First-order action and Euclidean quantum gravity

Tomáš Liko and David Sloan

Institute for Gravitation and the Cosmos, Pennsylvania State University, University Park, PA 16802, USA

E-mail: liko@gravity.psu.edu and sloan@gravity.psu.edu

Received 27 October 2008, in final form 29 May 2009
Published 19 June 2009
Online at stacks.iop.org/CQG/26/145004

Abstract

We show that the on-shell path integral for asymptotically flat Euclidean spacetimes can be given in the first-order formulation of general relativity, without assuming the boundary to be isometrically embedded in Euclidean space and without adding infinite counter-terms. For illustrative examples of our approach, we evaluate the first-order action for the four-dimensional Euclidean Schwarzschild and NUT-charged spacetimes to derive the corresponding on-shell partition functions, and show that the correct thermodynamic quantities for the solutions are reproduced.

PACS number: 04.70.Bw; 04.20.Cv

1. Introduction

To study the thermodynamics of any system from a quantum-mechanical point of view, the quantity of interest is the partition function. In the operator representation, this is

\[ Z = \text{Tr} \left[ \exp \left( -\beta \hat{H} \left[ \phi \right] \right) \right], \] (1)

for a system of fields \( \phi \) at finite temperature \( T = 1/\beta \) with Hamiltonian \( \hat{H} \left[ \phi \right] \). In the path integral representation (1) is equivalent to the expression [1]

\[ Z = \int D[\phi] \exp (-\tilde{I} [\phi]), \] (2)

where \( \tilde{I} [\phi] \) is the Euclidean action.

Evaluation of \( Z \), however, is quite difficult in practice: the integration and the measure \( D[\phi] \) are difficult to construct. The problem is that the integration is over all fields \( [\phi] \), not just the classical fields \( \phi_0 \) that satisfy the equations of motion \( \delta I[\phi_0] = 0 \). For physical applications, however, it is reasonable to expect that the dominant contributions to the partition
function will come from fields that are close to the classical fields, i.e. the stationary-phase approximation. So for a field \( \phi = \phi_0 + \delta \phi \) the action can be expanded in a Taylor series such that

\[
\tilde{I}[\phi_0 + \delta \phi] = \tilde{I}[\phi_0] + \delta \tilde{I}[\phi_0, \delta \phi] + \delta^2 \tilde{I}[\phi_0, \delta \phi] + \cdots .
\]  

(3)

For a generic field \( \phi \) the first term \( \tilde{I}[\phi_0] \) is assumed to be finite, the linear term \( \delta \tilde{I} \) is assumed to vanish, while the quadratic term \( \delta^2 \tilde{I} \) is assumed to be positive definite [2]. If these three properties hold, then the on-shell partition function may be approximated to the expression

\[
Z = \exp(-\tilde{I}[\phi_0]).
\]  

(4)

This form of the partition function is known as the stationary-phase approximation to the path integral (2). The standard quantities of thermodynamics can then be calculated. In particular, the average energy \( \langle E \rangle \) and entropy \( S \) are given by

\[
\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} \quad \text{and} \quad S = \beta \langle E \rangle + \ln Z.
\]  

(5)

The physical meaning of the energy may differ based on the boundary conditions that are used, i.e. holding the pressure or volume constant.

For gravitational objects that satisfy the Einstein equations, the partition function of interest is evaluated for the Einstein–Hilbert action on a \( D \)-dimensional manifold \( \mathcal{M} \) in a bounded region [1, 3, 4]:

\[
I[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} R d^D V + \frac{1}{\kappa} \oint_{\partial \mathcal{M}} K d^{D-1} V,
\]  

