FOLD–SADDLE BIFURCATION IN NON–SMOOTH VECTOR FIELDS ON THE PLANE.

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Abstract. This paper presents results concerning bifurcations of 2D piecewise–smooth dynamical systems governed by vector fields. Generic three–parameter families of a class of Non–Smooth Vector Fields are studied and its bifurcation diagrams are exhibited. Our main results describe the unfolding of the so called Fold – Saddle singularity.

1. Introduction

The general purpose of this article is to present some aspects of the geometric and qualitative theory of a class of planar non–smooth systems. Our main concern is to discuss the behavior of such systems around typical singularities that appear generically in three–parameter families. We mention that certain phenomena in control systems, impact in mechanical systems and nonlinear oscillations are the main sources of motivation of our study concerning the dynamics of those systems that emerge from differential equations with discontinuous right–hand sides.

The codimension zero and codimension one singularities were discussed in [4] and [5] respectively. In [3] codimension two singularities were studied. The specific topic addressed in this paper is the complete characterization of the Fold–Saddle bifurcation diagram. Those papers give the necessary basis for the development of our approach.

Let $K \subseteq \mathbb{R}^2$ be a compact set and $\Sigma \subseteq K$ given by $\Sigma = f^{-1}(0)$, where $f : K \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$) such that $\partial K \cap \Sigma = \emptyset$ or $\partial K \ni \Sigma$. Clearly $\Sigma$ is the separating boundary of the regions $\Sigma_+ = \{q \in K | f(q) \geq 0\}$ and $\Sigma_- = \{q \in K | f(q) \leq 0\}$. We can assume that $\Sigma$ is represented, locally around a point $q = (x, y)$, by the function $f(x, y) = y$.

Designate by $\chi^r$ the space of $C^r$ vector fields on $K$ endowed with the $C^r$–topology with $r \geq 1$ or $r = \infty$, large enough for our purposes. Call
\[ \Omega^r = \Omega^r(K, f) \] the space of vector fields \( Z : K \setminus \Sigma \to \mathbb{R}^2 \) such that

\[
Z(x, y) = \begin{cases} 
X(x, y), & \text{for } (x, y) \in \Sigma_+ , \\
Y(x, y), & \text{for } (x, y) \in \Sigma_- ,
\end{cases}
\]

where \( X = (f_1, g_1), Y = (f_2, g_2) \) are in \( \chi^r \). We write \( Z = (X, Y) \), which we will accept to be multivalued in points of \( \Sigma \). The trajectories of \( Z \) are solutions of \( \dot{q} = Z(q) \), which has, in general, discontinuous right-hand side. The basic results of differential equations, in this context, were stated by Filippov in [2]. Related theories can be found in [4, 6, 8].

In what follows we will use the notation

\[
X.f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad Y.f(p) = \langle \nabla f(p), Y(p) \rangle .
\]

1.1. Setting the problem. Let \( X_0 \) be a smooth vector field defined in \( \Sigma_+ \). We say that a point \( p_0 \in \Sigma \) is a \( \Sigma \)-fold point of \( X_0 \) if \( X_0.f(p_0) = 0 \) but \( X_0^2.f(p_0) \neq 0 \). Moreover, \( p_0 \in \Sigma \) is a visible (respectively invisible) \( \Sigma \)-fold point of \( X_0 \) if \( X_0.f(p_0) = 0 \) and \( X_0^2.f(p_0) > 0 \) (resp. \( X_0^2.f(p_0) < 0 \)). In this universe, \( \Gamma^N_{\Sigma} \), a \( \Sigma \)-fold point has codimension zero. Since \( f(x, y) = y \) we derive the following generic normal forms \( X_0(x, y) = (\alpha_1, \beta_1 x) \) with \( \alpha_1 = \pm 1 \) and \( \beta_1 = \pm 1 \).

Let \( Y_0 \) be a smooth vector field defined in \( \Sigma_- \). Assume that \( Y_0 \) has a hyperbolic saddle point \( S_{Y_0} \) on \( \Sigma \) and that the eigenspaces of \( DY_0(S_{Y_0}) \) are transverse to \( \Sigma \) at \( S_{Y_0} \). In this universe, \( \Gamma^N_{\Sigma} \), a saddle point \( S_{Y_0} \) has codimension one. Since \( f(x, y) = y \) we derive the following generic normal forms \( Y_0(x, y) = (\alpha_2 y, \alpha_2 x) \) with \( \alpha_2 = \pm 1 \) and its generic unfolding \( Y_{\beta} = (\alpha_2(y + \beta), \alpha_2 x) \) where \( \beta \in \mathbb{R} \). Let \( U \) be a small neighborhood of \( Y_0 \) in \( \Gamma^N_{\Sigma} \). Then:

(a) There exists a smooth function \( L : U \to \mathbb{R} \), such that \( DL_{Y_0} \) is surjective.

(b) The correspondence \( Y \to S_Y \) is smooth, where \( S_Y \) is a saddle point of \( Y \).

(c) If \( L(Y) > 0 \) then \( S_Y \in \Sigma_- \).

(d) If \( L(Y) = 0 \) then \( S_Y \in \Sigma \).

(e) If \( L(Y) < 0 \) then \( S_Y \in \Sigma_+ \).

In this paper we are concerned with the bifurcation diagram of systems \( Z_0 = (X_0, Y_0) \) in \( \Omega^r \) such that \( p_0 = S_{Y_0} \in \Sigma \). This singularity will be called Fold - Saddle singularity (see Figures 1 and 2).

We depart from \( Z_0^i, Z_0^u \in \Omega^r \) written in the following forms:

\[
Z_0^i = \begin{cases} 
X_0^i = \begin{pmatrix} 1 \\ -x \end{pmatrix} & \text{if } y \geq 0 , \\
Y_0 = \begin{pmatrix} -y \\ -x \end{pmatrix} & \text{if } y \leq 0 ,
\end{cases}
\]

and
Note that $X^i_0$ presents an invisible $\Sigma$–fold point on its phase portrait and $X^v_0$ presents a visible one. Following the techniques developed in [7], we are able to prove that there exists a smooth mapping $F^\tau : \Omega^\tau, Z^\tau_0 \rightarrow \mathbb{R}^3, 0$ where $\tau = i$ or $v$ such that:

1- $(DF^\tau)Z^\tau_0$ is surjective (So $M^\tau = (F^\tau)^{-1}(0)$ is locally, around $Z^\tau_0$, an imbedded differentiable manifold).

