The connective $K$-theory of the Eilenberg–MacLane space $K(\mathbb{Z}_p, 2)$

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Abstract

We compute $ku^*(K(\mathbb{Z}_p, 2))$ and $ku_2(K(\mathbb{Z}_p, 2))$, the connective $KU$-cohomology and connective $KU$-homology groups of the mod-$p$ Eilenberg–MacLane space $K(\mathbb{Z}_p, 2)$, using the Adams spectral sequence. We obtain a striking interaction between $b_0$-extensions and exotic extensions. The mod-$p$ connective $KU$-cohomology groups, computed elsewhere, are needed in order to establish higher differentials and exotic extensions in the integral groups.

1. Introduction

Algebraic topologists try to turn homotopy theory questions into algebraic ones. We do this by assigning algebraic objects to topological spaces. There are many standard topological spaces that occur all the time and several algebraic theories that are in standard use. Eilenberg–MacLane spaces are important building blocks in homotopy theory, and any new information about them is potentially useful. This paper focuses on the second mod $p$ Eilenberg–MacLane space, $K_2 = K(\mathbb{Z}_p, 2)$. We use $\mathbb{Z}_p$ to denote $\mathbb{Z}/p$, the integers mod $p$. The algebraic tool we use is complex $K$-theory. It has long been known that $KU^*(K_2)$ is trivial [2]. Although interesting, this gives limited information. But if we move to the connective version of complex $K$-theory, $ku^*(-)$, we suddenly obtain an overwhelming amount of new information about $K_2$.

Because $KU^*(K_2)$ is trivial, we know that the homotopy maps $[K_2, BU]$ and $[K_2, U]$ are trivial. Consider the connective Omega spectrum for $BU$, $bu_n$ with $bu_n = \mathbb{Z} \times BU$. We have $ku^*(X) \simeq [X, bu_n]$ and $bu_n$ is $(n-1)$-connected for $n > 0$.

Let $v \in ku^{-2}$ be the Bott periodicity element. It gives maps $bu_{n+2} \to bu_n$. In this paper, we give a complete computation of $ku^*(K_2)$. Our result shows that there are many nontrivial elements in most $[K_2, bu_n]$, but mapping any such element a finite number of times with $v$ results in the trivial map.

To simplify our discussion, let $K_n = K(\mathbb{Z}_p, n)$ and $K(\mathbb{Z}_p)$ be the stable Eilenberg–MacLane spectrum.

There are a couple of interesting directions in which this research could go. First, $ku^*(K_1)$ is well known and has no $v$-torsion, so the suspension map $ku^*(K_2) \to ku^*(K_1)$ is trivial ($ku^*(K_2)$ is all $v$-torsion). On the other hand, it is easy to compute the stable result $ku^*(K(\mathbb{Z}_p))$. Every element here is killed by multiplication with a single $v$, so the suspension image must lie in the trivial part of $ku^*(K_2)$, a part to which we pay little attention. However, it is easy to see that only one element is in the image and it is in degree $2p + 2$. Our computation of $ku^*(K_2)$ is just the first step in interpolating between $ku^*(K_1)$ and $ku^*(K(\mathbb{Z}_p))$. The results and the suspension maps would be most interesting.

With such results, one could go after $ko^*(K_n)$ and $ko_* (K_n)$ using the exact sequences that come from the usual maps:

$$\cdots \to bo_{n+1} \to bo_n \to bu_n \to bo_{n+2} \to \cdots$$
In [14] and [6], the authors use very partial results to give new information about non-immersions of spin manifolds. More complete results would allow us to go much further on this problem.

Our computation of $ku^*(K_2)$ is done with the Adams spectral sequence (ASS), but we have a second tool to use as well. We already know the mod $p$ connective complex $K$-theory of $K_2$ from [8]. Many (perhaps most) ASS computations result only in an associated graded object because solving the extension problems for the multiplication by $p$ can be very difficult. However, using the long exact sequence for $ku^*(-)$ and its mod $p$ version, we are able to solve all of these extension problems giving an unusually complete answer.

In general, the more algebraic invariants we have for standard spaces in homotopy theory, the better off we are.

In [14] and [6], the authors initiated a partial computation of the connective $KU$-homology groups, $ku_*(K(Z_2, 2))$, of the mod-2 Eilenberg–MacLane space $K(Z_2, 2)$ in separate studies of Stiefel–Whitney classes of manifolds. We eventually turned to the associated cohomology groups, $ku^*(K(Z_2, 2))$, and were able to give a complete determination, via the ASS. This generalized nicely to the odd primes, and then we found a duality result ([5]) relating these homology and cohomology groups which enabled us to determine the homology groups $ku_*(K(Z_p, 2))$.

**Notation 1.1.** We need to establish some notation. Whenever we have $ku$, we mean it to be localized at the prime $p$. Adjustments must be made for odd primes because we don’t work directly with $ku$, but with an Adams’ summand. It is well known that $BU$ splits at an odd prime. This splitting lifts $bu$. The original source for $BU$ is [1, Corollary 8, p. 91]. A stable version is proven in [9, Proposition 2.7]. We’ll skip Adams’ notation. In the literature, the stable cohomology summand is often denoted by $\ell$. In a context where $BP(n)$ is around for all $n$, the summand is naturally called $BP(1)$. We want something that reflects the obvious connection to $ku^*(-)$, and so we adopt for our notation $kup^*(-)$ for the stable summand. This gives an Omega spectrum, $\{bup\}$ with $kup^*(X) \simeq [X, bup]$. With this notation, Adams’ original theorem says

$$BU \simeq \underbrace{bup_2 \times bup_4 \times \cdots \times bup_{2p-2}}.$$  

There is a corresponding stable splitting:

$$bu \simeq \underbrace{bup \times \Sigma^2 bup \times \Sigma^4 bup \times \cdots \times \Sigma^{2p-4} bup}.$$  

Consequently, if we compute $kup^*(X)$, we also know $ku^*(X)$. Note that for $p = 2$, there is no splitting. At $p = 2$, $ku$ localized is $kup$. Because we are working with a $p$-local space, $K_2$, it is not really necessary to localize $ku$ as well. But for us to work with just the one summand, it is. Again, we repeat $ku$ and $kup$ are always localized at a prime $p$.

We begin with a description of the $kup^*$-module $kup^*(K_2)$. Note that $kup^* = \mathbb{Z}_{(p)}[v]$ with $|v| = -2(p - 1)$. We find that depiction via ASS charts is the most insightful way to envision the groups. There is a very nice interplay between extensions (multiplication by $p$) seen in Ext ($h_0$-extensions) and exotic extensions. We depict the ASS with cohomological (co)degrees increasing from right to left. We write $|x| = d$ if $x \in kup^*(K_2)$ or the associated $E_2$-term.

In $kup^*(K_2)$, there is a trivial submodule whose Poincaré series when $p = 2$ is described at the end of Section 2. It plays no role and will be ignored from now on. As a $kup^*$-module, $kup^*(K_2)$ is generated by certain products of elements of $E_2^*$:

$$y_0', y_i = y_0^{p^i}, \text{ with } |y_i| = 2p^i, \quad (1.2)$$

$$z_j \text{ for } j \geq 0 \text{ with } |z_j| = 2(p^{j+1} + 1), \quad (1.3)$$

and

$$q \text{ with } |q| = 9 \text{ if } p = 2 \text{ and } |q| = 4p - 1 \text{ if } p \text{ is odd}. \quad (1.4)$$
We give two descriptions of our answer. In Theorem 1.16, we give the $E_\infty$-term of the ASS and then describe the exotic extensions from multiplication by $p$. Our preferred description is to incorporate them together. That is done in Theorems 1.8 and 1.15.

Let $TP_i[v] := \mathbb{Z}_p[v]/(v^i)$, the truncated polynomial algebra. The even-graded part $kup^*(K_2)$ is formed from shifted copies of $kup^*$-modules $A_k$ and $B_k$, which can be defined inductively as follows.

**Definition 1.5.** Let $k_0 = 1$ if $p$ is odd, and $k_0 = 2$ if $p = 2$. Let $B_{k_0-1} = 0$. Let $A_0 = \langle z_0 \rangle$ for all $p$. Inductively

$$B_k \text{ is built from } z_{k-1}^{p-1}B_{k-1}, \ TP_{p-1}^{k-1}[v]z_k, \text{ and } y_{k-1}^{p-1}B_{k-1}, \text{ if } k \geq k_0$$

and

$$A_k \text{ is built from } z_{k-1}^{p-1}B_{k-1}, \ TP_{p-1}^{k-1}[v]z_k, \text{ and } y_{k-1}^{p-1}A_{k-1}, \text{ if } k \geq 1$$

with extensions determined by:

$$pz_k = v^ep_{k-1} \text{ for } k \geq 2, \text{ and } py_{k-1}^{p-1}z_{k-1} = v^{p-1}(y-1)z_k. \quad (1.6)$$

When we write something like $zB$, we mean that all elements of $B$ are multiplied by the element $z$. Saying “is built from” means that these are successive quotients in a filtration as a $kup^*$-module. The extension formulas are only asserted up to multiplication by a unit in $\mathbb{Z}$. An empty product is defined to equal 1.

**Theorem 1.16.** We give two descriptions of our answer. In Theorem 1.16, we give the $E_\infty$-term of the ASS and then describe the exotic extensions from multiplication by $p$. Our preferred description is to incorporate them together. That is done in Theorems 1.8 and 1.15.

The following theorem explains how the portion of $kup^*(K_2)$ in even gradings is a direct sum of shifted versions of $A_k$ and $B_k$.

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1 Colors are present in online versions, but not in the print version.
**Theorem 1.8.** Let $M_p[S]$ denote the set of monomials in the elements of a set $S$ raised to powers $< p$. Let

$$\mathcal{M}_k = \left( M_p[z_k, y_k] - \{ z_k^{p-1}, y_k^{p-1} \} \right) \cdot M_p[z_i, y_i : i > k],$$

where $M_p[z_k, y_k] - \{ z_k^{p-1}, y_k^{p-1} \} = \{ z_k^i y_k^j : 0 \leq i, j \leq p - 1 \text{ and } \{ i, j \} \neq \{ 0, p - 1 \} \}$, which is a set with $p^2 - 2$ elements. Let $\mathcal{M}_k^A$ be the set of monomials in $\mathcal{M}_k$ with no $z$-factors, and $\mathcal{M}_k^B = \mathcal{M}_k - \mathcal{M}_k^A$. Then,

$$\text{kup}^*(K_2) = \bigoplus_{k \geq 1} \left( \bigoplus_{M \in \mathcal{M}_k^A} M \cdot A_k \bigoplus \bigoplus_{M \in \mathcal{M}_k^B} M \cdot B_k \right)$$

plus a trivial kup$^*$-module.
Figure 2. Schematic of $A_3$ and $B_3$ for odd $p$.

Note that the monomial 1 is in $\mathcal{M}_k^A$, so $A_k$ appears by itself, but $B_k$ does not. For example, if $p = 2$, copies of $B_k$ appear multiplied by each monomial of the form:

\[ z_k^\varepsilon y_k^{\delta_k} z_{k+1}^{\delta_{k+1}} y_{k+1}^{\delta_{k+1}} \cdots \text{ such that } \varepsilon_k = \delta_k \text{ and } \sum \varepsilon_j \geq 1. \]
Now we describe the portion of $kup^r(K_2)$ in odd gradings. Let $P[S]$ denote the polynomial algebra on a set $S$, and $TP_i[S] = P[S]/(s' : s \in S)$, the truncated polynomial algebra. Let $\Lambda_i = TP_i[z_i : i \geq j]$. Note that if $p = 2$, $\Lambda_j$ is an exterior algebra. For $i \leq j$, let

$$z_{ij} = z_i(z_j \cdots z_{j-1})^{p-1}. \quad (1.12)$$

If $j = i$, then $z_{ij} = z_i$.

Definition 1.13. For $\ell > k \geq 1$, let $S_{k,\ell} = TP_{k+1}[v]\langle z_{4\ell}, \ldots, z_{\ell-k+1+4\ell} \rangle$ with $p_{z_i,\ell} = v_{z_i-1,\ell}$ and $p_{z_0,\ell} = 0$.

For example, $S_{5,8}$ with $p = 2$ is depicted in Figure 3.

The following result describes the portion of $kup^r(K_2)$ in odd gradings. The exponent of $p$ in an integer $i$ is denoted simply by $v(i)$; the prime $p$ is implicit. The element $q$ here has grading 9 or $4p - 1$, as mentioned earlier.

Theorem 1.15. There is an isomorphism of $kup^r$-modules:

$$kup^\text{odd}(K_2) \approx \bigoplus_{\ell \geq 1} \bigoplus_{t \geq v(0)+2} q_{v}^{t-1}S_{v(0)+1,\ell} \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$  

The nonvisual, formulaic form of our result is as follows.

Theorem 1.16. The $kup^r$-module $kup^r(K_2)$ is isomorphic to a trivial $kup^r$-module plus a module whose associated graded is

$$P[y_1] y_0^{p-1} z_0 \oplus \bigoplus_{\ell \geq 1} TP_{p^\ell} [v] \otimes P[y_1] z_\ell \quad (1.17)$$

$$\oplus \bigoplus_{t \geq 0} TP_{p^\ell - v} [v] \otimes P[y_1] z_\ell \Lambda_{\ell} \quad (1.18)$$

$$\oplus \bigoplus_{t \geq 0} TP_{p^\ell + v} [v] q_{v}^{t-1} z_{k_0 + \ell + v(0) + 2} \Lambda_{\ell + v(0) + 2}. \quad (1.19)$$

Multiplication by $p$ in (1.17) and (1.18) is determined by (1.6) and in (1.19) as in Definition 1.13.

Our initial interest in this project was $kup_*(K_2)$ ([14,6]), but we first achieved success in computing $kup^r(K_2)$. In [5, Example 3.4], the following result was proved.

Theorem 1.20. There is an isomorphism of $kup_*$-modules $kup_*(K_2) \approx (kup^{*+2p}_*(K_2))^\vee$.

Here, $M^\vee = \text{Hom}(M, \mathbb{Z}/p^\infty)$, the Pontryagin dual, localized at $p$. A homotopy chart for $kup_*(K_2)$ could be thought of as a shifted version of the homotopy chart of $kup^r(K_2)$ viewed upside-down and backward. For example, the element of $kup^{108}(K_2)^\vee$ dual to the element $v^3y^3z_3z_4$ in Figure 1 corresponds to the generator of a $\mathbb{Z}_4$ in $kup_{104}(K_2)$ on which $v^4$ acts nontrivially. This element can be seen in Figure 4.
Figure 4. Portion of kup₄(K₂) corresponding to B₅ and A₅.

A remarkable property, for which one explanation is given in Section 7, is that Bₖ is self-dual as a kup∗-module. One way of stating this is to let \( \tilde{B}_k \) denote \( B_k \) with its indices negated. Then there is an isomorphism of kup∗-modules:

\[
\Sigma (p^{i+1}p^{j}+(i+1)p^{j-1}) \tilde{B}_k \approx B_k'.
\] (1.21)

For example, with \( p = 2 \), the second smallest generator \( Y \) of \( \Sigma^{208} \tilde{B}_5 \) is in grading \( 208 - 134 = 74 \) and has \( 2Y \neq 0 \) and \( \nu^4Y \neq 0 \) (see Figure 1). The second generator \( Z \) of \( B_5' \) is dual to the class in position \((74, 4)\) in Figure 1 and also satisfies \( 2Z \neq 0 \) and \( \nu^4Z \neq 0 \). The isomorphism (1.21) can be proved by induction on \( k \) using Definition 1.5.

