MAYER-VIETORIS TRIANGLES FOR MOTIVES WITH MODULUS

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Abstract. We construct “MV squares” in the category $\text{MCor}$ of modulus pairs which was introduced in [5]. They allow us to describe the category $\text{MDM}_{\text{gm}}^{\text{eff}}$ of loc. cit. in a similar way as Voevodsky’s category $\text{DM}_{\text{gm}}^{\text{eff}}$ in [13], thus sharpening the results of [5].

Contents

Introduction 1
1. Review of modulus pairs 5
2. Reduction to a cofinality statement 6
3. Further reductions 10
4. Cofinality of universally minimal squares 12
5. Existence of partial compactifications 17
6. Existence of MV-compactifications 31
Appendix A. Some remarks on the sup of Cartier divisors 43
Appendix B. Elementary Lemmas 46
Appendix C. Complements on pro-adjoints 48
References 52

Introduction

In [13], Voevodsky defines his triangulated category $\text{DM}_{\text{gm}}^{\text{eff}}$ of geometric motives over a field $k$ “by generators and relations”: generators are motives $M(X)$ of smooth $k$-varieties $X$, and relations are of two kinds:

$\mathbb{A}^1$-invariance: $M(X \times \mathbb{A}^1) \sim M(X)$ for any $X$;
Mayer-Vietoris exact triangles: for any Zariski cover $X = U \cup V$, the sequence

$$M(U \cap V) \rightarrow M(U) \oplus M(V) \rightarrow M(X)$$

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1
yields an exact triangle in $\text{DM}_{\text{gm}}^{\text{eff}}$.

When $k$ is perfect, one gets more general exact triangles, associated to elementary Nisnevich covers. This is a highly non-trivial theorem of Voevodsky, and it is more reasonable to refund his theory by imposing these latter relations even when the field $k$ is not perfect: this renders part of this theory more elementary [4, §4].

This is the approach which is adopted in [5] to construct a triangulated category $\text{MDM}_{\text{gm}}^{\text{eff}}$ of “motives with modulus”. Unfortunately, the situation is not so simple. Namely, one first constructs in [5, §6.2] a larger triangulated category $\overline{\text{MDM}}_{\text{gm}}^{\text{eff}}$ “à la Voevodsky”: its generators are motives of modulus pairs whose total space is not necessarily proper, and relations are parallel to those of Voevodsky:

- □-invariance: $M(\mathcal{X} \otimes \Box) \sim M(\mathcal{X})$ for any modulus pair $\mathcal{X}$;
- Mayer-Vietoris exact triangles: for any elementary Nisnevich cover

\[
\begin{array}{ccc}
\mathcal{W} & \longrightarrow & \mathcal{V} \\
\downarrow & & \downarrow \\
\mathcal{U} & \longrightarrow & \mathcal{X}
\end{array}
\]

the sequence

$M(\mathcal{W}) \to M(\mathcal{U}) \oplus M(\mathcal{V}) \to M(\mathcal{X})$

yields an exact triangle in $\overline{\text{MDM}}_{\text{gm}}^{\text{eff}}$.

Here, $\Box$ is the modulus pair $(\mathbb{P}^1, \infty)$ (completing $\mathbb{A}^1$), and “elementary Nisnevich covers” are defined in a suitable category of modulus pairs, in a way parallel to the classical case. The category $\text{MDM}_{\text{gm}}^{\text{eff}}$ is then the full triangulated subcategory of $\overline{\text{MDM}}_{\text{gm}}^{\text{eff}}$ generated by motives of proper modulus pairs [5, §6.9].

In $\text{MDM}_{\text{gm}}^{\text{eff}}$, the □-invariance relation makes sense, because the modulus pair □ is proper. This is not true for the Mayer-Vietoris relation, because the use of elementary Nisnevich covers forces us to leave the world of proper varieties. Nevertheless, in [5, §7.5] we exhibit exact triangles in $\text{MDM}_{\text{gm}}^{\text{eff}}$ “of Mayer-Vietoris type” in a certain sense.

Are there enough such triangles to present $\text{MDM}_{\text{gm}}^{\text{eff}}$ in these terms? The main result of the present paper is a positive answer to this question. More precisely, write $\text{MCor}$ for the category of (proper) modulus pairs, as in [5, Def. 1.3.1], and $\text{MPST}$ for the category of $\text{MCor}$-modules as in [5, Def. 2.1.1]. Then we have:

**Theorem 1.** Let $\text{CI}$ be the collection of complexes of the form $\mathcal{X} \otimes \Box \rightarrow \mathcal{X}$ for $\mathcal{X} \in \text{MCor}$, and let $\text{MV}$ denote the collection of complexes of
the form

\[ N(00) \to N(10) \oplus N(01) \to N(11) \]

where \( \mathcal{N} = \{ N(ij) \mid i, j \in \{0, 1\} \} \) is an MV square in \( \mathbf{MCor} \) as in Definition 2.8 a) below. Then in the naturally commutative square

\[
\begin{array}{ccc}
\left( K^b(\mathbf{MCor})/\langle CI + MV \rangle \right)^{\mathbb{Z}} & \xrightarrow{\alpha} & D(\mathbf{MPST})/\langle \mathbb{Z}_{tr}(CI) + \mathbb{Z}_{tr}(MV) \rangle^{\text{loc}} \\
\downarrow_{\gamma} & & \downarrow_{\delta} \\
\mathbf{MDM}_{\text{eff}}^{\text{gm}} & \xrightarrow{\beta} & \mathbf{MDM}_{\text{eff}}^{\text{gm}}
\end{array}
\]

both vertical functors are equivalences of categories. Here, \( (\ )^{\text{loc}} \) means “localising subcategory generated by”.

The proof of Theorem 1 can be sketched as follows. We first reduce it to a cofinality statement, Theorem 2.10 (see Prop. 2.11). In [5, §4.3], a strategy was implicitly suggested to prove Theorem 2.10. We broadly follow this strategy, but the story turns out to be more complicated. Namely, we reduce in Proposition 3.4 the proof of Theorem 2.10 to two statements, Theorems 3.2 and 3.3. They are respectively Nisnevich generalizations of [5, Lemma 4.3.5] and [5, Prop. 4.3.10] (which only concern elementary Zariski squares), except that each one is much stronger.

Let us now give some ideas of the proofs of Theorems 3.2 and 3.3. Since their statements are quite technical, we offer here simplified (but weaker) statements. This helps explain our strategy.

Theorem 3.2 may be simplified as follows:

(A) Let \( \mathcal{X} \to \mathcal{X}' \) be an open immersion of modulus pairs and assume that \( \mathcal{X}' \) is proper. Then, any elementary Nisnevich cover of \( \mathcal{X} \) can be extended to an elementary Nisnevich cover of \( \mathcal{X}' \).

On the other hand, Theorem 3.3 may be simplified as follows:

(B) Let \( N(00) \) be a proper modulus pair. Any elementary Nisnevich square

\[
\begin{array}{ccc}
\mathcal{W} & \to & \mathcal{V} \\
\downarrow & & \downarrow \\
\mathcal{U} & \to & N(11)
\end{array}
\]

(0.2)
can be embedded into an MV-square
\[
\begin{align*}
N(00) & \longrightarrow N(01) \\
| & | \\
N(10) & \longrightarrow N(11).
\end{align*}
\]

The proof of Theorem 3.2 will be done in §5. Its basic strategy is to follow the case of a Zariski cover [5, Lemma 4.3.5]. But in the Nisnevich case, we need to control the fibers of the étale morphisms. This forced us to take care of (basically) set-theoretical problems, which made the proof much more technical and complicated. However, the statement of the theorem itself is not so surprising and easy to understand, so we do not step further into the proof here.

The proof of Theorem 3.3 will be done in §6, which is heart of the proof of Theorem 1. First, it is easy to compactify a square (0.2) to a square of the form (0.3) such that each corner of the square is a proper modulus pair. However, in general, such a compactification is not MV. So, in §6.2, we provide a recipe to produce an MV-square out of a square of the form (0.3). The idea is quite simple: increase the multiplicity of the boundary of the north-east corner \(N(01)\) (after enlarging the total space by a sequence of blowing-ups). After this modification, we obtain a square, denoted by \(N_1\), which satisfies the properties in Proposition 6.7. Then, our task is to prove that \(N_1\) is an MV-square (Theorem 6.8), which will be done in the rest of the §6.

The key point is the exactness of the sequence
\[
\begin{align*}
\text{MCor}(M, N_1(00)) & \rightarrow \text{MCor}(M, N_1(10)) \oplus \text{MCor}(M, N_1(01)) \\
& \rightarrow \text{MCor}(M, N_1(11)),
\end{align*}
\]
where \(M\) is any modulus pair. Take any element \((\alpha', \alpha)\) in the middle term which goes to zero in \(\text{MCor}(M, N_1(11))\), and write \(\alpha = \sum \alpha_i\), where \(\alpha_i\) are irreducible components of the algebraic cycle \(\alpha\). The proof is relatively easy if we assume that the image of the components \(\alpha_i\) in \(\text{MCor}(M, N_1(11))\) are distinct (see [5, Proof of Prop. 4.3.10] and §6.4 for details). An essentially new difficulty appears if they are not distinct: for example, suppose that \(\alpha = \alpha_1 - \alpha_2\), and each \(\alpha_i\) goes to the same cycle \(\beta\). Then, the image of \(\alpha\) is equal to zero, which implies that \(\alpha' = 0\). Therefore, we lose, a priori, a way to catch the information of the boundary of \(N_1(10)\), which makes it difficult to prove that \(\alpha\) comes from \(\text{MCor}(M, N_1(00))\). A special case of this situation was solved in [5, Rk. 4.3.14]; however, we don’t use ideas from loc. cit. Rather, we prove the following surprising result, which we call “resurgence principle”:
The assumption that two distinct cycles $\alpha_i$'s go to the same cycle $\beta$ automatically implies that $\alpha_i$'s come from $\mathcal{M}\text{Cor}(M, N_1(00))$.

For the precise statement, see Proposition 6.9, whose proof is given in §6.3.

Throughout the proofs, we have repeatedly used rather similar constructions; we made no serious attempt to spell them out systematically, except for Lemma 4.7, which is used several times in §4.4; even in its case, variants of it are used in §5, but we have not tried to work out a general statement. Similarly, we use several times the “pull-back” of a square along a morphism to its lower right corner, and the fact that such pull-backs preserve elementary Nisnevich squares. We hope that this looseness of exposition will not disturb the reader too much.

1. Review of modulus pairs

We denote by $\text{Sch}$ the category of separated $k$-schemes of finite type, and by $\text{Sm}$ the full subcategory of smooth $k$-schemes.

According to [5, Def. 1.1.1], a modulus pair is a pair $M = (\overline{M}, M^\infty)$ where $\overline{M} \in \text{Sch}$ (the total space), $M^\infty \subset \overline{M}$ (the boundary) is an effective Cartier divisor, and $M^\circ := \overline{M} - M^\infty$ (the interior) is smooth. The modulus pair $M$ is proper if $\overline{M}$ is proper.

Remark 1.1. By [5, Remark 1.1.2 (3)], the total space $\overline{M}$ is reduced and $M^\circ$ is dense in $\overline{M}$.

Let $N$ be another modulus pair, and consider an irreducible finite correspondence $\alpha \in \text{Cor}(M^\circ, N^\circ)$ as in [8, Lect. 1]. We say that $\alpha$ is admissible if the following condition holds:

$M^\infty|_{\overline{\alpha}^N} \geq N^\infty|_{\overline{\alpha}^N}$

where $\overline{\alpha}$ is the closure of $\alpha$ in $\overline{M} \times \overline{N}$, $\overline{\alpha}^N$ is its normalization and $-|_{\overline{\alpha}^N}$ means “pull-back (of Cartier divisors) to $\overline{\alpha}^N$”.

We say that $\alpha$ is minimal if equality holds in (1.1)

We also say that $\alpha$ is left proper if the projection of $\overline{\alpha}$ to $\overline{M}$ is proper. A general finite correspondence in $\text{Cor}(M^\circ, N^\circ)$ is admissible (resp. left proper) if all its components are. One shows [5, Prop. 1.2.3] that left proper admissible correspondences can be composed, whence an additive category $\mathcal{M}\text{Cor}$; its full subcategory consisting of proper modulus pairs is denoted by $\mathcal{M}\text{Cor}$. There is a forgetful functor

$\omega : \mathcal{M}\text{Cor} \to \text{Cor}, \quad M \mapsto M^\circ$

(which extends to $\mathcal{M}\text{Cor}$). We have the following important result [5, Th. 1.6.2 and Lemma 1.11.3 (2)]:
Theorem 1.2. The full embedding $\tau : \mathbf{MCor} \to \underline{\mathbf{MCor}}$ has a pro-left adjoint $\tau^!$, given by the formula

$$\tau^! M = \lim_{\leftarrow} \text{Comp}_{1} (N)$$

where, for $M \in \mathbf{MCor}$, $\text{Comp}_{1}(M)$ is the category whose objects are arrows $M \xrightarrow{\theta} \tau(N)$, with $N \in \mathbf{MCor}$, such that

- $\theta^o : M^o \to N^o$ is the identity;
- $\theta$ defines an open immersion on the total spaces;
- $\theta$ is minimal.

Morphisms between two objects in $\text{Comp}_{1}(M)$ are given by commutative triangles.

(See [SGA4-I, I.8.11.5] and [5, §A.2] for pro-left adjoints.)

We write $\mathbf{MSm} \subset \mathbf{MCor}$ and $\underline{\mathbf{MSm}} \subset \underline{\mathbf{MCor}}$ for the subcategories with the same objects, but morphisms restricted to (graphs of) scheme-theoretic morphisms on the interiors. Let $M, N \in \mathbf{MSm}$ and $f \in \mathbf{MSm}(M, N)$. We write $f \in \mathbf{MSm}^{\text{fin}}(M, N)$ if the rational map $\underline{M} \to \underline{N}$ defined by $f$ is a morphism; this defines a subcategory $\underline{\mathbf{MSm}}^{\text{fin}}$ of $\mathbf{MSm}$, with the same objects [5, Def. 1.10.1].

We have the notion of elementary Nisnevich square in $\mathbf{MSm}^{\text{fin}}$: it is a cartesian square such that all edges are minimal morphisms and the induced square on the total spaces is upper distinguished in the sense of [8, Def. 12.5]; see [5, Def. 3.5.6] for a more spelled-out definition.

The categories of modules [additive presheaves of abelian groups] over $\mathbf{MCor}$ and $\underline{\mathbf{MCor}}$ are respectively denoted by $\mathbf{MPST}$ and $\underline{\mathbf{MPST}}$ [5, Def. 2.1.1]. We denote by $\mathbf{Z}_{tr}$ the Yoneda embeddings $\mathbf{MCor} \to \mathbf{MPST}$ and $\underline{\mathbf{MCor}} \to \underline{\mathbf{MPST}}$.

Finally, one defines a notion of Nisnevich sheaves in $\underline{\mathbf{MPST}}$ and $\underline{\mathbf{MPST}}$: they form respective full subcategories $\underline{\mathbf{MNST}}$ and $\mathbf{MNST}$ [5, Def. 3.5.1 and 3.7.1]. The category $\mathbf{MSm}^{\text{fin}}$ plays a crucial rôle in these definitions and, as can be expected, an elementary Nisnevich square in $\underline{\mathbf{MSm}}^{\text{fin}}$ plays the same rôle as in the classical case. The functor $\mathbf{Z}_{tr}$ takes its values in $\underline{\mathbf{MNST}}$ and $\mathbf{MNST}$ respectively [5, Prop. 3.5.3 and 3.7.3].

2. Reduction to a cofinality statement

In this section, we reduce Theorem 1 to Theorem 2.10 below (see §2.2).

2.1. Categories of diagrams. Let $\mathcal{C}, \Delta$ be two categories, with $\Delta$ small: we write as usual $\mathcal{C}^\Delta$ for the category of functors from $\Delta$ to $\mathcal{C}$. Clearly, a functor $u : \mathcal{C} \to \mathcal{D}$ induces a functor $u^\Delta : \mathcal{C}^\Delta \to \mathcal{D}^\Delta$. 

In the sequel, we shall mainly consider the case where $\Delta = \text{Sq}$: the category with 4 objects, morphisms being given by the scheme

\[
\begin{array}{ccc}
00 & \longrightarrow & 01 \\
\downarrow & & \downarrow \\
10 & \longrightarrow & 11.
\end{array}
\]

(Note that $\text{Sq}$ is just the cartesian square of the category $[0] = \{0 \to 1\}$.) Thus, in [5, §4], a square (4.1) is a certain object of $(\text{MSm}^\text{fin})^\text{Sq}$, a square (4.3) is an object of $\text{MSm}^\text{sq}$, etc.

**Definition 2.1.** Let $\mathcal{P}$ be a property of morphisms of the category $\text{MCor}$. Then, a morphism $f: \underline{M} \to \underline{N}$ in the category $\text{MCor}^{\text{sq}}$ has $\mathcal{P}$ if, for any $i, j \in \{0, 1\}$, the morphism $f(ij): M(ij) \to N(ij)$ has $\mathcal{P}$.

**Example 2.2.** A morphism $f: \underline{M} \to \underline{N}$ in the category $\text{MCor}^{\text{sq}}$ is minimal if all the morphisms $f(ij): M(ij) \to N(ij)$ are minimal.

In the next definition, we use the comma notation $\downarrow$ of Mac Lane [6, Ch. II, §6].

**Definition 2.3.** For $M \in \text{MCor}^{\text{sq}}$, $\text{Comp}_1(M)$ is the full subcategory of $\underline{M} \downarrow \text{MCor}^{\text{sq}}$ consisting of those objects $\underline{M} \overset{f}{\to} \tau^\text{sq}(N)$ such that $f(\delta) \in \text{Comp}_1(M(\delta))$ for any $\delta \in \text{Ob}(\text{Sq})$.

We now apply the theory of Appendix C with $\mathcal{C} = \text{MCor}$, $\mathcal{D} = \text{MCor}$, $u = \tau$ and $\Delta = \text{Sq}$. From Lemma C.4 and [5, Lemmas 1.8.3 and 1.11.3], we get:

**Proposition 2.4.** The functor $\tau^\text{sq}: \text{MCor}^{\text{sq}} \to \text{MCor}^{\text{sq}}$ has a pro-left adjoint, which is represented by $M \mapsto \text{Comp}_1(M)$.

**Proof.** By definition, $\text{Comp}_1(M)$ is full in $\underline{M} \downarrow \tau$ for any $M \in \text{MCor}$. By Lemma C.4 b), $\text{Comp}_1(M)$ coincides with the category $I(M)$ introduced there, and Proposition 2.4 follows from part c) of this lemma. □

For any additive category $\mathcal{A}$, write $C(\mathcal{A})$ for the category of chain complexes on $\mathcal{A}$. Let $C \in C(\text{MPST})$. Viewing $C$ as a functor $\text{MCor}^{\text{op}} \to C(\text{Ab})$, we get the functor $C^{\text{sq}}: (\text{MCor}^{\text{op}})^{\text{sq}} \to C(\text{Ab})^{\text{sq}}$. For $\underline{M} \in \text{MCor}^{\text{sq}}$, we therefore get the complex

\[ D = \text{Tot} \tau^\text{sq}_1 C^{\text{sq}}(\underline{M}) \in C(\text{Ab}). \]

**Corollary 2.5.** The natural map

\[ \lim_{\longrightarrow} \text{Tot} C^{\text{sq}}(\underline{N}) \to D \]

is an isomorphism.
Proof. First a word on the “natural map”: For any \( M \to \tau_{\Sq}^* \in \text{Comp}_1(M) \), applying the functor \( \tau_{\Sq}^* \) , we get a composite map in \( C(\Ab)_{\Sq} \)

\[
C_{\Sq}(N) \to \tau_{\Sq}^* C(\tau_{\Sq}(N)) \to \tau_{\Sq}^* C(M)
\]

where the first map is given by Yoneda. By Proposition 2.4, the induced map

\[
\lim_{N \in \text{Comp}_1(M)} C_{\Sq}(N) \to \tau_{\Sq}^* C(M)
\]

is an isomorphism, and the conclusion follows from the fact that Tot commutes with filtering colimits. \( \square \)

For the sequel, we need the following definition and (trivial) lemma.

