Increasing fretting resistance of flexible element pack for rotary machine flexible coupling
Part 2. The influence of coupled shafts misalignment on flexible coupling flexible elements stress-strain state

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Abstract. It has been performed the analysis of stress-strain state of flexible element pack for flexible coupling. Plots of internal force factors arising in the flexible elements (plates) of the pack due to the transmitted torque and misalignments caused by the assembly errors when coupling the shafts have been constructed. The analytical dependences of the deflections and rotations of the sections have been obtained. The pressure between the plates, the friction forces in the pack and the loss of power because of strain and friction have been determined.

1. Introduction

In part 1 of the paper, there is represented an analysis of the causes of the rotor machine flexible coupling (FC) failure. The analysis has shown that the period of the flexible coupling trouble-free operation is limited by the flexible element (FE) pack durability, which in turn depends on the fretting resistance indices of the separate flexible elements (plates).

As a result of the study under an electron microscope of the individual sections of the flexible element surfaces damaged to varying degrees due to the fretting corrosion process, it has been set that there are at least three stages of their destruction, namely, the formation of microfractures in the form of caverns at separate points of the destruction zone, which microfractures preserve microparticles of the destroyed metal in a confined space; weakening of the compression force affecting the plates, and the wear product appearance in the form of abrasive particles from the fracture site; the occurrence of the abrasive wear which results in the complete destruction of the flexible element.

To develop technological and design recommendations aimed at reducing the fretting corrosion process between the contacting surfaces of the individual plates, it is necessary to analyze their stress-strain state and to determine the geometric and deformation parameters of the strain zone.

The couplings of the MSC type with flexible annular elements are used to transmit a torque and compensate for the inconsistency and relative axial misalignments of the shafts which are being coupled. They have high radial flexibility and transmit a great torque.

The coupling consists of two half-couplings connected by the FE packs and a cylindrical spacer (Figure 1). The torque is transmitted from the flange of the drive half-coupling to the pack and then to the spacer flange by means of the clamping bolts arranged in a circle, and which alternately (next but
one) attach the pack to one and then to the other flange. The flexible elements are made of wear resistant stainless steel of high strength. However, in the process of operation, the flexible elements (plates) are bolted only at certain points, and therefore, they have slight displacements relative to each other in the areas between the bolts, which fact causes both mechanical damage and fretting corrosion of the flexible elements (plates) surfaces.

![Figure 1. The design of the coupling with flexible metal elements](image)

Based on the analysis of production experience, the investigations of the reasons for the loss of the coupling capacity were performed in [1]. In the first part of this work, there was disclosed the analysis of the conditions of the coupling operation and represented the results of the studies of the flexible elements worn out. At the same time, there are no thorough investigations, in particular theoretical ones, which could explain the causes of fretting corrosion and identify the factors that most strongly influence the wear and tear of the flexible elements.

In this work, on the basis of the analysis of the stress-strain state of the expansion coupling flexible elements, an attempt is made to estimate the pressure forces between the plates and the relative displacements of the contact points in the course of the load transfer by the coupling, which fact is one of the reasons for fretting corrosion appearance.

2. Basic research material

2.1. Drawing up a calculation scheme

Flexible elements (plates) of the plate coupling transmit torques (sometimes up to 100,000 N·m) from the drive shaft to the driven shaft, which fact results in considerable forces between them. The coupling plate occurs in difficult conditions when operating in bending with torsion, stretching and compressing by eccentrically applied force, which provokes a possible loss of its stability. There are two main factors that affect the plate strain:

1. Mounting and operating deviations and misalignment of the shafts being coupled, this can be considered known from the art.
2. Circumferential force from the torque transmitted by the coupling.
In the course of the coupling rotation, the misalignment of the shafts being coupled causes the periodically variable displacements $\Delta A$ (Figure 2) of each of the clamping bolts in the direction of the axis $z$ of the movable coordinate system $xyz$ associated with the plate. These displacements correspond to the force $P_1$ being periodically varied by a symmetric cycle, which fact causes a bend of the plates in the plane $xz$.

The circumferential force $P_2$ brings about compressing or stretching as well as bending in two planes and also twisting the jumpers between the two adjacent bolts. This force can be considered constant if one does not take into account the torque fluctuations within one shaft revolution.

The displacement $\Delta A$ causes the membranes (plates) to bend in the direction of the shaft axis. The $ABC$ element (Figure 2a) can be considered as a cantilever beam in the length of

$$l = R (1 - \cos \alpha),$$

where $R$ is the radius of the circle of the holes for the bolts; $\alpha = \frac{360^\circ}{n}$ is the angle between the bolts; $n$ is the number of the holes for the bolts in the diaphragm (plate).

