Total insecurity of communication via strong converse for quantum privacy amplification

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Abstract

Quantum privacy amplification is a central task in quantum cryptography. Given shared randomness, which is initially correlated with a quantum system held by an eavesdropper, the goal is to extract uniform randomness which is decoupled from the latter. The optimal rate for this task is known to satisfy the strong converse property and we provide a lower bound on the corresponding strong converse exponent. In the strong converse region, the distance of the final state of the protocol from the desired decoupled state converges exponentially fast to its maximal value, in the asymptotic limit. We show that this necessarily leads to totally insecure communication by establishing that the eavesdropper can infer any sent messages with certainty, when given very limited extra information. In fact, we prove that in the strong converse region, the eavesdropper has an exponential advantage in inferring the sent message correctly, compared to the achievability region. Additionally we establish the following technical result, which is central to our proofs, and is of independent interest: the smoothing parameter for the smoothed max-relative entropy satisfies the strong converse property.

1 Introduction

Finding optimal rates of information-theoretic tasks, such as compression of information for efficient storage, transmission of information through noisy quantum channels, privacy amplification and entanglement manipulation, are of fundamental interest in quantum information theory. Depending on the specific task at hand, the optimal rate is either an optimal gain, quantifying the maximum rate at which a desired target resource\(^1\) can be produced in the process, or an optimal cost, quantifying the minimum rate at which an available resource is consumed in accomplishing the task. Any rate which lies below (resp. above) the optimal gain (resp. cost) is said to be an achievable rate. This is because, for any such rate, there is a corresponding protocol for accomplishing the task successfully, i.e., such that the error, \(\varepsilon_n\), incurred in the protocol for \(n\) successive uses of the underlying resource vanishes in the so-called asymptotic limit \((n \to \infty)\). In contrast, for protocols with non-achievable rates, the error does not vanish asymptotically. The optimal rate of an information-theoretic task is said to satisfy the strong converse property if any sequence of protocols with a non-achievable rate fails with certainty in the asymptotic limit. That is, the

\(^{1}\)A quantum information source, a noisy quantum channel, shared-entanglement or randomness, are examples of resources in quantum information theory.
incurred error, $\varepsilon_n$, is not only bounded away from zero but necessarily converges to one in the asymptotic limit. Often this convergence can be shown to be exponential in $n$, i.e.

$$\varepsilon_n \sim 1 - 2^{-rn}$$

for some positive constant $r$. In this case the smallest such constant is called the strong converse exponent of the task.

The above mentioned optimal rates are typically evaluated in the so-called asymptotic, memoryless setting, namely, a setting in which one assumes that (i) there is no correlation between successive uses of the underlying resource and (ii) the latter is available for arbitrarily many uses; moreover, one demands that the error vanishes in the asymptotic limit. The optimal rates are given by entropic quantities stemming from the quantum relative entropy $[21]$, which hence serves as a parent for these quantities. In contrast, in the so-called one-shot or finite blocklength setting, one considers just a single use (resp. a finite number of uses) of the resource, and hence it is unrealistic to demand that the error vanishes. So one allows for a small but non-zero error (say, $\varepsilon \in (0, 1)$). The analogous entropic quantities in this case are given by smoothed entropic functions, stemming from corresponding parent quantities which are smoothed generalised divergences, with the smoothing parameter being given by the error threshold $\varepsilon$. Examples of such divergences are the smoothed max- and min-relative entropies$^2$. For any $\varepsilon \in (0, 1)$, the smoothed divergence between two tensor power states (e.g. $\rho^{\otimes n}$ and $\sigma^{\otimes n}$) has been shown to reduce to the quantum relative entropy, $D(\rho\|\sigma)$, in the limit $n \to \infty$. This important property is called the quantum asymptotic equipartition property (AEP). This property and the fact that the smoothing parameter corresponds to an error, leads to a natural definition of a strong converse exponent of a smoothed generalised divergence, without reference to any information-theoretic task. For example, let us consider the smoothed max-relative entropy, $D_{\text{max}}^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n})$, of a pair of tensor power (or i.i.d.) states $\rho^{\otimes n}$ and $\sigma^{\otimes n}$. It is known that if this divergence is constrained to be greater than $nr$ for any fixed $r > D(\rho\|\sigma)$, then the smoothing parameter is known to vanishes asymptotically as $n \to \infty$. Recently, the exact exponent with which it vanishes, was evaluated by Li, Yao and Hayashi $[12]$.$^3$ In this paper, we show that in contrast, if one constrains $D_{\text{max}}^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n})$ to be less $nr$ for some $r < D(\rho\|\sigma)$, then the smoothing parameter converges to one exponentially. We call the exact exponent of this convergence the strong converse exponent of the smoothed max-relative entropy, and provide a lower bound on it which is tight if $\rho$ and $\sigma$ commute.

In this paper we focus on the task of quantum privacy amplification (also known as randomness extraction) which is of central importance in quantum cryptography. In it two distant parties (say, Alice and Bob) initially share some randomness, given by a random variable $X$, which is only partially secure, in the sense that an adversary (say, Eve) holds a system, $E$, which is correlated with $X$. The aim of Alice and Bob is to distil uniform randomness (or shared secret key) from $X$, using one-way communication, such that the resulting key is uncorrelated with Eve’s system, and is hence secure. This key can then be employed for secure communication between Alice and Bob. The case in which Eve’s system $E$ is a classical random variable, was studied in $[4, 8, 3]$. The case in which Eve is a quantum adversary, that is, her system $E$ is a quantum system, was studied in $[9, 16, 17]$. The optimal rate of privacy amplification (or secret key distillation) in both these cases, in the asymptotic memoryless setting, was shown to be given by the conditional entropy $H(X|E)$. Any attempt to distil uniform randomness at rates greater than this conditional entropy, fails with

$^2$This is also known as the hypothesis testing relative entropy $[22]$.

$^3$This was done in the case in which the smoothed divergence is defined in terms of the purified distance.
certainty, with the error incurred in the protocol converging exponentially to one. In this paper, we evaluate the strong converse exponent for this task, i.e. the speed of this exponential convergence, in the case in which Eve is a quantum adversary.

As mentioned above, a privacy amplification protocol is considered to be successful if the final shared randomness between Alice and Bob is close to uniform, and Eve has negligible information about it. Mathematically, in the asymptotic memoryless setting, one requires that the distance between the final state of the protocol and a decoupled state, in which the classical system is completely mixed (and hence corresponds to uniform randomness) and is uncorrelated with Eve’s system, vanishes in the asymptotic limit. It is clear that this ensures that the generated key is secure and hence can be used subsequently for secure communication between Alice and Bob. In contrast, in the strong converse region, this distance converges exponentially to one. Hence one would expect that if Alice employs keys which are generated at an asymptotic rate greater than $H(X|E)$ to send messages to Bob, then the communication is totally insecure. In this paper we additionally provide a precise mathematical meaning to the notion of total insecurity by establishing that with very little extra information Eve can infer Alice’s messages with certainty in the asymptotic limit. Thus we prove a strong converse theorem for secure communication.

1.1 The setup of quantum privacy amplification

Let the random variable $X \sim p_x$, $x \in \mathcal{X}$, denote the common shared randomness between two distant parties (Alice and Bob) at the start of a privacy amplification process. It is initially correlated with a quantum system, $E$, held by an eavesdropper (Eve). The initial state of the process is hence represented by the classical-quantum (c-q) state
\[
\rho_{XE} = \sum_{x \in \mathcal{X}} p_x |x\rangle \langle x| \otimes \rho_x^E.
\] (1)

Alice and Bob apply a (hash) function $f : \mathcal{X} \rightarrow \mathcal{Z}$, with $|\mathcal{Z}| < |\mathcal{X}|$, with the aim of extracting a random variable $Z$ which is uniformly distributed and independent of the state of $E$. The goal is to

(i) maximize the size of the extracted randomness, given by $|\mathcal{Z}|$, and
(ii) minimize the conversion error, i.e., the distance between the resulting state, $\rho_{fZE}^f$, of the protocol and the desired decoupled state $\frac{|\mathcal{Z}|}{|\mathcal{Z}|} \otimes \rho_E$. Here we have denoted Eve’s average state by $\rho_E = \sum_x p_x \rho_x^E$.