2- Each $Z \in U^\tau$, with $F^\tau(Z) = 0$ and $U^\tau$ a small neighborhood of $Z^\tau_0$ in $\Omega^\tau$ is $C^0$–equivalent to $Z^\tau_0$.

The main question is to exhibit the bifurcation diagram of $Z^\tau_0$. So, we have to consider generic imbeddings $\sigma^\tau : \mathbb{R}^3, 0 \rightarrow \Omega^\tau, Z^\tau_0$ (3–parameter families).

They are transversal imbeddings to $M^\tau$ at $Z^\tau_0$.

Consider $Z^\tau_0 = (X^\tau_0, Y_0) \in U^\tau$. Roughly speaking, we derive that:

1- There is a canonical imbedding $F^\tau_0 : \mathbb{R}^2, 0 \rightarrow \chi^\tau, Z^\tau_0$ such that $F^\tau_0(\lambda, \beta) = Z^\tau_{\lambda,\beta}$ expressed by:

$$Z^\tau_{\lambda,\beta} = \begin{cases} X^\tau_{\lambda} = \left( \frac{\alpha_1(\tau)(x - \lambda)}{1} \right) & \text{if } y \geq 0, \\ Y_{\beta} = \left( \frac{-y + \beta}{-x} \right) & \text{if } y \leq 0, \end{cases}$$

where $\lambda, \beta \in (-1, 1)$, $\alpha_1(i) = -1$ and $\alpha_1(v) = 1$. Moreover, its bifurcation diagram of $Z^\tau_{\lambda,\beta}$ is exhibited (see Figures 18 and 28). We observe that there are some typical topological types nearby $Z^\tau_0$ that do not appear in the bifurcation diagram of $Z^\tau_{\lambda,\beta}$. For example, when $\tau = i$ the configurations in Figures 3 and 4 are excluded and when $\tau = v$ the configuration in Figure 5 also is excluded.
II- We add an auxiliary parameter \( \mu \) in the following way:

\[
Z_{\lambda, \mu, \beta} = \begin{cases} 
X_{\lambda} = \left( \frac{1}{\alpha_1(\tau)}(x - \lambda) \right) & \text{if } y \geq 0, \\
Y_{\mu, \beta} = \left( \frac{\mu}{2} x + \frac{\mu - 2}{2} (y + \beta) \right) & \text{if } y \leq 0,
\end{cases}
\]

where \( \lambda, \beta \in (-1, 1) \), \( \alpha_1(i) = -1 \), \( \alpha_1(v) = 1 \) and \( \mu \in (-\varepsilon_0, \varepsilon_0) \) with the real number \( \varepsilon_0 > 0 \) being sufficiently small. By means of this late unfolding its bifurcation diagram cover all topological types near \( Z_{0,0,0} \).

In this universe, \( \Gamma_{Z_{0,0,0}} \), a Fold–Saddle singularity has codimension three. Since \( f(x,y) = y \) we derive the generic normal forms \( Z_{\lambda, \mu, \beta} \) with \( \mu_0 = \pm \varepsilon_0/2 \) and its generic unfolding \( Z_{\lambda, \mu, \beta} \) given by (4). Therefore, there is a codimension three bifurcation (global) branch terminating at \( Z_{0,0,0} \). In fact, note that we can obtain Equation (4) (respectively (2)) from Equation (4) taking \( \tau = i \) (respectively \( \tau = v \)), \( \lambda = 0 \), \( \mu = 0 \) and \( \beta = 0 \).

Of course, we can take another generic normal form of one or both vector fields \( X_0 \) and \( Y_0 \). In this paper we consider just the cases described in Equations (1) and (2). For the other cases a similar approach can be done.

It is worth mentioning that we detect branches of “canard cycles” in the bifurcation diagram of \( Z_{\lambda, \mu, \beta} \). Recall that, a canard cycle is a closed path composed by pieces of orbits of \( X \), \( Y \) and \( Z^\Sigma \) (see Figures [7, 8 and 9]). In Section 2 a precise definition will be given.

**Example 1.** Equations (1) and (2) appear in problems related to Control Theory, more specifically, in Relay Systems. In fact, consider the function \( \varphi : \mathbb{R} \to \mathbb{R} \) given by

\[
\varphi(y) = \begin{cases} 
-1, & \text{for } y \geq 0, \\
-y, & \text{for } y \leq 0,
\end{cases}
\]

and \( u(y) = -\varphi(y) \text{sign}(y) \). So (1) and (2) are represented by \( \tilde{Z}_0(x,y) = (u(y), \varphi(\tau)x) \) where \( \tau = i \) or \( v \), \( \varphi(i) = -1 \) and \( \varphi(v) = \text{sign}(y) \).

**1.2. Statement of the Main Results.** Our results are now stated. Theorems 1, 2 and 3 are intermediate steps towards Theorem A and Theorems 4, 5 and 6 are intermediate steps towards Theorem B.
Theorem 1. Take \( \tau = i \) in Equation (3) or equivalently, take \( \tau = i \) and \( \mu = 0 \) in Equation (4). The \((\lambda, \beta)\)-plane contains essentially 17 distinct typical configurations representing 5 distinct topological behaviors on its bifurcation diagram (see Figure 18).

It is easy to see that the cases covered by Theorem 1 do not represent the full unfolding of the (Invisible) Fold–Saddle singularity. Because of this, the next two theorems are necessary. Each one of them describes a distinct generic codimension two singularity.

Theorem 2. Take \( \tau = i \) and \( 0 < \mu < \varepsilon_0 \) in Equation (4). The \((\lambda, \beta)\)-plane contains essentially 19 distinct typical configurations representing 7 distinct topological behaviors on its bifurcation diagram (see Figure 20).

Theorem 3. Take \( \tau = i \) and \( -\varepsilon_0 < \mu < 0 \) in Equation (4). The \((\lambda, \beta)\)-plane contains essentially 19 distinct typical configurations representing 7 distinct topological behaviors on its bifurcation diagram (see Figure 22).

Theorem 4. Take \( \tau = v \) in Equation (3) or equivalently, take \( \tau = v \) and \( \mu = 0 \) in Equation (4). The \((\lambda, \beta)\)-plane contains essentially 13 distinct typical configurations representing 7 distinct topological behaviors on its bifurcation diagram (see Figure 28).

