Finite-dimensional observer-based PI regulation control of a reaction-diffusion equation

Hugo Lhachemi and Christophe Prieur

Abstract—This paper investigates the output feedback setpoint regulation control of a reaction-diffusion equation by means of boundary control. The considered reaction-diffusion plant may be open-loop unstable. The proposed control strategy consists of the coupling of a finite-dimensional observer and a PI controller in order to achieve the boundary setpoint regulation control of various system outputs such as the Dirichlet and Neumann traces. In this context, it is shown that the order of the finite-dimensional observer can always be selected large enough, with explicit criterion, to achieve both the stabilization of the plant and the setpoint regulation of the system output.

Index Terms—Reaction-diffusion equation, finite-dimensional observer, output feedback, PI control, boundary regulation control, boundary measurement.

I. INTRODUCTION

The problem of controlling the output of a system so as to achieve asymptotic tracking of prescribed trajectories is one of the most fundamental problems in control theory. In the general context of finite-dimensional linear time-invariant (LTI) control systems, the problem of setpoint regulation control is very classical and has been widely investigated. One possible way to solve this problem is based on the augmentation of the state-space representation of the plant with an integral component of the tracking error and the use of the separation principle by exploiting separately a Luenberger observer (which allows the estimation of the state based on the measurement only) and a stabilizing full-state feedback (see, e.g., [10]). Even if this approach has reached a very high level of maturity for finite-dimensional systems, its possible extension to infinite-dimensional systems, as those considered in this paper, is still an open problem.

Infinite-dimensional systems emerge in many practical applications due to the occurrence of delays, reaction-diffusion dynamics, or even flexible behavior (see, e.g., [12], [16], [17] for introductory textbooks on dedicated control theory for infinite-dimensional systems). While many efficient control design methods have been reported for the stabilization of distributed parameter systems, very few have been extended to the problem of output regulation. The main reason is that all techniques that have been developed for finite-dimensional LTI systems cannot be easily generalized to infinite-dimensional plants. For instance, the frequency domain approach has been generalized to the infinite-dimensional setting, but it requires to deal with an infinite number of poles, yielding an infinite-dimensional pole allocation problem. The state-space approach is followed in this work.

We propose, for the first time, an output feedback control design procedure to achieve the setpoint regulation control of reaction-diffusion systems by means of a finite-dimensional observer coupled with a PI controller. The considered reaction-diffusion plant, which might be unstable, is modeled by a Sturm-Liouville operator as those classical introduced in the context of parabolic partial differential equation (PDE). The case of PI regulation of this system by means of a state feedback was reported in [14] (see also [3], [6], [8], [12], [15], [20], [21]–[25], [28] for various approaches about PI control design for different types of PDEs). Here we go beyond by designing an output feedback PI control strategy. Even if the proposed procedure also applies to bounded control inputs and bounded observations, we focus the presentation on boundary controls and boundary measurements. This is because these configurations are the most interesting for practical applications and also the most challenging since they involve unbounded control and observation operators (see, e.g., [7] for further explanations). We study several cases for the input-to-output map, covering Dirichlet control inputs (easily extendable to Neumann control inputs as discussed in conclusion) along with Dirichlet and/or Neumann to-be-regulated outputs and measured outputs. We also show that our procedure can be used to regulate a system output that is distinct of the measured one. Therefore, our approach gives a complete framework to study every associated input-to-output maps.

The proposed control design strategy consists of an adequate integral component coupled with a finite-dimensional observer. The design of finite-dimensional observer-based controllers for distributed parameter plants is challenging due to the fact that the separation principle, that is classically used for finite-dimensional systems, does not apply for infinite-dimensional systems [2], [6], [9], [22]. Taking advantage of spectral reduction approaches [8], [21] and using the control architecture initially reported in [22], a LMI-based procedure for solving this stabilization problem for reaction-diffusion PDEs was reported in [11] in the case were the either control or observation operator is bounded. This approach was extended in [13] to the case were both control and observation operators are unbounded, including both Dirichlet and Neumann settings. The present work, taking advantage of [13], goes beyond the simple problem of closed-loop stabilization by embracing the issue of output setpoint regulation control. Since the designed observer only estimates a finite number of modes of the infinite-dimensional system, there is an inherent mismatch between the actually measured system output and its estimation as soon as the output is to be regulated to a non-zero value. Hence, one of the main challenges is to account for this mismatch in the dynamics of the observer and then in the subsequent stability analysis. An other challenge is to couple this finite-dimensional observer with a suitable integral component, inspired by the one described in [14] for a state-feedback, in an output feedback setting. Our approach is based on Lyapunov direct methods and the main results take the form of explicit sufficient conditions ensuring both stability and setpoint regulation control of the closed-loop plant. We assess that these conditions are always feasible provided the order of the observer is selected large enough. Therefore, we show in a constructive manner that the setpoint regulation control of reaction-diffusion PDEs can always be achieved by the coupling of a PI and a finite-dimensional observer.

The paper is organized by considering successively different input-output maps for the reaction-diffusion equation depending on the selected boundary measured output, the to-be-regulated output, and the control input. After recalling classical notations and properties for the Sturm-Liouville operators in Section II, the case of a Dirichlet
observation and a Dirichlet control input is considered in Section [III]. The case of a Neumann measurement and a Dirichlet control input is considered in Section [IV]. While the to-be-regulated output and the measured output are the same in the two latter sections, a crossed configuration is considered in Section [V]. The regulation problem is solved for a Dirichlet measured output, a Neumann to-be-regulated output, and a Dirichlet control input. This final result completes the picture and gives a full study of the different cases for the input-to-output map of the considered class of distributed parameter systems. Some numerical simulations are given in Section [VII] for this final result. Section [VII] also collects some concluding remarks.

II. NOTATION AND PROPERTIES

Spaces $\mathbb{R}^n$ are endowed with the Euclidean norm denoted by $\| \cdot \|$. The associated induced norms of matrices are also denoted by $\| \cdot \|$. Given two vectors $X$ and $Y$, $\cos(X,Y)$ denotes the vector $[X^T, Y^T]^T$. $L^2(0,1)$ stands for the space of square integrable functions on $(0,1)$ and is endowed with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$ with associated norm denoted by $\| \cdot \|_{L^2}$. For an integer $m \geq 1$, the Sobolev space of order $m$ is denoted by $H^m(0,1)$ and is endowed with its usual norm denoted by $\| \cdot \|_{H^m}$. For a symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P \succeq 0$ (resp. $P \succ 0$) means that $P$ is positive semi-definite (resp. positive definite).

