Center of mass integral in canonical general relativity

D. Baskaran,1,* S. R. Lau,2,† and A. N. Petrov1,‡

1Sternberg Astronomical Institute, Moscow State University,
Universitet Prospect 13, Moscow 119899, RUSSIA
2Applied Mathematics Group, Department of Mathematics,
University of North Carolina, Chapel Hill, NC 27599-3250 USA

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Abstract

For a two-surface $B$ tending to an infinite–radius round sphere at spatial infinity, we consider the Brown–York boundary integral $H_B$ belonging to the energy sector of the gravitational Hamiltonian. Assuming that the lapse function behaves as $N \sim 1$ in the limit, we find agreement between $H_B$ and the total Arnowitt–Deser–Misner energy, an agreement first noted by Braden, Brown, Whiting, and York. However, we argue that the Arnowitt–Deser–Misner mass–aspect differs from a gauge invariant mass–aspect by a pure divergence on the unit sphere. We also examine the boundary integral $H_B$ corresponding to the Hamiltonian generator of an asymptotic boost, in which case the lapse $N \sim x^k$ grows like one of the asymptotically Cartesian coordinate functions. Such a two–surface integral defines the $k$th component of the center of mass for (the initial data belonging to) a Cauchy surface $\Sigma$ bounded by $B$. In the large–radius limit, we find agreement between $H_B$ and an integral introduced by Beig and ó Murchadha as an improvement upon the center–of–mass integral first written down by Regge and Teitelboim. Although both $H_B$ and the Beig–ó Murchadha integral are naively divergent, they are in fact finite modulo the Hamiltonian constraint. Furthermore, we examine the relationship between $H_B$ and a certain two–surface integral which is linear in the spacetime Riemann curvature tensor. Similar integrals featuring the curvature appear in works by Ashtekar & Hansen, Penrose, Goldberg, and Hayward. Within the canonical 3+1 formalism, we define gravitational energy and center–of–mass as certain moments of Riemann curvature.

*Now at Department of Physics and Astronomy, University of Wales, Cardiff, CF24 3YB, Wales, UK;
Electronic address: spxdb@astro.cf.ac.uk
†Electronic address: lau@amath.unc.edu
‡Electronic address: anpetrov@rol.ru
I. INTRODUCTION

Realization of the Lie algebra $\mathfrak{g}$ of the Poincaré group $G$ as an algebra of differential operators features

$$M^{xt} = x\partial/\partial t + t\partial/\partial x$$

as the generator of a pure boost in the $x$–$t$ plane. A more remarkable realization of $\mathfrak{g}$ arises in classical, canonical, and special relativistic field theory, a setting where field integrals play the role of group generators and Poisson bracket serves as Lie bracket. The incarnation of the generator (1) in this setting is [1, 2]

$$M^{xt} = -\int_{\Sigma} d^{3}x xT^{tt}(x, t) + tP^{x},$$

where $T^{\mu\nu}$ is the Belinfante tensor, the same as the stress–energy–momentum tensor for simple theories, and

$$P^{x} = \int_{\Sigma} d^{3}x T^{tx}(x, t)$$

is the $x$–component of the canonical field momentum. Integration in these expressions is over an inertial three–dimensional hyperplane $\Sigma \simeq E^{3}$ determined by fixation of the coordinate time $t$. Notice that the first term on the RHS of Eq. (2), the one built with the material energy density $T^{tt}(x, t)$, defines (minus) the $x$–component of the field’s $\Sigma$ center of mass, whence the numerical value for $M^{tx} = -M^{xt}$ obtained via evaluation of the integrals is generally not equal to this center–of–mass component. Via the Poisson bracket $M^{xt}$ generates the change \{\Psi(x, t), M^{xt}\} in the classical field $\Psi(x, t)$ corresponding to the infinitesimal $x$–$t$ boost, and for points $x$ close to the origin this change is chiefly governed by $tP^{x}$ in (2). So indeed this term must be present.

One encounters a somewhat different situation in canonical general relativity (GR) and the arena of spacetimes asymptotically flat at spatial infinity. Although in general such spacetimes possess no group of isometries, one can nevertheless realize $G$ as an asymptotic isometry group by writing down gravitational Hamiltonians which generate the asymptotic symmetries. [3, 4] We draw attention to a salient feature of the resulting description. Namely, the generator of an asymptotic boost —say the one corresponding to $M^{tx}$ above— has a form incorporating only the first term on the RHS of Eq. (1), or Eq. (2) for that matter. (These remarks pertain only to boost generators in GR, as the remaining generators of asymptotic translations and rotations look just like they should.) Moreover, the numerical
value for the \( M^{tx} \) generator in GR equals the \( x \)-component of the gravitational field’s center of mass. Rather symbolically, the \textit{value} of the said generator takes the form

\[
M^{tx} = \int_B d^2x \sqrt{\sigma} x \tau^{tt},
\]

where \( B \) is a large two-surface with spherical topology, \( d^2x \sqrt{\sigma} \) is the proper area element of \( B \), and \( \tau^{tt} \) is an energy density stemming from a “boundary stress-energy-momentum tensor” \( \tau^{\alpha\beta} \). [5]

The following heuristic argument sheds light on this salient feature. For the scenario of asymptotic flatness towards spatial infinity, “asymptotic” crudely means fixed-time and \textit{arbitrarily large} radial separation from the “origin,” perhaps actually a closed two-surface with spherical topology. In \( E^3 \) such arbitrarily large radial separation corresponds —except on a set of measure zero, the \( y-z \) plane— to large values of \( x \); therefore, the asymptotic modifier roughly indicates that \( x \partial/\partial t \) is the dominant term in Eq. (1). The ramifications of these simple observations for GR stem from the following key point: the numerical value (determined by evaluation on a classical solution) of a canonical generator in GR is a surface integral at infinity, that is to say a boundary integral like the one in Eq. (4) over a two-surface surface \( B \) enclosing the “origin” and whose points are uniformly separated from it by an arbitrarily large radial distance. Therefore, the integration in Eq. (4) is quite unlike the integration in Eq. (2) in the following sense. The subset of \( B \) on which \( x \) is small is itself \textit{arbitrarily} small (essentially just a “great circle” around \( B \)), suggesting that the expression (4) —analogous to (minus) \( x \partial/\partial t \) only— is permissible. In the end, of course, a test for whether one has chosen the correct boost generator is whether it serves its part in a consistent representation of \( \mathfrak{g} \) under the Dirac algebra determined by the Poisson bracket. The asymptotic boost generator we consider below has long since measured up on this count. [3, 4] Our simple discussion here is meant only to draw attention to the discussed feature, one seemingly neglected in the literature. However, we do point out that within the framework of the Lagrangian (rather than Hamiltonian) field formulation of GR, Ref. [6] considered integrals of motion at spatial infinity which were in the spirit of Eq. (2) and special relativistic theory, while (on classical solutions) their numerical values would be of the form Eq. (4). Since a Lagrangian approach does not select preferred \( \Sigma \) hyperplanes, both types of integrals would thus be treated on the same footing.

Brown and York have written down a geometric expression for the integral appearing in
Eq. (4). They consider the following boundary term belonging to the “energy sector” of the gravitational Hamiltonian: [5, 7]

\[ H_B = \frac{1}{8\pi} \int_B d^2x \sqrt{\sigma} \; N \left(k - k|^{\text{ref}}\right), \]  \hspace{1cm} (5)

with \( B \) and \( d^2x \sqrt{\sigma} \) as before. Note that \( B \) is (perhaps one element of) the boundary \( \partial \Sigma \) of a hypersurface \( \Sigma \). \( N \) is a smearing lapse function, \( k \) is the mean curvature associated with the embedding of \( B \) in \( \Sigma \), and \( k|^{\text{ref}} \) is the reference mean curvature of \( B \) associated with an isometric embedding of \( B \) in an auxiliary Euclidean three-space \( \Sigma \simeq E^3 \). That is to say, whether \( B \) is viewed as a surface in \( \Sigma \) or in \( \Sigma \), it has the same two-metric \( \sigma_{ab} \). Hence, the integral (5) is essentially the difference of the total mean curvatures for the two embeddings.

As shown in [5], such a surface integral must be added to the smeared Hamiltonian constraint

\[ H_\Sigma = \int_\Sigma d^3x N\mathcal{H}, \]  \hspace{1cm} (6)

in order to obtain a \( \Sigma \) hypersurface functional \( H = H_\Sigma + H_B \) which is differentiable on the standard gravitational phase space. The \( H \) functional is differentiable provided \( k|^{\text{ref}} \) is determined solely by the \( B \) metric. Other choices for \( k|^{\text{ref}} \) are possible [8, 9], but except for one passing remark will not be considered here. In appropriate limiting scenarios it is known that \( H_B \) agrees with the standard Arnowitt–Deser–Misner (ADM) notion [10] of total energy for spacetimes asymptotically flat at spatial infinity, the Trautman–Bondi–Sachs notion [11] of total energy–momentum for spacetimes asymptotically flat at null infinity, and the Abbott–Deser notion [12] of conserved mass for spacetimes asymptotically anti–de Sitter at infinity. [9, 13, 14, 15]

In this paper we consider the the scenario of asymptotic flatness towards spatial infinity (spi), in which case the \( \Sigma \) gravitational initial data \((h_{ij}, K_{ij})\), spatial metric and extrinsic curvature, obey certain fall-off conditions specified below. If we adopt this setting and assume that \( B \) tends to an infinite–radius round sphere at spi, then we may obtain physical characterizations of the initial data in terms of the integral (5). For instance, if the lapse obeys \( N \sim 1 \) in the said limit, then \( H \) generates a pure time translation asymptotically, and the “on–shell” (meaning on solutions to \( \mathcal{H} = 0 \)) value \( E_\infty \equiv \lim_{r\to\infty} H_B \) of \( H \) defines the total \( \Sigma \) energy. Braden et al. [7] and later Hawking and Horowitz [13] have noted that such a definition of energy agrees with the standard ADM notion of total energy, although here we establish this equivalence in much more detail. We argue that the Arnowitt–Deser–Misner
mass–aspect differs from a gauge invariant mass–aspect by a pure divergence on the unit sphere.

Moreover, if the lapse grows like \( N \sim x^k \), where \( x^k \) is one of the Cartesian coordinate functions near \( \text{sp} \), then \( H \) generates an asymptotic boost, and the on–shell value \( M_{\infty}^{\perp k} \equiv \lim_{r \to \infty} H_B \) of \( H \) defines the \( k \)th component of the \( \Sigma \) center of mass. [3, 4] For this asymptotic lapse behavior, we find agreement between \( M_{\infty}^{\perp k} \) and the integral introduced by Beig and Ó Murchadha in Ref. [4] (BÓM hereafter) as an improvement upon the center of mass integral first written down by Regge and Teitelboim in Ref. [3] (RT hereafter). This is our first main result. We also establish the relationship between \( H_B \) and a certain two–surface integral which is linear in the spacetime Riemann curvature tensor. This is our second main result. Similar two–surface integrals featuring the curvature appear in works on gravitational energy–momentum by Ashtekar & Hansen [17], Penrose [18], Goldberg [19], and S. Hayward [20], among others. We believe our results to be of relevance for comparison between the standard 3+1 approach to spatial infinity and more formal treatments based on compactification arguments. [4, 21] The results of this work complement those given in the seminal Refs. [3, 4], as well as those given in Ref. [16] which mentioned that the present work would appear.

II. PRELIMINARIES

A. Fall–off for metric and extrinsic curvature

Recall the definition of a spacetime which is asymptotically flat towards spatial infinity given by BÓM. Such a spacetime possesses spatial sections on which there are so–called asymptotically Cartesian coordinates \( x^k \) with corresponding polar coordinates \((r, \theta, \phi)\). Further, with respect to these coordinates, the large–\( r \) perturbations \( \delta h_{ij} \) and \( \delta K_{ij} \) of the \( \Sigma \) three-metric and extrinsic curvature tensor are defined by

\[
\begin{align*}
\delta h_{ij} & = f_{ij} + \delta f_{ij} \sim f_{ij} + a_{ij}(\nu^k)r^{-1} + b_{ij}r^{-1-\varepsilon} \quad (7a) \\
\delta K_{ij} & = 0_{ij} + \delta K_{ij} \sim d_{ij}(\nu^k)r^{-2} + e_{ij}r^{-2-\varepsilon} \quad (7b)
\end{align*}
\]

where \( 0 < \varepsilon \leq 1 \). In (7a) the flat metric \( f_{ij} \) is \( \text{diag}(1,1,1) \), \( \nu^k = x^k/r \), the \( O(1) \) function \( a_{ij} \) is of even parity [that is to say, \( a_{ij}(-\nu^k) = a_{ij}(\nu^k) \)], and the \( O(1) \) angular function \( b_{ij} \).
is of undetermined parity. In (7b) $0_{ij}$ is the zero tensor, $d_{ij}$ is a function of odd parity [that is to say, $d_{ij}(-\nu^k) = -d_{ij}(\nu^k)$], and the $O(1)$ angular function $e_{ij}$ is of undetermined parity. The three-metric expansion considered by RT is of the same form, but with $\varepsilon = 1$ in (7a) and (7b). Cartesian differentiation of (7) yields behavior obtained via term–by–term multiplication by $r^{-1}$ and parity reversal on leading terms. Considering (7a), one can say that asymptotically Cartesian coordinates define their own Euclidean background $E^3$ at $\text{sp1}$, with respect to which the perturbations $\delta h_{ij}$ are defined.

In this paper we work with the expansion

$$h_{ij} = f_{ij} + \delta h_{ij} \sim f_{ij} + a_{ij}(\nu^k)r^{-1} + b_{ij}r^{-1-\varepsilon} + c_{ij}r^{-2},$$

(8)

with $c_{ij}$ also an $O(1)$ angular function of undetermined parity. Hence, we are allowing for the possibility of a single power, say for example $r^{-3/2}$, lying between $r^{-1}$ and $r^{-2}$. We could of course adopt a more general expansion, say

$$h_{ij} = f_{ij} + \delta h_{ij} \sim f_{ij} + a_{ij}(\nu^k)r^{-1} + \sum_{q=1}^{N} b_{ij}^{(q)}r^{-1-\varepsilon_q} + c_{ij}r^{-2},$$

(9)

with up to $N$ terms lying between the $r^{-1}$ and $r^{-2}$ orders. In this case we would have $0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_N \leq 1$ and $b_{ij}^{(q)}$ of undetermined parity for each $q$. However, adoption of such a more general expansion needlessly complicates the analysis. Furthermore, we believe that all of our main results are also valid for the expansion (9).

Let us comment on the meaning of the asymptotic symbol $\sim$ in this paper. In the asymptotic expansions we write down, if the last term written is $O(r^p)$ with $p$ some integer, then the next unwritten term is $O(r^{p-\gamma})$ where $0 < \gamma \leq 1$. We have switched from $\varepsilon$ to $\gamma$ now in order to emphasize following. For the next unwritten $O(r^{-2-\gamma})$ term in an equation like (8), the $\gamma$ need not be the same as the $\varepsilon$ in the third term on the RHS. Furthermore, in an equation such as (7a) the $\sim$ indicates that the next unwritten term is $O(r^{-2})$, that is to say the next integral power of $1/r$ following $r^{-1-\varepsilon}$. Sometimes we include the order symbol $O$ and write equalities for the sake of clarity.
B. BóM integral and asymptotic scenarios

Beig and ó Murchadha [4] define a center of mass integral associated with the Hamiltonian generator of an asymptotic boost. Namely,

\[ M^\perp_\infty(N) = \frac{1}{16\pi} \oint_B d^2x \sqrt{\sigma} n^k \left[ N h^{ij} (\partial_j h_{ik} - \partial_k h_{ij}) + (h^i_k h^{ij} - h^i_k h^{lj})(h_{ij} - f_{ij}) \partial_l N \right], \tag{10} \]

where \( \oint_B \) is shorthand for \( \lim_{r \to \infty} \int_B \) with \( B \) a level–\( r \) two–surface and \( n^k \) its outward–pointing normal. Furthermore, here, as it should for a boost [3], the lapse behaves as

\[ N = \beta^\perp k x^k + \alpha^- + O(r^{-\varepsilon}), \tag{11} \]

with the \( \beta^\perp k = -\beta_k^\perp \) constants and \( \alpha^- \) an \( O(1) \) angular function of odd parity. The expression above (10) is written in terms of asymptotically Cartesian coordinates and ordinary partial derivatives, but it is easy to render the expression covariant.

Were the lapse to go only as \( N = 1 + O(r^{-\varepsilon}) \), then the BóM integral would be the ADM energy. Indeed, one recognizes the first term in the integrand (10) as the familiar ADM expression,

\[ E_\infty = \frac{1}{16\pi} \oint_B d^2x \sqrt{\sigma} n^k h^{ij} (\partial_j h_{ik} - \partial_k h_{ij}). \tag{12} \]

We examine two scenarios in this paper; these being the energy scenario, in which the lapse behaves as \( N \sim 1 \), and the center–of–mass scenario, in which the lapse behaves as in Eq. (11).

