Autonomous Rotor Heat Engine

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The triumph of heat engines is their ability to convert the disordered energy of thermal sources into useful mechanical motion. In recent years, much effort has been devoted to generalizing thermodynamic notions to the quantum regime, partly motivated by the promise of surpassing classical heat engines. Here, we instead adopt a bottom-up approach: we propose a realistic autonomous heat engine that can serve as a testbed for quantum effects in the context of thermodynamics. Our model draws inspiration from actual piston engines and is built from closed-system Hamiltonians and weak bath coupling terms. We analytically derive the performance of the engine in the classical regime via a set of nonlinear Langevin equations. In the quantum case, we perform numerical simulations of the master equation and find that free dispersion and measurement backaction noise generally lower the engine’s performance.

I. INTRODUCTION

Historically, the initial goal of thermodynamics was to understand how to convert heat into useful mechanical motion, and it was only once this goal was achieved with the rise of steam engines that there was an interest in exploring the theoretical limitations to their efficiency [1]. As our ability to control quantum systems progresses, it has now become interesting to study heat engines where quantum effects are relevant [2–5]. However, in this case, the historical order has been reversed, with a prevalence of theoretical studies on the thermodynamic limitations of quantum machines [6–8]. In particular, there has been significant interest in determining the extent to which quantum effects may help surpass classical limits such as the Carnot efficiency [9–11], or complicate classical notions such as work [12, 13].

In this work, we instead consider the problem of designing a quantum heat engine that unambiguously achieves its goal of converting heat into the useful mechanical motion of a system. In particular, our goal is to devise a self-contained engine that autonomously converts heat from a thermal bath into motion in a single rotational degree of freedom. We choose to study a rotor to benefit from the useful features of rotational motion: it is inherently periodic, it can in principle be coupled through other systems such as gears and pistons into other types of motion, and it can be used to drive electric generators. Additionally, unlike the motion of oscillators, rotational motion has a meaningful sense of directionality [14] through the distinction between clockwise and counter-clockwise rotation. The rotation frequency is not upper bounded, as would be the case in finite dimensional systems, and it unambiguously displays the amount of useful energy stored in the rotor degree of freedom.

We design a simple opto-mechanical engine where the rotor is coupled to a single harmonic mode through radiation pressure, which is in turn coupled linearly to two baths at different temperatures. In analogy to the hot gas pushing down the piston of an actual car engine, the mode serves as the working medium transferring heat from the hot to the cold bath in sync with the angular motion of the rotor. We characterize analytically the engine’s operation in the classical regime and we find that it functions as desired. We then study the quantum case by simulating the dynamics numerically and find that inherent quantum effects, particularly dispersion of the rotor wavefunction and backaction noise, negatively affect its functioning. Indeed, for this quantum heat engine, the ideal regime of operation is the classical limit. Our results suggest that significant scrutiny is necessary...
to understand whether quantum effects are advantageous or not in the context of autonomous heat engines.

In the following, we outline our engine model in detail and describe the equations of motion that govern its dynamics both in the classical and quantum settings. We continue by describing how these equations of motion can be solved, including the challenges involved in doing so. We then study the dynamics for different values of the relevant parameters and discuss the significance they have in the performance of the engine and the role of quantum effects.

II. ROTOR HEAT ENGINE

**Design.**—The guiding principle in our design of the rotor heat engine is that it should unambiguously achieve its intended goal of converting heat into useful mechanical motion of the rotor. To be precise, we demand that (i) the engine is autonomous, (ii) the rotor draws energy exclusively from a thermal source, and (iii) the rotor undergoes useful directional motion, i.e. it has a well-defined angular momentum increasing with time. We do allow for an initialization of the rotor at a well localized angle, but not the use of external control fields and time-dependent Hamiltonians for the dynamics, contrary to previous works in the literature, e.g. Refs. [5, 7, 9–11].

