AN ERGODIC THEOREM FOR THE QUASI-REGULAR REPRESENTATION OF THE FREE GROUP

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Abstract. In [BM11], an ergodic theorem à la Birkhoff-von Neumann for the action of the fundamental group of a compact negatively curved manifold on the boundary of its universal cover is proved. A quick corollary is the irreducibility of the associated unitary representation. These results are generalized [Boy15] to the context of convex cocompact groups of isometries of a CAT(-1) space, using Theorem 4.1.1 of [Rob03], with the hypothesis of non arithmeticity of the spectrum. We prove all the analog results in the case of the free group \( F_r \) of rank \( r \) even if \( F_r \) is not the fundamental group of a closed manifold, and may have an arithmetic spectrum.

1. Introduction

In this paper, we consider the action of the free group \( F_r \) on its boundary \( B \), a probability space associated to the Cayley graph of \( F_r \) relative to its canonical generating set. This action is known to be ergodic (see for example [FTP82] and [FTP83]), but since the measure is not preserved, no theorem on the convergence of means of the corresponding unitary operators had been proved. Note that a close result is proved in [FTP83, Lemma 4, Item (i)]. We formulate such a convergence theorem in Theorem 1.2. We prove it following the ideas of [BM11] and [Boy15] replacing [Rob03, Theorem 4.1.1] by Theorem LM.

1.1. Geometric setting and notation. We will denote \( F_r = \langle a_1, \ldots, a_r \rangle \) the free group on \( r \) generators, for \( r \geq 2 \). For an element \( \gamma \in F_r \), there is a unique reduced word in \( \{ a_1^{\pm 1}, \ldots, a_r^{\pm 1} \} \) which represents it. This word is denoted \( \gamma_1 \cdots \gamma_k \) for some integer \( k \) which is called the length of \( \gamma \) and is denoted by \( |\gamma| \). The set of all elements of length \( k \) is denoted \( S_k \) and is called the sphere of radius \( k \). If \( u \in F_r \) and \( k \geq |u| \), let us denote \( Pr_{u}(k) := \{ \gamma \in F_r \mid |\gamma| = k, u \text{ is a prefix of } \gamma \} \).

Let \( X \) be the Cayley graph of \( F_r \) with respect to the set of generators \( \{ a_1^{\pm 1}, \ldots, a_r^{\pm 1} \} \), which is a 2\( r \)-regular tree. We endow it with the (natural) distance, denoted by \( d \), which gives length 1 to every edge; for this distance, the natural action of \( F_r \) on \( X \) is isometric and freely transitive on the vertices; the space \( X \) is uniquely geodesic, the geodesics between vertices being finite sequences of successive edges. We denote by \( [x, y] \) the unique geodesic joining \( x \) to \( y \).

We fix, once and for all, a vertex \( x_0 \) in \( X \). For \( x \in X \), the vertex of \( X \) which is the closest to \( x \) in \( [x_0, x] \), is denoted by \( [x] \) because the action is free, we can identify \( [x] \) with the element \( \gamma \) that brings \( x_0 \) on it, and this identification is an isometry.

The Cayley tree and its boundary. As for any other CAT(-1) space, we can construct a boundary of \( X \) and endow it with a distance and a measure. For a general construction, see [Boy95]. The construction we provide here is elementary.

Let us denote by \( B \) the set of all right-infinite reduced words on the alphabet \( \{ a_1^{\pm 1}, \ldots, a_r^{\pm 1} \} \). This set is called the boundary of \( X \).

We will consider the set \( \overline{X} := X \cup B \).

For \( u = u_1 \cdots u_l \in F_r \setminus \{ e \} \), we define the sets

\[ X_u := \{ x \in X \mid u \text{ is a prefix of } [x] \} \]

\[ B_u := \{ \xi \in B \mid u \text{ is a prefix of } \xi \} \]
\[ C_u := X_u \cup B_u \]

We can now define a natural topology on \( \overline{X} \) by choosing as a basis of neighborhoods

1. for \( x \in X \), the set of all neighborhoods of \( x \) in \( X \)
2. for \( \xi \in B \), the set \( \{ C_u \mid u \text{ is a prefix of } \xi \} \)

For this topology, \( \overline{X} \) is a compact space in which the subset \( X \) is open and dense. The induced topology on \( X \) is the one given by the distance. Every isometry of \( X \) continuously extend to a homeomorphism of \( \overline{X} \).

