Strange weak values of photon number operators in the optomechanical interaction

Sergio Carrasco¹ and Miguel Orszag¹,²
¹ Instituto de Física, Pontificia Universidad Católica de Chile, Casilla 306, Santiago, Chile
² Universidad Mayor, Avenida Alonso de Córdoba 5495, Las Condes, Santiago, Chile
E-mail: sjcarras@uc.cl, morszag@fis.puc.cl

April 2018

Abstract. An interferometric experiment is proposed in order to measure the radiation force exerted by a single photon on a nano mechanical oscillator placed in the middle of a Fabry-Pérot cavity. The force is measured by observing the shift given to the probability density function of the position of the oscillator, using weak measurements and post-selection of photons. We show that this shift is proportional to the weak value of the operator that accounts for the difference of photons between both sides of the cavity. For single-photon states, this operator has eigenvalues 1 or −1, depending on the side of the cavity in which is the photon, but the weak value can exceed by far this range. Thus, by preparing and post-selecting the photon in two nearly orthogonal states the weak value is increased and the radiation effect of a single photon turns out to be much larger than the pressure exerted by a photon without post-selection. This opens the possibility to study weak values of number operators in optomechanics and the radiation effect in pre- and post-selected ensembles.

PACS numbers: 03.65.-w, 03.65.Ta, 42.50.Xa, 42.50.-p, 42.50.Pq

Keywords: Weak Value, Weak Measurements, Optomechanics, Quantum Optics, Single Photons and Quantum Effects, Two-State Vector Formalism of Quantum Mechanics.
1. Introduction

The weak value (WV) of a physical variable, formally introduced in [1], is a property of a system between two complete measurements. Ensembles of particles that begin in an initial state $|a\rangle$ and are later found in a state $|b\rangle$, where $|a\rangle$ and $|b\rangle$ are two eigenvectors of the observables measured at the initial and final times, are called pre- and post-selected ensembles. This time symmetric formulation of quantum mechanics, in which new information from a second measurement is included, is known as Two-State Vector Formalism (TSVF) [2] and started with the seminal work [3].

WV’s can be measured in pre- and post-selected ensembles using weak measurements [4], which correspond to standard von Neumann’s measurements satisfying certain criteria of weakness. In general, weak values are complex quantities (a physical interpretation for the general case is given in [5]). Real weak values can exceed the range of eigenvalues of a physical variable and, in this case, are sometimes referred as anomalous [6], superweak [7] or strange [8].

This feature has been extensively used for detection of small effects, mostly in optics. In [9] the first measurement of a weak value was performed by coupling the polarization of a laser beam to its transverse momentum in a birefringent crystal, amplifying the separation of the two linear polarization components of the beam after passing through the crystal. Similarly, in [10] the polarization dependent displacement of a laser beam, perpendicular to the gradient of the refractive index of the medium, was enhanced up to four orders of magnitude, allowing to observe the quantum Spin Hall Effect of Light. An imaginary weak value was measured in [11] using a Sagnac interferometer, where the which-path degree of freedom of a photon was coupled to its transverse position through a tilted mirror, allowing to amplify the deflection of a laser beam. Using the same type of coupling, in [12] a small frequency change was measured by using a prism with frequency-dependent refractive index, showing the usefulness of weak value amplification also for precision frequency metrology. Using the spin-orbit coupling between polarization and orbital angular momentum, rotations in the position and angular momentum representations were measured on a He-Ne laser beam by performing measurements of complex weak values [13]. The advantages of weak value amplification for the estimation of small parameters in the presence of different noise models, as compared to conventional measurement methods, are analyzed in [14, 15].

In all the optical experiments described above the system and the measurement device were the same object, e.g. a photon, and it has been shown that in this case the effects can be explained semiclassically using Maxwell equations [16]. Nevertheless, in [17] for the first time the system and the meter were separate objects, two entangled photons. In this experiment the weak value of a polarization operator, the Stokes $\hat{S}_1$ operator, was measured by entangling the photons using a non deterministic quantum non demolition
Strange weak values of photon number operators in the optomechanical interaction

measurement (QND) scheme [18].

Weak measurements and WV’s have been also employed to propose an alternative method to quantum state tomography. By using single photons obtained by spontaneous parametric down-conversion, sequential weak measurements of the transverse position, followed by post-selection of photons with zero momentum, allowed to perform a direct observation of the transverse wave function, and a new method for constructing the quantum state of a system was described [19]. Other new interesting applications of weak value amplification include gravitational waves detection [20] and quantum control [21].

On the other hand, weak values and weak measurements have been used to explore fundamental questions in quantum mechanics. The Leggett-Garg inequalities [22], which appeared more or less at the same time that weak values and test the assumptions of macroscopic realism and non invasive measurements, were generalized for a system undergoing weak measurements in [23, 24]. In [25] (see also [26]) it is shown that a violation of the generalized Leggett-Garg inequality is equivalent to an observation of a strange weak value. These violations and their correspondence with strange weak values were experimentally measured in solid-state devices [27], via deterministic coupling of two transmon qubits to a bus resonator, and in optical setups where the polarizations of two photons became entangled through a non deterministic interaction in a controlled sign (CS) gate [28]. In [29] weak measurements of the transverse momentum of photons emitted by an InGaAs quantum dot and sent through a Handbury Brown-Twiss interferometer were used to reconstruct the trajectories of sub ensembles of photons arriving at a particular location, allowing to observe the average trajectories of single photons in a two-slit interferometer. Hence, pre- and post-selected ensembles provide an appealing perspective for studying interference phenomena [30] and wave-particle duality [31]. Another remarkable feature of pre- and post-selected ensembles refers to the possibility of separating a system from one of its properties [32]. In [33], using weak measurements, neutrons were sent through one path of an interferometer while their magnetic moment “went through the other arm”.

Since their formulation in 1988 weak values have overcome different controversies. In [34] Ferrie and Combes argued that weak value amplification could be explained classically. However, it turns out that it is a truly quantum effect [35]. Other controversies consider it usefulness for detection of small parameters, whether WV’s constitute a property of a system or not, and its interpretations as contextual values, all of which are summarized in [36]. Extensive reviews on weak values and weak measurements can be found in [37, 38, 39, 40, 41].
In this paper we are interested in the study of WV’s of number operators. In [42] a gedanken experiment involving trajectories of electrons and positrons along Mach-Zehnder type interferometers is described in order to prove Bell’s theorem [43] and to show that realistic local quantum theories, that are also Lorentz invariant, lead to paradoxes. In [44] these paradoxes were explored using weak measurements leading to negative weak values of number operators. In [45] these negative occupation numbers were measured using photons. In the “three-box paradox” [46] a particle can be in three separate boxes A, B and C. By using pre- and post-selection one can arrive to the surprising conclusion that, when box A is opened (and none of the others), the particle is found with certainty there, and that the same will occur if box B is opened (and none of the others). Furthermore, the weak value of the number of particles in box C is −1. An experimental realization of the three-box paradox was done in [47]. WV’s of number operators also appear in [48], where the nature of quantum correlations and interactions is studied by showing a quantum violation of the “Pigeonhole principle” and proposing interferometric experiments using atom optics in order to verify these violations.

In [49] an experiment is proposed where a single photon enters an interferometer but the measurement of the photon number in one of the arms turns out to be larger than 1. This is accomplished through a photon-photon interaction in a non linear Kerr-type medium, where the phase shift given to a beam probe is proportional to the weak value of the number of photons in one of the arms of the interferometer. In [50] the amplification of the phase shift caused by a single photon was experimentally verified. The novelty of this research is twofold since it involves deterministic coupling of photons and due to the fact that the measured number of photons is bigger than 1 in a system where only one photon was injected. In [51] new experiments are proposed where the number of atoms may be larger than 1 (or even negative) when only one atom enters the system.

In the present work an experimental proposal is described where the measured number of photons is also larger than 1 when only one photon is introduced into the system, but instead of using photon-photon interaction in a non linear medium, we use the optomechanical interaction between a single photon and a mechanical oscillator placed in the middle a Fabry-Pérot cavity. More precisely, in our experiment the measurement of the difference of photons between both sides of the cavity is larger than one, whereas for pre-selected ensembles this value lies between −1 and 1 (when only one photon enters the cavity). The difference of photons, which is proportional to the radiation force, is measured by observing the effect on the position of the mechanical oscillator, whose mean value is shifted after the measurement.
Strange weak values of photon number operators in the optomechanical interaction

Optomechanical systems have been proposed to prepare non classical optical and mechanical states [52] and to study the decoherence process of macroscopic objects [53, 54, 55, 56]. Although weak measurements in pre- and post-selected ensembles have been studied to amplify the radiation pressure exerted by a single photon [57, 58, 59], no weak values arising from the optomechanical interaction have been reported to date. Therefore, in the present paper we describe for the first time an experiment where the weak value of a photon number operator can be measured in this type of interaction. This allows to explain the amplification of the radiation pressure in the sense of [1], i.e. due to the fact that the weak value of a number operator can exceed the range of eigenvalues. Therefore we will conclude that the radiation effect of a pre- and post-selected photon can be much larger than the one produced by an only pre-selected photon. More surprisingly, we will arrive to the conclusion that a properly pre- and post-selected photon can exert a force directed inwards over the walls of the recipient that contains it.

The structure of this article is as follows. In section 2 we review the concepts of weak values and weak measurements. In section 3 we present a simple description of how the radiation force of a single photon can be very large (or directed inwards) when post-selection is included, while section 4 presents an experiment where these effects can be tested. In section 5 we explain the Hamiltonian model, the Hilbert space of states is described and the time evolution operator is calculated. In section 6 the unitary evolution of the system is performed. Section 7 analyses the case when no post-selection of photons is done. In section 8 post-selection is included and the appearance of weak values of number operators is shown. In section 9 we discuss the amplification effect, summarize our conclusions and further research lines are briefly commented.

2. Weak measurements and weak values

According to [60] the measurement process of a physical variable $\hat{A}$ of a system can be described by a Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + g(t) \hat{A} \hat{p},$$

(1)

where $\hat{H}_0$ is the unperturbed Hamiltonian of the system and the apparatus, while the second term describes the interaction that occurs between them during the measurement. The coupling $g(t)$ is a function that is switched on during the interaction and then turned off, while $\hat{p}$ is a physical variable of the measurement device whose conjugate variable is $\hat{q}$.

The system has initially no correlation with the apparatus and consequently the initial state is described by the product $|\Phi(0)\rangle = |\psi_i\rangle |\psi_i\rangle_m$, where $|\psi_i\rangle$ is the initial state of the system and $|\psi_i\rangle_m$ is the initial state of the apparatus or meter. Notice that a state $|\psi_i\rangle$ can be assigned to the system because it has been experimentally prepared in a
specific way, e.g. passing a light beam through a polarizer or atoms through a Stern-Gerlach apparatus. This means that there is some information available about the system, revealed because a measurement has been performed prior to the experiment. The state \( |\psi_i\rangle \) can be described in the orthonormal basis given by the eigenstates of \( \hat{A} \) as \( |\psi_i\rangle = \sum_k c_k |a_k\rangle \) where \( |a_k\rangle \) is one of the discrete eigenvectors of the measured variable with eigenvalue \( a_k \). Typically, the initial state of the apparatus is described by a Gaussian state according to

\[
|\psi_i\rangle_m = \int_{-\infty}^{\infty} \psi(q) |q\rangle \, dq,
\]

\[
\psi(q) = \left( \frac{1}{\sqrt{2\pi}\sigma_q} \right)^{1/2} \exp \left( -\frac{q^2}{4\sigma_q^2} \right),
\]

where \( \sigma_q^2 \) is the dispersion of the variable \( \hat{q} \). Since the initial state of the measuring device saturates the uncertainty principle the dispersion of the conjugate variable \( \hat{p} \) is

\[
\sigma_p^2 = \hbar^2/(4\sigma_q^2).
\]

When the interaction term commutes with \( \hat{H}_0 \), in the interaction picture, the time evolution operator is \( \hat{U} = \exp (-i\frac{\hbar}{\hbar} \hat{A} \hat{p}) \), where \( g \) is just the integration of \( g(t) \) over its compact support. When the unperturbed hamiltonian does not commute with the interaction term, then the evolution given by \( \hat{H}_0 \) has to be taken into account. In our setup this will be the case but for this introduction it will be assumed, for simplicity, that both terms commute.

After the interaction, the system and the measuring device get correlated according to

\[
|\Phi\rangle = \hat{U} |\psi_i\rangle |\psi_i\rangle_m = \sum_k c_k |a_k\rangle |\psi_k\rangle_m,
\]

where \( |\psi_k\rangle_m = \int_{-\infty}^{\infty} \psi(q - ga_k) |q\rangle \, dq \). The probability density function of the position of the meter, \( f(q) \), is obtained according to quantum mechanics, namely,

\[
f(q) = |\langle q |\Phi\rangle|^2 = \sum_k |c_k|^2 \psi(q - ga_k)^2,
\]

which corresponds to a Gaussian mixture. When the apparatus has a sufficiently narrow wave packet in the \( q \)-representation to resolve the different eigenvalues, i.e. when the spread \( \sigma_q \) is much smaller than the difference between all the \( a_i \)'s (multiplied by \( g \)), each Gaussian is sharply localized around \( ga_i \) and the overlap among them is negligible. Consequently, the eigenvalue \( a_i \) (multiplied by \( g \)) will be read with probability \( |c_i|^2 \). Each time the eigenvalue \( a_k \) is obtained, i.e. the apparatus is observed to be around the position \( ga_k \), according to quantum mechanics, the system and the measuring device will be left in the unnormalized state

\[
\int_{ga_k-\delta}^{ga_k+\delta} dq \langle q |\Phi\rangle = \sum_m c_m |a_m\rangle \int_{ga_k-\delta}^{ga_k+\delta} dq \psi(q - ga_m) |q\rangle
\]

\[
\approx c_k |a_k\rangle \int_{ga_k-\delta}^{ga_k+\delta} dq \psi(q - ga_k) |q\rangle,
\]

and the normalized state of the system after the observation is \( |a_k\rangle \). Therefore, under this model, the following two processes produce the same results: a) the observation of
Strange weak values of photon number operators in the optomechanical interaction

a system, excluding the measuring device, and b) the unitary evolution of a composite system, in which the measuring device has been included, followed by an observation of the measuring device. This is the central idea of the model proposed by von Neumann and it is important for not violating the so-called principle of the psycho-physical parallelism [61].

It is interesting to study the opposite limit, when \( g_{D_{\text{max}}} / \sigma_q \ll 1 \), where \( D_{\text{max}} \) represents the maximum distance between consecutive eigenvalues. This regime is achievable by choosing the parameter \( g/\sigma_q \) to be sufficiently small, e.g. \( g/\sigma_q \ll 1 \). In this case, the time evolution operator may be expanded as \( \hat{U} \approx 1 - i(g/\hbar) \hat{A} \hat{p} \) to a good approximation.

Using the result from Appendix J, the state of the system and the apparatus can be written as

\[
|\Phi\rangle = \frac{1}{\sqrt{N}} \left[ 1 - i(g/\hbar) \langle \hat{A} \rangle \hat{p} \right] |\psi_i\rangle |\psi_i\rangle_m - \frac{i}{\sqrt{N}} (g/\hbar) \sqrt{\langle \Delta \hat{A}^2 \rangle} |\psi_i^\perp\rangle \hat{p} |\psi_i\rangle_m ,
\]

where \( N = 1 + (1/4)(g/\sigma_q)^2 \langle \hat{A}^2 \rangle \) is a normalization factor, \( |\psi_i^\perp\rangle \) is any state of the system orthogonal to \( |\psi_i\rangle \), and all the expectation values are calculated in the state \( |\psi_i\rangle \).

In this scenario the probability of disturbing the system after the measurement is given by

\[
P_{\text{dist}} = 1 - |\langle \psi_i | \Phi \rangle|^2 = \frac{(1/4)(g/\sigma_q)^2 \langle \Delta \hat{A}^2 \rangle}{1 + (1/4)(g/\sigma_q)^2 \langle \hat{A}^2 \rangle},
\]

and therefore, for \( g/\sigma_q \) small enough, it is clear that \( P_{\text{dist}} = 0 \). Surprisingly, the probability density function for the meter becomes

\[
f(q) \approx \frac{1}{\sqrt{2\pi} \sigma_q} e^{-q^2/2\sigma_q^2} \sum_k |c_k|^2 \left( 1 - \frac{q g_k g}{\sigma_q^2} \right)
\]

\[
= \frac{1}{\sqrt{2\pi} \sigma_q} e^{-q^2/2\sigma_q^2} \left( 1 - q \langle \hat{A} \rangle g/\sigma_q^2 \right) \approx \frac{1}{\sqrt{2\pi} \sigma_q} \exp \left[ -\frac{(q - g \langle \hat{A} \rangle)^2}{2\sigma_q^2} \right],
\]

meaning that in this regime, although the state of the system remains unchanged (\( P_{\text{dist}} \) is negligible at first order), the measurement still provides some information since \( \langle \hat{A} \rangle \) can be computed. In other words, information goes linearly in \( g/\sigma_q \), while perturbations behave quadratically in this parameter. This method, in which \( g/\sigma_q \) is weakened, constitutes a type of weak measurement. Each reading of the meter will be normally distributed around \( \langle \hat{A} \rangle \) with large dispersion since \( g \langle \hat{A} \rangle \ll \sigma_q \), and the outcome of the measurement is the average of all the readings, i.e. \( \langle \hat{A} \rangle \). This type of weak measurement is called statistical method without disturbance [30]. It is important to point out that there are also other types of weak measurements [62].
Trivially, it is possible to post-select only readings with large values of \( q \) since the Gaussian distribution \( (8) \) has no vanishing tails and to change the outcome of the measurement. Nevertheless, supposing that post-selection is allowed to be performed only on the system and not on the apparatus, in [1] the following question was answered: how then can we maximize the outcome of the average of \( q \)? Assuming that the state \( |\psi_f\rangle \) is post-selected, with \( \langle \psi_i | \psi_f \rangle \neq 0 \) (see [63, 64] for the case of orthogonal post-selection), then the unnormalized state of the apparatus may be expressed as

\[
|\psi_f\rangle_m = \langle \psi_f | \hat{U} | \psi_i \rangle |\psi_i\rangle_m \\
\approx \langle \psi_f | 1 - i(g/\hbar)\hat{A} \hat{p} | \psi_i \rangle |\psi_i\rangle_m \\
= \langle \psi_f | \psi_i \rangle [1 - i(g/\hbar)A_w \hat{p}] |\psi_i\rangle_m \\
\approx \langle \psi_f | \psi_i \rangle \exp [-i(g/\hbar)A_w \hat{p}] |\psi_i\rangle_m.
\]

(9)

The conditions under which the approximations above hold are analyzed in [65] and in [Appendix I]. The complex number \( A_w \) is defined as

\[
A_w = \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle}
\]

(10)

and it is called the weak value of the operator \( \hat{A} \). Sometimes it is also expressed as \( \langle \hat{A} \rangle_w \).

