On Minkowski diagonal continued fraction

by Nikolay Moshchevitin

Abstract. We study some properties of the function $\mu_\alpha(t)$ associated with the Minkowski diagonal continued fraction for real $\alpha$.

1. Irrationality measure function, Lagrange and Dirichlet spectra.
For a real $\alpha$ we consider the irrationality measure function

$$\psi_\alpha(t) = \min_{1 \leq x \leq t, x \in \mathbb{Z}} ||x\alpha||$$

(here $|| \cdot ||$ stands for the distance to the nearest integer). For irrational $\alpha$ put

$$\lambda(\alpha) = \liminf_{t \to +\infty} t \cdot \psi_\alpha(t)$$

and

$$d(\alpha) = \limsup_{t \to +\infty} t \cdot \psi_\alpha(t).$$

The Lagrange spectrum $L$ is defined as

$$L = \{ \lambda \in \mathbb{R} : \text{there exists } \alpha \in \mathbb{R} \text{ such that } \lambda = \lambda(\alpha) \}.$$ 

Here we should note that

$$\lambda \left( \frac{1 + \sqrt{5}}{2} \right) = \frac{1}{\sqrt{5}} = 0.4472^+.$$ 

is the maximal element of $L$. Also we note that Lagrange spectrum has a “discrete part”

$$\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \ldots$$

which is related to Markoff numbers and there exist a whole segment $[0, \lambda^*] \subset L$ which is known as Hall’s ray.

The Dirichlet spectrum $D$ is defined as

$$D = \{ d \in \mathbb{R} : \text{there exists } \alpha \in \mathbb{R} \text{ such that } d = d(\alpha) \}.$$ 

The maximal element of $D$ is 1. The minimal element from $D$ is

$$d \left( \frac{1 + \sqrt{5}}{2} \right) = \frac{1}{2} + \frac{1}{2\sqrt{5}}.$$ 

It was found in [13] (see also [11]). Dirichlet spectrum also has a “discrete part” and there exists a segment $[d^*, 1] \subset D$.

A lot of results related to Lagrange spectra one can find in [4]. Dirichlet spectra was studied in [2], [12], [7], [5], [6]. An interesting survey one can find in [10].

2. Continued fractions.

\[1\text{research is supported by RFBR grant No.12-01-00681-a}\]
For irrational \( \alpha \) represented as a continued fraction
\[
\alpha = [a_0; a_1, a_2, \ldots, a_t, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_t + \cdots}}}, \quad a_0 \in \mathbb{Z}, \ a_j \in \mathbb{N}.
\]
we define
\[
\alpha_\nu = [a_\nu; a_{\nu+1}, \ldots], \quad \alpha^*_\nu = [0; a_\nu, a_{\nu-1}, \ldots, a_1].
\]
We consider continued fraction’s convergents
\[
\frac{p_\nu}{q_\nu} = [a_0; a_1, \ldots, a_\nu].
\]
We should note that
\[
\frac{q_{\nu-1}}{q_\nu} = [0; a_\nu, \ldots, a_1]. \tag{4}
\]
\[
\xi_\nu = ||q_\nu \alpha|| = |q_\nu \alpha - p_\nu|.
\]
It is a well-known fact that for irrational \( \alpha \) the values of \( \lambda(\alpha) \) and \( d(\alpha) \) may be expressed as follows:
\[
\lambda(\alpha) = \lim \inf_{\nu \to \infty} L(\alpha^*_\nu, \alpha_{\nu+1}), \quad \text{where} \quad L(x, y) = \frac{1}{x + y}. \tag{5}
\]
\[
d(\alpha) = \lim \sup_{\nu \to \infty} D(\alpha^*_\nu+1, \alpha_{\nu+2}), \quad \text{where} \quad D(x, y) = \frac{y}{x + y}. \tag{6}
\]

3. Functions in two variables.
In the sequel we consider two functions
\[
G(x, y) = \frac{x + y + 1}{4}
\]
and
\[
F(x, y) = \frac{(1 - xy)^2}{4(1 + xy)(1 - x)(1 - y)} \tag{7}
\]
in variables \( x, y \). It is clear that
\[
\min_{(x,y): x, y \geq 0} G(x, y) = \frac{1}{4}, \quad \max_{(x,y): x + y \leq 1} G(x, y) = \frac{1}{2}. \tag{8}
\]

The following lemma is obvious.

