Anomalous Long-Range Fluctuations in Homogeneous Fluid Phase of Active Brownian Particles

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We numerically investigate active Brownian particles with a finite mass for a wide range of activities. When the mass is sufficiently large, the motility-induced phase separation is suppressed even when the activities are large, and the system remains spatially homogeneous. Thus obtained uniform fluids are an ideal system to study the non-equilibrium fluctuations. We find that the density as well as velocity correlation becomes long-ranged. Escort by the growth of the density fluctuations, the giant number fluctuation arises. Our findings suggest the universality and ubiquitousness of long-range fluctuations in the homogeneous active fluids.

Active matters refer to a broad class of many-body systems consisting of self-propelling constituents, such as flocks of birds, herds of animals, bacterial colonies, or even self-propelled colloidal particles [1–3]. In the last decade, we have witnessed tremendous progress in the studies of active matters. Active matter systems exhibit many nontrivial phenomena that are prohibited in equilibrium systems. Representative examples include anomalous increases of the particle number fluctuation known as the giant number fluctuations (GNF) [4–6], spatiotemporal chaotic patterns of velocities fields reminiscent of turbulence [7, 8], and spontaneous separation of constituent particles into dense and dilute phases that is called the motility induced phase separation (MIPS) [9, 10].

The active Brownian particle (ABP) is one of the simplest theoretical models of active matters [10]. ABP is an emblematic model to exhibit MIPS [10], and there are a lot of theoretical [11–19] and numerical studies [10, 20–26]. MIPS resembles the liquid-vapor phase separation in equilibrium systems, and some efforts were made to understand MIPS by mapping ABP and other active fluids into the effective equilibrium system [13, 16, 17, 27–29]. Recently, however, it has been realized that MIPS of ABP is accompanied by intrinsically non-equilibrium phenomena, such as negative surface tension [28, 30], reversal of the Ostwald process [31, 32], and spatial long-range velocity correlation [33]. In particular, the long-range velocity correlation is not only observed inside the MIPS phase but also in the high-density regimes including the crystalline [34, 35], amorphous [36, 37], and even in the dense fluid phases [34, 38, 39]. The long-range velocity correlation is manifested as the vortex-shaped patterns of the velocity field, and it suggests a deep connection with the active turbulence [39]. Furthermore, even the strong enhancement of the particle number fluctuations has been reported in the MIPS phase of ABP [10, 21, 25]. This observation suggests the emergence of GNF even in simple spherical models such as ABP, which is somewhat surprising because, in many studies, GNF is observed in the active fluids with the intrinsic polar or nematic orders [5, 40, 41]. It is, however, difficult to judge whether the observation is due to a bona fide GNF or originated from heterogeneous fluctuations induced by the phase separation.

The goal of this letter is to demonstrate that ABP exhibits all archetypal anomalies of the active matter mentioned above: the turbulence-like vortex patterns, the long-range velocity correlations, and even the giant number fluctuation if MIPS is avoided. For this purpose, we consider the ABP model with an inertia term or a mass. It is known that MIPS is suppressed if the inertia term is added to the overdamped Langevin equation for the original ABP model [42]. If the mass is sufficiently large, the system apparently remains homogeneous without a sign of phase separation even at high activities, and it enables one to investigate intrinsically non-equilibrium fluctuations without being impeded by unwanted heterogeneous patterns caused by MIPS. In this letter, we show that even in the fluid phase at intermediate densities, the velocity fluctuations become long-ranged, and their longitudinal and transverse modes are characterized by distinct two correlation lengths. Surprisingly, the longitudinal velocity fluctuations escort the growth of the density fluctuations, which inevitably results in the emergence of GNF. The emerging correlation lengths were explained by recent linearized theories [38, 43], but we show that the growing fluctuations are caused predominantly by the nonlinear effects. These results suggest that anomalous fluctuations in active matters are more general than previously thought [2].

We consider two-dimensional active Brownian particles with a finite mass, which we shall refer to as the inertial Active Brownian Particles (iABP). The Langevin equation which iABP obeys is written as

$$m \frac{d^2 r_j}{dt^2} = -\zeta \frac{dr_j}{dt} - \nabla_j \sum_{k<l} U(r_{kl}) + \zeta v_0 e(\phi_j),$$

where $r_j$ is the position of the $j$-th particle. $m$ is the mass of a particle, $\zeta$ is the friction coefficient, $U(r_{kl})$ is the pairwise interaction potential between the particles $k$ and $l$. $r_{kl} = |r_k - r_l|$ is the distance between the two particles. The last term of the right-hand side of
Eq. (1) is the active noise. Its strength is characterized by the self-propelling speed \( v_0 \), and the direction is by a unit vector \( \mathbf{e}(\phi_j) = (\cos \phi_j, \sin \phi_j) \). The dynamics of orientation \( \phi_j \) of the particle \( j \) is described by

\[
\frac{d\phi_j(t)}{dt} = \sqrt{\frac{2}{\tau_p}} \eta_j(t),
\]

where \( \eta_j(t) \) is the Gaussian white noise that satisfies \( \langle \eta_j(t) \rangle = 0 \) and \( \langle \eta_j(t) \eta_k(t') \rangle = \delta_{jk} \delta(t - t') \). The symbol \( \langle \cdots \rangle \) denotes the ensemble average. \( \tau_p \) is the persistence time, an essential parameter characterizing how far the system is from equilibrium. In the \( \tau_p \to 0 \) limit, Eq. (1) becomes the equilibrium Langevin equation with the effective temperature \( T_{\text{eff}} = v_0^2 \tau_p \zeta/2 \). Note that Mandal et al. [42] employed the iABP with both the translational thermal noise and the rotational inertial term for \( \phi_j \), which we do not consider here for simplicity.

