LAMBDA NUMBER OF THE ENHANCED POWER GRAPH OF A FINITE GROUP

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Abstract. The enhanced power graph of a finite group $G$ is the simple undirected graph whose vertex set is $G$ and two distinct vertices $x, y$ are adjacent if $x, y \in (z)$ for some $z \in G$. An $L(2,1)$-labeling of graph $\Gamma$ is an integer labeling of $V(\Gamma)$ such that adjacent vertices have labels that differ by at least 2 and vertices distance 2 apart have labels that differ by at least 1. The $\lambda$-number of $\Gamma$, denoted by $\lambda(\Gamma)$, is the minimum range over all $L(2,1)$-labelings. In this article, we study the lambda number of the enhanced power graph $P_e(G)$ of the group $G$. This paper extends the corresponding results, obtained in [X. Ma, M. Feng, and K. Wang. Lambda number of the power graph of a finite group. J. Algebraic Combin., 53(3):743–754, 2021], of the lambda number of power graphs to enhanced power graphs. Moreover, for a non-trivial simple group $G$ of order $n$, we prove that $\lambda(P_e(G)) = n$ if and only if $G$ is not a cyclic group of order $n \geq 3$. Finally, we compute the exact value of $\lambda(P_e(G))$ if $G$ is a finite nilpotent group.

1. Introduction

For non-negative integers $j$ and $k$, an $L(j, k)$-labeling for the graph $\Gamma$ is an integer valued function $f$ on the vertex set $V(\Gamma)$ such that $|f(u) - f(v)| \geq k$ whenever $u$ and $v$ are vertices of distance two and $|f(u) - f(v)| \geq j$ whenever $u$ and $v$ are adjacent. The span of $f$ is the difference between the maximum and minimum of $f$. It is convenient to assume that the minimum of $f$ is 0, we regard the span of $f$ as the maximum of $f$. The $L(j, k)$-labeling number $\lambda_{j,k}(\Gamma)$ of the graph $\Gamma$ is the minimum span over all $L(j, k)$-labelings for $\Gamma$. The classical work of the $L(j, k)$ labeling problem is when $j = 2$ and $k = 1$. The $L(2, 1)$-labeling number of a graph $\Gamma$ is also called the $\lambda$-number of $\Gamma$.

The radio channel assignment problem [12] and the study of the scalability of optical networks [30], motivated the researchers to investigate the problem related to $L(j, k)$-labelings of a graph. The concept of $L(2, 1)$-labeling of a graph was introduced by Griggs and Yeh [11], and they showed that for a general graph, $L(2, 1)$-labeling is NP-complete. Georges and Mauro [3] later presented a generalization of the concept. The $L(j, k)$-labeling in particular $L(2, 1)$-labeling, has been studied extensively by various authors (see [9] [10] [20] [33]). A survey of results and open problems related to the $L(j, k)$-labeling of a graph can be found in [34].

Graphs associated with groups and other algebraic structures have been studied by various researchers as they have valuable applications and are related to the automata theory (cf. [16] [17] [18] [19]). Zhou [30] investigated $L(j, k)$-labeling of Cayley graphs of abelian groups. Kelarev et al. [16] showed connections between the structure of a semigroup and the minimum spans of distance labelings of its Cayley graphs. In 2021, Ma et al. [22] studied the $L(2, 1)$-labeling of the power graph of a finite group. Recently, Sarkar [31] investigated the lambda number of power graph of finite simple group. Mishra [24] studied the lambda number of power graphs of finite $p$-groups.

In order to measure how much the power graph is close to the commuting graph of a group $G$, Aalipour et al. [1] introduced a new graph called enhanced power graph of the group $G$. The enhanced power graph of a group $G$ is the simple undirected graph whose vertex set is $G$ and two distinct vertices $x, y$ are adjacent if $x, y \in (z)$ for some $z \in G$. Indeed, the enhanced power graph contains the power graph and is a spanning subgraph of the commuting graph. The study of enhanced power graphs has received the considerable attention by various researchers. Aalipour et al. [1] characterized the finite group $G$, for which equality holds for either two of the three graphs viz. power graph, enhanced power graph and commuting graph of $G$. Bera et al. [2] characterized the abelian groups and the non abelian $p$-groups having dominatable enhanced power graphs. A complete description of finite groups with enhanced power graphs admitting a perfect code have been studied in [23]. Ma et al. [25] investigated the metric dimension of an enhanced power graph of finite groups. Hamzeh et al. [13] derived the automorphism groups of enhanced power graphs of finite groups. Zahiroofić et al. [35] proved that two finite abelian groups are isomorphic if their enhanced power graphs are isomorphic. Also, they supplied a characterization of finite nilpotent groups whose enhanced power graphs are perfect. Recently, Panda et al. [28] studied the graph-theoretic properties, viz. minimum degree,
independence number, matching number, strong metric dimension and perfectness, of enhanced power graphs over finite abelian groups. Moreover, the enhanced power graphs associated to non-abelian groups such as semidihedral, dihedral, dicyclic, U_{6n}, V_{6n} etc., have been studied in [6 28]. Bera et al. [4] gave an upper bound for the vertex connectivity of enhanced power graph of any finite abelian group. Moreover, they classified the finite abelian groups whose proper enhanced power graphs are connected. Results related to the connectivity, dominating vertices and the spectral radius of proper enhanced power graph have been investigated by Bera et al. in [3]. Recently, other graph theoretic properties, namely: regularity, vertex connectivity and the Wiener index, of the enhanced power graphs of finite groups have been studied in [29]. For a comprehensive list of results and open questions on enhanced power graphs of groups, we refer the reader to [23].

The lambda number of the power graphs of finite groups has been studied in [22 27 31]. In this paper, we study the lambda number of the enhanced power graph of a finite group G. In Section 2 we recall the necessary definitions, results and fixed our notations which we used throughout the paper. Section 3 comprises the main results of the present paper.

