The classification of non-local chiral CFT
with $c < 1$

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Dedicated to Hans-Jürgen Borchers on the occasion of his 80th birthday

Abstract
All non-local but relatively local irreducible extensions of Virasoro chiral CFTs with $c < 1$ are classified. The classification, which is a prerequisite for the classification of local $c < 1$ boundary CFTs on a two-dimensional half-space, turns out to be 1 to 1 with certain pairs of $A-D-E$ graphs with distinguished vertices.

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1 Introduction

Non-local chiral conformal quantum field theories have gained renewed interest because they give rise to local CFT on the two-dimensional Minkowski halfspace $x > 0$ (boundary CFT, BCFT), and vice versa [9].

More precisely, a BCFT contains chiral fields which generate a net $A$ of local algebras on the circle, such that $A_+(O) = A(I) \vee A(J)$ are the chiral BCFT observables localized in the double cone $O = I \times J \equiv \{(t, x) : t + x \in I, t - x \in J\}$ where $I > J$ are two open intervals of the real axis (= the pointed circle). The two-dimensional local fields of the BCFT define a net of inclusions $A_+(O) \subset B_+(O)$ subject to locality, conformal covariance, and certain irreducibility requirements.

If $A$ is assumed to be completely rational [7], then there is a 1 to 1 correspondence [9] between Haag dual BCFTs associated with a given chiral net $A$, and non-local chiral extensions $B$ of $A$ such that the net of inclusions $A(I) \subset B(I)$ is covariant, irreducible and relatively local, i.e., $A(I)$ commutes with $B(J)$ if $I$ and $J$ are disjoint. The correspondence is given by the simple relative commutant formula

$$B_+(O) = B(K)' \cap B(L)$$

where $O = I \times J$ as before, $K$ is the open interval between $I$ and $J$, and $L$ is the interval spanned by $I$ and $J$. Conversely,

$$B(L) = \bigvee_{I \subset L, J \subset L, I > J} B_+(O).$$

BCFTs which are not Haag dual are always intermediate between $A_+$ and a Haag dual BCFT.

The classification of local BCFTs on the two-dimensional halfspace is thus reduced to the classification of non-local chiral extensions, which in turn [8] amounts to the classification of $Q$-systems (Frobenius algebras) in the $C^*$ tensor category of the superselection sectors [2] of $A$. The chiral nets $A = \text{Vir}_c$ defined by the stress-energy tensor (Virasoro algebra) with $c < 1$ are known to be completely rational, so the classification program just outlined can be performed.

Local chiral extensions of $\text{Vir}_c$ with $c < 1$ have a direct interpretation as local QFT models of their own. Their classification has been achieved previously ([5], see Remark 2.3) by imposing an additional condition [8] on the $Q$-system involving the braided structure (statistics [2]) of the tensor category. Of course, the present non-local classification contains the local one.

As in [5] we exploit the fact that the tensor subcategories of the “horizontal” and of the “vertical” superselection sectors of $\text{Vir}_c$ with $c < 1$ are isomorphic with the tensor categories of the superselection sectors of $SU(2)$ current algebras. (The braiding is different, however.) We therefore first classify the $Q$-systems in the latter categories (Sect. 1), and then proceed from $Q$-systems in the subcategories to $Q$-systems in the tensor categories of all sectors of $\text{Vir}_c$ (Sect. 2). Thanks to a cohomological triviality result [6], the classification problem simplifies considerably, and essentially
reduces to a combinatorial problem involving the Bratteli diagrams associated with the
local subfactors \( A(I) \subset B(I) \), combined with a "numerological" argument concerning
Perron-Frobenius eigenvalues.

In the last section, we determine the vacuum Hilbert spaces of the non-local ex-
tensions and of the associated BCFT's thus classified.

2 Classification of irreducible non-local extensions
of the \( SU(2)_k \)-nets

As an easy preliminary, we first classify all irreducible, possibly non-local, extensions
of the \( SU(2)_k \)-nets on the circle. Consider the representation category of the \( SU(2)_k \)-
net and label the irreducible DHR sectors as \( \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k \) as usual. (The vacuum
sector is labeled as \( \lambda_0 \).) Label this net as \( A \) and an irreducible extension as \( B \). Since
\( A \) is completely rational in the sense of [7], the index \([B : A]\) is automatically finite by
[5, Proposition 2.3]. We need to classify the irreducible \( B-A \) sectors \( B_i A \), where \( i \) is the
inclusion map. Note that the \( A-A \) sector \( A_i A \) gives the dual canonical endomorphism
of the inclusion and this decomposes into a direct sum of \( \lambda_j \) by [8]. Suppose we have
such a sector \( B_i A \), and consider the following sequence of commuting squares.

