Torsors on semistable curves and degenerations

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Abstract. In this paper, we answer two long-standing questions on the classification of $G$-torsors on curves for an almost simple, simply connected algebraic group $G$ over the field of complex numbers. The first is the construction of a flat degeneration of the moduli of $G$-torsors on smooth projective curves when the smooth curve degenerates to an irreducible nodal curve and the second one is to give an intrinsic definition of (semi)stability for a $G$-torsor on an irreducible projective nodal curve. A generalization of the classical Bruhat–Tits group schemes to two-dimensional regular local rings and an application of the geometric formulation of the McKay correspondence provide the key tools.

Keywords. Torsors; group schemes; semistability; stacks.

Mathematics Subject Classification. 14L15, 14D20, 14D23.

1. Introduction

Let $G$ be an almost simple and simply connected algebraic group or the linear group $GL(n)$, over the field $k = \mathbb{C}$ of complex numbers. Let $A = \text{Spec} k[[t]]$ and $K = \text{Spec} k((t))$. Let $o \in A$ be the closed point and let $C_A \to A$ be a proper, flat family with generic fibre $C_k$ a smooth projective curve of genus $g \geq 1$ and closed fibre $C_o$ an irreducible nodal curve $(C, c)$ with a single node $c \in C$. Let $\text{Bun}_G(C_K)$ denote the stack of $G$-bundles on the curve $C_K$. These stacks do not satisfy the valuative criterion for properness and one needs to impose suitable semistability conditions to get a separated Artin stack with a coarse space which is the proper moduli space of ‘slope’ semistable principal $G$-bundles. These moduli spaces were constructed by Ramanathan in 1975 [40]. The first examples of these spaces is in the case when $G = GL(n)$ which give the Mumford–Seshadri moduli spaces of (semi)stable vector bundles (where we need to fix the degree of the bundles).

The primary purpose of the present work is two-fold (see Section 1.1 below for an outline of the basic idea in the work). In the first part, we construct a flat degeneration of the stack $\text{Bun}_G(C_K)$. This question has remained open due to the lack of a suitable analogue for the notion of torsion-free sheaves on curves in the realm of $G$-bundles (see [17, page 489] and [18, page 347]).

This paper is dedicated to the memory of Prof. C S Seshadri.
The approach in this paper is to replace the node by a bubbling. This replaces the nodal curve $C$ by certain semistable curves $C^{(d)}$ whose stable model is $C$. We first construct a degeneration of the group $G$ to a parahoric group scheme over the semistable limits and then we define the limiting objects as certain torsors for these group schemes (see §3 for details). Thus, torsion-free sheaves on $C$ are replaced by triples $(C^{(d)}, \mathcal{E}, \mathcal{E})$. This basic idea in the case of $\text{GL}(n)$ goes back to the work of Gieseker [24] (for $\text{GL}(2)$) and those of Nagaraj and Seshadri [38,39] and Ivan Kausz [27–29] for $\text{GL}(n)$, $n > 2$. The novelty of the present approach is in relating this to torsors under parahoric degenerations of $G$.

The key idea for this generalization is the extension of the notion of Bruhat–Tits group schemes to the setting of regular 2-dimensional local rings (see §3). This local construction is based on a close study of the geometric McKay correspondence as in [25]. Loosely put, this allows us to set up a kind of Fourier–Mukai transform for group schemes and torsors in two steps. In the first step, one begins with an equivariant group scheme with fibre isomorphic to $G$ on the affine plane. By a Fourier–Mukai-like operation, we construct an affine ‘parahoric’ group scheme on the minimal resolution of singularities of the analytic normal surface $k[x, y, t]/(xy - t^d)$. The second step is then to set-up a Fourier–Mukai between the category of certain equivariant $G$-torsors on the affine plane and the category of torsors for the parahoric group scheme on the minimal resolution of singularities of the analytic normal surface $k[x, y, t]/(xy - t^d)$. To build the stack, the next hard step is to build a parahoric group scheme on certain standard models for degenerations of smooth curves using the ones already constructed on the minimal resolutions for the surfaces $k[x, y, t]/(xy - t^d)$. Here we rely heavily on the work of Li [33] on expanded degenerations and using these we build local models for Gieseker bundles (see §4). Once this is done, one can build the stack of Gieseker-torsors for the parahoric group schemes and prove the relevant properties using standard techniques (7.7) and (7.5). The new idea is now to view these as logarithmic schemes. Bundles with parabolic structures on the generic points of the normal crossing divisors appear naturally and give shape to the objects.

The resulting moduli stack over $A := \text{Spec } k[t]$ has a closed fibre which is a s.n.c. divisor with $\ell + 1$ smooth components, $\ell = \text{rank}(G)$, indexed by the extended Dynkin diagram of $G$. This degeneration is therefore a semistable degeneration in the sense of Mumford [31] (see also [9]). Towards the very end of [31], Mumford constructs a relative compactification $\bar{G}_A$ of $G \times A$ where $\bar{G}_K = G_K$. The closed fibre $\bar{G}_0$ is a union of complete varieties meeting with simple normal crossing singularities. These are indexed by the vertices of the affine Dynkin diagram. This is an algebro-geometric model of the Bruhat–Tits building in the relative case. He remarks at the close of the book that his compactification can be viewed as a kind of ‘Néron model with corners’ of the semisimple group scheme over the local ring. The degeneration in the present work can be viewed as a precise analogue for $\text{Bun}_G(C_A)$.

The stack, locally analytically, gives a resolution of singularities for analogues of certain matrix type singularities. These occur on the stack of torsion-free sheaves and their links to the theory of local models and PEL Shimura varieties were already seen by Faltings [19]. This we hope would make our stacks wider in their appeal. We describe in outline the structure of the closed fibre of the stack and elaborate it in the case when $G = \text{SL}(2)$ (8.1) (8.2).

In Part II and Part III, we work towards the coarse space. To get a separated and proper stack in the limit we require a definition of semistability of certain torsors under the parahoric group scheme on semistable curves. Here, even the basic case of a principal $G$-bundle on an irreducible nodal curve itself was not well understood.
A notion of $\mu$-semistability which appropriately generalizes Ramanathan’s definition, has been open and presents serious difficulties (see [17,22]). In the third of a series of papers [21–23] on principal $G$-bundles on elliptic curves and singular curves, Friedman and Morgan wrote that “There are many remaining open questions. One of the deepest is the problem of finding an intrinsic definition of semistability for $G$-bundles on a singular curve, and of a generalized form of $S$-equivalence, which would be broad enough to include those bundles coming from the parabolic construction”. Moreover, for the construction of coarse spaces, this notion should have a GIT interpretation.

For this study, we concentrate on a single nodal curve. The first point is to recognize that any definition of semistability of a principal $G$-bundle on an irreducible nodal curve would require one to already expand the notion of a $G$-bundle to our torsors for the parahoric group scheme. This phenomenon shows up even for vector bundles where the test objects for semistability could be torsion-free sheaves. As a step towards achieving this, we express $C$ as a coarse space of a twisted curve $\mathcal{C}_d$ in the sense of [3] and express the semistability condition on torsion-free sheaves on $C$ in terms of torsors on $\mathcal{C}_d$. To get a good notion of ‘degree’ of line bundles on these Deligne–Mumford stacks, we set up a ‘Fourier–Mukai’-like correspondence between torsors on $\mathcal{C}_d$ and certain objects, which we term laced torsors, on the normalization $\tilde{C}$ of $C$. These are parahoric torsors on $\tilde{C}$ with parahoric structures at the two points above the node, along with a ‘descent datum’. The notion of a (semi)stability of $G$-torsors on $\mathcal{C}_d$, which is ‘equivalent’ to the (semi)stability of torsion-free sheaves when $G = \text{GL}(n)$, is then achieved (11.2). We term this notion $\mathfrak{f}$-(semi)stability. The task then onwards is to show that this notion is a GIT notion (13.5) which defines a good moduli problem. Here we draw on the work of Schmitt [45] to express our moduli problem in his terms and solve it using GIT methods.

Finally, we relate the Gieseker torsors on semistable curves $C^{(d)}$ to laced torsors on $\tilde{C}$ by restricting the torsor to $\tilde{C}$ and then use this to define $\mathfrak{f}$-(semi)stability of Gieseker torsors. Note that at the back of this notion is the fact that there is a ‘morphism’ from the stack of Giesker-torsors to a ‘virtual’ space analogous to the space of torsion-free sheaves. This needs to be carefully placed on a rigorous footing and Schmitt’s construction plays the role for this.

Using the $\mathfrak{f}$-(semi)stability and a relative polarization for this morphism, we invoke a classical principle due to Seshadri to finally get a more refined notion, that of C-(semi)stability for Gieseker torsors. The GIT approach then shows how these notions give the construction of a coarse space for the open substack of semistable Gieseker torsors. We summarize the main results of the paper in the following:

**Theorem** (see (7.5), (7.7), (14.5)).

1. The stack $\text{Gies}_G(C_A)$ of Gieseker torsors (7.1) is an algebraic stack locally of finite type, which is regular and flat over $A$. Over $K$ we have an identification $\text{Gies}_G(C_K) = \text{Bun}_G(C_K)$ with the stack of $G$-torsors on the smooth projective curve $C_K$. Further the closed fibre $\text{Gies}_G(C_0) \subset \text{Gies}_G(C_A)$ is a divisor with normal crossings with $\ell + 1$ smooth components indexed by the extended Dynkin diagram.

2. The open substack $\text{Gies}_G(C_A)^{\text{L-s}}$ of $\mathfrak{f}$-(semi)stable Gieseker torsors (14.4.8) has a coarse space which parametrizes $S$-equivalence classes of Gieseker torsors and which provides a proper flat degeneration of the moduli scheme of $\mu$-(semi)stable $G$-torsors on $C_K$. 

The layout of the paper is as follows. In §3, we make the basic construction using the McKay correspondence. In §4, we construct the group objects on Li’s standard models. In §5, we discuss admissibility for the GL($n$) case and in §6, we discuss the general $G$ case. In §7, we prove the stack-theoretic properties and in §8, we look at some examples and describe the closed fibre of the degeneration. From §9 till §12, we work with a single nodal curve and define the semistability of Gieseker torsors on semistable curves. In §13, we complete the construction of the degeneration of the moduli space of $G$-bundles.

1.1 The basic idea in outline

We work with one parameter degenerations and with the family of curves $C_A$. In the discussion below, by the limiting fibre we mean objects over the closed fibre $C_o = (C, c)$. To tackle the degeneration problem for vector bundles, in the early eighties, Gieseker and Seshadri approached the problem in two different ways. Seshadri in [49] took the approach of degenerating (slope semistable) vector bundles to torsion-free sheaves on the nodal limit $(C, c)$, an approach which was initiated in a paper by Mayer–Mumford and Oda–Seshadri in the degeneration of the Picard variety. Seshadri’s limiting moduli space was slope semistable torsion-free sheaves of fixed rank and degree and his strategy was GIT. Gieseker [24] approached the problem again by GIT but his strategy had its seed in his approach to the construction of moduli of bundles on curves and surfaces. This was by studying smooth curves embedded in Grassmannians using vector bundles generated by sections. The Hilbert scheme of such curves had a natural action of a suitable linear group and GIT on the Hilbert scheme gave rise to ‘semistable’ curves embedded in the Grassmannian; these were limits of smooth curves. This was Gieseker’s GIT construction of the coarse space for Deligne–Mumford compactification of $\bar{M}_g$. The tautological vector bundle on the Grassmannian when restricted to the limiting curves gave the ‘semistable’ objects in the problem. The limiting moduli over the nodal curve was then classified as a ‘list’ of semistable curves $C^{(d)}$ (with a fixed stable model $(C, c)$) together with a class of vector bundles on them. These bundles on the chain of $\mathbb{P}^1$’s in $C^{(d)}$, were from a fixed list of vector bundles which Gieseker called ‘standard’; they were bundles whose direct summands on each $\mathbb{P}^1$ had only $O$ or $O(1)$. Gieseker’s approach had an added feature, viz, the ‘stack’ of objects he obtained (in the modern language) was regular over $k$ and the limiting fibre was reduced with normal crossing singularities.

Gieseker’s approach faced a serious block in going to higher rank bundles, since identifying the semistable limits by GIT became very unwieldy when the rank of the vector bundle exceeded 2. Nagaraj and Seshadri [39] and Kausz [27] combined the two approaches, namely Seshadri’s and Gieseker’s, to solve the higher rank degeneration problem with s.n.c property. En route, they obtain the standard list by two bits of data (this is my interpretation): (1) a local data, i.e., the local types of the torsion-free sheaves on $(C, c)$, which was encoded in the number of summands of the maximal ideal and (2) a data of realizing a torsion-free sheaf on $(C, c)$ as limit of vector bundles on $C_K$. The idea of bubbling was then to blow-up the torsion-free sheaves on the surface gotten by base changing $C_A$ by $t \mapsto t^d$. One obtains new one-parameter families of curves where the original family of vector bundles now had vector bundles as limits. The valuative criterion for the functor needs to account for such base change. The two bits of data gave the Gieseker-type list of semistable curves $C^{(d)}$, together with ‘standard’ vector bundles on them which came with a configuration of $O$’s and $O(1)$’s on the chain of rational curves.
The problem for $G$-bundles and their degeneration was that, on irreducible nodal curves there was no satisfactory solution à la Seshadri. More precisely, there was no torsion-free analogue except in the classical case of the symplectic and orthogonal cases both of which were exploited by Faltings [17]; in either case, there is a basic representation and the problem gets resolved as one on torsion-free sheaves equipped with degenerate forms. My approach is to work around this lacuna and get to the bubbling directly, i.e., by circumventing the torsion-free route. The basic principle was to identify the two bits of data for ‘possible limits’ of principal $G$-bundles. The list was essentially local data on the $\mathbb{P}^1$-chains which were then glued to $G$-bundles on the normalization $\tilde{C}$ of $C$ at two marked end points of the chain (Fig. 1).

In the 1980’s, Gonzalez-Springberg and Verdier [25] and Artin and Verdier [6] studied reflexive sheaves on normal surface singularities in the context of a geometric McKay correspondence. These objects had been studied in depth in the paper by Lipman [35]. From my standpoint, the paper [25] gives an alternate approach to reaching the bubbling data and vector bundles on chains. This was done by simultaneously considering the minimal resolution of the local normal singularities and viewing them also as quotient singularities by actions of finite Kleinian groups on affine planes. The minimal resolution of singularities $\mathcal{N}^{(d)}$ (see Fig. 2) of the normal singularity was realized as a ‘minimal platificateur’ (see (2.2.6) and [25, Corollaire 7, page 448]). The bundles were obtained by a ‘Fourier–Mukai’ from equivariant bundles on the affine plane to bundles on the minimal resolutions; the scheme $D^{(d)}$ in (2.2.3) acts as a correspondence for a pull-back and an invariant push-forward from $D$ to $\mathcal{N}^{(d)}$, indeed $\mathcal{N}^{(d)}$ can be identified with a certain ‘Hilbert
scheme’ classifying equivariant zero-cycles on $D$, and $D^{(d)}$ the universal space (work of Ito–Nakamura).

In my paper with Seshadri [10], we had studied the principal bundle analogue of parabolic vector bundles. The parahoric group schemes were realized via what we termed ‘invariant direct images’ from equivariant affine group schemes on ramified covers, or more precisely ‘orbifold stacks’. Invariant direct images of group schemes were simply taking Weil restrictions of scalars under Galois coverings and then taking invariants by the Galois group; this process works well in characteristic zero and also in the ‘tame’ cases.

How does all this come together in the degeneration question? The idea is to first get to the ‘basic list’ by a Fourier–Mukai like construction of affine group schemes on minimal resolution of singularities of normal surface singularities of type $A_d$; these singularities were simply $A^2/\mu_d$. The basic list is essentially local analytic in its content when viewed on the regular surface, being data along the rational curve-chains. The new group schemes, which we term 2BT group schemes, comes by the following process.

We begin by considering equivariant affine group schemes under the action of $\mu_d$ on $D$ (which was the analytic disc at the origin 0 in $A^2$). The $d$ are allowed to vary. The action is essentially given by the data of conjugacy class of representations $\rho : \mu_d \to G$ which we call ‘type $\tau$’ following an old terminology due to Weil–Seshadri. First, take the trivial $G$-bundle on $D$ with a twisted action by $\mu_d$, i.e., $\mathcal{T}_D := D \times^\rho G$ and then take the ‘adjoint group scheme’ $\mathcal{T}_D(G)$ (3.0.2), where $G$ acts on itself by inner conjugation. This gives the basic equivariant group schemes for each local type $\tau$. We then perform a ‘Fourier–Mukai’ to these group schemes to get smooth affine group schemes on regular analytic surfaces $\mathcal{N}^{(d)}$ (3.0.5). These group schemes now get ‘parahoric structures’ at the generic points of the rational chain. The minimal platificateur property mentioned above implies that the map $D^{(d)} \to \mathcal{N}^{(d)}$ is finite flat, and this is essential here. Each local type $\tau$ gives an affine 2BT group scheme $\mathcal{K}_{\tau,\mathcal{N}^{(d)}}^G$ on the regular surface $\mathcal{N}^{(d)}$ (3.1) and we arrive at the standard list of group schemes $\left\{\mathcal{K}_{\tau,\mathcal{N}^{(d)}}^G\right\}_{\tau}$ on the regular surface $\mathcal{N}^{(d)}$ (see (3.5) for the nomenclature).

Bruhat–Tits theory has been studied extensively on discrete valuation rings, but there is, as of now, no approach for higher dimensional regular local rings. Our objects give a large class of examples of such group schemes and from the philosophy of Bruhat–Tits, knowing the group schemes gives a hold on the possible ‘parahoric’ subgroups.

This basic list then gives global group algebraic spaces on regular surfaces $S^{(d)}$ which are proper over $A$ (2.2.7); these group objects are obtained by a ‘gluing’ (3.3), the 2BT group schemes to constant group schemes. These group algebraic spaces give degenerations of the constant group scheme $G$ on the generic fibre $C_K$ to non-reductive limits on $C^{(d)} \subset S^{(d)}$, for varying $d$. The next step was to replace ‘torsion-free’ sheaves on $(C, c)$ by torsors for the affine non-reductive group schemes on semistable curves $C^{(d)}$ with fixed stable model being the curve $(C, c)$. The torsors are also obtained by the Fourier–Mukai operation which were used to construct the group schemes, i.e. begin with equivariant torsors on the disc $D$ for the group scheme $\mathcal{E}_D(G)$ and realize them as torsors for the 2BT group schemes $\mathcal{K}_{\tau,\mathcal{N}^{(d)}}^G$ on the regular surface $\mathcal{N}^{(d)}$. The pairs $\left\{\mathcal{K}_{\tau,\mathcal{N}^{(d)}}^G, E\right\}_{\tau}$ on the regular surface $\mathcal{N}^{(d)}$ give the ‘admissible’ list of objects. These can be globalized by gluing.

A basic off-shoot which emerges even in the case of $G = GL(n)$ is a certain ‘Tannakian’ principle. A priori the admissible list of vector bundles on the rational chains, which allows only $\mathcal{O}$ and $\mathcal{O}(1)$ as summands, is clearly not closed under ‘tensor’ operations on vector bundles. However, there is an underlying ‘parabolic structure’ on these bundles when
they are viewed as restrictions of bundles on the regular surface \( N^{(d)} \). The rational chain is a normal crossing divisor (revealing a logarithmic structure) and the admissible data becomes a ‘parabolic data’. This gives rise to a ‘parabolic tensor structure’ which explains the phenomenon. This observation plays a central role in eventually constructing the stack.

How does one do GIT for these objects? The approach is similar. Firstly, the local picture (2.2.3) at the level of surfaces, when restricted to curves, gives global objects. More precisely, the disc \( D \) can be replaced by a proper Deligne–Mumford stack called twisted curves (10) [3], and the minimal resolution gets replaced by the normalization \( \tilde{C} \) of \( C \) (12.0.1). One then sets up a Fourier–Mukai machinery between pairs consisting of [group scheme, torsor] on twisted curves and pairs on \( \tilde{C} \) which we call [balanced group schemes, laced torsors]. In [38,39], semistability of Gieseker bundles on \( C^{(d)} \) had two ingredients: semistability of the torsion-free sheaf on \( (C, c) \) (obtained by taking direct images via \( C^{(d)} \to (C, c) \)) and a vertical component along the fibre of Gieseker vector bundles over a fixed torsion-free sheaf. We follow this approach. The torsion-free component is missing in the \( G \)-bundle setting and we replace it with \( G \)-torsors on twisted curves \( \mathcal{E}_d \to C \). A heuristic ‘semistability’ (11.2) which abstractly captures the semistability of the ‘underlying’ torsion-free object is then defined on the twisted curve \( \mathcal{E}_d \). To make these heuristic objects concrete, we take the Fourier–Mukai path to get to the normalization \( \tilde{C} \) of \( C \). Here we can define numerical invariants such as parabolic degrees which allow us to concretize the slope semistability. Finally, ideas from GIT provide the precise \( \mathcal{L} \)-(semi)stability for Gieseker torsors, the \( \mathcal{L} \) standing for an ample line bundle on a suitable ‘Quot-scheme’-like space which is an ‘atlas’ for the stack. To be precise, we define \( \mathcal{L} \)-(semi)stability for Gieseker torsors, which are pairs \( \langle G^G_{\tau,C^{(d)}}, \mathcal{E} \rangle \) consisting of [group scheme, torsor] on semistable curves \( C^{(d)} \) together with a technical ‘admissibility’ condition. We restrict \( \langle G^G_{\tau,C^{(d)}}, \mathcal{E} \rangle \) to the normalization \( \tilde{C} \subset C^{(d)} \) which give rise to laced torsors on \( \tilde{C} \). The \( \mathcal{L} \)-(semi)stability of these laced torsors is then used to define the notion for the pair \( \langle G^G_{\tau,C^{(d)}}, \mathcal{E} \rangle \).

1.2 Related works

In the early nineties, in several papers, Bhosle introduced the notion of ‘generalised parabolic bundles’ as a very useful tool to study the moduli space of torsion-free sheaves by working with objects on the normalization of the singular curve, but this had an intrinsic problem, that it was not amenable to the question of degeneration (see [13,45]). Teixidor in several papers considered the moduli of bundles on singular curves (see [53]). Abe [1] solved the Gieseker construction for SL(2), and Schmitt [43] constructed the universal Gieseker moduli over \( \mathcal{M}_q \), M. Thaddeus had also considered the SL(2) case in his thesis. The paper by Kiem and Li [32] studied more explicit geometry of the Gieseker spaces towards applications. There is also a preprint by Solis [50] which should be of some interest.

In [38], Nagaraj and Seshadri had made some conjectures towards the problem for the case of SL(n) in terms of the ‘determinant’ morphism on the moduli space of torsion-free sheaves. These conjectures were answered fully by Sun in [51,52]. In 2000–2003, Friedman and Morgan [21,22] and Friedman et al. [23] wrote several important papers on \( G \)-torsors on elliptic curves and singular curves. In 2004–2005, Schmitt, in a series of papers [43–45], brought back the focus on the question of moduli space of \( G \)-torsors on singular curves and introduced some new ideas on ‘decorated bundles’ and their slope (semi)stability.
1.3 Notations and conventions

Throughout this paper, unless otherwise stated, we have the following notations and assumptions:

(a) We work over an algebraically closed field \( k \) of characteristic zero and without loss of generality, we can take \( k \) to be the field of complex numbers \( \mathbb{C} \).

(b) \((C, c)\) will be an irreducible projective nodal curve over \( k \) with node \( c \in C \) and \( \nu : \tilde{C} \to C \) the normalization.

(c) Let \( G \) be an almost simple, simply connected affine algebraic group defined over \( k \) of rank \( \text{rank}(G) = \ell = \dim(T) \), where \( T \subset G \) is a fixed maximal torus; let \( X(T) = \text{Hom}(T, \mathbb{G}_m) \) be the group of characters of \( T \) and \( Y(T) = \text{Hom}(\mathbb{G}_m, T) \) be the group of all one-parameter subgroups of \( T \). Fix a Borel subgroup \( B \) containing \( T \), and a set \( \Delta \) of simple roots \( \{\alpha_1, \ldots, \alpha_\ell\} \). Let \( \check{\mathcal{R}} = R(T, G) \) denote the root system of \( G \). Thus for every \( r \in \check{\mathcal{R}} \), there is the root homomorphism \( u_r : \mathbb{G}_a \to G \). The standard affine apartment \( \mathcal{A}_T \) is the affine space under \( Y(T) \otimes_{\mathbb{Z}} \mathbb{R} \). and we shall identify \( \mathcal{A}_T \) with \( Y(T) \otimes_{\mathbb{Z}} \mathbb{R} \) (see [10, §2]).

(d) All group schemes considered in this paper are affine.

(e) Let \( A := \text{Spec} \, k[t] \) and \( K := \text{Spec} \, k((t)) \) and \( o \in A \) be the closed point. Let \( \pi : C_A \to A \) be such that \( \pi K : C_K \to K \) is a smooth projective curve of genus \( g \geq 2 \) and \( C_o \simeq (C, c) \). We assume that \( C_A \) is regular over \( k \). Let \( U_c \subset C_A \) be an analytic neighbourhood of the node \( c \in C_A \).

(f) Let \( d > 0 \) be a positive integer and let \( \mu_d = \langle \gamma \rangle \) be the cyclic group of order \( d \). The group \( \mu_d \) is considered as a subgroup of \( \text{SL}(2, k) \) generated by \( g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \), where \( \zeta = e^{i2\pi/d} \) is a primitive \( d \)-th root of unity.

(g) If \( \rho : \mu_d \to G \) is a representation, \( \tau \) will stand for its type \((2.0.2)\) and represent the conjugacy class of \( \rho \).

(h) The \( S^{(d)}(2.2.7) \) are smooth surfaces with a projective morphism to \( A \). These are minimal desingularizations of the surfaces with normal singularities with local equation \( x \cdot y = t^d \) obtained by base change from \( C_A \). The exceptional fibre \( C^{(d)} \) is the semistable curve with \( d \) − 1-chain of rational curves glued to the normalization \( \tilde{C} \). \( N^{(d)} \) is the local analytic neighbourhood of the exceptional divisor in \( S^{(d)} \).

(i) \( W[d] \) \((4.0.2)\) are Jun Li’s standard models and \( Z[d] \) the local standard models \((4.3)\).

(j) The map \( \nu : (\tilde{C}, c) \to (C, c) \) denotes the normalization of \( C \), where \( \nu^{-1}(c) = c \) and \( c \) stands for the pair of points \( \{c_1, c_2\} \).

(k) \( \mathcal{C}_d \) is a twisted curve \((10)\) in the sense of \([3]\).

(l) \( E_d \) are \( \mathcal{A} \subset \mathcal{C}_d \) torsors on the normalization \( \tilde{C} \) \((12.3)\) \((12.4)\).

(m) \( \mathfrak{g}^G_{\tau, N^{(d)}} \) are the 2BT-group schemes on the regular surface \( N^{(d)} \) \((3.1)\).

Part I

2. Preliminaries

We recall the obvious identification \([10, 2.2.8] \)

\[
\text{Hom}(\mu_d, T) \simeq \frac{Y(T)}{d \cdot Y(T)}.
\]

Let \( \rho : \mu_d \to G \) be a representation. Since \( \mu_d \) is cyclic, we can suppose that the representation \( \rho \) of \( \mu_d \) in \( G \) factors through \( T \) (by a suitable conjugation). The cocharacter
\(\alpha^*_\rho\) associated to \(\rho\) by (2.0.1) gives a tuple of integers \(\{a_1, a_2, \ldots, a_\ell\}\) determined uniquely modulo \(d\) and in terms of the canonical cocharacters \(\{\alpha^*_j \in Y(T), j = 1, \ldots, \ell\}\) dual to the simple roots \(\alpha_j\). We have

\[
\alpha^*_\rho = \sum_{j=1}^\ell a_j \alpha^*_j.
\] (2.0.2)

We will call the tuple \(\tau := (a_1, a_2, \ldots, a_\ell)\) the type of the representation \(\rho\) and denote the association in (2.0.1) by

\[
\rho \mapsto \theta_\tau,
\] (2.0.3)

where \(\theta_\tau \in Y(T)/dY(T)\). We view \(\theta_\tau\) as a point in the affine apartment \(\mathcal{A}_T\).

### 2.1 The geometric setting and assumptions

**Notation 2.1.** Let \(v : (\tilde{C}, c) \to (C, c)\) denote the normalization of \(C\) and let \(v^{-1}(c) = \{c_1, c_2\}\). Let \(E(1) \cup E(2)\) be the normalization of the analytic neighbourhood of \(c \in C\).

**DEFINITION 2.1**

A scheme \(E^{(m)}\) is called a chain of rational curves if

\[
E^{(m)} = \bigcup_{i=1}^{m-1} E_i,
\]

with \(E_i \simeq \mathbb{P}^1\), and if \(i \neq j\),

\[
E_i \cap E_j = \begin{cases} \text{singleton} & \text{if } |i - j| = 1 \\ \emptyset & \text{otherwise.} \end{cases}
\] (2.1.1)

**DEFINITION 2.2**

Let \(C^{(d)}\) denote the reducible nodal curve with components being the normalization \(\tilde{C}\) of \(C\) and a chain \(E^{(d)}\) of projective lines of length \(d - 1\) attached to \(\tilde{C}\) at \(c_1\) and \(c_2\). Equivalently, it is a semistable curve which has \(C\) as its stable model. If \(p : C^{(d)} \to C\) denotes the canonical morphism, the inverse image \(p^{-1}(c)\) is the chain \(E^{(d)}\).

