The early history of the integrable chiral Potts model and the odd–even problem

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Abstract

In the first part of this paper I shall discuss the round-about way of how the integrable chiral Potts model was discovered about 30 years ago. As there should be more higher-genus models to be discovered, this might be of interest. In the second part I shall discuss some quantum group aspects, especially issues of odd versus even \(N\) related to the Serre relations conjecture in our quantum loop subalgebra paper of 5 years ago and how we can make good use of coproducts, also borrowing ideas of Drinfeld, Jimbo, Deguchi, Fabricius, McCoy and Nishino.

Keywords: chiral Potts model, Yang–Baxter equation, star-triangle equation

1. Introduction

It is a great honor to be asked to contribute to the special issue honoring the 75th birthday of Professor Rodney Baxter and it may be proper to finally give a review of the early history of how the integrable chiral Potts model came into being. First of all, Baxter has given many important contributions to this topic, as is also evidenced by his contribution in the special issue [1]. Secondly, the early development of the theory went through many unexpected twists and turns and missed opportunities not reported in the published papers; reporting on these should encourage others seeking to discover new solvable models and further properties of existing ones.

2. Part 1: early history of the integrable chiral Potts model

2.1. Discovery of the Yang–Baxter integrable chiral Potts model

It really started in the summer of 1986, when Barry McCoy, Mulin Yan, Helen Au-Yang and I decided to start a project on parafermions. The work of Zamolodchikov and Fateev [2]
seemed to us to relate parafermionic conformal quantum field theory to the phase transition separating regimes I and II in the RSOS model of Andrews et al. [3], especially, since Huse [4] had already suggested that this transition is in the universality class of the chiral clock model [5, 6]. We split the project into two parts: Au-Yang and Yan were to look at a preprint of Alcaraz and Lima Santos, now published as [7], while McCoy and I would try to find Painlevé-type equations that could be used to describe parafermionic correlations, extending the conformal theory into the massive regime, and also try a variety of perturbation expansions of correlation functions to use for that purpose.

Au-Yang soon discovered that the full three-state self-dual case solves the quantum Lax pair equations, as formulated by Bashilov and Pokrovsky [8]; there are no conditions on the corresponding spin-chain Hamiltonian for a family of transfer matrices commuting with it to exist, even if the Boltzmann weights in [7] are allowed to be chiral! Chiral comes from the Greek word χειρ, hand, so that it means handed, not reflection invariant. The four of us looked briefly at Au-Yang’s solution and found that the spectral variables lie on an elliptic curve, but that the only physical two-dimensional classical spin model was the three-state critical Potts model. McCoy and I decided for the moment to continue with Painlevé and series expansions.

Au-Yang and Yan went on to analyze the non-self-dual three-state chiral Potts model with Boltzmann weights $W(a \rightarrow b)$ and $\overline{W}(a \rightarrow b)$ for horizontal and vertical nearest-neighbor pair interactions of spins $a$ and $b$ with values 1 to 3 (or 0 to 2) mod 3, writing

$$I_n = \frac{\sum_{j=0}^{N-1} \omega^{-jn} W(j)}{\sum_{j=0}^{N-1} W(j)}, \quad \tilde{I}_n = \frac{\overline{W}(n)}{\overline{W}(0)}, \quad \omega = e^{2\pi i/N}, \quad N = 3, \quad (1)$$

as used in the quantum Lax pair approach. A logical thing to do seemed to rewrite the star-triangle equations [10, 11] in these variables, assuming three versions of the weights for the three different positions in these equations, namely $W$ and $\overline{W}$, $W'$ and $\overline{W}'$, or $W''$ and $\overline{W}''$. As $I_n$ involves a discrete Fourier transform, such a transform was also applied to the star-triangle equations, which then became, with proper normalizations of all $W$’s and $\overline{W}$’s, which we can choose to be $\sum_{j=0}^{N-1} W(j) = \overline{W}(0) = 1$,

$$V_{ab} X_b = \overline{V}_{ba} X_a, \quad (2)$$

where

$$V_{ab} = \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} \omega^{am+bk+mk} I^a_{m} I^b_{k} = \sum_{k=0}^{N-1} \omega^{bk} W(a+k) \overline{W}'(k), \quad (3)$$

$$\overline{V}_{ab} = \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} \omega^{am+bk+mk} \tilde{I}_{m} \tilde{I}_{b} = \sum_{k=0}^{N-1} \omega^{mk} W(k) \overline{W}'(b+k), \quad (4)$$

Francisco Alcaraz told us later that he had had plans to also go in that direction, but was warned to stay away from breaking parity by one of the authorities in the field. This, while in some sense Wu and Wang had already introduced the one-dimensional classical chiral Potts model in a short paper on duality transformations [9] and the chiral clock model was introduced a few years after that [5, 6].
and\footnote{For general normalizations of the weights one has to absorb the scalar factor $R$ in the star-triangle equation into $X_{a}$.} \begin{equation}
abla_{a} = \sum_{k=0}^{N-1} \omega^{ak} l_{k}^{a} = \sum_{k=0}^{N-1} \omega^{ak} \overline{W}^{a}(k), \quad \overline{\nabla}_{a} = \sum_{k=0}^{N-1} \omega^{ak} \tilde{l}_{k}^{a} = W^{a}(a). \end{equation}

Eliminating $X$ and $\overline{X}$ from (2), one gets the consistency equations
\begin{equation}
\frac{V_{ab} V_{00}}{V_{0b} V_{00}} = \frac{V_{ba} V_{00}}{V_{0a} V_{00}}, \quad 1 \leq a, b \leq N - 1.
\end{equation}

Note that no difference-variable assumption is made on rapidities (spectral variables). In the self-dual case $\bar{l}_{n} = l_{n}$, $V_{ab} = V_{ba}$, one can restrict oneself to $a < b$; then for $N = 3$ there is only one equation left and choosing $l_{1}$ and $l_{2}$ arbitrarily, the equation then determines a relation between $l_{1}^{a}$ and $l_{2}^{a}$, parametrizing a commuting family of transfer matrices.

