Extremals for the Singular Moser-Trudinger Inequality via $n$-Harmonic Transplantation

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Abstract

The Moser-Trudinger embedding has been generalized by Adimuzhi and Sandeep to the following weighted version: if $\Omega \subset \mathbb{R}^n$ is smooth and bounded, $\omega_{n-1}$ is the $\mathcal{H}^{n-1}$ measure of the unit sphere, then for $\alpha > 0$ and $\beta \in [0, n)$,

$$\sup_{u \in \mathcal{B}_1} \frac{e^{\alpha|u|^{n/(n-1)}}}{|x|^\beta} \leq C \Leftrightarrow \frac{\alpha}{\alpha_n} + \frac{\beta}{n} \leq 1,$$

where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$ and $\mathcal{B}_1 = \{ u \in W^{1,n}_0(\Omega) \mid \int_\Omega |\nabla u|^n \leq 1 \}$. We prove that the supremum is attained on any domain $\Omega$. The paper also gives rigorous proofs to all of the statements contained in [Lin K.C., Extremal functions for Moser’s inequality, Trans. of. Am. Math. Soc., 384 (1996), 2663–2671], which deals with the case $\beta = 0$.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. The Moser-Trudinger imbedding, which is due to Trudinger [39] and in its sharp form to Moser [32], states that the following supremum is finite

$$\sup_{u \in W^{1,n}_0(\Omega), \|\nabla u\|_L^n \leq 1} \left( e^{n\omega_{n-1}^{1/(n-1)} \|\nabla u\|^n} - 1 \right) < \infty.$$

First it has been shown by Carleson-Chang [8] that the supremum is actually attained, if $\Omega$ is a ball. In [36], Struwe proved for $n = 2$ that the result remains true if $\Omega$ is close to a ball. Using the harmonic transplantation method Flucher [17] generalized this result to arbitrary bounded domains

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in $\mathbb{R}^2$. If $n = 2$ and $\Omega$ is the unit disk some new proofs have been obtained of the Moser-Trudinger inequality, respectively of the Carleson-Chang’s result in Mancini-Martinazzi [28]. Other results dealing with critical points and extremal functions of the Moser-Trudinger inequality, or its sharper version, have also been obtained in Malchiodi-Martinazzi [27] (for $\Omega$ equal the unit disk), Adimurthi-Druet [31] and Yang [40], [41] (for a version on manifolds) and [42]. These results use blow up analysis and are all restricted to 2 dimensions. See also Adimurthi-Tintarev [3], Mancini-Sandeep [29], [30], [31], [37] and Yang [43] and the references in these papers for other recent developments on the subject. However, the only result dealing with the extremal functions in higher dimension and more general bounded domains than balls, is by Lin [25], who gives an outline of a method how Flucher’s proof can be generalized to $n \geq 3$.

Setting $\beta = 0$ in this present paper we implicitly give rigorous proofs to all of Lin’s statements.

The Moser-Trudinger embedding has been generalized by Adimurthi-Sandeep [2] to a singular version, which reads as the following: If $\alpha > 0$ and $\beta \in [0, n)$ is such that

$$\frac{\alpha}{\alpha_n} + \frac{\beta}{n} \leq 1, \quad \text{where } \alpha_n = n \omega_n^{1/(n-1)}, \tag{1}$$

$\omega_n$ is the $H^{n-1}$ measure of the unit sphere, then the following supremum is finite

$$\sup_{v \in W_0^{1,n}(\Omega) \atop \|\nabla v\|_{L^n} \leq 1} \int_\Omega e^{\alpha |v|^{\frac{n}{n-1}}} \frac{1}{|x|^\beta} - 1 < \infty. \tag{2}$$

In the case $n = 2$ it was proven first in Csató-Roy [11], [12] that the supremum is attained. Afterwards, using blow-up analysis, the same result and some refinements have also been obtained by Yang and Zhu [44]. Their proof is based on classification theorems in 2 dimensions by Chen and Li [13], [14]. Recently, also in 2 dimensions, a third proof was given by Lula-Mancini [26], also depending on [13]. We point out that there is no available Chen-Li type classification result in higher dimensions. The aim of this paper is to generalize [11] to higher dimensions and we prove the following theorem.

**Theorem 1** Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $\alpha > 0$ and $\beta \in [0, n)$ be such that (1) is satisfied. Then there exists $u \in W_0^{1,n}(\Omega)$ such that $\|\nabla u\|_{L^n} \leq 1$, $u \geq 0$ and

$$\sup_{v \in W_0^{1,n}(\Omega) \atop \|\nabla v\|_{L^n} \leq 1} \int_\Omega e^{\alpha |v|^{\frac{n}{n-1}}} \frac{1}{|y|^\beta} - 1 = \int_\Omega e^{\alpha u^{\frac{n}{n-1}}} \frac{1}{|y|^\beta} - 1.$$

It is an immediate consequence of this theorem, that $u \in W_0^{1,n}$ gives a weak solution, for some $\mu > 0$, to the following semilinear elliptic equation involving the $n$-Laplacian

$$\begin{cases}
-\Delta_n u = \mu u^{\frac{n}{n-1}} e^{\alpha u^{\frac{n}{n-1}}} \frac{1}{|y|^\beta} & \text{in } \Omega \\
u \geq 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

satisfying moreover $\int_{\Omega} |\nabla u|^n = 1$. 

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The proof of Theorem 1 follows the ideas of Flucher [17], respectively Lin [25] and we have given an overview of the method in Csató-Roy [11], which we will not repeat here. The case $0 \notin \Omega$ is rather elementary and exactly as in the 2-dimensional case, see [11], and the main difficulty is the case $0 \in \Omega$. The proof of Theorem 1, roughly speaking, consists in reducing the problem to a kind of isoperimetric problem with density involving the $n$-Green’s function. The $n$-Green’s function $G_{\Omega,0}$ of a general domain $\Omega$ containing the origin has the form

$$G_{\Omega,0}(y) = -\frac{1}{\omega_{n-1}} \frac{1}{y^{n-1}} \log |y| - H(y),$$

where $H$ is the regular part. The conformal incenter at $0$ is then defined by

$$I_{\Omega}(0) = e^{-\omega_{n-1}/n} H(0).$$

This is usually called the conformal inradius at $0$, see Flucher [18] and we have adopted the name conformal incenter at $0$ in [11] mistakenly. However, in this paper, we will stick to this name for consistency with [11], [12]. The isoperimetric problem to which the question of the existence of extremal function is reduced is the following inequality:

$$\omega_1 I_{\Omega}^{2-\beta} \leq \int_{\partial \Omega} \frac{1}{|y|^\beta |\nabla G_{\Omega,0}(y)|} dH^{n-1}(y) \quad \text{for any } \Omega \text{ with } 0 \in \Omega. \quad (3)$$

There is equality for balls centered at the origin. Let us now point out the two main differences and difficulties compared to the 2-dimensional case:

1. The reduction of the problem to (3) uses the $n$-harmonic transplantation. This uses existence, regularity and other properties of the $n$-Laplace equation, which is a linear equation only if $n = 2$. Solutions to the $n$-Laplace equation have moreover weaker regularity properties if $n > 2$. The difficulties are more of technical type and are mostly relevant in Section 7.

2. If $n = 2$ then (3) was proven by the following three inequalities (where $R_\Omega$ is the radius of $\Omega^*$, i.e. $\pi R_\Omega^2 = |\Omega|$)

$$\omega_1^2 I_{\Omega}^{2-\beta} \leq (a) \omega_1^2 R_\Omega^{2-\beta} = \omega_1^2 \left( \frac{|\Omega|}{\pi} \right)^{2-\beta} \leq (b) \left( \int_{\partial \Omega} \frac{1}{|y|^\beta |\nabla G_{\Omega,0}(y)|} dH^{n-1}(y) \right)^2 \leq (c) \int_{\partial \Omega} \frac{1}{|y|^\beta |\nabla G(y)|} dH^{n-1}(y).$$

The estimate (a), i.e. $I_{\Omega} \leq R_\Omega$ is standard, (b) is a weighted isoperimetric inequality and (c) is just Hölder inequality and the elementary property $\int_{\partial \Omega} |\nabla G| = 1$. This is the method followed by Flucher [17], Csató-Roy [11]. If $\beta = 0$, then the same steps work in any dimension, just by using the classical isoperimetric inequality and this was what was used by Lin [25]. If $\beta \neq 0$, then the corresponding weighted isoperimetric inequality was proven by Csató [9] if $n = 2$. However, a higher dimensional version of the appropriate weighted isoperimetric inequality fails, see Csató [10] Theorem 9 (ii) and Example 11. Example 11 shows precisely that such an isoperimetric inequality even fails for any ball not centered at the origin. Moreover, if $n \geq 3$ both estimates (a) and (c) are too generous to prove (3), as numerical evidence suggests. So a completely new method had to be developed to prove the higher dimensional case, without using any of the estimates (a)-(c).
This new method relies on a careful analysis of the properties of $G_{\Omega, 0}$ and on a different weighted isoperimetric inequality by Alvino, Brock, Chiacchio, Mercaldo and Posteraro \[4\] Theorem 1.1 (ii), which reads as

$$
\int_{\Omega} |y|^{-\beta} dy \leq C \left( \int_{\partial \Omega} |y|^{-\frac{n-1}{n-\beta}} dH^{n-1}(y) \right)^{-\frac{n}{n-1}}
$$

for any smooth set $\Omega \subset \mathbb{R}^n$, where $C$ is such that equality holds for balls centered at the origin. The proof of (3) is contained in Section 3.

2 Notations and Definitions

Throughout this paper $n \geq 2$ is an integer and $\Omega \subset \mathbb{R}^n$ will denote a bounded open set with smooth boundary $\partial \Omega$. Its $n$-dimensional Lebesgue measure is written as $|\Omega|$. The $(n-1)$-dimensional Hausdorff measure is denoted by $H^{n-1}$. Balls with radius $R$ and center at $x$ are written $B_R(x) \subset \mathbb{R}^n$; if $x = 0$, we simply write $B_R$. The space $W^{1,n}(\Omega)$ denotes the usual Sobolev space of functions and $W^{1,n}_0(\Omega)$ those Sobolev functions with vanishing trace on the boundary. Throughout this paper $\alpha, \beta \in \mathbb{R}$ are two constants satisfying $\alpha > 0$, $\beta \in [0, n)$ and

$$
\frac{\alpha}{\alpha_n} + \frac{\beta}{n} \leq 1, \text{ where } \alpha_n = n\omega_{n-1}^{1/(n-1)},
$$

where $\omega_{n-1}$ is the $H^{n-1}$ measure of the unit sphere.

- We define the functionals $F_\Omega, J_\Omega : W^{1,n}_0(\Omega) \to \mathbb{R}$ by

\[
F_\Omega(u) = \int_{\Omega} e^{\alpha|u|^{\frac{n}{n-1}}} \frac{|x|^\beta}{|x|^n} - 1 \, dx,
\]

\[
J_\Omega(u) = \int_{\Omega} \left( e^{\alpha|u|^{\frac{n}{n-1}}} - 1 \right) dx.
\]

- We say that a sequence $\{u_i\} \subset W^{1,n}_0(\Omega)$ concentrates at $x \in \overline{\Omega}$ if

$$
\lim_{i \to \infty} \|\nabla u_i\|_{L^n} = 1 \quad \text{and} \quad \forall \, \epsilon > 0 \lim_{i \to \infty} \int_{\Omega \setminus B_{\epsilon}(x)} |\nabla u_i|^n = 0.
$$

This definition is equivalent to the convergence $|\nabla u_i|^n dx \to \delta_x$ weakly in measure, where $\delta_x$ is the Dirac measure at $x$. We will use the following well known property of concentrating sequences: if $\{u_i\}$ concentrates, then $u_i \to 0$ in $W^{1,n}(\Omega)$. In particular

$$
u_i \to 0 \quad \text{in } L^q(\Omega) \quad \text{for all } q < \infty. \quad (4)
$$

- We define the sets

$$
W^{1,n}_{0, rad}(B_1) = \left\{ u \in W^{1,n}_0(B_1) \mid u \text{ is radial} \right\}
$$

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We define \( J_u \) which we will use throughout. In particular we will use frequently and without further comment \( \mathbb{R}^n \) in \( u \).

For basic properties of the Schwarz symmetrization we refer to Kesavan [20], Chapters 1 and 2, and analogously over concentrating sequences, we write \( \sup_{x \in \Omega} F \).

Proposition 2

\[ \text{respectively Pólya-Szegö theorem, stated in the next proposition.} \]

We define in an analogous way \( J_u \) is defined in an analogous way, replacing \( \Omega \) by \( \Omega' \) and \( u \) by \( u' \).

Finally we define

\[ F^\sup_{\Omega} = \sup_{u \in B_1(\Omega)} F_{\Omega}(u). \]

\( J^\sup_{\Omega} \) is defined in an analogous way, replacing \( F \) by \( J \).

If \( \Omega = B_1 \), then we define

\[ F^\sup_{B_1,rad} = \sup_{u \in W^{1,n}_{0,rad}(B_1)} F_{B_1}(u), \]

\[ F^\sup_{B_1,rad}(0) = \sup_{\{u_i \} \subset B_1(\Omega) \text{ concentrates at } x} \left\{ \lim_{i \to \infty} \sup_{x \in B_1(\Omega)} F_{\Omega}(u_i) \right\}. \]

We define \( J^\sup_{B_1,rad} \) and \( J^\sup_{B_1,rad}(0) \) in an analogous way.