**Definition 2.6.** A morphism \( f : M \to M' \) in \( \text{MCor}_{\Sq} \) is an open immersion if \( f \in \text{MSm}^{\text{fin}} \) [5, Def. 1.10.1], \( f \) is minimal (ibid., Def. 1.10.2 a)) and \( f : M \to M' \) is an open immersion (cf. [5, Def. 3.1.1]).

**Lemma 2.7.** Let \( f : M \to N \) be a morphism in \( \text{MCor}_{\Sq} \). Suppose that, for any \( \delta \in \Sq \), \( f(\delta) \) is an open immersion in the sense of Definition 2.6 and \( \omega(f(\delta)) = 1_{M(\delta)} \). (In other words, \( f(\delta) \) verifies the conditions of [5, Lemma 1.11.3 (1)], except for the properness of \( N(\delta) \).) Then \( f \) induces a functor \( f^* : \text{Comp}_1(N) \to \text{Comp}_1(M) \). \( \square \)

### 2.2. Cofinality of MV-squares.

**Definition 2.8.**

a) A square \( N \in \text{MCor}_{\Sq} \) is MV if

- \( \omega(\Sq)(N) \in \text{Cor}_{\Sq} \) is an upper distinguished (=elementary Nisnevich) square;
- the complex \( \text{Tot} Z_{\Sq}(N) \) [5, (4.4)] is exact in \( \text{MNST} \).

b) Consider an elementary Nisnevich square

\[
S = \begin{array}{ccc}
S(00) & \xrightarrow{a} & S(01) \\
\downarrow c & & \downarrow d \\
S(10) & \xrightarrow{b} & S(11)
\end{array}
\]

as in [5, (4.1)]. An object \( S \to \tau_{\Sq}^* N \in \text{Comp}_1(S) \) is an MV completion of \( S \) if \( N \) is MV. We write \( \text{Comp}_1^{MV}(S) \) for the full subcategory of \( \text{Comp}_1(S) \) consisting of MV completions.

**Remark 2.9.** In Definition 2.8 a), Condition (i) was put for philosophical reasons and will not be used in the proofs. This condition is automatic for MV completions in the sense of Definition 2.8 b), and all MV squares which will appear in this paper are of this form.
Theorem 2.10. If \( S^{(11)} \) is normal, the category \( \text{Comp}^{MV}_1(S) \) is cofinal in \( \text{Comp}_1(S) \).

Proposition 2.11. Theorem 1 follows from Theorem 2.10.

Proof. Consider the square in Theorem 1. By [5, Ex. A.11.6 and Th. A.11.9], \( \alpha \) is fully faithful; by loc. cit., Prop. 6.9.1 (2), \( \gamma \) is essentially surjective, and \( \beta \) is a (full) embedding by definition (loc. cit., Def. 6.9.3); therefore it suffices to show that \( \delta \) is an equivalence of categories. We argue similarly to the proof of [5, Prop. 6.3.2]. First, \( D(a_{\text{Nis}}) : D(\text{MPST}) \to D(\text{MNST}) \) is a localisation functor by the same argument (referring to prop. 3.7.3 rather than prop. 3.5.3). Next, \( Z_{\text{tr}}(MV) \) is a set of compact objects of \( D(\text{MPST}) \), contained in \( \text{Ker} D(a_{\text{Nis}}) \) by definition of \( MV \) (see [5, Lemmas 4.3.2, 4.3.3 and Def. 4.3.7]). It suffices to show that \( \text{Ker} D(a_{\text{Nis}}) \) is generated by \( Z_{\text{tr}}(MV) \), or equivalently by [5, Th. A.11.7], that the right orthogonal of these objects in \( \text{Ker} D(a_{\text{Nis}}) \) is \( 0 \). Consider the naturally commutative diagram

\[
\begin{array}{c}
\text{Ker } D(a_{\text{Nis}}) \\
\downarrow \\
D(\text{MPST})
\end{array} 
\xrightarrow{D(\tau)} 
\begin{array}{c}
D(\text{MNST}) \\
\downarrow \\
D\tau_{\text{Nis}}(\text{MNST})
\end{array}
\]

Note that \( D(\tau) \) sends \( Z_{\text{tr}}(MV) \) to compact objects of \( D(\text{MPST}) \). Since this functor is fully faithful, it suffices to show: \( \square \)

Lemma 2.12. Let \( C \in D(\text{MPST}) \). If \( C \) is right orthogonal to \( Z_{\text{tr}}(MV) \), then \( D(\tau)C \) is right orthogonal to \( Z_{\text{tr}}(MV) \), where \( MV \) are the MV relations in \( \text{MPST} \) (denoted by \( \text{MV2} \) in [5, §6.3]).

Proof. Consider an elementary Nisnevich square \( S(2.1) \). Since \( Z_{\text{tr}}(S(00)), \ldots, Z_{\text{tr}}(S(11)) \) are projective objects of \( \text{MCor} \), we have to show that the complex of abelian groups

\[
(2.2) \quad \tau C(S(00)) \to \tau C(S(01)) \oplus \tau C(S(10)) \to \tau C(S(11))
\]

is acyclic.

Lemma 2.13. We may assume that \( S^{(11)} \) is normal.

Proof. For any \( i, j = 0, 1 \), let \( \pi_{ij} : S(ij)^{\infty} \to S(ij) \) be the normalization and let \( S(ij)^{\infty} = (\overline{S}(ij)^N, \pi_{ij}^* S(ij)^\infty) \). Then, \( \pi_{ij} \) defines an
isomorphism of modulus pairs $S(ij)^N \sim S(ij)$ in $\mathbf{MCor}$ for each $i,j$ since it is minimal, proper and induces an isomorphism $(S(ij)^N)^o \sim S(ij)^0$. Since the maps $\overline{S(ij)} \to \overline{S}(11)$ are étale for all $i,j$, we have $\overline{S}(ij)^N = \overline{S}(ij) \times_{\overline{S}(11)} \overline{S}(11)^N$. Therefore the $S(ij)^N$’s form an elementary Nisnevich square $\overline{S}^N$, and the natural morphism $\overline{S}^N \to \overline{S}$, induced by $\pi_{ij}$’s, is an isomorphism of squares. Therefore, we may replace everything by the pullbacks along $\pi$. □

Assume $\overline{S}(11)$ normal. Then Corollary 2.5 and Theorem 2.10 yield an isomorphism

$$\lim_{\mathcal{N} \in \text{Comp}_{MV}(\mathcal{S})} C^{\text{Sq}}(\mathcal{N}) \sim \tau_!^{\text{Sq}} C(\mathcal{S}).$$

The acyclicity of (2.2) follows, since Tot obviously commutes with colimits and the complex

$$\text{Tot} C^{\text{Sq}}(\mathcal{N}) = C(N(00)) \to C(N(01)) \oplus C(N(10)) \to C(N(11))$$

is acyclic for any $\mathcal{N} \in \text{Comp}_{MV}(\mathcal{S})$ by hypothesis. □

Remark 2.14. if $\overline{S}(11)$ is normal, so is $\overline{S}(ij)$ for all $(i,j)$ since the morphisms $\overline{S}(ij) \to \overline{S}(11)$ are étale.

Remark 2.15. Theorem 2.10 says that there are “enough” MV squares in $\mathbf{MCor}^{\text{Sq}}$. A sufficient condition for a square to be MV is given in Theorem 6.8. This raises the question: can one describe all MV squares?

3. Further reductions

In this section, we reduce Theorem 2.10 to the following two statements. For the first one, we need a definition:

Definition 3.1.

1. A morphism $\alpha : M \to N$ in $\mathbf{MCor}$ is called
   - entire if $\alpha \in \text{Cor}(M^o, N^o)$ is a morphism of schemes $M^o \to N^o$, and extends to a morphism of schemes $f_{\alpha : M^o} \to N^o$;
   - minimal if $\alpha$ is entire and $M^\infty = f_{\alpha : N}^* N^\infty$;
   - an extension if $\alpha$ is minimal, $f_{\alpha}$ is an open immersion, and induces an isomorphism $M^o \sim N^o$.

2. Let $\mathcal{S} \in \mathbf{MCor}^{\text{Sq}}$. An extension of $\mathcal{S}$ is a morphism of squares $f : \mathcal{S} \to \mathcal{S}'$ such that $f(ij)$ is an extension for all $(i,j)$. An extension of $\mathcal{S}$ is strict if, for all $i,j \in \{0,1\}$, we have $\mathcal{S}(ij) = \overline{S}(ij) \times_{\overline{S}(11)} \overline{S}(11)$.
Note that, in particular, any object of $\text{Comp}_1(S)$ defines an extension of $S$.

**Theorem 3.2.** Let $S$ be an elementary Nisnevich square, with $\overline{S}(11)$ normal. Let $S \xrightarrow{\alpha} N_0 \in \text{Comp}_1(S)$ be a compactification of $S$. Then, there exists a commutative diagram in $\text{MCor}^{\text{Sq}}$

\[
\begin{array}{ccc}
S' & \xrightarrow{c} & N_1 \\
\uparrow{b} & & \downarrow{d} \\
S & \xrightarrow{\alpha} & N_0
\end{array}
\]

such that

(i) $S'$ is an elementary Nisnevich square;
(ii) $S'(11)$ is proper and normal;
(iii) $b$ is a strict extension;
(iv) $c b : S \to N_1$ is an object of $\text{Comp}_1(S)$;
(v) $c$ is minimal.

The second one is:

**Theorem 3.3.** Theorem 2.10 is true when $\overline{S}(11)$ is proper.

**Proposition 3.4.** Theorem 3.2 and Theorem 3.3 imply Theorem 2.10.

**Proof.** We start from $a : S \to N_0$ in $\text{Comp}_1(S)$, and give ourselves a commutative diagram (3.1) satisfying the conditions of Theorem 3.2. Choose $[S' \xrightarrow{g} N_2] \in \text{Comp}_1(S')$. Define a square of schemes $\overline{N}_3$ as follows: $\overline{N}_3(ij)$ is the scheme-theoretic image of $S'(ij)$ in $\overline{N}_1(ij) \times \overline{N}_2(ij)$ (a closed subscheme, [EGA1, Ch. I, Def. 6.10.1]). By [EGA1, Ch. I, Prop. 6.10.5], it exists, and the corresponding morphism $f(ij) : S'(ij) \to \overline{N}_3(ij)$ is scheme-theoretically dominant. Write $p_\varepsilon : \overline{N}_3 \to \overline{N}_\varepsilon$ ($\varepsilon = 1, 2$) for the projection. Applying Lemma B.2 to the diagram

\[
\begin{array}{ccc}
\overline{N}_3(ij) & \xrightarrow{f(ij)} & \overline{N}_2(ij) \\
\downarrow{p_2(ij)} & & \\
S'(ij) & \xrightarrow{g(ij)} & \overline{N}_2(ij)
\end{array}
\]

we see that $f(ij)$ is an open immersion because $g(ij)$ is. Defining $N_3(ij)^\infty = p_1(ij)^* N_1(ij)^\infty$ we get $[S' \to N_3] \in \text{Comp}_1(S')$, thanks to (v).

By Theorem 3.3, choose $[S' \to N_4] \in \text{Comp}_1^{\text{MV}}(S')$ mapping to $[S' \to N_3]$. By Lemma 2.7, $[S \to N_4] \in \text{Comp}_1^{\text{MV}}(S)$ and maps to $[S \to N_1]$, hence to $[S \to N_0]$. □
Remark 3.5. In [EGA1, Ch. I, Prop. 6.10.5], there are two sufficient conditions for the existence of $\overline{N}_3(ij)$; (1) the morphism $\overline{S}(ij) \to \overline{N}_1(ij) \times \overline{N}_2(ij)$ is quasi-compact and quasi-separated, or (2) $\overline{S}(ij)$ is reduced. Both are true, the second one by Remark 1.1.

4. Cofinality of universally minimal squares

The aim of this section is to prove Proposition 4.2 below. It will be used in §6.

4.1. Universally minimal squares.

Definition 4.1 (cf. [5, Def. 4.3.9]). A square $\mathcal{N} \in \text{MCor}^{\text{Sq}}$ whose morphisms are entire (Definition 3.1) is called universally minimal if the following condition holds: write

$$
\begin{array}{ccc}
\mathcal{N}(00) & \xrightarrow{h_u} & \mathcal{N}(01) \\
v_l & \downarrow & v_r \\
\mathcal{N}(10) & \xrightarrow{h_d} & \mathcal{N}(11).
\end{array}
$$

Then, for any normal $k$-scheme $Y$ and for any $k$-morphism $f : Y \to \mathcal{N}(00)$ such that the pullback $f^*\mathcal{N}(00)^\infty$ is well-defined, we have

$$
f^*\mathcal{N}(00)^\infty = \sup(f^*h_u^*\mathcal{N}(01)^\infty, f^*v_l^*\mathcal{N}(10)^\infty),
$$

where $\sup$ is taken as Weil-divisors. Note that, since $|h_u^*\mathcal{N}(01)^\infty| \cup |v_l^*\mathcal{N}(10)^\infty| \subset |\mathcal{N}(00)^\infty|$, the pullbacks appearing on the right hand side of the equality are automatically well-defined. Write $\text{Comp}_1^{\text{min}}(S)$ for the full subcategory of $\text{Comp}_1^{\text{mor}}(S)$ consisting of those objects $X : S \to \tau^{\text{Sq}}\mathcal{N}$ such that $\mathcal{N}$ is universally minimal.

Proposition 4.2 (Cofinality of minimal squares). Let $\mathcal{S}$ be an elementary Nisnevich square, and assume that $\mathcal{S}(11)$ is normal. Then, $\text{Comp}_1^{\text{min}}(S)$ is cofinal in $\text{Comp}_1(S)$.

4.2. Three elementary lemmas.

Lemma 4.3. Let $\text{MCor}^{\text{Sq},\text{mor}}$ be the full subcategory of $\text{MCor}^{\text{Sq}}$ formed of those squares $\mathcal{N}$ in which all morphisms are given by entire morphisms (Definition 3.1). Then the full embedding

$$
\text{MCor}^{\text{Sq},\text{mor}} \hookrightarrow \text{MCor}^{\text{Sq}}
$$

is an equivalence of categories.
Proof. Let $\mathcal{N} \in \mathbf{MCor}^{\mathbf{Sq}}$. Using Notation (4.1), define a new square of schemes $\mathcal{N}'$ as follows:

$\mathcal{N}'(11) = \mathcal{N}(11)$;
$\mathcal{N}'(10) = \Gamma_{h_d}$;
$\mathcal{N}'(01) = \Gamma_{v_r}$;
$\mathcal{N}'(00) = \Gamma_{\varphi}$

where $\Gamma_f$ denotes the closure of the graph of a rational map $f$, and $\varphi$ is the rational map $\mathcal{N}(00) \to \mathcal{N}(01) \times \mathcal{N}(11) \to \mathcal{N}(10)$ determined by $h_d$ and $v_r$.

Then the rational maps of (4.1) "resolve" to morphisms in $\mathcal{N}'$, and we have a canonical morphism $\pi(ij) : \mathcal{N}'(ij) \to \mathcal{N}(ij)$ for all $(i,j)$. Define

$\mathcal{N}'(ij)^\infty = \pi(ij)^*\mathcal{N}(ij)^\infty$.

This defines modulus pairs $\mathcal{N}'(ij)$ and isomorphisms $\pi(ij) : \mathcal{N}'(ij) \isom \mathcal{N}(ij)$ in $\mathbf{MSm}$. The latter fact implies that the morphisms of $\mathcal{N}'$ are admissible, whence an object $\mathbf{MCor}^{\mathbf{Sq, mor}}$ together with an isomorphism $\pi : \mathcal{N}' \isom \mathcal{N}$ in $\mathbf{Comp}_1(S)$.

\[\square\]

Remark 4.4. This lemma, and its proof, extend to the case where $\mathbf{Sq}$ is replaced by any finite category without loops.

Lemma 4.5. Let $S$ be an elementary Nisnevich square, and let $\mathbf{Comp}^{\mathbf{mor}}_1(S)$ be the full subcategory of $\mathbf{Comp}_1(S)$ formed of those squares $\mathcal{N}$ in which all morphisms are given by entire morphisms (Definition 3.1). Then the full embedding $\mathbf{Comp}^{\mathbf{mor}}_1(S) \hookrightarrow \mathbf{Comp}_1(S)$ is an equivalence of categories. In particular, $\mathbf{Comp}^{\mathbf{mor}}_1(S)$ is cofinal in $\mathbf{Comp}_1(S)$.

Proof. This follows immediately from Lemma 4.3. \[\square\]

Lemma 4.6. Let $S$ be an elementary Nisnevich square, and let $\mathcal{N} \in \mathbf{Comp}^{\mathbf{mor}}_1(S)$. Name those morphisms as in the diagram (4.1) of Definition 4.2. Then, we have

$\varphi^{-1}(\mathcal{S}(01)) \cap v_l^{-1}(\mathcal{S}(10)) = \mathcal{S}(00)$.

Proof. By the universality of the fiber product, we obtain a unique morphism $f : \mathcal{N}(00) \to \mathcal{N}(01) \times_{\mathcal{N}(11)} \mathcal{N}(10)$ such that $h_u$ and $v_l$ factor through $f$. The restriction of $f$ to $\mathcal{S}(00)$ factors through the map

$\mathcal{S}(00) \to \mathcal{S}(01) \times_{\mathcal{N}(11)} \mathcal{S}(10) = \mathcal{S}(01) \times_{\mathcal{N}(11)} \mathcal{S}(10)$

which is an isomorphism by the hypothesis on $S$. Since $f$ is separated, Lemma B.2 then implies that

\[f^{-1}(\mathcal{S}(01) \times_{\mathcal{N}(11)} \mathcal{S}(10)) = f^{-1}(f(\mathcal{S}(00))) = \mathcal{S}(00).\]
Since we have
\[ \overline{S}(01) \times_{\overline{N}(11)} \overline{S}(10) = (\overline{S}(01) \times_{\overline{N}(11)} \overline{N}(10)) \cap (\overline{N}(01) \times_{\overline{N}(11)} \overline{S}(10)), \]
the left hand side of (4.2) can be rewritten as
\[
\begin{align*}
f^{-1}(\overline{S}(01) \times_{\overline{N}(11)} \overline{S}(10)) \\
= f^{-1}((\overline{S}(01) \times_{\overline{N}(11)} \overline{N}(10)) \cap f^{-1}((\overline{N}(01) \times_{\overline{N}(11)} \overline{S}(10))) \\
= h_u^{-1}(\overline{S}(01)) \cap v_l^{-1}(\overline{S}(10)).
\end{align*}
\]
This finishes the proof of Lemma 4.6. □

4.3. A useful construction.

Lemma 4.7. Let \( S \) be an elementary Nisnevich square such that \( \overline{S}(11) \) is normal, and take any \( \overline{N}_0 \in \text{Comp}_1(S) \). Let \( W \subset \overline{N}_0(00) \) be a closed subscheme whose restriction \( W \times_{\overline{N}_0(00)} \overline{S}(00) \) to the open subset \( \overline{S}(00) \) is an effective Cartier divisor on \( \overline{S}(00) \). Consider the modulus pair \( \overline{N}_0'(00) = (\text{Bl}_W(\overline{N}_0(00)))^N, \text{ the pullback of } \overline{N}_0(00)^\infty \).