\[
\text{End}(\text{id}_A) \subset \text{End}(A_{\lambda 1A}) \subset \text{End}(A_{\lambda 2A}) \subset \cdots \subset \text{End}(B_{\lambda 1A}) \subset \text{End}(B_{\lambda 2A}) \subset \cdots
\]

The Bratteli diagram of the first row arises from reflections of the Dynkin diagram
\( A_{k+1} \) as in usual subfactor theory, that is, it looks like Fig. 1, where each vertex
is labeled with irreducible sectors appearing in the irreducible decomposition of \( \lambda_k \)
for \( k = 0, 1, 2, \ldots \). (See [3 Section 9.6] for appearance of such graphs in subfactor
theory.)
The Bratteli diagram of the second row also arises from reflections of some Dynkin diagrams having the same Coxeter number as $A_{k+1}$ and starts with a single vertex because of the irreducibility of $B_{k+1}$. This gives a bipartite graph $G$, one of the $A$-$D$-$E$ Dynkin diagram and its initial vertex $v$ as an invariant of $\iota$, but the vertex $v$ is determined only up to the graph automorphism, so we denote the orbit of a vertex $v$ under such automorphisms by $[v]$. (Note that the Dynkin diagrams $A_n$, $D_n$, and $E_6$ have non-trivial graph automorphisms of order 2.) Also note that the graph $G$ is bipartite by definition. A $B$-$A$ sector corresponding to an even vertex of $G$ might be equivalent to another $B$-$A$ sector corresponding to an odd vertex of $G$. Later it turns out that this case does not occur in the $SU(2)_k$-case, but it does occur in the Virasoro case below. Fig. 2 shows an example of $G$ and $v$ where $G$ is the Dynkin diagram $E_6$.

**Theorem 2.1.** The pair $(G,[v])$ gives a complete invariant for irreducible extensions of nets $SU(2)_k$, and an arbitrary pair $(G,[v])$ arises as an invariant of some extension.

**Proof** As in the proof of [1, Proposition A.3], we know that the paragroup generated by $\iota$ is uniquely determined by $(G,[v])$ and it is isomorphic to the paragroup of the Goodman-de la Harpe-Jones subfactor given by $(G,[v])$, which was defined in [4, Section 4.5]. Then we obtain uniqueness of the $Q$-system for the extension, up to unitary equivalence, as in [4, Theorem 5.3]. (We considered only a local extension of $SU(2)_{28}$ corresponding to $E_8$ and its vertex having the smallest Perron-Frobenius eigenvector entry there, but the same method works for any $(G,[v])$.)

Any combination of $(G,[v])$ is possible as in the proof of [4, Lemma A.1]. (We considered only the case of $E_7$ and its vertex having the smallest Perron-Frobenius eigenvector entry there, but the same method works for any $(G,[v])$. \qed

**Remark 2.2.** Note that the pair $(G,[v])$ uniquely corresponds to an (isomorphism class of) irreducible Goodman-de la Harpe-Jones subfactor. So we may say that irreducible extensions of nets $SU(2)_k$ are labeled with irreducible Goodman-de la Harpe-Jones subfactors $N \subset M$ such that the inclusions $A(I) \subset B(I)$ of the localized algebras are isomorphic to $N \subset M$ tensored with a type III factor.
3 Classification of non-local extensions of the Virasoro nets $\text{Vir}_c$ with $c < 1$

Let $A$ be the Virasoro net with central charge $c < 1$. As studied in [5 Section 3], it is a completely rational net. We would like to classify all, possibly non-local, irreducible extensions of this net. For $c = 1 - 6/m(m+1)$, $m = 3, 4, 5, \ldots$, we label the irreducible DHR sectors of the net $A$ as follows. We have $\sigma_{j,k}$, $j = 0, 1, \ldots, m-2$, $k = 0, 1, \ldots, m-1$, with identification of $\sigma_{j,k} = \sigma_{m-2-j,m-1-k}$. (Our notation $\sigma_{j,k}$ corresponds to $\lambda_{j+1,k+1}$ in [5, Section 3].) Our labeling gives that the identity sector is $\sigma_{0,0}$ and the statistical dimensions of $\sigma_{1,0}$ and $\sigma_{0,1}$ are $2 \cos(\pi/m)$ and $2 \cos(\pi/(m+1))$, respectively. We have $m(m-1)/2$ irreducible DHR sectors. We again need to classify the irreducible $B$-$A$ sectors $B^\iota A$, where $\iota$ is the inclusion map. Take such $B^\iota A$ for a fixed $\text{Vir}_c$ with $c = 1 - 6/m(m+1)$ and we obtain an invariant as follows.

