SEMISTRUCT MODELS OF CONNECTED 3-TYPES AND TAMSAMANI’S WEAK 3-GROUPOIDS

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Abstract. Homotopy 3-types can be modelled algebraically by Tamsamani’s weak 3-groupoids as well as, in the path connected case, by cat$^3$-groups. This paper gives a comparison between the two models in the path-connected case. This leads to two different semistrict algebraic models of connected 3-types using Tamsamani’s model. Both are then related to Gray groupoids.

1. Introduction

The problem of modelling homotopy types is relevant to both homotopy theory and higher category theory. In homotopy theory, various models exist for the path-connected case: the cat$^n$-group model, introduced by Loday [14] and later developed in [4],[18] generalized the earlier work of Whitehead on crossed modules [24]; another model was built by Carrasco and Cegarra [6].

In higher category theory, homotopy models serve as a “test” for a good definition of weak higher category, which should give a model of $n$-types in the weak $n$-groupoid case. This property has been proved in [22],[23] for the Tam Asmani’s model of weak $n$-categories.

Tamsamani’s model and cat$^n$-groups are both multi-simplicial models but they have distinctly different features: the first is a strict but cubical higher categorical structure, the second is a weak and globular one. Yet they both encode (path-connected) $n$-types. Their differences as well as similarities stimulated our interest in searching for a direct comparison between the two. This paper solves the problem for the case $n = 3$; that is, for connected 3-types.

As a result of this comparison we find a model of connected 3-types through a subcategory $\mathcal{H}$ of Tamsamani’s weak 3-groupoids whose objects are not, in general, strict, but which are ‘less weak’ than the ones used by Tamsamani to model connected 3-types (Definition 5.1). Thus objects of $\mathcal{H}$ are ‘semistrict’ structures. Yet the subcategory $\mathcal{H}$ is not isomorphic to the category of Gray groupoids with one object (Remark 5.5). Thus our comparison yields a new semistrict model of connected 3-types and at the same time gives a refinement of Tamsamani’s result in this case (Theorem 5.3 and Corollary 5.4).

In a generic Tamsamani’s weak 3-groupoid, there are two fixed directions in the 3-simplicial set in which the Segal maps are equivalences rather than isomorphisms. In the semistrict model $\mathcal{H}$ the Segal maps in one fixed direction between the two are isomorphisms. Choosing the other direction for the isomorphisms of the Segal maps leads to a different semistrict subcategory $\mathcal{K}$ (Definition 6.3). It is then natural to ask whether $\mathcal{K}$ constitutes another semistrict model of connected 3-types. The answer is positive (Theorem 6.4). Finally, using simultaneously the semistrict models $\mathcal{H}$ and $\mathcal{K}$, the connection with Gray groupoids (with one object) is established (Theorem 7.2).

Date: 9 July 2006.

2000 Mathematics Subject Classification. 55P15 (18D05,18G50).

Key words and phrases. Homotopy types, weak 3-groupoids, Gray groupoids.
The paper is organized as follows. In Section 2 we give a review of the definitions and main properties of the Tamsamani’s model and of the cat$^2$-group model which we are going to use. In Section 3 we prove a general fact about internal categories in categories of groups with operations. In Section 4 we specialize this to the case where $\mathcal{C}$ is the category Cat (Gp) of internal categories in groups. This yields Proposition 4.2, which is the key to the passage from a strict cubical structure to a weak globular one. Proposition 4.2 says that we can represent a connected 3-type, up to homotopy, through a cat$^2$-group with the property that, as an internal category in Cat (Gp), its object of objects is projective in Cat (Gp).

The form of the projective objects in Cat (Gp), known in the literature, allows one to associate to a cat$^2$-group $\mathcal{G}$ of the type above a bisimplicial group $dsN\mathcal{G}$, which we call the discrete multinerve of $\mathcal{G}$. The discrete multinerve of $\mathcal{G}$ has the following properties (Lemma 4.3).

i) For each $n \geq 0$, $(dsN\mathcal{G})_n : \Delta^op \to \text{Gp}$ is the nerve of an object of Cat (Gp).

ii) $(dsN\mathcal{G})_0$ is a constant functor.

iii) The Segal maps are weak equivalences of simplicial groups.

Further, the classifying spaces of $\mathcal{G}$ and of its discrete multinerve are weakly homotopy equivalent.

In general, we call a bisimplicial group with properties i), ii), iii), an internal 2-nerve (Definition 4.4). We can think of an internal 2-nerve as a version of Tamsamani’s weak 2-nerves internalized in the category of Groups. Proposition 4.2 allows use of the discrete multinerve to define the functor discretization

$$\text{disc} : \text{Cat}^2(\text{Gp})/\sim \to \mathcal{D}/\sim$$

from the localization of cat$^2$-groups with respect to the weak equivalences to the localization of internal 2-nerves with respect to weak equivalences. As summarized in Proposition 4.5, the discretization functor preserves the homotopy type.

In Section 4 we realize the passage from cat$^2$-group to the semistrict subcategory $\mathcal{H}$ of Tamsamani’s weak 3-groupoids (Theorem 5.3). As explained in the proof of Theorem 5.3, this is achieved by composing the functor $\text{disc}$ with the functor $\bar{N} : \mathcal{D}/\sim \to \mathcal{H}/\sim^{\text{ext}}$ induced by the nerve functor Gp $\to \text{Set}$.

In this proof, and later in the paper, we use the fact that a morphism of weak 3-groupoids is an external equivalence if and only if it induces a weak homotopy equivalence of classifying spaces; in one direction, this fact was proved in [22, Proposition 11.2]. However, the opposite direction of it does not seem to be stated explicitly in the literature. In the Appendix we have provided a proof of this fact for the subcategory $\mathcal{S}$ of Tamsamani’s weak 3-groupoids which we use in this paper.

Section 6 considers a different semistrictification of Tamsamani’s weak 3-groupoids for the path-connected case, through the subcategory $\mathcal{K}$ (Definition 6.3 and Theorem 6.4). Our method uses a strictification functor from Tamsamani’s weak 2-groupoids to Tamsamani’s strict 2-groupoids (Definition 6.1). Section 6 is independent on the comparison with cat$^2$-groups. However, in order to relate $\mathcal{K}$ to Gray groupoids, the subcategory $\mathcal{H}$ is used (Lemma 7.1). We then associate to objects of $\mathcal{H}$ and $\mathcal{K}$, up to homotopy, a Gray groupoid (with one object) having the same homotopy type (Theorem 7.2).

Acknowledgements I am grateful to Michael Batanin and Clemens Berger for helpful conversations. I also thank the members of the Australian Category Seminar for support and encouragement. This work is supported by an Australian Research Council Postdoctoral Fellowship (project no. DP0558598).
2. Preliminaries.

In this section we review the main definitions and properties of the Tamsamani model of connected 3-types and of the cat^2-group model. Useful references for these topics are [3], [4], [5], [14], [18], [22], [23].

Let \( \Delta \) be the simplicial category; we denote by \( \mathcal{C}^{\Delta^\text{op}} \) the category of simplicial objects in \( \mathcal{C} \), and by \( \Lambda^{n,\text{op}} \) the product of \( n \) copies of \( \Delta^\text{op} \) so that \( \mathcal{C}^{\Delta^{n,\text{op}}} \) is the category of \( n \)-simplicial objects in \( \mathcal{C} \).

A 1-nerve is a simplicial set which is the nerve of a small category. The category of 1-nerves is isomorphic to \( \text{Cat} \). A 2-nerve is a bisimplicial set \( \phi : \Delta^{2,\text{op}} \rightarrow \text{Set} \), such that each \( \phi_{n,*} \) is a 1-nerve, \( \phi_{0,*} \) is constant and the Segal maps \( \phi_{n,*} \rightarrow \phi_1 \times \phi_{n-1} \times \cdots \times \phi_{0} \phi_{1,*} \) are equivalences of categories. A morphism of 2-nerves is a morphism of the underlying bisimplicial sets. We denote by \( \mathcal{N}_2 \) the category of Tamsamani’s 2-nerves.

For a 2-nerve \( \phi \), let \( T\phi_{n,*} \) be the set of isomorphism classes of objects of the category corresponding to \( \phi_{n,*} \). Then the simplicial set \( T\phi : \Delta^\text{op} \rightarrow \text{Set} \), \( T\phi([n]) = T\phi_{n,*} \) is a 1-nerve.

A 2-nerve \( \phi \) is a weak 2-groupoid when each \( \phi_{n,*} \) and \( T\phi \) are nerves of groupoids. We denote by \( \mathcal{T}_2 \) the category of the weak 2-groupoids. A 2-nerve is a strict 2-groupoid if it is a weak 2-groupoid and all Segal maps are isomorphisms. We denote by \( \mathcal{T}_2^{st} \) the category of Tamsamani’s strict 2-groupoids.

Let \( 2\text{-}gpd \) be the category of 2-groupoids in the ordinary sense, that is the subcategory of the category \( \text{2-cat} \) of those 2-categories whose cells of positive dimension are invertible. Hence \( 2\text{-}gpd \) is the category of groupoid objects in groupoids whose object of objects is discrete. By taking the nerve, we therefore have an isomorphism

\[
\nu : 2\text{-}gpd \rightarrow \mathcal{T}_2^{st}.
\]

A morphism \( f : \phi \rightarrow \phi' \) of 2-nerves is an external equivalence if for all \( x, y \in \phi_{0,*} \), the maps \( f_{(x,y)} : \phi_{(x,y)} \rightarrow \phi'_{(f(x),f(y))} \) and \( T\phi \) are equivalences of categories. Recall that \( \phi_{(x,y)} \) are “hom-categories” and

\[
\prod_{x,y \in \phi_{0,*}} \phi_{(x,y)} = \phi_1.
\]

A 3-nerve is a 3-simplicial set \( \psi : \Delta^{3,\text{op}} \rightarrow \text{Set} \) such that each \( \psi([n],-,-) \) is a 2-nerve, \( \psi([0],-,-) \) is constant and the Segal maps are equivalence of 2-nerves. A morphism of 3-nerves is a morphism of the underlying 3-simplicial sets.

For a 3-nerve \( \psi \), the bisimplicial set \( T\psi \) is a 2-nerve so we define \( T^2\psi = T(T\psi) \), which is a 1-nerve. A weak 3-groupoid is a 3-nerve \( \psi \) such that each \( \psi([n],-,-) \) is a weak 2-groupoid and \( T^2\psi \) is a groupoid.

A morphism \( g : \psi \rightarrow \psi' \) of weak 3-groupoids is an external equivalence if for each \( x, y \in \psi([0],-,-) \) the map \( g_{(x,y)} : \psi_{(x,y)} \rightarrow \psi'_{(f(x),f(y))} \) is an external equivalence of weak 2-groupoids and if \( T^2\psi \) is an equivalence of categories. We denote by \( \mathcal{T}_3 \) the category of Tamsamani’s weak 3-groupoids. A 3-nerve is a strict 3-groupoid if it is a weak 3-groupoid and all Segal maps are isomorphisms.

Tamsamani showed ([22, Theorem 8.0]) that the homotopy category of 3-types is equivalent to the localization of \( \mathcal{T}_3 \) with respect to external equivalences. Recall that a 3-type (resp. connected 3-type) is a topological space (resp. connected topological space) with trivial homotopy groups in dimension \( i > 3 \); we denote by \( \text{Top}^{(3)} \) the category of connected 3-types.