Our engine model is sketched in Fig. 1. In its initial configuration, the harmonic working mode is in contact with a hot reservoir, which causes it to thermalize to the average excitation number \( n_H \) at the rate \( \kappa \). Radiation pressure then pushes on the piston, which exerts torque on the attached rotor and causes it to spin clockwise. Once the piston passes its bottom turning point, the radiation pressure starts to push against the spin, which would drive the system into pendular motion. To prevent this, the working mode is now brought into contact with a cold reservoir that decreases the average excitation number to \( n_C \) in order to suppress radiation pressure until the upper turning point is reached and the thermal contact switches from the cold to the hot bath again. This modulated, angle-dependent thermal coupling between the working mode and the two reservoirs is what keeps the wheel spinning clockwise, gaining momentum with every round-trip. Note that this modulation does not conflict with our aim of building an autonomous engine. Indeed, our engine is analogous to a car engine where a crankshaft spins due to contact with an oscillating piston, and a synchronized camshaft controls the opening of the inlet and exhaust valves. In our heat engine, the rotor plays the role of the crankshaft while the modulation of the coupling to the bath plays the role of the camshaft.

**Model.**—Classically, the rotor degree of freedom is described by an angle variable \( \varphi \in [0, 2\pi] \) and an angular momentum component \( L_z \) perpendicular to the plane of rotation. However, in the quantum version of the rotor, the bound spectrum of the angle operator \( \hat{\varphi} \) implies a discrete spectrum of \( \hat{L}_z \), and Hermitianity must be enforced by imposing periodic boundary conditions [15]. That is, we must work with strictly periodic functions of the angle.

The Hamiltonian of the engine in a frame rotating at the mode frequency is given by

\[
\hat{H}_S = \frac{\hat{I}_z^2}{2I} + \hbar g \hat{a}^\dagger \hat{a} \cos(\hat{\varphi}),
\]

where \( I \) is the moment of inertia of the rotor, \( g \) is the opto-mechanical coupling strength and \( \hat{a} \) is the annihilation operator of the mode. Importantly, the mode frequency does not enter the description as the radiation pressure term is only proportional to the mode occupation \( \hat{a}^\dagger \hat{a} \). Hence, we are free to match the mode frequency to the requirements of weak linear bath coupling and of the preferred physical implementation.

The Hamiltonian \( \hat{H}_S \) may for instance describe an optical Fabry-Perot cavity where one of the mirrors is rigidly attached to the rotor and allowed to move along the \( x \) direction [16]. The radiation pressure acting on the mirror will then always try to push the rotor away from the cavity, in proportion to the number of photons in the cavity.

A direct opto-mechanical realization of the engine model can be envisaged, given the recent experimental advances in the rotational control of nanorods trapped in a cavity field [17–19]. Other physical realizations are also conceivable, where the working mode is not restricted to an optical field mode and where the angular variable might be associated to a phase degree of freedom instead of the mechanical gear depicted in Fig. 1.

We can now describe the interaction of the mode with its environment, consisting of the hot (H) and cold (C) baths. Specifically, we describe each bath as an ensemble of harmonic oscillators

\[
\hat{H}_T = \int_{-\infty}^{\infty} d\omega \hbar \omega \hat{b}_T^\dagger(\omega) \hat{b}_T(\omega),
\]

with correlation functions

\[
\begin{align*}
\langle \hat{b}_T^\dagger(\omega) \hat{b}_T(\omega') \rangle &= n_T \delta(\omega - \omega'), \\
\langle \hat{b}_T(\omega) \hat{b}_T^\dagger(\omega') \rangle &= (n_T + 1) \delta(\omega - \omega'),
\end{align*}
\]

where \( T = H, C \) and \( n_T \) is the associated thermal occupation at the mode frequency. The interaction between the mode and the two baths is then described by

\[
\hat{H}_{B-S} = i\hbar \sum_{T = H, C} \int_{-\infty}^{\infty} d\omega \gamma f_T(\hat{\varphi}) \left[ \hat{b}_T^\dagger(\omega) \hat{a} - \hat{a}^\dagger \hat{b}_T(\omega) \right],
\]

where we neglect the variation of the coupling constant \( \gamma \) and of the thermal occupation \( n_T \) around the mode frequency. At this point, the only non-standard feature of our model for the baths is the modulation of the coupling via the functions \( f_T(\hat{\varphi}) \). This synchronicity is what breaks the symmetry, allowing the engine to be pushed harder than it is slowed down. Naturally, this internal
clock cannot be used to construct a perpetual machine of the second kind, namely a heat engine that would run on a single heat bath at a fixed temperature. Indeed, the presence of the cold bath is of utmost importance in order to extract heat from the mode and hence lower the radiation pressure. We note that the crucial role of this internal clock for building autonomous machines has also been pointed out recently in the context of solar cells [20].