Consider the following sequence of commuting squares as in the Section 2

\[
\begin{array}{ll}
\text{End}(A\text{id}_A) & \subset \text{End}(A\sigma_{1,0,A}) \subset \text{End}(A\sigma_{1,0,A}^2) \subset \cdots \\
\cap & \cap \\
\text{End}(B^\iota A) & \subset \text{End}(B^\iota\sigma_{1,0,A}) \subset \text{End}(B^\iota\sigma_{1,0,A}^2) \subset \cdots
\end{array}
\]

From the Bratteli diagram of the second row, we obtain a graph $G_1$ and its vertex $v_1$ as in Section 2. The graph $G_1$ is one of the $A$-$D$-$E$ Dynkin diagrams and has the Coxeter number $m$. We also use $\sigma_{0,1}$ instead of $\sigma_{1,0}$ in this procedure and obtain a graph $G_2$ and its vertex $v_2$. The graph $G_2$ is one of the $A$-$D$-$E$ Dynkin diagrams and has the Coxeter number $m+1$. The quadruple $(G_1, [v_1], G_2, [v_2])$ is an invariant for $\iota$. (The notation $[\cdot]$ means the orbit under the graph automorphisms as in Section 2.) Note that one of the graphs $G_1, G_2$ must be of type $A$ because the $D$ and $E$ diagrams have even Coxeter numbers. We then prove the following classification theorem.

**Theorem 3.1.** The quadruple $(G_1, [v_1], G_2, [v_2])$ gives a complete invariant for irreducible extensions of nets $\text{Vir}_c$, and an arbitrary quadruple, subject to the conditions on the Coxeter numbers as above, arises as an invariant of some extension.

We will distinguish certain sectors by their dimensions. For this purpose, we need the following technical lemma on the values of dimensions, which we prove before the proof of the above theorem.

**Lemma 3.2.** Let $m$ be a positive odd integer and $G$ one of the $A$-$D$-$E$ Dynkin diagrams having a Coxeter number $n$ with $|n-m| = 1$. Take a Perron-Frobenius eigenvector $(\mu_a)_a$ for the graph $G$, where $a$ denotes a vertex of $G$. Set $d_j = \sin(j\pi/m)/\sin(\pi/m)$ for $j = 1, 2, \ldots, m-1$. Then the sets

\[
\{\mu_a/\mu_b \mid a, b \text{ are vertices of } G\}
\]

and

\[
\{d_2, d_3, \ldots, d_{m-2}\}
\]

are disjoint.
Proof. If \( m = 1, 3 \), then the latter set is empty, so we may assume \( m \geq 5 \). Note that the value 1 is not in the latter set.

Suppose a number \( \omega \) is in the intersection and we will derive a contradiction. Then \( \omega \) is in the intersection of the cyclotomic fields \( \mathbb{Q}(\exp(\pi i/m)) \) and \( \mathbb{Q}(\exp(\pi i/n)) \), which is \( \mathbb{Q} \) since \( (2m, 2n) = 2 \) and \( \mathbb{Q}(\exp(2\pi i/2)) = \mathbb{Q} \). Suppose \( d_j \) is equal to this \( \omega \). We may and do assume \( 2 \leq j \leq (m - 1)/2 \). We have

\[
\omega = \frac{\zeta^j - \zeta^{-j}}{\zeta - \zeta^{-1}} = \zeta^{j-1} + \zeta^{j-3} + \zeta^{j-5} + \ldots + \zeta^{-j+1},
\]

where \( \zeta = \exp(2\pi i/(2m)) \).

First assume that \( j \) is even. We note \( (m - 2, 2m) = 1 \) since \( m \) is odd. Then the map \( \sigma : \zeta^k \mapsto \zeta^{k(m-2)} \) for \( k = 0, 1, \ldots, 2m - 1 \) gives an element of the Galois group for the cyclotomic extension \( \mathbb{Q} \subset \mathbb{Q}(\zeta) \). We have

\[
\sigma(\omega) = \zeta^{(j-1)(m-2)} + \zeta^{(j-3)(m-2)} + \zeta^{(j-5)(m-2)} + \ldots + \zeta^{(-(j+1)(m-2)}}.
\]

Here the set

\[
\{ \zeta^{(j-1)(m-2)}, \zeta^{(j-3)(m-2)}, \zeta^{(j-5)(m-2)}, \ldots, \zeta^{(-(j+1)(m-2)} \}
\]

has \( j \) distinct roots of unity containing \( \zeta^{m-2} \) and it is a subset of

\[
Z = \{ \zeta^k \mid k = 1, 3, 5, \ldots, 2m - 1 \}.
\]