More generally, assuming the existence of a one-parameter family of solutions leading to a commuting family of diagonal-to-diagonal transfer matrices $T(u)$, we should by Baxter’s well-known argument \cite{Baxter} take the derivative of the logarithm of $T(u)$ at a shift point $u = 0$, where
\begin{equation}
l_{n} = \alpha_{n} u + \beta_{n} u^{2} + O(u^{3}), \quad \tilde{l}_{n} = \bar{\alpha}_{n} u + \bar{\beta}_{n} u^{2} + O(u^{3}),
\end{equation}
for $1 \leq n \leq N - 1$, $l_{0} = \tilde{l}_{0} = 1$, so that
\begin{equation}
T(u) = T(0)[1 + u \mathcal{H} + O(u^{2})].
\end{equation}

It was easy to check that this leads to a spin-chain Hamiltonian $\mathcal{H}$ of the form
\begin{equation}
\mathcal{H} = \sum_{j=1}^{L} \left( \sum_{n=1}^{N-1} \alpha_{n} X_{j}^{n} + \sum_{n=1}^{N-1} \bar{\alpha}_{n} Z_{j}^{n} \right) + c \mathbf{1}_{N^{4}},
\end{equation}
where
\begin{equation}
Z_{j} = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{Z} \otimes \mathbf{1} \cdots \otimes \mathbf{1}, \quad X_{j} = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{X} \otimes \mathbf{1} \cdots \otimes \mathbf{1},
\end{equation}
with \cite{DiFrancesco}
\begin{equation}
X \equiv \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad Z \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \omega & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega^{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{N-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \omega^{N-1} \end{pmatrix},
\end{equation}
satisfying
\begin{equation}
ZX = \omega XZ, \quad \omega = \exp(2\pi i/N), \quad \omega^{N} = 1,
\end{equation}
and $c$ an irrelevant constant that equals 0 with the above normalization $l_{0} = \overline{l}_{0} = 1$.

For the non-self-dual $N = 3$ case there is one condition
\begin{equation}
\frac{\alpha_{1} \alpha_{2}}{\alpha_{1} \alpha_{2}} = \frac{\alpha_{1} \alpha_{2}}{\alpha_{1} \alpha_{2}}.
\end{equation}
This was first derived from the quantum Lax pair approach and then rederived using (6) and substituting (7) and a similar equation with $u$ replaced by $u'$ for $l_{n}$ and $\overline{l}_{n}$.\footnote{For general normalizations of the weights one has to absorb the scalar factor $R$ in the star-triangle equation into $X_{a}$.}
Expanding only $W'$ and $\tilde{W}'$ at the shift point $u' = 0$ one has two equivalent sets of $(N - 1)^2$ equations determining the commutation of a transfer matrix with a Hamiltonian:

$$\alpha_m \sum_{k=0}^{N-1} \frac{l_{k-m}}{l_k} \omega^{mk} = \tau_n \sum_{k=0}^{N-1} \frac{l_{k-n}}{l_k} \omega^{nk}, \quad (1 \leq m, n \leq N - 1), \quad (14)$$

$$\alpha_m \sum_{k=0}^{N-1} \frac{S_{k+m}}{S_k} \omega^{nk} = \tau_n \sum_{k=0}^{N-1} \frac{S_{k+n}}{S_k} \omega^{mk}, \quad (1 \leq m, n \leq N - 1), \quad (15)$$

with

$$S_m = \sum_{k=0}^{N-1} \omega^{mk} l_k = W(m), \quad S_m = \sum_{k=0}^{N-1} \omega^{mk} \tilde{l}_k = \sum_{k=0}^{N-1} \omega^{mk} \tilde{W}(k), \quad (16)$$

the Fourier duals of the $\tilde{l}_m$ and $l_m$. Without conditions on the $\alpha_n$ and $\tau_n$, neither (14) nor (15) allow a one-parameter family of transfer matrices, unless $N = 2$ or $N = 3$ selfdual. Now (14) is linear and homogeneous in the alphas, so that the coefficient determinant should vanish. For $N = 3$ this was one way to derive (13).

Au-Yang then proceeded to eliminate $l_1$ and $l_2$ from both systems (14) and (15) for the case $N = 3$ by the Euclidean algorithm, assuming the consistency relation (13). In the meantime McCoy and I were still wrestling with our half of the project, trying to find some nonlinear Painlevé-type equation trying to generalize the conformal field theory equations of Zamolodchikov and Fateev [2] to the massive regime. We expanded some correlations of the ABF model and tried to fit them to quadratic or cubic relations. We also tried several extensions of the Sato–Miwa–Jimbo approach. We did not get very far in spite of massive computations. Part of what we found was written up at the end of 1986 [14], but the major spin-off would come two years later.

Everything changed early October 1986, when Au-Yang came to us with the curve

$$\frac{\tau_3 - \tau_2^2}{\tau_3 - \alpha_2^2} \frac{\alpha_1 \alpha_2}{\tau_1 \tau_2} 3 \sqrt{3} i(l_1^3 + l_2^3 - l_1 l_2 - l_1^2 l_2 l_3)(1 - l_1 l_2)$$

$$= (1 + l_1^3 + l_2^3 - 3l_1 l_2)(l_1^3 + l_2^3 + l_1^2 l_2 - 3l_1^2 l_2^2)$$

$$+ \frac{\alpha_1^3 + \alpha_2^3}{\alpha_1^3 - \alpha_2^3}(l_1^3 - l_2^3)(1 + l_1^3 l_2^3 - l_1^3 - l_2^3). \quad (17)$$

None of us could recognize this curve and McCoy showed it to Sah and Kuga in the Stony Brook mathematics department. Several days later they told us that the genus of the curve (17) was 10. This violated the folklore that solutions of the quantum Yang–Baxter equations are parametrized by curves of genus at most 1!

