Vortex Dynamics: Quantum Versus Classical Regimes

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Abstract For many years the classical Hall-Vinen-Iordanski (HVI) equation has been used to analyse vortex dynamics in superfluids. Here we discuss the extension of the theory of vortex dynamics to the quantum regime, in which the characteristic vortex frequency is higher than the temperature. At the same time we justify, in the low-frequency classical regime, the use of the HVI equation, provided an inertial mass term and a noise fluctuation term are added to it. The crossover to the quantum regime is discussed, and an intuitive picture is given of the vortex dynamics, which in general is described by 2 equations (one for the vortex coordinate, and one for its quantum fluctuations); we also discuss the simple equation of motion found in the extreme quantum regime.

Keywords Superfluid vortex dynamics

1 Introduction

Very soon after the discovery of quantum vortices in superfluid $^4$He by Vinen [1], an equation of motion for a vortex was proposed by Hall and Vinen [2]; a few years later Iordanskii [3] added an extra term, to produce what is commonly known as the Hall-Vinen-Iordanski (HVI) equation of motion. The HVI equation, occasionally supplemented by an inertial mass term and by a noise fluctuation term, has been used in the last 60 years to analyze thousands of experiments in superfluids and superconductors.
However it has been controversial, and for the last 20 years a strenuous debate has
been going on over its validity. Key questions concern the value of the vortex effec-
tive mass $M_v$ (estimates range from zero to infinity [4]) and the vortex-quasiparticle
coupling coefficients $D_o(T)$, $D'_o(T)$. Indeed, Thouless et al. [5] find $D'_o(T) = 0$
for all $T$; and scattering analyses[3, 6–9] give various different results for $D_o(T)$.

In what follows we wish to shift the focus of this discussion. In our view, the key
question is how to give a proper quantum-mechanical description of the vortex dy-
namics. This turns out to require two equations of motion, one for a vortex ‘centre
of mass’ coordinate, and the other for the quantum fluctuations around this coordi-
nate [10]. A key parameter in the theory is the dimensionless ratio $\tilde{\Omega} = \hbar \Omega / kT$,
where $T$ is the temperature, and $\Omega$ the characteristic frequency of the vortex dy-
namics. In the classical limit, where $\tilde{\Omega} \to 0$, one actually recovers the HVI equation with
added inertial and noise terms [10]. While key questions remain (notably the role of
boundaries and normal fluid velocity [11], and the calculation of the high frequency
effective mass [4]), we believe that the main points at issue have now been settled in
the classical regime.

However the quantum regime remains relatively unexplored. It possesses a number
of fascinating features, both theoretical and experimental. The theory gives an illumina-
ting picture of vortex-quasiparticle interactions, classical analyses of which have
been subtle and controversial, because of the long-range nature of the interaction,
and the difficulty of accounting for vortex ‘recoil’ in the scattering. A quantum anal-
ysis immediately makes clear that a key feature of the interaction (which is properly
described as an interaction between quantum soliton and quasiparticle field excita-
tions in $2 + 1$ dimensions), is the distortion of the quasiparticle part of the superfluid
wave-function by the vortex ‘zero mode’ part of this wave-function. On the experi-
mental side the results suggest new kinds of experiment, and show that most previous
experiments have been in the classical regime.

In this paper we present a more intuitive picture of the quantum regime for vor-
tices in a Bose superfluid, highlighting the new features. On the theoretical side we
first discuss the main features of a quantum description of the vortex, and of vortex-
quasiparticle interactions. We then briefly summarize the resulting equations of mo-
tion for the vortex, and give an intuitive picture of them in real time (as opposed to
frequency space). Almost all details of the calculations are eschewed—they are quite
lengthy and will appear elsewhere [12, 13]. We then very briefly discuss how one
might experimentally probe the quantum regime in a Bose superfluid. We emphasize
that this paper is about vortices in neutral Bose superfluids—the problem of vortex
dynamics in Fermi superfluids (both charged and neutral) is rather different (and is,
we believe, still open). All the results we quote here are for the low-energy hydro-
dynamic regime, where the normal fluid density $\rho_n(T) \ll \rho$, and where the vortex
velocity $\dot{\mathbf{R}}_v$ and frequency $\Omega$ are hydrodynamic (i.e., $|\mathbf{R}_v| \ll c$, the sound velocity).
Thus, e.g., in $^4$He we assume $kT < 0.6$–0.7 K, and $\dot{\mathbf{R}}_v \ll 240$ ms$^{-1}$. Therefore we
do not treat, e.g., sound emission from fast-moving vortices, or the high-frequency
corrections to the vortex effective mass, which strictly lie outside this low-energy
regime.
2 Quantum-Mechanical Description of a Superfluid Vortex

In a classical description, a superfluid vortex is described by its position $\mathbf{R}_v(t)$ (and time derivatives of this position), and its dynamics are specified by an equation of motion for $\mathbf{R}_v(t)$.