Since we are dealing in this paper with the path-connected case, we restrict our attention to the full subcategory \( \mathcal{S} \) of \( \mathcal{T}_3 \) consisting of those weak 3-groupoids
ψ : Δ³ → Set satisfying the condition that the constant functor ψ([0], -, -) : Δ³ → Set takes values in the one-element set.

In fact, from the construction of the fundamental 3-groupoid functor Π_3 : Top → T_3 given in [22, Theorem 6.4] it is immediate that, when restricting to the subcategory Top_* ⊂ Top of path-connected topological spaces, one obtains a functor Π_3 : Top_* → S. Hence, in the path-connected case, Tamsamani’s result becomes

**Theorem 2.1.** [22] There is an equivalence of categories

\[ \mathcal{S}/\sim^{ext} \simeq \mathcal{H}o(\text{Top}_*^{(3)}) \]

In Section 7 we need some results relating Tamsamani’s 2-nerves and bicategories, in order to construct a strictification functor from \( T_2 \) to \( T_2^{st} \). In the following theorem, \( N \text{Bicat} \) denotes the 2-category of bicategories, normal homomorphisms and op-lax natural transformations with identity components (see [12])

**Theorem 2.2.** [12, Th. 7.2] There is a fully faithful 2-functor \( N : N \text{Bicat} \to \mathcal{N}_2 \) with a left 2-adjoint \( \text{Bic} \). The counit \( \text{Bic} N \to 1 \) is invertible and each component \( u : X \to N \text{Bic} X \) of the unit is a pointwise equivalence, and \( u_0, u_1 \) are identities.

We now turn to the cat²-group model. We begin by recalling some general notions.

Given a category \( C \) with pullbacks, an internal category in \( C \) consists of a diagram

\[
\begin{array}{ccc}
\phi : C_1 \times_{C_0} C_1 & \xrightarrow{c} & C_1 \\
\downarrow d_0 & & \downarrow d_1 \\
C_0 & \xrightarrow{1} & C_0
\end{array}
\]

where \( d_0, d_1 \) are “source” and “target” maps, \( i \) is the “identity” map, \( c \) is the “composition” map. These data satisfy the axioms of a category; that is, the following identities hold, where \( \pi_0, \pi_1 : C_1 \times_{C_0} C_1 \to C_1 \) are the two projections:

\[
\begin{align*}
d_0 i &= 1_{C_0} = d_1 i, & d_1 \pi_1 &= d_1 c, & d_0 \pi_0 &= d_0 c \\
c \left( \frac{1_{C_1}}{1_{C_0}} \right) &= 1_{C_1} = \left( \frac{id_1}{1_{C_1}} \right), & c(1_{C_1} \times_{C_0} c) &= c(c \times_{C_0} 1_{C_1}).
\end{align*}
\]

Given internal categories \( \phi \) and \( \phi' \), an internal functor \( F : \phi \to \phi' \) consists of a pair of morphisms \( F_0 : C_0 \to C'_0, \ F_1 : C_1 \to C'_1 \) satisfying the conditions:

\[
\begin{align*}
d_0 F_1 &= F_0 d_0, & d_1 F_1 &= F_0 d_1 \\
F_1 i &= i F_0, & F_1 c &= c(F_1 \times_{F_0} F_1).
\end{align*}
\]

Let \( \text{Cat} \mathcal{C} \) be the category of internal categories in \( \mathcal{C} \) and internal functors.

In this paper we use the category \( \text{Cat} \mathcal{C} \) in the case where \( \mathcal{C} \) is a category of groups with operations in the sense of [16], [17]. Recall that this consists of a category of internal groups with a set of additional operations \( \Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \), where \( \Omega_i \) is the set of i-ary operations in \( \Omega \), such that the group operations of identity, inverse and multiplication (denotes 0, +, -) are elements of \( \Omega_0, \Omega_1, \Omega_2 \) respectively; it is \( \Omega_0 = \{0\} \) and certain compatibility conditions hold (see [16], [17]). Further, there is a set of identities which includes the groups laws.

When \( \mathcal{C} \) is a category of groups with operations, every object of \( \text{Cat} \mathcal{C} \) is an internal groupoid, as every arrow is invertible.

An example of a category of groups with operations which will be important for us is the category \( \text{Cat} (\text{Gp}) \) of internal categories in the category of groups. The fact that this forms a category of groups with operations follows easily from its equivalence with cat¹-groups. In fact, recall [14] that a cat¹-group consists of a group \( G \) with two endomorphisms \( d, t : G \to G \) satisfying

\[
ds d = t, \quad t d = d, \quad \ker d, \ker t \mid 1. \tag{1}
\]
A morphism of cat\(^1\)-groups \((G, d, t) \to (G', d', t')\) is a group homomorphism \(f : G \to G'\) such that \(fd = \varepsilon f\), \(ft = t'f\). The equivalence of Cat (Gp) with cat\(^1\)-groups is realized by associating to an object of Cat (Gp)

\[
C_1 \times_{C_0} C_1 \xrightarrow{c} C_1 \xrightarrow{d_0} C_1 \xrightarrow{d_1} C_0
\]

the cat\(^1\)-group \((C_1, i d_0, i d_1)\) (see [14]).

The category of cat\(^1\)-groups is a category of groups with operations by setting \(\Omega_0 = \{0\}\), \(\Omega_1 = \{-\} \cup \{t, d\}\), \(\Omega_2 = \{+\}\) and requiring the set of identities to comprise the group laws and the identities (1). An alternative proof that Cat (Gp) is a category of groups with operations is found in [5], using the language of crossed modules. In particular, [5] contains a characterization of objects in the algebraic category Cat (Gp) which are projective with respect to the class of regular epimorphisms. This characterization will be important in Section 4 and will be recalled explicitly there.

When \(\mathcal{C} = \text{Cat} (\text{Gp})\) the category Cat \(\mathcal{C}\) is the category Cat (Cat (Gp)) of double categories internal to Groups. This is equivalent to the category of cat\(^2\)-groups, originally introduced by Loday [14].

Recall that a cat\(^2\)-group consists of a group \(G\) together with four endomorphisms \(t_i, d_i : G \to G\) \(i = 1, 2\) such that, for all \(1 \leq i, j \leq 2\)

\[
\begin{align*}
d_i t_i &= t_i, & t_i d_i &= d_i, \\
d_i t_j &= t_j d_i, & d_i d_j &= d_j d_i, & t_i t_j &= t_j t_i, & i \neq j \\
|\ker d_i, \ker t_i| &= 1
\end{align*}
\]

A morphism of cat\(^2\)-groups \((G, d_i, t_i) \to (G', d_i', t_i')\) consists of a group homomorphism \(f : G \to G'\) such that \(f d_i = d_i' f\), \(f t_i = t_i' f\), \(1 \leq i \leq 2\). It is well known that the category of cat\(^2\)-groups is isomorphic to the category Cat (Cat (Gp)). Because of this, when in this paper we talk about the category of cat\(^2\)-groups, we always mean the category Cat (Cat (Gp)), which we denote by Cat \(^2\)(Gp).

Given an object \(\mathcal{G}\) of Cat (Cat (Gp)), by applying the nerve functor twice one obtains a bisimplicial group \(\mathcal{N}\mathcal{G}\), called the multinerve of \(\mathcal{G}\). The classifying space \(BG\) of the cat\(^2\)-group \(\mathcal{G}\) is, by definition, the classifying space of its multinerve. It can be shown [4] that \(BG\) is a connected 3-type.

A morphism of cat\(^2\)-groups is a weak equivalence if it induces a weak homotopy equivalence of classifying spaces.

In [4] a fundamental cat\(^2\)-group functor \(\mathcal{P} : \text{Top}_* \to \text{Cat}^2 (\text{Gp})\) is constructed. Further, it is shown in [4] that the functors \(B\) and \(\mathcal{P}\) induce an equivalence of categories

\[
\overline{B} : \frac{\text{Cat}^2 (\text{Gp})}{\sim} \simeq \mathcal{H}_0 (\text{Top}_*^{(3)} : \mathcal{P}) \tag{2}
\]

between the localization of cat\(^2\)-groups with respect to weak equivalences and homotopy category of connected 3-types.

Given a cat\(^2\)-group \(\mathcal{G}\) it is shown in [4] that there is a zig-zag of weak equivalences in Cat \(^2\)(Gp) between \(\mathcal{G}\) and \(\mathcal{P}BG\). We denote by \([\_] : \text{Cat}^2 (\text{Gp}) \to \text{Cat}^2 (\text{Gp})/\sim\) the localization functor. Hence \([\mathcal{G}] = [\mathcal{P}BG]\) for each \(\mathcal{G} \in \text{Cat}^2 (\text{Gp})\). In general, \([\mathcal{G}] = [\mathcal{G}']\), if and only if there is a zig-zag of weak equivalences in Cat \(^2\)(Gp) between \(\mathcal{G}\) and \(\mathcal{G}'\).

3. Internal categories in categories of groups with operations.

In this section we prove a general fact about internal categories in a category \(\mathcal{C}\) of groups with operations. We first recall some preliminary notions.
Internal categories in a category $C$ of groups with operations are internal groupoids, and are equivalent to reflexive graphs

$$
\begin{array}{c}
C_1 \\
\downarrow d_0 \\
C_0 \\
\downarrow d_1 \\
\downarrow \sigma_0
\end{array}
$$
satisfying the condition $[\ker d_0, \ker d_1] = 1$.

There is a full and faithful nerve functor

$$
\mathcal{N} : \text{Cat} \xrightarrow{} C^{\Delta^{op}}
$$

which has a left adjoint $\mathcal{P} : C^{\Delta^{op}} \rightarrow \text{Cat}$ given by

$$
\mathcal{P}(H_\ast) : H_1/d_2(H_2) \xrightarrow{\pi_0} H_0.
$$

The unit of the adjunction $u_{H_\ast} : H_\ast \rightarrow \mathcal{N}\mathcal{P}(H_\ast)$ induces isomorphisms of homotopy groups $\pi_i$ for $i = 0, 1$. We say that a map $f$ in $\text{Cat} C$ is a weak equivalence if $\mathcal{N}f$ is a weak equivalence in $C^{\Delta^{op}}$; that is, it induces isomorphisms of homotopy groups. We denote by $\text{Cat} C \rightarrow \text{Cat} C/\sim$ the localization functor.

The above notion of weak equivalence in $C^{\Delta^{op}}$ is part of the Quillen model category structure on simplicial objects in an algebraic category given in [19], [21]; in this model structure, if $X_\ast \in C^{\Delta^{op}}$ is cofibrant, then each $X_n$ is projective in $C$ with respect to the class of regular epimorphisms (see [19]).

**Lemma 3.1.** Let $C$ be a category of groups with operations and let $\mathcal{G} \in \text{Cat} C$. There exists $\phi_\mathcal{G} \in \text{Cat} C$ whose object of objects $\phi_0$ is projective in $C$ and a weak equivalence $u_\mathcal{G} : \phi_\mathcal{G} \rightarrow \mathcal{G}$ in $\text{Cat} C$. Further, given a morphism $f : \mathcal{G} \rightarrow \mathcal{G}'$ in $\text{Cat} C$, then there is a morphism $\phi_f : \phi_\mathcal{G} \rightarrow \phi_\mathcal{G}'$ making the following diagram commute

$$
\begin{array}{ccc}
\phi_\mathcal{G} & \xrightarrow{\phi_f} & \phi_\mathcal{G}' \\
\downarrow u_\mathcal{G} & & \downarrow u_{\mathcal{G}'} \\
\mathcal{G} & \xrightarrow{f} & \mathcal{G}'.
\end{array}
$$

Let $\mathcal{G} \xrightarrow{f} \mathcal{G}' \xrightarrow{f'} \mathcal{G}''$ be morphisms in $\text{Cat} C$. Then $[\phi_{f'f}] = [\phi_{f'}][\phi_f]$ and $[\phi_{id}] = [id]$.

