Nonunitary quantum circuit

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Abstract

A quantum circuit is generalized to a nonunitary one whose constituents are nonunitary gates operated by quantum measurement. It is shown that a specific type of one-qubit nonunitary gates, the controlled-\textit{NOT} gate, as well as all one-qubit unitary gates constitute a universal set of gates for the nonunitary quantum circuit, without the necessity of introducing ancilla qubits. A reversing measurement scheme is used to improve the probability of successful nonunitary gate operation. A quantum \textit{NAND} gate and Abrams-Lloyd’s nonlinear gate are analyzed as examples. Our nonunitary circuit can be used to reduce the qubit overhead needed to ensure fault-tolerant quantum computation.

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1 Introduction

Quantum computation \cite{1} is usually described by unitary operations because the time evolution of a closed system is described by unitary transformations. However, real systems interact with the environment, which entails decoherence and errors in quantum computation. To cope with the problem of decoherence, quantum error-correcting schemes \cite{2,3,4} have been proposed in which redundant qubits are introduced to ensure fault tolerance. Unfortunately, this qubit overhead is too demanding, since the number of available qubits will be severely restricted in the foreseeable future. To circumvent this problem, a probabilistic quantum error-correcting scheme without redundancy has recently been proposed \cite{5} using a reversing measurement.
scheme \([\text{6}]\). This scheme involves quantum measurement and is therefore described by nonunitary operations.

In this paper, we explore the possibility of a general quantum information processing based on nonunitary operations, i.e., quantum circuits that involve not only unitary but also nonunitary gates, the latter of which are implemented by quantum measurements. In a sense, our nonunitary quantum circuit is a generalization of the conventional unitary quantum circuit, because the latter also invokes quantum measurement at the end of computation. However, in our scheme measurements are exploited not only at the end but in the course of computation. Of course, even in the usual quantum computer, projective measurements are routinely used during computation. For example, Knill, Laflamme, and Milburn \([\text{7}]\) have shown that projective measurements can eliminate the need for nonlinear couplings in an optical quantum computer. Gilchrist et al. \([\text{8}]\) have also shown that an atomic measurement in an optical quantum computer corresponds to a nonunitary operator that is optically nonlinear and is approximately unitary in a Hilbert subspace for a single mode. Moreover, Raussendorf and Briegel \([\text{9}]\), and Nielsen \([\text{10}]\) have recently proposed two different schemes of quantum computation that consist entirely of projective measurements. Such measurements are intended to simulate unitary gates, and thus unitary operators connect the output states with the input states. In contrast, nonunitary operators are the connectors in our nonunitary quantum circuit, based on a general framework of quantum measurement. In other words, our gates are nonunitary at the logical level as well as at a physical level. This point emerges more clearly by comparing our circuit with an optical quantum computer \([\text{11}]\) that utilizes coherent states \(\{|-\alpha\rangle, |\alpha\rangle\}\) as the qubit states \(\{|0\rangle, |1\rangle\}\). These qubit states are only approximately orthogonal for large \(|\alpha|\), since \(\langle 0|1\rangle = \exp(-2|\alpha|^2)\). Due to this non-orthogonality, a unitary logical gate is physically implemented by a nonunitary gate in a wider Hilbert space using projective measurements. For example, the Hadamard gate \(H\), which acts as \(H|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\) and \(H|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}\), should be nonunitary at the physical level if the coherent-state qubit is used because \(\langle 0|H^\dagger H|1\rangle \neq \langle 0|1\rangle\), while the Hadamard gate itself is unitary at the logical level.

A natural question then arises as to whether or not a universal set of nonunitary gates exists for the nonunitary quantum circuit, since it is well known that a set of unitary gates is universal for the unitary quantum circuit \([\text{12} \text{13} \text{14} \text{15} \text{16} \text{17}]\). We will show that a set of nonunitary gates is universal for the nonunitary quantum circuit, without the necessity of intro-
ducing ancilla qubits. As a consequence of invoking quantum measurement, the nonunitary gate operation is necessarily probabilistic; however, we can be sure whether or not the gate operation is successful. We will discuss a reversing measurement scheme to increase the probability of successful nonunitary gate operation to the maximum allowable value. However, the total probability of successful operation of a nonunitary quantum circuit decreases exponentially with the number of nonunitary gates. This is a tradeoff for the reduction of qubit overhead, because unsuccessful measurements destroy the quantum state and halt the computation. We will show that if we could apply nonunitary gates with unit probability by some quantum dynamics, we could solve NP-complete problems in polynomial time. This parallels the result shown by Abrams and Lloyd [18], in which a hypothetical non-linear quantum theory implies a polynomial-time solution for NP-complete problems.