Consider now a quantum description of a superfluid with a single vortex in it. One begins with the $N$-particle wave-function $\Psi(\{r_j\}) = \langle \{r_j\}|\Psi \rangle$, where $j = 1, 2, \ldots, N$ and $|r_j\rangle$ is a position state for the $j$-th particle; or, equivalently, from the $N$-particle density matrix $\rho_N(\{r_j\}, \{r'_j\}) = \langle \{r_j\} | \hat{\rho}_N | \{r'_j\} \rangle$, where $\hat{\rho}_N = |\Psi\rangle \langle \Psi|$. The state-vector $|\Psi\rangle$ and the wave-function $\Psi(\{r_j\})$ are assumed to satisfy Bose symmetrization over permutations of the particles. We now add a vortex soliton to the system, with circulation $q_v \kappa$, where $q_v = \pm 1, \pm 2, \ldots$, and assume that in the $N$-particle wave-function the vortex node is at a point $\mathbf{R}(t)$ in the plane. Both the $N$-particle wave-function and the total density matrix $\hat{\rho}_N$ must then depend explicitly on the parameter $\mathbf{R}(t)$. One is now free to make a change of variables to a set $\{\mathbf{R}; q_k\}$ of ‘collective coordinates’ [14, 15], with $k = 1, 2, \ldots, N - 1$, wherein the vortex coordinate, formerly just a parameter in the wave-function, is now elevated to the status of a quantum variable associated with a state vector $|\mathbf{R}\rangle$ and with a ‘zero mode’, which we discuss below. All other degrees of freedom must now be properly orthogonalized, both to the zero mode and to each other [14, 15]. The corresponding $N$-particle density matrix is written

$$\rho_N(\{\mathbf{R}, q_k\}; \{\mathbf{R}', q'_k\}) = \langle \{\mathbf{R}; q_k\} | \hat{\rho}_N | \{\mathbf{R}', q'_k\} \rangle.$$  

We also define a vortex reduced density matrix by integrating out the $\{q_k\}$:

$$\tilde{\rho}_v(\mathbf{R}, \mathbf{R}', t) = \text{Tr}_{q_k} \rho_N(\{\mathbf{R}, q_k\}; \{\mathbf{R}', q_k\})$$

in terms of which all physical quantities associated with the vortex may be defined and evaluated, provided we have sufficient information about $\hat{\rho}_N$.

One may now, by fairly well-established manoeuvres, derive a field theory for the Bose-condensed system, starting from the 1-particle reduced density matrix [16–18], denoted by $\rho_1(\mathbf{r}, \mathbf{r}', t)$. Note that this is a different object from the vortex reduced density matrix we have defined above, and is obtained in the usual way by integrating over the coordinates of all the particles in the full density matrix except for one of them (in a fully symmetrized way that takes account of the Bose statistics). As is well known, one can characterize the result in terms of a quantum phase field $\Phi(\mathbf{r}, t)$ and a density field $\rho(\mathbf{r}, t)$; they are assumed to have Bose commutation relations (and in a classical approximation, these fields are amalgamated into a macroscopic wave-function $\psi(\mathbf{r}, t) \sim \rho(\mathbf{r}, t)e^{i\Phi(\mathbf{r}, t)}$, and the commutation relations are dropped). It is common in the literature to separate out a slowly-varying ‘texture’ in the two fields, and write

$$\Phi(\mathbf{r}, t) = \Phi_s + \phi(\mathbf{r}, t)$$

$$\rho(\mathbf{r}, t) = \rho + \eta(\mathbf{r}, t)$$
where the ‘quasiparticle’ variables $\phi(r, t)$ and $\eta(r, t)$ describe fluctuations about the texture. However once we introduce a vortex into the superfluid, we have to be a little more careful. The vortex solution breaks the global translational symmetry of the superfluid, leading to a new quantum mode associated with this broken symmetry, the so-called vortex zero mode. Without specifying a particular Hamiltonian $\mathcal{H}$ for the superfluid, we can nevertheless say that the general equations that admit the vortex are:

$$ \frac{\delta \mathcal{H}}{\delta \phi} \bigg|_V = \frac{\delta \mathcal{H}}{\delta \eta} \bigg|_V = 0 $$

for a fixed vortex. The perturbed Hamiltonian in the presence of the vortex is quadratic in the phase and density variations $\phi, \eta$ and leads to the coupled equations:

$$ -\frac{\hbar}{m_0} \dot{\phi} = \frac{\delta^2 \mathcal{H}}{\delta \phi \delta \eta} \bigg|_V \phi + \frac{\delta^2 \mathcal{H}}{\delta \eta^2} \bigg|_V \eta $$

$$ -\frac{\hbar}{m_0} \dot{\eta} = \frac{\delta^2 \mathcal{H}}{\delta \phi \delta \eta} \bigg|_V \eta + \frac{\delta^2 \mathcal{H}}{\delta \phi^2} \bigg|_V \phi $$

