On solutions for stochastic differential equations with Hölder continuous coefficients

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Abstract. In this paper, we prove the strong Feller property and the existence of probability density for a class of stochastic differential equations with Hölder continuous coefficients. Moreover, if the weak derivative of diffusion coefficients are in some sorts of Sobolev space, we also derive the pathwise uniqueness, Hölder continuity and weak differentiability for solutions.

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1. Introduction

Consider the following stochastic differential equation (SDE) in $\mathbb{R}^d$:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t > 0, \quad X_0 = x \in \mathbb{R}^d,$$  \hspace{1cm} (1.1)

where $\{W_t\}_{t \geq 0} = \{(W_{1,t}, \ldots, W_{d,t})\}_{t \geq 0}$ is a $d$-dimensional standard Wiener process defined on a given stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, and the coefficients $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ are Borel measurable. When $\sigma$ is Lipschitz continuous in $x$ uniformly in $t$ and $b$ is bounded measurable, Veretennikov [26] first proved the existence of a unique strong solution for SDE (1.1). Since then, Veretennikov’s result was strengthened in different forms under the same assumption on $b$. For instance, when $\sigma = I_{d \times d}$, Mohammed, Nilssen and Proske in [19] not only proved the existence and uniqueness of strong solutions, but also obtained that the unique strong solution forms a Sobolev differentiable stochastic flows; Davie showed in [3]...

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that for almost every Wiener path $W$, there was a unique continuous $X$ satisfying the integral equation (also see [6]). When $\sigma = I_{d \times d}$ and $b$ satisfied the Ladyzhenskaya-Prodi-Serrin (LPS) condition (see [17, 22, 23]):

$$b \in L^q(0, T; L^p(\mathbb{R}^d; \mathbb{R}^d)),$$  

$p, q \in [2, \infty), \ \forall \ T > 0,$ (1.2) and

$$\frac{2}{q} + \frac{d}{p} < 1,$$ (1.3)

in view of Girsanov’s transformation and the Krylov estimates, Krylov and Röckner [15] showed the existence and uniqueness of strong solutions for (1.1). Recently, following [15], Fedrizzi and Flandoli [5] derived the $\beta$-Hölder continuity of $x \mapsto X_t(x)$ for every $\beta \in (0, 1)$. More recently, for non-constant diffusion, if $\sigma(t, x)$ was continuous in $x$ uniformly with respect to $t$, $\sigma\sigma^\top$ met uniformly elliptic condition and $|\nabla \sigma| \in L^q_{loc}(\mathbb{R}^d; L^p(\mathbb{R}^d))$ with $p, q$ satisfying (1.3), Zhang [30] demonstrated the existence and local uniqueness of strong solutions to (1.1). Moreover, there are many other excellent research works devoted to studying the existence and uniqueness for strong solutions under various non-Lipschitz conditions on coefficients, we refer to see [1, 4, 9, 29].

If one turns the attention to weak solutions for (1.1), the restrictions on $b$ and $\sigma$ can be relaxed. In fact, if $b$ is bounded measurable and $\sigma$ is bounded continuous such that $\sigma\sigma^\top$ satisfies uniformly elliptic condition, then (1.1) exists a unique weak solution [24, 25]. This result was generalized by Jin [11] to the case of $b \in FK^c_{d-1} (c \in (0, 1/2))$ when $\sigma = I_{d \times d}$. Some other related works on time independent/dependent $b$ can also be founded in [2, 20]. However, to the best of our knowledge, no matter strong solutions and weak solutions, there are few investigations to argue the case of (1.2) with $q \leq 2$ since the condition (1.3) is no longer true in the present case. To get an analogue with $q \leq 2$ we assume that the coefficients lie in

$$L^q_{loc}(\mathbb{R}^d; C^\alpha_b(\mathbb{R}^d; \mathbb{R}^d)),$$ (1.4)

When $q = \infty$ and $\sigma = I_{d \times d}$, the existence as well as uniqueness for strong solution has been established by Flandoli, Gubinelli and Priola [7] (also see [8] for unbounded drift coefficients). As we known, there are still few research works concerned with $q \leq 2$ in the case of (1.4). This problem is the main driving source for us to work out the present paper.

The aim in this paper is two-fold: first we shall prove existence and uniqueness of weak solutions for SDE (1.1), and then discuss the the strong Feller property and the existence of probability density once SDE (1.1) has a unique weak solution. The key point is to transform the original SDE (1.1) to an equivalent new SDE by using Ito-Tanack’s trick and this will done in Sections 2 and 3. Secondly we shall study pathwise uniqueness, Hölder continuity and weak differentiability for solutions using the approach mentioned above for SDE with coefficients in Lebesgue-Hölder space and these results are established in Section 4. In summary, we outline our main results as follows.
SDEs with Hölder continuous coefficients

Theorem 1.1. Let \( b \in L^1_{\text{loc}}(\mathbb{R}^+_t;C_{bu}(\mathbb{R}^d;\mathbb{R}^d)) \), and let \( \sigma = (\sigma_{i,j}) \) be a \( d \times d \) matrix valued function such that \( \sigma_{i,j} \in L^2_{\text{loc}}(\mathbb{R}^+_t;C_{bu}(\mathbb{R}^d)) \) \((C_{bu}(\mathbb{R}^d))\) is the space consisting of all bounded uniformly continuous functions on \( \mathbb{R}^d \).

(i) Then there exists a weak solution to SDE (1.1).

(ii) We suppose further that \( \alpha \in (0, 1) \), \( b \in L^2_{\text{loc}}(\mathbb{R}^+_t;C^\alpha_{bu}(\mathbb{R}^d;\mathbb{R}^d)) \) and \( \sigma_{i,j} \in C(\mathbb{R}^+_t;C_{bu}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+_t;C^\alpha_{bu}(\mathbb{R}^d)) \), and \( a = (a_{i,j}) = \sigma \sigma^\top = (\sigma_{i,k}\sigma_{j,k}) \) meets uniformly elliptic condition. Then, for every \( T > 0 \), all weak solutions for SDE (1.1) have the same probability law on \( d \)-dimensional classical Wiener space \((W^d([0, T]), \mathcal{B}(W^d([0, T])))\), and all weak solutions are strong Markov processes. We use \( \mathbb{P}_x \) and \( P(x, t, dy) \) to denote the unique probability law on \((W^d([0, T]), \mathcal{B}(W^d([0, T])))\) and the transition probabilities, respectively. For every \( f \in L^\infty(\mathbb{R}^d) \), we define
\[
P_t f(x) := \mathbb{E}_{x}^f f(w(t)) = \int_{\mathbb{R}^d} f(y)P(x, t, dy), \quad t > 0, \tag{1.5}
\]
where \( w(t) \) is the canonical realisation of a weak solution \( \{X_t\} \) with initial data \( X_0 = x \in \mathbb{R}^d \) on \((W^d([0, T]), \mathcal{B}(W^d([0, T])))\). Then, \( \{P_t\} \) has strong Feller property, i.e. \( P_t \) maps a bounded function to a bounded continuous function for every \( t > 0 \). Moreover, \( P(x, t, dy) \) admits a density \( p(x, t, y) \) for almost all \( t \in [0, T] \). Besides, for every \( t_0 > 0 \) and for every \( s \in [1, \infty) \),
\[
\int_{t_0}^{T} \int_{\mathbb{R}^d} |p(x, t, y)|^s dy dt < \infty. \tag{1.6}
\]

(iii) With the same condition of (ii). We assume further that \( |\nabla \sigma| \in L^2_{\text{loc}}(\mathbb{R}^+_t;L^\infty(\mathbb{R}^d)) \), then the strong uniqueness holds for (1.1). The random field \( \{X_t(x), t > 0, x \in \mathbb{R}^d\} \) has a continuous modification \( \tilde{X} \), which is \( \beta \)-Hölder continuous in \( x \) for \( \beta \in (0, 1) \). Moreover, for every \( s \geq 1 \), and every \( T > 0 \),
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \left( \sup_{x \neq y} \left| \frac{\tilde{X}_t(x) - \tilde{X}_t(y)}{|x - y|^{\beta}} \right| \right)^s \right] < \infty. \tag{1.7}
\]

(iv) With the same condition of (iii). For almost all \( \omega \in \Omega \), every \( t > 0 \), \( x \mapsto X_t(x) \) is a homeomorphism on \( \mathbb{R}^d \). Moreover, \( X_t(x) \) is differentiable in \( x \) in the sense that: \( \{e_i\}_{i=1}^d \) is the canonical basis of \( \mathbb{R}^d \), for every \( x \in \mathbb{R}^d \), every \( T > 0 \) and \( 1 \leq i \leq d \), the limit
\[
\lim_{\delta \to 0} \frac{X_t(x + \delta e_i) - X_t(x)}{\delta} \tag{1.8}
\]
exists in \( L^2(\Omega \times (0, T)) \).

Remark 1.2. The proof for the above theorem is applicable to the drift \( b \in L^q_{\text{loc}}(\mathbb{R}^+_t;C^\alpha_{bu}(\mathbb{R}^d;\mathbb{R}^d)) \), with \( q > 2 \). Particularly, when \( \sigma = I_{d \times d} \) and \( q > 2/\alpha \) we conclude that: there is a unique strong solution \( X_t(x) \) of (1.1), which forms a stochastic flow of \( C^{1,\beta} \) \((0 < \beta < \alpha - 2/q)\) diffeomorphisms. In this point, we recover the result [7, Theorem 5], but only assuming \( q > 2/\alpha \).
In this paper, the summation convention is enforced, wherein summation is understood with respect to repeated indices. When there is no ambiguity, we use $C$ to denote a constant whose true value may vary from line to line and use $\nabla$ to denote the gradient of a function with respect to the space variable. As usual, $\mathbb{N}$ stands for the set of all natural numbers, $\mathbb{R}_+ = [0, \infty)$, $B_R = \{x \in \mathbb{R}^d; |x| \leq R\}$ for $R > 0$.