Let $p \in C^1([0,1])$ and $q \in C^0([0,1])$ with $p > 0$ and $q > 0$. Let the Sturm-Liouville operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ be defined by $Af = -(pf')' + qf$ on the domain $D(A) \subset L^2(0,1)$ given by either $D(A) = \{ f \in H^2(0,1) : f(0) = f(1) = 0 \}$ or $D(A) = \{ f \in H^2(0,1) : f'(0) = f(1) = 0 \}$. The eigenvalues $\lambda_n$, $n \geq 1$, of $A$ are simple, non-negative, and form an increasing sequence with $\lambda_n \to +\infty$ as $n \to +\infty$. Moreover, the associated unit eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis. We also have $D(A) = \{ f \in L^2(0,1) : \sum_{n \geq 1} |\lambda_n|^2 |\phi_n|^2 \}$. Thus $Af = \sum_{n \geq 1} \lambda_n \phi_n(0) \phi_n$ for any $f \in D(A)$.

Let $p, p', q' \in \mathbb{R}$ be such that $0 < p_s \leq p(x) \leq p_\ast$ and $0 \leq q(x) \leq q_\ast$ for all $x \in [0,1]$, then it holds [13]:

$$0 \leq \pi^2(n-1)^2 p_s \leq \lambda_n \leq \pi^2 n^2 p_\ast + q_\ast$$

for all $n \geq 1$. Assuming further that $p \in C^2([0,1])$, we have for any $x \in (0,1)$ that $\phi_n(x) = O(1)$ and $\phi'_n(x) = O(\sqrt{\lambda_n})$ as $n \to +\infty$ [18].

Finally, one can check that, for all $f \in D(A)$,

$$\sum_{n \geq 1} \lambda_n \phi_n(0) \phi_n = \int_0^1 \phi(x) f(x) \,dx.$$  

Moreover, for any $f \in D(A)$, we have $f(x) = \sum_{n \geq 1} \phi_n(0) \phi_n(x)$ and $f'(x) = \sum_{n \geq 1} \phi'_n(0) \phi_n(x)$ for all $x \in [0,1]$ (see, e.g., [13]).

III. DIRICHLET MEASUREMENT AND REGULATION CONTROL

We consider the reaction-diffusion system with Dirichlet boundary observation described for $t > 0$ and $x \in (0,1)$ by

$$z_t(t,x) = (p(x)z_x(t,x))_x + (q_c - q(x))z(t,x) \quad (3a)$$

$$z_x(t,0) = 0, \quad z(t,1) = u(t) \quad (3b)$$

$$z(0,x) = z_0(x) \quad (3c)$$

$$y(t) = z(t,0) \quad (3d)$$

in the case $p \in C^2([0,1])$. Here $q_c \in \mathbb{R}$ is a constant, $u(t) \in \mathbb{R}$ is the command input, $y(t) \in \mathbb{R}$ is the measurement, $z_0 \in L^2(0,1)$ is the initial condition, and $z(t,\cdot) \in L^2(0,1)$ is the state. The control design objective is to design a finite-dimensional observer-based controller to achieve both stabilization and setpoint regulation control of $y(t)$ to some prescribed reference signal $r(t)$. By setpoint tracking, we mean that our objective is to ensure that $y(t) \to r_e$ when $t \to +\infty$ as soon as $r(t) \to r_e$ when $t \to +\infty$ for any arbitrary given $r_e \in \mathbb{R}$.

A. Spectral reduction

As classically done in the context of boundary control systems (see [7] Sec. 3.3 for details), we start by transforming the non-homogeneous problem (3) into an equivalent homogeneous one by introducing the change of variable:

$$w(t,x) = z(t,x) - x^2 u(t)$$

that gives the equivalent representation:

$$w_t(t,x) = (p(x)w_x(t,x))_x + (q_c - q(x))w(t,x) + a(x)u(t) + b(x)u(t) \quad (5a)$$

$$w_x(t,0) = 0, \quad w(t,1) = 0 \quad (5b)$$

$$w(0,x) = w_0(x) \quad (5c)$$

$$\tilde{y}(t) = w(t,0) \quad (5d)$$

with $a, b \in L^2(0,1)$ defined by $a(x) = 2p(x) + 2px'(x) + (q_c - q(x))x^2$ and $b(x) = -x^2$, respectively. $\tilde{y}(t) = y(t)$, and $w_0(x) = z_0(x) - x^2 u(0)$. Introducing the auxiliary command input $v(t) = u(t)$, we infer that

$$\tilde{u}(t) = v(t) \quad (6a)$$

$$\frac{dw}{dt}(t,\cdot) = -Aw(t,\cdot) + q_c w(t,\cdot) + au(t) + bv(t) \quad (6b)$$

with $D(A) = \{ f \in H^2(0,1) : f'(0) = f(1) = 0 \}$. We introduce the coefficients of projection $w_n(t) = \langle w(t,\cdot), \phi_n \rangle$, $a_n = \langle a, \phi_n \rangle$, and $b_n = \langle b, \phi_n \rangle$. Considering classical solutions associated with any $z_0 \in H^2(0,1)$ and any $u_0 \in \mathbb{R}$ such that $z_0(0) = 0$ and $z_0(1) = u(0)$ (their existence for the upcoming closed-loop dynamics is an immediate consequence of [19] Chap. 6, Thm. 1.7), we have $w(t,\cdot) \in D(A)$ for all $t \geq 0$ and we infer that

$$\tilde{u}(t) = v(t) \quad (7a)$$

$$w_n(t) = (-\lambda_n + q_c)w_n(t) + a_n u_n(t) + b_n v(t), \quad n \geq 1 \quad (7b)$$

$$\tilde{y}(t) = \sum_{n \geq 1} \phi_n(0)w_n(t) \quad (7c)$$

B. Control design

Let $\delta > 0$ be the desired exponential decay rate for the setpoint regulation. We fix $N_0 \geq 1$ so that $-\lambda_0 + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. The integer $N_0$, which is definitely fixed for the rest of the control design procedure, can be interpreted as the number of modes that will be “actively” modified by the feedback. We now introduce an arbitrary integer $N \geq N_0 + 1$ which will be further constrained later. Inspired by [22], we design as in [13] an observer to estimate the $N$ first modes of the plant while the state-feedback is performed on the $N_0$ first modes of the plant. In this framework, the estimation of the modes ranging from $N_0 + 1$ to $N$ will solely be used to improve the estimate of the system output (see [13]).

Introducing $W^{N_0}(t) = \begin{bmatrix} w_0(t) & \ldots & w_{N_0}(t) \end{bmatrix}^T$, $A_0 = \operatorname{diag}(-\lambda_1 + q_c, \ldots, -\lambda_{N_0} + q_c)$, $B_{0,a} = \begin{bmatrix} a_1 & \ldots & a_{N_0} \end{bmatrix}^T$, and $B_{0,b} = \begin{bmatrix} b_1 & \ldots & b_{N_0} \end{bmatrix}^T$, we have

$$W^{N_0}(t) = A_0 W^{N_0}(t) + B_{0,a} u(t) + B_{0,b} v(t) \quad (8)$$

Our objective is to introduce an integral component to achieve the setpoint regulation control of the system output $y(t)$. To do so, we first consider the following classical integral component: $z_i(t) = y(t) - r(t) = \sum_{n \geq 1} \phi_n(0)w_n(t) = -r(t)$. This $z_i$-dynamics, which involves all the modes $w_n$ for $n \geq 1$, cannot be embedded into the reduced model (8) that only involves the modes $w_n$ for $1 \leq n \leq N_0$. To circumvent this issue, we follow the idea developed in [14] by introducing $\xi_p(t) = z_i(t) - \sum_{n \geq N_0 + 1} \frac{\phi_n(0)}{x_n - q_c} w_n(t)$. 

