Testing and Support Recovery of Correlation Structures for Matrix-Valued Observations with an Application to Stock Market Data

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Abstract

Estimation of the covariance matrix of asset returns is crucial to portfolio construction. As suggested by economic theories, the correlation structure among assets differs between emerging markets and developed countries. It is therefore imperative to make rigorous statistical inference on correlation matrix equality between the two groups of countries. However, if the traditional vector-valued approach is undertaken, such inference is either infeasible due to limited number of countries comparing to the relatively abundant assets, or invalid due to the violations of temporal independence assumption. This highlights the necessity of treating the observations as matrix-valued rather than vector-valued. With matrix-valued observations, our problem of interest can be formulated as statistical inference on covariance structures under sub-Gaussian distributions, i.e., testing non-correlation and correlation equality, as well as the corresponding support estimations. We develop procedures that are asymptotically optimal under some regularity conditions. Simulation results demonstrate the computational and statistical advantages of our procedures over certain existing state-of-the-art methods for both normal and non-normal distributions. Application of our procedures to stock market data reveals interesting patterns and validates several economic propositions via rigorous statistical testing.

Keywords— Kronecker product; Matrix sub-Gaussian distribution; Portfolio construction; Covariance matrix; Testing of non-correlation; One-sample and two-sample.

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1 Introduction

Understanding the covariance matrix of asset returns is of paramount importance as asset pricing theories dictate that the distribution of returns are related to the business cycle and consumption states, which affect the demands for holding financial assets and generate time-varying risk premia (Moskowitz 2003). However, characterizing the covariance structure of returns can be challenging, particularly when the number of assets is large.

We now take the perspective of a global investor and consider an even more difficult challenge that the stock returns are from various industries in multiple countries across the world. To enable estimation, we employ a country-industry Kronecker structure model given the short sample period relative to the tremendous cross section of asset returns. However, cautions need to be taken, as these countries can be categorized into emerging markets and developed countries, assuming the same covariance matrix of industry returns for these two groups could lead to undesirable consequences in making optimal investment decision. As pointed out by Bekaert and Harvey (1995), developed markets are more financially integrated while emerging markets are more financially segmented, thus industries may have different amount of systematic risk depending on the level of segmentation.

As the market return is an aggregation of all industries returns, therefore an industry’s systematic risk is simply the value-weighted average of its covariance with other industries, over its own variance. Therefore, we focus on the correlation matrices and conjecture that they should be significantly different between the two groups of countries. Specifically, emerging markets are characterized by frequent regime switches, and sudden changes of fiscal, monetary and trade policies (Aguiar and Gopinath 2007). When these economic policies change frequently in an unanticipated way in emerging markets, they tend to make cyclical sectors more pro-cyclical than those in developed markets. Furthermore, Kohn et al. (2018) show that emerging economies produce more commodities than they consume while developed markets do not. Therefore we also expect to detect larger comovements of commodity industries returns with others in emerging countries. Finally, due to different demographic patterns (DellaVigna and Pollet 2007), we also expect in emerging countries, recreative business industries co-vary more with the market returns.

Combined, these propositions all highlight the necessity of rigourous statistical testing of the equality of the correlation matrices from the two groups of countries. In the literature, researchers
often treat the returns of multiple industries as vector-valued observations over time in each country \cite{Fama:1997, Hong:2007}. With the vector-valued approach, as typically the number of assets far exceeds the length of the time series and the number of countries, estimating the correlation matrix can be so challenging that certain potentially problematic assumption of the temporal independence must be made. In this case, adopting the vector-based approach of two sample test, such as \cite{Li:2012, Cai:2013, Cai:2016, Chang:2017, Zheng:2019}, will be either infeasible due to the small number of observations, or invalid due to the violation of temporal independence.

The goal of this article is to perform hypothesis test of the equality of the two correlation matrices by considering observations that are matrix-valued. Compared to the conventional vector-valued observations, the matrix-valued observations add one more dimension, corresponding to the time domain. The temporal dimension is allowed to have a wide range of dependence, which is more flexible than in the vector-based approach. The addition of the temporal dimension also alleviates the problem of small sample size and insufficient length of the time series as seen later.

To be specific, let $g = 1, 2$ correspond to the two groups of countries, emerging and developed, respectively. There are $n_1$ (resp. $n_2$) countries in the emerging (resp. developed) group. Denote $X^{(g)}_k$, for $k = 1, \ldots, n_g$, the matrix of returns for country $k$ in group $g$. Each matrix is of size $p \times q$ when there are $p$ industries and $q$ time points. We are interested in the inference on the correlation matrix $\text{Cor}(\text{vec}(X^{(g)}_k))$, where $\text{vec}(\cdot)$ is the vectorization operation that stacks the columns of a matrix into a long vector.

As the correlation matrix is of enormous size $pq \times pq$ while the sample size is only $n_g$, it is necessary to make further assumption. We consider the Kronecker product model for the covariance matrix $\Sigma^{(g)} = \text{Cov}(\text{vec}(X^{(g)}_k)) = B^{(g)} \otimes A^{(g)}, g = 1, 2$. More information on the Kronecker product model, such as references, motivation, and applications, is given in Section 1.2. Here, $A^{(g)}$ is the covariance matrix of size $p \times p$ for the covariances between $p$ industries in group $g$ and $B^{(g)}$ is the $q \times q$ covariance matrix along the temporal dimension in group $g$. As such, the covariance of the observed matrix $X^{(g)}_k$ is decomposed into the product of the industrial covariance and temporal covariance. This Kronecker product structure reduces the number of unknown parameters from $O(p^2 q^2)$ to $O(p^2 + q^2)$. Furthermore, much smaller sample size is needed: without such structural assumption, a sample size of $n_g \geq pq$ is necessary to make the sample covariance matrix full rank,
while with the structure, the sample size is sufficient as long as $n_g p \geq q$ and $n_g q \geq p$.

Under the Kronecker product model, our goal is to test the equality of the correlation matrices of the industries by considering $B^{(g)}$ as nuisance parameters. Consider the correlation matrices: $R_A^{(g)} = (D_A^{(g)})^{-1/2} A^{(g)} (D_A^{(g)})^{-1/2}$, for $g = 1, 2$, where $D_A^{(g)}$ is the diagonal matrix consisting of the diagonal entries of $A^{(g)}$. As such, we test

$$H_0 : R_A^{(1)} = R_A^{(2)} \text{ versus } H_1 : R_A^{(1)} \neq R_A^{(2)}.$$  \hfill (1)

This is referred to as the two-sample hypothesis test of the equality of the correlation matrices of the two groups with matrix-valued observations.

Furthermore, it is also of interest to test whether the columns of the matrix-valued observations are uncorrelated. This is important because, if indeed there is no temporal correlation, then the vector-based approach can be implemented. This goal can be achieved by testing

$$H_{0,B,g} : B^{(g)} \text{ is diagonal versus } H_{1,B,g} : B^{(g)} \text{ is not diagonal,}$$  \hfill (2)

within group $g$. Similarly, non-correlation of the industries within either group can also be tested via $H_{0,A,g} : A^{(g)}$ is diagonal. These are referred to as the one-sample hypothesis test of the non-correlation of the columns, or rows, respectively, of the matrix-valued observations.

Moreover, when the null hypothesis of the one-sample hypothesis test is rejected, it is of further interest to identify which months or which industries have non-zero correlations; similarly, when the null hypothesis of the two-sample hypothesis test is rejected, it is important to further identify which industries have significantly different correlations among emerging countries versus developed countries. These are referred to as support recovery problems.

As a prelude, in our real data application, the one-sample null hypothesis, $H_{0,B,g} : B^{(g)}$ is diagonal, is rejected by our method introduced in Section 2, suggesting the existence of significant temporal correlation. This implies that we cannot use the aforementioned vector-based two-sample tests, and manifests the need of developing a method for the two-sample hypothesis test directly using matrix-valued observations (Section 3). According to our analysis (Section 6), the two-sample null hypothesis, $H_0 : R_A^{(1)} = R_A^{(2)}$, is also rejected, so we indeed identify significant differences in correlations across the two groups of countries. Furthermore, our support recovery analysis finds consistent evidence with existing economic propositions while vector-based method ignoring the temporal correlation contradicts with economic theories.
1.1 Literature Review

For the estimation and inference on the covariance/correlation/precision matrix, there have been numerous efforts and the list below is far from being comprehensive. A succinct summary for vector-valued observations is in place. From the aspect of estimation, a number of methods were proposed to estimate the covariance/correlation matrix (Bickel and Levina, 2008a; Rothman et al., 2009; Cai and Liu, 2011; Cai and Zhou, 2012; Han and Liu, 2013; Cai and Zhang, 2016) e.g.). Meanwhile, various methods of estimating the precision matrix have also been proposed (Meinshausen and Bühlmann, 2006; Yuan and Lin, 2007; Friedman et al., 2008; Ravikumar et al., 2011; Cai et al., 2011) e.g.), and some other works extend the single precision matrix to multiple precision matrices (Danaher et al., 2014; Zhu et al., 2014; Cai et al., 2016) e.g.). From the other aspect of inference, hypothesis testing procedures for vector data have been developed recently. In particular, Cai and Jiang (2011); Li and Chen (2012); Cai et al. (2013); Cai and Ma (2013); Cai and Zhang (2016); Chang et al. (2017); Zheng et al. (2019), for example, considered the one-sample or two-sample covariance/correlation matrix testing problem in high-dimensions. To investigate the graphical models, Liu (2013) and Xia et al. (2015), for example, proposed procedures to test the property of the precision matrix under one-sample or two-sample settings.

For matrix-valued observations, to the best of our knowledge, most of the existing works require matrix normal distribution, which combines the Kronecker product model and normality. Under this distribution, Leng and Tang (2012); Yin and Li (2012); Zhou (2014); Qiu et al. (2016); Han et al. (2016); Zhu and Li (2018) proposed methods to estimate the precision matrix and inspect the graphical structure; Xia and Li (2017, 2018) studied the one-sample and two-sample hypothesis testing of the structure of the precision matrices respectively; Dawid (1981); Dutilleul (1999); Werner et al. (2008); Hoff (2011) considered estimation of covariance matrices.

Table I summarizes the status of literature related to the hypothesis testing on covariance or precision matrix for both vector-valued and matrix-valued data under one-sample and two-sample regimes. The table in Online Supplement A.5 is more comprehensive with key references listed for each scenario. This article fills in the blank of hypothesis testing of correlation structures for the matrix-valued observations. There is a pressing need for such hypothesis testing in various finance and economics studies, such as the portfolio management challenge faced by a global investor.

In addition, for matrix-valued observations, the previous literature all adopted matrix normal
distribution, while we extend the framework to matrix sub-Gaussian distribution (see Section 4 for the details) for theoretical development. Technically, such extension makes the article considerably different from those methods that aim at testing of precision/partial correlation matrices and are thus highly dependent on the Gaussian assumption. As a result, the technical tools are much more involved and are not directly available in the literature due to the non-equivalence of “uncorrelated” and “independent”. A variation of Hanson-Wright inequality is derived, which is of separate interest to the literature of matrix-valued and tensor-valued observations in general.

Numerically, our proposed methods are more advantageous than the twist of precision matrix based methods: under the one-sample normal setting, our methods are similar to those existing ones statistically; under the two-sample normal setting, our methods are more powerful than them; under the one-sample and two-sample heavy-tailed settings, existing methods cannot control the size while our methods can; lastly, we offer much better computational performance in all settings.

|                 | Vector-valued data | Matrix-valued data |
|-----------------|--------------------|--------------------|
| Covariance or   | One-sample         | √                  |
| Correlation matrix | Two-sample        | √                  |
| Precision matrix | One-sample         | √                  |
|                 | Two-sample         | √                  |

Table 1: Summary of the literature on the hypothesis testing for both vector-valued and matrix-valued data under one-sample and two-sample regimes.

1.2 The Kronecker Product Model

Matrix-valued or tensor-valued data are ubiquitous nowadays. When dealing with such data, and sometimes even vector-valued data, Kronecker product structure has been a powerful tool because of its ability to approximate an arbitrary matrix (Cai et al., 2019) and reduce dimensionality. Hafner et al. (2020) used Kronecker product to approximate the covariance matrix for vector-valued data and aimed to estimate the approximated covariance matrix. Chen et al. (2020) investigated matrix autoregressive models where the coefficient matrix has Kronecker product structure. For tensor-valued time series, Wang et al. (2019); Chen et al. (2019a); Chen and Chen (2019); Chen et al. (2019b) assumed that the tensor factor model has a signal that exhibits Kronecker structure.
Aston et al. (2017); Constantinou et al. (2017) performed a test of the separability of terms in the Kronecker product. Molstad and Rothman (2019) proposed an algorithm to fit the linear discriminant analysis model with Kronecker product. There are countless works on tensor regression and tensor decomposition with Kronecker product. These articles demonstrate a wide range of applications in finance, economics, engineering, neuroimaging, geophysics, and many more.

The Kronecker product structure naturally arises when data have multiplicative structure. Suppose for a generic country, its return follows the interactive effect model of Bai (2009), \( x_{ij} = u_i v_j \), where \( x_{ij} \) is the return of industry \( i \) at time \( j \). This is equivalent to matrix \( X = uv' \) where \( u = (u_1, \ldots, u_p)' \) and \( v = (v_1, \ldots, v_q)' \), which implies that vec(\( X \)) = \( v \otimes u \). Suppose both \( u \) and \( v \) are random, mean zero, and independent of each other, then Cov(vec(\( X \))) = Cov(v) \otimes Cov(u). Here \( u \) is the industry factor and \( v \) is the time series factor. Hence the covariance of vec(\( X \)) is separable and consists of the product of cross-sectional industry dependence and temporal dependence. The argument above applies to other applications whenever the multiplicative random factor model holds. More generally, there might be multiple independent components/factors, in which case the covariance matrix can be decomposed into the summation of Kronecker products. Such extension remains an interesting and open question.

The following two concrete examples offer more insights on the applicability of the Kronecker product model in asset pricing studies. Brandt and Santa-Clara (2006) consider augmented asset space in optimal portfolio problem. Then solutions to the optimal portfolio weights naturally involve estimates of Kronecker product of covariance of returns and covariance of firm characteristics. Brandt et al. (2009) specify portfolio weights as linear functions of firm characteristics. To accommodate possible time variation in the coefficients of the portfolio policy, the impact of the characteristics on the portfolio weight may be allowed to vary with the realization of the state variables. Again, with the independence between firms characteristics and business cycle variables, estimating the portfolio weights also involves Kronecker product of covariances of these two sets of variables.

The random multiplicative factor model is only a sufficient, not necessary, condition. For example, Hafner et al. (2020) estimates a Kronecker product of covariance matrix of S&P500 stock returns where an apparent Kronecker structure is clearly missing. Still, it is interesting to see that they find an under-specified model with Kronecker structure performs as well as conventional
shrinkage estimator’s (Ledoit and Wolf, 2004) when building minimum variance portfolios. In such case, the Kronecker product of the covariance matrix can be viewed as an approximation of the true yet high dimensional covariance matrix.

The Kronecker product assumption in the stock market application mentioned above is indeed verified via the hypothesis testing method on separability of Aston et al. (2017). See Section 6.2 for the details.

1.3 Roadmap

The rest of the article is organized as follows. Section 2 is devoted to the one-sample global hypothesis testing on the non-correlation of the columns or the rows of matrix-valued observations and the recovery of the dependent entries when the global hypothesis test is rejected. Section 3 is dedicated to the two-sample global hypothesis testing of the equality of two correlation matrices along one dimension of the matrix-valued observations (in two groups), and the support recovery of the difference of the two correlation matrices. Section 4 establishes the theoretical properties of these procedures on hypothesis tests for both one-sample and two-sample settings. The numerical comparison of our procedures with existing ones via simulation is provided in Section 5 and the real data analysis of the aforementioned stock returns data is given in Section 6. Section 7 concludes. More theorems on support recovery methods, the proofs, and additional simulations are delegated to the Online Supplement.

2 One-Sample Testing of Non-correlation

To formulate the stock return example in terms of the matrix-valued two-sample hypothesis testing problem as introduced in [1], we shall first check whether the non-correlation assumption holds for the temporal dimension. Hence, we start with the one-sample testing of (2), which is easier to comprehend due to its simple structure and notation. We omit the superscript that denotes the group membership.

Suppose there are \( n \) i.i.d. centered random matrix-valued observations \( \{X_1, ..., X_n\} \), each with dimension \( p \times q \), from the matrix sub-Gaussian distribution as defined in Section 4 with the \( p \times p \) matrix \( A \) and \( q \times q \) matrix \( B \) being the covariance matrices associated with the rows and columns respectively. The vectorization \( \text{vec}(X_k) \) is a vector of length \( pq \) following a multivariate
sub-Gaussian distribution with mean zero and a covariance matrix of the form $\Sigma = \sigma^2 B \otimes A$. Denote $A = (a_{i,j})_{p \times p}$ and $B = (b_{i,j})_{q \times q}$. Without loss of generality (WLOG), we derive the testing procedure below for testing non-correlation relating to the matrix $A$ and assume that $\sigma = 1$. Note that we can simply transpose the observation $X_k$ so that the roles of $A$ and $B$ are switched and the procedure to test $A$ can be used to test $B$ after the transpose.

Our goals are to test the null hypothesis globally

$$H_0: \ A \text{ is diagonal versus } H_1: \ A \text{ is not diagonal},$$

and to identify nonzero entries $a_{i,j} \neq 0$, both of which are invariant up to a constant. As such, even though $A$ and $B$ are not identifiable as $c^{-1}A$ and $cB$ will lead to the same distribution for any positive scalar $c$, this has no effect on the global hypothesis testing procedure of Section 2.1 and the support recovery approach of Section 2.2. Throughout the paper, we use $c, c', c_0, c_1$ to denote constants whose values may change from line to line.

### 2.1 Global Testing Procedure

To test the property of $A$, it is necessary to construct the test statistic based on an estimate of $A$. A naive estimate of $A$ is $\hat{A} = \frac{1}{nq} \sum_{k=1}^{n} X_k X_k' = \frac{1}{nq} \sum_{k=1}^{n} \sum_{l=1}^{q} X_k,l X_k,l'$, where $X_k,l$ denotes the $l$-th column of matrix $X_k$. This naive estimate is the same as the sample covariance matrix for vector-valued observations if we treat $X_k,l$, for $k = 1, \ldots, n$ and $l = 1, \ldots, q$, as $nq$ i.i.d. observations. But note that these $nq$ observations are only uncorrelated when $B$ is a multiple of an identity matrix, which requires no temporal correlation. According to the structural assumption, the covariance matrix of any column $X_k,l$ is proportional to the matrix $A$. It follows that, for the naive estimate $\hat{A}$, there exists a constant $c > 0$ such that $\hat{A}/c$ is an unbiased estimate of $A$. Similarly, $\hat{B}/c'$ is an unbiased estimate of $B$ with a proper $c'$, where

$$\hat{B} = \frac{1}{np} \sum_{k=1}^{n} X_k' X_k.$$

(4)

However, the above naive estimation is not efficient and can be improved further as follows.

Consider $Z_k = X_k B^{-1/2}$, for $k = 1, \ldots, n$, where right-multiplying matrix $B^{-1/2}$ can be thought of as a step of pre-whitening. Because of the assumed structure, we obtain that all of the columns of $Z_k$ are uncorrelated with covariance $A$. Therefore, when $B$ is known, $\frac{1}{nq} \sum_{k=1}^{n} Z_k Z_k'$ is the most
efficient and oracle estimate of $A$. In practice, $B$ is often unknown, in which case, plugging in a legitimate estimate of $B$, such as $\tilde{B}/c'$, is a natural approach, which leads to the following estimate of $A$,

$$\hat{a}_{i,j} =: \hat{A} = \frac{1}{nq} \sum_{k=1}^{n} X_k (\tilde{B}/c')^{-1} X_k' ,$$

(5)

where $\tilde{B}$ is defined in (4). Note that when $np > q$, $\tilde{B}$ defined above is invertible with probability one.

To test whether $A$ is diagonal in (3), it is tempting to consider the magnitudes of all the off-diagonal entries of $\hat{A}$ in (5). However, they cannot be used directly because of different levels of variability. To make them comparable, it is necessary to standardize. Since (5) can be re-expressed as $\hat{A} = \frac{1}{nq} \sum_{k,l} \left( X_k (\tilde{B}/c')^{-1/2} \right)_i \left( X_k (\tilde{B}/c')^{-1/2} \right)_{i,l}'$, it has the oracle counterpart when $B$ is known $\hat{A}^o = \frac{1}{nq} \sum_{k,l} \left( X_k B^{-1/2} \right)_i \left( X_k B^{-1/2} \right)_{i,l}'$, whose entries are $\hat{a}_{i,j}^o = \frac{1}{nq} \sum_{k,l} \left( X_k B^{-1/2} \right)_i \left( X_k B^{-1/2} \right)_{j,l}'.

Then, it is natural to define the relevant population variances as

$$\theta_{i,j} = \text{Var}\left( (X_k B^{-1/2})_{i,l} (X_k B^{-1/2})_{j,l} \right) = \text{Var}\left( (Z_k)_{i,l} (Z_k)_{j,l} \right),$$

(6)

for all $i, j$. Note that the definition of $\theta_{i,j}$ above does not depend on $l = 1, \ldots, q$ nor $k = 1, \ldots, n$. The sample estimates of these variances can be obtained by

$$\hat{\theta}_{i,j} = \frac{1}{nq} \sum_{k=1}^{n} \sum_{l=1}^{q} \left( (X_k (\tilde{B}/c')^{-1/2})_{i,l} (X_k (\tilde{B}/c')^{-1/2})_{j,l} - \hat{a}_{i,j} \right)^2 .$$

(7)

So the variance of $\hat{a}_{i,j}$ can be estimated by $\hat{\theta}_{i,j}/(nq)$. Similar spirit of the estimation of the variances has been used in Cai and Liu (2011) and Cai et al. (2013), where the observations are vector-valued and do not need pre-whitening or the plugged-in estimate $\tilde{B}/c'$, while ours are matrix-valued and the estimation is more involved.

The standardized statistics are readily defined as

$$M_{i,j} = \frac{\hat{a}_{i,j}^2}{\hat{\theta}_{i,j}/(nq)} , \quad 1 \leq i < j \leq p,$$

(8)

where $\hat{a}_{i,j}$ and $\hat{\theta}_{i,j}$ are defined in (5) and (7) respectively. The $M_{i,j}$'s are on the same scale and can be compared together. It is also seen that $M_{i,j}$ doesn’t depend on $c'$ as the constant $c'$ in the numerator and denominator of (8) is cancelled. WLOG, we set $c' = 1$ for the rest of the article.

