Mutually unbiased bases in dimension six containing a product-vector basis

Lin Chen1,2 · Li Yu3

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Abstract
Excluding the existence of four MUBs in $\mathbb{C}^6$ is an open problem in quantum information. We investigate the number of product-vectors in the set of four mutually unbiased bases (MUBs) in dimension six, by assuming that the set exists and contains a product-vector basis. We show that in most cases the number of product-vectors in each of the remaining three MUBs is at most two. We further construct the exceptional case in which the three MUBs respectively contain at most three, two and two product-vectors. We also investigate the number of vectors mutually unbiased to an orthonormal basis.

Keywords Mutually unbiased bases · Quantum tomography · Quantum cryptography

1 Introduction
Deciding the maximum number of mutually unbiased bases (MUBs) in $\mathbb{C}^6$ is a well-known open problem in quantum information theory. The study of MUBs has various applications in quantum cryptography and quantum tomography, and in fundamental problems such as the construction of Wigner functions. It has been shown that there are three MUBs in $\mathbb{C}^6$, and it has been widely conjectured that four MUBs in $\mathbb{C}^6$ do not exist. Recent progress on MUBs and the conjecture can be found in [1–18]. We adopt the notation that a unitary matrix corresponds to an orthonormal basis consisting of the column vectors of the matrix. In particular, we refer to identity as the identity matrix corresponding to the computational basis. We have investigated in [17] the
conjecture in terms of the product-vectors and Schmidt rank of the unitary matrices corresponding to the bases. We review the main result of [17] as follows.

**Lemma 1** Suppose the set of four MUBs in $\mathbb{C}^6$ exists. If it contains the identity, then

(i) any other MUB in the set contains at most two product column vectors;

(ii) the other three MUBs in the set contains totally at most six product column vectors.

The identity is a product-vector basis in $\mathbb{C}^2 \otimes \mathbb{C}^3$. In this paper we extend Lemma 1 by replacing the identity by an arbitrary product-vector basis in $\mathbb{C}^2 \otimes \mathbb{C}^3$. The latter is classified into three subsets $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ in Lemma 3(ii). We shall show in Proposition 8(ii) that if one of the product-vector basis and the set of product states in another MUB is not from $\mathcal{P}_1$ up to local unitaries then the claims (i) and (ii) in Lemma 1 both hold. Otherwise, namely if the product-vector basis and the set of product states in another MUB are both from $\mathcal{P}_1$ up to local unitaries, then the number of product states in the MUB is at most three, as we show in Proposition 8(i). If the number is exactly three, then we show in Proposition 9(i) that the remaining two non-product-vector MUBs of the four MUBs in $\mathbb{C}^6$ each has at most two product-vectors. So the number of product-vectors in such a set of four MUBs is at most $6 + 3 + 2 + 2 = 13$. This does not mean that there cannot be four MUBs with more product-vectors. We have not excluded the possibility that there are 4 product-vectors in each of the four bases. On the other hand, if a basis contains 5 product-vectors, then it contains 6 product-vectors, according to [17, Lemma 6(xx)]. We investigate the expressions of product states and the remaining entangled states in the MUBs in Proposition 9(ii) and (iii).

To obtain the above results, we start by introducing the preliminary Lemma 3. Then we investigate the number of vectors mutually unbiased to an orthonormal basis. It is a more general problem than the existence of MUBs. We list the vectors when $d = 2, 3$ in Lemma 4, and prove some statements about product-vectors unbiased to some other product states when $d = 6$. Then we reiterate some results on MUBs and complex Hadamard matrices in Lemma 5.

It is known that two quantum states in $\mathbb{C}^d$ are MU when their inner product has modulus $\frac{1}{\sqrt{d}}$. Two orthonormal basis in $\mathbb{C}^d$ are MU if their elements are all MU. We can similarly define $n$ orthonormal basis in $\mathbb{C}^d$ as $n$ MUBs in $\mathbb{C}^d$, if any two of them are MU. It is easy to see that if the first MUB is the identity matrix then all other MUBs must be complex Hadamard matrices (CHM), i.e., unitary matrices whose entries all have the same modulus $1/\sqrt{d}$. From now on we regard any order-6 CHM as a $2 \times 3$ bipartite unitary operation. The latter has been recently extensively studied in terms of the entangling power, assisted entangling power of bipartite unitaries and the relation to controlled unitary operations [17,19–24]. The first two quantities quantitatively characterize the maximum amount of entanglement increase when the input states are respectively a product state and arbitrary pure states. The maximum amount of entanglement increase over all input states is a lower bound of the entanglement cost for implementing bipartite unitaries under local operations and classical communications. Further, controlled unitary operations such as CNOT gates are fundamental ingredients in quantum computing.

The rest of the paper is organized as follows. We introduce the notations and preliminary results in Sect. 2. We include the equivalent MUBs, the MUB trio and linear
algebra. We investigate the number of vectors mutually unbiased to an orthonormal basis in Sect. 3. We construct our main results on the maximum number of product-vectors in MUBs in Sect. 4. Finally we conclude in Sect. 5.

2 Preliminaries

In this section we introduce the notations and preliminary facts used in the paper. Let $I_d$ (abbreviated as $I$ when $d$ is known) denote the order-$d$ identity matrix. Let $|i, j⟩, i = 1, \ldots, d_A, j = 1, \ldots, d_B$ be the computational basis states of the bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$. We shall refer to $|a⟩$ and $|a^⊥⟩$ as two orthonormal states. For $d = pq$ with $p, q > 1$, the basis of $\mathbb{C}^d$ consisting of product-vectors in $\mathbb{C}^p \otimes \mathbb{C}^q$ is called a product-vector basis. We say that $n$ unitary matrices form $n$ product-vector MUBs when the column vectors of these matrices are all product vectors, and they form $n$ MUBs. Next, a square matrix $C$ is a direct-product matrix if $C = F \otimes G$ where $F$ and $G$ are square matrices of order greater than one. The subunitary matrix is matrix proportional to a unitary matrix. We review the following definition. This is a simplified version of Definition 1 from [17]. We shall refer to the complex permutation matrix as a unitary matrix each of whose rows has exactly one nonzero entry.