whose time derivative is given by $\dot{\xi}(t) = \sum_{n=1}^{N_0} \phi_n(0)w_n(t) + \alpha_0u(t) + \beta_0v(t) - r(t)$ with

$$
\alpha_0 = - \sum_{n \geq N_0+1} \frac{a_n \phi_n(0)}{-\lambda_n + q_c}, \quad \beta_0 = - \sum_{n \geq N_0+1} \frac{b_n \phi_n(0)}{-\lambda_n + q_c}.
$$

(9)

The main benefit is that the $\xi_n$-dynamics only involves the modes $w_n$ for $1 \leq n \leq N_0$ while achieving the same equilibrium condition than the $\xi_n$-dynamics. However, in this work and in sharp contrast with the state-feedback setting of [14], the modes $w_n$ are not measured. Hence we need to replace them in the dynamics of the integral component by their estimated version $\hat{w}_n$, which will be described below. Hence, the employed integral component is described by:

$$
\dot{\xi}(t) = \sum_{n=1}^{N_0} \phi_n(0)\hat{w}_n(t) + \alpha_0u(t) + \beta_0v(t) - r(t).
$$

(10)

We now define for $1 \leq n \leq N$ the observer dynamics:

$$
\dot{\hat{w}}_n(t) = (-\lambda_n + q_c)\hat{w}_n(t) + a_nu(t) + b_nv(t) - l_n \left( \sum_{i=1}^{N_0} \phi_i(0)\hat{w}_i(t) - \alpha_1 u(t) - \gamma(t) \right)
$$

with

$$
\alpha_1 = \sum_{n \geq N+1} \frac{a_n \phi_n(0)}{-\lambda_n + q_c}
$$

(12)

and where $l_n \in \mathbb{R}$ are the observer gains. We set $l_n = 0$ for $N_0 + 1 \leq n \leq N$. Compared to the stabilization problem studied in [13], we introduce the additional term $-\alpha_1 u(t)$ in the observer dynamics (11). This term is added to compensate the inherent steady-state mismatch between the actually measured system output $\tilde{y}(t)$ and its estimation $\sum_{i=1}^{N_0} \phi_i(0)\hat{w}_i(t)$, obtained from the observer that estimates the only $N$ first modes of the plant, as soon as the output is to be regulated to a non-zero value. Note that this latter estimate of the output improves as the dimension of the observer $N$ increases.

We define for $1 \leq n \leq N$ the observation error as $e_n(t) = w_n(t) - \hat{w}_n(t)$. Hence we have

$$
\dot{e}_n(t) = (-\lambda_n + q_c)e_n(t) + a_nu(t) + b_nv(t) + l_n \sum_{i=1}^{N_0} \frac{\phi_i(0)e_i(t)}{\sqrt{\lambda_i}} + l_n \alpha_1 u(t) + l_n \xi(t)
$$

(13)

with $\zeta(t) = \sum_{n \geq N+1} \phi_n(0)w_n(t)$ and $\xi(t) = \sum_{n \geq N+1} \phi_n(0)w_n(t)$.

Hence, introducing $\hat{W}^N(t) = [\hat{w}(t) \ldots \hat{w}_{N_0}(t)]^T$, $E^N(t) = [e(\hat{w}(t)) \ldots e(w_{N_0}(t))]^T$, $\xi^N(t) = \sum_{n \geq N+1} \phi_n(0)w_n(t)$, we obtain that

$$
\dot{\hat{W}}^N(t) = A_{\alpha}(\xi(t))\hat{W}^N(t) + B_{\alpha}u(t) + B_{\beta}v(t) + LC_0E^N(t) + LC_1\xi^N(t) + \alpha_1 Lu(t) + L \xi(t).
$$

(14)

With

$$
\hat{W}^N_0(t) = \text{col}(u(t), \hat{W}^N(t), \xi(t)),
$$

(15)

we deduce that

$$
\dot{\hat{W}}^N_0(t) = A_{\alpha}(\xi(t))\hat{W}^N_0(t) + B_{\alpha}u(t) - B_{\beta}v(t) + LC_0E^N_0(t) + LC_1\hat{W}^N_0(t) + \alpha_1 Lu(t) + L \xi(t).
$$

(17)

Setting the auxiliary command input as

$$
v(t) = K\hat{W}^N_0(t),
$$

(18)

and defining

$$
A_{\alpha}(\alpha_1) = A_1 + B_1L + \alpha_1L \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right],
$$

(19)

we obtain that

$$
\dot{\hat{W}}^N_0(t) = A_{\alpha}(\alpha_1)\hat{W}^N_0(t) + B_{\alpha}u(t) + B_{\beta}v(t) + LC_0\hat{W}^N_0(t) + LC_1\hat{W}^N_0(t) + \alpha_1 Lu(t) + L \xi(t).
$$

(20)

and, from (8) and (14),

$$
\dot{E}^N_0(t) = (A_0 - LC_0)E^N_0(t) - LC_1\hat{W}^N_0(t) - \alpha_1 L \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right] \hat{W}^N_0 - L \xi(t).
$$

(21)

We now define $\hat{W}^N_{-N_0}(t) = [\hat{w}_{N_0+1}(t) \ldots \hat{w}_N(t)]^T$, $A_2 = \text{diag}(-\lambda_{N_0+1}, \ldots, -\lambda_N, -\lambda_N + q_c)$. $B_{2,\alpha} = [a_{N_0+1} \ldots a_N]^T$, $B_{2,\beta} = [b_{N_0+1} \ldots b_N]^T$. We obtain from (11) with $l_n = 0$ for $N_0 + 1 \leq n \leq N$ that

$$
\dot{\hat{W}}^{-N_0}_N(t) = A_2\hat{W}^{-N_0}_N(t) + B_{2,\alpha}u(t) + B_{2,\beta}v(t)
$$

(22)

and, using (7b) and (11),

$$
\dot{\hat{E}}^{-N_0}_N(t) = A_2\hat{E}^{-N_0}_N(t).
$$

(23)

Putting now together (20), (23) while introducing

$$
\tilde{X}(t) = \text{col} \left( \hat{W}^N_0(t), E^N_0(t), \hat{W}^{-N_0}_N(t), \hat{E}^{-N_0}_N(t) \right),
$$

(24)

we obtain that

$$
\dot{\tilde{X}}(t) = FX(t) + LC(t) - L \tau(t)
$$

(25)

where

$$
F = \begin{bmatrix}
A_{\alpha}(\alpha_1) & \tilde{L}C_0 & 0 & \tilde{L}C_1 \\
-\alpha_1 L & A_0 - LC_0 & 0 & -LC_1 \\
B_{2,\alpha} & 0 & 0 & A_2 \\
0 & 0 & 0 & A_2 \\
\end{bmatrix},
$$

$$
L = \text{col}(\tilde{L}, -L, 0, 0), \quad \tilde{L} = \text{col}(B_{1,0}, 0, 0, 0).
$$

Defining $E = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$ and $\tilde{K} = \begin{bmatrix} K & 0 & 0 & 0 \end{bmatrix}$, we obtain from (15), (18), and (24) that

$$
u(t) = EX(t), \quad v(t) = \tilde{K}X(t)
$$

(26)

and we can introduce

$$
G = \|a\|_2^2E^T + \|b\|_2^2\tilde{K}^T \leq gI
$$

(27)

with $g = \|a\|_2^2 + \|b\|_2^2\|K\|^2$ a constant independent of $N$.

