Evaluation complexity bounds
for smooth constrained nonlinear optimization
using scaled KKT conditions, high-order models and the
criticality measure $\chi$

C. Cartis$^*$ N. I. M. Gould$^\dagger$ and Ph. L. Toint$^\ddagger$

1 October 2015

Abstract

Evaluation complexity for convexly constrained optimization is considered and it is shown first that the complexity bound of $O(\epsilon^{-3/2})$ proved by Cartis, Gould and Toint (IMA JNA 32(4) 2012, pp. 1662-1695) for computing an $\epsilon$-approximate first-order critical point can be obtained under significantly weaker assumptions. Moreover, the result is generalized to the case where high-order derivatives are used, resulting in a bound of $O(\epsilon^{-(p+1)/p})$ evaluations whenever derivatives of order $p$ are available. It is also shown that the bound of $O(\epsilon_p^{-1/2}\epsilon_D^{-3/2})$ evaluations ($\epsilon_p$ and $\epsilon_D$ being primal and dual accuracy thresholds) suggested by Cartis, Gould and Toint (SINUM, 2015, to appear) for the general nonconvex case involving both equality and inequality constraints can be generalized to a bound of $O(\epsilon_p^{1/p}\epsilon_D^{(p+1)/p})$ evaluations under similarly weakened assumptions. This paper is variant of a companion report (NTR-11-2015, University of Namur, Belgium) which uses a different first-order criticality measure to obtain the same complexity bounds.

Keywords: complexity theory, nonlinear optimization, constrained problems, high-order models, convex constraints.

1 Introduction

In [3] and [7], we examined the worst-case evaluation complexity of finding an $\epsilon$-approximate first-order critical point for smooth nonlinear (possibly nonconvex) optimization problems for methods using both first and second derivatives of the objective function. The case where constraints are defined by a convex set was considered in the first of these references while the general case (with equality and inequality constraints) was discussed in the second.

$^*$Mathematical Institute, Oxford University, Oxford OX2 6GG, Great Britain. Email: coralia.cartis@maths.ox.ac.uk.
$^\dagger$Numerical Analysis Group, Rutherford Appleton Laboratory, Chilton OX11 0QX, Great Britain. Email: nick.gould@stfc.ac.uk.
$^\ddagger$Namur Center for Complex Systems (naXys) and Department of Mathematics, University of Namur, 61, rue de Bruxelles, B-5000 Namur, Belgium. Email: philippe.toint@unamur.be. Ph. Toint would like to dedicate this paper to Thaïs, his first grand-daughter.
It was shown in [3] that at most $O(\epsilon^{-3/2})$ evaluations of the objective function and its derivatives are needed to compute such an approximate critical point. This result, which is identical in order to the best known result for the unconstrained case, comes at the price of potentially restrictive technical assumptions: it was assumed that an approximate first-order critical point of a cubic model subject to the problem’s constraints can be obtained for the subproblem solution in a uniformly bounded number of descent steps that is independent of $\epsilon$, that all iterates remains in a bounded set and that the gradient of the objective function is also Lipschitz continuous (see [3] for details). The analysis of [7] then built on the result of the convex case by first specializing it to convexly constrained nonlinear least-squares and then using the resulting complexity bound in the context of a two-phase algorithm for the problem involving general constraints. If $\epsilon_P$ and $\epsilon_D$ are the primal and the dual criticality thresholds, respectively, it was suggested that at most $O(\epsilon_P^{-1/2}\epsilon_D^{-3/2})$ evaluations of the objective function and its derivatives are needed to compute an approximate critical point in that case, where the Karush-Kuhn-Tucker (KKT) conditions are scaled to take the size of the Lagrange multipliers into account. Because this bound is based on that obtained for the convex case, it suffers from the same limitations (not to mention an additional constraint on the relative sizes of $\epsilon_P$ and $\epsilon_D$, see [7]).

More recently, Birgin, Gardenghi, Martínez, Santos and Toint [2] provided a new regularization algorithm for the unconstrained problem with two interesting features. The first is that the model decrease condition used for the subproblem solution is weaker than that used previously, and the second is that the use of problem derivatives of order higher than two is allowed, resulting in corresponding reductions in worst-case complexity. In addition, the same authors also analyzed the worst-case evaluation complexity of the general constrained optimization problem in [1] also allowing for high-order derivatives and models in a framework inspired by that of [5,7]. At variance with the analysis of these latter references, their analysis considers unscaled approximate first-order critical points in the sense that such points satisfy the standard unscaled KKT conditions with accuracy $\epsilon_P$ and $\epsilon_D$.

This paper is variant of a companion report [6] which uses a different first-order criticality measure to obtain the same complexity bounds. The first purpose of both papers is to explore the potential of the proposals made in [2] to overcome the limitations of [3] and to extend its scope by considering the use of high-order derivatives and models. A second objective is to use the resulting worst-case bounds to establish strengthened evaluation complexity bounds for the general nonlinearly constrained optimization problem in the framework of scaled KKT conditions, thereby improving [7]. This paper, like its companion, is thus organized in two main sections, Section 2 covering the convexly constrained case and Section 3 that allowing general nonlinear constraints. The results obtained are finally discussed in Section 4.

2 Convex constraints

The first problem we wish to solve is formally described as

$$\min_{x \in \mathcal{F}} f(x)$$

where we assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is $p$-times continuously differentiable, bounded from below, and has Lipschitz continuous $p$-th derivatives. For the $q$-the derivative of a function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ to be Lipschitz continuous on the set $\mathcal{S} \subseteq \mathbb{R}^n$, we require that i.e. there exists a
constant $L_{h,q} \geq 0$ such that, for all $x, y \in S$,

$$\| \nabla^q_x h(x) - \nabla^q_x h(y) \|_T \leq (p - 1)! L_{h,p} \|x - y\|$$

where $\| \cdot \|$ is the standard Euclidean norm on $\mathbb{R}^n$ and $\| \cdot \|_T$ is recursively induced by this norm on the space of $q$-th order tensors. We also assume that the feasible set $F$ is closed, convex and non-empty. Note that this formulation covers unconstrained optimization ($F = \mathbb{R}^n$), as well as standard inequality (and linear equality) constrained optimization in its different forms: the set $F$ may be defined by simple bounds, and/or by polyhedral or more general convex constraints. We are tacitly assuming here that the cost of evaluating values and derivatives of the constraint functions possibly involved in the definition of $F$ is negligible.