**Lemma 1.** Put
\[
\Omega = \{(x, y) \in \mathbb{R}^2 : \ 0 \leq x, y < 1, \ x + y^{-1} \geq 2, x^{-1} + y \geq 2\}.
\]
Then for the maximal value of \( F(x, y) \) we have
\[
\max_{(x,y) \in \Omega} F(x, y) = \frac{1}{2}
\]
(the maximum here attains at any point satisfying \( x + y^{-1} = 2 \) or \( x^{-1} + y = 2 \)). For the minimal value of \( F(x, y) \) we have
\[
\min_{(x,y) \in \Omega} F(x, y) = F(0, 0) = \frac{1}{4}.
\]
4. A function associated with Minkowski diagonal continued fraction.

Given \( \alpha \in \mathbb{R} \) we define a function \( \mu_\alpha(t) \) which corresponds to Minkowski diagonal continued fraction representation of \( \alpha \) (see [11]). To do this we recall the Legendre theorem on continued fractions. This theorem says that if

\[
\left| \alpha - \frac{A}{Q} \right| < \frac{1}{2Q^2}, \quad (A, Q) = 1
\]

then the fraction \( \frac{A}{Q} \) is a convergent fraction for the continued fraction expansion of \( \alpha \). The converse statement is not true. It may happen that \( \frac{A}{Q} \) is a convergent to \( \alpha \) but (9) is not valid. We consider the sequence of the denominators of the convergents to \( \alpha \) for which (9) is true. Let this sequence be

\[
Q_0 < Q_1 < \cdots < Q_n < Q_{n+1} < \cdots.
\]

Then for \( \alpha \not\in \mathbb{Q} \) the function \( \mu_\alpha(t) \) is defined by

\[
\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n} \cdot ||Q_n\alpha|| + \frac{t - Q_n}{Q_{n+1} - Q_n} \cdot ||Q_{n+1}\alpha||, \quad Q_n \leq t \leq Q_{n+1}.
\]

Here we should note that for every \( \nu \) one of the consecutive convergent fractions \( \frac{p_\nu}{q_\nu} \) to \( \alpha \) satisfies (9). So either

\[
(Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1})
\]

for some \( \nu \) and

\[
||Q_n\alpha|| = \xi_\nu, \quad ||Q_{n+1}\alpha|| = \xi_{\nu+1},
\]

or

\[
(Q_n, Q_{n+1}) = (q_{\nu-1}, q_{\nu+1})
\]

for some \( \nu \) and

\[
||Q_n\alpha|| = \xi_{\nu-1}, \quad ||Q_{n+1}\alpha|| = \xi_{\nu+1},
\]

It is obvious that

\[
\liminf_{t \to +\infty} t \cdot \mu_\alpha(t) = \lambda(\alpha)
\]

where the value \( \lambda(\alpha) \) is defined in (1). Analogously to \( d(\alpha) \) defined in (2) we consider the value

\[
m(\alpha) = \limsup_{t \to +\infty} t \cdot \mu_\alpha(t).
\]

We give here a result analogous to formulas (5) and (6).

**Theorem 1.** Put

\[
m_n(\alpha) = \begin{cases} G(\alpha_\nu, \alpha_{\nu+1}^{-1}), & \text{if } (Q_n, Q_{n+1}) = (q_{\nu-1}, q_{\nu+1}) \text{ with some } \nu, \\ F(\alpha_{\nu+1}, \alpha_{\nu+2}^{-1}), & \text{if } (Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1}) \text{ with some } \nu. \end{cases}
\]

Then

\[
m(\alpha) = \limsup_{n \to +\infty} m_n(\alpha),
\]

We give a proof of Theorem 1 in Sections 6, 7, 8.

