Entanglement Transfer from Bosonic Systems to Qubits

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We study the entanglement of a pair of qubits resulting from their interaction with a bosonic system. Here we restrict our discussion to the case where the set of operators acting on different qubits commute. A special class of interactions inducing entanglement in an initially separable two qubit system is discussed. Our results apply to the case where the initial state of the bosonic system is represented by a statistical mixture of states with fixed particle number.

I. INTRODUCTION

The quantum correlations between subsystems present in entangled states are indispensable for many quantum communication protocols [1]. However, these correlations cannot be created by local operations and classical communication (LOCC). Therefore, in order to entangle two systems \( A_1 \) and \( A_2 \), it is necessary to apply a global operation on the joint system \( A_1 A_2 \). A simple global operation consists in letting systems \( A_1 \) and \( A_2 \) interact. In general, as a result of direct interactions, systems \( A_1 \) and \( A_2 \) become quantum correlated. On the other hand, entanglement can also be transferred from a third system \( B_1 B_2 \) to \( A_1 A_2 \). If systems \( B_1 \) and \( B_2 \) are entangled, then one can apply local operations on the pairs of systems \( (A_1 B_1) \) and \( (A_2 B_2) \). As a result of these operations it is possible to transfer the entanglement, originally in \( B_1 B_2 \), to the joint system \( A_1 A_2 \) (see Fig. 1(a)). This approach is useful in the case where \( A_1 \) and \( A_2 \) represent two distant systems or when they interact weakly (or do not interact at all). The entanglement transfer from flying qubits to localized qubits has been extensively studied (see [2] and references therein). Entanglement transfer from two qubit systems to two qubit systems was investigated in [3]. Moreover, in Fig. 1(a) the systems \( A_i, B_i, (i = 1, 2) \) may represent different degrees of a freedom of single particle, such as spin and momentum. In fact, in [4], it was shown that the entanglement between the momenta of two different particles can be transferred to the spins of the particles under Lorentz transformations. In the last few years, entanglement transfer from many body systems and relativistic quantum field has been considered. In [5], a scheme was proposed to extract entanglement from a quantum gas to a pair of qubits via local interactions. Also, it was shown in [6] that two qubits interacting locally with a quantum field can become entangled even when the qubits remain in causally disconnected regions throughout the whole interaction process. The previously described scenarios are depicted in Fig. 1(b) where systems \( A_1 \) and \( A_2 \) are coupled to a common system \( B \).

In the present paper we investigate the entanglement transfer scheme illustrated in Fig. 1(b) where \( A_1 \) and \( A_2 \) represent a pair of two level systems (qubits). Here, we assume that the group of operators involved in the interaction between \( A_1 \) and \( B \) commute with the operators coupling \( A_2 \) to \( B \). By doing so, we mimic the entanglement transfer scheme described in Fig. 1(a). The difference between both cases lies in the fact that the Hilbert space \( \mathcal{H}_B \) of system \( B \) may not have the structure \( \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \). This point requires further explanation; for example, when systems \( A_1, A_2 \) are coupled to different segments of a chain of coupled harmonic oscillators, then one can identify the Hilbert spaces

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\( \mathcal{H}_B \) and \( \mathcal{H}_B \). However, when \( A_1 \) and \( A_2 \) couple to a quantum field via local field operators then it is not clear what should be taken as the subsystems \( B_1 \) and \( B_2 \). On the other hand, since operators acting on different Hilbert spaces commute, it is clear that the situation portrayed in Fig.1(a) is a particular case of the scheme we investigate. The paper is organized as follows. In section (II) we present some necessary conditions that the operators coupling the systems must satisfy in order to allow entanglement transfer from B to \( A_1 A_2 \). Also, we derive an expression for the two qubit reduced density matrix. In section (III), we introduce a special class of Hamiltonians describing the interaction between qubits and bosonic systems. For this class of Hamiltonians (being a generalization of the Jaynes-Cummings Hamiltonian), the reduced density matrix of the qubits assumes a particularly simple form. In section (IV) we expand the entanglement measure (negativity) in terms of the coupling strength. We express the first nonvanishing contribution to this expansion in terms of 2-point and 4-point correlation functions involving the operators acting on system B. We study the entanglement of qubits in the weak coupling approximation (or equivalently, for short interaction times), for different N-particle excitations of the bosonic system. We also investigate the entanglement of the qubits in the case where system B is in a mixed state of the form \( \rho_B = \sum_i p_i |N_i \rangle \langle N_i | \). Finally, in sections (V) and (VI) we compute the exact density matrix for the qubits and discuss their entanglement for different N-particle states and operators.

II. FORMULATION OF THE PROBLEM

Consider the interaction between system B, with Hilbert space \( \mathcal{H}_B \), and system \( A_1 A_2 \), with Hilbert space \( \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \). This interaction induces the following global operation on the system \( BA_1 A_2 \):

\[
\rho \rightarrow U \rho U^\dagger.
\] (1)

Assume that initially system A is in a separable state \( \rho_A = \sum_n p_n \rho_{A_1,n} \otimes \rho_{A_2,n} \) while system B is in the state \( \rho_B \). In addition, we assume that systems A and B are uncorrelated i.e. \( \rho = \rho_A \otimes \rho_B \). After the interaction, the state of A is described by the reduced density matrix

\[
\rho^A = \text{Tr}_B(U \rho_A \otimes \rho_B U^\dagger)
\] (2)

obtained by tracing out the degrees of freedom corresponding to system B. Making use of the spectral decomposition \( \rho_B = \sum_l \lambda_{B,l} |\Phi_{B,l} \rangle \langle \Phi_{B,l}| \) and choosing an orthonormal basis \( \{|\Phi_{B,j}\rangle\} \) for \( \mathcal{H}_B \), one can write the operator-sum representation \( \rho \) of the operation \( \rho_A \rightarrow \rho^A \)

\[
\rho^A = \sum_{j,l} A_{j,l} \rho_A A_{j,l}^\dagger
\] (3)

The above expression shows that the positivity of the density matrix \( \rho_A \) is preserved by the operation \( \rho_A \rightarrow \rho^A \). The operators \( A_{j,l} \) are given by

\[
A_{j,l} = \sqrt{\lambda_{B,l}} |\Phi_{B,j} \rangle \langle \Phi_{B,l}| U
\] (4)

and they satisfy the relation \( \sum_{j,l} A_{j,l} A_{j,l}^\dagger = I_A \) which guarantees that \( \text{Tr}(\rho^A) = 1 \). Let \( U = e^{-itH} \) where \( H \) is a Hermitian operator of the form

\[
H = \sum_k A_{1,k} \otimes B_{1,k} + \sum_l A_{2,l} \otimes B_{2,l}.
\] (5)

Here, \( A_{i,k} \) and \( B_{i,k} \) are Hermitian operators acting on \( \mathcal{H}_{A_i} \), \( i = (1,2) \) and \( \mathcal{H}_B \), respectively. In addition, we assume that \( [B_{1,k}, B_{2,l}] = 0 \) which implies that the evolution operator factorizes as \( U = U_1 \cdot U_2 \) where \( U_i = e^{-it\sum_k A_{i,k} \otimes B_{i,k}} \), \( i = (1,2) \). It turns out that if all the operators coupled to one of the systems, say \( A_1 \), commute i.e.

\[
[B_{1,k}, B_{1,k'}] = 0
\] (6)

then the state \( \rho^A \) will remain separable. In order to prove this fact, we choose \( \{|\Phi_j \rangle\} \) in expression (4) to be a basis in which all the operators \( B_{1,k} \) are diagonal, that is, \( B_{1,k} |\Phi_{B,j} \rangle = b_{1,kj} |\Phi_j \rangle \). Then, the operators \( A_{j,l} \) assume the form

\[
A_{j,l} = U_{1,j} \otimes A_{2,j,l}
\] (7)
with $U_{1,j} = e^{-i t \sum b_{1,k,j} A_{1,k}}$ and $A_{2,j,l} = \sqrt{\lambda_{B,j,l}} (\Phi_{B,j,l} | U_{2} \lambda_{B,j,l})$. It is clear that operators of the form (7) map a separable state into another separable state. For example, if $\rho_A = \rho_{A_1} \otimes \rho_{A_2}$ then using (4) one obtains

$$\rho_{A_1} \otimes \rho_{A_2} \rightarrow \sum_{j,l} \text{Tr}_{A_2}(A_{2,j,l} \rho_{A_2} A_{2,j,l}^\dagger)U_{1,j} \rho_{A_1} U_{1,j}^\dagger \frac{A_{2,j,l} \rho_{A_2} A_{2,j,l}^\dagger}{\text{Tr}_{A_2}(A_{2,j,l} \rho_{A_2} A_{2,j,l}^\dagger)} \tag{8}$$

which is a convex sum of density matrices i.e. $\rho^A = \sum_{j,l} P_{j,l} \rho_{A_1,j,l} \otimes \rho_{A_2,j,l}$, with $\sum_{j,l} P_{j,l} = 1$. Therefore, in order to entangle systems $A_1$ and $A_2$ each group of operators $\{B_{1,k}\}$ and $\{B_{2,k}\}$ in (6) must contain at least one pair of noncommuting operators. In particular, this fact rules out operations of the form (5). If one incorporates the free evolution (9) as

$$|K_{1}⟩ = T_r \rho_{A} |K_{2}⟩ \tag{12}$$

we will neglect the free evolution of the systems A and B. Furthermore, we will consider the situation where $A_1$ and $A_2$ are two-level systems (qubits) whereas $B$ is a bosonic system.

