Bayes Estimation under Conjugate Prior for the Case of Laplace Double Exponential Distribution

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Abstract

The Bayesian estimation approach is a non-classical device in the estimation part of statistical inference which is very useful in real world situation. The main objective of this paper is to study the Bayes estimators of the parameter of Laplace double exponential distribution. In Bayesian estimation loss function, prior distribution and posterior distribution are the most important ingredients. In real life we try to minimize the loss and want to know some prior information about the problem to solve it accurately. The well known conjugate priors are considered for finding the Bayes estimator. In our study we have used different symmetric and asymmetric loss functions such as squared error loss function, quadratic loss function, modified linear exponential (MLINEX) loss function and non-linear exponential (NLINEX) loss function. The performance of the obtained estimators for different types of loss functions are then compared among themselves as well as with the classical maximum likelihood estimator (MLE). Mean Square Error (MSE) of the estimators are also computed and presented in graphs.

Keywords: Squared Error Loss Function; Modified Linear Exponential Loss Function; Non-Linear Exponential Loss Function; Maximum Likelihood Estimator; Bayes Estimator Under Quadratic Loss Function.
1. Introduction

The Laplace distribution is a continuous probability distribution named after Pierre-Simon Laplace. It is also sometimes called the double exponential distribution, because it can be thought of as two exponential distributions (with an additional location parameter) spliced together back-to-back. The Laplace distribution can be used to model various real world problems. The distribution can find most interesting and successful application in modeling of financial data such as to model growth rates, stock prices, annual gross domestic production, interest and
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forex rates [1]. It has generally two parameters. One is location parameter and other is scale parameter. Here only scale parameter is considered to estimate.

A continuous random variable $X$ is said to have a generalised Laplace double exponential distribution if its p.d.f is given by [2].

$$f(X; \theta, \lambda) = \frac{1}{2\lambda} e^{-\frac{|x-\theta|}{\lambda}} \quad ; -\infty < x < \infty, \lambda > 0, -\infty < \theta < \infty$$

(1)

where $\lambda$ is the scale parameter and $\theta$ is the location parameter.

1.1. Prior and Posterior Density Function

For Bayesian estimation we need to specify a prior and posterior distribution for the parameter. We have from equation (1)

$$f(X; \theta, \lambda) = \frac{1}{2\lambda} e^{-\frac{|x-\theta|}{\lambda}}$$

For convenience replacing $\frac{1}{\lambda}$ by $p$, we get

$$f(x; \theta, p) = \frac{p}{2} e^{-p|x-\theta|} \quad ; p, x > 0$$

(2)

Consider a Gamma prior for $p$ having pdf

$$g(p) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta p} p^{\alpha-1} \quad ; \alpha, \beta, p > 0$$

(3)

Now the Posterior density function of $p$ for the given random sample $X$ is given by [3]

$$f(p|x) = \frac{\left(\frac{p}{2}\right)^n e^{-p|X-\theta|} e^{-\beta p} p^{\alpha-1}}{\int e^{-p|X-\theta|} e^{-\beta p} p^{\alpha-1} dp}$$

$$= \frac{e^{-p|X-\theta|+\beta}}{\int e^{-p|X-\theta|+\beta} p^{\alpha+n-1} dp}$$
\[ f(p|x) = \frac{[\sum|x-\theta|+\beta]^{\alpha+n}}{\Gamma(\alpha+n)} e^{-p[\sum|x-\theta|+\beta]} p^{\alpha+n-1} \]  

Which implies that \( f(p|x) \sim G(\alpha + n, \sum|x-\theta| + \beta) \), since prior and posterior distribution belongs to the same family hence the prior is conjugate prior.

2. Different Estimators of Parameter \( \lambda \)

In this section Bayes estimators of parameter \( \lambda \) for different loss functions along with maximum likelihood estimator have been determined.

