Orthogonality and minimality in the homology of locally finite graphs

Reinhard Diestel Julian Pott

May 7, 2014

Abstract

Given a finite set $E$, a subset $D \subseteq E$ (viewed as a function $E \to \mathbb{F}_2$) is orthogonal to a given subspace $F$ of the $\mathbb{F}_2$-vector space of functions $E \to \mathbb{F}_2$ as soon as $D$ is orthogonal to every $\subseteq$-minimal element of $F$. This fails in general when $E$ is infinite.

However, we prove the above statement for the six subspaces $F$ of the edge space of any 3-connected locally finite graph that are relevant to its homology: the topological, algebraic, and finite cycle and cut spaces. This solves a problem of [5].

1 Introduction

Let $G$ be a 2-connected locally finite graph, and let $\mathcal{E} = \mathcal{E}(G)$ be its edge space over $\mathbb{F}_2$. We think of the elements of $\mathcal{E}$ as sets of edges, possibly infinite. Two sets of edges are orthogonal if their intersection has (finite and) even cardinality. A set $D \in \mathcal{E}$ is orthogonal to a subspace $F \subseteq \mathcal{E}$ if it is orthogonal to every $F \in F$.

The topological cycle space $\mathcal{C}_{\text{top}}(G)$ of $G$ is the subspace of $\mathcal{E}(G)$ generated (via thin sums, possibly infinite) by the circuits of $G$, the edge sets of the topological circles in the Freudenthal compactification $|G|$ of $G$. This space $\mathcal{C}_{\text{top}}(G)$ contains precisely the elements of $\mathcal{E}$ that are orthogonal to $\mathcal{B}_{\text{fin}}(G)$, the finite-cut space of $G$ [4]. The algebraic cycle space $\mathcal{C}_{\text{alg}}(G)$ of $G$ is the subspace of $\mathcal{E}$ consisting of the edge sets inducing even degrees at all the vertices. It contains precisely the elements of $\mathcal{E}$ that are orthogonal to the skew cut space $\mathcal{B}_{\text{skew}}(G)$ [3], the subspace of $\mathcal{E}$ consisting of all the cuts of $G$ with one side finite. The finite-cycle space $\mathcal{C}_{\text{fin}}(G)$ is the subspace of $\mathcal{E}$ generated (via finite sums) by the finite circuits of $G$. This space $\mathcal{C}_{\text{fin}}(G)$ contains precisely the elements of $\mathcal{E}$ that are orthogonal to $\mathcal{B}(G)$, the cut space of $G$ [4,5]. Thus,

\[ \mathcal{C}_{\text{top}} = \mathcal{B}_{\text{fin}}^\perp, \quad \mathcal{C}_{\text{alg}} = \mathcal{B}_{\text{skew}}^\perp, \quad \mathcal{C}_{\text{fin}} = \mathcal{B}^\perp. \]

Conversely,

\[ \mathcal{C}_{\text{top}}^\perp = \mathcal{B}_{\text{fin}}, \quad \mathcal{C}_{\text{alg}}^\perp = \mathcal{B}_{\text{skew}}, \quad \mathcal{C}_{\text{fin}}^\perp = \mathcal{B}. \]

Thus, for any of the six spaces $F$ just mentioned, we have $F^\perp \perp = F$. 

1

arXiv:1307.0728v2 [math.CO] 13 Aug 2013
Proofs of most of the above six identities were first published by Casteels and Richter [3], in a more general setting. Any remaining proofs can be found in [5], except for the inclusion $\mathcal{C}_{\text{alg}}^+ \supseteq \mathcal{B}_{\text{skew}}$, which is easy.

The six subspaces of $E$ mentioned above are the the ones most relevant to the homology of locally finite infinite graphs. See [5], Diestel and Sprüssel [6], and Georgakopoulos [7, 8]. Our aim in this note is to facilitate orthogonality proofs for these spaces by showing that, whenever $F$ is one of them, a set $D$ of edges is orthogonal to $F$ as soon as it is orthogonal to the minimal nonzero elements of $F$.

This is easy when $F$ is $\mathcal{C}_{\text{fin}}$ or $\mathcal{B}_{\text{fin}}$ or $\mathcal{B}_{\text{skew}}$.

**Proposition 1.** Let $F$ be a subspace of $E$ all whose elements are finite sets of edges. Then $F$ is generated (via finite sums) by its $\subseteq$-minimal nonzero elements.