The beam is rigidly clamped by the BC side, and the free end (point A) is periodically displaced from the axis of the beam by the magnitude of $\Delta A$ to both directions with respect to the plane of the selected $ABC$ element. The force $P_1$ (Figure 2, d), which corresponds to the displacement $\Delta A$, can be approximately determined from the known dependence:

![Figure 2. The force and strain state of the flexible element bounded by three adjacent bolts: a - view of the plate in the $xy$ plane; b - bending of the jumper between the two bolts; c - longitudinal forces in the jumper between the bolts; d - bending of the triangular element of the plate from the force $P_1$; e - coupling the drive shaft and the driven shaft with the help of two packs of the coupling flexible elements and the spacer.](image)
\[ \Delta_A = \frac{P_1 \epsilon}{3EI} \]  

Wherefrom:

\[ P_1 = \frac{3EI\Delta_A}{\epsilon^3}, \]  

Where \( E \) is a modulus of elongation for the membrane (plate) material; \( I = \frac{bh^2}{6} \) is a moment of inertia for the cross-sectional area; \( 2b = \frac{2h}{\sin \frac{180^\circ}{n}} \) is a total width of the beam section; \( h \) is a bandwidth (jumper between the two adjacent bolts) taken as a constant for the safety margin, as in this case, it does not make sense to complicate the decision if the width \( h \) would be accepted variable; \( \delta \) is the thickness of the plate.

The circumferential force acting on one bolt can be determined as follows:

\[ P_2 = \frac{M_{xR}}{Rn}, \]  

Where \( M_{xR} \) (torque) is the torque transmitted by the coupling.

The force \( P_2 \) is defined for one (1) half coupling bolt assuming that the force is evenly distributed between the bolts. The problem is complicated by the fact that this force is applied eccentrically (\( \Delta_A \)) with respect to the axis of a rod-jumper. It compresses the BC rod, while the longitudinal force is equal to the \( R_B \) reaction, and stretches the rod of the second AC half-coupling by a longitudinal force equal to the \( R_C \) reaction. Since the triangle ABC is symmetric, then these reactions are modularly equal (Figure 2, c):

\[ R_B = R_C = 0.5 \frac{P_2}{\cos \frac{180^\circ}{n}}. \]  

In addition, due to the eccentricity \( \Delta_A \) of the force \( P_1 \), there appears bending in both planes (\( xy \) and \( xz \)), as well as twisting by the torque (Figure 3), which is equal to:

\[ M = P_2 \Delta_A = P_2 \Delta_A \sin \frac{180^\circ}{n}. \]  

\[ \text{Figure 3. Deformed AB jumper.} \]

In addition to all the strains, a certain danger is caused by the longitudinal compression of the AB rod by the force of \( N = P_1 \cos \frac{180^\circ}{n} \), which leads to the loss of stability of this rod in the \( xz \) plane,
wherein the AB jumper has minimal rigidity. As a calculation scheme, it could be taken a rod rigidly fixed on one side (support B) and pivotally supported at point A. The pack of the flexible elements (plates) has sufficient rigidity and, therefore, does not lose stability. But each plate, although being limited with the neighboring ones, tends to deflect due to the compression force, and, therefore, it presses on or attempts to move away from the adjacent plate.

Here we find the minimum value of the critical force at which the AB rod loses its stability due to longitudinal bending in the plane of the least rigidity of $xz$. This force will be later used in determining the Euler force.

\[
P_{kp} = \frac{\pi^2 EI_{\text{min}}}{(v^2)} = \frac{3.14^2 \cdot 2 \cdot 10^3 \cdot 0.0147}{(0.5 \cdot 83.5)^2} = 16.65 \text{ N}
\]

(7)

Where $E=2.1 \cdot 10^5$ MPa is for the corrosion resistant steel plates; $I_{\text{min}} = \frac{h^3}{12} = \frac{22 \cdot 0.2^3}{12} = 0.0147 \text{ mm}^4$ is the moment of inertia of the jumper cross section; $v=0.5$ is the coefficient of the rod length adjustment in the case of the accepted conditions for its fixation; $l = 83.5$ mm is the length of the AB jumper (as shown in Figure 3, it is $L$).

It should be noted the membrane (plate) strength for the jumper compression by the yield strength:

\[
P_1 = \sigma_1 F = \sigma_1 \delta h = 800 \cdot 0.2 \cdot 22 = 3520 \text{ N},
\]

(8)

Where $\sigma_1 = 800$ MPa is the yield strength for the plate material.

A real compression force for a single plate. If the 6-pin coupling contains $i = 50$ plates and transmits the torque of $M_{KR}(M_{eq}) = 400$ N·m, the real compression force for a single plate will be:

\[
P_{2x} = \frac{P_1 \cos 180^\circ}{2in} \cdot \frac{180^\circ}{n} = \frac{M_{KP} \cos 180^\circ}{2inR} \cdot \frac{180^\circ}{n} = \frac{400 \cdot 10^3}{2 \cdot 50 \cdot 6 \cdot 83.5} = 0.866 = 6.9 \text{ N}
\]

(9)

Obviously, the strength of the plates is far from being exhausted, while the plates lose their strength when affected by a load much less than the load transferred. The loss of stability is almost instantaneous, and it is accompanied by bulging of the compressed plate. The conclusion arises that the main reason for the pressure of the plates on each other is the loss of their stability. If they did not work in a bundled pack, then they would not be able to work at all.

Analysis of the influence of different power factors on the amount of contact loads between the plates. Let us analyze the influence of the other power factors in the light of the issue under study. First of all, it is necessary to pay attention to the fact that the displacement of $\Delta A$ is infinitesimal as compared to the absolute dimensions of the plates. Therefore, it is possible to neglect the torque of $P_{2y} \Delta A$ and the bending moment in the $xz$ plane due to the $P_{2x}$ force, which is equal to $P_{2x} \Delta A$. The bending moments due to the force of $P_{2y}$ in the $xy$ plane may be significant, but since the moment of resistance of the jumper cross section to bending in this plane, which is $W_i = \frac{h \delta^2}{6}$, significantly exceeds the moment of resistance in the $xz$ plane, which is $W_j = \frac{h \delta^2}{6}$, in this case, the stresses due to the $P_{2y}$ force can be neglected. These stresses are less than the stresses due to the force of the same magnitude, which bends the jumper in the $xy$ plane, i.e. in the case of this example, they are 110 times less.