A period of extensive checking by all four of us followed and we studied the conditions on the alphas for $N = 4$, concluding that when $N$ is not prime the solution is not unique. We found that the double Fourier transforms of (14) and (15),

$$\sum_{m=1}^{N-1} \frac{\omega^{mp}}{l_q} \omega^{-mp} = \sum_{n=1}^{N-1} \frac{\omega^{-nq}}{l_p} \omega^{nq}, \quad (0 \leq p, q \leq N - 1), \quad (18)$$

$(\alpha_0 = \tau_0 = 0)$, are easier to deal with for larger $N$. One can even subtract from (18) the same equation with $p = q = 0$. Expanding the $l_m$ and $\tilde{l}_n$, one finds equations with only $\alpha_n$ and $\tau_n$. Once we got a result for $N = 5$, Au-Yang and I guessed a general solution
\begin{equation}
\alpha_n = \frac{e^{i(2n-N)\phi/N}}{\sin(\pi n/N)}, \quad \pi_n = \lambda \frac{e^{i(2n-N)\phi/N}}{\sin(\pi n/N)}, \quad \cos \phi = \lambda \cos \tilde{\phi},
\end{equation}

and verified that it satisfies all equations for all \(N \geq 2\). McCoy and I also did an extensive literature search and found among others a paper by von Gehlen and Rittenberg [15], who had found the special case of (19) with \(\phi = \tilde{\phi} = \frac{\pi}{2}\), causing us to present (19) in the above form. We were surprised not to have found out earlier that [15] is cited in [7].

At some point early 1987 McCoy also got his student Shuang Tang involved in the checking before the paper was submitted, as he wanted to be absolutely sure about the conclusion before the submission of the letter [16]. Tang would be very involved in the next stage of the project. As the principal author of the work, Au-Yang was supposed to speak about it at the Rutgers meeting May 7–8, but she wanted me to do it [17]. There were two back-to-back talks scheduled, one by McGuire, ‘There are no higher-genus solutions of the star triangle relations’ and mine ‘Commuting transfer matrices in the chiral Potts models and solutions of star-triangle equations with genus larger than one.’ Before the talks McGuire and I compared notes and found no contradiction, as he had assumed that the weights depend only on rapidity differences, forcing the genus of the rapidity manifold to be 0 or 1. He changed his title and his talk became a good introduction to my talk [17, p 407].

At Summer Research Institute Theta Functions—Bowdoin, July 1987, I reported our results in more detail, adding several other observations that I had made, such as the equivalence of the quantum Lax pair and star-triangle equation approaches and that the Dolan–Grady criterion [18] implies the existence of an Onsager algebra, and I submitted a handwritten manuscript for publication. The proceedings came out only two years later [19], so that I had to include an update, modifying the last sentence and adding two further paragraphs, as a lot had happened since.

### 2.2. Parametrizing the N-state self-dual case

At Theta Functions—Bowdoin, Barry McCoy reported on our next nearly finished work [20]. We had noted that in (6), (14), (15) and (18) we have \((N-1)^2\) equations in the general case, (i.e. 4 for \(N = 3\), 9 for \(N = 4\), but only \(\frac{1}{2}(N-1)(N-2)\) equations in the self-dual case, (or 1 for \(N = 3\), 3 for \(N = 4\), 6 for \(N = 5\)). Therefore, we decided first to extract the curve for the integrable manifold of the \(N = 4\) self-dual case using the Euclidean algorithm both by hand and by computer using Wolfram’s SMP. Tang, McCoy and I thus obtained a curve for the 4-state self-dual case and Han Sah recognized it as a Fermat curve, mapping it to \(x^3 + y^3 = z^3\) in homogeneous coordinates [20].

Next we applied the same method to the five-state self-dual case using SMP, obtaining some curves that took several computer screens to display. But we managed to extract their rather simple common factor and using a map patterned after the \(N = 4\) case, we found a parametrization of the Boltzmann weights in terms of the curve \(x^3 + y^3 = z^3\). It did then not take much imagination to conjecture the answer for general \(N\) in terms of the Fermat curve \(x^N + y^N = z^N\). More precisely, we wrote the product form

\begin{equation}
\frac{l_n}{l_0} = b^{2n} \prod_{k=1}^{n} \frac{\omega^{-(k-1)/2} - \omega^{(N-k)/2} - \omega(N-k)/2}{\omega^{-(N-k)/2} - \omega(N-k)/2}, \quad \tilde{l}_n = l_n,
\end{equation}

with the Fermat curve given as

\begin{equation}
b^{-N}(x^N - z^N) = b^N(y^N - z^N), \quad b = e^{i\phi/N}, \quad \lambda = 1,
\end{equation}

involving a rescaling of the homogeneous coordinates \((x, y, z)\). Generically, the genus of the curve (21) is \(\frac{1}{2}(N-1)(N-2)\). For \(b = 1\), (20) reduces to the genus-zero solution of Fateev
and Zamolodchikov [21, equation (11)]. The new self-dual result was submitted in October 1987 as part of our contribution to the Sato Festschrift [22].

It was clear that we did not have the computer power to do the next step, the $N = 4$ non-self-dual case, the same way. Nevertheless, progress came soon after, during five weeks following a one week conference in Canberra in November 1987, when Rodney Baxter, Helen Au-Yang and I got together to work out the general case.

2.3. Star-triangle equation and full parametrization

The first thing to be decided was a proper notation for the Boltzmann weights. We clearly needed to incorporate Baxter’s $Z$-invariance [23, 24], but we decided to use notations from [25], see figure 1. In analogy with relativistic scattering theory we decided to call the spectral variables $p$, $q$, $r$ rapidities and to put arrows on the rapidity lines. The chirality of the interactions between spins, $a$, $b$, ..., is indicated by arrows. The Potts nature means that the weights depend on differences $a - b \mod N$. Not to give the same figures every time, I have here also given the equivalent checkerboard vertex model representation, representing the star-triangle equations as in figure 2.