The algorithm considered in this paper is iterative. Let $T_p(x_k, s)$ be the $p$-th order Taylor-series approximation to $f(x_k + s)$ at some iterate $x_k \in \mathbb{R}^n$, and define the local regularized model at $x_k$ by

$$m_k(x_k + s) \overset{\text{def}}{=} T_p(x_k, s) + \frac{\sigma_k}{p + 1} \|s\|^{p+1},$$

(2.2)

where $\sigma_k > 0$ is the regularization parameter. Note that $m_k(x_k) = T_p(x_k, 0) = f(x_k)$. The approach used in [3] (when $p = 2$) seeks to define a new iterate $x_{k+1}$ from the preceding one by computing an approximate solution of the subproblem

$$\min_{x \in F} m_k(x_k + s)$$

(2.3)

using a modified version of the Adaptive Regularization with Cubics (ARC) method for unconstrained minimization. By contrast, we now examine the possibility of modifying the ARp algorithm of [2] with the aim of inheriting its interesting features. As in [3], the modification involves a suitable continuous first-order criticality measure for the constrained problem of minimizing a given function $h : \mathbb{R}^n \to \mathbb{R}$ on $F$. For an arbitrary $x \in F$, this criticality measure is given by

$$\chi_h(x) \overset{\text{def}}{=} \left| \min_{x + d \in F, \|d\|_\chi \leq 1} \langle \nabla_x h(x), d \rangle \right|,$$

(2.4)

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $\| \cdot \|_\chi$ is any fixed norm, possibly chosen to make the computation of $\chi_h(x)$ easier. Let $\kappa_n > 0$ be the norm equivalence constant such that

$$\|v\| \leq \kappa_n \|v\|_\chi \quad \text{for all} \quad v \in \mathbb{R}^n.$$ (2.5)

Observe that $\chi_h(x)$ depends on the geometry of $F$ only (and not on its possible parametrization using constraint functions) and that $x$ is a first-order critical point of problem (2.1) if and only if $\chi_f(x) = 0$. Also note that $\chi_h(x) = \|\nabla_x h(x)\|$ whenever $F = \mathbb{R}^n$ and $\| \cdot \|_\chi = \| \cdot \|$.

We now describe our algorithm as the ARpCC algorithm (ARp for Convex Constraints) on the following page.

We first state a useful property of the ARpCC algorithm, which ensures that a fixed fraction of the iterations $1, 2, \ldots, k$ must be either successful or very successful.
Algorithm 2.1: Adaptive Regularization using $p$-th order models for convex constraints (AR$p$CC)

A starting point $x_{-1}$, an initial and a minimal regularization parameter $\sigma_0 \geq \sigma_{\text{min}} > 0$, algorithmic parameters $\theta > 0$, $\gamma_3 \geq \gamma_2 > 1 > \gamma_1 > 0$ and $1 > \eta_2 \geq \eta_1 > 0$, are given, as well as an accuracy threshold $\epsilon \in (0, 1]$. Compute $x_0 = P_F[x_{-1}]$, the projection of $x_{-1}$ onto $F$, and evaluate $f(x_0)$ and $\nabla_x f(x_0)$.

For $k = 0, 1, \ldots$, do:

1. Evaluate $\nabla_x f(x_k)$. If
   \[ \chi_f(x_k) \leq \epsilon, \] \hspace{1cm} (2.6)
   terminate with $x_\epsilon = x_k$. Otherwise compute derivatives of $f$ of order 2 to $p$ at $x_k$.

2. Compute a step $s_k$ by approximately minimizing $m_k(x_k + s)$ over $s \in F$ so that
   \[ x_k + s_k \in F, \] \hspace{1cm} (2.7)
   \[ m_k(x_k + s_k) < m_k(x_k) \] \hspace{1cm} (2.8)
   and
   \[ \chi_{m_k}(x_k + s_k) \leq \theta \|s_k\|^p. \] \hspace{1cm} (2.9)

3. Compute $f(x_k + s_k)$ and
   \[ \rho_k = \frac{f(x_k) - f(x_k + s_k)}{T_p(x_k, 0) - T_p(x_k, s_k)}. \] \hspace{1cm} (2.10)
   If $\rho_k \geq \eta_1$, set $x_{k+1} = x_k + s_k$. Otherwise set $x_{k+1} = x_k$.

4. Set
   \[ \sigma_{k+1} \in \begin{cases} 
   \{\max(\sigma_{\text{min}}, \gamma_1 \sigma_k)\sigma_k \} & \text{if } \rho_k > \eta_2 \text{ [very successful iteration]} \\
   \{\sigma_k, \gamma_2 \sigma_k\} & \text{if } \eta_1 \leq \rho_k \leq \eta_2 \text{ [successful iteration]} \\
   \{\gamma_2 \sigma_k, \gamma_3 \sigma_k\} & \text{otherwise. [unsuccessful iteration]}, 
   \end{cases} \] \hspace{1cm} (2.11)
   and go to step 2 if $\rho_k < \eta_1$. 

\[ \]
Lemma 2.1 [2 Lem.2.4, 5 Thm.2.2]. Assume that, for some $\sigma_{\text{max}} > 0$, $\sigma_j \leq \sigma_{\text{max}}$ for all $0 \leq j \leq k$. Then the ARpCC algorithm ensures that

$$k \leq \kappa_u |S_k|,$$

where $\kappa_u \overset{\text{def}}{=} \left(1 + \frac{\log \gamma_1}{\log \gamma_2}\right) + \frac{1}{\log \gamma_2} \log \left(\frac{\sigma_{\text{max}}}{\sigma_0}\right),$ \hspace{1cm} (2.12)

where $S_k$ is the number of successful and very successful iterations, in the sense of (2.11), up to iteration $k$.

We start our worst-case analysis by formalizing our assumptions, using $\mathcal{L}(x_0) = \{x \in \mathcal{F} \mid f(x) \leq f(x_0)\}$

**AS.1** The objective function $f$ is $p$ times continuously differentiable on an open set containing $\mathcal{L}(x_0)$.

**AS.2** The $p$-th derivative of $f$ is Lipschitz continuous on $\mathcal{L}(x_0)$.

**AS.3** The feasible set $\mathcal{F}$ is closed, convex and non-empty.

The AR$p$CC algorithm is required to start from a feasible $x_0 \in \mathcal{F}$, which, together with the fact that the subproblem solution in Step 2 involves minimization over $\mathcal{F}$, leads to AS.3.

We now recall some simple results whose proof can be found in [2] in the context of the original AR$p$ algorithm.

Lemma 2.2 Suppose that AS.1 and AS.2 hold. Then, for each $k \geq 0$,

(i) \hspace{1cm} $f(x_k + s_k) \leq T_p(x_k, s_k) + \frac{L_{f,p}}{p}\|s_k\|^{p+1};$ \hspace{1cm} (2.13)

and \hspace{1cm} $\|\nabla_x f(x_k + s_k) - \nabla s T(x_k, s_k)\| \leq L_{f,p}\|s_k\|^p$; \hspace{1cm} (2.14)

(ii) \hspace{1cm} $T_p(x_k, 0) - T_p(x_k, s_k) \geq \frac{\sigma_k}{p+1}\|s_k\|^{p+1};$ \hspace{1cm} (2.15)

(iii) \hspace{1cm} $\sigma_k \leq \sigma_{\text{max}} \overset{\text{def}}{=} \max \left[\sigma_0, \gamma_3 L_{f,p}(p + 1)\right].$ \hspace{1cm} (2.16)
Proof. See [2] for the proofs of (2.13) and (2.14), which crucially depend on AS.1 and AS.2 being valid on the segment $[x_k, x_k + s_k]$. Observe also that (2.2) and (2.8) ensure (2.15). Assume now that
\[
\sigma_k \geq \frac{L_{f,p}(p+1)}{p(1-\eta_2)}.
\]
(2.17)
Using (2.13) and (2.15), we may then deduce that
\[
|\rho_k - 1| \leq \frac{|f(x_k + s_k) - T_{p}(x_k, s_k)|}{|T_{p}(x_k, 0) - T_{p}(x_k, s_k)|} \leq \frac{L_{f,p}(p+1)}{p\sigma_k} \leq 1 - \eta_2
\]
and thus that $\rho_k \geq \eta_2$. Then iteration $k$ is very successful in that $\rho_k \geq \eta_2$ and $\sigma_{k+1} \leq \sigma_k$. As a consequence, the mechanism of the algorithm ensures that (2.16) holds.