The cases covered by Theorem 4 do not represent the full unfolding of the (Visible) Fold–Saddle singularity. Because of this, the next two theorems are necessary. Each one of them describes a distinct generic codimension two singularity.

Theorem 5. Take \( \tau = v \) and \( 0 < \mu < \varepsilon_0 \) in Equation (4). The \((\lambda, \beta)\)-plane contains essentially 13 distinct typical configurations representing 7 distinct topological behaviors on its bifurcation diagram (see Figure 28).

Theorem 6. Take \( \tau = v \) and \( -\varepsilon_0 < \mu < 0 \) in Equation (4). The \((\lambda, \beta)\)-plane contains essentially 13 distinct typical configurations representing 7 distinct topological behaviors on its bifurcation diagram (see Figure 28).

Finally, we are able to state the main results of the paper.

Theorem A. Equation (4) with \( \tau = i \) generically unfolds the (Invisible) Fold–Saddle singularity. Moreover, its bifurcation diagram exhibits 55 distinct typical configurations representing 11 distinct topological behavior (see Figure 28).
Theorem B. Equation \( 4 \) with \( \tau = v \) generically unfolds the (Visible) Fold–Saddle singularity. Moreover, its bifurcation diagram exhibits 39 distinct typical configurations representing 21 distinct topological behavior (see Figure 30).

The paper is organized as follows: in Section 2 we give the basic theory about Non-Smooth Vector Fields on the Plane, in Section 3 we prove Theorem 1, in Section 4 we prove Theorem 2, in Section 5 we prove Theorem 3, in Section 6 we prove Theorem A and present the Bifurcation Diagram of \( Z_{\lambda,\mu,\beta} \), in Section 7 we prove Theorem 4, in Section 8 we prove Theorem 5, in Section 9 we prove Theorem 6 and in Section 10 we prove Theorem B and present the Bifurcation Diagram of \( Z_{\lambda,\mu,\beta} \).

2. Preliminaries

We distinguish the following regions on the discontinuity set \( \Sigma \):

(i) \( \Sigma_1 \subseteq \Sigma \) is the sewing region if \( (X.f)(Y.f) > 0 \) on \( \Sigma_1 \).

(ii) \( \Sigma_2 \subseteq \Sigma \) is the escaping region if \( (X.f) > 0 \) and \( (Y.f) < 0 \) on \( \Sigma_2 \).

(iii) \( \Sigma_3 \subseteq \Sigma \) is the sliding region if \( (X.f) < 0 \) and \( (Y.f) > 0 \) on \( \Sigma_3 \).

Consider \( Z \in \Omega' \). The sliding vector field associated to \( Z \) is the vector field \( Z^s \) tangent to \( \Sigma_3 \) and defined at \( q \in \Sigma_3 \) by \( Z^s(q) = m - q \) with \( m \) being the point where the segment joining \( q + X(q) \) and \( q + Y(q) \) is tangent to \( \Sigma_3 \) (see Figure 6). It is clear that if \( q \in \Sigma_3 \) then \( q \in \Sigma_2 \) for \( -Z \) and then we can define the escaping vector field on \( \Sigma_2 \) associated to \( Z \) by \( Z^e = -(Z)^s \).

In what follows we use the notation \( Z^\Sigma \) for both cases.

We say that \( q \in \Sigma \) is a \( \Sigma \)-regular point if

(i) \( (X.f(q))(Y.f(q)) > 0 \) or

(ii) \( (X.f(q))(Y.f(q)) < 0 \) and \( Z^\Sigma(q) \neq 0 \) (that is \( q \in \Sigma_2 \cup \Sigma_3 \) and it is not a singular point of \( Z^\Sigma \)).

The points of \( \Sigma \) which are not \( \Sigma \)-regular are called \( \Sigma \)-singular. We distinguish two subsets in the set of \( \Sigma \)-singular points: \( \Sigma^t \) and \( \Sigma^p \). Any \( q \in \Sigma^p \)

\[ \text{Figure 6. Fillipov's convention.} \]
is called a pseudo equilibrium of $Z$ and it is characterized by $Z^\Sigma(q) = 0$. Any $q \in \Sigma^t$ is called a tangential singularity and is characterized by $Z^\Sigma(q) \neq 0$ and $X.f(q)Y.f(q) = 0$ ($q$ is a contact point of $Z^\Sigma$).

A pseudo equilibrium $q \in \Sigma^p$ is a $\Sigma-$saddle provided one of the following condition is satisfied: (i) $q \in \Sigma_2$ and $q$ is an attractor for $Z^\Sigma$ or (ii) $q \in \Sigma_3$ and $q$ is a repeller for $Z^\Sigma$. A pseudo equilibrium $q \in \Sigma^p$ is a $\Sigma-$repeller (resp. $\Sigma-$attractor) provided $q \in \Sigma_2$ (resp. $q \in \Sigma_3$) and $q$ is a repeller (resp. attractor) equilibrium point for $Z^\Sigma$.

**Definition 1.** Consider $Z \in \Omega^r$.

1. A curve $\Gamma$ is a canard cycle if $\Gamma$ is closed and
   - $\Gamma$ contains arcs of at least two of the vector fields $X|_{\Sigma^+}$, $Y|_{\Sigma^-}$ and $Z^\Sigma$ or is composed by a single arc of $Z^\Sigma$;
   - the transition between arcs of $X$ and arcs of $Y$ happens in sewing points;
   - the transition between arcs of $X$ (or $Y$) and arcs of $Z^\Sigma$ happens through $\Sigma-$fold points or regular points in the escape or sliding arc, respecting the orientation. Moreover if $\Gamma \neq \Sigma$ then there exists at least one visible $\Sigma-$fold point on each connected component of $\Gamma \cap \Sigma$.

2. Let $\Gamma$ be a canard cycle of $Z$. We say that
   - $\Gamma$ is a canard cycle of kind I if $\Gamma$ meets $\Sigma$ just in sewing points;
   - $\Gamma$ is a canard cycle of kind II if $\Gamma = \Sigma$;
   - $\Gamma$ is a canard cycle of kind III if $\Gamma$ contains at least one visible $\Sigma-$fold point of $Z$.

In Figures 7, 8 and 9 arise canard cycles of kind I, II and III respectively.

