Towards Noncommutative Quantum Reality

Otto C. W. Kong

Department of Physics and
Center for High Energy and High Field Physics,
National Central University, Chung-li, Taiwan 32054

Abstract

The implications of the physical theory of quantum mechanics on the question of realism is much a subject of sustaining interest, while the background questions among physicists on how to think about all the theoretical notion and ‘interpretation’ of the theory remains controversial. Through a careful analysis of the theoretical notions with the help of modern mathematical perspectives, we give here a picture of quantum mechanics, as the basic theory for ‘nonrelativistic’ particle dynamics, that can be seen as being as much about the physical reality as classical mechanics itself. The key is to fully embrace the noncommutativity of the theory and see it as a notion about the reality of physical quantities. Quantum reality is then just a noncommutative version of the classical reality.

Keywords : Quantum Mechanics, Quantum Reality

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1 Introduction

Quine (1960) stated that “What reality is like is the business of scientists”. Among various disciplines of sciences, fundamental theories in physics seem to have a privileged position in informing us about the ‘basic essence of’ reality. Quantum mechanics (QM) is the quantum version of the Newtonian theory of particle dynamics which fails to describe the ‘reality’ of the atomic scale and beyond as successful as the former. However, almost a century after Heisenberg first put up the theory, what is the true meaning of that ‘quantumness’, or what ‘quantum reality’ is like, remains controversial among physicists. Its conceptual foundations is still an area of active research. On the whole, what the basic situation seems to be is well illustrated from the following quotes from the introduction chapter of a book by Christopher Norris (2000) on the subject matter:

“the orthodox ‘Copenhagen’ interpretation of QM has influenced current anti-realist or ontological relativist approaches to philosophy of science”

“there are clear signs that some philosophers – including Hilary Putnam – have retreated from a realist position very largely in response to just these problems with the interpretation of quantum mechanics”

“any alternative (realist) construal should have been so often and routinely ruled out as a matter of orthodox QM wisdom”

For a more comprehensive picture including more recent perspectives, readers may consult Lombardi et al. (2019) and French and Saatsi (2020). It suffices to say that the importance of the question sure maintains. In the latter book, for example, French and Saatsi stated right from the beginning, (for “quantum physics”) that

“what kind of knowledge does it provide us? This question gains significance from weighty epistemological issues that forcefully arise in this context – issues that are also at the heart of a more general debate on ‘scientific realism’ in the philosophy of science.”

It is not however our intent to take a definite stand for scientific realism here. Our thesis is that

There is a way to look at quantum mechanics which makes it as much about a description of the physicists’ reality as the classical, Newtonian theory, hence as intuitive or in line with common sense as the latter.

Hence, we want to argue against the “notion of quantum mechanics as having destroyed the case for scientific realism” (Norris, 2000), through discussions on a notion of noncommutative quantum reality which we will argue to be
a sensible natural way to look at the theory. Note that we are referring to classical physics broadly as Newtonian. To the extent that a background philosophical perspective is relevant, we have to take no more than one of methodological naturalism. It should be emphasized that our discussions are mostly focused only on the so-called ‘nonrelativistic’ particle theory, on which we have a solid mathematical description of a notion of noncommutative values of physical quantities. To the extent that our argument stands, a parallel story for the ‘relativistic’ and even field theory counterparts is in principle not difficult to think of, though establishing solid details of the mathematics and physics picture will take more effort. More on the philosophical side, however, the picture of reality offered by quantum field theory in itself is a quite nontrivial topic, especially as the theory has been presented only with a picture of perturbation about free or noninteracting states. As individual particles may be created or annihilated in the process, no set of particles can be taken as the entities the theory describes. Taking the theory seriously as a field theory with the fields as a sort of entities ‘living’ in spacetime, as the commonly adopted picture of a classical field theory is also far from satisfactory. Quantum fields are formulated as operators which may not be Hermitian, hence may not even be taken as physical observables. So, they are neither observables nor states. Yet, all quantum fields in the theory are involved in any state, at least as quantum fluctuations. The states, in our opinion, can only be seen as states of the spacetime possessing different amount of the conserved quantities in different configurations. But no solid picture of such a model of spacetime has been given. Note that other than the electromagnetic field, no other quantum field actually has a corresponding classical field theory. And the ontological nature of the electromagnetic field is far from unchallenged either (Lazarovici, 2018). Nevertheless, whatever physical or ontological picture one can otherwise give to the states of the theory, the notion of noncommutative reality can be applied to the ontology of the states and their observables as in quantum mechanics.

A word of caution to the readers is in order. The article mostly adopts a philosophical tone which may be seen to have taken some implicit assumptions about what we want a physical theory to be, or at least how we generally see the classical theory as. These include the mathematical entities and formalism of a physical theory represent parts of physical reality, with the basic conceptually issues as intuitive or compatible with the scientifically based intuition about nature (without devoting to any particular mathematical theoretical models of such concepts). We see achieving all that as the
basic goal of theoretical physics, though past development on ‘interpretation’ of quantum mechanics has ‘forced’ some physicists to give up part of that. We make no apology for taking such a stand. However, readers may certainly take a different stand there, or at least see all that as up to debate and justifications. We emphasize again that we only present here our case for quantum mechanics being as much about the physical reality and as intuitive as classical mechanics, and hope that readers will not bother too much about our philosophical tone on the classical theory being essentially intuitive and truly describing reality.

Mathematics is the language of physical science, which gives rigorous logic of a theory and the required precision in its practical applications and hence verification. While the conceptual foundation or interpretation of the theory of quantum mechanics has been in ‘crisis’, the theory is otherwise mathematically well defined. The debate on how to think about the conceptual picture of the theory keeps going on and on. To look into such conceptual issues, a careful analysis about the basic mathematical notions and the possible logically sounded conceptual thinking about them, plausibly, and arguably most probably, beyond the old conceptual framework is called for. Readers of the current article are more likely to be quite familiar with most of such mathematical notions in the theory of quantum mechanics, and its classical counterpart, than otherwise. We will however address some of those very carefully to help set the stage right for our discussions on the ‘quantum reality’, and in so doing hopefully makes the article more accessible to a broader readership.

It is important to note, however, that the theory of quantum mechanics as a dynamical theory provides no dynamical description for the process of measurement. The ‘standard’ picture gives only a postulate about what is considered a successful measurement of a ‘quantum observable’ on a particular ‘state’ of the physical system being measured, namely one that gives a real number, an eigenvalue, as the result in apparently the same way as a measurement of a physical quantity in classical, Newtonian, mechanics does. The postulate says the ‘state’ will ‘collapse into’ an eigenstate of the ‘observable’ the corresponding eigenvalue being that result obtained. The kind of measurement processes are called von Neumann, or projective, measurements. The von Neumann measurements are far from the ideal kind, which are assumed not to change a state, and begs a dynamical theory for the successful description of the process itself. Achieving that within the basic framework of quantum mechanics is the task the decoherence theory (Zurek,
2003, 2018) sets for itself.

