INCREASE OF THE HAWKING RADIATION FOR SPINOR PARTICLES FROM SCHWARZSCHILD BLACK HOLES BY DIRAC MONOPOLES

YU. P. GONCHAROV
Theoretical Group, Experimental Physics Department, State Technical University Sankt-Petersburg 195251, Russia

N. E. FIRSOVA
Institute for Mechanical Engineering, Russian Academy of Sciences Sankt-Petersburg 199178, Russia

Received 13 November 2001

An algorithm for numerical computation of the barrier transparency for the potentials surrounding Schwarzschild black holes is described for massless spinor particles. It is then applied to calculate the all configurations (including the contributions of twisted field configurations connected with Dirac monopoles) luminosity for the Hawking radiation from a Schwarzschild black hole. It is found that the contribution due to monopoles can be of order 22 % of the all configurations luminosity.

1. Introductory Remarks

While studying quantum geometry of fields on black holes the nontrivial topological properties of the latters may play essential role. The black holes can actually carry the whole spectrum of topologically inequivalent configurations (TICs) for miscellaneous fields, in the first turn, complex scalar and spinor ones. The mentioned TICs can markedly modify the Hawking radiation from black holes. Physically, the existence of TICs should be obliged to the natural presence of magnetic U(N)-monopoles (with $N \geq 1$) on black holes though the total (internal) magnetic charge (abelian or nonabelian) of black hole remains equal to zero. Up to now, however, only influence of the TICs of complex scalar field on Hawking radiation has been studied more or less (see Ref. 1 and references therein for more details). The description of TICs for spinors was obtained in Ref. 2 but the detailed analysis of the TICs contribution to Hawking radiation requires knowledge of the conforming $S$-matrices which regulate the spinor particle passing through the potential barrier surrounding black hole. Those $S$-matrices for the Schwarzschild (SW) black holes have only recently been explored in Ref. 3 which allow us to obtain an algorithm to calculate the $S$-matrix elements numerically since for physical results to be obtained one needs to apply the numerical methods. In the massless case the $S$-matrices discussed are simpler to treat and the present paper will contain a description of the algorithm and will apply it to calculate the all configurations luminosity for massless spinor particles.
for a SW black hole. The results obtained can serve as an estimate, in the first turn, of the electron-positron Hawking radiation and also for the neutrino one from the SW black holes. The more exact computation should take into account the particle masses and will require a more complicated algorithm for calculating the corresponding $S$-matrices. The case of complex scalar field shows, however, that the massless limit can be a good approximation to the more realistic massive case, in particular, when the particles masses are small enough. As a result, the task under consideration here makes sense.

We write down the black hole metric under discussion (using the ordinary set of local coordinates $t, r, \theta, \phi$) in the form

$$ds^2 = adt^2 - a^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

with $a = 1 - 2M/r$ and $M$ is the black hole mass.

Throughout the paper we employ the system of units with $\hbar = c = G = 1$, unless explicitly stated otherwise. Finally, we shall denote $L_2^2(F)$ the set of the modulo square integrable complex functions on any manifold $F$ furnished with an integration measure while $L_2^n(F)$ will be the $n$-fold direct product of $L_2^2(F)$ endowed with the obvious scalar product.

2. Description of algorithm

As was discussed in Ref. 2, TICs of a spinor field on black holes are conditioned by the availability of a countable number of the twisted spinor bundles over the $\mathbb{R}^2 \times \mathbb{S}^2$-topology underlying the 4D black hole physics. From a physical point of view the appearance of spinor twisted configurations is linked with the natural presence of Dirac monopoles that play the role of connections in the complex line bundles corresponding to the twisted spinor bundles. Under the circumstances each TIC corresponds to sections of the corresponding spinor bundle $E$, which can be characterized by its Chern number $n \in \mathbb{Z}$ (the set of integers). Using the fact that all the mentioned bundles can be trivialized over the chart of local coordinates $(t, r, \theta, \phi)$ covering almost the whole manifold $\mathbb{R}^2 \times \mathbb{S}^2$ one can obtain a suitable Dirac equation on the given chart for TIC $\Psi$ with mass $\mu_0$ and Chern number $n \in \mathbb{Z}$ that looks as follows

