Let $C$ be a smooth projective curve of genus 0. Let $B$ be the variety of complete flags in an $n$-dimensional vector space $V$. Given an $(n-1)$-tuple $\alpha$ of positive integers one can consider the space $Q_\alpha$ of algebraic maps of degree $\alpha$ from $C$ to $B$. This space has drawn much attention recently in connection with Quantum Cohomology (see e.g. [Giv], [Kon]). The space $Q_\alpha$ is smooth but not compact (see e.g. [Kon]). The problem of compactification of $Q_\alpha$ proved very important. One compactification $Q^K_\alpha$ was constructed in loc. cit. (the space of stable maps). Another compactification $Q^L_\alpha$ (the space of quasiflags), was constructed in [Lau]. However, historically the first and most economical compactification $Q^D_\alpha$ (the space of quasimaps) was constructed by Drinfeld (early 80-s, unpublished). The latter compactification is singular, while the former ones are smooth. Drinfeld has conjectured that the natural map $\pi : Q^L_\alpha \to Q^D_\alpha$ is a small resolution of singularities. In the present note we prove this conjecture after the necessary recollections. In fact, the proof gives some additional information about the fibers of $\pi$. It appears that every fiber has a cell decomposition, i.e. roughly speaking, is a disjoint union of affine spaces. This permits to compute not only the stalks of $IC$ sheaf on $Q^D_\alpha$ but, moreover, the Hodge structure in these stalks. Namely, the Hodge structure is a pure Tate one, and the generating function for the $IC$ stalks is just the Lusztig's $q$-analogue of Kostant's partition function (see [Lus]).

In conclusion, let us mention that the Drinfeld compactifications are defined for the space of maps into flag manifolds of arbitrary semisimple group, and we plan to study their small resolutions in a paper to follow.

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1. The space of maps into flag variety

1.1. Notations. Let $G$ be a complex semisimple simply-connected Lie group, $H \subset B$ its Cartan and Borel subgroups, $N$ the unipotent radical of $B$, $Y$ the lattice of coroots of $G$ (with respect to $H$), $I = \{i_1, i_2, \ldots, i_l\}$ the set of simple coroots, $R^+$ the set of positive coroots, $X$ the lattice of weights, $X^+$ the cone of dominant weights, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_l\}$ the set of fundamental weights ($\langle \omega_k, i_l \rangle = \delta_{kl}$), $B = G/B$ the flag variety and $C$ a smooth projective curve of genus 0. Recall that there are canonical isomorphisms

$$H_2(B, \mathbb{Z}) \cong Y \quad H^2(B, \mathbb{Z}) \cong X.$$ 

For $\lambda \in X$ let $L_\lambda$ denote the corresponding $G$-equivariant line bundle on $B$.

---

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The map $\varphi : C \to B$ has degree $\alpha \in \mathbb{N}[I] \subset Y$ if the following equivalent conditions hold:

1. $\varphi_*(C) = \alpha$;
2. for any $\lambda \in X$ we have $\deg(\varphi^*L_\lambda) = (\lambda, \alpha)$.

We denote by $Q_\alpha$ the space of algebraic maps from $C$ to $B$ of degree $\alpha$. It is known that $Q_\alpha$ is smooth variety and $\dim Q_\alpha = 2|\alpha| + \dim B$. In this paper we compare two natural compactifications of the space $Q_\alpha$, which we presently describe.

### 1.2. Drinfeld’s compactification.**

The Plücker embedding of the flag variety $B$ gives rise to the following interpretation of $Q_\alpha$.

For any irreducible representation $V_\lambda$ ($\lambda \in X^+$) of $G$ we consider the trivial vector bundle $V_\lambda = V_\lambda \otimes \mathcal{O}_C$ over $C$.

For any $G$-morphism $\psi : V_\lambda \otimes V_\mu \to V_\nu$ we denote by the same letter the induced morphism $\psi : V_\lambda \otimes V_\mu \to V_\nu$.

Then $Q_\alpha$ is the space of collections of line subbundles $L_\lambda \subset V_\lambda$, $\lambda \in X^+$ such that:

a) $\deg L_\lambda = -(\lambda, \alpha)$;

b) For any nonzero $G$-morphism $\psi : V_\lambda \otimes V_\mu \to V_\nu$ such that $\nu = \lambda + \mu$ we have $\psi(L_\lambda \otimes L_\mu) = L_\nu$;

c) For any $G$-morphism $\psi : V_\lambda \otimes V_\mu \to V_\nu$ such that $\nu < \lambda + \mu$ we have $\psi(L_\lambda \otimes L_\mu) = 0$.

**Remark 1.2.1.** Certainly, the property b) guarantees that in order to specify such a collection it suffices to give $L_{\omega_\lambda}$ for the set $\Omega$ of fundamental weights.

If we replace the curve $C$ by a point, we get the Plücker description of the flag variety $B$ as the space of collections of lines $L_\lambda \subset V_\lambda$ satisfying conditions of type (b) and (c) (thus $B$ is embedded into $\coprod_{\lambda \in X^+} \mathbb{P}(V_\lambda)$). Here, a Borel subgroup $B$ in $B$ corresponds to a system of lines $(L_\lambda, \lambda \in X^+)$ if lines are the fixed points of the unipotent radical of $B$, $L_\lambda = (V_\lambda)^{G}$, or equivalently, if $N$ is the common stabilizer for all lines $N = \bigcap_{\lambda \in X^+} G_{L_\lambda}$.

The following definition in case $G = SL_2$ appeared in [Dr].

**Definition 1.2.2 (V. Drinfeld).** The space $Q_\alpha^D$ of quasimaps of degree $\alpha$ from $C$ to $B$ is the space of collections of invertible subsheaves $L_\lambda \subset V_\lambda$, $\lambda \in X^+$ such that:

a) $\deg L_\lambda = -(\lambda, \alpha)$;

b) For any nonzero $G$-morphism $\psi : V_\lambda \otimes V_\mu \to V_\nu$ such that $\nu = \lambda + \mu$ we have $\psi(L_\lambda \otimes L_\mu) = L_\nu$;

c) For any $G$-morphism $\psi : V_\lambda \otimes V_\mu \to V_\nu$ such that $\nu < \lambda + \mu$ we have $\psi(L_\lambda \otimes L_\mu) = 0$.