The simulation setting is as follows. We employ the Weeks-Chandler-Andersen potential as the pairwise potential [44]:

\[
U(r_{kl}) = 4\epsilon \left\{ \left( \frac{\sigma}{r_{kl}} \right)^{12} - \left( \frac{\sigma}{r_{kl}} \right)^{6} + \frac{1}{4} \right\} \theta(2^{1/6} \sigma - r_{kl}),
\]

where, \( \theta(x) \) is the Heaviside step function, and \( \sigma \) is the diameter of a particle. We choose \( \tau_v = \sigma/v_0 \) and \( \sigma \) as the units of time and length, respectively. The number density is fixed at \( \rho = 0.5 \) and the system size is \( L = \sqrt{N/\rho} \). Control parameters in the simulation are the Péclet number defined by \( Pe = \tau_p/\tau_v = \tau_p v_0/\sigma \) and the dimensionless mass defined by \( M = m/(\zeta \tau_v) \). We carry out the Brownian dynamics simulation for iABP with the periodic boundary condition. The number of particles \( N = 1 \times 10^4, 4 \times 10^4 \), and \( 1 \times 10^5 \) are chosen to check the system size effect. Details of the simulation are described in the supplementary material [45].

Figure 1 shows snapshots of particles configurations in \((M, Pe)\) space. At \( M = 1 \), the inertia effect is negligible, and the system undergoes MIPS at \( Pe \gtrsim 50 \) as reported for the overdamped ABP [10, 20, 25]. As \( M \) increases, the MIPS phase boundary line shifts to a larger \( Pe \) continuously, and at the largest \( M \approx 80 \), the system remains in the homogeneous fluid phase even at \( Pe = 300 \). Now that we successfully generated a homogeneous fluid with large Péclet numbers, we explore the properties of non-equilibrium fluctuations of ABP without being intervened by unwanted inhomogeneity induced by MIPS.

Figure 2 presents typical snapshots of the density field \( \rho(\mathbf{r}) \) [(a)-(c)], velocity field \( \mathbf{v}(\mathbf{r}) \) [(d)-(f)], and vorticity field \( \Omega(\mathbf{r}) = \nabla \times \mathbf{v}(\mathbf{r}) \) [(g)-(i)] for \( Pe = 1, 1.50, \) and 200 at \( M = 80 \). First, we focus on the velocity and vorticity field. The colors in Figures 2(d), (e), and (f), represent the angle of vector \( \mathbf{v}(\mathbf{r}) \) with respect to the \( x \)-axis. For \( Pe = 1 \), where the system is close to equilibrium, the velocity pattern is uniform. As \( Pe \) increases, velocity-aligned domains appear and their sizes grow. Concomitantly, the vorticity field \( \Omega(\mathbf{r}) \) develops as shown in Figures 2(g), (h), and (i). The sizes of the patterns, however, are appreciably smaller than those of \( \mathbf{v}(\mathbf{r}) \). To quantify these spatial patterns, we define the longitudinal and transverse velocity correlation function by [38]

\[
\omega_{\parallel}(q) = \frac{1}{N} \langle |\mathbf{J}_{\parallel}(q)|^2 \rangle, \quad \omega_{\perp}(q) = \frac{1}{N} \langle |\mathbf{J}_{\perp}(q)|^2 \rangle,
\]

where \( \mathbf{J}_{\parallel}(q) \) and \( \mathbf{J}_{\perp}(q) \) are the longitudinal and the transverse part of the current defined by \( \mathbf{J}(q) = \sum_j \hat{\mathbf{r}}_j e^{-iq \cdot \mathbf{r}_j} \). \( \omega_{\parallel}(q) \) is a good measure to probe the extent of the alignment of the velocity of particles, whereas \( \omega_{\perp}(q) \) probes the development of the vorticity pattern. Figures 3(a) and (b) show the \( q \)-dependence of \( \omega_{\parallel}(q) \) and \( \omega_{\perp}(q) \) for various \( Pe \) at \( M = 80 \). Both \( \omega_{\parallel}(q) \) and \( \omega_{\perp}(q) \) grow significantly at small wavenumber. This behavior indicates the development of the long-range correlations of both the longitudinal and transverse velocity correlations. We extract correlation lengths by fitting with the Ornstein-Zernike function \( \omega_{\mu}(q) = \omega_\mu(1 + (\zeta_\mu q)^2) \), \( \mu = \parallel, \perp \), for the two correlation functions (see insets of Figures 3(a) and (b)). The fitting range is \( q < 0.05 \) for \( \omega_{\parallel}(q) \) and \( q < 0.18 \) for \( \omega_{\perp}(q) \). Figure 3(c) shows the correlation length as a function of the Péclet number. We find that the two correlation lengths are distinct; the longitudinal length \( \xi_\parallel \) is much longer than the transverse counterpart \( \xi_\perp \). \( \xi_\parallel \) and \( \xi_\perp \) are comparable to the sizes of patterns of the velocity and the vorticity shown in Figures 2(e), (f), (h) and (i). Furthermore, \( \xi_\parallel \) is proportional to \( \sqrt{Pe} \), whereas \( \xi_\perp \) is almost independent of \( Pe \). These observations are qualitatively consistent with the numerical results and the prediction of the continuum theory applied for the fluid phase of ABP by Schamel and Flemer [38] and the re-
FIG. 2. Density (a)-(c), velocity (d)-(f), and vorticity field (g)-(i) for Pe = 1, 50, 200 at M = 80. Each box in the panels (d)-(f) corresponds to the plot range of the panels (g)-(i), respectively. The ticks on each panel are the x and y-coordinates with the length unit $\sigma$. Colors represent the local density in panels (a)-(c), the angle of local velocity with respect to the x-axis in panels (d)-(f), and the local vorticity in panels (g)-(i), respectively. Black arrows in panels (g)-(i) represent the direction of local velocity. Note that the horizontal length of the white scale bar in panels (e), (f), (h), and (i) denotes the longitudinal and transverse correlation length obtained by the velocity correlation functions.