3. Let $\Gamma$ be a canard cycle. We say that $\Gamma$ is hyperbolic if
   - $\Gamma$ is of kind I and $\eta'(p) \neq 1$, where $\eta$ is the first return map defined on a segment $T$ with $p \in T \cap \gamma$;
   - $\Gamma$ is of kind II;
   - $\Gamma$ is of kind III and or $\Gamma \cap \Sigma \subseteq \Sigma_1 \cup \Sigma_2$ or $\Gamma \cap \Sigma \subseteq \Sigma_1 \cup \Sigma_3$.

**Remark 1.** The expression “canard” is used here because these orbits are limit periodic sets of singular perturbation problems (see [1]).

**Definition 2.** Consider $Z \in \Omega^r$. A point $q \in \Sigma$ is a $\Sigma-$center if there is a neighborhood $U$ of $q$ such that an one parameter family of canard cycles encircles $q$ and foliates $U$.

**Definition 3.** Consider $Z \in \Omega^r$. A closed path $\Delta$ is a $\Sigma-$graph if it is a union of equilibria, pseudo equilibria, tangential singularities of $Z$ and arcs of $Z$ joining these points in such a way that $\Delta \cap \Sigma \neq \emptyset$. Like for canard
cycles, we say that $\Delta$ is a $\Sigma$–graph of kind I if $\Delta \cap \Sigma \subset \Sigma_1$, $\Delta$ is a $\Sigma$–graph of kind II if $\Delta \cap \Sigma = \Delta$ and $\Delta$ is a $\Sigma$–graph of kind III if $\Delta \cap \Sigma \subsetneq \Sigma_2 \cup \Sigma_3$.

In what follows, in order to simplify the calculations, we take $\mu = \alpha + 1$ in (4) and obtain the following expression

$$Z_{\tau,\alpha,\beta} = \begin{cases} X_{\lambda} = \frac{1}{\alpha_1(\tau)}(x - \lambda) & \text{if } y \geq 0, \\ Y_{\alpha,\beta} = \left(\frac{1+\alpha}{2}x + \frac{1+\alpha}{2}(y + \beta)\right) & \text{if } y \leq 0, \end{cases}$$

where $\lambda, \beta \in (-1, 1)$, $\alpha \in (-1 - \varepsilon_0, -1 + \varepsilon_0)$, $\tau = i$ or $v$, $\alpha_1(i) = -1$ and $\alpha_1(v) = 1$. When it does not produce confusion, in order to simplify the notation we use $Z = (X, Y)$ or $Z_{\lambda,\alpha,\beta} = (X, Y)$ instead $Z_{\tau,\lambda,\alpha,\beta} = (X_{\lambda}, Y_{\alpha,\beta})$.

Given $Z = (X, Y)$, we describe some properties of both $X = X_{\lambda}$ and $Y = Y_{\alpha,\beta}$.

The real number $\lambda$ measures how the $\Sigma$–fold point $d = (\lambda, 0)$ of $X$ is translated away from the origin. More specifically, if $\lambda < 0$ then $d$ is translated to the left hand side and if $\lambda > 0$ then $d$ is translated to the right hand side.

Some calculations show that the curve $Y.f = 0$ is given by $y = \frac{(1-\alpha)}{(1+\alpha)}x - \beta$. So the points of this curve are equidistant from the separatrices when $\alpha = -1$. It become closer to the stable separatrix of the saddle point $S = S_{\alpha,\beta}$ when $\alpha \in (-1, -1+\varepsilon_0)$. It become closer to the unstable separatrix of $S$ when $\alpha \in (-1 - \varepsilon_0, -1)$. Moreover, the smooth vector field $Y$ has distinct types of contact with $\Sigma$ according with the particular deformation considered. In this way, we have to consider the following behaviors:

- $Y^{-}$: In this case $\beta < 0$. So $S$ is translated to the $y$–direction with $y > 0$ (and $S$ is not visible for $Z$). It has a visible $\Sigma$–fold point $e = e_{\alpha,\beta} = \left(\frac{(1+\alpha)}{(1-\alpha)}\beta, 0\right)$ (see Figure 10).
- $Y^{0}$: In this case $\beta = 0$. So $S$ is not translated (see Figure 11).
• $Y^+$: In this case $\beta > 0$. So $S$ is translated to the $y$–direction with $y < 0$. It has an invisible $\Sigma$–fold point $i = (i_1, i_2) = i_{\alpha, \beta} = \left(\frac{1+\alpha}{1-\alpha} \beta, 0\right)$. Moreover, we distinguish two points: $h = h_\beta = (-\beta, 0)$ which is the intersection between the unstable separatrix with $\Sigma$ and $j = j_\beta = (\beta, 0)$ which is the intersection between the stable separatrix with $\Sigma$ (see Figure 11).

In Figure 11 we distinguish the arcs $\sigma_1$ of $Y$ joining the saddle point $S$ of $Y$ to $h$ and $\sigma_2$ of $Y$ joining $j$ to the saddle point $S$ of $Y$.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure10.png}
\caption{Case $Y^-$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure11.png}
\caption{Case $Y^+$.}
\end{figure}

3. Proof of Theorem 1

In $(a, b) \subset \Sigma_2 \cup \Sigma_3$, consider the point $c = (c_1, c_2)$, the vectors $X(c) = (d_1, d_2)$ and $Y(c) = (e_1, e_2)$ (as illustrated in Figure 12). The straight segment passing through $c + X(c)$ and $c + Y(c)$ meets $\Sigma$ in a point $p(c)$. We define the $C^r$–map

$$p: (a, b) \longrightarrow \Sigma$$

$$z \longmapsto p(z).$$

We can choose local coordinates such that $\Sigma$ is the $x$–axis; so $c = (c_1, 0)$ and $p(c) \in \mathbb{R} \times \{0\}$ can be identified with points in $\mathbb{R}$. According with this identification, the direction function on $\Sigma$ is defined by

$$H: (a, b) \longrightarrow \mathbb{R}$$

$$z \longmapsto p(z) - z.$$
• if $H(c) = 0$ then $c \in \Sigma^p$;
• if $H(c) > 0$ then the orientation of $Z^\Sigma$ in a small neighborhood of $c$ is from $a$ to $b$.

Simple calculations show that $p(c_1) = \frac{e_2(d_1 + c_1) - d_2(e_1 + c_1)}{e_2 - d_2}$ and consequently,

\begin{equation}
H(c_1) = \frac{e_2d_1 - d_2e_1}{e_2 - d_2}.
\end{equation}

We now in position to prove Theorem 1.