2. Preliminaries

Assume $\alpha \in (0, 1)$ and $q \in [1, 2]$. We denote $L^q_{loc}(\mathbb{R}_+; C^\alpha_b(\mathbb{R}^d))$ to be the set of all $C^\alpha_b(\mathbb{R}^d)$-valued Borel functions $h$ such that for every $T \in (0, \infty)$,

$$
\|h\|_{L^q(0,T; C^\alpha_b(\mathbb{R}^d))} = \left\{ \int_0^T \left[ \max_{x \in \mathbb{R}^d} |h(t, x)| + \sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|h(t, x) - h(t, y)|}{|x - y|^{\alpha}} \right]^q dt \right\}^{\frac{1}{q}} 
$$

$$
= \left\{ \int_0^T \left[ \|h(t)\|_{L^q} + |h(t)|_{\alpha} \right]^q dt \right\}^{\frac{1}{q}} < \infty. 
$$

We suppose that $T > 0$, $g \in L^q(0,T; C^2_{\alpha}^\infty(\mathbb{R}^d; \mathbb{R}^d))$ and $h \in L^q(0,T; C^\alpha_b(\mathbb{R}^d))$. Consider the following Cauchy problem for $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$,

$$
\begin{align*}
\partial_t u(t, x) &= \frac{1}{2}\Delta u(t, x) + g(t, x) \cdot \nabla u(t, x) \\
&\quad + h(t, x), \quad (t, x) \in (0,T) \times \mathbb{R}^d, \\
u(0, x) &= 0, \quad x \in \mathbb{R}^d. 
\end{align*}
$$

(2.1)

$u$ is called to be a generalized solution of (2.1) if it lies in $L^q(0,T; C^2_{\alpha}^\infty(\mathbb{R}^d)) \cap W^{1,q}(0,T; C^\alpha_b(\mathbb{R}^d))$ such that for every test function $\varphi \in C^\infty_0([0,T] \times \mathbb{R}^d)$,

$$
- \int_0^T \int_{\mathbb{R}^d} u(t, x) \partial_t \varphi(t, x) dxdt - \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} u(t, x) \Delta \varphi(t, x) dxdt \\
= \int_0^T \int_{\mathbb{R}^d} g(t, x) \cdot \nabla u(t, x) \varphi(t, x) dxdt + \int_0^T \int_{\mathbb{R}^d} h(t, x) \varphi(t, x) dxdt.
$$

The following lemma is standard, and for more details one consults to [31] Proposition 3.5, we omit its proof here.

**Lemma 2.1.** Let $q, \alpha$ and $T$ be real numbers, which are in $[1, 2]$, $(0, 1)$ and $(0, \infty)$, respectively. We assume that $h \in L^q(0,T; C^\alpha_b(\mathbb{R}^d))$, $u \in L^\infty(0,T; C^1_b(\mathbb{R}^d))$ and $g \in L^q(0,T; C^\alpha_b(\mathbb{R}^d; \mathbb{R}^d))$. Then the following statements are equivalent:

(i) $u$ is a generalized solution of (2.1);

(ii) for every $\psi \in C^\infty_0(\mathbb{R}^d)$, and every $t \in [0,T)$,

$$
\int_{\mathbb{R}^d} u(t, x) \psi(x) dx = \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u(r, x) \Delta \psi(x) dxdr \\
+ \int_0^t \int_{\mathbb{R}^d} g(r, x) \cdot \nabla u(r, x) \psi(x) dxdr + \int_0^t \int_{\mathbb{R}^d} h(r, x) \psi(x) dxdr;
$$

$$
\int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left( \int_{\mathbb{R}^d} u(t, x) \psi(x) dx \right) dt = \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left( u(t, x) \Delta \psi(x) dx \right) dt \\
+ \int_0^T \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left( g(t, x) \cdot \nabla u(t, x) \psi(x) dx \right) dt + \int_0^T \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left( h(t, x) \psi(x) dx \right) dt.
$$

$$
\int_{\mathbb{R}^d} u(t, x) dx = \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left( u(t, x) \Delta \psi(x) dx \right) dt \\
+ \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left( g(t, x) \cdot \nabla u(t, x) \psi(x) dx \right) dt + \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left( h(t, x) \psi(x) dx \right) dt.
$$
(iii) for every $t \in [0, T]$ and for almost everywhere $x \in \mathbb{R}^d$, $u$ fulfils the integral equation
\[
 u(t, x) = \int_0^t K(r, \cdot) \ast (g(t-r, \cdot) \cdot \nabla u(t-r, \cdot))(x) dr
 + \int_0^t (K(r, \cdot) \ast h(t-r, \cdot))(x) dr,
\]
where $K(r, x) = (2\pi r)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2r}}, r > 0, x \in \mathbb{R}^d$.

In preparation to the next section, the following lemma will play an important role later on.

**Lemma 2.2.** Let $q, \alpha, h$ and $g$ be described in Lemma [2.1]. We assume further that
\[
 \theta := 1 + \alpha - 2/q > 0.
\]
Then

(i) if $g = 0$, the Cauchy problem [2.1] has a unique generalized solution $u$. Moreover $u \in C([0, T]; C^1_b(\mathbb{R}^d)) \cap C^2_b([0, T]; C^1_b(\mathbb{R}^d))$ and
\[
 \|u\|_{C([0, T]; C^1_b(\mathbb{R}^d))} + \|u\|_{C^2_b([0, T]; C^1_b(\mathbb{R}^d))} \leq C\|h\|_{L^q(0, T; C^0_b(\mathbb{R}^d))};
\]

(ii) for a general function $g$, one assumes $g \in L^2(0, T; C^0_b(\mathbb{R}^d; \mathbb{R}^d))$ in addition, then [2.1] exists a unique generalized solution. Moreover, the unique solution $u$ belongs to $C([0, T]; C^1_b(\mathbb{R}^d)) \cap C^2_b([0, T]; C^1_b(\mathbb{R}^d))$ and
\[
 \|u\|_{C([0, T]; C^1_b(\mathbb{R}^d))} \leq C\|h\|_{L^q(0, T; C^0_b(\mathbb{R}^d))},
\]
where the constant $C$ in [2.5] only depends on $\alpha, d$ and $\|g\|_{L^2(0, T; C^0_b(\mathbb{R}^d))}$.

**Proof.** (i) With the help of [14] Theorems 3.1 and 3.3, the existence and uniqueness of generalized solution is clear. It remains to prove the Hölder continuity of the gradient of $u$ for $x$ and $t$. Since the proof for time regularity is similar to the space regularity, it suffices to show the regularity for the space variables. By virtue of Lemma [2.1] it needs to check: for every $1 \leq i \leq d$,
\[
 \partial_{x_i} u(t, x) = \partial_{x_i} \left[ \int_0^t K(r, \cdot) \ast h(t-r, \cdot) dr \right](x) \in C([0, T]; C^0_b(\mathbb{R}^d)).
\]

For every $x, y \in \mathbb{R}^d$, $t \in [0, T)$, $\delta > 0$ ($t + \delta < T$) and $1 \leq i \leq d$, we divide the quantity
\[
 [\partial_{x_i} u(t + \delta, x) - \partial_{x_i} u(t, x)] - [\partial_{y_i} u(t + \delta, y) - \partial_{y_i} u(t, y)]
\]
into $\sum_{i=1}^8 I_i^\delta(t)$ with
\[
 I_1^\delta(t) = \int_0^t \int_{|x-z| \leq 2|x-y|} \partial_{x_i} K(r, x-z)[h^\delta(t-r, z) - h^\delta(t-r, x)] dz,
\]
\[ I^δ_3(t) = \int_{0}^{t} \int_{|x-z|\leq 2|x-y|} \partial_{y_i} K(r, y - z)[h^δ(t - r, y) - h^δ(t - r, z)]dz, \]
\[ I^δ_4(t) = \int_{0}^{t} \int_{|x-z|> 2|x-y|} \partial_{y_i} K(r, y - z)[h^δ(t - r, y) - h^δ(t - r, x)]dz, \]
\[ I^δ_5(t) = \int_{0}^{t} \int_{|x-z|> 2|x-y|} [\partial_{x_i} K(r, x - z) - \partial_{y_i} K(r, y - z)] \times [h^δ(t - r, z) - h^δ(t - r, x)]dz, \]
\[ I^δ_6(t) = \int_{t}^{t+\delta} \int_{|x-z|\leq 2|x-y|} \partial_{x_i} K(r, x - z)h^δ(t - r, z, x)dz, \]
\[ I^δ_7(t) = \int_{t}^{t+\delta} \int_{|x-z|\leq 2|x-y|} \partial_{y_i} K(r, y - z)h^δ(t - r, y, z)dz, \]
\[ I^δ_8(t) = \int_{t}^{t+\delta} \int_{|x-z|> 2|x-y|} \partial_{y_i} K(r, y - z)h^δ(t - r, y, x)dz, \]
\[ I^δ_9(t) = \int_{t}^{t+\delta} \int_{|x-z|> 2|x-y|} [\partial_{x_i} K(r, x - z) - \partial_{y_i} K(r, y - z)] \times h^δ(t - r, z, x)dz, \]

where
\[ h^δ(t - r, x) = h(t + \delta - r, x) - h(t - r, x), \forall x \in \mathbb{R}^d, \]
and
\[ h^δ(t - r, x_1, x_2) = h(t + \delta - r, x_1) - h(t + \delta - r, x_2), \forall x_1, x_2 \in \mathbb{R}^d. \]

We first calculate the term \( I^δ_1 \):
\[ |I^δ_1(t)| \leq C \int_{0}^{t} \int_{|x-z|\leq 2|x-y|} [h^δ(t - r)]_\alpha |x - z|^\alpha e^{-\frac{|x-z|^2}{2r}} r^{-\frac{d+1}{2}} dzdr \]
\[ \leq C\|h^δ\|_{L^q(0,t;C^\alpha_\delta(\mathbb{R}^d))} \int_{|x-z|\leq 2|x-y|} |x - z|^\alpha \times \left[ \int_{0}^{t} e^{-\frac{|x-z|^2}{2r}} r^{-\frac{(d+1)\alpha'}{2}} dr \right]^{\frac{1}{q'}} dz \]
\[ \leq C\|h^δ\|_{L^q(0,t;C^\alpha_\delta(\mathbb{R}^d))} \int_{|x-z|\leq 2|x-y|} |x - z|^{1+\alpha - \frac{d}{q'}} dz \]
\[ \leq C\|h^δ\|_{L^q(0,t;C^\alpha_\delta(\mathbb{R}^d))} |x - y|^{\theta}, \quad (2.7) \]

where in the second line we have used the Hölder inequality and the Minkowski integral inequality, and \( 1/q' + 1/q = 1 \).