In principle, one can obtain the reduced density matrix $\rho^A$ from Kraus representation (4). However, for our purposes it is convenient to write equation (2) as

$$\langle a | \rho^A | a' \rangle = \text{Tr}_{B}(\rho_B \langle \phi_A | U^\dagger | a' \rangle \langle a | U | \phi_A \rangle) \tag{9}$$

which explicitly shows that the matrix elements of $\rho^A$ are given by expectation values of operators acting on system B. Since the unitary operator $U$ factorizes i.e. $U = U_1 \cdot U_2$, one can define the operators $K_i \equiv \langle 0 | U_i | 0 \rangle$ and $N_i \equiv \langle 1 | U_i | 0 \rangle$ for $i = 1,2$. Assuming that the qubits are initially in the separable state $\phi_A = |0,0 \rangle$ and choosing the basis $\{ |a_1 \rangle = |0,0 \rangle, |a_2 \rangle = |0,1 \rangle, |a_3 \rangle = |1,0 \rangle, |a_4 \rangle = |1,1 \rangle \}$ for $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$, one can write the reduced density matrix (9) as

$$\rho^A = \begin{pmatrix} \langle K_{1} | K_{2} | K_{3} | K_{4} \rangle & \langle K_{1} | K_{2} | N_{2} | K_{2} \rangle & \langle K_{1} | K_{2} | N_{3} | K_{2} \rangle & \langle K_{1} | K_{2} | N_{4} | K_{2} \rangle \\ \langle K_{1} | N_{2} | K_{2} | K_{2} \rangle & \langle K_{1} | N_{2} | N_{2} | K_{2} \rangle & \langle K_{1} | N_{2} | N_{3} | K_{2} \rangle & \langle K_{1} | N_{2} | N_{4} | K_{2} \rangle \\ \langle K_{1} | N_{3} | K_{2} | K_{2} \rangle & \langle K_{1} | N_{3} | N_{2} | K_{2} \rangle & \langle K_{1} | N_{3} | N_{3} | K_{2} \rangle & \langle K_{1} | N_{3} | N_{4} | K_{2} \rangle \\ \langle K_{1} | N_{4} | K_{2} | K_{2} \rangle & \langle K_{1} | N_{4} | N_{2} | K_{2} \rangle & \langle K_{1} | N_{4} | N_{3} | K_{2} \rangle & \langle K_{1} | N_{4} | N_{4} | K_{2} \rangle \end{pmatrix} \tag{10}$$

where we used the notation $\langle \bar{B} \rangle = \text{Tr}_{B}(\rho_B \bar{B})$. Notice that the operators $\{N_i, N_i^\dagger, K_i, K_i^\dagger\}$ satisfy the relations $K_i^\dagger K_i + N_i^\dagger N_i = I_B$, $(i = 1, 2)$ \Rightarrow $\text{Tr}(\rho^A) = 1$. Thus, from the matrix (10) we see that all the properties of state $\rho^A$ (in particular, separability) depend on the interplay of the different correlation functions of the operators $\{N_i, N_i^\dagger, K_i, K_i^\dagger\}$.

### III. THE INTERACTION MODEL

The most general form of the interaction between qubit $A_i$ and the bosonic system is given by the Hamiltonian $\mathcal{H}_i = \mathbb{I}_i \otimes \mathcal{B}_{0,i} + \sigma_x \otimes \mathcal{B}_{x,i} + \sigma_y \otimes \mathcal{B}_{y,i} + \sigma_z \otimes \mathcal{B}_{z,i}$ where $\mathcal{B}_{k,i}$ are Hermitian operators. In this paper we restrict our discussion to the case where $\mathcal{B}_{0,i} = \mathcal{B}_{z,i} = 0$, $i = 1, 2$. Under this assumption, the Hamiltonian of the system can be written as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ where

$$\mathcal{H}_i = \begin{pmatrix} 0 & F_i \\ F_i^\dagger & 0 \end{pmatrix}, \quad i = 1, 2. \tag{12}$$

The evolution operator factorizes (i.e. $U = U_1 \cdot U_2$) if the set of operators coupled to $A_1$ commute with those coupled to $A_2$. Consequently, we impose the following conditions:

$$[F_1, F_2] = [F_1^\dagger, F_2^\dagger] = [F_1^\dagger, F_2] = 0. \tag{13}$$

Now, one can express the operators $K_i \equiv \langle 0 | U_i | 0 \rangle$ and $N_i \equiv \langle 1 | U_i | 0 \rangle$ in terms of $F_i$ and $F_i^\dagger$. From (12), one obtains

$$K_i = \langle 0 | U_i | 0 \rangle = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} 2^{2k} F_i^k F_i^\dagger = \text{cos}(\sqrt{F_i^\dagger F_i} t) \tag{14}$$

$$N_i = \langle 1 | U_i | 0 \rangle = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} 2^{2k+1} F_i F_i^\dagger = -i F_i^\dagger \frac{\sin(\sqrt{F_i^\dagger F_i} t)}{\sqrt{F_i^\dagger F_i}} \tag{15}$$

Thus, all the properties of state $\rho^A$ and the free evolution (9) depend on the interplay of the different correlation functions of the operators $\{N_i, N_i^\dagger, K_i, K_i^\dagger\}$. 


Since $B$ is a bosonic system, one may write the operators $F_1$ in their second quantization form. We will restrict our discussion to the class of operators expressible as

$$F(a, a^\dagger) = p(a^\dagger a)a^n + q(a^\dagger a)a^m$$

where $p(a^\dagger a)$ and $q(a^\dagger a)$ are functions of $a^\dagger a$ and $n, m \geq 1$. Notice that while the above class includes field operators it does not include 1-body operators. If the eigenstates of the density matrix $\rho_B$ are also eigenstates of the particle number operator $\hat{N} = \sum_n a^\dagger_B n a_B$ then from \[11\], \[13\] and \[15\] we see that some of the matrix elements of $\rho^A$ vanish. In fact, under these assumptions $\rho^A$ takes the form

$$\rho^A = \begin{pmatrix}
\rho_{11} & 0 & 0 & \rho_{14} \\
0 & \rho_{22} & \rho_{23} & 0 \\
0 & \rho_{32} & \rho_{33} & 0 \\
\rho_{41} & 0 & 0 & \rho_{44}
\end{pmatrix}.$$  

(17)

The eigenvalues of $\rho^A$ are given by $\lambda_{1,\pm} = \frac{1}{2}(\rho_{11} + \rho_{44} \pm \sqrt{(\rho_{11} + \rho_{44})^2 - 4(\rho_{22}\rho_{33} - |\rho_{44}|^2)})$ and $\lambda_{2,\pm} = \frac{1}{2}(\rho_{22} + \rho_{33} \pm \sqrt{(\rho_{22} + \rho_{33})^2 - 4(\rho_{22}\rho_{33} - |\rho_{23}|^2)})$. Making use of Schwarz inequality ($\langle x|y \rangle < ||x|| \cdot ||y||$), one easily proves the relations $\rho_{11}\rho_{44} \geq |\rho_{44}|^2$ and $\rho_{22}\rho_{33} \geq |\rho_{23}|^2$ which ensure the positivity of the density matrix $\rho_A$. On the other hand, the partial transpose \[8\] of $\rho^A$ is defined as

$$\langle a_1, a_2| \rho^{T_A} | a'_1, a'_2 \rangle = \langle a'_1, a'_2| \rho^A | a_1, a_2 \rangle$$

and its matrix representation is

$$\rho^{T_A} = \begin{pmatrix}
\rho_{11} & 0 & 0 & \rho_{14}^* \\
0 & \rho_{22} & \rho_{23} & 0 \\
0 & \rho_{32} & \rho_{33} & 0 \\
\rho_{41} & 0 & 0 & \rho_{44}^*
\end{pmatrix}.$$  

(18)

Notice that $\rho^{T_A}$ can be obtained from $\rho^A$ replacing $\rho_{23}$ by $\rho_{14}^*$ and $\rho_{14}$ by $\rho_{23}^*$. Hence the matrix $\rho^{T_A}$ will have a negative eigenvalue if either

$$n_{23} \equiv |\rho_{23}|^2 - \rho_{11}\rho_{44} > 0 \quad \text{or} \quad n_{14} \equiv |\rho_{14}|^2 - \rho_{22}\rho_{33} > 0.$$  

(19)

If one of the above inequalities is fulfilled, then according to the PPT criterion \[9\], the two qubit system $A_1A_2$ will be in a nonseparable (entangled) state. Furthermore, in this particular case, one can show that $\rho^A > 0$ implies that $n_{23}$ and $n_{14}$ cannot be positive at the same time. For example, suppose that $n_{23} > 0$, then one can write the inequalities

$$\rho_{22}\rho_{33} \geq |\rho_{23}|^2 > \rho_{11}\rho_{44} \geq |\rho_{14}|^2$$

which imply $n_{14} < 0$. Therefore the partial transpose $\rho^{T_A}$ can only have one negative eigenvalue. In fact, it can be proved that in $2 \times 2$ dimensions, the partial transpose of any density matrix can have at most one negative eigenvalue \[10\].