**Remark 1.2.3.** Here is another version of the Definition, also due to V. Drinfeld. The principal affine space $\mathcal{A} = G/\mathfrak{N}$ is an $H$-torsor over $B$. We consider its affine closure $\mathfrak{A}$, that is, the spectrum of the ring of functions on $\mathcal{A}$. The action of $H$ extends to $\mathfrak{A}$ but it is not free anymore. Consider the quotient stack $\tilde{B} = \mathfrak{A}/H$. The flag variety $B$ is an open substack in $\tilde{B}$. A map $\phi : C \to \tilde{B}$ is nothing else than an $H$-torsor $\Phi$ over $C$ along with an $H$-equivariant morphism $f : \Phi \to \mathfrak{A}$. The degree of this map is defined as follows.
Let $\chi_\lambda : H \to \mathbb{C}^*$ be the character of $H$ corresponding to a weight $\lambda \in X$. Let $H_\lambda \subset H$ be the kernel of the morphism $\chi_\lambda$. Consider the induced $\mathbb{C}^*$-torsor $\Phi_\lambda = \Phi/H_\lambda$ over $C$. The map $\phi$ has degree $\alpha \in \mathbb{N}[I]$ if for any $\lambda \in X$ we have $\deg(\Phi_\lambda) = \langle \lambda, \alpha \rangle$.

**Definition 1.2.4.** The space $Q^D_\alpha$ is the space of maps $\tilde{\phi} : C \to \tilde{B}$ of degree $\alpha$ such that the generic point of $C$ maps into $B \subset \tilde{B}$.

The equivalence of $1.2.2$ and $1.2.4$ follows immediately from the Plücker embedding of $A$ into $\prod_{\lambda \in X^+} V_\lambda$.

**Proposition 1.2.5.** $Q^D_\alpha$ is a projective variety.

**Proof.** The space $Q^D_\alpha$ is naturally embedded into the space

$$\prod_{k=1}^l \mathbb{P}(\text{Hom}(O_C(-\langle \omega_k, \alpha \rangle), V_{\omega_k}))$$

and is closed in it. \qed

1.3. **The stratification of the Drinfeld’s compactification.** In this subsection we will introduce the stratification of the space of quasimaps.

**Configurations of $I$-colored divisors.**

Let us fix $\alpha \in \mathbb{N}[I] \subset Y$, $\alpha = \sum_{k=1}^N a_k i_k$. Consider the configuration space $C^\alpha$ of colored effective divisors of multidegree $\alpha$ (the set of colors is $I$). The dimension of $C^\alpha$ is equal to the length $|\alpha| = \sum_{k=1}^N a_k$.

Multisubsets of a set $S$ are defined as elements of some symmetric power $S^{(m)}$ and we denote the image of $(s_1, \ldots, s_m) \in S^m$ by $\{\{s_1, \ldots, s_m\}\}$. We denote by $\Gamma(\alpha)$ the set of all partitions of $\alpha$, i.e. multisubsets $\Gamma = \{\{\gamma_1, \ldots, \gamma_m\}\} \subset \mathbb{N}[I]$ with $\sum_{r=1}^m \gamma_r = \alpha$, $\gamma_r > 0$.

For $\Gamma \in \Gamma(\alpha)$ the corresponding stratum $C^\alpha_\Gamma$ is defined as follows. It is formed by configurations which can be subdivided into $m$ groups of points, the $r$-th group containing $\gamma_r$ points; all the points in one group equal to each other, the different groups being disjoint. For example, the main diagonal in $C^\alpha$ is the closed stratum given by partition $\alpha = \alpha$, while the complement to all diagonals in $C^\alpha$ is the open stratum given by partition

$$\alpha = \sum_{k=1}^N (i_k + i_k + \ldots + i_k)$$

Evidently, $C^\alpha = \bigsqcup_{\Gamma \in \Gamma(\alpha)} C^\alpha_\Gamma$.

**Normalization and defect of subsheaves.**

Let $F$ be a vector bundle on the curve $C$ and let $E$ be a subsheaf in $F$. Let $F/E = T(E) \oplus L$ be the decomposition of the quotient sheaf $F/E$ into the sum of its torsion subsheaf and a locally free sheaf, and let $\tilde{E} = \text{Ker}(F \to L)$ be the kernel
of the natural map $F \to L$. Then $\tilde{E}$ is a vector subbundle in $F$ which contains $E$ and has the following universal property:

for any subbundle $E' \subset F$ if $E'$ contains $E$ then $E'$ contains also $\tilde{E}$.

Moreover, $\text{rank } \tilde{E} = \text{rank } E$, $\tilde{E}/E \cong T(E)$ and $c_1(\tilde{E}) = c_1(E) + \ell(T(E))$ (for any torsion sheaf on $C$ we denote by $\ell(T)$ its length).

**Definition 1.3.1.** We will call $\tilde{E}$ the *normalization* of $E$ in $F$ and $T(E)$ the *defect* of $E$.

**Remark 1.3.2.** If $\tilde{E}$ is the normalization of $E$ in $F$ then $\Lambda^k(\tilde{E})$ is the normalization of $\Lambda^kE$ in $\Lambda^kF$.

For any $x \in C$ and torsion sheaf $T$ on $C$ we will denote by $\ell_x(T)$ the length of the localization of $T$ in the point $x$.

**Definition 1.3.3.** For any quasimap $\varphi = (L_\lambda \subset V_\lambda)_{\lambda \in X^+} \in Q^D_\alpha$ we define the *normalization* of $\varphi$ as follows:

$$\tilde{\varphi} = (\tilde{L}_\lambda \subset V_\lambda),$$

and the *defect* of $\varphi$ as follows:

$$\text{def}(\varphi) = (T(L_\lambda))$$

(the defect of $\varphi$ is a collection of torsion sheaves).