Similarly, we get
\[ |I^δ_2(t)| \leq C\|h^δ\|_{L^q(0,t;C^\alpha_\delta(\mathbb{R}^d))} |x - y|^{\theta}, \quad t \in [0, T), \ t + \delta \leq T. \quad (2.8) \]
For the term \( I_3^\delta(t) \), with the aid of Gauss-Green’s formula, the Hölder inequality, and the Minkowski integral inequality, we conclude that
\[
|I_3^\delta(t)| = \left| \int_0^t \int |x-z|=2|x-y| K(r, y - z) n_i [h^\delta(t - r, y) - h^\delta(t - r, x)] dS \right|
\leq C \|h^\delta\|_{L^q(0, t; C^\alpha_b(\mathbb{R}^d))} |x - y| \left[ \int_0^t \left( \int |x-z|=2|x-y| r^{-\frac{q}{4}} e^{-\frac{|x-z|^2}{2r}} ds \right)^{\frac{q}{q'}} dr \right] \frac{1}{\delta^q}
\leq C \|h^\delta\|_{L^q(0, t; C^\alpha_b(\mathbb{R}^d))} |x - y| \int |y - z|^{-d - \frac{2}{q} + 2} dz
\leq C \|h^\delta\|_{L^q(0, t; C^\alpha_b(\mathbb{R}^d))} |x - y|^{\theta}. \tag{2.9}
\]

To estimate \( I_4^\delta \), using the Hölder inequality first, the Minkowski integral inequality next, it yields that
\[
|I_4^\delta(t)| \leq C \|h^\delta\|_{L^q(0, t; C^\alpha_b(\mathbb{R}^d))} \int |x-z|^{\alpha}
\times \left( \int_0^t |\partial r, K(r, x - z) - \partial y, K(r, y - z)|^{\theta'} dr \right)^{\frac{1}{\theta'}} dz.
\]
For every \( \eta \in [x, y] \), a segment of \( x \) and \( y \), due to \( |x-z| > 2|x-y| \), then
\[
\frac{1}{2} |x-z| \leq |\eta - z| \leq 2|x-z|.
\]
By virtue of mean value inequality and the property of second order partial derivatives of the heat kernel \( K(t, x) \), it is easy to see that
\[
|I_4^\delta(t)| \leq C \|h^\delta\|_{L^q(0, t; C^\alpha_b(\mathbb{R}^d))} |x - y| \int |x-z|^{\alpha} \times \left( \int_0^\infty r^{-\frac{(d+2)\theta'}{2}} e^{-\frac{q|x-z|^2}{2r}} dr \right)^{\frac{1}{\theta'}} dz
\leq C \|h^\delta\|_{L^q(0, t; C^\alpha_b(\mathbb{R}^d))} |x - y| \int |x-z|^{\alpha-d-2+\frac{2}{q}} dz
\leq C \|h^\delta\|_{L^q(0, t; C^\alpha_b(\mathbb{R}^d))} |x - y|^{\theta}. \tag{2.10}
\]

Proceeding as the calculations from (2.7) to (2.10) lead to
\[
\max\{|I_3^\delta(t)|, |I_6^\delta(t)|, |I_7^\delta(t)|, |I_8^\delta(t)|\}
\leq C \|h\|_{L^q(0, \delta; C^\alpha_b(\mathbb{R}^d))} |x - y|^{\theta}, \quad t, t + \delta \in [0, T]. \tag{2.11}
\]
Since \( 1 \leq i \leq d \), combining (2.7) to (2.11), one asserts that
\[
\left[ \nabla u(t + \delta) - \nabla u(t) \right]_\theta
\leq C \|[h^\delta]\|_{L^q(0, t; C^\alpha_b(\mathbb{R}^d))} + \|h\|_{L^q(0, \delta; C^\alpha_b(\mathbb{R}^d))}, \quad t, t + \delta \in [0, T]. \tag{2.12}
\]
Analogue calculations from (2.7) to (2.11) also hints
\[ \|u(t + \delta) - u(t)\|_{C^1_b(\mathbb{R}^d)} \leq C[\|h^\delta\|_{L^q(0, \delta, C^0_b(\mathbb{R}^d))} + \|h\|_{L^q(0, t, C^0_b(\mathbb{R}^d))}], \quad t, t + \delta \in [0, T]. \]  
(2.13)

From (2.12) and (2.13), it follows that
\[ \|u(t + \delta) - u(t)\|_{C^1_b(\mathbb{R}^d)} \leq C[\|h^\delta\|_{L^q(0, \delta, C^0_b(\mathbb{R}^d))} + \|h\|_{L^q(0, t, C^0_b(\mathbb{R}^d))}], \quad t, t + \delta \in [0, T]. \]  
(2.14)

By letting \( \delta \) tend to zero in (2.14), we prove that as a \( C^1_b(\mathbb{R}^d) \) valued function, \( u \) is right continuous in \( t \). If one replaces \( \delta \) by \( -\delta \), by repeating above calculations, we conclude the left continuity of \( u \) in \( t \).

(ii) We proceed to show \( q = 2 \) first. Set a mapping \( \mathcal{T} \) on \( C([0, T]; C^1_b(\mathbb{R}^d)) \)
\[ \mathcal{T}v(t, x) = \int_0^t K(r, \cdot) * (g(t - r, \cdot) \cdot \nabla v(t - r, \cdot))(x)ds \]
\[ + \int_0^t (K(s, \cdot) * h(t - r, \cdot))(x)dr. \]  
(2.15)

Then \( \mathcal{T}v \in L^q(0, T; C^2_b(\mathbb{R}^d)) \cap W^{1, q}(0, T; C^0_b(\mathbb{R}^d)). \) To finish the proof, it suffices to show that the mapping is contractive on \( C([0, T]; C^1_b(\mathbb{R}^d)) \), so there is a unique \( u \in C([0, T]; C^1_b(\mathbb{R}^d)) \) satisfying \( u = \mathcal{T}u \). This fact combining an argument as \( q = 0 \) implies the existence of generalized solutions of the Cauchy problem (2.1).

Let \( t > 0 \) be given, \( v_1, v_2 \in C([0, T]; C^1_b(\mathbb{R}^d)) \), by utilising (2.4), then
\[ \|\mathcal{T}v_1 - \mathcal{T}v_2\|_{C([0, t]; C^1_b(\mathbb{R}^d))} \leq C\|\nabla v_1 - \nabla v_2\|_{L^2(0, t; C^0_b(\mathbb{R}^d))} \]
\[ \leq C\|g\|_{L^2(0, t; C^0_b(\mathbb{R}^d))}\|v_1 - v_2\|_{C([0, t]; C^1_b(\mathbb{R}^d))}, \]
which suggests that if \( t > 0 \) is sufficiently small, then \( \mathcal{T} \) is contractive, so there is a unique \( u \in C([0, t]; C^1_b(\mathbb{R}^d)) \) such that \( \mathcal{T}u = u \) and the inequality (2.5) holds clearly on \([0, t]\). We then repeat the preceding arguments to extend the solution to the time interval \([t, 2t]\). Continuing this procedure with finitely many steps, we construct a solution on \([0, T]\) for every given \( T > 0 \) and get the inequality (2.5) on \([0, T]\).

Second, we show \( h \in L^q(0, T; C^0_b(\mathbb{R}^d)) \) \((q < 2)\). Let \( u \) be given by (2.7). By suitable modification of the calculations from (2.7) to (2.10), we assert that the unique solution \( u \) belongs to \( L^{q/(2-q)}(0, T; C^0_b(\mathbb{R}^d)) \). If \( g \in L^2(0, T; C^0_b(\mathbb{R}^d; \mathbb{R}^d)) \), then for almost every \( t \in [0, T] \),
\[ \|g(t, \cdot) \cdot \nabla u(t, \cdot)\|_{C^0_b(\mathbb{R}^d)} \leq \|g(t)\|_{C^0_b(\mathbb{R}^d)} \|\nabla u(t)\|_{C^0_b(\mathbb{R}^d)}. \]

Observing that \( q \in [1, 2] \), \( \|g(\cdot)\|_{C^0_b(\mathbb{R}^d)} \in L^{q/(2-q)}(0, T) \) and \( \|\nabla u(\cdot)\|_{C^0_b(\mathbb{R}^d)} \in L^{q/(2-q)}(0, T) \), so
\[ \|g(\cdot)\|_{C^0_b(\mathbb{R}^d)} \|\nabla u(\cdot)\|_{C^0_b(\mathbb{R}^d)} \in L^q(0, T), \]
and thus \( g \cdot \nabla u \in L^q(0, T; \mathcal{C}^\alpha_b(\mathbb{R}^d)) \).

If we set \( T \) on \( L^{2q/(2-q)}(0, T; \mathcal{C}^{1,\alpha}_b(\mathbb{R}^d)) \) by (2.15), the following estimate holds. Repeating the discussion as \( q = 2 \), we accomplish the proof.

\[ \| T v_1 - T v_2 \|_{L^{2q/(2-q)}(0, T; \mathcal{C}^{1,\alpha}_b(\mathbb{R}^d))} \leq C \| g \|_{L^2(0, T; \mathcal{C}^\alpha_b(\mathbb{R}^d))} \| v_1 - v_2 \|_{L^{2q/(2-q)}(0, T; \mathcal{C}^{1,\alpha}_b(\mathbb{R}^d))} \]

From Lemma 2.2 we have

**Corollary 2.3.** Suppose that \( q, \alpha, h \) and \( \theta \) are given in Lemma 2.2. Let \( g \in L^2(0, T; \mathcal{C}^\alpha_b(\mathbb{R}^d; \mathbb{R}^d)) \). For \( \lambda \geq 0 \), consider the Cauchy problem

\[
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) &= \frac{1}{2} \Delta u(t, x) + g(t, x) \cdot \nabla u(t, x) \\
&\quad + h(t, x) - \lambda u(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
u(0, x) &= 0, \quad x \in \mathbb{R}^d.
\end{aligned}
\]

Then there is a unique generalized solution to (2.16). Moreover, if \( \lambda > 0 \), there is a real number \( \varepsilon > 0 \) such that

\[
\| \nabla u \|_{C((0, T), \mathcal{C}^\alpha_b(\mathbb{R}^d))} \leq C \lambda^{-\varepsilon} \| h \|_{L^q(0, T; \mathcal{C}^\alpha_b(\mathbb{R}^d))},
\]

where the constant \( C \) in (2.17) only depends on \( \alpha, q, d \) and \( \| g \|_{L^2(0, T; \mathcal{C}^\alpha_b(\mathbb{R}^d))} \).