**Proof.** Let $c : \chi \rightarrow \mathcal{N}\mathcal{G}$ be a cofibrant replacement in $C^{\Delta^{op}}$. We have a commutative diagram in $C^{\Delta^{op}}$

$$
\begin{array}{c}
\mathcal{N}\mathcal{P}\chi \\
\downarrow u_\chi \\
\chi \\
\downarrow c \\
\mathcal{N}\mathcal{G}.
\end{array}
$$

Since $\mathcal{P}\mathcal{N} = \text{id}$, $\mathcal{N}\mathcal{P}\mathcal{N}\mathcal{G} = \mathcal{N}\mathcal{G}$ and $u_{\mathcal{N}\mathcal{G}} = \text{id}$. Since $\pi_i\mathcal{N}\mathcal{G} = 0 = \pi_i\chi$ for $i > 1$ and $u_\chi$ induces isomorphisms of $\pi_0, \pi_1$, it follows that $u_\chi$ is a weak equivalence. Hence, in the diagram above, $c$, $u_\chi$ and $u_{\mathcal{N}\mathcal{G}}$ are weak equivalences. It follows that $\mathcal{N}\mathcal{P}\chi$ is a weak equivalence. Let $\phi_\mathcal{G} = \mathcal{P}\chi$, then $\phi_0 = (\mathcal{P}\chi)_0 = \chi_0$ is projective in $C$ (since $\chi$
is cofibrant in $\mathcal{C}^{\Delta^{op}}$ and $\mathcal{P}_C : \mathcal{P}_C \to \mathcal{G}$ is a weak equivalence in $\text{Cat} \mathcal{C}$. Let $\nu_f$ be a cofibrant approximation of the map $\mathcal{N}f$, so that the following diagram commutes:

$$
\begin{array}{ccc}
\chi & \xrightarrow{\nu_f} & \chi' \\
\downarrow c & & \downarrow c' \\
\mathcal{N}\mathcal{G} & \xrightarrow{\mathcal{N}f} & \mathcal{N}\mathcal{G}'.
\end{array}
$$

Applying to this diagram the functor $\mathcal{P}$, (3) follows, with $\phi_f = \mathcal{P}\nu_f$, $u_\mathcal{G} = \mathcal{P}c$, $u_\mathcal{G}' = \mathcal{P}c'$. Let $\mathcal{C}^{\Delta^{op}}\leq_\mathcal{C}$ be the full subcategory of $\mathcal{C}^{\Delta^{op}}$ whose objects $\psi$ are such that $\pi_i\psi = 0$ for $i > 1$ and let $i : \mathcal{C}^{\Delta^{op}}\leq_\mathcal{C} \to \mathcal{C}^{\Delta^{op}}$ be the inclusion. Then the functor $[\cdot] : \mathcal{C}^{\Delta^{op}}\leq_\mathcal{C} \to \text{Cat} / \sim$ sends weak equivalences to isomorphisms. Given morphisms in $\text{Cat} \mathcal{C}$, $\mathcal{G} \xrightarrow{f} \mathcal{G}' \xrightarrow{f'} \mathcal{G}''$, let $\chi \xrightarrow{\nu_f} \chi' \xrightarrow{\nu_{f'}} \chi''$ be as above. Notice that $\chi, \chi', \chi'' \in \mathcal{C}^{\Delta^{op}}\leq_\mathcal{C}$ and, by [21, p. 67-68] there is a right homotopy between $\nu_{f/1}$ and $\nu_{f'/1}$ and between $\nu_{id}$ and $\nu_{id}$. It follows from a general fact [8, Lemma 8.3.4] that $[\mathcal{P}\nu_{f/1}] = [\mathcal{P}\nu_{f'/1}]$ and $[\mathcal{P}\nu_{id}] = [id]$. □

4. The discretization functor.

We are going to apply the result of Section 3 to the case where $\mathcal{C} = \text{Cat} (\text{Gp})$. In this case $\text{Cat} \mathcal{C}$ is the category of cat$^2$-groups and there is a classifying space functor $B : \text{Cat}^2(\text{Gp})\to \text{Top}_*$ obtained by composition

$$
\text{Cat}^2(\text{Gp}) \xrightarrow{\mathcal{N}} \text{Gp}^{\Delta^{op}} \xrightarrow{\text{Ner}^*} \text{Set}^{\Delta^{op}} \xrightarrow{\text{diag}} \text{Set}^{\Delta^{op}} \xrightarrow{| \cdot |} \text{Top}_*.
$$

Here $\mathcal{N}$ is the multinerve, $\text{Ner}^*$ is induced by the nerve functor $\text{Ner} : \text{Gp} \to \text{Set}^{\Delta^{op}}$, $\text{diag}$ is the multidiagonal and $| \cdot |$ is the geometric realization.

**Lemma 4.1.** Let $\mathcal{C} = \text{Cat} (\text{Gp})$ and let $f : \mathcal{G}' \to \mathcal{G}$ be a weak equivalence in $\text{Cat} (\mathcal{C})$. Then $Bf$ is a weak homotopy equivalence.

**Proof.** By hypothesis $\mathcal{N}f$ induces isomorphisms of simplicial groups $\pi_i(\mathcal{N}\mathcal{G}') \cong \pi_i(\mathcal{N}\mathcal{G})$ $i = 0, 1$, and therefore isomorphisms of groups

$$
\pi_i(\mathcal{N}\mathcal{G}')_q \cong \pi_i(\mathcal{N}\mathcal{G})_q,
$$

for each $q \geq 0$, $i = 0, 1$. The map of simplicial spaces $\{B(\mathcal{N}\mathcal{G}')_q, \} \to \{B(\mathcal{N}\mathcal{G})_q, \}$ is therefore a levelwise weak equivalence, hence it induces a weak homotopy equivalence of classifying spaces. Since $B\mathcal{G}'$ is weakly homotopy equivalent to the classifying space of $\{B(\mathcal{N}\mathcal{G}')_q, \}$, and similarly for $\mathcal{G}$, the result follows. □

Given a cat$^2$-group $\chi$, we denote by $\chi_0 \in \text{Cat} (\text{Gp})$ the object of objects of the internal category in $\text{Cat} (\text{Gp})$ corresponding to $\chi$. From Lemma 3.1 and Lemma 4.1 we immediately deduce the following proposition.

**Proposition 4.2.** Let $\mathcal{G} \in \text{Cat}^2(\text{Gp})$. There exists $\phi \in \text{Cat}^2(\text{Gp})$ with $\phi_0$ projective in $\text{Cat} (\text{Gp})$ such that $B\phi$ is weakly homotopy equivalent to $BG$.

Projective objects in $\text{Cat} (\text{Gp})$ have been characterized in [5]; they have the form:

$$
\phi_0 : F_1 \times F_2 \xrightarrow{d_0} F_2
$$

where $j : F_1 \to F_2$ is a normal inclusion, $F_1$, $F_2$, $F_2/j(F_1)$ are free groups, $d_0(x, y) = y$, $d_1(x, y) = j(x)y$, $i(z) = (1, z)$ and $F_2$ acts on $F_1$ by $zy = j(x)yj(x^{-1})$. 
Let $\phi^d_0$ denote the discrete internal category

$$\phi^d_0 : F_2/j(F_1) \xrightarrow{id} F_2/j(F_1).$$

If $q_0 : F_2 \to F_1/j(F_1)$ is the quotient map and $q_1 : F_1 \times F_2 \to F_2/j(F_1)$ is defined by $q_1(x, y) = q_0(y)$, then (as easily checked) the map $d = (q_1, q_0) : \phi_0 \to \phi^d_0$ is a weak equivalence in $\text{Cat}(\text{Gp})$. Since $F_2/j(F_1)$ is free, $q_0$ has a section $q_0' : F_2/j(F_1) \to F_2$, $q_0'q_0 = \text{id}$. If $i : F_2 \to F_1 \times F_2$ is the inclusion, the map $t : (i'q_0, q_0') : \phi^d_0 \to \phi_0$ is also a weak equivalence in $\text{Cat}(\text{Gp})$ and $dt = \text{id}$.

Let $\phi \in \text{Cat}^2(\text{Gp})$ with $\phi_0$ projective in $\text{C}$. Let $\phi^d_0$, $d$, $t$ be as above and let $\text{Ner} : \text{Cat}(\text{Gp}) \to \text{Gp}^{\Delta^{op}}$ be the nerve functor; the multinerve of $\phi$, as a simplicial object in $\text{Gp}^{\Delta^{op}}$ has

$$(\mathcal{N}\phi)_p = \begin{cases} \text{Ner} \phi_0 & p = 0, \\ \text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \cdots \times_{\text{Ner} \phi_0} \text{Ner} \phi_1 & p > 0. \end{cases}$$

Let $\partial_0, \partial_1 : \text{Ner} \phi_1 \to \text{Ner} \phi_0$ and $\sigma_0 : \text{Ner} \phi_0 \to \text{Ner} \phi_1$ be face and degeneracy maps and let $d : \text{Ner} \phi_0 \to \text{Ner} \phi^d_0$ and $t : \text{Ner} \phi^d_0 \to \text{Ner} \phi_0$ be induced by the weak equivalences $d : \phi_0 \to \phi^d_0$ and $t : \phi^d_0 \to \phi_0$.

We construct a bisimplicial group, which we call the discrete multinerve of $\phi$, denoted $ds\mathcal{N}\phi$, as follows. As an object of $(\text{Gp}^{\Delta^{op}})^{\Delta^{op}}$ it is given by

$$(ds\mathcal{N}\phi)_p = \begin{cases} \text{Ner} \phi^d_0 & p = 0, \\ \text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \cdots \times_{\text{Ner} \phi_0} \text{Ner} \phi_1 & p > 0. \end{cases}$$

The face and degeneracy operators are $d\partial_0, d\partial_1 : (ds\mathcal{N}\phi)_1 \to (ds\mathcal{N}\phi)_0, \sigma_0 : (ds\mathcal{N}\phi)_0 \to (ds\mathcal{N}\phi)_1$. All other faces and degeneracy operators are as in $\mathcal{N}\phi$.

Since $dt = \text{id}$, it is straightforward to check that $ds\mathcal{N}\phi$ is a simplicial object in $\text{Gp}^{\Delta^{op}}$.

Notice that $\mathcal{N}\phi$ depends on the choice of the section $t$. However since $F_2 \cong F_1 \times F_2/j(F_1)$, two different choices of section lead to isomorphic bisimplicial groups.

The following Lemma describes the properties of the discrete multinerve.

**Lemma 4.3.** Let $\phi \in \text{Cat}^2(\text{Gp})$ with $\phi_0$ projective in $\text{Cat}(\text{Gp})$. Then

i) $B(ds\mathcal{N}\phi) = B\phi$.

ii) For each $n \geq 2$ the Segal map

$$(ds\mathcal{N}\phi)_n \longrightarrow (ds\mathcal{N}\phi)_1 \times (ds\mathcal{N}\phi)_0 \times \cdots \times (ds\mathcal{N}\phi)_0 (ds\mathcal{N}\phi)_1$$

is a weak equivalence in $\text{Gp}^{\Delta^{op}}$.

