ASYMPTOTIC STABILITY OF RAREFACTION WAVES FOR
COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH RELAXATION

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Abstract. The asymptotic stability of rarefaction wave for 1-d relaxed compressible isentropic
Navier-Stokes equations is established. For initial data with different far-field values, we show
that there exists a unique global in time solution. Moreover, as time goes to infinity, the ob-
tained solutions are shown to converge uniformly to rarefaction wave solution of p-system with
corresponding Riemann initial data. The proof is based on $L^2$ energy methods.

Keywords: Compressible Navier-Stokes equations; Rarefaction waves; Stability
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1. Introduction

In this paper, we consider the system of one-dimensional isentropic compressible Navier Stokes
equations in Lagrange coordinates as

\begin{equation}
\begin{cases}
v_t = u_x, \\
u_t + p(v)x = \tilde{S}_x,
\end{cases}
\end{equation}

(1.1)

with

\begin{equation}
\tau \tilde{S}_t + \tilde{S} = \mu \frac{u_x}{v},
\end{equation}

(1.2)

where $v, u, p, \tilde{S}$ denote the specific volume per unit mass, fluid velocity, pressure and stress tensor,
respectively. The equations (1.1) are the consequences of conservation of mass and balance of
momentum, respectively. The viscosity coefficient $\mu$ is assumed to be a positive constant as well
as the relaxation parameter $\tau$. Moreover, we assume the pressure $p$ satisfy $p(v) = av^{-\gamma}$ with $\gamma > 1$
being the adiabatic index.

In the constitutive relation (1.2), $\tau$ is the relaxation time describing the time lag in the response of
the stress tensor to the velocity gradient. In fact, even in simple fluid, water for example, the “time
lag” exists, but it is very small ranging from 1 ps to 1 ns, see [14, 19]. However, Pelton et al.
[17] showed that such a “time lag” cannot be neglected, even for simple fluids, in the experiments
of high-frequency (20GHZ) vibration of nano-scale mechanical devices immersed in water-glycerol
mixtures. It turned out that, cp. also [2], equation (1.2) provides a general formalism to charac-
terize the fluid-structure interaction of nano-scale mechanical devices vibrating in simple fluids. A
similar relaxed constitutive relation was already proposed by Maxwell in [15], in order to describe
the relation of stress tensor and velocity gradient for a non-simple fluid.

The system (1.1) is coupled with the initial conditions

\begin{equation}
(v, u, \tilde{S})(0, x) = (v_0, u_0, \tilde{S}_0)(x), \quad x \in \mathbb{R},
\end{equation}

(1.3)

with

\[
\lim_{x \to \pm \infty} (v_0, u_0, \tilde{S}_0)(x) = (v_\pm, u_\pm, 0).
\]
Here, we assume, for the far field conditions, $v_+ \neq v_-, u_+ \neq u_-$ in general. Furthermore, $(v_-, u_-)$ and $(v_+, u_+)$ are supposed to be the Riemann initial data which generates a centered rarefaction wave for the following $p-$system

$$
\begin{align*}
v_t &= u_x, \\
u_t + p(v)x &= 0.
\end{align*}
$$

That means, there exists a continuous weak solution with the form $(v^r, u^r)(x/t)$ of $p-$ system (1.4) with

$$
(v^r, u^r)(0, x) = \begin{cases} (v_-, u_-), & x < 0 \\ (v_+, u_+), & x > 0 \end{cases}
$$

The main purpose of this paper is devoted to establish the asymptotic stability of the rarefaction wave $(v^r, u^r, 0)$ defined above. More precisely, we assume the initial data $(v_0, u_0, \tilde{S}_0)$ is close to $(v_0^r, u_0^r, 0)$ in a suitable sense and the amplitude $\delta = |v_+ - v_-| + |u_+ - u_-|$ of the rarefaction wave is sufficiently small. Then, we will show that there exists a unique global defined solution $(v, u, \tilde{S})$ to problem (1.1)-(1.3) which approaches the rarefaction wave $(v^r, u^r, 0)$ uniformly in $x$ as $t \to \infty$. That is, we have

$$
\| (v - v^r, u - u^r, \tilde{S}) \|_{L^\infty} \to 0, \quad t \to \infty.
$$

See Theorem 2.1 for more details.

Note that, when $\tau = 0$, the system (1.1), (1.2) reduce to the classical isentropic compressible Navier-Stokes equations, which read as

$$
\begin{align*}
v_t &= u_x, \\
u_t + p(v)x &= \left( \frac{\mu u_x}{v} \right)_x.
\end{align*}
$$

For such system, the well-posedness and asymptotic stability results have been widely studied with various initial data, see [7, 8, 12, 13]. In particular, Matsumura and Nishihara [12] first established the asymptotic stability of rarefaction waves with small amplitude in 1986. Later, they [13] extended their results to large data cases. The half-space problem was conducted by Kawashima and Zhu [8]. A similar asymptotic stability result was also obtained for the full system of the compressible Navier-Stokes equations, see [9].

The asymptotic stability of rarefaction waves were also shown for other related system, see [11] for the Broadwell model, [10] for a model system of radiating gas, [16, 1] for a model of hyperbolic balance law.

One should note that it is not obvious that the results which hold for the classical systems also hold for the relaxed system. Indeed, and for example, Hu and Wang [5] and Hu, Racke and Wang [6] showed that, for the one-dimensional isentropic and/or non-isentropic compressible Navier-Stokes system, solutions exist globally with arbitrary large initial data for classical system, while solutions blow up in finite time with some large initial data for the corresponding relaxed system. A similar qualitative change was observed before for certain thermoelastic systems, where the non-relaxed system is exponentially stable, while the relaxed one is not, see Quintanilla and Racke resp. Fernández Sare and Muñoz Rivera [18, 3] for plates, and Fernández Sare and Racke [4] for Timoshenko beams.

The paper is organized as follows. Some preliminaries and main theorem (Theorem 2.1) are given in Section 2. In Section 3, following to [12, 13], we construct smooth approximations of rarefaction waves. The original problem is reformulated in Section 4 and there the statements of asymptotic stability results for the reformulated problems are given (Theorem 4.1). The a priori estimates are also given in Section 4 (Proposition 4.2). Last, the proof of the a priori estimates (Proposition 4.2) are given in Section 5.
Notations: $L^p(\mathbb{R})$ and $W^{s,p}(\mathbb{R})$ $(1 \leq p \leq \infty)$ denote the usual Lebesgue and Sobolev spaces over $\mathbb{R}$ with the norm $\| \cdot \|_{L^p}$ and $\| \cdot \|_{W^{s,p}}$, respectively. Note that, when $s = 0$, $W^{0,p} = L^p$. For $p = 2$, $W^{s,2}$ are abbreviated to $H^s$ as usual. Let $T$ and $B$ be a positive constant and a Banach space, respectively. $C^k(0, T; B)(k \geq 0)$ denotes the space of $B$-valued $k$-times continuously differentiable functions on $[0, T]$, and $L^p(0, T; B)$ denotes the space of $B$-valued $L^p$-functions on $[0, T]$. The corresponding space $B$-valued functions on $[0, \infty)$ are defined similarly.