We have the diagram

\[
\begin{array}{ccc}
(\tilde{C}, c) & \xleftarrow{v} & (C^{(d)}, E^{(d)}) \\
\downarrow{v} & & \downarrow{p} \\
(C, c) & & (C^{(d)}, E^{(d)})
\end{array}
\] (2.2.1)

Let \(C_{d,A} := C_A \times_{\text{Spec} \, k[t]} \text{Spec} \, k[t] \) via the map \(t \mapsto t^d\). Here and elsewhere, \(A = \text{Spec} \, k[t] \) and \(o \in A\) the closed point. Let \(N_d = \text{Spec} \, A_{(x,y-t^d)}^{[x,y]} \) be the analytic
neighbourhood of \( c \) in \( C_{d, A} \) which lies above the analytic neighbourhood \( U_c \subset C_A \). We recall [39, page 191] that \( N_d \) is a normal surface with an isolated singularity at \( c \) of type \( A_d \). By the generality of \( A_d \)-type singularities, one can realize \( N_d \) as a quotient \( \sigma : D \to N_d \) of \( D := \text{Spec } \frac{A[u, v]}{(u, v - t)} \) by the cyclic group \( \mu_d \), where \( \mu_d \) acts on \( D \) as follows:

\[
\gamma \cdot (u, v) = (\zeta \cdot u, \zeta^d - 1 \cdot v)
\]

and \( x = u^d, y = v^d \). We consider the following basic diagram for all \( d > 0 \) (see [25]):

\[
\begin{array}{ccc}
D^{(d)} & \xrightarrow{f} & N^{(d)} \\
\downarrow q & & \downarrow p_d \\
0 \in D & \xrightarrow{\sigma} & N_d \ni c
\end{array}
\]

where \( p_d : N^{(d)} \to N_d \) is the minimal resolution of singularities of \( N_d \) obtained by successively blowing up the singularity, with the exceptional divisor \( E^{(d)} = p_d^{-1}(c) \), and

\[
D^{(d)} := (D \times_{N_d} N^{(d)})_{\text{red}}.
\]

The closed fibre \( F^{(d)} = N^{(d)}_0 \) of the canonical morphism \( N^{(d)} \to A \) looks like

\[
F^{(d)} = E^{(d)} \cup E(1) \cup E(2).
\]

Thus, \( F^{(d)} \subset N^{(d)} \) is a normal crossing divisor with \( d + 1 \) components. By [25, Proposition 2.4], the morphism

\[
f : D^{(d)} \to N^{(d)}
\]

is finite and flat, the minimal platificateur in the sense of Grothendieck [25, Cor. 7, page 448]. Since \( N^{(d)} \) is smooth, this implies that \( f \) is ramified at the generic point of each of the \( d - 1 \) rational components of the exceptional divisor \( E^{(d)} = p_d^{-1}(c) \subset F^{(d)} \). Let

\[
p_d : S^{(d)} \to C_{d, A}
\]

be the minimal smooth model for \( C_K \). Then we see that \( N^{(d)} \to S^{(d)} \) gives an analytic neighbourhood of the exceptional fibre \( p_d^{-1}(c) \).

**Remark 2.3** (The étale picture). The surface \( C_A \) over \( A = \text{Spec } \mathbb{k}[t] \) is assumed to be regular and hence the analytic local ring at the node \( c \in C \) is \( A[x, y]/(xy - t) \). It is well known that \( \text{Spec } A[x, y]/(xy - t) \) is the analytic local ring for a versal deformation of the simple node. By [20, Proposition 2.8, page 184], which is somewhat delicate, one can in fact obtain an étale neighbourhood \( U(c) \) of \( c \) in \( C_A \) which is isomorphic to an étale neighbourhood of the origin \( 0 \) in \( \text{Spec } A[x, y]/(xy - t) \). By a base change by the map \( \mathbb{k}[t] \to \mathbb{k}[t] \) given by \( t \mapsto t^d \), we see that there is an étale neighbourhood \( N_{\text{ét}, d} \) of \( c \) in \( C_{d, A} \) which is isomorphic to an étale neighbourhood of the origin \( 0 \) in \( \text{Spec } A[x, y]/(xy - t^d) \). In other words, in the étale topology, we can express the neighbourhood \( N_{\text{ét}, d} \) of \( c \) in \( C_{d, A} \) as a quotient \( \mathbb{A}^2/\mu_d \) for the affine space \( D_{\text{ét}} := \mathbb{A}^2 = \text{Spec } \mathbb{C}[u, v] \) for the action (2.2.2). This gives the following étale picture corresponding to (2.2.3):
where $N_{\text{et}}^{(d)}$ is the minimal desingularization of the normal surface $N_{\text{et},d}$ and

$$D_{\text{et}}^{(d)} := (D_{\text{et}} \times N_{\text{et},d} N_{\text{et}}^{(d)})_{\text{red}}.$$  

Observe that $f_{\text{et}} : D_{\text{et}}^{(d)} \to N_{\text{et}}^{(d)}$ is finite and flat since the map $f$ in (2.2.3) is so. If $p_d$ is as in (2.2.7), then we see that $N_{\text{et}}^{(d)} \to S^{(d)}$ gives an étale neighbourhood of the exceptional fibre $p_d^{-1}(c)$.

**Remark 2.4 (Balanced action).** The action of $\mu_d$ is balanced in the sense of [2] (see also [28, 2.5]), i.e., the action of a generator $\xi$ on the tangent spaces to each branch are inverses to each other. For the corresponding dual action in the neighbourhood $D'_0$ (the component with local coordinate $v$), the action is by $\xi^{-1}$ (see 10.0.1). If we begin with a representation $\rho : \mu_d \to G$ of local type $\tau$ at a point in a branch, then corresponding local type for the dual action at the point in the second branch is denoted by $\bar{\tau}$. See (B1.2) for the expression of ‘dual weights’ when $G = \text{GL}(n)$.

### 2.2 Outline of proof strategy

We give the broad steps of the proof:

1. We work with $F^{(d)}$ (2.2.5) inside the analytic surface $N^{(d)}$, i.e. the basic models are built on smooth analytic surfaces.
2. The geometric McKay correspondence is then used to define the local models for the group schemes in §3.
3. Unlike the vector bundle case, admissibility will be defined on the analytic surface $N^{(d)}$ and then extended to more general ‘modifications’ defined in (6.1).
4. These group schemes on local analytic models get realized as invariant push-forwards from Kawamata covers (3.2). The logarithmic structure on the surfaces becomes significant.
5. Use the local analytic model to define group schemes on Jun Li’s local standard models $Z[d]$ (4.3). Realize these also from Kawamata covers of $Z[d]$ (4.4), (4.3).
6. Globalize and define group algebraic spaces on Jun Li’s standard models $W[d]$ via Kawamata covers of $W[d]$ (4.5).
7. Local models for torsors on $N^{(d)}$ are made by pulling back equivariant torsors and then taking invariant push-forwards (5.12), (5.9).
8. We globalize and define group algebraic spaces on smooth surfaces $S^{(d)}$ (projective over $A$) by gluing and invariant push-forwards. Kawamata covers of $S^{(d)}$ can be defined using the ramification data on $N^{(d)}$.
9. Torsors for these group algebraic spaces on $S^{(d)}$ are obtained by invariant push-forwards of equivariant torsors on these Kawamata covers.
10. Admissible pairs on $S^{(d)}$ (5.12) are torsors obtained by invariant push forwards of equivariant torsors using specific local data.
11. Define \textit{parabolically associated} \eqref{5.11} vector bundles to admissible pairs. This process gets done by taking associated equivariant vector bundles on covers and then taking invariant push forwards. These parabolically associated vector bundles are the quasi-admissible vector bundles on the standard models \(W[d]\).

12. Define admissible pairs on arbitrary modifications using Jun Li’s expanded degenerations and local effectivity \eqref{6.6} and \eqref{6.7} and show that this definition is intrinsic.

3. McKay correspondence and 2Bruhat–Tits group schemes

The aim of this section is to construct certain smooth affine group schemes on the \(N^{(d)}\) which are \textit{generically split}, i.e., a product over the open subset \(N^{(d)} - E^{(d)}\) \eqref{2.2.3} with fibre \(G\) and which degenerate \textit{parahorically}. More precisely, these group schemes are 2-dimensional generalizations of the classical Bruhat–Tits group schemes associated to parahoric subgroups of \(G(K)\) (see \eqref{3.5}). I make these constructions using the geometric McKay correspondence of Gonzalez-Springberg and Verdier \cite{25}.

Let \(T_D\) be a \((\mu_d, G)\)-torsor on \(D\) \eqref{2.2.3} (see \eqref{3.0.2} below). Assume that it is given by a homomorphism \(\rho : \mu_d \to G\) (in fact, it is easy to see that this is always the case on \(D\)). This gives a homomorphism \(\rho : \mu_d \to T\) into the maximal torus \(T\) of \(G\) of type \(\tau = (a_1, a_2, \ldots, a_\ell)\) in the sense of \eqref{2.0.2}. In other words, we have a \(\mu_d\)-action on \(D \times G\) given by

\[
\gamma \cdot (u, v, g) = (\xi \cdot u, \xi^{-1} \cdot v, \rho(\gamma) \cdot g)
\]

and

\[
\mathcal{T}_D \simeq D \times^\rho G. \tag{3.0.1}
\]

We observe that since the action of \(\mu_d\) is \textit{balanced} \eqref{2.4} at the two marked points \(z_1\) (resp. \(z_2\)) above the origin \(0 \in D\), the local type of the action on a \(G\)-torsor at these points are \(\tau\) (resp. \(\tau\)). Throughout the paper, we fix this \(G\)-torsor \(\mathcal{T}_D\) on \(D\) of local type \(\tau\).

Consider the adjoint group scheme \(\mathcal{T}_D(G) := \mathcal{T}_D \times^{G, \text{Ad}} G\) on \(D\), where \(G\) acts on itself by inner conjugation. We define the equivariant group scheme:

\[
E(G, \tau) := q^*(\mathcal{T}_D(G)) \tag{3.0.3}
\]

on \(D^{(d)}\) of local type \(\tau\) in the sense that it comes with a \(\mu_d\)-action via a representation \(\rho : \mu_d \to G\). Since the morphism \(f : D^{(d)} \to N^{(d)}\) is also \textit{finite and flat}, we can take the Weil restriction of scalars:

\[
f_*E(G, \tau)_{D^{(d)}} := \mathcal{R}_{D^{(d)}} / N^{(d)}(E(G, \tau)_{D^{(d)})}. \tag{3.0.4}
\]

and since \(E(G, \tau)_{D^{(d)}} \to D^{(d)}\) is a smooth (affine) group scheme, the basic properties of Weil restriction of scalars \cite[Lemma 2.2]{16} show that \(f_*E(G, \tau)\) is a smooth group scheme on \(N^{(d)}\), coming with a \(\mu_d\)-action. By taking invariants under the action of \(\mu_d\) and noting that we are over characteristic zero, by \cite[Prop 3.4]{16}, we obtain the smooth (affine) group scheme obtained by what we shall term \textit{invariant direct image}:

\[
\mathcal{R}^G_{\tau, N^{(d)}} = (\text{Inv} \circ f_*)(E(G, \tau)_{D^{(d)})}) := \left[\mathcal{R}_{D^{(d)}} / N^{(d)}(E(G, \tau)_{D^{(d)})}\right]^{\mu_d}. \tag{3.0.5}
\]

on \(N^{(d)}\) (see \cite[Definition 4.1.3]{10}).
DEFINITION 3.1

Let $G$ be an almost simple, simply connected algebraic group of rank $\ell$ and let $\theta_\tau \in \mathcal{A}_T$ be a weight in the affine apartment (2.0.3) arising from a representation $\rho$, and $d$ a positive integer. The $2\mathbb{BT}$-group scheme of type $\tau$ with generic fibre $G$ of singularity type $A_d$ associated to $\theta_\tau$ is defined to be the affine group scheme $\mathcal{X}_{\tau,N(d)}^G$ (3.0.5) on the regular surface $N(d)$ (see (3.5) for the nomenclature). This process defines a distinguished collection of $2\mathbb{BT}$-group scheme $\mathcal{X}_{\tau,N(d)}^G$ indexed by the type $\tau$.

As an instance of what we would be doing subsequently over more general modifications (6.1), we show that we can obtain such a group algebraic space by a different more general geometric construction (this is possible since we are over characteristic 0). This is via the Kawamata covering lemma (14.1.1). Note that $D(d)$ in (2.2.3) is only Cohen–Macaulay, in general.

**Theorem 3.2.** The group scheme $\mathcal{X}_{\tau,N(d)}^G$ can be realized as an invariant direct image (3.0.5) of a group scheme with fibres $G$, from a global smooth ramified covering $\tilde{N}(\tau)$ of the smooth surface $N(d)$.

**Proof.** The closed fibre $E^{(d)}(\tau) \subset N(d)$ (2.2.5) is a reduced divisor with normal crossing singularities. Further, the covering $f : D^{(d)} \to N(d)$ (2.2.3) of the analytic neighbourhood of the closed fibre is ramified at the generic points of the chain of rational curves $E^{(d)} \subset F^{(d)}$ with a choice of ramification indices. We fix this ramification data.

Recall the Kawamata covering lemma (14.1.1) (see [30, Theorem 17], [54, Lemma 2.5, page 56]). We have a Galois covering $h : V_1 \to N(d)$ with $V_1$ smooth, which is ramified along the irreducible components $E^{(d)}$ for the fixed ramification data dictated by the local type $\tau$ of the representation. Let $\Gamma = \text{Gal}(h)$ be the Galois group of the covering $h$. The stabilizer subgroups of $\Gamma$ at each of the generic points of the rational components $R_j$’s of $E^{(d)}$ is precisely the cyclic group $\mu_{d_j}$.

We now consider the two coverings $h : V_1 \to N(d)$ and $f : D^{(d)} \to N(d)$. The covering $f$ is étale over the complement of $E^{(d)}$. Hence by [54, Corollary 2.6], we have a smooth finite covering $\tilde{N}(\tau) \to D^{(d)}$ such that there is an étale morphism $j : \tilde{N}(\tau) \to V_1$ which can be assumed to be Galois (by going to the canonical Galois closure if need be). The group scheme $E(\tau, G)$ on $D^{(d)}$ pulled back by $\tilde{N}(\tau) \to D^{(d)}$ gives a group scheme $E(\tau, G)$ on $\tilde{N}(\tau)$ and hence on $V_1$. Note that the composite $h \circ j$ gives a Galois cover.

Thus we can again take the Weil restriction of scalars and invariants under the composite $h \circ j = \varphi : \tilde{N}(\tau) \to N(d)$ and we get a group scheme $(\mathcal{X} \times \mathbb{V}_\tau \circ \varphi_\tau)(E(\tau, G'))$. It is now easily checked that this group scheme is isomorphic to $\mathcal{X}_{\tau,N(d)}^G$.

3.1 Global constructions

Fix a $G$-torsor $E_A^0$ on $C_A - c$ and let $E_{d,A}^0$ be the pull-back to $C_{d,A} - c$. Let $\mathcal{D}_{d\ell}$ be the $(\mu_{d\ell}, G)$-torsor on $D_{d\ell} = \mathbb{A}^2$ as in (3.0.2) given by $\rho : \mu_{d\ell} \to G$ of type $\tau$ i.e., we have a $\mu_{d\ell}$-equivariant trivialization $\mathcal{D}_{d\ell} \simeq D_{d\ell} \times^\rho G$. We can again consider the adjoint group scheme $\mathcal{D}_{d\ell}(G)$ which is an equivariant group scheme on $D_{d\ell}$. Note that we can view $\mathcal{D}_{d\ell}$ as an $\mathcal{D}_{d\ell}(G)$-torsor as well. As in the analytic case, we see that we have a
The closed fibre \( C^{(d)} \subseteq S^{(d)} \) is a reduced divisor with normal crossing singularities. We can therefore get a smooth ramified covering \( \tilde{S}(\tau) \rightarrow S^{(d)} \) which is ramified over the precise locus in \( C^{(d)} \) (depending on \( \tau \)) which local analytically gives the Kawamata cover \( \tilde{N}(\tau) \rightarrow N^{(d)} \). In fact, this can be done in the étale setting and we get an equivariant group algebraic space over \( \tilde{S}(\tau) \) whose invariant direct image is \( \mathcal{G}_{\tau,S^{(d)}} \). We recall that Weil restriction of scalars sends algebraic spaces to algebraic spaces.

\[ \square \]

### 3.2 The McKay correspondence revisited

Before going to the salient feature of the group schemes \( \mathcal{G}_{\tau,N^{(d)}} \), we recall the geometric interpretation of the McKay correspondence given by Gonzalez-Springberg and Verdier [25]. Let \( \text{Irr}^s(\mu_d) \subset \text{Irr}(\mu_d) \) be the nontrivial irreducible representations of \( \mu_d \) and let \( \text{Irr}(E^{(d)}) \) denote the set of irreducible rational components of the exceptional divisor of the minimal resolution \( p_d : N^{(d)} \rightarrow N_d \) (2.2.3). Let \( \psi \) be a non-trivial character of \( \mu_d = \langle \gamma \rangle \). Then \( \psi \) corresponds to \( \zeta \mapsto \zeta^s \), where \( \zeta \) corresponds to a primitive \( d \)-th root of 1 and \( 1 \leq s \leq d - 1 \). Let \( L_\psi \) be the equivariant line bundle on \( D \) where \( \mu_d \) acts on \( D \times k \) as \( \gamma \cdot (u, v, a) = (\zeta \cdot u, \zeta^{d-1} \cdot v, \zeta^s \cdot a), \ a \in k \). A \( \mu_d \)-invariant section \( \delta \) of this line bundle is given by the relation \( \delta(\gamma \cdot (u, v)) = \zeta^s \delta(u, v) \), and hence the \( \mu_d \)-invariant sections are...
generated by \( u^s \) and \( u^{d-s} \). From this, it is easily checked that the invariant direct image of \( L_{\psi} \) under \( \sigma : D \rightarrow N_d \) is given by
\[
(\text{inv} \circ \sigma_*)(L_{\psi}) = (x, t^{d-s}),
\]
where \((x, t^{d-s})\) is an ideal sheaf on \( N_d = \text{Spec} \ k[x, y, t]/(x \cdot y - t^d) \) (2.2.2). Let \( \mathcal{L}_\psi := \text{inv} \circ f_* (q^* (L_{\psi})) \) be the induced line bundle on \( N^{[d]} \). This is a line bundle since \( f \) is finite and flat.

**Theorem 3.4** [25]. There is a bijection \( \text{Irr}^a (\mu_d) \rightarrow \text{Irr}(E^{(d)}), \psi \mapsto E_\psi \), such that for any \( E_j \in \text{Irr}(E^{[d]}) \), we have
\[
c_1(\mathcal{L}_\psi) \cdot E_j = \begin{cases} 0 & \text{if } E_j \neq E_\psi \\ 1 & \text{if } E_j = E_\psi \end{cases}.
\]

The statement in (3.4) implies that the first Chern class \( c_1(\mathcal{L}_\psi) \) can be represented by a divisor which meets \( F^{(d)} \) transversally at a unique point which lies in \( E_\psi \). More precisely, consider the divisor \( q^{-1}(0) \subset D^{(d)} \), with \( 0 \in D \) as in (2.2.3). Consider the reduced fibre \( \tilde{E} := q^{-1}(0)_{\text{red}} \). The group \( \mu_d \)-fixes the divisor \( q^{-1}(0) \) and hence its reduced subscheme \( \tilde{E} \). Thus, given \( \psi \in \text{Irr}^a (\mu_d) \), there is a unique component \( E_\psi \) of \( \tilde{E} \) such that the line bundle \( q^* (\mathcal{L}_\psi) \) gets a nontrivial linearization by the \( \mu_d \) action at the generic point of \( E_\psi \).

### 3.3 A brief description of the group scheme

Suppose that we are given a homomorphism \( \rho : \mu_d \rightarrow T \) of local type \( \tau \). For the simple roots \( \{\alpha_j\}_{j=1}^\ell \) of \( G \), let
\[
C_\rho := \{\alpha_j \circ \rho \mid \alpha_j \circ \rho \neq 1\}.
\]
Then \( C_\rho \) gives a subset of \( \text{Irr}^a (\mu_d) \). The McKay correspondence says that to each \( \psi \in C_\rho \), we have a unique rational component \( E_\psi \subset F^{(d)} \). Furthermore, the covering \( f : D^{(d)} \rightarrow N^{(d)} \) is ramified precisely over the rational curves \( E_\rho := \{E_\psi \mid \psi \in C_\rho\} \) with ramification index dictated by the number of \( \psi \)'s which give independent characters of \( \mu_d \) and their multiplicities.

**Remark 3.5.** By Bruhat–Tits theory, for each facet \( \af \) in the apartment \( \mathcal{A}_\tau \) of the Bruhat–Tits building of \( G(k((t))) \), there is a smooth group scheme \( P_\af \) over \( \text{Spec} \ k[[t]] \) with connected fibers whose generic fiber is \( G \times_{\text{Spec} k} \text{Spec} k((t)) \). We call such a \( P_\af \) a Bruhat–Tits group scheme. Let \( P_\af \) be the functor \( R \rightarrow P_\af (R[[t]]) \), which is representable by a pro-algebraic group over \( k \). We call \( P_\af (k) \) a parahoric subgroup of \( G(k((t))) \). The conjugacy classes of parahoric subgroups of \( G(k((t))) \) are classified by proper subsets of the nodes of the extended Dynkin diagram of \( G \) or the facets of the Weyl alcove \( a \subset \mathcal{A}_\tau \). These group schemes are indexed by the rational points of the alcove \( a \) which are in turn given by the types \( \tau \) (2.0.3).

In summary, the \( 2\mathbb{B}T \)-group scheme \( \mathcal{X}^{G}_{\tau, N^{[d]}} \) is such that it has non-trivial parahoric structures prescribed by the McKay correspondence. These are at precisely the generic points of the rational curves in \( E_\rho \) with further degeneration at the nodes on \( E^{(d)} \). The local type \( \tau \) carries the information of ramification at these primes. This becomes the data for the Kawamata cover \( \tilde{N}(\tau) \rightarrow N^{(d)} \) (3.2) and gives the points \( a_j \) in the Weyl alcove and
the group schemes $P_{\text{adj}}$ at the generic points. The comments after (3.4.2) show that each group scheme $\kappa_{\tau,N}$ is non-trivial on at most $\ell = \text{rank}(G)$ number of rational curves on the exceptional divisor of $N^{(d)}$.

4. Group algebraic spaces on standard models

The aim of this section is to build the basic group algebraic spaces on the standard models $W[d]$ defined in [33].

4.1 Standard models for a semistable curve

Let $\hat{A}^d := \text{Spec} \ k[[t_1, \ldots, t_d]]$ be given a $A$-scheme structure via the morphism: $t \mapsto t_1 \cdots t_d$.

Gieseker in [24, Lemma 4.2, Proposition 4.1] constructed a miniversal family for the semistable curve $C^{(d)}$ with fixed stable model $(C, c)$. We will however follow the detailed construction of the expanded degenerations in the paper of Li [33].

We begin with the base family $C_A \to A$. We let $B[0] := A$ to start the inductive construction. Thus, we have the family $C_B[0] \to B[0]$. Let

$$B[d - 1] := \hat{A}^d$$

with $\hat{A}^d$ being given the $A$-scheme structure as above. Li constructed the standard models $W[d - 1]$ over $B[d - 1]$ inductively as a small resolution of the scheme $W[d - 2]$ (see [33, page 521] for the details). The scheme $W[d - 1]$ comes with a tautological projection:

$$W[d - 1] \to C_A \times_A \times B[d - 1].$$

The fibre of $W[d - 1]$ over $0 \in B[d - 1]$ is denoted by $W[d - 1]_0$, which is isomorphic to the projective curve $C^{(d)}$.

4.2 The special degeneration

For the main applications, we need the description of an étale neighbourhood of $W[d - 1]_0$ in $W[d - 1]$. This is done by looking at a special degeneration (see below (4.3)).

In [33, page 522], Li constructed the special degeneration:

$$\Gamma[d - 1] \to \mathbb{A}^d.$$ 

We briefly recall its description for our purposes (see [4] for a nice exposition).

The first observation is that the fibres of $\Gamma[d - 1] \to \mathbb{A}^d$ are not projective. Secondly, an étale neighbourhood of the fibre of the origin of this morphism coincides with an étale neighbourhood of $W[d - 1]_0$, which is the fibre of the origin of the morphism $W[d - 1] \to B[d - 1]$.

The fibre $\Gamma[d - 1]_0 = E^{(d)} \cup E'(1) \cup E'(2)$ over $0 \in B[d - 1]$ is a chain of $d + 1$ curves of which the first and last are $E'(1)$ and $E'(2)$ which are $\mathbb{A}^1$’s and the rest of the members are $\mathbb{P}^1$’s [33, Lemma 1.2, page 522]. The scheme $\Gamma[d - 1]$ can be covered by $d$-open subsets $U_1, \ldots, U_d$, each of which is isomorphic to $\mathbb{A}^{d+1}$. If the coordinates of $U_\lambda$, which is isomorphic to the affine space $\mathbb{A}^{d+1}$, are denoted by $(u_1^{(\lambda)}, \ldots, u_{d+1}^{(\lambda)})$, 

The fibre $\Gamma[d - 1]_0 = E^{(d)} \cup E'(1) \cup E'(2)$ over $0 \in B[d - 1]$ is a chain of $d + 1$ curves of which the first and last are $E'(1)$ and $E'(2)$ which are $\mathbb{A}^1$’s and the rest of the members are $\mathbb{P}^1$’s [33, Lemma 1.2, page 522]. The scheme $\Gamma[d - 1]$ can be covered by $d$-open subsets $U_1, \ldots, U_d$, each of which is isomorphic to $\mathbb{A}^{d+1}$. If the coordinates of $U_\lambda$, which is isomorphic to the affine space $\mathbb{A}^{d+1}$, are denoted by $(u_1^{(\lambda)}, \ldots, u_{d+1}^{(\lambda)})$, 

The fibre $\Gamma[d - 1]_0 = E^{(d)} \cup E'(1) \cup E'(2)$ over $0 \in B[d - 1]$ is a chain of $d + 1$ curves of which the first and last are $E'(1)$ and $E'(2)$ which are $\mathbb{A}^1$’s and the rest of the members are $\mathbb{P}^1$’s [33, Lemma 1.2, page 522]. The scheme $\Gamma[d - 1]$ can be covered by $d$-open subsets $U_1, \ldots, U_d$, each of which is isomorphic to $\mathbb{A}^{d+1}$. If the coordinates of $U_\lambda$, which is isomorphic to the affine space $\mathbb{A}^{d+1}$, are denoted by $(u_1^{(\lambda)}, \ldots, u_{d+1}^{(\lambda)})$, 


the transition function from \(u^{(\delta)}\) to \(u^{(\delta+1)}\) on \(U_\delta \cap U_{\delta+1}\) is given by \(u^{(\delta+1)}_j = u^{(\delta)}_j\) for \(j = 1, \ldots, \delta - 1, \delta + 1, \delta + 2, \ldots, d + 1\). For the three coordinates with indices \(j = \delta, \delta + 1, \delta + 2\), we have the following transition relations:

\[
\begin{align*}
    u^{(\delta+1)}_\delta &= u^{(\delta)}_\delta \cdot u^{(\delta)}_{\delta+1}, \\
    u^{(\delta+1)}_{\delta+1} &= 1/u^{(\delta)}_{\delta+1}, \\
    u^{(\delta+1)}_{\delta+2} &= u^{(\delta)}_{\delta+1} \cdot u^{(\delta)}_{\delta+2}.
\end{align*}
\]

(4.0.4)
(4.0.5)
(4.0.6)

Remark 4.1. A fact which will be used later is that the action of \(G[d] := \mathbb{G}_m^d\) (see [33, page 525]) on the open subsets \(U_\delta\) covering \(Z[d - 1]\) is given by

\[
\sigma \cdot (u^{(\delta)}_1, \ldots, u^{(\delta)}_{d+1}) := (\ldots, \tilde{\sigma}_{\delta-2} u^{(\delta)}_{\delta-2}, \tilde{\sigma}_{\delta-1} u^{(\delta)}_{\delta-1}, \sigma_{\delta-1} u^{(\delta)}_{\delta}, \sigma_{\delta-1} u^{(\delta)}_{\delta+1}, \tilde{\sigma}_{\delta+1} u^{(\delta)}_{\delta+2}, \ldots),
\]

(4.1.1)

where \(\tilde{\sigma}_i = \sigma_i / \sigma_{i-1}\) with the convention that \(\sigma_i = 1\) for \(i = 0, d\).