If \( \Omega \subset \mathbb{R}^n \) then \( \Omega^* \) is its symmetric rearrangement, that is \( \Omega^* = B_R(0) \), where \( |\Omega| = R^n|\omega_{n-1}|/n \). If \( u \in W^{1,n}_0(\Omega) \), then \( u^* \in W^{1,n}_{0,rad}(B_R(0)) \) will denote the Schwarz symmetrization of \( u \). For basic properties of the Schwarz symmetrization we refer to Kesavan [20], Chapters 1 and 2, which we will use throughout. In particular we will use frequently and without further comment that if \( u \in W^{1,2}_0(\Omega) \), then \( u^* \) satisfies

\[ F_{B_R}(u) \leq F_{B_R}(u^*) \text{ if } \Omega = B_R \text{ and } \|\nabla u^*\|_{L^p(B_R)} \leq \|\nabla u\|_{L^p(\Omega)}. \]

We will additionally need, as in Flucher [17], a slight modification of the Hardy-Littlewood, respectively Pólya-Szegö theorem, stated in the next proposition.

Proposition 2 (i) Let \( f \in L^p(\Omega) \) and \( g \in L^q(\Omega) \), where \( 1/p + 1/q = 1 \). Then for any \( a \in \mathbb{R} \)

\[ \int_{\{f \geq a\}} fg \leq \int_{\{f \geq a\}} f^*g^*. \]

(ii) Let \( u \in W^{1,n}_0(\Omega) \) such that \( u \geq 0 \). Then for any \( t \in (0, \infty) \)

\[ \int_{\{u^* \leq t\}} |\nabla u^*|^n \leq \int_{\{u \leq t\}} |\nabla u|^n \text{ and } \int_{\{u^* \geq t\}} |\nabla u^*|^n \leq \int_{\{u \geq t\}} |\nabla u|^n. \]
3 $n$-Green’s function

If $x \in \Omega$, then $G_{\Omega,x}$ will denote the $n$-Green’s function of $\Omega$ with singularity at $x$. It is the unique function defined on $\Omega \setminus \{x\}$ such that the principal value of the integral

$$
\int_{\Omega} |\nabla G_{\Omega,x}(y)|^{n-2} \langle \nabla G_{\Omega,x}(y); \nabla \varphi(y) \rangle \, dy = \varphi(x) \quad \text{for all } \varphi \in \mathcal{C}_c^1(\Omega)
$$

and $G_{\Omega,x} = 0$ on $\partial \Omega$. It can always be decomposed in the form

$$
G_{\Omega,x}(y) = -\frac{1}{\omega_{n-1}^{1/(n-1)}} \log(|x-y|) - H_{\Omega,x}(y), \quad y \in \Omega \setminus \{x\},
$$

where $H_{\Omega,x}$ is a continuous function on $\Omega$ and is $C_{\text{loc}}^{1,\alpha}$ in $\Omega \setminus \{x\}$. This result is due to [21] (see therein Theorem stated in (0.10) and (0.11) or Theorem 2.1 and in particular Remark 1.4). Another useful reference on the $n$-Green’s function is [45].

The conformal incenter $I_{\Omega}(x)$ of $\Omega$ at $x$ is defined by

$$
I_{\Omega}(x) = e^{-\omega_{n-1}^{1/(n-1)} H_{\Omega,x}(x)}.
$$

Before stating the next proposition we need the following definition.

**Definition 3** We say that a sequence of sets $\{A_i\} \subset \mathbb{R}^n$ are approximately small balls at $x \in \mathbb{R}^n$ (of radius $\tau_i$) as $i \to \infty$ if there exists sequences $\tau_i, \sigma_i > 0$ such that

$$
\lim_{i \to \infty} \sigma_i = 0
$$

and

$$
B_{\tau_i-\sigma_i}(x) \subset A_i \subset B_{\tau_i+\sigma_i}(x) \quad \text{for all } i \text{ big enough}.
$$

We will need the following properties of the $n$-Green’s function. For what follows it is convenient to abbreviate, for $\beta \in [0, n)$

$$
\alpha_{n,\beta} = (n-\beta)^{1/(n-1)}.
$$

**Proposition 4** Let $x \in \Omega$. Then $G_{\Omega,x}$ and $I_{\Omega}(x)$ have the following properties:

(a) For every $t \in [0, \infty)$

$$
\int_{\{G_{\Omega,x} < t\}} |\nabla G_{\Omega,x}(y)|^n \, dy = t.
$$

(b) For every $t \in [0, \infty)$

$$
\int_{\{G_{\Omega,x} = t\}} |\nabla G_{\Omega,x}(y)|^{n-1} \, dH^{n-1}(y) = 1.
$$

(c)

$$
\lim_{t \to \infty} \frac{n |\{G_{\Omega,x} > t\}|}{\omega_{n-1} e^{-\omega_{n-1}^{1/(n-1)} t}} = (I_{\Omega}(x))^n.
$$
(d) If \( B_R = \Omega^* \) is the symmetrized domain, then for any \( x \in \Omega \)
\[
I_\Omega(x) \leq I_{B_R}(0) = R.
\]

(e) If \( t_i \geq 0 \) is a given sequence such that \( t_i \to \infty \), then the sets \( \{ G_{t_i, x} > t_i \} \) are approximately small balls at \( x \) of radius \( \tau_i \)
\[
\tau_i = I_\Omega(x)e^{-\omega_1/(n-1)t_i}. \]

In particular
\[
\lim_{t \to \infty} \frac{n - \beta}{\omega_1/(n-1)} e^{-\alpha_\omega \beta t} \int_{\{G_{t, x} > t\}} |y - x|^{-\beta} = I_\Omega(x)^{n-\beta}.
\]

**Proof** In all statements we can assume without loss of generality that \( x = 0 \in \Omega \) and abbreviate \( G_{\Omega, x} = G \), \( H_{\Omega, x} = H \).

(a) By approximation (5) holds also for any \( \varphi \in W_0^{1, \infty}(\Omega) \). Thus (a) follows by taking \( \varphi(y) = \min\{t, G(y)\} \).

(b) Observe that by (a) and the coarea formula
\[
t = \int_{\{G < t\}} |\nabla G(y)|^{n-1} |\nabla G(y)| dy = \int_0^t \left( \int_{\{G > s\}} |\nabla G(y)|^{n-1} dH^{n-1}(y) \right) ds.
\]

Hence, (b) follows from (a) by derivation.

(c) Step 1. Write \( G \) as
\[
G(y) = -\frac{1}{\omega_1/(n-1)} \log(|y|) - H(0) + H(0) - H(y) = -\frac{1}{\omega_1/(n-1)} \log \left( \frac{|y|}{I_\Omega(0)} \right) + H(0) - H(y)
\]
Let \( \|H\|_\infty = \sup\{|H(y)| : y \in \Omega\} \) and define
\[
S(t) = \left\{ y \in \Omega : |y| \leq I_\Omega(0)e^{-\omega_1/(n-1)(t-2\|H\|_\infty)} \right\},
\]
and set
\[
m(t) = \max_{y \in S(t)} |H(y) - H(0)|
\]
By the continuity of \( H \) it holds that
\[
\lim_{t \to \infty} m(t) = 0.
\]
We now define also the sets
\[
P(t) = \left\{ y \in \Omega : |y| < I_\Omega(0)e^{-\omega_1/(n-1)(t+m(t))} \right\}
\]
\[
Q(t) = \left\{ y \in \Omega : |y| < I_\Omega(0)e^{-\omega_1/(n-1)(t-m(t))} \right\},
\]
and claim that for all \( t \geq 2\|H\|_\infty \)
\[
P(t) \subset \{G > t\} \subset Q(t).
\]
Let $y \in P(t)$, then also $y \in S(t)$. So using (9), (8) and finally the definition of $P(t)$ we get

$$G(y) \geq -\frac{1}{\omega_{n-1}^1} \log \left( \frac{|y|}{I_{\Omega}(0)} \right) - m(t) = -\frac{1}{\omega_{n-1}^1} \log \left( \frac{|y|}{I_{\Omega}(0)} \right) - (m(t) + t) + t > t.$$ 

This shows $P(t) \subset \{ G > t \}$. If $y \in \{ G > t \}$, then using again (9) we obtain

$$|y| < I_{\Omega}(0)e^{-\omega_{n-1}^1(t-H(0)+H(y))}.$$ 

As $t - H(0) + H(y) \geq t - 2\| H \|_\infty$ it holds that $y \in S(t)$ and hence $|H(0) - H(y)| \leq m(t)$. This implies that $y \in Q(t)$ and proves the claim (10). It now follows from (10) that

$$\frac{\omega_{n-1}^1}{n}I_{\Omega}(0)^n e^{-\omega_{n-1}^1(t+m(t))n} \leq |\{ G > t \}| \leq \frac{\omega_{n-1}^1}{n}I_{\Omega}(0)^n e^{-\omega_{n-1}^1(t-m(t))n}.$$ 

Using (9) proves (c).

(d) We refer to Flucher [18] Lemma 8.2 page 64, or Lin [25] Lemma 2.

(e) is deduced from (10) in the following way: write

$$I_{\Omega}(0)e^{-\omega_{n-1}^1(t+m(t))} = \tau(t) - \sigma(t), \quad I_{\Omega}(0)e^{-\omega_{n-1}^1(t-m(t))} = \tau(t) + \sigma(t),$$

where

$$\tau(t) = I_{\Omega}(0)e^{-\omega_{n-1}^1t}, \quad \sigma(t) = I_{\Omega}(0)e^{-\omega_{n-1}^1t} \left( 1 - e^{-\omega_{n-1}^1m(t)} \right)$$

and

$$\sigma(t) = I_{\Omega}(0)e^{-\omega_{n-1}^1t} \left( e^{\omega_{n-1}^1m(t)} - 1 \right).$$

Using (9)

$$\lim_{t \to \infty} \frac{\sigma(t)}{\tau(t)} = \lim_{t \to \infty} \frac{\sigma(t)}{\tau(t)} = 0,$$

and the first statement follows by setting $\tau_i = \tau(t_i)$, $\sigma_i = \max\{ \sigma(t), \sigma(t_i) \}$. The second statement follows from

$$\int_{B_{\tau_i}(t) - \sigma(t)} |y|^{-\beta} dy \leq \int_{\{ G > t \}} |y|^{-\beta} dy \leq \int_{B_{\tau_i}(t) + \sigma(t)} |y|^{-\beta} dy,$$

calculating explicitly the first and last integral, and using the first statement of (e).

We will need the following result.

**Lemma 5** Let $\Omega$ be any smooth bounded domain of $\mathbb{R}^n$ and $x \in \Omega$. Suppose $\beta \in [0, n)$, then it holds

$$\int_\Omega |y - x|^{-\beta} dy \geq \frac{\omega_{n-1}^1}{n - \beta} I_{\Omega}(x)^{n-\beta}. \quad (11)$$

**Proof** It is enough to prove (11) for $x = 0$ and assume $0 \in \Omega$. We start the proof by recalling a sharp weighted isoperimetric inequality from [11] Theorem 1.1 (ii).

$$\int_A |y|^{-\beta} dy \leq \frac{1}{\alpha_{n, \beta}} \left( \int_{\partial A} |y|^\frac{-n+1}{n-\beta} d\mathcal{H}^{n-1}(y) \right)^{-\frac{n}{n-1}} \quad \text{for any smooth set } A \subset \mathbb{R}^n. \quad (12)$$
Applying (12) to the set \( \{ G_{\Omega,0} > t \} \) and using Hölder inequality and Proposition 3 (b), we have
\[
\int_{\{ G_{\Omega,0} > t \}} |y|^{-\beta} \, dy \leq \frac{1}{\alpha_{n,\beta}} \left( \int_{\{ G_{\Omega,0} = t \}} \frac{\| \nabla G_{\Omega,0}(y) \|^{n-1}}{|\nabla G_{\Omega,0}(y)|^{\frac{n-1}{n}}} \, dH^{n-1}(y) \right)^{\frac{n}{n-1}}
\leq \frac{1}{\alpha_{n,\beta}} \left( \int_{\{ G_{\Omega,0} = t \}} |\nabla G_{\Omega,0}(y)|^{n-1} \, dH^{n-1}(y) \right)^{\frac{n}{n-1}} \left( \int_{\{ G_{\Omega,0} = t \}} \frac{|y|^{-\beta}}{|\nabla G_{\Omega,0}(y)|} \, dH^{n-1}(y) \right)^{\frac{n-1}{n}}
= \frac{1}{\alpha_{n,\beta}} \int_{\{ G_{\Omega,0} = t \}} \frac{|y|^{-\beta}}{|\nabla G_{\Omega,0}(y)|} \, dH^{n-1}(y).
\] (13)

By co-area formula, we have
\[
\int_{\{ G_{\Omega,0} > t \}} |y|^{-\beta} \, dy = \int_t^\infty \int_{\{ G_{\Omega,0} = s \}} \frac{|y|^{-\beta}}{|\nabla G_{\Omega,0}(y)|} \, dH^{n-1}(y) \, ds.
\]

Whence (13) can be rewritten as
\[
\frac{d}{dt} \left( e^{\alpha_{n,\beta} t} \int_{\{ G_{\Omega,0} > t \}} |y|^{-\beta} \, dy \right) \leq 0.
\]

In other words, \( t \to e^{\alpha_{n,\beta} t} \int_{\{ G_{\Omega,0} > t \}} |y|^{-\beta} \, dy \) is a non-increasing function. Using Proposition 3 (c) we get
\[
\int_{\Omega} |y|^{-\beta} \, dy \geq \lim_{t \to \infty} e^{\alpha_{n,\beta} t} \int_{\{ G_{\Omega,0} > t \}} |y|^{-\beta} \, dx = \frac{\omega_{n-1}}{n-\beta} I_{\Omega}(0)^{n-\beta},
\]
as wanted. \( \blacksquare \)

The following proposition and its corollary are the main results of this section. We have included also the case \( \beta = n \), although this is not needed for the application to the singular Moser-Trudinger functional.