We denote by $l^\varepsilon(X|E)$ the largest key length, $\log |\mathcal{Z}|$, for which the conversion error can be made less than or equal to a fixed $\varepsilon \geq 0$. In the so-called asymptotic i.i.d. setting, $n$ copies of the c-q state (1) are available and one is interested in the asymptotic behaviour of the key length $l^\varepsilon(X^n|E^n)$ as $n$ goes to infinity. It is well-known [16, 10] that the optimal key rate is given by the conditional entropy of the state $\rho_{XE}$, i.e.,
\[
\lim_{n \rightarrow \infty} \frac{l^\varepsilon(X^n|E^n)}{n} = H(X|E),
\] (2)
for all $0 < \varepsilon < 1$. Hence, $H(X|E)$ has the operational interpretation in the context of privacy amplification of being the optimal rate at which secret keys can be generated which are uniformly distributed and independent of Eve’s system in the asymptotic limit.

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\footnote{This is a classical-quantum state with the classical part corresponding to the final shared randomness and the quantum part is with Eve.}
From (2) we can already infer the strong converse property of privacy amplification: If Alice and Bob apply a hash function $f : X^n \rightarrow Z_n$ with $|Z_n| = 2^{nR}$ and the privacy amplification rate, $R$, satisfies $R > H(X|E)$, then the conversion error necessarily tends to one in the asymptotic limit.

## 2 Main results

Below we summarize the main results of our paper.

- **Strong converse exponent of privacy amplification:** We prove that the conversion error for a privacy amplification scheme with rate $R > H(X|E)$ converges to 1 exponentially fast, in the asymptotic limit. Moreover, we provide a lower bound on the corresponding strong converse exponent\(^5\). In particular, if $Z_n$ is the output classical register with $|Z_n| = 2^{nR}$, and $\Delta_P(X^n \rightarrow Z_n)$ is the minimal conversion error, i.e. smallest attainable (purified) distance of the final state from the desired decoupled state, we show that the strong converse rate, $sc_P(R)$, satisfies:

$$
sc_P(R) := \liminf_{n \rightarrow \infty} \frac{-\log (1 - \Delta_P(X^n \rightarrow Z_n))}{n} \geq \sup_{0<\alpha<1} (1 - \alpha) (R - H_\alpha(X|E)),
$$

with $H_\alpha(X|E) := \frac{1}{1-\alpha} \log \text{Tr}(\rho^\alpha_X(1_X \otimes \rho_E)^{1-\alpha})$ being the $\alpha$-conditional Rényi entropy.

- **Strong converse of secure communication:** Most of the literature on privacy amplification was concerned with establishing security proofs showing that if the distance to the decoupled state can be made small using a suitable hash function, the probability with which Eve can guess the generated key is also small. On the other hand, the implications of large distance to decoupled state, as is the case in the strong converse region discussed above, are less obvious.

In order to clarify in what sense large conversion error implies insecurity in the communication with the generated key, we consider Alice encrypting a message using the key, and evaluate the extra side information Eve needs to have in order to guess the message with certainty. We consider the scenario in which Alice chooses her message from some subset $M_n \subset Z_n$. Here the set $M_n$ is known by Eve and its size determines Eve’s additional side information: Eve’s uncertainty about Alice’s message increases with the cardinality $|M_n|$.

We show below that whenever the minimal conversion error is strictly smaller than 1, Eve needs to have very strong additional side information about the set of messages, i.e. $|M_n|$ needs to be finite (uniformly in $n$) in order for Eve to be able to guess Alice’s message with certainty.

In contrast, in the strong converse region, in which the conversion error approaches 1 exponentially fast, Eve can infer the message even for sets of messages $M_n$ which grow exponentially.

\(^5\)Note that bounds on the strong converse exponent for privacy amplification were also obtained in [11] under a different choice of the figure of merit. Moreover, note that in a concurrent and independent work [18], a comparable bound for the average conversion error (over strongly 2-universal hash functions) in trace distance is obtained. See Section 5 for more details.
in $n$. In fact, we show that almost all\footnote{Here 'almost all' means that the proportion of subsets $M_n$ with size constraint $|M_n| \ll 2^{n \text{sc} P(R)/2}$ for which the statement holds approaches 1 as goes to $n \to \infty$.} sets with size $|M_n| \ll 2^{n \text{sc} P(R)/2}$ are such that if Alice picks a message $M$ out of $M_n$, the probability with which Eve is able to guess $M$ correctly approaches 1 as $n \to \infty$. Moreover, this convergence can be shown to happen exponentially fast.

These results imply that any communication from Alice to Bob, using keys generated at a rate $R > H(X|E)$, is totally insecure, in the following sense: Eve, with very limited additional information about the set $M_n \subset X_n$ from which Alice chooses her messages, is able to perfectly infer her messages in the asymptotic limit.

- **Strong converse exponent for smoothed max-relative entropy**: As a technical result which is used in the proof of (3), we show that the smoothing parameter in the smoothed max-relative entropy $D_{\text{max}}^e(\rho\|\sigma)$ of two states $\rho, \sigma$ necessarily converges to 1 in the asymptotic limit, if one demands that the quotient $D_{\text{max}}^e(\rho^{\otimes n}\|\sigma^{\otimes n})/n$ is less than the relative entropy $D(\rho\|\sigma)$. We call this the strong converse property of the smoothed max-relative entropy. Moreover, we establish that this convergence happens exponentially fast and provide lower bounds on the corresponding strong converse exponent which can be shown to be tight in the case of commuting states. This result can be seen as the corresponding converse statement of the recent result [12, Theorem 6] of Li, Yao and Hayashi. For the proof of this result we employ the quantum Hoeffding bound [7, 14, 2] from binary quantum hypothesis testing.

## 3 Preliminaries

### 3.1 Purified distance and generalised trace distance

Let $\mathcal{H}$ be a finite dimensional Hilbert space. We will denote the set of positive semi-definite operators on $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$ and moreover by $S_{\leq}(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H}) | \text{Tr}(\rho) \leq 1\}$ and $S(\mathcal{H}) = \{\rho \in \mathcal{P}(\mathcal{H}) | \text{Tr}(\rho) = 1\}$ the set of sub-normalised and normalised states respectively. For $\rho, \sigma \in S_{\leq}(\mathcal{H})$ the generalised trace distance is defined as

$$D(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1 + \frac{1}{2}|\text{Tr}(\rho - \sigma)|.$$

Note that it can be expressed as

$$D(\rho, \sigma) = \max_{0 \leq \Lambda \leq 1} |\text{Tr} (\Lambda(\rho - \sigma))|.$$  \hfill (4)

Moreover, the purified distance [19] is defined as

$$P(\rho, \sigma) := \sqrt{1 - F(\rho, \sigma)^2},$$

where

$$F(\rho, \sigma) := \text{Tr} \left( \sqrt{\sqrt{\rho} \sqrt{\sigma} \sqrt{\rho}} + \sqrt{1 - \text{Tr}(\rho)}(1 - \text{Tr}(\sigma)) \right).$$
is the generalised fidelity. Both the generalised trace distance as well as the purified distance are metrics on the set \( S_{\leq}(\mathcal{H}) \). From the Fuchs-van de Graaf inequalities [6], the relations between the generalised trace distance and the purified distance follow [19, Lemma 6]:

\[
D(\rho, \sigma) \leq P(\rho, \sigma) \leq \sqrt{2D(\rho, \sigma)}.
\] (5)

