EXISTENCE OF CRYSTAL BASES FOR
KIRILLOV-RESHETIKHIN MODULES OF TYPE D

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1. INTRODUCTION

Let $g$ be an affine algebra and let $U'_q(g)$ be the corresponding quantum affine algebra without degree operator. Among irreducible finite-dimensional $U'_q(g)$-modules there exists a distinguished family called Kirillov-Reshetikhin modules (KR modules for short). They were introduced in [KR] in connection with a certain conjectural formula of multiplicities of irreducible $U_q(g_0)$-modules in a tensor product of those modules. Here $g_0$ stands for the finite-dimensional simple Lie algebra whose Dynkin diagram is obtained by removing the 0-vertex, that is prescribed in [Kac], from that of $g$. It is known [D, CP] that irreducible finite-dimensional $U'_q(g)$-modules are classified by $n$-tuples of polynomials called Drinfeld polynomials, where $n$ is the rank of $g_0$.

Let us define KR modules by the Drinfeld polynomials. Let $k \in \{1, 2, \ldots, n\}$, $l \in \mathbb{Z} > 0$ and $a$ an invertible element of $\mathbb{Q}(q)$. A KR module $\tilde{W}^{(k)}_{l,a}$ is defined to be the unique irreducible finite-dimensional $U'_q(g)$-module that has

$$P_i(u) = \begin{cases} (1 - aq_i^{-1}u)(1 - aq_i^{-3}u) \cdots (1 - aq_i^{-l}u) & \text{if } i = k, \\ 1 & \text{otherwise} \end{cases}$$

as Drinfeld polynomials. (See section 2.1 for the definition of $q_i$.) When $l = 1$ it is also called a fundamental representation. Since a fundamental representation is known to have a crystal base [K2], we choose $a^l$ so that $\tilde{W}^{(k)}_{l,a}$ has a crystal base and redefine $W^{(k)}_{l,a} = \tilde{W}^{(k)}_{l,a}$. $W^{(k)}_{l,1}$ is also denoted simply by $W^{(k)}_l$.

The above-mentioned conjectural multiplicity formula was shown when $g$ is non-twisted by combining two results. The first one is the proof of certain algebraic relations among characters of KR modules, called $Q$-systems, presented in [NI] for simply-laced cases and in [H] for all non-twisted cases. The second one is a derivation of the multiplicity formula, called fermionic formula, in [Ki] for type $A$ and in [HKOTY] for all non-twisted cases, by using the $Q$-systems. However, it deserves to emphasize that there is also a $q$-analogue of the conjecture, called $X = M$ conjecture [HKOTY, HKOTT]. The definition of $X$ requires the existence of the crystal base of a KR module. Despite many efforts as in [KMN, KKM, Y, Ko, JMO, K2, NS, SS, BFKL], this existence problem is yet to be settled. For type $D$ for instance, the crystal base has been shown to exist for $W^{(k)}_l$ where $k = 1, n-1, n; l \in \mathbb{Z} > 0$ in [KMN] and for $W^{(k)}_l$ for arbitrary $k$ in [K2].

In this paper we prove the following theorem, thereby settling the problem for type $D$. 

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Theorem 1.1. For $2 \leq k \leq n - 2$ and $l \geq 1$, the $U_q'(D_n^{(1)})$-module $W_l^{(k)}$ has a crystal pseudobase.

Here $(L, B)$ is said to be a crystal pseudobase if $(L, B/\{\pm 1\})$ is a crystal base. (See Definition 2.1 for the definition of a crystal base.) Let us give a short sketch of our proof. We follow the technique already developed in [KMN], namely, (Proposition 2.9) to the constructed module $W_l^{(k)}$. Practically, we need to check two conditions (2.21) and (2.22). Checking the second one is not difficult, if once we find out the vectors $\{u_j\}$. Checking the first one requires information on the image $W$ and the kernel $N$ of the $R$-matrix $R(x, y) : W_l^{(k)} \otimes W_{1, x}^{(k)} \rightarrow W_{1, y}^{(k)} \otimes W_{1, x}^{(k)}$ at $x/y = q^2$. Up to now such information was obtained by computing the spectral decomposition of the $R$-matrix when dealing with $W_l^{(k)}$ for higher $l$. It seems to be the reason why showing the existence of crystal bases $W_l^{(k)}$ for higher $k$ has not been succeeded so far, since the calculation of the corresponding $R$-matrix is too complicated. However, thanks to the result by Nakajima [N1], we are now able to identify $W$ and $N$ with tensor products of KR modules (Lemma 3.1). Using the crystal base of $W$ and a property of a bilinear form between $W_l^{(k)} \otimes W_{1, q^{-1}}^{(k)}$ and $W_{1, q^{-1}} \otimes W_{1, q}$, we can check the first condition of the criterion. It is known that a $U_q(g)$-module with a connected crystal base is irreducible. Therefore, once $W_l^{(k)}$ is shown to have a crystal pseudobase, it follows that $W_l^{(k)}$ is an irreducible finite-dimensional module with the desired Drinfeld polynomials, since it is a simple quotient of $W_{1, q^{-1}}^{(k)} \otimes W_{1, q^{-1}}^{(k)} \otimes \cdots \otimes W_{1, q^{-1}}^{(k)}$.

After the author finished the manuscript, he learned from Kashiwara that the module $W_l^{(k)}$ can be shown to be irreducible by Theorem 9.2 of [K2]. Once it is established, the character is known by [C]. Hence it turns out that there is no need to prove the inequality of the character in (i) just after Proposition 3.7. However, this does not seem to prove that $W_l^{(k)}$ is isomorphic to a module of the form of $V^\otimes l/\sum_{i=0}^{l-2} V^\otimes i \otimes N \otimes V^\otimes (l-2-i)$. The author was also informed from Nakajima that the existence of a polarization on the fundamental representation $W_l^{(k)}$ was shown in [VV] (see also [N2] [BN] for more general results). Hence similar calculations of the prepolarization as in section 3 will give a proof of the existence of crystal bases for KR modules of other quantum affine algebras.

2. Crystal base and fusion construction

2.1. Crystal base. In this subsection we briefly recall the definition of crystal bases. For more details along with the definition of $U_q(g)$, refer to [K1].

Let $g$ be a symmetrizable Kac-Moody Lie algebra and let $M$ be a $U_q(g)$-module. $M$ is said to be integrable if $M = \bigoplus_{\lambda \in \delta} M_\lambda$, dim $M_\lambda \leq \infty$ for any $\lambda$, and for any $i$, $M$ is a union of finite-dimensional $U_q(g_i)$-modules. Here $P$ is the weight lattice of $g$, $M_\lambda$ is the weight space of $M$ of weight $\lambda$ and $U_q(g_i)$ is the subalgebra generated by Chevalley generators $e_i$ and $f_i$. If $M$ is integrable, we have

\begin{equation}
M = \bigoplus_{0 \leq n \leq \langle h_i, \lambda \rangle} f_i^{(n)}(\text{Ker } e_i \cap M_\lambda).
\end{equation}
Note that we use the following notations: \([m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1}), [n]_i! = \prod_{m=1}^{\infty} [m]_i, f_i^{(n)} = f_i^n/[n]_i! with q_i = q^{\alpha_i, \alpha_i}\), where \(( , )\) is an invariant bilinear form on \(P\). We define the endomorphisms \(\tilde{e}_i, \tilde{f}_i\) of \(M\) by

\[
\tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u \quad \text{and} \quad \tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}u
\]

for \(u \in \text{Ker} \ e_i \cap M_\lambda \) with \(0 \leq n \leq \langle h_i, \lambda \rangle\). Similarly, we have

\[
M = \bigoplus_{0 \leq n \leq -\langle h_i, \mu \rangle} e_i^{(n)}(\text{Ker} \ f_i \cap M_\mu).
\]

These two decompositions are related as follows:

1. If \(0 \leq n \leq \langle h_i, \lambda \rangle\) and \(u \in \text{Ker} \ e_i \cap M_\lambda\), then \(v = f_i^{(\langle h_i, \lambda \rangle)}u\) belongs to \(\text{Ker} \ f_i \cap M_{s_i(\lambda)}\) and \(f_i^{(n)}u = e_i^{(\langle h_i, \lambda \rangle - n)}v\).