2.1 MLE OF \( \lambda \)

Let \( X = (X_1, X_2, ..., X_n) \) be a random sample of size \( n \) drawn from the Laplace double exponential distribution defined in (1). Then the likelihood function of the parameter \( \lambda \) for the random sample \( X \) is given by [3]

\[ L(\lambda|X) = \prod_{i=1}^{n} f(x_i|\lambda) \]

\[ \Rightarrow L(\lambda|X) = \left(\frac{1}{2\lambda}\right)^n e^{-\sum_{i=1}^{n} \frac{|x_i-\theta|}{\lambda}} \]  

Taking log on both sides of (5) we get

\[ \log L(\lambda|X) = -n \log(2\lambda) - \sum_{i=1}^{n} \frac{|x_i-\theta|}{\lambda} \]

\[ \Rightarrow \log L(\lambda|X) = -n \log 2 - n \log \lambda - \sum_{i=1}^{n} \frac{|x_i-\theta|}{\lambda} \]

MLE of \( \lambda \) will be the solution of the equation

\[ \frac{d \log L(\lambda|X)}{d\lambda} = 0 \]

\[ \Rightarrow -n \frac{1}{\lambda} + \sum_{i=1}^{n} \frac{|x_i-\theta|}{\lambda^2} = 0 \]
\[ \Rightarrow -\frac{n\lambda + \sum |x_i - \theta|}{\lambda^2} = 0 \]
\[ \Rightarrow n\lambda = \sum |x_i - \theta| \]

Hence \( \hat{\lambda}_{MLE} = \frac{\sum |x_i - \theta|}{n} \), is the MLE of \( \lambda \) where \( \theta \) known.

### 2.2. Bayes Estimator of \( \lambda \) for Squared Error Loss Function

Here we have determined Bayes estimator of \( \lambda \) for squared error loss function defined as [4]

\[ L(t; p) = (t - p)^2 \]

(6)

For squared error loss function Bayes estimator is the mean of posterior density function. From (4) posterior density function is a Gamma distribution with parameter \( (\alpha + n) \) and \( (\sum |x - \theta| + \beta) \). Hence the mean of posterior density function is \( \frac{(\alpha + n)}{(\sum |x - \theta| + \beta)} \). Therefore the Bayes estimator of \( p \) is given by

\[ \hat{p}_{BSE} = \frac{\alpha + n}{\sum |x - \theta| + \beta} \]

We have \( \frac{1}{\lambda} = p \), hence \( \hat{\lambda}_{BSE} = \frac{1}{\hat{p}_{BSE}} = \frac{\sum |x - \theta| + \beta}{\alpha + n} \), is the Bayes estimator of parameter \( \lambda \) under SE loss function.

### 2.3. Bayes Estimator of \( \lambda \) for Quadratic Loss Function

Now suppose the loss function is quadratic, which is defined as [5]

\[ L(t; p) = \left( \frac{t - p}{p} \right)^2 \]

(7)
Under quadratic loss function Bayes estimator of $p$ is obtained by solving the following equation

$$\frac{d}{dt} \int \left(\frac{t-p}{p}\right)^2 f(p|x)dp = 0$$

$$\Rightarrow \int \frac{2(t-p)}{p^2} f(p|x)dp = 0$$

$$\Rightarrow \frac{\sum[x_i - \theta + \beta]^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-p[\sum_i x_i - \theta + \beta]} p^{\alpha+n-2-1} dp =$$

$$\frac{\sum[x_i - \theta + \beta]^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-p[\sum_i x_i - \theta + \beta]} p^{\alpha+n-1-1} dp$$

$$\Rightarrow t = \frac{\Gamma(\alpha+n-2)}{\sum[x_i - \theta + \beta]^{\alpha+n-2}} \frac{\Gamma(\alpha+n-1)}{\sum[x_i - \theta + \beta]^{\alpha+n-1}}$$

$$\Rightarrow t = \frac{(\alpha+n-2)}{\sum[x_i - \theta + \beta]} \frac{\sum[x_i - \theta + \beta]}{\sum[x_i - \theta + \beta]}$$

$$\Rightarrow \hat{p}_{BQL} = \frac{(\alpha+n-2)}{\sum[x_i - \theta + \beta]}$$

Since we have $\frac{1}{\lambda} = p$, then $\hat{\lambda}_{BQL} = \frac{\sum[x_i - \theta + \beta]}{(\alpha+n-2)}$, is the Bayes estimator of parameter $\lambda$ under quadratic loss function.