Periodic bending of the plates according to a symmetric cycle, due to the variable displacement of point A within $\pm \Delta A$ during one revolution of the shaft, is accompanied by the mutual displacements of adjacent plates relative to each other. But the plates are bolted. Each element under consideration is in the form of a triangle ABC clamped with three bolts. According to the theory of elasticity, bolted plates work as a whole. There should be no displacement. Due to the large length of the cantilever $l$, slight displacements can be expected in the middle of the beam or closer to the support B where the curvature of the bent axis arises. Relative displacements of the adjacent plates are also possible in
radial directions in the areas where they have free end surfaces. However, even in the presence of any
displacements, it would seem that in this case one should not expect the appearance of contact
pressures, since, according to the classical theory of bending, it is assumed that the longitudinal fibers
and layers of the beam do not press against each other. However, this is true under pure bending. In
our case, bending is transverse. There are no displacements at the ends of the jumpers due to the
presence of the bolts, and in the middle part of the beam, they may appear due to the presence of
transverse force. This fact can be clarified through prolonged testing on the stand performed up to the
appearance of wear traces.

Thus, to study the plate strain caused by the $P_1$ force, several calculation schemes have been
outlined:
1. Broken rod ABC bending combined with occurring torsion (Figure 2b);
2. Bending of the cantilever fixed triangle ABC (Figure 2, d);
3. Bending of the jumper AB.

The following computational schemes can be considered taking into account the circumferential
force $P_2$ (Figure 3):
1. The jumper AB bending combined with occurring torsion;
2. The jumper AB resistance.

2.2. Bending of plates caused by displacement $\Delta A$

**Bending of the triangle ABC as a bundled pack.** Of the above three calculation scheme options, we
will first study the second option, wherein the triangle ABC is considered as a cantilever beam of
constant cross-section, the area of which is equal to:

$$F = 2\delta b = 2\delta \frac{h}{\sin \frac{180}{n}} \quad (10)$$

This cantilever has its own features. Point A at the end of the cantilever has no free displacement
caused by a load on its length. Point A moves in the plane of rotation of the coupling and spacer, so in
addition to the force $P_1$, there must be a pair of forces that rotates the extreme section by the angle of
$\theta_A$ (Figure 4). When the point A is in the position $A_1$, the beam axis will have a double curvature
with a positive radius of curvature at portion I and a negative radius at portion II with a point of inflection
at the boundary of these portions. In this intersection, the bending moment is zero, which can be an
additional condition in determining the equation of the bent plate.

The basic displacements in this system are determined by bending, so the displacement and
compression of the plates are neglected and only the bending moments are taken into account.

![Figure 4. Plots of force factors for determining the unknown reactions of the supports of the triangle ABC as a coupled pack.](image-url)
The above mentioned beam is statically indeterminate. The two equilibrium equations of static contain 4 unknowns (the degree of static uncertainty is 2):

\[ \Sigma y = -V + P_1 = 0; \]  
\[ \Sigma M_B = P_1 \ell - m_A - m_{BC} = 0. \]

(11)  
(12)

On determining reactions, the force \( P_1 \) can be considered parallel to the \( y \) axis due to the small angle \( \theta_A \), although in reality it is deviated from the vertical by the angle \( \theta_A \). To draw up the additional equations of strain compatibility, we here use the force method [2]. The necessary plots are presented in Figure 4.

Canonical strain compatibility equations:

\[ X_1 \delta_{11} + X_2 \delta_{12} + \Delta_{1P} = \Delta_1; \]  
\[ X_1 \delta_{21} + X_2 \delta_{22} + \Delta_{2P} = \Delta_2; \]

(13)  
(14)

Contain coefficients at unknowns \( X_1 = P_1 \) and \( X_2 = m_A \), which have the forms:

\[ \delta_{11} = \frac{1}{EI} \left( -\frac{1}{2} \ell \ell^2 \right) = \frac{\ell^3}{3EI}; \]
\[ \delta_{22} = \frac{1}{EI} \left( -\frac{1}{2} \ell \ell^2 \right) = \frac{\ell^3}{EI}; \]
\[ \delta_{12} = \delta_{21} = \frac{1}{EI} \left( -\frac{1}{2} \ell \ell^2 \right) = \frac{\ell^3}{2EI}. \]

(15)  
(16)  
(17)

And the absolute terms of these equations, due to the absence of an external load, \( \Delta_{1P} = 0; \Delta_{2P} = 0. \) The external load in this case is present in a hidden form and is expressed through the displacements of \( \Delta_1 = \Delta_A; \Delta_2 = -\theta_A \). The minus sign here means that the direction of the displacement \( \Delta_2 \) is opposite to the direction of the moment \( m_A \). Then the canonical strain compatibility equations take the form:

\[ \frac{1}{3} P_1 \ell^3 - \frac{1}{2} m_A \ell^2 = EI \Delta_A; \]
\[ -\frac{1}{2} P_1 \ell^2 - \frac{1}{2} m_A \ell = -EI \theta_A. \]

