Distributed estimation of generalized matrix rank: Efficient algorithms and lower bounds

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Abstract

We study the following generalized matrix rank estimation problem: given an \( n \times n \) matrix and a constant \( c \geq 0 \), estimate the number of eigenvalues that are greater than \( c \). In the distributed setting, the matrix of interest is the sum of \( m \) matrices held by separate machines. We show that any deterministic algorithm solving this problem must communicate \( \Omega(n^2) \) bits, which is order-equivalent to transmitting the whole matrix. In contrast, we propose a randomized algorithm that communicates only \( \tilde{O}(n) \) bits. The upper bound is matched by an \( \Omega(n) \) lower bound on the randomized communication complexity. We demonstrate the practical effectiveness of the proposed algorithm with some numerical experiments.

1 Introduction

Given a parameter \( c \geq 0 \), the generalized rank of an \( n \times n \) positive semidefinite matrix \( A \) corresponds to the number of eigenvalues that are larger than \( c \). It is denoted by \( \text{rank}(A, c) \), with the usual rank corresponding to the special case \( c = 0 \). Estimating the generalized rank of a matrix is useful for many applications. In the context of large-scale principal component analysis (PCA) [11, 15], it is overly expensive to compute the full eigendecomposition before deciding when to truncate it. Thus, an important first step is to estimate the rank of the matrix of interest in order to determine how many dimensions will be sufficient to describe the data. The rank also provides useful information for determining the tuning parameter of robust PCA [4] and collaborative filtering algorithms [26, 24]. In the context of numerical linear algebra, a number of eigensolvers [27, 23, 25] for large-scale scientific applications are based on divide-and-conquer paradigms. It is a prerequisite of these algorithms to know the approximate number of eigenvalues located in a given interval. Estimating the generalized rank of a matrix is also needed in the context of sampling-based methods for randomized numerical linear algebra [13, 21]. For these methods, the rank of a matrix determines the number of samples required for a desired approximation accuracy.

Motivated by large-scale data analysis problems, in this paper we study the generalized rank estimation problem in a distributed setting, in which the matrix \( A \) can be decomposed as the the sum of \( m \) matrices

\[
A := \sum_{i=1}^{m} A_i, \tag{1}
\]
where each matrix $A_i$ is stored on a separate machine $i$. Thus, a distributed algorithm needs to communicate between $m$ machines to perform the estimation. There are other equivalent formulations of this problem. For example, suppose that machine $i$ has a design matrix $X_i \in \mathbb{R}^{n \times N_i}$ and we want to determine the rank of the aggregated design matrix

$$X := (X_1, X_2, \ldots, X_m) \in \mathbb{R}^{n \times N} \quad \text{where} \quad N := \sum_{i=1}^m N_i.$$  

Recall that the singular values of matrix $X$ are equal to the square root of the eigenvalues of the matrix $XX^T$. If we define $A_i := X_iX_i^T$, then equation (1) implies that

$$A = \sum_{i=1}^m A_i = \sum_{i=1}^m X_iX_i^T = XX^T.$$  

Thus, determining the generalized rank of the matrix $X$ reduces to the problem of determining the rank of the matrix $A$. In this paper, we focus on the formulation given by equation (1).

The standard way of computing the generalized matrix rank, or more generally of computing the number of eigenvalues within a given interval, is to exploit Sylvester’s law of inertia [12]. Concretely, if the matrix $A - cI$ admits the decomposition $A - cI = LDL^T$, where $L$ is unit lower triangular and $D$ is diagonal, then the number of eigenvalues of matrix $A$ that are greater than $c$ is the same as the number of positive entries in the diagonal of $D$. While this method yields an exact count, in the distributed setting it requires communicating the entire matrix $A$. Due to bandwidth limitations and network delays, the $\Theta(n^2)$ communication cost is a significant bottleneck on the algorithmic efficiency. For a matrix of rank $r$, the power method [12] can be used to compute the top $r$ eigenvalues, which reduces the communication cost to $\Theta(rng)$. However, this cost is still prohibitive for moderate sizes of $r$. Recently, Napoli et al. [10] studied a more efficient randomized approach for approximating the eigenvalue counts based on Chebyshev polynomial approximation of high-pass filters. When applying this algorithm to the distributed setting, the communication cost is $\Theta(pm)$, where $p$ is the degree of Chebyshev polynomials. However, the authors note that polynomials of high degree can be necessary.

In this paper, we study the communication complexity of distributed algorithms for the problem of generalized rank estimation, in both the deterministic and randomized settings. We establish upper bounds by deriving practical, communication-efficient algorithms, and we also establish complexity-theoretic lower bounds. Our first main result shows that no deterministic algorithm is efficient in terms of communication. In particular, communicating $\Omega(n^2)$ bits is necessary for all deterministic algorithms to approximate the matrix rank with constant relative error. That such algorithms cannot be viewed as efficient is due to the fact that by communicating $O(n^2)$ bits, we are able to compute all eigenvalues and the corresponding eigenvectors. In contrast to the inefficiency of deterministic algorithms, we propose a randomized algorithm that approximates matrix rank by communicating $\tilde{O}(n)$ bits. When the matrix is of rank $r$, the relative approximation error is $1/\sqrt{r}$. Under the same relative error, we show that $\Omega(n)$ bits of communication is necessary, establishing the optimality of our algorithm. This is in contrast with the $\Omega(rn)$ communication complexity lower bound for randomized PCA [16]. The difference shows that estimating the eigenvalue count using a randomized algorithm is easier than estimating the top $r$ eigenpairs.

The research on communication complexity has a long history, dating back to the seminal work of Yao [30] and Abelson [1]. Characterizing the communication complexity of linear algebraic operations is a fundamental question. For the problem of rank testing, Chu and Schnitger [5, 6]
prove the $\Omega(n^2)$ communication complexity lower bound for deterministically testing the singularity of integer-valued matrices. A successful algorithm for this task is required to distinguish two types of matrices—the singular matrices and the non-singular matrices with arbitrarily small eigenvalues—a requirement that is often too severe for practical applications. Luo and Tsitsiklis [20] prove an $\Omega(n^2)$ lower bound for computing one entry of $A^{-1}$, applicable to exact algorithms (with no form of error allowed). In contrast, our deterministic lower bound holds even if we force the non-zero eigenvalues to be bounded away from zero and allow for approximation errors, making it more widely applicable to the inexact algorithms used in practice. For randomized algorithms, Li et al. [28, 19] prove $\Omega(n^2)$ lower bounds for the problems of rank testing, computing a matrix inverse, and solving a set of linear equations over finite fields. To the best of our knowledge, it is not known whether the same lower bounds hold for matrices in the real field. In other related work, Clarkson and Woodruff [7] give an $\Omega(r^2)$ space lower bound in the streaming model for distinguishing between matrices of rank $r$ and $r - 1$. However, such a space lower bound in the streaming model does not imply a communication complexity lower bound in the two-way communication model studied in this paper.

2 Background and problem formulation

In this section, we begin with more details on the problem of estimating generalized matrix ranks, as well as some background on communication complexity.

2.1 Generalized matrix rank

Given an $n \times n$ positive semidefinite matrix $A$, we use $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0$ to denote its ordered eigenvalues. For a given constant $c \geq 0$, the generalized rank of order $c$ is given by

$$\text{rank}(A, c) = \sum_{k=1}^{n} [\sigma_k(A) > c], \quad (2)$$

where $[\sigma_k(A) > c]$ is a 0-1-valued indicator function for the event that $\sigma_k(A)$ is larger than $c$. Since $\text{rank}(A, 0)$ is equal to the usual rank of a matrix, we see the motivation for using the generalized rank terminology. We assume that $\|A\|_2 = \sigma_1(A) \leq 1$ so that the problem remains on a standardized scale.

In an $m$-machine distributed setting, the matrix $A$ can be decomposed as a sum $A = \sum_{i=1}^{m} A_i$, where the $n \times n$ matrix $A_i$ is stored on machine $i$. We study distributed protocols, to be specified more precisely in the following section, in which each machine $i$ performs local computation involving the matrix $A_i$, and the machines then exchange messages so to arrive at an estimate $\hat{r}(A) \in [n] := \{0, \ldots, n\}$. Our goal is to obtain an estimate that is close to the rank of the matrix in the sense that

$$(1 - \delta)\text{rank}(A, c_1) \leq \hat{r}(A) \leq (1 + \delta)\text{rank}(A, c_2), \quad (3)$$

where $c_1 > c_2 \geq 0$ and $\delta \in [0, 1)$ are user-specified constants. The parameter $\delta \in [0, 1)$ upper bounds the relative error of the approximation. The purpose of assuming different thresholds $c_1$ and $c_2$ in bound (3) is to handle the ambiguous case when the matrix $A$ has many eigenvalues smaller but very close to $c_1$. If we were to set $c_1 = c_2$, then any estimator $\hat{r}(A)$ would be strictly...
prohibited to take these eigenvalues into account. However, since these eigenvalues are so close to the threshold, distinguishing them from other eigenvalues just above the threshold is obviously difficult (but for an uninteresting reason). Setting \( c_1 > c_2 \) allows us to expose the more fundamental sources of difficulty in the problem of estimating generalized matrix ranks.