In the famous EPR paper (Einstein, Podolsky, & Rosen, 1935), it is stated that

“In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system.”

Predicting a physical quantity with certainty is not the same as being able to measure or obtain its value with certainty, and the authors were wise with their choice of words. While not being able to obtained the value within a required uncertainty limit even in principle sure renders ‘predicting it with certainty’ meaningless, not being able to do that with a single measurement of a particular kind is a very different story. That is a point we want our readers to bear in mind. Obsessions with a direct real number reading from an apparatus as what nature tells us about the value of an observable are sure to be counter-productive. Hardly any precision measurement in physics nowadays shows a simple output of the kind the understanding of which is trivial and transparent. The upshot is as a single von Neumann measurement is far from all one can do to extract information about a physical quantity (Wiseman & Milburn, 2010), how to think about the ‘reality’ of the kind of measurement is actually a question of secondary importance to the key

2The decoherence theory (Zurek, 2003, 2018) shows that when a quantum particle is put to interact with an experimental apparatus for the measurement of an observable to result in the apparatus settling into a ‘pointer state’, i.e. giving a reading essentially like what we would have in a classical physics measurement, corresponding to an eigenvalue of the observable, the state of the particle, the apparatus, and the environment around it generally evolves quantum mechanically into some entangled (micro)states. An entangled state of a system, nonexisting in classical physics, is one for which the notion of the state for a subsystem, here the particle, becomes ill-defined (Horodecki, Horodecki, Horodecki, & Horodecki, 2009). The exact microstate at any moment cannot be obtained theoretically or experimentally in any practical sense due to the very large number of degrees of freedom beyond our control. The picture hence becomes one of (quantum) statistical mechanics, where one knows only the macrostate as described by the ‘pointer state’. The best one can do for describing the ‘state of the particle’ is to get to the reduced state, which is a mixed state as a statistical distribution. The theory shows the postulated von Neumann distribution could be obtained, without assuming the Born probability interpretation for the initial state of the particle measured. This is really a dynamical picture of the wavefunction collapse. The particular eigenstate corresponding to the eigenvalue measured is more about the final state of the particle after interacting with the apparatus and the environment than the initial state one sets out to measure. Modern experiments [see for a specially illustrative example Minev et.al. (2019)] have essentially verified the
question about what ‘reality’ is described by quantum mechanics as a physical theory. The so-called Heisenberg uncertainty principle is often seen as giving a theoretical uncertainty limit to the value of the physical quantities obtained from the theory. We will illustrate that is not quite true. *Quantum mechanics as a theory gives exact predictions about all physical quantities without any theoretical uncertainty, though there is difficulty for describing the values of such physical quantities in the same way as we do in classical mechanics. Hence, we suggest the proper description of such values to be done in the proper way, as the noncommutative values.*

Lastly, since we are going to rely much on the mathematical notions in the formulations of quantum mechanics to be the logical background for the conceptualization of the *noncommutative quantum reality*, a few words on *mathematical reality* are in order. In line with assuming only methodological naturalism, we further adopt *mathematical fictionalism*. We accept, for example, Mary Leng’s (2010) statement “reasons to believe in mathematical objects” as being real “do not come from pure mathematics in empirical applications”. Mathematical objects, including the real numbers, are only abstract symbols useful for book keeping and logic reasoning, in relation to our scientific efforts to appreciate and manipulate phenomena in Nature.

Starting from the next section, we analyze the proper notion of particle, space, and time, and then states and observables, leading to the central notion of the value of an observable for a state, for quantum mechanics from those of classical mechanics as the background. The key modern mathematical perspective of algebraic geometry is the idea of a duality between an (observable) algebra and a geometric space (of its states). A state having fixed values for all observables is to be seen as giving an evaluation map of the full observable algebra. For a noncommutative algebra, like the algebra of quantum observables, that suggests a notion of noncommutative geometry beyond the picture of real number geometries. We look at how such an evaluation map can be best formulated for the quantum observable algebra to fully realize the duality, first to its space of pure states as the quantum phase space, which leads to the introduction and discussion about the notion of noncommutative values of the physical quantities in section 3. The section finishes with the noncommutative coordinate, hence noncommutative geometric, picture of the quantum space and the identification of it as the proper model for the physical space itself. Some further discussions and

collapse picture as compatible with the decoherence theory.
conclusions are given in the last section.

2 Particle Dynamics Classical to Quantum

2.1 Particle, Space and Time

In the standard physics presentations, a Newtonian particle is a point mass — it is an object characterized by a single attribute or basic properties called mass which occupies a single point in space. The notion of space, or our physical space, and a position in it, is an intuitive one. So is time. To give the physical theory of particle dynamics, one needs to ‘quantify’ position and time. As Quine put it, “To be is to be the value of a variable”. We have in classical mechanics the position variable $x$ and time variable $t$. In Newton’s time, there was essentially only one mathematical model of the physical space, that of a three-dimensional Euclidean geometry, which he adopted. Hence, at every possible point in the physical space, the position is given by three real number values of its three coordinate variables $x_i$ ($i = 1, 2, 3$), and time is taken to have the value of a single real number.

If there is only one mathematical model for the physical space, there is hardly a notion of right or wrong about it as a model. In fact, one would take it as more than a model. It is then not difficult to appreciate the great philosopher Kant’s rejecting the conception that space and time in Newtonian mechanics as physical entities, whether in his early Leibnizian “relationalist” period or the latter transcendent idealism. The best physical way to look at the space from the Newtonian theory, and arguably from any theory of particle dynamics, is that it is the totality of all possible positions for a (free) particle. Beyond that, the kind of physical theories cannot say anything more about the physical space, within its own logic. If there are different choice of available mathematical models, our logic here then would imply that the particular model for the physical space adopted by, or rather obtained from, a physical theory may only be as good as the theory itself. When the Newtonian theory of classical mechanics is to be superseded, there is no good reason to assume it may be good to keep the Newtonian model of the physical space.

Einstein’s theory of special relativity replaces the Newtonian space and

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3 Note that we used the term ‘Newtonian’ throughout the article in the way it has been used in the modern physics literature, as essentially equivalent to ‘classical nonrelativistic’.
time by a $1+3$ dimensional pseudo-Euclidean, or Minkowski, spacetime. His theory of general relativity further changes that to a $1+3$ dimensional Riemannian geometry with a dynamical curvature. Apart from the Minkowski zero curvature limit, it is generally non-Euclidean. Quantum mechanics, however, keeps the classical notion of particle, space and time more or less untouched, except that there is a problem.

The problem, or difficulty, quantum mechanics has with the Newtonian notion of particle, space and time is that a quantum particle cannot be required to always have a definite value for its single position in space. It is at least so stated almost in any literature on the subject matter. However, there is a caveat. The statement should be more exactly stated as a quantum particle cannot be required to always have a definite value for its single position in the Newtonian model of the physical space. Whether the problem can be overcome in an alternative model for the physical space becomes then a legitimate question and the theory itself may point at the right candidate for such a quantum model of the physical space. The latter seems like a simple logical possibility that has not been considered much in the literature on the physics of quantum mechanics or its background philosophy.

