Forward Kinematics Analysis and Tension Distribution of a Cable-Driven Sinking Winches Mechanism

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This paper concerns the forward kinematics and tension distribution of sinking winches mechanism, which is a type of four-cable-driven partly constrained parallel robot. Conventional studies on forward kinematics of cable-driven parallel robot assumed that all cables are taut. Actually, given the lengths of four cables, some cables may be slack when the platform is in static equilibrium. Therefore, in this paper, the tension state (tautness or slackness) of cables is considered in the forward kinematics model. We propose Traversal-Solving-Algorithm, which can indicate the tension state of cables, and further determine the pose of the platform, if the lengths of four cables are given. The effectiveness of the algorithm is verified by four examples. The results of this paper can be used to control sinking winches mechanism to achieve the level and stable motion of the platform, and to make the tension distribution of cables as uniform as possible.

Keywords: forward kinematics, tension distribution, sinking winches mechanism, cable-driven parallel robot

1 Introduction
In the coal mining industry, constructing a vertical shaft to transport miners and equipment down to a mine (and also to lift coal out of it) is the first step to mine the coal. The main equipment to construct a vertical shaft is a sinking winches mechanism, as shown in Fig. 1 which consists of four winches, four cables, a derrick and a platform. The platform, supplying the workspace for constructors, needs to be lowered down in digging process (or lifted up when cast the wall of the shaft) by rolling out (or rolling in) four cables on the winches.

The lengths of four cables from the winches to the platform determines the pose of the platform; the platform will remain horizontal if these lengths are equal. It is crucial to synchronize the motion of the winches so as to keep the platform leveled horizontally during operation, since a tilted platform in motion may collide with the wall of the shaft. However, the absolutely synchronized motion of the winches is impossible, and there always exists synchronization error (length difference of cables). To avoid the collision of the platform and shaft wall, length difference of cables must be limited. To obtain the limitation of the length difference requires the determination of the pose of the platform given the lengths of four cables. This is the forward kinematics problem that we address in this paper.

A sinking winches mechanism can be considered as a
cable-driven partially constrained parallel robot. The forward kinematics of parallel robots with rigid active links has been studied by many researchers. In addition to the well-established methods (e.g., the elimination method \[1\]2\[3\], the continuation method \[4\], the Gröbner basis method \[6\]7\] and the interval analysis method \[8\]), new approaches for dealing with this problem have been developed recently. These include methods based on neural networks \[9\], differential evolution \[10\], and meta-heuristic search technique \[11\], and the Gauss-Newton iterative method based on chaos and hyper-chaos \[12\].

For forward kinematics of cable-driven parallel robots, few results are available. Most studies focused on determining the pose of the end-effector of a cable-driven fully constrained parallel robot. This is generally accomplished by solving the forward kinematics equations using numerical method (such as Newton-Raphson method \[13\]) and analytical method based on the tetrahedron characteristic of the mechanism \[14\]15\]16\], and only one solution was solved. Using the tool of interval analysis, Merlet \[17\] obtained multiple solutions for the forward kinematics equations of a robot named MARIONET.

Compared to cable-driven fully constrained parallel robots, partially constrained parallel robots (usually suspending a platform as the end-effector) represent a more difficult case in forward kinematics analysis. This difficulty arises mainly from the fact that static equilibrium equations of the platform needs to be included in the forward kinematics model \[18\]. Merlet et al \[19\] studied the forward kinematics problem of a four-cable-driven parallel robot, which had the similar configuration with the sinking winches mechanism described in Figure 1 and obtained multiple poses of the platform by interval analysis; however, the multiple poses were solved based on assumption that all cables are taut. This assumption is usually unpractical since, arbitrarily giving the lengths of four cables, the platform may be suspended by one, two, three, or four cables when the platform is in static equilibrium. Therefore, the tension state (tautness and slackness) of cables must be taken into account in the forward kinematics analysis. To investigate the forward kinematics of a sinking winches mechanism in this context is the main focus of this paper.

The remainder of this paper is organized as follows. Section 2 establishes the forward kinematics model of the sinking winches mechanism. Section 3 presents our methods for solving the forward kinematics problem, while Section 4 discusses the solving procedure. Section 5 presents four examples. Section 6 summarizes the results, and discusses their implications.

## 2 Forward Kinematics Model

In this section we construct a forward kinematics model of a sinking winches mechanism driven by four cables. This model addresses geometrical constraints and static equilibrium; it incorporates the tension state, but ignores the masses and the elasticity, of the cables.

### 2.1 Description of the Sinking Winches Mechanism

As shown in Fig. 1, the pose of the platform is determined by the lengths of the four cables, each extending from a tangent point on the head sheave to an attachment point on the platform. Fig. 2 illustrates the relationship between the cable lengths and the pose of the platform, where \(A_i\) and \(B_i\), \(i = 1, 2, 3, 4\) are the tangent and attachment points, respectively, and \(A_1A_2A_3A_4\) and \(B_1B_2B_3B_4\) are identical rectangles with sides of \(2a\) and \(2b\). An inertial frame \((oxyz)\) is defined at the centroid of \(A_1A_2A_3A_4\), while the local frame \((OXYZ)\) is located at the centroid of \(B_1B_2B_3B_4\). Hence, the coordinates of \(B_i\) in the local frame are identical to those of \(A_i\), with \(i \in \{1, 2, 3, 4\}\), in the inertial frame; these coordinates are: \((a, b, 0), (-a, b, 0), (-a, -b, 0)\) and \((a, -b, 0)\).