### 3.2 Quantum relative entropies

The quantum relative entropy [21] of \( \rho \in S_{\leq}(\mathcal{H}) \) with respect to \( \sigma \in \mathcal{P}(\mathcal{H}) \) is defined as

\[
D(\rho \parallel \sigma) := \text{Tr} \left( \rho \left( \log \rho - \log \sigma \right) \right).
\]

Moreover, for \( \alpha \in (0,1) \cup (1,\infty) \) the Petz Rényi relative entropy [15] of order \( \alpha \) is defined as

\[
D_\alpha(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left( \rho^\alpha \sigma^{1-\alpha} \right)
\]

and the sandwiched Rényi relative entropy [13, 23] of order \( \alpha \) as

\[
\tilde{D}_\alpha(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left( \sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}} \right).
\]

The max-relative entropy [5] is defined as

\[
D_{\text{max}}(\rho \parallel \sigma) := \inf \left\{ \lambda \mid \rho \leq 2^\lambda \sigma \right\}.
\]

These relative entropies fulfill the following relations

\[
\tilde{D}_\alpha(\rho \parallel \sigma) \leq D_\alpha(\rho \parallel \sigma), \quad \lim_{\alpha \to 1} \tilde{D}_\alpha(\rho \parallel \sigma) = \lim_{\alpha \to 1} D_\alpha(\rho \parallel \sigma) = D(\rho \parallel \sigma), \quad \lim_{\alpha \to \infty} \tilde{D}_\alpha(\rho \parallel \sigma) = D_{\text{max}}(\rho, \sigma).
\]

Moreover, both the Petz- and sandwiched Rényi relative entropies are monotonically increasing in the parameter \( \alpha \).

For \( \varepsilon \geq 0 \) and \( d \) denoting a metric on \( \mathcal{P}(\mathcal{H}) \) we define the smoothed max-relative entropy [5] to be

\[
D_{\text{max}}^{\varepsilon,d}(\rho \parallel \sigma) := \inf_{\tilde{\rho} \in B_\varepsilon^d(\rho)} D_{\text{max}}(\tilde{\rho} \parallel \sigma).
\]

Here, we have denoted the ball of sub-normalised states with radius \( \varepsilon \) around \( \rho \) by \( B_\varepsilon^d(\rho) = \{ \tilde{\rho} \in S_{\leq}(\mathcal{H}) \mid d(\tilde{\rho}, \rho) \leq \varepsilon \} \). We will consider the metric \( d \) to be either the generalised trace distance or the purified distance, i.e. \( d \in \{ D, P \} \).

Using the above definitions of the various relative entropies, we can define the corresponding conditional entropies for a bipartite state \( \rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \) as follows

\[
H(A \mid B) := -D(\rho_{AB} \parallel 1_A \otimes \rho_B), \quad H_\alpha(A \mid B) := -D_\alpha(\rho_{AB} \parallel 1_A \otimes \rho_B),
\]

\[
\tilde{H}_\alpha(A \mid B) := -\tilde{D}_\alpha(\rho_{AB} \parallel 1_A \otimes \rho_B), \quad H_{\text{max}}^{\varepsilon,d}(A \mid B) := -D_{\text{max}}^{\varepsilon,d}(\rho_{AB} \parallel 1_A \otimes \rho_B).
\]

6
3.3 The Hoeffding bound of binary quantum hypothesis testing

In binary hypothesis testing the task is to discriminate between two quantum states $\rho$ and $\sigma$ using a measurement, i.e. a POVM $\{\Lambda, 1 - \Lambda\}$ with $0 \leq \Lambda \leq 1$. Here, we interpret $\text{Tr}((1 - \Lambda)\rho)$ as the type I error probability, i.e. the probability that $\rho$ was wrongly inferred to be $\sigma$, and $\text{Tr}(\Lambda\sigma)$ as the type II error probability, i.e. the probability that $\sigma$ was wrongly inferred to be $\rho$.

In the asymptotic i.i.d. setting with $n$ copies of the states available, assuming $\rho \neq \sigma$, both type I and type II error probabilities can be made exponentially small in $n$ choosing a suitable measurement. Here, for $s \geq 0$ the optimal optimal type I error exponent under the constraint that the type II error exponent is greater or equal to $s$ is given by

$$B(s|\rho\|\sigma) := \sup_{\{\Lambda_n\} \in \mathcal{P}_{\text{i.i.d.}}} \left\{ \liminf_{n \to \infty} \frac{-\log \left(\frac{\text{Tr}(\Lambda_n \rho^\otimes n)}{n}\right)}{n} \right\} \geq s.$$  

This quantity is called the quantum Hoeffding bound and it has been proven for $s > 0$ that $B(s|\rho\|\sigma)$ can be expressed using the Petz Rényi relative entropy $[7, 14, 2]$

$$B(s|\rho\|\sigma) = \sup_{0 \leq \alpha \leq 1} \frac{\alpha - 1}{\alpha} (s - D_\alpha(\rho\|\sigma)).$$  

(7)

For the proof of our Theorem 2 we will for $r \in \mathbb{R}$ fixed use the quantity

$$s_r := \sup_{0 \leq \alpha \leq 1} (\alpha r - (\alpha - 1)D_\alpha(\rho\|\sigma)),$$

which can be easily seen to be the unique solution to the equation $B(s|\rho\|\sigma) = s - r$. It can be shown that the so-called the Neyman-Pearson test is optimal in (6), which is summarised in the following lemma (see e.g. [1, Theorem 1.4]7)

**Lemma 1 (Asymptotics of Neyman-Pearson tests)** Let $r \in \mathbb{R}$ and $s_r$ be such that $B(s_r|\rho\|\sigma) = s_r - r$. Denote by $T_n = \{\rho^\otimes n \leq 2^{nr}\sigma^\otimes n\}$ the Neyman-Pearson test, i.e. the projector onto the non-negative subspace of $2^{nr}\sigma^\otimes n - \rho^\otimes n$. Then

$$\lim_{n \to \infty} \frac{-\log \text{Tr}(T_n \rho^\otimes n)}{n} = B(s_r|\rho\|\sigma), \quad \lim_{n \to \infty} \frac{-\log \text{Tr}((1 - T_n)\sigma^\otimes n)}{n} = s.$$  

(8)

4 Strong converse exponent in smoothed max-relative entropy

For $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{P}(\mathcal{H})$, $r \in \mathbb{R}$ and $d \in \{D, P\}$ we define by $\varepsilon^d(\rho\|\sigma, r)$ the optimal, i.e. smallest exponent such that the corresponding smoothed max-relative entropy is less or equal to $r$, i.e. as

$$\varepsilon^d(\rho\|\sigma, r) = \inf \left\{ d(\rho, \tilde{\rho}) \left| \tilde{\rho} \leq 2^r\sigma, \tilde{\rho} \in \mathcal{S}_r(\mathcal{H}) \right. \right\} = \inf \left\{ \varepsilon \geq 0 \left| D_{\max}^{\varepsilon,d}(\rho\|\sigma) \leq r \right. \right\}.$$  

(9)

In their recent work [12], Li, Yao and Hayashi proved that for $r > D(\rho\|\sigma)$ the optimal exponent in the i.i.d. setting, $\varepsilon^P(\rho^\otimes n\|\sigma^\otimes n, nr)$ converges exponentially fast to 0 as $n \to \infty$. Moreover, they found that the corresponding exponential rate is given by

$$\lim_{n \to \infty} \frac{-\log \varepsilon^P(\rho^\otimes n\|\sigma^\otimes n, nr)}{n} = \sup_{\alpha > 1} \frac{\alpha - 1}{2} (r - \tilde{D}_\alpha(\rho\|\sigma)).$$  

(10)

7Note that in [1] the divergence $H_s(\rho\|\sigma)$ satisfies in our convention $H_s(\rho\|\sigma) = B(s|\sigma\|\rho)$. 

