Magnetic screening in the hot gluon system

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Abstract

The gluon transverse self–energy of the pure Yang–Mills system at high–temperature is analysed in the static limit and at fourth order in the coupling. Possible contributions to this function are collected, seen to be gauge–fixing independent subsets and shown to vanish all, except those which are either regulators or constituents of the self–energy of Euclidean 3D Yang–Mills theory at zero temperature. The latter self–energy, in turn, is known from the non–perturbative analysis by Karabali and Nair.

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1. Introduction

Twenty years ago it was observed by Linde [1] that the perturbative treatment of the high-temperature Yang–Mills system runs into a serious problem. If a magnetic mass $\tau$, the system might be able to generate thermally, falls short of $g^2 T$ in magnitude, the series would diverge, and a phase of deconfined gluons could not exist. But even if $\tau \sim g^2 T$, the perturbation series becomes an (unknown) numerical series. Due to this phenomenon [2], no one was able so far to calculate the pressure at order $g^6$ or the gluon self-energy at $g^4$ – a shame for analytical theoretical physics. Today, however, there is a way out. It is provided by the non-perturbative analysis of Karabali, Kim and Nair (referred to as KKN in the sequel) of 2+1D Yang–Mills theory at zero temperature [3, 4, 5]. In this note, by studying static magnetic screening, which is presumably the simplest example, it is shown how to use KKN’s results in an otherwise perturbative treatment.

The ”Linde sea” of diagrams is easily understood from figure 1. If one more line is added to an arbitrary skeleton diagram, e.g. in the manner shown in the figure, then, in the sense of power counting, it has two more 3–vertices ($\sim p^2 g^2$), three more propagators ($\sim (p^2 + \tau^2)^{-3}$) and one more loop integration ($T \int d^3 p$, if reduced to the term with zero Matsubara frequency). Thus, the (n+1)–loop and n–loop differ by a factor $\sim g^2 T \int d^3 p p^2 (p^2 + \tau^2)^{-3} \sim g^2 T/\tau$. For $\tau \sim g^2 T$ this factor has order 1 in magnitude. Once the zero–frequency modes become relevant, all skeletons contribute with the same order of magnitude. Any finite–n–loop calculation of the magnetic mass is thus inconsistent.

Figure 1: An arbitrary 2–leg n–loop skeleton diagram with one line added: the half circle on top, say, or, equivalently, the one below. The outer momentum $Q$ is static ($Q_0 = 0$) and supersoft ($q \sim \tau$).

Bosonic fields live on a cylinder with circumference $\beta = 1/T$. Each loop integration $\sum_P \equiv T \sum_n \int_P^3, \int_P^3 \equiv (2\pi)^{-3} \int d^3 p$, has its zero–frequency part $T \int_P^3$. A field depending on $P$ looses dependence on its time coordinate in this part. Irrespective of the physical quantity under study, the subset of contributions with $P_0 = 0$ in all loops might be the full set of an Euklidean physics at $T = 0$ in three dimensions [4, 7, 8]. However, this theory needs regulators to be derived from the underlying 4D setup.

All about the regularized 3D Euklidean Yang–Mills theory is inherent in its 2+1D version, and this is the system treated by KKN. For an outline of the main argument see [4]. Appropriate shortcuts are found in [9], and for a low–level introduction see [10]. Working in Weyl gauge and after splitting off the remaining gauge volume,
KKN quantize the physical degrees of freedom in the Schroedinger functional picture. A point–splitting regularization enables them to write scalar products of functional states as correlators of the hermitean WZW model. Conformal field theory then reveals the structure of normalizable wave functionals. Eigenstates of the Hamiltonian are constructed. The mass in the 3D gluon propagator is found to have the value \( g^2NT/(2\pi) \) at its leading order.