When the weak value is real the operator \( \exp [-i(g/\hbar)A_w \hat{p}] \) produces a translation of the wave function in the \( q \)-representation by an amount of \( gA_w \) and hence the normalized state of the apparatus after the measurement is

\[
|\psi_f\rangle_m = \int_{-\infty}^{\infty} \psi(q - gA_w) |q\rangle dq.
\]

(11)

The wave function of the meter has not changed its Gaussian shape but only translated. This is important since we do not want the probability density function of the meter to change after a measurement (in that case the process would resemble more to an interaction than to a measurement), but only to shift it by a quantity that may be understood as an element of physical reality of the system under observation [66]. The shift of the position grows as the initial and final states come close to orthogonality and can exceed the range of eigenvalues of \( \hat{A} \). However, pre- and post-selected ensembles constructed upon two nearly orthogonal states are difficult to obtain since the probability of post-selecting the state \( |\psi_f\rangle \) is given by \( |\langle \psi_f | \psi_i \rangle|^2 \) and approaches to zero as the states become orthogonal. The weak value depends therefore on two kinds of information. The first is obtained by a measurement performed in order to prepare the system for the experiment and allows to assign an initial quantum state \( |\psi_i\rangle \). The second corresponds to new information coming from post-selection, i.e. from a second measurement carried out after the experiment that permits to assign a final state \( |\psi_f\rangle \).
3. Amplification of the radiation force

Let us consider a single photon that can be in two boxes, $A$ and $B$, which are separated by a wall that does not permit any interaction between both sides but is allowed to move due to the force $\hat{F}$ exerted by the photon. The way in which we measure the force is by making the photon weakly interact with the wall, in the sense explained in the last section. Then, we observe the position of the moving wall and repeat the same experiment many times, recording the position of the wall each time. Finally, we average all the results. As we have explained in the last section, when there is no post-selection the average position of the wall is proportional to $\langle \hat{F} \rangle$, but is proportional to $\langle \hat{F} \rangle_w$ when post-selection is performed.

Let $\hat{a}^\dagger \hat{a}$ and $\hat{b}^\dagger \hat{b}$ be the number operators that correspond to the number of photons in the boxes $A$ and $B$, respectively, and let us define the single-photon states by $\hat{a}^\dagger \hat{a} |1, 0\rangle = |1, 0\rangle$ and $\hat{b}^\dagger \hat{b} |0, 1\rangle = |0, 1\rangle$. Photons in box $A$ exert a force to the right over the moving wall equal to $\hat{F}_A = \hbar G \hat{a}^\dagger \hat{a}$ and photons in the other box exert a force $\hat{F}_B = \hbar G \hat{b}^\dagger \hat{b}$, to the left. Consequently, the total force over the moving wall is $\hat{F} = \hat{F}_A - \hat{F}_B = \hbar G (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$. This situation is represented in figure 1.

![Figure 1](image)

**Figure 1.** A photon in the left box exerts a force $\hbar G \hat{a}^\dagger \hat{a}$ to the right, while a photon in the right box externs a force $\hbar G \hat{b}^\dagger \hat{b}$ in the other direction. The net force over the moving wall is $\hbar G (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})$.

The parameter $G$ is proportional to the frequency $\omega_0$ of the photon and inversely proportional to the length $l$ of each box. The operator $\hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}$ corresponds to the total number of photons and notice that $\hbar G \hat{N} = \hat{F}_A + \hat{F}_B$.

Let us assume firstly that a single photon is prepared in the state $|\psi_i\rangle = \cos(\theta_i/2) |1, 0\rangle + \sin(\theta_i/2) |0, 1\rangle$, which is a superposition of being in the box $A$ or in the box $B$. In this case the measurement (the average displacement of the wall) will be proportional to

$$\langle \hat{F} \rangle = \hbar G \cos(\theta_i),$$

which is bounded by $\pm \hbar G$. Since $\langle \hat{N} \rangle = 1$, then $\langle \hat{F}_A \rangle + \langle \hat{F}_B \rangle = \hbar G$. Also, $\langle \hat{F} \rangle = \langle \hat{F}_A \rangle - \langle \hat{F}_B \rangle$. Therefore, we can conclude that $\langle \hat{F}_A \rangle = \hbar G \cos^2(\theta_i/2)$ and
\[ \langle \hat{F}_B \rangle = \hbar G \sin^2(\theta_i/2) \]. It is clear then that the forces are positive and bounded, e.g. if \( \theta_i = \pi/3 \) then \( \langle \hat{F}_A \rangle = 3\hbar G/4 \) and \( \langle \hat{F}_B \rangle = \hbar G/4 \). Photons in box A will exert an average force of \( 3\hbar G/4 \) to the right and photons in the left side will exert a force of \( \hbar G/4 \) to the left, while the total average force will be \( \langle \hat{F} \rangle = \hbar G/2 \). This case is illustrated in figure 2.

Let us imagine now a different situation in which the photon is post-selected in the state \( |\psi_f\rangle = \cos(\theta_f/2) |1,0\rangle - \sin(\theta_f/2) |0,1\rangle \). In this case, we will see an average displacement of the moving wall proportional to the weak value of the force, namely,

\[ \langle \hat{F} \rangle_w = \hbar G \frac{\cos\left(\frac{\theta_i - \theta_f}{2}\right)}{\cos\left(\frac{\theta_i + \theta_f}{2}\right)}, \]  

(13)

which is always greater or equal than \( \hbar G \) (or lesser or equal than \( -\hbar G \)). Since \( \langle \hat{N} \rangle_w = 1 \), then \( \langle \hat{F}_A \rangle_w + \langle \hat{F}_B \rangle_w = \hbar G \). Also, \( \langle \hat{F} \rangle_w = \langle \hat{F}_A \rangle_w - \langle \hat{F}_B \rangle_w \). Hence, we conclude that

\[ \langle \hat{F}_A \rangle_w = \hbar G \frac{\cos(\theta_i/2) \cos(\theta_f/2)}{\cos\left(\frac{\theta_i + \theta_f}{2}\right)}, \quad \langle \hat{F}_B \rangle_w = -\hbar G \frac{\sin(\theta_i/2) \sin(\theta_f/2)}{\cos\left(\frac{\theta_i + \theta_f}{2}\right)}. \]

(14)

We can see that the forces \( \langle \hat{F}_A \rangle_w \) and \( \langle \hat{F}_B \rangle_w \) are unbounded and may be negative, i.e. a pre- and post-selected photon can exert a force directed inwards over the walls of the box that contains it. As an example, if \( \theta_i = \pi/2 \) and \( \theta_f = \pi/3 \) then \( \langle \hat{F}_A \rangle_w = \hbar G \sqrt{6}/(\sqrt{6} - \sqrt{2}) \approx 2.4\hbar G \) and \( \langle \hat{F}_B \rangle_w = -\hbar G \sqrt{2}/(\sqrt{6} - \sqrt{2}) \approx -1.4\hbar G \), i.e. the total force is \( \approx 3.4\hbar G \). Figure 3 represents this case.

Hence, we will see a four times bigger displacement than the maximum value we could have achieved without post-selection. More interesting, photons in box A will exert a force of \( 2.4\hbar G \) to the right while photons in box B will exert a force \( 1.4\hbar G \) also to the right. Consequently, photons in box B behave as if they could push the moving wall to the right. The experiment proposed in this article is a measurement of the weak value of \( \hat{F} \), i.e. of the difference of photons in a two-sided container.
Strange weak values of photon number operators in the optomechanical interaction

Figure 3. The displacement of the wall is proportional to the weak value of the force, which is unbounded.

4. General description of the experiment

The proposed setup consists of an optomechanical system joining two arms of a Michelson interferometer (figure 4). The optomechanical system is a cavity with a moving mirror in the middle.

One single photon, with horizontal polarization, is injected into the system through the input port $CD$. After being reflected by a polarizing beam splitter (PBS), with 100\% reflectivity for horizontal polarization and 100\% transmissivity for the vertical component, the photon enters the interferometer through a polarization dependent beam splitter (PDBS). The photon travels then along the arms of the interferometer, enters the cavity, interacts with the mechanical oscillator and returns back to the PDBS in an entangled state with the vibrating mirror. The photon comes back with vertical polarization due to the action of the $\lambda/4$ plates mounted on each arm of the interferometer.

After exiting, the photon can be detected at $D_1$ or $D_2$, which disentangles the photon from the mirror. $D_1$ is called the dark port since, when the PDBS is symmetric for both polarization components, no light arrives at this point. Only the cases when detector $D_1$ clicks should be considered, disregarding the other cases. This corresponds to the post-selection of photons. By collecting data of the position of the mirror, only for the cases of successful post-selection, the average position of the oscillator can be calculated. It will be shown that in this setup the mean position of the vibrating mirror is proportional to the weak value of the difference of photons between both arms of the interferometer.
Strange weak values of photon number operators in the optomechanical interaction

Figure 4. Optomechanical system joining two arms of a Michelson interferometer. Post-selection of photons is done at detector $D_1$. Preparation of photons is performed in the source and after passing through a polarizer oriented in the horizontal direction.

5. Optomechanical model

In this section we present a hamiltonian model for the physical process that occurs in the branch $KN$. We assume that this process has relativistic space-like separation with respect to the processes occurring in the PDBS and in all other parts of the setup.

The optomechanical system consist of a high finesse Fabry-Pérot cavity with a nano mechanical oscillator in the middle. Nano mechanical devices have masses in the order of $10^{-12}$ kg, contain about $10^7$ atoms and have mode frequencies around $10^6$ Hz [67]. The oscillator is considered to be a perfectly reflecting vibrating mirror, while the mirrors of the cavity have losses, allowing interaction with the external field. In this model, the cavity field has two modes (one mode for each side), while the external field is considered to be a four-mode field, with one mode describing propagation towards the cavity and another one describing propagation away from it, for each arm of the interferometer. This is naturally an approximation for describing a nearly monochromatic photon. In practise, the external field comprises a continuous range of modes over a narrow interval of frequencies $[\omega_0 - \Delta\omega_{ph}/2, \omega_0 + \Delta\omega_{ph}/2]$, such that $\Delta\omega_{ph} \ll \gamma_C$, where $\Delta\omega_{ph}$ is the
bandwidth of the photon and $\gamma_C$ is the cavity field decay rate associated with the interfaces between the external field and the optical resonator, i.e. not considering the internal losses of the cavity. Notice that the range of frequencies is centered around the cavity resonance frequency, i.e. the external field is resonant with the cavity as it is shown in figure 5.

Figure 5. A two-sided cavity with a mechanical oscillator in the middle. The oscillator is considered to be a perfectly reflecting moving mirror, while the mirrors of the cavity have losses. This allows photon exchange to occur between the fields. The external field is a four-mode field, while the cavity field has two modes. The external field is resonant with the cavity.

5.1. Hamiltonian model

The hamiltonian for the system can be written as a sum of different contributions, namely,

$$\hat{H} = \hat{H}_{\text{cav}} + \hat{H}_{\text{ext}} + \hat{H}_m + \hat{H}_{\text{cav-ext}} + \hat{H}_{\text{OM}}.$$  (15)

The first term is the energy of the electromagnetic field inside the cavity and is given by

$$\hat{H}_{\text{cav}} = \hbar \omega_0 (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2).$$  (16)

The operators $\hat{a}_1$ ($\hat{a}_1^\dagger$) and $\hat{a}_2$ ($\hat{a}_2^\dagger$) are boson annihilation (creation) operators for the cavity modes of the left and right sides, respectively. Since the mirror is put exactly in the middle, both modes have the same resonant frequency $\omega_0 = c\pi n_0/l$, where $c$ is the speed of light, $l$ is the length of each side of the cavity and $n_0$ is the integer mode number.

The second term is the energy of the electromagnetic field in the arms of the interferometer and is expressed as

$$\hat{H}_{\text{ext}} = \hbar \omega_0 \sum_{i=1}^{2} (\hat{b}_{i,k_0}^\dagger \hat{b}_{i,k_0} + \hat{b}_{i,-k_0}^\dagger \hat{b}_{i,-k_0}).$$  (17)

The operators $\hat{b}_{1,k_0}$ ($\hat{b}_{1,k_0}^\dagger$) and $\hat{b}_{2,k_0}$ ($\hat{b}_{2,k_0}^\dagger$) are annihilation (creation) operators for travelling modes, with wavenumber $k_0 = \omega_0/c$, propagating towards the cavity through the
left and right arms, respectively. Analogously, \( b_{1,-k_0} (\hat{b}_{1,-k_0}^\dagger) \) and \( b_{2,-k_0} (\hat{b}_{2,-k_0}^\dagger) \) are annihilation (creation) operators for modes with wavenumber \( k_0 \) propagating away from the cavity in the left and right arms, respectively. These operators satisfy the usual boson commutation relations, i.e. \([\hat{b}_{i,k_0}, \hat{b}_{j,k_0}^\dagger] = \delta_{ij}\) and \([\hat{b}_{i,k_0}, \hat{b}_{j,k_0}] = [\hat{b}_{i,k_0}^\dagger, \hat{b}_{j,k_0}^\dagger] = 0\), for \( i = 1, 2 \) and \( j = 1, 2 \). Analogous commutation relations are satisfied by operators associated to modes propagating away from the cavity. It is also clear that \([\hat{b}_{i,k_0}, \hat{b}_{j,-k_0}] = [\hat{b}_{i,k_0}^\dagger, \hat{b}_{j,-k_0}^\dagger] = 0\), for \( i = 1, 2 \) and \( j = 1, 2 \).

The third term is the energy of the mechanical oscillator and is given by \( \hbar \omega_m \hat{c} \hat{c}^\dagger \), where \( \hat{c} (\hat{c}^\dagger) \) is the mechanical annihilation (creation) boson operator and \( \omega_m \) is the mode frequency of the oscillator.

The fourth term represents the interaction energy between the cavity field and the external field, and is given by

\[
\hat{H}_{cav-ext} = \hbar g \sum_{i=1}^{2} \left[ \hat{a}_i (\hat{b}_{i,k_0}^\dagger + \hat{b}_{i,-k_0}^\dagger) + \hat{a}_i^\dagger (\hat{b}_{i,k_0} + \hat{b}_{i,-k_0}) \right].
\]

Where \( g \) is the photon-hopping interaction strength between the cavity and the external field at the resonant frequency, which accounts for transmission losses through the input mirrors. Over the narrow bandwidth of the external field, this coupling is assumed to be constant. We are also assuming that this bandwidth is shorter than the cavity free spectral range, i.e. \( \Delta \omega_{ph} \leq \omega_{FSR} \), and therefore only one mode is excited in each side of the cavity.

The last term represents the optomechanical interaction, through radiation pressure, between the cavity and the mechanical oscillator, and is given by

\[
\hat{H}_{OM} = -\hbar \kappa (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) (\hat{c}^\dagger + \hat{c}),
\]

where \( \kappa = x_0 G \) is the optomechanical single-photon coupling strength, \( G = \omega_0/l \) is the frequency shift per displacement that characterizes the cavity (frequency pull parameter), \( x_0 = \sqrt{\hbar/2M\omega_m} \) is the mechanical zero-point fluctuation and \( M \) is the mass of the oscillator (see [68] for a derivation of the optomechanical hamiltonian and [69] for a review on cavity optomechanics). A minus sign appears between the number operators because the movement of the mirror in one direction shortens the effective length of one side of the cavity while enlarges the other. The radiation force is given by \( \hat{F}_{rad} = \hbar G (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \) [70] and consequently this term can be expressed as \(-\hat{F}_{rad} \hat{\hat{q}}\), where \( \hat{\hat{q}} \) is the position of the oscillator.

Firstly, let us define for each arm of the interferometer, i.e. for \( i = 1, 2 \), the following operators:

\[
\hat{a}_i = \frac{\hat{b}_{i,k_0} + \hat{b}_{i,-k_0}}{\sqrt{2}}, \quad \hat{a}_i^\dagger = \frac{\hat{b}_{i,k_0}^\dagger + \hat{b}_{i,-k_0}^\dagger}{\sqrt{2}}, \quad \hat{c}_i = \frac{\hat{b}_{i,k_0} - \hat{b}_{i,-k_0}}{\sqrt{2}}, \quad \hat{c}_i^\dagger = \frac{\hat{b}_{i,k_0}^\dagger - \hat{b}_{i,-k_0}^\dagger}{\sqrt{2}}.
\]
Strange weak values of photon number operators in the optomechanical interaction

These operators obey the same boson commutation relations as the travelling mode operators, but annihilate (create) excitations of standing-wave modes. Following [74] we say that the operators \( \hat{a}_i \) (\( \hat{d}_i^\dagger \)) annihilate (create) excitations of cosine-type modes (even modes) while \( \hat{c}_i \) (\( \hat{c}_i^\dagger \)) annihilate (create) excitations of sine-type modes (odd modes).

Going to an interaction picture I with respect to \( \hat{H}_{\text{cav}} + \hat{H}_{\text{ext}} \) the hamiltonian becomes

\[
\hat{H}_I = \hbar \sqrt{2g} \sum_{i=1}^2 \left( \hat{a}_i^\dagger \hat{d}_i + \hat{a}_i \hat{d}_i^\dagger \right) + \hat{H}_m + \hat{H}_{\text{OM}}. \tag{21}
\]

Notice that the cavity couples only to even modes with a coupling constant strengthened by a factor of \( \sqrt{2} \). For simplicity, this factor will be absorbed into \( g \). Next, for each arm of the interferometer, the following operators are defined:

\[
\begin{align*}
\hat{J}_{x1} &= \frac{\hbar}{2} (\hat{a}_1 \hat{d}_1^\dagger + \hat{d}_1 \hat{a}_1^\dagger), \\
\hat{J}_{x2} &= \frac{\hbar}{2} (\hat{a}_2 \hat{d}_2^\dagger + \hat{d}_2 \hat{a}_2^\dagger), \\
\hat{J}_{y1} &= \frac{i\hbar}{2} (\hat{a}_1 \hat{d}_1^\dagger - \hat{d}_1 \hat{a}_1^\dagger), \\
\hat{J}_{y2} &= \frac{i\hbar}{2} (\hat{a}_2 \hat{d}_2^\dagger - \hat{d}_2 \hat{a}_2^\dagger), \\
\hat{J}_{z1} &= \frac{\hbar}{2} (\hat{a}_1^\dagger \hat{a}_1 - \hat{d}_1^\dagger \hat{d}_1), \\
\hat{J}_{z2} &= \frac{\hbar}{2} (\hat{d}_2^\dagger \hat{d}_2 - \hat{a}_2^\dagger \hat{a}_2). \tag{22}
\end{align*}
\]

Each set of operators generates the \( SU(2) \) algebra and acts separately on each side of the system. Operators acting jointly on both sides are defined as \( \hat{\mathbf{J}}_x = \hat{\mathbf{J}}_{x1} + \hat{\mathbf{J}}_{x2} \), \( \hat{\mathbf{J}}_y = \hat{\mathbf{J}}_{y1} + \hat{\mathbf{J}}_{y2} \) and \( \hat{\mathbf{J}}_z = \hat{\mathbf{J}}_{z1} + \hat{\mathbf{J}}_{z2} \). They are the generators of the \( SU(2) \) algebra on the tensor product space spanned by the number states of the cavity modes and even modes of both sides.

