Quantum noise enhanced extreme events in nonlinear optics

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Extreme events and rogue waves are a subject of interest during the last decade. They are especially fascinating because they can lead to catastrophic changes in the system despite being quite rare. However, their probability is still much higher than one expects from the normal or exponential distributions. A signature of rogue-wave behavior is a ‘heavy tail’ of the probability distribution. The latter is observed in a plethora of phenomena, especially in nonlinear optical effects. Here we show that by pumping these effects with strongly amplified quantum noise such as bright squeezed vacuum, one obtains a huge enhancement of rogue-wave behavior. We generate tremendously fluctuating light through two nonlinear processes: optical harmonics and supercontinuum generation. We present a simple description of the phenomena in terms of heavy-tail characteristics of probability distributions and normalized intensity correlation functions. In particular, for supercontinuum generated from bright squeezed vacuum we observe power-law (Pareto) probability distributions having unprecedentedly low tail exponents (0.3), leading to a non-existing mean value as well as higher moments.

I. INTRODUCTION

Extreme events, or ‘rogue waves’, are a subject of active research in many fields of physics. A distinguishing feature of a rogue wave is that its amplitude is almost impossible under Gaussian statistics with the same mean energy. This effect has many facets and manifestations, from waves in ocean to optical pulses [1–8]. In optics, rogue-wave phenomena has been observed for mode-locked lasers, delayed feedback in nonlinear optics, laser filamentation and most frequently, for supercontinuum generation in optical fibers [9]. The study of such effects and their dynamics, apart of being fundamentally interesting, is important for preventing them in cases where they are destructive. On the other hand, these effects can have useful applications, some of which we discuss in this work.

Here we show that extreme events appear in nonlinear optical effects pumped by strongly fluctuating quantum light, i.e. bright squeezed vacuum (BSV) [10]. This state of light can be produced from a strongly pumped unseeded parametric amplifier, and can reach a brightness comparable with lasers (more than 10^{13} photons per mode) [11, 12]. Despite the common opinion that large intensity noise is detrimental for any applications, these amplified quantum fluctuations turn out to be very useful for pumping multiphoton effects. Indeed, multiphoton absorption [13–15], multiphoton ionization [16, 17], generation of optical harmonics [18–20], and modulation instability (MI) [21] are hugely enhanced if pumped by fluctuating light. In our experiments we use BSV to pump the generation of optical harmonics and supercontinuum. In both processes, we observe much more ‘extreme’ behavior than in any of the cases described in the literature.

Extreme events are characterized by heavy tails of intensity probability distributions. We observe these distributions for the second and third harmonics of BSV and show that they are heavy-tailed but have finite moments. Meanwhile, supercontinuum generation from BSV leads to a heavy tailed Pareto-like intensity distribution with indefinite moments. The latter means an arbitrary large bunching parameter \( g^{(2)} = g^{(2)}(0) \), normalized second-order correlation function (CF) at zero delay [22]. For supercontinuum generated from BSV, we therefore measure unprecedentedly high values of \( g^{(2)} \).

The paper is organized as follows. In Section II, we briefly describe the statistics of BSV. Section III studies the statistics of the second and third optical

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harmonics generated from BSV and demonstrates the existence of extreme events and heavy-tailed photon-number distributions. Section IV deals with the supercontinuum generation from BSV. The obtained photon-number distributions show extremely heavy tails and very high $g^{(2)}$ values. In Section V, we discuss some applications of bright light with such intensity fluctuations. Finally, Section VI concludes the paper. Detailed analysis of the probability distributions, including the effects of noise and multiple modes, is moved to the Appendices.

II. BRIGHT SQUEEZED VACUUM AND ITS PHOTON STATISTICS

Squeezed vacuum can be generated via parametric down-conversion (PDC) or four-wave mixing (FWM). The number of photons in this state is

$$N = \sinh^2 G,$$

where the parametric gain $G$ scales as the pump field amplitude for PDC and squared amplitude for FWM. In the case of weak pumping, squeezed vacuum is known as light consisting of photon pairs. But under strong pumping, the parametric gain can reach values $G \gg 1$ [23]. In this case the output light has high brightness, although optical quadratures have zero mean values. It is therefore called bright squeezed vacuum (BSV) [10].

