Apply Ant Colony Algorithm to Search All Extreme Points of Function

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Abstract

To find all extreme points of multimodal functions is called extremum problem, which is a well known difficult issue in optimization fields. Applying ant colony optimization (ACO) to solve this problem is rarely reported. The method of applying ACO to solve extremum problem is explored in this paper. Experiment shows that the solution error of the method presented in this paper is less than $10^{-8}$.

Keywords: Extremum Problem; Ant Colony Optimization (ACO)
I. INTRODUCTION

A. Extremum Problem

Multimodal function refers to the function which has more than one extreme point. To find all extreme points is called extremum problem, which is a well known difficult issue in optimization fields. Many practical engineering problems can be converted as this problem, such as the detection of multiple objects in military field. Therefore, solving the extremum problem is a useful study topic. To solve the extremum problem, many methods of optimization are applied, such as genetic algorithm (GA) [1], simulated annealing (SA) [2], particle swarm optimization algorithm (PSO) [3, 4], immune algorithm (IA) [5], and so on. However, currently there is rare report that applying Ant Colony Optimization (ACO) to solve the extremum problem. The motivation of this paper is to apply ACO to search all extreme points of function.

B. Introduction of Ant Colony Optimization (ACO)

Ant Colony Optimization (ACO) was first proposed by Dorigo (1991) [6, 7, 8]. The inspiring source of ACO is the foraging behavior of real ants. When ants search for food, they initially explore the area surrounding their nest in a random manner. As soon as an ant finds a food source, it remembers the route passed by and carries some food back to the nest. During the return trip, the ant deposits pheromone on the ground. The deposited pheromone, guides other ants to the food source. And the feature has been shown, indirect communication among ants via pheromone trails enables them to find the shortest routes between their nest and food sources. ACO imitates this feature and it becomes an effective algorithm for the optimization problems [9]. It has been successfully applied to many combinatorial optimization problems [10, 11, 12], such as Traveling Salesman Problem (TSP) [13, 14], Quadratic Assignment Problem(QAP) [15], Job-shop Scheduling Problem(JSP) [16], Vehicle Routing Problem(VRP) [17, 18], Data Mining(DM) [19] and so on.

The application of ACO pushes the study of ACO theory, and its two main study topics are the analysis of convergence and runtime. M. Birattari proves the invariance of ACO and introduced three new ACO algorithms [20]. Convergence is one of focus study of ACO. Walter J. Gutjahr studied the convergence of ACO firstly in 2000 [21]. T. Stüezele and
M. Dorigo proved the existence of the ACO convergence under two conditions, one is to only update the pheromone of the shortest route generated at each iteration step, the other is that the pheromone on all routes has lower bound \[22\]. C.-Y. Pang and et.al. found a potential new view point to study ACO convergence under general condition, that is entropy convergence in 2009 \[23\]. In ref. \[23\], the following conclusion is get: ACO may not converges to the optimal solution in practice, but its entropy is convergent. The other study focus of ACO is time complexity. ACO has runtime \(O(tmN^2)\), where \(t\), \(m\) and \(N\) refers to the number of iteration steps, ants, cities and \(M = \lceil \frac{N}{15} \rceil\) in general. To reduce runtime, cutting down parameter \(t\) and \(N\) is the main way possibly. In 2008, Walter J. Gutjahr presented some theoretical results about ACO runtime \[24, 25\]. Since runtime is proportional to the square of \(N\), parameter \(N\) is the key factor of runtime. Through cutting down \(N\) to reduce runtime is a choice. And Pang and et al. do the following attempt \[26\]. Firstly, cluster all cities into some classes (group) and let ACO act on these small classes respectively to get some local TSP routes. And then joint these local route to form the whole TSP route. If class is compact, the length of local route got at every iteration step will change continually possibly, where compactness refers to all data cluster in a small region tightly. And this property result in the conclusion: the convergence criterion \(\frac{|L_t - L_{t+1}|}{L_t} \to 0\) is the marker of ACO convergence, where \(L_t\) is the length of local route at \(t\) – th iteration step. Thus, minimum iteration number \(t\) can be estimated by the marker approximately.