(18)  
(19)

As a result, we obtain:

\[ m_A = \frac{2EI}{\ell^2} \left( 3 \Delta_A - 2 \theta_A \ell \right); \]
\[ P_1 = V = \frac{6EI}{\ell^2} \left( 2 \Delta_A - \theta_A \ell \right). \]

(20)  
(21)

And from the basic equation of statics \( \Sigma M_B = 0 \), we obtain the unknown moment for the support BC:

\[ m_{BC} = P_1 \ell - m_A = \frac{2EI}{\ell^2} \left( 3 \Delta_A - 2 \theta_A \ell \right). \]

(22)

Twice integrating the differential equation of the bent beam axis

\[ EI \frac{d^2 w}{dx^2} = m_{BC} - Vx, \]

we obtain the equations of rotation angles and deflections of the sections:
\[ EI\theta_{(x)} = \frac{m_{bc}x^2}{2} \cdot \frac{Vx^2}{2} \];

\[ EIw_{(x)} = \frac{m_{bc}x^2}{2} - \frac{Vx^2}{6} \].

For the section of A \((x = \ell)\), we have an additional system of two equations:

\[ EI\Delta_A = \frac{m_{bc} \ell^2}{2} - \frac{V \ell^3}{6} ; \]

\[ EI\theta_A = m_{bc} \ell - \frac{1}{2} V \ell^2 , \]

on solving which together with the equation of static equilibrium \(\Sigma MB = 0\), we can find the unknown reactions:

\[ V = P_1 = \frac{6EI}{\ell^3} \left(2\Delta_A - \theta_A \ell \right) ; \]

\[ m_A = \frac{2EI}{\ell^2} \left(3\Delta_A - 2\theta_A \ell \right) ; \]

\[ m_{bc} = \frac{2EI}{\ell^2} \left(3\Delta_A - \theta_A \ell \right) . \]

Consider a specific example. Let us assume that for the coupling (Figure 2): \(n = 6\); \(E = 2\cdot10^5\) MPa; \(\Delta_A = 0.5\) mm; number of plates \(i = 50\); \(\delta = 0.2\) mm. Taking the spacer length \(\ell_s = 400\) mm (Figure 2, e), we can find:

\[ \theta_A \approx \sin\theta_A = \Delta_A / \ell_s = 0.5/400 = 0.00125 \text{ rad} . \]

Then we calculate

\[ m_A = \frac{2 \cdot 2 \cdot 10^5 \cdot 7333}{41,75^3} \left(3 \cdot 0.5 - 2 \cdot 0.00125 \cdot 41,75 \right) = 2,349 \cdot 10^6 \text{ N} \cdot \text{mm} ; \]

\[ m_{bc} = \frac{2 \cdot 2 \cdot 10^5 \cdot 7333}{41,75^3} \left(3 \cdot 0.5 - 0.00125 \cdot 41,75 \right) = 2,436 \cdot 10^6 \text{ N} \cdot \text{mm} ; \]

\[ V = P_1 = \frac{6 \cdot 2 \cdot 10^5 \cdot 7333}{41,75^3} \left(2 \cdot 0.5 - 0.00125 \cdot 41,75 \right) = 0.1146 \cdot 10^6 \text{ N} . \]

Apparently, the forces obtained are very large, while the assembled pack of 50 plates is easily strained by a slight manual push. Therefore, the correct choice of the calculation scheme will be a pack of the plates which can be displaced relative to each other along the entire length, except the points where they are compressed by bolts. The latter can be taken into account, assuming real boundary conditions: zero displacements and rotation angles for sections B and C and previously accepted displacements of section A. Then the force per one plate of an unbundled pack of 50 plates will be 50 times less than in the case of the bundled pack.

The abscissa of the inflection point of the bent axis can be found from the condition:

\[ M_{x=x_i} = m_{bc} - V \ell_1 = 0 , \]

wherefrom:

\[ \ell_1 = \frac{1}{3} \left( \frac{3\Delta_A - \theta_A \ell}{2\Delta_A - \theta_A \ell} \right) = \frac{1}{3} \left( \frac{3 \cdot 0.5 - 0.00125 \cdot 41,75}{2 \cdot 0.5 - 0.00125 \cdot 41,75} \right) \cdot 41,75 = 20.5 \text{ mm} . \]

Verification confirms the result:
\[ \ell_1 = \frac{m_{ae}}{V} = \frac{2.436 \cdot 10^6}{0.115 \cdot 10^6} = 21.18 \text{ mm}. \] (38)

The error is about 3.2%, which is quite possible in view of some lengthy calculations.