### 2.2 Basics of communication complexity

To orient the reader, here we provide some very basic background on communication complexity theory; see the books [18, 17] for more details. The standard set-up in multi-party communication complexity is as follows: suppose that there are \( m \) players (equivalently, agents, machines, etc.), and for \( i \in \{1, \ldots, m\} \), player \( i \) holds an input string \( x_i \). In the standard form of communication complexity, the goal is to compute a joint function \( F(x_1, \ldots, x_m) \) of all \( m \) input strings with as little communication between machines as possible. In this paper, we analyze a communication scheme known as the public blackboard model, in which each player can write messages on a common blackboard to be read by all other players. A distributed protocol \( \Pi \) consists of a coordinated order in which players write messages on the blackboard. Each message is constructed from the player’s local input and the earlier messages on the blackboard. At the end of the protocol, some player outputs the value of \( F(x_1, \ldots, x_m) \) based on the information she collects through the process. The communication cost of a given protocol \( \Pi \), which we denote by \( C(\Pi) \), is the maximum number of bits written on the blackboard given an arbitrary input.

In a deterministic protocol, all messages must be deterministic functions of the local input and previous messages. The deterministic communication complexity computing function \( F \), which we denote by \( D(F) \), is defined by

\[
D(F) := \min \left\{ C(\Pi) : \Pi \text{ is a deterministic protocol that correctly computes } F \right\}.
\]  

In other words, the quantity \( D(F) \) is the communication cost of the most efficient deterministic protocol.

A broader class of protocols are those that allow some form of randomization. In the public randomness model, each player has access to an infinite-length random string, and their messages are constructed from the local input, the earlier messages and the random string. Let \( \mathcal{P}_\epsilon(F) \) be the set of randomized protocols that correctly compute the function \( F \) on any input with probability at least \( 1 - \epsilon \). The randomized communication complexity of computing function \( F \) with failure probability \( \epsilon \) is given by

\[
R_\epsilon(F) := \min \left\{ C(\Pi) : \Pi \in \mathcal{P}_\epsilon(F) \right\}.
\]  

In the current paper, we adopt the bulk of the framework of communication complexity, but with one minor twist in how we define “correctness” in computing the function. For our problem, each machine is a player, and the \( i \)th player holds the matrix \( A_i \). Our function of interest is given by \( F(A_1, \ldots, A_m) = \text{rank}(\sum_{i=1}^m A_i) \). The public blackboard setting corresponds to a broadcast-free model, in which each machine can send messages to a master node, then the master node broadcasts the messages to all other machines without additional communication cost.

Let us now clarify the notion of “correctness” used in this paper. In the standard communication model, a protocol \( \Pi \) is said to correctly compute the function \( F \) if the output of the protocol is exactly equal to \( F(A_1, \ldots, A_m) \). In this paper, we allow approximation errors in the computation,
as specified by the parameters \((c_1, c_2)\), which loosen the matrix rank to the generalized matrix ranks, and the tolerance parameter \(\delta \in (0, 1)\). More specifically, we say:

**Definition 1.** A protocol \(\Pi\) correctly computes the rank of the matrix \(A\) up to tolerances \((c_1, c_2, \delta)\) if the output \(\hat{r}(A)\) satisfies inequality (3).

Given this definition of correctness, we denote the deterministic communication complexity of the rank estimation problem by \(D(c_1, c_2, \delta)\), and the corresponding randomized communication complexity by \(R_\epsilon(c_1, c_2, \delta)\). The goal of this paper is to study these two quantities, especially their dependence on the dimension \(n\) of matrices.

In addition to allowing for approximation error, our analysis—in contrast to most classical communication complexity—allows the input matrices \(\{A_i\}_{i=1}^m\) to take real values. However, doing so does not make the problem substantially harder. Indeed, in order to approximate the matrices in elementwise \(\ell_\infty\)-norm up to \(\tau\) rounding error, it suffices to discretize each matrix entry using \(O(\log(1/\tau))\) bits. As we discuss in more detail in the sequel, this type of discretization has little effect on the communication complexity.

### 3 Main results and their consequences

This section is devoted to statements of our main results, as well as discussion of some of their consequences.

#### 3.1 Bounds for deterministic algorithms

We begin by studying the communication complexity of deterministic algorithms. Here our main result shows that the trivial algorithm—the one in which each machine transmits essentially its whole matrix—is optimal up to logarithmic factors. In the statement of the theorem, we assume that the \(n\)-dimensional matrix \(A\) is known to have rank in the interval \([r, 2r]\) for some integer \(r \leq n/4\).

**Theorem 1.** For matrices \(A\) with rank in the interval \([r, 2r]\):

(a) For all \(0 \leq c_2 < c_1\) and \(\delta \in (0, 1)\), we have \(D(c_1, c_2, \delta) = O\left(mrn \log \left(\frac{mrn}{c_1-c_2}\right)\right)\).

(b) For two machines \(m = 2\), constants \(0 \leq c_2 < c_1 < 1/20\) and \(\delta \in (0, 1/12)\), we have \(D(c_1, c_2, \delta) = \Omega(rn)\).

When the matrix \(A\) has rank \(r\) that grows proportionally with its dimension \(n\), the lower bound in part (b) shows that deterministic communication complexity is surprisingly large: it scales as \(\Theta(n^2)\), which is as large as transmitting the full matrices. Up to logarithmic factors, this scaling is matched by the upper bound in part (a). It is proved by analyzing an essentially trivial algorithm: for each index \(i = 2, \ldots, m\), machine \(i\) encodes a reduced rank representation of the matrix \(A_i\), representing each matrix entry by \(\log_2\left(\frac{12mrn}{c_1-c_2}\right)\) bits. It sends this quantized matrix \(\tilde{A}_i\) to the first machine. Given these received messages, the first machine then computes the matrix sum \(\tilde{A} := A_1 + \sum_{i=2}^m \tilde{A}_i\), and it outputs \(\hat{r}(\tilde{A})\) to be the largest integer \(k\) such that \(\sigma_k(\tilde{A}) > (c_1 + c_2)/2\).

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1We use an interval assumption, as the problem becomes trivial if the rank is fixed exactly.
On the other hand, in order to prove the lower bound, we consider a two-party rank testing problem. Consider two agents holding matrices $A_1$ and $A_2$, respectively, such that the matrix sum $A := A_1 + A_2$ has operator norm at most one. Suppose that exactly one of the two following conditions are known to hold:

- the matrix $A$ has rank $r$, or
- the matrix $A$ has rank between $\frac{6r}{5}$ and $2r$, and in addition its $(6r/5)^{th}$ eigenvalue is lower bounded as $\frac{\sigma_{6r/5}(A)}{20}$.

The goal is to decide which case is true by exchanging the minimal number of bits between the two agents. Denoting this problem by $\text{RankTest}$, the proof of part (a) proceeds by showing first that $D(\text{RankTest}) = \Omega(rn)$, and then reducing from the $\text{RankTest}$ problem to the matrix rank estimation problem. See Section 4.1 for the proof.

3.2 Bounds for randomized algorithms

We now turn to the study of randomized algorithms, for which we see that the communication complexity is substantially lower. In Section 3.2.1, we propose a randomized algorithm with $\mathcal{O}(n)$ communication cost, and in Section 3.2.3, we establish a lower bound that matches this upper bound in various regimes.

3.2.1 Upper bounds via a practical algorithm

In this section, we present an algorithm based on uniform polynomial approximations for estimating the generalized matrix rank. Let us first provide some intuition for the algorithm before defining it more precisely. For a fixed pair of scalars $c_1 > c_2 \geq 0$, consider the function $H_{c_1,c_2} : \mathbb{R} \rightarrow [0,1]$ given by

$$H_{c_1,c_2}(x) := \begin{cases} 
1 & \text{if } x > c_1 \\
0 & \text{if } x < c_2 \\
\frac{x - c_2}{c_1 - c_2} & \text{otherwise.} 
\end{cases}$$

As illustrated in Figure 1, it is a piecewise linear approximation to a step function. The squared function $H_{c_1,c_2}^2$ is useful in that it can be used to sandwich the generalized ranks of a matrix $A$. In particular, given a positive semidefinite matrix $A$ with ordered eigenvalues $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_n(A) \geq 0$, observe that we have

$$\text{rank}(A, c_1) \leq \sum_{i=1}^{n} H_{c_1,c_2}^2(\sigma_i(A)) \leq \text{rank}(A, c_2).$$

Our algorithm exploits this sandwich relation in estimating the generalized rank.