### 2.4. Bayes Estimator of $\lambda$ for MLINEX Loss Function

Now let us consider the MLINEX loss function defined as [5]

$$L(\hat{p}; p) = w \left[ \left(\frac{\hat{p}}{p}\right)^c - \log \left(\frac{\hat{p}}{p}\right) - 1 \right], w>0, c\neq 0$$

For MLINEX loss function Bayes estimator of $p$ is obtained from [5]

$$\hat{p}_{BML} = \left[ E(p^{-c}|x) \right]^{\frac{-1}{c}}$$

Here $E(p^{-c}) = \int_0^\infty p^{-c} f(p|x)$
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\[ = \frac{\sum |x_i - \theta| + \beta}{\Gamma(\alpha + n)} \int_0^\infty e^{-p[\sum |x_i - \theta| + \beta]} p^{\alpha + n - c - 1} \, dp \]

\[ = \frac{\sum |x_i - \theta| + \beta}{\Gamma(\alpha + n - c)} \int_0^\infty \frac{\Gamma(\alpha + n - c)}{[\sum |x_i - \theta| + \beta]^{\alpha + n - c}} \, dp \]

\[ \Rightarrow E(p^{-c}) = \frac{\Gamma(\alpha + n - c)}{\Gamma(\alpha + n)} \left[ \sum |x_i - \theta| + \beta \right]^c \]

Therefore from (9) we get \( \hat{p}_{BML} = \left[ \frac{\Gamma(\alpha + n - c)}{\Gamma(\alpha + n)} \right]^{-\frac{1}{c}} \frac{1}{[\sum |x_i - \theta| + \beta]} \).

We have \( \frac{1}{\lambda} = p \), hence \( \hat{\lambda}_{BML} = \left[ \frac{\Gamma(\alpha + n - c)}{\Gamma(\alpha + n)} \right]^{-\frac{1}{c}} \left[ \sum |x_i - \theta| + \beta \right], \) is the Bayes estimator of parameter \( \lambda \) under MLINEX loss function.

### 2.5 Bayes Estimator of \( \lambda \) for NLINEX Loss Function

Let us consider the following NLINEX loss function of the form [6]

\[ L(D) = k \left[ \exp(c D) + c D^2 - cD - 1 \right], \quad k > 0, \ c > 0 \] (10)

where \( D \) represents the estimation error.i.e. \( D = p - p \)

For NLINEX loss function Bayes estimator of \( p \) is [1]

\[ \hat{p}_{BNL} = -\left[ \ln E_p\{\exp(-cp)\} - 2E_p(p) \right] / (c + 2) \] (11)

where \( E_p(.) \) stands for posterior expectation.

Now, \( E_p\{\exp(-cp)\} = \int_0^\infty e^{-cp} f(p|x) \, dp \)

\[ = \frac{[\sum |x_i - \theta| + \beta]^{\alpha + n}}{\Gamma(\alpha + n)} \int_0^\infty e^{-p[c + \sum |x_i - \theta| + \beta]} p^{\alpha + n - 1} \, dp \]

\[ = \frac{[\sum |x_i - \theta| + \beta]^{\alpha + n}}{[c + \sum |x_i - \theta| + \beta]^{\alpha + n}} \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + n)} \]

\[ = \left( 1 + \frac{c}{[\sum |x_i - \theta| + \beta]} \right)^{-(\alpha + n)} \]

Hence \( \ln E_p\{\exp(-cp)\} = -(\alpha + n) \ln \left( 1 + \frac{c}{[\sum |x_i - \theta| + \beta]} \right) \) (12)

Again \( E_p(p) = \int_0^\infty p \, f(p/x) \, dp \)
\[
\begin{align*}
&= \frac{[\Sigma x_i - \theta + \beta]^{\alpha+n}}{\Gamma(\alpha+n)} \int_0^\infty e^{-p[\Sigma x_i - \theta + \beta]} p^{\alpha+n+1} \, dp \\
&= \frac{[\Sigma x_i - \theta + \beta]^{\alpha+n}}{\Gamma(\alpha+n+1)} \\
\Rightarrow \quad E_p(p) &= \frac{(\alpha+n)}{[\Sigma x_i - \theta + \beta]} \\
\end{align*}
\]