The set

\[
\{ \zeta^{j-1}, \zeta^{j-3}, \zeta^{j-5}, \ldots, \zeta^{-j+1} \}
\]

is the unique subset having \( j \) distinct elements of \( Z \) that attains the maximum of \( \Re \sum_{k=1}^j \alpha_k \) among all subsets \( \{ \alpha_1, \alpha_2, \ldots, \alpha_j \} \) having \( j \) distinct elements of \( Z \). However, we have \( m > 3 \), which implies \( j \leq (m - 1)/2 < m - 2 \), thus the complex number \( \zeta^{m-2} \) is not in the above unique set, and thus the sum

\[
\zeta^{(j-1)(m-2)} + \zeta^{(j-3)(m-2)} + \zeta^{(j-5)(m-2)} + \ldots + \zeta^{(-(j+1)(m-2)}
\]

cannot be equal to

\[
\zeta^{j-1} + \zeta^{j-3} + \zeta^{j-5} + \ldots + \zeta^{-j+1},
\]

which shows that \( \omega \) is not fixed by \( \sigma \), so \( \omega \) is not an element of \( \mathbb{Q} \), which is a contradiction.

Next we assume that \( j \) is odd. We now have that

\[
\zeta^{2(j-1)/2} + \zeta^{2(j-3)/2} + \ldots + \zeta^{2(1-j)/2} \in \mathbb{Q}.
\]

Since \( (m, (m - 1)/2) = 1 \), the map \( \sigma : \zeta^{2k} \mapsto \zeta^{k(m-1)} \) for \( k = 0, 1, \ldots, m - 1 \) gives an element of the Galois group for the cyclotomic extension \( \mathbb{Q} \subset \mathbb{Q}(\zeta^2) \). Since \( m - 1 > j - 1 \), \( \sigma(\omega) \) contains a term \( \zeta^{m-1} \) which does not appear in \( \omega \). Then by an argument similar to the above case of even \( j \), we obtain a contradiction. \( \square \)
We now start the proof of Theorem 3.1.

**Proof**  Without loss of generality, we may assume that $m$ is odd and hence that the graph $G_1$ is $A_{m-1}$. (Otherwise, the graph $G_2$ is $A_m$, and we can switch the symmetric roles of $G_1$ and $G_2$.) Note that a sector corresponding to an even vertex of $A_{m-1}$ can be equivalent to another sector corresponding to an odd vertex of $A_{m-1}$.

The tensor category having the irreducible objects $\{\sigma_{0,0}, \sigma_{1,0}, \ldots, \sigma_{m-2,0}\}$ is isomorphic to the representation category of $SU(2)_{m-2}$, thus each of the irreducible objects is labeled with a vertex of the Dynkin diagram $A_{m-1}$. Let $\sigma_{j,0}$ be one of the two sectors corresponding to $[v_1]$. We choose $j$ to be even, and then $j$ is uniquely determined. Let $\Delta$ be the set of the irreducible $B$-$A$ sectors arising from the decomposition of $B \mu \sigma_{1,0}^{2k}$ for all $k$. Note that $\Delta$ is a subset of the vertices of $G_1$. Let $\tilde{i}$ be one of the $B$-$A$ sectors in $\Delta$ having the smallest dimension. By the Perron-Frobenius theory and the definition of the graph $G_1$, which is now $A_{m-1}$, we know that such $\tilde{i}$ is uniquely determined and that the set

$$\{d(\lambda)/d(\tilde{i}) \mid \lambda \in \Delta\}$$

is equal to

$$\{\sin(k\pi/m)/\sin(\pi/m) \mid k = 1, 2, \ldots, (m - 1)/2\}.$$  

The situation is illustrated in Fig. 3 where we also have the graph $G$ which will be defined below. The vertices corresponding to the elements in $\Delta$ are represented as larger circles.

Now we consider the Bratteli diagram for $\text{End}(\tilde{i}\sigma_{0,1}^n)$ for $n = 0, 1, 2, \ldots$, and obtain a graph $G$ and an orbit $[v]$ whose reflection gives this Bratteli diagram. The graph $G$ is one of the $A$-$D$-$E$ Dynkin diagrams and its Coxeter number differs from $m$ by 1.

