Distance 4 Curves on Closed Surfaces of Arbitrary Genus

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Abstract. Let $S_g$ denote a closed, orientable surface of genus $g \geq 2$ and $C(S_g)$ be the associated curve complex. The mapping class group of $S_g$, $\text{Mod}(S_g)$ acts on $C(S_g)$ by isometries. Since Dehn twists about certain curves generate $\text{Mod}(S_g)$, one can ask how Dehn twists move specific vertices in $C(S_g)$ away from themselves. We show that if two curves represent vertices at a distance 3 in $C(S_g)$ then the Dehn twist of one curve about another yields two vertices at distance 4. This produces many tractable examples of distance 4 vertices in $C(S_g)$. We also show that the minimum intersection number of any two curves at a distance 4 on $S_g$ is at most $(2g-1)^2$.

Keywords: Curve complex, Minimal intersection number, Distance 4 curves, Filling pairs of curves
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1. Introduction

Let $S_g$ denote a closed, orientable surface of genus $g \geq 2$. Throughout this article, a curve on $S_g$ will mean an essential simple closed curve on it. For two curves, $\alpha, \beta$ in $S_g$, $i(\alpha, \beta)$ denotes their geometric intersection number and $T_\alpha(\beta)$ denotes the Dehn twist of the curve $\beta$ about the curve $\alpha$. In [9], Harvey associated with $S_g$ a simplicial complex called the complex of curves which is denoted by $C(S_g)$ and defined as follows. The 0-skeleton, $C^0(S_g)$ of this complex is in one-to-one correspondence with isotopy classes of essential simple closed curves on $S_g$. Two vertices span an edge in $C(S_g)$ if and only if these vertices have mutually disjoint representatives. $C^0(S_g)$ can be equipped with a metric, $d$ by defining the distance between any two vertices to be the minimum number of edges in any edge path between them in $C(S_g)$. By the distance between two curves on $S_g$, we mean the distance between the corresponding vertices in $C(S_g)$. The minimal intersection number between any two curves on $S_g$ which are at a distance $n$ is denoted by $i_{\text{min}}(g,n)$.

Masur and Minsky, in their seminal paper [12], proved that $C(S)$ is a $\delta$-hyperbolic space with the metric $d$. Later, it was shown that the $\delta$ can be chosen to be independent of the surface $S_g$, see [1, 5, 6, 10, 14]. The coarse geometry of the curve complex has various applications to 3-manifolds, Teichmüller theory and mapping class groups. One can see [13] for many such applications.

[16, 18, 17] and [4] gave algorithms to compute distances between two vertices in $C(S_g)$. The algorithm in [4], although for closed surfaces is by far the most effective in calculating distances in $C(S_g)$. Given vertices, $\nu, \mu$ in $C(S_g)$, the algorithm works by giving an initially efficient geodesic, $\nu_0 = \nu, \nu_1, \ldots, \nu_n = \mu$ where $\nu_1$ is chosen from a list of $n^{6g-6}$ possible vertices.

In [3], Aougab and Huang gave all $\text{Mod}(S_g)$ orbits of minimally intersecting distance 3 curves in $C(S_g)$ by giving the curves as permutations which are solutions
to a system of equation in the permutation group of order $8g - 4$. They further showed that for genus, $g \geq 3$, $i_{\text{min}}(g, 3) = 2g - 1$ and that all pairs of curves which intersect at $2g - 1$ points and cuts up $S_{g\geq 3}$ into a disc are at distance 3.

For vertices at distance 4 in $C(S_g)$, we found only limited pictures (like, Figure 2) of such curves in $S_{g\leq 3}$ and none for $S_{g>3}$. In [8], the authors gave a test to determine when two vertices in $C(S_g)$ are at a distance $\geq 4$ using the efficient geodesic algorithm of [4]. Using the MICC software, they showed that $i_{\text{min}}(2, 4) = 12$ by giving all minimally intersecting pairs of curves at distance 4. They also gave examples of curves which are at a distance 4 on a surface of genus 3 and concluded that $i_{\text{min}}(3, 4) \leq 29$ (refer [8] Theorem 1.8).

Although $i_{\text{min}}(g, 4)$ is still not known, in [2], Aougab and Taylor proved that $i_{\text{min}}(g, 4) = O(g^3)$ by answering a more general question by Dan Margalit that $i_{\text{min}}(g, n) = O(g^{n-2})$.

In this article, we give a method to construct examples of vertices at distance 4 in $C(S_g)$ and improve the known upper bound of $i_{\text{min}}(g, 4)$ to $(2g - 1)^2$. In particular, using any of the minimally intersecting filling pairs of curves on $S_3$, as described in [3], one can get examples of curves at distance 4 in $C(S_3)$ which intersect at 25 points. This implies that $i_{\text{min}}(3, 4) \leq 25$. Using a minimally intersecting pair
of curves (Figure 1) described in [3] as a permutation of 28 symbols, we use our method to construct a pair of distance 4 curves in $C(S_4)$ as in Figure 2.

Such workable examples of distance 4 curves were a result of trying to investigate the effect of certain Dehn twists on distances in $C(S_g)$ as follows:

**Remark 1.** If $d(\alpha, \gamma) = 1$ or 2, then $d(\alpha, T_\gamma(\alpha)) = d(\alpha, \gamma)$.

In general, one can ask the following question:

**Question 1.** If $d(\alpha, \gamma) > 2$, then what is the relation between $d(\alpha, \gamma)$ and $d(\gamma, T_\gamma(\alpha))$?

In this article, we prove that if $d(\alpha, \gamma) \geq 3$ then $d(\alpha, T_\gamma(\alpha)) \geq 3$. We then answer the question 1 for $d(\alpha, \gamma) = 3$ and show that $d(\alpha, T_\gamma(\alpha)) = d(\alpha, \gamma) + 1$.

### 2. Setup

For any ordered index in this work, we follow cyclical ordering. For instance, if $i \in \{1, 2, \ldots, k\}$, $i = k + 1$ will indicate $i = 1$.