The number of interacting photons in the left side is defined as \( \hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1 + \hat{d}_1 \hat{d}_1^\dagger \) and the number of interacting photons in the right side is \( \hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2 + \hat{d}_2^\dagger \hat{d}_2 \). The term “interacting” is used since odd modes operators do not appear in the interaction terms and therefore eigenstates of \( \hat{c}_i^\dagger \hat{c}_i \) evolve freely. The difference of interacting photons between both sides will be denoted by \( \Delta \hat{N} = \hat{N}_1 - \hat{N}_2 \). With all these notations, hamiltonian (21) can be expressed as

\[
\hat{H}_I = 2g \hat{\mathbf{J}}_x + \hbar \omega_m \hat{c}^\dagger \hat{c} - \kappa \left( \frac{\hbar}{2} \Delta \hat{N} + \hat{J}_z \right) \left( e \hat{c}^\dagger + \hat{c} \right). \tag{23}
\]

Recall that the first term is \( \hat{H}_{\text{cav-ext}} \), the second corresponds to \( \hat{H}_m \) while the third is \( \hat{H}_{\text{OM}} \). Notice that the optomechanical interaction can be further split into two terms, i.e. \( \hat{H}_{\text{OM}} = \hat{H}_{\text{OM1}} + \hat{H}_{\text{OM2}} \), where \( \hat{H}_{\text{OM1}} = -\frac{\hbar \kappa}{2} \Delta \hat{N} \left( e \hat{c}^\dagger + \hat{c} \right) \) and \( \hat{H}_{\text{OM2}} = -\kappa \hat{J}_z \left( e \hat{c}^\dagger + \hat{c} \right) \).

In order to see the time dependence of the second process we go to a rotating frame II with respect to \( \hat{H}_{\text{cav-ext}} + \hat{H}_m + \hat{H}_{\text{OM1}} \). In this picture the time-ordered expansion of the time evolution operator can be terminated at zero order, i.e. the evolution operator in this frame is just the identity and accordingly the evolution given by \( \hat{H}_{\text{OM2}} \) may be disregarded from \( \hat{H}_I \), when the condition

\[
\kappa \ll \omega_m \ll g \tag{24}
\]
is fulfilled (see Appendix A). Therefore, the Hamiltonian becomes

$$\hat{H}_1 = 2g\hat{J}_x + \hbar \omega_m \hat{c}^\dagger \hat{c} - \frac{\hbar \kappa}{2} \Delta \hat{N}(\hat{c}^\dagger + \hat{c}).$$

(25)

Notice that in this regime the oscillator does not couple to the difference of intensities inside the cavity but to the difference of photons between both sides of the system, i.e. including the arms of the interferometer. Mathematically, the “original” radiation force $\hat{F}_{\text{rad}} = \hbar G(\hat{a}_1^\dagger \hat{a} - \hat{a}_2^\dagger \hat{a}_2)$ was replaced by $\hat{F}_{\text{rad}} = \hbar G\left(\frac{\hat{a}_1^\dagger \hat{a}_{1i} + \hat{d}_1^\dagger \hat{d}_1 - \hat{a}_2^\dagger \hat{a}_{2i} + \hat{d}_2^\dagger \hat{d}_2}{2}\right)$. When the optomechanical strength $\kappa$ is much smaller than the hopping strength $g$, the energy flow from and into the cavity occurs at a rate $g$, as if no moving mirror was inside the cavity. For the mirror, whose characteristic time scale is given by $\omega_m$, this energy exchange occurs very fast when $\omega_m \ll g$. Therefore, it only “sees” a time-averaged intensity inside the cavity. This average corresponds to $(\hat{a}_{1i}^\dagger \hat{a}_i + \hat{d}_1^\dagger \hat{d}_i)/2$ for the side $i = 1, 2$. See Appendix B for a discussion of this regime in the Heisenberg picture. Condition (24) defines the range of parameters for which the process of entrance and output of photons into and from the cavity can be uncoupled from the interaction of the photon inside the cavity with the oscillator, i.e. $[\hat{H}_{\text{cav-ext}}, \hat{H}_m + \hat{H}_{\text{OM1}}] = 0$.

5.2. Space of states

The Hilbert space of the entire system is a tensor product space $\mathcal{H} = \mathcal{H}_{\text{EM}} \otimes \mathcal{H}_m$, where $\mathcal{H}_{\text{EM}}$ is the Hilbert space of the electromagnetic field and $\mathcal{H}_m$ is the mechanical space of states. As in section 2, state vectors in $\mathcal{H}_m$ are denoted by $|\psi\rangle_m$.

The Hilbert space of the electromagnetic field, in turn, is a tensor product space $\mathcal{H}_{\text{EM}} = \mathcal{H}_{\text{ext}} \otimes \mathcal{H}_{\text{cav}}$, where $\mathcal{H}_{\text{ext}}$ is the space of states of the external field and $\mathcal{H}_{\text{cav}}$ the Hilbert space for the cavity. The electromagnetic field inside the cavity is a two-mode field. The single-photon states are defined as

$$\hat{a}_1^\dagger \hat{a}_1 |1, 0\rangle_{\text{cav}} = |1, 0\rangle_{\text{cav}}, \quad \hat{a}_2^\dagger \hat{a}_2 |0, 1\rangle_{\text{cav}} = |0, 1\rangle_{\text{cav}},$$

and the two-mode vacuum state of the cavity will be denoted by $|0, 0\rangle_{\text{cav}}$.

The external field is a four-mode field. Regarding the field inside the interferometer two sets of mode operators have been defined. One set is given by $\hat{d}_1, \hat{d}_2, \hat{e}_1, \hat{e}_2$ and their hermitian conjugate operators. The corresponding number operators allow to define single-photon states as

$$\hat{d}_1^\dagger \hat{d}_1 |1, 0, 0, 0\rangle_S = |1, 0, 0, 0\rangle_S, \quad \hat{d}_2^\dagger \hat{d}_2 |0, 1, 0, 0\rangle_S = |0, 1, 0, 0\rangle_S,$$

$$\hat{e}_1^\dagger \hat{e}_1 |0, 0, 1, 0\rangle_S = |0, 0, 1, 0\rangle_S, \quad \hat{e}_2^\dagger \hat{e}_2 |0, 0, 0, 1\rangle_S = |0, 0, 0, 1\rangle_S.$$  

(27)

This set of eigenvectors defines a basis for expressing single-photon states of the external field that will be called standing wave basis. The subscript $S$ will be used to label these four-mode number states.
Strange weak values of photon number operators in the optomechanical interaction

The other set is given by the \( \hat{b}_{1,k_0} \), \( \hat{b}_{2,k_0} \), \( \hat{b}_{1,-k_0} \), \( \hat{b}_{2,-k_0} \) and their hermitian conjugate operators. Single-photon states can also be expressed as solutions to the following eigenvalue equations:

\[
\begin{align*}
\hat{b}_{1,k_0}^\dagger \hat{b}_{1,k_0} |1,0,0,0\rangle_T &= |1,0,0,0\rangle_T, \\
\hat{b}_{2,k_0}^\dagger \hat{b}_{2,k_0} |0,1,0,0\rangle_T &= |0,1,0,0\rangle_T, \\
\hat{b}_{1,-k_0}^\dagger \hat{b}_{1,-k_0} |0,0,1,0\rangle_T &= |0,0,1,0\rangle_T, \\
\hat{b}_{2,-k_0}^\dagger \hat{b}_{2,-k_0} |0,0,0,1\rangle_T &= |0,0,0,1\rangle_T.
\end{align*}
\]

(28)

These eigenvectors define another basis for expressing single-photon states of the external field, which will be called \textit{travelling wave} basis and the subscript \( T \) will be employed. Conversion between vectors of each basis can easily be done using (20) and some examples are given in Appendix C.

On the other hand, regarding the electromagnetic field \textit{outside the interferometer}, let us define the operators \( \hat{j}_{1,k_0} \) and \( \hat{j}_{1,k_0}^\dagger \) as annihilation and creation operators for travelling modes propagating \textit{into} the interferometer through the paths \( AB \) and \( EF \). Similarly, the travelling mode operators \( \hat{j}_{1,-k_0} \) and \( \hat{j}_{1,-k_0}^\dagger \) will describe propagation \textit{away} from the interferometer along the arms \( AB \) and \( EF \). For the \( UV \) branch the operators are defined equivalently, with \( \hat{j}_{2,k_0} \) and \( \hat{j}_{2,k_0}^\dagger \) describing propagation into the interferometer and the operators \( \hat{j}_{2,-k_0} \) and \( \hat{j}_{2,-k_0}^\dagger \) describing propagation away from it. The eigenvalue equations that define the single-photon states outside the interferometer are:

\[
\begin{align*}
\hat{j}_{1,k_0}^\dagger \hat{j}_{1,k_0} |1,0,0,0\rangle_T^{\text{out}} &= |1,0,0,0\rangle_T^{\text{out}}, \\
\hat{j}_{2,k_0}^\dagger \hat{j}_{2,k_0} |0,1,0,0\rangle_T^{\text{out}} &= |0,1,0,0\rangle_T^{\text{out}}, \\
\hat{j}_{1,-k_0}^\dagger \hat{j}_{1,-k_0} |0,0,1,0\rangle_T^{\text{out}} &= |0,0,1,0\rangle_T^{\text{out}}, \\
\hat{j}_{2,-k_0}^\dagger \hat{j}_{2,-k_0} |0,0,0,1\rangle_T^{\text{out}} &= |0,0,0,1\rangle_T^{\text{out}}.
\end{align*}
\]

(29)

The operators for the field outside the interferometer are related to the operators inside the interferometer through a unitary transformation that ensures that they satisfy boson commutation relations. This unitary transformation is given by the beam splitter and, since conserves the total number of photons, allows transformation between photon states outside and inside the interferometer. The beam splitter transformation is described in Appendix D.

The four-mode vacuum state inside the interferometer will be denoted by \( |0,0,0,0\rangle_S = |0,0,0,0\rangle_T \) in the standing and travelling wave representations, respectively. The vacuum state outside the interferometer will be denoted by \( |0,0,0,0\rangle_T^{\text{out}} \).

As an important remark, it should be pointed out that all modes propagating into the interferometer along the paths \( EF \) and \( UV \) or into the cavity along \( GH \) and \( ST \) have horizontal polarization, while modes coming back through these branches are vertically polarized. The polarization changes as they pass the \( \lambda/4 \) plates. Nevertheless, the polarization degree of freedom will not be put explicitly in the states but commented whenever is necessary.
5.3. Time evolution operator

The exponential operator for the time evolution can be easily disentangled since $\Delta \hat{N}$ is invariant under rotations, i.e. $[\Delta \hat{N}, \hat{J}_x] = [\Delta \hat{N}, \hat{J}_y] = [\Delta \hat{N}, \hat{J}_z] = 0$. Hence,

$$\hat{U}_1(t) = \exp \left[ -i \omega_m t \hat{c}^\dagger \hat{c} + i \left( \frac{\kappa}{2} \right) t \Delta \hat{N} (\hat{c}^\dagger + \hat{c}) \right] \exp \left[ -i (2gt/\hbar) \hat{J}_x \right].$$  \hspace{1cm} (30)

The first term corresponds to the evolution given by $\hat{H}_m + \hat{H}_{OM1}$ while the second represents the evolution produced by $\hat{H}_{cav-ext}$. After working out the first term (see Appendix E), the evolution operator becomes

$$\hat{U}_1(t) = \exp \left\{ \bar{k}(t) \Delta \hat{N} [e^{i\alpha(t)} \hat{c}^\dagger - e^{-i\alpha(t)} \hat{c}] \right\} \exp \left[ -i (2gt/\hbar) \hat{J}_x \right] \exp \left(-i \omega_m t \hat{c}^\dagger \hat{c} \right),$$ \hspace{1cm} (31)

where $\bar{k}(t) = \bar{k} \sin(\omega_m t/2)$, $\bar{k} = \kappa/\omega_m$ is a scaled coupling parameter and $\alpha(t) = (\pi - \omega_m t)/2$. Notice that $\bar{k}(t)$ oscillates between 0 and $\bar{k}$ with frequency $\omega_m/2$ and the phase factor rotates at the same frequency.

The first term in (31) entangles the photon to the vibrating mirror and will be denoted by $\hat{U}_{OM}(t)$. The second operator describes the photon exchange process between the external even modes and the cavity modes and will be called $\hat{U}_{ex}(t)$. The third term is just the free evolution of the mechanical oscillator, and will be denoted by $\hat{U}_m(t)$. Therefore, the time evolution operator can be compactly expressed as $\hat{U}_1(t) = \hat{U}_{OM}(t) \hat{U}_{ex}(t) \hat{U}_m(t)$, where $\hat{U}_{OM}(t)$ acts over $\hat{H}$, $\hat{U}_{ex}(t)$ over $\hat{H}_{EM}$ and $\hat{U}_m(t)$ over $\hat{H}_m$.

Notice that the term $\hat{U}_{ex}(t) = \exp [-i (2gt/\hbar) \hat{J}_x]$ will produce Rabi oscillations between the cavity and the external field with frequency $2g$, i.e. the photon will be emitted and absorbed by the cavity in a cycle that will repeat indefinitely. This occurs because we have treated the whole system as a closed system, which has no stationary state. In order to make the system an open system, the damping should be taken into account, which appears when the external field is a continuum (or a discrete set of infinite modes). The way for dealing with this issue will be to simply “stop” the interaction at time $t = T$, where $T = 2\pi/\Delta \omega_{ph}$ and $\Delta \omega_{ph}$ is the narrow, but not zero, bandwidth of the photon.

6. Unitary evolution

The process starts with one photon in a travelling mode at the input port. The photon has horizontal polarization and is completely reflected by the polarizing beam splitter (PBS) into the arm $\text{EF}$. Hence, the state of the external field is given by $|1, 0, 0, 0\rangle_\text{out}$. The photon enters the interferometer through a polarizing dependent beam splitter (PDBS). Using the beam splitter transformation (D.5) the field in the arms $\overline{GH}$ and $\overline{TS}$ is in a path-entangled state described by $-\cos(\theta_1/2) |1, 0, 0, 0\rangle_T + \sin(\theta_1/2) |0, 1, 0, 0\rangle_T$. After the quarter wave plates the field changes its polarization state from horizontal to circular. Then, the field in the arm 2 is reflected twice in $M3$ and $M2$, while the field
 Strange weak values of photon number operators in the optomechanical interaction

in the other arm is reflected once in M1. Therefore, the state in the arms $\overline{KL}$ and $\overline{MN}$ is given by $\cos(\theta/2) |1, 0, 0, 0\rangle_T + \sin(\theta/2) |0, 1, 0, 0\rangle_T$. The cavity is initially empty and begins in the vacuum state $|0, 0\rangle_{cav}$. Therefore, the initial state of the electromagnetic field is given by $\cos(\theta/2) |1, 0, 0, 0\rangle_{cav} + \sin(\theta/2) |0, 1, 0, 0\rangle_{cav}$. It is convenient to express this initial state using the standing wave basis for the external field. Hence,

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left( \cos(\theta/2) |1, 0, 0, 0\rangle_S |0, 0\rangle_{cav} + \sin(\theta/2) |0, 1, 0, 0\rangle_S |0, 0\rangle_{cav} \right) + \frac{1}{\sqrt{2}} \left( \cos(\theta/2) |0, 0, 1, 0\rangle_S |0, 0\rangle_{cav} + \sin(\theta/2) |0, 0, 0, 1\rangle_S |0, 0\rangle_{cav} \right). \quad (32)$$

In this way the initial optical state is expressed as a linear combination of eigenstates of $\Delta \hat{N}$. In fact, the state $|1, 0, 0, 0\rangle_S |0, 0\rangle_{cav}$ has eigenvalue +1 (one interacting photon in the left side), the state $|0, 1, 0, 0\rangle_S |0, 0\rangle_{cav}$ is an eigenstate with eigenvalue $-1$ (one interacting photon in the right side), while the states $|0, 0, 1, 0\rangle_S |0, 0\rangle_{cav}$ and $|0, 0, 0, 1\rangle_S |0, 0\rangle_{cav}$ have both eigenvalue 0 (no interacting photons).