We now prove that, at successful iterations, the step at iteration $k$ must be bounded below by a multiple of the $p$-th root of the criticality measure at iteration $k + 1$.

**Lemma 2.3** Suppose that AS.1–AS.3 hold. Then
\[
\|s_k\| \geq \left(\frac{\chi_f(x_{k+1})}{2\kappa_n(L_{f,p} + \theta + \sigma_{\max})}\right)^{\frac{1}{p}} \quad \text{for all} \quad k \in S.
\]
(2.18)

**Proof.** Since $k \in S$ and by definition of the trial point, we have that $x_{k+1} = x_k + s_k$. Observe now that (2.14) and (2.16) imply that
\[
\|\nabla f(x_{k+1}) - \nabla x m_k(x_{k+1})\| \leq L_{f,p}\|s_k\|^p + \sigma_k\|s_k\|^p \leq (L_{f,p} + \sigma_{\max})\|s_k\|^p,
\]
(2.19)
and also that
\[
\chi_f(x_{k+1}) \overset{\text{def}}{=} |\langle \nabla x f(x_{k+1}), d_{k+1} \rangle| \leq |\langle \nabla x f(x_{k+1}) - \nabla x m_k(x_{k+1}), d_{k+1} \rangle| + |\langle \nabla x m_k(x_{k+1}), d_{k+1} \rangle|,
\]
(2.20)
where the first equality defines the vector $d_{k+1}$ with
\[
\|d_{k+1}\|_\chi \leq 1.
\]
(2.21)
Assume now, for the purpose of deriving a contradiction, that (2.18) fails at iteration $k \in S$. Using the Cauchy-Schwarz inequality, (2.5), (2.21), (2.19), the failure of (2.18) and the first part of (2.20) successively, we then obtain that
\[
\langle \nabla x m_k(x_{k+1}), d_{k+1} \rangle - \langle \nabla x f(x_{k+1}), d_{k+1} \rangle \leq |\langle \nabla x f(x_{k+1}), d_{k+1} \rangle - \langle \nabla x m_k(x_{k+1}), d_{k+1} \rangle| \leq \|\nabla x f(x_{k+1}) - \nabla x m_k(x_{k} + s_l)\| \|d_{k+1}\|
\]
\[
\leq \kappa_n(L_{p,f} + \sigma_{\max})\|s_k\|^p \leq \kappa_n(L_{p,f} + \theta + \sigma_{\max})\|s_k\|^p \leq \frac{1}{2}\chi_f(x_{k+1}) = -\frac{1}{2}\langle \nabla x f(x_{k+1}), d_{k+1} \rangle,
\]
(2.22)
which in turn ensures that
\[ \langle \nabla s m_k(x_{k+1}), d_{k+1}\rangle \leq \frac{1}{2} \langle \nabla f(x_{k+1}), d_{k+1}\rangle < 0. \]
Moreover, \( x_{k+1} + d_{k+1} \in F \) by definition of \( \chi f(x_{k+1}) \), and hence, using (2.21),
\[ |\langle \nabla s m_k(x_{k+1}), d_{k+1}\rangle| \leq \chi m_k(x_{k+1}). \]
(2.22)
We may then substitute this inequality in (2.20) and use the Cauchy-Schwarz inequality, (2.5) and (2.21) again to deduce that
\[ \chi f(x_{k+1}) \leq \| \nabla f(x_{k+1}) - \nabla s m_k(x_{k+1}) \| + \chi m_k(x_{k+1}) \leq \kappa_n(L_p + \alpha + \sigma_{\text{max}})\| s_k \|^p \]
(2.23)
where the last inequality results from (2.19), the identity \( x_{k+1} = x_k + s_k \) and (2.9). But this contradicts our assumption that (2.18) fails. Hence (2.18) must hold. \( \Box \)

We now consolidate the previous results by deriving a lower bound on the objective function decrease at successful iterations.

**Lemma 2.4** Suppose that AS.1–AS.3 hold. Then, if iteration \( k \) is successful,
\[ f(x_k) - f(x_{k+1}) \geq \frac{1}{\kappa^f} \chi f(x_{k+1})^{\frac{p+1}{p}} \]
where
\[ \kappa^f \overset{\text{def}}{=} \frac{p+1}{\eta_1 \sigma_{\text{min}}} \left[ 2\kappa_n(L_{f,p} + \theta + \sigma_{\text{max}}) \right]^{\frac{p+1}{p}}. \]
(2.24)

**Proof.** If iteration \( k \) is successful, we have, using (2.10), (2.15), (2.11), (2.18) and (2.16) successively, that
\[ f(x_k) - f(x_{k+1}) \geq \eta_1 \left[ T_p(x_k, 0) - T_p(x_k, s_k) \right] \]
\[ \geq \frac{\eta_1 \sigma_{\text{min}}}{p+1} \| s_k \|^{p+1} \]
\[ \geq \frac{\eta_1 \sigma_{\text{min}}}{(p+1)[2\kappa_n(L_{f,p} + \theta + \sigma_{\text{max}})]^{\frac{p+1}{p}}} \chi f(x_{k+1})^{\frac{p+1}{p}}. \]
(2.24)
It is important to note that the validity of this lemma does not depend on the history of the algorithm, but is only conditional to the smoothness assumption on the objective function holding along the step from \( x_k \) to \( x_{k+1} \). We will make use of that observation in Section 3.

Our worst-case evaluation complexity results can now be proved by combining this last result with the fact that \( \chi f(x_k) \) cannot be smaller than \( \epsilon \) before termination.
Theorem 2.5 Suppose that AS.1–AS.2 hold and let $f_{\text{low}}$ be a lower bound on $f$ on $\mathcal{F}$. Then, given $\epsilon > 0$, the ARpCC algorithm applied on problem (2.1) needs at most
\[ \left\lfloor \frac{\kappa_f f(x_0) - f_{\text{low}}}{\epsilon^{p+1}} \right\rfloor \]
successful iterations (each involving one evaluation of $f$ and its $p$ first derivatives) and at most
\[ \kappa_u \left\lfloor \frac{\kappa_f f(x_0) - f_{\text{low}}}{\epsilon^{p+1}} \right\rfloor \]
iterations in total to produce an iterate $x_\epsilon$ such that $\chi_f(x_\epsilon) \leq \epsilon$, where $\kappa_u$ is given by (2.12) with $\sigma_{\text{max}}$ defined by (2.16).