Two curves, $\nu_1$ and $\nu_2$ on $S_g$ are said to be a filling pair if every component of $S_g \setminus \{\nu_1, \nu_2\}$ is a disk. A component, $D$ of $S_g \setminus \{\nu_1, \nu_2\}$ is said to be an $2n$-gon if its boundary comprises of $n$ arcs of $\nu_1$ and $\nu_2$. Consider a pair, $\alpha, \beta$, of filling curves on $S_g$ with geometric intersection number, $i(\alpha, \beta) = k$. Let their set of intersection points be $\{w_1, \ldots, w_k\}$. Define a triangulation, $G$ of $S$ as follows:

Let $\{w'_1, \ldots, w'_k\}$ be the 0-skeleton of $G$. There is an edge between $w'_i$ and $w'_j$ in $G$ if and only if there is an arc of $\alpha$ or, $\beta$ between the two intersection points, $w_i, w_j$ in $S$. As $\alpha$ and $\beta$ are filling pairs, for each disk component, $D$ of $S_g \setminus \{\alpha, \beta\}$ attach a disk to the cycle formed by the edges of $G$ corresponding to the arcs of $\alpha, \beta$ that form the boundary of $D$. Using the Euler characteristic of a 2-dimensional complex, the number of faces, $f$ of $G$ is given by $f = k + 2 - 2g$.

Consider a geodesic, $v_0, \ldots, v_N$ of length $N$ in $C^0(S)$. An arc, $\omega$ in $S$ is a reference arc for the triple $v_0, \nu_1, \nu_N$ if $\omega$ and $\nu_1$ are in minimal position and the interior of $\omega$ is disjoint from $v_0 \cup v_N$. The oriented geodesic $v_0, \ldots, v_N$ is said to be initially efficient if $i(\nu_1, \omega) \leq N - 1$ for all choices of reference arc, $\omega$. The authors of [4] prove that there exists an initially efficient geodesic between any two vertices of $C(S)$.

The following theorem from [8] gives a criterion for detecting vertices in $C(S)$ at distance at-least 4.

**Theorem 1 (Theorem 1.3, [8]).** For the filling pair, $\kappa, \omega$, let $\Gamma \subset C^0(S)$ be the collection of all vertices such that the following hold:

![Figure 3](image-url)
(1) for \( \gamma \in \Gamma \), \( d(\kappa, \gamma) = 1 \); and
(2) for \( \gamma \in \Gamma \); for each segment, \( b \subset \omega \setminus \kappa \), \( i(\gamma, b) \leq 1 \).

Then \( d(\kappa, \omega) \geq 4 \) if and only if \( d(\gamma, \omega) \geq 3 \) for all \( \gamma \in \Gamma \).

Let \( \lambda \) and \( \mu \) be two simple closed curves on \( S_g \) and let \( R_\lambda \) and \( R_\mu \) be closed regular neighborhoods of \( \lambda \) and \( \mu \) respectively. We say that the 4-tuple \((\lambda, \mu, R_\lambda, R_\mu)\) is amenable to Dehn twist in special position if the following hold:

1. \( \lambda \) and \( \mu \) intersect transversely and minimally on \( S_g \).
2. \( \lambda \) and \( \mu \) fill \( S_g \).
3. the number of components of \( R_\lambda \cap R_\mu \) is equal to the number of components of \( S \setminus (\lambda \cup \mu) \) and each of these components is a disc.

When \( \lambda \) and \( \mu \) fill \( S_g \), by considering a Euclidean model of \( S_g \), it is easy to see that a 4-tuple \((\lambda, \mu, R_\lambda, R_\mu)\) amenable to Dehn twist in special position always exists.

Consider a 4-tuple \((\lambda, \mu, R_\lambda, R_\mu)\) which is amenable to Dehn twist in special position. Let \( i(\lambda, \mu) = k \) and \( K := \{1, 2, ..., k\} \). We construct a curve in the isotopy class of \( T_\lambda(\mu) \) which we call \( T_\lambda(\mu) \) in special position w.r.t. the 4-tuple \((\lambda, \mu, R_\lambda, R_\mu)\). Start at any one of the components of \( R_\lambda \cap R_\mu \) and label it as \( A_1 \).

Since \( \mu \) intersects \( \lambda \) transversely, the arc \( \mu_1 \) of \( \mu \) contained in \( A_3 \) has its endpoints \( X \) and \( Y \) on boundary arcs of \( R_\lambda \) such that \( X \) and \( Y \) lie on distinct boundary components of \( \partial R_\lambda \). We call the component of \( \partial R_\lambda \) containing \( X \) to be \( \partial_x R_\lambda \) and the other component containing \( Y \) to be \( \partial_y R_\lambda \). Equip \( A_1 \) with the Euclidean metric such that it is a square in the \( xy \)-plane. Two opposite sides of \( A_1 \) are formed from the arcs of \( \partial R_\lambda \) and the two remaining sides formed from arcs of \( \partial R_\mu \) and the \( x \)-axis lies along \( \mu_1 \) and the value of the \( x \)-coordinate increases from \( X \) to \( Y \). Orient \( \mu_1 \) from \( X \) to \( Y \). This induces an orientation on \( \mu \). Next we pick \( k \) distinct points \( \{q_1, q_2, ..., q_k\} \) in the interior of \( \mu_1 \) such that the \( x \)-coordinate of \( q_i \), \( i > j \), is greater than the \( x \) coordinate of \( q_j \), whenever \( i > j \) and \( i, j \in K \). For each \( i \in K \), let \( \lambda_i \) be a curve in \( R_\lambda \) which is isotopic to \( \lambda \) and passes through \( q_i \). Further for each \( i, j \in K, \ i \neq j \) let \( \lambda_i \) and \( \lambda_j \) be disjoint.

Orient \( \lambda_1 \) such that the \( y \)-coordinate on \( \lambda_1 \) increases when following this orientation in the disk \( A_1 \). Starting with \( A_1 \), label the subsequent disk components, \( R_3 \cap R_\mu \), as \( A_2, A_3, ..., A_4 \), in the orientation of \( \lambda_1 \). For each \( i \in K \), \( A_i \) contains a unique arc of \( \mu \) which we label as \( \mu_i \). \( \mu_i \) gets an induced orientation from \( \mu \). For each \( i \in K \), equip \( A_i \) with Euclidean metric and assume it to be a square in the \( xy \)-plane where \( \mu_i \) lies along the \( x \)-axis with the \( x \)-coordinate increasing along the orientation of \( \mu_i \). Assume \( A_i \) to be positioned such that \( \mu_i \) is the line segment joining the mid-points of the left and right sides of the square. In this orientation, call the component of \( \partial R_\mu \) which appears above \( \mu_i \) as \( \partial_x R_\mu \) and the component of \( \partial R_\mu \) below \( \mu_i \) as \( \partial_y R_\mu \). However, note that the side of \( A_i \) which is formed of the arcs of \( \partial_x R_\lambda \) could either be to the right or to the left of this square. Accordingly, the side of \( A_4 \) which is formed of the arcs of \( \partial_y R_\lambda \) could either be to the left or to the right of this square. For \( i, j \in K \), by an isotopy inside \( A_i \), we can assume that all the arcs of \( \lambda_i \) in \( A_i \) are straight lines.