For several days the three of us made attempts to generalize the conjectured product form (21) for the self-dual case. At some point I proposed to look at the Ising model and to write the theta functions of the differences of $p$ and $q$ as sums of products of functions of $p$ and $q$ separately. We could use either Onsager’s paper [26] or our two papers [24, 25]. Baxter and Au-Yang thought that I was crazy to guess a product from a single factor and I was to finish
the calculation I was doing. The next day we went for an outing in Tidbinbilla, but the morning after Baxter came up with the desired product forms

\[ W_n = \frac{\prod_{j=1}^{\infty} (d_p b_q - a_p c_q \omega^j)}{\prod_{j=1}^{\infty} (b_p d_q - c_p a_q \omega^j)} \]

Periodicity mod \( N \), \( W_p(n + N) = W_p(n) \), \( W_p(n + N) = W_p(n) \), led to the chiral Potts curve condition for the rapidities \( x_p = (a_p, b_p, c_p, d_p) \),

\[ a_p^N + k' b_p^N = k d_p^N, \quad k' a_p^N + b_p^N = k c_p^N, \quad k^2 + k'^2 = 1, \]

using a suitable normalization. We did extensive checking using Fortran on the new little Macintosh computers at Australian National University and the scalar factor \( R \) in the star-triangle equation was guessed also from Ising and checked by Fortran. Our letter \[ 27 \] contains many more results, and we obtained a proof that the star-triangle equations are satisfied using recurrences based on variations of (2). This proof was published a few years later in the appendix of \[ 28 \].

2.4. Free energy, order parameters and functional equation

During these six weeks in Canberra, I made a brief visit to Melbourne which was also very significant, because Paul Pearce gave me a copy of a preprint by Bazhanov and Reshetikhin. This preprint contained a cubic functional relation, that was omitted in the published version \[ 29 \]. As the Onsager algebra \[ 19 \] and the paper of von Gehlen and Rittenberg \[ 15 \] indicated that the chiral Potts model is a cyclic version of quantum \( \text{sl}(2) \), we conjectured that chiral Potts would have a similar functional relation. Barry McCoy thus made his students Tang and

\[ \text{Figure 2. The star-triangle equations represented as checkerboard Yang–Baxter equations. For the chiral Potts model both equations (interchanging gray and white colorings of the faces) are equivalent.} \]
Albertini solve the eigenvalue problem for small systems so that soon numerical support for our conjecture would be available.

In the meantime, Baxter obtained the first free-energy result by the ‘399th’ method [30] to solve the Ising model and he found the specific heat exponent to be $\alpha = 1 - 2/N$ for the two-dimensional classical case [31].

The small chain results for the von Gehlen–Rittenberg special case were particularly simple and McCoy coined the term ‘superintegrable’ for this case with extra Onsager-algebra integrability, as several concepts were being ‘supered’ by our high-energy colleagues in Stony Brook. Therefore, as I had some traveling to do, I wrote a couple of self-submitting batch jobs in SMP on the three Ridge Unix computers of the Institute for Theoretical Physics that gave us iteratively the ground state energy of the superintegrable chain in the commensurate phase for chains of considerable length, leading to long series expansions for $N = 3, 4, 5$ and general $N$. At some higher orders I had to work around multiplication errors in the SMP program.

Also our earlier labor doing series expansions trying to get Painlevé-type results paid off: One day McCoy told us to look again at the paper by Howes et al [32], as they had conjectured a particularly simple result for the order parameter of the three-state superintegrable quantum chain in the ordered ground state, namely $(1 - \lambda^3)^{1/3}$ [32, equation (3.13)]. They stated that they had done a series expansion that reproduced this conjecture to thirteenth order! Soon we obtained the leading terms in the expansion of the order parameters for general $N$, so that we could generalize their conjecture as

$$\langle Z_n^k \rangle = (1 - \lambda^3)^{1(N-k)/2N^2}. \quad (24)$$

Invoking Baxter’s $Z$-invariance [23], this result should apply also to the ordered phase of the full integrable chiral Potts model, both for the one-dimensional quantum chain and the two-dimensional classical case.

However, this was based on only very few terms. Luckily, just walking by, I noted a preprint of Henkel and Lacki [33] on the very top of a pile of discarded preprints in one of the garbage bins of the ITP. This gave further confirmation, as Henkel and Lacki had expanded the sum of the order parameters to one more order than we had done. Unfortunately for them, for $N \geq 4$ this sum does not have the binomial form that it has for $N = 2$ and 3. Our conjecture was submitted May 1988 in a paper with several other results [35].

Early October 1988 we received a preprint of Baxter [36] in which he solved several properties of the superintegrable case by the inversion relation. About a week later we submitted our paper [37] with our cubic functional equation that had been verified by Albertini for several chain lengths using Fortran. As a result, we found that there had to exist a commensurate–incommensurate phase transition in the superintegrable three-state chain [37] for $\lambda < 1$, so that the conjectured phase diagram in [32, figure 2] with a Lifshitz point at $\lambda = 1$ is not quite correct.

At the Taniguchi conference, October 1988, Miwa made me present a detailed proof that the weights (22) satisfy the star-triangle equation. Each time I had filled his high-tech whiteboard he printed a copy of what I had written. A more elaborate paper appears in the proceedings [28], with the proof given in the appendix. McCoy presented details [38] about the appearance of the incommensurate phase in the three-state superintegrable quantum chain. Multi-particle excitations were also studied to estimate size of the incommensurate phase [39].

3 Later Henkel and Lacki published [34], citing the preprint of [35].
2.5. Representation theoretical understanding, outlook and some prehistory

There were many further developments, especially the representation theory explanation of Bazhanov and Stroganov, valid for odd $N$ [40, 41], and an alternative approach valid for all $N$ [42]. The difference of these two approaches is discussed in the next section. These two works made the relation with cyclic representations of quantum groups as described later by De Concini and Kac [43] explicit, confirming our earlier thoughts on it.