4.3 \(\mathcal{BT}\)-group schemes on the local standard model

From here onwards we work with the base change of the special degeneration \(\Gamma'[d - 1] \to \mathbb{A}^d\) to the analytic neighbourhood of 0 in \(\mathbb{A}^d\). We denote this by \(Z[d - 1] \to \mathbb{A}^d\) and call it the local standard model. Thus, \(Z[d - 1]_0 = E[d] \cup E(1) \cup E(2) = F[d]\) as in (2.2.5) and \(Z[d - 1]\) can be identified with the analytic neighbourhood of the closed fibre \(W[d - 1]_0\) of \(W[d - 1]\).

The scheme \(Z[d - 1]\) is smooth with a divisor \(\mathcal{D} := Z[d - 1] \times_A 0 \subset Z[d - 1]\) (4.1.2), with normal crossing singularities having \(d + 1\) irreducible components; \(\mathcal{D} = \bigcup_{j=0}^d \mathcal{D}_j\), where \(\mathcal{D}_j, j = 1, \ldots, d\) are the smooth irreducible components of \(\mathcal{D}\) intersecting transversally along the \(d + 1\) disjoint, smooth, codimension two subvarieties \(\{D_j\}_{j=1}^d\), respectively (see [33, page 538] and [34, page 207], where these are expressed in the setting of logarithmic schemes). This is precisely the configuration which matches the configuration of the nodal structure of the central fibre \(Z[d - 1]_0 \simeq F[d]\). We have the following structure of \(Z[d - 1]\):

\[
\begin{align*}
    Z[d - 1]_0 \longrightarrow & \quad Z[d - 1] \supset \mathcal{D} \\
    \{0\} \longrightarrow & \quad \mathbb{A}^d \\
    \text{origin} \quad & \quad t_0 + \cdots + t_d \quad \mathcal{D} \\
    \end{align*}
\]

(4.1.2)

Let \(\mathbb{X}^G_{\tau, N[d]}\) on be the group scheme of local type \(\tau\) on \(N[d]\) (2.2.3). Let

\[
\mathbb{X}^G_{\tau} := \mathbb{X}^G_{\tau, F[d]},
\]

(4.1.3)

denote the restriction of \(\mathbb{X}^G_{\tau, N[d]}\) to \(F[d]\) the closed fibre (2.2.5).

In this subsection, we prove the following result which will play a central role in what follows.
PROPOSITION 4.2

Fix a group scheme \( \mathcal{X}_{G_t} \) on the fibre \( Z[d - 1]_0 \simeq F^{(d)} \) of \( Z[d - 1] \rightarrow \mathbb{A}^d \). There exists a group scheme \( \mathcal{X}_{G_t} \) on \( Z[d - 1] \) which restricts to \( \mathcal{X}_{G_t} \) on \( Z[d - 1]_0 \) and which is isomorphic to the split group scheme with fibre group \( G \) on the complement of the divisor \( \mathcal{D} \) (4.1.2).

Proof. We recall that the minimal smooth resolution \( N^{(d)} \rightarrow N_d \) has a description analogous to the one given for \( Z[d - 1] \). The closed fibre (2.2.5) \( F^{(d)} \) is identified with \( Z[d - 1]_0 \).

The scheme \( N^{(d)} \) can be covered by \( d \)-open subsets \( B_1, \ldots, B_d \), each of which is isomorphic to \( \mathbb{A}^2 \). If the coordinates of \( B_\delta \) are denoted by \((a_\delta, b_\delta)\) in its identification with \( \mathbb{A}^2 \), the transition functions from \( (a_\delta, b_\delta) \) to \((a_{\delta+1}, b_{\delta+1})\) on \( B_\delta \cap B_{\delta+1} \) are given by

\[
a_{\delta+1} = a_\delta^2 b_\delta, \tag{4.2.1}
b_{\delta+1} = 1/a_\delta. \tag{4.2.2}
\]

We firstly show that there is a natural morphism

\[
q_d : Z[d - 1] \rightarrow N^{(d)} \tag{4.2.3}
\]

which has the following key property: for each \( j \) the intersection of the divisor \( \mathcal{D} \subset Z[d - 1] \) with the open set \( U_j \) is mapped by \( q_d \) precisely to the intersection of the normal crossing divisor (2.2.5) \( F^{(d)} \subset N^{(d)} \), with the open set \( B_j \).

We begin by defining the morphism for \( d = 2 \):

\[
q_2 : Z[1] \rightarrow B_1 \cup B_2 = N^{(2)} \tag{4.2.4}
\]

by sending (for \( \delta = 1, 2 \)),

\[
(u_1^{(\delta)}, u_2^{(\delta)}, u_3^{(\delta)}) \mapsto (a_\delta, b_\delta), \tag{4.2.5}
\]

where

\[
a_1 = u_2^{(1)}, \quad b_1 = u_1^{(1)} u_3^{(1)}, \tag{4.2.6}
a_2 = u_2^{(2)} u_3^{(2)}, \quad b_2 = u_2^{(2)}. \tag{4.2.7}
\]

We now define the morphism (4.2.3) recursively as follows: if \( \phi_\delta : U_\delta \rightarrow B_\delta \) sends \( \phi_\delta(u_1^{(\delta)}, \ldots, u_d^{(\delta)}) = (a_\delta, b_\delta) \), then recursively \( \phi_{\delta+1}(u_1^{(\delta+1)}, \ldots, u_d^{(\delta+1)}) = (a_{\delta+1}, b_{\delta+1}) \), where

\[
a_{\delta+1}(u_1^{(\delta+1)}, \ldots, u_d^{(\delta+1)}) = a_\delta^2 b_\delta, \tag{4.2.8}
b_{\delta+1}(u_1^{(\delta+1)}, \ldots, u_d^{(\delta+1)}) = 1/a_\delta. \tag{4.2.9}
\]

The explicit formulae can be obtained by using the transition data (4.0.4). For example, the maps \( U_j \rightarrow B_j \), \( j = 1, 2, 3 \), look as follows:

\[
a_1(u_1^{(1)}, \ldots, u_d^{(1)}) = u_2^{(1)}, \quad b_1(u_1^{(1)}, \ldots, u_d^{(1)}) = u_1^{(1)} u_3^{(1)}. \tag{4.2.10}
\]
\[ a_2(u_1^{(2)} \ldots u_{d+1}^{(2)}) = u_1^{(2)} u_3^{(2)}, \quad b_2(u_1^{(2)} \ldots u_{d+1}^{(2)}) = u_2^{(2)}, \quad (4.2.11) \]
\[ a_3(u_1^{(3)} \ldots u_{d+1}^{(3)}) = (u_1^{(3)})^{-1} u_2^{(3)} (u_3^{(3)})^{-1}, \quad (4.2.12) \]

That the morphism \( q_d \) has the stated property can be seen by a simple check using [33, Lemma 1.2(i)] and the computations above.

We define

\[ \mathcal{G}^G_{\tau, Z[d-1]} := q_d^*(\mathcal{G}^G_{\tau, N(d)}). \quad (4.2.13) \]

The group scheme \( \mathcal{G}^G_{\tau, Z[d-1]} \) can be easily seen to satisfy the stated properties.

4.4 The Kawamata cover of \( Z[d - 1] \)

**Theorem 4.3.** The group scheme \( \mathcal{G}^G_{\tau, Z[d-1]} \) on \( Z[d - 1] \) can be realized as invariant direct image \( (3.0.5) \) from a global smooth ramified covering \( \tilde{Z}_\tau[d - 1] \to Z[d - 1] \).

**Proof.** By [33, Lemma 1.2(i)] and the computations above, it follows by a simple check that for each \( j \), the intersection of the divisor \( D \subset Z[d - 1] \) with the open set \( U_j \) maps precisely to the intersection of the normal crossing divisor \( (2.2.5) F^{(d)} \subset N^{(d)} \), with the open set \( B_j \). By the general theory, (14.1.1) [30] or [54, Lemma 2.5], there exist a Kawamata covering of \( Z[d - 1] \) with the prescribed ramification data.

Let \( \tilde{N}(\tau) \to N^{(d)} \) be as in (3.2) (see also (3.3)). Pulling back by the morphism \( Z[d - 1] \to N^{(d)} \) and taking reduced scheme structure, we get a covering \( Z'' \to Z[d - 1] \). The scheme \( Z'' \) need not be smooth but which is étale over the complement of the divisor \( D \). This can be rectified by [54, proof of Corollary 2.6], where we can get the required Kawamata covering \( \tilde{Z}_\tau[d - 1] \) of \( Z[d - 1] \) as a finite covering of \( Z'' \) (see (3.2) for similar arguments). The ramification data is controlled by the local type \( \tau \). More importantly, we can conclude as in (3.2), that we can realize the group scheme \( \mathcal{G}^G_{\tau, Z[d-1]} \) on \( Z[d - 1] \) as an invariant direct image \( (3.0.5) \) of an equivariant group scheme on the Kawamata covering \( \tilde{Z}_\tau[d - 1] \).

4.5 Group algebraic spaces on \( W[d - 1] \)

As in (4.4), by [33, page 538], we see that \( W[d - 1] \) with its tautological projection \( (4.0.2) \) also has a canonically defined divisor \( D := W[d - 1] \times_A 0 \) defined by \( (4.1.2) \), with normal crossing singularities and we have a Kawamata covering \( \tilde{W}_\tau[d - 1] \to W[d - 1] \) with the precise ramification data (given by \( \tau \)) at the generic points of the smooth irreducible components \( \{D_j\} \)(14.1.1). We can obtain group algebraic spaces \( \mathcal{G}^G_{\tau, W[d-1]} \) of local type \( \tau \) by gluing the group scheme with constant fibre \( G \) away from the divisor with \( \mathcal{G}^G_{\tau, Z[d-1]} \). The gluing is done using the pull-back of \( E^*_A(G) \) on \( C_A - c \) as in (3.1). It is not hard to check that there is a group algebraic space \( \mathcal{G}_{\tilde{W}} \) on the covering \( \tilde{W}_\tau[d - 1] \) such that the group algebraic space \( \mathcal{G}^G_{\tau, W[d-1]} \) is obtained as an invariant direct image \( (3.0.5) \) of \( \mathcal{G}_{\tilde{W}} \) as in (4.3).
Remark 4.4. By (3.5), we see that the Kawamata cover $\tilde{W}_\tau[d - 1]$ of $W[d - 1]$ is ramified at the generic points of at most $\ell$ of the divisors $\mathcal{D}_j$’s. Likewise, the group algebraic space $\mathfrak{g}_{\tau,W[d-1]}^G$ has non-trivial parahoric structure at the generic points of at most $\ell$ components of the divisor $\mathcal{D}$.

5. Admissible pairs on standard models

The aim of this section is to define admissible pairs consisting of basic group algebraic spaces and a special class of torsors on the standard models $W[d]$. This is done as before by building it first on analytic surfaces $N^{(d)}$ and from there to more general objects.

5.1 Quasi-admissibility of vector bundles and McKay correspondence

In this subsection, we assume $G = \text{GL}(n)$. The aim is to understand the notion of (quasi)admissibility of vector bundles (A1.1) as an outcome of the McKay correspondence. This gives the new perspective on Gieseker’s objects.

**DEFINITION 5.1**

A vector bundle $\mathcal{V}$ of rank $n$ on the smooth surface $N^{(d)}$ is called quasi-admissible (see (A1.1)) if the restriction $\mathcal{V} |_{E^{(d)}}$ to the chain $E^{(d)} \subset F^{(d)}$ in the closed fibre (2.2.5) is standard (A1.1), and furthermore the direct image $p_d_* (\mathcal{V})$ is torsion-free on $N_d$, where $p_d$ is as in (5.1.1).

We consider representations $\rho : \mu_d \to \text{GL}(n)$ of type $\tau$. We fix a maximal torus $T_n \subset \text{GL}(n)$ (which can be taken as the $(n \times n)$-diagonal matrices by choice of a basis). Recall that the isomorphism classes of $(\mu_d, \text{GL}(n))$-bundles on $D$ are classified by the equivalence classes of representations $\rho : \mu_d \to T_n$, which we term the local type $\tau$ of $\rho$. In particular, this is given by writing $\rho$ as $\rho(\gamma) = \text{diag}(\zeta^{m_1}, \ldots, \zeta^{m_j})$, where $\gamma$ is a generator of the cyclic group $\mu_d$, and $\zeta$ is the primitive $d$-th root of unity defined by $\gamma \cdot z = \zeta \cdot z$, with $z = (u, v)$ being local coordinates in $D$, and $0 \leq m_1 < m_2 \ldots < m_j \leq d - 1$. Each $m_i$ repeats $r_i$-times so that $\sum_i r_i = n$.

The basic motivation for our study is to relate this definition to the notion of quasi-admissibility of vector bundles on the smooth surface $N^{(d)}$. Recall the basic diagram:

$$
\begin{array}{ccc}
D^{(d)} & \xrightarrow{f} & N^{(d)} \\
q \downarrow & \quad & \quad \downarrow p_d \\
D & \xrightarrow{\sigma} & N_d
\end{array}
$$

(5.1.1)

**PROPOSITION 5.2 (Fourier–Mukai)**

A vector bundle $\mathcal{V}$ on $N^{(d)}$ is quasi-admissible if and only if there is a $\mu_d$-equivariant vector bundle $V$ on $D$ arising from a representation $\rho$ such that $\mathcal{V} \simeq (\Pi \cap \nu \circ f_\ast)(q^* (V))$. In particular, the vector bundle $\mathcal{V}$ is non-trivial on at most $n$ rational curves in the exceptional divisor of $N^{(d)}$. 

Proof. Let \( V \) be a \( \mu_d \)-vector bundle on \( D \) coming from a representation \( \rho : \mu_d \to \text{GL}(n) \) of local type \( \tau \). The invariant direct image \( \text{Inv} \circ \sigma_\ast(V) \) of \( V \) under \( \sigma : D \to N_d \) gives a reflexive sheaf \( \mathcal{F} \) on the normal surface \( N_d \) which takes the form

\[
\mathcal{F} \simeq \bigoplus_{i=1}^{j} (x, t^{m_i})^{\otimes r_i}.
\]  

(5.2.1)

Each summand is the invariant push-forward of the equivariant line bundle associated to a character of \( \mu_d \) given by the \( m_i \)'s with multiplicity \( r_i \) in terms of the representation \( \rho \).

In [8, Proposition 4.2], it was shown that quasi-admissible vector bundles on \( N^{(d)} \) are precisely those with the property that the direct image \( (p_d)_\ast(\mathcal{V}) \) is the reflexive sheaf \( \mathcal{F} \) as in (5.2.1). Indeed, we can identify the quasi-admissible vector bundle with \( p_d^\ast(\mathcal{F})/(\text{tors}) \).

On the other hand, the result of Gonzalez-Springberg and Verdier [25, Theorem 2.2] proves that any reflexive sheaf \( \mathcal{F} \) on \( N_d \) can be expressed as \( \text{Inv} \circ \sigma_\ast(V) \), for \( V \) a vector bundle on \( D \) of local type \( \tau \) and conversely (see also [8,39]). Moreover, one has the isomorphism [25, Theorem 2.2, Proposition 2.8]:

\[
(\text{Inv} \circ f_\ast)(q^\ast(V)) \simeq p_d^\ast(\mathcal{F})/(\text{tors}).
\]  

(5.2.2)

Whence by (5.2.2), the process \( (\text{Inv} \circ f_\ast)(q^\ast(V)) \) gives quasi-admissible vector bundles on \( N^{(d)} \) and conversely. \( \Box \)

Remark 5.3. Let \( V \) be a quasi-admissible vector bundle on \( F^{(d)} \), i.e., \( V \) is firstly standard and further, if \( p : F^{(d)} \to E_c \) be the projection to the analytic neighbourhood of \( c \in C \), the sheaf \( p_\ast(V) \) is torsion-free on the reducible curve \( E_c \subset N_d \). At the node, \( p_\ast(V) \) gets a decomposition \( \mathcal{O}^a \oplus \mathcal{O}^b \) for some \( a, b \geq 0 \). Hence it is the restriction of a reflexive sheaf \( \mathcal{F} \) on \( N_d \) given as in (5.2.1), where the \( m_i \) come from the rational component \( E_\chi \) labelled by the character \( \chi : \zeta \mapsto \zeta^{m_i} \) as dictated by McKay correspondence and the multiplicity \( r_i \) is precisely the number of copies of \( \mathcal{O}(1) \) in the restriction of \( V \) to \( E_\chi \). Whence, \( V \) is isomorphic to the restriction of the quasi-admissible vector bundle \( \mathcal{V} \) on \( N^{(d)} \).

5.2 Quasi-admissible vector bundles on \( Z[d - 1] \) and parabolic structures

Let \( \mathcal{V} \) be a vector bundle on the local standard model \( Z[d - 1] \to \hat{\mathbb{A}}^d \). Then \( \mathcal{V} \) is called quasi-admissible (A1.1) if it is so on each closed fibre. The fibre over \( 0_{\hat{\mathbb{A}}^d} \) is the chain \( F^{(d)} \) with \( d + 1 \)-components, \( d - 1 \) of which are \( \mathbb{P}^1 \)'s and the deformations are by smoothing of nodes. A quasi-admissible bundle \( \mathcal{V}_0 \) has no deformations along the fibre since the summands are \( \mathcal{O} \) or \( \mathcal{O}(1) \). Therefore the lift \( \mathcal{V}_0 \), in an analytic or étale neighbourhood of the central fibre, is uniquely determined by \( \mathcal{V}_0 \). By following the construction in (4.3), we can construct the vector bundle \( \mathcal{V}_0 \) on \( Z[d - 1] \) knowing \( \mathcal{V}_0 \). In other words, we can realize \( \mathcal{V}_0 \) as the invariant direct image of an equivariant vector bundle of local type determined by \( \mathcal{V}_0 \), on the Kawamata cover \( \tilde{Z}_\tau[d - 1] \) of \( Z[d - 1] \). This model of local type determined by \( \mathcal{V}_0 \) on the local standard model then allows us to construct quasi-admissible bundles on the projective family given by the standard model \( W[d - 1] \) by gluing the local quasi-admissible ones with bundles on the complement of the divisor \( \mathcal{D} \subset W[d - 1] \). Whence all quasi-admissible vector bundles on \( W[d - 1] \) with fixed local type \( \tau \) can also be obtained as invariant direct images of certain equivariant vector bundles on \( \tilde{W}_\tau[d - 1] \).
COROLLARY 5.4

Let $\mathcal{V}$ be a quasi-admissible vector bundle on $W[d - 1]$. Then $\mathcal{V}$ gets a canonical parabolic structure at the generic points of the s.n.c. divisor $\mathcal{D} \subset W[d - 1]$ with weights determined by the local type $\tau$ of $\mathcal{V}_0$. The number of components at the generic points of which $\mathcal{V}$ has non-trivial parabolic structure is bounded above by the rank of $\mathcal{V}$.

Proof. Since $\mathcal{V}$ comes as an invariant direct image of an equivariant bundle on the Kawamata cover, by [11], it gets canonical parabolic structure at the generic points of $\mathcal{D}_j$ and the number of these components is bounded by the rank of $\mathcal{V}$. □

5.3 Admissible pairs for the GL(n) case

Our first task is to get a ‘group scheme’ plus ‘torsor’ equivalent of these definitions in the case when $G = \text{GL}(n)$. Fix a $(\mu_d, \text{GL}(n))$-torsor $\mathcal{T}_D$ of local type $\tau$, and let $\mathcal{E}_{\tau, N[d]}^{\text{GL}(n)}$ be as in (3.0.5) and $\mathcal{E}_{\tau, S(d)}^{\text{GL}(n)}$ the group algebraic space on $S(d)$ (coming from a fixed gluing). We keep the diagram (2.2.3) in mind for the next definitions. Let $E$ be a $\mathcal{T}_D(\text{GL}(n))$-torsor on $D$. Then the pull-back $q^*(E)$ gets the structure of an $E(G, \tau)$-torsor on $D(d)$ (12.0.5).

DEFINITION 5.5

Let $\mathcal{E}$ be a $\mathcal{E}_{\tau, N[d]}^{\text{GL}(n)}$-torsor on $N(d)$. The pair $(\mathcal{E}_{\tau, N[d]}^{\text{GL}(n)}, \mathcal{E})$ is called admissible if $\mathcal{E}$ arises as $(\text{Inv} \circ f_*)q^*(E))$ for a $\mathcal{T}_D(\text{GL}(n))$-torsor $E$ on $D$. A pair $(\mathcal{E}_{\tau, S(d)}^{\text{GL}(n)}, \mathcal{E})$ on $S(d)$ is called admissible if its restriction to $N(d)$ is so.

Remark 5.6. By [10, Theorem 4.1.6], the process of taking ‘invariant direct images’ extends to torsors as well and gives an isomorphism of stacks. In other words, $(\text{Inv} \circ f_*)q^*(E))$ above does give a $\mathcal{E}_{\tau, S(d)}^{\text{GL}(n)}$-torsor and this process can be carried out for all groups $G$, not merely $\text{GL}(n)$.

Remark 5.7. By using the Kawamata covering $h: \tilde{S}(\tau) \rightarrow S(d)$ (with local ramification given by $\tau$) (3.3) we can check that in an admissible pair $(\mathcal{E}_{\tau, S(d)}^{\text{GL}(n)}, \mathcal{E})$ the torsor $\mathcal{E}$ can be realized as $(\text{Inv} \circ h_*)(E'))$ for an equivariant torsor $E'$ on $\tilde{S}(\tau)$ for a group algebraic space with fibre $G$. Moreover, $E'$ agrees with $E$ on the inverse image of $N(d)$.

A parabolic vector bundle on $N(d)$.

DEFINITION 5.8

Let $(\mathcal{E}_{\tau, N[d]}^{\text{GL}(n)}, \mathcal{E})$ be an admissible pair on $N(d)$. The vector bundle $\mathcal{E}^{\text{par}}(k^n)$ parabolically-associated to $\mathcal{E}$ is defined as

$$\mathcal{E}^{\text{par}}(k^n) := \text{Inv} \circ f_*(q^*(E(k^n))).$$ (5.8.1)

Let $(\mathcal{E}_{\tau, S(d)}^{\text{GL}(n)}, \mathcal{E})$ be an admissible pair on $S(d)$. Define the vector bundle $\mathcal{E}^{\text{par}}(k^n) := \text{Inv} \circ h_*(E'(k^n))$ with $E'$ on $\tilde{S}(\tau)$. 
The vector bundle $E_{\text{par}}(k^n)$ defined above does indeed come with canonical parabolic structures in the sense of Seshadri. These structures are at the generic points of the rational components of $C^{(d)} \subset S^{(d)}$. We note that it is this phenomenon which makes sure that the line bundles in the decomposition along the chain of $\mathbb{P}^1$’s in $E^{(d)}$ remains $\mathcal{O}$ or $\mathcal{O}(1)$. The terminology comes from the similarity of the phenomenon with the process of taking ‘parabolic tensor products’ of parabolic vector bundles on curves. Taking usual tensor products will increase the degree of the line bundles on the rational components.

5.4 Equivalence of various notions of admissibility

The next result ties up the various notions of admissibility.

PROPOSITION 5.9 (‘Fourier–Mukai’ on group schemes and torsors)

Let $(\mathfrak{g}_{\tau, N^{(d)}}, \mathcal{E})$ be an admissible pair on $N^{(d)}$ (5.12). Then the parabolically-associated vector bundle $E_{\text{par}}(k^n)$ is a quasi-admissible vector bundle on $N^{(d)}$ (5.1). Conversely, if $\mathcal{V}$ is a quasi-admissible vector bundle of rank $n$ on $N^{(d)}$, then there exists an admissible pair $(\mathfrak{g}_{\tau, N^{(d)}}, \mathcal{E})$ on $N^{(d)}$ and an admissible torsor $E$ such that $\mathcal{V} = E_{\text{par}}(k^n)$.

Proof. This is immediate from (5.2) and the fact that giving a $\mu_d$-vector bundle on $D$ coming from a representation $\rho$ is equivalent to giving a $G$-torsor $E$.

COROLLARY 5.10

A vector bundle $\mathcal{V}$ on $S^{(d)}$ is quasi-admissible if and only if $\mathcal{V} \simeq E_{\text{par}}(k^n)$ for an admissible pair $(\mathfrak{g}_{\tau, S^{(d)}}, \mathcal{E})$.

5.5 Torsors and vector bundles on $W[d - 1]$

Let $(\mathfrak{g}_{\tau, Z[d-1]}^{\text{GL}(n)}, \mathcal{E})$ be the group scheme constructed on the local standard model $Z[d-1]$ (4.3). By (4.3), these group schemes can be realized as invariant direct images (3.0.5) of group schemes $\mathcal{G}_Z$ with fibres $G = \text{GL}(n)$, from the Kawamata cover $\kappa : \tilde{Z} \rightarrow Z[d-1]$. Let $\mathcal{E}$ be a $\mathfrak{g}_{\tau, Z[d-1]}^{\text{GL}(n)}$-torsor. Then by [10, Theorem 4.1.6, page 24], there exist a unique $\mathcal{G}_Z$-torsor $E'$ on $\tilde{Z}_\tau[d-1]$ such that the invariant push-forward of $E'$ is $\mathcal{E}$. Hence we can define the parabolically associated vector bundle $E_{\text{par}}(k^n) := \text{Inv} \circ \kappa_* (E'(k^n))$. Observe that all these considerations make sense on the standard model $W[d-1]$.

Following (5.9), we can therefore have the following definition.

DEFINITION 5.11

The pair $(\mathfrak{g}_{\tau, W[d-1]}^{\text{GL}(n)}, \mathcal{E})$ is called admissible if the parabolically-associated vector bundle $E_{\text{par}}(k^n)$ is quasi-admissible on $Z[d-1]$. Likewise, for any group algebraic space $\mathfrak{g}_{\tau, W[d-1]}^{\text{GL}(n)}$ of local type $\tau$ on $W[d-1]$, a pair $(\mathfrak{g}_{\tau, W[d-1]}^{\text{GL}(n)}, \mathcal{E})$ is called admissible if the parabolically-associated vector bundle $E_{\text{par}}(k^n)$ is quasi-admissible.
5.6 Admissible pairs for general $G$

Let $E$ be a $\mathcal{Z}(G)$-torsor on $D$. Then the pull-back $q^*(E)$ gets the structure of an $E(G, \tau)$-torsor on $D^{(d)}$ (12.0.5). Let $\mathbb{H}^G_{\tau, N^{(d)}}$ be the group scheme of local type $\tau$ on $N^{(d)}$.

DEFINITION 5.12

Let $\mathscr{E}$ be a $\mathbb{H}^G_{\tau, N^{(d)}}$-torsor on $N^{(d)}$. A pair $(\mathbb{H}^G_{\tau, N^{(d)}}, \mathscr{E})$ on $N^{(d)}$ is called admissible if $\mathscr{E}$ arises as $(\mathcal{I}_{\mathcal{V}} \circ f_\tau)(q^*(E))$ (compare (5.6) where $G = \text{GL}(n)$, and also [10, Theorem 4.1.6]). A pair $(\mathbb{H}^G_{\tau, S^{(d)}}, \mathscr{E})$ on $S^{(d)}$ is called admissible if its restriction to $N^{(d)}$ is so.

The notion of admissibility behaves well under extension of structure groups. More precisely, let $\eta : G \hookrightarrow \text{GL}(n)$ be a faithful representation. Fix a maximal torus $T_\eta \subset \text{GL}(n)$ such that $\eta : T \hookrightarrow T_\eta$. Observe that the group scheme $\mathbb{H}^G_{\tau, N^{(d)}}$ comes with a canonical inclusion

$$\eta : \mathbb{H}^G_{\tau, N^{(d)}} \hookrightarrow \mathbb{H}^{\text{GL}(n)}_{\tau, N^{(d)}}, \quad \eta : \mathbb{H}^G_{\tau, S^{(d)}} \hookrightarrow \mathbb{H}^{\text{GL}(n)}_{\tau, S^{(d)}},$$

(5.12.1)

by taking invariant push-forwards of inclusions of group schemes (with fibre $G$ and $\text{GL}(n)$) induced by $\eta$, via the Kawamata covering $\tilde{N}(\tau) \to N^{(d)}$ (resp. $\tilde{S}(\tau) \to S^{(d)}$) (3.2) and (3.3), we get the following statement which is easily seen using (5.9).

Lemma 5.13. A pair $(\mathbb{H}^G_{\tau, N^{(d)}}, \mathscr{E})$ is admissible if and only if the associated pair $(\mathbb{H}^{\text{GL}(n)}_{\tau, N^{(d)}}, \eta_*(\mathscr{E}))$ is so for any $\eta$.

We now work with $Z[d - 1]$ and $W[d - 1]$ just as we did with $N^{(d)}$ and $S^{(d)}$. By the construction of the group scheme $\mathbb{H}^G_{\tau, Z^{[d-1]}}$ on $Z[d - 1]$, it is clear that $\eta$ induces a canonical inclusions

$$\mathbb{H}^G_{\tau, Z^{[d-1]}} \hookrightarrow \mathbb{H}^{\text{GL}(n)}_{\tau, Z^{[d-1]}}, \quad \mathbb{H}^G_{\tau, W^{[d-1]}} \hookrightarrow \mathbb{H}^{\text{GL}(n)}_{\tau, W^{[d-1]}},$$

(5.13.1)

Modelling after (5.13) and using (5.11), we get the following.