The pose of the platform is represented by the Cartesian coordinates of \(B_1, B_2, B_3\) and \(C\) in the inertia frame. Since \(B_1, B_2, B_3\), and \(B_4\) are coplanar and form a rectangle, the coordinates of \(B_4\) can be expressed in terms of the other three points as follows:

\[
B_4 = B_1 - B_2 + B_3. \quad (1)
\]
describes the fixed distances among $B_1$, $B_2$, $B_3$ and $C$, i.e.,

\[
\begin{align*}
|B_1B_2| &= 2a \\
|B_2B_3| &= 2b \\
|B_1B_3| &= 2\sqrt{a^2 + b^2} \\
|CB_1| &= r_1 \\
|CB_2| &= r_2 \\
|CB_3| &= r_3
\end{align*}
\]

(2)

where, $r_1$, $r_2$ and $r_3$ denote the distances from $C$ to $B_1$, $B_2$ and $B_3$, respectively, and can be calculated as follows:

\[
\begin{align*}
r_1 &= \sqrt{(1 - k_1)^2a^2 + (1 - k_2)^2b^2 + h^2} \\
r_2 &= \sqrt{(1 + k_1)^2a^2 + (1 - k_2)^2b^2 + h^2} \\
r_3 &= \sqrt{(1 + k_1)^2a^2 + (1 + k_2)^2b^2 + h^2}
\end{align*}
\]

The second group of constraint equations expresses the lengths of taut cables in terms of the coordinates of $B_1$, $B_2$ and $B_3$. When in static equilibrium, the platform can be suspended by one, two, three, or all four cables. If the platform is suspended by a single cable, then the length of that cable can be expressed as

\[
|A_iB_i| = l_i \quad i \in \{1, 2, 3, 4\},
\]

(3)

where $l_i$ denotes the length of cable $i$. Similarly, if the platform is suspended by two, three or four cables, then the constraint equations are

\[
|A_iB_i| = l_i, \quad |A_jB_j| = l_j \\
i, j \in \{1, 2, 3, 4\}, i \neq j;
\]

(4)

Clearly the second set equations are determined by the tension state of the cables.

### 2.3 Static Equilibrium

The platform is in static equilibrium under its external wrench ($F$) and the tension of cables ($T$). The static equilibrium can be expressed as [17]:

\[
J^T \cdot T = F,
\]

(7)

where $J^T$ is the structure matrix of sinking winches mechanism, and denoted as

\[
J^T = \begin{bmatrix}
u_1 & u_2 & u_3 & u_4 \\
\begin{bmatrix}CB_1 \times u_1 & CB_2 \times u_2 & CB_3 \times u_3 & CB_4 \times u_4\end{bmatrix}
\end{bmatrix},
\]

(8)

where $u_i$ is the unit vector of $A_i$, and calculated by $A_iB_i/|A_iB_i|$.

Since only gravity is applied on the platform, external wrench can be expressed as $F = [0, 0, -mg, 0, 0, 0]^T$. It should be emphasized that the tension vector $T = [\tau_1, \tau_2, \tau_3, \tau_4]^T$, where $\tau_i$ is the tension of $i$ th cable, is determined by the tension state of cables: if the $i$ th cable is slack, $\tau_i$ must be zero.

To sum up, an example can illustrate how to construct the forward kinematics model of sinking winches mechanism. Supposing cables $\{1, 2, 3\}$ are taut when the platform is in static equilibrium with the given lengths of four cables, Eqn. (2), Eqn. (3) (with $i = 1, j = 2, k = 3$) and Eqn. (7) (with $T = [\tau_1, \tau_2, \tau_3, 0]^T$) build up the forward kinematics model. We totally obtain 15 nonlinear equations with 15 unknowns, which are tensions $(\tau_1, \tau_2, \tau_3)$ and coordinates of $B_1$, $B_2$, $B_3$ and $C$. Clearly, the second geometrical constraints Eqn. (3) and static equilibrium Eqn. (7) are determined by tension state of cables.

### 3 Solving the Forward Kinematics

As described in section 2, the forward kinematics model of sinking winches mechanism is constructed by a system of nonlinear equations. There exists at most six reasonable solutions for these equations since, for sinking winches mechanism, maximum six configurations (shown in Fig. 3) satisfy the condition that the platform is in static equilibrium. However, this paper isn’t dedicated to all reasonable solutions but focuses on the solution in the context of the first configuration (see Fig. 3(a)), because it is the actual configuration of sinking winches mechanism.
The following sections give detailed analysis on reasons and propose some approaches to overcome them.

3.1 Single Cable Suspending the Platform

If a cable is much shorter than other three cables, the platform may be solely suspended by the shortest cable when it is in static equilibrium. Without loss of generality, we assume that the platform is suspended by cable ①, and cables ②, ③, ④ are slack; the tension vector is expressed as \( T = [\tau_1, 0, 0, 0]^T \). When the platform is in static equilibrium, the tangent point (A1), attachment point (B1), and the platform’s center of gravity (C) form a plumb line, since external wrench applied on the platform only includes gravity. Subsequently, in inertial frame, the coordinates of \( B_1 \) and \( C \) are \((a, b, -l_1)\) and \((a, b, -l_1 - r_1)\). Thus, \( u_1 = A_1B_1/|A_1B_1| = [0, 0, -1]^T \) and \( CB_1 \times u_1 = 0 \), since \( CB_1 \) and \( u_1 \) are collinear; the static equilibrium equation is expressed as

\[
\tau_1 \cdot [0, 0, -1, 0, 0, 0]^T = [0, 0, -mg, 0, 0, 0]^T. \quad (9)
\]

Equation (9) can be reduced as \( \tau_1 = mg \cdot g \), and the effective number of Eqn. (9) is reduced form six to one. Combining Eqns. (2), (3) and (9), we end up with eight effective equations for the model of forward kinematics with thirteen unknowns. The insufficient equations for unknowns causes the ineffective and incorrect convergence of Trust-Region Dogleg Algorithm.