Proof of Theorem 1. In Cases 1, 2 and 3 we assume that $Y$ presents the behavior $Y^-$. In Cases 4, 5 and 6 we assume that $Y$ presents the behavior $Y^+$. In these cases canard cycles are not allowed.

- Case 1. $d < e$, Case 2. $d = e$ and Case 3. $d > e$: The points of $\Sigma$ outside the interval $(d, e)$ belong to $\Sigma_1$. The points inside this interval, when it is not degenerated, belong to $\Sigma_3$ in Case 1 and to $\Sigma_2$ in Case 3. In both cases $H(z) > 0$ for all $z \in (d, e)$. See Figure 13.

- Case 4. $d < S$, Case 5. $d = S$ and Case 6. $d > S$: The points of $\Sigma$ outside the interval $(d, S)$ belong to $\Sigma_1$. The points inside this interval, when it is not degenerated, belong to $\Sigma_3$ in Case 4 and to $\Sigma_2$ in Case 6. In both cases $H(z) > 0$ for all $z \in (d, S)$. See Figure 14.

In Cases 7 to 17 we assume that $Y$ presents the behavior $Y^+$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{cases123}
\caption{Cases 1, 2 and 3.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{cases456}
\caption{Cases 4, 5 and 6.}
\end{figure}
Case 7. \( \lambda < -\beta \), Case 8. \( \lambda = -\beta \), Case 9. \( -\beta < \lambda < -\beta/2 \), Case 10. \( \lambda = -\beta/2 \) and Case 11. \( -\beta/2 < \lambda < 0 \): The points of \( \Sigma \) outside the interval \((d, i)\) belong to \( \Sigma_1 \). The points inside this interval belong to \( \Sigma_3 \). The direction function \( H \) assumes positive values in a neighborhood of \( d \), negative values in a neighborhood of \( i \) and \( H(\lambda\beta/(1 + \beta)) = 0 \). So, by (6), the \( \Sigma \)-attractor \( P = (\lambda\beta/(1 + \beta), 0) \), nearby \((0, 0)\), is the unique pseudo equilibrium. In these cases canard cycles are not allowed. See Figure 15.

\[
\begin{align*}
\lambda < -\beta & \quad \lambda = -\beta & \quad -\beta < \lambda < -\beta/2 & \quad \lambda = -\beta/2 & \quad -\beta/2 < \lambda < 0 \\
\text{Figure 15. Cases 7} - 11.
\end{align*}
\]

Case 12. \( \lambda = 0 \): Since \( \alpha = -1 \) and \( d = i \), it is straightforward to show that each point \( Q \in (h, i) \) belongs to a closed curve composed by an arc of \( X \) and an arc of \( Y \). So \( d = i \) is a \( \Sigma \)-center. See Figure 16.

\[
\begin{align*}
\text{Figure 16. Case 12}.
\end{align*}
\]

Case 13. \( 0 < \lambda < \beta/2 \), Case 14. \( \lambda = \beta/2 \), Case 15. \( \beta/2 < \lambda < \beta \), Case 16. \( \lambda = \beta \) and Case 17. \( \lambda > \beta \): The points of \( \Sigma \) outside the interval \((i, d)\) belong to \( \Sigma_1 \) and the points inside this interval belong to \( \Sigma_2 \). The direction function \( H \) assumes positive values in a neighborhood of \( d \), negative values in a neighborhood of \( i \) and \( H(\lambda\beta/(1 + \beta)) = 0 \). So, by (6), the \( \Sigma \)-repeller \( P = (\lambda\beta/(1 + \beta), 0) \), nearby \((0, 0)\), is the unique pseudo equilibrium. In these cases canard cycles are not allowed. See Figure 17.

The bifurcation diagram is illustrated in Figure 18. □
0 < \lambda < \beta/2  \quad \lambda = \beta/2  \quad \beta/2 < \lambda < \beta  \quad \lambda = \beta  \quad \lambda > \beta

\[ \begin{array}{c}
13_1 \\
15_1 \\
14_1 \\
16_1
\end{array} \]

\begin{center}
\textbf{Figure 17.} Cases 13\_1 - 17\_1.
\end{center}

\begin{center}
\textbf{Figure 18.} Bifurcation Diagram of Theorem 1.
\end{center}

4. Proof of Theorem 2

Proof of Theorem 2. In Cases 1\_2, 2\_2 and 3\_2 we assume that \( Y \) presents the behavior \( Y^- \). In Cases 4\_2, 5\_2 and 6\_2 we assume that \( Y \) presents the behavior \( Y^0 \). In Cases 7\_2 - 19\_2 we assume that \( Y \) presents the behavior \( Y^+ \).

\[ \begin{array}{c}
\Diamond \text{ Case 1\_2. } d < e, \text{ Case 2\_2. } d = e, \text{ Case 3\_2. } d > e, \text{ Case 4\_2. } d < S, \text{ Case 5\_2. } d = S \text{ and Case 6\_2. } d > S: \text{ Analogous to Cases 1\_1, 2\_1, 3\_1, 4\_1, 5\_1 and 6\_1.}
\end{array} \]

\[ \begin{array}{c}
\Diamond \text{ Case 7\_2. } \lambda < -\beta, \text{ Case 8\_2. } \lambda = -\beta, \text{ Case 9\_2. } -\beta < \lambda < -\beta/(1 - \alpha), \text{ Case 10\_2. } \lambda = -\beta/(1 - \alpha) \text{ and Case 11\_2. } -\beta/(1 - \alpha) < \lambda < 0: \text{ Analogous to Cases 7\_1 - 11\_1 changing } -\beta/2 \text{ by } -\beta/(1 - \alpha) = -\text{dist}(h, i)/2, \text{ where dist}(h, i) \text{ is the distance between } h \text{ and } i. \text{ The unique pseudo equilibrium}
\end{array} \]
occurs in $P = (p^−, 0)$ where