A complete description of the kup∗-module kup₄(K₂) is immediate from Theorems 1.8, 1.15, and 1.20. However, one might like a complete description of its ASS. We can write formulas for the \( E_2 \)-term and differentials but will not do so here. In Theorem 1.23, we give a complete description of the \( E_\infty \)-term of the ASS of kup₄(K₂) with exotic extensions included, in terms of the charts described in Section 1.

In [5], a comparison was made of a chart for \( A_3 \) and its kup₄ analog. Here, we present in Figure 4 the kup₄ analog of Figure 1. This presents the portion of the ASS of kup₄(K₂) dual to \( A_5 \) with \( p = 2 \) under the
isomorphism of Theorem 1.20. The ASS chart dual to $B_3$ is obtained from this by removing the classes connected by dashed lines and lowering the remaining tower so that the bottom is in filtration 0. The resulting chart is isomorphic to the $B_3$ part of Figure 1.

We observe that in even gradings of the ASS for $kup_*(K_2)$, $h_0$-extensions exactly correspond to exotic extensions in the ASS of $kup^{+2p}(K_2)$, and vice versa. As a typical example of the duality, the summands of $kup^{82}(K_2)$, $kup^{82}(K_2)^\vee$, and $kup_{78}(K_2)$ in Figures 1 and 4 are all isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$. But for the $kup_*$-module structure, it is $kup^{82}(K_2)^\vee$ and $kup_{78}(K_2)$ that correspond to, since in both, the element that is divisible by 4, in position $(82,0)$ and $(78,7)$, resp., is also divisible by $v^7$ for $A_5$ and by $v^4$ for $B_5$.

**Theorem 1.23.** The $E_\infty$-term of the ASS of $kup_*(K_2)$ with exotic extensions included contains exactly the following.

- There is a trivial $kup_*$-module, which when $p = 2$ has generators corresponding to those enumerated at the end of Section 2 with gradings decreased by 4, and similarly when $p$ is odd.
- For every $S_{k,\ell}$ occurring in a summand of Theorem 1.15, there is a chart of the same form as Figure 3 with $v$-towers of height $k + 1$ on generators in gradings $2p^{i+1} + 2(p - 1)(i - k_0 - 1)$ for $1 \leq i \leq \ell - k$. One must add to this the grading of the other factors accompanying $S_{k,\ell}$ in Theorem 1.15.
- For each occurrence of $B_k$ in Theorem 1.8, there is a summand
  \[ \sum 2p^{i+1} + p^{i+2} + p^{i+k+1} \tilde{B}_k \]
  with gradings increased by those of other factors accompanying $B_k$ in 1.1. Here, $\tilde{B}_k$ is as defined prior to (1.21).
- For each summand $y^*A_k$ in Theorem 1.8, there is a variant of \[ \sum 2p^{i+1} + p^{i+k+1} \tilde{B}_k \] with gradings increased by $2ep^i$. In this variant, the initial $v$-towers are pushed up by $k$ filtrations and surrounded with a triangle of classes of the sort appearing in the lower left corner of Figure 4. See Remark 1.22.

**Proof.** Theorem 1.20 and our results for $kup^*(K_2)$ give the $kup_*$-module structure of $kup_*(K_2)$, but that is not the same as the ASS picture. Expanding on work done in [6] and [14] and using methods such as those in Section 2, we were able to write the $E_2$-term of the ASS for $kup_*(K_2)$ and had conjectured the differentials (but not the extensions) prior to embarking on our $kup$-cohomology project. We were unable to prove the differentials, probably because we had not taken sufficient advantage of the exact sequence with $k(1)_*(K_2)$. Now that we know the 2-orders and $v$-heights of generators (by grading, at least, if not by name), it is straightforward to see that the differentials must be as we expected. The isomorphism (1.21) plays an important role here; the left-hand side gives the ASS form of the right-hand side.

**Remark 1.24.** Regarding the unusual portion of the ASS chart for part of $kup_*(K_2)$ in the lower left of Figure 4, this is obtained from [6, Figure 4.2] with $d_*$-differentials on all odd-graded towers. For $A_5$, it will be a triangle going up to filtration $k$, with all but the first two dots on the top row being part of $B_k$.

The structure of the rest of the paper is as follows. In Section 2, we compute the $E_2$-term of the ASS for $kup^*(K_2)$. In Section 3, we determine the differentials in this ASS. In order to do so, we need to compare with $k(1)^*(K_2)$, where $k(1)$ is a summand of the spectrum for mod-$p$ connective $KU$-theory, using the exact sequence:

\[ \to k(1)^{-1}(K_2) \to kup^*(K_2) \stackrel{p}{\to} kup^*(K_2) \to k(1)^*(K_2) \to kup^{p+1}(K_2) \to kup^{p+1}(K_2) \to \ldots \]  \hspace{1cm} (1.25)

In Section 3, we restate results about $k(1)^*(K_2)$ from [8]. At the end of Section 3, we show how the descriptions of $kup^*(K_2)$ in Theorems 1.8 and 1.15 are obtained once we know the differentials and extensions. This exact sequence is also used in determining the exotic extensions of (1.6), which is done
in Section 4. In Section 5, we propose complete formulas for the exact sequence (1.25), and then in Section 6, we show that our proposed formulas account for all elements of \( k(1)^*(K_2) \) exactly once.

The main point of Section 6 is to prove that there are no additional exotic extensions in \( kup^*(K_2) \). An exotic extension \( p \cdot A = B \) implies that \( A \) is not in the image from \( k(1)^{-1}(K_2) \) and \( B \) does not map nontrivially to \( k(1)^*(K_2) \), so once we have shown that all elements are accounted for, there can be no more extensions. Many of our formulas in Section 5 are forced by naturality. However, many others occur in regular families, but with surprising filtration jumps. We could probably prove that the homomorphisms must be as we claim, by showing that there are no other possibilities, but we prefer to forgo doing that. In the optional Section 7, we discuss in more detail how the charts are obtained and provide an explanation for the duality result (1.21).

2. The \( E_2 \)-term of the ASS for \( kup^*(K_2) \)

We will need some notation. By \( H^*K_2 \), we understand \( H^*(K(Z_p, 2); Z_q) \). Let \( E \) denote an exterior algebra, \( P \) a polynomial algebra, and \( TP_n[x] = P[x]/(x^n) \) the truncated polynomial algebra. In all cases these will be over \( Z_p \), the integers mod \( p \). Let \( \overline{E} \) denote the augmentation ideal of an exterior algebra, and \( E_1 = E[Q_0, Q_1] \), where \( Q_i \) are the Milnor primitives. Because \( Q_i^2 = 0 \) we have homology groups, \( H_*(\cdot; Q_i) \), defined for \( E_1 \)-modules. We let \( \langle y_1, y_2, \ldots \rangle \) denote the \( Z_p \)-span of classes \( y_i \).

The ASS for \( kup^*(K_2) \) has \( E_2 = Ext_{\overline{E}}^*(H^*(bup), H^*K_2) \), where \( \overline{A} \) is the mod \( p \) Steenrod algebra and \( H^*(bup) \approx A/\overline{A}(Q_0, Q_1) \). Using a standard change of rings theorem [10], this is \( Ext_{E_1}^*(Z_p, H^*K_2) \). This converges to \( kup^{-I-r}(K_2) \). We depict this with \( E_2 \) in position \((i - s, s)\) as usual but label the axis with codegrees, the negative of the homotopical degree, so the left side of the chart will have positive gradings and refer to cohomological grading. In an attempt to avoid confusion, we rewrite this as \( G_r^{(r-s),s} \). With this notation, the differentials are \( d_i : G_r^{a,b} \rightarrow G_r^{a+1,b+r} \), multiplication by the element \( v \in kup^{-2(p-1)} \) (also considered in \( G_r^{-2(p-1),1} \)) is \( v : G_r^{a,b} \rightarrow G_r^{a-2(p-1),b+1} \), and multiplication by the element representing \( p \in kup^0 \), \( (h_0 \in G_r^{1,1}) \), is \( h_0 : G_r^{a,b} \rightarrow G_r^{a,b+1} \).

In the paragraph preceding Remark 2.17, we will define elements \( z_j \in G_2^2(p+1,0) \) for \( j \geq 0 \) and elements:

\[
z_{ij} \in G_2^{2(p+1)+1+i(p-1)(j-i),0}
\]

as in (1.12) satisfying the properties in Definition 1.13.

**Definition 2.1.** For \( j \geq k_0 \), we define \( W_j = \langle z_{ij}, z_{i-j,j}, \ldots, z_{k_0,j} \rangle \).

We also have \( y_i \in G_2^{2p,0} \) for \( i \geq 0 \), and

\[
q \in G_2^{0,0} \text{ if } p = 2, \text{ and in } G_2^{4p-1,0} \text{ if } p \text{ is odd. (2.2)}
\]

Cf. (1.3), (1.2), and (2.2). One last definition, let \( \Lambda_{j+1} = TP_p[z_i; i \geq j + 1] \).

A picture of \( P[v] \otimes W_2 \) as a \( P[v, h_0]-\text{module} \) with \( p = 2 \) appears in Figure 5.

**Figure 5.** A depiction of \( P[v] \otimes W_5 \).
The remainder of this section is devoted to the proof of the following result.

**Theorem 2.4.** The $E_2$ term of the Adams spectral sequence for the $kup^*(K_2)$ is isomorphic as a $P[h_0, v]$-module to

$$P[v, y_1] \otimes E[q] \otimes \left( \bigoplus_{j \geq 0} (W_j \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}) \right)$$

$$\oplus \left( P[h_0, v, y_1] \otimes E[v^h q] \right) \oplus \left( P[y_1] \otimes \begin{cases} \{v_0^{p-1}z_0\} & \text{if } p \text{ odd} \\ \{v_0z_0, z_1, h_0 y_0 z_0 = v z_1\} & p = 2. \end{cases} \right)$$

plus a trivial $P[h_0, v]$-module.

Some of the algebra structure of this $E_2$ will be useful later. For example, the product structure among the $z_j$’s will be clear, and also the formula

$$(v^2q)^2 = v^4z_2,$$  \hspace{1cm} (2.5)

holds when $p = 2$ since, as we shall see, in $H^*(K_2)$, $x_0^2 - Q_6x_{17} \in \text{im}(Q_1)$.

We will give a detailed proof when $p = 2$ and then sketch the minor changes for odd $p$. There are two parts to proving this theorem. First, we must give a complete description of the $E_1$-module structure of $H^*K_2$. Second, we have to compute $\text{Ext}^*_E(\mathbb{Z}_2, -)$ of this. We begin the first part.

Serre ([11]) showed that $H^*K_2$ is a polynomial algebra on classes $u_{2j+1}$ in degree $2^j + 1$ for $j \geq 0$ defined by $u_2 = t_2$ and $u_{2j+1} = Sq^j u_{2j+1}$ for $j \geq 0$. We easily have

$$Q_0(u_2) = u_3, \quad Q_0(u_3) = 0, \quad Q_0(u_{2j+1}) = u_{2j-1+1}^2 \text{ for } j \geq 2,$$

and

$$Q_1(u_2) = u_5, \quad Q_1(u_3) = u_5^2, \quad Q_1(u_5) = 0, \quad Q_1(u_{2j+1}) = u_{2j-2+1}^4 \text{ for } j \geq 3.$$  

Let $x_5 = u_5 + u_2u_3$ and write $H^*K_2$ as an associated graded object:

$$P[u_2^2] \otimes E[x_5] \otimes (E[u_2] \otimes P[u_3]) \otimes_{j \geq 2} \left( E[u_{2j+1}] \otimes P[(u_{2j+1})^2] \right)$$

From this, we can read off

**Lemma 2.6.**

$$H_*(H^*K_2; Q_0) = P[u_2^2] \otimes E[x_5]$$

Letting $x_9 = u_9 + u_3^3$ and $x_{17} = u_{17} + u_2u_3^3$, we rewrite again as:

$$P[u_2^2] \otimes TP_4[x_9] \otimes TP_4[x_{17}] \otimes_{j \geq 4} E[(u_{2j+1})^2]$$

$$\otimes (E[u_2] \otimes P[u_3]) \otimes (E[u_5] \otimes P[u_5^2]) \otimes_{j \geq 4} \left( E[u_{2j+1}] \otimes P[(u_{2j+2})^4] \right).$$

Again we read off

**Lemma 2.7.**

$$H_*(H^*K_2; Q_1) = P[u_2^2] \otimes TP_4[x_9] \otimes TP_4[x_{17}] \otimes_{j \geq 4} E[(u_{2j+1})^2]$$

An associated graded version of this is

**Lemma 2.8.**

$$H_*(H^*K_2; Q_1) = P[u_2^2] \otimes E[x_9] \otimes E[x_{17}] \otimes_{j \geq 2} E[(u_{2j+1})^2]$$

The bulk of the work here is finding a nice splitting of $H^*K_2$ as an $E_1$-module.
Let $N$ be the $E_1$-submodule with single nonzero elements in gradings 5, 7, 8, 9, and 10 with generators $x_5 = u_5 + u_2u_5$, $x_7 = u_2u_5$, and $x_9 = u_9 + u_3^4$, satisfying $Q_0x_7 = Q_1x_5$ and $Q_0x_9 = Q_1x_7 = x_{10}$. It has a $Q_0$-homology class $x_5$ and a $Q_1$-homology class $x_9$. This class $x_9$ is called $q$ in Theorem 2.4 and in all other sections. A picture of $N$ is in Figure 6. In pictures such as this, straight lines indicate $Q_0$-modules and curved lines $Q_1$.

The $E_1$-submodule $P[u_2^2] \oplus P[u_2^3] \otimes N$ carries the $Q_0$-homology of $H^*K_2$, while the remaining $Q_1$-homology is, written in our usual way as an associated graded version,

$$P[u_2^2] \otimes E[x_5] \otimes E[x_{17}, u_{2j+1}^2, j > 2].$$

(2.10)

We will exhibit a $Q_0$-free $E_1$-submodule $R$ whose $Q_1$-homology is exactly the above $E$. Moreover, $N \otimes R$ contains an $E_1$-split summand $S$ which maps isomorphically to $\langle x_9 \rangle \otimes R$.

It is premature to state this because we haven’t defined $R$ and $S$ yet, but for the record:

**Proposition 2.11.** As an $E_1$ module, $\tilde{H}^*K_2$ is isomorphic to $T \oplus F$ where $F$ is free over $E_1$ and $T$ is

$$P[u_2^2] \otimes \langle \langle u_2^2 \rangle \oplus N \oplus R \oplus S \rangle$$

A start on $R$ and $S$. For this to make sense, we need to find $R$ and $S$. The module $R$ is a direct sum of shifted versions of modules $L_k$, $k \geq 0$, which have generators $g_{2i}$, $0 \leq i \leq k$, with $Q_1g_{2i} = Q_0g_{2i+2}$ for $0 \leq i < k$, $Q_0g_0 \neq 0$, and $Q_1g_{2k} = 0$. For example, $L_1$ is depicted in Figure 7.