For each \( i, j \in K \), let \( u_{i,j} := A_i \cap \lambda_j \cap \partial_x R_\mu \) and \( v_{i,j} := A_i \cap \lambda_j \cap \partial_y R_\mu \). Also for each \( i \in K \) let the left end point of \( \mu_i \) in the square \( A_i \) be \( u_{i,0} \) and the right end point of \( \mu_i \) in the square \( A_i \) be \( u_{i,k+1} \). Construct the Dehn twist of \( \mu \) about \( \lambda \) as follows: For each \( j \in K \), by an isotopy inside \( A_j \), we can assume that all the arcs of \( \lambda_j \) are straight lines. For each \( i, j \in K \), \( i \neq j \), draw line segments, \( \theta_{i,j} \), connecting \( v_{i,j} \) to \( u_{i,j+1} \. T_\lambda(\mu) \) is the curve

\[
((\mu \cup (\bigcup_{i \in K} \lambda_i))) \cap (S \setminus (\bigcup_{i \in K} A_i)) \cup (\bigcup_{i,j \in K} \theta_{i,j}).
\]

The schematic, figure 6 shows \( A_i \) before and after this transformation. In the complement of \( A_i \)'s the transformation described above does not disturb the curves \( \lambda_i \)'s and \( \mu \). In [7], an algorithm to obtain the Dehn twist, \( T_\lambda(\mu) \) has been described.
such that the curves in the discs of transformation are as in figure 5. The line segments in figure 4 are isotopic to the corresponding curves in 5 which shows that the above transformation indeed results in $T\lambda(\mu)$. When $T\lambda(\mu)$ is constructed as above and as shown in figure 4, we say that $T\lambda(\mu)$ is in special position w.r.t. $\lambda$ and $\mu$. We call the $k$ copies of $\lambda, \lambda_i, i \in K$, and $\mu$ to be the scaffolding for $T\lambda(\mu)$. We call the Euclidean disks $A_i, i \in K$, along with the line segments $\theta_{i,j}$'s for $j \in K$ to be the disks of transformation for $T\lambda(\mu)$. The points $u_{i,j}$'s, $v_{i,j}$'s, $u_{i,k+1}$ and $v_{i,0}$ for $i, j \in K$ shall hold their meaning as defined in the context of the disks of transformations. So, using these phrases, when $T\lambda(\mu)$ is in special position w.r.t. $\lambda$ and $\mu$, the scaffolding of $T\lambda(\mu)$ remains unchanged outside its disks of transformation. Inside the disks of transformation for $T\lambda(\mu)$, the schematic in figure 4 describes the changes to its scaffolding.

3. Distance 4 curves in $C(S_{g \geq 2})$

**Theorem 2.** Let $S$ be a surface of genus $g \geq 2$. Let $\alpha$ and $\gamma$ be two curves on $S$ with $d(\alpha, \gamma) = 3$. Then, $d(T\gamma(\alpha), \alpha) = 4$.

**Proof.** Let $\nu_0, \nu_1, \nu_2, \nu_3$ be a geodesic from the vertex $\nu_0$ corresponding to $\alpha$ to the vertex $\nu_3$ corresponding to $\gamma$ in $C(S)$. Let $T\gamma(\nu_0)$ be the vertex in $C(S)$ corresponding to $T\gamma(\alpha)$. The existence of the path $T\gamma(\nu_0), T\gamma(\nu_1), T\gamma(\nu_2) = \nu_2, \nu_1$, $\nu_0$ gives that $d(T\gamma(\alpha), \alpha) \leq 4$. We prove that $d(T\gamma(\alpha), \alpha) \geq 4$ by using Theorem 1 with $\kappa = T\gamma(\alpha)$ and $\omega = \alpha$, hence showing that $d(T\gamma(\alpha), \alpha) = 4$.

**Claim 1.** $T\gamma(\alpha)$ and $\alpha$ fill $S$.

**Proof of claim 2** Let $i(\gamma, \alpha) = k, K := \{1, 2, \ldots, k\}, K_{-1} := \{1, 2, \ldots, k-1\}$ and $K_{2-2g} := \{1, 2, \ldots, k+2-2g\}$. We refer to section 2 for the terminology used here. Since $\alpha$ and $\gamma$ fill $S$, there is a 4-tuple $(\alpha, \gamma, R_0, R_\gamma)$ which is amenable to
Dehn twist in special position. Let \( T_\gamma(\alpha) \) be in special position w.r.t to \( \alpha \) and \( \gamma \). We denote the disks of transformation of \( T_\gamma(\alpha) \) by \( A_i \) for \( i \in K \). By an isotopy we assume the curve \( \alpha \) to be disjoint from \( T_\gamma(\alpha) \setminus A_i \) for \( i \in K \) and in each \( A_i \) we further assume the arc \( \alpha_i := \alpha \cap A_i \) to be a straight line segment below the segment connecting \( v_{i,0} \) and \( u_{i,k+1} \).

For \( i \in K \), let \( g_i \) along with \( \alpha \) be the scaffolding for \( T_\gamma(\alpha) \). For \( j \in K-1 \), one of the components of \( S \setminus \{g_j, g_{j+1}\} \) is an annulus, \( G_j \). Any component of \( G_j \setminus \alpha \) is a 4-gon which we call as a rectangle of the scaffolding for \( T_\gamma(\alpha) \). Figure 8 shows an example of such a rectangle of the scaffolding. The disks \( A_i \), \( i \in K \), further divide each rectangle of the scaffolding into three components. There is a unique \( i \in K \) such that \( A_i \) and \( A_{i+1} \) intersect a given rectangle of the scaffolding. Denote a rectangle of the scaffolding formed out of \( G_j \) with its arcs of \( \alpha \) lying in \( A_i \) and \( A_{i+1} \) by \( B_{i,j} \). Denote the sub-rectangles \( B_{i,j} \cap A_i \), by \( C'_{i,j} \) and \( B_{i,j} \cap A_{i+1} \), by \( C''_{i+1,j} \). Also let \( B_{i,j} := B_{i,j} \setminus (C'_{i,j} \cup C''_{i+1,j}) \). Let

\[ B = \bigcup_{i=1}^{K} \bigcup_{j=1}^{K-1} B_{i,j}. \]

\( S \setminus (\alpha \cup \gamma) \) has \( k+2-2g \) disk components by Euler characteristic considerations. If \( F_p \) is a disk component of \( S \setminus (\alpha \cup \gamma) \), for some \( p \in K_{2-2g} \), then \( F_p' := F_p \setminus B \) is a single disk as \( B \) intersects any \( F_p \) only in disks which contain a boundary arc of \( F_p \), namely arcs of \( \gamma \). The components of \( S \setminus (\alpha \cup g_1 \cup \cdots \cup g_k) \) comprise of \( k(k-1) \) rectangles of the scaffolding for \( T_\gamma(\alpha) \), namely \( B_{i,j} \) where \( i \in K \), \( j \in K-1 \), and \( k+2-2g \) even sided polygonal discs, namely \( F_p' \), where \( p \in K_{2-2g} \). Let \( F'_p \) denote \( F_p' \setminus R_\alpha \) for \( p \in K_{2-2g} \).