While taking KKN’s results for granted, the subject of this letter is the perturbation theory for magnetic screening, i.e. for the static limit of a ”dynamical” quantity. Dynamics rests on linear response theory. The response of the gluon medium to infinitesimal perturbations is obtained by analytical continuation \( Q_0 \rightarrow \omega + i\varepsilon \) from Matsubara frequencies \( Q_0 = i2\pi nT \). The breakthrough in understanding the gluon dynamics came in 1990, because the ”zeroth approximation” of high–T QCD was established only then \([1, 12]\) and given the form of an effective action \([13, 14, 15, 16]\).

In the whole \( \omega-\tilde{\omega} \) space gauge fixing independence is guaranteed \([17]\) only along the dispersion lines, longitudinal and transverse. In a diagram \( \omega^2 \over q^2 \), which is figure 1 in \([18]\), the transverse line runs down near the light–cone but then deviates from the Dirac–spectrum shape to reach the plasmon frequency \( m^2 = g^2T^2/9 \) at \( q^2 = 0 \). Crossing this point, \( q^2 \) becomes negative. The transverse line then changes slope and turns towards the origin.

The static limit refers to zero frequency, i.e. to the lower border of the \( \omega^2 > 0 \) half–plane. According to the above, only two points on this border line have physical meaning, namely the end points \( -q^2 = \sigma^2 \) (Debye screening) and \( -q^2 = \tau^2 \) (magnetic screening) of the two dispersion lines. Here, \( \sigma^2 \) is the squared Debye mass including its correction \( \sim g^3 \) \([19]\). Its leading term is \( 3m^2 \). The transverse polarization function \( \Pi_t(Q) \), on the other hand, vanishes at \( Q_0 = 0 \) to leading order \( \sim g^2 \) as well as in next–to–leading order \([19, 18]\), thereby giving rise to the magnetic mass problem. It is resolved at order \( g^4 \) where the solution \( -q^2 = \tau^2 \) to the equation \( 0 = q^2 + \Pi_t(Q_0 = 0, q) \) attains a positive value.

The purpose of this note is threefold:

1. Though it is rather plausible, that the 4D object \( \Pi_t(Q_0 = 0, q) \) may be identified with \( \Pi_t(q) \) of the zero–temperature 3D Yang–Mills system, we like to remove possible doubts in this step.
2. It has to be seen with detail how diagrammatic contributions reduce to the correct 3D Euclidean rest.
3. Regulators are to be prepared out of the 4D thermal theory.

We shall analyse \( \Pi_t(Q_0 = 0, q) \) up to order \( g^4 \) only. Momenta of order \( T, gT, g^2T \) in magnitude are called hard, soft and supersoft, respectively. Then, with an upper–left
index denoting the number of loops,

\[-q^2 = 1 \Pi_t^{\text{bare}} + 1 \Pi_t^{\text{next-to}} + 2 \Pi_t^{\text{hard-hard}} + 2 \Pi_t^{\text{hard-supersoft}} + \geq 2 \Pi_t^{\text{all momenta supersoft}} \quad (1)\]

is all we have to include. Diagrams with more than two loops enter through the Linde mechanism, i.e. only if all momenta are supersoft. Hence, the above last term is the two- and higher loop part of the 3D Euklidean theory. Its 1-loop pieces are to be taken from the first two terms of (1). But once they are combined with the last term, nothing might be left in the spirit of the above conjecture. In fact, after such regroupings, our final result may be written as

\[-q^2 = 0 + 0 + 0 + \Pi_t^{\text{regularized} \text{ 3D, } T=0}. \quad (2)\]

where each number zero corresponds to a gauge-fixing independent subset, and the index refers to the \(g\) power to which it vanishes. As the 3D theory is a physics by itself, even the non-vanishing term of (2) refers to a gauge-fixing independent subset.

In sections 2, 3, 4 and 5 the first four terms of (1) are studied, respectively.

**2. One loop diagrams with bare lines**

While re-examining the transverse polarization function \(\Pi_t\) at one-loop order we shall take care of its \(g^4\) contributions.