A trivial solution to the perturbed Hamiltonian can be found by taking the gradient of the original vortex equations (4):

$$ \nabla \frac{\delta \mathcal{H}}{\delta \eta} \bigg|_V = \frac{\delta^2 \mathcal{H}}{\delta \phi \delta \eta} \bigg|_V \nabla \phi + \frac{\delta^2 \mathcal{H}}{\delta \eta^2} \bigg|_V \nabla \rho_V = 0 $$

$$ \nabla \frac{\delta \mathcal{H}}{\delta \phi} \bigg|_V = \frac{\delta^2 \mathcal{H}}{\delta \phi \delta \eta} \bigg|_V \nabla \rho_V + \frac{\delta^2 \mathcal{H}}{\delta \phi^2} \bigg|_V \nabla \phi_V = 0 $$

where $\phi_V$ and $\rho_V$ are the ‘texture’ solutions in the presence of the vortex. Comparing with the equations of motion resulting from the perturbed Hamiltonian, we see that the derivative of the original vortex profile satisfies them at zero frequency. The zero modes are then $\phi_0 = \nabla \phi_V \cdot \hat{n}, \eta_0 = \nabla \rho_V \cdot \hat{n}$ where the derivatives have been projected onto an arbitrary direction $\hat{n}$.

The zero mode can also be found by considering a small translation of the vortex. Expanding the vortex solution about a shifted position $r + \delta r$, we have:

$$ \Phi_V(r + \delta r) \approx \Phi_V(r) + \delta r \cdot \nabla \Phi_V(r) $$

$$ = \Phi_V(r) - \delta r \sin(\theta - \theta_d) \frac{1}{r} \partial_\theta \Phi_V(r) $$

$$ \rho_V(r + \delta r) \approx \rho_V(r) + \delta r \cdot \nabla \rho_V(r) $$

$$ = \rho_V(r) + \delta r \cos(\theta - \theta_d) \partial_r \rho_V(r) $$

The zero mode corresponds to the prefactor of the small translation $\delta r$, i.e., the zero mode generates translations of the vortex, and indeed, the zero mode degrees of freedom are equivalent to the vortex degrees of freedom.
There are two key observations that follow from this discussion. First, and rather obviously, we must now exclude the zero modes when defining the quasiparticle excitations. Thus, the quasiparticle wave-functions must now be redefined so as to be orthogonal at all times to the zero modes, so that as the vortex moves, the quasiparticle wave-functions must continuously adapt to the changing position of the vortex (indeed, they are excluded from the vortex core, and phase-shifted far from the vortex). Second, although there will be an interaction between the vortex and the new perturbed quasiparticles, this interaction cannot have any term linear in the quasiparticle variables. This is because the vortex soliton is itself a minimum action solution to the equations of motion, and so any fluctuations about this solution (corresponding to the perturbed quasiparticles) are at lowest order quadratic in the fluctuation variables.

These points are familiar in the discussion of quantum solitons in 1 + 1-d field theories [14, 15]; a well-known example is the quantum Sine-Gordon model. However 1 + 1-dimensional field theories are in many ways rather unique (many indeed are integrable), and quantum soliton problems in higher dimensions were sometimes thought to be intractable. In fact this is not the case [10]; however, the vortex problem does bring in some interesting new features, notably:

(i) unlike most of the interesting 1 + 1-dimensional models, the quasiparticle spectrum is gapless here. This, along with the long-range nature of the interaction between vortices and quasiparticles, emphasizes the infra-red part of their coupling—indeed we expect to find divergences in the coupling to the unperturbed quasiparticles (so that any perturbative or diagrammatic expansion in powers of this coupling will be unreliable, or even meaningless). However, as we shall see, the coupling to the perturbed quasiparticles is not IR divergent.

(ii) the perturbed quasiparticles differ from the unperturbed plane wave quasiparticles not only in the spatial form of their wave-function—they are also now chiral excitations, with angular momentum defined relative to the vortex position.