2. Preliminaries and main results

Let’s first recall the definition of rarefaction waves of p-system (1.4) with Riemann initial data (1.5). For fixed $(v_-, u_-)(v_- > 0, u_- \in \mathbb{R})$, we define a proper neighbourhood $\omega \subset \mathbb{R}_{v,u}^2 := \mathbb{R}_+ \times \mathbb{R}$ as

$$R_1(v_-, u_-) := \{(v, u) \in \omega | u = u_- - \int_{v_-}^v \lambda_1(s)ds, u \geq u_- \},$$

$$R_2(v_-, u_-) := \{(v, u) \in \omega | u = u_- - \int_{v_-}^v \lambda_2(s)ds, u \geq u_- \}$$

and

$$RR(v_-, u_-) := \{(v, u) \in \omega | u \geq u_- - \int_{v_-}^v \lambda_1(s)ds, u \geq u_- - \int_{v_-}^v \lambda_2(s)ds \},$$

where $\lambda_1(v) = -\sqrt{-p'(v)}$, $\lambda_2(v) = \sqrt{-p'(v)}$ are eigenvalues of the matrix

$$\begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}.$$ 

If $(v_+, u_+) \in RR(v_-, u_-)$, then Riemann problem (1.4), (1.5) admits a continuous weak solution of the form $(v^r, u^r)(x/t)$ which is called centered rarefaction wave.

Let

$$I_0 := \|(v_0 - v_0^r, u_0 - u_0^r)\|_{L^2} + \|((v_0)_x, (u_0)_x)\|_{H^1} + \|\tilde{S}_0\|_{H^2}.$$ 

Our main result is stated as follows.

Theorem 2.1. Let $(v_+, u_+) \in RR(v_-, u_-)$ and $\delta = |v_+ - v_-| + |u_+ - u_-|$. Assume the initial data $(v_0, u_0, \tilde{S}_0)$ satisfy

$$(v_0 - v_0^r, u_0 - u_0^r, \tilde{S}_0) \in L^2,$$

$$((v_0)_x, (u_0)_x) \in H^1, \tilde{S}_0 \in H^2$$

and there exists a positive constant $\epsilon_0$ such that if $I_0 + \delta < \epsilon_0$, then the initial value problem (1.4) - (1.5) has a unique global solution in time satisfying

$$\begin{cases} (v - v_0^r, u - u_0^r, \tilde{S}) \in C^0(0, +\infty, L^2), \\
(v_x, u_x) \in C^0(0, +\infty, H^1) \cap L^2(0, +\infty, H^1), S \in C^0(0, +\infty, H^2) \cap L^2(0, +\infty, H^2). \end{cases} \tag{2.1}$$

Moreover, this solution approaches the rarefaction wave $(v^r, u^r, 0)$ uniformly in $x \in \mathbb{R}$ as $t \to +\infty$:

$$\|(v(t,x) - v^r(x/t), u(t,x) - u^r(x/t), \tilde{S}(t,x))\|_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty. \tag{2.2}$$
## 3. Smooth approximation of rarefaction waves

We start from the inviscid Burgers equation

\[
\begin{aligned}
& w_t + w w_x = 0, \\
& w(0, x) = w_0(x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases}
\end{aligned}
\]  

(3.1)

Let \( w_- < w_+ \), then (3.1) has a continuous weak solution \( w'(x/t) \) with

\[
w'(x/t) = \begin{cases} w_-, & x/t \leq w_- , \\ x/t, & w_- \leq x/t \leq w_+ , \\ w_+, & \xi \geq w_+. \end{cases}
\]  

(3.2)

This solution is called central rarefaction wave connecting the two constant state \( w_- \) and \( w_+ \).

Next, we consider the following system whose solutions approximate \( w'(x/t) \) smoothly.

\[
\begin{aligned}
& w_t + w w_x = 0, \\
& w(0, x) = w_0(x) = \hat{w} + \hat{w} k_q \int_0^x (1 + y^2)^{-q}dy,
\end{aligned}
\]  

(3.3)

where \( \hat{w} = \frac{w_++w_-}{2}, \hat{w} = \frac{w_+-w_-}{2}, \epsilon > 0 \) is a constant. \( k_q \) is a constant satisfying \( k_q \int_0^\infty (1+y^2)^{-q}dy = 1 \) for \( q > \frac{3}{2} \). Then we have the following lemma

**Lemma 3.1.** \([12][13]\) Let \( w_- < w_+ \), then there exists a smooth solution \( w(t, x) \) of system (3.3) satisfying

1) \( w_- < w(t, x) < w_+, w_x(t, x) > 0, \ \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R} \)

2) \( \forall 1 \leq p \leq \infty, \) there exists a constant \( C_p \) such that

\[
\|w_x\|_{L^p} \leq C_p \min\left( \epsilon^{1 - \frac{q}{2}} w, \epsilon^{1 - \frac{q}{2}} t^{-1 + \frac{q}{2}} \right),
\]

\[
\|\partial_x^2 w\|_{L^p} \leq C \min\left( \epsilon^{2 - \frac{q}{2}} w, \epsilon^{(1 - \frac{q}{2}) (1 - \frac{q}{2})} w \right) t^{-1 - \frac{q}{2}}.
\]

3) if \( w_- > 0 \), then

\[
|w(t, x) - w_-| \leq C \hat{w} \left( 1 + \left( \epsilon x \right)^2 \right)^{-\frac{q}{2}} \left( 1 + \left( \epsilon w_- t \right)^2 \right)^{-\frac{q}{2}},
\]

\[
|w_x| \leq C \epsilon \hat{w} \left( 1 + \left( \epsilon x \right)^2 \right)^{-\frac{q}{2}} \left( 1 + \left( \epsilon w_- t \right)^2 \right)^{-\frac{q}{2}}.
\]

4) if \( w_+ < 0 \), then

\[
|w(t, x) - w_-| \leq C \hat{w} \left( 1 + \left( \epsilon x \right)^2 \right)^{-\frac{q}{2}} \left( 1 + \left( \epsilon w_+ t \right)^2 \right)^{-\frac{q}{2}},
\]

\[
|w_x| \leq C \epsilon \hat{w} \left( 1 + \left( \epsilon x \right)^2 \right)^{-\frac{q}{2}} \left( 1 + \left( \epsilon w_+ t \right)^2 \right)^{-\frac{q}{2}}.
\]

5) \( \lim_{t \to \infty} \sup_{\mathbb{R}} |w(t, x) - w'(x/t)| = 0 \).

