i=1}^n \sigma(y_i \bar{b}) \sigma(x_i^2 a^2)}
\]

is symmetric in the union of the coordinates of \( x \) and \( y \).

To treat the refined enumeration problem introduce the function

\[
f_U(n; u) = \sigma(u)^{4n-2} \sigma(u^2) Z'_U(n; (u, 1, \ldots, 1)).
\]

**Lemma 19** The function \( f_U(n; u) \) has the following properties:

(a) The function \( u^{6n-2} f_U(n; u) \) is a polynomial of degree \( 6n - 2 \) in \( u^2 \).

(b) The function \( f_U(n; u) \) satisfies the relation

\[
f(n; \bar{u}) = -f(n; u).
\]

(c) The function \( f_U(n; u) \) obeys the equality

\[
f_U(n; u) + f_U(n; a^2 u) + f_U(n; a^4 u) = 0.
\]

(d) The Laurent polynomial \( f_U(n; u) \) is divisible by \( \sigma(u)^{4n-2} \) and by \( \sigma(u^2) \).

The property (b) follows now, for example, from the determinant representation for \( Z'_U(n; u) \) supplied by Theorem 17. All other properties are evident.

Since \( \sigma(u^2) = (u + \bar{u}) \sigma(u) \), the function \( f_U(n; u) \) is actually divisible by \( \sigma(u)^{4n-1} \). Comparing now Lemmas 19 and 12 and taking into account Lemma 13 we conclude that the function \( f_U(n; u) \) is proportional to the function \( \varphi(2n; u) \). Let us find the corresponding coefficient. To this end consider equality (12) for \( a = \exp(i \pi / 3) \). Taking into account an evident property

\[
A_U(2n, 1; 1, y) = y A_U(2n - 2; 1, y),
\]

we obtain the relation

\[
A_U(2n - 2; 1, y) = \frac{Z_U(n; 1, (a, 1, \ldots, 1))}{\sigma(a)^{2n^2} \sigma(a b)^{n-1} \sigma(a b)}
\]

which after some simple calculations gives

\[
f_U(n; u) = A_U(2n - 2; 1, y) \frac{\sigma(a)^{2n^2 + 3n-1} \sigma(a b)^{n-1}}{\sigma(b)^{n-1}} \varphi(2n; u).
\]
Using this relation, it is not difficult to demonstrate that equality (12) for $a = \exp(i \pi / 3)$ can be written as

$$\frac{1}{A_U(2n-2; 1, y)} \sum_{r=1}^{2n} A_U(2n, r; 1, y) t^{r-1} = \frac{\sigma(b \bar{u}) \sigma(a)^{6n-2} \varphi(2n; u)}{\sigma(ba) \sigma(a u)^{2n-1} \sigma(u)^{4n-2} \sigma(u^2)}.$$ \hspace{1cm} (14)

Comparing (14) with (11), we come to

**Theorem 20** The following equality

$$\frac{1}{A_U(2n-2; 1, y)} \sum_{r=1}^{2n} A_U(2n, r; 1, y) t^{r-1} = \frac{1}{A(2n-1)} \frac{t + y}{t + 1} \sum_{r=1}^{2n} A(2n, r) t^{r-1},$$ \hspace{1cm} (15)

is valid.

The statement of Theorem 20 for $t = 1$ takes the form

$$\frac{A_U(2n, 1, y)}{A_U(2n-2, 1, y)} = \frac{1}{2} \left(1 + \frac{y}{1 + y}\right) \frac{A(2n)}{A(2n-1)}.$$ \hspace{1cm} (16)

For $n = 1$ square ice with U-turn boundary has two states, one state with upward U-turn and

one with downward U-turn, see Figure 10. Therefore, one has

$$A_U(2; 1, y) = (1 + y),$$

and equality (16) gives

$$A_U(2n; 1, y) = \frac{1}{2^n} (1 + y)^n \prod_{k=1}^{n} \frac{A(2k)}{A(2k-1)}.$$ \hspace{1cm} (17)

Using the famous relation

$$\frac{A(n)}{A(n-1)} = \frac{(3n-2)! (n-1)!}{(2n-1)! (2n-2)!},$$

one can rewrite the above formula for $A_U(2n; 1, y)$ as

$$A_U(2n; 1, y) = \frac{1}{2^n} (1 + y)^n \prod_{k=1}^{n} \frac{(6k-2)! (2k-1)!}{(4k-1)! (4k-2)!}.$$ \hspace{1cm} (17)

Let us discuss now the connection of ASMs with U-turn boundary and vertically symmetric ASMs. It is not difficult to see that the central column of a $(2n + 1) \times (2n + 1)$ VSASM consists
of alternating 1’s and −1’s, and it is the same for any such matrix. Hence, a VSAM is uniquely
classified by its left-most $(2n + 1) \times n$ submatrix. Therefore, one can associate with a
VSASM a state of the square ice $(2n + 1) \times n$ region with the boundary condition defined as in
at the left side, out at the top and bottom and alternating boundary at the right side, see the first
picture in Figure 11. Note also that the top and bottom rows of a VSASM are fixed. Therefore,
the orientation of the edges belonging to the corresponding vertices is fixed as well. This fact
is reflected in Figure 11. Actually one can remove the top and bottom rows from consideration.
However, it is customary to preserve them, and we will obey this agreement.