In particular, suppose that we can find a polynomial function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \approx H_{c_1,c_2}$, and which is extended to a function on the cone of PSD matrices in the standard way. Observe that if $\sigma$ is an eigenvalue of $A$, then the spectral mapping theorem \cite{spectral_mapping} ensures that $f(\sigma)$ is an eigenvalue
of $f(A)$. Consequently, letting $g \sim N(0, I_{n \times n})$ be a standard Gaussian vector, we have the useful relation

$$E \left[ \left\| f(A)g \right\|_2^2 \right] = \sum_{i=1}^{n} f^2(\sigma_i(A)) \approx \sum_{i=1}^{n} H_{c_1,c_2}^2(\sigma_i(A)). \quad (8)$$

Combined with the sandwich relation (7), we see that a polynomial approximation $f$ to the function $H_{c_1,c_2}$ can be used to estimate the generalized rank.

If $f$ is a polynomial function of degree $p$, then the vector $f(A)g$ can be computed through $p$ rounds of communication. In more detail, in one round of communication, we can first compute the matrix-vector product $Ag = \sum_{i=1}^{m} A_i g$. Given the vector $Ag$, a second round of communication suffices to compute the quantity $A^2g$. Iterating a total of $p$ times, the first machine is equipped with the collection of vectors \{g, Ag, A^2g, \ldots, A^pg\}, from which it can compute $f(A)g$.

Let us now consider how to obtain a suitable polynomial approximation of the function $H_{c_1,c_2}$. The most natural choice is a Chebyshev polynomial approximation of the first kind: more precisely, since $H_{c_1,c_2}$ is a continuous function with bounded variation, classical theory \cite[Theorem 5.7]{22} guarantees that the Chebyshev expansion converges uniformly to $H_{c_1,c_2}$ over the interval $[0, 1]$. Consequently, we may assume that there is a finite-degree Chebyshev polynomial $q_1$ of the first kind such that

$$\sup_{x \in [0,1]} |q_1(x) - H_{c_1,c_2}(x)| \leq 0.1. \quad (9a)$$

By increasing the degree of the Chebyshev polynomial, we could reduce the approximation error (set to 0.1 in the expansion (9a)) to an arbitrarily small level. However, a very high degree could be necessary to obtain an arbitrary accuracy. Instead, our strategy is to start with the Chebyshev polynomial $q_1$ that guarantees the 0.1-approximation error (9a), and then construct a second polynomial $q_2$ such that the composite polynomial function $f = q_2 \circ q_1$ has an approximation error, when measured over the intervals $[0, c_2]$ and $[c_1, 1]$ of interest, that converges linearly in the degree of function $f$. More precisely, consider the polynomial of degree $2p + 1$ given by

$$q_2(x) = \frac{1}{B(p+1,p+1)} \int_{0}^{x} t^p(1-t)^p dt \quad \text{where } B(\cdot, \cdot) \text{ is the Beta function.} \quad (9b)$$

Figure 1: An illustration of the function $x \mapsto H_{c_1,c_2}(x)$ with $c_1 = 0.5$ and $c_2 = 0.1$. 
Figure 2. Comparison of the composite polynomial approximation in Algorithm 2 with the Chebyshev polynomial expansion. The error is measured with the $\ell_\infty$-norm on the interval $[0, c_2] \cup [c_1, 1]$. The composite polynomial approximation achieves a linear convergence rate as the degree is increased, while the Chebyshev expansion converges at a much slower rate.

Lemma 1. Consider the composite polynomial $f(x) := q_2(q_1(x))$, where the base polynomials $q_1$ and $q_2$ were previously defined in equations (9a) and (9b) respectively. Then $f(x) \in [0, 1]$ for all $x \in [0, 1]$, and moreover

$$|f(x) - H_{c_1,c_2}(x)| \leq 2^{-p} \quad \text{for all } x \in [0, c_2] \cup [c_1, 1].$$

See Appendix A for the proof.

Figure 2 provides a comparison of the error in approximating $H_{c_1,c_2}$ for the standard Chebyshev polynomial and the composite polynomial. In order to conduct a fair comparison, we show the approximations obtained by Chebyshev and composite polynomials of the same final degree, and we evaluate the $\ell_\infty$-norm approximation error on interval $[0, c_2] \cup [c_1, 1]$—namely, for a given polynomial approximation $h$, the quantity

$$\text{Error}(h) := \sup_{x \in [0, c_2] \cup [c_1, 1]} |h(x) - H_{c_1,c_2}(x)|.$$

As shown in Figure 2 shows, the composite polynomial function achieves a linear convergence rate with respect to its degree. In contrast, the convergence rate of the Chebyshev expansion is sublinear, and substantially slower than that of the composite function. The comparison highlights the advantage of our approach over the method only based on Chebyshev expansions.

Given the composite polynomial $f = q_2 \circ q_1$, we first evaluate the vector $f(A)g$ in a two-stage procedure. In the first stage, we evaluate $q_1(A)g, q_1^2(A)g, \ldots, q_1^{2p+1}(A)g$ using the Clenshaw recurrence [8], a procedure proven to be numerically stable [22]. The details are given in Algorithm 1. In the second stage, we substitute the coefficients of $q_2$ so as to evaluate $q_2(q_1(A))b$. The overall procedure is summarized in Algorithm 2.

The following result provides a guarantee for the overall procedure (combination of Algorithm 1 and Algorithm 2) when run with degree $p = \lceil \log_2(2n) \rceil$:

Theorem 2. For any $0 \leq \delta < 1$, with probability at least $1 - 2\exp\left(-\frac{T(\hat{r}(A),c_1)}{32}\right)$, the output of Algorithm 2 satisfies the bounds

$$(1 - \delta)\text{rank}(A, c_1) - 1 \leq \hat{r}(A) \leq (1 + \delta)(\text{rank}(A, c_2) + 1).$$

(11)
Algorithm 1: Evaluation of Chebyshev Polynomial

**Input:** $m$ machines hold $A_1, A_2, \ldots, A_m \in \mathbb{R}^{n \times n}$; vector $v \in \mathbb{R}^d$; Chebyshev polynomial expansion $q(x) = \frac{1}{2}a_0T_0(x) + \sum_{i=1}^{d}a_iT_i(x)$.

**Output:** matrix-vector product $q(A)v$.

1. Initialize vector $b_{d+1} = b_{d+2} = 0 \in \mathbb{R}^n$.
2. For $j = d, \ldots, 1, 0$: the first machine broadcasts $b_{j+1}$ to all other machines. Machine $i$ computes $A_i b_{j+1}$ and sends it back to the first machine. The first machine computes 
   \[ b_j := \left( 4 \sum_{i=1}^{m} A_i b_{j+1} \right) - 2b_{j+1} - b_{j+2} + a_j v. \]
3. Output $\frac{1}{2}(a_0 v + b_1 - b_3)$.

Algorithm 2: Randomized Algorithm for Rank Estimation

**Input:** Each of $m$ machines hold matrices $A_1, A_2, \ldots, A_m \in \mathbb{R}^{n \times n}$. Tolerance parameters $(c_1, c_2)$, polynomial degree $p$, and number of repetitions $T$.

1. (a) Find a Chebyshev expansion $q_1$ of the function $H_{c_1, c_2}$ satisfying the uniform bound (9a).
   (b) Define the degree $2p + 1$ polynomial function $q_2$ by equation (9b).
2. (a) Generate a random Gaussian vector $g \sim N(0, I_{n \times n})$.
   (b) Apply Algorithm 1 to compute $q_1(A)g$, and sequentially apply the same algorithm to compute $q_2^2(A)g, \ldots, q_2^{2p+1}(A)g$.
   (c) Evaluate the vector $y := f(A)g = q_2(q_1(A))g$ on the first machine.
3. Repeat Step 2 for $T$ times, obtaining a collection of $n$-vectors $\{y_1, \ldots, y_T\}$, and output the estimate $\hat{\tau}(A) = \frac{1}{T} \sum_{i=1}^{T} \|y_i\|_2^2$.

Moreover, we have the following upper bound on the randomized communication complexity of estimating the generalized matrix rank:

\[ \mathcal{R} (c_1, c_2, 1/\sqrt{\text{rank}(A, c_1)}) = \tilde{O}(mn). \]  

(12)

We show in Section 3.2.3 that the upper bound (12) is unimprovable up to the logarithmic pre-factors. For now, let us turn to the results of some numerical experiments using Algorithm 2, which show that in addition to being an order-optimal algorithm, it is also practically useful.

3.2.2 Numerical experiments

Given $m = 2$ machines, suppose that machine $i$ (for $i = 1, 2$) receives $N_i = 1000$ data points of dimension $n = 1000$. Each data point $x$ is independently generated as $x = a + \varepsilon$, where $a \sim N(0, \lambda \Sigma)$ and $\varepsilon \sim N(0, \sigma^2 I_{n \times n})$ are random Gaussian vectors. Here $\Sigma \in \mathbb{R}^{n \times n}$ is a low-rank covariance matrix.
of the form $\Sigma := \sum_{i=1}^{r} u_i u_i^T$, where $\{u_i\}_{i=1}^{r}$ are an orthonormal set of vectors in $\mathbb{R}^n$ drawn uniformly at random. The goal is to estimate the rank $r$ from the observed $N_1 + N_2 = 2000$ data points.