Note that the null hypothesis $H_0 : A$ is diagonal is equivalent to $H_0 :$ all of the off-diagonal entries of $A$ are zero, and hence further equivalent to $H_0 :$ the maximum of all the off-diagonal
entries is zero, i.e., $H_0 : \max_{1 \leq i < j \leq p} |a_{ij}| = 0$. Therefore, it is natural to construct the following test statistic,

$$M_n = \max_{1 \leq i < j \leq p} M_{i,j}, \quad (9)$$

where $M_{i,j}$ is the standardized statistic for the $i,j$-th entry in (8). Under the alternative hypothesis, there exists at least one non-zero off-diagonal entry $a_{i,j} \neq 0$, whose associated statistic $M_{i,j}$ is large, and the maximum test statistic $M_n$ will be large. Therefore, the null hypothesis should be rejected for large value of the test statistic $M_n$.

To perform hypothesis test based on the test statistic $M_n$, we further need to establish its null distribution. The exact theoretical property of its limiting behavior will be discussed in details in Section 4. For now, we can still obtain some intuition of the critical value. Roughly speaking, under the null hypothesis, each $M_{i,j}$ is approximately the square of a standard normal random variable due to standardization, and under certain conditions, the $M_{i,j}$’s are only weakly correlated with each other. So loosely speaking, the test statistic $M_n$ is the maximum of $\binom{p}{2}$ squared normals that are weakly dependent. Since the extreme value of the square of $n$ i.i.d. normal random variables is close to $2 \log n$, $M_n$ is close to $2 \log \binom{p}{2} \approx 4 \log p$ under $H_0$. To be precise, theorems in Section 4 will show rigourously that under the null distribution $H_0$ and certain regularity assumptions, $M_n - 4 \log p + \log \log p$ converges to a Gumbel distribution. Due to this limiting distribution, for any significance level $0 < \alpha < 1$, we can define the global test $\Phi_\alpha$ by

$$\Phi_\alpha = I(M_n \geq q_\alpha + 4 \log p - \log \log p), \quad (10)$$

where $I(\cdot)$ is the indicator function. Here, the quantity

$$q_\alpha = -\log(8\pi) - 2 \log \log(1 - \alpha)^{-1}, \quad (11)$$

is the $1 - \alpha$ quantile of the Gumbel distribution with the cumulative distribution function (cdf) $\exp(-(8\pi)^{-1/2} \exp(-x/2))$. The null hypothesis $H_0 : A$ is diagonal is rejected whenever $\Phi_\alpha = 1$.

**Remark 1.** There are many appropriate choices for the estimation of $B$ besides the simple sample estimator as long as it satisfies the equation (41) in our proof. This may lead us to use, for example, the banded estimator in Rothman et al. (2010), the adaptive thresholding estimator in Cai and Liu (2011), etc., if we have the prior information on the structure of $B$. 

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Remark 2. Note that we define the test statistic $M_n$ based on the standardized statistics $M_{i,j}$. The standardization achieves two purposes simultaneously: first, it eliminates the necessity of estimation of $c'$, even though the Kronecker product is only identifiable up to a constant; second, the statistics $M_{i,j}$’s are scale-free and maximum of them can be taken. The current definition of $M_{i,j}$ is the ratio of the squared covariance estimate $\hat{\sigma}_{i,j}^2$ and its associated variance estimate. An alternative definition is the ratio of the squared correlation estimate $\hat{r}_{i,j}^2 = \hat{\sigma}_{i,j}^2 / (\hat{\sigma}_{i,i} \hat{\sigma}_{j,j})$ and its corresponding variance estimate $\hat{\theta}_{i,j} / (nq) = \hat{\theta}_{i,j} / (\hat{\sigma}_{i,i} \hat{\sigma}_{j,j}) / (nq)$, which is exactly the same as the covariance version of the definition due to cancellation of $\hat{\sigma}_{i,i} \hat{\sigma}_{j,j}$ in both the numerator and denominator. Therefore, for the one-sample case, the covariance version and the correlation version are equivalent. Nevertheless, as a prelude, in Section 3 on the two-sample test of the correlation matrix equality, the standardized statistics are defined with the correlation version, not the covariance version. This is because we are interested in the equality of the two correlation matrices, and in the two-sample case the two versions of definitions are not equivalent any more since difference is taken before standardization.

Remark 3. Since $M_n$ is the maximum of $M_{i,j}$, the test $\Phi_\alpha$ is best suited for the case when the alternative hypothesis is sparse, that is, when only a small number of the off-diagonal entries of the covariance matrix are large. As long as one of the off-diagonal entries is large enough, the test will reject the null hypothesis. This test does not assume any other structure of the alternative hypothesis. In Section 4, we will show that this test is optimal against sparse alternatives. Note that, when the alternative is dense and many small off-diagonal entries exist, the proposed test $\Phi_\alpha$ is less capable of rejecting the null. Nevertheless, the large body of literature on portfolio construction typically assumes i.i.d excess returns and all serial correlations are zero (for a survey, see Brandt (2009)). In practice, the temporal correlations are more apparent in daily or even weekly returns due to non-synchronous trading or the bid-ask bounce effect, but much less so at monthly frequency so most of them may not be different from zero (Campbell et al. 1997).

Remark 4. For simplicity of notation, we have assumed that the observations $\{X_1, \cdots, X_n\}$ are from distributions with mean zero. The proposed methods in this article can be easily extended to the cases with nonzero means (Chen and Liu, 2019, e.g.).
2.2 Support Recovery Procedure

We have focused on the test of the non-correlation of the rows of \( X_k \) by testing globally whether all of the off-diagonal entries of the row covariance matrix \( A \) are zeros in Section 2.1 If the null hypothesis is rejected, it is of great value to locate the places where the covariances are not zero. Taking the stock return data for example, if the non-correlation of the months is rejected (the matrix-valued observations need to be transposed before feeding into the testing procedure), one may want to identify which months are highly correlated, and if the non-correlation of industries is rejected, it might be interesting to know which industries are correlated. Another example is brain imaging analysis, where the matrix-valued observations for patients are spatial-temporal data (Xia and Li, 2017, 2018 e.g.), and it is worthwhile investigating further how voxels of the brains are correlated after the rejection of non-correlation of voxels.

This problem of support recovery can be thought of as simultaneous testing of whether the off-diagonal entries of the covariance matrix \( A \) are zero. Let the support of \( A \), neglecting the diagonal entries, be
\[
\Psi = \Psi(A) = \{(i, j) : a_{i,j} \neq 0, 1 \leq i < j \leq p\}.
\]

The same intuition that leads to (10) can be used to construct the support estimate with the following threshold,
\[
\hat{\Psi}(\tau) = \{(i, j) : M_{i,j} \geq \tau \log p, \ 1 \leq i < j \leq p\},
\]
where the \( M_{i,j} \)'s are previously defined in (8), and \( \tau \) is a threshold constant. Online supplement Section A.1 will show that: when \( \tau = 4 \), the probability of exact recovery goes to 1 asymptotically if the nonzero entries are large enough; a smaller choice \( \tau < 4 \) will fail to recover the support under certain conditions; therefore \( \tau = 4 \) is optimal.

Remark 5. We aim for the asymptotic exact recovery of the support in this and the following sections for the matrix-valued scenarios. The direct application of vector-valued support recovery methods in existing literatures will lead to poor results as shown in the simulation section.

3 Two-Sample Testing of Correlation Matrix Equality

Having derived the procedure for the (one-sample) testing of non-correlation, we can extend the approach to the two-sample scenario of testing the equality of two correlation matrices. Following
the same notation as in the introduction, we have i.i.d. matrix-valued observations from matrix sub-Gaussian distribution for two groups $g = 1, 2$. Considering the definition of the correlation matrices for the two groups in the introduction, we wish to test

$$H_0^* : \mathbf{R}_A^{(1)} = \mathbf{R}_A^{(2)} \text{ versus } H_1^* : \mathbf{R}_A^{(1)} \neq \mathbf{R}_A^{(2)}. \quad (14)$$

Hereafter, we use the superscript $^*$ to distinguish the quantities that are of relevance to the two-sample case from the one-sample case.

Given the centered observations $\{\mathbf{X}_1^{(1)}, \cdots, \mathbf{X}_{n_1}^{(1)}\}$ and $\{\mathbf{X}_1^{(2)}, \cdots, \mathbf{X}_{n_2}^{(2)}\}$, as discussed for the one-sample case in Section 2, we can construct the estimates of the covariance and correlation matrices for the two groups separately,

$$\hat{a}_{i,j}^{(g)} = \frac{1}{n_g q} \sum_{k=1}^{n_g} \mathbf{X}_k^{(g)} (\hat{\mathbf{B}}^{(g)})^{-1} (\mathbf{X}_k^{(g)})', \quad (15)$$

$$\hat{r}_{i,j}^{(g)} = \frac{\hat{a}_{i,j}^{(g)}}{\left(\hat{a}_{i,i}^{(g)} \hat{a}_{j,j}^{(g)}\right)^{1/2}}, \quad (16)$$

where $\hat{\mathbf{B}}^{(g)} = \frac{1}{n_g p} \sum_{k=1}^{n_g} (\mathbf{X}_k^{(g)})' (\mathbf{X}_k^{(g)})$ is the naive estimate of $\mathbf{B}^{(g)}$. Again, we cannot directly make inference based on $\hat{r}_{i,j}^{(1)} - \hat{r}_{i,j}^{(2)}$, because they are heteroscedastic. To make them homoscedastic, define the entry-wise population variance and the sample counterpart similarly as in (6) and (7),

$$\theta_{i,j}^{(g)} = \text{Var} \left( \mathbf{X}_k^{(g)} (\mathbf{B}^{(g)})^{-1/2} i,l (\mathbf{X}_k^{(g)} (\mathbf{B}^{(g)})^{-1/2} j,l) \right),$$

$$\hat{\theta}_{i,j}^{(g)} = \frac{1}{n_g q} \sum_{k=1}^{n_g} \sum_{l=1}^{q} \left[ \mathbf{X}_k^{(g)} (\hat{\mathbf{B}}^{(g)})^{-1/2} i,l (\mathbf{X}_k^{(g)} (\hat{\mathbf{B}}^{(g)})^{-1/2} j,l) - \hat{\theta}_{i,j}^{(g)} \right]^2.$$

As such, the variance of $\hat{r}_{i,j}^{(g)}$ can be estimated by $\hat{\theta}_{i,j}^{(g)}/(n_g q)$, where $\hat{\theta}_{i,j}^{(g)} = \hat{\theta}_{i,j}^{(g)}/\hat{\theta}_{i,i}^{(g)} \hat{\theta}_{j,j}^{(g)}$. Consequently, the variance of $\hat{r}_{i,j}^{(1)} - \hat{r}_{i,j}^{(2)}$ can be estimated by $\hat{\theta}_{i,j}^{(1)}/(n_1 q) + \hat{\theta}_{i,j}^{(2)}/(n_2 q)$. Note that, for vector-valued observations, to test the equality of the correlations from two populations, Cai and Zhang (2016) estimated the variance by a careful investigation of the Taylor expansion in the calculation of correlation from covariance, and Cai and Liu (2016) introduced a variance stabilization method based on Fisher’s $z$-transformation. Our approach is different from both methods.

When we focus on a single entry of the hypothesis in (14) such as $\hat{r}_{i,j}^{(1)} = \hat{r}_{i,j}^{(2)}$, in accordance with the two-sample $t$-test with unequal variances for i.i.d. random variables, it is natural to define
the standardized statistic as
\[ M_{i,j}^* = \frac{\left( \hat{r}_{i,j}^{(1)} - \hat{r}_{i,j}^{(2)} \right)^2}{\hat{\sigma}_{i,j}^{(1)} / (n_1q) + \hat{\sigma}_{i,j}^{(2)} / (n_2q)} , \] (17)
and the maximum test statistic as
\[ M_n^* = \max_{1 \leq i < j \leq p} M_{i,j}^*. \] (18)

Because the diagonal entries of the correlation matrix are all 1, the maximum is only taken over off-diagonal entries. The rest of the two-sample case proceeds exactly the same as the one-sample case. The \( M_n^* \) in the two-sample scenario has similar properties as the \( M_n^0 \) in the one-sample scenario. Section 4 proves that \( M_n^* - 4 \log p + \log \log p \) also converges to a Gumbel distribution under \( H_0^* \) and certain regularity assumptions. Therefore, for a given significance level \( 0 < \alpha < 1 \), the test \( \Phi_0^* \) can be defined in parallel as (10),
\[ \Phi_0^* = I(M_n^* \geq q_\alpha + 4 \log p - \log \log p). \] (19)

The hypothesis \( H_0^* : R_A^{(1)} = R_A^{(2)} \) is rejected whenever \( \Phi_0^* = 1 \).

To find which industries have correlations that are significantly different between emerging countries and developed countries, we need to recover the support of the difference of the correlation matrices between the two groups of countries. Denote the support of \( R_A^{(1)} - R_A^{(2)} \) by
\[ \Psi^* = \Psi^*(R_A^{(1)}, R_A^{(2)}) = \{ (i,j) : \hat{r}_{i,j}^{(1)} \neq \hat{r}_{i,j}^{(2)}, 1 \leq i < j \leq p \}. \] (20)

We threshold the entry-wise statistic \( M_{i,j}^* \) in (17) at an appropriate level to obtain the estimated support as
\[ \hat{\Psi}^*(\tau) = \{ (i,j) : M_{i,j}^* \geq \tau \log p, \ 1 \leq i < j \leq p \}, \] (21)
where the optimal choice of the threshold constant is \( \tau = 4.0 \).

4 Theoretical Properties

We present the theoretical properties of the testing procedures for the one-sample case in Section 4.1 and the two-sample case in Section 4.2. The theorems on support recovery are delegated to Online Supplement.
The following notational conventions are adopted. For a length $p$ vector $\mathbf{a} = (a_1, \ldots, a_p)' \in \mathbb{R}^p$, denote its Euclidean norm by $\| \mathbf{a} \|_2 = \sqrt{\sum_{j=1}^{p} a_j^2}$. For a size $p \times q$ matrix $\mathbf{A} = (a_{i,j}) \in \mathbb{R}^{p \times q}$, denote its Frobenius norm by $\| \mathbf{A} \|_F = \sqrt{\sum_{i,j} a_{i,j}^2}$ and its spectral norm by $\| \mathbf{A} \|_2 = \sup_{\| \mathbf{x} \|_2 \leq 1} \| \mathbf{A} \mathbf{x} \|_2$. For a matrix $\mathbf{A} \in \mathbb{R}^{p \times p}$, let $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$ be its largest and smallest eigenvalues respectively. Denote its matrix 1-norm as $\| \mathbf{A} \|_{L_1} = \max_{1 \leq j \leq p} \sum_{i=1}^{p} |a_{i,j}|$. For two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, write $a_n = O(b_n)$ (respectively $a_n \sim b_n$) if there exists a constant $c$ such that $|a_n| \leq c |b_n|$ (respectively $1/c \leq |a_n|/|b_n| \leq c$) holds for all sufficiently large $n$ and write $a_n = o(b_n)$ if $\lim_{n \to \infty} a_n/b_n = 0$.

To deal with matrix-valued observations, an intuitive assumption of matrix normal distribution has been predominantly adopted; see for example Leng and Tang (2012); Yin and Li (2012); Zhou (2014); Qiu et al. (2016); Han et al. (2016); Zhu and Li (2018). Under such assumption, the two key ingredients are normality and the Kronecker product structure of the covariance matrix $\Sigma^{(g)} = \text{Cov}(\text{vec}(\mathbf{X}_k^{(g)})) = \sigma^2 \mathbf{B}^{(g)} \otimes \mathbf{A}^{(g)}$, $g = 1, 2$. The motivation and applicability of Kronecker product are stated in Section 1.2. The matrix normal distribution is equivalent to assuming that the pre-whitened matrix $(\mathbf{A}^{(g)})^{-1/2} \mathbf{X}_k^{(g)} (\mathbf{B}^{(g)})^{-1/2}$ has i.i.d. Gaussian entries. Unlike the aforementioned papers, this article relaxes the Gaussian assumption and studies more general settings such that the pre-whitened matrix has i.i.d. sub-Gaussian entries. Under such matrix sub-Gaussian distributions, the covariance matrix preserves the Kronecker product structure. Furthermore, the matrix sub-Gaussian distribution will reduce to the matrix normal distribution if the pre-whitened entries are normally distributed.

### 4.1 Theoretical Properties for Testing of Non-correlation

The theoretical properties of the one-sample global testing procedure will be established from two perspectives: the size and the power. Specifically, to study the asymptotic size of the test, we prove the asymptotic distribution of the test statistic under the null hypothesis; to analyze the power, we consider the sparse alternatives where only a small subset of the entries are nonzero.

The regularity conditions are as follows.

(C1) Assume that $\log p = o((nq)^{1/5})$, $np > q$, and there are some constant $c_0, c_1 > 0$ such that,

$$c_0^{-1} \leq \lambda_{\min}(\mathbf{A}) \leq \lambda_{\max}(\mathbf{A}) \leq c_0, \text{ and } c_1^{-1} \leq \lambda_{\min}(\mathbf{B}) \leq \lambda_{\max}(\mathbf{B}) \leq c_1.$$

(C2) Let $\mathbf{X}_k \overset{d}{=} \mathbf{X} =: (x_{i,j})$. Assume that the entries $x_{i,j}$ satisfy the sub-Gaussian-type assumption,
i.e., there exist some constants $\eta > 0$ and $K > 0$ such that

$$
\mathbb{E} \exp(\eta \tau_{i,j}^2) \leq K, \quad \text{for all } i, j.
$$

(C3) Assume that $q^3 \log q \log^3 \max(p, q, n) = o(np)$.

**Remark 6.** Condition (C1) on the eigenvalues of the covariance matrices is commonly assumed in the high-dimensional setting. It implies that the majority of the variables are not highly correlated with the others in either the row direction or the column direction. Intuitively, to make a valid inference for $p$ variables of interest, the dimension $p$ should not grow too fast compared to $nq$, and that is why we need the condition $\log p = o((nq)^{1/5})$ to obtain rates of similar form as $\{(\log p)/(nq)\}^{1/2}$. Condition (C2) is the moment condition on $X$ which is much weaker than the Gaussian assumption.

A few comments on Condition (C3) are in order. Firstly, Condition (C3) requires that $q$ does not grow too fast compared to $np$. Secondly, it is assumed to ensure that $\tilde{B}^{-1}$ defined in (4), as the estimator of the inverse of the nuisance covariance $B^{-1}$, is reasonably accurate. As such, the oracle estimate $\hat{A}^\circ$ will be close to the estimate $\hat{A}$ in (5) as shown in the proof. Thirdly, this is applicable to any other estimator of $B$ as long as it satisfies $\|\tilde{B} - B\|_\infty = O_p[\{\log q/(np)\}^{1/2}]$. Hence we can use, for example, the methods in Remark 1 given certain prior knowledge on the structure of $B$. One can also estimate $B^{-1}$ directly and all results still hold as long as $\|\tilde{B}^{-1} - B^{-1}\|_2 = O_p[\{\log q/(np)\}^{1/2}]$ or $\|\tilde{B}^{-1} - B^{-1}\|_\infty = O_p[\{\log q/(np)\}^{1/2}]$. This can be satisfied by many precision matrix estimates such as the CLIME estimator in Cai et al. (2011). Finally, if we have prior information on the structure of $B$, say the AR(1) model where the off-diagonal elements decay exponentially as they get further away from the diagonal as in the simulation section (i.e. $B^{-1}$ is banded), or the moving average model with banded $B$, then $\|\tilde{B}^{-1} - B^{-1}\|_2$ has a faster rate of convergence and Condition (C3) can be further relaxed to $q \log q \log^3 \max(p, q, n) = o(np)$.

Under Conditions (C1)-(C3), as mentioned in Section 2, Theorem 1 shows that $M_n - 4 \log p + \log \log p$ indeed converges weakly to a Gumbel distribution under the null hypothesis.

**Theorem 1.** Suppose that the regularity conditions (C1)-(C3) hold. Then under $H_0$, for any $t \in \mathbb{R}$,

$$
P\left(M_n - 4 \log p + \log \log p \leq t\right) \rightarrow \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{t}{2}\right)\right), $$

(22)
as \( nq, p \to \infty \). Furthermore, under \( H_0 \), the convergence in (22) is uniform for all \( \{X_k, k = 1, \ldots, n\} \) satisfying (C1)-(C3).

We next turn to the power analysis of the test \( \Phi_\alpha \). In order to perform the power analysis, we focus on sparse alternative hypothesis, as explained in Section 2.1. Define the following class of covariance matrices associated with the row direction of the matrix-valued observations:

\[
U(c) = \left\{ A = (a_{i,j})_{p \times p} : \max_{1 \leq i < j \leq p} \frac{|a_{i,j}|}{\sqrt{\theta_{i,j}/(nq)}} \geq c \sqrt{\log p} \right\},
\]

(23)

where \( \theta_{i,j} \) was defined previously in (6). Note that this class of covariance matrices only requires one element to be large enough, \( |a_{i,j}|/\sqrt{\theta_{i,j}/(nq)} \geq c \sqrt{\log p} \). As \( \theta_{i,j} = O(1) \), it essentially requires only one off-diagonal entry of \( A \) to be larger than \( c \sqrt{\log p}/(nq) \). For such matrices with \( c = 4 \) as the alternative hypothesis, Theorem 2 shows that \( \Phi_\alpha \) can distinguish the alternative hypothesis from the null hypothesis, where the off-diagonal entries of \( A \) are all zero, asymptotically. In other words, \( H_0 \) is rejected by \( \Phi_\alpha \) with probability tending to 1 if \( A \in U(4) \).

**Theorem 2.** Suppose that Conditions (C1)-(C3) hold. As \( nq, p \to \infty \), we have

\[
\inf_{A \in U(4)} P(\Phi_\alpha = 1) \to 1.
\]

Theorem 3 further demonstrates that the lower bound of \( 4\sqrt{\log p} \) in the definition of the class of covariance matrices is rate optimal. Let \( T_\alpha \) be the set of level \( \alpha \) tests, i.e., we have \( P(T_\alpha = 1) \leq \alpha \) under the null hypothesis for any test \( T_\alpha \in T_\alpha \).

**Theorem 3.** Suppose that \( \log p = o(nq) \). Let \( \alpha, \beta > 0 \) and \( \alpha + \beta < 1 \). There exists some constant \( c_0 > 0 \) such that for all sufficiently large \( nq \) and \( p \),

\[
\inf_{A \in U(c_0)} \sup_{T_\alpha \in T_\alpha} P(T_\alpha = 1) \leq 1 - \beta.
\]

The above theorem implies that, when \( c_0 \) is small enough, with probability going to one, any level \( \alpha \) test cannot reject the null hypothesis uniformly over \( U(c_0) \). As a consequence, the rate \( \sqrt{\log p} \) as the lower bound of \( |a_{i,j}|/\sqrt{\theta_{i,j}/(nq)} \) cannot be improved.