Here \(s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i\). Hence we obtain

\[
\tilde{f}_i(e_i^{(n)}v) = e_i^{(n-1)}v \quad \text{and} \quad \tilde{e}_i(e_i^{(n)}v) = e_i^{(n+1)}v
\]

for \(v \in \text{Ker} \ f_i \cap M_\mu\) with \(0 \leq n \leq -\langle h_i, \mu \rangle\).

Let us now look at the definition of a crystal base. Let \(A\) be the subring of \(\mathbb{Q}(q)\) consisting of rational functions without poles at \(q = 0\). Let \(M\) be an integrable \(U_q(\mathfrak{g})\)-module.

**Definition 2.1.** A pair \((L, B)\) is called a crystal base of \(M\) if it satisfies the following 6 conditions:

\[
\begin{align*}
(2.5) & \quad L \text{ is a free sub-}A\text{-module of } M \text{ such that } M \simeq \mathbb{Q}(q) \otimes_A L, \\
(2.6) & \quad B \text{ is a base of the } \mathbb{Q}\text{-vector space } L/QL, \\
(2.7) & \quad \tilde{e}_iL \subset L \text{ and } \tilde{f}_iL \subset L \text{ for any } i.
\end{align*}
\]

By (2.7), \(\tilde{e}_i\) and \(\tilde{f}_i\) act on \(L/QL\).

\[
\begin{align*}
(2.8) & \quad \tilde{e}_iB \subset B \cup \{0\} \text{ and } \tilde{f}_iB \subset B \cup \{0\}, \\
(2.9) & \quad L = \bigoplus_{\lambda \in P} L_\lambda \text{ and } B = \bigcup_{\lambda \in P} B_\lambda
\end{align*}
\]

where \(L_\lambda = L \cap M_\lambda\) and \(B_\lambda = B \cap (L_\lambda/QL_\lambda)\).

\[
(2.10) \quad \text{For } b, b' \in B, b' = \tilde{f}_i b \text{ if and only if } \tilde{e}_i b' = b.
\]

Standard notations are in order. For \(b \in B\) we set

\[
\begin{align*}
(2.11) & \quad \varepsilon_i(b) = \max\{m \in \mathbb{Z}_{\geq 0} | \tilde{e}_i^m b \neq 0\}, \quad \varphi_i(b) = \max\{m \in \mathbb{Z}_{\geq 0} | \tilde{f}_i^m b \neq 0\}, \\
(2.12) & \quad \varepsilon(b) = \sum_i \varepsilon_i(b) \Lambda_i, \quad \varphi(b) = \sum_i \varphi_i(b) \Lambda_i, \\
(2.13) & \quad \text{wt } b = \varphi(b) - \varepsilon(b).
\end{align*}
\]

Here \(\{\Lambda_i\}\) stands for the set of fundamental weights of \(\mathfrak{g}\).

The crystal base behaves nicely under the tensor product. Let \((L_j, B_j)\) be the crystal base of an integrable \(U_q(\mathfrak{g})\)-module \(M_j\) \((j = 1, 2)\). Set \(L = L_1 \otimes_A L_2\) and
$B = \{ b_j \otimes b_2 \mid b_j \in B_j (j = 1, 2) \}$. Then $(L, B)$ is a crystal base of $M_1 \otimes M_2$. Moreover, the action of $\tilde{e}_i$ and $\tilde{f}_i$ becomes very simple as

\begin{align}
\tilde{e}_i (b_1 \otimes b_2) &= \begin{cases} 
\delta_{i,1} b_1 \otimes b_2 & \text{if } \varphi_i (b_1) \geq \varepsilon_i (b_2), \\
b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i (b_1) < \varepsilon_i (b_2),
\end{cases} \tag{2.14}
\tilde{f}_i (b_1 \otimes b_2) &= \begin{cases} 
\delta_{i,1} b_1 \otimes b_2 & \text{if } \varphi_i (b_1) > \varepsilon_i (b_2), \\
b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i (b_1) \leq \varepsilon_i (b_2).
\end{cases} \tag{2.15}
\end{align}

Here $0 \otimes b$ and $b \otimes 0$ are understood to be $0$. We denote this $B$ by $B_1 \otimes B_2$. $\varepsilon_i, \varphi_i$ and $wt$ are given by

\begin{align}
\varepsilon_i (b_1 \otimes b_2) &= \max (\varepsilon_i (b_1), \varepsilon_i (b_1) + \varepsilon_i (b_2) - \varphi_i (b_1)), \\
\varphi_i (b_1 \otimes b_2) &= \max (\varphi_i (b_2), \varphi_i (b_1) + \varphi_i (b_2) - \varepsilon_i (b_2)), \\
wt (b_1 \otimes b_2) &= wt b_1 + wt b_2. \tag{2.16, 2.17, 2.18}
\end{align}

Next lemma is used later in section $3$.

**Lemma 2.2.** Let $(L, B)$ be a crystal base. Assume that $\tilde{e}_i^3 b = \tilde{f}_i^3 b = 0$ for any $b \in B$. Let $v \in L$ be such that $v \equiv b \mod qL$. Then we have

$$
e_i v \equiv q_i^{-\varphi_i(b)} \tilde{e}_i v \mod qq_i^{-\varphi_i(b)}L, \\
f_i v \equiv q_i^{-\varepsilon_i(b)} \tilde{f}_i v \mod qq_i^{-\varepsilon_i(b)}L.$$

In particular, $e_i v \equiv 0$ (resp. $f_i v \equiv 0$) if $\varepsilon_i (b) = 0$ (resp. $\varphi_i (b) = 0$).

**Proof.** We prove the second relation. Let $\lambda$ be the weight of $v$. From the assumption it suffices to check the relation for the following cases, since the other cases are trivial.

(i) $\varepsilon_i (b) = 0, \langle h_i, \lambda \rangle = 1$ or 2,

(ii) $\varepsilon_i (b) = 0$ or 1, $\langle h_i, \lambda \rangle = 0$.

In case (i) we have $f_i v = \tilde{f}_i v$ by (2.11) and (2.2). In case (ii) let us write $v = f_i v_1 + v_2$ with $v_j \in \text{Ker} e_i \cap L$ $(j = 1, 2)$ such that $\langle h_i, wt v_1 \rangle = 2, \langle h_i, wt v_2 \rangle = 0$. Then we have $f_i v = f_i^2 v_1 = [2]_i f_i v \equiv q_i^{-1} \tilde{f}_i v \mod qq_i^{-1}L$.

The first relation can be checked similarly by using (2.10) and (2.11). $\square$

**2.2. Polarization.** We define a total order on $\mathbb{Q}(q)$ by

$$f > g \text{ if and only if } f - g \in \bigsqcup_{n \in \mathbb{Z}} \{ q^n (c + qA) \mid c > 0 \}$$

and $f \geq g$ if $f > g$ or $f = g$.

Let $M$ and $N$ be $U_q (g)$-modules. A bilinear form $(\phantom{2}, \phantom{2}) : M \otimes_{\mathbb{Q}(q)} N \to \mathbb{Q}(q)$ is called an admissible pairing if it satisfies

\begin{align}
(q^h u, v) &= (u, q^h v), \\
(e_i u, v) &= (u, q_i^{-1} t_{-i}^{-1} f_i v), \\
f_i u, v) &= (u, q_i^{-1} t_i e_i v),
\end{align} \tag{2.19}

for all $u \in M$ and $v \in N$. (2.19) implies

\begin{align}
(e_i^{(n)} u, v) &= (u, q_i^{-n^2} t_{-i}^{-n} f_i^{(n)} v), \\
f_i^{(n)} u, v) &= (u, q_i^{-n^2} t_i e_i^{(n)} v).
\end{align} \tag{2.20}
A symmetric bilinear form \((\ , \)\) on \(M\) is called a preporlarization of \(M\) if it satisfies \(2.10\) for \(u, v \in M\). A preporlarization is called a polarization if it is positive definite with respective to the order on \(\mathbb{Q}(q)\).