In the previous section, we assumed that the initial state of system $A_1A_2$ was $|\phi_A \rangle = |a_1 \rangle = |0, 0 \rangle$. Consequently, we wrote down the matrix \[11\] representing the final state of the qubits. However, if system $A_1A_2$ is initially in a different state then, obviously, the matrix elements of $\rho^A$ will be different from those in expression \[11\]. Let us consider, for example, the case $|\phi_A \rangle = |1, 1 \rangle = \sigma_x \otimes \sigma_x |0, 0 \rangle$. Now, equation \[9\] can be written as

$$\langle a| \rho^A |a' \rangle = Tr_B(\rho_B(a_1| \sigma_x^\dagger U_k^1 \sigma_x \otimes \sigma_x^\dagger U_k^2 \sigma_x |a' \rangle \langle a| \sigma_x^\dagger U_k^1 \sigma_x \otimes \sigma_x^\dagger U_k^2 \sigma_x |a_1 \rangle)$$

(20)

where $|\tilde{a} \rangle$ is obtained from $|a \rangle$ by flipping both qubits, i.e. $|\tilde{a} \rangle = \sigma_x \otimes \sigma_x |a \rangle$. Since

$$\sigma_x^\dagger W_k^4 \sigma_x = \begin{pmatrix}
0 & 0 \\
F_{k}^1 & 0
\end{pmatrix},$$

(21)

it is easy to conclude that if initially system $A$ was known to be in the state $|a_4 \rangle = |1, 1 \rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ then the final density matrix $\rho^A$ is given by

$$\rho^A(|a_4 \rangle, \mathcal{H}) = \begin{pmatrix}
\hat{\rho}_{14} & 0 & 0 & \hat{\rho}_{41} \\
0 & \hat{\rho}_{33} & \hat{\rho}_{32} & 0 \\
0 & \hat{\rho}_{32} & \hat{\rho}_{22} & 0 \\
\hat{\rho}_{41} & 0 & 0 & \hat{\rho}_{11}
\end{pmatrix}.$$  

(22)
where the matrix elements $\rho_{ij}$ are determined by means of the expressions (10), (13), (15) with the operators $F_k$ replaced by the operators $F^\dagger_k$ and vice-versa. Treating $\rho^A$ as a function of the initial state $|\phi_A\rangle$ and the operator $U$, one can summarize the previous discussion as

$$\rho^A(|a_4\rangle, U(F_k)) = V \rho^A(|a_1\rangle, U(F^\dagger_k)) V^\dagger \quad (23)$$

where $V = \sigma_x \otimes \sigma_x = |a_1\rangle \langle a_4| + |a_4\rangle \langle a_1| + |a_2\rangle \langle a_3| + |a_3\rangle \langle a_2|$ is the unitary transformation corresponding to the basis permutation $(1, 2, 3, 4) \rightarrow (4, 3, 2, 1)$. Similarly, one can write

$$\rho^A(|a_2\rangle, U(F, F_2)) = V \rho^A(|a_1\rangle, U(F_1, F_2)) V^\dagger \quad (24)$$

with $V = I \otimes \sigma_x$.

**IV. SERIES EXPANSION FOR $N(\rho^A)$**

In general, for a given set of operators $\{F_i, F_i^\dagger\}$, it may not be possible to determine the matrix elements of $\rho^A$ in closed form. On the other hand, it is interesting to explore the different terms in the series expansion of $n_{23}, n_{14}$ (see 19) in terms of the operators $\{F_i, F_i^\dagger\}$. From the general expression (10) for the reduced density matrix, we expect this series expansion to contain correlation functions of the operators $\{F_i, F_i^\dagger\}$. Introducing a coupling constant, i.e. $F_i \rightarrow g_i F_i$, one obtains a series expansion of the form

$$n(g_1, g_2) = n_1 g_1^2 g_2^2 + n_2 (g_1^2 g_2^4 + g_1^4 g_2^2) + \ldots \quad (25)$$

where $n$ denotes either $n_{23}$ or $n_{14}$. The above form can be justified as follows. Each term of the series expansion should be symmetric in $g_1$ and $g_2$. Terms containing odd powers of $g_1$, e.g. $g_1 g_2^3 + g_1^3 g_2$, should not be present since then one could change the sign of $n$ by simply changing the signs of $g_1$ or $g_2$. Clearly the entanglement in $A_1 A_2$ should not depend on the sign of $g_1$ or $g_2$. Finally, if we switch off one of the interactions ($g_1 = 0$ or $g_2 = 0$) then both $n_{23}$ and $n_{14}$ vanish (see expressions (23)). This rules out terms of the form $g_1^4 + g_2^4$. Now, we proceed to express the first nonvanishing contribution to the expansion (25) in terms of $F_i$ and $F_i^\dagger$. Making use of the series expansion (14) and (15) for $K_i$ and $N_i$, one writes $K_i = I - \frac{4}{3} (F_i | F_i^\dagger) t^2 + \frac{1}{3!} (F_i | F_i^\dagger)^2 t^4 \ldots$ and $N_i = -i F_i t (I - \frac{1}{3!} F_i^\dagger F_i^\dagger t^2) + \ldots$. From (14) one obtains the following series expansion for the diagonal matrix elements $\rho_{kk}$:

$$\rho_{11} = 1 - \langle F_1 \rangle t^2 - \langle F_2 \rangle t^2 + \frac{1}{3} \langle (F_i^\dagger F_i^\dagger)^2 \rangle t^4 + \langle F_1 F_2 \rangle t^4, \quad \rho_{22} = \langle F_2 \rangle t^2 - \frac{1}{3} \langle (F_i^\dagger F_i^\dagger)^2 \rangle t^4 - \langle F_1 F_2 \rangle t^4, \quad \rho_{33} = \langle F_1 \rangle t^2 - \frac{1}{3} \langle (F_i^\dagger F_i^\dagger)^2 \rangle t^4 - \langle F_1 F_2 \rangle t^4, \quad \rho_{44} = \langle F_1 F_2 \rangle t^4 \quad (26)$$

where $F_i \equiv (F_i^\dagger F_i)$. Similarly, one expands the $\rho_{23}$ and $\rho_{14}$ in series of $\{F_i, F_i^\dagger\}$ obtaining the following expressions:

$$\rho_{23} = \langle F_1 F_2 \rangle t^2 - \frac{1}{2} \langle (F_i^\dagger F_i^\dagger F_i^\dagger F_i^\dagger) \rangle t^4 - \frac{1}{2} \langle (F_i^\dagger) t^2 F_i F_2 \rangle t^4 - \frac{1}{3!} \langle F_i^\dagger F_j F_i^\dagger F_j F_i F_2 \rangle t^4 \quad (28)$$

$$\rho_{14} = -\langle F_i^\dagger F_i^\dagger \rangle t^2 + \frac{1}{2} \langle (F_i^\dagger F_i^\dagger)^2 F_2 \rangle t^4 + \frac{1}{2} \langle (F_i^\dagger F_i^\dagger) F_i F_2 \rangle t^4 + \frac{1}{3!} \langle F_i^\dagger F_j F_i^\dagger F_j F_i F_2 \rangle t^4 \quad (29)$$

Thus up to fourth order one has

$$\rho_{11} \rho_{44} = \langle F_1 F_2 \rangle t^4 = \langle F_1^\dagger F_1^\dagger F_i^\dagger F_i^\dagger \rangle t^4 \geq 0 \quad (30)$$

$$|\rho_{14}|^2 = |\langle F_i^\dagger F_i^\dagger \rangle|^2 t^4 \quad (31)$$

$$\rho_{22} \rho_{33} = \langle F_1 \rangle \langle F_2 \rangle t^4 = \langle F_1^\dagger F_1^\dagger \rangle \langle F_2 F_2 \rangle t^4 \geq 0 \quad (32)$$

$$|\rho_{23}|^2 = |\langle F_i^\dagger F_i^\dagger \rangle|^2 t^4. \quad (33)$$
For consistency, one can check the density matrix is positive. In fact, the identity $\langle AA^\dagger \rangle \geq |\langle A \rangle|^2$ with $A = F_1^\dagger F_2^\dagger$ implies that $\rho_{11}\rho_{44} \geq |\rho_{14}|^2$ while the inequality $\rho_{22}\rho_{33} \geq |\rho_{23}|^2$ follows from Schwarz inequality. Clearly, the partial transpose $\rho^{T_{A_1}}$ may be negative. In fact, substituting the above approximations for $\rho^A$ in the expressions (14) for $n_{23}$ and $n_{14}$ one obtains:

$$n_{23} = t^4(|\langle F_1^\dagger F_2 \rangle|^2 - |\langle F_1^\dagger F_1 F_2^\dagger F_2 \rangle|^2)$$

$$n_{14} = t^4(|\langle F_1^\dagger F_2^\dagger \rangle|^2 - |\langle F_1^\dagger F_1^\dagger F_2 F_2^\dagger \rangle|^2).$$

For small values of $t$, the above expressions can be used to detect the presence of entanglement in the system $A_1A_2$. If one of the above quantities ($n_{23}$ or $n_{14}$) is positive then the state of qubits $A_1$ and $A_2$ is nonseparable. Notice that if we choose the operators $F_i$ to be field operators acting on $B$, then $n_{23}$ and $n_{14}$ will contain 2-point and 4-point correlation functions of these operators. On the other hand, if the operators $\{F_1, F_2\}$ are normal (i.e. $[F_1, F_2] = 0$) then $n_{23}$ and $n_{14}$ are be negative. This is a reflection of the fact that in order to entangle system $A_1$ with system $A_2$ we must have non commuting operators. In the particular case where the operators $F_i$ are linear combinations of creation and annihilation operators we have

$$[F_i, F_i^\dagger] = c_i \quad \text{for} \quad (i = 1, 2),$$

where $c_i$ is a $c$-number. In this case one can show that $n_{23}$ and $n_{14}$ are negative if either $c_1 < 0$ or $c_2 < 0$. For example, let $c_2 < 0$, then Schwarz inequality leads to

$$n_{14} = t^4(|\langle F_1^\dagger F_2^\dagger \rangle|^2 - |\langle F_1^\dagger F_1 F_2^\dagger F_2 \rangle|^2 + c_2 (\langle F_1^\dagger F_1 \rangle) \leq c_2 t^4 (\langle F_1^\dagger F_1 \rangle) \leq 0$$

$$n_{23} = t^4(|\langle F_1^\dagger F_2^\dagger \rangle|^2 - |\langle F_1^\dagger F_2 (F_1^\dagger F_2)^\dagger \rangle|^2 + c_2 (\langle F_1^\dagger F_1 \rangle)) \leq c_2 t^4 (\langle F_1^\dagger F_1 \rangle) \leq 0.$$ (37) (38)

The relation (38) is obtained using the inequality

$$\langle AA^\dagger \rangle \geq |\langle A \rangle|^2.$$ (39)