Proof. At each successful iteration, we have, using Lemma 2.4, that
\[ f(x_k) - f(x_{k+1}) \geq (\kappa_f) f(x_{k+1})^{p+1} \geq (\kappa_f) \epsilon^{p+1}, \]
where we used the fact that $\chi_f(x_{k+1}) \geq \epsilon$ before termination to deduce the last inequality. Thus we deduce that, as long as termination does not occur,
\[ f(x_0) - f(x_{k+1}) = \sum_{j \in S_k} [f(x_j) - f(x_j + s_j)] \geq \left| S_k \right| \frac{\epsilon^{p+1}}{\kappa_f}, \]
from which the desired bound on the number of successful iterations follows. Lemma 2.1 is then invoked to compute the upper bound on the total number of iterations. \qed

3 The general constrained case

We now consider the general smooth constrained problem in the form
\[ \min_{x \in \mathcal{F}} f(x) \quad \text{subject to} \quad c(x) = 0 \quad (3.1) \]
where $c : \mathbb{R}^n \to \mathbb{R}^m$ is sufficiently smooth and $f$ and $\mathcal{F}$ are as above. Note that this formulation covers the general problem involving both equality and inequality constraints, the latter being handled using slack variables and the inclusion of the associated simple bounds in the definition of $\mathcal{F}$. In order to revise our smoothness assumptions, we first define, for some parameter $\beta > 0$, the neighbourhood of the feasible set given by
\[ C_\beta = \{ x \in \mathcal{F} \mid \| c(x) \| \leq \beta \}, \]
where our revised assumptions on the objective function have to hold (we continue to assume AS.3).

AS.4 The objective function $f$ is $p$ times continuously differentiable on an open set containing $C_\beta$. 
All derivatives of $f$ of order 1 to $p$ are uniformly bounded and Lipschitz continuous in $C_\beta$.

For each $i = 1, \ldots, m$, the constraint function $c_i$ is $p$ times continuously differentiable on an open set containing $\mathcal{F}$.

All derivatives of order 1 to $p$ of each $c_i$ ($i = 1, \ldots, m$) are uniformly bounded and Lipschitz continuous in $\mathcal{F}$.

There exists constants $f_{\text{low}} \leq f_{\text{up}}$ such that $f(x) \in [f_{\text{low}}, f_{\text{up}}]$ for all $x \in \mathcal{C}_\beta$.

Note that AS.3, AS.5 and AS.7 allow us to apply the ARpCC algorithm to the problem

$$\min_{x \in \mathcal{F}} \frac{1}{2} \|c(x)\|^2.$$  \hspace{1cm} (3.2)

for any $\epsilon \leq \beta$. If an approximately feasible point is found, then, because of AS.3–AS.7, the same ARpCC may then be applied to approximately solve the problem

$$\min_{x \in \mathcal{F}} \mu(x, t_k) \overset{\text{def}}{=} \frac{1}{2} \|r(x, t_k)\|^2 \overset{\text{def}}{=} \frac{1}{2} \left\| \left( \begin{array}{c} c(x) \\ f(x) - t_k \end{array} \right) \right\|^2$$  \hspace{1cm} (3.3)

for some monotonically decreasing sequence of “targets” $t_k$ ($k = 1, \ldots$). This suggests that we might solve the problem (3.3) using a two-phase algorithm much in the spirit of that proposed by Cartis et al. [4,5,7] and Birgin et al.\cite{2}. It is described on the next page.

Observe that the recomputations of $\chi_{\mu}(x_{k+1}, t_{k+1})$ in Step 2.(b) do not require re-evaluating $f(x_{k+1})$ or $c(x_{k+1})$ or any of their derivatives.

We now start our analysis by examining the complexity of Phase 1.

**Lemma 3.1** Suppose that AS.3, AS.4 and AS.6 hold. Then Phase 1 of the ARpGC algorithm terminates after at most

$$\left\lfloor \kappa_{\text{CC}}^c \|c(x_0)\| \epsilon_p^{-\frac{1}{p}} \epsilon_D^{-\frac{p+1}{p}} \right\rfloor$$

evaluations of $c$ and its derivatives, where $\kappa_{\text{CC}}^c \overset{\text{def}}{=} \frac{1}{2} \kappa_u \kappa_s \frac{1}{2} \|c\|^2 \frac{1}{\delta} \frac{1}{p}$.

**Proof.** Let us index the iteration of the ARpCC algorithm applied on problem (3.2) by $j$. Assume that iteration $j$ is successful and that

$$\|c(x_j)\| > \epsilon_p - \epsilon_p^\frac{p+1}{p} = \epsilon_p (1 - \epsilon_p^{1/p}) \geq \delta^{-1} \epsilon_p,$$  \hspace{1cm} (3.8)
Algorithm 3.1: Adaptive Regularization using $p$-th order models for general constraints (AR$_p$GC)

A constant $\beta$ defining $C_\beta$, a starting point $x_{-1}$, a minimum regularization parameter $\sigma_{\min} > 0$, an initial regularization parameter $\sigma_0 \geq \sigma_{\min}$ are given, as well as a constant $\delta > 1$. The primal and dual tolerances

$$0 < \epsilon_p \leq \min \left[ \beta, \left( \frac{\delta - 1}{\delta} \right)^p , 1 \right] \quad \text{and} \quad 0 < \epsilon_D < 1$$

are also given.

Phase 1:

Starting from $x_0 = P_F(x_{-1})$, apply the AR$_p$CC algorithm to minimize $\frac{1}{2} \|c(x)\|^2$ subject to $x \in F$ until a point $x_1 \in F$ is found such that

$$\|c(x_1)\| \leq \epsilon_p - \epsilon_p^p \quad \text{or} \quad \chi_1^p \|c(x)\| \leq \epsilon_D \|c(x_1)\|. \quad (3.4)$$

If $\|c(x_1)\| > \epsilon_p - \epsilon_p^p$, terminate with $x_\epsilon = x_1$.

Phase 2:

1. Set $t_1 = f(x_1) - \sqrt{\epsilon_p^2 - \|c(x_1)\|^2}$.
2. For $k = 1, 2, \ldots$, do:
   (a) Starting from $x_k$, apply the AR$_p$CC algorithm to minimize $\mu(x, t_k)$ as a function of $x \in F$ until an iterate $x_{k+1} \in F$ is found such that

   $$\|r(x_{k+1}, t_k)\| \leq \epsilon_p - \epsilon_p^p \quad \text{or} \quad \chi_{k+1} \leq \epsilon_D \|c(x_1)\|. \quad (3.5)$$

   (b) i. If $\|r(x_{k+1}, t_k)\| < \epsilon_p - \epsilon_p^p$, define $t_{k+1}$ according to

   $$t_{k+1} = f(x_{k+1}) - \sqrt{\epsilon_p^2 - \|c(x_{k+1})\|^2}. \quad (3.6)$$

   and terminate with $(x_\epsilon, t_\epsilon) = (x_{k+1}, t_{k+1})$ if $\chi_{k+1, t_{k+1}} \leq \epsilon_D$.

   ii. If $\|r(x_{k+1}, t_k)\| \geq \epsilon_p - \epsilon_p^p$ and $f(x_{k+1}) < t_k$, define $t_{k+1}$ according to

   $$t_{k+1} = 2f(x_{k+1}) - t_k \quad (3.7)$$

   and terminate with $(x_\epsilon, t_\epsilon) = (x_{k+1}, t_{k+1})$ if $\chi_{k+1, t_{k+1}} \leq \epsilon_D$.

