ON THE STOCHASTIC LIE ALGEBRA

MANUEL GUERRA, ANDREY SARYCHEV

Abstract. We study the structure of the Lie algebra $\mathfrak{s}(n,\mathbb{R})$ corresponding to the so-called stochastic Lie group $S(n,\mathbb{R})$. We obtain the Levi decomposition of the Lie algebra, classify Levi factor and classify the representation of the factor in $\mathbb{R}^n$. We discuss isomorphism of $S(n,\mathbb{R})$ with the group of invertible affine maps $\text{Aff}(n-1,\mathbb{R})$. We prove that $\mathfrak{s}(n,\mathbb{R})$ is generated by two generic elements.

1. Stochastic Lie group and stochastic Lie algebra

Let $S_0^+(n,\mathbb{R})$ denote the space of transition matrices of size $n$, i.e., the space of real $n \times n$ matrices with all entries non-negative and row sums equal to 1.

One important motivation for the study of such matrices is their relation to Markov processes: It is easy to see that for any Markov process $X$ with $n$ possible states, the family

$$P(s, t) = [p_{i,j}(s, t)]_{1 \leq i, j \leq n}, \quad 0 \leq s \leq t < +\infty,$$

where $p_{i,j}(s, t)$ is the probability of $X_t = j$, conditional on $X_s = i$, is a family of transition matrices such that

$$(1.1) \quad P_{s,t} = P_{u,t}P_{s,u}, \quad \forall 0 \leq s \leq u \leq t < +\infty.$$

Conversely, the Kolmogorov extension theorem (see e.g. [2], Theorem IV.4.18), states that for every family $\{P(s, t) \in S_0^+(n,\mathbb{R})\}_{0 \leq s \leq t < +\infty}$ satisfying (1.1), there exists a Markov process $X$ such that $p_{i,j}(s, t) = \text{Pr}\{X_t = j | X_s = i\}$ for every $i,j \leq n$ and every $0 \leq s \leq t < +\infty$.

Let $S^+(n,\mathbb{R})$ denote the space of nonsingular transition matrices. It is clear that $S_0^+(n,\mathbb{R})$ is a semigroup with respect to matrix multiplication, and $S^+(n,\mathbb{R})$ is a subsemigroup. However, $S^+(n,\mathbb{R})$ is not a group, since the inverse of a transition matrix is not, in general, a transition matrix.

The smallest group containing $S^+(n,\mathbb{R})$ is denoted by $S(n,\mathbb{R})$. Due to the considerations above, this is called the stochastic group [3]. It can be shown that

$$S(n,\mathbb{R}) = \{ P \in \mathbb{R}^{n \times n} : \text{Det}(P) \neq 0, \text{P1 = 1} \},$$

Manuel Guerra was partly supported by FCT/MEC through the project CEMAPRE UID/MULTI/00491/2013.
where \( \mathbf{1} \) is the \( n \)-dimensional vector with all entries equal to 1. It follows that \( \mathcal{S}(n, \mathbb{R}) \), provided with the topology inherited from the usual topology of \( \mathbb{R}^{n \times n} \), is a \( n \times (n - 1) \) dimensional analytic Lie group.

The Lie algebra of \( \mathcal{S}(n, \mathbb{R}) \) is called stochastic Lie algebra, and is denoted by \( \mathfrak{s}(n, \mathbb{R}) \). Notice that \( \mathfrak{s}(n, \mathbb{R}) \) is isomorphic to the tangent space of \( \mathcal{S}(n, \mathbb{R}) \) at the identity
\[
\mathfrak{s}(n, \mathbb{R}) \sim T_{Id} \mathcal{S}(n, \mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} : A \mathbf{1} = 0 \},
\]
\( \mathfrak{s}(n, \mathbb{R}) \) is provided with the matrix commutator
\[
[A, B] = AB - BA.
\]

We introduce the subset
\[
\mathfrak{s}^+(n, \mathbb{R}) = \{ A \in \mathfrak{s}(n, \mathbb{R}) : a_{i,j} \geq 0, \forall i \neq j \}.
\]

It is clear that \( \mathfrak{s}^+(n, \mathbb{R}) \) is not a subalgebra of \( \mathfrak{s}(n, \mathbb{R}) \), but it is a convex cone with nonempty interior in \( \mathfrak{s}(n, \mathbb{R}) \). Since \( \mathcal{S}^+(n, \mathbb{R}) \) is invariant under the flow by ODE's of type
\[
\dot{P}_t = P_t A,
\]
with \( A \in \mathfrak{s}^+(n, \mathbb{R}) \), it follows that \( \mathcal{S}^+(n, \mathbb{R}) \) has nonempty interior in \( \mathcal{S}(n, \mathbb{R}) \).