2.3. Fusion construction. In what follows we assume that \(\mathfrak{g}\) is of affine type. Let \(P\) be the weight lattice, \(\{\Lambda_i\}\) the set of fundamental weights and \(\delta\) the generator of null roots of \(\mathfrak{g}\). Then we have \(P = \bigoplus_i \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta\). We set

\[ P_{cl} = P/\mathbb{Z} \delta. \]

Similar to the quantum algebra \(U_q(\mathfrak{g})\) which is associated with \(P\), we can also consider \(U'_q(\mathfrak{g})\), which is associated with \(P_{cl}\), namely, the subalgebra of \(U_q(\mathfrak{g})\) generated by \(e_i, f_i, q^h (h \in (P_{cl})^*)\).

Let \(K\) be a commutative ring containing \(\mathbb{Q}(q)\) and let \(x\) be an invertible element of \(K\). We introduce a \(K \otimes_{\mathbb{Q}(q)} U'_q(\mathfrak{g})\)-module \(V_x\) by replacing the actions of \(e_i, f_i\) with \(x^{\delta w} e_i, x^{-\delta w} f_i\). The action of \(q^h\) is not changed. Let \(y\) also be an invertible element of \(K\). A \(K \otimes_{\mathbb{Q}(q)} U'_q(\mathfrak{g})\)-linear map

\[ R(x, y) : V_x \otimes V_y \rightarrow V_y \otimes V_x \]

is called a \(R\)-matrix. Here we need to specify the coproduct \(\Delta\) of \(U_q(\mathfrak{g})\) we use in this paper. Our choice is the “lower” one (see [K1]) given by

\[ \Delta(q^h) = q^h \otimes q^h \quad \text{for} \quad h \in (P_{cl})^*, \]

\[ \Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \]

\[ \Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i. \]

For a finite-dimensional \(U'_q(\mathfrak{g})\)-module \(V\) we assume the following.

(2.21) \(V\) is irreducible.

(2.22) There exists \(\lambda_0 \in P_{cl}\) such that \(wt V \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i\) and \(\dim V_{\lambda_0} = 1\).

Here \(\{\alpha_i\}\) is the set of simple roots. Under these assumptions it is known that there exists a unique \(R\)-matrix up to a scalar multiple. Moreover, \(R(x, y)\) depends only on \(x/y\). Take a non zero vector \(u_0\) from \(V_{\lambda_0}\). We normalize \(R(x, y)\) in such a way that \(R(x, y)(u_0 \otimes u_0) = u_0 \otimes u_0\). It is known in [K2] that if \(V\) is a “good” module then the normalized \(R\)-matrix does not have a pole at \(x/y = a \in A\).

Next we review the fusion construction following section 3 of [KMN]. Let \(\ell\) be a positive integer and \(S_l\) the \(l\)-th symmetric group. Let \(s_i\) be the simple reflection which interchanges \(i\) and \(i + 1\), and let \(\ell(w)\) be the length of \(w \in S_l\). Let \(V\) be a finite-dimensional \(U'_q(\mathfrak{g})\)-module. Let \(R(x, y)\) denote the \(R\)-matrix for \(V \otimes V\). For any \(w \in S_l\) we construct a map \(R_w(x_1, \ldots, x_l) : V_{x_1} \otimes \cdots \otimes V_{x_l} \rightarrow V_{x_w(1)} \otimes \cdots \otimes V_{x_w(l)}\) by

\[ R_1(x_1, \ldots, x_l) = 1, \]

\[ R_{s_i}(x_1, \ldots, x_l) = \left( \bigotimes_{j < i} \text{id}_{V_{x_j}} \right) \otimes R(x_i, x_{i+1}) \otimes \left( \bigotimes_{j > i+1} \text{id}_{V_{x_j}} \right), \]

\[ R_{w w'}(x_1, \ldots, x_l) = R_w(x_{w(1)}, \ldots, x_{w(l)}) \circ R_w(x_1, \ldots, x_l) \]

for \(w, w'\) such that \(\ell(w w') = \ell(w) + \ell(w')\).
Fix \( r \in \mathbb{Z}_{>0} \). For each \( l \in \mathbb{Z}_{>0} \), we put
\[
R_l = R_{w_l}(q^{r(l-1)}, q^{r(l-3)}, \ldots, q^{-r(l-1)});
\]
\[
V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}} \rightarrow V_{q^{r-(l-1)}} \otimes V_{q^{-r(l-3)}} \otimes \cdots \otimes V_{q^{r(l-1)}},
\]
where \( w_l \) is the longest element of \( \mathfrak{S}_l \). Then \( R_l \) is a \( U'_q(\mathfrak{g}) \)-linear homomorphism. Define
\[
V_l = \text{Im} \, R_l.
\]
Let us denote by \( W \) the image of
\[
R(q^r, q^{-r}) : V_{q^r} \otimes V_{q^{-r}} \rightarrow V_{q^r} \otimes V_{q^{-r}}
\]
and by \( N \) its kernel. Then we have
\[
(2.23) \quad V_l \text{ considered as a submodule of } V^{\otimes l} = V_{q^{r-(l-1)}} \otimes \cdots \otimes V_{q^{r(l-1)}}
\]
is contained in \( \bigcap_{i=0}^{l-2} V^{\otimes i} \otimes W \otimes V^{\otimes(l-2-i)} \).

Similarly, we have
\[
(2.24) \quad V_l \text{ is a quotient of } V^{\otimes l} / \sum_{i=0}^{l-2} V^{\otimes i} \otimes N \otimes V^{\otimes(l-2-i)}. \]

2.4. Preliminary propositions. In this subsection, following [KMN] we define a repolarization on \( V_l \) and prepare a necessary proposition to show the main theorem. First we recall

Lemma 2.3. Let \( M_j \) and \( N_j \) be \( U'_q(\mathfrak{g}) \)-modules and let \( ( \ , \ )_j \) be an admissible pairing between \( M_j \) and \( N_j \) (\( j = 1, 2 \)). Then the pairing \( ( \ , \ ) \) between \( M_1 \otimes M_2 \) and \( N_1 \otimes N_2 \) defined by \( (u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2) \) for all \( u_j \in M_j \) and \( v_j \in N_j \) is admissible.

Let \( V \) be a finite-dimensional \( U'_q(\mathfrak{g}) \)-module satisfying (2.21), (2.22). Suppose \( V \) has a polarization. The polarization on \( V \) gives an admissible pairing between \( V_x \) and \( V_{x^{-1}} \). Hence it induces an admissible pairing between \( V_{x_1} \otimes \cdots \otimes V_{x_l} \) and \( V_{x_1^{-1}} \otimes \cdots \otimes V_{x_l^{-1}} \).