**Lemma 1:** $(A_2, B_1)$ is controllable and $(A_0, C_0)$ is observable.

**Proof:** From [14] Lem. 2, $(A_2, B_1)$ is controllable if and only if

$$
\begin{bmatrix}
0 & 0 & 0 & 1 \\
B_{2,\alpha} & A_0 & 0 & B_{2,\beta} \\
0 & 0 & A_0 & 0 \\
\alpha_0 & C_0 & \beta_0 & 0
\end{bmatrix}
$$

(28)

satisfies the Kalman condition and the matrix $T = \begin{bmatrix} 0 & 0 & 1 \\
B_{2,\alpha} & A_0 & 0 & B_{2,\beta} \\
0 & 0 & A_0 & 0 \\
\alpha_0 & C_0 & \beta_0 & 0
\end{bmatrix}$ is invertible. The former condition was assessed in [13]. Hence we focus on the latter one. Let

$$
\begin{bmatrix}
u_e & w_{1,e} & \ldots & w_{N_0,e} & v_e
\end{bmatrix}^T \in \text{ker}(T). \quad \text{We obtain that}
$$

$$
u_e = 0,
$$

(29a)

$$
a_nu_e + (-\lambda_n + q_c)w_{n,e} = 0, \quad 1 \leq n \leq N_0,
$$

(29b)
\begin{equation}
\alpha_0 u_e + \sum_{n=1}^{N_0} \phi_n(0) w_{n,e} = 0.
\end{equation}

Defining for \( n \geq N_0 + 1 \) the quantity \( w_{n,e} = -\frac{\alpha_n}{\lambda_n + q_e} u_e \), we have \((-\lambda_n + q_e) w_{n,e} + \alpha_n u_e = 0\) for all \( n \geq 1 \). Hence \( (w_{n,e})_{n \geq 2} = (\lambda_n w_{n,e})_{n \geq 2} \in \mathbb{F}^2(\mathbb{N}) \) ensuring that \( w_e \triangleq \sum_{n \geq 2} w_{n,e} \phi_n \in D(A) \) and \( A w_e = \sum_{n \geq 2} \lambda_n w_{n,e} \phi_n \). This shows that \(-A w_e + q_e w_e + \alpha_n u_e = 0\). Moreover, from \( (29c) \) and \( (9) \), we infer that \( w_e = 0 \). From the two last identities, we have that \((p u'_e(0) + (q_e - q) u_e + a u_e = 0, w_e(0) = w'_e(0) = 0\), and \( \hat{z} = 1 \).

Introducing the change of variable \( z_e(x) = w_e(x) + x^2 u_e \), we deduce that \( (p u'_e(0) + (q_e - q) z_e = 0, \hat{z}(0) = \hat{z}'(0) = 0\), and \( \hat{z} = 1 \).

By Cauchy uniqueness, we infer that \( \hat{z} = 1 \).

Thus we have \( w_e = z_e^2 u_e = 0 \) hence \( w_e = 0 \) for all \( n > 1 \). We deduce that ker \( (T) \) = \{0\}. Overall, we have shown that \((A_1, B_1)\) is controllable. Finally, the pair \((A_0, C_0)\) is observable because \( A_0 \) is diagonal with simple eigenvalues, 2) by Cauchy uniqueness, \( \phi_0(0) \neq 0 \) for all \( n \geq 1 \).

In the sequel we select, once for all and independently of the dimension \( N \) of the observer, the gains \( K \in \mathbb{R}^{1 \times (N_0+2)} \) and \( L \in \mathbb{R}^{N_0} \) so that \( A_1 + B_1 K \) and \( A_0 - LC_0 \) are Hurwitz with eigenvalues that have a real part strictly less than \(-\delta < 0\).

### C. Equilibrium condition and dynamics of deviations

We aim at characterizing the equilibrium condition of the closed-loop system composed of the reaction-diffusion system \((3)\), the auxiliary command input dynamics \( (9) \), the integral action \( (10) \), the observer dynamics \( (11) \), and the state-feedback \( (13) \). To do so let \( r(t) = r_e \in \mathbb{R} \) be arbitrary. We must solve the system of equations:

\begin{align}
 0 &= (-\lambda_n + q_e) w_{n,e} + \alpha_n u_e + b_ne = 0, \quad n \geq 1, \\
 0 &= r_e = K W_{n,e}^0, \\
 0 &= \sum_{n=1}^{N_0} \phi_n(0) \hat{w}_{n,e} + \alpha_0 u_e + \beta_0 u_e - r_e, \\
 0 &= (-\lambda_n + q_e) \hat{w}_{n,e} + \alpha_n u_e + b_ne, \quad 1 \leq n \leq N_0, \\
 0 &= (-\lambda_n + q_e) \hat{w}_{n,e} + \alpha_n u_e + b_ne, \quad N_0 + 1 \leq n \leq N, \\
 \hat{y}_e &= \sum_{n \geq 1} \phi_n(0) w_{n,e}.
\end{align}

We first note from \( (30b) \) that \( r_e = 0 \). Then, from \( (30a) \) we have \( w_{n,e} = -\frac{\alpha_n}{\lambda_n + q_e} u_e \) for all \( n \geq N_0 + 1 \). In particular, from \( (30a) \), we have \( w_{n,e} = w_{n,e} = -\frac{\alpha_n}{\lambda_n + q_e} u_e \) for all \( N_0 + 1 \leq n \leq N \). Defining \( e_n = w_{n,e} - \hat{w}_{n,e} \) and \( \zeta_e = \sum_{n \geq N_0 + 1} \phi_n(0) w_{n,e} \), we obtain that \( e_n = 0 \) for all \( N_0 + 1 \leq n \leq N \). Hence, from \( (30a) \), we infer that \( 0 = (-\lambda_n + q_e) w_{n,e} + \alpha_n u_e + b_ne, \quad 1 \leq n \leq N_0, \)

\begin{align}
 0 &= (-\lambda_n + q_e) \hat{w}_{n,e} + \alpha_n u_e + b_ne, \quad N_0 + 1 \leq n \leq N, \\
 \hat{y}_e &= \sum_{n \geq 1} \phi_n(0) w_{n,e}.
\end{align}

D. Stability analysis and regulation assumption

We define the constant \( M_{1, \phi} = \sum_{n \geq N_0 + 1} \phi_n(0)^2 \), which is finite when \( p \in C^2([0,1]) \) because \( \phi_n(0) = O(1) \) as \( n \to +\infty \) and \( \delta > 0 \).