We want to show that the pairs $(G_2, [v_2])$ and $(G, [v])$ are equal as follows. We first claim that the irreducible decomposition of $\overline{\tilde{i}}$ contains only sectors among $\sigma_{0,k}$, $k = 0, 1, \ldots, m - 1$. Suppose that $\overline{\tilde{i}}$ contains $\sigma_{2l,k}$ with $l > 0$ on the contrary. By the Frobenius reciprocity, we have

$$0 < \langle \overline{\tilde{i}}, \sigma_{2l,k} \rangle = \langle i\sigma_{0,k}, \overline{\tilde{i}}\sigma_{2l,0} \rangle.$$
By the above description of the graph $G_1$, we know that $i\sigma_{2,0}$ is irreducible and distinct from $i$. The assumption that $i\sigma_{2,0}$ appears in the irreducible decomposition of $\tilde{\sigma}_{0,k}$ means that the graphs $G$ and $G_1$ have a common vertex other than $i$ and this is impossible by Lemma 3.2. We have thus proved that the irreducible decomposition of $\tilde{t}$ contains only sectors among $\sigma_{0,k}$, $k = 0, 1, \ldots, m - 1$.

We know $i\sigma_{j,0}$ and $\iota$ are equivalent sectors since they both are irreducible. To show $(G_2, [v_2]) = (G, [v])$, we therefore need to compare the irreducible decompositions of $i\sigma_{j,0}\sigma_{0,1}^k$ and $i\sigma_{0,1}^k$ for $k = 0, 1, 2, \ldots$. Suppose that $\lambda$ is an irreducible sector appearing in the decomposition of $i\sigma_{0,1}^k$ for some $k$. We have

$$\langle \lambda \sigma_{j,0}, \lambda \sigma_{j,0} \rangle = \langle \lambda \lambda, \sigma_{j,0}^2 \rangle.$$ 

Now the decomposition of $\tilde{\lambda} \lambda$ contains only sectors among $\sigma_{0,l}$, $l = 0, 1, \ldots, m - 1$ as above. Thus the only irreducible sector appearing in decompositions of both $\tilde{\lambda} \lambda$ and $\sigma_{j,0}^2$ is the identity sector, which appears exactly once in the both. We conclude that $\lambda \sigma_{j,0}$ is also irreducible. Thus the irreducible decompositions of $i\sigma_{j,0}\sigma_{0,1}^k$ and $i\sigma_{0,1}^k$ for $k = 0, 1, 2, \ldots$ are described by the same Bratteli diagram and $(G, [v])$ and $(G_2, [v_2])$ are equal.

Then $\tilde{\iota}_{\tilde{t}}$ decomposes into some irreducible sectors among $\sigma_{0,0}, \sigma_{0,1}, \ldots, \sigma_{0,m-1}$. Since the tensor category having the irreducible objects

$$\{\sigma_{0,0}, \sigma_{0,1}, \ldots, \sigma_{0,m-1}\}$$

is isomorphic to the representation category of $SU(2)_{m-1}$, we obtain uniqueness of $\tilde{t}$ for a given $(G, [v])$ hence $(G_2, [v_2])$. Then $\iota = i\sigma_{j,0}$ determines a $Q$-system uniquely, up to unitary equivalence.

We next prove a realization of a given $(G_1, [v_1], G_2, [v_2])$. We continue to assume that $G_1$ is $A_{m-1}$. Let $\sigma_{j,0}$ be one of the two sectors corresponding to $[v_1]$ as above. Using the tensor category having the irreducible objects $\{\sigma_{0,0}, \sigma_{0,1}, \ldots, \sigma_{0,m-1}\}$, we have $\iota$ corresponding to $(G_2, [v_2])$ as in the proof of Theorem 2.1. Set $\iota = i\sigma_{j,0}$. Then one can verify that this $\iota$ produces the quadruple $(G_1, [v_1], G_2, [v_2])$ by the same argument as in the above one showing $G = G_2$.

**Remark 3.3.** By [5, Theorem 4.1], we already know that the local extensions among the above classification are labeled with $(A_{n-1}, A_n)$, $(A_{4n}, D_{2n+2})$, $(D_{2n+2}, A_{4n+2})$, $(A_{10}, E_6)$, $(E_6, A_{12})$, $(A_{28}, E_8)$ and $(E_8, A_{30})$ for $(G_1, G_2)$ and the vertices $v_1, v_2$ are those having the smallest Perron-Frobenius eigenvector entries.

**Remark 3.4.** As in Remark 2.2 we may say that irreducible extensions of the Virasoro nets with $c < 1$ are labeled with pairs of irreducible Goodman-de la Harpe-Jones subfactors having the Coxeter numbers differing by 1.

**Remark 3.5.** The graphs $G_1$ and $G_2$ are by definition bipartite, thus excluding the tadpole graphs which also have Frobenius norm $< 2$. Tadpole diagrams arise by pairwise identification of the vertices of $A_m$ diagrams when $m = 2n$ is even. Indeed,
when \( \iota : A \rightarrow B \) equals \( \sigma_{n-1,j} : A \rightarrow A \), the even vertices of \( G_2 = A_{2n} \) pairwise coincide as \( B-A \) sectors with the odd vertices, so that the fusion graph for multiplication by \( \sigma_{0,1} \) is \( T_n \). The invariant \( G_2 \) in these cases is \( A_m \), nevertheless.

