Geophysical turbulence and the duality of the energy flow across scales

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The ocean and the atmosphere, and hence the climate, are governed at large scale by interactions between pressure gradient, Coriolis and buoyancy forces. This leads to a quasi-geostrophic balance in which, in a two-dimensional-like fashion, the energy injected by solar radiation, winds or tides goes to large scales in what is known as an inverse cascade. Yet, except for Ekman friction, energy dissipation and turbulent mixing occur at small scale implying the formation of such scales associated with breaking of geostrophic dynamics through wave-eddy interactions [1,2] or frontogenesis [3,4], in opposition to the inverse cascade. Can it be both at the same time? We exemplify here this dual behavior of energy with the help of three-dimensional direct numerical simulations of rotating stratified Boussinesq turbulence. We show that efficient small-scale mixing and large-scale coherence develop simultaneously in such geophysical and astrophysical flows, both with constant flux as required by theoretical arguments, thereby clearly resolving the aforementioned contradiction.

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Geophysical balance, in which nonlinearities are neglected, leads to simplified quasi-bi-dimensional behavior with energy flowing to large scales, and reduced small-scale dissipation, contrary to observations [5]: vertical mixing can decrease water density, contributing to the (upward) closing of the ocean global circulation [1]. It is identified with breaking of internal gravity waves [6], and it can potentially control the amplitude of the mesoscales.

Such flows are neither three-dimensional (3D) nor two-dimensional (2D), since at small scales, 3D eddies may prevail. Considering the system dimensionality \( D_S \) proved essential when examining critical phenomena which simplify in higher dimensions, due to more mode interactions as \( D_S \) grows. Fluid turbulence is vastly different in two or three dimensions, because of the strong constraint imposed by the new 2D invariants (such as the integrated powers of vorticity). This leads to energy flowing towards the largest scales, ending up in a condensate [7]; it can take the form of features such as jets, observed in the atmosphere of planets, or in the oceans as striations [8]. Thus, geophysical turbulence is anisotropic, quasi-2D at large scale and quasi-3D at small scale [9].

However, traditional three-dimensional homogeneous isotropic turbulence (HIT) is known to break structures (meso-scale eddies, clouds) into progressively smaller entities which will be dissipated at small scale, enhancing mixing of tracers such as pollutants [10] or biota [11]. Whereas the fate of energy in 3D is modeled through an enhanced viscosity \( \nu_{\text{turb}} > 0 \), the 2D evolution leading to large-scale structures can be related to a destabilizing transport coefficient, e.g. \( \nu_{\text{turb}} \leq 0 \). Since the direction of the cascade is known to affect the amount of energy available to irreversible processes of dissipation and mixing, it is thus an essential parameter in the overall energy budget of the atmosphere and ocean [12].

A transition from 2D to 3D in turbulence has been investigated in various contexts. For example, is there a critical dimension for which \( \nu_{\text{turb}} \) changes sign, indicative of a change of behavior in the overall flow dynamics? Using two-layer quasi-geostrophic (QG) models with bottom friction, it was shown recently that when adding, in a somewhat ad-hoc fashion, a horizontal eddy-viscosity mimicking coupling to smaller scales and thereby presumably changing locally the sign of \( \nu_{\text{turb}} \), both a direct and inverse energy cascades were obtained [13].

More formally, starting from two-point turbulence closure, space dimensionality appears through incompressibility. The critical dimension that separates 2D from 3D behavior can be computed and is found to be \( \approx 2.05 \) [14] (see also [15]). A simple model which is a local version (in modal space) of the closure equations, derived in [16], describes the energy flux to the small scales and the large scales by introducing an (unsigned) parameter which represents the ratio of inverse to direct flux,

\[
R \equiv |\epsilon/\epsilon_D| ;
\]

it is found to be a smooth monotonic function of \( D_S \), in a fashion similar to critical phenomena, thus providing a path between 2D and 3D behavior. In order to model the anisotropy of geophysical flows, one can alternatively introduce an anisotropic scale contraction/dilation. This allows to break the geostrophy constraint by considering explicitly the production of horizontal vorticity by horizontal or vertical eddies; it leads to a fractal dimension of turbulence, close to 2.55 for stratified flows [17].

Furthermore, an inverse energy cascade can also occur in 3D-HIT. On the one hand, when restricting nonlinear interactions in 3D to those between helical waves of the same polarization, energy is found to flow to large scale, with helicity (velocity-vorticity correlations) populating the small scales [18]. In reality, cross-polarization interactions dominate, but the tendency for strong inverse transfer is clearly displayed in this restricted model.

On the other hand, taking a purely 2D input of energy and a fluid with a variable aspect ratio \( A_r \), energy again has an increased tendency to flow to large scale as \( A_r \) becomes small, with a transition at \( A_r \approx 1/2 \) (\( A_r \) is defined as the ratio of the vertical resolution to the forcing...
scale) [19]. A clear dual energy cascade obtains, with $R_{10}$ a decreasing function of $A_r$. Also, inverse transfer in thick layers (with now $A_r \approx 0.78$) is observed experimentally, the suppression of vertical motions being attributed to interactions with vertical shear for eddies whose timescale is larger than the characteristic shear time [20].