(6)

where \( \kappa = 8\pi \) (with \( G_D = 1 \)), \( \partial \mathcal{M} \) is the boundary of \( \mathcal{M} \), \( R \) is the Ricci scalar of the spacetime metric \( g \) and \( K \) is the trace of the extrinsic curvature of the boundary \( \partial \mathcal{M} \), \( d^D V \) is the volume element determined by \( g \) and \( d^{D-1} V \) is the volume element determined by the induced metric \( h \) on \( \partial \mathcal{M} \). The surface term in (6) can be understood to arise in the action principle for general relativity because the Lagrangian density depends on the second derivatives of the metric. As a result the first derivatives of \( g \) in addition to \( g \) itself must be held fixed on \( \partial \mathcal{M} \). This is in contrast to the usual form of variational principles for which only the field itself is held fixed. If \( \mathcal{M} \) is spatially compact then a well-defined variational principle for the action (6) exists. In this case (and in this case only), the partition function is well defined and the stationary-phase approximation gives

\[
Z = \exp(-I[g_0]).
\]  

(7)

From here one can then calculate the average energy and entropy of the spacetime with metric \( g_0 \).

The above prescription for finding the thermodynamic properties of gravitational objects works well if \( \mathcal{M} \) is spatially compact, but contains the seeds of many problems if \( \mathcal{M} \) is asymptotically flat: in the latter case the action (6) is infinite, even in the flat limit. This would imply that the partition function (4) evaluated on asymptotically flat Euclidean spacetimes is ill-defined. The solution (for Lorentzian spacetimes) is to isometrically embed the boundary manifold (\( \partial \mathcal{M}, h \)) in Minkowski spacetime, calculate the extrinsic curvature \( K_0 \) of \( \partial \mathcal{M} \) defined by the Minkowski metric and subtract the resulting quantity from the boundary integral in (6). Thus the action [4]

\[
I[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} R d^D V + \frac{1}{\kappa} \oint_{\partial \mathcal{M}} (K - K_0) d^{D-1} V
\]  

(8)

gives a well-defined action principle for asymptotically flat spacetimes. One may then proceed to find the partition function for asymptotically flat spacetimes such as the Schwarzschild
or Kerr solutions, and hence the thermodynamic quantities of interest. The problem with the infinite subtraction in the action (8) is that the embedding scheme does not work for generic spacetimes in dimensions $D \geq 4$ [1, 5–7]. This is because the Gauss–Codazzi equation (in $D > 3$ dimensions) cannot be generically solved for the extrinsic curvature. In addition, the infinite subtraction method crucially depends on the topology of the system under consideration.

Since the original work of Gibbons and Hawking [4] many proposals for counter-terms have been proposed. See e.g. [2, 5, 8–12]. These counter-term methods resolve the limitations of the infinite subtraction method and have led to an understanding of the thermodynamics of a number of solutions that was not possible before. Physically, however, it is desirable to employ a framework that generically produces finite quantities without the need of adding any counter-terms. As was recently shown [6, 7], the first-order formulation of general relativity based on orthonormal co-frames and Lorentz connections as independent fields does provide such a framework for asymptotically flat spacetimes in $D \geq 4$ dimensions. The purpose of this paper is to show that a partition function can be given for asymptotically flat Euclidean manifolds in this framework.

2. Finiteness of the first-order action

We proceed with our discussion of the properties of the first-order action parallel to that in [7]. The Cartesian coordinates $x^a$ of the flat metric $g_{ab} = \delta_{IJ}^0 e_a^I \otimes e_b^J$ and the associated radial coordinates $(r, \Phi^I)$ (with $r^2 = \delta_{ab} x^a x^b$ and $\Phi^I$ being the standard angular coordinates on hyperboloids defined by $r =$ constant) will be used in asymptotic expansions. Detailed analysis shows that to define the angular momentum one needs $e_a^I$ to admit an expansion to order $D - 2$. Therefore, we will assume that $e_a^I$ can be expanded as

$$e = e_0^I(\Phi) + \frac{D-3}{r} e_1(\Phi) + \frac{D-2}{r^2} e(\Phi) + \mathcal{O}(r^{D-1}),$$

with a reflection symmetric $D-3$-form $e(\Phi)$.