   iii. If $\|r(x_{k+1}, t_k)\| \geq \epsilon_p - \epsilon_p^p$ and $f(x_{k+1}) \geq t_k$, terminate with $(x_\epsilon, t_\epsilon) = (x_{k+1}, t_{k})$.
where the last inequality follows from the bound on $\epsilon_p$ as a function of $\delta$. Then, using the decreasing nature of the sequence $\{\|c(x_j)\|\}$, Lemma 2.4 and the second part of (3.4), we obtain that

$$(\|c(x_j)\| - \|c(x_{j+1})\|) \|c(x_j)\| \geq \frac{1}{2}\|c(x_j)\|^2 - \frac{1}{2}\|c(x_{j+1})\|^2 \geq \left(\frac{1}{\kappa_8} \frac{1}{2}\|c\|^2\right)^{-1} (\epsilon_D \|c(x_j)\|)^{\frac{p+1}{p}}$$

and thus that

$$\|c(x_j)\| - \|c(x_{j+1})\| \geq \left(\frac{1}{\kappa_8} \frac{1}{2}\|c\|^2\right)^{-1} \|c(x_j)\| \epsilon_D^{\frac{1}{p}} \epsilon_p^{\frac{p+1}{p}} \delta^{-\frac{1}{p}} \epsilon_p \epsilon_D^{\frac{p+1}{p}}$$

where we have used (3.8) to derive the last inequality. As in Theorem 2.5, we then deduce that the number of successful iterations required for the ARpCC algorithm to produce a point $x_1$ satisfying (3.4) is bounded above by

$$\frac{1}{2}\kappa_8 \frac{1}{2}\|c\|^2 \delta^{-\frac{1}{p}} \|c(x_0)\| \epsilon_D^{\frac{1}{p}} \epsilon_p \epsilon_D$$

The desired conclusion follows by using Lemma 2.1.

We now partition the Phase 2 outer iterations into two subsets whose indexes are given by

$$K_+ \overset{\text{def}}{=} \{ k \geq 0 \mid (3.6) \text{ holds} \} \quad \text{and} \quad K_- \overset{\text{def}}{=} \{ k \geq 0 \mid (3.7) \text{ holds} \}. \quad (3.9)$$

This partition allows us to prove the following technical results.

**Lemma 3.2** The sequence $\{t_k\}$ is monotonically decreasing. Moreover, in every Phase 2 iteration of the ARpGC algorithm of index $k \geq 1$, we have that

$$f(x_k) - t_k \geq 0, \quad (3.10)$$

$$\|r(x_{k+1}, t_{k+1})\| = \epsilon_p \quad \text{for} \quad k \in K_+, \quad (3.11)$$

$$\|r(x_{k+1}, t_{k+1})\| = \|r(x_{k+1}, t_k)\| \leq \epsilon_p \quad \text{for} \quad k \in K_-, \quad (3.12)$$

$$\|c(x_k)\| \leq \epsilon_p \quad \text{and} \quad f(x_k) - t_k \leq \epsilon_p, \quad (3.13)$$

$$t_k - t_{k+1} \geq \epsilon_p^{\frac{p+1}{p}} \quad \text{for} \quad k \in K_+. \quad (3.14)$$

Moreover, if AS.8 holds, then, for $k \geq 1$,

$$t_k \in [f_{\text{low}} - 1, f_{\text{up}}]. \quad (3.15)$$

Finally, at termination of the ARpGC algorithm,

$$\|r(x_\epsilon, t_\epsilon)\| \geq \epsilon_p - \epsilon_p^{\frac{p+1}{p}} \quad \text{and} \quad f(x_\epsilon) \geq t_\epsilon \quad \text{and} \quad \chi_\mu(x_\epsilon, t_\epsilon) \leq \epsilon_p \epsilon_D. \quad (3.16)$$
The inequality (3.10) follows from (3.6) for \( k - 1 \in \mathcal{K}_+ \) and from (3.7) for \( k - 1 \in \mathcal{K}_- \). (3.11) is also deduced from (3.6) while (3.7) implies the equality in (3.12), the inequality in that statement resulting from the decreasing nature of \( \|r(x, t_k)\| \) during inner iterations in Step 2.(a) of the ARpGC algorithm. The inequalities (3.13) then follow from (3.10), (3.11) and (3.12). We now prove (3.14), which only occurs when \( \|r(x_{k+1}, t_k)\| \leq \epsilon_{p} - \epsilon_{p}^{p+1} \), that is when

\[
(f(x_{k+1}) - t_k)^2 + \|c(x_{k+1})\|^2 \leq \left( \epsilon_{p} - \epsilon_{p}^{p+1} \right)^2. \tag{3.17}
\]

From (3.6), we then have that

\[
t_k - t_{k+1} = -(f(x_{k+1}) - t_k) + \sqrt{\|r(x_k, t_k)\|^2 - \|c(x_{k+1})\|^2}. \tag{3.18}
\]

Now taking into account that the global minimum of the problem

\[
\min_{(f, c) \in \mathbb{R}^2} \psi(f, c) \overset{\text{def}}{=} -f + \sqrt{\epsilon_{p}^2 - c^2} \quad \text{subject to} \quad f^2 + c^2 \leq \omega^2,
\]

for \( \omega \in [0, \epsilon_{p}] \) is attained at \((f_c, c_c) = (\omega, 0)\) and it is given by \( \psi(f_c, c_c) = \epsilon_{p} - \omega \) (see Lemma 5.2), we obtain from (3.17) and (3.18) (setting \( \omega = \epsilon_{p} - \epsilon_{p}^{p+1} \)) that

\[
t_k - t_{k+1} \geq \epsilon_{p} - \omega = \epsilon_{p}^{p+1},
\]

for \( k \in \mathcal{K}_+ \), which is (3.14). Note that, if \( k \in \mathcal{K}_- \), then we must have that \( t_k > f(x_{k+1}) \) and thus (3.7) ensures that \( t_{k+1} < t_k \). This observation and (3.14) then allow us to conclude that the sequence \( \{t_k\} \) is monotonically decreasing.

The inclusion (3.15) is deduced from Step 1 of Phase 2 of the ARpGC algorithm, the decreasing nature of the sequence \( \{t_k\} \), (3.10), (3.13) and AS.8.

In order to prove (3.16), we need to consider, in turn, each of the three possible cases where termination occurs in Step 2.(b). In the first case (i), \( \|r(x_{k+1}, t_k)\| \) is small (in the sense that the first inequality of (3.5) holds) and (3.6) is then used, implying that (3.11) holds and that \( f(x_{k+1}) > t_{k+1} \). If termination occurs because \( \chi(x_{k+1}, t_{k+1}) \leq \epsilon_{p} \epsilon_{D} \), then (3.16) clearly holds at \((x_{k+1}, t_{k+1})\). In the second case (ii), \( \|r(x_{k+1}, t_k)\| \) is large (the first inequality of (3.5) fails), but \( f(x_{k+1}) < t_k \) and \( t_{k+1} \) is then defined by (3.7), ensuring that \( f(x_{k+1}) > t_{k+1} \) and, because of (3.12), that \( \|r(x_{k+1}, t_{k+1})\| \) is also large. As before (3.16) holds at \((x_{k+1}, t_{k+1})\) if termination occurs because \( \chi(x_{k+1}, t_{k+1}) \leq \epsilon_{p} \epsilon_{D} \). The third case (iii) is when \( \|r(x_{k+1}, t_k)\| \) is sufficiently large and \( f(x_{k+1}) \geq t_k \). But (3.5) then guarantees that \( \chi(x_{k+1}, t_k) \leq \epsilon_{p} \epsilon_{D} \), and the inequalities (3.16) are again satisfied at \((x_{k+1}, t_k)\) \( \square \).