Lemma 2.4. If \( x_j = x_{j+1}^{-1} \) for \( j = 1, \ldots, l \), then for any \( u, u' \in V_{x_1} \otimes \cdots \otimes V_{x_l} \), we have
\[
(u, R_{w_0}(x_1, \ldots, x_l)u') = (u', R_{w_0}(x_1, \ldots, x_l)u).
\]

By taking \( x_1 = q^{r(l-1)}, x_2 = q^{r(l-3)}, \text{ etc.}, \) we obtain the admissible pairing \( ( \ , \ ) \) between \( W = V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}} \) and \( W' = V_{q^{-r(l-1)}} \otimes V_{q^{r-(l-3)}} \otimes \cdots \otimes V_{q^{r(l-1)}} \) that satisfies
\[
(2.25) \quad (w, R_l(u')) = (w', R_l w) \quad \text{for any } w, w' \in W.
\]
This allows us to define a repolarization \( ( \ , \ )_l \) on \( V_l \) by
\[
(R_l u, R_l u')_l = (u, R_l u')
\]
for \( u, u' \in V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}} \).
Next we introduce a \( \mathbb{Z} \)-form of \( U'_q(\mathfrak{g}) \). Recall that \( A \) is the subring of \( \mathbb{Q}(q) \) consisting of rational functions without poles at \( q = 0 \). We introduce the subalgebras \( A_Z \) and \( K_Z \) of \( \mathbb{Q}(q) \) by

\[
A_Z = \{ f(q)/g(q) \mid f(q), g(q) \in \mathbb{Z}[q], g(0) = 1 \}, \\
K_Z = A_Z[q^{-1}].
\]

Then we have

\[
K_Z \cap A = A_Z, \quad A_Z/qA \simeq \mathbb{Z}.
\]

We then define \( U'_q(\mathfrak{g})_{K_Z} \) as the \( K_Z \)-subalgebra of \( U'_q(\mathfrak{g}) \) generated by \( e_i, f_i, q^h \) (\( h \in (P_\mathbb{Q})^* \)). Set \( V_{K_Z} = U'_q(\mathfrak{g})_{K_Z} u_0 \) and assume

\[
(V_{K_Z})_{\lambda_0} = K_Z u_0.
\]

Let us further set

\[
(V_{i})_{K_Z} = R_l((V_{K_Z})_{\otimes}) \cap (V_{K_Z})_{\otimes}.
\]

Then one can show

**Proposition 2.5.**

(i) \( (\ , \ )_l \) is a nondegenerate prepolarization on \( V_l \).

(ii) \( (R_l(u_0^{\otimes}), R_l(u_0^{\otimes}))_{l} = 1 \).

(iii) \( ((V_l)_{K_Z}, (V_l)_{K_Z})_l \subset K_Z \).

Let \( I \) be the index set of the vertices of the Dynkin diagram of \( \mathfrak{g} \) with the vertex \( 0 \) as in [Kac]. Let \( \mathfrak{g}_0 \) be the finite-dimensional simple Lie algebra whose Dynkin diagram is obtained by removing the 0-vertex from that of \( \mathfrak{g} \). Let \( P_+ \) be the set of dominant integral weights of \( \mathfrak{g}_0 \) and \( V(\lambda) \) be the irreducible highest weight \( U_q(\mathfrak{g}_0) \)-module of highest weight \( \lambda \) for \( \lambda \in P_+ \). The following proposition, which is essentially stated in Proposition 2.6.1 and 2.6.2 of [KMN], is a key to prove the main theorem.

**Proposition 2.6.** Let \( M \) be a finite-dimensional integrable \( U'_q(\mathfrak{g}) \)-module. Let \( (\ , \ ) \) be a prepolarization on \( M \), and \( M_{K_Z} \) a \( U'_q(\mathfrak{g})_{K_Z} \)-submodule of \( M \) such that \( (M_{K_Z}, M_{K_Z}) \subset K_Z \). Let \( \lambda_1, \ldots, \lambda_m \in P_+ \), and we assume the following conditions.

\[
\dim M_{\lambda_k} \leq \sum_{j=1}^m \dim V(\lambda_j)_{\lambda_k} \text{ for } k = 1, \ldots, m.
\]

\[
\text{There exist } u_j \in (M_{K_Z})_{\lambda_j} \text{ (} j = 1, \ldots, m \text{)} \text{ such that } (u_j, u_k) \in \delta_{jk} + qA, \text{ and } (e_iu_j, e_iu_j) \in qq_i^{-2(1+(h_i, \lambda_j))} A \text{ for any } i \in I \setminus \{0\}.
\]

Set \( L = \{ u \in M \mid (u, u) \in A \} \) and set \( B = \{ b \in M_{K_Z} \cap L/M_{K_Z} \cap qL \mid (b, b)_0 = 1 \} \). Here \( (\ , \ )_0 \) is the \( \mathbb{Q} \)-valued symmetric bilinear form on \( L/qL \) induced by \( (\ , \ ) \). Then we have the following.

(i) \( (\ , \ ) \) is a polarization on \( M \).

(ii) \( M \simeq \bigoplus_{j} V(\lambda_j) \) as \( U_q(\mathfrak{g}_0) \)-modules.

(iii) \( (L, B) \) is a crystal pseudobase of \( M \).
3. KR module of type $D$

3.1. KR module $W^1_1(k)$. First we review the Dynkin datum of type $D_n^{(1)}$. Let $I = \{0, 1, \ldots, n\}$ be the index set of the Dynkin diagram, $\{\alpha_i\}_{i \in I}$ the set of simple roots, $\{\Lambda_i\}_{i \in I}$ the set of fundamental weights. The standard null root $\delta$ is given by

$$\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n. \quad (3.1)$$

We denote the weight lattice by $P$, that is, $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$. The sublattice $\overline{P} = \bigoplus_{i \in I_0} \mathbb{Z}\Lambda_i$ can be viewed as the weight lattice for $D_n$. Here $I_0 = I \setminus \{0\}$ and $\overline{\Lambda}_i = \Lambda_i - a_i\Lambda_0$ with $a_i$ being the coefficient of $\alpha_i$ in $(3.1)$. It is sometimes useful to introduce an orthonormal basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ of $\mathbb{Q} \otimes \mathbb{Z}\overline{P}$ in such a way that we have

$$\alpha_i = \begin{cases} 
\epsilon_i - \epsilon_{i+1} & (i = 1, \ldots, n-1) \\
\epsilon_{n-1} + \epsilon_n & (i = n),
\end{cases} \quad \overline{\alpha}_i = \begin{cases} 
\epsilon_1 + \cdots + \epsilon_i & (i = 1, \ldots, n-2) \\
(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2 & (i = n-1) \\
(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2 & (i = n).
\end{cases} \quad (3.2)$$

Then we have $\alpha_0 = \delta - \epsilon_1 - \epsilon_2$. Since the lengths of the simple roots are all equal, we have $q_i = q$ for any $i \in I$. Hence we shall abbreviate $i$ from $[m]_i$ or $[m]_i!$.

Let $W^1_1(k)$ be the $k$-th fundamental representation of $U_q'(D_n^{(1)})$. It is known that it has the following decomposition into $U_q(D_n)$-modules.

$$W^1_1(k) \simeq \begin{cases} 
V(\overline{\Lambda}_k) \oplus V(\overline{\Lambda}_{k-2}) \oplus \cdots \oplus V(\overline{\Lambda}_1) & \text{if } 1 \leq k \leq n-2, \\
V(\overline{\Lambda}_k) & \text{if } k = n-1, n.
\end{cases} \quad (3.2)$$

On $W^1_1(k)$ the following results are known.

**Proposition 3.1.**

1. $W^1_1(k)$ is “good” in the sense of Kashiwara. In particular, it has a crystal base.
2. $W^1_1(k)$ has a polarization.

The first claim is due to Kashiwara [K2] and the second to Koga [K9], who got the result by exploiting the fusion construction among the spin representations.

3.2. Crystal of $W^1_1(k)$. We denote the crystal of $W^1_1(k)$ by $B^{k,1}$. We review in this subsection Schilling’s variation of Koga’s result on the crystal structure of $B^{k,1}$. First we treat the case of $k = 1$. The crystal graph of $B^{1,1}$ is depicted as follows.

```
1 −→ 2 −→ 3 −→ \cdots −→ n-2 −→ n-1 −→ n
0
1
```

Here for $b, b' \in B^{1,1}$, $b \rightarrow^i b'$ means $\tilde{f}_i b = b'$ (i.e., $b = \tilde{e}_i b'$).