When engineering the modulating function $f_H(\varphi)$, it is sufficient to ensure that the mode is coupled strongly with the hot bath in the interval $0 < \varphi < \pi$ but only weakly coupled in the interval $\pi < \varphi < 2\pi$, and vice versa for $f_C(\varphi)$. For simplicity, we consider the following modulating functions

$$f_H(\varphi) = \frac{1 + \sin(\varphi)}{2}, \quad f_C(\varphi) = \frac{1 - \sin(\varphi)}{2}.$$  \hspace{0.5cm} (5)

This specific choice is motivated by the requirement of working with periodic functions of the angle in the quantum regime. Nevertheless, the results obtained will not be qualitatively affected by a different choice of functions, as long as they alternate with negligible overlap as described above. A somewhat closer resemblance to the engine of a car can be achieved by working with coupling functions of narrow support, say, on small windows around the angles $\varphi = 0$ (H) and $\pi$ (C). Efficient operation would then require sufficiently fast thermalization within the respective time windows.

To conclude the presentation of our model, we emphasize that while the choice of modulating the coupling rate via the rotor’s angular position is genuinely novel, the Langevin equations and the master equation can still be obtained following the original derivations presented in [21]. This will allow us to trace out the bath degrees of freedom and describe their influence in terms of an effective thermalization rate $\kappa = 2\pi\gamma^2$ [22]. For this treatment to be valid, the latter is assumed to be small compared to the (freely adjustable) mode frequency and the spectral variation in the bath coupling.

### III. CLASSICAL REGIME

**Nonlinear stochastic dynamics.**–We start by studying the dynamics of the rotor heat engine in the classical regime. To this end, we consider the classical limit of the quantum Langevin equations for the rotor coordinates and the complex mode amplitude [21],

$$d\varphi = L_z/I \, dt, \quad dL_z = h\varphi|a|^2\sin(\varphi) \, dt,$$

$$da = -[ig\cos(\varphi) + \kappa(\varphi)/2]a \, dt - \sum_{T=H,C} \sqrt{\kappa f_T^2(\varphi)\bar{n}_T} \, dW_T. \hspace{1cm} (6)$$

The third line is a stochastic differential equation in Itô form [23] which describes the thermalization of the mode with the two baths. Here $W_H$ and $W_C$ are complex Wiener processes, i.e. continuous stochastic processes with independent time increments that take complex values following a normal distribution. They correspond to the noise incoming from the baths, with

$$\kappa(\varphi) = \kappa \left[ f_H^2(\varphi) + f_C^2(\varphi) \right]$$  \hspace{0.5cm} (7)

the overall decay rate of the mode intensity. For our choice of functions, we have $\kappa/2 \leq \kappa(\varphi) \leq \kappa$, such that the mode is always in contact with a thermal bath for any position of the rotor. It is worth noting that the classical description (6) does not include the measurement back-action term due to the angle-dependent coupling to the baths, which is a purely quantum effect.

In describing the dynamics of the engine, we are interested in the evolution of statistical quantities like the averages and variances of the random variables described in the equations of motion (6). However, solving them is not a straightforward matter, especially given the nonlinear form of the radiation pressure term driving the angular momentum $L_z$. In order to tackle this problem, we proceed in three steps: (i) reduce the two Wiener processes to a single one (ii) derive the equation of motion for the mode intensity, from which we can simulate efficiently the dynamics (iii) perform the adiabatic elimination of the mode, which will allow us to obtain compact analytical results.

As a first step, we thus make use of the fact that the sum of two independent Wiener processes can be expressed as a single effective Wiener process $W_{\text{eff}}$, namely

$$\sum_{T=H,C} \sqrt{\kappa f_T^2(\varphi)\bar{n}_T} \, dW_T = \sqrt{\kappa(\varphi)\bar{n}(\varphi)} \, dW_{\text{eff}}. \hspace{1cm} (8)$$

In the classical regime, the two baths can thus be reduced to a single bath with an effective thermal occupation modulated by the rotor’s position

$$\bar{n}(\varphi) = \frac{f_H^2(\varphi)\bar{n}_H + f_C^2(\varphi)\bar{n}_C}{f_H^2(\varphi) + f_C^2(\varphi)}. \hspace{1cm} (9)$$

As we shall see later, this intuitive simplification does not generalize to the quantum regime where coherence between different angles may occur.