Any state of square ice with VS boundary can be transformed into a state of square ice with
U-turn boundary via the following two steps. First, we remove the bottom-most row of vertices
as it is shown on the second picture in Figure 11. Second, we connect pairwise the alternating
edges on the right side, see the third picture in Figure 11. Note that the resulting state has only
downward oriented U-turn vertices. It is clear that any state of square ice with U-turn boundary
which has only downward oriented U-turns can be uniquely transformed into a state of square
ice with VS boundary. Thus, the set of VSASMs can be identified with a subset of VSASMs.

Returning to enumeration problems, write the equality

$$A_V(2n+1, r) = A_U(2n, r; 1, 0)$$

which follows directly from the relation of VSASMs and UASMs discussed just above. Having
this equality in mind, we see that relation (17) implies

$$A_V(2n+1) = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)! (2k-1)!}{(4k-1)! (4k-2)!},$$

and we can write

$$A_U(2n; 1, y) = (1 + y)^n A_V(2n + 1).$$

This equality is a partial case of the equality

$$A_U(2n; x, y) = (1 + y)^n A_V(2n + 1; x)$$
proved by Kuperberg [11]. Formula (19) is equivalent to the recurrent relation
\[
\frac{A_V(2n+1)}{A_V(2n-1)} = \frac{(6n-2)}{2(4n-1)}
\]
conjectured by Robbins [21]. It can be shown that relation (19) is equivalent to the formula for
\[
A_V(2n+1)
\]
obtained by Kuperberg [11].

Consider now the statement of Theorem 20 for general \( t \). Rewrite (15) as
\[
A_U(2n-1)(t + 1) \sum_{r=1}^{2n} A_U(2n, r; 1, y) t^{r-1} = A_U(2n - 2; 1, y) (t + y) \sum_{r=1}^{2n} A(2n, r) t^{r-1}
\]
and equate coefficients at different powers of \( t \). As the result one obtains the following equalities
\[
A(2n - 1) [A_U(2n, r - 1; 1, y) + A_U(2n, r; 1, y)] = A_U(2n - 2; 1, y) [A(2n, r - 1) + y A(2n, r)], \quad r = 2, 3, \ldots, 2n. \quad (22)
\]
Using relations (13) and (20), we obtain the following solution to this recurrent relation:
\[
A_U(2n, r; 1, y) = \frac{(1 + y)^{n-1} A_V(2n - 1)}{A(2n - 1)} \left[ y A(2n; r) + \sum_{k=1}^{r-1} (-1)^{r+k-1} (1 - y) A(2n; k) \right].
\]
For \( y = 1 \) we have
\[
A_U(2n; r) = 2^{n-1} \frac{A_V(2n - 1)}{A(2n - 1)} A(2n; r).
\]
Consider, for example, the case of \( n = 2 \). In this case there are 12 UASMs depicted in Figure 12. The numbers \( A_U(2n, r) \) are 2, 4, 4, 2. The corresponding numbers \( A(2n, r) \) are 7, 14, 14, 7. One can easily find that \( A_V(3) = 1 \) and \( A(3) = 7 \). Hence, our relation is valid in this case.

Relation (22) for \( y = 0 \) gives the recurrent relation
\[
A(2n - 1) [A_V(2n + 1, r - 1) + A_V(2n + 1, r)] = A_V(2n - 1) A(2n, r - 1)
\]
which has the solution
\[ A_V(2n + 1, r) = A_V(2n - 1) \frac{1}{A(2n - 1)} \sum_{k=1}^{r-1} (-1)^{r+k-1} A(2n, k). \]

Using the relation \([26, 23]\)
\[ \frac{A(n, r)}{A(n - 1)} = \frac{1}{(2n - 2)!} \frac{(n + r - 2)! (2n - r - 1)!}{(r - 1)! (n - r)!}, \]
we come to the equality
\[ A_V(2n + 1, r) = \frac{A_V(2n - 1)}{(4n - 2)!} \sum_{k=1}^{r-1} (-1)^{r+k-1} \frac{(2n + k - 2)! (4n - k - 1)!}{(k - 1)! (2n - k)!}. \]

Note that in terms of the generating functions one has the equality
\[ \frac{1}{A_V(2n - 1)} \sum_{r=1}^{2n} A_V(2n + 1; r) t^{r-1} = \frac{1}{A(2n - 1)} \frac{t}{t + 1} \sum_{r=1}^{2n} A(2n, r) t^{r-1}, \tag{23} \]
which follows immediately from the statement of Theorem 20 if we put \( y = 0 \) and take into account equality (18).