sult of the linearized fluctuating hydrodynamic theory by Marconi et al. [43]. However, the size of fluctuations is larger than the prediction of these linearized theories by several factors, which implies the enhanced fluctuations are of intrinsically nonlinear origin (see also supplementary material [45]). The vortex pattern in Figures 2(h) and (i) and its long-range correlation developed at high Pe are reminiscent of the active turbulence reported in various active matter systems [7, 8, 46–54]. We evaluate the energy spectrum $E(q)$, which is obtained from the velocity correlation function $\omega(q) = \omega_{||}(q) + \omega_{\perp}(q)$, exhibits a weak power-law behavior and the slight deviation of velocity distribution from Gaussian distribution (see supplemental material [45]). However, the power-law exponent of $E(q)$ is small compared with previous studies [7, 8, 46–54]. Seeking a link between the active turbulence and the long-range correlation is out of the scope of the present study and left for future work.

In figure 3(d), we show the density correlation function, or the static structure factor, defined by $S(q) = \langle \delta \rho(q) \delta \rho^*(q) \rangle / N$, where $\delta \rho(q) = \rho(q) - \langle \rho(q) \rangle$ is the fluctuation of the Fourier transformed density field $\rho(q) = \sum_j e^{-iq \cdot r_j}$. $S(q)$ at small wavenumber is almost constant at Pe = 1 but rises significantly as Pe increases, meaning that the density correlation becomes spatially long-ranged. Note that a large increase of $S(q)$ at small
wavenumbers is distinct from that reported for $S(q)$ in the MIPS phase for the conventional ABP where $M = 0$. In the latter case, $S(q)$ is well fitted by $S(q) \propto q^{-(d+1)}$, which is called Porod’s law [55, 56], and it is a natural consequence of the domains created by the phase separation (see also supplementary material [45]). On the contrary, the system in our study is spatially uniform and the increase of $S(q)$ observed in Figure 3(d) is induced by the emerging long-range correlation of the longitudinal velocity field. The observed enhancement of $S(q)$ at small $q$’s is larger than the prediction of the linearized fluctuating hydrodynamics by more than an order of magnitude, which implies that nonlinear fluctuations play an essential role in this phenomenon [45].

Finally, we investigate the particle number fluctuation defined by $\Delta N = \sqrt{\langle (N - \langle N \rangle)^2 \rangle}$. We measure $\Delta N$ and the average number of particles $\langle N \rangle$ in the sub-box with the side length $\ell$ in the whole system. In equilibrium systems, $\Delta N$ should be proportional to $\langle N \rangle^{1/2}$. In some active matters, however, GNF that is characterized by $\Delta N \propto \langle N \rangle^\alpha$ with a larger exponent $\alpha > 0.5$ is observed [5, 6, 40, 41, 57–62]. The iABP model is ideal for examining GNF because MIPS is absent even at large Péclet numbers. In Figure 4, we plot $\Delta N$ as a function of $\langle N \rangle$ for several $Pe$ for a fixed $M (= 80)$. $\Delta N$ behaves as $\langle N \rangle^\alpha$ with exponents $\alpha$ is greater than 0.5 for all $Pe$. We chose the fitting range as $\langle N \rangle \in [100, 1000]$ to extract the exponent $\alpha$. Dot-dashed lines in Figure 4 are the power-law fit of the simulation data. Interestingly, the size of the sub-box $\ell = \sqrt{\langle N \rangle}/\rho$ at which $\Delta N$ deviates from the power-law almost agrees with $\xi_\parallel$, obtained from $\omega_\parallel(q)$ (see Figure 3), as indicated by vertical dashed lines in Figure 4. The dependence of the exponent $\alpha$ on $Pe$ is plotted in the inset of Figure 4. Starting from the smallest value of $\alpha \simeq 0.5$ at $Pe = 1$, $\alpha$ increases with $Pe$, up to $\alpha \simeq 0.84$ at the largest $Pe$, which is close to the values reported for the polar active matters [61, 62].

The particle number fluctuation $\Delta N$ is related to the static structure factor $S(q)$ by $S(q) \to 0 = \Delta N^2/\langle N \rangle^2$ at large $q$. This implies that a power-law $\Delta N \propto \langle N \rangle^\alpha$ should be reflected in the scale-free power law in the reciprocal space $q^{-\beta}$ for $S(q)$ and the two exponents are related by $\beta = 4 \alpha - 2$ [4, 63]. On the other hand, it is natural to expect that the density fluctuation is characterized by the same correlation length, $\xi_{\parallel}$. Thus, we assume the scaling form as $S(q) = \xi_{\parallel}^{-\beta} f(q\xi_{\parallel})$ where the scaling function satisfies $f(x) \sim$ const. for $x < 1$ and $f(x) \sim x^{-\beta}$ for $x > 1$. The inset of Figure 3(d) is the rescaled plot of the $S(q)$, and we can confirm the validity of the above scaling ansatz, which means that the emergence of GNF and behavior of $S(q)$ are consistent. Puzzling is the fact that the velocity correlation $\omega_\parallel(q)$

FIG. 3. Velocity correlation function and static structure factor at $M = 80$ in the Fourier space. (a) longitudinal, (b) transverse part, and (d) static structure factor. Bullet and triangle symbols represent data of the $N = 4 \times 10^4$ and $N = 1 \times 10^3$ simulation, respectively. Longitudinal and transverse correlation lengths $\xi_\parallel$ and $\xi_\perp$, which are found by fitting, as a function of the Péclet number, are indicated on the panel (c). Insets of panels (a) and (b) are fitting by the Ornstein-Zernike function. Rescaled curves of the $S(q)$ are indicated on the inset of panel (d).

