An improved upper bound on the adjacent vertex distinguishing chromatic index of a graph

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Abstract
An adjacent vertex distinguishing coloring of a graph $G$ is a proper edge coloring of $G$ such that any pair of adjacent vertices are incident with distinct sets of colors. The minimum number of colors needed for an adjacent vertex distinguishing coloring of $G$ is denoted by $\chi'_a(G)$. In this paper, we prove that $\chi'_a(G) \leq \frac{5}{2}(\Delta + 2)$ for any graph $G$ having maximum degree $\Delta$ and no isolated edges. This improves a result in [S. Akbari, H. Bidkbori, N. Nosrati, $r$-Strong edge colorings of graphs, Discrete Math. 306 (2006), 3005-3010], which states that $\chi'_a(G) \leq 3\Delta$ for any graph $G$ without isolated edges.

Keywords: Adjacent vertex distinguishing coloring, maximum degree, edge-partition

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1 Introduction

All graphs considered in this paper are finite and without self-loops or multiple edges.

In order to avoid trivialities, we also assume that every graph has no isolated vertices.

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Let $V(G)$ and $E(G)$ denote the vertex and the edge sets of $G$, respectively. Let $N_G(v)$ denote the set of neighbors of $v$ in $G$ and $d_G(v) = |N_G(v)|$ the degree of $v$ in $G$. A vertex $v$ is called a $k$-vertex if $d_G(v) = k$. Let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degree of a vertex in $G$, respectively. An edge $k$-coloring of a graph $G$ is a function $\phi : E(G) \rightarrow \{1, 2, \ldots, k\}$ such that any two incident edges receive different colors. The chromatic index, denoted by $\chi'(G)$, of a graph $G$ is the smallest integer $k$ such that $G$ has an edge $k$-coloring.

Zhang, Liu and Wang [20] first introduced and investigated the adjacent vertex distinguishing edge coloring (adjacent strong edge coloring in their terminology) of graphs. They proposed the following conjecture.

**Conjecture 1** If a connected normal graph $G$ is different from a 5-cycle and satisfies $|V(G)| \geq 3$, then $\chi'_a(G) \leq \Delta(G) + 2$.

Balister et al. [3] confirmed Conjecture 1 for all normal graphs $G$ that are bipartite or satisfy $\Delta(G) = 3$. In particular, we need the following statement in the sequel.

**Theorem 1.1** For any normal graph $G$ with $\Delta(G) \leq 3$, $\chi'_a(G) \leq 5$.

They further proved that $\chi'_a(G) \leq \Delta(G) + O(\log k)$, where $k$ is the (vertex) chromatic number of the normal graph $G$. It follows from Brooks’ Theorem that $\chi'_a(G) \leq 2\Delta(G)$ for $G$ with sufficiently large $\Delta(G)$. Hatami [12] showed that every normal graph $G$ with $\Delta(G) > 10^{20}$ has $\chi'_a(G) \leq \Delta(G) + 300$ by the probabilistic method. Edwards et al. [11] proved that $\chi'_a(G) \leq \Delta(G) + 1$ if $G$ is a planar bipartite normal graph with $\Delta(G) \geq 12$. Wang and Wang [18] verified Conjecture 1 for a class of graphs with small maximum average degree. Their results were further extended by Hocquard and Montassier [13, 14]. Recently, it has been characterized in [19] which of the two cases $\chi'_a(G) = \Delta(G)$ and $\chi'_a(G) = \Delta(G) + 1$ holds for a $K_4$-minor-free normal graph $G$ with $\Delta(G) \geq 5$.
An adjacent vertex distinguishing edge coloring of a graph $G$ is a special case of a vertex distinguishing edge coloring, which requires that every pair of vertices be incident with distinct color sets. This more general notion was introduced by Burris and Schelp [9], and independently by Horňák and Soták [15], and Černý et al. [10] (under the name observability). The reader is referred to [2, 3, 5–8, 17] for relevant results.

The aim of this paper is to improve the following upper bound obtained in [1].

**Theorem 1.2** For any normal graph $G$, $\chi'_a(G) \leq 3\Delta(G)$.

The proof of our main theorem in Section 2 is based on an edge-partition result. The details will be supplied in the last section. In Section 3, the new upper bound is further reduced for regular graphs.

## 2 An improved upper bound

For a graph $G$ and any $S \subseteq E(G)$, the edge-induced subgraph $G[S]$ is the subgraph of $G$ whose edge set is $S$ and whose vertex set consists of all end vertices of edges in $S$. We only deal with subgraphs that are edge-induced subgraphs unless otherwise stated. For a subgraph $H$ of $G$, we use $\overline{H}$ to denote the edge-induced subgraph $G[E(G)\setminus E(H)]$ and call it the complement of $H$ in $G$. An edge-partition of a graph $G$ into subgraphs $G_1, G_2, \ldots, G_m$ is a decomposition of $G$ that satisfies $V(G) = \bigcup_{i=1}^{m} V(G_i)$, $E(G) = \bigcup_{i=1}^{m} E(G_i)$ and $E(G_i) \cap E(G_j) = \emptyset$ for any pair $i \neq j$. Clearly, a subgraph $H$ of $G$ together with its complement $\overline{H}$ constitute an edge-partition of $G$. This edge-partition is said to be induced by the subgraph $H$. The proof of the following is deferred to Section 4.