The study of ACO theory speeds up its application again. ACO not only can be applied to solve discrete optimization problems, but also to continuous ones. The first method for continuous-space optimization problems, called Continuous ACO (CACO), was proposed by Bilchev and Parmee (1995) \[29\], and later it was used by some others \[30, 31, 32, 33\]. In general, the application of ACO to continuous optimization problems need to transform a continuous search space to a discrete one. Other methods include that, in 2002, Continuous Interacting Ant Colony (CIAC) was proposed by Dreo and Siarry \[34\], and in 2003, an adaptive ant colony system algorithm for continuous-space optimization problems was proposed by Li Yan-jun \[35\], and so on.
C. Framework of ACO

Traveling Salesman Problem (TSP) is a famous combinatorial problem, it can be stated very simply:

A salesman visit \( N \) cities cyclically provided that he visits each city just once. In what order should he visit them to minimize the distance traveled?

The typical application of ACO is to solve TSP, and its basic idea is stated as below:

When an ant passes through an edge, it releases pheromone on this edge. The shorter the edge is, the bigger the amount of pheromone is. And the pheromone induces other ants to passes through this edge. At last, all ants select a unique route, and this route is shortest possibly.

The framework of ACO is introduced as below:

Step1 (Initialization): Posit \( M \) ants at different \( M \) cities randomly; Pre-assign maximum iteration number \( t_{\text{max}} \); Let \( t = 0 \), where \( t \) denotes the \( t – th \) iteration step; Initialize amount of pheromone of every edge.

Step2 While\((t < t_{\text{max}})\)

{  

Step2.1: Every ant selects its next city according to the transition probability. The transition probability from the \( i – th \) city to the \( j – th \) city for the \( k – th \) ant is defined as Eq.[I]

\[
p_{ij}^{(k)}(t) = \begin{cases}  
\frac{\tau_{ij}(t)\eta_{ij}}{\sum_{s \in \text{allowed}_k} \tau_{is}^{(s)}(t)\eta_{is}}, & \text{if } j \in \text{allowed}_k \\
0, & \text{otherwise}
\end{cases}
\quad (1)
\]

, where \( \text{allowed}_k \) denotes the set of cities that can be accessed by the \( k – th \) ant; \( \tau_{ij}(t) \) is the pheromone value of the edge \((i,j)\); \( \eta_{ij} \) is local heuristic function defined as

\[
\eta_{ij} = \frac{1}{d_{ij}}
\]

, where \( d_{ij} \) is the distance between the \( i – th \) city and the \( j – th \) city; the parameters \( \alpha \) and \( \beta \) determine the relative influence of the trail strength and the heuristic information respectively.

Step2.2: After all ants finish their travels, all pheromone values are updated according to Eq.[2]
\[ \tau_{ij}(t+1) = (1 - \rho) \cdot \tau_{ij}(t) + \Delta \tau_{ij}(t) \quad (2) \]

\[ \Delta \tau_{ij}(t) = \sum_{k=1}^{m} \Delta \tau_{ij}^{(k)}(t) \]

\[ \Delta \tau_{ij}^{(k)}(t) = \begin{cases} 
Q \frac{L^{(k)}(t)}{L^{(k)}(t)} & \text{if the } k-th \text{ ant pass edge } (i,j) \\
0, & \text{otherwise}
\end{cases} \]

where \( L^{(k)}(t) \) is the length of the route passed by the \( k-th \) ant during the \( t-th \) iteration; \( \rho \) is the persistence percentage of the trail (thus, \( 1 - \rho \) corresponds to the evaporation); \( Q \) denotes constant quantity of pheromone.