In order to verify the expressions for the reactive forces, we here calculate the known deflection and the rotation angle of the intersection A:

\[ \theta_\alpha = \frac{1}{EI} \left( m_{ae} \ell - \frac{V f^2}{2} \right) = \frac{1}{2 \cdot 10^3 \cdot 7333} \times \]

\[ \times \left( 2.436 \cdot 10^6 \cdot 41.75 - \frac{0.1146 \cdot 10^6 \cdot 41.75^3}{2} \right) = 0.001249 \text{ rad} \approx 0.00125 \text{ rad}; \] (39)

\[ \Delta_\alpha = \frac{1}{EI} \left( m_{ae} \ell^2 - \frac{V f^3}{6} \right) = \frac{1}{2 \cdot 10^3 \cdot 7333} \times \]

\[ \times \left( 2.436 \cdot 10^6 \cdot 41.75 - \frac{0.1146 \cdot 10^6 \cdot 41.75^3}{6} \right) = 0.4998 \text{ mm} \approx 0.5 \text{ mm}. \] (40)

**Bending the jumper between the bolts.** The above said beam cannot be used to determine the contact forces between the plates, because its axis passes through void, and the sections, being perpendicular to the axis of the beam, are inclined at the angle of 180°/n to the longitudinal axis of the jumper between the bolts. There is need in considering the stress-strain state precisely of the jumper itself. The calculation scheme will be the same as in the case considered above. The difference is only in the angle \( \theta_\alpha \). For the new calculation scheme (Figure 5), it is equal to the projection of the angle \( \theta_\alpha \) on the plane perpendicular to the axis of the jumper \( \theta_\alpha = 0.00125 / \cos 60° = 0.0025 \text{ rad} \). Here we take the vertical displacement of point A equal to \( \Delta_\alpha = 0.5 \text{ mm} \). The rod length (jumper) \( \ell = 41.75 / \sin 30° = 83.5 \text{ mm} \); cross-sectional area \( F = \delta h = 0.2 \cdot 22 = 4.4 \text{ mm}^2 \); the moment of inertia and the moment of resistance of the cross section of the unbundled pack relative to the neutral axis of the entire cross section are:

\[ I = \frac{22}{12} \left( \frac{50 - 0.2}{50} \right)^3 = 0.0147 \text{ mm}^4; \] (41)

\[ W = \frac{22}{6} \left( \frac{50 - 0.2}{50} \right)^2 = 0.147 \text{ mm}^3. \] (42)

The same values can be obtained for a single plate width \( h = 22 \text{ mm} \) and thickness \( \delta = 0.2 \text{ mm} \):

\[ I = \frac{h \delta^3}{12} = \frac{22 \cdot 0.2^3}{12} = 0.0147 \text{ mm}^4; \] (43)

\[ W = \frac{h \delta^2}{6} = \frac{22 \cdot 0.2^2}{6} = 0.147 \text{ mm}^3. \] (44)

As is well known, the projection of a segment or a flat figure onto a plane is equal to the length of the segment or the area of the figure times the cosine of the angle between the projected element and the plane. When determining the projection of the plane angle onto the plane, one must divide the angle for which the projection is determined by the cosine of the angle between the planes. This can be seen from Figure 5, where angle \( \alpha \) located in plane 1 is projected onto plane 2 rotated relative to plane 1 by angle \( \beta \). Then \( \tan \alpha = A_1 B_1 / O B_1 = AB / (O B \cos \beta) \). For small values of \( \alpha \) and \( \alpha 1, \alpha 1 = \alpha / \cos \beta \) is quite accurate.
Determination of reactive forces. In this case, the calculation scheme remains the same as when calculating the triangle ABC (Figure 6). Its differences are only in the values of the applied forces. Therefore, to find the reactions, one can use the formulas already known for P1, mA, and mB. One can calculate the reactive forces, taking into account that the force P1 bends two jumpers:

\[
V = P_1 = \frac{6 \cdot 2 \cdot 10^5 \cdot 0.0147}{83.5^3} (2 \cdot 0.5 - 2 \cdot 0.0025 \cdot 83.5) = 0.024 \text{ N};
\]

(45)

\[
m_A = \frac{2 \cdot 2 \cdot 10^5 \cdot 0.0147}{83.5^2} (3 \cdot 0.5 - 2 \cdot 0.0025 \cdot 83.5) = 0.932 \text{ N} \cdot \text{mm};
\]

(46)

\[
m_B = \frac{2 \cdot 2 \cdot 10^5 \cdot 0.0147}{83.5^2} (3 \cdot 0.5 - 0.0025 \cdot 83.5) = 1.089 \text{ N} \cdot \text{mm}.
\]

(47)

Figure 5. To the definition of the projection of an angle onto a plane.

Figure 6. To the calculation of an unbundled pack of plates between two adjacent bolts affected by the force P1.

Bending moment and transverse force in an arbitrary cross-section of the jumper:

\[
M_{(z)} = m_B \cdot V_x; \quad 0 \leq x \leq f; \quad Q_{(z)} = -V.
\]

(48)
In this case, the abscissa of the deflection point of the beam bent axis will be equal to:

\[ \ell' = \frac{m_b}{V} \times \frac{1.089}{0.024} = 45.375 \text{ mm}. \]  
(49)

The function of angles of rotation and deflections of intersections can be obtained after double integrating of the differential equation of the bent axis \( EI \frac{d^2w}{dx^2} = M(x) \):

\[ EI \theta_{(s)} = m_b x - \frac{V_x^2}{2}; \quad EI w_{(s)} = m_b x^2 - \frac{V_x^3}{6}. \]  
(50)

Figure 6 depicts all four graphs, namely, of the power factor plots and the section displacement plots: angles of rotation and deflections.