Next, we construct smooth solutions which approximate the weak solutions \( (w', u')(x/t) \) of system (1.4)-(1.5) by using \( w(t, x) \). Firstly, suppose \( (v_+, u_+) \in RR(v_-, u_-) \), then there exists a unique pair \( (\bar{v}, \bar{u}) \) satisfying

\[
(\bar{v}, \bar{u}) \in R_1(v_-, u_-), \quad (v_+, u_+) \in R_2(\bar{v}, \bar{u}).
\]

Moreover, the continuous weak solution of system (1.4)-(1.5) can be expressed as

\[
(v', u')(\xi) = (v'_1 + v'_2 - \bar{v}, u'_1 + u'_2 - \bar{u})(\xi), \quad \xi := x/t,
\]  

(3.4)
where
\[ \lambda_1(v_1^i(\xi)) = w_1^i(\xi), \lambda_2(v_2^i(\xi)) = w_2^i(\xi), \]
\[ u_1^i(\xi) = u_2 - \int_{v_1^i}^{v_2^i} \lambda_1(s)ds, w_2^i(\xi) = \bar{u} - \int_{\bar{v}}^{v_2^i} \lambda_2(s)ds, \]
and \( w_1^i(\xi), w_2^i(\xi) \) are given by \( 3.2 \) with
\[ w_1^- = \lambda_1(v_-), w_2^- = \lambda_1(\bar{v}), w_2^+ = \lambda_2(\bar{v}), w_2^+ = \lambda_2(v_+). \]

Now, we define \((V, U)\) as
\[
\begin{cases}
(V, U)(t, x) = (V_1 + V_2 - \bar{v}, U_1 + U_2 - \bar{u})(t, x), \\
\lambda_1(V_1) = w_1(t, x), \lambda_2(V_2) = w_2(t, x), \\
U_1 = u_1 - \int_{v_1}^{V_1} \lambda_1(s)ds, U_2 = \bar{u} - \int_{\bar{v}}^{V_2} \lambda_2(s)ds
\end{cases}
\]
(3.5)
where \( w_1(t, x), w_2(t, x) \) are solutions of \( 3.3 \). Note that \((V_1, U_1)\) and \((V_2, U_2)\) defined above are exact solution of \( p \)-system \( 3.4 \) and \((V, U)\) satisfies
\[
\begin{cases}
V_t - U_x = 0, \\
U_t + p(V)x = g(V),
\end{cases}
\]
(3.6)
where \( g(V) = p(V) - p(V_1) - p(V_2) + p(\bar{v}) \).

By use of Lemma \( 3.4 \) we have

**Lemma 3.2.** \( 12, 13, 16 \) \((V, U)\) defined in \( 3.5 \) satisfy
1) \( V_t > 0, \forall (x, t) \in \mathbb{R}^+ \times \mathbb{R} \),
2) There exists a constant \( C \) such that
\[
|V_x| \leq CV_t, \quad V_t \leq C\delta, \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}
\]
3) Denote \( \tilde{w}_1 = \frac{w_2 - w_1}{2} \), then \( \forall t \in \mathbb{R}^+ \),
\[
\|g(V)x\|_{L^p} \leq C(1 + (\delta w_2 - t)^{-\phi}) + (1 + (\delta w_1 + t)^{1+\beta})
\]
and
\[
\int_0^\infty \|g(V)x\|_{L^p} dx \leq C\delta^2 \epsilon^{-\frac{\phi}{2}}.
\]
4) \( \forall t \in \mathbb{R}^+ \),
\[
\|V_x\|_{L^p}, \|U_x\|_{L^p} \leq C \min\{\delta w_1^{1-\phi}, \delta w_2^{1+\phi}(1 + t)^{-1+\phi}\},
\]
5) \( \forall t \in \mathbb{R}^+ \) and \( k = 2, 3, 4 \),
\[
\|\partial_x^k V\|_{L^p}, \|\partial_x^k U\|_{L^p} \leq C \left( \delta w_2^{1-\phi} \epsilon^{1+\phi} (1 + t)^{-1+\phi} + \delta w_1^{1+\phi} \right)
\]
and \( \forall p > 1 \),
\[
\int_0^\infty (\|\partial_x^k V\|_{L^p} + \|\partial_x^k U\|_{L^p}) dt \leq C\delta^2 \epsilon^{-\frac{\phi}{2}}.
\]
6) \( \lim_{t \to \infty} \sup_{\mathbb{R}} |(V, U)(t, x) - (v^*, u^*)(x/t)| = 0. \)
4. Reformulation of the problem

Let \((v^r, u^r)(x/t)\) be centered rarefaction wave for system \((1.4)-(1.5)\) which is given in \((3.4)\), and let \((V, U)(t, x)\) be the smooth approximations of \((v^r, u^r)(x/t)\). Note that \((V, U)\) are the functions defined in \((3.5)\) and satisfying \((3.6)\). Now, let \(\varphi = v - V, \psi = u - U, S = S - \mu \frac{U_0}{V_0}\) be perturbations, then

\[
\begin{align*}
\varphi_t &= \psi_x, \\
\psi_t + [p(\varphi + V) - p(V)]_x - S_x &= \mu \left( \frac{U_0}{V_0} \right)_x - g(V)_x, \\
\tau S_t + S - \frac{\mu}{\psi} \psi_x &= -\tau \mu \left( \frac{U_0}{V_0} \right)_t - \frac{U_0}{V_0} \psi_x,
\end{align*}
\]

with

\[
(\varphi, \psi, S)(x, 0) = (\varphi_0, \psi_0, S_0),
\]

where

\[
\varphi_0 = v_0 - V_0, \psi_0 = u_0 - U_0, S_0 = S_0 - \mu \frac{U_0}{V_0}. \tag{4.3}
\]

For this reformulated problem \((4.1)-(4.2)\), we have the following theorem of global existence and asymptotic stability.

**Theorem 4.1.** Let \((v_+, u_+) \in RR(v_-, u_-)\) and \(\delta = |v_+ - v_-| + |u_+ - u_-|\). Assume the initial data \((\varphi_0, \psi_0, S_0) \in H^2\) and let \(E_0 := \|\varphi_0, \psi_0, S_0\|_{H^2}\). Then, there exists a positive constant \(\delta_1\) such that if \(E_0 + \delta < \delta_1\), the initial value problem \((4.1)-(4.2)\) has a unique global solution in time satisfying

\[
(\varphi, \psi, S)(t, x) \in C^0(0, +\infty, H^2),
\]

\[
(\varphi_x, \psi_x)(t, x) \in L^2(0, +\infty, H^1), \quad S(t, x) \in L^2(0, +\infty, H^2). \tag{4.5}
\]

Moreover, this solution decay to \((0, 0, 0)\) uniformly in \(x\) as \(t \to +\infty\):

\[
\| (\varphi, \psi, S) \|_{W^{1, \infty}} \to 0, \quad \text{as} \quad t \to +\infty. \tag{4.6}
\]

To prove Theorem 4.1, the key point is to show the a priori estimate of solutions to the problem \((4.1)-(4.2)\). To do this, we introduce the energy norm \(E(t)\) as follows

\[
E(t) = \sup_{0 \leq s \leq t} \| (\varphi(s), \psi(s), S(s)) \|_{H^2} + \int_0^t \left( \| (\varphi(s), \psi(s), S(s)) \|_{H^1} + \| \sqrt{\psi} \|_2 + \| S \|_2^2 \right) dt.
\]

The a priori estimate result is stated as follows.

**Proposition 4.2.** Let \(T > 0\) and \((\varphi, \psi, S)(t, x)\) be a solution to the problem \((4.1)-(4.2)\) such that \((\varphi, \psi, S)(t, x) \in C^0([0, T], H^2) \cap C^1([0, T], H^1)\). Then, there exists a positive constant \(\delta_2\) which is independent of \(T\) such that if

\[
E(T) + \delta < \delta_2, \tag{4.7}
\]

then the solution \((\varphi, \psi, S)\) satisfies

\[
E(t) \leq C(E_0 + E^\theta_{\delta_2}(t) + \delta^\theta) \tag{4.8}
\]

for \(t \in (0, T)\) with \(\theta = \min\{\frac{1}{2}, \frac{3}{2} - \frac{1}{q}\}\).