The insufficient equations for unknowns also results in multiple solutions existing for coordinates of \( B_2 \) and \( B_3 \), since tension \( \tau_1 \) and coordinates of \( B_1 \) and \( C \) are definite. Therefore, the pose of the platform is not definite (or out of control) if the platform is suspended by single cable. This motivates us to propose the criteria to check whether this case happens with the given lengths of four cables; furthermore, we can avoid this case by control the lengths of cables.

In fact, if the platform is suspended by single cable, the other three slack cables allow the platform freely rotating around the single cable within a rotational range. Therefore, whether single cable suspends the platform can be deduced by checking if the rotational range exists. We first define critical lengths of slack cables, and then illustrate the criteria.

3.1.1 Critical Length

Assuming that cable ① is shortest and solely suspend the platform, we define the critical lengths of cables ②, ③ and ④ as follows.

1) Rotating the platform around cable ① (line \( A_1B_1 \)) by \( 2\pi \), the trajectory of \( B_2 \) is a circle with its center locating at line \( A_1B_1 \), as shown in Fig. 4.
2) Arbitrarily select a point \( B_2 \) on the circle, let \( d_2 \) be the normal vector from \( B_2 \) to line \( A_1B_1 \), and \( \theta_2 \) is the included angle of \( d_2 \) with respect to x-axis of inertial frame;
3) The distance of \( A_2B_2 \) is a function of \( \theta_2 \), and denoted as \( l_2(\theta_2) = |A_2B_2|\);
4) We define \( l_{2min} = \min[l_2(\theta_2)] \) and \( l_{2max} = \max[l_2(\theta_2)] \) as the minimal and maximal critical length of cable ②, respectively.

The same definitions can be made for cables ③ and ④, which are denoted as \( l_{3min}, l_{3max}, l_{4min}, \) and \( l_{4max} \).

It is hard to analytically solve these nonlinear equations, since the number of equations is at least 13 (if one cable is in tension). Thus, numerical method Trust-Region Dogleg Algorithm is selected to solve them. However, there exists three cases that cause the numerical algorithm non-convergence, or converge to wrong solutions, which are listed as follows.

1) Single cable suspends the platform;
2) Four cables have equal lengths;
3) Adjacent cables have equal lengths.

The following sections give detailed analysis on reasons and propose some approaches to overcome them.
Next we deduce the values of $l_{2\text{min}}$ and $l_{2\text{max}}$. Based on geometry, the $l_{2\text{min}}$ and $l_{2\text{max}}$ are available only if $A_1B_1$ and $A_2B_2$ are coplanar. As shown in Fig. 4, let $B_{21}$, $B_{22}$ be the intersections between the circle trajectory of $B_2$ and the coplane determined by $A_1B_1$ and $A_2B_2$, coordinates of $B_{21}$ and $B_{22}$ are solved by the following equations

$$
\begin{cases}
(A_1B_1 \times B_1B_2) \times (A_2B_2 \times B_1B_2) = 0 \\
|B_1B_2| = 2a \\
|CB_2| = r_2
\end{cases}
$$

And the coordinates of $B_{21}$ and $B_{22}$ are solved as

$$
\begin{align}
x_{21} &= a - \sqrt{4a^2 - (r_2^2 - 4a^2 - r_1^2)/(2r_1))^2} \\
y_{21} &= b \\
z_{21} &= (r_2^2 - 4a^2 - r_1^2)/(2r_1) - l_1
\end{align}
$$

and

$$
\begin{align}
x_{22} &= a + \sqrt{4a^2 - (r_2^2 - 4a^2 - r_1^2)/(2r_1))^2} \\
y_{22} &= b \\
z_{22} &= (r_2^2 - 4a^2 - r_1^2)/(2r_1) - l_1
\end{align}
$$

Finally, with the fixed coordinates of $A_2(-a,b,0)$, the minimal and maximal critical lengths of cable $2$ are calculated as $l_{2\text{min}} = |A_2B_{21}|$ and $l_{2\text{max}} = |A_2B_{22}|$. And the critical lengths of cables $3$ and $4$ can be solved by similar approach as cable $2$.

### 3.1.2 Criteria

As discussed previously, the platform is solely suspended by the shortest cable $1$, if the lengths of cables $2$, $3$ and $4$ allow the platform freely rotating around cable $1$ within a rotational range. This criteria can be mathematically described as: let $\Phi_2$, $\Phi_3$ and $\Phi_4$ be the rotational ranges that the cables $2$, $3$ and $4$ respectively allow the platform rotating around cable $1$; define $\Phi = \Phi_2 \cap \Phi_3 \cap \Phi_4$; the platform is solely suspended by cable $1$ if $\Phi$ is not empty.

From the definition of critical length, we conclude:

1) If $l_j \leq l_{j\text{min}}$, $j \in \{2,3,4\}$, the platform is not solely suspended by cable $1$, since cable $j$ does not allow the platform freely rotating around cable $1$ if the length of cable $j$ is less or equal to its minimal critical length, i.e., $\Phi = \emptyset$;
2) If $l_j \geq l_{j\text{max}}$, $j = 2,3,4$, the platform is solely suspended by cable $1$, since the platform can rotate around cable $1$ by $2\pi$ if the three cables $(2,3,4)$ are larger or equal to their maximal critical lengths, i.e., $\Phi = [0,2\pi]$.

However, what if $l_j \in (l_{j\text{min}}, l_{j\text{max}})$, $j \in \{2,3,4\}$? In these cases, $\Phi_j$, $j \in \{2,3,4\}$ need to be calculated respectively to deduce whether $\Phi$ is empty. And $\Phi_j$ can be calculated as follows.