Our metrics is \(+ -- - -\). We shall need three members of the four-fold Lorentz matrix basis \([3]\):

\[\mathcal{A} = \frac{(0, \vec{p}) \circ (0, \vec{p})}{p^2} + g - U \circ U, \quad \mathcal{B} = \frac{(p^2, P_0 \vec{p}) \circ (p^2, P_0 \vec{p})}{-P^2p^2}, \quad \mathcal{D} = \frac{P \circ P}{P^2}, \quad (3)\]

where \(U = (1, 0)\). The Lagrangian is written as

\[\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu} a - \frac{1}{2\alpha} \left(\partial^\mu A^a_\mu\right)^2 + \frac{1}{2} A^{\mu a}(Y A)^a_\mu + c_2^a \partial^\mu D^{ab} b^b - \frac{1}{2} A^{\mu a}(Y A)^a_\mu \quad . (4)\]

with \(F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\) and \(D^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A^c_\mu\). Renormalization is understood to be done at \(T = 0\). For the present, we may account for \(Z\) factors, which are missing in (4), by omitting non-Bose-function parts in the evaluation of frequency sums.

At the right end of (4) the mass term \(\mathcal{Y}\) is subtracted to reinstall the original theory. The bare only indicates its use at one loop higher order. Through \((Y A)^a_\mu = \sum_\nu e^{-iP\nu} Y^{\mu\nu}(P)A^a_\nu(P)\) this mass may be chosen momentum dependent [20]. By

\[Y^{\mu\nu}(P) = \mathcal{A}^{\mu\nu}(P) \tau^2 \delta_{P_0,0} + \mathcal{B}^{\mu\nu}(P) \sigma^2 \delta_{P_0,0} \quad (5)\]
we adopt the minimal version of infrared regularisation, which was found useful particularly in thermodynamics \[21, 22\]. We are allowed to work with simple mass terms, in place of the full effective action \[15\], because there is no hard–thermal–loop dressing of vertices at \( \omega = 0 \). The Kronecker version \((5)\) of these masses needs restriction to the static limit as well. In principle, in place of \( \sigma^2 \) there is some function of \( p \) to be determined consistently. But we know from \[20\] that it becomes the longitudinal self–energy to the order of interest. Even \( \tau^2 \) is inserted into \((5)\) as the expected outcome by summing over the Linde sea. The possible dependence of \( \tau^2 \) on \( p \) is of no relevance in the sequel. But according to KKN, and anticipating the announced three zeros, there might be no such dependence, at least not at supersoft \( p \).

The one–loop diagrams for \( \Pi^{\mu\nu} \) are tadpole, loop and ghost–loop. With view to \((5)\) it would be natural to decompose \( \sum P_0 \) into its \( P_0 = 0 \) and \( P_0 \neq 0 \) part. However, two divergent pieces would arise this way. Therefore, but also to keep contact with earlier work \[23, 24, 18, 25\], we better distinguish bare from dressed lines, start with bare ones and defer the difference (giving \( g^3 \) terms) to the next section.

With bare gluon lines \( G_0(P) = g^{\mu\nu}/P^2 + (\alpha - 1) P^\mu P^\nu/P^4 \), the transverse function \( \Pi_t = \frac{1}{2} \text{Tr}(\Pi) \), is conveniently split into three terms

\[ h_{\Pi_{\text{bare}}} = \Pi(0) + \Pi(1) + \Pi(2) \quad . \] (6)

For the last term see \((12)\) below. The familiar loop integration

\[ \Pi(0) + \Pi(1) = g^2 N \sum_P \left( -\frac{2}{P^2} - \frac{2 \left[ p^2 - \frac{(q \cdot p)^2}{q^2} \right]}{P^2(P - Q)^2} \right) \] (7)

shares the "true zeroth order" of the hot gluon system: \( \Pi(0) \sim g^2 \) sets the scale. Watching \( g^4 \) terms, however, \((8)\) is not identical with its leading order part. The latter,

\[ \Pi(0) = g^2 N T q^2 \left[ z^2 - (z^2 - 1) \frac{z}{2} \ln \left( \frac{z + 1}{z - 1} \right) \right] , \quad z \equiv \frac{Q_0}{q} \quad , \] (8)

is the well known transverse self–energy at leading order (see e.g. \[23\], Appendix B). It is gauge–fixing independent, and for \( Q_0 \to \omega = 0 \) (\( z = 0 \)) it clearly vanishes. Let this zero contribution be the first number zero in equation \((4)\).