**Proposition 6** Let \( \Omega \subset \mathbb{R}^n \) be any smooth bounded set and \( x \in \Omega \). Suppose \( \beta \in [0, n] \), then it holds that
\[
\omega_{n-1}^{n/(n-1)} I_{\Omega}(x)^{n-\beta} \leq \int_{\partial \Omega} \frac{|x-y|^{-\beta}}{|\nabla G_{\Omega,x}(y)|} \, dH^{n-1}(y).
\]

**Proof** Let us first assume that \( \beta \in [0, n) \). We can assume without loss of generality that \( x = 0 \in \Omega \) and write again \( G = G_{\Omega,x} \). Applying (13) to \( t = 0 \), we have
\[
\int_{\Omega} |y|^{-\beta} \, dy \leq \frac{1}{\alpha_{n,\beta}} \int_{\partial \Omega} \frac{|y|^{-\beta}}{|\nabla G(y)|} \, dH^{n-1}(y).
\]

It then follows from (11) that
\[
\omega_{n-1}^{n/(n-1)} I_{\Omega}(0)^{n-\beta} = \alpha_{n,\beta} \frac{\omega_{n-1}}{n-\beta} I_{\Omega}(0)^{n-\beta} \leq \int_{\partial \Omega} \frac{|y|^{-\beta}}{|\nabla G_{\Omega,x}(y)|} \, dH^{n-1}(y),
\]
which proves the proposition in the present case. The case \( \beta = n \) is deduced by a continuity argument from the case \( \beta < n \). \( \blacksquare \)

For our application in Section 6 the following extension to the level sets of \( G_{\Omega,x} \) is crucial. This corollary generalizes Lemma 3 in Lin 25 to the singular case \( \beta \neq 0 \).
Corollary 7 Let $n \geq 2$, $\beta \in [0, n]$ and $\Omega$ be a bounded open smooth subset of $\mathbb{R}^n$ with $x \in \Omega$. Then all level sets $A_r$ of $G_{\Omega,x}$

$$A_r = \left\{ y \in \Omega : G_{\Omega,x}(y) > -\frac{1}{\omega^{1/(n-1)}_{n-1}} \log r \right\}, \quad r \in (0, 1]$$

satisfy the inequality

$$\frac{\omega_{n-1}}{n-1} I_{\Omega}^{n-\beta}(x) \leq \frac{1}{\omega^{1/(n-1)}_{n-1}} \int_{\partial A_r} \frac{|x - y|^{-\beta}}{|\nabla G_{\Omega,x}(y)|} dH^{n-1}(y). \quad (14)$$

Remark 8 It can be shown that the inequality tends to an equality when $r \to 0$, but this is not required for the proof of Theorem 1. This follows from the fact that $\lim_{y \to x} |y - x| |\nabla H_{\Omega,x}(y)| = 0$ (see (1.2) in [21]) and therefore

$$|\nabla G_{\Omega,x}(y)| = \frac{1}{\omega^{1/(n-1)}_{n-1}} (1 + o(1)) \quad \text{as } y \to x.$$ 

The proof is then similar to that of [25] Lemma 1 (d) and Lemma 3, where it seems that it has been assumed that $\nabla H_{\Omega,x}$ is bounded near $x$. The boundedness of $\nabla H_{\Omega,x}$ has been conjectured in [21] Remark 1.4, but we are not aware whether this has been proven.

Proof We can assume without loss of generality that $x = 0 \in \Omega$. Apply Proposition 6 to the set $\Omega = A_r$. Note that $0 \in A_r$ for all $r \in (0, 1]$,

$$G_{A_r,0}(y) = G_{\Omega,0}(y) + \frac{1}{\omega^{1/(n-1)}_{n-1}} \log r \quad \text{and} \quad H_{A_r,0}(y) = H_{\Omega,0}(y) - \frac{1}{\omega^{1/(n-1)}_{n-1}} \log r.$$ 

In particular this implies that $\nabla G_{A_r,0} = \nabla G_{\Omega,0}$ and $I_{A_r}(0) = r I_{\Omega}(0)$. This proves (14). ■

4 Some Preliminary Results

We first note that it is sufficient to work with non-negative smooth maximizing sequences. More precisely we have the following lemma, which we will use in Section 7 in a crucial way.

Lemma 9 Let $\{u_i\} \subset B_1(\Omega)$ be a sequence such that the limit $\lim_{i \to \infty} F_{\Omega}(u_i)$ exists. Then there exists a sequence $\{w_i\} \subset B_1(\Omega) \cap C^\infty_c(\Omega)$ such that

$$\liminf_{i \to \infty} F_{\Omega}(w_i) \geq \lim_{i \to \infty} F_{\Omega}(u_i).$$

Moreover, if $u_i$ concentrates at $x_0 \in \partial \Omega$, then also $w_i$ concentrates at $x_0$. In particular maximizing sequences for $F_{\Omega}^{\sup}$ and $F_{\Omega}(x_0)$ can always be assumed to be smooth and non-negative.

Proof The proof is exactly the same as in the case $n = 2$, see Lemma 4 in [11]. ■
Lemma 10 (compactness in interior) Let $0 < \eta < 1$ and suppose $\{u_i\} \subset W^{1,n}_0(\Omega)$ is such that
\[
\limsup_{i \to \infty} \|\nabla u_i\|_{L^n} \leq \eta \quad \text{and} \quad u_i \rightharpoonup u \text{ in } W^{1,n}(\Omega)
\]
for some $u \in W^{1,n}(\Omega)$. Then for some subsequence
\[
e^{\alpha u_i^n/n} |x|^\beta \to e^{\alpha u^n/n} |x|^\beta \quad \text{in } L^1(\Omega)
\]
and in particular $\lim_{i \to \infty} F_\Omega(u_i) = F_\Omega(u)$.

Proof The idea of the proof is to apply Vitali convergence theorem. We can assume that, up to a subsequence, that $u_i \to u$ almost everywhere in $\Omega$ and that
\[
\|\nabla u_i\|_{L^n} \leq \frac{1 + \eta}{2} < 1 \quad \forall i \in \mathbb{N}.
\]
We can therefore define $v_i = u_i/\theta \in B_1(\Omega)$, which satisfies $\|\nabla v_i\|_{L^n} \leq 1$ for all $i$. Moreover let us define $\overline{\alpha} = \alpha \theta^n/n < \alpha$, such that
\[
\frac{\overline{\alpha}}{\alpha_n} + \frac{\beta}{n} < 1.
\]
Let $E \subset \Omega$ be an arbitrary measurable set. We use H"older inequality with exponents $r$ and $s$, where
\[
r = \frac{\alpha_n}{\alpha} > 1 \quad \text{and} \quad \frac{1}{s} = 1 - \frac{1}{r} > \frac{\beta}{n},
\]
to obtain that
\[
\int_E e^{\alpha u_i^n/n} |x|^\beta \leq \left( \int_E e^{\overline{\alpha} v_i^n/n} |x|^\beta \right)^{\frac{n}{\beta}} \left( \int_E 1/|x|^\beta \right)^{\frac{n}{\beta}}.
\]
Let $\epsilon > 0$ be given. In view of the Moser-Trudinger inequality and using that $1/|x|^\beta s \in L^1(\Omega)$, we obtain that for any $\epsilon > 0$ there exists a $\delta > 0$ such that
\[
\int_E e^{\alpha u_i^n/n} |x|^\beta \leq \epsilon \quad \forall |E| \leq \delta \text{ and } i \in \mathbb{N}.
\]
This shows that the sequence $e^{\alpha u_i^n/n} |x|^\beta /|x|^\beta$ is equi-integrable and the Vitali convergence theorem yields convergence in $L^1(\Omega)$. This proves the lemma.

The proof of the next theorem can be found in Lions [24], Theorem I.6.

Theorem 11 (Concentration-Compactness Alternative) Let $\{u_i\} \subset B_1(\Omega)$. Then there is a subsequence and $u \in W^{1,n}_0(\Omega)$ with $u_i \rightharpoonup u$ in $W^{1,n}(\Omega)$, such that either
(a) $\{u_i\}$ concentrates at a point $x \in \Omega$,
or
(b) the following convergence holds true
\[
\lim_{i \to \infty} F_\Omega(u_i) = F_\Omega(u).
\]
The proof of the next two propositions is the same as in the 2-dimensional case, see [11].

**Proposition 12** Let $\beta > 0$, $\{u_i\} \subset B_1(\Omega)$ and suppose that $u_i$ concentrates at $x_0 \in \Omega$, where $x_0 \neq 0$. Then one has that, for some subsequence, $u_i \rightharpoonup 0$ in $W^{1,n}(\Omega)$ and

$$\lim_{i \to \infty} F_\Omega(u_i) = F_\Omega(0) = 0.$$

In particular $F_\Omega^s(x_0) = 0$.

**Remark 13** However, if $\alpha/\alpha_n + \beta/n = 1$ then $F_\Omega^s(0) > 0$. To see this assume that $\epsilon > 0$ is such that $B_\epsilon(0) \subset \Omega$ and define

$$u_i(x) = \begin{cases} i & \text{if } 0 \leq |x| \leq \epsilon e^{-\omega_1/(n-1) \log \left( |x| \right)/\epsilon} \\
\frac{1}{\omega_{n-1}^{1/(n-1)} \log \left( |x| \right)/\epsilon} & \text{if } \epsilon e^{-\omega_1/(n-1) \log \left( |x| \right)/\epsilon} \leq |x| \leq \epsilon \\
0 & \text{if } |x| \geq \epsilon. \end{cases}$$

One verifies by explicit calculation that $\{u_i\} \subset B_1(\Omega)$ and it concentrates at $0$. Let us show that $\lim \inf_{i \to \infty} F_\Omega(u_i) > 0$. First we make the estimate

$$F_\Omega(u_i) = \int_\Omega e^{\alpha u_i} \leq \int_{B_{a_i}(0)} e^{\alpha u_i} - 1, \text{ where } a_i = \epsilon e^{-\omega_1/(n-1) \log \left( |x| \right)/\epsilon}.$$

To conclude it is sufficient to show that

$$\lim_{i \to \infty} \int_{B_{a_i}(0)} e^{\alpha u_i} \frac{1}{|x|^\beta} - 1 = \omega_{n-1} e^{-\beta} > 0.$$

We can use that $|x|^{-\beta}$ is integrable and hence

$$\lim_{i \to \infty} \int_{B_{a_i}(0)} |x|^{-\beta} = 0.$$

One calculates that

$$\int_{B_{a_i}(0)} e^{\alpha u_i} \frac{1}{|x|^\beta} = \omega_{n-1} e^{-\beta} e^{\frac{\alpha}{\alpha_n} \left( \alpha - (n-\beta) \omega_1 \right)} = \omega_{n-1} e^{-\beta} \text{ if } \alpha/\alpha_n + \beta/n = 1.$$

This shows that $F_\Omega^s(0) > 0$.

We first prove Theorem [11] for some simple cases, which is the content of the next proposition. The proof is exactly the same as in the 2-dimensional case, see [11].

**Proposition 14** There exists $u \in B_1(\Omega)$ such that $F_\Omega(u) = F_\Omega^{sup}$ in the following cases:

(i) $0 \notin \Omega$ or (ii) $\alpha/\alpha_n + \beta/n < 1$. 

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5 The Case $\Omega = B_1(0)$.

In this section we deal with the case where $\Omega$ is the unit ball. The following lemma is essentially due to Adimurthi-Sandeep [2]. There one can find a proof, which is similar to the 2-dimensional case, see [11].

**Lemma 15** Let $0 < a < \infty$, and $u$ be radial function on $B_1$. Define

$$T_a(u)(x) = a^{\frac{n-1}{n}} u \left( \frac{|x|}{a} \right).$$

Then $T_a$ satisfies that

$$T_a : W^{1,n}_{0,\text{rad}}(B_1) \rightarrow W^{1,n}_{0,\text{rad}}(B_1).$$

$T_a$ is invertible with $(T_a)^{-1} = T_{1/a}$ and it satisfies

$$\|\nabla (T_a(u))\|_{L^n} = \|\nabla u\|_{L^n}, \quad \forall u \in W^{1,n}_{0,\text{rad}}(B_1).$$

Moreover if $a = 1 - \frac{\beta}{n}$, then

$$F_{B_1}(u) = \frac{1}{a} J_{B_1}(T_a(u)), \quad \forall u \in W^{1,n}_{0,\text{rad}}(B_1). \quad (15)$$

The following corollary follows easily from Lemma [15]

**Corollary 16** Let $a = 1 - \beta/n$. Then the following identities hold true

$$\sup_{u \in W^{1,n}_{0,\text{rad}}(B_1) \cap B_1(B_1)} F_{B_1}(u) = \frac{1}{a} \sup_{u \in W^{1,n}_{0,\text{rad}}(B_1) \cap B_1(B_1)} J_{B_1}(u),$$

and

$$F_{B_1}^{\sup} = \frac{1}{a} J_{B_1}^{\sup}.$$

**Proof** The first equality follows directly from Lemma [15]. By Schwarz symmetrization, the two equalities of the corollary are equivalent.