**Proposition 1.3.4.** For any $\varphi \in Q^D_\alpha$ there exists $\beta \leq \alpha \in \mathbb{N}[I]$, partition $\Gamma = (\gamma_1, \ldots, \gamma_m) \in \Gamma(\alpha - \beta)$ and a divisor $D = \sum_{r=1}^m \gamma_r x_r \in \mathbb{P}^{\alpha - \beta}$ such that

$$\tilde{\varphi} \in Q^{\beta}, \quad \ell_x(\text{def}(\varphi)_\lambda) = \begin{cases} \langle \lambda, \gamma_r \rangle, & \text{if } x = x_r \\ 0, & \text{otherwise} \end{cases}$$

**Proof.** Clear. \qed

**Definition 1.3.5.** The pair $(\beta, \Gamma)$ will be called the *type of degeneration* of $\varphi$. We denote by $D_{\beta, \Gamma}$ the subspace of $Q^D_\alpha$ consisting of all quasimaps $\varphi$ with the given type of degeneration.

**Remark 1.3.6.** Note that $D_{0, \emptyset} = Q_\alpha$.

We have

$$Q^D_\alpha = \bigcup_{\beta \leq \alpha} D_{\beta, \Gamma} \quad (1)$$

The map $d_{\beta, \Gamma} : D_{\beta, \Gamma} \to Q^{\beta} \times \mathbb{P}^{\alpha - \beta}$ which sends $\varphi$ to $(\tilde{\varphi}, D)$ (see 1.3.4) is an isomorphism. The inverse map $\sigma_{\beta, \Gamma}$ can be constructed as follows. Let $\varphi = (L_\lambda) \in Q^D_\beta$. Then

$$\sigma_{\beta, \Gamma}(\varphi, D) \overset{\text{def}}{=} (L'_\lambda), \quad L'_\lambda \overset{\text{def}}{=} \bigcap_{r=1}^m m_{x_r}^{(\lambda, \gamma_r)} \cdot L_\lambda,$$

where $m_x$ denotes the sheaf of ideals of the point $x \in C$. 
1.4. Laumon’s compactification. Let $V$ be an $n$-dimensional vector space. From now on we will assume that $G = SL(V)$ (in this case certainly $l = n - 1$). In this case there is the Grassmann embedding of the flag variety, namely

$$B = \{(U_1, U_2, \ldots, U_{n-1}) \in G_1(V) \times G_2(V) \times \cdots \times G_{n-1}(V) \mid U_1 \subset U_2 \subset \cdots \subset U_{n-1}\},$$

where $G_k(V)$ is the Grassmann variety of $k$-dimensional subspaces in $V$. This embedding gives rise to another interpretation of $Q_\alpha$.

We will denote by $V$ the trivial vector bundle $V \otimes \mathcal{O}_C$ over $C$. Let $\alpha = \sum_{k=1}^{n-1} a_k i_k$, where $i_k$ is the simple coroot dual to the highest weight $\omega_k$ of representation $G$ in $\Lambda^k V$.

Then $Q_\alpha$ is the space of complete flags of vector subbundles

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset V \quad \text{such that} \quad c_1(E_k) = -\langle \omega_k, \alpha \rangle = -a_k.$$

**Definition 1.4.1** (Laumon, [Lau, 4.2]). The space $Q^L_\alpha$ of quasiflags of degree $\alpha$ is the space of complete flags of locally free subsheaves

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset V \quad \text{such that} \quad c_1(E_k) = -\langle \omega_k, \alpha \rangle = -a_k.$$

It is known that $Q^L_\alpha$ is a smooth projective variety of dimension $2|\alpha| + \dim B$ (see loc. cit., Lemma 4.2.3).

1.5. The stratification of the Laumon’s compactification. There is a stratification of the space $Q^L_\alpha$ similar to the above stratification of $Q^D_\alpha$.

**Definition 1.5.1.** For any quasiflag $E_\bullet = (E_1, \ldots, E_{n-1})$ we define its normalization as

$$\tilde{E}_\bullet = (\tilde{E}_1, \ldots, \tilde{E}_{n-1}),$$

where $\tilde{E}_k$ is the normalization of $E_k$ in $V$ and defect

$$\text{def}(E_\bullet) = (\tilde{E}_1/E_1, \ldots, \tilde{E}_{n-1}/E_{n-1}).$$

Thus, the defect of $E_\bullet$ is a collection of torsion sheaves.

**Proposition 1.5.2.** For any $E_\bullet \in Q^L_\alpha$ there exist $\beta \leq \alpha \in \mathbb{N}[I]$, partition $\Gamma = (\gamma_1, \ldots, \gamma_m) \in \Gamma(\alpha - \beta)$ and a divisor $D = \sum_{r=1}^{m} \gamma_r x_r \in C^\alpha_\Gamma - \beta$ such that

$$\tilde{E}_\bullet \in Q^L_\beta,$$

$$\ell_x(\text{def}(E_k)) = \begin{cases} \langle \omega_k, \gamma_r \rangle, & \text{if } x = x_r \\ 0, & \text{otherwise} \end{cases}$$

**Definition 1.5.3.** The pair $(\beta, \Gamma)$ will be called the type of degeneration of $E_\bullet$. We denote by $Q^L_{\beta, \Gamma}$ the subspace in $Q^L_\alpha$ consisting of all quasiflags $E_\bullet$ with the given type of degeneration.

**Remark 1.5.4.** Note that $Q^L_{0, \emptyset} = Q_\alpha$.

We have

$$Q^L_\alpha = \bigsqcup_{\beta \leq \alpha \atop \Gamma \in \Gamma(\alpha - \beta)} Q^L_{\beta, \Gamma}.$$  \hfill (2)
1.6. The map from \( Q^L_\alpha \) to \( Q^D_\alpha \). Consider the map \( \pi : Q^L_\alpha \to Q^D_\alpha \) which sends a quasiflag of degree \( \alpha \) \( E_\bullet \in Q^L_\alpha \) to a quasimap given by the collection \( (L_\omega_k)^{n-1}_{k=1} \) (see Remark 1.2.1) where \( L_\omega_k = \Lambda^k E_k \subset \Lambda^k V = V_\omega_k \).

