Signatures of Prelocalized States in Classically Chaotic Systems

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We investigate the statistics of eigenfunction intensities $P(\mid\psi\mid^2)$ in dynamical systems with classical chaotic diffusion. Our results contradict some recent theoretical considerations which challenge the applicability of field theoretical predictions, derived in a different framework for diffusive disordered samples. For two-dimensional systems, the tails of $P(\mid\psi\mid^2)$ contradict the results of the optimal fluctuation method, but agree very well with the predictions of the non-linear $\sigma$-model.

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The statistical properties of wavefunction intensities have sparked a great deal of research activity in recent years. These studies are not only relevant for mesoscopic physics, but also for understanding phenomena in areas of physics, ranging from nuclear and atomic microwave physics and optics. Experimentally, using microwave cavity techniques it is possible to probe the microscopic structure of electromagnetic wave amplitudes in chaotic or disordered cavities. Recently, the interest in this problem was renewed when new effective field theoretic techniques were developed for the study of the distribution of eigenfunction intensities $P(\mid\psi\mid^2)$ of random Hamiltonians. As the disorder increases, these results predict that, the eigenfunctions become increasingly non-uniform, leading to an enhanced probability of finding anomalously large eigenfunction intensities in comparison with the random matrix theory (RMT) prediction. Thus, the notion of prelocalized states has been introduced to explain the appearance of long tails in the distributions of the conductance and other physical observables.

Up to now all theoretical predictions and numerical calculations apply to disordered systems and are based on an ensemble averaging over disorder realizations. Their validity, however, for a quantum dynamical system (with a well defined classical limit) that behaves diffusively is not evident. Furthermore, based on an argument put forward, the far tail of $P(\mid\psi\mid^2)$ is due to rare realizations of the disorder potential, and therefore requires an exponentially large number of eigenfunctions, which can only be accounted by disorder averaging. Here instead we study the statistical properties of eigenfunctions in a dynamical model without introducing any ensemble averaging. Our main conclusion is that in a generic dynamical system with classical diffusion, $P(\mid\psi\mid^2)$ is described quite well by the nonlinear $\sigma$-model (NLSM). We point out here that between the various theoretical works there is a considerable disagreement about the parameters that control the shape of $P(\mid\psi\mid^2)$ and their dependence on time-reversal symmetry (TRS). More specifically, the NLSM suggests that the tail of $P(\mid\psi\mid^2)$ in two dimensions $(2d)$ is sensitive to TRS, while a direct optimal fluctuation (DOF) method predicts a symmetry independent result. Recent numerical calculations on the Anderson model seem to support the latter theory. This controversy, was an additional motivation for the present work.

In the present article, we numerically study the distribution of intensities of the Floquet- states of the kicked rotor (KR) on a torus and its $2d$ generalization. Our system is defined by the time-dependent Hamiltonian

$$H = H_0 + kV \sum_m \delta(t - mT) , H_0(\{\mathcal{L}_i\}) = \sum_{i=1}^d \frac{\tau_i}{2}(\mathcal{L}_i + \gamma_i)^2$$

where $\mathcal{L}_i$ denotes the angular momentum and $\theta_i$ the conjugate angle of one rotor. The kick period is $T$, $k$ is the kicking strength, while $\tau_i$ is a constant inversely proportional to the moment of inertia of the rotor. The standard KR corresponds to $d = 1$ (with $\theta_0 = 0$) whereas for $d = 2$ we have a two-dimensional generalization. The parameter $\alpha$ breaks TRS, the parameters $\gamma_i$ are irrational numbers whose meaning will be explained below. The Hamiltonian describes a system which is kicked periodically in time and is integrable in the absence of the kicking potential. The motion generated by is classically chaotic and for a sufficiently strong kicking strength there is diffusion in momentum space with diffusion coefficient $D = \lim_{t\to\infty} < L^2(t) > / t \sim k^2 / 2T$ (within the random phase approximation)

If the $\mathcal{L}_i$ are taken mod($2\pi m_i / T \tau_i$) where $m_i$ are integers, Eq. defines a dynamical system on a torus. The quantum mechanics of this system is described by a finite-dimensional time evolution operator for one period

$$U = \exp(-iH_0(\{\mathcal{L}_i\})T) \exp(-iV(\{\theta_i\}))$$

where we put $\hbar = 1$. Upon quantization, additional symmetries associated with the discreteness of the momentum show up, which can be destroyed by introducing irrational values for the parameters $\gamma_i$'s. The most striking consequence of quantization is the suppression of classical diffusion in momentum space due to quantum dynamical
tering events may be treated semiclassically. This limit

\[ U_{mn} \Psi_k(n) = e^{i\omega_k T} \Psi_k(n). \]