The upshot of all of this is that we must now distinguish between the original quasiparticles, described by the field variables \( \phi(r, t) \) and \( \eta(r, t) \), and the new perturbed quasiparticles. We can describe the low-energy dynamics of the system by defining these new variables as excitations about the vortex texture, i.e., we write:

\[
\Phi(r, t) = \Phi_V(r - R(t)) + \tilde{\phi}(r, t|R(t))
\]

\[
\rho(r, t) = \rho_V(r - R(t)) + \tilde{\eta}(r, t|R(t))
\]

where the notation makes clear that the quasiparticles are tied to the vortex position. We can then write \( \tilde{\phi} \) and \( \tilde{\eta} \) in cylindrical components centered at the instantaneous position \( R(t) \) of the vortex, which we write as

\[
\tilde{\phi}(r, \theta, t) = \tilde{\phi}_{lk}(r) \sin(\omega_k t + l\theta)
\]

\[
\tilde{\eta}(r, \theta, t) = -\tilde{\eta}_{lk}(r) \cos(\omega_k t + l\theta)
\]

1 Another example of a higher-dimensional quantum soliton system in which quasiparticle-soliton interactions have been treated (in this case using semi-classical methods) is the Skyrme model—see, e.g. [19–21].
We see that the time and angular dependence cannot be separated in the perturbed quasiparticles; they are now chiral modes tied to the background vortex.

From this discussion one might imagine that we can now completely forget about the original plane wave quasiparticles. In an isolated system this would indeed be the case. However in experiments one can do something rather interesting, which is to inject ‘external’ plane wave quasiparticles—in effect, one can irradiate the vortex with an external quasiparticle wind. These quasiparticles are not orthogonal to the vortex ‘zero mode’ wave-function, and they will interact linearly with it. Below, we discuss the experimental implications of this point.

3 Vortex-Quasiparticle Interaction

Formally, we may now write the expansion of the superfluid action in terms of the perturbed (tilded) quasiparticles in the form:

\[ S = \tilde{S}_v^0[R(t)] + \tilde{S}_{qp}\{[\tilde{\phi}, \tilde{\eta}] \} + \Delta S_{int}\{[\tilde{\phi}, \tilde{\eta}] \} \]  

where the zero mode is accounted for in \( R(t) \); here \( \tilde{S}_v^0[R(t)] \) is the vortex action and \( \tilde{S}_{qp}\{[\tilde{\phi}, \tilde{\eta}] \} \) is the quasiparticle action (both written in terms of perturbed quasiparticles), and \( \Delta S_{int}^{(2)}\{[\tilde{\phi}, \tilde{\eta}] \} \) is the interaction term (where the superscript indicates that it is quadratic in the perturbed quasiparticle variables). Rather than give a lengthy discussion of how \( \Delta S_{int}^{(2)}\{[\tilde{\phi}, \tilde{\eta}] \} \) is calculated, let us instead discuss the result, which can be portrayed in terms of the Feynman diagrams for the final form of the vortex-quasiparticle interaction. One may give these results either in terms of unperturbed ‘external’ quasiparticles, or in terms of the perturbed (tilded) quasiparticles—here we focus on the perturbed quasiparticles.

The key question is of course to understand the form of the interaction between the vortex and the perturbed quasiparticles, which is incorporated in \( \Delta S_{int}^{(2)}\{[\tilde{\phi}, \tilde{\eta}] \} \). To give an intuitive feel for this interaction, we sketch here the form of the effective field theory which describes it, in diagrammatic terms.

In a large system, the quasiparticle propagators are the same for perturbed or unperturbed quasiparticles. We define the quasiparticle matrix propagator \( G_{km}^\sigma(\omega) \) by

\[ G_{km}^\sigma(\omega) \left( \frac{\hbar^2 k^2}{m_0 \omega}, \frac{m_0 \omega}{\hbar^2 \chi} \right) = 1 \]  

where \( \chi \) is the superfluid compressibility. In the same way one may define a propagator for the vortex itself, starting from \( \tilde{S}_v^0[R(t)] \). Consider now the diagrams for the interaction between the quantum zero mode and the perturbed quasiparticles. One of the vertices involved is shown in Fig. 1.

The expression for the total coupling \( \Lambda_{kq}^{\sigma l} \), between the vortex and a pair of quasiparticles having momenta \( k \) and \( q \) respectively, takes the following form:

\[ \Lambda_{kq}^{\sigma l} = \int \frac{dr}{2m_0} \left[ r(\tilde{\phi}_{lk} \partial_r \tilde{\eta}_{l+\sigma,q} + \tilde{\eta}_{lk} \partial_r \tilde{\phi}_{l+\sigma,q}) + \sigma(l + \sigma)(\tilde{\phi}_{lk} \tilde{\eta}_{l+\sigma,q} + \tilde{\eta}_{lk} \tilde{\phi}_{l+\sigma,q}) \right] \]  