One of the crucial ingredients of the proof is the following result of Carleson and Chang [8]. Essential is the strict inequality in the following theorem. The second equality is an immediate consequence of the properties of Schwarz symmetrization.

**Theorem 17 (Carleson-Chang)** The following strict inequality holds true

$$J_{B_1,\text{rad}\gamma}^{\delta}(0) < J_{B_1,\text{rad}\gamma}^{\sup} = J_{B_1}^{\sup}.$$ 

**Remark 18** The result in Carleson and Chang is actually more precise, stating that

$$e^{1+\frac{1}{2}+\cdots+\frac{1}{n}} |B_1| = \sup_{x \in B_1} J_{B_1,\text{rad}\gamma}^{\delta}(x) < J_{B_1,\text{rad}\gamma}^{\sup},$$

but for our purpose we only need an estimate for the concentration level at 0.
From Lemma 15 and Theorem 17 we easily deduce the following proposition.

**Lemma 19** Let \( \{ u_i \} \subset B_1(B_1) \) be a sequence which concentrates at 0. If \( \{ u_i^* \} \) also concentrates at 0, then the following strict inequality holds true

\[
\limsup_{i \to \infty} F_{B_1}(u_i) < F_{B_1}^{\sup}.
\]

**Proof** The proof is identical to the 2-dimensional case, see [11], with the only difference that one sets \( a = 1 - \beta/n \).

A consequence of Lemma 19 is the following theorem, stating that the supremum of \( F_{B_1} \) is attained. The proof is here also identical to the 2-dimensional case.

**Theorem 20** The following strict inequality holds

\[
F_{B_1}^B(0) < F_{B_1}^{\sup}.
\]

In particular there exists \( u \in B_1(B_1) \) such that \( F_{B_1}^{\sup} = F_{B_1}(u) \).

6 Ball to Domain Construction

In view of Proposition 14, it remains to prove Theorem 1 for general domain with \( 0 \in \Omega \), when \( \alpha/\alpha + \beta/n = 1 \), and we can also take \( \beta > 0 \). Hence from now on we always assume that we are in this case. In addition, we assume in this section and Section 7 that \( 0 \in \Omega \). The ball to domain construction is given by the following definition: for \( v \in W^{1,n}_{0,rad}(B_1) \) and \( x \in \Omega \), define \( P_x(v) = u : \Omega \setminus \{ x \} \to \mathbb{R} \) by

\[
P_x(v)(y) = v \left( e^{-\frac{1}{\omega_{n-1}} G_{\Omega,x}(y)} \right) = v \left( (G_{B_1,0})^{-1} (G_{\Omega,x}(y)) \right),
\]

where, by abuse of notation, we have identified \( v \) and \( G_{B_1,0} \) with the corresponding radial function, for instance:

\[
G_{B_1,0}(z) = \frac{1}{\omega_{n-1}} \log z, \quad \text{if } z \in (0, 1].
\]

The main result of this section is the following theorem.

**Theorem 21** Assume \( \Omega \subset \mathbb{R}^n \) is a bounded open smooth set with \( 0 \in \Omega \). For any \( v \in W^{1,n}_{0,rad}(B_1) \cap B_1(B_1) \) define \( u = P_0(v) \). Then \( u \in B_1(\Omega) \) and it satisfies

\[
F_\Omega(u) \geq I_\Omega(0)^{\alpha-\beta} F_{B_1}(v).
\]

In particular the following inequality holds true

\[
F_{\Omega}^{\sup} \geq I_\Omega(0)^{\alpha-\beta} F_{B_1}^{\sup}.
\]

Moreover if \( \{ v_i \} \subset W^{1,n}_{0,rad}(B_1) \) concentrates at 0, then \( u_i = P_0(v_i) \) concentrates at 0.

The following lemma holds true for any domain, whether containing the origin or not. So we state this general version, although we will use it with \( x = 0 \).
Lemma 22 Let $x \in \Omega$ and let $v \in W^{1,n}_{0,rad}(B_1)$. Then $P_x(v) \in W^{1,n}_{0}(\Omega)$ and in particular
\[
\|\nabla(P_x(v))\|_{L^n(\Omega)} = \|\nabla v\|_{L^n(B_1)}.
\] (16)
Moreover if \( \{v_i\} \subset W^{1,n}_{0,rad}(B_1) \) concentrates at 0, then $P_x(v_i)$ concentrates at $x$.

Proof Step 1. We write $G = G_{\Omega,x}$. Let $h$ be defined by $h(y) = e^{-\omega_{n-1}/(n-1)}G(y)$ and hence $u(y) = v(h(y))$. In particular
\[
\nabla u(y) = v'(h(y)) \nabla h(y).
\]
Note that, since $G \geq 0$ in $\Omega$ we get that if $y \in h^{-1}\{(t)\} \cap \Omega$, then $t \in [0,1]$. Thus the coarea formula gives that
\[
\int_{\Omega} |\nabla u|^n = \int_{\Omega} |v'(h(y))|^n |\nabla h(y)|^{n-1} |\nabla h(y)| dy
= \int_0^1 \left[ \int_{h^{-1}\{(t)\} \cap \Omega} |v'(h(y))|^n |\nabla h(y)|^{n-1} d\mathcal{H}^{n-1}(y) \right] dt.
\]
Using that $|\nabla h(y)| = \omega^{1/(n-1)} h(y) |\nabla G(y)|$, gives
\[
\int_{\Omega} |\nabla u|^n = \int_0^1 \omega_{n-1}^{1/n-1} |v'(t)|^n \left[ \int_{h^{-1}\{(t)\} \cap \Omega} |\nabla G(y)|^{n-1} d\mathcal{H}^{n-1}(y) \right] dt.
\]
Note that $h^{-1}\{(t)\} \cap \Omega$ is a level set of $G$. Thus we obtain from Proposition (4) (b) that
\[
\int_{h^{-1}\{(t)\} \cap \Omega} |\nabla G(y)|^{n-1} d\mathcal{H}^{n-1}(y) = 1 \quad \forall \ t \in (0,1),
\]
which implies that
\[
\int_{\Omega} |\nabla u|^n = \int_0^1 |v'(t)|^n \omega_{n-1}^{1/n-1} dt = \int_{B_1} |\nabla v|^n.
\]
This proves (16).

Step 2. Let us now assume that $\{v_i\}$ concentrates at 0 and let $\epsilon > 0$ be given. We know from Proposition (4) (e), that for some $M > 0$ big enough $\{G > M\} \subset B_1(x)$. Thus we obtain exactly as in Step 1 that
\[
\int_{\Omega \setminus B_1(x)} |\nabla u_i|^n \leq \int_{|G \leq M|} |\nabla u_i|^n = \int_0^1 \left[ \int_{G^{-1}\{(t)\} \cap \Omega} \left( \frac{e^{\alpha n/(n-1)}}{|y|^2} - 1 \right) |\nabla G(y)|^{n-1} d\mathcal{H}^{n-1}(y) \right] dt.
\]
The right hand side goes to 0, since $v_i$ concentrates. This proves that $u_i$ concentrates too. ■

We are now able to prove the main theorem.

Proof (Theorem 21). We abbreviate again $G = G_{\Omega,0}$. From Lemma 22 we know that $u \in B_1(\Omega)$. Using the coarea formula we get
\[
F_{\Omega}(u) = \int_{\Omega} \left( \frac{e^{\alpha n/(n-1)}}{|y|^2} - 1 \right) |\nabla G(y)|^{n-1} d\mathcal{H}^{n-1}(y) = \int_0^\infty \left[ \int_{G^{-1}\{(t)\} \cap \Omega} \left( \frac{e^{\alpha n/(n-1)}}{|y|^2} - 1 \right) |\nabla G(y)|^{n-1} d\mathcal{H}^{n-1}(y) \right] dt
= \int_0^\infty \left( e^{\alpha n/(n-1)} - 1 \right) \left[ \int_{G^{-1}\{(t)\} \cap \Omega} \left( \frac{e^{\alpha n/(n-1)}}{|y|^2} - 1 \right) |\nabla G(y)|^{n-1} d\mathcal{H}^{n-1}(y) \right] dt.
\]

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We now use Corollary 7 and set
\[ r(t) = e^{-\omega_{n-1} t}, \]
to obtain
\[
F_\Omega(u) \geq I_\Omega(0)^{n-\beta} \int_0^\infty \frac{e^{\alpha v_n t}}{r(t)^\beta} \omega_{n-1} \omega_{n-1} (r(t))^n dt
\]
\[ = - I_\Omega(0)^{n-\beta} \int_0^\infty \frac{e^{\alpha v_n t}}{r(t)^\beta} \omega_{n-1} (r(t))^{n-1} r'(t) dt
\]
\[ = I_\Omega(0)^{n-\beta} \int_0^1 \frac{e^{\alpha v_n r}}{r^\beta} \omega_{n-1} r^{n-1} dr = I_\Omega(0)^{n-\beta} F_{B_1}(v). \]
This proves the first claim of the theorem. The statement about the concentration follows directly from Lemma 22.

7 Domain to Ball Construction

The aim of this section is to prove the inequality \( F^\delta_{\Omega}(0) \leq I_\Omega(0)^{n-\beta} F^\delta_{B_1}(0). \) The main difficulty compared to the 2-dimensional case is that we have to deal with the \( n \)-Laplace equation, which becomes a nonlinear partial differential equation with weaker regularity properties. We summarized the results on \( n \)-harmonic functions that we need in an Appendix. Recall that we assume \( 0 \in \Omega \).

**Theorem 23 (Concentration Formula)** Suppose \( \Omega \) contains the origin. Then the following formula holds
\[
F^\delta_{\Omega}(0) = I_\Omega(0)^{n-\beta} F^\delta_{B_1}(0).
\]
The proof of this result will be a consequence of the following proposition, which allows to construct a concentrating sequence in the ball from a given concentrating sequence in \( \Omega \).

**Proposition 24** Let \( \{u_i\} \subset B_1(\Omega) \cap C^\infty(\Omega) \) be a sequence which concentrates at 0 and is a maximizing sequence for \( F^\delta_{\Omega}(0) \). Then there exists a sequence \( \{v_i\} \subset W^{1,n}_{0,rad}(B_1) \cap B_1(B_1) \) concentrating at 0, such that
\[
F^\delta_{\Omega}(0) = \lim_{i \to \infty} F_\Omega(u_i) \leq I_\Omega^{n-\beta}(0) \liminf_{i \to \infty} F_{B_1}(v_i).
\]
**Proof (Theorem 23).** From Lemma 9 and Proposition 24 we immediately obtain that
\[
F^\delta_{\Omega}(0) \leq I_\Omega^{n-\beta}(0) F^\delta_{B_1}(0).
\]
The reverse inequality follows from Theorem 21.

The proof of Proposition 24 is long and technical. We split it into several intermediate steps. To make the presentation less cumbersome, we assume in what follows that \( 0 \in \Omega \). However, we actually need this, and the fact that concentration occurs at 0, only in Step 6 in the proof of Lemma 30. The proof of the next two Lemmas is identical to the 2-dimensional case, see [11].
Lemma 25 Suppose \( \{u_i\} \subset B_1(\Omega) \) concentrates at \( x_0 \in \Omega \) and let \( \{r_i\} \subset \mathbb{R} \) be such that \( r_i > 0 \) for all \( i \) and \( \lim_{i \to \infty} r_i = 0 \). Then there exists a subsequence \( u_{j_i} \) such that

\[
\lim_{i \to \infty} F_{\Omega}(u_i) = \lim_{i \to \infty} \int_{B_{2r_i}(x_0)} e^{\frac{\alpha u_{ij_i}^n}{|x|^\beta} - 1} dx = \lim_{i \to \infty} \int_{B_{2r_i}(x_0)} e^{\frac{\alpha u_{ij_i}^n}{|x|^\beta} - 1} dx.
\]

Moreover any subsequence of \( u_{j_i} \) will also satisfy the above equality.

Lemma 26 Suppose \( \{u_i\} \) is a sequence of measurable non-negative functions such that \( u_i \to 0 \) almost everywhere in \( \Omega \). Let \( \{s_i\} \subset \mathbb{R} \) be a bounded sequence. Then

\[
\lim_{i \to \infty} \int_{\{u_i \leq s_i\}} e^{\frac{\alpha u_i^*}{|x|^\beta} - 1} dx = 0.
\]

Lemma 27 Suppose \( \{u_i\} \subset B_1(\Omega) \cap C^\infty(\Omega) \) concentrates at \( 0 \in \Omega \) and satisfies

\[
\lim_{i \to \infty} F_{\Omega}(u_i) = F^{\delta}_{\Omega}(0). \tag{17}
\]

Then for any \( r > 0 \) there exists \( j \in \mathbb{N} \) and \( k_j \in [1, 2] \) such that

\[
\{u_j \geq k_j\} \cap B_r(0) \neq \emptyset \tag{18}
\]

and all connected components \( A \) of \( \{u_j \geq k_j\} \) will have the property:

\[
\text{If } A \cap B_r(0) \neq \emptyset \text{ then } A \subset B_{2r}(0). \tag{19}
\]

Moreover \( A \) has smooth boundary.

Proof It is sufficient to prove that there exists \( j \in \mathbb{N} \) such that (18) is satisfied for \( k_j = 2 \) and (19) holds for the connected components of \( \{u_j \geq 1\} \), i.e. \( k_j = 1 \). This implies that (18) and (19) also hold for any \( k \in [0, 1] \), and hence, using Sard’s theorem, one can choose \( k_j \in [1, 2] \) appropriately such that \( A \) has smooth boundary in addition.