A splitting map, $\langle x_9 \rangle \otimes L_k \to N \otimes L_k$, for the epimorphism $N \otimes L_k \to \langle x_9 \rangle \otimes L_k$ is defined by:

$$x_9g_{2i} \mapsto x_9 \otimes g_{2i} + x_7 \otimes g_{2i+2} + x_5 \otimes g_{2i+4}$$

for $0 \leq i \leq k - 2$.

$$x_9g_{2k-2} \mapsto x_9 \otimes g_{2k-2} + x_7 \otimes g_{2k}, \text{ and } x_9 \otimes g_{2k} \mapsto x_9 \otimes g_{2k}.$$

**The $E_1$-module $M_j$.** Let

$$x_{2j+1}^2 = u_{2j+1}^2 + \begin{cases} u_2u_3^3 & j = 4 \\ u_2u_3u_3^3u_3^3 & j = 5 \\ u_2u_3u_3^3u_3^3 & j = 6 \\ u_3^2u_3^3u_3^3 & j = 6 \\ 0 & j > 6 \end{cases}$$

and $w_{2j-1} = \begin{cases} u_2u_3u_3^3 & j = 4 \\ u_2u_3u_3^3 & j = 5 \\ 0 & j > 5. \end{cases}$

Then $Q_0x_{2j+1} = u_{2j-1+1}^2$, so $Q_0x_{2j+1}$ and $u_{2j-1+1}^2$ represent the same $Q_1$-homology class. Define $E_1$-modules $M_j$ inductively by $M_3 = 0$, and for $j \geq 4$ there is a short exact sequence of $E_1$-modules:

$$0 \to u_{2j-2+1}^2M_{j-1} \to M_j \to M'_j \to 0,$$

(2.13)

where $M'_j = \langle x_{2j+1}, Q_0x_{2j+1} \rangle$ and $Q_1x_{2j+1} = u_{2j-2+1}^2Q_0x_{2j-1+1}$. The above definitions of the $x_{2j+1}$ are necessary to get this formula to work right.
There is an isomorphism of $E_1$-modules $M_j \cong \Sigma^{i+1}L_{j-i}$ given by:

$$\Sigma^{i+1}g_2 \mapsto \begin{cases} x_{2i+1} & i = 0 \\ u_{2i-2}^i x_{2i-1} & i = 1 \\ u_{2i-2}^i u_{2i-3}^i x_{2i-2} & i = 2 \\ u_{2i-2}^i u_{2i-3}^i \cdots u_{2j-2}^i x_{2j-1} & 2 < i < j - 4 \end{cases}$$  
(2.14)

And we have

$$H_*(M_j; Q) = \begin{cases} \{u_{j+1}^1, u_{j+1}^0\} & j = 4 \\ \{u_{j+1}^2, u_{j+1}^0 u_{j+1}^0\} & j = 5 \\ \{u_{j+1}^3, u_{j+1}^0 u_{j+1}^1\} & j = 6 \\ \{u_{j+1}^0, u_{j+2}^0 \cdots u_{j+1}^0 x_{j+1}^0\} & j > 6 \end{cases}$$  
(2.15)

The $E_1$-module $R$. Let

$$R = \bigoplus_{j \geq 4} M_j \otimes E[u_{2j+1}^0, u_{2j+1}^2, \ldots].$$  
(2.16)

Then $H_*(R; Q) = E[x_{17}, u_{2}^0, u_{2}^2, \ldots]$ since monomials in $E$ without $x_{17}$ appear from a first term (of the two in (2.15)) in $H_*(M_j \otimes E; Q)$, where $j$ is minimal such that $u_{2j+1}^0$ appears in the monomial, while those with $x_{17}$, and also containing a product $u_{2}^0 \cdots u_{2}^2 x_{17}$ of maximal length, occur as a second term in $H_*(M_j \otimes E; Q)$.

**Proof of Proposition 2.11.** We have the $E_1$-submodule $T$ given in Proposition 2.11. Because this contains all of the $Q_0$ and $Q_1$ homology, what remains must be free over $E_1$ by [13].

**Proof of Theorem 2.4.** We compute Ext$_{E_1}(\mathbb{Z}_2, T)$ with $T$ as in Proposition 2.11. We will not be concerned with the free $E_1$-module $F$, but later we will give the Poincaré series for it. Each copy of $E_1$ in $F$ gives a $\mathbb{Z}_2$ in $G^0$ that corresponds to $Q_0Q_1$. That

$$\text{Ext}^{*,*}_{E_1}(\mathbb{Z}_2, P[u_2^0]) = P[v, h_0, y_1]$$

with $y_1 \in G^0_{2,0}$ should be clear, given our labeling conventions. We normally work with the reduced cohomologies, so the $y_1^0$ generator above would be ignored. The $y_1$ notation is particularly useful when we consider all primes $p$. It is $y_1^0$ where $y_0 \in G^0_{2,0}$. So, $|y_1| = 2p$.

We compute Ext$_{E_1}(\mathbb{Z}_2, N)$ in two ways using two different filtrations of $N$. From this, we see that the generator of the towers can be thought of either as $v^2x_9$ or $h_0^2x_5$.

Using Figure 6 as our guide, our first filtration is $(x_5, x_9), (x_7, x_{10}),$ and $(x_9).$ The Ext on $x_9 \in G^0_9$ is just $P[v, h_0].$ For the other two, we get $h_0$-towers on $x_{10} \in G^0_{10}$ and $x_9 \in G^0_{8,0}.$ The extensions in $N$ show these two $h_0$-towers are connected by multiplication by $v$. In addition, a $d_1$ is forced on us by the extensions. Figure 8 describes this completely.
Again referring to Figure 6, our second filtration is \( \langle x_9, x_{10} \rangle, \langle x_7, x_8 \rangle, \) and \( \langle x_5 \rangle \). Now our Ext groups are \( P[v, h_0] \) on \( x_5 \in G^{5,0} \) and \( P[v] \) on \( x_9 \in G^{8,0} \) and \( x_{10} \in G^{10,0} \). Again, the \( d_1 \) is forced by the extensions in \( N \). Figure 9 describes the result.

This concludes the computation of Ext for \( P[u_2^2] \otimes (\langle u_2^2 \rangle \oplus N) \) of Proposition 2.11. The result is the second line of Theorem 2.4.

We need to compute Ext for \( P[u_2^2] \otimes (R \oplus S) \) and show it is the same as the top line in Theorem 2.4. Since \( S \approx \langle x_9 \rangle \otimes R \), all we need to do is \( P[u_2^2] \otimes R \) and ignore the \( E[x_9] \). Similarly, we can ignore the \( P[u_2^2] \) and the \( P[y_1] \) because for every power of \( u_2^2 \) we will have a copy of the answer indexed by powers of \( y_1 \). All we have left now is \( R \), but \( R \) is just many copies of the various \( M_j \) and the indexing for the number of copies is given by the \( \Lambda_{j+1} \).

All that remains is to show that \( \text{Ext}_{E_1}(\langle z_2, M_j \rangle \approx P[v] \otimes W_{j-2} \) with \( W_{j-2} \) as in Definition 2.1.2. Recall that \( M_j = \Sigma^{2j+4} L_{j-4} \). We can filter \( L_{j-4} \) into pairs of elements \( g_{2i}, Q_0 g_{2i} \), for \( 0 \leq i \leq j - 4 \). Then \( \text{Ext}_{E_1}(\langle z_2, M_j \rangle \) has a \( P[v] \) on each element \( \Sigma^{2j+4} Q_0 g_{2i} \) which we denote by \( z_{j-1-2, j-2} \in G^{2j+2+2i,0} \). The element \( z_{j-1-2, j-2} \) is often called \( z_{j-2} \). There is no \( d_1 \), but undoing the filtration does solve the extension problem and gives us \( h_0 z_{j-2, j-2} = \nu z_{j-1, j-2} \). This completes our computation and thus our proof.

Remark 2.19. To illustrate the last computation in the proof, consider the generators of the \( v \)-towers for \( \text{Ext}_{E_1}(\langle z_2, M_j \rangle \). They are \( z_5, z_8, z_{10} \), and \( z_{12, 0} \), which is what we have called \( z_{5, 5}, z_4, z_{3, 5} \), and \( z_{2, 5} \), as pictured in Figure 5. For future reference, we note that (with \( v \) meaning homologous)

\[
z_{j} = Q_0 u_{2j+3+1} \sim u_{2j+3+1} = Q_0 u_{2j+3+1} = Q_0 Q_{j+2} t_2 = Q_{j+2} Q_0 t_2.
\]

(2.20)

We now describe briefly the changes required when \( p \) is odd. We have

\[
H^*(K_2) = P[y_0] \otimes P[g_1, g_2, \ldots] \otimes E[u_0, u_1, \ldots],
\]

with \( |y_0| = 2, |g_i| = 2(p^i + 1), |u_i| = 2p^i + 1, Q_0 y_0 = u_0, Q_0 u_i = g_i, Q_1 y_0 = u_1, Q_1 u_i = g_1, \) and \( Q_i u_1 = g_{i-1}^p \), \( i \geq 2 \). Let \( y_1 = y_0^p \). Then, similarly to the case \( p = 2 \),

\[
H_*(H^* K_2, Q_0) = P[y_1] \otimes E[y_0^{p-1} u_0].
\]

Let \( N = \{ y_0^{p-1} u_0, q = y_0^{p-1} u_1, Q_0 q = Q_1 (y_0^{p-1} u_0) \} \). Then, \( P[y_1] \otimes P[y_1] \otimes N \) carries the \( Q_0 \)-homology and part of the \( Q_1 \)-homology. Similarly to (2.10), the rest of the \( Q_1 \)-homology is

\[
P[y_1] \otimes E[q] \otimes E[w_1] \otimes TP[p, g_2, g_3, \ldots],
\]

where \( w_1 = u_2 + u_0 g_1^{p-1} \). There are \( E_1 \)-submodules \( M_j \) for \( j \geq 2 \), defined inductively by \( M_2 = \langle w_1, g_2 = Q_0 w_1, M' \rangle = \langle u_j, g_j = Q_0 u_j \rangle \) for \( j \geq 3 \), and for \( j \geq 3 \), there exists a short exact sequence of \( E_1 \)-modules:

\[
0 \rightarrow g_{j-1}^{p-1} M_{j-1} \rightarrow M_j \rightarrow M'_j \rightarrow 0,
\]

with \( Q_1 u_i = g_i^{p-1} \). There is an isomorphism of \( E_1 \)-modules \( M_j \otimes \Sigma^{2j+4} L_{j-2} \), where \( L_j \) is similar to Figure 7, but with \( i \)th generator \( (i \geq 0) \) in grading \( 2(p-1)i \) rather than \( 2i \).

\footnote{The reason for this awkward shift is that the gradings for \( z_j \) which give the elegant statements in Definition 1.5 and elsewhere are not particularly convenient in developing the \( E_2 \) statement.}
Let

\[ R = \bigoplus_{j \geq 2} M_j \otimes TP_{p-1}[g_j] \otimes TP_p[g_{j+1}, \ldots]. \]

Then \( H_s(R; Q_1) = E[w_1] \otimes TP_p[g_2, g_3, \ldots] \), and so, similarly to Proposition 2.11, up to free \( E_1 \)-modules:

\[ H^*K_2 \approx P[y_1] \otimes (\langle y_1 \rangle \oplus N \oplus R \oplus qR). \] (2.21)

Similarly to Figure 9, \( \text{Ext}_{E_1}(\mathbb{Z}_p, N) \) can be read off from Figure 10. This gives the third summand and \( vq \) part of the second summand in Theorem 2.4, while the \( \langle y_1 \rangle \) part of (2.21) gives the non-\( vq \) part of the second summand. For the first summand in Theorem 2.4, we replace \( vq \) part of the second summand in Theorem 2.4, while the free part, all we have to do to get the PS for these elements is to multiply by higher \( \text{Ext} \).

More on the \( E_1 \)-free part when \( p = 2 \). If we compute the \( \text{Ext}_{E_1}(\mathbb{Z}_2, F) \) for the \( E_1 \) free part of \( H^*K_2 \), we just get a \( \mathbb{Z}_2 \) corresponding to the top element for each copy of \( E_1 \). If we find the Poincaré series (PS) for the free part, all we have to do to get the PS for these elements is to multiply by \( x^t \). The Poincaré series for free part is obtained by subtracting the PS for the non-free part of Proposition 2.11 from that of \( H^*K_2 \). This is

\[
\prod_{k \geq 0} \frac{1}{(1 - x^{2^k+1})} - \frac{1}{(1 - x^4)} \left( 1 + x^5 + x^7 + x^8 + x^9 + x^{10} \right) - \frac{1}{(1 - x^2)(1 - x^4)} \left( \bigoplus_{j \geq 2} (x^{2^j+1}(1 + x^9)(1 + x)(1 - x^{2j-6}) \prod_{k \geq j} (1 + x^{2^{k+1}+2})) \right)
\]

The first term is the PS for \( H^*K_2 \). The second is the PS for \( P[u_2^2] \otimes (\langle 1 \rangle \oplus N) \). The last term is more complicated but does the \( S \) and \( R \) terms. The \( (1 - x^4) \) in the denominator is for the \( P[u_2^2] \). The \( x^9 \) is the shift that takes \( R \) to \( S \). The \( (1 + x) \) is because they are \( Q_0 \) free. The \( x^{2^j+1} (1 - x^{2j-6}) / (1 - x^2) \) is for the odd part of \( M_j \) and the remainder is for \( \Lambda \).

This is easy to put into a computer and calculate. For example, the number of free generators in degree 79 is 245.
3. Differentials in the ASS of $k^p_*(K_2)$

The main theorem of this section determines the differentials in the ASS for $k^p_*(K_2)$.

**Theorem 3.1.** The differentials in the spectral sequence whose $E_2$-term was given in Theorem 2.1 are as follows. All $v$-towers are involved, either as source or target, in exactly one of these. Here, $M$ refers to any monomial (possibly $= 1$) in the specified algebra. Recall that $\Lambda_j = TP_{p}[z_i : i \geq j]$, which is an exterior algebra if $p = 2$. Also, recall $y_i = y_i^{p-1}$. We give reference numbers to the differentials when $p$ is odd, but references to these also apply to the corresponding differential when $p = 2$, as the proofs are extremely similar.