In [1], it is stated that the Levi decomposition
\[
(1.2) \quad \mathfrak{s}(n, \mathbb{R}) = \mathfrak{l} \oplus \mathfrak{r},
\]
has the following components:

a) The radical \( \mathfrak{r} \) is the linear subspace generated by the matrices
\[
(1.3) \quad \dot{R}_i = E_i(n) - E_n(n), \quad i = 1, \ldots, n - 1, \quad \dot{Z} = Id - \frac{1}{n} J_n,
\]
where \( E_i(n) \) are the matrices with the elements in the \( i \)-th column equal to 1 and all other elements equal to zero, \( J_n \) is the matrix with all elements equal to 1;

b) The Levi subalgebra \( \mathfrak{l} \) is the linear subspace of real traceless matrices with all row and column sums equal to zero.

The result is correct but the respective proof of [1 Proposition 3.3] seems to contain a logical gap in what regards the semisimplicity of \( \mathfrak{l} \) and the maximality of \( \mathfrak{r} \).

In what follows, we present an orthonormal basis for \( \mathfrak{s}(n, \mathbb{R}) \) which has interesting properties with respect to the Lie algebraic structure of \( \mathfrak{s}(n, \mathbb{R}) \). In particular, it allows for the explicit computation of the Killing form and therefore we prove semisimplicity of \( \mathfrak{l} \) by application of Cartan criterion. We also obtain the Dynkin diagram of \( \mathfrak{l} \), showing that it is isomorphic to \( \mathfrak{sl}(n - 1, \mathbb{R}) \).

2. Basis for the Lie algebra \( \mathfrak{s}(n, \mathbb{R}) \)

Choose an orthonormal basis \( v_1, \ldots, v_{n-1} \) of the hyperplane
\[
\Pi_n = \{ x \in \mathbb{R}^n : x_1 + \ldots + x_n = 0 \},
\]
and set $v_0 = \frac{1}{\sqrt{n}}(1, \ldots, 1) \in \mathbb{R}^n$. Recall that for $a, b \in \mathbb{R}^n$, the dyadic product $a \otimes b$ is the matrix:

$$
\begin{pmatrix}
    a_1 \\
    \vdots \\
    a_n
\end{pmatrix} \otimes \begin{pmatrix}
    b_1 & \cdots & b_n
\end{pmatrix} = 
\begin{pmatrix}
    a_1 b_1 & \cdots & a_1 b_n \\
    \vdots & \ddots & \vdots \\
    a_n b_1 & \cdots & a_n b_n
\end{pmatrix}.
$$

The matrices

$$
Z = \frac{1}{\sqrt{n-1}} (I_n - v_0 \otimes v_0),
$$

$$
R_i = v_0 \otimes v_i, \quad i = 1, \ldots, n - 1
$$

span the same linear subspace as the matrices $(2.3)$.

We take the $(n - 1)(n - 2)$-dimensional linear subspace

$$\mathcal{A} = \text{span} \{ A_{ij}, \ i = 1, \ldots, n - 1, \ j = 1, \ldots, n - 1, \ i \neq j \},$$

spanned by the rank-1 matrices

$$A_{ij} = v_i \otimes v_j.$$  

Since $v_i \in \Pi$, there holds $v_0^* (v_i \otimes v_j) = (v_0 \cdot v_i) v_j^* = 0$. Similarly, $(v_i \otimes v_j) v_0 = 0$. Hence the matrices $A_{ij}$ have zero row and column sums. Since $\text{Tr}(v_i \otimes v_j) = v_i \cdot v_j = 0$, the matrices $A_{ij}$ are traceless.

Now, consider the linear subspace

$$\mathcal{H} = \left\{ H = \sum_{\ell=1}^{n-1} \gamma_{\ell} (v_{\ell} \otimes v_{\ell}) \left| \sum_{\ell=1}^{n-1} \gamma_{\ell} = 0 \right. \right\}.$$ 

The row and column sums of each $(v_{\ell} \otimes v_{\ell})$ are zero, and the trace of $H \in \mathcal{H}$ equals $\sum_{\ell=1}^{n-1} \gamma_{\ell} = 0$.

We set

$$l = \mathcal{A} \oplus \mathcal{H}.$$ 

We introduce a basis of $\mathcal{H}$:

$$H_k = \sum_{\ell=1}^{n-1} \gamma_{k,\ell}^\ell (v_{\ell} \otimes v_{\ell}), \quad k = 1, \ldots, (n - 2),$$

where $\gamma^k = (\gamma_1^k, \ldots, \gamma_{n-1}^k)$, $k = 1, \ldots, (n - 2)$, form an orthonormal basis for the subspace

$$\Pi_{n-1} = \{ x \in \mathbb{R}^{n-1} : x_1 + \ldots + x_{n-1} = 0 \}.$$

Using the definition of dyadic product and elementary properties of the trace, it is straightforward to check that the matrices

$$Z, \quad R_i \ (i = 1, \ldots, n - 1),$$

$$A_{ij} \ (i, j = 1, \ldots, n - 1, i \neq j), \quad H_i \ (i = 1, \ldots, n - 2)$$
form an orthonormal system with respect to the matrix scalar product
\[ \langle A, B \rangle = \text{Tr}(AB^\ast). \]
The following Lemma presents the multiplication table for our basis. Its
proof is accomplished by a direct computation.