DEFINITION 5.14

Say that a pair $(\mathbb{H}^G_{\tau, W^{[d-1]}}, \mathscr{E})$ is admissible if the associated pair $(\mathbb{H}^{\text{GL}(n)}_{\tau, W^{[d-1]}}, \eta_*(\mathscr{E}))$ is admissible (5.11), i.e., the parabolically-associated vector bundle $\mathscr{E}^{\text{par}}(k^n)$ (more accurately $(\eta_*(\mathscr{E}))^{\text{par}}(k^n)$) is a quasi-admissible vector bundle on $W[d - 1]$.

Remark 5.15. Observe that we could alternately have defined admissibility of a pair $(\mathbb{H}^G_{\tau, Z^{[d-1]}}, \mathscr{E})$ on $Z[d - 1]$ by simply saying that it is isomorphic to the pull-back of an admissible pair on $N^{(d)}$ (5.13). I noted this when I was giving the talks, so it does not occur in the published version of this paper! Having done this, the admissibility of a pair $(\mathbb{H}^G_{\tau, W^{[d-1]}}, \mathscr{E})$ on $W[d - 1]$ would then simply be the condition that its restriction to $Z[d - 1]$ is admissible. The equivalence of these definitions are immediate from the discussions above. In particular, the dependence on the faithful representation $\eta$ is quickly dispensed with.
6. Admissible pairs on general modifications

Modifications or expanded degenerations of curves have been used by Gieseker and others to study degenerations of moduli spaces of vector bundles on smooth curves. In the previous section, we defined and studied the notion of admissibility of pairs on standard models $W_d$ which are special examples of modifications but having certain local versal properties. The aim of this section is to work with very general modifications of $CA$ and define admissible pairs on them.

**Definition 6.1** (cf. [33, Definition 1.9, page 531], see also [27, Definition 3.8] and [29])

For every $A$-scheme $T$, a modification or an expanded degeneration of $CT$ over $T$ is a pair $(M, \pi)$, where $M$ is a flat family of projective curves over $T$ together with a $T$-projection:

$$\pi : M \rightarrow CT = CA \times_A T \tag{6.1.1}$$

with the following property: there is an open covering $T_\alpha$ of $T$ in the étale topology and morphisms $\zeta_\alpha : T_\alpha \rightarrow B[d_\alpha]$ which induces an isomorphism $\xi_\alpha : M_\alpha \cong W[d_\alpha] \times_{B[d_\alpha]} T_\alpha$ compatible with the projection $\pi : M \rightarrow CT$, where $M_\alpha := M \times_T T_\alpha$.

An arrow $M \rightarrow M'$ consists of an $A$-morphism $T \rightarrow T'$ and an $T$-isomorphism $M \rightarrow M' \times_{T'} T$ which is compatible with their tautological projections to $CT$.

Two modifications $M_T$ and $M'_T$ are isomorphic if there is a $T$-isomorphism $M_T \rightarrow M'_T$ compatible with the projections $M_T \rightarrow CT$ and $M'_T \rightarrow CT$. The groupoid of modifications is a stack (see [33, Proposition 1.10]).

**6.1 Universal construction when $G = GL(n)$**

We work in the setting of vector bundles on modifications. Here we rely on [27,38]. This has been summarized with some small variations in the appendix (A1.2) below. The notations are as in the Appendix.

For vector bundles of rank $n$, by the representability of the functor $G^G_N$ (see (A1.2) and [39, Proposition 8]), we have a flat $A$-scheme $Y \rightarrow A$ which is quasi-projective and regular over $k$ and which is obtained as a $PGL(N)$-invariant open subscheme of a suitable Hilbert scheme. Its generic fibre is smooth and the closed fibre $Y_o \subset Y$ over $o \in A$ is a reduced divisor with normal crossing singularities. Indeed, $Y_o$ is known to be irreducible (see [38, Proposition 8 and page 200]).

We also get a universal family of curves $M \rightarrow Y$ of chain length bounded by the rank $n$. Let $\{ M_\alpha \rightarrow Y_\alpha \}$ together with the morphisms $\zeta_\alpha, \xi_\alpha$ provided by (6.1) define the cover for the modification $M$ in terms of the $W[d_\alpha]$.

In fact, at each closed point of $x \in Y_o$ corresponding to a semistable curve $C^{(d)}$, there is an étale neighbourhood $T(x)$ and a smooth morphism $T(x) \rightarrow B[d-1]$ such that the pull back of the versal space $W[d-1]$ is isomorphic to the restriction of $M$ to the neighbourhood. Since the $W[d]'s$ have a divisor with simple normal crossing singularities, we see that the universal family of curves $M$ also has a simple normal crossing divisor $M_{Y_o}$ which is étale locally the pull-back of the normal crossing divisor on the $W[d]$ (4.5).
PROPOSITION 6.2

Let \( V \) be the universal quasi-admissible vector bundle on \( M \) of rank \( n \) (see (A1.2) and the remarks there). There exists a group algebraic space \( \mathfrak{X}^{GL(n)}_{univ} \) together with a torsor \( \mathcal{P} \) on \( M \) such that \((\mathfrak{X}^{GL(n)}_{univ}, \mathcal{P})\) gives the “universal” admissible pair, i.e.,

\[
\mathcal{P}_{\text{par}}(k^n) \simeq V.
\]

**Proof.** By [24, proof of Proposition 4.1, page 183], the restriction \( V_\alpha \) to \( M_\alpha \) of the universal quasi-admissible bundle \( V \), is the pull-back \( = \xi_\alpha^*(V_\alpha) \) of an admissible vector bundle \( V_\alpha \) on \( W[d_\alpha] \). The local type of \( V_\alpha \) dictates the choice of the Kawamata cover \( \tilde{W}_\alpha \to W[d_\alpha] \) which realizes \( V_\alpha \) as the invariant direct image of an equivariant vector bundle on \( \tilde{W}_\alpha \) (4.5). This can be done for each \( \alpha \).

By [24], in the case of the universal space \( M \), the morphisms \( \zeta_\alpha : Y_\alpha \to B[d_\alpha] \) and the induced \( \xi_\alpha : M_\alpha \to W[d_\alpha] \) are smooth morphisms. Hence, the pull-backs \( \tilde{M}_\alpha := \xi_\alpha^*(\tilde{W}_\alpha) \) are such that \( \kappa_\alpha : \tilde{M}_\alpha \to \tilde{M}_\alpha \) give finite covers of the smooth quasi-projective schemes \( \tilde{M}_\alpha \) which are unramified away from the divisors in \( M_\alpha \).

Since \( V \) is defined on \( M \), the quasi-admissible vector bundles obtained by restricting \( V \) agree on the intersection \( M_{\alpha\beta} := M_\alpha \cap M_\beta \). Hence we have two covers \( \tilde{M}_{\alpha\beta} \to M_{\alpha\beta} \) and \( \tilde{M}_{\beta\alpha} \to M_{\alpha\beta} \) with same ramification data. By [54, Proof of Corollary 2.6, page 56], it follows that we can go to a larger cover \( \tilde{M} \) which is étale over both.

For each \( \alpha \), the bundle \( V_\alpha \) on \( M_\alpha \) comes as an invariant direct image \((\text{Inv} \circ \kappa_{\alpha*})(V_\alpha')\) for equivariant vector bundles \( V_\alpha' \) on \( \tilde{M}_\alpha \). Whence, by going to \( \tilde{M} \) dominating \( \tilde{M}_{\beta\alpha} \) we can identify the pull-backs of \( V_\alpha' \) and \( V_\beta' \).

Now each \( V_\alpha \) comes as \( \mathcal{E}_\alpha^{\text{par}}(k^n) \) for admissible pairs \((\mathfrak{X}_{\text{univ}}^{GL(n)}(t,W[d_\alpha]), \mathcal{E}_\alpha)\) on \( W[d_\alpha] \). Furthermore, each \( \mathcal{E}_\alpha \) comes as invariant push-forwards of equivariant torsors \( \mathcal{E}_\alpha' \) on \( \tilde{W}_\alpha \). Let \( E_\alpha' = \xi_\alpha^*(E_\alpha') \) on \( M_\alpha \). Then, we have \( E_\alpha'(k^n) = V_\alpha' \).

Thus, we can glue together the invariant push-forward \((\text{Inv} \circ \kappa_{\alpha*})(E_\alpha')\) to construct the universal group algebraic space \( \mathfrak{X}^{GL(n)}_{univ} \) together with a torsor \( \mathcal{P} \) on \( M \) such that \((\mathfrak{X}^{GL(n)}_{univ}, \mathcal{P})\) gives the ‘universal’ admissible pair. This clearly has the property that

\[
\mathcal{P}_{\text{par}}(k^n) \simeq V.
\]

\[\square\]

6.2 Universal constructions for \( G \)

Fix a faithful representation \( \eta : G \to \text{GL}(n) \). We have a smooth \( k \)-scheme \( \tilde{M} \) with a group algebraic space \( \mathfrak{X}^{GL(n)}_{\text{univ}} \). Further, the inclusion \( \eta \) gives a constant subgroup scheme with fibre type \( G \) away from the divisor \( M_{\eta\alpha} \). By taking flat closure at the generic point of the divisor \( M_{\eta\alpha} \), this subgroup scheme extends to a subgroup algebraic space over an open subscheme \( \tilde{M}_{\eta} \subset \tilde{M} \) with complement of codimension \( \geq 2 \). We may assume that \( \tilde{M}_{\eta} \) is the maximal such open subset to which the subgroup scheme extends. Let \( \mathfrak{X}^{G}_{\text{univ}} \) denote this subgroup algebraic space over \( \tilde{M}_{\eta} \) and

\[
\eta_{\text{univ}} : \mathfrak{X}^{G}_{\text{univ}} \hookrightarrow \mathfrak{X}^{GL(n)}_{\text{univ}}
\]

the canonical inclusion over \( \tilde{M}_{\eta} \).
**Remark 6.3.** We wish to emphasize that at them moment we only have the group algebraic space $\mathcal{K}^{G}_{\text{univ}}$ on $M_\eta$ and not the universal torsor.

### 6.3 Admissible pairs on modifications

We make two definitions, the first one (6.4) makes use of a faithful representation and is for practical purposes and applications. The second one (6.5) is independent of the choice of any representations. However this involves some other choices, but the definition will be shown to be independent of these choices.

The two will be reconciled in (6.7). Let $M \rightarrow Y$ be as in (6.1) and (6.2).

**Definition 6.4**

Let $\eta : G \hookrightarrow \text{GL}(n)$ be a faithful representation. An $\eta$-admissible pair $(G_t, M, E)$ on a modification $q : M \rightarrow T$ consists of a diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\phi_t} & M \\
q \downarrow & & \downarrow \\
T & \xrightarrow{\psi_t} & Y
\end{array}
\]  

(6.4.1)

where $\phi_t$ factors as $\phi_t : M \rightarrow M_\eta \subseteq M$ and such that the following hold:

- The group algebraic space $\mathcal{K}^{G}_{t, M}$ is obtained as a pull-back, i.e., $\mathcal{K}^{G}_{t, M} := \phi_t^*(\mathcal{K}^{G}_{\text{univ}})$.

Define the group space $\mathcal{K}^{\text{GL}(n)}_{t, M} := \phi_t^*(\mathcal{K}^{\text{GL}(n)}_{\text{univ}})$ and let $\eta : \mathcal{K}^{G}_{t, M} \hookrightarrow \mathcal{K}^{\text{GL}(n)}_{t, M}$ be the inclusion obtained by pulling back the inclusion (6.2.3).

- The $\mathcal{K}^{G}_{t, M}$-torsor $E$ on $M$ is such that $(\mathcal{K}^{\text{GL}(n)}_{t, M}, \eta^*(E))$ is isomorphic to the pull-back $\phi_t^*(\mathcal{K}^{\text{GL}(n)}_{\text{univ}}), \mathcal{P})$, where $\eta^*(E)$ is the $\mathcal{K}^{\text{GL}(n)}_{t, M}$-torsor obtained by extension of structure group via $\eta$.

**Definition 6.5**

Let $M \rightarrow T$ be a modification as in (6.1). A pair $(\mathcal{K}^{G}_{t, M}, E)$ is called admissible if for each $\alpha$, there is an admissible pair $(\mathcal{K}^{G}_{t, W[d_\alpha]}, \mathcal{P})$ on the standard model $W[d_\alpha]$ such that $\xi_\alpha^*(\mathcal{K}^{G}_{t, W[d_\alpha]}, \mathcal{P}) \simeq (\mathcal{K}^{G}_{t, M}, E) |_{M_\alpha}$.

The fact that the definition (6.5) is independent of the choice of the covering $\{T_\alpha\}$, is derived as a consequence from (6.7) below. Recall the notion of an effective degeneration of $C_A$ [33, page 527], i.e., it is a modification $M \rightarrow T$ as in (6.1) such that there is a morphism $\xi : T \rightarrow B[d]$ for a single $d$, and such that isomorphism $M \simeq \xi^*(W[1])$ is compatible with the morphism to $C_T$.

**Proposition 6.6**

Let $M \rightarrow T$ be a modification which is made effective by two morphisms $\xi_i : T \rightarrow B[d_i], i = 1, 2$ and arrows $\xi_i$. Suppose further that we have admissible pairs...
Proof. By (5.14), the faithful representation \( \eta : G \hookrightarrow \text{GL}(n) \), there is a unique \( \text{A-morphism } \psi : T \to Y \) and a corresponding morphism \( \phi : M \to \mathbb{M}_\eta \subset \mathbb{M} \) (with a diagram (6.4.1)), such that the pair \((\mathcal{M}_G, \mathcal{E}) \) is \( \eta \)-admissible (6.4).

By [33, Lemma 1.8], for each \( p \in T \) there is an étale neighbourhood \( T_o \) such that the isomorphism between \( (\xi^*_i(W[d_1]))_a \simeq (\xi^*_i(W[d_2]))_a \) on \( T_o \) is induced by a sequence of effective arrows [33, page 527], and this holds for each \( a \). As in the proof of [33, Lemma 1.8], we may assume that on \( T_o \), we have two morphisms \( \xi_i : T_o \to B[d] \) such that \( \xi_i(p) = 0 \in B[d] \), which induces the isomorphism via the \( \xi_i \)'s.

The assumption, \( \xi_1(p) = \xi_2(p) = 0 \in B[d] \), forces the effective arrow inducing the isomorphism between \( \xi^*_1(W[d]) \) and \( \xi^*_2(W[d]) \) on \( T_o \) to be the one that is induced by an automorphism of the fibre \( W[d]_0 \) which commutes with the canonical projection \( W[d]_0 \to C \). This is induced by an element of the group \( G[d+1] \) (4.1.1) by [33, Corollary 1.4]. By [33, Lemma 1.2], this therefore lifts to an action on \( W[d] \) preserving the configuration of the s.c.n divisor.

The isomorphism \( \xi^*_1(\mathcal{M}_G, \mathcal{E}_1) \simeq \xi^*_2(\mathcal{M}_G, \mathcal{E}_2) \) shows that the restricted to the single semistable curve \( C^{d+1} \simeq W[d]_0 \), the admissible pairs are isomorphic. Recall that the Kawamata coverings \( \tilde{W}_{r_1}[d] \to W[d] \) (with Galois group \( \mathfrak{G} \)) induced by the pairs \((\mathcal{M}_G, \mathcal{E}_i), i = 1, 2 \) are completely determined by the admissible pair at the central fibre, i.e., the local types are completely determined, which in turn determines the ramification data. Thus, we conclude that the Kawamata covers are isomorphic. By pulling back using \( \xi_i \), we get a finite flat cover \( \mathbb{M}_o \to \mathbb{M}_o \) together with an action of \( \mathfrak{G} \). This allows us to take Weil restrictions and invariants.

The isomorphism of the pairs \( \xi^*_1(\mathcal{M}_G, \mathcal{E}_1) \) and \( \xi^*_2(\mathcal{M}_G, \mathcal{E}_2) \) when restricted to \( \mathbb{M}_o \) is recovered as invariant push-forwards of isomorphism of pairs on \( \tilde{M}_o \). By taking associated vector bundles and invariant push-forwards, we can first realize the admissible vector bundles \( \xi^*_i(\mathcal{E}^{\text{par}}(k^n)) \) on \( \mathbb{M}_o \) as invariant push-forwards from \( \tilde{M}_o \). Then the isomorphism of the pairs induces an isomorphism of these associated parabolic bundles \( \xi^*_i(\mathcal{E}^{\text{par}}(k^n)) \) over \( \mathbb{M}_o \). These are isomorphic quasi-admissible vector bundles on the modification \( \mathbb{M}_o \). Thus, they induce the same morphisms from \( T_o \) to the universal space \( Y \) (by uniqueness). All in all, we can conclude that on \( \mathbb{M}_o \to T_o \), the restrictions of the morphisms \( \psi_1, \phi_1 \) coincide and hence they coincide everywhere on \( T \). The remaining statements are straightforward.

**Theorem 6.7 (Tannakian).** Let \( \mathbb{M} \to T \) be a modification as in (6.1). A pair \((\mathcal{M}_G, \mathcal{E}) \) is admissible (6.5) if and only if it is \( \eta \)-admissible (6.4) for any faithful \( \eta \). In particular, the definition (6.5) does not depend on the covering \{\( T_o \}\).

**Proof.** (6.4) implies (6.5) is easily seen. For the other direction, on each \( T_o \) we have unique morphisms \( \psi_\alpha, \phi_\alpha \) with the added properties. The intersection \( T_{o, \beta} := T_o \cap T_\beta \) satisfies the assumption in (6.6) and hence the uniqueness forces that \( \psi_\alpha, \phi_\alpha \) agree on the intersections and hence glue to give unique morphisms \( \psi, \phi \) which does the job.
7. The stack of Gieseker torsors

Using the admissible pairs constructed on modifications, we define the stacks of Gieseker torsors and study the basic properties.

DEFINITION 7.1

For a scheme $T$ over $A$, a Gieseker torsor on $T$ is a datum $(M, G_t M, E)$, consisting of a modification $M \to T$, and an admissible pair $(G_t M, E)$ on $M$ (6.5).

Two Gieseker torsors $(M_1, G_{t,1} M_1, E_1)$ and $(M_2, G_{t,2} M_2, E_2)$ on $C_T$, $j = 1, 2$ are called isomorphic if there exists an $A$-isomorphism $\delta : T_1 \to T_2$ and a diagram

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\epsilon} & M_2 \\
\downarrow & & \downarrow \\
T_1 & \xrightarrow{\delta} & T_2
\end{array}
$$

(7.1.1)

compatible with the tautological projections $M_j \to C_T$, $j = 1, 2$, an isomorphism

$$
\epsilon^* (G_{t,2} M_2, E_2) \cong (G_{t,1} M_1, E_1)
$$

(7.1.2)
of admissible pairs on $M_1$.

DEFINITION 7.2

Let $Gies_G(C_A)$ be the category over $\text{Sch}/A$, whose objects are Gieseker torsors $(M, G_{t,M}, E)$. The functor $\mathfrak{f} : Gies_G(C_A) \to \text{Sch}/A$ which sends $(M, G_{t,M}, E) \mapsto T$ realizes it as a fibered category.

An arrow between two objects $\Upsilon_1 = (M, G_{t,M}, E)$ and $\Upsilon_2 = (M', G_{t,M'}, E')$ over $T$ and $T'$ consists of (1) an $A$-morphism $T \to T'$, (2) an isomorphism of modifications $M \to M' \times_{T'} T$ and (3) an isomorphism over $T$ of Gieseker torsors $(M, E)$ and $(M' \times_{T'} T, E' \times_{T'} T)$.

Since pull-backs of modifications (resp. group algebraic spaces, torsors) are modifications (resp. group algebraic spaces, torsors), and arrows between two objects are as defined above and are fiber diagrams, the category $Gies_G(C_A)$ is fibered in groupoids under $\mathfrak{f}$. We have the following straightforward result.

PROPOSITION 7.3

The category $Gies_G(C_A)$ is a stack.

Proof. It suffices to show the following:

(1) For any $T \in \text{Sch}/A$ and two objects $\Upsilon_1, \Upsilon_2 \in Gies_G(C_A)(T)$, the functor

$$
\text{Isom}_T (\Upsilon_1, \Upsilon_2) : \text{Sch}_T \to \text{Sets}
$$

(7.3.1)

which associates to any morphism $\phi : T' \to T$ the set of isomorphisms in $Gies_G(C_A)$ between $\phi^* (\Upsilon_1)$ and $\phi^* (\Upsilon_2)$, is a sheaf in the étale topology.
(2) Effective descent. Let \( \{ T_i \to T \} \) be a covering of \( T \) in the étale topology. Let \( \Upsilon_i \in \text{Gies}_G(C_A)(T_i) \) and let \( \phi_{ij} : \Upsilon_i|T_i \times_\mathcal{T} T_j \to \Upsilon_j|T_i \times_\mathcal{T} T_j \) be isomorphisms in \( \text{Gies}_G(C_A)(T_i \times_{\mathcal{T}} T_j) \) satisfying the cocycle condition. Then there is an \( \Upsilon \in \text{Gies}_G(C_A)(T) \) with isomorphisms \( \psi_i : \Upsilon|T_i \to \Upsilon_i \) so that

\[
\phi_{ij} = \psi_i \circ \psi_j^{-1}.
\]  

(7.3.2)

Each object \( \Upsilon = (\mathcal{M}, \mathfrak{g}^G_{t,M}, \mathcal{E}) \) consists of three components. Modifications or expanded degenerations form a stack [33, 1.10, page 531] or [27, Proposition 3.16].

For the second item, this follows since the sheaf property and effective descent is automatic for morphisms.

The sheaf property of the third component, namely the torsor \( \mathcal{E} \) is immediate since isomorphisms of \( G_t,M \)-torsors \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) is given by a section of \( (\mathfrak{g}^G \times \mathfrak{g}^G_\Delta) \). Effective descent of torsors holds in the category of algebraic spaces, and the action maps descend by [14, Theorem 6, Section 6.1]. The admissibility of the descended pair is immediate since it holds on each \( T_i \).

**Theorem 7.4.** When \( G = GL(n) \), we have isomorphisms:

\[
\text{Gies}_{GL(n)}(C_A) \simeq \text{GVB}_n(C_A).
\]  

(7.4.1)

In particular, \( \text{Gies}_{GL(n)}(C_A) \) is an algebraic \( A \)-stack, locally of finite type.

**Proof.** We show that there is a canonical functor \( \epsilon : \text{Gies}_{GL(n)}(C_A) \to \text{GVB}_n(C_A) \) defined over \( \text{Sch}/\mathcal{A} \), which is an equivalence of fibered categories. Let \( T \in \text{Sch}/\mathcal{A} \). Let \( (\mathcal{M}, \mathfrak{g}^{GL(n)}_{t,M}, \mathcal{E}) \in \text{Gies}_{GL(n)}(C_A)(T) \). By the definition of an admissible pair (6.4), we have the parabolically-associated vector bundle \( \mathcal{E}^{\text{par}}(k^n) := \phi^*_t(\mathcal{E}^{\text{par}}(k^n)) \) as in (6.2.2), and clearly \( \mathcal{E}^{\text{par}}(k^n) \) is a quasi-admissible vector bundle. Define

\[
\epsilon(\mathcal{M}, \mathfrak{g}^{GL(n)}_{t,M}, \mathcal{E}) := (\mathcal{M}, \mathcal{E}^{\text{par}}(k^n)).
\]  

(7.4.2)

The functor \( \epsilon \) is essentially surjective for each \( T \in \text{Sch}/\mathcal{A} \). Let \( \mathcal{V} \) be an admissible bundle on \( \mathcal{M} \) of rank \( n \). The universal property of \( (\mathcal{M}, \mathcal{V}) \) shows that there is a morphism \( \phi : \mathcal{M} \to \mathcal{M} \) such that \( \phi^*(\mathcal{V}) = \mathcal{V} \). We get back \( (\mathfrak{g}^{GL(n)}_{t,M}, \mathcal{E}) \) by pulling back \( (\mathfrak{g}^{GL(n)}_{t,M}, \mathcal{P}) \). That \( \epsilon \) is fully-faithful is immediate from the definitions of isomorphisms of the objects.

**Theorem 7.5.** The stack \( \text{Gies}_G(C_A) \) is an algebraic \( C \)-stack, locally of finite type. For the fixed nodal curve \( (C, c) \) over \( \mathcal{C} \), \( \text{Gies}_G(C) \) is an algebraic \( \mathcal{C} \)-stack, locally of finite type.

**Proof.** By the definition of Gieseker torsors, we have a morphism:

\[
\eta_* : \text{Gies}_G(C_A) \to \text{Gies}_{GL(n)}(C_A),
\]  

(7.5.1)

\[
(\mathcal{M}, \mathfrak{g}^G_{t,M}, \mathcal{E}) \mapsto (\mathcal{M}, \mathfrak{g}^{GL(n)}_{t,M}, \eta_*(\mathcal{E})),
\]  

(7.5.2)

where \( \eta : \mathfrak{g}^G_{t,M} \to \mathfrak{g}^{GL(n)}_{t,M} \) is the canonical inclusion.
This morphism of stacks is representable, locally of finite presentation. To see this we follow [12]; let \( T \) be a \( \mathcal{A} \)-scheme and let \( P \) be an admissible \( \mathcal{X}_{t, M}^{\GL(n)} \)-torsor on a modification \( M \to T \) (6.4).

The quotient \( \mathcal{X}_{t, M}^{\GL(n)}/\mathcal{X}_{t, M}^{G} \) exists as an algebraic space with a \( \mathcal{X}_{t, M}^{\GL(n)} \) action. We identify the associated space \( P(\mathcal{X}_{t, M}^{\GL(n)}/\mathcal{X}_{t, M}^{G}) \) with \( P/\mathcal{X}_{t, M}^{G} \). Let \( q : M \to T \) be the arrow defining the modification. Then we have a 2-cartesian diagram of \( A \)-stacks

\[
\begin{array}{ccc}
q_{*}(P/\mathcal{X}_{t, M}^{G}) & \to & T \\
\downarrow & & \downarrow \\
\text{Gies}_G(C_A) & \to & \text{Gies}_{\GL(n)}(C_A)
\end{array}
\]

(7.5.3)

By [41, Corollary 2.17], \( P/\mathcal{X}_{t, M}^{G} \) is an algebraic space of finite presentation. Thus, by using the theory of Hilbert schemes for algebraic spaces as in [5, Section 6], we see that \( q_{*}(P/\mathcal{X}_{t, M}^{G}) \) is also an algebraic \( T \)-space of finite presentation. Hence the morphism (7.5.1) is locally of finite presentation.

By (7.4), the stack \( \text{Gies}_{\GL(n)}(C_A) \) is an algebraic \( A \)-stack locally of finite type. Hence by (7.5.3), we conclude that \( \text{Gies}_G(C_A) \) is an algebraic stack locally of finite type over \( A \). It is immediate that \( \text{Gies}_G(C) \) is an algebraic \( k \)-stack and also locally of finite type being the closed fibre of \( \text{Gies}_G(C_A) \).

\[\Box\]

### 7.1 Deformations of Gieseker torsors

We follow [24] and [39, Appendix]. We work in the setting of the diagram (6.4.1) with \( \eta : G \hookrightarrow \GL(n) \). Let \( \mathcal{M} \to \mathcal{Y} \) be the universal modification (6.1) and \( \mathcal{M}_{t} \) as in (6.2). Let \( \psi_{t} : T \to \mathcal{Y} \) be a \( \mathcal{Y} \)-scheme and \( q : M \to T \) a modification such that the morphism \( M \to \mathcal{M} \) factors as \( M \to \mathcal{M}_{t} \subset \mathcal{M} \) as in (6.4.1). Let \( \phi_{t} : M \to \mathcal{M}_{t} \) be the induced morphism.

Let \( P_{M} = \phi_{t}^{*}(\mathcal{P}) \) which is therefore a \( \mathcal{X}_{t, M}^{\GL(n)} \)-torsor on \( M \). Further, we have the group algebraic space \( \mathcal{X}_{t, M}^{G} \) and an inclusion of group algebraic spaces \( \mathcal{X}_{t, M}^{G} \hookrightarrow \mathcal{X}_{t, M}^{\GL(n)} \) over \( M \).

**DEFINITION 7.6**

Define the functor \( \mathcal{G}_{A}^{G} : \text{Sch}_{\mathcal{Y}} \to \text{Sets} \),

\[
T \mapsto \{(P_{M}, \zeta) | \zeta \in \Gamma(M, (P_{M}(\mathcal{X}_{t, M}^{\GL(n)})/\mathcal{X}_{t, M}^{G}))\}
\]

(7.6.1)

i.e. \( \mathcal{G}_{A}^{G}(T) \) consists of isomorphism classes of pairs \((P_{M}, \zeta)\) on the modification \( M \to T \), where \( \zeta \) is a reduction of structure group of \( P_{M} \) to \( \mathcal{X}_{t, M}^{G} \).