Using the results of this lemma allows us to bound the number of outer iterations in \( \mathcal{K}_+ \).

**Lemma 3.3** Suppose that AS.3, AS.4, AS.6 and AS.8 hold. Then

\[
|\mathcal{K}_+| \leq [f_{up} - f_{low} + 1] \epsilon_{p}^{-\frac{p+1}{p}}.
\]


Proof. We know from Lemma 3.2 that \( t_k \) decreases monotonically with, by (3.14), a decrease of at least \( \epsilon_p \) for \( k \in K_+ \). Hence the desired conclusion follows from (3.15). \( \square \)

We now state a very useful consequence of (3.15), which is of interest for the analysis of the inner iterations.

**Lemma 3.4** Suppose that AS.4-AS.8 hold. Then there exists a constant \( L_{\mu,p} \geq 0 \) such that the \( p \)-th derivative of \( \mu(x,t_k) \) with respect to \( x \) is Lipschitz continuous with Lipschitz constant \( L_{\mu,p} \) for all values of \( t_k \) computed by the ARpGC algorithm. Furthermore, there exists a constant \( \sigma_{\mu,\text{max}} > \sigma_{\text{min}} \) such that all regularization parameters arising in the ARpCC algorithm within Step 2.(a) of the ARpGC algorithm are bounded above by \( \sigma_{\mu,\text{max}} \).

Proof. Because of AS.4-AS.7, we obtain that, for any \( t \in [f_{\text{low}} - 1, f_{\text{up}}] \), \( \mu(x,t) \) is \( p \) times continuously differentiable in \( C_p \) and its \( p \)-th derivative is Lipschitz continuous. Moreover, since \( \mu(x,t) \) and its derivatives depend continuously on \( t \in [f_{\text{low}} - 1, f_{\text{up}}] \), we may deduce the existence of \( L_{\mu,p} \), which is an upper bound on the \( p \)-th derivative Lipschitz constant associated with each \( t \). This proves the first part of the proposition. The second is then derived by introducing \( L_{\mu,p} \) in (2.16) as specified by Lemma 2.2 to obtain \( \sigma_{\mu,\text{max}} \). \( \square \)

The main consequence of this result is that we may apply the ARpCC algorithm to the minimization of \( \mu(x,t_k) \) in Step 2.(a) of the ARpGC algorithm and use all the properties of the former (as derived in the previous section) using problem constants valid for every possible \( t_k \), because of (3.15).

Consider now \( x_k \) for \( k \in K_+ \) and denote by \( x_{k+\ell(k)} \) the next iterate such that \( k+\ell(k) \in K_+ \). Two cases are then possible (assuming termination does not occur at \( x_{k+\ell(k)} \)): either a single pass in Step 2.(a) of the ARpGC algorithm is sufficient to obtain \( x_{k+\ell(k)} \) \( (\ell(k) = 1) \) or two or more passes are necessary, with iterations \( k, \ldots, k+\ell(k)-1 \) belonging to \( K_- \). Assume now that the iterations of the ARpCC algorithm at Step 2.(a) of the outer iteration \( j \) are numbered \( (j,0), (j,1), \ldots, (j,e_j) \) and note that the mechanism of the ARpGC algorithm ensures that iteration \( (j,e_j) \) is successful for all \( j \). Now define, for \( k \in K_+ \),

\[
I_k \overset{\text{def}}{=} \{(k,0), \ldots, (k,e_k), \ldots, (j,0), \ldots, (j,e_j), \ldots, (k+\ell(k)-1,0), \ldots, (k+\ell(k)-1,e_{k+\ell(k)-1})\}
\]

(3.19)

the index set of all inner iterations necessary to deduce \( x_{k+\ell(k)} \) from \( x_k \). Observe that, by the definitions (3.9) and (3.19), the index set of all inner iterations before termination is given by \( \cup_{k \in K_+} I_k \), and therefore that the number of evaluations of problem’s functions required to terminate in Phase 2 is bounded above by

\[
| \bigcup_{k \in K_+} I_k | + 1 \leq \left( [f_{\text{up}} - f_{\text{low}} + 1] \epsilon_p \times \max_{k \in K_+} |I_k| \right) + 1,
\]

(3.20)

where we added 1 to take the final evaluation into account and where we used Lemma 3.3.
to deduce the inequality. We now invoke the complexity properties of the ARpCC algorithm applied on problem \([3.3]\) to obtain an upper bound on the cardinality of each \(I_k\).

\[
|I_k| \leq \kappa_{CC}^\mu \epsilon_p \epsilon_D^{\frac{p+1}{p}},
\]

where \(\kappa_{CC}^\mu\) is independent of \(\epsilon_p\) and \(\epsilon_D\) and captures the problem-dependent constants associated with problem \([3.3]\) for all values of \(t_k\) satisfying \([3.15]\).

**Proof.** Observe first that \([3.11]\) implies that, for each \(k\), \(\mathcal{L}(x_k) \subseteq C_{\beta}\). Hence, because of Lemma \([3.4]\) we may apply the ARpCC algorithm for the minimization of \(\mu(x, t_j)\) for each \(j\) such that \(k \leq j < k + \ell(k)\). Observe that \([3.12]\) guarantees the decreasing nature of the sequence \(\{\|r(x_j, t_j)\|\}_{j=k}^{k+\ell(k)}\) and hence of the sequence \(\{\|r(x_j, s, t_j)\|\}_{(j,s) \in \mathcal{I}_k}\). For each \(k \in \mathcal{K}_{+}\), this minimization starts from the initial value \(\frac{1}{2}\|r(x_{k,0}, t_k)\|^2 = \frac{1}{2}\epsilon^2\) and is carried out for all iterations with index in \(\mathcal{I}_k\) at worst down to the value \(\frac{1}{2}(\epsilon_p - \epsilon_D^{(p+1)/p})^2\) (see the first part of \([3.5]\)). We may then invoke Lemmas \([3.4]\) and \([2.4]\) to deduce that, if \((j, s) \in \mathcal{I}_k\) is the index of a successful inner iteration and as long as the third part of \([3.5]\) does not hold,

\[
\frac{1}{2}\|r(x_j, s, t_j)\|^2 - \frac{1}{2}\|r(x_j, s+1, t_j)\|^2 \geq \kappa_{CC}^\mu (\epsilon_p \epsilon_D) \frac{p+1}{p}, \quad \text{for } 0 \leq s < e_j, \text{ and }
\]

\[
\frac{1}{2}\|r(x_j, e_j, t_j)\|^2 - \frac{1}{2}\|r(x_{j+1,0}, t_{j+1})\|^2 \geq \kappa_{CC}^\mu (\epsilon_p \epsilon_D) \frac{p+1}{p}
\]

for some constant \(\kappa_{CC}^\mu > 0\) independent of \(\epsilon_p, \epsilon_D, s\) and \(j\). As a consequence, the number of successful iterations of the ARpCC algorithm needed to compute \(x_{k+\ell(k)}\) from \(x_k\) cannot exceed

\[
k_{CC}^\mu \left[ \frac{\epsilon_p^2 - (\epsilon_p - \epsilon_D^{(p+1)/p})^2}{2(\epsilon_p \epsilon_D)} \right] = \kappa_{CC}^\mu \left[ \frac{2\epsilon_p \epsilon_D^{(p+1)/p} - \epsilon_p^{p+1}}{2\epsilon_p^{(p+1)/p} \epsilon_D^{p+1}} \right] < \kappa_{CC}^\mu \epsilon_p \epsilon_D^{\frac{p+1}{p}}.
\]