Given that the phase of the mode does not impact the dynamics of the rotor heat engine, the model can be reduced further by solely considering the mode intensity $|a|^2$, whose Itô stochastic differential equation can be derived from the Fokker-Planck equation as [23, 24].

$$d|a|^2 = -\kappa(\varphi) \left[ |a|^2 - \bar{n}(\varphi) \right] \, dt + \sqrt{2|a|^2\kappa(\varphi)\bar{n}(\varphi)} \, dW, \hspace{1cm} (10)$$

where $W$ is a real Wiener process. Note that the Itô calculus used here implies that the dynamical variables $|a|^2(t)$, $L_z(t)$ and $\varphi(t)$ are non-anticipating functions of the noise [23], i.e. they are independent of the behaviour of the Wiener process $W$ in the future of $t$.

**Numerical simulation.**–Typical results from a numerical Monte Carlo integration of the classical engine model (6,10) are shown in Fig. 2 for exemplary cases of fast
and slow thermalization. Starting from the rotor at rest ($L_z(0) = 0$, $\varphi(0) = \pi/2$), in the fast case the evolution of the average angular momentum ($\langle L_z \rangle$) [green line in panel (a)] shows that the engine is accelerating clockwise. Moreover, the noise (shaded area) is relatively small and does not impact significantly the performance of the engine, which is in contrast to the case of slow thermalization (grey). Looking at the angle variable in panel (b) yields the same conclusion, with an additional subtlety. Indeed, while it is true that the one-dimensional unbounded coordinate $\varphi$ grows with negligible relative noise, the actual angle coordinate of the rotor is defined up to a multiple of $2\pi$. As a consequence, as soon as the standard deviation $\Delta \varphi$ is of order $\pi$, the distribution of angles will essentially appear flat. Physically speaking, this means that one will not be able to infer the exact angle of the rotor from a known value that lies several cycles in the past. This can be seen in Fig. 2(c) where we plot the two-time correlation function of the periodic angle variable [25]

$$S_{\varphi\varphi}(t_1, t_2) = \frac{R[\varphi(t_1) - \varphi(t_2)] - R[\varphi(t_1) + \varphi(t_2)]}{\sqrt{(1 - R[2\varphi(t_1)])(1 - R[2\varphi(t_2)])}}$$

(11)

where $R[\phi] = \langle \cos \phi \rangle^2 + \langle \sin \phi \rangle^2$. The width of $S_{\varphi\varphi}$ with respect to $|t_1 - t_2|$ determines the time window of phase stable motion, before the angular position is completely diffused. However, that the phase stability of the motion is bounded by thermal fluctuations would pose a practical limitation only when they were able to stop or reverse the average spinning direction. In other words, the relevant condition for a steady operation of the engine is that the relative spread $\Delta L_z / \langle L_z \rangle$ in the angular momentum must be small and two-time correlations $S_{\varphi\varphi}$ extend over more than one cycle of rotation.

Adiabatic elimination. The equations for the classical description are exact so far. In order to characterize analytically the engine’s operation, we now adiabatically eliminate the mode variable $|a|^2(t)$. Specifically, we assume that thermalization occurs on a much shorter timescale than the motion of the rotor such that the mode intensity will assume its mean value $\langle 9 \rangle$ almost instantaneously for each angle $\varphi(t)$. To be explicit, let us separate the small noise deviations from the mean in the mode intensity, $|a|^2(t) = \bar{n}[\varphi(t)] + \varepsilon_a(t)$, and insert it into the equation of motion (10). We obtain $d\varepsilon_a = -\kappa(\varphi)\varepsilon_a dt + 2\kappa(\varphi)\bar{n}(\varphi)(\bar{n}(\varphi) + \varepsilon_a) dW$, a random variable that contains information of the rotor trajectory integrated over a time scale of $1/\kappa$. At low rotation speed $L_z / I \ll \kappa$, we can neglect this short-time memory effect and assume that any function of the angle $\varphi(t)$ is non-anticipating for $\varepsilon_a(t)$.

This approximation, which we use to derive the analytical results below, corresponds to the desired regime of operation for the engine. Indeed, it is precisely when the mode is given sufficient time to thermalize with the baths that heat can be extracted to create the required bias in radiation pressure. Optimal performance is thus reached in the limit $\kappa \rightarrow \infty$, whereas $\kappa$-dependent corrections are expected to appear for $\kappa \sim g$ or $\kappa \sim \langle L_z \rangle / I$. As the rotor accelerates, it will eventually enter this latter regime where the approximation breaks down. This situation would be analogous to a car engine that could not follow the opening and closing of its valves given the fast rotation of the camshaft. However, a car engine would rarely operate in such a regime as the targeted velocities under load are kept well below the intrinsic thermalization rates. Additionally, other effects would start to play a role at such high speeds, such as friction of the rotor which is not included in the present model, but could be done following the results of Ref. [26].