**Proposition 1.6.1.** Let \( E_\bullet \) be a quasiflag of degree \( \alpha \) and let \( (\beta, \Gamma) \) be its type of degeneration. Then \( \pi(E_\bullet) \) is a quasimap of degree \( \alpha \) and its type of degeneration is \( (\beta, \Gamma) \).

**Proof.** Obviously we have \( \deg L_\omega_k = \deg \Lambda^k E_k = c_1(E_k) = -\langle \omega_k, \alpha \rangle \) which means that \( \pi(E_\bullet) \in Q^D_\alpha \). According to the Remark 1.3.2, \( L_\omega_k = \Lambda^k \tilde{E}_k \) (i.e. \( (L_\omega_k) \in Q^\beta \)), hence

\[
\ell_x(\tilde{L}/L) = \ell_x(\tilde{E}/E).
\]

This proves the Proposition. \( \square \)

**Remark 1.6.2.** Note that (3) implies that \( \pi \) preserves not only \( \beta \) and \( \Gamma \) but also \( D \) (see 1.3.4, 1.5.2).

Recall that a proper birational map \( f : X \to Y \) is called small if the following condition holds: let \( Y_m \) be the set of all points \( y \in Y \) such that \( \dim f^{-1}(y) \geq m \).

Then for \( m > 0 \) we have

\[
\text{codim } Y_m > 2m.
\]

**Main Theorem.** The map \( \pi \) is a small resolution of singularities.

2. The fibers of \( \pi \)

2.1. We fix \( E_\bullet \in Q^\beta \), a partition \( \Gamma \in \Gamma(\alpha - \beta) \), and a divisor \( D \in C^{\alpha - \beta}_\Gamma \). Then \( (E_\bullet, D) \in Q^\beta \). We define \( F(E_\bullet, D) \) as \( \pi^{-1}(E_\bullet, D) \).

Let \( D = \sum_{r=1}^{m} \gamma_r x_r \). We define the space \( F(E_\bullet, D) \) of commutative diagrams

\[
\begin{array}{cccccc}
E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_{n-1} \\
\varepsilon_1 & & \varepsilon_2 & & \cdots & & \varepsilon_{n-1} \\
T_1 & \longrightarrow & T_2 & \longrightarrow & \cdots & \longrightarrow & T_{n-1} \\
\tau_1 & & \tau_2 & & \cdots & & \tau_{n-1} \\
\end{array}
\]

such that

a) \( \varepsilon_k \) is surjective,

b) \( T_k \) is torsion,

c) \( \ell_x(T_k) = \begin{cases} 
(\omega_k, \gamma_r), & \text{if } x = x_r \\
0, & \text{otherwise}
\end{cases} \)

**Lemma 2.1.1.** We have an isomorphism

\[ F(E_\bullet, D) \cong F(E_\bullet, D). \]

**Proof.** If \( E_\bullet \in F(E_\bullet, D) \) then by the 1.5.2 the collection \( (T_1, \ldots, T_{n-1}) = \text{def}(E_\bullet) \) satisfies the above conditions.

Vice versa, if the collection \( (T_1, \ldots, T_k) \) satisfies the above conditions, then consider \( E_k = \text{Ker}(E_\bullet \xrightarrow{\varepsilon_k} T_k) \).
Since the square
\[ \begin{diagram}
\mathcal{E}_k & \xrightarrow{\varepsilon_k} & T_k \\
\downarrow & & \downarrow \tau_k \\
\mathcal{E}_{k+1} & \xrightarrow{\varepsilon_{k+1}} & T_{k+1}
\end{diagram} \]
commutes, we can extend it to the commutative diagram
\[ \begin{diagram}
0 & \rightarrow & E_k & \rightarrow & \mathcal{E}_k & \rightarrow & T_k & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \tau_k & & \downarrow & & \downarrow \\
0 & \rightarrow & E_{k+1} & \rightarrow & \mathcal{E}_{k+1} & \rightarrow & T_{k+1} & \rightarrow & 0
\end{diagram} \]
The induced morphism \( E_k \rightarrow E_{k+1} \) is injective because \( \mathcal{E}_k \rightarrow \mathcal{E}_{k+1} \) is, and
\[
c_1(E_k) = c_1(\mathcal{E}_k) - l(T_k) = -\langle \omega_k, \beta \rangle - \sum_{x \in C} \ell_x(T_k) =
-\langle \omega_k, \beta \rangle - \sum_{r=1}^{m} \langle \omega_k, \gamma_r \rangle = -\langle \omega_k, \beta + (\alpha - \beta) \rangle = -\langle \omega_k, \alpha \rangle
\]
This means that \( E_\bullet \in F(\mathcal{E}_\bullet, D) \).

**Proposition 2.1.2.** If \( D = \sum_{r=1}^{m} \gamma_r x_r \) is a decomposition into disjoint divisors then
\[
F(\mathcal{E}_\bullet, D) \cong \prod_{r=1}^{m} F(\mathcal{E}_\bullet, \gamma_r x_r).
\]

**Proof.** Recall that if \( T \) is a torsion sheaf on the curve \( C \) then
\[
T = \bigoplus_{x \in C} T_x,
\]
where \( T_x \) is the localization of \( T \) in the point \( x \). This remark together with Lemma 2.1.1 proves the Proposition.

The above Proposition implies, that in order to describe general fiber \( F(\mathcal{E}_\bullet, D) \) it is enough to have a description of the fibers \( F(\mathcal{E}_\bullet, \gamma x) \), which we will call simple fibers.

2.2. The stratification of a simple fiber. We will need the following obvious Lemma.

**Lemma 2.2.1.** Let \( \mathcal{E} \) be a vector bundle on \( C \). Let \( \mathcal{E}' \subset \mathcal{E} \) be a vector subbundle, and let \( E \subset \mathcal{E} \) be a (necessarily locally free) subsheaf. Then \( E' = \mathcal{E}' \cap E \) is a vector subbundle in \( E \).