**Theorem 1:** Let \( p \in C^2([0,1]) \) with \( p > 0 \), \( q \in C^1([0,1]) \) with \( q > 0 \), and \( q_e \in \mathbb{R} \). Consider the reaction-diffusion system described by \((3)\). Let \( N_0 \geq 1 \) and \( \delta > 0 \) be given such that \(-\lambda_n + q_e < -\delta < 0\) for all \( n \geq N_0 + 1 \). Let \( K \in \mathbb{R}^{1 \times (N_0+2)} \) and \( L \in \mathbb{R}^{N_0} \) be such that \( A_1 + B_1 K \) and \( A_0 - LC_0 \) are Hurwitz with eigenvalues that have a real part strictly less than \(-\delta < 0\). For a given \( N_0 + 1 \), assume that there exist \( \lambda > 0, \alpha > 1, \delta, \beta \) such that \( \Theta_1 = \begin{bmatrix} F T P & P F + 25P + \alpha \gamma G & P L^2 \end{bmatrix} - \beta \end{bmatrix} < 0 \), \( \Theta_2 = 2 \gamma \begin{bmatrix} - \frac{1}{\alpha} \lambda_{N+1} + q_e + \delta \end{bmatrix} + \beta M_{1, \phi} < 0 \). Then, for any \( \eta \in [0,1] \), there exists \( M_0 > 0 \) such that, for any \( z_0 \in H^2(\mathbb{R}^2,1) \) and \( u_0(\xi))w_0(n) \in \mathbb{R} \) such that \( z_0(0) = 0 \) and \( \zeta_0(1) = u(0) \), the classical solution of the closed-loop system composed of the plant \((3)\), the integral actions \( (9) \) and \( (10) \), the observer dynamics \( (11) \), and the state-feedback \( (13) \) satisfies

\[
\begin{align}
\Delta u(t)^2 + \Delta \xi(t)^2 &+ \sum_{n=1}^{N_0} \Delta \hat{w}_n(t)^2 + \|\Delta z(t)\|_{H^1}^2 \\
&\leq M e^{-2\delta t} \left( \Delta u(0)^2 + \Delta \xi(0)^2 + \sum_{n=1}^{N_0} \Delta \hat{w}_n(0)^2 + \|\Delta z(0)\|_{H^1}^2 \right) \\
&+ M \sup_{r \in [0,t]} e^{-2\delta (t-r)} \Delta r(t)^2
\end{align}
\]

for all \( t \geq 0 \). Moreover, the above constraints are always feasible for \( N \) large enough.

**Proof.** Let \( P > 0 \) and \( \gamma > 0 \) and consider the Lyapunov function candidate defined by

\[
V(\Delta X, \Delta \phi) = \Delta X^T P \Delta X + \gamma \sum_{n \geq N_0 + 1} \lambda_n \langle \Delta w, \phi_n \rangle^2.
\]

with \( \Delta X \in \mathbb{R}^{N_0+2} \) and \( \Delta \phi \in D(A) \). The first term accounts for the dynamics of the truncated model \((25)\) while the series, which in view of \((2)\) is related to the \( H^2 \)-norm of the PDE trajectories, is used to handle the modes \( w_n \) for \( n \geq N_0 + 1 \) of the PDE plant. Proceeding
Hence the assumptions imply that  
\[
\dot{\theta} \leq \begin{cases} -\frac{1}{2} \lambda_n + q_n + \delta & \text{for } n \geq N + 1, \\
\beta M_1, \phi & \text{for } n = N + 1, \end{cases}
\]
with \(\Gamma = 2 \gamma \{-\frac{1}{2} \lambda_n + q_n + \delta\} + \beta M_1, \phi\) for \(n \geq N + 1\), \(\alpha > 1\) and \(\beta > 0\) arbitrary, and where, with a slight abuse of notation, \(V(t)\) denotes the time derivative of \(V(X(t), w(t))\) along the system trajectories \([11]\). Since \(\alpha > 1\) we have \(\Gamma_n \leq \Theta_2 \leq 0\) for all \(n \geq N + 1\). From \([32a]\), there exists \(\epsilon > 0\) such that \(\Theta_1 \leq -\epsilon I\). Hence the assumptions imply that 
\[
\dot{V}(t) + 2\delta V(t) \leq \frac{\Delta X(t)}{\Delta \zeta(t)}^T \Theta_1 \frac{\Delta X(t)}{\Delta \zeta(t)} - 2\Delta X(t)^T P \bar{L} C \dot{r}(t) + \sum_{n \geq N + 1} \lambda_n \Gamma_n \Delta w_n(t)^2
\]
for all \(t \geq 0\).

Proof. Recalling that \(y_n = r_n\), one has \(|y(t) - r(t)| \leq |\Delta y(t)| + |\Delta r(t)|\). From \([31l]\) and Cauchy-Schwarz inequality, we infer that 
\[
|\Delta y(t)| \leq \sqrt{\sum_{n \geq 1} \frac{q_n^2}{\lambda_n}} \sqrt{\sum_{n \geq 1} \lambda_n \Delta w_n(t)^2}
\]
Using now \([22]\) we infer the existence of a constant \(M_2 > 0\) such that \(|\Delta y(t)| \leq M_2\|\Delta w(t)\|\). The proof is completed by invoking the change of variable \([31a]\) and the stability result \([33]\).

IV. NEUMANN MEASUREMENT AND REGULATION CONTROL

We now consider the reaction-diffusion system with Neumann boundary observation described for \(t > 0\) and \(x \in (0, 1)\) by

\[
\begin{align*}
\frac{\partial^2 z}{\partial x^2}(t, x) &= \left( p(x) z_x(x, t) \right)_x + (q_n - q(x)) z(x, t) \\
&\quad + a(x) u(t) + b(x) \dot{u}(t)
\end{align*}
\]

We obtain
\[
\begin{align*}
w(t, 0) &= 0, \quad w(t, 1) = 0 \\
w(0, x) &= w_0(x) \\
\hat{y}(t) &= w_x(t, 0)
\end{align*}
\]
for all \(t \geq 0\).