**Proof.** For a contradiction suppose that some $F \in F$ is not a finite sum of finitely many minimal nonzero elements of $F$. Choose $F$ with $|F|$ minimal. As $F$ is not minimal itself, by assumption, it properly contains a minimal nonzero element $F'$ of $F$. As $F$ is finite, $F + F' = F \setminus F' \in F$ has fewer elements than $F$, so there is a finite family $(M_i)_{i \leq n}$ of minimal nonzero elements of $F$ with $\sum_{i \leq n} M_i = F + F'$. This contradicts our assumption, as $F' + \sum_{i \leq n} M_i = F$. \(\square\)

**Corollary 2.** If $F \in \{\mathcal{C}_{\text{fin}}, \mathcal{B}_{\text{fin}}, \mathcal{B}_{\text{skew}}\}$, a set $D$ of edges is orthogonal to $F$ as soon as $D$ is orthogonal to all the minimal nonzero elements of $F$. \(\square\)

When $F \in \{\mathcal{C}_{\text{top}}, \mathcal{C}_{\text{alg}}, \mathcal{B}\}$, the statement of Corollary 2 is generally false for graphs that are not 3-connected. Here are some examples.

For $F = \mathcal{B}$, let $G$ be the graph obtained from the $\mathbb{N} \times Z$ grid by doubling every edge between two vertices of degree 3 and subdividing all the new edges. The set $D$ of the edges that lie in a $K^3$ of $G$ is orthogonal to every bond $F$ of $G$: their intersection $D \cap F$ is finite and even. But $D$ is not orthogonal to every element of $F = \mathcal{B}$, since it meets some cuts that are not bonds infinitely.

For $F = \mathcal{C}_{\text{top}}$, let $B$ be an infinite bond of the infinite ladder $H$, and let $G$ be the graph obtained from $H$ by subdividing every edge in $B$. Then the set $D$ of edges that are incident with subdivision vertices has a finite and even intersection with every topological circuit $C$, finite or infinite, but it is not orthogonal to every element of $F = \mathcal{C}_{\text{top}}$, since it meets some of them infinitely.

For $F = \mathcal{C}_{\text{alg}}$ we can re-use the example just given for $\mathcal{C}_{\text{top}}$, since for 1-ended graphs like the ladder the two spaces coincide.

However, if $G$ is 3-connected, an edge set is orthogonal to every element of $\mathcal{C}_{\text{top}}, \mathcal{C}_{\text{alg}}$, or $\mathcal{B}$ as soon as it is orthogonal to every minimal nonzero element:

**Theorem 3.** Let $G = (V, E)$ be a locally finite 3-connected graph, and $F, D \subseteq E$.

(i) $F \in \mathcal{C}_{\text{top}}^+$ as soon as $F$ is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\text{top}}$, the topological circuits of $G$.

(ii) $F \in \mathcal{C}_{\text{alg}}^+$ as soon as $F$ is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\text{alg}}$, the finite circuits and the edge sets of double rays in $G$. 2
(iii) $D \in \mathcal{B}^\perp$ as soon as $D$ is orthogonal to all the minimal nonzero elements of $\mathcal{B}$, the bonds of $G$.

Although Theorem 3 fails if we replace the assumption of 3-connectedness with 2-connectedness, it turns out that we need a little less than 3-connectedness. Recall that an end $\omega$ of $G$ has (combinatorial) vertex-degree $k$ if $k$ is the maximum number of vertex-disjoint rays in $\omega$. Halin [9] showed that every end in a $k$-connected locally finite graph has vertex-degree at least $k$. Let us call an end $\omega$ of $G$ $k$-padded if for every ray $R \in \omega$ there is a neighbourhood $U$ of $\omega$ such that for every vertex $u \in U$ there is a $k$-fan from $u$ to $R$ in $G$, a subdivided $k$-star with centre $u$ and leaves on $R$. If every end of $G$ is $k$-padded, we say that $G$ is $k$-padded at infinity. Note that $k$-connected graphs are $k$-padded at infinity. Our proof of Theorem 3(i) and (ii) will use only that every end has vertex-degree at least 3 and that $G$ is 2-connected. Similarly, and in a sense dually, our proof of Theorem 3(iii) uses only that every end has vertex-degree at least 2 and $G$ is 3-connected at infinity.

**Theorem 4.** Let $G = (V, E)$ be a locally finite 2-connected graph.

(i) If every end of $G$ has vertex-degree at least 3, then $F \in \mathcal{C}_{\text{top}}^\perp$ as soon as $F$ is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\text{top}}$, the topological circuits of $G$.

(ii) If every end of $G$ has vertex-degree at least 3, then $F \in \mathcal{C}_{\text{alg}}^\perp$ as soon as $F$ is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\text{alg}}$, the finite circuits and the edge sets of double rays in $G$.