Moreover, the commutative square
\[ \begin{diagram}
E' & \rightarrow & E \\
\downarrow & & \downarrow \\
\mathcal{E}' & \rightarrow & \mathcal{E}
\end{diagram} \]
can be extended to the commutative diagram

\[
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow & & \downarrow \\
E' & \longrightarrow & E \\
\downarrow & & \downarrow \\
E'/E' & \longrightarrow & E/E'
\end{array}
\]

in which both the rows and the columns form the short exact sequences.

The sheaf in the lower-right corner of the diagram will be called cointersection of \(E\) and \(E'\) inside \(E\) and denoted by \(\nabla_E (E, E')\).

Let

\[
\gamma = \sum_{k=1}^{n-1} c_k \delta_k.
\]

For every \(E_* \in F(E_*, \gamma x)\) we define

\[
\mu_{pq}(E_*) \overset{\text{def}}{=} l \left( \frac{E_q}{E_p \cap E_q} \right) \quad (1 \leq q \leq p \leq n - 1),
\]

(6)

\[
\nu_{pq}(E_*) = \begin{cases} 
\mu_{pq}(E_*) - \mu_{p+1,q}(E_*), & \text{if } 1 \leq q \leq p < n - 1 \\
\mu_{pq}(E_*), & \text{if } 1 \leq q \leq p = n - 1
\end{cases}
\]

(7)

\[
\kappa_{pq}(E_*) = \begin{cases} 
\nu_{pq}(E_*) - \nu_{p,q-1}(E_*), & \text{if } 1 < q \leq p \leq n - 1 \\
\nu_{pq}(E_*), & \text{if } 1 = q \leq p \leq n - 1
\end{cases}
\]

(8)

Remark 2.2.2. The transformations (7) and (8) are invertible, so the numbers \(\mu_{pq}\) can be uniquely reconstructed from \(\nu_{pq}\) or \(\kappa_{pq}\). Namely,

\[
\nu_{pq} = \sum_{r=1}^{q} \kappa_{pr}; \quad \mu_{pq} = \sum_{s=p}^{n-1} \nu_{sq} = \sum_{r \leq q \leq p \leq s} \kappa_{sr}.
\]

(9)

Lemma 2.2.3. We have

\[
\nu_{pq}(E_*) = l \left( \frac{E_q \cap E_{p+1}}{E_q \cap E_p} \right).
\]

(10)

\[
\kappa_{pq}(E_*) = l \left( \nabla_{E_q \cap E_{p+1}} (E_q \cap E_p, E_{q-1} \cap E_{p+1}) \right).
\]

(11)

Proof. The commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \longrightarrow & E_q \cap E_p \\
\downarrow & & \downarrow \\
E_q & \longrightarrow & E_q \\
\downarrow & & \downarrow \\
0 & \longrightarrow & E_q \cap E_{p+1} \\
\end{array}
\]

implies (10). In order to prove (11) note that

\[
E_{q-1} \cap E_p = (E_q \cap E_p) \cap (E_{q-1} \cap E_{p+1})
\]
Proof. See (11), (8), (7) and compare the definition of

Applying (8), (15) and (9) we get

\textbf{Lemma 2.2.5}. For any numbers \( \gamma \)

\[ \text{Lemma follows from (14).} \]

\textbf{Corollary 2.2.4}. Numbers \( \mu_{pq}, \nu_{pq} \) and \( \kappa_{pq} \) satisfy the following inequalities:

\begin{align*}
0 &\leq \kappa_{pq} &\quad (12) \\
0 &\leq \nu_{p1} \leq \nu_{p2} \leq \cdots \leq \nu_{pp} &\quad (13) \\
0 &\leq \mu_{n-1,q} \leq \mu_{n-2,q} \leq \cdots \leq \mu_{qq} = c_q. &\quad (14)
\end{align*}

\textbf{Proof}. See (11), (8), (7) and compare the definition of \( \mu_{pq} \) with 2.1.1.

We will denote by \( [p, q] \) the positive coroot

\[ [p, q] = \sum_{k=q}^{p} i_k \in \mathbb{R}^+ \]  

\textbf{Lemma 2.2.5}. For any \( E_\ast \in F(E_\ast, \gamma x) \) we have

\[ \sum_{1 \leq q \leq p \leq n-1} \kappa_{pq}(E_\ast)[p, q] = \gamma. \]

\textbf{Proof}. Applying (8), (13) and (4) we get

\[ \sum_{1 \leq q \leq p \leq n-1} \kappa_{pq}[p, q] = \sum_{1 \leq q \leq p \leq n-1} [\nu_{pq} - \nu_{p, q-1}][p, q] = \sum_{1 \leq q \leq p \leq n-1} [\nu_{pq} - [p, q + 1]] = \sum_{1 \leq q \leq p \leq n-1} \nu_{pq} = \sum_{q=1}^{n-1} \nu_{pq} i_q = n \sum_{q=1}^{n-1} \nu_{pq} i_q. \]

Now Lemma follows from (14).}

Let \( \mathcal{R}(\gamma) \) be the set of all partitions of \( \gamma \in \mathbb{N}[I] \) into the sum of positive coroots:

\[ \gamma = \sum_{s=1}^{t} \delta_s \], where \( \delta_s \in \mathbb{R}^+ \) (note that \( \mathcal{R}(\gamma) \neq \Gamma(\gamma) \)). In other words, since every positive coroot for \( G = SL(V) \) is equal to \( [p, q] \) for some \( p, q \),

\[ \mathcal{R}(\gamma) = \{(\kappa_{pq})_{1 \leq q \leq p \leq n-1} \mid \kappa_{pq} \geq 0 \ \text{and} \ \sum_{1 \leq q \leq p \leq n-1} \kappa_{pq}[p, q] = \gamma \}. \]

Let \( \mathcal{M}(\gamma) \) denote the set of all collections \( (\mu_{pq}) \) which can be produced from some \( (\kappa_{pq}) \in \mathcal{R}(\gamma) \) as in (13).