First with $p = 2$.

\[
\begin{align*}
\delta_{v(i)+2}(y_i^j) &= h_0 v^i q_1^{j-1}, \quad i \geq 1; \\
\delta_{v(i)+2}(y_i^j z_j M) &= v^{i+2} q_1^{j-1} z_j v^{-i} M, \quad j \geq v(i) + 2, \ M \in \Lambda_j; \\
\delta_{2-t}(h_0^2 v^2 q_1^{x-1} M) &= v^2 z_t M, \quad t \geq 2, \ M \in P[y_i]; \\
\delta_{2-t}(q_1^{x-1} z_j M) &= v^{2-t} z_t z_j M, \quad j \geq t \geq 2, \ M \in P[y_i] \otimes \Lambda_{j+1}.
\end{align*}
\]

Now with $p$ odd.

\[
\begin{align*}
\delta_{v(i)+2}(y_i^j) &= h_0 v^{i+1} q_1^{j-1}, \quad i \geq 1; \tag{3.2} \\
\delta_{v(i)+2}(y_i^j z_j M) &= v^{i+2} q_1^{j-1} z_j v^{-i} M, \quad j \geq v(i) + 2, \ M \in \Lambda_j; \tag{3.3} \\
\delta_{p-t}(h_0^{-1} v q_1^{p-1} M) &= v^p z_t M, \quad t \geq 1, \ M \in P[y_i]; \tag{3.4} \\
\delta_{p-t}(q_1^{p-1} z_j M) &= v^{p-t} z_t z_j M, \quad j \geq t \geq 1, \ M \in P[y_i] \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}. \tag{3.5}
\end{align*}
\]

The proof occupies the rest of this section, except that at the end of the section we explain briefly how this leads to our description of $k^p_*(K_2)$ in Section 1, except for the exotic extensions.

By [12, Theorem A], $Q_0 Q_0_{+1}$ is in the image from $BP^*_0(K_2)$ and hence must be a permanent cycle in our ASS. Thus by (2.20), $z_j$ is a permanent cycle, and so (3.3) follows from (3.2), and (3.5) follows from (3.4), using $pi_{+d} = v^{z_{i-1},d}$, as noted in 1.13.

The differentials (3.2) follow from the result of [3] that $H^{2p+1}_*(K_2; \mathbb{Z}) \approx \mathbb{Z}/p^{x(i)+2} \oplus \mathbb{Z}_p$. See also [4, Proposition 1.3.5] when $p = 2$. The ASS converging to $H^*_0(K_2)$ has $E_2 = Ext_{A_0}([Z_0, H^* K_2])$, where $A_0 = (1, Q_0)$. We depict this $E_2$ similarly to our ASS for $k^p_*(K_2)$. It has an $h_0$-tower for each element of $H_*(H_0 K_2, Q_0)$, which was described in Lemma 2.6. These come in pairs in grading $2pi$ and $2pi + 1$ corresponding to $y_i$ and $y_i^{-1} y_0^{p-1} u_0$. In order to get the $\mathbb{Z}/p^{x(i)+2}$, there must be a $\delta_{v(i)+2}$-differential, as pictured on the right-hand side of Figure 11.
Similarly to Figures 8 and 9, we have, for \( p = 2 \) and \( i \geq 1 \), an \( h_0 \)-tower in the ASS for \( kup^*(K_2) \) arising from \( G_{i+1,2} \), called either \( h_0y_1^{i-1}x_5 \) or \( v^2y_1^{i-1}q \). There is also an \( h_0 \)-tower arising from \( y_1^i \in G_{i,0}^2 \). The classes \( y_1 \) and \( x_5 \) correspond to cohomology classes \( u_2^i \) and \( u_5 + u_5u_3 \). Under the morphism \( kup^*(K_2) \to H^*(K_2; \mathbb{Z}) \), these towers map across, as suggested in Figure 11. We deduce the \( d_{n+2} \)-differential claimed in (3.2), promulgated by the action of \( v \). Note that \( x_5 = q \).

The situation when \( p \) is odd is extremely similar, using Figure 10. The difference is that the \( h_0 \)-tower in \( 2p+1 \) in the \( kup^* \) ASS starts in filtration 1 rather than 2. Its generator can be called \( vy_1^{i-1}q \).

In Figure 12, we depict many of the differentials asserted in Theorem 3.1 in grading \( \leq 36 \) when \( p = 2 \). Regarding the third (final) summand in Theorem 2.4, which is \( P[y_1] \otimes A_1 \) when \( p = 2 \), we have included \( y_1A_1, y_1^2A_1, \) and \( y_1^3A_1 \). Not included are the portions involving (3.2) and (3.3) when \( i \) is odd, as this portion self-annihilates. What is shown is (3.2) for \( i = 2, 4, \) and 6, (3.4) for \( (i, k) = (1, 0), (1, 1), (1, 2), \) and \( (2, 0) \), and (3.5) with \( t = 1, k = 0, \) and \( j = 4 \).

In order to establish the remaining differentials, we will need the following description of \( k(1)^*(K_2) \), which is proved in [8]. We shift by 1 the subscripts of the classes \( z_j \) and \( w_j \) used there. The formulas for \( r(j) \) and \( r'(j) \) are as in [8]. We recapitulate some of their properties. Those stated here but not there are easily proved by induction.

**Proposition 3.8.** [8] For \( j \geq 0 \), \( z_j \) is the reduction of the class in \( kup^*(K_2) \) and satisfies \( |z_j| = 2(p^{i+1} + 1) \). The classes \( w_j \) satisfy \( |w_1| = 2p^2 + 1, |w_2| = 2p^3 - 2p^2 + 6p - 3, \) and \( w_{j+2} = y_1^{p-1}w_{j+1}^{p-1} \). The integers \( r(j) \) and \( r'(j) \) satisfy the following properties:

\[
\begin{align*}
    r(0) = 1, & \quad r(1) = p, \quad r(j+2) = r(j) + p^{i+1}(p-1) + 1; \\
    r'(0) = p - 1, & \quad r'(1) = p^2 - p, \quad (3.9) \\
    r'(j+2) = r'(j) + p^{i+2}(p-1) - 1, & \quad (3.10) \\
    r(j) - r'(j-1) = j, & \quad (3.11) \\
    r(j) + r'(j) = p^{i+1}, & \quad (3.12) \\
    r(j+2) + r'(j) = p^{i+2} + 1, & \quad (3.13) \\
    (p - 1)(r(j-1) + j - 1) < p', & \quad (3.14) \\
    p^{i+1} - p' \leq r'(j) < p^{i+1} - p' - 1. & \quad (3.15)
\end{align*}
\]
Figure 12. Some differentials with $p = 2$. 
Theorem 3.16. [8] For any $p$, $k(1)^*(K_2)$ is a trivial $k(1)^*$-module plus

$$\bigoplus_{j=0} TP_{\nu j}[v] \otimes P[y_{j+1}] \otimes TP_{p-1}[y_j] \otimes E[w_j] \otimes E[w_{j+1}] \otimes \Lambda_{j+1}$$

$$\bigoplus_{j\geq 1} TP_\nu(y-j-1)[v] \otimes P[y_j] \otimes E[w_j] \otimes TP_\nu[z_j] \otimes \Lambda_{j+1}$$

$$\bigoplus P[y_1] \otimes \left( E[y_0^{-1} z_0] \otimes \begin{cases} E[z_1] & p = 2 \\ 0 & p \text{ odd} \end{cases} \right) \bigoplus_{j\geq 1} P[y_1] \otimes E[q] \otimes E[z_j^p] \otimes \Lambda_{j+1}.$$ 

The last line was not discussed in [8]; it is from free $E[Q_1]$ summands which are not part of free $E_1$ summands and plays a very important role.

Now we continue the proof of Theorem 3.1. We have already proved (3.2) and (3.3). As already noted, the $z_j$’s are infinite cycles by [12], and so the differentials in (3.5) are implied as soon as the corresponding differential in (3.4) is proved.

As a warmup, we consider the cases $t = 2$ and $3$ of (3.4) when $p = 2$. We make extensive use of the exact sequence (1.25). Referring to Figure 12 is useful.

In even gradings $\leq 14$, $k(1)^*(K_2) = 0$ in positive filtration, by Theorem 3.16. Thus, the map $kup^*(K_2) \rightarrow k(1)^*(K_2)$ implies that in the ASS for $kup^*(K_2)$, $v^t z_2$ must be hit by a differential or divisible by 2 for $s \geq 2$. In grading $< 8$, there is nothing that can divide it, and the only odd-grading $\nu$-tower in that range is on $v^\nu y_1 q$. Thus, $d_2(v^\nu y_1 q) = v^\nu z_2$, the case $t = 2$, $M = 1$ of (3.4). Since $d_2(v^\nu y_1 q) = 0$ by (3.2), the case $t = 2$ of (3.4) follows for any $M$ by the derivation property. An analogous argument does not work at the odd primes.

Similarly $v^t z_2$ must be hit or divisible for $s \geq 4$, and examination of options in Figure 12 shows that we must have $d_3(h_0 v^\nu y_1^s q) = v^\nu z_3$, preceded by extensions. Since $d_3(y_1^s q) = h_0^s v^\nu y_1^s q$, we deduce the case $t = 3$, $M \in P[y^8]$ of (3.4) using the derivation property (2.5) and $h_0 z_2 = 0$. We do not have a priori knowledge that $y_1^4 z_3$ is a permanent cycle in the ASS of $kup^*(K_2)$. However, if it supported a nonzero differential, then the tower of $v$-height 4 on $y_1^t z_3$ in the ASS of $k(1)^*(K_2)$ would have to map to $v^t C$ for $0 \leq t \leq 3$ for some $C$ in positive filtration in grading 51 in the ASS of $kup^*(K_2)$. Then, $v^t C$ must be $d_i(B)$ with $r \geq 5$ and $B$ in filtration 0 in grading 42. ($B$ cannot have higher filtration since everything is $\nu$-towers, and $v^t C$ cannot be hit.) But the only possible $B$ is $y_1^s z_2$, and we already know that $v^{pt} y_1^s z_2 \in \text{im}(d_3)$. (Ordinarly this would not preclude the possibility of $B$ supporting a differential, but it does since everything is $\nu$-towers.) Thus, $y_1^4 z_3$ is a permanent cycle, and consideration of its image in $k(1)^*(K_2)$ implies that $v^t y_1^4 z_3$ is hit by a differential for some $s \geq 4$. The only element in odd grading $< 42$ not yet accounted for is $h_0 v^2 y_1^s q$ in grading 33, and so this must be the source of the differential. This is the case $t = 3$, $M = y_1^s$ of (3.4). The validity for all $M = y_1^{s+t}$ (and $t = 3$) now follows similarly to what we did for $M = y_1^s$ at the beginning of this paragraph.

Now we switch our attention to the odd primes. The situation when $p = 2$ is extremely similar. We want to prove the following version of (3.4):

$$d_{p-1}\left(h_0^{p-1} v^q y_1^{i+1} y_1^{p^{r-1}-1}\right) = v^p y_1^{p^{r-1}-1} z_i. \quad (3.17)$$

Now we work toward proving this. We illustrate with $p = 5$, but it should be clear how it generalizes to an arbitrary prime. One new thing is the Divisibility Criterion as invoked in [8]. Each mod $(p-1)$ value of $i$ can be considered separately. We will consider (3.17) with $p = 5$ and $i = 4\ell$; other congruences follow similarly. We index the differential (3.17) by $(\ell, t)$. We write $T$ (for vertical tower) for the class $h_0^{p-1} v^q y_1^{(4\ell+1)5^{r-1}-1}$, and $M$ (for Monomial) is $y_1^{45^{r-1}} z_i$. We will often afflict $T$ and $M$ with the parameters $(\ell, t)$. We write $|T|$ for $\frac{1}{2}(|T| + 1)$. The $\frac{1}{2}$ avoids extraneous factors of 2 that always cancel out. The $+1$ is so that this indicates the grading (times $\frac{1}{2}$) of the class that it hits. $|M|$ denotes $\frac{1}{2}$ times the grading of $M$, and $M'$ equals $\frac{1}{2}$ times the grading of $v^h M$, where $h$ is the $v$-height of $M$ in $k(1)^*(K_2)$. We wish to show that the differentials must be as claimed.
There are three types of constraints on the differentials involving these classes. Constraint C1 is that if \( T \to M \) (by which we mean that a certain \( T \) class supports a differential hitting \( v^i M \) for some \( i \) and a certain monomial \( M \)), then \( |T| \leq M' \). (This says that the \( v \)-tower on \( M \) cannot be hit while its image in \( k(1)^\ast(K_2) \) is nonzero.)

Constraint C2 says that if \( T(5\ell + 1, t - 1) \to M_1 \) and \( T(\ell, t) \to M_2 \), then \( |M_2| > |M_1| \). Since \( |T(5\ell + 1, t - 1)| = |T(\ell, t)| \), this says that as you move up an \( h_0 \) tower,\(^3\) differentials must get longer (unless they are hitting into an \( h_0 \) tower, which is not the case here.)

Constraint C3 says that if \( T_2 \to M_1 \), then there exists \( M_3 \) such that \( |M_1| \geq |M_3| \geq M'_1 \) and either
\[
M'_1 \leq |T_2|
\]
or
\[
T_3 \to M_3 \text{ has already been proved, and } |T_3| \leq |T_2|.
\]
The reason for C3 is that there must be extensions into the \( M_1 \)-tower from grading \( M'_1 \) to \(|T_2| + 4\). The nonzero classes on the \( v \)-tower (on \( M_1 \)) supporting the extensions must go to at least \(|T_2| + 4\), and it has nonzero classes at least to \(|M'_1| + 4\), and if \( T_3 \to M_3 \) was already proved, it has nonzero classes to \(|T_3| + 4\).

Note that we are saying that the \( v \)-tower on \( M_1 \) maps to 0 in \( k(1)^\ast(K_2) \) once we get to grading \( M'_1 \) (and hence in gradings \( \leq |M'_1| \) it is either hit by differentials or is divisible by \( p \)). There might be classes of higher filtration in \( k(1)^\ast(K_2) \) to which it could map, but, if so, we can modify the generator of the \( M_1 \) tower by the class on the tower sitting above it. Also note that it is possible that extensions from the tower \( M_3 \) don’t start from the generator, if there are \( h_0 \)-extensions on the tower for awhile. See Figure 13. There is an exception to the C3 requirement for \( T(\ell, 1) \to M(\ell, 1) \). Here, the extension into \( v^i y^4 z^1 \) is obtained from the special class \( y^4 z^1 v^{10} \).

With the above conventions, we have \( |T| = 5'(4\ell + 1) + 1 \), \( |M| = 5'(4\ell + 5) + 1 \), and \( M' = |M| - 4r'(t - 1) \), where \( 4r'(t - 1) \) has the values 16, 80, 412, and 2076 for \( t = 1, 2, 3, \) and 4. Increasing from \( t \) to \( t + 2 \) increases this by \( 4^2 \cdot 5^{t+1} - 4 \). We consider the cases in order of increasing \( |M| \) and, for equal values of \( |M| \), increasing \( \ell \). We tabulate a representative sample in Table 1. We omit listing values of \( \ell \equiv 3, 4 \mod 5 \) because they behave similarly to \( \ell \equiv 2 \).

Before presenting a general argument, we illustrate with an example, starting with \( M_1 = M(1, 3) \). We will see that it builds a chart which is \( y_1^{100} \) times Figure 2. In Table 1, we have \(|M_1| = 1126\). Its \( v \)-tower is truncated at height \( p^3 = 125 \) by a differential on \( T(1, 3) \), with \(|T(1, 3)| = 626\), using our grading conventions. Playing the role of \( M_3 \) is \( M(6, 2) \) with \(|M_3| = 726\). We have \( M'_1 = 714 \). It is \( v^3 M_3 \) which supports the extension in “grading” 714. Note that for \( 0 \leq i \leq 2 \), \( h_0 v^i M_3 \neq 0 \), and so \( p \cdot v^i M_3 \) is not a \( v \)-multiple of \( M_1 \). (In Figure 2, the class \( y^2 z \) corresponds to \( M_3 \).) From Table 1, we see that \( M'_1 = 646 \), which means that in “grading” \( \leq 646 \), the \( v \)-tower on \( M_3 \) is either hit by a differential or divisible by \( p \). Table 1 says it is hit by a differential in 626. In “gradings” from 646 to 630, it is divisible by \( p \). It has its own, distinct, \( M_3 \) class, namely \( M(31, 1) \). In Figure 2, this latter class corresponds to \( y^2 z_2 \).