2.2 States and Observables

A state of the classical particle is given by the six real number values of its position and momentum variables $(x_i, p_i)$. The collection of all possible states forms the phase space, which is the six-dimensional Euclidean geometry with the $x_i$ and $p_i$ as Cartesian coordinates. The six coordinates are the basic observables, or physical quantities.

Classical mechanics is a theory of real number valued observables each of which is represented by a real variable. Such variables, for the single particle, can be seen as functions of the six basic variables $x_i$ and $p_i$. When the six real numbers are known, the theoretical value of any one of its observables as a known function of the six basic coordinate variables $f(x_i, p_i)$ is determined in principle, without uncertainty. We have an algebra of observables as the collection of all observables, for which the sum of two of its elements, the product of a real number and an element, as well as the product of any two elements are all its elements. The algebra of observables for a classical
A physical quantity in the quantum theory, the quantum observable, is described by a different kind of mathematical object, the physical meaning of which is the crucial question we are after. The implications of all that about ‘quantum reality’ actually hinges on the ‘actual values’ of the observables for a particular state. For the quantum theory, whether a theoretical notion which is not an ‘observable’ is anything physical is much the core of the sustaining controversial debate. The key focus is the question if the quantum state is observable, and whether it represents any ‘realistic’ properties of the particle. A very recent article by Gao (2020) gives a nice summary about the debate on ‘ontic versus epistemic’ state pictures or models. We are adding support to that ‘ontic’ stand, from the conceptual point of view, by presenting a new picture on the relation between the states and the observables deemed impossible in much of the literature. We see that relation as not so different conceptually for quantum mechanics as it is for classical mechanics, for which there is never such a debate. It should be noted that, from the practical side, the modern quantum trajectory theory (Brun, 2002; Carmichael, 1993; Jacobs & Steck, 2006; Plenio & Knight, 1998; Wiseman, 1996) traces the evolution of a quantum state through its deterministic Schrödinger dynamics and probabilistic collapse well and the theory has been very successfully applied to experimental settings. In particular, quite precise real-time monitoring of such trajectories in superconducting qubits (Minev et al., 2019) verified a ‘realistic’ picture of the quantum state compatible with the decoherence theory. If a state of a physical system, in a ‘nonrelativistic’ theory, is a specification of its properties at an instant in time, a quantum/classical state should then be considered ‘real’ if it gives the complete mathematical description of the mathematical content of the observables. Particle states in Newtonian mechanics sure has no problem there. We will see that the quantum particle state actually is also real in the same sense, that its mathematical description uniquely specified the mathematical content of all the observables and such prediction can be experimentally verified.

The geometric structure of the phase space, its *symplectic geometry*, encodes all possible dynamics for the particle. The mathematical structure can be taken simply as admitting Hamiltonian dynamics. Symplectic geo-
metric structure can be used to formulate all classical dynamical systems and beyond, including special and general relativity, Maxwell’s theory of electromagnetism, and actually quantum mechanics, as in the Hamiltonian formulations.

The phase space for a particle in the theory of quantum mechanics is what is known as the projective Hilbert space (Bengtsson & Życzkowski, 2006). Each ray of the infinite dimensional Hilbert space corresponds to a point in the projective Hilbert space. The latter is a symplectic geometry of infinite (real) dimension. The picture is really better clarified only starting from about the 1980’s. Very good description of that is available from Cirelli, Manià, and Pizzocchero (1990) [see also Schilling (1996) and Roberts (2014)], within which there are very important further results we will get back to below. Details about Hamiltonian formulation of dynamics and their symplectic geometric picture is not necessary, in order to follow the present discussions. It suffices to note that the time evolution of the quantum state of a particle as given by the Schrödinger equation is exactly equivalent to an infinite pairs of differential equations, one for each canonical pair of ‘position’ and ‘momentum’ variables, all in the same mathematical form as those for the three pairs of $x_i$-$p_i$ variables for classical mechanics. All the ‘position’ and ‘momentum’ variables, denoted by $(q_n, s_n)$ ($n$ takes any positive integer value), are real variables, that is each of them takes the value of a real number on a fixed state. Looking at the phase space, quantum mechanics is not so different from classical mechanics at all except that the theory really suggests the position of a quantum particle to be given by an infinite number of real numbers instead of only three.

We introduced the state of a quantum particle as a point on the specific geometric space instead of something to be described by the Schrödinger wavefunction $\psi(x_i)$. The notion of the wavefunction is a source of much confusion. Its form as a complex function of some $x_i$ variables, hence like a function on the Newtonian space, reinforces the idea of the theory being a theory on the Newtonian model of the physical space. It is also the background for the Born probability interpretation of the orthodox ‘Copenhagen’ picture, which gives up the intuitive notion that a particle has a definite position at all. The mathematics gives $\psi(x_i)$ as an infinite collection of complex number valued coordinates for the quantum phase space, but not that of the Born interpretation. If Schrödinger himself did start thinking about his formulation of quantum mechanics with the wavefunction $\psi(x_i)$ being a function on the physical space, he remained standing with Einstein against
the orthodox ‘Copenhagen’ picture of the theory.

The physical variables, or observables, of the theory is the part that has little controversy. While the collection of all classical observables form a commutative algebra, that of the quantum observables form a noncommutative algebra. There is otherwise essentially a one-to-one correspondence between a classical and a quantum observable, at least so long as the standard usage of the terms in physics is concerned. The basic quantum observables are $\hat{X}_i$ and $\hat{P}_i$, which serve as the quantum position and momentum (coordinate) observables and the exact counterparts of the classical $x_i$ and $p_i$, respectively. They satisfy the Heisenberg commutation relation

$$[\hat{X}_i, \hat{P}_j] \equiv \hat{X}_i \hat{P}_j - \hat{P}_j \hat{X}_i = i\hbar \delta_{ij}$$

where $\delta_{ij}$ is unity for $i = j$ and zero otherwise, and $\hbar$ the Planck’s constant divided by $2\pi$. We can still think about all other observables as elements of the noncommutative quantum algebra of observables to be given by a formal ‘function’ of the basic observables, $f(\hat{X}_i, \hat{P}_i)$. One may naively project that $\hat{X}_i$ and $\hat{P}_i$ are like coordinates of the quantum phase space. Our question is like to what extent and how one can possibly make sense out of such a picture. A more concrete mathematical picture takes the variables as linear operators on the Hilbert space.