At this point, we have all the necessary information to derive the rates at which the average angular momentum

FIG. 2. Classical simulation of the engine dynamics. Panels (a) and (b) depict, respectively, the angular momentum and the unwrapped angle of the rotor for two exemplary cases of fast ($\kappa = 100g$) and slow ($\kappa = g$) thermalization. The lines represent the mean values for a numerical sample over 100,000 trajectories, the shaded areas cover two standard deviations. Red and blue areas on the top indicate the sign of $\sin(\varphi(t))$ for $\kappa = 100g$, i.e. where the working mode is mostly in contact with the hot and the cold bath. Panel (c) shows the angular two-time correlation function $S_{\varphi\varphi}(t_1, t_2)$ for $\kappa = 100g$. $S_{\varphi\varphi}(t_1, t_2) = 1$ implies that the angular position at time $t_2$ can be predicted from its value at time $t_1$ (and vice versa) while $S_{\varphi\varphi}(t_1, t_2) = 0$ corresponds to uncorrelated angular random variables. All simulations start with the rotor at rest ($L_z(0) = 0$, $\varphi(0) = \pi/2$, $I = 5/g$), and reservoirs at $(\bar{n}_H, \bar{n}_C) = (1, 0)$. 

\[0\]}
\[ \langle L_z \rangle \] and its variance \( \Delta L_z^2 \) increase as a function of time. They read

\[ \langle L_z \rangle = \frac{h}{2} \sin(\varphi) \bar{n}(\varphi) \]  \( \text{(12)} \)

\[ \Delta L_z^2 = 2h^2 \langle \sin(\varphi) \bar{n}(\varphi) \rangle \left( L_z - \langle L_z \rangle + \frac{h}{2} \frac{\sin(\varphi) \bar{n}(\varphi)}{\kappa(\varphi)} \right) \]

\[ \text{F.R.} \quad \frac{\hbar^2 g^2}{\kappa} \left( 1 - \frac{1}{2} (\bar{n}_H - \bar{n}_C)^2 + \frac{3}{8\sqrt{2}} (\bar{n}_H - \bar{n}_C)^2 \right), \]

where \( \text{F.R.} \) stands for the limit when the gain in angular momentum per cycle is small enough so that the quantities can be averaged over one round-trip of free rotation, \( \langle g(\varphi) \rangle \rightarrow 1/2\pi \int_0^{2\pi} d\varphi g(\varphi) \). This yields compact analytical expressions in spite of the nonlinear dynamics (see Fig. 3 for a comparison with the exact dynamics).

Following our intended goal, the average angular momentum \( \langle L_z \rangle \) is driven in proportion to the difference in thermal occupation of the baths, which in turn drives the average angle coordinate \( \varphi \). Inevitably, heat also enters the system in the form of noise, limiting the phase stability and accumulating uncertainty in the angle as a function of time. Classically, this happens at finite temperatures even in the absence of driving, \( \bar{n}_H - \bar{n}_C = 0 \). Indeed, the angular momentum variance \( \Delta L_z^2 \) acquires contributions from the difference as well as the sum in thermal occupation of the baths. However, it grows linearly in time as does \( \langle L_z \rangle \), which implies that the relative noise \( \Delta L_z/\langle L_z \rangle \) decreases over time. In other words, the rotor behaves as a clock with a steadily improving signal-to-noise ratio.

**Performance.**—We have shown that the classical version of the rotor heat engine is achieving its goal of extracting useful directional motion from the heat transfer between the baths. How well it performs its task can be further illustrated as follows. The heat stored in the harmonic mode as thermal excitation \( \bar{n}(\varphi) \) translates into a mean torque (or energy) \( \hbar g \bar{n}(\varphi) \) per cycle that is in principle available to accelerate the rotor. Comparing this to the actually exerted net torque \( \langle L_z \rangle \) per rotation cycle gives a measure of the engine’s performance,

\[ \eta = \frac{1}{t_{\text{cyc}}} \int_{t_{\text{cyc}}} dt' \frac{\langle \dot{L}_z(t') \rangle}{\hbar g \langle \bar{n}(\varphi(t')) \rangle} \]

\[ \text{F.R.} \quad 2(1 - \frac{1}{\sqrt{2}}) \frac{\bar{n}_H - \bar{n}_C}{\bar{n}_H + \bar{n}_C}. \]

