Aharonov-Bohm oscillations in twisted bilayer graphene

C. De Beule,1 F. Dominguez,1 and P. Recher1,2

1Institute for Mathematical Physics, TU Braunschweig, 38106 Braunschweig, Germany
2Laboratory for Emerging Nanometrology, 38106 Braunschweig, Germany

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We investigate transport in the network of valley Hall states that emerges in twisted bilayer graphene under interlayer bias, supporting a triplet of chiral zigzag modes. In particular, we analyze how the coupling between zigzag modes affects magnetotransport. Remarkably, we find that scattering between parallel zigzag channels gives rise to Aharonov-Bohm oscillations that are robust against temperature. On the other hand, coupling between zigzag modes propagating in different directions leads to Shubnikov-de Haas oscillations due to the formation of network Landau levels and a Hofstadter pattern appears in the magnetoconductance.

Twisted bilayer graphene (TBG) consists of two graphene layers stacked with a relative twist, leading to a moiré pattern of alternating stacking domains which substantially changes the electronic structure [1–3]. Over the past few years, TBG attracted great interest due to the discovery of exotic phenomena in magic-angle TBG [4–11]. At tiny twist angles $\theta \sim 0.1^\circ$, TBG also exhibits interesting physics as the moiré pattern grows so large that the lattice favors expanding Bernal-stacking domains, as illustrated in Fig. 1(a) [12, 13]. In this case, a potential bias between layers induces a local gap in the Bernal-stacking domains. Furthermore, due to the difference in band topology of the AB and BA domains [14, 15], the low-energy physics is governed by a topological network of valley Hall states when the Fermi energy is tuned in the local gap [16–19]. Moreover, the large coherence length of the topological network modes accommodates the observation of interference oscillations in transport measurements [20]. In particular, Aharonov-Bohm (A-B) oscillations were observed in the presence of a magnetic field perpendicular to the graphene layers [21]. In contrast to other topological systems where A-B oscillations arise due to interference between one-dimensional (1D) topological channels at the boundaries [22–26], the network in TBG extends over the whole system. To our knowledge, a transport theory is lacking, most likely due to the large computational cost of standard methods at such small twist angles since the number of atoms per moiré cell is of the order of $10^4 \theta ^{-2}$.

In this paper, we investigate transport in the network of valley Hall states that emerges in TBG under interlayer bias. To this aim, we use a two-channel network model that respects all symmetries of the system and captures the low-energy physics in the network regime [27]. In particular, we consider the case for which the network gives rise to three families of 1D chiral zigzag (ZZ) modes [28, 29] for each valley and spin, as illustrated in Fig. 1(b). We find that there are two transport regimes depending on the type of coupling between the ZZ modes: a quasi-1D regime where only parallel ZZ channels are coupled, and a 2D regime for which ZZ modes propagating along different directions are coupled. When a magnetic field is applied perpendicularly to the graphene layers, the two-terminal conductance displays A-B oscillations in the quasi-1D regime. While in the 2D regime the A-B oscillations are accompanied by Shubnikov-de Haas (SdH) oscillations due to the formation of network Landau levels. Moreover, a Hofstadter pattern appears in the magnetoconductance. At finite temperatures, the A-B oscillations are robust, while the SdH oscillations are washed out such that A-B oscillations always dominate at sufficiently high temperatures.

The network model consists of scattering nodes, characterized by a unitary $S$-matrix, and links between the nodes along which chiral modes propagate freely [30]. In the case of TBG, the links are given by AB/BA domain...
walls supporting two chiral channels per valley and spin, and the nodes correspond to AA regions as illustrated in Fig. 1(a). We consider intra- and interchannel forward scattering with probabilities $P_{f1}$ and $P_{f2}$, respectively, and we assume that intra- and interchannel deflections have the same probability $P_d$. Current conservation at the nodes requires $P_{f1} + P_{f2} + 4P_d = 1$. In the absence of forward scattering, the $S$-matrix depends only on the phase shift $0 \leq \phi \leq \pi/2$ after $120^\circ$ deflections, which controls the coupling between ZZ modes [27]. In the limit $\phi \to 0$, the different ZZ branches are decoupled and forward scattering only couples parallel ZZ channels with probability $P_f = P_{f1} + P_{f2}$ (quasi-1D regime). While for finite $\phi$, different ZZ branches are coupled (2D regime).