If a quantum observable is not given as a real valued function of the basic variables, it certainly is not supposed to have a value of a single real number as the full information of the observable for a particular state. Mathematically, the question is what can be taken as the value, the full information, of an operator for a particular state. We will argue that it should be a noncommutative value. The noncommutative phase space coordinated by $\hat{X}_i$ and $\hat{P}_i$ is then exactly the space of all possible (noncommutative) values of the coordinate variables, a noncommutative geometry. Note that the Heisenberg uncertainty principle is really about the lack of precision in using single real numbers to model the values of the quantum observables, hence not at all in conflict with the latter having definite noncommutative values. The picture would agree perfectly with the theory if that phase space can actually be identified with the projective Hilbert space mathematically. What amount effectively to a transformation of coordinates between the infinite number of complex number coordinates $z_n = q_n + is_n$ and the three pairs of noncommutative coordinates $\hat{X}_i$ and $\hat{P}_i$ has been presented in Kong and Liu (2021b), achieving that identification. Our notion of noncommutative values
for coordinates is not otherwise available in the literature of noncommutative geometry. In particular, Huggett, Lizzi, and Menon (2021) illustrates the “invalidity of the notion of localizability or a point” for noncommutative geometry as the “undefinability” of “arbitrarily small separations”, however, with smallness as to be characterized by a real number, exactly demonstrating the incompatibility of the notion of a real value for observables, coordinates or distance between points, with the geometry. The noncommutative values of physical quantities gives an alternative notion of ‘noncommutative point’ which offers a coordinate picture for noncommutative geometry.

2.3 A Quantum State Can be Determined by Measurements.

A quantum state can actually be represented by an observable. It is called the projection operator onto the state, or a density matrix for a ‘pure state’. So, a quantum state is essentially an observable. Can that observable really be observed? We actually have a direct positive answer to that from the domain of quantum optics, where physicists have been performing such measurements since the 1990’s (Smithey, Beck, Raymer, & Faridani, 1993; Leonhardt, 1997, see also Minev et.al., 2019). It is unfortunate that the line of works seems to be unknown to many theorists questioning the reality of the quantum state. Actually, that is about measuring the state of light in terms of what is called the Wigner distribution which is a representation of the projection operator for the state as a function of six real variables that look like those of the classical phase space variables $x_i$ and $p_i$. Measuring a function is certainly a lot more than measuring a number, but not anything experimental physicists are unfamiliar with. Mathematically, a function is like a set of infinite number of real numbers. Practically, for any finite precision approximate determination of it, obtaining a finite number of real number values for the function suffices. Recall that the standard geometric picture for the states is that each of them is a point in an infinite dimensional symplectic manifold, which can be described by the infinite number of values for its real number coordinates.

As a quantum state can be determined experimentally, through its projection operator or otherwise, up to any required finite precision at least in principle, it is of course as real as a classical state. Other important theoretical efforts to help ‘establishing’ the ‘reality’ of the quantum state include

12
analysis based on the protective measurements (Gao, 2017, 2020), as well as the ‘ontic’ versus ‘epistemic’ state analysis starting from the inspiring paper of Pusey, Barrett, and Rudolph (2012). Interested readers may also consult Hardy (2013) and the references therein. We are illustrating here what we consider an intuitive and logically consistent and complete conceptual framework to look at the reality of quantum mechanics.

2.4 On The Value of an Observable for a State

It is of crucial importance to carefully consider the role of a state in the notion of any kind of value for an observable. The latter is a variable because its value is not determined until we specify the state the system is in. Some philosophers and physicists seem to have forgotten that in their discussions about quantum reality, or the lack of. If there is no reality in the state, the notion of the values for physical observables as part of reality is at best an ambiguous one. Mathematicians indeed define a state of an algebra as a functional, let us call it an evaluation functional. An evaluation functional is a map that sends each element of the algebra to a real number. For the case of the observables algebra for classical mechanics, that real number is the usual value the state has for the particular observable. Each state can be identified exactly as one such evaluation functional. More explicitly, we can use \([x_i, p_i]\) to denote the evaluation functional of a generic state, with the understanding that a definite state is to be given by definite real number values of the \(x_i\) and \(p_i\) in \([x_i, p_i]\). Then we can write

\[ [x_i, p_i](f) = f(x_i, p_i), \]

where for a particular state \([x_i, p_i]\), any observable \(f\) [or as the function \(f(x_i, p_i)\) with \(x_i\) and \(p_i\) as variables] is evaluated to have the real number answer of \(f(x_i, p_i)\) at the specific values of \(x_i\) and \(p_i\). This is the famous Gel’fand transform, which is the backbone of modern algebraic geometry and noncommutative geometry (Connes, 1994), on which we will come back to below.

One can start with the quantum observable algebra, and define ‘states’ as functionals of it. There is though a mismatch of exact terminology one has to be careful of. The physicists’ state as we have mostly been using the term above actually corresponds to a special class of functionals, called extremal states in mathematics or pure states in statistical physics. They
send unity as an element of the algebra to the real number 1, and cannot be written as a nontrivial convex linear combination of other functionals (Alfsen & Shultz, 2001). The mathematicians’ generic state is used only in statistical physics with the name mixed states. Each of them describes a statistical distribution of pure states. Actually, with von Neumann as one of its pioneer, theory of operator algebras has found its applications in statistical quantum mechanics and quantum field theory (Brattelli & Robinson, 1987; Emch, 2009; Swanson, 2020) since not too long after the publication of his book on the mathematical foundation of quantum mechanics (von Neumann, 1955).

The pure and mixed state terminology has been getting more and more popular in physics (Bengtsson & Życzkowski, 2006), though most discussions about fundamental theories still used the term state without the word pure, as we do above. In fact, as we do not have to care much about the mixed states beyond this paragraph, we will mostly simply keep using the term state for a pure state below, when the context leads to no confusion. What is important to note is that the collection of mixed states is just the collection of convex linear combinations of pure states. Statistical states of course can be constructed as such from the single particle states.

The mathematical notion of (pure) states for the quantum observable algebra agrees exactly with the physical states of the theory. Each point \((q_n, s_n)\) of the projective Hilbert space can be seen as an evaluation functional that maps every observable given by the operator \(\hat{A}\) to the definite real number value \([q_n, s_n](\hat{A})\). The number is the expectation value of the observable for the state. Practically, it is the average of the statistical distribution of eigenvalue results from von Neumann measurements. From the theoretical point of view, the expectation value is certainly the best candidate for a single real number value of a quantum observable, and is definitely predicted by the theory. Moreover, it can be experimentally determined up to any required precision, at least from von Neumann measurements. A direct measurement of it can also be implemented as a protective measurement (Aharonov, Anandan, & Vaidman, 1993, 1996).