**Step 2.3:** Increase iteration step: \( t \leftarrow t + 1 \)

**Step 3:** End procedure and select the shortest route as output from the routes traveled by the ants.

**II. APPLY ACO TO SEARCH ALL EXTREME POINTS OF FUNCTION**

**A. Basic Idea**

Assume that the function is \( f(x) \), \( x \) is real number and it belongs to a closed interval \([a, b]\). The task of this paper is to extract all extreme points that the corresponding value is minimal locally. The basic idea of this paper is stated roughly as below:

Divide interval \([a, b]\) into many tiny intervals with equal size. Suppose these small interval are \( \{I_1, I_2, \ldots, I_n\} \) and the center of interval \( I_i \) is denoted by \( x_i \). Suppose the neighbor interval of interval \( I_i \) is \( I_{i+1} \) (or \( I_{i-1} \)). And an ant is put at the center of each small interval. If \( f(x_i) > f(x_{i+1}) \), the ant at interval \( I_i \) will move to interval \( I_{i+1} \) possibly, just liking there is an virtual edge between \( I_i \) and \( I_{i+1} \). And assume the virtual edge is \( e(I_i, I_{i+1}) \). The weight (virtual distance) of edge \( e(I_i, I_{i+1}) \) is proportional to \( f(x_i) - f(x_{i+1}) \). That is, the bigger \( f(x_i) - f(x_{i+1}) \) is, the more possibly the ant moves to \( I_{i+1} \) from \( I_i \). When the ant moves to \( I_{i+1} \), it releases pheromone at \( I_{i+1} \), and the pheromone is proportional to value \( f(x_i) - f(x_{i+1}) \). The pheromone depositing on \( I_{i+1} \) will attract other ants move to it.
After some iteration steps, some intervals contain many ants while other ones contain no ant. The intervals containing ants include extreme points possibly, while other ones include no extreme point possibly. And then keep the intervals which contain ants, and divide them into much more small intervals, repeat the same procedure again until the size of intervals is sufficient small. At last all ants will stay around extreme points. The centers of these sufficient small intervals are the approximations of extreme points.

From above discussion, it can be seen that the realization of the basic idea consists of four parts: partition of interval \([a, b]\) and initialization, rule of ant moving, rule of pheromone updating, and keeping the intervals containing ants. The contents of the four parts are stated as below.

**B. Partition of Interval and Initialization**

Suppose interval \([a, b]\) is partitioned into \(n\) small intervals with equal size, which are denoted by \(\{I_1, I_2, \cdots, I_n\}\), where \(n\) is a pre-assigned number. Then each interval has size

\[
\delta = \frac{b - a}{n}
\]

\(I_i = [a + (i - 1)\delta, a + i\delta]\)

The \(i\)\(-th\) interval \(I_i\) has center \(x_i\).

\[
x_i = a + (i - \frac{1}{2})\delta
\]

Suppose \(t\) denotes the \(t\)\(-th\) iteration step of ACO and it is initialized as zero (i.e., \(t = 0\)). Put \(n\) ants at the centers of the \(n\) intervals, and each interval has only one ant. Suppose these ants are denoted by \(a_1, a_2, \cdots, a_n\) respectively, and ant \(a_i\) is associated with the \(i\)\(-th\) interval \(I_i\). Each ant will release an initial pheromone at its associated interval \(I_i\) (i.e., \(\tau_i(0) = \text{const}\), \(\text{const}\) is a constant number). In addition, set the increment of the pheromone of each interval to zero (i.e., \(\Delta\tau_i(0) = 0\))

Figure 1 shows a diagram of initialization.
FIG. 1: Initialization: Divide the function domain into intervals and put an ant at the center of every interval.