First note that for all \( m \in \mathbb{N} \) there exists a \( j \geq m \) such that (18) must hold. If this is not the case, then Lemma 25 and Lemma 26 imply that

\[
\lim_{i \to \infty} \int_{B_r} e^{\frac{\alpha u_{ij_i}^n}{|x|^\beta} - 1} dx \leq \lim_{i \to \infty} \int_{\{u_i \leq 2\}} e^{\frac{\alpha u_{ij_i}^n}{|x|^\beta} - 1} dx = 0,
\]

which is a contradiction to (17) (Recall that \( F^{\delta}_{\Omega}(0) > 0 \), see Remark 13).

Suppose now that (19) does not hold. We show that this leads to a contradiction, using a capacity argument in dimension \( n - 1 \) (following the idea in [18] Equation (2.12) page 15). In that case there exists for all \( j \in \mathbb{N} \) a connected component \( D_j \) of \( \{u_j \geq 1\} \) and \( a, b \in \Omega \) such that

\[
a \in D_j \cap B_r \quad \text{and} \quad b \in D_j \cap \Omega \setminus B_{2r}.
\]

For what follows we fix \( j \) and omit the explicit dependence on \( j \) (Note that \( a \) and \( b \) depend on \( j \)). Without loss of generality we can assume, by rotating the domain, that \( b = (b_1, 0) \) and \( b_1 \geq 2r \).
Therefore, since $D_j$ is connected, for all $x_1 \in [r, 2r]$ there exists a $X' = (x_2, \ldots, x_n) = X'(x_1) \in \mathbb{R}^{n-1}$ such that $x = (x_1, X'(x_1)) \in D_j$. In particular $u_j(x) \geq 1$. Since $\Omega$ is bounded, there exists an $M > 0$, which is independent of the rotation of the domain (and hence of $j$), such that $\Omega \subset B_M(0)$. In particular this implies that

$$|X'(x_1)| \leq M \quad \text{for all } x_1 \in [r, 2r]. \tag{20}$$

Let us extend $u_j$ by zero in $\mathbb{R}^n \setminus \Omega$. Denote by

$$B'_R(y') \text{ ball of radius } R \text{ in } \mathbb{R}^{n-1} \text{ centered at } y' \in \mathbb{R}^{n-1} \quad \text{and} \quad \nabla' u = \left( \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

With this notation, using (20), we have $u_j(x_1, y) = 0$ for $y$ outside of $B_{2M}'(X'(x_1))$ for all $x_1 \in (r, 2r)$. Moreover $u_j(x_1, X'(x_1)) \geq 1$. Using now the properties of $n$-capacity in $n-1$ dimension, see for instance [19] Example 2.12 pages 35–36,

$$\int_{B_{2M}'(X'(x_1))} |\nabla' u_j(x_1, y')|^n \, dy' \geq \text{cap}_n (\{X'(x_1)\}, B_{2M}'(X'(x_1)))$$

$$= \text{cap}_n (\{0\}, B_{2M}'(0)) = c(n, M) > 0, \quad \text{for all } x_1 \in (r, 2r),$$

for some positive constant $c(n, M)$ depending only on $n$ and $M$. Hence, using also that $\Omega$ intersected with any plane where first coordinate equals $x_1$ is contained in $\{x_1\} \times B_{2M}'(0)$, we get

$$\int_{\Omega \setminus B_r(0)} |\nabla u_j|^n \geq \int_{\Omega \setminus B_r(0)} |\nabla' u_j|^n \geq \int_r^{2r} \int_{\Omega \cap \{y_1 = x_1\}} |\nabla' u_j(x_1, y')|^n \, dy' \, dx_1$$

$$= \int_r^{2r} \int_{B_{2M}'(0)} |\nabla' u_j(x_1, y')|^n \, dy' \, dx_1 = \int_r^{2r} \int_{B_{2M}'(X'(x_1))} |\nabla' u_j(x_1, y')|^n \, dy' \, dx_1$$

$$\geq c(n, M) r.$$

This implies that

$$rc(n, M) \leq \int_{\Omega \setminus B_r} |\nabla u_j|^n.$$

But this cannot hold true for all $j$, since $u_j$ concentrates at 0. □

The next lemma is about the first modification of the sequence $\{u_i\}$ given in Proposition [24].

**Lemma 28** Let $\{u_i\} \subset B_1(\Omega) \cap C^\infty(\Omega)$ be a sequence which concentrates at 0 $\in \Omega$ and satisfies

$$\lim_{i \to \infty} F_\Omega(u_i) = F_\Omega^3(0).$$

Then there exists a sequence $\{v_i\} \subset B_1(\Omega)$ and sequences $r_i > 0$, with $r_i \to 0$ and $\{k_i\} \subset [1, 2]$ such that

$$\{v_i \geq k_i\} \subset B_{2r_i}, \quad \Delta_n v_i = 0 \quad \text{in} \quad \{v_i < k_i\}.$$
Moreover $v_i$ has the properties: there exist a sequence $\{\lambda_i\} \subset \mathbb{R}$, $\lambda_i > 0$ such that

(i) $\lim_{i \to \infty} \lambda_i = \infty$
(ii) $\lim_{i \to \infty} v_i(y) = 0$ for all $y$ in $\Omega \setminus \{0\}$
(iii) $\lambda_i v_i \to G_{\Omega, 0}$ in $C^1_{\text{loc}}(\Omega \setminus \{0\})$
(iv) $\lim_{i \to \infty} F_{\Omega}(v_i) = F_{\Omega}(0)$.

**Proof**

**Step 1.** Take a sequence of positive real numbers $r_i$ such that $\lim_{i \to \infty} r_i = 0$ and choose a subsequence of $u_i$, using Lemma 25, such that

$$F_{\Omega}(0) = \lim_{i \to \infty} F_{\Omega}(u_i) = \lim_{i \to \infty} \int_{B_{r_i}} e^{\alpha u_i \frac{|x|^\beta}{r_i^n}} - 1. \tag{21}$$

Choosing again a subsequence we can assume by Lemma 27 that there exist $k_i \in [1, 2]$ such that all connected components $A$ of $\{u_i \geq k_i\}$ which intersect $B_{r_i}$ are contained in $B_{2r_i}$. We define $A_i$ as the union of all such $A$. We also know from Lemma 27 that $A_i$ is not empty. Let $w_i \in W^{1, n}(\Omega \setminus A_i)$ be the solution of, see Theorem 32,

$$\Delta_n w_i = 0 \quad \text{in} \quad \Omega \setminus A_i$$
$$w_i = 0 \quad \text{on} \quad \partial \Omega, \quad w_i = k_i \quad \text{on} \quad \partial A_i.$$

We now define $v_i \in W^{1, n}_0(\Omega)$ as

$$v_i = \begin{cases} 
 u_i & \text{in} \ A_i \\
 w_i & \text{in} \ \Omega \setminus A_i.
\end{cases}$$

Since $n$-harmonic functions minimize the $n$-Dirichlet integral we have $\|
abla v_i\|_{L^n} \leq \|
abla u_i\|_{L^n}$. Thus we have constructed a sequence which has the properties: (we have used Theorem 35 in the second property)

$$\{v_i \geq k_i\} \subset B_{2r_i}, \quad \Delta_n v_i = 0 \quad \text{in} \ \{v_i < k_i\} \quad \text{and} \quad \|
abla v_i\|_{L^n} \leq 1.$$

**Step 2.** We will show in this Step that for all $y \in \Omega \setminus \{0\}$ we have $v_i(y) > 0$ for all $i$ large enough and $\lim_{i \to \infty} v_i(y) = 0$. The fact that $v_i(y) > 0$ follows from the maximum principle Theorem 35. Since $\Omega$ is bounded there exists $M > 0$ such that $\overline{\Omega} \subset B_M$. Define $W_i = B_M \setminus B_{2r_i}$ and let $\psi_i$ be the solution of

$$\Delta_n \psi_i = 0 \quad \text{in} \ W_i$$
$$\psi_i = 2 \quad \text{on} \ \partial B_{2r_i} \quad \text{and} \quad \psi_i = 0 \quad \text{on} \ \partial B_M.$$

The function $\psi_i$ can be given explicitly:

$$\psi_i = \frac{2}{\log \left( \frac{2r_i}{M} \right)} \log \left( \frac{|x|}{M} \right).$$

Recall that $k_i \in [1, 2]$ and note that

$$v_i > 0 \quad \text{and} \quad v_i = 0 \quad \text{on} \ \partial \Omega,$$
$$v_i = 2 \quad \text{and} \quad v_i < k_i \leq 2 \quad \text{on} \ \partial B_{2r_i}.$$
and thus $\psi_i - v_i > 0$ on $\partial W_i$. Since $v_i$ is also harmonic in $W_i$ the comparison principle (Theorem 33) implies that $v_i \leq \psi_i$ in $W_i$. For $i$ big enough $y \in W_i$ and the claim of Step 2 follows from the fact that $\lim_{i \to \infty} \psi_i(y) = 0$.

**Step 3.** Choose $y \in \Omega \setminus \{0\}$ and define $\lambda_i$ by
\[
\lambda_i = \frac{G_{\Omega,0}(y)}{v_i(y)} \quad \Leftrightarrow \quad \lambda_i v_i(y) = G_{\Omega,0}(y)
\]
In view of Step 2 this is well defined, $\lambda_i > 0$ and
\[
\lim_{i \to \infty} \lambda_i = \infty.
\]
Let $y \in K_1 \subset \Omega \setminus \{0\}$ be a compact set. Choose another compact set $K_2$, such that $K_1 \subset \subset K_2 \subset \Omega \setminus \{0\}$. Applying Harnack inequality (Theorem 34) on $K_2$ we get that there exist $c_1, c_2 > 0$, such that
\[
c_1 |G_{\Omega,0}(y)| \leq |\lambda_i v_i(x)| \leq c_2 |G_{\Omega,0}(y)|, \quad \forall x \in K_2 \text{ and } \forall i \text{ large enough.}
\]
Hence the sequence $\lambda_i v_i$ is uniformly bounded in the $C^0(K_2)$ norm. It follows from Theorem 36 that
\[
\lambda_i v_i \text{ is uniformly bounded in the } C^{1,\alpha}(K_1) \text{ norm}
\]
for some $0 < \alpha$. Using the compact embedding $C^{1,\alpha}(K_1) \hookrightarrow C^1(K_1)$ we obtain that there exists $g \in C^1(K_1)$ and a subsequence $v_i$ with
\[
g_i := \lambda_i v_i \to g \quad \text{in } C^1(K_1).
\]
It follows from (22) and Corollary 38 that $g = G_{\Omega,0}$, once we have shown that $g = 0$ on $\partial \Omega$. We prove this claim in the next step.

**Step 4.** We show now that $g = 0$ on $\partial \Omega$. Define $\Omega_\epsilon$ as
\[
\Omega_\epsilon = \{x \in \Omega : 0 < \text{dist}(x, \partial \Omega) < \epsilon\},
\]
where $\epsilon$ will be chosen later small enough, and
\[
\partial \Omega_\epsilon = \partial \Omega \cup \Gamma_\epsilon \quad \text{where} \quad \Gamma_\epsilon = \{x \in \Omega : \text{dist}(x, \partial \Omega) = \epsilon\}.
\]

**Step 4.1.** We claim that $g_i$ are uniformly bounded on $\Omega_\epsilon$. Note that $g_i = 0$ on $\partial \Omega$. So for small enough $\epsilon$ it follows from Lemma 40 and Remark 41 (ii) (as in the proof of Proposition 39) that
\[
\|g_i\|_{L^\infty(\Omega_\epsilon)} \leq C(\Omega_\epsilon)\|g_i\|_{L^\infty(\Omega_\epsilon)}.
\]
(25) We now fix such an $\epsilon > 0$ for which (25) holds and we can also assume that $\Omega_\epsilon$ is a smooth set. It follows from (24) that there exists $\Lambda_1 = \Lambda_1(K_1, K_2)$ (choosing $K_1$ such that $\Gamma_\epsilon \subset \subset K_1$), such that
\[
\|g_i\|_{C^{1,\alpha}(\Gamma_\epsilon)} \leq \Lambda_1 \quad \text{for all } i \text{ big enough,}
\]
and hence also for some $\Lambda_2 > 0$
\[
\|g_i\|_{W^{1,\frac{1}{n}}(\Gamma_\epsilon)} \leq \Lambda_2 \quad \text{for all } i \text{ big enough.}
\]
Chose now a bounded right inverse $T$ of the trace operator on $\partial \Omega_e$ as

$$T : W^{1,\frac{1}{2}}(\partial \Omega_e) \to W^{1,\infty}(\Omega_e)$$

and apply it to $g_i$ restricted to $\partial \Omega_e$. Hence there exists $h_i \in W^{1,\infty}(\Omega_e)$, $h_i = T(g_i)$ such that

$$h_i = g_i \quad \text{on } \partial \Omega_e$$

and

$$\|h_i\|_{W^{1,\infty}(\Omega_e)} \leq C_1(\Omega_e) \|g_i\|_{W^{1,\frac{1}{2}}(\partial \Omega_e)} \leq C_1(\Omega_e) A_2. \quad (26)$$

As in the proof of Proposition 39, since $\Delta h_i = 0$ in $\Omega_e$ for $i$ big enough,

$$\|g_i - h_i\|_{L^\infty(\Omega_e)} \leq C_2(\Omega_e) \|\nabla g_i - \nabla h_i\|_{L^\infty(\Omega_e)} \leq C_2 \left( \|\nabla g_i\|_{L^\infty(\Omega_e)} + \|\nabla h_i\|_{L^\infty(\Omega_e)} \right) \leq 2C_2 \|\nabla h_i\|_{L^\infty(\Omega_e)} \leq 2C_1C_2A_2.$$

Since the $h_i$ are also uniformly bounded by 26 it follows the $g_i$ are uniformly bounded in the $L^\infty(\Omega_e)$ norm. This shows Step 4.1.