(7)
\[
p^− = \frac{1}{2(\alpha + 1)}\left(\frac{1}{2}(1 − \alpha)(1 + \beta) + \lambda(1 + \alpha) + \sqrt{((1 − \alpha)(1 + \beta) + \lambda(1 + \alpha))^2 - 4\beta(1 + \alpha)(1 + \alpha + \lambda(1 − \alpha))}\right).
\]

\[\Box\] Case $12_2$. $\lambda = 0$: The points of $\Sigma$ outside the interval $(d, i)$ belong to $\Sigma_1$ and the points inside this interval belong to $\Sigma_3$. The direction function $H$ assumes positive values in a neighborhood of $d$, negative values in a neighborhood of $i$ and $H(p^+_0, 0) = 0$ where $p^+_0$ is given by (7) with $\lambda = 0$. So $P = (p^+_0, 0)$ is a $\Sigma$-attractor. Since $e = 0$, it is easy to see that there is an arc $\gamma^X_1$ of $X$ connecting the points $h$ and $j$. It generates a $\Sigma$-graph $\Gamma = \gamma^X_1 \cup \sigma_2 \cup S \cup \sigma_1$ of kind I. Since $-1 < \alpha < -1 + \varepsilon_0$, it is straightforward to show that the First Return Map $\eta = \varphi_Y \circ \varphi_X$, where

\[
\varphi_X : \Sigma \rightarrow \Sigma \\
z = (x, 0) \mapsto (-x + 2\lambda, 0)
\]

and

\[
\varphi_Y : (i, j) \subset \Sigma \rightarrow (h, i) \subset \Sigma \\
z = (x, 0) \mapsto \left(\frac{x(i_1 + \beta) - 2i^2_1}{\beta - i_1}, 0\right),
\]

has derivative bigger than 1 in the interval $(h, d)$. By consequence, $\Gamma$ is a repeller for the trajectories inside it and in this case canard cycles are not allowed. See Figure 19.

\[\Box\] Case $13_2$. $0 < \lambda < i_1$: The distribution of the connected components of $\Sigma$ and the behavior of $H$ are the same of Case $12_2$ with $P = (p^+_\lambda, 0)$ where $p^+_\lambda$ is given by (7). Since $0 < \lambda < i_1$, there is an arc $\gamma^X_1$ of $X$ connecting the point $j$ to a point $k_1 \in (h, d)$. Also there is an arc $\gamma^Y_1$ of $Y$ connecting the point $k_1$ to a point $l_1 \in (i, j)$. Repeating this argument, we can find an increasing sequence $(k_i)_{i \in \mathbb{N}}$. We can prove that there is an interval $I \subset (k_1, d)$ such that $\eta' = (\varphi_Y \circ \varphi_X)' < 1$. As $P$ is a $\Sigma$-attractor, there is an interval $J \subset (k_1, d)$ such that $\eta' > 1$. Moreover, there exists an
unique point $Q \in (k_1, d)$ given by $Q = ((-\alpha_1^2 + \lambda(i_1 + \beta))/\beta, 0)$ such that $\eta' = 1$.

By $Q$ passes a repeller canard cycle $\Gamma$ of kind I. See Figure 19.

- **Case 14.** $\lambda = i_1$: Every point of $\Sigma$ belongs to $\Sigma_1$ except the point $d = i$. As in the previous case, we can construct sequences $(k_i)_{i \in \mathbb{N}}$ and $(l_i)_{i \in \mathbb{N}}$. Since $e = i_1$, we have that $k_i \to d$ and $l_i \to d$. So $d$ is a non generic tangential singularity of repeller kind. In this case canard cycles are not allowed. See Figure 19.

- **Case 15.** $i_1 < \lambda < \alpha\beta/(1 - \alpha)$, Case 16. $\lambda = \alpha\beta/(1 - \alpha)$, Case 17. $\alpha\beta/(1 - \alpha) < \lambda < \beta$: Analogous to Cases 13.2 - 17.1 changing $\beta/2$ by $\alpha\beta/(1 - \alpha) = -\text{dist}(i, j)/2$. The unique pseudo equilibrium occurs in $P = (p^-, 0)$ where $p^-$ is given by (7).

\[ \begin{array}{c}
\text{Figure 20. Bifurcation Diagram of Theorem 2.} \\
\end{array} \]

The bifurcation diagram is illustrated in Figure 20. □

5. PROOF OF THEOREM 3

**Proof of Theorem 3.** In Cases 13.3, 23.3 and 33.3 we assume that $Y$ presents the behavior $Y^-$. In Cases 43.3, 53 and 63 we assume that $Y$ presents the behavior $Y^0$. In Cases 73.3 - 19.3 we assume that $Y$ presents the behavior $Y^+$.

- **Case 13.** $d < e$, Case 23. $d = e$, Case 33. $d > e$, Case 43. $d < S$, Case 53. $d = S$ and Case 63. $d > S$: Analogous to Cases 1.1, 2.1, 3.1, 4.1, 5.1 and 6.1.

- **Case 73.** $\lambda < -\beta$, Case 83. $\lambda = -\beta$, Case 93. $-\beta < \lambda < -\beta/(1 - \alpha)$, Case 103. $\lambda = -\beta/(1 - \alpha)$ and Case 113. $-\beta/(1 - \alpha) < \lambda < i_1$: Analogous to Cases 7.1 - 11.1 changing $-\beta/2$ by $-\beta/(1 - \alpha) = -\text{dist}(h, i)/2$. The unique pseudo equilibrium occurs in $P = (p^-, 0)$ where $p^-$ is given by (7).

- **Case 123.** $\lambda = i_1$: Analogous to Case 14.2 except that here $d$ is an attractor, i.e., there is a change of stability. See Figure 21.
○ Case 13. $i_1 < \lambda < 0$: Analogous to Case 13.2 except that there is a change of stability on $P = (p^-,0)$, which is a $\Sigma-$repeller, and on $\Gamma$, which is an attractor canard cycle of kind I. See Figure 21.

○ Case 14. $\lambda = 0$: Analogous to Case 12.2 except that occurs a change of stability on $P = (p^-,0)$, which is a $\Sigma-$repeller, and on $\Gamma$, which is an attractor for the trajectories inside it. See Figure 21.

○ Case 15. $0 < \lambda < \alpha \beta / (1 - \alpha)$, Case 16. $\lambda = \alpha \beta / (1 - \alpha)$, Case 17. $\alpha \beta / (1 - \alpha) < \lambda < \beta$, Case 18. $\lambda = \beta$ and Case 19. $\lambda > \beta$: Analogous to Cases 13.1 $-$ 17.1 changing $\beta / 2$ by $\alpha \beta / (1 - \alpha) = -\text{dist}(i,j)/2$. The unique pseudo equilibrium occurs in $P = (p^-,0)$.