We will give the proof of Proposition 4.2 in Section 5.

**Proof of Theorem 4.1.** Firstly, we choose \(\delta_2\) such that \(C\delta_2 \leq \frac{1}{4}\). Then, under the assumption \((4.7)\) and using \((4.8)\), we have

\[
E(t) \leq 2C(E_0 + \delta^\theta). \tag{4.9}
\]
Now, we choose $\epsilon_0$ small enough such that $2C(\epsilon_0 + \epsilon_0^2) + \epsilon_0 < \frac{\delta}{2}$, which closes the assumption (4.7) by noting that $E_0 + \delta < \epsilon_0 < \frac{\delta}{2}$. Therefore, based on the local existence theorem and the a priori estimate (4.9), we can get a global solution in time with regularity (4.4) and (4.5) by the classical continuation methods.

Now, we show the convergence result (4.6). Firstly, by Sobolev interpolation theorem, we have

$$\|(\varphi, \psi, S)\|_{L^\infty} \leq C\|(\varphi, \psi, S)\|_{L^2}^{\frac{1}{2}}\|(\varphi_x, \psi_x, S_x)\|_{L^2}^{\frac{1}{2}},$$

$$\|(\varphi_x, \psi_x, S_x)\|_{L^\infty} \leq C\|(\varphi, \psi, S)\|_{L^2} \|(\varphi_x, \psi_x, S_x)\|_{L^2}.$$ 

Thus, the convergence $\|(\varphi_x, \psi_x, S_x)\|_{L^2} \to 0$ as $t \to +\infty$ implies (4.6) immediately. Let $\Psi(t) := (\varphi_x, \psi_x, S_x)(t, -)\|_{L^2}^2$. Then, by (4.5), $\Psi(t) \in L^1(0, +\infty)$. From the system (4.1), we can easily get $\Psi(t) \in L^1(0, +\infty)$. Therefore, we get $\Psi(t) \in W^{1,1}(0, +\infty)$ which implies $\Psi(t) \to 0$ as $t \to +\infty$. Thus the proof of Theorem (4.1) is complete.

Finally, by using Theorem (4.1) we are able to show Theorem (2.1) hold.

**Proof of Theorem 2.1.** Assume that $I_0 + \delta$ is suitable small where

$$I_0 = \|(v_0 - v_0', u_0 - u_0')\|_{L^2} + \|(v_0, u_0)\|_{H^1} + \|\tilde{S}_0\|_{H^2},$$

then we have

$$\|(\varphi_0, \psi_0)\|_{L^2} \leq \|(v_0 - v_0', u_0 - u_0')\|_{L^2} + \|(v_0' - v_0', u_0' - u_0')\|_{L^2} \leq I_0 + C\delta;$$

$$\|\partial_x(\varphi_0, \psi_0)\|_{H^1} + \|S_0\|_{H^2} \leq \|\partial_x(v_0, u_0)\|_{H^1} + \|\tilde{S}_0\|_{H^2} + \|\partial_x(V_0, U_0)\|_{H^1} + \|\mu\partial_x^2 U_0\|_{H^2} \leq I_0 + C\delta.$$ 

Therefore, we derive that $E_0 + \delta \leq I_0 + C\delta$ which can be sufficiently small. Therefore, by using Theorem (4.1) we get a unique global solution $(\varphi, \psi, S)$ to the problem (4.1), (4.2). Then, the functions defined by $(v, u, \tilde{S}) = (\varphi + V, \psi + U, S + \mu\ell)$ solves the original problem (1.1) - (1.3). To show the convergence (2.2) hold, we note that

$$\|(v - v', u - u', \tilde{S})\|_{L^\infty} \leq \|(\varphi, \psi, S)\|_{L^\infty} + \|(V - v', U - u', \mu\ell)\|_{L^\infty} \to 0$$

as $t \to +\infty$, where we used the convergence result (4.6) and Lemma (3.2). This completes the proof of Theorem (2.1).

5. **Proof of Proposition (4.2)**

In the following lemmas, we always assume that $E(t) + \delta \leq \delta_2$ for some small $\delta_2$ and $\epsilon = \delta^3$. In particular, we have $0 < V_\pm \leq V \leq V_+$ and $0 < v_- \leq v \leq v_+$. Firstly, we get the following $L^2$ estimates.

**Lemma 5.1.** There exists a constant $C$ such that for $0 \leq t \leq T$,

$$\|\varphi\|^2_{L^2} + \|\psi\|^2_{L^2} + \|S\|^2_{L^2} + \int_0^t (\|\sqrt{\ell}/\varphi\|^2_{L^2} + \|S\|^2_{L^2})dt \leq C(E_0 + \delta^0),$$

where $\theta = \min\left\{\frac{1}{2}, \frac{\beta}{2}, \frac{1}{\tau_1}\right\}$.

**Proof.** Multiplying the equations (4.1), (4.2), (4.3) by $p(V) - p(\varphi + V)$, $\psi$ and $\varphi S$, respectively, and integrating over $\mathbb{R}$ with respect to $x$, we get

$$\frac{d}{dt}E_1(\varphi, \psi, S) + E_2(\varphi, S) = \int_{\mathbb{R}} \left(\mu\psi \frac{U_x}{V} - \psi g(V)_x + \tau_0 S \left(\frac{U_x}{V}\right)_t + \frac{\tau_v}{2\mu} S^2\right)dx,$$

where

$$E_1(\varphi, \psi, S) = \int_{\mathbb{R}} \left(p(V)\varphi - \int_V^{V + \varphi} p(\xi)\frac{\psi}{2} d\xi + \frac{1}{2} \psi^2 + \frac{\tau v}{2\mu} S^2\right)dx.$$
Moreover, using (3.6) and Lemma 3.2, we have
\[ p(V)\varphi - \int_V^{V+\varphi} p(\xi) d\xi \geq C\varphi^2, \]
since \( p(\cdot) \) is a convex function on \((0, +\infty)\). Therefore, we derive that
\[ E_1(\varphi, \psi, S) \geq C\|\varphi, \psi, S\|_{L^2}^2. \tag{5.3} \]
On the other hand, let \( p(V + \varphi) - p(V) - p'(V)\varphi = f(v, V)\varphi^2 \), then \( f(v, V) \geq C > 0 \). Since \( V_1 \leq C\epsilon \delta \) where \( \epsilon \) and \( \delta \) are sufficiently small, we have
\[ \frac{v}{\mu} S^2 - \frac{V_1\varphi S}{V} + (p(V + \varphi) - p(V) - p'(V)\varphi)V_1 \geq C(S^2 + V_1\varphi^2). \tag{5.4} \]