We note that the integral (10) is naively divergent, although its actual finiteness is ensured by the particular choice (11) of lapse \( N \), the even parity of \( a_{ij} \) built into the asymptotic \( \Sigma \) metric (7a), and on account of the integral \( H_\Sigma \) of the corresponding constraint \( \mathcal{H} \). As Beig and ó Murchadha show in Appendix C of their Ref. [4], addition of \( M^\perp_\infty(N) \) to \( H_\Sigma \) given in (6) yields a total expression which is explicitly finite. (We also analyze the integral \( H_\Sigma \) in our Appendix C, isolating its divergent contribution via a method different than the BóM one.) Therefore, the BóM integral is finite on–shell, that is to say, given the vanishing of the Hamiltonian constraint \( \mathcal{H} \) and in turn the volume term \( H_\Sigma \). Let us use \( M^\perp_B(N) \) to denote the integral corresponding to (10) before the \( r \to \infty \) limit is taken. The asymptotic expansion

\[ M^\perp_B(N) \sim -1 M^\perp_B r + (-1+\varepsilon) M^\perp_B r^{1-\varepsilon} + 0 M^\perp_B r^0 \tag{13} \]
for the integral (corresponding now to a large but finite \( B \) two–surface) particularly elucidates the issues at hand. The parity conditions built into Eqs. (7a,11) automatically ensure that the constant \(-\frac{1}{\mathcal{M}}\) in fact vanishes. However, invocation of the Hamiltonian constraint is required to ensure that the coefficient \((\frac{1}{\mathcal{M}} + \epsilon)\mathcal{M}\) vanishes. With \( rT \) fall–off the integral (10) is explicitly finite without appeal to the Hamiltonian constraint, as the \( r^{1-\epsilon} \) order in the (13) is absent. Whatever the specified fall–off is, in the end \( \mathcal{M}(N) = 0 \mathcal{M}(N) \).

\[ \text{C. Curvature integral} \]

The other aforementioned two–surface integral, written in terms of the Riemann tensor, we wish to consider is the following:

\[
\mathcal{M}_R \equiv \frac{1}{16\pi} \sqrt{\frac{A}{4\pi}} \int_B d^2x \sqrt{\sigma} N \sigma^{\mu\nu} \sigma^{\lambda\kappa} \mathcal{R}_{\mu\lambda\nu\kappa}.
\] (14)

In this equation \( \mathcal{R}_{\mu\nu\lambda\kappa} \) is the spacetime Riemann tensor, \( A \) is the area of \( B \), and the square root factor outside of the integral is asymptotic to a single power of the coordinate radius \( r \). Furthermore, the spacetime representation of the \( B \) two–metric \( \sigma^{\mu\nu} \) serves here to project free indices into \( B \). With the future–pointing normal of \( \Sigma \) in spacetime denoted by \( u^\mu \) and the out–pointing normal of \( B \) in \( \Sigma \) denoted by \( n^\mu \), we have \( \sigma^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - n^\mu n^\nu \). As for a physical interpretation of the integral (14), consider timelike geodesics whose tangents at points of the hypersurface \( \Sigma \) are given by \( u^\mu \). In the absence of matter \( \sigma^{\mu\nu} \sigma^{\lambda\kappa} \mathcal{R}_{\mu\lambda\nu\kappa} = -2u^\mu n^\lambda u^\nu n^\kappa \mathcal{R}_{\mu\lambda\nu\kappa} \), and the latter quantity controls the rate of change of the vector joining any two nearby geodesics and describing the radial displacement between them. Hence, the integrand in (14) dictates the radial component of the geodesic deviation vector, as one would expect for a “mass–aspect.” [21] We show below that \( H_B \) and \( M_R \) also agree in the \( r \to \infty \) limit, again for both energy and center–of–mass scenarios. The on–shell finiteness of both \( H_B \) and \( M_B \) as \( r \to \infty \) in the weaker \( b_{\text{om}} \) fall–off setting is intimately tied to the fact that these are correct surface integrals to add to (6), thereby achieving a functionally differentiable total Hamiltonian \( H = H_S + H_B \). On the other hand, \( M_R \) in (14) has no particular relation to the canonical 3+1 Hamiltonian; therefore, its on–shell finiteness is far from obvious. See Ref. [20] for a discussion of how an integral very similar to \( M_R \) stems from a dual–null 2+2 Hamiltonian description of general relativity.

Our comparison of \( H_B \) with \( M_R \) leads to the following alternative definitions for energy
and center of mass for gravitational initial data sets. For the energy we find

\[ E_\infty = \frac{1}{16\pi} \oint d\Omega^3 [\sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa}] , \tag{15} \]

where the superscript 3 means take the coefficient of the \( O(r^{-3}) \) term in the radial expansion of the curvature component \( \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \) and here the averaging is over the unit sphere. This definition may be easily generalized to a “superenergy” —the charge integral corresponding to a general time supertranslation at \( \text{SPT} \)— by placing an angle–dependent smearing function in front of the mass–aspect \( 3[\sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa}] / 4 \), in other words using \( N = O(1) \) rather than \( N = 1 \) in Eq. (15). We stress that the ADM mass–aspect (determined from the integrand lifted from the expression for the ADM energy) differs from this manifestly gauge–invariant mass–aspect by a pure divergence on the unit sphere. See the concluding remarks in Section V for further details. Arguably, the ADM mass–aspect does not define a valid superenergy, since such a definition not would generally vanish for Minkowski spacetime. For the coordinates \( M_{\infty}^k \) of the data set’s center of mass, we find

\[ M_{\infty}^k = \frac{1}{16\pi} \oint d\Omega^4 [\sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa}] \nu^k , \tag{16} \]

with a similar meaning for the superscript 4.

III. \( H_B \) AND THE BEIG–Ó MURCHADHA INTEGRAL

In this section we consider the \( H_B \) integral (5), with \( B \) a level–\( r \) surface, showing that its \( r \to \infty \) limit coincides with the BóM integral (10). To set up and compute the limit, we shall introduce a background metric \( h_{ij} \) on \( \Sigma \) via a certain embedding of \( B \) into an auxiliary Euclidean three–space \( \Sigma \simeq E^3 \). All quantities associated with the background will be denoted by either sans serif or boldface letters. In particular, we use \( k \) to represent \( k|^{\text{ref}} \), or more precisely an approximation to \( k|^{\text{ref}} \) of sufficient accuracy to compute the limit.

A. Key identities

Let us first lay some groundwork necessary to obtain useful identities for both \( k - k \) and \( N(k - k) \). Assume that we have a three–dimensional slice \( \Sigma \), equipped with two distinct proper Riemannian three–metrics: \( h_{ij} \) with compatible covariant derivative \( D_i \) and \( h_{ij} \)
with compatible covariant derivative $D_i$. Viewing $h_{ij}$ as the physical metric and $h_{ij}$ as the background metric, we set

$$h_{ik} = h_{ik} + \Delta h_{ik}, \quad h^{ik} = h^{ik} + \Delta q^{ik},$$

(17)

where the terms $\Delta h_{ik}$ and $\Delta q^{ik}$ contain all orders of perturbations.

Assume that a radial coordinate $s$ foliates $\Sigma$ into a nested family of smooth closed two-surfaces. Later, $s$ will be the radial coordinate $r = (x_k x^k)^{-1/2}$ determined by asymptotically Cartesian coordinates $x^k$. We use $s$ at this point to emphasize the fact that here our calculations are not tied to a particular type of two-surface. Loosely, we use the letter $B$ to represent both the $s$-foliation of $\Sigma$ itself and a particular slice (or leaf) of the foliation determined by setting $s$ equal to a constant value $s_0$. Respectively, let $\sigma_{ab}$ and $\sigma_{ab}$ denote the metrics induced on a generic level-$s$ slice by the $h_{ij}$ and $h_{ij}$ metrics. Our central assumption is that the metrics $\sigma_{ab}$ and $\sigma_{ab}$ agree in a sense made precise below. The background metric introduced in (17) will not be arbitrary, rather it will be defined by an (essentially) isometric embedding of $B$ into Euclidean space and as such will be the flat metric belonging to a Euclidean three-space $\Sigma \simeq E^3$.

Let $n_k = M \partial_k s$ be the outward-pointing normal covector for $B$ as a submanifold of the Riemannian space $(\Sigma, h_{ij})$. Likewise, let $n_k = M \partial_k s$ be the outward-pointing normal covector for $B$ as a submanifold of the Riemannian space $(\Sigma, h_{ij})$. The function $M = [h^{ij}(\partial_i s)(\partial_j s)]^{-1/2}$ ensures that $n_k n^k = 1$, and likewise $M$ ensures the $n_k n^k = 1$. Above and in what follows, indices on physical objects are lowered and raised with $h_{ij}$ and its inverse $h^{ij}$, whereas indices on background objects are lowered and raised with $h_{ij}$ and its inverse $h^{ij}$. Again, we let $\Delta n^i = n^i - n^i$ stand for all orders of perturbations. In $\Sigma$ coordinates the induced (projection) metrics for the $s$ foliation of our two Riemannian spaces are

$$\sigma^{ik} = h^{ik} - n^i n^k, \quad \sigma^{ik} = h^{ik} - n^i n^k.$$

(18)

Now our requirement that the metrics induced on $B$ by $h_{ij}$ and $h_{ij}$ agree can be rewritten as

$$\Delta \sigma^{ij} = 0,$$

(19)

where $\Delta \sigma^{ik} = \sigma^{ik} - \sigma^{ik}$. Actually, this condition is too strong. Below, when $s$ is $r = (x_k x^k)^{1/2}$ we shall only demand that $\Delta \sigma^{ij} = O(r^{-2-\varepsilon})$. However, as we shall retain $\Delta \sigma^{ij}$ in our calculations, there is no harm in considering this stronger agreement for the time-being,
and doing so more clearly demonstrates the reason for working with the inverse two–metric. We stress that while working with three dimensional indices, one must enforce equality of the inverse two–metrics as above in order to ensure that $\sigma^{ab} = \sigma^{ab}$ on $B$ (the same, of course, as $\sigma_{ab}$ and $\sigma^{ab}$ agreeing on $B$). Indeed, if we choose $s$ as the first coordinate $x^1$, then the index $a$ on the $B$ coordinate $x^a$ runs over 2, 3 and we have

$$\sigma^{1k} = \sigma^{1k} = 0.$$  \hspace{1cm} (20)

That is to say, $\sigma^{ik}$ and $\sigma^{ik}$ have only 2, 3 components (i. e. $B$ components). In contrast, $\sigma_{1k} \neq 0$, $\sigma_{1k} \neq 0$ and $\sigma_{1k} \neq \sigma_{1k}$ in general on $B$. Therefore, $\sigma_{ik}$ and $\sigma_{ik}$ need not agree on $B$. In other words, the equation with lower three dimensional indices which is analogous to Eq. (19) is generally not valid, although, of course, $\sigma^{ab} = \sigma^{ab}$ on $B$.

With the assumptions spelled out in the preceding paragraph, we now collect the promised identities. Let $N$ be a smearing function and

$$k = -D_in^i, \quad k = -D_in^i.$$  \hspace{1cm} (21)

We then have the following identities:

$$k - k = \frac{1}{2}n^i n^j (D_k \Delta h_{ij} - D_i \Delta h_{kj}) + n_k D_i \left(n^j [\Delta n^k]ight)$$  \hspace{1cm} (22a)

$$+ \frac{1}{2}n^i \Delta h_{ij} D_k \Delta q^{kl} + \frac{1}{2}n_i D_k (\Delta n^i \Delta n^k)$$

$$- \frac{1}{2} h^{kl} \Delta n^k D_i \Delta h_{kl} - \frac{1}{2} \Delta \sigma^{ik} D_i n_k,$$

$$N(k - k) = \frac{1}{2} N n^i n^j (D_k \Delta h_{ij} - D_i \Delta h_{kj}) - \frac{1}{2} \left(n^k h^{il} \Delta h_{ik} - n^l h^{ik} \Delta h_{ik}\right) D_l N$$  \hspace{1cm} (22b)

$$+ n_k D_i \left(N n^j [\Delta n^k]\right) + \frac{1}{2} N n^i \Delta h_{ij} D_k \Delta q^{kl} + \frac{1}{2} N n_i D_k (\Delta n^i \Delta n^k)$$

$$- \frac{1}{2} N h^{kl} \Delta n^j D_i \Delta h_{kl} + \frac{1}{2} \left(n^l h_{ik} \Delta n^i \Delta n^k - n_i \Delta n^i \Delta n^l\right) D_l N$$

$$- \frac{1}{2} N \Delta \sigma^{ik} D_i n_k + \frac{1}{2} n^l h_{ik} \Delta \sigma^{ik} D_l N,$$

with the first identity following from the second upon assuming that $N$ is constant and unity. We derive these identities in Appendix A.1. Note that $\Delta \sigma^{ij}$ could be replaced in these identities via use of the appendix Eq. (A1).

**B. Construction of $h_{ij}$ and various coordinates**

We construct a diffeomorphism between $\Sigma$ and $\Sigma \simeq E^3$ as follows. Take a level–$s$ surface $B$ in $\Sigma$, say the one determined by $s = s_0$, and embed it in $\Sigma$. At this point we make no
assumption that this embedding is isometric. In $\Sigma$ assume that $B$ is also a level–$s$ coordinate surface of value $s_0$. Label the points on this level surface in $\Sigma$ by their coordinate values $x^a$ inherited from $B$ in $\Sigma$. Now extend the the coordinates off of $B$ to a system $(s, x^a)$ on $\Sigma$ in a region surrounding $B$. One way of doing this would be to construct Riemann normal coordinates. The construction described gives us a diffeomorphism, as $(s, x^a)$ label points in both $\Sigma$ and $\Sigma$. Further, this diffeomorphism identifies level–$s$ surfaces in $\Sigma$ with corresponding ones in $\Sigma$, providing us with the set–up in Section III.A where we can work with a single $\Sigma$ equipped with two distinct proper Riemannian three–metrics. We stress that with this construction $h_{ij}$ is a flat Euclidean metric, although it need not be the trivial metric diag$(1, 1, 1)$. Let us make a few comments here meant to highlight the exceptional nature of the foregoing construction that occurs when $s$ is the radius $r = (x_k x^k)^{1/2}$ stemming from asymptotically Cartesian coordinates. Suppose that the embedding into $\Sigma$ of the original two–surface $B$ defined by $s = s_0$ were an exact isometry, which we can guarantee via rather mild assumptions on the Ricci scalar $\mathcal{R}$ of $B$. Then one would not expect that level–$s$ surfaces in $\Sigma$ neighboring this initial surface would also be isometric to their counterparts in $\Sigma$. Indeed, were this the case, one would have a foliation of (an annulus of) flat Euclidean space in which an infinite number of slices were exactly isometric to slices belonging to a foliation of the non–trivial Riemannian space $(\Sigma, h_{ij})$. This would seem to us an overly restrictive situation to achieve. However, while working with the coordinate $r$, we shall find it possible to nearly achieve this situation by relaxing the requirement that the isometries are exact, instead assuming that they hold approximately through some appropriate order in the small parameter $1/r$. Even subject to this relaxation, $k$ and $k^{\text{ref}}$ will agree to an accuracy sufficient to compute the $r \to \infty$ limit of (5) with $N(k - k)$ in place of $N(k - k^{\text{ref}})$.

Equation (17) has been viewed in the system $(s, x^a)$ of coordinates just discussed,

$$h_{ij}(s, x^a) = h_{ij}(s, x^a) + \Delta h_{ij}(s, x^a).$$

(23)

The splitting above depends on (i) the initial choice of $B$ two–surface through the embedding of $B$ in $\Sigma$ and on (ii) how the coordinates are extended off of $B$ once the embedding is carried out. (Let us just loosely say that the splitting depends on the $B$ embedding.) Nevertheless, our calculations are covariant on $\Sigma$, and we can go to any other arbitrary coordinates. For example, from the system $(s, x^a)$ on $\Sigma$, we may transform to a truly Cartesian system $X^k$ on $\Sigma$. Via the constructed diffeomorphism, the system $X^k$ may also be placed on $\Sigma$. Since
by definition coordinate transformation to the system $X^k$ makes $h_{ij}$ the Kronecker delta (we denote this diagonal flat–space metric by $f_{ij}$), adoption of the system $X^k$ on $\Sigma$ yields the splitting

$$h_{ij}(X^k) = f_{ij} + \Delta h_{ij}(X^k),$$

which is quite similar to the type of decomposition used by RT and BôM in the system $x^k$ of coordinates, namely Eq. (8).

Now assume that the coordinate $s$ is in fact the radius $r$ stemming from asymptotically Cartesian coordinates $x^k$. Then we may write Eq. (23) as

$$h_{ij}(r, \theta, \phi) = h_{ij}(r, \theta, \phi) + \Delta h_{ij}(r, \theta, \phi).$$

As mentioned, now the situation will be that all large level–$r$ surfaces will have essentially the same intrinsic geometry $\sigma_{ab}$, whether induced by $h_{ij}$ or by $h_{ij}$. We show this in the next subsection. Notice that under the transformation $(r, \theta, \phi) \rightarrow x^k$ in Eq. (25), one does not recover Eq. (8), of course. Rather one obtains

$$h_{ij}(x^k) = h_{ij}(x^k) + \Delta h_{ij}(x^k),$$

where $h_{ij}(x^k) \neq f_{ij}$ in general. Eqs. (25) and (26) are in fact the same unique splitting with the flat background $\Sigma$ defined by the $B$ embedding, only the coordinates differ. Eq. (8) is a metric splitting with respect to the flat space $E^3$ defined by asymptotically Cartesian coordinates, while Eq. (26) represents a different decomposition into background and perturbation parts, one defined with respect to the different Euclidean space $\Sigma$. Later we will have need to consider the fall–off for $\Delta h_{ij}(x^k)$ in (26) as $r = (x^k x^k)^{1/2} \rightarrow \infty$ which is also defined in terms of $\delta h_{ij}$ through the as yet unknown expansion

$$h_{ij}(x^k) = f_{ij} + \delta h_{ij}(x^k),$$

with respect to $f_{ij}$ in $x^k$ coordinates. Now, it is not evident that the fall–off for $\delta h_{ij}(x^k)$ coincides with $\delta h_{ij}(x^k)$ in (8). However, as shown below, these fall-offs are qualitatively the same.

### C. Isometric embedding of a slightly deformed two–sphere into a flat space

In this subsection we solve the problem of removing a distant large–$r$ two–sphere from an asymptotically flat slice $\Sigma$ and isometrically embedding it into a flat space $\Sigma$. Our
method of solution is a perturbative one. Solution of this problem defines the transformation between asymptotically Cartesian coordinates $x^k$ belonging to $\Sigma$ and Cartesian coordinates $X^k$ belonging to the flat space $\Sigma$ of the embedding.