C. Rule of Ant Moving

Let \( \text{Neighbor}(I_i) \) be a set of neighboring intervals of \( I_i \). Take one-dimensional function for example

\[
\text{Neighbor}(I_i) = \begin{cases} 
\{I_{i+1}\}, & i = 1 \\
\{I_{i-1}, I_{i+1}\}, & i = 2 \cdots n - 1 \\
\{I_{i-1}\}, & i = n
\end{cases}
\]

As it is discussed in section A, the ant staying at interval \( I_i \) will move to neighbor interval denoted by \( I_j \), just like there is a virtual edge \( e(I_i, I_j) \). The weight of the virtual edge is \( |f(x_i) - f(x_j)| \). Then the heuristic factor is

\[
\eta_{ij} = |f(x_i) - f(x_j)|
\]

Suppose interval \( I_i \) contains ant \( a_k \). If \( f(x_i) > f(x_j) \), ant \( a_k \) is allowed to move to neighbor interval \( I_j \). Otherwise, it is forbidden to move. Suppose all intervals allowed to be accessed by ant \( a_k \) is marked as \( \text{allowed}_k \). The transition probability of ant \( a_k \) is defined as

\[
p_{ij}^{(k)}(t) = \begin{cases} 
\frac{\tau_{ij}^\alpha(t) \eta_{ij}^\beta}{\sum_{h \in \text{allowed}_k} \tau_{ih}^\alpha(t) \eta_{ih}^\beta}, & \text{if } f(x_i) > f(x_j) \\
0, & \text{otherwise}
\end{cases}
\]  (3)
In Eq.3, $\alpha$ is the relative influence of the trail strength; $\beta$ is the heuristic information; $\tau_j(t)$ is the pheromone of interval $I_j$.

D. The Rule of Pheromone Updating

Suppose $a_k$ ant is staying at interval $I_i$ and it will move to neighbor interval $I_j$. After it moves to $I_j$, releases pheromone at $I_j$. The pheromone amount is denoted by $\Delta \tau_j^{(k)}(t)$

$$\Delta \tau_j^{(k)}(t) = C_1(f(x_i) - f(x_j))$$

, where $C_1$ is a positive constant.

The bigger $f(x_i) - f(x_j)$ is, the higher the amount of released pheromone is, the more possibly that other ants will be attracted to interval $I_j$.

Not only ant $a_k$ arrives $I_j$ and releases pheromone, but also other ants which move to interval $I_j$ and release pheromone too. Suppose there are $q$ ants will move to interval $I_j$ during the $t-th$ iteration step, which are denoted by $a_{j_1}, a_{j_2}, \cdots, a_{j_q}$. The sum of released pheromone by $a_{j_1}, a_{j_2}, \cdots, a_{j_q}$ is $\Delta \tau_j(t)$. Then

$$\Delta \tau_j(t) = \sum_{p=1}^{q} \Delta \tau_j^{(j_p)}(t)$$

When all ants $a_{j_1}, a_{j_2}, \cdots, a_{j_q}$ move to $I_j$, the amount of pheromone $\tau_j$ is changed as Eq.4

$$\tau_j(t+1) = (1-\rho) \cdot \tau_j(t) + \Delta \tau_j(t)$$

, where $\rho$ is the evaporation percentage of the trail (thus, $1-\rho$ corresponds to persistence).

E. Keeping Only the Intervals Containing Ants to Cut Down Search Range

The intervals that have smaller function values depositing much more pheromone, and it will attract ants more powerfully. After several iterations, the distribution of ants has the feature that all ants stay at the intervals that have smaller function values and other intervals contain no ants. That is, extreme points are included in the intervals containing ants. And then keep the intervals having ant and delete other intervals to update search
range. Thus, the updated search range became smaller. Dividing the updated search range into smaller intervals will result in much smaller search range at next iteration step. When intervals becoming sufficient small, all ants will stay around extreme points, the centers of the intervals are their approximations.

Figure 2 shows the distribution feature of ants.

In addition, as it is well known, ACO runs slow, which is the bottleneck of application. And it evades this bottleneck that keeping only the intervals having ants to cut down search range.

FIG. 2: The Feature of Ant Distribution: The intervals including extreme points have more pheromone to attract ants. After several iteration steps, all ants will stay at the intervals containing extreme points while other intervals are empty. Then search range become smaller, which consists of the intervals containing ants. At last all ants stay around extreme points.
F. Method of Searching All Extreme Points

**Step 1 (Initialization):** Divide domain \([a, b]\) into many small intervals and put an ant in each interval; Do other initialization. The detail is shown at section B. Suppose \(\delta\) is the length of interval and \(\varepsilon\) is a stop threshold.

