A Note on the Picard–Fuchs Equations for
N=2 Seiberg–Witten Theories

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March 1997

Abstract: A concise presentation of the PF equations for N=2 Seiberg–Witten theories for the classical groups of rank \( r \) with \( N_f \) massless hypermultiplets in the fundamental representation is provided. For \( N_f = 0 \), all \( r \) PF equations can be given in a generic form. For certain cases with \( N_f \neq 0 \), not all equations are generic. However, in all cases there are at least \( r - 2 \) generic PF equations. For these cases the classical part of the equations is generic, while the quantum part can be formulated using a method described in a previous paper by the authors, which is well suited to symbolic computer calculations.

1Supported in part by Ministerio de Educación y Ciencia, Spain. Home address: Dep. de Física de Partículas, Universidad de Santiago, 15706 Santiago, Spain.

2Supported by the DOE under grant DE–FG02–92ER40706

3Supported by the DOE under grant DE–FG02–91ER40688–Task A

4Research supported in part by the DOE under grant DE–FG02–92ER40706.

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1. Introduction

Enormous advances have been made in understanding the low-energy properties of super-symmetric gauge theories [1]. In particular, the exact solution for the low-energy properties for the Coulomb phase of \( N=2 \) gauge theories with \( N_f \) matter multiplets in the fundamental representation is given in principle by a hyperelliptic curve [1]-[17]. In fact a great deal of analysis is required to extract the strong-coupling physics from the curve characterizing the theory in question. To do so, one defines the Seiberg–Witten (SW) period integrals

\[
\vec{\pi} = \begin{pmatrix}
\vec{a}_D \\
\vec{a}
\end{pmatrix},
\]

which are related to the prepotential \( \mathcal{F}(\vec{a}) \) characterizing the low-energy effective Lagrangian by

\[
a^i_D = \partial \mathcal{F}/\partial a^i.
\]

One strategy for obtaining the necessary information is to derive a set of Picard–Fuchs (PF) equations for the SW period integrals.\(^7\) The PF equations have been derived in a number of special cases for \( N_f = 0 \) and massless multiplets for \( N_f \neq 0 \) [10, 21, 22, 23, 28], using the strategy of Klemm, et al. [2], or Isidro, et al. [23] [Examples of PF equations for massive multiplets are also known [24, 27, 31], but this situation will not be the concern of this paper.] A systematic method for finding PF equations, which is particularly convenient for symbolic computer computations, was given by Isidro, et al. [23]. Solutions to these equations for low rank have also been considered [10, 21, 25, 26, 27], and studies of softly broken \( N=2 \) theories [20] have also been carried out.

Although one can compute explicit PF equations for particular cases by the methods of Klemm, et al., [2] or Isidro, et al., [23] it would be preferable to have an explicit, generic\(^8\) set of PF

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\(^7\) For alternate strategies see refs. [18] and [19].

\(^8\) By generic we mean PF equations which are applicable without specializing to any particular rank \( r \). For example, see (2.17). Sometimes we will refer to this as an analytic set of PF equations.
equations valid for each of the classical groups, and those $N_f$ consistent with asymptotic freedom. One expects $r$ PF equations, involving $r$ moduli for classical groups of rank $r$. A set of $(r - 1)$ equations independent of the quantum scale for classical groups with $N_f = 0$ have been formulated by Ito and Sasakura [22], while a derivation of the remaining PF equation for $N_f = 0$ for $A_r$, $B_r$ and $D_r$ was provided by Alishahiha [28].

It is the purpose of this paper to extend these results so as to consider a complete set of $r$ PF equations for the classical groups of rank $r$, with $N_f$ massless hypermultiplets, with all equations generic if possible. The motivation for deriving such a set of equations is ultimately to extract general features of the low-energy strong-coupling behavior of $N=2$ theories, without resorting to a case by case evaluation of the solutions of the equations. Generic solutions for the prepotentials in the weak-coupling region have been given by D’Hoker, et al., [18] and Matone, et al., [19] but no comparable results are known for the strong-coupling regions, although we are presently considering such solutions to the equations presented in this paper.

In section 2, we review the PF equations available in the literature [22, 28] for the case $N_f = 0$, and derive analogous results for $C_r$. However, these methods also require some generalization to obtain the full set of PF equations for all cases.

In section 3, we derive the $r$ PF equations for $N_f \neq 0$, consistent with asymptotic freedom. In certain cases, not all $r$ PF equations are generic, although there are always at least $r - 2$ generic equations. For the remaining required PF equations we appeal to the $M$-matrix method of Isidro, et al., to complete the discussion. Thus, these equations must be dealt with on a case by case basis. However, the $M$-matrix methods are only required for the $\Lambda$-dependent part of the additional PF equations, where $\Lambda$ is the quantum scale of the theory.

We note that this paper provides a concise and complete presentation of the PF equations.
presently available for all classical groups, with $N_f$ consistent with asymptotic freedom and with $m = 0$. We emphasize that it is not our purpose in this paper to actually exhibit the explicit computer calculations of the method of [23], whenever its use is required here. Instead we wish to expose the maximal information available in generic form in the PF equations. It is our hope that this information will facilitate the study of the Coulomb phase of $N = 2$ theories in the strong-coupling region.