To appropriate leading orders, $A_a^{IJ}$ can be required to be compatible with $e_a^I$ on the boundary $\partial M$ of $M$. This leads us to require that $A_a^{IJ}$ is asymptotically of order $D - 1$,

$$A = A_0(\Phi) + \frac{1}{r} A_1(\Phi) + \cdots + \frac{D-1}{r^{D-1}} A(\Phi) + \mathcal{O}(r^{D-1}).$$

Compatibility of $A$ with $e$ and flatness of $\eta$ enable us to set $A_0 = \cdots = D-3 A = 0$ and express $D-2 A$ as

$$D-2 A_a^{IJ}(\Phi) = 2 r^{D-2} \eta^{(r^3-D_3 e_a^I)}.$$  

We consider a $D$-dimensional Euclidean manifold bounded by two spacelike Cauchy surfaces, $M_1$ and $M_2$, which are asymptotically related by a time translation. In the first-order formulation of general relativity the action is given by (see e.g. [7])

$$I[e, A] = \frac{1}{2\kappa} \int_M \Sigma_{IJ} \wedge \Omega^{IJ} - \frac{1}{2\kappa} \oint_{\partial M} \Sigma_{IJ} \wedge A^{IJ}.$$

This action depends on the co-frame $e^I$ and the $SO(D)$ connection $A^{IJ}$. The co-frame determines the metric $g_{ab} = \delta_{IJ} e_a^I \otimes e_b^J$, $(D-2)$-form $\Sigma_{IJ} = [1/(D-2)!] e_I^K e_J^L \wedge \eta^{KL}$, and
\[ \epsilon^K = e^K_{\mu_0} \wedge \cdots \wedge e_{D-1}^{D} \] is the totally antisymmetric Levi-Civita tensor. The connection determines the curvature 2-form

\[ \Omega^I_J = dA^I_J + A^I_K \wedge A^K_J = \frac{1}{2} R^I_{JKLM} e^K \wedge e^L, \] (13)

with \( R^I_{JKLM} \) as the Riemann tensor. Internal indices \( I, J, \ldots \in \{0, \ldots, D-1\} \) are raised and lowered using the flat metric \( \delta_{IJ} = \text{diag}(1, \ldots, 1) \).

Now, as was explicitly shown in \([6, 7]\), ‘power counting’ arguments are sufficient to show that \( \tilde{I} \) is finite and that \( \delta_{I} = 0 \) on shell. Thus the limitations of the infinite subtraction method discussed in section 1 are resolved as a consequence of the natural boundary conditions for asymptotic flatness in the first-order formalism, without having to add counter-terms. Therefore the on-shell partition function in the first-order framework is

\[ Z = \exp(-I_0[e_0, A_0]). \] (14)

This is precisely the zero-loop contribution to the path integral (4). The partition function (14) is well defined because of two important features of the first-order action:

(i) It does not require, or make any reference to, the embedding of \( \partial M \) in Euclidean space.
(ii) It is finite for asymptotically flat Euclidean manifolds without any counter-terms.

These key properties are important for Euclidean quantum gravity techniques to be applicable to generic spacetimes in dimensions \( D \geq 4 \), regardless of topology.

3. Correspondence between the first-order and second-order boundary terms

An important feature of the boundary term in the action (12) is its inequivalence to the Gibbons–Hawking–York (GHY) boundary term in the action (8) for generic spacetimes with boundaries. This can be understood in the following way. First, fix an internal gauge once and for all such that \( n_a = n_I e^I_a \) is the unit normal to \( \partial M \sim M_1 \cup M_2 \cup \tau_\infty \), and that \( \partial_a n_I = 0 \) on \( M_1 \) and \( M_2 \). Now we note that the first-order boundary term, in components, can be simplified to

\[ \oint_{\partial M} \Sigma_{IJ} \wedge A^{IJ} = 2 \oint_{\partial M} \epsilon^{ai} A_{ai}^J n_J, \] (15)