**Step 4.2.** We now conclude that $g = 0$ on $\partial \Omega$. Fix some $a > 0$ so that $B_a(0) \subset \subset \Omega$ and define $\Omega_a = \Omega \setminus \overline{B_a(0)}$. Note that $g_i$ is uniformly bounded on $\partial B_a(0)$ in the $C^{1,\alpha}$ norm, using again 24, i.e. for some $\Lambda_3 > 0$ we have

$$\|g_i\|_{C^{1,\alpha}(\partial B_a(0))} \leq \Lambda_3.$$

On the compact set $\Omega \setminus (B_a(0) \cup \Omega_e)$ $g_i$ is uniformly bounded by 24. Thus, together with Step 4.1 this shows that there exists a constant $M_0$ independent of $i$ such that

$$\|g_i\|_{L^\infty(\Omega_a)} \leq M_0.$$

Note that

$$\Delta_a g_i = 0 \quad \text{in } \Omega_a.$$

So it follows from Theorem 32 that

$$\|g_i\|_{C^{1,\alpha}(\Omega_a)} \leq C(M_0, \Lambda_3).$$

It follows that for some subsequence $g_i \to g$ in $C^1(\overline{\Omega_a})$ from which it follows that $g = 0$ on $\partial \Omega$.

**Step 5.** It remains to prove (iv). Recall that $v_i \leq k_i$ in $\Omega \setminus A_i$. We therefore obtain, using Lemma 26 twice and the definition of $A_i$ that

$$\lim_{i \to \infty} \int_{\Omega} e^{\alpha u_i} \frac{\nabla u_i}{|x|^\beta} - 1 = \lim_{i \to \infty} \int_{A_i} e^{\alpha u_i} \frac{\nabla u_i}{|x|^\beta} - 1 \geq \lim_{i \to \infty} \int_{A_i \cap B_r} e^{\alpha u_i} \frac{\nabla u_i}{|x|^\beta} - 1,$$

where we have used 21 in the last equality. ■

The next lemma is about the second modification of the sequence $\{u_i\}$ given in Proposition 24 following the first modification given by Lemma 28.
Lemma 29 Let \( \{u_i\} \subset W^{1,n}_0(\Omega) \) be a sequence and \( \lambda_i \) a sequence in \( \mathbb{R} \) such that \( \lambda_i \to \infty \),
\[ \lambda_i u_i \to G_{\Omega,0} \] in \( C^0_{loc}(\Omega \setminus \{0\}) \) and \( \Delta_n u_i = 0 \) in \( \{u_i < 1\} \).

Then there exists a subsequence \( \lambda_{i_l} \) and a sequence \( \{v_l\} \subset W^{1,n}_0(\Omega) \) such that the following properties hold true:

(a) \( \lambda_{i_l} \geq l \)
(b) The sets \( \{v_l \geq l/\lambda_{i_l}\} \) are approximately small balls at 0 as \( l \to \infty \).
(c) \( v_l(x) \to 0 \) as \( l \to \infty \) for every \( x \in \Omega \setminus \{0\} \).
(d) For every \( l \)
\[ \int_{\Omega} |\nabla v_l|^n \leq \int_{\Omega} |\nabla u_{i_l}|^n. \]
(e) The inequality \( v_l \geq u_{i_l} \) holds in \( \Omega \). In particular \( F_{\Omega}(v_l) \geq F_{\Omega}(u_{i_l}) \).

Proof The proof is exactly the same as in the 2-dimensional case, since one can use the strong maximum principle, Theorem 35, for \( n \)-harmonic functions.

After having modified the sequence \( \{u_i\} \) given in Proposition 24 in the two previous lemmas, we finally construct the appropriate corresponding sequence \( \{v_i\} \subset W^{1,n}_{0,rad}(B_1) \). This is contained in the following lemma.

Lemma 30 Let \( \{u_i\} \subset W^{1,n}_0(\Omega) \) and \( \{s_i\} \subset \mathbb{R} \) be sequences with the following properties:
\[ s_i \leq 1 \quad \forall i \in \mathbb{N}, \]
the sets \( \{u_i \geq s_i\} \) are approximately small balls at 0 as \( i \to \infty \) and moreover suppose that pointwise \( u_i(x) \to 0 \) for all \( x \in \Omega \setminus \{0\} \). Then there exists a sequence \( \{v_i\} \subset W^{1,n}_{0,rad}(B_1) \) such that for all \( i \)
\[ ||\nabla v_i||_{L^n(B_1)} \leq ||\nabla u_i||_{L^n(\Omega)} \]
and, assuming that the left hand side limit exists,
\[ \lim_{i \to \infty} F_{\Omega}(u_i) \leq I_{\Omega}(0)^{n-\beta} \lim_{i \to \infty} \inf F_{B_1}(v_i). \]

Moreover \( v_i(x) \to 0 \) for all \( x \in B_1 \setminus \{0\} \) and if \( v_i \) concentrates at some \( x_0 \in B_1 \), then \( x_0 = 0 \).

Proof Throughout this proof \( G = G_{\Omega,0} \) shall denote the \( n \)-Green’s function of \( \Omega \) with singularity at 0. Recall that by assumption there exists real positive numbers \( \rho_i \) and \( \epsilon_i \) such that for \( i \to \infty \)
\[ \rho_i \to 0 \quad \text{and} \quad \frac{\epsilon_i}{\rho_i} \to 0, \]
satisfying for all \( i \) the following inclusion
\[ B_{\rho_i - \epsilon_i} \subset \{u_i \geq s_i\} \subset B_{\rho_i + \epsilon_i}, \]
(28)

Step 1. Let us define \( \lambda_i \), implicitly, by the following equation:
\[ \rho_i = I_{\Omega}(0)e^{-\omega_{n-1}/\lambda_i}, \]
(29)
that is
\[ \lambda_i = -\frac{1}{\omega_n^{1/(n-1)}} \log \left( \frac{\rho_i}{I_\Omega(0)} \right). \]
Note that \( \lambda_i \to \infty \) as \( i \to \infty \). We claim that there exists \( t_i \geq \lambda_i \) such that
\[ \lim_{i \to \infty} (t_i - \lambda_i) = 0 \quad (30) \]
and
\[ \{ G \geq t_i \} \subset \{ u_i \geq s_i \}. \quad (31) \]
To show this we use Proposition 4 (e), which states that if \( t_i \geq 0 \) is given such that \( t_i \to \infty \), then there exists \( \sigma_i \geq 0 \) such that
\[ \lim_{i \to \infty} \sigma_i / \tau_i = 0 \]
and
\[ B_{\tau_i - \sigma_i} \subset \{ G \geq t_i \} \subset B_{\tau_i + \sigma_i}, \]
where \( \tau_i = I_\Omega(0) e^{-\omega_n^{1/(n-1)} t_i} \). In view of (28) it is therefore sufficient to choose \( t_i \) such that
\[ \tau_i + \sigma_i = \rho_i - \epsilon_i. \quad (32) \]
It remains to show that with this choice (30) is also satisfied. Using (29) and solving the previous equation for \( t_i \) explicitly gives that
\[ t_i = \lambda_i - \frac{1}{\omega_n^{1/(n-1)}} \log \left( 1 - \frac{\epsilon_i + \sigma_i}{\rho_i} \right). \]
Since we know from (27) that \( \epsilon_i / \rho_i \to 0 \), it is sufficient to show that \( \sigma_i / \rho_i \to 0 \). We obtain from (32) that
\[ \frac{\sigma_i}{\tau_i} = \frac{\sigma_i}{\rho_i - \epsilon_i - \sigma_i} = \frac{\sigma_i}{\rho_i \left( 1 - \frac{\epsilon_i}{\rho_i} - \frac{\sigma_i}{\rho_i} \right)}. \]
Solving this equation for \( (\sigma_i / \rho_i) \) and using that \( \epsilon_i / \rho_i \to 0 \) and \( \sigma_i / \tau_i \to 0 \) shows that also \( (\sigma_i / \rho_i) \to 0 \). This proves (30).

**Step 2.** In this step we will show that
\[ \int_{\{ u_i < 1 \}} |\nabla u_i|^n \geq \frac{s^n}{t_i^{n-1}}. \quad (33) \]
Let us denote
\[ U = \{ u_i \geq s_i \} \quad \text{and} \quad V = \{ G \geq t_i \}. \]
From Step 1 we know that \( V \subset U \) and since \( u_i = 0 \) on \( \partial \Omega \) we also have that \( \overline{U} \subset \Omega \). Let \( h_i \in W^{1,n}(\Omega \setminus V) \) be the unique solution of the problem
\[ \Delta_n h_i = 0 \quad \text{in} \quad \Omega \setminus V \]
\[ h_i = 0 \quad \text{on} \quad \partial \Omega \quad \text{and} \quad h_i = 1 \quad \text{on} \quad \partial V. \]
We see that this is satisfied precisely by \( h_i = G/t_i \). Let us define \( w_i \in W^{1,n}(\Omega \setminus V) \) by

\[
w_i = \begin{cases} \\ u_i & \text{in } \Omega \setminus U \\ 1 & \text{in } U \setminus V.
\end{cases}
\]

Note that \( w_i \) has the same boundary values as \( h_i \) on the boundary of \( \Omega \setminus V \). Since \( h_i \) is the unique minimizer of the functional

\[
J(h) = \int_{\Omega \setminus V} |\nabla h|^n
\]

among all functions with these fixed boundary values, we get that

\[
\int_{\Omega \setminus V} |\nabla h_i|^n \leq \int_{\Omega \setminus V} |\nabla w_i|^n = \int_{\Omega \setminus U} |\nabla w_i|^n = \int_{\{u_i < s_i\}} |\nabla u_i|^n.
\]

From Proposition 4 (a) we know that

\[
\int_{\Omega \setminus V} |\nabla h_i|^n = \int_{\{G < tai\}} \left| \nabla \left( \frac{G}{t_i} \right) \right|^n = \frac{1}{t_i^{n-1}}.
\]

Setting this into the previous inequality proves (33).

**Step 3.** In this step we will define \( v_i \in W^{1,n}_{0, \text{rad}}(B_1) \). Let \( \Omega^* = B_R \) be the symmetrized domain and \( u_i^* \in W^{1,n}_{0, \text{rad}}(B_R) \) be the radially decreasing symmetric rearrangement of \( u_i \). Then there exists \( 0 < a_i < R \) such that

\[
\{u_i^* \geq s_i\} = B_{a_i}.
\]

Moreover define \( 0 < \delta_i < 1 \) by

\[
\delta_i = e^{-s_i/|\omega_{n-1}| t_i}.
\]

At last we can define \( v_i \) as

\[
v_i(x) = \begin{cases} \\ -\frac{s_i}{\omega_{n-1}} \log(|x|) & \text{if } x \geq \delta_i \\ u_i^* \left( \frac{x}{\delta_i} \right) & \text{if } x \leq \delta_i.
\end{cases}
\]

Note that \( v_i \) belongs indeed to \( W^{1,n}(B_1) \) since

\[
u_i^*(a_i) = s_i = -\frac{s_i}{\omega_{n-1}} \log(\delta_i).
\]

**Step 4.** In this Step we will show that \( \|\nabla v_i\|_{L^n(B_1)} \leq \|\nabla u_i\|_{L^n(\Omega)} \). Let us denote

\[
A_i = \int_{B_1 \setminus B_{a_i}} |\nabla v_i|^n \quad \text{and} \quad D_i = \int_{B_{a_i}} |\nabla v_i|^n.
\]

A direct calculation gives that

\[
A_i = \frac{\omega_{n-1} s_i^n}{\omega_{n-1}^{n/(n-1)}} \int_{\delta_i}^{1} \frac{1}{r} dr = \frac{s_i^n}{t_i^{n-1}}.
\]
Using a change of variables and Proposition 2(ii) gives that
\[
D_i = \int_{B_{a_i}} |\nabla u_i|^n = \int_{\{u_i \geq s_i\}} |\nabla u_i|^n \leq \int_{\{u_i \geq s_i\}} |\nabla u_i|^n.
\]

Finally we get that, using (33), that
\[
\int_{B_1} |\nabla v_i|^n = D_i + A_i \leq \int_{\Omega} |\nabla u_i|^n - \int_{\{u_i < s_i\}} |\nabla u_i|^n + \frac{s_i^n}{t_i^n} \leq \int_{\Omega} |\nabla u_i|^n.
\]

**Step 5.** In this step we show that
\[
\lim_{i \to \infty} \frac{a_i}{\delta_i} = I_\Omega(0).
\]

Using the fact that \(|\{u_i \geq s_i\}| = \{|u_i \geq s_i|\}|, we get from (34) and the hypothesis (28) we get that
\[
\rho_i - \epsilon_i \leq a_i \leq \rho_i + \epsilon_i.
\]
From this equation we obtain that
\[
\frac{\rho_i}{\delta_i} \left(1 - \frac{\epsilon_i}{\rho_i}\right) \leq \frac{a_i}{\delta_i} \leq \frac{\rho_i}{\delta_i} \left(1 + \frac{\epsilon_i}{\rho_i}\right).
\]
From the hypothesis (27) we know that \(\epsilon_i/\rho_i \to 0\). It is therefore sufficient to calculate the limit of \(\rho_i/\delta_i\). In view of the definitions of \(\rho_i, \delta_i\) and (30) this is indeed equal to
\[
\lim_{i \to \infty} \frac{\rho_i}{\delta_i} = \lim_{i \to \infty} I_\Omega(0) e^{\frac{1}{n-1}(\kappa_i - \lambda_i)} = I_\Omega(0),
\]
which proves the statement of this step.