Definition 2

(i) Let $U_1, \ldots, U_n$ be $n$ unitary matrices of order $d$. They form $n$ MUBs if and only if for an arbitrary unitary matrix $X$, and arbitrary complex permutation matrices $P_1, \ldots, P_n$, the $n$ matrices $XU_1P_1, \ldots, XU_nP_n$ form $n$ MUBs. Let $U_1, \ldots, U_n$ be product-vector MUBs such that $U_j = (\ldots, |a_{jk}\rangle, b_{jk}\rangle, \ldots)$ where $|a_{jk}\rangle \in \mathbb{C}^p$ and $|b_{jk}\rangle \in \mathbb{C}^q$. Let $U_{jA}^Γ$ and $U_{jB}^Γ$ both denote $U_j$ except that $|a_j⟩$ and $|b_j⟩$ are respectively replaced by their complex conjugates. Then we say that any two of the following four sets $U_1, \ldots, U_n, U_{1A}^Γ, \ldots, U_{nA}^Γ, U_{1B}^Γ, \ldots, U_{nB}^Γ$, $XU_1P_1, \ldots, XU_nP_n$, are LU-equivalent product-vector MUBs, where $X$ is a direct-product matrix.

(ii) Let $U$, $V$ and $W$ be three CHMs of order six. Determining the existence of four MUBs in $\mathbb{C}^6$ is equivalent to asking whether $I$, $U$, $V$ and $W$ can form four MUBs, i.e., whether $U^†V$, $V^†W$ and $W^†U$ are still CHMs. If they do, then we denote the set of $U, V$ and $W$ as an MUB trio.

(iii) Two CHMs $X$ and $Y$ are equivalent when there exist two complex permutation matrices $C$ and $D$ such that $X = CYD$. Further, $X$ and $Y$ are locally equivalent when $C$ is a direct-product matrix.

The following lemma contains some statements about linear algebra, and the first of them is from [17, Lemma 6].

Lemma 3

(i) Suppose an orthonormal basis in $\mathbb{C}^6$ contains $k$ product states with $k = 0, 1, \ldots, 5$. Then the remaining $6 - k$ states in the basis span a subspace spanned by orthogonal product vectors.
(ii) Any product-vector basis in $\mathbb{C}^2 \otimes \mathbb{C}^3$ is equivalent to a member of one of the following three sets of orthonormal bases,

$$\mathcal{P}_1 := \{ |0, 0\rangle, |0, 1\rangle, |0, 2\rangle, |1, a_0\rangle, |1, a_1\rangle, |1, a_2\rangle \};$$  \hspace{1cm} (1)

$$\mathcal{P}_2 := \{ |0, 0\rangle, |0, 1\rangle, |1, b\rangle, |1, b^\perp\rangle, |c, 2\rangle, |c^\perp, 2\rangle \};$$  \hspace{1cm} (2)

$$\mathcal{P}_3 := \{ |0, 0\rangle, |1, 0\rangle, |d, 1\rangle, |d^\perp, 1\rangle, |e, 2\rangle, |e^\perp, 2\rangle \}. $$  \hspace{1cm} (3)

Here $\{ |a_i\rangle \}$ is an orthonormal basis in $\mathbb{C}^3$, $\{ |b, |b^\perp\rangle \}$ is an orthonormal basis in $\mathbb{C}^2$, the first row and column of the matrix $|[a_0], |a_1], |a_2]\rangle$ are all $1/\sqrt{3}$, $|b, |b^\perp\rangle$, $|c, |c^\perp\rangle, |d, |d^\perp\rangle$ are all real, the first elements of $|e\rangle$ and $|e^\perp\rangle$ are both real.

(iii) Let $|a|^2 + |b|^2 = 1, |v\rangle, |w\rangle \in \mathbb{C}^3$, and $a|v\rangle + b|w\rangle$ be of elements of modulus $1/\sqrt{3}$. Then $ab = 0$ or $|v\rangle, |w\rangle$ is a matrix of size $3 \times 2$ equivalent to a real matrix.

Evidently, the computational basis $\{ |0, 0\rangle, |0, 1\rangle, |0, 2\rangle, |1, 0\rangle, |1, 1\rangle, |1, 2\rangle \} = \mathcal{P}_1 \cap \mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_3$. We claim that up to phases,

$$\{ |0, 0\rangle, |0, 1\rangle, |1, b\rangle, |1, b^\perp\rangle, |0, 2\rangle, |1, 2\rangle \} = \mathcal{P}_1 \cap (W \otimes X)\mathcal{P}_2, \forall W, X;$$  \hspace{1cm} (4)

$$\{ |0, 0\rangle, |0, 1\rangle, |0, 2\rangle, |1, 0\rangle, |1, 1\rangle, |1, 2\rangle \} = \mathcal{P}_1 \cap (U \otimes V)\mathcal{P}_3, \forall U, V$$

\hspace{2.5cm} $$= \mathcal{P}_1 \cap (W \otimes X)\mathcal{P}_2 \cap (U \otimes V)\mathcal{P}_3, \forall W, X, U, V; $$ \hspace{1cm} (5)

where $W, U$ are order-2 unitary matrices and $X, V$ are order-3 unitary matrices. To prove (4), we can see that the lhs of (4) belongs to the rhs of (4) by assuming $W|c\rangle = |0\rangle$ and $X$ as the identity matrix. Next suppose $x \in \mathcal{P}_1 \cap (W \otimes X)\mathcal{P}_2$. We obtain that $W|c\rangle = |0\rangle$ or $|1\rangle$. So $x$ belongs to the lhs of (4). We have proved that the lhs and rhs of (4) are the same. So (4) holds. One can similarly prove (5).