2. Preliminaries

In this section, first we recall the graph theoretic notions from [32]. A graph Γ is a pair Γ = (V, E), where V(Γ) and E(Γ) are the set of vertices and edges of Γ, respectively. Two distinct vertices u1 and u2 are adjacent, denoted by u1 ∼ u2, if there is an edge between u1 and u2. Otherwise, we write it as u1 ∼ u2. Let Γ be a graph. A subgraph Γ′ of Γ is the graph such that V(Γ′) ⊆ V(Γ) and E(Γ′) ⊆ E(Γ). A subgraph Γ′ of graph Γ is said to be a spanning subgraph of Γ if V(Γ′) = V(Γ'). For X ⊆ V(Γ), the subgraph of Γ induced by the set X is the graph with vertex set X and its two distinct vertices are adjacent if and only if they are adjacent in Γ. The complement Γ1 of Γ is a graph with same vertex set as Γ and distinct vertices u, v are adjacent in Γ1 if they are not adjacent in Γ. A graph Γ is said to be complete if any two distinct vertices are adjacent. We denoteKn by the complete graph of n vertices. A graph Γ is said to be k-partite if the vertex set of Γ can be partitioned into k subsets, such that no two vertices in the same subset of the partition are adjacent. A complete k-partite graph, denoted byK_{n_1,n_2,\ldots,n_k}, is a k-partite graph having its parts sizes n_1, n_2, \ldots, n_k such that every vertex in each part is adjacent to all the vertices of all other parts of K_{n_1,n_2,\ldots,n_k}. A vertex v of Γ is said to be a dominating vertex if v is adjacent to all the vertices of Γ. We denote Dom(Γ) by the set of all dominating vertices of the graph Γ. A walk λ in Γ from the vertex u to the vertex w is a sequence of vertices u = v_1, v_2,\ldots, v_m = w (m > 1) such that v_i ∼ v_{i+1} for every i ∈ {1, 2, \ldots, m - 1}. A walk is said to be a path if no vertex is repeated. A graph Γ is connected if each pair of vertices has a path in Γ. Otherwise, Γ is disconnected. The distance between u, v ∈ V(Γ), denoted by d(u, v), is the number of edges in a shortest path connecting them. A path covering C(Γ) of a graph Γ is a collection of vertex-disjoint paths in Γ such that each vertex in V(Γ) is contained in a path of C(Γ). The path covering number c(Γ) of Γ is the minimum cardinality of a path covering of Γ.

Let G be a group. The order of an element x in G is the cardinality of the subgroup generated by x and it is denoted by o(x). For a positive integer n, ϕ(n) denotes the Euler’s totient function of n. Consider the set π_G = {o(g) : g ≠ e ∈ G}. The exponent of a group is defined as the least common multiple of the orders of all elements of the group. For any x, y ∈ G, define a relation ρ such that xρy if and only if o(xy) = o(x) + o(y) - o(xy). Note that ρ is an equivalence relation and the equivalence class of x is denoted by ρ_x. For d ∈ π_G, C_d denotes the number of equivalence classes that consists of elements of order d. Moreover, we denote τ_d = {g ∈ G : o(g) = d}. For n ≥ 3, the dihedral group D_{2n} is a group of order 2n defined in terms of generators and relations as D_{2n} = ⟨x, y : x^n = y^2 = e, xy = yx^{-1}⟩. For n ≥ 2, the dicyclic group Q_{4n} is a group of order 4n is defined in terms of generators and relations as Q_{4n} = ⟨a, b : a^{2n} = e, a^n = b^2, ab = ba^{-1}⟩. A cyclic subgroup of a group G is called a maximal cyclic subgroup if it is not properly contained in any cyclic subgroup of G other than itself. If G is a cyclic group, then G is the only maximal cyclic subgroup of G. We denote M(G) by the set of all maximal cyclic subgroups of G. Also, M ∈ M(G), we write G_M = {x ∈ G : ⟨x⟩ = M} and G_M(G) = {x ∈ G : ⟨x⟩ ∈ M(G)}. Let G be a group and H, K be subgroups of G. The subgroup [H, K] of G is defined as the subgroup generated by all elements of the form [h, k] := h^{-1}k^{-1}hk, where h ∈ H, k ∈ K. The lower central series of subgroups of G is the descending sequence

G ≥ G^{(2)} ≥ G^{(3)} ≥ \cdots ≥ G^{(i)} ≥ G^{(i+1)} ≥ \cdots

of normal subgroups of G given by G^{(2)} := [G, G] and G^{(i+1)} := [G^{(i)}, G] for every i ≥ 2. If for a group G, this descending sequence contains only finitely many non-trivial terms then G is said to be a nilpotent group. Every finite p-group is a nilpotent group.
A finite $p$-group of order $p^n$ is said to be of maximal class if $G^{(n-1)} \neq \{e\}$ and $G^{(n)} = \{e\}$. In this case, $G/G^{(2)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $G^{(i)}/G^{(i+1)} \cong \mathbb{Z}_p$ for all $2 \leq i \leq n - 1$. The following result says a lot more about the class numbers of a finite $p$-group $G$.

**Theorem 2.1.** \cite{[1],[11],[21],[20]} Let $G$ be a finite $p$-group of exponent $p^k$. Assume that $G$ is not cyclic for an odd prime $p$, and for $p = 2$, it is neither cyclic nor of maximal class. Then

(i) $C_p \equiv 1 + p \pmod{p^2}$.
(ii) $p | C_p^i$ for every $2 \leq i \leq k$.

**Corollary 2.2.** \cite{[27]} Let $G$ be a finite $p$-group of exponent $p^k$. Then $C_p^i = 1$ for some $1 \leq i \leq k$ if and only if one of the following occurs:

(i) $G \cong \mathbb{Z}_{p^k}$ and $C_p^i = 1$ for all $1 \leq j \leq k$, or
(ii) $p = 2$ and $G$ is isomorphic to one of the following 2-groups:

(i) **dihedral 2-group**
$$ \mathbb{D}_{2k^i} = \langle x, y : x^{2^k} = 1, y^2 = 1, y^{-1}xy = x^{-1} \rangle, \quad (k \geq 1) $$
where $C_2 = 1 + 2^k$ and $C_{2^j} = 1$ for all $(2 \leq j \leq k)$.

(ii) **generalized quaternion 2-group**
$$ \mathbb{Q}_{2k^i+1} = \langle x, y : x^{2^k} = 1, x^{2^{k-1}} = y^2, y^{-1}xy = x^{-1} \rangle, \quad (k \geq 2) $$
where $C_4 = 1 + 2^{k-1}$ and $C_{2^j} = 1$ for all $1 \leq j \leq k$ and $j \neq 2$.

(iii) **semi-dihedral 2-group**
$$ \mathbb{S}\mathbb{D}_{2k^i+1} = \langle x, y : x^{2^k} = 1, y^2 = 1, y^{-1}xy = x^{-1+2^{k-1}} \rangle, \quad (k \geq 3) $$
where $C_2 = 1 + 2^{k-1}$, $C_4 = 1 + 2^{k-2}$ and $C_{2^j} = 1$ for all $3 \leq j \leq k$.

**Theorem 2.3.** \cite{[27]} Let $G$ be a finite group. Then the following statements are equivalent:

(i) $G$ is a nilpotent group.
(ii) Every Sylow subgroup of $G$ is normal.
(iii) $G$ is the direct product of its Sylow subgroups.
(iv) For $x, y \in G$, $x$ and $y$ commute whenever $o(x)$ and $o(y)$ are relatively primes.

Any non-cyclic nilpotent group $G$ is of one of the following forms

(1) $G \cong G' \times \mathbb{Z}_n$, where $G'$ is a non-trivial nilpotent group of odd order having no cyclic Sylow subgroup and $\gcd(n, |G'|) = 1$.