(iii) If $G$ is 3-padded at infinity, then $D \in \mathcal{B}^\perp$ as soon as $D$ is orthogonal to all the minimal nonzero elements of $\mathcal{B}$, the bonds of $G$.

In general, our notation follows [1]. In particular, given an end $\omega$ in a graph $G$ and a finite set $S \subseteq V(G)$ of vertices, we write $C(S, \omega)$ for the unique component of $G - S$ that contains a ray $R \in \omega$. The vertex-degree of $\omega$ is the maximum number of vertex-disjoint rays in $\omega$. The mathematical background required for this paper is covered in [5, 6]. For earlier results on the cycle and cut space see Bruhn and Stein [1, 2].

## 2 Finding disjoint paths and fans

Menger’s theorem that the smallest cardinality of an $A$–$B$ separator in a finite graph is equal to the largest cardinality of a set of disjoint $A$–$B$ paths trivially extends to infinite graphs. Thus in a locally finite $k$-connected graph, there are $k$ internally disjoint paths between any two vertices. In Lemmas 5 and 6 we

---

1For example, if $G$ is the union of complete graphs $K_1, K_2, \ldots$ with $|K_i| = i$, each meeting the next in exactly one vertex (and these are all distinct), then the unique end of $G$ is $k$-padded for every $k \in \mathbb{N}$. 

3
show that, for two such vertices that are close to an end \( \omega \), these connecting paths need not use vertices too far away from \( \omega \).

In a graph \( G \) with vertex sets \( X, Y \subseteq V(G) \) and vertices \( x, y \in V(G) \), a \( k \)-fan from \( X \) (or \( x \)) to \( Y \) is a subdivided \( k \)-star whose center lies in \( X \) (or is \( x \)) and whose leaves lie in \( Y \). A \( k \)-linkage from \( x \) to \( y \) is a union of \( k \) internally disjoint \( x-y \) paths. We may refer to a sequence \( (v_i)_{i \in \mathbb{N}} \) simply by \( (v_i) \), and use \( \bigcup (v_i) := \bigcup_{i \in \mathbb{N}} \{v_i\} \) for brevity.

**Lemma 5.** Let \( G \) be a locally finite graph with an end \( \omega \), and let \( (v_i)_{i \in \mathbb{N}} \) and \( (w_i)_{i \in \mathbb{N}} \) be two sequences of vertices converging to \( \omega \). Let \( k \) be a positive integer.

(i) If for infinitely many \( n \in \mathbb{N} \) there is a \( k \)-fan from \( v_n \) to \( \bigcup (w_i) \), then there are infinitely many disjoint such \( k \)-fans.

(ii) If for infinitely many \( n \in \mathbb{N} \) there is a \( k \)-linkage from \( v_n \) to \( w_n \), then there are infinitely many disjoint such \( k \)-linkages.

**Proof.** For a contradiction, suppose \( k \in \mathbb{N} \) is minimal such that there is a locally finite graph \( G = (V, E) \) with sequences \( (v_i)_{i \in \mathbb{N}} \) and \( (w_i)_{i \in \mathbb{N}} \) in which either (i) or (ii) fails. Then \( k > 1 \), since for every finite set \( S \subseteq V(G) \) the unique component \( C(S, \omega) \) of \( G-S \) that contains rays from \( \omega \) is connected and contains all but finitely many vertices from \( \bigcup (v_i) \) and \( \bigcup (w_i) \).

For a proof of (i) it suffices to show that for every finite set \( S \subseteq V(G) \) there is an integer \( n \in \mathbb{N} \) and a \( k \)-fan from \( v_n \) to \( \bigcup (w_i) \) avoiding \( S \). Suppose there is a finite set \( S \subseteq V(G) \) that meets all \( k \)-fans from \( \bigcup (v_i) \) to \( \bigcup (w_i) \). By the minimality of \( k \), there are infinitely many disjoint \( (k-1) \)-fans from \( \bigcup (v_i) \) to \( \bigcup (w_i) \) in \( C := C(S, \omega) \). Thus, there is a subsequence \( (v'_i)_{i \in \mathbb{N}} \) of \( (v_i)_{i \in \mathbb{N}} \) in \( C \) and pairwise disjoint \( (k-1) \)-fans \( F_i \subseteq C \) from \( v'_i \) to \( \bigcup (w_i) \) for all \( i \in \mathbb{N} \). For every \( i \in \mathbb{N} \) there is by Menger’s theorem a \((k-1)\)-separator \( S_i \) separating \( v'_i \) from \( \bigcup (w_i) \) in \( C \), as by assumption there is no \( k \)-fan from \( v'_i \) to \( \bigcup (w_i) \) in \( C \). Let \( C_i \) be the component of \( G - (S \cup S_i) \) containing \( v'_i \).