Transport in the network model — We consider a network strip as shown in Fig. 1(c) with length $L = Nl$ or $(N - 1/2)l$ ($N = 1, 2, \ldots$) and width $W \gg L$, where $l \approx 14 (\theta^*)^{-1}$ nm is the moiré lattice constant. Incoming modes of each node are related to outgoing modes by the $S$-matrix, $\psi_{\text{out}} = S \psi_{\text{in}}$. However, for transport calculations, it is preferable to work with the transfer matrix that relates amplitudes between two sides, e.g. $\psi_{\text{right}} = M \psi_{\text{left}}$ [31]. The total transfer matrix $T$ links one lead to the other lead, e.g. $\psi_{N+1} = T \psi_0$ with [30]

$$T = \begin{cases} BT_N \cdots T_1 & L = Nl, \\ T_N \cdots T_1 & L = (N - \frac{1}{2})l, \end{cases}$$

where $T_n = C_n D A_n B$ with $A_n$, $B$, $C_n$, and $D$ defined in Fig. 1(c). We impose periodic boundary conditions along the $y$ direction which introduces the transverse momentum $0 \leq k < 2\pi/\sqrt{3}l$. The transfer matrices of the unit cell can then be written as $B = \text{diag}(I_2, M)$ and

$$D(k) = \begin{pmatrix} M_{22} & 0 & M_{21} e^{i k \sqrt{3}l} \\ 0 & I_2 & 0 \\ M_{12} e^{-i k \sqrt{3}l} & 0 & M_{11} \end{pmatrix},$$

where we defined the submatrices

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

with $M_{11}$ and $M_{22}$ square matrices of dimension 2 and 4, respectively. In the absence of disorder and magnetic fields, $A_n = C_n = \text{diag}(\lambda^*, \lambda^2, \lambda^*, \lambda^2) \otimes I_2$ with $\lambda = e^{i \pi E_i/E_F}$ the dynamical phase, independent of $n$. Here, $E$ is the energy and $E_i = 2\pi \hbar v/l$ where $v$ is the velocity of the valley Hall states. The transmission probability is then calculated by imposing boundary conditions on the leads. For transport in the positive $x$ direction, we have $\psi_{\text{in}}(t) = (r_{1\alpha}, a_{1r}, a_{2\alpha}, a_{2r})^t$ and $\psi_{\text{out}}(t) = (0, t_{1\alpha}, 0, t_{2\alpha})^t$ where $r_i$ ($t_i$) are two-component reflection (transmission) amplitudes, and $\alpha$ labels the incoming modes with amplitudes $a_{11} = a_{23} = (1, 0)$ and $a_{12} = a_{24} = (0, 1)$ and $(0, 0)$ otherwise. The transmission probability for a transverse mode with momentum $k$ is then given by $T_k = 1 - (1/4) \sum_{\alpha=1}^4 r_{1\alpha}^* r_{\alpha}$ with

$$T_k = \frac{1}{1 - (1/4) \sum_{\alpha=1}^4 r_{1\alpha}^* r_{\alpha}}.$$

FIG. 2. (color online) Zero-field conductance of the network strip shown in Fig. 1(c) with length $L = 30l$ and width $W \gg L$ for $P_{f1,2} = 0.02$ (solid) and $P_{f1} = 0.2$, $P_{f2} = 0.5$ (dashed) calculated with the method of Ref. 32 for temperature $k_B T/E_F = 0.002$. The inset shows the Fermi surface at zero energy in the moiré Brillouin zone for $P_{f1,2} = 0.02$ and $\phi = 0$ (left) or $\phi = 0.2$ (right), where the $K$ ($K'$) valley corresponds to the dark (light) contour.

$$r_{\alpha} = (r_{1\alpha}, r_{2\alpha})^t.$$ The total transmission probability becomes $T = (\sqrt{3}l/2\pi) \int_{2\pi/\sqrt{3}l}^{2\pi} T_k dk$. Up to now, we considered a single valley. For the other valley, the propagation direction of valley Hall states shown in Fig. 1 is reversed, as the valleys are related by time reversal. Hence, the transmission for $K'$ in the positive $x$ direction is given by the transmission of $K$ in the negative $x$ direction. At zero temperature, the zero-bias differential conductance is given by

$$G \frac{G_0}{G_0} = N_l T_{K'} + \frac{T_k}{2}, \quad N_l = \frac{4W}{\sqrt{3}l}.$$
there now exist interfering paths with different lengths which accumulate a different dynamical phase. In addition, forward scattering allows for even more possible paths leading to more gap openings.