We show that this functor is representable by a \( \mathcal{Y} \)-scheme, following [40, page 424–425], i.e., by embedding the homogeneous space \( \mathcal{X}_{t, M}^{\GL(n)}/\mathcal{X}_{t, M}^{G} \) in a vector bundle over \( M \). By Chevalley’s theorem on semi-invariants we obtain an embedding

\[
\GL(n)/G \hookrightarrow W
\]

(7.6.2)

in a \( \GL(n) \)-module \( W \).
By following (6.1) and using the GL(n)-module $W$, it is straightforward to see that we can take the associated vector bundles $E'_\alpha(W)$ on $\overline{\mathbb{M}}_\alpha$ with its natural equivariant structure and define the invariant push-forward $\mathcal{W}_\alpha := (\text{Inv} \circ \kappa_*)(E'_\alpha(W))$. These glue up to give a vector bundle $\mathcal{W} := \mathcal{P}\text{par}(W)$ on $\overline{\mathbb{M}}$. Since $\mathcal{P}$ is a $\mathfrak{X}_{\text{univ}}^{\text{GL}(n)}$-torsor, by restricting to $\overline{\mathbb{M}}_{\eta}$ (6.2), we can consider the algebraic space $\mathcal{P}\left(\mathfrak{X}_{\text{univ}}^{\text{GL}(n)}/\mathfrak{X}_{\text{univ}}^{\text{GL}(n)}\right)$ over $\overline{\mathbb{M}}_{\eta}$.

Restricting $\mathcal{W}$ to $\overline{\mathbb{M}}_{\eta} \subseteq \overline{\mathbb{M}}$, we get an embedding

$$\mathcal{P}\left(\mathfrak{X}_{\text{univ}}^{\text{GL}(n)}/\mathfrak{X}_{\text{univ}}^{\text{GL}(n)}\right) \hookrightarrow \mathcal{W}. \quad (7.6.3)$$

Pulling back using the morphism $\phi_1: \mathbb{M} \to \overline{\mathbb{M}}_{\eta}$, we see that we have the embedding

$$P_M\left(\mathfrak{X}_{t,M}^{\text{GL}(n)}/\mathfrak{X}_{t,M}^{\text{GL}(n)}\right) \hookrightarrow \mathcal{W}_M, \quad (7.6.4)$$

where $\mathcal{W}_M := \phi_1^*(\mathcal{W})$.

In other words, we can realize the functor $\mathcal{F}_A^G$ as a closed subfunctor of the functor $T \mapsto H^0(\mathcal{W}_T)$. By [39, Proposition 8] and (A1.4), $\mathcal{Y}$ is a reduced scheme and hence the functor $T \mapsto H^0(\mathcal{W}_T)$ is representable by a linear scheme; therefore, there exists a $\mathcal{Y}$-scheme $\mathcal{Y}^G$ which represents $\mathcal{F}_A^G$.

We can also describe the $T$-points of $\mathcal{F}_A^G(T)$ as $[(\mathbb{M}, \epsilon, \mathfrak{X}_{t,M}^{\text{GL}(n)}, \mathcal{E})]$, where $(\mathbb{M}, \epsilon)$ are as in (A1.4), and $\mathcal{E}$ is a $\mathfrak{X}_{t,M}^{\text{GL}(n)}$-torsor. Equivalently, one could describe it as $[(\mathbb{M}, \epsilon, \eta_*(\mathcal{E})), \zeta_T]$, where $(\mathfrak{X}_{t,M}^{\text{GL}(n)}, \eta_*(\mathcal{E}))$ is an admissible pair and $\zeta_T$ is a reduction of structure group to the subgroup scheme $\mathfrak{X}_{t,M}^{\text{GL}(n)}$.

Notice that $\mathcal{E}_{\text{par}}(k^n)$ is a quasi-admissible vector bundle. Let $\mathcal{V}_T := \mathcal{E}_{\text{par}}(k^n)$ (7.6.5) be the vector bundle on $\mathbb{M}$ which is parabolically-associated to the torsor $\eta_*(\mathcal{E})$. Thus, giving the representation $\eta: G \hookrightarrow \text{GL}(n)$ also induces a morphism $\mathcal{F}_A^G \to \mathcal{F}_N^G$ (more precisely, by taking the associated vector bundle plus a twisting of the vector bundles by a positive $m$ to ensure that the first cohomology vanishes and the sections generate the bundle, see [39, page 176, Remark 4(4)])

Let $\mathcal{F}_A'$ be the functor defined as

$$\mathcal{F}_A' : \text{Sch}_Y \to \text{Sets}, \quad (7.6.6)$$

$$T \mapsto \mathbb{M} \quad (7.6.7)$$

such that $\mathbb{M} \to C \times_A T$ is a modification.

Thus, we have a morphism from $\mathcal{F}_N^G$ to $\mathcal{F}_A'$ obtained by forgetting the condition (1) in Definition A1.2 namely, the embeddings into the Grassmannians (see [39, Appendix, page 197]). Composing with the morphism $\mathcal{F}_A^G \to \mathcal{F}_N^G$ we have the induced forget morphism

$$\mathcal{F}_A^G \to \mathcal{F}_A', \quad [(\mathbb{M}, \epsilon, \mathfrak{X}_{t,M}^{\text{GL}(n)}, \mathcal{E})] \mapsto \mathbb{M}. \quad (7.6.8)$$

The functors $\mathcal{F}_A^G$, $\mathcal{F}_A'$ are defined with a fixed choice of the fibered surface $C_A \to A$. Further, the functor $\mathcal{F}_A'$ defined above parametrizes semistable curves with a fixed stable model, being the irreducible nodal curve $(C, c)$ with a single node. Gieseker [24, page 183] (see also [39, Appendix]) shows that the canonical map $B[d] \to \mathcal{F}_A'$ defined by the point $W[d] \in \mathcal{F}_A'(B[d])$ is formally smooth.
Theorem 7.7. The algebraic stack $\text{Gies}_G(C_A)$ is regular and flat over $A$; further, $\text{Gies}_G(C) \subset \text{Gies}_G(C_A)$ is a divisor with normal crossings. More precisely, the morphism (7.6.8) is formally smooth.

Proof. Let $T$ be the spectrum of an Artin local ring, and $T_0 \subset T$ the subscheme defined by an ideal of dimension 1. Let $M \in \mathcal{G}_A(T)$ be such that the restriction $M_0 \in \mathcal{G}_A(T_0)$ can be lifted to an element of $\mathcal{G}_A^G(T_0)$, then we need to show that $M$ itself can be lifted to an element of $\mathcal{G}_A^G(T)$. Let $M$ be defined by the family of curves $M \to T$, and by the modification $M \to C \times_A T$.

The lifting of the family $M_0 \to T_0$ to $M \to T$ comes with information which we require: there is a morphism $\phi_0 : T_0 \to B[d]$ such that pull-backs by $\phi_0$ of the versal families $W[d] \to B[d]$ and $W[d] \to C_B[d]$ coincide with the datum given by the point $M_0 \in \mathcal{G}_A^G(T_0)$. Gieseker then showed that we have a diagram

\[
\begin{array}{ccc}
T_0 & \xrightarrow{\phi_0} & T \\
\downarrow & & \downarrow \phi \\
B[d] & \xrightarrow{\phi_0} & B[d]
\end{array}
\] (7.7.1)

such that the pull-backs of $W[d] \to B[d]$ and $W[d] \to C_B[d]$ give the family $M \to T$ and the point $M \in \mathcal{G}_A^G(T)$.

The lifting of $M_0$ to an element of $\mathcal{G}_A^G(T_0)$ defines an admissible pair $(\mathcal{X}_{M_0}^G, o\mathcal{T}_0)$ on the restriction $M_0$ of $M$ to $T_0$. The problem is

1. to extend the pair $(\mathcal{X}_{M_0}^G, o\mathcal{T}_0)$ to an admissible pair $(\mathcal{X}_M^G, o\mathcal{T})$ on $M$,
2. to lift the morphism $M_0 \to T_0 \times W(N, n)$ to a morphism $M \to T \times W(N, n)$.

1. By versality, the group algebraic space $\mathcal{X}_{M_0}^G$ is isomorphic to the pull-back of a group algebraic space $\mathcal{X}_{T_0}^G, W[d]$ on $W[d]$ by the morphism $M_0 \to W[\{\text{cl}\}]$ induced by $\phi_0$. The diagram (7.7.1) thus gives the group algebraic space $\mathcal{X}_M^G$ on $M$ extending $\mathcal{X}_{M_0}^G$, namely the pull-back of $\mathcal{X}_{T_0}^G, W[d]$ by the morphism $M_0 \to W[\{\text{cl}\}]$ induced by $\phi$.

Let $E$ be the restriction of $\mathcal{T}_0$ to the closed fibre $C(d)$ of $M_0 \to T_0$. Let $\mathcal{X}_M^G$ be the restriction of $\mathcal{X}_{M_0}^G$ to $C(d)$. It is standard that the obstruction to lifting $\mathcal{T}_0$ to $\mathcal{T}$ is simply the group $H^2(C(d), E(\text{Lie}(\mathcal{X}_M^G)))$ which vanishes since we are in dimension 1.

2. For proving the second item, by the definition of $W(N, n)$ (A1.1) and by what has been discussed above regarding the versal property, it remains to extend the sections of the vector bundle $\mathcal{Y}_T(m)$ $(m \gg 0)$ which defines the given map $M_0 \to T_0 \times \text{Grass}(N, n)$ to the sections of $\mathcal{V}_T(m)$ so as to define the lift $M \to T \times \text{Grass}(N, n)$. The second item is therefore possible since the obstruction to lifting of sections lies in $H^1(M_T, \mathcal{V}_T(m))$ and this group vanishes by (A1.5)). Thus we conclude that the morphism (7.6.8) is formally smooth. It is shown in [24] that there is a formally smooth morphism from the versal space $B[d]$ to the functor $\mathcal{G}_A''$.

We deduce (using 4.1) that the scheme $\mathcal{Y}_A$ which represents $\mathcal{G}_A^G$ has all the stated properties. One then concludes (following [27, Proposition 3.24]) that the stack $\text{Gies}_G(C_A)$ has all the stated properties. \qed
7.2 Some remarks on a weak properness

Let \( C_{d,A} := C_A \times_{\text{Spec } k[t]} \text{Spec } k[t] \) via the map \( t \mapsto t^d \) (2.2.1), and let \( E_L \) be a \( G \)-torsor on the generic fibre \( C_{d,L} \) of \( C_{d,A} \). It is not hard to see that the \( G \)-bundle extends to \( C_{d,A} - c \). Locally, we have a \( G \)-bundle on \( N_d - c \). By going to \( N_0 = \text{Spec } k[[u,v]] \) with coordinates \( u, v \), and using a Hartogs like argument, we can extend the \( G \)-torsor to a \((\mu_d, G)\)-torsor of local type \( \tau \) or equivalently a \( \mathcal{E}_D(G) \)-torsor (3.0.2). Then it is straightforward to obtain a group algebraic space \( \mathbb{G}_T \) on the minimal desingularization \( S^{(d)} \to C_{d,A} \) and an admissible pair \((\mathbb{G}_T, \mathcal{E})\) such that over the generic fibre we have an isomorphism \( \mathcal{E}_L \simeq E_L \). As we have seen earlier, the group space \((\mathbb{G}_T, \mathcal{E})\) is non-trivial parahoric at the generic points of at most \( \ell = \text{rank}(G) \) number of projective lines in the exceptional divisor. The closed fibre of the moduli stack is describable in terms of \( \text{laced parahoric torsors} \) on \( \tilde{C} \) with parahoric structures at two points (12.4). To work in families, we fix a faithful representation \( \eta : G \hookrightarrow \text{GL}(V) \), where \( \text{dim}(V) = n \) then the associated \( \text{parabolic bundle}, \mathcal{E}_{\text{par}}(V) \) on \( S^{(d)} \) which is quasi-admissible is non-trivial in at most \( \text{dim}(V) \) rational curves on the exceptional divisor. Thus, the number \( m(G) \), being the minimal dimension of a faithful representation of \( G \), gives an upper bound for \( d \). More precisely, the stack of Gieseker torsors can be obtained by taking torsors on semistable curves \( C^{(d)} \) with \( d \) bounded above by \( m(G) \).

8. The closed fibre of the stack and examples

By Bruhat–Tits theory, for each facet \( a \) in each apartment \( \mathcal{A}_T \) of the Bruhat–Tits building of \( G(k((t))) \), there is a smooth group scheme \( P_a \) over \( k[[t]] \) with connected fibers whose generic fiber is \( G \times_{\text{Spec } k} \text{Spec } k((t)) \). We call such \( P_a \) a Bruhat–Tits group scheme. Let \( P_{\mathcal{A}}(K) := P_a(k[[t]]) \). We call \( P_{\mathcal{A}} \) a parahoric subgroup of \( G(k((t))) \). The conjugacy classes of parahoric subgroups of \( G(k((t))) \) are classified by proper subsets of the nodes of the extended Dynkin diagram of \( G \) or the facets of the Weyl alcove \( a \subset \mathcal{A}_T \). Let \( P_\mathcal{X} \) denote the group schemes associated to the maximal parahoric subgroups which are indexed by the vertices \( \mathcal{X} \) of \( a \). We summarize two results, the details of which is work in progress.

**Theorem 8.1.**

(1) The closed fibre \( \text{Gies}_G(C) \) of \( \text{Gies}_G(C_A) \) is a divisor with simple normal crossing singularities. It has \( \ell + 1 \) irreducible smooth components \( \mathcal{G}_j \) indexed by the vertices of the extended Dynkin diagram.

(2) Let \( P(\mathcal{X}) \) be the parahoric group scheme on the smooth projective curve \( \tilde{C} \) which restricts to the maximal parahoric \( P_\mathcal{X} \) at the two marked points. In each component \( \mathcal{G}_j \), the open locus of Gieseker torsors can be identified with \( P(\mathcal{X}) \)-torsors on \( \tilde{C} \) for varying vertices \( \mathcal{X} \).

(3) The minimal stratum will be torsors under \( P_I \) on \( \tilde{C} \) defined by the Iwahori group scheme at the two marked points on \( \tilde{C} \).

**Theorem 8.2.**

(1) Let \( Q_\mathcal{X} \) be the smooth group scheme (maximal parahoric) on the nodal curve \( (C, c) \) obtained by identifying the closed fibres of the group scheme \( P(\mathcal{X}) \) on \( \tilde{C} \). Then the stack \( \text{Bun}_C^+(Q_\mathcal{X}) \) of laced \( P(\mathcal{X}) \) torsors (12.4) (12.3) on \( \tilde{C} \) is a regular Artin stack.
(2) This stack is isomorphic to the stack of admissible torsors on $C^{(d)}$ coming from representations $\rho : \mu_d \to G$ which give the maximal parahoric.

(3) The dimension of the stacks $\text{Bun}_{\mathbb{L}}^L(C)_{\mathbb{Q}_v}$ and $\text{Bun}_C(G)$ coincide and hence these constitute the components of the normal crossing divisor.

Let $e(\theta_\alpha)$ be as in [10, 7.2.1] where $\theta_\alpha = v_\alpha$ are the vertices of the Weyl alcove. Recall $e(\theta_\alpha) := 2 \cdot (\dim_C(G/P_\alpha) - \mu(\alpha))$ (8.2.1) where $P_\alpha$ is the maximal parabolic subgroup of $G$ associated to $\alpha$ and

$$\mu(\alpha) = \# \left\{ r \in R^+ \left| r = c_\alpha \cdot \alpha + \sum_{\beta \neq \alpha} x_\beta \cdot \beta \right. \right\}. \quad (8.2.2)$$

Let $L_\alpha$ be the Levi subgroup of the closed fibre of the Bruhat–Tits group scheme $P_v$. Recall that $L_\alpha$ are all semisimple.

**Lemma 8.3.** We have the following relation: for each simple root $\alpha$,

$$\dim(G) = e(\theta_\alpha) + \dim(L_\alpha) \quad (8.3.1)$$

and hence

$$\dim(\text{Bun}_C^\mathbb{L}(\mathbb{Q}_v)) = \dim(G)(g - 1). \quad (8.3.2)$$

**Proof.** This is computational and I have used the tables. I do not see any general argument for this. □

**Remark 8.4.** The group scheme $\mathbb{Q}_v$ on the irreducible nodal curve will be non-reductive for the case of non-hyperspecial $v$, while the hyperspecial cases will be semisimple group schemes which are not globally split. The groups of type $A_n$ will not give the exotic examples since all maximal parahorics are hyperspecial.

**Remark 8.5 (A bit imprecise!).** The component $\mathcal{G}_0$ consists of torsors on $C^{(d)}$ which are $G$-bundles with parabolic structures at the nodes. In terms of the surface $S^{(d)}$, the admissible pairs $(\mathbb{R}^G_{r,S}, \mathcal{E})$ are such that the representation $\rho : \mu_d \to G$ is of type $\tau$. Consider the set of Giseker torsors $\mathcal{E}$ on $C^{(d)}$ for $1 \leq d \leq \ell$ such that $\mathcal{E} \mid_{\tilde{C}} \simeq \tilde{C} \times G$. This set is bijective to the stack theoretic compactification of $G$ constructed by Marten and Thaddeus [36]. By [36, page 94] and [50, Remark 5], it follows that stable $G$-bundle chains on $C^{(d)}$ correspond to admissible torsors $(\mathbb{R}^G_{\tau,S}, \mathcal{E})$ when $G = GL(n)$. One can assume firstly that $d \leq \ell$ and then with a bit more work, one can check that the Martens–Thaddeus stack is a closed substack of the principal component $\mathcal{G}_0$. It seems likely that the principal component is isomorphic to a bundle on $\text{Bun}_G(\tilde{C})$ with fibres the Martens–Thaddeus stack for $G$.

**8.1 Example: $G = SL(2)$**

Let $\eta : SL(2) \hookrightarrow GL(2)$ be the standard inclusion. We list the basic representation types.

Let $\rho_d : \mu_d \to SL(2)$ be given by $\xi_d \mapsto \begin{pmatrix} \xi_d & 0 \\ 0 & \xi_d^{-1} \end{pmatrix}$, where $\xi = e^{2\pi i/d}$. 
\[ d = 2: \text{The representation } \xi_2 \mapsto \begin{pmatrix} \xi_2 & 0 \\ 0 & \xi_2 \end{pmatrix}. \] This is central and hence the group scheme \( \mathfrak{m}^G_{\Gamma, \mathcal{N}(1)} \) on \( \mathcal{N}(1) \), restricted to \( \mathcal{C}^{(1)} \) is obtained by gluing a parahoric group scheme \( \mathcal{P} \) on \( \tilde{C} \) (which is the maximal parahoric group scheme \( \mathcal{P}_\mathcal{Y} \) near the two marked points) with the constant group scheme \( \mathcal{P}_{\mathcal{Y}, \mathcal{O}} \times \mathbb{P}^1 \) on the single rational component, where \( \mathcal{P}_{\mathcal{Y}, \mathcal{O}} \) is the closed fibre of the parahoric group scheme \( \mathcal{P}_\mathcal{Y} \). Being hyperspecial, in these cases, \( \mathcal{P}_{\mathcal{Y}, \mathcal{O}} \) is isomorphic to \( \text{SL}(2) \).

Torsors are obtained from equivariant torsors \( E \) on \( A^2 \) for the \( \mu_2 \)-action by the process \( \text{Inv} \circ f_\ast (E) \). These in turn give laced torsors on the normalization \( \tilde{\mathcal{C}} \). Since the parahoric is hyperspecial, the lacing is simply an isomorphism of the fibres of the torsors on the normalization which in turn give an object in \( \text{Bun}_C^L (\mathbb{Q}_\mathcal{Y}) \).

Remark 8.6. If the parahoric is non-hyperspecial, the identification of the fibres is via the centralizer of the image of \( \rho \) and this translates as identification of the associated \( L_\alpha \)-torsor and hence an element in the adjoint group of \( L_\alpha \). Since the \( L_\alpha \)'s are semisimple the dimension is that of \( L_\alpha \).

\[ d = 3: \text{The representation } \xi_3 \mapsto \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3^2 \end{pmatrix}. \] This case gives the bundles on the surface \( \mathcal{N}(2) \) with closed fibre \( \mathcal{F}^{(2)} \) having two \( \mathbb{P}^1 \)'s. The simple root \( \alpha \) on the maximal torus of \( \text{SL}(2) \) sends \( \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3^2 \end{pmatrix} \to \xi_3^2 \). The induced character \( \chi \) of \( \Gamma_3 \) is \( \xi_3 \mapsto \xi_3^2 \) and this corresponds to second \( \mathbb{P}^1 \) on \( C^{(2)} \). In other words, the group scheme \( \mathfrak{m}^G_{\Gamma, \mathcal{N}(2)} \) has non-trivial parahoric structure on \( E_2 \) and is the constant group scheme on \( E_1 \). The representation \( \xi_3 \mapsto \begin{pmatrix} \xi_3^2 & 0 \\ 0 & \xi_3 \end{pmatrix} \) will produce the other case.

The associated rank 2 vector bundle has \( \mathcal{O} \oplus \mathcal{O}(1) \) on each \( \mathbb{P}^1 \), where the \( \mathcal{O}(1) \) on the first \( \mathbb{P}^1 \) is associated to the character \( \xi_3 \mapsto \xi_3 \) and on the second to the character \( \xi_3 \mapsto \xi_3^2 \) using the McKay correspondence. Torsors on the closed fibre correspond to torsors on \( \mathcal{C} \) for the group scheme \( \mathcal{P}_L \), the Iwahori at the two marked points together with the lacing data. The flags are full flags and the lacing data disappear since the Levi is the maximal torus and ‘modulo centre’ is trivial.

So the dimension of this stratum is: dimension of the space of torsors for the group scheme which is the Iwahori at two points on \( \tilde{\mathcal{C}} \) = \( \dim(G)(g - 2) + 2\dim(G/B) = \dim(G)(g - 1) - \ell \), where \( \ell = n - 1 \) in this case, i.e., the stratum is of codimension \( \ell \).

In the case of \( \text{SL}(2) \), the closed fibre of the stack is the union of two smooth components, which meet at the last stratum of the torsors for the group scheme on \( \mathcal{C} \) which is the Iwahori at the two points. The smoothness of the components can also be deduced by deformation arguments. The miniversal space for a Gieseker torsor \( (C^{(d)}, \mathfrak{m}^G_{\Gamma}, \mathcal{O}) \) will be such that \( d = \ell \) (in the Iwahori situation) and in the case of \( \text{SL}(2) \) it will be \( C^{(1)} \), i.e. with a single \( \mathbb{P}^1 \). Hence, the number of components meeting the Iwahori-type bundles is 2, corresponding to the two nodes on \( C^{(1)} \).

9. On Mumford’s toroidal realization of buildings

With notations as before, we work with the split group scheme \( G_A = G \times A \) and \( T_A \subset G_A \) a split torus over \( A \), i.e., \( T_A \simeq \mathbb{G}_m^\ell \). In [31], towards the very end, Mumford gives a
beautiful construction of the geometric realization of both the absolute and the relative case of Tits buildings via toroidal embeddings. We will talk of the relative case alone here.

The choice of \( T \) entails a choice of an apartment \( \mathcal{A}_T \) and the choice of the root system entails a choice of an origin in \( \mathcal{A}_T \). Let \( \mathcal{R} = \mathcal{R}^+ \cup \mathcal{R}^- \) be the decomposition of the roots \( \mathcal{R} \) into positive and negative roots. An alcove is given by

\[
a := \{ x \in \mathcal{A}_T \mid 0 \leq (\alpha, x) \leq 1, \alpha \in \mathcal{R}^+ \}. \tag{9.0.1}
\]

The alcoves give the top dimensional simplices of the polyhedral decomposition of \( \mathcal{A}_T \) in terms of the affine hyperplanes. Define \( \sigma \subset \mathcal{A}_T \times \mathbb{R} \) to be the cone over \( a \times (1) \). This gives an affine torus embedding \( T_A \subset T_\sigma \). Hence we get the fibre bundle \( G_A \times^{T_A} T_\sigma \) associated to the principal \( T_A \)-bundle \( G_A \to G_A/T_A \). On the generic fibre we have the identification \( T_K = (T_\sigma)_K \) and hence \( G_K = (G_A \times^{T_A} T_\sigma)_K \). Mumford then defines the relative embedding

\[
\tilde{G}_A := \bigcup_{x \in G(k(t))} (G_A \times^{T_A} T_\sigma) \cdot x, \tag{9.0.2}
\]

where notation for the action of \( G(k(t)) \) on \( G_A \times^{T_A} T_\sigma \) stands for embeddings of \( G_K \) by a translation by \( x \). These can be suitably glued to get a separated scheme over \( A \) (see [31, 206]).

The salient feature of the toroidal embedding \( G_A \subset \tilde{G}_A \) is that \( G_K = \tilde{G}_K \), and for each \( x \in G(K) \), the right multiplication by \( x \) extends to give an automorphism of \( \tilde{G}_A \). Finally, the strata of \( \tilde{G}_A \setminus G_A \) correspond precisely to the parahoric subgroups of \( G(K) \). This bijection extends to an isomorphism of the graph of the embedding \( G_A \subset \tilde{G}_A \) with the Bruhat–Tits building of \( G_A \). The aim of the present section is to give one point of contact between this construction and the stack Giesek(C_A) constructed earlier.

Let \( (C^{(\ell)}, x_G^{(\ell)}, s) \) be a Gieseker torsor, where \( \tau \) is the type of \( \rho : \mu_\ell \to G \) which is such that all the characters of \( \mu_\ell \) occur precisely once in \( \rho \). With the choice of the maximal torus \( T \subset G \) and the root system \( \mathcal{R} \) we see that \( \tau \) gives a point \( \theta_{\tau} \in a \). The group scheme \( x_G^{(\ell)} \) when restricted to the normalization \( \tilde{C} \) has the property that in the analytic neighbourhood of both the points \( c_1, c_2 \), the group scheme is the Iwahori group scheme. The Iwahori structure is a consequence of the distribution of characters of \( \mu_\ell \) in \( \rho \).

The Gieseker torsor \( (C^{(\ell)}, x_G^{(\ell)}, s) \) gives a point of the scheme \( Y^G(7.1) \). Furthermore, in an étale neighbourhood of this point we have a morphism to \( B[\ell] \). Note that the standard model \( W[\ell] \) in our setting is such that \( B[\ell] \simeq \mathbb{A}^{\ell+1} \) but it can be defined over the affine space \( \mathbb{A}^{\ell+1} \). We work in the latter setting here.

The base \( \mathbb{A}^{\ell+1} \) of the standard model \( W[\ell] \to \mathbb{A}^{\ell+1} \) is an affine toric variety and as a toric variety over \( C \) we can identify it with \( T_\sigma \). The big cell in \( G_A \) gives an open subset where the principal \( T_A \)-bundle \( G_A \to G_A/T_A \) is a product. Thus, in an open subset we can identify the associated fibre space \( G_A \times^{T_A} T_\sigma \) as a \( C \)-scheme with \( U^- \times \mathbb{A}^{\ell+1} \times U^+ \). The family \( W[\ell] \) pulled-back by the projection thus gives a modification on the open subset \( U^- \times \mathbb{A}^{\ell+1} \times U^+ \). The Gieseker torsor \( (C^{(\ell)}, x_G^{(\ell)}, s) \) spreads to a formal neighbourhood of the origin in \( \mathbb{A}^{\ell+1} \). This gives a morphism \( U^- \times \mathbb{A}^{\ell+1} \times U^+ \to Y^G \) which send 0 to \( (C^{(\ell)}, x_G^{(\ell)}, s) \). All in all this shows that \( Y^G \) is formally smooth to the Mumford embedding (9.0.2) at the point \( (C^{(\ell)}, x_G^{(\ell)}, s) \).
Part II

10. Twisted curves and torsion-free sheaves

10.1 Goals of Part II

In this part of the paper, we focus on the single nodal curve \((C, c)\) and its normalization \(\nu: (\tilde{C}, \tilde{c}) \to (C, c)\) (2.1). The final aim is to describe the Gieseker torsors on the nodal curves \(C^{(d)}\) in terms of its restriction to \(\tilde{C}\) and thereby get a notion of ‘semistability’ for them. Recall that in [27,38], the direct image under \(p: C^{(d)} \to C\) relates Gieseker vector bundles to torsion-free sheaves and gives a morphism of stacks. (Semi)stability for Gieseker vector bundles is then defined in terms of two ingredients, the (semi)stability of torsion-free sheaves and a (semi)stability using a relative polarization for this morphism. Our approach is modelled after this one. The first goal is to understand the classical (semi)stability of torsion-free sheaves from a new standpoint. The aim is to avoid going to sub-objects to test stability.

10.2 The basic setup

Let \(N_c = \text{Spec } \frac{k[[x,y]]}{(x,y)}\) be the analytic neighbourhood at the node on \((C, c)\) and let \(N_0\) denote \(\text{Spec } \frac{k[u,v]}{(u,v)}\) with coordinates \(u, v\). We can express \(N_0 = D_0 \cup D_0'\), where \(D_0\) and \(D_0'\) are identified with discs with 0 as origin. Let \(\mu_d\) act on \(N_0\) by sending

\[
    u \mapsto \zeta \cdot u, \quad v \mapsto \zeta^{-1} \cdot v,
\]

where \(\zeta\) is a primitive \(d\)-th root of unity and \(v\) is the local coordinate of \(D_0\) and \(u\) for \(D_0'\). The quotient morphism

\[
    \sigma: N_0 \to N_c = N_0/\mu_d = D_0/\mu_d \cup D_0'/\mu_d
\]

is given by \(\sigma(u) = u^d = x, \sigma(v) = v^d = y\).