Using (12) and (13) in (11) we get
\[
\hat{\lambda}_{BNL} = -[-(\alpha + n) \ln \left(1 + \frac{c}{[\Sigma x_i - \theta + \beta]}\right) - 2 \frac{(\alpha+n)}{[\Sigma x_i - \theta + \beta]}] / (c + 2)
\]
\[
\Rightarrow \hat{\lambda}_{BNL} = (\alpha + n) \left[\ln \left(1 + \frac{c}{[\Sigma x_i - \theta + \beta]}\right) + \frac{2}{[\Sigma x_i - \theta + \beta]}\right] / (c + 2)
\]
We have \( \frac{1}{\lambda} = \rho \), hence \( \hat{\lambda}_{BNL} = \frac{\lambda}{(\alpha+n)} \left[\ln \left(1 + \frac{c}{[\Sigma x_i - \theta + \beta]}\right) + \frac{2}{[\Sigma x_i - \theta + \beta]}\right] \), is the Bayes estimator of parameter \( \lambda \) under NLINEX loss function.

3. Empirical Study

To compare the estimators \( \hat{\lambda}_{MLE}, \hat{\lambda}_{BSE}, \hat{\lambda}_{BQL}, \hat{\lambda}_{BML}, \) and \( \hat{\lambda}_{BNL} \) we have considered the MSE of the estimators. The MSE of an estimator \( \lambda \) is defined as
\[
\text{MSE (\hat{\lambda})} = E \left[ (\lambda - \hat{\lambda})^2 \right] = \text{Var (\hat{\lambda})} + \text{Bias (\hat{\lambda})}^2
\]
To obtain the variance of \( \hat{\lambda} \), we have used the true value of the parameter \( \lambda \) under consideration. We have obtained the estimated value and MSE of the estimator by using the Monte Carlo simulation method [7] from the Laplace double exponential
distribution. Five thousand samples have taken for each case. The results and their graphs are presented below.

Table 1. Estimated value and MSE of different estimators of the parameter $\lambda$ of Laplace double exponential distribution when $\alpha=1$, $\beta=2$, $\theta=1$, $\lambda=1$ and $c=1$

| n  | Criteria | $\hat{\lambda}_{MLE}$ | $\hat{\lambda}_{BSE}$ | $\hat{\lambda}_{BQL}$ | $\hat{\lambda}_{BML}$ | $\hat{\lambda}_{BNL}$ |
|----|----------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 5  | Estimated value | 0.6989 | 0.9150 | 1.3760 | 1.0867 | 0.9386 |
|    | MSE      | 0.2181 | 0.0917 | 0.3446 | 0.1292 | 0.0890 |
| 10 | Estimated value | 0.6928 | 0.8118 | 0.9893 | 0.8901 | 0.8275 |
|    | MSE      | 0.1563 | 0.0860 | 0.0753 | 0.0723 | 0.0816 |
| 15 | Estimated value | 0.6943 | 0.7716 | 0.8896 | 0.8208 | 0.7826 |
|    | MSE      | 0.1359 | 0.0874 | 0.0614 | 0.0720 | 0.0817 |
| 20 | Estimated value | 0.6930 | 0.7601 | 0.8293 | 0.7919 | 0.7633 |
|    | MSE      | 0.1251 | 0.0859 | 0.0616 | 0.0736 | 0.0840 |
| 25 | Estimated value | 0.6933 | 0.7442 | 0.8074 | 0.7747 | 0.7485 |
|    | MSE      | 0.1180 | 0.0881 | 0.0643 | 0.0760 | 0.0854 |
| 30 | Estimated value | 0.6945 | 0.7371 | 0.7866 | 0.7598 | 0.7445 |
|    | MSE      | 0.1137 | 0.0888 | 0.0674 | 0.0788 | 0.0843 |

From the above table it is seen that the MSE of $\hat{\lambda}_{MLE}$ is largest for almost all cases except for sample size 5. On the other hand, among the non-classical estimators $\hat{\lambda}_{BQL}$ shows smallest MSE for different sample size only, it is the highest when sample size is 5 (Figure-1).