1) Suppose the platform rotating around cable $1$ by $\theta$ from a initial pose;
2) Calculate the distances of $A_jB_j$, $j \in \{2,3,4\}$, which are expressed by $l_j(\theta)$;
3) The rotational ranges $\Phi_j$ are solved by the inequality $l_j(\theta) \leq l_j$ with $\theta \in [0,2\pi]$, since the distance of $A_jB_j$ is less or equal to the length of cable $j$ if $\theta \in \Phi_j$.

The following gives detailed description on above approach.

Supposing the platform rotates around cable $1$ by $\theta$ from the initial pose that minimal critical length of cable $2$ is available, as shown in Fig. 5, we calculate the distance $l_j(\theta) = |A_jB_j|$, $j \in \{2,3,4\}$. Since the coordinates of fixed points $A_j$ are known, we just need to express the coordinates of $B_j$ in term of $\theta$. At the initial pose, we define $d_j$, $j \in \{2,3,4\}$, the normal vector from point $B_j$ to line $A_1B_1$, whose magnitude is $d_j$, and included angel with respect to $x$-axis is $\theta_j$. If the platform counterclockwise rotates by $\theta$, the coordinates of $B_j$, $j \in \{2,3,4\}$, are expressed as

$$
\begin{align}
x_j &= a + d_j \cos(\theta_j + \theta) \\
y_j &= b + d_j \sin(\theta_j + \theta) \\
z_j &= z_j
\end{align}
$$

since the trajectory of $B_j$ is a circle around $A_1B_1$ in parallel with plane $xoy$; and $z_j$ keeps the same during the rotation of the platform.

In the following, we calculate $d_j$, $\theta_j$ and $z_j$, $j \in \{2,3,4\}$, when the platform is at the initial pose. At the initial pose, the coordinates of $B_1$ are $(a,b,-l_1)$, $B_2$ are expressed in Eqn. (10). The coordinates of $B_3$ are solved by the following equations

$$
\begin{cases}
|B_1B_3| = 2\sqrt{a^2 + b^2} \\
|B_2B_3| = 2|b| \\
|CB_3| = r_3
\end{cases}
$$

There exists two set of solutions for coordinates of $B_3$, and select the set of solutions satisfying the initial pose as

$$
\begin{align}
x_3 &= [3a^2 + x_{21}^2 + (z_3 - z_{21})^2 - (z_3 + l_1)^2]/[2(x_{21} - a)] \\
y_3 &= b - \sqrt{4a^2 + 4b^2 - (z_3 + l_1)^2 - (x_3 - a)^2} \\
z_3 &= [(r_2^2 - 4a^2 - 4b^2 - r_1^2)/(2r_1)] + l_1
\end{align}
$$

Fig. 4. Critical lengths of cable 2
Subsequently, the coordinates of $B_4$ are calculated by substituting the coordinates of $B_1$, $B_2$ and $B_3$ into Eqn. (1). At this point, the coordinates of $B_i$, $i \in \{1,2,3,4\}$, have been calculated, and $z_j$, $j \in \{2,3,4\}$, can be expressed as

\[
\begin{align*}
  z_2 &= \frac{(r_2^2 - 4a^2 - r_1^2)}{(2r_1)} - l_1 \\
  z_3 &= \frac{(r_3^2 - 4a^2 - 4b^2 - r_1^2)}{(2r_1)} - l_1 \\
  z_4 &= \frac{(r_4^2 - 4a^2 - 4b^2)}{(2r_1)} - l_1.
\end{align*}
\]

With the coordinates of the fixed point $A_1$ and $B_i$, $i \in \{1,2,3,4\}$, $d_j$ and $\theta_j$, $j \in \{2,3,4\}$, can be calculated based on analytical geometry. Let $B_j'$ be the projection point of $B_j$ on line $A_1B_1$, and the coordinates of $B_j'$ in the initial frame $oxyz$ can be calculated as follows:

\[
B_j' = B_1 + B_1 A_1 \cdot \frac{B_i A_i \cdot B_j B_j}{|B_i A_i|^2}, \quad j \in \{2,3,4\}.
\]

Thus, the normal vector is expressed as $d_j = B_j - B_j'$, whose magnitude $d_j$ and angles $\theta_j$ can subsequently calculated. Substitution of $d_j$, $\theta_j$, and $z_j$ into Eqn. (1) yields coordinates of $B_j$ in term of $\theta$. Consequently, the distance $l_j(\theta)$ can be expressed by $l_j(\theta) = |A_1B_1|$.

With the expression of $l_j(\theta)$, $j \in \{2,3,4\}$, the rotational range $\phi_j$ can be solved by the inequality $l_j(\theta) \leq l_j$ with $\theta \in [0,2\pi]$. To solve the inequality, we first solve the equation $l_j(\theta) = l_j$ with $\theta \in [0,2\pi]$. The equation with an interval can be solved by interval analysis [20][21]. And two solutions exist in the interval, which are denoted as $\theta_{21}$ and $\theta_{22}$ with $\theta_{21} < \theta_{22}$. Thus, the interval $[0,2\pi]$ is separated into three sub-intervals $[0, \theta_{21}]$, $[\theta_{21}, \theta_{22}]$ and $[\theta_{22}, 2\pi]$. The solutions of the inequality $l_j(\theta) \leq l_j$ are selected from the three sub-intervals by midpoint method, i.e., the sub-interval is the solution of the inequality if the its midpoint satisfies the inequality. And rotational range $\phi_j$, $j \in \{2,3,4\}$ is obtained, thus, $\phi = \phi_2 \cap \phi_3 \cap \phi_4$. We can deduce whether the platform is solely suspended by cable $i$ by checking if $\phi$ is empty.