As announced, \((8)\) is not \((7)\). With a bit analysis, and turning to the static limit, we obtain

\[ \Pi_{(1)}^{Q=0} = g^2 N \frac{T q}{16} = g^2 N T \int_\rho^3 \left( \frac{4}{3} \frac{1}{p^2} - \frac{2 \left[ p^2 - \frac{(q \cdot p)^2}{q^2} \right]}{p^2(p - q)^2} \right) \quad . \] (9)

For the right 3D Euclidean Yang–Mills theory we must beware the \( P_0 = 0 \) part of \((7)\). But if taken at \( Q_0 = P_0 = 0 \), \( \sum \to T \int_\rho^3 \), \((7)\) diverges. The integral in \((9)\), on
the other hand, is convergent, but the first term comes with the wrong factor. The resolution to this puzzle is by adding to (9) the following identity, valid for all $M$:

\[
0 = \int_p \frac{2 M^2}{3 p^2} \partial_p \left( \frac{p^2}{M^2 + p^2} \right) = \int_p \left( \frac{2}{3} \frac{1}{p^2} - \frac{2}{M^2 + p^2} + \frac{4}{3 (M^2 + p^2)^2} \right) .
\]

(10)

Then (9) turns into

\[
\Pi^{Q_0=0}_{(1)} = g^2 N T \int_p \left( \frac{2}{p^2} - \frac{2}{M^2 + p^2} - \frac{2 \left[ p^2 - \left( \frac{\vec{q} \cdot \vec{p}}{q^2} \right)^2 \right]}{p^2 (\vec{p} - \vec{q})^2} + \frac{4}{3 (M^2 + p^2)^2} \right) .
\]

(11)

Note that, for large $p$ and due to angular integration, the square bracket may be replaced by $2 p^2 / 3$. Hence, the terms to be preserved are supplied with regulation in the UV. Obviously, apart from $M \gg g^2 T$, the value of the regulator mass $M$ may be chosen at will.

There remains to notice the third contribution to $\Pi^\text{bare}_t$:

\[
\Pi_{(2)} = g^2 N \sum_p \left( \frac{2 Q^2}{P^2 (P - Q)^2} + (\alpha - 1) \left[ - \frac{Q^2}{P^4} + \frac{Q^4}{P^4 (P - Q)^2} - \frac{\left[ p^2 - \left( \frac{\vec{q} \cdot \vec{p}}{q^2} \right)^2 \right] Q^2}{P^4 (P - Q)^2} \right] \right. \\
\left. - (\alpha - 1)^2 \frac{\left[ p^2 - \left( \frac{\vec{q} \cdot \vec{p}}{q^2} \right)^2 \right] Q^4}{4 P^4 (P - Q)^4} \right) .
\]

(12)

Such terms are known from the next-to-leading order analysis at soft scale. There, for the order $g^3$, they are all irrelevant \([23]\) (§4 there). But here, where terms of order $g^4 T^2$ are to be maintained, they are all relevant — in spite of the supersoft $Q^2 = \tilde{q}^2$ in the numerators. This fact is easily realized by power counting with a supersoft IR regulator $\sim \tau$ in mind. Nonzero $P_0$ make $\Pi_{(2)} \sim g^6$. So, the Matsubara sum in (12) may be reduced to its $P_0 = 0$ term, hence $\sum$ replaced by $T \int_p^3$, immediately.