(2) $G \cong G' \times P \times \mathbb{Z}_n$, where $G'$ is a nilpotent group of odd order having no cyclic Sylow subgroup, $P$ is a 2-group which is neither cyclic nor of maximal class and $\gcd(n, |G'|) = \gcd(2, n) = 1$.

(3) $G \cong G' \times \mathbb{Q}_{2k^i+1} \times \mathbb{Z}_n$, where $G'$ is described as in (2), $\mathbb{Q}_{2k^i+1}$ is a generalized quaternion group of order $2^{k+1}$ and $\gcd(n, |G'|) = \gcd(2, n) = 1$.

(4) $G \cong G' \times \mathbb{D}_{2k^i+1} \times \mathbb{Z}_n$, where $G'$ is described as in (2), $\mathbb{D}_{2k^i+1}$ is a dihedral group of order $2^{k+1}$ and $\gcd(n, |G'|) = \gcd(2, n) = 1$.

(5) $G \cong G' \times \mathbb{S}\mathbb{D}_{2k^i+1} \times \mathbb{Z}_n$, where $G'$ is described as in (2), $\mathbb{S}\mathbb{D}_{2k^i+1}$ is a dihedral group of order $2^{k+1}$ and $\gcd(n, |G'|) = \gcd(2, n) = 1$.

The following result characterizes the dominating vertices of enhanced power graph of a finite nilpotent group and we use this result explicitly in this paper without referring to it.

**Theorem 2.4.** \cite{[3]} Theorem 4.1] Let $G$ be a finite non-cyclic nilpotent group and let $D_1 = \{(e', e_2, x) : x \in \mathbb{Z}_n\}$, $D_2 = \{(e', y, x) : y \in \mathbb{Q}_{2k^i+1}, x \in \mathbb{Z}_n\}$ and $o(y) = 2$. Then
$$ \text{Dom}(P_E(G)) = \begin{cases} 
\{(e', x) : x \in \mathbb{Z}_n\}, & \text{if } G = G' \times \mathbb{Z}_n \text{ and } \gcd(|G'|, n) = 1 \\
\{(e', e_1, x) : x \in \mathbb{Z}_n\}, & \text{if } G = G' \times P \times \mathbb{Z}_n \text{ and } \gcd(|G'|, n) = \gcd(n, 2) = 1 \\
\{(e', e_3, x) : x \in \mathbb{Z}_n\}, & \text{if } G = G' \times \mathbb{Q}_{2k^i+1} \times \mathbb{Z}_n \text{ and } \gcd(|G'|, n) = \gcd(n, 2) = 1 \\
\{(e', e_4, x) : x \in \mathbb{Z}_n\}, & \text{if } G = G' \times \mathbb{D}_{2k^i+1} \times \mathbb{Z}_n \text{ and } \gcd(|G'|, n) = \gcd(2, n) = 1, \\
\end{cases} $$
where $e', e_i$’s, $1 \leq i \leq 4$, are the identity elements of the respective groups in $G$. 

Lemma 2.5. Let $G = P_1 \times P_2 \times \cdots \times P_r$ be a finite nilpotent group of order $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Suppose $x, y \in G$ such that $o(x) = s$ and $o(y) = t$. Then there exists an element $z \in G$ such that $o(z) = \text{lcm}(s,t)$.

Proof. Let $x = (x_1, x_2, \ldots, x_r), y = (y_1, y_2, \ldots, y_r) \in G$. It follows that $s = p_1^{\beta_1}p_2^{\beta_2} \cdots p_r^{\beta_r}$ and $t = p_1^{\gamma_1}p_2^{\gamma_2} \cdots p_r^{\gamma_r}$, where $p_i^{\beta_i} = o(x_i), p_i^{\gamma_i} = o(y_i)$ and $0 \leq \beta_i, \gamma_i \leq \alpha_i$. Consequently, $\text{lcm}(s,t) = p_1^{\delta_1}p_2^{\delta_2} \cdots p_r^{\delta_r}$, where $\delta_i = \max\{\beta_i, \gamma_i\}$. Consider $z = (z_1, z_2, \ldots, z_r)$ such that

$$z_i = \begin{cases} x_i & \text{if } \beta_i \geq \gamma_i, \\ y_i & \text{if } \beta_i < \gamma_i. \end{cases}$$

Clearly, $z \in G$ and $o(z) = \prod_{i=1}^{r} o(z_i) = \prod_{i=1}^{r} p_i^{\delta_i}$. Thus, the result holds. \hfill \Box

Remark 2.6. Let $G$ be a finite group. Then $G = \bigcup_{M \in \mathcal{M}(G)} M$ and the generators of a maximal cyclic subgroup does not belong to any other maximal cyclic subgroup of $G$. Consequently, if $|M_i|$ is a prime number then for distinct $M_i, M_j \in \mathcal{M}(G)$, we have $M_i \cap M_j = \{e\}$.

The following results will be useful for further study.

Lemma 2.7. If $G$ is a finite group, then $|\mathcal{M}(G)| \neq 2$.

Proof. On contrary, assume that the group $G$ has two maximal cyclic subgroups $M_1$ and $M_2$. Then every element of $G$ belongs to at least one of the maximal cyclic subgroup of $G$ and $e \in M_1 \cap M_2$. It follows that $|M_1| + |M_2| \geq o(G)+1$. Since $M_1$ and $M_2$ are proper subgroups of a finite group $G$, by Lagrange’s theorem, we have

$$|M_1| \leq \frac{o(G)}{2}$$

and $|M_2| \leq \frac{o(G)}{2}$.

Consequently, we get $o(G) + 1 \leq |M_1| + |M_2| \leq o(G)$, which is not possible. Hence, $|\mathcal{M}(G)| \neq 2$. \hfill \Box

Lemma 2.8. [11] Lemma 2.1 Let $G$ be a finite non-cyclic simple group. Then for any $d \in \pi_G$, we have $C_d \geq 2$.

Theorem 2.9. [2] Theorem 2.4 The enhanced power graph $P_E(G)$ of the group $G$ is complete if and only if $G$ is cyclic.

Theorem 2.10. [10] Theorem 14 Let $G$ be a graph of order $n$.

(i) Then $\lambda(G) = n - 1$ if and only if $c(G) = 1$.

(ii) Let $r$ be an integer at least 2. Then $\lambda(G) = n + r - 2$ if and only if $c(G) = r$.

3. Main Results

In this section, we present the main results of the paper. Recall that if $G$ is a finite cyclic group then the enhanced power graph $P_E(G)$ is complete. First we obtain the bounds for $\lambda(P_E(G))$, where $G$ is a finite group. Then we classify finite simple groups $G$ such that $\lambda(P_E(G)) = |G|$. Since the set of dominating vertices in $P_E(G)$ is known (see Theorem 2.14), we obtain the lambda number of the enhanced power graphs of nilpotent groups (see Theorem 3.9 and Theorem 3.10).