**Magnetoconductance** — Given the effective dimensionality of these two regimes, one expects a different behavior in the presence of a magnetic field perpendicular to the layers. The magnetic field introduces an additional Peierls phase accumulated by the valley Hall states during propagation along links. In the Landau gauge $A = B x e_y$, the Peierls phase along a link starting at $x = m l/2 \ (m \in \mathbb{Z})$ is zero for horizontal links [see Fig. 1(c)] and for an upward or downward diagonal link,

$$\Phi_{\pm}(m) = \mp \pi \left( m + \frac{1}{2} \right) \frac{\Phi}{\Phi_0},$$

respectively, where $\Phi_0 = h/e$ is the flux quantum and $\Phi = B A$ is the flux through the moiré cell, comprising an AB and BA triangle, with $B$ the magnetic field and $A = \sqrt{3} l^2 / 2$ the moiré cell area. If we assume that the magnetic field is sufficiently small such that the structure of the $S$-matrix is unchanged, the transfer matrices $B$ and $D$ remain the same, while $A_n(\Phi) = Q_{2n-2}(\Phi)$ and $C_n(\Phi) = Q_{2n-1}(\Phi)$ with $Q_m = \text{diag} \left( \lambda^+, \lambda^2 e^{i \Phi}, (\lambda^+)^*, (\lambda^2 e^{i \Phi})^- \right) \otimes I_2$.

The magnetoconductance in the quasi-1D regime is shown as a function of the flux $\Phi$ in Fig. 3. We find that the conductance depends only on the total forward scattering probability $P_f = P_{f1} + P_{f2}$ and observe periodic resonances on a constant background. This plateau is again due to ZZ modes that are chiral in the transport direction (e.g. $I_3$ for $K$). Hence, they are always fully transmitted and cannot contribute to A-B oscillations as illustrated in Fig. 4(a). On the other hand, when forward scattering is present, parallel ZZ channels are coupled so that ZZ modes propagating oppositely to the transport direction (e.g. $I_{1,2}$ for $K$) can also contribute to the conductance, giving rise to A-B oscillations as illustrated in Fig. 4(b). Perturbatively, by only including interfering paths of lengths $2 L$ and $2 L + 3 l$, we find

$$\frac{G}{G_0} \approx \left( 2 + 2 P_f^{2L/l+1} [1 + P_f (4 P_d)^2 F_L(\Phi)^2] \right) \frac{W}{\sqrt{3} l},$$

where $F_L(\Phi) = \sin[\pi(2L/l - 1)\Phi/\Phi_0]/\sin(\pi\Phi/\Phi_0)$, reproducing the main A-B peak and background, where the peak width is given by $\Delta \Phi = 2 \Phi_0/(2L/l - 1)$. The main A-B period corresponds to one flux quantum per moiré cell as paths encircling a single AB or BA triangle do not occur in the quasi-1D regime. In general, we find that A-B resonances occur for $\Phi = \Phi_0/n A$ with $n$ an integer, due to interfering paths encircling an area equal to $n$ times the moiré cell, as illustrated in Fig. 4(b). Note that the conductance remains independent of $E_F$ as interfering paths always have the same length in the quasi-1D regime, as we mentioned before, such that the A-B resonances are robust against temperature.

The A-B oscillations persist for finite $\phi$ (2D regime) as can be seen by the vertical bright lines in Fig. 5, where we show the conductance as a function of the flux $\Phi$ and $E_F$ for $\phi = 0.6$, but they are accompanied by ShH oscillations resulting from network Landau levels. Note that the flux periodicity is doubled due to the inclusion of paths encircling a single AB or BA triangle. Furthermore, a crossed pattern appears in the magnetoconductance because of intersecting electron-like and hole-like Landau fans, as observed in magnetotransport measurements.