**Figure 1.** Graph of MSEs of different estimates of parameter $\lambda$ of Laplace double exponential distribution when $\alpha=1$, $\beta=2$, $\theta=1$, $\lambda=1$ and $c=1$
Table 2. Estimated value and MSE of different estimators of the parameter $\lambda$ of Laplace double exponential distribution when $\alpha = 1.5$, $\beta = 2$, $\theta = 1$, $\lambda = 1.5$ and $c = 1$

| $n$ | Criteria | $\hat{\lambda}_{MLE}$ | $\hat{\lambda}_{BSE}$ | $\hat{\lambda}_{BQL}$ | $\hat{\lambda}_{BML}$ | $\hat{\lambda}_{BNL}$ |
|-----|-----------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 5   | Estimated value MSE | 1.0483 | 1.1130 | 1.6093 | 1.3154 | 1.1402 |
|     | Estimated value MSE | 0.4907 | 0.3660 | 0.3660 | 0.2605 | 0.3029 |
| 10  | Estimated value MSE | 1.0370 | 1.0776 | 1.3048 | 1.1767 | 1.0880 |
|     | Estimated value MSE | 0.3526 | 0.2839 | 0.1909 | 0.2287 | 0.2724 |
| 15  | Estimated value MSE | 1.0415 | 1.0679 | 1.2101 | 1.1373 | 1.0641 |
|     | Estimated value MSE | 0.3042 | 0.2670 | 0.1836 | 0.2196 | 0.2633 |
| 20  | Estimated value MSE | 1.0643 | 1.0624 | 1.1575 | 1.1123 | 1.0723 |
|     | Estimated value MSE | 0.2509 | 0.2523 | 0.1892 | 0.2152 | 0.2437 |
| 25  | Estimated value MSE | 1.0387 | 1.0554 | 1.1390 | 1.1002 | 1.0625 |
|     | Estimated value MSE | 0.2675 | 0.2461 | 0.1864 | 0.2133 | 0.2383 |
| 30  | Estimated value MSE | 1.0418 | 1.0575 | 1.1247 | 1.0890 | 1.0597 |
|     | Estimated value MSE | 0.2558 | 0.2384 | 0.1896 | 0.2129 | 0.2355 |

Table 2 represents largest value of MSE for $\hat{\lambda}_{MLE}$ in all cases. It is also clear from table 2 that MSE of $\hat{\lambda}_{BQL}$ is smallest than others estimators for different sample size (figure 2).

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**Figure 2.** Graph of MSEs of different estimates of parameter $\lambda$ of Laplace double exponential distribution when $\alpha = 1.5$, $\beta = 2$, $\theta = 1$, $\lambda = 1.5$ and $c = 1$
Table 3. Estimated value and MSE of different estimators of the parameter \( \lambda \) of Laplace double exponential distribution when \( \alpha = 0.5, \beta = 1, \theta = -1, \lambda = 2 \) and \( c = 2 \)

| n  | Criteria | \( \hat{\lambda}_{MLE} \) | \( \hat{\lambda}_{BSE} \) | \( \hat{\lambda}_{BQL} \) | \( \hat{\lambda}_{BML} \) | \( \hat{\lambda}_{BNL} \) |
|-----|----------|----------------|----------------|----------------|----------------|----------------|
| 5   | Estimated value | 1.3955 | 1.4312 | 2.2456 | 2.0144 | 1.5344 |
|     | MSE      | 0.8551 | 0.7389 | 1.0067 | 0.8070 | 0.6539 |
| 10  | Estimated value | 1.3755 | 1.4259 | 1.7602 | 1.6564 | 1.4640 |
|     | MSE      | 0.6379 | 0.5558 | 0.3879 | 0.4280 | 0.5118 |
| 15  | Estimated value | 1.3784 | 1.4179 | 1.6143 | 1.5547 | 1.4390 |
|     | MSE      | 0.5448 | 0.4967 | 0.3590 | 0.3907 | 0.4735 |
| 20  | Estimated value | 1.3808 | 1.4032 | 1.5546 | 1.5022 | 1.4290 |
|     | MSE      | 0.5023 | 0.4767 | 0.3433 | 0.3847 | 0.4458 |
| 25  | Estimated value | 1.3861 | 1.3981 | 1.5135 | 1.4873 | 1.4190 |
|     | MSE      | 0.4743 | 0.4561 | 0.3460 | 0.3675 | 0.4289 |
| 30  | Estimated value | 1.3869 | 1.3995 | 1.4951 | 1.4702 | 1.4165 |
|     | MSE      | 0.4558 | 0.4403 | 0.3479 | 0.3662 | 0.4198 |

Table 3. shows the variation in the performance of the estimator for different sample size. More or less similar pattern are observed here as previous tables that is MSE of \( \hat{\lambda}_{MLE} \) is higher than all other estimators. MSE of \( \hat{\lambda}_{BQL} \) is least in the class of Bayes estimators for increasing size of samples (Figure 3).