This paper is organized as follows. Section 2 formulates nonunitary gates and Sec. 3 discusses a universal set of nonunitary gates for the nonunitary quantum circuit. Section 4 shows a reversing measurement scheme to increase the probability of success of a nonunitary gate. Section 5 considers two examples of nonunitary gates: a quantum NAND gate and Abrams-Lloyd’s nonlinear gate. Section 6 summarizes our results.

\section{Nonunitary Gates}

We first define the nonunitary gate as a generalization of the unitary gate. A unitary gate is described as $|\psi\rangle \rightarrow U|\psi\rangle$, where $U$ is a unitary operator satisfying $U^\dagger U = UU^\dagger = I$, with $I$ being the identity operator. In the computational basis for $n$ qubits, the unitary operator $U$ is represented by a complex-valued $2^n \times 2^n$ matrix that satisfies the unitary condition \cite{1}. We define a nonunitary gate operation as

$$|\psi\rangle \rightarrow \frac{N|\psi\rangle}{\sqrt{\langle\psi|N^\dagger N|\psi\rangle}} \quad (1)$$

where $N$ is a nonunitary operator to be specified later. In the computational basis for $n$ qubits, $N$ is represented by a complex-valued $2^n \times 2^n$ matrix, without being subject to the unitary condition. Since a linear operation in a finite Hilbert space is always bounded and the normalization of $N$ does
not affect the state after the gate operation, we normalize $N$ so that the maximum eigenvalue of $N^\dagger N$ is unity:

$$\max_{|\psi\rangle} \langle \psi | N^\dagger N | \psi \rangle = 1. \quad (2)$$

To implement this nonunitary gate, we utilize a general framework of quantum measurement, in which a general measurement is described by a set of measurement operators $\{M_m\}$ \cite{19}. If the system is initially in a state $|\psi\rangle$, the probability for outcome $m$ is given by $p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$, and the corresponding postmeasurement state is given by

$$\frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m | \psi \rangle}}. \quad (3)$$

Since the total probability $\sum_m p(m)$ is 1, the measurement operators must satisfy $\sum_m M_m^\dagger M_m = I$. This means that all the eigenvalues of $M_m^\dagger M_m$ must be less than or equal to 1. That is, for any $|\psi\rangle$,

$$\langle \psi | M_m^\dagger M_m | \psi \rangle \leq 1. \quad (4)$$

Using this general measurement, we implement the nonunitary gate $N$ as follows: We perform a measurement $\{M_0, M_1\}$ with two outcomes, 0 and 1, such that

$$M_0 = c N, \quad M_1 = \sqrt{I - M_0^\dagger M_0}, \quad (5)$$

where $c$ is a normalization constant. It follows from Eqs. (2) and (4) that $|c| \leq 1$. We assume that the successful measurement corresponds to outcome 0 and the unsuccessful measurement corresponds to outcome 1. With the probability given by

$$p(|\psi\rangle; c) = \langle \psi | M_0^\dagger M_0 | \psi \rangle = |c|^2 \langle \psi | N^\dagger N | \psi \rangle, \quad (6)$$

the measurement is successful and then the state of the system becomes

$$|\psi\rangle \rightarrow \frac{M_0 |\psi\rangle}{\sqrt{\langle \psi | M_0^\dagger M_0 | \psi \rangle}} = \frac{N |\psi\rangle}{\sqrt{\langle \psi | N^\dagger N | \psi \rangle}}. \quad (7)$$

Comparing this equation with definition (1), we find that this measurement implements a nonunitary gate $N$ with the probability of success $p(|\psi\rangle; c)$.
given in Eq. (6). This probability is less than or equal to $|c|^2$ from Eq. (2). While the measurement may be unsuccessful, we can be sure whether or not the gate operation is successful by checking the measurement outcome. In terms of the quantum operations formalism [1], the nonunitary gate $N$ is described by a quantum operation:

$$
\mathcal{E}(|\psi\rangle\langle\psi|) = M_0|\psi\rangle\langle\psi|M_0^\dagger.
$$

(8)

Since this quantum operation does not include a summation over the measurement outcomes, a pure state remains pure during the gate operation [20].