We now use Lemma \([3.4]\) again and invoke Lemma \([2.1]\) to account for possible unsuccessful inner iterations, yielding that the total number of successful and unsuccessful iterations of the ARpCC algorithm necessary to deduce \(x_{k+\ell(k)}\) from \(x_k\) is bounded above by

\[
k_u \kappa_{CC}^\mu \epsilon_p \epsilon_D^{\frac{p+1}{p}} \triangleq \kappa_{CC}^\mu \epsilon_p \epsilon_D^{\frac{p+1}{p}}.
\]

We finally combine our results in a final theorem stating our evaluation complexity bound for the ARpGC algorithm applied on the general smooth nonlinear optimization problem.
Theorem 3.6 Suppose that AS.3–AS.8 hold. Then, for some constants \( \kappa_{CC}^c \) and \( \kappa_{CC}^\mu \) independent of \( \epsilon_P \) and \( \epsilon_D \), the ARpGC algorithm applied on problem (3.1) needs at most
\[
\left[ \kappa_{CC}^c \|c(x_0)\|^2 + \kappa_{CC}^\mu (f_{\text{up}} - f_{\text{low}} + 1) \right]^{-\frac{1}{p}} \epsilon_P^{-\frac{p+1}{p}} \epsilon_D^{-\frac{p+1}{p}}
\]
(3.21)
evaluations of \( f, c \) and their derivatives up to order \( p \) to compute a point \( x_\epsilon \) such that
\[
\|c(x_\epsilon)\| > \epsilon_P - \epsilon_P^{\frac{p+1}{p}} \quad \text{and} \quad \chi_{\frac{1}{2}||c||^2}(x_\epsilon) \leq \epsilon_D \|c(x_\epsilon)\| \quad \text{(3.22)}
\]
or
\[
\|c(x_\epsilon)\| \leq \epsilon_P \quad \text{and} \quad \chi_{\Lambda}(x_\epsilon, y_\epsilon) \leq \delta \epsilon_D \|y_\epsilon, 1\|, \quad \text{(3.23)}
\]
where \( \Lambda(x, y) \defeq f(x) + y^T c(x) \) is the Lagrangian with respect to the equality constraints and \( y_\epsilon \) is a vector of Lagrange multipliers associated with the equality constraints.

Proof. If the ARpGC algorithm terminates in Phase 1, we immediately obtain that (3.22) holds, and Lemma 3.1 then ensures that the number of evaluations of \( c \) and its derivatives cannot exceed
\[
\kappa_{CC}^c \|c(x_0)\|^{-\frac{1}{p}} \epsilon_P^{-\frac{p+1}{p}} \epsilon_D^{-\frac{p+1}{p}}. \quad \text{(3.24)}
\]
The conclusions of the theorem therefore hold in this case.

Let us now assume that termination does not occur in Phase 1. Then the ARpGC algorithm must terminate after a number of evaluations of \( f \) and \( c \) and their derivatives which is bounded above by the upper bound on the number of evaluations in Phase 1 given by (3.24) plus the bound on the number of evaluations of \( \mu \) given by (3.20) and Lemma 3.5. This yields the combined upper bound
\[
\kappa_{CC}^c \|c(x_0)\|^2 \epsilon_P^{-\frac{1}{p}} \epsilon_D^{-\frac{p+1}{p}} + \left[ \kappa_{CC}^\mu \epsilon_P \epsilon_D^{-\frac{p+1}{p}} \right] \left[ (f_{\text{up}} - f_{\text{low}} + 1) \epsilon_P^{-\frac{p+1}{p}} \right],
\]
and (3.21) follows. Remember now that (3.16) holds at termination of Phase 2, and therefore that
\[
\epsilon_P \geq \|r(x_\epsilon, t_\epsilon)\| \geq \epsilon_P - \epsilon_P^{\frac{p+1}{p}} = \epsilon_P \left( 1 - \epsilon_P^{\frac{1}{p}} \right) \geq \delta^{-1} \epsilon_P. \quad \text{(3.25)}
\]
Moreover, we also obtain from (3.16) that
\[
\chi_{\mu}(x_\epsilon, t_\epsilon) \leq \epsilon_P \epsilon_D \leq \delta \epsilon_D \|r(x_\epsilon, t_\epsilon)\|. \quad \text{(3.26)}
\]
Assume first that \( f(x_\epsilon) = t_\epsilon \). Then, using (2.4) and the definition of \( r(x, t) \), we deduce that
\[
\chi_{\frac{1}{2}||c||^2}(x_\epsilon) = \chi_{\mu}(x_\epsilon, t_k) \leq \delta \epsilon_D \|c(x_\epsilon)\|
\]
and (3.22) is again satisfied. Assume now that \( f(x_\epsilon) > t_\epsilon \) (the case where \( f(x_\epsilon) < t_\epsilon \) is excluded by (3.16)). Defining now
\[
y_\epsilon \defeq \frac{c(x_\epsilon)}{f(x_\epsilon) - t_\epsilon}.
\]
and successively using the definition of $\Lambda(x, y)$, the inequality $f(x_\varepsilon) \geq t_\varepsilon$, the linearity of $\chi_\Lambda$ for positive multiples of $\nabla x \Lambda(x, y)$ (see (2.4)), (3.26) and the definition of $r(x, t)$, we deduce that

$$
\chi_\Lambda(x_\varepsilon, y_\varepsilon) = \frac{\chi_{\mu}(x_\varepsilon, t_\varepsilon)}{f(x_\varepsilon) - t_\varepsilon} \leq \delta e^{\mu} \|r(x_\varepsilon, t_\varepsilon)\| = \delta e^{\mu} \|y_\varepsilon, 1\|.
$$

This finally implies (3.23) since $\|c(x_\varepsilon)\| \leq \|r(x_\varepsilon, t_\varepsilon)\| \leq \varepsilon_P$. \qed

Note that the bound (3.21) is $O(\varepsilon^{-p+2})$ whenever $\varepsilon_p = e_D = \varepsilon$. Note also that, because of (3.25), the first inequality in (3.22) ensures that $|c(x_\varepsilon)| \geq \delta^{-1} \varepsilon_P$.

4 Discussion

We have first shown in Section 2 that, if derivatives of the objective function up to order $p$ can be evaluated and if the $p$-th one is Lipschitz continuous, then the AR$p$CC algorithm applied to the convexly constrained problem (2.1) needs at most $O(\varepsilon^{-p+1})$ evaluations of $f$ and its derivatives to compute an $\varepsilon$-approximate first-order critical point. This worst-case bound corresponds to that obtained in [3] when $p = 2$, but with significantly weaker assumptions. Indeed, the present proposal no longer needs any assumption on the number of descent steps in the subproblem solution, the iterates are no longer assumed to remain in a bounded set and the Lipschitz continuity of the gradient is no longer necessary. That these stronger results are obtained as the result of a considerably simpler analysis is an added bonus. While we have not developed here the case (covered for $p = 2$ in [3]) where the $p$-th derivative is only known approximately (in the sense that $\nabla^p x f(x_k)$ is replaced in the model’s expression by some tensor $B_k$ such that the norm of $(\nabla^p x f(x_k) - B_k)$ applied $p - 1$ times to $s_k$ must be $O(\|s_k\|^p)$), the generalization of the present proposal to cover this situation is easy.