Figure 3. Graph of MSEs of different estimates of parameter \( \lambda \) of Laplace double exponential distribution when \( \alpha = 0.5, \beta = 1, \theta = -1, \lambda = 2 \) and \( c = 2 \)
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Table 4. Estimated value and MSE of different estimators of the parameter \( \lambda \) of Laplace double exponential distribution when \( \alpha =1, \beta =1, \theta =1, \lambda =1 \) and \( c=1 \)

| n | Criteria | \( \hat{\lambda}_{MLE} \) | \( \hat{\lambda}_{BSE} \) | \( \hat{\lambda}_{BQL} \) | \( \hat{\lambda}_{BML} \) | \( \hat{\lambda}_{BNL} \) |
|---|---|---|---|---|---|---|
| 5 | Estimated value | 0.6867 | 0.7491 | 1.1225 | 0.9008 | 0.7714 |
| | MSE | 0.2198 | 0.1514 | 0.2053 | 0.1398 | 0.1377 |
| 10 | Estimated value | 0.6930 | 0.7223 | 0.8779 | 0.7903 | 0.7349 |
| | MSE | 0.1550 | 0.1288 | 0.0893 | 0.1048 | 0.1216 |
| 15 | Estimated value | 0.6943 | 0.7134 | 0.8104 | 0.7629 | 0.7180 |
| | MSE | 0.1350 | 0.1195 | 0.0820 | 0.0980 | 0.1142 |
| 20 | Estimated value | 0.6943 | 0.7067 | 0.7814 | 0.7434 | 0.7142 |
| | MSE | 0.1240 | 0.1126 | 0.0819 | 0.0966 | 0.1091 |
| 25 | Estimated value | 0.6922 | 0.7094 | 0.7634 | 0.7337 | 0.7134 |
| | MSE | 0.1184 | 0.1081 | 0.0834 | 0.0966 | 0.1050 |
| 30 | Estimated value | 0.6919 | 0.7048 | 0.7529 | 0.7305 | 0.7100 |
| | MSE | 0.1153 | 0.1063 | 0.0821 | 0.0936 | 0.1032 |

For different sample size table 4 also shows minimum values of MSE for \( \hat{\lambda}_{BQL} \). On the other hand \( \hat{\lambda}_{MLE} \) keep its tradition as previous cases (Figure 4).
Table 5. Estimated value and MSE of different estimators of the parameter $\lambda$ of Laplace double exponential distribution when $\alpha=2$, $\beta=2$, $\theta=-2$, $\lambda=1.5$ and $c=2$

| n  | Criteria     | $\hat{\lambda}_{MLE}$ | $\hat{\lambda}_{BSE}$ | $\hat{\lambda}_{BQL}$ | $\hat{\lambda}_{BML}$ | $\hat{\lambda}_{BNL}$ |
|----|--------------|-------------------------|-------------------------|-------------------------|-------------------------|-------------------------|
| 5  | Estimated value | 1.0473 | 1.0289 | 1.4422 | 1.3110 | 1.0926 |
|    | MSE          | 0.4826 | 0.3674 | 0.2846 | 0.2620 | 0.3120 |
| 10 | Estimated value | 1.0398 | 1.0395 | 1.2363 | 1.1876 | 1.0753 |
|    | MSE          | 0.3554 | 0.3097 | 0.2055 | 0.2225 | 0.2772 |
| 15 | Estimated value | 1.0384 | 1.0369 | 1.1676 | 1.1345 | 1.0587 |
|    | MSE          | 0.3058 | 0.2856 | 0.2020 | 0.2201 | 0.2646 |
| 20 | Estimated value | 1.0396 | 1.0366 | 1.1369 | 1.1131 | 1.0583 |
|    | MSE          | 0.2818 | 0.2727 | 0.2005 | 0.2152 | 0.2524 |
| 25 | Estimated value | 1.0342 | 1.0399 | 1.1250 | 1.0963 | 1.0545 |
|    | MSE          | 0.2716 | 0.2612 | 0.1968 | 0.2165 | 0.2457 |
| 30 | Estimated value | 1.0401 | 1.0401 | 1.1073 | 1.0881 | 1.0558 |
|    | MSE          | 0.2585 | 0.2504 | 0.1991 | 0.2135 | 0.2386 |