To sum up, Theorems 2 and 3 suggest that the test \( \Phi_\alpha \) defined in Section 2 has asymptotic level \( \alpha \), it has power one asymptotically under certain sparse alternative hypothesis, and the rate requirement on the sparse alternative is the weakest possible one.
4.2 Theoretical Properties for Testing of Correlation Matrix Equality

For the two-sample testing of correlations, we assume the sample sizes from the two groups are comparable, \( n_1 \approx n_2 \), and write \( n = \max(n_1, n_2) \) in this section.

The Conditions (C1)-(C3) in the one-sample case need to be replaced by the following conditions for the two-sample case.

(C1∗) Assume that \( \log p = o((nq)^{1/5}) \), \( n_q p > q \), and there are some constant \( c_0, c_1 > 0 \) such that,
\[
 c_0^{-1} \leq \lambda_{\min}(\mathbf{A}^{(g)}) \leq \lambda_{\max}(\mathbf{A}^{(g)}) \leq c_0, \quad \text{and} \quad c_1^{-1} \leq \lambda_{\min}(\mathbf{B}^{(g)}) \leq \lambda_{\max}(\mathbf{B}^{(g)}) \leq c_1, \quad \text{for } g = 1, 2.
\]

(C2∗) Let \( \mathbf{X}_k^{(g)} \equiv \mathbf{X}^{(g)} =: (x_{i,j}^{(g)}) \). Assume that the entries \( x_{i,j}^{(g)} \) satisfy the sub-Gaussian-type assumptions, i.e., there exist some constants \( \eta > 0 \) and \( K > 0 \) such that
\[
 \mathbb{E} \exp(\eta x_{i,j}^{(1)})^2 \leq K
\]
\[
 \mathbb{E} \exp(\eta x_{i,j}^{(2)})^2 \leq K, \quad \text{for all } i, j.
\]

(C3∗) Assume that \( q^3 \log q \log^3 \max(p, q, n) = o(n p) \).

(C4∗) Let \( \mathbf{X}_k^{(g)}(\mathbf{B}^{(g)})^{-1/2} : = \mathbf{Z}_k^{(g)} =: (z_{i,j}^{(g)}) \). Assume that there exist \( \kappa_1, \kappa_2 \geq \frac{1}{3} \) such that for any \( j, k, l, m \in \{1, 2, \cdots, p\} \) and \( i \in \{1, 2, \cdots, q\}, \)
\[
 \mathbb{E} z_{j,i}^{(1)} z_{k,i}^{(1)} z_{l,i}^{(1)} z_{m,i}^{(1)} = \kappa_1 (a_{j,k}^{(1)}a_{l,m}^{(1)} + a_{j,l}^{(1)}a_{k,m}^{(1)} + a_{j,m}^{(1)}a_{k,l}^{(1)}),
\]
\[
 \mathbb{E} z_{j,i}^{(2)} z_{k,i}^{(2)} z_{l,i}^{(2)} z_{m,i}^{(2)} = \kappa_2 (a_{j,k}^{(2)}a_{l,m}^{(2)} + a_{j,l}^{(2)}a_{k,m}^{(2)} + a_{j,m}^{(2)}a_{k,l}^{(2)})
\]

(C5∗) There exists some \( \gamma > 0 \) such that \( |A_{\gamma}| = o(p^{1-\nu}) \) for any sufficiently small constant \( \nu > 0 \), where the set is defined as
\[
 A_{\gamma} = \{(i, j) : |r_{i,j}^{(g)}| \geq (\log p)^{-1-\gamma}, 1 \leq i < j \leq p, \text{ for } g = 1 \text{ or } 2\}.
\]

Note that, Conditions (C1∗) and (C2∗) are the two-sample analogue of the one-sample conditions (C1) and (C2), and Condition (C3∗) is the same as Condition (C3). Condition (C4∗) holds for the elliptically contoured distributions with \( \kappa_g = \text{Kurtosis}(z_{i,j}^{(g)})/3, \ g = 1, 2 \). (C5∗) ensures that most of the variables are not highly correlated with each other.

Under appropriate regularity conditions, Theorems 4-6 are the two-sample counterparts of the one-sample Theorems 1-3. In particular, Theorem 4 shows the limiting distribution of \( M_n^{*} \) (18).
under the null hypothesis and proves that $\Phi^*_\alpha$ has level $\alpha$ asymptotically, Theorem 5 provides the power analysis of $\Phi^*_\alpha$, and Theorem 6 demonstrates the optimality of the test.

**Theorem 4.** Suppose that Conditions (C1*)-(C5*) hold. Then under $H^*_0$ in (14), for any $t \in \mathbb{R}$,

$$P(M^*_n - 4 \log p + \log \log p \leq t) \to \exp \left( - \frac{1}{\sqrt{8\pi}} \exp \left( - \frac{t}{2} \right) \right),$$

(24)
as $nq, p \to \infty$. Furthermore, under $H^*_0$, the convergence in (24) is uniform for all $\{X^{(1)}_k, k = 1, \ldots, n_1\}$ and $\{X^{(2)}_k, k = 1, \ldots, n_2\}$ satisfying (C1*)-(C5*).

To analyze the power of $\Phi^*_\alpha$, in parallel with (23), define the following class of matrices:

$$\mathcal{U}^*(c) = \left\{ (R^{(1)}_A, R^{(2)}_A) : \max_{1 \leq i < j \leq p} \frac{|r^{(1)}_{i,j} - r^{(2)}_{i,j}|}{\vartheta^{(1)}_{i,j}/(n_1q) + \vartheta^{(2)}_{i,j}/(n_2q)} \geq c \sqrt{\log p} \right\},$$

(25)

where $\vartheta^{(g)}_{i,j} = \theta^{(g)}_{i,j}/(a^{(g)}_{i,i}a^{(g)}_{j,j}).$ We have the following result.

**Theorem 5.** Suppose that Conditions (C1*) - (C3*) hold. As $nq, p \to \infty$, we have

$$\inf_{(R^{(1)}_A, R^{(2)}_A) \in \mathcal{U}^*(c)} P(\Phi^*_\alpha = 1) \to 1.$$

Note that $\Phi^*_\alpha$ is able to distinguish the alternative from the null so long as one entry satisfies the requirement $|r^{(1)}_{i,j} - r^{(2)}_{i,j}|/(\vartheta^{(1)}_{i,j}/(n_1q) + \vartheta^{(2)}_{i,j}/(n_2q))^{1/2} \geq 4\sqrt{\log p}$.

The above rate is optimal because of the next theorem. Let $T^*_\alpha$ be the set of all $\alpha$-level tests, i.e., $P(T_\alpha = 1) \leq \alpha$ under $H^*_0$ for any $T_\alpha \in T^*_\alpha$.

**Theorem 6.** Suppose that $\log p = o(nq)$. Let $\alpha, \beta > 0$ and $\alpha + \beta < 1$. There exists some constant $c_0 > 0$ such that for all large $nq$ and $p$,

$$\inf_{(R^{(1)}_A, R^{(2)}_A) \in \mathcal{U}^*(c_0)} \sup_{T_\alpha \in T^*_\alpha} P(T_\alpha = 1) \leq 1 - \beta.$$

5 Simulation Studies

In this section, we investigate numerical performance of the proposed approaches and compare them with relevant existing procedures via simulation studies. We examine the global hypothesis test $\Phi_\alpha$ for the one-sample case under matrix normal distribution in Section 5.1 and the global test $\Phi^*_\alpha$ for the two-sample case under matrix normal distribution in Section 5.2. Online Supplement
A.4.1 provides additional hypothesis test experiment under normal distribution; Online Supplement A.4.2 lists the simulation results under $t$ distribution; Online Supplement A.4.3 is devoted to the simulation results for support recovery including $\hat{\Psi}(\tau)$ (13) and $\hat{\Psi}^*(\tau)$ (21); Pseudo simulation based on real data application is given in Online Supplement A.4.4.

5.1 One-Sample Testing of Non-correlation

Our estimators and competing methods. To perform the one-sample global hypothesis testing $H_0: A$ is diagonal in (3), we compare seven methods, three of which are ours and the remaining four are based on certain twists of existing methods. To the best of our knowledge, there is no existing literature that targets directly at the same goal as ours.

The first three are our methods described in Section 2 and their variants. To implement the test $\Phi_\alpha$ in (10), the statistics $M_{i,j}$ and $M_n$ in (8) and (9) need to be constructed and they depend on $\hat{a}_{i,j}$ and $\hat{\theta}_{i,j}$ in (5) and (7). Based on the definitions of $\hat{a}_{i,j}$ and $\hat{\theta}_{i,j}$, it is required to plug in an estimate of the covariance matrix $B$. We experiment with three choices: (i) the oracle procedure when the true $B$ is known and plugged in (denoted as “One sample cov: oracle”), (ii) the procedure when the sample estimate $\tilde{B}$ (4) is plugged in (denoted as “One sample cov: sample-est”), and (iii) the procedure when a banded estimate of $B$ (Rothman et al., 2010) is plugged in (denoted as “One sample cov: banded-est”). The “One sample cov: oracle” method can serve as a benchmark to see the effectiveness of the other two methods. When $B$ is the covariance matrix associated with temporal or spatial dimension, those measurements that are far apart in the ordering are most often weakly correlated and $B$ tends to exhibit some banded structure. For this type of data, a banded estimate of $B$ is more accurate than the naive sample estimate $\tilde{B}$ (Rothman et al., 2010; Bickel and Levina, 2008b), so “One sample cov: banded-est” is expected to perform better or no worse than “One sample cov: sample-est”.

The remaining four methods are related to existing approaches. Although they are not designed to test the covariance matrix of matrix-valued observations in (3), they can be tweaked to achieve the same goal and are hence potential competitors.

Three of them are originally designed to test the precision matrix, the inverse of the covariance matrix, of matrix-valued observations with matrix normal distribution (Xia and Li 2017). They are relevant because of the following observation: when the covariance matrix $A$ is diagonal, its
inverse $A^{-1}$, the precision matrix, is also diagonal. So we can make inference of $A$ by considering an equivalent hypothesis testing problem $H_{0,\text{precision}} : A^{-1} \text{ is diagonal versus } H_{1,\text{precision}} : A^{-1} \text{ is not diagonal.}$ The latter problem was solved by [Xia and Li (2017)]. In their approach, to construct the test statistic that is related to $A^{-1}$, it is also necessary to plug in an estimate of $B$, which then leads to three versions as well: (iv) “One sample pre: oracle”, (v) “One sample pre: sample-est” and (vi) “One sample pre: banded-est”.

The last method, (vii) “One sample vector”, is based on the covariance matrix testing approach ([Cai et al. 2013], for the vector-valued observations, where we simply ignore the temporal correlation captured by the column covariance matrix $B$ and treat the $n$ matrix-valued observations $X_k$ as $nq$ vector-valued observations $X_{k,l}$ that are i.i.d. random vectors.

**Simulation setup.** We simulate the data according to the following model, which is also adopted in [Xia and Li (2017)]. The data follow matrix normal distribution $\mathcal{MN}_{pq}(0, A, B)$. The nuisance covariance matrix $B$ has the structure of a time series autoregressive model where the off-diagonal elements decay exponentially as they get further away from the diagonal, that is, $b_{i,j} = 0.4^{|i-j|}, 1 \leq i, j \leq q$. The targeted covariance matrix $A$ is set up differently under the null and alternative hypotheses. Under the null hypothesis, to evaluate the size of the test, we set $A = I$. Under the alternative hypothesis, to evaluate the power of the test, we set $A = (I + U + \delta I)/(1 + \delta)$, where $\delta$ is a scalar with value $\delta = |\lambda_{\min}(I + U)| + 0.05$, and $U$ is a sparse and symmetric matrix with eight nonzero entries. Four of these nonzero entries are located randomly in the lower triangle of the matrix, have random magnitudes that follow uniform distribution on $[2\{\log p/(nq)\}^{1/2}, 4\{\log p/(nq)\}^{1/2}]$, and possess random positive or negative signs. This definition of $A$ can be thought of as a perturbation of $I$, which is diagonal under the null, with $U$ as the perturbation. The terms related to $\delta$ are to ensure the positive-definiteness of the covariance matrix. We examine a range of matrix dimensions and sample sizes. Specifically, combinations of $p = \{50, 200\}$, $q = \{50, 200\}$, and $n = \{10, 50\}$ are considered. 1000 replications are conducted in each configuration.

**Simulation results and interpretation.** Table 2 summarizes the empirical size and power, respectively, of the aforementioned seven methods with significance level $\alpha = 0.05$. Overall, the six methods (i)-(vi), that are based on matrix-valued observations, can control the sizes no larger than 0.05, but the vector-based approach (vii) has serious size distortion across all configurations.
Regarding to the power performance, no matter whether covariance matrix based or precision matrix based methods are considered, the “banded-est” methods are as powerful as the “oracle” ones, which is unsurprising as the covariance matrix $B$ indeed exhibits banded structure. The “sample-est” ones are typically slightly worse than the “banded-est” and “oracle” ones, which is expected as the estimation of $B$ is worse.

It is seen that “One sample cov: sample-est” is sometimes undersized. This is because the size depends on the accuracy of the estimation of $B$, which is driven by how large $np$ is compared with $q$ (recall Remark 6 on Condition (C3)). It is hence expected that when $q$ is smaller, or $n$ and $p$ are bigger, the size is closer to the nominal level. For instance, when $q = 50$, $p = 200$, $n = 50$, the size is close to 0.05 while when $q = 200$, $p = 50$, $n = 10$, the size is smaller. Additional simulation result for even larger sample size $n = 500$ is given in Table 5 in Online Supplement, which shows that the “one sample cov:sample-est” is no longer undersized. Note that the theoretical condition (C3) may be relaxed for “one sample cov: banded-est”, whose size is close to the oracle and nominal level. Moreover, this phenomenon occurs for “one sample pre:sample-est” as well because of the same reason.

It is also observed that our methods based on covariance matrix (i)-(iii) are comparable as those methods [Xia and Li 2017] based on precision matrix (iv)-(vi). This phenomenon of comparable performance can be understood as follows. The power of the covariance matrix based methods depends on the largest magnitude of the correlations, whereas the power of the precision matrix based methods depends on the largest magnitude of the partial correlations. In the model setting, the perturbation matrix $U$ is extremely sparse with only eight nonzero entries out of $50 \times 50$ or $200 \times 200$ entries with magnitude $[2\{\log p/(nq)\}^{1/2}, 4\{\log p/(nq)\}^{1/2}]$. With very high probability, the random locations of the nonzero entries will guarantee the largest magnitudes of the correlation and the partial correlation being similar to each other. Put these altogether, the levels of difficulty in detecting the nonzero off-diagonal entries of the correlation matrix and the partial correlation matrix are nearly identical, which makes (i)-(iii) and (iv)-(vi) similar in statistical performance.

However, our methods (i)-(iii) and the methods by Xia and Li (2017) (iv)-(vi) have different computational performance as shown in Table 3. When comparing (i) vs (iv), similarly (ii) vs (v) and (iii) vs (vi), we can exclude the computation time of estimating $B$ because the same estimation strategies are adopted by Xia and Li (2017) and our proposed methods. We only need to compare
Table 2: The empirical size and power of the testing procedures for the one-sample case under normal distribution based on 1000 replications. The percentages are shown with the standard errors provided in parentheses. The significant level is $\alpha = 5\%$. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n = \{10, 50\}$. 

| $p$ | methods          | $q = 50$         | $q = 200$         | $n = 10$         | $n = 50$         |
|-----|------------------|------------------|------------------|------------------|------------------|
|     | Empirical size   |                  |                  |                  |                  |
| 50  | One sample cov: oracle | 3.9(0.6) | 4.5(0.7) | 4.2(0.6) | 5.3(0.7) |
|     | One sample cov: sample-est | 1.9(0.4) | 0.1(0.1) | 3.9(0.6) | 1.8(0.4) |
|     | One sample cov: banded-est | 3.3(0.6) | 4.7(0.7) | 5.0(0.7) | 4.6(0.7) |
| 50  | One sample pre: oracle | 4.4(0.6) | 5.0(0.7) | 4.8(0.7) | 5.6(0.7) |
|     | One sample pre: sample-est | 1.5(0.4) | 0(0) | 3.8(0.6) | 1.8(0.4) |
|     | One sample pre: banded-est | 3.6(0.6) | 4.6(0.7) | 4.8(0.7) | 4.9(0.7) |
|     | One sample vector | 35.8(1.5) | 41.1(1.6) | 38.3(1.5) | 44.6(1.6) |
| 200 | One sample cov: oracle | 3.6(0.6) | 5.1(0.7) | 5.1(0.7) | 4.8(0.7) |
|     | One sample cov: sample-est | 3.4(0.6) | 1.0(0.3) | 5.3(0.7) | 3.3(0.6) |
|     | One sample cov: banded-est | 3.6(0.6) | 4.9(0.7) | 5.5(0.7) | 4.9(0.7) |
| 200 | One sample pre: oracle | 4.3(0.6) | 4.5(0.7) | 5.0(0.7) | 4.7(0.7) |
|     | One sample pre: sample-est | 2.7(0.5) | 0.9(0.3) | 4.8(0.7) | 3.3(0.6) |
|     | One sample pre: banded-est | 3.6(0.6) | 4.2(0.6) | 5.2(0.7) | 4.8(0.7) |
|     | One sample vector | 58.5(1.6) | 66.9(1.5) | 68.4(1.5) | 69.5(1.5) |
|     | Empirical power   |                  |                  |                  |                  |
| 50  | One sample cov: oracle | 84.9(1.1) | 70.0(1.4) | 67.0(1.5) | 61.1(1.5) |
|     | One sample cov: sample-est | 65.2(1.5) | 1.8(0.4) | 63.5(1.5) | 49.8(1.6) |
|     | One sample cov: banded-est | 83.0(1.2) | 67.0(1.5) | 66.1(1.5) | 59.7(1.6) |
| 50  | One sample pre: oracle | 87.4(1.0) | 70.9(1.4) | 67.9(1.5) | 60.9(1.5) |
|     | One sample pre: sample-est | 67.8(1.5) | 2.6(0.4) | 64.0(1.5) | 41.9(1.6) |
|     | One sample pre: banded-est | 84.7(1.1) | 67.3(1.5) | 67.2(1.5) | 60.0(1.5) |
|     | One sample vector | 90.3(0.9) | 82.7(1.2) | 81.6(1.2) | 80.6(1.3) |
| 200 | One sample cov: oracle | 92.5(0.8) | 72.8(1.4) | 70.4(1.4) | 58.6(1.6) |
|     | One sample cov: sample-est | 89.1(1.0) | 46.1(1.6) | 68.5(1.5) | 53.8(1.6) |
|     | One sample cov: banded-est | 92.1(0.9) | 72.0(1.4) | 70.4(1.4) | 58.8(1.6) |
| 200 | One sample pre: oracle | 94.9(0.7) | 74.1(1.4) | 70.8(1.4) | 58.7(1.6) |
|     | One sample pre: sample-est | 92.6(0.8) | 48.4(1.6) | 69.6(1.5) | 54.1(1.6) |
|     | One sample pre: banded-est | 94.3(0.7) | 72.7(1.4) | 71.2(1.4) | 58.5(1.6) |
|     | One sample vector | 97.1(0.5) | 91.7(0.9) | 91.6(0.9) | 87.5(1.0) |
Table 3: The average computation time of method (i) “One sample cov: oracle” and (iv) “One sample pre: oracle” in seconds per replication for the one-sample case under normal distribution. The number of observations and the dimensions of the matrices vary: \(p = \{50, 200\}\), \(q = \{50, 200\}\), and \(n = \{10, 50\}\).

| \(p\) | methods                      | \(q = 50\) | \(q = 200\) | \(q = 50\) | \(q = 200\) |
|-------|------------------------------|-------------|-------------|-------------|-------------|
| 50    | One sample cov: oracle       | 0.04        | 0.12        | 0.24        | 0.46        |
|       | One sample pre: oracle       | 0.09        | 0.41        | 0.58        | 1.90        |
| 200   | One sample cov: oracle       | 0.70        | 2.22        | 6.61        | 12.85       |
|       | One sample pre: oracle       | 1.85        | 7.38        | 10.70       | 35.46       |

the computation time of the testing procedures with given \(B\), either known or estimated. As such, Table 3 only presents the average computation times of (i) and (iv) in seconds per replication. Our method is much faster than the method of Xia and Li (2017). The computational advantage becomes even more pronounced as the dimensions increase. This is expected as Xia and Li (2017) performs column-wise LASSO which is time-consuming. Given the similar statistical performance and different computation performances, our covariance matrix based methods are preferred.

5.2 Two-Sample Testing of Correlation Matrix Equality

Our estimators and competing methods. To perform the two-sample global hypothesis testing \(H_0^* : \mathbf{R}_A^{(1)} = \mathbf{R}_A^{(2)}\) in (14), we compare seven methods. These include our method \(\Phi^*_\alpha\) proposed in Section 3, where the covariance matrices \(\hat{\mathbf{B}}^{(g)}\) from both groups can be the ground truth, the sample estimate, or the banded estimate, which again lead to three approaches to be compared: (i) “Two sample cov: oracle”, (ii) “Two sample cov: sample-est”, and (iii) “Two sample cov: banded-est”.

Similar to the one-sample case in Section 5.1, we can also twist the method of Xia and Li (2018) which was initially designed for the two-sample partial correlation matrix test under matrix normal distribution. To draw the connection, recall the model setup \(X_k^{(g)} \sim \mathcal{MN}_{pq}(0, \mathbf{A}^{(g)}, \mathbf{B}^{(g)})\). Xia and Li (2018) is interested in the precision matrices \(\Omega_A^{(g)} = (\mathbf{A}^{(g)})^{-1}\). The precision matrices are not identifiable either under the Kronecker product model. So the attention was given to the partial correlation matrix \(\mathbf{P}_A^{(g)} = (\mathbf{D}_A^{\text{precision}})^{-1/2} \Omega_A^{(g)} (\mathbf{D}_A^{\text{precision}})^{-1/2}\), where \(\mathbf{D}_A^{\text{precision}}\) is a diagonal matrix consisting of the diagonal entries of the precision matrix \(\Omega_A^{(g)}\). The partial correlation matrix under the Kronecker product model is identifiable. Xia and Li (2018) eventually tested
\( H_{0, \text{precision}}^* : \mathbf{P}_A^{(1)} = \mathbf{P}_A^{(2)} \) versus \( H_{1, \text{precision}}^* : \mathbf{P}_A^{(1)} \neq \mathbf{P}_A^{(2)} \). Because the correlation matrix and partial correlation matrix fully determine each other, \( H_{0, \text{precision}}^* \) in (14) is equivalent to \( H_{0, \text{precision}}^* \). By analogy, (iv) “Two sample pre: oracle”, (v) “Two sample pre: sample-est”, and (vi) “Two sample pre: banded-est” correspond to the three choices of the nuisance matrices \( \mathbf{B}^{(g)} \) in Xia and Li (2018).