Theorem 3.1. Let $G$ be a finite group of order $n$. Then $\lambda(P_E(G)) \geq n$ with equality holds if and only if $P_E(G) \setminus \{e\}$ contains a Hamiltonian path.

Proof. It is well known that for a finite group $G$, the graph $P_G(G)$ is a spanning subgraph of the power graph $P(G)$ and note that the lambda number is a monotone parameter. By [22] Theorem 3.1], the result holds. \hfill \Box

Theorem 3.2. Let $G$ be a finite non-cyclic group of order $n$. Suppose $M_1, M_2, \ldots, M_r$ be the maximal cyclic subgroups of $G$ such that $m_1 \geq m_2 \geq \cdots \geq m_r$, where $m_i = \phi(|M_i|)$ for $1 \leq i \leq r$. Then

$$\lambda(P_E(G)) \leq \begin{cases} 2n - |\mathcal{M}(G)| - 1; & \text{if } m_1 \leq \sum_{i=2}^{r} m_i, \\
2(n - m_1 - 1); & \text{Otherwise.} \end{cases}$$
Proof. We prove this result by finding an upper bound of the path covering number of $P_E(G)$. We discuss the following two possible cases.

Case-1: $m_1 \leq \sum_{i=2}^{r} m_i$. We discuss this case into two subcases.

Subcase-1.1: $m_1 = m_2$. Now we provide a Hamiltonian path in the subgraph of $P_E(G)$ induced by the set $G_{M_i(G)}$. Since the generators of two distinct maximal cyclic subgroups are adjacent in $P_E(G)$, note that the path $P : x_{1,1} \sim x_{1,2} \sim \cdots \sim x_{1,r} \sim x_{2,1} \sim x_{2,2} \sim \cdots \sim x_{m_i,s}$, where $(x_{i,j}) = M_j, 1 \leq i \leq m_j$ and $s = \max\{t : 2 \leq t \leq r, m_i = m_1\}$, covers all the vertices of $G_{M_i(G)}$ in $P_E(G)$. It follows that $c(P_E(G)) \leq n - |G_{M_i(G)}| + 1$. By Theorem 2.10, we have $\lambda(P_E(G)) \leq 2n - |G_{M_i(G)}| - 1.$

Subcase-1.2: $m_1 > m_2$. Since $G_{M_i(G)} = \bigcup_{i=1}^{r} G_{M_i}$, we consider $A_1 = \{a_1, a_2, \ldots, a_{m_1}, \ldots, m_2\} \subseteq G_{M_1}$. In $A_2$, we collect the $m_1 - m_2$ elements starting from $G_{M_i}$. If $m_r \geq m_1 - m_2$, then we take $A_2 \subseteq G_{M_r}$, such that $|A_2| = m_1 - m_2$. Otherwise, we collect remaining $(m_1 - m_2 - m_r)$ elements from $G_{M_{r-1}}$ and then choose remaining elements, if required, such that $|A_2| = m_1 - m_2$, from $G_{M_{r-2}}, G_{M_{r-3}}$ and so on. Write $A_2 = \{b_1, b_2, \ldots, b_{m_1 - m_2}\}$. Further, consider the set $A_3 = G_{M_i(G)} \setminus (A_1 \cup A_2)$. In view of the above given partition of $G_{M_i(G)}$, now we provide a Hamiltonian path in subgraph of $P_E(G)$ induced by the set $G_{M_i(G)}$. Since the generators of distinct maximal cyclic subgroups are adjacent in $P_E(G)$. Thus, we have a Hamiltonian path $P : a_1 \sim b_1 \sim a_2 \sim a_2 \sim \cdots \sim a_{m_1} \sim a_{m_2} \sim b_{m_1 - m_2}$ in the subgraph induced by the set $A_1 \cup A_2$. Notice that the subgraph $\Gamma$ induced by the set $A_3$ in $P_E(G)$ is a complete $r$-partite graph, where $t = \max\{i : G_{M_i} \cap A_3 \neq \emptyset\}$ and the partition set of $\Gamma$ is $G_{M_1} \setminus A_1, G_{M_2}, \ldots, G_{M_t}$. Since $|G_{M_1} \setminus A_1| = |G_{M_i} \geq |G_{M_i}|| \leq 3 \leq i \leq t$, we have a Hamiltonian path $H'$ of $\Gamma$ with initial vertex $x$ belongs to $G_{M_i}$. Since $b_{m_1 - m_2} \in G_{M_i}$ for some $k$, where $t \leq k \leq r$, we have $b_{m_1 - m_2} \sim x$. Consequently, we get a Hamiltonian path in the subgraph induced by the set $G_{M_i(G)}$ in $P_E(G)$. Thus $c(P_E(G)) \leq n - |G_{M_i(G)}| + 1$. By Theorem 2.10, $\lambda(P_E(G)) \leq 2n - |G_{M_i(G)}| - 1.$

Case-2: $m_1 > \sum_{i=2}^{r} m_i$. Since $G$ is a non-cyclic group, it implies that $M_1$ is a proper subgroup of $G$. By consequence of Lagrange’s theorem, $|M_1| < \frac{n}{2}$ and so $|G_{M_i}| < \frac{n}{2}$. Notice that each element of $G_{M_i}$ is adjacent to every element of $G \setminus M_1$ in $P_E(G)$. Thus, for $<x_i>$, $i \neq y_i \in G \setminus M_1$, we have a path $P : y_1 \sim x_1 \sim y_2 \sim \cdots \sim x_{m_1} \sim y_{m_1 + 1}$ of length $2m_1 + 1$ in $P_E(G)$. Consequently, $c(P_E(G)) \leq n - 2m_1$. Hence, by Theorem 2.10, $\lambda(P_E(G)) \leq 2n - 2m_1 - 2.$ \hfill $\square$

In view of Lemma 2.7, we have the following corollary of Theorem 5.2.

Corollary 3.3. Let $G$ be a finite non-cyclic group of order $n$. Then $\lambda(P_E(G)) \leq 2n - 4$, with equality holds if and only if $G$ is isomorphic to $Z_2 \times Z_2$.

Proof. Since $G$ is a non-cyclic group, by Lemma 2.7, we get $|G_{M_i(G)}| \geq 3$. Consequently, by Theorem 5.2, $\lambda(P_E(G)) \leq 2n - 4$. If $G$ is isomorphic to $Z_2 \times Z_2$, then $\lambda(P_E(G)) \geq 4 = 2\sigma(G) - 2$ and so $\lambda(P_E(G)) = 2n - 4$. We now suppose that $\lambda(P_E(G)) = 2n - 4$. This is possible only when $|G_{M_i(G)}| = 3$. Note that for a non-cyclic group $G$, $|G_{M_i(G)}| = 3$ if and only if $G$ has exactly three maximal cyclic subgroups each with having only one generator. By Remark 2.6, $o(G) = 4$. Thus, we must have $G \cong Z_2 \times Z_2$. \hfill $\square$

Now we classify finite simple groups $G$ such that $\lambda(P_E(G)) = |G|$. For this purpose, first we derive the following two lemmas.