It should be noted that above criteria is proposed based on the assumption that cable $i$ is shortest. For other situations, such as cable $2$ (or $3$, or $4$) is shortest, we can take the similar analysis. Moreover, to avoid single cable suspending the platform, the lengths of cables should be controlled so as to guarantee the rotational range $\phi$ is empty.

### 3.2 Four Cables Having Equal Lengths

#### 3.2.1 Pose of Platform and Tension of Cables

For sinking winches mechanism, if four cables have equal length $l$, the platform will remain horizontal and stay at the pose that tangent point ($A_i$) and attachment point ($B_i$) of cable $i,i \in \{1,2,3,4\}$, form a plump line. At this pose, the coordinates of $B_i,i \in \{1,2,3,4\}$ and $C$ in inertial frame are $(a,-b,-l)$, $(a,-b,-l)$, $(a,-b,-l)$ and $(k_1a,k_2b,-l-h)$, respectively. At this point, the pose of the platform is obtained.

With the coordinates of $B_i,i \in \{1,2,3,4\}$ and $C$, the vectors $\mathbf{u}_i$ and $\mathbf{CB}_i$ can be calculated, which are introduced into Eqn. (3), and the structure matrix $J^T$ is obtained. Substituting $J^T$ into Eqn. (7), the static equilibrium of the platform is expressed as:

\[
\begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
 -1 & -1 & -1 & -1 \\
 (k_2-1)b & (k_2-1)b & (k_2+1)b & (k_2+1)b \\
 (1-k_1)a & (1-k_1)a & (1-k_1)a & (1-k_1)a
\end{bmatrix}
\begin{bmatrix}
  \tau_1 \\
  \tau_2 \\
  \tau_3 \\
  \tau_4
\end{bmatrix}
\begin{bmatrix}
  0 \\
  0 \\
  -mg \\
  0
\end{bmatrix}
\]

There are infinite solutions for above equation, since the ranks of $J^T$ and $J^T | F$ are both three while the number of unknowns are four. That is why Trust-Region Dogleg Algorithm can’t converge effectively and correctly. However, the above equation can be analytically solved as

\[
\begin{bmatrix}
  \tau_1 \\
  \tau_2 \\
  \tau_3 \\
  \tau_4
\end{bmatrix}
= \begin{bmatrix}
  (1+k_1)mg/2 \\
  (k_2-k_1)mg/2 \\
  (1-k_2)mg/2 \\
  0
\end{bmatrix}
+ \begin{bmatrix}
  -1 \\
  1 \\
  -1 \\
  1
\end{bmatrix}.
\]

Considering that the tension of cables are non-negative, i.e., $\mathbf{\tau} \geq \mathbf{0}$, from which we deduce the value of $\tau_4$ is limited in the following intervals.

1) $[0, (1-k_2)mg/2]$ if $(k_1 \in (-1,1), k_2 \in (0,1), |k_1| \leq |k_2|)$;
2) $[0, (1+k_1)mg/2]$ if $(k_1 \in (-1,0), k_2 \in (-1,1), |k_1| \geq |k_2|)$;
3) $[(k_1-k_2)mg/2,(1+k_1)mg/2]$ if $(k_1 \in (-1,1), k_2 \in (-1,0), |k_1| \leq |k_2|)$;
4) $[(k_1-k_2)mg/2, (1-k_2)mg/2]$ if $(k_1 \in (0,1), k_2 \in (-1,1), |k_1| \geq |k_2|)$.

As can be seen, the coefficients $k_1$ and $k_2$ determine the range of $\tau_4$, subsequently the tension scales of all cables, but can
not determine a definite tension distribution. Meanwhile, $k_1$ and $k_2$ also describe the location of center of gravity ($C$) on the platform, more specially, the $X$ and $Y$ coordinates of $C$ in local frame as $(k_1a, k_2b)$, $k_1, k_2 \in (-1, 1)$. Therefore, location of center of gravity affects the tension scales of all cables. Among the tension scales, we are interested to the even tension distribution, i.e., tension difference among cables is minimal. This tension distribution has practical sense for the operation and maintenance of sinking winches mechanism, since it lets four cables bear even load and elastic deformation, which can extend the cables’ lives and help the platform keep horizontal during the motion.

### 3.2.2 Even Tension Distribution

In this section, we investigate the relation between even tension distribution and the location of center of gravity. This relation can be used to guide the designing of the platform to make tension distribution as uniform as possible.

The tension difference can be defined as

$$
\Delta T = \left[ \sum_{i \neq j} (\tau_i - \tau_j)^2 \right]^{\frac{1}{2}}, \quad i, j \in \{1, 2, 3, 4\}.
$$

Substitution of Eqn. (12) into above equation yields

$$
\Delta T = \left[ 16\tau_1^2 - (8 + 8k_1 - 8k_2)mg \cdot \tau_4 + \right. \\
\left. \left( \frac{1 + 2k_1 - k_2}{2mg} \right)^2 + \left( \frac{k_1 + k_2}{2mg} \right)^2 + \right. \\
\left. \frac{2k_2 - 1 - k_1}{2mg} \right)^2 + \left( \frac{1 + k_1}{2mg} \right)^2 + \right. \\
\left. \left( \frac{1 - k_2}{2mg} \right)^2 + \left( \frac{k_1 - k_2}{2mg} \right)^2 \right]^{\frac{1}{2}}.
$$

The above expression implies that $\Delta T$ is minimal if $\tau_4 = \tau_4^\star = (1 + k_1 - k_2)mg/4$. Besides, $\tau_4$ must be limited in its interval to guarantee the positive tension of cables. Therefore, the minimum of $\Delta T$ is co-determined by $\tau_4^\star$ and the interval of $\tau_4$.