Moreover, for sufficiently small twist angles one can thread a flux quantum through the moiré cell $[A \approx 172(\theta^*)^{-2} \text{nm}^2]$ with realistic magnetic fields, giving rise to Hofstadter physics, as can also be seen in
FIG. 5. Magnetoconductance as a function of magnetic flux $\Phi$ and Fermi energy $E_F$, for a strip of length $L = 10l$ and width $W \gg L$ for $P_{f1} = P_{f2} = 0.4$ and $\phi = 0.6$. Here, a magnetic flux $\Phi = \Phi_0$ corresponds to $B_0 \approx 24(\theta^\circ)^2$ T.

Fig. 5 although the resolution of the Hofstadter pattern is limited by the system length in this case [31].

Our findings in the quasi-1D regime exhibit the same qualitative features as the experiment in Ref. 21 where resonances in the magnetoresistivity were observed at integer fractions of the main period on top of a plateau slightly above $h/8e^2 \approx 3.2$ kΩ. This increase might be attributed to small nonzero $\phi$. However, in Ref. 21 the main A-B period was interpreted in terms of interference between paths encircling a single AB or BA triangle. In our model, such paths are only allowed in the 2D regime and do not give rise to A-B resonances. Moreover, paths encircling an odd number of AB or BA triangles pick up a relative dynamical phase and their contributions are therefore washed out at finite temperatures, as shown in Fig. 6. Hence, this discrepancy can be resolved if instead the observed main period corresponds to paths encircling a moiré cell. This would be consistent with the observation of A-B resonances over a wide range of temperatures below the gap, and is equivalent to an increase of the twist angle reported in Ref. 21 by a factor $\sqrt{2}$. The twist angle was also determined from the difference in carrier density between subsequent maxima of the resistivity at zero magnetic field. In the network model, we find that for large $P_f$ and nonzero $\phi$, the distance between consecutive minima in the conductance is given by $\Delta E \approx E_1/6$ as can be seen in Fig. 2. This energy difference corresponds to filling one network band, instead of two network bands as reported in the experiment, yielding the same increase in twist angle. This ambiguity in the determination of the twist angle was also reported in earlier experimental works on magic-angle TBG [5].

We further propose a means to extract the forward scattering probability $P_f$ and the phase shift $\phi$ from the difference between the zero-field resistivity and the plateau at small magnetic fields. To match Ref. 21, for example, we find that $\phi \sim 0.1$ and $P_f \gtrsim 0.6$. Such a large value for $P_f$ is justified, since the AB/BA domain walls become atomically sharp for tiny twist angles [34], leading to stronger confinement of the valley Hall states [35]. Disorder due to non-uniformity in the twist angle and Fermi energy can reduce the number of A-B resonances, and can be introduced in the network model by adding random phases to the dynamical phase picked up during propagation along links [30].

Conclusions — We investigated magnetotransport in the network of valley Hall states that emerges in twisted bilayer graphene under interlayer bias. We found that there are two transport regimes characterized by the effective dimensionality of the network zigzag modes, depending on the hierarchy of couplings. In the quasi-1D regime, only parallel zigzag channels are coupled and the Aharonov-Bohm oscillations are accompanied by Shubnikov-de Haas oscillations. Remarkably, we find that the Aharonov-Bohm oscillations dominate at sufficiently high temperatures since the Shubnikov-de Haas oscillations are washed out. In conclusion, we demonstrated how magnetotransport oscillations emerge
from the topological network in minimally twisted bilayer graphene under interlayer bias.