The quantities \( \omega_k \) are known as quasi-energies, and their
density is \( \rho = T/2\pi \). The corresponding mean quasi-
energy spacing is \( \Delta = 1/(\rho L^d) \), where \( L \) is the linear size
of the system. The Heisenberg time is \( t_H = 2\pi/\Delta \) while \( t_D = L^2/D \) is the diffusion time (Thouless time). Now
one can formally define a dimensionless conductance as
\[ \Gamma = t_H/t_D = D_L L^{-2} \] where \( D_L = TD \) is the diffusion
coefficient measured in the number of kicks. Four length
scales are important here: the wavelength \( \lambda \), the mean
free path \( l_M \), the linear extent of the system \( L \), and the
localization length \( \xi \). According to Refs. [16,17] the field
theoretical predictions are derived under the conditions
\[ \lambda \ll l_M \ll L \ll \xi. \]

The first condition ensures that transport between scatter-
ing events may be treated semiclassically. This limit
 can be achieved for our system \( \psi \) when \( k \to \infty \), \( T \to 0 \)
while the classical parameter \( K = kT \) remains constant.
When \( l_M \ll L \) as long as the motion is not localized
(i.e. \( L \ll \xi \)) it is diffusive, since a particle scatters many
times before it can traverse the system. The resulting
mean free path for our system \( \psi \) is \( l_M \approx \sqrt{D_k} \) while the
localization length \( \xi \) for \( d = 1 \) is \( \xi \approx D_k/2 \) [16] and for
\( d = 2 \) is \( \xi \approx l_M e^{D_k/2} \) [17,18].

Here we calculate the distribution function \( P(t =
L^d|\Psi_k(n)|^2) \) by using a direct diagonalization of the
Floquet operator \( \psi \). The TRS is broken entirely for \( \alpha = 5.749 \).
In order to test the issue of dynamical correlations, we randomize the phases of the kinetic term of the evolution operator \( \psi \) and calculate the result-
ning \( P(t) \). This model will be referred to as Random
Phase KR (RPKR). Since all our eigenfunctions have the
same statistical properties (in contrast to the Anderson
cases where one should pick up only eigenfunctions hav-
ing eigenenergies within a small energy interval [16,18]) we make use of all of them in our statistical analysis.
The classical parameter \( K \) is large enough in all cases
to exclude the existence of any stability islands in phase
space. The classical diffusion coefficient \( D_k \) is calculated
numerically by iterating the classical map obtained from
\( \psi \). Below we present our numerical results and compare
them to the predictions of Refs. [16,17].

1d Kicked Rotor. It was shown in [13], that the ef-
effective field theory describing the semiclassical physics
of the system is precisely the NLSM for quasi-one di-

mensional (1d) metallic wires. Such a mapping however,
requires an averaging over an ensemble of rotors having
the same classical limit. We point out again that in the
calculations below we do not adopt such an averaging
procedure.

![FIG. 1. Corrections to the distribution intensities \( \delta P_\beta(t) \) for the kicked rotator model i.e. Eq. \( \psi \) for \( d = 1 \). The system size is \( L = 1024 \), \( \alpha \beta = 1 \), \( \beta = 2 \). The solid (dashed) lines are the best fit of \( \beta \) to the numerical data: (a) \( D_k \approx 1800 \) and (b) \( D_k \approx 3150 \); (c) Shows the extracted diffusion propagator \( \kappa_\beta \) vs. \( L/D_k \). The NLSM for quasi-1d systems can be solved ex-
acting for the distribution function \( P_\beta(t) \), using a transfer
matrix approach [3,4]. In the ballistic regime (where \( g \to \infty \)) RMT is applicable and one finds \( P_{RMT}^{\beta=1}(t) = \exp(-t/2)/\sqrt{2\pi t} \) and \( P_{RMT}^{\beta=2}(t) = \exp(-t/2) \) [20]. Here \( \beta \)
denotes the corresponding Dyson ensemble (\( \beta = 1 \) for
preserved (broken) TRS). As localization increases, the
deviations from the RMT results of the body and the
tails of the distribution \( P_\beta(t) \) become noticeable and can
be parameterized by a single parameter which is the di-

mensionless conductance \( g = D_k/L \).

For \( t < \sqrt{D_k/L} \), according to all studies [3,4] \( P(t) \)
is just the RMT result with polynomial corrections in
powers of \( L/D_k \), i.e. \( P_\beta(t) = P_{RMT}^{\beta=1}(t) + \delta P_\beta(t) \).

The leading term of this expansion is given by

\[ \delta P_\beta(t) \simeq \kappa \left\{ \begin{array}{ll}
3/4 - 3t/2 + t^2/4 \quad & \text{for } \beta = 1 \\
1 - 2t + t^2/2 \quad & \text{for } \beta = 2
\end{array} \right\}, \]

where \( \kappa \sim 1/g \) is the 1d diffusion propagator, which is
identical for \( \beta = 1 \) and \( \beta = 2 \) since it is a classical quan-
tity.