Now we start the proof. We begin with a lemma.

**Lemma 3.19.** For \( M = M(\ell', t') \) with \( |M(5\ell + 1, t - 1)| < |M| < |M(\ell, t)| \), we have \( t' < t \), \(|T(\ell, t)| < |T(\ell', t')| \), and \(|M(5\ell + 1, t - 1)| < M' \).

\(^3\)Note that \( h_0 T(5\ell + 1, t - 1) = T(\ell, t) \).
Proof. The given inequalities quickly force $t' < t$. The inequality $|T(\ell, t)| < |T(\ell', t')|$ follows immediately. Finally, the given inequalities prevent $M' \leq |M(5\ell + 1, t - 1)|$.

To prove the differentials, we use induction on our ordering of the $M'$'s. If the differentials are not as posed, consider the smallest $|M|$ such that $T(\ell, t) \to M$ with $M \neq M(\ell, t)$. We cannot have $|M| > |M(\ell, t)|$, because $|M(\ell, t)|$ would contradict the minimality of $|M|$. We cannot have $|M| \leq |M(5\ell + 1, t - 1)|$ by constraint C2.

If $|M(5\ell + 1, t - 1)| < |M| < |M(\ell, t)|$, by constraint C3 and the lemma, we must have $M_3$ with $|M(5\ell + 1, t - 1)| < M' \leq |M_3| < |M|$. Because $|M_3| < |M|$, we know $T_3 \to M_3$ by induction. From the lemma, we get $|T(\ell, t)| < |T_3|$, but that contradicts constraint C3.

We must have $T(\ell, t) \to M(\ell, t)$, and $M(5\ell + 1, t - 1)$ is eligible for our $M_3$. This completes most of the proof of (3.17) and hence of Theorem 3.1.

Underlying the above analysis has been an assumption that the $M$-classes are always hit by $T$-classes. We show now that it could not have occurred that an $M$-class supported a differential. Assume that $M = y_1^{p-1}z_1$ is the $M$-class of lowest grading which supports a differential. We now revert to letting $|x|$ denote the actual grading of a class $x$, not divided by 2.

### Table 1. Cases in order.

| $\ell$ | $t$ | $|T|$ | $|M|$ | $M'$ | $\ell$ | $t$ | $|T|$ | $|M|$ | $M'$ |
|--------|-----|------|------|------|--------|-----|------|------|------|
| 0      | 1   | 6    | 26   | 10   | 36     | 1   | 726  | 746  | 730  |
| 1      | 1   | 26   | 46   | 30   | 37     | 1   | 746  | 766  | 750  |
| 2      | 1   | 46   | 66   | 50   | 7      | 2   | 726  | 826  | 746  |
| 0      | 2   | 26   | 126  | 46   | 40     | 1   | 806  | 826  | 810  |
| 5      | 1   | 106  | 126  | 110  | 41     | 1   | 826  | 846  | 830  |
| 6      | 1   | 126  | 146  | 130  | 42     | 1   | 846  | 866  | 850  |
| 7      | 1   | 146  | 166  | 150  | 8      | 2   | 826  | 926  | 846  |
| 1      | 2   | 126  | 226  | 146  | 45     | 1   | 906  | 926  | 910  |
| 10     | 1   | 206  | 226  | 210  | 46     | 1   | 926  | 946  | 930  |
| 11     | 1   | 226  | 246  | 230  | 47     | 1   | 946  | 966  | 950  |
| 12     | 1   | 246  | 266  | 250  | 9      | 2   | 926  | 1026 | 946  |
| 2      | 2   | 226  | 326  | 246  | 50     | 1   | 1006 | 1026 | 1010 |
| 15     | 1   | 306  | 326  | 310  | 51     | 1   | 1026 | 1046 | 1030 |
| 16     | 1   | 326  | 346  | 330  | 52     | 1   | 1046 | 1066 | 1050 |
| 17     | 1   | 346  | 366  | 350  | 1      | 3   | 626  | 1126 | 714  |
| 3      | 2   | 326  | 426  | 346  | 10     | 2   | 1026 | 1126 | 1046 |
| 20     | 1   | 406  | 426  | 410  | 55     | 1   | 1106 | 1126 | 1110 |
| 21     | 1   | 426  | 446  | 430  | 56     | 1   | 1126 | 1146 | 1130 |
| 22     | 1   | 446  | 466  | 450  | 57     | 1   | 1146 | 1166 | 1150 |
| 4      | 2   | 426  | 526  | 446  | 11     | 2   | 1126 | 1226 | 1146 |
| 25     | 1   | 506  | 526  | 510  | 60     | 1   | 1206 | 1226 | 1210 |
| 26     | 1   | 526  | 546  | 530  | 61     | 1   | 1226 | 1246 | 1230 |
| 27     | 1   | 546  | 566  | 550  | 62     | 1   | 1246 | 1266 | 1250 |
| 0      | 3   | 126  | 626  | 214  |        | 3086 | 3106 | 3090 |
| 5      | 2   | 526  | 626  | 546  | 154    | 1   | 3086 | 3106 | 3090 |
| 30     | 1   | 606  | 626  | 610  | 0      | 4   | 626  | 3126 | 1050 |
| 31     | 1   | 626  | 646  | 630  | 5      | 3   | 2626 | 3126 | 2714 |
| 32     | 1   | 646  | 666  | 650  | 30     | 2   | 3026 | 3126 | 3046 |
| 6      | 2   | 626  | 726  | 646  | 155    | 1   | 3106 | 3126 | 3110 |
| 35     | 1   | 706  | 726  | 710  | 156    | 1   | 3126 | 3146 | 3130 |
In \(k(1)^*(K_2)\), \(M\) supports a \(v\)-tower of \(v\)-height \(r'(t - 1)\) by 3.2. We will show at the end of the proof that there is a number \(\Delta \leq t\) such that \(v' M\) maps nontrivially to \(kup^{*+1}(K_2)\) if and only if \(i \leq r'(t - 1) - \Delta\). (Usually \(\Delta = 1\).) The image of \(M\) in \(kup^{M+1}(K_2)\) is a class \(C\) of positive filtration such that \(v'(u-1-\Delta)C \neq 0\) and \(v'(u-1-\Delta+1)C = 0 \in kup^*(K_2)\), so there must be a differential in the ASS of \(kup^*(K_2)\) from a filtration-0 class hitting a class of filtration \(\geq r'(t - 1) - \Delta + 2\) in grading \(|M| + 2(p-1)(r'(t - 1) - \Delta + 1)\). (The reason that the differential must start from filtration 0 is that in even gradings, \(E_2\) consists entirely of \(v\)-towers starting in filtration 0.) This differential cannot come from another such \(M\) because of our lowest-grading assumption. It cannot come from a product of one or more \(z\)'s times one of these \(M\)'s because \(z\)'s are infinite cycles. We must rule out the possibility that this differential is one of type (3.3). They are distinguished by having the smallest \(z\)-subscript at least 2 greater than the \(p\)-exponent of the exponent of \(y_1\).

The differential to \(C\) has subscript \(\geq r'(t - 1) - \Delta + 2\), and so the class in (3.3) would be \(y_1^{(p'^{(u-1)-\Delta})} Z\) for some positive integer \(\ell\), where \(Z\) is a product of \(z_j\)'s with \(j \geq r'(t - 1) - \Delta + 2\), and each \(j\) appears at most \(p - 1\) times, except that the smallest \(j\) might appear \(p\) times. Equating this grading with \(|M| - 2(p-1)(r'(t - 1) - \Delta + 1)\) and canceling a common factor 2 from all terms yield

\[
\ell p'^{(u-1)-\Delta+1} + \sum_j (p'^{j+1} + 1) = ip' + p'^{j+1} + 1 - (p-1)(r'(t - 1) - \Delta + 1). \tag{3.20}
\]

Using (3.12) and (3.14) and \(\Delta \leq t\), the right-hand side of (3.20) equals \(p'(i + 1) + (p-1)(r(t - 1) + \Delta - 1) + 1 \equiv (p-1)(r(t - 1) + \Delta - 1) + 1 \mod p'\), with \((p-1)(r(t - 1) + \Delta - 1) + 1 \leq p'\) (strict if \(t > 2\)). Since \(r'(t - 1) - \Delta > t\), this implies that the \(\sum_j\) on the left-hand side of (3.20) must contain at least \((p - 1)(r(t - 1) + \Delta - 1) + 1\) summands. We obtain

\[
\sum_p p' \geq p \cdot p'^{(u-1)-\Delta+2} + (p-1)\left(p'^{(u-1)-\Delta+3} + \cdots + p'^{(u-1)+r(t-1)}\right)
\]

\[
= p'^{(u-1)+r(t-1)+1} = p'^{j+1},
\]

so \(\sum_j p'^{j+1} \geq p'^{j+2}\), and hence \(p'(i + 1) > p'^{j+2}\). Thus \(i \geq p'^{j+2} > p'^{j-2r}\).

Since \(d_{p'-i+1}(y_1^{p'^{j-1}})\) is defined,

\[
d_r(y_1^{p'^{j-1}}) = 0 \text{ for } r \leq p' - t, \tag{3.21}
\]

and by the lowest-grading assumption, \(d_{p'-t}(y_1^{(i-p'^{-2}+1)p'^{j-1}-1}) = v' y_1^{(i-p'^{-2}+1)p'^{j-1}-1} z_t y_1^{(i-p'^{-2}+1)p'^{j-1}-1})\) is a permanent cycle. Since

\[
y_1^{p'^{j-1}} z_t = y_1^{(i-p'^{-2}+1)p'^{j-1}-1} z_t y_1^{(i-p'^{-2}+1)p'^{j-1}-1},
\]

we deduce that \(y_1^{p'^{j-1}} z_t\) survives to \(E_{p'-t}\), and (3.17), using the derivation property of differentials.

Now we consider the need for \(\Delta\) in the above argument. The worry is that maybe part of the \(v\)-tower on \(M\) in \(k(1)^*(K_2)\) might be in the image from \(kup^*(K_2)\), due to a filtration jump from a lower tower, as sketched in Figure 14, so that only a smaller part of the \(M\)-tower in \(k(1)^*(K_2)\) maps to \(kup^{*+1}(K_2)\).
The monomials $M_i = y_0^εz_0^ε$ ($ε = 1, 2$) have $|M_i| = 2(p^ε(i_ε + p) + 1)$ and are truncated in $k(1)^*(K_2)$ in grading $M'_i = |M_i| - 2(p - 1)r(t_i - 1)$. In $kup^ε(K_2)$, $M_2$ is truncated in grading $|T_2| = |v^εr^εM_2| = 2(p^ε(i_2 + 1) + 1)$. In Figure 14, elements $c$ are in grading $M'_2$, and $c'$ is in grading $M'_1 + 2(p - 1)$. The necessary condition for nontrivial image in $k(1)^*(K_2)$ (and hence $Δ > 1$) is

$$|T_2| + 2(p - 1) ≤ M'_1 + 2(p - 1) ≤ M'_2.$$  \hspace{1cm} (3.23)

If this occurs, then we might have $Δ$ as large as $\frac{M'_2 - M'_1}{2(p - 1)} + 1$. We now show in Lemma 3.24 that if (3.23) holds, then $(M'_2 - M'_1)/(2(p - 1)) < t$, establishing the claim made earlier about $Δ ≤ t$.

We restrict to $p = 5$, $i = 4ℓ$ for simplicity, and so that the reader can refer to Table 1 as an aid. The argument easily generalizes to any prime and any congruence. We divide everything by 2 as was done above and also subtract off the $+1$ which occurs in formulas for $|M|$ and $|T|$, so the numbers will be 1 smaller than those in the table.

**Lemma 3.24.** If $t_1 > t_2$ and

$$5^c(4\ell_2 + 1) + 4 \leq 5^c(4\ell_1 + 5) - 4r'(t_1 - 1) + 4 \leq 5^c(4\ell_2 + 5) - 4r'(t_2 - 1),$$

then

$$\frac{1}{4}(5^c(4\ell_2 + 5) - 4r'(t_2 - 1) - (5^c(4\ell_1 + 5) - 4r'(t_1 - 1))) < t_1 - 1.$$  \hspace{1cm} (3.24)

**Proof.** If there is a counterexample to this, then there is one with $\ell_1 = 0$, since $\ell_2$ could be decreased by $5^{h-2}\ell_1$, so it suffices to use $\ell_1 = 0$. Let $Q(k) = (5^k - 1)/24$ (called $q(k)$ in [8, Lemma 5.3]). Then, using [8, Lemma 5.5] for $t = 2k + δ$ with $δ = 1$ or 2,

$$5^{c+1} - 4r'(t - 1) = 5^{c+δ} + 16 \cdot 5^δQ(k) + 4k + 4 \cdot 5^{h-1}.$$  \hspace{1cm} (3.25)

Since $16 \cdot 5^δQ(k) + 4k + 4 \cdot 5^{h-1} < 3 \cdot 5^{2δ+δ}$, the hypothesis of the lemma says that $5^{c+1} - 4r'(t_1 - 1)$ mod $4 \cdot 5^c$ lies in the mod-$\cdot(4 \cdot 5^c)$ interval $[5^c, 5^c+1 - 4r'(t_2 - 1) - 4]$.

Let $t_1 = 2k_1 + δ_1$ and $t_2 = 2k_2 + δ_2$. The condition is restated as:

$$5^{2k_1+δ_1} + 16 \cdot 5^δQ(k_1) + 4k_1 + 4 \cdot 5^{h-1}$$  \hspace{1cm} (3.26)

lies in the mod-$\cdot(4 \cdot 5^c)$ interval:

$$[5^c, 16 \cdot 5^δQ(k_2) + 4k_2 + 4 \cdot 5^{k_2-1} - 4].$$  \hspace{1cm} (3.27)

Let $δ_2 = 1$. The reduction mod $4 \cdot 5^c$ of (3.26) is

$$5^{2k_2} + 16 \cdot 5^hQ(k_2) + 4k_2 + 4 \cdot 5^{h-1}.$$  \hspace{1cm} (3.27)

Let $δ_1 = 2$. Then, $5^{2k_2} + 16 \cdot 5^hQ(k_2) > 4 \cdot 5^c$ and equals $5^{2k_2+2} - (2000Q(k_2 - 1) + 100)$, so (3.27) will first be in the interval (3.26) when $4k_2 + 20 = 2000Q(k_2 - 1) + 100$, hence $k_1 = 500Q(k_2 - 1) + 20$, so $t_1 = 1000Q(k_2 - 1) + 42$. The left-hand side of the conclusion of the lemma is $\frac{1}{8}(M'_2 - M'_1)$ with $M'_1$ and $M'_2$ as in (3.23). For $k_1 = 500Q(k_2 - 1) + 20$, the value of $M'_1$ is at the left end of the interval (3.26), and so $\frac{1}{8}(M'_2 - M'_1)$ equals $\frac{1}{8}$ times the length plus 4 of (3.26), which is

$$20Q(k_2) + k_2 + 1 = 500Q(k_2 - 1) + k_2 + 21 = \frac{1}{2}t_1 + k_2.$$  \hspace{1cm} (3.27)

Since $k_2 << t_1$, this is less than $t_1 - 1$, as claimed. If $k_1$ is increased from the value $500Q(k_2 - 1) + 20$, the value of $t_1$ increases, while $M'_1 - M'_2$ decreases, since $M'_1$ is moving through the interval, so the inequality asserted in the lemma is satisfied more strongly.