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Supplemental Material

S1. S-MATRIX AND TRANSFER MATRIX FOR A SINGLE NODE

The S-matrix for a scattering node is defined by \( (b,b')^t = S(a,a')^t \), where the amplitudes of incoming and outgoing modes, \( (a,a')^t \) and \( (b,b')^t \) respectively, are defined in Fig. S1. The symmetries of twisted bilayer graphene under interlayer bias impose the following conditions on the S-matrix:

\[
C_3 : \quad S = C_3 S C_3^{-1},
\]

\[
C_2 : \quad S_{K'} = S_K, \quad \text{(2)}
\]

\[
T : \quad S_{K'} = (S_K)^t, \quad \text{(3)}
\]

\[
C_2 T : \quad S = S'. \quad \text{(4)}
\]

We previously demonstrated that under these symmetry conditions, the S-matrix can be written as [27]

\[
S = \begin{pmatrix}
\sqrt{iP_1} & \sqrt{iP_1} & -\sqrt{iP_2} & -\sqrt{iP_2} \\
\sqrt{iP_1} & \sqrt{iP_1} & -\sqrt{iP_2} & -\sqrt{iP_2} \\
-\sqrt{iP_2} & -\sqrt{iP_2} & \sqrt{iP_3} & -\sqrt{iP_3} \\
-\sqrt{iP_2} & -\sqrt{iP_2} & -\sqrt{iP_3} & \sqrt{iP_3}
\end{pmatrix}, \quad \text{(5)}
\]

with \( \chi = (P_{d2} - P_{d1})/2 \sqrt{P_{f1} P_{f2}} \) such that \( \chi \) is real, i.e. \( 2 \sqrt{P_{f1} P_{f2}} \geq |P_{d2} - P_{d1}| \). Here, we assume equal probabilities for intrachannel processes, and also for interchannel deflections to the left and right. Current conservation then requires \( 2(P_{d1} + P_{d2}) + P_{f1} + P_{f2} = 1 \), where \( P_{f1} \) (\( P_{d1} \)) and \( P_{f2} \) (\( P_{d2} \)) are the probabilities for intra- and interchannel forward scattering (deflections), respectively.

It is preferable to work in the zigzag basis, i.e. \( U(b,b')^t = (USU^t) (a,a')^t \) with \( U = [\mathbb{1}_6 - i \sigma_y e^{i \phi z} \otimes \mathbb{1}_3] / \sqrt{2} \) and \( U(a,a')^t = (a_-, a_+) \) with \( a_\pm = (a \pm a' e^{\pm i \phi}) / \sqrt{2} \). For \( P_d = P_{d1} = P_{d2} \), we have

\[
USU^t = \begin{pmatrix}
\sqrt{f} \cos \phi & 0 & 2i \sqrt{P_d} \sin \phi & -i f^* e^{-i \phi} \sin \phi & 2 \sqrt{P_d} e^{-i \phi} \cos \phi & 0 \\
2i \sqrt{P_d} \sin \phi & 0 & 2i \sqrt{P_d} \cos \phi & 0 & 2i \sqrt{P_d} e^{-i \phi} \cos \phi & i f^* e^{-i \phi} \sin \phi \\
0 & 2i \sqrt{P_d} \sin \phi & f \cos \phi & 0 & 2 \sqrt{P_d} e^{-i \phi} \cos \phi & 0 \\
2i \sqrt{P_d} e^{i \phi} \sin \phi & 0 & i f e^{i \phi} \sin \phi & 2 \sqrt{P_d} e^{-i \phi} \cos \phi & 0 & 0 \\
0 & 2i \sqrt{P_d} e^{i \phi} \cos \phi & i f e^{i \phi} \sin \phi & 0 & 2i \sqrt{P_d} \sin \phi & 0 \\
0 & 0 & 2i \sqrt{P_d} \sin \phi & i f e^{i \phi} \sin \phi & 0 & -f^* \cos \phi
\end{pmatrix}, \quad \text{(6)}
\]

where \( f = \sqrt{P_{f2}} + i \sqrt{P_{f1}} \). The transfer matrix for a single node \( \mathcal{M} \) is then found from solving

\[
(b_{3-}, b_{1-}, a_{1-}, a_{1+}, b_{2-}, b_{2+})^t = \mathcal{M} (a_{2-}, a_{2+}, b_{1-}, b_{1+}, a_{3-}, b_{3+})^t. \quad \text{(7)}
\]

where unitarity is now expressed as \( \mathcal{M}^t J \mathcal{M} = J \), where \( J = \text{diag} (1,1,-1,-1,1,1) \) is the current operator for a single node, so that \( |\det \mathcal{M}| = 1 \).