Lastly, (vii) “Two sample vector” based on the modified correlation version of Cai et al. (2013) is implemented in a similar fashion as in Section 5.1.

**Simulation setup.** We next describe the model to generate the data. Again, the matrix-valued observations from two groups follow matrix normal distribution, \( \mathbf{X}_k^{(g)} \sim MN_{pq}(\mathbf{0}, \mathbf{A}^{(g)}, \mathbf{B}^{(g)}) \). The covariance matrices \( \mathbf{B}^{(g)} \) are the autocorrelation matrices of AR(1) process with coefficient 0.8 and 0.9 respectively. Under the null hypothesis, the two covariance matrices are identical \( \mathbf{A}^{(1)} = \mathbf{A}^{(2)} \), and they are set to be the \( \Sigma^{(1)} \) in Model 1 of Cai et al. (2013), i.e., \( \Sigma^{* (1)} = (\Sigma^{* (1)})_i,j \), where \( \Sigma^{* (1)}_i,i = 1 \), \( \Sigma^{* (1)}_i,j = 0.5 \) for \( 5(k - 1) + 1 \leq i \neq j \leq 5k \) with \( k = 1, \ldots, p/5 \) and \( \Sigma^{* (1)}_i,j = 0 \) otherwise, and \( \Sigma^{(1)} = \mathbf{D}^{1/2} \Sigma^{* (1)} \mathbf{D}^{1/2} \), where \( \mathbf{D} = (d_{i,j}) \) is a diagonal matrix with diagonal elements \( d_{i,i} = \text{Unif}(0.5, 2.5) \) for \( i = 1, \ldots, p \). Under the alternative hypothesis, we set \( \mathbf{A}^{(1)-1} = (\Sigma^{(1)} + \delta \mathbf{I})/(1 + \delta) \) and \( \mathbf{A}^{(2)-1} = (\Sigma^{(1)} + \mathbf{U} + \delta \mathbf{I})/(1 + \delta) \), where \( \delta = \min\{\lambda_{\min}(\Sigma^{(1)}), \lambda_{\min}(\Sigma^{(1)} + \mathbf{U})\} + 0.05 \). Here, the perturbation matrix \( \mathbf{U} \) is symmetric and has ten random nonzero entries, five of which are located randomly in the lower triangle with random sign and random nonzero magnitude on the interval \([3\{\log p/(nq)\}^{1/2}, 5\{\log p/(nq)\}^{1/2}] \).

**Simulation results and interpretation.** Under the two-sample model setup, Table 4 presents the empirical size and power, respectively, of Methods (i-vii) based on 1000 replications with significance level \( \alpha = 0.05 \). Considering the sizes, Table 4 for the two-sample case shares the same message as Table 2 for the one-sample case, where the vector-based method (vii) cannot control the Type I error, while our methods (i-iii) and the precision matrix based methods (iv-vi) in Xia and Li (2018) behave well.

Note that “two sample cov: sample-est” and “two sample pre: sample-est” are undersized sometimes. The reason for this phenomenon is similar to that of the one-sample case. In essence, the size will be closer to the nominal level when the temporal covariance estimation is more accurate. Moreover, it is observed in (Cai et al. 2013, Proposition 1) that the empirical sizes can be smaller than the nominal level due to the correlation among the variables. Hence, the sizes may vary depending on the model setting (the correlations among the variables). Table 6 in Online
Supplement A.4.1 provides the simulation results when $\Sigma^{(2)}$, Model 2, of Cai et al. (2013) is used instead of $\Sigma^{(1)}$, Model 1; see A.4.1 for the details of the simulation setup. Both Tables 4 and 6 are the two-sample test results for normal distribution. The difference between them is that Table 4 is based on data generating with $\Sigma^{(1)}$ while Table 6 with $\Sigma^{(2)}$. The sizes in Table 6 are generally larger that those in Table 4 and closer to 0.05. This demonstrates that the sizes vary depending on the model setting, although both our methods and Xia and Li (2018) can control the size by 0.05. In contrast, considering the powers, the messages from Tables 2 and 4 are not always the same between the one-sample and two-sample cases. First of all, the relative performances of the procedures with different plugged-in estimates of $B$ are consistent. That is, both tables show that the “oracle” methods and the “banded-est” methods perform similarly, which dominate the “sample-est” methods most of the time.

However, the relative performances of the covariance matrix based methods and the precision matrix based methods are different between the two cases. In particular, in the one-sample case (Table 2), our methods are similar as the precision matrix based methods; but in the two-sample case (Table 4), our tests are more powerful than those based on the precision matrix. Here is the fundamental reason for this phenomenon: for the two precision matrices, $(A^{(1)})^{-1} = (\Sigma^{(1)} + \delta I)/(1 + \delta)$ and $(A^{(2)})^{-1} = (\Sigma^{(1)} + U + \delta I)/(1 + \delta)$, which differ by a sparse matrix, the magnitude of the maximum difference between the corresponding correlation matrices can be dramatically different from the magnitude of the maximum difference between the corresponding partial correlation matrices, primarily because of the non-diagonal structure of $\Sigma^{(1)}$. This is different from the one-sample case, where the two covariance matrices, $I$ and $(I + U + \delta I)/(1 + \delta)$, also differ by a sparse matrix, but the above mentioned two magnitudes are about the same primarily because of the diagonal structure under the null hypothesis.

The comparison of the computation time of our methods and Xia and Li (2018) is not provided for the two-sample case because the takeaway is the same as in the one-sample case.

In addition, we have tried experiments with Laplace and gamma distributions, the simulation results are qualitatively quite similar to the normal cases, and hence omitted. Online Supplement A.4.2 provides the simulation results under $t$ distribution, in which case, the precision matrix based approaches (which depend on normality heavily) in both one-sample and two-sample scenarios are no longer valid, while our covariance matrix based approaches (which do not rely on normality)
Table 4: The empirical size and power of the testing procedures for the two-sample case under normal distribution based on 1000 replications. The percentages are shown with the standard errors provided in parentheses. The $\Sigma^{(1)}$ matrix adopts the form of Model 1 in Cai et al. (2013). The significant level is $\alpha = 5\%$. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n_1 = n_2 = n = \{10, 50\}$.
Moreover, borrowing the configuration of the stock data, Online Supplement A.4.4 further demonstrates the validity and power of our methods and the invalidity of the precision matrix related methods.

6 Real Data Analysis

In our real data application, we employ a comprehensive sample of 30 industry sector returns from 43 countries around the world from 2001:07~2017:12.

We obtain the data on stock market returns and accounting items for international public firms from the Thomson-Reuters Datastream and Worldscope databases. Our sample covers 22 emerging markets and 21 developed countries from 2001:07 up to 2017:12. All returns are denominated in USD and excess return is calculated after subtracting the U.S. risk free rate. We categorize firms into 30 industry groups by their SIC (Standard classification code). We use the last December market capitalization denominated in USD as portfolio weight, then aggregate each stocks into industry portfolios for each country.

6.1 Economic Propositions

The well-documented evidence on the financial market segmentation suggests that in general, asset returns co-move with the market return differently across countries. In an integrated financial market, industry sectors have the same amount of systematic risk no matter from which country. However, this is not true for a segmented market. Obviously, with more rigid exchange rate policies, higher investment barrier, and sovereign risks, emerging countries are more segmented while developed countries are more integrated to global financial markets (Bekaert and Harvey, 1995). The measure of systematic risk, the market beta, is just the ratio of covariance between the stock’s excess return and market excess return, over the variance of the latter. As we standardize

*22 emerging countries include: Argentina, Brazil, Chile, China, Czech, Greece, Hungary, India, Indonesia, Israel, Malaysia, Mexico, Pakistan, Peru, Philippines, Poland, Portugal, South Africa, South Korea, Taiwan, Thailand, and Turkey. 21 Developed countries include: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Hong Kong, Ireland, Italy, Japan, Netherlands, New Zealand, Norway, Singapore, Spain, Sweden, Switzerland, the U.K., and the U.S.

†These industries are: 1 Food, 2 Beer, 3 Smoke, 4 Games, 5 Books, 6 Hshld, 7 Clths, 8 Hlth, 9 Chems, 10 Txtls, 11 Cnstr, 12 Steel, 13 FabPr, 14 EleEq, 15 Autos, 16 Carry, 17 Mines, 18 Coal, 19 Oil, 20 Util, 21 Telcm, 22 Srvcs, 23 BusEq, 24 Paper, 25 Trans, 26 Whsl, 27 Retail, 28 Meals, 29 Fin, 30 Other. See French http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/det_30_ind_port.html.

‡There are a small number of missing industries in certain emerging countries. We augment our data by approximating the country-industry return as a portfolio of that country's specific value-weighted return and that industry's specific value-weighted return, and use different portfolio weights to ensure the robustness of our results. In our main results, we assume an equally weighted portfolio of country and industry returns. Alternatively, we regress all non-missing country-industry returns on both country and industry returns and obtain weights of 0.79 and 0.21 after scaling. We also conduct such a regression with constraint of weights summing up to one, and obtain weights of 0.553 and 0.447. All the results are qualitatively similar.
the variances and market is just an aggregation of all industries, we therefore conjecture that there should be significant differences in correlations comparing developed to emerging countries. In particular, we make the following three propositions.

**Proposition 1.** In general, developed countries are more industrialized, and its financial markets are also more developed, usually accompanied with a better information environment. For example, [Campbell et al. (2001)](Campbell2001) show that in the U.S., there is a growing trend of idiosyncratic volatility in stock returns. In other words, stock returns generally co-move less than in the past. In contrast, emerging countries are more vulnerable to disasters, sudden change in supply or demand of natural resources, and population displacement. Stock returns then co-move more closely, and achieve higher returns in periods of economic prosperity and expansion and lower in periods of economic downturn and contraction.

Therefore, we conjecture that some industries are more pro-cyclical in emerging countries than in developed countries. For example, industries are more pro-cyclical in the production of durable goods, such as raw materials and heavy equipment. Also, airline industry may prosper more at the economic boom when people travel more. Developed economies are more open and free market-oriented, characterized by a relatively stable trend, and industry-specific shock tends not to affect other industries in the economic system. In contrast, emerging markets are characterized by frequent regime switches, a premise motivated by the dramatic reversals in fiscal, monetary and trade policies ([Aguiar and Gopinath 2007](Aguiar2007)). When these policies change frequently in an unanticipated way, they tend to have bigger and longer impacts on the entire economy and make procyclical sectors more cyclical than those in developed markets.

**Proposition 2.** [Kohn et al. (2018)](Kohn2018) show that emerging economies produce more commodities than they consume; in contrast, developed economies produce and consume commodities in similar amounts. They document these sectoral imbalances in the trade of commodities and manufactures in emerging economies are positively correlated with business cycle volatility. Therefore we also expect to detect larger co-movements of commodity industries returns with others.

**Proposition 3.** One more major difference between emerging markets and developed economies is the demographic patterns. [DellaVigna and Pollet (2007)](DellaVigna2007) argue that different goods have distinctive age profiles of consumption, and changes in the age distribution can forecast shifts in demand for various goods. These shifts in demand induce predictable changes in profitability for industries
that are not perfectly competitive. Given that emerging countries account for 90% of the global population aged under 30, we also expect in emerging countries, recreative business industries co-vary higher with the market returns.

6.2 Economic Interpretation of the Testing Results

The matrix sub-Gaussian distribution is reasonable for our real data application. First of all, it is crucial for the stock return data to satisfy the Kronecker product assumption for the covariance matrix. To this end, we use the test in Aston et al. (2017), where the null hypothesis is that the covariance matrix indeed adopts a Kronecker product structure. The bootstrap method in Aston et al. (2017) fails to reject the null, which supports our fundamental assumption. Second, our observations are monthly value-weighted industry portfolio returns. As shown by Campbell et al. (1997), monthly value-weighted index returns are less leptokurtic than their daily counterparts, and even less than daily individual stock returns. They conclude that fat-tailed distributions can be suited for shorter-horizon return, but the Central Limit Theorem applies and drives longer-horizon returns towards normality.

Moreover, the matrix sub-Gaussian distribution is also consistent with the broad mean-variance optimal portfolio literature. For example, Okhrin and Schmid (2006) study distributional properties for minimum variance portfolio weights, where all results for finite samples are made assuming normally distributed returns. Also, in studying high-frequency returns, Cai et al. (2020) propose two consistent estimators of the minimum risk of the global minimum variance portfolio, where the second estimator also assumes returns are i.i.d. normally distributed. Similarly, Frahm and Membel (2010) also make i.i.d. normal assumptions in deriving two shrinking estimators of minimum variance portfolio, and concur that normality seems a reasonable assumption for monthly stock returns.

Recall the notations from the introduction - we have matrix-valued observations from two groups of countries: \( n_1 = 22 \) and \( n_2 = 21 \) for emerging and developed countries respectively. \( X^{(g)}_k \in \mathbb{R}^{30 \times 198} \) is the matrix of returns for 30 industries over 198 months for country \( k \) in group \( g \).

One-sample test of temporal non-correlation. We first perform one-sample hypothesis test on the temporal covariance matrix within each group of countries, \( H_{0,B,g} : B^{(g)} \) is diagonal, by feeding
the transposed matrices \( \left( X_k^{(g)} \right)' \), \( k = 1, \ldots, n_g \), into the algorithms. We apply three methods including “One sample cov: sample-est”, “One sample pre: sample-est” \cite{Xia2017}, and “One sample vector” \cite{Cai2013} described in Section 5.1. The oracle procedures cannot be applied, because the covariance matrices of the industries \( A^{(g)} \) are unknown. Neither is “One sample cov: banded-est” nor “One sample pre: banded-est” applied, because it is not reasonable to assume the covariance matrices of the industries \( A^{(g)} \) have banded structure. All three methods reject the null hypothesis within each group respectively. These results unanimously show strong temporal dependence in both emerging and developed countries and serve as a warning that the vector-based approaches might be problematic.

**One-sample test of industrial non-correlation.** We also implement all one-sample methods in Section 5.1 except for the oracle ones, to test the covariance matrices of the industries, \( H_{0,A,g} : A^{(g)} \) is diagonal. Again, all five methods reject the null hypotheses. Furthermore, most methods reveal that almost all pairwise covariances are significantly larger than zero, conveying the fact that almost all industries are positively correlated with each other in both emerging and developed countries.

**Two-sample test and support recovery.** To verify the economic propositions in Section 6.1, we apply the two-sample global hypothesis testing described in Section 5.2 and two-sample support recovery methods. All five methods, including “Two sample cov: sample-est”, “Two sample cov: banded-est”, “Two sample pre: sample-est”, “Two sample pre: banded-est” \cite{Xia2018}, and “Two sample vector” \cite{Cai2013}, reject the null, \( H_{0}^* : R_A^{(1)} = R_A^{(2)} \). This result suggests that the overall correlation structures of the industries between emerging and developed countries are significantly different. To dive into details of the discrepancy, two-sample support recovery techniques are carried out. The approaches based upon precision matrix are not included as they uncover the pattern of partial correlation, not correlation. Since the correlation matrix associated with the temporal dimension normally has a banded structure, we present the support recovery results by our method “Two sample cov: banded-est” and “Two sample vector” in Figures 1 and 2 respectively.

In these two figures, red, yellow, and blue cells indicate whether the difference \( (R_A^{(1)})_{(i,j)} - (R_A^{(2)})_{(i,j)} \) is significantly larger than, equal to, or significantly smaller than zero. Again, \( g = 1, 2 \) correspond to emerging and developed countries, respectively. Note that the orderings of industries, columns and rows, in these two figures are different, as each is generated according to a hierarchical
clustering algorithm based on the value of the sign matrix of $R_A^{(1)} - R_A^{(2)}$ for better visualization.

We find that by using our method, there are no blue entries, suggesting that overall correlations in the emerging countries are no smaller than those in the developed countries. In contrast, the modified correlation version of the method in Cai et al. (2013) ignores the temporal correlation and potentially destroys the matrix structure due to vectorization. Therefore, this method produces some counter-intuitive results in Figure 2, where quite a few blue cells appear. Among them, Household (consumer goods), a pro-cyclical industry, can be seen to significantly co-move less with many other pro-cyclical industries in emerging countries. This is also the case with Coal, a commodity industry, and Textile, another consumer goods industry. These findings are generally inconsistent with the literature and the propositions of Section 6.1, casting some doubts on the method of Cai et al. (2013). As the hypothesis test of temporal non-correlation is already rejected, these findings further highlight the weakness of implementing the vector-based approach for matrix-valued data.

Now we focus on Figure 1 and examine our economic propositions. Based on the procedures detailed in Section 3, we find that for many industries, correlations (off-diagonal terms) do not differ. Still, there are exceptions. In Figure 1, the clustering algorithm separates the 30 industries into three groups and 9 industries stand out in the rightmost group: #27 Retail, #20 Utilities, #10 Textile, #19 Oil, #15 Automobiles and Trucks, #17 Mines, #14 Electrical Equipment, #16 Aircraft, and #4 Games. For these industries, all the correlations between them and the other industries are larger for the emerging countries than the developed markets.

In particular, consistent with Proposition 1 that in emerging countries pro-cyclical industries co-move more, we find that among these 9 industries, there are indeed more pro-cyclical industries, such as #14 Electrical Equipment, #15 Automobiles and Trucks, #16 Aircraft, ships, and railroad equipment, and #20 Utilities. The average of the estimated correlations between each of these industries and all the other industries in the developed countries are respectively, 0.218, 0.279, 0.235, and 0.292. In comparison, in the emerging countries, the corresponding values are separately, 0.402, 0.418, 0.399, and 0.423. Thus the magnitudes of the differences are also large.

Also consistent with Proposition 2 that commodity industries may co-move with other industries differently, we find that the correlations are larger for the emerging countries in commodity industries such as #17 Mines and #19 Oil. The average of the correlations in the developed countries
Figure 1: Support recovery result by our method “Two sample cov: banded-est”. Heat map of the sign matrix of $R_A^{(1)} - R_A^{(2)}$. The columns and rows of the 30 industries are ordered according to a hierarchical clustering algorithm with the clustering result shown on the left and the top.
Figure 2: Support recovery result by ignoring the temporal correlation and implementing the two-sample hypothesis testing method for vector-valued data: "Two sample vector". Heat map of the sign matrix of $\mathbf{R}_A^{(1)} - \mathbf{R}_A^{(2)}$. The columns and rows of the 30 industries are ordered according to a hierarchical clustering algorithm with the clustering result shown on the left and the top.
for these industries are separately, 0.229, and 0.265. In comparison, the average of the correlations in the emerging countries are separately, 0.409, and 0.402.

Finally, also consistent with the argument per [DellaVigna and Pollet (2007)] in Proposition 3, we find industries that are more related to discretionary consumer goods co-move differently, reflecting demographic pattern differences, such as #4 Games, #10 Textile, and #27 Retail. The average of the correlations in the developed countries for these industries are separately, 0.255, 0.223, and 0.267. In comparison, the average of the correlations in the emerging countries are separately, 0.403, 0.349, and 0.392.

Given all the statistically significantly larger correlations, we conclude that these 9 industries command more systematic risk in emerging countries than in developed countries.

**Remark 7.** Even though our real data application uses monthly industry returns, which tend to have light tail since they are low frequency and aggregated according to [Campbell et al. (1997)] there are other financial and economic applications, which are heavy-tailed sometimes. To offer additional assurance, besides simulations with normal, gamma, Laplace, and $t$ distributions for different combinations of $n, p, q$, we further performed two experiments in Online Supplement [A.4.4]. 1. we use the same setting of $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$ as in Section 5 with $n, p, q$ similar as the real data under $t_3$ distribution, and our methods are still valid under the null and powerful under the alternative; 2. we did a pseudo simulation with $t_3$ distribution and parameters $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$ estimated from the real data, our methods have power one. It is worth mentioning that our stock market data does not meet Condition (C3*), i.e., $q^3 \log q \log^2 \max(p, q, n) = o(np)$, with $n \approx 20, p = 30, q \approx 200$, but it is close to the weaker condition $q \log q \log^3 \max(p, q, n) = o(np)$ for “Two sample cov: banded-est” under the AR model as discussed in Remark [6].

### 6.3 Impact on Portfolio Construction

We now consider an investor aiming to build the global minimum variance (GMV) portfolio with 30 industry sector stock returns on the international equity market. We show that the differences in correlation matrix between emerging and developed countries have direct implications for the GMV portfolio weights, which are proportional to $\Sigma^{-1} \iota$, where $\iota$ is a vector of ones and the covariance matrix of the return is the only input that needs to be estimated. As noted by [Jagannathan and Ma (2003)] stocks that have high covariances with other stocks tend to receive small or even negative
portfolio weights. The rationale is that such stocks may be more likely to be redundant in the efficient portfolio construction as the remaining stocks can achieve the risk reduction already at the absence of the former stocks. Put it differently, these stocks can be better hedged by other stocks, and their returns may be spanned by other returns [Stevens 1998; Goto and Xu 2015].

To ensure that differences on the GMV portfolio weights are not driven by differences in variances (diagonal terms of the covariance matrix), we scale the covariance matrices and make their diagonal terms identical, so that any differences in covariances (off-diagonal terms) must be the only source of different portfolio weights. Specifically, we take the estimates of the covariance matrix of the emerging countries industry returns $A^{(1)}$ and the correlation matrix of the developed market industry returns $R^{(2)}$ from (15) and (16), and then obtain two covariance matrices that are directly comparable: $\Sigma_1 = A^{(1)}$, and $\Sigma_2 = (D_A^{(1)})^{1/2}R^{(2)}(D_A^{(1)})^{1/2}$. Then we can build two GMV portfolios both using 30 industry returns, and calculate the portfolio weights as $\Sigma_1^{-1}\iota/\iota^\top\Sigma_1^{-1}\iota$ and $\Sigma_2^{-1}\iota/\iota^\top\Sigma_2^{-1}\iota$, separately for the emerging and developed markets.