**Proof.**

i) The classifying space of any bisimplicial group can be obtained by composition

$$\text{Gp}^{\Delta^{op}} \xrightarrow{\text{diag}} \text{Gp}^{\Delta^{op}} \xrightarrow{N} \text{Set}^{\Delta^{op}} \xrightarrow{\text{diag}} \text{Set}^{\Delta^{op}} \xrightarrow{\cdot} \text{Top}^*,$$

where $\text{diag}$ are diagonal functors, $N$ is induced by the nerve functor $\text{Gp} \to \text{Set}^{\Delta^{op}}$ and $\cdot$ is the geometric realization.

Let $\psi$ be a bisimplicial group and denote by $\delta^h_i, \delta^v_i, \sigma^h_i, \sigma^v$ the horizontal and vertical face and degeneracy operators. Let $U : \text{Gp} \to \text{Set}$ be the forgetful functor. Then

$$(N\text{diag} \psi)_p = \begin{cases} \{\cdot\} & q = 0, \\ U\psi_{pp} \times \cdot & q > 0. \end{cases}$$

The horizontal face and degeneracies in $N\text{diag} \psi$ are induced by those in $\text{diag} \psi$, while the vertical face and degeneracies in $N\text{diag} \psi$ are the ones given by the nerve construction.
Let \( \chi \) be another bisimplicial group, with face and degeneracies \( \mu^h_i, \mu^v_i, \nu^h_i, \nu^v_i \). Suppose that, for all \( p > 0 \) and \( q \geq 0 \),

\[
\begin{align*}
\chi_{pq} &= \psi_{pq} \\
\delta^h_i &= \mu^h_i : \psi_{p,q+1} \rightarrow \psi_{pq} \\
\delta^v_i &= \mu^v_i : \psi_{p+1,q} \rightarrow \psi_{pq} \\
\sigma^h_i &= \nu^h_i : \psi_{p,q} \rightarrow \psi_{p,q+1} \\
\sigma^v_i &= \nu^v_i : \psi_{p,q} \rightarrow \psi_{p+1,q}.
\end{align*}
\]

We claim that the bisimplicial sets \( N\text{diag} \psi \) and \( N\text{diag} \chi \) have the same diagonal. In fact, by hypothesis (4) \( \psi_{nn} = \chi_{nn} \) for \( n \geq 1 \) so that \( (N\text{diag} \psi)_{00} = (N\text{diag} \chi)_{00} \) and \( (N\text{diag} \psi)_{pq} = (N\text{diag} \chi)_{pq} \) for \( p > 0, q \geq 0 \). Further, by hypothesis (4) the face and degeneracy operators \( (\text{diag} \psi)_{n+1} \rightarrow (\text{diag} \psi)_n \) and \( (\text{diag} \psi)_n \rightarrow (\text{diag} \psi)_{n+1} \) coincide with the respective ones for \( \text{diag} \chi \) for \( n > 0 \). This implies that the face and degeneracy operators

\[
\begin{align*}
(N\text{diag} \psi)_{p+1,q} &\rightarrow (N\text{diag} \psi)_{p,q} \\
(N\text{diag} \psi)_{p,q+1} &\rightarrow (N\text{diag} \psi)_{p,q} \\
(N\text{diag} \psi)_{p,q} &\rightarrow (N\text{diag} \psi)_{p+1,q} \\
(N\text{diag} \psi)_{p,q} &\rightarrow (N\text{diag} \psi)_{p,q+1}
\end{align*}
\]

for \( p > 0, q \geq 0 \) coincide with the respective ones for \( N\text{diag} \chi \).

Clearly \( (\text{diag}N\text{diag} \psi)_{k} = (\text{diag}N\text{diag} \chi)_{k} \) for all \( k \geq 0 \). From above, all face and degeneracy maps in positive dimension of \( \text{diag}N\text{diag} \psi \) coincide with the respective ones for \( \text{diag}N\text{diag} \chi \). The face maps \( U\psi_{11} \rightarrow \{\cdot\} \) are unique as \( \{\cdot\} \) is the terminal object and the degeneracy maps \( \{\cdot\} \rightarrow U\psi_{11} \) and \( \{\cdot\} \rightarrow U\chi_{11} \) coincide as they both send \( \{\cdot\} \) to the unit of the group \( \psi_{11} = \chi_{11} \). In conclusion \( \text{diag}N\text{diag} \psi = \text{diag}N\text{diag} \chi \). It follows that \( B\psi = B\chi \).

Let \( \phi \) be a cat\(^2\)-group satisfying the hypothesis of the lemma, \( N\phi \in \text{Gp}^{\Delta^\omega_{op}} \) be its multinerve and \( d\text{s}N\phi \) the discrete multinerve. By definition of \( d\text{s}N\phi \), the two bisimplicial groups \( N\phi \) and \( d\text{s}N\phi \) satisfy hypothesis (4). It follows from above that \( BN\phi = Bd\text{s}N\phi \). By definition \( B\phi = BN\phi \), hence the result.

ii) We need to show that for \( n \geq 2 \),

\[
\text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \cdots \times_{\text{Ner} \phi_0} \text{Ner} \phi_1 \quad \text{and} \quad \text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \cdots \times_{\text{Ner} \phi_0} \text{Ner} \phi_1
\]

are weakly equivalent simplicial groups.

We proceed by induction on \( n \). Since \( \partial_0, \partial_1 : \text{Ner} \phi_1 \rightarrow \text{Ner} \phi_0 \) are fibrations, the pullback \( \text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \text{Ner} \phi_1 \) is weakly equivalent to the homotopy pullback (see [8]); since every simplicial group is fibrant this is weakly equivalent to the homotopy limit of the diagram

\[
\text{Ner} \phi_1 \quad \partial_0 \quad \text{Ner} \phi_0 \quad \partial_1 \quad \text{Ner} \phi_1
\]

By the homotopy invariance property of homotopy limits (see [8]), since

\[
\text{Ner} \phi_0 \quad d \quad \text{Ner} \phi_0^d
\]

is a weak equivalence, then

\[
\text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \cdots \times_{\text{Ner} \phi_0} \text{Ner} \phi_1 \quad \text{and} \quad \text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \cdots \times_{\text{Ner} \phi_0} \text{Ner} \phi_1
\]

are weakly equivalent.

Inductively, suppose

\[
\text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \cdots \times_{\text{Ner} \phi_0} \text{Ner} \phi_1 \quad \text{and} \quad \text{Ner} \phi_1 \times_{\text{Ner} \phi_0} \cdots \times_{\text{Ner} \phi_0} \text{Ner} \phi_1
\]

are weakly equivalent.
Notice that $\text{Ner} \phi_1 \times \text{Ner} \phi_0 \times \cdots \times \text{Ner} \phi_0 \times \text{Ner} \phi_1$ is the pullback of the diagram $\text{Ner} \phi_1 \times \text{Ner} \phi_0 \times \cdots \times \text{Ner} \phi_0 \times \text{Ner} \phi_1 \to \text{Ner} \phi_0 \leftarrow \text{Ner} \phi_1$. As before, this is weakly equivalent to the homotopy limit of this diagram; the homotopy invariance property of homotopy limits and the induction hypothesis then imply that

$$\text{Ner} \phi_1 \times \text{Ner} \phi_0 \times \cdots \times \text{Ner} \phi_0 \times \text{Ner} \phi_1$$

$\text{Ner} \phi_1$ are weakly equivalent.

\[\square\]

**Definition 4.4.** The category $\mathcal{D}$ of internal 2-nerves is the full subcategory of bisimplicial groups $\psi : \Delta^{op} \to \text{Gp}$ such that:

i) Each $\psi_n : \Delta^{op} \to \text{Gp}$ is the nerve of an object of $\text{Cat} (\text{Gp})$.

ii) $\psi_0 : \Delta^{op} \to \text{Gp}$ is constant.

iii) The Segal maps $\psi_n \to \psi_1 \times \psi_0 \cdots \times \psi_0 \times \psi_1$ are weak equivalences of simplicial groups.

By Lemma 4.3 ii) the discrete multinerve $ds \mathcal{N} \phi$ is an internal 2-nerve. We say that a morphism $f$ in $\mathcal{D}$ is a weak equivalence if $Bf$ is a weak homotopy equivalence.

We aim to define a functor, which we call discretization

$$\text{disc} : \text{Cat}^2 (\text{Gp})/\sim \longrightarrow \mathcal{D}/\sim.$$  

Let $\text{Cat}^2 (\text{Gp})_p$ be the full subcategory of $\text{Cat}^2 (\text{Gp})$ whose objects $\phi$ are such that $\phi_0$ is projective in $\text{Cat} (\text{Gp})$. Let $[\cdot] : \text{Cat}^2 (\text{Gp}) \to \text{Cat}^2 (\text{Gp})/\sim$ be the localization functor. From Lemma 3.1 there is a functor $\mathcal{S} : \text{Cat}^2 (\text{Gp}) \to \text{Cat}^2 (\text{Gp})_p/\sim$ defined on objects by $S(\mathcal{G}) = [\phi_0]$ and on morphisms by $S(f) = [\phi_f]$. Also, by Lemma 3.1, $\mathcal{S}$ sends weak equivalences to isomorphisms. Therefore $\mathcal{S}$ induces a unique functor $\overline{\mathcal{S}} : \text{Cat}^2 (\text{Gp})/\sim \to \text{Cat}^2 (\text{Gp})_p/\sim$ with $\overline{\mathcal{S}} \circ [\cdot] = \mathcal{S}$. On the other hand, the discrete multinerve defines a functor $ds \mathcal{N} : \text{Cat}^2 (\text{Gp})_p \to \mathcal{D}$. On objects, this associates to $\phi$ its discrete multinerve $ds \mathcal{N} \phi$. Given a morphism in $\text{Cat}^2 (\text{Gp})_p$

$$F : \phi \to \phi',$$

let $ds \mathcal{N}F : ds \mathcal{N} \phi \to ds \mathcal{N} \phi'$ be given by $(ds \mathcal{N}F)_n = F_n$ for $n \geq 1$, $(ds \mathcal{N}F)_0 = T_0$, where $T_0 : \phi_0 \to \phi_0'$ is induced by $F_0$. Since $dt = id$, it is easily checked that $T_0 d_t = d' T_0$, $F_0 t = t' F_0$; this implies that $T_0 d_i = d' T_0$, $i = 0, 1$ and $F_1 t F_0 = \sigma_1 t' T_0$, so that $ds \mathcal{N}F$ is a morphism in $\mathcal{D}$. By Lemma 4.3 i), $ds \mathcal{N}$ preserves weak equivalences, hence it induces a functor $\overline{ds \mathcal{N}} : \text{Cat}^2 (\text{Gp})_p/\sim \to \mathcal{D}/\sim$. Define $\text{disc}$ to be the composite $disc = ds \mathcal{N} \circ \overline{\mathcal{S}}$.

**Proposition 4.5.** There is a commutative diagram

$$\begin{array}{ccc}
\text{Cat}^2 (\text{Gp}) & \xrightarrow{\text{disc}} & \mathcal{D} \\
\sim & \searrow & \sim \\
 & \mathcal{H}o (\text{Top}^{(3)}) & \swarrow \\
\end{array}$$

**Proof.** Given $[\mathcal{G}] \in \text{Cat}^2 (\text{Gp})/\sim$, choose a weak equivalence in $\text{Cat}^2 (\text{Gp})$, $\phi \to \mathcal{G}$ with $\phi_0$ projective in $\text{Cat} (\text{Gp})$. By definition of $\text{disc}$ and by Lemma 4.3 i), we have

$$B\text{disc} [\mathcal{G}] = B(ds \mathcal{N} \phi) = [B ds \mathcal{N} \phi] = [B \phi] = [B \mathcal{G}] = B [\mathcal{G}].$$

$\Box$

5. FROM $\text{cat}^2$-GROUPS TO TAMSAHAMI’S WEAK $3$-GROUPOIDS.

In this section we connect $\text{cat}^2$-groups with a subcategory of Tamsamani’s weak 3-groupoids.