Dropping the lowest order term, we rewrite the line–element (8) associated with the general $\Sigma$ metric $h_{ij}$ in terms of the polar coordinates $(r, \theta, \phi)$ associated with $x^k$. Then, fixing $r = r_0$, with $r_0 \gg 1$ some large constant, we define a large two–surface $B(r_0)$, which we refer to as a slightly deformed two–sphere. From the $\Sigma$ line–element rewritten in polar coordinates, we may obtain the line element for our slightly deformed two–sphere,

$$
\begin{aligned}
    ds^2_B &\sim r_0^2 \left( 1 + \frac{2\alpha(\theta, \phi)}{r_0} + \frac{2A(\theta, \phi)}{r_0^{1+\varepsilon}} \right) d\theta^2 + r_0^2 \left( \frac{2\gamma(\theta, \phi)}{r_0} + \frac{2G(\theta, \phi)}{r_0^{1+\varepsilon}} \right) d\theta d\phi \\
    &+ r_0^2 \sin^2 \theta \left( 1 + \frac{2\beta(\theta, \phi)}{r_0} + \frac{2B(\theta, \phi)}{r_0^{1+\varepsilon}} \right) d\phi^2.
\end{aligned}
$$

(28)

It is easy to conclude that under the parity transformation $P(x^k) = -x^k$ the spherical coordinates behave as $P(r, \theta, \phi) = (r, \pi - \theta, \phi + \pi)$. Then, keeping in mind that the $a_{ij}$ in (8) are of even parity and the coefficients in (28) are held as independent, one sees that $\alpha$ and $\beta$ are of the even parity, whereas $\gamma$ is of odd parity.

Next, we write down the line–element for the Euclidean space $\Sigma$ in the corresponding spherical polar coordinates,

$$
    ds^2_\Sigma = dR^2 + R^2 (d\Theta^2 + \sin^2 \Theta d\Phi^2).
$$

(29)

Our goal is to isometrically embed $B(r_0)$ defined by (28) into the flat space $\Sigma$ with line–element (29). We posit the existence of a coordinate transformation with asymptotic form

$$
\begin{aligned}
    R/r &\sim 1 + \frac{f(\theta, \phi)}{r} + \frac{F(\theta, \phi)}{r^{1+\varepsilon}} \\
    \Theta &\sim \theta + \frac{g(\theta, \phi)}{r} + \frac{G(\theta, \phi)}{r^{1+\varepsilon}} \\
    \Phi &\sim \phi + \frac{h(\theta, \phi)}{r} + \frac{H(\theta, \phi)}{r^{1+\varepsilon}},
\end{aligned}
$$

(30a,b,c)

which features as yet undefined angular functions. Substitution of the transformation (30) into (29) yields another expression for the flat metric consistent with BôM fall-off. We again fix $r = r_0$ in the resulting expression to define a two–surface $B(r_0)$ and demand that its
corresponding line–element
\begin{align}
\text{d}s^2_B & \sim r_0^2 \left[ 1 + \frac{2}{r_0} \left( f + \frac{\partial g}{\partial \theta} \right) + \frac{2}{r_0^{1+\varepsilon}} \left( F + \frac{\partial G}{\partial \theta} \right) \right] \text{d}\theta^2 \\
& + r_0^2 \left[ \frac{2}{r_0} \left( \frac{\partial g}{\partial \phi} + \sin^2 \theta \frac{\partial h}{\partial \theta} \right) + \frac{2}{r_0^{1+\varepsilon}} \left( \frac{\partial G}{\partial \phi} + \sin^2 \theta \frac{\partial H}{\partial \theta} \right) \right] \text{d}\theta \text{d}\phi \\
& + r_0^2 \sin^2 \theta \left[ 1 + \frac{2}{r_0} \left( f + g \cot \theta + \frac{\partial h}{\partial \phi} \right) + \frac{2}{r_0^{1+\varepsilon}} \left( F + G \cot \theta + \frac{\partial H}{\partial \phi} \right) \right] \text{d}\phi^2
\end{align}

matches the line–element (28) just given. Such matching is tantamount to solving the isometric embedding problem. As $1/r_0$ is here a small parameter, we shall –after peeling off an overall $r_0^2$ factor– explicitly consider the $r_0^{-1}$ and $r_0^{-1-\varepsilon}$ orders and implicitly consider the $r^{-2}$ order. Hence our solution of the isometric embedding problem will only be approximate.

Balancing terms at order $r_0^{-1}$, we obtain the system
\begin{align}
f + \frac{\partial g}{\partial \theta} &= \alpha \\
f + g \cot \theta + \frac{\partial h}{\partial \phi} &= \beta \\
\frac{\partial g}{\partial \phi} + \sin^2 \theta \frac{\partial h}{\partial \theta} &= \gamma,
\end{align}
while for the case $\varepsilon < 1$ a similar balance of terms at order $r_0^{-1-\varepsilon}$ yields the system
\begin{align}
F + \frac{\partial G}{\partial \theta} &= \mathcal{A} \\
F + G \cot \theta + \frac{\partial H}{\partial \phi} &= \mathcal{B} \\
\frac{\partial G}{\partial \phi} + \sin^2 \theta \frac{\partial H}{\partial \theta} &= \mathcal{G}.
\end{align}

In the case $\varepsilon = 1$ the system is at order $r_0^{-2}$ and the RHS of each equation in (33) is modified by addition of known functions of $f, g, h$ and their derivatives: $\mathcal{A} \mapsto \mathcal{A} + \mathcal{A}'(f, g, h)$, $\mathcal{B} \mapsto \mathcal{B} + \mathcal{B}'(f, g, h)$, $\mathcal{G} \mapsto \mathcal{G} + \mathcal{G}'(f, g, h)$. Of course, we would assume that $f, g, h$ were obtained via resolution of the first system (32) before turning to solve the modified second system. In the case $\varepsilon < 1$ we may also obtain a system at order $r_0^{-2}$, yet a third system and one of the same type. This is possible because we have included the extra terms $b_{ij}$ and $c_{ij}$ in Eq. (8) and therefore use a more detailed expansion than the one Bŏm use. Considering all of these systems together, including the possible third unwritten system at order $r_0^{-2}$, we may solve the embedding equations asymptotically to a high enough order to ensure that $\sigma_{ab} - \sigma_{ab} = O(r^{-\varepsilon})$, in turn giving the following crucial result: $\Delta \sigma^{ij} = O(r^{-2-\varepsilon})$. 

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We may consider our isometric embedding problem solved (to the required accuracy) if there is a combined solution to the systems (32) and (33) as well as the third system we just mentioned. In Appendix A.2 we examine the system (32), finding a solution assuming simple compatibility conditions for the coefficients in (28). The remaining systems are formally similar and may be examined analogously. Note that conditions of regularity on the coefficients $\alpha$, $\beta$, and $\gamma$ (intrinsic $B$ geometry) are expected and should mirror conditions on the full metric $\sigma_{ab}$ which are necessary and sufficient for existence of a (suitably unique) solution to the full embedding equations. If the Ricci scalar (twice the Gaussian curvature) of $B$ is everywhere positive (as is the case here), the task of isometrically embedding $B$ in Euclidean space is Weyl’s problem, a classic embedding problem of differential geometry in the large. In the most robust formulation of the problem known to us, that due to Heinz [22], existence of such an embedding is guaranteed if the $B$ metric coefficients are twice continuously differentiable. Uniqueness of the embedding (up to Euclidean motions) follows from the Cohn–Vosson theorem. [23]

First performing the trivial transformation from the system $(R, \Theta, \Phi)$ to the corresponding Cartesian coordinates $X^k$, we now seek, subject to the assumption of Bôm fall–off, the behavior of the transformation between $X^k$ and $x^k$. In fact, the transformation (30) can be rewritten as [see (A31) in Appendix A.2 and the text thereafter] as

$$X^k \sim x^k + 0 \xi^k(\nu^l) + \varepsilon \xi^k r^{-\varepsilon} + 1 \xi^k r^{-1},$$

(34)

whence

$$\partial X^k / \partial x^i \sim \delta^k_i + A^k_{i} (\nu^l) r^{-1} + B^k_{i} r^{-1-\varepsilon} + C^k_{i} r^{-2}.$$  

(35)

The angular functions $0 \xi^k(\nu^l)$ and $A^k_{i}(\nu^l)$ are of odd and even parity respectively, while the angular functions $\varepsilon \xi^k$, $1 \xi^k$, $B^k_{i}$, and $C^k_{i}$ are not of definite parity. Note that the $A^k_{i}$ and $B^k_{i}$ here are not the $A$ and $B$ appearing in the system (33).
D. \textbf{Asymptotic expansions for }$\Delta h_{ij}$, $\Delta q^{ij}$, and $\Delta n^{i}$

With respect to the b"om metric splitting (8), we first collect the following standard and easily derived formulae:

\begin{align}
    h_{ij} &= f_{ij} + \delta h_{ij} \sim f_{ij} + a_{ij} r^{-1} + b_{ij} r^{-1-\varepsilon} + c_{ij} r^{-2} \quad (36a) \\
    h^{ij} &= f^{ij} + \delta q^{ij} \sim f^{ij} - a^{ij} r^{-1} - b^{ij} r^{-1-\varepsilon} - (c^{ij} - a^{i} a^{j}) r^{-2} \quad (36b) \\
    M &= 1 + \delta M \sim 1 + \frac{1}{2} a_{\nu \nu} r^{-1} + \frac{1}{2} b_{\nu \nu} r^{-1-\varepsilon} + \frac{1}{2} (c_{\nu \nu} - a^{i}_{\nu} a_{i \nu} + \frac{3}{4} a^{2}_{\nu \nu}) r^{-2} \quad (36c) \\
    n^{i} &= \nu^{i} + \delta n^{i} \sim \nu^{i} + \frac{1}{2} (\nu^{j} a_{\nu \nu} - 2 a^{i}_{\nu}) r^{-1} + \frac{1}{2} (\nu^{j} b_{\nu \nu} - 2 b^{i}_{\nu}) r^{-1-\varepsilon} \\
    &\quad + \frac{1}{2} \left[ \nu^{i} (c_{\nu \nu} - a_{\nu k} a_{k \nu} + \frac{3}{4} a^{2}_{\nu \nu}) - 2 c^{i}_{\nu} + 2 a^{ik} a_{k \nu} - a^{i}_{\nu} a_{\nu \nu} \right] r^{-2} \quad (36d)
\end{align}

with the equalities indicating that $\delta h_{ij}$, $\delta q^{ij}$, $\delta M$, and $\delta n^{i}$ contain all orders of perturbations. Moreover, on the rightmost side of the equations all indices have been raised or lowered with the trivial flat $f_{ij}$ metric and a subscript $\nu$ indicates contraction with $\nu^{i}$ (for example $a^{i}_{\nu} = a^{ik} \nu_{k}$).

With $h_{ij}(X^{k}) = f_{ij}$ and the expansion (35), we find the sought–for expansion (27) of b"om–type for the $h_{ij}$ background metric,

\begin{equation}
    h_{ij} \sim f_{ij} + a_{ij} (\nu^{k}) r^{-1} + b_{ij} r^{-1-\varepsilon} + c_{ij} r^{-2}, \quad (37)
\end{equation}

where now all subdominant terms are pure–gauge and given by

\begin{align}
    a_{ij} &= A_{ij} + A_{ji} \quad (38a) \\
    b_{ij} &= B_{ij} + B_{ji} \quad (38b) \\
    c_{ij} &= A_{ki} A^{k}_{,j} + C_{ij} + C_{ji} \quad (38c)
\end{align}

Note that $a_{ij}$ has even parity, while both $b_{ij}$ and $c_{ij}$ are not of definite parity. From the expansion (37) we obtain identities for $h_{ij}$, $h^{ij}$, $M$, and $n^{i}$ which are token identical to those given in Eqs. (36). Moreover, these mirror identities also have the same parity behavior as those found in Eqs. (36). Substitution of Eqs. (8) and (37) into Eq. (26) and these considerations show that

\begin{align}
    \Delta h_{ij} &= \delta h_{ij} - \delta h_{ij} \quad (39a) \\
    \Delta q^{ij} &= \delta q^{ij} - \delta q^{ij} \quad (39b) \\
    \Delta n^{i} &= \delta n^{i} - \delta n^{i} \quad (39c)
\end{align}
each possess precisely the same fall–off and parity properties as $\delta h_{ij}, \delta q^{ij},$ and $\delta n^i$, although of course they do differ term–by–term from these.

### E. Energy integral

We turn now to the detailed comparison between the Brown–York and BóM integrals. If we naively insert the identities (22) in terms of $\Delta$ perturbations into the Brown–York integral $H_B$, then we arrive at expressions appearing to be the ADM and BóM integrals. However, one cannot make such a direct comparison, as we have expressed $H_B$ in terms of $\Delta$ perturbations with respect to reference Euclidean space $\Sigma$, whereas the BóM integral is expressed in terms of $\delta$ perturbations with respect to the Euclidean space $E^3$ defined by the asymptotically Cartesian coordinates. An elegant way to achieve the comparison would appeal to the technique of gauge (inner) transformations [24] based on the theory of continuous transformations [25]. However, we adopt a more cumbersome but straightforward approach.

Let us first consider energy scenario. As demonstrated later in the text around Eq. (86) below, the proper area–element for a level–$r$ two–surface $B$ obeys

$$d^2 x \sqrt{\sigma} = d\Omega r^2 \left[1 + \frac{1}{2} (a^i_i - a_{\nu\nu}) r^{-1} + O(r^{-1-\epsilon})\right], \quad (40)$$

where again $d\Omega$ denotes the area–element of the unit sphere. Only the leading behavior of this equation is relevant for the energy scenario at hand; however, the next–to–leading order is relevant for the center–of–mass scenario, since a lapse $N$ growing like $r$ will "sample" this next–to–leading order. Hence we keep it here for future use. Our results from the last subsection further imply that

$$\sqrt{\sigma} - \sqrt{\sigma} = O(r^{-\epsilon}). \quad (41)$$

Therefore, as far as all energy scenario limits and even center–of–mass scenario limits are concerned, we may view $d^2 x \sqrt{\sigma}$ as the area–element induced either by $h_{ij}$ or $h_{\mu\nu}$. Since the area–element grows like $r^2$ and for now $N \sim 1$, we see that only leading $O(r^{-2})$ terms in the expression (22a) for $k - k$ will contribute to the $r \to \infty$ limit of $H_B$. It is readily seen that the last four terms in (22a) can not contribute to the limit. Indeed, symbolically $D = \partial + \Gamma$, where $\Gamma$ Christoffel terms are $O(r^{-2})$, a fact stemming from the derived fall–off (37) for $h_{ij}$. Therefore, $D_k$ differentiation–like $\partial_k$ differentiation– drops fall–off by one power
of inverse radius (and reverses parity). This shows that the third, fourth and fifth terms are \( O(r^{-3}) \). Finally, our perturbative solution to the embedding equations has shown the sixth and final term to be \( O(r^{-3-\varepsilon}) \).

Now considering the remaining two factors (the first two) on the RHS of (22a), we drop some more subdominant terms and write

\[
k - k \sim \frac{1}{2} n^i h^{kl} \left( \partial_k \Delta n_{il} - \partial_i \Delta h_{kl} \right) + \nu_k \partial_l \left( n^i \Delta n^{kl} \right). \tag{42}
\]

The product of the last factor and the leading contribution to the area–element (40) integrates to zero via Stokes’ theorem. Whence for the energy scenario we have that

\[
H_B \sim \frac{1}{16\pi} \int_B d^2 x \sqrt{\sigma} n^i h^{kl} \left( \partial_k \Delta n_{il} - \partial_i \Delta h_{kl} \right), \tag{43}
\]

a result which yields

\[
H_B \sim \frac{1}{16\pi} \int_B d^2 x \sqrt{\sigma} n^i h^{kl} \left( \partial_k h_{il} - \partial_i h_{kl} \right) - \frac{1}{16\pi} \int_B d^2 x \sqrt{\sigma} n^i h^{kl} \left( \partial_k h_{il} - \partial_i h_{kl} \right) \tag{44}
\]

after some replacements of \( n^i \) and \( h^{ij} \) by \( n^i \) and \( h^{ij} \) at the expense of introducing subdominant terms which are then discarded. Apart from the fact that the integration is over the finite surface \( B \) and not the sphere at infinity, the first integral on the RHS is the ADM energy belonging to \( \Sigma \). Likewise, in the limit the second integral is the ADM energy belonging to the vacuum slice \( \Sigma \approx E^3 \). To see this, merely replace \( \sqrt{\sigma} \) with \( \sqrt{\sigma} \) at the expense of introducing subdominant terms. Because we work with physically reasonable BôM or RT fall–off, we expect that this second integral then vanishes in the limit. That it indeed does may also be verified via direct calculation. With Eqs. (35,37,38) we get

\[
h_{ij} \sim f_{ij} + \partial_i^0 \xi_j + \partial_j^0 \xi_i. \tag{45}
\]

Whence to leading order the integrand of the second integral in Eq. (44) is \( \nu^i \partial^l (\partial_l^0 \xi_i) \), and again the leading order of the product of this term with the area–element integrates to zero via Stokes’ theorem.

**F. Center–of–mass integral**

Evaluation of the integral (5) for the scenario with \( N \) behaving as in Eq. (11) is quite a bit more subtle. The integral is naively divergent in the \( r \to \infty \) limit, so we must keep
track of both divergent terms (which only vanishes upon invocation of the Hamiltonian constraint) and finite terms. Let us turn now to the identity (22b) for $N(k - k)$ and first dispatch the terms which most obviously do not contribute to the limit. First, as the last two terms are $O(r^{-2-\varepsilon})$ they do not contribute. The fourth, fifth, sixth, and seventh terms are each $O(r^{-2})$. Nevertheless, none of these terms contribute to the limit, since for each parity conditions ensure that all leading terms integrate to zero. For example, consider the fourth term, $\frac{1}{2} N_i^j \Delta h_{ij} D_k \Delta q^{kl}$. As argued just above, $D_k$ differentiation drops fall–off and reverses parity, so $D_k \Delta q^{kl}$ is $O(r^{-2})$ and leading odd–parity. The product $n^i \Delta h_{il}$ is clearly $O(r^{-1})$ and also leading odd–parity. Since the lapse here is $O(r^{-1})$ and leading odd–parity, we see that the full term is leading odd–parity, whence the product of this term with the leading contribution to the area–element integrates to zero.