**Distance and measure on the boundary.** For \( \xi_1 \) and \( \xi_2 \) in \( B \), we define the Gromov product of \( \xi_1 \) and \( \xi_2 \) with respect to \( x_0 \) by

\[ (\xi_1|\xi_2)_{x_0} := \sup \{ k \in \mathbb{N} \mid \xi_1 \text{ and } \xi_2 \text{ have a common prefix of length } k \} \]

and

\[ d_{x_0}(\xi_1, \xi_2) := e^{-(\xi_1|\xi_2)_{x_0}}. \]

Then \( d \) defines an ultrametric distance on \( B \) which induces the same topology; precisely, if \( \xi = u_1 u_2 u_3 \cdots \), then the ball centered in \( \xi \) of radius \( e^{-k} \) is just \( B_{u_1 \cdots u_k} \).

On \( B \), there is at most one Borel regular probability measure which is invariant under the isometries of \( X \) which fix \( x_0 \); indeed, such a measure \( \mu_{x_0} \) must satisfy

\[ \mu_{x_0}(B_u) = \frac{1}{2r(2r-1)^{|u|-1}} \]

and it is straightforward to check that the \( \ln(2r-1) \)-dimensional Hausdorff measure verifies this property.

If \( \xi = u_1 \cdots u_n \cdots \in B \), and \( x, y \in X \), then \( (d(x, u_1 \cdots u_n) - d(y, u_1 \cdots u_n))_{n \in \mathbb{N}} \) is stationary.

We denote this limit \( \beta_{\xi}(x, y) \). The function \( \beta_{\xi} \) is called the Busemann function at \( \xi \).

Let us denote, for \( \xi \in B \) and \( \gamma \in \mathbb{F}_r \), the function

\[ P(\gamma, \xi) := (2r-1)^\beta_{\xi}(x_0, \gamma x_0) \]

The measure \( \mu_{x_0} \) is, in addition, quasi-invariant under the action of \( \mathbb{F}_r \). Precisely, the Radon-Nikodym derivative is given for \( \gamma \in \Gamma \) and for a.e. \( \xi \in B \) by

\[ \frac{d\gamma_*\mu_{x_0}}{d\mu_{x_0}}(\xi) = P(\gamma, \xi), \]

where \( \gamma_*\mu_{x_0}(A) = \mu_{x_0}(\gamma^{-1} A) \) for any Borel subset \( A \subset B \).

**The quasi-regular representation.** Denote the unitary representation, called the quasi-regular representation of \( \mathbb{F}_r \) on the boundary of \( X \) by

\[ \pi : \mathbb{F}_r \to \mathcal{U}(L^2(B)) \]

\[ \gamma \mapsto \pi(\gamma) \]

defined as

\[ (\pi(\gamma)g)(\xi) := P(\gamma, \xi)^{\frac{1}{2}} g(\gamma^{-1} \xi) \]

for \( \gamma \in \mathbb{F}_r \) and for \( g \in L^2(B) \). We define the Harish-Chandra function

\[ \Xi(\gamma) := \langle \pi(\gamma)1_B, 1_B \rangle = \int_B P(\gamma, \xi)^{\frac{1}{2}} d\mu_{x_0}(\xi), \]

where \( 1_B \) denotes the characteristic function on the boundary.

For \( f \in C(\overline{X}) \), we define the operators

\[ M_n(f) : g \in L^2(B) \mapsto \frac{1}{|S_n|} \sum_{\gamma \in S_n} f(\gamma x_0) \pi(\gamma)g(\gamma^{-1} \xi) \in L^2(B). \]

We also define the operator

\[ M(f) := m(f_{|B})P_{1_B} \]

where \( m(f_{|B}) \) is the multiplication operator by \( f_{|B} \) on \( L^2(B) \), and \( P_{1_B} \) is the orthogonal projection on the subspace of constant functions.
Results. The analog of Roblin’s equidistribution theorem for the free group is the following.