Quantum mechanics has definite notions of the physical quantities as a noncommutative observable algebra and the states as evaluation functions giving the definite expectation values. That is what the mathematical logic of the theory presents. Why is that not good enough? There has been the prejudice of taking the single eigenvalue result from a von Neumann measurement as the true value, which is tied with the Born probability interpretation of the quantum state, adhering to taking the Newtonian model for the phys-
ical space. More importantly, the picture cannot be easily connected to its classical approximation and the classical notion of the position and momentum observables. We can consider all possible expectation values of the six $\hat{X}_i$ and $\hat{P}_i$. The collection does not give a picture of the phase space. The best one can do is to associate each set of the six values to a coherent state (Peremolov, 1977). One can easily see that different states can have exactly the same set of the expectation values. This is an indication that the expectation value does not carry the full information of the observable on the state, which can also be easily appreciated from the statistical distribution it is associated with. Actually, the full distribution is definitely predicted by the theory, and it is possible to be determined up to any required precision experimentally, at least in principle. Such a distribution certainly carries more information about than only its expectation value. And if two distributions measured for two states share the same expectation values but are otherwise very different, we sure do not want to say they share the same value for the observable. To describe that difference, we need more than the expectation value. Maybe we can think about taking as the value the full distribution, say by noting the infinite set of moments of the distribution. The noncommutative value is something in the direction, only better.

3 The Noncommutative Values of Physical Quantities and the Noncommutative Space

3.1 The Quest for the Noncommutative Value and the Evaluation Homomorphism

We have brought up a couple of limitations of the notion of states as evaluation functionals in quantum mechanics. While each state as a point in the phase space corresponds to such a functional for each observable, their real number values for the position and momentum observables are not enough to fully distinguish the state. Such a value does not give full information about the observable for the state, for which one may need an infinite number of real numbers. A careful thinking would reveal a further deficiency of the evaluation functionals for the quantum observable algebra compared to the classical one when one looks at the relation between such functionals. The classical state as an evaluation map is really a homomorphism; that is to say
that it maps each observable as a real variable onto a real number value in such a way that the whole observable algebra is mapped into a subalgebra of the algebra of real numbers with the algebraic relations among the variables preserved by their values. Mathematically, we have the variable $fg$ as a product of variables $f$ and $g$ as an example,

$$
[x_i, p_i](fg) = fg(x_i, p_i) = f(x_i, p_i) g(x_i, p_i) = [x_i, p_i](f) [x_i, p_i](g) .
$$

$fg(x_i, p_i)$ as a function equals to the product of functions $f(x_i, p_i)$ and $g(x_i, p_i)$ by definition, and that simply applies to the values of the functions at any point. In fact, a relation of this kind is very important in physics, any relationship between the observables predicted theoretically can only be verified by checking their values. The evaluation map for the quantum observables taken as given by the evaluation functionals giving the expectation values obviously fails to maintain that. In fact, no functional can do that. Functionals have real numbers as values, which commute among themselves. But the quantum observables do not. If one can find a functional $[\omega]$ that, for example, satisfies

$$
[\omega](\hat{X}_i \hat{P}_i) = [\omega](\hat{X}_i) [\omega](\hat{P}_i) ,
$$

to which the expectation value functional certainly fails, we would have

$$
[\omega](\hat{P}_i) [\omega](\hat{X}_i) = [\omega](\hat{X}_i) [\omega](\hat{P}_i) ,
$$

giving the result

$$
[\omega](\hat{X}_i \hat{P}_i) = [\omega](\hat{P}_i \hat{X}_i)
$$

for the operator products $\hat{X}_i \hat{P}_i$ and $\hat{P}_i \hat{X}_i$ evaluated on any state $[\omega]$. The latter is as good as the statement that the observables $\hat{X}_i \hat{P}_i$ and $\hat{P}_i \hat{X}_i$ are really the same. However, the basic structure of quantum observable algebra is exactly the nontrivial commutation relation.

Following the above, we can see that it would be a good idea to generalize the notion of an evaluation functional with a real number value to a noncommutative analog with a value that is an element of a noncommutative algebra keeping the evaluation map as a homomorphism between the quantum observable algebra and the algebra of its values. Moreover, we would like each such noncommutative value to have the information content of infinite number of real numbers covering all information in the probability/statistical
distribution obtainable from von Neumann measurements, with the expectation value playing an important role inside. One can even think about the algebra of all such noncommutative values as like the set of ‘noncommutative numbers’. It is of interest to note that the preface of one of Takesaki’s books (2003) on the theory of operator algebra starts with the sentence

“The author believes that the theory of operator algebras should be viewed as a number theory in analysis.”

We see that with the notion of the noncommutative values, the operators, or quantum observables, can be seen rather as noncommutative number variables. Together with the noncommutative value notion, a complete picture of ‘noncommutative number theory’ may evolve.

Furthermore, the six $\hat{X}_i$ and $\hat{P}_i$ observables to serve as a system of noncommutative coordinates of the quantum phase space as the projective Hilbert space, we need again each of the noncommutative coordinates to have a value carrying the information of an infinite number of real numbers. One can think about the value of a quantum observable as a piece of quantum information (Braunstein & van Loock, 2005), mathematical seen as an element of a noncommutative algebra, each of which can be seen as like an infinite number of classical information (i.e. real numbers);

“a single qubit can substitute for an infinite number of classical bits”, as Hardy (2013) put it (Galvão and Hardy, 2003).

### 3.2 A Solid Answer to Our Call

Though the notion of the noncommutative value has not been introduced by other authors, the solution candidate has been available. For each quantum state, a one-to-one homomorphism, between observables in quantum mechanics and a noncommutative algebra of so-called ‘symmetry data’ with each element as a set of infinite number of real numbers, has been given by Schilling (1996), in a Ph.D dissertation under the supervision of Ashtekar. We want to skip much of the technical details here only to sketch the essence of the story. *We emphasize that each noncommutative value is one mathematical quantity*, an element of a noncommutative algebra much like a real number is an element of a commutative algebra. *It is not a necessity to think about it as the set of infinite number of real numbers*, though a convenience before we learn to deal with the noncommutative values directly.

The Ashtekar-Schilling homomorphism is a map for each state that takes a quantum observable $\hat{A}$ as an operator essentially to the set of values of all
the derivatives of expectation value function $[\omega](\hat{A})$ taken as a function of the state $\omega$, say, explicitly as functions of the real coordinates $(q_n, s_n)$. Let us put it as $[\omega](\hat{A}) = f_\lambda(q_n, s_n)$. By all derivatives here, we include the zeroth derivative, which is just the expectation value function itself. Actually, only the derivatives up to the second order are needed as the higher order derivatives for the class of functions can be expressed in terms of the zeroth, first, and second order ones, and the known metric of phase space. With infinite number of coordinates ($n$ counting from 1 to $\infty$), we have an infinite set of real number values. The derivatives of the expectation value function for the operator product $f_{\hat{A}\hat{B}}(q_n, s_n)$ can be given in terms of the derivatives of $f_\lambda(q_n, s_n)$ and $f_\rho(q_n, s_n)$ that verifies the evaluation map we put as $[[\omega]]$ sending an observable $\hat{A}$ to $[[\omega]](\hat{A})$, the latter being the noncommutative value of ‘symmetry data’, as a homomorphism between the observable algebra $\hat{A}$ and the noncommutative algebra of $[[\omega]](\hat{A})$. For example, with the noncommutative, but associative, product as the rules to give the $f_{\hat{A}\hat{B}}(q_n, s_n)$ derivatives marked by $\star_\kappa$, we have

$$[[\omega]](\hat{A}\hat{B}) = [[\omega]](\hat{A}) \star_\kappa [[\omega]](\hat{B}),$$

and

$$[[\omega]](\hat{A}\hat{B}) - [[\omega]](\hat{B}\hat{A}) = [[\omega]](\hat{A}) \star_\kappa [[\omega]](\hat{B}) - [[\omega]](\hat{B}) \star_\kappa [[\omega]](\hat{A}),$$

for each $[[\omega]]$ on any observables $\hat{A}$ and $\hat{B}$. The map is of course a linear one. The noncommutative value to be identified as the set of infinite number of real numbers should each be seen directly as what it is, namely an element in a noncommutative algebra.