$$\mathcal{D}_n \Psi = \mu_0 \Psi,$$

with the twisted Dirac operator $\mathcal{D}_n = i\gamma^\mu \nabla_\mu$ and we can call (standard) spinors corresponding to $n = 0$ untwisted while the rest of the spinors with $n \neq 0$ should be referred to as twisted. Referring for details and for explicit form of $\mathcal{D}_n$ to Ref. 2, it should be noted here that in $L_2^2(\mathbb{R}^2 \times \mathbb{S}^2)$ there is a basis from the solutions of (2) in the form

$$\Psi_{\lambda m} = \frac{1}{\sqrt{2\pi \omega}} e^{i\omega t} r^{-1} \begin{pmatrix} F_1(r, \omega, \lambda) \Phi_{\lambda m} \\ F_2(r, \omega, \lambda) \sigma_1 \Phi_{\lambda m} \end{pmatrix},$$

where $\sigma_1$ is the Pauli matrix, the 2D spinor $\Phi_{\lambda m} = \Phi_{\lambda m}(\theta, \phi) = (\Phi_{1\lambda m}, \Phi_{2\lambda m})$ is the eigenspinor of the twisted euclidean Dirac operator with Chern number $n$ on
the unit sphere with the eigenvalue \( \lambda = \pm \sqrt{(l + 1)^2 - n^2} \) while \(-l \leq m \leq l + 1\), \(l \geq |n|\). As said above, in the given paper we consider \( \mu_0 = 0 \) and then the functions \( F_{1,2} \) obey the system of equations

\[
\begin{align*}
\sqrt{a} \partial_r F_1 + \left( \frac{1}{2} \frac{d}{dr} r \right) F_1 &= -icF_2, \\
\sqrt{a} \partial_r F_2 + \left( \frac{1}{2} \frac{d}{dr} r \right) F_2 &= -icF_1
\end{align*}
\]

with \( c = \omega / \sqrt{a} \) and \( a \) of (1). The explicit form of the 2D spinor \( \Phi_{\lambda m} \) is inessential in the given paper and can be found in Ref. 2. One can only notice here that they can be subject to the normalization condition at \( n \) fixed

\[
\int_0^{2\pi} \int_0^\pi (|\Phi_{1\lambda m}|^2 + |\Phi_{2\lambda m}|^2) \sin \vartheta d\vartheta d\varphi = 1
\]

and these spinors form an orthonormal basis in \( L^2(\mathbb{S}^2) \) at any \( n \in \mathbb{Z} \).

By passing on to the Regge-Wheeler variable \( r^* = r + 2M \ln(r/2M - 1) \) and by going to the quantities \( x = r^*/M, y = r/M, k = \omega M, \) we shall have \( x = y + 2 \ln(0.5y - 1), \) so that \( y(x) \) is given implicitly by the latter relation (i.e., \(-\infty < x < \infty, 2 \leq y < \infty\)) with

\[
y' = dy/dx = 1 - 2/y = (y - 2)/y = a_0(x)
\]

and the system (4) can be rewritten as follows

\[
\begin{align*}
E_1' + a_1 E_1 &= b_1 E_2, \\
E_2' + a_2 E_2 &= b_2 E_1
\end{align*}
\]

with \( E_1 = E_1(x, k, \lambda) = F_+(Mx), F_+ (r^*) = F_1[r(r^*)] \), \( E_2 = E_2(x, k, \lambda) = iF_-(Mx), F_- (r^*) = F_2[r(r^*)] \) and

\[
a_{1,2} = \frac{1}{2y^2} \pm \frac{\lambda}{y} \sqrt{a_0},
\]

\[
b_{1,2} = \mp k.
\]

To evaluate luminosity of the Hawking radiation for spinor particles it is necessary to know the asymptotics of the functions \( E_{1,2} \) at \( x \to +\infty \). As was shown in Ref. 3, the latter asymptotics look as follows

\[
E_1 \sim \sqrt{-k}s_{11}(k, \lambda)e^{ikx}, \quad x \to +\infty,
\]

\[
E_2 \sim \frac{i ks_{11}(k, \lambda)}{\sqrt{-k}}e^{ikx}, \quad x \to +\infty,
\]

where \( s_{11}(k, \lambda) \) is an element of the \( S \)-matrix connected with some scattering problem for the Schrödinger-like equation

\[
u'' + k^2 u = qu
\]
with potential
\[ q(x, \lambda) = \frac{\lambda^2}{y^2(x)} \sqrt{a_0 + \frac{y(x) - 3}{\lambda y(x)}}. \]

(12)