(15)
This interaction is zero unless $\sigma = \pm 1$: the moving vortex thus only couples the renormalized modes $\tilde{\phi}_l k$ and $\tilde{\eta}_l q$ to each other if $|l - l'| = 1$, transferring angular momentum $\hbar \sigma = \hbar (l' - l)$ between them. Because of the long-range vortex field, we focus on the ‘long-wavelength’ regime where $ka_0 \ll 1$. In this regime, only the transitions between $l = 0$ and $l = \pm 1$ contribute to linear order in $a_0 k$. Using the anti-symmetry of $\Lambda$ under exchange of the initial state $(k, l)$ and the final state $(q, l + \sigma)$, viz., $\Lambda_{kq}^{\sigma l} = -\Lambda_{qk}^{-\sigma,l+\sigma}$, we can fully express the total coupling by the $l = 0$ term, to get:

$$
\Lambda_{kq}^{\sigma 0} = \frac{k + q}{4\sqrt{kq}} \delta(k - q) + \frac{a_0}{4} \left\{ \frac{\sigma \sqrt{kq}}{k + q} \frac{k}{\sqrt{kq}} \frac{q}{k(q-k)} \right\} \text{if } k < q \\
\Lambda_{kq}^{\sigma 0} = \frac{k + q}{16\sqrt{kq}} a_0 q + \frac{\sigma}{2} \left( \frac{q}{k} \right)^2 + \frac{\sigma \sqrt{kq}}{k(q-k)} \frac{q}{k(q-k)} \right\} \text{if } q \leq k
$$

(16)

valid for $q_v = 1$.

The form of this interaction is interesting. As noted above, it is not IR divergent—this is because the perturbed quasiparticle wave-functions have adjusted to the vortex zero mode wave-function. Nevertheless it is not analytic about the zero momentum point, and part of it changes sign with the angular momentum transfer $\sigma$. Insofar as one believes the long-wavelength description of the Bose superfluid, the result is exact in the long wavelength limit.

There have of course been many attempts in the past to derive the form of the interaction between a vortex and the quasiparticles in a Bose superfluid, and it is useful to compare the result (16) with some of the forms found in previous work on this problem. These fall mainly into 2 categories. The first set of calculations attempts to calculate a scattering amplitude for quasiparticles interacting with a classical vortex potential. Such calculations automatically yield a quadratic coupling, ie., a coupling to quasiparticle pairs. However, the long-range nature of the vortex field creates infra-red divergences in this scattering amplitude, which require careful discussion [6, 22]; moreover, the potential also carries an effective Aharonov-Bohm flux [6, 9]. It actually turns out to be quite difficult to compare the details of such calculations with the results given here, mainly because (i) almost all of these calculations deal with the scattering of plane wave excitations off a static vortex (with no recoil); and (ii) the vortex itself is not treated quantum-mechanically.

A second class of calculations employs a Hamiltonian of form [23, 24]

$$
H = \frac{1}{2M_v} \left[ P - q_v A(r) \right]^2 + \sum_k \left( c_k q_k \cdot R + \frac{1}{2m_k} \left[ p_k^2 + m_k \omega_k^2 q_k^2 \right] \right)
$$

(17)
where the vector potential $\mathbf{A}$ yields a ‘field’ $\nabla \times \mathbf{A}(r) = \pi \hbar \rho_s \mathbf{\hat{z}}$, which is responsible for the Magnus force. This Hamiltonian takes the Feynman-Vernon/Caldeira-Leggett form [25, 26], with couplings $c_k$ to quasiparticle coordinates $\{q_k\}$ (having conjugate momenta $\{p_k\}$), which are linear in the $\{q_k\}$. As noted above, such a linear interaction does exist between the vortex and the unperturbed plane-wave quasiparticles. However, as we have already explained, no linear interaction to correctly orthogonalized quasiparticles can exist.

Note however that this does not stop us from writing down a Hamiltonian like (17) containing a linear interaction between the vortex and pairs of quasiparticles—such forms have been employed in other cases involving quasiparticle-soliton interactions. However the interaction now depends on the momenta of both quasiparticles, and is usually strongly temperature-dependent; considerable care is needed to evaluate it. In our view the only reliable way to carry out such a manoeuvre is to first derive the interaction between the vortex and the true orthogonalized quasiparticles, as above, and then from this derive the form of the interaction to a bath of effective oscillators.

As shown in Ref. [10], one can in fact carry through a fully non-perturbative derivation of the time dynamics of the vortex system, by dealing directly with the superfluid action, which incorporates the infra-red convergent interaction (16) between the vortex and the perturbed quasiparticle field. By then integrating out the quasiparticles one finds an equation of motion for the vortex reduced density matrix - there is no need to deal directly with the vortex scattering problem at all.