**Lemma 2.1.** For meaningful values of the indexes \( i, j, k, \ell \) there holds:
\[
\begin{align*}
[Z, R_i] &= \frac{1}{n-1} R_i; \\
[Z, A_{ij}] &= 0; \\
[Z, H_i] &= 0; \\
[R_i, R_j] &= 0; \\
[R_i, A_{j,k}] &= \begin{cases} R_k, & \text{if } i = j, \\ 0, & \text{if } i \neq j; \end{cases} \\
[R_i, H_j] &= \gamma_i^j R_i; \\
[A_{ij}, A_{k\ell}] &= \begin{cases} A_{k\ell}, & \text{if } i \neq \ell, j = k, \\ -A_{kj}, & \text{if } i = \ell, j \neq k, \\ 0, & \text{if } i \neq \ell, j \neq k; \end{cases} \\
[A_{ij}, H_k] &= \left( \gamma_j^k - \gamma_i^k \right) A_{ij}; \\
[H_i, H_j] &= 0. \quad \Box
\end{align*}
\]

**Remark 2.2.** Lemma 2.1 shows that the orthogonal subspaces \( \mathcal{A}, \mathcal{H} \) possess
remarkable properties:
1. \( \mathcal{H} \) is a Cartan subalgebra of \( \mathfrak{l} \).
2. \( \mathcal{[H, A]} \subset \mathcal{A} \). The adjoint action of \( \mathcal{H} \) on \( \mathcal{A} \) is diagonal, for \( H \in \mathcal{H} \):
\[
\text{ad} H A_{ij} = (\gamma_i - \gamma_j) A_{ij}.
\]
3. \( A_{ij}, A_{ji} = v_i \otimes v_i - v_j \otimes v_j = H_{ij} \in \mathcal{H} \).
4. For \( i \neq j \), \( \{ A_{ij}, A_{ji}, [A_{ij}, A_{ji}] \} \) spans a 3-dimensional Lie subalgebra:
\[
[H_{ij}, A_{ij}] = 2 A_{ij}, \quad [H_{ij}, A_{ji}] = -2 A_{ji}.
\]
5. For any \((i,j), (k,\ell)\) the commutator \( [A_{ij}, A_{k\ell}] = \text{ad} A_{ij} A_{k\ell} \) is orthogonal
to \( A_{k\ell} \) with respect to the matrix scalar product. \( \Box \)

3. Semsimplicity of \( \mathfrak{l} \)

In this section, we prove semisimplicity of \( \mathfrak{l} \) by direct computation of the
Killing form \( \mathfrak{B} \).

**Proposition 3.1.** The Killing form \( \mathfrak{B} \) satisfies:
\[
\begin{align*}
\mathfrak{B}(A, \mathcal{H}) &= 0, \\
\mathfrak{B}(H_i, H_j) &= 2(n-1) \langle H_i, H_j \rangle, \quad \text{for } i, j = 1, \ldots, n-2, \\
\mathfrak{B}(A_{ij}, A_{k\ell}) &= \begin{cases} 0, & \text{if } (i,j) \neq (\ell, k), \\ 2(n-1), & \text{if } (i,j) = (\ell, k). \end{cases} \quad \Box
\end{align*}
\]

According to Cartan criterion for semisimplicity, we get
Corollary 3.2. The Killing form $\mathfrak{B}$ is non-degenerate and the algebra $\mathfrak{l}$ is semisimple. Q.E.D.

Proof of Proposition 3.1. (i) Take $A_{ij}$, $H_k$ from the basis of $\mathcal{A}$ and $\mathcal{H}$, respectively.

Since $\mathcal{H}$ is Abelian, $(\text{ad}A_{ij}\text{ad}H_k)_{\mathcal{H}} = 0$.

Due to Lemma 2.1, for any $A_{\ell \ell}$, $(\text{ad}A_{ij}\text{ad}H_kA_{\ell \ell} = C\text{ad}A_{ij}A_{\ell \ell}$. By property 5 in Remark 2.2, the last matrix is orthogonal to $A_{\ell \ell}$ and therefore the trace of the restriction $(\text{ad}A_{ij}\text{ad}H_k)_{\mathcal{A}}$ is null, and we can conclude that $\mathfrak{B}(\mathcal{A}, \mathcal{H}) = 0$.

(ii) Choose $H_k, H_\ell \in \mathcal{H}$. As far as $(\text{ad}H_k\text{ad}H_\ell)_{\mathcal{H}} = 0$, we only need to compute the trace of $(\text{ad}H_k\text{ad}H_\ell)_{\mathcal{A}}$.