A. Control design

Introducing the change of variable
\[
w(t, x) = z(t, x) - xu(t)
\]
we obtain
\[
\begin{align*}
w_1(t, x) &= (p(x) w_1(x, t))_x + (q_n - q(x)) w(t, x) \\
&\quad + a(x) u(t) + b(x) \dot{u}(t)
\end{align*}
\]
with \(a, b \in L^2(0, 1)\) defined by \(a(x) = p(x) + (q_c - q(x))x\) and \(b(x) = -x\), respectively, \(\hat{y}(t) = y(t) - u(t)\), and \(w_0(x) = z_0(x) - xu_0\). Introducing the auxiliary command input \(v(t) = \dot{u}(t)\), we infer that \([31a]\) still holds but the domain of \(A\) is now replaced by \(D(A) = \{f \in H^2(0, 1) : f(0) = f(1) = 0\}\). Then, considering classical solutions associated with any \(z_0 \in H^2(0, 1)\) and any \(w_0 \in \mathbb{R}\) such that \(z_0(0) = 0\) and \(z_0(1) = v(0)\) (their existence for the upcoming closed-loop dynamics is an immediate consequence of \([19\text{ Chap. 6, Thm. 1.7}]\). \([7a, 7b]\) is still valid while \([7c]\) is replaced by

\[
\hat{y}(t) = \sum_{i=1}^N \phi_i(0) w_i(t).
\]

Based on similar motivations than the ones reported in Section III we consider the integral component
\[
\dot{\xi}(t) = \sum_{n=1}^N \phi_n(0) \dot{w}_n(t) + a_0 u(t) + \beta_0 v(t) - r(t).
\]

with
\[
\begin{align*}
\alpha_0 &= 1 - \sum_{n \geq N + 1} \frac{a_n \phi_n(0)}{-\lambda_n + q_n}, \\
\beta_0 &= -\sum_{n \geq N + 1} \frac{b_n \phi_n(0)}{-\lambda_n + q_n}
\end{align*}
\]
and where the observation dynamics, for \(1 \leq n \leq N\), take the form
\[
\dot{w}_n(t) = (-\lambda_n + q_n) w_n(t) + a_n u(t) + b_n v(t)
\]
with
\[
\alpha_1 = \sum_{n \geq N + 1} \frac{a_n \phi_n'(0)}{-\lambda_n + q_n}
\]
and where $l_n ∈ R$ are the observer gains. We set $l_n = 0$ for $N_0 + 1 ≤ n ≤ N$. Proceeding now as in Section III but with the updated versions of the matrices $C_0$ and $C_1$ now given by $C_0 = [φ_t(0) , \ldots , φ_t(N_0)]$ and $C_1 = [φ_t(0), \ldots , φ_t(N)]$, while redefining $κ_0(t)$ and $κ(t)$ as $κ_0(t) = λ_n κ_0(t)$ and $κ(t) = \sum_{n ≥ N_0+1} φ_t(n)w_n(t)$, we infer that (25) holds.

**Lemma 2:** ($A_1, B_1$) is controllable and ($A_0, C_0$) is observable.

The proof of this Lemma is analogous to the one of Lemma 1 and is thus omitted. We select in the sequel $K ∈ R^{1×(N_0+2)}$ and $L ∈ R^{N_0}$ such that $A_1 + B_1K$ and $A_0 - LC_0$ are Hurwitz.

B. Equilibrium condition and dynamics of deviations

Proceeding similarly to Section III, we can characterize the equilibrium condition of the closed-loop system composed of the reaction-diffusion system (38), the auxiliary command dynamics (38a), the integral action (38b), the observer dynamics (44), and the state-feedback (18). In particular, setting $r(t) = r_ε ∈ R$, it can be shown that there exists a unique solution to:

$$0 = (−λ_n + q_ε)w_n + a_nu + b_nv, \quad n ≥ 1, \quad (46a)$$

$$v = KW_{N_0}, \quad (46b)$$

$$0 = \sum_{n=1}^{N_0} φ_t(n)w_n + a_nu + b_nv - r_ε, \quad n ≥ 1, \quad (46c)$$

$$0 = (−λ_n + q_ε)w_n + a_nu + b_nv - λ_n w_n - a_n u_n - b_nv, \quad n ≥ 1, \quad (46d)$$

$$y_ε = \sum_{n=1}^{N_0} φ_t(n)w_n \quad (46f)$$

Moreover we can define $w_ε \triangleq \sum_{n=1}^{N_0} w_n ∈ D(A)$. Introducing the change of variable $z_ε = w_ε + \xi_ε$, $z_ε$ is a static solution of (38a) associated with the constant control input $u(t) = u_ε$. Denoting by $y_ε = z_ε$, we also infer that $y_ε = r_ε$, achieving the desired reference tracking. This allows the introduction of the dynamics of deviation of the different quantities w.r.t. the equilibrium condition characterized by $r_ε ∈ R$. We have:

$$Δw(x,t) = Δz(x,t) - xΔu(t), \quad (47a)$$

$$ΔX(t) = FΔX(t) + Δξ(t) - L_εΔr(t), \quad (47b)$$

$$Δz(t) = \sum_{n ≥ N_0+1} φ_t(n)Δw_n(t), \quad (47c)$$

$$Δw_n(t) = (−λ_n + q_ε)Δw_n(t) + a_nΔu(t) + b_nΔv(t), \quad n ≥ 1, \quad (47d)$$

$$Δv(t) = KΔw_{N_0}(t), \quad (47e)$$

$$Δy_ε(t) = Δy(t) - Δu(t) = \sum_{n≥1} φ_t(n)Δw_n(t). \quad (47f)$$

C. Stability analysis and regulation assessment

We define, for any $ε ∈ (0, 1/2]$, the constant $M_{ε, φ_t}(ε) = \sum_{n ≥ N_0+1} \frac{φ_t(n)}{ε^2 + λ_n^2}$, which is finite when $p ∈ C^2([0, 1])$ because we recall that $φ_t(n) = O(1/n)$ as $n → +∞$ and $1$ hold.

**Theorem 3:** Let $p ∈ C^2([0, 1])$ with $p > 0$, $q ∈ C^2([0, 1])$ with $q ≥ 0$, and $ε ∈ R$. Consider the reaction-diffusion system described by (38). Let $N_0 ≥ 1$ and $δ > 0$ be given such that $−λ_n + q_ε < −δ < 0$ for all $n ≥ N_0+1$. Let $K ∈ R^{1×(N_0+2)}$ and $L ∈ R^{N_0}$ be such that $A_1 + B_1K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $−δ < 0$. For a given $N ≥ N_0+1$, assume that there exist $P > 0$, $ε ∈ (0, 1/2]$, $α > 1$, and $β, γ > 0$ such that $Θ_1 < 0$, where $Θ_2$ is defined by (32a).

$$Θ_2 = 2γ \left\{ (1 - \frac{1}{α}) λ_n + q_ε + δ \right\} + βM_{ε, φ_t}(ε)λ_n^2 ε^2 + 2 < 0,$$

Then, for any $ε ∈ (0, 1)$, there exists $M > 0$ such that, for any $z_0 ∈ H^2(0, 1)$ and $u(0), ξ(0), w_n(0) ∈ R$ such that $z_0(0) = 0$ and $z_0(1) = u(0)$, the classical solution of the closed-loop system composed of the plant (35), the integral actions (38a) and (47), the observer dynamics (44), and the state-feedback (18) satisfies (33) for all $t ≥ 0$. Moreover, the above constraints are always feasible for $N$ large enough.