Let $\tau_{4L}$ and $\tau_{4R}$ be the left and right endpoint of the interval, respectively. From the expression of $\Delta T$, we conclude:

1) $\Delta T$ is minimal with $\tau_4 = \tau_4^\star$ if $\tau_4^\star \leq \tau_4 \leq \tau_4^\star$,
2) $\Delta T$ is minimal with $\tau_4 = \tau_4^\star$ if $\tau_{4L} \leq \tau_4 \leq \tau_4^\star$,
3) $\Delta T$ is minimal with $\tau_4 = \tau_4^\star$ if $\tau_4^\star \geq \tau_{4R}$.

For example, the sinking winches mechanism has $k_1 = 0.25$ and $k_2 = 0.2$, which determine $\tau_4$ lying in the fourth interval $[(k_1 - k_2)mg/2, (1 - k_2)mg/2] = [0.025mg, 0.4mg]$, thus $\tau_{4L} = 0.025mg$ and $\tau_{4R} = 0.4mg$. $\tau_4^\star = (1 + k_1 - k_2)mg/4 = 0.2625mg$ implies $\tau_{4L} \leq \tau_4^\star \leq \tau_{4R}$, thus the minimum of $\Delta T$ is obtained with $\tau_4 = \tau_4^\star = 0.2625mg$. Substitution of $\tau_4$ into Eqn. (12) yields the tension distribution with minimal difference as: $\tau_1 = 0.3625mg$, $\tau_2 = 0.2375mg$, $\tau_3 = 0.1375mg$ and $\tau_4 = 0.2625mg$.

To illustrate the relation of the even tension distribution and the location of center of gravity ($C$), we calculate $\Delta T_{\min}$ with respect to $k_1, k_2 \in (-1, 1)$, and the results are shown in Fig. 6. As can be seen, only if $C$ locates at the centroid of platform ($k_1 = 0, k_2 = 0$), cables can have uniform tension distribution ($\Delta T_{\min} = 0$). The farther $C$ leaves from the centroid, the larger difference exists. Therefore, center of gravity should be designed at the centroid of the platform to achieve an uniform tension distribution of cables.

### 3.3 Adjacent Cables Having Equal Lengths

For sinking winches mechanism, if adjacent cables have equal lengths and all cables are taut, there exists two configurations, i.e., $(l_1 = l_2, l_1 = l_4, l_1 \neq l_3)$ and $(l_1 = l_4, l_2 = l_3, l_1 \neq l_2)$. For these two configurations, the structure matrix $J^T$ is nearly singular, which will be illustrated by example in section 5. The nearly singularity of $J^T$ results in the ineffective and incorrect convergence of Trust-Region Dogleg Algorithm when the algorithm is used to solve the static equilibrium equation. To overcome the defect, we firstly simplify the spacial sinking winches mechanism as a planar one because of the symmetry of its configuration; and then solve the forward kinematics of the planar mechanism; finally calculate the pose of platform and tension of cables from the forward kinematics solutions of planar mechanism.

Without loss of generality, taking the first configuration $(l_1 = l_2, l_1 = l_4, l_1 \neq l_3)$ as example, the simplified planar mechanism is shown in Fig. 7. This planar mechanism can be regarded as the projection of sinking winches mechanism on plane $oyz$ of inertia frame. And points $A_i, B_i, i \in \{1, 2, 3, 4\}$, and $C$ projects at $A^p_i, B^p_i, B^p_1, B^p_4, C^p; \; \text{besides, the projections of } B_1 \text{ and } B_2 \text{ are overlapped, so are } B_3 \text{ and } B_4$. The coordinates of projection points in frame $oyz$ are shown in Fig. 8. Thus, the geometrical constraints are expressed as

$$
\begin{pmatrix}
A^p_1 & B^p_1 \\
A^p_2 & B^p_4 \\
B^p_1 & B^p_4 \\
C^p & B^p_1
\end{pmatrix} = \begin{pmatrix}
l_1 \\
l_4 \\
2b \\
r_1^p
\end{pmatrix},
$$

where $r_1^p$ and $r_4^p$ are the lengths of projections of $CB_1$ and
CB4 on plane yoz, and expressed as $r'_t = \sqrt{(1 - k_2)^2b^2 + h^2}$ and $r'_z = \sqrt{(1 + k_2)^2b^2 + h^2}$.

To build up the static equilibrium equation of the planar mechanism, we need to express the structure matrix $J_P^T$, the wrench applied on the platform $F_P$, and tension of cables $T_P$. $J_P^T$ are defined as:

$$
J_P^T = \begin{bmatrix}
\frac{y_1 - b}{|z_1|} & \frac{y_4 + b}{|z_4|} \\
\frac{(y_1 - b)^2 + z_1^2}{z_1} & \frac{(y_4 + b)^2 + z_4^2}{z_4} \\
\frac{(y_1 - b)^2 + z_1^2}{z_1} & \frac{(y_4 + b)^2 + z_4^2}{z_4}
\end{bmatrix}
$$

where $z^*_1 = (y_1 - y_c)z_1 - (y_1 - b)(z_1 - z_c)$ and $z^*_4 = (y_4 - y_c)z_4 - (y_4 + b)(z_4 - z_c)$; $F_P = [0, -mg, 0]^T$; $T_P = [\tau_{12}, \tau_{34}]^T$, where $\tau_{12}$ is resultant tension of cables 1 and 2, and $\tau_{34}$ is result tension of cables 3 and 4. Substitution of $J_P^T$, $T_P$, and $F_P$ into Eqn. (7) yields the static equilibrium equation

$$
\begin{bmatrix}
\frac{y_1 - b}{\sqrt{(y_1 - b)^2 + z_1^2}} & \frac{y_4 + b}{\sqrt{(y_4 + b)^2 + z_4^2}} \\
\frac{(y_1 - b)^2 + z_1^2}{z_1} & \frac{(y_4 + b)^2 + z_4^2}{z_4} \\
\frac{(y_1 - b)^2 + z_1^2}{z_1} & \frac{(y_4 + b)^2 + z_4^2}{z_4}
\end{bmatrix} \begin{bmatrix}
\tau_{12} \\
\tau_{34}
\end{bmatrix} = \begin{bmatrix}
-0 \\
-mg \\
0
\end{bmatrix}
$$