**Theorem 2.1** Let $G$ be a normal graph with $\Delta(G) \geq 6$. Then there is an edge-partition of $G$ induced by a subgraph $H$ such that the following conditions hold.

1. Both $H$ and $\overline{H}$ are normal.
2. $\Delta(H) \leq 3$.
3. $\Delta(\overline{H}) \leq \Delta(G) - 2$.

**Theorem 2.2** Let $G$ be a normal graph with $\Delta(G) \geq 4$. Then there is an edge-partition of $G$ into subgraphs $G_0, G_1, \ldots, G_k$, $k \leq \lfloor \Delta(G)/2 \rfloor - 2$, such that the following hold.
1. Every \( G_i \) is a normal subgraph.

2. \( \Delta(G_i) \leq 3 \) for \( 1 \leq i \leq k \).

3. \( \Delta(G_0) \leq 5 \).

**Proof.** The proof proceeds by induction on \( \Delta(G) \). If \( \Delta(G) \leq 5 \), the result holds trivially. Let \( G \) be a normal graph with \( \Delta(G) \geq 6 \). By Theorem 2.11 there is an edge-partition of \( G \) induced by a subgraph \( H \) such that both \( H \) and \( H' \) are normal, \( \Delta(H) \leq 3 \) and \( \Delta(H') \leq \Delta(G) - 2 \). Clearly, \( \Delta(H') \geq 3 \). If \( \Delta(H') = 3 \), then \( \Delta(G) = 6 \).

Let \( G_0 = H \) and \( G_1 = H' \). If \( \Delta(H') \geq 4 \), by the induction hypothesis, there is an edge-partition of \( H' \) into subgraphs \( G_0, G_1, \ldots, G_k, k \leq \lfloor \Delta(H')/2 \rfloor - 2 \), such that properties 1, 2 and 3 hold. Now let \( G_{k+1} = H \). Then \( G_0, G_1, \ldots, G_k, G_{k+1} \) form an edge-partition of \( G \). Note that \( k + 1 \leq \lfloor \Delta(H)/2 \rfloor - 2 + 1 \leq \lfloor (\Delta(G) - 2)/2 \rfloor - 1 = \lfloor \Delta(G)/2 \rfloor - 2 \) and we are done.

**Lemma 2.3** If a normal graph \( G \) has an edge-partition into two normal subgraphs \( G_1 \) and \( G_2 \), then \( \chi'_a(G) \leq \chi'_a(G_1) + \chi'_a(G_2) \).

**Proof.** For \( i = 1, 2 \), let \( \phi_i \) be an adjacent vertex distinguishing edge coloring of \( G_i \) satisfying \( |C_{\phi_i}| = \chi'_a(G_i) \) and \( C_{\phi_1} \cap C_{\phi_2} = \emptyset \). The union of \( \phi_1 \) and \( \phi_2 \) forms a proper edge coloring \( \phi \) of \( G \) with color set \( C_{\phi_1} \cup C_{\phi_2} \). Let \( uv \in E(G) \) with \( d_G(u) = d_G(v) \). Since \( E(G_1) \cap E(G_2) = \emptyset \), we may assume that \( uv \in E(G_1) \setminus E(G_2) \) with \( d_{G_1}(u) \geq d_{G_1}(v) \). Since \( G_1 \) is normal, \( uv \) is not an isolated edge of \( G_1 \), i.e., \( d_{G_1}(u) \geq 2 \). By definition of \( \phi_1 \), there exists a \( c \in C_{\phi_1}(u) \setminus C_{\phi_2}(v) \). Since \( C_{\phi_1} \cap C_{\phi_2} = \emptyset \), it follows that \( c \in C_{\phi}(u) \setminus C_{\phi}(v) \), and hence \( C_{\phi}(u) \neq C_{\phi}(v) \). Consequently, \( \chi'_a(G) \leq |C_{\phi_1} \cup C_{\phi_2}| = |C_{\phi_1}| + |C_{\phi_2}| = \chi'_a(G_1) + \chi'_a(G_2) \).