**Proof:** Let $P > 0$ and $γ > 0$ and consider the Lyapunov function candidate defined by (34). Then, proceeding as in (13) but taking into account the extra contribution of the reference signal appearing in (25), we obtain that (35) holds for all $t ≥ 0$ with $Γ_n = 2γ \left\{ (1 - \frac{1}{α}) λ_n + q_ε + δ \right\} + βM_{ε, φ_t}(ε)λ_n^2 ε^2 + 2$. Since $ε ∈ (0, 1/2]$, we have $λ_n^2 ε^2 + 2 ≤ λ_n^2/λ_n ≤ λ_n$. Hence, for all $n ≥ N_0+1$, we have that $Θ_2 ≥ 0$. Since $ε ≥ 1$ and $α > 1$, the stability estimate (33) is analogous to the one reported in the proof of Theorem 1.

It remains to show that we can always select $M ≥ N_0+1, P > 0, ε ∈ (0, 1/2]$, $α > 1$, and $β, γ > 0$ such that $Θ_1 < 0$, $Θ_2 < 0$, and $Θ_3 ≥ 0$. To handle the constraint $Θ_1 < 0$, we proceed as in the last part of the proof of Theorem 1. This is allowed because $∥C_1∥ = O(1)$ as $N → +∞$. We set $ε = 1/8$ and we arbitrarily fix $α > 1$. Setting $β = N^{1/8}$ and $γ = N^{3/16}$, we deduce the existence of an integer $N ≥ N_0+1$ large enough such that $Θ_3 ≥ 0$, $Θ_2 < 0$, and $Θ_3 ≥ 0$.

We are now in position to assess the setpoint regulation control of the left Dirichlet trace.

**Theorem 4:** Under both assumptions and conclusions of Theorem 3, for any $ε ∈ (0, 1)$, there exists $M_ε > 0$ such that

$$|y(t) - r(t)| ≤ M_ε e^{-δt} \left[ |Δu(0)| + |Δξ(t)| + \sum_{n=1}^{N_0} |Δw_n(0)| \right]$$

$$+ \left[ |Δz_0(0)| + |Δu(0)| \right] + M_ε sup_{r∈[0,t]} e^{-δ(t-r)} |Δr(t)| \quad (48)$$

for all $t ≥ 0$ where $Δz_0 = Δz_0 - xΔu(0)$.

**Proof:** Recalling that $y_ε = r_ε$, one has $|y_ε(t) - r(t)| ≤ |Δy(t)| + |Δr(t)|$. We infer from (47f) and Cauchy-Schwarz inequality that $|Δy(t)| ≤ \sqrt{\sum_{n=1}^{N_0} φ_t(n)^2}/\sqrt{\sum_{n=1}^{N_0} λ_n^2 Δw_n(t)^2} + |Δu(t)|$. In view of the stability estimate (33) provided by Theorem 3, we only need to study the term $\sum_{n=1}^{N_0} φ_t(n)^2/λ_n^2$. This can be done as in (14).

**Proof of Theorem 2**, yielding the claimed estimate (45).

V. DIRICHLET MEASUREMENT AND NEUMANN REGULATION CONTROL

We now consider the reaction-diffusion system described by (33)–(45), still in the case $p ∈ C^2([0, 1])$, but this time with the boundary measurement $y_n(t)$ and the distinct and unmeasured output to-be-regulated $y_ε(t)$ described by:

$$y_n(t) = (z(t, 0), y_ε(t) = z_ε(t, 1). \quad (49)$$

A. Control design

Using the change of variable (4), we obtain that (32) still hold while (33) is replaced by:

$$\tilde{y}_n(t) = w(t, 0) = z(t, 0) = y_n(t) \quad (50a)$$
Based on similar motivations that the ones reported in Section III, we infer that \( \phi_n(1) \) is controllable. We set \( \phi_n(1) = \phi(0) \), achieving the desired reference tracking. Consequently, we obtain the following dynamics of deviations:

\[
\Delta w(t, x) = \Delta z(t, x) - x^2 \Delta u(t), \quad \Delta \bar{x}(t) = F \Delta x(t) + L \Delta \zeta(t) - L \Delta r(t), \quad \Delta \zeta(t) = \sum_{n \geq N+1} \phi_n(0) \Delta w_n(t), \quad \Delta \bar{w}_n(t) = (-\lambda_n + q_n) \Delta w_n(t) + \alpha_n u(t) + b_n v(t), \quad \Delta \bar{v}_n(t) = K \Delta \bar{w}_n(t), \quad \Delta \bar{y}_n(t) = \sum_{n \geq 1} \phi_n(1) \Delta w_n(t). \]

C. Stability analysis and regulation assessment

The proof of the following theorem directly follows from the proofs reported in the previous sections.

**Theorem 5:** Under the assumption of Lemma 3, the stability result stated by Theorem 4 also applies to the closed-loop system composed of the plant (20-22), with (49), the integral actions \( \bar{w}_n \) and \( \bar{v}_n \), the observer dynamics (52), and the state feedback (18). In particular, setting \( r(t) = r_e \in \mathbb{R} \), it can be shown that there exists a unique solution to:

\[
0 = (-\lambda_n + q_n) w_n + \alpha_n u_n + b_n v_n = 0, \quad n \geq 1, \quad 0 = v_e = kW_{a,e}, \quad 0 = \sum_{n=1}^{N_0} \phi_n(1) \bar{w}_n + \alpha_n u_n + b_n v_n, \quad 0 = (-\lambda_n + q_n) \bar{w}_n + \alpha_n u_n + b_n v_n, \quad N_0 + 1 \leq n \leq N, \quad (56) \]

Moreover we can define \( u_e \triangleq \sum_{n \geq 1} w_n \phi_n \in D(A) \). Introducing the change of variable \( z_e = u_e + x^2 \bar{u}_e \), \( z_e \) is a static solution of (42) associated with the constant control input \( u(t) = u_e \). Denoting by \( y_r, e = \Delta z_e(1) \), we also infer that \( y_r, e = r_e \) achieving the desired reference tracking. Consequently, we obtain the following dynamics of deviations:

\[
\Delta w_0(t) = \Delta z_0(t) - x^2 \Delta u_0(t), \quad \Delta \bar{x}(t) = F \Delta x(t) + L \Delta \zeta(t) - L \Delta r(t), \quad \Delta \zeta(t) = \sum_{n \geq N+1} \phi_n(0) \Delta w_n(t), \quad \Delta \bar{w}_n(t) = (-\lambda_n + q_n) \Delta w_n(t) + \alpha_n u(t) + b_n v(t), \quad \Delta \bar{v}_n(t) = K \Delta \bar{w}_n(t), \quad \Delta \bar{y}_n(t) = \sum_{n \geq 1} \phi_n(1) \Delta w_n(t). \]

VI. NUMERICAL ILLUSTRATION

We illustrate the result of Section V for Dirichlet measurement and Neumann regulation using a modal approximation that captures the 50 dominant modes of the reaction-diffusion PDE. We set \( p = 1, q = 0, \) and \( q_c = 3 \) for which the open-loop plant is unstable. Selecting \( \delta = 0.5 \), we obtain \( N_0 = 1 \), the feedback gain \( K = [-10.4134 \quad -11.3747 \quad 2.3100] \), and the observer gain \( L = 1.4373 \). The conditions of Theorem 5 are found feasible for \( N = 3 \). The time-domain evolution of the closed-loop system trajectories are depicted in Fig. 1 confirming the theoretical predictions.
We proposed the design of a finite-dimensional observer-based PI controller to achieve the dual output stabilization and regulation control of reaction-diffusion PDEs. Even if presented for a Dirichlet boundary control input, the presented results extend easily to Neumann/Robin boundary control by modifying the change of variable formula to obtain an homogeneous PDE, giving different to Neumann/Robin boundary control (by modifying the change of Dirichlet boundary control input, the presented results easily extend to Neumann boundary control). Even if presented for a reaction-diffusion system (3a-3c) the same approach provided the satisfaction of adequate observability conditions. While we have adopted in this paper an early lumping approach, future research directions for finite-dimensional PI regulation control of reaction-diffusion PDEs may be concerned with the study of late lumping approaches [1] in the framework of backstepping control design for PDE-ODE cascades [23, 27].