Since \( F_i \) is a subdivided \( |S_i| \)-star, \( S_i \subseteq V(F_i) \). Hence for all \( i \neq j \), our assumption of \( F_i \cap F_j = \emptyset \) implies that \( F_i \cap S_j = \emptyset \), and hence that \( F_i \cap C_j = \emptyset \). But then also \( C_i \cap C_j = \emptyset \), since any vertex in \( C_i \cap C_j \) could be joined to \( v'_i \) by a path \( P \) in \( C_i \) and to \( v'_j \) by a path \( Q \) in \( C_j \), giving rise to a \( v'_i - \bigcup (w_i) \) path in \( P \cup Q \cup F_i \) avoiding \( S_i \), a contradiction.

As \( S \cup S_i \) separates \( v'_i \) from \( \bigcup (w_i) \) in \( G \) and there is, by assumption, a \( k \)-fan from \( v'_i \) to \( \bigcup (w_i) \) in \( G \), there are at least \( k \) distinct neighbours of \( C_i \) in \( S \cup S_i \). Since \( |S_i| = k-1 \), one of these lies in \( S \). This holds for all \( i \in \mathbb{N} \). As \( C_i \cap C_j = \emptyset \) for distinct \( i \) and \( j \), this contradicts our assumption that \( G \) is locally finite and \( S \) is finite. This completes the proof of (i).

For (ii) it suffices to show that for every finite set \( S \subseteq V(G) \) there is an integer \( n \in \mathbb{N} \) such that there is a \( k \)-linkage form \( v_n \) to \( w_n \) avoiding \( S \). Suppose there is a finite set \( S \subseteq V(G) \) that meets all \( k \)-linkages from \( v_i \) to \( w_i \) for all \( i \in \mathbb{N} \). By the minimality of \( k \) there is an infinite family \( (L_i)_{i \in I} \) of disjoint \( (k-1) \)-linkages \( L_i \) in \( C := C(S, \omega) \) from \( v_i \) to \( w_i \). As earlier, there are pairwise disjoint \( (k-1) \)-sets \( S_i \subseteq V(L_i) \) separating \( v_i \) from \( w_i \) in \( C \), for all \( i \in I \). Let
$C_i, D_i$ be the components of $C - S_i$ containing $v_i$ and $w_i$, respectively. For no $i \in I$ can both $C_i$ and $D_i$ have $\omega$ in their closure, as they are separated by the finite set $S \cup S_i$. Thus for every $i \in I$ one of $C_i$ or $D_i$ contains at most finitely many vertices from $\bigcup_{i \in I} L_i$. By symmetry, and replacing $I$ with an infinite subset of itself if necessary, we may assume the following:

The components $C_i$ with $i \in I$ each contain only finitely many vertices from $\bigcup_{i \in I} L_i$. \hfill (1)

If infinitely many of the components $C_i$ are pairwise disjoint, then $S$ has infinitely many neighbours as earlier, a contradiction. By Ramsey’s theorem, we may thus assume that

$$C_i \cap C_j \neq \emptyset \text{ for all } i, j \in I. \hfill (2)$$

Note that if $C_i$ meets $L_j$ for some $j \neq i$, then $C_i \supseteq L_j$, since $L_j$ is disjoint from $L_i \supseteq S_i$. By (1), this happens for only finitely many $j > i$. We can therefore choose an infinite subset of $I$ such that $C_i \cap L_j = \emptyset$ for all $i < j$ in $I$. In particular, $(C_i \cup S_i) \cap S_j = \emptyset$ for $i < j$. By (2), this implies that

$$C_i \cup S_i \subseteq C_j \text{ for all } i < j. \hfill (3)$$

By assumption, there exists for each $i \in I$ some $v_i - w_i$ linkage of $k$ independent paths in $G$, one of which avoids $S_i$ and therefore meets $S$. Let $P_i$ denote its final segment from its last vertex in $S$ to $w_i$. As $w_i \in C \cap (C_i \cup S_i)$ and $P_i$ avoids both $S_i$ and $S$ (after its starting vertex in $S$), we also have

$$P_i \cap C_i = \emptyset. \hfill (4)$$

On the other hand, $L_i$ contains $v_i \in C_i \subseteq C_{i+1}$ and avoids $S_{i+1}$, so $w_i \in L_i \subseteq C_{i+1}$. Hence $P_i$ meets $S_j$ for every $j \geq i + 1$ such that $P_i \subseteq S \cup C_j$. Since the $L_j \supseteq S_j$ are disjoint for different $j$, this happens for only finitely many $j > i$. Deleting those $j$ from $I$, and repeating that argument for increasing $i$ in turn, we may thus assume that $P_i \subseteq S \cup C_{i+1}$ for all $i \in I$. By (3) and (4) we deduce that $P_i \cap S$ are now disjoint for different values of $i \in I$. Hence $S$ contains a vertex of infinite degree, a contradiction.