**Step 2** While \((\delta > \varepsilon)\)

\[
\text{Step 2.1: All ants move to new intervals according the rule shown at section C} \\
\text{Step 2.2: Update pheromone according the rule shown at section D} \\
\text{Step 2.3: Update search range according to section E and divide it into smaller intervals (Suppose the number of these intervals is } n) . \text{ Calculate the size of interval and set it to } \delta. \\
\]

**Step 3** Extract all intervals that contain ants, the centers of the intervals are the approximations of extreme points.

If argument \(x\) is multi-dimensional vector, divide the range of every component of vector into smaller intervals, the combination of these intervals forms many small lattices. And then put an ant in each lattice, apply the above method, all extreme points can be extracted.

G. An Example

To understand above method easily, an simple example is stated as below:

Assume that the domain of 1-dimensional function is divided into 3 intervals \(I_1, I_2, I_3\), which associated center is \(x_1, x_2\) and \(x_3\) respectively. Initially ant \(a_1, a_2,\) and \(a_3\) is put at \(x_1, x_2\) and \(x_3\) respectively.

Check the first ant: If \(f(x_1) > f(x_2)\), ant \(a_1\) moves to interval \(I_2\). Otherwise, do nothing.

Check the 2nd ant: If their is unique interval (e.g. \(I_3\) ) such that \(f(x_2) > f(x_3)\), ant \(a_2\) moves to interval \(I_3\). If \(f(x_2)\) is smaller than both \(f(x_1)\) and \(f(x_3)\), do nothing. If \(f(x_2)\) is bigger than both \(f(x_1)\) and \(f(x_3)\), it is uncertain that ant \(a_2\) moves to \(I_1\) or \(I_3\). And ant \(a_2\) will select its visiting interval randomly according to its transition probability defined at Eq.3.

Check the 3rd ant using same way.

After all ants are checked, update their associated interval (position) and interval
pheromone. Repeat above processing until all ants can not move.

Then keep the intervals which contains ants, and delete other blank intervals. And divided the intervals containing ants into smaller interval, repeat above process until the size of interval is sufficient small. And then all interval centers are the approximations of extreme points.

III. EXPERIMENT

In this section, several functions will be tested. The parameters are listed as below:

\[ const = 10, \alpha = 1, \beta = 1, C_1 = 1, \rho = 0.3, \varepsilon = 0.0001 \]

Two performances are considered, which are error (ratio of inaccuracy) \( r \) and runtime. Error \( r \) is defined as

\[
    r = \left( \left| \frac{f(x'_0) - f(x_0)}{f(x_0)} \right| \right) \times 100\%
\]

, where \((x_0, f(x_0))\) denotes the true extreme point on theory and \((x'_0, f(x'_0))\) is its approximation calculated by the method presented in this paper.

In addition, the hardware condition is: notebook PC DELL D520, CPU 1.66 GHZ.

A. Instance 1 (see table 1 and Fig.3):

\[ f_1(x) = \sin^6(5.1\pi x + 0.5), \quad x \in [0, 1] \]

In the experiment, additional parameter is \( n = 20 \) and \( n1 = 10 \). Table 1 and Fig.3 show all extreme points (local maximal points) of \( f_1(x) \). Only 0.7106 seconds is cost.

Other functions are tested, their errors are less than \( 10^{-8} \) except the boundary. And runtime is less than 1 second (see appendix I).

B. Instance 2 (see Fig.4):

\[ f_2(x) = 5e^{-0.5x} \sin(30x) + e^{0.2x} \sin(20x) + 6, \quad x \in [0, 8] \]

Instance 2 is a typical test function, which include many extreme points and any small change of argument \( x \) will result in big change. In addition, the theoretical calculation of extreme points of instance 2 is difficult.
The additional parameters are \( n = 480 \) and \( n1 = 10 \). Fig.4 shows all the calculated extreme points, and the real numbers are listed at appendix (see appendix II).