$$[[\omega]](\hat{A}\hat{B}) = [[\omega]](\hat{A}) \star_\kappa [[\omega]](\hat{B}).$$

Explicitly, more conveniently expressed in terms of derivatives with respect to the complex coordinates $z_n = q_n + is_n$ taking value at $z_n = \tilde{z}_n = \tilde{q}_n + i\tilde{s}_n$ characterizing a particular $[[\omega]]$, we have a description of the noncommutative value as $[[\omega]](\hat{A}) = \{f_\lambda, \frac{\partial f_\lambda}{\partial z_1}, \frac{\partial f_\lambda}{\partial z_2}, \ldots, \frac{\partial f_\lambda}{\partial z_1 \partial z_1}, \frac{\partial f_\lambda}{\partial z_1 \partial z_2}, \ldots, \frac{\partial f_\lambda}{\partial z_2 \partial z_1}, \ldots\}$, where $\tilde{z}_n = q_n - is_n$ and all functions inside the expression are to be evaluated at $z_n = \tilde{z}_n = \tilde{q}_n + i\tilde{s}_n$. Only up to second order derivatives are needed. Getting the noncommutative product $[[\omega]](\hat{A}\hat{B})$ from $[[\omega]](\hat{A})$ and $[[\omega]](\hat{B})$ is just above calculating the expressions of the (complex number values) of $f_{\hat{A}\hat{B}}$, $\frac{\partial f_{\hat{A}\hat{B}}}{\partial z_m}$ and $\frac{\partial f_{\hat{A}\hat{B}}}{\partial z_m \partial z_n}$, for $m, n = 1$ to $\infty$, from those of $f_\lambda$ and $f_\rho$ following the
noncommutative product $\star_\kappa$ as essentially explicitly given by Cirelli, Manià, and Pizzocchero (1990). The complex number values in the sequence of complex coordinate derivatives can certainly be expressed in term of real number values for the real coordinates $q_n$ and $s_n$, only that the expression is formally more complicated. Conceptually, we mostly talk about them here in terms of the real ones. From the real number values of $\hat{f}_A(q_n, s_n)$ and $\hat{f}_B(q_n, s_n)$ at a point, one cannot get the corresponding value for $\hat{f}_{\hat{A}\hat{B}}(q_n, s_n)$. However, from the values of all the derivatives of $\hat{f}_A(q_n, s_n)$ and $\hat{f}_B(q_n, s_n)$, one can retrieve from the noncommutative product $\star_\kappa$ the values of all the derivatives of $\hat{f}_{\hat{A}\hat{B}}(q_n, s_n)$. That infinite set of values for the derivatives is what is to be seen as a single noncommutative value.

3.3 The Noncommutative Geometric Picture

Although Ashtekar and Schilling were apparently not aware of the important work of Cirelli, Manià, and Pizzocchero (1990) when performing their study, quite a part of their results have been given in the latter paper with, in a way, a more direct picture. For example, Ashtekar and Schilling mostly worked with commutators and anticommutators instead of the products. The latter authors have also given an interesting isomorphism, which can be seen as a logical precursor of the Ashtekar-Schilling map. The Cirelli-Manià-Pizzocchero isomorphism involves essentially a noncommutative associative product, called the Kähler product, between the expectation value functions $\hat{f}_A(q_n, s_n)$, which we write as

$$\hat{f}_{\hat{A}\hat{B}}(q_n, s_n) = \hat{f}_A(q_n, s_n) \star_\kappa \hat{f}_B(q_n, s_n)$$

or equivalently

$$[\omega](A\hat{B}) = [\omega](\hat{A}) \star_\kappa [\omega](\hat{B}) .$$

The product involves differentiation with respect to the coordinates. Having the product between the functions of course implies relations between the derivatives and hence their values. While the quantum observables as operators naively cannot be seen as functions $f(q_n, s_n)$, the expectation value functions $\hat{f}_A(q_n, s_n)$ can be written as $\hat{f}_A(q_n, s_n) \star_\kappa 1$, hence each corresponds to a differential operator $\hat{f}_A(q_n, s_n)\star_\kappa$ involving the derivatives of $f_A(q_n, s_n)$. The algebra of operators $\hat{f}_A(q_n, s_n)\star_\kappa$ is isomorphic to the observable algebra of $\hat{A}$. In fact, they should really be seen as the same algebra described differently. With the noncommutative Kähler product, the quantum observable
algebra may then be seen as an algebra of functions on the quantum phase space as its space of pure states. That fulfills the basic idea of noncommutative geometry, except that the actual geometry does not look, naively, any noncommutative.

To see that quantum phase space as a noncommutative geometry, recall that $A = A(\hat{X}_i, \hat{P}_i)$. Then, the alternative description of $A$ as $f_i(q_n, s_n)$ is exactly saying that the six $\hat{X}_i$ and $\hat{P}_i$ may be seen as an alternative set of coordinate variables in the place of the infinite set of $q_n$ and $s_n$. So, each point of the quantum phase space can be specified by the infinite real number values of the real coordinate variables $q_n$ and $s_n$, or the six noncommutative values of noncommutative coordinate variables $\hat{X}_i$ and $\hat{P}_i$. Another way to see that is to think about the infinite number of the derivatives of the six $f_i(q_n, s_n)$ and $f_n(q_n, s_n)$ as the alternative infinite number of real number coordinates. Physicists and mathematicians are very familiar with describing the same geometry with different choices of coordinate systems and making transformations among them, including having complex number valued coordinates each of which has the information content of two real number coordinates. Going between the commutative and noncommutative coordinates (Kong & Liu, 2021b) may simply be a new manifestation of the kind.