2. PF Equation for $N_f = 0$

A. Curves

The hyperelliptic curves for the classical groups with $N_f = 0$ can be put in the following form

$$y^2 = p^2(x) - G(x) \quad (2.1)$$

with SW differential [1, 3, 5, 9, 14]

$$\lambda = \frac{x}{y} \left[ \frac{G'}{G} \cdot \frac{p}{2} - p \right] dx . \quad (2.2)$$

In particular

$$A_r = SU(r + 1) \quad (2.3)$$

$$p(x) = x^{r+1} - \sum_{i=2}^{r+1} u_i x^{r+1-i}$$

$$G(x) = \Lambda^{2r+2}$$

$$\quad (2.4)$$

$$B_r = SO(2r + 1) \quad (2.5)$$

$$p(x) = P_r(x) ,$$

where

$$P_r(x) = x^{2r} - \sum_{i=1}^{r} u_{2i} x^{2r-2i} \quad (2.6)$$

throughout this paper, and

$$G(x) = x^2 \Lambda^{4r-2} \quad (2.7)$$
\( C_r = Sp(2r) \) \[14, 22\]

\[
p(x) = x P_r(x) + \frac{1}{x} \Lambda^{2r+2} \tag{2.8}
\]

\[
G(x) = \frac{1}{x^2} \Lambda^{4r+4} \tag{2.9}
\]

or equivalently

\[
y^2 = x^2 P_r^2(x) + 2P_r(x)\Lambda^{2r+2} \tag{2.10}
\]

\( D_r = SO(2r) \) \[4, 6\]

\[
p(x) = P_r(x) \tag{2.11}
\]

\[
G(x) = x^4 \Lambda^{4r-4} \tag{2.12}
\]

**B. PF Equations**

Define \( \partial_i = \partial / \partial u_i \) for convenience.

**A**

\[
\partial_i \lambda = - \frac{x^{r+1-i}}{y} dx + d(*) \quad (i = 2 \text{ to } r + 1) \tag{2.13}
\]

\[
\partial_i \partial_j \lambda = - \frac{x^{2r+2-i-j}}{y^3} p(x) dx + d(*) \tag{2.14}
\]

which implies that \( \mathcal{L}_{i,j,p,q} \vec{\pi} = 0 \) where

\[
\mathcal{L}_{i,j,p,q} = \partial_i \partial_j - \partial_p \partial_q \tag{2.15}
\]

such that \( i + j = p + q \). Then

\[
\frac{d}{dx} \left( \frac{x^n}{y} \right) = n \frac{x^{n-1}}{y} - \left( \frac{x^n p'(x)}{y^3} \right) p(x) \tag{2.16}
\]

Using (2.3), (2.13) and (2.14) one obtains \( \mathcal{L}_n \vec{\pi} = 0 \), where

\[
\mathcal{L}_0 = (r+1) \partial_2 \partial_r - \sum_{j=2}^{r+1} (r+1-j) u_j \partial_j \partial_{r+1} \tag{2.17}
\]
\[
\mathcal{L}_n = -n \partial_{r+2-n} + (r+1)\partial_2 \partial_{r-n}
\]
\[
- \sum_{j=2}^{r+1} (r+1-j)u_j \partial_{r+2-n} \partial_j \quad (n = 1 \text{ to } r-2)
\]  
(2.18)

Equations (2.17) and (2.18) provide \( r-1 \) PF equations \[22\]. The additional PF equation follows from the strategy laid out in equations (10) and (11) of ref. \[28\], and our generalization thereof.

From (2.16) one may write

\[
D = -(r+1) d \left( \frac{x^{r+2}}{y} \right) + \sum_{j=2}^{r+1} (r+1+j)u_j d \left( \frac{x^{r+2-j}}{y} \right)
\]

\[
= \lambda - \sum_{j=2}^{r+1} j(j-2)u_j \frac{x^{r+1-j}}{y} + \sum_{i,j=2}^{r+1} ij u_i u_j \frac{x^{2r+2-i-j}}{y^3} p(x)
\]

\[
- (r+1)^2 \frac{\Lambda^{2r+2}}{y^3} p(x) \right) dx .
\]  
(2.19)

Using equations (2.13) and (2.14), this leads to

\[
\mathcal{L}_{r-1} = 1 + \sum_{j=2}^{r+1} j(j-2)u_j \partial_j + \sum_{i,j=2}^{r+1} ij u_i u_j \partial_i \partial_j
\]

\[
- (r+1)^2 \Lambda^{2r+2} \partial_{r+1}^2 ,
\]  
(2.20)

which was conjectured in equation (68) of ref. \[22\] and given in ref. \[28\].

\[
B_r
\]

\[
\partial_{2i} \lambda = -x^{2r-2i} \frac{dx + d(*)}{y} \]

\[
\partial_{2i} \partial_{2j} \lambda = -x^{4r-2i-2j} \frac{p(x)dx + d(*)}{y^3} \]  
(2.21)

(2.22)

which implies \( \mathcal{L}_{2i;2j;2p;2q} \bar{\pi} = 0 \) where

\[
\mathcal{L}_{2i;2j;2p;2q} = \partial_{2i} \partial_{2j} - \partial_{2p} \partial_{2q}
\]  
(2.23)
with \( i + j = p + q \). Also
\[
\frac{d}{dx} \left( \frac{x^{2n+1}}{y} \right) = \frac{2nx^{2n}}{y} - \frac{[x^{2n+1}p'(x) - x^{2n}p(x)]}{y^3} p(x) . \tag{2.24}
\]

From (2.21) and (2.22), one has
\[
\mathcal{L}_n = 2n\partial_{2r-2n} - (2r - 1)\partial_2 \partial_{2r-2n-2} + \sum_{j=1}^{r} (2r - 2j - 1)u_{2j} \partial_{2j} \partial_{2r-2n} \quad (n = 0 \text{ to } r - 2) \tag{2.25}
\]