Then the projection \( \pi : \overline{N}_0'(00) \to \overline{N}_0(00) \) is an isomorphism, and the morphism \( S(00) \to \overline{N}_0(00) \) lifts to a morphism \( S(00) \to N \) in \( \text{MSm}^{\text{mor}} \). Moreover, \( \pi^{-1}(W) \) is an effective Cartier divisor on \( \overline{N}_0'(00) \), with same restriction to \( \overline{S}(00) \) as \( W \).

Proof. The projection \( \text{Bl}_W(\overline{N}_0(00))^N \to \text{Bl}_W(\overline{N}_0(00)) \to \overline{N}_0(00) \) is an isomorphism over \( \overline{S}(00) \) by the universal property of blow-up and by the assumption that \( \overline{S}(00) \) is normal. The last claim is obvious. □

Remark 4.8. Let \( W_1, \ldots, W_r \) be closed subschemes of \( \overline{N}_0(00) \) verifying the hypothesis of Lemma 4.7. Applying its construction repeatedly, we obtain a modulus pair \( N_0^{(r)}(00) \) with \( \overline{N}_0^{(r)}(00) \) normal and a morphism \( \pi_r : N_0^{(r)}(00) \to \overline{N}_0(00) \) in \( \text{MSm}^{\text{mor}} \) which is an isomorphism in \( \text{MSm} \), such that \( S(00) \to N_0(00) \) lifts to \( N_0^{(r)}(00) \) and, for any \( i, \pi_r^{-1}(W_i) \) is a Cartier divisor with same restriction to \( \overline{S}(00) \) as \( W_i \). Of course, \( N_0^{(r)}(00) \) depends in general on the ordering on the \( W_i \)'s.

In the next subsection, we will make use of Lemma 4.7 and Remark 4.8 several times for various \( W_i \)'s.

4.4. Proof of Proposition 4.2. Take any \( \overline{N}_0 \in \text{Comp}_1(S) \). By Lemma 4.5, we may assume that \( \overline{N}_0 \in \text{Comp}_1^{\text{mor}}(S) \).

For each \( i, j \in \{0, 1\} \), define \( D_{ij} := \overline{N}_0(ij) \setminus \overline{S}(ij) \), and regard \( D_{ij} \) as a closed subscheme of \( \overline{N}_0(ij) \) with the reduced scheme structure.
Recall the notation in Diagram (4.1), and set
\[ p := v_r \circ h_u = h_d \circ v_l. \]
We set
\[ \tilde{N}_0(00)^\infty := N_0(00)^\infty - p^* N_0(11)^\infty. \]
By the admissibility of \( p \), the Cartier divisor \( \tilde{N}_0(00)^\infty \) is effective. Since the restrictions of \( N_0(00)^\infty \) and \( p^* N_0(11)^\infty \) to \( \overline{S}(00) \) both equal \( S(00)^\infty \), we have
\[ \tilde{N}_0(00)^\infty \times_{\overline{\pi}_0(00)} \overline{S}(00) = 0, \]
hence
\[ |\tilde{N}_0(00)^\infty| \subset \overline{N}_0(00) \setminus \overline{S}(00) = |D_{00}|. \]
Moreover, by applying §4.3, we may reduce to the case where the closed subschemes \( v_i^* D_{10}, h_u^* D_{01} \) and \( v_i^* D_{10} \times_{\overline{\pi}_0(00)} h_u^* D_{01} \) are effective Cartier divisors on \( \overline{N}_0(00) \), because their restrictions to \( \overline{S}(00) \) are empty. Then, by Lemma 4.6, we have
\[ |D_{00}| = \overline{N}_0(00) \setminus \overline{S}(00) = \overline{N}_0(00) \setminus (h_u^{-1}(\overline{S}(01)) \cap v_l^{-1}(\overline{S}(10))) \]
\[ = (\overline{N}_0(00) \setminus h_u^{-1}(\overline{S}(01))) \cup (\overline{N}_0(00) \setminus v_l^{-1}(\overline{S}(10))) \]
\[ = |h_u^* D_{01}| \cup |v_i^* D_{10}| \]
\[ = |\sup(h_u^* D_{01}, v_i^* D_{10})|, \]
where the sup is taken as Weil divisors, but it is also the sup as Cartier divisors by Lemma A.3 and by the assumption that \( v_i^* D_{10} \times_{\overline{\pi}_0(00)} h_u^* D_{01} \) is an effective Cartier divisor. Therefore, by Lemma B.1, there exists a positive integer \( m > 0 \) such that
\[ \tilde{N}_0(00)^\infty \leq msup(v_i^* D_{10}, h_u^* D_{01}) = sup(mv_i^* D_{10}, mh_u^* D_{01}). \]
Again by §4.3, we may assume that the closed subscheme
\[ (p^* N_0(11)^\infty + mv_i^* D_{10}) \times_{\overline{\pi}_0(00)} (p^* N_0(11)^\infty + mh_u^* D_{01}) \]
is an effective Cartier divisor on \( \overline{N}_0(00) \), because its restriction to \( \overline{S}(00) \) is the effective Cartier divisor \( S(00)^\infty \). Since \( \tilde{N}_0(00)^\infty = N_0(00)^\infty - p^* N_0(11)^\infty \) by definition, we obtain
\[ N_0(00)^\infty \leq sup(p^* N_0(11)^\infty + mv_i^* D_{10}, p^* N_0(11)^\infty + mh_u^* D_{01}), \]
where the sup is taken as Weil divisors, but it is also the sup as Cartier divisors by Lemma A.3, and by the assumption that \( (p^* N_0(11)^\infty + mv_i^* D_{10}) \times_{\overline{\pi}_0(00)} (p^* N_0(11)^\infty + mh_u^* D_{01}) \) is an effective Cartier divisor.
By \( p = v_r \circ h_u = h_d \circ v_l \), and by the admissibility of \( h_d \) and \( v_r \), we have
\[ p^* N_0(11)^\infty = v_i^* h_d^* N_0(11)^\infty \leq v_i^* N_0(10)^\infty, \]
\[ p^* N_0(11)^\infty = h_u^* v_l^* N_0(11)^\infty \leq h_u^* N_0(01)^\infty. \]
Still by §4.3, we may assume that the closed subscheme $(v_i^*(N_0(10)^\infty + mD_{10})) \times_{N_0(00)} (h_u^*(N_0(01)^\infty + mD_{01}))$ is an effective Cartier divisor on $N_0(00)$, because its restriction to $S(00)^\infty$. Combining the inequalities above, we obtain
\begin{equation}
N_0(00)^\infty \leq \sup(v_i^*(N_0(10)^\infty + mD_{10}), h_u^*(N_0(01)^\infty + mD_{01}))
\end{equation}
where the sup is a sup of Cartier divisors, as previously.

Define modulus pairs $N_1(ij), i,j \in \{0,1\}$, by
\begin{align*}
N_1(11) &:= N_0(11) = (N_0(11), N_0(11)^\infty), \\
N_1(10) &:= (N_0(10), N_0(10)^\infty + mD_{10}), \\
N_1(01) &:= (N_0(01), N_0(01)^\infty + mD_{01}), \text{ and} \\
N_1(00) &:= (N_0(00), \sup\{v_i^*(N_0(10)^\infty + mD_{10}), h_u^*(N_0(01)^\infty + mD_{01})\}).
\end{align*}
First, we check that these modulus pairs form a square. The admissibility of the morphisms $N_0(10) \to N_0(11)$ and $N_0(01) \to N_0(11)$ imply that we have admissible morphisms $N_1(10) \to N_1(11)$ and $N_1(01) \to N_1(11)$. Moreover, since $N_1(00)^\infty \geq h_u^*N_1(01)^\infty$ and $N_1(00)^\infty \geq v_i^*N_1(10)^\infty$ by definition, we have admissible morphisms $N_1(00) \to N_1(10)$ and $N_1(00) \to N_1(01)$. Therefore, we obtain a square $N_1 \in \text{MCor}^{\text{Sq}}$.

Next, we check that $N_1$ belongs to $\text{Comp}_{\min}^1(S)$. Since $N_1(11)^\infty = N_0(11)^\infty$ by definition, we have $N_1(11)^\infty|_{S(11)} = S(11)^\infty$ by the minimality of $S(11) \to N_0(11)$. Next, let $(i,j)$ be either $(1,0)$ or $(0,1)$. Since $D_{ij} \cap S(ij) = \emptyset$ by the definition of $D_{ij}$, we have $N_1(ij)^\infty|_{S(ij)} = N_0(ij)^\infty|_{S(ij)} = S(ij)^\infty$. Finally, we have
\begin{align*}
N_1(00)^\infty|_{S(00)} &= \sup\{h_u^*N_1(01)^\infty, v_i^*N_1(10)^\infty\}|_{S(00)} \\
&= \sup\{N_1(01)^\infty|_{S(00)}, N_1(10)^\infty|_{S(00)}\} \\
&= \sup\{S(00)^\infty, S(00)^\infty\} \\
&= S(00)^\infty.
\end{align*}
Therefore, we have $N_1 \in \text{Comp}_1(S)$. The universal minimality of $N_1$ holds by the definition of $N_1$, by the assumption that $(v_i^*(N_0(10)^\infty + mD_{10})) \times_{N_0(00)} (h_u^*(N_0(01)^\infty + mD_{01}))$ is an effective Cartier divisor, and by Lemma A.3.

Finally, we check that $N_1$ dominates $N_0$. For any $i,j$, noting that $N_1(ij) = N_0(ij)$, the identity morphism on total spaces induces an admissible morphism $N_1(ij) \to N_0(ij)$ since $N_1(ij)^\infty \geq N_0(ij)^\infty$ by definition if $(i,j) \neq (0,0)$ and by (4.3) if $(i,j) = (0,0)$. This finishes the proof of Proposition 4.2.
5. Existence of partial compactifications

This section is devoted to the proof of Theorem 3.2.

5.1. A first reduction. In this subsection, we prove the following result:

**Proposition 5.1.** Let $\mathcal{S}$ be an elementary Nisnevich square, and take any $S'(11) \in \text{Comp}_1(S(11))$. Then, there exists a cartesian diagram of schemes

\[
\begin{array}{ccc}
\mathcal{S}(01) & \longrightarrow & V \\
\downarrow f & \quad \square & \quad \downarrow q_V \\
\mathcal{S}(11) & \longrightarrow & S'(11),
\end{array}
\]

satisfying the following properties:

1. The horizontal arrows are open immersions.
2. The map $f$ (resp. $g$) is the (underlying) structure map of the square $\mathcal{S}$ (resp. of the morphism $S'(11) \to S'(11)$).
3. Set $Z(00) := \overline{\mathcal{S}(01)} \setminus \overline{\mathcal{S}(00)}$. Then, the closure $\overline{Z(00)}$ of $Z(00)$ in $V$ is proper over $\mathcal{S}'(11)$.
4. The morphism $q_V$ is flat, and $V$ is separated.

In the following, we prove Proposition 5.1 in several steps.

5.1.1. A compactification. Applying Nagata’s theorem to the morphism $\varphi : \overline{\mathcal{S}(01)} \to \overline{\mathcal{S}(11)}$, we obtain a commutative diagram

\[
\begin{array}{ccc}
\overline{\mathcal{S}(01)} & \longrightarrow & T \\
\downarrow f & \quad \quad \downarrow p \\
\overline{\mathcal{S}(11)} & \longrightarrow & \overline{\mathcal{S}'(11)},
\end{array}
\]

where the horizontal morphisms are open immersions, and $p$ is a proper surjective morphism. (Since $\varphi$ is étale, in particular, quasi-finite, we could also apply Zariski’s main theorem.) Set $Z(10) := \overline{\mathcal{S}(11)} \setminus \overline{\mathcal{S}(10)}$.

Consider the closure $\overline{Z(10)}$ in $\overline{\mathcal{S}(11)}$, given the reduced scheme structure. Then, in particular, the induced morphism $p_Z : p^{-1}(\overline{Z(10)}) \to \overline{Z(10)}$ is proper.

Moreover, set $Z(00) := \overline{\mathcal{S}(01)} \setminus \overline{\mathcal{S}(00)}$. 
By the assumption that $S$ is an elementary Nisnevich square, we have the following canonical identifications of schemes:

$$Z(00) \xrightarrow{\sim} f^{-1}(Z(10)) := Z(10) \times_{\overline{S}(11)} \overline{S}(01) \xrightarrow{\sim} Z(10).$$

5.1.2. Shrinking $T$: elimination of the fibers in $p^{-1}(Z(10))$ outside $Z(00)$.

**Lemma 5.2.** The natural inclusion $Z(00) \to p^{-1}(Z(10))$ is an open and closed immersion. In particular, there exists an open closed subscheme $W \subset p^{-1}(Z(10))$ such that

$$p^{-1}(Z(10)) = Z(00) \sqcup W. \tag{5.1}$$

**Proof.** Consider the commutative diagram

$$
\begin{array}{ccc}
Z(00) & \longrightarrow & p^{-1}(Z(10)) \\
\downarrow & & \downarrow \\
Z(10) & \searrow & \text{separated} \\
& & T
\end{array}
$$

The horizontal map is open by the equation

$$p^{-1}(Z(10)) \cap \overline{S}(01) = Z(00)$$

and proper, since the morphism $p^{-1}(Z(10)) \to Z(10)$ is separated and an isomorphism is proper. This finishes the proof. \hfill \square

Since $p^{-1}(Z(10))$ is a closed subset of $p^{-1}(\overline{S}(11))$, we obtain two disjoint closed subsets $Z(00)$ and $W$ of $p^{-1}(\overline{S}(11))$. Let $\overline{Z}(00)$ and $\overline{W}$ be their closures in $T$ (with the reduced scheme structures). Noting that $\overline{Z}(00) \cap \overline{W} \cap p^{-1}(\overline{S}(11)) = \emptyset$ by (5.1), the blow-up

$$\pi : T' := \text{Bl}_{\overline{Z}(00) \cap \overline{W}} T \to T$$

is an isomorphism over $p^{-1}(\overline{S}(11))$. Denote by $\overline{Z}(00)$ and $\overline{W}$ the strict transforms of $\overline{Z}(00)$ and $\overline{W}$ along $\pi$, respectively. Then, we have

$$\overline{Z}(00) \cap \overline{W} = \emptyset.$$

Moreover, regard $Z(00)$ and $W$ as closed subschemes of $\pi^{-1}(p^{-1}(\overline{S}(11)))$, through the isomorphism

$$\pi^{-1}(p^{-1}(\overline{S}(11))) \to p^{-1}(\overline{S}(11))$$

induced by $\pi$. Then, the closed subschemes $\overline{Z}(00)$ and $\overline{W}$ of $T'$ are equal to the closures of $Z(00)$ and $W$ in $T'$. Since $\pi$ is an isomorphism over $p^{-1}(\overline{S}(11))$, it is in particular an isomorphism over $\overline{S}(01)$ because
$\overline{S}(01) \subset p^{-1}(\overline{S}(11))$. Therefore, the open immersion $\overline{S}(01) \to T$ lifts to an open immersion $\overline{S}(01) \to T'$. Therefore, we may replace $T$ by $T'$. After this replacement, we obtain

(5.2) \quad \overline{Z(00)} \cap \overline{W} = \emptyset.

Define an open subset $U \subset T$ by

$U := T \setminus \overline{W},$

and let

$q_U : U \to T \to \overline{S}(11)$

be the natural map.

**Proposition 5.3.**

1. We have $q_U^{-1}(Z(10)) = Z(00)$.
2. The closure of $Z(00)$ in $T$ is contained in $U$; in particular, it is proper over $\overline{S}(11)$.

**Proof.** The map $q_U$ restricted over $\overline{S}(11)$ is equal to

$q_U|_{\overline{S}(11)} : p^{-1}(\overline{S}(11)) \setminus W \hookrightarrow p^{-1}(\overline{S}(11)) \to \overline{S}(11).$

Therefore, since $p^{-1}(\overline{S}(Z(10)) \setminus W = Z(00)$ by definition of $W$, we have

$q_U^{-1}(Z(10)) = (q_U|_{\overline{S}(11)})^{-1}(Z(10)) = Z(00).$

This proves (1), and (2) follows from (5.2) and the properness of $p$. □

**5.1.3. “Separation” of two closed subsets in $U$.** Define a closed subset $Z(01) \subset q_U^{-1}(\overline{S}(11))$ by

$Z(01) := q_U^{-1}(\overline{S}(11)) \setminus \overline{S}(01).$

Denote by $\overline{Z(01)}$ and $\overline{Z(00)}$ the closures of $Z(01)$ and $Z(00)$ in $U$ with the reduced scheme structures, respectively. Then, we have the following

**Lemma 5.4.**

$\overline{Z(00)} \cap \overline{Z(01)} \cap q_U^{-1}(\overline{S}(11)) = \emptyset.$

**Proof.** Note that $Z(01)$ is closed in $q_U^{-1}(\overline{S}(11))$ by definition. Moreover, $Z(00)$ is also closed in $q_U^{-1}(\overline{S}(11))$, since by Proposition 5.3, we have

$q_U^{-1}(Z(10)) = Z(00).$

Therefore, it suffices to show that $Z(00) \cap Z(01) = \emptyset$. But this is obvious by $Z(00) \subset \overline{S}(01)$, and by $Z(01) = q_U^{-1}(\overline{S}(11)) \setminus \overline{S}(01)$. This finishes the proof. □
Consider now the blow-up
\[ \pi_U : U' := \text{Bl}_{Z(00) \cap Z(01)} U \to U. \]

The strict transforms \( \widetilde{Z(01)} \) and \( \widetilde{Z(00)} \) along \( \pi_U \) are equal to the closures of \( Z(01) \) and \( Z(00) \) in \( U' \), and we have
\[
(5.3) \quad \widetilde{Z(00)} \cap \widetilde{Z(01)} = \emptyset.
\]

By Lemma 5.4, \( \pi_U \) is an isomorphism over \( q_U^{-1}(\overline{S(11)}) \). Regard \( Z(01) \) and \( Z(00) \) as closed subschemes of \( \pi_U^{-1}q_U^{-1}(\overline{S(11)}) \). Then, by Proposition 5.3 (1), we have
\[
\pi_U^{-1}q_U^{-1}(Z(10)) = Z(00).
\]

Moreover, the strict transform \( \widetilde{Z(00)} \) is proper over \( \overline{Z(00)} \), hence proper over \( \overline{S'}(11) \) by Proposition 5.3 (2). Therefore, Proposition 5.3 remains true after we replace \( U \) by \( U' \) and \( q_U \) by \( q_U \circ \pi_U \), respectively. After this replacement, the equality (5.3) implies that
\[
(5.4) \quad \overline{Z(00)} \cap \overline{Z(01)} = \emptyset.
\]

Define an open subset \( V_0 \subset U \) by
\[
V_0 := U \setminus \overline{Z(01)},
\]
and let
\[
q_V : V_0 \to \overline{S'}(11)
\]
be the natural map. Then, we have the following

**Proposition 5.5.** The map \( q_V \) satisfies the following conditions:

1. \( q_V^{-1}(\overline{S(11)}) = \overline{S(01)} \). In particular, \( q_V \) is étale over \( \overline{S(11)} \).
2. The closure of \( Z(00) \) in \( V_0 \) is proper over \( \overline{S'}(11) \).