FIG. 4. Number fluctuation $\Delta N$ as a function of the expectation value of the number of particles $\langle N \rangle$ at $M = 80$. Solid and dot-dash lines are simulation data fitting lines, respectively. Vertical dash lines represent the value of $\langle N \rangle$ at $\ell = \xi_{\parallel}$. The exponent of number fluctuation $\alpha$ is indicated on the inset.
does not follow the same scaling law as $S(q)$ but is rather well fitted by the Ornstein-Zernike function (see the inset of Figure 3(a)). This may be due to the relatively small nonlinear effect in the velocity field. The simulation with larger Péclet numbers and larger system size will be required to resolve this conundrum, which is left for future work.

In summary, we studied the ABP model with the inertia term, which suppresses MIPS. The model allows us to investigate the non-equilibrium fluctuations of the active fluid with very large activities, unimpeded by MIPS. We have shown that velocity and density correlation become long-ranged, and concomitantly GNF appears. The velocity correlation functions are characterized by two distinct correlation lengths, the longitudinal and the transverse ones. The transverse correlation length is associated with the size of vorticities patterns, which is reminiscent of active turbulence. The longitudinal velocity correlation, on the other hand, enhances the density fluctuations, which share the same correlation length. GNF arises as a natural consequence of the long-range density and longitudinal-velocity correlation. In the homogeneous ordered phase of the Vicsek model [40, 61], the active nematic [5, 59], or the self-propelled rod [41], GNFs are understood as the “infection” of Goldstone mode to the density field [2, 4, 64, 65]. In our model, the aligned velocity fluctuations couple with density mode which grows rise to GNFs. To the best of our knowledge, this is the first study which relates the velocity fluctuations, density fluctuations, and GNF in simple active fluids. Furthermore, the fact that the vortex patterns and long-ranged density fluctuations coexist simultaneously in the homogeneous phase suggests that the active turbulence and GNF, which have been studied independently in previous studies, are intimately related at a deep level and eventually described by a unified theory.

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In this study, we performed the Brownian dynamics simulation for the two-dimensional ABPs. We integrated the Langevin equation, Eqs. (1) and (2), in the main text using the Euler method. We chose $\tau_v = \sigma/v_0$, the diameter of a particle $\sigma$, and the Lennard-Jones unit $\epsilon$, as the unit of time, length, and energy scale, respectively. Control parameters of this system are the Péclet number $Pe = \tau_p/\tau_v$, dimensionless mass $M = m/(\zeta \tau_v)$, energy ratio $\epsilon/(\zeta v_0 \sigma)$, and number density $\rho$. The energy ratio and number density are fixed as $\epsilon/(\zeta v_0 \sigma) = 100$ and $\rho = 0.5$, respectively. We performed the simulation up to $t = 1 \times 10^5$. The results in the main text are obtained from the trajectories at $t \in [1 \times 10^4, 6 \times 10^4]$ to ensure that the system is in a stationary state.

II. SYSTEM SIZE DEPENDENCE OF PHASE BEHAVIOR

![Phase diagram](image)

**FIG. S1.** The “phase diagram” of iABP for the system size $N = 1 \times 10^4$. Each panel is the snapshot of particle configuration for corresponding parameters $(M, Pe)$. For the parameters indicated by the red-colored panels, we observed that the system undergoes MIPS for larger system size $N = 4 \times 10^4$.

In Figure 1 of the main text, we have shown the “phase diagram” in $(M, Pe)$-space and the phase boundary between the MIPS and homogeneous phase. We have chosen a relatively large simulation size of $N = 4 \times 10^4$ because the phase boundary is sensitive to the system size. In Figure S1, we show the phase diagram obtained from the smaller system $N = 1 \times 10^4$. The red-colored panels are configurations for the parameters in which the system undergoes MIPS at a larger system size of $N = 4 \times 10^4$, as shown in Figure 1 of the main text. Furthermore, we have also observed that MIPS disappears at very large $Pe \geq 500$ for small system sizes. This re-entrant transition is reminiscent of the results shown by Mandal *et al.* [1], in which the inertia of both the position and the rotation (of the active noises) as well as the thermal noise are taken into account. We address that this re-entrance is the artifact due to the small system size for our current model.

III. CALCULATION OF COARSE-GRAINED DENSITY, VELOCITY, AND VORTICITY FIELD

We have shown the coarse-grained density, velocity, and vorticity field in Figure 2 of the main text. These quantities are calculated as follows. The local density $\rho(\mathbf{r})$ is obtained by averaging the number of particles in a circle with
radius $3\sigma$ placed on every node of a square-lattice with the lattice constant $\sigma$. The velocity field $v(r)$ was obtained by taking the Gaussian-weighted average in a circle with radius $3\sigma$. The value of the variance of the Gaussian function is chosen in such a way that the Gaussian function is 0.1 at $r = 3\sigma$. The vorticity field $\Omega(r)$ is calculated as $\Omega(r) \approx \sum_{\text{cell}} v(r) \cdot \delta r / \delta S_{\text{cell}}$, a line-integral along the circumference of a square cell with a side length $0.25\sigma$. $\delta S_{\text{cell}}$ is the area of the cell.