The Lemma 2.2.5 implies that for any \( E_\ast \in F(E_\ast, \gamma x) \) we have \( (\mu_{pq}(E_\ast)) \in \mathcal{M}(\gamma) \).

Define the stratum \( \mathcal{S}((\mu_{pq})_{1 \leq q \leq p \leq n-1}, (E_k)_{k=1}^{n-1}) \) as follows:

\[ \mathcal{S}((\mu_{pq})_{1 \leq q \leq p \leq n-1}, (E_k)_{k=1}^{n-1}) = \{ E_\ast \in F(E_\ast, \gamma x) \mid \mu_{pq}(E_\ast) = \mu_{pq} \}. \]

To unburden the notations in the cases when it is clear which flag \( E_\ast \) is used we will write just \( \mathcal{S}_\mu \). We have obviously

\[ F(E_\ast, \gamma x) = \bigsqcup_{\mu \in \mathcal{M}(\gamma)} \mathcal{S}_\mu. \]  

\textbf{Remark 2.2.6}. We will also use the similar varieties \( \mathcal{S}((\mu_{pq})_{1 \leq q \leq p \leq N}, (E_k)_{k=1}^{N}) \) that can be defined in the same way for any short flag \( (E_k)_{k=1}^{N} \) (that is the flag of subbundles \( E_1 \subset \cdots \subset E_N \subset \mathcal{V} \) with rank \( E_k = k \)).
2.3. The strata $\mathcal{S}_\mu$. In order to study $\mathcal{S}_\mu$ we will introduce some more varieties.

For every $1 \leq N \leq n-1$, a short flag of subbundles $(\mathcal{E}_k)_{k=1}^N$ (see Remark 2.2.6) and a collection of numbers $(\nu_k)_{k=1}^N$ such that $0 \leq \nu_1 \leq \cdots \leq \nu_N$, we define the space $\mathfrak{T}((\nu_k)_{k=1}^N, (\mathcal{E}_k)_{k=1}^N)$ as follows:

$$\mathfrak{T}((\nu_k)_{k=1}^N, (\mathcal{E}_k)_{k=1}^N) = \{ E \in \mathcal{E}_N \mid \text{rank}(E) = N \quad \text{and} \quad \ell\left(\frac{\mathcal{E}_k}{\mathcal{E}_k \cap E}\right) = \nu_k \} \quad (17)$$

We define pseudoaffine spaces by induction in dimension. First, the affine line $\mathbb{A}^1$ is a pseudoaffine space. Now a space $A$ is called pseudoaffine if it admits a fibration $A \to B$ with pseudoaffine fibers and pseudoaffine $B$.

**Theorem 2.3.1.** The space $\mathfrak{T}((\nu_k)_{k=1}^N, (\mathcal{E}_k)_{k=1}^N)$ is pseudoaffine of dimension $\sum_{k=1}^{N-1} \nu_k$.

**Proof.** We use induction in $N$. The case $N = 1$ is trivial. There is only one subsheaf $E$ in the line bundle $\mathcal{E}_1$ with $\ell(\mathcal{E}_1/E) = \nu_1$, namely $E = m_1^\nu_1 \cdot \mathcal{E}_1$. This means that $\mathfrak{T}(\nu_1, \mathcal{E}_1)$ is a point and the base of induction follows.

If $N > 1$ then consider the map

$$\tau : \mathfrak{T}((\nu_k)_{k=1}^N, (\mathcal{E}_k)_{k=1}^N) \to \mathfrak{T}((\nu_k)_{k=1}^{N-1}, (\mathcal{E}_k)_{k=1}^{N-1}),$$

which sends $E \in \mathfrak{T}((\nu_k)_{k=1}^N, (\mathcal{E}_k)_{k=1}^N)$ to $E' = E \cap \mathcal{E}_{N-1} \in \mathfrak{T}((\nu_k)_{k=1}^{N-1}, (\mathcal{E}_k)_{k=1}^{N-1})$.

**Lemma 2.3.2.** Let $L = m_{E_{N-1}}^{\nu_{N-1}} \cdot \left(\frac{\mathcal{E}_N}{\mathcal{E}_{N-1}}\right)$. For any $E' \in \mathfrak{T}((\nu_k)_{k=1}^{N-1}, (\mathcal{E}_k)_{k=1}^{N-1})$ there is an isomorphism

$$\tau^{-1}(E') \cong \text{Hom}(L, \mathcal{E}_{N-1}/E') \cong h^{\ell(\mathcal{E}_{N-1}/E')} = h^{\nu_{N-1}}.$$

Thus, the space $\mathfrak{T}((\nu_k)_{k=1}^N, (\mathcal{E}_k)_{k=1}^N)$ is affine fibration over a pseudoaffine space, hence it is pseudoaffine and its dimension is equal to

$$\dim \mathfrak{T}((\nu_k)_{k=1}^{N-1}, (\mathcal{E}_k)_{k=1}^{N-1})) + \nu_{N-1} = \sum_{k=1}^{N-2} \nu_k + \nu_{N-1} = \sum_{k=1}^{N-1} \nu_k.$$

The Theorem is proved.