C. Instance 3 (see Fig.5):

\[
f_3(x_1, x_2) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2), \quad x_1, x_2 \in [-1, 1]
\]

The interval \([-1, 1] \otimes [-1, 1]\) is divided into \( n = 40 \times 40 \) intervals initially. And at next iteration steps, search domain is divided into \( n1 = 20 \times 20 \) small intervals. 36 extreme points are got and shown at Fig.5 and the digital solutions are listed at appendix (see appendix III).

Instance 3 is a 2-dimensional function and 3203.2968 seconds is cost. And it is slower than 1-dimensional function instance 1 and instance 2. To improve running speed is the next work.

Many functions are tested by the authors, and some experiments results are listed at appendix. These tests demonstrate that solution error is less than \( 10^{-8} \) except the special case that extreme point is at the border of the domain. In addition, these testes also demonstrate that the method is very fast for 1-dimensional function.
TABLE 1. The Extreme Points of Function $f_1(x)$

| Extreme Points      | Theory Value  | Calculated Value  | Error (%) |
|---------------------|---------------|-------------------|-----------|
| $\left( x_0, f(x_0) \right)$ | $(x'_0, f(x'_0))$ | $r$               |
| 1                   | 0.066832364, 1| 0.066827175, 0.999999979 | 2.1e-06     |
| 2                   | 0.262910795, 1| 0.262914975, 0.999999987 | 1.3e-06     |
| 3                   | 0.458989227, 1| 0.458990475, 0.999999999 | 1.0e-07     |
| 4                   | 0.655067658, 1| 0.655066175, 0.999999998 | 2.0e-07     |
| 5                   | 0.851146090, 1| 0.851148550, 0.999999995 | 5.0e-07     |
| 6                   | 1             | 0.999998400, 0.147800654 | 0.0145      |

Conclusion: Error of the method presented this paper is low except the border point (6th point). And runtime is fast (0.7106s). Notice: The 6 points are shown at Fig.3

Notice: From the Table 1, we can see that the border point has big error because the value calculated is the center of the interval, not boundary. To evade this drawback, the function value at boundary can be calculated directly.

IV. CONCLUSION

To find all extreme points of multimodal functions is called extremum problem, which is a well known difficult issue in optimization fields. It is reported rarely that applying Ant Colony Optimization (ACO) to solve the problem. And the motivation of this paper is to explore ACO application method to solve it. In this paper, the following method is presented:

Divide the domain of function into many intervals and put an ant in each interval. And then design rule such that every ant moves to the interval containing extreme point near by. At last all ants stay around extreme points.

The method presented in this paper has following three advantages:
1. Solution accuracy is high. Experiment shows that solution error is less than $10^{-8}$.

2. Solution calculated is stable (robust). Ant only indicates the interval containing extreme point, not the accurate position of extreme point. It is easy for ant to find a interval although finding a special point in interval is difficult.

3. The method is fast for 1-dimensional function. ACO is slow. But some feature is
FIG. 5: The Extreme Points Calculated of Function $f_3(x_1, x_2)$.

found to speed ACO (see section 2.5)

Acknowledgments

The authors appreciate the discussion from the members of Gene Computation Group, J. Gang, X. Li, C.-B. Wang, W. Hu, S.-P. Wang, Q. Yang, J.-L. Zhou, P. Shuai, L.-J. Ye. The authors appreciate the help from Prof. J. Zhang, Z.-Lin Pu, X.-P. Wang, J. Zhou, and Q. Li.