Let us now describe how to estimate the rank using the covariance matrix of the samples. Notice that $E[xx^T] = \lambda^2 \Sigma + \sigma^2 I_{n \times n}$, of which there are $r$ eigenvalues equal to $\lambda + \sigma^2$ and the remaining eigenvalues are equal to $\sigma^2$. Letting $x_{i,j} \in \mathbb{R}^n$ denote the $j$-th data point received by machine $i$, that machine can compute the local sample covariance matrix

$$A_i = \frac{1}{N_1 + N_2} \sum_{j=1}^{N_i} x_{i,j} x_{i,j}^T, \quad \text{for } i = 1, 2.$$  

The full sample covariance matrix is given by the sum $A := A_1 + A_2$, and its rank can be estimated using Algorithm 2.

In order to generate the data, we choose the parameters $r = 100$, $\lambda = 0.4$ and $\sigma^2 = 0.1$. These choices motivate the thresholds $c_1 = \lambda + \sigma^2 = 0.5$ and $c_2 = \sigma^2 = 0.1$ in Algorithm 2. We illustrate the behavior of the algorithm for three different choices of the degree parameter $p$—specifically, $p \in \{0, 1, 5\}$—and for a range of repetitions $T \in \{1, 2, \ldots, 30\}$. Letting $\hat{r}(A)$ denote the output of the algorithm, we evaluate the mean squared error, $E[(\hat{r}(A) - r)^2]$, based on 100 independent runs of the algorithm.

We plot the results of this experiment in Figure 3. Panel (a) shows the distribution of eigenvalues of the matrix $A$. In this plot, there is a gap between the large eigenvalues generated by the low-rank covariance matrix $\Sigma$, and small eigenvalues generated by the random Gaussian noise, showing that the problem is relatively easy to solve in the centralized setting. Panel (b) shows the estimation error achieved by the communication-efficient distributed algorithm; notice how the estimation error stabilizes after $T = 30$ repetitions or iterations. We compare our algorithm for $p \in \{0, 1, 5\}$, corresponding to polynomial approximations with degree in $\{4, 12, 44\}$. For the case $p = 0$, the polynomial approximation is implemented by the Chebyshev expansion. For the case $p = 1$ and $p = 5$, the approximation is achieved by the composite function $f$. As a baseline method, we also implement Napoli et al.’s algorithm [10] in the distributed setting. In particular, their method replaces the function $f$ in Algorithm 2 by a Chebyshev expansion of the high-pass filter $\mathbb{I}(x \geq \frac{c_1+c_2}{2})$. 

Figure 3. Panel (a): distribution of eigenvalues of matrix $A$. Panel (b): mean squared error of rank estimation versus the number of iterations for the baseline method by Napoli et al. [10], and three versions of Algorithm 2 (with parameters $p \in \{0, 1, 5\}$).
It is observed that both the Chebyshev expansion with \( p = 0 \) and the baseline method incur a large bias in the rank estimate, while the composite function’s estimation errors are substantially smaller. After \( T = 30 \) iterations, Algorithm 2 with \( p = 1 \) achieves a mean squared error close to 10, which means that the relative error of the estimation is around 3%.

### 3.2.3 Lower Bound

It is natural to wonder if the communication efficiency of Algorithm 2 is optimal. The following theorem shows that, in order to achieve the same \( 1/\sqrt{r} \) relative error, it is necessary to send \( \Omega(n) \) bits. As in our upper bound, we assume that the matrix \( A \) satisfies the spectral norm bound \( \|A\|_2 \leq 1 \). Given an arbitrary integer \( r \) in the interval \([16, n/4]\), suppose that the generalized matrix ranks satisfy the sandwich relation \( r \leq \text{rank}(A, c_1) \leq \text{rank}(A, c_2) \leq 2r \). Under these conditions, we have the following guarantee:

**Theorem 3.** For any \( c_1, c_2 \) satisfying \( c_1 < 2c_2 \leq 1 \) and any \( \epsilon \leq \epsilon_0 \) for some numerical constant \( \epsilon_0 \), we have

\[
\mathcal{R}_\epsilon(c_1, c_2, 1/\sqrt{r}) = \Omega(n).
\]

(13)

See Section 4.3 for the proof of this lower bound.

According to Theorem 3, for matrices with true rank in the interval \([16, n/2]\), the communication complexity for estimating the rank with relative error \( 1/\sqrt{r} \) is lower bounded by \( \Omega(n) \). This lower bound matches the upper bound provided by Theorem 2. In particular, choosing \( r = 16 \) yields the worst-case lower bound

\[
\mathcal{R}_\epsilon(c_1, c_2, 1/4) = \Omega(n),
\]

showing that \( \Omega(n) \) bits of communication are necessary for achieving a constant relative error. This lower bound is not trivial relative to the coding length of the correct answer: given that the matrix rank is known to be between \( r \) and \( 2r \), this coding length scales only as \( \Omega(\log r) \).

There are several open problems suggested by the result of Theorem 3. First, it would be interesting to strengthen the lower bound (13) from \( \Omega(n) \) to \( \Omega(mn) \), incorporating the natural scaling with the number of machines \( m \). Doing so requires a deeper investigation into the multi-party structure of the problem. Another open problem is to lower bound the communication complexity for arbitrary values of the tolerance parameter \( \delta \), say as small as \( 1/r \). When \( \delta \) is very small, communicating \( O(mn^2) \) bits is an obvious upper bound, and we are not currently aware of better upper bounds. On the other hand, whether it is possible to prove an \( \Omega(n^2) \) lower bound for small \( \delta \) remains an open question.

### 4 Proofs

In this section, we provide the proofs of our main results, with the proofs of some more technical lemmas deferred to the appendices.

#### 4.1 Proof of Theorem 1

Let us begin with our first main result on the deterministic communication complexity of the generalized rank problem.
4.1.1 Proof of lower bound

We first prove the lower bound stated in part (a) of Theorem 1. Let us recall the RankTest problem previously described after the statement of Theorem 1. Alice holds a matrix $A_1 \in \mathbb{R}^{n \times n}$ and Bob holds a matrix $A_2 \in \mathbb{R}^{n \times n}$ such that the matrix sum $A := A_1 + A_2$ has operator norm at most one. Either the matrix $A$ has rank $r$, or the matrix $A$ has rank between $\frac{6r}{5}$ and $2r$, and in addition its $6r/5$ eigenvalue is lower bounded as $\sigma_{6r/5}(A) > \frac{1}{2r}$. The RankTest problem is to decide which of these two mutually exclusive alternatives holds. The following lemma provides a lower bound on the deterministic communication complexity of this problem:

Lemma 2. For any $r \leq n/4$, we have $D(\text{RankTest}) = \Omega(rn)$.

We use Lemma 2 to lower bound $D(c_1, c_2, \delta)$, in particular by reducing to it from the RankTest problem. Given a RankTest instance, since there are $m \geq 2$ machines, the first two machines can simulate Alice and Bob, holding $A_1$ and $A_2$ respectively. All other machines hold a zero matrix. Suppose that $c_1 \leq 1/20$ and $\delta \leq 1/12$. If there is an algorithm achieving the bound (3), then if $A = A_1 + A_2$ is of rank $r$, then

$$\tilde{r}(A) \leq (1 + \delta)\text{rank}(A, c_2) \leq \left(1 + \frac{1}{12}\right)r = \frac{13r}{12}. \quad (14a)$$

Otherwise, the $\frac{6r}{5}$-th eigenvalue of $A$ is greater than $1/20$, so that

$$\tilde{r}(A) \geq (1 - \delta)\text{rank}(A, c_1) \geq (1 - \frac{1}{12})(\frac{6r}{5}) = \frac{11r}{10} > \frac{13r}{12}. \quad (14b)$$

In conjunction, inequality (14a) and (14b) show that we can solve the RankTest problem by testing whether or not $\tilde{r}(A) \leq \frac{13r}{12}$. Consequently, the deterministic communication complexity $D(c_1, c_2, \delta)$ is lower bounded by the communication complexity of RankTest.

In order to complete the proof of Theorem 1(a), it remains to prove Lemma 2, and we do so using a randomized construction. Let us say that a matrix $Q \in \mathbb{R}^{r \times n}$ is sampled from the orthogonal ensemble if it is sampled in the following way: let $U \in \mathbb{R}^{n \times n}$ be a matrix uniformly sampled from the group of orthogonal matrices, then $Q$ is the sub-matrix consisting of the first $r$ rows of $U$. We have the following claim.

Lemma 3. Given matrices $Q_1 \in \mathbb{R}^{r \times n}$ and $Q_2 \in \mathbb{R}^{r \times n}$ independently sampled from the orthogonal ensemble, we have $\sigma_{6r/5}(Q_1^T Q_1 + Q_2^T Q_2) > \frac{1}{10}$ with probability at least $1 - e^{-\frac{3rn}{100}}$.

See Appendix B for the proof.