From (2.24) we have
\[
D = - (2r - 1) d \left( \frac{x^{2r+1}}{y} \right) + \sum_{j=1}^{r} (2r + 2j - 1)u_{2j} d \left( \frac{x^{2r+1-2j}}{y} \right) = \lambda - \left[ 4 \sum_{j=1}^{r} j(j - 1)u_{2j} \frac{x^{2r-2j}}{y} \right. \\
+ 4 \sum_{i,j=1}^{r} ij u_{2i} u_{2j} \frac{x^{4r-2i-2j}}{y^3} p(x) \\
- (2r - 1)^2 \Lambda^{4r-2} \frac{x^2}{y^3} p(x) \right] dx . \tag{2.26}
\]

Using (2.21) and (2.22), this gives
\[
\mathcal{L}_{r-1} = 1 + 4 \sum_{j=1}^{r} j(j - 1)u_{2j} \partial_{2j} + \sum_{i,j=1}^{r} ij u_{2i} u_{2j} \partial_{2i} \partial_{2j} \\
- (2r - 1)^2 \Lambda^{4r-2} \partial_{2r} \partial_{2r-2} . \tag{2.27}
\]

\[
\mathcal{C}_r \quad \partial_{2i} \lambda = \frac{-x^{2r-2i+1}}{y} dx + d(*) \tag{2.28}
\]
\[
\partial_{2i} \partial_{2j} \lambda = \frac{-x^{4r-2i-2j+2}}{y^3} p(x)dx + d(*) \tag{2.29}
\]
so that (2.23) is valid for $C_r$. 

$$
\frac{d}{dx} \left( \frac{x^{2n+2}}{y} \right) = \frac{(2n+3)x^{2n+1}}{y} - \left[ \frac{x^{2n+2}p'(x) + x^{2n+1}p(x)}{y^3} \right] p(x) .
$$

(2.30)

Thus

$$
\mathcal{L}_n = (2n+3)\partial_{2r-2n} - (2r+2)\partial_2 \partial_{2r-2n-2} \\
+ \sum_{j=1}^{r} (2r+2-2j)u_{2j} \partial_{2j} \partial_{2r-2n} \quad (n = 0 \text{ to } r-2)
$$

(2.31)

From (2.30) one can write

$$
D = -(2r+2) \frac{d}{dx} \left( \frac{x^{2r+2}}{y} \right) \\
+ \sum_{j=1}^{r} (2r+2+2j)u_{2j} \frac{d}{dx} \left( \frac{x^{2r+2-2j}}{y} \right) \\
- (2r+2)^2 \Lambda^{2r+2} d(1/y) \\
= \lambda - \left[ 4 \sum_{j=1}^{r} j(j-1)u_{2j} \frac{x^{2r-2j+1}}{y} + 4 \sum_{i,j=1}^{r} ij u_{2i} u_{2j} \frac{x^{4r-2i-2j+2}}{y^3} p(x) \\
+ (2r+2)^2 \Lambda^{2r+2} \sum_{j=1}^{r} (2r - 2j) u_{2j} \frac{x^{2r-2j}}{y^3} p(x) \\
- 2r(2r+2)^2 \Lambda^{2r+2} x^{2r} \frac{p(x)}{y^3} \right] dx
$$

(2.32)

[It should be noted that equation (2.32) is a generalization of the method of ref. [28].] From (2.28) and (2.29)

$$
\mathcal{L}_{r-1} = 1 + 4 \sum_{j=1}^{r} j(j-1)u_{2j} \partial_{2j} \\
+ 4 \sum_{i,j=1}^{r} ij u_{2i} u_{2j} \partial_{2i} \partial_{2j} \\
+ (2r+2)^2 \Lambda^{2r+2} \sum_{j=1}^{r} (2r - 2j) u_{2j} \partial_{2j+2} \partial_{2r} \\
- 2r(2r+2)^2 \Lambda^{2r+2} \partial_2 \partial_{2r}.
$$

(2.33)
\[ D_r \]

\[
\partial_{2i} \lambda = -\frac{x^r - 2i}{y} \frac{dx}{y} + d(*) \quad (2.34)
\]

\[
\partial_{2i} \partial_{2j} \lambda = -\frac{x^r - 2i - 2j}{y^3} p(x) dx + d(*) \quad (2.35)
\]

and (2.23) is valid for \( D_r \) as well. We have

\[
\frac{d}{dx} \left( \frac{x^{2n+1}}{y} \right) = (2n - 1) x^{2n} - \frac{[x^{2n+1} p'(x) - 2x^{2n} p(x)]}{y^3} p(x) \quad (2.36)
\]

which from (2.34), (2.35) gives

\[
\mathcal{L}_n = (2n - 1) \partial_{2r} - (2r - 2) \partial_{2r} \partial_{2r - 2n - 2} + \sum_{j=1}^{r} (2r - 2 - 2j) u_{2j} \partial_{2j} \partial_{2r - 2n} \quad (n = 0 \text{ to } r - 2) \quad (2.37)
\]

From (2.36)

\[
D = -(2r - 2) \frac{d}{dx} \left( \frac{x^{2r+1}}{y} \right) + 2 \sum_{j=1}^{r} (r - 1 + j) u_{2j} \frac{d}{dx} \left( \frac{x^{2r+1 - 2j}}{y} \right)
\]

\[
= \lambda \left[ 4 \sum_{j=1}^{r} j(j - 1) u_{2j} \frac{x^{2r - 2j}}{y} + 4 \sum_{i,j=1}^{r} i j u_{2i} u_{2j} \frac{x^{4r - 2i - 2j}}{y^3} p(x) \right] dx , \quad (2.38)
\]

which, when combined with (2.34) and (2.35), provides the remaining equation

\[
\mathcal{L}_{r-1} = 1 + 4 \sum_{j=1}^{r} j(j - 1) u_{2j} \partial_{2j} + 4 \sum_{i,j=1}^{r} i j u_{2i} u_{2j} \partial_{2i} \partial_{2j} - (2r - 2)^2 \Lambda^{4r-4} \partial_{2r-2} \quad (2.39)
\]

In this section we have presented the \( r \) PF equations for \( A_r, B_r, C_r, \) and \( D_r \) with \( N_f = 0 \) in analytic form. Equations (2.31)–(2.33) for \( C_r \) are new.
C. **Alternate Curves**

It is understood that the particular form of the hyperelliptic curve for a given theory is not unique. A number of different curves might represent the same physics, perhaps with a redefinition of moduli in such a way that different definitions of moduli agree in the semi-classical region, though in the strong-coupling region they may differ considerably. The point we make is that not all versions of curves for a given theory lead to a complete set of PF equations in analytic form.