**Step 6 (equality of functional limit).** Let us first show that both \(u_i\) and \(v_i\) converge to zero almost everywhere. For \(u_i\) this holds true by hypothesis. So let \(x \in B_1 \setminus \{0\}\) be given and note that for all \(i\) big enough
\[
x \geq e^{-\omega_n^{1/(n-1)}\sqrt{t_i}} \geq e^{-\omega_n^{1/(n-1)}t_i} = \delta_i.
\]
Therefore we obtain from the definition of \(v_i\) that
\[
v_i(x) \leq \frac{s_i}{\omega_n^{1/(n-1)}t_i} \log \left(e^{-\omega_n^{1/(n-1)}\sqrt{t_i}}\right) = \frac{s_i}{\sqrt{t_i}} \to 0,
\]
which shows the claim also for \(v_i\). In view of lemma 26 it is therefore sufficient to show that
\[
\lim_{i \to \infty} \int_{\{u_i \geq s_i\}} \frac{e^{\alpha u_i^{n/(n-1)}}}{|x|^\beta} - 1 = I_\Omega^{n-\beta}(0) \lim_{i \to \infty} \int_{\{v_i \geq s_i\}} \frac{e^{\alpha v_i^{n/(n-1)}}}{|x|^\beta} - 1.
\]
(36)

From Proposition 3(i) and the properties of symmetrization we get that for every \(i\)
\[
\int_{\{u_i \geq s_i\}} \frac{e^{\alpha u_i^{n/(n-1)}}}{|x|^\beta} - 1 \leq \int_{\{u_i \geq s_i\}} \frac{e^{\alpha (u_i)^{n/(n-1)}}}{(|x|^\beta)^\kappa} - 1 = \int_{B_{a_i}} \frac{e^{\alpha (u_i)^{n/(n-1)}}}{(|x|^\beta)^\kappa} - 1.
\]
Note that if $\beta = 0$ then the inequality can actually be replaced by an equality (see Kesavan, page 14, equation (1.3.2)). For $i$ big enough $B_{a_i} \subset \Omega$, and then $\left(\|x\|^\beta\right)^* = \|x\|^\beta$ for all $x \in B_{a_i}$. Making the substitution $x = (a_i/\delta_i) y$ gives

$$
\int_{\{u_i \geq s_i\}} \frac{e^{\alpha u_i/(n-1)} - 1}{\|x\|^\beta} \leq \left(\frac{a_i}{\delta_i}\right)^{n-\beta} \int_{B_{a_i}} \frac{e^{\alpha v_i/(n-1)} - 1}{\|y\|^\beta} = \left(\frac{a_i}{\delta_i}\right)^{n-\beta} \int_{\{v_i \geq s_i\}} \frac{e^{\alpha v_i/(n-1)} - 1}{\|x\|^\beta}.
$$

From Step 5 we therefore get that

$$
\lim_{i \to \infty} \int_{\{u_i \geq s_i\}} \frac{e^{\alpha u_i/(n-1)} - 1}{\|x\|^\beta} \leq I_{\Omega}(0)^{n-\beta} \liminf_{i \to \infty} \int_{\{v_i \geq s_i\}} \frac{e^{\alpha v_i/(n-1)} - 1}{\|x\|^\beta},
$$

which proves (36).

Step 7. The last statement of the lemma ($v_i$ can concentrate only at 0) follows from (35), because $\lim_{i \to \infty} A_i = 0$ (using that $t_i \to \infty$). Thus $v_i$ cannot concentrate at any other point than 0.

We are now able to prove the main proposition of this section.

**Proof (Proposition 24).** The proof is exactly the same as in the 2-dimensional case, see details in [11]. One modifies successively the given sequence $u_i$, using Lemma 28 and Lemma 29, and construct finally a sequence $\{v_i\} \subset W^{1,n}_{0, rad}(B_1) \cap B_1(B_1)$, using Lemma 30, such that

$$
F_{\Omega}^0(0) \leq \liminf_{i \to \infty} F_{\Omega}(u_i) \leq I_{\Omega}^n(0)^{n-\beta} \liminf_{i \to \infty} F_{B_1}(v_i).
$$

The sequence $\{v_i\}$ has to concentrate at 0, because it goes to zero almost everywhere. (use last statement of Lemma 30, Theorem 11 and Remark 13).

8 Proof of the Main Result

We now prove Theorem 11. We assume that $0 \in \Omega$, the other cases are proven exactly as in the two dimensional setting. For the case $0 \in \partial \Omega$ use [45] Proposition 2.4 (3), which implies that $I_{\Omega_m}(0)$ tends to zero if $\{\Omega_m\}$ is a sequence of sets whose boundary approaches the origin.

**Proof** Let $\{u_i\}_{i \in \mathbb{N}} \subset B_1(\Omega)$ be a maximizing sequence for $F_{\Omega}$. Then by Theorem 11 for some subsequence, either

$$
\lim_{i \to \infty} F_{\Omega}(u_i) = F_{\Omega}(u),
$$

or $\{u_i\}$ concentrates at some point $x \in \Omega$. In the first case $u$ is an extremal function and we are done. It remains to exclude concentration. Assume, by contradiction, we have concentration at $x$. By Proposition 12 we must have $x = 0$ (because we have assumed $\{u_i\}$ is a maximizing sequence and $F_{\Omega}^{\text{sup}} \neq 0$). Thus we get

$$
F_{\Omega}^{\text{sup}} = \lim_{i \to \infty} F_{\Omega}(u_i) = F_{\Omega}^0(0).
$$

By Theorem 23 it holds

$$
F_{\Omega}^0(0) = I_{\Omega}(0)^{n-\beta} F_{\Omega}^0(0),
$$

and by Theorem 20

$$
F_{B_1}^0(0) < F_{\Omega}^{\text{sup}}.
$$

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Finally Theorem 21 states

$$F_{B_i}^\sup I_\Omega(0)^{n-\beta} \leq F_{\Omega}^\sup.$$  \hspace{1cm} (40)

Combining (37)-(40) gives the contradiction $F_{\Omega}^\sup < F_{\Omega}^\sup$, and therefore concentration cannot occur.

\section{Appendix: $n$-harmonic functions}

We summarize the results on $n$-harmonic functions that we have used. We only state them under the more restrictive hypothesis that are sufficient for our construction of the $n$-harmonic transplantation and we refer to references for more general versions.

\textbf{Definition 31} \hspace{1cm} Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in W^{1,n}(\Omega)$. we say that $f$ is $n$-harmonic, that is

$$\Delta_n f = 0 \quad \text{in} \quad \Omega,$$

if

$$\int_{\Omega} |\nabla f|^{n-2} \langle \nabla f, \nabla \varphi \rangle = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega).$$

The existence of $W^{1,n}$ solutions to the $n$-Laplace equation follows easily by direct methods of the calculus of variations. The difficult part of the next theorem is the $C^{1,\alpha}$ regularity. For the interior regularity see [15], [38] or [16]. Lieberman [22] proves regularity up to the boundary under the assumption that $f$ is bounded. In this paper we only need the regularity result for very special boundary values (constants on different parts of the boundary). In this case the proof is simple, so we have included a proof of the boundedness, see Proposition 39. For a more general version, not assuming the boundedness of $f$, we have not found a satisfactory reference.

\textbf{Theorem 32 (Existence and Regularity)} \hspace{1cm} There exists $0 < \alpha < 1$, depending only on $n$, with the following property. Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set, $g \in C^{1,\alpha}(\partial \Omega)$ such that $\|g\|_{C^{1,\alpha}(\partial \Omega)} \leq t_0$. Then there exists a unique $f \in W^{1,n}(\Omega)$ such that

$$\Delta_n f = 0 \quad \text{in} \quad \Omega$$

$$f = g \quad \text{on} \quad \partial \Omega.$$

The solution $f$ satisfies

$$\int_{\Omega} |\nabla f|^n \leq \int_{\Omega} |\nabla \varphi|^n \quad \forall \varphi \in W^{1,n}(\Omega) \text{ with } \varphi = g \quad \text{on} \quad \partial \Omega.$$ \hspace{1cm} (41)

Moreover if $\|f\|_{L^\infty(\Omega)} \leq M_0$ then $f \in C^{1,\alpha}(\overline{\Omega})$ and there exists a constant $C$ depending on $n, \Omega, M_0$ and $t_0$ such that

$$\|f\|_{C^{1,\alpha}(\overline{\Omega})} \leq C(n, \Omega, M_0, t_0)$$

A reference for the next three results is for instance [23] Theorems 2.15, 2.20 and Corollary 2.21.

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Theorem 33 (Comparison Principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Suppose $f, g \in W^{1,n}(\Omega)$ and $\Delta_n f = \Delta_n g = 0$ in $\Omega$. If
$$f \leq g \quad \text{on } \partial \Omega,$$
then
$$f \leq g \quad \text{in } \Omega.$$

Theorem 34 (Harnack Inequality) Let $V$ be a compact set and $U$ open with $V \subset U$. Then there is a constant $C = C(V)$ such that
$$\sup_V u \leq C \inf_V u$$
for all $u \in W^{1,n}(U)$ such that $\Delta_n u = 0$ in $U$ and $u \geq 0$ in $U$.

Theorem 35 (Strong maximum principle) Let $\Omega$ be a bounded open set with smooth boundary $\partial \Omega$ and $u \in W^{1,n}(\Omega)$. Suppose $\Delta_n u = 0$ in $\Omega$, $u$ is not constant and $u \geq 0$. Then the following holds
$$\inf_{\partial \Omega} f < f(x) < \sup_{\partial \Omega} f \quad \text{for all } x \in \Omega.$$

The next theorem follows from [38] Theorem 1.

Theorem 36 Let $V$ be a compact set $V \subset U$, where is $U$ open. Then there exists $\alpha \in (0, 1)$ and a constant $C > 0$ depending on $V, U, n$, and a variable $g_0$ such that
$$\|u\|_{C^{1,\alpha}(V)} \leq C(V, U, n, g_0)$$
for all $u \in W^{1,n}(U)$ with $\Delta_n u = 0$ in $U$ and $= \|u\|_{L^{\infty}(U)} \leq g_0$.

The next theorem is contained in [34] Theorem 10 and [35] Theorem 3, combined with Theorem 32. This is the generalization of Bocher’s theorem (see [5] Theorem 3.9 page 50) to $n$-harmonic functions.

Theorem 37 Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set with $0 \in \Omega$, $u \in W^{1,n}_{\text{loc}}(\Omega \setminus \{0\})$ with the properties
$$\Delta_n u = 0 \quad \text{in } \Omega \setminus \{0\}.$$
Then
(a) either $u \in L^{\infty}(\Omega)$, in which case $u$ is continuous in $\Omega$. Moreover $u \in W^{1,n}(\Omega)$ and it solves the equation
$$\Delta_n u = 0 \quad \text{in } \Omega,$$
and hence $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $0 < \alpha < 1$.
(b) or there exists a constant $k \in \mathbb{R}$ such that
$$\int_{\Omega} |\nabla u|^{n-2} \nabla u, \nabla \varphi = k \varphi(0) \quad \text{for all } \varphi \in C^1_c(\Omega).$$
Corollary 38 Suppose \( u \in W^{1,n}(\Omega \setminus \{0\}) \), \( \Delta_n u = 0 \) in \( \Omega \setminus \{0\} \) and \( u = 0 \) on \( \partial \Omega \). Then either \( u \) is a constant multiple of the n-Green’s function, i.e. \( u = kG_{\Omega,0}(y) \) or \( u \) vanishes identically.

The next lemma states that the hypothesis of the regularity result of Lieberman \[22\] is satisfied, that is, the boundedness of \( f \) by \( M_0 \) in Proposition \[32\]

**Proposition 39** Let \( \Omega \subset \mathbb{R}^n \) be a bounded opens smooth set such that \( \partial \Omega = \bigcup_{i=1}^{L} \Gamma_i \) where \( \Gamma_i \) are the connected smooth components of \( \partial \Omega \). Let \( k_i \in \mathbb{R}, i = 1, \ldots, L \) be constants such that \( |k_i| \leq k \).

Then there exists a constant \( M_0 = M_0(\Omega, n, k) \) such that

\[
\|u\|_{L^\infty(\Omega)} \leq M_0(\Omega, n, k)
\]

for all \( u \in W^{1,n}(\Omega) \) satisfying \( \Delta_n u = 0 \) in \( \Omega \) and \( u = k_i \) on \( \Gamma_i \).

The following lemma is required for the proof of Proposition \[39\] and has also been used in Lemma \[28\]

**Lemma 40** Let \( \Omega, \Gamma_i, u \) and \( k_i \) be as in Proposition \[39\]. Then there exists constants \( C = C(\Omega, n) > 0 \), \( q = q(\Omega, n) > 1 \) such that for any \( x \in \Gamma_i \) and \( 0 < r < R \) with \( B_R(x) \cap \partial \Omega \subseteq \Gamma_i \) (that is the ball of radius \( R \) does not intersect other connected parts of the boundary) one has

\[
\|(u - k_i)_+\|_{L^\infty(B_r(x) \cap \Omega)} \leq \frac{C}{(R - r)^q} \|(u - k_i)_+\|_{L^\infty(B_R(x) \cap \Omega)}.
\]

The same holds true for \((u - k_i)_-\) instead of \((u - k_i)_+\).