Finally, the entanglement measure we use in this paper is the negativity $N(\rho^A)$ defined as twice the absolute value of the negative eigenvalue of $\rho^{T_{A_1}}$. In our case, the eigenvalues of the partial transpose $\rho^{T_{A_1}}$ (18) are (up to fourth order in $t$):

$$\lambda_{1,-}^{T_{A_1}} = \frac{1}{2}(\rho_{11} + \rho_{44} - \sqrt{(\rho_{11} + \rho_{44})^2 + 4n_{23}}) \approx -\frac{n_{23}}{\rho_{11} + \rho_{44}} \approx -n_{23}$$

$$\lambda_{2,-}^{T_{A_1}} = \frac{1}{2}(\rho_{22} + \rho_{33} - \sqrt{(\rho_{22} + \rho_{33})^2 + 4n_{14}}) \approx -\frac{n_{14}}{\rho_{22} + \rho_{33}}.$$ (40) (41)

A. Examples

Utilizing expressions (34) and (35) one can calculate the entanglement transferred from system $B$ to the pair of qubits $A_1A_2$. In this subsection we compute the quantities $n_{23}$ and $n_{14}$ for interactions of the form (10) i.e. $F_i(a, a^\dagger) = f(a^\dagger a)a^n + g(a^\dagger a)a^{-m}$ and statistical mixtures of $N$-particle states $\rho_B = \sum_N p_N |N\rangle \langle N|$. Most of the examples that we present here can be computed exactly using (10). Nevertheless we believe that the first nonvanishing contribution to entanglement gives us some hints about the states and interactions that induce entanglement in system $A_1A_2$. Moreover, from a technical point of view, expressions of the form $\langle F_1^\dagger F_2 \rangle$, $\langle F_1^\dagger F_1^\dagger F_2 F_2^\dagger \rangle$... can be easily computed using Wick Theorem [15].

First, we consider the JC [12] describing a system of two two-level atoms interacting with a set of electromagnetic modes. Let the creation and annihilation operators for these modes be $a_k$ and $a_k^\dagger$. We assume that the Hamiltonian for the two atoms is of the form $H = H_1 + H_2$ with $H_i$, $(i = 1, 2)$ given by

$$H_i = \sum_k g_i,k (\sigma_+ a_k + \sigma_- a_k^\dagger), \quad \sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$ (42)
Finally, notice that a thermal-like state of density matrices having eigenstates with fixed particle number. Thus, from equations (44) and (45), one obtains
\[ \mathbb{H}_i = g_i(\sigma_+ a(\phi_i) + \sigma_- a^\dagger(\phi_i)) = g_i \begin{pmatrix} 0 & a^\dagger(\phi_i) \\ a(\phi_i) & 0 \end{pmatrix} \]  

(43)

where \( g_i = \sqrt{\sum_k g_{i,k}^2} \). This Hamiltonian is of the form \( \mathbb{H}_i = \begin{pmatrix} 0 & F_i^\dagger \\ F_i & 0 \end{pmatrix} \) with \( F_i = g_i a^\dagger(\phi_i) \). These operators satisfy \([F_1, F_2] = 0, [F_1, F_1^\dagger] = g_1^2\) and \([F_1, F_2^\dagger] = g_1 g_2 \langle \phi_1 | \phi_2 \rangle \). If we assume that the states \(|\phi_1\rangle\) and \(|\phi_2\rangle\) are orthogonal, then relations (13) will be satisfied and we can apply, the results from previous sections. From the physical point of view, the orthogonality of \(|\phi_1\rangle\) and \(|\phi_2\rangle\) could describe the situation in which two atoms are sensible to different modes i.e. they absorb/emmit particles in different modes. If initially the qubits are prepared in the state \(|\Phi_B\rangle = \frac{1}{\sqrt{N!}} (a^\dagger(\phi_B))^N |0\rangle \) (representing N bosons occupying the same state \(|\phi_B\rangle\)) one easily finds that
\[ \langle F_1^\dagger F_2 \rangle = g_1 g_2 \langle a_1^\dagger a_2 \rangle = g_1 g_2 N \langle \phi_B | \phi_1 \rangle \langle \phi_2 | \phi_B \rangle \]  

(44)
\[ \langle F_1^\dagger F_1 F_2^\dagger F_2 \rangle = g_1^2 g_2^2 \langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle = g_1^2 g_2^2 N(N-1) \langle \phi_1 | \phi_B \rangle^2 | \langle \phi_2 | \phi_B \rangle|^2 \].  

(45)

Substituting these results in expressions (44) and (45) one obtains:
\[ n_{23} = N g_1^2 g_2^2 t^4 | \langle \phi_1 | \phi_B \rangle |^2 | \langle \phi_2 | \phi_B \rangle |^2 > 0 \]  

(46)
\[ n_{14} = -N^2 g_1^2 g_2^2 t^4 | \langle \phi_1 | \phi_B \rangle |^2 | \langle \phi_2 | \phi_B \rangle |^2 < 0 \].

From the above expressions we conclude that system \( A_1 A_2 \) is always entangled except for the cases where \( u_1 \equiv \langle \phi_1 | \phi_B \rangle = 0 \) or \( u_2 \equiv \langle \phi_2 | \phi_B \rangle = 0 \). One can also consider the situation where the initial state of system B is a mixed state of the form
\[ \rho_B = \sum_{N=0}^{\infty} p_N |N\rangle \langle N|, \quad \sum_{N=0}^{\infty} p_N = 1, \]  

(47)

and \(|N\rangle = \frac{1}{\sqrt{N!}} (a^\dagger(\phi_B))^N |0\rangle \). As discussed in section (11), all the expressions we have derived so far hold for the class of density matrices having eigenstates with fixed particle number. Thus, from equations (14) and (15), one obtains
\[ n_{23} = g_1^2 g_2^2 t^4 |u_1|^2 |u_2|^2 \left( \sum_{N=0}^{\infty} p_N N^2 - \sum_{N=0}^{\infty} p_N N(N-1) \right) \]  

(48)

which implies that \( \rho^A \) will be nonseparable for probability distributions \( p_N \) satisfying
\[ \left( \sum_{N=0}^{\infty} p_N N^2 \right) > \sum_{N=0}^{\infty} p_N N(N-1). \]  

(49)

It is interesting to note that the above condition implies that the probability distribution \( \{ p_N \} \) must be sub-poissionian \( [14] \). i.e. \( N > N^2 - (N) = \sigma_N^2 \). An example of a distribution satisfying \( N > \sigma_N^2 \) is the binomial distribution \( p_N = \binom{M}{N} p^N (1-p)^{M-N} \) (for \( N = 0, 1, \ldots, M \)) having \( N = Mp \) and \( \sigma_N^2 = Mp(1-p) \). In this particular case one obtains
\[ n_{23} = g_1^2 g_2^2 t^4 |u_1|^2 |u_2|^2 M p^2 > 0. \]  

(50)

Finally, notice that a thermal-like state \( \rho_B = (1-z) \sum_N z^N |N\rangle \langle N| \) yields a separable state for \( A_1 A_2 \). If the initial state of the two-level systems is \( |\phi_A\rangle = |1, 1\rangle \) then according to section (11), the entanglement of \( A_1 A_2 \) is determined by \( n_{23} \) and \( n_{14} \) with the operators \( F_i \) being replaced by \( F_i^\dagger \). As expected, in this case we obtain a separable state. In fact, if we denote \( u_i = \langle \phi_i | \phi_B \rangle \), then we have
\[ | \langle F_1 F_2^\dagger \rangle |^2 - \langle F_1 F_1^\dagger F_2 F_2^\dagger \rangle = -g_1^2 g_2^2 N (|u_1|^2 + |u_2|^2 - |u_1|^2 |u_2|^2 + \frac{1}{N}) < 0. \]  

(51)
One can also consider a state of the form

$$|\Phi_B\rangle = \frac{1}{\sqrt{N_1 N_2}} (a_1^\dagger (\phi_{B_1}))^{N_1} (a_2^\dagger (\phi_{B_2}))^{N_2} |0\rangle \equiv |N_1, N_2\rangle$$

(52)

describing the situation where $N_1$ bosons occupy the state $|\phi_{B_1}\rangle$ and $N_2$ bosons occupy the state $|\phi_{B_2}\rangle$ orthogonal to $|\phi_{B_1}\rangle$, i.e. $\langle \phi_{B_1} | \phi_{B_2} \rangle = 0$. Straightforwardly, one finds that

$$a_i |\Phi_B\rangle = \sqrt{N_i} u_{i,1} |N_i - 1, N_2\rangle + \sqrt{N_2} u_{i,2} |N_1, N_2 - 1\rangle$$

(53)

$$|\langle F_1^\dagger F_2 \rangle|^2 = g_1^2 g_2^2 |N_1 u_{1,1}^* u_{2,1} + N_2 u_{1,2}^* u_{2,2}^2|^2$$

(54)

where $u_{i,k} \equiv \langle \phi_i | \phi_{B_k} \rangle$. On the other hand, $\langle F_1^\dagger F_2 \rangle$ reads:

$$\langle F_1^\dagger F_2 \rangle = g_1^2 g_2^2 (N_1(N_1 - 1)|u_{1,1} u_{2,1}|^2 + N_2(N_2 - 1)|u_{2,1} u_{2,2}|^2 + N_1 N_2 |u_{1,2} u_{1,2} + u_{2,2} u_{1,1}|^2).$$

(55)

Combining the above results one obtains

$$n_{23} = g_1^2 g_2^2 t^4 (N_1 |u_{1,1} u_{2,1}|^2 + N_2 |u_{1,2} u_{2,2}|^2 - N_1 N_2 (|u_{1,2} u_{1,2}|^2 + |u_{2,2} u_{1,1}|^2)).$$

(56)

Here, two things are worth mentioning. First of all, equation (56) indicates, that in this approximation, no entanglement will be transferred to $A_1 A_2$ for large values of $N_1$ and $N_2$. Also notice that when $N_1 = N_2 = 1$, $|\langle F_1^\dagger F_2 \rangle|^2 = g_1^2 g_2^2 |N_1 u_{1,1} u_{2,1} + N_2 u_{1,2} u_{2,2}|^2$ vanishes when the $2 \times 2$ matrix $u_{i,k} = \langle \phi_i | \phi_{B_k} \rangle$ is unitary implying $n_{23} < 0$. This case will be studied in detail in section (VI).