For example for

\[
\alpha = \frac{1 + \sqrt{5}}{2} = [1; 1, 1, ...]
\]
we have $Q_n = q_n$ and

$$m\left(\frac{1 + \sqrt{5}}{2}\right) = F\left(\frac{\sqrt{5} - 1}{2}, \frac{\sqrt{5} - 1}{2}\right) = \frac{1}{4} + \frac{1}{2\sqrt{5}} = 0.4736^+,$$

and for

$$\alpha = \sqrt{2} = [1; 2, 2, 2,...]$$

one has

$$m(\sqrt{2}) = F(\sqrt{2} - 1, \sqrt{2} - 1) = \frac{1}{4} + \frac{1}{4\sqrt{2}} = 0.4267^+. \tag{13}$$

But if we consider

$$\alpha = \frac{1 + \sqrt{3}}{2} = [1; 2, 1, 2, 1, 2, 1,...]$$

then

$$m\left(\frac{1 + \sqrt{3}}{2}\right) = G\left(\frac{\sqrt{3} - 1}{2}, \frac{\sqrt{3} - 1}{2}\right) = \frac{\sqrt{3}}{4} = 0.4330^+. \tag{14}$$

Alanogously to Lagrange and Dirichle spectra $L$ and $D$ we consider the set

$$M = \{ m \in \mathbb{R} : \exists \alpha \in \mathbb{R} \text{ such that } m = m(\alpha) \}.$$

**Theorem 2.** For the minimal and the maximal element of $M$ one has

$$\min M = \frac{1}{4}, \quad \max M = \frac{1}{2}. \tag{15}$$

We give a proof of Theorem 2 in Sections 9.

5. Oscillating property.

In [8] it was proved that for any two different irrational numbers $\alpha, \beta$ such that $\alpha \pm \beta \not\in \mathbb{Z}$ the difference function

$$\psi_\alpha(t) - \psi_\beta(t)$$

changes its sign infinitely many times as $t \to +\infty$.

The situation with oscillating property of the difference

$$\mu_\alpha(t) - \mu_\beta(t) \tag{16}$$

is quite different. In [9] the it is shown that there exist real $\alpha$ and $\beta$ such that they are linearly independent over $\mathbb{Z}$ together with 1 and

$$\mu_\alpha(t) > \mu_\beta(t), \quad \forall t \geq 1.$$  

However as it is shown in the same paper [9] for almost all pairs $(\alpha, \beta) \in \mathbb{R}^2$ (in the sense of Lebesgue measure) the difference (16) does oscillate as $t \to \infty$.

The consideration of the values $\lambda(\alpha)$ and $m(\alpha)$ leads obviously to the following result.

**Proposition 1.** Suppose that real $\alpha, \beta$ satisfy

$$m(\alpha) < \lambda(\beta).$$

Then there exists $t_0$ such that

$$\mu_\alpha(t) < \mu_\beta(t), \quad t \geq t_0.$$
In particular, according to \((3,13,14)\) one has
\[
\mu \sqrt{2}(t) < \mu_{\frac{1+\sqrt{5}}{2}}(t), \quad \mu_{\frac{1+\sqrt{3}}{2}}(t) < \mu_{\frac{1+\sqrt{5}}{2}}(t)
\]
for all \(t\) large enough. This example shows that the conjecture from \([9]\) is false. Here we should note that J. Chaika \([3]\) was the first to understand that the conjecture from \([9]\) is false.

We would like to formulate here a positive result on oscillating of the difference \((16)\).

**Theorem 3.** Suppose that \(\alpha\) and \(\beta\) are quadratic irrationalities such that they are linearly independent, together with 1, over \(\mathbb{Z}\) and

\[
\lambda(\beta) < \lambda(\alpha) < m(\beta).
\]

Then the difference \((16)\) changes its sign infinitely often as \(t \to \infty\).