**Theorem 2.4** If \( G \) is a normal graph, then \( \chi'_a(G) \leq \frac{5}{2}(\Delta(G) + 2) \).

**Proof.** The result can be derived immediately from Theorem 1.1 when \( \Delta(G) \leq 3 \). Now assume that \( \Delta(G) \geq 4 \). By Theorem 2.2 there is an edge-partition of \( G \) into subgraphs \( G_0, G_1, \ldots, G_k, k \leq \lfloor \Delta(G)/2 \rfloor - 2 \), such that properties 1, 2 and 3 hold. Using Lemma 2.3 and Theorem 1.1 repeatedly, we have

\[
\chi'_a(G) \leq \chi'_a(G_0) + \chi'_a(G_1) + \cdots + \chi'_a(G_k) \\
\leq \chi'_a(G_0) + 5k \\
\leq \chi'_a(G_0) + 5(\lfloor \Delta(G)/2 \rfloor - 2).
\]
By Theorem 2.2, \( \Delta(G_0) \leq 5 \). It follows from Theorem 1.2 that \( \chi'_a(G) \leq 15 + 5(\lceil \Delta(G)/2 \rceil - 2) \leq \frac{5}{2}(\Delta(G) + 2) \). ■

3 Regular graphs

Theorem 2.4 can be further improved for regular graphs. We first establish an auxiliary edge-partition lemma. We need the following well-known result of Vizing [16] on chromatic index.

**Theorem 3.1** For every graph \( G \), \( \chi'(G) \leq \Delta(G) + 1 \).

**Lemma 3.2** Let \( G \) be a regular graph of degree \( r \geq 5 \). Then there is an edge-partition of \( G \) into normal subgraphs \( G_1, G_2, \ldots, G_k \) such that one of the following conditions holds.

1. If \( r \equiv 2 \pmod{3} \), then \( k = (r + 1)/3 \) and \( \Delta(G_i) \leq 3 \) for \( 1 \leq i \leq k \).

2. If \( r \equiv 1 \pmod{3} \), then \( k = (r - 1)/3 \), \( \Delta(G_i) \leq 4 \) for \( 1 \leq i \leq 2 \) and \( \Delta(G_i) \leq 3 \) for \( 3 \leq i \leq k \).

3. If \( r \equiv 0 \pmod{3} \), then \( k = r/3 \) and \( \Delta(G_1) \leq 4 \) and \( \Delta(G_i) \leq 3 \) for \( 2 \leq i \leq k \).

**Proof.** By Theorem 3.1, \( E(G) \) can be partitioned into \( r + 1 \) disjoint color classes \( E_1, E_2, \ldots, E_{r+1} \) such that each \( E_i \) is a matching of \( G \). Let \( H \) be a subgraph of \( G \) edge-induced by \( m \), \( 3 \leq m \leq r \), of these color classes. Obviously, \( \Delta(H) \leq m \). For any given vertex \( v \) of \( G \), exactly one color is not used on any edge incident with \( v \) since \( G \) is \( r \)-regular. Therefore \( d_H(v) \geq 2 \), and hence \( H \) is a normal graph.

If \( r \equiv 2 \pmod{3} \), let \( k = (r + 1)/3 \). Then we define \( G_1 = G[E_1 \cup E_2 \cup E_3] \), \( G_2 = G[E_4 \cup E_5 \cup E_6] \), \ldots, \( G_k = G[E_{r-1} \cup E_r \cup E_{r+1}] \). Then \( G_1, G_2, \ldots, G_k \) form an edge-partition of \( G \) satisfying condition 1.

If \( r \equiv 1 \pmod{3} \), let \( k = (r - 1)/3 \). Then we define \( G_1 = G[E_1 \cup E_2 \cup E_3 \cup E_4] \), \( G_2 = G[E_5 \cup E_6 \cup E_7 \cup E_8] \), \( G_3 = [E_9 \cup E_{10} \cup E_{11}] \), \ldots, \( G_k = G[E_{r-1} \cup E_r \cup E_{r+1}] \). Then \( G_1, G_2, \ldots, G_k \) form an edge-partition of \( G \) satisfying condition 2.

If \( r \equiv 0 \pmod{3} \), let \( k = r/3 \). Then we define \( G_1 = G[E_1 \cup E_2 \cup E_3 \cup E_4] \), \( G_2 = G[E_5 \cup E_6 \cup E_7] \), \( G_3 = [E_8 \cup E_9 \cup E_{10}] \), \ldots, \( G_k = G[E_{r-1} \cup E_r \cup E_{r+1}] \). Then \( G_1, G_2, \ldots, G_k \) form an edge-partition of \( G \) satisfying condition 3. ■

**Theorem 3.3** Let \( G \) be a regular graph of degree \( r \geq 2 \). Then \( \chi'_a(G) \leq (5r + 37)/3 \).
Proof. If $2 \leq r \leq 4$, the result follows from Theorems 1.1 and 1.2. Assume that $r \geq 5$. By Lemma 3.2 there is an edge-partition of $G$ into normal subgraphs $G_1, G_2, \ldots, G_k$ such that one of the stated conditions 1, 2 or 3 holds.

If condition 1 holds, by Lemma 2.3, Theorems 1.1 and 1.2 we have $\chi'_a(G) \leq \sum_{i=1}^{k} \chi'_a(G_i) \leq 5k = 5(r + 1)/3 < (5r + 37)/3$.

If condition 2 holds, then $\chi'_a(G) \leq \chi'_a(G_1) + \chi'_a(G_2) + \sum_{i=3}^{k} \chi'_a(G_i) \leq 12 + 12 + 5(k - 2) = 5(r - 1)/3 + 14 = (5r + 37)/3$.