**IV. VELOCITY CORRELATION FUNCTION IN REAL SPACE**

In Figure S2(a), we show the velocity correlation function in real space [2, 3], that defined by

$$C(r) = \frac{1}{N} \left\langle \sum_{i,j,k} v_j \cdot v_k \delta(r - r_j + r_k) \right\rangle.$$  

This function is related to $\omega(q) = \omega_{\parallel}(q) + \omega_{\perp}(q)$ by the Fourier transformation. The fact that spatial velocity correlation becomes long-ranged as increasing $Pe$ can be confirmed even in the real space. The velocity correlation function, in Fourier space, behaves as Ornstein-Zernike function in low-wavenumber, means that, in real space, $C(r)$ behaves as

$$C(r) \approx Ar^{-1/2}e^{-r/\lambda},$$

in large $r$ [4]. Dashed lines in Figure S2(d) are fitting results by this function. The fitting range is chosen as $r > 5$ for any $Pe$. The triangle symbol in Figure S2(b) represents the correlation length $\lambda$ that is found by the fitting of $C(r)$. It is natural that $C(r)$ is dominated by the transverse part because $\xi_{\perp} \ll \xi_{\parallel}$ as seen in Figure S2(b).

**V. ENERGY SPECTRUM AND VELOCITY DISTRIBUTION**

In the main text, we observed the development of the vortex structures whose spatial patterns are reminiscent of the fluid turbulence. In the standard inertial turbulence of fluids at high Reynolds numbers, the fingerprint of the turbulence is the universal scale-free behavior of the energy spectrum, known as the Kolmogorov law [5]. A similar power law is also found in the energy spectrum in various active matters [6–16]. Such behaviors are aptly called active turbulence. However, the exponent of the power law depends on systems. Little is known about the universality of the active turbulence. Here we show the energy spectrum of the inertial ABP model studied in the main text. In two-dimension, the energy spectrum is related to the velocity correlation function $\omega(q) = \omega_{\parallel}(q) + \omega_{\perp}(q)$ by

$$E(q) = 2\pi q \omega(q).$$

In Figure S3(a), the energy spectra $E(q)$ for several $Pe$’s at $M = 80$ are shown. One observes a faint sign of the power law with the amplitudes increasing with $Pe$ at intermediate wavevectors at $q \gtrsim 0.1$. A crude estimate of the exponent...
FIG. S3. (a) Energy spectrum $E(q)$ for $Pe = 50$, 100, and 200 as a function of the wavevector $q$. The mass is fixed at $M = 80$. The black broken line of $q^{-0.6}$ is a guide for the eyes. (b) Velocity distribution function $P(v)$ for $M = 80$. The white bullet symbol with blue and orange edge color denotes the numerical results in $Pe = 1$ and 200, respectively. Solid lines are the fits by the Gaussian distribution corresponding to each data.

$\gamma$ of the power law $E(q) \sim q^{-\gamma}$ is about 0.6, which is much smaller than any value reported in the past [6–16]. Data is too noisy and scant, and it is premature to draw any conclusions from Figure S3(a).

Recently, the non-Gaussian distribution of the velocity distribution has been reported in ABP and the active Ornstein-Uhlenbeck (AOUP) at high densities and high $Pe$ [17, 18]. We evaluated the velocity distribution to check if such deviation is also observed for low densities. In Figure S3(b), we show the velocity distribution for $Pe = 1$ and 200 at $M = 80$. The solid lines are the corresponding Gaussian distribution defined by

$$P(v) = \sqrt{\frac{M}{2 \pi T_{\text{kin}}}} \exp \left( -\frac{M v^2}{2 T_{\text{kin}}} \right),$$

(S3)

where $T_{\text{kin}} = M \langle v_x^2 + v_y^2 \rangle / 2$ is the kinetic temperature. For both $Pe$'s, the observed distribution functions are well fitted by the Gaussian. However, one sees a small but discernible deviation from the Gaussian at large $v$’s for $Pe = 200$. Such small deviation was also reported for the dense ABP by Caprini et al. [17]. On the other hand, most numerical studies on the active turbulence do not report deviations from the Gaussian [7, 9, 15]. The deviation in Figure S3(b) is very small and more works need to be done to conclude the deviation is the smoking gun of the active turbulence.

### VI. DENSITY CORRELATION AT MIPS

It is known that the system undergoing the phase separation with smooth surfaces develops the peak in the static structure factor characterized by a power-law tail, $S(q) \propto q^{-(d+1)}$ in the low-wavevector regime. This is called Porod’s law [19, 20]. Porod’s law is also observed in MIPS of active matters [21, 22]. We show that Porod’s law is also observed for iABP when the system undergoes MIPS. Figure S4(a), (b), and (c) are snapshots of the system undergoing MIPS for several $M$’s. The color represents the density calculated by averaging the number of particles in a circle with radius $3 \sigma$. For the smallest inertia, $M = 1$, the phase boundary is sharp, and their surface is smooth. When $M = 30$ and 80, on the other hand, the phase boundary becomes diffusive, and the surfaces are blurred. This behavior might be related to the difference in (effective) temperatures between the dense and gas phase in the presence of inertia [1].

In Figure S4(d), we show the static structure factor $S(q)$ for $M = 1$, 30 and 80. For $M = 1$, Porod’s law, i.e, $S(q) \propto q^{-3}$, is clearly observed at low $q$’s [21, 22]. For the higher inertia, $M = 30$ and 80, $S(q)$ deviates from Porod’s law, although the heights of $S(q)$ at low $q$’s are unaltered. This behavior should be the consequence of the change in the sharpness of the phase boundaries.

In any case, we address that the development of the peak of $S(q)$ at low $q$’s reported in the main text is distinct from trivial Prod’s law of MIPS.
FIG. S4. Colored snapshots by density, (a) $M = 1$, $Pe = 300$, (b) $M = 30$, $Pe = 300$, and (c) $M = 80$, $Pe = 500$. The number of particles is $N = 4 \times 10^4$. The static structure factor $S(q)$ for each parameter is indicated on the panel (d).