Focus now on the third term, $n_k D_l (N n^l \Delta n^k)$, on the RHS of (22b), writing it as

$$n_k D_l (N n^l \Delta n^k) = n_k \partial_l (N n^l \Delta n^k) + n_k (N n^p \Delta n^k) \Gamma^l_{pl}.$$ (46)

The term involving the Christoffel symbol is $O(r^{-2})$ and with leading odd parity, essentially because the $\Gamma^k_{pl}$ are themselves $O(r^{-2})$ and of leading odd parity. Therefore, this term does not contribute to the limit. The first term on the RHS also makes no contribution. To show this, first recall the discussion above and introduce the odd–parity term

$$A^i = \frac{1}{2} (\nu^i a_{\nu\nu} - 2 a^i_{\nu} - \nu^i a_{\nu\nu} + 2 a^i_{\nu}),$$ (47)

so that $\Delta n^i \sim A^i r^{-1}$. Next, expand $N n^l \Delta n^k$, the factor within the parenthesis, as follows

$$N n^l \Delta n^k = \beta^+_{\nu} \nu^l A^k + \lambda^{[lk]} + \rho^{[lk]} + O(r^{-1-\varepsilon}),$$ (48)

where $\beta^+_{\nu} = \beta^+_{-j} \nu^j$ is $O(r^0)$ and $\lambda^{[lk]}$ and $\rho^{[lk]}$ are respectively $O(r^{-\varepsilon})$ and $O(r^{-1})$. From the last equation, infer that

$$n_k \partial_l (N n^l \Delta n^k) = n_k \partial_l (\beta^+_{\nu} \nu^l A^k) + \nu_k \partial_l (\lambda^{[lk]} + \rho^{[lk]}) + O(r^{-2-\varepsilon}).$$ (49)

With the analog of (36c) for $M$ in $n_k = M\nu_k$, the fact that $\beta^+_{\nu}$ is $O(1)$ and of odd parity, and Eq. (47), we conclude that on the RHS the first and only worrisome term includes $O(r^{-1})$ and $O(r^{-2})$ pieces (both potentially contributing to the integral) which are of odd parity and thus integrate to zero.

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We drop those terms in Eq. (22b) shown to make no contribution to the limit, thereby reaching

\[ H_B \sim \frac{1}{16\pi} \int_B d^2x \sqrt{\sigma} \left[ Nn^i h^{kl} (\partial_k \Delta h_{il} - \partial_i \Delta h_{kl}) - (n^k h^{il} \Delta h_{ik} - n^l h^{ik} \Delta h_{ik}) \partial_l N \right]. \] (50)

To get this last equation we have also replaced \( D_i \) differentiation with \( \partial_i \) differentiation, a step easily seen as permissible via fall–off and parity arguments. We now write \( \Delta h_{ij} = \delta h_{ij} - \delta h_{ij} \), in order to cast the last equation into the form

\[ H_B \sim \frac{1}{16\pi} \int_B d^2x \sqrt{\sigma} \left[ Nn^i h^{kl} (\partial_k h_{il} - \partial_i h_{kl}) - (n^k h^{il} \delta h_{ik} - n^l h^{ik} \delta h_{ik}) \partial_l N \right]. \] (51)

To reach this equation, we have made several swaps of \( h^{ij} \) for \( h^{ij} \) and \( n^i \) for \( n^i \). Such swaps are permissible, since they result in the introduction of \( O(r^{-2}) \) terms in the integrand which are then seen to integrate to zero via parity arguments. One recognizes the first integral as the \( \text{BôM} \) integral \( M^B_B(N) \) belonging to \( \Sigma \). It is easy to infer that the second integral on the \( \text{RHS} \) of Eq. (51) must vanish in the \( r \to \infty \) limit. First replace \( \sqrt{\sigma} \) with \( \sqrt{\sigma} \), a step which does not affect the limit. The resulting integral \( M^B_B(N) \) is the \( \text{BôM} \) integral for \( h_{ij} \) and the vacuum slice \( \Sigma \simeq E^3 \); hence we also expect this integral to vanish in the limit as \( h_{ij} \) obeys physically reasonable fall–off.

Let us quickly establish the vanishing of \( \text{BôM} \) integral when evaluated on the flat metric \( h_{ij} \) belonging to the vacuum slice \( \Sigma \). Now, the full integrand, the sum of four terms, is \( O(r^{-1}) \) and of leading odd parity. Consider the product of its leading term with the area–element (40). It is only this leading term of the integrand which “sees” the next–to–leading term in the area element; that is to say, the product of the leading term of the integrand with the next–to–leading term of the area element potentially contributes to the integral. But we see that this potential contribution integrates to zero via parity arguments, whence in the calculations to follow we may work solely with the leading round–sphere term \( d\Omega r^2 \) of the area element.

Without affecting the \( r \to \infty \) limit, the integrand for the \( \text{BôM} \) integral of \( h_{ij} \) may be
replaced by the sum of the following four terms:

\[
\begin{align*}
\text{term1} &= (\beta^\perp l x^l) \nu^k f^{ij} \partial_j h_{ik} \\
\text{term2} &= - (\beta^\perp l x^l) \nu^k f^{ij} \partial_k h_{ij} \\
\text{term3} &= - \beta^\perp l \nu^l f^{ij} \delta h_{ij} \\
\text{term4} &= \beta^\perp l \nu^l f^{ij} \delta h_{ij}.
\end{align*}
\] (52a-b-c-d)

To see why this is the case, consider, for example, the term

\[
N n^k h^{ij} \partial_j h_{ik} = N \nu^k f^{ij} \partial_j h_{ik} + N \delta n^k f^{ij} \partial_j h_{ik} + N \nu^k \delta q^{ij} \partial_j h_{ik} + N \delta n^k \delta q^{ij} \partial_j h_{ik}.
\] (53)

On the RHS the last term is \(O(r^{-3})\) while the second and third terms are each \(O(r^{-2})\) and off odd parity, whence none of these terms contribute to the limit. Finally, the first term on the RHS is

\[
N \nu^k f^{ij} \partial_j h_{ik} = (\beta^\perp l x^l) \nu^k f^{ij} \partial_j h_{ik} + \alpha - \nu^k \partial^j (\partial_i \xi_j) + O(r^{-2-\epsilon}),
\] (54)

where the middle term on the RHS is \(O(r^{-2})\) and of odd parity, so on the RHS the sole limit–contributing term is the first one.

Eqs. (35,37,38c) allow us to write

\[
h_{ij} = f_{ij} + L_{\xi} f_{ij} + (\text{E.P.T.}) r^{-2} + O(r^{-2-\epsilon})
\]

\[
= f_{ij} + \partial_i \xi_j + \partial_j \xi_i + (\text{E.P.T.}) r^{-2} + O(r^{-2-\epsilon}).
\] (55)

Again, (E.P.T.) stands for generic terms of even parity, and the one here stems from the \(A_{ki} A^k_j\) in Eq. (38c). When coupled with simple parity and fall–off arguments, this expression for \(h_{ij}\) shows that we may replace \(\delta h_{ij}\) terms in Eqs. (52) with \(L_{\xi} f_{ij}\). Therefore, up to terms not contributing in the \(r \to \infty\) limit,

\[
\text{term1} + \text{term2} + \text{term3} + \text{term4}
\]

\[
= 2(\beta^\perp l x^l) \nu^k f^{ij} \partial_j [\partial_i \xi_k] - 2 \beta^\perp l \nu^l f^{ij} (\partial_i \xi_j + \partial_j \xi_i) + 2 \beta^\perp l \nu^l f^{ij} \partial_i \xi_j
\]

\[
= 2 \nu^k \partial_i [(\beta^\perp l x^l) f^{ij} \partial_j \xi_k] - 2 \beta^\perp l \nu^l f^{ij} \partial_j \xi_i + 2 \beta^\perp l \nu^l f^{ij} \partial_i \xi_j
\]

\[
= 2 \nu_k \partial_i [(\beta^\perp l x^l) \partial^j [\xi]^k] - 2 \beta^\perp l [\xi]^k,
\] (56)

whence —to the relevant order— the full integrand integrates to zero via Stokes’ theorem.
IV. \( H_B \) AND THE CURVATURE INTEGRAL \( M_R \)

In this section we again evaluate the main surface integral (5) in the large–sphere limit towards SPI, this time showing that

\[
H_B \sim \frac{1}{16\pi} \sqrt{\frac{A}{4\pi}} N \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa}
\]

as \( r \to \infty \). We have discussed the integral on the RHS in the paragraph after Eq. (14) where it is denoted \( M_R \). This result holds for both the energy scenario with \( N \sim 1 \) and the center–of–mass scenario with \( N \) given by Eq. (11) discussed in Section II.

We will verify (57) by establishing the following results. First,

\[
k - k |_{\text{ref}} \sim 3 \left[ \frac{1}{2} \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \right] r^{-2},
\]

where the superscript 3 means take the coefficient of the leading \( O(r^{-3}) \) term in a radial expansion of the curvature component \( \frac{1}{2} \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \). The coefficient \( 3 \left[ \frac{1}{2} \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \right] \) happens to be of even parity. Second, Eq. (40) can be written

\[
d^2 x \sqrt{\sigma} = r^2 d\Omega \left[ 1 + (\text{E.P.T.}) r^{-1} + O(r^{-1}) \right],
\]

where we use E.P.T. to stand for generic terms of even parity. Important in itself, this result also shows that \( \sqrt{A/4\pi} \sim r + O(r^{-1}) \). Third,

\[
k - k |_{\text{ref}} \sim (\text{E.P.T.}) r^{-2} + (3 + \varepsilon) \left[ \frac{1}{2} \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \right] r^{-2} - \varepsilon + (\text{E.P.T.}) + 4 \left[ \frac{1}{2} \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \right] r^{-3}.
\]

The E.P.T. coefficient sitting before the \( r^{-2} \) is again \( 3 \left[ \frac{1}{2} \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \right] \), and the term \( (3 + \varepsilon) \left[ \frac{1}{2} \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \right] \) is not of definite parity. When coupled with by now standard arguments, these three results establish Eq. (57) subject to the assumptions of either the energy scenario or center–of–mass scenario.

Before turning to detailed calculations, let us lay some groundwork and fix more notations. We may write the \( \Sigma \) metric in an ADM–like form

\[
h_{ij} dx^i dx^j = M^2 dr^2 + \sigma_{ab} (dx^a + W^a dr) (dx^b + W^b dr),
\]

where \( M \) and \( W^a \) are respectively the radial lapse function and radial shift vector. Take \( x^a = (\zeta, \bar{\zeta}) \) as coordinates on a \( B \) surface. Here \( \zeta \) —a single complex coordinate on \( B \) — is the stereographic coordinate \( e^{i\phi} \cot(\theta/2) \) belonging to the asymptotically Cartesian coordinates.
$x^k$, and $\bar{\zeta}$ is the complex conjugate of $\zeta$. Introduce a complex null dyad $m^a \partial/\partial x^a$ and $\bar{m}^a \partial/\partial x^a$ on $B$, chosen so that $m_a \bar{m}^a = 1$ and $m_a m^a = 0$. In terms of the null dyad the $B$ metric is written as

$$\sigma_{ab} = m_a \bar{m}_b + \bar{m}_a m_b.$$  \hspace{1cm} (62)

One advantage of working with a complex null $B$ dyad is that results derived for one frame leg $\hat{e}_3 = m^a \partial/\partial x^a$ yield corresponding results for the other leg $\hat{e}_4 = \bar{m}^a \partial/\partial x^a$ under complex conjugation. (See Appendix B.1 for an explanation as to why we use $\hat{3}$ and $\hat{4}$ as the name indices here.) Another is that the components $m^a m^b k_{ab}$ and $2m^a \bar{m}^b k_{ab}$ respectively capture the trace–free and trace pieces of the extrinsic curvature tensor $k_{ab}$ associated with the radial foliation of $\Sigma$ into nested $B$ two–surfaces. As an orthonormal co–triad on $\Sigma$ take

$$e^r = M dr$$ \hspace{1cm} (63a)

$$e^3 = \bar{m}_a (dx^a + W^a dr) = \bar{m}_\zeta d\zeta + \bar{m}_\bar{\zeta} d\bar{\zeta} + W_m dr,$$ \hspace{1cm} (63b)

where $W_m = \bar{m}_a W^a$. We may obtain the one–form $e^4$ from $e^3$ via conjugation. The spatial triad $e_\hat{k}$ dual to and the connection coefficients $\omega_{j\hat{k}\hat{l}}$ determined by this co–triad are listed respectively in the appendix Eqs. (B28) and (B31). Triad indices $j, k, l, \cdots$ run over the values $\mp, \hat{3}, \hat{4}$.

A. Asymptotic expansions

Recall that $\nu^k = x^k/r$, and with the stereographic coordinate define the complex direction

$$\mu^i = \sqrt{\frac{1}{2} P^{-1} \left( 1 - \bar{\zeta}^2, -i (1 + \zeta^2), 2\bar{\zeta} \right)}$$ \hspace{1cm} (64)

discussed in Appendix B.1. The Cartesian components $\mu^i$ obey the relationship

$$f^{ij} = \mu^i \bar{\mu}^j + \bar{\mu}^i \mu^j + \nu^i \nu^j.$$ \hspace{1cm} (65)

With $\mu^i$ and $\nu^i$, we define the following $O(r^0)$ quantities (and their complex conjugates):

$$a_{\nu \nu} = \nu^i \nu^j a_{ij}$$ \hspace{1cm} (66a)

$$a_{\nu \mu} = \nu^i \mu^j a_{ij}$$ \hspace{1cm} (66b)

$$a_{\mu \bar{\mu}} = \mu^i \bar{\mu}^j a_{ij}$$ \hspace{1cm} (66c)

$$a_{\mu \mu} = \mu^i \mu^j a_{ij}.$$ \hspace{1cm} (66d)
Both $a_{\nu\nu}$ and $a_{\mu\bar{\mu}}$ are in fact of even parity, although this is not obvious for the latter quantity, while $a_{\nu\mu}$ and $a_{\mu\nu}$ are not of definite parity. Below we discuss the behavior of $a_{\nu\mu}$, $a_{\mu\bar{\mu}}$, and $a_{\mu\mu}$ under the parity operation.

The expansions for the metric functions $M$ and $W_m$ are

$$M \sim 1 + \frac{1}{2}a_{\nu\nu}r^{-1} \quad (67a)$$
$$W_m \sim a_{\nu\mu}r^{-1}. \quad (67b)$$

To motivate similar expansions for $m_\zeta$ and $m_{\bar{\zeta}}$, let us first note that

$$\mu_i dx^i = r\sqrt{2}P^{-1}d\zeta, \quad (68)$$

which may be demonstrated via the chain rule. Therefore, $\mu_\zeta = \mu_i \partial x^i / \partial \zeta = 0$ and $\mu_{\bar{\zeta}} = \mu_i \partial x^i / \partial \bar{\zeta} = r\sqrt{2}P^{-1}$, although both $\partial x^i / \partial \zeta$ and its conjugate are $O(r)$. These results determine the leading behavior in the expansions for the components of the $B$ co–frame,

$$m_\zeta \sim 0 \cdot r + \sqrt{\frac{1}{2}}P^{-1}a_{\mu\mu} \quad (69a)$$
$$m_{\bar{\zeta}} \sim \sqrt{2}P^{-1}r + \sqrt{\frac{1}{2}}P^{-1}a_{\mu\bar{\mu}}. \quad (69b)$$

We have carefully tailored the next–to–leading order behavior in the expansions (69) to ensure that

$$m_b = \mu_b + \frac{1}{2}(a_{\mu\mu}\bar{\mu}_b + a_{\mu\bar{\mu}}\mu_a) r^{-1} + O(r^{-\varepsilon}). \quad (70)$$

In this equation note that $\mu_a = \mu_i \partial x^i / \partial x^a$ is $O(r)$. Now construct $\sigma_{bc}$ via Eq. (62), thereby finding

$$\sigma_{bc} = f_{bc} + a_{bc}r^{-1} + O(r^{1-\varepsilon}), \quad (71)$$

where $f_{bc} = \mu_b\bar{\mu}_c + \bar{\mu}_b\mu_c$ is the $O(r^2)$ metric on a radius–$r$ round sphere and

$$a_{bc} = (\mu^i\bar{\mu}_b + \bar{\mu}^i\mu_b)(\mu^j\bar{\mu}_c + \bar{\mu}^j\mu_c)a_{ij}$$

is the $O(r^2)$ projection of $a_{ij}$ into the said radius–$r$ sphere. Note that $\partial x^i / \partial x^b = \mu^i\bar{\mu}_b + \bar{\mu}^i\mu_b$ as can be shown by Eq. (65). We see therefore that our expansions (69) for the null co–frame are consistent, since contraction of Eq. (7a) upon the $O(r^2)$ projector $$(\partial x^i / \partial x^b)(\partial x^j / \partial x^c)$$
also yields Eq. (71).