**Definition 5.1.** Let $\mathcal{H}$ be the subcategory of Tamsamani’s weak 3-groupoids $\psi : \Delta^{op} \to \text{Set}$ satisfying the additional conditions:
a) The constant functor \( \psi([0], \cdot, \cdot) : \Delta^{2op} \to \text{Set} \) takes values in the one-element set.

b) For each \( m \geq 2 \) the Segal maps \( \psi([m], \cdot, \cdot) \to \psi([1], \cdot, \cdot) \times \cdots \times \psi([1], \cdot, \cdot) \) are bijections.

Note that objects of \( \mathcal{H} \) are not strict 3-groupoids because, in general, \( \psi([m], \cdot, \cdot) \) are weak, not strict, 2-nerves. We say a morphism \( f : \psi \to \psi' \) in \( \mathcal{H} \) is an external equivalence if it is an external equivalence in \( T_3 \). By definition this amounts to requiring that \( \psi_{1*} \to \psi'_{1*} \) is an external equivalence of weak 2-groupoids and that \( T^2 \psi \to T^2 \psi' \) is an equivalence of categories.

Let \( \mathcal{S} \) be as in Section 2; the following Lemma characterizes external equivalences in \( \mathcal{S} \).

**Lemma 5.2.**

a) Let \( f : \phi \to \phi' \) be a morphism of weak 2-groupoids. Then \( f \) is an external equivalence if and only if \( Bf \) is a weak homotopy equivalence.

b) Let \( g : \psi \to \psi' \) be a morphism in \( \mathcal{S} \). Then \( g \) is an external equivalence if and only if \( Bg \) is a weak homotopy equivalence.

**Proof.** See Appendix.

Let \( \mathcal{H}_0(\mathcal{S}) \) be the full subcategory of \( \mathcal{S}/\sim^{ext} \) whose objects are in \( \mathcal{H} \). Notice that \( \mathcal{H}/\sim^{ext} \subseteq \mathcal{H}_0(\mathcal{H}) \).

**Theorem 5.3.** There is a functor

\[
\text{Cat}^2(\text{Gp})_{\sim} \xrightarrow{\sim} \mathcal{H}_{\sim^{ext}} \xrightarrow{F} \mathcal{H}_{\sim^{ext}} \xrightarrow{\mathcal{H}_0(\text{Top}_3^{(3)})} \mathcal{D}
\]

making the following diagram commute

Further, \( F \) induces an equivalence of categories

\[
\text{Cat}^2(\text{Gp})_{\sim} \simeq \mathcal{H}_0(\mathcal{H}).
\]

**Proof.** Let \( N : \mathcal{D} \to \text{Set}^\Delta_{\text{op}} \) be induced by the nerve functor \( \text{Gp} \to \text{Set}^\Delta_{\text{op}} \) and let \( U : \text{Gp} \to \text{Set} \) be the forgetful functor. If \( \mathcal{G} = \{\psi_{1*}\} \in \mathcal{D} \) then

\[
N(\mathcal{G})_{pqr} = \begin{cases} 
\{\cdot\} & p = 0, \\
U\psi_q \times \cdots \times U\psi_q & p > 0.
\end{cases}
\]

We are going to show that \( N(\mathcal{G}) \in \mathcal{H} \). We claim that \( N([1], \cdot, \cdot) = U\psi_{1*} \) is a weak 2-groupoid. In fact, since \( \psi \in \mathcal{D} \), for each \( n \geq 0 \), \( U\psi_{n*} \) is the nerve of a groupoid and \( U\psi_{0*} \) is a constant simplicial set. For \( n \geq 2 \), the map \( U\psi_{n*} \to U\psi_{1*} \times U\psi_{0*} \times \cdots \times U\psi_{0*} \) induces isomorphisms for each \( i \geq 0 \),

\[
\pi_iBU\psi_{n*} \cong \pi_iB\psi_{n*} \cong \pi_iB(\psi_{1*} \times \psi_{0*} \times \cdots \times \psi_{0*} \psi_{1*}) = \\
= \pi_iB(U\psi_{1*} \times U\psi_{0*} \times \cdots \times U\psi_{0*} U\psi_{1*}).
\]

Hence the groupoids \( U\psi_{1*} \) and \( U\psi_{1*} \times U\psi_{0*} \times \cdots \times U\psi_{0*} U\psi_{1*} \) have weakly homotopy equivalent classifying spaces. On the other hand, being the underlying simplicial
This proves the first part of the theorem.

Hence $TU\psi_{n*}$ is the underlying simplicial set of the nerve of a $\text{Cat}(Gp)$, hence it is the nerve of a groupoid. This completes the proof that $U\psi_{n*} = N([1], \cdots, -)$ is a weak 2-groupoid. Since, for $p \geq 2$, $N([p], \cdots, -) = N([1], \cdots, \times N([1], \cdots))$. $N([p], \cdots, -)$ is a weak 2-groupoid for all $p$ and condition b) in the definition of $\mathcal{H}$ holds. Clearly condition a) also holds.

In order to show that $N(\mathcal{G}) \in \mathcal{H}$ it remains to check that $T^2 N(\mathcal{G})$ is a groupoid. This follows from the fact that

$$T^2 N(\mathcal{G})[p] = \begin{cases} \{\}\quad p = 0, \\ U\pi_0\pi_0\psi \times \cdots \times U\pi_0\pi_0\psi \quad p > 0. \end{cases}$$

The functor $N$ induces a functor

$$\overline{N} : \mathcal{D}/\sim \longrightarrow \mathcal{H}/\sim_{\text{ext}}$$

with $\overline{N}[\mathcal{G}] = [N(\mathcal{G})]$. In fact, if $f$ is a weak equivalence in $\mathcal{D}$, then $Nf$ is a morphism in $\mathcal{H}$ inducing a weak homotopy equivalence of classifying spaces. Therefore, by Lemma 5.2 b), $Nf$ is an external equivalence.

Define $F = \overline{N} \circ \text{disc}$. Let $[\mathcal{G}] \in \text{Cat}^2(\text{Gp})/\sim$ and choose a weak equivalence of $\text{cat}^2$-groups $\phi \rightarrow \mathcal{G}$ with $\phi_0$ projective in $\text{Cat}(\text{Gp})$ (Proposition 4.2). Then

$$BF[\mathcal{G}] = B\overline{N}\text{disc}[\mathcal{G}] = B\overline{N}\text{dsN}\phi = [BN\text{dsN}\phi] = [B\text{dsN}\phi] = [B\phi] = [BG] = B[\mathcal{G}].$$

This proves the first part of the theorem.

Let $R : \text{Ho}_S(\mathcal{H}) \rightarrow \text{Cat}^2(\text{Gp})/\sim$ be given by $R[\chi] = \overline{P}B\chi$ where

$$\overline{P} : \text{Ho}(\text{Top}^{(3)}_\ast) \rightarrow \text{Cat}^2(\text{Gp})/\sim$$

is induced by the fundamental $\text{cat}^2$-group functor $P : \text{Top}_\ast \rightarrow \text{Cat}^2(\text{Gp})$ (see [4]).

Let $i : \mathcal{H}/\sim_{\text{ext}} \hookrightarrow \text{Ho}_S(\mathcal{H})$ be the inclusion. We are going to show that the pair of functors $(iF, R)$ is an equivalence of categories.

Let $[\mathcal{G}] \in \text{Cat}^2(\text{Gp})/\sim$ and let $\phi \rightarrow \mathcal{G}$ be as above. By Lemma 4.3 i), $B\phi = B\text{dsN}\phi$ so that

$$RiF[\mathcal{G}] = \overline{P}Bi\overline{N}\text{disc}[\mathcal{G}] = \overline{P}B\overline{N}\text{dsN}\phi = [PB\text{dsN}\phi] = [PB\phi].$$

On the other hand, by [4], $[PB\phi] = [\phi]$; so that, in conclusion,

$$RiF[\mathcal{G}] = [\phi] = [\mathcal{G}].$$

Let $\Pi_3 : \text{Ho}(\text{Top}^{(3)}_\ast) \rightarrow \mathcal{S}$ be the fundamental groupoid functor from [22]. Recall that $\Pi_3$ sends weak equivalences to external equivalences and, further, for any $\mathcal{G} \in \mathcal{S}$ there is an external equivalence $\mathcal{G} \rightarrow \Pi_3 B\mathcal{G}$ [22, Proposition 11.4].

Let $[\psi] \in \text{Ho}_S \mathcal{H}$. Since $B\psi \simeq BPB\psi$ and there is an external equivalence in $\mathcal{S}$, $\psi \rightarrow \Pi_3 B\psi$ (see [22]), we deduce

$$[\psi] = \Pi_3 B\psi = \Pi_3 BPB\psi$$

in $\mathcal{S}/\sim_{\text{ext}}$. Choose a weak equivalence in $\text{Cat}^2(\text{Gp})$ $\phi \rightarrow PB\psi$ with $\phi_0$ projective in $\text{Cat}(\text{Gp})$. Then

$$[\Pi_3 B\phi] = \Pi_3 BPB\psi$$

(6)
in $S/\sim_{ext}$. By Lemma 4.3 i) we have

$$[NdsN\phi] = [\Pi_3 BdsN\phi] = [\Pi_3 B\phi]$$

in $S/\sim_{ext}$.

In conclusion (5), (6), (7) imply

$$[\psi] = [N dsN\phi]$$

in $\mathcal{H}as(\mathcal{H})$. On the other hand we have

$$i FR[\psi] = i F[PB\psi] = i N disc [PB\psi] = i N[dsN\phi] = [N dsN\phi]$$

so that $i FR[\psi] = [\psi]$. This completes the proof that $(iF,R)$ is an equivalence of categories.

From the previous theorem, we deduce:

**Corollary 5.4.** Every object of $S$ is equivalent to an object of $\mathcal{H}$ through a zig-zag of external equivalences.

**Proof.** Given $G \in S$, by Theorem 5.3, $\lbrack BPBG \rbrack = \lbrack BFPBG \rbrack$ in $\mathcal{H}o(Top^{(3)}_*)$. Therefore by Theorem 2.1 $\lbrack \Pi_3 BPBG \rbrack = \lbrack \Pi_3 BFPBG \rbrack$ in $S/\sim_{ext}$. On the other hand, as in the proof of Theorem 5.3, we have, in $S/\sim_{ext}$:

$$\lbrack \Pi_3 BPBG \rbrack = \lbrack \Pi_3 BG \rbrack = \lbrack G \rbrack, \quad \lbrack \Pi_3 BFPBG \rbrack = \lbrack FPG \rbrack.$$

Therefore $\lbrack G \rbrack = \lbrack FPG \rbrack$ in $S/\sim_{ext}$, and since $FPG \in \mathcal{H}$, this proves the result. \qed

Since the subcategory $\mathcal{H}$ of $S$ is strictly contained in $S$, Corollary 5.4 gives a refinement of Tamsamani’s result (Theorem 2.1) showing that every object of $S$, representing a connected 3-type, can be “semistrictified” to an object of $\mathcal{H}$.