Theorem 1.1. We have, in $C(\overline{X} \times \overline{X})^*$, the weak-\* convergence
\[
\frac{1}{|S_n|} \sum_{\gamma \in S_n} D_{\gamma x_0} \otimes D_{\gamma^{-1} x_0} \rightharpoonup \mu_{x_0} \otimes \mu_{x_0}
\]
where $D_x$ denotes the Dirac measure on a point $x$.

Remark 1. It is then straightforward to deduce the weak-\* convergence
\[
\|m_I\|e^{-\delta n} \sum_{|\gamma| \leq n} D_{\gamma x_0} \otimes D_{\gamma^{-1} x_0} \rightharpoonup \mu_{x_0} \otimes \mu_{x_0}
\]

$m_I$ denoting the Bowen-Margulis-Sullivan measure on the geodesic flow of $\overline{X}/\Gamma$ (where $\overline{X}$ is the “unit tangent bundle”) and $\delta$ denoting $\ln(2r - 1)$, the Hausdorff measure of $B$.

1. Notice that in our case, the spectrum is $\mathbb{Z}$ so the geodesic flow is not topologically mixing, according to [Dal99] or directly by [CT01, Ex 1.3].

2. Notice also that our multiplicative term is different of that of [Rob03, Theorem 4.1.1], which shows that the hypothesis of non-arithmeticity of the spectrum cannot be removed.

We use the above theorem to prove the following convergence of operators.

Theorem 1.2. We have, for all $f$ in $C(\overline{X})$, the weak operator convergence
\[
M_n(f) \rightharpoonup \pi(f).
\]
In other words, we have, for all $f$ in $C(\overline{X})$ and for all $g, h$ in $L^2(B)$, the convergence
\[
\frac{1}{|S_n|} \sum_{\gamma \in S_n} f(\gamma x_0) \langle \pi(\gamma) g, h \rangle_{\Xi(\gamma)} \rightharpoonup \langle \pi(f) g, h \rangle.
\]

We deduce the irreducibility of $\pi$, and give an alternative proof of this well known result (see [FTPS82, Theorem 5]).

Corollary 1.3. The representation $\pi$ is irreducible.

Proof. Applying Theorem 1.2 to $f = 1_B$ shows that the orthogonal projection onto the space of constant functions is in the von Neumann algebra associated with $\pi$. Then applying Theorem 1.2 to $g = 1_B$ shows that the vector $1_B$ is cyclic. Then, the classical argument of [Gar14, Lemma 6.1] concludes the proof.

Remark 2. For $\alpha \in \mathbb{R}_+^*$, let us denote by $W_\alpha$ the wedge of two circles, one of length 1 and the other of length $\alpha$. Let $p : T_\alpha \to W_\alpha$ the universal cover, with $T_\alpha$ endowed with the distance making $p$ a local isometry. Then $\mathbb{F}_2 \simeq \pi_1(W_\alpha)$ acts freely properly discontinuously and cocompactly on the 4-regular tree $T_\alpha$ (which is a CAT(-1) space) by isometries. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the analog of Theorem 1.2 for the quasi-regular representation $\pi_\alpha$ of $\mathbb{F}_2$ on $L^2(\partial T_\alpha, \mu_\alpha)$ for a Patterson-Sullivan measure associated to a Bourdon distance is known to hold ([Boy15]) because [Rob03, Theorem 4.1.1] is true in this setting. Now if $\alpha_1$ and $\alpha_2$ are such that $\alpha_1 \neq \alpha_2^{-1}$, then the representations $\pi_\alpha$ are not unitarily equivalent ([Gar14, Theorem 7.5]). For $\alpha \in \mathbb{Q}_+^* \setminus \{1\}$, it would be interesting to formulate and prove an equidistribution result like Theorem 1.1 in order to prove Theorem 1.2 for $\pi_\alpha$.