Let us obtain the asymptotic expansion for the action of $\bar{\partial}$ on an $s$ spin–weighted scalar. Using Eqs. (69) in the appendix Eq. (B28b) for the dyad leg $e_3$ or via simple inference, one
finds the not unexpected result,

$$m^b = \mu^b - \frac{1}{2} \left( a_{\mu \bar{\mu}} \mu^b + a_{\mu \bar{\mu}} \mu^b \right) r^{-1} + O(r^{-2-\epsilon}), \quad (73)$$

where, for example, $\mu^a = f^{ab} \mu_b = O(1/r)$. Now define the operator $\delta = m^a \partial / \partial x^a$, and let $\delta_0 = r \mu^a \partial / \partial x^a$ be the corresponding operator on the unit sphere. On scalars, these operators agree with the ones discussed in Appendix B.1. Contraction of Eq. (73) on $\partial / \partial x^b$ then gives

$$\delta \sim \delta_0 r^{-1} - \frac{1}{2} \left( a_{\mu \bar{\mu}} \delta_0 + a_{\mu \bar{\mu}} \delta_0 \right) r^{-2}. \quad (74)$$

Next, we insert the co–frame expansions (69) into the appendix result (B31b) for the $B$ connection form $\omega = \omega_{\hat{3} \hat{4} \hat{3}}$, thereby reaching

$$\omega \sim \omega_0 r^{-1} + \frac{1}{2} \left( \delta_0 [a_{\mu \bar{\mu}}] - \delta_0 [a_{\mu \bar{\mu}}] - \omega_0 a_{\mu \bar{\mu}} - \bar{\omega}_0 a_{\mu \bar{\mu}} \right) r^{-2}, \quad (75)$$

where $\omega_0 = -\tilde{\zeta} / \sqrt{2}$ is the connection form on the unit sphere. Now, by definition $\bar{\sigma} = \delta - s \omega$, whence we find

$$\bar{\sigma} \sim \bar{\delta}_0 r^{-1} - \frac{1}{2} \left[ a_{\mu \bar{\mu}} \bar{\delta}_0 + a_{\mu \bar{\mu}} \bar{\delta}_0 + s (\bar{\delta}_0 a_{\mu \bar{\mu}}) - s (\bar{\delta}_0 a_{\mu \bar{\mu}}) \right] r^{-2} \quad (76)$$

as the sought–for $\bar{\sigma}$ expansion. To get the corresponding expansion for $\bar{\sigma}$, simply complex conjugate (76) and then send $s \to -s$.

Let us obtain expansions for the dyad components of the $B$ extrinsic curvature tensor. As a connection form, the trace is given by the formula $k = 2k_{\hat{3} \hat{4}} = -2\omega_{\hat{3} \hat{3}}$. So we insert the expansions (67), (69), and (76) into appendix formula (B31c) for $\omega_{\hat{3} \hat{3}}$, obtaining

$$k \sim -2r^{-1} + (a_{\nu \nu} + a_{\mu \bar{\mu}} + \bar{\omega}_0 a_{\nu \mu} + \bar{\delta}_0 a_{\nu \mu}) r^{-2} \quad (77)$$

as the desired expansion. Similarly, starting with $k_{\hat{3} \hat{3}} = -\omega_{\hat{3} \hat{3}}$ and Eq. (B31d), we get

$$k_{mm} \sim (\frac{1}{2} a_{\mu \bar{\mu}} + \bar{\delta}_0 a_{\nu \mu}) r^{-2} \quad (78)$$

as the expansion for the trace–free part of $k_{ab}$.

Finally, let us obtain the asymptotic expansion for the scalar curvature $\mathcal{R}$ of $B$. Into the appendix formula (B34) for $\mathcal{R}$, we insert the expansions (74) and (75), with result

$$\mathcal{R} \sim 2r^{-2} + (\bar{\delta}_0^2 a_{\mu \bar{\mu}} + \bar{\delta}_0^2 a_{\mu \bar{\mu}} - 2\bar{\delta}_0 \bar{\delta}_0 a_{\mu \bar{\mu}} - 2a_{\mu \bar{\mu}}) r^{-3}. \quad (79)$$
Our expansions for $k = 2k_{\tilde{m}\tilde{m}}$, $k_{mm}$, $\bar{\sigma}$, and $\mathcal{R}$ determine the leading order behavior in radial expansions for some components of the $\Sigma$ Riemann tensor. Indeed from the Gauß-Codazzi-Mainardi embedding equations, [8, 26]

\begin{align}
(k_{\tilde{m}\tilde{m}})^2 - k_{mm}k_{\tilde{m}\tilde{m}} - \frac{1}{2}\mathcal{R} &= R_{m\tilde{m}m\tilde{m}} \quad (80a) \\
\bar{\sigma}k_{mm} - \partial k_{m\tilde{m}} &= R_{mmm\tilde{m}} \quad (80b)
\end{align}

we may infer the explicit expressions for the coefficients $^3R_{m\tilde{m}m\tilde{m}}$ and $^3R_{m\tilde{m}m\tilde{m}}$ in the expansions

\begin{align}
R_{m\tilde{m}m\tilde{m}} &\sim ^3R_{m\tilde{m}m\tilde{m}}r^{-3} \quad (81a) \\
R_{m\tilde{m}m\tilde{m}} &\sim ^3R_{m\tilde{m}m\tilde{m}}r^{-3}. \quad (81b)
\end{align}

Eq. (80a) determines that

\[ ^3R_{m\tilde{m}m\tilde{m}} = -\frac{1}{2}^3\mathcal{R} - k , \quad (82) \]

whence we obtain

\[ ^3R_{m\tilde{m}m\tilde{m}} = -a_{\nu\nu} - \bar{\sigma}_0a_{\nu\bar{\mu}} - \bar{\sigma}_0a_{\nu\mu} - \frac{1}{2}\bar{\sigma}_0^2a_{\bar{\mu}\bar{\nu}} - \frac{1}{2}\bar{\sigma}_0^2a_{\mu\nu} + \bar{\sigma}_0\bar{\sigma}_0a_{\mu\bar{\nu}} . \quad (83) \]

Likewise, Eq. (80b) determines that

\[ ^3R_{m\tilde{m}m\tilde{m}} = \bar{\sigma}_0^2k_{mm} - \bar{\sigma}_0^2k_{\tilde{m}\tilde{m}} , \quad (84) \]

whence

\[ ^3R_{m\tilde{m}m\tilde{m}} = \frac{1}{2}a_{\nu\mu} - \frac{1}{2}\bar{\sigma}_0a_{\nu\nu} + \frac{1}{2}\bar{\sigma}_0a_{\mu\nu} - \frac{1}{2}\bar{\sigma}_0^2a_{\nu\bar{\mu}} - \frac{1}{2}\bar{\sigma}_0^2a_{\bar{\mu}\nu} + \frac{1}{2}\bar{\sigma}_0\bar{\sigma}_0a_{\nu\mu} . \quad (85) \]

To reach the last equation, we have appealed to the commutator equation $2[\bar{\sigma}_0, \partial_0]a_{\nu\mu} = a_{\nu\mu}$ discussed in Appendix B.1.

The following consistency check affords some confidence in our results. First, show that

\[ \sqrt{\sigma} = i(|m_\zeta|^2 - |m_\zeta|^2) = i2r^2P^{-2} \left[ 1 + a_{\mu\bar{\nu}}r^{-1} + O(r^{-1-\epsilon}) \right] , \quad (86) \]

where the $i$ is necessary for $d^2x\sqrt{\sigma}$ to be real as $d\zeta \wedge d\bar{\zeta}$ is pure imaginary. Now the area form on a radius–$r$ round sphere is

\[ i\bar{\mu}_a\mu_b\, dx^a \wedge dx^b = i2r^2P^{-2}d\zeta \wedge d\bar{\zeta} ; \quad (87) \]
and with this fact and Eq. (79), we perform a direct calculation showing that

$$\int_B d^2x \sqrt{\sigma} R = 8\pi + O(r^{-1-\epsilon}).$$

(88)

That is to say, the Gauß–Bonnet Theorem holds to the same level of accuracy as our approximations thus far. We point out that the presence of the sole term in $^3R$ which is not a unit–sphere divergence, namely $-2a_{\mu\bar{\mu}}$, plays a crucial role in this agreement, as it cancels a similar such term in $\sqrt{\sigma}$. Note that our discussion here has also established the last section’s Eq. (40) as well as Eq. (59) at the beginning of this section. Do note, however, that the coordinates used here are $(\zeta, \bar{\zeta})$, whereas in Section III the $B$ coordinates used were $(\theta, \phi)$. These two systems are related by a non–trivial (in fact imaginary) Jacobian. Therefore, $d^2x$ here is not the $d^2x$ from Section III, although of course $d^2x \sqrt{\sigma}$ is the same here as there.

B. Parity

Define the action of the parity operator $P$ via $P(x^i) = -x^i$, whence it follows that $P(\nu^k) = -\nu^k$. Moreover, one can show that $P(\zeta) = -1/\bar{\zeta}$, and from this result and (64) that

$$P(\mu^i) = -(\bar{\zeta}/\zeta)\bar{\mu}^i.$$  

(89)

We can then immediately write

$$P(a_{\nu\nu}) = a_{\nu\nu}, \quad P(a_{\nu\mu}) = (\bar{\zeta}/\zeta)a_{\nu\bar{\mu}},$$  

(90a)

$$P(a_{\mu\bar{\mu}}) = a_{\mu\bar{\mu}}, \quad P(a_{\mu\mu}) = (\bar{\zeta}/\zeta)^2a_{\bar{\mu}\bar{\mu}}.$$  

(90b)

Moreover,

$$P(\delta_0) = (\bar{\zeta}/\zeta)\bar{\delta}_0, \quad P(\omega_0) = -(\zeta)^{-2}\bar{\omega}_0,$$

(91)

where the first of these formulae follows since $P(\delta_0) = P(r\mu^i\partial/\partial x^i)$. Note that the minus sign difference between the formulae for $P(\delta_0)$ and $P(\mu^i)$ stems from $P(\partial/\partial x^i) = -\partial/\partial x^i$.

With the formulae amassed so far, it is fairly easy to establish the following:

$$P(\bar{\delta}_0^2a_{\mu\bar{\mu}}) = \bar{\delta}_0^2a_{\mu\bar{\mu}}$$  

(92a)

$$P(\bar{\delta}_0a_{\nu\mu}) = \bar{\delta}_0a_{\nu\mu}$$  

(92b)

$$P(\bar{\delta}_0a_{\nu\bar{\mu}}) = (\bar{\zeta}/\zeta)^2\bar{\delta}_0a_{\nu\bar{\mu}}$$  

(92c)

$$P(\bar{\delta}_0\bar{\delta}_0a_{\mu\bar{\mu}}) = \bar{\delta}_0\bar{\delta}_0a_{\mu\bar{\mu}}.$$  

(92d)
These results and their complex conjugates in turn show that $3\mathcal{R}$, $2k$, $3R_{\bar{m}\bar{m}m\bar{m}}$, and $2k_{mm}2k_{\bar{m}\bar{m}}$ are all of even parity.

C. Energy integral

Let us return to the surface integral (5) and total energy scenario. Consider the expression (82) derived from the Gauß–Codazzi–Mainardi equations. A similar formula is associated with the isometric embedding of $B$ in $\Sigma$ and the Ansatz

$$k|_{\text{ref}}^\sim \sim -2r^{-1} + 2k|_{\text{ref}}^\sim r^{-2}. \quad (93)$$

Namely,

$$2k|_{\text{ref}}^\sim = -\frac{1}{2} 3\mathcal{R}. \quad (94)$$

We give a more careful derivation of this result in the next subsection. Therefore, appealing to Eq. (82) we get

$$k - k|_{\text{ref}}^\sim \sim -3R_{\bar{m}\bar{m}m\bar{m}} r^{-2}, \quad (95)$$

and, noting that

$$2R_{\bar{m}\bar{m}m\bar{m}} = -\sigma^{ij} \sigma^{kl} R_{ikjl}, \quad (96)$$

we may also write this as

$$k - k|_{\text{ref}}^\sim \sim 3 \left[ \frac{1}{2} \sigma^{ij} \sigma^{kl} R_{ikjl} \right] r^{-2}. \quad (97)$$

Moreover, for the $\Sigma$ components of the spacetime Riemann tensor, one has [26]

$$\mathcal{R}_{ijkl} = R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}. \quad (98)$$

Since $K_{ij} = O(r^{-2})$, the terms quadratic in $K_{ij}$ are in fact $O(r^{-4})$ terms, whence we have established Eq. (58).

D. Center–of–mass integral

When considering the surface integral (5) in the center–of–mass scenario, we keep more powers of inverse radius in the $k$ expansion than we kept in Eq. (77). We now write

$$k \sim -2r^{-1} + 2kr^{-2} + (2+\varepsilon)kr^{-2-\varepsilon} + 3kr^{-3}, \quad (99)$$
with the previously given expression for $2k$ still valid. From the embedding equation (80a) we infer that

$$2^{(2+\varepsilon)}k_{\bar{m}\bar{n}} = -\frac{1}{2}^{(3+\varepsilon)}\mathcal{R} - (3+\varepsilon)R_{mn\bar{m}\bar{n}}$$

and so —with the results of the parity subsection— we have

$$k \sim -2r^{-1} + (\text{E.P.T.})r^{-2} - \left(\frac{1}{2}^{(3+\varepsilon)}\mathcal{R} + (3+\varepsilon)R_{mn\bar{m}\bar{n}}\right)r^{-2-\varepsilon} - (\text{E.P.T.} + \frac{1}{2}^{4}\mathcal{R} + 4R_{mn\bar{m}\bar{n}})r^{-3}.$$  

Again, E.P.T. stands for generic even parity terms. We claim that

$$k|_{\text{ref}} \sim -2r^{-1} + (\text{E.P.T.})r^{-2} - \left(\frac{1}{2}^{(3+\varepsilon)}\mathcal{R} + (3+\varepsilon)R_{mn\bar{m}\bar{n}}\right)r^{-2-\varepsilon} - (\text{E.P.T.} + \frac{1}{2}^{4}\mathcal{R} + 4R_{mn\bar{m}\bar{n}})r^{-3},$$  

and will establish this result below. We note in passing that the expansion (102) agrees with the “lightcone reference” $-\sqrt{2\mathcal{R}}$ derived in Ref. [9]. Combination of these equations with Eq. (96) gives

$$k - k|_{\text{ref}} \sim (\text{E.P.T.})r^{-2} + (3+\varepsilon)\left[\frac{1}{2}\sigma^{ij}\sigma^{kl}R_{ikjl}\right]r^{-2-\varepsilon} + (\text{E.P.T.} + \frac{1}{2}^{4}\sigma^{ij}\sigma^{kl}R_{ikjl})r^{-3}.$$  

Now, since $K_{ij} \sim d_{ij}(\nu^{k})r^{-2}$ with $d_{ij}$ of odd parity, the quadratic extrinsic curvature terms in (98) are of leading $r^{-4}$ order and even parity. Hence, we have

$$^{(3+\varepsilon)}\left[\frac{1}{2}\sigma^{\mu\nu}\sigma^{\lambda\kappa}\mathfrak{g}_{\mu\lambda\nu\kappa}\right] = ^{(3+\varepsilon)}\left[\frac{1}{2}\sigma^{ij}\sigma^{kl}R_{ikjl}\right]$$

and this result along with (103) establishes Eq. (60).

Let us now verify (102). We must consider the Gauß–Codazzi–Mainardi equations associated with the isometric embedding of $B$ in $\Sigma \simeq E^{3}$. Using $k_{ab} = (k|_{\text{ref}})_{ab}$ as a shorthand, these are

$$(k_{\bar{m}\bar{n}})^{2} - k_{mn}k_{\bar{m}\bar{n}} - \frac{1}{2}\mathcal{R} = 0$$

and

$$\bar{\partial}k_{mn} - \partial k_{mn} = 0.$$  

We solve these equations —in the sense of asymptotic expansions— for $k_{mn}$ and $k_{\bar{m}\bar{n}}$, viewing the intrinsic $B$ geometry, both $\partial$ and $\mathcal{R}$, as fixed. We make the following Ansätze:

$$k_{\bar{m}\bar{n}} \sim -r^{-1} + 2k_{mn}r^{-2} + (2+\varepsilon)k_{mn}r^{-2-\varepsilon} + 3k_{mn}r^{-3}$$

and

$$k_{mn} \sim 2k_{mn}r^{-2}.$$
We remark that consistency of the Ansätze adopted here follows from the fact that — as shown in Section III— the non-trivial flat background metric $h_{ij}$ admits an expansion analogous to $h_{ij}$. Therefore, one expects all of the asymptotic expansion calculated in Section IV.A to carry over for $h_{ij}$. However, here we endeavor to follow an independent approach. From Eqs. (106) we first algebraically find that

$$2k_{mm} = -\frac{1}{4} \mathcal{R}$$

$$\quad (2+\varepsilon)k_{mm} = -\frac{1}{4} (3+\varepsilon) \mathcal{R},$$

establishing in particular the claim made earlier in Eq. (94). Clearly then —as the previous parity analysis shows— $\mathcal{P}(2k_{mm}) = 2k_{mm}$, which justifies the E.P.T. term next to the $r^{-2}$ on the rhs of Eq. (102). Also algebraically, we obtain

$$2^3k_{mm} = (2k_{mm})^2 - 2k_{mm}^2k_{m\bar{m}} - \frac{1}{2} 4^3 \mathcal{R},$$

where —as seen from Eqs. (105b) and (107a)— the weight–two scalar $2k_{mm}$ solves the PDE

$$\bar{\partial}_0 2k_{mm} = -\frac{1}{4} \bar{\partial}_0 3^3 \mathcal{R}.$$  

**Lemma.** $\mathcal{P}(2k_{mm}) = (\bar{\zeta}/\zeta)^2 (2k_{m\bar{m}})$, whence $2k_{mm}^2k_{m\bar{m}}$ is of even parity [which is the remaining needed piece to verify Eq. (102)].