Now, we estimate the right-hand-side of (5.2). Firstly, using Lemma 3.2 we get
\[ \int_R |\mu \psi \left( \frac{U_x}{V} \right)_x| dx \leq C \int_R (|\psi U_{xx}| + |\psi U_x V_x|) dx \leq C\|\psi\|_{L^2} (\|U_{xx}\|_{L^2} + \|U_x\|_{L^4} \|V_x\|_{L^4}) \leq C \left( \delta^{\frac{3}{2} - \frac{1}{4}}(1 + t)^{-1 - \frac{1}{4}} + \delta^{\frac{3}{2}}(1 + t)^{-\frac{3}{2}} \right) \|\psi\|_{L^2} \]
and
\[ \int_R |\psi g(V)_{xx}| dx \leq \|\psi\|_{L^2} \|g(V)_{xx}\|_{L^2}. \]
Moreover, using (5.6) and Lemma 3.2 we have
\[ \int_R |\tau_0 S \left( \frac{U_x}{V} \right)_t| dx \leq C \int_R S(|g(V)_{xx}| + |p(V)_{xx}| + U_x^2) dx \leq \varepsilon \int_R S^2 dx + C(\varepsilon)(\|V_x\|_{L^4}^4 + \|V_{xx}\|_{L^2}^2 + \|U_x\|_{L^4}^4) \leq C\varepsilon \int R S^2 dx + C\delta(1 + t)^{-3} + (\delta^{\frac{3}{2}} - \frac{1}{4})(1 + t)^{-1 - \frac{1}{4}} + \delta^{\frac{3}{2}}(1 + t)^{-\frac{3}{2}})^2. \]

Besides, note that \( \|v_t\|_{L^\infty} \leq \|V_1 + \varphi_2\|_{L^\infty} \leq C(\epsilon \delta + \delta_1) \), we derive that
\[ \int_R \frac{\tau v_t}{2\mu} S^2 dx \leq C(\delta^4 + \delta_2) \int_R S^2 dx, \]
which can be absorbed by \( E_2(\varphi, S) \) for suitable small \( \delta \) and \( \delta_2 \). Combining the above estimates, and integrating (5.2) over \((0, t)\), we derive that
\[ \|\varphi\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|S\|_{L^2}^2 + \int_0^t (\|\sqrt{V_1} \varphi\|_{L^2}^2 + \|S\|_{L^2}^2) ds \leq C E_0 + C \int_0^t (\delta^{\frac{3}{2}}(1 + t)^{-\frac{3}{4}} + \delta^{\frac{3}{2}} - \frac{1}{4})(1 + t)^{-1 - \frac{1}{4}} + \|g(V)_{xx}\|_{L^2}) \|\psi\|_{L^2} ds + C\delta + C\delta^{3 - \frac{3}{4}} \leq C(E_0 + \delta^\theta), \]
where we have used the fact that \( \int_0^{+\infty} \|g(V)_{xx}\|_{L^2} dt \leq C\delta^{2}e^{-\frac{t}{2}} = C\delta^\frac{3}{4} \) from Lemma 5.2. Therefore, the proof of this lemma is completed. \( \square \)

**Lemma 5.2.** There exists a constant \( C \) such that for \( 0 \leq t \leq T \), we have
\[ \|\varphi_x\|_{H^1}^2 + \|\psi_x\|_{H^1}^2 + \|S_x\|_{H^1}^2 + \int_0^t \|S_x\|_{H^1}^2 dt \leq C(E_0 + \delta^\theta + E^\frac{3}{4}(t)). \tag{5.5} \]
Secondly, we have for

\[
\text{Multiplying the above equations by } -p'(\varphi + V) \frac{\partial^k}{\partial x^k} \varphi, \frac{\partial^k}{\partial x^k} \psi, \frac{\partial^k}{\partial t^k} S, \text{ respectively, and integrating over } \mathbb{R}, \text{ we get}
\]

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( -\frac{1}{2} p'(\varphi + V)(\partial_x^k \varphi)^2 + \frac{1}{2} (\partial_x^k \psi)^2 + \frac{\tau v}{2 \mu} (\partial_x^k S)^2 \right) dx + \int_{\mathbb{R}} \left( \frac{v}{\mu} - \frac{\tau}{2} \partial_t v \right) (\partial_x^k S)^2 dx =: \sum_{j=1}^{8} R_k^j,
\]

where

\[
R_k^1 = -\int_{\mathbb{R}} \frac{1}{2} p''(\varphi + V)((\varphi_t + V_t)(\partial_x^k \varphi)^2 + (\varphi_x + V_x) \partial_x^k \varphi \partial_x^k \psi) dx,
\]

\[
R_k^2 = -\int_{\mathbb{R}} (\partial_x^k (p'(\varphi + V) \varphi_x) - p'(\varphi + V) \partial_x^k \varphi) \partial_x^k \psi dx,
\]

\[
R_k^3 = -\int_{\mathbb{R}} \partial_x^k (p'(\varphi + V) - p'(V)) V_x \partial_x^k \psi dx, \quad R_k^4 = \int_{\mathbb{R}} \mu \partial_x^k + \left( \frac{U_x}{V} \right) \partial_x^k \psi dx V_x,
\]

\[
R_k^5 = -\int_{\mathbb{R}} \partial_x^k g(V) \partial_x^k \psi dx, \quad R_k^6 = -\int_{\mathbb{R}} \tau \mu \partial_t \partial_x^k \left( \frac{U_x}{V} \right) \partial_x^k S dx, \quad R_k^7 = \int_{\mathbb{R}} \partial_x^k \left( \frac{U_x \varphi}{V} \right) \partial_x^k S dx,
\]

\[
R_k^8 = \int_{\mathbb{R}} (\partial_x^k (\frac{\mu}{v} \psi_x) - \frac{\mu}{v} \partial_x^k \psi_x) \frac{v}{\mu} \partial_x^k S dx.
\]

We shall show that for \(1 \leq j \leq 8, k = 1, 2,\)

\[
\int_{0}^{t} R_k^j(t) dt \leq C(\delta^\theta + E_{\delta}(t)) + \varepsilon \int_{0}^{t} \|S_x\|_{L^1} dt + C \int_{0}^{t} (1 + t)^{-\delta}(\|\varphi_x\|_{H^1}^2 + \|\psi_x\|_{H^1}^2) dt. \quad (5.7)
\]

Firstly,

\[
\int_{0}^{t} R_k^1 dt \leq C \int_{0}^{t} \int \|\varphi_x| + |\varphi_x| + |V_x| |(\partial_x^k \varphi)^2 + (\partial_x^k \psi)^2| dx dt \leq C(E_{\delta}(t) + \delta^4).
\]

Secondly, we have for \(k = 1\) that

\[
\int_{0}^{t} R_k^2(t) dt = \int_{0}^{t} \int \frac{p''(\varphi + V)(\varphi_x + V_x) \varphi_x \psi_x dx dt \leq C(E_{\delta}(t) + \delta^4).
\]