In its turn, the correct statement of the mentioned scattering problem for Eq. (11) consists in searching for two solutions \( u^+(x, k, \lambda), u^-(x, k, \lambda) \) of the equation (11) obeying the following conditions
\[
\begin{align*}
  u^+(x, k, \lambda) &= \begin{cases} 
  e^{ikx} + s_{12}(k, \lambda)e^{-ikx} + o(1), & x \to -\infty, \\
  s_{11}(k, \lambda)e^{ikx} + o(1), & x \to +\infty,
  \end{cases} \\
  u^-(x, k, \lambda) &= \begin{cases} 
  s_{22}(k, \lambda)e^{-ikx} + o(1), & x \to -\infty, \\
  e^{-ikx} + s_{21}(k, \lambda)e^{ikx} + o(1), & x \to +\infty.
  \end{cases}
\end{align*}
\]

(13)

It is seen that there arises some \( S \)-matrix with elements \( s_{ij}, i, j = 1, 2 \). After this one can obtain the luminosity \( L(n) \) with respect to the Hawking radiation for TIC with the Chern number \( n \) in the form (for more details see Ref. 2 and 3)
\[
L(n) = A \sum_{\pm \lambda} \sum_{l=|n|}^{\infty} 2(l + 1) \int_0^\infty \frac{|s_{11}(k, \lambda)|^2}{e^{8\pi k} + 1} dk = \\
A \sum_{l=|n|}^{\infty} 2(l + 1) \int_0^\infty \frac{|s_{11}(k, \lambda)|^2 + |s_{11}(k, -\lambda)|^2}{e^{8\pi k} + 1} dk,
\]

(14)

where \( A = \frac{c^5}{16\pi G} \left( \frac{\alpha}{\pi} \right)^{1/2} \approx 0.251455 \cdot 10^{55} \text{erg} \cdot \text{s}^{-1} \cdot M^{-1} \) and \( M \) in g. Luminosity \( L(n) \) can be interpreted, as usual, \(^1,^2\) as an additional contribution to the Hawking radiation due to the additional spinor particles leaving the black hole because of the interaction with monopoles and the conforming radiation can be called the monopole Hawking radiation. \(^2\)

Under this situation, for the all configurations luminosity \( L \) of black hole with respect to the Hawking radiation concerning the spinor field to be obtained, one should sum up over all \( n, i.e. \)
\[
L = \sum_{n \in \mathbb{Z}} L(n) = L(0) + 2 \sum_{n=1}^{\infty} L(n)
\]

(15)

since \( L(-n) = L(n) \).

It should be emphasized that in (14), generally speaking, \( s_{11}(k, \lambda) \neq s_{11}(k, -\lambda) \) and that is clear from the form of potential \( q \) of (12) which satisfies the condition\(^3\)
\[
\int_{-\infty}^{+\infty} |q(x, \lambda)| dx < \infty.
\]

As an example, in Figs. 1, 2 the numerically computed potentials \( q(x, \lambda) \) with \( \lambda > 0 \) and \( \lambda < 0 \) are shown.
Fig. 1. Typical potential barrier for $\lambda > 0$.

Fig. 2. Typical potential barrier for $\lambda < 0$. 
One can notice that when $\lambda < 0$, $q(x, \lambda) > 0$ at any $x \in [-\infty, \infty]$ while at $\lambda > 0$, $q(x, \lambda)$ can change the sign.

Obviously, for neutrino there exists only $L(0)$ since neutrino does not interact with monopoles (when neglecting a possible insignificantly small magnetic moment of neutrino). Also it is evident that for numerical computation of all the above luminosities one needs to have some algorithm for calculating $s_{11}(k, \lambda)$. This can be extracted from the results of Ref. 3. Namely

$$s_{11}(k, \lambda) = 2ik/[f^-(x, k, \lambda), f^+(x, k, \lambda)] ,$$

where $[,]$ signifies the Wronskian of functions $f^-, f^+$, the so-called Jost type solutions of Eq. (11). In their turn, these functions and their derivatives obey the certain integral equations. Since the Wronskian does not depend on $x$ one can take the following form of the mentioned integral equations

$$f^-(x_0, k, \lambda) = e^{-ikx_0} + \frac{1}{k} \int_{-\infty}^{x_0} \sin[k(x_0 - t)]q(t, \lambda)f^- (t, k, \lambda)dt ,$$

$$f^-(x_0, k, \lambda) = -ike^{-ikx_0} + \int_{-\infty}^{x_0} \cos[k(x_0 - t)]q(t, \lambda)f^- (t, k, \lambda)dt ,$$

$$f^+(x_0, k, \lambda) = e^{ikx_0} + \frac{1}{k} \int_{x_0}^{+\infty} \sin[k(x_0 - t)]q(t, \lambda)f^+(t, k, \lambda)dt ,$$

$$f^+(x_0, k, \lambda) = ike^{ikx_0} + \int_{x_0}^{+\infty} \cos[k(x_0 - t)]q(t, \lambda)f^+(t, k, \lambda)dt .$$