On the other hand, the oscillator starts in the squeezed vacuum state $|\xi\rangle_m = \hat{S}(\xi) |0\rangle_m$, where $\hat{S}(\xi) = \exp(\xi \hat{c}^\dagger \hat{c} - \xi^* \hat{c} \hat{c}^\dagger)$ is the squeezed displacement operator, $\xi = r \exp(i \theta_S)$ is the squeezing parameter, and $|0\rangle_m$ is the mechanical vacuum state. Therefore, $|\psi(0)\rangle_m = |\xi\rangle_m$. Consequently, the initial state of the optomechanical system is given by

$$|\Phi(0)\rangle = |\psi(0)\rangle_m |\psi(0)\rangle_m. \quad (33)$$

The state of the system after a time $t$ is obtained by applying the time evolution operator over the initial state and is given by

$$|\Phi(t)\rangle = \frac{\cos(gt)}{\sqrt{2}} \left[ \cos(\theta/2) |1, 0, 0, 0\rangle_S |0, 0\rangle_{cav} \hat{\kappa}(t) e^{i\alpha(t)} |\xi'\rangle_m + \sin(\theta/2) |0, 1, 0, 0\rangle_S |0, 0\rangle_{cav} - \hat{\kappa}(t) e^{i\alpha(t)} |\xi'\rangle_m \right] - \frac{i \sin(gt)}{\sqrt{2}} |0, 0, 0, 0\rangle_S \left[ \cos(\theta/2) |1, 0\rangle_{cav} \hat{\kappa}(t) e^{i\alpha(t)} |\xi'\rangle_m + \sin(\theta/2) |0, 1\rangle_{cav} - \hat{\kappa}(t) e^{i\alpha(t)} |\xi'\rangle_m \right] + \frac{1}{\sqrt{2}} \left[ \cos(\theta/2) |0, 0, 1, 0\rangle_S |0, 0\rangle_{cav} + \sin(\theta/2) |0, 0, 0, 1\rangle_S |0, 0\rangle_{cav} \right] |\xi'\rangle_m. \quad (34)$$

The details of this calculation are included in Appendix E. The mechanical states of the form $|\alpha, \beta\rangle_m$ denote a coherent squeezed state, i.e. $|\alpha, \beta\rangle_m = \hat{D}(\alpha) \hat{S}(\beta) |0\rangle_m$, where $\hat{D}(\alpha)$ is a Glauber displacement operator and $\hat{S}(\beta)$ a squeeze operator. Also, $\xi' = e^{-2i\omega_m t} \xi$. 
Let us analyze the interaction at time \( T \). At this time we want the photon to be outside the cavity and, consequently, \( T = N\pi/g \), where \( N \) is a large odd integer number. Notice that this entails \( \Delta \omega_{\text{ph}} \propto g/N \). In a realistic situation (an open system) this condition would mean that the time duration of the photon has to be equal or larger than the cavity storage time \( 1/\gamma_C \). The state of the system at time \( T \) is described by

\[
|\Phi(T)\rangle = -\frac{1}{\sqrt{2}} \left[ \cos(\theta_t/2) |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} |\tilde{\kappa}(T)e^{i\alpha(T)}, \xi'\rangle_m + 
\sin(\theta_t/2) |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} \right] + \frac{1}{\sqrt{2}} \left[ \cos(\theta_t/2) |0, 0, 1, 0\rangle_S |0, 0\rangle_{\text{cav}} + \sin(\theta_t/2) |0, 0, 0, 1\rangle_S |0, 0\rangle_{\text{cav}} \right] |\xi\rangle_m .
\]

Recall that the initial state was a linear combination of states of the form \( |\lambda\rangle |\xi\rangle_m \), where \( |\lambda\rangle \) is one the eigenstates of \( \Delta \hat{N} \). The unitary evolution given by \( \hat{U}_t(T) \) transformed the states \( |\lambda\rangle |\xi\rangle_m \) into states of the form \( |\lambda\rangle |\xi(T)\rangle_m \); i.e. the final mechanical states act as markers for the eigenvalues of \( \Delta \hat{N} \). From this perspective, the optomechanical interaction can be understood as a measurement of \( \Delta \hat{N} \), although it does not have the impulsive character of a standard measurement.

If \( \tilde{\kappa}(T) = 0 \), which occurs if \( T \) is an exact multiple of the vibrational period, then \( |\pm \tilde{\kappa}(T)e^{i\alpha(T)}, \xi'\rangle_m = |\xi\rangle_m \) and state (35) becomes a product described by \(-[\cos(\theta_t/2) |0, 0, 1, 0\rangle_T + \sin(\theta_t/2) |0, 0, 0, 1\rangle_T] |0, 0\rangle_{\text{cav}} |\xi\rangle_m \), where we have switched to the travelling wave representation. In this case, there is no entanglement between the oscillator and the photon. Let us assume now the opposite situation, i.e. that \( \tilde{\kappa}(T) \) achieves its maximum value. This occurs when

\[
T = \left(n + \frac{1}{2}\right) T_m ,
\]

where \( T_m = 2\pi/\omega_m \) is the vibrational period of the oscillator and \( n \) is an integer number. Since \( \omega_m \ll g \) this condition implies that \( N \) is a large number and, hence, \( \Delta \omega_{\text{ph}} \ll g \), i.e. the duration of the photon should be much larger than the cavity storage time. Under this condition, \( \tilde{\kappa}(T) \exp[i\alpha(T)] = 1 \), \( \xi' = \xi \) and the state (35) becomes

\[
|\Phi(T)\rangle = -\frac{1}{\sqrt{2}} \left[ \cos(\theta_t/2) |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} |\tilde{\kappa}, \xi\rangle_m + \sin(\theta_t/2) |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} \right] + \frac{1}{\sqrt{2}} \left[ \cos(\theta_t/2) |0, 0, 1, 0\rangle_S |0, 0\rangle_{\text{cav}} + \sin(\theta_t/2) |0, 0, 0, 1\rangle_S |0, 0\rangle_{\text{cav}} \right] |\xi\rangle_m .
\]

The wave functions in the position representation of the mechanical states \( |\pm \tilde{\kappa}, \xi\rangle_m \) and \( |\xi\rangle_m \) (see Appendix G) are Gaussians centered on \( \pm 2x_0\tilde{\kappa} \) and zero, respectively. The position along each peak is normally distributed with standard deviation \( \sigma_q \). The Gaussians wave functions do not overlap when their mean values are much greater than the standard deviation. If \( \theta_S = \pi \), then \( \sigma_q = x_0e^{x} \). Therefore, when \( \tilde{\kappa} \gg e \) there is a negligible overlap between the wave functions and the states \( |\pm \tilde{\kappa}, \xi\rangle_m \) and \( |\xi\rangle_m \) become

\[\sigma_q^2 = x_0^2[e^{-2x} \cos(\theta_S/2) + e^{2x} \sin(\theta_S/2)].\]
orthogonal. This situation resembles a standard von Neumann’s measurement. In this case, when the position of the mechanical oscillator is observed, the photon is projected into one of the eigenstates of $\Delta \hat{N}$, e.g. if the position of the oscillator is observed to be near $2x_0\tilde{\kappa}$ then the photon is projected into the state $|1, 0, 0, 0\rangle_{\text{cav}} |0, 0\rangle$. Nevertheless, condition (24) demands $\kappa \ll 1$ and consequently we are outside the regime of a strong measurement. In our regime we obtain rather a weak measurement, a situation where the wave functions are almost completely overlapped, since the following condition is satisfied

$$x_0\tilde{\kappa} \ll \sigma_q \iff \tilde{\kappa} \ll e^\pi. \quad (38)$$

By using the travelling wave representation, state (37) can be expressed according to

$$|\Phi(T)\rangle = \cos(\theta_i/2) \left[ |1, 0, 0, 0\rangle_T |0, 0\rangle_{\text{cav}} \left( |\xi\rangle_m - |\tilde{\kappa}, \xi\rangle_m \right) / 2 - |0, 0, 1, 0\rangle_T |0, 0\rangle_{\text{cav}} \left( |\xi\rangle_m + |\tilde{\kappa}, \xi\rangle_m \right) / 2 \right] + \sin(\theta_i/2) \left[ |0, 1, 0, 0\rangle_T |0, 0\rangle_{\text{cav}} \left( |\xi\rangle_m - |\pm\tilde{\kappa}, \xi\rangle_m \right) / 2 - |0, 0, 0, 1\rangle_T |0, 0\rangle_{\text{cav}} \left( |\xi\rangle_m + |\pm\tilde{\kappa}, \xi\rangle_m \right) / 2 \right]. \quad (39)$$

Under the weak measurement condition (38) the approximations $|\xi\rangle_m - |\pm\tilde{\kappa}, \xi\rangle_m \approx 0$ and $|\xi\rangle_m + |\pm\tilde{\kappa}, \xi\rangle_m \approx 2|\pm\tilde{\kappa}/2, \xi\rangle_m$ may be performed (Appendix H) and the final state becomes

$$|\Phi(T)\rangle = \left[ \cos(\theta_i/2) |0, 0, 1, 0\rangle_T |\tilde{\kappa}/2, \xi\rangle_m + \sin(\theta_i/2) |0, 0, 0, 1\rangle_T |\pm\tilde{\kappa}/2, \xi\rangle_m \right] |0, 0\rangle_{\text{cav}}, \quad (40)$$

disregarding a global phase factor. In this state the photon is “weakly” entangled to the oscillator.

7. Weak measurement without post-selection

In this section the mechanical oscillator is observed without post-selection, i.e. without using the detectors. In this case there is only information about the initial state of the photon, i.e. about how does it travels to the cavity, but ignorance regarding the final state. The expectation value of the position of the oscillator in the state (40) is given by

$$\langle \hat{q} \rangle = 2x_0\tilde{\kappa} \langle \psi(0)| \Delta \hat{N} |\psi(0)\rangle = 2x_0\tilde{\kappa} \langle \Delta \hat{N} \rangle = x_0\tilde{\kappa} \cos(\theta_i), \quad (41)$$

where $\langle \Delta \hat{N} \rangle$ is the expectation value of the operator $\Delta \hat{N}$ in the state $|\psi(0)\rangle$. Since $\hat{F}_{\text{rad}} = \hbar G \Delta \hat{N}/2$, expression (41) can be equivalently expressed in terms of the radiation force as

$$\langle \hat{q} \rangle = 2\langle \hat{F}_{\text{rad}} \rangle \omega_m^2 M. \quad (42)$$

The mean position of the oscillator is therefore proportional to the mean value of $\Delta \hat{N}$. This is what we expected since in a weak measurement without post-selection the

‡ If $\theta_S$ were chosen to be 0, then $\sigma_q = x_0 e^{-r}$ and a strong measurement could be achieved by choosing a sufficiently large value for $r$. Nevertheless, in our case $\theta_S = \pi$ in order to further weaken the measurement as $r$ is increased.
probability distribution of the meter is simply shifted by an amount proportional to the expectation value of the variable being measured, according to expression (8) from section 2. The expectation value of $\Delta \hat{N}$ is bounded between $\pm 1/2$ because the initial state contains one photon in a non-interacting mode. Otherwise, it would lie between $\pm 1$. Therefore, in this system the mean position can have a maximum shift of $x_0 \tilde{\kappa}$, which occurs every time we know through which arm the photon has travelled (particle behaviour). Notice that, if $\theta_i = \pi/2$, then the photon enters the interferometer in an equal superposition in both arms and exerts no net force over the mirror. In this case, there is no change in the mean position of the oscillator.

8. Weak measurement with post-selection

In this situation the mechanical oscillator is observed after post-selecting the photon in a specific state, which is done using the detectors. In this case, there is knowledge about the initial state of the photon and also about the final state, i.e. how does the photon return to the beam splitter. In this section we firstly show which are the post-selection states and obtain the probabilities of detecting a photon at $D_1$ and $D_2$. Then, it is shown that, for post-selected ensembles obtained by successful detection at $D_1$, the shift of the probability distribution of the position of the oscillator is proportional to the weak value of $\Delta \hat{N}$.

8.1. Post-selection state and probability of detection

The field in state (40) has circular polarization since the photon is in the branches $KL$ or $MN$. Next, the photon propagates back to the beam splitter and then into the detectors. The state in the branches $GH$ and $ST$ is given by $\cos(\theta_i/2) |0, 0, 1, 0\rangle_T |0, 0\rangle_{cav} |\tilde{\kappa}/2, \xi\rangle_m - \sin(\theta_i/2) |0, 0, 1\rangle_T |0, 0\rangle_{cav} |\tilde{\kappa}/2, \xi\rangle_m$ and due to the quarter wave plates the photon has acquired vertical polarization. Using the beam splitter transformation (D.6), the state of the system after the photon has exit the interferometer, i.e in the branches $\overline{EF}$ and $\overline{UV}$, is given by

$$
|0, 0, 1, 0\rangle_T^{out} |0, 0\rangle_{cav}^{in} \left[ \cos(\theta_i/2) \cos(\theta_i/2) |\tilde{\kappa}/2, \xi\rangle_m - \sin(\theta_i/2) \sin(\theta_i/2) |\tilde{\kappa}/2, \xi\rangle_m \right] - \\
|0, 0, 0, 1\rangle_T^{out} |0, 0\rangle_{cav}^{in} \left[ \cos(\theta_i/2) \sin(\theta_i/2) |\tilde{\kappa}/2, \xi\rangle_m + \sin(\theta_i/2) \cos(\theta_i/2) |\tilde{\kappa}/2, \xi\rangle_m \right].
$$

(43)

The field in the path $\overline{EF}$ is directly transmitted into $\overline{AB}$ since it is vertically polarized. From this expression it is clear that, when detector $D_1$ clicks, the oscillator is left in the (unnormalized) conditional state $\cos(\theta_i/2) \cos(\theta_i/2) |\tilde{\kappa}/2, \xi\rangle_m - \sin(\theta_i/2) \sin(\theta_i/2) |\tilde{\kappa}/2, \xi\rangle_m$. Analogously, when detector $D_2$ clicks the vibrating mirror is left in the (unnormalized) conditional state $\cos(\theta_i/2) \sin(\theta_i/2) |\tilde{\kappa}/2, \xi\rangle_m + \sin(\theta_i/2) \cos(\theta_i/2) |\tilde{\kappa}/2, \xi\rangle_m$. Therefore, detecting a photon at $D_1$ is equivalent to project state (40) into the state

$$
|\psi_f\rangle = \cos(\theta_i/2) |0, 0, 1, 0\rangle_T |0, 0\rangle_{cav} - \sin(\theta_i/2) |0, 0, 0, 1\rangle_T |0, 0\rangle_{cav},
$$

(44)
while detecting a photon at $D_2$ is equivalent to project into 
\[ |\psi^+_f\rangle = \sin(\theta_f/2) |0, 0, 1, 0\rangle_{\text{cav}} + \cos(\theta_f/2) |0, 0, 0, 1\rangle_{\text{cav}}, \]
\[ |\psi^+_i\rangle = \sin(\theta_i/2) |0, 0, 1, 0\rangle_{\text{cav}} + \cos(\theta_i/2) |0, 0, 0, 1\rangle_{\text{cav}}, \]
i.e. both detectors are orthogonal. Consequently, the probability of detecting a photon at $D_1$ is given by 
\[ P_1 = |\langle \psi_1 | \Phi(T) \rangle|^2 = \cos^2 \left( \frac{\theta_1 + \theta_f}{2} \right). \]

Notice that the inner products $m \langle -\kappa/2, \xi | \kappa, \xi \rangle_m$ and $m \langle \kappa/2, -\xi | \kappa, \xi \rangle_m$ are both equal to one due to the weak measurement condition (38). For the general case $m \langle -\kappa/2, \xi | \kappa, \xi \rangle_m = m \langle \kappa/2, -\xi | \kappa, \xi \rangle_m = \exp(-2|\beta|^2)$, where $|\beta|^2 = (\kappa e^{-r})^2/4$. Under condition (38) $\kappa e^{-r} \ll 1$ and therefore $\exp(-2|\beta|^2) \approx 1$. On the other hand, the probability of detecting a photon at $D_2$ is $|\langle \psi^+_f | \Phi(T) \rangle|^2 = 1 - P_1$.

**8.2. Weak value of $\Delta \hat{N}$**

At this point, it is useful to define the states 
\[ |\psi^d_i\rangle = \cos(\theta_i/2) |1, 0, 0, 0\rangle_{\text{s}} |0, 0\rangle_{\text{cav}} + \sin(\theta_i/2) |0, 1, 0, 0\rangle_{\text{s}} |0, 0\rangle_{\text{cav}}, \]
\[ |\psi^e_i\rangle = \cos(\theta_i/2) |0, 0, 1, 0\rangle_{\text{s}} |0, 0\rangle_{\text{cav}} + \sin(\theta_i/2) |0, 0, 0, 1\rangle_{\text{s}} |0, 0\rangle_{\text{cav}}, \]
where the superscript $d$ or $e$ indicates which type of modes contains photons. This notation allows to express the initial state $|\psi(0)\rangle$ as 
\[ |\psi(0)\rangle = \frac{|\psi^d_i\rangle + |\psi^e_i\rangle}{\sqrt{2}}. \]

The state of the system after a time $t$ is given by 
\[ |\Phi(t)\rangle = \frac{1}{\sqrt{2}} \hat{U}_{\text{OM}}(t) \hat{U}_{\text{ex}}(t) |\psi^d_i\rangle \hat{U}_m(t) |\psi_i\rangle_m + \frac{1}{\sqrt{2}} |\psi^e_i\rangle \hat{U}_m(t) |\psi_i\rangle_m. \]

For time $T = N\pi/g$, which is an odd multiple of $T_m/2$ by condition (36), $\hat{U}_m(T) |\psi_i\rangle_m = |\psi_i\rangle_m = |\xi\rangle_m$. Also, $\hat{U}_{\text{ex}}(T) = \exp(-\frac{i}{2} 2\pi \hat{J}_c)$, which corresponds to a rotation of $2\pi$. Its action over $|\psi^d_i\rangle$ is $\hat{U}_{\text{ex}}(T) |\psi^d_i\rangle = -|\psi^d_i\rangle$ (in order to see this, set $t = N\pi/g$ in equation (F.4)). Additionally, $\hat{U}_{\text{OM}}(T) = \exp[\kappa \Delta \hat{N}(\hat{c}^\dagger - \hat{c})]$. Therefore, state (49) can be written as 
\[ |\Phi(T)\rangle = \hat{U}_{\text{OM}}(T) \left( \frac{|\psi^d_i\rangle - |\psi^e_i\rangle}{\sqrt{2}} \right) |\xi\rangle_m, \]
disregarding a global phase factor of $-1$. Consequently, the action of $\hat{U}_{\text{ex}}(T)$ transforms the initial state $|\psi(0)\rangle$ into $(|\psi^d_i\rangle - |\psi^e_i\rangle)/\sqrt{2}$. Regarding the optomechanical term $\hat{U}_{\text{OM}}(T)$, this state can be accounted as an “initial state” because this part of the evolution is due to the interaction with the external field and not to the optomechanical coupling. Therefore, we define 
\[ |\psi_i\rangle \equiv \frac{|\psi^d_i\rangle - |\psi^e_i\rangle}{\sqrt{2}} = \cos(\theta_i/2) |0, 0, 1, 0\rangle_T |0, 0\rangle_{\text{cav}} + \sin(\theta_i/2) |0, 0, 0, 1\rangle_T |0, 0\rangle_{\text{cav}}, \]
and $|\Phi(T)\rangle = \hat{U}_{\text{OM}}(T) |\psi_i\rangle |\xi\rangle$. Notice that $|\psi_i\rangle$ has exactly the same form as $|\psi(0)\rangle$ but the photon is “now” propagating away from the cavity. The normalized state of the
Strange weak values of photon number operators in the optomechanical interaction

oscillator after the photon is detected at the dark port, i.e. after projecting state $|\Phi(T)\rangle$ into state $|\xi\rangle$,

$$|\psi_f\rangle_m = \frac{1}{\sqrt{P_1}} \langle \psi_f | \hat{U}_{\text{OM}}(T) |\psi_i\rangle |\xi\rangle_m. \quad (52)$$

Under condition (38) the operator $\hat{U}_{\text{OM}}(T)$ can be expanded up to first order to a good approximation, as follows

$$|\psi_f\rangle_m = \frac{1}{\sqrt{P_1}} \langle \psi_f | \left(1 + \bar{\kappa} \Delta \hat{\mathcal{N}} (\hat{c}^\dagger - \hat{c})\right) |\psi_i\rangle |\xi\rangle_m. \quad (53)$$