Lemma 3.4. Let $G$ be a finite non-cyclic simple group. Then for any $d \in \pi_G$, there exists a Hamiltonian path in subgraph of $P_E(G)$ induced by the set $\tau_d$.

Proof. In view of Lemma 2.3, suppose $C_d = s$, where $s \geq 2$. Let $H_d = \{\rho_1, \rho_2, \ldots, \rho_s\}$ be the set of all cyclic classes of elements of order $d$. Let $x, y \in \rho_i$, where $i \in \{1, 2, \ldots, s\}$. Then $x, y \in <x>$ and so $x \sim y$ in $P_E(G)$. Consequently, $x \sim y$ in $P_E(G)$. Also for $i \neq j$, let $x \in \rho_i$ and $y \in \rho_j$. Let $x \sim y$ in $P_E(G)$. Then there exists $z \in G$ such that $x, y \in <z>$. Since $o(x) = o(y) = d$, we obtain $<x> = <y>$, which is not possible. Thus, $x \sim y$ in $P_E(G)$ and so $x \sim y$ in $P_E(G)$. It follows that the subgraph of $P_E(G)$ induced by the set $\tau_d$ is a complete $s$-partite graph such that the size of each partition set is $\phi(d)$. Hence, the result holds. \hfill $\square$

Lemma 3.5. Let $G$ be a finite non-cyclic simple group and $d_1, d_2 \in \pi_G$. For each $x \in \tau_{d_1}$, there exists $y \in \tau_{d_2}$ such that $x \sim y$ in $P_E(G)$.

Proof. To prove this result, it is sufficient to show that there exists $y \in \tau_{d_2}$ such that $xy \neq yx$ so that $x \sim y$ in $P_E(G)$. Let $xy' = y'x$ for all $y' \in \tau_{d_2}$. Note that $\langle \tau_{d_2} \rangle$ is a subgroup of $G$. For $g \in G$ and $x' = x_1x_2 \cdots x_k \in \langle \tau_{d_2} \rangle$, where
Let \( G \) be a non-trivial finite simple group of order \( n \). Then \( \lambda(P_E(G)) = n \) if and only if \( G \) is not a cyclic group of order \( n \geq 3 \).

**Proof.** If \( G \) is cyclic, then by Theorem 2.5 \( P_E(G) \) is a complete graph. Consequently, \( \lambda(P_E(G)) = 2n - 2 \). Thus \( \lambda(P_E(G)) = n \) if and only if \( n = 2 \). We may now suppose that \( G \) is a non-cyclic group. To prove our result, it is sufficient to show that the graph \( P_E(G) \setminus \{e\} \) has a Hamiltonian cycle (see Theorem 2.10). Let \( \pi_G = \{d_1, d_2, \ldots, d_k\} \).

Then \( G \setminus \{e\} = \bigcup_{i=1}^{k} \tau_d \). By Lemma 3.4 for each \( i \in [k] \), we have a Hamiltonian path in the subgraph induced by the set \( \tau_d \) in \( P_E(G) \) and by Lemma 5.3 we get a Hamiltonian path in \( P_E(G) \setminus \{e\} \). Thus, the result holds. \( \square \)

In the remaining part of the paper, we obtain the lambda number of enhanced power graphs of finite nilpotent groups.

**Lemma 3.7.** Let \( G' \) be a non-trivial nilpotent group of odd order having no Sylow subgroup which is cyclic. If \( G \cong G' \times \mathbb{Z}_m \), where \( m \geq 1 \) and \( \text{gcd}(m, |G'|) = 1 \), then \( C_d(x) \geq 3 \) for each \( x \in G \setminus \text{Dom}(P_E(G)) \).

**Proof.** Let \( x = (x', y') \) be an arbitrary element of \( G' \text{Dom}(P_E(G)) \). Since \( G' \) is a nilpotent group, and so \( G' = P_1 \times P_2 \times \cdots \times P_r \), where \( P_i's \) are Sylow subgroups of \( G \). Consequently, \( x \in \{x_1, x_2, \ldots, x_r, y'\} \), where \( x_i \in P_i \) for each \( i \in [r] \). It follows that \( x_i \neq e \) for some \( j \in [r] \) because \( x \notin \text{Dom}(P_E(G)) \). Consider \( y = (x_1, x_2, \ldots, x_j, x_j, \ldots, x_{j+1}, \ldots, x_r, y') \), where \( y_j, z_j \in P_j \) such that \( o(x_j) = o(y_j) = o(z_j) \) and the cyclic subgroups \( \langle x_j \rangle, \langle y_j \rangle, \langle z_j \rangle \) of \( P_j \) are distinct (cf. Theorem 2.11). Clearly, \( o(x) = o(y) = o(z) \). Note that the cyclic subgroups \( \langle x \rangle, \langle y \rangle, \langle z \rangle \) of \( G \) are distinct. Without loss of generality, let \( x \leq y \). Then there exists \( m \in \mathbb{N} \) such that \( x^m = y \). Now consider \( l = o(x_1)o(x_2)\cdots o(x_j-1)o(x_{j+1})\cdots o(x_r) \). Then \( x^m = y^l \) and it follows that \( x^m = y^l \).

Since \( \text{gcd}(o(y), l) = 1 \), we obtain \( o(y_j) = o(y_j^l) \). Consequently, \( \langle y \rangle = \langle y_j \rangle = \langle x^m \rangle \leq \langle x_j \rangle \); a contradiction. Thus, the result holds. \( \square \)

**Lemma 3.8.** Let \( G' \) be a non-trivial nilpotent group of odd order having no Sylow subgroup which is cyclic. If \( G \cong G' \times \mathbb{Z}_m \), where \( m \geq 1 \) and \( \text{gcd}(m, |G'|) = 1 \), then for each \( d \in D \), there exists a Hamiltonian path in the subgraph of \( P_E(G) \) induced by the set \( \tau_d \), where \( D = \{o(x) : x \in G \setminus \text{Dom}(P_E(G)) \} \).

**Proof.** Let \( d \in D \). Then by Lemma 3.7 \( C_d(s) \geq 3 \). Notice that the subgraph induced by the set \( \tau_d \) in \( P_E(G) \) is a complete s-partite graph with exactly \( \phi(d) \) vertices in each partition set. Thus, we get a Hamiltonian path between any two elements of \( \tau_d \). \( \square \)

**Theorem 3.9.** Let \( G' \) be a non-trivial nilpotent group of odd order having no Sylow subgroup which is cyclic. If \( G \cong G' \times \mathbb{Z}_m \), where \( m \geq 1 \) and \( \text{gcd}(m, |G'|) = 1 \), then \( \lambda(P_E(G)) = |G| + |\text{Dom}(P_E(G))| - 1 \).