As an example, consider the case \( m = 4 \), that is, \( c = 7/10 \). In this case, we have six irreducible DHR sectors for the net \( \text{Vir}_{7/10} \). The graphs \( G_1 \) and \( G_2 \) are automatically \( A_3 \) and \( A_4 \), respectively, so we have four possibilities for the invariant \( (G_1, [v_1], G_2, [v_2]) \). If \( (G_1, [v_1], G_2, [v_2]) \) is as in case (1) of Fig. 4, then the sector \( \iota \) is given by \( \sigma_{1,1} \), thus the four vertices of the graph \( A_4 \) give only two mutually inequivalent \( B-A \) sectors. That is, the fusion graph of the \( B-A \) sectors for multiplication by \( \sigma_{0,1} \) is the tadpole graph as in Fig. 5.

In case (2) of Fig. 4 we have four mutually inequivalent \( B-A \) sectors for the graph \( G_2 = A_4 \) for the sector \( \iota \) given by \( \sigma_{0,1} \), and the fusion graph is also \( A_4 \).

### 4 The canonical endomorphism

We want to determine, viewed as a representation of the subtheory \( A = \text{Vir}_c \), the vacuum Hilbert space of the local boundary conformal QFT associated with each of the non-local extensions \( B \), classified in the previous section. This representation is given by a DHR endomorphism \( \theta \) of \( A \) whose restriction to a local algebra \( A(I) \) (where \( \theta \) is localized in the interval \( I \)) coincides with the canonical endomorphism \( \bar{\iota} \) of the subfactors \( A(I) \subset B(I) \) classified above. We are therefore interested in the computation of \( \bar{\iota} \).
Figure 6: The graph $G_1 \times G_2$ with $G_1 = A_6$ and $G_2 = D_5$. Different vertices may represent the same $B$-$A$-sector. $\iota$ may be any vertex of $G$.

By the equality of local and global intertwiners, and by reciprocity, the multiplicity of each irreducible DHR sector $\sigma$ within $\theta$ equals the multiplicity of $\iota$ within $i\sigma$. We therefore need to control the decomposition of $i\sigma$ into irreducibles (“fusion”) for all DHR sectors. Because every irreducible sector is a product $\sigma_{j,k} = \sigma_{j,0}\sigma_{0,k}$, and $\sigma_{j,0}$ and $\sigma_{0,k}$ are obtained from the generators $\sigma_{1,0}$ and $\sigma_{0,1}$ by the recursion $\sigma_{0,k+1} = \sigma_{0,k}\sigma_{0,1} \ominus \sigma_{0,k-1}$ and likewise for $\sigma_{j+1,0}$, it suffices to control the fusion with the generators.

We know from the preceding section that the fusion of $\iota$ with the generators $\sigma_{1,0}$ and $\sigma_{0,1}$ separately can be described in terms of the two bi-partite graphs $G_1$ and $G_2$ such that the vertices of the graphs represent irreducible $B$-$A$-sectors and two vertices are linked if the corresponding sectors are connected by the generator. $\iota$ corresponds to a distinguished vertex in both graphs. Moreover, we have seen that the fusion of $\iota$ with both generators can be described by the “product graph” $G = G_1 \times G_2$ with vertices $\lambda = (v_1 \in G_1, v_2 \in G_2)$ and “horizontal” edges linking $(v_1, v_2)$ with $(v_1', v_2)$ if $v_1$ and $v_1'$ are linked in $G_1$, and likewise for “vertical” edges according to the graph $G_2$. Again, the vertices $\lambda$ represent irreducible $B$-$A$-sectors, and $\iota$ is a distinguished vertex of the product graph. See Fig. 6 for an example.

From the product graph $G$, the fusion of each of its vertices with any DHR sector can be computed in terms of vertices of $G$, i.e., $\lambda\sigma$ can be decomposed into irreducibles represented by the vertices of $G$. But different vertices may represent identical $B$-$A$-sectors; we only know that within each horizontal or vertical subgraph, the even vertices represent pairwise inequivalent sectors, and so do the odd vertices, cf. Remark 3.5. In order to compute the canonical endomorphism, we have to determine all identifications between vertices of $G$ as $B$-$A$-sectors.