To start the proof, let us show that there exists a unique solution $2k_{mm}$ to the PDE given in Eq. (109), where we view the rhs as a prescribed source determined by Eq. (79). As is well–known, for an equation of the form $\bar{\partial}_0 f = g$ with $g$ a prescribed weight–one source, a unique inverse $\bar{\partial}_0^{-1}$ to the operation $\bar{\partial}_0$ exists if and only if

$$\oint d\Omega \bar{Y}_{1m} g = 0,$$

where again $\oint d\Omega$ denotes average over the unit sphere $S^2$ and the $1Y_{1m}$ are spin–1 spherical harmonics. Using Eq. (B25) from Appendix B.1, we see that for our equation the issue at hand is whether or not

$$\oint d\Omega _{-1} Y_{1m} \bar{\partial}_0 3^3 \mathcal{R}$$

vanishes for $m = -1, 0, 1$. Simple integration by parts will not show that the integral vanishes, since $\bar{\partial}_0 (-1Y_{1m}) \neq 0$. However, for any scalar curvature $\mathcal{R}$, there exists a weight–two scalar $Q$ such that $\partial \mathcal{R} = \bar{\partial} Q$. [28] More precisely, letting $\psi$ denote the conformal factor
relating $\sigma_{ab}$ to the line–element $ds_0^2$ of the unit sphere, as in

$$ds_0^2 = \psi^2 \sigma_{ab} dx^a dx^b,$$

(112)

Tod shows that [28]

$$Q = 4\bar{\sigma}^2 \log \psi - 4(\bar{\sigma} \log \psi)^2.$$

(113)

Assume $\psi \sim r^{-1} + 2\psi r^{-2}$ and that $^2\psi$ is of even parity. Since the log terms are all differentiated we may write

$$\log \psi = - \log r + \log \left[ 1 + 2\psi r^{-1} + O(r^{-1-\varepsilon}) \right]$$

$$= - \log r + 2\psi r^{-1} + O(r^{-1-\varepsilon}),$$

(114)

whence (113) in tandem with the expansion (76) for $\bar{\sigma}$ leads to

$$Q \sim 4\bar{\sigma}^2 (2\psi) r^{-3}.$$

(115)

This is a consistent result since $\bar{\sigma}Q = \bar{\sigma}R = O(r^{-4})$, the leading $2r^{-2}$ term in $R$ being annihilated by the angular derivatives in $\bar{\sigma}$. We have then the leading-order identity $\bar{\sigma}^3 R = 4\bar{\sigma}\bar{\sigma}^2 (2\psi)$, and so —dropping a factor of 4— the integral (111) just above becomes

$$\oint d\Omega_{-1} Y_{1m} \bar{\sigma}_0 \bar{\sigma}_0^2 (2\psi) = 0.$$

(116)

The last equality follows by integration by parts and $\bar{\sigma}_0(\bar{Y}_{1m}) = 0$. Thus, the uniqueness of $\bar{\sigma}^{-1}$ is established. Whence we have

$$^2k_{mm} = -\frac{1}{4} \bar{\sigma}_0^{-1} \bar{\sigma}_0^3 R$$

$$= -\frac{1}{4} \bar{\sigma}_0^{-1} [4\bar{\sigma}_0 \bar{\sigma}_0^2 (2\psi)]$$

$$= -\bar{\sigma}_0^2 (2\psi).$$

(117)

Since $^2\psi$ is an even parity scalar function, the lemma then follows by simple calculations as outlined in the parity subsection.

V. CONCLUDING REMARKS

Our investigation has shown the mass–aspect to be the following: [cf. Eqs. (15,83, 96,98)]

$$\frac{1}{4} \left[ \sigma^{\mu\nu} \sigma^{\lambda\kappa} R_{\mu\lambda\nu\kappa} \right] = \frac{1}{4} (2a_{\nu\nu} + 2\bar{\sigma}_0 a_{\nu\bar{\mu}} + 2\bar{\sigma}_0 a_{\nu\mu} + \bar{\sigma}_0^2 a_{\bar{\mu}\bar{\mu}} + \bar{\sigma}_0^2 a_{\mu\mu} - 2\bar{\sigma}_0 \bar{\sigma}_0 a_{\mu\bar{\mu}}).$$

(118)
Upon proper averaging over the unit sphere, that is to say \((4\pi)^{-1} \oint d\Omega\) integration, the mass–aspect yields the total energy. This mass–aspect may be compared with the one stemming from the ADM integral (12). Namely,

\[
\frac{1}{4} \pi \rho n^k h^{ij} (\partial_j h_{ik} - \partial_k h_{ij}) = \frac{1}{4} \left( 2a_{\nu\nu} + \delta_0 a_{\nu\mu} + \bar{\delta}_0 a_{\nu\mu} \right).
\]  

(119)

(118) and (119) differ by a pure divergence on the unit sphere. We note that the reference term \(k|^{\text{ref}}\) in Eqs. (5,93) plays a crucial role in yielding a mass–aspect (118) whose proper unit–sphere average agrees with the ADM energy. Indeed, not only does \(k|^{\text{ref}}\) remove the leading order divergent contribution to the “unreferenced energy” [the proper B integral of \((8\pi)^{-1}k\) alone which blows up like \(r\)], at the next order it removes a dangerous factor of \(a_{\mu\bar{\mu}}\) from \(2k\) in Eq. (77). Indeed, the total energy \((8\pi)^{-1} \oint d\Omega a_{\nu\nu}\) is not equal to \((8\pi)^{-1} \oint d\Omega 2k\) in general.

In stark contrast with the energy scenario, we point out that the results of Appendix C.1 (in particular the lemmas as they pertain to coefficients of the \(\mathcal{R}\) expansion) show that the \(k|^{\text{ref}}\) term in Eqs. (5,102) in fact makes no contribution at all to the \(r \to \infty\) limit in the center–of–mass scenario! This suggests that the bôm integral might be compared directly to the proper \(B\) integral of \((8\pi)^{-1}Nk\) as an alternate way of checking the correspondence between \(H_B\) and \(M_B^\perp\) in the \(r \to \infty\) limit, and one which bypasses the issue of the reference term and the solution to the embedding equations altogether. (Using Section IV techniques, we have performed such a check. The calculation amounts to a tedious exercise in perturbation theory.) However, this alternate way requires that one somehow know in advance that the reference term makes no contribution to the center–of–mass limit, so the reasoning would seem circular. Furthermore, we note that our asymptotic solution to the embedding equations presented in Appendix A.2 and the analysis of Section III justify the Ansätze (106) of Section IV, ultimately showing that we have needed to use this solution and that analysis in verifying that the reference term does not contribute to the center of mass after all. By way of comparison with the last sentence of the preceding paragraph, we mention that —knowing the reference term makes no contribution— the \(k\)th center–of–mass Cartesian component is indeed \((8\pi)^{-1} \oint d\Omega 3k\nu^k\), and we find \((8\pi)^{-1} \oint d\Omega (c_{\nu\nu} + 2c_{\mu\bar{\mu}} + \delta_0 c_{\nu\bar{\mu}} + \bar{\delta}_0 c_{\nu\mu})\nu^k\) as the explicit expression for this component. Moreover, under changes in \(c_{ij}\) induced by coordinate transformations on \(h_{ij}\) of the form (34,35), we find that this unit–sphere integral is an invariant.
These considerations also highlight the difference between the Brown–York and BóM integrals. Indeed, Eqs. (58,103,104) show explicitly that even the integrand of $H_B$ in (5) vanishes to a high order in $1/r$ for trivial initial data. Actually, of course, by definition $k − k[^\text{ref}]$ is identically zero to all orders if $h_{ij}$ is a Euclidean metric and $\Sigma$ is $E^3$. However, as we have seen in Section III.F, the BóM integrand can be non-zero even for trivial data, in which case the vanishing of the full integral relies on the integration itself. This difference may well be relevant when supertranslations are brought into play, and we hope that our extremely careful treatment of the reference term will prove useful in future investigations.

VI. ACKNOWLEDGMENTS

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Note added in proof: As we have often noted, our calculations have been carried out with our more detailed version (8) of Beig–ó Murchadha fall–off [4]. Recently, Szabados has carefully analyzed the Poincare structure of asymptotically flat spacetimes and found Beig–ó Murchadha fall–off to be the weakest possible fall–off which ensures finiteness of the angular momentum and center of mass [35].

APPENDIX A: KEY IDENTITIES AND EMBEDDING EQUATIONS

1. Derivation of key identities

Let us now establish the main identities given in Eqs. (22a,22b). We begin by collecting some preliminary identities needed to get the main ones. Subtracting the two equations in
we find
\[ \Delta \sigma^{ik} = \Delta q^{ik} - n^i \Delta n^k - \Delta n^j n^k - \Delta n^i \Delta n^k, \quad (A1) \]
which upon contraction with \( n_i n_k \) yields
\[ n_i \Delta n^i = \frac{1}{2} n_i n_k (\Delta q^{ik} - \Delta \sigma^{ik}) - \frac{1}{2} \left( n_i \Delta n^i \right)^2. \quad (A2) \]
Next, contracting Eq. (A1) on \( \frac{1}{2} \sigma_{ik} \) we get
\[ \frac{1}{2} \sigma_{ik} (\Delta q^{ik} - \Delta \sigma^{ik} - \Delta n^i \Delta n^k) = 0, \quad (A3) \]
and so addition of (A3) to the RHS of (A2) gives
\[ n_i \Delta n^i = \frac{1}{2} h_{ik} \left( \Delta q^{ik} - \Delta \sigma^{ik} \right) - \frac{1}{2} h_{ik} \Delta n^i \Delta n^k. \quad (A4) \]
Equating the righthand sides of Eqs. (A2,A4), we then have
\[ n_i n_k (\Delta q^{ik} - \Delta \sigma^{ik}) = h_{ik} (\Delta q^{ik} - \Delta \sigma^{ik}) + \left( n_i \Delta n^i \right)^2 - h_{ik} \Delta n^i \Delta n^k. \quad (A5) \]
Finally, we contract (A1) on \( n_i \) and insert Eq. (A2) into the result, thereby obtaining
\[ \Delta n^k = n_i (\Delta q^{ik} - \Delta \sigma^{ik}) - \frac{1}{2} \Delta h_{ik} \Delta n^i \Delta n^k + \frac{1}{2} n_i (\Delta q^{ik} - \Delta \sigma^{ik}) \quad (A6) \]
\begin{align*}
\text{Let us now turn to the expression } (k - k). \text{ Straightaway, we have} \\
\quad k - k = -D_i \Delta n^i - n^i \Delta \Gamma^l_{li} - \Delta n^i \Delta \Gamma^l_{li}, \quad (A7)
\end{align*}
where the difference in Christoffel symbols is \( \Delta \Gamma^l_{li} = \Gamma^l_{li} - \Gamma^l_{li} \). Now, standard formulae show both that \( \Delta \Gamma^l_{li} = \frac{1}{2} h^{kl} D_i \Delta h_{kl} \) and \( h^{kl} D_k \Delta h_{kl} = -h_{il} D_k \Delta q^{kl} \), and with these we rewrite the last equation as
\[ k - k = \frac{1}{2} n^i h^{kl} (D_k \Delta h_{il} - D_i \Delta h_{kl}) - D_i \Delta n^i \\
\quad + \frac{1}{2} n^i h_{il} D_k \Delta q^{kl} - \frac{1}{2} h^{kl} \Delta n^i D_i \Delta h_{kl}. \quad (A8) \]
Focus attention on the next to last term in this equation, which we rewrite as
\[ \frac{1}{2} n^i h_{il} D_k \Delta q^{kl} = \frac{1}{2} n^i \Delta h_{il} D_k \Delta q^{kl} + \frac{1}{2} n_i D_k \Delta q^{kl}. \quad (A9) \]
An appeal to Eq. (A1) then shows that

\[
\frac{1}{2} n^i h_{il} D_k \Delta q^{kl} = \frac{1}{2} n^i \Delta h_{il} D_k \Delta q^{kl} + \frac{1}{2} n_i D_k \Delta \sigma^{ik} \\
+ \frac{1}{2} n_i D_k (\Delta n^i \Delta n^k) + n_i D_k (n^i \Delta n^k). \tag{A10}
\]

We have seen before that \( n_i \) and \( n_i \) are proportional (as both are proportional to \( \partial_i s \)) and so \( n_i \Delta \sigma^{ik} = 0 \). Hence, we may freely shift the derivative \( D_k \) off of \( \Delta \sigma^{ik} \) and onto \( n_i \). Next, for the last term on the RHS of Eq. (A10), we use \( n_i D_k n^i = 0 \) (following from the normalization of \( n^i \)) and find

\[
\frac{1}{2} n^i h_{il} D_k \Delta q^{kl} = \frac{1}{2} n^i \Delta h_{il} D_k \Delta q^{kl} - \frac{1}{2} \Delta \sigma^{ik} D_k n_i \\
+ \frac{1}{2} n_i D_k (\Delta n^i \Delta n^k) + n_i D_i (n^i \Delta n^k) + D_k \Delta n^k. \tag{A11}
\]

Finally, plugging (A11) into (A8), we have the first identity (22a).

To obtain, the second identity (22b), first simply multiply the first identity (22a) by the smearing function \( N \), thereby obtaining a new equation which has \( N n_k D_l (n^l \Delta n^k) \) as one of its RHS terms. Let us reexpress this term to get the result. Straightaway,

\[
N n_k D_l (n^l \Delta n^k) = n_k D_l (N n^l \Delta n^k) + \frac{1}{2} n_k (n^k \Delta n^l - n^l \Delta n^k) D_l N, \tag{A12}
\]

and into this equation we twice substitute (A6), thereby reaching

\[
N n_k D_l (n^l \Delta n^k) = n_k D_l (N n^l \Delta n^k) + \frac{1}{2} [n_i (\Delta q^{il} - \Delta \sigma^{il}) - n^l n_i n_k (\Delta q^{ik} - \Delta \sigma^{ik}) \\
+ n^l (n_i \Delta n^l)^2 - n_i \Delta n^i \Delta n^l] D_l N. \tag{A13}
\]

Next, appealing to identity (A5), we find

\[
N n_k D_l (n^l \Delta n^k) = n_k D_l (N n^l \Delta n^k) + \frac{1}{2} [n^k h_{ik} \Delta q^{il} - n^l h_{ik} (\Delta q^{ik} - \Delta \sigma^{ik}) \\
+ n^l h_{ik} \Delta n^i \Delta n^k - n_k \Delta n^i \Delta n^l] D_l N. \tag{A14}
\]

The last equation transforms exactly into

\[
N n_k D_l (n^l \Delta n^k) = n_k D_l (N n^l \Delta n^k) - \frac{1}{2} (n^k h^{il} \Delta h_{ik} - n^l h^{ik} \Delta h_{ik}) D_l N \\
+ \frac{1}{2} (n^l h_{ik} \Delta n^i \Delta n^k - n_i \Delta n^i \Delta n^l) D_l N + \frac{1}{2} n^l h_{ik} \Delta \sigma^{ik} D_l N, \tag{A15}
\]

and using this result in the new equation obtained from (22a) via multiplication by \( N \) we get the desired identity (22b).
2. Asymptotic solution to the embedding equations

We now show how to formally solve the system (32), although our discussion also pertains to the system (33) as well as a third system at the next order. As we pointed out in Section III.C, the three systems are formally the same. Let us rewrite (32), eliminating $f$ and making the substitution $g = \sin \theta \tilde{g}$, as

$$\sin \theta \frac{\partial \tilde{g}}{\partial \theta} - \frac{\partial h}{\partial \phi} = \alpha - \beta, \quad \text{(A16a)}$$

$$\sin \theta \frac{\partial \tilde{g}}{\partial \phi} + \sin^2 \theta \frac{\partial h}{\partial \theta} = \gamma. \quad \text{(A16b)}$$

Applying $\partial/\partial \theta$ and $(1/\sin^2 \theta) \partial/\partial \phi$ to the first and the second lines of (A16) respectively, we find upon elimination of $h$ that

$$\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \tilde{g} = \psi(\theta, \phi) \quad \text{(A17)}$$

with

$$\psi = \frac{1}{\sin \theta} \left( \frac{\partial \alpha}{\partial \theta} - \frac{\partial \beta}{\partial \phi} \right) + \frac{1}{\sin^3 \theta} \frac{\partial \gamma}{\partial \phi}. \quad \text{(A18)}$$

Whence we have transformed the system (32) to (A17), that is to say the Poisson equation

$$\Delta \tilde{g} = \psi, \quad \text{(A19)}$$

where now and from now on $\Delta$ is the Laplacian on the unit sphere. We may similarly establish that

$$\Delta h = \chi, \quad \text{(A20)}$$

with

$$\chi = \frac{1}{\sin^2 \theta} \left( \frac{\partial \beta}{\partial \phi} - \frac{\partial \alpha}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\gamma}{\sin \theta} \right). \quad \text{(A21)}$$

Solvability of these $S^2$ Poisson equations is ensured if the following compatibility conditions are satisfied:

$$\oint d\Omega \psi = 0 = \oint d\Omega \chi. \quad \text{(A22)}$$

These conditions simply state that the sources $\psi$ and $\chi$ lie in the range of the Laplacian. A careful treatment of this issue would relate these compatibility conditions to the regularity of $\alpha$, $\beta$, and $\gamma$. Indeed, in the axially symmetric case these conditions are tantamount to the statement that the $B$ metric $\sigma_{ab}$ is free of conical singularities at the north and south pole at leading and next–to–leading order. In the general case we expect that the conditions
(A22) are also related to the absence of canonical singularities at the poles. Recall that our
discussion here is also meant to address the system (33) as well as a third system at the
next order. Such compatibility conditions will also crop up when examining these systems.
We shall require whatever conditions [on the coefficients $A$, $B$, and $G$ for the system (33)
and on similar coefficients for the next order] are necessary in order that these systems are
solvable.

A solution to (A19) is formally a solution to the original system (A16). To verify that
this is indeed the case, integrate Eq. (A16a) to reach
$$h = \int d\phi \left( \sin \theta \frac{\partial \tilde{g}}{\partial \theta} + \beta - \alpha \right).$$
Substitution of this equation into Eq. (A16b) then yields
$$\int d\phi \sin^3 \theta (\Delta \tilde{g} - \psi) = 0,$$
which clearly holds for solutions to (A19).

We solve (A19) via standard methods. We expand $\psi$ with respect to spherical harmonics:
$$\psi = \sum_{l,m} \psi_{lm} Y_{lm}(\theta, \phi),$$
and are assuming that as a function $\psi$ at least lies in the space $L_2(S^2, d\Omega)$ of square integrable
functions. But this is certainly the case, as we expect that the metric coefficients $\alpha$, $\beta$, and
$\gamma$ are $C^2(S^2)$. As discussed below, $\psi$ is of even parity and, moreover, must satisfy (A22).
Therefore, the summation in (A25) is over even values of $l$ and with $\psi_{00} = 0$ as the constant
mode so that $l$ is never zero in the sum.