VII. FLUCTUATING HYDRODYNAMIC DESCRIPTION

In this section, we consider the fluctuating hydrodynamics description for iABP. We shall show that the long-range correlation can be derived from the linearized fluctuating hydrodynamic equations, as has been shown in the overdamped version of ABP [23]. First, we translate the equation of motion for iABP to the equation of the field variables using the method to derive a fluctuating hydrodynamic equation from the microscopic Langevin equations [24, 25]. Then, we calculate the longitudinal velocity and density correlation function by linearizing the fluctuating hydrodynamic equations.

A. Derivation from active Brownian particles with inertia

Our stating point is the equation of motion for the inertial active Brownian particles (iABP) in two-dimension:

\[
\frac{d\mathbf{r}_j(t)}{dt} = \mathbf{v}_j(t),
\]

\[
m \frac{d\mathbf{v}_j(t)}{dt} = -\zeta \mathbf{v}_j(t) - \sum_{k=1}^{N} \nabla_j U(\mathbf{r}_j - \mathbf{r}_k) + \zeta v_0 \mathbf{e}(\phi_j(t)),
\]

\[
\frac{d\phi_j(t)}{dt} = \sqrt{\frac{2}{\tau_p}} \eta_j(t).
\]

Here, $\eta_j(t)$ is a white noise that satisfies $\langle \eta_j(t) \rangle = 0$ and $\langle \eta_j(t) \eta_k(t') \rangle = \delta_{j,k} \delta(t - t')$. $U(\mathbf{r})$ is the pairwise potential $U(\mathbf{r})$. We assume that $\nabla U(\mathbf{0}) = 0$ for simplicity. $\mathbf{e}(\phi) = (\cos \phi, \sin \phi)$ is the unit vector pointing to the direction of
the active random force. Hydrodynamic fields of this system are the number of density

$$\rho(r, t) = \sum_{j=1}^{N} \delta(r - r_j(t)), \quad (S7)$$

density current

$$\mathbf{J}(r, t) = \sum_{j=1}^{N} \mathbf{v}_j(t) \delta(r - r_j(t)), \quad (S8)$$

and polarization

$$\mathbf{p}(r, t) = \sum_{j=1}^{N} e(\phi_j(t)) \delta(r - r_j(t)). \quad (S9)$$

By differentiating these hydrodynamic fields with respect to time, we obtain the following set of equations. For the density, it is the continuum equation;

$$\partial_t \rho(r, t) = -\nabla \cdot \mathbf{J}(r, t). \quad (S10)$$

For the current and the polarization fields,

$$m \partial_t \mathbf{J}(r, t) = -\nabla \cdot \mathbf{M}^{\text{vv}}(r, t) - \zeta \mathbf{J}(r, t) - \rho(r, t) \nabla U(r - r_k) + \zeta v_0 \mathbf{p}(r, t), \quad (S11)$$

$$\partial_t \mathbf{p}(r, t) = -\nabla \cdot \left( \mathbf{M}^{e\nu}(r, t) \right)^T - \sum_{j=1}^{N} \frac{de(\phi_j(t))}{dt} \delta(r - r_j(t)), \quad (S12)$$

where tensors $\mathbf{M}^{\text{vv}}(r, t)$ and $\mathbf{M}^{e\nu}(r, t)$ are defined by

$$\mathbf{M}^{\text{vv}}(r, t) := m \sum_{j=1}^{N} \mathbf{v}_j(t) \mathbf{v}_j(t) \delta(r - r_j(t)), \quad (S13)$$

$$\mathbf{M}^{e\nu}(r, t) := \sum_{j=1}^{N} e(\phi_j(t)) \mathbf{v}_j(t) \delta(r - r_j(t)). \quad (S14)$$

These tensors can be rewritten in terms of hydrodynamic fields, following the procedure discussed in Ref.[25], as

$$\mathbf{M}^{\text{vv}}(r, t) = \frac{m \mathbf{J}(r, t) \mathbf{J}(r, t)}{\rho(r, t)}, \quad (S15)$$

$$\mathbf{M}^{e\nu}(r, t) = \frac{\mathbf{p}(r, t) \mathbf{J}(r, t)}{\rho(r, t)}. \quad (S16)$$

The potential part in the right-hand side of Eq. (S11) can be expressed as

$$\nabla U(r - r_k) = \nabla \frac{\delta \mathcal{F}[\rho(\cdot, t)]}{\delta \rho(r, t)}, \quad (S17)$$

where functional $\mathcal{F}[\rho]$ is defined by

$$\mathcal{F}[\rho(\cdot, t)] := \frac{1}{2} \int_V d^2r \int_V d^2r' \rho(r, t) \rho(r', t) U(r - r'). \quad (S18)$$

Substituting these expression, Eq. (S11) becomes

$$m \partial_t \mathbf{J}(r, t) = -\nabla \cdot \left( \frac{m \mathbf{J}(r, t) \mathbf{J}(r, t)}{\rho(r, t)} \right) - \zeta \mathbf{J}(r, t) - \rho(r, t) \nabla \frac{\delta \mathcal{F}[\rho(\cdot, t)]}{\delta \rho(r, t)} + \zeta v_0 \mathbf{p}(r, t). \quad (S19)$$
Next, we derive the closed equation for polarization. The time derivative of the unit vector $\mathbf{e}(\phi_j(t))$ in right-hand side of Eq. (S12) is given by

$$\frac{d\mathbf{e}(\phi_j(t))}{dt} = \sqrt{\frac{2}{\tau_p}} \left( \begin{array}{c} -\sin \phi_j(t) \\ \cos \phi_j(t) \end{array} \right) \odot \eta_j(t)$$

$$= -\frac{1}{\tau_p} e_j(t) + \sqrt{\frac{2}{\tau_p}} \left( \begin{array}{c} -\sin \phi_j(t) \\ \cos \phi_j(t) \end{array} \right) \bullet \eta_j(t),$$  \hspace{1cm} (S20)