The angle of rotation and the deflection of the intersection at the point of inflection of the bent axis are:

\[ \theta_{s(\ell')} = \frac{1}{EI} \left( m_b \ell' - \frac{V_x^2}{2} \right) = \frac{1}{2 \times 10^9 \cdot 0.0147} \times \left( 1.089 \cdot 45.4 - 0.024 \cdot 45.4^2 \right), \]
\[ = 0.0086 \text{ rad}; \]  
(51)

\[ w_{s(\ell')} = \frac{1}{EI} \left( m_b \ell' - \frac{V_x^3}{6} \right) = \frac{1}{2 \times 10^9 \cdot 0.0147} \times \left( 1.089 \cdot 45.4^2 - 0.024 \cdot 45.4^3 \right), \]
\[ = 0.264 \text{ mm}. \]  
(52)

In the unbundled pack, each plate operates independently without being fastened to other plates, except for its extreme bolted locations. We here believe that the force \( P_1 \) is evenly divided between the two packs converging into one bolt, and the force \( P_1 \) is also evenly distributed between the plates of the pack. In this regard, the calculation of the unbundled pack of the plates is adequate to the calculation of a single plate.

It is now possible to proceed to the examination of the strained state of a single plate, and to determine the operation of the external friction forces that accompany the strain of the plates.

2.3. Strained state of an individual plate. Friction and strain work

**Strained length of the plate.** Under the condition of bending a pack of unbundled plates clamped at the edges, each plate receives a longitudinal strain \( \Delta \ell = \ell d - \ell \) (Figure 7), where the strained length \( \ell d \) is determined from expression [2]:

\[ \ell_{jd} = \int dS = \int \sqrt{dx^2 + dw^2} dx = \sqrt{1 + \left( \frac{dw}{dx} \right)^2} dx \]
(53)

If the friction between the plates is not taken into account, the longitudinal tensile strength of the plate is equal to:

\[ N = EF \frac{\Delta \ell}{\ell} = EF \varepsilon, \]  
(54)

where \( \varepsilon = \Delta \ell/\ell \) – elongation of the plate.

Knowing the longitudinal force \( N \), one can find the specific pressure \( q(x) \):

\[ 2N \sin \frac{d\theta}{2} = q(x) hdS, \]  
(55)
Figure 7. On determining the length of a bent plate.

wherefrom

$$q(x) = \frac{Nd\theta}{hdS}.$$ \hfill (56)

Here, \(\sin \frac{d\theta}{2} = \frac{d\theta}{2}\), since the angle \(\theta\) is very small.

$$EI \theta_{(s)} = EI \frac{dw_{(s)}}{dx} = m_{x} x - \frac{Vx^{2}}{2} = \frac{2EI}{\ell^{2}} (3\Delta_{A} - \theta_{A} \ell) x - \frac{6EI}{\ell^{3}} (2\Delta_{A} - \theta_{A} \ell) x^{2},$$ \hfill (57)

wherefrom

$$\frac{dw_{(s)}}{dx} = -\frac{3}{\ell^{3}} (2\Delta_{A} - \theta_{A} \ell) x^{2} + \frac{2}{\ell^{2}} (3\Delta_{A} - \theta_{A} \ell) x = ax^{2} + bx,$$ \hfill (58)

where \(a = -\frac{3}{\ell^{3}} (2\Delta_{A} - \theta_{A} \ell)\); \(b = \frac{2}{\ell^{2}} (3\Delta_{A} - \theta_{A} \ell)\).

Then the expression for the length \(\ell_{(s)}\) of the strained plate can be written as:

$$\ell_{(s)} = \int \sqrt{a^{2}x^{4} + 2abx^{3} + b^{2}x^{2} + 1} \, dx$$ \hfill (59)

It is an elliptic integral which is reduced to the sum of elementary functions and three so-called normal elliptic integrals.

It is not possible to obtain a solution in general form and, moreover, to use it for the subsequent calculations of the friction forces and the power consumed to overcome them, because of the complexity of the expression for \(\ell_{(s)}\). In this regard, the simplification of the integrand \(\ell_{(s)}\) is necessary. In this example, in accordance with the received input data, the ratio of the coefficients \(a / b = 0.011\), i.e. the coefficient \(a\) is only 1.1% of the value of the coefficient \(b\). If we neglect the coefficient \(a\), then the integral for \(\ell\) (x) is greatly simplified and its solution will be:

$$\ell_{(s)} = \int \sqrt{1 + bx} \, dx = \frac{2}{3b} \sqrt{(1 + bx)^{3}} + C$$ \hfill (60)

Determining the integration constant \(C\) from the condition \((x = 0, \ell_{(s)} = 0)\), we here obtain:

$$\ell_{(s)} = \frac{2}{3b} \sqrt{(1 + bx)^{3}} - 1$$ \hfill (61)

Of course, the ratio between the coefficients \(a\) and \(b\) depends on the initial values \(\Delta_{A}, \theta_{A}\) and \(\ell\). Therefore, the issue of accepting the proposed simplification should be decided on a case-by-case basis. As a rule, in all practical cases it is quite possible.
The full length of the strained plate corresponds to \( x = \ell \). The elongation of the plate at an arbitrary value of \( x \) is \( \ell(x) - x \).

However, we are interested in the relative displacement of the two adjacent plates. Their radii of curvatures differ only by the thickness of one plate, so the displacement of the contacting surfaces of the tangent plates is extremely small. But they are the reason for the wear of the plates.