In addition, the photon number has large fluctuations. If BSV is generated as two distinguishable beams (signal and idler), each beam taken separately is in a thermal state. The photon number (intensity) is distributed exponentially [24]:

$$P_{th}(N) = \frac{1}{\langle N \rangle} e^{-\frac{N}{\langle N \rangle}}.$$

where $\langle N \rangle$ is the mean photon number (Fig. 1a, black line). Its variance $\text{Var}(N) = \langle N \rangle^2$ [25] is much larger than for coherent light, for which the photon-number distribution is Poissonian (Gaussian at $\langle N \rangle \gg 1$).

The way a probability distribution decays, whether it is heavy-tailed or not, can be analyzed with the help of the complementary cumulative distribution function (CCDF, see Appendix A). The CCDF analysis shows that thermal distribution (2) has a heavier tail and so is not tail equivalent [26] to any Gaussian distribution. For single-mode thermal light, the second-order normalized CF of order $m$, defined as $g^{(m)} = \langle N^m \rangle / \langle N \rangle^m$ [27], is $g^{(m)}_th = m!$. It is larger than for coherent light, for which $g^{(m)} = 1$; this phenomenon is usually referred to as ‘photon bunching’.

For degenerate BSV, where the signal and idler beams coincide, the photon-number distribution is even broader than in the thermal case. Here we will ignore its fine structure, related to the fact that BSV contains only even photon numbers. The envelope of the photon-number distribution is [28, 29]

$$P_{sb}(N) = \frac{1}{\sqrt{2\pi\langle N \rangle N}} e^{-\frac{N}{2\langle N \rangle}}.$$

This is a Gamma distribution with a shape parameter 1/2 and a scale parameter 1/(2$\langle N \rangle$). Its tail is heavier (see Appendix A and Fig. 1b, black solid
line) and the variance is twice larger than in the case of thermal light (2), Var\((N) = 2\langle N\rangle^2\). The normalized CFs are also higher than for thermal light, \(g_{sb}^{(m)} = (2m - 1)!!\). For this reason, degenerate BSV is often called 'superbunched light' [23, 30, 31] as it has even larger photon fluctuations than thermal 'bunched light'. However, neither distribution (2) nor distribution (3) can be strictly called a heavy-tailed distribution (see Appendix A).

Experimentally, we produce superbunched BSV at 1600 nm and thermal light at 1585 nm via PDC in a 10 mm BBO crystal with type-I phase-matching in collinear degenerate and non-degenerate regime, respectively (Fig. 2) [12]. As a pump we use a Ti:sapphire laser at a wavelength of 800 nm, with 1.6 ps pulse duration, 5 kHz repetition rate, and an energy per pulse of 0.5 mJ. After cutting off the pump with a dichroic mirror (DM), we measure the probability distributions using a p-i-n diode-based infrared charge-integrating photodetector (IR PD), whose signal scales linearly with the number of photons in the pulse [20]. The photodetector has neither spatial nor temporal resolution within a pulse. In order to trace the intensity fluctuations, PDC is filtered spatially with a slit and spectrally with a 4\(f\) monochromator (4f) down to a single mode [20]. When taking into account the noise distribution (see Appendix B) of our photodetector, the experimentally obtained photon-number histograms for thermal and superbunched light (Fig. 1, red points) perfectly agree with the theoretical probability distributions [(2), (3)] with \(\langle N_\omega\rangle\) taken from experimental data. The corresponding values of \(g^{(2)}\) and \(\langle N_\omega\rangle\) are listed in Table I.

In the next two sections, we use BSV, both nondegenerate (thermal) and degenerate (superbunched) for pumping nonlinear optical effects and study the statistics of light at the output.

### III. OPTICAL HARMONIC GENERATION

In the absence of pump depletion, for the \(n\)th harmonic the number of generated photons \(N_{n\omega}\) scales as the \(n\)th power of the number of photons in the fundamental radiation,

\[
N_{n\omega} = KN_{\omega}^{n},
\]

where \(K\) is related to the conversion efficiency.

Using relation (4), the distribution \(P_{n\omega}(N_{n\omega})\) for the harmonic radiation can be obtained from the distribution of the fundamental \(P_{\omega}(N_{\omega})\) [19, 28, 32] as

\[
P_{n\omega}(N_{n\omega}) = P_{\omega}\left(\frac{\sqrt{N_{n\omega}}/K}{n\sqrt{KN_{\omega}^{1-1/n}}}\right)^n.
\]

It follows that the \(n\)th moment of the harmonics is proportional to the \(mn\)th moment of the fundamental radiation, \(\langle N_{n\omega}^m \rangle = K^n\langle N_{\omega}^{mn} \rangle\). As a result, the CFs \(g_{n\omega}^{(m)}\) can be obtained from the fundamental radiation CFs [19, 20, 33].