Notice that $\sqrt{P_1} = |\langle \psi_f | \psi_i \rangle|$. The weak value of $\Delta \hat{\mathcal{N}}$ is defined as

$$\langle \Delta \hat{\mathcal{N}} \rangle_w \equiv \frac{\langle \psi_f | \Delta \hat{\mathcal{N}} |\psi_i\rangle}{\langle \psi_f | \psi_i \rangle} = \frac{1}{2} \cos \left(\frac{\theta_i - \theta_f}{2}\right) \cos \left(\frac{\theta_i + \theta_f}{2}\right) \in \mathbb{R}, \quad (54)$$

and the state of the mirror becomes

$$|\psi_f\rangle_m = \left[1 + \kappa \langle \Delta \hat{\mathcal{N}} \rangle_w (\hat{c}^\dagger - \hat{c})\right] |\xi\rangle_m. \quad (55)$$

Moreover, when a new (and stronger) condition is imposed, namely,

$$\bar{\kappa} \langle \Delta \hat{\mathcal{N}} \rangle_w \ll e^r, \quad (56)$$

then the exponential operator can be recovered (see Appendix I) and

$$|\psi_f\rangle_m = \exp \left[\bar{\kappa} \langle \Delta \hat{\mathcal{N}} \rangle_w (\hat{c}^\dagger - \hat{c})\right] |\xi\rangle_m = |\bar{\kappa} \langle \Delta \hat{\mathcal{N}} \rangle_w, \xi\rangle_m, \quad (57)$$

i.e. the mechanical squeezed vacuum is displaced by $\bar{\kappa} \langle \Delta \hat{\mathcal{N}} \rangle_w$. The expectation value of the position of the oscillator in this state is

$$\langle \hat{q}\rangle = 2 x_0 \bar{\kappa} \langle \Delta \hat{\mathcal{N}} \rangle_w = x_0 \bar{\kappa} \frac{\cos \left(\frac{\theta_i - \theta_f}{2}\right)}{\cos \left(\frac{\theta_i + \theta_f}{2}\right)}. \quad (58)$$

As a first observation let us indicate that for pre-selected ensembles we have found that

$$\langle \hat{q}\rangle = 2 x_0 \bar{\kappa} \langle \Delta \hat{\mathcal{N}} \rangle \quad (41),$$

while for pre- and post-selected ensembles

$$\langle \hat{q}\rangle = 2 x_0 \bar{\kappa} \langle \Delta \hat{\mathcal{N}} \rangle_w.$$ Contrary to the standard expectation value $\langle \Delta \hat{\mathcal{N}} \rangle$, which is bounded by $\pm 1/2$, the weak value $\langle \Delta \hat{\mathcal{N}} \rangle_w$ is unbounded. As the initial and final states become orthogonal, the weak value gets larger and the shift of the meter is increased. Therefore, the radiation pressure effect of a single photon is amplified when post-selection is employed. In other words, the position of the apparatus (the mechanical oscillator) is shifted as if many photons were inside the cavity when no post-selection is performed.
In figure 6 the amplification of the shift caused by one photon is shown for different post-selection states. For the black curve, the probability density function of the oscillator is shifted as if 3 photons were inside the cavity, while for the blue dotted curve the shift is equivalent to the displacement caused by nearly 60 photons. Notice that, as the shift gets larger, the density function begins to deform since condition (56) starts to break down. When the oscillator begins in a state with larger uncertainty in momentum, bigger displacements can be achieved without deforming the density function (see dotted curves, for which squeezing was added). Nevertheless, the weak value can be increased at the cost of decreasing the detection probability as it is illustrated in table 1 and figure 7.

![Figure 6. Probability density of the X - quadrature of the mechanical oscillator after post-selection. For all curves the initial state is \( \theta_i = \pi/2 \) and \( \kappa = 0.01 \). The displacement of the position is proportional to the weak value: \( \langle \Delta \hat{N} \rangle_w = 3 \) (black), 10 (blue), 15 (green), 32 (red dotted) and 58 (blue dotted). Only for the dotted curves squeezing was added in order to fulfil condition (56), using \( r = 0.65 \) (red dotted) and \( r = 1.4 \) (blue dotted).](image)

Secondly, notice that every time the trajectory is known with certainty, which occurs when \( \theta_i = 0, \pi \), or \( \theta_f = 0, \pi \), then the weak value equals the mean value, i.e. \( \langle \Delta \hat{N} \rangle_w = \langle \hat{N} \rangle = \pm 1/2 \). Nevertheless, when the trajectory is completely unknown, which happens when \( \theta_i = \theta_f = \pi/2 \), then the mean value \( \langle \Delta \hat{N} \rangle \) is zero while the weak value \( \langle \Delta \hat{N} \rangle_w \) diverges to \( \pm \infty \). In general, when \( |\psi_i\rangle \) and \( |\psi_f\rangle \) are orthogonal states, the weak value diverges. This occurs when \( \theta_f = \pi - \theta_i \) for \( \theta_i \neq 0 \land \theta_i \neq \pi \) as it is shown in figure 8.
Strange weak values of photon number operators in the optomechanical interaction

Table 1. Weak value and detection probability at the dark port for different final states ($\theta_f$). The initial state is fixed by choosing $\theta_i = \pi/2$, i.e. the photon starts in a maximally path-entangled state.

| $\theta_f$ | Weak Value ($\langle \Delta \hat{N} \rangle_w$) | Detection Probability ($P_1$) |
|------------|---------------------------------|-------------------------------|
| 50°        | 1.4                             | 11.7 %                        |
| 60°        | 1.9                             | 6.7 %                         |
| 70°        | 2.8                             | 3.0 %                         |
| 75°        | 3.8                             | 1.7 %                         |
| 80°        | 5.7                             | 0.8 %                         |
| 85°        | 11.0                            | 0.2 %                         |

Figure 7. The weak value $\langle \Delta \hat{N} \rangle_w$ (blue line) and the probability of detection $P_1$ (red line) are plotted as a function of the final state ($\theta_f$), for a fixed initial state $\theta_i = \pi/2$.

As a third remark, it is worth to mention that, in order for the shift to reach the level of the zero-point fluctuations, then $\langle \Delta \hat{N} \rangle_w = 1/(2\tilde{\kappa})$. For $\tilde{\kappa} = 5 \cdot 10^{-3}$ the weak value should be equal to 100, which occurs with a very small probability $\ll 1\%$ (of the order of $10^5$ trials). When $\tilde{\kappa} = 5 \cdot 10^{-2}$ then $\langle \Delta \hat{N} \rangle_w$ should be equal to 10 which happens with probability $\approx 0.25\%$ (2 or 3 successes over $10^3$ trials). Disregarding these small probabilities, as $\tilde{\kappa} \langle \Delta \hat{N} \rangle_w$ approaches to 1/2, condition (56) can be only satisfied by adding squeezing, i.e. choosing $r \neq 0$, as it is shown in figure 9.

9. Discussion

In this section we discuss the fact that in this experiment the weak value of $\Delta \hat{N}$ is anomalous, i.e. it is always greater or equal than the standard expectation value $\langle \Delta \hat{N} \rangle$. For that purpose, we analyse the conditional quantum state of the oscillator for post-
Strange weak values of photon number operators in the optomechanical interaction

Figure 8. The weak value of the operator $\Delta \hat{N}$ is plotted as a function of the final state $(\theta_f)$, for different initial states, $\theta_i = 3\pi/4$ (blue line), $\pi/2$ (green line), $\pi/4$ (red line) and $0, \pi$ (grey dotted lines). On the other hand, the standard mean value is bounded, i.e. $-1/2 \leq \langle \Delta \hat{N} \rangle \leq 1/2$.

Figure 9. Wave functions of state (52) in the $X$ - quadrature representation for different values of $\tilde{\kappa} \langle \Delta \hat{N} \rangle_w$: 0.2 (blue line), 0.5 (purple line) and 0.7 (green line). For all curves squeezing was added ($r = 1.5$).

selected photons at $D_1$. Consequently, let us consider the normalized mechanical state (40) when it is projected into the post-selection state (44), namely,

$$|\psi_f\rangle_m = \frac{\cos(\theta_i/2) \cos(\theta_f/2)}{\cos \left( \frac{\theta_i + \theta_f}{2} \right)} |\tilde{\kappa}/2, \xi \rangle_m - \frac{\sin(\theta_i/2) \sin(\theta_f/2)}{\cos \left( \frac{\theta_i + \theta_f}{2} \right)} |\tilde{\kappa}/2, \xi \rangle_m,$$

that consists of a superposition of two non-orthogonal coherent squeezed states. For simplicity, it is convenient to name the terms that depend on the parameters of the
Strange weak values of photon number operators in the optomechanical interaction

beam splitter, defining
\[
A = \frac{\cos(\theta_1/2) \cos(\theta_1/2)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}, \quad \quad B = -\frac{\sin(\theta_1/2) \sin(\theta_1/2)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}.
\]

(60)

When we ignore the path followed by the photon, then \(A\) and \(B\) are not zero. Also, notice that \(A\) and \(B\) have different signs \((A > 0 \text{ and } B < 0)\). In the position representation the wave function of the oscillator is
\[
\psi_1(q) = A\psi_+ (q) + B\psi_-(q),
\]
where \(\psi_{\pm}(q)\) are Gaussian functions given by
\[
\psi_{\pm}(q) = \left(\frac{1}{\sqrt{2\pi}\sigma_q}\right)^{1/2} \exp\left[-(q \mp x_0\lambda^2)^2\right], \quad \sigma_q = x_0 e^r,
\]
i.e. each Gaussian is just a translation of the original wave function. In terms of its Fourier transform, the wave function can be expressed as
\[
\psi_1(q) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \psi_1(p) S(p) e^{ipq/h},
\]
where \(\psi_1(p)\) is the initial wave function of the meter in momentum representation, and
\[
S(p) = Ae^{-i\omega_0(p/h)} + Be^{i\omega_0(p/h)}.
\]
(64)

Notice that \(|A + B| = 1\). Without lost of generality, let us assume that \(A + B = 1\).

Notice also that \(A - B = 2 \langle \Delta \hat{N}\rangle\). Therefore,
\[
S(p) = \left(\frac{1}{2} + \langle \Delta \hat{N}\rangle\right) e^{-i\omega_0(p/h)} + \left(\frac{1}{2} - \langle \Delta \hat{N}\rangle\right) e^{i\omega_0(p/h)}.
\]
(65)

From here we see that, when the weak value is anomalous, the Fourier coefficients \(A\) and \(B\) have different signs. Notice also that this function has wavelength \(\lambda = 2\pi/(\tilde{\kappa}x_0)\), but if we look close to the origin, we find out that
\[
S(p) \approx 1 - i2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w (p/h) - \frac{1}{2} (x_0\tilde{\kappa})^2 (p/h)^2
\]
(66)
\[
= 1 - i2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w (p/h) - (2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w)^2 (p/h)^2 + \left[\langle \Delta \hat{N}\rangle_w^2 - \frac{1}{4}\right] (2x_0\tilde{\kappa})^2 (p/h)^2.
\]
(67)

Since the weak value is anomalous the quadratic terms cancel out and
\[
S(p) = 1 - i2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w (p/h) \approx \exp\left[-i2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w (p/h)\right],
\]
(68)
i.e. the function has a much shorter wavelength over a region in momentum space such that \(p/h < 1/(2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w)\). Therefore, \(S(p)\) is a superoscillatory function [31].

The superscillatory behaviour of \(S(p)\) appears in the wave function of the meter when \(\sigma_p/h < 1/(2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w)\), or, equivalently, when \(2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w < \sigma_q\), which corresponds to condition [56]. This behaviour produces a “super” shift in the position representation, namely,
\[
\psi_1(q) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \psi_1(p) \exp [i(q - 2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w)(p/h)] = \psi_1(q - 2x_0\tilde{\kappa} \langle \Delta \hat{N}\rangle_w).
\]
(69)
Strange weak values of photon number operators in the optomechanical interaction

Consequently, as a result of the weak measurement, the meter is left in a coherent superposition of two non-orthogonal states, which has superoscillatory behaviour in momentum space when the weak value is anomalous and the uncertainty in the position is large enough. The appearance of superoscillations in weak measurements of a spin-like operator is described in [72]. This effect is also at the root of quantum random walks, introduced in 1993 by Y. Aharonov, L. Davidovich and N. Zagury [73] (see also [74]).

Let us examine now the physical meaning of an anomalous weak value in our system. The initial state of the field is \( |\psi_0\rangle = \cos(\theta_i/2) |0, 0, 1, 0\rangle_T + \sin(\theta_i/2) |0, 0, 0, 1\rangle_T \), while the final state is given by \( |\psi_f\rangle = \cos(\theta_f/2) |0, 0, 1, 0\rangle_T - \sin(\theta_f/2) |0, 0, 0, 1\rangle_T \) (the cavity is in the vacuum state and therefore we can restrict our analysis to the subspace \( \mathcal{H}_{\text{ext}} \)). In the initial state there is no relative phase between the states \( |0, 0, 1, 0\rangle_T \) and \( |0, 0, 0, 1\rangle_T \), while in the final state there is a relative phase of \( \pi \) between \( |0, 0, 1, 0\rangle_T \) and \( |0, 0, 0, 1\rangle_T \). In this situation, whatever are the values of \( \theta_i \) and \( \theta_f \), except for 0 and \( \pi \) (no knowledge about the trajectory), the weak value will exceed the standard expectation value. Mathematically, this will produce Fourier coefficients \( A \) and \( B \) with different signs. On the contrary, for post-selected photons at \( D_2 \) the final state is \( |\psi_f^+\rangle = \sin(\theta_f/2) |0, 0, 1, 0\rangle_T + \cos(\theta_f/2) |0, 0, 0, 1\rangle_T \), which, like the initial state, has no relative phase between both paths. In this case, the weak value has the same range as the standard expectation value whatever are the values of \( \theta_i \) and \( \theta_f \) (in this case the Fourier coefficients will have the same sign). Consequently, pre- and post-selected ensembles in which the field initially propagates on phase in both arms but returns with a phase difference of \( \pi \), or vice versa, show an anomalous force, that allows to displace the oscillator as if there were many photons in the no post-selection scenario.

Finally, let us consider a simple example to illustrate more concretely the amplification of the force exerted by a single photon. Pre- and post-selected ensembles in which \( \theta_i = 90^\circ \) and \( \theta_f = 70^\circ \) can be constructed with probabilities around 3\%. In these ensembles the shift given by a single photon is equivalent to the shift produced by 3 photons (no post-selection scenario), i.e. \( \langle \Delta \hat{N} \rangle_w \approx 3 \) which is the same as saying that the weak radiation force is \( 3\hbar G \). This is already surprising since without post-selection the maximum radiation force achievable is \( \hbar G \). Since the weak value of the total number of photons is 1 and because the weak value of a subtraction of operators is just the difference between the weak values, it follows that

\[
\langle \hat{d}_1^\dagger \hat{d}_1 \rangle_w - \langle \hat{d}_1^\dagger \hat{d}_2 \rangle_w = 3,
\langle \hat{d}_1^\dagger \hat{d}_1 \rangle_w + \langle \hat{d}_1^\dagger \hat{d}_2 \rangle_w = 1.
\tag{70}
\]

Which means that \( \langle \hat{d}_1^\dagger \hat{d}_1 \rangle_w = 2 \) and \( \langle \hat{d}_2^\dagger \hat{d}_2 \rangle_w = -1 \), i.e. on average a photon in the left side pushes the mirror with weak force \( 2\hbar G \), to the right, while a photon in the other side pushes with force \( \hbar G \), but also to the right. In [75] the weak velocity of a particle can be faster than the speed of light and in [76] the weak kinetic energy may be negative. In a similar way, in this context the weak pressure exerted by a single photon can push
Strange weak values of photon number operators in the optomechanical interaction

the wall inwards.

Summarizing, in this article we have proposed an experiment in which the weak value of the difference of photons between two sides of a cavity can be measured by observing the shift of the average position of a mechanical oscillator. We have shown that this is theoretically possible since the optomechanical interaction couples the difference of photons between both sides of the system with one of the quadratures of the mirror. This coupling occurs when the incident photon has a much longer duration than the cavity storage time, the optomechanical strength is smaller than the mechanical mode frequency and the hopping strength between the cavity and the external field is greater than the normal frequency of the oscillator.

Therefore, when post-selection is performed and certain criteria of weakness in the measurement are satisfied, the weak value can be observed. We have thus explained the amplification of the force exerted by a single photon due to the appearance of strange weak values of number operators.

The relaxation of the cavity field due to internal losses and the decoherence of the oscillator are assumed to occur much slowly than the time scales of the problem. Additionally, since this type of interaction is not impulsive, the weak value depends on time. It has been shown that the time of interaction should be close to odd multiples of half the mechanical period in order to entangle the photon to the oscillator. This requires photodetectors with time resolution in the order of $10^{-9}$ – $10^{-6}$ seconds which is altogether possible.

Under these conditions, with probabilities in the range of $1\% - 7\%$, ensembles can be constructed in which the shift given to the oscillator is equivalent to the shift produced by 2 to 6 photons when no post-selection is performed. This shift can be even bigger at the cost of decreasing post-selection probabilities. In order to approach the level of zero-point fluctuations, it has been explained that starting the oscillator in the squeezed vacuum would permit to preserve its Gaussian form and only shift it by a quantity proportional to the weak value.

In a future work the damping might be taken into account. This could be done in two manners. One considers the use of master equations, Langevin equations and similar tools to treat open quantum systems. In this case, there is no post-selection and strange weak values might arise as classical variables in the dynamics of coherence terms, in a similar way as it is indicated in [77]. Another possibility is to solve the Schrödinger equation of the system following the methods used in [78, 79, 80]. This treatment would allow us to engineer the photon wave function and to study whether weak value amplification of the radiation pressure may occur for single photons with a wider frequency content. In the monochromatic limit we expect to recover the results.
Strange weak values of photon number operators in the optomechanical interaction

predicted in this article. Future research will also concern on novel applications of weak measurements in other types of interactions.

Acknowledgments

We thank the financial support of Conicyt with the project Fondecyt #1180175.