**Remark 5.5.** Let $(T_2, \times)$ be the category of Tamsamani’s weak 2-groupoids equipped with the cartesian monoidal structure. We observe that $\mathcal{H}$ is isomorphic to a full subcategory of the category $\text{Mon}(T_2, \times)$ of monoids in $(T_2, \times)$. In fact, by general theory, using the reduced bar construction, one can show that $\text{Mon}(T_2, \times)$ is isomorphic to the category of simplicial objects $\phi$ in $T_2$ such that $\phi_0$ is trivial and the Segal maps are isomorphisms. On the other hand, by its definition the category $\mathcal{H}$ is the full subcategory of simplicial objects $\phi$ in $T_2$ with $\phi_0$ trivial and Segal maps isomorphisms such that $T\phi$ is a groupoid.

## 6. Another semistrictification of Tamsamani’s weak 3-groupoids with one object

In the previous section we showed that every object of $S$ can be “semistrictified” to an object of $\mathcal{H}$ having the same homotopy type. The goal of this section is to show that we can perform a different semistrictification from $S$ to another subcategory $\mathcal{K}$ of $S$ in such a way that the homotopy type is preserved; objects of $\mathcal{K}$ are “semistrict” 3-groupoids, but the directions in which the Segal maps are isomorphisms rather than equivalences are different from the ones for $\mathcal{H}$. In the next section, both $\mathcal{H}$ and $\mathcal{K}$ will be used in establishing the connection with Gray groupoids.

We start by defining a strictification functor from Tamsamani’s weak 2-groupoids to Tamsamani’s strict 2-groupoids, and by establishing its properties.

Let $\text{Bic} : N_2 \to \text{Bicat}$ and $\nu : 2-gpd \to T_2^{st}$ be as in Section 2. Let $\text{Bigpd}$ be the category of bigroupoids and their homomorphisms. Then $\text{Bic}$ restricts to a functor $\text{Bic} : T_2 \to \text{Bigpd}$. Let $st : \text{Bicat} \to 2\cdot\text{cat}$ be the strictification functor described in [7]. Then $st$ restricts to a functor $st : \text{Bigpd} \to 2\cdot\text{gpd}$.

**Definition 6.1.** Let $st : T_2 \to T_2^{st}$ be the composite functor

$$T_2 \xrightarrow{\text{Bic}} \text{Bigpd} \xrightarrow{st} 2\cdot\text{gpd} \xrightarrow{\nu} T_2^{st}.$$
Proposition 6.2. Let $\phi_1, \ldots, \phi_n \in T_2$.

i) $St : T_2 \to T_2^{st}$ preserves external equivalences.

ii) There is an external equivalence $\bar{g} : St(\phi_1 \times \cdots \times \phi_n) \to St \phi_1 \times \cdots \times St \phi_n$.

iii) For each $\phi \in T_2$ there is a functorial zig-zag of weak equivalences in $T_2$ between $\phi$ and $St \phi$.

Proof.

i) The functor $Bic$ sends external equivalences to biequivalences; from [7] the latter are sent by $st$ to equivalences of 2-categories which, in turn, are sent to external equivalences by $\nu$. Hence the result.

ii) It is sufficient to prove the statement for $n = 2$, from which the general case follows easily. Since $\nu$ is an isomorphism and $Bic$ preserves products (this follows from the explicit description of $Bic$ given in [12]), it is enough to show that there is an equivalence of 2-groupoids

$$g : st (Bic \phi_1 \times Bic \phi_2) \to st Bic \phi_1 \times st Bic \phi_2.$$ 

For any bicategory $X$, let $\eta_X : X \to st X$ be the biequivalence, natural in $X$ defined in [7]. There is a commutative diagram

![Diagram]

where the map $g$ is uniquely determined by $st pr_1$, $st pr_2$ ($p_1, p_2, pr_1, pr_2$ are product projections). By universality, it follows that

$$g \eta_{Bic \phi_1 \times Bic \phi_2} = (\eta_{Bic \phi_1}, \eta_{Bic \phi_2}),$$

hence

$$B \eta_{Bic \phi_1 \times Bic \phi_2} = B \eta_{Bic \phi_1} \times B \eta_{Bic \phi_2}.$$ 

Since $B \eta_{Bic \phi_1 \times Bic \phi_2}$ and $B \eta_{Bic \phi_1} \times B \eta_{Bic \phi_2}$ are weak homotopy equivalences, so is $B \eta$. But $B \eta = B \eta g$, so by Lemma 5.2, $\eta g$ is an external equivalence, so $g$ is an equivalence of 2-groupoids, as required.

iii) By Theorem 2.2 for each $\phi \in T_2$ there is a map $\phi \to N Bic \phi$, natural in $\phi$, which is a levelwise equivalence of categories, hence in particular it is a weak homotopy equivalence. Also, by [7] there is a biequivalence $Bic \phi \to st Bic \phi$, natural in $\phi$, which gives rise to a weak homotopy equivalence $N Bic \phi \to N st Bic \phi$. Also by [12], for each $\psi \in 2$-$cat$, $Bic \psi = \psi$ and, by Theorem 2.2, we have a weak homotopy equivalence $St \phi = \nu st Bic \phi \to N Bic \nu st Bic \phi = N st Bic \phi$. In conclusion we obtain a functorial zig-zag of weak equivalences

$$\phi \to N Bic \phi \to N st Bic \phi \leftarrow St \phi$$

\[\square\]

Definition 6.3. Let $K$ be the subcategory of Tamsamani’s weak 3-groupoids $\psi : \Delta^{2op} \to Set$ satisfying the additional conditions:

a) The constant functor $\psi([0], \cdot, \cdot) : \Delta^{2op} \to Set$ takes values in the one-element set.

b) For each $n$ $\psi([n], \cdot, \cdot) : \Delta^{2op} \to Set$ is an object of $T_2^{st}$, that is for $m \geq 2$ the Segal maps

$$\psi([n], [m], \cdot) : \psi([n], [1], \cdot) \times \psi([n], [0], \cdot) \times \cdots \times \psi([n], [0], \cdot) \psi([n], [1], \cdot)$$
Hence, by Proposition 6.2, an external equivalence in $\mathcal{T}_3$. Let $\mathcal{H}_{\mathcal{S}}(\mathcal{K})$ be the full subcategory of $\mathcal{S}/\sim^{\text{ext}}$ whose objects are in $\mathcal{K}$. Notice that $\mathcal{K}/\sim^{\text{ext}} \subseteq \mathcal{H}_{\mathcal{S}}(\mathcal{K})$.

**Theorem 6.4.** There is a functor

$$\overline{\mathcal{S}}: \overline{\mathcal{S}}(\mathcal{S}/\sim^{\text{ext}}) \to \mathcal{K}/\sim^{\text{ext}}$$

making the following diagram commute:

$$\begin{align*}
\mathcal{S}/\sim^{\text{ext}} & \xrightarrow{\overline{\mathcal{S}}} \mathcal{K}/\sim^{\text{ext}} \\
\mathcal{S}/\sim^{\text{ext}} \downarrow & \quad \downarrow \mathcal{K}/\sim^{\text{ext}} \\
\mathcal{H}_{\mathcal{S}}(\mathcal{S}) & \xrightarrow{\mathcal{H}(\mathcal{T}_{\infty})} \mathcal{H}_{\mathcal{S}}(\mathcal{K})
\end{align*}$$

Further, $F$ induces an equivalence of categories

$$\overline{\mathcal{S}}(\mathcal{S}/\sim^{\text{ext}}) \cong \mathcal{H}_{\mathcal{S}}(\mathcal{K})$$

**Proof.** Given an object $\psi: \Delta^{op} \to \mathcal{S}$, let $\overline{\mathcal{S}}\psi: \Delta^{op} \to \mathcal{S}$ be given by

$$(\overline{\mathcal{S}}\psi)_{n*} = \mathcal{S}\psi_{n*}.$$

We claim that $\overline{\mathcal{S}}\psi \in \mathcal{K}$. Clearly for each $n$ $(\overline{\mathcal{S}}\psi)_{n*} \in \mathcal{T}_{\infty}$ and $(\overline{\mathcal{S}}\psi)_{0*} = \{\cdot\}$. It remains to check that the Segal maps

$$\nu_n: (\overline{\mathcal{S}}\psi)_{n*} \to (\overline{\mathcal{S}}\psi)_{1*} \times \cdots \times (\overline{\mathcal{S}}\psi)_{1*}$$

are external equivalences in $\mathcal{T}_{\infty}$. Since $\psi$ is a weak 3-groupoid, the Segal maps $\delta_n: \psi_{n*} \to \psi_{1*} \times \cdots \times \psi_{1*}$ are external equivalences in $\mathcal{T}_2$. On the other hand, by definition of the map $g: \mathcal{S}\psi_{1*} \times \cdots \times \mathcal{S}\psi_{1*} \to \mathcal{S}\psi_{1*} \times \cdots \times \mathcal{S}\psi_{1*}$ it is easily checked that $\nu_n = g \circ \mathcal{S}\delta_n$. Since, by Proposition 6.2, $g$ and $\mathcal{S}\delta_n$ are external equivalences, so is $\nu_n$. This completes the proof that $\overline{\mathcal{S}}\psi \in \mathcal{K}$.

If $f: \psi \to \phi$ is an external equivalence in $\mathcal{S}$, by definition $f_1: \psi_{1*} \to \phi_{1*}$ is an external equivalence in $\mathcal{T}_2$ and $TF: T\psi \to T\phi$ is an equivalence of categories. Hence, by Proposition 6.2, $\mathcal{S}f_1 = (\overline{\mathcal{S}}f)_1$ is an external equivalence and, since $(T\psi)_n = \pi_0 \psi_{n*} \cong \pi_0 \mathcal{S}\psi_{n*}$, we have $T\psi \cong T\overline{\mathcal{S}}\psi$ and similarly $T\phi \cong T\overline{\mathcal{S}}\phi$; therefore $T\overline{\mathcal{S}}\psi$ is also an equivalence of categories and in conclusion $\overline{\mathcal{S}}f$ is an external equivalence.

Thus $\overline{\mathcal{S}}$ induces a functor $\overline{\mathcal{S}}: \mathcal{S}/\sim^{\text{ext}} \to \mathcal{K}/\sim^{\text{ext}}$. We claim that there is a weak homotopy equivalence of classifying spaces

$$B\overline{\mathcal{S}}\psi \simeq B\psi.$$  

In fact, by Proposition 6.2 iii), for each $n \geq 0$ there is a functorial zig-zag of weak equivalences between $\psi_n$ and $\mathcal{S}\psi_n$. In turn this gives rise to a zig-zag of maps between $\psi$ and $\overline{\mathcal{S}}\psi$; each of these maps is a weak homotopy equivalence, since it is so levelwise. Thus (9) follows. We conclude that $[B\overline{\mathcal{S}}\psi] = [B\psi]$ in $\mathcal{H}(\mathcal{T}_{\infty})$, showing that (8) commutes.