While the constant $c$ does not affect the postmeasurement state, it does affect the probability of success. The maximal probability of success is attained by the measurement with $|c| = 1$. When this optimal measurement is not available, we can still improve the probability of success so that it is arbitrarily close to the maximum allowable value by applying a reversing measurement scheme to a non-optimal measurement $|c| < 1$, as will be shown later.

There are, however, two problems with the nonunitary gate. First, if $\det N$ is zero, then there exists a state $|\psi_W\rangle$ such that $N|\psi_W\rangle = 0$. For this state, the measurement never succeeds because $p(|\psi_W\rangle; c) = 0$. We can circumvent this problem by excluding the wrong states from the input state, or by choosing $N$ such that $\det N$ is nonzero. If $\det N \neq 0$, the gate $N$ is said to be logically reversible [6] in the sense that the input state can be calculated from the output state. In other words, a logically reversible gate preserves all pieces of information about the input state during the gate operation. An example of the logically irreversible gate is the projective measurement, by the action of which the information about the states orthogonal to the projector is completely lost. The second problem is that the total probability of success of a quantum circuit involving nonunitary gates decays exponentially with the number of nonunitary gates, since an unsuccessful measurement in an intermediate gate forces us to restart from the first gate. Nevertheless, a nonunitary quantum circuit has the advantage of reducing the number of qubits in some situations. We shall discuss these two issues below, using a quantum NAND gate.
3 Universality

We consider a universal set of gates for the nonunitary quantum circuit both with and without ancilla qubits.

3.1 With ancilla qubits

An arbitrary quantum measurement can be simulated by the projective measurement and unitary operation with the use of ancilla qubits. Therefore, if ancilla qubits are available, the controlled-NOT (CNOT) gate, all one-qubit unitary gates, and the one-qubit projective measurement constitute a universal set for the nonunitary quantum circuit. We begin by proving this theorem.

Consider a \(2^n \times 2^n\) nonunitary matrix \(N\) representing a nonunitary quantum circuit for \(n\) qubits. We then make the singular value decomposition of \(N\),

\[
N = U D(d_1, d_2, \ldots, d_{2^n}) V,
\]

where \(U\) and \(V\) are unitary matrices and \(D(d_1, d_2, \ldots, d_{2^n})\) is a diagonal matrix whose diagonal components \(\{d_i\}\) satisfy \(0 \leq d_i \leq 1\). The unitary matrices \(U\) and \(V\) can be further decomposed into CNOT gates and one-qubit unitary gates, since this set of gates is universal for the unitary quantum circuit [16]. We thus concentrate on the diagonal matrix \(D(d_1, d_2, \ldots, d_{2^n})\). Using the NOT gate \(X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), this matrix can be factorized into \(2^n\) matrices that have the form of \(D(1, 1, \ldots, 1, d_i)\). Each of these matrices corresponds to a controlled-\(N_1(a)\) gate with \(n-1\) control qubits (denoted by \(C^{n-1}[N_1(a)]\)), where \(N_1(a)\) is a one-qubit nonunitary gate given by

\[
N_1(a) = \begin{pmatrix}
1 & 0 \\
0 & a
\end{pmatrix}, \quad 0 \leq a < 1.
\]

The case of \(n = 2\) is illustrated in Fig. 1. This \(C^{n-1}[N_1(a)]\) gate can be implemented by one \(N_1(a)\) gate and two \(C^n[X]\) gates (i.e., two CNOT gates with \(n\) control qubits) with the help of an ancilla qubit prepared in state \(|0\rangle\). Figure 2 shows the case of \(n = 3\). Since the \(C^n[X]\) gate is unitary, it can be decomposed into CNOT gates and one-qubit unitary gates [16]. On the other hand, as shown in Fig. 3 the \(N_1(a)\) gate can be decomposed into one \(N_1(0)\) gate and one controlled-\(U_1(a)\) gate by using an ancilla qubit, where \(U_1(a)\) is
Figure 1: Circuit for a $D(d_1, d_2, d_3, d_4)$ gate.