In particular, we focus on the weights of the following industries whose return correlations have systematically exhibited differences between the two groups of countries:

| Industry | Games | Txtls | ElcEq | Autos | Carry | Mines | Oil | Util | Rtail |
|----------|-------|-------|-------|-------|-------|-------|-----|------|-------|
| Emerging | 0.078 | 0.025 | 0.047 | 0.005 | 0.075 | 0.009 | -0.013 | 0.023 | -0.001 |
| Developed | 0.121 | 0.059 | 0.136 | 0.063 | 0.132 | 0.085 | 0.032 | 0.052 | 0.051 |

In the first row we present the GMV portfolio weights for the emerging markets, and in the second row the counterparts for the developed countries. Obviously, for these 9 specific industries, their portfolio weights are all larger for the developed markets, indicating they all contribute to the GMV portfolio in a non-trivial way. In contrast, all the weights are smaller for the emerging countries, including two less than 1% (Autos and Mines) and two negative (Oil and Rtail) weights. Therefore, these industries can be more likely to be spanned by the remaining industries. When we calculate the sum of absolute positions (bets on both long and short positions), we find emerging and developed markets GMV portfolios obtain very similar values, 144.78% and 144.94%, which suggests very similar level of leverage. This fact further highlights that the contribution of these 9 industry returns in the emerging markets is overshadowed by that of the others.

In practice of conducting out-of-sample tests, when an investor faces huge estimation uncertainty, a long-only (no-short-sale) constraint is typically assumed. In this case, $-0.013$ (Oil) and
−0.001 (Retail) can simply be excluded in the efficient portfolio construction. Furthermore, given the high transaction cost, trivial weights such as 0.005 (Automobiles and trucks) and 0.009 (Mines) can also fail the threshold due to their tiny magnitudes.

Alternatively, we could also fix the diagonals of the covariance matrix of the developed market industry returns, then blend with the correlation matrix of the emerging market industry returns: $\Sigma_1 = (D_A^{(2)})^{1/2}R^{(1)}(D_A^{(2)})^{1/2}$, and $\Sigma_2 = A^{(2)}$. Similarly, we calculate the GMV portfolio weights and single out these 9 industries:

| Industry | Games | Txtls | ElcEq | Autos | Carry | Mines | Oil | Util | Rtail |
|----------|-------|-------|-------|-------|-------|-------|-----|------|-------|
| Emerging | -0.064 | -0.009 | -0.079 | -0.086 | -0.033 | -0.087 | -0.024 | 0.045 | -0.035 |
| Developed | 0.004 | 0.026 | 0.022 | -0.011 | 0.025 | -0.012 | 0.004 | 0.084 | 0.021 |

Again, the portfolio weights for these 9 specific industries are all larger for the developed markets. This time, for the emerging market industry portfolio, all except one industry (Util) receive negative weights (short positions), ranging from −0.9% to −8.7%. Again, these industries make contributions to the construction of the GMV portfolio only through short sellings. As short sale is generally difficult and costly, taking large short positions can be undesirable. Furthermore, the sum of absolute positions is 190.24% and 119.25% separately for the emerging and developed markets, indicating additional leverage, thus extra risk for the GMV portfolio using the emerging market industry returns. Therefore, these results are consistent with the literature that in emerging countries when stocks tend to move together, diversification gains can be achieved using a smaller set of investible assets.

7 Conclusions

Under matrix sub-Gaussian distribution, we have proposed one-sample and two-sample tests of the correlation structures for matrix-valued observations. Theoretically, the proposed methods are asymptotically optimal, and empirically, they outperform the methods based on precision matrices both computationally and statistically for normal and other heavy-tailed distributions. Applying the proposed testing methods to the stock return data endorses the economic theory rigorously and the support recovery method unveils interesting structural difference of the correlations among different industries between developed and developing countries.
Further theoretical extension to include structural estimation of $B$ is worth investigating in the future. It will also be fruitful to propose methodology with theoretical guarantees to deal with heavy-tailed distribution by implementing robust strategy such as thresholding, Huber loss, and rank-based statistic, so that more financial and economical data are applicable. Higher-order tensorial observations, which are frequently encountered in many fields, exhibit even more challenges to analyze the theoretical properties. The current method relies upon one-step estimation of the nuisance parameter $B$. Empirically, iterative estimations of $B$ are more accurate and can generate more powerful tests for finite sample. However, the theoretical analysis of the iterative approach is much more involved and currently under study.

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References

Aguiar, M. and Gopinath, G. (2007). Emerging market business cycles: The cycle is the trend. *Journal of Political Economy*, 115(1):69–102.

Aston, J. A., Pigoli, D., and Tavakoli, S. (2017). Tests for separability in nonparametric covariance operators of random surfaces. *The Annals of Statistics*, 45(4):1431–1461.

Bai, J. (2009). Panel data models with interactive fixed effects. *Econometrica*, 77(4):1229–1279.

Bekaert, G. and Harvey, C. R. (1995). Time-varying world market integration. *The Journal of Finance*, 50(2):403–444.
Bickel, P. J. and Levina, E. (2008a). High dimensional inference and random matrices —— covariance regularization by thresholding. *The Annals of Statistics*, 36(6):2577–2604.

Bickel, P. J. and Levina, E. (2008b). Regularized estimation of large covariance matrices. *The Annals of Statistics*, 36(1):199–227.

Brandt, M. (2009). Portfolio choice problems. Handbook of Financial Econometrics, North-Holland.

Brandt, M. W. and Santa-Clara, P. (2006). Dynamic portfolio selection by augmenting the asset space. *The Journal of Finance*, 61(5):2187–2217.

Brandt, M. W., Santa-Clara, P., and Valkanov, R. (2009). Parametric portfolio policies: Exploiting characteristics in the cross-section of equity returns. *The Review of Financial Studies*, 22(9):3411–3447.

Cai, C., Chen, R., and Xiao, H. (2019). Kopa: Automated kronecker product approximation. *arXiv preprint arXiv:1912.02392*.

Cai, T. and Liu, W. (2011). Adaptive thresholding for sparse covariance matrix estimation. *Journal of the American Statistical Association*, 106(494):672–684.

Cai, T., Liu, W., and Luo, X. (2011). A constrained ℓ1 minimization approach to sparse precision matrix estimation. *Journal of the American Statistical Association*, 106(494):594–607.

Cai, T., Liu, W., and Xia, Y. (2013). Two-sample covariance matrix testing and support recovery in high-dimensional and sparse settings. *Journal of the American Statistical Association*, 108(501):265–277.

Cai, T. T., Hu, J., Li, Y., and Zheng, X. (2020). High-dimensional minimum variance portfolio estimation based on high-frequency data. *Journal of Econometrics*, 214(2):482–494.

Cai, T. T. and Jiang, T. (2011). Limiting laws of coherence of random matrices with applications to testing covariance structure and construction of compressed sensing matrices. *The Annals of Statistics*, 39(3):1496–1525.

Cai, T. T., Li, H., Liu, W., and Xie, J. (2016). Joint estimation of multiple high-dimensional precision matrices. *Statistica Sinica*, 26(2):445.

Cai, T. T. and Liu, W. (2016). Large-scale multiple testing of correlations. *Journal of the American Statistical Association*, 111(513):229–240.

Cai, T. T. and Ma, Z. (2013). Optimal hypothesis testing for high dimensional covariance matrices. *Bernoulli*, 19(5B):2359–2388.
Cai, T. T. and Zhang, A. (2016). Inference for high-dimensional differential correlation matrices. *Journal of multivariate analysis*, 143:107–126.

Cai, T. T. and Zhou, H. H. (2012). Optimal rates of convergence for sparse covariance matrix estimation. *The Annals of Statistics*, 40(5):2389–2420.

Campbell, J. Y., Lettau, M., Malkiel, B. G., and Xu, Y. (2001). Have individual stocks become more volatile? an empirical exploration of idiosyncratic risk. *The Journal of Finance*, 56(1):1–43.

Campbell, J. Y., Lo, A. W., and MacKinlay, A. C. (1997). *The Econometrics of Financial Markets*. Princeton University Press, 41 William Street Princeton, New Jersey 08540 USA.

Chang, J., Zhou, W., Zhou, W. X., and Wang, L. (2017). Comparing large covariance matrices under weak conditions on the dependence structure and its application to gene clustering. *Biometrics*, 73(1):31–41.

Chen, E. Y. and Chen, R. (2019). Modeling dynamic transport network with matrix factor models: with an application to international trade flow. *arXiv preprint arXiv:1901.00769*.

Chen, E. Y., Tsay, R. S., and Chen, R. (2019a). Constrained factor models for high-dimensional matrix-variate time series. *Journal of the American Statistical Association*, pages 1–37.

Chen, H. and Xia, Y. (2021). A normality test for high-dimensional data based on a nearest neighbor approach. *Journal of the American Statistical Association*, (just-accepted):1–35.

Chen, R., Xiao, H., and Yang, D. (2020). Autoregressive models for matrix-valued time series. *Journal of Econometrics*.

Chen, R., Yang, D., and Zhang, C.-h. (2019b). Factor models for high-dimensional tensor time series. *arXiv preprint arXiv:1905.07530*.

Chen, S. X., Zhang, L.-X., and Zhong, P.-S. (2010). Tests for high-dimensional covariance matrices. *Journal of the American Statistical Association*, 105(490):810–819.

Chen, X. and Liu, W. (2019). Graph estimation for matrix-variate gaussian data. *Statistica Sinica*, 29(1):479–504.

Constantinou, P., Kokoszka, P., and Reimherr, M. (2017). Testing separability of space-time functional processes. *Biometrika*, 104(2):425–437.

Danaher, P., Wang, P., and Witten, D. M. (2014). The joint graphical lasso for inverse covariance estimation across multiple classes. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(2):373–397.
Dawid, A. P. (1981). Some matrix-variate distribution theory: notational considerations and a bayesian application. *Biometrika*, 68(1):265–274.

DellaVigna, S. and Pollet, J. M. (2007). Demographics and industry returns. *The American Economic Review*, 97(5):1667–1702.

Dutilleul, P. (1999). The mle algorithm for the matrix normal distribution. *Journal of statistical computation and simulation*, 64(2):105–123.

Fama, E. F. and French, K. R. (1997). Industry costs of equity. *Journal of Financial Economics*, 43(2):153–193.

Frahm, G. and Memmel, C. (2010). Dominating estimators for minimum-variance portfolios. *Journal of Econometrics*, 159(2):289 – 302.

Friedman, J., Hastie, T., and Tibshirani, R. (2008). Sparse inverse covariance estimation with the graphical lasso. *Biostatistics*, 9(3):432–441.

Goto, S. and Xu, Y. (2015). Improving mean variance optimization through sparse hedging restrictions. *The Journal of Financial and Quantitative Analysis*, 50(6):1415–1441.

Guo, X. and Tang, C. (2020). Specification tests for covariance structures in high-dimensional statistical models. *Biometrika*.

Hafner, C. M., Linton, O. B., and Tang, H. (2020). Estimation of a multiplicative correlation structure in the large dimensional case. *Journal of Econometrics*, 217(2):431–470.

Han, F., Han, X., Liu, H., and Caffo, B. (2016). Sparse median graphs estimation in a high-dimensional semiparametric model. *The Annals of Applied Statistics*, 10(3):1397–1426.

Han, F. and Liu, H. (2013). Optimal rates of convergence for latent generalized correlation matrix estimation in transelliptical distribution. *arXiv preprint arXiv:1305.6916*, 34.

Hoff, P. D. (2011). Separable covariance arrays via the tucker product, with applications to multivariate relational data. *Bayesian Analysis*, 6(2):179–196.

Hong, H., Torous, W., and Valkanov, R. (2007). Do industries lead stock markets? *Journal of Financial Economics*, 83(2):367 – 396.

Jagannathan, R. and Ma, T. (2003). Risk reduction in large portfolios: Why imposing the wrong constraints helps. *Journal of Finance*, 58(4):1651–1683.

Kohn, D., Leibovici, F., and Tretvoll, H. (2018). Trade in commodities and business cycle volatility. *Available at SSRN 2654792*.  

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42
Ledoit, O. and Wolf, M. (2004). A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis*, 88(2):365 – 411.

Leng, C. and Tang, C. Y. (2012). Sparse matrix graphical models. *Journal of the American Statistical Association*, 107(499):1187–1200.

Li, J. and Chen, S. X. (2012). Two sample tests for high-dimensional covariance matrices. *The Annals of Statistics*, 40(2):908–940.

Liu, W. (2013). Gaussian graphical model estimation with false discovery rate control. *The Annals of Statistics*, 41(6):2948–2978.

Meinshausen, N. and Bühlmann, P. (2006). High-dimensional graphs and variable selection with the lasso. *The Annals of Statistics*, 34(3):1436–1462.

Molstad, A. J. and Rothman, A. J. (2019). A penalized likelihood method for classification with matrix-valued predictors. *Journal of Computational and Graphical Statistics*, 28(1):11–22.

Moskowitz, T. J. (2003). An analysis of covariance risk and pricing anomalies. *The Review of Financial Studies*, 16(2):417–457.

Narayan, M. and Allen, G. I. (2016). Mixed effects models for resampled network statistics improves statistical power to find differences in multi-subject functional connectivity. *Frontiers in neuroscience*, 10:108.

Okhrin, Y. and Schmid, W. (2006). Distributional properties of portfolio weights. *Journal of Econometrics*, 134(1):235 – 256.

Qiu, H., Han, F., Liu, H., and Caffo, B. (2016). Joint estimation of multiple graphical models from high dimensional time series. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 78(2):487–504.

Ravikumar, P., Wainwright, M. J., Raskutti, G., and Yu, B. (2011). High-dimensional covariance estimation by minimizing $\ell_1$-penalized log-determinant divergence. *Electronic Journal of Statistics*, 5:935–980.

Rothman, A. J., Levina, E., and Zhu, J. (2009). Generalized thresholding of large covariance matrices. *Journal of the American Statistical Association*, 104(485):177–186.

Rothman, A. J., Levina, E., and Zhu, J. (2010). A new approach to cholesky-based covariance regularization in high dimensions. *Biometrika*, 97(3):539–550.

Rudelson, M. and Vershynin, R. (2013). Hanson-wright inequality and sub-gaussian concentration.
Schott, J. R. (2007). A test for the equality of covariance matrices when the dimension is large relative to the sample sizes. *Computational Statistics & Data Analysis*, 51(12):6535–6542.

Srivastava, M. S. and Yanagihara, H. (2010). Testing the equality of several covariance matrices with fewer observations than the dimension. *Journal of Multivariate Analysis*, 101(6):1319–1329.

Stevens, G. V. G. (1998). On the inverse of the covariance matrix in portfolio analysis. *Journal of Finance*, 53(5):1821–1827.

Wang, D., Liu, X., and Chen, R. (2019). Factor models for matrix-valued high-dimensional time series. *Journal of Econometrics*, 208(1):231–248.

Werner, K., Jansson, M., and Stoica, P. (2008). On estimation of covariance matrices with kronecker product structure. *IEEE Transactions on Signal Processing*, 56(2):478–491.

Xia, Y., Cai, T., and Cai, T. T. (2015). Testing differential networks with applications to the detection of gene-gene interactions. *Biometrika*, 102(2):247–266.

Xia, Y. and Li, L. (2017). Hypothesis testing of matrix graph model with application to brain connectivity analysis. *Biometrics*, 73(3):780–791.

Xia, Y. and Li, L. (2018). Matrix graph hypothesis testing and application in brain connectivity alternation detection. *Statistica Sinica, to appear*.

Yin, J. and Li, H. (2012). Model selection and estimation in the matrix normal graphical model. *Journal of multivariate analysis*, 107:119–140.

Yuan, M. and Lin, Y. (2007). Model selection and estimation in the gaussian graphical model. *Biometrika*, 94(1):19–35.

Zheng, S., Cheng, G., Guo, J., and Zhu, H. (2019). Test for high-dimensional correlation matrices. *The Annals of Statistics*, 47(5):2887–2921.

Zhou, S. (2014). Gemini: Graph estimation with matrix variate normal instances. *The Annals of Statistics*, 42(2):532–562.

Zhu, Y. and Li, L. (2018). Multiple matrix gaussian graphs estimation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80(5):927–950.

Zhu, Y., Shen, X., and Pan, W. (2014). Structural pursuit over multiple undirected graphs. *Journal of the American Statistical Association*, 109(508):1683–1696.
Supplementary Material to “Testing and Support Recovery of Correlation Structures for Matrix-Valued Observations with an Application to Stock Market Data”

The Online Supplement is organized as follows: Online Supplement A.1 states the theorems related to the support recovery for both the one-sample and the two-sample cases. Proofs of Theorems 1-6 in the main text and Theorems 7-8 in Online Supplement A.1 are given in Section A.2. Lemmas and their proofs are shown in Online Supplement A.3. Online Supplement A.4 provides more simulation results. A more comprehensive version of Table 1 with key references listed is provided in Online Supplement A.5.

A.1 Theorems on Support Recovery for the One/Two-Sample Cases

To study the theoretical property of the support recovery procedure \( \hat{\Psi} \) in (13), recall the definition of the true support of \( A \) in (12) and define the following class of covariance matrices in parallel with (23):

\[
W(c) = \left\{ A = (a_{i,j})_{p \times p} : \min_{(i,j) \in \Psi} \frac{|a_{i,j}|}{\sqrt{\theta_{i,j}/(nq)}} \geq c \sqrt{\log p} \right\}.
\]

Note that \( U(c) \) requires the maximum of \( |a_{i,j}|/\sqrt{\theta_{i,j}/(nq)} \) to be lower bounded by \( c \sqrt{\log p} \) while \( W(c) \) requires the minimum of \( |a_{i,j}|/\sqrt{\theta_{i,j}/(nq)} \) over the support is lower bounded by the same quantity. This requirement essentially means that all of the entries over the support are sufficiently large and thus can be distinguished from the noise. Then Theorem 7 below shows that the estimator \( \hat{\Psi}(4) \) with threshold constant \( \tau = 4 \) recovers the support \( \Psi \) perfectly with probability going to 1 when the magnitudes of all the nonzero off-diagonal entries are above certain thresholds as in \( W(4) \).

**Theorem 7.** Suppose that Conditions (C1)-(C3) hold. As \( nq, p \to \infty \), we have

\[
\inf_{A \in W(4)} P\left( \hat{\Psi}(4) = \Psi \right) \to 1.
\]

**Remark 8.** With the same reasoning as in [Cai et al. (2013)], it can be easily verified that the choice of the threshold constant \( \tau = 4 \) is optimal. As a matter of fact, for any \( \tau < 4 \), the probability of exact recovery of the support goes to zero. The failure of exact recovery is because the small threshold of \( \tau \log p \) will estimate some of the zero entries by nonzero values, i.e., the estimated support will be larger than the true support. In addition, the rate of \( \sqrt{\log p/(nq)} \) as the requirement of the nonzero entries of \( A \) cannot be relaxed.
Turning to the support recovery procedure $\hat{\Psi}^*$ in (21) as an estimate of $\Psi^*$ in (20) for the two-sample case, construct the set of matrices whose support has the rate defined in (25), namely, 

$$
W^*(c) = \left\{ (R_A^{(1)}, R_A^{(2)}) : \min_{(i,j)\in \Psi^*} \frac{|r_{i,j}^{(1)} - r_{i,j}^{(2)}|}{\sqrt{\vartheta_{i,j}^{(1)} / (n_1 q) + \vartheta_{i,j}^{(2)} / (n_2 q)}} \geq c \sqrt{\log p} \right\}.
$$

Theorem 8 claims that $\hat{\Psi}^*(4)$ can recover such matrices exactly with probability tending to one.

**Theorem 8.** Suppose that Conditions (C1$^*$), (C2$^*$) and (C3$^*$) hold. As $nq, p \to \infty$, we have

$$
\inf_{(R_A^{(1)}, R_A^{(2)}) \in W^*(4)} P\left( \hat{\Psi}^*(4) = \Psi^* \right) \to 1.
$$

**A.2 Proofs of Theorems**

**Proof of Theorem 1**

WLOG, throughout this section, we assume $a_{i,i} = 1$ for $i = 1, \cdots, p$. Define the following oracle quantities when $B$ is known,

$$
(\hat{a}_{i,j}^o) = \hat{A}^o = \frac{1}{nq} \sum_{k=1}^{n} X_k B^{-1} X_k',
$$

$$
\hat{\theta}_{i,j}^o = \frac{1}{nq} \sum_{k=1}^{n} \sum_{l=1}^{q} \left( (X_k B^{-1/2})_{i,l} (X_k B^{-1/2})_{j,l} - \hat{a}_{i,j}^o \right)^2,
$$

and define two maximum statistics that are based on these oracle quantities,

$$
\hat{M}_n^o = \max_{1 \leq i < j \leq p} \frac{\hat{a}_{i,j}^2}{\hat{\theta}_{i,j}^o / (nq)},
$$

$$
M_n^o = \max_{1 \leq i < j \leq p} \left( \frac{\hat{a}_{i,j}^o}{} \right)^2 / \theta_{i,j}^{o} / (nq).
$$

Theorem 1 is readily proved when combining the following facts. Under the Conditions (C1)-(C3),

$$
P(M_n^o - 4 \log p + \log \log p \leq t) \to \exp \left( -\frac{1}{\sqrt{8\pi}} \exp(-\frac{t}{2}) \right), \quad (26)
$$

$$
|M_n - M_n^o| = o_p(1), \quad (27)
$$

$$
|M_n^o - \hat{M}_n| = o_p(1), \quad (28)
$$

where (26) is proved by Lemma 4 and (27) and (28) are proved by Lemma 5 in Section A.3 □
Proof of Theorem 2

We first prove for any constant $c > 0$,

$$P(M_n^o \geq c + q_\alpha + 4 \log p - \log \log p) \to 1.$$  

Let

$$M_{n,d}^{o} = \max_{1 \leq i < j \leq p} \frac{(\hat{a}_{i,j}^o - a_{i,j})^2}{\hat{\theta}_{i,j}^o/nq}.$$  

By Lemmas 1 and 2,

$$P(M_{n,d}^{o} \leq 4 \log p - \frac{1}{2} \log \log p) \to 1,$$  

as $nq, p \to \infty$. By Lemma 1, the inequalities

$$\max_{1 \leq i < j \leq p} \frac{a_{i,j}^2}{\theta_{i,j}^o/nq} \leq 2M_{n,d}^{o} + 2M_n^o,$$

and

$$\max_{1 \leq i < j \leq p} \frac{a_{i,j}^2}{\theta_{i,j}^o/nq} \geq 16 \log p,$$

we conclude that $P(M_n^o \geq c + q_\alpha + 4 \log p - \log \log p)$ $\to 1$ as $nq, p \to \infty$.