For each \( j \in K \) let \( w_{i,j} := \theta_{i,j} \cap \alpha_i \). For each \( i \in K \) and \( j \in K-1 \), let \( D''_{i,j} \) be the parallelogram with vertices \( v_{i,j}, v_{i,j+1}, w_{i,j} \) and \( w_{i,j+1} \) and \( D'_{i,j+1} \) be the parallelogram with vertices \( w_{i,j}, w_{i,j+1}, u_{i,j+1}, u_{i,j+2} \). In each disk \( A_i \), for \( i \in K \), there is a pentagon, \( P_{i,1} \), which is above \( \alpha_i \) and bounded by the lines \( \theta_{i,0} \), \( \partial R_\gamma \), \( \alpha_i \), \( \theta_{i,1} \) and the line segment of \( \partial R_\gamma \) between \( u_{i,1} \) and \( u_{i,2} \). Likewise, in each disk \( A_i \), for \( i \in K \), there is a triangle, \( T_{i,k+1} \), which is bounded by the lines \( \alpha_i \), \( \theta_{i,k} \) and \( \partial R_\gamma \). Figure 7 shows a schematic before and after the transformation to the disk \( A_i \); the figure to the left shows the rectangles \( C'_{i,1} \) and \( C''_{i,k} \) and the figure on the right shows \( P_{i,1} \) and \( T_{i,k+1} \).

Figure 8 shows a schematic of \( R_\alpha \) before and after the transformation to the scaffolding of \( T_\gamma(\alpha) \). The shaded region in the figure on the left shows \( C'_{i,j} \) and \( C''_{i,j-1} \) for some indices \( i, j \). The shaded region in the figure on the right shows \( D'_{i,j} \) and \( D''_{i,j-1} \) for some indices \( i, j \).
DISTANCE 4 CURVES ON CLOSED SURFACES OF ARBITRARY GENUS

Figure 7 The disk of transformation for $T_\gamma(\alpha)$: the figure on the left shows the portion of the scaffolding for $T_\gamma(\alpha)$; the figure on the right shows the pentagon $P_{i,1}$, the triangle $T_{i,k+1}$ and the parallelograms formed due to $\alpha_i$ and $T_\gamma(\alpha)$.

Figure 8 A schematic of $R_\alpha$ (figure on the left) and after (figure on the right) the Dehn twist.

Figure 9 Two adjacent disks of transformation in $R_\alpha$.

For $i \in K$, note that all the disks $A_i$, occur in some sequence in the annulus $R_\alpha$ when moving along $\alpha$. So, a disk $A_i$ is connected to some disk $A_j$ on the left and to some other disk $A_p$ on the right by a single arc of $\alpha \setminus R_\gamma$, for some distinct indices $i, j, p \in K$. The schematic for two disks $A_i$ and $A_j$, for some $i, j \in K$, which are connected via a single arc of $\alpha \setminus R_\gamma$ and an arc of $T_\gamma(\alpha) \setminus R_\gamma$ is as shown in the figure 9. Note that this schematic is generic since for every $j \in K$, there is a distinct $i \in K$ such that $A_j$ occurs to the left of $A_i$, in the sense mentioned above.

Figure 9 is a schematic of a portion of figure 8. In any of the cases, viz. $\partial_+ R_\gamma$ and $\partial_+ R_\gamma$ face each other, $\partial_+ R_\gamma$ and $\partial_- R_\gamma$ face each other or $\partial_- R_\gamma$ and $\partial_- R_\gamma$ face each other. In this schematic, we see that the pentagon $P_{i,1}$ of the disk $A_i$ is connected to the triangle $T_{j,k+1}$ of $A_j$ via an arc of $\alpha \setminus R_\gamma$, $\omega_{i,j}$, and an arc of $T_\gamma(\alpha)$, $\eta_{i,j}$. The disk, $R_{i,j}$ outside $R_\gamma$ bounded by $\omega_{i,j}$, $\eta_{i,j}$ and two arcs of $\partial R_\gamma$, will be called a conduit. Equip the conduit with the Euclidean metric and assume that $R_{i,j}$ is a rectangle with two opposite sides $\omega_{i,j}$ and $\eta_{i,j}$. Now $P_{i,1} \cup R_{i,j} \cup T_{j,k+1}$ is
a 4-gon bounded by four arcs viz. (i) \( \theta_{i,0} \cup \eta_{i,j} \cup \theta_{j,k} \), (ii) \( \alpha_j \cup \omega_{i,j} \cup \alpha_i \), (iii) \( \theta_{i,1} \) and (iv) the arc of \( \partial_1 R_0 \) between \( u_{i,1} \) and \( u_{i,2} \). This protracted 4-gon will be denoted by \( D'_{i,1} \).

Let \( S' = S \setminus R_0 \). The components of \( S' \setminus (\alpha \cup T_{\gamma}(\alpha)) \) are the components of \( S' \setminus T_1(\alpha) \) and the components of \( R_0 \setminus (\alpha \cup T_{\gamma}(\alpha)) \) glued at the boundary of \( R_0 \). Since the changes to the scaffolding of \( T_{\gamma}(\alpha) \) is restricted to \( R_0 \), the components of \( S' \setminus T_{\gamma}(\alpha) \) are precisely the disc components of \( S' \setminus (g_1 \cup \cdots \cup g_k) \).

The components of \( S' \setminus (g_1 \cup \cdots \cup g_k) \) are \( B'_{i,j}, i \in \{1, 2 \ldots k \}, j \in K-1 \), along with disks \( F''_p, p \in K_2-(2q) \), as explained above. The components of \( R_0 \setminus (\alpha \cup T_{\gamma}(\alpha)) \) will be examined using the schematic figure of a portion of \( R_0 \). There are four kinds of regions in \( R_0 \). The upper disk regions, like \( R_1 \) in the schematic figure, the lower disk regions, like \( R_2 \) in the schematic figure and the disks \( D'', D''_{i,j} \), \( i \in K, j \in K-1 \). Figure shows how the upper and lower disk regions are glued to disks \( F''_p \) for \( p \in K_2-(2q) \). For each \( p \in K_2-(2q) \), after gluing the lower disk regions and the upper disk regions to the respective disks \( F''_p \), we get disks which we denote by \( F'''_p \). We know that \( F'''_p \) is a disk because the upper and the lower disk regions are disjoint, except for the points \( u_{i,j} \) on the boundary and share a single arc of \( \partial R_0 \) with a unique \( F''_p \). For each \( p \in K_2-(2q) \), we call \( F'''_p \) to be the modified disk corresponding to the initial disk \( F_p \).