Appendix A. Hamiltonian approximation - mathematical description

In the interaction picture II with respect to \( \hat{H}_{\text{cav-ext}} + \hat{H}_m + \hat{H}_{\text{OM1}} \) the Hamiltonian is given by

\[
\hat{H}_\text{II}(t) = -\hat{J}_z \left[ \hat{c}^\dagger A(t) + \hat{c} A^*(t) + \Delta \hat{N} f(t) \right] - \hat{J}_y \left[ \hat{c}^\dagger B(t) + \hat{c} B^*(t) + \Delta \hat{N} g(t) \right].
\]

(A.1)

The time dependent coefficients (with units of frequency) are

\[
A(t) = \kappa \cos(2gt) \exp \left( i \omega_m t \right), \quad f(t) = \frac{\kappa^2}{\omega_m} \cos(2gt) \left[ 1 - \cos(\omega_m t) \right],
\]

\[
B(t) = \kappa \sin(2gt) \exp \left( i \omega_m t \right), \quad g(t) = \frac{\kappa^2}{\omega_m} \sin(2gt) \left[ 1 - \cos(\omega_m t) \right],
\]

(A.2)

and \( A^* \) and \( B^* \) denote complex conjugation. The time evolution operator in this frame is given by the Dyson series, that is,

\[
\hat{U}_\text{II}(t) = 1 - \frac{i}{\hbar} \int_0^t dt_1 \hat{H}_\text{II}(t_1) + \left( \frac{-i}{\hbar} \right)^2 \int_0^t dt_2 \hat{H}_\text{II}(t_2) \int_0^{t_2} dt_1 \hat{H}_\text{II}(t_1) + \ldots
\]

(A.3)

The first term, \( \frac{-i}{\hbar} \int_0^t \hat{H}_\text{II}(t_1) dt_1 \), is given by

\[
\begin{align*}
\hat{U}_\text{II}(t) &= \hat{U}_\text{II}(0) \exp \left[ \frac{-i}{\hbar} \int_0^t dt_1 \hat{H}_\text{II}(t_1) \right] \hat{U}_\text{II}(t),
\end{align*}
\]

The overline indicates integration in time of the coefficients, namely,

\[
\hat{A}(t) = \int_0^t A(z) dz = -\frac{i}{\hbar} \left( \frac{\kappa}{2g} \right) \left( \frac{\omega_m}{2g} \right) \left[ \frac{1}{1 - \left( \omega_m/2g \right)^2} \right] \left[ 1 - \cos(2gt) e^{i \omega_m t} \right]
\]

\[
+ \left( \frac{\kappa}{2g} \right) \left[ \frac{1}{1 - \left( \omega_m/2g \right)^2} \right] \sin(2gt) e^{i \omega_m t},
\]

(A.5)

\[
\hat{B}(t) = \int_0^t B(z) dz = \left( \frac{\kappa}{2g} \right) \left[ \frac{1}{1 - \left( \omega_m/2g \right)^2} \right] \left[ 1 - \cos(2gt) e^{i \omega_m t} \right]
\]

\[
+ i \left( \frac{\kappa}{2g} \right) \left( \frac{\omega_m}{2g} \right) \left[ \frac{1}{1 - \left( \omega_m/2g \right)^2} \right] \sin(2gt) e^{i \omega_m t},
\]

(A.6)

\[
\hat{f}(t) = \int_0^t f(z) dz = \left( \frac{\kappa}{\omega_m} \right) \left( \frac{\kappa}{2g} \right) \sin(2gt) - \left( \frac{\kappa}{2g} \right) \left( \frac{\kappa}{\omega_m} \right) \left[ \frac{1}{1 - \left( \omega_m/2g \right)^2} \right] \cos(\omega_m t) \sin(2gt)
\]

\[
+ \left( \frac{\kappa}{2g} \right)^2 \left[ \frac{1}{1 - \left( \omega_m/2g \right)^2} \right] \cos(2gt) \sin(\omega_m t),
\]

(A.7)
\[ \tilde{g}(t) = \int_{0}^{t} g(z) \, dz = \left( \frac{\kappa}{\omega_m} \right) \left( \frac{\kappa}{g} \right) \sin^{2}(2gt) - \left( \frac{\kappa}{2g} \right) \left( \frac{\kappa}{\omega_m} \right) \left[ \frac{1}{1 - (\omega_m/2g)^{2}} \right] \]

\[ + \left( \frac{\kappa}{\omega_m} \right) \left( \frac{\kappa}{2g} \right) \left[ \frac{1}{1 - (\omega_m/2g)^{2}} \right] \cos(\omega_m t) \cos(2gt) \]

\[ + \left( \frac{\kappa}{2g} \right)^{2} \left[ \frac{1}{1 - (\omega_m/2g)^{2}} \right] \sin(2gt) \sin(\omega_m t). \] (A.8)

There are three expansion parameters appearing in the terms above, which are \( \kappa/g, \kappa/\omega_m \) and \( \omega_m/g \). We will assume that all satisfy \( \kappa/g \ll 1, \kappa/\omega_m \ll 1 \) and \( \omega_m/g \ll 1 \), which can be summarized in the condition \( \kappa \ll \omega_m \ll g \). Notice that this condition implies that \( \kappa/g \ll \kappa/\omega_m \) and \( \kappa/g \ll \omega_m/g \), i.e. the parameter \( \kappa/g \) is the smallest term. We will work at first order in \( \kappa/\omega_m \) and consequently neglect the terms of order \( \kappa/g \). This entails that \( \tilde{A}(t) = \tilde{B}(t) = \tilde{f}(t) = \tilde{g}(t) = 0 \) and accordingly all the remaining terms of the Dyson series are also negligible. Thus, the evolution operator in this frame, at first order in \( \kappa/\omega_m \), is just the identity operator. Since kets in this frame do not change, the evolution given by \( \tilde{H}_{OM2} \) can be disregarded from the hamiltonian in the previous picture.

### Appendix B. Hamiltonian approximation - physical description

From (15) the Heisenberg hamiltonian is given by

\[ \tilde{H}_{H}(t) = \hbar \omega_0 \sum_{i=1}^{2} \left[ \hat{a}^\dagger_i(t)\hat{a}_i(t) + \hat{d}^\dagger_i(t)\hat{d}_i(t) + \hat{c}^\dagger_i(t)\hat{c}_i(t) \right] + \hbar g \sum_{i=1}^{2} \left[ \hat{a}^\dagger_i(t)\hat{a}_i(t) + \hat{a}_i(t)\hat{d}^\dagger_i(t) \right] \]

\[ + \hbar \omega_m \hat{c}^\dagger \hat{c} - \hbar G \hat{F}_{rad}(t) \hat{q}(t), \] (B.1)

where the operators \( \hat{X}(t) \) are the Heisenberg operators associated to the Schrödinger operators \( \hat{X} \). The equations of motion for the momentum \( \hat{p}(t) \) and position \( \hat{q}(t) \) of the oscillator are given by

\[ \ddot{\hat{p}} = -\omega_m^2 \hat{p} + \frac{d}{dt} \hat{F}_{rad}(t), \quad \ddot{\hat{q}} = -\omega_m^2 \hat{q} + \hat{F}_{rad}(t)/M. \] (B.2)

In a closed cavity \( \hat{F}_{rad}(t) = \hbar G[\hat{a}^\dagger_1(t)\hat{a}_1(t) - \hat{a}^\dagger_2(t)\hat{a}_2(t)] \) is time independent and the equations above describe a simple harmonic oscillator with a constant force. In an open cavity the situation is different since the radiation force is changing. Let us examine first the time dependence of the field intensity inside the cavity, e.g. the field in side 1. The equation of motion for \( \hat{a}^\dagger_1(t)\hat{a}_1(t) \) is

\[ \frac{d}{dt} \hat{a}^\dagger_1(t)\hat{a}_1(t) = -i\hbar[\hat{a}^\dagger_1(t)\hat{d}_1(t) - \hat{d}^\dagger_1(t)\hat{a}_1(t)]. \] (B.3)

The equation of motion for the term in the right hand, which is proportional to the energy flow from (or into) side 1 of the cavity, is given by

\[ \frac{d}{dt} [\hat{a}^\dagger_1(t)\hat{d}_1(t) - \hat{a}_1(t)\hat{d}^\dagger_1(t)] = -2i\hbar \left\{ [\hat{a}^\dagger_1(t)\hat{a}_1(t) - \hat{d}^\dagger_1(t)\hat{d}_1(t)] + \frac{G\hat{q}(t)}{2g} [\hat{a}^\dagger_1(t)\hat{d}_1(t) + \hat{a}_1(t)\hat{d}^\dagger_1(t)] \right\}. \] (B.4)
Notice that the position of the oscillator appears in this equation, in the second term of the right side. By expressing $\dot{q}(t)$ as $x_0[\dot{c}(t) + \dot{c}(t)]$, the parameter $x_0G/g = \kappa/g$ appears, which in turn can be neglected since we are working at first order in $\kappa/\omega_m$. This allows to uncouple the equations for the oscillator from the equations for the cavity fields. In other words, when the optomechanical strength $\kappa$ is much smaller than the strength $g$, then the energy flow from (or into) the cavity is slightly perturbed by the movement of the mirror.

Next, taking into account the equation for $\hat{a}^\dagger_1(t)\hat{a}_1(t) - \hat{d}^\dagger_1(t)\hat{d}_1(t)$, the following coupled equations are obtained:

\[
\frac{d}{dt}[\hat{a}^\dagger_1(t)\hat{d}_1(t) - \hat{a}_1(t)\hat{d}^\dagger_1(t)] = 2g[\hat{a}^\dagger_1(t)\hat{a}_1(t) - \hat{d}^\dagger_1(t)\hat{d}_1(t)],
\]

\[
\frac{d}{dt}[\hat{a}_1(t)\hat{d}^\dagger_1(t) - \hat{a}^\dagger_1(t)\hat{d}_1(t)] = -2g[i\hat{a}^\dagger_1(t)\hat{d}_1(t) - \hat{a}_1(t)\hat{d}^\dagger_1(t)].
\]  

(B.5)

These equations can be readily uncoupled. Solving the corresponding equations allows to work out the equation for the cavity field (B.3) and to find out that

\[
\hat{a}^\dagger_1(t)\hat{a}_1(t) = i[\hat{a}_1(0)\hat{d}^\dagger_1(0) - \hat{a}^\dagger_1(0)\hat{d}_1(0)]\sin(2gt) + \left[\frac{\hat{a}^\dagger_1(0)\hat{a}(0) - \hat{d}^\dagger_1(0)\hat{d}(0)}{2}\right]\cos(2gt)
\]

\[+ \frac{\hat{a}^\dagger_1(0)\hat{a}(0) + \hat{d}^\dagger_1(0)\hat{d}(0)}{2}.
\]  

(B.6)

Recall that the Heisenberg operators at $t = 0$ coincide with the Schrödinger operators. In the time scale of the oscillator, when the natural frequency $\omega_m$ is much slower than the frequency $g$, all time dependent terms are averaged out (are non-secular terms) and, consequently,

\[
\hat{a}^\dagger_1\hat{a}_1(t) = \frac{\hat{a}^\dagger_1\hat{a} + \hat{d}^\dagger_1\hat{d}}{2}.
\]  

(B.7)

Hence, for the mirror, the intensity in the side 1 of the cavity is constant and corresponds to an average over many periods of energy exchange between the cavity and the external field. Figure [B1] represents pictorially the general case and our regime of interest.

Appendix C. Standing and travelling wave basis

Transformation between basis vectors can be done using (20), e.g. the state $|0, 1, 0, 0\rangle_T$ can be expressed in terms of standing wave basis vectors in the following way:

\[
|0, 1, 0, 0\rangle_T = \hat{b}^\dagger_{2, k_0} |0, 0, 0, 0\rangle_T = \frac{\hat{d}^\dagger_2 + \hat{e}^\dagger_2}{\sqrt{2}} |0, 0, 0, 0\rangle_S
\]

\[= |0, 1, 0, 0\rangle_S + |0, 0, 0, 1\rangle_S.
\]

(C.1)

In a similar manner, the state $|0, 0, 1, 0\rangle_S$ can be written in terms of travelling wave basis vectors as follows.

\[
|0, 0, 1, 0\rangle_S = \hat{e}^\dagger_1 |0, 0, 0, 0\rangle_S = \frac{\hat{b}^\dagger_{1, k_0} - \hat{b}^\dagger_{1, -k_0}}{\sqrt{2}} |0, 0, 0, 0\rangle_T
\]
Strange weak values of photon number operators in the optomechanical interaction

Figure B1. Figure (a) shows the exact case, in which the radiation force is time dependent and the cavity fields are coupled to the position and momentum of the oscillator and to the external fields. In figure (b) the energy flow from and into the cavity occurs at frequency $g$, as if no mirror was inside the cavity, because the optomechanical strength is much weaker than the hopping strength. The mirror oscillates independently from the fields since the radiation force is a constant of motion.

$$= \frac{|1, 0, 0, 0\rangle_T - |0, 0, 1, 0\rangle_T}{\sqrt{2}}. \quad (C.2)$$

Appendix D. Beam splitter transformation

For horizontal polarization the beam splitter has classical reflection coefficients given by $r^H_1 = -\cos(\theta_i/2)$ and $r^H_2 = \cos(\theta_i/2)$, where $\theta_i \in [0, \pi]$ is a mixing parameter for horizontal polarization. Notice that the photon reflected in the side 1 of the PDBS gets a phase of $\pi$ (see figure D1). On the other hand, the transmission coefficients for horizontal polarization are $t^H_1 = t^H_2 = \sin(\theta_i/2)$. When $\theta_i = 0$ the beam splitter acts like a 100% mirror, while for $\theta_i = \pi$ it is perfectly transmissive.

The following description of beam splitters is based on [82] and additional information can be found in [83, 84]. The equations for the field operators are

$$\hat{b}_{1,k_0} = -\cos(\theta_i/2) \hat{j}_{1,k_0} + \sin(\theta_i/2) \hat{j}_{2,k_0},$$
Strange weak values of photon number operators in the optomechanical interaction

\[ b_{2,k_0} = \sin(\theta_t/2) j_{1,k_0} + \cos(\theta_t/2) j_{2,k_0}, \]  
(D.1)

and, conversely,

\[ \hat{j}_{1,k_0} = -\cos(\theta_t/2) \hat{b}_{1,k_0} + \sin(\theta_t/2) \hat{b}_{2,k_0}, \]

\[ \hat{j}_{2,k_0} = \sin(\theta_t/2) \hat{b}_{1,k_0} + \cos(\theta_t/2) \hat{b}_{2,k_0}. \]  
(D.2)

On the other hand, for vertical polarization, the beam splitter transmission coefficients are \( t_1^\text{V} = -\sin(\theta_t/2), t_2^\text{V} = \sin(\theta_t/2) \) and the reflection coefficients are \( r_1^\text{V} = r_2^\text{V} = \cos(\theta_t/2) \), with \( \theta_t \in [0, \pi] \). Notice that a phase of \( \pi \) is given to the field transmitted from \( \overrightarrow{GH} \) to \( \overrightarrow{UV} \). If \( \theta_t = 0 \) the field in arm 1 is reflected into detector \( D_1 \) and the field in arm 2 is reflected into \( D_2 \). If \( \theta_t = \pi \), the field in arm 1 is transmitted into \( D_2 \) and the field in arm 2 is transmitted into \( D_1 \). In these two cases, every time one of the detectors clicks, the path followed by the photon is known with certainty. The equations for the field operators are

\[ \hat{j}_{1,-k_0} = \cos(\theta_t/2) \hat{b}_{1,-k_0} + \sin(\theta_t/2) \hat{b}_{2,-k_0}, \]

\[ \hat{j}_{2,-k_0} = -\sin(\theta_t/2) \hat{b}_{1,-k_0} + \cos(\theta_t/2) \hat{b}_{2,-k_0}. \]  
(D.3)

and

\[ \hat{b}_{1,-k_0} = \cos(\theta_t/2) \hat{j}_{1,-k_0} - \sin(\theta_t/2) \hat{j}_{2,-k_0}, \]

\[ \hat{b}_{2,-k_0} = \sin(\theta_t/2) \hat{j}_{1,-k_0} + \cos(\theta_t/2) \hat{j}_{2,-k_0}. \]  
(D.4)

The unitary transformation given by the relations (D.1), (D.2), (D.3) and (D.4) is depicted in figure [D1]. These relations and their corresponding hermitian conjugate equations allow conversion between photon number states outside and inside the interferometer. We use the fact that an input vacuum transforms into an output vacuum as follows

\[ |1,0,0,0\rangle_{\text{out}}^\text{out} = \hat{j}_{1,k_0}^\dagger |0,0,0,0\rangle_{\text{T}}^\text{T} \rightarrow [-\cos(\theta_t/2) \hat{b}_{1,k_0}^\dagger + \sin(\theta_t/2) \hat{b}_{2,k_0}^\dagger] |0,0,0,0\rangle_{\text{T}} + \cos(\theta_t/2) |1,0,0,0\rangle_{\text{T}} \]  
(D.5)

Let us consider now the state of the system \( \cos(\theta_t/2) |0,0,1,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} - \sin(\theta_t/2) |0,0,0,1\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} \). The photon propagates outside the interferometer according to

\[ \cos(\theta_t/2) |0,0,1,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} - \sin(\theta_t/2) |0,0,0,1\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} = \]

\[ \cos(\theta_t/2) \hat{b}_{1,k_0}^\dagger |0,0,0,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} - \sin(\theta_t/2) \hat{b}_{2,k_0}^\dagger |0,0,0,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} \]

\[ \cos(\theta_t/2) \left[ \cos(\theta_t/2) \hat{j}_{1,k_0}^\dagger - \sin(\theta_t/2) \hat{j}_{2,k_0}^\dagger \right] |0,0,0,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} - \sin(\theta_t/2) \left[ \sin(\theta_t/2) \hat{j}_{1,k_0}^\dagger + \cos(\theta_t/2) \hat{j}_{2,k_0}^\dagger \right] |0,0,0,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} = \]

\[ \cos(\theta_t/2) \left[ \cos(\theta_t/2) |0,0,1,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} - \sin(\theta_t/2) |0,0,0,1\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} \right] - \sin(\theta_t/2) \left[ \sin(\theta_t/2) |0,0,1,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} + \cos(\theta_t/2) |0,0,0,1\rangle_{\text{T}} |0,0\rangle_{\text{cav}} |\bar{k}/2,\xi\rangle_{m} \right] = \]

\[ |0,0,1,0\rangle_{\text{T}} |0,0\rangle_{\text{cav}} \left[ \cos(\theta_t/2) \cos(\theta_t/2) |\bar{k}/2,\xi\rangle_{m} - \sin(\theta_t/2) \sin(\theta_t/2) |\bar{k}/2,\xi\rangle_{m} \right] - \]
\[ |0, 0, 0, 0, 1 \rangle^{\text{out}}_{\text{T}} |0, 0 \rangle_{\text{cav}} \left[ \cos(\theta_i/2) \sin(\theta_f/2) |\kappa/2, \xi \rangle_{m} + \sin(\theta_i/2) \cos(\theta_f/2) |-\kappa/2, \xi \rangle_{m} \right]. \] (D.6)

**Figure D1.** The beam splitter transformation of fields. Figure (a) shows the operation for incoming fields and (b) the operation for outgoing fields.