\[
g_{n\omega}^{(m)} = g_{\omega}^{(mn)}\left(\frac{g_{\omega}^{(n)}}{g_{\omega}^{(1)}}\right)^m.
\]

In the simplest case of a bright coherent light pump, \(P_{\omega}(N_{\omega})\) is a Gaussian distribution. Since it decreases significantly faster than the exponential

| \(g^{(2)}\) | \(\langle N_\omega\rangle\), photons/pulse |
|----------------|----------------------------------|
| thermal light  | 2.00 ± 0.02 | 1.33 × 10^5 |
| BSV           | 3.02 ± 0.04 | 7.6 × 10^4 |
| 2\(\omega\) from thermal | 6.08 ± 0.07 | 1.15 × 10^5 |
| 2\(\omega\) from BSV   | 11.4 ± 0.2  | 1.59 × 10^5 |
| 3\(\omega\) from BSV   | 33.1 ± 1.6  | 1.08 × 10^5 |

Table I. Measured characteristics of the obtained thermal light, superbunched BSV, and the second and third harmonics generated from them.
distribution (2) for the same mean photon number, the probability of extreme events for the harmonics remains negligible.

The situation is very different for input light with larger photon-number fluctuations [(2), (3)]. For the \( n \)th harmonic of thermal light, we get the photon-number distribution

\[
P_{n\omega}(N_{\omega}) = \frac{\sqrt{n}!}{n \sqrt{(N_{\omega})} N_{\omega}^{1-1/n}} e^{-\sqrt{n}! \frac{N_{\omega}}{N_{\omega}}}, \tag{7}
\]

while pumping with superbunched BSV gives

\[
P_{n\omega}(N_{\omega}) = \frac{\sqrt{2n-1}!}{n \sqrt{2\pi} \sqrt{(N_{\omega})} N_{\omega}^{1-1/2n}} e^{-\frac{1}{2} \sqrt{2n-1}! \frac{N_{\omega}}{N_{\omega}}}. \tag{8}
\]

Distribution (7) is a Weibull distribution [34] with the shape parameter \( 1/n \) and the scale parameter \( \langle N_{\omega} \rangle/n! \). Distribution (8) is a generalized Gamma distribution [35] with the shape parameters \( 1/2 \) and \( 1/n \) and the scale parameter \( 2^n \langle N_{\omega} \rangle/(2n-1)! \). Both (7) and (8) are heavy-tailed distributions, according to the definition in Ref. [26]. However, all their moments are finite. The use of the intensity CF to analyze such distributions is therefore fully justified.

Experimentally, superbunched and thermal states presented in Section II are used as pumps to generate the second and third optical harmonics. The harmonics are produced in a 1 mm LiNbO\(_3\) crystal without phase matching [20], see Fig. 2. After the separation from the fundamental radiation using a short-pass filter (SP), they are detected by a visible-range charge-integrating photodetector (PD) [36]. The histograms for the measured harmonics are shown in Fig. 3 by red points. They are in perfect agreement with the theoretical distributions (solid lines) taking into account the noise distribution (see Appendix B). On a semi-logarithmic scale, they demonstrate large deviation from the straight line scaling of thermal light (Fig. 3, lower insets, dashed black lines). They show enhanced photon-number fluctuations, the \( g^{(2)}_{\omega\omega} \) values for the second harmonic (Table 1) are in agreement with Eq. (6) [37].

The top right insets in Fig. 3 show 0.1 s time traces containing pulses with the highest photon numbers \( N_{\omega\max} \) detected within a 100 s of total measurement, the probability of such events being about \( 10^{-6} \). The numbers \( N_{\omega\max} \) exceed the mean values \( \langle N_{\omega} \rangle \) by more than two orders of magnitude and differ from them by more than 40 standard deviations. This behavior is especially pronounced for the third harmonic (Fig. 3c), \( N_{3\omega\max} = 675 \langle N_{3\omega} \rangle \). For comparison, the probability of events with \( N > 670 \langle N \rangle \) for thermal light (2) is less than \( 10^{-290} \).
IV. SUPERCONTINUUM GENERATION

Even more pronounced are extreme events in supercontinuum generation, usually observed with coherent pumping [1, 9]. A fluctuating pump can enhance the rogue-wave behavior; such enhancement was observed in seeded MI with partially incoherent pump [3, 21]. Here we describe the unseeded supercontinuum generation with strongly fluctuating pump, i.e. thermal and superbunched BSV.