We illustrate this issue with a discussion of $C_r$ with $N_f = 0$. The version of the curve we use in (2.8)–(2.10) is from Ito and Sasakura [22]. We observe from (2.31) and (2.33) that a complete set of $r$ PF equations in analytic form are available for this case. However, we can also consider the curve for $C_r$ used by D’Hoker, et al., [18]

$$y^2 = p^2(x) - \Lambda^{4r+4} \quad (2.40)$$

where $p(x) = x^2P_r(x)$. Then defining, analogous to (2.32),

$$D = -(2r + 2)d \left( \frac{x^{2r+3}}{y} \right) + \sum_{j=1}^{r} (2r + 2 + 2j)u_{2j}d \left( \frac{x^{2r+3-2j}}{y} \right), \quad (2.41)$$

one finds that this cannot be related to $\lambda$ so as to provide a generic PF equation.

The lesson is that not all distinct versions of a curve representing the same prepotential lead to a complete set of $r$ PF equations in generic form. For our purposes (2.10) is preferred.

3. **PF Equations for $N_f \neq 0$**

A. **Curves**

The hyperelliptic curves for the classical groups with $N_f \neq 0$ are also described by equations (2.1) and (2.2). In particular

$$A_r; 1 \leq N_f < (r + 1) \quad \mathbb{3}$$

$$p(x) = x^{r+1} - \sum_{i=2}^{r+1} u_i x^{r+1-i} \quad (3.1)$$

$$G(x) = \Lambda^{2r+2-N_f} x^{N_f} \quad (3.2)$$
\[ A_r; \quad (r + 1) \leq N_f < (2r + 2) \]

\[
p(x) = x^{r+1} - \sum_{i=2}^{r+1} u_i x^{r+1-i} + \frac{1}{4} \Lambda^{2r+2-N_f} x^{N_f-r-1}
\]

(3.3)

and \( G(x) \) as in (3.2).

\[ B_r; \quad 1 \leq N_f < (r - 1) \]

\[
p(x) = P_r(x)
\]

(3.4)

\[
G(x) = \Lambda^{4r-2-2N_f} x^{2+2N_f}
\]

(3.5)

\[ B_r; \quad (r - 1) \leq N_f < (2r - 1) \]

\[
p(x) = P_r(x) + \Lambda^{4r-2-2N_f} x^{2N_f-2r+2}
\]

\[
G(x) = \Lambda^{4r-2-2N_f} x^{2+2N_f}
\]

(3.6)

\[ C_r; \quad 1 \leq N_f < 2r + 2 \]

\[
p(x) = x P_r(x)
\]

(3.7)

\[
G(x) = \Lambda^{4r+4-2N_f} x^{2N_f-2}
\]

(3.8)

\[ D_r; \quad 1 \leq N_f < (r - 2) \]

\[
p(x) = P_r(x)
\]

(3.9)

\[
G(x) = \Lambda^{4r-4-2N_f} x^{4+2N_f}
\]

(3.10)

\[ D_r; \quad (r - 2) \leq N_f < (2r - 2) \]

The coefficient 1/4 in (3.3) is subject to testing by means of microscopic instanton calculations. In fact, for \( SU(3) \) with \( N_f = 3, 4, \) and 5, this coefficient is actually 1/16, 5/32, and 17/64 respectively. We keep the coefficient 1/4 in (3.3) and (3.22) ff, recognizing that this coefficient is readily replaced as more information becomes available. The same issue applies to the curves corresponding to other gauge groups for large \( N_f \).
\[ p(x) = P_r(x) + \Lambda^{4r-4-2N_f} x^{2N_f-2r+4} \]

\[ G(x) = \Lambda^{4r-4-2N_f} x^{4+2N_f} \] \quad (3.11)

**B. PF Equations**

\[ A_r : 1 \leq N_f < (r + 1) \]

From

\[
\frac{d}{dx} \left( \frac{x^n}{y} \right) = \left( n - \frac{N_f}{2} \right) \frac{x^{n-1}}{y} - \left( r + 1 - \frac{N_f}{2} \right) \frac{x^{r+n}}{y^3} p(x) + \sum_{j=2}^{r+1} \left( r + 1 - j - \frac{N_f}{2} \right) u_j \frac{x^{r+n-j}}{y^3} p(x) \] \quad (3.12)

and equations (2.13) and (2.14), one has

\[
\mathcal{L}_n = - \left( n - \frac{N_f}{2} \right) \partial_{r+2-n} + \left( r + 1 - \frac{N_f}{2} \right) \partial_2 \partial_{r-n} - \sum_{j=2}^{r+1} \left( r + 1 - j - \frac{N_f}{2} \right) u_j \partial_{r+2-n} \partial_j , \] \quad (3.13)

valid for \( 1 \leq n \leq r - 2 \). Although (3.12) is applicable for \( n = 0 \), some work will be required to obtain an equation in terms of \( \lambda \).