By Lemma 2.1, $\text{ad}H_k\text{ad}H_\ell A_{ij} = \text{ad}H_k(\gamma_i^k - \gamma_j^k)A_{ij} = (\gamma_i^k - \gamma_j^k)(\gamma_i^k - \gamma_j^k)A_{ij}$.

Hence,

$$\mathfrak{B}(H_k, H_\ell) = \sum_{i,j} (\gamma_i^k - \gamma_j^k)(\gamma_i^k - \gamma_j^k) =$$

$$(n - 1) \sum_i \gamma_i^k - \sum_i \gamma_i^k \sum_j \gamma_j^k - \sum_j \gamma_j^k \sum_i \gamma_i^k + (n - 1) \sum_j \gamma_j^k \gamma_j^k.$$

Since $\sum_i \gamma_i^k = 0$, it follows that

$$\mathfrak{B}(H_k, H_\ell) = 2(n - 1) \sum_i \gamma_i^k \gamma_i^k = 2(n - 1)\langle H_k, H_\ell \rangle.$$

(iii) Pick $A_{ij}$, $A_{kl}$. For every $H_m$

$$\text{ad}A_{ij}\text{ad}A_{kl}H_m = \text{ad}A_{ij}(\gamma_i^m - \gamma_j^m)A_{kl},$$

lies in $\mathcal{A}$ whenever $(k, \ell) \neq (j, i)$. This implies

$$\text{Tr} (\text{ad}A_{ij}\text{ad}A_{kl})_{\mathcal{H}} = 0, \text{ for } (k, \ell) \neq (j, i).$$

To compute $\text{Tr} (\text{ad}A_{ij}\text{ad}A_{kl})_{\mathcal{A}}$, notice that

$$\langle A_{\alpha\beta}, \text{ad}A_{ij}\text{ad}A_{kl}A_{\alpha\beta} \rangle = v^*_{\alpha}(A_{ij}\text{ad}A_{kl}A_{\alpha\beta} - \text{ad}A_{ij}A_{\alpha\beta})v_{\beta} =$$

$$(v_{\alpha} \cdot v_i)v^*_j (A_{kl}A_{\alpha\beta} - A_{\alpha\beta}A_{ij})v_{\beta} - (v_{\beta} \cdot v_j)v^*_\alpha (A_{kl}A_{\alpha\beta} - A_{\alpha\beta}A_{kl})v_i.$$

Since $i \neq j$ and $k \neq \ell$, $v^*_\alpha A_{\alpha\beta}A_{ij}v_{\beta} = v^*_\alpha A_{ij}A_{\alpha\beta}v_i = 0$, and therefore

$$\langle A_{\alpha\beta}, \text{ad}A_{ij}\text{ad}A_{kl}A_{\alpha\beta} \rangle =$$

$$(v_{ij} \cdot v_k)(v_i \cdot v_{\alpha}) + (v_i \cdot v_k)(v_j \cdot v_{\beta})(v_k \cdot v_{\beta}),$$

which is zero whenever $(k, \ell) \neq (j, i)$.

For $(k, \ell) = (j, i)$, the equality (3.1) and Lemma 2.1 yield

$$\langle H_m, \text{ad}A_{ij}\text{ad}A_{ji}H_m \rangle = (\gamma_i^m - \gamma_j^m)\langle H_m, \text{ad}A_{ij}A_{ji} \rangle =$$

$$(\gamma_i^m - \gamma_j^m)^2\langle H_m, v_i \otimes v_i - v_j \otimes v_j \rangle = (\gamma_i^m - \gamma_j^m)^2,$$

and $\text{Tr} (\text{ad}A_{ij}\text{ad}A_{ji})_{\mathcal{H}} = \sum_{m=1}^{n-2} (\gamma_i^m - \gamma_j^m)^2.$
To compute the last expression, let us form the matrix

\[
\Gamma = \begin{pmatrix}
\gamma_1^1 & \cdots & \gamma_1^{n-2} \\
\vdots & \ddots & \vdots \\
\gamma_1^{n-1} & & \gamma_1^{n-2}
\end{pmatrix},
\]

Then \(\Gamma \Gamma^*\) is the matrix of the orthogonal projection of \(\mathbb{R}^{n-1}\) onto the subspace \(\Pi_{n-1}\). Take a standard basis \(e_1, \ldots, e_{n-1}\) in \(\mathbb{R}^{n-1}\), and note that 
\(e_i - e_j \in \Pi_{n-1}\). Then

\[
\text{Tr} (\text{ad} A_{ij} \text{ad} A_{ji})|_H = \sum_{m=1}^{n-2} (\gamma_m^i - \gamma_m^j)^2 =
\]

\[
= (e_i - e_j)^* \Gamma \Gamma^* (e_i - e_j) = (e_i - e_j)^* (e_i - e_j) = 2.
\]