Taking Lemma 3 as given, introduce the shorthand $N = \lfloor \frac{rn}{100} \rfloor$. Suppose that we independently sample $2^N$ matrices of dimensions $r \times n$ from the orthogonal ensemble. Since there are $2^N(2^N - 1)/2$ distinct pairs of matrices in our sample, the union bound in conjunction with Lemma 3 implies that

$$\mathbb{P}\left[ \forall i \neq j : \sigma_{6r/5}(Q_i^T Q_i + Q_j^T Q_j) > \frac{1}{10} \right] \geq 1 - \frac{2^N(2^N - 1)}{2} \exp\left( -\frac{3rn}{100} \right). \quad (15)$$
With our choice of \( N \), it can be verified that the right-hand side of inequality (15) is positive. Thus, there exists a realization of orthogonal matrices \( Q_1, \ldots, Q_{2N} \in \mathbb{R}^{r \times n} \) such that for all \( i \neq j \) we have \( \sigma_{op}(Q_i^T Q_i + Q_j^T Q_j) > \frac{1}{10} \).

We use this collection of orthogonal matrices in order to reduce the classical Equality problem to the rank estimation problem. In the Equality problem, Alice has a binary string \( x_1 \in \{0,1\}^N \) and Bob has another binary string \( x_2 \in \{0,1\}^N \), and their goal is to compute the function

\[
\text{Equality}(x_1, x_2) = \begin{cases} 
1 & \text{if } x_1 = x_2; \\
0 & \text{otherwise}; 
\end{cases}
\]

It is well-known [17] that the deterministic communication complexity of the Equality problem is \( D(\text{Equality}) = N + 1 \).

In order to perform the reduction, given binary strings \( x_1 \) and \( x_2 \) of length \( N \), we construct two matrices \( A_1 \) and \( A_2 \) such that their sum \( A = A_1 + A_2 \) has rank \( r \) if and only if \( x_1 = x_2 \). Since both \( x_1 \) and \( x_2 \) are of length \( N \), each of them encodes an integer between 1 and \( 2^N \). Defining \( A_1 = \frac{Q_1^T Q_1}{2} \) and \( A_2 = \frac{Q_2^T Q_2}{2} \), the triangle inequality guarantees that

\[
\|A\|_2 \leq \|A_1\|_2 + \|A_2\|_2 = \frac{\|Q_1^T Q_{x_1}\|_2 + \|Q_2^T Q_{x_2}\|_2}{2} \leq 1,
\]

showing that \( A \) satisfies the required operator norm bound. If \( x_1 = x_2 \), then \( A = Q_{x_1}^T Q_{x_1} \), which is a matrix of rank \( r \). If \( x_1 \neq x_2 \), then by our construction of \( Q_{x_1} \) and \( Q_{x_2} \), we know that the matrix \( A \) has rank between \( \frac{6r}{5} \) and \( 2r \) and moreover that \( \sigma_{op}(A) > \frac{1}{20} \). Thus, we can output \( \text{Equality}(x_1, x_2) = 1 \) if we detect the rank of matrix \( A \) to be \( r \) and output \( \text{Equality}(x_1, x_2) = 0 \) otherwise. Using this protocol, the Equality evaluation is always correct. As a consequence, the deterministic communication complexity of \( \text{RankTest} \) is lower bounded by that of Equality. Finally, noting that \( D(\text{RankTest}) \geq D(\text{Equality}) = N + 1 > \frac{c_1}{60} \) completes the proof.

### 4.1.2 Proof of upper bound

In order to prove the upper bound stated in part (b), we analyze the algorithm described following the theorem statement. If the matrix \( A = \sum_{i=1}^m A_i \) has rank at most \( 2r \), then given the PSD nature of the component matrices, each matrix \( A_i \) also has rank at most \( 2r \). Consequently, we can find a factorization of the form \( A_i = B_i B_i^T \) where \( B_i \in \mathbb{R}^{n \times r} \). Let \( \tilde{B}_i \) be a quantization of the matrix \( B_i \), allocating \( \log_2 \left( \frac{12mn}{c_1 - c_2} \right) \) bits to each entry. Note that each machine must transmit at most \( rn \log_2 \left( \frac{12mn}{c_1 - c_2} \right) \) bits in order to convey the quantized matrix \( \tilde{B}_i \).

Let us now analyze the approximation error. By our choice of quantization, we have

\[
\|\tilde{B}_i - B_i\|_{op} \leq \|\tilde{B}_i - B_i\|_F \leq \sqrt{2rn} \frac{c_1 - c_2}{12mn} = \frac{c_1 - c_2}{6m} \sqrt{2rn}.
\]

Defining \( \tilde{A}_i = \tilde{B}_i \tilde{B}_i^T \) we have

\[
\|\tilde{A}_i - A_i\|_F \leq \|\tilde{B}_i - B_i\|_F \sqrt{2rn} \left( \|B_i\|_{op} + \|\tilde{B}_i\|_{op} \right) \leq \frac{c_1 - c_2}{6m} \left( 2 + \frac{c_1 - c_2}{6m} \sqrt{2rn} \right) \leq \frac{c_1 - c_2}{2m},
\]

13
where the final inequality follows as long as \( \frac{c_1 - c_2}{6m \sqrt{2m}} \leq 1 \).

Consequently, the sum \( \tilde{A} = \sum_{i=1}^{m} \tilde{A}_i \) satisfies the bound
\[
\| \tilde{A} - A \|_F \leq \sum_{i=2}^{m} \| \tilde{A}_i - A_i \|_F \leq \frac{(c_1 - c_2)}{2}.
\]

Applying the Wielandt-Hoffman inequality \([14]\) yields the upper bound
\[
|\sigma_k(\tilde{A}) - \sigma_k(A)| \leq \| \tilde{A} - A \|_F \leq (c_1 - c_2)/2 \quad \text{for all } k \in [n].
\]

Recalling that \( \hat{r}(A) \) is the largest integer \( k \) such that \( \sigma_k(\tilde{A}) > (c_1 + c_2)/2 \), inequality \((16)\) implies that
\[
(c_1 + c_2)/2 \geq \sigma_{\hat{r}(A) + 1}(\tilde{A}) \geq \sigma_{\hat{r}(A) + 1}(A) - (c_1 - c_2)/2,
\]
which implies \( \sigma_{\hat{r}(A) + 1}(A) \leq c_1 \). This upper bound verifies that \( \hat{r}(A) \geq \hat{r}(A, c_1) \). On the other hand, inequality \((16)\) also yields
\[
(c_1 + c_2)/2 < \sigma_{\hat{r}(A)}(\tilde{A}) \leq \sigma_{\hat{r}(A)}(A) + (c_1 - c_2)/2,
\]
which implies \( \sigma_{\hat{r}(A)}(A) > c_2 \) and \( \hat{r}(A) \leq \hat{r}(A, c_2) \). Combining the above two inequalities yields the claim \((3)\).

### 4.2 Proof of Theorem 2

We split the proof into two parts, corresponding to the upper bounds \((11)\) and \((12)\) respectively.

**Proof of upper bound \((11)\):** Let \( \lambda_j \) be the \( j \)-th largest eigenvalue of \( A \) and let \( v_j \) be the associated eigenvector. Let function \( f \) be defined as \( f(x) := q_2(q_1(x)) \). Using basic linear algebra, we have
\[
\|y\|_2^2 = \sum_{j=1}^{n} f^2(\lambda_j)(v_j^T g)^2.
\]

Since \( g \) is an isotropic Gaussian random vector, the random variables \( Z_j = (v_j^T g)^2 \) are i.i.d., each with \( \chi^2 \) distribution with one degree of freedom. To analyze the concentration behavior of \( Z \) variables, we recall the notion of a sub-exponential random variable.

A random variable \( Y \) is called sub-exponential with parameter \( (\sigma^2, \beta) \) if \( E[Y] = 0 \) and the moment generating function is upper bounded as \( \mathbb{E}[e^{tY}] \leq e^{t^2 \sigma^2/2} \) for all \( |t| \leq 1/\beta \). The following lemma, proved in Appendix C, characterizes some basic properties of sub-exponential random variables.

**Lemma 4.** (a) If \( Z \sim \chi^2 \), then both \( Z - 1 \) and \( 1 - Z \) are sub-exponential with parameter \((4, 4)\).