Equations (13) and (14) build up the forward kinematics model of the planar mechanism. Solving this model with Trust-Region Dogleg Algorithm yields the coordinates of $B_1^p$, $B_2^p$ and $C^p$ in frame oyz, which are used to deduce the coordinates of $C$ and $B_i, i \in \{1, 2, 3, 4\}$, in inertial frame oxyz. Since $B_1^p$ is the co-projection of $B_1$ and $B_2$ on plane yoz, the coordinates of $B_1$ and $B_2$ are $B_1 = (a, y_1, z_1)$ and $B_2 = (-a, y_1, z_1)$. Similarly, $B_3$ and $B_4$ are deduced from $B_1^p$ as $B_3 = (-a, y_4, z_4)$ and $B_4 = (a, y_4, z_4)$. And $C$ is implied from $C^p$ as $C = (k_1a, y_c, z_c)$. With coordinates of $C$ and $B_i, i \in \{1, 2, 3, 4\}$, the structure matrix of sinking winches mechanism $J^T$ can also be calculated through Eqn. (3). Substituting $J^T$ into Eqn. (7), the tension distribution of cables is calculated by the expression $T = (J^T)^+ \cdot F$, where $(J^T)^+$ is the pseudoinverse of $J^T$. At this point, the pose of the platform and tension distribution of cables have been solved.

### 4 Program for Solving the Forward Kinematics

This section presents the procedure for solving the forward kinematics model (i.e., a system of nonlinear equations). Except for the three special configurations discussed in section 3 forward kinematics models for other configurations can be effectively solved by Trust-Region Dogleg Algorithm. This algorithm starts from a guess of solution to search for the solution of nonlinear equations. As discussed in section 3 there exists at most six sets of reasonable solutions. Therefore, to get the unique set of solutions in the context of the first configuration, the guess of solutions are provided as follows.

1) With the given lengths of four cables $(l_1, l_2, l_3, l_4)$, calculate the average length as $l_{ave} = (l_1 + l_2 + l_3 + l_4)/4$;
2) Supposing the platform is suspended by four cables with equal length $l_{ave}$, we take the pose of platform and even tension distribution of cables as the guess of solution.

As discussed in section 3.2 if four cables have equal length $l_{ave}$, the coordinates of reference points $B_1, B_2, B_3, C$ representing the pose of platform, and tension of cables $(\tau_{12}, \tau_{34})$. To get the solutions in context of the first configuration, the guess of solutions are provided as follows.

1) With the given lengths of four cables $(l_1, l_2, l_3, l_4)$, calculate the average length as $l_{ave} = (l_1 + l_2 + l_3 + l_4)/4$;
2) Supposing the platform is suspended by four cables with equal length $l_{ave}$, we take the pose of platform and even tension distribution of cables as the guess of solution.

As discussed in section 3.2 if four cables have equal length $l_{ave}$, the coordinates of reference points are $B_1 = (a, b, -l_{ave}), B_2 = (-a, b, -l_{ave}), B_3 = (-a, -b, -l_{ave})$ and $C = (k_1a, k_2b, -l_{ave} - h)$, and even tension distribution of cables are denoted as $(\tau_{12}, \tau_{34})_{ave}$.

### 4.2 Traversal Solving Algorithm

Section 3.1 has elaborated that the forward kinematics model is determined by the tension state (tautness and slackness) of cables, therefore, the guess for tension need to be modified according to the tension state. For example, if cables 1, 2 and 3 are taut when the platform is in static equilibrium, the guess for tension is modified as $(\tau_{12}, \tau_{34})_{ave}$.
then establish and solve the forward kinematics model corresponding to the tension state; finally check whether the assumption holds with the solutions of forward kinematics model. If the assumption holds, output the solutions; else assume another tension state, repeat above process until finding the tension state of cables.

An example can illustrate the Try-and-Error progress: assume cables 1 and 2 being taut when the platform is in static equilibrium; Eqn. 2, Eqn. 3 with \( i = 1, j = 2 \) and Eqn. 7 with \( \tau_1 = \tau_3 = 0 \) build up the forward kinematics model; solve the model by Trust-Region Dogleg Algorithm with the guess of solutions proposed in section 3.1.1; calculate the distances of the four assumption holds, go to step (viii), otherwise, go to step (iii);

(iii) Check whether the platform is suspended by single cable referring to section 3.1.1. If the assumption holds, output message “The platform is suspend by single cable, an definite pose can’t be solved”, go to step (viii). Otherwise, go to step (iv);

(iv) Suppose the platform is suspended by two cables, which include six tension states: \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}. For each tension state, establish and solve the forward kinematics model, and check whether the assumption holds. If any of the six assumption holds, go to step (viii), otherwise, go to step (v);

(v) Suppose the platform is suspended by three cables, which include four tension states: \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}. For each tension state, establish and solve the forward kinematics model, and check whether the assumption holds. If any of the four assumption holds, go to step (viii), otherwise, go to step (vii);

(vi) If adjacent cables have equal lengths, solve forward kinematics following the method described in section 3.3, go to step (viii); else, go to step (vii);

(vii) Solve the forward kinematics model corresponding to the tension state that four cables are taut, and go to step (viii);

(viii) Output the solutions, and exit the procedure.

5 Examples

In this section, four examples are presented to examine the Traversal Solving Algorithm. The lengths of four cables are given in Tab. 1.