**Proof.** The map \( q_V \) restricted over \( \overline{S(11)} \) is equal to
\[
q_U^{-1}(\overline{S(11)}) \setminus Z(01) \to q_U^{-1}(\overline{S(11)}) \to \overline{S(11)}.
\]

Therefore, by definition of \( Z(01) \), we have \( q_V^{-1}(\overline{S(11)}) = \overline{S(01)} \). This proves the assertion (1). The assertion (2) follows from Proposition 5.3 and the equality (5.4). This finishes the proof. \( \square \)
5.1.4. “Platification” of $q_V$. In this final step, we construct a replacement of $q_V$ so that it becomes flat. Proposition 5.5 (1) implies that $q_V$ is étale (in particular, flat) over $\overline{S}(11)$. Therefore, by the theorem of “platification” of Raynaud-Gruson [10, Th. 5.2.2], we can find a blow-up

$$\pi_S : \overline{S}(11) \to \overline{S}(11)$$

satisfying the following properties:

(1) Let

$$\pi_V : V \to V_0$$

be the strict transform of $V_0$ along $\pi_S$. Then, the morphism

$$\tilde{q}_V : V \to \overline{S}(11),$$

which is induced by the universal property of blow-up, is flat.

(2) The blow up $\pi_S$ is an isomorphism over $\overline{S}(11)$.

Since $\pi_S$ is an isomorphism over $\overline{S}(11)$, the strict transform $\pi_V$ is an isomorphism over $\overline{q}_V^{-1}(\overline{S}(11)) = \overline{S}(01)$, where the equality follows from Proposition 5.5 (1). Regard $Z(00)$ as a closed subscheme of $\pi_V^{-1}(\overline{S}(01))$ through the isomorphism

$$\pi_V^{-1}(\overline{S}(01)) \sim \overline{S}(01)$$

induced by $\pi_V$. Let $\overline{Z}(00)$ be the closure of $Z(00)$ in $V_0$, and let $\overline{Z}(00)$ be the closure of $Z(00)$ in $V$. Then, $\overline{Z}(00)$ is a strict transform of $\overline{Z}(00)$. Therefore, we conclude that

$$(5.5) \quad \text{the strict transform of } Z(00) \text{ in } V \text{ is proper over } \overline{S}(11).$$

Moreover, we have

$$(5.6) \quad \overline{q}_V^{-1}(\overline{S}(11)) = 1 \overline{q}_V^{-1} \pi_S^{-1}(\overline{S}(11)) = 2 \pi_V^{-1} \overline{q}_V^{-1}(\overline{S}(11)) = 3 \pi_V^{-1}(\overline{S}(01)) = 4 \overline{S}(01),$$

where $=1$ follows from Property (2) above, $=2$ follows from the equality $q_V \pi_V = \pi_S \tilde{q}_V$, $=3$ follows from Proposition 5.5 (1), and $=4$ follows from the fact that $\pi_V$ is an isomorphism over $\overline{q}_V^{-1}(\overline{S}(11))$, which contains $\overline{S}(01)$.

The assertions (5.5) and (5.6) imply that Proposition 5.5 remains true after we replace $V_0$, $\overline{S}(11)$ and $q_V$ by $V$, $\overline{S}(11)$ and $\tilde{q}_V$, respectively. Therefore, after this replacement, we obtain the flatness of $q_V$. 
Finally, $V$ is separated, since it is obtained from $T$ (proper) in §5.1.1 by a composition of open immersions and proper morphisms. This completes the proof of Proposition 5.1.

5.2. Construction of a strict extension $S \to S'$. In this subsection, we use Proposition 5.1 to prove

**Proposition 5.6.** Let $S$ be an elementary Nisnevich square, and take any $N_0(11) \in \text{Comp}_1(S(11))$. Then, there exists a strict extension $S \to S'$ such that $S'$ is an elementary Nisnevich square, $S'(11) \in \text{Comp}_1(S(11))$, $S'(11)$ is normal and $S'(11)$ dominates $N_0(11)$.

In the statement of Proposition 5.1, choose $S'(11)$ dominating $N_0(11)$. We then have the following diagram

\[
\begin{array}{ccc}
S(01) & \longrightarrow & V \\
\downarrow f & & \downarrow q_V \\
S(11) & \longrightarrow & S'(11)
\end{array}
\]

where the left square is the one given in Proposition 5.1, and $Z(10)$ is the closure of $Z(10) := S(11) \setminus S(10)$ in $S(11)$ with the reduced scheme structure. The scheme $q_V^{-1}(Z(10))$ is defined to be the fiber product $Z(10) \times_{S(11)} V$. We need the following lemma (compare [11, Cor. 2.2] and [12, Th. 2.7]):

**Lemma 5.7.** Let $\varphi : X \to S$ be a morphism of schemes and $U \subset S$ an open subset. Assume the following conditions hold:

(i) $\varphi$ is separated, quasi-finite, of finite presentation and flat.
(ii) $\varphi^{-1}(U) \to U$ is an isomorphism.
(iii) The inclusion $U \to S$ is quasi-compact and scheme-theoretically dense.

Then, $\varphi$ is an open immersion.

A proof of Lemma 5.7 is given in the Stacks Project\textsuperscript{1}. In fact, the assumption (i) can be replaced by the following weaker condition: $\varphi$ is separated, locally of finite type and flat. However, for our purpose, Lemma 5.7 is enough. We reconstruct a proof in §B.4 for the reader’s convenience.

**Proposition 5.8.** The map

\[ q_z : q_V^{-1}(Z(10)) \to Z(10) \]

is an isomorphism.

\textsuperscript{1}Lemma 37.11.5, \url{https://stacks.math.columbia.edu/tag/081M}
Proof. Since $q_V$ is flat by Proposition 5.1 (4), so is $q_Z$. Since $S(11) \times_{S(11)} V \cong S(01)$ by (5.7), we have

$$Z(10) \times_{S(11)} V \cong Z(10) \times_{S(11)} S(01) \cong Z(10),$$

where the isomorphism $\cong^{\dagger}$ is obtained by the assumption that $S$ is an elementary Nisnevich square. Therefore, the morphism $q_Z$ is an isomorphism over the dense open subset $Z(10) \subset Z(10)$. Since $Z(10)$ is given the reduced scheme structure, the open subset $Z(10) \subset Z(10)$ is scheme-theoretically dense. Moreover, $q_Z$ is quasi-finite (since it is flat and birational), and separated since $V$ is (see [EGA1, Ch. I, §5.3]). Finally, $q_Z$ is of finite presentation since it is a morphism between schemes of finite type (hence of finite presentation) over a field. Therefore, Lemma 5.7 implies that $q_Z$ is an open immersion.

By Proposition 5.1 (3), the map $q_Z$ induces a proper morphism $\overline{Z(00)} \to \overline{Z(10)}$. Since $Z(00) \to Z(10)$ is an isomorphism, the morphism $Z(00) \to Z(10)$ is dominant and proper, hence surjective. This implies that $q_Z$ is surjective, because $\overline{Z(00)} \subset q^{-1}_V(\overline{Z(10)})$. This concludes the proof. \hfill \square

Lemma 5.9. The flat morphism $q_V$ is étale over an open neighborhood of $\overline{Z(10)}$. In particular, the étale locus of $q_V$ contains $q_V^{-1}(\overline{Z(10)})$.

Proof. By Proposition 5.8, $q_V$ is unramified over $\overline{Z(10)}$. Since the unramified locus is open, $q_V$ is unramified over an open neighborhood of $\overline{Z(10)}$. Since $q_V$ is flat, this finishes the proof. \hfill \square

Definition 5.10. Define an open subset $\overline{S'}(01) \subset V$ by

$$\overline{S'}(01) := (\text{the étale locus of } q_V) \subset V.$$

Set

$$\overline{S'}(10) := \overline{S'}(11) \setminus \overline{Z(10)}$$

and

$$\overline{S'}(00) := \overline{S'}(10) \times_{\overline{S'}(11)} \overline{S'}(01).$$

Define for each $(i,j) \in \{(1,0), (0,1), (0,0)\}$

$$S'(ij) := (\overline{S'}(ij), S(11)_{\overline{S'}(ij)}).$$

Lemma 5.11. The square

$$\begin{array}{ccc}
S'(00) & \longrightarrow & S'(01) \\
\downarrow & & \downarrow \\
\overline{S'}(10) & \longrightarrow & S'(11)
\end{array}$$
is an elementary Nisnevich square extending $\mathcal{S}$. Moreover, for all $i, j \in \{0, 1\}$, we have
\begin{equation}
(5.8) \quad \mathcal{S}(ij) = \mathcal{S}'(ij) \times_{\mathcal{S}(11)} \mathcal{S}(11).
\end{equation}

**Proof.** The map $\mathcal{S}'(01) \to \mathcal{S}'(11)$ is étale by definition. By Lemma 5.8 and Lemma 5.9, it is an isomorphism over $\mathcal{Z}(10) = (\mathcal{S}'(11) \setminus \mathcal{S}'(10))_{\text{red}}$. This shows that the square is an elementary Nisnevich square. We prove the equality $\mathcal{S}(ij) = \mathcal{S}'(ij) \times_{\mathcal{S}(11)} \mathcal{S}(11)$. The case $(i, j) = (0, 0)$ is trivial. The case $(i, j) = (1, 0)$ follows from $\mathcal{S}'(10) \cap \mathcal{S}(11) = \mathcal{S}(11) \setminus \mathcal{Z}(10) = \mathcal{S}(10)$. 

The case $(i, j) = (0, 1)$ follows from $\mathcal{S}(11) \times_{\mathcal{S}(11)} \mathcal{V} \cong \mathcal{S}(01)$ (see (5.7)). Finally, the case $(i, j) = (0, 0)$ follows from 
\begin{align*}
\mathcal{S}(00) \times_{\mathcal{S}(11)} \mathcal{S}(11) &= (\mathcal{S}(10) \times_{\mathcal{S}(11)} \mathcal{S}(11)) \times_{\mathcal{S}(11)} (\mathcal{S}(01) \times_{\mathcal{S}(11)} \mathcal{S}(11)) \\
&= \mathcal{S}(10) \times_{\mathcal{S}(11)} \mathcal{S}(01) = \mathcal{S}(00).
\end{align*}

This finishes the proof. \hfill $\square$

Thus, we have constructed $\mathcal{S}'$ satisfying the properties in Proposition 5.6, except for the normality of $\mathcal{S}'(11)$. To obtain the latter, consider modulus pairs $\mathcal{S}'(ij) := (\mathcal{S}'(ij))^N$, the pullback of $\mathcal{S}'(ij)^\infty$ in $\mathcal{S}'(11)$. By Lemma B.4, we have $\mathcal{S}'(ij)^N \cong \mathcal{S}(ij) \times_{\mathcal{S}(11)} \mathcal{S}'(11)^N$. Therefore, $\mathcal{S}'(ij)^N$'s form an elementary Nisnevich square $\mathcal{S}'N$. Since $\mathcal{S}(ij)$ are normal, the strict extension $\mathcal{S} \to \mathcal{S}'$ extends to a strict extension $\mathcal{S} \to \mathcal{S}'N$. Thus, $\mathcal{S}'N$ satisfies the properties in Proposition 5.6. This finishes the proof.

5.3. **Construction of a minimal morphism $\mathcal{S}' \to N_0$.** Take $N_0$ as in the statement of Theorem 3.2, and $\mathcal{S}'$ as given by Proposition 5.6 with respect to $N_0(11)$. We will modify $\mathcal{S}'$ so that it admits a minimal morphism $\mathcal{S}' \to N_0$. By Proposition 5.6, we already have an admissible morphism $\mathcal{S}'(11) \to N_0(11)$. However, it is not necessarily the case for the other corners of the square. To solve this problem, we consider the following construction.

At least, for each $i, j$, we have a birational map $\mathcal{S}'(ij) \dashrightarrow N_0(ij)$, which is defined on $\mathcal{S}(ij)$. Denote by $\Gamma_{ij}$ its graph, i.e., the closure of the graph of $\mathcal{S}(ij) \to N_0(ij)$ in $\mathcal{S}'(ij) \times N_0(ij)$, with the reduced
scheme structure. Then, we obtain a diagram
\[
\overline{S}(ij) \leftarrow \Gamma_{ij} \rightarrow N_0(ij).
\]
Consider the composite morphism
\[
f_{ij} : \Gamma_{ij} \rightarrow \overline{S}(ij) \rightarrow \overline{S}'(11).
\]

**Lemma 5.12.** We have \( f_{ij}^{-1}(\overline{S}(11)) = \overline{S}(ij) \). In particular, the morphism \( f_{ij} \) is étale over \( \overline{S}(11) \).

**Proof.** This follows from the fact that the morphism \( \Gamma_{ij} \rightarrow \overline{S}(ij) \) is an isomorphism over \( \overline{S}(ij) \), and that \( \overline{S} \rightarrow \overline{S}' \) is a strict extension. □

Lemma 5.12 shows that the coproduct
\[
f := \bigsqcup_{(i,j) \neq (1,1)} f_{ij} : \bigsqcup_{(i,j) \neq (1,1)} \Gamma_{ij} \rightarrow \overline{S}'(11)
\]
is étale (in particular, flat) over \( \overline{S}(11) \). Then the platification theorem [10, Th. 5.2.2] applied to \( f \) shows that there exists a closed subscheme \( Z \hookrightarrow \overline{S}(11) \setminus \overline{S}(11) \) such that the morphism induced between the blow-ups
\[
\bigsqcup_{(i,j) \neq (1,1)} \tilde{f}_{ij} : \bigsqcup_{(i,j) \neq (1,1)} \text{Bl}_{Z \times \overline{S}'(11)}(\Gamma_{ij}) \rightarrow \text{Bl}_{Z}(\overline{S}'(11))
\]
is flat. Consider the following commutative diagram, induced by the universal property of blowing up:

\[
\begin{array}{ccc}
\text{Bl}_{Z \times \overline{S}'(11)}(\Gamma_{ij}) & \xrightarrow{h_{ij}} & \text{Bl}_{Z \times \overline{S}'(11)}(\overline{S}(ij)) \\
\downarrow \tilde{f}_{ij} & & \downarrow g_{ij} \\
\text{Bl}_{Z}(\overline{S}'(11)) & & \text{Bl}_{Z}(\overline{S}(11))
\end{array}
\]

where \( g_{ij} \) is étale since we have
\[
(5.9) \quad \text{Bl}_{Z \times \overline{S}'(11)}(\overline{S}(ij)) \cong \overline{S}(ij) \times_{\overline{S}(11)} \text{Bl}_{Z}(\overline{S}'(11))
\]
by the étaleness of the map \( \overline{S}(ij) \rightarrow \overline{S}'(11) \) and by the (trivial) compatibility of blow-up and flat base change. Since \( f_{ij} \) is flat and \( g_{ij} \) is étale, the horizontal morphism \( h_{ij} \) is flat (see for example [EGA4-IV, Prop. 18.4.9]).

However, \( h_{ij} \) is a proper birational morphism induced by \( \Gamma_{ij} \rightarrow \overline{S}(ij) \), hence it is an isomorphism over the open dense subset \( \overline{S}(ij) \subset \text{Bl}_{Z \times \overline{S}'(11)}(\overline{S}(ij)) \). Therefore, Lemma 5.7 implies that \( h_{ij} \) is an isomorphism.
Define modulus pairs $S'_1(ij) = (S'_1(ij), S'_1(ij)\infty)$ by

$$S'_1(ij) := \overline{S}(ij) \times_{\overline{S}(11)} \operatorname{Bl}_{Z_{\overline{S}(11)}} S(11) \cong^\dagger \operatorname{Bl}_{Z_{\overline{S}(11)}} S(11) \times_{\overline{S}(11)} \overline{S}(ij),$$

$$S'_1(ij)\infty := S'_1(ij)\infty \times_{\overline{S}(ij)} S'_1(ij) = S'_1(ij)\infty \times_{\overline{S}(11)} \overline{S}(11),$$

where $\cong^\dagger$ follows from (5.9). Then, the modulus pairs $S'_1(ij)$ form an elementary Nisnevich square $S'_1$. Since the blow-up $\overline{S}_1(11) \to \overline{S}(11)$ is an isomorphism over $S(11)$ by the construction, there is a natural strict extension $S \to S'_1$. Moreover, we have natural maps on the total spaces

$$\overline{S}_1(ij) \cong \operatorname{Bl}_{Z_{\overline{S}(11)}} S(ij) \times_{\overline{S}(11)} \overline{S}(11) \xrightarrow{h_{ij}^{-1}} \operatorname{Bl}_{Z_{\overline{S}(11)}} S(ij) \to S(ij) \to \overline{N}_0(ij).$$

Moreover, by taking normalization everywhere, we can realize the condition that $\overline{S}(11)$ is normal. Here, note that the compatibility between normalization and étale base change (see Lemma B.4) ensures that the normalization preserves the property to be an elementary Nisnevich square.

Therefore, by replacing $\overline{S}'$ by $\overline{S}_1$, we may assume that

for each $i, j$, the total space $\overline{S}(ij)$ maps to $\overline{N}_0(ij)$.

However, it is not always the case that these maps induce minimal (or admissible) morphisms $S'(ij) \to N_0(ij)$. Our task in the next subsection is to adjust the moduluses of $S'$ and $N_0$ to ensure that these maps on the total spaces induce minimal morphisms.

5.4. Modification of moduluses of $S'$ and $N_0$. The argument will be divided into several steps. First, note that we may assume the following condition without loss of generality:

$$\overline{S}(11) = \overline{N}_0(11).$$

Indeed, consider the modulus pairs

$$N'_0(ij) := (\overline{N}_0(ij) \times_{\overline{N}_0(11)} \overline{S}(11),$$

the pullback of $N_0(ij)\infty$, $(i, j = 0, 1)$.

These modulus pairs naturally form a square $N'_0 \in \text{Comp}_1(\overline{S})$ dominating $N_0$, because the projection $\overline{N}_0(ij) \to \overline{N}_0(ij)$ is an isomorphism over $\overline{S}(ij)$ for each $i, j$ since $\overline{S}(11) \to \overline{N}_0(11)$ is an isomorphism over $\overline{S}(11)$ by Lemma B.2. Moreover, the morphisms of schemes $\overline{S}(ij) \to \overline{N}_0(ij)$ canonically lift to $\overline{S}(ij) \to \overline{N}_0(ij)$. Therefore, by replacing $N_0$ by $N'_0$, we obtain the equality above.
5.4.1. **Enlargement of the modulus of** $S'$. First, we enlarge the modulus of $S'$, preserving the minimality of $S 	o S'$, as follows.

Define $Z := \overline{S}'(11) \setminus \overline{S}(11)$, and regard $Z$ as a closed subscheme of $\overline{S}'(11) = \overline{N}_0(11)$ with the reduced scheme structure. Consider the blow-up

$$\text{Bl}_Z \overline{N}_0(11) \to \overline{N}_0(11),$$

which is an isomorphism over $\overline{S}(11)$ since $Z \cap \overline{S}(11) = \emptyset$. Replacing the total spaces $\overline{N}_0(ij)$ and $\overline{S}(ij)$ by their strict transforms along this blow-up, and pulling back every modulus, we may assume that $Z$ is an effective Cartier divisor on $\overline{N}_0$. Here, note that for each $i,j$, the strict transform $\overline{S}'(ij)$ of $\overline{S}(ij)$ is given by the fiber product $\overline{S}'(ij) \times_{\overline{N}_0(11)} \text{Bl}_Z \overline{N}_0(11)$, which ensures that $\overline{S}'(ij)$’s form an elementary Nisnevich square.