For each \( i \in K \) and \( j \in K-1 \), the line segment of \( \partial R_0 \) between \( u_{i,j} \) \( u_{i,j+1} \) is the common boundary of \( C_{i,j} \) and \( D'_{i,j} \). Likewise, for each such \( i, j \), the line segment of \( \partial R_0 \) between \( v_{i,j} \) \( v_{i,j+1} \) is the common boundary of \( C''_{i,j} \) and \( D''_{i,j} \). So, for such \( i, j \), when considering the components of \( S \setminus \{g_1 \cup \cdots \cup g_k\} \), the rectangular core \( B''_{i,j} \) is connected to \( C''_{i,j} \) along the boundary segment \( u_{i,j} \) \( u_{i,j+1} \) and to \( C''_{i,j} \) along the boundary segment \( v_{i,j} \) \( v_{i,j+1} \), whereas when considering the components of \( S \setminus (\alpha \cup T_{\gamma}(\alpha)) \), the rectangular core \( B''_{i,j} \) is connected to \( D''_{i,j} \) along the boundary segment \( u_{i,j} \) \( u_{i,j+1} \) and \( D''_{i,j} \) along the boundary segment \( v_{i,j} \) \( v_{i,j+1} \). So the rectangles of the scaffolding for \( T_{\gamma}(\alpha) \), \( B_{i,j} \), which are components of \( S \setminus \{g_1 \cup \cdots \cup g_k\} \), after the transformation in the disks of transformation for \( T_{\gamma}(\alpha) \) result in disks \( E_{i,j} = B''_{i,j} \cup D''_{i,j} \cup D''_{i,j+1} \) which are components of \( S \setminus (\alpha \cup T_{\gamma}(\alpha)) \). For each \( p \in K_2-(2q) \), \( F'''_p \) is a disk as seen earlier. The components of \( S \setminus (\alpha \cup T_{\gamma}(\alpha)) \) are precisely the disks \( F'''_p \) and \( E_{i,j} \) where \( p \in K_2-(2q), i \in K \) and \( j \in K-1 \). This proves that the components of \( S \setminus (\alpha \cup T_{\gamma}(\alpha)) \) are all disks and hence proving Claim.

The components of \( R_\gamma \setminus T_{\gamma}(\alpha) \) are disks and their boundary consists of two arc segments of \( T_{\gamma}(\alpha) \) and one each of \( \partial_1 R_0 \) and \( \partial_2 R_0 \). We call these disks as rectangular tracks. The word tracks derives its motivation from how these tracks appear in \( R_\gamma \). Figure shows \( R_\gamma \) and rectangular tracks inside \( R_\gamma \).

Since \( i(\alpha, \gamma) = k \), there are \( k \) components of \( \alpha \cap R_\gamma \). Every component of \( \alpha \setminus T_{\gamma}(\alpha) \) is either contained in \( R_\gamma \) or, has a sub-arc which is contained in \( R_\gamma \). For any \( i \in K, \alpha_i \) intersects the rectangular tracks.

Let \( i_0 \in K \). In the schematic figure, \( A_{i_0} \) has exactly \( k + 1 \) arcs of \( T_{\gamma}(\alpha) \). Call \( \theta_{i_0,0} \) to be the leftmost arc of \( A_{i_0} \) and \( \theta_{i_0,k} \) to be the rightmost arc of \( A_{i_0} \). Let us consider one component of \( T_{\gamma}(\alpha) \cap R_\gamma \), call it \( \rho_{i_0} \), which intersects \( A_{i_0} \) in its leftmost arc. This \( \rho_{i_0} \) intersects \( A_{i_0} \) precisely in the arcs \( \theta_{i_0,0} \) and \( \theta_{i_0,k} \) and it intersects \( A_j \) for every \( j \in K \setminus \{i_0\} \) in the arcs \( \theta_{j,m} \) where \( m = (j - i_0)(\text{mod } k) \). This is easily seen from the construction of \( T_{\gamma}(\alpha) \) in special position w.r.t. \( \alpha \) and \( \gamma \). From this discussion it is clear that \( \rho_{i_0} \) intersects each \( \alpha_j \), for \( j \in K \), exactly once. It is also clear that, for \( j \in K \), the points of \( \rho_{i_0} \cap \alpha_j \) lie on \( \rho_{i_0} \), in the order \( \alpha_{i_0+1}, \ldots, \alpha_k, \alpha_1, \ldots, \alpha_{i_0-1}, \alpha_{i_0} \) when \( \rho_{i_0} \) is traversed from \( \partial_1 R_0 \) to \( \partial_2 R_0 \). We now consider two arc components, \( \rho_{i_0} \) and \( \rho_{i_0+1} \), of \( T_{\gamma}(\alpha) \cap R_\gamma \) and the rectangular track, \( T_{i_0} \), which is enclosed by these two components in \( R_\gamma \). We equip this rectangular
Figure 10 The rectangular tracks shown inside the annulus $R_γ$

Figure 11 A rectangular track $T_i$ along with arcs of $α_i$ in it

Figure 12 $A_i$ shown inside $R_γ$ in the two possible ways: the figure on the left shows $α_i$ oriented from top to bottom; the figure on the right shows $α_i$ oriented from bottom to top

tracks $T_{i0}$ with the Euclidean metric so that the boundary arcs $ρ_{i0}$, $ρ_{i0}+1$, and the arcs of $T_{i0} \cap \partial R_γ$ are all straight lines and so that $T_{i0}$ is a rectangle. We refer to $T_{i0} \cap \partial R_γ$ as the left end of the rectangle and $T_{i0} \cap \partial_+ R_γ$ as the right end of this rectangular track. We can draw the arcs of $α_j$, for $j \in K$, as straight line segments in the rectangular tracks $T_{i0}$. Figure [11] shows a schematic of $T_i$ where $i \in K$.

From this schematic, at both the left and right end of this rectangular track $T_i$, $a_i$ is a common boundary to a triangle and a pentagon. We call $α_i$ the starting arc of this rectangular track $T_i$.

Figure [12] shows two possible schematics when $A_i$ is pictured in $R_γ$.