**Proof of Lemma 2.3.2.** Let $E \in \tau^{-1}(E')$. Since $E' = E \cap \mathcal{E}_{N-1}$ we can apply Lemma 2.2.1 which gives the following commutative diagram:

$$
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow & \downarrow & \downarrow \\
E_{N-1} & \longrightarrow & \mathcal{E}_N \\
\downarrow & \downarrow & \downarrow \\
T_{N-1} & \longrightarrow & T_N
\end{array}
$$

(Note that since $\mathcal{E}_N/\mathcal{E}_{N-1}$ is a line bundle and $\ell(T_N/T_{N-1}) = \ell(T_N) - \ell(T_{N-1}) = \nu_N - \nu_{N-1}$ the kernel of natural map $\mathcal{E}_N/\mathcal{E}_{N-1} \to T_N/T_{N-1}$ is isomorphic to $L$.) Let $\hat{\mathcal{E}}_N = \psi^{-1}(L)$. Then $E$ is contained in $\hat{\mathcal{E}}_N$ and we have the following commutative
This means that the points of $\tau^{-1}(E')$ are in one-to-one correspondence with maps $\varepsilon: \tilde{E}_N \to T_{N-1}$ such that $\varepsilon \cdot j$ is equal to the canonical projection from $\tilde{E}_N$ to $T_{N-1}$. Applying the functor $\text{Hom}(\bullet, T_{N-1})$ to the middle row of the above diagram we get an exact sequence:

$$0 \to \text{Hom}(L, T_{N-1}) \to \text{Hom}(\tilde{E}_N, T_{N-1}) \xrightarrow{j^\ast} \text{Hom}(\mathcal{E}_{N-1}, T_{N-1}) \to \text{Ext}^1(L, T_{N-1}).$$

The last term in this sequence is zero because $L$ is locally free and $T_{N-1}$ is torsion. This means that the space of maps $\varepsilon$ which we need to describe is a torsor over the group $\text{Hom}(L, T_{N-1})$. Hence this space can be identified with the group. Thus, we have proved that $\tau^{-1}(E') \cong \text{Hom}(L, T_{N-1})$ is an affine space.

Now,

$$\dim(\tau^{-1}(E')) = \dim \text{Hom}(L, T_{N-1}) = \dim H^0(T_{N-1}) = \ell(T_{N-1}) = \nu_{N-1}.$$ 

The Lemma is proved.

\[\blacksquare\]

**Theorem 2.3.3.** The space $\mathfrak{S}((\mu_{pq})_{1 \leq q \leq p \leq N}, (\mathbb{E}_k)_{k=1}^N)$ is a pseudoaffine space of dimension $\mu_1 + \mu_2 + \cdots + \mu_{N,N-1}$.

**Proof.** We use induction in $N$. If $N = 1$ then $\mathfrak{S}_\mu$ is a point and the base of induction follows.

If $N > 1$ consider the map

$$\sigma: \mathfrak{S}((\mu_{pq})_{1 \leq q \leq p \leq N}, (\mathbb{E}_k)_{k=1}^N) \to \mathfrak{S}((\mu_{N,k})_{k=1}^N, (\mathbb{E}_k)_{k=1}^N),$$

which sends $(\mathbb{E}_k)_{k=1}^N$ to $E_N \subset \mathbb{E}_N$.

**Lemma 2.3.4.** Let $E \in \mathfrak{S}((\nu_{N,k})_{k=1}^N, (\mathbb{E}_k)_{k=1}^N)$. Consider $\tilde{\mathbb{E}}_k = \mathbb{E}_k \cap E$ ($1 \leq k \leq N - 1$) and set $\tilde{\mu}_{pq} = \mu_{pq} - \mu_{Nq}$ ($1 \leq q \leq p \leq N - 1$). Then $(\tilde{\mathbb{E}}_k)_{k=1}^{N-1}$ is a short flag of subbundles and for any $E \in \mathfrak{S}((\nu_{N,k})_{k=1}^N, (\mathbb{E}_k)_{k=1}^N)$

$$\sigma^{-1}(E) \cong \mathfrak{S}((\tilde{\mu}_{pq})_{1 \leq q \leq p \leq N-1}, (\tilde{\mathbb{E}}_k)_{k=1}^{N-1}).$$

Thus $\mathfrak{S}((\mu_{pq})_{1 \leq q \leq p \leq N}, (\mathbb{E}_k)_{k=1}^N)$ is a fiber space with pseudoaffine base and fiber, therefore it is pseudoaffine.

Now, the calculation of the dimension

$$\dim \mathfrak{S}((\mu_{pq})_{1 \leq q \leq p \leq N}, (\mathbb{E}_k)_{k=1}^N)) = \sum_{k=1}^{N-1} \mu_{N,k} + \sum_{k=1}^{N-2} \tilde{\mu}_{k+1,k} = \sum_{k=1}^{N-1} \mu_{N,k} + \sum_{k=1}^{N-2} (\mu_{k+1,k} - \mu_{N,k}) = \mu_{N,N-1} + \sum_{k=1}^{N-2} \mu_{k+1,k} = \sum_{k=1}^{N-1} \mu_{k+1,k},$$

finishes the proof of the Theorem. \[\blacksquare\]
Proof of Lemma 2.3.4. Assume that \((E_k)_{k=1}^N \in \mathfrak{S}_\mu\) and \(E_N = E\). The commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{E}_q \cap E_p & \longrightarrow & \mathcal{E}_q & \longrightarrow & \frac{\mathcal{E}_q}{\mathcal{E}_q \cap E_p} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{E}_q \cap E & \longrightarrow & \mathcal{E}_q & \longrightarrow & \frac{\mathcal{E}_q}{\mathcal{E}_q \cap E} & \longrightarrow & 0
\end{array}
\]

implies that

\[
l \left( \frac{\mathcal{E}_q \cap E}{\mathcal{E}_q \cap E_p} \right) = l \left( \frac{\mathcal{E}_q}{\mathcal{E}_q \cap E_p} \right) - l \left( \frac{\mathcal{E}_q}{\mathcal{E}_q \cap E} \right) = \mu_{pq} - \mu_{Nq} = \tilde{\mu}_{pq}, \tag{19}\]

hence \((E_k)_{k=1}^{N-1} \in \mathfrak{S}_{\tilde{\mu}}\).

Vice versa, if \((E_k)_{k=1}^{N-1} \in \mathfrak{S}_{\tilde{\mu}}\) then the above commutative diagram along with (19) implies that \((E_k)_{k=1}^N \in \mathfrak{S}_\mu\), where we have put \(E_{Nq} = \mathcal{E}_q\).

2.4. The cohomology of the simple fiber. Now we will compute the dimension of the strata \(\mathfrak{S}_\mu\) in terms of the partition \(\kappa\).

Definition 2.4.1. A space \(X\) is called cellular if it admits a stratification with pseudoaffine strata.