**Proof.** In view of Theorem 2.10 to prove our result it is sufficient to show that \( \lambda(P_E(G)) = |\text{Dom}(P_E(G))| + 1 \). For \( x \in \text{Dom}(P_E(G)) \), clearly \( x \) is an isolated vertex in \( P_E(G) \). Thus, it is sufficient to show that the subgraph of \( P_E(G) \) induced by the non-dominating vertices of \( P_E(G) \) has a Hamiltonian path. Let \( G \setminus \text{Dom}(P_E(G)) \) has elements of order \( d_1, d_2, \ldots, d_t \). Consider \( S = \{d_1, d_2, \ldots, d_t\} \) where \( d_1 < d_2 < \cdots < d_t \).

**Claim:** There exists an ordered set \( S' = \{\beta_1, \beta_2, \ldots, \beta_t\} \), where \( \beta_i \in S \), such that either \( \beta_i | \beta_{i+1} \) or \( \beta_{i+1} | \beta_i \) for each \( i \in [t-1] \).

**Proof of claim:** If for each \( i \in [t-1] \), either \( d_i | d_{i+1} \) or \( d_{i+1} | d_i \), then \( S = S' \). Otherwise, choose the smallest \( l \) such that \( d_i | d_{i+1} \) or \( d_{i+1} | d_i \). By Lemma 2.5 \( d_{i+1} = \text{lcm}(d_i, d_{i+1}) \in \pi_G \) for some \( j \geq 2 \). Now let \( x = (x_1, x_2) \in G \setminus \text{Dom}(P_E(G)) \) such that \( o(x) = d_i \). Clearly, \( o(x_1) > 1 \) and \( o(x) = o(x_1)o(x_2) \). Suppose \( z = (z_1, z_2) \in G \) such that \( o(z) = d_{i+1} \). Since \( d_i | d_{i+1} \), it follows that \( o(x_1) | o(z_1) \). Consequently, \( o(z_1) > 1 \) and so \( d_{i+1} | S \). Let \( \gamma_1, \gamma_2, \ldots, \gamma_t \) be elements of \( G' \) such that \( \gamma_i | \gamma_{i+1} \) or \( \gamma_{i+1} | \gamma_i \) for each \( i \in \{1, 2, \ldots, l+1 \} \). If for each \( i \in \{l+2, l+3, \ldots, t-1\} \), either \( \gamma_i | \gamma_{i+1} \) or \( \gamma_{i+1} | \gamma_i \), then \( S_1 = S' \). Otherwise, choose the smallest \( l' \in \{l+2, l+3, \ldots, t-1\} \) and repeat the above process. On continuing this process, we get desired ordered set \( S' \).
Now by Lemma 3.11, for each \(i \in [l]\), the subgraph of \( \mathcal{P}_E(G) \) induced by the set \( \tau_{\beta_i} \) is a complete \( s \)-partite graph with \( \phi(\beta_i) \) vertices in each partition set. Observe that \( G \setminus \text{Dom}(\mathcal{P}_E(G)) = \bigcup_{i=1}^{l} \tau_{\beta_i} \). Then there exist paths \( H_1, H_2, \ldots, H_t \) which covers all the vertices of \( \tau_{\beta_1}, \tau_{\beta_2}, \ldots, \tau_{\beta_t} \), respectively. Now we shall show that for each \( i \in [t-1] \), the end vertex of \( H_i \) is adjacent to the initial vertex of \( H_{i+1} \) in \( \mathcal{P}_E(G) \) through the following two cases:

**Case-1:** \( \beta_i|\beta_{i+1} \). Let \( x \in \tau_{\beta_i} \) and \( y \) be the initial vertex of \( H_{i+1} \). If \( x \sim y \) in \( \mathcal{P}_E(G) \), then we choose \( x \) to be the end vertex of \( H_i \) so that \( x \sim y \) in \( \mathcal{P}_E(G) \). Now we may assume that \( x \sim y \) in \( \mathcal{P}_E(G) \). Then there exists \( z \in G \) such that \( x, y \in \langle z \rangle \). Since \( \beta_i|\beta_{i+1} \), we get \( \langle x \rangle \subset \langle y \rangle \). Let \( x' \in \tau_{\beta_i} \) such that \( \langle x \rangle \neq \langle x' \rangle \). Thus \( x' \sim y \) in \( \mathcal{P}_E(G) \). Therefore, we can choose \( x' \) as the end vertex of \( H_i \).

**Case-2:** \( \beta_{i+1}|\beta_i \). Let \( x \) be the end vertex of \( H_i \) and \( y \in \tau_{\beta_{i+1}} \). If \( x \sim y \) in \( \mathcal{P}_E(G) \), then we choose \( y \) to be the initial vertex of \( H_{i+1} \) so that \( x \sim y \) in \( \mathcal{P}_E(G) \). Otherwise, there exists \( z \in G \) such that \( x, y \in \langle z \rangle \). Since \( \beta_{i+1}|\beta_i \), it follows that \( \langle y \rangle \subset \langle x \rangle \). Let \( y' \in \tau_{\beta_{i+1}} \) such that \( \langle y \rangle \neq \langle y' \rangle \). Now if \( x \sim y' \) in \( \mathcal{P}_E(G) \) then \( \langle y' \rangle \subset \langle x \rangle \), which is not possible as \( \langle x \rangle \neq \langle y' \rangle \). Thus \( x \sim y' \) in \( \mathcal{P}_E(G) \). Therefore, consider \( y' \) as the initial vertex of \( H_{i+1} \).

Hence, we get a Hamiltonian path in subgraph of \( \mathcal{P}_E(G) \) induced by the set \( G \setminus \text{Dom}(\mathcal{P}_E(G)) \).

**Lemma 3.10.** Let \( G' \) be a nilpotent group of odd order having no Sylow subgroup which is cyclic. If \( G \cong G' \times P \times \mathbb{Z}_n \), where \( P \) is a non-cyclic 2-group and \( \gcd(n, |G'|) = \gcd(2, n) = 1 \), then for each \( x \in S' = \{ (g_1, g_2, g_3) \in G \mid g_1 \neq e_{G'} \} \), we have \( C_{\phi(x)} \geq 3 \).

**Proof.** After taking \( S' \) in place of \( G \setminus \text{Dom}(\mathcal{P}_E(G)) \), the proof is similar to the proof of Lemma 3.11. Hence, we omit the details.

**Theorem 3.11.** Let \( G' \) be a nilpotent group of odd order having no Sylow subgroup which is cyclic. If \( G \cong G' \times P \times \mathbb{Z}_n \), where \( P \) is a non-cyclic 2-group and \( \gcd(n, |G'|) = \gcd(2, n) = 1 \), then \( \lambda(\mathcal{P}_E(G)) = |G| + |\text{Dom}(\mathcal{P}_E(G))| - 1 \).