The relation (5) shows that the computational basis is the unique element in the intersection of three sets of product-vector bases in $\mathbb{C}^2 \otimes \mathbb{C}^3$. The computational basis corresponds to the identity matrix in the four MUBs in $\mathbb{C}^6$, and we have studied the case in [17]. So the case is the basis of studying the four MUBs in $\mathbb{C}^6$ containing a product-vector basis, as we will see in Sect. 4.

3 The number of vectors mutually unbiased to an orthonormal basis

We say that a vector is dephased if it is a zero vector or its first nonzero element is real and positive. For any order-$d$ unitary $U$, we denote an MU vector of $U$ as a dephased normalized vector unbiased to all column vectors of both $I_d$ and $U$. Let $N_v(U)$ denote the number of such vectors. They provide examples of the so-called zero noise and zero disturbance (ZNZD) states for two orthonormal bases consisting of the column vectors of $I_d$ and $U$ [25]. The $N_v$ is not an invariant under local unitary operations. A counterexample is as follows. Let $U = I_2 \otimes F_3$, where $F_3$ is the order-3 Fourier matrix. Then $N_v(U) = \infty$, though there is an order-2 unitary $X$ such that $N_v[(X \otimes I_3)U]$ is
finite. The problem of finding MU vectors is a more general problem than constructing MUBs, because sometimes the found MU vectors do not form an orthonormal basis or some set of MUBs. Finding four MUBs in $\mathbb{C}^6$ requires to find out 18 unit vectors MU to a given orthonormal basis, and these 18 vectors need to form three MUBs. To study $N_v(U)$ we construct a preliminary lemma. We shall refer to the complex number $\omega := e^{2\pi i/3}$ throughout the paper.

Lemma 4  
(i) If $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, then $N_v(U) = 2$, and the two vectors are 
$(1, i)/\sqrt{2}$ and $(1, -i)/\sqrt{2}$.

(ii) If $U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$, then $N_v(U) = 6$, and the six vectors are the column vectors in the matrix $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \omega & \omega^2 & 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 & \omega^2 & 1 & \omega \end{bmatrix}$.

(iii) Suppose two orthogonal product-vectors $|a, b\rangle$, $|a, b^\perp\rangle$ are MU to another two orthogonal product vectors $|c, d\rangle$, $|c, d^\perp\rangle$, where $|b\rangle$, $|b^\perp\rangle$, $|d\rangle$, $|d^\perp\rangle$ are 3-dimensional vectors of elements of modulus $1/\sqrt{3}$. Then $|a\rangle$ and $|c\rangle$ are MU, and $|b\rangle$, $|b^\perp\rangle$ and $|d\rangle$, $|d^\perp\rangle$ are also MU. Further if $|b\rangle$, $|b^\perp\rangle$ are two column vectors 
with the form $\begin{bmatrix} 1 \\ \omega^m \\ \omega^n \end{bmatrix}$ with some integers $m, n$, then so are $|d\rangle$, $|d^\perp\rangle$.

Proof Assertion (i) and (ii) follow from Eqs. (2.6) and (2.10) in [5].

It remains to prove (iii). We can find a complex permutation matrix $P$ such that 
$P|b\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $P|b^\perp\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$. They are MU to the two orthogonal product-vectors $P|d\rangle \propto \frac{1}{\sqrt{3}} \begin{bmatrix} x_1 \\ \omega x_1 \\ y_1 \end{bmatrix}$ and $P|d^\perp\rangle \propto \frac{1}{\sqrt{3}} \begin{bmatrix} x_2 \\ \omega^2 x_2 \\ y_2 \end{bmatrix}$ where $|x_j| = |y_j| = 1$. Suppose $|a\rangle$, $|c\rangle \in \mathbb{C}^n$. The hypothesis implies that

$$|1 + x_1 + y_1| = |1 + x_2 + y_2| = |1 + x_1 \omega^2 + y_1 \omega| = |1 + x_2 \omega^2 + y_2 \omega| = \sqrt{3} \frac{|\langle a|c \rangle|}{\sqrt{n}}.$$  \hspace{1cm} (6)

The orthogonality implies $1 + x_1 x_2^* + y_1 y_2^* = 0$. So $(x_1 x_2^*, y_1 y_2^*) = (\omega, \omega^2)$ or $(\omega^2, \omega)$.

In either case, (6) implies that the two vectors $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_1 \\ y_1 \end{bmatrix}$ and $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_2 \\ y_2 \end{bmatrix}$ are both MU to the column vectors of $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$. It follows from Lemma 4(ii) that the
two vectors are from the column vectors of \( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \omega & \omega^2 & 1 & \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \end{bmatrix} \). So (6) implies that \( |\langle a|c \rangle| = 1/\sqrt{n} \). We have proved the first assertion of (iii). The second assertion follows from the use of \( P \) in the above proof. This completes the proof. \( \square \)

We claim that there exist an integer \( d \) and order-\( d \) unitary matrices \( U \), such that \( N_v(U) = \infty \) or 2. An example for the former claim is any non-identity permutation matrix \( U \), and an example for the latter claim is \( U = [\cos \alpha \sin \alpha \sin \alpha \cos \alpha] \) where \( \alpha \in (0, \pi/2) \). On the other hand if \( U \) is a CHM, it is known that \( N_v(U) \) is finite when \( d = 2, 3 \) and 5, and infinite when \( d = 4 \) \[17\]. More results on the minimum number of MUBs and \( N_v(U) \) can be found in \[18, Table 1\]. Lemma 4 will be used in the proof of Proposition 8. The following lemma has been proved in \[17\].