**Proof.** It is sufficient to show (2.17). Since the proof is standard, and to make the proof clearer and without loss of generality, we pay our attention to \( g = 0 \). When \( g = 0 \), the unique solution is represented by

\[ u(t, x) = \int_0^t e^{-\lambda(t-r)} K(r, \cdot) \ast h(t - r, \cdot) dr. \]

For every \( x \in \mathbb{R}^d \) and \( 1 \leq i \leq d \),

\[
|\partial_{x_i} u(t, x)| \\
= \int_0^t dr \int_{\mathbb{R}^d} |\partial_{x_i} K(r, x - z)| e^{-\lambda(t-r)} [h(t - r, z) - h(t - r, x)] dz \\
\leq \int_0^t e^{-\lambda(t-r)} [h(t - r)]_\alpha dr \int_{\mathbb{R}^d} |x - z|^\alpha e^{-|x - z|^2/2r} r^{-d + \frac{1}{2}} dz \\
\leq C \int_0^t e^{-\lambda(t-r)} r^{-\frac{\alpha}{2}} [h(t - r)]_\alpha dr \\
\leq C \| h \|_{L^q(0, T; \mathcal{C}^\alpha_b(\mathbb{R}^d))} \left[ \int_0^t r^{-\frac{(1-\alpha) p_1}{2}} dr \right]^{\frac{1}{p_1}} \left[ \int_0^t e^{-\lambda \varepsilon r} dr \right]^{\frac{1}{\varepsilon}} \\
\leq C \| h \|_{L^q(0, T; \mathcal{C}^\alpha_b(\mathbb{R}^d))} \lambda^{-\varepsilon},
\]

where \( p_1 = 1/(1 - \alpha) + q/(2q - 2) \), \( \varepsilon = 1 - 1/q - 1/p_1. \) □

Lemma 2.2 and Corollary 2.3 and standard methods of the theory of parabolic differential equations (see, for instance, Krylov [13]) allow us to conclude that if \( a_{i,j}(t, x), i, j = 1, \ldots, d \) are real-valued functions such that
there exists a unique generalized solution of
\[ \xi \in C([0,T]; C^2_b(\mathbb{R}^d)) \]
holds, for every \( t, x \in (0, T) \times \mathbb{R}^d \) there is a constant \( \Lambda > 0 \) such that
\[ \Lambda|\xi|^2 \leq a_{i,j}(t, x)\xi_i\xi_j \leq \frac{1}{\Lambda}|\xi|^2, \quad \forall \xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d, \quad (2.18) \]
there exists a unique generalized solution of
\[ \begin{aligned}
\partial_t u(t, x) &= \frac{1}{2} a_{i,j}(t, x) \partial_{x_i} \partial_{x_j} u(t, x) + g(t, x) \cdot \nabla u(t, x) \\
&\quad + h(t, x) - \lambda u(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d,
\end{aligned} \quad (2.19) \]
and \( u \in C([0, T]; C^1_b(\mathbb{R}^d)) \) such that \( \text{[2.17]} \) holds. In summary, we have

**Theorem 2.4.** Let \( q, \alpha, h, g \) and \( \theta \) be described in Corollary \( \text{[2.3]} \) Let \( (a_{i,j}) \) be a symmetric \( d \times d \) matrix valued function whose components \( a_{i,j} \) are in \( L^\infty(0, T; C^0_b(\mathbb{R}^d)) \), and let \( \text{[2.18]} \) hold. Then there is a unique generalized solution to \( \text{[2.19]} \). Moreover, for all \( \lambda > 0 \), \( \text{[2.17]} \) holds.

**Remark 2.5.** For the Cauchy problem \( \text{[2.19]} \), when \( g \in L^\infty(0, T; C^0_b(\mathbb{R}^d; \mathbb{R}^d)) \), the existence and unique of generalized solution has been proved by Krylov \( \text{[14]} \), and when \( g \in B([0, T]; C^0_b(\mathbb{R}^d; \mathbb{R}^d)) \),
\[ \sup_{t \in [0, T]} \|g(t)\|_{C^0_b(\mathbb{R}^d; \mathbb{R}^d)} < \infty, \]
the existence and unique of \( B([0, T]; C^2_b(\mathbb{R}^d)) \) solution is established by Lorenzi \( \text{[18]} \). Noticing that, here we only assume that \( g \in L^2(0, T; C^0_b(\mathbb{R}^d; \mathbb{R}^d)) \), so we extend Krylov and Lorenzi’s results \( \text{[14 18]} \). This result plays a central role in proving the uniquenss of weak and strong solutions since it yields the following Itô formula.

**Theorem 2.6.** Suppose that \( X_t(x) \) satisfies \( \text{[17]} \) with \( b \in L^2(0, T; C^0_b(\mathbb{R}^d; \mathbb{R}^d)) \) and \( (\sigma_{i,j}) \in L^\infty(0, T; C^0_b(\mathbb{R}^d; \mathbb{R}^{d \times d})) \). Let \( q, \alpha \) and \( a = (a_{i,j}) = (\sigma_{i,k}\sigma_{j,k}) \) be described in Theorem \( \text{[2.2]} \) and let \( u \in L^2(0, T; C^2_b(\mathbb{R}^d)) \cap W^{1,2}(0, T; C^0_b(\mathbb{R}^d)). \) Then the following Itô formula
\[ u(t, X_t(x)) = u(0, x) + \int_0^t \partial_s u(s, X_s(x)) + \nabla u(s, X_s(x)) \cdot b(s, X_s(x)) \, ds \]
\[ + \frac{1}{2} \int_0^t a_{i,j}(s, X_s(x)) \partial_{x_i} \partial_{x_j} u(s, X_s(x)) \, ds \]
\[ + \int_0^t \partial_X u(s, X_s(x)) \sigma_{i,k}(s, X_s(x)) dW_{k,s} \quad (2.20) \]
holds, for every \( t \in [0, T] \).

**Proof.** We follow the proof of \( \text{[7] Lemma 3} \). For \( 0 < \varepsilon < 1 \), a given function \( h \in L^2(0, T; C^0_b(\mathbb{R}^d)) \), and every \( t \in [0, T] \), we set
\[ h_\varepsilon(t, x) = \int_0^1 h(t + r\varepsilon, x) \, dr. \]
Then, random variable \( h_\varepsilon(t, X_t(x)) \) converges to \( h(t, X_t(x)) \) \( \mathbb{P} \)-almost surely as \( \varepsilon \downarrow 0 \) for almost every \( t \in [0, T] \). Let \( u \) be stated in Theorem 2.6 if we replace \( h \) with \( u \), then \( u_\varepsilon(t, x) \) is continuous and differentiable in \( t \) and random variable \( u_\varepsilon(t, X_t(x)) \) converges to \( u(t, X_t(x)) \) almost surely as \( \varepsilon \downarrow 0 \). We apply the classical Itô formula to \( u_\varepsilon(t, X_t) \) and get

\[
\begin{align*}
|u_\varepsilon(t, X_t(x))| &= |u_\varepsilon(0, x) + \int_0^t \left[ \partial_s u_\varepsilon(s, X_s(x)) + \nabla u_\varepsilon(s, X_s(x)) \cdot b(s, X_s(x)) \right] ds \\
&\quad + \frac{1}{2} \int_0^t a_{i,j}(s, X_s(x)) \partial_{x_i,x_j}^2 u_\varepsilon(s, X_s(x)) ds \\
&\quad + \int_0^t \partial_{X_i} u_\varepsilon(s, X_s(x)) \sigma_{i,k}(s, X_s(x)) dW_{k,s}. 
\end{align*}
\]

For a general function \( h \) in \( L^2(0, T; \mathcal{C}_b^0(\mathbb{R}^d)) \), we estimate \( h_\varepsilon(t, X_t(x)) \) by

\[
|h_\varepsilon(t, X_t(x))| \leq \int_0^1 \|h(t + r\varepsilon)\|_{\mathcal{C}_b(\mathbb{R}^d)} dr \\
\leq \sup_{0<\varepsilon<1} \frac{1}{t+\varepsilon} \int_0^{t+\varepsilon} \|h(r)\|_{\mathcal{C}_b(\mathbb{R}^d)} dr =: g(t).
\]

With the aid of the property for Hardy-Littlewood maximum function, then \( g \in L^2(0, T) \). By applying the dominated convergence theorem, random variable \( \int_0^t \zeta(s) h_\varepsilon(s, X_s(x)) ds \) converges to \( \int_0^t \zeta(s) h(s, X_s(x)) ds \) \( \mathbb{P} \)-almost surely as \( \varepsilon \downarrow 0 \) for every \( \zeta \in L^2(0, T) \), which suggests that the second term and the third term in the right hand side of (2.21) converge to the second term and the third term in the right hand side of (2.20), respectively.

We calculate the difference for the last terms in (2.21) and (2.20) by

\[
\mathbb{E} \left| \int_0^t \left[ \partial_{X_i} u_\varepsilon(s, X_s(x)) - \partial_{X_i} u(s, X_s(x)) \right] \sigma_{i,k}(s, X_s(x)) dW_{k,s} \right|
\]

\[
= \sum_{k=1}^d \mathbb{E} \int_0^t \left| \left[ \partial_{X_i} u_\varepsilon(s, X_s(x)) - \partial_{X_i} u(s, X_s(x)) \right] \sigma_{i,k}(s, X_s(x)) \right|^2 ds
\]

\[
\leq C \mathbb{E} \int_0^t \left| \nabla u_\varepsilon(s, X_s(x)) - \nabla u(s, X_s(x)) \right|^2 ds. \tag{2.22}
\]

Clearly, \( \nabla u_\varepsilon(s, X_s(x)) - \nabla u(s, X_s(x)) \) converges to 0 \( \mathbb{P} \)-almost surely as \( \varepsilon \downarrow 0 \) for almost every \( s \in [0, T] \), and

\[
\left| \nabla u_\varepsilon(s, X_s(x)) - \nabla u(s, X_s(x)) \right|
\]

\[
\leq \sup_{0<\varepsilon<1} \frac{1}{s+\varepsilon} \int_s^{s+\varepsilon} \left| \nabla u(r) \right|_{\mathcal{C}_b(\mathbb{R}^d)} dr + \left| \nabla u(s) \right|_{\mathcal{C}_b(\mathbb{R}^d)} \in L^2(0, T).
\]

Thus, the last term in the right hand side of (2.22) vanishes if \( \varepsilon \) tends to 0, and it also suggests that random variable \( \int_0^t \partial_{X_i} u_\varepsilon(s, X_s(x)) \sigma_{i,k}(s, X_s(x)) dW_{k,s} \) converges to \( \int_0^t \partial_{X_i} u(s, X_s(x)) \sigma_{i,k}(s, X_s(x)) dW_{k,s} \) \( \mathbb{P} \)-almost surely up to choosing a unlabelled subsequence. \( \square \)
3. Stochastic differential equations: weak solutions

Firstly, we present an approximating result.

**Lemma 3.1.** (i) Let $T > 0$ be a real number, and let $h \in L^q(0,T;C_{bu}(\mathbb{R}^d))$ with $q \in [1,2]$. We set $h_n(t,x) = (h(t,\cdot) * \rho_n)(x), n \in \mathbb{N}$, where $*$ stands for the usual convolution and $\rho_n(x) = n^d \rho(nx)$ with

$$0 \leq \rho \in C_0^\infty(\mathbb{R}^d), \quad \text{support}(\rho) \subset B_0(1), \quad \int_{\mathbb{R}^d} \rho(x)dx = 1. \quad (3.1)$$

Then

$$\lim_{n \to \infty} \|h_n - h\|_{L^1(0,T;C_{bu}(\mathbb{R}^d))} = \lim_{n \to \infty} \int_0^T \sup_{x \in \mathbb{R}^d} |h_n(t,x) - h(t,x)|^q dt = 0. \quad (3.2)$$

(ii) Let $W_t, W^n_t, n = 1,2,...$ be $d$-dimensional standard Wiener processes on a same stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ for which $W^n$ converges to $W$, $\mathbb{P}$-almost surely. Assume $g \in L^2(0,T)$, then

$$\lim_{n \to \infty} \mathbb{E} \left| \int_0^T g(t)d[W^n_t - W_t] \right|^2 = 0. \quad (3.3)$$

**Proof.** (i) For every $n \in \mathbb{N}$ and $t \in [0,T]$,

$$h_n(t,x) - h(t,x) = \int_{\mathbb{R}^d} [h(t,x - \frac{y}{n}) - h(t,x)] \rho(y)dy,$$

which suggests that

$$\sup_{x \in \mathbb{R}^d} |h_n(t,x) - h(t,x)|^q \leq C \int_{|y| \leq 1} \sup_{x \in \mathbb{R}^d} |h(t,x - \frac{y}{n}) - h(t,x)|^q \rho(y)dy.$$ 

Therefore, (3.2) is true.