There should be some models that have rapidities on higher-genus curves other than the one of chiral Potts. Martins [44] claims to have found a model parametrized by a K3 surface recently and it would be interesting to investigate this further.

To conclude this section, we should mention some early works of Krichever and Korepanov, of which we were not aware until some time in 1993 when we received some copies in Russian in the mail. In [45, 46] Krichever proved his theorem 1 stating that generically the genus of the curve coming from vacuum vectors of an $N$-state model related to the six-vertex model had to be $(N - 1)^2$, but he only worked out the case $N = 2$ in detail. Korepanov followed this up studying the cases $N = 3$ and greater [47], discovering thus the Boltzmann weight of some $\tau_2$ model, in agreement with Krichever’s theorem and with [45, equation (13)], [46, equation (10)]. This way, though unknown outside the Soviet Union for many years, Korepanov gave the first explicit demonstration of a solution of the quantum Yang–Baxter equation with a higher-genus parametrization, several months before the discovery of the integrable chiral Potts model.

However, Korepanov did not discover the integrable chiral Potts model, nor did he construct the $R$-matrix intertwining two cyclic representations. That construction had to wait until [40, 41]. Only with the complete construction does one know that both the horizontal and vertical transfer matrices of the $\tau_2$ model form commuting families with one set of spectral parameters (rapidities) taken from the high-genus curve and the other set from the genus-zero curve of the six-vertex model. Until [40, 41] the meaning of Korepanov’s discovery was veiled.

Finally, in the introduction of our recent paper [48] one can find some other references related to parafermions that are of historical interest, including papers on generalized Clifford algebras.

3. Part 2: odd or even

3.1. Ising case $N = 2$: Onsager algebra and Jordan–Wigner transformation

When $N = 2$, the integrable chiral Potts model becomes the Ising model. The chiral Potts spin-chain Hamiltonian (9) reduces to

$$\mathcal{H} = \sum_{j=1}^{L} (\alpha_j \sigma^z_j \sigma^z_{j+1} + \alpha_j X_j) = -\sum_{j=1}^{L} (J \sigma^x_j \sigma^x_{j+1} + B \sigma^z_j),$$

(25)

identifying $X = \sigma^x$ and $Z = \sigma^z$ as Pauli matrices, and $\alpha_1 = -J$, $\alpha_1 = -B$. This is the transverse-field Ising chain Hamiltonian, now so popular in quantum information circles. As said before, the connection with the Ising model for $N = 2$ has been very important for us to find the high-genus solutions of the star-triangle equations [27, 28].

In the Ising case of spin-$\frac{1}{2}$, the spin operators have mixed commutation relations: commuting if at different sites, but anticommuting at the same site. This was first addressed in the Ising model context by Bruria Kaufman [49], who introduced the Clifford algebra spinors,
A generalization to spin-$\frac{1}{2}$ XXZ models at roots of unity was introduced by Deguchi, Fabricius and McCoy [50]. Nishino and Deguchi [51] found a further generalization applicable to the superintegrable $\tau_2$-model when $N$ is odd, as required in [41]. This was followed by a series of papers on the superintegrable $\tau_2$ and chiral Potts models [52–59] for general $N$ using [42]. In these papers we constructed the eigenvectors in the ground state sector and the order parameters using a generalized Jordan–Wigner transform.

In these superintegrable models there is additional $\text{sl}(2)$ loop group symmetry, supporting representations of the Onsager algebra [19, 26],

$$[A_j, A_k] = 4G_{j-k}, \quad [G_m, A_j] = 2A_{j+m} - 2A_{j-m}, \quad [G_j, G_k] = 0,$$

but with a more complicated closure relation than in the Ising model. It should be noted that von Gehlen and Rittenberg [15] had already constructed the superintegrable chiral Potts quantum chain in 1985, using the Dolan–Grady criterion [18, 19, 61],

$$H = A_0 + \lambda A_1, [A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1],$$

$$[A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0].$$

which is a kind of Serre relation implying the existence of the Onsager algebra.

### 3.2. Bazhanov–Stroganov construction

A quantum group construction of the integrable chiral Potts model has first been given by Bazhanov and Stroganov [41] for odd $N$, starting from an $R$-matrix of the six-vertex model, the intertwiner of two highest-weight spin-$\frac{1}{2}$ representations. They constructed next the intertwiner of a spin-$\frac{1}{2}$ and a cyclic representation ($\tau_2$-model weights). Finally, the square of four chiral Potts weights as given in [27] and figure 3 was shown to intertwine two cyclic representations.

The more general the six-vertex $R$-matrix is chosen, the more easy it is to arrive at the chiral Potts model. Korepanov [47] had chosen $R_{00}^{00} = R_{11}^{11}$, $R_{01}^{01} = R_{10}^{10}$ and $R_{00}^{11} = R_{11}^{00}$. Bazhanov and Stroganov [41] made a special gauge choice with $R_{01}^{01} = R_{10}^{01}$, whereas [42] also ended up with $R_{01}^{01} = R_{00}^{00}$, see [60, equation (1.7)] for the comparison.

### 3.3. Baxter–Bazhanov–Perk construction

In order to get a construction valid for all $N$, Baxter, Bazhanov and Perk [42] started in the opposite direction with a star of chiral Potts model weights as in figure 4, with Boltzmann weights defined by (22) and figure 1. Summing out the central spin gives an interaction-round-a-face (IRF) model weight $U_{pp'qq'}(a, b, c, d)$, which can also be viewed as a vertex model weight using the well-known map assigning spin differences mod $N$ to the line pieces\(^4\).