VII. CONCLUSION

We proposed the design of a finite-dimensional observer-based PI controller to achieve the dual output stabilization and regulation control of reaction-diffusion PDEs. Even if presented for a Dirichlet boundary control input, the presented results extend easily to Neumann/Robin boundary control (by modifying the change of variable formula to obtain an homogeneous PDE, giving different $a, b \in L^2(0, 1)$). In-domain measurements can also be handled with the same approach provided the satisfaction of adequate observability conditions. While we have adopted in this paper an early lumping approach, future research directions for finite-dimensional PI regulation control of reaction-diffusion PDEs may be concerned with the study of late lumping approaches [1] in the framework of backstepping control design for PDE-ODE cascades [23, 27].

REFERENCES

[1] J. Auriol, K. A. Morris, and F. Di Meglio, “Late-lumping backstepping control of partial differential equations,” *Automatica*, vol. 100, pp. 247–259, 2019.
[2] M. J. Balas, “Finite-dimensional controllers for linear distributed parameter systems: exponential stability using residual mode filters,” *Journal of Mathematical Analysis and Applications*, vol. 133, no. 2, pp. 283–296, 1988.
[3] M. Barreau, F. Gouaisbaut, and A. Seuret, “Practical stability analysis of a drilling pipe under friction with a PI-controller,” *IEEE Transactions on Control Systems Technology*, 2019.
[4] J.-M. Coron and A. Hayat, “PI controllers for 1-D nonlinear transport equation,” *IEEE Transactions on Automatic Control*, vol. 64, no. 11, pp. 4570–4582, 2019.
[5] J.-M. Coron and E. Trélat, “Global steady-state controllability of one-dimensional semilinear heat equations,” *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 549–569, 2004.
[6] R. Curtain, “Finite-dimensional compensator design for parabolic distributed systems with point sensors and boundary input,” *IEEE Transactions on Automatic Control*, vol. 27, no. 1, pp. 98–104, 1982.
[7] R. F. Curtain and H. Zwart, An introduction to infinite-dimensional linear systems theory: A State-Space Approach.* Springer, 2020, vol. 71.
[8] V. Dos Santos, G. Bastin, J.-M. Coron, and B. d’Andréa Novel, “Boundary control with integral action for hyperbolic systems of conservation laws: Stability and experiments,” *Automatica*, vol. 44, no. 5, pp. 1310–1318, 2008.
[9] C. Harkort and J. Deutscher, “Finite-dimensional observer-based control of linear distributed parameter systems using cascaded output observers,” *International journal of control*, vol. 84, no. 1, pp. 107–122, 2011.
[10] J. P. Hespanha, *Linear systems theory.* Princeton university press, 2018.
[11] R. Katz and E. Fridman, “Constructive method for finite-dimensional observer-based control of 1-D parabolic PDEs,” *Automatica*, vol. 122, p. 109285, 2020.
[12] M. Krstic and A. Smyshlyaev, Boundary control of PDEs: A course on backstepping designs. *SIAM, 2008.
[13] H. Lhachemi and C. Prieur, “Finite-dimensional observer-based boundary stabilization of reaction-diffusion equations with either a Dirichlet or Neumann boundary measurement,” *Automatica*, vol. 135, p. 109955, 2022.
[14] H. Lhachemi, C. Prieur, and E. Trélat, “PI regulation of a reaction-diffusion equation with delayed boundary control,” *IEEE Transactions on Automatic Control*, vol. 66, no. 4, pp. 1573–1587, 2020.
[15] ——, “PI regulation control of a 1-D semilinear wave equation,” *SIAM Journal on Control and Optimization*, 2021, in press.
[16] T. Meurer, Control of higher–dimensional PDEs: Flatness and backstepping designs.* Springer Science & Business Media, 2012.
[17] K. A. Morris, Controller Design for Distributed Parameter Systems.* Springer, 2020.
[18] Y. Orlov, “On general properties of eigenvalues and eigenfunctions of a Sturm–Liouville operator: comments on ’ISS with respect to boundary disturbances for 1-D parabolic PDEs’,” *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5970–5973, 2017.
[19] A. Pazy, *Semigroups of linear operators and applications to partial differential equations.* Springer Science & Business Media, 2012, vol. 44.
[20] S. Pohjolainen, “Robust multivariable PI-controller for infinite dimensional systems,” *IEEE Transactions on Automatic Control*, vol. 27, no. 1, pp. 17–30, 1982.
[21] D. L. Russell, “Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions,” *SIAM Review*, vol. 20, no. 4, pp. 639–739, 1978.
[22] Y. Sakawa, “Feedback stabilization of linear diffusion systems,” *SIAM journal on control and optimization*, vol. 21, no. 5, pp. 667–676, 1983.
[23] S. Tang and C. Xie, “State and output feedback boundary control for a coupled PDE–ODE system,” *Systems & Control Letters*, vol. 60, no. 8, pp. 540–545, 2011.
[24] A. Terrand-Jeanne, V. Andrieu, V. D. S. Martins, and C. Xu, “Adding integral action for open-loop exponentially stable semigroups and application to boundary control of PDE systems,” *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4481–4492, 2020.
[25] A. Terrand-Jeanne, V. Andrieu, M. Tayakout-Fayolle, and V. D. S. Martins, “Regulation of inhomogeneous drilling model with a PI controller,” *IEEE Transactions on Automatic Control*, vol. 65, no. 1, pp. 58–71, 2020.
[26] N.-T. Trinh, V. Andrieu, and C.-Z. Xu, “Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations,” *IEEE Transactions on Automatic Control*, vol. 62, no. 9, pp. 4527–4536, 2017.
[27] J. Wang and M. Krstic, “Output feedback boundary control of a heat pde sandwiched between two ODEs,” *IEEE Transactions on Automatic Control*, vol. 64, no. 11, pp. 4653–4660, 2019.
[28] C.-Z. Xu and H. Jerbi, “A robust PI-controller for infinite-dimensional systems,” *International Journal of Control*, vol. 61, no. 1, pp. 33–45, 1995.