**Remark 41** (i) The proof of the lemma shows that \( 1 < q < \infty \) can be choosen freely, in which case \( C \) depends also on \( q \).

(ii) The Lemma is a local result and requires \( u \) to be \( n \)-harmonic only in a neighborhood of the ball \( B_R(x) \) and constant on \( \partial \Omega \cap B_R(x) \).

**Proof (Proposition 39).** By standard interior regularity, see for instance \[23\] Lemma 3.6, the estimate \[42\] holds true also for balls \( B_R(x) \) contained in \( \Omega \) (with \( k_i \) replaced by 0). Since \( \partial \Omega \) is smooth and compact and its connected components \( \Gamma_i \) have positive distance between eachother, we can cover the boundary by finitely many balls \( B_r(x), x \in \partial \Omega \) satisfying the estimate \[42\]. Summing up these estimates we obtain that there exists a constant \( C = C(\Omega) \) such that

\[
\|u\|_{L^\infty(\Omega)} \leq C \left( \|u\|_{L^\infty(\Omega)} + k \right).
\]

Choose a function \( g \in W^{1,n}(\Omega) \) such that \( g = k_i \) on \( \Gamma_i \) (take for instance a bounded linear extension operator \( T : W^{1-1/n,n}(\partial \Omega) \to W^{1,n}(\Omega) \) ) so that

\[
\|g\|_{W^{1,n}(\Omega)} \leq C(\Omega)k.
\]

It follows from Poincaré inequality \((u - g = 0 \text{ on } \partial \Omega)\) and using \[11\] that

\[
\|u - g\|_{L^\infty(\Omega)} \leq C\|\nabla u - \nabla g\|_{L^\infty(\Omega)} \leq C \left( \|\nabla u\|_{L^\infty(\Omega)} + \|\nabla g\|_{L^\infty(\Omega)} \right) \leq 2C\|\nabla g\|_{L^\infty(\Omega)} \leq 2Ck
\]

From this we obtain the desired estimate for \( \|u\|_{L^\infty} \) which we plug into \[43\] to conclude. \( \blacksquare \)
We now prove Lemma \[40\] by a standard Moser iteration method.

**Proof (Lemma \[40\])** Since the Lemma is a local result and $\Delta_n (u - k_i) = 0$, we can assume without loss of generality that $k_i = 0$ and $u = 0$ on $\Gamma_i$. The estimate \[42\] is clearly also valid for $(u - k_i)_+$ instead of $(u - k_i)_+$, by applying a posteriori the Lemma \[41\] to $-u$.

**Step 1.** We shall first prove that if $\chi > 1$ and $n \geq n$. Then there exists a constant $C = C(\Omega, n, \chi) > 0$ such that for any $x \in \Gamma_i$ and $0 < t < T$ such that $B_T (x) \cap \partial \Omega \subset \Gamma_i$

$$
\|u\|_{L^\gamma (B_t(x) \cap \Omega)} \leq \left( \frac{C \gamma}{T - t} \right)^{\frac{n}{\gamma}} \|u\|_{L^\gamma (B_T (x) \cap \Omega)},
$$

provided the right hand side is finite. We fix and write henceforth for simplicity

$$
B_T (x) = B_T, \quad B_t (x) = B_t \quad \text{and} \quad \Omega_T = \Omega \cap B_T (x), \quad \Omega_i = \Omega \cap B_t (x)
$$

**Step 1.1.** By assumption we have that

$$
\int_{\Omega} \langle \nabla u \rangle_n, \nabla \varphi \rangle = 0 \quad \text{for all } \varphi \in W^{1, n}_0 (\Omega).
$$

Let $u_+ = \max \{0, u\}$. Let $\xi \in C^1_c (B_T)$. We define for $m \in \mathbb{N}$ and $\beta \geq n - 1$

$$
u_m (y) = \begin{cases} 
  u_+ (y) & \text{if } u(y) < m, \\
  m & \text{if } u(y) \geq m,
\end{cases}
$$

and $\varphi = \xi u_m u \in W^{1, n}_0 (\Omega)$, which we plug into \[43\]. Deriving we obtain

$$
\nabla \varphi = \xi^n u_m \nabla u + \beta \xi^n u_m^{\beta - 1} u \nabla u_m + n \xi^{n - 1} u_m^{\beta} u \nabla \xi.
$$

This gives that

$$
\int_{\Omega_T} \xi^n u_m \nabla u + \beta \int_{\Omega_T} \xi^n u_m^{\beta - 1} u \nabla u_m \nabla u |\nabla u|^{n - 2} = -n \int_{\Omega_T} \xi^{n - 1} u_m^{\beta} u (\nabla \xi, \nabla u) |\nabla u|^{n - 2}.
$$

In the second term $\nabla u_m$ appears, which is zero if $u \geq m$, and therefore we can replace everywhere $u$ by $u_m$ to get

$$
\int_{\Omega_T} \xi^n u_m \nabla u + \beta \int_{\Omega_T} \xi^n u_m^{\beta} |\nabla u_m| |\nabla u_m|^{n - 2} = -n \int_{\Omega_T} \xi^{n - 1} u_m^{\beta} u (\nabla \xi, \nabla u) |\nabla u|^{n - 2}.
$$

Take $\epsilon > 0$, which we will soon chose appropriately. We use the estimate

$$
ab \leq \frac{(n - 1) a^{n/(n - 1)}}{n} + \frac{b^n}{n},
$$

with $a = \xi^{n - 1} |\nabla u|^{n - 1} u_m^{(\beta - 1)(n - 1)/n} \epsilon$ and $b = u_m^{\beta / n} |\nabla \xi| / \epsilon$ to estimate the right side of \[43\]

$$
-n \int_{\Omega_T} \xi^{n - 1} u_m^{\beta} u (\nabla \xi, \nabla u) |\nabla u|^{n - 2} \leq (n - 1) \int_{\Omega_T} \xi^n u_m^{\beta} |\nabla u|^{n - 2} + \int_{\Omega_T} u_m^{\beta} u |\nabla \xi|^{1 / \epsilon^n}.
$$

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So for an appropriate choice of $\epsilon = \epsilon(n)$ we obtain a constant $C = C(n)$ such that

$$
\int_{\Omega_T} \xi^n u_m^\beta |\nabla u|^n + \beta \int_{\Omega_T} \xi^n u_m^\beta |\nabla u_m|^n \leq C \int_{\Omega_T} u_m^\beta u^n |\nabla \xi|^n. \quad (47)
$$

We now set

$$
w = u_m^{\frac{\beta}{n}} u, \quad \Rightarrow \quad \nabla w = \frac{\beta}{n} u_m^{\frac{\beta}{n} - 1} \nabla u_m u + u_m \nabla u.
$$

Recall that if in any product $\nabla u_m$ appears, then $u = u_m$ in that product, so we obtain

$$
|\nabla w|^n \leq \left( \frac{\beta}{n} \frac{\beta^2}{n} |\nabla u_m| + u_m |\nabla u| \right)^n \leq 2^{n-1} \left( \left( \frac{\beta}{n} \frac{\beta^2}{n} u_m^\beta |\nabla u_m|^n + u_m^\beta |\nabla u|^n \right) \right)
$$

$$
= 2^{n-1} \left( \frac{\beta}{n} \frac{\beta^2}{n} |\nabla u_m|^n + u_m^\beta |\nabla u|^n \right)
$$

$$
\leq 2^{n-1} \left( 1 + \frac{\beta}{n} \frac{\beta^2}{n} \right) (\beta u_m^\beta |\nabla u_m|^n + u_m^\beta |\nabla u|^n)
$$

We use this in (47) to obtain that, renaming $2^{n-1} C$ by $C$ (as we will do henceforth)

$$
\int_{\Omega_T} \xi^n |\nabla w|^n \leq C \left( 1 + \frac{\beta}{n} \frac{\beta^2}{n} \right) \int_{\Omega_T} w^n |\nabla \xi|^n,
$$

Now use that

$$
|\nabla (\xi w)|^n \leq (|\xi| |\nabla w| + |\nabla \xi| |w|)^n \leq 2^{n-1} (|\xi|^n |\nabla w|^n + |\nabla \xi|^n |w|)^n.
$$

We thus get that

$$
\int_{\Omega_T} |\nabla (\xi w)|^n \leq C \left( 1 + \frac{\beta}{n} \frac{\beta^2}{n} \right) \int_{\Omega_T} w^n |\nabla \xi|^n, \quad (48)
$$

Note that $\xi w \in W_0^{1,n}(\Omega)$ and that $W_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 \leq q < \infty$ by the Sobolev embedding. In particular there exists a constant $C = C(\chi, \Omega, n)$ such that

$$
\left( \int_{\Omega} |\xi w|^n \right)^{\frac{1}{n}} \leq C(\chi, \Omega, n) \int_{\Omega} |\nabla (\xi w)|^n = C(\chi, \Omega, n) \int_{\Omega_T} |\nabla (\xi w)|^n.
$$

So we obtain from (48)

$$
\left( \int_{\Omega_T} |\xi w|^n \right)^{\frac{1}{n}} \leq C \left( 1 + \frac{\beta}{n} \frac{\beta^2}{n} \right) \int_{\Omega_T} w^n |\nabla \xi|^n
$$

We now choose $\xi$ such that $\xi = 1$ in $B_t$ and $|\nabla \xi| \leq 2/(T-t)$. We also assume that $u$ has been extende by zero outside of $\Omega$ (more precisely on the other side of $\Gamma_i$). Thus, using the definition of $w = u_m^{\beta/n} u$ we get

$$
\left( \int_{B_t} \left( u_m^{\beta/n} u \right)^n \right)^{\frac{1}{n}} \leq C \left( 1 + \frac{\beta}{n} \frac{\beta^2}{n} \right) \frac{2^n}{(T-t)^n} \int_{B_T} u_m^\beta u^n \leq C \left( 1 + \frac{\beta}{n} \frac{\beta^2}{n} \right) \frac{2^n}{(T-t)^n} \int_{B_T} u_m^{\beta+n}
$$
We now let $m \to \infty$ and obtain that
\[
\left( \int_{B_t} u_{+}^{\gamma(n+1)} \right)^{\frac{1}{n}} \leq C \left( 1 + \frac{\beta n^{-1}}{n^{n}} \right) \frac{2n}{(T-t)^{n}} \int_{B_T} u_{+}^{\beta n}.
\]
Set now $\gamma = \beta + n$ and we get that
\[
\left( \int_{B_t} u_{+}^{\gamma} \right)^{\frac{1}{n}} \leq C \left( 1 + \frac{(\gamma - n)n^{-1}}{n^{n}} \right) \frac{1}{(T-t)^{n}} \int_{B_T} u_{+}^{\gamma} \leq C^{\gamma} \frac{1}{(T-t)^{n}} \int_{B_T} u_{+}^{\gamma}.
\]
From this follows.

**Step 2.** We now use (44) iteratively. Set
\[
t_j = r + \frac{R-r}{2^j}, \quad \gamma_j = \chi^j n \quad \text{for} \quad j = 0, 1, 2, \ldots.
\]
Note that $t_0 = R$, $t_j > t_{j+1}$ and $\gamma_0 = n$, $\gamma_{j+1} = \chi \gamma_j$. Moreover $t_j - t_{j+1} = (R-r)/2^{j+1}$. So it follows from (44) (by induction the right hand side of the next inequality is bounded for each $j$) that
\[
\| u_{+} \|_{L^{\gamma_j}(B_{t_{j+1}})} \leq \left( \frac{C^{\gamma_j}}{R-r} \right)^{\frac{1}{\gamma_j}} \frac{2^{j+1}}{n} \| u_{+} \|_{L^{\gamma_j}(B_{t_j})}.
\]

Note that
\[
\left( \frac{C^{\gamma_j}}{R-r} \right)^{\frac{1}{\gamma_j}} \frac{2^{j+1}}{n} \leq D \left( R-r \right)^{-\frac{1}{\frac{\gamma_j}{n}}},
\]
for some constant $D = D(\Omega, n, \chi)$. We now set $a_j = \| u_{+} \|_{L^{\gamma_j}(B_{t_j})}$ and notice $a_0 = \| u_{+} \|_{L^{n}(B_R)}$. Thus we obtain from (49) that
\[
a_{j+1} \leq D \left( R-r \right)^{-\frac{1}{\frac{\gamma_j}{n}}} a_j.
\]
It follows by induction that
\[
a_{j+1} \leq D \sum_{l=0}^{j} \left( R-r \right)^{-\frac{l}{\gamma_j}} a_0.
\]
Notice that for all $j$ we have $\| u_{+} \|_{L^{\gamma_j}(B_r)} \leq a_{j+1}$. The sums
\[
\sum_{l=0}^{j} \frac{l}{\chi^l} \quad \text{and} \quad \sum_{l=0}^{j} \frac{1}{\chi^l}
\]
are convergent. So letting $j \to \infty$ we obtain that
\[
\| u_{+} \|_{L^{\infty}(B_r)} \leq C \frac{1}{(R-r)^{\delta}} \| u_{+} \|_{L^{n}(B_R)},
\]

which proves the lemma.

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