Figure 2: Circuit for a $C^2[N_1(a)]$ gate using an ancilla qubit (represented by the bottom line).

Figure 3: Circuit for a $N_1(a)$ gate using an ancilla qubit (represented by the bottom line).
Figure 4: Circuit for a $C^2[N_1(a)]$ gate with no ancilla qubit, where $a' = \sqrt{a}$.

a one-qubit unitary gate defined by

$$U_1(a) \equiv \begin{pmatrix} \frac{a}{\sqrt{1-a^2}} & \frac{1}{a} \\ \sqrt{1-a^2} & -a \end{pmatrix}. \quad (11)$$

Since the $N_1(0)$ gate corresponds to the one-qubit projective measurement $|0\rangle\langle0|$, the theorem is proved.

### 3.2 Without ancilla qubits

In view of necessity of reducing the number of qubits, we next consider the case where no ancilla qubits are available. In this case, we cannot find a set of gates with which to exactly construct an arbitrary nonunitary circuit. We therefore apply a definition of universality in a broad sense in which two gates are regarded as identical if they differ only by a normalization factor. Note that the normalization of a nonunitary gate does not affect the state after the gate operation, though it affects the probability of success. With this proviso, we here prove that if ancilla qubits are not available, the CNOT gate, all one-qubit unitary gates, and the $N_1(a)$ gates ($0 \leq a < 1$) constitute a universal set for the nonunitary circuit.

The proof goes as follows. As shown in the preceding section, an arbitrary nonunitary matrix can be decomposed into the $C^{m-1}[N_1(a)]$ gates and unitary matrices. When $a \neq 0$, each $C^{m-1}[N_1(a)]$ gate can be further decomposed into controlled-$N_1(a')$ gates, controlled-$N_1(1/a')$ gates, and CNOT gates without ancilla qubits as in the unitary case [10], where $a' = a^{\frac{1}{2m-2}}$. Figure 4 illustrates the case of $n = 3$. Moreover, as shown in Fig. 4, the controlled-$N_1(a')$ gate can be implemented by two $N_1(\bar{a})$ gates, one $N_1(1/\bar{a})$ gate, and two CNOT gates, where $\bar{a} = \sqrt{a'}$. Similarly, the controlled-$N_1(1/a')$ gate can
be implemented by two $N_1(1/\bar{a})$ gates, one $N_1(\bar{a})$ gate, and two CNOT gates. Therefore, we are left with the $N_1(\bar{a})$ and $N_1(1/\bar{a})$ gates ($0 < \bar{a} < 1$), apart from the CNOT gate and the wire $N_1(1)$. However, the $N_1(1/\bar{a})$ gate is not a gate in the strict sense, because it does not satisfy the normalization condition \[a \bar{a} > 1\] due to $1/\bar{a} > 1$. We must renormalize it using the identification up to a normalization factor. Namely, via the equation

$$N_1(1/\bar{a}) \propto X N_1(\bar{a}) X,$$

we express the $N_1(1/\bar{a})$ gate by one $N_1(\bar{a})$ gate and two NOT gates.

On the other hand, when $a = 0$, the $C^{n-1}[N_1(a)]$ gate becomes the $D(1,1,\cdots,1,0)$ gate, which is paired with the $D(0,0,\cdots,0,1)$ gate through the relation $M_0^\dagger M_0 + M_1^\dagger M_1 = I$. This means that the unsuccessful operation of the $D(0,0,\cdots,0,1)$ gate is identical to the successful operation of the $C^{n-1}[N_1(0)]$ gate, and vice versa. We thus construct the $D(0,0,\cdots,0,1)$ gate instead of the $C^{n-1}[N_1(0)]$ gate. Note that the $D(0,0,\cdots,0,1)$ gate corresponds to the $n$-qubit projective measurement $|11\cdots1\rangle\langle11\cdots1|$. Since the one-qubit projective measurement $|1\rangle\langle1|$ corresponds to the $X N_1(0)X$ gate, the operation of the $D(0,0,\cdots,0,1)$ gate can be implemented by the action of the $X N_1(0)X$ gate on each qubit. Consequently, we can construct any nonunitary quantum circuit from the CNOT gate, all one-qubit unitary gates, and the $N_1(a)$ gates with $0 \leq a < 1$.