7
Here, we are interested in the behaviour of the exponents $\varepsilon^D(\rho^{\otimes n}\|\sigma^{\otimes n}, nr)$ in the region $r < D(\rho\|\sigma)$. We will see that in this case the exponents converge exponentially fast to 1 and by that establishing the corresponding strong converse property. Moreover, we provide a lower bound on the exponential rate of convergence.

**Theorem 2** For $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{P}(\mathcal{H})$ and $r \in \mathbb{R}$ we have

$$\liminf_{n \to \infty} \frac{- \log(1 - \varepsilon^D(\rho^{\otimes n}\|\sigma^{\otimes n}, nr))}{n} \geq \sup_{0 \leq \alpha \leq 1} (\alpha - 1) (r - D_\alpha(\rho\|\sigma)), \quad (11)$$

$$\liminf_{n \to \infty} \frac{- \log(1 - \varepsilon^P(\rho^{\otimes n}\|\sigma^{\otimes n}, nr))}{n} \geq \sup_{0 \leq \alpha \leq 1} (\alpha - 1) (r - D_\alpha(\rho\|\sigma)). \quad (12)$$

Moreover, in the case in which $\rho$ and $\sigma$ commute, we have

$$\lim_{n \to \infty} \frac{- \log(1 - \varepsilon^D(\rho^{\otimes n}\|\sigma^{\otimes n}, nr))}{n} = \sup_{0 \leq \alpha \leq 1} (\alpha - 1) (r - D_\alpha(\rho\|\sigma)). \quad (13)$$

Note that indeed $\sup_{0 \leq \alpha \leq 1} (\alpha - 1) (r - D_\alpha(\rho\|\sigma))$ is positive if and only if $r < D(\rho\|\sigma)$ which hence establishes the strong converse property of the exponent of the smoothed max-relative entropy.

**Proof of Theorem 2.** We first show the achievability bound (11). Note that (12), which involves the purified distance, then follows immediately since using (5) we have the inequality $\varepsilon^D(\rho\|\sigma, r) \leq \varepsilon^P(\rho\|\sigma, r)$.

Let now $\tilde{\rho}_n \in \mathcal{S}_\leq(\mathcal{H}^{\otimes n})$ be such that $\tilde{\rho}_n \leq 2^{nr} \sigma^{\otimes n}$. Hence, by (4)

$$1 - D(\rho^{\otimes n}, \tilde{\rho}_n) = \min_{0 \leq \Lambda_n \leq 1} (1 - |\text{Tr}(\Lambda_n(\rho^{\otimes n} - \tilde{\rho}_n))|)$$

$$= \min_{0 \leq \Lambda_n \leq 1} \min \{1 - \text{Tr}(\Lambda_n(\rho^{\otimes n} - \tilde{\rho}_n)), 1 - \text{Tr}(\Lambda_n(\tilde{\rho}_n - \rho^{\otimes n}))\}$$

$$= \min_{0 \leq \Lambda_n \leq 1} (\text{Tr}(\rho^{\otimes n}(1 - \Lambda_n)) + \text{Tr}(\tilde{\rho}_n \Lambda_n))$$

$$\leq \min_{0 \leq \Lambda_n \leq 1} \left(\text{Tr}(\rho^{\otimes n}(1 - \Lambda_n)) + 2^{nr} \text{Tr}(\sigma^{\otimes n} \Lambda_n)\right). \quad (14)$$

Now, by (7) there exists for all $s, \varepsilon > 0$ a sequence $0 \leq \Lambda_n(s, \varepsilon) \leq 1$ such that

$$\liminf_{n \to \infty} \frac{- \log(\text{Tr}(\sigma^{\otimes n} \Lambda_n(s, \varepsilon))))}{n} \geq s, \quad \liminf_{n \to \infty} \frac{- \log(\text{Tr}(\rho^{\otimes n}(1 - \Lambda_n(s, \varepsilon))))}{n} \geq B(s\|\sigma) - \varepsilon.$$

Choosing $s = s_r$ where

$$s_r := \sup_{0 \leq \alpha \leq 1} (\alpha r - (\alpha - 1) D_\alpha(\rho\|\sigma))$$

and noting that $s_r$ is the unique solution to the equation $B(s\|\sigma) = s - r$, we get for all $\varepsilon > 0$

$$\liminf_{n \to \infty} \frac{- \log(2^{nr} \text{Tr}(\sigma^{\otimes n} \Lambda_n(s_r, \varepsilon))))}{n} \geq s_r - r, \quad \liminf_{n \to \infty} \frac{- \log(\text{Tr}(\rho^{\otimes n}(1 - \Lambda_n(s_r, \varepsilon))))}{n} \geq s_r - r - \varepsilon.$$
Therefore, using now (14) we get
\[
\liminf_{n \to \infty} -\log \left(1 - D(\rho^{\otimes n}, \tilde{\rho}_n)\right) \geq \liminf_{n \to \infty} \frac{-\log \left(\text{Tr} \left(\rho^{\otimes n} (1 - \Lambda_n(s_r, \varepsilon))\right) + 2nr \text{Tr} \left(\sigma^{\otimes n} \Lambda_n(s_r, \varepsilon)\right)\right)}{n} \\
\geq s_r - r - \varepsilon.
\]
Hence, by definition
\[
\liminf_{n \to \infty} -\log \left(1 - D(\rho^{\otimes n}, \tilde{\rho}_n)\right) \geq s_r - r - \varepsilon = \sup_{0 \leq \alpha \leq 1} (\alpha - 1)(r - D_\alpha(\rho\|\sigma)) - \varepsilon,
\]
which finishes the proof of (11) since \(\varepsilon > 0\) was arbitrary.

We now consider \(\rho\) and \(\sigma\) to be commute and establish the converse in (13). For that we define the sub-normalised states \(\tilde{\rho}_n = T_n \rho^{\otimes n} T_n\), where \(T_n = \{\rho^{\otimes n} \leq 2nr \sigma^{\otimes n}\}\) denotes the Neyman-Pearson test. Using that \(\rho\) and \(\sigma\) commute we see by definition \(\tilde{\rho}_n \leq 2nr \sigma^{\otimes n}\).

Moreover, the generalised trace distance can be written as
\[
D(\rho^{\otimes n}, \tilde{\rho}_n) = \frac{1}{2} \left\| \rho^{\otimes n} - \tilde{\rho}_n \right\|_1 + \frac{1}{2} \left| \text{Tr}(\rho^{\otimes n}) - \text{Tr}(\tilde{\rho}_n) \right| \\
= \frac{1}{2} \left\| (1 - T_n) \rho^{\otimes n} (1 - T_n) \right\|_1 + \frac{1}{2} \left(1 - \text{Tr}(T_n \rho^{\otimes n})\right) \\
= 1 - \text{Tr}(T_n \rho^{\otimes n}).
\]
Here we have used for the first equality that since \(\rho^{\otimes n}\) commutes with \(\sigma^{\otimes n}\) and therefore also with \(T_n\), the cross terms vanish, i.e. \(T_n \rho^{\otimes n} (1 - T_n) = (1 - T_n) \rho^{\otimes n} T_n = 0\). Using now Lemma 1 we see that
\[
\lim_{n \to \infty} \frac{-\log(1 - D(\rho^{\otimes n}, \tilde{\rho}_n))}{n} = B(s_r|\rho\|\sigma) = \sup_{0 \leq \alpha \leq 1} (\alpha - 1)(r - D_\alpha(\rho\|\sigma)),
\]
and therefore
\[
\limsup_{n \to \infty} \frac{-\log(1 - \varepsilon D(\rho^{\otimes n}\|\sigma^{\otimes n}, nr))}{n} \leq \sup_{0 \leq \alpha \leq 1} (\alpha - 1)(r - D_\alpha(\rho\|\sigma)),
\]
which finishes the proof.