The bifurcation diagram is illustrated in Figure 22.
6. Proof of Theorem A

Proof of Theorem A. Since in Equation (5) we can take \( \alpha \) in the interval \((-\infty, 0)\), from Theorems 1, 2 and 3 we derive that this equation, with \( \tau = i \), unfolds generically the (Invisible) Fold–Saddle singularity.

Observe that the bifurcation diagram contain all the typical configurations and all the distinct topological behavior described in Theorems 1, 2 and 3. So, the number of typical configurations is 55 and the number of distinct topological behaviors is 11. Moreover, each topological behavior can be represented respectively by the Cases 1\(_1\), 4\(_1\), 7\(_1\), 12\(_1\), 13\(_1\), 12\(_2\), 13\(_2\), 14\(_2\), 12\(_3\), 13\(_3\) and 14\(_3\).

The full behavior of the three-parameter family of non-smooth vector fields presenting the normal form (5), with \( \tau = i \), is illustrated in Figure 23 where we consider a sphere around the point \((\lambda, \mu, \beta) = (0, 0, 0)\) with a small ray and so we make a stereographic projection defined on the entire sphere, except the south pole. Still in relation with this figure, the numbers pictured correspond to the occurrence of the cases described in the previous theorems. As expected, the cases 5\(_1\) and 5\(_2\) are not represented in this figure because they are, respectively, the center and the south pole of the sphere. \(\square\)

7. Proof of Theorem 4

Proof of Theorem 4. Since \(X\) has a unique \(\Sigma\)-fold point which is visible we conclude that canard cycles are not allowed.

In Cases 1\(_4\), 2\(_4\) and 3\(_4\) we assume that \(Y\) presents the behavior \(Y^-\). In Cases 4\(_4\), 5\(_4\) and 6\(_4\) we assume that \(Y\) presents the behavior \(Y^0\). In these cases, when it is well defined, the direction function \(H\) assumes positive values.

\(\diamond\) Case 1\(_4\), \(d < e\): The points of \(\Sigma\) inside the interval \((d, e)\) belong to \(\Sigma_1\). The points on the left of \(d\) belong to \(\Sigma_3\) and the points on the right of \(e\) belong to \(\Sigma_2\). See Figure 24.

\(\diamond\) Case 2\(_4\), \(d = e\): Here \(\Sigma_1 = \emptyset\). The vector fields \(X\) and \(Y\) are linearly dependent on \(d = e\) which is a tangential singularity. Moreover, it is an attractor for the trajectories of \(Z\) crossing \(\Sigma_3\) and a repeller for the trajectories of \(Z\) crossing \(\Sigma_2\). See Figure 24.

\(\diamond\) Case 3\(_4\), \(d > e\): The points of \(\Sigma\) inside the interval \((e, d)\) belong to \(\Sigma_1\). The points on the left of \(e\) belong to \(\Sigma_3\) and the points on the right of \(d\) belong to \(\Sigma_2\). See Figure 24.

\(\diamond\) Case 4\(_4\), \(d < S\): The points of \(\Sigma\) inside the interval \((d, S)\) belong to \(\Sigma_1\). The points on the left of \(d\) belong to \(\Sigma_3\) and the points on the right of \(S\) belong to \(\Sigma_2\). See Figure 24.

\(\diamond\) Case 5\(_4\), \(d = S\): Here \(\Sigma_1 = \emptyset\) and \(S\) is an attractor for the trajectories of \(Z\) crossing \(\Sigma_3\) and it is a repeller for the trajectories of \(Z\) crossing \(\Sigma_2\). See Figure 25.
\[ -1 < \alpha < 0 < \alpha < -1 \]

\[ \lambda > 0 \]

\[ \lambda = 0 \]

\[ \lambda < 0 \]

\[ \beta < 0 \]

\[ \beta = 0 \]

\[ \beta > 0 \]

Figure 23. Bifurcation diagram of the (Invisible) Fold–Saddle singularity.

Figure 24. Cases 1, 2 and 3.

\[ \diamond \text{Case 6}_{1}. \ d > S: \] The points of \( \Sigma \) inside the interval \((d, S)\) belong to \(\Sigma_{1}\). The points on the left of \(S\) belong to \(\Sigma_{3}\) and the points on the right of \(d\) belong to \(\Sigma_{2}\). See Figure 25.
In Cases 7\textsubscript{4} – 13\textsubscript{4} we assume that $Y$ presents the behavior $Y^+$. 

- **Case 7**: $d < h$; **Case 8**: $d = h$; **Case 9**: $h < d < i$: The points of $\Sigma$ inside the interval $(d, i)$ belong to $\Sigma_1$. The points on the left of $d$ belong to $\Sigma_3$ and the points on the right of $i$ belong to $\Sigma_2$. The direction function $H$ assumes positive values on $\Sigma_3$ and negative values in a neighborhood of $i$. Moreover, $H(\beta \lambda /(-1 + \beta)) = 0$ and the $\Sigma$–repeller $P = (\beta \lambda /(-1 + \beta), 0)$ is the unique pseudo equilibrium. See Figure 26.

- **Case 10**: $d = i$: Here $\Sigma_1 = \emptyset$. The vector fields $X$ and $Y$ are linearly dependent on the tangential singularity $d = i$. A straightforward calculation shows that $H(z) = (1 - \beta) / 2 \neq 0$ for all $z \in \Sigma /\{d\}$. So $d = i$ is an attractor for the trajectories of $Z$ crossing $\Sigma_3$ and a repeller for the trajectories of $Z$ crossing $\Sigma_2$. Moreover, $\Delta = \{d\} \cup dj \cup \sigma_2 \cup \{S\} \cup \sigma_1 \cup \sigma d$ is a $\Sigma$–graph of kind III in such a way that each $Q$ in its interior belongs to another $\Sigma$–graph of kind III passing through $d$. See Figure 26.

- **Case 11**: $i < d < j$; **Case 12**: $d = j$; **Case 13**: $j < d$: The points of $\Sigma$ inside the interval $(i, d)$ belong to $\Sigma_1$. The points on the left of $i$ belong to $\Sigma_3$ and the points on the right of $d$ belong to $\Sigma_2$. The direction function $H$ assumes positive values on $\Sigma_2$ and negative values in a neighborhood of $i$. Moreover, $H(\beta \lambda /(-1 + \beta)) = 0$ and the $\Sigma$–attractor $P = (\beta \lambda /(-1 + \beta), 0)$ is the unique pseudo equilibrium. See Figure 26.