with \( \epsilon \) being the volume form on \( \partial M \) (for useful identities that make this proof simple, see section 2.3.1 of \([13]\)). Now we note that the extrinsic curvature, in general, is given by

\[ K_{ab} = h^{cd} \nabla_c n_d, \]

\[ = h^{cd} (\partial_a n_d + A_{ai}^J n_J) e_d^I, \] (16)

where we used the relation \( n_a = n_I e^I_a \) and the compatability of \( e^I_a \). Taking the trace of the extrinsic curvature therefore gives the identity

\[ e^{ai} A_{ai}^J n_J = K - e^{ai} \partial_a n_J. \] (17)

It follows that on \( \partial M \) the first-order boundary term is given by

\[ \oint_{\partial M} \Sigma_{IJ} \wedge A^{IJ} = 2 \oint_{\partial M} \epsilon(K - K'_{0}), \] (18)

where we defined the quantity \( K'_{0} = e^{ai} \partial_a n_J \). From this correspondence we can make the following two conclusions:

(i) On \( M_1 \) and \( M_2 \) we have that \( \Sigma_{IJ} \wedge A^{IJ} = 2 \epsilon K \), which is precisely the GHY term.
(ii) On \( \tau_\infty \) the first-order boundary term will coincide with the GHY term in (8) if and only if there exists an isometric embedding of the boundary \( \tau_\infty \) in Euclidean space (i.e. where \( e = 0 e(\Phi) \)).
To summarize, we can recover a second-order action by enforcing compatibility between co-frame and connection in the first-order action. The resulting second-order action reproduces exactly the GHY boundary term if the boundary can be isometrically embedded in Euclidean space, as is required in the GHY prescription. By contrast, no such embedding is required in the first-order framework. This important feature will be illustrated in the following section, where we will reproduce the correct thermodynamical quantities for NUT-charged spacetimes.

We conclude this section by commenting on the gauge invariance of the first-order boundary term in (12). At first glance it may seem that this boundary term breaks gauge invariance because of the explicit dependence of the connection in the integrand. However, this is not the case. In particular, the gauge invariance on $M_1$ and $M_2$ follows trivially from the preceding discussion: on $M_1$ and $M_2$ we have that $\Sigma_{IJ} \wedge A^{IJ} = 2\epsilon K$ which is manifestly gauge invariant. On the other hand, on $\tau_\infty$ the co-frame $e$ tends to $0$ while permissible gauge transformations of $A$ tend to identity (i.e. under infinitesimal gauge transformations). Therefore our boundary term is also gauge invariant on $\tau_\infty$.

4. Partition functions and thermodynamics

As illustrative examples of our formalism, we will now derive the partition functions and hence the thermodynamic quantities for specific solutions to the Einstein equations. For concreteness, we will evaluate the first-order action (12) for the Euclidean Schwarzschild, Taub-NUT and Taub-bolt spacetimes in four dimensions. All three examples are vacuum solutions, so that the corresponding bulk actions do not contribute.

To evaluate the boundary terms, the standard prescription is to evaluate separately the contributions from the inner and outer boundaries by calculating the integrals on constant-$r$ hypersurfaces and taking the limits as $r$ goes to the horizon and to infinity; for all three examples the contribution from the inner limit is zero. Therefore in what follows we will only provide details of the contribution at $\tau_\infty$. In the first-order formalism, the calculation of $\tau_\infty$’s contribution amounts to calculating the $^2A$ contribution to the boundary integral, which can be obtained by expanding the co-frame in powers of $r^{-1}$ and substituting the $^1e$ term into equation (11).

4.1. Schwarzschild solution

The metric for four-dimensional Schwarzschild spacetime with Euclidean time $\tau$ has line element [14]
\[ ds^2 = f(r) \, d\tau^2 + \frac{dr^2}{f(r)} + r^2( d\theta^2 + \sin^2 \theta \, d\phi^2), \]
(19)
with $f(r) = 1 - 2M/r$ and $M$ being the mass of the source. Regularity of the metric at the point singularity $r = 2M$ requires that $\tau$ have a period $\beta = 8\pi M$.