3. One loop : the next-to-leading order

Contributions of order $g^3$ to $\Pi_t$ (its "true first approximation") arise from 1-loop diagrams due dressing of lines and vertices \([11, 23, 24, 18]\). They are prepared by forming the difference

\[
1\Pi^\text{next-to} \rightarrow = \Pi^\text{dressed}_t - \Pi^\text{bare}_t .
\]

(13)

Thanks to the static limit (no vertex dressing) and to the economical IR regularization \(\Box\), terms with non-zero $P_0$ drop out in this difference. Moreover, the outer momentum has $Q_0 = 0$. Hence,

\[
G^{\mu \nu}(P_0 = 0, \vec{p}) = -\mathcal{A}^{\mu \nu}_{\mu \nu \sigma} \Delta \tau - B^{\mu \nu \sigma}_{\mu \nu \sigma \tau} \Delta \sigma - \alpha D^{\mu \nu \sigma \tau}_{\mu \nu \sigma \tau} \Delta_0
\]

(14)
is all we need about the 4D dressed propagator. In (14), the 4D partial propagators (such as e.g. \(1/(P^2 - \sigma^2\delta_{P,0})\) have turned -- under change of sign -- into the 3D Euklidean propagators

\[
\Delta_{\sigma} = \frac{1}{p^2 + \sigma^2}, \quad \Delta_{\tau} = \frac{1}{p^2 + \tau^2}, \quad \Delta_0 = \frac{1}{p^2} \quad \text{and} \quad \Delta_\sigma = \frac{1}{(p-q)^2 + \sigma^2} \quad \text{etc.} \quad \text{(15)}
\]

The calculation of \(\Pi_t^{\text{next-to}}\) is straightforward, although a bit lengthy. It is much simplified by the following four observations:

(a) Since not dressed, the ghost–loop needs not be included.

(b) At \(P_0 = 0\) the matrix \(B(P)\) becomes \(U \circ U\). Hence, as the matrix \(A\) has no zeroth components (moreover: \(A_{P_0=0} = A\)), products of \(A\) with \(B\) vanish.

(c) Terms \(\sim Q^\mu, \sim Q^\nu\) or \(\sim B^{\mu\nu}\) may be omitted, since \(\Pi^{\mu\nu}\) will be traced with \(A(Q)\), which is a projector with respect to \(Q\).

(d) For inner momenta we have \(A(P)P = 0\) and \(A(P-Q)P = A(P-Q)Q\).

According to (b) and (c) only very few terms with \(\Delta_\sigma\) survive, namely one in the tadpole diagram (due \(B^\lambda_\lambda = 1\)) and one in the loop (due \(U^\lambda_\mu U^\mu_\nu U^\nu_\rho = 1\)). The result is

\[
\Pi_t^{\text{next-to}} = g^2 NT \int_p^3 \left\{ \Delta_{\sigma} - \Delta_0 - \left[ p^2 - \frac{(\overrightarrow{q} \overrightarrow{p})^2}{q^2} \right] \left( \Delta_{\sigma} \Delta_{\sigma} - \Delta_0 \Delta_0 \right) + \frac{4}{3} \left( \Delta_{\tau} - \Delta_0 \right) \right. \\
- \frac{1}{4} \left[ p^2 - \frac{(\overrightarrow{q} \overrightarrow{p})^2}{q^2} \right] \left( 6 + 2p^2 \Delta_0 + 12q^2 \Delta_0 + q^4 \Delta_0 \right) \left( \Delta_{\tau} \Delta_{\tau} - \Delta_0 \Delta_0 \right) \\
+ \alpha \left[ 2 \left[ p^2 - \frac{(\overrightarrow{q} \overrightarrow{p})^2}{q^2} \right] \Delta_0 - 1 \right] (p^2 - q^2)^2 \Delta_0 \Delta_0 \left( \Delta_{\tau} - \Delta_0 \right) \left\} \right. \quad \text{(16)}
\]

As always, we first look, whether the subset (16) depends on the gauge-fixing parameter. It does. However, thanks to the last factor, the term with \(\alpha\) is of order \(g^2 T \int_p^3 (\Delta_{\tau} - \Delta_0) \sim g^2 T \tau \sim g^4 T^2\) in magnitude. Hence, the \(g^3\) contribution to \(\Pi_t\) at \(Q_0 = 0\), which is fully contained in (16), is gauge-fixing independent.