We finally show that any $N_1(a)$ gate can be approximated to arbitrary accuracy by only two fixed nonunitary gates together with the NOT gate. For a real number $\alpha$ and an irrational number $\gamma$, we consider the $N_1(\alpha)$ and $N_1(\alpha^\gamma)$ gates. Note that for any real numbers $a$ and $\epsilon$, there exist integers $m$ and $l$ such that

$$|\log_\alpha a - (m\gamma + l)| < \epsilon.$$  

(13)
Figure 6: A reversing measurement scheme. If the measurement \{M_0, M_1\} (solid arrows) fails, then a reversing measurement \{R_0, R_1\} (dashed arrows) probabilistically reverts the postmeasurement state \(M_1|\psi\rangle\) back to the original state \(|\psi\rangle\).

Using these \(m\) and \(l\), the \(N_1(a)\) gate is approximately written as

\[
N_1(a) \sim [N_1(\alpha^\gamma)]^m [N_1(\alpha)]^l.
\] (14)

If \(m < 0\) or \(l < 0\), we use Eq. (12) to make the power positive; ignoring the normalization factor, we obtain

\[
[N_1(\alpha)]^l \propto X [N_1(\alpha)]^{-l} X,
\] (15)

\[
[N_1(\alpha^\gamma)]^m \propto X [N_1(\alpha^\gamma)]^{-m} X.
\] (16)

In this way, we can approximate any \(N_1(a)\) gate by only the \(N_1(\alpha)\) and \(N_1(\alpha^\gamma)\) gates together with the NOT gate.

4 Optimization by Reversing Measurement

To implement a nonunitary gate \(N\), we must prepare the measurement \{\(M_0, M_1\)\} defined in Eq. (5). The probability of success of this measurement depends on the value of \(c\). When the optimal measurement \(|c| = 1\) is not available, the probability of success of the gate operation is reduced. However, we can improve the probability of success so that it is arbitrarily close to the maximum allowable value by applying a reversing measurement scheme [6] to a non-optimal measurement \(|c| < 1\) (see Fig. 6). More specif-
ically, if the measurement \( \{M_0, M_1\} \) fails, we perform another measurement \( \{R_0, R_1\} \) that satisfies
\[
R_0 M_1 = qI, \quad R_1 = \sqrt{I - R_0^\dagger R_0},
\]
where \( q \) is a constant. Note that \( R_0 \) exists when \( |c| < 1 \) and is proportional to \( M_1^{-1} \). Therefore, if this measurement is successful, the postmeasurement state becomes the original one \( |\psi\rangle \) and we can then try the measurement \( \{M_0, M_1\} \) again to increase the probability of success of the nonunitary gate operation. Of course, the reversing measurement \( \{R_0, R_1\} \) also fails with a nonzero probability. The joint probability for \( M_1 \) followed by \( R_0 \) is given by
\[
|q|^2,
\]
which does not depend on the measured state \( |\psi\rangle \). Note that \( |q| \) cannot be set to 1, since the maximum eigenvalue of \( R_0^\dagger R_0 \) is \( |q|^2/(1 - |c|^2) \) which must be less than 1. In order for \( R_0 \) to be a measurement operator, Eq. (4) requires that
\[
0 < |q| \leq \sqrt{1 - |c|^2}. \tag{18}
\]