In Fig. 1a,b we report our numerical results for \( \delta P_\beta(t) \)
for two representative values of \( D_k \). One can clearly see
that the agreement with the theoretical prediction \( \psi \)
becomes better as \( D_k \) increases. This is due to the fact
that by increasing \( D_k \) we are approaching the semiclas-
sical region and therefore Eqs. (3) are better satisfied.
At the same time higher order corrections in \( \delta P_\beta(t) \) be-
come negligible with respect to the leading term given by
Eq. (5). The resulting \( \kappa_1 \) and \( \kappa_2 \) obtained by the
best fit of our data to Eq. (5) are found to be equal
and in excellent agreement with the theoretical value (see
Fig. 1c). We therefore conclude, that in a generic dynam-

ical system, the only parameter that controls the shape
of the deviations $\delta P_\beta(t)$ is the classical diffusion propagator. Moreover, our results are in excellent agreement with the recent NLSM predictions derived in the framework of diffusive disordered systems. Finally in Fig. 1c we also report the outcome of the RPKR model. The results remain essentially the same indicating that $P_\beta(t)$ for quasi-1d systems are insensitive to dynamical correlations.

$\ln P(t)$ is the 21 for two representative values of $D$. We have also calculated the stretched exponential form

$$A \text{ nice agreement with the results obtained from the real } D \text{ coefficients (6). We have also calculated the stretched exponential form}$$

$$A \text{ results remain essentially the same indicating that}\beta \text{ for } q=1(2).$$

$$C_\beta = \beta \sqrt{D_k/L}$$

where $A_\beta$ is a symmetry dependent constant. Our numerical results agree nicely with Eq. (6). In Fig. 2a we present an example of $P_\beta(t)$. By fitting our data to Eq. (6) the coefficients $C_1, C_2$ can be extracted. In Fig. 2b we report the extracted stretched exponential coefficients $C_\beta$ from the best fit of (6) as a function of the square root of the dimensionless conductance $g = D_k/L$. A nice linear behavior is observed. The best linear fit $C_\beta = A_\beta \sqrt{D_k/L} + B_\beta$ yields $A_{\beta=1} = 0.41 \pm 0.05$ and $A_{\beta=2} = 0.82 \pm 0.05$. The resulting ratio $R = A_2/A_1 = 2$ is in excellent agreement with the theoretical prediction (2).

We have also calculated the stretched exponential coefficients $C_\beta$ for the RPKR model. The results for various $D_k$ values are summarized in Fig. 2b and show a nice agreement with the results obtained from the real Hamiltonian.

2d Kicked Rotor. According to Ref. [3], corrections to the body of $P_\beta^{RMT}$ are still given by Eq. (1), but now $\kappa$ is the 2d diffusion propagator.

Figures 3a,b show corrections to $P_\beta^{RMT}$ for $g = D_k \gg 1$ for two representative values of $D_k$. We find again that the form of the deviations are very well described by Eq. (3) and the agreement becomes better for larger values of the diffusion constant. In Fig. 3c we summarize our results for various $D_k$ values. The extracted $\kappa_\beta$ values are obtained by the best fit of the data to Eq. (4).

Again we find that $\kappa_\beta$ depends linearly on $1/D_k$. However, contrary to the 1d-KR, here $\kappa_1$ and $\kappa_2$ are different. Moreover the best fit with $\kappa_\beta = A_\beta D_k^{1/2}$ yields $A_{\beta=1} = 5.44 \pm 0.03$ and $A_{\beta=2} = 10.84 \pm 0.04$ indicating that the ratio $R = A_2/A_1$ is close to 2, a value that could be explained on the basis of ballistic effects (3). Taking the latter into account leads to an additional term in the classical propagator $\kappa_\beta = \kappa_{diff} + \frac{1}{2}\kappa_{ball}$. The first term is the one discussed previously and is associated with long trajectories which are of diffusive nature while the latter one is associated with short ballistic trajectories which are self-tracing (4). Thus, when $\kappa_{diff} \ll \kappa_{ball}$ we get $R = 2$. The calculation with the RPKR model shows, however, that the corresponding ratio is $R \simeq 1$ in agreement with the theoretical prediction for disordered systems with a pure diffusion. This indicates that dynamical correlations can be important in the 2d case.

**FIG. 2.** (a) Tails of the distribution $P_\beta(t > D_k/L)$ for the model (1) for $d = 1$ with $L = 1024, D_k \simeq 2625$ and for $\beta = 1$ \((\circ)\) and $\beta = 2$ \((\diamondsuit)\). The solid (dashed) lines are the best fit of (6) for $\beta = 1(2)$ to our data; (b) Coefficients $C_\beta$ vs. $\sqrt{D_k/L}$. The solid (dashed) lines are the best fits to $C_\beta = A_\beta \sqrt{D_k/L} + B_\beta$ for $\beta = 1(2)$.