For \(k = 2,\)

\[
\int_{0}^{t} R_k^3(t) dt = \int_{0}^{t} \int \frac{p''(\varphi + V)(\varphi_x + V_x)^2 \varphi_x + p''(\varphi + V)(3\varphi_x \varphi_x + V_x \varphi_x + 2V_x \varphi_x)}{\psi_x dx dt}
\]

\[
\leq C \int_{0}^{t} \|(|\varphi_x|, V_x, V_x)\|_{L^1}(\|\varphi_x\|_{H^1}^2 + \|\psi_x\|_{H^1}^2) dt
\]

\[
\leq C(E_{\delta}(t) + \delta^4 + \delta^3 - \delta^4) + C \int_{0}^{t} (1 + t)^{-\delta}(\|\varphi_x\|_{H^1}^2 + \|\psi_x\|_{H^1}^2) dt.
\]
Similarly, for $k = 1$, we have

\[
\int_0^t R_1^3(t)dt
\]

\[
= \int_0^t \int_{\mathbb{R}} \left( p''(\varphi + V)(\varphi_x + V_x) - p''(V)V_x + (p'(\varphi + V) - p'(V))V_{xx} \right) \psi_x dx dt
\]

\[
\leq C \int_0^t \left( \|V_x\|_{L^\infty}^2 \|\varphi_x\|^2_{H^1} + \|\psi_x\|^2_{H^1} \right) + (\|V_x\|^2_{L^2} + \|V_{xx}\|_{L^2}) \|\partial_x \psi\|_{L^2} dt
\]

\[
\leq C(\delta^\frac{4}{3} + \delta^\frac{2}{3} + \delta^\frac{1}{2})
\]

and for $k = 2$,

\[
\int_0^t R_2^3(t)dt
\]

\[
\leq C \int_0^t \left( (E^k(t) + \|V_x\|_{L^\infty} + \|V_{xx}\|_{L^\infty})(\|\varphi_x\|^2_{H^1} + \|\psi_x\|^2_{H^1} + C(\|V_x\|^3\|L^2 + \|V_{xx}V_x\|_{L^2})\|\psi_{xx}\|_{L^2} \right) dt
\]

\[
\leq C(E^k(t) + \delta^4 + \delta^3 - \frac{3}{4} + \delta^\frac{1}{2} + \delta^\frac{1}{4} - \frac{1}{4}).
\]

Now, we estimate $R_k^1$. For $k = 1$,

\[
\int_0^t R_k^1(t)dt = \int_0^t \int_{\mathbb{R}} \mu \left( \frac{U_x}{V} \right)_{xx} \psi_x dx
\]

\[
= \int_0^t \int_{\mathbb{R}} \mu \left( \frac{U_{xxx}}{V} - \frac{2U_{xx}V_x}{V^2} - \frac{U_xV_{xx}}{V^2} + \frac{2V_x^2U_x}{V^3} \right) \psi_x dx
\]

\[
\leq C \int_0^t \left( \|U_{xxx}\| + \|U_{xx}\|_{L^4}\|V_x\|_{L^4} + \|U_{xx}\|_{L^4}\|U_x\|_{L^4} + \|U_x^2\|_{L^4}\|U_x\|_{L^4} \right) \|\psi_x\| dt
\]

\[
\leq C \int_0^t (\delta^\frac{2}{3} - \frac{1}{3}(1 + t)^{-1} - \frac{1}{3} + \delta^\frac{2}{3}(1 + t)^{-\frac{1}{2}}) \|\psi_x\| dt \leq C(\delta^\frac{2}{3} - \frac{1}{3} + \delta^\frac{1}{2}).
\]

For $k = 2$, some tedious calculations give

\[
\partial^3 \left( \frac{U_x}{V} \right)
\]

\[
= \frac{\partial_x^3 U V_x + 3 \partial_x^2 U V_{xx} + \partial_x^2 U_{xx} V + 6 \partial_x (V_x)^2 + 6 \partial_x V_{xx} V_x}{V^3} - \frac{6U_x (V_x)^3}{V^4}.
\]

So, we have

\[
\int_0^t R_k^1(t)dt = \int_0^t \int_{\mathbb{R}} \mu \partial_x^3 \left( \frac{U_x}{V} \right) \psi_{xx} dx
\]

\[
\leq C \int_0^t \left( \|\partial_x^2 U\|_{L^2} + \|\partial_x^3 U_{xx}\|_{L^2} + \|U_{xx} V_{xx}\|_{L^2} + \|U_x \partial_x^2 V\|_{L^2} + \|U_x V_{xx} V_x\|_{L^2} + \|U_x (V_x)^3\|_{L^2} \right) \|\psi_{xx}\|_{L^2} dt
\]

\[
\leq C \int_0^t (\delta^{\frac{2}{3} - \frac{1}{3}}(1 + t)^{-1} - \frac{1}{3} + \delta^2 - \frac{1}{3}(1 + t)^{-\frac{1}{2}} - \frac{1}{3} + \delta^\frac{2}{3} - \frac{1}{3}(1 + t)^{-\frac{1}{2}} + \delta^\frac{1}{3} - \frac{1}{3}(1 + t)^{-\frac{1}{2}}) \|\psi_{xx}\|_{L^2} dt
\]

\[
\leq C(\delta^{\frac{2}{3} - \frac{1}{3}} + \delta^\frac{1}{2}).
\]
Similarly, for $k = 1, 2$, we can get

$$
\int_0^t R_k^1 dt \leq C \int_0^t ((V_x)^2, (V_x))^\|_{L^2} + ((V_{xx}, \partial_x^2 V)^{\|_{L^2}} + (V_{x, V_{xx}})^{\|_{L^2}})((\partial_x, \partial_x^2 \psi)^{\|_{L^2}} dt
$$

$$
\leq C \int_0^t (\delta^{3-\frac{3}{4}}(1 + t)^{-1/2} + \delta^{\frac{1}{2}}(1 + t)^{-1/4})((\partial_x, \partial_x^2 \psi)^{\|_{L^2}} dt \leq C(\delta^{3-\frac{3}{4}} + \delta^{\frac{1}{2}}).
$$

On the other hand, $\int_0^t R_k^0(t) dt$ and $\int_0^t R_k^1(t) dt$ are estimated as follows. Note that for $k = 1$, we have

$$
\left( \frac{U_x}{V} \right)_t = \frac{(U_{xx} V + U_{xx} V_x - U_{xx} V_1 - U_{xx} V_x) V^2 - (U_{xx} V - U_x V_1) 2VV_x}{V^4},
$$

$$
\left( \frac{U_x V}{V} \right)_t = \frac{(U_{xx} V + U_{xx} V_x) V - U_x V (V_x V + V_x)}{V^2 V^2}.
$$

So, we derive that

$$
\int_0^t R_k^0(t) dt \leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + C(\varepsilon) \int_0^t \int \left( \frac{U_x}{V} \right)^2_t dx dt
$$

$$
\leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + C(\varepsilon) \int_0^t \int \left( V_{xx} + V_{xx} V_x + V_x + U_{xx} U_x + U_x V_x \right)^2 dx dt
$$

$$
\leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + C(\varepsilon) \int_0^t \left( (V_x)^6 + (V_x)^3 + (V_{xx})^2 + (V_{xx})^2 + (U_{xx})^2 + (V_{xx})^2 \right) dx dt
$$

$$
\leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + \int_0^t (\delta(1 + t)^{-3} + \delta^{3-\frac{3}{2}}(1 + t)^{-\frac{7}{2}} + \delta^{3-\frac{3}{4}}(1 + t)^{-\frac{1}{4}}) dt
$$

$$
\leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + \delta + \delta^{3-\frac{3}{4}}.
$$

and

$$
\int_0^t R_k^1(t) dt \leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + C(\varepsilon) \int_0^t \int \left( U_{xx} \varphi + U_x \varphi_x + U_x V_x + U_x \varphi_x \right)^2 dx dt
$$

$$
\leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + C(\varepsilon)((E^2(t) + \delta^{\frac{1}{2}} + \delta^{3-\frac{3}{4}}) + \int_0^t (1 + t)^{-2} ||x||^2_{L^2}) dt).
$$

The estimates for $\int_0^t R_k^0(t) dt$ and $\int_0^t R_k^1(t) dt$ can be done in a similar way, we omit the details.