Using the reversing measurement once, the probability of success of the nonunitary gate operation increases to \( p(|\psi\rangle; c) + |q|^2 p(|\psi\rangle; c) \), where the first and second terms result from the process \( M_0 \) and the process \( M_1 \to R_0 \to M_0 \), respectively. The other processes, \( M_1 \to R_1 \) and \( M_1 \to R_0 \to M_1 \), are unsuccessful gate operations. However, we can repeatedly perform the reversing measurement when the process ends with \( M_1 \) in order to further increase the probability of success. By repeating the reversing measurement, at most \( k \) times as long as the reversing measurement succeeds, we increase the probability of success to
\[
\tilde{p}_k(|\psi\rangle; c) = \left[ 1 + |q|^2 + \cdots + |q|^{2k} \right] p(|\psi\rangle; c)
= \frac{1 - |q|^{2k+2}}{1 - |q|^2} p(|\psi\rangle; c), \tag{19}
\]
since we can repeat the process \( M_1 \to R_0 \) \( l \) times with probability \( |q|^{2l} \). Substituting the maximum value \( \sqrt{1 - |c|^2} \) for \( |q| \), we find
\[
\tilde{p}_k(|\psi\rangle; c) = \frac{1 - (1 - |c|^2)^{k+1}}{|c|^2} p(|\psi\rangle; c). \tag{20}
\]
In the limit of \( k \to \infty \), we obtain
\[
\tilde{p}_\infty(|\psi\rangle; c) = \frac{1}{|c|^2} p(|\psi\rangle; c) = p(|\psi\rangle; 1). \tag{21}
\]
Table 1: The truth table of the NAND gate.

| Inputs | Output |
|--------|--------|
| 0 0    | 1      |
| 0 1    | 1      |
| 1 0    | 1      |
| 1 1    | 0      |

This shows that even if the measurement to implement a nonunitary gate is not optimal, i.e., $|c| < 1$, we can, in principle, increase the probability of success so that it is arbitrarily close to the optimal value with $|c| = 1$ by utilizing the reversing measurement scheme, provided that the optimal reversing measurement with $|q| = \sqrt{1 - |c|^2}$ is available. Note that the optimal value with $|c| = 1$ does not mean a deterministic nonunitary gate, as can be seen from Eq. (3). We cannot increase the probability of success to 1, since the reversing measurement is only successful in a probabilistic way. Nevertheless, it is worthwhile to note that the reversing measurement scheme can improve the probability of success to some extent if the original one is not optimal.

5 Examples

Finally we discuss two examples of the nonunitary gate. The first one is a quantum NAND gate, which can reduce the work space required to perform a specific type of quantum computation. The second one is Abrams-Lloyd’s nonlinear gate, which could solve the NP-complete problem in polynomial time if its probability of success were 1.

5.1 Quantum NAND gate

The classical NAND gate (Table 1) is a universal gate for irreversible classical computation. However, a quantum version of this gate cannot be a unitary gate, due to the irreversibility of the NAND operation. In contrast, we can make a quantum version of the NAND gate as a nonunitary gate.

Consider a two-qubit nonunitary gate represented in a computational
basis,

\[ |00⟩ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01⟩ = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |10⟩ = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11⟩ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

as

\[ N = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (22)

This gate transforms the computational basis as

\[ N|00⟩ = N|01⟩ = N|10⟩ = |10⟩ \quad \text{and} \quad N|11⟩ = |00⟩, \]

which yields the truth table of the NAND gate as in Table I if the second qubit of the output state is ignored. We thus call \( N \) a quantum NAND gate. Note that the second qubit always becomes \( |0⟩ \) after the gate operation in order not to become entangled with the first qubit.

To implement this nonunitary gate, we prepare a measurement \( \{M_0, M_1\} \) with two outcomes, 0 and 1, as in Eq. (5). For the states in the computational basis, the probabilities of success are equal, \( p(|x⟩; c) = |c|^2/3 \) for \( x \in \{00, 01, 10, 11\} \). When the initial state is

\[ |ψ_{\text{max}}⟩ = \frac{|00⟩ + |01⟩ + |10⟩}{\sqrt{3}}, \] (23)

the probability of success becomes maximal due to constructive interference: \( p(|ψ_{\text{max}}⟩; c) = |c|^2 \). Therefore, \( c \) must satisfy \( 0 < |c| \leq 1 \). Otherwise \( M_1^TM_1 = I - M_0^TM_0 \) fails to be positive semidefinite. An explicit form of \( M_1 \) is given by

\[ M_1 = \frac{1}{3} \begin{pmatrix} 2 + a & -1 + a & -1 + a & 0 \\ -1 + a & 2 + a & -1 + a & 0 \\ -1 + a & -1 + a & 2 + a & 0 \\ 0 & 0 & 0 & 3b \end{pmatrix}, \] (24)

where \( a = \sqrt{1 - |c|^2} \) and \( b = \sqrt{1 - (|c|^2/3)} \).