5 Strong converse rates of privacy amplification against a quantum adversary

The initial state of a privacy amplification protocol is given by the classical-quantum state
\[
\rho_{XE} = \sum_{x \in X} p_x |x\rangle \langle x| \otimes \rho^x_E,
\]
with the classical system \(X\) belonging to Alice and Bob and the quantum system \(E\) belonging to Eve.
The objective of Alice and Bob is to apply a hash function \( f : \mathcal{X} \rightarrow \mathcal{Z} \) to decouple their part of the system from Eve’s system, the latter playing the role of quantum side information. Their operation results in the state

\[
\rho_{ZE}^f = \sum_{z \in \mathcal{Z}} |z\rangle \langle z| \otimes \sum_{x \in \mathcal{X} : x \in f^{-1}(z)} p_x \rho_E^x.
\]  

(16)

The minimal errors in this conversion, measured in trace distance and purified distance, respectively, are given by

\[
\Delta_1(\mathcal{X} \rightarrow \mathcal{Z}) = \min_f \frac{1}{2} \left\| \rho_{ZE}^f - \frac{1}{|\mathcal{Z}|} \otimes \rho_E \right\|_1,
\]

\[
\Delta_P(\mathcal{X} \rightarrow \mathcal{Z}) = \min_f P \left( \rho_{ZE}^f, \frac{1}{|\mathcal{Z}|} \otimes \rho_E \right),
\]

where the minimum is over all functions \( f : \mathcal{X} \rightarrow \mathcal{Z} \). The optimal key length which can be distilled from a c-q state \( \rho_{XE} \) in that manner with \( \varepsilon \geq 0 \) error measured in metric \( d \in \{D, P\} \) is given by

\[
l_{\varepsilon,d}(\mathcal{X} | \mathcal{E}) = \sup \left\{ \log |\mathcal{Z}| \mid \Delta_d(\mathcal{X} \rightarrow \mathcal{Z}) \leq \varepsilon \right\}.
\]

Note that \( l_{\varepsilon,P}(\mathcal{X} | \mathcal{E}) \) is essentially given by the smoothed conditional min-entropy as we have for every \( 0 < \eta \leq \varepsilon \leq 1 \) the relation [20, Theorem 8]

\[
H_{\min}^{\varepsilon,P}(\mathcal{X} | \mathcal{E}) \geq H_{\min}^{\varepsilon-P}(\mathcal{X} | \mathcal{E}) - \log \left( \frac{1}{\eta^4} \right) - 3.8
\]

(17)

Let us go to the \( n \)-copy setting in which Alice and Bob possess strings \( x^{(n)} = (x_1, \ldots, x_n) \in \mathcal{X}^n \) with the \( x_i \)'s being values taken by a sequence of i.i.d. random variables with common p.m.f. \( p_x \), \( x \in \mathcal{X} \). The initial state of the privacy amplification process is in this case:

\[
\rho_{X^nE^n} := \sum_{x^{(n)} \in \mathcal{X}^n} p_{x^{(n)}} |x^{(n)}\rangle \langle x^{(n)}| \otimes \rho_{E^n}^{x^{(n)}},
\]

(18)

and the desired final state is \( \frac{1}{|\mathcal{Z}|} \otimes \rho_{E^n}^{\otimes n} \).

For a privacy amplification rate \( R \geq 0 \) we consider \( \mathcal{Z}_n \) to be such that \( |\mathcal{Z}_n| = \lfloor 2^{nR} \rfloor \). The next theorem shows that if the randomness extraction rate in a transformation from \( \mathcal{X}^n \) to \( \mathcal{Z}_n \) is larger than the conditional entropy \( H(\mathcal{X} | \mathcal{E}) \), the distance of the corresponding decoupled state to any possible \( \rho_{Z^nE^n}^f \), defined analogously to (16), goes exponentially fast to 1 as \( n \) goes to infinity. By that it provides the strong converse of privacy amplification. Moreover, it provides bounds on the corresponding strong converse rates.

---

\( ^8 \) Note that in [20] the result is proven for a different version of the smoothed conditional min-entropy, namely, \( H_{\min}^{\varepsilon,P}(\mathcal{X} | \mathcal{E}) := \sup_{\sigma_E \in S(\mathcal{H}_E)} -D_{\min}^{\varepsilon,P}(\rho_{XE} \| \mathbb{1}_X \otimes \sigma_E) \). Since clearly \( H_{\min}^{\varepsilon,P}(\mathcal{X} | \mathcal{E}) \geq H_{\min}^{\varepsilon,P}(\mathcal{A} | \mathcal{B}) \), their lower bound on \( l_{\varepsilon,P}(\mathcal{X} | \mathcal{E}) \) implies the one in (17). Moreover, using the result [12, Proposition 10] and the same proof as in [20, Theorem 8], the upper bound on \( l_{\varepsilon,P}(\mathcal{X} | \mathcal{E}) \) in (17) also follows.
Theorem 3 Let $\rho_{XE}$ be a c-q state, $R \geq 0$ and $Z_n$ with $|Z_n| = 2^{nR}$. Then we have
\[
\liminf_{n \to \infty} \frac{-\log (1 - \Delta_1(X^n \to Z_n))}{n} \geq \sup_{0 \leq \alpha \leq 1} \frac{1 - \alpha}{2}(R - H_\alpha(X|E)), \tag{19}
\]
\[
\liminf_{n \to \infty} \frac{-\log (1 - \Delta_P(X^n \to Z_n))}{n} \geq \sup_{0 \leq \alpha \leq 1} (1 - \alpha)(R - H_\alpha(X|E)). \tag{20}
\]

Remark 4 Note that the right hand side of both (20) and (19) are strictly positive if only if $R > H(X|E)$. This in turn proves the full strong converse, since $H(X|E)$ is the optimal achievable extraction rate of privacy amplification against a quantum adversary (c.f. (2)). In other words, the optimal extraction rate of privacy amplification against a quantum adversary satisfies the so-called strong converse property: i.e., an attempt to extract private bits at a rate higher than $H(X|E)$ leads to the conversion error going to 1 exponentially fast as $n \to \infty$.

Remark 5 Note that in the concurrent and independent work [18], the lower bound
\[
\frac{-\log (1 - \overline{\Delta}_1(X^n \to Z_n))}{n} \geq \sup_{0 \leq \alpha \leq 1} (1 - \alpha)(R - H_\alpha(X|E)), \tag{21}
\]
has been established. Here, $\overline{\Delta}_1(X^n \to Z_n) = \mathbb{E}_{h_n} \frac{1}{2} \| Z_n || Z_n \otimes \rho_E^{\otimes n} ||_1$ denotes the average trace distance from the decoupled state, with $\mathbb{E}_{h_n}$ being the average over all strongly 2-universal hash functions (see [18] for more details). Note that due to the factor of 1/2 the right-hand side of our (19) is smaller than the right-hand side of (21). This factor arises solely from the fact that we use our result for purified distance, (20), together with the Fuchs-van de Graaf inequality [6]. If the following conjectured inequality (22) holds:
\[
H_{\min}^{\varepsilon,D}(X^n|E^n)_{\rho_{XE^n}} \geq H_{\min}^{\varepsilon,D}(Z_n|E^n)_{\rho_{Z_n,E^n}^n},
\]
which is the analogue of (22) but with smoothing considered in terms of the generalised trace distance instead of purified distance, we can obtain a lower bound on the strong converse exponent (19) with an improvement of a factor of 2. The resulting bound would then match the lower bound (21) from [18]. In fact, since we consider the minimal conversion error, $\overline{\Delta}_1(X^n \to Z_n)$, rather than the average one $\overline{\Delta}_1(X^n \to Z_n)$, this would give a stronger result on the strong converse exponent of privacy amplification.