We continue to assume that $m$ is odd and hence $G_1 = A_{m-1}$. The Coxeter number $m' = 2n$ of $G_2$ is either $m+1$ or $m-1$. We exploit the fact that $\sigma_{m-2,0}$ and $\sigma_{0,m'-2}$ represent the same DHR sector $\tau$, and that $\tau$ is simple (it has dimension 1).
Hence fusion with \( \tau \), as a horizontal sector \( \sigma_{m-2,0} \), yields an automorphism \( \alpha_1 \) of the graph \( G_1 \) such that \( \alpha_1(v_1) = v_1 \sigma_{m-2,0} \), and, as a vertical sector, similarly yields an automorphism \( \alpha_2 \) of \( G_2 \). It follows that the vertices \( \lambda = (v_1, v_2) \) and \( \alpha(\lambda) = (\alpha_1(v_1), \alpha_2(v_2)) \) of the product graph represent the same sectors.

Because \( \tau \) connects even vertices of \( G_1 \) with odd ones, \( \alpha_1 \) must be the unique non-trivial automorphism of \( G_1 \). To determine \( \alpha_2 \), one may use the above-mentioned recursion to compute the fusion of the vertices of \( G_2 \) with \( \tau = \sigma_{0,m'-2} \). We find that \( \alpha_2 \) is the unique non-trivial automorphism if \( G_2 \) is either an \( A \) graph or \( E_6 \) or \( D_{2n+1} \), and it is trivial if \( G_2 \) is \( E_7 \), \( E_8 \), or \( D_{2n} \).

We now claim that the identifications due to \( \alpha \) give all pairs of vertices of \( G \) which represent the same \( B-A \)-sector.

**Proposition 4.1.** The graph

\[
(G_1 \times G_2)/(\alpha_1 \times \alpha_2)
\]

is the fusion graph of \( \iota \) with respect to \( \sigma_{1,0} \) and \( \sigma_{0,1} \), i.e., its vertices represent inequivalent irreducible \( B-A \)-sectors, and its horizontal and vertical edges correspond to fusion with the two generators.

**Proof** By the Perron-Frobenius theory, the dimensions of the \( B-A \)-sectors represented by the vertices \( \lambda = (v_1, v_2) \) of \( G \) are common multiples of \( \nu(v_1) \mu(v_2) \) where \( \nu(v_1) \) and \( \mu(v_2) \) are the components of the Perron-Frobenius eigenvectors \( \nu \) of \( G_1 \) and \( \mu \) of \( G_2 \). Let now \( \lambda = (v_1, v_2) \) and \( \lambda' = (v'_1, v'_2) \) be two vertices of \( G \) which represent the same \( B-A \)-sector. Then clearly

\[
\nu(v_1) \mu(v_2) = \nu(v'_1) \mu(v'_2).
\]

If \( v_1 \) it at distance \( j \) from an extremal vertex \( \tilde{v}_1 \) of \( G_1 \), then \( \tilde{\lambda} = (\tilde{v}_1, v_2) \) is a subsector of \( \lambda \sigma_{j,0} \) and consequently of \( \lambda' \sigma_{j,0} \). Hence \( \tilde{\lambda} \) is equivalent to some subsector \( (\tilde{v}'_1, v'_2) \) of \( \lambda' \sigma_{j,0} \), implying \( \mu(v_2)/\mu(v'_2) = \nu(\tilde{v}'_1)/\nu(\tilde{v}_1) = d_k \) for some \( k \). Lemma 3.2 tells us that this is only possible if \( d_k = 1 \). It follows that \( \mu(v'_2) = \mu(v_2) \) and \( \nu(v'_1) = \nu(v_1) \).

This means in particular that \( v'_1 = v_1 \) or \( v'_1 = \alpha_1(v_1) \), and that \( v_2 \) and \( v'_2 \) and \( \alpha_2(v_2) \) are all even or all odd. If \( v'_1 = v_1 \), then \( \lambda \) and \( \lambda' \) are two even or two odd vertices within the same vertical subgraph representing the same sector. This is only possible if \( \lambda = \lambda' \). If on the other hand \( v'_1 = \alpha_1(v_1) \), then the same argument applies to \( \alpha(\lambda) \) and \( \lambda' \), giving \( \lambda' = \alpha(\lambda) \). \( \square \)

Having determined the fusion graph, it is now straightforward to compute (as described above) the canonical endomorphism for every possible position of \( \iota \) as a distinguished vertex of the fusion graph, and hence to determine the vacuum Hilbert space for each local boundary conformal QFT with \( c < 1 \).