**Proof.** Using Lemma 3.10 and by the similar argument used in the proof of Theorem 3.11, the result holds.

**Theorem 3.12.** Let \( G' \) be a nilpotent group of odd order having no Sylow subgroup which is cyclic. If \( G \cong G' \times P \times \mathbb{Z}_n \), where \( P \) is a non-cyclic 2-group and \( \gcd(n, |G'|) = \gcd(2, n) = 1 \), then there exists a Hamiltonian path in the subgraph of \( \mathcal{P}_E(G) \) induced by the set \( S' \).

**Proof.** In view of Theorem 3.10, we show that the subgraph of \( \mathcal{P}_E(G) \) induced by the set of all non-dominating vertices of \( \mathcal{P}_E(G) \) has a Hamiltonian path. Let \( c_1, c_2, c_3 \) be the identity elements of the groups \( G', P, \) and \( \mathbb{Z}_n \), respectively. By Theorem 3.11 let \( H' \) be a Hamiltonian path in the subgraph of \( \mathcal{P}_E(G) \) induced by the set \( S' \) and let \( g' = (x', y', z') \) be the end vertex of \( H' \). By the proof of Theorem 3.11 notice that the order of \( g' = (x', y', z') \) is maximum. Suppose that the exponent of the group \( P ) is 2^k. Consequently, \( o(y') = 2^k \). Further, observe that if \( y \sim y' \) in the graph \( \mathcal{P}_E(P) \) then \( (x, y, z) \sim (x', y', z') \) in \( \mathcal{P}_E(G) \). In view of Theorem 2.1 and Corollary 2.2, we have the following cases:

**Case-1:** \( P \) is not of maximal class. Consider the set \( S'' = \{ (c_1, y, z) \in G : y \neq c_2 \} \). Notice that the sets \( S' \), defined in Lemma 3.10, \( S'' \) and \( \text{Dom}(\mathcal{P}_E(G)) \) forms a partition of \( G \). Now we provide a Hamiltonian path of the subgraph of \( \mathcal{P}_E(G) \) induced by the set \( S' \cup S'' \). To do this, first we drive a Hamiltonian path of the subgraph of \( \mathcal{P}_E(G) \) induced by the set \( S'' \). By Theorem 2.1 for \( 2 \leq j \leq k \), we have \( C_{2^j} \geq 2 \) and \( C_{2^j} \geq 3 \) in \( P \). For \( 1 \leq j \leq k \), let \( t_j = C_{2^j} \) and let \( T_j = \{ C_1^{(j)}, C_2^{(j)}, \ldots, C_{2^j}^{(j)} \} \) denotes the cyclic classes of \( P \) containing the elements of order \( 2^j \). Observe that each class in \( T_j \) is of cardinality \( 2^{j-1} \). Notice that each element of \( P \) belongs to exactly one cyclic class of \( T_j \) for \( j \in [k] \). Further note that, for \( i \neq s \), if \( x_i \in C_i^{(j)} \) and \( y_s \in C_s^{(j)} \), then \( x_i \sim y_s \) in \( \mathcal{P}_E(P) \). We label a class in \( T_k \) by \( C_1^{(k)} \) such that \( y_{1} \notin C_1^{(k)} \). For \( 2 \leq j \leq k \), let \( u_j \in C_j^{(j)} \) be an arbitrary element. Now \( u_j \) can be adjacent to at most one of the cyclic classes in \( T_j \) in \( \mathcal{P}_E(P) \). If possible, let \( u_j \sim v_1 \) and \( u_j \sim v_2 \), where \( v_1 \in C_i^{(j-1)} \) and \( v_2 \in C_j^{(j-1)} \). Then there exist elements \( u_1, w_2 \in P \) such that \( u_j, v_1 \in \langle w_1 \rangle \) and \( u_j, v_2 \in \langle w_2 \rangle \). Since \( o(v_1)o(u_j) \) and \( o(v_2)o(u_j) \), we get \( \langle v_1 \rangle = \langle v_2 \rangle \). It follows that \( t_1 = t_2 \). By \( t_1 \geq 2 \), we obtain that the elements of \( C_1^{(j)} \) is not adjacent to at least one of the cyclic class in \( T_j \) in \( \mathcal{P}_E(P) \). By relabelling, if necessary, we may assume that each element of \( C_1^{(j)} \) is not adjacent to every element of \( C_1^{(j)} \) in \( \mathcal{P}_E(P) \). Since \( t_1 \geq 3 \), we can label a class in \( T_1 \setminus C_1^{(1)} \) by \( C_1^{(1)} \) in which for \( x \in C_1^{(1)} \) and \( y \in C_1^{(k)} \), we have \( \langle x \rangle \notin \langle y \rangle \). It implies that \( x \sim y \) in \( \mathcal{P}_E(P) \). Let \( z_1, z_2, \ldots, z_n \) be the elements of \( \mathbb{Z}_n \). Then for \( y_{p,q}^{(r)} \), the \( q \)-th element of \( C_1^{(r)} \), the path \( H'' \) given below
(e_1, y_{1, z_1}^k) \sim (e_1, y_{2, z_1}^{k}) \sim \ldots \sim (e_1, y_{t_1, z_1}^{(k)}) \sim (e_1, y_{1, z_1}^{(k)}) \sim (e_1, y_{2, z_1}^{(k)}) \sim \ldots \sim (e_1, y_{2, z_1}^{(k)}) \sim (e_1, y_{1, z_1}^{(k)}) \sim (e_1, y_{2, z_1}^{(k)}) \sim \ldots \sim (e_1, y_{1, z_1}^{(k)})$, where $1 \leq r \leq k$, $1 \leq p \leq t$, and $1 \leq q \leq 2^{k-1}$, is a Hamiltonian path in the subgraph of $\overline{P}_G(G)$ induced by the set $S''$. Since $y' \not\in c^{(k)}$, we get $y' \sim y_{1, z_1}^{(k)}$ in $P_G(P)$. Consequently, $(x', y', z') \sim (e_1, y_{1, z_1}^{(k)})$ in $P_G(G)$ and so $(x', y', z') \sim (e_1, y_{1, z_1}^{(k)})$ in $P_G(G)$. Thus, we get a Hamiltonian path in the subgraph of $\overline{P}_G(G)$ induced by the set $S' \cup S''$.

Case-2: $P$ is of maximal class. In view of Corollary 2, we discuss this case into three subcases.