(b) Given an independent sequence \( \{Y_i\}_{i=1}^{n} \) in which \( Y_i \) is sub-exponential with parameter \((\sigma_i^2, \beta_i)\), then for any choice of non-negative weights \( \{\alpha_i\}_{i=1}^{n} \), the weighted sum \( \sum_{i=1}^{n} \alpha_i Y_i \) is sub-exponential with parameters \( \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2, \max_{i \in [n]} \{\alpha_i \beta_i\} \).
(c) If $Y$ is sub-exponential with parameter $(\sigma^2, \beta)$, then

$$
\mathbb{P}[Y \geq t] \leq e^{-\frac{t^2}{2\sigma^2}} \quad \text{for all } t \in [0, \frac{\sigma^2}{\beta}].
$$

We consider $\|y\|^2_2$ as well as the associated lower bound $L = \sum_{i=1}^{\text{rank}(A,c_1)} f^2(\lambda_i) (v^T_i b)^2$. By parts (a) and (b) of Lemma 4, the variable $\|y\|^2_2 - \mathbb{E}[\|y\|^2_2]$ is sub-exponential with parameter $(4 \sum_{i=1}^{n} f^2(\lambda_i), 4)$, and the variable $\mathbb{E}[L] - L$ is sub-exponential with parameter $(4 \sum_{i=1}^{\text{rank}(A,c_1)} f^2(\lambda_i), 4)$. In order to apply part (c) of Lemma 4, we need upper bounds on the sum $\sum_{i=1}^{n} f^2(\lambda_i)$, as well as upper/lower bounds on the sum $\sum_{i=1}^{\text{rank}(A,c_1)} f^2(\lambda_i)$. For the first sum, we have

$$
\sum_{j=1}^{n} f^2(\lambda_j) = \sum_{j=1}^{\text{rank}(A,c_2)} f^2(\lambda_j) + \sum_{j=\text{rank}(A,c_2)+1}^{n} f^2(\lambda_j)
$$

$$
\leq \text{rank}(A,c_2) + n 2^{-p}
$$

$$
\leq \text{rank}(A,c_2) + 1.
$$

(18)

where the last two inequalities use Lemma 1 and the fact that $p = \lceil \log_2(2n) \rceil$. For the second sum, using Lemma 1 implies that

$$
\text{rank}(A,c_1) \geq \sum_{i=1}^{\text{rank}(A,c_1)} f^2(\lambda_i) \geq \text{rank}(A,c_1)(1 - 2^{-p})^2
$$

$$
\geq \text{rank}(A,c_1)(1 - 1/(2n))^2 \geq \text{rank}(A,c_1) - 1.
$$

where inequality (i) follows since $2^{-p} \leq 1/(2n)$; inequality (ii) follows since $(1 - 1/(2n))^2 \geq 1 - 1/n$. Thus, we have

$$
\mathbb{E}[\|y\|^2_2] \leq \text{rank}(A,c_2) + 1 \quad \text{and} \quad \mathbb{E}[L] \geq \text{rank}(A,c_1) - 1.
$$

(19)

Putting together the pieces, we see that $\|y\|^2_2 - \mathbb{E}[\|y\|^2_2]$ is sub-exponential with parameter $(4(\text{rank}(A,c_2) + 1), 4)$ and $\mathbb{E}[L] - L$ is sub-exponential with parameter $(4 \text{rank}(A,c_1), 4)$.

Let $\tilde{r}$ be the average of $T$ independent copies of $\|y\|^2_2$, and let $\tilde{r}_L$ be the average of $T$ independent copies of $L$. By Lemma 4 (b), we know that $\tilde{r} - \mathbb{E}[\tilde{r}]$ is sub-exponential with parameter $(4(\text{rank}(A,c_2)+1)/T, 4/T)$, and $\mathbb{E}[\tilde{r}_L] - \tilde{r}_L$ is sub-exponential with parameter $(4 \text{rank}(A,c_1)/T, 4/T)$. Plugging these parameters into Lemma 4 (c), for any $0 \leq \delta < 1$, we find that

$$
\mathbb{P}[\tilde{r} \leq \mathbb{E}[\tilde{r}] + \delta(\text{rank}(A,c_2) + 1)] \geq 1 - \exp\left(-\frac{T\delta^2(\text{rank}(A,c_2)+1)}{32}\right)
$$

(20a)

$$
\mathbb{P}[\tilde{r}_L \geq \mathbb{E}[\tilde{r}_L] - \delta\text{rank}(A,c_1)] \geq 1 - \exp\left(-\frac{T\delta^2\text{rank}(A,c_1)}{32}\right).
$$

(20b)

Combining inequalities (19), (20a), and (20b) yields

$$
\mathbb{P}\left[(1 - \delta)\text{rank}(A,c_1) - 1 \leq \tilde{r}_L \leq \tilde{r} \leq (1 + \delta)(\text{rank}(A,c_2) + 1)\right] \leq 1 - 2e^{-\frac{T\delta^2\text{rank}(A,c_1)}{32}},
$$

(21)

which completes the proof of inequality (11).
Proof of upper bound (12): It remains to show to establish the upper bound (12) on the randomized communication complexity. The subtle issue is that in a discrete message model, we cannot calculate \(f(A)g\) without rounding errors. Indeed, in order to make the rounding error of each individual message bounded by \(\tau\), each machine needs \(O(n \log(1/\tau))\) bits to encode a message. Consequently, the overall communication complexity scales as \(O(Tmdp n \log(1/\tau))\), where \(T\) is the number of iterations of Algorithm 2; \(m\) is the number of machines, the quantities \(d\) and \(p\) are the degrees of \(q_1\) and \(q_2\), and \(n\) is the matrix dimension. With the choices given, we have \(d = O(1)\) and \(p = O(\log n)\). In order to make inequality (11) hold with probability at least \(1 - \epsilon\), the upper bound (21) suggests choosing \(T = \Theta(1/\epsilon)\).

Finally, we need to upper bound the quantity \(O(\log(1/\tau))\). In order to do so, let us revisit Algorithm 2 to see how rounding errors affect the final output. For each integer \(k = 1, \ldots, 2p + 1\), let us denote by \(\delta_k\) the error of evaluating \(q_1^k(A)g\) using Algorithm 1. It is known [22, Chapter 2.4.2] that the rounding error of evaluating a Chebyshev expansion is bounded by \(md\tau\). Thus, we have \(\delta_{k+1} \leq \|q_1(A)\|_2\delta_k + md\tau\). Since \(\|q_1(A)\|_2 \leq 1.1\) by construction, we have the upper bound

\[
\delta_k \leq 10(1.1^{k+1} - 1)md\tau. \tag{22}
\]

For a polynomial of the form \(q_2(x) = \sum_{i=0}^{2p+1} a_i x^i\), we have \(y = \sum_{i=0}^{2p+1} a_i q_1^i(A)b\). As a consequence, there is a universal constant \(C\) such that error in evaluating \(y\) is bounded by

\[
C \sum_{i=0}^{2p+1} \delta_i |a_i| \leq C' (1.1)^{2p+1}md\tau \sum_{i=0}^{2p+1} |a_i|.
\]

By the definition of the polynomial \(q_2\) and the binomial theorem, we have

\[
\sum_{i=0}^{2p+1} |a_i| \leq \frac{2^p}{B(p+1, p+1)} = \frac{2^p (2p+1)!}{(p!)^2} \leq 2^{3p}.
\]

Putting the pieces together, in order to make the overall error small, it suffices to choose \(\tau\) of the order \((mdn)^{-1}2^{-4p}\). Doing so ensures that \(\log(1/\tau) = O(p \log(mdn))\), which when combined with our earlier upper bounds on \(d, p\) and \(T\), establishes the claim (12).

### 4.3 Proof of Theorem 3

In order to prove Theorem 3, it suffices to consider the two-player setting, since the first two machines can always simulate the two players Alice and Bob. Our proof proceeds via reduction from the 2-SUM problem [29], in which Alice and Bob have inputs \((U_1, \ldots, U_r)\) and \((V_1, \ldots, V_r)\), where each \(U_i\) and \(V_i\) are subsets of \(\{1, \ldots, L\}\). It is promised that that for every index \(i \in \{1, \ldots, r\}\), the intersection of \(U_i\) and \(V_i\) contains at most one element. The goal is to compute the sum \(\sum_{i=1}^{r} |U_i \cap V_i|\) up to an additive error of \(\sqrt{\tau}/2\). Woodruff and Zhang [29] showed that randomized communication complexity of the 2-SUM problem is lower bounded as \(\Omega(rL)\).

We note here that when \(r \geq 16\), the same communication complexity lower bound holds if we allow the additive error to be \(2\sqrt{\tau}\). To see this, suppose that Alice and Bob have inputs of length \(r/16\) instead of \(r\). By replicating their inputs 16 times, each of Alice and Bob can begin with an input of length \(r\). Assume that by using some algorithm, they can compute the 2-SUM for the replicated input with additive error at most \(2\sqrt{\tau}\). In this way, they have computed the 2-SUM
for the original input with additive error at most $\sqrt{r}/8$. Note that $\sqrt{r/8} = \sqrt{r/16}/2$. The lower bound on the 2-SUM problem implies that the communication cost of the algorithm is $\Omega(rL/16)$, which is on the same order of $\Omega(rL)$.

To perform the reduction, let $L = \lfloor n/r - 1 \rfloor$. Since $r \leq n/2$, we have $L \geq 1$. Suppose that Alice and Bob are given subsets $(U_1, \ldots, U_r)$ and $(V_1, \ldots, V_r)$, which define an underlying instance of the 2-SUM problem. Based on these subsets, we construct two $n$-dimensional matrices $A_1$ and $A_2$ and the matrix sum $A := A_1 + A_2$; we then argue that any algorithm that can estimate the generalized matrix rank of $A$ can solve the underlying 2-SUM problem.