Now, with $δ_2 = 1$ continuing, let $δ_1 = 1$. Since $k_1 > k_2$, (3.27) lies outside the interval (3.26) until $80Q(k_2) + 4k_1 + 4 = 4 \cdot 5^c$, so

$$k_1 = 5^{2k_2+1} - 20Q(k_2) - 1 = 100Q(k_2) + 4$$
and $t_1 = 200Q(k_2) + 9$. Again $\frac{1}{8}(M'_2 - M'_1) = 200Q(k_2) + k_2 + 1 \approx \frac{1}{10} t_2 + k_2$, so the conclusion of the lemma is satisfied more strongly.

A similar analysis works when $\delta_2 = 2$. In this case, $\frac{1}{8}(M'_2 - M'_1) \approx \frac{1}{2} t_1 + k_2$ if $\delta_1 = 1$, and $\frac{1}{8}(M'_2 - M'_1) \approx \frac{1}{10} t_1 + k_2$ if $\delta_1 = 2$.

We close this section by explaining how Theorems 2.4 and 3.1 lead to the descriptions of $kup^*(K_2)$ given in Theorems 1.8 and 1.15, modulo exotic extensions. We begin with the portion in even gradings and restrict our attention to odd $p$. All elements in the $P[h_0, v, y_1]$ part of Theorem 2.4 support differentials of type (3.2). Note that $y_0^{p-1} = y_0^{p-1} = \prod_{j=0}^{k-1} y_j^{p-1}$. The first is easiest to write, the second occurs in Theorem 2.4, and the third in 1.5 and Figure 2. From 1.5, $y_0^{p-1} z_0$ is in $A_k$ for $k \geq 1$, the bottom right element in Figure 2. Then,

$$P[y_1] y_0^{p-1} z_0 = \bigoplus \mathcal{M}^4_k \cdot y_0^{p-1} z_0 \subset \bigoplus \mathcal{M}^4_k A_k.$$  (3.28)

The first part occurs in Theorem 2.4 and the last part in Theorem 1.8.

Now we consider $P[y_1] \otimes \bigoplus_{i=1}^t W_i \otimes TP_{p-1} z_i \otimes \Lambda_{j+1}$ in Theorem 2.4. The $\bigoplus$ part is all monomials $z_i M$ with $\ell \geq 1$ and $M \in \Lambda_\ell$. From Theorem 3.1, $y_i^j z_i M$ supports a differential (3.3) if $\ell \geq \nu(i) + 2$, while those with $\nu(i) \geq \ell - 1$ are hit by differentials (3.4) and (3.5), yielding $\nu$-towers with heights as given in 1.5. These are all monomials in $\bigoplus_{i=1}^t P[y_i, y_{i+1}, \ldots] z_i \Lambda_\ell$. From 1.5 or (1.7), the generators of the $\nu$-towers in $B_k$ are all

$$z_j \prod_{i=j}^{k-1} \left\{ z_i^{p-1}, y_i^{p-1} \right\}, \quad 1 \leq j \leq k.$$

Let $(z_i M)$ be the $y_i^j z_i^p$ factors of $M$. Then, $\mathcal{M}_k B_k$ consists of all monomials $z_i M$ such that $(z_i M)$, equals $y_i^{p-1}$ or $z_i^{p-1}$ for $\ell \leq i < k$, but not for $i = k$, and so every monomial $z_i M$ is in a unique $\mathcal{M}_k B_k$. From Theorem 3.1, $z_i M$ has $\nu$-height $p^i$ if and only if $M$ contains no $z$-factors, which explains the split into $\mathcal{M}_k^4$ and $\mathcal{M}_k^8$ in Theorem 1.8.

Now we address the odd gradings. The $P[h_0, v, y_1] v q$ part of Theorem 2.4 is totally removed either as sources (3.4) or targets (3.2) of differentials. See grading 17 in Figure 12 for a nice illustration. The $q y_1^{p-1} S_{\nu(i)+1} \ell$ part of Theorem 1.15 is formed from $TP_{\nu(i)+2} v q y_1^{p-1} W_i$ in 2.1 using (3.3). The generators of $S_{\nu(i)+1} \ell$ are $z_1, \ell, \ldots, z_{\nu(i)-1}, \ell, 1$, but to see the differential from (3.3), one should write $z_{i, \ell} = z_{\nu(i)+1}^{i+1} Z_{i+1}^{i+1}$, where

$$Z_i^j = (z_i \cdots z_{j-1})^{p-1} \text{ for } j > i, \quad \text{with } Z_0^i = 1.$$  (3.29)

The remaining generators of $q y_1^{p-1} W_\ell$, namely $q y_1^{p-1} z_{i, \ell}$ with $\ell - \nu(i) \leq j \leq \ell$, support differentials (3.5). There can be no unexpected exotic extensions among these summands for the reason noted at the end of Section 1. The ker $(p)$ elements in the $S$ summands play a very important role in the exact sequence.

4. Exotic extensions

In this section, we prove the following expansion of (1.6).

**Theorem 4.1.** If $i \geq 0$ and $k \geq k_0$,

$$p y_{k} y_{k-1}^{p-1} z_{k-1} = v^{p-1} y_{k}^{p-1} z_{k-2}$$

with an additional term $v y_{k}^{p-1} z_{k-2}^{p}$ if $k \geq k_0 + 2$.

The additional term is seen in Ext and will be ignored in the rest of this section. We have included the factor $y_k$, which is not automatic since $y_k$ is not a permanent cycle. Since, for example, $y_{k+1} = y_k^p$, we need not consider $y_i$ for $i > k$. It is automatic that this formula can be multiplied by $z_i$'s, since they do survive the spectral sequence.
The extension is deduced from the exact sequence:
\[ k\uparrow^*(K_2) \xrightarrow{p} k\uparrow^*(K_2) \rightarrow k(1)^*(K_2) \]
and the fact that \( v^{r(k-1)}y^e_kz_k = 0 \) in \( k(1)^*(K_2) \) with \( r(k-1) \geq p^k(p-1) \). Thus, \( v^{r(k-1)}y^e_kz_k \) must be divisible by \( p \) in \( k\uparrow^*(K_2) \), and, as we will show, the \( v \)-tower on \( y^e_ky^{p-1}_{k-1}z_{k-1} \) provides the only classes that can do the dividing. Once we know the division formula toward the end of the \( v \)-tower, we can deduce that it holds earlier in the tower, as well. For example, \( r(2) = p^3 - p^2 + p - 2 \), which is the height in the top \( v \)-tower in Figure 2 where the extensions into it do not also involve an \( h_0 \)-extension. We deduce the extensions from the earlier part of the \( v \)-tower on \( y^e_ky^{p-1}_{k-1}z_{k-1} \) by naturality.

We illustrate in Figure 15, using the notation of the preceding section. Thus, \( T_i \) is the class satisfying \( d_i(T_i) = v^iM_i \). Here, the portion of the top tower to the right of \( M' \) must be divisible by \( p \). The tower providing the extension must have \( M_1 \leq |M_2| < |M_1| \) and \( |T_2| \leq |T_1| \).

As we did for the differentials in the previous section, we will perform the argument for \( p = 5 \). It will be clear that it generalizes to an arbitrary odd prime, and with minor modification to \( p = 2 \). Also, we use \( i = 4\ell \) in Theorem 1.15. If instead we used \( i = 4\ell + d \) for \( 1 \leq d \leq 5 \), it will just add the same amount to the quantities \( |M|, \ |T|, \) and \( M' \) involved in the argument. We can use Table 1 to envision the analysis, with the \( t \) there replaced by \( k \). For a monomial \( M(\ell, k) = y^e_kz_k \), we have, after dividing by \( 2, \ |M| = 5^\ell(4\ell + 5) + 1, \ |T| = 5^\ell(4\ell + 1) + 1, \) and \( 5^\ell(4\ell + 1.16) + 1 < M' \leq 5^\ell(4\ell + 1.8) + 1, \) using (3.15). With \( M_1 \) and \( M_2 \) as in Figure 15, we will show that \( M_2(5\ell + 1, k - 1) \) is the unique monomial satisfying the inequalities stated just before Figure 15 for \( M_i(\ell, k) \). Note that \( M(5\ell + 1, k - 1) = y^e_ky^{p-1}_{k-1}z_{k-1} \).

We omit the +1 in all the formulas.

The inequalities are satisfied by \( M_2(5\ell + 1, k - 1) \) since
\[ 5^\ell(4\ell + 1.8) \leq 5^{k-1}(4(5\ell + 1) + 5) < 5^\ell(4\ell + 5) \text{ and } 5^{k-1}(4(5\ell + 1) + 1) \leq 5^k(5\ell + 1). \]

If \( k_2 \geq k \), then the first inequality, after dividing by \( 5^k \), becomes
\[ 4\ell + 1.8 \leq 5^{k-\ell}(4\ell + 5) < 4\ell + 5, \]
which cannot be satisfied since the middle term is \( \equiv 1 \) mod 4. If \( k_2 < k - 1 \), then
\[ M' - |T_1| > 5^{k-1} \cdot 1.16 \geq 4 \cdot 5^{k_2} = |M_2| - |T_2|, \]
which is inconsistent with two of the inequalities. Let \( k_2 = k - 1 \). If \( \ell_2 < 5\ell + 1 \), then
\[ |M_2| = 5^{k-1}(4\ell_2 + 5) \leq 5^{k-1}(4 \cdot 5\ell + 5) < 5^k(4\ell + 1.16) < M', \]
contradicting one of the inequalities. If \( k_2 = k - 1 \) and \( \ell_2 > 5\ell + 1 \), then
\[ |T_2| \geq 5^{k-1}(4(5\ell + 2) + 1) > 5^k(4\ell + 1) = |T_1|, \]
contradicting one of the inequalities.

We deduce that \( M_2 = y^e_ky^{p-1}_{k-1}z_{k-1} \), as claimed. We should perhaps have noted that the extensions could not have come from classes with more than one \( z_j \)-factor, because these are \( z_j \) times a class on which the extensions have already been determined.
5. Proposed formulas for the exact sequence (1.25)

In this section, we propose what we conjecture must be the correct complete formulas for the exact sequence (1.25). Some homomorphisms are forced by naturality, but many others involve significant filtration jumps. However, they all occur in several families with nice properties. The 10-term exact sequence (5.2) shows how the $S_{i+1}$ portions and the exotic extensions yield compatibility of the differing $v$-tower heights in $kup^*(K_2)$ and $(1)^*(K_2)$. In Section 6, we show that all elements of $(1)^*(K_2)$ are accounted for exactly once in these homomorphisms, which implies that there can be no more exotic extensions. This does not require us to prove that our homomorphism formulas are actually correct, as discussed at the end of Section 1. We will focus on the case when $p$ is odd. We could incorporate all primes together at the expense of involving the parameter $k_0$, but things are complicated enough without that. In an earlier version of this paper ([7]), a thorough analysis when $p = 2$ was performed.

We propose that (1.25) can be split into exact sequences of length 4 and 10 (not including 0's at the end). There are subgroups of $(1)^*(K_2)$ called $G_k^j$ and $G_k^j$ for $k \geq 1$ and $G_k^j$ for $3 \leq i \leq 6$ and $1 \leq k < \ell$ such that there are exact sequences:

$$0 \rightarrow G_k^j \rightarrow A_k \xrightarrow{p} A_k \rightarrow G_k^2 \rightarrow 0 \quad (5.1)$$

for $k \geq 1$, and, for $1 \leq k < \ell$,

$$0 \rightarrow G_k^{j} \rightarrow y_kB_kZ_k^j \xrightarrow{p} y_kB_kZ_k^j \rightarrow G_k^{j} \rightarrow y_k^j qS_{k,\ell} \rightarrow p \rightarrow y_k^{j-1} qS_{k,\ell} \rightarrow G_k^{j} \rightarrow B_kz_k \xrightarrow{p} B_kz_k \rightarrow G_k^{j} \rightarrow 0, \quad (5.2)$$

with $Z_k^j$ as defined in (3.29). The sequence (5.1) can be tensored with $TP^p_{p-1}[y_k] \otimes P[y_k+1]$, while (5.2) can be tensored with $TP^p_{p-1}[y_k] \otimes P[y_k+1] \otimes TP^p_{p-1}[z_k] \otimes \Lambda_{k+1}$. If $p$ is odd, there are also exact sequences:

$$0 \rightarrow G_k^{j} \rightarrow B_kz_k \xrightarrow{p} B_kz_k \rightarrow G_k^{j} \rightarrow 0 \quad (5.3)$$

for $k \geq 1$ and $1 \leq e \leq p - 2$. This can be tensored with $P[y_k] \otimes \Lambda_{k+1}$.

One can verify that the totality of $A_k$ and $B_k$ groups in these exact sequences agrees with that in Theorem 1.8. We will study these exact sequences by breaking them up into short exact sequences and isomorphisms involving kernels and cokernels of $p$.

Let $K^A_k = \ker (p|A_k)$, $K^B_k = \ker (p|B_k)$, $C^1_k = \text{coker} (p|A_k)$, and $C^2_k = \text{coker} (p|B_k)$. There are important elements $g_k \in K^A_k$ and $K^B_k$ defined (up to unit coefficients) by $g_1 = z_1$, $g_2 = v^{p-2}z_2$, and, for $k \geq 1$,

$$g_{k+2} = v^{k-1} z_{k+2} + g_k y_k^{p-1} z_{k+1}^{p-1} \quad (5.4)$$

To see that this is in $\ker(p)$, we use (1.6) to see that $p \cdot v^{k-1} z_{k+2} = v^{k-1} z_{k+1}^{p-1}$, and that the $v^{k-2} z_{k+1}^{p-1}$ term in $g_k$ yields $v^{k-2} z_{k+1}^{p-1}$ in $p \cdot g_k y_k^{p-1} z_{k+1}^{p-1}$. Using (3.10), these terms cancel. Other terms in $p \cdot g_k y_k^{p-1} z_{k+1}^{p-1}$ yield 0 since $g_k \in \ker(p)$.