4 Equations of Motion

As we noted at the beginning of this paper, the classical description of a vortex involves an equation of motion for the classical coordinate $\mathbf{R}_v(t)$. As we will discuss below, the correct classical vortex equation of motion turns out to be

$$M_v \dddot{\mathbf{R}}_v - f_M - f_{qp} - f_{ac}(t) = f^{(cl)}_{fl}(t)$$

where $f_{ac}(t)$ is some driving force, $M_v$ is the hydrodynamic vortex mass (whose value depends on the geometry of the system [4]), $f_M = \rho_s \kappa \times (\dot{\mathbf{R}}_v - \mathbf{v}_s)$ is the Magnus force for a vortex with circulation $\kappa = \mathbf{\hat{z}} h/m$, and the quasiparticle force $f_{qp}$ is

$$f_{qp} = D_o (v_n - \dot{\mathbf{R}}_v) + D'_o \mathbf{\hat{z}} \times (v_n - \dot{\mathbf{R}}_v)$$

in which the longitudinal drag $D_o(T)$ and the transverse term $D'_o(T)$ depend strongly on the temperature $T$. The classic discussion of Iordanskii yields

$$D'_o(T) = -\kappa \rho_n(T)$$

Finally, $f^{(cl)}_{fl}(t)$ is the classical limit of a ‘fluctuational noise’ term $\mathbf{F}_{fl}(t)$, whose behaviour is defined by its correlator $\chi_{ij}(t - t', T) = \langle F^i_{fl}(t, T) F^j_{fl}(t', T) \rangle$. In the

\[M_v = \pi \rho_s a_o^2 [\ln(R_o/a_o) + \gamma_E + 1/4].\]
classical regime this correlator takes the form
\[
\chi_{ij}(t-t', T) \rightarrow \chi_{ij}^{(cl)}(t-t', T) \sim \chi_{\parallel o}(T)\delta_{ij}\delta(t-t')
\] (21)
i.e., it is entirely longitudinal, and displays Markovian white noise—we discuss the temperature dependence below.

Now Eq. (18) is just the original HVI equation [2, 3], but with an added inertial mass term and a longitudinal noise term. It is actually a local (in spacetime) equation, i.e., it can be written in the form
\[
\hat{L}(t)\dot{R}_v(t) = f(t),
\]
where \(\hat{L}(t)\) is a local differential operator acting at time \(t\), involving forces and an inertial term which act on \(R_v(t)\) at time \(t\) only. There is actually no reason why the classical dynamics need to be local—one could easily have, e.g., ‘memory’ terms in the dissipation of form \(\int dt' \Gamma_{ij}(t-t')\dot{R}_j(t')\), and quite generally one could have an equation of form \(\hat{L}(t, t')R_v(t')\), where \(\hat{L}\) is now some integrodifferential operator function of \(t\) and \(t'\). However, we will see that the correlation times in the classical regime are very short, and a local equation is a very accurate approximation to the truth.

Consider now the quantum dynamics. This has to be written in terms of an equation of motion for the reduced density matrix \(\bar{\rho}_v(R, R', t)\). Now the range of possible different forms for an equation of motion for \(\bar{\rho}_v(R, R', t)\) is very large. Quite generally we might expect the equation to be non-local in the variables \(R, R'\), and \(t\). Moreover, there is no reason to assume that we will be able to write the equation of motion in terms of simple forces, which are a classical notion of limited applicability in quantum mechanics.

It is then refreshing to find that the actual time dynamics of \(\bar{\rho}_v(R, R', t)\) do assume a fairly simple form, even in the quantum regime. For a discussion of experiments it is convenient to transform the equations of motion to the frequency domain, and we discuss this in the next section. But it is also of interest to look at them in the real time domain, which we do here. The derivation of these results is described in Refs. [10, 12, 13]. The key assumption is that we can make a Born-Oppenheimer expansion, consistent with our low-energy hydrodynamic assumption that the vortex velocity is small compared to the sound velocity in the superfluid. For the derivation of the forces acting on the vortex this expansion is perfectly well behaved, and we can thus have confidence in the results. However the derivation of the effective mass is more subtle and the Born-Oppenheimer expansion misses the ‘radiation reaction’ terms (which also exist classically). Consequently the Born-Oppenheimer expansion yields a hydrodynamic mass \(M^o_v\) for the vortex, without frequency-dependent corrections or higher time derivatives (e.g., terms proportional to \(R_v\)).