On the other hand, the minimum probability of success is

\[ \min_{|ψ⟩} p(|ψ⟩; c) = 0, \] (25)
since the minimum eigenvalue of $N^\dagger N$ is zero. Two eigenvectors correspond to the zero eigenvalue,

$\frac{|00⟩ - |01⟩}{\sqrt{2}}, \frac{|01⟩ - |10⟩}{\sqrt{2}}$. (26)

This means that the measurement never succeeds for the states in the two-dimensional subspace spanned by these vectors, due to destructive interference. For example, $N(\frac{|00⟩ - |01⟩}{\sqrt{2}}) = 0$. Note that $N$ is not logically reversible because $\det N = 0$. When using $N$, we must exclude these wrong states from the input state.

Since $M_1$ is logically reversible ($\det M_1 \neq 0$) in the non-optimal case $|c| < 1$, the reversing measurement scheme can be utilized to improve the probability of success. We can thus perform the reversing measurement $\{R_0, R_1\}$ defined by Eq. (17). The explicit form of $R_0$ is

$$R_0 = \frac{q}{3a} \begin{pmatrix} 1 + 2a & 1 - a & 1 - a & 0 \\ 1 - a & 1 + 2a & 1 - a & 0 \\ 1 - a & 1 - a & 1 + 2a & 0 \\ 0 & 0 & 0 & 3a/b \end{pmatrix}. \quad (27)$$

It is easy to confirm that $R_0M_1$ is equal to $qI$ and that the eigenvalues of $R_0^\dagger R_0$ are less than or equal to 1 if $|q| \leq \sqrt{1 - |c|^2}$. The reversing measurement scheme then increases the probability of success as in Eq. (20) if $|q| = \sqrt{1 - |c|^2}$. For the states in the computational basis $|x⟩$ with $x \in \{00, 01, 10, 11\}$, we obtain

$$\tilde{p}_k(|x⟩; c) = 1 - \frac{(1 - |c|^2)^{k+1}}{3}, \quad (28)$$

and for the maximally successful state we obtain

$$\tilde{p}_k(|\psi_{\text{max}}⟩; c) = 1 - (1 - |c|^2)^{k+1}. \quad (29)$$

In the limit of $k \to \infty$, they become the maximum allowable values, 1/3 and 1, respectively.

As an application of the quantum NAND gate, we consider computing

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x⟩|0⟩ \to \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x⟩|f(x)⟩ \quad (30)$$
for a given function \( f(x) \). (Consider, for example, the modular exponentiation in Shor’s algorithm [21].) In conventional quantum computers, we build up a unitary quantum circuit for this computation by the following steps [1]:

(i) We construct an irreversible classical circuit to calculate \( x \rightarrow f(x) \) using classical \text{NAND} gates, since these gates are universal in classical computation.

(ii) We replace the classical \text{NAND} gates with classical Toffoli gates to make this classical circuit reversible, by adding ancilla bits. (iii) We translate this reversible classical circuit into a quantum one by replacing the classical Toffoli gates with quantum Toffoli gates. Note that the resultant circuit needs more qubits than the irreversible classical circuit, due to step (ii).

Because all coefficients of the linear combination in Eq. (30) are positive, no destructive interference occurs in operating the quantum \text{NAND} gate. We thus utilize the quantum \text{NAND} gate to reduce the number of qubits needed. Instead of steps (ii) and (iii), we directly replace the classical \text{NAND} gates with quantum \text{NAND} gates. This procedure allows us to reduce the number of qubits needed to perform calculation (30), because the quantum \text{NAND} gate is a two-qubit gate, unlike the quantum Toffoli gate. However, the quantum \text{NAND} gate is probabilistic, since it is implemented by quantum measurement. When all the classical \text{NAND} gates are replaced with quantum \text{NAND} gates, the probability of success becomes exponentially small as the number of \text{NAND} gates increases. Thus, in practice, we replace only some classical \text{NAND} gates with quantum ones. After dividing the function \( f \) into two functions, \( g_1 \) and \( g_2 \), i.e.,

\[
x \rightarrow g_1(x) \rightarrow g_2(g_1(x)) = f(x),
\]

we calculate \( g_1 \) using quantum \text{NAND} gates and \( g_2 \) using quantum Toffoli gates. If \( g_1 \) contains \( m \) quantum \text{NAND} gates, this method can save \( m \) qubits with the probability of success \( (|c|^2/3)^m \). By checking the measurement outcome, we can be sure whether or not the gate operations are successful.