Subcase-2.1: $P = Q_{2k+1} = \langle x, y : x^2 = e_2, x^{2^k} = y^2, y^{-1}xy = x^{-1} \rangle$, where $k \geq 2$. Consider the set $S'' = \{e_1, b, c\} \in G : b \neq e_2, x^{2^{k-1}} \rangle$. Note that the sets $S'$, $S''$ and Dom($P_G(G)$) forms a partition of the group $G$. Observe that $Q_{2k+1}$ has one maximal cyclic subgroup of order $2^k$ and $2^{k-1}$ maximal cyclic subgroup of order 4 (see [23]). Let $M' = \langle x \rangle$ be the maximal cyclic subgroup of order $2^k$ and let for $1 \leq i \leq 2^{k-1}$, $M_i = \{e_2, x^{2^k-i}, x^iy, x^{2^{k-1}+i} \}$ be the maximal cyclic subgroups of order 4. For $1 \leq j \leq 2^{k-1}$, note that $x^jy$ is a generator of a maximal cyclic subgroup of $Q_{2k+1}$. Consequently, $x^jy \sim a$ in $P_G(Q_{2k+1})$, where $a \in Q_{2k+1} \setminus \langle x^jy \rangle$. Since $o(y') = 2^k$, for $k \geq 3$, we have $y' \in M'$. Thus, the Hamiltonian path $H''$ in the subgraph of $P_G(G)$ induced by the set $S''$ can be given as $(e_1, x, z_1) \sim (e_1, x, y_2, z_1) \sim (e_1, x^2, z_1) \sim \ldots \sim (e_1, x^{2^{k-1}+1}, y, z_1) \sim (e_1, x^{2^{k-1}+1}, z_1) \sim \ldots \sim (e_1, x^{2^{k-1}+1}, z_1) \sim (e_1, x, y_2, z_1) \sim (e_1, x, z_1) \sim (e_1, x, y_2, z_1) \sim (e_1, x, y_2, z_1) \sim \ldots \sim (e_1, y, z_1)$, where $z_1, z_2, \ldots, z_n \in Z_0$. We have a Hamiltonian path $H''$ in the subgraph induced by the set $S'$ with end vertex $(x', y', z')$ and also $H''$ is a Hamiltonian path induced by $S''$ with initial vertex $(e_1, x, y_2, z_1)$. Moreover, $(x', y', z') \sim (e_1, x, y_2, z_1)$. Thus, we get a Hamiltonian path $H$ in the subgraph induced by $S' \cup S''$. If $k = 2$ and $y' \in M_1$ then again we have a Hamiltonian path by interchanging the vertices $(e_1, x, y_2, z_1)$ and $(e_1, x, y_2, z_1)$ of $H$.

Subcase-2.2: $P = D_{2k+1} = \langle x, y : x^k = e_2, y^2, y^{-1}xy = x^{-1} \rangle$, where $k \geq 1$. Consider the set $S'' = \{e_1, b, c\} \in G : b \neq e_2)$. Observe that the sets $S'$, $S''$ and Dom($P_G(G)$) forms a partition of the group $G$. Also notice that $M' = \langle x \rangle$ is the only maximal cyclic subgroup of order $2^k$ in $D_{2k+1}$ and for $1 \leq i \leq 2^{k-1}$, $M_i = \{e_2, x^i, y, x^{2^k-i} \}$ be the maximal cyclic subgroups of order 2 in $D_{2k+1}$. By [23] Figure 1], $x^jy$, where $1 \leq j \leq 2^{k-1}$, is not adjacent to any non-identity element of $D_{2k+1}$ in $P_G(D_{2k+1})$. Since $o(y') = 2^k$, for $k \geq 2$, we have $y' \in M'$. Thus, the Hamiltonian path in the subgraph of $P_G(G)$ induced by the set $S''$ is $H'' : (e_1, x, y_2, z_1) \sim (e_1, x^2, y_2, z_1) \sim (e_1, x^2, y_2, z_1) \sim \ldots \sim (e_1, y, z_1)$, where $z_1, z_2, \ldots, z_n \in Z_0$. We have a Hamiltonian path $H''$ in the subgraph induced by the set $S'$ with end vertex $(x', y', z')$ and also $H''$ is a Hamiltonian path induced by $S''$ with initial vertex $(e_1, x, y_2, z_1)$. Furthermore, $(x', y', z') \sim (e_1, x, y_2, z_1)$. Consequently, we get a Hamiltonian path $H$ in the subgraph induced by $S' \cup S''$. If $k = 1$ and $y' \in M_1$, then again we have a Hamiltonian path by interchanging the vertices $(e_1, x, y_2, z_1)$ in $(e_1, x, y_2, z_1)$ of $H$.

Subcase-2.3: $P = S_{2k+1} = \langle x, y : x^k = e_2, y^2, y^{-1}xy = x^{-1+2^{k-1}} \rangle$, where $k \geq 3$. Consider the set $S'' = \{e_1, b, c\} \in G : b \neq e_2)$. Notice that the sets $S'$, $S''$ and Dom($P_G(G)$) forms a partition of the group $G$. Also note that $S_{2k+1}$ has one maximal cyclic subgroup of order $2^k$, $2^{k-1}$ cyclic subgroup of order 2 and $2^{k-2}$ maximal cyclic subgroup of order 4. Let $M' = \langle x \rangle$ be the maximal cyclic subgroup of order $2^k$ and for $1 \leq i \leq 2^{k-2}$, $M_i = \{e_2, x^{2^k-i}, x^{2^{k-2}+i} \}$ be the maximal cyclic subgroups of $S_{2k+1}$ of order 4 and for $1 \leq j \leq 2^{k-1}$, $M_j = \{e_2, x^{2^k}y \}$ be the maximal cyclic subgroups of $S_{2k+1}$ of order 2. Now for $1 \leq t \leq 2^k$, $x^{2^k}$ is a generator of a maximal cyclic subgroup of $S_{2k+1}$. It follows that $x^{2^k} \sim b \in P_G(S_{2k+1})$, where $b \in S_{2k+1} \setminus \langle x^{2^k} \rangle$ (see [23] Figure 2]). Let $z_1, z_2, \ldots, z_n$ be the elements of $Z_0$. Thus, the Hamiltonian path $H''$ in the subgraph of $P_G(G)$ induced by the set $S''$ is as follows

$(e_1, x, y_2, z_1) \sim (e_1, x, y_2, z_1) \sim (e_1, x^2, y_2, z_1) \sim \ldots \sim (e_1, x^{2^k-2}, y_2, z_1) \sim (e_1, x^{2^k-2}, y_2, z_1) \sim (e_1, x, z_1) \sim (e_1, x^2, z_1)$.

Moreover, we have a Hamiltonian path $H''$ in the subgraph induced by the set $S'$ with end vertex $(x', y', z')$. Since $o(y') = 2^k$, we have $y' \in M'$. It follows that $y' \sim x$ and so $(x', y', z') \sim (e_1, x, y_2, z_1)$ in $P_G(G)$. Thus, the subgraph of $\overline{P}_G(G)$ induced by the set $G \setminus \text{Dom}(P_G(G))$ has a Hamiltonian path.

Corollary 3.13. For the group $G = Q_{2k+1}$, we have $\lambda(P_G(G)) = 2^{k+1} + 1$.

Corollary 3.14. For the group $G \in \{D_{2k+1}, S_{2k+1}\}$, we have $\lambda(P_G(G)) = 2^{k+1}$.
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