**Lemma 5**

(i) Any set of four MUBs in \( \mathbb{C}^6 \) contains at most one product-vector basis. Equivalently, any two of four MUBs in \( \mathbb{C}^6 \) contain at most ten product-vectors.

(ii) An order-6 CHM is a member of some MUB trio if and only if so is its adjoint matrix, if and only if so is its complex conjugate, and if and only if so is its transpose. Further, if \( k \leq 3 \) then \( k \) order-6 CHMs are the members of some MUB trio if and only if so are their complex conjugate.

(iii) The product state \( |a, b \rangle \) in \( \mathbb{C}^d \) is MU to an orthogonal product-vector basis \( \{|x_i, y_i\} \}_{i=1,...,d} \) if and only if \( |a \rangle \) is MU to \( \{|x_i\} \}_{i=1,...,d} \) and \( |b \rangle \) is MU to \( \{|y_i\} \}_{i=1,...,d} \).

(iv) Any MUB trio contains none of the order-6 CHMs \( Y_1, \ldots, Y_6 \) where

1. \( Y_1 \) contains an order-3 subunitary matrix.
2. \( Y_2 \) contains a submatrix of size 3 \( \times \) 2 and rank one.
3. \( Y_3 \) contains an order-3 submatrix whose one column vector is orthogonal to the other two column vectors.
4. Two column vectors of \( Y_4 \) are product-vectors \( |a, b \rangle \) and \( |a, c \rangle \).
5. \( Y_5 \) contains an order-3 singular submatrix.
6. \( Y_6 \) contains a real submatrix of size 3 \( \times \) 2.

### 4 Mutually unbiased bases containing a product-vector basis

In this section we present the main results of this paper, namely Proposition 8 and 9. In Lemma 5(i), we have investigated when two of four MUBs in \( \mathbb{C}^6 \) have at most 10 product column vectors. We hope to decrease the number. This is a problem different from Lemma 4, in which the two MUBs may not belong to a set of four MUBs. The motivation of the problem is as follows. We have met many sets of four MUBs in which an MUB consists of product column vectors. The product column vectors in other three MUBs may decide the structure of the four MUBs or their existence. An approach to the problem is assuming that up to local unitaries, an MUB is from \( P_1, P_2 \) and \( P_3 \) in Lemma 3(ii). If it corresponds to the identity matrix, each of the other three MUB contains at most two product-vectors by Lemma 1. On the other hand, we have
shown that the identity matrix is a subcase of the product-vector basis in Lemma 3(ii). So studying the MUBs under the assumption extends Lemma 1 more generally helps understand the existence of four MUBs in \( \mathbb{C}^6 \). We begin by presenting the following two lemmas.

**Lemma 6** Suppose there are four MUBs in \( \mathbb{C}^6 \), and the first one of them is \( \mathcal{P}_j \) for \( j \in \{1, 2, 3\} \). Then any product state in other three MUBs has elements of modulus 1/\( \sqrt{6} \). That is, up to global phases the product state has the expression \((1, u)^T / \sqrt{2} \otimes (1, v, w)^T / \sqrt{3}\) where \(|u| = |v| = |w| = 1\).

Suppose one of the other three MUBs contains exactly \( n \) product states. We have

(i) if \( j = 2 \), then \( n \leq 2 \); 
(ii) if \( j = 3 \), then \( n \leq 2 \).

**Proof** Let the product state be \((a, b)^T \otimes (c, d, e)^T\). It follows from Lemma 3(ii) and Lemma 5(iii) that \(|a| = |b| = 1/\sqrt{2}\) and \(|c| = |d| = |e| = 1/\sqrt{3}\). We have proved the first assertion. Next we prove the second assertion consisting of (i) and (ii). Suppose the second MUB of the four MUBs is the order-6 unitary matrix \( U \), and it contains exactly \( n \) product states. Using the first assertion, we may assume that one of the \( n \) product states is

\[
(1, u)^T / \sqrt{2} \otimes (1, v, w)^T / \sqrt{3},
\]

(7)

where \(|u| = |v| = |w| = 1\).

(i) It follows from Lemma 3(ii) that the first MUB is \( \mathcal{P}_2 = \{|0, 0\}, |0, 1\}, |1, 0\}, \{|1, 0\}, |1, 1\}, \{|c_0, 2\}, \{|c_1, 2\}\) with real orthonormal states \(|b_0\), \(|b_1\) \in \( \mathbb{C}^2 \) and real orthonormal states \(|c_0\), \(|c_1\) \in \( \mathbb{C}^2 \). If both of them are the basis \(|0\), \(|1\) then \( n \leq 2 \) follows from Lemma 1. Suppose \(|b_0\), \(|b_1\) is not the basis \(|0\), \(|1\). It follows from Lemma 5(iii) and (7) that \(|b_0\), \(|b_1\) are both MU to \((1, v, w)^T / \sqrt{3}\). Hence \( v = i \) or \(-i\). We can assume that the upper left submatrix of \( U \) of size 2 \( \times \) \( n \) is \( V = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & \cdots & 1 \\ p_1 i & \cdots & p_n i \end{bmatrix} \)

where \( p_j = 1 \) or \(-1\). Let \( I_2 + W \) be an order-6 unitary such that \((I_2 + W)\mathcal{P}_2 = I_6\), and the upper left submatrix of \((I_2 + W)U\) of size 2 \( \times \) \( n \) is still \( V\). Since \( \mathcal{P}_2 \) and \( U \) are two members of four MUBs, \((I_2 + W)U\) is a member of some MUB trio. We have \( n \leq 2 \) by Lemma 5(ii) and the matrix \( Y_6 \) in (iv).