Next in view of [K2] we recall the $U_q(D_n)$-crystal structure of $B(\overline{\Lambda}_i)$, the crystal of $U_q(D_n)$-module $V(\overline{\Lambda}_i)$, by [KN]. Consider the alphabet $\mathcal{A} = \{1, 2, \ldots, n, \overline{\Lambda}, n-1, \ldots, \overline{1}\}$ consisting of the crystal elements of $B^{1,1}$. It is given the following (partial)
order.

\[ 1 \prec 2 \prec \cdots \prec n - 1 \prec \frac{n}{n - 1} \prec \cdots \prec 1. \]

Then, for \( 1 \leq l \leq n - 2 \) \( B(\overline{\Lambda}_l) \) is identified with the set of columns

| \( m_1 \) |
| \( m_2 \) |
| \( \vdots \) |
| \( m_l \) |

of height \( l \) satisfying

\( m_j \neq m_{j+1} \) for \( j = 1, \ldots, l - 1 \),

(3.3)

if \( m_a = p \) and \( m_b = p \), then \( \text{dist}(p, p) \leq p \).

Here \( \text{dist}(p, p) = a + l + 1 - b \) if \( m_a = p \) and \( m_b = p \). The column tableau as above is also written as \( m_1m_2 \cdots m_l \). Note that we allow \( (m_j, m_{j+1}) = (n, \overline{n}) \) and \( (\overline{n}, n) \).

The actions of \( \tilde{e}_i \) and \( \tilde{f}_i \) are defined by

\[
\begin{align*}
B(\overline{\Lambda}_l) & \mapsto (B^{1,1})^{\otimes l} \\
m_1m_2 \cdots m_l & \mapsto m_1 \otimes m_2 \otimes \cdots \otimes m_l
\end{align*}
\]

and apply the tensor product rule of crystals on the r.h.s. \( B(0) \) is realized as \( \{ \phi \} \) with the trivial actions of \( \tilde{e}_i \) and \( \tilde{f}_i \) (\( i \in I_0 \)), that is, \( \tilde{e}_i \phi = \tilde{f}_i \phi = 0 \).

For \( 1 \leq k \leq n - 2 \), we are to represent \( B^{k,1} \) as the set of column tableaux of height \( k \) satisfying (3.3). By (3.2) \( B^{k,1} \) is the union of the sets corresponding to \( B(\overline{\Lambda}_l) \) with \( 0 \leq l \leq k \) and \( l \equiv k \) (mod 2). In [S] maps from \( B(\overline{\Lambda}_l) \) to column tableaux of height \( k \) were defined. If \( b \in B(\overline{\Lambda}_l) \), then fill the column of height \( l \) of \( b \) successively by a pair \((i_j, \overline{j})\) for \( 1 \leq j \leq (k - l)/2 \) in the following way to obtain a column of height \( k \). Set \( i_0 = 0 \). Let \( i_{j-1} \prec i_j \) be minimal such that

1. neither \( i_j \) or \( \overline{i_j} \) is in the column;
2. adding \( i_j \) and \( \overline{i_j} \) to the column we have \( \text{dist}(i_j, \overline{i_j}) \geq i_j + j \);
3. adding \( i_j \) and \( \overline{i_j} \) to the column, all other pairs \((a, \overline{a})\) in the new column with \( a > i_j \) satisfy \( \text{dist}(a, \overline{a}) \leq a + j \).

The filling map and \( \tilde{f}_i \) for \( i \in I_0 \) commute. Denote the filling map to height \( k \) by \( F_k \) or simply \( F \). Let \( D_k \) or \( D \), the dropping map, be the inverse of \( F_k \). Explicitly, given a one-column tableau of \( b \) of height \( k \), let \( i_0 = 0 \) and successively find \( i_j \geq i_{j-1} \) minimal such that the pair \((i_j, \overline{i_j})\) is in \( b \) and \( \text{dist}(i_j, \overline{i_j}) \geq i_j + j \). Drop all such pairs \((i_j, \overline{i_j})\) from \( b \). Thus we have

\[ B^{k,1} \simeq \bigoplus_{0 \leq l \leq k, \ l \equiv k (2)} F_k(B(\overline{\Lambda}_l)) \text{ as } U_q(D_n)\text{-crystals.} \]

It is the set of all column tableaux of height \( k \) satisfying (3.3) only.

We are left to give the rule of the actions of \( \tilde{e}_0 \) and \( \tilde{f}_0 \). For this purpose we need slight variants of \( F_k \) and \( D_k \), denoted by \( \tilde{F}_k \) and \( \tilde{D}_k \), respectively, which act on columns that do not contain \( 1, 2, \overline{2}, 3, \overline{3} \). On these columns \( \tilde{F}_k \) and \( \tilde{D}_k \) are defined by replacing \( i \mapsto i - 2 \) and \( \overline{i} \mapsto \overline{i - 2} \), then applying \( F_k \) and \( D_k \), and finally replacing \( i \mapsto i + 2 \) and \( \overline{i} \mapsto \overline{i + 2} \). The following proposition is given in [S].
Proposition 3.2. For $b \in B^{k,1}$, \[ c_0b = \begin{cases} F_k(\hat{D}_{k-2}(x)) & \text{if } b = 12x \\ \hat{F}_{k-1}(x) & \text{if } b = 12\mathcal{T} \\ \hat{F}_{k-1}(x) & \text{if } b = 12\mathcal{P} \\ \hat{F}_{k-2}(x) & \text{if } b = 12x2\mathcal{P} \\ F_k(\hat{D}_{k-1}(x)\mathcal{T}) & \text{if } b = 2x \\ x2\mathcal{P} & \text{if } b = 1x\mathcal{T} \text{ and } \hat{D}_{k-2}(x) = x \\ 0 & \text{otherwise} \end{cases} \]

\[ f_0b = \begin{cases} F_k(\hat{D}_{k-2}(x)) & \text{if } b = x2\mathcal{T} \\ 2\hat{F}_{k-1}(x) & \text{if } b = 2x2\mathcal{T} \\ \hat{F}_{k-1}(x) & \text{if } b = 1x2\mathcal{T} \\ 12\hat{F}_{k-2}(x) & \text{if } b = 12x2\mathcal{P} \\ F_k(2\hat{D}_{k-1}(x)) & \text{if } b = x\mathcal{T} \\ F_k(1\hat{D}_{k-1}(x)) & \text{if } b = x\mathcal{P} \\ 12x & \text{if } b = 1x\mathcal{T} \text{ and } \hat{D}_{k-2}(x) = x \\ 0 & \text{otherwise} \end{cases} \]

where $x$ does not contain $1, 2, \mathcal{T}, \mathcal{P}$.

3.3. Existence of crystal pseudobase for $W_l^{(k)}$. In this subsection we prove our main theorem by using Proposition 2.6. We prepare several lemmas and propositions.

Lemma 3.3. Let $R(x/y)$ be the $R$-matrix from $W_{1,x}^{(k)} \otimes W_{1,y}^{(k)}$ to $W_{1,y}^{(k)} \otimes W_{1,x}^{(k)}$. Then it has the following form.

\[ R(z) = F_{\mathcal{P}} + \frac{z - q^2}{1 - q^2z} P_{\mathcal{P}} + \cdots \]

Here $z = x/y, \mathcal{P} = \mathcal{P}^\mathcal{P}$ for $0 \leq j \leq n - 1$, and $P_{\mathcal{P}}$ stands for the projector onto the irreducible $U_q(D_n)$-module $V(\lambda)$ in $(W_{1}^{(k)})^\otimes$.