We give a sketch of a proof for Theorem 3 in Section 10.

6. **Identities with continued fractions.**

**Lemma 2.**

(i) The following identities are valid:

\[
q_\nu \xi_\nu = \frac{1}{\alpha_\nu + \alpha_{\nu+1}} = \frac{1}{(\alpha_{\nu+1}^* \alpha_{\nu+2})^{-1} + (\alpha_{\nu+2})^{-1}} = \frac{\alpha_{\nu+1}^* \alpha_{\nu+2}}{\alpha_{\nu+1}^* + \alpha_{\nu+2}}.
\]

\[
\frac{\xi_\nu}{\xi_{\nu+1}} = \alpha_{\nu+2}.
\]

(ii) Suppose that \(a_{\nu+1} = 1\). Then

\[
\frac{\xi_{\nu-1}}{\xi_{\nu+1}} = \alpha_{\nu+2} + 1.
\]

Proof. The first equality from \((18)\) is well known (see \([13]\), Ch.1). To obtain the second one we should note that

\[
\alpha_\nu^* + \alpha_{\nu+2} = \alpha_{\nu}^* + a_{\nu+1} + \frac{1}{\alpha_{\nu+2}} = \frac{1}{\alpha_{\nu+1}^*} + \frac{1}{\alpha_{\nu+2}}.
\]

Equality \((18)\) is proved.

To prove \((19)\) we observe that

\[
\frac{\xi_\nu}{\xi_{\nu+1}} = \frac{q_\nu \xi_\nu}{q_{\nu+1} \xi_{\nu+1}} = (\alpha_{\nu+1}^* + \alpha_{\nu+2}) \times \frac{\alpha_{\nu+1}^* \alpha_{\nu+2}}{\alpha_{\nu+1}^* + \alpha_{\nu+2}} \times \frac{1}{\alpha_{\nu+1}^*} = \alpha_{\nu+2}
\]

(here we use \((1)\) and equalities from \((18)\)).

To prove the statement from (ii) we observe that

\[
\frac{\xi_{\nu-1}}{\xi_{\nu+1}} = \frac{\xi_{\nu-1}}{\xi_{\nu}} \frac{\xi_{\nu}}{\xi_{\nu+1}} = \alpha_{\nu+1} \alpha_{\nu+2}.
\]

Now we take into account that

\[
\alpha_{\nu+1} = 1 + \frac{1}{\alpha_{\nu+2}},
\]

and \((20)\) follows. □

Define

\[
d_{1,\nu} = \sqrt{q_{\nu+1} - q_{\nu-1}}, \quad d_{2,\nu} = \sqrt{q_{\nu+1} - q_{\nu}},
\]

\[
q_{\nu+1} - q_{\nu-1} = \xi_{\nu-1} - \xi_{\nu+1},
\]

\[
q_{\nu+1} - q_{\nu} = \xi_{\nu} - \xi_{\nu+1}.
\]

5
and
\[ M_{1,\nu} = \frac{1}{4} \left( d_{1,\nu} \xi_{\nu-1} + \frac{q_{\nu-1}}{d_{1,\nu}} \right)^2, \quad M_{2,\nu} = \frac{1}{4} \left( d_{2,\nu} \xi_{\nu} + \frac{q_{\nu}}{d_{2,\nu}} \right)^2. \]