If one now chooses $(x'_q, y'_q, \mu_q') = (y_q, \omega^2 x_q, \mu_q^{-1})$, then $U_{pp'qq'}(a, b, c, d) = 0$ for $0 \leq a - d \leq 1$ and $2 \leq b - c \leq N - 1$, so that $U$ is triangular and the leading block is the $R$-matrix of a $\tau_2$ model. If one also chooses $(x'_q, y'_q, \mu_q') = (y_q, \omega^2 x_q, \mu_q^{-1})$, then also $U_{pp'qq'}(a, b, c, d) = 0$ for $0 \leq d - c \leq 1$ and $2 \leq a - b \leq N - 1$, and we receive a six-vertex $R$-matrix.

\(^4\) We could also have started with a square as in figure 3, as that setup differs by a Fourier duality transform from the one in figure 4.
We consider from now on the superintegrable case $(x'_p, y'_p, \mu'_p) = (y, x, \mu^{-1})$. Then, dropping some overall factors we can write the $2^2 \times 2^2$ $R$-matrix as

$$R(x, y) = \begin{pmatrix} 1 - \frac{x}{y} \omega^{-1} & 0 & 0 & 0 \\ 0 & 1 - \frac{x}{y} \omega^{-1} & \frac{x}{y}(1 - \omega^{-1}) & 0 \\ 0 & 1 - \omega^{-1} & 1 - \frac{x}{y} \omega^{-1} & 0 \\ 0 & 0 & 0 & 1 - \frac{x}{y} \omega^{-1} \end{pmatrix}$$

and the $2N \times 2N$ $\tau_2$ $R$-matrix as

$$U(x) = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} = \begin{pmatrix} 1 - \omega x Z & -\omega x (1 - Z) X \\ X^{-1}(1 - Z) & \omega(Z - x \mathbf{1}) \end{pmatrix}$$

Here $x$ and $y$ are genus-0 rapidities of the six-vertex model, $X$ and $Z$ are the $N \times N$ matrices defined before.

Note that $R(x, y)$ is a linear combination of only $1$, $\frac{x}{y}$, $\omega^{-1}$ and $\frac{x}{y} \omega^{-1}$. This makes that choice particularly amenable for further analysis.
3.4. The monodromy matrix and its expansion coefficients

We can string $L_2$-model $R$-matrices together to form a monodromy matrix. Writing

$$
\mathcal{L}(x) = \begin{pmatrix}
    \mathcal{L}_{00} & \mathcal{L}_{01} \\
    \mathcal{L}_{10} & \mathcal{L}_{11}
\end{pmatrix} = \begin{pmatrix}
    A(x) & B(x) \\
    C(x) & D(x)
\end{pmatrix},
$$

(31)

following notations of the Faddeev school [62]. Then two such monodromy matrices $\mathcal{L}(x)$ and $\mathcal{L}(y)$ satisfy a Yang–Baxter equation with the six-vertex $R$-matrix. This is so, both for the finite-dimensional case with $\omega$ and the infinite-dimensional case with generic $q$ that we shall introduce later.

The monodromy matrix $\mathcal{L}(x)$ is a polynomial in $x$ of degree $L$. So, we write

$$
\mathcal{L}(x) = \sum_{l=0}^{L} A_l x^l, \quad B_l = \sum_{l=1}^{L} B_l x^l, \quad C_l = \sum_{l=0}^{L-1} C_l x^l, \quad D_l = \sum_{l=0}^{L} D_l x^l.
$$

(32)

In each of these four series all coefficients commute. It is easy to work out some of these coefficients explicitly [52, 56]

\begin{align*}
A_0 &= D_L = 1, \quad A_L = D_0 \omega^{-L} = \prod_{j=1}^{L} Z_j, \quad C_L = B_0 = 0, \\
B_L &= \frac{B_L}{1-\omega} = \sum_{j=1}^{L} \left( \prod_{m=1}^{j-1} Z_m \right) f_j, \quad C_0 = \frac{C_0}{1-\omega} = \sum_{j=1}^{L} \left( \prod_{m=1}^{j-1} (\omega Z_m) \right) e_j, \\
B_1 &= \frac{B_1}{1-\omega} = \sum_{j=1}^{L} f_j \left( \prod_{m=j+1}^{L} (\omega Z_m) \right), \quad C_{L-1} = \frac{C_{L-1}}{1-\omega} = \sum_{j=1}^{L} e_j \left( \prod_{m=j+1}^{L} Z_m \right).
\end{align*}

(33)

We note that some of these look like coproducts. The $e_j$ and $f_j$ are defined by having

$$
e = \frac{X^{-1}(1-Z)}{1-\omega} \quad \text{and} \quad f = \frac{(1-Z)X}{1-\omega}
$$

(34)

acting only on position $j$.

The eigenstates of the superintegrable $\tau_2$ model are highly degenerate and we can define creation and annihilation operators within each sector by

$$
x_{\alpha}^{+} = C_0^{(\alpha)} B_1^{(\alpha)}, \quad x_{\alpha}^{-} = C_0^{(\alpha)} B_1^{(\alpha)}, \quad (Q = 0, \ldots, N - 1). \quad (35)
$$

Here we defined

$$
B_1^{(n)} = \frac{B_1^n}{(n)_q}, \quad C_0^{(n)} = \frac{C_0^n}{(n)_q}
$$

(36)

using the $q$-factorial and $q$-integers

$$
(n)_q! = (n)_q (n-1)_q \cdots (1)_q, \quad (n)_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},
$$

(37)

with $q = \omega$.

For each $Q$ between 0 and $N-1$, $x_{0,Q}^{+}$ and $x_{1,Q}^{-}$ generate an $sl(2)$ (sub)algebra provided the Serre relations

$$
[x_{0,Q}^{+}, [x_{0,Q}^{+}, x_{1,Q}^{-}]] = 0, \quad [x_{1,Q}^{-}, [x_{1,Q}^{-}, x_{0,Q}^{+}]] = 0.
$$

(38)

hold.
3.5. Infinite-dimensional representation

In order to properly define all this, it will be important to analytically continue in $\omega$. But then we need the infinite-dimensional representation

$$X_q \equiv \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad Z_q \equiv \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & q & 0 & \ldots & 0 & 0 \\ 0 & 0 & q^2 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & 0 & q^{N-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

(39)

satisfying $Z_qX_q = qX_qZ_q$.