The \(g^3\) subset even vanishes. To rederive this known fact (19) from (16), we select its \(g^3\) terms as

\[
\Pi_t^{\text{next-to}} \bigg|_{g^3} = g^2 NT \int_p^3 \left( \Delta_{\sigma} - \Delta_0 - \frac{2}{3} p^2 \left( \Delta_{\sigma} \Delta_{\sigma} - \Delta_0 \Delta_0 \right) \right) = 0 \quad \text{(17)}
\]

and recall the identity (10). (17) is the second number zero in (2). But note that (16) also shows what is left at order \(g^4\).

All \(g^4\) contributions in 1-loop diagrams are now collected from (16), (11) and (12), the latter taken at \(Q_0 = P_0 = 0\). Even for the order \(g^4\) of interest, the large-mass propagator \(\Delta^-\) in (16) may be replaced by \(\Delta_\sigma\). With \(\Re\) denoting regulator terms, the result may be written as

\[
\Pi_t^{\text{1-loop}} = g^2 NT \int_p^3 \left\{ \Re + \frac{4}{3} \Delta_{\tau} + \frac{1}{2} \left[ p^2 - \frac{(\overrightarrow{q} \overrightarrow{p})^2}{q^2} \right] \Delta_0 \Delta_0 \right. \\
\]

\[\]
\[-\frac{1}{2} \left[ p^2 - \frac{(q \cdot p)^2}{q^2} \right] \left( 3 + p^2 \Delta_0 \right) \Delta_\tau \Delta_\tau - 3q^2 \left[ p^2 - \frac{(q \cdot p)^2}{q^2} \right] \Delta_0 \Delta_\tau \Delta_\tau \]

\[-\frac{1}{4} q^4 \left[ p^2 - \frac{(q \cdot p)^2}{q^2} \right] \Delta_0 \Delta_0 \Delta_\tau \Delta_\tau + \text{terms } \sim \alpha, \alpha^2 \]

The regulators are

\[ \Re = \Delta_\sigma - 2 \Delta_M + \frac{4}{3} p^2 \Delta_2^2 - \frac{2}{3} p^2 \Delta_\sigma^2 = -\Delta_\sigma + \frac{2}{3} p^2 \Delta_\sigma^2 \]

with the right end valid for the choice \( M = \sigma \). Hence, as the reminder in the wavy bracket \([18]\) behaves as \( \frac{1}{3} \Delta_0 \) for large \( p \), the regulators prevent \([18]\) from diverging linearly. Remember, that independence on gauge–fixing can not be required, because now (being at order \( g^4 \)) consistency is only achieved by including the whole Linde sea. Therefore the terms \( \sim \alpha, \alpha^2 \) in \([18]\), which are UV finite, need not be detailed.

The expression \([18]\) is precisely what one obtains at 1–loop order for the 3D Yang–Mills theory at zero temperature and with coupling \( e^2 = g^2 T \). We have done this 3D calculation in the metrics \(+ – –\), by using the propagator \( A_{\mu\nu}/(K^2 - \tau^2) + \alpha D_{\mu\nu}/K^2 \), by rotating the zeroth momentum at the end and by tracing with \( \frac{1}{2} A(Q) \). Because \([18]\) has order \( g^4 \) in magnitude, the 3D self–energy starts with \( e^4 \).