**Lemma 3.** Suppose that \( a_{\nu+1} = 1 \). Then
\[ M_{1,\nu} = G(\alpha_{\nu}^*, \alpha_{\nu+2}^{-1}). \quad (21) \]

Proof. Calculations show that
\[ d_{1,\nu}^2 \xi_{\nu-1}^2 = \frac{q_{\nu} q_{\nu-1} \xi_{\nu-1}}{q_{\nu-1} (1 - \xi_{\nu-1})} = \frac{(\alpha_{\nu+2}^{-1} + 1)^2}{\alpha_{\nu} + \alpha_{\nu+2}^{-1} + 1} \quad (22) \]
and
\[ \frac{q_{\nu-1}^2}{d_{1,\nu}^2} = \frac{q_{\nu-1} q_{\nu-1} \xi_{\nu-1}}{q_{\nu} \left( 1 - \frac{\xi_{\nu-1}}{\xi_{\nu-1}} \right)} = \frac{(\alpha_{\nu}^*)^2}{\alpha_{\nu} + \alpha_{\nu+2}^{-1} + 1} \quad (23) \]
(here we use the equality (4), equality (18) for \( q_{\nu-1} \xi_{\nu-1} \) and (20)). Note that from (18) it follows that
\[ q_{\nu-1} \xi_{\nu-1} = \frac{1}{1 + \alpha_{\nu}^* + 1} = \frac{\alpha_{\nu}^* (\alpha_{\nu+2}^{-1} + 1)}{\alpha_{\nu} + \alpha_{\nu+2}^{-1} + 1}. \quad (24) \]

We combine (22, 23, 24) to get (21). Lemma is proved. \( \square \)

**Lemma 4.**
\[ M_{2,\nu} = F(\alpha_{\nu+1}^*, \alpha_{\nu+2}^{-1}). \quad (25) \]

Proof. From the definition of \( M_{2,\nu} \) and (18) we have
\[ M_{2,\nu} = \frac{1}{4 \left( (\alpha_{\nu+1}^*)^{-1} + (\alpha_{\nu+2}^{-1})^{-1} \right)} \left( g + 1 \right)^2, \quad g = \frac{q_{\nu+1}}{q_{\nu}} - 1 \leq \frac{q_{\nu+1}}{q_{\nu+1} \xi_{\nu}} \cdot \frac{q_{\nu}}{q_{\nu+1}} \]
By (4) and (19) we get
\[ g = \frac{(\alpha_{\nu+1}^*)^{-1} - 1}{1 - \left( \frac{\alpha_{\nu+1} + \alpha_{\nu+2}}{\alpha_{\nu+1}} \cdot \alpha_{\nu+1} \right) \cdot \alpha_{\nu+1}^*} = \frac{(\alpha_{\nu+1}^*)^{-1} - 1}{1 - (\alpha_{\nu+2})^{-1}}, \]
and lemma follows by an easy calculation. \( \square \)

7. Inequalities with continued fractions.

**Lemma 5.** If \( a_{\nu+1} = 1 \) then
\[ d_{1,\nu}^2 \cdot \frac{\xi_{\nu+1}}{q_{\nu+1}} \leq 1 \leq d_{1,\nu}^2 \cdot \frac{\xi_{\nu-1}}{q_{\nu-1}}. \quad (26) \]

Proof. As \( a_{\nu+1} = 1 \) by (4) and (20) we see that
\[ \frac{q_{\nu+1} - q_{\nu-1}}{q_{\nu+1}} = \frac{q_{\nu}}{q_{\nu+1}} = \alpha_{\nu+1}^* \leq 1 < \alpha_{\nu+2} = \frac{\xi_{\nu-1} - \xi_{\nu+1}}{\xi_{\nu+1}}. \]
The left inequality from (26) follows. To obtain the right inequality we should use the inequality
\[ 1 < \frac{1}{\alpha_{\nu+1}^*} < \frac{1}{\alpha_{\nu+1}} + \frac{1}{\alpha_{\nu+2} + 1} \]
which by (4,19) leads to
\[ \frac{q^\nu_{+1} - q^\nu_{-1}}{q^\nu_{-1}} = \frac{q^\nu}{q^\nu_{-1}} = \frac{1}{\alpha^*_\nu} > 1 - \frac{1}{\alpha^\nu_{+2} + 1} = \frac{\xi^\nu_{-1} - \xi^\nu_{+1}}{\xi^\nu_{-1}}. \]

This gives right inequality from (26). □

**Lemma 6.** Suppose that \( \alpha^*_\nu + \alpha^\nu_{+1} > 2 \) and \( \alpha^*_\nu + \alpha^\nu_{+2} > 2 \). Then
\[ d^2_{2,\nu} \cdot \frac{\xi^\nu_{+1}}{q^\nu_{+1}} \leq 1 \leq d^2_{2,\nu} \cdot \frac{\xi^\nu}{q^\nu}. \]