If condition 3 holds, then $\chi'_a(G) \leq \chi'_a(G_1) + \sum_{i=2}^{k} \chi'_a(G_i) \leq 12 + 5(k - 1) = 5r/3 + 7 < (5r + 37)/3$.

Note that the upper bound in Theorem 3.3 is better than the upper bound in Theorem 2.4 when $r \geq 14$.

4 Proof of Theorem 2.1

We devote this section to a complete proof of Theorem 2.1.

Assume that $G$ is a normal graph with $\Delta(G) \geq 6$. We abbreviate $\Delta(G)$ and $d_G(v)$ to $\Delta$ and $d(v)$, respectively. Let $\mathcal{H}(G)$ be the collection of subgraphs $M$ of $G$ that satisfy the following conditions.

1. $\Delta(M) \leq 3$.
2. If $d(v) = \Delta$, then $d_M(v) \geq 2$.
3. If $d(v) = \Delta - 1$, then $d_M(v) \geq 1$.

We first show that $\mathcal{H}(G) \neq \emptyset$. By Theorem 3.1 $E(G)$ can be partitioned into $\Delta + 1$ disjoint color classes $E_1, E_2, \ldots, E_{\Delta+1}$ such that each $E_i$ is a matching of $G$.

Let $M = G[E_1 \cup E_2 \cup E_3]$. Then $\Delta(M) \leq 3$. For a $\Delta$-vertex $x$ of $G$, at most one among $E_1, E_2, E_3$ contains no edge incident with $x$. For a $(\Delta - 1)$-vertex $y$ of $G$, at most two among $E_1, E_2, E_3$ contain no edge incident with $y$. Thus $M \in \mathcal{H}(G)$.

For any $M \in \mathcal{H}(G)$, it is easy to see that $\Delta(M) \leq \Delta - 2$. Now let $I(M)$ and $\overline{I(M)}$ denote the sets of isolated edges of $M$ and $\overline{M}$, respectively, and write $i(M) = |I(M)|$ and $\overline{i(M)} = |\overline{I(M)}|$. Among all subgraphs $M$ that attain the minimum for $i(M) + i(\overline{M})$, we pick and fix an $H$ that has minimum number of edges.

We are going to show that the edge-partition of $G$ induced by this $H$ satisfies conditions 1, 2 and 3 of Theorem 2.1. If $i(H) + i(\overline{H}) = 0$, then we are done. Now we assume that $i(H) + i(\overline{H}) > 0$.

We first classify some of the vertices of $G$ into two types.
A vertex \( v \in V(G) \) is classified as type-I if \( 1 \leq d_H(v) \leq 2, d(v) \geq \Delta - 1 \), and for every \( u \in N_{\overline{G}}(v) \), one of the following three conditions holds.

1. \( d_H(u) = 3 \).
2. \( d_H(u) = d_{\overline{G}}(u) = 2 \).
3. \( d_H(u) \leq 1, d_{\overline{G}}(u) = 2 \), and, for the unique \( w \in N_{\overline{G}}(u) \setminus \{v\} \), both \( d_{\overline{G}}(w) = 1 \) and \( d_H(w) = 3 \).

**Claim 1.** Suppose that \( vv' \in I(H) \) with \( d(v) \geq d(v') \). Then \( d(v) = \Delta - 1 \) and \( v \) is a type-I vertex.

**Proof.** Since \( H \in \mathcal{H}(G) \) and \( vv' \) is an isolated edge of \( H \), \( d_H(v) = 1 \) and \( d(v) \leq \Delta - 1 \).

If \( d(v) \leq \Delta - 2 \), then \( H' = H \setminus \{vv'\} \in \mathcal{H}(G) \). Note that \( i(H') = i(H) - 1 \) and \( i(\overline{H'}) \leq i(\overline{H}) \) since \( vv' \notin I(\overline{H}) \). The subgraph \( H' \) contradicts the choice of \( H \).

Consequently, \( d(v) = \Delta - 1 \).

Assume to the contrary that \( v \) is not a type-I vertex. Then there exists a particular \( u \in N_{\overline{G}}(v) \) that satisfies none of (1), (2) or (3). Thus, the following three statements hold for this \( u \).

(a) \( d_H(u) \neq 3 \), and hence \( d_H(u) \leq 2 \).

(b) If \( d_H(u) = 2 \), then \( d_{\overline{G}}(u) \neq 2 \).

(c) If \( d_H(u) \leq 1 \) and \( d_{\overline{G}}(u) = 2 \), then, for the unique \( w \in N_{\overline{G}}(u) \setminus \{v\} \), \( d_{\overline{G}}(w) = 1 \) implies \( d_H(w) \neq 3 \), and hence \( d_H(w) \leq 2 \).