The remaining case is that \(|b_0\), \(|b_1\) is the basis \(|0\), \(|1\), and \(|c_0\), \(|c_1\) is not the basis \(|0\), \(|1\). It follows from Lemma 5(iii) and (7) that \(|c_0\), \(|c_1\) are both MU to \((1, u)^T / \sqrt{2}\). Hence \( u = i \) or \(-i\). We can assume that the 2 \( \times \) \( n \) submatrix formed by the first and fourth rows of \( U \) is \( \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & \cdots & 1 \\ p_1 i & \cdots & p_n i \end{bmatrix} \) where \( p_j = 1 \) or \(-1\). Let \( X = I_2 \otimes I_2 + W \otimes |2\rangle \langle 2| \) be an order-6 unitary such that \( XP_2 = I_6\). So \( XU \) is a member of some MUB trio. We have \( n \leq 2 \) by Lemma 5(ii) and the matrix \( Y_6 \) in (iv).

(ii) It follows from Lemma 3(ii) that the first MUB is \( \mathcal{P}_3 = \{|0, 0\}, |1, 0\}, \{|d_0, 1\}, |d_1, 1\}, \{|e_0, 2\}, |e_1, 2\}\). Here \(|d_i\}) and \(|e_i\}) are all orthonormal bases in \( \mathbb{C}^2 \), \(|d_i\}) and the first elements of \(|e_i\}) are both real. If both of them are the basis \(|0\), \(|1\) then \( n \leq 2 \) follows from Lemma 1. If one of \(|d_0\), \(|d_1\}) and \(|e_0\), \(|e_1\}) is the basis \(|0\), \(|1\), then \( \mathcal{P}_3 \) is locally equivalent to some \( \mathcal{P}_2 \). So \( n \leq 2 \) follows
from (i). Suppose neither of $|d_0\rangle$, $|d_1\rangle$ and $|e_0\rangle$, $|e_1\rangle$ is the basis $|0\rangle$, $|1\rangle$. It follows from Lemma 5(iii) and (7) that $|d_0\rangle$, $|d_1\rangle$ are both MU to $(1, u)^T/\sqrt{2}$. Hence $u = i$ or $-i$. Using (7), we can assume that the $2 \times n$ submatrix formed by the first and fourth rows of $U$ is $V = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & \cdots & 1 \\ p_1i & \cdots & p_ni \end{bmatrix}$ where $p_j = 1$ or $-1$. Let $X = I_2 \otimes |0\rangle\langle 0| + W \otimes |1\rangle\langle 1| + W' \otimes |2\rangle\langle 2|$ be an order-6 unitary such that $X P_3 = I_6$. So $X U$ is a member of some MUB trio, and the $2 \times n$ submatrix formed by the first and fourth rows of $X U$ is $V$. We have $n \leq 2$ by Lemma 5(ii) and the matrix $Y_6$ in (iv). This completes the proof. \hfill \qed

**Lemma 7** Suppose there are four MUBs in $\mathbb{C}^6$, the first one of them is $P_1$ and the second one of them contains exactly $n$ product states. We have

(i) if the n product states are from $P_1$ up to local unitaries then $n \leq 3$. Further if $n = 3$ then the three product states are $|0, 0\rangle$, $|0, 1\rangle$, $|1, a_0\rangle$ up to local unitaries, and $\langle j | a_0 \rangle \neq 0$ for $j = 0, 1, 2$;

(ii) if the n product states are from $P_2$ or $P_3$ up to local unitaries then $n \leq 2$.

**Proof** Equation (1) says that $P_1 = \{ |0, 0\rangle, |0, 1\rangle, |0, 2\rangle, |1, a_0\rangle, |1, a_1\rangle, |1, a_2\rangle \}$, where $|a_0\rangle$, $|a_1\rangle$ and $|a_2\rangle$ form an orthonormal basis in $\mathbb{C}^3$. Suppose the second one of the four MUBs is $U = \{ |w_0, x_0\rangle, \ldots, |w_{n-1}, x_{n-1}\rangle, |0, y_n\rangle + |1, z_n\rangle, \ldots, |0, y_5\rangle + |1, z_5\rangle \}$. Let $V$ be an order-3 unitary matrix such that $V |a_i\rangle = |i\rangle$ for $i = 0, 1, 2$. Then $(I_3 \oplus V) U$ is a member of some MUB trio, and it is an order-6 CHM. So any $|w_j\rangle$ with $j \leq n - 1$ is of elements of modulus $1/\sqrt{2}$, any $|x_j\rangle$ with $j \leq n - 1$ is of elements of modulus $1/\sqrt{3}$, and any $|y_j\rangle$ with $n \leq j \leq 5$ is of elements of modulus $1/\sqrt{6}$. The upper left submatrix of size $3 \times n$ of $(I_3 \oplus V) U$ is $X := \{ |0, y_n\rangle + |1, z_n\rangle, \ldots, |0, y_5\rangle + |1, z_5\rangle \}$ span a $(6 - n)$-dimensional subspace spanned by orthogonal product-vectors $|w_n, x_n\rangle, \ldots, |w_5, x_5\rangle$. The states $|0, y_j\rangle + |1, z_j\rangle$ is entangled because $U$ contains exactly $n$ product states. Lemma 3(ii) implies that $Z := \{ |w_0, x_0\rangle, \ldots, |w_5, x_5\rangle \}$ is from some $P_j$ up to local unitaries. So the states $|x_0\rangle, \ldots, |x_5\rangle$ are equal to the 3-dimensional states in the product states of $P_j$ up to local unitaries.

Suppose $Z$ is from $P_2$ up to local unitaries. If $n \geq 3$ then $X$ contains three column vectors which are linearly dependent, or one of which is orthogonal to the other two. It is a contradiction with $Y_3$ and $Y_5$ in Lemma 5(iv), because $X$ is a submatrix of $(I_3 \oplus V) U$ which is a member of some MUB trio. Hence $n \leq 2$. One can similarly show that if $Z$ is from $P_3$ up to local unitaries then $n \leq 2$. So we have proved (ii).