### B. Algebraic Construction of Operators

One can also consider the entanglement induced in system $A_1 A_2$ when the operators $F_i$ are linear combinations of creation and annihilation operators. Let

$$F_i = g_i (a_i + |\beta_i| e^{i \theta_i} a_i^\dagger), \quad i = (1, 2), \quad \text{with} \quad [a_1, a_2^\dagger] = 0 \quad \text{and} \quad |\beta_i| < 1.$$  

(57)

If system $B$ is in a 1-particle state $a_B^\dagger |0\rangle$, then

$$\langle F_1^\dagger F_2 \rangle = \langle 0 | a_B F_1^\dagger F_2 a_B^\dagger |0\rangle = g_1 g_2 (u_1^* u_2 + u_1 u_2^* |\beta_1 |\beta_2| e^{-i(\theta_1 - \theta_2)})$$

(58)

$$\langle F_1^\dagger F_2 F_1^\dagger F_2 \rangle = \langle 0 | a_B F_1^\dagger F_2 F_1^\dagger F_2 a_B^\dagger |0\rangle$$

$$= g_1^2 g_2^2 (|\beta_1|^2 |\beta_2|^2 + |u_1|^2 (1 + |\beta_1|^2)|\beta_2|^2 + |u_2|^2 (1 + |\beta_2|^2)|\beta_1|^2).$$

(59)

In the case where $u_1 = u_2 = \frac{1}{\sqrt{2}}$ and $\beta_1 = \beta_2 = \beta$ we have

$$\langle F_1^\dagger F_2 \rangle - \langle F_1^\dagger F_2 F_1^\dagger F_2 \rangle > 0 \quad \text{for} \quad |\beta| < |\beta_{\text{max}}| = \sqrt{\frac{2\sqrt{2} - 1}{7}} = 0.51$$

(60)

and $n_{14} = t^4 \langle F_1^\dagger F_2 \rangle - \langle F_1^\dagger F_2 F_1^\dagger F_2 \rangle < 0$ for $\beta \in (0, 1)$. Thus, one can mix creation and annihilation operators as in equation (57), obtaining an entangled state for $A_1 A_2$ for $|\beta| < |\beta_{\text{max}}|$. We conclude this section by constructing a set of operators inducing entanglement in system $A_1 A_2$ when $B$ is the particle vacuum state $|0\rangle$. This problem was studied in (6) where it was shown using perturbation theory that a two-level systems can become entangled after having locally interacted with a scalar field. Here, we assume an effective interaction between the qubits and system $B$ of the form

$$H_i = \begin{pmatrix} 0 & F_i^\dagger \\ F_i & 0 \end{pmatrix}$$

with $F_i = g_i (a_i (\phi_i) + \beta_i^* a_i^\dagger (\psi_i)).$  

(61)

We need to accommodate conditions (13) into this picture. Notice that if $\langle \phi_1 | \phi_2 \rangle = \langle \psi_1 | \psi_2 \rangle = 0$ then $[F_1, F_2^\dagger] = 0$. On the other hand, $[F_1, F_2] = 0$ holds if we impose the condition

$$\beta_1 \langle \psi_1 | \phi_2 \rangle = \beta_2 \langle \psi_2 | \phi_1 \rangle.$$  

(62)
Now, we have

$$
\langle N| F_2^\dagger F_2 | N \rangle = g_1 g_2 (\beta_1 \langle \psi_1 | \phi_2 \rangle + N \langle \langle \phi_B | \phi_1 \rangle \beta_2 \langle \psi_2 | \phi_B \rangle + \langle \phi_B | \phi_2 \rangle \beta_1 \langle \psi_1 | \phi_B \rangle)
$$

$$
\langle N| F_1^\dagger F_1 | N \rangle = g_2^2 (|\beta_1|^2 + N (\langle \langle \phi_B | \phi_1 \rangle |^2 + |\beta_2|^2 ) \langle \phi_B | \psi_1 \rangle |^2 )
$$

(63)

(64)

If $|\Phi_B\rangle = |0\rangle$, we set $N = 0$ in the above equations and compute $n_{14\text{vac}}$ from (35) obtaining the expression

$$
n_{14\text{vac}} = g_1^2 g_2^2 t^4 |\beta_1|^2 (|\langle \psi_1 | \phi_2 \rangle|^2 - |\beta_2|^2)
$$

(65)

which indicates that the system $A_1A_2$ is entangled for $0 < |\beta_2| < |\langle \psi_1 | \phi_2 \rangle| \text{ (or equivalently, } 0 < |\beta_1| < |\langle \psi_2 | \phi_1 \rangle| \text{).}

We close this section by presenting a situation in which operators of the form $F_i = g_i (\alpha(\phi_i) + \beta_i^* a^i (\psi_i))$ appear in the effective interaction between the qubits and the bosonic system. Consider two qubits interacting with a relativistic scalar field $\Phi$. Let the Hamiltonian of the system be

$$
\mathcal{H} = \mathcal{H}_0 + \sum_{i=1,2} g_i(t) \sigma_x i \hat{\phi}_i, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

(66)

where $\mathcal{H}_0 = \frac{i}{2} \sum_{i=1,2} \omega_A_i \sigma_z i + \sum_i \omega_k (a_k^\dagger a_k + \frac{1}{2})$ is the free Hamiltonian of the qubits+field system. The functions $g_1(t)$ and $g_2(t)$ describe time dependent coupling strengths and $\hat{\phi}_i = \int O_i d^3 r f_i(t) \phi(t, \vec{r})$, $\hat{\phi}_2 = \int O_2 d^3 r f_i(t) \phi(t, \vec{r})$ are average fields on the spatial regions $O_1$ and $O_2$ where the qubits are located. If the interaction is fast compared to the free evolution of the qubits and the field (by taking the average fields we introduce a cut-off for the field frequencies) one can make use of Magnus approximation [14] and write the time evolution operator as $\hat{U} = e^{-iH_{eff}t}$ where $H_{eff}(t)$ is the interaction picture Hamiltonian. Following [3], we assume that the qubits remain causally disconnected throughout the whole interaction process. The time evolution operator can be written as $\hat{U} = e^{-iH_{eff}t}$ where

$$
H_{eff} = \sum_{i=1,2} \int dt g_i(t) \left( \begin{array}{cc} 0 & e^{-i\omega_A_i t} \\ e^{i\omega_A_i t} & 0 \end{array} \right) \hat{\phi}_i(t) = \sum_{i=1,2} \left( \begin{array}{cc} 0 & F_i \ F_i^\dagger \end{array} \right),
$$

(67)

Here the operators $F_i$ read

$$
F_i = \int_0^\infty dt g_i(t) e^{i\omega_A_i t} \hat{\phi}_i(t) = \int \frac{d^3 k}{(2\pi)^2} \frac{1}{\sqrt{2\omega_k}} \hat{f}_i(k) (\hat{g}_i(\omega_k - \omega_{A_i}) e^{ik \cdot \vec{r}} a_k + \hat{g}_i^*(\omega_k + \omega_{A_i}) e^{-i\vec{k} \cdot \vec{r}} a_k^\dagger)
$$

(68)

Thus the necessary conditions [33] for the operators $F_i$ are satisfied. Notice that if we neglect the free evolution of the qubits (we set $\omega_{A_i} = 0$) $F_1$ and $F_2$ become normal operators and the state of the qubits remains separable after the interaction. The final entanglement of the qubit system depends on the form of the functions $g_i(t)$, $i = (1, 2)$ and it will not be discussed here. See [3] for a detailed discussion.

V. ENTANGLEMENT FROM N PARTICLES OCCUPYING A SINGLE 1-PARTICLE STATE.

In the previous section we found that for interactions of the form [12] with $F_i = g_i a(\phi_i)$, the system $A_1A_2$, initially in the state, $|a_1\rangle = |0, 0\rangle$, becomes instantaneously entangled when $B$ contains $N$ particles occupying a single 1-particle state $|\phi_B\rangle$. The only requirement for the state $|\phi_B\rangle$ to overlap with $|\phi_1\rangle$ and $|\phi_2\rangle$ i.e. $u_1 = \langle \phi_1 | \phi_B \rangle \neq 0$ and $u_2 = \langle \phi_2 | \phi_B \rangle \neq 0$. Since the states $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal, any 1-particle state can be written as $|\phi_B\rangle = \sum_{i=1,2} u_i a_i^\dagger |0\rangle + u_T |\phi_T\rangle$ where $\langle \phi_i | \phi_T \rangle = 0$ for $i = (1, 2)$. Therefore the state $|\Phi_B\rangle = \frac{1}{\sqrt{N!}} a^i (\phi_B)^N |0\rangle$ can be written in occupation number representation as

$$
|\Phi_B\rangle = \sum_{n_1, n_2} \sqrt{\frac{N!}{n_1! n_2! (N - n_1 - n_2)!}} u_1^{n_1} u_2^{n_2} u_T^{N-n_1-n_2} |n_1, n_2, N - n_1 - n_2\rangle
$$

(70)
Now making use of (14), (15) and (10), we compute the reduced density matrix for $A_1 A_2$ as a function of time. For example, the diagonal element

$$
\rho_{11} = \langle K_1^\dagger K_1 K_2^\dagger K_2 \rangle = \langle \Phi_B | \cos^2(\sqrt{F_1} F_1 t) \cos^2(\sqrt{F_2} F_2 t) | \Phi_B \rangle
$$