Next, by the proof of Lemma 5, it can be easily shown that, under Conditions (C1)-(C3), $|M_n - M_n^o| = o_p \left( \frac{M_n^o}{\log p} \right)$. Notice that for a constant $c' > 0$, we have

$$P(M_n \geq q_\alpha + 4 \log p - \log \log p) \geq P \left( M_n^o \geq \frac{c'}{3} \sqrt{\frac{M_n^o}{\log p}} + q_\alpha + 4 \log p - \log \log p \right) - P \left( M_n - M_n^o \geq \frac{c'}{3} \sqrt{\frac{M_n^o}{\log p}} \right),$$

Consider the event $A = \{M_n^o \geq c' + q_\alpha + 4 \log p - \log \log p\}$, then we have $P(A) \to 1$. Since

$$P \left( M_n^o - M_n \geq \frac{c'}{3} \sqrt{\frac{M_n^o}{\log p}} \right) \to 0,$$

and

$$P \left( M_n^o \geq \frac{c'}{3} \sqrt{\frac{M_n^o}{\log p}} + q_\alpha + 4 \log p - \log \log p \right) \geq P \left( M_n^o \geq \frac{c'}{3} \sqrt{\frac{M_n^o}{\log p}} + q_\alpha + 4 \log p - \log \log p | A \right) P(A) \to 1.$$

3
The results thus follow. ■

**Proof of Theorem 3**

This theorem is essentially proved in [Cai et al. (2013)], we skip the proof here.

**Proof of Theorem 4**

WLOG, assume $a^{(g)}_{i,i} = 1$ for $g = 1, 2$ and $i = 1, \cdots, p$. Similar to the proof of Theorem 1, define the oracle quantities,

$$
(a^{o(g)}_{i,j}) = \hat{A}^{o(g)} = \frac{1}{nq} \sum_{k=1}^{n} X^{(g)}_k (B^{(g)})^{-1}(X^{(g)}_k)',
$$

$$
(p^{o(g)}_{i,j}) = \hat{R}^{o(g)} = \left(\frac{a^{o(g)}_{i,j}}{(a^{o(g)}_{i,i} a^{o(g)}_{j,j})^{1/2}}\right),
$$

$$
\hat{\theta}_{i,j}^{o(g)} = \frac{1}{nq} \sum_{k=1}^{n} \sum_{l=1}^{q} \left[(X^{(g)}_k (B^{(g)})^{-1/2})_{i,l} (X^{(g)}_k (B^{(g)})^{-1/2})_{j,l} - a^{o(g)}_{i,j}\right]^2,
$$

$$
\hat{\varrho}_{i,j}^{o(g)} = \frac{\hat{\theta}_{i,j}^{o(g)}}{(a^{o(g)}_{i,i} a^{o(g)}_{j,j})^{1/2}}
$$

for $g = 1, 2$ and define oracle test statistics

$$
M_{n}^{*,o} = \max_{1 \leq i < j \leq p} \frac{(r^{(1)}_{i,j} - r^{(2)}_{i,j})^2}{\hat{\varrho}_{i,j}^{o(1)}/(n_1 q) + \hat{\varrho}_{i,j}^{o(2)}/(n_2 q)},
$$

$$
M_{n}^{*,o} = \max_{1 \leq i < j \leq p} \frac{(r^{o(1)}_{i,j} - r^{o(2)}_{i,j})^2}{\hat{\varrho}_{i,j}^{o(1)}/(n_1 q) + \hat{\varrho}_{i,j}^{o(2)}/(n_2 q)}.
$$

Theorem 4 is proved because of the following. Under Conditions (C1\textsuperscript{*})-(C5\textsuperscript{*}),

$$
P(M_{n}^{*,o} - 4 \log p + \log \log p \leq t) \to \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{t^2}{2}\right)\right),
$$

$$
|M_{n}^{*} - \hat{M}_{n}^{*,o}| = o_p(1),
$$

$$
|M_{n}^{*,o} - \hat{M}_{n}^{*,o}| = o_p(1),
$$

where (30) is proved by Lemma 6 and (31) and (32) are proved by Lemma 7 in Section A.3. ■

**Proof of Theorem 7**

From Theorem 1, we have that, uniformly for $A \in \mathcal{W}(4),

$$
P\left(\max_{(i,j) \notin \Psi} M_{i,j} \geq 4 \log p\right) \to 0.
$$
By (29), Lemma 1 and the inequality
\[
\min_{(i,j) \in \Psi} M_{i,j}^o \geq \min_{(i,j) \in \Psi} \frac{1}{2} \cdot \frac{a_{i,j}^2}{\theta_{i,j}^2/(nq)} - M_{n,d}^o,
\]
we can easily get for any constant \( c > 0 \) and uniformly for \( \mathbf{A} \in \mathcal{W}(4) \),
\[
P \left( \min_{(i,j) \in \Psi} M_{i,j}^o \geq c + 4 \log p \right) \to 1,
\]
where \( M_{i,j}^o = \left( \hat{a}_{i,j}^o \right)^2 \hat{\theta}_{i,j}^o/(nq) \). Similar to the proof of Theorem 2, we have
\[
P \left( \min_{(i,j) \in \Psi} M_{i,j}^o \geq 4 \log p \right) \to 1.
\]
which implies the desired results. ■

**Proofs of Theorems 5, 6, and 8**

The proofs of Theorems 5, 6, and 8 are similar to those of Theorems 2, 3, and 7. We skip the proofs here.

**A.3 Lemmas and Their Proofs**

Lemma 1 is the large deviation bound for the oracle estimate of variance \( \hat{\theta}_{i,j}^o \), whose proof is given in Cai et al. (2013).

**Lemma 1.** There exists some constant \( C > 0 \) such that
\[
P \left( \max_{i,j} |\hat{\theta}_{i,j}^o - \theta_{i,j}|/(a_{i,j}a_{i,j}) \geq C \frac{\epsilon_{nq}}{\log p} \right) = O(p^{-1}),
\]
where \( \epsilon_{nq} = \max((\log p)^{1/6}/(nq)^{1/2}, (\log p)^{-1}) \to 0 \) as \( nq, p \to \infty \).

Lemma 2 is the large deviation bound for the maximum of the oracle estimate of covariance \( \hat{a}_{i,j}^o \) in the one-sample case and \( \hat{a}_{i,j}^{o,(g)} \) in the two-sample case, whose proof is given in Cai et al. (2013).

**Lemma 2.** Let \( \Lambda \) be any subset of \( \{(i,j) : 1 \leq i \leq j \leq p\} \) and \( |\Lambda| = \text{Card}(\Lambda) \). We have for some constant \( C > 0 \) that
\[
P \left( \max_{(i,j) \in \Lambda} \frac{(\hat{a}_{i,j}^o - a_{i,j})^2}{\theta_{i,j}^2/(nq)} \geq x^2 \right) \leq C|\Lambda|(1 - \Phi(x)) + O(p^{-1}),
\]
\[
P \left( \max_{(i,j) \in \Lambda} \frac{(\hat{a}_{i,j}^{o,(1)} - a_{i,j}^{o,(1)} - a_{i,j}^{o,(2)} + a_{i,j}^{o,(2)})^2}{\theta_{i,j}^{o,(1)}/(n1q) + \theta_{i,j}^{o,(2)}/(n2q)} \geq x^2 \right) \leq C|\Lambda|(1 - \Phi(x)) + O(p^{-1}),
\]
uniformly for \( 0 \leq x \leq (8 \log p)^{1/2} \) and \( \Lambda \subseteq \{(i,j) : 1 \leq i \leq j \leq p\} \).
Lemma 3. Let \( X = (X_1, \ldots, X_n)^T \) and \( Y = (Y_1, \ldots, Y_n)^T \in \mathbb{R}^n \) be two random vectors with independent components each and they satisfy \( \mathbb{E}X_i = \mathbb{E}Y_i = 0 \) and \( \|X_i\|_{\psi^2}, \|Y_i\|_{\psi^2} \leq K \). Here \( \|\cdot\|_{\psi^2} \) refers to the Orlicz norm with \( \psi^2(x) = e^{x^2} - 1 \), where the Orlicz norm of a random variable \( X \) is defined as \( \|X\|_{\psi} = \inf\{M > 0 : \mathbb{E}\psi(|X|/M) \leq 1\} \). Let \( A \) be and \( n \times n \) matrix. Then, for every \( t > 0 \),

\[
P(|X^T AY - \mathbb{E}X^T AY| > t) \leq 2 \exp \left[-c \min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|_2}\right)\right],
\]

for some constant \( c > 0 \).

Remark 9. We require \( \{X_i, i = 1, \ldots, n\} \) are independent with each other and also \( \{Y_i, i = 1, \ldots, n\} \) are independent with each other but we allow the correlations between \( X \) and \( Y \). Note that if \( Y = X \), this is the famous Hanson-Wright inequality.

Lemma 4 is the oracle version of Theorem 1.

Lemma 4. Suppose that Conditions (C1) and (C2) hold, then under \( H_0 \), for any \( t \in \mathbb{R} \),

\[
P(M_n^o - 4 \log p + \log \log p \leq t) \to \exp \left(-\frac{1}{\sqrt{8\pi}} \exp \left(-\frac{t}{2}\right)\right),
\]

as \( nq, p \to \infty \). Furthermore, under \( H_0 \), the convergence above is uniform for all \( \{X_k, k = 1, \ldots, n\} \) satisfying (C1) and (C2).

Lemma 5 shows the distance between oracle \( M_n^o \) and the test statistic \( M_n \).

Lemma 5. Uniformly for all \( \{X_k, k = 1, \ldots, n\} \) satisfying Conditions (C1)-(C3), we have that, under \( H_0 \),

\[
|M_n^o - \hat{M}_n^o| = o_p(1), \tag{33}
\]

\[
|M_n - M_n^o| = o_p(1), \tag{34}
\]

Lemma 6 is the oracle version of Theorem 4.

Lemma 6. Suppose that Conditions (C1\(^*\)), (C2\(^*\)), (C4\(^*\)) and (C5\(^*\)) hold, then under \( H_0^* \), for any \( t \in \mathbb{R} \),

\[
P(M_n^{*,o} - 4 \log p + \log \log p \leq t) \to \exp \left(-\frac{1}{\sqrt{8\pi}} \exp \left(-\frac{t}{2}\right)\right), \tag{35}
\]

as \( nq, p \to \infty \). Furthermore, under \( H_0^* \), the convergence in (35) is uniform for all \( \{X_k^{(1)}, k = 1, \ldots, n_1\} \) and \( \{X_k^{(2)}, k = 1, \ldots, n_2\} \) satisfying (C1\(^*\)), (C2\(^*\)), (C4\(^*\)) and (C5\(^*\)).
Lemma 7 shows the distance between oracle $M_{n}^{*,o}$ and the test statistic $M_{n}^{*}$.

**Lemma 7.** Uniformly for all $\{X_{k}^{(1)}, k = 1, \ldots, n_{1}\}$ and $\{X_{k}^{(2)}, k = 1, \ldots, n_{2}\}$ satisfying $(C1^*)$-$$(C5^*)$, we have that, under $H_{0}^{*}$

$$|M_{n}^{*,o} - M_{n}^{*,o}| = o_{p}(1).$$  \hspace{1cm} (36)

$$|M_{n}^{*} - M_{n}^{*,o}| = o_{p}(1).$$  \hspace{1cm} (37)

**Proof of Lemma 3**

WLOG, we assume $K = 1$. Following the proofs of Theorem 1.1 in [Rudelson and Vershynin (2013)](#), we represent

$$X^{T}AY - E X^{T}AY = \sum_{i} a_{ii}(X_{i}Y_{i} - E X_{i}Y_{i}) + \sum_{i,j;i \neq j} a_{ij}X_{i}Y_{j}.$$

Assume

$$P(X^{T}AY - E X^{T}AY > t) \leq P(\sum_{i} a_{ii}(X_{i}Y_{i} - E X_{i}Y_{i}) > t/2) + P(\sum_{i,j;i \neq j} a_{ij}X_{i}Y_{j} > t/2) =: p_{1} + p_{2}.$$ 

Note that $X_{i}Y_{i} - EX_{i}Y_{i}$ are independent mean-zero subexponential random variables, and

$$\|X_{i}Y_{i} - EX_{i}Y_{i}\|_{\psi_{1}} \leq 2\|X_{i}Y_{i}\|_{\psi_{1}} \leq 2\|X_{i}\|_{\psi_{2}}\|Y_{i}\|_{\psi_{2}} \leq 2,$$

where $\| \cdot \|_{\psi_{1}}$ refers to the Orlicz norm with $\psi_{1}(x) = e^{x} - 1$. Thus, by using a Bernstein-type inequality we can obtain

$$p_{1} \leq \exp \left[ -c \min \left( \frac{t^{2}}{\|A\|_{F}^{2}}, \frac{t}{\|A\|_{2}} \right) \right].$$

We use the decoupling method to bound the off-diagonal sum

$$S := \sum_{i,j;i \neq j} a_{ij}X_{i}Y_{j}.$$ 

Consider independent Bernoulli random variables $\delta_{i} \in \{0, 1\}$ with $E\delta_{i} = 1/2$. We have

$$S = 4E\delta S_{\delta},$$

where

$$S_{\delta} = \sum_{i,j} \delta_{i}(1 - \delta_{j})a_{ij}X_{i}Y_{j}.$$
Here $\mathbb{E}_\delta$ denotes the expectation with respect to $\delta = (\delta_1, \ldots, \delta_n)$. For any $\lambda > 0$, Jensen’s inequality yields
\[
\mathbb{E}_{X,Y} \exp(\lambda S) \leq \mathbb{E}_{X,Y,\delta} \exp(4\lambda S_\delta),
\]
where $\mathbb{E}_{X,Y,\delta}$ denotes the expectation with respect to $X, Y$ and $\delta$. Consider the set of indices $\Lambda_\delta = \{i \in [n] : \delta_i = 1\}$, we can express
\[
S_\delta = \sum_{i \in \Lambda_\delta, j \in \Lambda_\delta} a_{ij}X_iY_j = \sum_{j \in \Lambda_\delta} Y_j \left( \sum_{i \in \Lambda_\delta} a_{ij}X_i \right).
\]
Now we condition on $\delta$ and $(X_i)_{i \in \Lambda_\delta}$. We can obtain
\[
\mathbb{E}_{(Y_j)_{j \in \Lambda_\delta}} \exp(4\lambda S_\delta) \leq \exp(\mathcal{C}\lambda^2 \|S_\delta\|_{\psi_2}^2) \leq \exp(C\lambda^2 \sigma_\delta^2),
\]
where $\sigma_\delta^2 = \sum_{j \in \Lambda_\delta} (\sum_{i \in \Lambda_\delta} a_{ij}X_i)^2$. Taking expectations of both sides again, we have
\[
\mathbb{E}_{X,Y} \exp(4\lambda S_\delta) \leq \mathbb{E}_{X,Y} \exp(C\lambda^2 \sigma_\delta^2) =: E_\delta,
\]
which holds for any fixed $\delta$. It remains to estimate $E_\delta$. The rest of the proof is the same as proof of Theorem 1.1 in [Rudelson and Vershynin (2013)]. We skip the details here. ■

**Proof of Lemma 4**

Consider
\[
\tilde{M}_n^o = \max_{1 \leq i < j \leq p} \left( \frac{\hat{\theta}_{i,j}^o)^2}{\theta_{i,j}/(nq)} \right).
\]
From Lemma 1, we have $|\hat{\theta}_{i,j}^o/\theta_{i,j} - 1| = o_p\left( \frac{1}{\log p} \right)$, which leads to
\[
|M_n^o - \tilde{M}_n^o| = o_p\left( \frac{1}{\log p} \right) \tilde{M}_n^o = o_p(1).
\]
Thus it suffices to prove that for any $t \in \mathbb{R}$,
\[
P(\tilde{M}_n^o - 4 \log p + \log \log p \leq t) \to \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{t}{2}\right)\right).
\]
Let $X_k B^{-1/2} = Z_k = (Z_{k,i,j})$, where the columns of $Z_k$ are independent. We arrange the indices $\{(i, j) : 1 \leq i < j \leq p\}$ in any ordering and set them as $\{(i_m, j_m) : m = 1, \ldots, s\}$ with $s = p(p - 1)/2$. Let $\theta_m = \theta_{i_m,j_m}$ and define $Y_{k,m,l} = Z_{k,i_m,l} Z_{k,j_m,l}$ for $1 \leq k \leq n$ and $1 \leq l \leq q$, $V_m = (nq\theta_m)^{-1/2} \sum_{k=1}^n \sum_{l=1}^q Y_{k,m,l}$, and $\hat{V}_m = (nq\theta_m)^{-1/2} \sum_{k=1}^n \sum_{l=1}^q \hat{Y}_{k,m,l}$, where $\hat{Y}_{k,m,l} =$
\[ Y_{k,m,l} \mid |Y_{k,m,l}| \leq \tau_n \] 
\[ - \mathbb{E}\{Y_{k,m,l} \mid |Y_{k,m,l}| \leq \tau_n\}, \quad \text{and} \quad \tau_n = 32 \log(p + nq). \]

Note that \( \tilde{M}_n^a = \max_{1 \leq m \leq s} V_m^2. \)

The rest of the proof is completely the same as the proof of Theorem 1 in [Xia and Li (2017)].

We skip the details here. ■

**Proof of Lemma 5**

For a matrix \( A \), denote the element-wise infinity norm as \( \|A\|_\infty = \max_{1 \leq i,j \leq p} |a_{i,j}| \) and denote by \( \|A\|_2 \) the spectral norm of \( A \).

We first show \( \|\tilde{B} - B\|_\infty = O_p(\{\log q/np\}^{1/2}). \)

Let \( \tilde{B} =: (\tilde{b}_{i,j}) \) and \( B =: (b_{i,j}) \) and recall we have assumed \( \mathbb{E}\tilde{B} = B \). Let \( H_k =: A^{-1/2}X_k \) and \( h_i^{(k)} \) is the i-th column of \( H_k \). It is easy to know the components of \( h_i^{(k)} \) are independent. Then, we have

\[
\tilde{b}_{i,j} = \frac{1}{np} \sum_{k=1}^{n} (h_i^{(k)})^T A h_j^{(k)}
\]

Let \( h_i^T := ((h_i^{(1)})^T, (h_i^{(2)})^T, \cdots, (h_i^{(n)})^T) \) and

\[
A^{(n)} := \begin{pmatrix}
A & A \\
\cdots & \cdots \\
A & A
\end{pmatrix},
\]

we obtain

\[
\tilde{b}_{i,j} = \frac{1}{np} h_i^T A^{(n)} h_j.
\]

By Condition (C2) and Lemma 3, together with the fact that \( \|A^{(n)}\|_F^2 \leq Cnp \) for some constant \( C > 0 \), we have \( \|\tilde{B} - B\|_\infty = O_p(\{\log q/np\}^{1/2}) \). This yields that \( \|\tilde{B}^{-1} - B^{-1}\|_2 = \|B^{-1}(\tilde{B} - B)\tilde{B}^{-1}\|_2 = O_p(\{\log q/(np)\}^{1/2}), \) which further implies that

\[
\|B^{1/2}(\tilde{B}^{-1} - B^{-1})B^{1/2}\|_\infty \leq \|B^{1/2}(\tilde{B}^{-1} - B^{-1})B^{1/2}\|_2 = O_p(\{\log q/(np)\}^{1/2}). \quad (38)
\]
Notice that
\[
\|\hat{A} - \hat{A}^0\|_\infty = \left\| \frac{1}{nq} \sum_{k=1}^{n} X_k B^{-1/2} B^{1/2} \left( \tilde{B}^{-1} - B^{-1} \right) B^{1/2} (X_k B^{-1/2})' \right\|_\infty \\
\leq \max_{1 \leq i, j \leq p} \|B^{1/2} (\tilde{B}^{-1} - B^{-1}) B^{1/2}\|_\infty \frac{1}{nq} \sum_{k=1}^{n} \left( \sum_{l=1}^{q} Z_{k,i,l} \right) \left( \sum_{l=1}^{q} Z_{k,j,l} \right).
\]

By (38) and independence of \(Z_{k,i,l}\), we have
\[
\|\hat{A} - \hat{A}^0\|_\infty = O_p(q \log \max(p, q, n) \{\log q/(np)\}^{1/2}).
\]

Combined with Condition (C3) we can conclude
\[
|\sqrt{M_n^0} - \sqrt{\tilde{M}_n^0}| = O_p(\sqrt{nq}) \max_{1 \leq i, j \leq p} |\hat{a}_{i,j} - \hat{a}^0_{i,j}| = o_p((\log p)^{-1/2}),
\]

Notice that when Conditions (C1) and (C2) hold, \(M_n^0 = O_p(\log p)\). Thus we have
\[
|M_n^0 - \tilde{M}_n^0| = o_p(1),
\]

then (33) is obtained.

To prove (34), we first show that
\[
\max_{1 \leq i, j \leq p} |\hat{\theta}_{i,j} - \hat{\theta}^0_{i,j}| = O_p(\{q^3 \log q \log \max(p, q, n)/(np)\}^{1/2}). \quad (39)
\]

Note that
\[
\hat{\theta}_{i,j} - \hat{\theta}^0_{i,j} = \\
\frac{1}{nq} \sum_{k=1}^{n} \sum_{l=1}^{q} \left[ (X_k \tilde{B}^{-1/2})_{i,l} (X_k (\tilde{B}^{-1/2} - B^{-1/2}))_{j,l} + (X_k (\tilde{B}^{-1/2} - B^{-1/2}))_{i,l} (X_k B^{-1/2})_{j,l} \right] \\
\left[ (X_k \tilde{B}^{-1/2})_{i,l} (X_k \tilde{B}^{-1/2})_{j,l} + (X_k B^{-1/2})_{i,l} (X_k B^{-1/2})_{j,l} - (\hat{a}_{i,j} + \hat{a}^0_{i,j}) \right],
\]

it suffices to prove that uniformly for \(1 \leq i, j \leq p, \)
\[
\frac{1}{nq} \sum_{k=1}^{n} \sum_{l=1}^{q} (X_k \tilde{B}^{-1/2})_{i,l} (X_k (\tilde{B}^{-1/2} - B^{-1/2}))_{j,l} ((X_k B^{-1/2})_{i,l} (X_k B^{-1/2})_{j,l} - \hat{a}^0_{i,j}) \\
= O_p(\{q^3 \log q \log \max(p, q, n)/np\}^{1/2}). \quad (40)
\]

Since
\[
\|\tilde{B}^{-1} - B^{-1}\|_2 = O_p(q \{\log q/np\}^{1/2}), \quad (41)
\]

then (34) is obtained.
by Lemma 1 of [Chen and Xia (2021)], we have that \( \| \tilde{B}^{-1/2} - B^{-1/2} \|_2 = O_p(q\{\log q/(np)\}^{1/2}) \), then we have that
\[
\| X_k(\tilde{B}^{-1/2} - B^{-1/2}) \|_\infty = O_p(\{q^3 \log q \log \max(p, q, n)/(np)\}^{1/2}),
\]
and uniformly in \( 1 \leq i, j \leq p \),
\[
\frac{1}{nq} \sum_{k=1}^n \sum_{l=1}^q Z_{k,i,l}^2 Z_{k,j,l} = O_p(1),
\]
where \( Z_{k,i,l} = (Z_k)_{i,l} \) and \( X_k B^{-1/2} = Z_k \). Thus it is easy to show that
\[
\frac{1}{nq} \sum_{k=1}^n \sum_{l=1}^q (X_k \tilde{B}^{-1/2})_{i,l} (X_k (\tilde{B}^{-1/2} - B^{-1/2}))_{j,l} (X_k B^{-1/2})_{i,l} (X_k B^{-1/2})_{j,l}
\]
\[
= O_p(\{q^3 \log q \log \max(p, q, n)/(np)\}^{1/2}).
\]
Similarly,
\[
\frac{1}{nq} \sum_{k=1}^n \sum_{l=1}^q (X_k \tilde{B}^{-1/2})_{i,l} (X_k (\tilde{B}^{-1/2} - B^{-1/2}))_{j,l} \hat{a}_{i,j}^2 = O_p(\{q^3 \log q \log \max(p, q, n)/(np)\}^{1/2}).
\]
Hence (40) is proved and in turn (39) is proved.