Remark 6 In [11], the authors obtained alternative bounds on the strong converse exponent for quantum privacy amplification, using a figure of merit which is different from the ones considered in this paper. The figure of merit in [11] is the fidelity (between the final c-q state of privacy amplification protocol and a decoupled state corresponding to uniform shared randomness of the classical system and any possible quantum state of Eve) optimized over all possible states of Eve.

Proof of Theorem 3. We first prove (20) which essentially follows by the same method as in [12, Theorem 8], but we still carry it out for completeness. From [12, Proposition 10] we know for all $\varepsilon \geq 0$ and $f_n : X^n \to Z_n$
\[
H_{\min}^{\varepsilon,D}(X^n|E^n)_{\rho_{XE^n}} \geq H_{\min}^{\varepsilon,D}(Z_n|E^n)_{\rho_{Z_n,E^n}^n}. \tag{22}
\]
This gives for all $r \in \mathbb{R}$

$$
\varepsilon^P \left( \rho_{Z^n E^n} \| 1_{Z_n} \otimes \rho_{E^n}, r \right) \geq \varepsilon^P \left( \rho_{X^n E^n} \| 1_{X_n} \otimes \rho_{E^n}, r \right).
$$

(23)

By definition we have

$$
P \left( \rho_{Z^n E^n}^{f_n} \left| \frac{1_{Z_n}}{|Z_n|} \right| \otimes \rho_{E}^{n} \right) \geq \varepsilon^P \left( \rho_{Z^n E^n}^{f_n} \left| 1_{Z_n} \otimes \rho_{E}, - \log |Z_n| \right| \right) \geq \varepsilon^P \left( \rho_{Z^n E^n}^{f_n} \left| 1_{Z_n} \otimes \rho_{E}, - nR \right| \right),
$$

where we have used $\log |Z_n| = \log \left\lfloor 2^{nR} \right\rfloor \leq nR$ for the last inequality. Combining this with (23) and using that $f_n$ was arbitrary gives

$$
\Delta P(\mathcal{X}^n \rightarrow \mathcal{Z}_n) \geq \varepsilon^P \left( \rho_{X^n E^n} \| 1_{X_n} \otimes \rho_{E^n}, - nR \right).
$$

Using now Theorem 2 gives

$$
\liminf_{n \rightarrow \infty} \frac{- \log (1 - \Delta P(\mathcal{X}^n \rightarrow \mathcal{Z}_n))}{n} \geq \sup_{0 \leq \alpha \leq 1} (1 - \alpha) \left( R + D_\alpha (\rho_{XE} \| 1_X \otimes \rho_E) \right) = \sup_{0 \leq \alpha \leq 1} (1 - \alpha) \left( R - H_\alpha \left( X \left| E \right. \right) \right).
$$

For (19) we use that by the Fuchs-van de Graaf inequality [6] we have for all functions $f_n : \mathcal{X}^n \rightarrow \mathcal{Z}_n$

$$
1 - \frac{1}{2} \left\| \rho_{Z^n E^n}^{f_n} - \frac{1_{Z_n}}{|Z_n|} \otimes \rho_{E}^{n} \right\|_1 \leq \sqrt{1 - P^2 \left( \rho_{Z^n E^n}^{f_n} \left| \frac{1_{Z_n}}{|Z_n|} \otimes \rho_{E}^{n} \right| \right)}
$$

$$
= \sqrt{\left( 1 + P \left( \rho_{Z^n E^n}^{f_n} \left| \frac{1_{Z_n}}{|Z_n|} \otimes \rho_{E}^{n} \right| \right) \right) \left( 1 - P \left( \rho_{Z^n E^n}^{f_n} \left| \frac{1_{Z_n}}{|Z_n|} \otimes \rho_{E}^{n} \right| \right) \right)} \leq \sqrt{2 \left( 1 - P \left( \rho_{Z^n E^n}^{f_n} \left| \frac{1_{Z_n}}{|Z_n|} \otimes \rho_{E}^{n} \right| \right) \right)}
$$

and therefore by (20)

$$
\liminf_{n \rightarrow \infty} \frac{- \log (1 - \Delta_1(\mathcal{X}^n \rightarrow \mathcal{Z}_n))}{n} \geq \frac{1}{2} \liminf_{n \rightarrow \infty} \frac{- \log \left( 1 - \Delta P(\mathcal{X}^n \rightarrow \mathcal{Z}_n) \right)}{n} \geq \frac{1}{2} \liminf_{n \rightarrow \infty} - \log \left( 1 - \Delta P(\mathcal{X}^n \rightarrow \mathcal{Z}_n) \right)
$$

$$
\geq \sup_{0 \leq \alpha \leq 1} \frac{(1 - \alpha)}{2} \left( R - H_\alpha \left( X \left| E \right. \right) \right)
$$

which finishes the proof.

6 Strong converse for secure communication

In the following we consider $Z_n$ to be a sequence of classical systems, $E_n$ a sequence of quantum systems. Moreover, we consider sequences of c-q states denoted by

$$
\rho_{Z^n E^n} = \sum_{z_n \in \mathcal{Z}_n} p_{z_n} \left| z_n \right\rangle \langle z_n \right| \otimes \rho_{E^n}^{z_n},
$$

(24)

12
where we assume the system $Z_n$ to be held by Alice and $E_n$ to be held by an evesdropper, Eve. we write $\rho_{E_n} = \text{Tr}_{Z_n}(\rho_{Z_n E_n}) = \sum_{z_n \in Z_n} p_{z_n} \rho_{Z_n n}^{z_n}$ for the corresponding reduced states of Eve’s system.

Here, we want to understand in what sense the condition

$$
\lim_{n \to \infty} \frac{1}{2} \left\| \rho_{Z_n E_n} - \frac{1}{|Z_n|} \otimes \rho_{E_n} \right\|_1 = 1
$$

implies insecurity of the keys Alice can generate from $\rho_{Z_n E_n}$. The example we have in mind is $\rho_{Z_n E_n}$ being the resulting state after privacy amplification with privacy amplification rate $R > H(X|E)$, where we have already seen that the convergence in (25) happens exponentially fast (c.f. Theorem 3).

We consider the scenario in which Alice is using a generated key $z_n \in Z_n$ to encode a message $m$, where the message is taken out of a subset $M_n \subset Z_n$. For the encoding she uses an encryption scheme, i.e. a map

$$
\mathcal{E} : Z_n \times Z_n \to Z_n
$$

$$(z_n, m) \mapsto \mathcal{E}_{z_n}(m),
$$

which is bijective in both entries for the other entry fixed. The encoded message $\mathcal{E}_{z_n}(m)$ is then sent publicly. Given a party has access to the key $z_n$, they can then decode $m$ due to bijectivity of the function $\mathcal{E}_{z_n}$. As an example of such an encryption scheme we can think of $Z_n$ being a set of bit strings and take the one-time pad encoding $\mathcal{E}_{z_n}(m) = z_n \oplus m$, where $\oplus$ denotes the component-wise addition modulo 2.