In what regards \(\text{Tr} (\text{ad} A_{ij} \text{ad} A_{ji})|_A\), then by (3.2):

\[
(A_{\alpha\beta}, \text{ad} A_{ij} \text{ad} A_{ji} A_{\alpha\beta}) = (v_i \cdot v_\alpha) + (v_j \cdot v_\beta).
\]

Hence,

\[
\text{Tr} (\text{ad} A_{ij} \text{ad} A_{ji})|_A = \sum_{\alpha, \beta \leq n-1, \alpha \neq \beta} ((v_i \cdot v_\alpha) + (v_j \cdot v_\beta)) = 2(n - 2),
\]

and therefore \(\text{Tr} (\text{ad} A_{ij} \text{ad} A_{ji}) = 2(n - 1)\). \(\square\)

4. Classification of the Levi subalgebra \(\mathfrak{l}\)

Now we wish to prove the following result concerning the type of the semisimple subalgebra \(\mathfrak{l}\).

**Theorem 4.1.** The Levi subalgebra \(\mathfrak{l}\) is isomorphic to the special linear Lie algebra \(\mathfrak{sl}(n - 1, \mathbb{R})\). \(\square\)

**Proof.** As stated in Remark 2.2, \(\mathcal{H}\) is a Cartan subalgebra of \(\mathfrak{l}\). From Lemma 2.1, we see that the nonzero characteristic functions of \(\mathfrak{l}\) with respect to \(\mathcal{H}\) are the linear functionals \(\alpha_{ij} : \mathcal{H} \rightarrow \mathbb{R}\) such that

\[
\alpha_{ij}(H_k) = \gamma_k^i - \gamma_k^j, \quad \text{for } 1 \leq k \leq n - 2, \ 1 \leq i, j \leq n - 1, \ i \neq j,
\]

and the corresponding characteristic spaces are

\[
A_{ij} = \{tA_{ij} : t \in \mathbb{R}\} \quad 1 \leq i, j \leq n - 1, \ i \neq j.
\]

Thus, \(\mathfrak{l}\) is split as

\[
\mathfrak{l} = \mathcal{H} \oplus \bigoplus_{i \neq j} A_{ij}.
\]

Hence the set, \(\mathcal{R} = \{\alpha_{ij} : 1 \leq i \leq n - 1, \ 1 \leq j \leq n - 1, \ i \neq j\}\) is a root system of \(\mathfrak{l}\).
Since the Killing form restricted to $\mathcal{H}$ is diagonal, the dual space $\mathcal{H}^*$ is provided with the inner product uniquely defined by
\[
\langle \alpha_{ij}, \alpha_{\ell m} \rangle = \sum_{k=1}^{n-2} \left( \gamma_i^k - \gamma_j^k \right) \left( \gamma_{\ell m}^k - \gamma_{\ell}^k \right) = (e_i - e_j)^*(e_\ell - e_m)
\]
for every $\alpha_{ij}, \alpha_{\ell m} \in \mathcal{R}$. Thus, $\mathcal{R}$ is isomorphic to the root system $\mathcal{E} = \{e_i - e_j : 1 \leq i \leq n - 1, 1 \leq j \leq n - 1, \ i \neq j\}$ on the hyperplane $\Pi_{n-1}$. Since
\[
e_{\ell} - e_m = \begin{cases} 
\sum_{i=\ell}^{m-1} (e_i - e_{i+1}), & \text{if } \ell < m, \\
\sum_{i=m}^{\ell-1} (e_i - e_{i+1}), & \text{if } \ell > m,
\end{cases}
\]
it follows that the set $\Delta = \{\alpha_{12}, \alpha_{23}, \alpha_{34}, \ldots, \alpha_{(n-2)(n-1)}\}$ is a system of positive simple roots. Further,
\[
\frac{2 \langle \alpha_{(i+1)(i)}, \alpha_{(i+1)(j+1)} \rangle}{\langle \alpha_{(i+1)(i)}, \alpha_{(i+1)(j+1)} \rangle} = \begin{cases} 
-1 & \text{if } |i - j| = 1, \\
0 & \text{if } |i - j| > 1.
\end{cases}
\]
Thus, the Dynkin diagram of $l$ is of type $A_{n-2}$, and therefore, $l$ is isomorphic to $\mathfrak{sl}(n-1, \mathbb{R})$ (see, e.g., [6, Chapter 14]).