**FIG. 3.** Corrections to the distribution intensities $\delta P_\beta(t)$ for the kicked rotator model (1) for $d = 2$. The system size is $L = 90$, \((\circ)\) $\beta = 1$, \((\diamondsuit)\) $\beta = 2$. The solid (dashed) lines are the best fit of (6) for $\beta = 1(2)$ to the numerical data: (a) $D_k \approx 34$ and (b) $D_k \approx 53$ ; (c) Fit parameters $\kappa_\beta$ vs. $D_k^{-1}$. The solid (dashed) lines are the best fits to $\kappa_\beta = A_\beta D_k^{-1} + B_\beta$ for $\beta = 1(2)$.

For the tails of the distributions, the result of the NLSM within a saddle-point approximation (5) is

$$P_\beta(t) \simeq \exp[-C_\beta^2 (\ln t)^2], \quad C_\beta^2 = \frac{1}{2} \frac{\kappa_{diff}}{\kappa_{ball}}$$

Note that the decay in the tails of Eq. (7) depends on $\beta$, as in the 1d-KR case (see Eq. (1)). Recently, a DOF method was used to calculate the tails of $P_\beta(t)$ (6). It was found that the tails are still given by Eq. (7) but with a log-normal coefficient $C$ which is independent of the parameter $\beta$. 

$\beta$

$C$

$\ln P(t)$

$C_{1.2}$

$C_{2.2}$

$\beta$

$(D_k/L)^{3/2}$

$C_{1}$

$C_{1(RPKR)}$

$\beta$

$\kappa_\beta$

$\kappa_{diff}$

$\kappa_{ball}$

$\kappa_0$

$\kappa_{diff}$

$\kappa_{ball}$

$\kappa_0$

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$\kappa_{ball}$

$\kappa_0$
\[ C_{\text{DOF}} = \pi^2 \rho \frac{D}{\ln(L/\lambda)}. \] 

(8)

![Graph](image)

**FIG. 4.** (a) Tails of the distribution \( P_{\beta=1}(t > D_k) \) for the model [1] for \( d = 2 \) and \( D_k \geq 35 \). The system size is \( L = 80 \), \((\langle \rangle) \beta = 1\), \((\langle \rangle) \beta = 2 \). The solid (dashed) lines are the best fit of [1] for \( \beta = 1(2) \) to the numerical data; (b) Fitted log-normal coefficients \( C_\beta \) versus the classical diffusion coefficient \( D_k \). The solid (dashed) lines are the best fits to \( C_\beta = A_\beta D_k + B_\beta \) for \( \beta = 1(2) \).

Figure 4a shows a representative case of \( P_{\beta=1}(t > D_k) \). The tails show a log-normal behavior predicted by Eq. (8). In Fig. 4b we report the log-normal coefficients \( C_\beta \) extracted from the best fit to our numerical data, versus the classical diffusion coefficient. A pronounced linear behavior is observed in agreement with both theories. However one clearly sees that \( C_1 \) differs from \( C_2 \) in contrast to the DOF prediction [8] and to recent numerical calculations done for the 2d Anderson model [9]. We point out here that in [8] the authors were not able to go to large enough values of conductance \( g \) (in comparison to our study) where the theory can really be tested. In contrast, the NLSM predicts a value of 2 for the ratio \( R = C_\beta^2 / C_1^2 \). We note that \( C_\beta^2 \) is only the leading term in \( D_k \). In order to calculate this ratio, we performed a fit to our data with \( C_\beta = A_\beta D_k + B_\beta \). The resulting ratio was found to be \( R = A_2 / A_1 = 1.97 \pm 0.03 \) in perfect agreement with the NLSM predictions. Finally in Fig. 4b we also present our results for the RPKR model (using the same data as the one in Fig. 3d). Again we found that the ratio \( R = 1.96 \pm 0.03 \approx 2 \). Thus \( P(t > D_k) \) depends on TRS and is described by the NLSM.

In summary, we have performed a detailed numerical analysis of the statistical properties of the wavefunction intensities \( P(t) \) of the standard KR on a torus and its 2d generalization. Based on these results, we concluded that the distribution \( P(t) \) of generic quantum dynamical systems with diffusive classical limit is affected by the existence of prelocalized states. The deviations from RMT are well described by field theoretical methods developed for disordered systems. In particular, in a clarifying way we have resolved the controversy between DOF and NLSM by demonstrating that the dependence of the tails of \( P_\beta(t) \) on TRS is described correctly by the latter theoretical approach.

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