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V. APPENDIX

A. Appendix I

1. Instance 4:

\[ f_4(x) = (x + 1)(x + 2)(x + 3)(x + 4)(x + 5) + 5 \quad x \in [-5, 0] \]
The additional parameters are $n = 30$ and $n_1 = 10$.

| Extreme Points | Theory Value          | Calculated Value          | Error (%) |
|----------------|-----------------------|---------------------------|-----------|
| $(x_0, f(x_0))$ | $(x'_0, f(x'_0))$     |                           |           |
| 1              | $-5.0000000000000000$, | $-4.99999893333333$,      | 5.12e-004 |
|                | $5.0000000000000000$  | $5.00002559994310$        |           |
| 2              | $-3.5439122590234$,   | $-3.54391549166667$,      | 1.91e-009 |
|                | $3.58130337441708$    | $3.58130337448565$        |           |
| 3              | $-1.35556713184173$,  | $-1.35557000833333$,      | 1.20e-008 |
|                | $1.36856779155116$    | $1.36856779171500$        |           |

Notice: 1. Runtime is 0.5280s. 2. From the table, we can see that only the boundary maximum value has big error because $x'_0$ is the center of interval, not the boundary. To evade this drawback, the function value at boundary can be calculated directly.

FIG. 6: The Extreme Points Calculated of Function $f_4(x)$. The red stars are the extreme points.
2. **Instance 5:**

\[ f_5(x) = (x + 2) \cos(9x) + \sin(7x) \quad x \in [0, 4] \]

The additional parameters are \( n = 95 \) and \( n_1 = 10 \)

| Table 3. Extreme Points of Function \( f_5(x) \) |
|---|---|---|---|
| Extreme Points \( (x_0, f(x_0)) \) | Calculated Value \( (x'_0, f(x'_0)) \) | Error (%) \( r \) |
| 1 | 0.00000000000000, 2.00000000000000 | 0.00000134736842, 2.00001077880032 | 5.39e-004 |
| 2 | 0.38762423574739, -1.8300394960224 | 0.38762006315789, -1.83003949456305 | 7.97e-008 |
| 3 | 1.03491254831578, -2.19650045294125 | -1.03490865263158, -2.19650045140718 | 16.98e-008 |
| 4 | 1.72787330641709, -4.135970121530 | 1.72787490526316, -4.13597012080927 | 9.82e-009 |
| 5 | 2.44888001781347, -5.43427465397202 | 2.44888418947368, -5.43427465041009 | 6.55e-008 |
| 6 | 3.16064032539339, -5.21793460286144 | 3.16064037894737, -5.21793460286083 | 1.17e-001 |
| 7 | 3.84493263713282, -4.86068863199737 | 3.84493433684211, -4.86068863183223 | 1.27e-008 |

Notice: 1. The runtime is 0.9030s. 2. The boundary maximum value has big error because \( x'_0 \) is the center of the interval, not the boundary. To evade this drawback, the function value at boundary can be calculated directly.

B. **Appendix II.**

\[ f_2(x) = 5e^{-0.5x} \sin(30x) + e^{0.2x} \sin(20x) + 6, \quad x \in [0, 8] \]

The additional parameters are \( n = 480 \) and \( n_1 = 10 \).
FIG. 7: The Extreme Points Calculated of Function $f_5(x)$
## Table 4. Extreme Points of Function $f_2(x)$