10.3 Twisted curves

Let \(d \leq m(G)\) (7.2). Let \(\mathcal{E}_d\) be a twisted nodal curve in the sense of [3, Definition 2.1]. Note that \(\mathcal{E}_d\) is an algebraic stack with \(C\) as its coarse space, and we have the morphism \(\sigma: \mathcal{E}_d \to C\) (see [3]). Assume that analytic locally at the node \(c \in C\) it is given by \(N_0 \to [N_0/\mu_d] \to N_c\).

Fix \(\mathcal{E}_d\) the \((\mu_d, G)\)-torsor on \(N_0 \subset D\) obtained by restricting \(\mathcal{E}_D\) (12.0.5), given by a representation \(\rho: \mu_d \to G\) of local type \(\tau\). On \(N_0 \times G\), the \(\mu_d\)-action is given by

\[
    \gamma \cdot (u,g) = (\zeta \cdot u, \rho(\gamma) \cdot g), \quad \gamma \cdot (v,g) = (\zeta^{-1} \cdot v, \rho(\gamma) \cdot g).
\]

Let \(\mathcal{E}_0(G)\) be the equivariant group scheme on \(N_0\) of type \(\tau\). This is therefore fixed throughout.
DEFINITION 10.1

A $G$-torsor $\mathcal{E}$ of local type $\tau$ on $\mathcal{C}_d$ is the datum $(E', E_0, g')$, where

- $E'$ is a $G$-torsor on the punctured curve $C - c$,
- $E_0$ is a $\mathcal{E}_0(G)$-torsor,
- $g'$ is a $\mu_d$-invariant gluing function

\[ g' : N_0^\tau \rightarrow G \]  \hspace{1cm} (10.1.1)

which gives an isomorphism: $\sigma^* (E' |_{N_0^\tau}) \simeq E_0 |_{N_0^\tau}$.

We observe that $E_0$ being a $\mathcal{E}_0(G)$-torsor encapsulates the statement that $E_0$ is a $(\mu_d, G)$-torsor of type $\tau$.

There is an obvious notion of isomorphism of such torsors. We also note that for sheaves on the twisted curve $\mathcal{C}_d$, we have the natural push-forward $\sigma_*$. Locally on $N_0$ this is the invariant push-forward (Inv $\circ \sigma_*$).

10.4 Torsion-free sheaves on $(C, c)$ as $GL(n)$-bundles on twisted curves $\mathcal{C}_d$

In this subsection, we make some remarks on torsion-free sheaves on $C$ and their lifts to the stack $\mathcal{C}_d$. These remarks are key to the idea behind the general $\mathfrak{t}$-$\mathfrak{f}$-semistability of Gieseker torsors which we define later.

Let $\mathscr{A}$ be a torsion-free sheaf on $C$ of rank $n$. We view $\mathscr{A}$ as the datum $(V', F_0, g')$, where $V'$ is a vector bundle of rank $n$ on $C - c$, $F_0$ a torsion-free sheaf on $N_c$ of rank $n$ and $g' : V' |_{N_c^\tau} \simeq F_0 |_{N_c^\tau}$ a gluing function. We can express this equivalently as the datum of principal bundles as $(E', F_0, g')$, where $E'$ is the frame bundle of $V'$ and $F_0 |_{N_c^\tau}$ (by an abuse of notation) stands for the frame bundle of the vector bundle $F_0 |_{N_c^\tau}$ and $g'$ is an isomorphism of principal bundles on $N_c^\tau$.

Recall that a torsion-free sheaf $F_0$ of rank $n$ on $N_c$ is isomorphic to a direct sum $E_i^a \oplus m^b$ with $a + b = n$, where $m$ is the maximal ideal sheaf at $c$. It is easily seen that $m$ can be realized as an invariant direct image of a $\mu_d$-line bundle on $N_0$ associated to a non-trivial character of $\mu_d$. Whence, the local type of $F_0$ gives us a $(\mu_d, GL(n))$-bundle $E_0$ coming from a representation $\rho : \mu_d \rightarrow T_n \subset GL(n)$ such that Inv $\circ \sigma_*(E_0(k^n)) = F_0$. Thus we get a triple $(E', E_0, g')$ with $g' = \sigma^*(g')$. Equivalently, we get a $GL(n)$-torsor $\mathcal{E}$ on $\mathcal{C}_d$ from the torsion-free sheaf $\mathscr{A}$ and there is nothing unique about $\mathcal{E}$ except that $\sigma_*(\mathcal{E}(k^n)) \simeq \mathscr{A}$.

By the general remarks in the appendix below (see (B2.1)), giving a 1-PS of $GL(V)$ produces weighted filtrations on vector spaces $V$. From the standpoint of the datum above, given a filtration of $\mathscr{A}$ by saturated subsheaves, the associated graded sheaf can be recovered from data on the torsor $\mathcal{E}$. More precisely, suppose we are given the weighted filtration

\[ \mathscr{A}_\lambda^*: 0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_s \subsetneq A_{s+1} = \mathscr{A}, \]  \hspace{1cm} (10.1.2)

and let $\lambda : G_m \rightarrow GL(n)$ be the 1-PS coming from this datum. Further, let $P = P(\lambda) \subset GL(n)$ be the induced parabolic subgroup. Restricting the filtration (10.1.2) to $C - c$, gives a weighted filtration of the locally free sheaf $E'(k^n) = \mathscr{A}|_{C-c}$:

\[ 0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_s \subsetneq A_{s+1} = \mathscr{A}|_{C-c}, \]  \hspace{1cm} (10.1.3)
which is equivalent to giving a reduction of structure group $E'_p \subset E'$ to $P$. By saturating this filtration in the sheaf $\mathcal{A}$ we get back the weighted filtration (10.1.2) on $C$. In other words, a reduction of structure group $E'_p \subset E'$ gives a saturated filtration of $\mathcal{A}$ by saturated subsheaves and a tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_s)$ of positive rational numbers which are recovered from $\lambda$ (B2.1).

How does one recover the associated graded sheaf $gr_{\lambda}(\mathcal{A}) = \bigoplus J_{j+1}/J_j$? Observe that this is also torsion-free, but its local type, i.e., its type restricted to $N_c$ at the node, depends on the filtration (10.1.3), which in turn comes by a process of saturation.

We proceed as follows: firstly, locally, from the torsion-free sheaf $gr_{\lambda}(\mathcal{A})$, we can get a $\mu_d$-vector bundle $V_{0,p}$ on $N_0$ such that

$$\text{Inv} \circ \sigma_*(V_{0,p}) \simeq gr_{\lambda}(\mathcal{A})|_{N_c}. \quad (10.1.4)$$

The $\mu_d$-vector bundle $V_{0,p}$ comes from a representation $\varphi : \mu_d \to T_n \hookrightarrow \text{GL}(n)$ and this gives a $(\mu_d, T_n)$-bundle $E_0^\varphi$ on $N_0$; since the $\mu_d$-action on $N_0^*$ is free, we have a canonical isomorphism:

$$E_0^\varphi|_{N_0^*} \simeq E_0|_{N_0^*}. \quad (10.1.5)$$

The filtration (10.1.2) on $\mathcal{A}$ produces an obvious filtration on $gr_{\lambda}(\mathcal{A})$ by deleting at each step a summand. These coincide on $C - c$, i.e., this filtration, when restricted to $C - c$ produces a gluing function $\varphi : \text{gr}_{\lambda}(E'_p|_{N_c}) \simeq E_0^\varphi(P)|_{N_0^*}$, such that it induces $\varphi$ when the structure group is extended from $P$ to $G$. Whence, we firstly get a $P$-torsor $(E'_p, E_0^\varphi, \partial^\varphi)$ from $gr_{\lambda}(\mathcal{A})$.

The recipe to recover $gr_{\lambda}(\mathcal{A})$ is as follows: Let $H = H(\lambda)$ be the canonical Levi quotient of $P$. By the remarks made after (B2.2), it follows easily that

$$\sigma_*((E'_{H}, E_0^{\varphi}, \partial^\varphi)) = gr_{\lambda}(\mathcal{A}), \quad (10.1.6)$$

where $(E'_{H}, E_0^{\varphi}, \partial^\varphi)$ is a $H$-torsor of local type $\varphi$ on $\mathcal{C}_d$, the objects being natural extension of structure groups to $H$.

A weighted slope is assigned by Schmitt [43] to the weighted filtration (10.1.2):

$$L(\mathcal{A}_\lambda^*, \epsilon) := \sum_{i=1}^s \epsilon_i \deg_C(\mathcal{A}) \cdot \text{rk} \mathcal{A}_i - \deg_C(\mathcal{A}_i) \cdot \text{rk} \mathcal{A}_i. \quad (10.1.7)$$

and an obvious weighted slope $L(gr_{\lambda}(\mathcal{A}^*), \epsilon)$ for the associated graded sheaf $gr_{\lambda}(\mathcal{A})$.

It is easy to see that

$$L(\mathcal{A}_\lambda^*, \epsilon) = L(gr_{\lambda}(\mathcal{A}^*), \epsilon). \quad (10.1.8)$$

We have the obvious but useful reformulation of semi-stability which circumvents going to sub-objects.

**Lemma 10.2.** A torsion-free sheaf $\mathcal{A}$ on $C$ is semi-stable if and only if for every weighted filtration (10.1.2), $L(gr_{\lambda}(\mathcal{A}^*), \epsilon) \geq 0$.

### 11. On semistability of $G$-torsors on twisted curves

Let $\mathcal{E} = (E', E_0, \partial)$, be a $G$-torsor on $\mathcal{C}_d$ of local type $\tau$ coming from a representation $\rho : \mu_d \to T \subset G$ (see (10.1)). Thus, $E_0$ can be taken as a $(\mu_d, T)$-bundle coming
from $\rho$ and $q : \sigma^*(E' | N^*_c) \simeq E_0 | N^*_0$. The $(\mu_d, T)$-bundle $E_0$ gives a $\mu_d$-line bundle decomposition $\oplus E_0(\alpha_i \circ \rho)$ coming from the simple roots $\alpha_i : T \to \mathbb{G}_m$ and we get a rank $\ell$ torsion-free sheaf

$$\mathcal{F}_0 := \bigoplus \text{Inv} \circ \sigma_*(E_0(\alpha_i \circ \rho))$$

(11.0.1)
on $N_c$. Let $\lambda : \mathbb{G}_m \to T \subset G$ be a 1-PS which gives a filtration

$$0 \subset F_1 \subset \cdots \subset F_s \subset F_{s+1} = \mathcal{F}_0 | N^*_c$$

(11.0.2)
of the locally free sheaf $\mathcal{F}_0 | N^*_c$. By saturation in $\mathcal{F}_0$, we get a filtration

$$0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s \subset \mathcal{F}_{s+1} = \mathcal{F}_0,$$

(11.0.3)
by subsheaves (with quotients torsion-free) and we also get the associated graded sheaf $\mathfrak{g}x_\lambda(\mathcal{F}_0)$ on $N_c$. We now follow the earlier procedure. Firstly, the local type of $\mathfrak{g}x_\lambda(\mathcal{F}_0)$ gives a new $\varphi : \mu_d \to T$ (dependent on the 1-PS $\lambda$) and a $(\mu_d, T)$-bundle $E_0^\varphi$ on $N_0$. The induced bundle $\oplus E_0^\varphi(\alpha_i \circ \varphi)$ recovers $\mathfrak{g}x_\lambda(\mathcal{F}_0)$ as $\bigoplus (\text{Inv} \circ \sigma_*(E_0^\varphi(\alpha_i \circ \varphi)))$.

Let $\mathcal{E} = (E', E_0, q)$ be a $G$-torsor on $\mathcal{E}_d$ with $E_0$ an $E_0(G)$-torsor coming from $\rho : \mu_d \to T \subset G$ (10.1). So $E_0$ comes with a choice of reduction of structure group to $T$. A reduction of structure group $\mathcal{E}_P$ of $\mathcal{E}$ to $P = P(\lambda)$ comprises of the datum $(E'_P, E_0, q_P)$, where $E'_P$ is a reduction of structure group of $E'$ to the parabolic subgroup $P$ on $C - c$ and $q_P : N^*_c \to P$ is a function which gives an identification

$$q_P : \sigma^*(E'_P | N^*_c) \simeq (E_0 \times_T P) | N^*_0$$

(11.0.4)
which glues the $P$-torsors along $N^*_0$ with the constraint that the composite $q_P : N^*_0 \to P \hookrightarrow G$ equals $q$.

The 1-PS $\lambda$ gave rise to a new $\varphi : \mu_d \to T$ and a new $(\mu_d, T)$-bundle $E_0^\varphi$ on $N_0$. Since the $\mu_d$-action is free away from $0$, it follows that we have canonical identification

$$E_0 | N^*_0 \simeq E_0^\varphi | N^*_0.$$  

(11.0.5)

Since $T \subset P$, by extending structure groups via $P \to H$ and by (11.0.5), we get an $H$-torsor

$$\mathcal{E}_H^\varphi := (E'_H, E_0^\varphi, q_H)$$

(11.0.6)
of local type $\varphi$ on $\mathcal{E}_d$, where

$$q_H : \sigma^*(E'_H | N^*_c) \simeq E_0^\varphi(H) | N^*_0$$

(11.0.7)
is the one induced by $q_P$. Moreover, as we saw above,

$$\mathfrak{g}x_\lambda(\mathcal{F}_0) := \bigoplus_i (\text{Inv} \circ \sigma_*(E_0^\varphi(\alpha_i \circ \varphi))).$$

(11.0.8)

**DEFINITION 11.1**

We call $\mathcal{E}_H^\varphi$ (11.0.6) the twisted $H$-torsor of local type $\varphi$ associated to $\mathcal{E}_P$.

Let $\eta : G \hookrightarrow \text{GL}(W)$ be a faithful representation and let $T_W \subset \text{GL}(W)$ be a maximal torus such that $\eta : T \hookrightarrow T_W$. Let $\mathcal{E}$ be a $G$-torsor on $\mathcal{E}_d$. Then we get the associated
GL(W)-torsor $\mathcal{E}(GL(W))$ and associated vector bundle $\mathcal{E}(W)$. We can get torsion-free sheaves on $C$ via $\sigma : \mathcal{E}_d \rightarrow C$. Let $\mathcal{F}_W := \sigma_* (\mathcal{E}(W))$.

Let $\lambda : \mathbb{G}_m \rightarrow T \subset G$ be a one-parameter subgroup, which gives parabolic subgroups $P = P(\lambda)$ and Levi $H$ and induced 1-PS $\eta \circ \lambda : \mathbb{G}_m \rightarrow T \leftrightarrow T_W$ and canonical inclusions $P \subset P_W$ and $H \leftrightarrow H_W$.

When the torsion-free sheaf $\mathcal{F}_W$ is restricted to $C - c$, the one-parameter subgroup $\eta \circ \lambda$ gives a weighted filtration

$$0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m \subsetneq F_{m+1} = \mathcal{F}_W|_{C-c} \quad (11.1.1)$$

of the locally free sheaf $\mathcal{F}_W|_{C-c}$ which by saturation gives a filtration

$$0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_m \subsetneq \mathcal{F}_{m+1} = \mathcal{F}_W, \quad (11.1.2)$$

by subsheaves (with torsion-free quotients). This gives the associated graded torsion-free sheaf

$$\text{gr}(\mathcal{F}_W) := \bigoplus_{j=1}^{m} \mathcal{F}_{j+1}/\mathcal{F}_j. \quad (11.1.3)$$

Note that, as $H_W$-modules (and hence as $H$-modules), we have $W \simeq \text{gr}(\mathcal{F}(W))$ (filtered via $\eta \circ \lambda$). Note further that the weighted filtration on $W$ induced by $\lambda$ also gives a canonical weighted filtration $(\text{gr}(W), \epsilon)$ on $\text{gr}(W)$. Thus we get $\mathcal{E}_H^\psi(\text{gr}(W)) = \bigoplus_{j=1}^{m} \mathcal{E}_H^\psi(W_{j+1}/W_j)$. Therefore, for each $\varphi$ we have an associated vector bundle $\mathcal{E}_H^\psi(\text{gr}(W))$. By taking push-forward by $\sigma : \mathcal{E}_d \rightarrow C$, we have the isomorphism

$$\sigma_* (\mathcal{E}_H^\psi(\text{gr}(W))) \simeq \bigoplus_{j=1}^{m} \mathcal{E}_H^\psi(W_{j+1}/W_j), \quad (11.1.4)$$

of torsion-free sheaves on $C$. We also get the obvious weighted filtration matching (11.1.2) term by term as

$$0 \subsetneq \mathcal{E}_H,1 \subsetneq \cdots \subsetneq \mathcal{E}_H,m \subsetneq \mathcal{E}_H,m+1 = \mathcal{E}_H^\psi(\text{gr}(W)). \quad (11.1.5)$$

If we have a notion of ‘degree’ of associated vector bundle on the twisted curve $\mathcal{E}_d$, then following Schmitt [43], we also have a weighted slope

$$L(\mathcal{E}_H^\psi(\text{gr}(W)))$$

$$:= \sum_{i=1}^{s} \epsilon_i \left\{ \text{deg}_{\mathcal{E}_d}(\mathcal{E}_H^\psi(\text{gr}(W))) \cdot \text{rk} \mathcal{E}_H,i - \text{deg}_{\mathcal{E}_d}(\mathcal{E}_H^\psi(\text{gr}(W))) \cdot \text{rk} (\mathcal{E}_H^\psi(\text{gr}(W))) \right\}. \quad (11.1.6)$$

By (B2.3), associated to $\lambda$, we have an anti-dominant character $\chi_{\lambda}$ of the parabolic subgroup $P$ (and the Levi $H$) ‘dual’ to $\lambda$. Suppose further that the ‘degree’ satisfies the equality (see (13.1)):

$$\text{deg}_{\mathcal{E}_d}(\mathcal{E}_H^\psi(\chi_{\lambda})) = L(\mathcal{E}_H^\psi(\text{gr}(W))). \quad (11.1.7)$$

By (B2.2), for any anti-dominant character $\chi$ of $P$, there is a positive rational $r$ such that $\chi_{\lambda} = r \chi$ and hence by (11.1.7), we deduce that for each anti-dominant $\chi$, $L(\mathcal{E}_H^\psi(\text{gr}(W)))$ and $\text{deg}_{\mathcal{E}_d}(\mathcal{E}_H^\psi(\chi))$ have the same sign. Thus, with our hypothetical ‘degree’ plus (11.1.7) (see (12.5)), we have the following definition.
DEFINITION 11.2

(1) A $G$-torsor $\mathcal{E}$ of local type $\tau$ on the twisted curve $C_d$ is called $\tau \varphi$-semi(stable) if for every $1$-PS $\lambda : G_m \to T$, and reduction of structure group $\mathcal{E}_p$, we have $\deg_{\mathcal{E}_p} \left( \mathcal{E}_p^H (\mathcal{X}) \right) (\geq) 0$.

(2) $\mathcal{E}$ is $\eta$-semi(stable) if for every $1$-PS $\lambda : G_m \to T \subset T_W$, we have $L \left( \mathcal{E}_p^H (g \mathcal{X}(W)) \right) (\geq) 0$.

Our observations show then that we have a theorem analogous to the classical theorem of Ramanathan.

Theorem 11.3. A $G$-torsor $\mathcal{E}$ of local type $\tau$ on the twisted curve $C_d$ is $\tau \varphi$-semi(stable) if and only if it is $\eta$-semi(stable) for every $\eta : G \to GL(W)$.

The next task is to show that there is a well-defined notion of ‘degree’ on $C_d$ (12.5) with some properties like (11.1.7) (see (13.1)), and that the above notion is geometric invariant theoretic. To achieve this we traverse a ‘parabolic path’ via a Fourier–Mukai from bundles on $C_d$ to laced ones on the normalization $\tilde{C}$ which comes with a balanced parabolic structure.

12. Laced torsors via Fourier–Mukai for torsors on twisted curves

Let $q : (C_d, z) \to (C_d, o)$ be the normalization, i.e., the maximal reduced substack of $C_d \times C \tilde{C}$. Then $C_d$ is a twisted curve with two markings, the normalization $\tilde{C}$ of $C$ is its coarse space with the canonical morphism $f : (C_d, z) \to (\tilde{C}, e)$, and we have a diagram

$$
\begin{array}{ccc}
C_d & \xrightarrow{f} & \tilde{C} \\
q \downarrow & & \downarrow v \\
C_d & \xrightarrow{\sigma} & C
\end{array}
$$

and the corresponding local picture

$$
\begin{array}{ccc}
[N/\mu_d] & \xrightarrow{f} & \tilde{U} \\
q \downarrow & & \downarrow v \\
[N_0/\mu_d] & \xrightarrow{\sigma} & N_c
\end{array}
$$

An analytic neighbourhood of $C_d$ at $z_1$ (resp. $z_2$) gets identified with $[D_0/\mu_d]$ (resp. $[D'_0/\mu_d]$) with action given by (10.0.1) and similarly analytic neighbourhood of $\tilde{C}$ at $c_1$ (resp. $c_2$) gets identified with $D_0/\mu_d$ (resp. $D'_0/\mu_d$). Given the ramification data at $c_i$, we can get a smooth projective Kawamata cover

$$
f' : Z \to \tilde{C}
$$

with the same local ramification data as $f$. The pull-back $q^*(\mathcal{E})$ gives a $G$-torsor on $C_d$ plus a ‘descent datum’, i.e., a $(\mu_d, G)$-isomorphism

$$
i : P_{z_1} \simeq P_{z_2}^t.
$$
where $\mathcal{P}^\dagger$ is the $(\mu_d, G)$-torsor in a neighbourhood of $z_2$ given by the local type $\tau$. Equivalently, we can take the ‘adjoint’ group scheme $\mathcal{E}(G)$ on $\tilde{\mathcal{C}}_d$ and let

$$\mathcal{E}(G, \tau) = q^*(\mathcal{E}(G)).$$

(12.0.5)

Then $q^*(\mathcal{E})$ is simply a $\mathcal{E}(G, \tau)$-torsor on the twisted curve $\tilde{\mathcal{C}}_d$.

Remark 12.1 (On the descent datum). Observe that a $(\mu_d, G)$-isomorphism $\iota : P_{z_1} \cong \mathcal{P}^\dagger_{z_2}$ gives a $G$-isomorphism of the quotients $\mathcal{E}_{z_1}/I_{\rho} \cong \mathcal{E}_{z_2}/I_{\rho}$, where $I_{\rho} := \text{Im}(\rho) \subset G$.

The descent datum (12.0.4) in the case of the group scheme $\mathcal{E}_0(G, \tau)$ translates as a $G$-isomorphism:

$$\iota_h : G/I_{\rho} \rightarrow G/I_{\rho},$$

(12.1.1)

given by an inner automorphism induced by an element $h \in \text{Cent}_G(I_{\rho})$ modulo the center of $\text{Cent}_G(I_{\rho})$.

For instance, when $\rho : \mu_d \rightarrow G$ sends a generator $\zeta$ to a Borel-de Seibenthal element $g_\alpha$, we see that $\text{Cent}_G(I_{\rho})$ is precisely the Levi quotient of the closed fibre of the maximal parahoric group scheme $P_{\theta_\alpha}$. Hence a $\text{balanced}$ maximal parahoric group scheme on $\tilde{\mathcal{C}}$ (12.3) is given by a maximal parahoric group scheme $P_{\theta_\alpha}$ at the two marked points with an isomorphism of the Levi quotients of the closed fibres, modulo the center. Instead, if $\rho$ gives the Iwahori structure, then descent datum becomes trivial since the Levi is abelian.

DEFINITION 12.2

For $\theta_\tau \in \mathcal{A}_\tau$, we define the $\text{balanced}$ parahoric group scheme on $(\tilde{\mathcal{C}}, c)$ as: $\mathcal{G}(\theta_\tau) := f_*(\mathcal{E}(G, \tau))$, with $\mathcal{E}(G, \tau)$ as in (12.0.5).

We could use the Kawamata cover (12.0.3) to see that $\mathcal{G}(\theta_\tau) := \text{Inv} \circ f_*(\mathcal{E}(G, \tau))$. Clearly, the group scheme $f_*(\mathcal{E}(G, \tau))$ on $\tilde{\mathcal{C}}$ is obtained by gluing two Bruhat–Tits group schemes $P_{\theta_\tau}, P_{\bar{\theta}_\tau}$ at $c_1, c_2$ (with $\bar{\tau}$ as in Remark 2.4) together with the datum of an isomorphism of the Levi factors of the closed fibres $P_{ci}$.

PROPOSITION 12.3

Let $\mathcal{G}_\tau^{\mathcal{G}}$ be a group scheme over $C^{(d)}$ coming from a representation $\rho : \mu_d \rightarrow G$ together with a gluing datum. Then the restriction

$$\mathcal{G}(\theta_\tau) := \mathcal{G}_\tau^{\mathcal{G}}|_{\tilde{\mathcal{C}}}$$

(12.3.1)

to $\tilde{\mathcal{C}} \subset C^{(d)}$ is a balanced group scheme.

Proof. The question is obviously local along the components of $C^{(d)}$. The process of taking invariant direct images of group schemes, i.e., Weil restriction followed by invariants, commutes with base change. Hence by (12.0.5) and (3.0.5), we get

$$\left(\text{Inv} \circ f_*\right)v\left(E(G, \tau)|_{\tilde{\mathcal{N}}}\right) \simeq f_*(\mathcal{E}(G, \tau))|_{\tilde{\mathcal{G}}}. \quad (12.3.2)$$

The proof of (12.3.1) follows immediately from (3.0.5) and (12.3.2). \qed
We are in the setting of (12.0.1).

**Definition 12.4 (Fourier–Mukai)**

Let \( G(\theta_\tau) \) be a balanced parahoric group scheme. A \( G(\theta_\tau) \)-torsor \( E_\varphi \) on \((\tilde{C}, c)\) is called \textit{laced} if \( E_\varphi \simeq f_*(q^*(\mathcal{E})) \) for a \( G \)-torsor \( \mathcal{E} \) of local type \( \tau \) on \( \mathcal{C}_d \) for some \( d \).

12.1 **Twisted Levi torsor associated to laced \( G(\theta_\tau) \)-torsors**

Let \( E_\varphi \) be the laced torsor associated to \( E_\varphi \). Given a reduction of structure group \( E_P \) of \( E_\varphi \) on \( \mathcal{C}_d \), by pulling back \( E_\varphi \) with \( q : \tilde{C}_d \rightarrow \mathcal{C}_d \), we obtain a balanced \( H \)-torsor \( q^*(\mathcal{E}^H) \) on \( \tilde{C}_d \). We can then take the push-forward (invariant push-forward) by the morphism \( f : \tilde{C}_d \rightarrow \tilde{C} \) and we get the \textit{laced} torsor

\[
E_\varphi^{\psi, H} := f_*(q^*(\mathcal{E}^H))
\]

which is a torsor under the invariant direct image group scheme with generic fibre \( H \). We term this the \textit{laced} Levi torsor associated to the \( P \)-torsor \( E_\varphi \). Let \( \chi : P \rightarrow \mathbb{G}_m \) be any anti-dominant character. This is indeed a character of \( H \) as well. Given a reduction of structure group of \( E_\varphi \) to \( P \), to the twisted \( H \)-torsor \( E_\varphi \) we get line bundles \( E_\varphi(\mathcal{E}(\chi)) \) on \( \mathcal{C}_d \).

These define parabolic line bundles on \( \tilde{C} \) with a balanced parabolic structure at \( \{c_1, c_2\} \) as below:

\[
E_\varphi^{\psi, H}(\chi) := f_*(q^*(\mathcal{E}^H(\chi))).
\]

12.2 **Equivariant degree of line bundles on \( \mathcal{C}_d \) and semistability**

We now have a well-defined \( t \times \psi \)-(semi)stability of \( G \)-torsors \( E_\varphi \) on \( \mathcal{C}_d \) by combining (11.3) by using (12.5) for \( \deg \mathcal{G}_d \).

**Definition 12.5**

Let \( L \) be a line bundle on the twisted curve \( \mathcal{C}_d \). Define the equivariant degree \( \deg_q \mathcal{G}_d(L) := \pardeg_C f_*(q^*(L)) \).

13. **\( t \times \psi \)-semistability via GIT**

Let \( \eta : G \hookrightarrow \text{GL}(W) \) be a faithful representation and let \( T_W \subset \text{GL}(W) \) be a maximal torus such that \( \eta : T \hookrightarrow T_W \). Let \( \mathcal{E} \) be a \( G \)-torsor on \( \mathcal{C}_d \). Then we get the associated \( \text{GL}(W) \)-torsor \( \mathcal{E}(\text{GL}(W)) \) and associated vector bundle \( \mathcal{E}(W) \). Likewise, if \( E_\varphi = f_*(q^*(\mathcal{E})) \), then \( \eta \) gives the associated \textit{laced} \( \text{GL}(W) \)-torsor \( E_\varphi(\text{GL}(W)) := f_*(q^*(\mathcal{E}(\text{GL}(W)))) \) and similarly the associated \textit{laced} vector bundle \( E_\varphi(W) \) on \((\tilde{C}, c)\). Given a 1-PS \( \lambda \), we get graded versions and isomorphisms

\[
f_* \circ q^* \left[ \mathcal{E}_H^\psi(\mathcal{G}_\tau(W)) \right] = f_* \circ q^* \left[ \bigoplus_{j=1}^m \mathcal{E}_H^\psi(W_{j+1}/W_j) \right]
\]

\[
\simeq \bigoplus_{j=1}^m E_\varphi^{\psi, H}(W_{j+1}/W_j) = E_\varphi^{\psi, H}(\mathcal{G}_\tau(W)).
\]

(13.0.1)
Weighted filtrations (11.1.5) on associated objects by an application of $f_\ast(q^\ast)$ term by term give an obvious filtration for the associated graded:

$$0 \subsetneq E_{\wp,1} \subsetneq \cdots \subsetneq E_{\wp,m} \subsetneq E_{\wp,m+1} = E_{\wp,H}^\wp(\mathfrak{g} \tau(W)).$$ (13.0.2)

**Lemma 13.1.** Let $\chi_\lambda$ be as in (B2.1). Then we have the equality

$$\text{par.deg}(E_{\wp,H}^\wp(\chi_\lambda)) = L(E_{\wp,H}^\wp(\mathfrak{g} \tau(W))).$$ (13.1.1)

In particular, (11.1.7) holds for degree as defined in (12.5).