It remains to prove (i). Suppose $Z$ is from $P_1$ up to local unitaries. Let $n \geq 4$. Lemma 5(i) shows that $n = 4$. The argument for (ii) shows that $|w_0\rangle = |w_1\rangle = |w_4\rangle$, $|w_2\rangle = |w_3\rangle = |w_5\rangle$, and $|x_0\rangle, |x_1\rangle, |x_4\rangle$ and $|x_2\rangle, |x_3\rangle, |x_5\rangle$ are two orthonormal basis of $\mathbb{C}^3$. Since $|w_0\rangle, \ldots, |w_3\rangle$ are all of elements of modulus $1/\sqrt{2}$, so are $|w_4\rangle$ and $|w_5\rangle$. Since $|x_0\rangle, \ldots, |x_3\rangle$ are all of elements of modulus $1/\sqrt{3}$, so are $|x_4\rangle$ and $|x_5\rangle$. Recall that $|0, y_4\rangle + |1, z_4\rangle$ and $|0, y_5\rangle + |1, z_5\rangle$ span a 2-dimensional subspace spanned by
\[ |w_4, x_4 \rangle \text{ and } |w_5, x_5 \rangle. \] There are two complex numbers \( \alpha, \beta \) such that \(|\alpha|^2 + |\beta|^2 = 1\) and \(\alpha ((0, y_4) + |1, z_4 \rangle) + \beta (|0, y_5 \rangle + |1, z_5 \rangle) = |w_4, x_4 \rangle. \) So

\[ \alpha |y_4 \rangle + \beta |y_5 \rangle = \frac{1}{\sqrt{2}} |x_4 \rangle. \] (8)

Recall that \(|y_4 \rangle\) and \(|y_5 \rangle\) both have elements of modulus \(1/\sqrt{6}\). Applying Lemma 3(iii) to (8) we obtain \(\alpha \beta = 0\) or that \([|y_4 \rangle, |y_5 \rangle]\) is equivalent to a real matrix of size \(3 \times 2\). The former results in the fifth product-vector in \(U\), and the latter gives us a contradiction with \(Y_6\) in Lemma 5(iv). So both are excluded. We have proved \(n \leq 3\).

The last assertion of (i) except the claim \( \langle j|a_0 \rangle \neq 0 \) for \( j = 0, 1, 2 \) follows from the above argument.

It remains to prove the above-mentioned claim. Suppose \( n = 3 \). If \( \langle j|a_0 \rangle = 0 \) for some \( j \) then \( X \) contains three column vectors which are linearly dependent, or one of which is orthogonal to the other two. It is a contradiction with \(Y_3\) and \(Y_5\) in Lemma 5(iv), because \(X\) is an order-3 submatrix of \((I_3 \oplus V)U\) which is a member of some MUB trio. Hence \(\langle j|a_0 \rangle \neq 0 \) for \( j = 0, 1, 2 \). We have proved (i). This completes the proof. \( \square \)

The last condition in Lemma 7(i), namely \( \langle j|a_0 \rangle \neq 0 \) for \( j = 0, 1, 2 \) implies that the three product states in Lemma 7(i) are not from \( P_2 \) or \( P_3 \) up to local unitaries. So Lemma 7(i) and (ii) investigate different MUBs.

Based on the above results, we characterize below the four MUBs in \( \mathbb{C}^6 \) containing a product-vector basis. We shall study the first two MUBs in Proposition 8, and all MUBs in Proposition 9.

**Proposition 8** Suppose there are four MUBs in \( \mathbb{C}^6 \), the first MUB consists of six product states and the second MUB contains exactly \( n \) product states. We have

(i) if the first MUB and the \( n \) product states are both from \( P_1 \) up to local unitaries then \( n \leq 3 \). Further if \( n = 3 \) then up to local unitaries the first MUB is \( I_3 \oplus U \) where

\[ U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \] (9)

and at the same time the second MUB consists of three product states

\[ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \omega \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \omega \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}, \] (10)
and three Schmidt-rank-two entangled states

\[ u_{j0} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix} + u_{j1} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} x\omega \\ y\omega^2 \end{bmatrix} + u_{j2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} x\omega^2 \\ y\omega \end{bmatrix}, \tag{11} \]

and \(|\alpha| = |\beta| = |x| = |y| = 1, j = 0, 1, 2, \) and \([u_{jk}]\) is an order-3 unitary matrix.

(ii) If one of the first MUB and the \(n\) product states in the second MUB is not from \(\mathcal{P}_1\) up to local unitaries then \(n \leq 2\).

(iii) If the first MUB is \(\mathcal{P}_j\) for \(j \in \{1, 2, 3\}\), then the product state in the second MUB has the expression \((1, u)^T/\sqrt{2} \otimes (1, v, w)^T/\sqrt{3}\) where \(|u| = |v| = |w| = 1\).

(iv) In (i), \(\alpha, x, y\) are not equal to \(\omega^m\) for any integer \(m\). Further \([u_{jk}]\) has no zero entries.

**Proof** (i, ii, iii) The first assertion of (i), and assertion (ii) and (iii) follow from Lemma 6 and 7. Using Lemma 7 we may assume that the first MUB is \(I_3 \oplus V\), where \(V\) is an order-3 unitary matrix. Using (iii) and Lemma 7(i) we may assume that the three product-vectors in the second MUB are locally equivalent to the product-vectors in (10). Let the operation for local equivalence be \(Q \otimes R\). It can be chosen as a diagonal unitary matrix, since the three product-vectors in the second MUB are unbiased to the first three column vectors in the first MUB. Then the first MUB and the product vectors in the second MUB are, respectively and simultaneously, locally equivalent to \((Q \otimes R)(I \otimes V)\) and the product-vectors in (10). Since the MUB is an order-6 unitary matrix, we can obtain (11) using (10). So we have proven (10) and (11).