**4. Two loops — both inner momenta hard**

We turn to the third number zero in \([2]\). Due to its generic prefactor \( g^4 \), 2–loop diagrams with hard inner momenta appear to give the natural contributions to the order \( g^4 \) of interest. In an other context, the 13 diagrams of Figure 2 were analysed in \([23]\), but in the ”wrong” limit \( q \rightarrow 0 \) first) and in Feynman gauge \( \alpha = 1 \) only. Here, instead, we ask two other questions: first, whether the hard–hard diagrams form a gauge invariant subset in the static limit, and second, whether its contribution vanishes. Hence, lines are associated with the propagators \( G_0(P) = g^{\mu\nu}/P^2 + (\alpha - 1) P^\mu P^\nu/P^4 \).

Application of the diagrammatic rules and the colour sums were done by hand. But MAPLE programs were helpful in Lorentz contracting and sorting the various terms.

![Figure 2: The two–loop diagrams with symmetry factors in front of each. Dotted lines refer to ghost propagators, normal lines to gluons.](image)

For convenience the result was split according to powers of \((\alpha - 1)\) and traced with \( \mathbb{A} \). Terms \( \sim (\alpha - 1)^5 \) occur in only two diagrams and drop out in each, immediately.
Terms $\sim (\alpha - 1)^4$ to $2$ were seen to vanish on the transversal line in the static limit, i.e. at $Q_0 = 0$ and $q \to 0$ in numerators. This somewhat laborious step involves several symmetry arguments and, finally, the suppression of all $q^2$ or $\hat{q}$, which remained in numerators. The powers $(\alpha - 1)^4$ and $(\alpha - 1)^0$ require more detail. By working as indicated above we first obtain

$$
\Pi_{t}^{2-\text{loop hh}} \bigg|_{\sim (\alpha - 1)} = 2g^4N^2 \left( - J_0(0) I_0 - J_0(0) I_1(Q) - I_0 J_1(Q) + 2L(Q) \right)
$$

with

$$
I_0 = \sum_P \frac{1}{P^2} = -\frac{T^2}{12}, \quad J_0(Q) = \sum_P \frac{1}{P^2(P - Q)^2}
$$

and

$$
I_1(Q) = \sum_P \frac{[p^2 - (\hat{q} \hat{p})]}{P^2(P - Q)^2}, \quad J_1(Q) = \sum_P \frac{[p^2 - (\hat{q} \hat{p})]}{P^2(P - Q)^2},
$$

$$
L(Q) = \sum_P \sum_K \frac{(PK)[\hat{p} \hat{k} - (\hat{q} \hat{p})(\hat{q} \hat{k})]}{P^2(P - Q)^2(K - Q)^2} = -\frac{1}{2} I_1(Q) J_1(Q)
$$

where a bit of spherical trigonometry has led to the last expression. The remaining angular integrations can be done as well. Now, in a last step, the limit $\omega \to 0$ (first) and $q \to 0$ (afterwards) is performed. The subleties of this limit are detailed in the Appendix, where we obtain that $I_1(Q) \to -I_0$. Hence

$$
\Pi_{t}^{2-\text{loop hh}} \bigg|_{\sim (\alpha - 1)} = -2g^4N^2 \left( J_0(0) + J_1(Q) \right) \left( I_0 + I_1(Q) \right) \to 0.
$$

Thus, in this limit, the hard–hard 2–loop contribution is a gauge–fixing independent subset, indeed.

The contribution itself, which is now given by the $(\alpha - 1)^0$ term, is obtained to be

$$
\Pi_{t}^{2-\text{loop hh}} = -4g^4N^2 \left( I_1(Q) J_0(Q) - 2I_0 J_1(Q) \right) \to 0
$$

by virtue of $-2J_1(Q) \to J_0(Q)$ in the static limit. (24) is the third number zero in (3).

5. Two loops — one hard, one supersoft

There is, of course, also the $g^4$ contribution in the 2–loop diagrams, which arises from the supersoft region (of both momenta) and may be addressed to the 3D Euclidean theory. Due to the vanishing of the hard–hard part, we may assume, that the Euclidean part is anyhow regularized. The very details of this behaviour are under present study.