(ii) Since $g \in L^2(0,T)$, we approximate it by a sequence of smooth functions $g_\varepsilon \in W^{1,2}(0,T)$ such that $g_\varepsilon(T) = 0$ and $g_\varepsilon \to g$ in $L^2(0,T)$ as $\varepsilon \downarrow 0$. Then

$$\lim_{n \to \infty} \mathbb{E} \left| \int_0^T g(t)d[W^n_t - W_t] \right|^2$$

$$\leq 2 \lim_{n \to \infty} \mathbb{E} \left| \int_0^T [g(t) - g_\varepsilon(t)]d[W^n_t - W_t] \right|^2$$

$$+ 2 \lim_{n \to \infty} \mathbb{E} \left| \int_0^T g_\varepsilon(t)d[W^n_t - W_t] \right|^2. \quad (3.4)$$
In view of Itô’s isometry and the integration by parts for Wiener’s integral, from (3.4), it yields that
\[
\lim_{n \to \infty} \mathbb{E} \left| \int_0^T g(t) d[W_t^n - W_t] \right|^2 \\
\leq C \lim_{n \to \infty} \mathbb{E} \int_0^T |g(t) - g_\varepsilon(t)|^2 dt + 2 \lim_{n \to \infty} \mathbb{E} \left| \int_0^T g'_\varepsilon(t) [W_t^n - W_t] dt \right|^2 \\
= C \mathbb{E} \int_0^T |g(t) - g_\varepsilon(t)|^2 dt.
\]
So (3.3) holds by letting \( \varepsilon \downarrow 0 \).

We are now in a position to state and prove our main result on the existence of weak solutions to SDE (1.1).

**Theorem 3.2.** Let \( T > 0 \) be a given real number. Assume that the drift \( b \) belongs to \( L^1(0, T; C_{bu} \mathbb{R}^d) \), that the diffusion \( \sigma = (\sigma_{i,j}) \) is a \( d \times d \) matrix valued function for which \( \sigma_{i,j} \in L^2(0, T; C_{bu} \mathbb{R}^d) \) \( (C_{bu} \mathbb{R}^d) \) is the space consisting of all bounded uniformly continuous functions on \( \mathbb{R}^d \). There is a filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}) \), two processes \( \tilde{X}_t \) and \( \tilde{W}_t \) defined for \( t \in [0, T] \) on it such that \( \tilde{W}_t \) is a \( d \)-dimensional Wiener process and \( \tilde{X}_t \) is an \( \{\mathcal{F}_t\} \)-adapted, continuous, \( d \)-dimensional process and for almost surely,

\[
\tilde{X}_t = x + \int_0^t b(r, \tilde{X}_r) dr + \int_0^t \sigma(r, \tilde{X}_r) d\tilde{W}_r, \quad \forall \ t \in [0, T].
\] (3.5)

**Proof.** We follow the proof of [12] Theorem 1, p.87. Firstly, we smooth out \( b \) and \( \sigma \) using the convolution: \( b^n(t, x) = (b(t, \cdot) * \rho_n)(x) \), \( \sigma^n(t, x) = (\sigma(t, \cdot) * \rho_n)(x) \) with \( \rho_n \) given by (3.1).

According to (3.2), as \( n \to \infty \),

\[
\|b^n - b\|_{L^1(0, T; C_{bu} \mathbb{R}^d)} \to 0, \quad \|\sigma^n - \sigma\|_{L^2(0, T; C_{bu} \mathbb{R}^d)} \to 0. \] (3.6)

Moreover, for every \( n \geq 1 \), and almost every \( t \in [0, T] \),

\[
\|b^n(t)\|_{C_{bu} \mathbb{R}^d} \leq \|b(t)\|_{C_{bu} \mathbb{R}^d}, \quad \|\sigma^n(t)\|_{C_{bu} \mathbb{R}^d} \leq \|\sigma(t)\|_{C_{bu} \mathbb{R}^d}. \] (3.7)

Therefore, there are two sequences of square-integrable functions \( h^n \) and \( l^n \) on \([0, T]\) such that

\[ |b^n(t, x) - b^n(t, y)| \leq h^n(t)|x - y|, \quad \forall \ x, y \in \mathbb{R}^d \]
and

\[ |\sigma^n(t, x) - \sigma^n(t, y)| \leq l^n(t)|x - y|, \quad \forall \ x, y \in \mathbb{R}^d. \]

By Cauchy-Lipschitz’s theorem, there is a unique \( \{\mathcal{F}_t\} \)-adapted, continuous, \( d \)-dimensional process \( X^n_t \) defined for \( t \in [0, T] \) on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}) \) such that

\[
X^n_t = x + \int_0^t b^n(r, X^n_r) dr + \int_0^t \sigma^n(r, X^n_r) dW_t. \] (3.8)
With the help of (3.7), for every $0 \leq t_1 < t_2 \leq T$,
\[
\sup_n \mathbb{E} \int_{t_1}^{t_2} |b^n(t, X^n_t)| dt \leq \int_{t_1}^{t_2} \sup_{x \in \mathbb{R}^d} |b(t, x)| dt
\]  \hspace{1cm} (3.9)
and
\[
\sup_n \mathbb{E} \left[ \int_{t_1}^{t_2} \sigma^n(t, X^n_t) dW_t \right]^2 \leq \int_{t_1}^{t_2} \sup_{x \in \mathbb{R}^d} |\sigma(t, x)|^2 dt.
\]  \hspace{1cm} (3.10)

Combining (3.8), (3.9) and (3.10), for every $\epsilon > 0$, one concludes that
\[
\lim_{c \to \infty} \sup_n \mathbb{P}\{|X^n_t| > c\} = 0
\]  \hspace{1cm} (3.11)
and
\[
\lim_{h \downarrow 0} \sup_{n \geq 1} \sup_{0 \leq t \leq T} \mathbb{P}\{|X^n_{t_1} - X^n_{t_2}| > \epsilon\} = 0.
\]  \hspace{1cm} (3.12)

From (3.11) and (3.12), along with Prohorov’s theorem, there is a subsequence still denoted by itself such that $(X^n, W^n)$ weakly converge. Next, Skorohod’s representation theorem implies that there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}, \tilde{\mathbb{P}})$ and random processes $(\tilde{X}^n_t, \tilde{W}^n_t), (\tilde{X}_t, \tilde{W}_t)$ on this probability space such that
(i) the finite dimensional distributions of $(\tilde{X}^n_t, \tilde{W}^n_t)$ coincide with the corresponding finite dimensional distributions of $(X^n_t, W_t)$.
(ii) $(\tilde{X}^n_t, \tilde{W}^n_t)$ converges to $(\tilde{X}_t, \tilde{W}_t)$, $\tilde{\mathbb{P}}$-almost surely.

In particular, $\tilde{W}$ is still a Wiener process and
\[
\tilde{X}^n_t = x + \int_0^t b^n(r, \tilde{X}^n_r) dr + \int_0^t \sigma^n(r, \tilde{X}^n_r) d\tilde{W}^n_r.
\]  \hspace{1cm} (3.13)

For every $k \in \mathbb{N}$ be fixed, then
\[
\tilde{\mathbb{E}} \left( \int_0^T \left| b^n(r, \tilde{X}^n_r) - b(r, \tilde{X}_r) \right| dr \right)
\leq \tilde{\mathbb{E}} \left( \int_0^T \left| b^n(r, \tilde{X}^n_r) - b^k(r, \tilde{X}^n_r) \right| dr \right) + \tilde{\mathbb{E}} \left( \int_0^T \left| b^k(r, \tilde{X}^n_r) - b^k(r, \tilde{X}_r) \right| dr \right)
\leq C \left[ \|b^n - b^k\|_{L^1(0,T;C_b(\mathbb{R}^d))} + \|b^k - b\|_{L^1(0,T;C_b(\mathbb{R}^d))} \right]
\]  \hspace{1cm} (3.14)

We approach $n \to \infty$ first, $k \to \infty$ next, from (3.6) and (3.14), it follows that
\[
\lim_{n \to \infty} \int_0^t b^n(r, \tilde{X}^n_r) dr = \int_0^t b(r, \tilde{X}_r) dr, \ \tilde{\mathbb{P}} - a.s.
\]  \hspace{1cm} (3.15)
Similar manipulation also hints

\[ \hat{E}\left| \int_0^T \sigma^n(r, \tilde{X}_r^n)d\tilde{W}_r^n - \int_0^T \sigma(r, \tilde{X}_r)d\tilde{W}_r \right|^2 \]

\[ \leq C\hat{E}\left| \int_0^T \sigma^n(r, \tilde{X}_r^n)d\tilde{W}_r^n - \int_0^T \sigma(r, \tilde{X}_r)d\tilde{W}_r \right|^2 \]

\[ + C\hat{E}\left| \int_0^T \sigma(r, \tilde{X}_r)d\tilde{W}_r^n - \int_0^T \sigma(r, \tilde{X}_r)d\tilde{W}_r \right|^2 \]

\[ \leq C\hat{E}\int_0^T \left| \sigma^n(r, \tilde{X}_r^n) - \sigma(r, \tilde{X}_r) \right|^2dr \]

\[ + C\hat{E}\int_0^T \left| \sigma(r, \tilde{X}_r)d\tilde{W}_r^n - \int_0^T \sigma(r, \tilde{X}_r)d\tilde{W}_r \right|^2 \]

\[ =: J_1^n + J_2^n. \]

We adopt the same procedure as in (3.14) to assert that \( J_1^n \to 0 \) as \( n \to \infty \).