Finally, let’s estimate the term $\int_0^t R_k^2(t) dt$. Firstly, for $k = 1$, we have

$$
\int_0^t R_k^2(t) dt = - \int_0^t \int \frac{w_x \psi_x S_x}{v} dx dt
$$

$$
\leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + C(\varepsilon) \int_0^t \int (V_x^2 + \varphi_x^2) \psi_x^2 dx dt
$$

$$
\leq \varepsilon \int_0^t \int \frac{1}{\mu} S_x^2 dx dt + C(\varepsilon)(E^2(t) + \delta^8).
$$
and for $k = 2$,
\[
\int_0^t R_x^2(t) dt = \int_0^t \int_{\mathbb{R}} \frac{2v_x^2 \partial_x \psi - 2vv_x \psi_x x}{v^2} S_x dx dt \\
\leq \varepsilon \int_0^t \int_{\mathbb{R}} \frac{u^2}{\mu} S_{xx}^2 + C(\varepsilon) \int_0^t \int_{\mathbb{R}} ((|V_x|^4 + |\varphi_x|^4)|\psi_x|^2 + (|V_x|^2 + |\varphi_x|^2)|\psi_{xx}|^2) dx dt \\
\leq \varepsilon \int_0^t \int_{\mathbb{R}} \frac{v^2}{\mu} S_{xx}^2 dx dt + C(\varepsilon)(E^2(t) + \delta^8).
\]

Combining the above estimates and using the Gronwall’s inequality, we get the desired results.

The next lemma gives the dissipative estimates of $\varphi_x$ and $\psi_x$.

**Lemma 5.3.** There exists a constant $C$ such that for $0 \leq t \leq T$, we have
\[
\int_0^t \left( \|\varphi_x\|^2_{H^1} + \|\psi_x\|^2_{H^1} \right) dt \leq C(E_0 + \delta^6 + E^2(t)).
\]  

**Proof.** Multiplying the equation (5.6) by $\partial_x^{k+1} \varphi$ for $k = 0$ or 1, and integrating over $(0, t) \times \mathbb{R}$, we get
\[
\int_0^t \int_{\mathbb{R}} -p'(\varphi + V)(\partial_x^{k+1} \varphi)^2 dx dt = \sum_{j=1}^6 M_k^j,
\]
where
\[
M_k^1 = \int_0^t \int_{\mathbb{R}} \partial_t \partial_x^k \psi \cdot \partial_x^{k+1} \varphi dx dt, \\
M_k^2 = \int_0^t \int_{\mathbb{R}} (\partial_x^k (p' (\varphi + V) \varphi_x)) - p' (\varphi + V) \partial_x^{k+1} \varphi \cdot \partial_x^{k+1} \varphi dx dt,
\]
\[
M_k^3 = \int_0^t \int_{\mathbb{R}} \partial_x^k ((p' (\varphi + V) - p' (V))V_x) \cdot \partial_x^{k+1} \varphi dx dt, \\
M_k^4 = \int_0^t \int_{\mathbb{R}} \partial_x^{k+1} S \cdot \partial_x^{k+1} \varphi dx dt, \\
M_k^5 = - \int_0^t \int_{\mathbb{R}} \mu \partial_x^{k+1} \left( \frac{U} {V} \right) \partial_x^{k+1} \varphi dx dt, \\
M_k^6 = \int_0^t \int_{\mathbb{R}} \partial_x^{k+1} g(V) \partial_x^{k+1} \varphi dx dt.
\]
We shall show that
\[
\sum_{j=1}^6 M_k^j \leq \varepsilon \int_0^t \int_{\mathbb{R}} (\partial_x^{k+1} \varphi)^2 dx dt + C \int_0^t \int_{\mathbb{R}} (\partial_x^{k+1} \psi)^2 dx dt + C(E^2(t) + E_0 + \delta^6).
\]  

Firstly, by doing integration by part and using equation (5.6)1, we get
\[
\int_0^t \int_{\mathbb{R}} \partial_t \partial_x^k \psi \partial_x^{k+1} \varphi dx dt \\
= \int_{\mathbb{R}} \partial_x^k \psi(t) \partial_x^{k+1} \varphi(t) dx - \int_{\mathbb{R}} \partial_x^k \psi_0 \partial_x^{k+1} \varphi_0 dx + \int_0^t \int_{\mathbb{R}} \partial_x^{k+1} \psi_0 \partial_t \partial_x^k \varphi dx dt \\
\leq \int_0^t \int_{\mathbb{R}} (\partial_x^{k+1} \psi)^2 dx dt + C(E^2(t) + \delta^6 + E_0).
Secondly, for \( k = 0 \), \( M_0^2 \) vanishes and for \( k = 1 \),
\[
M_1^2 = \int_0^t \int_R \phi''(\varphi + V)(\varphi_x + V_x)\varphi_{xx}dxdt
\]
\[
\leq \varepsilon \int_0^t \int_R (\varphi_{xx})^2dxdt + C \int_0^t \int_R (\varphi_x + V_x^2\varphi_x^2)dx
\]
\[
\leq \varepsilon \int_0^t \int_R (\varphi_{xx})^2dxdt + C(E^2(t) + \delta^6).
\]
Similarly, we get for \( k = 0 \),
\[
M_0^3 \leq \varepsilon \int_0^t \int_R (\varphi_x)^2dxdt + C \int_0^t \int_R \varphi_x^2V_x^2dxdt
\]
\[
\leq \varepsilon \int_0^t \int_R (\varphi_x)^2dxdt + \sup_{\varphi} \|\varphi\|_2^2 \int_0^t \|V_x\|_4^2 dt
\]
\[
\leq \varepsilon \int_0^t \int_R (\varphi_x)^2dxdt + C(E_0 + \delta),
\]
and \( k = 1 \),
\[
M_1^3 \leq \varepsilon \int_0^t \int_R (\varphi_{xx})^2dxdt + \int_0^t \int_R (V_{xx}^2 + V_x^4 + \varphi_x^2V_x^2)dxdt
\]
\[
\leq \varepsilon \int_0^t \int_R (\varphi_{xx})^2dxdt + C(\delta^{\frac{3}{4}} + \delta).
\]
By using the dissipation of \( S_x \), we have
\[
M_k^4 \leq \varepsilon \int_0^t \int_R (\varphi_{xx})^2dxdt + C \int_0^t \int_R (\varphi_x^{k+1}S)^2dt
\]
\[
\leq \varepsilon \int_0^t \int_R (\varphi_x^{k+1})^2dxdt + C(E_0^2(t) + E_0 + \delta^6).
\]
\( M_k^5 \) and \( M_k^6 \) can be estimated in the same way as in Lemma 5.2. Indeed, we have
\[
M_0^5 \leq C \int_0^t \int_R (|\varphi_{xx}U| + |U_x||V_x||\varphi_x|)dxdt
\]
\[
\leq \varepsilon \int_0^t \int_R |\varphi_x|^2dxdt + C \int_0^t (|\varphi_{xx}U|_2^2 + |U_x|_4^4 + |V_x|_4^4)dt
\]
\[
\leq \varepsilon \int_0^t \int_R |\varphi_x|^2dxdt + C(\delta^{3-\frac{4}{3}} + \delta),
\]
and
\[
M_1^5 \leq C \int_0^t (|U_{xxx}| + |U_x||V_x||L^4 + |V_x||L^4 + |U_x||L^4 + |V_x^2||L^4 + |U_x||L^4 + |V_x||L^4||U_x|_L^4)|\varphi_{xx}^2dxdt
\]
\[
\leq \varepsilon \int_0^t \int_R (\varphi_{xx}^2)^2dxdt + C(\delta^{3-\frac{4}{3}} + \delta^{3-\frac{4}{3}} + \delta).
\]
Note that \( \int_0^t |\varphi_xg(V)||L^2 dt \leq C\delta^{\frac{1}{2}} \), we get
\[
M_0^6 \leq \sup_{\varphi} \|\varphi_x\|_L^2 \int_0^t |\varphi_xg(V)||L^2 dt \leq C\delta^{\frac{1}{2}}.
\]
Besides, we have
\[ M_1^6 \leq \int_0^t \int_{\mathbb{R}} (|V_{xx}| + |V_x|^2) |\varphi_{xx}| \, dx \, dt \leq \varepsilon \int_0^t \int_{\mathbb{R}} (\varphi_{xx})^2 \, dx \, dt + C(\delta^{3/4} + \delta). \]
Combining the above estimates, summing up \( k \) from 0 to 1, we finally get
\[ \int_0^t \| \varphi_x \|_{H^1}^2 \, dt \leq C \int_0^t \| \psi_x \|_{H^1}^2 \, dt + C(E^\frac{3}{4}(t) + E_0 + \delta^6). \]  
(5.10)