Since the encoded message is sent publicly, Eve has access to the c-q state

$$
\rho_{Z_n E_n}^m = \sum_{Z_n \in Z_n} p_{Z_n} |\mathcal{E}_{z_n}(m)\rangle \langle \mathcal{E}_{z_n}(m)| \otimes \rho_{Z_n E_n}^n = (U_m \otimes 1_{E_n}) \rho_{Z_n E_n} (U_m^* \otimes 1_{E_n}).
$$

Here, we have denoted the unitary $U_m$ on system $Z_n$ defined by $U_m |z_n\rangle = |\mathcal{E}_{z_n}(m)\rangle$. In order to infer which message $m \in M_n \subset Z_n$ has been sent, Eve needs to distinguish the states $(\rho_{Z_n E_n}^m |M\rangle \langle M|)^{m \in M_n}$ by picking a suitable POVM $\Lambda \equiv (\Lambda_m)_{m \in M_n}$. Given that Alice choses the message $M = m$, the probability that Eve’s guess, denoted by $M$, is correct is given by

$$
P_{\Lambda_n}(M = m | M = m) = \text{Tr}(\Lambda_m \rho_{Z_n E_n}^m |M\rangle \langle M|).
$$

Moreover, if Alice message is distributed by some distribution $M \sim q_m$ on $M_n$ then the optimal probability with which Eve guesses correctly on average, the guessing probability, is given by

$$
p_{\text{guess}}(M | \mathcal{E}_{z_n}(M), E_n) = \max_{(\Lambda_m)_{m \in M_n} \text{POVM}} \sum_{M \in M_n} q_m P_{\Lambda_n}(M = m | M = m)
$$

$$
= \max_{(\Lambda_m)_{m \in M_n} \text{POVM}} \sum_{m \in M_n} q_m \text{Tr}(\Lambda_m \rho_{Z_n E_n}^m |M\rangle \langle M|).
$$

However, in order for Eve to be able to pick the POVM sensefully, i.e. the maximiser in (26), she needs to have additonal side information, which is knowledge of the set $M_n$ and the distribution $q_m$. Here, we say her additional side information is large if $M_n$ is small. In the following we will
quantify how much additional side information Eve needs to have in order to guess Alice’s message with certainty.

We first consider the case in which (25) does not hold, i.e. the trace distance between $\rho_{Z_n E_n}$ and the decoupled state is strictly smaller than 1 uniformly in $n$. In that case, it is only possible for Eve to have certainty of the sent message if she has strong additional side information which is that $M_n \subset Z_n$ is finite uniformly in $n$, i.e. $\sup_{n \in \mathbb{N}} |M_n| < \infty$.

**Proposition 7** Let $\delta \in [0, 1)$ and $(\rho_{Z_n E_n})_{n \in \mathbb{N}}$ be a sequence of c-q states such that

$$\sup_{n \in \mathbb{N}} \frac{1}{2} \left\| \rho_{Z_n E_n} - \frac{1}{|Z_n|} \otimes \rho_{E_n} \right\|_1 \leq \delta. \tag{27}$$

Moreover, assume Alice uses an encryption scheme $g$ as defined above and chooses to send a message $M$ which she picks uniformly out of the set $M_n \subset Z_n$, i.e. $q_m = 1/|M_n|$ for all $m \in M_n$. Then Eve’s guessing probability is bounded by

$$p_{\text{guess}}(M | \mathcal{E}_{Z_n}(M), E_n) \leq \delta + \frac{1}{|M_n|}.$$  

In particular, if $\sup_{n \in \mathbb{N}} |M_n| = \infty$ then

$$\limsup_{n \to \infty} p_{\text{guess}}(M | \mathcal{E}_{Z_n}(M), E_n) \leq \delta < 1.$$  

**Proof.** From (28) and unitary invariance of the trace norm we get for all $m \in Z_n$

$$\sup_{n \in \mathbb{N}} \frac{1}{2} \left\| \rho_{\mathcal{E}_{Z_n}(M) E_n}^m - \frac{1}{|Z_n|} \otimes \rho_{E_n} \right\|_1 = \sup_{n \in \mathbb{N}} \frac{1}{2} \left\| \rho_{Z_n E_n} - \frac{1}{|Z_n|} \otimes \rho_{E_n} \right\|_1 \leq \delta.$$  

Using this we get

$$p_{\text{guess}}(M | \mathcal{E}_{Z_n}(M), E_n) = \max_{\{\Lambda_m\}_{m \in M_n} \in \text{POVM}} \frac{1}{|M_n|} \sum_{m \in M_n} \text{Tr} \left( \Lambda_m \rho_{\mathcal{E}_{Z_n}(M) E_n}^m \right)$$

$$\leq \max_{\{\Lambda_m\}_{m \in M_n} \in \text{POVM}} \frac{1}{|M_n|} \sum_{m \in M_n} \text{Tr} \left( \Lambda_m \frac{1}{|Z_n|} \otimes \rho_{E_n} \right) + \delta$$

$$= \frac{1}{|M_n|} + \delta.$$  

In the case where (25) does hold, the situation is drastically different to the one described in Proposition 7. Let $c_n \in [0, 1]$ be the speed of convergence in (25), i.e.

$$1 - \frac{1}{2} \left\| \rho_{Z_n E_n} - \frac{1}{|Z_n|} \otimes \rho_{E_n} \right\|_1 \leq c_n \tag{28}$$

and $\lim_{n \to \infty} c_n = 0$. We will show that for all encryption schemes and almost all sets of messages with $|M_n| \ll c_n^{-1}$, Eve’s guessing probability will converge to 1 as $n \to \infty$ (c.f. Proposition 8). In particular, in the strong converse region of privacy amplification, i.e. for privacy amplification rate
\( R > H(X|E) \), Theorem 3 gives that \( c_n \) converges exponentially fast to 0 with exponential decay rate being bounded from below as
\[
\liminf_{n \to \infty} -\frac{\log c_n}{n} \geq \sup_{0 \leq \alpha \leq 1} \frac{1}{2} (R - H_\alpha(X|E)).
\]
Hence, the sets of messages for which Eve can guess correctly can be made even exponentially large in \( n \), which can be considered as small additional side information needed. Moreover, it can be shown that the convergence of Eve’s guessing probability towards 1 happens exponentially fast in that region.

Therefore, to conclude, we see by Proposition 7 that in the achievable region of privacy amplification for privacy amplification rate \( R \leq H(X|E) \) in which
\[
\lim_{n \to \infty} \Delta_1(X^n \to Z_n) = 0,
\]
(compare [16]) Eve can infer the sent message correctly only if she already had complete knowledge of it to start with (i.e. \(|\mathcal{M}_n| = 1\)). Whereas, the following Proposition 8 gives in the strong converse region for \( R > H(X|E) \), Eve only needs very limited additional side information (\(|\mathcal{M}_n| \) possibly scaling exponentially in \( n \)) while still being able to infer the message correctly. Hence, our result serves as a strong converse for secure communication.

**Proposition 8** Let \( (\rho_{Z_n E_n})_{n \in \mathbb{N}} \) be a sequence of c-q states such that
\[
\lim_{n \to \infty} \frac{1}{2} \left\| \rho_{Z_n E_n} - \frac{1}{|Z_n|} \otimes \rho_{E_n} \right\|_1 = 1,
\]
with speed of convergence \( c_n \in [0, 1] \), i.e.
\[
\left\| 1 - \frac{1}{2} \left\| \rho_{Z_n E_n} - \frac{1}{|Z_n|} \otimes \rho_{E_n} \right\|_1 \right\| \leq c_n
\]
and \( \lim_{n \to \infty} c_n = 0 \). Let \( \varepsilon > 0 \) and \( \mathcal{M}_n \subset Z_n \) be a set of messages with \( |\mathcal{M}_n| \leq c_n^{\varepsilon - 1} \) chosen uniformly at random, which Alice encodes using an encryption scheme \( \mathcal{E} \) as above. Then Eve can find a measurement \( \Lambda \) such that for all \( m \in \mathcal{M}_n \)
\[
P_{\Lambda,n}(\hat{M} = m|M = m) > 1 - c_n^{\varepsilon/2}
\]
with probability greater than \( 1 - 2c_n^{\varepsilon/2} \). Hence, in particular for all \( m \in \mathcal{M}_n \) we have \( \lim_{n \to \infty} P_{\Lambda,n}(\hat{M} = m|M = m) = 1 \) almost surely.