On the other hand, thanks to the definition of stochastic integral, then

\[ \hat{E}\left| \int_0^t \sigma(r, \tilde{X}_r)d[\tilde{W}_r^n - \tilde{W}_r] \right|^2 \]

\[ = \lim_{k \to \infty} \hat{E} \left| \sum_{i=1}^k \sigma(r_i \wedge t, \tilde{X}_{r_i})(\tilde{W}_{r_{i+1}} - \tilde{W}_{r_i})^2 \right| \]

\[ = \lim_{k \to \infty} \hat{E} \left| \sum_{i=1}^k |\sigma(r_i \wedge t, \tilde{X}_{r_i})|^2(\tilde{W}_{r_{i+1}} - \tilde{W}_{r_i})^2 \right| \]

\[ \leq \lim_{k \to \infty} \hat{E} \left| \sum_{i=1}^k \sup_{x \in \mathbb{R}^d} |\sigma(r_i \wedge t, x)|^2(\tilde{W}_{r_{i+1}} - \tilde{W}_{r_i})^2 \right| \]

\[ = \hat{E} \left| \int_0^t \sup_{x \in \mathbb{R}^d} |\sigma(r, x)|d[\tilde{W}_r^n - \tilde{W}_r] \right|^2. \quad (3.16) \]

According to (3.3), from (3.16), so \( J_2^n \to 0 \) as \( n \to \infty \). Therefore,

\[ \lim_{n \to \infty} \int_0^t \sigma^n(r, \tilde{X}_r^n)d\tilde{W}_r^n = \int_0^t \sigma(r, \tilde{X}_r)d\tilde{W}_r, \quad \mathbb{P} - a.s. \quad (3.17) \]

Combining (3.13), (3.15) and (3.17), one reaches at

\[ \tilde{X}_t = x + \int_0^t b(s, \tilde{X}_s)ds + \int_0^t \sigma(r, \tilde{X}_r)d\tilde{W}_r. \]

From this one ends the proof. \( \square \)
Consider SDE (1.1). If \((X_t, W_t)\) is a weak solution on a probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), for every \(f \in C^2_b(\mathbb{R}^d)\), by Itô’s formula
\[
f(X_t) - f(x) - \int_0^t b(s, X_s) \cdot \nabla f(X_s)ds - \frac{1}{2} \int_0^t a_{i,j}(s, X_s) \partial_{x_i,x_j}^2 f(X_s)ds = \int_0^t \sigma_{i,k}(s, X_s) \partial_{x_i} f(X_s)dW_{k,s}.
\]
For every \(T > 0\), then
\[
f(X_t) - f(x) - \int_0^t b(s, X_s) \cdot \nabla f(X_s)ds - \frac{1}{2} \int_0^t a_{i,j}(s, X_s) \partial_{x_i,x_j}^2 f(X_s)ds \in \mathcal{M}^c_2([0, T]),
\]
if \(\int_0^T |b(s, X_s)|ds < \infty\) and \(\int_0^T |\sigma(s, X_s)|^2ds < \infty\), \(\mathbb{P}\)-a.s., where \(\mathcal{M}^c_2([0, T])\) is the set of all continuous \(F_t\)-adapted \(L^2(0, T)\) martingale processes. Conversely, if a \(d\)-dimensional continuous adapted process \(\{X_t\}_{t \geq 0}\) defined on a probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) satisfies (3.18) for every \(T > 0\), then on an extension \((\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \geq 0}, \bar{\mathbb{P}})\), we can find a \(d\)-dimensional \(\{\bar{W}_t\}_{t \geq 0}\)-Wiener process \(\{\bar{W}_t\}_{t \geq 0}\) such that \((X, \bar{W})\) is a weak solution of (1.1) (see [10] pp168-169). And if \(X\) meets (1.1), its probability law \(\mathbb{P}_x = \mathbb{P} \circ X^{-1}\) on \(d\)-dimensional Wiener space \((W^d([0, T]), \mathcal{B}(W^d([0, T])))\) satisfies
\[
f(w(t)) - f(x) - \int_0^t b(s, w(s)) \cdot \nabla f(w(s))ds - \frac{1}{2} \int_0^t a_{i,j}(s, w(s)) \partial_{x_i,x_j}^2 f(w(s))ds \in \mathcal{M}^c_2,
\]
for every \(f \in C^2_b(\mathbb{R}^d)\).

In summary, we have

**Lemma 3.3.** ([10] Proposition 2.1, p169) The existence of a weak solution of (1.1) is equivalent to the existence of a \(d\)-dimensional process \(X\) satisfying (3.18), and this is also equivalent to the existence of a probability \(\mathbb{P}\) on \((W^d([0, T]), \mathcal{B}(W^d([0, T])))\) satisfying (3.17).

For the convenience of the reader, we present other useful lemmas, which will serve us well later when we prove the uniqueness, the Feller property and the existence of density.

**Lemma 3.4.** ([10], Corollary, p206) If \((X, W)\) and \((X', W')\) are weak solutions (1.1). Then \(\mathbb{P}_x = \mathbb{P}'_x\) is equivalent to
\[
\int_{W^d([0, T])} f(w(t)) \mathbb{P}_x(dw) = \int_{W^d([0, T])} f(w(t)) \mathbb{P}'_x(dw), \tag{3.20}
\]
for every \(t \in [0, T]\) and every \(f \in C_b(\mathbb{R}^d)\).

**Lemma 3.5.** ([25]) Consider SDE (1.1). Suppose that \(b\) is bounded and Borel measurable, \(\sigma\) is bounded continuous and \((a_{i,j}) = (\sigma_{i,k} \sigma_{i,j})\) is uniformly continuous and uniformly elliptic. Then there is a unique weak solution of (1.1),
which is a strong Markov process. Let $P_t$ and $P(x,t,dy)$ be defined by \((1.3)\), then for every $T > 0$, we have the following claims:

(i) $P_t f(x)$ is continuous in $x$ for $t > 0$.

(ii) $P(x,t,dy)$ has a density $p(x,t,y)$ for almost all $t \in [0,T]$, which satisfies \((1.6)\) for every $s \in [1, \infty)$ provided $t_0 > 0$.

We now give our second result.

**Theorem 3.6.** Assume that $\alpha \in (0, 1)$ and $b \in L^2(0,T; C^\alpha_b(\mathbb{R}^d; \mathbb{R}^d))$, that $\sigma = (\sigma_{i,j})$ is a $d \times d$ matrix valued function, and $\sigma_{i,j} \in C(\mathbb{R}^d; C^\alpha_b(\mathbb{R}^d)) \cap L^\infty(0,T; C^\alpha_b(\mathbb{R}^d))$, and \((2.15)\) holds with $a = (a_{i,j}) = \sigma \sigma^\top = (\sigma_{i,k} \sigma_{j,k})$. Then for every $T > 0$, all weak solutions for SDE \((1.1)\) have the same probability law on $d$-dimensional classical Wiener space $(W^d([0,T]), B(W^d([0,T])))$. If one uses $\mathbb{P}_x$ to denote the unique probability law on $(W^d([0,T]), B(W^d([0,T])))$ corresponding to the initial value $x \in \mathbb{R}^d$. For every $f \in L^\infty(\mathbb{R}^d)$, we define $P_t f(x)$ by \((1.3)\). Then, $\{P_t\}$ has strong Feller property, i.e. $P_t$ maps a bounded function to a bounded and continuous function for every $t > 0$. Moreover, $P(x,t,dy)$ admits a density $p(x,t,y)$ for almost every $t \in [0,T]$. Besides, for every $t_0 > 0$ and for every $s \in [1, \infty)$, \((1.6)\) holds.

**Proof.** Recalling Lemmas 3.3 and 3.4 to show the uniqueness in probability laws, it is equivalent to show that \((3.20)\) holds true for every $t \in [0,T]$ and every $f \in C_b(\mathbb{R}^d)$, where $\mathbb{P}_x$ and $\mathbb{P}'_x$ are the probability laws of solutions $X_t$ and $X'_t$ on $(W^d([0,T]), B(W^d([0,T])))$ corresponding to the same initial value $x \in \mathbb{R}^d$.

We proceed to show the identity \((3.20)\) by using Itô-Tanack's trick (see [7]). Consider the following vector valued Cauchy problem on $(0,T) \times \mathbb{R}^d$

\[
\partial_t U(t,x) = \frac{1}{2} a_{i,j}(T-t,x) \partial^2_{x_i,x_j} U(t,x) + b_i(T-t,x) \partial_{x_i} U(t,x) - \lambda U(t,x) + b(T-t,x), \quad (3.21)
\]

with initial data $U(0,x) = 0$.

Because of Theorem 2.4, there is a unique $U \in L^2(0,T; C^2_b(\mathbb{R}^d)) \cap W^{1,2}(0,T; C^\alpha_b(\mathbb{R}^d))$ solving \((3.21)\). Moreover, $U \in C([0,T]; C^\alpha_b(\mathbb{R}^d; \mathbb{R}^d))$ and there is a real number $\varepsilon > 0$ such that

\[
\|U\|_{C([0,T]; C^\alpha_b(\mathbb{R}^d))} \leq C \lambda^{-\varepsilon}.
\]

So,

\[
\|U\|_{C([0,T]; C^\alpha_b(\mathbb{R}^d))} < \frac{1}{2}, \quad \text{if } \lambda > (2C)^{1/2}.
\]

Let $\lambda$ be big enough ($\lambda > (2C)^{1/\varepsilon}$) and fixed. We define

\[
\Phi(t,x) = x + U(T-t,x). \quad (3.22)
\]

Obviously, $\Phi$ forms a non-singular diffeomorphism of class $C^1$ uniformly in $t \in [0,T]$ and

\[
\frac{1}{2} < \|\nabla \Phi\|_{C([0,T]; C^\alpha_b(\mathbb{R}^d))} < \frac{3}{2}, \quad \frac{2}{3} < \|\nabla \Psi\|_{C([0,T]; C^\alpha_b(\mathbb{R}^d))} < 2, \quad (3.23)
\]
where $\Psi(t, \cdot) = \Phi^{-1}(t, \cdot)$. Moreover, the measurable function $\Phi(t, x) - x$ belongs to $L^2(0, T; C^2_b(\mathbb{R}^d; \mathbb{R}^d)) \cap W^{1,2}(0, T; C^1_b(\mathbb{R}^d; \mathbb{R}^d))$.

As a result of Itô’s formula (Theorem 2.6), we assert
\[
d\Phi(t, X_t) = -\partial_t u(T - t, X_t) dt + b_i(t, X_t)\partial_{x_i} U(T - t, X_t) dt \\
+ \frac{1}{2} a_{i,j}(t, x) \partial^2_{x_i, x_j} u(T - t, X_t) dt \\
+ \partial_{x_i} U(T - t, X_t) \sigma_{i,j} dW_{jt} + b(t, X_t) dt + \sigma(t, X_t) dW_t \\
= (\nabla U(T - t, X_t) + I) \sigma(t, X_t) dW_t + \lambda U(T - t, X_t) dt.
\]

Denote $Y_t = X_t + U(T - t, X_t)$, it yields that
\[
dY_t = \lambda U(T - t, \Psi(t, Y_t)) dt + (I + \nabla U(T - t, \Psi(t, Y_t)) \sigma(t, \Psi(t, Y_t))) dW_t \\
= : \tilde{b}(t, Y_t) dt + \tilde{\sigma}(t, Y_t) dW_t,
\]
with $Y_0 = y = \Phi(0, x)$. Therefore, if $(X_t, W_t)$ is a weak solution of \((1.1)\), then $(Y_t, W_t)$ is a weak solution of \((3.24)\) and vice versa.