Recall that $G$ is $k$-padded at an end $\omega$ if for every ray $R \in \omega$ there is a neighbourhood $U$ such that for all vertices $u \in U$ there is a $k$-fan from $u$ to $R$ in $G$. Our next lemma shows that, if we are willing to make $U$ smaller, we can find the fans locally around $\omega$:

**Lemma 6.** Let $G$ be a locally finite graph with a $k$-padded end $\omega$. For every ray $R \in \omega$ and every finite set $S \subseteq V(G)$ there is a neighbourhood $U \subseteq C(S, \omega)$ of $\omega$ such that from every vertex $u \in U$ there is a $k$-fan in $C(S, \omega)$ to $R$.

**Proof.** Suppose that, for some $R \in \omega$ and finite $S \subseteq V(G)$, every neighbourhood $U \subseteq C(S, \omega)$ of $\omega$ contains a vertex $u$ such that $C(S, \omega)$ contains no $k$-fan from $u$ to $R$. Then there is a sequence $u_1, u_2, \ldots$ of such vertices converging to $\omega$. As $\omega$ is $k$-padded there are $k$-fans from infinitely many $u_i$ to $R$ in $G$. By Lemma 5(i) we may assume that these fans are disjoint. By the choice of $u_1, u_2, \ldots$, all these disjoint fans meet the finite set $S$, a contradiction.
3 The proof of Theorems 3 and 4

As pointed out in the introduction, Theorem 4 implies Theorem 3. It thus suffices to prove Theorem 4, of which we prove (i) first. Consider a set \( F \neq \emptyset \) of edges that meets every circuit of \( G \) evenly. We have to show that \( F \in C_{\text{top}} \), i.e., that \( F \) is a finite cut. (Recall that \( C_{\text{top}} \) is known to equal \( B_{\text{mn}} \), the finite-cut space [5].) As \( F \) meets every finite cycle evenly it is a cut, with bipartition \((A, B)\) say. Suppose \( F \) is infinite. Let \( R \) be a set of three disjoint rays that belong to an end \( \omega \) in the closure of \( F \). Every \( R-R' \) path \( P \) for two distinct \( R, R' \in R \) lies on the unique topological circle \( C(R, R', P) \) that is contained in \( R \cup R' \cup P \cup \{\omega\} \). As every circuit meets \( F \) finitely, we deduce that no ray in \( R \) meets \( F \) again and again. Replacing the rays in \( R \) with tails of themselves as necessary, we may thus assume that \( F \) contains no edge from any of the rays in \( R \). Suppose \( F \) separates \( R \), with the vertices of \( R \in R \) in \( A \) and the vertices of \( R', R'' \in R \) in \( B \) say. Then there are infinitely many disjoint \( R-(R' \cup R'') \) paths each meeting \( F \) at least once. Infinitely many of these disjoint paths avoid one of the rays in \( B \), say \( R' \). The union of these paths together with \( R \) and \( R' \) contains a ray \( W \in \omega \) that meets \( F \) infinitely often. For every \( R''-W \) path \( P \), the circle \( C(W, R'', P) \) meets \( F \) in infinitely many edges, a contradiction. Thus we may assume that \( F \) does not separate \( R \), and that \( G[A] \) contains \( \bigcup R \).

As \( \omega \) lies in the closure of \( F \), there is a sequence \((v_i)_{i \in \mathbb{N}} \) of vertices in \( B \) converging to \( \omega \). As \( G \) is 2-connected there is a 2-fan from each \( v_i \) to \( \bigcup R \) in \( G \). By Lemma 3 there are infinitely many disjoint 2-fans from \( \bigcup(v_i) \) to \( \bigcup R \). We may assume that every such fan has at most two vertices in \( \bigcup R \). Then infinitely many of these fans avoid some fixed ray in \( R \), say \( R \). The two other rays plus the infinitely many 2-fans meeting only these together contain a ray \( W \in \omega \) that meets \( F \) infinitely often and is disjoint from \( R \). Then for every \( R-W \) path \( P \) we get a contradiction, as \( C(R, W, P) \) is a circle meeting \( F \) in infinitely many edges.