---

**Figure 1.** Dynkin diagram of $l$

---

5. Representation of the Levi factor $l$ in $V = \mathbb{R}^n$

Considering $l$ as a subalgebra of the stochastic (matrix) algebra $\mathfrak{sl}(n, \mathbb{R})$ defines its representation $\phi : l \to \mathfrak{gl}(n)$ in $V = \mathbb{R}^n$. To characterize it, let us pick the basis $v_0, v_1, \ldots, v_{n-1}$, introduced in Section 2 and consider the matrix $M \in \mathbb{R}^{n \times n}$:
\[
M = \begin{pmatrix} v_0 | v_1 | \cdots | v_{n-1} \end{pmatrix}.
\]

By construction, $M$ is orthogonal and the mapping
\[
\forall y \in l : \ y \mapsto M^* \phi(y) M,
\]
defines an isomorphic representation of $l$ in $V = \mathbb{R}^n$.

Note that the subspace $V_1 = \text{span}\{v_1, \ldots, v_{n-1}\}$ is invariant under $\phi(l)$ and therefore we get:
\[
\forall y \in l : \ M^* \phi(y) M = \begin{pmatrix} 0 & 0 \\
0 & M_1^* \phi(y) M_1 \end{pmatrix},
\]
where $M_1 = \begin{pmatrix} v_1 | v_2 | \cdots | v_{n-1} \end{pmatrix} \in \mathbb{R}^{n \times (n-1)}$. 

---

\[\text{STOCHASTIC LIE ALGEBRA}\]

7
The mapping
\[ y \mapsto \phi_1(y) = M_1^t \phi(y) M_1 \]
is a faithful representation of \( I \) in \( V_1 = \mathbb{R}^{n-1} \).

Formula (5.1) identifies the representation of the semisimple Levi factor \( I \) in \( \mathbb{R}^n \) by stochastic matrices with a direct sum of the faithful representation \( \phi_1 \) in \( \mathbb{R}^{n-1} \) and the null 1-dimensional representation.

Besides
\[
M_1^t A_{ij} M_1 = e_i \otimes e_j \quad \text{for } i, j \in \{1, 2, \ldots, n-1\}, \; i \neq j,
\]
\[
M_1^t H_i M_1 = \text{diag}(\gamma^i) \quad \text{for } i = 1, 2, \ldots, n-2.
\]

Therefore \( \phi_1 \) maps isomorphically the Cartan subalgebra \( \mathcal{H} \) onto the space of traceless diagonal \( (n-1) \times (n-1) \) matrices, while \( \phi_1(A) \) coincides with the space of \( (n-1) \times (n-1) \) matrices with vanishing diagonal.

6. Affine group and affine Lie algebra

It is noticed in [5] that the group of \( \mathcal{S}(n, \mathbb{R}) \) is isomorphic to the group \( \text{Aff}(n-1, \mathbb{R}) \) of the affine maps \( S : x \to Ax + B, \; x \in \mathbb{R}^{n-1} \). We wish to discuss this relation, in the light of the results obtained above. We also discuss the relation between the elements of \( \mathcal{S}(n, \mathbb{R}) \) and finite state space Markov processes outlined in Section 1.

Let \( (\mathbb{R}^n)^* \) be the dual of \( \mathbb{R}^n \). As usual, elements of \( \mathbb{R}^n \) are identified with column vectors, and elements of \( (\mathbb{R}^n)^* \) are identified with row vectors. Further, we identify any vector \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) with the function \( x : i \mapsto x_i \), with the domain \( D_n = \{1, 2, \ldots, n\} \), and identify any dual vector \( p = (p_1, p_2, \ldots, p_n) \in (\mathbb{R}^n)^* \) with the (signed) measure \( p \) on the set \( D_n \) such that \( p\{i\} = p_i \), for \( i = 1, 2, \ldots, n \). Thus, the product \( px \) is identified with the integral \( \int_{D_n} x \, dp \).

Each \( S \in \mathcal{S}(n, \mathbb{R}) \) can be identified either with the linear endomorphism of \( \mathbb{R}^n \), \( S : x \mapsto Sx \) or with the linear endomorphism of \( (\mathbb{R}^n)^* \), \( S : p \mapsto pS \).

Let \( Y \) be a \( D_n \)-valued Markov process and \( S \in \mathcal{S}^+(n, \mathbb{R}) \) be defined by \( s_{ij} = \Pr \{ Y_t = j \mid Y_s = i \} \) for every \( i, j \in D_n \) (\( 0 \leq s \leq t < +\infty \), fixed). Then the vector \( Sx \) is identified with the function \( i \mapsto \mathbb{E} |x(Y_t)| \mid Y_s = i \), while the covector \( pS \) is identified with the probability law of \( Y_t \) assuming the probability law of \( Y_s \) is \( p \).