Table 5 gives smaller values of MSE for $\hat{\lambda}_{BQL}$ than all other estimators in the study and in some cases it is very near to that of $\hat{\lambda}_{BML}$ (figure 5).

Figure 5. Graph of MSEs of different estimates of parameter $\lambda$ of Laplace double exponential distribution when $\alpha=2$, $\beta=2$, $\theta=-2$, $\lambda=1.5$ and $c=2$
The performance of the estimators for different values of parameter theta (θ), alpha (α), Beta (β), are also shown in the subsequent tables along with their graphical presentation.

Table 6. Estimated value and MSE of different estimators of the parameter λ of Laplace double exponential distribution for different values of θ and n=10, α =1, β=1, λ =1 and c=1

| θ  | Criteria | $\hat{\lambda}_{MLE}$ | $\hat{\lambda}_{BSE}$ | $\hat{\lambda}_{BQL}$ | $\hat{\lambda}_{BML}$ | $\hat{\lambda}_{BNL}$ |
|----|----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| -1 | Estimated value | 0.6909 | 0.7173 | 0.8820 | 0.7901 | 0.7378 |
|    | MSE | 0.1539 | 0.1330 | 0.0906 | 0.1050 | 0.1202 |
| -1.5 | Estimated value | 0.6903 | 0.7212 | 0.8809 | 0.7905 | 0.7380 |
|    | MSE | 0.1560 | 0.1277 | 0.0885 | 0.1027 | 0.1191 |
| -2 | Estimated value | 0.6953 | 0.7206 | 0.8870 | 0.8009 | 0.7329 |
|    | MSE | 0.1529 | 0.1265 | 0.0917 | 0.1038 | 0.1238 |
| 1 | Estimated value | 0.6951 | 0.7283 | 0.8825 | 0.7918 | 0.7376 |
|    | MSE | 0.1558 | 0.1250 | 0.0870 | 0.1040 | 0.1207 |
| 1.5 | Estimated value | 0.6889 | 0.7185 | 0.8806 | 0.7909 | 0.7409 |
|    | MSE | 0.1546 | 0.1289 | 0.0895 | 0.1061 | 0.1201 |
| 2 | Estimated value | 0.6909 | 0.7185 | 0.8815 | 0.8003 | 0.7354 |
|    | MSE | 0.1557 | 0.1297 | 0.0902 | 0.1019 | 0.1209 |

Table 6. also shows smaller MSE of $\hat{\lambda}_{BQL}$ for different values of θ (figure 6).
Table 7. Estimated value and MSE of different estimators of the parameter $\lambda$ of Laplace double exponential distribution for different values of $\alpha$ and $n=15, \theta=2, \beta=2, \lambda=1.5, c=2$

| $\alpha$ | Criteria | $\hat{\lambda}_{MLE}$ | $\hat{\lambda}_{BSE}$ | $\hat{\lambda}_{BQL}$ | $\hat{\lambda}_{BML}$ | $\hat{\lambda}_{BNL}$ |
|--------|----------|-------------------------|------------------------|------------------------|------------------------|------------------------|
| 0.5    | Estimated value MSE | 1.0415 | 1.1284 | 1.3084 | 1.2582 | 1.1703 |
|        | Estimated value MSE | 0.3027 | 0.2236 | 0.1524 | 0.1670 | 0.1976 |
| 1      | Estimated value MSE | 1.0411 | 1.1034 | 1.2530 | 1.2063 | 1.1207 |
|        | Estimated value MSE | 0.3028 | 0.2375 | 0.1637 | 0.1839 | 0.2246 |
| 1.5    | Estimated value MSE | 1.0373 | 1.0633 | 1.2170 | 1.1741 | 1.0920 |
|        | Estimated value MSE | 0.3043 | 0.2636 | 0.1809 | 0.1952 | 0.2442 |
| 2      | Estimated value MSE | 1.0479 | 1.0322 | 1.1783 | 1.1322 | 1.0656 |
|        | Estimated value MSE | 0.2994 | 0.2886 | 0.2012 | 0.2237 | 0.2623 |