For a proof of (ii), note first that the minimal elements of \( C_{\text{alg}} \) are indeed the finite circuits and the edge sets of double rays in \( G \). Indeed, these are clearly in \( C_{\text{alg}} \) and minimal. Conversely, given any element of \( C_{\text{alg}} \), a set \( D \) of edges inducing even degrees at all the vertices, we can greedily find for any given edge \( e \in D \) a finite circuit or double ray with all its edges in \( D \) that contains \( e \). We may thus decompose \( D \) inductively into disjoint finite circuits and edge sets of double rays, since deleting finitely many such sets from \( D \) clearly produces another element of \( C_{\text{alg}} \), and including in each circuit or double ray chosen the smallest undeleted edge in some fixed enumeration of \( D \) ensures that the entire set \( D \) is decomposed. If \( D \) is minimal in \( C_{\text{alg}} \), it must therefore itself be a finite circuit or the edge set of a double ray.

Consider a set \( F \) of edges that fails to meet some set \( D \in C_{\text{alg}} \) evenly; we have to show that \( F \) also fails to meet some finite circuit or double ray evenly. If \(|F \cap D| \) is odd, then this follows from our decomposition of \( D \) into disjoint finite circuits and edges sets of double rays. We thus assume that \( F \cap D \) is infinite. Since \( |G| \) is compact, we can find a sequence \( e_1, e_2, \ldots \) of edges in \( F \cap D \) that
converges to some end \( \omega \). Let \( R_1, R_2, R_3 \) be disjoint rays in \( \omega \), which exist by our assumption that \( \omega \) has vertex-degree at least 3. Subdividing each edge \( e_i \) by a new vertex \( v_i \), and using that \( G \) is 2-connected, we can find for every \( i \) a 2-fan from \( v_i \) to \( W = V(R_1 \cup R_2 \cup R_3) \) that has only its last vertices and possibly \( v_i \) in \( W \). By Lemma \[3\] with \( w_1, w_2, \ldots \) an enumeration of \( W \), some infinitely many of these fans are disjoint. Renaming the rays \( R_i \) and replacing \( e_1, e_2, \ldots \) with a subsequence as necessary, we may assume that either all these fans have both endvertices on \( R_1 \), or that they all have one endvertex on \( R_1 \) and the other on \( R_2 \). In both cases all these fans avoid \( R_3 \), so we can find a ray \( R \) in the union of \( R_1, R_2 \) and these fans (suppressing the subdividing vertices \( v_i \) again) that contains infinitely many \( e_i \) and avoids \( R_3 \). Linking \( R \) to a tail of \( R_3 \) we thus obtain a double ray in \( G \) that contains infinitely many \( e_i \), as desired.

To prove (iii), let \( D \subseteq E \) be a set of edges that meets every bond evenly. We have to show that \( D \in B^+ \), i.e., that \( D \) has an (only finite and) even number of edges also in every cut that is not a bond.

As \( D \) meets every finite bond evenly, and hence every finite cut, it lies in \( R_{\text{fin}} = C_{\text{top}} \). We claim that

\[
D \text{ is a disjoint union of finite circuits.} \quad (\ast)
\]

To prove \((\ast)\), let us show first that every edge \( e \in D \) lies in some finite circuit \( C \subseteq D \). If not, the endvertices \( u, v \) of \( e \) lie in different components of \((V, D \setminus \{e\})\), and we can partition \( V \) into two sets \( A, B \) so that \( e \) is the only \( A \sim B \) edge in \( D \). The cut of \( G \) of all its \( A \sim B \) edges is a disjoint union of bonds \([4]\), one of which meets \( D \) in precisely \( e \). This contradicts our assumption that \( D \) meets every bond of \( G \) evenly.

For our proof of \([4]\), we start by enumerating \( D \), say as \( D = \{e_1, e_2, \ldots \} =: D_0 \). Let \( C_0 \subseteq D_0 \) be a finite circuit containing \( e_0 \), let \( D_1 := D_0 \setminus C_0 \), and notice that \( D_1 \), like \( D_0 \), meets every bond of \( G \) evenly (because \( C_0 \) does). As before, \( D_1 \) contains a finite circuit \( C_1 \) containing the edge \( e_i \) with \( i = \min \{j \mid e_j \in D_1 \} \). Continuing in this way we find the desired decomposition \( D = C_1 \cup C_2 \cup \ldots \) of \( D \) into finite circuits. This completes the proof of \([4]\).

As every finite circuit lies in \( B^+ \), it suffices by \([1]\) to show that \( D \) is finite. Suppose \( D \) is infinite, and let \( \omega \) be an end of \( D \) in its closure. Let us say that two rays \( R \) and \( R' \) hug \( D \) if every neighbourhood \( U \) of \( \omega \) contains a finite circuit \( C \subseteq D \) that is neither separated from \( R \) by \( R' \) nor from \( R' \) by \( R \) in \( U \). We shall construct two rays \( R \) and \( R' \) that hug \( D \), inductively as follows.