Let $\mathcal{K}/\sim^{\text{ext}} \xrightarrow{i} \mathcal{H}_{\mathcal{S}}(\mathcal{K}) \xrightarrow{\mathcal{H}} \mathcal{S}/\sim^{\text{ext}}$ be inclusions. We claim that the pair of functors $(i\mathcal{S}, j)$ is an equivalence of categories between $\mathcal{H}_{\mathcal{S}}(\mathcal{K})$ and $\mathcal{S}/\sim^{\text{ext}}$. In fact, let $[\psi] \in \mathcal{S}/\sim^{\text{ext}}$. Since $[B\overline{\mathcal{S}}\psi] = [B\psi]$ in $\mathcal{H}(\mathcal{T}_{\infty})$ we have

$$[\Pi_3 B\overline{\mathcal{S}}\psi] = [\Pi_3 B\psi]$$
in $S/\sim_{\text{ext}}$. On the other hand, by [22], we have
\[ [\text{St}\psi] = [\Pi_3 B\text{St}\psi], \quad [\psi] = [\Pi_3 B\psi] \]
in $S/\sim_{\text{ext}}$. Therefore
\[ j i \text{St} [\psi] = [\text{St}\psi] = [\psi]. \]
Finally, let $[\phi] \in \mathcal{H}_0 S(K)$, $\phi \in K$. Then
\[ i \text{St} j [\phi] = i [\text{St} \phi] = [i \text{St} \phi] = [\phi]. \]
Therefore $(i \text{St}, j)$ is an equivalence of categories. $\square$

7. The comparison with Gray groupoids

It is well known that Gray groupoids are semi-strict algebraic models of homotopy 3-types; see [10], [13], [1], [11]. Recall that $\text{Gray}$ is the category of 2-categories with monoidal structure given by the Gray tensor product. A Gray category is a category enriched in $\text{Gray}$. A Gray groupoid is a Gray category whose cells of positive dimension are invertible. Denote by $\text{Gray-}gpd_0$ the category of Gray groupoids with one object.

In this section we compare our semistrict models of connected 3-types to Gray groupoids; given objects of $\mathcal{H}_0 S(H)$ and of $\mathcal{H}_0 S(K)$, we will associate to them an object of $\mathcal{H}_0 (\text{Gray-}gpd_0)$ representing the same homotopy type. The functor $\text{St}: S \to K$ described in the proof of Theorem 6.4 gives by restriction a functor $\overline{\text{St}}: \mathcal{H} \to K$. Let $\overline{\text{St}} \mathcal{H}$ be the full subcategory of $K$ whose objects have the form $\overline{\text{St}} \phi$, $\phi \in \mathcal{H}$. Notice that, in general, objects of $\overline{\text{St}} \mathcal{H}$ are weak rather than strict 3-groupoids. In fact, given $\phi \in \mathcal{H}$, although the Segal maps $\phi_{n**} \to \phi_{1**} \times \cdots \times \phi_{1**}$ are bijections, the maps $\text{St}(\phi_{n**}) \cong \text{St}(\phi_{1**} \times \cdots \times \text{St}(\phi_{1**}) \to \text{St}(\phi_{1**}) \times \cdots \times \text{St}(\phi_{1**})$ are merely external equivalences in $T_2^{\text{st}}$; this follows from the fact that the functor $\text{st}: \text{Bicat} \to \text{2-Cat}$ does not preserve products strictly but only up to equivalences of $2\text{-cat}$, as easily seen from its definition.

Lemma 7.1. Every object of $K$ is equivalent to an object of $\overline{\text{St}} \mathcal{H}$ through a zig-zag of external equivalences.

Proof. Let $[\phi] \in \mathcal{H}_0 S(K)$. By [22], $[\phi] = [\Pi_3 B\phi]$ in $S/\sim_{\text{ext}}$. Also, $[B\phi] = [B\mathcal{P} \phi]$ in $\mathcal{H}_0 (\text{Top}_{(3)}^*)$, hence by [22] $[\Pi_3 B\phi] = [\Pi_3 B\mathcal{P} \phi]$ in $S/\sim_{\text{ext}}$. Therefore
\[ [\phi] = [\Pi_3 B\mathcal{P} \phi] \quad (10) \]
in $S/\sim_{\text{ext}}$.

By Theorem 5.3, $[B\mathcal{P} \mathcal{P} \phi] = [B\mathcal{P} \phi]$ in $\mathcal{H}_0 (\text{Top}_{(3)}^*)$, therefore
\[ [\Pi_3 B\mathcal{P} \mathcal{P} \phi] = [\Pi_3 B\mathcal{P} \phi] \quad (11) \]
in $S/\sim_{\text{ext}}$.

By Theorem 6.4, $[B\mathcal{P} \mathcal{P} \phi] = [B\text{St} \mathcal{P} \mathcal{P} \phi]$ in $\mathcal{H}_0 (\text{Top}_{(3)}^*)$, therefore
\[ [\Pi_3 B\mathcal{P} \mathcal{P} \phi] = [\Pi_3 B\text{St} \mathcal{P} \mathcal{P} \phi] \quad (12) \]
in $S/\sim_{\text{ext}}$. By [22]
\[ [\Pi_3 B\text{St} \mathcal{P} \mathcal{P} \phi] = [\text{St} \mathcal{P} \mathcal{P} \phi] \quad (13) \]
in $S/\sim_{\text{ext}}$. Hence (10), (11), (12), (13) imply
\[ [\phi] = [\text{St} \mathcal{P} \mathcal{P} \phi] \]
in $\mathcal{H}_0 S(K)$. Since $\text{St} \mathcal{P} \mathcal{P} \phi \in \overline{\text{St}} \mathcal{H}$ this proves the result. $\square$
Theorem 7.2. There are functors \( S : \mathcal{H}_S(\mathcal{H}) \to \mathcal{H}_0(\text{Gray-gpd}) \) and \( T : \mathcal{H}_S(\mathcal{K}) \to \mathcal{H}_0(\text{Gray-gpd}) \) making the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{H}_S(\mathcal{H}) & \xrightarrow{S} & \mathcal{H}_0(\text{Gray-gpd}) \\
\downarrow B & & \downarrow T \\
\mathcal{H}_0(\text{Top}_3^{(3)}) & \xrightarrow{B} & \mathcal{H}_0(\text{Gray})
\end{array}
\]

Proof. Let \( \phi : \Delta^{op} \to \text{Set} \) be an object of \( \mathcal{H} \). Denote \( \phi_n = \phi_{n**} \). By remark 5.5, \( \mathcal{H} \) is isomorphic to a full subcategory of the category \( \text{Mon}(\mathcal{T}_2, \times) \) of monoids in \( (\mathcal{T}_2, \times) \). The monoid corresponding to \( \phi \) is \( (\phi_1, \partial_{02}, \sigma_0) \) where \( \partial_{02} : \phi_2 \cong \phi_1 \times \phi_1 \to \phi_1 \) is the face operator induced by \([1] \to [2] 0 \to 0 1 \to 2 \) and \( \sigma_0 : \phi_0 \to \phi_1 \) is a degeneracy operator.

The functor \( \text{Bic} : \mathcal{T}_2 \to \text{Bicat} \) preserves products and the terminal object, hence it is a monoidal functor from \( (\mathcal{T}_2, \times) \) to \( (\text{Bicat}, \times) \). In [7] it is shown that the functor \( st : \text{Bicat} \to 2\text{-cat} \) is in fact a monoidal functor \( st : (\text{Bicat}, \times) \to \text{Gray} \). Denote by \( \eta_{X,Y} : st X \otimes st Y \to st (X \times Y) \) the tensor product constraint. It follows that the composite functor

\[ st \circ \text{Bic} : (\mathcal{T}_2, \times) \to \text{Gray} \]

is monoidal. Therefore, from a general fact, a monoid in \( (\mathcal{T}_2, \times) \) is sent by \( st \circ \text{Bic} \) to a monoid in \( \text{Gray} \). More precisely, the monoid \( (\phi_1, \partial_{02}, \sigma_0) \) in \( (\mathcal{T}_2, \times) \) is sent to the monoid \( (st \text{Bic} \phi_1, \mu, n) \) in \( \text{Gray} \) where

\[ \mu = (st \text{Bic} \partial_{02})(\eta_{\text{Bic}\phi_1, \text{Bic}\phi_1}), \quad n = st \text{Bic} \sigma_0. \]

The category \( \text{Mon}(\text{Gray}) \) of monoids in \( \text{Gray} \) is isomorphic to the full subcategory of Gray categories consisting of those with just one object. Thus we can view \( (st \text{Bic} \phi_1, \mu, n) \) as a Gray category with one object and 2-categorical hom-set given by \( st \text{Bic} \phi_1 \); composition is the 2-functor

\[ \mu : st \text{Bic} \phi_1 \otimes st \text{Bic} \phi_1 \to st \text{Bic} \phi_1. \]

It is immediate to see that all cells in positive dimension of this Gray category are invertible, so this is in fact a Gray groupoid. We denote this Gray groupoid by \( G_\phi \).

We are now going to show that there is a weak homotopy equivalence of classifying spaces \( BG_\phi \simeq B\phi \).

Let \( W_\phi : \Delta^{op} \to 2\text{-gpd} \) be the reduced bar construction for the monoid \( (st \text{Bic} \phi_1, \mu, n) \), so that \( \nu W_\phi : \Delta^{op} \to \mathcal{T}_2^{st} \). We observe that there is a map \( l : \nu W_\phi \to \mathcal{S}t \phi \), where \( l_n = \text{id} \) for \( n = 0, 1 \) while for \( n > 1 \) \( l_n \) is given by

\[
(\nu W_\phi)_n = \nu(st \text{Bic} \phi_1 \otimes \cdots \otimes st \text{Bic} \phi_1) \xrightarrow{\eta}_{st} \nu st (\text{Bic} \phi_1 \times \cdots \times \text{Bic} \phi_1) = \nu st (\phi_1 \times \cdots \times \phi_1) = \mathcal{S}t (\phi)_n.
\]

We claim that \( l \) induces isomorphisms of homotopy groups \( \pi_i BG_\phi \cong \pi_i B\phi \) for all \( i \geq 0 \). In fact in [1, p.63] the homotopy groups of a Gray groupoid \( G \) (called there lax 3-groupoid) with respect to a base point \( * \) are described algebraically as follows:

\[
\begin{align*}
\pi_1(G) &= \text{Aut}_G(*)/\sim \\
\pi_2(G) &= \text{Aut}_G(1_*)/\sim \\
\pi_3(G) &= \text{Aut}_G(1_*)
\end{align*}
\]

where the equivalence relation is induced by the cells of the next higher dimension.
When \( G = \mathcal{G}_\phi \), recalling the algebraic expression of the homotopy groups of a strict 2-groupoid as in [15], we therefore find:

\[
\begin{align*}
\pi_1(\mathcal{G}_\phi) &= \text{Aut}_{\mathcal{G}_\phi}(\ast) / \sim = \pi_0(st\text{ Bic } \phi_1) \\
\pi_2(\mathcal{G}_\phi) &= \text{Aut}_{\mathcal{G}_\phi}(1,) / \sim \cong \pi_0\text{Hom}_{st\text{ Bic } \phi_1}(*,*) = \pi_1\text{ st Bic } \phi_1 \\
\pi_3(\mathcal{G}_\phi) &= \text{Aut}_{\mathcal{G}_\phi}(1_{1,}) \cong \text{Aut}_{st\text{ Bic } \phi_1}(1,*) = \pi_2\text{ st Bic } \phi_1
\end{align*}
\]

By Proposition 6.2, \( \pi_i\text{ st Bic } \phi_1 \cong \pi_i\phi_1 \) for all \( i \) and, by the proof of Lemma 5.2 and by Proposition 6.2, \( \pi_i\text{ st } \mathcal{F} \cong \pi_i\phi \equiv \pi_{i-1}\phi_1 \) for all \( i \). Therefore from (15) the claim follows.