One may suspect that a contribution is omitted so far, namely the admixture of one loop momentum hard and the other supersoft and regularized. The system might
“know” of a soft scale only through these regulators. Such contributions $2\Pi^\text{hard-supersoft}$, however, can be ruled out either (a) by taking into account the one-loop-higher subtraction of $\nabla$ terms or (b) by power counting.

For the mechanism (a), consider the first six diagrams in the second line of figure 2 with the lower loop at soft momentum. They combine with 1-loop diagrams containing a $\nabla$ insertion. The sum of these diagrams vanishes to order $g^4$.

For the power counting (b), consider for example the setting sun diagram, which is the first in figure 2. Reducing one loop integration to the zero mode and supplying it with a Pauli–Villars regulator, one has

$$\sim g^4 \sum_K \frac{1}{K^4} \left( \frac{1}{p^2 + \tau^2} - \frac{1}{p^2 + \sigma^2} \right) \sim g^4 T \sigma \sim g^5 T^2,$$

hence one $g$ order below the one of present interest.

It is now tempting to speculate on how to go to higher orders in the perturbative treatment. "Towers” (i.e. HTL vertex insertions) in the Linde sea are to be included – one in each diagram, if the order $g^5$ is studied. Their combinatorics of positions and the question for an effective action could be future problems.

6. Conclusions

The magnetic screening mass has no other contributions than those with zero Matsubara frequency in each loop. It is thus given, to its leading order, by the Karabali–Nair value $g^2NT/(2\pi)$. The Linde problem is going to be overcome.

Appendix

Here the non–commutativity of limits, long–wavelength versus static, is illustrated with the sum–integral

$$I_1(Q) = \sum_p \frac{[p^2 - \frac{1}{2}(\hat{q}^2)]}{P^2(P - Q)^2} = -\frac{Q^2}{4\pi^2 q^2} \int_0^\infty dp \, p \, n(p) \left\{ 1 - \frac{Q}{q} \ln \left( \frac{Q}{Q_0 - q} \right) - \frac{Q_0}{2q} \ln \left( \frac{(Q_0 - q)^2 - 4p^2}{(Q_0 + q)^2 - 4p^2} \right) + \frac{p}{2q} \left( 1 + \frac{Q^2}{4p^2} \right) \ln \left( \frac{Q_0^2 - (2p + q)^2}{Q_0^2 - (2p - q)^2} \right) \right\}. \tag{A.1}$$

at hand. A temperature independent piece was omitted to the right and addressed to renormalization at $T = 0$. $Q_0$ may still attain the Matsubara values. Analytical continuation $Q_0 = \omega + i\eta$ reveals several cuts on the real axis. Maintaining $\pm Q_0$ symmetry, one cut inevitably extends from $\omega = -q$ to $\omega = q$ (if $q$ is real). The others lie outside, with inner end points of order $p \sim T$ in magnitude. Hence, when
concentrating on soft or supersoft $\omega, q$, (A.1) may be simplified to

$$I_1(\omega + i\eta, \vec{q}) = \frac{T^2}{24} \left[ 2 - \frac{\omega + i\eta}{q} \ln \left( \frac{\omega + q + i\eta}{\omega - q + i\eta} \right) \right] \left( 1 - \frac{(\omega + i\eta)^2}{q^2} \right)$$  \hspace{1cm} (A.2)

with the only cut from $-q$ to $q$. When reaching the plasmon frequency $\omega \to m$ with $q \to 0$ ($\eta = 0$), (A.2) attains the "usual" value of $T^2/36$. For negative $q^2$, however, the cut extends along the imaginary axis. The limit $\omega \to 0$ (again $\eta = 0$) is now possible, while $|q|$ remains finite, though supersoft. Hence, in the static limit, (A.2) becomes $T^2/12$. This is also the value of $I_1$ at $Q_0 = 0$ ($|q| \ll T$) before continuing analytically.

Figure 3: The complex $Q_0 = \omega + i\eta$ plane and the two limits encountered at long wavelength with the real dispersion (left part of the figure) and in the static limit (right part).

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