For any of the two possible cases observed in figure [12] a portion of one of the two pentagons of $T_i$ appears in the $A_i$ which is between $α_i$ and $∂_+ R_α$, where $α_i$ is the starting arc of this track. We call this pentagon the upper pentagon of the
rectangular track $T_i$, owing to the viewpoint that $\partial_j R_\alpha$ is the upper boundary of $R_\alpha$. A portion of the other pentagon of $T_i$ appears in $A_i$ which is between $\alpha_i$ and $\partial_j R_\alpha$. We call this pentagon the lower pentagon of the rectangular track. Likewise, we define the upper triangle and the lower triangle of a rectangular track $T_i$.

Let $\gamma \in \Gamma$ as in the statement of the Theorem. We prove that $d(\gamma, \alpha) \geq 3$ by showing that $\gamma$ and $\alpha$ fill $S$. By Theorem 4 this will imply that $d(T_\gamma(\alpha), \alpha) \geq 4$.

It can be observed that $i(\gamma \cap \alpha) \neq 0$ because if $\gamma$ is disjoint from both $\alpha$ and $T_\gamma(\alpha)$ then $\gamma$ is non-essential as it will lie completely in one of the disc components of $S \setminus (T_\gamma(\alpha) \cup \alpha)$. Since $d(\gamma, \gamma) \geq d(T_\gamma(\alpha), \gamma) - d(\gamma, T_\gamma(\alpha)) = d(T_\gamma(\alpha), T_\gamma(\gamma)) - d(\gamma, T_\gamma(\alpha)) = 3 - 1 = 2$, we also conclude that $i(\gamma \cap \gamma) \neq 0$. Since $\gamma$ intersects $\gamma$, it intersects $R_\alpha$. It cannot be completely contained in $R_\alpha$ because every simple closed curve contained in an annulus bounds a disk or is isotopic to the core curve of the annulus. Since neither of these is true, it follows that $\gamma$ intersects $R_\alpha$ in arcs. Since $i(\gamma, T_\gamma(\alpha)) = \phi$, each component of $\gamma \cap R_\alpha$ has to be completely contained in one of the rectangular tracks described by $T_\gamma(\alpha)$. Such a component arc of $\gamma$ could either be boundary reducible or essential in $R_\alpha$.

We consider an isotopy $I_1$ of $\gamma$, as follows: In the case that a component arc of $\gamma$ in $R_\alpha$ is boundary reducible in $R_\alpha$, we can perform the boundary reduction of $\gamma$ preserving its minimal intersection position with $\alpha$ and $T_\gamma(\alpha)$. This is possible because an arc of $\gamma$ which is boundary reducible in $R_\alpha$, and is contained in the disk $T_i$ will bound a bigon with one boundary arc of $R_\alpha$ in $T_i$. Also, since $\gamma$ was already in minimal intersection position with $\alpha$, it does not bound bigons with the arcs $\alpha_j$ inside $T_i$. Call the isotopy of $\gamma$ which reduces all the boundary-reducible arcs of $\gamma \cap R_\alpha$ as $I_1$. After the isotopy $I_1$, we can assume that all the arcs of $\gamma$ in $R_\alpha$ are essential. We know that there is at least one component of $\gamma \cap R_\gamma$ which is an essential arc of $R_\alpha$, as $\gamma$ cannot be disjoint from $R_\alpha$. By the hypothesis that $i(\gamma, b) \leq 1$ for $b \subset \alpha \setminus T_\gamma(\alpha)$ each rectangular track can contain at most one component of $\gamma \cap R_\gamma$.

Next, we describe an isotopy $I_2$ of $\gamma$ such that all the points of $\gamma \cap \alpha$ will lie inside $R_\alpha$ and so that no new boundary reducible arc components of $\gamma \cap R_\alpha$ are introduced and $\gamma$’s minimal intersection position with $\alpha$ and $T_\gamma(\alpha)$ is retained. To this end, suppose that a point of $\gamma \cap \alpha$ lies outside $R_\alpha$.

Following the construction of the disk $D_{i,j}$ described above using figure 9, we see that the upper pentagon of the rectangular track $T_i$ is connected to the upper triangle of the rectangular track $T_j$ via a conduit $R_{i,j}$ where $i, j \in K$ are such that $A_j$ is to the left of $A_i$ in $R_\alpha$ as in schematic 9.

If a point of $\gamma \cap \alpha$, $x_0$, lies outside $R_\alpha$, then it has to lie on $\omega_{i,j}$ for some $i$ and $j$ such that $i, j \in K$, $i 

Since the intersection of $\gamma$ and $\alpha$ is transverse, an arc of $\gamma$, call it $\delta$, lies on the two sides of the conduit $R_{i,j}$, one inside and one outside $R_{i,j}$. The endpoint $P$ of the arc $\delta$ inside $R_{i,j}$ is also the endpoint of some other arc of $\gamma$ as $\gamma$ is a closed curve. If $P$ connects to an arc of $\gamma$ lying in the upper triangular region of the track $T_j$, then an essential arc $\delta_1$ of $\gamma \cap R_\alpha$ lies in $T_j$ with its endpoint $Q$ on $\partial R_\alpha$ in the upper triangle of $T_j$ so that $\delta$, the arc $PQ$ and $\delta_1$ together form a bigon with $\alpha$ contradicting the minimal intersection position of $\gamma$ with $\alpha$. So, $P$ connects to an arc of $\gamma$ in the upper pentagon in the track $T_i$ as is the dotted line in figure 13. Consider an isotopy $I_2$ which slides the point $x_0$ onto $\alpha_i$. The image of the arc component of $\gamma \cap R_\alpha$ which is in $T_i$, under $I_2$ has its endpoint in the lower triangle of $T_i$ and the image of $x_0$ lies in $R_\alpha$. A schematic for this isotopy $I_2$ is shown in figure 13.