The point $x_0$ should be chosen from the considerations of the computational convenience. The relations (17)–(20) can be employed for numerical calculation of $s_{11}(k, \lambda)$.

Finally, we should touch upon the convergence of the series (15) over $n$. For this aim we denote

$$c_l(\pm \lambda) = \int_0^{\infty} \frac{|s_{11}(k, \pm \lambda)|^2}{e^{8\pi \pm k} + 1}dk ,$$

and we represent the coefficients $c_l(\pm \lambda)$ in the form (omitting the integrand)

$$c_l(\pm \lambda) = c_{l1}(\pm \lambda) + c_{l2}(\pm \lambda) = \int_0^{(\ln 1)/2\pi} + \int_{(\ln 1)/2\pi}^{\infty} ,$$

so that $L(n)$ of (14) is equal to $L_1(n) + L_2(n)$ respectively. Also it should be noted, as follows from the results of Ref. 3, we have for barrier transparency $\Gamma(k, \lambda) = |s_{11}(k, \lambda)|^2$ that $\Gamma(0, \lambda) = 0$ and at $k \to +\infty$

$$|s_{11}(k, \lambda)|^2 = \Gamma(k, \lambda) = 1 + O(k^{-1}) .$$
Further we have the asymptotic behaviour for large \( l \) and \( k \ll l \)

\[
|s_{11}(k, l, n)| = C \frac{e^{-\sqrt{n + l + 1}}}{\sqrt{n + l + 1}} k[1 + o(1)]
\]  

(23)

with some constant \( C \). To obtain it we should work using the equations (17)–(20) by conventional methods of mathematical analysis so that the detailed derivation of (23) lies somewhat out of the scope of the present paper and will be considered elsewhere. Using (21) and (23) we find

\[
c_{11}(\pm \lambda) \leq B \frac{e^{-2\sqrt{n + l + 1}}}{n + l + 1} \int_0^{(\ln l)/2\pi} \frac{k^3}{e^{8\pi k} - 1} \, dk
\]

\[
\leq B \frac{e^{-2\sqrt{n + l + 1}}}{n + l + 1} \int_0^{\infty} \frac{k^3}{e^{8\pi k} - 1} \, dk = B \frac{e^{-2\sqrt{n + l + 1}}}{n + l + 1} \frac{3\zeta(4)}{(8\pi)^3}
\]

with some constant \( B \), where we used the formula (for natural \( p > 0 \))

\[
\int_0^\infty \frac{t^p \, dt}{e^t - 1} = p! \zeta(p + 1)
\]

with the Riemann zeta function \( \zeta(s) \), while \( \zeta(4) = \pi^4/90 \). As a consequence, one may consider

\[
L_1(n) \sim \int_n^\infty \frac{2(l + 1)}{n + l + 1} e^{-2\sqrt{n + l + 1}} \, dl \sim 2 \int_0^\infty e^{-2t} \left( t - \frac{n}{l} \right) \, dt
\]

\[
\sim e^{-2\sqrt{n + l + 1}} \left( \sqrt{2n + 1} - \frac{1}{\sqrt{2n + 1}} \right),
\]

(24)

where we employed the asymptotical behaviour of the incomplete gamma function4

\[
\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} \, dt \sim x^{\alpha-1} e^{-x}, \, x \to +\infty,
\]

so that the series \( \sum_{n=1}^\infty L_1(n) \) is evidently convergent.

At the same time due to (22)

\[
c_{12}(\pm \lambda) \leq D \int_{(\ln l)/2\pi}^\infty k e^{-8\pi k} \, dk \sim D \frac{\ln l}{l^4}
\]

(25)

with some constant \( D \). This entails

\[
L_2(n) \sim \int_n^\infty \frac{2(l + 1) \ln l}{l^4} \, dl \sim \frac{\ln n}{n^2}.
\]

(26)
and the series $\sum_1^\infty L_2(n)$ is also convergent. Under this situation we obtain that the series of (15) is convergent, i.e., $L < +\infty$. Consequently, we can say that all the further computed luminosities are well defined and exist.