These are idealized physical systems, modeling complex fluids under rather restrictive conditions. However, the link between large scales and small scales (or nonlocal interactions between Fourier modes) is embodied in coherent structures such as chlorophyll filaments [21], water vapor, ozone, temperature or salinity tracer fronts, and in magnetohydrodynamics, current sheets, plasmoids and Alfven vortices [22]. These structures have one dimension comparable to the integral scale of the flow or larger, and one close to the dissipative scale. One element altering the way such structures arise and evolve is the ideal invariants, and in particular whether or not they involve gradients. Finally, if one expects the symmetries of the primitive equations to recover at small scale, using a statistical argument based on the large number of modes, this recovery may be impeded by the presence of large-scale shear [23]. For example, direct coupling between large scales (at which the inertio-gravity waves reside) and small scales (at which turbulence resides) was demonstrated in [21], providing a progressive destruction of shear layers together with propagation, over the layer depth, of efficient mixing induced by the turbulence.

Stratified turbulence is not 2D in the traditional sense: it has strong vertical shearing [9, 25, 29], allowing for the efficient creation of small scales, as well as of large scales in the presence of rotation [30]. What is perhaps not well recognized is that the 3D Boussinesq equations, including rotation and stratification as in the atmosphere and oceans, can produce both large scale and small scale energy excitation, both with constant flux. Numerous numerical studies suffer from a lack of resolving both the large and the small eddies: because of the inherent cost of such computations, a divide-and-conquer approach has been successfully followed, analyzing either the direct or the inverse cascade itself. However, in all these studies, the smallness of the forcing wavenumber ($\approx 4$ or $5$) does not allow for a clear conclusion concerning the existence of the inverse cascade itself.

We thus now bring numerical evidence of the simultaneous generation of large-scale and small-scale flows, both with constant flux, using direct numerical simulations (DNS) of the Boussinesq equations (see Table I).

### Methods

Oceanic turbulence is studied in the idealized context of the incompressible stably stratified rotating Boussinesq primitive equations, with $\mathbf{u}$ the velocity and $\theta$ the density fluctuations in units of velocity. Solid-body rotation of strength $I_0$ (with $f = 2I_0$) is imposed in the vertical ($z$) direction with unit vector $\hat{z}$, as well as anti-aligned gravity $g$; isotropic three-dimensional forcing $\mathbf{F}$ is included; $\nabla \cdot \mathbf{u} = 0$ ensures incompressibility:

$$
\partial_t \mathbf{u} - \nu \Delta \mathbf{u} + N \theta \hat{z} + \mathbf{F} + \nabla p - f \mathbf{u} \times \hat{z} = -\mathbf{u} \cdot \nabla \theta,
$$

and

$$
\partial_t \theta - \kappa \Delta \theta - N w = -\mathbf{u} \cdot \nabla \theta,
$$

where $w$ is the vertical velocity, $p$ the pressure, $\nu$ the viscosity, and $\kappa = \nu$ the thermal diffusivity. The square Brunt-Vaisälä frequency is given by $N^2 = -(\gamma / \theta) (d \theta / dz)$, where $d \theta / dz$ is the imposed background stratification, assumed to be linear and constant. In the ideal case ($\nu = 0$, $\mathbf{F} = 0$), the total (kinetic plus potential) energy $E_V = \frac{1}{2} \langle |\mathbf{u}|^2 + \theta^2 \rangle = E_V + E_P$ is conserved and the point-wise potential vorticity $F_V = -N u_z + f \theta = \omega \cdot \nabla \theta$ is a material invariant. No modeling of small-scale dynamics is included.

The numerical code, GHOST (Geophysical High Order Suite for Turbulence), uses a pseudo-spectral method and is tri-periodic, with $n_3^3$ grid points; it is parallelized with a hybrid MPI/Open-MP method and scales linearly up to 98,000 processors for grid of up to 6144$^3$ points [30]. Forcing is introduced in the momentum equation as a random field centered in the wavenumber band $k_F \in [10, 11]$. The largest resolved scale is adimensionalized to $L_0 = 2\pi$, corresponding to a minimum wavenumber $k_{min} = 1$; the smallest resolved scale is $2\pi/k_{max} = 6\pi/n_{p}$. Initial conditions are zero for the density and random for $\mathbf{u}$.

Three dimensionless parameters characterize the flow: the Reynolds number $Re = U_0 L_F / \nu$, the Rossby number $Ro = U_0 / (L_F f)$ and the Froude number, $Fr = U_0 / (L_F N)$; $U_0$ is the rms velocity. $L_F = 2\pi/k_F$ is the forcing scale; finally, $\epsilon_{TV} \equiv dE_V/dt = -(\mathbf{u} \cdot \mathbf{F})$ is the kinetic energy injection rate. Note that in order to resolve the Ozmidov scale, at which the eddy turn-over time and $1/N$ become equal and isotropisation recoveries, one can show that $R_B \geq 1$ where $R_B = ReFr^2$ is the
Fig. 1 shows full 2D cuts of vertical velocity.