Proof. Let $u_0$ be a $U_q(D_n)$-highest weight vector of $W_{1}^{(k)}$ of highest weight $\mathcal{P}_k (= \mathcal{P})$. Since $V(\mathcal{P}_k + \mathcal{P}_{k-1})$ is multiplicity free, a unique highest weight vector up to a scalar is given by

\[ v = u_0 \otimes f_k u_0 - q f_k u_0 \otimes u_0. \]

Since $f_i u_0 = 0$ for $i \in \{1 \ldots (k-1)\}$, we have

\[ F^{(1)} v = u_0 \otimes F^{(1)} u_0 - q F^{(1)} u_0 \otimes u_0 \]

where $F^{(1)} = f_k \cdot \cdots f_n f_{n-2} f_{n-3} \cdots f_1 f_1 \cdots f_k$. Hence we have

\[ F^{(2)} v = q u_0 \otimes F^{(2)} u_0 - q F^{(2)} u_0 \otimes u_0 + (\text{unwanted terms}) \]

where $F^{(2)} = f_2 \cdots f_k F^{(1)}$ and we know $f_0(\text{unwanted terms}) = 0$ by weight consideration. Hence we have

\[ F^{(3)} v = q^{-1} y^{-1} u_0 \otimes F^{(3)} u_0 - q x^{-1} F^{(3)} u_0 \otimes u_0 \] on $V_x \otimes V_y$. 

where $F^{(3)} = f_0 F^{(2)}$ and $V = W_1^{(k)}$. Note that $F^{(3)}u_0 = \alpha u_0$ with some $\alpha \neq 0$, since the corresponding crystal element is not killed from Proposition 3.2. Thus we have

$$F^{(3)}v = \alpha(q^{-1} y^{-1} - qx^{-1})u_0 \otimes u_0.$$ 

Now let

$$R(z) \propto \varphi(z)P_2 \varphi_k + \varphi'(z)P_{\varphi_{k+1} + \varphi_{k-1}} + \cdots .$$

Then we have

$$R(z) F^{(3)}v = \alpha(q^{-1} y^{-1} - qx^{-1})\varphi(z)(u_0 \otimes u_0)$$

$$= F^{(3)} R(z)v = \varphi'(z) F^{(3)}v' = \alpha(q^{-1} x^{-1} - qy^{-1})\varphi'(z)(u_0 \otimes u_0).$$

Here by $v'$ we mean that it is considered to be in $V_y \otimes V_z$. Thus we have

$$\varphi'(z)/\varphi(z) = \frac{z - q^2}{1 - q^2 z}.$$

\[\square\]

Set $W = \text{Im } R(q^2), N = \text{Ker } R(q^2)$. They are $U'_q(D_n^{(1)})$-modules. Using the main result of [N1] one can show the following.

**Lemma 3.4.** We have

$$W \simeq W_2^{(k)}, \quad N \simeq \bigotimes_{j \sim k} W_j^{(j)}$$

as $U'_q(D_n^{(1)})$-modules. Here $j \sim k$ means that the corresponding vertices are tied by an edge in the Dynkin diagram. Moreover, both $W$ and $N$ are irreducible.

**Proof.** In [N1] it is shown that there exists an exact sequence of $U'_q(D_n^{(1)})$-modules

$$0 \to \bigotimes_{j \sim k} W_j^{(j)} \to W_1^{(k)} \otimes W_1^{(k)} \to W_2^{(k)} \to 0.$$ (An acute reader should have noticed that the exact sequence is different from [N1]. It is because the definition of the KR modules and the choice of the coproduct are different.) Moreover, it is also known that $\bigotimes_{j \sim k} W_j^{(j)}$ and $W_2^{(k)}$ are irreducible.

Set $W' = \bigotimes_{j \sim k} W_j^{(j)}$ and consider $N \cap W'$. Since $W'$ is irreducible, we have $N \cap W' = \{0\}$ or $W'$. Recall that $W_1^{(k)} \otimes W_1^{(k)}$ contains a unique irreducible $U'_q(D_n)$-module $V(\varphi_{k+1} + \varphi_{k-1})$. From the previous lemma and (3.2) we know it is contained both in $N$ and in $W'$. Hence we have $N \supset W'$. Now suppose $N \nsubseteq W'$. Then we have a surjective $U'_q(D_n^{(1)})$-linear map

$$W_1^{(k)} \otimes W_1^{(k)} / W' \to W_1^{(k)} \otimes W_1^{(k)} / N.$$

Since the l.h.s. is irreducible, $N = W'$ or $W_1^{(k)} \otimes W_1^{(k)}$. Since $N$ cannot be the second choice by the previous lemma. One obtains $N = W'$ and $W \simeq W_2^{(k)}$. \[\square\]

Since $W$ is known to be a KR module by the previous lemma, we have

**Lemma 3.5.** As a $U_q(D_n)$-module $W$ has the following decomposition.

$$W \simeq \bigoplus_{0 \leq m_1 \leq m_2 \leq \lfloor k/2 \rfloor} V(\bar{\alpha}_{k-2m_1} + \bar{\alpha}_{k-2m_2})$$
We set \( B = B^{k, 1} \). We fix a basis \( \{ v_I \}_{I \in B} \) of \( W_1^{(k)} \) in such a way that \( v_I \text{ mod } qL = I \) as an element of \( B \).

**Proposition 3.6.** \( N \) contains a vector of the form

\[
v_{I_1} \otimes v_{I_2} - \sum_{J_1 \otimes J_2 \in B_1} a_{J_1, J_2} v_{J_1} \otimes v_{J_2} \quad (a_{J_1, J_2} \in A)
\]

for any \( I_1, I_2 \) such that \( I_1 \otimes I_2 \in B^{\otimes 2} \setminus B_1 \).

See [4.2], [4.4] for the definition of \( B_1 \).

We now apply the fusion construction in section [2.3] to \( V = W_1^{(k)} \) with \( r = 1 \). The assumptions (2.21), (2.22) are valid with \( \lambda_0 = \Lambda_k \). (2.20) can also be checked. Other necessary properties are guaranteed by Proposition [3.1]. For \( l \in \mathbb{Z}_{>0} \) we define \( W_l^{(k)} = \text{Im } R_l \). Let \( k' = [k/2] \). Let \( c = (c_1, c_2, \ldots, c_{k'}) \) be a sequence of integers such that \( l \geq c_1 \geq c_2 \geq \cdots \geq c_{k'} \geq 0 \). For such \( c \) we define a vector \( u_m \) \((0 \leq m \leq k')\) in \( W_l^{(k)} \) inductively by

\[
u_m = (e_{c_{l-2m}}^m \cdots e_{c_3}^m e_{c_1}^m)(e_{c_{l-2m}+1}^m \cdots e_2^m e_2^m)u_{m-1},
\]

where \( u_0 \) here is \( u_0^{(1)} \) in \( (W_1^{(k)})^{(1)} \). Set \( u(c) = u_{c_{k'}} \). The weight of \( u(c) \) is given by

\[
\lambda(c) = \sum_{j=0}^{k'} (c_j - c_{j+1})\Lambda_{k-2j},
\]

where we have set \( c_0 = l, c_{k'+1} = 0 \). For \( l, m \in \mathbb{Z}_{\geq 0} \) such that \( m \leq l \) we define the \( q \)-binomial coefficient by

\[
\left[ \frac{l}{m} \right] = \frac{[l]!}{[m]![l-m]!}.
\]

The following proposition calculates values of prepolarsations necessary to prove the main theorem.

**Proposition 3.7.**

1. \((u(c), u(c))_l = \prod_{j=1}^{k'} q^{c_j(2l-c_j)} \left[ \frac{2l}{c_j} \right], \)

2. \((e_ju(c), e_ju(c))_l = 0 \) unless \( k-j \in 2\mathbb{Z}_{>0} \). If \( k-j \in 2\mathbb{Z}_{>0} \), then setting \( p = (k-j)/2 + 1 \), \((e_ju(c), e_ju(c))_l \) is given by

\[
q^{2l-c_p-1} \prod_{j=1}^{k'} q^{(c_j-\delta_{j,p})(2l-c_j)} \left[ \frac{2l-\delta_{j,p}}{c_j-\delta_{j,p}} \right].
\]

Proofs of these propositions are given in subsequent sections.

The rest of this section is devoted to the proof of Theorem [1.1]. From Proposition [2.6] it suffices to show

(i) \( \text{ch} W_{l}^{(k)} \leq \sum_{c_1 \geq \cdots \geq c_{k'} \geq 0} \text{ch} V(\lambda(c)) \), where \( V(\lambda) \) is the irreducible \( U_q(D_n) \)-module with highest weight \( \lambda \) and \( \text{ch} \) \( V \) stands for the formal character of \( V \).