Define \( H' = H \cup \{uv\} \) for case (b) or when \( d_{\overline{G}}(w) \neq 1 \) for case (c). Define \( H' = H \cup \{uw, uw\} \) when \( d_{\overline{G}}(w) = 1 \) for case (c). It is easy to check that \( H' \in \mathcal{H}(G) \). Since \( d_{\overline{G}}(v) = d(v) - d_{H'}(v) = (\Delta - 1) - 2 > 2 \), no new isolated edge is created in \( \overline{H'} \). Yet \( i(H') = i(H) - 1 \). This contradicts the choice of \( H \).

A vertex \( u \in V(G) \) is classified as type-II if \( d_H(u) = 3 \), or \( d_H(u) = d_{\overline{G}}(u) = 2 \), and for every \( v \in N_H(u) \), one of the following two conditions holds.

4. \( 1 \leq d_H(v) \leq 2 \) and \( d(v) \geq \Delta - 1 \).

5. \( d_H(v) = 2, d(v) < \Delta - 1 \), and, for the unique \( w \in N_H(v) \setminus \{u\} \), both \( d_H(w) = 1 \) and \( d(w) = \Delta - 1 \).

**Claim 2.** Suppose that \( uu' \in I(\overline{G}) \) with \( d(u) \geq d(u') \). Then \( d_H(u) = 3 \) and \( u \) is a type-II vertex.

**Proof.** Since \( uu' \) is an isolated edge of \( \overline{G} \) and \( G \) has no isolated edges, it follows that \( d_H(u) \geq 1 \). If \( d_H(u) \leq 2 \), then \( H' = H \cup \{uu'\} \in \mathcal{H}(G) \). Note that \( i(H') \leq i(H) \).
and $i(\overline{H'}) = i(H) - 1$. The subgraph $H'$ contradicts the choice of $H$. Consequently, $d_H(u) = 3$.

Assume to the contrary that $u$ is not a type-II vertex. Then there exists a particular $v \in N_H(u)$ that satisfies neither (4) nor (5). Thus, the following two statements hold for this $v$.

(d) If $1 \leq d_H(v) \leq 2$, then $d(v) < \Delta - 1$.

(e) If $d_H(v) = 2$, $d(v) < \Delta - 1$, then, for the unique $w \in N_H(v) \setminus \{u\}$, $d_H(w) = 1$ implies $d(w) \neq \Delta - 1$, and hence $d(w) < \Delta - 1$.

If $d_H(v) = 1$ or $d_H(v) = 2$ and $d_H(w) \geq 2$, let $H' = H \setminus \{uv\}$. If $d_H(v) = 2$ and $d_H(w) = 1$, let $H' = H \setminus \{uv,vw\}$. Thus, the subgraph $H' \in \mathcal{H}(G)$ and satisfies $i(H') \leq i(H)$ and $i(\overline{H'}) = i(\overline{H}) - 1$, contradicting the choice of $H$.

We observe that no vertex can be classified both as type-I and type-II since $1 \leq d_H(z) \leq 2$ and $d(z) \geq \Delta - 1 \geq 5$ for a type-I vertex $z$, while $d_H(w) = 3$ or $d_H(w) = d_{\overline{H'}}(w) = 2$ for a type-II vertex $w$.

An $H$-chain emanating from a vertex $u$ is a path from $u$ to a $v \in N_H(u)$ when $v$ satisfies (4), or through $v$ to the unique $w \in N_H(v) \setminus \{u\}$ when $v$ satisfies (5). We write $u \to x$ for an $H$-chain emanating from $u$ and terminating at $x$. An $\overline{H}$-chain emanating from a vertex $v$ is a path from $v$ to a $u \in N_{\overline{H}}(v)$ when $u$ satisfies (1) or (2), or through $u$ to the unique $w \in N_{\overline{H}}(u) \setminus \{v\}$ when $u$ satisfies (3). We write $v \sim y$ for an $\overline{H}$-chain emanating from $v$ and terminating at $y$. A path $P$ of $G$ is called an alternating chain if $P$ is a concatenation of $H$-chains and $\overline{H}$-chains such that they appear alternately and the terminating vertex of one chain is the emanating vertex of the next chain.

Claim 3. If $vv' \in I(H)$ satisfies $d(v) \geq d(v')$, then the two ends of each $H$-chain or $\overline{H}$-chain of an alternating chain $P$ beginning with $v$ are of different types.

Proof. Let $v_0 = v$. By Claim 1, $v_0$ is a type-I vertex. By the definition of an alternating chain, we may assume that $P$ is $v_0 \sim u_1 \to v_1 \sim \cdots \sim v_{s-1} \sim u_s$ or $P$ is $v_0 \sim u_1 \to v_1 \sim \cdots \sim u_s \to v_s$, where $s \geq 1$. It suffices to prove by induction that $v_1, v_2, \ldots, v_s$ are type-I vertices and $u_1, u_2, \ldots, u_s$ are type-II vertices. Equivalently, for each $1 \leq k \leq s$, the following statements (A) and (B) are true.