Observing that $\tilde{b}$ is bounded Borel measurable, $\tilde{\sigma}$ is bounded uniformly continuous on $[0, T] \times \mathbb{R}^d$ and $a$ is uniformly elliptic, by Lemma 3.5 then:

(i) there is a unique weak solution of \((3.24)\);

(ii) if one uses $\tilde{P}(y, t, dz)$ to stand the transition probabilities and for every bounded function $f$, to define
\[
\tilde{P}_t f(y) = \int_{\mathbb{R}^d} f(y) \tilde{P}(y, t, dz),
\]
$\tilde{P}_t f(y)$ is continuous in $y$ for $t > 0$;

(iii) $\tilde{P}(y, t, dz)$ has a density $\tilde{p}(y, t, z)$ for almost all $t \in [0, T]$, which satisfies
\[
\int_{t_0}^T \int_{\mathbb{R}^d} |\tilde{p}(y, t, z)|^s dz dt < \infty,
\]
for every $s \in [1, \infty)$ provided $t_0 > 0$.

On the other hand, the relationships of $P_x$ and $P_y, P'_x$ and $P'_y$ are given by $P_y = P_x \circ \Phi^{-1}$, and $P'_y = P'_x \circ \Phi^{-1}$, respectively. Hence, for every $f \in C_b(\mathbb{R}^d)$, and every $t \in [0, T]$,
\[
\int_{W^d([0, T])} f(w(t)) P_x(dw) = \int_{W^d([0, T])} f(\Psi(t, \Phi(t, w(t)))) P_x(dw) \\
= \int_{W^d([0, T])} f(\Psi(t, w(t))) P_y(dw)
\]
and
\[
\int_{W^d([0, T])} f(w(t)) P'_x(dw) = \int_{W^d([0, T])} f(\Psi(t, \Phi(t, w(t)))) P'_x(dw) \\
= \int_{W^d([0, T])} f(\Psi(t, w(t))) P'_y(dw).
\]
Since $P_y = P'_y$, and for every $t \in [0, T]$, $f \circ \Psi(t, \cdot) \in C_b(\mathbb{R}^d)$, from (3.26) and (3.27) one ends up with (3.20), which means (1.1) has uniqueness in probability laws. Moreover, $P(x, t, dz)$ has a density $p(x, t, y)$, which is given by $p(x, t, y) = \tilde{p}(\Phi(0, x), t, \Phi(t, y)) |\nabla \Phi(t, y)|$. Hence, $\{P_t\}$ has strong Feller property and (1.6) is true by using (3.23). □

**Remark 3.7.** When $\sigma = I_{d \times d}$ and $b(T - \cdot, \cdot) \in C_0^q((0, T]; L^p(\mathbb{R}^d))$ with $2/q + d/p = 1$, Theorems 3.2 and 3.6 hold true as well (see [27]). Since the proof of Theorem 3.6 is similar to the proof [27, Theorems 4.1, 4.2], we skip some details.

4. Stochastic differential equation: strong solutions

Before stating the main result in this section, we need two useful lemmas. The first one is concerned with a Kolmogorov’s criterion and the second is discussing the non-confluent property of strong solutions for SDEs on $d = 1$ with non-Lipschitz coefficients.

**Lemma 4.1.** Let $\{X_t(x), x \in [0, 1]^d, t \in [0, 1]\}$ be a random field for which there exist three strictly positive constants $s, c, \varepsilon$ such that

$$
\mathbb{E}[\sup_{0 \leq t \leq 1} |X_t(x) - X_t(y)|^s] \leq c|x - y|^{d+\varepsilon}. 
$$

(4.1)

Then there is a modification $\tilde{X}$ of $X$ such that

$$
\mathbb{E}[\sup_{0 \leq t \leq 1} \left( \sup_{x \neq y} \frac{\tilde{X}_t(x) - \tilde{X}_t(y)}{|x - y|^\beta} \right)^s] < \infty
$$

(4.2)

for every $\beta \in [0, \varepsilon/s)$. In particular, the paths of $\tilde{X}$ are Hölder continuous in $x$ of order $\beta$.

**Proof.** Let $D_m$ be the set of points in $[0, 1]^d$ whose components are equal to $2^{-m}i$ for some integer $i \in [0, 2^m]$. The set $D = \cup_m D_m$ is the set of dyadic numbers. Let further $\Delta_m$ be the set of pairs $(x, y)$ in $D_m$ such that $|x - y| = 2^{-m}$. There are $2^{(m+1)d}$ such pairs in $\Delta_m$.

Let us finally set $K_i(t) = \sup_{(x, y) \in \Delta_i} |X_t(x) - X_t(y)|$. The hypothesis entails that for a constant $J$,

$$
\mathbb{E}[\sup_{0 \leq t \leq 1} K_i(t)^s] \leq \sum_{(x, y) \in \Delta_i} \mathbb{E}[\sup_{0 \leq t \leq 1} |X_t(x) - X_t(y)|^s] 
\leq c2^{(i+1)d}2^{-i(d+\varepsilon)} = J2^{-i\varepsilon}.
$$

For a point $x$ (respect to $y$) in $D$, there is an increasing sequences $\{x_m\}$ (respect to $\{y_m\}$) of points in $D$ such that $x_m$ (respect to $y_m$) is in $D_m$. 

for each $m$, $x_m \leq x$ ($y_m \leq y$) and $x_m = x$ ($y_m = y$) from some $m$ on. If $|x - y| \leq 2^{-m}$, then either $x_m = y_m$ or $(x_m, y_m) \in \Delta_m$ and in any case
\[
X_t(x) - X_t(y) = \sum_{i=m}^{\infty} (X_t(x_{i+1}) - X_t(x_i)) + X_t(x_m) - X_t(y_m)
\]
\[= \sum_{i=m}^{\infty} (X_t(y_{i+1}) - X_t(y_i)),
\]
where the series are actually finite sums. It follows that
\[
|X_t(x) - X_t(y)| \leq K_m + 2 \sum_{i=m+1}^{\infty} K_i(t) \leq 2 \sum_{i=m}^{\infty} K_i(t).
\]
As a result, setting
\[
M_\beta(t) = \sup \left\{ \frac{|X_t(x) - X_t(y)|}{|x - y|^\beta}, \ x, y \in D, \ x \neq y \right\},
\]
we have
\[
M_\beta(t) \leq \sup_{m \in \mathbb{N}} \left\{ 2^{m\beta} \sup_{|x - y| \leq 2^{-m}} |X_t(x) - X_t(y)|, \ x, y \in D, \ x \neq y \right\}
\]
\[\leq 2^{m\beta+1} \sum_{i=m}^{\infty} K_i(t) \leq 2 \sum_{i=0}^{\infty} 2^{i\beta} K_i(t).
\]
For $s \geq 1$ and $\beta < \varepsilon/s$, we get with $J' = 2J$,
\[
[\mathbb{E} \sup_{0 \leq t \leq 1} M_\beta(t)^s]^{\frac{1}{s}} \leq 2 \sum_{i=0}^{\infty} 2^{i\beta} [\mathbb{E} \sup_{0 \leq t \leq 1} K_i(t)^s]^{\frac{1}{s}} \leq J' \sum_{i=0}^{\infty} 2^{i(\beta - \frac{s}{2})} < \infty.
\]
For $s < 1$, the same reasoning applies to $[\mathbb{E} \sup_{0 \leq t \leq 1} M_\beta(t)^s]$ instead of $[\mathbb{E} \sup_{0 \leq t \leq 1} M_\alpha(t)^s]^{1/s}$.

It follows in particular that for almost every $\omega$, $X_t(\cdot)$ is uniformly continuous on $D$ and it is uniformly in $t$, so it make sense to set
\[
\tilde{X}_t(x, \omega) = \lim_{y \to x, y \in D} X_t(y, \omega).
\]

By Fatou’s lemma and the hypothesis, $\tilde{X}_t(x) = X_t(x)$ a.s. and $\tilde{X}$ is clearly the desired modification. \hfill \Box

**Lemma 4.2.** ([21, Theorem 3.2]) Suppose that the bounded measurable functions $b, \sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfy the following hypotheses:

(H1) there exists an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that
\[
(\sigma(t, x) - \sigma(t, y))^2 \leq (x - y)(\varphi(x) - \varphi(y))
\]
for $x \geq y, x, y \in \mathbb{R}, t \in \mathbb{R}^+$;

(H2) there exists a positive constant $\varepsilon > 0$ such that $|\sigma(t, x)| \geq \varepsilon$ for $x \in \mathbb{R}, t \in \mathbb{R}^+$;
(H3) there exists an increasing function \( \phi : \mathbb{R} \to \mathbb{R} \) such that
\[
|b(t, x) - b(t, y)| \leq |\phi(x) - \phi(y)|
\]
for \( x, y \in \mathbb{R}, t \in \mathbb{R}_+ \).

Then, the non-confluent proper of solutions of SDE (1.1) on \( d = 1 \):
\[
|x - y| > 0 \implies \mathbb{P}\{\omega, \ |X_t(x) - X_t(y)| > 0, \ \forall \ t \in [0, T]\} = 1
\]
holds.

We now give a uniqueness result.

**Theorem 4.3.** Let \( \alpha \in (0, 1) \) such that \( b \in L^2_{loc}(\mathbb{R}^d; C_b^\alpha(\mathbb{R}^d; \mathbb{R}^d)) \). Let \( \sigma = (\sigma_{i,j}) \) be a \( d \times d \) matrix valued function such that \( |\nabla \sigma_{i,j}| \in L^2_{loc}(\mathbb{R}^d; L^\infty(\mathbb{R}^d)) \) and \( \sigma_{i,j} \in L^\infty_{loc}(\mathbb{R}^d; C^\alpha_b(\mathbb{R}^d)) \). Suppose (2.18) holds with \( a = (a_{i,j}) = \sigma \sigma^\top = (\sigma_{i,j} \sigma_{j,i}). \) Then we have

(i) there is a unique strong solution \( X_t(x) \) to (1.1) for every \( x \in \mathbb{R}^d \).

The random field \( \{X_t(x), t > 0, x \in \mathbb{R}^d\} \) has a continuous modification \( \tilde{X} \), which is \( \beta \)-Hölder continuous in \( x \) for \( \beta \in (0, 1) \). Moreover, for every \( p \geq 1 \), and every \( T > 0 \),
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \left( \sup_{x \neq y} \frac{|X_t(x) - X_t(y)|}{|x - y|^{\beta}} \right)^p \right] < \infty; \tag{4.3}
\]

(ii) for almost all \( \omega \in \Omega \), every \( t > 0 \), \( x \to X_t(x) \) is a homeomorphism on \( \mathbb{R}^d \).