Now, we derive the dissipative estimates of \( \psi_x \). Multiplying the equation by \( \partial_x^{k+1} \psi \), we derive that
\[ \int_0^t \int_{\mathbb{R}} \frac{\mu}{V} (\partial_x^{k+1} \psi)^2 \, dx \, dt =: \sum_{j=1}^5 N_k^j, \]
where
\[ N_k^1 = \int_0^t \int_{\mathbb{R}} \tau \partial_t \partial_x^k S \partial_x^{k+1} \psi \, dx \, dt, \]
\[ N_k^2 = \int_0^t \int_{\mathbb{R}} \partial_x^k S \partial_x^{k+1} \psi \, dx \, dt, \]
\[ N_k^3 = - \int_0^t \int_{\mathbb{R}} \left( \partial_x^k \left( \frac{\mu}{V} \partial_x \psi \right) - \frac{\mu}{V} \partial_x^{k+1} \psi \right) \partial_x^{k+1} \psi \, dx \, dt, \]
\[ N_k^4 = \int_0^t \int_{\mathbb{R}} \tau \mu \partial_t \partial_x^k \left( \frac{U_x}{V} \right) \partial_x^{k+1} \psi \, dx \, dt, \]
we shall show that
\[ \sum_{j=1}^5 N_k^j \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\mu}{V} (\partial_x^{k+1} \psi)^2 \, dx \, dt + \varepsilon \int_0^t \| \partial_x \psi \|_{H^1}^2 \, dt + C(E^\frac{3}{4} + E_0 + \delta^6), \]  
(5.11)
which implies that
\[ \int_0^t \| \psi_x \|_{H^1}^2 \, dt \leq \varepsilon \int_0^t \| \varphi_x \|_{H^1}^2 \, dt + C(E^\frac{3}{4}(t) + E_0 + \delta^6). \]  
(5.12)
Combining (5.10) and (5.12) together, we get the desired result. Therefore, we only need to show (5.11) hold.

Firstly, using Lemma 5.2 and integrating by part with respect to \( x \) and \( t \), we have
\[ N_k^1 \leq C(E^\frac{3}{4}(t) + \delta^6 + E_0) + \int_0^t \int_{\mathbb{R}} \partial_x^{k+1} S \partial_t \partial_x^k \psi \, dx \, dt, \]
while, using equation (5.6) and Lemma 5.2 and exploiting the same methods as above (estimates of \( M_j^1 \), \( j = 2, \ldots, 6 \), for example), we get
\[ \int_0^t \int_{\mathbb{R}} \tau \partial_x^{k+1} S \partial_t \partial_x^k \psi \, dx \, dt \]
\[ = \int_0^t \int_{\mathbb{R}} \tau \partial_x^{k+1} S \left( - \partial_x^k (p'(\varphi + V) \varphi_x) - \partial_x^k ((p'(\varphi + V) - p'(V))V_x) \right) \]
\[ + \tau \partial_x^{k+1} S + \mu \partial_x^{k+1} \left( \frac{U_x}{V} \right) - \partial_x^{k+1} g(V) \right) \, dx \, dt \]
\[ \leq \varepsilon \int_0^t \int_{\mathbb{R}} (\partial_x^{k+1} \varphi)^2 \, dx \, dt + C(\varepsilon) \int_0^t \int_{\mathbb{R}} (\partial_x^{k+1} S)^2 \, dx \, dt + C(E^\frac{3}{4}(t) + \delta^6 + E_0) \]
\[ \leq \varepsilon \int_0^t \int_{\mathbb{R}} (\partial_x^{k+1} \varphi)^2 \, dx \, dt + C(E^\frac{3}{4}(t) + \delta^6 + E_0), \]
which gives the estimates of $N_k^1$. Moreover, it’s easy to see that
\[
N_k^2 \leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \frac{\mu}{v} (\partial_x k^1 \psi)^2 dx dt + C \int_0^t ||S||_{L^1} dt
\]
\[
\leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \frac{\mu}{v} (\partial_x k^1 \psi)^2 dx dt + C(E_0^2(t) + \delta^8 + E_0).
\]

Now, we estimate the term $N_k^j, j = 3, 4, 5$. Note that, for $k = 0$, $N_0^3$ vanishes, and for $k = 1$,
\[
N_1^3 = \int_0^t \int_{\mathbb{R}} \frac{\mu v}{v^2} \psi_x \psi_x dx dt
\]
\[
\leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \frac{\mu}{v} (\psi_x x^2)^2 dx dt + C \int_0^t \int_{\mathbb{R}} (|V_x|^2 + |\varphi_x |^2) |\psi_x|^2 dx dt
\]
\[
\leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \frac{\mu}{v} (\psi_x x^2)^2 dx dt + C(E_0^2(t) + \delta^8).
\]

On the other hand, for $k = 0$, using the equation (3.10) and Lemma 5.2, we have
\[
N_0^4 + N_0^5 \leq C \int_0^t \int_{\mathbb{R}} (|f(V)_{xx} | + |p(V)_{xx} | + |U_x|^2 + |U_x||\varphi |) |\psi_x| dx dt
\]
\[
\leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \frac{\mu}{v} |\psi_x|^2 dx dt + C(E_0 + \delta + \delta^3 \delta^2 4),
\]
and for $k = 1$, with the estimates of $R_0^6$ and $R_1^7$, we do have
\[
N_1^4 + N_1^5 \leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \frac{\mu}{v} |\psi_x x^2|^2 dx dt + C(E_0^2(t) + E_0 + \delta^8).
\]

This finish the proof of the Lemma. □

Combining the results of Lemma 5.1-5.3, the proof of the Proposition 4.2 is finished.

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