![Figure S1](image-url)  

**FIG. S1.** Incoming \((a,a')\) and outgoing \((b,b')\) amplitudes at a node of the triangular network.
FIG. S2. Feynman paths in the quasi-1D regime of the $l_2$ ZZ modes for a network strip of length $L = 3l/2$, where $f = \sqrt{P_{d1} + i\sqrt{P_{d2}}}$ is the forward scattering amplitude in the ZZ basis between solid arrows and $\Phi$ is the flux through a moiré cell.

S2. QUASI-1D REGIME

A. Analytical result for the conductance

In the quasi-1D regime ($P_{d} = P_{d1} = P_{d2}$ and $\phi = 0$), there are three separate contributions to the conductance from zigzag (ZZ) modes along the $l_{1,2,3}$ directions for a given valley and spin. In the setup shown in Fig. 1(c) of the main text, the contribution from $l_3$ is always given by $T_{l_3} = 2$, while the transmission for $l_{1,2}$ are identical. The latter vanish in the absence of forward scattering and can be calculated straightforwardly for an infinitely wide system of length $L/l = 1/2, 1, 3/2$ by summing Feynman paths. The results are

$$G|_{L=\frac{3}{2}} = (2 + 2P_f^3) \frac{G_0 W}{\sqrt{3l}},$$

$$G|_{L=1} = \left(2 + \frac{2P_f^3}{1 - P_f (4P_d)^2}\right) \frac{G_0 W}{\sqrt{3l}},$$

$$G|_{L=\frac{3}{2}} = \left(2 + \frac{2P_f^4}{1 - P_f (4P_d)^2 \cos^2(\pi\Phi/\Phi_0)}\right) \frac{G_0 W}{\sqrt{3l}},$$

where $P_f = P_{f1} + P_{f2}$, which is illustrated in Fig. S2 for $L = 3l/2$. Note that these expressions only depend on the total forward scattering probability $P_f$, which we find is always the case in the quasi-1D regime. The conductivity is shown in Fig. S3 at zero magnetic field as a function of $P_f$ for several lengths. For larger systems, the Feynman paths are more involved due to contributions from paths that have segments that cut horizontal along the sample, opposite
to the transport direction. Up to second order in the path length, we find

$$G \simeq \left( 2 + 2P_f^{2L/l+1} \left[ 1 + P_f(4P_d)^2 \left( \frac{\sin[\pi(2L/l - 1)\Phi/\Phi_0]}{\sin(\pi\Phi/\Phi_0)} \right)^2 \right] \right) \frac{G_0W}{\sqrt{3l}},$$

(11)

which only includes contributions from paths with lengths $2L$ and $2L + 3l$ and corresponds to Eq. (6) of the main text. We find that Eq. (11) gives rise to the main A-B peak at $\Phi = n\Phi_0$ with $n$ an integer.

Remarkably, there also exist formal analytical expressions for $L \geq 2l$ which can be obtained by explicit calculation of the total transfer matrix and integration over transverse momentum, which essentially acts as a bookkeeping variable for Feynman paths. We find that the momentum-dependent transmission can be written as

$$T_{l_1, l_2}(k) = \frac{P_f^{2L/l+1}}{\sum_{n=0}^{m_L} a_n \cos^n k},$$

(12)

where $m_L = \text{floor}(L/l)$ and the coefficients $a_n$ are functions of $P_f$ and the flux $\Phi$. We have observed this explicitly up to $L = 2l$ for finite $\Phi$ and $L = 7l/2$ for zero $\Phi$ and we expect it holds for all $L$. For example, for $L = 2l$ we find that

$$a_0 = 1 - 4P_dP_f(1 + P_f^2) + 8(4P_d)^2P_f\cos^2\frac{\pi\Phi}{\Phi_0}\cos\frac{2\pi\Phi}{\Phi_0};$$

(13)

$$a_1 = 2\sqrt{P_f} \left[ 1 + (P_f - 2)P_f^2 \right] \left( 1 + 2\cos\frac{2\pi\Phi}{\Phi_0} \right);$$