Under Condition (C3), \( \{q^3 \log q \log \max(p, q, n)/(np)\}^{1/2} = o(1/\log p) \), so \( |\widehat{\theta}_{i,j}/\hat{\theta}_{i,j}^\circ - 1| = o_p(1/\log p) \).

By Lemma 4 and (33), we have \( |M_n - \hat{M}_n^\circ| = o_p(1/\log p) \hat{M}_n^\circ = o_p(1) \). Hence the proof of (34) is completed. 

Proof of Lemma 6

WLOG, assume \( a_{i,i}^{(g)} = 1 \) for \( g = 1, 2 \) and \( i = 1, \ldots, p \). Let \( n = \max(n_1, n_2) \) and \( A = \{(i, j) : 1 \leq i \leq j \leq p\} \). Define
\[
\tilde{M}_{i,j}^{\star, \circ} = \frac{(\hat{r}_{i,j}^{(1)} - \hat{r}_{i,j}^{(2)})^2}{\hat{\theta}_{i,j}^{(1)}/(n_1 q) + \hat{\theta}_{i,j}^{(2)}/(n_2 q)},
\]
and
\[
\tilde{M}_n^{\star, \circ} = \max_{1 \leq i, j \leq p} \tilde{M}_{i,j}^{\star, \circ}.
\]
where \( \theta_{ij}^{(g)} = \theta_{ij}^{(g)}/(a_{ij}^{(g)} a_{j,j}^{(g)}) = \theta_{ij}^{(g)} \). From Lemma 1, we have \( |\hat{\theta}_{ij}^{(g)} / \theta_{ij}^{(g)} - 1| = o_p(1 / \log p) \). Thus it suffices to prove that for any \( t \in \mathbb{R} \),

\[
P(\hat{M}_{n}^{*,o} - 4 \log p + \log \log p \leq t) \rightarrow \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{t^2}{2}\right)\right).
\]

Let \( X_k^{(g)} (B^{(g)})^{-1/2} = Z_k^{(g)} \), where the columns of \( Z_k^{(g)} \) are independent for \( g = 1, 2 \). Define

\[
V_{i,j} = \frac{U_{i,j}^{(1)} - U_{i,j}^{(2)}}{\left(\hat{\vartheta}_{i,j}^{[1]} / (n_1 q) + \hat{\vartheta}_{i,j}^{[2]} / (n_2 q)\right)^{1/2}},
\]

where

\[
U_{i,j}^{(g)} = \frac{1}{n_g q} \sum_{k=1}^{n_g} \sum_{l=1}^{q} \left(Z_{k,i}^{(g)} Z_{k,j}^{(g)} \theta_{ij}^{(g)} - a_{i,j}^{(g)}\right) = \hat{a}_{i,j}^{(g)} - a_{i,j}^{(g)},
\]

for \( g = 1, 2 \). Note that \( \max_{1 \leq i,j \leq p} |\hat{a}_{i,j}^{(g)} - a_{i,j}^{(g)}| = O_p(\sqrt{\log p / n_q}) \) and \( O_p(\sqrt{\log p / n_q}) \). Thus it is easy to obtain \( \max_{1 \leq i,j \leq p} |\hat{a}_{i,j}^{(g)} - a_{i,j}^{(g)}| = o_p((nq \log p)^{-1/2}) \) by the assumption (C1*), we can obtain

\[
\hat{r}_{i,j}^{(g)} - r_{i,j}^{(g)} = a_{i,j}^{(g)} \left(\frac{1 - \hat{a}_{i,j}^{(g)} a_{j,j}^{(g)}}{\sqrt{a_{i,j}^{(g)} a_{j,j}^{(g)}} (1 + \sqrt{a_{i,j}^{(g)} a_{j,j}^{(g)}})}\right) + U_{i,j}^{(g)}
\]

\[
= \frac{1}{2} a_{i,j}^{(g)} (1 - \hat{a}_{i,j}^{(g)} + 1 - \hat{a}_{j,j}^{(g)}) + U_{i,j} + o_p((nq \log p)^{-1/2}),
\]

uniformly for \( 1 \leq i \leq j \leq p \). Thus we have

\[
\hat{r}_{i,j}^{(g)} - r_{i,j}^{(g)} = U_{i,j}^{(g)} + O_p((\log p/(nq))^{1/2}) r_{i,j}^{(g)} + o_p((nq \log p)^{-1/2}).
\]

Note that \( |\hat{r}_{i,j}^{(g)} - r_{i,j}^{(g)}| = o_p(1 / \log p) \) and that for \( (i,j) \in A \setminus A_r \), we have \( |r_{i,j}^{(g)}| = o((\log p)^{-1}) \). Thus from (44), it is easy to obtain \( \max_{(i,j) \in A \setminus A_r} |\hat{M}_{i,j}^{*,o} - V_{i,j}| = o_p((\log p)^{-1/2}) \). For \( (i,j) \in A_r \), by (43) we have

\[
\sqrt{M_{i,j}^{*,o}} = V_{i,j} + t_{i,j} + o_p((\log p)^{-1/2}),
\]

where \( t_{i,j} = \frac{1}{2} a_{i,j} (\hat{a}_{i,i}^{(1)} - a_{i,i}^{(1)} - (\hat{a}_{j,j}^{(1)} - a_{j,j}^{(1)})) / (\hat{r}_{i,j}^{(1)}/(n_1 q) + \hat{r}_{i,j}^{(2)}/(n_2 q))^{1/2} \). Here \( a_{i,j} = a_{i,j}^{(1)} = a_{i,j}^{(2)} \) under the null hypothesis \( H_0^* \). By Condition (C4*), under the null hypothesis \( H_0^* \) we can calculate \( \hat{r}_{i,j}^{(1)} = (2\kappa_1 - 1)a_{i,j}^{2} + \kappa_1, \hat{r}_{i,j}^{(2)} = (2\kappa_2 - 1)a_{i,j}^{2} + \kappa_2 \) and \( \hat{r}_{i,j}^{(1)} = \hat{r}_{i,j}^{(2)} = 3\kappa_1 - 1, \hat{r}_{i,j}^{(2)} = \hat{r}_{i,j}^{(2)} = 3\kappa_2 - 1 \). Since

\[
\frac{(2\kappa_1 - 1)a_{i,j}^{2} + \kappa_1}{a_{i,j}^{2}} \geq 3\kappa_1 - 1 \quad \text{and} \quad \frac{(2\kappa_2 - 1)a_{i,j}^{2} + \kappa_2}{a_{i,j}^{2}} \geq 3\kappa_2 - 1,
\]

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we can obtain

\[
|t_{i,j}| \leq \frac{1}{2} \left( \frac{|\hat{a}_{i,i}^{o,(1)} - \hat{a}_{i,i}^{o,(2)}|}{(\hat{\vartheta}_{i,i}^{(1)}/(n_1 q) + \vartheta_{i,i}^{(2)}/(n_2 q))^{1/2}} + \frac{|\hat{a}_{j,j}^{o,(1)} - \hat{a}_{j,j}^{o,(2)}|}{(\hat{\vartheta}_{j,j}^{(1)}/(n_1 q) + \vartheta_{j,j}^{(2)}/(n_2 q))^{1/2}} \right),
\]

and this inequality further implies that

\[
P(\max_{(i,j)\in A_{\gamma}} |\hat{M}_{i,j}^{*o} - \hat{M}_{i,j}^{o}| \geq 4 \log p - \log \log p + t) \leq \text{Card}(A_{\gamma})\{ P(V_{i,j}^2 \geq (2 - 2\nu) \log p) + P(t_{i,j}^2 \geq (2 - 2\nu) \log p) \} = o(1),
\]

where the last equality is a direct result of Lemma 2 and Condition (C5"). Thus it suffices to prove that for any \( t \in \mathbb{R} \),

\[
P(\max_{(i,j)\in A\setminus A_{\gamma}} V_{i,j}^2 - 4 \log p + \log \log p \leq t) \to \exp\left(-\frac{1}{\sqrt{8\pi}} \exp\left(-\frac{t}{2}\right)\right)
\]

The rest of the proof is essentially proved in Xia et al. (2015). We skip the details here. □

**Proof of Lemma 7**

Notice that \( \max_{1 \leq i,j \leq p} |\hat{r}_{i,j}^{(g)} - \hat{r}_{i,j}^{o,(g)}| = O_p(\max_{1 \leq i,j \leq p} |\hat{a}_{i,j}^{(g)} - \hat{a}_{i,j}^{o,(g)}|) \) for \( g = 1, 2 \). Similar to the proof of (33), we can get (36). On the other hand, \( \max_{1 \leq i,j \leq p} |\hat{\vartheta}_{i,j}^{o,(g)} - \hat{\vartheta}_{i,j}^{(g)}| = O_p(\max_{1 \leq i,j \leq p} |\hat{\theta}_{i,j}^{o,(g)} - \hat{\theta}_{i,j}^{(g)}|) \) for \( g = 1, 2 \). Thus similar to the proof of (34), we can get (37). □

### A.4 Additional Simulations

In this section, we show some additional simulations. Online Supplement A.4.1 provides additional hypothesis test experiment under normal distribution; Online Supplement A.4.2 lists the simulation results under \( t \) distribution; Online Supplement A.4.3 is devoted to the simulation results for support recovery; Pseudo simulation based on real data application is given in Online Supplement A.4.4.

#### A.4.1 Additional Simulation with Normal Distribution

In Section 5.1, we performed the simulation for one-sample test under normal distribution for \( p = \{50, 200\} \), \( q = \{50, 200\} \), and \( n = \{10, 50\} \), whose results is given in Table 2. Table 5 offers additional result on the empirical size when sample size is \( n = 500 \) for our covariance matrix based methods. It can be seen that as sample size gets larger, the proposed tests are no longer undersized.

In Section 5.2 on two-sample test, we generated the data according to \( X^{(g)}_k \sim \mathcal{MVN}(0, A^{(g)}, B^{(g)}) \), where \( B^{(g)} \) are the autocorrelation matrices of AR(1) process. For the target covariance matrix,
Table 5: The empirical size of the testing procedures for the one-sample case under normal distribution based on 1000 replications. The percentages are shown with the standard errors provided in parentheses. The significant level is $\alpha = 5\%$. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n = \{500\}$.

| $p$  | methods                | $q = 50$ | $q = 200$ |
|------|------------------------|----------|-----------|
| 50   | One sample cov: oracle | 6.0(0.8) | 5.5(0.7)  |
|      | One sample cov: sample-est | 5.1(0.7) | 5.1(0.7)  |
|      | One sample cov: banded-est | 5.9(0.7) | 4.9(0.7)  |
| 200  | One sample cov: oracle | 3.7(0.6) | 4.7(0.7)  |
|      | One sample cov: sample-est | 3.9(0.6) | 5.0(0.7)  |
|      | One sample cov: banded-est | 3.8(0.6) | 4.5(0.7)  |

under the null, we set $A^{(1)} = A^{(2)} = \Sigma^{(1)}$; under the alternative, we set $(A^{(1)})^{-1} = (\Sigma^{(1)} + \delta I)/(1 + \delta)$ and $(A^{(2)})^{-1} = (\Sigma^{(1)} + U + \delta I)/(1 + \delta)$. In this section, we use $\Sigma^{(2)}$, Model 2 of Cai et al. (2013), to replace $\Sigma^{(1)}$, Model 1 of Cai et al. (2013). Here, $\Sigma^{(2)} = D^{1/2}\Sigma^{* (2)} D^{1/2}$ and $\Sigma^{* (2)} = (\sigma_{i,j}^{* (2)})$, where $\sigma_{i,j}^{* (2)} = 0.5|i-j|$ for $1 \leq i, j \leq p$, and $D = (d_{i,j})$ is a diagonal matrix with diagonal elements $d_{i,i} = \text{Unif}(0.5, 2.5)$ for $i = 1, \ldots, p$.

Table 6 shows the simulation result for $\Sigma^{(2)}$ corresponding to Table 4 for $\Sigma^{(1)}$. The key message is almost identical to 4. But sizes in Table 6 are generally larger that those in Table 4 and closer to 0.05. This demonstrates that the sizes vary depending on the model setting: $\Sigma^{(1)}$ has exact sparsity while $\Sigma^{(2)}$ has weak sparsity with many nonzero entries but small magnitude.

### A.4.2 Simulation with $t_3$ Distribution

We perform simulation study under very heavy tail distribution $t_4$ for both the one-sample and the two-sample cases in this section.

For the one-sample case, we simulate the data according to $X_k = A^{1/2}W_kB^{1/2}$ for $k = 1, \ldots, n$. Here, $W_k$, $k = 1, \ldots, n$, are independent random matrices of dimension $p \times q$, where all of the entries are iid with $t_3$ distribution. This way, the covariance matrix adopts the Kronecker product form $\text{Cov}(\text{vec}(X_k)) = B \otimes A$. $B$ and $A$ are designed in the same fashion as in Section 5.1, except when under the alternative, $A = (I + U + \delta I)/(1 + \delta)$, where the magnitude of the nonzero entries of $U$ follow uniform distribution on $[3\{\log p/(nq)\}^{1/2}, 5\{\log p/(nq)\}^{1/2}]$. The remaining settings are
Table 6: The empirical size and power of the testing procedures for the two-sample case under normal distribution based on 1000 replications. The percentages are shown with the standard errors provided in parentheses. The $\Sigma^{(2)}$ matrix adopts the form of Model 2 in Cai et al. (2013). The significant level is $\alpha = 5\%$. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n_1 = n_2 = n = \{10, 50\}$.
identical to Section 5.1. Table 7 is the counterpart of Table 2 when normal distribution is changed to $t$ distribution. It provides the size of the seven relevant methods and power of our methods with significance level $\alpha = 0.05$. It can be seen that the three methods (iv)-(vi), that are proposed by Xia and Li (2017), and the vector-based approach (vii), cannot control the sizes while ours (i)-(iii) can. This magnifies the strong dependence of precision matrix based methods on the Gaussian assumption.

For the two-sample case, we simulate the data accordingly, $X_k^{(g)} = (A^{(g)})^{1/2} \mathbf{W}_k^{(g)} (B^{(g)})^{1/2}$ for $k = 1, \ldots, n_g$, $g = 1, 2$. Again, $\mathbf{W}_k^{(g)}$ for $k = 1, \ldots, n_g$, $g = 1, 2$ are independent random matrices of size $p \times q$, where all of the entries are iid with $t_3$ distribution. $B^{(g)}$ is generated in the same way as in Section 5.2. When $A^{(g)}$ is generated according to Model 1 in Section 5.2 Table 8 presents the results, which is the counterpart of Table 4. When $A^{(g)}$ is generated according to Model 2 in Section A.4.1, Table 9 presents the results, which is the counterpart of Table 6. Both Tables 8 and 9 show again that the precision matrix based methods are not valid with sizes much larger than 0.05. Furthermore, comparing these two tables, it is proved again that the relative sizes of our methods depend on the true model settings. Model 1 tends to lead to more undersized results comparing to Model 2.

A.4.3 Simulation for Support Recovery

In this section, we perform simulation experiments on support recovery for both one-sample and two-sample cases under both normal and $t_3$ distributions. We only demonstrate the performance of our methods (i)-(iii) and the vector-based method (vii) in Cai et al. (2013). The precision matrix based methods (iv)-(vi) are excluded. This is because the support recovered by them is the locations of the nonzero partial correlations, not the nonzero correlations; hence they are not comparable with ours.

Following Cai et al. (2013), the accuracy of the support recovery is evaluated by the similarity measure $s(\hat{\Psi}, \Psi)$, defined as

$$s(\hat{\Psi}, \Psi) = \frac{|\hat{\Psi} \cap \Psi|}{\sqrt{|\hat{\Psi}||\Psi|}},$$

where $\Psi$ is the true support, $\hat{\Psi}$ is the estimated support, and $|\cdot|$ denotes the cardinality. Note that the similarity measure takes value between zero and one: $s(\hat{\Psi}, \Psi) = 1$ indicates perfect recovery, and $s(\hat{\Psi}, \Psi) = 0$ implies complete failure.
Table 7: The empirical size and power of the testing procedures for the one-sample case under $t_3$ distribution based on 1000 replications. The percentages are shown with the standard errors provided in parentheses. The significant level is $\alpha = 5\%$. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n = \{10, 50\}$. 

| $p$ | methods                            | $n = 10$ | $n = 50$ |
|-----|------------------------------------|----------|----------|
|     |                                    | $q = 50$ | $q = 200$| $q = 50$ | $q = 200$ |
|-----|------------------------------------|----------|----------|
|     | **Empirical size**                 |          |          |
| 50  | One sample cov: oracle             | 0.5(0.2) | 1.6(0.4) | 1.5(0.4) | 3.7(0.6) |
|     | One sample cov: sample-est         | 0.9(0.3) | 0(0)     | 1.8(0.4) | 1.8(0.4) |
|     | One sample cov: banded-est         | 1.8(0.4) | 1.4(0.4) | 1.9(0.4) | 3.2(0.6) |
| 200 | One sample pre: oracle             | 51.6(1.6)| 42.9(1.6)| 41.8(1.6)| 31.6(1.5)|
|     | One sample pre: sample-est         | 11.2(1)  | 0(0)     | 22.0(1.3)| 6.7(0.8) |
|     | One sample pre: banded-est         | 20(1.3)  | 3.6(0.6) | 23.9(1.3)| 10.8(1.0)|
|     | One sample vector                  | 25.8(1.4)| 31.3(1.5)| 27.9(1.4)| 38.1(1.5)|
|     | **Empirical power**                |          |          |
| 50  | One sample cov: oracle             | 93.1(0.8)| 90.2(0.9)| 90.6(0.9)| 87.8(1.0)|
|     | One sample cov: sample-est         | 87.2(1.1)| 8.4(0.9) | 89.1(1.0)| 77.1(1.3)|
|     | One sample cov: banded-est         | 95.0(0.7)| 83.5(1.2)| 91.7(0.9)| 87.9(1.0)|
| 200 | One sample cov: oracle             | 93.3(0.8)| 92.0(0.9)| 91.2(0.9)| 89.4(1.0)|
|     | One sample cov: sample-est         | 91.2(0.9)| 77.8(1.3)| 91.2(0.9)| 86.9(1.1)|
|     | One sample cov: banded-est         | 92.8(0.8)| 91.5(0.9)| 91.4(0.9)| 89.4(1.0)|
| $p$ | methods                  | $n = 10$ | $n = 50$ |
|-----|--------------------------|----------|----------|
|     |                          | $q = 50$ | $q = 200$| $q = 50$ | $q = 200$|
|-----|--------------------------|----------|----------|
|     | Empirical size           |          |          |
| 50  | Two sample cov: oracle   | 1.0(0.3) | 2.5(0.5) | 3.4(0.6) | 2.7(0.5) |
|     | Two sample cov: sample-est| 0.5(0.2) | 0.1(0.1) | 2.4(0.5) | 0.6(0.2) |
|     | Two sample cov: banded-est| 1.3(0.4) | 0.8(0.3) | 2.8(0.5) | 1.7(0.4) |
| 200 | Two sample pre: oracle   | 39.5(1.5)| 32.6(1.5)| 32.0(1.5)| 30(1.4)  |
|     | Two sample pre: sample-est| 10.5(1.0)| 1.3(0.4) | 17.2(1.2)| 8.4(0.9) |
|     | Two sample pre: banded-est| 14.4(1.1)| 8.8(0.9) | 16.0(1.2)| 8.9(0.9) |
|     | Two sample vector        | 100(0.0) | 100(0.0) | 100(0.0)| 100(0.0) |
|     | Empirical power          |          |          |
| 50  | Two sample cov: oracle   | 2.2(0.5) | 2.0(0.4) | 1.7(0.4) | 3.3(0.6) |
|     | Two sample cov: sample-est| 1.5(0.4) | 0.4(0.2) | 1.1(0.3) | 1.4(0.4) |
|     | Two sample cov: banded-est| 2.3(0.5) | 1.5(0.4) | 1.3(0.4) | 3.5(0.6) |
| 200 | Two sample pre: oracle   | 94.0(0.8)| 93.4(0.8)| 92.2(0.8)| 85.0(1.1)|
|     | Two sample pre: sample-est| 76.8(1.3)| 45.8(1.6)| 82.7(1.2)| 58.0(1.6)|
|     | Two sample pre: banded-est| 76.6(1.3)| 53.6(1.6)| 83.2(1.2)| 58.9(1.6)|
|     | Two sample vector        | 100(0.0) | 100(0.0) | 100(0.0)| 100(0.0) |
|     | Empirical power          |          |          |
| 50  | Two sample cov: oracle   | 52.7(1.6)| 65.4(1.5)| 63.7(1.5)| 68.2(1.5)|
|     | Two sample cov: sample-est| 36.9(1.5)| 1.3(0.4) | 60.4(1.5)| 52.2(1.6)|
|     | Two sample cov: banded-est| 41.2(1.6)| 38.9(1.5)| 60.6(1.5)| 63.2(1.5)|
| 200 | Two sample cov: oracle   | 83.6(1.2)| 83.9(1.2)| 84.2(1.2)| 85.8(1.1)|
|     | Two sample cov: sample-est| 84.0(1.2)| 75.7(1.4)| 84.9(1.1)| 86.6(1.1)|
|     | Two sample cov: banded-est| 84.4(1.1)| 83.6(1.2)| 84.8(1.1)| 87.3(1.1)|

Table 8: The empirical size and power of the testing procedures for the two-sample case under $t_3$ distribution based on 1000 replications. The percentages are shown with the standard errors provided in parentheses. The $\mathbf{\Sigma}^{(1)}$ matrix adopts the form of Model 1 in [Cai et al. (2013)](http://example.com). The significant level is $\alpha = 5\%$. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n_1 = n_2 = n = \{10, 50\}$. 