The complexity of the problem is that in the unbundled pack, each plate works as a separate beam, and the bundled pack works as a whole. In our case, the plates are not clamped over the entire length of the jumper, so they can freely displace relative to one another. The presence of tightening bolts at the ends of the jumper does not make this displacement entirely free. Thus, it can be assumed that each section in the span moves as a whole, with each plate still displacing relative to the neighboring one. In addition, it can be assumed that there is a neutral axis of the pack.

![Neutral axis of the pack](image)

**Figure 8.** Relative displacement of contacting plates.

Fig. 8 shows that the displacement of adjacent plates at an arbitrary point (at an arbitrary value of \( x \)) is equal to:

\[
d(\Delta) = (\rho_i + \delta) \theta_{(x)} - \rho_i \theta_{(x)} - \delta(ax^2 + bx),
\]

(62)

Where \( \rho_i = \rho_0 + y \) and \( \rho_{i+1} = \rho_i + \delta_y \) are the radii of curvatures of the tangent plates; \( \rho_0 \) is the radius of curvature of the neutral layer of the pack; \( y \) is the distance from the neutral axis of the pack to the \( i \)-plate.

As can be seen, the displacements of any adjacent plates of equal thickness are equal to each other and depend only on the abscissa \( x \) and the parameters \( a \) and \( b \), which in turn are determined by the initial data of \( \theta_A \) and \( \Delta_A \).

**Contact pressure between plates.** Determine the contact pressure between adjacent plates, which will obviously vary along the axis of the pack, and also depend on the distance \( y \) between the surface of the test plate and the neutral axis of the intersection of the pack. From the previously obtained ratios (Figure 7), we here obtain the expression for the specific contact pressure at any point of each plate:

\[
q(x) = \frac{N}{S} \cdot d\theta = \frac{N}{S} \cdot \frac{EF \cdot \ell(x) - x}{h \left( \frac{EI}{M(x)} + y \right)},
\]

(63)

Where \( F = \delta h \) and \( I = h \delta^2 / 12 \) are, respectively, the cross-sectional area and the moment of inertia of the section of the individual plate, provided that the bending moment \( M(x) = m_B - V \) is calculated for one plate, where \( m_B = \frac{2EI}{\ell^2} (3\Delta_A - \theta_A \ell) \); \( V = \frac{P}{2\ell} = \frac{6EI}{\ell^3} (2\Delta_A - \theta_A \ell) \).
Elemental work of friction forces at the length of the plate $\ell(x) - x$ will be equal to:

$$dA = f q(x) h \cdot dx \cdot (\ell(x) - x),$$

(64)

where $f$ is a coefficient of friction.

The work of friction forces on one contact surface is

$$A_i = \int_0^{\ell_i} f \cdot q(x) \cdot h \cdot \left(\ell_i - x\right) dx.$$  

(65)

Finally, the whole work of the friction forces for all moving plates of one pack is:

$$A_{fr} = \sum_{i=1}^{0.5n} A_i.$$  

(66)

To evaluate the power spent only on the friction between the plates, it is necessary to take into account that within one revolution of the coupling, all the power parameters also change in time.

**The friction power between the plates.** Based on the accepted version of the work of friction between the plates, it (the friction power between the plates for one revolution of the coupling) will consist of four equivalent terms (components), each of which corresponds to a quarter of the total revolution of the coupling. In this case, we obtain the same results for two pairs of plates equidistant from the neutral axis of the pack on both sides of the coupling. In addition, it should be noted that the parameters $\Delta \Lambda$ and $\theta_A$ also vary within each quarter of the coupling revolution from zero to the maximum value, as well as the fact that all $n$ coupling jumpers are exposed to this effect during the full revolution period. With this in mind, the work of friction forces per revolution is equal to:

$$A_{fr} = 4 n F E I \sum_{i=1}^{0.5n} \left(\ell_i(x) - x\right)^2 \frac{dx}{EI_y + y_i}.$$  

(67)

The friction power between the plates is:

$$N_{fr} = A_{fr} / t.$$  

(68)

Were $t$ is a period of time for one revolution of the coupling.

Here is the bending moment:

$$M_i = m_i \cdot x - \frac{Vx^2}{2} = \frac{2EI}{\ell^2} \left(3\Delta \Lambda - \theta_A\ell\right) x - \frac{6EI}{\ell^3} \left(2\Delta \Lambda - \theta_A\ell\right) x^2 = ax^2 + bx.$$  

(69)

As can be seen, it depends on $\Delta \Lambda$ and $\theta_A$, which vary from zero to maximum every quarter revolution of the coupling. This complicates integration.

It is almost impossible to integrate such a function without necessary simplifications. For example, in the expression for $q_i(x)$, in our case, the minimum radius of curvature of the bent plate is equal to $\rho_{max} = \frac{EI}{M_i} = \frac{2 \cdot 10^7 \cdot 0.0147}{1.089} \approx 2700$ mm, and the largest distance to the neutral axis is $y_{max} = 0.5i\delta = 0.5 \cdot 25 \cdot 0.2 = 2.5$ mm. Obviously, the formula for $q_i(x)$ can be simplified by adopting $y = 0$. This allows replacing the sum of the integrals which differ only by the value of $y$, with the integrals of the same value in the amount of 0.5i:

$$q_i(x) = \frac{FM_i \left(\ell_i - x\right)}{2hx},$$  

(70)

Then
\[ A_{tp} = 4 fnF \int_0^L \left( \ell(x) - x \right)^2 \frac{M(x)}{x} \, dx. \] (71)