(ii) \( (u(c), u(c'))_l \in \delta_{cc'} + qA \) and \((e_ju(c), e_ju(c'))_l \in q^{-1-2(h_j, \lambda(c))} A \) for \( j \neq 0 \).

Let us show (i). First notice that \( \sum_{c_1 \geq \cdots \geq c_{k'} \geq 0} \text{ch} V(\lambda(c)) = \sum_{0 \leq m_1 \leq \cdots \leq m_l \leq k'} \text{ch} V(\Lambda_{k-2m_1}, \cdots, \Lambda_{k-2m_l}) \). In view of (2.24) and Proposition [3.6], \( W_{l}^{(k)} \) is a quotient of a module generated by the set of vectors

\[
\{ v_{I_1} \otimes v_{I_2} \otimes \cdots \otimes v_{I_l} | I_j \otimes I_{j+1} \in B_1 \text{ for } j = 1, \ldots, l-1 \}.
\]
Assume $I_j \in B(\overline{\Lambda}_k-2m_{i+1}) \subset B$. Then $I_1 \otimes I_2 \otimes \cdots \otimes I_l$ belongs to
\[
\bigcap_{j=1}^{l-1} B(\overline{\Lambda}_k-2m_l) \otimes \cdots \otimes B(\overline{\Lambda}_k-2m_{j+2}) \otimes B(\overline{\Lambda}_k-2m_{j+1} + \overline{\Lambda}_k-2m_j) \\
\otimes B(\overline{\Lambda}_k-2m_{j-1}) \otimes \cdots \otimes B(\overline{\Lambda}_k-2m_1).
\]

However, the above crystal is known to be identified with $B(\overline{\Lambda}_k-2m_1 + \cdots + \overline{\Lambda}_k-2m_l)$ (see Proposition 2.2.1 of \cite{KN}). This fact verifies (i).

For the proof of (ii) note that
\[
[m] \in q^{1-m} A, \quad \left[ \begin{array}{c} m \\ n \end{array} \right] \in q^{-n(m-n)} A.
\]

If $c \neq c'$, $(u(c), u(c'))_l = 0$ since the weights of $u(c)$ and $u(c')$ are different. $(u(c), u(c'))_l \in 1 + qA$ by Proposition 3.7 (1). For the second part it suffices to notice that $\langle h_i, \lambda(c) \rangle \geq 0$. The proof is completed.

4. Proof of Proposition 3.6

We prepare several lemmas. The next one is a direct consequence of Proposition 3.2.

Lemma 4.1. Suppose $k \geq 2$. Set $k' = [k/2]$. For elements $b, b'$ in $B^{k,1}$ let $b \xrightarrow{\varepsilon_0} b'$ mean $\varepsilon_0 b = b'$. Then we have the following rules of $0$-actions. In (2)-(4) $\bullet$ stands for a crystal element whose explicit form is not used later.

\begin{enumerate}
\item
\[
\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & \varepsilon_0 & \vdots & \varepsilon_0 \\
\vdots & \vdots & \ddots & \vdots \\
k & k & 2 & 1
\end{array}
\]
\item Let $1 \leq m \leq k' - 1$.
\[
\begin{array}{cccc}
1 & 2 & \cdots & k- m-1 \\
2 & \vdots & & \varepsilon_0 \\
\vdots & \vdots & \ddots & \varepsilon_0 \\
k-m & \varepsilon_0 & k- m-1 & \varepsilon_0 \\
k- m & \cdots & & \bullet \\
k-2m+1 & k-2m+1 & 3 & 1
\end{array}
\]
\end{enumerate}
(3) Let \( 0 \leq m_1 \leq m_2 \leq k' - 1, m_1 \leq p \leq \min(m_{21}, m_s - 1) \), where \( m_{21} = m_2 - m_1, m_s = m_1 + m_2 \).

\[
\begin{array}{c|c}
1 & 2 \\
2 & \\
\vdots & \vdots \\
k - m_{21} - p & k - 2p + 1 \\
k - 2p + 1 & \varepsilon_0 \\
\vdots & \vdots \\
k - 2m_1 & k - m_{21} - p - 1 \\
k - m_{21} - p & \\
\vdots & \\
k - 2m_2 + 1 & \frac{p}{1}
\end{array}
\]

(4) Let \( 1 \leq m_1 \leq m_2 \leq k' - 1, m_{21} + 1 \leq p \leq m_2 \), where \( m_{21} = m_2 - m_1 \). Set \( m_s = m_1 + m_2 \).

\[
\begin{array}{c|c}
1 & 2 \\
2 & \\
\vdots & \vdots \\
k - m_s + p & k - m_s + p - 1 \\
k - m_s + p & \varepsilon_0 \\
\vdots & \vdots \\
k - 2m_1 + 1 & k - 2m_1 - 1 \\
k - 2p & k - 2p \\
\vdots & \\
k - 2m_2 + 1 & \frac{p}{1}
\end{array}
\]

**Lemma 4.2.** Let \( V \) be a \( U_q(g_0) \)-module with a crystal base \((L, B)\). Let \( W \) a submodule of \( V \). Let \( \{b_j \mid j \in I\} \subset B \). Suppose \( v \) is a vector in \( W \) such that \( v \equiv \sum_{j \in I} b_j \) \( \text{mod } qL \). Decompose \( I \) as \( I = I_1 \sqcup I_2 \) by

\[
I_1 = \{ j \in I \mid \hat{e}_i b_j = 0 \text{ for any } i \}, \quad I_2 = \{ j \in I \mid \hat{e}_i b_j \neq 0 \text{ for some } i \}.
\]

Then there exits a highest weight vector \( w \) in \( W \) such that \( w \equiv \sum_{j \in I_1} b_j \) \( \text{mod } qL \).

**Proof.** By applying \( \hat{f}_i \hat{e}_i \) we know that there exists a vector \( v' \) in \( W \) such that \( v' \equiv \sum_{j \in I'} b_j \) \( \text{mod } qL \), where \( I' = \{ j \in I_2 \mid \hat{e}_i b_j \neq 0 \} \). Hence there also exists a vector \( v'' \) in \( W \) such that \( v'' \equiv \sum_{j \in I''} b_j \) \( \text{mod } qL \). Continuing this with different \( i \)'s, we obtain a vector \( v''' \) in \( W \) such that \( v''' \equiv \sum_{j \in I''} b_j \) \( \text{mod } qL \). Hence we can write \( v''' \) as \( v''' = w + w' \) in such a way that \( w \equiv \sum_{j \in I_1} b_j \) \( \text{mod } qL \) is a highest weight vector and \( w' \equiv qL \) is not, but we can remove \( w' \) from \( v''' \).

In what follows in this section, by abuse of notation we represent a basis vector \( v_I \) in \( W_1^{(k)} \) also as \( I \).
Lemma 4.3. Let $0 \leq m \leq k$. A highest weight vector of $V(2\Lambda_{k-2m})$ in $W$ is given by

$$
\begin{array}{cccccc}
1 & 1 & & & & \\
2 & 2 & & & & \\
\vdots & \vdots & & & & \\
(k-m) & (k-m) & \otimes & \otimes & \mod qL. \\
\end{array}
$$

Proof. A highest weight vector of $V(2\Lambda_k)$ is given by

$$
\begin{array}{cccccc}
1 & 1 & & & & \\
2 & 2 & & & & \\
\vdots & \vdots & & & & \\
k & k & & & & \\
\end{array}
$$

Noting that $W$ is a submodule of $W_{1, q^{-1}} \otimes W_{1, q}$, apply $e_0^2$ and use Lemma 2.2 and 4.1 (1). We obtain

$$
\begin{array}{cccccc}
3 & 1 & 1 & & & \\
4 & 1 & 3 & 3 & 2 & 3 & 3 \\
\vdots & \vdots & \otimes & \otimes & \vdots & \otimes & \otimes & \mod qL \\
k & k & k & k & k & k \\
3 & 1 & 1 & & & \\
\end{array}
$$

as a vector in $W$. Apply further $e_{k-2}^{(2)} \cdots e_2^{(2)} e_1^{(2)} e_{k-1}^{(2)} \cdots e_3^{(2)} e_2^{(2)}$, one obtains

$$
\begin{array}{cccccc}
1 & 1 & & & & \\
2 & 2 & & & & \\
\vdots & \vdots & & & \mod qL \\
(k-1) & (k-1) & & & & \\
\end{array}
$$

as a vector in $W$. 