(A) If $v_1, v_2, \ldots, v_{k-1}$ are type-I vertices and $u_1, u_2, \ldots, u_{k-1}$ are type-II vertices, then $u_k$ is a type-II vertex.
(B) If \( v_1, v_2, \ldots, v_{k-1} \) are type-I vertices and \( u_1, u_2, \ldots, u_k \) are type-II vertices, then \( v_k \) is a type-I vertex.

In order to show (A), assume to the contrary that \( u_k \) is not a type-II vertex. Since \( v_{k-1} \sim u_k \) and \( v_{k-1} \) is a type-I vertex, \( d_H(u_k) = 3 \), or \( d_H(u_k) = \Delta(u_k) = 2 \). Then there exists a vertex \( x \in N_H(u_k) \) such that the following two statements hold for this \( x \).

\[
\begin{align*}
(d') & \text{ If } 1 \leq d_H(x) \leq 2, \text{ then } d(x) < \Delta - 1. \\
(e') & \text{ If } d_H(x) = 2, \text{ for the unique } y \in N_H(x) \setminus \{u_k\}, d_H(y) = 1 \text{ implies } d(y) < \Delta - 1.
\end{align*}
\]

Since \( v_0, v_1, \ldots, v_{k-1} \) are type-I vertices by the induction hypothesis, \( 1 \leq d_H(v_i) \leq 2 \) and \( d(v_i) \geq \Delta - 1 \) for all \( 0 \leq i \leq k - 1 \). Since \( d_H(x) = 3 \), or \( d(x) < \Delta - 1 \), it follows that there exists a vertex \( z \) such that \( z \notin \{v_0, v_1, \ldots, v_{k-1}\} \). We next show that \( x \notin \{u_1, u_2, \ldots, u_k\} \).

Assume to the contrary that there is an index \( i \) such that \( x = u_i \). Since \( u_i \) is a type-II vertex and \( u_k \in N_H(u_i) \), it follows that \( d_H(u_k) \leq 2 \). We have already known that \( d_H(u_k) = 3 \), or \( d_H(u_k) = \Delta(u_k) = 2 \). Hence, \( d_H(u_k) = 2 \) and \( d(u_k) = 4 \).

Let \( z \in N_H(u_k) \setminus \{u_1\} \). Define

\[
H' = (H \cup \bigcup_{j=0}^{i-1} E(v_j \sim u_{j+1})) \setminus \bigcup_{j=1}^{i-1} (S \cup \bigcup_{j=1}^{i-1} E(u_j \rightarrow v_j)),
\]

where \( S = \{u_1u_k, u_kz\} \) if \( d_H(z) = 1 \); or \( S = \{u_kz\} \) otherwise. It is straightforward to check that \( H' \in \mathcal{H}(G) \), such that \( i(H') = i(H) - 1 \) and \( i(\overline{H'}) = i(\overline{H}) \), which contradicts the choice of \( H \).

Suppose that \( d_H(x) = 1 \) or \( d_H(x) = 2 \) and \( d_H(y) > 1 \) in \( (e') \). If \( d_H(u_k) = 3 \), then let \( H' = H \setminus \{xu_k\} \). It is obvious that \( H' \in \mathcal{H}(G) \). Since \( xu_k \) is adjacent to an edge in \( v_{k-1} \sim u_k \), \( xu_k \) can not be an isolated edge of \( \overline{H} \). Thus, \( i(H') = i(H) \) and \( i(\overline{H'}) = i(\overline{H}) \). However, \( |E(H')| = |E(H)| - 1 \), which contradicts the choice of \( H \).

If \( d_H(u_k) = \Delta(u_k) = 2 \), define

\[
H' = (H \cup \bigcup_{i=0}^{k-1} E(v_i \sim u_{i+1})) \setminus \bigcup_{i=1}^{k-1} E(u_i \rightarrow v_i) \cup \{xu_k\}).
\]

Note that \( d_H(u_i) = d_H(u_i) \) and \( d_H(v_i) = d_H(v_i) \) for \( 1 \leq i \leq k \), \( d_H(v_0) = d_H(v_0) + 1 = 2 \), \( d_H(v_0) = (\Delta - 1) - 2 \geq 3 \), and hence \( v'v_0 \notin I(H') \). It follows that \( i(H') = i(H) - 1 \) and \( i(\overline{H'}) = i(\overline{H}) \), which contradicts the choice of \( H \).

Next consider the case \( d_H(y) = 1 \) in \( (e') \). Then \( y \notin \{v_0, v_1, \ldots, v_{k-1}\} \) since \( d(y) < \Delta - 1; y \notin \{u_1, u_2, \ldots, u_{k-1}\} \) for each type-II vertex \( u_i \) (\( 1 \leq i \leq k - 1 \)) has \( d_H(u_i) \geq 2 \).
Define
\[ H' = (H \cup \bigcup_{i=0}^{k-1} E(v_i \sim u_{i+1})) \setminus \bigcup_{i=1}^{k-1} E(u_i \rightarrow v_i) \cup \{xy, xu_k\}. \]

Then \( H' \in \mathcal{H}(G) \). Reasoning as before, we see that \( i(H') = i(H) - 1 \) and \( i(\overline{H'}) = i(\overline{H}) \), which contradicts the choice of \( H \).