(27)

**Proof.** From the conditions of lemma we have
\[ \frac{1}{\alpha^*_\nu + 1} + \frac{1}{\alpha^\nu_{+2}} = \alpha^*_\nu + \alpha^\nu_{+1} > 2. \]

So by (4) and (19) we get
\[ \frac{q^\nu_{+1} - q^\nu}{q^\nu} = \frac{1}{\alpha^*_\nu + 1} - 1 > 1 - \frac{\xi^\nu_{+1}}{\xi^\nu} = \frac{\xi^\nu - \xi^\nu_{+1}}{\xi^\nu} \]

and hence
\[ \frac{\xi^\nu}{q^\nu} > \frac{\xi^\nu - \xi^\nu_{+1}}{q^\nu_{+1} - q^\nu}. \]

The last inequality coincides with the right inequality from (27).

To prove the left inequality from (27) we observe that
\[ \frac{q^\nu_{+1} - q^\nu}{q^\nu_{+1}} = 1 - \alpha^*_\nu + 1 < \alpha^\nu_{+2} - 1 = \frac{\xi^\nu}{\xi^\nu_{+1} - 1} = \frac{\xi^\nu - \xi^\nu_{+1}}{\xi^\nu_{+1}} \]

(here we use the condition \( \alpha^*_\nu + \alpha^\nu_{+2} > 2 \) and (4,19)). □

8. Segments.

In \( \mathbb{R}^2 \) with coordinates \((t, \mu)\) we consider segments
\[ \mathcal{I}_\nu = [A^\nu, A^\nu_{+1}], \quad \mathcal{J}_\nu = [A^\nu_{-1}, A^\nu_{+1}], \quad A^\nu_j = (q^\nu_j, \xi^\nu_j) \in \mathbb{R}^2, \quad j = \nu - 1, \nu, \nu + 1. \]

**Lemma 7.** Suppose that \( a^\nu_{+1} = 1 \). Then
\[ \max_{(t, \mu) \in \mathcal{I}_\nu} t \cdot \mu = G(\alpha^*_\nu, \alpha^{-1}_{\nu_{+2}}). \]

**Proof.** We consider the hyperbolic rotation
\[ D_{1, \nu} = \begin{pmatrix} d_{1, \nu}^{-1} & 0 \\ 0 & d_{1, \nu} \end{pmatrix}. \]

This rotation preserves each hyperbola \( t \cdot \mu = \omega \) and translates the segment \( \mathcal{J}_\nu \) into the segment \([D_{1, \nu}A^\nu_{-1}, D_{1, \nu}A^\nu_{+1}]\) which is orthogonal to the diagonal \( \{ \mu = t \} \) (this is the reason for the choice of the parameter \( d_{1, \nu} \)). By Lemma 5 the endpoints of the segment \([D_{1, \nu}A^\nu_{-1}, D_{1, \nu}A^\nu_{+1}]\) lie by the different sides of the line \( \{ t = \mu \} \). So the maximal value of the form \( t \cdot \mu \) on the segment occurs at the point with \( \mu = t \). Easy calculation shows that this maximum is equal to \( M_{1, \nu} \). Now one should apply Lemma 3 and everything is proved. □
Lemma 8. Suppose that $\alpha_\nu^* + \alpha_{\nu+1} > 2$ and $\alpha_{\nu+1}^* + \alpha_{\nu+2} > 2$. Then

$$\max_{(t, \mu) \in \mathcal{J}_\nu} t \cdot \mu = F(\alpha_{\nu+1}^*, \alpha_{\nu+2}^{-1})$$

Proof. The proof is similar to the proof of Lemma 7. One should consider the hyperbolic rotation

$$D_{2,\nu} = \begin{pmatrix} d_{2,\nu}^{-1} & 0 \\ 0 & d_{2,\nu} \end{pmatrix}$$

applied to the segment $\mathcal{I}_\nu$ and take into account Lemmas 4, 6. □

Now we should note that Theorem 1 immediately follows from the definition of $m(\alpha)$ and Lemmas 7,8.