A suitable tetrad of co-frames for this spacetime is given by
\[ e^0 = \sqrt{f} \, dr, \quad e^1 = \frac{1}{\sqrt{f}} \, d\tau, \quad e^2 = r \, d\theta, \quad e^3 = r \sin \theta \, d\phi. \]
(20)
Expanding this tetrad in powers of $r^{-1}$, we find that the only non-zero $^1e$ components are
\[ ^1e_0^0 = -M, \quad ^1e_1^1 = M, \]
(21)
and substituting these into (11) gives
\[ ^2A_0^{01} = M, \quad ^2A_1^{01} = -M. \]
(22)
We therefore find that the Euclidean action is given by
\[ \tilde{I} = \frac{1}{\kappa} \oint_{\tau_0} e_2 e_3 A_{01} \frac{r}{r^2} \partial_1 r \]
\[ = \frac{\beta^2}{16\pi}. \]  
(23)
Substituting this into (14) then gives the partition function
\[ Z = \exp \left( -\frac{\beta^2}{16\pi} \right). \]  
(24)
The thermodynamic quantities can now be calculated. From (5) we have that
\[ \langle E \rangle = M \quad \text{and} \quad S = 4\pi M^2. \]  
(25)
Replacing \( M = r_e / 2 \) and noting that the surface area \( A \) of the horizon is \( A = 4\pi r_e^2 \) we get \( S = A/4 \) as should have been expected.

4.2. NUT-charged solutions

The metric for four-dimensional Taub-NUT spacetime with Euclidean time \( \tau \) has line element
\[ ds^2 = V(r) \left[ d\tau + 2N \cos \theta d\phi \right]^2 + \frac{dr^2}{V(r)} + (r^2 - N^2)(d\theta^2 + \sin^2 \theta d\phi^2), \]  
(26)
with \( V(r) = (r^2 - 2Mr + N^2)/(r^2 - N^2) \) and \( N \) the NUT parameter. Regularity of the metric requires that \( \tau \) has a period \( \beta = 8\pi N \).

A suitable tetrad of co-frames for this spacetime is given by
\[ e_0 = \sqrt{V} d\tau + 2\sqrt{V} N \cos \theta d\phi, \quad e_1 = \frac{1}{\sqrt{V}} dr, \quad e_2 = \sqrt{r^2 - N^2} d\theta, \]
\[ e_3 = \sqrt{r^2 - N^2} \sin \theta d\phi. \]  
(27)
Expanding this tetrad in powers of \( r^{-1} \), we find that the non-zero \( e \) components are
\[ 1e_0^0 = -M, \quad 1e_1^0 = 2N \cos \theta, \quad 1e_1^1 = M, \]  
(28)
and substituting these into (11) gives
\[ 2A_{01} = M, \quad 2A_{01}^1 = 2N \cos \theta, \quad 2A_{01}^3 = -2N \cos \theta, \quad 2A_{01}^2 = -2N \cos \theta. \]  
(29)
The Euclidean action for the Taub-NUT spacetime is therefore given by
\[ \tilde{I} = \frac{1}{\kappa} \oint_{\tau_0} e_2 e_3 A_{01} \frac{r}{r^2} \partial_1 r \]
\[ = 4\pi MN. \]  
(30)
Substituting this into (14) then gives the partition function
\[ Z = \exp \left( -4\pi MN \right). \]  
(31)
This result agrees exactly with the partition function that was computed by Astefanesei et al [12] using the Mann–Marolf counter-term method. The thermodynamic quantities can now be calculated. In particular, substituting \( M = N \) into (31) we find the average energy and entropy for the ‘NUT’ charge given by
\[ \langle E \rangle = N \quad \text{and} \quad S = 4\pi N^2, \]  
(32)
while substituting \( M = 5N/4 \) into (31) we find the average energy and entropy for the ‘bolt’ charge given by
\[ \langle E \rangle = \frac{5N}{4} \quad \text{and} \quad S = 5\pi N^2. \]  
(33)
Therefore we find agreement with previous results due to Mann [9] and Astefanesei et al [12].
5. Discussion