The proposed worst-case evaluation bound also generalizes that of [2] for unconstrained optimization to the case of set-constrained problems, under very weak assumptions on the feasible set. As was already the case for $p \leq 2$, it is remarkable that the complexity bound for the considered class of problems (which includes the standard bound constrained case) is, for all $p \geq 1$, identical in order to that of unconstrained problems.

The present framework for handling convex constraints is however not free of limitations, resulting from the choice to transfer difficulties associated with the original problem to the subproblem solution, thereby sparing precious evaluations of $f$ and its derivatives. The first is that we need to compute values of $\chi_f$ and $\chi_m$. While this is straightforward for simple convex sets such boxes, the process might be more intensive for the general case, although the $\|\cdot\|\chi$ norm may be chosen to simplify this computation. The second limitation is that the approximate solution of the subproblem may also be very expensive in terms of internal calculations (we do not consider here suitable algorithms for this purpose). Observe nevertheless that, crucially, neither the computation of the criticality measures nor the subproblem solution involve evaluating the objective function or its derivatives: despite their potential computational drawbacks, they have therefore no impact on the evaluation complexity of the original problem. Moreover, as the cost of evaluating any constraint function/derivative possibly necessary for computing $\chi_f$ and $\chi_m$ is neglected by the present approach, it must therefore be seen as a suitable framework to handle ”cheap inequality constraints” such as simple bounds.
We have also shown in Section 3 that the evaluation complexity of finding an approximate first-order scaled critical point for the general smooth nonlinear optimization problem involving both equality and inequality constraints is at most \(O(\epsilon_p^{-1/p} \epsilon_d^{-(p+1)/p})\) evaluations of the objective function, constraints and their derivatives up to order \(p\). We refer here to an "approximate scaled critical point" in that such a point is required to satisfy \((3.22)\) or \((3.23)\), where the accuracy is scaled by the size of the constraint violation or that of the Lagrange multipliers. Because this bound now only depends on the assumptions necessary to prove the evaluation complexity bound for the \(AR_pCC\) algorithm in Section 2, it therefore strengthens and generalizes that of [7] since the latter directly hinges on [3].

Interestingly, an \(O(\epsilon_p \epsilon_d^{-(p+1)/p} \min[\epsilon_d, \epsilon_p]^{-(p+1)/p})\) evaluation complexity bound was also proved by Birgin, Gardenghi, Martínez, Santos and Toint in [1] for unscaled, standard KKT conditions and in the least expensive of three cases depending on the degree of degeneracy identifiable by the algorithm. Even if the bounds for the scaled and unscaled cases coincide in order when \(\epsilon_p \leq \epsilon_d\), comparing the two results is however not straightforward. On one hand the scaled conditions take into account the possibly different scaling of the objective function and constraints. On the other hand the same scaled conditions may result in earlier termination with \((3.23)\) if the Lagrange multipliers are very large, as \((3.23)\) is then consistent with the weaker requirement of finding a John’s point. But the framework discussed in the present paper also differs from that of [1] in additional significant ways. The first is that the present one provides a potentially stronger version of the termination of the algorithm at infeasible points (in Phase 1): indeed the second part of \((3.22)\) can be interpreted as requiring that the size of the feasible linear decrease of \(\|c(x)\|\) is below \(\epsilon_d\), while [1] considers the gradient of \(\|c(x)\|^2\) instead. The second is that, if termination occurs in Phase 2 for an \(x_\epsilon\) such that \(\chi_{\|c\|^2}(x_\epsilon) = \|J(x_\epsilon)^T c(x_\epsilon)\|\) is itself of order \(\epsilon_p \epsilon_d\) (thereby covering the case where \(f(x_\epsilon) = t_k\) discussed in Theorem 3.6), then Birgin et al. show that the Łojaciewicz inequality [8] must fail for \(c\) in the limit for \(\epsilon_p\) and \(\epsilon_d\) tending to zero (see [1] for details). This observation is interesting because smooth functions satisfy the Łojaciewicz inequality under relatively weak conditions, implying that termination in these circumstances is unlikely. The same information is also obtained in [1], albeit at the price of worsening the evaluation complexity bound mentioned above by an order of magnitude in \(\epsilon_d\). We also note that the approach of [1] requires the minimization, at each iteration, of a residual whose second derivatives are discontinuous, while all functions used in the present paper are \(p\) times continuously differentiable. A final difference between the two approaches is obviously our introduction of \(\chi_A\) and \(\frac{1}{2} \|c\|^2\) in the expression of the criticality condition in Theorem 3.6 for taking the inequality constraints into account.

We conclude by recalling that parallel results are obtained in [6] using

\[\pi_f(x) = \|P_F[x - \nabla f(x)] - x]\]

as an alternative criticality measure replacing \(\chi_f(x)\).

References

[1] E. G. Birgin, J. L. Gardenghi, J. M. Martínez, S. A. Santos, and Ph. L. Toint. Evaluation complexity for nonlinear constrained optimization using unscaled KKT conditions and high-order models. SIAM Journal on Optimization, 26(2):951–967, 2016.
[2] E. G. Birgin, J. L. Gardenghi, J. M. Martínez, S. A. Santos, and Ph. L. Toint. Worst-case evaluation complexity for unconstrained nonlinear optimization using high-order regularized models. *Mathematical Programming, Series A*, 163(1):359–368, 2017.

[3] C. Cartis, N. I. M. Gould, and Ph. L. Toint. An adaptive cubic regularization algorithm for nonconvex optimization with convex constraints and its function-evaluation complexity. *IMA Journal of Numerical Analysis*, 32(4):1662–1695, 2012.

[4] C. Cartis, N. I. M. Gould, and Ph. L. Toint. On the complexity of finding first-order critical points in constrained nonlinear optimization. *Mathematical Programming, Series A*, 144(1):93–106, 2013.

[5] C. Cartis, N. I. M. Gould, and Ph. L. Toint. On the evaluation complexity of cubic regularization methods for potentially rank-deficient nonlinear least-squares problems and its relevance to constrained nonlinear optimization. *SIAM Journal on Optimization*, 23(3):1553–1574, 2013.

[6] C. Cartis, N. I. M. Gould, and Ph. L. Toint. Evaluation complexity bounds for smooth constrained nonlinear optimization using scaled KKT conditions and high-order models. Technical Report naXys-11-2015(R1), Namur Center for Complex Systems (naXys), University of Namur, Namur, Belgium, 2015.

[7] C. Cartis, N. I. M. Gould, and Ph. L. Toint. On the evaluation complexity of constrained nonlinear least-squares and general constrained nonlinear optimization using second-order methods. *SIAM Journal on Numerical Analysis*, 53(2):836–851, 2015.

[8] S. Lojasiewicz. Ensembles semi-analytiques. Technical report, Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, France, 1965. Available online at http://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf.