Mesoscopic entanglement from central potential interactions

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The generation and detection of mesoscopic entanglement would have major fundamental and applicative implications. In this work, we evaluate the entanglement arising between two homogeneous levitated nanobeads interacting through the Coulomb potential, Newton’s gravitational potential, and the Casimir potential. We identify such levitated quantum systems as good candidates for measuring entanglement due to the Casimir effect, and discuss the challenges involved in probing gravitational entanglement with them.

Introduction. – The mastery over levitated optomechanical systems in the laboratory is becoming increasingly refined. With advances in cooling to the ground-state [1–4] and in the preparation of squeezed states [5], compounded by the ability to implant charges into levitated nanobeads [6], setups are reaching unprecedented levels of control. As optomechanical systems have shown significant potential for sensing applications [7, 8], especially with regards to measuring gravitational parameters [9–11], the engineering of schemes to entangle multiple levitated oscillators and, more broadly, mesoscopic systems, is being identified as a major medium-term milestone of the field [12]. In fact, the entanglement for a number of mesoscopic systems has already been demonstrated experimentally [13–15], although a more thorough and exhaustive understanding of the conditions under which entanglement may be generated is required to move on to applications. Entanglement also acts as an unambiguous hallmark of non-classicality, and entanglement between mesoscopic systems can aid the quest of mapping the transition from the quantum to classical scale [16, 17]. Recent proposals concerning the fundamental nature of gravity have considered quantum entanglement as generated by a Newtonian potential between two massive quantum systems [18, 19]. In general, the detection of entanglement as generated by gravity has significant ramifications, in the attempt to feed theories of quantum gravity with new empirical evidence at low energies [20, 21].

Is it possible to entangle two levitated systems through direct interaction? In this work, we consider fundamental or effective central potentials of the form $1/r^n$, for integer $n$ and where $r = |\mathbf{r}_1 - \mathbf{r}_2|$ with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ acting between two spherically symmetric quantum systems. Crucially, we ask whether minimal initial state preparation, such as squeezing, as opposed to severe requirements of preparing the highly non-Gaussian states (as, for example, envisaged in [18]) is enough, and whether the witnessing of the generated entanglement is possible simply by measuring position–momentum correlations. To address these questions, we quantify the leading-order contribution to entanglement within the continuous-variable (CV) framework for both the dynamical generation of entanglement from initial squeezed states of the interacting oscillators and for the system’s steady-state in the presence of noise. In addition, we propose an entanglement witness to detect such entanglement. We shall discuss three specific cases (see Fig. 1a): (i) the Coulomb potential between charged spheres, (ii) the Newtonian potential for two gravitating spheres, which we treat as point masses, and (iii) the Casimir potential between two homogeneous, spherical objects.

Dynamics. — We begin by considering two optomechanical spheres trapped next to each other. The non-interacting system Hamiltonian $\hat{H}_0$ describing the harmonic motion for both spheres in the trap is given by

$$H_0 = \frac{1}{2}m_1 \omega_1^2 \hat{x}_1^2 + \frac{1}{2}m_2 \omega_2^2 \hat{x}_2^2 + \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2},$$

(1)

where $m_1$ and $m_2$ are the system masses, $\omega_1$ and $\omega_2$ are the respective mechanical trapping frequencies for each sphere, and $\hat{x}_i$ and $\hat{p}_i$ with $i = 1, 2$ are the position and momentum operators for system 1 and 2, respectively.

We now consider a generic central potential of the form $\alpha/|\mathbf{r}_1 - \mathbf{r}_2|^n$, where $\alpha$ is the coupling constant and $\mathbf{r}_1$ and $\mathbf{r}_2$ are position vectors. To derive the Hamiltonian interaction term, we assume that the movement of the spheres is constrained in all but the $x$-direction. We consider small perturbations to $\mathbf{r}_1$ and $\mathbf{r}_2$, such that $\mathbf{r}_1 = x_1 + \Delta x_1$ and $\mathbf{r}_2 = x_2 + \Delta x_2$, with $x_1 \ll r_1$ and $x_2 \ll r_2$. By denoting $r = r_1 - r_2$ and $\Delta r = x_1 - x_2$, we Taylor-expand the interaction to second order in $\Delta x$:

$$\frac{1}{(r - \Delta r)^n} = \frac{1}{r^n} + \frac{\Delta x}{r^{n+1}} + \frac{n(n+1)}{2r^{n+2}}(\Delta x)^2 + \ldots,$$

(2)

where we have ignored the dimensional prefactor $\alpha$ and all $O[(\Delta x)^3]$. We then quantise the positions of the two masses around their equilibrium positions and promote the position coordinates to operators: $x_i \rightarrow \hat{x}_i$. For gravity, this step incorporates the assumption that gravity is a quantum force which can path-entangle two quantum systems [18].

The interaction term in the Hamiltonian thus becomes

$$\hat{H}_I = \alpha \frac{n}{r^{n+1}}(\hat{x}_1 - \hat{x}_2) + \alpha \frac{n(n+1)}{2r^{n+2}}(\hat{x}_1 - \hat{x}_2)^2,$$

(3)

where we have ignored the constant term. Since all quadratic Hamiltonians are invariant up to a displacement term [22], we can discard the first term in Eq. (3).
The final term, however, contains a mixing of the position operators in the form $\dot{x}_1, \dot{x}_2$, which will generate entanglement between the two states.

Since the full Hamiltonian is quadratic, all initial Gaussian states exclusively evolve into Gaussian states. Furthermore, all Gaussian states are uniquely defined by their first and second moments, which allows us to model this system completely within the covariance matrix framework [22]. We introduce the $4 \times 4$ two-mode covariance matrix $\sigma$, defined as $\sigma(t) = \text{Tr} \left\{ \left[ \mathbf{r}, \mathbf{r}^\dagger \right] \rho(t) \right\}$ for the state $\rho(t)$ and the vector of operators $\mathbf{r} = (\dot{x}_1, \dot{p}_1, \dot{x}_2, \dot{p}_2)^\dagger$. The covariance matrix $\sigma(t)$ evolves as

$$\dot{\sigma} = A(t)\sigma + \sigma A^T(t) + D,$$

where $A(t) = \Omega H(t) - \kappa \mathbb{I}_4/2$ is a drift matrix that incorporates the Hamiltonian matrix $H$ defined by $\dot{H} = \sigma^T H \sigma/2$, and $D = (2N_{\text{th}} + 1)\kappa \mathbb{I}_4$, with $\kappa$ being the phonon dissipation rate, $N_{\text{th}}$ the number of thermal phonons present in the system and $\mathbb{I}_4$ the $4 \times 4$ identity matrix. Finally, $\Omega$ is the symplectic form defined in this basis as $\Omega = \bigoplus_{i=1}^n \omega$ with $\omega = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. To determine the Hamiltonian matrix, we write $\dot{x}_i$ and $\dot{p}_i$ as $\dot{x}_i = \sqrt{h/m_i}\omega_i \dot{x}_i'$ and $\dot{p}_i = \sqrt{h/m_i}\omega_i \dot{p}_i'$. We now assume that $m_1 = m_2 = m$, and $\omega_1 = \omega_2 = \omega_m$ for notational simplicity. The full Hamiltonian now reads

$$\dot{H} = \frac{\hbar \omega_m}{2} \left( \dot{x}_1'^2 + \dot{p}_1'^2 \right) + \frac{\hbar \omega_m}{2} \left( \dot{x}_2'^2 + \dot{p}_2'^2 \right) + \frac{\alpha \hbar}{\omega_m m} n(n + 1) \left( \dot{x}_1' - \dot{x}_2' \right)^2.$$

In what follows, we will rescale the laboratory–time $t$ by $\omega_m$, such that the rescaled time $\tau$ is given by $\tau = \omega_m t$. This yields the Hamiltonian matrix $\tilde{H} = H/(\hbar \omega_m)$:

$$\tilde{H} = H_0 + 2\alpha \tilde{H}_1^{(1)} - 4 \alpha \tilde{H}_1^{(2)},$$

where for convenience we have defined the dimensionless coupling $\tilde{\alpha} = \alpha n(n + 1)/(2 \omega_m^2 m r^{n+2})$, and where $H_0 = \mathbb{I}_4$ is the Hamiltonian matrix governing the free evolution with $H_i^{(z)}$ denoting the Hamiltonian matrices responsible for the interaction. They will induce the entanglement and are given by

$$H_i^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_i^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We note that $H_i^{(2)}$ is of the form of two-mode squeezing, which implies that the system will not display periodic behaviour.

Entanglement in the CV framework can be computed by dividing $\sigma$ into submatrices $\sigma_A, \sigma_B$ and $\sigma_{AB}$ as such:

$$\sigma = \begin{pmatrix} \sigma_A & \sigma_{AB} \\ \sigma_{AB}^T & \sigma_B \end{pmatrix}.$$

We define the symplectic invariant quantity $\Delta = \det \sigma_A + \det \sigma_B + 2 \det \sigma_{AB}$. In this basis, the partial transpose is equivalent to setting $\hat{p}_r \rightarrow -\hat{p}_r$ for one subsystem, which implies $\Delta \rightarrow \Delta = \det \sigma_A - \det \sigma_B - 2 \text{det} \sigma_{AB}$. The positive partial transpose (PPT) criterion for two-mode Gaussian states can thus be compactly expressed as $\det \sigma - \Delta + 1 \leq 0$ [22]. For bipartite states, this is a necessary and sufficient condition [23]. To quantify the entanglement, we calculate the logarithmic negativity $E_N$ of the state, defined by $E_N(\sigma) = \max(0, -\log_2 \tilde{\nu}_+$), where $\tilde{\nu}_+$ are the symplectic eigenvalues of the state, defined for bipartite systems as $\tilde{\nu}_+ = \left( \Delta + \sqrt{\Delta^2 - 4 \det \sigma} \right)/2$.

As an initial state, we consider a two-mode mechanically squeezed state $\sigma_S = \text{diag}(z, z^{-1}, z^{-1}, z)$. Note that the squeezing occurs in the opposite quadrature. When $z = 1$, the state is a coherent state with $\sigma_S = \text{diag}(1, 1, 1, 1)$. There are many ways to mechanically squeeze the system, including controlling the trap frequency or using a Duff non-linearity [24], and thermal squeezing has been experimentally realised in [5].

Results. — For the system to be thermally stable, we require that the rescaled Hamiltonian matrix $\tilde{H}$ in Eq. (6) obeys $\tilde{H} > 0$, meaning all the eigenvalues of $\tilde{H}$ must be positive [22]. The eigenvalues $\lambda_i$ for $i = 1, 2, 3, 4$ of $\tilde{H}$ are

FIG. 1: (a) Comparison between the different couplings $\tilde{\alpha}$ for the Coulomb, Newtonian, and Casimir potentials given the values in Table I. (b) $E_N$ as a function of time $\tau$ for $z = 1$ and different values of $\tilde{\alpha}$. (c) $E_N$ as a function of time $\tau$ for different squeezing parameters $z$ and $\tilde{\alpha} = 0.01$. (d) $\text{det} M$ as a function of time $\tau$ for different $z$ and $\tilde{\alpha} = -0.1$. (e) $E_N$ as a function of time $\tau$ for increasingly noisy systems with decoherence rate $\tilde{\kappa}$ at $\tilde{\alpha} = -0.1$. (f) $E_N$ as a function of time $\tau$ for different values of squeezing $z$ with decoherence rate $\tilde{\kappa} = 0.05$ at $\tilde{\alpha} = -0.1$. 

We define the symplectic invariant quantity $\Delta = \det \sigma_A + \det \sigma_B + 2 \det \sigma_{AB}$. In this basis, the partial transpose is equivalent to setting $\hat{p}_r \rightarrow -\hat{p}_r$ for one subsystem, which implies $\Delta \rightarrow \Delta = \det \sigma_A - \det \sigma_B - 2 \text{det} \sigma_{AB}$. The positive partial transpose (PPT) criterion for two-mode Gaussian states can thus be compactly expressed as $\det \sigma - \Delta + 1 \leq 0$ [22]. For bipartite states, this is a necessary and sufficient condition [23]. To quantify the entanglement, we calculate the logarithmic negativity $E_N$ of the state, defined by $E_N(\sigma) = \max(0, -\log_2 \tilde{\nu}_+)$, where $\tilde{\nu}_+$ are the symplectic eigenvalues of the state, defined for bipartite systems as $\tilde{\nu}_+ = \left( \Delta + \sqrt{\Delta^2 - 4 \det \sigma} \right)/2$.

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Results. — For the system to be thermally stable, we require that the rescaled Hamiltonian matrix $\tilde{H}$ in Eq. (6) obeys $\tilde{H} > 0$, meaning all the eigenvalues of $\tilde{H}$ must be positive [22]. The eigenvalues $\lambda_i$ for $i = 1, 2, 3, 4$ of $\tilde{H}$ are
given by $\lambda_{1,2} = 1$, $\lambda_3 = 1 - 2\bar{a}$, and $\lambda_4 = 1 + 6\bar{a}$. Thus the overall restriction for stable dynamics is $\bar{a} < 1/2$ if $\bar{a}$ is positive, or $|\bar{a}| < 1/6$ if $\bar{a}$ is negative. In this work, we will always consider $\bar{a} \in (-1/6, 1/2)$.

We now proceed to compute the entanglement $E_N$. While an analytic expression for $E_N$ is available, it is too long to reproduce here. Instead, we plot the entanglement $E_N$ for different values of $\bar{a}$ in Fig. 1b for coherent states with $z = 1$. Next, in Fig. 1c we plot the same general dynamics for $\bar{a} = -0.1$ but for different $z$. It is evident that entanglement is affected by both the value of $\bar{a}$ and the squeezing $z$.

**Entanglement witness.** — A CV entanglement witness can be constructed by considering the subdeterminants of matrices constructed out of expectation values $[22, 25]$. For a linearised central-potential interaction, the following CV witness will detect entanglement:

$$M = \begin{pmatrix}
1 & \langle \hat{a} \rangle & \langle \hat{a}^\dagger \rangle & \langle \hat{b} \rangle \\
\langle \hat{a}^\dagger \rangle & \langle \hat{a} \rangle^2 & \langle \hat{a} \hat{b}^\dagger \rangle & \langle \hat{a}^2 \rangle \\
\langle \hat{b} \rangle & \langle \hat{a} \hat{b}^\dagger \rangle & \langle \hat{b}^2 \rangle & \langle \hat{a} \hat{b} \rangle \\
\langle \hat{b} \rangle & \langle \hat{a} \hat{b} \rangle & \langle \hat{b}^2 \rangle & \langle \hat{a} \rangle^2
\end{pmatrix},$$

(9)

where $\hat{a}, \hat{a}^\dagger$ are operators acting on the first subsystem and $\hat{b}, \hat{b}^\dagger$ are operators acting on the second subsystem. The state is entangled when $\det M < 0$. These expectation values can be measured experimentally by noting that $\hat{a} = (\hat{x}_1 + \hat{p}_1)/\sqrt{2}$. The position is generally measured to great accuracy for levitated nanobeads by interferometric techniques $[4, 26]$, while momentum can be measured as position after a quarter of a time period of free evolution of a harmonic oscillator.

To compute the expectation values in Eq. (9) from $\sigma(\tau)$, we define the characteristic function $\chi_G$ of a Gaussian state $G = e^{-\frac{1}{2}r^T\Gamma r}e^{i\vec{r}\cdot\vec{\xi}}e^{i\vec{r}\cdot\vec{\xi}}$, where $\vec{r}$ is a vector of first moment coordinates. The expectation values are related to $\chi_G$ by $\langle \hat{a}^{m}\hat{a}^{n}\rangle = \text{Tr}(\hat{a}^{m}\hat{a}^{n}\rho) = (\partial_{\vec{\xi}})^m(\partial_{\vec{\xi}})^n\chi_G(\vec{\xi})_{\vec{\xi}=0}$. In our basis with $\hat{x}_1$ and $\hat{p}_1$, it follows that $x_1 = (\xi^* + \xi)/\sqrt{2}$ and $p_1 = i(\xi^* - \xi)/\sqrt{2}$. We have plotted det $M$ in Fig. 1d for $\bar{a} = -0.1$ and different values of $z$.

**Noisy dynamics.** — We consider two types of noisy dynamics: damping of the oscillator motion in terms of phonon decay, where we denote the rescaled phonon decay rate by $\bar{\kappa} = \kappa/\omega_m$, and the number of thermal phonons $N_{\text{th}}$ present in the system. We evolve the system through Eq. (4) to find that the entanglement slowly decreases with time $\tau$. Fig. 1e shows $E_N$ as a function of rescaled time $\tau$ for a noisy environment for different $\bar{\kappa}$ at $\bar{a} = -0.1$. Similarly, in Fig. 1f, we have plotted $E_N$ as a function of rescaled time $\tau$ for different squeezing values $z$. We note that while increasing $z$ causes $E_N$ to grow at first, higher squeezing rates also makes the system more sensitive to noise over time $[27]$. The different curves bunch together around the second peak near $\tau = 2\pi$, showing that squeezing is effective only in the short-term.

**Steady state entanglement.** — We can now insert our Hamiltonian matrix $H$ and solve the matrix equation $A\sigma_\infty + \sigma_\infty A^T + D = 0$ for the steady state $\sigma_\infty$. We find the following elements of $\sigma_\infty$:

$$\sigma_{11}^{(\infty)} = \frac{96\bar{\alpha}^2 - 12\bar{\Gamma} \bar{\alpha} - \bar{\Gamma}^2}{192\bar{\alpha}^2 - 16\bar{\Gamma} \bar{\alpha} - \bar{\Gamma}^2}(2N_{\text{th}} + 1)$$

$$\sigma_{22}^{(\infty)} = \frac{192\bar{\alpha}^3 - 8(5\bar{\kappa}^2 - 16)\bar{\alpha}^2 - 20\bar{\Gamma} \bar{\alpha} - \bar{\Gamma}^2}{192\bar{\alpha}^2 - 16\bar{\Gamma} \bar{\alpha} - \bar{\Gamma}^2}(2N_{\text{th}} + 1)$$

$$\sigma_{12}^{(\infty)} = -\frac{2\bar{\alpha} \bar{\kappa}(24\bar{\alpha} - \bar{\Gamma})}{192\bar{\alpha}^2 - 16\bar{\Gamma} \bar{\alpha} - \bar{\Gamma}^2}(2N_{\text{th}} + 1)$$

$$\sigma_{13}^{(\infty)} = -\frac{8\bar{\alpha} \bar{\Gamma}}{192\bar{\alpha}^2 - 16\bar{\Gamma} \bar{\alpha} - \bar{\Gamma}^2}(2N_{\text{th}} + 1)$$

$$\sigma_{14}^{(\infty)} = -\frac{4 \bar{\alpha} \bar{\kappa} \bar{\Gamma}}{192\bar{\alpha}^2 - 16\bar{\Gamma} \bar{\alpha} - \bar{\Gamma}^2}(2N_{\text{th}} + 1),$$

(10)

where $\bar{\Gamma} = \bar{\kappa}^2 + 4$. All other elements follow by symmetry from the fact that $\sigma_\infty^T = \sigma_\infty$.

Let us analyse the covariance matrix elements in Eq. (10) to draw conclusions about the entanglement present in the steady state $\sigma_\infty$. The off-diagonal covariance matrix elements $\sigma_{13}$ and $\sigma_{14}$ scale asymptotically with $\bar{\kappa}^{-2}$, and $\bar{\kappa}^{-1}$ respectively, since $\bar{\Gamma} \propto \bar{\kappa}^2$ for $\bar{\kappa} \gg 1$. As information about entanglement is contained in the off-diagonal elements of $\sigma_\infty$, it follows that the logarithmic negativity $E_N$ will decrease as $\bar{\kappa}$ increases. We have plotted this behaviour in Fig. 2a for a range of interaction strengths $\bar{a} \in (-0.1, 0.1)$. As $\bar{\kappa}$ increases, the logarithmic negativity $E_N$ decreases rapidly. Furthermore, all elements in Eq. (10) are proportional to the term $(2N_{\text{th}} + 1)$. To compute the quantity $\hat{\Delta}$, we take the determinants of all the submatrices, meaning that $\hat{\Delta} \propto (2N_{\text{th}} + 1)^2$. To see how this affects the entanglement, we write the PPT criterion as $\det \sigma + 1 \geq \Delta$, meaning that the larger $\Delta$
is, the more difficult it is to violate the inequality. Thus, increasing \(N_{th}\) has a severe effect on the entanglement. This becomes evident in Fig. 2b, where we plot \(E_N\) as a function of \(N_{th}\) and \(k\) at \(\alpha = -0.1\). We note that very low noise-levels are required to detect entanglement in the steady state. The same holds in Fig. 2c and 2d, where we plot \(\det M\) for the steady state. In 2d, we have set all values of \(\det M > 0\) to zero as well for clarity.

Discussion and implementation. — We now specialise to the following three potentials: (i) the Coulomb potential, (ii) the Newtonian potential and (iii) the Casimir potential. They are given by, respectively:

\[
\alpha_{CI} = \frac{q_1 q_2}{4\pi \varepsilon_0 r^3 m \omega_m^2}, \quad \alpha_{NW} = -\frac{Gm}{r^3 \omega_m^2},
\]

\[
\alpha_{Cs} = -\frac{28}{(4\pi \varepsilon_0)^2} \frac{23 \hbar c R^6}{4 \pi r^3 m \omega_m^2} \left(\frac{\epsilon - 1}{\epsilon + 2}\right)^2,
\]

where \(q_1\) and \(q_2\) are the charges on the bipartite system, \(\varepsilon_0\) is the vacuum permittivity, \(\hbar\) is Planck’s reduced constant, \(c\) is the speed of light, \(R\) is the radius of the spheres, and \(\epsilon\) is the relative permittivity of the system. Finally, \(G\) is Newton’s constant, and \(m\) is mass of both systems. We have plotted these potentials as functions of distance \(r\) in Fig. 1a using the values in Table I.

Coulomb potential. — We consider two optomechanical spheres with charges \(q_1 = -q_2 = e\). The number of charges on each sphere can be controlled to exquisite precision by using ultraviolet light [28]. Linearised Coulomb interactions are commonly considered in trapped ions, and have also led to the generation of entanglement [29] but have not yet been implemented for optomechanical systems, although a proposal was put forward in [30]. With the parameters in Table I, it is possible to achieve a coupling \(\alpha_{CI} \approx -0.115\), meaning entanglement due to direct Coulombic interactions should be readily implementable for levitated systems.

Newtonian potential. — We first compute the entanglement for the parameters suggested in [18]. With \(m = 10^{-14}\) kg, \(r = 200 \times 10^{-6}\) m and \(\omega_m = 10^2\) Hz, we find \(\alpha_{NW} = 8.342510^{-16}\). This will not yield any detectable entanglement within this scheme. Furthermore, even if these figures were more lenient, there is the added complication that the Casimir potential will typically dominate over gravity for all parameter choices. Increasing \(m\) also increases the radius \(R = [3m/(4\pi \rho)]^{1/3}\), which strengthens the Casimir potential. Nevertheless, we compute \(\alpha_{NW}\) from the parameters found in Table I, where we make the optimistic assumption that \(\omega_m\) can be made as small as 10 Hz. For these values, we still only find \(\alpha_{NW} = -8.77 \times 10^{-7}\). Again, entanglement, given this interaction strength is not likely to be detectable in the near term.

Casimir potential. — Finally, the Casimir potential is the most promising effect to be measured with levitated nanobeads; see Fig. 1a, where the Casimir effect dominates at small \(r\). Crucial for applications in nanoscale devices [31], especially as a source of non-contact fric-

| Parameter Symbol | Value |
|------------------|-------|
| Mechanical frequency | \(\omega_m\) | \(10^2\) Hz |
| Charge | \(q_1, q_2\) | \(1.602 \times 10^{-19}\) C |
| Oscillator mass | \(m\) | \(2 \times 10^{-14}\) kg |
| Separation | \(r\) | \(10^{-5}\) m |
| Coupling strength | \(\alpha_{CI}\) | -0.115 |

| Parameter Symbol | Value |
|------------------|-------|
| Mechanical frequency | \(\omega_m\) | \(10\) Hz |
| Oscillator mass | \(m\) | \(10^{-14}\) kg |
| Separation | \(r\) | \(10^{-5}\) m |
| Coupling strength | \(\alpha_{NW}\) | \(-8.77 \times 10^{-7}\) |

| Parameter Symbol | Value |
|------------------|-------|
| Mechanical frequency | \(\omega_m\) | \(10^2\) Hz |
| Oscillator mass | \(m\) | \(10^{-14}\) kg |
| Separation | \(r\) | \(5 \times 10^{-4}\) m |
| Relative permittivity | \(\epsilon\) | 5.7 |
| Density | \(\rho\) | 3539 kgm\(^{-3}\) |
| Sphere radius | \(R\) | \(8.77 \times 10^{-7}\) m |
| Coupling strength | \(\alpha_{Cs}\) | -0.114 |

*TABLE I: Values used to compute the strength of the Coulomb, Newtonian and Casimir potentials.*
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