We suppose, as it is commonly accepted, that the elementary physical process behind the supercontinuum generation is FWM. Assuming, as above, that the pump is single-mode and undepleted, the gain \( G \) in Eq. (1) is in this case

\[
G = \kappa N_p.
\]

Here \( N_p \) is the pump photon number and \( \kappa \) characterizes the interaction strength.

Similar to the harmonics case, from the pump photon-number distribution \( P(N_p) \) we obtain the FWM photon-number distribution. We get

\[
P_{\text{FWM}}(N) = \frac{e^{-\arcsinh N/\kappa(N_p)}}{2\kappa(N_p)\sqrt{N(1+N)}} \tag{9}
\]

for thermal pumping and

\[
P_{\text{FWM}}(N) = \frac{e^{-\arcsinh N/2\kappa(N_p)}}{\sqrt{8\pi\kappa(N_p)N(1+N)\arcsinh N}} \tag{10}
\]

for superbunched BSV pumping (Fig. 4). Here \( \langle N_p \rangle \) is the mean pump photon number.

As shown in Appendix A, distributions (9) and (10) are very different from the ones for optical harmonics. For large \( N \), both distributions have asymptotic scaling typical for the Pareto distribution [26]

\[
P(N) \sim \frac{1}{N^{1+k}} \tag{11}
\]

with the tail exponent (Pareto index) \( k > 0 \). The latter tends to zero for both distributions at \( \kappa \langle N_p \rangle \gg 1 \) (see Appendix A). In this case, all moments of the distribution are indefinite. Therefore a statistical description based on them is not applicable, contrary to the harmonic case. Hence, strictly speaking we cannot use correlation functions defined by the moments. However, experimentally the distributions are truncated due to the finite time of measurement as well as the saturation of the photodetectors.

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In experiment, we generate a supercontinuum in a 5 m single-mode fused silica fiber from superbunched and thermal BSV (Fig. 5). Both are produced via PDC in two 3 mm BBO crystals pumped by frequency doubled radiation (400 nm) from the same laser as in Section III. After the pump is cut off with a dichroic mirror (DM), BSV is filtered by a bandpass filter (BP) and coupled into the single-mode fiber (SMF). The resulting BSV light is therefore spatially single mode due to the filtering realized by the fiber, but contains a few spectral modes which depends on the BP filter bandwidth. At the fiber output, the supercontinuum radiation is measured by a CCD array spectrometer (Avantes ULS3648) or by a visible charge-integrating photodetector (PD) [36] after 1 nm filtering required for single-mode detection.
Fig. 6 shows an average output spectrum obtained under pumping with low-power (1 µW, black line) and high-power (200 µW, red line) thermal (a) and superbunched (b) BSV. Under stronger pumping, the spectrum shows considerable broadening with respect to the initial spectral width. The latter corresponds to the weak pumping regime.

![Fig. 6. Spectra of supercontinuum pumped by thermal light (a) and superbunched BSV (b) at 1 µW (black) and 200 µW (red) power.](image)

Fig. 7, left panels, shows various single-pulse spectra of supercontinuum pumped by superbunched BSV. At high power (200 µW, bottom) the spectra are much broader than at moderate power (120 µW, top). Individual spectra differ considerably from each other because of independent intensity fluctuations at different wavelengths λ. These fluctuations are correlated; one can characterize them using the second-order CF

\[
g^{(2)}(\lambda, \lambda') = \frac{\langle S(\lambda)S(\lambda') \rangle}{\langle S(\lambda) \rangle \langle S(\lambda') \rangle}. \quad (12)
\]

Here \( S(\lambda) \) is the spectral density measured for a single pulse and the angular brackets denote averaging over many pulses. The resulting two-dimensional (2D) distributions of \( g^{(2)}(\lambda, \lambda') \) are shown in the right panels of Fig. 7.

One can see both the auto- (diagonal part of the distribution) and cross- (anti-diagonal) correlations for signal and idler intensities. As the BSV power increases, the correlations get more pronounced at wavelengths detuned further from the pump. The auto-CF in the anti-Stokes range displays the largest values. The CF values increase with the BSV pump power until a certain limit and decrease thereafter. The highest value is about 170, it exceeds all previously reported values [20, 38]. At the same time, in the central part of the 2D distribution, no correlations are visible and the CF (12) is close to unity. This behavior is also pronounced in the single-shot spectra; at wavelengths close to the pump, intensity fluctuations are suppressed. The range of this suppression gets broader as the pump gets stronger. It happens because near the pump wavelength the supercontinuum is so strong that it produces FWM sidebands. The latter involves pairwise annihilation of photons, which reduces the intensity fluctuations [39, 40].