From Eq. (3.12), with \( n = 0 \)

\[
\frac{d}{dx} \left( \frac{1}{y} \right) = - \frac{N_f}{2} \left( \frac{1}{x} \right) - \left( r + 1 - \frac{N_f}{2} \right) \frac{x^r}{y^3} p(x) + \sum_{j=2}^{r+1} \left( r + 1 - j - \frac{N_f}{2} \right) u_j \frac{x^{r-j}}{y^3} p(x) \] \quad (3.14)

It follows from the curve (3.1) that

\[
\frac{d}{dx} \left( \frac{1}{y} \right) = - (r + 1) \frac{x^r}{y^3} p(x) \]
\[ \sum_{j=2}^{r+1} (r + 1 - j) u_j \frac{x^{r-j}}{y^3} p(x) \]
\[ + \frac{N_f}{2} \Lambda^{2r+2-N_f} \frac{x^{N_f-1}}{y^3} . \]  
(3.15)

Using (2.13) and (2.14)
\[ d \left( \frac{1}{y} \right) = (r + 1) \partial_2 \partial_r \lambda \]
\[ - \sum_{j=2}^{r+1} (r + 1 - j) u_j \partial_{j+1} \partial_{r+1} \lambda \]
\[ + \frac{N_f}{2} \Lambda^{2r+2-N_f} \frac{x^{N_f-1}}{y^3} dx . \]  
(3.16)

It is very satisfying that (3.16) coincides with (2.17) in the limits \( N_f = 0 \) or \( \Lambda = 0 \), as it must.

However, the term \( x^{N_f-1}/y^3 \) is not immediately expressible in terms of \( \lambda \), as one cannot apply (2.13) or (2.14) directly. Instead we appeal to the method of Isidro, et al. \[23, 29, 31\].

Define, following equation (2.9) of ref. \[23\],
\[ \Omega_m^{(\mu)} = (-1)^{\mu+1} \Gamma(\mu + 1) \int \frac{x^m}{y^{2(\mu+1)}} \, dx \]  
(3.17)
where we have suppressed the dependence of \( \Omega_m^{(\mu)} \) on the moduli and the fixed 1-cycle \( \gamma \). Thus, we are interested in relating \( \Omega_m^{(1/2)} \) to \( \Omega_m^{(-1/2)} \). One can relate \( \Omega_{m'}^{(1/2)} \) to \( \Omega_m^{(-1/2)} \) for values of \( m \) and \( m' \) belonging to the basic range \( R \): \( m = 0 \) to \( r - 1; r + 1 \) to \( 2r \). Since \( \Omega_{N_f-1}^{(1/2)} \) belongs to the basic range for \( 1 \leq N_f \leq r \), one may use Eq. (2.19) of Isidro et al., \[23\] to write the matrix equation
\[ \Omega^{(-1/2)} = M \cdot \Omega^{(1/2)} . \]  
(3.18)

Thus \( \int dx \frac{x^{N_f-1}}{y^3} \) is related to the periods \( \int dx \frac{x^m}{y} \); \( m \in R \), in terms of the coefficient matrix \( M \), which is given in terms of polynomials in the moduli and the quantum scale \( \Lambda \).

Note that the period is not yet given in terms of \( \lambda \). To complete this task, one may follow the method of Sec. 3.3 of Isidro, et al. \[23\]. We do not give the details here, but sketch the main
ideas. Equation (3.18) relates $\Omega_{N,1}^{(1/2)}$ to the periods $\Omega_m^{(-1/2)}$, $m \in R$. Then from (2.13)

$$
\Omega_m^{(-1/2)} = -i \Gamma(1/2) \partial_{r+1-m} \int dx \lambda \quad \text{for} \quad m = 0 \text{ to } r - 1.
$$

(3.19)

However, if $m = r + 1$ to $2r$, we need the analogue of (3.9) of Isidro, et al. [23]. In our notation, schematically

$$
\begin{pmatrix}
\Omega_{r+1}^{(-1/2)} \\
\vdots \\
\Omega_{2r}^{(-1/2)}
\end{pmatrix}
= -B_i^{-1} \left( \frac{\partial}{\partial u_i} + A_i \right)
\begin{pmatrix}
\Omega_0^{(-1/2)} \\
\vdots \\
\Omega_{r-1}^{(-1/2)}
\end{pmatrix}
\quad (i = 2 \text{ to } r + 1)
$$

(3.20)

[The matrices $A_i$ and $B_i$ depend on the moduli and the quantum scale $\Lambda$. No summation over $i$ is implied in the above equation. It holds for any $i$ from 2 to $r + 1$, and any convenient value of $i$ is sufficient for our purposes.] Finally, one uses (3.19) in (3.20) to express the periods in terms of $\int \lambda$. Clearly, the last term in (3.16) leads to first and second order modular derivatives of $\lambda$. Thus (3.16) gives a second-order PF equation.

The steps (3.18) to (3.20) do not have a generic solution, although the iterative algorithms of Isidro, et al., are well suited to symbolic computer calculations. They must be dealt with on a case by case basis. However, observe that the computer calculations are only needed for the $\Lambda$-dependent part of (3.16).

We need one more PF equation. Using the curve (3.1), (3.2), the generalization of (2.19) and (2.20) gives us

$$
\mathcal{L}_{r-1} = 1 + \sum_{j=2}^{r+1} j(j - 2)u_j \partial_j + \sum_{i,j=2}^{r+1} i j u_i u_j \partial_i \partial_j - \left[ \left( r + 1 - \frac{N_f}{2} \right)^2 \Lambda^{2r+2-N_f} \partial_i \partial_j \right]_{i+j=2r+2-N_f}
$$

(3.21)

which completes the set of $r$ PF equations.
The curve given by (3.3) allows the replacement

\[ u_{2r+2-N_f} \rightarrow \bar{u}_{2r+2-N_f} \]

\[ = u_{2r+2-N_f} - \frac{1}{4} \Lambda^{2r+2-N_f}, \]  \hspace{1cm} (3.22)

for \( r+1 \leq N_f \leq 2r \). Thus, (3.13) gives \( r-2 \) PF equations, with the replacement (3.22) understood for these flavors.