Define effective Cartier divisors on $\overline{S}'(ij)$ by

$$Z_{ij} := Z \times_{\overline{S}(11)} \overline{S}'(ij).$$

Note that

$$|Z_{ij}| = (\overline{S}(11) \setminus \overline{S}(11)) \times_{\overline{S}(11)} \overline{S}'(ij) = \overline{S}'(ij) \setminus \overline{S}(ij),$$

where the equality $=\dagger$ follows from (5.8).

**Proposition 5.13.** There exists a positive integer $m$ satisfying

$$N_0(ij)^\infty|_{\overline{S}(ij)} \leq S'(ij)^\infty + mZ_{ij}$$

for each $i,j$, where $N_0(ij)^\infty|_{\overline{S}(ij)}$ denotes the pullback of the effective Cartier divisor $N_0(ij)^\infty$ by the dominant morphism $\overline{S}(ij) \to \overline{N}_0(ij)$.

**Proof.** The restrictions of the Cartier divisor $S'(ij)^\infty - N_0(ij)^\infty|_{\overline{S}(ij)}$ to $\overline{S}(ij)$ is zero. Therefore, by the equality (5.11), Lemma B.1 implies that we can find positive integers $m_{ij}$ such that

$$N_0(ij)^\infty|_{\overline{S}(ij)} \leq S'(ij)^\infty + m_{ij}Z_{ij}$$

holds. Take $m := \max_{i,j} m_{ij}$. This finishes the proof. \(\square\)

Take $m$ as in Proposition 5.13. Define modulus pairs $S'_1(ij)$ by

$$S'_1(ij) := \left(\overline{S}(ij), S'(ij)^\infty + mZ_{ij}\right).$$

Since by definition we have $\overline{S}(ij) \times_{\overline{S}(11)} Z_{11} = Z_{ij}$, the modulus pairs $S'_1(ij)$’s form an elementary Nisnevich square $S'_1$. 


Proposition 5.14. The map $\mathcal{S} \to \mathcal{S}_1'$ is a strict extension, and we have

$$N_0(ij)^\infty|_{\mathcal{S}_1'(ij)} \leq S_1'(ij)^\infty,$$

where $N_0(ij)^\infty|_{\mathcal{S}_1'(ij)}$ denotes the pullback of the Cartier divisor $N_0(ij)^\infty$ by the dominant morphism $\mathcal{S}_1'(ij) \to \mathcal{N}_0(ij)$. Hence the morphism $\mathcal{S} \to \mathcal{N}_0$ extends to a morphism $\mathcal{S}_1' \to \mathcal{N}_0$.

Proof. The minimality of the map follows from $|Z_{ij}| \cap \mathcal{S}(ij) = \emptyset$ and the minimality of $\mathcal{S} \to \mathcal{S}'$. The equality $\mathcal{S}_1'(ij) \times_{\mathcal{S}_1'(11)} \mathcal{S}(11) = \mathcal{S}(ij)$ follows from $\mathcal{S}_1'(ij) = \mathcal{S}(ij)$. Therefore, $\mathcal{S} \to \mathcal{S}_1'$ is a strict extension. The last assertion is immediate by Proposition 5.13. □

Up to pulling back $\mathcal{N}_0$ by (5.10), we may assume that the condition $\mathcal{S}_1'(11) = \mathcal{N}_0(11)$ continues to hold.

5.4.2. Creation of Cartier divisors.

Lemma 5.15. For all $i, j \in \{0, 1\}$, define a closed subscheme $B_{ij}$ of $N_0(ij)^\infty$ by

$$B_{ij} := S_1'(11)^\infty \times_{\mathcal{N}_0(11)} N_0(ij)^\infty.$$

Then, for each $i, j \in \{0, 1\}$, we have

$$B_{ij}|_{\mathcal{S}_1'(ij)} = N_0(ij)^\infty|_{\mathcal{S}_1'(ij)}.$$

In particular, the left hand side of the equality is an effective Cartier divisor on $\mathcal{S}_1'(ij)$.

Proof. Since $S_1'$ is an elementary Nisnevich square, we have

$$S_1'(11)^\infty|_{\mathcal{S}_1'(ij)} = S_1'(ij)^\infty$$

and the claim follows from Proposition 5.14. □

Lemma 5.16. There exists a morphism $\tilde{\mathcal{N}}_0 \xrightarrow{\pi} \mathcal{N}_0$ in $\text{Comp}_{\text{mor}}^1(S)$ such that

1. The morphism $\mathcal{S}_1' \to \mathcal{N}_0$ lifts to $\tilde{\mathcal{N}}_0$.
2. $\pi^*B_{ij}$ is an effective Cartier divisor for all $i, j$.
3. The total space of $\tilde{\mathcal{N}}_0(ij)$ is normal for any $(i, j)$.

Proof. The equality $\mathcal{S}_1'(11) = \mathcal{N}_0(11)$ implies the inequality $S_1'(11)^\infty \geq N_0(11)^\infty$. Thus, $B_{11} = N_0(11)^\infty$ is already an effective Cartier divisor on $\mathcal{N}_0(11)$. We set $\tilde{\mathcal{N}}_0(11) = N_0(11)$.
Next, let \((i, j)\) be one of \((1, 0)\) or \((0, 1)\), and define modulus pairs \(N_0(ij)\) and \(N_0(00)\) by
\[
N_0(ij) := (\text{Bl}_{B_{ij}} \overline{N}_0(ij), \text{the pullback of } N_0(ij)^\infty),
\]
\[
N_0(00) := (\overline{N}_0(00) \times_{\overline{N}_0(ij)} \overline{N}_0(ij), \text{the pullback of } N_0(00)^\infty).
\]
Then, by Lemma 5.15, the morphism \(S'_1(ij) \to N_0(ij)\) lifts to \(S'_1(ij) \to \overline{N}_0(ij)\). Therefore, the morphism \(S'_1(00) \to N_0(00)\) lifts to \(S'_1(00) \to \overline{N}_0(00)\). Moreover, we have
\[
B_{ij} \times_{\overline{N}_0(ij)} \overline{N}_0(ij) = S'_1(11)^\infty \times_{\overline{N}_0(11)} \overline{N}_0(ij)^\infty.
\]
Denote by \(\overline{\overline{N}_0}\) the square obtained from \(\overline{\overline{N}_0}\) by replacing \(N_0(ij)\) and \(N_0(00)\) by \(\overline{N}_0(ij)\) and \(\overline{N}_0(00)\), respectively. Then, \(\overline{\overline{N}_0}\) dominates \(\overline{N}_0\), and the map \(S'_1 \to \overline{N}_0\) lifts to a morphism \(S'_1 \to \overline{N}_0\). Therefore, by replacing \(\overline{N}_0\) by \(\overline{\overline{N}_0}\), we may assume that \(B_{ij}\) is an effective Cartier divisor on \(\overline{N}_0(ij)\).

After this replacement, we apply the same procedure to \((i', j') \in \{(1, 0), (0, 1)\} - \{(i, j)\}\). Then, we may assume that \(B_{ij}\) is an effective Cartier divisor on \(\overline{N}_0(00)\) for \((i, j) = (1, 0), (0, 1)\).

Now, we reset the notation, and treat the case \((i, j) = (0, 0)\). Define a modulus pair \(\overline{N}_0(00)\) by
\[
\overline{N}_0(00) := (\text{Bl}_{B_{00}} \overline{N}_0(00), \text{the pullback of } N_0(00)^\infty).
\]
Then, by Lemma 5.15, the morphism \(S'_1(00) \to N_0(00)\) lifts to \(S'_1(00) \to \overline{N}_0(00)\). Moreover, we have
\[
B_{00} \times_{\overline{N}_0(00)} \overline{N}_0(00) = S'_1(11)^\infty \times_{\overline{N}_0(11)} \overline{N}_0(00)^\infty.
\]
Therefore, replacing \(N_0(00)\) by \(\overline{N}_0(00)\), we may assume that \(B_{00}\) is an effective Cartier divisor on \(\overline{N}_0(00)\).

Finally, we may assume that \(N_0(ij)\) is normal without loss of generality, just by replacing each \(N_0(ij)\) by
\[
N_0(ij)^N := (N_0(ij))^N, \text{the pullback of } N_0(ij)^\infty.
\]
Note that \(N_0(ij)^N\)’s form a square \(N_0^N\) which dominates \(N_0\), and that the morphism \(S'_1 \to N_0\) lifts to a morphism \(S'_1 \to N_0^N\) by the normality of \(S'_1(ij)\)’s. □

5.4.3. Enlargement of the modulus of \(N_0\). In the following, we replace \(\overline{N}_0\) by \(\overline{\overline{N}_0}\) as in Lemma 5.16, hence assume that the subscheme \(B_{ij}\) of Lemma 5.15 is an effective Cartier divisor on \(\overline{N}_0(ij)\) and that the latter is normal, for each \(i, j \in \{0, 1\}\).
Define modulus pairs $N_1(ij) = (\overline{N}_1(ij), N_1(ij)\infty)$ by

\[
\overline{N}_1(ij) := \overline{N}_0(ij),
\]

\[
N_1(ij)\infty := \sup(S'_1(11)\infty \times_{\overline{N}_0(11)} \overline{N}_0(ij), N_0(ij)\infty),
\]

where the sup is taken as Weil divisors, but it is also the sup as Cartier divisors by Lemma A.3, the normality of $\overline{N}_0(ij)$ and the assumption that $B_{ij}$ is an effective Cartier divisor on $\overline{N}_0(ij)$.

**Lemma 5.17.** For each $i, j \in \{0, 1\}$, we have

\[
N_1(ij)\infty|_{\overline{S}(ij)} = S'_1(ij)\infty,
\]

where $N_1(ij)\infty|_{\overline{S}(ij)}$ denotes the pullback of the Cartier divisor $N_1(ij)\infty$ by the dominant morphism $\overline{S}'_1(ij) \to \overline{N}_1(ij) = \overline{N}_0(ij)$.

**Proof.** By Lemma A.3 (3), we have

\[
N_1(ij)\infty|_{\overline{S}(ij)} = \sup(S'_1(11)\infty \times_{\overline{N}_0(11)} \overline{N}_0(ij)|_{\overline{S}(ij)}, N_0(ij)\infty|_{\overline{S}(ij)})
\]

\[
= S'_1(ij)\infty
\]

where the equality $=1$ follows from

\[
S'_1(11)\infty \times_{\overline{N}_0(11)} \overline{N}_0(ij)|_{\overline{S}(ij)} = S'_1(11)\infty \times_{\overline{N}_0(11)} \overline{S}'_1(ij),
\]

and from the minimality of the morphism $S'_1(ij) \to S'_1(11)$. The equality $=2$ follows from Proposition 5.14. This finishes the proof. \qed

We summarize the results of this subsection in the following proposition.

**Proposition 5.18.** The modulus pairs $N_1(ij)$’s form a square $\overline{N}_1$ in $\text{MCor}^{\text{Sq}}$ such that $\overline{N}_1 \in \text{Comp}_1^{\text{mor}}(S)$ and there exists a morphism $\overline{S}_1(ij) \to \overline{N}_0(ij)$ in $\text{Comp}_1(S)$. Moreover, the maps on the total spaces $\overline{S}_1(ij) \to \overline{N}_0(ij) = \overline{N}_1(ij)$ induce a minimal morphism $\overline{S}_1 \to \overline{N}_1$.

**Proof.** Let $(i, j) \to (i', j')$ be a morphism in $\text{Sq}$, and let $f : \overline{N}_0(ij) \to \overline{N}_0(i'j')$ be the corresponding morphism on total spaces. Then, we have $N_0(ij)\infty \geq f^*N_0(i'j')\infty$ since $\overline{N}_0 \in \text{MCor}^{\text{Sq}}$. This implies that

\[
N_1(ij)\infty \geq \sup\{f^*(S'_1(11)\infty \times_{\overline{N}_0(11)} \overline{N}_0(i'j')), f^*N_0(i'j')\infty\} = f^*N_1(i'j')\infty,
\]

where the last equality follows from Lemma A.3 (3). This proves that the square $\overline{N}_1$ is well-defined. The existence of the map $\overline{N}_1 \to \overline{N}_0$ is obvious by the construction of $\overline{N}_1$. Lemma 5.17 implies that the
morphism $S'_{11} \to N_{11}$ is minimal. In particular, $S \to N_{11}$ is also minimal, which implies $N_{11} \in \text{Comp}_1(S)$. This finishes the proof. □

Thus, we have finished the proof of Theorem 3.2.

6. Existence of MV-compactifications

The aim of this section is to prove Theorem 3.3. Thus, throughout, $S$ is an elementary Nisnevich square with $S(11)$ proper.

6.1. A remark and a lemma.

Remark 6.1. We recall a basic discussion from [5, Lemma 4.3.2, Lemma 4.3.3]. Consider any compactification $N \in \text{Comp}_1(S)$. Then, the associated sequence

$$0 \to \mathbb{Z}_{tr}N(00) \to \mathbb{Z}_{tr}N(10) \oplus \mathbb{Z}_{tr}N(01) \to \mathbb{Z}_{tr}N(11) \to 0$$

is automatically exact at $\mathbb{Z}_{tr}N(00)$ and $\mathbb{Z}_{tr}N(11)$ in MNST. Indeed, the injectivity of $\mathbb{Z}_{tr}N(00) \to \mathbb{Z}_{tr}N(10)$ is trivial, and the surjectivity of $\mathbb{Z}_{tr}N(10) \oplus \mathbb{Z}_{tr}N(01) \to \mathbb{Z}_{tr}N(11)$ follows from the surjectivity of $\mathbb{Z}_{tr}S(10) \oplus \mathbb{Z}_{tr}S(01) \to \mathbb{Z}_{tr}S(11) = \mathbb{Z}_{tr}N(11)$, where the equality $S(11) = N(11)$ is a consequence of the properness of $S(11)$.

Let $N_0 \in \text{Comp}_1(S)$ be a compactification. We must construct an MV-compactification $N_{11} \in \text{Comp}_1(S)$ which admits a morphism $N_{11} \to N_0$ in $\text{MCor}^{\text{Sq}}$. By Lemma 4.2, we may assume that $N_{0} \in \text{Comp}_1^{\text{min}}(S)$, where we take the notation of (4.1) (see Definition 4.1). Before beginning the proof, we prepare an elementary lemma.

Lemma 6.2. Let $X, Y$ and $Z$ be smooth schemes over $k$, $\alpha \in \text{Cor}(X, Y)$ an elementary finite correspondence, and $f : Y \to Z$ be a morphism of $k$-schemes. Denote by $\Gamma_f \in \text{Cor}(Y, Z)$ the graph of $f$, which is regarded as an elementary finite correspondence. Set $\beta := (\text{id}_X \times f)(\alpha) \subset X \times Z$. Then, $\beta$ coincides with the support of the divisor $\Gamma_f \circ \alpha \in \text{Cor}(X, Z)$, where $\circ$ denotes the composition in the category of finite correspondences $\text{Cor}$.

Proof. The composition $\Gamma_f \circ \alpha$ is defined as the pushforward of the cycle $(\alpha \times Z) \cdot (X \times \Gamma_f) \subset X \times Y \times Z$ by the projection $X \times Y \times Z \to X \times Z$, where $\cdot$ denotes the intersection product (see for example [8, §1]). Therefore, the support $|\Gamma_f \circ \alpha|$ is equal to the image of the set-theoretic map $(\alpha \times Z) \cap (X \times \Gamma_f) \to X \times Y \times Z \to X \times Z$, which is nothing but $\beta$. This finishes the proof. □
6.2. **Construction of a compactification.** In this subsection, we construct an object $\overline{N}_1 \in \text{Comp}_1(S)$ dominating $\overline{N}_0$; the main result will be that it is an MV-compactification as in the statement of Theorem 3.3.

6.2.1. A splitting. For the reader’s convenience, we reproduce the square (4.1) here:

\[
\begin{array}{ccc}
\overline{N}_0(00) & \xrightarrow{h_u} & \overline{N}_0(01) \\
v_l \downarrow & & v_r \downarrow \\
\overline{N}_0(10) & \xrightarrow{h_d} & \overline{N}_0(11).
\end{array}
\]  

(6.1)

As in Section 5, define closed subschemes

\[ Z(10) := \overline{S}(11) \setminus \overline{S}(10) \subset \overline{S}(11) \]

and

\[ Z(00) := \overline{S}(01) \setminus \overline{S}(00) \subset \overline{S}(01), \]

with the reduced scheme structures. Since $\overline{S}$ is an elementary Nisnevich square, we have the following identifications of schemes:

\[ Z(00) = Z(10) \times_{\overline{S}(11)} \overline{S}(01) \cong Z(10). \]  

(6.2)

Since $\overline{S}(11)$ is proper, we have $S(11) = N_0(11)$.

By the same proof as that of Lemma 5.2, we can find a closed open subset $W \subset v_r^{-1}(Z(10))$ such that

\[ v_r^{-1}(Z(10)) = Z(00) \sqcup W. \]  

(6.3)

**Lemma 6.3.** The set $W$ is a closed subset of $\overline{N}_0(01) \setminus \overline{S}(01)$.

**Proof.** Since $W$ is closed in $v_r^{-1}(Z(10))$, it is closed in $\overline{N}_0(01)$. So, it suffices to prove that $W \cap \overline{S}(01) = \emptyset$. Since $\overline{S}$ is an elementary Nisnevich square, we have

\[ v_r^{-1}(Z(10)) \cap \overline{S}(01) = Z(10) \times_{\overline{S}(11)} \overline{S}(01) = Z(00). \]

By (6.3), we obtain

\[ (Z(00) \sqcup W) \cap \overline{S}(01) = Z(00), \]

which implies $W \cap \overline{S}(01) = \emptyset$. This finishes the proof. $\square$
6.2.2. Creation of a Cartier divisor. We regard $W$ as a closed subscheme of $\mathcal{N}_0(01)$ with the reduced scheme structure. We reduce to the case where $W$ is an effective Cartier divisor on $\mathcal{N}_0(01)$, as follows. By Lemma 6.3, the blow-up $\tilde{\mathcal{N}}_0(01) := \text{Bl}_W(\mathcal{N}_0(01)) \to \mathcal{N}_0(01)$ is an isomorphism over $\mathcal{S}(01)$. Define $\tilde{\mathcal{N}}_0(00) := \mathcal{N}_0(00) \times_{\mathcal{N}_0(01)} \tilde{\mathcal{N}}_0(01)$. Then, the projection $\tilde{\mathcal{N}}_0(00) \to \mathcal{N}_0(00)$ is an isomorphism over $\mathcal{S}(00)$. Therefore, by replacing $\mathcal{N}_0(01)$ and $\mathcal{N}_0(00)$ with $\tilde{\mathcal{N}}_0(01)$ and $\tilde{\mathcal{N}}_0(00)$, respectively, and by pulling-back the moduluses, we may, and do, assume that $W$ is an effective Cartier divisor on $\mathcal{N}_0(01)$.