At wavelengths where the fluctuations are strong, the photon-number distribution has a pronounced heavy-tail shape. Fig. 8a shows this distribution for the supercontinuum pumped by 240 µW of BSV and detected by PD at 770 nm. The distribution demonstrates a linear behavior on a log-log scale within two orders of magnitude. For \( N > 10^6 \) photons the photodetector exhibits saturation and modifies the distribution.

The experimental data are well fitted by the Pareto distribution (11). The tail exponents \( k_e \), reported in Table II, were determined from a fit of the CCDF rather than the probability distribution as it yields lower uncertainties, \( \Delta k_e = 0.02 \) (see Appendix A for details).

The estimation of the theoretical tail exponents \( k_l \)
FIG. 8. Photon-number histograms for the supercontinuum pumped by 240 μW (a) and 150 μW (b) and (c) of superbunched BSV with bandwidth Δλp = 10 nm (red points). The histograms with different BSV bandwidth (3 nm) and attenuation (57% loss) are shown by black points in panels (b) and (c) respectively. The Pareto fits are shown by blue lines. The inset in panel (a) shows 0.1 s time trace with the largest extreme event.

Table II. Characteristics of the supercontinuum photon-number distributions from Fig. 8. P and Δλp are pump power and its bandwidth; k_ε and k_t are experimental and theoretical tail exponents. k_ε are derived from a fit of the CCDFs corresponding to the distributions in Fig. 8. Despite k < 1, the mean number of photons ⟨N⟩ exists for the measured distributions due to the photodetector saturation.

| Fig. 8 | P, Δλp | k_ε | k_t | ⟨N⟩, photons/pulse |
|--------|--------|-----|-----|---------------------|
| (a)    | 240 μW, 10 nm | 0.49 | 0.31 | 1.18 × 10^4 |
| (b)    | 150 μW, 10 nm | 0.64 | 0.5  | 6.5 × 10^3 |
|        | 150 μW, 3 nm  | 0.31 | 0.2  | 5.6 × 10^2 |
| (c)    | 150 μW, 10 nm | 0.64 | 0.5  | 6.5 × 10^3 |
|        | with 57% loss  | 0.63 | 0.5  | 2.8 × 10^3 |

V. POSSIBLE APPLICATIONS OF STRONGLY FLUCTUATING BRIGHT LIGHT

We see that supercontinuum generated from superbunched light has giant values of the normalized CF, up to g^(2) = 170. From the Pareto scaling of the observed probability distribution, P(N) ~ N^-1.3, g^(2) is undefined and will depend on the measurement time. However it is experimentally limited by the saturation of the photodetector.

Ghost imaging (GI) is one of possible applications of light with high g^(2) values. In the simplest scenario of GI [41], a beam with classical (usually, thermal) intensity fluctuations is split on a beamsplitter and an object to be imaged is placed into one of the output beams. Typically, it is a mask transmitting light within an area A and blocking it elsewhere. Light passing through the object is registered by a ‘bucket’ photodetector, which does not resolve the image but integrates over its area. The output signal of this photodetector is therefore proportional to

\[ S_b = s \int_A I(\vec{r})d\vec{r}^2, \]  

(13)
where $s$ is the proportionality coefficient, $\vec{r}'$ is the coordinate in the object channel and $I$ is the intensity. The other (reference) photodetector is placed in the conjugated channel and it has spatial resolution. Its output signal depends on the intensity distribution, $S_r(\vec{r}) = \kappa I(\vec{r'})$, where $\vec{r}$ is the coordinate in the reference channel.

The image is retrieved by measuring the CF

$$G^{(2)}(\vec{r}) = \langle S_b S_r(\vec{r}) \rangle.$$  \hspace{1cm} (14)

If the mean intensity in both channels is the same and homogeneous, the contrast of the image is

$$R = 1 + (g^{(2)} - 1) \frac{a}{A},$$  \hspace{1cm} (15)

where $a$ is the area where the CF $g^{(2)}$ is considerably different from the background.

From Eq. (15), one can see that for light with high $g^{(2)}$, the contrast of the image is proportional to it. This is why the original version of GI with two-photon light [42] has a higher contrast than the one employing thermal light. However, the normalized CF for two-photon light scales with the mean photon number $\langle N \rangle$ as $g^{(2)}(\langle N \rangle) = 2 + 1/\langle N \rangle$; therefore, a high image contrast can be achieved only for very faint light.