For every \( S \in \mathcal{S}(n, \mathbb{R}) \), the map \( p \mapsto pS \) preserves each affine space of the form \( \{ p \in (\mathbb{R}^n)^* : p1 = C \} \) (\( C \in \mathbb{R} \), fixed), which is the space of signed measures on \( D_n \) such that \( p(D_n) = C \). Note that, \( \{ t1 : t \in \mathbb{R} \} \) is the unique affine (linear) proper subspace of \( \mathbb{R}^n \) which is preserved by all the maps \( x \mapsto Sx \) with \( S \in \mathcal{S}(n, \mathbb{R}) \).
Now, consider the group of invertible affine maps $S : q \mapsto qA + B$, $q \in \mathbb{R}^{n-1}$. The group can be identified with the subgroup $A((\mathbb{R}^{n-1})^*)$ of $GL((\mathbb{R}^n)^*)$:

$$A((\mathbb{R}^{n-1})^*) = \left\{ \begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} : A \in \mathbb{R}^{(n-1)\times(n-1)} \text{ is nonsingular} \right\}.$$

The Lie algebra $\mathfrak{a}((\mathbb{R}^{n-1})^*)$ of $A((\mathbb{R}^{n-1})^*)$ consists of matrices

$$\begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix}.$$

Now, fix $S \in S(n, \mathbb{R})$. By the results of Section 2, $S$ can be written as

$$S = \beta_0 Z + \sum_{i=1}^{n-1} \beta_i R_i + A,$$

with $\beta_0, \beta_1, \ldots, \beta_{n-1} \in \mathbb{R}$, $A \in \mathfrak{l}$. Taking into account that

$$Zv_0 = 0,$$

$$Zv_i = \frac{1}{\sqrt{n-1}}v_i, \quad R_i v_j = \begin{cases} v_0, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \quad \text{for } i = 1, 2, \ldots, n-1,$$

we get

$$M^*SM = \begin{pmatrix} 0 & \beta^* \\ 0 & M_1^*AM_1 + \frac{\beta_0}{\sqrt{n-1}}Id \end{pmatrix},$$

where $\beta^* = (\beta_1, \beta_2, \ldots, \beta_{n-1})$. Thus, the similarity $S \mapsto M^*SM$ is an isomorphism from $S(n, \mathbb{R})$ into $\mathfrak{a}((\mathbb{R}^{n-1})^*)$. In particular, the radical of $\mathfrak{a}((\mathbb{R}^{n-1})^*)$ is the linear space of matrices

$$\begin{pmatrix} 0 & \beta^* \\ 0 & \beta_0 Id \end{pmatrix}, \quad \beta_0, \beta_1, \ldots, \beta_{n-1} \in \mathbb{R},$$

while the Levi subalgebra of $\mathfrak{a}((\mathbb{R}^{n-1})^*)$ consists of matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \quad A \in \mathfrak{sl}(n-1, \mathbb{R}).$$

Thus, the Levi splitting of $\mathfrak{a}((\mathbb{R}^{n-1})^*)$ corresponds to two connected Lie subgroups of $A((\mathbb{R}^{n-1})^*)$: The subgroup generated by the translations and rescalings of $(\mathbb{R}^{n-1})^*$, and the subgroup of orientation and volume preserving linear transformations in $(\mathbb{R}^{n-1})^*$.

\[1\] The mapping

$$q = (q_1, q_2, \ldots, q_{n-1}) \mapsto \left( C - \sum_{i=1}^{n-1} q_i, q_1, q_2, \ldots, q_{n-1} \right)$$

coordinates the affine subspace $\{ p \in (\mathbb{R}^n)^* : p1 = C \}$. 
Finally we prove

**Theorem 7.1.** The Lie algebra \( \mathfrak{s}(n, \mathbb{R}) \) is generated by two matrices. \( \square \)

The argument in our proof is an adaptation of the argument used in [4] to prove that every semisimple Lie algebra is generated by two elements. We will use the following lemma:

**Lemma 7.2.** For every integer \( n \geq 2 \) there is a vector \( \gamma \in \mathbb{R}^n \) such that

a) \( \sum_{i=1}^{n} \gamma_i = 0; \)
b) \( \gamma_i \neq 0, \quad i = 1, \ldots, n; \)
c) \( \gamma_i \neq \gamma_j, \quad \forall i, j \in \{1, \ldots, n\}, i \neq j; \)
d) \( \gamma_i - \gamma_j \neq \gamma_k - \gamma_{\ell}, \quad \forall i, j, k, \ell \in \{1, \ldots, n\}, i \neq j, k \neq \ell, (i, j) \neq (k, \ell). \)

For every \( \gamma \) satisfying (a)–(d) and every \( \lambda \in \mathbb{R} \setminus \{0\} \), \( \lambda \gamma \) satisfies (a)–(d). \( \square \)

**Proof.** For \( n = 2 \), the Lemma holds with \( \gamma = (1, -1) \).

Suppose that the Lemma holds for some \( n \geq 2 \), and fix \( \gamma \in \mathbb{R}^n \) satisfying (a)–(d). Let

\[ \tilde{\gamma} = (\gamma_1, \ldots, \gamma_{n-1}, \gamma_n - \varepsilon, \varepsilon). \]

Since there are only finitely many values of \( \varepsilon \) such that \( \tilde{\gamma} \) fails at least one condition (a)–(d), we see that the Lemma holds for \( n + 1 \).