(71)

can be easily computed using the occupation number representation (70) for the state $| \Phi_B \rangle$. Likewise, one may determine the remaining diagonal entries of $\rho^A$ which can written in the compact form

$$
\rho_{kk} = \sum_{n_1, n_2=0}^N \frac{N!}{n_1! n_2! (N - n_1 - n_2)!} P_{n_1, n_2}(u_1, u_2) F_{kk}(n_1, n_2)
$$

(72)

where $P_{n_1, n_2}(u_1, u_2) = |u_1|^{2n_1} |u_2|^{2n_2} (1 - |u_1|^2 - |u_2|^2)^{N-n_1-n_2}$ and

$$
F_{11}(n_1, n_2) = \cos^2(\sqrt{n_1} g_1 t) \cos^2(\sqrt{n_2} g_2 t), \quad F_{23}(n_1, n_2) = \sin^2(\sqrt{n_1} g_1 t) \cos^2(\sqrt{n_2} g_2 t),
$$

$$
F_{22}(n_1, n_2) = \cos^2(\sqrt{n_1} g_1 t) \sin^2(\sqrt{n_2} g_2 t), \quad F_{44}(n_1, n_2) = \sin^2(\sqrt{n_1} g_1 t) \sin^2(\sqrt{n_2} g_2 t).
$$

(73)

(74)

From equation (10), it can be easily seen that operators of the form $Q_1$ can be easily computed using the occupation number representation (70) for the state $| \Phi_B \rangle$.

$$
\rho_{23} = u_1 u_2 \sum_{n_1, n_2=0}^{N-1} \frac{N!}{n_1! n_2! (N - n_1 - n_2 - 1)!} Q_{n_1, n_2}(u_1, u_2) F_{23}(n_1, n_2)
$$

(75)

where the polynomial $Q_{n_1, n_2}(u_1, u_2) = |u_1|^{2n_1} |u_2|^{2n_2} (1 - |u_1|^2 - |u_2|^2)^{N-n_1-n_2-1}$ and

$$
F_{23}(n_1, n_2) = \cos(\sqrt{n_1} g_1 t) \sin(\sqrt{n_1 + 1} g_1 t) \cos(\sqrt{n_2} g_2 t) \sin(\sqrt{n_2 + 1} g_2 t).
$$

(76)

Unfortunately, it does not seem possible to express the sums (72) and (75) in closed form. However, numerical analysis indicates that one obtains entangled states for the qubits for all values of $N$ with larger values of negativity when $N$ is an odd number (see Fig 2(b)). The above results can be extended to the case where

$$
\mathbb{F}_i = g_a a_i^m, \quad i = (1, 2), \quad m \geq 1, \quad \text{and} \quad [a_1, a_2^\dagger] = 0.
$$

(77)

This type of operators describe the situation where $m$ particles are needed to flip (or excite one of the atoms) the state of one of the qubits [13]. The expressions (72a) and (72b) (corresponding to the case $m=1$) can be generalized to any value of $m$. Using the identities $a_i a_i^m = \frac{(a_i a_i + M)}{(a_i a_i)}$, $a_i^m a_i = \frac{(a_i a_i)}{(a_i a_i - m)}$ and writing the state $| \Phi_B \rangle$ as in equation (70), one arrives at the following expressions:

$$
\rho_{kk}(m) = \sum_{n_1, n_2=0}^N \frac{N!}{n_1! n_2! (N - n_1 - n_2)!} P_{n_1, n_2}(u_1, u_2) F_{kk}^{(m)}(n_1, n_2),
$$

(78)

$$
\rho_{23}(m) = (u_1 u_2)^m \sum_{n_1, n_2=0}^N \frac{N!}{n_1! n_2! (N - n_1 - n_2 - m)!} Q_{n_1, n_2}(u_1, u_2) F_{23}^{(m)}(n_1, n_2)
$$

(79)

where $F_{kk}^{(m)}(n_1, n_2) = F_{kk}^{(m)}(n_1, n_2), \quad Q_{n_1, n_2}(u_1, u_2) = |u_1|^{2n_1} |u_2|^{2n_2} (1 - |u_1|^2 - |u_2|^2)^{N-n_1-n_2-m}$ and

$$
F_{23}^{(m)}(n_1, n_2) = \frac{\cos(\sqrt{n_1 + 1} g_1 t) \sin(\sqrt{n_1 + 1} g_1 t) \cos(\sqrt{n_2} g_2 t) \sin(\sqrt{n_2 + 1} g_2 t)}{\sqrt{n_1 + 1}}.
$$

(80)

From equation (73) one finds that $\rho_{44}$ vanishes for states having $N = \{m, m + 1, \ldots, 2m - 1\}$ particles occupying the 1-particle state $| \Phi_B \rangle$. This is easy to understand; the matrix element $\rho_{44}$ corresponds to the process in which the
states of both qubits are flipped. Therefore, the presence of at least 2m particles is required to have $\rho_{44} \neq 0$. In our model, $\rho_{44}(t) = 0$ implies that the two qubits are entangled with negativity $N(\rho^A) = \sqrt{(\rho_{11})^2 + 4|\rho_{23}|^2} - \rho_{11} \geq 0$ for any value of t (except for those values of t for which $m > 1$). For $N = m$, $|u_1| = |u_2| = \frac{1}{\sqrt{2}}$ and $g_1 = g_2$, expressions (78) and (79) take the particularly simple form

$$
\rho_{23} = \frac{1}{2N} \sin^2(\sqrt{N}gt), \quad \rho_{kk} = \frac{N!}{2N} \sum_{n_1+n_2=N} \frac{(n_1, n_2)}{n_1!n_2!} \text{ for } k = (1, 2, 3) \text{ and } \rho_{44} = 0.
$$

Here, we notice that the maximum values of the negativity $N(\rho^A)$, behave like $1/2^{2N-1}$ for large values of N. For $m > 1$, this behavior can be improved. In fact, for $N = m + 1$ we have $\rho_{23} = \frac{N}{2N} \sin(\sqrt{N}gt) \sin(\sqrt{(m+1)}gt)$, $\rho_{44} = 0$ and as a result $N(\rho^A) \sim N/2^{2N-3}$ for large values of N. Graphs for the cases $m = 2$, $N = 2$ and $m = 2$, $N = 3$ are shown in Fig. 2(a) In what follows, we will restrict our discussion to the case $m = 1$.

### A. Entanglement from 1-Particle States

If the qubit system is initially in the state $|0,0\rangle$, we obtain from (78) and (79) the following density matrix

$$
\rho_{11} = 1 - |u_1|^2 \sin^2(g_1t) - |u_2|^2 \sin^2(g_2t), \\
\rho_{33} = |u_1|^2 \sin^2(g_1t), \\
\rho_{22} = |u_2|^2 \sin^2(g_2t), \\
\rho_{23} = u_1^*u_2 \sin(g_1t) \sin(g_2t).
$$

As we know from previous discussions, this state is always entangled with negativity $N(\rho^A) = \sqrt{(\rho_{11})^2 + 4|\rho_{23}|^2} - \rho_{11}$. Furthermore, if the amplitudes $u_i$ satisfy $|u_1| = |u_2| = \frac{1}{\sqrt{2}}$ and $g_1 = g_2$, the negativity oscillates between zero and its maximum value $N = 1$ with period $\frac{\pi}{g}$ (see Fig. 2(a)). However, this behavior changes substantially if the initial state of $A_1 A_2$ is $|a_1, a_2\rangle = |1, 1\rangle$. From section III, we know that the reduced density matrix can be obtained from the relation

$$
\rho^A(|a_1\rangle, U) = \begin{pmatrix}
\hat{\rho}_{44} & 0 & 0 & \hat{\rho}_{41} \\
0 & \hat{\rho}_{33} & \hat{\rho}_{32} & 0 \\
0 & \hat{\rho}_{32} & \hat{\rho}_{22} & 0 \\
\hat{\rho}_{14} & 0 & 0 & \hat{\rho}_{11}
\end{pmatrix} = V \rho^A(|a_1\rangle, U|F_1^1, F_2^1\rangle V^\dagger, \quad V = \sigma_x \otimes \sigma_x.
$$

![Negativity as a function of time for odd and even values of N. We have assumed $g_1 = g_2 = \frac{\pi}{g}$ and $u_1 = u_2 = \frac{1}{\sqrt{2}}$.](image)
In this case we also have \( \rho_{14} = 0 \) and therefore in order to quantify the entanglement in system \( A_1A_2 \) one needs the following matrix elements:

\[
\begin{align*}
\rho_{44} &= \cos^2(g_1t)\cos^2(g_2t) + |u_1|^2\cos^2(g_2t)(\cos^2(\sqrt{2}g_1t) - \cos^2(g_1t)) \\
&\quad + |u_2|^2\cos^2(g_1t)(\cos^2(\sqrt{2}g_2t) - \cos^2(g_2t)) \\
\rho_{11} &= \sin^2(g_1t)\sin^2(g_2t) - |u_1|^2\sin^2(g_2t)(\cos^2(\sqrt{2}g_1t) - \cos^2(g_1t)) \\
&\quad - |u_2|^2\sin^2(g_1t)(\cos^2(\sqrt{2}g_2t) - \cos^2(g_2t)) \\
\rho_{32} &= u_1u_2^*\cos(\sqrt{2}g_2t)\cos(\sqrt{2}g_1t)\sin(g_1t)\sin(g_2t).
\end{align*}
\]  

(83) The above expressions dictate the time dependence of the negativity (see Fig. 3(b)) and contrary to the situation in which the initial state of \( A_1A_2 \) was \( |0,0\rangle \), now system \( A_1A_2 \) exhibits periods of entanglement death and entanglement revivals. Nevertheless, for the symmetric scenario (maximally entangled state \((\rho_2 = 2|\psi^+\rangle\langle\psi^+|) \) one can obtain an almost maximally entangled state \( N \approx 1 \). Notice from (83, 84, 85) that one can simultaneously have \( \rho_{11} \approx 0, \rho_{44} \approx 0 \) and \( \rho_{23} \approx \frac{1}{2} \) when

\[
\sqrt{2}gt \approx n\pi \quad \text{and} \quad gt \approx (2m + 1)\frac{\pi}{2} \; \Rightarrow \; (2m + 1) \approx n\sqrt{2}.
\]

(86) The first three pairs of numbers of the form \((n,2m+1)\) satisfying (approximately) these equations are \( (5,7), (12,17) \) and \((29,41)\). \( 5\sqrt{2} = 7.07, \quad 12\sqrt{2} = 16.97, \quad 29\sqrt{2} = 41.01 \). The first two pairs correspond to the peaks with \( N \approx 1 \) in Fig 3(b) At these points the system \( A_1A_2 \) is in the state \( \rho^1 \approx |\psi^+\rangle\langle\psi^+| \) with \( |\psi^+\rangle = \frac{1}{2}(|a_2\rangle + |a_3\rangle) \).