where the product $\odot$ and $\bullet$ denote the Stratonovich and Itô product, respectively. We have adopted the Itô representation for the multiplicative noise to ensure that the average of the noise is zero [26]. Using Eq. (S20), Eq. (S12) is rewritten as

$$\partial_t \mathbf{p}(r, t) = -\frac{1}{\tau_p} \mathbf{p}(r, t) - \nabla \cdot \left( \frac{J(r, t)\mathbf{p}(r, t)}{\rho(r, t)} \right) + \mathbf{\Lambda}(r, t),$$  \hspace{1cm} (S21)

where the noise term $\mathbf{\Lambda}(r, t)$ is defined as

$$\mathbf{\Lambda}(r, t) := \sqrt{\frac{2}{\tau_p}} \sum_{j=1}^{N} \left( \begin{array}{c} -\sin \phi_j(t) \\ \cos \phi_j(t) \end{array} \right) \bullet \eta_j(t) \delta(r - r_j(t)).$$  \hspace{1cm} (S22)

We rewrite this noise as

$$\mathbf{\Lambda}(r, t) = \sqrt{\frac{\rho(r, t)}{\tau_p}} \mathbf{\Upsilon}(r, t),$$  \hspace{1cm} (S23)

which satisfies $\langle \mathbf{\Upsilon}_\alpha(r, t) \rangle = 0$ and

$$\langle \mathbf{\Upsilon}_\alpha(r, t) \mathbf{\Upsilon}_\beta(r', t') \rangle = \delta_{\alpha, \beta} \delta(r - r') \delta(t - t').$$  \hspace{1cm} (S24)

We can prove Eq. (S24), by calculating the each component of noise correlations and compare the results from Eq. (S22). For example, the $(x, x)$ component is calculated as

$$\langle A_x(r, t) A_x(r', t') \rangle = \frac{2}{\tau_p} \sum_{j=1}^{N} \langle \sin^2 \phi_j(t) \rangle \delta(r - r_j(t)) \delta(r - r') \delta(t - t').$$  \hspace{1cm} (S25)

Using Eq. (S6), the expectation value of $\sin^2 \phi_j(t)$ can be obtained as

$$\langle \sin^2 \phi_j(t) \rangle = \frac{1}{2} - \frac{1}{2} \cos(2\phi_j(0)) e^{-4t/\tau_p}.$$  \hspace{1cm} (S26)

The summation $\sum_{j=1}^{N} \cos(2\phi_j(0))$ becomes 0 in the limits of $N \to \infty$ because the initial value of angles $\phi_j(0)$ is completely random and the $\cos(2\phi_j(0))$ takes random values in the range of $[-1, 1]$. Hence, in the limit of $N \to \infty$, Eq. (S25) becomes

$$\langle A_x(r, t) A_x(r', t') \rangle = \frac{\rho(r, t)}{\tau_p} \delta(r - r') \delta(t - t').$$  \hspace{1cm} (S27)

The $(y, y)$ component of Eq. (S22) is also given by Eq. (S27) in the limit of $N \to \infty$. The correlation function between $x$ and $y$ component of Eq. (S22) becomes 0 by using the relation

$$\langle \sin \phi_j(t) \cos \phi_j(t) \rangle = \frac{1}{2} \sin(2\phi_j(0)) e^{-4t/\tau_p}.$$  \hspace{1cm} (S28)

It is noteworthy that the noise correlation for the polarization field is identical to those of another, or even simpler, active matter model known as the active Ornstein-Uhlenbeck (AOUP) model [27], in the continuum limit.
Below, we summarize the derived fluctuating hydrodynamic equation for iABP;

\[ \partial_t \rho(r, t) = -\nabla \cdot J(r, t), \quad (S29) \]

\[ m \partial_t J(r, t) = -\nabla \cdot P(r, t) - \zeta J(r, t) - \nabla \cdot \left( \frac{m J(r, t) J(r, t)}{\rho(r, t)} \right) + \zeta v_0 p(r, t), \quad (S30) \]

\[ \partial_t p(r, t) = -\frac{1}{\tau_p} p(r, t) - \nabla \cdot \left( \frac{J(r, t) p(r, t)}{\rho(r, t)} \right) + \sqrt{\frac{\rho(r, t)}{\tau_p}} \Upsilon(r, t), \quad (S31) \]

with the pressure tensor \( P(r, t) \) defined by

\[ \nabla \cdot P(r, t) := \rho(r, t) \nabla \frac{\delta F[\rho(\cdot, t)]}{\delta \rho(r, t)}. \quad (S32) \]

Now let us consider the linearization of the fluctuating hydrodynamics of iABP so that we can derive the correlation functions. We assume that the pressure tensor Eq. (S32) depends on only the density field. Up to the linear order in \( \rho \), recovered.

In the limit of \( \tau_p \to 0 \), the active noise becomes white noise and Eq. (S37) and the fluctuation dissipation relation is recovered.

Note that, recently, Marconi et al. [28] has derived fluctuating hydrodynamic equations for the underdamped ABP and AOUP model. They started from the BBGKY equation obtained from the Fokker-Planck equation and then constructed equations for the average of hydrodynamic variables as well as higher-order moments. Finally, by performing a closure approximation of BBGKY hierarchies, they arrived at a closed set of equations.