**Remark 9** Note that by Proposition 8 we immediately also get the lower bound on the average guessing probability (for all possible distributions \( q_m \) on \( \mathcal{M}_n \))
\[
p_{\text{guess}}(M|E_{Z_n}(M), E_n) > 1 - c_n^{\varepsilon/2}
\]
with probability greater than \( 1 - 2c_n^{\varepsilon/2} \) over the set \( \mathcal{M}_n \) chosen uniformly at random from the set of subsets of \( Z_n \) with cardinality constraint \( |\mathcal{M}_n| \leq c_n^{\varepsilon - 1} \).

15
Proof of Proposition 8. Using the well-known expression for the trace distance of two states $\rho$ and $\sigma$

$$\frac{1}{2}||\rho - \sigma|| = \max_{0 \leq \Lambda \leq 1} \text{Tr}(\Lambda(\rho - \sigma)) = \max_{\text{orthogonal projection}} \text{Tr}(\pi(\rho - \sigma)),$$

assumption (30) gives that there exists a sequence of orthogonal projections $(\pi_n)_{n \in \mathbb{N}}$ such that

$$\text{Tr}(\pi_n \rho_{Z_n} E_n) \geq 1 - c_n,$$

$$\text{Tr}\left(\frac{1}{Z_n} \otimes \rho_{E_n}\right) \leq c_n.$$

Moreover, define as above for $m \in \mathcal{Z}_n$

$$\rho_{E_n}^m(\mathcal{E}_n) = \sum_{z_n \in \mathcal{Z}_n} p_{z_n} |z_n \oplus m\rangle \langle z_n \oplus m| \otimes \rho_{E_n}^{z_n} = (U_m \otimes 1_{E_n}) \rho_{Z_n} E_n (U_m^\dagger \otimes 1_{E_n}),$$

with unitary $U_m$ on system $Z_n$ defined by $U_m|x\rangle = |\mathcal{E}_n (m)\rangle$. Furthermore, we write $\pi_n^m = (U_m^\dagger \otimes 1_{E_n}) \pi_n (U_m \otimes 1_{E_n})$. Note

$$\frac{1}{|\mathcal{Z}_n|} \sum_{m \in \mathcal{Z}_n} \rho_{E_n}^m(\mathcal{E}_n) = \sum_{z_n \in \mathcal{Z}_n} \frac{1}{Z_n} |z_n\rangle \langle z_n| \otimes \rho_{E_n}^{z_n} = \frac{1}{|\mathcal{Z}_n|} \otimes \rho_{E_n},$$

where we have used for the first equality that for every $z_n \in \mathcal{Z}_n$ the function $\mathcal{E}_n$ is a bijection. For $K_n \leq \lfloor 1/c_n^{(1-\varepsilon)} \rfloor$ being a natural number let $M_n = (m_1, \cdots, m_{K_n}) \in \mathcal{Z}_n^{K_n}$ be random vector such that each component of $M_n$ is picked uniformly at random from $\mathcal{Z}_n$. For every $k \in [K_n] := \{1, \cdots, K_n\}$ we can calculate the expectation

$$\mathbb{E}_{\mathcal{M}_n} \left[ \text{Tr}\left( \frac{1}{|\mathcal{Z}_n|} \sum_{l \in [K_n] \setminus \{k\}} \pi_n^m \right) \right] = \frac{1}{|\mathcal{Z}_n|} \sum_{m_1, \cdots, m_{K_n} \in \mathcal{Z}_n} \text{Tr}\left( \rho_{E_n}^m(\mathcal{E}_n) \sum_{l \in [K_n] \setminus \{k\}} \pi_n^m \right)$$

$$= \frac{1}{|\mathcal{Z}_n|^{K_n-1}} \sum_{m_1, \cdots, m_{K-1}, m_{K} \in \mathcal{Z}_n} \text{Tr}\left( \frac{1}{Z_n} \otimes \rho_{E_n} \sum_{l \in [K_n] \setminus \{k\}} \pi_n^m \right)$$

$$= \sum_{l \in [K_n] \setminus \{k\}} \text{Tr}\left( \frac{1}{Z_n} \otimes \rho_{E_n} \pi_n \right) \leq |\mathcal{M}_n| c_n \leq c_n^\varepsilon.$$

Define for all $m \in \mathcal{M}_n$ the subspaces of $\mathcal{H}_{Z_n} \otimes \mathcal{H}_{E_n}$ denoted by $V_m = \text{supp}(\pi_n^m) \cap \text{supp}(\sum_{m' \in \mathcal{M}_n \setminus \{m\}} \pi_n^{m'})^\perp$ and $\Lambda_n^m$ the orthogonal projection on $V_m$. Note, that by definition all $\Lambda_n^m$ have mutually orthogonal supports and hence $\sum_{m \in \mathcal{M}_n} \Lambda_n^m \leq 1$, which means we can extend the family $(\Lambda_n^m)_{m \in \mathcal{M}_n}$ to a POVM (i.e. by redefining $\Lambda_n^m = 1 - \sum_{m' \in \mathcal{M}_n \setminus \{m\}} \Lambda_n^{m'}$ for one fixed $m \in \mathcal{M}_n$).

Using the notation $P_V$ for the orthogonal projection onto a subspace $V \subset \mathcal{H}_{Z_n} \otimes \mathcal{H}_{E_n}$ we note for all $m \in \mathcal{M}_n$

$$\mathbb{E}_{\mathcal{M}_n} \left[ \mathbb{E}_\Lambda, m \right] = \mathbb{E}_{\mathcal{M}_n} \left[ \text{Tr}\left( \Lambda_n^m \rho_{E_n}^m(\mathcal{E}_n) \right) \right]$$

$$= \mathbb{E}_{\mathcal{M}_n} \left[ \text{Tr}\left( \pi_n^m \rho_{E_n}^m(\mathcal{E}_n) \right) - \text{Tr}\left( P_{\text{supp}(\pi_n^m) \cap \text{supp}(\sum_{m' \in \mathcal{M}_n \setminus \{m\}} \pi_n^{m'})} \rho_{E_n}^m(\mathcal{E}_n) \right) \right].$$
\[
\begin{align*}
&\geq \text{Tr} (\pi_n \rho_{Z_n} E_n) - \mathbb{E}_{M_n} \left[ \text{Tr} \left( P_{\text{supp} \left( \sum_{m'\in M_n \setminus \{m\}} \pi_{m'}^m \right)} \rho_{Z_n}^m (M) E_n \right) \right] \\
&\geq \text{Tr} \left( \pi_n^m \rho_{Z_n}^m (M) E_n \right) - \mathbb{E}_{M_n} \left[ \text{Tr} \left( \rho_{Z_n}^m (M) E_n \sum_{m'\in M_n, m'\neq m} \pi_{m'}^m \right) \right] \\
&\geq 1 - c_n - c_n^\varepsilon \geq 1 - 2c_n^\varepsilon.
\end{align*}
\]

Moreover, by Markov’s inequality, we know that for every $\delta > 0$

\[
\mathbb{P}_{M_n} \left[ \mathbb{P}_{\Lambda,n} (\hat{M} = m | M = m) \leq 1 - \delta \right] \leq \frac{\mathbb{E}_{M_n} \left[ 1 - \mathbb{P}_{\Lambda,n} (\hat{M} = m | M = m) \right]}{\delta} \leq \frac{2c_n^\varepsilon}{\delta}
\]

Hence, choosing $\delta = c_n^{\varepsilon/2}$ gives the desired result. ■

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