**Proof.** This follows immediately from (B2.4). □

We can get torsion-free sheaves on $\mathcal{C}$ via $\sigma : \mathcal{C}_d \to \mathcal{C}$ as well as via $\nu : \tilde{\mathcal{C}} \to \mathcal{C}$ by the functor $\nu_\ast : \text{Vect}_{\mathcal{C}(,e)}^L \to \text{Tf}_C$. Then, it is routine to check that

$$\nu_\ast(E_{\wp}(W)) \simeq \sigma_\ast(\mathcal{E}(W)) \simeq \mathcal{F}_W.$$ (13.1.2)

In fact, the parabolic degree of the laced vector bundle $E_{\wp}(W)$ coincides with the degree of the torsion-free sheaf $\mathcal{F}_W$ (B1.7).

**Lemma 13.2.** We have isomorphisms

$$\text{gr}_\mathcal{C}(\mathcal{F}_W) = \bigoplus_{j=1}^m \mathcal{F}_{j+1}/\mathcal{F}_j \simeq \bigoplus_{j=1}^m \nu_\ast(E_{\wp,H}^\wp(W_{j+1}/W_j)) = \nu_\ast(E_{\wp,H}^\wp(\mathfrak{g} \tau(W)))$$ (13.2.1)

of torsion-free sheaves on $\mathcal{C}$. Furthermore, for each $j$,

$$\text{par.deg}_\mathcal{C}[E_{\wp,H}^\wp(W_{j+1}/W_j)] = \text{deg}_\mathcal{C}(\mathcal{F}_{j+1}/\mathcal{F}_j).$$ (13.2.2)

**Proof.** The first statement is a summary of the previous discussion. The last statement follows from (B1.7). □

The next proposition is the primary reason why we go to the laced category. The balanced parabolic structure of laced bundles makes their parabolic degree coincide with the push down torsion free sheaf (B1.7).

**PROPOSITION 13.3**

We have the equality

$$L(\mathcal{F}_W^\bullet, \epsilon_\bullet) = L(E_{\wp,H}^\wp(\mathfrak{g} \tau(W))) \overset{(13.1)}{=} \text{par.deg}(E_{\wp,H}^\wp(\chi_\lambda)).$$ (13.3.1)

where $L(\mathcal{F}_W^\bullet, \epsilon_\bullet)$ is as in (10.1.7).

**Proof.** By (13.2) applied to each summand, it follows easily that

$$L(\mathcal{F}_W^\bullet, \epsilon_\bullet) \overset{(10.1.8)}{=} L(\mathfrak{g} \tau_\lambda(\mathcal{F}_W)) = L(E_{\wp,H}^\wp(\mathfrak{g} \tau(W))).$$ □
COROLLARY 13.4

Let \( E \) be a \( G \)-torsor on \( C \) and let \( \mathcal{E}_P \) be a reduction of structure group to \( P(\lambda) \subset G \). Let \( \eta : G \hookrightarrow GL(W) \). Then we have the equality

\[
L(\mathcal{F}_W, \epsilon) = L(\mathcal{E}_P^\eta(g \tau(W))) = \deg_{C}(\mathcal{E}_P^\eta(\chi_\lambda)).
\]

(13.4.1)

Proof. By (13.0.1), \( L(\mathcal{E}_P^\eta(g \tau(W))) = L(E_\varphi^{\eta,h}(g \tau(W))) \) and the rest follows immediately. \( \square \)

We recall from [42]. The torsion-free sheaf \( \mathcal{F}_W \) when restricted to \( C - c \) (or more precisely, the underlying frame bundle), comes with a reduction of structure group \( \nu \) to \( G \). Further, since \( G \) is semisimple, \( \deg_C(\mathcal{F}_W) = 0 \). Thus the pair \( (\mathcal{F}_W, \nu) \) is a singular principal bundle (14.1). By [42, 3.5.1], \( (\mathcal{F}_W, \nu) \) is semi(stable) if \( L(\mathcal{F}_W, \epsilon)(\geq 0) \). This definition is a GIT notion and we arrive at the following.

**Theorem 13.5 (tf-semistability is a GIT notion).** Let \( \eta : G \hookrightarrow GL(W) \). A \( G \)-torsor \( \mathcal{E} \) of local type \( \tau \) on \( C \) is semi-stable if and only if it is \( \eta \)-semi-stable and hence if and only if the associated singular principal bundle \( (\mathcal{F}_W, \nu) \) (13.1.2) is (semi)stable.

**Remark 13.6.** The importance of this notion stems from the fact that the moduli space of semi(stable) singular principal bundles \( M_\eta(\tilde{C}) \) is a projective scheme [42, Theorem 3.5.2]. Moreover, if \( \eta \) is chosen carefully, there is a projective scheme \( M \rightarrow A \), albeit non-flat, such that \( M_K \) is the Ramanathan moduli space over \( C_K \) and \( M_0 \simeq M_\eta(C) \).

13.1 Semistability of \( G \)-torsors on nodal curves

Recall that if \( E \) is a \( G \)-torsor on \( (C, c) \) then \( \sigma^*(E) \) is a \( G \)-torsor on \( \mathcal{C} \). The notion of \( \tau \)-semi(stability) of a \( G \)-torsor on \( \mathcal{C} \) therefore gives an intrinsic notion of (semi)stability of \( G \)-torsors on \( (C, c) \) thereby answering a long-standing question.

13.2 Gieseker torsors on \( C^{(d)} \) and laced torsors on \( \tilde{C} \)

Let \( \text{Lac}_{\tilde{C}}(\mathcal{G}_{(\theta)}) \) denote the set of isomorphism classes of \( \text{Laced}_{(\theta)} \)-torsors on \( (\tilde{C}, c) \).

**PROPOSITION 13.7**

We have a set-theoretic map from the isomorphism classes of Gieseker torsors on \( (C, c) \) to the \( \text{Laced}_{(\theta)} \)-torsors on \( \tilde{C} \) for varying types \( \tau \),

\[
|Gies_G(C)| \rightarrow \bigsqcup_{\tau} \text{Lac}_{\tilde{C}}(\mathcal{G}_{(\theta_{(\tau)})}).
\]

(13.7.1)

Proof. It is immediate from (12.3) that a Gieseker torsor on \( (C, c) \) when restricted to \( \tilde{C} \) gives a \( \text{Laced}_{(\theta)} \)-torsors on \( \tilde{C} \). \( \square \)

A laced torsor \( E_\varphi \) (12.4) is called \( \tau \)-semi(stable) if the \( G \)-torsor \( \mathcal{E} \) of local type \( \tau \) is so. Let \( \bigsqcup_{\tau} \text{Lac}_{\tilde{C}}(\mathcal{G}_{(\theta)})^{\tau \text{-ss}} \) be the subset of \( \tau \)-semi(stable) laced torsors.
DEFINITION 13.8

A Gieseker torsor on the semistable curve $C^{(d)}$ (for some $d > 0$) is called $\text{tf}$-(semi)stable if its image under (13.7.1) lies in $\bigsqcup \text{Lac}_{\mathcal{C}}(\mathcal{G}(\theta))^{\text{tf-ss}}$.

We have an obvious notion of families of $\text{tf}$-(semi)stable Gieseker torsors. Let $\text{Gies}_{G}(C_A)^{\text{tf-ss}}$ denote the substack of $\text{tf}$-(semi)stable Gieseker torsors on $C_A$.

For the openness property of this notion of $\text{tf}$-semistability, see (14.3) below.

Part III

14. The moduli construction

The aim of this part is to combine the results of Parts I and II to construct flat degenerations of the Ramanathan moduli space of slope (semi)stable $G$-torsors. When $G = \text{GL}(n)$, these are the precise analogues of degenerations via Gieseker bundles on modifications of the nodal curve [24,27,39].

14.1 The Bhosle–Schmitt spaces and associated Gieseker bundles

Fix a faithful representation $\rho: G \hookrightarrow \text{GL}(W)$ and let $\eta: G \hookrightarrow \text{GL}(W \oplus W^*)$. Let $2w := \dim(W \oplus W^*)$. In [45,46], Schmitt studies the algebraic $A$-stack $\text{Bun}_{\mathcal{G}}(\eta)^{\text{Sing}}(C_A)$ whose generic fibre is the stack $\text{Bun}_{C_K}(G)$ of $G$-torsors on $C_K$ and whose closed fibre has $T$-points which are families of singular principal $G$-bundles, i.e., of pairs $(\mathcal{F}, \delta)$,

- A torsion-free $\mathcal{O}_{C}$-module $\mathcal{F}$ with generic fibre type $W \oplus W^*$.
- A pseudo-$G$-structure $\delta$ which gives a reduction of structure group of the principal $\text{GL}(W \oplus W^*)$-bundle on $\bar{C}^* = \bar{C} - \{c_1, c_2\}$ underlying the locally free sheaf $\mathcal{F}|_{\bar{C}^*}$ to the subgroup $\eta: G \hookrightarrow \text{GL}(W \oplus W^*)$.

Note that, $G$-torsors on the generic fibre $C_K$ are also viewed as a pair $(\mathcal{F}_K, \delta_K)$ where $\mathcal{F}_K$ is a locally free sheaf of rank $2w$ on $C_K$ and $\delta_K$ a reduction of structure group of the frame $\text{GL}(2w)$-bundle of $\mathcal{F}_K$ to $G$. In particular, there is a forget $A$-morphism

$$\bar{j}: \text{Bun}_{\mathcal{G}}(\eta)^{\text{Sing}}(C_A) \rightarrow \text{Tfs}_{2w}(C_A), \quad (\mathcal{F}, \delta) \mapsto \mathcal{F}$$

(14.0.1)

into the algebraic stack $\text{Tfs}_{2w}(C_A)$ of relative torsion-free sheaves of rank $2w$ on the surface $C_A$. On the other hand, there is the isomorphism of stacks over $A$:

$$p^* : \text{GVB}_{2w}(C_A) \rightarrow \text{Tfs}_{2w}(C_A)$$

(14.0.2)

which is obtained by taking direct images under the canonical morphism $M \rightarrow C_T$ for varying $A$-schemes $T$. Thus we have a fibre square

$$\begin{array}{ccc}
\text{GVB}_{2w}(C_A) & \xrightarrow{p^*} & \text{GVB}_{2w}(C_A) \\
\downarrow & & \downarrow \\
\text{Bun}_{\mathcal{G}}(\eta)^{\text{Sing}}(C_A) & \xrightarrow{\bar{j}} & \text{Tfs}_{2w}(C_A)
\end{array}$$

(14.0.3)

The stack $\bar{j}^*(\text{GVB}_{2w}(C_A))$ parametrizes pairs $(V, \delta)$ of Gieseker vector bundles of rank $2w$ and a generic reduction of structure group $\delta$ of the underlying principal $\text{GL}(2w)$-bundle $V_{GL}$ to the subgroup $G$ via $\eta: G \hookrightarrow \text{GL}(2w)$. 
By combining this diagram with the isomorphism (7.4) and the morphism $\eta_*$ from (7.5.1) (obtained via extension of structure groups), we get a commutative diagram

\[ \begin{array}{ccc}
\text{Gies}_G(C_A) & \xrightarrow{\eta_*} & \text{Gies}_{GL(2w)}(C_A) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\bar{j}^* (\text{GVB}_{2w}(C_A)) & \xrightarrow{f} & \text{GVB}_{2w}(C_A) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\text{Bun}_{\bar{G}}^{\eta, \text{Sing}}(C_A) & \xrightarrow{\bar{j}} & \text{Tf}_{2w}(C_A)
\end{array} \]  

(14.0.4)

Loosely speaking, we may view the stack $\text{Gies}_G(C_A)$ as parametrizing pairs $(\mathcal{E}, \mathcal{S})$ with $\mathcal{E} \in \text{Gies}_C(\text{GL}(2w))(T)$ together with a reduction of structure group $\mathcal{S}$.

Thus, by composition with the direct image, we get a morphism $\text{Gies}_G(C_A) \to \text{Tf}_{2w}(C_A)$. This morphism factors via a vertical morphism $\text{Gies}_G(C_A) \to \text{Bun}_{\bar{G}}^{\eta, \text{Sing}}(C_A)$. To see this, we need to note the following on the torsion-free sheaf $p_*(E)$. The reduction of structure group $\mathcal{S}$ to $G$ away from the singularity on the normal surface $C_A$ extends to give a pseudo $G$-bundle $(p_*(E), \tau)$ (with notation as in [45, page 1428, Section 1.1]). This follows by the normality of the surface and a simple Hartogs argument (see for example, [7, Remark 2.5]).

On the other hand, the image under the unlabelled vertical morphism from $\text{Gies}_G(C_A) \to \bar{j}^* (\text{GVB}_{2w}(C_A))$ consists of pairs $(V_{\text{GL}}, \mathcal{S}')$ such that the $G$-torse given by the generic reduction of structure group $\mathcal{S}'$ extends to a full Gieseker torus over $M$ for a group algebraic space $\mathcal{G}_{(T),M}$ (which extends the semi-simple group scheme with fibre $G$). We denote the composite vertical $A$-morphism by

\[ \Upsilon_G : \text{Gies}_G(C_A) \to \text{Bun}_{\bar{G}}^{\eta, \text{Sing}}(C_A). \]  

(14.0.5)

which is analogous to taking direct images in the case of locally free sheaves.

14.2 A properness result

We recall the following definitions from [8, page 15].

**DEFINITION 14.1** (Horizontal properness)

Let $F, G : \text{Sch}_S \to \text{Sets}$ be two functors with $S = \text{Spec} \ A$ for a discrete valuation ring $A$ and quotient field $K$. Let $f : F \to G$ be a $S$-morphism. We say, $f$ is horizontally proper if the following property holds: let $B$ be a discrete valuation ring with function field $L$ such that $L$ is a finite extension of $K$ and $\text{Spec} \ B \to \text{Spec} \ A$ is surjective. Then for every map $\alpha \in F(L)$, if the composite $f(\alpha) \in G(L)$ extends to an element $G(A)$, then $\alpha$ also extends to an element in $F(A)$.

This definition becomes significant because of the following observation.

**Lemma 14.2.** Let $f : F \to G$ be a projective $S$-morphism of schemes of finite type such that $f_\zeta : F_\zeta \to G_\zeta$ over the generic point is proper. Suppose further that the structure morphisms $F \to S$ and $G \to S$ are surjective, that $F$ is $S$-flat and that $f$ is horizontally proper. Then $f$ is proper.
Using a stability parameter $\delta$, Schmitt [47, page 340] defines an open substack $\text{Bun}_{C}^{\eta, \text{Sing}}(G)^{ss}$ of $\delta$-(semi)stable pairs $(\mathcal{F}, \mathcal{S})$ and by using GIT (of decorated objects), he then goes on to construct the coarse moduli space $\mathcal{M}_{C}^{\eta, \text{Sing}}(G)^{ss}$ of this stack (as a subspace of a certain ‘big’ moduli space constructed by Bhosle [13]).

When $\delta$ is chosen ‘large’, Schmitt in [45, Theorem 1.1] showed that the generic fibre of Hartogs theorem, or by a theorem of Colliot-Thélène and Sansuc [15, Theorem 6.7], it is independent of the faithful representation $\eta$ and is in fact the moduli space of $S$-equivalence classes of stable $G$-torsors (in the sense of Ramanathan) and the special fibre is the moduli space $\mathcal{M}_{C}^{\eta, \text{Sing}}(G)^{ss}$ of semistable singular principal bundles. A small check using (13.5) shows that

$$\mathcal{A}_{G}^{\eta}(\text{Bun}_{G}^{\eta, \text{Sing}}(C_{A})^{ss}) \simeq \text{Gies}_{G}(C_{A})^{\text{ss}} \quad (14.2.1)$$

**Remark 14.3.** The identification (14.2.1) in particular shows that the substack $\text{Gies}_{G}(C_{A})^{\text{ss}}$ of $\text{Gies}_{G}(C_{A})$ of $\text{ss}$-semistable Gieseker torsors on $C_{A}$ is an open substack, thereby verifying the openness of the notion of $\text{ss}$-semistability as defined in (13.8).

**Theorem 14.4.** The $A$-morphism

$$\mathcal{A}_{G}^{\eta} : \text{Gies}_{G}(C_{A})^{\text{ss}} \to \text{Bun}_{G}^{\eta, \text{Sing}}(C_{A})^{ss} \quad (14.4.1)$$

is horizontally proper and an isomorphism over $C_{K}$. Over the closed point $a \in A$, it induces a morphism

$$\mathcal{A}_{G}^{\eta} : \text{Gies}_{G}(C)^{\text{ss}} \to \text{Bun}_{G}^{\eta, \text{Sing}}(C)^{ss} \quad (14.4.2)$$

**Proof.** Let $A' = \text{Spec } k[z]$ with $z^{d} = t$ let $L = \text{Spec } k(((z)))$ be the function field of $A'$. Let $E_{L}$ be a family of semistable $G$-bundles on the smooth curve $C_{L}$ degenerating to a semistable singular principal $G$-bundle $E_{0}$ on $(C, c)$. Equivalently, on the nodal curve $(C, c)$, there exists a pair $(\mathcal{F}, \mathcal{S})$, with a torsion-free $O_{C}$-module $\mathcal{F}$ of rank $2w$ together with a reduction of structure group $\mathcal{S}$ on $C - c$ to $G$ such that the family $E_{L}$ degenerates to $(\mathcal{F}, \mathcal{S})$ which is $\delta$-semistable. By definition, the reduction of structure group $\mathcal{S}$ gives rise to a $G$-torsor $E_{A' - c}$ on $C_{A' - c}$. The local type of the torsion-free sheaf gives also a diagram as in (2.2.3).

The diagram is also such that the pull-back $\sigma_{A'}^{*}(E_{A' - c})$ gives a $(\mu_{d}, G)$-torsor $P$ on $D - o$ (since the frame $GL(2w)$-bundle of $\mathcal{F}|_{C - c}$ comes with a reduction of structure group to $G$ in the complement of $c$ in $N(d)$). By the smoothness of $D$, and an application of Hartogs theorem, or by a theorem of Colliot-Thélène and Sansuc [15, Theorem 6.7], it follows that we get an extension of $P$ to a $(\mu_{d}, G)$-torsor $\tilde{P}$ on $D$. This is of some local type $\tau$.

Therefore, $q^{*}(\tilde{P})$ gives a $E(G, \tau)$-torsor on $D^{(d)}$ and by taking invariant direct images we get $\mathcal{P} = \text{Inv} \circ f_{*}(q^{*}(\tilde{P}))$ such that $(\mathcal{X}_{\tau, N(d)}, \mathcal{P})$ is an admissible pair on $N(d)$. This can be glued to the $G$-torsor pulled back from $C_{A' - c}$ to get an admissible pair $(\mathcal{X}_{\tau, C_{A' - c}}, \mathcal{P}_{A'})$ on $C_{A' - c}$.

By definition the parabolically associated vector bundle $\mathcal{D}_{A'}(W \oplus W^{*})$ via the homomorphism $\eta : G \to GL(W \oplus W^{*})$ is a quasi-admissible vector bundle on $C_{A'}^{(d)}$ and
furthermore, we have
\[ p_*(\mathcal{P}_o(W \oplus W^*)) = \mathcal{F}. \quad (14.4.3) \]

Since \((\mathcal{F}, \bar{s})\) is a semistable singular bundle, by (13.5), this implies that the \(\mathcal{P}_o\) is \(\tau\)-\(\mathfrak{f}\)-semistable. Clearly, \(\mathcal{P}_L \cong E_L\) and hence, the family \((k^G_{\tau, C_A}, \mathcal{P}_{A'})\) gives the required point in \(\text{Gies}_G(C_A)^{\tau, \mathfrak{f}-\text{ss}}(A')\) proving the horizontal properness. \[\square\]

14.3 The coarse moduli

Let \(\text{Bun}^\eta_{C_A} \rightarrow \mathcal{M}_{C_A}^\eta, \text{Sing}(G)^{ss}\) be the canonical morphism to the coarse space.

Let \(R^\eta_{C_A}(G)\) denote the total family so that the coarse space
\[ \mathcal{M}_{C_A}^\eta, \text{Sing}(G)^{ss} = R^\eta_{C_A}(G) // \text{PGL}_A \quad (14.4.4) \]
is realized as a GIT quotient of \(R^\eta_{C_A}(G)\) by a suitable reductive group \(\text{PGL}_A\).

We recall the definition of the functor \(\mathcal{G}_G\) as also the \(\tilde{C}\)-scheme \(\tilde{C}G\) which represents it (see (7.6)). The diagram (14.0.4) gives a diagram of total families
\[ \begin{diagram}
\tilde{C}G & \xrightarrow{\mathcal{G}} & \tilde{C} \\
\mathcal{G}_G \downarrow & & \downarrow p_* \\
\mathcal{G}_G^\eta, (G)^{ss} & \rightarrow & R^\eta_{C_A}(G) \\
\end{diagram} \quad (14.4.5) \]
where the morphism \(\mathcal{G}_G\) is horizontally proper (14.4). Further, since the objects are projective, by (14.2, 7.7), it follows that \(\mathcal{G}_G\) is proper.

At the level of total families we also have the identification
\[ (\tilde{C}G)^{\tau, \mathfrak{f}-\text{ss}} = \mathcal{G}_G^* (R^\eta_{C_A}(G)^{ss}) \quad (14.4.6) \]
as an open subscheme of \(\tilde{C}G\).

We are now in the precise setting of the theme in Nagaraj and Seshadri [39, Remark 6, page 180, 184]. We consider the polarization
\[ L_\epsilon := \mathcal{G}_G^* (\Theta_{C_A}^\eta (G)(1)) \otimes (\Theta_{\tilde{C}G}(\epsilon)), \quad (14.4.7) \]
where \(\Theta_{\tilde{C}G}(1)\) is the relative polarization for \(\mathcal{G}_G\). By choosing a ‘small’ \(\epsilon\) for the relative polarization, we get a natural notion of (semi)stability (which we term \(L_\epsilon\)-stability), on \((\tilde{C}G)^{\tau, \mathfrak{f}-\text{ss}}\) (and hence on \(\text{Gies}_G(C_A)^{\tau, \mathfrak{f}-\text{ss}}\)) which is such that \(L_\epsilon\)-stability implies \(\tau\)-\(\mathfrak{f}\)-(semi)stability and via (14.2.1) we get an inclusion
\[ \text{Gies}_G(C_A)^{L_\epsilon-\text{ss}} \subset \text{Gies}_G(C_A)^{\tau, \mathfrak{f}-\text{ss}} \quad (14.4.8) \]
(an inclusion which is a proper one in general (see [39, page 185])). In fact, by GIT, the \(L_\epsilon\)-(semi)stability constructs the actual separated coarse space \(\mathcal{M}_{C_A}^{L_\epsilon-\text{ss}}(k^G_{A}) = (\tilde{C}G)^{L_\epsilon-\text{ss}} // \text{PGL}_A\), for the Artin stack \(\text{Gies}_G(C_A)^{L_\epsilon-\text{ss}}\).

We summarise this discussion, using (7.7) and by following the arguments in [39], to arrive at the following main theorem.
Theorem 14.5.

1. The stack $\text{Gies}_G(C_A)^{L-ss} \subset \text{Gies}_G(C_A)$ is an algebraic stack, which is locally of finite type and flat over $A$.

2. The generic fibre $\text{Gies}_G(C_K)^{L-ss}$ is isomorphic to the algebraic stack $\text{Bun}_G(C_K)^{\mu-ss}$ of $\mu$-(semi)stable $G$-torsors on the smooth projective curve $C_K$ and the closed fibre $\text{Gies}_G(C)^{L-ss}$ is a divisor with normal crossings.

3. The closed fibre has an open subscheme comprising of (semi)stable $G$-torsors on the nodal curve $(C, c)$, where (semi)stability is the intrinsic one from (13.5).

4. The coarse space $\mathcal{M}^{L-ss}_{CA}(\mathfrak{g}_A^G) = (Y^G)^{L-ss}/\text{PGL}_A$, for the Artin stack $\text{Gies}_G(C_A)^{L-ss}$ therefore provides a proper and flat degeneration of the moduli space of $\mu$-(semi)stable $G$-torsors on smooth curves degenerating to a simple nodal curve.

5. There is a morphism of coarse spaces
   \[
   p_* : \mathcal{M}^{L-ss}_{CA}(\mathfrak{g}_A^G) \rightarrow \mathcal{M}^{\eta, \text{Sing}, (G)^{ss}}_{CA}(\mathfrak{g}_A^G)
   \]
   The closed fibre $\mathcal{M}^{L-ss}_{CA}(\mathfrak{g}_A^G)$ of the coarse moduli scheme $\mathcal{M}^{L-ss}_{CA}(\mathfrak{g}_A^G)$ parametrizes $\mathcal{S}$-equivalence classes of $L$-semistable Gieseker torsors. This scheme contains an open dense subscheme of $\mathcal{E}$-semistable $G$-torsors on the underlying nodal curve $(C, c)$.

Remark 14.6. Recall that the moduli scheme $\mathcal{M}_{CA}^{\eta, \text{Sing}, (G)^{ss}} \rightarrow S$ provides a degeneration of the moduli spaces $\mathcal{M}_{CA}(G)$, but the drawback with this construction is that $\mathcal{M}_{CA}^{\eta, \text{Sing}, (G)^{ss}}$ is not $A$-flat.

14.4 The orthogonal and symplectic case

Recall that Faltings [17] has constructed the moduli space of semistable orthogonal and symplectic torsion-free sheaves on nodal curves and gets a flat degeneration of the moduli space of semistable orthogonal and symplectic bundles on smooth curves when the curves degenerate to a simple nodal curve. By the comments in [45, page 1430, 1436], we see that under the faithful representation $\eta : G \hookrightarrow \text{GL}(W \oplus W^*)$ the image lies in the orthogonal (resp. symplectic) group $O(W \oplus W^*, q)$ (resp. $\text{Sp}(W \oplus W^*, q)$) where $W \oplus W^*$ is seen to be equipped with a canonical non-degenerate symmetric (resp. alternating) form $q$. One could more generally have worked with any pair $(W, q)$ where $W$ is equipped with a non-degenerate symmetric (resp. alternating) form $q$ and carried out the entire construction of the coarse spaces $\mathcal{M}^{L-ss}_{CA}(\mathfrak{g}_A^G)$. It is now easy to conclude from the main results that in the case when $G$ is either orthogonal or symplectic, then the moduli space $\mathcal{M}_{CA}^{\eta, \text{Sing}, (G)^{ss}}$ is the moduli space of [17] and the morphism $p_* : \mathcal{M}^{L-ss}_{CA}(\mathfrak{g}_A^G) \rightarrow \mathcal{M}_{CA}^{\eta, \text{Sing}, (G)^{ss}}$ is a surjection.

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Appendices

Appendix A. Appendix to Part I

A.1 Quasi-Gieseker bundles

In the first part of the appendix, we will outline a small variant of the theme developed in [24,27,38]. We recall the notion of an admissible vector bundle $V$ on a curve $C^{(d)}$ [27, Definition 3.11], [8, Definition 3.6] and add a variant, namely the notion of a quasi-admissible bundle. In fact, Kiem and Li in [32, Lemma 1.2(a)] just call these admissible bundles. In [38, Definition 1, page 167], we have the notion of a standard vector bundle on $C^{(d)}$ as a preliminary notion.

**DEFINITION A1.1**

Let $V$ be a vector bundle of rank $n$ on a chain $E^{(d)}$. Let $V|_{R_j} = \bigoplus_{j=1}^{n} \mathcal{O}(a_{ij})$, where the $R_j$ are the $\mathbb{P}^1$'s on the chain $E^{(d)}$. Say that $V$ is standard if the $a_{ij}$ are 0 or 1. The bundle $V$ is called strictly standard if moreover, for every $i$ there is an index $j$ such that $a_{ij} = 1$.

A vector bundle $V$ on $C^{(d)}$ of rank $n$ is called admissible (resp. quasi-admissible), if, for $d \geq 1$, the restriction $V|E^{(d)}$ is strictly standard (resp. standard) and the direct image $(p_d)_*(V)$ is a torsion-free $\mathcal{O}_C$-module, where $p_d : C^{(d)} \to C$ is the canonical morphism which contracts the chain to the node.