**Appendix E. Time evolution operator**

The method used here is the one employed by Bose et al \[52\]. Let us define a scaled time \( t' = \omega_m t \) and the adimensional parameter \( \tilde{\kappa} = \kappa/\omega_m \). In this calculation we will use just \( t \) for the scaled time. The first term in (30) can be written as

\[
\hat{X}(t) = \exp \left[ -i t \hat{c}^{\dagger} \hat{c} + i \left( \frac{\tilde{\kappa}}{2} \right) t \Delta \hat{N}(\hat{c}^{\dagger} + \hat{c}) \right].
\]

Secondly, the following unitary transformation is defined

\[
\hat{T} = \exp \left[ - \frac{\tilde{\kappa}}{2} \Delta \hat{N}(\hat{c}^{\dagger} - \hat{c}) \right], \tag{E.1}
\]

which generates a translation in the position of the oscillator by \( \tilde{\kappa} \Delta \hat{N} x_{0} \). Next, the operator \( \hat{X}(t) \) is worked out as follows.

\[
\hat{X}(t) = \hat{T}^{\dagger} \hat{T} \hat{X}(t) \hat{T}^{\dagger} \hat{T} = \hat{T}^{\dagger} \exp \left[ i \left( \frac{\tilde{\kappa}}{2} \right)^{2} \Delta \hat{N}^{2} \right] \exp (-i t \hat{c}^{\dagger} \hat{c}) \hat{T} = \exp \left[ i \left( \frac{\tilde{\kappa}}{2} \right)^{2} \Delta \hat{N}^{2} \right] \hat{T}^{\dagger} \exp (-i t \hat{c}^{\dagger} \hat{c}) \hat{T}.
\]
\[
\begin{align*}
&= \exp \left[ i \left( \frac{\tilde{\kappa}}{2} \right)^2 \Delta \hat{N}^2 \right] \hat{T}^\dagger \exp (-it\hat{c}^\dagger \hat{c}) \hat{T} \exp \left( it\hat{c}^\dagger \hat{c} \right) \exp (-it\hat{c}^\dagger \hat{c}) \\
&= \exp \left[ i \left( \frac{\tilde{\kappa}}{2} \right)^2 \Delta \hat{N}^2 \right] \hat{T}^\dagger \exp \left[ \frac{\tilde{\kappa} \Delta \hat{N}}{2} \left( \hat{c} e^{it} - \hat{c}^\dagger e^{-it} \right) \right] \exp (-it\hat{c}^\dagger \hat{c}). \quad (E.2)
\end{align*}
\]

In [E.2] the well known operator expansion for two non-commuting operators \( \hat{A} \) and \( \hat{B} \)
\[
\exp (\alpha \hat{A}) \hat{B} \exp (-\alpha \hat{A}) = \hat{B} + \alpha [\hat{A}, \hat{B}] + \frac{\alpha^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + ...
\]

has been used in addition with the property \( \hat{T} \hat{f}(\{ \hat{Z}_i \}) \hat{T}^\dagger = f(\{ \hat{Z}_i \hat{T} \}) \) for any unitary transformation \( \hat{T} \) and any analytic function \( f \) of a set of operators \( \{ \hat{Z}_i \} \) [55].

Using the property \( \exp (\hat{A} + \hat{B}) = \exp (\hat{A}) \exp (\hat{B}) \exp (-[\hat{A}, \hat{B}]/2) \) for any pair of operators \( \hat{A}, \hat{B} \) that satisfy \([\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0 \) it is straightforward to show that
\[
\hat{T}^\dagger \exp \left[ \frac{\tilde{\kappa} \Delta \hat{N}}{2} \left( \hat{c} e^{it} - \hat{c}^\dagger e^{-it} \right) \right] = \exp \left\{ \Delta \hat{N} [\hat{c}^\dagger \varphi(t) - \hat{c}^\dagger \varphi^*(t)] \right\} \exp \left[-i \left( \frac{\tilde{\kappa}}{2} \right)^2 \Delta \hat{N}^2 \sin(t) \right],
\]

where \( \varphi(t) = (\tilde{\kappa}/2)(1 - e^{-it}) \) and \( \varphi^*(t) \) its complex conjugate. This result allows to write expression [E.2] as
\[
\hat{X}(t) = \exp \left\{ i \left( \frac{\tilde{\kappa}}{2} \right)^2 [1 - \sin(t)] \Delta \hat{N}^2 \right\} \exp \left\{ \Delta \hat{N} [\hat{c}^\dagger \varphi(t) - \hat{c}^\dagger \varphi^*(t)] \right\} \exp (-it\hat{c}^\dagger \hat{c}). \quad (E.4)
\]

By defining \( \phi(t) = (\tilde{\kappa}/2)^2(1 - \sin t) \) the operator \( \hat{X}(t) \) becomes
\[
\hat{X}(t) = \exp \left\{ i \phi(t) \Delta \hat{N}^2 \right\} \exp \left\{ \Delta \hat{N} [\hat{c}^\dagger \varphi(t) - \hat{c}^\dagger \varphi^*(t)] \right\} \exp (-it\hat{c}^\dagger \hat{c}). \quad (E.5)
\]

The factor \( \exp [i \phi(t) \Delta \hat{N}^2] \) contains the so-called Kerr phase [56, 57]. For single-photon states \( \Delta \hat{N}^2 = 1 \) and this term will merely add a global phase factor that will be omitted.

The complex function \( \varphi(t) \) can be written as \( \tilde{\kappa}(t) \) \exp [i\alpha(t)], where \( \tilde{\kappa}(t) = \tilde{\kappa} \sin(t/2) \) and \( \alpha(t) = (\pi - t)/2 \). Returning to the original time \( \omega_m t \) the operator is finally expressed as
\[
\hat{X}(t) = \exp \left\{ \tilde{\kappa}(t) \Delta \hat{N} [e^{i\alpha(t)}\hat{c}^\dagger - e^{-i\alpha(t)}\hat{c}^\dagger] \right\} \exp (-i\omega_m t\hat{c}^\dagger \hat{c}). \quad (E.6)
\]

**Appendix F. Evolution of the initial state**

The state of the system after a time \( t \) is given by
\[
|\Phi(t)\rangle = \hat{U}_t(\xi) |\Phi(0)\rangle = \hat{U}_{OM}(t) \hat{U}_{ex}(t) \hat{U}_{in}(t) |\Phi(0)\rangle.
\]

The free evolution of the mirror, \( \hat{U}_{in}(t) \), transforms the state \( |\xi\rangle \) into \( |\xi'\rangle \), where \( \xi' = \xi \exp (-2i\omega_m t) \). \( \hat{U}_{ex}(t) \) has no action over the state in the second line of [32], while the action over the first ket is given by
\[
\hat{U}_{ex}(t) \frac{1}{\sqrt{2}} \left[ \cos(\theta/2) |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} + \sin(\theta/2) |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} \right] =
\]
\[
\cos(\theta/2) \frac{\sqrt{2}}{\sqrt{2}} \left[ \exp \left( -\frac{i}{\hbar} 2gt \hat{J}_{x1} \right) |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} + \right.
\]
\[
\sin(\theta/2) \frac{\sqrt{2}}{\sqrt{2}} \exp \left( -\frac{i}{\hbar} 2gt \hat{J}_{x2} \right) |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}}.
\]

(F.2)
Strange weak values of photon number operators in the optomechanical interaction

In this last expression the state of the mirror has been omitted since this operator acts only over $\mathcal{H}_{\text{EM}}$. In order to evolve each state in (F.2), let us consider the single-photon eigenstates of the operators $\hat{J}_z i$, $i = 1, 2$, which are given by

\[
\begin{align*}
\hat{J}_{z1} |0, 0, 0, 0\rangle_S |1, 0\rangle_{\text{cav}} &= |0, 0, 0, 0\rangle_S |1, 0\rangle_{\text{cav}}, \\
\hat{J}_{z1} |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} &= |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}}, \\
\hat{J}_{z2} |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} &= |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}}, \\
\hat{J}_{z2} |0, 0, 0, 0\rangle_S |1, 0\rangle_{\text{cav}} &= |0, 0, 0, 0\rangle_S |0, 1\rangle_{\text{cav}}. \quad (F.3)
\end{align*}
\]

The operator $2\hat{J}_{z1}/\hbar$ becomes the standard spin 1/2 Pauli matrix $\sigma_x$ when it is represented in the basis $\{|0, 0, 0, 0\rangle_S |1, 0\rangle_{\text{cav}}, |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}}\}$. This matrix will be denoted by $\sigma_{x1}$. Analogously, $2\hat{J}_{z2}/\hbar$ is the spin 1/2 Pauli matrix $\sigma_x$ when it is represented in the basis $\{|0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}}, |0, 0, 0, 0\rangle_S |0, 1\rangle_{\text{cav}}\}$. In this case, the matrix will be denoted by $\sigma_{x2}$. Consequently, expression (F.2) can be written as follows

\[
\begin{align*}
\cos(\theta/2)/\sqrt{2} \exp\left(-igt\sigma_{x1}\right) |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} + \\
\sin(\theta/2)/\sqrt{2} \exp\left(-igt\sigma_{x2}\right) |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} = \\
\cos(\theta/2)/\sqrt{2} \left[ \cos(gt) - i\sin(gt)\sigma_{x1} \right] |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} + \\
\sin(\theta/2)/\sqrt{2} \left[ \cos(gt) - i\sin(gt)\sigma_{x2} \right] |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}}. \quad (F.4)
\end{align*}
\]

The action of $\sigma_{x1}$ over one of the eigenstates of $\hat{J}_{z1}$ produces a spin flip into the other eigenstate, namely,

\[
\begin{align*}
\sigma_{x1} |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} &= |0, 0, 0, 0\rangle_S |1, 0\rangle_{\text{cav}}, \\
\sigma_{x2} |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} &= |0, 0, 0, 0\rangle_S |0, 1\rangle_{\text{cav}}. \quad (F.5)
\end{align*}
\]

Therefore, expression (F.4) becomes

\[
\begin{align*}
\cos(\theta/2)/\sqrt{2} \left[ \cos(gt) |1, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} |\xi^\prime\rangle - i\sin(gt) |0, 0, 0, 0\rangle_S |1, 0\rangle_{\text{cav}} |\xi^\prime\rangle \right] + \\
\sin(\theta/2)/\sqrt{2} \left[ \cos(gt) |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} |\xi^\prime\rangle - i\sin(gt) |0, 0, 0, 0\rangle_S |0, 1\rangle_{\text{cav}} |\xi^\prime\rangle \right].
\end{align*}
\]

Finally, in order to evolve the initial state, the operator $\hat{U}_{\text{OM}}(t)$ must be applied over the state

\[
\begin{align*}
\cos(\theta/2)/\sqrt{2} & \left[ \cos(gt) |0, 0, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} |\xi^\prime\rangle - i\sin(gt) |0, 0, 0, 0\rangle_S |1, 0\rangle_{\text{cav}} |\xi^\prime\rangle \right] + \\
\sin(\theta/2)/\sqrt{2} & \left[ \cos(gt) |0, 1, 0, 0\rangle_S |0, 0\rangle_{\text{cav}} |\xi^\prime\rangle - i\sin(gt) |0, 0, 0, 0\rangle_S |0, 1\rangle_{\text{cav}} |\xi^\prime\rangle \right] + \\
\frac{1}{\sqrt{2}} & \left[ \cos(\theta/2) |0, 0, 1, 0\rangle_S |0, 0\rangle_{\text{cav}} + \sin(\theta/2) |0, 0, 0, 1\rangle_S |0, 0\rangle_{\text{cav}} \right] |\xi^\prime\rangle. \quad (F.6)
\end{align*}
\]
This action can be readily computed by realizing that each of the photonic states appearing in (F.6) is an eigenstate of $\Delta \hat{N}$. In fact,
\[
\begin{align*}
\Delta \hat{N} |1, 0, 0, 0\rangle_S |0, 0\rangle_{cav} &= |1, 0, 0, 0\rangle_S |0, 0\rangle_{cav}, \\
\Delta \hat{N} |0, 0, 0, 0\rangle_S |1, 0\rangle_{cav} &= |0, 0, 0, 0\rangle_S |1, 0\rangle_{cav}, \\
\Delta \hat{N} |0, 1, 0, 0\rangle_S |0, 0\rangle_{cav} &= -|0, 0, 0, 0\rangle_S |0, 0\rangle_{cav}, \\
\Delta \hat{N} |0, 0, 0, 0\rangle_S |0, 1\rangle_{cav} &= -|0, 0, 0, 0\rangle_S |0, 1\rangle_{cav}, \\
\Delta \hat{N} |0, 0, 1, 0\rangle_S |0, 0\rangle_{cav} &= 0, \\
\Delta \hat{N} |0, 0, 0, 1\rangle_S |0, 0\rangle_{cav} &= 0.
\end{align*}
\] (F.7)

Therefore, acting over the states in (F.6), the operator $\hat{U}_{OM}(t) = \exp \{ \tilde{\kappa}(t) \Delta \hat{N}[e^{i\alpha(t)} \hat{c}^\dagger - e^{-i\alpha(t)} \hat{c}] \}$ becomes a Glauber displacement operator
\[
\hat{D}(\lambda \tilde{\kappa}(t)e^{i[\alpha(t)]}) = \exp \{ \tilde{\kappa}(t)\lambda [e^{i\alpha(t)} \hat{c}^\dagger - e^{-i\alpha(t)} \hat{c}] \},
\] (F.8)

where $\lambda = -1, 0, 1$, is the corresponding eigenvalue of each one of the photonic states. A displacement operator $\hat{D}(\alpha)$ acting over the squeezed mechanical vacuum $\hat{S}(\beta) |0\rangle_m$ produces a coherent squeezed state, i.e. $\hat{D}(\alpha)\hat{S}(\beta) |0\rangle_m = |\alpha, \beta\rangle_m$. Consequently, the state of the optomechanical system after a time $t$ is given by
\[
|\Phi(t)\rangle = \frac{\cos(\theta/2)}{\sqrt{2}} \left[ \cos(gt) |1, 0, 0, 0\rangle_S |0, 0\rangle_{cav} |\tilde{\kappa}(t)e^{i\alpha(t)}, \xi\rangle_m + i \sin(gt) |0, 0, 0, 0\rangle_S |1, 0\rangle_{cav} |\tilde{\kappa}(t)e^{i\alpha(t)}, \xi\rangle_m \right] + \\
\frac{\sin(\theta/2)}{\sqrt{2}} \left[ \cos(gt) |0, 1, 0, 0\rangle_S |0, 0\rangle_{cav} |\tilde{\kappa}(t)e^{i\alpha(t)}, \xi\rangle_m - i \sin(gt) |0, 0, 0, 0\rangle_S |0, 1\rangle_{cav} |\tilde{\kappa}(t)e^{i\alpha(t)}, \xi\rangle_m \right] + \\
\frac{1}{\sqrt{2}} \left[ \cos(\theta/2) |0, 0, 1, 0\rangle_S |0, 0\rangle_{cav} + \sin(\theta/2) |0, 0, 0, 1\rangle_S |0, 0\rangle_{cav} \right] |\xi\rangle_m.
\] (F.9)

Appendix G. Wave function of a coherent squeezed state in the position and momentum representations

The wave function in the position representation of a general coherent squeezed state of the form $|\varphi\rangle = \hat{D}(\alpha)\hat{S}(\beta) |0\rangle = |\alpha, \beta\rangle$, where $\hat{D}(\alpha)$ is a Glauber displacement operator, $\alpha = Ae^{i\theta}$, $\hat{S}(\beta)$ a squeeze operator with squeeze parameter $\beta = re^{i\theta_s}$, and $|0\rangle$ the vacuum state, is given by
\[
\varphi(q) = \left( \frac{1}{\sqrt{2\pi\sigma_q}} \right)^{1/2} \exp \left\{ -\frac{[q - 2x_0\Re(\alpha)]^2}{4\sigma_q^2} \right\} \exp \left\{ -i \frac{f_s}{4\sigma_q^2} q^2 \right\} \exp \left\{ i \frac{f_{ca}x_0}{\sigma_q^2} q \right\},
\] (G.1)

where $x_0$ corresponds to the zero-point fluctuations, $\Re(\alpha)$ denotes the real part of $\alpha$ and the other parameters are
\[
\sigma_q^2 = x_0^2 \left[ e^{2r} \sin^2(\theta_s/2) + e^{-2r} \cos^2(\theta_s/2) \right],
\] (G.2)
Strange weak values of photon number operators in the optomechanical interaction

\[ f_s = \sinh(2r) \sin(\theta_S), \quad (G.3) \]

\[ f_{cs} = A \left[ \cosh(2r) \sin \theta_C + \sinh(2r) \sin(\theta_S - \theta_C) \right]. \quad (G.4) \]

For the particular case when \( \alpha \in \mathbb{R} \) and \( \theta_S = 0 \) or \( \pi \), the wave function reduces to a real function

\[ \varphi(q) = \left( \frac{1}{\sqrt{2\pi} \sigma_q} \right)^{1/2} \exp \left[ -\left( \frac{q - 2x_0 \alpha}{4\sigma_q^2} \right)^2 \right], \quad (G.5) \]

with standard deviation \( \sigma_q = x_0 e^r \) for \( \theta_S = 0 \) or \( \sigma_q = x_0 e^r \) for \( \theta_S = \pi \).