The $v$-towers in $K^A_k$ are generated by:

$$g_k \text{ and } g_k y_k^{p-1} \prod_{i=j+1}^{k-1} \left\{ z_i^{-1}, y_i^{p-1} \right\}, \quad 1 \leq j \leq k - 1. \quad (5.5)$$

For example, using Figure 2 when $k = 3$, these are $g_3 = v^{p-1} z_3 + y_1^{p-1} z_2^{p-1}$, $g_2 y_2^{p-1} z_2^{-1}$, $g_1 z_2^{-1} = v^{p-2} z_2$, $g_1 y_2^{p-1} z_2^{-1}$, and $g_1 y_2^{p-1} z_2^{-1} y_2^{-1}$. The $v$-heights are $p^k - (r(k-2) - 1)$ for $g_k$, and $p^j - (r(j-2) - 1)$ for the others, since they are determined by $v$-heights of $z_i$ in $B_k$. The map $G_k^j \rightarrow K_k^j$ sends $w_k$ to $g_k$ and

$$w_j P \mapsto g_j P = z_j^{-1} \prod_{i=j+1}^{k-1} \left\{ z_i^{-1}, y_i^{p-1} \right\}, \quad (5.6)$$

with $w_j$ as in 3.8 and 3.16. The $v$-height of $w_j$ is $r(j)$ if it is not accompanied by $z_j$, and $r(j - 1)$ if it is. By (3.13) and ((3.12) and (3.11)), the $v$-heights agree, so (5.6) is an isomorphism on $v$-towers.
For \( L = K^a_k \) or \( K^b_k \) or \( C^a_k \) or \( C^b_k \), we say that a \( Z_p \) in \( L \) is a class of \( v \)-height 1 in \( L \) which is not part of a larger \( v \)-tower in \( L \). There is one \( Z_p \) in \( K^a_3 \), as can be seen in Figure 2. This is the element \( v^{p-2}y^p_1z^p_1z^p_2 \). Note that for \( i < p - 1 \), \( v^p y^p_{1}z^p_1z^p_2 + v^p y^p_{2}z^p_1z^p_2 \) is part of a \( v \)-tower in \( K^a_3 \), which continues with the elements \( v^p z^p_1 \) for \( i > p^2 - 3 \), but \( v^p y^p_{1}z^p_1z^p_2 \) itself is in \( K^a_3 \) only for \( i = p - 2 \). Using 1.5, we find that the \( Z_p \)'s in \( K^a_3 \) are

\[
v^{p^2-p-1}(y_1 \cdots y_{j-1})y^{p-1}_{j-1}z^{p-1}_{i} y^{p-1}_{j} \prod_{i=j+1}^{k-1} \{z^{p-1}_{i}, y^{p-1}_{i}\} \quad \text{for } 1 \leq t < j < k. \tag{5.7}
\]

For example, the elements \( v^{p^2-p}y^{p}y_{1}z^{p}_{1}z^{p}_{2} \) and \( v^{p^2-p}y^{p}_{2}z^{p}_{1}z^{p}_{2} \) in Figure 2 yield elements in \( K^a_3 \) after being multiplied by \( z^p_3 \). The basic formula for the homomorphism from part of \( k(1)^*(K_3) \) to \( Z_p \)'s in various \( K^a_3 \) and \( K^b_3 \), possibly tensored with other classes as in Theorem 1.8, is

\[
(q(y_1 \cdots y_{j})y^{p-1}_{j-1}z^{p-1}_{j-1}) \otimes P[y_j] \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1} \quad \text{for } j > t \geq 1. \tag{5.8}
\]

The domain elements are in the second half of the third line of Theorem 3.16. The ones that are in \( \text{in the isomorphism } G^3_k \rightarrow K^a_3 \) can be extracted using (5.7).

The isomorphism \( G^3_{k,0} \rightarrow y_k K^a_{k}Z^a_k \) in (5.2) is given using formulas analogous to (5.6) and (5.8). There are several minor differences. One is that the \( v \)-tower on \( y_k g_k Z^a_k \) is truncated due to \( v^{p^2-2}z_k = 0 \) in \( B_k \) (as opposed to \( v^{p^2}z_k = 0 \) in \( A_k \)). This is compatible with the fact that the \( v \)-height of \( w_k z_k \) in \( k(1)^*(K_3) \) is less than that of \( w_k \), using Theorem 3.16 and (3.11). The other is that \( K^a_3 \) has additional \( Z_p \)'s:

\[
v^{p^2-p-1}(y_1 \cdots y_{k-1})y^{p-1}_{k-1}z^{p-1}_t \quad \text{for } 1 \leq t < k - 1, \tag{5.9}
\]

as seen in Figure 2 when \( k = 3 \), but these are always multiplied by higher \( z \)'s, and so (5.8) applies.

The isomorphisms \( C^a_k \rightarrow G^3_k \) and \( C^b_k z_k \rightarrow G^3_k \) are defined simply by sending an element to one with the same name. Moreover, \( C^a_k = C^a_k \) except for \( (y_0 \cdots y_{k-1})y^{p-1}z_0 \in C^a_k - C^a_k \). When \( k = 3 \), we see that the \( Z_p \)'s in \( C^a_3 \) are \( \{z^{p}, y^{p}_{1}, y^{p}_{2}\} \) in Figure 2. For future reference,

\[
Z^a_p \text{'s in } C^a_k \text{ are } \left\{ z^{p}_t \prod_{i=t+1}^{k-1} \{z^{p-1}_i, y^{p-1}_i\} \mid 1 \leq t < k \right\}. \tag{5.10}
\]

The corresponding elements in \( k(1)^*(K_3) \) are from the third line of 3.2.

The \( v \)-towers in \( C^a_k = C^b_k \) are generated by:

\[
z_k \text{ and } y^{p-1}_{t}z^{p-1}_t \prod_{i=t+1}^{k-1} \{z^{p-1}_i, y^{p-1}_i\}, \quad 1 \leq t < k. \tag{5.11}
\]

We will show that the \( v \)-height of \( z_k \) in \( C^a_k \) is \( r'(k - 1) \), which equals its \( v \)-height in \( k(1)^*(K_3) \). It follows from 1.5 that the \( v \)-height of \( y^{p-1}_{t}z^{p-1}_t \prod_{i=t+1}^{k-1} \{z^{p-1}_i, y^{p-1}_i\} \) equals \( r'(t - 1) \), establishing the isomorphisms out of \( C^a_k \) and \( C^b_k \) in Figure 2, the \( v \)-height of \( z_k \) equalling \( p^2 - p + 2 = r'(2) \) is apparent.

The proof of the claim about \( v \)-heights is by induction. By (3.10), \( r'(k - 1) - r'(k - 3) = p^k - (p - 1) - 1 \). Let \( D = (|z_k| - |y^{p-1}_{k-1}z^{p-1}_{k-1}|) / (2(p - 1)) = p^{k-1}(p - 1) \). This is the filtration on the \( z_k \)-tower above the element \( y^{p-1}_{k-1}z^{p-1}_{k-1} \). We show that \( v^{r'(1)}D(z_k) \) is divisible by \( p \) if and only if \( v^{r'}z_{k-2} \) is divisible by \( p \). Thus, the difference of the \( v \)-heights in cokernels equals the difference of the corresponding \( r' \) values. From Theorem 4.1, we have

\[
pv^{r'-1}y^{p-1}_{k-1}z^{p-1}_{k-1} = v^{r'(1)}D(z_k) = y^{p-1}_{k-1}z^{p-1}_{k-2} + pv^{r'-1}y^{p-1}_{k-1}z^{p-1}_{k-2}.
\]

The claim follows, since \( v^{r'}y^{p-1}_{k-1}z^{p-1}_{k-2} \) is divisible by \( p \) if and only if \( v^{r'}z_{k-2} \) is, by 1.5.

The analysis of (5.3) is extremely similar.

Now \( S_{k,\ell} \) becomes involved. Let \( S^k_{R, k, \ell} = \ker \left( \langle p \rangle \cdot S_{k, \ell} \right) \) and \( S^C_{k, \ell} = \ker \left( \langle p \rangle \cdot S_{k, \ell} \right) \). Then, \( S^C_{k, \ell} \) consists of \( TP_{k+1}[v](z_{k, \ell}) \) plus \( Z_p \)'s on \( v^{p}z_{k, \ell} \) for \( 2 \leq t \leq \ell - k \), while \( S^k_{R, k, \ell} \) has \( TP_{k+1}[v](z_{k-2, \ell}) \) plus \( Z_p \)'s on \( z_{k, \ell} \) for

---

4The class \( y^{p-1}_{p+1}z^{p-1}_{p+1} \) should really be called \( y^{p-1}_{p+1}z^{p-1}_{p+1} + v^{r'(p-1)-1}z \) so that \( v \) times it is divisible by \( p \), hence 0 in \( C^a_3 \), but we will ignore this fine-tuning.
1 \leq i < \ell - k. Next, we consider the short exact sequence:

\[ 0 \to y_k C^{\ell}_k Z_k^l \overset{\phi}{\longrightarrow} G_{k,\ell}^4 \overset{\psi}{\longrightarrow} y_1^{\ell-1-1} qS_{k,\ell}^p \to 0. \tag{5.12} \]

The map \( \phi \) sends everything except the \( v \)-tower on \( y_k z_k Z_k^l \) to classes with the same name, and the heights of these \( v \)-towers agree, as seen above. The class \( y_k z_k Z_k^l = y_k z_{k,\ell} \) maps to a \( \mathbb{Z}_p \) with the same name in \( k(1)^*(K_2) \). We have \( \psi (w_k w_{k+1} Z_{k+1}^l) = q y_1^{\ell-1-1} z_{k,\ell} \). Then \( v^{k+1} w_k w_{k+1} Z_{k+1}^{l+1} \in \ker (\psi) \), and we have

\[ \phi (v y_k z_{k,\ell}) = v^{k+1} w_k w_{k+1} Z_{k+1}^l. \]

We illustrate this in the schematic Figure 16, in which \( X, \circ, \) and \( \bullet \) map to elements with the same symbol. The expressions at the end of the \( v \)-towers are their \( v \)-heights. In particular, \( v^{(k-1)} y_k z_k \neq 0 \) in \( y_k C^{\ell}_k Z_k^l \).

The \( v \)-heights agree by (3.11), and the gradings match by an induction proof. The \( \mathbb{Z}_p \)'s in \( y_1^{\ell-1-1} qS_{k,\ell}^p \) are hit by \( \psi (y_k z_{k+\ell-1,\ell}) = y_1^{\ell-1-1} q v^i z_{k,\ell} \), \( 2 \leq i \leq \ell - k \), another interesting filtration jump.

Finally, we consider the short exact sequence:

\[ 0 \to y_1^{\ell-1-1} qS_{k,\ell}^C \overset{\phi'}{\longrightarrow} G_{k,\ell}^5 \overset{\psi'}{\longrightarrow} K_{k,\ell}^n \to 0. \tag{5.14} \]

Similarly to (5.5), the generators of \( v \)-towers in \( K_{k,\ell}^n \) are \( g_k \) and, for \( 1 \leq j < k \), elements of the form

\[ g_{r,j} \prod_{i=1}^{p-1} \{ z_{r,j+1}^{-1}, y_i^{p-1} \}. \]

The morphism \( \psi' \) is determined by \( w_j \mapsto g_j \). The \( v \)-heights of the corresponding elements in \( k(1)^*(K_2) \) and \( K_{k,\ell}^n \) both equal \( r'(j-1) \) for \( j < k \). However, the \( v \)-height of \( w_k z_k \) is \( r(k) \), which is \( k \) greater than \( r'(k-1) \). We have \( \phi' (v y_1^{\ell-1-1} q z_{\ell-k,\ell}) = v^{r(k-1)} w_k z_{k,\ell} \). The class \( y_1^{\ell-1-1} q z_{\ell-k,\ell} \) at the base of the \( v \)-tower maps to a \( \mathbb{Z}_p \) with the same name. The picture is quite similar to Figure 16 with \( k + 1 \) and \( r'(k-1) \) interchanged.

The \( \mathbb{Z}_p \) classes \( y_1^{\ell-1-1} q z_{\ell,\ell} \) for \( 1 \leq i \leq \ell - k \) are mapped by \( \phi' \) to classes with the same name in \( G_{k,\ell}^5 \subset k(1)^*(K_2) \). The \( \mathbb{Z}_p \)'s in \( K_{k,\ell}^n z_{\ell,\ell} \) are of the same form as in (5.7) and are hit by analogs of (5.8).

6. All accounted for

In this section, we show that all elements of \( k(1)^*(K_2) \) are involved in exactly one of the homomorphisms involving some \( G \)-group described in the preceding section. As discussed earlier, this implies that there can be no exotic extensions in \( kup^*(K_2) \) other than those in (1.6), because an additional extension would decrease the number of elements in \( \ker (\mathbb{p}kup^*(K_2)) \) and \( \coker (\mathbb{p}kup^*(K_2)) \), and these must correspond to elements of \( k(1)^*(K_2) \). It also provides an excellent check on our analysis.
Let $p$ be odd, $G'_k$ and $G'_{k,\ell}$ as in (5.1) and (5.2), and

$$G' = \begin{cases} 
\bigoplus_{k \geq 1} G'_k \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] & 1 \leq i \leq 2 \\
\bigoplus_{1 \leq k < \ell \leq p-2} G'_{k,\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_{\ell}] \otimes \Lambda_{\ell+1} & 3 \leq i \leq 6 \\
\bigoplus_{k \geq 1} G'_{k,\ell} \otimes P[y_k] \otimes \Lambda_{k+1} & 7 \leq i \leq 8.
\end{cases}$$

**Theorem 6.1.** $G^1 \oplus \cdots \oplus G^8$ equals $k(1)^*(K_2)$, as described in Theorem 3.16.

As throughout the paper, $\mathbb{Z}_p$'s coming from $E_1$-free submodules of $H^*(K_2)$ are ignored here. The remainder of this section is devoted to the proof of Theorem 6.1. There are four parts of Theorem 3.16. We deal with one at a time.

**Case 1.** $P[y_1] y_0^{p-1} z_0$. In (3.28), it is shown that these classes form a subset of $\bigoplus M^i A_k$, and they map to classes with the same name in $G^2$.

**Case 2.** $\bigoplus_{j \geq 0} TP_{r(j)}[v] \otimes P[y_{j+1}] \otimes TP_{p-1}[y_j] \otimes E[w_j] \otimes E[w_{j+1}] \otimes \Lambda_{j+1}$. The generators of $v$-towers of height $r(j)$ occur in $G^1$, $G^4$, and $G^5$. From (5.6), only $w_j$ is in $G^1_j$. So $G^1$ has $TP_{p-1}[y_j] \otimes P[y_{j+1}] w_j$. From Figure 16, $G^4_{j,k}$ has $w_j w_{j+1} z_{j+1}^r$. Note that $\bigoplus_{r \leq 1} TP_{p-1}[z_{\ell}] \otimes \Lambda_{\ell+1} = \Lambda_{j+1}$, since the $\ell$-component gives the monomials whose smallest non-$(p-1)$-power is a power of $z_\ell$, so $G^4$ contains $P[y_{j+1}] \otimes TP_{p-1}[y_j] w_j w_{j+1} \otimes \Lambda_{j+1}$. From the analysis following (5.14), $G^5_{j,k}$ has only $w_j z_\ell$ of $v$-height $r(j)$, so $G^5$ will have $P[y_{j+1}] \otimes TP_{p-1}[y_j] w_j \otimes \Lambda_{j+1}$. Thus, $G^1 \oplus G^5$ contains the part without $w_{j+1}$, while $G^4$ contains the part with $w_{j+1}$.

**Case 3.** $\bigoplus_{j \geq 1} TP_{r(j-1)}[v] \otimes P[y_j] \otimes E[w_j] \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}$. The generators of $v$-towers of height $r'(j - 1)$ occur in each $G^i$ as follows.