### 5.2 Abrams-Lloyd’s gate

Abrams and Lloyd [18] showed that \text{NP}-complete problems could be solved in polynomial time if quantum theory were nonlinear at some level. Although the nonlinearity of quantum theory is hypothetical, their work establishes a new link between a physical law and the power of computing machines. We here describe their nonlinear gate as a nonunitary gate.
Let $F(x)$ be a function that maps an $n$-bit input to a single bit $\{0, 1\}$. Given an oracle to calculate $F(x)$, can we determine whether or not there exists an input value $x$ for which $F(x) = 1$? The integer $s$ is defined as the number of such input values and, to simplify the problem, is assumed to be either 0 or 1. In order to solve this NP-complete problem, Abrams and Lloyd first prepare $n$ qubits $|x\rangle$ and one flag qubit $|F(x)\rangle$ in an entangled state

$$|\psi_i\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |F(x)\rangle,$$  \hfill (32)

using the usual quantum computer. They then let a two-qubit quantum gate perform a nonlinear transformation

$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \rightarrow \frac{1}{\sqrt{2}} (|01\rangle + |11\rangle),$$

$$\frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \rightarrow \frac{1}{\sqrt{2}} (|01\rangle + |11\rangle),$$

$$\frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \rightarrow \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle),$$  \hfill (33)

successively on the first and flag qubits, on the second and flag qubits, and so on through the $n$th and flag qubits. The final state is given by

$$|\psi_f\rangle = \left( \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \right) |s\rangle,$$  \hfill (34)

in which the flag qubit is not entangled with the first $n$ qubits. Therefore, by measuring the flag qubit, the answer $s$ is found in polynomial time.

It is easy to see that our nonunitary gate can simulate Abrams-Lloyd’s nonlinear gate as

$$N_{AL} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$  \hfill (35)

even though the nonunitary gate is linear except for the normalization factor. This means that NP-complete problems could be solved in polynomial time if the $N_{AL}$ gate could be applied with probability 1 by some quantum dynamics. We can thus establish yet another new link between a physical law and the power of computing machines.
Unfortunately, as discussed in the preceding sections, the implementation of a nonunitary gate by quantum measurement is intrinsically probabilistic. Even if the implementing measurement is optimal (|c| = 1), the $N_{AL}$ gate succeeds only with probability $1/6$ for the states (33); otherwise an unsuccessful operator, e.g.,

$$M_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix},$$

(36)

is applied to the states. When the number of qubits is $n$, the total probability of success decays exponentially as $(1/6)^n$. To obtain a definite result, we must repeat the algorithm $6^n$ times, which consumes an exponential computation time.

6 Conclusions

We have formulated a nonunitary quantum circuit having nonunitary gates operated by quantum measurement. In contrast with recently proposed schemes on quantum computation using measurements [7, 8, 9, 10, 11], our gates utilize the nonunitarity fully in the sense that not only the physical implementation but also the logical operation is nonunitary. We have shown that the CNOT gate, a complete set of one-qubit unitary gates, and the $N_1(a)$ gates constitute a universal set of gates for the nonunitary quantum circuit without the necessity of introducing ancilla qubits, and have shown that a nonunitary gate can be optimized by a reversing measurement scheme. These results will be useful for the construction of a quantum computer equipped with probabilistic error correction by the reversing measurement. More generally, the nonunitary quantum circuit can reduce the number of qubits required to perform some kinds of quantum computation, as illustrated by the quantum NAND gate. Although we cannot reduce the number of qubits excessively, due to the probabilistic nature of the nonunitary gate, this approach would be useful for constructing a quantum computer as long as the number of available qubits is severely restricted. Moreover, apart from this practical interest, there may be an academic interest in extending quantum computation itself to include nonunitary operations. At least, using Abrams-Lloyd’s gate, the nonunitary quantum computer can solve NP-complete problems in polynomial time (if the probabilistic nature is ignored), whereas it is widely
believed that the usual unitary quantum computer cannot do so. It would be interesting to quantify how much nonunitarity is required to solve NP-complete problems in polynomial time.

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