Let \( S_0 = \emptyset \) and let \( R_0, R'_0 \) be disjoint rays in \( \omega \). (These exist as \( G \) is 2-connected \([3]\).) For step \( j \geq 1 \), assume that let \( S_i, R_i, \) and \( R'_i \) have been defined for all \( i < j \) so that \( R_i \) and \( R'_i \) each meet \( S_i \) in precisely some initial segment (and otherwise lie in \( C(S_i, \omega) \)) and \( S_i \) contains the \( i \)th vertex in some fixed enumeration of \( V \). If the \( j \)th vertex in this enumeration lies in \( C(S_{j-1}, \omega) \), add to \( S_{j-1} \) this vertex and, if it lies on \( R_{j-1} \) or \( R'_{j-1} \), the initial segment of that ray up to it. Keep calling the enlarged set \( S_{j-1} \). For the following choice of \( S \) we apply Lemma \([6]\) to \( S_{j-1} \) and each of \( R_{j-1} \) and \( R'_{j-1} \). Let \( S \supseteq S_{j-1} \) be a finite set such that from every vertex \( v \) in \( C(S, \omega) \) there are 3-fans in \( C(S_{j-1}, \omega) \) both
to \(R_{j-1}\) and to \(R'_{j-1}\). By \([1]\) and the choice of \(\omega\), there is a finite circuit \(C_j \subseteq D\) in \(C(S, \omega)\). Then \(C_j\) can not be separated from \(R_{j-1}\) or \(R'_{j-1}\) in \(C(S_{j-1}, \omega)\) by fewer than three vertices, and thus there are three disjoint paths from \(C_j\) to \(R_{j-1} \cup R'_{j-1}\) in \(C(S_{j-1}, \omega)\).

There are now two possible cases. The first is that in \(C(S_{j-1}, \omega)\) the circuit \(C_j\) is neither separated from \(R_{j-1}\) by \(R'_{j-1}\) nor from \(R'_{j-1}\) by \(R_{j-1}\). This case is the preferable case. In the second case one ray separates \(C_j\) from the other. In this case we will reroute the two rays to obtain new rays as in the first case. We shall then ‘freeze’ a finite set containing initial parts of these rays, as well as paths from each ray to \(C_j\). This finite fixed set will not be changed in any later step of the construction of \(R\) and \(R'\). In detail, this process is as follows.

If \(C(S_{j-1}, \omega)\) contains both a \(C_j-R_{j-1}\) path \(P\) avoiding \(R'_{j-1}\) and a \(C_j-R'_{j-1}\) path \(P'\) avoiding \(R_{j-1}\), let \(Q\) and \(Q'\) be the initial segments of \(R_{j-1}\) and \(R'_{j-1}\) up to \(P\) and \(P'\), respectively. Then let \(R_j = R_{j-1}\) and \(R'_j = R'_{j-1}\) and

\[S_j = S_{j-1} \cup V(P) \cup V(P') \cup V(Q) \cup V(Q').\]

This choice of \(S_j\) ensures that the rays \(R, R'\) constructed form the \(R_i\) and \(R'_i\) in the limit will not separate each other from \(C_j\), because they will satisfy \(R \cap S_j = R_j \cap S_j\) and \(R' \cap S_j = R'_j \cap S_j\).

If the ray \(R_{j-1}\) separates \(C_j\) from \(R'_{j-1}\), let \(P_j\) be a set of three disjoint \(C_j-R'_{j-1}\) paths avoiding \(S_{j-1}\). All these paths meet \(R_{j-1}\). Let \(P_1 \in P_j\) be the path which \(R_{j-1}\) meets first, and \(P_3 \in P_j\) the one it meets last. Then \(R_{j-1} \cup C_j \cup P_1 \cup P_3\) contains a ray \(R_j\) with initial segment \(R_{j-1} \cap S_{j-1}\) that meets \(C_j\) but is disjoint from the remaining path \(P_2 \in P\) and from \(R'_{j-1}\). Let \(R'_j = R'_{j-1}\), and let \(S_j\) contain \(S_{j-1}\) and all vertices of \(\bigcup P_j\), and the initial segments of \(R_{j-1}\) and \(R'_{j-1}\) up to their last vertex in \(\bigcup P\). Note that \(R_j\) meets \(C_j\), and that \(P_2\) is a \(C_j-R'_j\) path avoiding \(R_j\).