The notion of admissibility (resp. quasi-admissibility) extends obviously to vector bundles on any modification $\mathcal{M}$ over $T \in \text{Sch}/A$. Let $V$ be a standard vector bundle on $C^{(d)}$ of rank $(V) = n$. Then, by the discussions in [38, page 168-171], after twisting the vector bundles sufficiently to ensure the vanishing of the first cohomology and ensure generation by sections, we get a canonical morphism $\phi_V : C^{(d)} \to \text{Grass}(H^0(V), n)$. This morphism contracts the $R_j = \mathbb{P}^1$'s on the chain $E^{(d)} \subseteq C^{(d)}$ such that the restriction $V|_R_j$ is trivial. The condition that $V$ is strictly standard is shown to be equivalent to the morphism $\phi_V$ being a closed immersion.

Let $N := \dim(H^0(V))$. Let $W[j]$ be the $j$-th standard model with $C^{(j)} \subset W[j]$ as the central fibre (4.1). Recall that this a smooth quasi-projective scheme with a tautological morphism $W[j] \to C \times_A B[j]$. For each $j \leq n$, fix the coordinate plane embedding $\mathbb{A}^{j+1} \subset \mathbb{A}^{n+1}$ by the first coordinates. This gives an identification $W[j] \simeq W[n] \times B[n]$ compatible with the tautological morphism [33, page 526]. Define

$$W(N,n) := W[n] \times_A \text{Grass}(N,n).$$

If $V$ is a standard bundle on $C^{(j)}$ for some $j$, we get a closed immersion

$$C^{(j)} \subset W(N,n),$$

via the inclusions $\text{Graph}(\phi_V) \subset C^{(j)} \times \text{Grass}(N,n) \subset W(N,n)$.

Following [24, page 179] and [38, Definition 7, page 185], we have the definition.

**DEFINITION A1.2**

Let $\mathcal{G}_N^3 : \text{Sch}_A \to \text{Sets}$, be the functor defined as

$$\mathcal{G}_N^3(T) = (\mathcal{M}, \varepsilon),$$

(A1.3)
where

$$\epsilon : \mathcal{M} \hookrightarrow C_A \times_A T \times_k W(N, n)$$  \hfill (A1.4)

is a closed embedding in the product and such that, (a) the projection \( j : \mathcal{M} \to T \times_k W(N, n) \) is a closed immersion, (b) the projection \( \pi : \mathcal{M} \to C \times_A T \) is a modification as in Definition 6.1, and (c) the projection \( q_T : \mathcal{M} \to T \) is a flat family of curves \( \mathcal{M}, t \in T \) as in Definition 6.1. (d) Moreover, the chain lengths \( d \) occurring in \( \mathcal{M} \) is bounded above by \( n \).

Further, if \( V \) is the tautological quotient bundle of rank \( n \) on \( \text{Grass}(N, n) \) and \( V_T \) its pull-back to \( T \times W(N, n) \), then the pull-back \( \mathcal{V}_T \) is such that, \( \mathcal{V}_T \) is a quasi-admissible vector bundle of rank \( n \) (A1.1) for the modification \( \mathcal{M} \to C \times_A T \).

By the definition of \( \mathcal{V}_T \), for each \( t \in T \) we get a quotient morphism \( \mathcal{O}_\mathcal{M}^N \to \mathcal{V}_t \), and we assume that this map induces an isomorphism: \( H^0(\mathcal{O}_\mathcal{M}^N) \cong H^0(\mathcal{V}_t) \). In particular, we have \( \dim(H^0(\mathcal{V}_t)) = N \) and it follows that

$$H^1(\mathcal{V}_t) = 0.$$  \hfill (A1.5)

As in [24] and [39, Proposition 8], it is easily seen that this new functor \( \mathcal{G}_N^2 \) is also represented by a \( \text{PGL}(N) \)-invariant open subscheme \( \mathcal{Y} \) of the Hilbert scheme \( \text{Hilb}(C_A \times_A W(N, n)) \) for the natural polarization on \( W(N, n) \).

Let \( \mathcal{M} \subset C_A \times_A \mathcal{Y} \times_k W(N, n) \) be the universal object defining the functor \( \mathcal{G}_N^2 \). This defines a universal modification \( \mathcal{M} \to \mathcal{Y} \) together with a universal quasi-admissible vector bundle \( \mathcal{V} \) on \( \mathcal{M} \). The representability of the functor \( \mathcal{G}_N^2 \) implies that for any quasi-admissible vector bundle \( \mathcal{V} \) on a modification \( \mathcal{M} \) there exists a unique morphism \( \psi : T \to \mathcal{Y} \) and \( \phi : \mathcal{M}_T \to \mathcal{M} \) so that \( \phi^*(\mathcal{V}) = \mathcal{V} \).

The stack \( \text{GVB}_n^2(C_A) \) (cf. [27, Definition 3.11]): \( (\mathcal{M}, \mathcal{V}) \in \text{GVB}_n^2(C_A)(T) \) is such that (1) \( \mathcal{V} \) is a quasi-admissible vector bundle on the modification \( \mathcal{M} \) and (2) \( d \leq n \) for chains \( E^{(d)} \) in \( \mathcal{M} \). We may call \( (\mathcal{M}, \mathcal{V}) \) a quasi-Gieseker bundle. Modifications with bounded chain lengths is easily seen to be a stack and \( \text{GVB}_n^2(C_A) \) is easily checked to be an Artin stack.

As in [27, Definition 3.22], if we fix a very ample sheaf on \( C \). Then for a quasi-Gieseker vector bundle \( (\mathcal{M}, \mathcal{V}) \) for \( T \) a \( A \)-scheme and for an integer \( N' \) we have the quasi-admissible bundle \( \mathcal{V}(N') \) and for every pair of integers \( N \geq n, N' \geq 0 \), we have a canonical morphism of \( A \)-groupoids

$$\mathcal{G}_N^2 \to \text{GVB}_n^2(C_A).$$  \hfill (A1.6)

Analogous to [27, Lemma 3.23], given a quasi-Gieseker bundle \( (\mathcal{M} \to C \times_A T, \mathcal{V}) \), we again have an open subschemes \( T_{N,N'} \subset T \) which has properties (1) and (2) in [27, Lemma 3.23], with the added observation that the scheme \( W(N, n) \), which replaces the Grassmannian in loc cit, ensures that \( \forall t \in T_{N,N'} \), the induced morphism \( \mathcal{M}_t \to W(N, n) \) is a closed immersion.

For the analogue of [loc cit], we need to do a bit more.

**PROPOSITION A1.3**

The morphism of \( A \)-groupoids

$$\coprod_{N \geq n, N' \geq 0} \mathcal{G}_N^2 \to \text{GVB}_n^2(C_A)$$  \hfill (A1.7)
is smooth and surjective.

Proof. Let $T$ be a $A$-scheme and let $T \to \text{GVB}_n^q(C_A)$ a $T$-point on $\text{GVB}_n^q(C_A)$ given by a quasi-Gieseker bundle $(\mathcal{M} \to C \times_A T, \mathcal{Y})$. Let $Z$ be the $A$-groupoid defined by the cartesian square

\[
\begin{array}{ccc}
Z & \longrightarrow & \mathcal{G}_N^q \\
\downarrow & & \downarrow \\
T & \longrightarrow & \text{GVB}_n^q(C_A)
\end{array}
\]

Let $\{T_\alpha\}$ be en étale cover of $T$ so that we have a morphism $T_\alpha \to B[d_\alpha]$ and modification $\mathcal{M}_\alpha$ comes as a pull-back. For each quasi-Gieseker bundle $(\mathcal{M}_\alpha, \mathcal{Y}_\alpha)$, we again have open subschemes $T_{N,N',\alpha} \subset T_\alpha$ with properties as stated above. We in fact have a morphism $\mathcal{M} |_{T_{N,N',\alpha}} \longrightarrow T_{N,N',\alpha} \times \text{Grass}(N, n)$ and hence a morphism $\mathcal{M} |_{T_{N,N',\alpha}} \longrightarrow T_{N,N',\alpha} \times \text{Grass}(N, n)$. This morphism is proper and for each $\forall \in T_{N,N',\alpha}$, the induced morphism $\mathcal{M}_\alpha \to \text{Grass}(N, n)$ is a closed immersion. Hence by [27, Lemma 3.13], we get a closed immersion $\mathcal{M} |_{T_{N,N',\alpha}} \longrightarrow T_{N,N',\alpha} \times \text{Grass}(N, n)$.

Let $Z_\alpha = T_\alpha \times_T Z$. Then, following the arguments in [27, page 4913], we again have the identification $Z_\alpha = \text{Isom}(\mathcal{O}_{T_{N,N',\alpha}}, \pi_*(\mathcal{Y}_\alpha)(N') |_{T_{N,N',\alpha}})$, where $\pi : \mathcal{M}_\alpha \to T_\alpha$. Thus, $Z_\alpha$ is smooth and surjective over $T_{N,N',\alpha}$ and since the $T_{N,N',\alpha}$ cover $T_\alpha$ for each $\alpha$ we are done. \hfill $\square$

Remark A1.4. The analogues of [27, Theorem 3.21] hold without any serious difficulty. In particular, the deformation theory works to show that $\mathcal{Y}$ is regular, its generic fibre over $A$ is smooth while its special fibre $\mathcal{Y}_\circ$ is a divisor with normal crossings. The proof of (7.7) gets easily adapted to this case.

### A.2 Kawamata coverings

Let $X$ be a smooth quasi-projective variety and let $D = \sum_{i=1}^r D_i$ be the decomposition of the simple or reduced normal crossing divisor $D$ into its smooth components (intersecting transversally). The ‘covering lemma’ of Kawamata [30, Theorem 17] (see also [54, Lemma 2.5, page 56]) says that, given positive integers $N_1, \ldots, N_r$, there is a connected smooth quasi-projective variety $Z$ over $\mathcal{C}$ and a Galois covering morphism

$$\kappa : Z \to X$$

such that the reduced divisor $\kappa^*D := (\kappa^*D)_{\text{red}}$ is a normal crossing divisor on $Z$ and furthermore, $\kappa^*D_i = N_i.(\kappa^*D_i)_{\text{red}}$. Let $\Gamma$ denote the Galois group for the covering map $\kappa$.

The isotropy group of any point $z \in Z$, for the action of $\Gamma$ on $Z$, will be denoted by $\Gamma_z$. It is easy to see that the stabilizer at generic points of the irreducible components of $(\kappa^*D_i)_{\text{red}}$ are cyclic of order $N_i$. By an equivariant principal $G$-torsor $P$ on $Z$ of local type $\tau = \{\tau_i\}_{i=1}^r$, we mean

1. the restriction of the $G$-torsor $P_{U_z}$ to an étale neighbourhood at a generic point $z$ of an irreducible component of $(\kappa^*D_i)_{\text{red}}$ is given by a representation $\rho_i : \Gamma_z \to G$;
(2) for a general point \( y \) of an irreducible component of a ramification divisor for \( \kappa \) not contained in \((\kappa^*D)_{\text{red}}\), the action of \( \Gamma_y \) on \( P \) is the trivial action.

Such a \( P \) will always exist as an algebraic space with a \( G \)-action and can be obtained by gluing trivial \((\Gamma_z, G)\)-torsors given by \( \rho_i \), in \( U_z \) for the generic point \( z \) of \((\kappa^*D)_{\text{red}}\) with pull-backs of \( G \)-torsors on \( X \setminus D \) to \( Z \). By a Hartogs type argument, it is easily checked that equivariant \( G \)-torsors are uniquely defined on \( Z \) once given on a subscheme of codimension bigger than 1.

**Appendix B. Appendix to Part II**

**B.1 Laced vector bundles**

In this subsection we analyse the special case of laced torsors when \( G \) is the linear group. Much of the early material in this subsection is adapted from [48].

**Notation B1.1.** Let \( \text{Vect}^d_{(\tilde{\mathcal{C}}, z)} \) denote the category of vector bundles \( W \) on \((\tilde{\mathcal{C}}, z)\), i.e., balanced vector bundles on \((\tilde{\mathcal{C}}, z)\) with descent datum (12.0.4) which translates as an isomorphism \( V_{z_1} \cong V_{z_2}^* \).

**Definition B1.2**

A balanced parabolic structure on a vector bundle \( V \) of rank \( n \) on a doubly marked curve \((\tilde{\mathcal{C}}, c)\) is given by the following datum:

1. For \( 1 \leq s \leq n \), weights, \((\alpha_1, \ldots, \alpha_s)\), which are rational numbers such that
   \[ 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_s < 1. \]  
   (B1.1)

   and ‘dual weights’:
   \[
   (\beta_1, \ldots, \beta_s) = \begin{cases} 
   (1 - \alpha_s, 1 - \alpha_{s-1}, \ldots, 1 - \alpha_1) & \text{if } \alpha_1 \neq 0 \\
   (0, 1 - \alpha_s, \ldots, 1 - \alpha_2) & \text{if } \alpha_1 = 0.
   \end{cases}
   \]  
   (B1.2)

2. A balanced parabolic structure on \( V \) at \( c_j \), \( j = 1, 2 \), i.e., strictly decreasing flags
   \[ V_{c_j} = \mathcal{F}^1_{c_j} \supset \mathcal{F}^2_{c_j} \supset \cdots \supset \mathcal{F}^s_{c_j} \supset \mathcal{F}^{s+1}_{c_j} = 0, \quad j = 1, 2 \]  
   (B1.3)

   together with weights are given as follows:
   - The weight of \( \mathcal{F}^m_{c_1} \) is \( \alpha_m \), where \( \alpha_1, \ldots, \alpha_s \) as in (B1.1).
   - The weight of \( \mathcal{F}^m_{c_2} \) is \( \beta_m \), where \( \beta_1, \ldots, \beta_s \) are as in (B1.2).

Let \( \text{PVect}^\text{bal}_{(\tilde{\mathcal{C}}, c)} \) denote the category of vector bundles on \((\tilde{\mathcal{C}}, c)\) with balanced parabolic structure.

Let \( V \) be an object in \( \text{PVect}^\text{bal}_{(\tilde{\mathcal{C}}, c)} \).

1. The flag \( \mathcal{F}_{c_2} \) and \( V_{c_2} \) induces on the dual \( V_{c_2}^* \) of \( V_{c_2} \), the natural dual flag \( \mathcal{F}_{c_2}^* \) and the weights of \( \mathcal{F}_{c_2}^* \) are ‘dual’ to those of \( \mathcal{F}_{c_2} \), i.e., they coincide with \( \alpha_1, \ldots, \alpha_s \), the weights associated to \( \mathcal{F}_{y_1} \).
(2) For $i = 1, 2$, define

$$\text{gr}(V_{c_i}) := \bigoplus_m \text{gr}^m \mathcal{F}_{c_i}, \text{ with}$$

$$\text{gr}^m \mathcal{F}_{c_i} := \mathcal{F}_{c_i}^m / \mathcal{F}_{c_i}^{m+1}. \tag{B1.5}$$

The graded pieces, $\text{gr}(V_{c_2}^*)$, gets identified with $\text{gr}(V_{c_2})$ by a shifting of degrees as follows:

$$\begin{cases} 
\text{gr}^m \mathcal{F}_{c_2}^* = \text{gr}^{s+1-m} \mathcal{F}_{c_2}^* \text{ for } 1 \leq m \leq s, \text{ if } \alpha_1 \neq 0 \\
\text{gr}^1 \mathcal{F}_{c_2}^* = \text{gr}^1 \mathcal{F}_{c_2} \text{ and } \text{gr}^m \mathcal{F}_{c_2}^* = \text{gr}^{s+2-m} \mathcal{F}_{c_2} \text{ if } \alpha_1 = 0. 
\end{cases} \tag{B1.6}$$

**DEFINITION B1.3**

Let $V$ be an object in $\text{PVect}_\mathbb{C}^{\text{bal}}(\tilde{C}, c)$. A **lacing** on $V$ (or more precisely a $s$-**lacing**) is a $s$-tuple

$$\varrho := \{\varrho_m : \text{gr}^m \mathcal{F}_{c_1} \to \text{gr}^m \mathcal{F}_{c_2}^*\}_{m=1}^s \tag{B1.7}$$

of linear isomorphisms.

**DEFINITION B1.4**

A balanced parabolic vector bundle endowed with a lacing $\varrho$ will be called a **laced vector bundle**, i.e., given by the datum

$$V_\varrho := (V_*, \varrho), \tag{B1.8}$$

where $V_*$ is a balanced parabolic bundle on $(\tilde{C}, c)$.

**DEFINITION B1.5**

The parabolic degree of a laced bundle $V_\varrho$ is defined as

$$\text{par.deg}_\mathbb{C}(V_\varrho) := \text{par.deg}_\mathbb{C}(V_*). \tag{B1.9}$$

**Lemma B1.6.** Let $V_\varrho$ be a laced bundle on $(\tilde{C}, c)$ and let $k = k_1$ denote the multiplicity of the weight $\alpha_1$. Let $l = (n - k)$. Then

$$\text{par.deg}_\mathbb{C} V_\varrho = \deg V + (n - k) = \deg V + l. \tag{B1.10}$$

As a consequence, the parabolic degree of a laced bundle on $(\tilde{C}, c)$ does not depend on the choice of the parabolic weights.

**Proof.** [48]. By the definition of parabolic degree, we see that

$$\text{par.deg}_\mathbb{C} V_\varrho = \begin{cases} 
\deg V + \sum_{m=1}^s k_m \alpha_m + \sum_{m=1}^s k_m (1 - \alpha_m) \text{ if } \alpha_1 \neq 0 \\
\deg V + \sum_{m=2}^s k_m \alpha_m + \sum_{m=2}^s k_m (1 - \alpha_m) \text{ if } \alpha_1 = 0.
\end{cases} \tag{B1.11}$$
Hence
\[
\text{par.deg}_{\tilde{C}} V_\varphi = \begin{cases} 
\deg V + n & \text{if } \alpha_1 \neq 0 \\
\deg V + (n - k_1) & \text{if } \alpha_1 = 0.
\end{cases} \tag{B1.12}
\]
which gives the equation (B1.10).

We summarize the following from [48].

**PROPOSITION B1.7**

Let \( V_\varphi \) be a laced bundle on \( \tilde{C} \). Then the direct image \( v_*(V_\varphi) = \mathcal{F} \) is a torsion-free sheaf on \( C \) and conversely, \( v^*(\mathcal{F})/\text{tors} \) recovers the underlying vector bundle of \( V_\varphi \). Moreover,

\[
\text{par.deg}_{C}(V_\varphi) = \deg_{C}(\mathcal{F}). \tag{B1.13}
\]

### B.2 Some remarks on parabolic subgroups

**Remark B2.1.** Let \( \lambda : \mathbb{G}_m \to G \) be a one-parameter subgroup and \( P(\lambda) \) be the associated parabolic subgroup and \( H(\lambda) \) the Levi quotient which canonically defines a Levi subgroup \( L(\lambda) \) as the centralizer of \( \lambda \). Let \( \eta : G \hookrightarrow \text{GL}(W) \) be a faithful representation. Then the one-parameter subgroup given by the composition \( \eta \circ \lambda : \mathbb{G}_m \to \text{GL}(W) \) defines a parabolic and Levi subgroups \( P(\eta \circ \lambda) \) and \( L(\eta \circ \lambda) \) of \( \text{GL}(W) \).

We can view the parabolic subgroup \( P(\eta \circ \lambda) \) as the stabilizer of the flag
\[
(W_\bullet(\lambda), \epsilon_\bullet) : 0 \subsetneq W_1 \subsetneq W_2 \cdots W_s \subsetneq W_{s+1} = W \subsetneq W
\]
where \( W_i : = \bigoplus_{j=1}^i W^j \), with \( W^j \) being the eigenspace of the \( \mathbb{G}_m \)-action via \( \lambda \) for the character \( z \mapsto z^{\gamma_j} \), and \( \gamma_1 < \cdots < \gamma_{s+1} \) are the distinct weights which occur. Set \( \epsilon_i : = (\gamma_{i+1} - \gamma_i)/\dim(V) \), \( i = 1, \ldots, s \). The pair \( (W_\bullet(\lambda), \epsilon_\bullet) \) is called the associated weighted filtration of \( \lambda \). The weighted filtration \( (W_\bullet(\lambda), \epsilon_\bullet) \) has an associated graded:

\[
gr_{\lambda}(W) := \bigoplus_{j=1}^s W_{j+1}/W_j = \bigoplus_{j=1}^s W^j
\]
and it is easy to see that as \( H = H(\lambda) \)-modules, \( W \simeq gr_{\lambda}(W) \). Further, \( H \) fixes the \( \lambda \)-eigenspaces \( W^{j+1} \), i.e., the above decomposition is a decomposition of \( H \)-modules. We also have an obvious weighted filtration \( (gr_*(W)^\bullet, \epsilon) \) with the same weights \( \epsilon \):

The 1-PS \( \eta \circ \lambda \) also defines a canonical anti-dominant character \( \chi_{\eta \circ \lambda} : P(\eta \circ \lambda) \to H(\eta \circ \lambda) \to \mathbb{G}_m \) dual to \( \eta \circ \lambda \) [47, 2.4.9]. For instance, if \( m : = (m_1, \ldots, m_{s+1}) \) is a point of the Levi \( H(\eta \circ \lambda) \) as block matrices, then \( \chi_{\lambda}((m)) : = \otimes_j \det(m_j)^{\epsilon_j} \). This which restricts to an anti-dominant character \( \chi_{\lambda} \) of \( P(\lambda) \). We recall the following result.

**Lemma B2.2** ([47, Proposition 2.4.9.1]). Let \( \chi : P(\lambda) \to \mathbb{G}_m \) be any anti-dominant character. Then there is a positive rational number \( r \) such that \( \chi = r \cdot \chi_{\lambda} \).
Remark B2.3. Let $E$ be a $G$-torsor and suppose that we are given a reduction of structure group $E_P \subset E$ to the parabolic $P$. There is a canonical anti-dominant character $\chi_\lambda : P \rightarrow \mathbb{G}_m$ (B2.2) which defines a line bundle $E_P(\chi_\lambda)$ on $Y$.

Again, the representation $\eta$ gives a weighted filtration (B2.1) stabilized by $P(\eta \circ \lambda)$. We can take the associated vector bundle $E_P(W)$ which comes with its weighted filtration

$$(E_P(W)_\bullet, \epsilon) : 0 \subsetneq E_P(W_1) \subsetneq \cdots \subsetneq E_P(W_s) \subsetneq E_P(W_{s+1}) = E_P(W)$$

and the weighted slope defined by Schmitt [43]:

$$L((E_P(W)_\bullet, \epsilon_\bullet))$$

$$:= \sum_{i=1}^s \epsilon_i \{ \deg_C(E_P(W)) \cdot rk E_P(W_i) - \deg E_P(W_i) \cdot rk(E_P(W)) \}.$$  

(B2.4)

Claim.

$$\deg(E_P(\chi_\lambda)) = L((E_P(W)_\bullet, \epsilon_\bullet)),$$

$$\deg(E_H(\chi_\lambda)) = L((E_H(\text{gr}(W))_\bullet, \epsilon_\bullet)).$$  

(B2.5)

(B2.6)

To see this, note that the line bundle $E_H(\chi_\lambda) \simeq \bigotimes \det(E_H(W_i))^{-\epsilon_i}$ with $\epsilon_i := (\gamma_{i+1} - \gamma_i) / \dim(V)$ as above (see [47, Exercise 2.4.9.2, page 209]).

Remark B2.4. Let $Y$ be a smooth projective curve and let $P_Y$ be a parahoric group scheme generically split with fibre $G$, with parahoric structures $\nu := (v_j)$ at points $y_j \in Y$ given by a tuple of points in the affine apartment $\mathcal{A}_T$ [10]. Given a faithful representation $\eta : G \hookrightarrow \text{GL}(W)$, we get a corresponding group parahoric group scheme $P_{\text{GL}(W)}$ with generic fibre $\text{GL}(W)$. If $E$ is a $P_Y$-torsor then we get an associated parabolic vector bundle $E(W)_\bullet$ with parabolic structures at $y_j$. If $\lambda : \mathbb{G}_m \rightarrow G$ is a 1-PS, and the setting be as in the previous paragraph, then we have a parahoric Levi-type torsor $E_H$ for a parahoric group scheme with generic fibre isomorphic to $H$ and associated parabolic line bundles $E_H(\chi_\lambda)_\bullet$. The standard properties of degrees of direct sum of vector bundles in terms of the determinants obviously go through in the parabolic setting by replacing degrees with parabolic degrees and tensor products with parabolic tensor products. This follows by expressing parabolic bundles in terms of orbifold bundles and push-forwards. Thus the entire formalism goes through and we get a relation $\text{par.deg}(E_H(\chi_\lambda)_\bullet) = L((E_H(\text{gr}(W))_\bullet, \epsilon_\bullet))$ with parabolic degrees everywhere.

We apply it in the main paper for the laced bundle $E_\varphi$ on $\tilde{C}$ which has an underlying parahoric structure at the two points $c_i$.

B.3 A counter example to a simplistic generalization of Ramanathan’s definition in the nodal case

Let $E$ be a principal $G$-bundle on the nodal curve $C$. A naïve generalization of the usual definition along the lines of A. Ramanathan’s definition turns out to be false even when $G = \text{GL}(2)$. 


For every maximal parabolic subgroup $P \subset G$ and for every reduction of structure group $E_P$ of $E$ over $C - c$, consider the Lie algebra sub-bundle $E_P(p) \subset E(g)_{|C-c}$. Let $E_P(p)$ be the torsion-free sheaf which is the saturation of the sub-bundle $E_P(p)$ in $E(g)$ over $C$. The bundle $E$ is ‘conjecturally’ (semi)stable if
\[
\deg(E_P(p)) < 0(\leq 0).
\]
(B3.1)

For the failure of this ‘conjectural definition’ of (semi)stability of $G$-torsors on nodal curves even when $G = \text{GL}(2)$, we give the following counter-example which essentially comes from a remark due to Seshadri.

Let $L, M$ be torsion-free sheaves on $C$ of rank 1 and degree 0 which are not locally free. In particular, they are of local type $m$. Consider the group $\text{Ext}^1(L, M)$ of extensions of $M$ by $L$. We claim that there is a locally free sheaf $V$ such that
\[
0 \to L \to V \to M \to 0
\]
and hence automatically $V$ is semistable of degree 0. To see the existence of such a $V$, we consider the local-global spectral sequence for $\text{Ext}$ [26, Section 4.2] which gives (since $\dim(C) = 1$)
\[
H^1(C, \mathcal{H}\text{om}(L, M)) \to \text{Ext}^1(L, M) \to H^0(C, \mathcal{E}\text{xt}^1(L, M)) \to 0.
\]
(B3.3)

Note that $H^0(C, \mathcal{E}\text{xt}^1(L, M)) = \text{Ext}^1(C_E, \mathcal{M}_C)$, where $A = \mathcal{O}_{C,c} \simeq \mathbb{C}[x, y]/(xy)$. Locally we have $m = (x, y)$. Using these as generators, we have an embedding
\[
m \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C
\]
and hence an extension
\[
0 \to m \to \mathcal{O}_C \oplus \mathcal{O}_C\to m \to 0.
\]
(B3.4)

This gives an element in $\text{Ext}^1(C_E, \mathcal{M}_C)$ which lifts to give an element in $\text{Ext}^1(L, M)$. Clearly this extension is locally free since it is so at the node and we get the required $V$. This $V$ is semistable of degree 0.

Giving a reduction of structure group of the principal $\text{GL}(2)$-bundle underlying $V$ is expressing it in an exact sequence of vector bundles (B3.2) and the conjectural definition of semistability is equivalent to saying that for the sub-bundle $L \otimes M^* \subset V \otimes V^*$, we have
\[
\deg(L \otimes M^*) \leq 0,
\]
(B3.5)

where $\overline{L \otimes M^*}$ denotes the saturation in $V \otimes V^*$.

**Claim.**
\[
\deg(L \otimes M^*) = 1.
\]
(B3.6)

In particular, $V \otimes V^*$ is not semistable. Let $L'$ (resp. $M'$) denote $p^*(L)/\text{tors}$ (resp. $p^*(M)/\text{tors}$). Then the line sub-bundle of $p^*(V)$ (resp. $p^*(V^*)$) generated by $L'$ (resp. $M'$) is of the form $L'(y_1 + y_2)$ (resp. $M'(y_1 + y_2)$). We have $\deg L' = \deg M' = -1$, so that $\deg L'(y_1 + y_2) = \deg M'(y_1 + y_2) = 1$. Then we see that the line bundle
\[
N = (L'(y_1 + y_2) \otimes M'(y_1 + y_2)(-y_1 - y_2)
\]
(B3.7)
descends to a torsion free subsheaf of $V \otimes V^*$, which is the saturation $\overline{L \otimes M^*}$. Since $\deg N = 0$, we see that $\deg(L \otimes M^*) = 1$. 

Remark B3.1. The lesson is to avoid taking the saturation after taking tensor products. The degree exceeds the bound. Instead, one has to take some sort of a ‘parabolic tensor product’ and then take a saturation, both of these operations need to be carried out on the normalization $Y$. This can be made precise. We proceed differently in Section 13 to achieve this.

Abhyavasthāḥ praṇāyaṁte pra vavrer vavriś ciketa, upasthe mātur vi cashte
States upon states are born, covering over covering awakens to knowledge, in the lap of the universal mother he wholly sees.

Rig Veda, Mandala V, Hymn 19.1

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