![Negativity vs. time.](image)

**FIG. 3:** Negativity vs. time.

### B. Operators of the form \( F_i = g_i(a_i + |\beta_i|e^{i\theta_i}a_i^\dagger) \)

Another type of operators \( F_i \) belonging to the class \([16]\) are the operators of the form \( F_i = g_i(a_i + |\beta_i|e^{i\theta_i}a_i^\dagger) \) with \( |\beta_i| < 1 \) and \( |a_1,a_2| = 0 \). For simplicity, we assume that system \( A_1A_2 \) is initially in the state \( |0,0\rangle \). For these operators, one can sum the series \([14],[15]\) by means of a Bogoliubov transformation \([18]\). In fact, one may set \( \beta_i = \tanh(r_i) \) and write \( F_i = \tilde{g}_i b_i \) where \( b_i = \cosh(r_i)a_i + e^{i\theta_i}\sinh(r_i)a_i^\dagger \) and \( \tilde{g}_i = \frac{g_i}{\cosh(r_i)} \). Using the identities

\[
b_i = \mathcal{U}_i a_i \mathcal{U}_i^\dagger, \quad \text{with} \quad \mathcal{U}_i = e^{\frac{1}{2}z^*z}, \quad z = r_i e^{i\theta_i}
\]

(87) one can express the vacuum corresponding to the operators \( a_i \) in terms of the eigenstates of the operators \( n_i = b_i^\dagger b_i \). That is

\[
|\bar{0}\rangle = \sum_{n_1,n_2} c_{n_1,n_2} |2n_1,2n_2,0\rangle, \quad c_{n_i} = \frac{1}{\sqrt{\cosh(r_i)}} e^{in_i\theta_i} \frac{\tanh(n_i)(2n_i)!}{2^{n_i}n_i!}.
\]  

(88)
In this new basis the 1-particle excitation $|\Phi_B\rangle = a_B^\dagger |0\rangle$ with $u_1 = u_2 = \frac{1}{\sqrt{2}}$ assumes the form

$$|\Phi_B\rangle = \frac{1}{\sqrt{2}} \sum_{n_1,n_2} |c_{n_1} c_{n_2}(\sqrt{\frac{2n_1 + 1}{\cosh(r_1)}} |2n_1 + 1, 2n_2\rangle + \sqrt{\frac{2n_2 + 1}{\cosh(r_2)}} |2n_1, 2n_2 + 1\rangle).$$

(89)

Assuming $g_1 = g_2$ and taking into account the fact that the states $|n_i\rangle$ are eigenstates of the operators $F_i^\dagger F_i$ with eigenvalues $\tilde{g}_i^2 n_i$ we obtain:

$$\rho^A = \begin{pmatrix} AB & 0 & 0 & -EF \\ 0 & \frac{AD + BC}{2} & \frac{E^2 + F^2}{2} & 0 \\ 0 & \frac{E^2 + F^2}{2} & \frac{AD + BC}{2} & 0 \\ -EF & 0 & 0 & CD \end{pmatrix}$$

(90)

with $A, B, C, D, E, F$ given by the following series

$$A = \sum_n \frac{2n + 1}{\cosh^2(r)} |c_n|^2 \cos(\sqrt{2n + 1} \tilde{g} t), \quad B = \sum_n |c_n|^2 \cos^2(\sqrt{2n} \tilde{g} t)$$

(91)

$$C = \sum_n \frac{2n + 1}{\cosh^2(r)} |c_n|^2 \sin^2(\sqrt{2n + 1} \tilde{g} t), \quad D = \sum_n |c_n|^2 \sin^2(\sqrt{2n} \tilde{g} t)$$

(92)

and

$$E = \sum_n \frac{\sqrt{2n + 1}}{\cosh(r)} |c_n|^2 \cos(\sqrt{2n + 1} \tilde{g} t) \sin(\sqrt{2n + 1} \tilde{g} t)$$

(93)

$$F = \frac{\sinh(r)}{\cosh(r)^2} \sum_n \frac{2n + 1}{\sqrt{2(n + 1)} |c_n|^2 \cos(\sqrt{2n + 1} \tilde{g} t) \sin(\sqrt{2(n + 1)} \tilde{g} t)}$$

(94)

Notice, that in the limit $|\beta| = 1$, one has $[F_i, F_i^\dagger] = 0$. Hence, according to section (III), the entanglement in system A should disappear as we approach $|\beta| = 1$. In Fig.4 we present graphs of entanglement versus time for different values of $|\beta|$. Numerical analysis shows, that entanglement is more strongly deteriorated for values of $\beta$ close to 1. Thus we conclude that operators being mixtures of the form $a + \beta a^\dagger$ with $\beta < 1$ can also transfer a substantial amount of entanglement to the two qubit system.

FIG. 4: Negativity vs. time for different values of $\beta$. Here we assume $g_1 = g_2 = 1$ and $u_1 = u_2 = \frac{1}{\sqrt{2}}$. The cases $\beta = 0$, $\beta = 0.5$, and $\beta = 0.7$ are represented by dashed, solid, and thick solid lines respectively.
C. Entanglement from Mixed States

The expressions (72) and (75) can be also used to determine the final state of the qubits when system B is in a mixed state of the form \( \rho_B = \sum_N p_N |N \rangle \langle N |. \) In this case, the matrix elements of \( \rho^A \) read

\[
\rho^A_{ij} = \sum_N p_N \rho_{ij}(N)
\]

with \( \rho_{ij}(N) \) given by equations (72) and (75). The separability of the qubits \( A_1A_2 \) depends on the distribution \( \{ p_N \} \).

For example, consider the binomial distribution \( p_N = (\frac{N}{M}) p^N (1-p)^{M-N} \) (which was already discussed in section (IV)). In this case, equations (95), (72) and (75) yield

\[
\rho^A_{kk} = \frac{M!}{n_1!n_2!(M-n_1-n_2)!} |\sqrt{p}| u_1|^{2n_1} |\sqrt{p}| u_2|^{2n_2} \left( 1 - |\sqrt{p}| u_1|^2 - |\sqrt{p}| u_2|^2 \right)^{M-n_1-n_2} F_{k,k}(n_1, n_2).
\]

Similarly, we compute \( \rho^A_{11} \) from (75) to find that the matrix elements \( \rho^A_{ij} \) corresponding to the binomial distribution \( p_N = (\frac{M}{N}) p^N (1-p)^{M-N} \) have the same form as the \( \rho_{ij}(N) \) from (72) and (75) with \( N \) replaced by \( M \) and the amplitudes \( u_i \) replaced by \( \sqrt{p} u_i \). Therefore, this case reduces to the previously studied situation where \( \rho_B = |M \rangle \langle M | \).

On the other hand, since entanglement disappears as the amplitudes \( u_i = \langle \phi_i | \phi_B \rangle \) approach zero, one expects to obtain a separable state \( \rho^A \) for a Poissonian distribution i.e. \( \rho_B = p_N |N \rangle \langle N | \) with \( p_N = \frac{N^2}{2^N} e^{-N} \) (recall that Poisson distribution is the limit of the binomial distribution for \( M \to \infty, p \to 0 \) and \( Mp = \lambda \)). In fact, for a Poissonian distribution, one obtains from (95) (72) and (75) the state

\[
\rho^A = \begin{pmatrix}
c_1c_2 & 0 & 0 & 0 \\
c_1s_2 & m_1^* m_2 & 0 & 0 \\
0 & m_1 m_2^* & c_2 s_1 & 0 \\
0 & 0 & 0 & s_1 s_2
\end{pmatrix}
\]

where the functions \( c_i, s_i, \) and \( m_i \) given by

\[
c_i = e^{-\lambda |u_i|^2} \sum_{n_i} \frac{(\lambda |u_i|^2)^{n_i}}{n_i!} \cos^2(\sqrt{n_i} g_i t), \quad s_i = e^{-\lambda |u_i|^2} \sum_{n_i} \frac{(\lambda |u_i|^2)^{n_i}}{n_i!} \sin^2(\sqrt{n_i} g_i t),
\]

\[
m_i = \sqrt{\lambda} u_i e^{-\lambda |u_i|^2} \sum_{n_i} \frac{(\lambda |u_i|^2)^{n_i}}{n_i!} \frac{\cos(\sqrt{n_i} g_i t) \sin(\sqrt{n_i} + 1 g_i t)}{\sqrt{n_i} + 1}.
\]

A matrix with the structure of (97) must necessarily be separable. Notice that the entanglement condition \( |\rho_{23}|^2 > \rho_{11} \rho_{44} \) is not compatible with the positivity condition \( |\rho_{23}|^2 < \rho_{22} \rho_{33} \). Hence, \( \rho^A \) is separable.