2. Proofs

2.1. Proof of the equidistribution theorem. For the proof of Theorem 1.1 let us denote
\[
E := \left\{ f : C(\overline{X} \times \overline{X}) \mid \frac{1}{|S_n|} \sum_{\gamma \in S_n} f(\gamma x_0, \gamma^{-1} x_0) \to \int_{\overline{X} \times \overline{X}} f d(\mu_{x_0} \otimes \mu_{x_0}) \right\}
\]
The subspace $E$ is clearly closed in $C(\overline{X} \times \overline{X})$; it remains only to show that it contains a dense subspace of it.
Let us define a modified version of certain characteristic functions: for \( u \in \mathbb{F} \), we define
\[
\chi_u(x) := \begin{cases} 
\max\{1 - d(x, C_u), 0\} & \text{if } x \in X \\
0 & \text{if } x \in B \setminus B_u \\
1 & \text{if } x \in B_u
\end{cases}
\]

It is easy to check that the function \( \chi_u \) is a continuous function which coincides with \( \chi_{C_u} \) on \( \mathbb{F}, x_0 \) and \( B \).

The proof of the following lemma is straightforward.

**Lemma 2.1.** Let \( u \in \mathbb{F} \) and \( k \geq |u| \), then \( \chi_u - \sum_{\gamma \in P_{\nu(u)}} \chi_{\gamma} \) has compact support included in \( X \).

**Proposition 2.2.** The set \( \chi := \{ \chi_u \mid u \in \mathbb{F} \} \) separates points of \( B \), and the product of two such functions of \( \chi \) is either in \( \chi \), the sum of a function in \( \chi \) and of a function with compact support contained in \( X \), or zero.

**Proof.** It is clear that \( \chi \) separates points. It follows from Lemma 2.1 that \( \chi_u \chi_v = \chi_v \) if \( u \) is a proper prefix of \( v \), that \( \chi_u^2 - \chi_u \) has compact support in \( X \), and that \( \chi_u \chi_v = 0 \) if none of \( u \) and \( v \) is a proper prefix of the other. \( \square \)

**Proposition 2.3.** The subspace \( E \) contains all functions of the form \( \chi_u \otimes \chi_v \).

**Proof.** We make the useful observation that
\[
\frac{1}{|S_n|} \sum_{\gamma \in S_n} (\chi_u \otimes \chi_v)(\gamma x_0, \gamma^{-1} x_0) = \frac{|S_{n,u}^{a,v}|}{|S_n|}
\]
where \( S_{n,u}^{a,v} \) is the set of reduced words of length \( n \) with \( u \) as a prefix and \( v^{-1} \) as a suffix. We easily see that this set is in bijection with the set of all reduced words of length \( n - (|u| + |v|) \) that do not begin by the inverse of the last letter of \( u \), and that do not end by the inverse of the first letter of \( v^{-1} \). So we have to compute, for \( s, t \in \{ a_1^{\pm 1}, ..., a_{\ell + 1}^{\pm 1} \} \) and \( m \in \mathbb{N} \), the cardinal of the set \( S_m(s,t) \) of reduced words of length \( m \) that do not start by \( s \) and do not finish by \( t \).

Now we have
\[
S_m = S_{m}(s,t) \cup \{ x \mid |x| = m \text{ and starts by } s \} \cup \{ x \mid |x| = m \text{ and ends by } t \}
\]

Note that the intersection of the two last sets is the set of words both starting by \( s \) and ending by \( t \), which is in bijection with \( S_{m-2}(s^{-1}, t^{-1}) \).

We have then the recurrence relation:
\[
|S_m(s,t)| = 2r(2r-1)^{m-1} - 2(2r-1)^m + |S_{m-2}(s^{-1}, t^{-1})| = 2(r-1)(2r-1)^{m-1} + 2(r-1)(2r-1)^m - 2(2r-1)^{m-1} |S_{m-4}(s,t)|
\]
\[
= (2r-1)^m \left( \frac{(2r-1)^2 + 1}{(2r-1)^3} \right) + |S_{m-4}(s,t)|
\]