Search for a solution taking the form
$$\tilde{g} = \sum_{l,m} \tilde{g}_{lm} Y_{lm}(\theta, \phi).$$
Substitution of (A25) and (A26) into (A19) determines the sought-for solution as
$$\tilde{g} = -\sum_{l,m} \frac{\psi_{lm}}{l(l+1)} Y_{lm}(\theta, \phi),$$
since, of course, $\Delta Y_{lm} = -l(l+1) Y_{lm}$. Note that in this equation and the ones to follow,
we never divide by zero, since the constant mode $\psi_{00}$ vanishes. Similar steps lead to the
expansion
$$h = -\sum_{l,m} \frac{\chi_{lm}}{l(l+1)} Y_{lm}(\theta, \phi).$$
From Eq. (A27)
\[ g = -\sin \theta \sum_{l,m} \frac{\psi_{lm}}{l(l+1)} Y_{lm}(\theta, \phi). \] 
(A29)

Whence (32) gives the desired final result,
\[ f = \alpha + \sum_{l,m} \frac{\psi_{lm}}{l(l+1)} \left[ \cos \theta Y_{lm}(\theta, \phi) + \sin \theta \frac{\partial}{\partial \theta} Y_{lm}(\theta, \phi) \right]. \] 
(A30)

The solution must not destroy the approximation scheme. We expect a good solution provided that the metric coefficients \( \alpha, \beta, \) and \( \gamma \) both are \( C^2(S^2) \) and lead only to small corrections \( 3\mathcal{R}r^{-3} \) to the leading \( 2/r^2 \) behavior for the \( B \) Ricci scalar \( \mathcal{R} \).

Let us now consider the action of the parity operator \( \mathcal{P} \). Recalling its defining action \( \mathcal{P}(x^k) = -x^k \), one can show that \( \mathcal{P}(r, \theta, \phi) = (r, \pi - \theta, \phi + \pi) \) and that in (28) the terms \( \alpha \) and \( \beta \) are of the even parity while \( \gamma \) is of odd parity. Further, we can state that \( \cos \theta, \partial/\partial \phi, \psi \) from (A18), and \( \Delta \) are of even parity, while \( \sin \theta \) and \( \partial/\partial \theta \) are of odd parity. Then Eq. (A19) determines \( \mathcal{P}(\tilde{g}) = \tilde{g} \), and, consequently, \( g \) is of odd parity. In tandem with (32) this shows that \( f \) and \( h \) are of even parity. We now re-express the transformations (30) in Cartesian rather than spherical–polar form, adopting standard notation \( x^1 = x, x^2 = y, x^3 = z \), and likewise for the \( X^k \):

\[ X = x + r^{-1} \left[ xf - yh + \frac{xzg}{\sqrt{x^2 + y^2}} \right] + O(r^{-\varepsilon}), \] 
(A31a)

\[ Y = y + r^{-1} \left[ yf + xh + \frac{yzg}{\sqrt{x^2 + y^2}} \right] + O(r^{-\varepsilon}), \] 
(A31b)

\[ Z = z + r^{-1} \left[ zf - g\sqrt{x^2 + y^2} \right] + O(r^{-\varepsilon}). \] 
(A31c)

By inspection we see that all middle terms involving square brackets are \( O(r^0) \) and of odd parity. Therefore, the transformation (A31) may indeed be rewritten in the form \( X^k = x^k + 0\xi^k + O(r^{-\varepsilon}) \) with \( \xi^k \) an \( O(1) \) odd parity function of the angular variables. Via similar analysis we can in principle solve the other two systems we have mentioned [one (33) at \( O(r^{-1-\varepsilon}) \) involving \( \mathcal{A}, \mathcal{B}, \) and \( \mathcal{G} \), and the other —not written down— at \( O(r^{-2}) \)]. Whence, in more detail we find that (34) holds.

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APPENDIX B: δ OPERATOR AND FRAME AND CONNECTION

1. δ operator

Here we develop some standard formulae (drawing mostly from Refs. [29, 30]) necessary for and in the notation of this paper.

a. General two-surfaces

Suppose that $B$ is a Riemannian two-manifold equipped with metric $\sigma_{ab}$ and compatible covariant derivative operator $d_a$. Let $m^a$, with complex conjugate $\bar{m}^a$, be a complex null vector field on $B$, chosen such that $m_a\bar{m}^a = 1$. Then the set \{ $e_3 = m$, $\bar{e}_3 = \bar{m}$ \} is a complex null dyad. (We use $\hat{3}$ and $\hat{4}$ rather than $\hat{1}$ and $\hat{2}$ for the name indices, as we might assume our null dyad completes a Newman–Penrose null tetrad. [31] Were we to work with such a spacetime tetrad, we would respectively use $e_1$ and $e_2$ for the outgoing and ingoing (real) null normals to $B$.) Hatted indices $\hat{a}, \hat{b}, \ldots$ are null dyad indices, whereas $a, b, \ldots$ are general frame indices. The complex connection coefficient

$$\omega \equiv \omega_{\hat{3}\hat{3}} = e_4^a e_3^c d_c e_{\hat{3}a} = \bar{m}^a m^c d_c m_a \quad (B1)$$

is the only one which need be considered. Note that the coefficients $\omega_{\hat{a}\hat{b}\hat{c}}$ are not the connection coefficients $\Gamma_{abc}$ (which are Christoffel symbols if $a, b, \ldots$ are coordinate indices). Introduce the operator $\delta \equiv m^a d_a$. From Eq. (B1) we then have

$$\delta m^a = \omega m^a \quad (B2a)$$
$$\delta \bar{m}^a = -\omega \bar{m}^a \quad (B2b)$$

Also introduce a dyad derivative operator $\hat{d}_a$ which “sees” dyad indices, and whose action on a dyad vector $\nu^{\hat{a}}$ is defined as

$$\hat{d}_a \nu^{\hat{c}} = e_a [\nu^{\hat{c}}] + \nu^{\hat{b}} \omega^{\hat{c}}_{\hat{b}a} \quad (B3)$$

where $e_a = \partial_a$ if $a, b, \ldots$ are coordinate indices. The operator $m^a \hat{d}_a$ is known as $\tilde{\delta}$ in the compacted spin coefficient formalism. [30]

Consider a scalar defined on $B$ as follows:

$$\eta \equiv m^{a_1} \cdots m^{a_p+} \bar{m}^{c_1} \cdots \bar{m}^{c_p} T_{a_1 \cdots a_p+ c_1 \cdots c_p} \quad (B4)$$
where \( T_{a_1 \cdots a_p + s c_1 \cdots c_p} \) is a rank-\((2p+s)\) tensor field on \( B \) (note that \( s \) may be negative). Under the mapping
\[
m^a \mapsto e^{i\psi} m^a,
\]
\( \eta \) behaves as
\[
\eta \mapsto e^{is\psi} \eta.
\]
We say that \( \eta \) has \textit{spin–weight} \( s \), or symbolically \( \text{sw}(\eta) = s \). Notice that \( \eta \) is just a component
\[
T_{\hat{a}_1 \cdots \hat{a}_p + s \hat{c}_1 \cdots \hat{c}_p}
\]
of the tensor \( T_{ab \cdots d} \) with respect to the null dyad. More generally, we may define a scalar of spin–weight \( s \) by taking any rank–\( q \) covariant \( B \) tensor and contracting any \( \frac{1}{2}(q+s) \) of its indices on \( m^a \) and the remaining \( \frac{1}{2}(q-s) \) indices on \( \bar{m}^a \). For example, \( T_{\hat{a}_1 \cdots \hat{a}_p + s \hat{c}_1 \cdots \hat{c}_p} \) defines a spin–weight 1 scalar. Of course for our construction here \( \frac{1}{2}(q+s) \) should be positive and an integer. For example, with the tensor \( T_{abc} \) one can not obtain a component of spin–weight 2 or \(-2\) (components of spin–weight \(-3\), \(-1\), 1, and 3 are possible). Just for the sake of concreteness, let us continue the development of the formalism with a tensor \( T_{ab \cdots d} \) having the specific index structure given in Eq. (B4).

Now define the “eth” and “eth–bar” operators,
\[
\bar{\partial} \eta \equiv m^{a_1} \cdots m^{a_{p+s}} \bar{m}^{c_1} \cdots \bar{m}^{c_p} m^b d_b T_{a_1 \cdots a_{p+s} c_1 \cdots c_p}
\]
\[
\bar{\partial} \eta \equiv m^{a_1} \cdots m^{a_{p+s}} \bar{m}^{c_1} \cdots \bar{m}^{c_p} m^b d_b T_{a_1 \cdots a_{p+s} c_1 \cdots c_p}.
\]

By inspection we see that \( \text{sw}(\bar{\partial} \eta) = s + 1 \) and \( \text{sw}(\bar{\partial} \eta) = s - 1 \). Quick calculations using integration by parts followed by appeals to Eqs. (B2) and their complex conjugates show that
\[
\bar{\partial} \eta = \delta \eta - s \omega \eta
\]
\[
\bar{\partial} \eta = \delta \eta + s \bar{\omega} \eta.
\]
Moreover, the first equation in (B8), for example, may also be written as
\[
\bar{\partial} \eta = m^a \hat{d}_a T_{\hat{a}_1 \cdots \hat{a}_p + s \hat{c}_1 \cdots \hat{c}_p}.
\]

**Lemma.** The commutator of \( \partial \) and \( \bar{\partial} \) is
\[
(\bar{\partial} \partial - \partial \bar{\partial}) \eta = \frac{1}{2} s R \eta,
\]
where $\mathcal{R}$ is the Ricci scalar of $B$. The lemma is nothing more than the standard “Ricci identity” obeyed by covariant derivative operators. In lieu of a proof, we establish the identity for a particular illustrative example. With the tensor $T_{abc}$, build $\eta = T_{\hat{3}\hat{4}\hat{3}}$, a scalar of spin–weight 1. Write $\bar{\partial} = m^a \hat{d}_a = \hat{d}_3$ and similarly $\bar{\partial} = \bar{m}^a \hat{d}_a = \hat{d}_4$. Then we have

\[(\bar{\partial}\partial - \partial\bar{\partial})\eta = -2 \hat{d}_{[3} \hat{d}_{4]}T_{3\bar{4}\bar{3}}\]

\[= R^\hat{3}_{\bar{3}\bar{3}}T_{\bar{3}\bar{4}\bar{3}} + R^\hat{4}_{\bar{3}\bar{3}}T_{\bar{3}\bar{4}\bar{3}} + R^\hat{3}_{\bar{4}\bar{3}}T_{\bar{3}\bar{4}\bar{3}}
\]

\[= R^\bar{3}\bar{4}_{\bar{3}\bar{3}}T_{\bar{3}\bar{4}\bar{3}} + R^\bar{4}\bar{3}_{\bar{3}\bar{3}}T_{\bar{3}\bar{4}\bar{3}} + R^\bar{3}\bar{3}_{\bar{4}\bar{3}}T_{\bar{3}\bar{4}\bar{3}}, \tag{B12} \]

where $R_{\hat{a}\hat{b}\hat{c}\hat{d}}$ are the dyad components of the $B$ Riemann tensor. The dyad curvature $R_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is antisymmetric in its first (and last) pair of indices, and the Ricci scalar is $\mathcal{R} = -2R_{3\bar{4}\bar{3}}$. Hence, we indeed have $2[\bar{\partial}\partial, \partial\bar{\partial}]\eta = \frac{1}{2}\mathcal{R}\eta$.

b. Round spheres: moving frame and coordinates

Let us consider a round sphere of radius $r$ sitting in Euclidean three-space $E^3$. To highlight the fact that we are now working with a round sphere, let us use $\mu$ and $\bar{\mu}$ in place of $m$ and $\bar{m}$ for legs of the complex dyad. On $E^3$ we choose the moving frame $\{e_\nu, e_\mu, e_{\bar{\mu}}\} = \{\nu, \mu, \bar{\mu}\}$, with $\nu = \partial/\partial r$ and the complex leg $\mu^i \partial/\partial x^i = -\sqrt{\frac{1}{2}r^{-1}}e^{-i\phi}[\partial/\partial \theta + i(\sin \theta)^{-1}\partial/\partial \phi] = \sqrt{\frac{1}{2}r^{-1}}P\partial/\partial \zeta. \tag{B13}$

Here $\zeta = e^{i\phi} \cot(\theta/2)$ is the stereographic coordinate and $P = 1 + \zeta \bar{\zeta}$. These conventions agree with Dougan’s. [32] Clearly, $\mu^i$ points everywhere tangent to the foliation of $E^3$ into level–$r$ spheres. Restriction of $\mu^i$ to a particular level–$r$ sphere determines a $\mu^a$ as before (but before denoted $m^a$). With this choice of $\mu^a$ we find the following connection coefficient

\[\omega \equiv \omega_{\bar{3}\bar{4}\bar{3}} = -\sqrt{\frac{1}{2}r^{-1}}/r. \tag{B14} \]

When $B$ is the unit sphere $S^2$, we use the notation $\omega_0$ for this connection coefficient and also sometimes $\alpha^0 = \sqrt{\frac{1}{8}}\zeta$ for $-\frac{1}{2}\bar{\omega}_0$. Here the notation is seemingly odd, but we may adopt it to have agreement with the standard Newman–Penrose formalism. [31, 32] Adopting this
notation, we may consistently define
\[ \bar{\partial}_0 \eta \equiv (\delta_0 + 2s\alpha^0)\eta, \] (B15a)
\[ \tilde{\partial}_0 \eta \equiv (\bar{\delta}_0 - 2s\alpha^0)\eta, \] (B15b)
where \( \delta_0 = \sqrt{\frac{1}{2}} P \partial / \partial \zeta \). Now the commutator considered in the lemma above is
\[ (\tilde{\partial}_0 \bar{\partial}_0 - \bar{\partial}_0 \tilde{\partial}_0)\eta = s\eta, \] (B16)
since the Ricci scalar for the unit sphere is the constant function 2. Two other useful round-sphere formulae are
\[ \oint d\Omega \chi (\bar{\partial}_0 \eta) = -\oint d\Omega \eta (\bar{\partial}_0 \chi), \] where
\[ \sw(\chi) + \sw(\eta) = -1, \] and
\[ \partial_k V^k = (2V^\nu + \bar{\partial}_0 V_\mu + \tilde{\partial}_0 V_\mu) r^{-1} + \partial V^\nu / \partial r, \] where \( V^k \) is some Cartesian vector.

c. Spin–0 spherical harmonics

Let us now document some standard results concerning the unit sphere Laplacian \( \Delta \). First, when acting on scalars of zero spin–weight,
\[ 2\bar{\partial}_0 \partial_0 = \Delta \equiv \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial}{\partial \theta} \right] + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \] (B17)
The eigenfunctions of \( \Delta \) are the spherical harmonics \( Y_{lm}(\theta, \phi) \) (\( m \) runs from \(-l \) to \( l \) in integer steps), which we may also view as functions \( Y_{lm}(\zeta, \bar{\zeta}) \). These obey \( \sw(Y_{lm}) = 0 \) and \( \Delta Y_{lm} = -l(l+1)Y_{lm}. \) We may realize the spherical harmonics with the following construction.

Consider the standard stereographic projection. \[33\] Namely, given a complex number \( \zeta \), we produce a unit vector \( \nu^i = (\nu^1, \nu^2, \nu^3) \) via the formulae
\[ \nu^1 P = \zeta + \bar{\zeta}, \quad \nu^2 P = -i(\zeta - \bar{\zeta}), \quad \nu^3 P = P - 2, \] (B18)
whence we indeed have \( \delta_{ij}\nu^i \nu^j = 1 \). In these and following formulae, \( i, j, \ldots \) are \( E^3 \) Cartesian coordinate indices. Viewing \( \zeta \) as a complex coordinate on the unit sphere, we may then produce a Cartesian point \( x^i = r(\nu^1, \nu^2, \nu^3) \) given “polar coordinates” \((r, \zeta)\). The Cartesian components,
\[ \mu^i = \sqrt{\frac{1}{2}} r^{-1} P \partial x^i / \partial \zeta = \sqrt{\frac{1}{2}} P^{-1} \left( 1 - \bar{\zeta}^2, -i(1 + \bar{\zeta}^2), 2\bar{\zeta} \right), \] (B19)
of the vector (B13) are obtained via the chain rule. Now choose an arbitrary constant Cartesian vector field \( c^i \). Clearly then \( \sw(\mu^i c_i) = 1 \) and \( \sw(\nu^i c_i) = 0 \). It follows that,
viewed as functions on $S^2$, the components $\mu^i$ and $\nu^i$ have spin–weights 1 and 0 respectively. Therefore, computations based on the rules (B15) give the useful formulae

\[ \bar{\partial}_0 \nu^i = \mu^i, \quad \bar{\partial}_0 \mu^i = 0, \quad \bar{\partial}_0 \mu^i = -\nu^i. \]  

(B20)

With these formulae, it is easy to verify that $2\bar{\partial}_0 \bar{\partial}_0 \nu^i = -2\nu^i$; that is to say, the $\nu^i$ are essentially the $Y_{1m}$. Moreover, defining $f^{ij} = \nu^i \nu^j + \mu^i \bar{\mu}^j + \bar{\mu}^i \mu^j$, one can check with (B20) that

\[ 2\bar{\partial}_0 \bar{\partial}_0 (\nu^i \nu^j - \frac{1}{3} f^{ij}) = -6(\nu^i \nu^j - \frac{1}{3} f^{ij}), \]  

(B21)

thereby showing the $\nu^i \nu^j - \frac{1}{3} f^{ij}$ to be essentially the $Y_{2m}$. Naively, there are six $\nu^i \nu^j - \frac{1}{3} f^{ij}$, but only five $Y_{2m}$. However, notice that $\nu^i \nu^j - \frac{1}{3} f^{ij} = 0$ (one condition); hence, $\nu^i \nu^j - \frac{1}{3} f^{ij}$ has only five independent components as expected. As is well–known, one may continue building higher zero spin–weight harmonics as symmetric trace–free Cartesian tensors polynomial in $\nu^i$ and $f^{ij}$.

d. Spin–s spherical harmonics

Extension of the action of $\Delta \equiv 2\bar{\partial}_0 \bar{\partial}_0$ to scalars of arbitrary spin–weight is trivial, since $2\bar{\partial}_0 \bar{\partial}_0$ is defined on arbitrary scalars. Eq. (B16) shows that on an $s$ spin–weight scalar

\[ 2\bar{\partial}_0 \bar{\partial}_0 = \Delta - 2s. \]  

(B22)

The spin–s spherical harmonics $sY_{lm}$ are the $s$ spin–weight eigenfunctions of the Laplacian $\Delta$. Following Goldberg et. al. in Ref. [34], we define

\[ sY_{lm} \equiv [2^s(l - s)!/(l + s)!]^{1/2} \bar{\partial}_0^s Y_{lm} \]  

(B23a)

\[ -sY_{lm} \equiv [2^s(l - s)!/(l + s)!]^{1/2}(-1)^s \bar{\partial}_0^s Y_{lm}, \]  

(B23b)

here with the restriction $0 \leq s \leq l$. The discrepancies in factors of $2^{s/2}$ with the definitions in Ref. [34] stem from the fact that the $\bar{\partial}_0$ operator of Goldberg et. al. is $\sqrt{2}$ times our own. Notice that we are not allowed to increment the spin–weight beyond the range $-l$ to $l$ (the same restriction on the integer $m$). This is suggested by the considerations above. Indeed, the formulae in (B20) clearly show that both $\bar{\partial}_0^2$ and $\bar{\partial}_0$ annihilate $\nu^i$; that is to say, $\bar{\partial}_0^1 Y_{1m} = 0 = \bar{\partial}_0^{-1} Y_{1m}$. Furthermore, repeated use of these formulae shows that both $\bar{\partial}_0^3$
and $\bar{\partial}_0$ annihilate $\nu^i\nu^j - \frac{1}{3} f^{ij}$, our geometric representation of the $Y_{2m}$. This then implies $\partial_0 Y_{2m} = 0 = \bar{\partial}_{0-} Y_{2m}$. These concrete examples exhibit the idea behind the identities

$$\partial_0 Y_{lm} = 0 = \bar{\partial}_{0-} Y_{lm}. \quad (B24)$$

The standard identity $Y_{lm} = (-1)^m Y_{lm}$ along with the definitions of $\partial_0$ and $\bar{\partial}_0$ determines that

$$s Y_{lm} = (-1)^{m+s} Y_{l,-m}, \quad (B25)$$

now with $-l \leq s \leq l$. We have the main

**Lemma.** With the Laplacian $\Delta = 2\partial_0 \bar{\partial}_0$ and for $-l \leq s \leq l$,

$$\Delta_s Y_{lm} = [s(s+1) - l(l+1)] Y_{lm}. \quad (B26)$$

To prove the lemma, first establish the positive $s$ case via a simple induction argument, one using Eq. (B16). Next, obtain the negative $s$ case with the identities (B25) and (B22).