The results for the vortex dynamics can be written in terms of an equation of motion for \(\bar{\rho}_v(R, R', t)\). We define the sum and difference variables \(R_v = \frac{1}{2}(R + R')\) and \(\xi = R - R'\), so that \(R_v(t)\) denotes a ‘centre of mass’ coordinate for the vortex, and \(\xi(t)\) a ‘quantum fluctuation’ coordinate about the centre of mass coordinate. One can then write the results in terms of equations of motion for these 2 variables. For the centre of mass coordinate one gets an equation which can be written as
\[
M^o_v \ddot{R}_v(t) - f_M(\dot{R}_v) - F_{QP}[\dot{R}_v - v_n] = F_{f u l c}(t)
\] (22)
where \( f_M(\dot{R}_v) \) is again the Magnus force, and where the new quasiparticle force \( F_{QP}^R \) is now a non-local functional of the velocity of the vortex, relative to the normal velocity. The equation for \( \xi(t) \) takes a somewhat similar form:

\[
M_v \ddot{\xi}(t) - f_M(\dot{\xi}) - F_{QP}^R[\dot{\xi}(t)] = 0
\]  

(23)

where however now \( f_M(\dot{\xi}) = \rho_sq \kappa \times \dot{\xi}(t) \) (i.e., this force is like the Magnus force acting on the relative velocity \((\dot{R}_v(t) - \dot{v}_s)\), except it now acts simply on the ‘fluctuation velocity’ \(\dot{\xi}\)); where the ‘quasiparticle’ term does not depend on \(v_n\); and where there is no noise fluctuation term.

We do not have space here to look in detail at the equation of motion for \(\xi(t)\), but it is physically illuminating to look at the new quasiparticle force term \(F_{QP}^R[\dot{R}_v - \dot{v}_n]\).

To make the connection with the HVI classical force \(f_{qp}\) appearing in (19), let us write its quantum generalization as

\[
F_{QP}^R[\dot{R}_v - \dot{v}_n] = F_{\parallel}^R[\dot{R}_v - \dot{v}_n] + F_{\perp}^R[\dot{R}_v - \dot{v}_n]
\]  

(24)

where the idea is to separate out the two terms that in the classical limit lead to the drag force and the Iordanski force.

Let us consider in detail the ‘parallel’ term \(F_{\parallel}^R[\dot{R}_v - \dot{v}_n]\). It is a functional of the prior vortex velocity parallel to the normal fluid; in fact it takes the microscopic form:

\[
F_{\parallel}^R[\dot{R}_v - \dot{v}_n] = \frac{\hbar}{L_z} \sum_{m \sigma k q} \left( A_{k q}^{\sigma m} \right)^2 \Omega_{k q} (n_k - n_q) \times \int_{t_1}^t ds (\dot{R}_v(s) - \dot{v}_n) \cos[\Omega_{k q}(t - s)]
\]  

(25)

where we have assumed a quasi-2d film of thickness \(L_z\), and the frequency \(\Omega_{k q}\) is just the difference in energies between the two quasiparticles that interact with the vortex, i.e., \(\Omega_{k q} = \omega_k - \omega_q\), where we expect that in the long wavelength regime, \(\hbar \omega_k = c_s |k|\), where \(c_s\) is the sound velocity. We see that this force has just assumed a simple ‘memory’ form, and that its instantaneous value and direction depend on the previous path traced out by the vortex (more precisely, the component of the vortex velocity along that path that was parallel to \(v_n\)). However there is no requirement for \(F_{\parallel}^R[\dot{R}_v - \dot{v}_n]\) at a given time \(t\) to be parallel to \(v_n\) at the position \(R_v\) of the vortex—if \(v_n\) varies with time or with position, then this will not in general be the case.

Consider now the behaviour in time of this ‘memory term’. To do this, let us imagine a vortex following a straight line trajectory—we can then write the parallel force as

\[
F_{\parallel}^R[\dot{R}_v - \dot{v}_n] \to \int_{t_1}^t ds D_{\parallel}(t - s; T)(v_n - \dot{R}_v(s))
\]  

(26)

where the function \(D_{\parallel}(t - s; T)\) is a non-local (in time) generalization of the HVI coefficient \(D_o(T)\). Now this function turns out to depend only on the product \(kT(t - s)/\hbar\) (so that its Fourier transform, as advertised, depends only on the ratio \(\tilde{\Omega} = \hbar \tilde{\Omega} / kT\)). In Fig. 2 we show its behaviour as a function of renormalized time.
Fig. 2 The longitudinal damping time integral kernel $D_\parallel(t-s)$, shown as a function of the dimensionless variable $k_B T (t-s)/\hbar$: the coefficient decays roughly as $(t-s)^{-2}$ at long times, a relatively slow decay coming from the linear low-frequency behaviour of $D_\parallel(\Omega)$. The arrow at the origin denotes a local in time ($\delta$-function) damping contribution (Color figure online)