The reduction consists of the following steps. First, Alice constructs a matrix $X$ of dimensions $rL \times n$ as follows. For each $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, L\}$, define $t(i, j) = (i - 1)L + j$, and let $X_{t(i,j)}$ denote the associated row of $X$. Letting $e_{t(i,j)} \in \mathbb{R}^n$ denote the canonical basis vector (with a single one in entry $t(i,j)$), we define

$$X_{t(i,j)} = \begin{cases} e_{t(i,j)} & \text{if } j \in U_i \\ 0 & \text{otherwise.} \end{cases}$$

Second, Bob constructs a matrix $Y$ of dimensions $rL \times n$ following the same rule as Alice, but using the subset $(V_1, \ldots, V_L)$ in place of $(U_1, \ldots, U_L)$. Now define the $n \times n$ matrices

$$A_1 := c_2 \left( X^TX + \sum_{i=1}^{r} e_{rL+i}e_{rL+i}^T \right) \quad \text{and} \quad A_2 := c_2 \left( Y^TY + \sum_{i=1}^{r} e_{rL+i}e_{rL+i}^T \right).$$

With these definitions, it can be verified that $\|A\|_2 \leq 2c_2 \leq 1$, and moreover that all eigenvalues of $A$ are either equal to $2c_2$ or at most $c_2$. Since $c_1 < 2c_2$, the quantities $\text{rank}(A, c_1)$ and $\text{rank}(A, c_2)$ are equal, and equal to the number of eigenvalues at $2c_2$. The second term in the definition of $A_1$ and $A_2$ ensures that there are at least $r$ eigenvalues equal to $2c_2$. For all $(i,j)$ pairs such that $j \in U_i \cap V_i$, the construction of $X$ and $Y$ implies that there are two corresponding rows in $X$ and $Y$ equal to each other, and both of them are canonical basis vectors. Consequently, they create a $2c_2$ eigenvalue in matrix $A$. Overall, we have $\text{rank}(A, c_1) = \text{rank}(A, c_2) = r + \sum_{i=1}^{r} |U_i \cap V_i|$, Since the problem set-up ensures that $|U_i \cap V_i| \leq 1$, we conclude $r \leq \text{rank}(A, c_1) \leq 2r$.

Now suppose that there is a randomized algorithm estimating the rank of $A$ such that

$$(1 - \delta)\text{rank}(A, c_1) \leq \hat{r}(A) \leq (1 + \delta)\text{rank}(A, c_2).$$

Introducing the shorthand $s := \sum_{i=1}^{k} |U_i \cap V_i|$, when $\delta = 1/\sqrt{r}$, we have

$$r + s - (r + s)/\sqrt{r} \leq \hat{r}(A) \leq r + s + (r + s)/\sqrt{r}.$$ 

Thus, the estimator $\hat{r}(A) - r$ computes $s$ up to additive error $(r + s)/\sqrt{r}$, which is upper bounded by $2\sqrt{r}$. It means that the rank estimation algorithm solves the 2-SUM problem. As a consequence, the randomized communication complexity of the rank estimation problem is lower bounded by $\Omega(rL) = \Omega(n)$.

5 Discussion

In this paper, we have studied the problem of estimating the generalized rank of matrices. Our main results are to show that in the deterministic setting, sending $\Theta(n^2)$ bits is both necessary and
sufficient in order to obtain any constant relative error. In contrast, when randomized algorithms are allowed, this scaling is reduced to $\Theta(n)$.

Our work suggests an important problem, one whose resolution has a number of interesting consequences. In the current paper, we establish the $\Theta(n)$ scaling of communication complexity for achieving a relative error $\delta = 1/\sqrt{r}$ where $r$ is the matrix rank. Moreover, Algorithm 2 does not guarantee higher accuracies (e.g., $\delta = 1/r$), and as discussed in Section 3.2.3, it is unknown whether the $\Omega(n)$ lower bound is tight. The same question remains open even for the special case when all the matrix eigenvalues are either greater than constant $c$ or equal to zero. In this special case, if we were to set $c_1 = c$ and $c_2 = 0$ in Algorithm 2, then it would compute ordinary matrix rank with relative error $\delta = 1/\sqrt{r}$. Although the problem is easier in the sense that all eigenvalue are promised to lie in the subset $\{0\} \cup (c,1]$, we are currently not aware of any algorithm with $\tilde{O}(n)$ communication cost achieving better error rate. On the other hand, proving a tight lower bound for arbitrary $\delta$ remains an open problem.

The special case described above is of fundamental interest because it can be reduced to many classical problems in linear algebra and convex optimization, as we describe here. More precisely, if there is an algorithm solving any of these problems, then it can be used for computing the matrix rank with relative error $\delta = 0$. On the other hand, if we obtain a tight lower bound for computing the matrix rank, then it implies a lower bound for a larger family of problems. We list a subset of these problems giving a rough intuition for the reduction.

To understand the connection, we begin by observing that the problem of rank computation can be reduced to that of matrix rank testing, in which the goal is to determine whether a given matrix sum $A := A_1 + \cdots + A_m$ has rank at most $r - 1$, or rank at least $r$, assuming that all eigenvalues belong to $\{0\} \cup (c,\infty)$. If there is an algorithm solving this problem for arbitrary integer $r \leq n$, then we can use it for computing the rank. The reduction is by performing a series of binary searches, each step deciding whether the rank is above or below a threshold. In turn, the rank test problem can be further reduced to the following problems:

**Singularity testing:** The goal of singularity testing is to determine if the sum of matrices $B := B_1 + \cdots + B_m$ is singular, where machine $i$ stores the PSD matrix $B_i$. Algorithms for singularity testing can be used for rank testing. The reduction is by using a public random coin to generate a shared random projection matrix $Q \in \mathbb{R}^{r \times n}$ on each machine and then setting $B_i := QA_i Q^T$. The inclusion of the public coin only increases the communication complexity by a moderate amount [18], in particular by an additive term $\tilde{O}(\log(n))$. On the other hand, with high probability the matrix $A$ has rank at most $r - 1$ if and only if the matrix $B$ is singular.

**Solving linear equations:** Now suppose that machine $i$ stores a strictly positive definite matrix $C_i$ and a vector $y$. The goal is to compute the vector $x$ satisfying $Cx = y$ for $C := C_1 + \cdots + C_m$. Algorithms for solving linear equations can be used for the singularity test. In particular, let $C_i := B_i + \lambda I$ and take $y$ to be a random Gaussian vector. If the matrix $B$ is singular, then the norm $\|x\|_2 \to \infty$ as $\lambda \to 0$. Otherwise, it remains finite as $\lambda \to 0$. Thus, we can test for $\lambda = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ to decide if the matrix is singular. Note that the solution need not be exact, since we only test if the $\ell_2$-norm remains finite.

**Convex optimization:** Suppose that each machine has a strictly convex function $f_i$, and the overall goal is to compute a vector $x$ that minimizes the function $x \mapsto f(x) := f_1(x) + \cdots + f_m(x)$. The algorithms solving this problem can be used for solving linear equations. In particular, for a strictly positive definite matrix $C_i$, the function $f_i(x) := \frac{1}{2}x^T C_i x - \frac{1}{2}y^T x$ is strictly convex, and
with these choices, the function $f$ is uniquely minimized at $C^{-1}y$. (Since the linear equation solver doesn’t need to be exact, the solution here is also allowed to be approximate.)

This reduction chain suggests the importance of studying matrix rank estimation, especially for characterizing lower bounds on communication complexity. We hope the results in this paper are a meaningful first step in exploring this problem area.

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A Proof of Lemma 1

The function $q_2$ is monotonically increasing on $[0,1]$. In addition, we have $q_2(0) = 0$ and $q_2(1) = 1$, and hence $q_2(z) \in [0,1]$ for all $z \in [0,1]$. Let us refine this analysis on two end intervals: namely, $z \in [-0.1,0.1]$ and $z \in [0.9,1.1]$. For $z \in [-0.1,0.1]$, it is easy to observe from the definition of $q_2$ that $q_2(z) \geq 0$. Moreover, for $z \in [-0.1,0.1]$ we have $|z(1-z)| \leq 0.11$. Thus,

$$q_2(z) = \frac{\int_{0}^{z} t^p (1-t)^p dt}{\int_{0}^{1} t^p (1-t)^p dt} \leq \frac{\int_{0}^{0.1} t^p (1-t)^p dt}{\int_{0.1}^{1} t^p (1-t)^p dt} \leq \frac{0.1 \times (0.11)^p}{0.2 \times (0.24)^p} < 2^{-p}.$$  

The function $q_2$ is symmetric in the sense that $q_2(z) + q_2(1-z) = 1$. Thus, for $z \in [0.9,1.1]$, we have $q_2(z) = 1- q_2(1-z) \in [1-2^{-p},1]$. In summary, we have proved that

$$0 \leq q_2(z) \leq 1 \text{ for } z \in [-0.1,1.1],$$

$$q_2(z) \leq 2^{-p} \text{ for } z \in [-0.1,0.1],$$

$$q_2(z) \geq 1-2^{-p} \text{ for } z \in [0.9,1.1].$$  

(23a) (23b) (23c)