We display below the canonical endomorphism \( \theta_{\tilde{v}_1, \tilde{v}_2} \) whenever \( \tilde{v}_1 \) and \( \tilde{v}_2 \) are extremal vertices of \( G_1 \) and \( G_2 \). All other cases are then easily obtained by the following argument: If \( v_1 \) is at distance \( j \) from an extremal vertex \( \tilde{v}_1 \) of \( G_1 \), then \( v_1 = \tilde{v}_1 \sigma_{j,0} \).
If \( v_2 \) is at distance \( k \) from the extremal vertex \( \tilde{v}_2 \) on the same “leg” of \( G_2 \), then 
\[
v_2 = \tilde{v}_2 \sigma_{0,k}.
\]  
(If \( v_2 \) is the trivalent vertex of the \( D \) or \( E \) graphs, then this is true for each of the three legs.) It then follows that \((v_1, v_2) = (\tilde{v}_1, \tilde{v}_2) \sigma_{j,k} \), and hence
\[
\theta_{v_1, v_2} = \theta_{\tilde{v}_1, \tilde{v}_2} \sigma_{j,k}^2.
\]

The canonical endomorphisms \( \theta_{\tilde{v}_1, \tilde{v}_2} \) for all pairs of extremal vertices of \( G_1 \) and \( G_2 \) are listed in the following table.

| \( G_2 \) | \( m' \) | dist. | \( \theta_{\tilde{v}_1, \tilde{v}_2} \) |
|---|---|---|---|
| \( A_n \) | \( n + 1 \) | - | \( \sigma_{0,0} \) |
| \( D_n \) | \( 2n - 2 \) | 1 | \( \sigma_{0,0} \oplus \sigma_{0,4} \oplus \sigma_{0,8} \oplus \ldots \oplus \sigma_{0,4[n/2] - 4} \) |
| \( E_n \) | \( 2n - 2 \) | \( n - 3 \) | \( \sigma_{0,0} \oplus \sigma_{0,2n - 4} \) |
| \( E_6 \) | 12 | 1 | \( \sigma_{0,0} \oplus \sigma_{0,4} \oplus \sigma_{0,6} \oplus \sigma_{0,10} \) |
| \( E_6 \) | 12 | 2 | \( \sigma_{0,0} \oplus \sigma_{0,6} \) |
| \( E_7 \) | 18 | 1 | \( \sigma_{0,0} \oplus \sigma_{0,4} \oplus \sigma_{0,6} \oplus \sigma_{0,8} \oplus \sigma_{0,10} \oplus \sigma_{0,12} \oplus \sigma_{0,16} \) |
| \( E_7 \) | 18 | 2 | \( \sigma_{0,0} \oplus \sigma_{0,6} \oplus \sigma_{0,10} \oplus \sigma_{0,16} \) |
| \( E_7 \) | 18 | 3 | \( \sigma_{0,0} \oplus \sigma_{0,8} \oplus \sigma_{0,16} \) |
| \( E_8 \) | 30 | 1 | \( \sigma_{0,0} \oplus \sigma_{0,4} \oplus \sigma_{0,6} \oplus \sigma_{0,8} \oplus 2\sigma_{0,10} \oplus \sigma_{0,12} \oplus 2\sigma_{0,14} \oplus \sigma_{0,16} \oplus 2\sigma_{0,18} \oplus \sigma_{0,20} \oplus \sigma_{0,22} \oplus \sigma_{0,24} \oplus \sigma_{0,28} \) |
| \( E_8 \) | 30 | 2 | \( \sigma_{0,0} \oplus \sigma_{0,6} \oplus \sigma_{0,10} \oplus \sigma_{0,12} \oplus \sigma_{0,16} \oplus \sigma_{0,18} \oplus \sigma_{0,22} \oplus \sigma_{0,28} \) |
| \( E_8 \) | 30 | 4 | \( \sigma_{0,0} \oplus \sigma_{0,10} \oplus \sigma_{0,18} \oplus \sigma_{0,28} \) |

**Table 4.1.** The canonical endomorphisms \( \theta_{\tilde{v}_1, \tilde{v}_2} \) for all pairs of extremal vertices of \( G_1 \) and \( G_2 \). The entry in the third column indicates the distance of \( \tilde{v}_2 \) from the trivalent vertex, i.e., the length of the “leg” of \( G_2 \) on which \( \tilde{v}_2 \) is the extremal vertex.

The local chiral extensions classified earlier \[5\] are precisely those cases where \( G_2 \) is \( A, D_{2n}, E_6, \) or \( E_8 \), and both \( v_1 \) and \( v_2 \) are extremal vertices (on the respective longest leg in the \( D \) and \( E \) cases).

In the non-local cases, the local algebras of the associated BCFT on the half-space are the relative commutants as described in the introduction. Note that, in order to determine the resulting factorizing chiral charge structure \[9\] of the local fields, more detailed information about the DHR category and the \( Q \)-system is needed, than the simple combinatorial data exploited in this work.

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