Looking further into quantum dynamics from the perspective of the noncommutative coordinates, the theory can be seen as giving equations of motion for the time development of any observable, or rather the noncommutative value of the observable. That is the Heisenberg equation of motion. The equation can be ‘transformed’ into a form given in terms of any of the time dependent $f_i(q_n, s_n)$ function for the evolving observable, and the $f_n(q_n, s_n)$ for the physical Hamiltonian operator or energy observable. The resulted equation is the Poisson bracket form of equation of motion for a classical-like observable with the classical-like energy observable as the generator of time evolution in the usual Hamiltonian formulation (Johns, 2005). Hence in exactly the same mathematical form as for classical mechanics except with a different phase space. The generic equation applied to the functions $f_{q_n}(q_n, s_n) = q_n$ and $f_{s_n}(q_n, s_n) = s_n$ give essentially the Schrödinger equation of motion. Moreover, the Heisenberg equation of motion itself should be seen as exactly an equation of the kind with $\frac{1}{\hbar}$ times the operator commutator seen as the Poisson bracket for the quantum observables as $\hat{A} = A(\hat{X}_i, \hat{P}_i)$ (Cirelli, Manià, & Pizzocchero, 1990; Schilling, 1996). Note that expression for the quantum operator, or noncommutative variable, now appreciated as
the quantum Poisson bracket, was identified early to give the classical Poisson bracket at the classical limit by Dirac. Again, that simply suggests thinking about $\hat{X}_i$ and $\hat{P}_i$ as pairs of noncommutative canonical coordinates of the phase space, the position and momentum coordinate observables indeed for the phase space as a symplectic geometry. One can actually obtained from it the standard form of Hamilton’s equation of motion for $\hat{X}_i$ and $\hat{P}_i$:

$$\frac{d}{dt}\hat{X}_i = \frac{\partial \hat{H}}{\partial \hat{P}_i}, \quad \frac{d}{dt}\hat{P}_i = -\frac{\partial \hat{H}}{\partial \hat{X}_i},$$

where we have the physical Hamiltonian operator $\hat{H} = H(\hat{X}_i, \hat{P}_i)$, to be matched to

$$\frac{d}{dt}x_i = \frac{\partial H(x_i, p_i)}{\partial p_i}, \quad \frac{d}{dt}p_i = -\frac{\partial H(x_i, p_i)}{\partial x_i},$$

in classical mechanics with $H(x_i, p_i)$ as the Hamiltonian function (Kong & Liu, 2021b). From the mathematical studies, there are in fact good indications that noncommutative geometries, as the spaces of pure states, may all be in a way symplectic (Chen, 2014; Cirelli, Lanzavecchia, & Manià, 1983; Roberts & Teh, 2016; Shultz, 1982).

Physicists are familiar with the transformation between the Heisenberg picture and the Schrödinger picture of quantum dynamics. The standard interpretation of that is to depict the time evolution for the same expectation value of an observable on a state (vector) in terms of either the evolving observable in the former or the evolving state in the latter. With the Hamiltonian formulation perspective for both the Heisenberg and the Schrödinger equations of motion, they are simply about depicting the time evolution for the observable, or its noncommutative value, in terms of the noncommutative coordinates as $A(\hat{X}_i, \hat{P}_i)$ or the commutative coordinates as $f_A(q_n, s_n)$, respectively. That is exactly a coordinate transformation picture.

We note in passing that the notion of the operator ‘coordinate observables’, or noncommutative coordinate variables, $\hat{X}_i$ and $\hat{P}_i$ as coordinates for the noncommutative geometry is very much a pure physics one. It is based on quantum mechanics as the quantum version of classical mechanics. Mathematicians may otherwise not be particularly interested in the specific algebra. While the notion of local coordinates can be used to define a commutative geometry as a real manifold, which is locally (around any particular point) essentially a copy of a Euclidean geometry, noncommutative geometries are to be defined algebraically with a basic pure geometric picture under...
pursuit. What is a local picture of a noncommutative geometry? What is the noncommutative analog of an Euclidean space? One cannot even be sure that such questions are sensible ones. Physicists have however been working on noncommutative spacetime models essentially based on the idea of the noncommutative position, and time, coordinate variables (Doplicher, Fredenhagen, & Roberts, 1995; Wess & Zumino, 1990; Wess, 2007), commonly believed to be necessary for Planck scale physics. To keep a good control on the otherwise speculative nature of such physical theories, the nature of the physics for the noncommutative coordinate variables of quantum mechanics may be a very useful guideline, especially if that also gives a noncommutative model of the physical space.

3.4 The Quantum Physical Space

We have painted almost a full picture of the noncommutative quantum reality above. Quantum mechanics is to be seen as a theory with physical quantities described by noncommutative observable variables, taking noncommutative values on a state. The state of a particle is a point in the noncommutative symplectic geometry of its quantum phase space, to be fully characterized by the noncommutative values of the six noncommutative position and momentum coordinate observables. All of that is exactly like the case of classical mechanics if the adjectives ‘noncommutative’ are replaced by ‘commutative’, of course also ‘quantum’ replaced by ‘classical’. The classical theory is exactly the commutative approximation to the quantum one. One thing is however missing. That is the quantum or noncommutative picture of the physical space.

A naive answer to the question may be the physical space as seen from quantum mechanics is the totality of all possible noncommutative values for the three position observables $\hat{X}_i$. Not only that we do not have a commutative geometric picture about it, the idea has quite some problems. Considering the three $\hat{X}_i$ and ‘functions’ of them only, there is zero noncommutativity. And it is not clear what may be the role of the algebra in physics. On the other hand, there is actually a simple argument telling that the quantum phase space, unlike its classical counterpart, cannot be properly seen as a product of the space of positions and the space of momenta. It is so exactly in the same way the Minkowski spacetime as the spacetime model from Einstein’s theory of special relativity cannot be properly seen as a product of a three dimensional space and a one dimensional time. Technically, it is
a question of the spacetime model as a representation space of the relevant (relativity) symmetry (Chew, Kong, & Payne, 2017). Minkowski spacetime, seen as a vector space, corresponds to what is called an irreducible representation of the Lorentz symmetry. Its Newtonian approximation, with the Lorentz transformation symmetries replaced by the Galilean counterparts, is a reducible representation that reduces to Newtonian space and Newtonian time as irreducible components. For the quantum mechanics, part of the relevant symmetry is the Heisenberg-Weyl symmetry which incorporates the noncommutativity between the $\hat{X}_{i}$ and $\hat{P}_{i}$, the quantum Hilbert space is essentially the only representation space obtainable from an irreducible representation (Taylor, 1986). Hence, the quantum phase space is the only sensible model for the quantum physical space. While within any specific Lorentz frame of reference the Minkowski spacetime can be seen as a product of a Newtonian space and the geometric space/line of time, an admissible Lorentz boost to another frame of reference would take the time line to a new direction which is partly an original spatial direction. As all the Lorentz frames are on equal footing, no part or subspace, no line, in the Minkowski spacetime can be definitely identified as what time is all about. Hence the Minkowski spacetime has to be seen as a single ‘entity’. That is the notion of the Minkowski spacetime as an irreducible representation. Quantum mechanics as an analog case has a fundamental symmetry which gives a complex phase transformation to the Hilbert space vectors actually without observable consequence. It takes the real number coordinates $q_{n}$ and $s_{n}$, to $q'_{n}$ and $s'_{n}$ with $q'_{n} + is'_{n} = e^{i\theta}(q_{n} + is_{n})$. The configuration/position (sub)space of described then by the $q'_{n}$ coordinates is not the same as the one described by the $q_{n}$ coordinates. There is no reference frame independent way to identify a configuration/position subspace.