After finitely many such isotopies, we can now assume that all the points of $\gamma \cap \alpha$ lie inside $R_\alpha$. Now consider an isotopy $I_3$ of $\gamma$ as follows: If any of the
components of \( \overline{\tau} \cap R_i \) has its endpoint on the boundary of the upper triangle of \( T_j \), for some \( j \in K \), then by the above discussion, \( \overline{\tau} \) cannot intersect \( \omega_{i,j} \) or \( \eta_{i,j} \), for some \( i \in K \) such that the arcs of \( T_i \) and \( T_j \) form the opposite sides of a conduit \( R_{i,j} \). So \( \overline{\tau} \cap R_{i,j} \) is an arc \( MN \) which has its endpoints \( M \in T_j \) and \( N \in T_i \) on \( \partial R_i \). Further, since \( \overline{\tau} \) is a closed curve, \( \overline{\tau} \cap T_i \) is an arc with its endpoint as \( N \) such that \( N \) necessarily lies in the upper pentagon of \( T_i \). Conversely, if any of the components of \( \overline{\tau} \cap R_i \) has its endpoint, \( z_0 \), on the boundary of the upper pentagon of \( T_i \), then it should be connected to an arc, \( g \), of \( \overline{\tau} \) in the conduit \( R_{i,j} \). Note that the endpoints, \( z_0, z'_0 \) of \( g \) are on \( \partial R_i \). There exists an arc component of \( \overline{\tau} \cap R_i \), lying in \( T_j \) such that \( z'_0 \) is on the boundary of the upper triangle of \( T_j \), as the dotted line in figure 14 shows. If any such arc \( g \) of \( \overline{\tau} \) exists, consider an isotopy, \( I_3 \), of \( g \) such that the image, \( I_3(g) \), lies outside \( R_{i,j} \). A schematic of this is figure 14.

The component of \( \overline{\tau} \cap R_i \) in \( T_j \) now has an endpoint on the boundary of the lower pentagon of \( T_j \) and the component of \( \overline{\tau} \cap R_i \) in \( T_i \) has an endpoint on the boundary of the lower triangle of \( T_i \). Also the image of \( \overline{\tau} \cap \alpha \) under \( I_3 \) moves a point of \( \overline{\tau} \cap \alpha \) from the boundary of the upper triangle of \( T_j \) to the boundary of the lower pentagon of \( T_i \). We call \( I_3 \) to be a normalization move on \( \overline{\tau} \). After finitely many normalization moves performed on \( \overline{\tau} \), wherever applicable, we can assume that every component of \( \overline{\tau} \cap R_i \) is contained in a rectangular track \( T_i \) for some \( i \in K \) such that the endpoints of that component lie on the boundary of the lower triangle and the lower pentagon of \( T_i \). So a schematic of every component of \( \overline{\tau} \cap R_i \) inside \( T_i \) is as in figure 14.

After these isotopies \( I_1, I_2, I_3 \) of \( \overline{\tau} \), we say that \( \overline{\tau} \) is in a rectified position. We now prove that \( \overline{\tau} \) in rectified position and \( \alpha \) fill \( S \). From now on we assume that \( \overline{\tau} \) is in a rectified position.

For \( i \in K \), let \( H_i \) be the rectangular component of \( R_i \) \( \setminus (\cup_{i \in K} \alpha_i) \) containing the arcs \( \alpha_i \) and \( \alpha_{i+1} \) on its boundary. Each of these \( H_i \) contains a unique segment, \( \gamma_i \), of the core curve \( \gamma \). The schematic 16 shows \( H_i \) and \( \gamma_i \) for instance.

We say that an arc, \( g \) of \( \overline{\tau} \) covers \( \gamma_i \) if \( g \subseteq H_i \) has its end points on \( \alpha_i \) and \( \alpha_{i+1} \) and \( g \) is isotopic in \( H_i \) to \( \gamma_i \) through arcs whose end points stay on \( \alpha_i, \alpha_{i+1} \).
Since $\gamma$ and $\alpha$ form a filling pair, the set of essential arcs, \{\gamma_1, \ldots, \gamma_k\} fill $S \setminus \alpha$. It follows that $\gamma$ fills $S$ along with $\alpha$ if segments of $\gamma \setminus \alpha$ cover $\gamma_i$ for all $i$ with $i \in K$.

Since $\gamma$ is in rectified position, each component of $\gamma \cap R_\gamma$ already covers all $\gamma_i$ except one as in figure 15. More precisely, if a component of $\gamma \cap R_\gamma$ is in a rectangular track $T_i$, then $\gamma$ covers every $\gamma_j$ where $j$ is such that $1 \leq j \leq k$ and $j \neq i - 1$. So, if $\gamma \cap R_\gamma$ has two distinct components, then each component has to lie in $T_i$ for distinct $i$ and hence $\gamma$ covers $\gamma_j$ for $j \in \{1, 2, \ldots k\}$. We conclude that $\gamma$ and $\alpha$ fill $S$ in this case. Now it remains to show that if there is a single component of $\gamma \cap R_\gamma$, which is an essential arc of $R_\gamma$ and is contained in some rectangular track $T_i$, then $\gamma$ and $\alpha$ fill $S$. As in the previous case, $\gamma$ covers every $\gamma_j$ where $j$ is such that $1 \leq j \leq k$ and $j \neq i - 1$. The components of $S \setminus (\alpha \cup \gamma_1 \cup \ldots \cup \gamma_k)$ will be disks except possibly one which could be a cylinder. This can be seen as follows. Since $\alpha$ and $\gamma$ fill $S$, the components of $S \setminus \alpha \cup \gamma_1 \cup \ldots \cup \gamma_k$ are disks. Each segment of $\gamma_j \setminus \alpha$ for $j \in \{1, 2, \ldots k\}$ contributes to two distinct edges of a component $J_0$ or two separate components $J, J'$ of $S \setminus \alpha \cup \gamma_1 \cup \ldots \cup \gamma_k$.

Let $P_1 := \gamma \cap \alpha_1$ and $P_2 := \gamma \cap \alpha_{i-1}$ be points in $T_i$ which appear on the unique component of $\gamma \cap R_\gamma$. Let $[P_1, P_2]$ represent the arc of $\gamma$ in $R_\gamma$ with endpoints $P_1$ and $P_2$ and $\gamma_1 := \gamma \setminus [P_1, P_2]$. $\gamma_1$ is contained in all the components of $S \setminus \alpha \cup \gamma_1 \cup \ldots \cup \gamma_k$ which contain the arcs $\alpha_{i-1}$ and $\alpha_1$ on their boundary. We know that there is at least one such component because $\gamma_{i-1}$ is also such an arc which joins $\alpha_{i-1}$ to $\alpha_i$. If $\gamma_{i-1}$ is the boundary of $J, J'$, then it would have been an arc which connected $\alpha_{i-1}$ on one disk to $\alpha_i$ on another disk. Note that both $\alpha_i$ and $\alpha_{i-1}$ are also boundary arcs of both $J$ and $J'$. So, we would find $P_1$ on the disk containing $\alpha_i$ and $P_2$ on the disk containing $\alpha_{i-1}$. When we join $J$ and $J'$ along $\gamma_{i-1}$ we get a disk where $\gamma_{i-1}$ is an arc from $P_1$ to $P_2$ intersecting $\gamma_{i-1}$. Cutting along $\gamma_{i-1}$ still yields two different disks. The schematic, figure 17 shows this situation.