(13)

where \( t_{\text{cyc}} \) is the duration of the cycle. Note that this quantity is only meaningful in the regime of large signal-to-noise ratio. One can see from (13) that the parameter \( \eta \) does not reach unity, but at best \( \eta < 2 - \sqrt{2} \approx 0.6 \) (or \( \eta_H \lesssim 0.3 \)) if \( \bar{n}_C \ll \bar{n}_H \). This non-optimal behaviour is related to our choice of the coupling functions (5) covering a broad range of angles, while the mode occupation is exploited optimally by radiation pressure at \( \varphi = \pi/2 \).

IV. QUANTUM REGIME

**Master equation.**—The master equation governing the dynamics of the quantum heat engine is given by [21]

\[ \dot{\rho} = -\frac{i}{\hbar} \left[ \hat{H}_S, \hat{\rho} \right] \]

\[ + \sum_{T=H,C} \kappa(n_T + 1) D \left[ f_T(\varphi) \hat{a} \right] \hat{\rho} + \kappa n_T D \left[ f_T(\varphi) \hat{a}^\dagger \right] \hat{\rho}, \]

\[ \text{(14)} \]

where \( D[\hat{O}] \hat{\rho} = \hat{O} \rho \hat{O}^\dagger - \frac{1}{2} \left\{ \hat{O}^\dagger \hat{O}, \hat{\rho} \right\} \) is the Lindblad superoperator [27]. In contrast to the classical regime, the dissipative coupling of the mode to the reservoirs cannot be reduced to an effective single-bath term. This is a manifestation of quantum coherence in the angle coordinate \( \varphi \), since off-diagonal matrix elements \( \langle n, \varphi | \hat{p} | n', \varphi' \rangle \) are influenced by both reservoirs simultaneously.

**Initialization.**—For the rotor to start spinning in the right direction, we initialize its angular position in the region \( \varphi \in [0, \pi] \) where the mode is predominantly coupled to the hot bath. This implies that any convex combination of energy eigenstates – such as a thermal distribution – should be avoided, since in this case the rotor is completely delocalized. Alternatively, one could start with an initial displacement in momentum. However, this would require an external energy source, just like the battery of a starting car engine, which we avoid in our autonomous model.

Here we select an initial pure state given by the periodic von Mises wavefunction

\[ \psi_i(\varphi) = \frac{e^{ik \cos(\varphi - \mu)}}{\sqrt{2\pi I_0(2k)}}, \]

(15)
where \( I_0(2k) \) is a modified Bessel function. With this choice, the angular position of the rotor is localized around a mean value \( \mu \) with a spread determined by the parameter \( k \). For an angle sufficiently localized \( k \gg 1 \), the angle distribution is approximately Gaussian, with a standard deviation \( 1/\sqrt{2k} \ll \pi/2 \). The corresponding angular momentum distribution covers a spectrum of order \( \sqrt{k} \) quanta around its zero average. No further initialization is required, and any net rotation of the rotor will come entirely from the engine dynamics.

Ideally, the rotor will evolve in such a way that its average angular momentum \( \langle L_z \rangle \) increases with time, while the relative spread \( \Delta L_z/\langle L_z \rangle \) remains small. In contrast to the classical model, however, quantum mechanics will impose additional constraints for the performance of the engine. In particular, for the rotor to start spinning, the initial acceleration of the engine must exceed the free dispersion of the rotor. This is roughly the case if the initial spread in kinetic energy, \( \Delta E(0) \approx \hbar^2 k^2/2I \), is small compared to the gain \( h\bar{\nu}_H \) in potential energy during the first half-cycle, \( h\bar{\nu}_H/h \gg k \). The larger \( h\bar{\nu}_H/h \) is, the more the quantum rotor approaches classical behaviour and the less it is prone to free dispersion.