Various protocols have been proposed to increase the contrast of GI with classical light, such as putting a threshold on the intensity [43] or using higher-order CFs [44, 45]. For instance, in the latter case the increase in the contrast is given by the order $n$ of the CF. Meanwhile, the above reported values of $g^{(2)} > 170$ promise a much higher contrast of GI than any techniques reported in the literature so far.

It should be mentioned that all protocols of GI involving higher CF values have the disadvantage that the noise of the image is higher as well; therefore, despite a high contrast, the signal-to-noise ratio (SNR) is often the same as in the simplest scenario [41]. The SNR for GI with the supercontinuum discussed in our work has still to be investigated.

In addition, following an approach in terms of quantum Fisher information as in Ref. [46], one can conclude that light with a heavy-tail distribution and indefinite moments (11) might be promising for quantum illumination.

**Photon subtraction experiments** are discussed in the literature in connection with the implementation of quantum gates [47–50]. In such an experiment, a very weakly reflecting beamsplitter is placed into a beam of light, and the state in the transmitted path is analyzed conditioned on a single photon detection in the reflected path. If the initial state is $|\Psi\rangle$, the output state is then $|\Psi'\rangle = a|\Psi\rangle$, where $a$ is the photon annihilation operator.

Recently, it has been noted [51] that subtraction of a photon from a single-mode state of light with the mean photon number $\langle N \rangle$ creates a state with the mean photon number $\langle N' \rangle = g^{(2)}(\langle N \rangle)$. For this reason, subtraction of a photon from thermal light increases its mean photon number by a factor of two. According to our results, subtraction of a single photon from the supercontinuum pumped by superbunched BSV will increase the mean number of photons drastically, at least by a factor of 170 according to our measurements. This can be considered as the heralded preparation of a state orders of magnitude brighter than the initial one. According to the analysis of Ref. [52], the resulting state will be also extremely non-equilibrium and can be considered as an interesting resource for quantum communication and quantum thermodynamics [53].

**VI. CONCLUSION**

We have shown, theoretically and experimentally, that the generation of optical harmonics and supercontinuum from high-gain parametric down-conversion leads to a rogue-wave behavior and the creation of extreme events. This behavior is revealed by a heavy tail of the photon-number distribution. Especially striking in this case is the statistics of supercontinuum, featuring a Pareto probability distribution with the tail exponent as small as 0.3. Such a distribution does not have any finite statistical moments unless some truncation is made. In our experiment, the truncation naturally occurs due to the photodetector saturation. Still, we have measured a record value of $g^{(2)} = 170$. From the viewpoint of extreme events, we observe pulses with photon numbers exceeding the mean value by a factor of 675, which corresponds to almost 120 standard deviations. For comparison, in the case of
thermal statistics the probability of such an event is less than $10^{-290}$. For both harmonics and super-continuum generation, the rogue-wave behavior is more pronounced for pumping with superbunched BSV, whose photon-number fluctuations are much stronger than for thermal light.

It should be stressed that, although classical intensity fluctuations could be, in principle, created by means of an amplitude modulator [54], the fluctuations we discuss here are ultrafast and cannot be mimicked this way. In our experiment, typical fluctuation times are about 1 ps but they can be further reduced by using BSV with a larger bandwidth [20]. This promises interesting applications. One of them, as mentioned above, is pumping nonlinear effects. For instance, the supercontinuum radiation with $g^{(2)} = 170$ will be 170 times more efficient for second harmonic generation, FWM, or any other two-photon effect, than coherent light [20]. Other possible applications for such fluctuating light, as described in the last section, are ghost imaging, for which the value of $g^{(2)}$ determines the contrast of the image, and also photon subtraction experiments, with a considerable increase in the light brightness and some prospects for testing quantum thermodynamics of unstable systems.

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Appendix A: Complementary cumulative distribution function and tails of probability distributions

To characterize the tail of a probability distribution, one can use the complementary cumulative distribution function (CCDF) [34],

$$\tilde{C}(N) = \int_{N}^{\infty} P(N')dN'.$$  \hspace{1cm} (A1)

It is also known as the survival or reliability function and specifies the tail distribution. The analysis with CCDF is valid even if the moments diverge.

The tails of two distributions can be compared using the limit [34]

$$L_{12} = \lim_{N \to \infty} \frac{\tilde{C}_1(N)}{\tilde{C}_2(N)}.$$  \hspace{1cm} (A2)

If $L_{12} = 1$, the distributions are tail equivalent; if $L_{12}$ is diverging/equal to zero, then $\tilde{C}_1(N)$ has a heavier/lighter tail than $\tilde{C}_2(N)$.