The bifurcation diagram is illustrated in Figure 28. Each topological behavior can be represented respectively by Cases 1\textsubscript{4}, 2\textsubscript{4}, 4\textsubscript{4}, 5\textsubscript{4}, 7\textsubscript{4}, 10\textsubscript{4} and 11\textsubscript{4}. \hfill $\square$
8. PROOF OF THEOREM 5

Proof of Theorem 5. The direction function $H$ has a root $Q = (q,0)$ where

$$q = \frac{1}{2(\alpha + 1)}((-1 + \alpha)(1 - \beta) - \lambda(1 + \alpha) +$$

$$+\sqrt{((-1 + \alpha)(1 - \beta) - \lambda(1 + \alpha))^2 + 4\beta(1 + \alpha)(1 + \alpha + \lambda(-1 + \alpha)))}).$$

Moreover, $H$ assumes positive values on the right of $Q$ and negative values on the left of $Q$. Note that when $\alpha \to -1$ so $Q \to -\infty$ under the line $\{y = 0\}$ and it occurs the configurations showed in Theorem 4.

In Cases 15, 25 and 35 we assume that $Y$ presents the behavior $Y^-$. In Cases 45, 55 and 65 we assume that $Y$ presents the behavior $Y^0$. In Cases 75 - 135 we assume that $Y$ presents the behavior $Y^+$.

\(\Diamond\) Case 15. $d < e$, Case 25. $d = e$, Case 35. $d > e$, Case 45. $d < S$, Case 55. $d = S$ and Case 65. $d > S$: Analogous to Cases 14, 24, 34, 44, 54 and 64 respectively, except that here it appears the $\Sigma$–saddle $Q$ on the left of $d$ and $e$ or $S$. See Figure 29.
\[ \lambda < (1 + \alpha)\beta/(1 - \alpha) \quad \lambda = (1 + \alpha)\beta/(1 - \alpha) \quad \lambda > (1 + \alpha)\beta/(1 - \alpha) \]

**Figure 29.** Cases 15, 25 and 35.

- **Case 75.** \( d < h \), **Case 85.** \( d = h \), **Case 95.** \( h < d < i \): Analogous to Cases 74–94, except that here it appears the \( \Sigma \)-saddle \( Q \) on the left of \( d \) and \( i \). Here \( P = (p, 0) \) where

  \[
p = \frac{1}{2(\alpha + 1)}((-1 + \alpha)(1 - \beta) - \lambda(1 + \alpha) + \sqrt{((-1 + \alpha)(1 - \beta) - \lambda(1 + \alpha))^2 + 4\beta(1 + \alpha)(1 + \alpha + \lambda(-1 + \alpha)))}.\]

- **Case 105.** \( d = i \): Analogous to Case 104, except that here appear the \( \Sigma \)-saddle \( Q \) on the left of \( d = i \).

- **Case 115.** \( i < d < j \), **Case 125.** \( d = j \) and **Case 135.** \( j < d \): Analogous to Cases 114–134, except that here it appears the \( \Sigma \)-saddle \( Q \) on the left of \( d \) and \( i \).

The bifurcation diagram is illustrated in Figure 28. Each topological behavior can be represented respectively by Cases 15, 25, 45, 55, 75, 105 and 115.

\[ \square \]

### 9. Proof of Theorem 6

**Proof of Theorem 6.** The direction function \( H \) has a root \( Q = (q, 0) \) where \( q \) is given by (8). Moreover, \( H \) assumes positive values on the left of \( Q \) and negative values on the right of \( Q \). Note that when \( \alpha \to -1 \) so \( Q \to \infty \) under the line \( \{y = 0\} \) and it occurs the configurations showed in Theorem 4.

- **Case 16.** \( d < e \), **Case 26.** \( d = e \), **Case 36.** \( d > e \), **Case 46.** \( d < S \), **Case 56.** \( d = S \) and **Case 66.** \( d > S \), **Case 76.** \( d < h \), **Case 86.** \( d = h \), **Case 96.** \( h < d < i \), **Case 106.** \( d = i \), **Case 116.** \( i < d < j \), **Case 126.** \( d = j \) and **Case 136.** \( j < d \): Analogous to Cases 16–136, respectively, except that here the \( \Sigma \)-saddle \( Q \) takes place on the right of \( d, e, S \) and \( i \) when these points appear.

The bifurcation diagram is illustrated in Figure 28. Each topological behavior can be represented respectively by Cases 16, 26, 46, 56, 76, 106 and 116.

\[ \square \]
10. Proof of Theorem B

Proof of Theorem B. Since in Equation (5) we can take $\alpha$ in the interval $(-1 - \varepsilon_0, -1 + \varepsilon_0)$ we conclude that Theorems 4, 5 and 6 prove that this equation, with $\tau = v$, unfolds generically the (Visible) Fold–Saddle singularity. Its bifurcation diagram contains all typical configurations and all distinct topological behavior described in Theorems 4, 5 and 6. So, the number of typical configurations is 39 and the number of distinct topological behavior is 21. Moreover, each topological behavior can be represented respectively by the Cases $1_4, 1_5, 1_6, 2_4, 2_5, 2_6, 4_4, 4_5, 4_6, 5_4, 5_5, 5_6, 7_4, 7_5, 7_6, 10_4, 10_5, 10_6, 11_4, 11_5$ and $11_6$.

![Bifurcation diagram of the (Visible) Fold–Saddle singularity.](image)

**Figure 30.** Bifurcation diagram of the (Visible) Fold–Saddle singularity.

The full behavior of the three–parameter family of non–smooth vector fields presenting the normal form (5), with $\tau = v$, is illustrated in Figure
where we consider a sphere around the point $(\lambda, \mu, \beta) = (0, 0, 0)$ with a small ray and so we make a stereographic projection defined on the entire sphere, except the south pole. Still in relation with this figure, the numbers pictured correspond to the occurrence of the cases described in the previous theorems. As expected, the cases $5_4$ and $5_5$ are not represented in this figure because they are, respectively, the center and the south pole of the sphere.

Acknowledgments. The first and the third authors are partially supported by a FAPESP-BRAZIL grant 2007/06896-5. The second author is partially supported by a FAPESP-BRAZIL grant 2007/08707-5.

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