Suppose \(X = \bigsqcup_{\xi \in \Xi} S_\xi\) is a pseudoaffine stratification of a cellular space \(X\). For a positive integer \(j\) we define \(\chi(j) \overset{\text{def}}{=} \{\xi \in \Xi \mid \dim S_\xi = j\}\).

Lemma 2.4.2. The Hodge structure \(H^\bullet(X, \mathbb{Q})\) is a direct sum of Tate structures, and \(\mathbb{Q}(j)\) appears with multiplicity \(\chi(j)\). In other words,

\[
H^\bullet(X, \mathbb{Q}) = \bigoplus_{j \in \mathbb{N}} \mathbb{Q}(j)^{\chi(j)}.
\]

Proof. Evident.

Given a Tate structure \(H = \bigoplus_{j \in \mathbb{N}} \mathbb{Q}(j)^{\chi(j)}\) we consider a generating function

\[
P(H, t) = \sum_{j \in \mathbb{N}} \chi(j)t^j \in \mathbb{N}[t].
\]

For \(\kappa \in \mathfrak{K}(\gamma)\) we define \(K(\kappa) \overset{\text{def}}{=} \sum_{1 \leq q \leq p \leq n-1} \kappa_{pq}\) as the number of summands in the partition \(\kappa\). For \(\gamma \in \mathbb{N}[I]\) the following \(q\)-analog of the Kostant’s partition function was was introduced in [Lus]:

\[
K_\gamma(t) = t^{\lvert \gamma \rvert} \sum_{\kappa \in \mathfrak{K}(\gamma)} t^{-K(\kappa)}.
\]

Lemma 2.4.3. Let \(\kappa \in \mathfrak{K}(\gamma)\) and \(\mu \in \mathfrak{M}(\gamma)\) be defined as in (3). Then

\[
dim \mathfrak{S}_\mu = \sum_{k=1}^{n-2} \mu_{k+1,k} = \lvert \gamma \rvert - K(\kappa), \tag{21}\]

Proof. Evident.
Proof. Applying (8), we get

\[
\sum_{k=1}^{n-2} \mu_{k+1,k} = \sum_{k=1}^{n-2} \left( \sum_{1 \leq q \leq k} \kappa_{pq} \right) = \sum_{1 \leq q \leq p \leq n-1} (p-q) \kappa_{pq} = \sum_{1 \leq q \leq p \leq n-1} ((|p,q| - 1) \kappa_{pq} = |\gamma| - \sum_{1 \leq q \leq p \leq n-1} \kappa_{pq} = |\gamma| - K(\kappa).
\]

\]

Corollary 2.4.4. For any \(\gamma \in \mathbb{N}[I]\), \(x \in C\), the simple fiber \(F(\mathcal{E}_\bullet, \gamma x)\) is a cellular space, and the generating function of its cohomology is equal to the Lusztig–Kostant polynomial

\[
P(H^*(F(\mathcal{E}_\bullet, \gamma x)), t) = K_\gamma(t).
\]

Proof. Apply (16), 2.3.3, 2.4.2 and 2.4.3.

Corollary 2.4.5. Let \(D = \sum_{r=1}^{m} \gamma_r x_r \in C^\alpha_{\Gamma - \beta}\). The fiber \(F(\mathcal{E}_\bullet, D)\) is a cellular space and

\[
P(H^*(F(\mathcal{E}_\bullet, D)), t) = K_\Gamma \text{ def } = \prod_{r=1}^{m} K_{\gamma_r}(t).
\]

(22)

Proof. Apply 2.1.2, 2.4.2 and 2.4.4.

Lemma 2.4.6. Let \(D = \sum_{r=1}^{m} \gamma_r x_r\). We have

\[
\dim F(\mathcal{E}_\bullet, D) \leq \sum_{r=1}^{m} |\gamma_r| - m.
\]

Proof. Note that for any \(\kappa \in \mathbb{R}(\gamma_r)\) we have \(K(\kappa) \geq 1\), hence \(\deg K_{\gamma_r} \leq |\gamma_r| - 1\). Now, the Lemma follows from 2.4.5.

Proof of Main Theorem. Consider the stratum \(\mathcal{O}_{\beta, \Gamma}\) of \(Q^D_{\alpha}\). Its dimension is \(2|\beta| + \dim B + m\) and codimension is \(2|\alpha - \beta| - m\). The Lemma 2.4.6 implies that the dimension of the fiber of \(\pi\) over the stratum \(\mathcal{O}_{\beta, \Gamma}\) is less than or equal to \(|\alpha - \beta| - m\), which is strictly less than the half codimension of the stratum.

2.5. Applications. Let \(\mathcal{Q}\) denote the smooth constant Hodge irreducible module on \(Q^L_{\alpha}\) (as a constructible complex it lives in cohomological degree \(-2|\alpha| - \dim B\)). Let \(IC\) denote the minimal extension of a smooth constant irreducible Hodge module from \(Q_{\alpha}\) to \(Q^D_{\alpha}\). It is well known that the smallness of \(\pi\) implies the following corollary.

Corollary 2.5.1.

\[
IC = \pi_* \mathcal{Q}.
\]

Now we can compute the stalks of \(IC\) as cohomology of fibers of \(\pi\): for \(\varphi \in Q^D_{\alpha}\) we have

\[
IC_{(\varphi)} = H^*(\pi^{-1}(\varphi), \mathcal{Q})
\]
as graded Hodge structures.
Corollary 2.5.2 (Parity vanishing).

\[ IC_{(\varphi)}^j = 0 \quad \text{if} \quad j - \dim B \text{ is odd.} \]

Proof. Use 2.4.5.

Corollary 2.5.3. For \( \varphi \in \mathcal{D}_{\beta, \Gamma} \) we have

\[ IC_{(\varphi)}^{-(2|\beta| - \dim B) + 2j} = Q(j)^{t_{\Gamma}(j)}, \]

where \( t_{\Gamma}(j) \) is the coefficient of \( t^j \) in \( \mathcal{R}_\Gamma(t) \).

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