It remains to prove (9). Setting \(RV = W\) we obtain that the first MUB is locally equivalent to \(I_3 \oplus W\). Since MUBs are unitary matrices in \(\mathbb{C}^6\), the submatrix \(\frac{1}{\sqrt{3}} W^\dagger \begin{bmatrix} 1 & 1 \\ 1 & \omega \\ 1 & \omega^2 \end{bmatrix}\) have elements of modulus \(1/\sqrt{3}\). Since the two column vectors are orthogonal, we can find an order-3 complex permutation matrix \(P\) such that \(\frac{1}{\sqrt{3}} P^\dagger W^\dagger \begin{bmatrix} 1 & 1 \\ 1 & \omega \\ 1 & \omega^2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \alpha^* \\ 1 & \alpha^* \omega \\ 1 & \alpha^* \omega^2 \end{bmatrix}\) with some \(|\alpha| = 1\). So \(\frac{1}{\sqrt{3}} P^\dagger W^\dagger \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \alpha^* & \beta^* \\ 1 & \alpha^* \omega & \beta^* \omega^2 \\ 1 & \alpha^* \omega^2 & \beta^* \omega \end{bmatrix}\) with some \(|\beta| = 1\), because the matrices in the square brackets are unitary. Assuming \(U = WP\) implies the assertion, note that we are free to multiply any complex permutation matrix on the rhs of an MUB. We have ignored \(Q\) because its effect can be absorbed into multiplying each column vector of the first MUB by a phase, which means multiplying a special complex permutation matrix on the rhs of the first MUB. So we have proved (9).
(iv) Let \( M \) be the second MUB. Since the first MUB is \( I_3 \oplus U \), we obtain that \((I_3 \oplus U^\dagger)M\) is a member of some MUB trio. Recall that \( M \) contains the three product-vectors in (10), and \( U^\dagger \begin{bmatrix} 1 & 1 \\ 1 & \omega \\ 1 & \omega^2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \alpha^* \\ 1 & \alpha^* \omega \\ 1 & \alpha^* \omega^2 \end{bmatrix} \). So \((I_3 \oplus U^\dagger)M\) contains the two columns \((1, \ldots, 1)/\sqrt{6}\) and \((1, \omega, \omega^2, \alpha^*, \alpha^* \omega, \alpha^* \omega^2)/\sqrt{6}\). If \( \alpha = 1, \omega \) or \( \omega^2 \), then (by permuting the last three rows) these two columns are equivalent to two product-vectors \((1, 1)/\sqrt{2} \otimes (1, 1, 1)/\sqrt{3}\) and \((1, 1)/\sqrt{2} \otimes (1, \omega, \omega^2)/\sqrt{3}\). It is a contradiction with the matrix \( Y_4 \) in Lemma 5(iv). We have proved the assertion \( \alpha \neq 1, \omega, \omega^2 \).

It remains to prove that \((x, y) \neq (\omega^m, \omega^n)\) for \( m, n = 0, 1, 2 \). Note that when the assumption \( n = 3 \) holds, the other assumption that the first MUB and the \( n \) product states are both from \( P_1 \) up to local unitaries automatically holds, because we have proved assertion (ii). Then the matrix \((I_3 \oplus U^\dagger)M\) is a member of some MUB trio. If \( \frac{1}{\sqrt{3}} \begin{bmatrix} x \\ y \end{bmatrix} \) in (i) is a column vector of \( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega & \omega^2 & \omega \end{bmatrix} \), then the upper three rows of \((I_3 \oplus U^\dagger)M\) has a rank one matrix of size \( 3 \times 2 \), or an order-3 subunitary matrix. This is a contradiction with the matrices \( Y_1 \) and \( Y_2 \) in Lemma 5(iv). Suppose \( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \) in (i) is a column vector of \( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \omega & \omega^2 & 1 & \omega \end{bmatrix} \). One can verify that

\[
\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \omega & \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 & \omega \\ \omega & \omega & 1 & \omega \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} i & \omega^2 i & \omega^2 i & -i \\ \omega^2 i & i & \omega^2 i & -i \\ \omega^2 i & \omega^2 i & i & -i \\ \omega^2 i & \omega^2 i & \omega^2 i & -i \end{bmatrix} \cdot \text{diag} \left( i, \omega^2 i, -i, -\omega i, -\omega i \right). \tag{12}
\]

So \( \frac{1}{\sqrt{3}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & \omega \\ \omega & \omega^2 & \omega \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} x \\ y \end{bmatrix} \) is a vector in the second equation of (12). Note that \( U^\dagger = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} 0 & \alpha^* & 0 \\ 0 & 0 & \beta^* \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ \omega & \omega \end{bmatrix} \). Since the first MUB is \( I_3 \oplus U \) and the second MUB contains the product-vector \( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \)
$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$, we obtain that $U^\dagger \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ \omega \omega^2 \omega \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \alpha^* v_2 \\ \beta^* v_3 \end{bmatrix}$ is a vector of elements of modulus $1/\sqrt{3}$. It follows from Lemma 4(ii) that $\begin{bmatrix} v_1 \\ \alpha^* v_2 \\ \beta^* v_3 \end{bmatrix}$ is proportional to one of the six column vectors in $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \omega^2 \omega \omega^2 \omega \omega \end{bmatrix}$. It follows from the third equation of (12) that $\text{diag}(1, \alpha^*, \beta^*) \cdot \begin{bmatrix} 1 \\ 1 \\ \omega \omega^2 \omega \omega \omega^2 \omega \omega \end{bmatrix}$ is from the columns of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \omega \omega \omega^2 \omega \omega \omega \end{bmatrix}$. So $(\alpha, \beta) = (\omega^m, \omega^n)$ with $m, n = 0, 1, 2$. It is a contradiction with the fact that $\alpha \neq 1, \omega, \omega^2$ proved in the first part of (iv). We have proved that $(x, y) \neq (\omega^m, \omega^n)$ for $m, n = 0, 1, 2$.