Also, we really only need

$$e_q = \frac{X_q^{-1}(1 - Z_q)}{1 - q} \quad \text{and} \quad f_q = \frac{(1 - Z_q)X_q}{1 - q}$$

(40)

so that the extra 1, present in the finite-dimensional case in the upper-right corner of $X$ and making $X$ a cyclic matrix, cancels out.

Using the $q$-integers (37), we can write

$$e_q \equiv \begin{pmatrix} 0 & (1)_q & 0 & \ldots & 0 & 0 \\ 0 & 0 & (2)_q & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & 0 & (n)_q \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

$$f_q \equiv \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ (1)_q & 0 & 0 & \ldots & 0 & 0 \\ 0 & (2)_q & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & (n)_q & 0 \end{pmatrix}$$

(41)

or

$$(e_q)_{kl} = (k)_q \delta_{k+1,l}, \quad (f_q)_{kl} = (l)_q \delta_{k,l+1}, \quad (Z_q)_{kl} = q^{k-1} \delta_{k,l}.$$ 

(42)

Also

$$e_q f_q - q f_q e_q = \frac{1 - qZ_q^2}{1 - q}, \quad e_q Z_q = q Z_q e_q, \quad Z_q f_q = q f_q Z_q.$$ 

(43)
When \( q \) is an \( N \)th root of unity, \( (N)_q = 0 \) and the first \( N \times N \) block decouples from the rest, making \( \varepsilon_q \), \( f_q \) and \( Z_q \) block diagonal. Said differently, replacing \( q \) by \( \omega \), we only need to keep the first \( N \times N \) block.

### 3.6. Commutators and \( q \)-commutators

Replacing \( \omega \) by generic \( q \) in (29), the six-vertex \( R \)-matrix with the special choice of the gauge rapidities in the Yang–Baxter equation (see [11, equation (20)]) becomes

\[
R(x, y) = \begin{pmatrix}
1 - \frac{x}{y}q^{-1} & 0 & 0 & 0 \\
0 & 1 - \frac{x}{y} & \frac{x}{y}(1 - q^{-1}) & 0 \\
0 & 1 - q^{-1} & (1 - \frac{x}{y})q^{-1} & 0 \\
0 & 0 & 0 & 1 - \frac{x}{y}q^{-1}
\end{pmatrix},
\]

(44)

We can now take any monodromy matrix (31) associated with it and write the Yang–Baxter equation out in terms of the \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \) [62]. Then we get the following sixteen equations:

\[
[A(x), A(y)] = [B(x), B(y)] = [C(x), C(y)] = [D(x), D(y)] = 0,
\]

(45)

\[
\left(1 - \frac{x}{y}q^{-1}\right)B(x)A(y) = \frac{x}{y}(1 - q^{-1})B(y)A(x) + \left(1 - \frac{x}{y}q^{-1}\right)A(y)B(x),
\]

(46)

\[
\frac{x}{y}(1 - q^{-1})C(x)A(y) + \left(1 - \frac{x}{y}\right)A(x)C(y) = \left(1 - \frac{x}{y}q^{-1}\right)C(y)A(x),
\]

(47)

\[
\frac{x}{y}(1 - q^{-1})D(x)B(y) + \left(1 - \frac{x}{y}\right)B(x)D(y) = \left(1 - \frac{x}{y}q^{-1}\right)D(y)B(x),
\]

(48)

\[
\frac{x}{y}(1 - q^{-1})D(x)B(y) + (1 - \frac{x}{y})B(x)D(y) = \left(1 - \frac{x}{y}q^{-1}\right)B(y)D(x),
\]

(49)
\[
\frac{x}{y}(1 - q^{-1})\mathcal{D}(x)A(y) + \left(1 - \frac{x}{y}\right)B(x)\mathcal{C}(y) \\
= \frac{x}{y}(1 - q^{-1})\mathcal{D}(y)A(x) + \left(1 - \frac{x}{y}\right)q^{-1}\mathcal{C}(y)B(x),
\]
\[
\frac{x}{y}(1 - q^{-1})\mathcal{C}(x)B(y) + \left(1 - \frac{x}{y}\right)A(x)\mathcal{D}(y) \\
= \left(1 - \frac{x}{y}\right)\mathcal{D}(y)A(x) + (1 - q^{-1})\mathcal{C}(y)B(x), \tag{50}
\]
\[
\left(1 - \frac{x}{y}\right)q^{-1}\mathcal{D}(x)A(y) + (1 - q^{-1})B(x)\mathcal{C}(y) \\
= \frac{x}{y}(1 - q^{-1})B(y)\mathcal{C}(x) + \left(1 - \frac{x}{y}\right)q^{-1}A(y)\mathcal{D}(x),
\]
\[
\left(1 - \frac{x}{y}\right)q^{-1}\mathcal{C}(x)B(y) + (1 - q^{-1})A(x)\mathcal{D}(y) \\
= \left(1 - \frac{x}{y}\right)B(y)\mathcal{C}(x) + (1 - q^{-1})A(y)\mathcal{D}(x). \tag{51}
\]

Note that we have a symmetry under the simultaneous replacements \(A \leftrightarrow C\) and \(B \leftrightarrow D\), as \(46 \leftrightarrow 49\) and \(47 \leftrightarrow 48\); if we replace \(x \leftrightarrow y\) then also \(50 \leftrightarrow 51\).

At this point we stress that the special asymmetric gauge of \(R(x, y)\) causes all coefficients to be linear combinations of only \(1, \frac{x}{y} \omega^{-1}\) and \(\frac{x}{y} \omega^{-1}\). This we could have because we chose to use \([42]\) rather than \([41]\). It makes expansions in powers of \(x\) and \(y\) particularly attractive.