It is of interest to note that within the mathematics of noncommutative algebra, there is actually a notion of dimension (Mc Connell & Robson, 2001) relevant to the quantum picture of the physical space here. The (projective) Hilbert space taken as a cyclic module of the noncommutative ring of polyno-

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5The mathematical description of any system or object observing a certain set of symmetries is known to correspond essentially to a representation of the group as the mathematical description of the set of symmetries. A representation is reducible if it can be seen from the symmetry picture, as consisting of a few independent parts. Otherwise, it is irreducible. Hence, to the extent that space or spacetime is seen to have a certainly (relativity) symmetry, its mathematical description should correspond to such a representation (Elliott & Dawber, 1979).
mials in $\hat{X}_i$ and $\hat{P}_i$ has a (Gel’fand-Kirillov) dimension of three, which agrees with our intuitive notion of dimension for the physical space. In fact, the corresponding cyclic module for the algebra generated by $n$ pairs of $X_i-P_i$ has Gel’fand-Kirillov dimension of exactly $n$. The three pairs of $X_i-P_i$ are certainly (linear) independent, though each pair has the fixed Heisenberg commutation relation. That is at least how far we see that the notion of Gel’fand-Kirillov dimension matches the intuitive picture familiar in their classical, commutative, limit. In the latter, Newtonian, limit, however, $x_i$ and $p_i$ become independent too and the position space and momentum spaces exist as separable notions.

4 Further Discussions and Conclusions

Wess (2007) had remarked that

“That a change in the concept of space for very short distances might be necessary was already anticipated in 1854 by Riemann in his famous inaugural lecture.”

as he also quoted from Riemann (1854)

“Now it seems that the empirical notions on which the metric determinations of space are founded, the notion of a solid body and a light ray, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena . . .

. . . The answer to these questions can only be got by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. Researches starting from general notions, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices.”

He further noted that

“. . . from the discovery of quantum mechanics. There physics data forced us to introduce the concept of noncommutativity.”

and advocated the idea of noncommutative coordinates, only not so much for the phase space or the physical space of quantum mechanics itself. Of course
a consistent picture of the latter cannot be obtained without the notion of the noncommutative values. We echo, again, Quine’s “To be is to be the value of a variable” here. The “very short distances” most physicists working on noncommutative geometry have in mind are much shorter than the relevant scale of quantum mechanics, for which we already actually have “the notion of a solid body and a light ray” as classical physics knows them “cease to be valid”. The picture of noncommutative quantum reality does give a “simpler explanation of phenomena” even in line with our intuition so long as that is not “checked by traditional prejudices”.

Let us take up further on the “traditional prejudices” of physical quantities having real number values. Bohr emphasized a lot about all measurements as giving classical, here to be read as real number, results. Upon more careful thinking, the real numbers we have been reading out of our measuring apparatus are always really from the real number scales we put onto the apparatus during our calibrations. Even in reading a simple pointer position, we get the real number answer only because we use the real number geometry model to look at the physical space in which the pointer and the scale sit. Otherwise, the real number value is never indicated by Nature. In line with mathematical fictionalism (Leng, 2010) is Quine’s notion of “convenient fiction”. For the case of quantum physics, however, real numbers as part of that old fiction is not even convenient. The noncommutative values are the new convenient fiction. Maybe we should simply deal with the noncommutative values directly as pieces of quantum information. Indeed, all kinds of quantum information experiments can be seen as giving some measurements on a quantum system. We are extracting information about the system under study, which can be used, together with our theory, to get information about the initial state of the system. How one may directly deal with the noncommutative values or quantum information is a very nontrivial question which requires ingenious thinking in practical experimental settings, to which we are not capable of giving any specific suggestion at this point. The world is quantum hence all information obtainable from it of quantum nature, though a major part of those we have learned to manipulate can be well approximated by classical, real number, information. It is more like we have succeeded, with our classical physical theories, to deal with physical phenomena in which the model of real number valued physical quantities seems to offer a good enough description, as in classical physics measurements and projective measurements for quantum physics, rather than the our world being so much as apparently commutative. Of course technically
the quantum theory gives conditions when its classical counterpart works well enough as an approximation. Our science and technology is only starting to deal with quantum information is relatively simple but yet extremely unclassical settings. There is a very long way to go. A more humble approach is to determine the infinite sequence of complex numbers \([\omega]\)(\(\hat{A}\)) as a description of a noncommutative value up to a certain precision with probably a good number of real number measurements, which would not impose much difficulty in principle. Otherwise, a lot more effort on the part of theoretical and experimental physics may be needed for us to gain more understanding of the noncommutative value picture.

It is of interested to note that in 1934, Haldane (1934) spoke against quantum mechanics as a “refutation of materialism”, calling it “a refutation of . . . . . . spatialism”. Haldane’s spatialism is “the Cartesian view of matter as definitely localized in space”. Our noncommutative quantum reality picture presents the theory as not a refutation of spatialism either, but rather more like only a refutation of the simple Cartesian view of space, namely the Newtonian space model itself. Haldane also remarked about that “mental events are inexactly localized in space-time”, where the space-time of course means the Newtonian models of them. A puzzling question is then how well can we see mental events as localized in noncommutative quantum models of space-time.

The business of science is about describing ‘reality’, and fundamental physical theories try to do that with conceptual and mathematical models. Going from classical mechanics to quantum mechanics as a more successful theory, we have changed our models for the observables and the states. The mathematical perspective of noncommutative geometry or algebraic geometry says that the observable algebra and the phase space should be seen as dual descriptions of one another. Hence, the ‘reality’ is actually to be described by the relations among the values of all the observables, with the state being given by the values of the basic observables, the position and momentum ones. To properly understand quantum mechanics as the relations among the ‘real’ values of the observables, we also have to change the classical model for such values, to the noncommutative one. The evaluation homomorphism then fully illustrates the duality. The model for the physical space, with ‘positions’ or locations, points, to be described by the coordinate observables, has also to be changed from the classical Newtonian model. Only then we have a fully consistent story for that quantum reality. 

*The simple move to fully embrace the quantum noncommutativity of Nature*
actually allows us to see quantum mechanics as being as much about the non-commutative reality as classical mechanics is about the assumed commutative reality.\footnote{While the article focuses on the philosophical presentation of the notion of quantum reality as offered by a new theoretical perspective on the theory of quantum mechanics taking the noncommutativity more seriously than ever as the core feature and properties of Nature it depicts, the physics and mathematics of the key new conceptual notion of the noncommutative values of physical quantities has been studied in Kong (2020, 2021), Kong & Liu (2021a) in the meantime.}

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