(14)

$$a_2 = 16P_dP_f,$$

(15)

where $4P_d = 1 - P_f$. The total transmission function is given by

$$T_{l_1, l_2} = \left. \frac{P_f^{2L/l+1}}{2\pi} \int_0^{2\pi} \frac{dk}{\sum_{n=0}^{m_L} a_n \cos^n k} \right|_{\Phi = n\Phi_0} = \frac{P_f^{2L/l+1}}{2\pi} \oint \frac{(1/z)dz}{\sum_{n=0}^{m_L} \frac{a_n}{2^n} (z + 1/z)^n},$$

(16)

where $z = e^{ik}$ and the contour goes along the unit circle. The integral can be worked out and becomes

$$\frac{1}{2\pi i} \oint \frac{z^{m_L-1}dz}{\sum_{n=0}^{2m_L} b_n z^n} = \frac{2^{m_L}}{a_{2m_L}} \left( \sum_{|z_i| < 1} \frac{1}{2} \sum_{|z_i| = 1} \prod_{j \neq i} (z_i - z_j) \right),$$

(17)

where $z_i (i = 1, \ldots, 2m_L)$ are the roots of the polynomial of degree $2m_L$ in the denominator on the left-hand side of the equation and we assumed that the poles are simple. If this is not the case, the results can be obtained with the...
FIG. S4. (color online) Conductivity for a network strip with width $W \gg L$ in the quasi-1D regime ($\phi = 0$) as a function of the magnetic flux through the moiré unit cell for (a) $L = 5l$ and (b) $L = 15l$ and different values of $P_f = P_{f2}$.

A general residue theorem. Moreover, up to $L = 5l/2$, there exist closed-form expressions for the roots. For $L = 2$, we find there are only simple poles, given by

$$z_{\zeta,\xi} = -\frac{a_1 + \zeta \sqrt{a_1^2 - 4a_0a_2} + \xi \sqrt{2a_1 (a_1 + \zeta \sqrt{a_1^2 - 4a_0a_2}) - 4a_2(a_0 + a_2)}}{2a_2}$$

(18)

where $\zeta, \xi = \pm$. Furthermore, we find there are always two poles inside the unit circle, labeled $z_1$ and $z_2$ (for $\Phi = 0$, they are given by $z_{\pm,-}$), and two outside the unit circle, labeled $z_3$ and $z_4$, so that

$$T_{l,2}|_{L=2l} = \frac{4P_f^4}{a_2^2} \left[ \frac{z_1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{z_2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} \right].$$

(19)

B. Length dependence of magnetoconductance

The conductance as a function of the magnetic flux through the moiré unit cell $\Phi$ is shown in Fig. S4(a) and (b) for a network strips of length $L = 5l$ and $L = 15l$, respectively. For larger $L$, the width $\Delta \Phi \simeq 2\Phi_0/(2L/l - 1)$ [see Eq. (11)] of the A-B resonances is reduced and more resonances corresponding to paths encircling larger areas are present. Furthermore, when forward scattering is sufficiently large, the background increases as $2 + 2P_f^{2L/l+1}$ due to contributions of paths going straight through the sample.

S3. 2D REGIME

A. Hofstadter pattern

In Fig. S5 we show the magnetoconductance for the same parameters as in Fig. 5 of the main text but for larger lengths $L = 15l$ and $L = 20l$. 

FIG. S5. (color online) Magnetoconductance for the same parameters as in Fig. 5 of the main text but for larger lengths $L = 15l$ and $L = 20l$. 

(a) $P_f = 0.4$, $P_f = 0.6$, $P_f = 0.7$, $P_f = 0.8$ 
(b) $P_f = 0.4$, $P_f = 0.6$, $P_f = 0.7$, $P_f = 0.8$
FIG. S5. Magnetoconductance as a function of magnetic flux and Fermi energy, for a network strip of length $L = 15\ell$ (a) and $L = 20\ell$ (b) and width $W \gg L$ for $P_{f1} = P_{f2} = 0.4$ and $\phi = 0.6$. Here, a magnetic flux $\Phi = \Phi_0$ corresponds to $B_0 \approx 24(\theta^\circ)^2\ T$. 