The tail index of a distribution is defined as

$$\alpha = \lim_{N \to \infty} \frac{H(N)}{N},$$  \hspace{1cm} (A3)

where $H(N) = -\log \tilde{C}(N)$ is called the hazard function. If $\alpha = \text{const}$, the distribution decays exponentially. For diverging $\alpha$ the tail decays faster than exponential, for $\alpha = 0$ slower. The latter means a heavy-tailed distribution.

For thermal light (2), the CCDF is

$$\tilde{C}_{th}(N) = e^{-\frac{N}{\langle N \rangle}},$$  \hspace{1cm} (A4)

for superbunched BSV (3),

$$\tilde{C}_{sb}(N) = \text{Erfc} \left[ \sqrt{\frac{N}{2\langle N \rangle}} \right],$$  \hspace{1cm} (A5)

where Erfc is the complementary error function. Their tail indexes $\alpha$ are equal to $1/\langle N \rangle$ and $1/(2\langle N \rangle)$, respectively. Hence both distributions are not heavy-tailed. The experimental $H(N)/N$ curves fully demonstrate this behavior (Fig. 9); at large $N$ they tend to the corresponding theoretical $\alpha$. The thermal (black points and gray line) and BSV (red points and magenta line) distributions have the same tail index if the mean for the first one is twice as large as for the second one.
In the case of FWM, the CCDF is

$$\tilde{C}_{FWM}(N) = e^{-\frac{\text{arcsinh}\sqrt{N}}{N\langle N_p \rangle}} \quad (A8)$$

for the thermal pumping (9) and

$$\tilde{C}_{FWM}(N) = \text{Erfc} \left[ \frac{\text{arcsinh}\sqrt{N}}{2\kappa\langle N_p \rangle} \right] \quad (A9)$$

for the BSV superbunched pump (10). Similarly to the case of harmonics, $\alpha$ tends to zero, the distributions are heavy-tailed. However, they are not tail-equivalent to the ones for harmonics [(A6) and (A7)]. The experimental $H(N)/N$ (green points in Fig. 9) exhibits a faster tendency to zero than for any harmonic.

The CCDFs (A8) and (A9) are of the form $N^{-k}L(N)$, where $k$ is the tail exponent and $L(N)$ is a slowly varying function (i.e., $\lim_{N \to \infty} L(tN)/L(N) = 1$, for any $t > 1$ [26]). Therefore distributions (9) and (10) belong to regularly varying distributions with a finite tail exponent. They both are tail equivalent to the Pareto distribution with the same $k$. The tail exponents tend to $1/(2\kappa\langle N_p \rangle)$ for (9) and $1/(4\kappa\langle N_p \rangle)$ for (10). In other words, BSV pumping leads to a smaller $k$ than pumping with thermal light with the same power. In both cases $k$ goes below unity for a sufficiently large pump and then, all moments are indefinite. The first scaling is straightforward, the second one has been found using asymptotics $\text{Erfc}[z] \sim \exp(-z^2)/(\sqrt{\pi}z)$, where the numerator has the tail exponent $1/(4\kappa\langle N_p \rangle)$ and the denominator asymptotics is proportional to $\sqrt{\log N}$. From this analysis, it follows that any attenuation does not affect the tail exponent as shown in Fig. 8c and Table II.

Although a rigorous description of supercontinuum generation is not possible in our case, the FWM description [(A8) and (A9); (C5) and (C6)] explains the Pareto scaling. The experimental CCDF demonstrates this scaling within two orders of magnitude in $N$ (Fig. 10b). The corresponding fit gives $k = 0.49$ (see Table II), i.e., a value well below unity. It is to be noted that this value is well defined and is robust within a fit range between $10^4$ and $8 \times 10^5$ photons per pulse, avoiding the parts of the distribution affected by noise and saturation.
At high photon numbers the scaling fails due to the photodetector saturation, which truncates the distribution for $N > 10^6$ photons. At low photon numbers, the distribution is affected by the photodetector noise.

**Appendix B: Taking photodetector noise into account**

As one can see from Figs. 1 and 11a,b, the experimental histograms deviate from the theoretical distributions [(2) and (3)] at low photon numbers.

Our infrared photodetector has a noise distribution shown in Fig. 11c with the standard deviation $\sigma$ corresponding to 1600 photons per pulse. This noise is well described by the Gaussian distribution with zero mean

$$P(N) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{N^2}{2\sigma^2}}. \quad (B1)$$

The visible photodetector has the same distribution, but with $\sigma = 270$ photons per pulse. This noise is independent from the fluctuations of the detected light. Therefore the noise fluctuations just add to the light noise.