For \( N_f = 2r + 1 \), (3.22) does not describe an allowable shift. Nonetheless for this case one easily finds instead of (3.13)

\[
\mathcal{L}_n = -\left(n - r + 1 \right) \partial_{r+2-n} + \frac{1}{2} \partial_2 \partial_{r-n} \\
- \sum_{j=2}^{r+1} \frac{1}{2} - j \ u_j \partial_{r+2-n} \partial_j \\
+ \frac{1}{8} \Lambda \partial_{r+1-n} \partial_2 , \]  \hspace{1cm} (3.23)

valid for \( 1 \leq n \leq r - 2 \). [One can understand the additional term in (3.23) as the formal shift \( u_1 \rightarrow \bar{u}_1 = u_1 - \frac{1}{4} \Lambda \), with \( u_1 \) set to zero at the end.]

Since the relevant curve is (3.3), the equation \( \mathcal{L}_0 \) differs slightly from (3.14) to (3.16). Make the shift (3.22) in (3.14). Then note that the last term of (3.16) is

\[
\frac{N_f}{2} \Lambda^{2r+2-N_f} \frac{x^{N_f-1}}{y^3} . \]  \hspace{1cm} (3.24)

For \( r+2 \leq N_f \leq 2r + 1 \) the period integral \( \int dx \frac{x^{N_f-1}}{y^3} \) is in the basic range of \( \Omega_m^{(1/2)} \). Therefore, for these flavors the steps described in (3.17) to (3.20) are applicable to (3.24). This again requires a symbolic computer calculation.

For \( N_f = r+1 \), the situation is more complicated as \( \int x^r/y^3 \) is not in the basic range of \( \Omega_m^{(1/2)} \). However, this is exactly the problem dealt with in section 3.3 of ref. \[23\]. [Note that \( n = r + 1 \) in
that reference. One then follows the scheme described there to reduce the $2g \times (2g + 1)$ $M$ matrix to the required $2g \times 2g$ $M$ matrix. Only then will the reduced $M$ matrix be suitable for application of (3.17) to (3.20). All these steps can be carried out by symbolic computer calculations, as in ref. [23].

The last PF equation is obtained from the generalization of ref. [28].

$$D = \lambda + \sum_{j=2}^{r+1} j(j-2)\bar{u}_j \partial_j \lambda$$
$$+ \sum_{i,j=2}^{r+1} i j \bar{u}_i \bar{u}_j \partial_i \partial_j \lambda$$
$$+ (r + 1 - \frac{N_f}{2})^2 \Lambda^{2r+2-N_f} \frac{x^{N_f}}{y^3} p(x) dx$$

(3.25)

where $\bar{u}_i$ is defined by (3.22). Using (3.3), one must relate

$$\frac{x^{N_f}}{y^3} p(x) = \frac{x^{N_f+r+1}}{y^3} - \sum_{i=2}^{r+1} \bar{u}_i \frac{x^{N_f+r+1-i}}{y^3}$$

(3.26)

to $\lambda$. The terms of $\int \frac{x^m}{y^r}$ for $m > 2r$ are outside the basic range. These must be brought back to the basic range using section 2 of ref. [28], or the methods of refs. [10] and [27]. Once $\Omega^{(1/2)}_m$ lies in the basic range, then one applies (3.17)–(3.20) to relate the results to $\lambda$.

For $N_f = 2r + 1$ the shift (3.22) is not valid, and $u_i$, not $\bar{u}_i$, appears in (3.25). Finally the last term in (3.25) is now

$$\frac{\Lambda x^{2r+1}}{4 y^3} p(x) - \frac{\Lambda x^r}{8 y} - \frac{\Lambda^2 x^{3r+1}}{16 y^3} \frac{r+1}{8} i u_i \frac{x^{2r+1-i}}{y^3} p(x)$$

(3.27)

The method outlined in the previous paragraph must be followed to relate (3.27) to $\lambda$.

Note that for $N_f \geq r + 1$, only $r - 2$ PF equations have a generic presentation, while for $1 \leq N_f < r + 1$, there are $r - 1$ generic equations. This is to be contrasted with $N_f = 0$, where all $r$ PF equations are generic. Furthermore, we observe that the computer calculations are only required for the $\Lambda$-dependent term.
\[
B_r: 1 \leq N_f < (r-1)
\]

The SW differential satisfies (2.21) to (2.23). Equation (2.24) is replaced by
\[
\frac{d}{dx} \left( \frac{x^{2n+1}}{y} \right) = \frac{(2n - N_f)x^{2n}}{y} - \frac{[x^{2n+1}p' - (N_f + 1)x^{2n}p]}{y^3} \quad p(x)
\]

which implies
\[
\mathcal{L}_n = (2n - N_f)\partial_{2r-2n} - (2r - 1 - N_f)\partial_2 \partial_{2r-2n-2}
\]
\[
+ \sum_{j=1}^{r} (2r - 2j - 1 - N_f)u_{2j} \partial_{2j} \partial_{2r-2n} \quad (n = 0 \text{ to } r - 2)
\]