6.2.3. A majoration. Note that $Z(00)$ is also a closed subscheme of $\mathcal{N}_0(01)$ by (6.3). Define closed subschemes $\tilde{Z}(00), \tilde{W}$ of $\mathcal{N}_0(00)$ by
\begin{align}
\tilde{Z}(00) &:= h_u^{-1}(Z(00)) = Z(00) \times_{\mathcal{N}_0(01)} \mathcal{N}_0(00), \\
\tilde{W} &:= h_u^{-1}(W) = W \times_{\mathcal{N}_0(01)} \mathcal{N}_0(00).
\end{align}
Then, we have
\begin{align}
p^{-1}(Z(10)) = h_u^{-1}v_r^{-1}(Z(10)) = h_u^{-1}(Z(00) \cup W) = \tilde{Z}(00) \cup \tilde{W}.
\end{align}

Lemma 6.4. There exists a positive integer $m$ such that
\begin{align}
v_l^*\mathcal{N}_0(10)^\infty|_{\mathcal{N}_0(00) \setminus \tilde{Z}(00)} \leq h_u^*(\mathcal{N}_0(01)^\infty + mW)|_{\mathcal{N}_0(00) \setminus \tilde{Z}(00)}.
\end{align}

Proof. First, we prove

Claim 6.5. $v_l^*\mathcal{N}_0(10)^\infty|_{p^{-1}(\mathcal{S}(10))} \leq h_u^*\mathcal{N}_0(01)^\infty|_{p^{-1}(\mathcal{S}(10))}$.

Proof of Claim. Since the morphism $\mathcal{N}_0(01) \to \mathcal{N}_0(11)$ is admissible, we have $\mathcal{N}_0(01)^\infty \geq v_r^*\mathcal{N}_0(11)^\infty$. Since $p = v_r \circ h_u$, we obtain $h_u^*\mathcal{N}_0(01)^\infty \geq p^*\mathcal{N}_0(11)^\infty$. Therefore, it suffices to prove that
\begin{align}
p^*\mathcal{N}_0(11)^\infty|_{p^{-1}(\mathcal{S}(10))} = v_l^*\mathcal{N}_0(10)^\infty|_{p^{-1}(\mathcal{S}(10))}.
\end{align}
Since $p = h_d \circ v_l$, we are reduced to showing
\begin{align}
h_d^*\mathcal{N}_0(11)^\infty|_{h_d^{-1}(\mathcal{S}(10))} = \mathcal{N}_0(10)^\infty|_{h_d^{-1}(\mathcal{S}(10))}.
\end{align}
By applying Lemma B.2 to the morphism $h_d : \mathcal{N}_0(10) \to \mathcal{N}_0(11)$ and the dense open subset $\mathcal{S}(10) \subset \mathcal{N}_0(10)$, we have $h_d^{-1}(\mathcal{S}(10)) = \mathcal{S}(10)$. Therefore, it suffices to prove
\begin{align}
h_d^*\mathcal{N}_0(11)^\infty|_{\mathcal{S}(10)} = \mathcal{N}_0(10)^\infty|_{\mathcal{S}(10)}.
\end{align}
Since $N_0(11) = S(11)$, both sides of the above equality coincides with $S(10)\infty$ by the minimality of the morphism $S(10) \to S(11)$. This finishes the proof of Claim 6.5. □

By (6.6), we have
\[ p^{-1}(\overline{S}(10)) = p^{-1}(\overline{S}(11) \setminus Z(10)) = \overline{N}_0(00) \setminus p^{-1}(Z(10)) = \overline{N}_0(00) \setminus (\bar{Z}(00) \sqcup \bar{W}). \]

Therefore, Claim 6.5 says that
\[ (h^*_uN_0(01)\infty - v^*_lN_0(10)\infty)|_{\overline{N}_0(00) \setminus (\bar{Z}(00) \sqcup \bar{W})} \geq 0. \]
Then, Lemma B.1 implies that there exists a positive integer $m$ such that
\[ (h^*_uN_0(01)\infty + m\bar{W} - v^*_lN_0(10)\infty)|_{\overline{N}_0(00) \setminus \bar{Z}(00)} \geq 0. \]
This finishes the proof of Lemma 6.4. □

6.2.4. Construction of $\overline{N}_1$. In the following, we fix $m \geq 1$ as in Lemma 6.4.

Define an effective Cartier divisor $N_1(01)\infty$ on $\overline{N}_0(01)$ by
\[ (6.7) \quad N_1(01)\infty := N_0(01)\infty + mW \]
(see (6.3) for the definition of $W$, and Lemma 6.4 for $m$). Define a modulus pair $N_1(01)$ by
\[ (6.8) \quad N_1(01) := (\overline{N}_0(01), N_1(01)\infty). \]

Lemma 6.6. We have the following equality of closed subschemes of $\overline{S}(01)$:
\[ v^*_lN_0(10)\infty \cap \overline{S}(00) = S(00)\infty = h^*_uN_1(01)\infty \cap \overline{S}(00). \]
In particular, we have
\[ (v^*_lN_0(10)\infty \times_{\overline{N}_0(00)} h^*_uN_1(01)\infty) \cap \overline{S}(00) = S(00)\infty. \]

Proof. The equality $v^*_lN_0(10)\infty \cap \overline{S}(00) = S(00)\infty$ follows from the minimality of the morphism $S(00) \to S(10) \to N_0(10)$. By Lemma 6.3, the morphism $S(01) \to N_1(01)$ remains minimal. Therefore, the equality $h^*_uN_1(01)\infty \cap \overline{S}(00) = S(00)\infty$ follows from the minimality of the composite $S(00) \to S(01) \to N_1(01)$. This finishes the proof. □

By Lemma 6.6, the blow-up
\[ \text{Bl}_{v^*_lN_0(10)\infty \times_{\overline{N}_0(00)} h^*_uN_1(01)\infty} \overline{N}_0(00) \to \overline{N}_0(00) \]
is an isomorphism over $\overline{S}(01)$. Note that $\overline{S}(00)$ is normal by Remark 2.14. Therefore, the normalized blow-up of $\overline{N}_0(00)$ along the closed
subscheme $v_i^* N_0(10)^{\infty} \times_{\mathcal{N}_1(00)} h_u^* N_1(01)^{\infty}$ is an isomorphism over $\mathcal{S}(00)$. So, replacing $\mathcal{N}_0(00)$ by the blow-up and by pulling back the moduluses, we may assume that

$$v_i^* N_0(10)^{\infty} \times_{\mathcal{N}_1(00)} h_u^* N_1(01)^{\infty}$$

is an effective Cartier divisor on $\mathcal{N}_0(00)$.

Define modulus pairs $N_1(11), N_1(10)$ and $N_1(00)$ by

$$N_1(ij) = N_0(ij) \text{ for } (i, j) \in \{(1, 1), (1, 0)\},$$

$$N_1(00) = (\mathcal{N}_0(00), \sup\{v_i^* N_0(10)^{\infty}, h_u^* N_1(01)^{\infty}\})$$

(the modulus pair $N_1(01)$ is already defined in (6.7) and (6.8)). These modulus pairs obviously form a square $\mathcal{N}_1 \in \mathbf{MCor}^{\text{sq}}$, and the identity maps on the total spaces induce an admissible morphism

$$\mathcal{N}_1 \to \mathcal{N}_0.$$ 

Indeed, the existence of this map follows from the definition of $\mathcal{N}_1$ and the minimality of the square $\mathcal{N}_0$ (see Definition 4.1 for the definition of the minimality of squares).

Moreover, for each $(i, j)$, the minimal morphism $S(ij) \to N_0(ij)$ lifts to an admissible morphism $S(ij) \to N_1(ij)$, which is automatically minimal. Indeed, this is trivial by definition for $(i, j) = (1, 1), (1, 0)$. The minimality for $(i, j) = (0, 1)$ follows from Lemma 6.3. Finally, the minimality for $(i, j) = (0, 0)$ follows from

$$N_1(00)^{\infty}|_{\mathcal{S}(00)} = \sup\{v_i^* N_0(10)^{\infty}, h_u^* N_1(01)^{\infty}\}|_{\mathcal{S}(00)}$$

$$= \sup\{v_i^* N_0(10)^{\infty}|_{\mathcal{S}(00)}, h_u^* N_1(01)^{\infty}|_{\mathcal{S}(00)}\}$$

$$= S(00)^{\infty},$$

where the last equality follows from Lemma 6.6. Therefore, we conclude that

$$\mathcal{N}_1 \in \text{Comp}_1(\mathcal{S}).$$

**Proposition 6.7.** The square $\mathcal{N}_1$ has the following properties:

1. It is universally minimal.
2. We have

$$v_i^* N_1(10)^{\infty}|_{\mathcal{N}_1(00) \setminus \tilde{Z}(00)} \leq h_u^* N_1(01)^{\infty}|_{\mathcal{N}_1(00) \setminus \tilde{Z}(00)}.$$

(See (6.4) for the definition of $\tilde{Z}(00)$.)

**Proof.** (1) is obtained by combining (6.9), the definition of $N_1(00)$ and Lemma A.3. (2) is immediate by Lemma 6.4 and (6.8). \qed

To prove Theorem 3.3, it suffices to show the following Theorem.
Theorem 6.8. Any square \( N_1 \in \text{Comp}_1(S) \) having the properties of Proposition 6.7 is an MV-square.

The rest of this section will be devoted to the proof of Theorem 6.8.

6.3. Key Proposition. Take any modulus pair \( M = (\overline{M}, M^\infty) \). The key step is to prove the following proposition. The proof of Theorem 6.8 will be finished in §6.4.

Proposition 6.9 (Resurgence principle). Let \( \alpha_1 \) and \( \alpha_2 \) be two distinct elementary finite correspondences in \( \text{MCor}(M, N_1(01)) \). Denote by \( \rho \) the morphism \( 1_{M^o} \times v_0^o : M^o \times N_1(01)^o \to M^o \times N_1(11)^o \). Assume that the equality of sets
\[
\rho(\alpha_1) = \rho(\alpha_2) =: \beta
\]
holds. Note that by Lemma 6.2, \( \beta \) is an elementary finite correspondence in \( \text{Cor}(M^o, N_1(11)^o) \), and coincides with the support of the image of \( \alpha_i \) in \( \text{Cor}(M^o, N_1(11)^o) \) for \( i = 1, 2 \).

Then, we have
\[
\beta \in \text{Cor}(M^o, N_1(10)^o) \subset \text{Cor}(M^o, N_1(11)^o),
\alpha_1, \alpha_2 \in \text{Cor}(M^o, N_1(00)^o) \subset \text{Cor}(M^o, N_1(01)^o),
\]
and moreover
\[
\beta \in \text{MCor}(M, N_1(10)),
\alpha_1, \alpha_2 \in \text{MCor}(M, N_1(00)).
\]

The proof of Proposition 6.9 will be divided into several steps.

6.3.1. The separation lemma. Let \( \overline{\beta} \) be the closure of \( \beta \) in \( \overline{M} \times \overline{N}_1(11) \), and \( \overline{\beta}^N \) its normalization. Similarly, let \( \overline{\alpha}_i \) be the closure of \( \alpha_i \) in \( \overline{M} \times \overline{N}_1(01) \), and \( \overline{\alpha}_i^N \) its normalization. The proper morphism \( \overline{M} \times \overline{N}_1(01) \to \overline{M} \times \overline{N}_1(11) \) induces proper surjective morphisms \( \overline{\alpha}_i \to \overline{\beta} \), and the universality of normalization induces proper surjective morphisms \( \overline{\alpha}_i^N \to \overline{\beta}^N \).

Consider the morphisms
\[
\overline{\alpha}_i \times_{\overline{N}_1(01)} \overline{S}(01) \to \overline{\beta} \times_{\overline{N}_1(11)} \overline{S}(01) \quad (i = 1, 2).
\]

Note that \( \iota_i \) is closed immersion, because both morphisms \( \overline{\alpha}_i \times_{\overline{N}_1(01)} \overline{S}(01) \to \overline{M} \times \overline{S}(01) \) and \( \overline{\beta} \times_{\overline{N}_1(11)} \overline{S}(01) \to \overline{M} \times \overline{S}(01) \) are closed immersions. We have more precisely:

Lemma 6.10. For each \( i = 1, 2 \), \( \iota_i \) is the closed immersion of an irreducible component. These two irreducible components are distinct.
Proof. We note that \( \alpha_i \times N_1(01) \cong \alpha_i \times N_1(01) \) contains \( \alpha_i \), which is contained in \( M^o \times N_1(01) \), and \( \alpha_i \) is obviously dense in \( \alpha_i \times N_1(01) \) \( S(01) \) which is therefore irreducible. This also shows that \( \alpha_i \times N_1(01) \) \( N_1(01) \) \( S(01) \) is therefore irreducible. This also shows that \( \alpha_i \times N_1(01) \) \( S(01) \) \( (i = 1, 2) \) are distinct. To see that they are irreducible components, we use the following equalities:

\[
\dim \alpha_i \times N_1(01) = 1 \quad \dim \alpha_i = 2 \quad \dim \beta = 3 \quad \beta \times N_1(11) S(01),
\]

where \( =^1 \) is obvious, and the equalities \( =^2 \) and \( =^3 \) follow from the étaleness of the morphism \( S(01) \rightarrow N_1(11) \).

Lemma 6.11 (Separation Lemma). The composition

\[
(\alpha_1^N \sqcup \alpha_2^N) \times_{N_0(01)} Z(00) \rightarrow \alpha_1^N \sqcup \alpha_2^N \rightarrow \beta^N
\]

is a closed immersion.

Proof. Consider the following commutative diagram (recall that \( S(11) = N_1(11) \)):

\[
\begin{array}{ccc}
(\alpha_i \times N_1(01) S(01))^N & \longrightarrow & \alpha_i \times N_1(01) S(01) \\
\downarrow \iota_i & & \downarrow \iota_i \\
(\beta \times N_1(11) S(01))^N & \longrightarrow & \beta \times N_1(11) S(01) \longrightarrow S(01) \\
\downarrow \iota & & \downarrow \iota \\
\beta^N & \longrightarrow & \beta \longrightarrow N_1(11)
\end{array}
\]

where the south-west square is cartesian by the fact that normalization is compatible with étale base change (see Lemma B.4). For the existence of the north-west square, see [EGA2, Cor. 6.3.8] and Lemma 6.10.

Lemma 6.10 implies that \( (\alpha_1 \times N_1(01) S(01))^N \) and \( (\alpha_2 \times N_1(01) S(01))^N \) are distinct connected components of \( (\beta \times_{N_1(01)} S(01))^N \). In particular, the map

\[
(\alpha_1^N \sqcup \alpha_2^N) \times_{N_1(01)} S(01) = (\alpha_1 \times N_1(01) S(01))^N \sqcup (\alpha_2 \times N_1(01) S(01))^N
\]

\[
\xrightarrow{\sqcup \iota_i} (\beta \times_{N_1(11)} S(01))^N
\]

\[
= \beta^N \times_{N_1(11)} S(01)
\]

is a closed immersion. Taking the base change by the closed immersion \( Z(00) \rightarrow S(01) \), we conclude that the morphism

\[
(\alpha_1^N \sqcup \alpha_2^N) \times_{N_1(01)} Z(00) \rightarrow \beta^N \times_{N_1(11)} Z(00)
\]
is a closed immersion. Since $Z(00) \rightarrow Z(10) \subset \overline{S}(11)$ is a closed immersion, the morphism

$$\beta^N \times_{\overline{S}(11)} Z(00) \rightarrow \overline{\beta}^N$$

is a closed immersion. Therefore, the composite map

$$(\alpha^N_1 \sqcup \alpha^N_2) \times_{N_1(01)} Z(00) \rightarrow \beta^N \times_{\overline{S}(11)} Z(00) \rightarrow \overline{\beta}^N$$

is a closed immersion. This finishes the proof of Lemma 6.11. □

6.3.2. Lifting of the closures.

Lemma 6.12. The generic point $\eta$ of $\beta$ lies in $M^\circ \times N_1(10)^\circ$. Moreover, for each $i = 1, 2$, the generic point $\xi_i$ of $\alpha_i$ lies in $M^\circ \times N_1(00)^\circ$.

Proof. Note that $N_1(ij)^\circ = S(ij)^\circ$ for each $i, j \in \{0, 1\}$. First, we prove that $\eta \not\in M^\circ \times Z(10)$ for this, it suffices to prove that $\eta \not\in M^\circ \times Z(10)$ since $N_1(11)^\circ \setminus N_1(10)^\circ \subset Z(10)$. Suppose that $\eta$ lies in $M^\circ \times Z(10)$. Then, for each $i = 1, 2$, the generic point $\xi_i$ lies in $\alpha^N_i \times_{\overline{S}(11)} Z(10) \cong \alpha^N_i \times_{\overline{S}(01)} Z(00)$, and its image in $\beta^N$ is $\eta$. However, this contradicts Lemma 6.11. Therefore, we conclude that $\eta \not\in M^\circ \times N_1(10)^\circ$. Finally, since $N_1(00)^\circ \cong N_1(10)^\circ \times N_1(11)^\circ N_1(01)^\circ$, and since $\xi_i$ is over $\eta$ for each $i = 1, 2$, we have $\xi_i \in M^\circ \times N_1(00)^\circ$. This finishes the proof. □

Define closed subsets

$$\overline{\beta} := \text{the closure of } \{\eta\} \text{ in } M \times N_1(10),$$

$$\overline{\alpha}_i := \text{the closure of } \{\xi_i\} \text{ in } M \times N_1(00),$$

and regard them as integral closed subschemes with the reduced scheme structures. Then, the proper morphism $M \times N_1(10) \rightarrow M \times N_1(11)$ (resp. $M \times N_1(00) \rightarrow M \times N_1(01)$) induces a proper surjective morphism $\overline{\beta} \rightarrow \overline{\beta}$ (resp. proper surjective morphisms $\overline{\alpha}_i \rightarrow \overline{\alpha}_i$). Then, the universality of normalization induces a commutative diagram

$$\begin{array}{ccc}
\overline{\alpha}^N_i & \rightarrow & \overline{\alpha}^N_i \\
\downarrow & & \downarrow \\
\overline{\beta} & \rightarrow & \overline{\beta}
\end{array}$$

where all the arrows are proper surjective morphisms.
6.3.3. **Modulus condition on \( \tilde{\beta}^N \).** Now, we prove the following crucial result.

**Proposition 6.13.** The following inequality holds:
\[
(6.10) \quad \overline{M} \times N_1(10)^\infty|_{\beta^N} \leq M^\infty \times \overline{N}_1(10)|_{\beta^N}.
\]

In particular, \( \tilde{\beta} \cap M^\circ \times N_1(10)^\infty = \emptyset \).

**Proof.** Note that the last assertion follows readily from (6.10).

By the assumption that \( \alpha_i \in \text{MCor}(M, N_1(01)) \), we have
\[
\overline{M} \times N_1(10)^\infty|_{\pi^N} \leq M^\infty \times \overline{N}_1(01)|_{\pi^N}.
\]

Pulling back by the morphism \( \tilde{\alpha}_i^N \to \overline{\alpha}_i^N \), we obtain
\[
(6.11) \quad \overline{M} \times h^*_u N_1(01)^\infty|_{\overline{\alpha}_i^N} \leq M^\infty \times \overline{N}_1(00)|_{\overline{\alpha}_i^N}
\]
for each \( i = 1, 2 \).

Set
\[
(6.12) \quad \tilde{\alpha}_i^\phi := \tilde{\alpha}_i^N \setminus (\overline{M} \times \tilde{Z}(00))
\]
(see (6.4) for the definition of \( \tilde{Z}(00) \)). Then, we have
\[
(6.13) \quad \overline{M} \times \nu_i^* N_1(10)^\infty|_{\overline{\alpha}_i^\phi} \leq^1 \overline{M} \times h^*_u N_1(01)^\infty|_{\overline{\alpha}_i^N} \leq^2 M^\infty \times \overline{N}_1(00)|_{\overline{\alpha}_i^\phi},
\]
where \( \overline{\alpha}_i^\phi \) denotes the normalization of \( \tilde{\alpha}_i^\phi \), which is an open subscheme of \( \overline{\alpha}_i^N \). Here, the inequality \( \leq^1 \) follows from Proposition 6.7, and the inequality \( \leq^2 \) follows from (6.11).