Since the pioneering work of Gibbons and Hawking [4] on the Euclidean path integral methods of black-hole thermodynamics, calculations of partition functions have been almost exclusively done in the second-order formalism. In this formalism, however, the semiclassical approximation of considering small perturbations around a classical asymptotically flat solution faces two obstacles: (1) the action is not finite, even on-shell; and (2) the linear term in the perturbation need not vanish. These two problems can both be solved by adding counter-terms to the action [2, 5, 8–12], but these are model-specific post hoc additions.

By contrast, the first-order action is finite on-shell under the natural boundary conditions arising from asymptotic flatness [6, 7] and does not require the boundary to be isometrically embedded in Euclidean space. We have shown here that the Euclidean path integral in the first-order formalism yields a well-defined partition function without the need of adding by hand any counter-terms.

The first-order action, when evaluated on the Euclidean Schwarzschild, Taub-NUT and Taub-bolt solutions, agrees exactly with the counter-term methods in the second-order formalism. In turn, the corresponding partition functions for these solutions in the first-order and second-order frameworks are identical. The simplified manner by which this was achieved in the first-order formalism relative to the second-order formalism suggests that this provides a more solid basis for quantum theory.

Acknowledgments

We would like to thank Abhay Ashtekar and Andrew Randono for discussions. We also thank Abhay Ashtekar for suggesting the initial direction for the work presented here, and for commenting on an earlier version of the manuscript. Finally, we thank Mohammad Akbar, Daniel Grumiller and Robert McNees for important questions and comments. This work was supported in part by NSERC (TL), a Frymoyer scholarship (DS), NSF grant PHY0854743, The George A and Margaret M Downsbrugh Endowment and the Eberly research funds of Penn State.

References

[1] Hawking S W 1979 The path-integral approach to quantum gravity Quantum Gravity. An Einstein Centenary Survey ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
[2] Grumiller D and McNees R 2007 Thermodynamics of black holes in two (and higher) dimensions J. High Energy Phys. JHEP04(2007)074
[3] York J W Jr 1972 Role of conformal three geometry in the dynamics of gravitation Phys. Rev. Lett. 28 1082
[4] Gibbons G W and Hawking S W 1977 Action integrals and partition functions in quantum gravity Phys. Rev. D 15 2752
[5] Mann R B and Marolf D 2006 Holographic renormalization of asymptotically flat spacetimes Class. Quantum Grav. 23 29277
[6] Ashtekar A, Engle J and Sloan D 2008 Asymptotics and Hamiltonians in a first-order formalism Class. Quantum Grav. 25 095020
[7] Ashtekar A and Sloan D 2008 Action and Hamiltonians in higher dimensional general relativity: first-order framework arXiv:0808.2069 [gr-qc]
[8] Lau S R 1999 Light-cone reference for total gravitational energy Phys. Rev. D 60 104034
[9] Mann R B 1999 Misner string entropy Phys. Rev. D 60 104047
[10] Kraus P, Larsen F and Siebelink R 1999 The gravitational action in asymptotically ADS and flat spacetimes Nucl. Phys. B 563 259
[11] Mann R B, Marolf D and Virmani A 2006 Covariant counter-terms and conserved charges in asymptotically flat spacetimes Class. Quantum Grav. 23 6357
[12] Astefanesei D, Mann R B and Stelea C 2007 Note on counterterms in asymptotically flat spacetimes Phys. Rev. D 75 024007

[13] Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report Class. Quantum Grav. 21 R53

[14] Gibbons S W and Hawking S W 1979 Classification of gravitational instanton symmetries Commun. Math. Phys. 66 291