| Extreme Points | Calculated Value | Extreme Points | Calculated Value | Extreme Points | Calculated Value |
|----------------|------------------|----------------|------------------|----------------|------------------|
| $(x'_0, f(x'_0))$ | $(x'_0, f(x'_0))$ | $(x'_0, f(x'_0))$ | $(x'_0, f(x'_0))$ |
| 1               | 0.0000, 6.0005   | 12             | 2.2337, 5.5878   | 23             | 5.2547, 3.3639   |
| 2               | 0.1615, 1.3372   | 13             | 2.4521, 3.0483   | 24             | 5.5812, 2.7148   |
| 3               | 0.3627, 2.7441   | 14             | 2.6982, 4.2229   | 25             | 5.8862, 2.9274   |
| 4               | 0.5725, 1.2573   | 15             | 2.8368, 5.9928   | 26             | 6.2081, 2.3729   |
| 5               | 0.7926, 2.5460   | 16             | 3.0777, 3.2186   | 27             | 6.5164, 2.4487   |
| 6               | 0.9889, 3.9745   | 17             | 3.3398, 4.2595   | 28             | 6.8355, 1.9567   |
| 7               | 1.1995, 2.1270   | 18             | 3.7032, 3.2478   | 29             | 7.1485, 1.9215   |
| 8               | 1.4252, 3.3895   | 19             | 3.9822, 4.0777   | 30             | 7.4631, 1.4650   |
| 9               | 1.6133, 4.9025   | 20             | 4.3288, 3.1621   | 31             | 7.7748, 1.3366   |
| 10              | 1.8261, 2.7009   | 21             | 4.6206, 3.7560   | 32             | 8.0000, 7.1737   |
| 11              | 2.0601, 3.9371   | 22             | 4.9548, 2.9802   |                |                  |

Notice: Runtime is 4.3411s. The associated figure is Fig[4]

### C. Appendix III.

\[ f_3(x_1, x_2) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2), \quad x_1, x_2 \in [-1, 1] \]

The interval $[-1, 1] \otimes [-1, 1]$ is divided into $n = 40 \times 40$ intervals initially. And at next
iteration steps, search domain is divided into $n1 = 20 \times 20$ small intervals.

| Extreme Points $(x_1, x_2, f(x_1, x_2))$ | Extreme Points $(x_1, x_2, f(x_1, x_2))$ | Extreme Points $(x_1, x_2, f(x_1, x_2))$ |
|-----------------------------------------|-----------------------------------------|-----------------------------------------|
| 1 $-0.8781, -0.8781, 3.5326$ | 13 $-0.1810, -0.8725, 2.7873$ | 25 $0.5175, -0.8700, 3.0176$ |
| 2 $-0.8737, -0.5310, 3.0362$ | 14 $-0.1773, -0.5252, 2.3056$ | 26 $0.5200, -0.5200, 2.5367$ |
| 3 $-0.8725, -0.1810, 2.7873$ | 15 $-0.1756, -0.1756, 2.0613$ | 27 $0.5215, -0.1700, 2.2969$ |
| 4 $-0.8700, 0.1675, 2.7759$ | 16 $-0.1715, 0.1715, 2.0559$ | 28 $0.5252, 0.1773, 2.3056$ |
| 5 $-0.8700, 0.5175, 3.0176$ | 17 $-0.1700, 0.5215, 2.2969$ | 29 $0.5268, 0.5268, 2.5517$ |
| 6 $-0.8675, 0.8675, 3.4966$ | 18 $-0.1675, 0.8700, 2.7759$ | 30 $0.5310, 0.8737, 3.0362$ |
| 7 $-0.5310, -0.8737, 3.0362$ | 19 $0.1675, -0.8700, 2.7759$ | 31 $0.8675, -0.8675, 3.4966$ |
| 8 $-0.5268, -0.5268, 2.5517$ | 20 $0.1700, -0.5215, 2.2969$ | 32 $0.8700, -0.5175, 3.0176$ |
| 9 $-0.5252, -0.1773, 2.3056$ | 21 $0.1715, -0.1715, 2.0559$ | 33 $0.8700, -0.1675, 2.7759$ |
| 10 $-0.5215, 0.1700, 2.2969$ | 22 $0.1756, 0.1756, 2.0613$ | 34 $0.8725, 0.1810, 2.7873$ |
| 11 $-0.5200, 0.5200, 2.5367$ | 23 $0.1773, 0.5252, 2.3056$ | 35 $0.8737, 0.5310, 3.0362$ |
| 12 $-0.5175, 0.8700, 3.0176$ | 24 $0.1810, 0.8725, 2.7873$ | 36 $0.8781, 0.8781, 3.5326$ |

**Table 5. The Extreme Points of Function $f_3(x_1, x_2)$**

Notice: All points are shown at Fig[5] Runtime is 3203.2968s.