**Lemma 6.14.** The images of the proper morphisms
\[
\tilde{\alpha}_i^N \times_{\overline{N}_1(00)} \tilde{Z}(00) \to \tilde{\beta}^N \quad \text{for} \quad i = 1, 2
\]
do not intersect with each other. Here, recall that \( \overline{N}_1(00) = \overline{N}_0(00) \) by definition. For the definition of \( \tilde{Z}(00) \), see (6.4).

**Proof.** Consider the commutative diagram
\[
\begin{array}{ccc}
\tilde{\alpha}_i^N \times_{\overline{N}_0(00)} \tilde{Z}(00) & \longrightarrow & \overline{\alpha}_i^N \times_{\overline{N}_0(01)} Z(00) \\
\tilde{\pi}_i \downarrow & & \downarrow \pi_i \\
\tilde{\beta}^N & \longrightarrow & \overline{\beta}^N.
\end{array}
\]

By Lemma 6.11, the images of the maps \( \pi_1 \) and \( \pi_2 \) do not intersect with each other. Therefore, the commutativity of the diagram shows that the images of the maps \( \tilde{\pi}_1 \) and \( \tilde{\pi}_2 \) do not intersect with each other. This finishes the proof of Lemma 6.14. \( \square \)
We recall the following lemma from [2, Lemma 2.2]:

**Lemma 6.15.** Let $f : Y \to X$ be a surjective morphism between normal integral $k$-schemes. Let $D$ be a Cartier divisor on $X$. Then, $D$ is effective if and only if $f^*(D)$ is effective. □

For each $i = 1, 2$, define

$$B_i := \text{the image of the map } \tilde{\alpha}_i^N \times N_1(00) \to \tilde{\beta}^N.$$

Then, by Lemma 6.14, $B_1$ and $B_2$ are disjoint closed subsets of $\tilde{\beta}^N$. Therefore, it suffices to prove the inequality (6.10) on each open subset $\tilde{\beta}^N \setminus B_1$ and $\tilde{\beta}^N \setminus B_2$. For each $i = 1, 2$, define a closed subset $A_i \subset \tilde{\alpha}_i^N$ as the fiber of $B_i$ by the map $\tilde{\alpha}_i^N \to \tilde{\beta}^N$. Since the induced morphisms

$$\tilde{\alpha}_i^N \setminus A_i \to \tilde{\beta} \setminus B_i$$

are surjective morphisms between normal integral $k$-schemes, Lemma 6.15 reduces the proof of the inequality (6.10) to proving the following inequalities for $i = 1, 2$:

$$(6.14) \quad \tilde{\alpha}_i^N \times N_1(10)^\infty|_{\tilde{\alpha}_i^N \setminus A_i} \leq \tilde{M}^\infty \times N_1(00)|_{\tilde{\alpha}_i^N \setminus A_i}.$$

Since by definition we have

$$\tilde{\alpha}_i^N \times N_1(00) \tilde{Z}(00) \subset A_i,$$

we obtain

$$\tilde{\alpha}_i^N \setminus A_i \subset \tilde{\alpha}_i^{oN}$$

where $\tilde{\alpha}_i^{o}$ we defined in (6.12). Therefore, the inequality (6.14) immediately follows from (6.13). This finishes the proof of Proposition 6.13. □

6.3.4. **End of Proof of Proposition 6.9.** We are reduced to proving the following Claim 6.16 and Claim 6.17.

**Claim 6.16.** We have

$$\beta \in \mathbf{MCor}(M, N_1(10)).$$

**Proof.** First, we have to check that $\beta \in \text{Cor}(M^\circ, N_1(10)^\circ)$. Since $\beta$ is proper over $M^\circ$, we have $\beta = \overline{\beta} \times \overline{M} M^\circ$. Since $\overline{\beta} \to \overline{\beta}$ is surjective, so is the induced map $\overline{\beta} \times \overline{M} M^\circ \to \overline{\beta} \times \overline{M} M^\circ$. By Proposition 6.13, we have

$$\overline{\beta} \times \overline{M} M^\circ = \overline{\beta} \cap (M^\circ \times \overline{N}_1(10)) \subset M^\circ \times N_1(10)^\circ,$$

and

$$\overline{\beta} \times \overline{M} M^\circ \to \overline{\beta} \times \overline{M} M^\circ.$$
which implies that
\[ \beta = \overline{\beta} \times M^\circ \subset M^\circ \times N_1(10)^\circ. \]
Therefore, we have \( \beta \in \text{Cor}(M^\circ, N_1(10)^\circ) \). Then, the closure of \( \beta \) in \( \overline{M} \times \overline{N}_1(10) \) is (by definition) equal to \( \overline{\beta} \), and the inequality (6.10) shows that \( \beta \in \text{MCor}(M, N_1(10)) \). This finishes the proof of Claim 6.16. \( \square \)

**Claim 6.17.** We have \( \alpha_1, \alpha_2 \in \text{MCor}(M, N_1(00)). \)

**Proof.** Since we have canonical morphisms \( \overline{\alpha}_i^N \to \overline{\beta}^N \), the inequality (6.10) implies
\[ \overline{M} \times v_i^* N_1(10)^\infty|_{\overline{\alpha}_i^N} \leq M^\infty \times \overline{N}_1(00)|_{\overline{\alpha}_i^N} \]
for each \( i = 1, 2 \). By Proposition 6.7 (1), the square \( N_1 \) is universally minimal. Therefore, by combining (6.11) and (6.15), we obtain
\[ \overline{M} \times N_1(00)^\infty|_{\overline{\alpha}_i^N} \leq M^\infty \times \overline{N}_1(00)|_{\overline{\alpha}_i^N}. \]
In particular, \( \overline{\alpha}_i \times M^\circ = \overline{\alpha}_i \cap M^\circ \times \overline{N}_1(00) \) is contained in \( M^\circ \times N_1(00)^\circ \). Since the natural projection \( \overline{\alpha}_i \to \overline{\alpha}_i \) is surjective, we have
\[ \alpha_i = \overline{\alpha}_i \times M^\circ \subset M^\circ \times N_1(00)^\circ, \]
where the equality follows from the properness of \( \alpha_i \) over \( M^\circ \). Therefore, we have \( \alpha_i \in \text{Cor}(M^\circ, N_1(00)^\circ) \). The closure of \( \alpha_i \) in \( \overline{M} \times \overline{N}_1(00) \) is (by definition) equal to \( \overline{\alpha}_i \). Therefore, the inequality (6.16) shows that \( \alpha_i \in \text{MCor}(M, N(00)) \) for each \( i = 1, 2 \). This finishes the proof of Claim 6.17. \( \square \)

Thus, we have finished the proof of Proposition 6.9.

**6.4. End of Proof of Theorem 6.8.** Let \( M \in \text{MCor} \) be a modulus pair. By remark 6.1, it suffices to prove that the sequence
\[ \mathbb{Z}_{tr} N_1(00)(M) \to \mathbb{Z}_{tr} N_1(10)(M) \oplus \mathbb{Z}_{tr} N_1(01)(M) \to \mathbb{Z}_{tr} N_1(11)(M) \]
is exact at the middle term. Let \( \alpha \) be an element of \( \text{MCor}(M, N_1(01)) \), and write
\[ \alpha = \sum_{i \in I} m_i \alpha_i, \]
where \( I \) is a finite set, \( \alpha_i \) are prime cycles and \( m_i \) are non-zero integers. Denote by \( \rho \) the map \( \text{MCor}(M, N_1(01)) \to \text{MCor}(M, N_1(11)) \). Define
\[ I_1 := \{ i \in I | \exists j \in I \setminus \{ i \}, |\rho(\alpha_i)| = |\rho(\alpha_j)| \}, \]
\[ I_2 := I \setminus I_1. \]
Then, Proposition 6.9 implies that
\[ \alpha_i \in \text{MCor}(M, N_1(00)) \text{ for each } i \in I_1. \]

Assume that
\[ \rho(\alpha) \in \text{MCor}(M, N_1(10)) \subset \text{MCor}(M, N_1(11)). \]

By (6.17), we are reduced to showing the following claim.

Claim 6.18. For any \( i \in I_2 \), we have
\[ \alpha_i \in \text{MCor}(M, N_1(00)). \]

Proof of Claim. Take any \( i \in I_2 \). By definition of \( I_2 \), the coefficient of the integral cycle \( |h(\alpha_i)| \) in the cycle \( \rho(\alpha) \) is non-zero. Therefore, we have
\[ \beta := |\rho(\alpha_i)| \in \text{MCor}(M, N_1(10)), \]
where \( |\rho(\alpha_i)| \) denotes the irreducible support of the divisor. In particular, we have
\[ \beta \in \text{Cor}(M^\circ, N_1(10)^\circ), \]
\[ \alpha_i \in \text{Cor}(M^\circ, N_1(00)^\circ), \]
where the second claim follows from \( N_1(00)^\circ = N_1(10)^\circ \times N_1(11)^\circ \times N_1(01)^\circ \).

Let \( \overline{\alpha}_i \) be the closure in \( \overline{M} \times \overline{N}_1(00) \) and \( \overline{\alpha}_i^N \) its normalization. Similarly, let \( \overline{\beta} \) be the closure of \( \beta \) in \( \overline{M} \times \overline{N}_1(10) \) and \( \overline{\beta}^N \) its normalization. Then, (6.18) implies that
\[ \overline{M} \times N_1(10)^\circ \big|_{\overline{\beta}^N} \leq M^\circ \times \overline{N}_1(10)^\circ \big|_{\beta^N}. \]

Since the maps \( \overline{\alpha}_i \rightarrow \overline{\beta} \) induced by \( \text{id}_{\overline{M}} \times v_i : \overline{M} \times \overline{N}_1(00) \rightarrow \overline{M} \times \overline{N}_1(10) \) are dominant, the universality of normalization induces a morphism \( \overline{\alpha}_i^N \rightarrow \overline{\beta}^N \). And (6.19) implies
\[ \overline{M} \times v_i^* N_1(10)^\circ \big|_{\overline{\alpha}_i^N} \leq M^\circ \times \overline{N}_1(00)^\circ \big|_{\overline{\alpha}_i^N}. \]

On the other hand, the assumption that \( \alpha_i \in \text{MCor}(M, N_1(01)) \) implies that
\[ \overline{M} \times N_1(01)^\circ \big|_{\overline{\alpha}_i^N} \leq M^\circ \times \overline{N}_1(01)^\circ \big|_{\overline{\alpha}_i^N}, \]
where \( \overline{\alpha}_i \) denotes the closure of \( \alpha_i \) in \( \overline{M} \times \overline{N}_1(01) \), and \( \overline{\alpha}_i^N \) its normalization. Since \( \text{id}_{\overline{M}} \times h_u : \overline{M} \times \overline{N}_1(00) \rightarrow \overline{M} \times \overline{N}_1(01) \) and the universality of normalization induce a canonical morphism \( \overline{\alpha}_i^N \rightarrow \overline{\alpha}_i^N \), we obtain
\[ \overline{M} \times h_u^* N_1(01)^\circ \big|_{\overline{\alpha}_i^N} \leq M^\circ \times \overline{N}_1(00)^\circ \big|_{\overline{\alpha}_i^N}. \]
Since $N_1(00) = \sup\{v^* N_1(10), h^* N_1(01)\}$ by definition, the inequalities (6.20) and (6.21) imply
\[ \overline{M} \times N_1(00) \leq M' \times \overline{N}_1(00). \]
Therefore, we have $\alpha_i \in \text{MCor}(M, N_1(00))$. This finishes the proof of Claim 6.18.

Thus, we finished the proof of Theorem 6.8. Therefore, we have proven Theorem 3.3.

APPENDIX A. SOME REMARKS ON THE SUP OF CARTIER DIVISORS

A.1. Preliminary.

Lemma A.1. Let $X$ be a scheme. Suppose given three effective Cartier divisors $D_1, D_2$ and $E$ on $X$ such that $E \leq D_i$ for each $i = 1, 2$.

Then, we have:
\[ E = D_1 \times_X D_2 \iff |D_1 - E| \cap |D_2 - E| = \emptyset. \]

Remark A.2. The “inf” of two effective Cartier divisors might be zero even if $|D_1| \cap |D_2| \neq \emptyset$: for example, consider the case $X = \mathbb{A}^2 = \text{Spec}(k[x_1, x_2])$ and $D_i = \{x_i = 0\}$.

Proof of Lemma A.1. Regard the effective Cartier divisors $D_i - E$ as closed subschemes on $X$, and set
\[ Z := (D_1 - E) \times_X (D_2 - E). \]
For a closed subscheme $i : V \to X$, we set
\[ I_V := \text{Ker}(\mathcal{O}_X \to i_* \mathcal{O}_V). \]

Then, we have
\[ I_{D_1 \times_X D_2} = I_{D_1} + I_{D_2}. \]
Since $I_Z = I_{D_1 - E} + I_{D_2 - E} = I_{D_1} \cdot I_{E}^{-1} + I_{D_2} \cdot I_{E}^{-1}$, we have
\[ I_Z \cdot I_E = (I_{D_1} \cdot I_{E}^{-1} + I_{D_2} \cdot I_{E}^{-1}) \cdot I_E = I_{D_1} + I_{D_2}, \]
where $I_{E}^{-1}$ denotes the inverse of the invertible ideal sheaf $I_E$. Combining the above equalities, we obtain
\[ (A.1) \quad I_Z \cdot I_E = I_{D_1 \times_X D_2}. \]

Therefore, we have
\[ |D_1 - E| \cap |D_2 - E| = \emptyset \iff Z = \emptyset \iff I_Z = \mathcal{O}_X \iff I_{D_1 \times_X D_2} = I_E \]
\[ \iff D_1 \times_X D_2 = E, \]
where $\iff$ follows from (A.1) and the fact that $I_E$ is invertible. This finishes the proof of Lemma A.1. □
A.2. **Compatibility between pullback and sup, after normalized blow-up.** For two Weil divisors $Z_1$ and $Z_2$ on a scheme $X$, denote by

$$\text{sup}^{\text{Wei}}(Z_1, Z_2)$$

the smallest Weil divisor on $X$ which is larger than or equal to $Z_1$ and $Z_2$. For two Cartier divisors $D_1$ and $D_2$ on a scheme $X$, denote by

$$\text{sup}^{\text{Car}}(D_1, D_2)$$

the smallest Cartier divisor on $X$ which is larger than or equal to $D_1$ and $D_2$ (if it exists).

**Lemma A.3.** Let $X$ be a normal scheme, and let $D_1, D_2$ and $E$ be effective Cartier divisors on $X$ such that $E \leq D_i$ for each $i = 1, 2$. Assume one of the following equivalent conditions:

(a) $E = D_1 \times_X D_2$.

(b) $|D_1 - E| \cap |D_2 - E| = \emptyset$.

(The equivalence of these conditions follow from Lemma A.1.)

Then, the following assertions hold:

1. Regard $D_1$ and $D_2$ as Weil divisors on $X$ (since $X$ is normal, any Cartier divisor can be naturally regarded as a Weil divisor). Then, we have

$$\text{sup}^{\text{Wei}}(D_1, D_2) = D_1 + D_2 - E.$$  

2. $\text{sup}^{\text{Wei}}(D_1, D_2)$ is an effective Cartier divisor, and is equal to the sup as Cartier divisor:

$$\text{sup}^{\text{Wei}}(D_1, D_2) = \text{sup}^{\text{Car}}(D_1, D_2) = \text{sup}(D_1, D_2).$$

3. The square

$$(X, \text{sup}(D_1, D_2)) \quad (X, D_1)$$

$$\downarrow \quad \downarrow$$

$$(X, D_2) \quad (X, E)$$

is universally minimal in the sense of [5, Def. 4.3.9]. In other words: let $f : Y \to X$ be a morphism such that $Y$ is normal, and such that the pullback of Cartier divisors $D_1, D_2$ and $E$ are well-defined. Then, we have

$$f^* \text{sup}(D_1, D_2) = \text{sup}(f^*D_1, f^*D_2).$$

(See the assertion (2) for the definition of sup.)
Proof. We may calculate $\sup^{\text{Wei}} (D_1, D_2)$ as follows:

$$\sup^{\text{Wei}} (D_1, D_2) = ^1 \sup^{\text{Wei}} (D_1 - E, D_2 - E) + E = ^2 (D_1 - E) + (D_2 - E) + E = D_1 + D_2 - E,$$

where $^1$ is obvious, and $^2$ follows from the assumption. This proves the assertion (1).

Since the right hand side of $\sup^{\text{Wei}} (D_1, D_2) = D_1 + D_2 - E$ is Cartier, the assertion (2) is obvious.

Finally, we prove the assertion (3). Let $f : Y \to X$ be a morphism as in the statement. Note that the scheme

$$(A.2) \quad f^* D_1 \times_Y f^* D_2 = (D_1 \times_X Y) \times_Y (D_2 \times_X Y) = D_1 \times_X D_2 \times_Y Y = f^* (D_1 \times_X D_2).$$

is an effective Cartier divisor on $Y$. Then, we have

$$\sup^{\text{Wei}} (f^* D_1, f^* D_2) = ^1 f^* D_1 + f^* D_2 - (f^* D_1 \times_Y f^* D_2) = ^2 f^* D_1 + f^* D_2 - f^* (D_1 \times_X D_2) = ^3 f^* (D_1 + D_2 - D_1 \times_X D_2) = ^4 f^* \sup^{\text{Wei}} (D_1, D_2),$$

where $^1$ and $^4$ follow from the assertion (1), $^2$ follows from (A.2), and $^3$ follows from the compatibility between the addition and the pullback of Cartier divisors. This finishes the proof of Lemma A.3. □

**Proposition A.4.** Let $X \in \text{Sch}$, and let $D_1, D_2$ be effective Cartier divisors on $X$. Assume that $X \setminus |D_i|$ are smooth. Set $E := D_1 \times_X D_2$. Let $\pi : X' := (\text{Bl}_E X)^N \to \text{Bl}_E X \to X$ be the normalized blow-up along $E$. Set

$$D'_1 = D_i \times_X X', \quad E' := E \times_X X'.$$

Then, the pair $(X', E')$ forms a modulus pair, $S := \sup^{\text{Cat}} (D'_1, D'_2)$ exists, we have

$$(X', D'_i) \sim (X, D_i), \quad (X', S)^o = X \setminus (|D_1| \cup |D_2|), \quad (X', E')^o = X \setminus (|D_1| \cap |D_2|).$$

and the square

$$\begin{array}{ccc}
(X', S) & \longrightarrow & (X', D'_1) \\
\downarrow & & \downarrow \\
(X', D'_2) & \longrightarrow & (X', E')
\end{array}$$

(A.3)
is universally minimal and extends the elementary Zariski square
\[
(X' \setminus (|\tilde{D}'_1| \cup |\tilde{D}'_2|), E'|_{X' \setminus (|\tilde{D}'_1| \cup |\tilde{D}'_2|)}) \rightarrow (X' \setminus |\tilde{D}'_1|, E'|_{X' \setminus |\tilde{D}'_1|})
\]
\[
|\tilde{D}'_2|, E'|_{X' \setminus |\tilde{D}'_2|}) \rightarrow (X', E'),
\]
where we set $\tilde{D}'_i := D'_i \setminus E'$.