We set \( C := \frac{2(r-1)(2r-1)^2 + 1}{(2r-1)^3} \), \( n = 4k + j \) with \( 0 \leq j \leq 3 \) and we obtain
\[
|S_{4k+j}^{a,b}| = C(2r-1)^{4k+j} + |S_{4(k-1)+j}^{a,b}|
\]
\[
= C(2r-1)^{4k+j} + C(2r-1)^{4(k-1)+j} + |S_{4(k-2)+j}^{a,b}|
\]
\[
= C \sum_{i=1}^{k} (2r-1)^{4i+j} + |S_{4i+j}^{a,b}|
\]
\[
= C(2r-1)^{4j+1} \frac{(2r-1)^{4k} - 1}{(2r-1)^4 - 1} + |S_{j}(s,t)|
\]
\[
= (2r-1)^{4j+1} \frac{(2r-1)^{4k} - 1}{2r} + |S_{j}(s,t)|
\]

Now we can compute
Lemma 2.6. The function fixing \( x \) can be written as

\[
|S_{4k+j}^{n,v}| = \frac{|S_{4k+j-(|u|+|v|)}(u_{|u|}, v_{|v|})|}{|S_{4k+j}|} = \frac{(2r - 1)^{1+j}(2r - 1)^{4k-(|u|+|v|)} - 1}{2r^2(2r - 1)^{4k+j-1}} + \frac{1}{2r^2(2r - 1)^{4k+j-1} + o(1)} = \mu_{x_0}(B_u)\mu_{x_0}(B_v) + o(1)
\]
when \( k \to \infty \), and this proves the claim. \( \square \)

Corollary 2.4. The subspace \( E \) is dense in \( C(\overline{X} \times \overline{X}) \).

Proof. Let us consider \( E' \), the subspace generated by the constant functions, the functions which can be written as \( f \otimes g \) where \( f, g \) are continuous functions on \( \overline{X} \) and such that one of them has compact support included in \( X \), and the functions of the form \( \chi_u \otimes \chi_v \). By Proposition \( \ref{prop:stone-weierstrass} \), it is a subalgebra of \( C(\overline{X} \times \overline{X}) \) containing the constants and separating points, so by the Stone-Weierstraß theorem, \( E' \) is dense in \( C(\overline{X} \times \overline{X}) \). Now, by Proposition \( \ref{prop:stone-weierstrass} \), we have that \( E' \subseteq E \), so \( E \) is dense as well. \( \square \)

2.2. Proof of the ergodic theorem. The proof of Theorem \( \ref{thm:ergodic} \) consists in two steps:

Step 1: Prove that the sequence \( M_n \) is bounded in \( \mathcal{L}(C(\overline{X}), B(L^2(B))) \).

Step 2: Prove that the sequence converges on a dense subset.

2.2.1. Boundedness. In the following \( 1_{\overline{X}} \) denotes the characteristic function of \( \overline{X} \). Define

\[
F_n := [M_n(1_{\overline{X}})] 1_B.
\]

We denote by \( \Xi(n) \) the common value of \( \Xi \) on elements of length \( n \).

Corollary 2.5. The function \( \xi \mapsto \sum_{\gamma \in S_n} (P(\gamma, \xi))^\frac{1}{2} \) is constant equal to \( |S_n| \times \Xi(n) \).

Proof. This function is constant on orbits of the action of the group of automorphisms of \( X \) fixing \( x_0 \). Since it is transitive on \( B \), the function is constant. By integrating, we find

\[
\sum_{\gamma \in S_n} (P(\gamma, \xi))^\frac{1}{2} = \int_B \sum_{\gamma \in S_n} (P(\gamma, \xi))^\frac{1}{2} d\mu_{x_0}(\xi) = \sum_{\gamma \in S_n} \int_B (P(\gamma, \xi))^\frac{1}{2} d\mu_{x_0}(\xi) = \sum_{\gamma \in S_n} \Xi(n) = |S_n| \Xi(n),
\]

Lemma 2.6. The function \( F_n \) is constant, equal to \( 1_B \).

Proof. Because \( \Xi \) depends only on the length, we have that

\[
F_n(\xi) := \frac{1}{|S_n|} \sum_{\gamma \in S_n} (P(\gamma, \xi))^\frac{1}{2} \Xi(\gamma) = \frac{1}{|S_n| \Xi(n)} \sum_{\gamma \in S_n} (P(\gamma, \xi))^\frac{1}{2} = 1,
\]
and the proof is done. \( \square \)
It is easy to see that $M_n(f)$ induces continuous linear transformations of $L^1$ and $L^\infty$, which we also denote by $M_n(f)$.