Since the probability distribution of a sum of two independent variables is given by a convolution of their probability distributions, the detected light should be described by the convolution of light and noise probability distributions. The convolution of Eq. (B1) with Eqs. (2) or (3) perfectly coincides with the experimental histograms for thermal light and superbunched BSV (Figs. 1a,b and 11a,b). Also for generated harmonics (Fig. 3) the convolution provides perfect agreement.

Note that all lines in Figs. 1, 3, and 11 are not fits, they are exact theoretical distributions or their convolutions with $\langle N \rangle$ (or $\sigma$ for the noise) taken from the experimental data (Table I).
Appendix C: Effect of multimode pump

In the presence of multiple modes, the photon-number probability distribution for most states of light changes.

For $M$-mode thermal light, the probability distribution is

$$P_{th,M}(N) = \frac{N^{M-1}}{(M-1)!} \left( \frac{M}{\langle N \rangle} \right)^M e^{-\frac{MN}{\langle N \rangle}}$$

(C1)

where $\langle N \rangle$ is the total mean number of photons for all modes. The distribution approaches the Gaussian distribution for a large $M$. Superbunched BSV with $M$ modes has

$$P_{th,M}(N) = \frac{N^{M/2-1}}{\Gamma\left(\frac{M}{2}\right)} \left( \frac{M}{2\langle N \rangle} \right)^{M/2} e^{-\frac{MN}{\langle N \rangle}}$$

(C2)

where $\Gamma(x)$ is the gamma function.

Similarly to the single-mode case, from the pump multimode distributions (C1) and (C2) we obtain FWM photon-number distributions. We get

$$P_{FWM,M}(N) = \frac{e^{-\frac{\text{arcsinh}\sqrt{N}}{\kappa\langle N_p \rangle/M}} (\text{arcsinh}\sqrt{N})^{M-1}}{2\sqrt{N(N+1)}(M-1)! (\kappa\langle N_p \rangle/M)^M}$$

for $M$-mode thermal pumping and

$$P_{FWM,M}(N) = \frac{e^{-\frac{\text{arcsinh}\sqrt{N}}{\kappa\langle N_p \rangle/M}} (\text{arcsinh}\sqrt{N})^{M/2-1}}{2\sqrt{N(N+1)} \Gamma\left(\frac{M}{2}\right) (2\kappa\langle N_p \rangle/M)^M}$$

(C3)

for $M$-mode superbunched BSV one. The corresponding CCDFs are

$$C_{FWM,M}(N) = \frac{\Gamma\left(M\frac{\text{arcsinh}\sqrt{N}}{\kappa\langle N_p \rangle/M}\right)}{(M-1)!}$$

and

$$\tilde{C}_{FWM,M}(N) = \frac{\Gamma\left(M\frac{\text{arcsinh}\sqrt{N}}{\kappa\langle N_p \rangle/M}\right)}{\Gamma\left(\frac{M}{2}\right)}$$

(C5)

respectively. Here $\Gamma(s,x)$ is the upper incomplete Gamma function. The latter tends to $\Gamma(s,x) \sim x^{s-1}e^{-x}$, which leads to $k = M/(2\kappa\langle N_p \rangle)$ for (C3) and $k = M/(4\kappa\langle N_p \rangle)$ for (C4). The tail exponents linearly increase with $M$ for the same $\langle N_p \rangle$; in other words, $k$ is determined by the pump number of photons per mode $\langle N_p \rangle/M$.

The number of modes of the pump can be estimated using $g^{(2)}$ measurement. For any $M$-mode state of light with each mode having the same second-order CF $g_1^{(2)}$, the resulting CF is [24, 39]

$$g_M^{(2)} = 1 + \frac{g_1^{(2)} - 1}{M}.$$  

(C7)

We measure $g_M^{(2)}$ for weak superbunched BSV ($P = 15$ nW, $g_1^{(2)} = 3$) after the fiber using a visible-range charge-integrating photodetector. Under such weak pumping, the supercontinuum is not produced and the pump is not depleted. We get $g_M^{(2)} = 1.42 \pm 0.08$ for $\Delta\lambda_p = 10$ nm and $g_M^{(2)} = 2.0 \pm 0.1$ for $\Delta\lambda_p = 3$ nm, which corresponds to five and two modes, respectively.
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