**Proof.** Note that $X \setminus E = (X \setminus |D_1|) \cup (X \setminus |D_2|)$ is smooth. Therefore, $(X, E)$ is a modulus pair. The existence of $\sup \text{Car}(D'_1, D'_2)$ follows from Lemma A.3 (2). The next claims are easy. By construction, we have
\[
E' = D'_1 \times_{X'} D'_2,
\]
and $E'$ is an effective Cartier divisor. Therefore the statement about (A.3) follows from Lemma A.3 (3). Note that the second square is indeed elementary Zariski, since $|\tilde{D}'_1| \cap |\tilde{D}'_2| = \emptyset$. The normality of $X \setminus E$ implies that the birational morphism $\pi$ is an isomorphism over $X \setminus E$. Therefore we obtain the second assertion. This finishes the proof. \qed

**Appendix B. Elementary Lemmas**

**B.1. Modulus increasing lemma.** We recall the following lemma from [9, Lemma 3.16].

**Lemma B.1.** Let $X$ be a quasi-compact scheme and let $D, E$ be Cartier divisors on $X$ with $E \geq 0$. Assume that the restriction of $D$ to the open subset $X \setminus E \subset X$ is effective. Then, there exists a natural number $n_0 \geq 1$ such that $D + n \cdot E$ is effective for any $n \geq n_0$. \qed

**B.2. No extra fiber lemma.**

**Lemma B.2 (No extra fiber lemma).** Let
\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow{p} & & \downarrow \\
U & \xrightarrow{j} & X
\end{array}
\]
be a commutative triangle of schemes. We assume:

- $p$ is separated;
- $f$ is scheme-theoretically dominant [EGA1, Ch. 1, Def. 5.4.2].

Let $V = U \times_X Y$. Then $p' : V \rightarrow U$ is an isomorphism. If $j$ is an open immersion, so is $f$. 
Remark B.3. The morphism \( f : U \to Y \) is scheme-theoretically dominant if \( Y \) is reduced and \( f \) is dominant \([\text{EGA}1, \text{Ch. I, Prop. 5.4.3}]\).

Proof. The diagram yields a section \( s : U \to V \) of \( p' \), and we need to show that \( sp' \) is the identity of \( V \). Since \( f \) is scheme-theoretically dominant, so is \( s \); since \( p \) is separated, so is \( p' \). Then it suffices to show that \( sp' \) and \( 1_V \) agree after composition with \( s \), which is obvious. The last assertion follows from the fact that \( V \to Y \) is an open immersion if \( j \) is. \( \square \)

B.3. Normalization and smooth base change.

Lemma B.4. Let \( f : Y \to X \) be a smooth morphism, where \( X, Y \in \text{Sch}(k) \), and let \( X^N \to X \) be the normalization of \( X \). Then, \( f^N : Y \times_X X^N \to Y \) is the normalization of \( Y \).

Proof. Since \( Y \times_X X^N \to X^N \) is smooth, \( Y \times_X X^N \) is normal; since \( f^N \) is dominant, it factors through the normalization \( Y^N \to Y \). But \( f^N \) is finite since \( f \) is, hence the morphism \( Y \times_X X^N \to Y^N \) is an isomorphism. \( \square \)

B.4. Proof of Lemma 5.7. First, consider the case that \( f \) is finite. Then, the proof is easy. Indeed, since \( f \) is finite flat of finite presentation, it is finite locally free. Since \( f \) is an isomorphism over a dense open subset, the rank of \( f \) is equal to 1, which implies that \( f \) is an isomorphism.

Next, consider the case that \( f \) is quasi-finite, flat, separated of finite presentation. Since \( f \) is flat, the image \( f(X) \subset S \) is open. Therefore, we may assume that \( f \) is surjective. It suffices to prove that \( f \) is finite.

We need the following propositions from \([7, \text{§2.3 Prop. 8 (a), §2.5 Prop. 2}]\):

Proposition B.5 (étale localization of quasi-finite morphisms). Let \( f : X \to Y \) be locally of finite type. Let \( x \) be a point of \( X \), and set \( y := f(x) \).

If \( f \) is quasi-finite at \( x \), then there exists an étale neighborhood \( Y' \to Y \) of \( y \) such that the morphism \( f' : X' \to Y' \), obtained from \( f \) by the base change \( Y' \to Y \) induces a finite morphism \( f'|_{U'} : U' \to Y' \), where \( U' \) is an open neighborhood of the fiber of \( X' \to X \) above \( x \). In addition, if \( f \) is separated, \( U' \) is a connected component of \( X' \). \( \square \)

Proposition B.6 (compatibility between schematic images and flat base changes). Let \( f : X \to Y \) be an \( S \)-morphism which is quasi-compact and quasi-separated. Let \( g : S' \to S \) be a flat morphism, and...
denote by \( f' : X' \to Y' \) be the \( S' \)-morphism obtained from \( f \) by base change. Let \( Z \) (resp. \( Z' \)) be the schematic image of \( f \) (resp. \( f' \)). Then, \( Z \times_S S' \) is canonically isomorphic to \( Z' \).

Since the finiteness of \( f \) is Zariski local on \( S \), it suffices to check over an open neighborhood of a fixed point \( s \in S \). Take a point \( x \in X \) above \( s \). Take an étale neighborhood \( g : S' \to S \) of \( s \) as in Proposition B.5, and set \( X' := X \times_S S' \). Denote by \( f' \) the induced morphism \( X' \to S' \). Since \( f \) is separated, quasi-finite and locally of finite type, there exists a connected component \( V' \subset X' \) such that
\[
(f')_V : V' \to X' \to S'
\]
is finite, and \( V' \) is an open neighborhood of the fiber of \( x \). Since \( f \) is flat, so is \( f' \), hence the image \( f'(V') \subset S' \) is an open subset. By shrinking \( S' \), we may assume that \( V' \to S' \) is surjective. Since \( f \) is an isomorphism over \( U \subset S \), \( f' \) is an isomorphism over \( g^{-1}(U) \subset S' \).

Therefore, combining with the surjectivity of \( f'_{V'} \), we have
\[
(B.1) \quad (f')^{-1}(g^{-1}(U)) \subset V'.
\]
On the other hand, since the map \( g \circ f' \) is a flat morphism, Proposition B.6 implies that the open subset
\[
(B.2) \quad (f')^{-1}(g^{-1}(U)) \subset X' = V' \cup X'_1,
\]
is schematically dense, where \( X'_1 \) is an open and closed subset of \( X' \).

Therefore, \((B.1)\) shows that \( X'_1 = \emptyset \) and \( X' = V' \). Therefore, we have \( f' = f'_{V'} \), hence \( f' \) is finite.

By replacing \( S \) by the image of \( S' \to S \), we may assume that \( S' \to S \) is an fpqc-covering. Since finiteness is an fpqc-local property, we conclude that \( f \) is finite. This finishes the proof of Lemma 5.7.

### Appendix C. Complements on pro-adjoints

We keep the notation of [5, A.2].

**C.1. Canonical representation.** Let \( u : \mathcal{C} \to \mathcal{D} \) have a pro-adjoint \( v \): it is unique up to unique isomorphism of functors. For \( d \in \mathcal{D} \), we may write \( v(d) = \lim_{i \in I} c_i \) for some suitable inverse system \( I \to \mathcal{C} \).

By [SGA4-I, Exp. I, Prop. 8.1.6], we may choose \( I \) to be a cofiltering ordered set. More specifically, let us pick once and for all, for each \( d \in \mathcal{D} \), a cofiltering ordered set \( I(d) \) and a functor \( c : I(d) \to \mathcal{C} \) such that \( \lim_{i \in I(d)} c_i \in \text{pro-}\mathcal{C} \) corepresents the functor \( c' \mapsto \mathcal{D}(d, u(c')) \).

Then the pairs \((I(d), c)\) assemble to a functor isomorphic to \( v \). The unit of the adjunction yields a system of compatible morphisms
\[
(d \xrightarrow{f_i} u(c_i))_{i \in I(d)}
\]
which defines a subcategory $I'(d)$ of $d \downarrow u$, with an obvious functor $\varphi : I(d) \to I'(d)$, and we have the following tautology:

**Lemma C.1.** $\varphi$ is an isomorphism of categories. \hfill \Box

Thus we may identify $I'(d)$ with $I(d)$, thus view $I(d)$ as a subcategory of $d \downarrow u$, the functor $\varphi$ being given by the second projection.

**C.2. Categories of diagrams.** Let $\mathbf{Cat}$ be the 2-category of categories, and let $\Delta$ be a small category. We have a 2-functor

$$\Delta_* : \mathbf{Cat} \to \mathbf{Cat}$$

$$\mathcal{C} \mapsto C^\Delta := \text{Funct}(\Delta, \mathcal{C}).$$

In particular, if $v : \mathcal{D} \rightleftarrows \mathcal{C} : u$ is an adjunction, then $v^\Delta : \mathcal{D}^\Delta \rightleftarrows \mathcal{C}^\Delta : u^\Delta$ is also an adjunction, because the adjunction identities for $(v, u)$ yield adjunction identities for $(v^\Delta, u^\Delta)$.

This also applies to pro-adjoints as follows: let $u : \mathcal{C} \to \mathcal{D}$ have a pro-left adjoint $v : \mathcal{D} \to \text{pro-}\mathcal{C}$. Equivalently [5, Prop. A.2.1], pro-$u$ has a left adjoint $\tilde{v}$. Hence $(\text{pro-}u)^\Delta : (\text{pro-}\mathcal{C})^\Delta \to (\text{pro-}\mathcal{D})^\Delta$ has the left adjoint $\tilde{v}^\Delta$, which by composition yields a functor

$$(C.1) \quad v^\Delta : \mathcal{D}^\Delta \to (\text{pro-}\mathcal{C})^\Delta$$

which verifies the same identity as in [5, Prop. A.2.1 (ii)], and is actually the image of $v$ under the 2-functor $\Delta_*$. In particular, $v^\Delta$ is computed by simply applying $v$ to the relevant diagrams.

**C.3. Diagrams of pro-objects.** Keep the above notation. There is an obvious functor

$$(C.2) \quad P : \text{pro-}(C^\Delta) \to (\text{pro-}\mathcal{C})^\Delta$$

which sends a pro-object $(\varphi_i)_{i \in I}$ in $\mathcal{C}^\Delta$ to the diagram $\delta \mapsto (\varphi_i(\delta))_{i \in I}$.

**Proposition C.2.** a) If $\text{Ob}(\Delta)$ is finite, (C.2) is faithful.

b) If $\Delta$ is finite, (C.2) is full.

c) If moreover $\Delta$ has no loops, (C.2) is an equivalence of categories.

**Proof.** Let $\varphi = (\varphi_i)_{i \in I}, \psi = (\psi_j)_{j \in J}$ be two objects of $\text{pro-}(C^\Delta)$. A morphism $\theta : \varphi \to \psi$ is represented by a collection $(\theta_{i,i(j)})_{i,j \in J}$ where, for each $j$, $\theta_{j,i(j)} \in C^\Delta(\varphi_{i(j)}, \psi_j)$ for a suitable $i(j)$. Then $P(\theta)$ is the collection of diagrams of morphisms corresponding tautologically to these morphisms of diagrams. Consider another $\theta' : \varphi \to \psi$ and take a corresponding collection $(\theta'_{j,i'(j)})_{j \in J}$. Suppose that $P(\theta) = P(\theta')$. 

Then, for all \( j \in J \) and all \( \delta \in Ob(\Delta) \), there exists \( i(j, \delta) \geq \hat{i}(j) \) such that \( \theta_{j,i(j,\delta)}(\delta) = \theta'_{j,i(j,\delta)}(\delta) \), where \( \theta_{j,i(j,\delta)} \) is the composition

\[
\varphi_{i(j,\delta)} \rightarrow \psi_{j,\delta} \rightarrow \psi_j
\]

and similarly for \( \theta'_{j,i(j,\delta)} \). If \( Ob(\Delta) \) is finite, we may choose \( i(j, \delta) \) independent of \( \delta \). This shows that \( \theta = \theta' \), hence a).

Let \( \rho : P(\varphi) \rightarrow P(\psi) \) be a morphism. By definition, \( \rho \) is a morphism between the functors \( P(\varphi), P(\psi) : \Delta \rightarrow \text{pro–}\mathcal{C} \), given by its components \( \rho(\delta) \) subject to commutation with the morphisms of \( \Delta \). Let \( j \in J \); for each \( \delta \), we have a morphism \( \rho_{i(j,\delta),j} : \varphi_{i(j,\delta)}(\delta) \rightarrow \psi_j(\delta) \) for a suitable \( i(j, \delta) \in I \); as above, if \( Ob(\Delta) \) is finite we may choose \( i(j, \delta) \) independent of \( \delta \), i.e. \( i(j, \delta) = i(j) \). If \( \lambda : \delta_1 \rightarrow \delta_2 \) is a morphism, the naturality of \( \rho \) with respect to \( \lambda \) may be expressed in terms of the \( \rho_{i(j,\delta),j} \) at the cost of replacing \( i(j) \) by a larger, suitable \( i'(j, \lambda) \); if \( \Delta \) is finite, we may choose \( i'(j, \lambda) \) independent of \( \lambda \). Then \( \rho \) is of the form \( P(\tilde{\rho}) \), hence b).

Finally, c) (essential surjectivity) follows from [1, Appendix, Prop. 3.3].

C.4. A pro-adjoint with parameters. We continue to suppose that \( \Delta \) is finite and without loops. By Proposition C.2, the functor of (C.1) refines to a functor:

(C.3) \[ w : D^\Delta \rightarrow \text{pro–}(\mathcal{C}^\Delta). \]

**Lemma C.3.** The functor (C.3) is pro-adjoint to \( u^\Delta \). □

We shall now give an explicit and direct construction of this pro-adjoint, making Proposition C.2 possibly unnecessary. We give ourselves a system of subcategories \( (I(d) \subset d \downarrow u)_{d \in D} \) representing \( v \), as after Lemma C.1. Let \( d \in D^\Delta \).

**Lemma C.4.** Define a subcategory \( I(d) \) of \( d \downarrow u^\Delta \) as follows: an object \( X \) (resp. morphism \( f \)) of \( d \downarrow u^\Delta \) is in \( I(d) \) if and only if \( X(\delta) \) (resp. \( f(\delta) \)) is in \( I(d(\delta)) \) for all \( \delta \in \Delta \). Then:

a) The category \( I(d) \) is ordered and cofiltering for all \( d \in D^\Delta \).

b) If \( I(d) \) is full in \( d \downarrow u \) for all \( d \in D \), then \( I(d) \) is full in \( d \downarrow u^\Delta \) for all \( d \in D^\Delta \).

c) The family \( (I(d))_{d \in D^\Delta} \) corepresents (C.3).

**Proof.** a) Ordered is obvious. For cofiltering, induction on \( \#Ob(\Delta) \).

We may assume \( \Delta \) nonempty. The finiteness and “no loop” hypotheses imply that \( \Delta \) has an object \( \delta_0 \) such that no arrow leads to \( \delta_0 \); we call such an object minimal. Let \( \Delta' \) be the subcategory of \( \Delta \) obtained by removing \( \delta_0 \) and all the arrows leaving from \( \delta_0 \). Let \( X_1 : d \rightarrow u^\Delta(c_1) \),
$X_2: \underline{d} \to u^A(c_2)$ be two objects of $I(\underline{d})$. By induction, we may find $Y_3: \underline{d} | \Delta' \to u^{A'}(c_3') \in I(\underline{d} | \Delta')$ sitting above $X_1 | \Delta'$ and $X_2 | \Delta'$. Let $f: \delta_0 \to \delta$ be an arrow, with $\delta \in \Delta'$: by the functoriality of $v$, there exists a commutative diagram in $D$

$$
\begin{array}{ccc}
d(\delta_0) & \xrightarrow{\varphi(f)} & u(c(f)) \\
\downarrow d(f) & & \downarrow u(\psi(f)) \\
d(\delta) & \xrightarrow{Y_3(\delta)} & u(c'_3(\delta))
\end{array}
$$

with $\varphi(f) \in I(d(\delta_0))$. Since $I(d(\delta))$ is cofiltering, we may find an object $d(\delta_0) \xrightarrow{d} u(c) \in I(d(\delta))$ sitting above all $\varphi(f)$’s as well as $X_1(\delta_0)$ and $X_2(\delta_0)$. Then, together with $Y_3$, $X_3(\delta_0) =: g$ completes the construction of $X_3$ dominating $X_1$ and $X_2$.

b) is obvious. c) Let $\psi: \underline{d} \to u^A(c)$ be a morphism in $D^A$, with $c \in C^A$. To construct a morphism $\varphi: (I(\underline{d}), pr_2) \to c$ in pro $\Delta^A$, we proceed as in a). Suppose $\varphi | \Delta'$ has already been constructed. By definition, it means that an object $d(\delta_0) \xrightarrow{d} u(c(\delta_0)) \in I(d(\delta) | \Delta')$ and a compatible morphism $c_1 \to c | \Delta'$ have been given. Exactly as in a), we can complete this to a compatible pair $(d \to u^A(c_2), c_2 \to c)$ yielding $\varphi$. □

**Remark C.5.** Suppose that $C$ is essentially small and has finite limits. Then a functor $u: C \to D$ has a pro-adjoint if and only if it commutes with finite limits [1, App., Cor. 2.6]. If this is the case, then $C^A$ and $u^A$ verify the same hypotheses as soon as $\Delta$ is essentially small, and the existence of the pro-adjoint (C.3) is automatic. However, in the case of Proposition 2.4 below, $C = \text{MCor}$ does not have finite limits, for example no kernels (equalisers of two arrows) in general. Indeed, if it had kernels, so would $\text{Cor}$, since $\omega: \text{MCor} \to \text{Cor}$ has a pro-adjoint. But it is easy to give examples of nonrepresentable kernels in $\text{Cor}$, for example the equaliser of the two morphisms $f, g: \mathbb{A}^2 \to \mathbb{A}^1$ given by $f(x, y) = y^2$ and $g(x, y) = x^3$. So it seems that the passage through Proposition C.2 (and in particular the condition on $\Delta$) is necessary in this case. The hypothesis on finite limits in $C$ is dropped from [SGA4-I, I.8.11.4].

**C.5. Cofinality.** Keep the situation of Lemma C.4 a), and let $\Delta_1$ be a full subcategory of $\Delta$. We have an obvious functor

$$
\varphi: I(\underline{d}) \to I(\underline{d} | \Delta_1).
$$

The following lemma gives an abstract version of [5, Lemma 4.3.1]:
Lemma C.6. Suppose that $\Delta(\delta, \delta_1) = \emptyset$ for all $(\delta, \delta_1)$ such that $\delta \in \Delta - \Delta_1$ and $\delta_1 \in \Delta_1$ (in the terminology of [3, Def. 2.3.2], the inclusion $\Delta' \subseteq \Delta$ is cellular). Then $\varphi$ is cofinal.

Proof. Consider $\Delta - \Delta_1$ as a full subcategory of $\Delta$. Let $\delta_0 \in \Delta - \Delta_1$ be a minimal object (with respect to $\Delta - \Delta_1$) in the sense of the proof of Lemma C.4 a): the hypothesis on $\Delta_1$ shows that $\delta_0$ is also minimal with respect to $\Delta$. Setting $\Delta' = \Delta - \{\delta_0\}$ as in this proof, we are reduced by induction to the case where $\Delta_1 = \Delta'$. We conclude with the same reasoning as in the proof of Lemma C.4 a). \hfill \Box

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Acronyms

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