3. Numerical results

In view of (22), we have

$$\int_0^\infty \frac{\Gamma(k, \lambda) k dk}{e^{8\pi k} + 1} \sim \int_0^\infty \frac{k dk}{e^{8\pi k} - 1} = \frac{1}{(8\pi)^2} \zeta(2) = \frac{1}{6} \cdot \frac{82}{2},$$

(27)

whilst the latter integral can be accurately evaluated using the trapezium formula on the $[0, 3]$ interval. Having confined the range of $k$ to the $[0, 3]$ interval, accordingly, it is actually enough to restrict oneself to $0 \leq l, n \leq 15-20$ when computing $L(n), L$. We took $0 \leq l \leq 20, 0 \leq n \leq 15$. Since the Wronskian of (16) does not depend on $x$, the latter should be chosen in the region where the potentials $q(x, \pm \lambda)$ are already small enough. Besides, potentials $q$ are really equal to 0 when $x < -30$. We computed $\Gamma(k, \lambda)$ according to (16) at $x = x_0 = 300$, where $f^\pm$ were obtained from the Volterra integral equations, respectively, (17) and (19). For this aim, according to the methods developed for numerical solutions of integral equations (see e.g., Ref. 5), those of (17) and (19) have been replaced by systems of linear algebraic equations which can be gained when calculating the conforming integrals by the trapezium formula, respectively, for the $[-30, 300]$ and $[300, 400]$ intervals. The sought values of $f^\pm$ were obtained as the solutions to the above linear systems. After this the derivatives $(f^\pm)^\prime_x$ were evaluated in accordance with (18) and (20) while employing the values obtained for $f^\pm$. The typical behaviour of $\Gamma(k, \lambda)$ is presented in Fig. 3.

Fig. 3. Typical behaviour of barrier transparencies.

Finally, Fig. 4 represents the untwisted $L(0)/A$ and the all configurations $L/A$ luminosities with $A$ of (14) as functions of $k$. The areas under the curves give the
corresponding values of $L(0)/A$ and $L/A$

\[
\begin{align*}
L(0)/A &= 0.514490 \cdot 10^{-3}, \\
L/A &= 0.658571 \cdot 10^{-3}, \\
L(0)/L &= 0.781222,
\end{align*}
\]

so that contribution owing to Dirac monopoles amounts to $21.8778\% \sim 22\%$ of $L$. For inquiry, the similar contribution for scalar particles was of order $(17–18)\%$.

4. Concluding remarks

As was mentioned above, the results obtained can serve as an estimate, in the first turn, of the electron-positron Hawking radiation and also of that for neutrino from the SW black holes. Clearly, for neutrino there exists only $L(0)$ since neutrino does not interact with monopoles (when neglecting a possible insignificantly small magnetic moment of neutrino). The corresponding computation for other fundamental fermions ($\mu^\pm$-mesons and $\tau^\pm$-leptons) and also the more exact computation for $e^\pm$-particles should take into account the particle masses and will require a more complicated algorithm for calculating the corresponding $S$-matrices. We hope to get the numerical results in the massive case elsewhere.

Acknowledgements

The work of Goncharov was supported in part by the Russian Foundation for Basic Research (grant No. 01-02-17157).
References

1. Yu. P. Goncharov and N. E. Firsova, *Int. J. Mod. Phys.* **D5**, 419 (1996); *Nucl. Phys. B486*, 371 (1997); *Phys. Lett. B478*, 439 (2000).
2. Yu. P. Goncharov, *Pis’ma v ZhETF* **69**, 619 (1999); *Phys. Lett. B458*, 29 (1999).
3. N. E. Firsova, *Mod. Phys. Lett.* **A16**, 1573 (2001).
4. Eds. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D. C., 1964).
5. L. I. Turchak, *Foundations of Numerical Methods* (Nauka, 1987).
\[ q(x, \lambda) \]

\[ n = 0 \ (\lambda > 0) \]

Fig. 1
$q(x, \lambda)$

$n = 3 \ (\lambda < 0)$

Fig. 2
\[ \Gamma(k, \lambda) \]

Fig. 3

\[ n = 2 \ (\lambda < 0) \]
Fig. 4