The last statement in the Lemma is obvious, since the equations in conditions (a)–(d) are homogeneous. \( \square \)

**Proof of Theorem 7.1.** Pick a vector \( \gamma \in \mathbb{R}^{n-1} \) satisfying conditions (a)–(d) of Lemma 7.2, let \( \Gamma \) be the matrix \( (3.4) \), and \( \beta = (\beta_1, \ldots, \beta_{n-2}) = \gamma^T \Gamma \). Let \( Z, R_1, A_{ij}, H_i \) be elements of our basis of \( \mathfrak{s}(n, \mathbb{R}) \), and consider the matrices

\[ X = Z + \sum_{k=1}^{n-2} \beta_k H_k, \quad Y = R_1 + \sum_{i \neq j} A_{ij}. \]

Using the Lemma 2.1 we obtain

\[ \text{ad}XY = [Z, R_1] + \sum_{i \neq j} [Z, A_{ij}] + \sum_{k=1}^{n-2} \beta_k [H_k, R_1] + \sum_{k=1}^{n-2} \beta_k [H_k, A_{ij}] = \]

\[ = \left( -\frac{1}{n-1} \right) R_1 + 0 - \gamma_1 R_1 + \sum_{i \neq j} (\gamma_i - \gamma_j) A_{ij} = \]

\[ = - \left( \frac{1}{n-1} + \gamma_1 \right) R_1 + \sum_{i \neq j} (\gamma_i - \gamma_j) A_{ij}. \]
Multiplying $\gamma$ by an appropriate non zero constant we can make $\gamma_1 = \frac{-1}{n-1}$, and thus

$$\text{ad}XY = \sum_{i \neq j} (\gamma_i - \gamma_j)A_{ij}.$$ 

Iterating, we see that

$$\text{ad}^kXY = \sum_{i \neq j} (\gamma_i - \gamma_j)^kA_{ij} \quad \forall k \in \mathbb{N}.$$ 

Let $m = (n-1)(n-2)$. Since

$$\det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \gamma_1 - \gamma_2 & \cdots & \gamma_{n-1} - \gamma_{n-2} \\ 0 & 0 & (\gamma_1 - \gamma_2)^2 & \cdots & (\gamma_{n-1} - \gamma_{n-2})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & (\gamma_1 - \gamma_2)^m & \cdots & (\gamma_{n-1} - \gamma_{n-2})^m \end{pmatrix} \neq 0,$$

we see that the matrices $X, Y, \text{ad}XY, \ldots, \text{ad}^mXY$ span the same subspace as the matrices $X, R_1, A_{ij} \quad i, j \leq n-1, i \neq j$, and this subspace lies in $\mathfrak{Lie}\{X, Y\}$, the Lie algebra generated by $X, Y$.

By the Lemma \[2.1\] $[R_1, A_{ij}] = R_i$, for $i = 1, 2, \ldots, n-1$. Hence

$$\{R_2, \ldots, R_{n-1}\} \subset \mathfrak{Lie}\{X, Y\}.$$ 

Finally, also by the Lemma \[2.1\] $[A_{ij}, A_{ji}] = \sum_{r=1}^{n-2} (\gamma_i^r - \gamma_j^r)H_r$. This implies that $[A_{ij}, A_{ji}]$, $j = 2, \ldots, n-1$ are $n-2$ linearly independent elements of $\mathcal{H}$. Hence, $\mathcal{H} \subset \mathfrak{Lie}\{X, Y\}$ and $Z \in \mathfrak{Lie}\{X, Y\}$. \hfill $\square$

8. ACKNOWLEDGEMENT

The authors are grateful to A.A.Agrachev for stimulating remarks.

REFERENCES

1. Anadreas Boukas, Philip Feinsilver, Anargyros Fellouris. Structure and decompositions of the linear span of generalized stochastic matrices. Communications on Stochastic Analysis, Vol.9, No.2 (2015). 239–250.
2. Erhan Çinlar. Probability and stochastics. Springer (2011).
3. J.E. Humphreys. Introduction to Lie algebras and representation theory. Springer-Verlag, New York Inc (1978).
4. Masatake Kuranishi. On everywhere dense imbeddings of free groups in Lie groups. Nagoya Math. J., No.2 (1951). 63–71.
5. D.G. Poole. The Stochastic Group. The Amer. Mathem. Monthly, 102 (1995). 798–801.
6. Arthur A. Sagle, Ralph E. Walde. Introduction to Lie groups and Lie algebras. Academic Press (1973).
ISEG and CEMAPRE, ULisboa, Rua do Quelhas 6, 1200-781 Lisboa, Portugal, University of Florence, DiMAI, v. delle Pandette 9, Firenze, 50127, Italy
E-mail address: mguerra@iseg.ulisboa.pt, asarychev@unifi.it