9. Proof of Theorem 2.

To prove (15) we use equalities (8) for $G(x, y)$, Lemma 1 and Theorem 1. This gives

$$\min \mathbb{M} \geq \frac{1}{4}, \quad \max \mathbb{M} \leq \frac{1}{2}$$

To prove that there are just equalities one should consider examples

$$\alpha^- = [0; a_1, a_2, ..., a_n, ...], \quad a_n \to \infty, \ n \to \infty$$

and

$$\alpha^+ = [0; 1, 1, a_3, 1, 1, a_6, 1, 1, a_9, 1, 1, a_{12}, 1, 1, a_{15}, ...], \quad a_3 < a_6 < a_9 < a_{12} < a_{15} < ...$$

with

$$m(\alpha^-) = \lim_{\nu \to \infty} F(\alpha_{\nu+1}^*, \alpha_{\nu+2}^{-1}) = \frac{1}{4}$$

and

$$m(\alpha^+) = \lim_{\nu \to \infty} F(\alpha_{3\nu+2}^*, \alpha_{3(\nu+1)}^{-1}) = \lim_{\nu \to \infty} G(\alpha_{3\nu+1}^*, \alpha_{3(\nu+1)}^{-1}) = \frac{1}{2}.$$ 

10. Proof of Theorem 3.

From the conditions of Theorem 3 we see that $\beta \not\in \mathbb{Q}(\alpha)$. Consider fundamental units $\varepsilon_\alpha$ and $\varepsilon_\beta$ of the fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\alpha)$, respectively. Then

$$\frac{\log \varepsilon_\alpha}{\log \varepsilon_\beta} \not\in \mathbb{Q}. \quad (28)$$

Given positive $\eta$ one can construct three sequences of integers

$$Q_{\nu}^{[\beta],1}, \ Q_{\nu}^{[\beta],2}, \ Q_{\nu}^{[\alpha]}, \ \nu = 0, 1, 2, 3, ...$$

such that

$$|Q_{\nu}^{[\beta],1} \mu_\beta(Q_{\nu}^{[\beta],1}) - \lambda(\beta)| < \eta$$

and

$$Q_{\nu}^{[\beta],1} = C_{\beta,1} B_1^{\nu}(1 + o(1)), \ \nu \to +\infty,$$

where $C_{\beta,1}$ is positive and $B_1 = \varepsilon_\beta^{b_1}$ with a positive integer $b_1$;

$$|Q_{\nu}^{[\beta],2} \mu_\beta(Q_{\nu}^{[\beta],2}) - m(\beta)| < \eta$$

and

$$Q_{\nu}^{[\beta],2} = C_{\beta,2} B_2^{\nu}(1 + o(1)), \ \nu \to +\infty,$$
where \( C_{\beta,2} \) is positive and \( B_2 = \varepsilon_{\beta}^{b_2} \) with a positive integer \( b_2 \);

\[
|Q^{[\alpha]}_\nu \mu_\alpha(Q^{[\alpha]}_\nu) - \lambda(\alpha)| < \eta
\]

and

\[
Q^{[\alpha]}_\nu = C_\alpha A^{\nu} (1 + o(1)), \nu \to +\infty,
\]

where \( C_\alpha \) is positive and \( A = \varepsilon_\alpha^a \) with a positive integer \( a \).