To prove (B), assume to the contrary that \( v_k \) is not a type-I vertex. Since \( u_k \rightarrow v_k \) and \( u_k \) is a type-II vertex, \( 1 \leq d_H(v_k) \leq 2 \) and \( d(v_k) \geq \Delta - 1 \). Then there exists a vertex \( x \in N_{\overline{H'}}(v_k) \) such that the following three statements hold for this \( x \).

(a') \( d_H(x) \neq 3 \), and hence \( d_H(x) \leq 2 \).
(b') If \( d_H(x) = 2 \), then \( d_{\overline{H}}(x) \neq 2 \).
(c') If \( d_H(x) \leq 1 \) and \( d_{\overline{H}}(x) = 2 \), then, for the unique \( y \in N_{\overline{H'}}(x) \setminus \{v_k\} \), \( d_{\overline{H}}(y) = 1 \) implies \( d_H(y) \leq 2 \).

Since \( u_1, u_2, \ldots, u_k \) are type-II vertices by the induction hypothesis, we see that for \( 1 \leq i \leq k \), either \( d_H(u_i) = 3 \) or \( d_H(u_i) = d_{\overline{H}}(u_i) = 2 \). Therefore, \( x \notin \{u_1, u_2, \ldots, u_k\} \).

We next show that there is an index \( i \) (\( 0 \leq i \leq k - 1 \)) such that \( x = v_i \). Since \( v_i \) is a type-I vertex and \( v_k \in N_{\overline{H'}}(v_i) \), it follows that \( d_H(v_k) = 3 \) or \( d_H(v_k) = d_{\overline{H}}(v_k) = 2 \). However, \( d_H(v_k) \leq 2 \) and \( d(v_k) \geq \Delta - 1 \geq 5 \) since \( u_k \rightarrow v_k \). We have reached a contradiction.

Now assume \( d_{\overline{H}}(y) = 1 \) in (c'). Then \( y \notin \{u_1, u_2, \ldots, u_k\} \). We also have \( y \notin \{v, v_1, \ldots, u_{k-1}\} \), for otherwise it would imply \( d_{\overline{H}}(y) \geq 2 \). Define
\[ H' = (H \cup S \cup \bigcup_{i=0}^{k-1} E(v_i \sim u_{i+1})) \setminus \bigcup_{i=1}^{k-1} E(u_i \rightarrow v_i), \]
where \( S = \{xy, xu_k\} \) when \( d_{\overline{H}}(y) = 1 \) for case (c'); \( S = \{vx_k\} \) for case (b') or when \( d_{\overline{H}}(y) \neq 1 \) for case (c'). It is easy to check that \( H' \in \mathcal{H}(G) \) such that \( i(H') = i(H) - 1 \) and \( i(\overline{H'}) = i(\overline{H}) \). This contradicts the choice of \( H \).

\[ \blacksquare \]

**Claim 4.** If \( uu' \in I(\overline{H}) \) satisfies \( d(u) \geq d(u') \), then the two ends of each \( H \)-chain or \( \overline{H} \)-chain of an alternating chain \( P \) beginning with \( u \) are of different types.

**Proof.** Let \( u_1 = u \) which is a type-II vertex by Claim 2. By the definition of an alternating chain, we may assume that \( P = u_1 \rightarrow v_1 \sim u_2 \rightarrow \cdots \sim u_s \rightarrow v_s \) or \( P = u_1 \rightarrow v_1 \sim u_2 \rightarrow \cdots \sim v_{s-1} \sim u_s \), where \( s \geq 1 \). Similar to the proof of Claim 3, we may argue that, for each \( 1 \leq k \leq s \), the following statements (C) and (D) are true.

(C) If \( u_1, u_2, \ldots, u_k \) are type-II vertices and \( v_1, v_2, \ldots, v_{k-1} \) are type-I vertices, then \( v_k \) is a type-I vertex.
(D) If \( u_1, u_2, \ldots, u_{k−1} \) are type-II vertices and \( v_1, v_2, \ldots, v_{k−1} \) are type-I vertices, then \( u_k \) is a type-II vertex.

The proof of (B) in Claim 3 can be adapted to show the validity of (C). Here we define

\[
H' = (H \cup S) \cup \bigcup_{i=1}^{k−1} E(v_i \xrightarrow{u_i} v_{i+1}) \setminus \bigcup_{i=1}^{k−1} E(u_i \rightarrow v_i),
\]

where \( S = \{xy, xv_k\} \) if \( d_H(y) = 1 \); \( S = \{xv_k\} \) if \( d_H(y) > 1 \).

The proof of (A) in Claim 3 can be adapted to show the validity of (D). Here we define

\[
H' = (H \cup S) \cup \bigcup_{i=1}^{k−1} E(v_i \xrightarrow{u_i} v_{i+1}) \setminus (S \cup \bigcup_{i=1}^{k−1} E(u_i \rightarrow v_i)),
\]

where \( S = \{xy, xu_k\} \) if \( d_H(y) = 1 \); \( S = \{xu_k\} \) if \( d_H(y) > 1 \).