Of course, the 16 equations \((45)\) through \((51)\) with \(x \leftrightarrow y\) are also valid. Using all 32 equations we find the following ten commutator equations and six \(q\)-commutator equations:

\[
\begin{align*}
[A(x), A(y)] &= [B(x), B(y)] = [C(x), C(y)] = [D(x), D(y)] = 0, \\
[A(x), B(y)] &= [A(y), B(x)], \quad [C(x), D(y)] = [C(y), D(x)], \\
[A(x), D(y)] &= [A(y), D(x)],
\end{align*}
\]
\[
y[A(x), C(y)] = x[A(y), C(x)], \quad y[B(x), D(y)] = x[B(y), D(x)],
\]
\[
y[A(x), D(y)] + y[B(x), C(y)] = x[A(y), D(x)] + x[B(y), C(x)],
\]
\[
[C(x), A(y)]_q = [C(y), A(x)]_q, \quad [D(x), B(y)]_q = [D(y), B(x)]_q,
\]
\[
[D(x), A(y)]_q + [C(x), B(y)]_q = [D(y), A(x)]_q + [C(y), B(x)]_q,
\]
\[
x[A(x), B(y)]_q = y[A(y), B(x)]_q, \quad x[C(x), D(y)]_q = y[C(y), D(x)]_q,
\]
\[
x[C(x), B(y)]_q = y[C(y), B(x)]_q, \quad \text{where } [\mathcal{X}, \mathcal{Y}]_q = \mathcal{X}\mathcal{Y} - q^\mathcal{Y}\mathcal{X}. \tag{52}
\]

This structure is a consequence of the special choice \((44)\).

3.7. Quantum group and coproducts

The notion of quantum groups came about after many years of progress by many people. To me two papers by Zamolodchikov and Zamolodchikov \([63]\) and by Berg et al \([64]\) were very significant, as they gave me the idea that somehow \(R\)-matrices are associated with groups. At a conference in Kyoto, May 1981, Jimbo asked me how the models that I had presented in my
talk [65] fit into a larger classification. I answered him that I believed that they should be classified with the first series of Dynkin diagrams.

Sklyanin seems to be the first to have suggested that the correct mathematical framework associated with \( R \)-matrices is Hopf algebras, in a one page note in Russian [66] on his two earlier works on quantum algebra structures [67, 68]. The works of Drinfeld [69] and Jimbo [70, 71] describe a lot of the structure of what now is called quantum groups, a term first coined by Drinfeld [72]. Here I prefer to use Jimbo’s review [73], as it also addresses the chiral Potts model, albeit only for \( N \) odd, even though the case \( N \) even can be dealt with [60] also.

To make contact with the quantum group \( U_q(\mathfrak{sl}_2) \), we assume \( q \to \pm \omega^{1/2} \), defining a proper limiting process where it is needed. We define the generators to be

\[
e_0 = q \frac{Z_q - Z_{q^{-1}}} {q - q^{-1}}, \quad e_1 = f_0 \lambda^{-1},
\]

\[
f_0 = -q e_q Z_q^{-1} = -X^T \frac{Z_q - Z_{q^{-1}}} {q - q^{-1}}, \quad f_1 = e_0 \lambda,
\]

\[
k_0 = q Z_2^2, \quad k_1 = k_0^{-1},
\]

(53)

where \( \lambda \) is a complex parameter and \( e_q \) and \( f_q \) are defined in (40). It is easily verified that these generators satisfy the required relations

\[
e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_j^{-1}} {q - q^{-1}}, \quad (54)
\]

\[
\frac{e^n}{[n]_q!} \quad \text{and} \quad \frac{f^n}{[n]_q!}, \quad \text{Laurent polynomials in } q, \quad (55)
\]

\[
k_i^{1/2} e_i = q e_i k_i^{1/2}, \quad k_i^{-1/2} e_i = q^{-1} e_i k_i^{1/2},
\]

\[
k_i^{1/2} f_i = q f_i k_i^{1/2}, \quad k_i^{-1/2} f_i = q f_i k_i^{1/2},
\]

(56)

\[
e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i^2 - e_j e_i^3 = 0,
\]

\[
f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 = 0,
\]

(57)

where

\[
[n]_q = \frac{q^n - q^{-n}} {q - q^{-1}} \quad (58)
\]

is the \( q \)-integer now. The condition (55) follows from (41) with \((n)_q\) replaced by \([n]_q\).

Having two such representations we can prove that the following coproduct satisfies the same relations:

\[
\Delta(e_0) = e_0 \otimes q^{1/2} k_0^{1/2} + q^{-1/2} k_0^{-1/2} \otimes e_0,
\]

\[
\Delta(f_0) = f_0 \otimes q^{-1/2} k_0^{1/2} + q^{1/2} k_0^{-1/2} \otimes f_0,
\]

\[
\Delta(e_1) = e_1 \otimes q^{-1/2} k_1^{1/2} + q^{1/2} k_1^{-1/2} \otimes e_1,
\]

\[
\Delta(f_1) = f_1 \otimes q^{1/2} k_1^{1/2} + q^{-1/2} k_1^{-1/2} \otimes f_1,
\]

\[
\Delta(k_0) = k_0 \otimes k_0, \quad \Delta(k_1) = k_1 \otimes k_1,
\]

(59)
Thus this coproduct is indeed a quantum group homomorphism. By induction we can
define the coproduct $\Delta^{L-1}$ with $L$ factors consistently. We only need to check the con-
sistency for $\Delta^2$, namely that we get equal results whether we replace the first factors in (59)
by their coproduct or do this for the second factors. It follows then that the coproduct $\Delta^{L-1}$ is
also a quantum group homomorphism.

Hence, realizing that the operators in (33) are coproducts, we can use this fact to greatly simplify
checking their relations, as it is sufficient to check them for $L = 2$ or 3 only. This can be
used also for proving higher Serre relations in [56], for example, to which we may return in a
future paper.

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