**Numerical simulations.**– We employ two numerical methods to simulate the dynamics of the quantum engine. One is direct integration using the QuTiP package in Python over a truncated Hilbert space [28]. Specifically, we restrict the rotor Hilbert space to angular momentum quantum numbers \( m_{\text{min}} \leq m \leq m_{\text{max}} \), with suitably chosen bounds to cover the occupied spectrum at all times. The thermal mode is allowed to have at most 8 excitations, which is sufficient given that we operate with \( \langle \bar{c}_H, \bar{c}_C \rangle = (1, 0) \) throughout. Higher reservoir temperatures are computationally expensive, but do not provide further insight into the engine’s operation. The other method is a stochastic sampling of the master equation (14) in terms of piecewise deterministic jump trajectories [29]. We use direct integration to solve for the dynamics of the engine, and stochastic sampling to explore values of the engine parameters that lead to a good performance of the engine.

**Quantum vs. Classical.**– The results of the numerical simulations for both the classical and quantum models are summarized in Fig. 4 for two different choices of parameters. We have selected values of the moment of inertia \( I \) and the coupling strength \( g \) such that \( I g/h \gg k \) in one case, while \( I g/h = k \) in the other case. Additionally, these values are set such that the classical dynamics are essentially unchanged in both cases, allowing us to showcase how properties that are irrelevant classically become meaningful in a quantum setting.

As discussed previously in reference to Eq. (15), due to the uncertainty principle, it is impossible to perfectly localize both the angular position and angular momentum of a quantum rotor. In order to make a fair comparison and mimic this in the classical case, we initialize the rotor’s angular position and angular momentum in a Gaussian probability distribution, with mean \( \pi/2 \) and standard deviation \( 1/\sqrt{2k} \) for the angular position, and mean 0 and standard deviation \( \sqrt{k}/2 \) for the angular
momentum. This corresponds to the Gaussian approximation of the von Mises distribution (15), which holds for $k \gg 1$. This initialization of the classical rotor allows us to distinguish quantum effects that arise due to initialization from effects that originate from the rotor’s interaction with the working mode.

As seen in Fig. 4(a), the quantum engine shows almost identical behaviour for the angular frequency as in the classical case when $Ig/h \gg k$, as expected. However, for $Ig/h = k$ in panel (d), the quantum model yields a much larger variance around the mean value, even though the latter still increases steadily in time. This additional noise arises due to the combined effects of free dispersion and measurement backaction noise, which are features unique to the quantum rotor. In fact, in the two regimes we have explored, the amount of noise in the quantum case is strictly larger than in the classical one.

We can also examine the distribution of the angular position of the rotor as a function of time. Here we see that there is significantly less broadening when $Ig/h \gg k$ in panel (b), whereas the angular distribution is almost flat before it completes one revolution for $Ig/h = k$ in panel (e). Classically, this is simply explained by different spreads in angular frequency for a given spread in angular momentum. In the quantum case, however, the additional noise contributions not only broaden the angle distribution further, but also impact the phase stability of the rotor engine. This is shown in panels (c) and (f), where the symmetric two-time correlation function $S_{\phi^2}(t_1, t_2)$ is plotted for the quantum (upper triangle) and classical (lower triangle) cases. While the phase stability of the classical engine survives in the regime $Ig/h = k$, correlations drop almost instantaneously in the quantum case.

Overall, the quantum heat engine operates best when it approaches the classical limit of a large moment of inertia together with a large coupling strength. We emphasize that dispersion of the rotor wavefunction and backaction noise – which are purely quantum effects – are problematic for the proper functioning of the engine. Our results showcase the importance of studying the actual dynamics of heat engines and of addressing countermeasures to quantum sources of noise.

V. CONCLUSION

Inspired by actual piston engines, we have proposed an autonomous rotor heat engine described by standard Hamiltonians. By solving the underlying equations of motion, we have shown analytically that the engine functions as desired in the classical regime. We have also explored the role of quantum effects, and our results show that, in the case of our engine, quantum effects are detrimental to performance as they give rise to additional noise and dispersion. This suggests that it is still an open question whether there is a quantum advantage in the context of autonomous heat engines. To this end, our rotor heat engine provides a suitable testbed for the various notions and concepts that have been put forward in the context of quantum thermodynamics. For example, the study of Otto-cycle-type of modulating functions, as well as quantum-engineered initial states of the rotor, could open new room for improvement of the engine.

Finally, we note that an analogy can be drawn between our engine and the one-dimensional motion of a particle in a periodic potential generated by, say, a standing-wave cavity field [30]. In this context, the acceleration of the rotor can also be understood as the reverse of a Sisyphus cooling scheme, where the rotor runs effectively more downhill than uphill on the potential energy curve [31, 32].

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