In both cases, \( d_{H'}(u_1) = 3 − 1 = 2 \) and \( d_{H'}(u_1) = 2 \). It is easy to check that \( \overline{H'} \in \mathcal{H}(G) \) such that \( i(H') = i(H) \) and \( i(\overline{H'}) = i(\overline{H}) − 1 \). This contradicts the choice of \( H \).

Now we are ready to derive contradictions from the assumption \( i(H) + i(\overline{H}) > 0 \).

**Case 1** \( i(H) > 0 \).

Suppose that \( v_0v' \in I(H) \) with \( d(v_0) \geq d(v') \). Let \( C(v_0) \) be the set of alternating chains of \( G \) beginning with the vertex \( v_0 \). By Claims 1 and 3, \( C(v_0) \) is a nonempty set. Let \( V_I(P) \) and \( V_{II}(P) \), respectively, be the sets of type-I vertices and type-II vertices on an alternating path \( P \in C(v_0) \). Define \( V_I = \bigcup \{V_I(P) \mid P \in C(v_0)\} \) and \( V_{II} = \bigcup \{V_{II}(P) \mid P \in C(v_0)\} \).

For any vertex \( w \in V_{II} \), if \( x \in N_H(w) \), then either \( x \in V_I \), or \( d_H(x) = 2 \) and the unique vertex \( y \in N_H(x) \setminus \{w\} \) satisfies that \( d_H(y) = 1 \) and \( y \in V_I \). Thus

\[
\sum_{z \in V_I} d_H(z) \geq \sum_{w \in V_{II}} d_H(w).
\]

Since each vertex of \( V_I \) has degree at most two in \( H \), and each vertex of \( V_{II} \) has degree at least two in \( H \), we have

\[
2|V_I| \geq \sum_{z \in V_I} d_H(z) \geq \sum_{w \in V_{II}} d_H(w) \geq 2|V_{II}|.
\]

Thus, \( |V_I| \geq |V_{II}| \).
For any $z \in V_I$, we have $d_H(z) \leq 2$ and $d(z) \geq \Delta - 1$, and hence $d_{\overrightarrow{\Pi}}(z) \geq \Delta - 3$. From $d_H(v_0) = 1$ and $d(v_0) = \Delta - 1$, we know $d_{\overrightarrow{\Pi}}(v_0) = \Delta - 2$. Hence,

$$\sum_{z \in V_I} d_{\overrightarrow{\Pi}}(z) = d_{\overrightarrow{\Pi}}(v_0) + \sum_{z \in V_I \setminus \{v_0\}} d_{\overrightarrow{\Pi}}(z) \geq |V_I|(|\Delta - 3| + 1).$$

For any $w \in V_{II}$, we see that $d_{\overrightarrow{\Pi}}(w) = 3$ or $d_{\overrightarrow{\Pi}}(w) = d_{\overrightarrow{\Pi}}(w) = 2$. Thus $\Delta \geq 6$ implies

$$\sum_{w \in V_{II}} d_{\overrightarrow{\Pi}}(w) \leq |V_{II}|(|\Delta - 3|).$$

However, $|V_I| \geq |V_{II}|$ implies

$$\sum_{w \in V_{II}} d_{\overrightarrow{\Pi}}(w) < \sum_{z \in V_I} d_{\overrightarrow{\Pi}}(z).$$

Case 2 $i(\overrightarrow{\Pi}) > 0$.

Suppose that $u_1u' \in I(\overrightarrow{\Pi})$ with $d(u_1) \geq d(u')$. Let $D(u_1)$ be the set of alternating chains of $G$ beginning with the vertex $u_1$. By Claims 2 and 4, $D(u_1)$ is a nonempty set. Let $V_I(P)$ and $V_{II}(P)$, respectively, be the sets of type-I vertices and type-II vertices on an alternating path $P \in D(u_1)$. Define $V_I = \cup\{V_I(P) \mid P \in D(u_1)\}$ and $V_{II} = \cup\{V_{II}(P) \mid P \in D(u_1)\}$.

Similar to the proof of Case 1, we have that $|V_I| \geq |V_{II}|$ and

$$|V_I|(|\Delta - 3|) \leq \sum_{z \in V_I} d_{\overrightarrow{\Pi}}(z) \leq \sum_{w \in V_{II}} d_{\overrightarrow{\Pi}}(w).$$

However, since $d_{\overrightarrow{\Pi}}(u_1) = 1$ and $\Delta \geq 6$, we get

$$\sum_{w \in V_{II}} d_{\overrightarrow{\Pi}}(w) = d_{\overrightarrow{\Pi}}(u_1) + \sum_{w \in V_{II} \setminus \{u_1\}} d_{\overrightarrow{\Pi}}(w) < |V_{II}|(|\Delta - 3|).$$

A contradiction is produced. This completes the proof of Theorem 2.1.

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