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Table 9: The empirical size and power of the testing procedures for the two-sample case under $t_3$ distribution based on 1000 replications. The percentages are shown with the standard errors provided in parentheses. The $\Sigma^{(2)}$ matrix adopts the form of Model 2 in Cai et al. (2013). The significant level is $\alpha = 5\%$. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n_1 = n_2 = n = \{10, 50\}$.
Table 10: The support recovery performance of our methods (i)-(iii) and the vector-based method (vii) for the one-sample case under normal distribution based on 100 replications. The similarity measure is shown in percentage and the standard errors are provided in parentheses. The number of observations and the dimensions of the matrices vary: $p = \{50,200\}$, $q = \{50,200\}$, and $n = \{10,50\}$.

| p   | methods          | $n = 10$ | $q = 50$ | $q = 200$ | $n = 50$ | $q = 50$ | $q = 200$ |
|-----|-----------------|---------|----------|----------|---------|----------|----------|
| 50  | One sample cov: oracle | 99.8(0.1) | 99.5(0.1) | 99.6(0.1) | 96.6(0.3) |         |          |
|     | One sample cov: sample-est | 99.9(0)  | 70.8(0.8) | 99.5(0.1) | 93.6(0.4) |         |          |
|     | One sample cov: banded-est | 99.2(0.1) | 98.7(0.2) | 98.6(0.1) | 94.7(0.3) |         |          |
|     | One sample vector | 51.5(0.3) | 45.0(0.3) | 45.6(0.3) | 40.0(0.3) |         |          |
| 200 | One sample cov: oracle | 99.9(0)  | 99.8(0.1) | 98.8(0.1) | 97.7(0.2) |         |          |
|     | One sample cov: sample-est | 99.9(0)  | 99.6(0.1) | 98.8(0.1) | 97.2(0.2) |         |          |
|     | One sample cov: banded-est | 99.1(0.1) | 98.7(0.1) | 97.1(0.2) | 95.2(0.4) |         |          |
|     | One sample vector | 21.4(0.1) | 18.7(0.1) | 17.8(0.1) | 16.0(0.2) |         |          |

One-sample support recovery. Under normal distribution, we generate the data with the same model as in the one-sample global testing in Section 5.1 except that the covariance matrix $B$ is the autocorrelation matrix of AR(1) process with coefficient 0.8 and the symmetric perturbation matrix $U$ has 50 random nonzero entries of magnitude $5\{\log p/(nq)\}^{1/2}$. Under $t_3$ distribution, the same setting of $A$ and $B$ is implemented, and the data are produced with $X_k = A^{1/2}W_kB^{1/2}$ for $k = 1, \ldots, n$, where $W_k$ are iid random matrices with iid entries of $t_3$ distributions.

Tables [10] and [11] are for the normal and $t_3$ distributions respectively. The means and standard errors of $s(\hat{\Psi}, \Psi)$ for the four methods based on 100 replications are given. Our methods (i)-(iii) have similarity measures that are close to one while the vector-based method (vii) cannot recover the support well. If we further increase the signal strength $5\{\log p/(nq)\}^{1/2}$ to some larger value, our power will be even closer to one.

Two-sample support recovery. The data are simulated according to the same $\Sigma^{(1)}$ and $\delta$ as in the two-sample global testing in Section 5.2. However, in this case $A^{(1)} = (\Sigma^{(1)} + \delta I)/(1 + \delta)$
Table 11: The support recovery performance of our methods (i)-(iii) and the vector-based method (vii) for the one-sample case under $t_3$ distribution based on 100 replications. The similarity measure is shown in percentage and the standard errors are provided in parentheses. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n = \{10, 50\}$.
Table 12: The support recovery performance of our methods (i)-(iii) and the vector-based method (vii) for the two-sample case under normal distribution based on 100 replications. The $\Sigma^{(1)}$ matrix adopts the form of Model 1 in [Cai et al. 2013]. The similarity measure is shown in percentage and the standard errors are provided in parentheses. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n_1 = n_2 = n = \{10, 50\}$.

and $\mathbf{A}^{(2)} = (\Sigma^{(1)} + \mathbf{U} + \delta \mathbf{I})/(1 + \delta)$, where the perturbation matrix $\mathbf{U}$ has 50 nonzero entries with magnitude $5\{\log p/(nq)\}^{1/2}$. Tables 12 and 13 provide the means and the standard errors of the same measurement $s(\hat{\Psi}, \Psi)$ over 100 replications for normal and $t_3$ distributions respectively. Again, they demonstrate the advantage of preserving the matrix structure over the simple vectorization.

Note that in the experiments above, we set the signal strength, the magnitude of the support, to be $5\{\log p/(nq)\}^{1/2}$, which varies with the sample size $n$ and dimensions $p$, $q$. Such a setting was chosen because the theorems in Online Supplement A.1 imply the smallest signal over the support should be of rate $\{\log p/(nq)\}^{1/2}$. With this setting, the similarity measures in Tables 10-13 do not exhibit the features such that they increase with $n$ and $q$ and decrease with $p$. To make our intuitions right, Table 14 offers the results of support recovery of the one sample case under normal distribution, when the signal strength does not vary with $n, p, q$ and stays fixed at level 0.12. It is clearly seen that when $n$ or $q$ increases, similarity measure increases, and when $p$
Table 13: The support recovery performance of our methods (i)-(iii) and the vector-based method (vii) for the two-sample case under $t_3$ distribution based on 100 replications. The $\Sigma^{(1)}$ matrix adopts the form of Model 1 in [Cai et al., 2013]. The similarity measure is shown in percentage and the standard errors are provided in parentheses. The number of observations and the dimensions of the matrices vary: $p = \{50, 200\}$, $q = \{50, 200\}$, and $n_1 = n_2 = n = \{10, 50\}$.
Table 14: The support recovery performance of our methods (i)-(iii) and the vector-based method (vii) for the one-sample case under normal distribution with fixed-level signal based on 100 replications. The similarity measure is shown in percentage and the standard errors are provided in parentheses. The number of observations and the dimensions of the matrices vary: \( p = \{50, 200\}, q = \{50, 200\}, \) and \( n = \{10, 50\}. \)

| p    | methods                           | \( q = 50 \) | \( q = 200 \) | \( q = 50 \) | \( q = 200 \) |
|------|-----------------------------------|-------------|--------------|-------------|--------------|
| 50   | One sample cov: oracle            | 11.8(1)     | 38.9(0.9)    | 51.3(0.9)   | 99.6(0.1)    |
|      | One sample cov: sample-est        | 7.6(1.2)    | 20(0)        | 49.3(0.8)   | 99.4(0.1)    |
|      | One sample cov: banded-est        | 9.8(1)      | 39.6(0.9)    | 52.9(0.9)   | 99(0.1)      |
|      | One sample vector                 | 6.6(0.4)    | 17.9(0.5)    | 22.1(0.5)   | 46.6(0.3)    |
| 200  | One sample cov: oracle            | 6(1)        | 23.7(0.8)    | 29.3(0.9)   | 98.6(0.2)    |
|      | One sample cov: sample-est        | 8.6(1.1)    | 20.3(0.7)    | 29.5(0.8)   | 98.3(0.2)    |
|      | One sample cov: banded-est        | 2.6(0.6)    | 21(1.1)      | 32.4(1)     | 96.7(0.3)    |
|      | One sample vector                 | 1.4(0.1)    | 4.3(0.2)     | 5.8(0.2)    | 17(0.1)      |

increases, similarity measure decreases.

### A.4.4 Pseudo Simulation Based on Real Data Application

In this section, two pseudo experiments are performed with information from the real data.

Experiment 1: We use the sample sizes and dimensions from the real data to mimic the real world while keeping the model parameters \( A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)} \) the same as in Section 5.2 under \( t_3 \) distribution.

Table 15 provides the results for Experiment 1. The precision matrix based methods are invalid due to larger empirical sizes than nominal level. For our methods, “two sample cov: oracle” and “two sample: banded-est” have sizes close to 0.05 under the null and remain powerful under the alternative; “two sample cov: sample-est” is severely undersized under the null and not quite powerful under the alternative because with large \( q \approx 200 \) and small \( n \approx 20, p \approx 30, \) the sample estimate of \( B \in \mathbb{R}^{200 \times 200} \) is not very accurate. Since \( B^{(1)}, B^{(2)} \) correspond to the covariance
matrices of autoregressive time series, they have parsimonious banded structure, which makes banded estimation of $B$ much more accurate and hence “two sample: banded-est” is preferred over “two sample cov: sample-est” when $B$ corresponds to the temporal dimension in practice.

| Method                  | Empirical size | Empirical power |
|-------------------------|----------------|-----------------|
| Two sample cov: oracle  | 2.4(0.5)       | 91.3(0.9)       |
| Two sample cov: sample-est | 0(0)       | 14.5(1.1)       |
| Two sample cov: banded-est | 5.4(0.7)   | 74.8(1.4)       |
| Two sample pre: oracle  | 19.9(1.3)     | 75.3(1.4)       |
| Two sample pre: sample-est | 2.9(0.5)   | 10.8(1.0)       |
| Two sample pre: banded-est | 6.4(0.8)   | 49.5(1.6)       |
| Two sample vector       | 100(0)        | 100(0)          |

Table 15: The empirical size and power of the testing procedures for the two-sample case under $t_3$ distribution based on 1000 replications. The percentages are shown with the standard errors provided in parentheses. The $\Sigma^{(1)}$ matrix adopts the form of Model 1 in Cai et al. (2013). The significant level is $\alpha = 5\%$. The number of observations and the dimensions of the matrices are approximate to those of real data, that is $p = 30, q = 200, n_1 = n_2 = n = 20$.

Experiment 2. We estimate $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$ from the real data, simulate data with these parameters under $t_3$ distribution in the same fashion as in Online Supplement A.4.2, and implement all seven methods for the two-sample hypothesis testing. All seven methods have power one in this case.

A.5 Key References for Table 1

Table 16 provides the key references for Table 1.

References

Aguiar, M. and Gopinath, G. (2007). Emerging market business cycles: The cycle is the trend. *Journal of Political Economy*, 115(1):69–102.
|                         | Vector-valued data | Matrix-valued data |
|-------------------------|--------------------|-------------------|
| Covariance or correlation matrix |                    |                   |
| One-sample              | Chen et al. (2010) | This article      |
|                         | Cai and Jiang (2011) |                |
|                         | Zheng et al. (2019) |                |
| Two-sample              | Schott (2007)      | This article      |
|                         | Srivastava and Yanagihara (2010) |     |
|                         | Li and Chen (2012) |                |
|                         | Cai et al. (2013)  |                |
|                         | Cai and Ma (2013)  |                |
|                         | Cai and Zhang (2016) |             |
|                         | Chang et al. (2017) |               |
|                         | Zheng et al. (2019) |               |
| Precision matrix        |                    |                   |
| One-sample              | Liu (2013)         | Narayan and Allen (2016) |
|                         | Guo and Tang (2020) | Xia and Li (2017) |
|                         |                     | Chen and Liu (2019) |
| Two-sample              | Xia et al. (2015)  | Xia and Li (2018) |

Table 16: Summary of the literature on the hypothesis testing for both vector-valued and matrix-valued data under one-sample and two-sample regimes.
Aston, J. A., Pigoli, D., and Tavakoli, S. (2017). Tests for separability in nonparametric covariance operators of random surfaces. *The Annals of Statistics*, 45(4):1431–1461.

Bai, J. (2009). Panel data models with interactive fixed effects. *Econometrica*, 77(4):1229–1279.

Bekaert, G. and Harvey, C. R. (1995). Time-varying world market integration. *The Journal of Finance*, 50(2):403–444.

Bickel, P. J. and Levina, E. (2008a). High dimensional inference and random matrices —— covariance regularization by thresholding. *The Annals of Statistics*, 36(6):2577–2604.

Bickel, P. J. and Levina, E. (2008b). Regularized estimation of large covariance matrices. *The Annals of Statistics*, 36(1):199–227.

Brandt, M. (2009). Portfolio choice problems. Handbook of Financial Econometrics, North-Holland.

Brandt, M. W. and Santa-Clara, P. (2006). Dynamic portfolio selection by augmenting the asset space. *The Journal of Finance*, 61(5):2187–2217.

Brandt, M. W., Santa-Clara, P., and Valkanov, R. (2009). Parametric portfolio policies: Exploiting characteristics in the cross-section of equity returns. *The Review of Financial Studies*, 22(9):3411–3447.

Cai, C., Chen, R., and Xiao, H. (2019). Kopa: Automated kronecker product approximation. *arXiv preprint arXiv:1912.02392*.

Cai, T. and Liu, W. (2011). Adaptive thresholding for sparse covariance matrix estimation. *Journal of the American Statistical Association*, 106(494):672–684.

Cai, T., Liu, W., and Luo, X. (2011). A constrained l1 minimization approach to sparse precision matrix estimation. *Journal of the American Statistical Association*, 106(494):594–607.

Cai, T., Liu, W., and Xia, Y. (2013). Two-sample covariance matrix testing and support recovery in high-dimensional and sparse settings. *Journal of the American Statistical Association*, 108(501):265–277.

Cai, T. T., Hu, J., Li, Y., and Zheng, X. (2020). High-dimensional minimum variance portfolio estimation based on high-frequency data. *Journal of Econometrics*, 214(2):482 – 494.

Cai, T. T. and Jiang, T. (2011). Limiting laws of coherence of random matrices with applications to testing covariance structure and construction of compressed sensing matrices. *The Annals of Statistics*, 39(3):1496–1525.

Cai, T. T., Li, H., Liu, W., and Xie, J. (2016). Joint estimation of multiple high-dimensional
precision matrices. *Statistica Sinica*, 26(2):445.

Cai, T. T. and Liu, W. (2016). Large-scale multiple testing of correlations. *Journal of the American Statistical Association*, 111(513):229–240.

Cai, T. T. and Ma, Z. (2013). Optimal hypothesis testing for high dimensional covariance matrices. *Bernoulli*, 19(5B):2359–2388.

Cai, T. T. and Zhang, A. (2016). Inference for high-dimensional differential correlation matrices. *Journal of multivariate analysis*, 143:107–126.

Cai, T. T. and Zhou, H. H. (2012). Optimal rates of convergence for sparse covariance matrix estimation. *The Annals of Statistics*, 40(5):2389–2420.

Campbell, J. Y., Lettau, M., Malkiel, B. G., and Xu, Y. (2001). Have individual stocks become more volatile? An empirical exploration of idiosyncratic risk. *The Journal of Finance*, 56(1):1–43.

Campbell, J. Y., Lo, A. W., and MacKinlay, A. C. (1997). *The Econometrics of Financial Markets*. Princeton University Press, 41 William Street Princeton, New Jersey 08540 USA.

Chang, J., Zhou, W., Zhou, W. X., and Wang, L. (2017). Comparing large covariance matrices under weak conditions on the dependence structure and its application to gene clustering. *Biometrics*, 73(1):31–41.

Chen, E. Y. and Chen, R. (2019). Modeling dynamic transport network with matrix factor models: with an application to international trade flow. *arXiv preprint arXiv:1901.00769*.

Chen, E. Y., Tsay, R. S., and Chen, R. (2019a). Constrained factor models for high-dimensional matrix-variate time series. *Journal of the American Statistical Association*, pages 1–37.

Chen, H. and Xia, Y. (2021). A normality test for high-dimensional data based on a nearest neighbor approach. *Journal of the American Statistical Association*, (just-accepted):1–35.

Chen, R., Xiao, H., and Yang, D. (2020). Autoregressive models for matrix-valued time series. *Journal of Econometrics*.

Chen, R., Yang, D., and Zhang, C.-h. (2019b). Factor models for high-dimensional tensor time series. *arXiv preprint arXiv:1905.07530*.

Chen, S. X., Zhang, L.-X., and Zhong, P.-S. (2010). Tests for high-dimensional covariance matrices. *Journal of the American Statistical Association*, 105(490):810–819.

Chen, X. and Liu, W. (2019). Graph estimation for matrix-variate gaussian data. *Statistica Sinica*, 29(1):479–504.
Constantinou, P., Kokoszka, P., and Reimherr, M. (2017). Testing separability of space-time functional processes. *Biometrika*, 104(2):425–437.

Danaher, P., Wang, P., and Witten, D. M. (2014). The joint graphical lasso for inverse covariance estimation across multiple classes. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 76(2):373–397.

Dawid, A. P. (1981). Some matrix-variate distribution theory: notational considerations and a bayesian application. *Biometrika*, 68(1):265–274.

DellaVigna, S. and Pollet, J. M. (2007). Demographics and industry returns. *The American Economic Review*, 97(5):1667–1702.

Dutilleul, P. (1999). The mle algorithm for the matrix normal distribution. *Journal of statistical computation and simulation*, 64(2):105–123.

Fama, E. F. and French, K. R. (1997). Industry costs of equity. *Journal of Financial Economics*, 43(2):153–193.

Frahm, G. and Memmel, C. (2010). Dominating estimators for minimum-variance portfolios. *Journal of Econometrics*, 159(2):289 – 302.

Friedman, J., Hastie, T., and Tibshirani, R. (2008). Sparse inverse covariance estimation with the graphical lasso. *Biostatistics*, 9(3):432–441.

Goto, S. and Xu, Y. (2015). Improving mean variance optimization through sparse hedging restrictions. *The Journal of Financial and Quantitative Analysis*, 50(6):1415–1441.

Guo, X. and Tang, C. (2020). Specification tests for covariance structures in high-dimensional statistical models. *Biometrika*.

Hafner, C. M., Linton, O. B., and Tang, H. (2020). Estimation of a multiplicative correlation structure in the large dimensional case. *Journal of Econometrics*, 217(2):431–470.

Han, F., Han, X., Liu, H., and Caffo, B. (2016). Sparse median graphs estimation in a high-dimensional semiparametric model. *The Annals of Applied Statistics*, 10(3):1397–1426.

Han, F. and Liu, H. (2013). Optimal rates of convergence for latent generalized correlation matrix estimation in transelliptical distribution. *arXiv preprint arXiv:1305.6916*, 34.

Hoff, P. D. (2011). Separable covariance arrays via the tucker product, with applications to multivariate relational data. *Bayesian Analysis*, 6(2):179–196.

Hong, H., Torous, W., and Valkanov, R. (2007). Do industries lead stock markets? *Journal of
Jagannathan, R. and Ma, T. (2003). Risk reduction in large portfolios: Why imposing the wrong constraints helps. *Journal of Finance, 58*(4):1651–1683.

Kohn, D., Leibovici, F., and Tretvoll, H. (2018). Trade in commodities and business cycle volatility. *Available at SSRN 2654792.*

Ledoit, O. and Wolf, M. (2004). A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis, 88*(2):365 – 411.

Leng, C. and Tang, C. Y. (2012). Sparse matrix graphical models. *Journal of the American Statistical Association, 107*(499):1187–1200.

Li, J. and Chen, S. X. (2012). Two sample tests for high-dimensional covariance matrices. *The Annals of Statistics, 40*(2):908–940.

Liu, W. (2013). Gaussian graphical model estimation with false discovery rate control. *The Annals of Statistics, 41*(6):2948–2978.

Meinshausen, N. and Bühlmann, P. (2006). High-dimensional graphs and variable selection with the lasso. *The Annals of Statistics, 34*(3):1436–1462.

Molstad, A. J. and Rothman, A. J. (2019). A penalized likelihood method for classification with matrix-valued predictors. *Journal of Computational and Graphical Statistics, 28*(1):11–22.

Moskowitz, T. J. (2003). An analysis of covariance risk and pricing anomalies. *The Review of Financial Studies, 16*(2):417–457.

Narayan, M. and Allen, G. I. (2016). Mixed effects models for resampled network statistics improves statistical power to find differences in multi-subject functional connectivity. *Frontiers in neuroscience, 10*:108.

Okhrin, Y. and Schmid, W. (2006). Distributional properties of portfolio weights. *Journal of Econometrics, 134*(1):235 – 256.

Qiu, H., Han, F., Liu, H., and Caffo, B. (2016). Joint estimation of multiple graphical models from high dimensional time series. *Journal of the Royal Statistical Society: Series B (Statistical Methodology), 78*(2):487–504.

Ravikumar, P., Wainwright, M. J., Raskutti, G., and Yu, B. (2011). High-dimensional covariance estimation by minimizing ℓ1-penalized log-determinant divergence. *Electronic Journal of Statistics, 5*:935–980.
Rothman, A. J., Levina, E., and Zhu, J. (2009). Generalized thresholding of large covariance matrices. *Journal of the American Statistical Association*, 104(485):177–186.

Rothman, A. J., Levina, E., and Zhu, J. (2010). A new approach to cholesky-based covariance regularization in high dimensions. *Biometrika*, 97(3):539–550.

Rudelson, M. and Vershynin, R. (2013). Hanson-wright inequality and sub-gaussian concentration. *Electronic Communications in Probability*, 18.

Schott, J. R. (2007). A test for the equality of covariance matrices when the dimension is large relative to the sample sizes. *Computational Statistics & Data Analysis*, 51(12):6535–6542.

Srivastava, M. S. and Yanagihara, H. (2010). Testing the equality of several covariance matrices with fewer observations than the dimension. *Journal of Multivariate Analysis*, 101(6):1319–1329.

Stevens, G. V. G. (1998). On the inverse of the covariance matrix in portfolio analysis. *Journal of Finance*, 53(5):1821–1827.

Wang, D., Liu, X., and Chen, R. (2019). Factor models for matrix-valued high-dimensional time series. *Journal of Econometrics*, 208(1):231–248.

Werner, K., Jansson, M., and Stoica, P. (2008). On estimation of covariance matrices with kronecker product structure. *IEEE Transactions on Signal Processing*, 56(2):478–491.

Xia, Y., Cai, T., and Cai, T. T. (2015). Testing differential networks with applications to the detection of gene-gene interactions. *Biometrika*, 102(2):247–266.

Xia, Y. and Li, L. (2017). Hypothesis testing of matrix graph model with application to brain connectivity analysis. *Biometrics*, 73(3):780–791.

Xia, Y. and Li, L. (2018). Matrix graph hypothesis testing and application in brain connectivity alternation detection. *Statistica Sinica, to appear*.

Yin, J. and Li, H. (2012). Model selection and estimation in the matrix normal graphical model. *Journal of multivariate analysis*, 107:119–140.

Yuan, M. and Lin, Y. (2007). Model selection and estimation in the gaussian graphical model. *Biometrika*, 94(1):19–35.

Zheng, S., Cheng, G., Guo, J., and Zhu, H. (2019). Test for high-dimensional correlation matrices. *The Annals of Statistics*, 47(5):2887–2921.

Zhou, S. (2014). Gemini: Graph estimation with matrix variate normal instances. *The Annals of Statistics*, 42(2):532–562.
Zhu, Y. and Li, L. (2018). Multiple matrix gaussian graphs estimation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80(5):927–950.

Zhu, Y., Shen, X., and Pan, W. (2014). Structural pursuit over multiple undirected graphs. *Journal of the American Statistical Association*, 109(508):1683–1696.