From (28) we have

\[
\frac{\log B_j}{\log A} \notin \mathbb{Q}, \ j = 1, 2.
\]

By the Kronecker theorem given real \( \omega_2, \omega_1 \) and positive \( \eta \) there exist infinitely many pairs \((\nu_{1,n}, \kappa_{1,n})\) and \((\nu_{2,n}, \kappa_{2,n})\) such that

\[
|\nu_{1,n} \log B_1 - \kappa_{1,n} \log A - \omega_1| < \eta,
\]

and

\[
|\nu_{2,n} \log B_2 - \kappa_{2,n} \log A - \omega_2| < \eta,
\]

With the proper choice of \( \omega_2, \omega_1 \) this gives

\[
\frac{Q^{[\beta]}_1}{Q^{[\alpha]}_{\kappa_{1,n}}} \to 1, \quad \frac{Q^{[\beta]}_2}{Q^{[\alpha]}_{\kappa_{2,n}}} \to 1, \quad n \to \infty.
\]

Simple calculation shows that for small \( \eta \) and for \( n \) large enough one has

\[
\left| \frac{\mu_\beta(Q^{[\beta]}_{\nu_{j,n}})}{\mu_\beta(Q^{[\alpha]}_{\kappa_{j,n}})} - 1 \right| < 2\eta, \quad j = 1, 2.
\]

and

So for \( n \) large enough

\[
\frac{\mu_\beta(Q^{[\alpha]}_{\kappa_{1,n}})}{\mu_\alpha(Q^{[\alpha]}_{\kappa_{1,n}})} < \frac{\lambda(\beta)}{\lambda(\alpha)} (1 + 4\eta) < 1 < \frac{m(\beta)}{\lambda(\alpha)} (1 - 4\eta) < \frac{\mu_\beta(Q^{[\alpha]}_{\kappa_{2,n}})}{\mu_\alpha(Q^{[\alpha]}_{\kappa_{2,n}})}
\]

The last inequalities show that the difference (16) does oscillate. □

References

[1] H. Davenport and W. M. Schmidt, Dirichlet’s theorem on Diophantine approximation, Simposia Mathematica (INDAM, Rome, 1968/69), vol. IV, Academic Press, London 1970, pp. 113 - 132.

[2] B. Divis, B. Novak, A remark on the theory of diophantine approximations, Comment. Math. Univ. Carolinae, 12< No. 1 (1971), 127 - 141.

[3] J. Chaika, private communication, September 2011.

[4] T.W. Cusick, M.E. Flahive, The Markoff and Lagrange spectra, Math. Surveys Monogr., vol. 30, Amer. Math. Soc., Providence, RI 1989.

[5] V.A. Ivanov, Rational approximations of real numbers, Mathematical Notes, 1978, 23:1, 3 - 16.
[6] V.A. Ivanov, Dirichlet’s theorem in the theory of diophantine approximations, Mathematical Notes, 1978, 24:4, 747 - 755.

[7] V.A. Ivanov, Origin of the ray in the Dirichlet spectrum of a problem in the theory of Diophantine approximations, J. Math. Sci. 19:2 (1982), 1169 - 1183.

[8] I.D. Kan, N.G. Moshchevitin, Approximations to two real numbers, Uniform Distribution Theory 5 (2010), no.2, 79 - 86.

[9] I.D. Kan, N.G. Moshchevitin, J. Chaika, On Minkowski diagonal functions for two real numbers, in ’The Proceedings Diophantine Analysis and Related Fields 2011’, M. Amou and M. Katsurada (Eds.), AIP Conf. Proc. No. 1385, pp. 42 - 48 (2011), American Institute of Physics, New York.

[10] A.V. Malyshev, Markov and Lagrange spectra (survey of the literature), J. Math. Sci. 16:1 (1981), 767 - 788.

[11] H. Minkowski, Über die Annäherung an eine reelle Grösse durch ratiolale Zahlen, Math. Ann., 54 (1901), p. 91 - 124.

[12] S. Morimoto, Zur Theorie der Approximation einer irrationalen Zahl durch rationale Zahlen, Tohoku Math. J., 45 (1938), 177 - 187.

[13] W.M. Schmidt, Diophantine approximations, Lect. Notes Math., 785 (1980).

[14] G. Szekeres, On a problem of the lattice-plane, J. London Math. Soc (1937), s 1-12(2), 88 - 93.