Let us formulate the main enumeration results of this section as a corollary of Theorem 20:

**Corollary 21** The following equalities
\[ A_V(2n + 1) = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k - 2)! (2k - 1)!}{(4k - 1)! (4k - 2)!}, \]
\[ A_U(2n) = 2^n A_V(2n + 1), \]
\[ A_U(2n; r) = 2^{n-1} \frac{A_V(2n - 1)}{A(2n - 1)} A(2n; r), \]
\[ A_V(2n + 1, r) = \frac{A_V(2n - 1)}{(4n - 2)!} \sum_{k=1}^{r-1} (-1)^{r+k-1} \frac{(2n + k - 2)! (4n - k - 1)!}{(k - 1)! (2n - k)!} \]

are valid.

### 4 Off-diagonally symmetric ASMs

An off-diagonally symmetric ASM (OSASM) is an ASM which coincides with its transpose and has null diagonal. An example of the corresponding square ice model pattern is given in Figure 13. This model was posed by Kuperberg [11] who also found the following Pfaffian representation for the partition function:
\[ Z_O(n; \mathbf{u}) = \frac{\sigma(a^2)^n \prod_{\mu<\nu} \alpha(u_\mu u_\nu)}{\prod_{\mu<\nu} \sigma(\bar{u}_\mu u_\nu)} \text{Pf } M_O(n; \mathbf{u}), \]
where $M_O(n, x)$ is $2n \times 2n$ matrix with the matrix elements given by

$$M_O(n; u)_{\mu\nu} = \frac{\sigma(\bar{u}_\mu u_\nu)}{\alpha(u_\mu u_\nu)}.$$ 

It is not difficult to obtain the relation

$$\sum_{r=2}^{2n} A_O(2n, r; x) t^{r-2} = \frac{Z_O(n; \bar{u}, 1, \ldots, 1)}{\sigma(a)^{2n^2-2n+1} \sigma(a^2)^n \sigma(a u)^{2n-2}}.$$ (24)

where

$$x = \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^2, \quad t = \frac{\sigma(a \bar{u})}{\sigma(a u)}.$$

Assume again that $a = \exp(i\pi/3)$. In this case, using equality (5), we write the partition function as

$$Z_O(n; u) = (-1)^n \frac{\sigma(a^2)^n \prod_{\mu<\nu} \sigma(u_\mu^3 u_\nu^3) \prod_{\mu} \sigma(u_\mu^2)}{\prod_{\mu<\nu} \sigma(\bar{u}_\mu u_\nu) \prod_{\mu\leq \nu} \sigma(u_\mu u_\nu)} \text{Pf} M_O(n; u).$$

Introduce the function

$$F_O(n; u) = \sigma(a)^{-n} \prod_{\mu<\nu} \sigma(\bar{u}_\mu u_\nu) \prod_{\mu\leq \nu} \sigma(u_\mu u_\nu) Z_O(n; u),$$

which can be equivalently defined by

$$F_O(n; u) = (-1)^n \prod_{\mu<\nu} \sigma(u_\mu^3 u_\nu^3) \prod_{\mu} \sigma(u_\mu^2) \text{Pf} M_O(n; u).$$

**Lemma 22** The function $F_O(n; u)$ has the following properties.

(a) For every $\mu = 1, \ldots, 2n$ one has

$$F_O(n; (u_1, \ldots, u_\mu, \ldots, u_{2n}))$$

$$+ F_O(n; (u_1, \ldots, a^2 u_\mu, \ldots, u_{2n})) + F_O(n; (u_1, \ldots, a^4 u_\mu, \ldots, u_{2n})) = 0.$$
(b) For every $\mu = 1, \ldots, 2n$ the function $u_\mu^{6n-2} F_{O}(n, u)$ is a polynomial of degree $6n - 2$ in $u_\mu^2$.

(c) The function $F_{O}(n, u)$ turns to zero if for some $\mu \neq \nu$ either $u_\mu^2 = u_\nu^2$, or $u_\mu^2 = \bar{u}_\nu^2$, and if $u_\mu^4 = 1$ for some $\mu$.