$G^1$: $w_j z_j^{p-1} \bigoplus_{k \geq j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \bigoplus_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}$. This can be deduced from (5.6).

$G^2$: From (5.11),

$$z_j TP_{p-1}[y_j] \otimes P[y_{j+1}] \otimes y_j^{p-1} z_j \bigoplus_{j \geq j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

$G^3$: We use (5.5) and (5.6) and adapt some arguments used in Case 2 to obtain

$$w_j z_j^{p-1} \left( TP_{p}[y_j] \otimes P[y_{j+1}] \otimes \Lambda_{j+1} \otimes \bigoplus_{k \geq j+1} TP_{p}[y_k] P[y_{k+1}] z_k^{p-1} \Lambda_{k+1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} \right).$$

$G^4$: We use (5.11) and (5.12) to obtain

$$y_j^{p-1} z_j \bigoplus_{k \geq j+1} TP_{p}[y_k] \otimes P[y_{k+1}] z_k^{p-1} \Lambda_{k+1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

$G^5$: We use (5.14) and $\bigoplus_{\ell-k} z_\ell TP_{p-1}[z_{\ell}] \otimes \Lambda_{\ell+1} \simeq \Lambda_{k+1}$ to obtain

$$w_j z_j^{p-1} \bigoplus_{k \geq j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \Lambda_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$
$G^0$: We combine the analysis for $G^2$ and the observation used for $G^5$ to obtain

$$z_jTP_{p-1}[y] \otimes P[y_{j+1}] \otimes \Lambda_{j+1}$$

$$\oplus y_j^{p-1}z_j \bigoplus_{k>j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \Lambda_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}$$

$G^3$: Similarly to $G^0$, we have

$$\bigoplus_{e=1}^{p-2} \left( w_jz_j^e \otimes P[y_j] \otimes \Lambda_{j+1} \oplus w_jz_j^{p-1} \bigoplus_{k>j+1} z_j^e \otimes P[y_k] \otimes \Lambda_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} \right).$$

$G^5$: Using (5.11), we get

$$\bigoplus_{e=1}^{p-2} \left( z_j^e \otimes P[y_j] \otimes \Lambda_{j+1} \oplus y_j^{p-1}z_j \bigoplus_{k>j+1} z_j^e \otimes P[y_k] \otimes \Lambda_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} \right).$$

We begin by analyzing the portion including the factor $w_j$. We will show that

$$G^1 \oplus G^3 \oplus G^5 \oplus G^6 = P[y_j]w_j \otimes TP_p[z_j] \otimes \Lambda_{j+1}.$$ 

Here, and in the remainder of our analysis of Case 3, $G^r$ refers just to the relevant portion of $G^r$, here the part with $TP_{r(j-1)}[y]w_j$. The first part of $G^6$ gives all terms with $z_j^e$ for $1 \leq e \leq p - 2$. The remaining part has factors $w_jz_j^{p-1}$, which we will omit writing. Combining $G^1$ and $G^5$ removes the bar in $G^5$. The first part of $G^3$ gives the part with positive exponent of $y_j$, which we now omit.

Let $E_i = P[y_i] \otimes \Lambda_i$, thought of as monomials in $y_i$ and $z_i$ for $i \geq \ell$ with exponents $\leq p - 1$. The remaining parts of the $G^r$’s under consideration combine to

$$\bigoplus_{k>j+1} \left( TP_{p-1}[y_k] \oplus y_k z_k^{p-1} TP_{p-1}[y_k] \oplus \bigoplus_{e=1}^{p-2} z_j^e TP_p[y_k] \right) \otimes E_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}. \tag{6.2}$$

We wish to show this equals $E_{j+1}$. The portion in parentheses is all monomials in $TP_p[y_k, z_k]$ except $y_k^{p-1}$ and $z_k^{p-1}$. For a monomial $M$ in $E_{j+1}$, let $M_i$ denote its $y_j^i, z_j^i$ factor. The $k$-summand in (6.2) is all monomials $M$ in $E_{j+1}$ for which $k$ is the smallest $i$ such that $M_i$ is neither $y_i^{p-1}$ nor $z_i^{p-1}$. Thus, the sum over all $k$ yields all of $E_{j+1}$, as claimed.

A very similar argument shows that the $G^2 \oplus G^4 \oplus G^6 \oplus G^8$ part for Case 3 equals the portion which includes just the 1 in $E[w_j]$, that is, $P[y_j] \otimes TP_p[z_j] \otimes \Lambda_{j+1}$.

**Case 4.** $\bigoplus_{j \geq 1} P[y_j] \otimes E[q] \otimes \bar{E}[z_j^e] \otimes \Lambda_{j+1}$. We first consider the part without the $q$, and fix $j$ and omit writing the $z_j^e$. The desired answer is $P[y_j] \otimes \Lambda_{j+1}$. These come from the $\mathbb{Z}_p$’s in $G^2 \oplus G^4 \oplus G^6 \oplus G^8$. Similarly to Case 3, $G^2$ and $G^6$ combine to give

$$\bigoplus_{k>j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \Lambda_{k+1} \otimes \prod_{i=k+1}^{j-1} \{z_i^{p-1}, y_i^{p-1}\}.$$ 

This, together with the portion of $G^4$ from im(φ) in (5.12) obtained using (5.10), and the $\mathbb{Z}_p$’s in $G^8$ obtained using (5.10) give exactly (6.2), which we showed equals $P[y_{j+1}] \otimes \Lambda_{j+1}$.\footnote{Here, the classes in (6.2) are $\mathbb{Z}_p$’s and are multiplied by $z_j^e$, whereas in Case 3 they were multiplied by $w_jz_j^{p-1}$ and were generators of $\nu$-towers of height $\nu(j-1)$.} The element $X$ in Figure 16 with $k$ replaced by $j$, from $G^4$,

$$y_j TP_{p-1}[y] \otimes P[y_{j+1}] \otimes \bigoplus_{\ell \geq j} Z_{j+1}^{p-1} TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}$$

$$= y_j TP_{p-1}[y] \otimes P[y_{j+1}] \otimes \Lambda_{j+1},$$
which combines with the portion just obtained to yield $P[y_j] \otimes \Lambda_{j+1}$.

The last line of the $G^4_{k,\ell}$ discussion in Section 5 describes $\mathbb{Z}_p$’s in $G^4$ mapped by $\psi$ in (5.12). Those with a $z_j$ factor yield

$$\bigoplus_{k=1}^{j-1} y_k TP_{p-1}[y_k] P[y_{k+1}] \bigoplus_{\ell > j} Z_{j+1}^\ell TP_{p-1}[z_\ell] \Lambda_{\ell+1}$$

$$= \bigoplus_{k=1}^{j-1} (P[y_k] - P[y_{k+1}]) \otimes \Lambda_{j+1}$$

$$= (P[y_1] - P[y_j]) \otimes \Lambda_{j+1}.$$  

Combining this with the result of the preceding paragraph yields the desired $P[y_1] \otimes \Lambda_{j+1}$.

We finish this section by showing that the $\mathbb{Z}_p$’s including a factor $q$ are obtained exactly once. We omit writing the $q$'s in (5.12). The classes which we must obtain are $P[y_1] \bigoplus_{j=1} \mathbb{Z}_p \Lambda_{j+1}$. There are eight ways these appear in $G^4$-sets.

1. In $G^1$, using (5.7) and (5.8), for $1 \leq i < j < k$,

$$y_1^{p-1} - z_{ij} z_j z_{ij}^2 \prod_{s=j+1}^{k-1} \{ z_s^{p-1}, y_s^{p-1} \} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}].$$

2. In $G^3$, using (5.9) and (5.8), for $1 \leq i < k < \ell$,

$$y_1^{p-1} - y_k z_k z_k z_k^2 \prod_{s=j+1}^{k-1} \{ z_s^{p-1}, y_s^{p-1} \} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

3. In $G^3$, using (5.7) and (5.8), for $1 \leq i < j < k < \ell$,

$$y_1^{p-1} - y_k z_k z_k z_k^2 \prod_{s=j+1}^{k-1} \{ z_s^{p-1}, y_s^{p-1} \} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

4. From im $(\psi')$ in (5.14), for $1 \leq k < \ell$ and $1 \leq i \leq \ell - k$,

$$y_1^{p-1} - z_{i-\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

5. From $\psi'$ in (5.14), using (5.9) and (5.8), for $k < \ell$ and $\ell - k < i < \ell$,

$$y_1^{p-1} - z_{i-\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

6. From $\psi'$ in (5.14), using (5.7) and (5.8), for $i < j < k < \ell$,

$$y_1^{p-1} - z_{i-\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

7. From (5.3), using (5.9) and (5.8), for $i < k$ and $1 \leq e < p - 2$,

$$y_1^{p-1} - z_{i-\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \Lambda_{\ell+1}.$$

8. From (5.3), using (5.7) and (5.8), for $i < j < k$ and $1 \leq e < p - 2$,

$$y_1^{p-1} - z_{i-\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \Lambda_{\ell+1}.$$

First combine (1)+(6) to put a $\otimes \Lambda_{\ell+1}$ at the end of (1), and then, similarly to the simplification of (6.2), combine with (3)+(8) to get

$$\bigoplus_{i<j} y_1^{p-1} P[y_{j+1}] z_{ij} z_{ij}^2 \Lambda_{j+1}.$$

(6.3)
We combine and relabel (4)+(5) to give
\[ \bigoplus_{i<j} y_i^{p-1} T P_{p-1} [y_j] P [y_{j+1}] z_{i,j+1} \Lambda_{j+1} \] (6.4)

It together with
\[ \bigoplus_{i\geq j\geq 1} y_i^{p-1} T P_{p-1} [y_j] P [y_{j+1}] z_j^{p} \Lambda_{i+1} \] (6.5)

Let \( Y(s) = y_1^{p-1} T P_{p-1} [y_{s+1}] P [y_{s+2}] = \langle y_1^s; v(i+1) = s \rangle \). Then (6.5) is
\[ \bigoplus_{i \geq 0} Y(s) z_i^p \Lambda_{i+1} \] (6.6)

We simplify and relabel (2) to
\[ \bigoplus_{i<j} y_i^{p-1} T P_{p-1} [y_j] P [y_{j+1}] z_{i,j}^{p} \Lambda_{j+1} \] (6.7)

(6.3), (6.7), and (7) combine to give
\[ \bigoplus_{i<j} y_i^{p-1} P [y_j] z_{i,j} T P_{p-1} [z_j] \Lambda_{j+1} = \bigoplus_{i \leq j \leq t} Y(t) z_j T P_{p-1} [z_j] \Lambda_{j+1} \]

For any \( t \geq i \), the coefficient of \( Y(t) z_i^p \) in (6.4) plus this is
\[ Z_{i+1}^{t+2} \Lambda_{i+2} \bigoplus_{j=i+1}^{t+1} Z_{i+1}^{t+2} T P_{p-1} [z_j] \Lambda_{j+1} = \Lambda_{i+1}, \]

as the second part has all monomials not divisible by \( Z_{i+1}^{t+2} \). Combining this with (6.6) yields the desired result,
\[ \bigoplus_{x \geq 0} Y(s) \bigoplus_{i \geq 1} z_i^p \Lambda_{i+1}. \]

7. An explanation of self-duality of \( B_k \)

In this optional section, we discuss some observations about the ASS of \( kup^*(K_2) \) and \( kup_*(K_2) \) which, among other things, provide an explanation of the self-dual nature of the \( B_k \) summands which occur in both \( kup^*(K_2) \) and \( kup_*(K_2) \). We restrict to \( p = 2 \).

We first observe that, for \( k \geq 1 \), there is an \( E_1 \)-submodule, \( M_k \), of \( H^*(K_2) \) such that \( \text{Ext}_E \) (\( Z_2, M_k \)) (resp. \( \text{Ext}_E (M_k, Z_2) \)) is closed under the differentials in the ASS converging to \( kup^*(K_2) \) (resp. \( kup_*(K_2) \)), yielding the chart \( A_k \) (resp. the \( kup \)-homology analog of \( A_k \) discussed in Theorem 1.23). For example, with \( M_j \) as in (2.13) and \( N \) as in Figure 6, \( M_3 \) is as depicted in Figure 17.
The two ASSs for \( M_3 \) will yield the charts for \( A_3 \) and its homology analog pictured in [5].
Figure 19. Two ASSs for $\mathcal{M}_{2,3}$. 
The situation for $B_k$ is slightly more complicated. There is no $E_1$-submodule of $H^*(K_2)$ which, by itself, can give a chart $B_k z_4$. Some of the differentials that truncate $v$-towers in $B_k z_4$ come from classes that are part of a summand that includes $y_1^{3k-1} q S_{k,4}$. We find that, for $2 \leq k < \ell$, there is an $E_1$-submodule $M_{k,4}$ of $H^* K_2$ such that $\text{Ext}_{E_1}(\mathbb{Z}_2, M_{k,4})$ is closed under the differentials in the ASS converging to $kup^*(K_2)$ and yields the chart:

$$B_k z_4 \oplus y_1^{3k-1} q S_{k,4} \oplus y_B B_k z_4.$$

Note that these three subsets of $kup^*(K_2)$ appeared together in the 10-term exact sequence (5.2).

This $M_{k,4}$ is symmetric, that is, there is an integer $D$ such that $\Sigma^D M_{k,4}$ and $M_{k,4}$ are isomorphic $E_1$-modules, where $M_{k,4}$ is obtained from $M_{k,4}$ by negating gradings and dualizing $Q_0$ and $Q_1$. This implies that the $v$-towers in $\text{Ext}_{E_1}(\mathbb{Z}_2, M_{k,4})$ and $\text{Ext}_{E_1}(M_{k,4}, \mathbb{Z}_2)$ correspond nicely. Moreover, the differentials in the two ASS correspond to obtaining isomorphic charts, although the gradings in one decrease from left to right, while in the other they increase.

We illustrate with an example, $M_{3,4}$, and then discuss the implication for self-duality of $B_k$. In Figure 18, we depict $M_{3,4}$.

In Figure 19, we depict the ASS chart for both $\text{Ext}_{E_1}(\mathbb{Z}_2, M_{3,4})$ and $\text{Ext}_{E_1}(M_{3,4}, \mathbb{Z}_2)$. They are isomorphic except that, from left to right, the gradings start with 102 for the first and 70 for the second. We label the portions of the chart corresponding to the eight summands of $M_{3,4}$ just by the $M$-factor, since accompanying factors differ for the two versions. For example, the $M_5$ on the left-hand side is $z_3 M_5$ for the first spectral sequence and is $y_1^3 x_5 M_5$ for the second.

For the $kup^*(K_2)$ version, $B_3 z_4$ is on the left-hand side of Figure 19 and $y_3 B_3 z_3$ on the right-hand side, with $y_1^3 q S_{3,4}$ separating them. The duality isomorphism in Theorem 1.20 says that the Pontryagin dual of $B_3 z_4$ is isomorphic as a $kup_*$-module to $\Sigma^4$ of the right-hand side of the $kup_*(K_2)$ version of Figure 19, and we see that this is isomorphic to a shifted version of $B_3$ with indices negated. This is the self-duality statement that the Pontryagin dual of $B_k$ is isomorphic to a shifted version of $B_k$ with indices negated.

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