If the ray \(R'_{j-1}\) separates \(C_j\) from \(R_{j-1}\), reverse their roles in the previous part of the construction.

The edges that lie eventually in \(R_i\) or \(R'_i\) as \(i \to \infty\) form two rays \(R\) and \(R'\) that clearly hug \(D\).

Let us show that there are two disjoint combs, with spines \(R\) and \(R'\) respectively, and infinitely many disjoint finite circuits in \(D\) such that each of the combs has a tooth in each of these circuits. We build these combs inductively, starting with the rays \(R\) and \(R'\) and adding teeth one by one.

Let \(T_0 = R\) and \(T'_0 = R'\) and \(S_0 = \emptyset\). Given \(j \geq 1\), assume that \(T_i, T'_i\) and \(S_i\) have been defined for all \(i < j\). By Lemma \([1]\) there is a finite set \(S \supseteq S_{j-1}\) such that every vertex of \(C(S, \omega)\) sends a 3-fan to \(R \cup R'\) in \(C(S_{j-1}, \omega)\). As \(R\) and \(R'\) hug \(D\) there is a finite cycle \(C\) in \(C(S, \omega)\) with edges in \(D\), and which neither of the rays \(R\) or \(R'\) separates from the other. By the choice of \(S\), no one vertex of \(C(S_{j-1}, \omega)\) separates \(C\) from \(R \cup R'\) in \(C(S_{j-1}, \omega)\). Hence by Menger’s theorem there are disjoint \((R \cup R')-C\) paths \(P\) and \(Q\) in \(C(S_{j-1}, \omega)\). If \(P\) starts on \(R\) and \(Q\) starts on \(R'\) (say), let \(P' := Q\). Assume now that \(P\) and \(Q\) start on the same ray \(R\) or \(R'\), say on \(R\). Let \(Q'\) be a path from \(R'\) to \(C \cup P \cup Q\) in \(C(S_{j-1}, \omega)\) that avoids \(R\). As \(Q'\) meets at most one of the paths \(P\) and \(Q\), we
may assume it does not meet \( P \). Then \( Q' \cup (Q \setminus R) \) contains an \( R' - C \) path \( P' \) disjoint from \( P \) and \( R \). In either case, let \( T_j = T_{j-1} \cup P \), let \( T'_j = T'_{j-1} \cup P' \), and let \( S_j \) consist of \( S_{j-1} \), the vertices in \( C \cup P \cup P' \), and the vertices on \( R \) and \( R' \) up to their last vertex in \( C \cup P \cup P' \).

The unions \( T = \bigcup_{i \in \mathbb{N}} T_i \) and \( T' = \bigcup_{i \in \mathbb{N}} T'_i \) are disjoint combs that have teeth in infinitely many common disjoint finite cycles whose edges lie in \( D \). Let \( A \) be the vertex set of the component of \( G - T \) containing \( T' \), and let \( B := V \setminus A \). Since \( T \) is connected, \( E(A, B) \) is a bond, and its intersection with \( D \) is infinite as every finite cycle that contains a tooth from both these combs meets \( E(A, B) \) at least twice. This contradiction implies that \( D \) is finite, as desired.  

\[
\square
\]

References

[1] H. Bruhn and M. Stein. On end degrees and infinite circuits in locally finite graphs. *Combinatorica*, 27:269–291, 2007.

[2] Henning Bruhn and Maya Stein. Duality of ends. *Comb., Probab. Comput.*, 19:47–60, 2010.

[3] K. Casteels and B. Richter. The Bond and Cycle Spaces of an Infinite Graph. *J. Graph Theory*, 59(2):162–176, 2008.

[4] R. Diestel. *Graph Theory*. Springer, 4th edition, 2010.

[5] R. Diestel. Locally finite graphs with ends: a topological approach. *Discrete Math.*, 310–312: 2750–2765 (310); 1423–1447 (311); 21–29 (312), 2010–11. arXiv:0912.4213.

[6] R. Diestel and P. Sprüssel. The homology of locally finite graphs with ends. *Combinatorica*, 30:681–714, 2010.

[7] A. Georgakopoulos. Graph topologies induced by edge lengths. In Diestel, Hahn, and Mohar, editors, *Infinite graphs: introductions, connections, surveys*, volume *Discrete Mathematics* 311, pages 1523–1542, 2011.

[8] A. Georgakopoulos. Cycle decompositions: from graphs to continua. *Advances in Mathematics*, 229:935–967, 2012.

[9] R. Halin. A note on Menger’s theorem for infinite locally finite graphs. *Abh. Math. Sem. Univ. Hamburg*, 40:111–114, 1974.