In the momentum representation the wave function of a general coherent squeezed state \( |\alpha, \beta\rangle \) is given by

\[ \varphi(p) = \left( \frac{1}{\sqrt{2\pi} \sigma_p} \right)^{1/2} \exp \left\{ -\left[ \frac{p - 2(\frac{\hbar}{2x_0}) \Im(\alpha)}{4\sigma_p^2} \right]^2 \right\} \exp \left\{ \frac{i}{4\sigma_p^2} f_s p^2 \right\} \exp \left\{ \frac{-i}{2} g_{cs} \frac{\hbar}{2x_0} p^2 \right\}, \quad (G.6) \]

where \( \Im(\alpha) \) is the imaginary part of \( \alpha \) and the other parameters are

\[ \sigma_p^2 = \left( \frac{\hbar}{2x_0} \right)^2 \left[ e^{2r} \cos^2(\theta_S/2) + e^{-2r} \sin^2(\theta_S/2) \right], \quad (G.7) \]

\[ f_s = \sinh(2r) \sin(\theta_S), \quad (G.8) \]

\[ g_{cs} = A \left[ \cosh(2r) \cos \theta_C + \sinh(2r) \cos(\theta_S - \theta_C) \right]. \quad (G.9) \]

Appendix H. Weak interaction approximation

Let us define \( |\varphi_+\rangle_m \equiv (|\xi\rangle_m \pm |\tilde{\kappa}, \xi\rangle_m)/2 \). The norm of these vectors is given by

\[ |m \langle \varphi_+ | \varphi_+ \rangle_m | = \left( \frac{1 \pm \frac{1}{2} \langle \xi \rangle_{\tilde{\kappa}} \langle \xi \rangle_m}{2} \right)^{1/2} = \left\{ \frac{1}{2} \pm \frac{1}{2} \exp \left[ -\frac{1}{2} \left( \frac{\tilde{\kappa}}{e^r} \right)^2 \right] \right\}^{1/2}, \quad (H.1) \]

which is a function of the parameter \( \tilde{\kappa}/e^r \). By condition (38) we know that this parameter is much smaller than 1. Hence, to a good approximation,

\[ |m \langle \varphi_+ | \varphi_+ \rangle_m | \approx 1, \quad |m \langle \varphi_- | \varphi_- \rangle_m | \approx 0. \quad (H.2) \]

This means that the vector \( |\varphi_-\rangle_m \) has (approximately) zero norm and therefore \( |\varphi_-\rangle_m \approx 0 \). Let us prove now that \( |\varphi_+\rangle_m \approx |\tilde{\kappa}/2, \xi\rangle_m \). In the X-quadrature representation, this state is expressed as

\[ |\varphi_+\rangle_m = \frac{1}{2} \left( \frac{1}{\sqrt{2\pi} \sigma_x} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left[ -\frac{x^2}{4\sigma_x^2} \right] \exp \left[ -\frac{-(x - \tilde{\kappa}/2)^2}{4\sigma_x^2} \right] |x\rangle \, dx, \quad (H.3) \]

where \( \sigma_x = e^r/2 \). It is convenient to introduce a change of coordinates by defining \( z = x - \tilde{\kappa}/2 \). In this frame the state is expressed as

\[ |\varphi_+\rangle_m = \frac{1}{2} \left( \frac{1}{\sqrt{2\pi} \sigma_x} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left[ -\frac{(z + \tilde{\kappa}/2)^2}{4\sigma_x^2} \right] \exp \left[ -\frac{-(z - \tilde{\kappa}/2)^2}{4\sigma_x^2} \right] |z + \tilde{\kappa}/2\rangle \, dz. \]

Performing the approximations

\[ \exp \left[ -\frac{(z \pm \tilde{\kappa}/2)^2}{4\sigma_x^2} \right] \approx \exp[-z^2/(4\sigma_x^2)] \left( 1 \pm \frac{z\tilde{\kappa}}{8\sigma_x^2} \right), \quad (H.4) \]
we obtain
\[
|\varphi_+\rangle_m \approx \left(\frac{1}{\sqrt{2\pi\sigma_x}}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{4\sigma_x^2}\right) \left|z + \frac{\kappa}{2}\right| \, dz
\]
\[
= \left(\frac{1}{\sqrt{2\pi\sigma_x}}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left[-\frac{(x - \frac{\kappa}{2})^2}{4\sigma_x^2}\right] \left|x\right| \, dx = |\frac{\kappa}{2}, \xi\rangle_m. \tag{H.5}
\]
Hence, \( |\varphi_+\rangle_m = |\frac{\kappa}{2}, \xi\rangle_m \).

Appendix I. Weak value approximation

Let us consider the case in which we perform post-selection on the system, by projecting the composite system into the state \( |\psi_f\rangle \). The meter will be left in the unnormalized state
\[
|\psi_f\rangle_m = \langle \psi_f | \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-ig\hat{A}\hat{p}}{\hbar}\right)^n |\psi\rangle |\psi\rangle_m, \tag{I.1}
\]
and expanding the evolution operator at second order in \( g/\sigma_q \) we obtain
\[
|\psi_f\rangle_m \approx \langle \psi_f | \int_{-\infty}^{\infty} dp\psi(p) \left(1 - i\frac{g\hat{A}\hat{p}}{\hbar} - \frac{g^2\hat{A}^2\hat{p}^2}{2\hbar^2}\right) |\psi\rangle |p\rangle
\]
\[
= \int_{-\infty}^{\infty} dp\psi(p) \left(1 - i\frac{g\langle \hat{A}\rangle_w p}{\hbar} - \frac{g^2\langle \hat{A}^2\rangle_w p^2}{2\hbar^2}\right) |p\rangle. \tag{I.2}
\]
In [4] the weak uncertainty is defined as
\[
\Delta A^2_w = |\langle \hat{A}^2 \rangle_w - \langle \hat{A}\rangle^2_w|. \tag{I.3}
\]
We will focus on the case when \( \langle \hat{A}^2 \rangle_w - \langle \hat{A}\rangle^2_w < 0 \). In particular, for an involutory operator, i.e. \( \hat{A}^2 = 1 \), when \( |\langle \hat{A}\rangle_w| > 1 \) (as will occur for our experiment), this condition will be satisfied. Therefore, we can express the weak uncertainty as \( \Delta A^2_w = \langle \hat{A}^2 \rangle_w - \langle \hat{A}\rangle^2_w \).

Substituting in (I.2) we obtain
\[
|\psi_f\rangle_m = \int_{-\infty}^{\infty} dp\psi(p) \left(1 - i\frac{g\langle \hat{A}\rangle_w p}{\hbar} - \frac{g^2\langle \hat{A}\rangle^2_w p^2}{2\hbar^2} + \frac{g^2\Delta A^2_w p^2}{2\hbar^2}\right) |p\rangle
\]
\[
\approx \int_{-\infty}^{\infty} dp \exp\left[-\frac{p^2}{\hbar^2} \left(\sigma_q^2 - g^2\Delta A^2_w\right)\right] \exp\left(-i\frac{g\langle \hat{A}\rangle_w p}{\hbar}\right) |p\rangle. \tag{I.4}
\]
In the last step the normalization factor of the wave function of the meter has been omitted. When
\[
g^2\Delta A^2_w/\sigma_q^2 \ll 1, \tag{I.5}
\]
then
\[
|\psi_f\rangle_m \approx \exp\left(-i\frac{g\langle \hat{A}\rangle_w p}{\hbar}\right) |\psi\rangle_m. \tag{I.6}
\]
Since the weak value is real, the evolution operator produces a shift in the position of the meter by an amount of \( g \langle \hat{A}\rangle_w \). For an involutory operator, condition (I.5) can be replaced by \( g \langle \hat{A}\rangle_w \ll \sigma_q \). Notice that in this case we will have imprecise measurements since the uncertainty will be bigger than the shift.
Appendix J. General theorem of linear algebra

Let $|\psi\rangle$ be any vector from a vector space equipped with an inner product, and let $\hat{A}$ be any linear operator that acts on this space. Then, $\hat{A} |\psi\rangle = \langle A |\psi\rangle + \sqrt{\langle \Delta \hat{A}^2 \rangle} |\psi^\perp\rangle$, where $|\psi^\perp\rangle$ is any vector orthogonal to $|\psi\rangle$, $\Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle$, and all the expectation values are calculated in the state $|\psi\rangle$. A two-line proof of this theorem is presented in [4].

References

[1] Aharonov Y, Albert D and Vaidman L 1988 How the Result of a Measurement of a Component of the Spin of a Spin - 1/2 Particle Can Turn Out to be 100 Phys. Rev. Lett. 60 1351
[2] Aharonov Y and Vaidman L 2008 Time in Quantum Mechanics vol 1 ed J G Muga et al (Berlin Heidelberg, Springer-Verlag) pp 399–447
[3] Aharonov Y, Bergmann P and Lebowitz J 1964 Time Symmetry in the Quantum Process of Measurement Phys. Rev. B 134 1410
[4] Aharonov Y and Vaidman L 1990 Properties of a quantum system during the time interval between two measurements Phys. Rev. A 41 11
[5] Jozsa R 2007 Complex weak values in quantum measurement Phys. Rev. A 76 4
[6] Sokolovski D 2016 The meaning of “anomalous weak values” in quantum and classical theories Phys. Lett. A 379 1097
[7] Berry M V and Shukla P 2010 Typical weak and superweak values J. Phys. A: Math. Theor. 43 354024
[8] Hosoya A and Shikano Y 2010 Strange weak values J. Phys. A: Math. Theor. 43 385307
[9] Ritchie N W M, Story J G and Hulet R G 1991 Realization of a measurement of a “weak value” Phys. Rev. Lett. 66 1107
[10] Hosten O and Kwiat P 2008 Observation of the Spin Hall Effect of Light via Weak Measurements Science 319 787
[11] Dixon P B, Starling D J, Jordan A N and Howell J C 2009 Ultrase nsitive Beam Deflection Measurement via Interferometric Weak Value Amplification Phys. Rev. Lett. 102 173601
[12] Starling D J, Dixon P B, Jordan A N and Howell J C 2010 Precision frequency measurements with interferometric weak values Phys. Rev. A 82 063822
[13] Magaña-Loaiza O S, Mirhosseini M, Rodenburg B and Boyd R W 2014 Amplification of Angular Rotations Using Weak Measurements Phys. Rev. Lett. 112 200401
[14] Jordan A N, Martínez-Rincón J and Howell J C 2014 Technical Advantages for Weak-Value Amplification: When Less is More Phys. Rev. X 4 011031
[15] Sinclair J, Hallaji M, Steinberg A M, Tollaksen J and Jordan A N 2017 Weak-value amplification and optimal parameter estimation in the presence of correlated noise Phys. Rev. A 96 052128
[16] Suter D 1995 “Weak measurements” and the “quantum time-translation machine” in a classical system Phys. Rev. A 51 45
[17] Pryde G J, O’ Brien J L, White A G, Ralph T C and Wiseman HM 2005 Measurement of Quantum Weak Values of Photon Polarization Phys. Rev. Lett. 94 220405
[18] Pryde G J, O’ Brien J L, White A G, Bartlett S D and Ralph TC 2004 Measuring a Photonic Qubit without Destoying it Phys. Rev. Lett. 92 190402
[19] Lundeen J S, Sutherland B, Patel A, Stewart C and Bamber C 2011 Direct measurement of the quantum wave function Nature 474 188
[20] Hu M-J and Zhang Y-S 2017 Gravitational Waves Detection via Weak Measurements Amplification arXiv:1707.00866v2 [quant-ph]
Strange weak values of photon number operators in the optomechanical interaction

[21] Coto R, Montenegro V, Eremeev V, Mundarain D and Orszag M 2017 The power of a control qubit in weak measurements Sci. Rep. 7 6351
[22] Leggett A and Garg A 1985 Quantum Mechanics versus Macroscopic Realism: Is the Flux There when Nobody Looks? Phys. Rev. Lett. 54 857
[23] Ruskov R, Korotkov A N and Mizel A 2006 Signatures of Quantum Behavior in Single-Qubit Weak Measurements Phys. Rev. Lett. 96 200404
[24] Jordan A N, Korotkov A N and Büttiker M 2006 Leggett-Garg Inequality with a Kicked Quantum Pump Phys. Rev. Lett. 97 026805
[25] Williams N S and Jordan A N 2008 Weak Values and the Leggett-Garg Inequality in Solid-State Qubits Phys. Rev. Lett. 100 026804
[26] Williams N S and Jordan A N 2009 Erratum: Weak Values and the Leggett-Garg Inequality in Solid-State Qubits Phys. Rev. Lett. 103 089902
[27] Groen J P, Ristè D, Tornberg L, Cramer J, de Groot P C, Picot T, Johansson G and DiCarlo L 2013 Partial-Measurement Backaction and Nonclassical Weak Values in a Superconducting Circuit Phys. Rev. Lett. 111 090506
[28] Goggin M E, Almeida M P, Barbieri M, Lanyon B P, O’ Brien J L, White A G and Pryde G J 2011 Proc. Natl. Acad. Sci. U.S.A 108 1256
[29] Kocsis S, Braverman, Ravets S, Stevens M J, Mirin R P, Shalm L K and Steinberg A M 2011 Observing the Average Trajectories of Single Photons in a Two-Slit Interferometer Science 332 1170
[30] Tollaksen J, Aharonov Y, Casher A, Kaufherr T and Nussinov S 2010 Quantum Interference Experiments, Modular Variables and Weak Measurements New J. Phys. 12 013023
[31] Aharonov Y, Cohen E, Colombo F, Landsberger T, Sabadini I, Struppa D and Tollaksen J 2017 Finally making sense of the double-slit experiment Proc. Natl. Acad. Sci. U.S.A 114 6480
[32] Aharonov, Y, Popescu S, Rohrlich D and Skrzypczyk P 2013 Quantum Cheshire Cats New J. Phys. 15 113015
[33] Denkmayr T, Geppert H, Sponar S, Lemmel H, Matzkin A, Tollaksen J and Hasegawa Y 2014 Observation of a quantum Cheshire Cat in a matter-wave interferometer experiment Nat. Commun. 5 4492
[34] Ferrie C and Combes J 2014 How the Result of a Single Coin Toss Can Turn Out to be 100 Phys. Rev. Lett. 113 120404
[35] Mundarain D and Orszag M 2016 Quantunness of the anomalous weak measurement value Phys. Rev. A 93 032106
[36] Vaidman L 2017 Weak Value Controversy Philos. Trans. Royal Soc. A 375 2106
[37] Svensson B 2013 Pedagogical Review of Quantum Measurement Theory with an Emphasis on Weak Measurements Quanta 2 18
[38] Tamir B and Cohen E 2013 Introduction to Weak Measurements and Weak Values Quanta 2 7
[39] Kofman A G, Ashhab S and Nori F 2012 Nonperturbative theory of weak pre- and post-selected measurements Phys. Rep. 520 42
[40] Dressel J, Malik M, Miatto F, Jordan A and Boyd R 2014 Colloquium: Understanding quantum weak values: Basics and applications Rev. Mod. Phys. 86 307
[41] Aharonov Y and Rohrlich D 2005 Quantum Paradoxes (Weinheim, Wiley-VCH) pp 225–48
[42] Hardy L 1992 Quantum Mechanics, Local Realistic Theories and Lorentz-Invariant Realistic Theories Phys. Rev. Lett. 68 2981
[43] Bell J S 1964 On the Einstein Rosen Podolski Paradox Physics 1 195
[44] Aharonov Y, Botero A, Popescu S, Reznik B and Tollaksen J 2002 Revisiting Hardy’s paradox: counterfactual statements, real measurements, entanglement and weak values Phys. Lett. A 301 130
[45] Lundeen J S and Steinberg A M 2009 Experimental Joint Weak Measurement on a Photon Pair as a Probe of Hardy’s Paradox Phys. Rev. Lett. 102 020404
[46] Tavon T and Vaidman L 2007 The three-box paradox revisited J. Phys. A: Math. Theor. 40 2873
Strange weak values of photon number operators in the optomechanical interaction

[47] Resch K J, Lundeen J S and Steinberg A M 2004 Experimental realization of the quantum box problem Phys. Lett. A 324 125
[48] Aharonov Y, Colombo F, Popescu S, Sabadini I, Struppa D and Tollaksen J 2016 Quantum violation of the pigeonhole principle and the nature of quantum correlations Proc. Natl. Acad. Sci. U.S.A 113 532
[49] Feizpour A, Xing X and Steinberg AM 2011 Amplifying Single-Photon Nonlinearity Using Weak Measurements Phys. Rev. Lett. 107 133603
[50] Hallaji M, Feizpour A, Dmochowski G, Sinclair J and Steinberg AM 2017 Weak-value amplification of the nonlinear effect of a single photon Nat. Phys. 13 540
[51] Aharonov Y, Cohen E, Carmi A and Elitzur C 2017 Anomalous Weak Values Emerging from Strong Interaction between Light and Matter arXiv:1709.98475v2 [quant-ph]
[52] Bose S, Jacobs K and Knight P L 1997 Preparation of nonclassical states in cavities with a moving mirror Phys. Rev. A 56 4175
[53] Mancini S, Man’ko V I and Tombesi P 1997 Ponderomotive control of quantum macroscopic coherence Phys. Rev. A 55 3042
[54] Bose S, Jacobs K and Knight P L 1999 Scheme to probe the decoherence of a macroscopic object Phys. Rev. A 59 3204
[55] Marshall W, Simon C, Penrose R and Bouwmeester D 2003 Towards Quantum Superpositions of a Mirror Phys. Rev. Lett. 91 130401
[56] Pepper B, Gobradi R, Jeffrey E, Simon C and Bouwmeester D 2012 Optomechanical Superpositions via Nested Interferometry Phys. Rev. Lett. 109 023601
[57] Li G, Wang T and Song H-S 2014 Amplification effects in optomechanics via weak measurements Phys. Rev. A 90 013827
[58] Li G, Chen L-B, Lin X-M and Song H-S 2015 Weak measurement amplification in optomechanics via a squeezed coherent state pointer J. Phys. B: At. Mol. Opt. Phys. 48 165504
[59] Li G, Wang T, Ye M-Y and Song H-S 2015 Weak measurement combined with quantum delayed-choice experiment and implementation in optomechanical system Eur. Phys. J. D 69 266
[60] von Neumann J 1983 Mathematical Foundations of Quantum Mechanics, (Princeton, NJ: Princeton University Press) pp 417–45
[61] Bohr N 1929 Wirkungsquantum und Naturbeschreibung Naturwissen. 26 483
[62] Tollaksen J 2007 Robust weak measurements on finite samples J. Phys.: Conf. Ser. 70 012015
[63] Wu S and Li Y 2011 Weak measurements beyond the Aharonov-Albert-Vaidman formalism Phys. Rev. A 83 052106
Strange weak values of photon number operators in the optomechanical interaction

[75] Rohrlich D and Aharonov Y 2002 Cherenkov radiation of superluminal particles Phys. Rev. A 66 042102
[76] Aharonov Y, Popescu S, Rohrlich D and Vaidman L 1993 Measurements, errors, and negative kinetic energy Phys. Rev. A 48 4084
[77] Dressel J 2015 Weak values as interference phenomena Phys. Rev. A 91 032116
[78] Liao J-Q, Cheung H K and Law C K 2012 Spectrum of single-photon emission and scattering in cavity optomechanics Phys. Rev. A 85 025803
[79] Hong T, Miao H and Chen Y 2013 Open quantum dynamics of single-photon optomechanical devices Phys. Rev. A 88 023812
[80] Shen J and Fan S 2009 Theory of single-photon transport in a single-mode waveguide. I. Coupling to a cavity containing a two-level atom Phys. Rev. A 79 023837
[81] Aharonov Y, Columbo F, Sabadini I, Struppa D C and Tollaksen J 2011 Some mathematical properties of superoscillations J. Phys. A: Math. Theor. 44 365304
[82] Gerry C and Knight P 2005 Introductory Quantum Optics (New York, Cambridge University Press) pp 137–43
[83] Haroche S and Raimond J M 2006 Exploring the Quantum Atoms, Cavities and Photons (New York, Oxford University Press) pp 126–35
[84] Agarwal G S 2013 Quantum Optics (United Kingdom, Cambridge University Press) pp 103–13
[85] Orszag M 2008 Quantum Optics Including Noise Reduction, Trapped Ions, Quantum Trajectories and Decoherence (Berlin Heidelberg, Springer-Verlag) pp 363–66