VI. ENTANGLEMENT FROM N PARTICLES OCCUPYING DIFFERENT 1-PARTICLE STATES

So far we have considered N-particle excitations of system B with all the particles occupying the same 1-particle state. It is also interesting to study multiparticle states of the form

\[
|\Phi_B \rangle = \prod_{k=1}^N a_k^\dagger (\phi_{B,k}) |0 \rangle, \quad \langle \phi_{B,k'}| \phi_{B,k'} \rangle = \delta_{k,k'}
\]

representing N identical particles occupying mutually orthogonal 1-particle states. Again, we assume that the interactions between B and A, are of the form (12) with \( F_i = g_i a(\phi_i) \) and \( \langle \phi_1 | \phi_2 \rangle = 0 \). It is clear that now the entanglement transferred to \( A_1A_2 \) depends on the relative geometry of the set of states \( \{ \phi_{B_1}, \phi_{B_2}, \ldots, \phi_{B_N} \} \) and the states \( |\phi_1 \rangle \) and \( |\phi_2 \rangle \). One can find the density matrix for states of the form (98) using expressions (72) and (75). Taking the linear combinations \( |\phi_B \rangle = \sum_{k=1}^N x_k |\phi_{B,k} \rangle, \) \( |\tilde{\phi}_B \rangle = \sum_{k=1}^N y_k^* |\phi_{B,k} \rangle \) and defining the states \( |\Phi_B \rangle \equiv \frac{1}{\sqrt{N}} a^N (\phi_B) |0 \rangle, \)

\[
|\tilde{\Phi}_B \rangle \equiv \frac{1}{\sqrt{N}} a^N (\phi_B) |0 \rangle,
\]

one computes the auxiliary matrix element

\[
\tilde{\rho}_{11}(x,y) \equiv \langle \tilde{\Phi}_B | \tilde{K}_1^\dagger \tilde{K}_1 \tilde{K}_2^\dagger \tilde{K}_2 | \tilde{\Phi}_B \rangle = \sum_{n_1,n_2} \frac{N!}{n_1!n_2!(N-n_1-n_2)!} P_{n_1,n_2}(x,y) F_{11}(n_1,n_2).
\]

(99)
In the above expression, the polynomial \( P_{n_1, n_2}(x, y) \) is given by
\[
P_{n_1, n_2}(x, y) = (u_1 \tilde{u}_1)^{n_1}(u_2 \tilde{u}_2)^{n_2}(\langle \tilde{\phi}_B | \phi_B \rangle - u_1 \tilde{u}_1 - u_2 \tilde{u}_2)^{N-n_1-n_2}. \tag{100}
\]
with \( u_i = \sum_k x_k u_{i,k} \), \( \tilde{u}_i = \sum_k y_k u_{i,k}^* \) and \( u_{i,k} \equiv \langle \phi_i | \phi_{B_k} \rangle \). From the above expression we can extract the first diagonal element of the two qubit density matrix. Notice, that \( \rho_{11} \) (which corresponds to the original state \( |\Phi_B\rangle = \prod_{k=1}^N a^\dagger(\phi_{B_k})|0\rangle \)) is related to \( \tilde{\rho}_{11}(x, y) \) as follows:
\[
\rho_{11} = \frac{1}{N!} \frac{\partial^N}{\partial x_1 \ldots \partial x_N \partial y_1 \ldots \partial y_N} \tilde{\rho}_{11}(x, y)
\]
\[
= \sum_{n_1,n_2} \frac{1}{n_1! n_2! (N-n_1-n_2)!} F_{11}(n_1,n_2) \frac{\partial^N}{\partial x_1 \ldots \partial x_N \partial y_1 \ldots \partial y_N} P_{n_1,n_2}(x, y). \tag{101}
\]
Following the same steps, one obtains \( \rho_{22}, \rho_{33} \) and \( \rho_{44} \). Similarly, one finds that the off diagonal element \( \rho_{23} \) is
\[
\rho_{23} = \frac{1}{N!} \frac{\partial^N}{\partial x_1 \ldots \partial x_n \partial y_1 \ldots \partial y_N} \tilde{\rho}_{23}(x, y)
\]
\[
= \sum_{n_1,n_2} \frac{1}{n_1! n_2! (N-n_1-n_2-1)!} F_{23}(n_1,n_2) \frac{\partial^N}{\partial x_1 \ldots \partial x_n \partial y_1 \ldots \partial y_N} Q_{n_1,n_2}(x, y) \tag{102}
\]
with the polynomial \( Q_{n_1,n_2}(x, y) \) given by
\[
Q_{n_1,n_2}(x, y) = \tilde{u}_1^* u_2 (u_1 \tilde{u}_1)^{n_1}(u_2 \tilde{u}_2)^{n_2}(\langle \tilde{\phi}_B | \phi_B \rangle - u_1 \tilde{u}_1 - u_2 \tilde{u}_2)^{N-n_1-n_2-1}. \tag{103}
\]
Using the above equations one can express the density matrix \( \rho^A \) in terms of the 2 × N matrix \( u_{i,k} = \langle \phi_i | \phi_{B_k} \rangle \) and the constants \( g_1 \) and \( g_2 \). Let us study the particular case where \( |\phi_{B_1}\rangle \) and \( |\phi_{B_2}\rangle \) lie on the plane spanned by \( |\phi_1\rangle \) and \( |\phi_2\rangle \). Then they are related by an \( SU(2) \) transformation, i.e.
\[
\begin{pmatrix} \phi_{B_1} \\ \phi_{B_2} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) e^{i\eta} \\ -\sin(\theta) e^{-i\eta} & \cos(\theta) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{104}
\]
Making use of (102) one obtains
\[
\rho_{23}(t) = \frac{1}{2\sqrt{2}} \sin(4\theta) e^{i\eta} (\sin(g_1 t) \sin(\sqrt{2}g_2 t) \cos(g_2 t) - \sin(g_2 t) \sin(\sqrt{2}g_1 t) \cos(g_1 t)) \tag{105}
\]
which vanishes when either \( \sin(4\theta) = 0 \) or \( g_1 = g_2 \). Therefore, in this case, the entanglement transfer scheme works if the coupling constants are different. The diagonal elements \( \rho_{11} \) and \( \rho_{44} \) read
\[
\rho_{11}(t) = \frac{1}{2} \sin^2(2\theta)(\cos^2(\sqrt{2}g_1 t) + \cos^2(\sqrt{2}g_2 t)) + \cos^2(2\theta) \cos^2(g_1 t) \cos^2(g_2 t) \tag{106}
\]
\[
\rho_{44}(t) = \cos^2(2\theta) \sin^2(g_1 t) \sin^2(g_2 t). \tag{107}
\]

From the above equations, one finds that the maximum value of entanglement in \( A_1A_2 \) is achieved when \( |\sin(4\phi)| = 1 \). It is interesting to compare this situation with the case when both particles occupy the same state (see (72) and (73)); if \( g_1 \neq g_2 \), one can increase the entanglement transferred to the qubits preparing B in a state of the form (104) (see Fig 5).

**VII. CONCLUSIONS**

We have studied the entanglement induced in a two qubit system as a result of its interaction with a bosonic system. The operators coupling each of the qubits to the bosonic system were assumed to commute. As discussed throughout
FIG. 5: Negativity as a function of time. The dashed line corresponds to a state of the form \(|104\rangle\) with \(\theta = \frac{\pi}{8}\) and \(g_1 = \frac{3}{\sqrt{2}}\), while the solid line corresponds to a 2-particle excitation where each of the particles occupies the state \(|\phi_B\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle)\).

In this paper, these interactions appear in the situation when one couples each qubit to a different mode. More precisely, we considered operators of the form

\[
F_i = p_i(a_i^\dagger a_i)^n + q_i(a_i^\dagger a_i)^{m'}, \quad i = (1, 2) \quad \text{with} \quad [a_1, a_2^\dagger] = 0. \tag{108}
\]

In this case, the mechanism entangling the qubits is analogous to the mechanism responsible for the entanglement transfer from two qubit systems to two qubit systems (see Fig. 1(a)). From section (V), we know that a 1-particle state being a superposition of the modes \(|\phi_1\rangle, |\phi_2\rangle\) takes the form of the entangled state

\[
|\Phi_B\rangle = a^\dagger(\phi_B)|0\rangle = u_1|1, 0\rangle + u_2|0, 1\rangle \tag{109}
\]

when written in occupation number representation. However, the form of state \(|\Phi_B\rangle\) depends on the interaction between the qubits and system B. If one of the qubits interacts with mode \(|\phi_B\rangle\) while the other qubit interacts with mode \(|\phi'_B\rangle\) (orthogonal to \(|\phi_B\rangle\)), then the state \(|\Phi_B\rangle = a^\dagger(\phi_B)|0\rangle\) may be written as

\[
|\Phi_B\rangle = |1\rangle|0\rangle. \tag{110}
\]

Now, \(|\Phi_B\rangle\) has the form of a separable state. It is for this reason that we avoided talking about the entanglement between the modes. Instead, we computed the entanglement induced in the two qubit system as a result of the interaction with multiparticle systems. For all the N-particles states considered, we found an interaction inducing entanglement in the two qubit system. This situation changes dramatically if the bosonic system is in the coherent state \(|z\rangle = e^{z a(\phi_B)^\dagger - z^* a(\phi_B)} \sim e^{z a(\phi_B)}|0\rangle\). In fact, this state behaves like a separable state for operators of the form \(|108\rangle\). In section (I), we studied the series expansion of the negativity \(N(\rho^A)\). We computed the first nonvanishing contribution to \(N(\rho^A)\) in the case where the operators acting on B were different from those in \(|108\rangle\). We found that when system B is in the particle vacuum state \(|0\rangle\), the qubits may become entangled if the interaction Hamiltonian contains operators of the form \(F_i = g_i(a(\phi_i) + \beta a^\dagger(\psi_i))\). This type of interactions could be used to extract entanglement from a coherent state \(|z\rangle\) (entanglement extraction from coherent states has been discussed in \[5\]). We leave the this problem for future work.

ACKNOWLEDGEMENTS

The author is grateful to Professors Thomas Curtright and Luca Mezincescu for helpful comments. He would also like to thank Łukasz Cywinski and Dan Pruteanu for their interest in this work.

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