However, here the problem is the integration of expressions for \( \ell(x) \) and \( M(x) \), in which the coefficients \( a \) and \( b \) contain the variables \( \Delta_A \) and \( \theta_A \). These values can be replaced by their constant averages. In the systems which rotate uniformly, the following parameters change according to the law of sine: \( \Delta_A = \Delta_A \sin(\omega \cdot t) \) and \( \theta_A = \theta_A \sin(\omega \cdot t) \) (Figure 9), where \( \omega \) is the angular velocity (rotational rate) of the coupling rotation.

\[ \int_0^{2\pi} \Delta_A \sin \varphi \, d\varphi = \frac{\pi}{2} \Delta_A \text{average}, \]

\( \text{Figure 9. To the determination of the average value of } \Delta_A. \)

Since the functions \( \Delta_A \) and \( \theta_A \) are integrable, an average value, such as \( \Delta_A \text{average} \), must determine equal areas, according to Figure 9. This corresponds to: \( \int_0^{2\pi} \Delta_A \sin \varphi \, d\varphi = \frac{\pi}{2} \Delta_A \text{average} \), wherefrom \( \Delta_A \text{average} = 0.633 \Delta_A \).

Thus, the expression for \( q(x) \) and \( A \text{average} \) get the simplified form:

\[ q(x) = \frac{0.633 FM(x)}{2h} \left( \frac{\ell(x)}{M(x)} - x \right), \] (72)

\[ A_{tp} = 2.532 fnF \int_0^L \left( \frac{\ell(x)}{M(x)} - x \right) \frac{M(x)}{x} \, dx. \] (73)

**Plate strain power.** The coupling plates operate on continuous friction modes and receive strains that change over time. Friction forces are external so-called passive forces or resistance forces. To overcome them, the work we have identified above is spent. But also, there is a process of plate strain, which is accompanied by internal friction in the material, electromagnetic phenomena, and sound energy dissipation. If these side effects are neglected, then it can be assumed that the work of external forces spent for strain goes to create the potential strain energy. In our case, there are three internal force factors: the bending moment \( M(x) \), the longitudinal force \( N(x) \), and the transverse force \( Q(x) \). The potential strain energy \( W \) caused by these factors can be equated to the work of \( A \text{strain} \) external forces:

\[ A_{\text{strain}} = W = \int_0^L \frac{M^2(x)}{2EI} \, dx + \int_0^L \frac{N^2(x)}{2EI} \, dx + \int_0^L \frac{Q^2(x)}{2GI} \, dx, \] (74)

where \( G \) is the transverse modulus of elasticity for the material of the plates; \( K \) is the coefficient that depends on the shape of the cross-section of the plate (for rectangle \( K = 1.2 \)).

Practice shows that of the three components of the potential strain energy, the first, being from the bending moment \([3]\), is highly significant. Therefore, to evaluate the work spent on the strain of the plates, we take only the first term in the above expression, and further we use the principle of compiling the energy of all the plates of the pack for determining the work of the friction forces. However, due to the fact that in an unbundled pack each plate is considered as an independent beam
irrespective of the radius of curvature, all plates are on the same conditions. Then the work spent on the strain of all the plates of the coupling per its one revolution will be equal:

\[ A_{strain} = 2EIni \int_0^1 \left( ax^2 + bx \right)^2 dx , \]  

(75)

Here is the bending moment for one plate:

\[ M_{(b)} = \frac{Vx}{2} - \frac{2EI}{\ell^2} \left( 3\Delta_A - \theta_A \ell \right) - \frac{3EI}{\ell^3} \left( 2\Delta_A - \theta_A \right) x = EI(ax + b) . \]  

(76)

If one considers that \( a=0 \) and changes \( \Delta_A \) and \( \theta_A \) to their averages \( \Delta_A=0.633\Delta_A \) and \( \theta_A=0.633\theta_A \), then the final expression for the forces of strain will be:

\[ A_{strain} = \frac{2.532niEI}{\ell^3} \int_0^1 \left( 3\Delta_A - \theta_A \ell \right)^2 dx = \frac{2.532}{\ell^3} niEI \left( 3\Delta_A - \theta_A \ell \right)^2 . \]  

(77)

### 3. Conclusion

The study of the stress-strain state of the flexible elements of the MSC type coupling has been carried out, and the following problems have been solved:

- A comparative assessment of the flexibility of the coupling plate pack in the bundled and unbundled states of the plates has shown that, under real conditions, with a specified design of the coupling, the coupling plates operate as an unbundled pack;
- The method of forces was applied to determine the reaction of the supports for the pack, with each plate and the pack as a whole being considered as a twice statically indeterminate system;
- There were defined boundary conditions in the form of values of strain in the places of tightening the plates by bolts at the ends of the packs;
- It has been solved the problem of determining the plate stresses and strains resulted from the forces caused by the misalignment of the shafts connected by the coupling;
- The expressions were found out for the specific pressure between the plates;
- There has been obtained a formula for determining the work of the friction forces and the forces of elastic strain, which allows estimating the unproductive power and specifying the coefficient of efficiency for plants where the couplings of such types are used.

The overall conclusion is:

The main factors causing the wear of the plates due to friction are:

- bending forces induced by the inaccuracy of the shafts connected with the help of the coupling, and also
- longitudinal and transverse bounding resulted from the circumferential force.

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