Proof. To prove the statement (a) note that for $a = \exp(i \pi/3)$ one has

$$M_{O}(n; u)_{\mu\nu} = -\frac{\sigma(\bar{u}_\mu u_\nu) \sigma(u_\mu u_\nu)}{\sigma(u_\mu^3 u_\nu^3)}.$$ 

One can get convinced that

$$\sigma(u_\mu^2) M_{O}(n; u)_{\mu\nu} = -\frac{\sigma(u_\mu^2) \sigma(u_\nu^2) - \sigma(u_\mu^4)}{\sigma(u_\mu^3 u_\nu^3)}.$$ 

Taking into account the equality

$$\sigma(x) + \sigma(a^8 x) + \sigma(a^{16} x) = 0,$$

which follows from [8], one obtains

$$\sigma(u_\mu^2) M_{O}(n; (u_1, \ldots, u_{2n})_{1\nu}) + \sigma(a^4 u_\mu^2) M_{O}(n; (a^2 u_1, \ldots, u_{2n})_{1\nu}) + \sigma(a^8 u_\mu^2) M_{O}(n; (a^4 u_1, \ldots, u_{2n})_{1\nu}) = 0.$$ 

Since Pf $M_{O}(n; u)$ for every $\nu$ linearly depends on $M_{O}(n; u)_{1\nu}$, the above equality implies that

$$\sigma(u_\mu^2) \text{Pf } M_{O}(n; (u_1, \ldots, u_{2n})) + \sigma(a^4 u_\mu^2) \text{Pf } M_{O}(n; (a^2 u_1, \ldots, u_{2n})) + \sigma(a^8 u_\mu^2) \text{Pf } M_{O}(n; (a^4 u_1, \ldots, u_{2n})) = 0.$$ 

We see that the statement (a) of the lemma is valid for $\mu = 1$. For all other values of $\mu$ the proof is the same. The statements (b) and (c) are evident. \(\square\)

Comparing Lemmas [22] and [14] and taking into account Lemma [15] we see that the function $F_{O}(n; u)$ is proportional to $\det P_{U}(n; u)$.

**Theorem 23** For $a = \exp(i \pi/3)$ the function $Z_{O}(n; u)$ has the following determinant representation:

$$Z_{O}(n; u) = \frac{\sigma(a)^n}{\prod_{\mu<\nu} \sigma(u_\mu \bar{u}_\nu) \prod_{\mu \leq \nu} \sigma(u_\mu u_\nu)} \det P_{U}(n; u).$$

The proof of Theorem [23] is based on usage of Lemma [16] and the recurrent relation for the partition function $Z_{O}(n; u)$ found by Kuperberg [11].

**Corollary 24** The partition function $Z_{O}(n; u)$ coincides with the modified partition function $Z'_{U}(n; u)$.  

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To treat enumeration problems define the function

\[ f_O(n; u) = \sigma(u)4n^{-2}\sigma(u^2)Z_O(n; (u, 1, \ldots, 1)). \]

The function \( f_O(n; u) \) has the properties of the function \( f_U(n; u) \) described by Lemma \[19\]. Therefore, it is proportional to the function \( \varphi(2n; u) \). As for the case of UASMs we find

\[ f_O(n; u) = A_O(2n - 2)\sigma(a)^{2n^2 - 5n - 2}\varphi(2n; u). \]

From this equality and relation (24) it follows that for \( a = \exp(i\pi/3) \) one has

\[ \frac{1}{A_O(2n - 2)} \sum_{r=2}^{2n} A_O(2n, r) t^{r-2} = \frac{\sigma(a)6n-2\varphi(2n; u)}{\sigma(a u)^{2n-2}\sigma(u)^{4n-2}\sigma(u^2)}. \]

Comparing this equality with (11), we come to

**Theorem 25** The refined enumerations of OSASMs and ASMs are connected by the equality

\[ \frac{1}{A_O(2n - 2)} \sum_{r=1}^{2n} A_O(2n, r) t^{r-1} = \frac{1}{A(2n - 1)} t \sum_{r=1}^{2n} A_{2n, r} t^{r-1}. \]

Comparing the statement of this theorem with equality (23), and having in mind that

\[ A_O(2) = A_V(3) = 1, \]

we obtain

**Corollary 26** The refined enumerations of OSASMs and VSASMs coincide:

\[ A_O(2n, r) = A_V(2n + 1, r). \]

This is the conjecture made by Kutin and Yuen.

Note that the determinant representaions of Theorems \[10\], \[17\] and \[23\] can be obtained from the determinant representaions by Izergin–Korepin and Kuperberg using some of the equalities between determinants and Pfaffians found by Okada \[15\].

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