Suppose we write the longitudinal damping function $D_\parallel(t-s; T)$ as a sum of a local $\delta$-function contribution and a retarded term. Now at first glance, one’s intuitive expectation is that the $\delta$-function term will simply be equal to the classical $D_o(T)$ contribution. But this is not correct. Actually one finds that

$$D_\parallel(t-s, T) = \frac{1}{16} D_o(T) \delta(t-s) + \delta D_\parallel(t-s; T)$$

(27)

where the long-time tail term $\delta D_\parallel(t-s; T)$ behaves roughly as $\sim (t-s)^{-2}$. Thus as one tends to low temperatures or short times, such that $k_B T (t-s)/\hbar \ll 1$, one is left with the ‘quantum contribution’ to the longitudinal damping—but this is 16 times smaller than $D_o(T)$, and moreover, it behaves like a local term! It is only at high temperatures or long times, when one integrates over all of $D_\parallel(t-s, T)$, that the full contribution to the classical coefficient $D_o(T)$ is recovered.

From these remarks one sees that the crossover between the classical HVI equations and the fully quantum regime is going to be an interesting one. We have no space here to outline the behaviour of the other terms in the equation of motion—suffice it to say that both the transverse quasiparticle force and the fluctuation force have non-local memory terms (although the correction to the Iordanskii force in the transverse term turns out to be very small). Remarkably, once we have made the full crossover to the quantum regime (i.e., where $kT(t-s)/\hbar \ll 1$, or where $\hbar \Omega \ll kT$, one actually ends up again with a local equation of motion for the vortex, this time coming only from the $\delta$-function terms in the various memory kernels. This equation is

$$M_\nu \ddot{\mathbf{R}}_\nu - f_M - F_{qp}^{(Q)} - F_{ac}(t) = F_{fluc}^{(Q)}(t, T)$$

(28)

where the quasiparticle force $F_{QP}^{(Q)}$ is given by

$$F_{QP}^{(Q)} = \frac{1}{16} D_o(v_n - \dot{\mathbf{R}}_\nu) + D_o' \hat{\mathbf{z}} \times (v_n - \dot{\mathbf{R}}_\nu)$$

(29)
and where the fluctuation correlator is again Markovian and entirely longitudinal:

\[ \chi^{(Q)}_{ij}(t-s, T) = \frac{\zeta(5)}{4\zeta(4)} \chi_{ij}^\parallel (T) \delta_{ij} \delta(t-s) \]  

(30)

Thus the equation of motion in the extreme quantum regime has exactly the same form as the classical HVI equation, but with quite different coefficients (except for the Magnus and Iordanskii terms, which have exactly the same coefficients).

To summarize—one finds that the actual equations of motion for a quantum vortex are rather complicated except in the extreme classical regime, where they reduce to the standard HVI equation, and in the extreme quantum regime, where they reduce to equation (28). The intervening crossover regime is expected to show quite different behaviour from either of the two limiting cases.

5 Conclusions, and Remarks on Experiments

Condensed matter systems are populated by 3 different kinds of quantum excitation—extended quasiparticle modes, localized modes such as spins, or defects, and quantum solitons. It is obviously of great importance to understand how these different excitations interact, and the debate over the nature of vortex-quasiparticle interactions has assumed a central importance in the theory of superfluids over the years. In this work and in Refs. [10, 12, 13] we present what we think is a solution to this problem, obtained by extending the theory beyond the purely classical regime. We should however note some of the limitations of this work. First, it is only valid in the long wavelength limit—we ignore higher-order interquasiparticle interactions, and excitations like rotons in superfluid \(^4\)He (in \(^4\)He this confines us to \(T < 0.6–0.7\) K). It also means that we cannot deal with the crossover in the vortex flow field between the ‘near’ regions where normal fluid viscosity can be neglected, and the far region where the viscosity controls the flow. Thouless et al. [11], have shown there are subtle problems involved in this crossover, and in fact we believe that the question of the total circulating normal fluid to be found around a vortex still needs to be settled. A second limitation is the assumption of a slow vortex and a Born-Oppenheimer expansion, so that we cannot capture all terms contributing to the inertial forces on a vortex. Finally, the work here describes vortices in a Bose superfluid—vortices in Fermi superfluids have to be dealt with separately.

It is nevertheless interesting to speculate on how one might address the quantum regime in experiments. One needs high frequency vortex motion. The obvious candidates for experiments then include (i) vortex tunneling experiments (where the bounce frequency is very high) (ii) experiments on the dynamics of single vortices in 2-dimensional ‘pancake’ cold BEC gases—this problem is discussed by Cox and Stamp [27]; and (iii) very low-\(T\) experiments on turbulence (where vortex motions can be extremely rapid). The detailed theory of such experiments remains an interesting challenge.

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