**Proposition 2.7.** The operator $M_n(\mathbf{1}_X^\gamma)$, as an element of $\mathcal{L}(L^\infty, L^\infty)$, has norm 1; as an element of $\mathcal{B}(L^2(B))$, it is self-adjoint.

**Proof.** Let $h \in L^\infty(B)$. Since $M_n(\mathbf{1}_X^\gamma)$ is positive, we have that

\[ \| [M_n(\mathbf{1}_X^\gamma)] h \|_\infty \leq \| [M_n(\mathbf{1}_X^\gamma)] \mathbf{1}_B \|_\infty \| h \|_\infty = \| F_n \|_\infty \| h \|_\infty = \| h \|_\infty \]

so that $\| M_n(\mathbf{1}_X^\gamma) \|_{\mathcal{L}(L^\infty, L^\infty)} \leq 1$.

The self-adjointness follows from the fact that $\pi(\gamma^*) = \pi(\gamma^{-1})$ and that the set of summation is symmetric. \[\Box\]

Let us briefly recall one useful corollary of Riesz-Thorin’s theorem:

Let $(Z, \mu)$ be a probability space.

**Proposition 2.8.** Let $T$ be a continuous operator of $L^1(Z)$ to itself such that the restriction $T_2$ to $L^2(Z)$ (resp. $T_\infty$ to $L^\infty(Z)$) induces a continuous operator of $L^2(Z)$ to itself (resp. $L^\infty(Z)$ to itself).

Suppose also that $T_2$ is self-adjoint, and assume that $\|T_\infty\|_{\mathcal{L}(L^\infty(Z), L^\infty(Z))} \leq 1$.

Then $\|T_2\|_{\mathcal{L}(L^2(Z), L^2(Z))} \leq 1$.

**Proof.** Consider the adjoint operator $T^*$ of $(L^1)^* = L^\infty$ to itself. We have that

\[ \|T^*\|_{\mathcal{L}(L^\infty, L^\infty)} = \|T\|_{\mathcal{L}(L^1(Z), L^1(Z))} \]

Now because $T_2$ is self-adjoint, it is easy to see that $T^* = T_\infty$. This implies

\[ 1 \geq \|T^*\|_{\mathcal{L}(L^\infty, L^\infty)} = \|T\|_{\mathcal{L}(L^1(Z), L^1(Z))} \]

Hence the Riesz-Thorin’s theorem gives us the claim. \[\Box\]

**Proposition 2.9.** The sequence $(M_n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{L}(C(X), \mathcal{B}(L^2(B)))$.

**Proof.** Because $M_n(f)$ is positive in $f$, we have, for every positive $g \in L^2(B)$, the inequality

\[ -\|f\|_\infty [M_n(\mathbf{1}_X^\gamma)] g \leq [M_n(f)] g \leq \|f\|_\infty [M_n(\mathbf{1}_X^\gamma)] g \]

from which we deduce, for every $g \in L^2(B)$

\[ \| [M_n(\mathbf{1}_X^\gamma)] g \|_{L^2} \leq \|f\|_\infty [\mathcal{L}_2] \|M_n(\mathbf{1}_X^\gamma)] g \|_{L^2} \leq \|f\|_\infty \|M_n(\mathbf{1}_X^\gamma)] \|_{\mathcal{B}(L^2)} \| g \|_{L^2} \]

which allows us to conclude that

\[ \| M_n(f) \|_{\mathcal{B}(L^2)} \leq \| M_n(\mathbf{1}_X^\gamma)] \|_{\mathcal{B}(L^2)} \| f \|_\infty \].

This proves that $\| M_n \|_{\mathcal{L}(C(X), \mathcal{B}(L^2))} \leq \| M_n(\mathbf{1}_X^\gamma) \|_{\mathcal{B}(L^2)}$.