We conclude that \([B\mathcal{G}_\phi] = [B\phi]\) in \( \mathcal{H}o(Top^{s}(3)) \). Given \([\phi] \in \mathcal{H}o_{S}(\mathcal{H})\), \( \phi \in \mathcal{H} \), define

\[ S[\phi] = \mathcal{G}_\phi. \]

This functor is well defined since if \([\phi] = [\phi']\) in \( \mathcal{H}o_{S}(\mathcal{H}) \), then \( \phi \) and \( \phi' \) are externally equivalent, hence \( B\phi \simeq B\phi' \) so, from above \( B\mathcal{G}_\phi \simeq B\mathcal{G}_{\phi'} \), which implies \([\mathcal{G}_\phi] \simeq [\mathcal{G}_{\phi'}]\) in \( \mathcal{H}o(\text{Gray-gpd}_0) \).

Given \([\psi] \in \mathcal{H}o_{S}(\mathcal{K})\), by Lemma 7.1 there is \( \phi \in \mathcal{H} \) such that \([\psi] = [\text{st } \mathcal{F}] \phi\) in \( \mathcal{H}o_{S}(\mathcal{H}) \). Define

\[ T[\phi] = T[\text{st } \mathcal{F}] \phi = [\mathcal{G}_\phi]. \]

This is well defined because if \([\text{st } \mathcal{F}] \phi = [\text{st } \mathcal{F}] \phi'\) then \( B\text{ st } \mathcal{F} \phi = B\text{ st } \mathcal{F} \phi' \) hence \( B\phi \simeq B\phi' \) so \( B\mathcal{G}_\phi \simeq B\mathcal{G}_{\phi'} \), which implies \([\mathcal{G}_\phi] \simeq [\mathcal{G}_{\phi'}]\). It is immediate that (14) commutes. \( \square \)

**Appendix**

**Proof of Lemma 5.2**

a) Let \( f \) be an external equivalence. The fact that \( Bf \) is then a weak homotopy equivalence is proved in [22, Proposition 11.2]. Suppose, conversely, that \( Bf \) is a weak homotopy equivalence. Since \( \phi \) is a weak 2-groupoid, \( \pi_1(\phi) \) is the nerve of a group and \( \pi_n(\phi) = 0 \) for \( n > 1 \), then \( \phi \) satisfies the \( \pi_t\)-Kan condition in the sense of [2], for all \( t \geq 0 \). By [2, Theorem B.5] there is a first quadrant spectral sequence

\[
E_{s,t}^2 = \pi_s\pi_t\phi \Rightarrow \pi_{s+t}D
\]

where \( D = \text{diag } \phi \). Since each \( \phi_n\ast \) is the nerve of a groupoid, \( \pi_n\phi = 0 \) for \( t > 1 \), so that \( E_{s,t}^2 = 0 \) unless \( t = 0, 1 \). It follows that there is a long exact sequence

\[
\cdots \to \pi_{p+1}D \to E_{p+1,0}^2 \to E_{p,1}^2 \to \pi_pD \to E_{p,0}^2 \to \cdots
\]

On the other hand, since \( \phi \) is a weak 2-groupoid, for \( n > 1 \)

\[
\begin{align*}
\pi_0\phi_{n\ast} &\cong \pi_0(\phi_{1\ast} \times \phi_{0\ast} \cdots \times \phi_{0\ast} \phi_{1\ast}) \cong \pi_0\phi_{1\ast} \times \pi_0\phi_{0\ast} \cdots \times \pi_0\phi_{0\ast} \pi_0\phi_{1\ast} \\
\pi_1\phi_{n\ast} &\cong \pi_1(\phi_{1\ast} \times \phi_{0\ast} \cdots \times \phi_{0\ast} \phi_{1\ast}) \cong \pi_1\phi_{1\ast} \times \pi_1\phi_{1\ast} \cdots \times \pi_1\phi_{1\ast} \pi_1\phi_{1\ast}.
\end{align*}
\]

It follows that

\[
\begin{align*}
E_{s,0}^2 = \pi_s\pi_0\phi &= 0 \quad \text{for } s > 1 \\
E_{s,0}^2 = \pi_s\pi_1\phi &= 0 \quad \text{for } s = 0 \text{ and } s > 1.
\end{align*}
\]

From the above long exact sequence we deduce

\[
\pi_pD = \begin{cases}
E_{0,0}^2 = \pi_0\pi_0\phi & p = 0, \\
E_{1,0}^2 = \pi_1\pi_0\phi & p = 1, \\
E_{1,1}^2 = \pi_1\pi_1\phi = \pi_1\phi_{1\ast} & p = 2, \\
0 & p > 2.
\end{cases}
\]
By hypothesis, \( f \) induces isomorphisms of homotopy groups \( \pi_i B \phi \cong \pi_i B \phi' \), \( i \geq 0 \). Since \( \pi_i B \phi = \pi_i \text{diag} \phi \) from above we obtain isomorphisms
\[
\begin{align*}
\pi_0 \pi_0 \phi & \cong \pi_0 \pi_0 \phi' \\
\pi_1 \pi_0 \phi & \cong \pi_1 \pi_0 \phi' \\
\pi_0 \phi_1 & \cong \pi_0 \phi_1'.
\end{align*}
\]
(17)
The first two isomorphisms in (17) show that the map \( Tf : \pi_0 \phi \to \pi_0 \phi' \) is a weak homotopy equivalence. On the other hand, since \( \pi_0 \phi \) and \( \pi_0 \phi' \) are nerves of groupoids, they are fibrant simplicial sets. Since every simplicial set is cofibrant, \( Tf \) is a weak homotopy equivalence between fibrant and cofibrant simplicial sets, Thus, by a general result on model categories [9], \( Tf \) is a simplicial homotopy equivalence. Hence \( Tf \) corresponds to an equivalence of categories.

In order to show that \( f \) is an external equivalence of weak 2-groupoids it remains to check that, for all \( x, y \in \phi_0 \), the map
\[
f_{(x,y)} : \phi_{(x,y)} \to \phi'_{(f x, f y)}
\]
is an equivalence of categories. Since \( \phi_1 \) is a groupoid, its first homotopy group is isomorphic to the endomorphism group of any identity arrow, that is, for any \( z \in \phi_{10} \)
\[
\pi_1 \phi_1 \cong \text{Hom}_{\phi_1}(z, z).
\]
Further, as \( \phi_0 \) is a groupoid for any \( z, z' \in \phi_{10} \) there are isomorphisms
\[
\text{Hom}_{\phi_1}(z, z) \cong \text{Hom}_{\phi_1}(z, z'),
\]
and similarly for \( \phi_1' \).

Hence by the third isomorphism in (17) we deduce that for each \( z, z' \in \phi_{10} \)
\[
\text{Hom}_{\phi_1}(z, z') \cong \text{Hom}_{\phi_1'}(f z, f z').
\]
(18)
On the other hand recall that
\[
\prod_{x, y \in \phi_0} \phi_{(x,y)} = \phi_1, \quad \prod_{x', y' \in \phi'_0} \phi'_{(x',y')} = \phi_1'.
\]
Taking \( z, z' \) such that \( \partial_0 z = x = \partial_0 z' \) and \( \partial_1 z = y = \partial_1 z' \) we see that (18) restricts to an isomorphism
\[
\text{Hom}_{\phi_{(x,y)}}(z, z') \cong \text{Hom}_{\phi'_{(f x, f y)}}(f z, f z').
\]
showing that \( f_{(x,y)} \) is full and faithful.

Finally let \( z \in \phi'_{(f x, f y)}(0) \). Then \( f x \) and \( f y \) are in the same isomorphism class of objects of the category \( \pi_0 \phi' \). Since by (17) \( \pi_0 \pi_0 \phi \cong \pi_0 \pi_0 \phi' \), there exists \( w \in \phi_{(x,y)}(0) \) and an isomorphism \( f_{(x,y)}(w) \to z \). This shows that \( f_{(x,y)} \) is essentially surjective on objects. Hence \( f_{(x,y)} \) is an equivalence of categories.

b) The fact that \( B g \) is a weak homotopy equivalence when \( g \) is an external equivalence is proved in [22, Proposition 11.2]. Suppose, conversely, that \( B g \) is a weak homotopy equivalence. Consider the bisimplicial sets \( \phi \) and \( \phi' \) defined by
\[
\phi_{np} = (\text{diag} \psi_{n*})_p, \quad \phi'_{np} = (\text{diag} \psi'_{n*})_p.
\]
For each \( n \geq 1 \) the Segal maps \( \delta_n : \psi_{n*} \to \psi_{1*} \times \cdots \times \psi_{1*} \) induce maps of simplicial sets \( \overline{\delta}_n : \phi_{n*} \to \phi_1 \times \cdots \times \phi_1 \). Since \( \psi, \in \mathcal{S} \), \( \delta_n \) is an external equivalence of weak 2-groupoids, hence \( B \delta_n \) is a weak homotopy equivalence. Since \( B \psi_{n*} \cong B \phi_{n*} \), \( B \delta_n \) is also a weak homotopy equivalence. Therefore, for each \( i \geq 0 \)
\[
\pi_i \phi_{n*} = \begin{cases} 
0 & n = 0, \\
\pi_i \phi_1 \times \cdots \times \pi_i \phi_1 & n > 0.
\end{cases}
\]
Thus $\pi_i\phi_1$ is the nerve of a group, so it is fibrant. Hence the bisimplicial set $\phi$ satisfies the $\pi_*$-Kan condition. By [2] there is a spectral sequence

$$E^2_{s,t} = \pi_s\pi_t\phi \Rightarrow \pi_{s+t}(\text{diag } \phi).$$

We have

$$\pi_s\pi_t\phi = \begin{cases} 0 & s = 0, \ s > 1, \\ \pi_t\phi_{1s} & s = 1. \end{cases}$$

It follows that

$$\pi_n\text{ diag } \phi \cong \begin{cases} 0 & n = 0, \\ E^2_{1,n-1} = \pi_{n-1}\phi_{1s} & n > 0. \end{cases}$$

Similarly we have

$$\pi_n\text{ diag } \phi' \cong \begin{cases} 0 & n = 0, \\ E^2_{1,n-1} = \pi_{n-1}\phi'_{1s} & n > 0. \end{cases}$$

By hypothesis, $g$ induces isomorphisms of homotopy groups $\pi_iB\psi \cong \pi_iB\psi'$ for all $i$. Since $\pi_iB\psi \cong \pi_iB\phi \cong \pi_i\text{ diag } \phi$ and similarly for $\psi'$, from above we deduce $\pi_i\phi_{1s} \cong \pi_i\phi'_{1s}$ and therefore $\pi_iB\psi_{1**} \cong \pi_iB\psi'_{1**}$ for all $i$. Hence the map $g_1: \psi_{1**} \to \psi'_{1**}$ of weak 2-groupoids is a weak homotopy equivalence. By a), it is also an external equivalence of weak 2-groupoids.

To conclude the proof that $g$ is an external equivalence of weak 3-groupoids, it remains to show that $T^2g$ is an equivalence of categories. We have

$$T\psi([p], \cdot, \cdot) = \begin{cases} \{\} & p = 0, \\ \pi_0\pi_0\psi_{1**} \times \cdot \times \pi_0\pi_0\psi_{1**} & p > 0. \end{cases}$$

and similarly for $\psi'$.

Since $g_1: \psi_{1**} \to \psi'_{1**}$ is an external equivalence of weak 3-groupoids, $\pi_0g_1$ is an equivalence of categories, therefore $\pi_0\pi_0\psi_{1**} \cong \pi_0\pi_0\psi'_{1**}$. So $Tg$ is in fact an isomorphism, hence in particular an equivalence of the corresponding categories. □

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