With the formulae collected so far, we gather the results,

$$\partial_{0s} Y_{lm} = [(l-s)(l+s+1)/2]^{1/2} Y_{lm} \quad (B27a)$$

$$\bar{\partial}_{0s} Y_{lm} = -[(l+s)(l-s+1)/2]^{1/2} Y_{lm} \quad (B27b)$$

which augment those given in Eq. (B24). Hence we may view $\partial_0$ as a kind of raising operator and $\bar{\partial}_0$ as the corresponding lowering operator.

## 2. Frame and connection

The spatial frame

$$e_\tau = \frac{1}{M} \left( \frac{\partial}{\partial r} - W^a \frac{\partial}{\partial x^a} \right) = \frac{1}{M} \left( \frac{\partial}{\partial r} - W_m m^a \frac{\partial}{\partial x^a} - W_m \bar{m}^a \frac{\partial}{\partial x^a} \right) \quad (B28a)$$

$$e_3 = m^a \frac{\partial}{\partial x^a} = \frac{1}{|m_\zeta|^2 - |m_\bar{\zeta}|^2} \left( m_\zeta \frac{\partial}{\partial \zeta} - m_\bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} \right) \quad (B28b)$$

is dual to (63). Note that $\bar{m}_\zeta = \bar{m}_\zeta$ and $m_\zeta = m_\zeta$. The basic coframe variables are then seen to be $M, W_m, m_\zeta$, and $m_{\bar{\zeta}}$.

We may now use the method of Cartan to calculate the connection coefficients $\omega_{jkl}$ determined by the co–frame (63). The method starts with the no–torsion formula

$$d e^k = -\omega^k_l \wedge e^l \quad (B29)$$
and the identities
\begin{align}
\text{dr} \wedge \text{d}\zeta &= \frac{1}{M(|m_\zeta|^2 - |m_\zeta|^2)} \left(m_\zeta e^r \wedge e^3 - \bar{m}_\zeta e^r \wedge e^4\right) \\
\text{d}\zeta \wedge \text{d}\bar{\zeta} &= \frac{1}{M(|m_\zeta|^2 - |m_\zeta|^2)} \left(M e^3 \wedge e^4 + W_m e^r \wedge e^3 - W_{\bar{m}} e^r \wedge e^4\right)
\end{align}
prove useful in carrying out the necessary calculations. In any case, the list of \(\omega_{\bar{\jmath}k\bar{i}}\) is the following: [cf. Eq. (86)]
\begin{align}
\omega_{3-+} &= -\bar{\partial} \log M \\
\omega_{i\bar{j}\bar{k}} &= \frac{1}{|m_\zeta|^2 - |m_\zeta|^2} \left(\frac{\partial m_\zeta}{\partial \zeta} - \frac{\partial m_\zeta}{\partial \zeta}\right) \\
\omega_{3-i} &= \frac{1}{2M} \left[\left(\frac{|m_\zeta|^2 - |m_\bar{\zeta}|^2}{|m_\zeta|^2 - |m_\bar{\zeta}|^2}\right)' - \bar{\partial}W_m - \bar{\partial}W_m\right] \\
\omega_{3\bar{j}3} &= \frac{1}{M(|m_\zeta|^2 - |m_\zeta|^2)} \left[m_\zeta (m_\zeta)' - m_\zeta (m_\zeta)\right] - \frac{1}{M} \bar{\partial}W_m \\
\omega_{3\bar{j}3} &= \frac{(m_\zeta)^2(m_\zeta/m_\zeta)' - (m_\zeta)^2(m_\bar{\zeta}/m_\bar{\zeta})'}{2M(|m_\zeta|^2 - |m_\bar{\zeta}|^2)} \\
&\quad + \frac{1}{M} \left(\frac{1}{2} \bar{\partial}W_m - \frac{1}{2} \bar{\partial}W_m + W_m \omega_{i\bar{j}\bar{k}} - W_{\bar{m}} \omega_{3\bar{j}3}\right),
\end{align}
where the prime denotes differentiation by \(\partial/\partial r\). Moreover, with \(\omega \equiv \omega_{i\bar{j}\bar{k}}\), the action of \(\bar{\partial}\) is that of \(e_3, e_3 - \omega, e_3 + \omega\), on \(\log M, W_m, \text{and} W_{\bar{m}}\) respectively (consistent with the defining action of \(\bar{\partial}\) given above).

Let us obtain an expression for the scalar curvature \(\mathcal{R}\) of \(B\). Denote by \(\theta^3 = \bar{m}_a dx^a\) the pullback of the co-frame \(e^3\) to a \(B\) surface, where \(d\) is the \(B\) exterior derivative (the exterior derivative of \(\Sigma\) has appeared as \(d\)). Then the connection one–form on \(B\) is
\begin{align}
\omega_{i\bar{j}\bar{k}} \theta^3 + \omega_{i\bar{j}\bar{k}} \theta^1 = \omega \theta^3 - \bar{\omega} \theta^3 .
\end{align}
In terms of the dyad components of the \(B\) Riemann tensor, the \(B\) scalar curvature is \(\mathcal{R} = -2\mathcal{R}_{i\bar{j}\bar{k}\bar{l}}\), whence we may obtain \(\mathcal{R}\) from the formula
\begin{align}
\mathcal{R}_{i\bar{j}\bar{k}\bar{l}} \theta^4 \wedge \theta^3 = \text{Im} \left[d(\omega \theta^3)\right].
\end{align}
We find
\begin{align}
\mathcal{R} = -2(e_4[\omega] + e_3[\bar{\omega}] + 2\omega \bar{\omega})
\end{align}
as the result.

**APPENDIX C: DIVERGENCE OF THE BASE HAMILTONIAN**

Assuming both the fall–off given in Eqs. (7,8) and the center–of–mass scenario with lapse given by Eq. (11), in this appendix we examine the “base Hamiltonian” $H_\Sigma$, isolating its divergent contribution in the $r \to \infty$ limit. We point out that our calculations here complement those given in Appendix C. of Ref. [4], which examined the same issue via a different method. The method we adopt here brings our Section IV results into sharper focus. Moreover, the discussion in Section V is based in part on the results of this appendix. We now consider $H_\Sigma$ off–shell; we do not assume that the $\Sigma$ initial data obeys the scalar constraint $\mathcal{H} = 0$.

1. Geometric identities and two lemmas

Before turning to $H_\Sigma$, let us first collect some geometric identities and prove two lemmas. Consider the following expression for the $\Sigma$ scalar curvature:

$$R = \mathcal{R} + k^2 - k_{ab}k^{ab} + 2D_j(kn^j + b^j). \quad (C1)$$

In this equation we may write $k^2 - k_{ab}k^{ab} = \frac{1}{2}k^2 - 2k_{mm}k_{\bar{m}\bar{m}}$ with $k = 2k_{m\bar{m}}$ as before. Moreover, here $b_j = n^kD_kn_j = -\sigma_j^kD_k\log M$, where the last equality follows from the fact that the $\Sigma$ Levi–Civita connection is torsion–free. One way to verify it is to use $n_j = M D_jr$ in $n^kD_kn_j$ and then carry through with some algebra, along the way using the identities $D_jn_j = -k$; $D_jb^j = \sigma^i_jD_i b_j + n^i n^j D_i b_j = (\partial b_m + \bar{\partial} b_m) - b_\alpha b^\alpha$, where the term within the parenthesis is the $B$ divergence.
of the $B$ vector $b^n$; and

$$n^k \partial / \partial x^k = M^{-1} \left( \partial / \partial r - W_m \bar{\partial} - W_m \bar{\sigma} \right), \quad \text{(C3)}$$

which is just Eq. (B28a). In Eq. (C2) the terms involving $k_{mm}k_{\bar{m}\bar{m}}$ and $W_m \bar{\partial} k$ are each $O(r^{-4})$ and of even parity, as shown by the asymptotic expansions (67b,76,77,78) and the parity discussion found in Section IV.B. Moreover, from Eq. (36c) we infer that

$$M^{-1} \sim 1 - M r^{-1} - (1 + \varepsilon) M r^{-1-\varepsilon} - (2 M + \text{E.P.T.}) r^{-2}, \quad \text{(C4)}$$

where we recall that $1/M$ is of even parity. These considerations and our Section IV results show that $R$ has the expansion

$$R \sim 3 \varepsilon R - 4 \left( 1 + \bar{\sigma}_0 \bar{\sigma}_0 \right) M \quad \text{(C5a)}$$

$$^{(3+\varepsilon)} R = 2 (1 - \varepsilon) (2 + \varepsilon) R - 4 \left( 1 + \bar{\sigma}_0 \bar{\sigma}_0 \right) (1 + \varepsilon) M \quad \text{(C5b)}$$

$$4 R = 4 \varepsilon R - 4 \left( 1 + \bar{\sigma}_0 \bar{\sigma}_0 \right) 2 M + \text{E.P.T.} \quad \text{(C5c)}$$

The coefficient $3 \varepsilon$ is easily seen to be of even parity via Section IV results and arguments.

We now state the lemmas. Let $\beta^\perp_{\mu} = \beta^\perp_{\nu} \nu^i$ as before. Then we have the following:

**Lemma 1.**

$$\oint d\Omega \beta^\perp_{\mu} (3+\varepsilon) R = (1 - \varepsilon) \oint d\Omega \beta^\perp_{\nu} (3+\varepsilon) \left[ \sigma^\mu\nu \sigma^{\lambda\rho} \mathcal{R}_{\lambda\rho\mu\nu} \right] \quad \text{(C6)}$$

**Lemma 2.**

$$\oint d\Omega \beta^\perp_{\mu} 4 R = 0 \quad \text{(C7)}$$

To prove Lemma 1, first use Eqs. (96,100a,C5b) to obtain

$$\oint d\Omega \beta^\perp_{\mu} (3+\varepsilon) R = (1 - \varepsilon) \oint d\Omega \beta^\perp_{\nu} (3+\varepsilon) \left[ \sigma^{ij} \sigma^{kl} R_{ijkl} \right]$$

$$+ \oint d\Omega \beta^\perp_{\nu} \left[ (3+\varepsilon) \mathcal{R} - 4 \left( 1 + \bar{\sigma}_0 \bar{\sigma}_0 \right) (1+\varepsilon) M \right] \quad \text{(C8)}$$

and then with Eq. (98) replace the $\Sigma$ Riemann tensor on the RHS with the spacetime Riemann tensor. The second integral on the RHS vanishes. Indeed, double use of angular integration by parts along with Eq. (B20) shows that the unit–sphere average of $\beta^\perp_{\nu} (1 + \bar{\sigma}_0 \bar{\sigma}_0) (1+\varepsilon) M$ is zero. Moreover, the unit–sphere average of $\beta^\perp_{\nu} (3+\varepsilon) \mathcal{R}$ is also zero. Indeed, explicitly

$$(3+\varepsilon) \mathcal{R} = \bar{\sigma}_0^2 b_{\mu\bar{\mu}} + \bar{\sigma}_0^2 b_{\mu\mu} - 2 \bar{\sigma}_0 \bar{\sigma}_0 b_{\mu\bar{\mu}} - 2 b_{\mu\bar{\mu}} \quad \text{(C9)}$$
that is to say the result (79) only with $a_{ij}$ replaced by $b_{ij}$. This result can be verified with the techniques of Section IV. Angular integration by parts along with (B20) again shows that the average in question vanishes. (Note, however, that the unit–sphere average of $(3+\varepsilon)R$ need not vanish. The discussion in Section V of the mass–aspect for the energy scenario rests on the fact that the same is true for $3R$.) As for LEMMA 2, the same arguments apply, because we expect $4^4R$ to have the form

$$4^4R = \bar{\partial}_0^2c_{\mu\bar{\mu}} + \bar{\partial}_0^2c_{\mu\bar{\mu}} - 2\bar{\partial}_0\bar{\partial}_0c_{\mu\bar{\mu}} - 2c_{\mu\bar{\mu}} + E.P.T.$$  \hspace{0.5cm} (C10)

This result should follow from Eq. (C9) upon setting $\varepsilon = 1$ and including quadratic even–parity corrections.

2. Divergence of the base Hamiltonian

Let us now recall that $H_\Sigma$ is the volume integral given in Eq. (6), where

$$\mathcal{H} = \sqrt{h} \frac{16\pi}{16\pi} \left( K_{ij}K^{ij} - K^2 - R \right).$$  \hspace{0.5cm} (C11)

In what follows, let us assume for simplicity that the slice $\Sigma$ is topologically Cartesian three–space $R^3$, that we may ignore the issue of inner boundary terms. Upon integration against the product of the $\Sigma$ volume element $d^3x\sqrt{h} = d^2xd\sqrt{\sigma M}$ and lapse $N$, the terms in (C11) which are quadratic in the $\Sigma$ extrinsic curvature tensor $K_{ij}$ do not contribute to the limit. Indeed, these terms are $O(r^{-4})$ and of leading even parity, the volume element is $O(r^2)$ and of leading even parity, while $N$ is $O(r)$ and of course of leading odd parity. Then the product of all of these terms is $O(r^{-1})$ and upon integration over the radial coordinate would yield a logarithmic divergence were it not for the fact that this product is of leading odd parity.

Let us then focus on the term

$$-\frac{1}{16\pi} \int d^3x\sqrt{h}NR \sim \frac{1}{16\pi} \int d\Omega drr^3 \beta_{\mu}^{\perp}R,$$  \hspace{0.5cm} (C12)

where we have used Eqs. (11,36c,40) and the parity properties of leading terms to isolate on the RHS only those terms from the LHS which lead or could lead to an infinite limit. Next, we insert the radial expansion for $R$ and integrate in $r$ term–by–term, thereby reaching

$$-\frac{1}{16\pi} \int d^3x\sqrt{h}NR \sim \frac{1}{16\pi} \int d\Omega drr^3 \beta_{\mu}^{\perp}R - \frac{r^{1-\varepsilon}}{16\pi(1-\varepsilon)} \int d\Omega \beta_{\mu}^{\perp}(3+\varepsilon)R - \frac{\log r}{16\pi} \int d\Omega \beta_{\mu}^{\perp}4R.$$  \hspace{0.5cm} (C13)
The first term on the RHS vanishes due to the even parity of $3R$. These calculations, the argument of the previous paragraph, and the lemmas above then show that

$$H_\Sigma \sim -\frac{r^{1-\varepsilon}}{16\pi} \oint d\Omega \beta^\perp \nu (3+\varepsilon)[\sigma^{\mu\nu}\sigma^{\lambda\kappa}\mathcal{R}_{\mu\lambda\nu\kappa}] + 0 \cdot \log r.$$

(C14)

Note that the coefficient of $\log r$ has been found to vanish both by parity arguments and Lemma 2. Because the first divergent term in Eq. (C14) has the opposite sign from the corresponding term in the main text [cf. Eqs. (5,11,60)], we see that the full Hamiltonian $H_\Sigma + H_B$ is finite even off–shell. Whence $H_B$ must be finite on–shell.

Finally, as an aside we note that the results of this appendix show

$$M_{\infty}^{ik} = -\frac{1}{8\pi} \oint d\Omega \left[n^i n^j R_{ij}\right] \nu^k$$

(C15)

to be equivalent to Eq. (16) as a definition of center–of–mass coordinates for initial data sets. This expression in tandem with the Gauß–Codazzi–Mainardi splitting of the spacetime Ricci tensor $\mathcal{R}_{\mu\nu}$ might be used to derive other equivalent expressions for center of mass.

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