Now, it follows from Proposition 2.7 and Proposition 2.8 that the sequence $(M_n(\mathbf{1}_X^\gamma))_{n \in \mathbb{N}}$ is bounded by 1 in $\mathcal{B}(L^2)$, so we are done. \[\Box\]

2.2.2. Estimates for the Harish-Chandra function. The values of the Harish-Chandra are known (see for example [LPS2] Theorem 2, Item (iii)]). We provide here the simple computations we need.

We will calculate the value of

\[ \langle \pi(\gamma) \mathbf{1}_B, \mathbf{1}_B \rangle = \int_{B_n} P(\gamma, \xi) \frac{1}{d \mu_{x_0}} d \mu(\xi). \]
Lemma 2.10. Let $\gamma = s_1 \cdots s_n \in \mathbb{F}_r$. Let $l \in \{1, ..., |\gamma|\}$, and $u = s_1 \cdots s_{l-1} t_l t_{l+1} \cdots t_{|\gamma|}$, with $t_l \neq s_1$ and $k \geq 0$, be a reduced word. Then

$$\langle \pi(\gamma)1_B, 1_{B_u} \rangle = \frac{1}{2r(2r-1)^{\frac{|\gamma|}{2}}}$$

and

$$\langle \pi(\gamma)1_B, 1_{B_s} \rangle = \frac{2r-1}{2r(2r-1)^{\frac{|\gamma|}{2}}}$$

Proof. The function $\xi \mapsto \beta_\xi(x_0, \gamma x_0)$ is constant on $B_u$ equal to $2(l - 1) - |\gamma|$. So $\langle \pi(\gamma)1_B, 1_{B_u} \rangle$ is the integral of a constant function:

$$\int_{B_u} P(\gamma, \xi) \frac{1}{2} d\mu(x_0) = \mu(x_0(B_u)) e^{\log(2r-1)(l - 1) - |\gamma|}$$

$$= \frac{1}{2r(2r-1)^{\frac{|\gamma|}{2}}}.$$

The value of $\langle \pi(\gamma)1_B, 1_{B_s} \rangle$ is computed in the same way. \hfill \qed

Lemma 2.11. (The Harish-Chandra function)

Let $\gamma = s_1 \cdots s_n$ in $S_n$ written as a reduced word. We have that

$$\Xi(\gamma) = \left(1 + \frac{r-1}{r}|\gamma|\right)(2r-1)^{-\frac{|\gamma|}{2}}.$$

Proof. We decompose $B$ into the following partition:

$$B = \bigsqcup_{u \neq s_1} B_u \sqcup \left( \bigsqcup_{l=2} |\gamma| \bigsqcup \bigsqcup_{u=s_1 \cdots s_{l-1} t_l \in \{s_1 \cdots \} \cap S_n} B_u \right) \cup B_\gamma$$

and Lemma 2.10 provides us the value of the integral on the subsets forming this partition. A simple calculation yields the announced formula. \hfill \qed

The proof of the following lemma is then obvious:

Lemma 2.12. If $\gamma, w \in \mathbb{F}_r$ are such that $w$ is not a prefix of $\gamma$, then there is a constant $C_w$ not depending on $\gamma$ such that

$$\frac{\langle \pi(\gamma)1_B, 1_{B_w} \rangle}{\Xi(\gamma)} \leq C_w \frac{1}{|\gamma|}.$$

2.2.3. Analysis of matrix coefficients. The goal of this section is to compute the limit of the matrix coefficients $\langle M_n(\chi_u)1_{B_w}, 1_{B_w} \rangle$.

Lemma 2.13. Let $u, w \in \mathbb{F}_r$ such that none of them is a prefix of the other (i.e. $B_u \cap B_w = \emptyset$). Then

$$\lim_{n \to \infty} \langle M_n(\chi_u)1_B, 1_{B_w} \rangle = 0$$

Proof. Using Lemma 2.12 we get

$$\langle M_n(\chi_u)1_B, 1_{B_w} \rangle = \frac{1}{|S_n|} \sum_{\gamma \in S_n} \chi_u(\gamma x_0) \underbrace{\langle \pi(\gamma)1_B, 1_{B_w} \rangle}_{\Xi(\gamma)}$$

$$= \frac{1}{|S_n|} \sum_{\gamma \in C_w \cap S_n} \underbrace{\langle \pi(\gamma)1_B, 1_{B_w} \rangle}_{\Xi(\gamma)}$$

$$\leq \frac{1}{|S_n|} \sum_{\gamma \in C_w \cap S_n} C_w \frac{1}{|\gamma|}$$

$$= O\left(\frac{1}{n}\right)$$

\footnote{For $l = 1$, $s_1 \cdots s_{l-1}$ is $e$ by convention.}
Lemma 2.14. Let \( u, v \in \mathbb{F}_r \). Then
\[
\limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_B \rangle \leq \mu_{x_0}(B_u)\mu_{x_0}(B_v)
\]