If $\gamma_{i-1}$ were on the boundary of $J_0$ representing two edges of $J_0$ then it would have been an arc which connected $\alpha_{i-1}$ to $\alpha_i$. When we glue $J_0$ to itself along

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**Figure 15** The portion of $\gamma$ in rectified position inside $T_i$.

**Figure 16** Schematic showing $H_1$ and $\gamma_1$ in $R_\gamma$.
Figure 17 The figure on the left shows disks $J$ and $J'$ formed by cutting along $\gamma_{i-1}$; the figure on the right shows the new disks formed when $J \cup J'$ are cut along $\gamma_1$

Figure 18 The disk $J_0$ glued to itself along $\gamma_{i-1}$ and cut along $\gamma_1$

$\gamma_{i-1}$, we get a cylinder, $A$, where $\alpha_i$ and $\alpha_{i-1}$ will be arcs on different boundary components of $A$. So we would find $P_1$ and $P_2$ on distinct boundaries of $A$ and hence $\gamma_{i-1}$ would be an essential arc on $A$. So cutting $A$ along this arc $\gamma_{i-1}$ would yield a disk as shown in the schematic, figure 18.

In any case, we get disks by cutting $S \setminus \alpha$ along the arcs of $\gamma \setminus \alpha$. So this proves the theorem.

Claim 1 of Theorem 2 doesn’t use the hypothesis that $d(\alpha, \gamma) = 3$. So we have the following:

**Corollary 1.** If $\alpha$ and $\gamma$ are a pair of curves which fill $S$, then $\alpha$ and $T_1(\alpha)$ also fill $S$.

**Corollary 2.** For a surface of genus $g \geq 3$, $i_{\min}(g, 4) \leq (2g - 1)^2$.

**Proof.** Aougab and Huang [3] proved that $i_{\min}(g, 3) = 2g - 1$ for $g \geq 3$. Now, on $S_g$, for $g \geq 3$, suppose that $\alpha$ and $\beta$ are two such minimally intersecting curves with $d(\alpha, \beta) = 3$. Then $i(\alpha, T_1(\alpha)) = (2g - 1)^2$ and by Theorem 2, $d(\alpha, T_1(\alpha)) = 4$. So $i_{\min}(g, 4) \leq (2g - 1)^2$.



4. An initially efficient geodesic

**Lemma 1.** If $\alpha = \nu_0, \nu_1, \nu_2, \nu_3 = \gamma$ is an initially efficient geodesic then so is $T_1(\alpha), T_1(\nu_1), \nu_2, \nu_1, \alpha$.

**Proof.** For $p \in K_{2-g}$, let $F''_p$ be the components of $S \setminus \{\alpha, R_\gamma\}$ as in the proof of Theorem 2. Since the geodesic $\alpha, \nu_1, \nu_2, \gamma$ is an initially efficient one, each segment of $\nu_1$ intersects every reference arc in $E_i$ at most twice. In particular, arcs of $\partial(R_\gamma)$ that form the edges of $E_i$ intersect $\nu_1$ at most twice. It follows from here that there are at the most two segments of $\nu_1$ in each rectangular track $T_i$ as
There can be at most two distinct segments of $T_\gamma(\nu_1)$ in any rectangular component of $S \setminus (\alpha \cup T_\gamma(\alpha))$ in $R_\gamma$. A schematic of this is shown in figure 19. Further, since the interior of a reference arc is disjoint from $\alpha \cup T_\gamma(\alpha)$, it is sufficient to check for the initial efficiency of the geodesic, $T_\gamma(\alpha), T_\gamma(\nu_1), \nu_2, \nu_1, \alpha$ in the modified disks $F_p'$, abbreviated $F$, corresponding to $F_p$, abbreviated $E$.

Since $E$ and $F$ are homeomorphic to a $2g$-gon. Without loss of generality assume $E$ and $F$ to be a regular Euclidean regular polygon with $2g$ sides. Starting at any segment of $\alpha$ in $E$, we label the edge as $\alpha_1$. Label the edges of $E$ in a clockwise direction, starting at $\alpha_1$ as $\gamma_1, \alpha_2, \gamma_2, \ldots, \gamma_g$. Let $S' = S \setminus R_\gamma$. Since the components of $S' \setminus \{\alpha, \gamma\}$ and $S' \setminus \{\alpha, T_\gamma(\alpha)\}$ are the same, it follows that for every edge, $a_{j_0}$ in $F$ corresponding to $\alpha$, there exists a unique $i_0 \in \{1, \ldots, g\}$ such that $a_{j_0} \subset \alpha_{i_0}$. Index the edges, $a_{j_0}$ of $F$ such that $j_0 = i_0$. Label the edge of $T_\gamma(\alpha)$ in $F$ between $a_i$ and $a_{i+1}$ as $t_i$. Let $\omega$ be a reference arc in $F$ with end points on $t_p$ and $t_q$ for some $p, q \in \{1, \ldots, g\}$. Suppose to the contrary that $\omega \cap T_\gamma(\nu_1) \geq 3$. Then there exists three segments, $z_1, z_2, z_3$ of $T_\gamma(\nu_1)$ in $F$ such that $z_j \cap \omega \neq \phi$. For $j \in \{1, 2, 3\}$, let the end points of $z_j$ lie on $a_{j_1}$ and $a_{j_2}$. From our previous discussion on Dehn twist and figure 20 there exists arcs of $\nu_1$ in $E$ with end points on $\gamma_{j_1}$ and $\gamma_{j_2}$ for all $j \in \{1, 2, 3\}$. Consider a line segment, $\omega'$ in $F$ from an interior point of $a_p$ to an interior point of $a_q$. Then $\omega'$ is a reference arc for the triple, $\alpha, \nu_1, \gamma$ and $\omega' \cap \nu_1 \geq 3$. This contradicts that $\alpha, \nu_1, \nu_2, \gamma$ is an initially efficient geodesic. Hence, $\omega \cap T_\gamma(\nu_1) \leq 2$ for any choice of reference arc, $\omega$ for the triple $T_\gamma(\alpha), T_\gamma(\nu_1), \alpha$.

Since $T_\gamma(\alpha), T_\gamma(\nu_1), \nu_2, \nu_1, \alpha$ is already a geodesic we have that $d(T_\gamma(\nu_1), \alpha) = 3$. This gives that $T_\gamma(\nu_1)$ is an initially efficient geodesic of distance 4 from $T_\gamma(\alpha)$ to $\alpha$. □
DISTANCE 4 CURVES ON CLOSED SURFACES OF ARBITRARY GENUS

Figure 20 Initial efficiency of $T_\gamma(a_1)$ follows from the initial efficiency of $a_1$

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