**B. Velocity and density correlation functions**

From Eqs. (S34) and (S37), we can easily calculate the longitudinal velocity and density correlation function. By Fourier transforming in time and space, Eq. (S34) and (S37) are written as

\[ -i \omega \delta \tilde{p}(q, \omega) = -iq \delta \tilde{J}_\parallel(q, \omega), \quad (S39) \]

\[ -i \omega \delta \tilde{J}_\parallel(q, \omega) = -i q \delta \tilde{J}_\parallel(q, \omega) - \gamma \delta \tilde{J}_\parallel(q, \omega) + \frac{1}{m} \tilde{\Xi}_\parallel(q, \omega), \quad (S40) \]
where $\gamma = \zeta/m$ and $b = 1/(\rho c \chi)$. The variable with check symbol $\tilde{X}(q, \omega)$ is the Fourier transformed variable respect to $r$ and $t$. By eliminating the density field from Eq. (S40) and using the Wiener-Khinchin theorem, we obtain the dynamical longitudinal velocity correlation function in Fourier space:

$$
\omega_\parallel(q, \omega) = \frac{1}{N} \int_{-\infty}^{\infty} dt \left\langle \delta \tilde{J}_\parallel(q, t) \delta \tilde{J}_\parallel^*(q, 0) \right\rangle e^{i\omega t} = \frac{v_0^2 \gamma^2 D \omega^2}{((\omega^2 - \gamma bq^2)^2 + \gamma^2 \omega^2)(\omega^2 + D^2)},
$$

where $D = 1/\tau_p$ and the variables with tildes, $\tilde{X}(q, t)$, are the Fourier transformed variables with respect to $r$. By integrating Eq. (S41) over $\omega$, we obtain the equal time correlation function,

$$
\omega_\parallel(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \omega_\parallel(q, \omega) = \frac{\omega_0}{1 + (\xi_\parallel q)^2},
$$

with

$$
\omega_0 := \frac{v_0^2 \gamma}{2(D + \gamma)} = \frac{v_0^2 \tau_p}{2(\tau_m + \tau_p)}, \quad \xi_\parallel^2 := \frac{b\gamma}{D(D + \gamma)} = \frac{b\tau_p}{1 + \tau_m/\tau_p}.
$$

Here, $\tau_m = 1/\gamma$ is the inertial relaxation time. Next, we calculate the density correlation function. Using Eq. (S39), the dynamical structure factor is written as

$$
S(q, \omega) = \frac{q^2}{\omega^2} \omega_\parallel(q, \omega) = \frac{v_0^2 \gamma^2 D q^2}{((\omega^2 - \gamma b q^2)^2 + \gamma^2 \omega^2)(\omega^2 + D^2)}.
$$

By integrating over $\omega$, we obtain the static structure factor given by

$$
S(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega S(q, \omega) = \frac{S_0}{1 + (\xi_\parallel q)^2}
$$

with

$$
S_0 := \frac{v_0^2}{2bD} = \frac{1}{2} \frac{v_0^2 \tau_p \rho \chi}{\rho T_{\text{eff}} \chi} = \rho T_{\text{eff}} \chi,
$$

where we defined the effective temperature by $T_{\text{eff}} := v_0^2 \tau_p \zeta/2$. The longitudinal velocity correlation function Eq. (S42) and static structure factor Eq. (S45) is the Ornstein-Zernike type function and both of them have the same correlation length $\xi_\parallel$.

C. Comparison of the linearized theory with numerical results

Obviously, the results of the linearized fluctuating hydrodynamic equation, Eqs. (S43) and (S45) correctly reproduce the emerging correlation length, which is, at least qualitatively, consistent with the numerical results reported in the main text. We shall show here that the linearized equation fails to explain the simulation results quantitatively. In Figure S5(a), we compare a theoretical value of $\omega_0$ with $\omega_\parallel$ of Figures S5(a) and (b) to draw the differences between the two fitting lines are not large and it is too early to decide which scaling works better. We fit $\xi_\parallel = (br_c/\sigma^2)^{1/2}Pe/\sqrt{M + Pe}$ (see Eq. (S43)) with the simulation data and estimated the value of the unknown parameter $br_c/\sigma^2$ to be about 73. By using this value of $br_c/\sigma^2$, we compare theoretical prediction for the longitudinal velocity correlation function $\omega_\parallel(q)$ and that obtained from the simulation (see Figure 3(a) in the main text) in Figure S5(c). $M = 80$ is chosen for the mass. We use the data of $\omega_0$ and $\xi_\parallel$ of Figures S5(a) and (b) to draw the theoretical results. One finds that the numerical result is larger than the theoretical values by a factor of 3.4 at $Pe = 200$.

Likewise, Figure S5(d) shows the comparison of the static structure factor $S(q)$ obtained from the theory and simulation. The parameters used to draw the theoretical prediction are the same as those used for $\omega_\parallel(q)$. The values
FIG. S5. (a) Fitting values of $\omega_0$ versus the theoretical curve Eq. (S43). Blue and red bullets represent the longitudinal and transverse part, respectively. The black dashed line is the nondimensionalized form of the first equation of Eq. (S43). (b) Numerical values and theoretical curve [second equation of Eq. (S43)] of the longitudinal correlation length. Blue dots represent numerical values. The blue dashed line is the fitting curve of the second equation of Eq. (S43). The fitting parameter is a dimensionless compressibility $b\tau_v/\sigma^2$, and we found $b\tau_v/\sigma^2 \approx 73.1$. The black dot line represents $\xi_\parallel \propto Pe^{1/2}$. Panels (c) and (d) indicate numerical results versus theoretical curves of the longitudinal velocity correlation function and static structure factor, respectively. Colored dashed lines represent the Eq. (S42) and Eq. (S45). We have used the value $b\tau_v/\sigma^2 \approx 73.1$, which is found by fitting for $\xi_\parallel$, for Eq. (S45).

of the simulation data differ from the theoretical prediction by more than one order of magnitude. This indicates that the increasing fluctuations at large Pe’s are predominantly due to the nonlinear coupling of the fluctuations, which are completely absent in our theoretical analysis based on the linearized equation.

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