By the standard uniform Chebyshev approximation, we are guaranteed that $q_1(x) \in [-0.1,1.1]$ for all $x \in [0,1]$. Thus, inequality (23a) implies that $q_2(q_1(x)) \in [0,1]$ for all $x \in [0,1]$. If $x \in [0,c_2]$, then $q_1(x) \in [-0.1,0.1]$, and thus inequality (23b) implies $q_2(q_1(x)) \leq 2^{-p}$. If $x \in [c_1,1]$, then $q_1(x) \in [0.9,1.1]$, and thus inequality (23c) implies $q_2(q_1(x)) \geq 1-2^{-p}$. Combining the last two inequalities yields that

$$|q_2(q_1(x)) - H_{c_1,c_2}(x)| \leq 2^{-p} \quad \text{for all } x \in [0,c_2] \cup [c_1,1].$$

B Proof of Lemma 3

Let $q_t$ be the $t$-th row of $Q_2$, and let $Q^{(t)} \in \mathbb{R}^{r+t}$ be the matrix whose first $r$ rows are the rows of $Q_1$, and its remaining $t$ rows are $q_1,\ldots,q_t$. Let $q_{t+1}^\perp$ be the projection of $q_{t+1}$ to the subspace generated by the rows of $Q^{(t)}$ and let $q_{t+1}^\parallel := q_{t+1} - q_{t+1}^\perp$. We have

$$(Q^{(t+1)})^T Q^{(t+1)} = (Q^{(t)})^T Q^{(t)} + q_t^T q_t = (Q^{(t)})^T Q^{(t)} + (q_{t+1}^\parallel)^T q_{t+1}^\parallel + (q_{t+1}^\perp)^T q_{t+1}^\perp \geq (Q^{(t)})^T Q^{(t)} + (q_{t+1}^\perp)^T q_{t+1}^\perp.$$
This inequality yields the lower bound
\[ Q_1^T Q_1 + Q_2^T Q_2 \geq Q_1^T Q_1 + \sum_{t=1}^{r} (q_t^\perp)^T q_t^\perp, \tag{24} \]
where \(\succeq\) denotes ordering in the positive semidefinite cone. Note that the rows of \(Q_1\) and \(\{q_t^\perp\}_{t=1}^{r}\) are mutually orthogonal. To prove that the \(6/5\)-th largest eigenvalue of \(Q_1^T Q_1 + Q_2^T Q_2\) is greater than 1/10, it suffices to prove that there are at least \(r/5\) vectors in \(\{q_t^\perp\}_{t=1}^{r}\) which satisfy \(\|q_t^\perp\|_2^2 > 1/10\).

Let \(S_1\) be the linear subspace generated by \(q_1, \ldots, q_{t-1}\) and let \(S_1^\perp\) be its orthogonal subspace. The vector \(q_t\) is uniformly sampled from a unit sphere in \(S_1^\perp\). Let \(S_2\) be the linear subspace generated by the rows of \(Q^{(t-1)}\). Since \(Q^{(t-1)}\) has \(r + t - 1\) rows, the subspace has at most \(r + t - 1\) dimensions. Without loss of generality, we assume that \(S_2\) has \(r + t - 1\) dimensions (otherwise, we expand it to reach the desired dimensionality). We let \(S_2^\perp\) be the orthogonal subspace of \(S_2\). By definition, \(q_t^\perp\) is the projection of \(q_t\) to \(S_2^\perp\) (or a linear space that contains \(S_2^\perp\) if the subspace \(S_2\) has been expanded to reach the \(r + t - 1\) dimensionality). Let \(q_t'\) be the projection of \(q_t\) to \(S_1^\perp \cap S_2^\perp\), then we have
\[ \|q_t^\perp\|_2^2 \geq \|q_t'\|_2^2. \tag{25} \]

Note that \(S_1^\perp\) is of dimension \(n - t - 1\) and \(S_2^\perp\) is of dimension \(n - r - t + 1\). Thus, the dimension of \(S_1^\perp \cap S_2^\perp\) is at least \(n - r - 2t + 2\). Constructing \(q_t'\) is equivalent to projecting a random vector in the \((n - t + 1)\)-dimension sphere to a \((n - r - 2t + 2)\)-dimension subspace. It is a standard result (e.g. [9, Lemma 2.2]) that
\[ \Pr[\|q_t'\|_2^2 \leq \beta \cdot \frac{n - r - 2t + 2}{n - t + 1}] \leq \exp\left(\frac{n - r - 2t + 2}{2}(1 - \beta + \log(\beta))\right) \text{ for any } \beta < 1. \]

Setting \(\beta = 0.3\) and using the fact that \(t \leq r \leq n/4\), we find that
\[ \Pr[\|q_t'\|_2^2 \leq 1/10] \leq \exp\left(\frac{n - n/4 - n/2 + 2}{2}(1 - 0.3 + \log(0.3))\right) \leq \exp(-n/16). \tag{26} \]

Defining the event \(\mathcal{E}_i := \{\|q_t^\perp\|_2^2 \leq 1/10\}\), note that inequality (26) yields \(\Pr[\mathcal{E}_i] \leq \exp(-n/16)\). Since \(q_t'\) is the projection of a random unit vector to a subspace of constant dimension, the events \(\{\mathcal{E}_j\}_{j=1}^t\) are mutually independent, and hence
\[ \Pr[\text{at least } \frac{4r}{5} \text{ events in } \{\mathcal{E}_j\}_{j=1}^t \text{ occur}] \leq \left(\frac{r}{4r/5}\right) \exp(-n/16)^{4r/5} \leq \exp\left(\frac{r \log(r)}{5} - \frac{rn}{20}\right) \leq \exp\left(-\frac{3rn}{100}\right), \]

where the last inequality follows since any integer \(r\) satisfies \(\log(r) \leq \frac{2r}{5} \leq \frac{n}{10}\). Thus, with probability at least \(1 - \exp(-\frac{3rn}{100})\), there are at least \(r/5\) rows satisfying \(\|q_t^\perp\|_2^2 > 1/10\). Combining this result with inequality (24) and (25) completes the proof.

C Proof of Lemma 4

The claimed facts about sub-exponential random variables are standard [3], but we provide proofs here for completeness.
Part (a): Let $Z$ be $\chi^2$ variable with one degree of freedom. Its moment generating function takes the form
\[
E[\exp(t(Z - 1))] = (1 - 2t)^{-1/2}e^{-t} \quad \text{for } t < 1/2.
\]
Some elementary algebra shows that $(1 - 2t)^{-1/2}e^{-t} \leq e^{2t^2}$ for any $t \in [-1/4, 1/4]$. Thus, we have $E[\exp(t(Z - 1))] \leq e^{2t^2}$ for $|t| \leq 1/4$, verifying the recentered variable $X = Z - 1$ is sub-exponential with parameter $(4, 4)$. Also by the moment generating function of $Z$, we have
\[
E[\exp(t(1 - Z))] = (1 + 2t)^{-1/2}e^t \quad \text{for } t > -1/2.
\]
Replacing $t$ by $-t$ and comparing with the previous conclusion reveals that $1 - Z$ is sub-exponential with parameter $(4, 4)$.

Part (b): Suppose that $Z_1, \ldots, Z_n$ are independent and $Z_i$ is sub-exponential with parameter $(\sigma^2_i, \beta_i)$. By the definition of sub-exponential random variable, we have
\[
E\left[\exp\left(t\sum_{i=1}^{n} \alpha_i Z_i\right)\right] = \prod_{i=1}^{n} E[\exp(t\alpha_i Z_i)] \leq \prod_{i=1}^{n} \exp((t\alpha_i)^2\sigma^2_i/2) = \exp\left(t^2\sum_{i=1}^{n} \alpha_i^2\sigma^2_i\right)
\]
for all $t \leq \max_{i \in [n]}\{1/(\alpha_i \beta_i)\}$. This bound establishes that $\sum_{i=1}^{n} \alpha_i Z_i$ is sub-exponential with parameter $(\sum_{i=1}^{n} \alpha_i^2\sigma^2_i, \max_{i \in [n]}\{\alpha_i \beta_i\})$, as claimed.

Part (c): Notice that $P[Z \geq t] = P[e^{\lambda Z} \geq e^{\lambda t}]$ with any $\lambda > 0$. Applying Markov’s inequality yields
\[
P[Z \geq t] \leq \frac{E[\exp(\lambda Z)]}{e^{\lambda t}} \leq \exp\left(-\lambda t + \frac{\lambda^2\sigma^2}{2}\right) \quad \text{for } \lambda \leq 1/\beta,
\]
where the last step follows since $Z$ is sub-exponential with parameter $(\sigma^2, \beta)$. Notice that the minimum of $-\lambda t + \lambda^2\sigma^2/2$ occurs when $\lambda^* = t/\sigma^2$. Since $t < \sigma^2/\beta$, we have $\lambda^* < 1/\beta$, verifying the validness of $\lambda^*$. Plugging $\lambda^*$ in the previous inequality completes the proof.

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