In table 7, it is seen that as $\alpha$ increases all MSEs show rising trend except that of MLE which is parallel to the horizontal axis. Here MSE of $\hat{\lambda}_{BQL}$ increases at a smaller rate than all other estimators.
Table 8. Estimated value and MSE of different estimators of the parameter $\lambda$ of Laplace double exponential distribution for different values of $\beta$ and $\alpha = 1$, $\theta = 1$, $\lambda = 1$, $c = 1$

| $\beta$ | Criteria   | $\hat{\lambda}_{MLE}$ | $\hat{\lambda}_{BSE}$ | $\hat{\lambda}_{BQL}$ | $\hat{\lambda}_{BML}$ | $\hat{\lambda}_{BNL}$ |
|---------|------------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 1       | Estimated value MSE | 0.6946 | 0.7081 | 0.7779 | 0.7432 | 0.7120 |
|         | Estimated value MSE | 0.1247 | 0.1131 | 0.0821 | 0.0961 | 0.1106 |
| 2       | Estimated value MSE | 0.6886 | 0.7579 | 0.8270 | 0.7954 | 0.7607 |
|         | Estimated value MSE | 0.1270 | 0.0863 | 0.0629 | 0.0726 | 0.0848 |
| 3       | Estimated value MSE | 0.6935 | 0.8047 | 0.8877 | 0.8394 | 0.8132 |
|         | Estimated value MSE | 0.1253 | 0.0617 | 0.0454 | 0.0561 | 0.0623 |
| 4       | Estimated value MSE | 0.6936 | 0.8492 | 0.9378 | 0.8928 | 0.8550 |
|         | Estimated value MSE | 0.1252 | 0.0504 | 0.0364 | 0.0417 | 0.0487 |

Table 8. presents declining pattern of MSE for all non-classical estimators as $\beta$ increases.

In the above tables $\hat{\lambda}_{BQL}$ is also showing better performance according to MSE criterion.

Figure 8. Graph of MSEs of different estimates of parameter $\lambda$ of Laplace double exponential distribution when $n = 20$, $\alpha = 1$, $\theta = 1$, $\lambda = 1$ and $c = 1$.

The performance of the estimators is checked in the following table for different values of $\lambda$. 
Table 9. Estimated value and MSE of different estimators of the parameter $\lambda$ of Laplace double exponential distribution when $n=20$, $\alpha=2$, $\beta=3$, $\theta=2$ and $c=2$

| $\lambda$ | Criteria | $\hat{\lambda}_{MLE}$ | $\hat{\lambda}_{BSE}$ | $\hat{\lambda}_{BQL}$ | $\hat{\lambda}_{BML}$ | $\hat{\lambda}_{BNL}$ |
|-----------|----------|------------------------|------------------------|------------------------|------------------------|------------------------|
| 1         | Estimated value MSE | 0.6937 | 0.7657 | 0.8432 | 0.8247 | 0.7903 |
| 1.5       | Estimated value MSE | 1.0454 | 1.0789 | 1.1925 | 1.1629 | 1.1001 |
| 2         | Estimated value MSE | 1.3773 | 1.3949 | 1.5376 | 1.4971 | 1.4239 |
| 3         | Estimated value MSE | 2.0816 | 2.0238 | 2.2348 | 2.1585 | 2.0443 |

In this case also the MSE of $\hat{\lambda}_{BQL}$ is least among the estimators considered in the study (figure 9).

Conclusions

From the above simulation results analysis and graphical study we can conclude that except for few cases Bayes estimator under quadratic loss is better than other estimators in the study. Thus quadratic loss function can be suggested for the
estimation of parameters from Laplace Double Exponential Distribution in case of non-classical method.

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