Equations (2.26) and (2.27) trivially generalize to give
\[
\mathcal{L}_{r-1} = 1 + 4 \sum_{j=1}^{r} j(j-1)u_{2j} \partial_{2j}
\]
\[
+ 4 \sum_{i,j=1}^{r} ij u_{2i}u_{2j} \partial_{2i} \partial_{2j}
\]
\[
- (2r - 1 - N_f)^2 \Lambda^{4r-2-2N_f} \partial_{2r-2-2N_f}
\]

\[
B_r: (r-1) \leq N_f \leq 2r - 2
\]

Equations (2.21)–(2.23) and (3.29) are applicable. One may use equations (3.6) and (2.6) to shift the modulus
\[
u_{2i} \rightarrow u_{2i} - \Lambda^{4r-2-2N_f} = \bar{u}_{2i}
\]

where \(2i = 4r - 2N_f - 2\). With this redefinition, (3.29) is applicable, but with a (single) shifted modulus. That is,
\[
\mathcal{L}_n = (2n - N_f)\partial_{2r-2n} - (2r - 1 - N_f)\partial_2 \partial_{2r-2n-2}
\]
\[
+ \sum_{\substack{j \neq i \atop j = 1}}^{r} (2r - 2j - 1 - N_f)u_{2j} \partial_{2j} \partial_{2r-2n}
\]
\[
+ (2r - 2i - 1 - N_f)(u_{2i} - \Lambda^{4r-2-2N_f})u_{2i} \partial_{2i} \partial_{2r-2n}
\]

where \(i = 2r - N_f - 1\) and \(n = 0 \text{ to } r - 2\).
To obtain the last PF equation note that (3.30) is applicable for \( r - 1 \leq N_f \leq 2r - 3 \), but with the substitution (3.31).

It is more complicated to obtain the last PF equation for \( N_f = 2r - 2 \). In that case one has from the analogue of (3.30)

\[
\mathcal{L}_{r-1} \lambda = \lambda + 4 \sum_{j=1}^{r} j(j-1) \bar{u}_{2j} \partial_{2j} \lambda \\
+ 4 \sum_{i,j=1}^{r} ij \bar{u}_{2i} \bar{u}_{2j} \partial_{2i} \partial_{2j} \lambda \\
+ \Lambda^{2r} x^{4r-2} y^{-2} p(x) \ dx ,
\]

where \( \bar{u}_{2j} = u_{2j} \) for \( j \neq 1 \), and \( u_2 \) shifted by (3.31). One cannot use (2.21) for the last term, as this would not involve allowed modular derivatives. Since \( p(x) = x^{2r} - \sum_{i=1}^{r} \bar{u}_{2i} x^{2r-2i} \), the period \( \Omega_{m}^{(1/2)} \) is outside the basic range. One uses the recursion relations of ref. [23] to bring these periods back to the basic range. Then one uses (3.17)–(3.20) to relate the last term of (3.33) to \( \lambda \). Although there is no generic solution, note that the computer calculation is restricted to the \( \Lambda \)-dependent part of (3.33).

\(\mathcal{C}_r : 2 \leq N_f \leq 2r + 1\)

Equations (2.28) and (2.29) apply, together with

\[
\frac{d}{dx} \left( \frac{x^{2n+2}}{y} \right) = \frac{(2n + 3 - N_f)x^{2n+1}}{y} \\
- \frac{[x^{2n+2}p' + (1 - N_f)x^{2n+1}p]}{y^3} \ p(x)
\]

which implies

\[
\mathcal{L}_n = (2n + 3 - N_f)\partial_{2r-2n} - (2r + 2 - N_f)\partial_2\partial_{2r-2n-2} \\
+ \sum_{j=1}^{r} (2r + 2 - N_f - 2j)u_{2j} \partial_{2j} \partial_{2r-2n} \quad (n = 0 \text{ to } r - 2)
\]
Using the strategy of ref. [28]

\[ D = -(2r + 2 - N_f)d \left( \frac{x^{2r+2}}{y} \right) \]
\[ + \sum_{j=1}^{r} (2r + 2 - N_f + 2j)u_{2j}d \left( \frac{x^{2r+2-2j}}{y} \right) \]  

leads to the equation

\[ \mathcal{L}_{r-1} = 1 + 4 \sum_{j=1}^{r} j(j-1)u_{2j}\partial_{2j} \]
\[ + 4 \sum_{i,j=1}^{r} ij u_{2i} u_{2j}\partial_{2i}\partial_{2j} \]
\[ - (2r + 2 - N_f)^2 \Lambda^{4r+4-2N_f} \partial_{2r}\partial_{2r+4-2N_f} \]  

valid for \( 2 \leq N_f \leq 2r \).

For \( N_f = 2r + 1 \), we have instead of (3.37)

\[ \mathcal{L}_{r-1}\lambda = 1 + 4 \sum_{j=1}^{r} j(j-1)u_{2j}\partial_{2j}\lambda \]
\[ + 4 \sum_{i,j=1}^{r} ij u_{2i} u_{2j}\partial_{2i}\partial_{2j}\lambda \]
\[ + \Lambda^2 \frac{x^{4r}}{y^3} p(x) dx \]

where \( p(x) = x^{2r+1} - \sum_{i=1}^{r} u_{2i} x^{2r-2i+1} \). Since

\[ \frac{x^{4r}p(x)}{y^3} = \frac{x^{6r+1}}{y^3} - \sum_{i=1}^{r} u_{2i} \frac{x^{6r-2i+1}}{y^3} \]

lies outside the basic range, the period \( \Omega_m^{(1/2)} \) must be brought back to the basic range by the relations of ref. [28]. Then (3.17) to (3.20) will relate the last term of (3.38) to \( \lambda \). Once again the symbolic computer computation is only required for the quantum term in (3.38).

\[ C_r : \ N_f = 1 \]