**Proof.** (i) By Theorem 3.2 and Yamada-Watanabe’s theorem (see [28]), it suffices to prove the pathwise uniqueness. Consider the vector valued Cauchy problem \( (3.21) \) on \( (0, T) \times \mathbb{R}^d \) with \( U(0, x) = 0 \). Repeating the calculations from (3.21) to (3.23), then (3.24) holds. By virtue of Lemma 4.1 the scaling transformation and the continuity of \( X \) in \( t \), we need to check that for every \( p > 1 \), \( x, y \in \mathbb{R}^d \),
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq 1} |Y_t(x) - Y_t(y)|^p \right] \leq C|x - y|^p. \tag{4.4}
\]

Since \( |\nabla \sigma| \in L^2(0, 1; L^\infty(\mathbb{R}^d)) \), we know \( \tilde{b} \in C([0, 1]; C^1_0(\mathbb{R}^d; \mathbb{R}^d)) \), \( \tilde{\sigma} \in L^2(0, 1; W^{1,\infty}(\mathbb{R}^d)) \). From (3.24), by employing the Itô formula, there is a measurable function \( \kappa \) in \( L^1(0, 1) \) such that
\[
\mathbb{E}|Y_t(x) - Y_t(y)|^p \leq |x - y|^p + C \int_0^t |Y_r(x) - Y_r(y)|^p \, dr \\
+ \int_0^t \kappa(r)|Y_r(x) - Y_r(y)|^p \, dr. \tag{4.5}
\]
Then the Grönwall inequality is applied, for every \( p > 1 \), we have
\[
\sup_{0 \leq t \leq 1} \mathbb{E}|Y_t(x) - Y_t(y)|^p \leq C|x - y|^p. \tag{4.6}
\]
By (3.24), the Doob and BDG inequalities, we gain
\[
\mathbb{E} \sup_{0 \leq t \leq 1} |Y_t(x) - Y_t(y)|^p
\leq |x - y|^p + C \mathbb{E} \int_0^1 (1 + \kappa(r))|Y_r(x) - Y_r(y)|^p dr
\]
\[
+ C \left[ \mathbb{E} \int_0^1 |Y_r(x) - Y_r(y)|^{2p-2} |\tilde{\sigma}(r, Y_r(x)) - \tilde{\sigma}(r, Y_r(y))|^2 dr \right]^{\frac{1}{2}}
\]
\[
\leq |x - y|^p + C \mathbb{E} \int_0^1 (1 + \kappa(r))|Y_r(x) - Y_r(y)|^p dr
\]
\[
+ C \left[ \mathbb{E} \int_0^1 \kappa(r)|Y_r(x) - Y_r(y)|^{2p} dr \right]^{\frac{1}{2}}.
\]
(4.7)

Observing that (4.6) holds for every $p > 1$, from (4.7), (4.4) holds. Moreover, since $\tilde{b}$ and $\tilde{\sigma}$ are bounded, for every $0 \leq t, r \leq T$,
\[
\sup_{y \in \mathbb{R}^d} \mathbb{E}|Y_t(y) - Y_r(y)|^p \leq C|t - r|^\frac{p}{2}.
\]
(4.8)

(ii) By the relationship between $X_t$ and $Y_t$, it needs to prove that for almost all $\omega \in \Omega$, every $t > 0$, $y \rightarrow Y_t(y)$ is a homeomorphism on $\mathbb{R}^d$. Due to [16] Theorem 4.5.1 and (4.8), we should prove that: for every $T > 0$, $\tau \in \mathbb{R}$, all $y, x \in \mathbb{R}^d$ ($y \neq x$)
\[
\mathbb{E} \sup_{0 \leq t \leq T} (1 + |Y_t(y)|^2)^\tau \leq C(1 + |y|^2)^\tau
\]
(4.9)

and
\[
\sup_{0 \leq t \leq T} \mathbb{E}|Y_t(x) - Y_t(y)|^{2\tau} \leq C|x - y|^{2\tau}.
\]
(4.10)

Since $\tilde{b}$ and $\tilde{\sigma}$ are bounded, (4.9) is obvious. It remains to calculate (4.10). For $\epsilon > 0$, if one chooses $F(x) = f^\tau(x) = (\epsilon + |x|^2)^\tau$ and set $Y_t(x, y) := Y_t(x) - Y_t(y)$, then by utilising the Itô formula,
\[
F(Y_t(x, y))
\]
\[
= 2\tau \int_0^t f^{\tau-1}(Y_r(x, y))Y_r(x, y)(\tilde{b}(r, Y_r(x)) - \tilde{b}(r, Y_r(y))) dr
\]
\[
+ 2\tau \int_0^t f^{\tau-1}(Y_r(x, y))Y_r(x, y)(\tilde{\sigma}(r, Y_r(x)) - \tilde{\sigma}(r, Y_r(y))) dW_r
\]
\[
+ \tau \int_0^t f^{\tau-2}(Y_r(x, y))[f(Y_r(x, y))\delta_{i,j} + 2(\tau - 1)Y_{i,r}(x, y)Y_{j,r}(x, y)]
\times \tilde{\sigma}_{i,k}(r, Y_r(x)) - \tilde{\sigma}_{i,k}(r, Y_r(y))[\tilde{\sigma}_{j,k}(r, Y_r(x)) - \tilde{\sigma}_{j,k}(r, Y_r(y))] dr
\]
\[
\leq C|\tau| \int_0^t F(Y_r(x, y)) dr + C|\tau(\tau - 1)| \int_0^t \kappa(r)F(Y_r(x, y)) dr
\]
\[
+ 2\tau \int_0^t f^{\tau-1}(Y_r(x, y))Y_r(x, y)(\tilde{\sigma}(r, Y_r(x)) - \tilde{\sigma}(r, Y_r(y))) dW_r,
\]
(4.11)
where $\kappa$ is given in (4.5).

Thanks to (4.11) and the Grönwall inequality, one arrives at

$$\sup_{0 \leq t \leq T} \mathbb{E}[\epsilon + |Y_t(x) - Y_t(y)|^2] \leq C[\epsilon + |x - y|^2].$$

By letting $\epsilon \downarrow 0$, then (4.10) holds. $\square$

When $d = 1$, we also derive the pathwise uniqueness without assuming the Sobolev differentiability on $\sigma$ if $\alpha \geq 1/2$. Moreover, when $\alpha > 1/2$, the non-confluent property of the trajectories for (1.1) is true. Precisely, we have

**Theorem 4.4.** We suppose that $\alpha \in [1/2, 1)$, $b \in L^2(0,T;C^0_b(\mathbb{R}))$ and $\sigma \in L^\infty(0,T;C^\alpha(\mathbb{R}))$. We suppose further that there is positive constant $\delta$, $\sigma^2 > \delta$. Then we have

(i) there is a unique strong solution $X_t(x)$ to (1.1) for every $x \in \mathbb{R}^d$;
(ii) if $\alpha > 1/2$, then the trajectories of (1.1) are non-confluent, that is for every $x, y \in \mathbb{R}^d$,

$$|x - y| > 0 \implies \mathbb{P}\{\omega, \ |X_t(x) - X_t(y)| > 0, \forall t \in [0,T]\} = 1.$$  (4.12)

**Proof.** (i) Clearly, we need to check the pathwise uniqueness only. Observing the relationship between $X$ and $Y$, it suffices to prove the uniqueness for SDE (3.24). Noting that $\tilde{b} \in C([0,T];C^1_b(\mathbb{R}))$, $\tilde{\sigma} \in L^\infty(0,T;C^\alpha(\mathbb{R}))$, by Yamada-Watanabe’s theorem (see [28]), the pathwise uniqueness holds.

(ii) When $\alpha > 1/2$, with the help of Lemma 4.2, then (4.12) is true for all trajectories of solutions for SDE (3.24). Since $Y(t,x) = \Phi(t,X_t(x))$, then it yields that: for every $x, y \in \mathbb{R}^d$,

$$|\Phi(x) - \Phi(y)| > 0 \implies \mathbb{P}\{\omega, \ |\Phi(t,X_t(x)) - \Phi(t,X_t(y))| > 0, \forall t \in [0,T]\} = 1.$$  (4.13)

According to (3.23), from (4.13), then (4.12) is true. $\square$

Now let us discuss the Sobolev differentiable property for the solution.

**Theorem 4.5.** Let $b, \sigma$ and $\alpha$ be stated in Theorem 4.3, and let $X_t(x)$ be the unique strong solution of (1.1). Then $X_t(x)$ is differentiable in $x$ in the sense that: $\{e_i\}_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$, for every $x \in \mathbb{R}^d$ and $1 \leq i \leq d$, the limit

$$\lim_{\delta \to 0} \frac{X.(x + \delta e_i) - X.(x)}{\delta}$$  (4.14)

exists in $L^2(\Omega \times (0,T))$.

**Proof.** Clearly, it only needs to show

$$\lim_{\delta \to 0} \frac{Y.(y + \delta e_i) - Y.(y)}{\delta}$$  (4.15)
exists in $L^2(\Omega \times (0,T))$. Set $Y^\delta_t(y) := Y_t(y + \delta e_i) - Y_t(y)$, then by (3.24)

$$Y^\delta_t(y) = \delta e_i + \int_0^t \left[ \tilde{b}(r, Y_r(y + \delta e_i)) - \tilde{b}(r, Y_r(y)) \right]dr$$

$$+ \int_0^t \left[ \tilde{\sigma}(r, Y_r(y + \delta e_i)) - \tilde{\sigma}(r, Y_r(y)) \right]dW_r$$

$$= \delta e_i + \int_0^1 \int_0^t \nabla \tilde{b}(r, sY_r(y + \delta e_i) + (1-s)Y_r(y))Y^\delta_r(y)drds$$

$$+ \int_0^1 \int_0^t \tilde{\sigma}(r, sY_r(y + \delta e_i) + (1-s)Y_r(y))Y^\delta_r(y)dW_rds. \quad (4.16)$$

By virtue of BDG’s inequality, we achieve from (4.16) that

$$\mathbb{E}|Y^\delta_t(y)|^2 \leq 2|\delta|^2 + C\mathbb{E}\int_0^t |Y^\delta_r(y)|^2dr + \mathbb{E}\int_0^t \kappa(r)|Y^\delta_r(y)|^2dr,$$

which suggests that

$$\mathbb{E}\int_0^T \left| \frac{Y^\delta_t(y)}{\delta} \right|^2 dt \leq C.$$

Then by applying Fatou’s lemma, the desired result follows. \qed

**Remark 4.6.** From our proof, we also prove that: for every $x,e \in \mathbb{R}^d$, as $|e| \to 0$, the limit $(|X(x + e) - X(x)|)/|e|$ exists in $L^2(\Omega \times (0,T))$.

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