Results: Fig. 1 shows full 2D cuts of vertical velocity in the vertical and horizontal for run 15a; the forcing is roughly 1/10th of the box and one clearly observes both intense small-scale features where dissipation occurs, and organized patches significantly larger than the forcing scale, indicative of the dual flux of energy.

Results concerning scale-to-scale distribution in Fourier space are displayed in Fig. 2 for runs with N/f = 2 and 4, with the fluxes Π_{T}(k) (right) being averaged for 10 turn-over times after the peak of dissipation t_{p} ≈ 1.3, which also marks the onset of the inverse cascade. All runs listed in Table 1 display a clear direct energy cascade (k < k_{F}), with a negative flux, and with an approximate k^{-5/3} scaling [40], as expected from classical theory of two-dimensional (2D) turbulence [7, 34]. This inverse cascade to large scales in 2D was demonstrated using e.g. two-point closures of turbulence [35], or more recently high-resolution numerical simulations [36].

These runs also have a clear direct energy cascade (k > k_{F}), with a constant positive flux. Spectral indices α are defined through \hat{E}_{V}(k) ~ k^{α} where the fit is performed in the inertial range of wavenumber, k_{F} < k < k_{diss} with k_{diss} ≈ k_{max} marking the onset of the dissipation range. These exponents (see Table 1) vary between ≈ 3.99 and 4.18, the steeper the lower Re and \mathcal{R}_{B}. The shallower spectrum is close to a Kolmogorov solution α_{Kol} ≈ 5/3, expected (with small intermittency corrections) once the small scales recover isotropy for high enough \mathcal{R}_{B} (see [31] for the rotating case).

The inset in Fig. 2 gives the temporal variation of Π_{V} (solid lines) and (scaled) dissipation \mathcal{D}_{V} = 2\nu \langle \omega \rangle^{2} (dashed lines). The steady energy increase, after an initial transient, is typical of inverse cascades; The variation of the ratio of inverse to direct flux with the buoyancy Reynolds number is indicative of the increased effectiveness of turbulence as \mathcal{R}_{B} grows. One can expect this ratio to decrease as N/f increases since no inverse cascade occurs in the purely stratified case [30].

Such direct cascades of energy in rotating stratified turbulence have been analyzed using theoretical closure models of turbulence [37]. Dual cascades were also found when examining AVISO altimeter data for the Kuroshio current [13], with values of R_{B} approaching those of oceanic data for the largest imposed turbulent (horizontal) viscosity. Whereas these authors conclude to some ambiguity in the interpretation of their results due to the necessary filtering of the data, our DNS of the Boussinesq equations unambiguously show that dual energy cascades are realistic outcomes in a geophysical setting. The

buoyancy Reynolds number. Runs are performed with 2 < \mathcal{R}_{B} \leq 120 (see Table 1). Whether the Ozmidov scale is properly resolved or not may well alter the efficiency of mixing, and the properties of stratified turbulence, as advocated in [27] and as also observed here.

The right-hand sides of equations (??, ??) are used to derive the evolution of the total (kinetic + potential) energy density. Taking its Fourier transform (denoted by \hat{}, with * denoting complex conjugate) gives access to the spectral transfer which, upon integration over wavenum-
We have shown in this paper that a dual (direct and inverse) constant flux energy cascade is present in rotating stratified turbulence, thereby resolving the paradox noted by some authors (see, e.g., (13)) and thus adding credence to having both geostrophic balance and anomalous transport in geophysical turbulence. The computations clearly point out the possibility of the co-existence in the ocean and the atmosphere of idealized large-scale dynamics dominated by quasi-geostrophic motions, together with the production of small scales, essential to mixing (38).

More computations and data analysis are required to categorize in a quantitative way the mixing efficiency one can expect in such flows. For example, the variation of $R_H$ with the relevant dimensionless parameters, is an open problem which requires huge numerical as well as observational resources. Sub-grid scale modeling of small-scale dynamics may be introduced to study this phenomenon in a parametric fashion (see e.g. (39) for rotating flows). However, there are some indications of a dual flux, using quasi-geostrophy (13), or in more complex settings using a numerical oceanic model applied to the California coastal current (40). This somewhat paradoxical behavior of the energy directivity can be understood if one recalls that triadic energetic exchanges can be either positive or negative, and it is a delicate balance between the two that determines the overall sign of the flux, as also found for helical flows (18).

Physical descriptions beyond the Boussinesq equations can be used in modeling geophysical turbulence. For example, one can consider the evaporatively-driven (as opposed to radiatively driven) configurations of stratocumulus clouds, in which case the buoyancy term is altered by the existence of a threshold (in saturation mixture fraction), leading to a nonlinear equation of state. Similar phenomena may occur in the oceans, for which there is a complex set of state relations between temperature, density and salinity which may lead to distorted isopycnal surfaces. However, using the Boussinesq framework, it is clear that, beyond the energy cascades with small-scale or (exclusive) large-scale constant fluxes, other mixed solutions are found that explain how the oceanic and atmospheric systems are in quasi-geostrophic balance at large scale and yet have a sufficient production of small scales leading to enhanced mixing.

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