In example 1, the lengths of four cables are not equal, the Traversal Solving Algorithm thus check whether the platform is solely suspended by the shortest cable 1 according to the criteria proposed in section 3.1.2. And the interval \( \phi \) is \([0, 0.715] \cup [5.65, 6.28]\) rad, where \( \phi \) is the rotational range that cables 2, 3, and 4 allow the platform freely rotating around cable 1. The non-empty of \( \phi \) implies that the platform is solely suspended by cable 1. Therefore, a definite pose of the platform can not be solved. As for examples 2–4, the intervals \( \phi \) are empty. Consequently, the pose of platform and tension distribution of cables can be solved with Traversal Solving Algorithm, and the solutions are list in Tab. 2

In example 2, since \( l_1 = l_2, l_3 = l_4, \) and \( l_1 \neq l_3, \) adjacent cables have equal lengths. Moreover, all cables are taut, since the assumptions that two or three cables are taut do not hold. As described in section 3.3, the forward kinematics is

| Example | \( l_1 \) (m) | \( l_2 \) (m) | \( l_3 \) (m) | \( l_4 \) (m) |
|---------|----------------|----------------|----------------|----------------|
| 1       | 20             | 21             | 22             | 21.5           |
| 2       | 20             | 20             | 20.1           | 20.1           |
| 3       | 20             | 20             | 21             | 21             |
| 4       | 20.3           | 20.1           | 20.5           | 20.2           |

| Example | 2 | 3 | 4 |
|---------|---|---|---|
| \( B_{1X} \) | 2.000 | 2.000 | 1.996 |
| \( B_{1Y} \) | 2.499 | 2.500 | 2.499 |
| \( B_{1Z} \) | -19.999 | -20.000 | -20.299 |
| \( B_{2X} \) | -2.000 | -2.000 | -1.999 |
| \( B_{2Y} \) | 2.499 | 2.500 | 2.499 |
| \( B_{2Z} \) | -19.999 | -20.000 | -20.099 |
| \( B_{3X} \) | -2.000 | -2.000 | -1.995 |
| \( B_{3Y} \) | -2.499 | -2.403 | -2.499 |
| \( B_{3Z} \) | -20.099 | -20.981 | -20.000 |
| \( C_X \) | 0.500 | 0.500 | -0.001 |
| \( C_Y \) | 0.700 | 2.500 | 0.299 |
| \( C_Z \) | -30.038 | -30.198 | -30.170 |
| \( \tau_1 \) (kN) | 39.201 | 61.250 | 5.856 |
| \( \tau_2 \) (kN) | 23.520 | 38.750 | 49.018 |
| \( \tau_3 \) (kN) | 13.229 | 0 | 0 |
| \( \tau_4 \) (kN) | 22.049 | 0 | 43.126 |
solved by simplifying the spacial configuration as a planar one. With the coordinates of reference points $B_1$, $B_2$, $B_3$ and $C$, the structure matrix $J^T$ is calculated as

$$J^T = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-0.00002 & -0.00002 & 0.00003 & 0.00003 \\
-1 & -1 & -1 & -1 \\
-1.79942 & -1.79942 & 3.19908 & 3.19908 \\
1.5 & -2.5 & -2.5 & 1.5 \\
-0.00003 & 0.00004 & -0.00008 & 0.00005
\end{bmatrix}.$$ 

The singular values of $J^T$ are $5.43464$, $4.08118$, $1.32388$, $1.4932$, $0.1457$, and $0.00008$. Since there exist a singular value (0.00008) closing to 0, the $J^T$ is nearly singular, which results in the ineffective and incorrect convergence of Trust-Region Dogleg Algorithm.

In example 3, even if $l_1 = l_2$, $l_3 = l_4$ and $l_1 \neq l_3$, the platform is hanged only by cables $\{1, 2\}$, since cables $\{3, 4\}$ are much longer than $\{1, 2\}$. As for example 4, the platform is hanged by cables $\{1, 2, 3\}$ and $\{4\}$ when it is in static equilibrium.

6 Conclusions

This paper examines the forward kinematics and tension distribution of sinking winches mechanism. The tension state of cables is considered in the forward kinematics model. And the tension state affects both geometrical constraints and statical equilibrium equations. When the platform is in static equilibrium, there exist at most six configurations, i.e., maximum six reasonable solutions exist for the forward kinematics model. However, this paper only focus on the solution in the context of the first configuration, which agrees with the configuration of sinking winches mechanism.

Traversal Solving Algorithm is proposed to solve the forward kinematics model. In particular, three special configurations that cause the ineffective and uncorrect convergence of the numerical algorithm are detailed discussed:

1) A definite pose of the platform can not be solved from the forward kinematic model if the platform is suspended by single cable; and the criteria is proposed to check whether this case happens;

2) If four cables’ lengths are equal, a definite tension distribution can not be obtained from the forward kinematic analysis, but the even tension distribution is unique;

3) When adjacent cables have equal lengths and all cables are taut, the forward kinematics can be analyzed by simplifying the spacial mechanism as a planar one.

The analysis of forward kinematics yields the pose of platform and tension distribution of cables if the lengths of four cables are given. With the pose, we can check whether the platform collide with the wall, furthermore control the lengths of cables to achieve the level and stable motion of the platform. To achieve an uniform tension distribution, the platform’s center of gravity must be designed at the centroid of the platform.

In this paper, the static equilibrium of platform is considered in the forward kinematics analysis. Actually, during the motion of platform, the inertial force of the platform can not be neglected, because the mass of platform take much account of the whole mechanism. Furthermore, the mass and elasticity of cable play important role in the motion of the mechanism. Therefore, the dynamics model of the mechanism is our future work.

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