Equations (3.28) and (3.29) are applicable. Note that (3.37) does not apply to this case, as the last term would not involve legitimate derivatives with respect to moduli. Instead of (3.37) we
have

\[ L_{r-1} \lambda = \lambda + 4 \sum_{j=1}^{r} j(j - 1)u_{2j} \partial_{2j} \lambda + 4 \sum_{i,j=1}^{r} i j u_{2i}u_{2j} \partial_{2i} \partial_{2j} \lambda + (2r + 1)^2 \Lambda^{4r+2} \frac{p(x)}{y^3} \, dx \]  

(3.40)

where \( p(x) \) is given by (2.6) and (3.7). Thus

\[ \frac{p(x)}{y^3} = \frac{x^{2r+1}}{y^3} - \sum_{j=1}^{r} u_{2j} \frac{x^{2r+1-2j}}{y^3} \].  

(3.41)

Using the definition (3.17), we consider the periods required by (3.41), i.e.,

\[ \Omega_{2r+1-2j}^{(1/2)} \quad (j = 0 \text{ to } r) \]

where the periods are “odd” in the terminology of ref. \[23\]. Equation (3.18) relates periods \( \Omega_{m'}^{(1/2)} \) to periods \( \Omega_m^{-1/2} \). The periods \( \Omega_{2m+1}^{-1/2} \), with \( m = 0 \) to \( r - 1 \) can be related directly to \( \lambda \), by means of (2.28). That is

\[ \Omega_{2m+1}^{-1/2} = -i \Gamma(1/2) \partial_{2r-2m} \int dx \lambda \quad \text{for } m = 0 \text{ to } r - 1 \].  

(3.42)

However, for \( m = r \) to \( 2r - 1 \) one must use the analogue of (3.20). Then the periods on the right side of (3.20) are expressed in terms of \( \lambda \) by means of (3.42). Thus (3.40) to (3.42), with (3.18) and (3.20), provides us with the final second order PF equation for \( N_f = 1 \). The computer calculations are only needed for the \( \Lambda \)-dependent part of (3.40).

\[ D_r : 1 \leq N_f < (r - 2) \]

The SW differentials satisfy (2.34) and (2.35). We also have

\[ \frac{d}{dx} \left( \frac{x^{2n+1}}{y} \right) = \frac{(2n - 1 - N_f)x^{2n}}{y} - \frac{[x^{2n+1}p' - (2 + N_f)x^{2n}p]}{y^3} p(x) \].  

(3.43)
Hence

\[ \mathcal{L}_n = (2n - 1 - N_f) \partial_{2r-2n} - (2r - 2 - N_f) \partial_2 \partial_{2r-2n-2} \]
\[ + \sum_{j=1}^{r} (2r - 2 - 2j - N_f) u_{2j} \partial_{2j} \partial_{2r-2n} \quad (n = 0 \text{ to } r - 2) \]  \hspace{1cm} (3.44)

Equations (2.38) and (2.39) generalize to

\[ \mathcal{L}_{r-1} = 1 + 4 \sum_{j=1}^{r} j(j-1) u_{2j} \partial_{2j} \]
\[ + 4 \sum_{i,j=1}^{r} ij u_{2i} u_{2j} \partial_{2i} \partial_{2j} \]
\[ - \left[ (2r - 2 - N_f)^2 \Lambda^{4r-4-2N_f} \partial_{2k} \partial_{2l} \right]_{k+l=r-2-N_f}. \]  \hspace{1cm} (3.45)

\[ D_r: \quad (r - 2) \leq N_f \leq 2r - 3 \]

From (2.6) and (3.11) we can shift

\[ u_{2i} \rightarrow u_{2i} - \Lambda^{4r-4-2N_f} \tilde{u}_{2i} \]  \hspace{1cm} (3.46)

where \( 2i = 4r - 2N_f - 4 \). With this redefinition, there are \( r - 1 \) PF equations identical to (3.44), but with the single shifted modulus (3.46).

When \( r - 2 \leq N_f \leq 2r - 4 \) the last PF equation is obtained from (3.45) with the substitution (3.46).

\[ N_f = 2r - 3 \] must be considered separately. Instead of (3.45), one has

\[ \mathcal{L}_{r-1} \lambda = \lambda + 4 \sum_{j=1}^{r} j(j-1) \tilde{u}_{2j} \partial_{2j} \lambda \]
\[ + 4 \sum_{i,j=1}^{r} ij \tilde{u}_{2i} \tilde{u}_{2j} \partial_{2i} \partial_{2j} \lambda \]
\[ + \Lambda^2 \frac{x^{4r-2}}{y^3} p(x) dx \]  \hspace{1cm} (3.47)

with \( \tilde{u}_{2j} = u_{2j} \) for \( j \neq 1 \), and \( u_2 \) given by (3.46). Then with (3.11) for \( p(x) \) one uses the strategy outlined in (3.33) ff. It is obvious once again that the computer calculation is only required for the \( \Lambda \)-dependent part of (3.47).
4. Concluding Remarks

We have provided a systematic presentation of the PF equations for $N=2$ theories with massless multiplets presently available. For $N_f = 0$ all $r$ equations can be put in generic form. When $N_f \neq 0$ there are at least $r - 2$ equations which can be formulated generically, with only the quantum part requiring a case-by-case analysis. Although these symbolic computer calculations are exhibited here, the structure of the remaining 1 or 2 equations is quite explicit for the classical part. It is our belief that this compilation of PF equations will facilitate further understanding of the Coulomb phase of these theories in the strong-coupling regions.

We wish to thank S. Naculich and H. Rhedin for helpful conversations.

After this paper was completed we received the paper [32] which has considerable overlap with our work.

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