It remains to prove the last assertion of (iv). Suppose $u_{jk} = 0$ for some $j, k$. Since (11) is a column vector of the second MUB, [23, Lemma 1] and the above observation imply that $u_{jk} = 0$ for some $k' \neq k$. It is a contradiction with the fact that (11) is entangled. This completes the proof. □

**Proposition 9** Suppose a set of four MUBs in $\mathbb{C}^6$ contains a product-vector MUB.

(i) If one of the remaining three MUBs in the set has exactly three product-vectors then either of the other two MUBs has at most two product-vectors.

(ii) The number of product-vectors in the set is at most $6+3+2+2=13$. It is achievable only if up to local unitaries, the first MUB is $I_3 \oplus U$ with $U$ in (9), and the remaining three MUBs respectively have the following product vectors,

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \omega \omega^2 \omega \omega \omega \omega^2 \omega \omega \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix},
\]

(13)

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ u \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_1 \\ y_1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -u \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_2 \\ y_2 \end{bmatrix},
\]

(14)

\[
\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ v \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_3 \\ y_3 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -v \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_4 \\ y_4 \end{bmatrix}
\]

(15)
with $|x| = |y| = |u| = |v| = |x_j| = |y_j| = 1$.

(iii) The remaining three MUBs in (ii) respectively have the following Schmidt-rank-two entangled states,

\[
a_{j0} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \omega^2 \end{bmatrix} + a_{j1} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x\omega \end{bmatrix} \\
+a_{j2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x\omega^2 \end{bmatrix},
\]

(16)

\[
b_{j0} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ u \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_1\omega \end{bmatrix} + b_{j1} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ u \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ y_1\omega \end{bmatrix} \\
+b_{j2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -u \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_2\omega \end{bmatrix} \\
+b_{j3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -u \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ y_2\omega \end{bmatrix},
\]

(17)

and

\[
c_{j0} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ v \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_3\omega \end{bmatrix} + c_{j1} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ v \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ y_3\omega \end{bmatrix} \\
+c_{j2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -v \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_4\omega \end{bmatrix} \\
+c_{j3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -v \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ y_4\omega \end{bmatrix},
\]

(18)

where $[a_{jk}]$ is an order-3 unitary matrix, $[b_{jk}]$ and $[c_{jk}]$ are two order-4 unitary matrices.

**Proof** (i) Using Proposition 8 and local unitaries, we may assume that the first MUB is $P_1 = I_3 \oplus U$ with $U$ given in (9), and the second MUB has three product-vectors in (10), i.e., $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ \omega \end{bmatrix}$, and $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x \end{bmatrix}$ with $|x| = |y| = 1$. Suppose the third MUB has three product vectors. It follows from
Proposition 8(iii) that they are all of elements of modulus $1/\sqrt{6}$. Using Proposition 8(i) we may assume that the three product-vectors are $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ x_1 \\ y_1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_3 \\ y_3 \end{bmatrix}$, and $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ x_2 \\ y_2 \end{bmatrix}$, with $|v| = |x_j| = |y_j| = 1$. Thus $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_1 \\ y_1 \end{bmatrix}$ and $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_2 \\ y_2 \end{bmatrix}$ are orthogonal. Since the vectors from the second and third MUBs are MU, the first two expressions in Eq. (10) together with Lemma 4(ii) imply that $v = i$ or $-i$, and $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_1 \\ y_1 \end{bmatrix}$ and $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_2 \\ y_2 \end{bmatrix}$ are orthogonal column vectors in the matrix $M = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \omega & \omega^2 & 1 & \omega^2 & \omega & 1 \\ \omega & \omega & \omega^2 & 1 & \omega \end{bmatrix}$. The same reason implies that $(x, y) = (\omega^m, \omega^n)$ for some integers $m, n$. It is a contradiction with Proposition 8(iv), so the third MUB have at most two product-vectors. If it is achievable, then the above argument implies that they are $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ x_1 \\ y_1 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_3 \\ y_3 \end{bmatrix}$ and $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ x_2 \\ y_2 \end{bmatrix} \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ x_3 \\ y_3 \end{bmatrix}$ with $|v| = |x_j| = |y_j| = 1$.

The claim (ii) follows from (i) and its proof. The claim (iii) follows from (ii). This completes the proof.

We have tried to show the claim that if a set of four MUBs in $\mathbb{C}^6$ contains a product-vector basis then any of the other three bases contains at most two product-vectors. Although we have found indications that some specific choices of $u, v, x_j, y_j$ cannot appear in Proposition 9(ii), a proof to the claim is still missing. We believe that the expressions of MUBs in Proposition 9 will give more constraints and result in a contradiction with the existence of the four MUBs containing a product-vector basis.

5 Conclusions

We have investigated the existence of sets of four MUBs in $\mathbb{C}^6$, by assuming that four MUBs in $\mathbb{C}^6$ containing a product-vector basis exist. We have shown that if the product-vector basis (regarded as the first basis among the four) and the set of product states in another MUB is not from $\mathcal{P}_1$ up to local unitaries then any other MUB in the set contains at most two product-vectors. In any case, the number of product states in the second MUB is at most three. We also have showed that when the number is exactly three, the remaining two non-product-vector MUBs in the set of four MUBs each contains at most two product-vectors. We have also investigated the expressions of product states and the remaining entangled states in the MUBs. A possible next step in the study of this problem is to find tighter upper bounds on the number of product-
vectors in four MUBs in $\mathbb{C}^6$ containing a product-vector basis (or in particular, the standard basis).

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