Proof. For all \( u, v \in \mathbb{F}_r \), we first show the inequality
\[
\langle M_n(\chi_u)1_{B_v}, 1_B \rangle = \frac{1}{|S_n|} \sum_{\gamma \in S_n} \chi_u(\gamma^{-1}x_0) \frac{\langle \pi(\gamma)1_{B_v}, 1_B \rangle}{\Xi(\gamma)} \leq \frac{1}{|S_n|} \sum_{\gamma \in S_n} \chi_u(\gamma^{-1}x_0) \chi_v(\gamma x_0) + \frac{1}{|S_n|} \sum_{\gamma \in S_n} \chi_u(\gamma^{-1}x_0) \langle \pi(\gamma)1_{B_v}, 1_B \rangle \Xi(\gamma)
\]

Hence, by taking the \( \limsup \) and using Theorem 1, we obtain the desired inequality.

Proposition 2.15. For all \( u, v, w \in \mathbb{F}_r \), we have
\[
\lim_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle = \mu_{x_0}(B_u \cap B_w)\mu_{x_0}(B_v)
\]

Proof. We first show the inequality
\[
\limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle \leq \mu_{x_0}(B_u \cap B_w)\mu_{x_0}(B_v).
\]
If none of \( u \) and \( w \) is a prefix of the other, we have nothing to do according to Lemma 2.13. Let us assume that \( u \) is a prefix of \( w \) (the other case can be treated analogously). We have, by Lemma 2.14, that
\[
\mu_{x_0}(B_u)\mu_{x_0}(B_v) \geq \limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_B \rangle \geq \limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle \geq \limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle + \sum_{\gamma \in \mathbb{P}_{ru}(|w|) \setminus \{w\}} \limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle = \limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle
\]
We now compute the expected limit. Let us define
\[
S_{u,v,w} := \{(u', v', w') \in \mathbb{F}_r \mid |u| = |u'|, |v| = |v'|, |w| = |w'|\}.
\]
Then
\[
1 = \liminf_{n \to \infty} \langle M_n(1_{B_v})1_{B_w} \rangle \leq \liminf_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle + \sum_{(u', v', w') \in S_{u,v,w} \setminus \{u,v,w\}} \limsup_{n \to \infty} \langle M_n(\chi_{u'})1_{B_{v'}}, 1_{B_{w'}} \rangle \leq \limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle + \sum_{(u', v', w') \in S_{u,v,w} \setminus \{u,v,w\}} \mu_{x_0}(B_{u' \cap B_{w'}})\mu_{x_0}(B_{v'}) \leq \mu_{x_0}(B_u \cap B_w)\mu_{x_0}(B_v) + \sum_{(u', v', w') \in S_{u,v,w} \setminus \{u,v,w\}} \mu_{x_0}(B_{u' \cap B_{w'}})\mu_{x_0}(B_{v'}) = 1
\]
This proves that all the inequalities above are in fact equalities, and moreover proves that the inequalities
\[
\liminf_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle \leq \limsup_{n \to \infty} \langle M_n(\chi_u)1_{B_v}, 1_{B_w} \rangle \leq \mu_{x_0}(B_u \cap B_w)\mu_{x_0}(B_v)
\]
are in fact equalities.

Proof of Theorem 1.2. Because of the boundedness of the sequence \((M_n)_{n \in \mathbb{N}}\) proved in Proposition 2.9, it is enough to prove the convergence for all \((f, h_1, h_2)\) in a dense subset of \(C(X) \times L^2 \times L^2\), which is what Proposition 2.15 asserts.

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