For \( m > 1 \), we prove by induction on \( m \). By Lemma 2.2 and 4.1 (2), one has

\[
\begin{array}{cccc}
1 & 1 & & \\
2 & 2 & & \\
\vdots & \vdots & & \\
\end{array}
\]

\[
\begin{array}{cccc}
e_{k-2m-2}^{(2)} \cdots e_2^{(2)} e_1^{(2)} e_{k-2m-1}^{(2)} \cdots e_3^{(2)} e_2^{(2)} e_0^{(2)} & k-m & \otimes & k-m \\
\vdots & \vdots & & \\
\end{array}
\]

\[
\begin{array}{cc}
k-2m+1 & k-2m+1 \\
\end{array}
\]

as a vector in \( W \). Lemma 4.2 completes the proof. \( \square \)

**Lemma 4.4.** Let \( 0 \leq m_1 \leq m_2 \leq k' \). Set \( m_{21} = m_2 - m_1, m_s = m_1 + m_2, M = \max(m_1, m_{21}) \). A highest weight vector of \( V(\Lambda_{k-2m_1} + \Lambda_{k-2m_2}) \) in \( W \) is given by

\[
(4.1) \sum_{p=m_1}^{M} \frac{k-p}{k-p} \otimes \frac{k-m_{21} - p}{k-2p+1} + \sum_{p=M+1}^{m_2} \frac{k-p}{k-p} \otimes \frac{k-m_s + p}{k-2m_1 + 1} \]

\mod qL.

**Proof.** We prove by induction on \( m_2 \). The case of \( m_2 = m_1 \) is proved in the previous lemma. Assume \( m_1 > 0 \). Apply \( e_{k-2m_2-2} \cdots e_1 e_{k-2m_2-1} \cdots e_2 e_0 \) to (4.1) and use Lemma 2.2 and 4.1 (2), (3), (4). Since one can always neglect terms corresponding to the crystal elements that are not killed by \( \tilde{e}_i \) for some \( i \neq 0 \) by Lemma 4.2 we
Lemma 4.5. Let \((\ , \ )\) be the admissible pairing between \(W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)}\) and \(W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)}\) induced from the polarization of \(W_{1}^{(k)}\). Then \(u \in N\) if and only if \((u, v) = 0\) for any \(v \in W\).

The following lemma is an easy consequence of (2.25) with \(l = 2\) and the non-degeneracy of the admissible pairing.

Lemma 4.5. Let \((\ , \ )\) be the admissible pairing between \(W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)}\) and \(W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)}\) induced from the polarization of \(W_{1}^{(k)}\). Then \(u \in N\) if and only if \((u, v) = 0\) for any \(v \in W\).

Now we are to prove Proposition 3.6. Set

\[ b(\lambda_{k-2m}) = 12 \cdots (k-m)(k-m) \cdots (k-2m+1) \in B. \]

\(b(\lambda_{k-2m})\) is a highest weight vector of \(V(\lambda_{k-2m})\) in \(W_{1}^{(k)}\). We define the following subsets of \(B^\otimes\).

\[ B_{1}^{h} = \{ b(\lambda_{k-2m}) \otimes b(\lambda_{k-2m}) \mid 0 \leq m_{1} \leq m_{2} \leq k' \}, \]

\[ B_{2}^{h} = \{ \text{elements appearing in the summand of } (4.1) \} \setminus B_{1}^{h}, \]

\[ B_{a} = \left( \bigcup_{i_{1}, \ldots, i_{m} \in I_{0}} \hat{f}_{i_{1}} \cdots \hat{f}_{i_{m}} B_{a}^{h} \right) \setminus \{ 0 \} \quad \text{for } a = 1, 2, \]

\[ B_{3} = B^\otimes \setminus (B_{1} \cup B_{2}), \quad B_{3}^{h} = \{ b \in B_{3} \mid \hat{c}_{i} b = 0 \text{ for any } i \neq 0 \}. \]
Let Lemma 5.1.

The claim follows from the fact that \((\epsilon^a, \epsilon_i) > l\) for some \(i \in \{1, \ldots, n\}\), then \(v = 0\).

Proof. The claim follows from the fact that \((\text{wt} \, u, \epsilon_i) \leq 1\) for a nonzero weight vector \(u\) in \(W_1^{(k)}\) and \(W_l^{(k)}\) is a subspace of \(W_1^{(k)}\). \(\square\)

The following formula will be used frequently.

\[
\binom{a}{b} = \min(a, b) 
\binom{(b-j)}{j} \left\{ \prod_{k=1}^{l} (q^1 - k t - q^{k-1} t^{-1}) / (q^k - q^{-k}) \right\},
\]

where \(\left\{ \binom{a}{j} \right\} = \prod_{k=1}^{l} (q^1 - k t - q^{k-1} t^{-1}) / (q^k - q^{-k})\).
Lemma 5.2.

\[ |u_m|^2 = q^{c_m(2l-c_m)} \begin{bmatrix} 2l \\ c_m \end{bmatrix} |u_{m-1}|^2. \]

Proof. Since the other case is similar, we prove when \( k \) is even. Using (2.20), we have

\[ |u_m|^2 = (e_{k-2m-1} \cdots e_1)(e_{k-2m+1} \cdots e_2) c_0 u_{m-1}, \]

By (5.1) we obtain

\[ f_{k-2m} u_m = \sum_j e_{k-2m-j} f_{k-2m-j} |j| (e_{k-2m-1} \cdots e_1)(e_{k-2m+1} \cdots e_2) c_0 u_{m-1}. \]

Note that \( wt f_{k-2m} = (e_{k-2m-1} \cdots e_1)(e_{k-2m+1} \cdots e_2) c_0 u_{m-1} = wt u_m - (2c_m - j) \alpha_{k-2m}. \) From Lemma 5.1 the summand in the r.h.s. of (5.2) becomes 0 unless \( j = c_m. \) Hence we have

\[ |u_m|^2 = |(e_{k-2m-1} \cdots e_1)(e_{k-2m+1} \cdots e_2) c_0 u_{m-1}|^2. \]

Similar calculations continue until we arrive at \( |u_m|^2 = |c_0 u_{m-1}|^2. \) Using (2.20), (5.1) and Lemma 5.1 again, we this time have \( |u_m|^2 = q^{c_m(2l-c_m)} \begin{bmatrix} 2l \\ c_m \end{bmatrix} |u_{m-1}|^2. \]

Lemma 5.3.

1. \( e_j u_{k'} = 0 \) when \( k \) is even, if \( j \geq k + 1 \) or \( j = 1 \) when \( k \) is odd.
2. \( |f_j u_p|^2 = q^{c_p(2l-1-c_p)} |c_p-1||c_p-1|^2 |u_p|^2 \) if \( j = k-2p+2, p = 1, \ldots, k'. \)
3. \( |f_j u_p|^2 = 0 \) if \( j = k-2p+1, p = 1, \ldots, k'. \)

Proof. (1) Write \( u_{k'} = E u_0. \) If \( j \geq k+1, e_j \) commutes with \( E. \) The claim follows from \( e_j u_0 = 0. \) When \( k \) is odd, \( e_1 u_{k'} = 0 \) follows from Lemma 5.1.

(2) When \( c_p = 0, \) the equality is shown as follows.

\[ |f_j u_p|^2 = |f_{k-2p+2} u_{p-1}|^2 \]

\[ = (u_{p-1}, q^{-1} t_{k-2p+2} e_{k-2p+2} f_{k-2p+2} u_{p-1}) \]

\[ = q^{c_{p-1}} |c_{p-1}| |u_{p-1}|^2. \]

Here we have used the relation \( e_{k-2p+2} u_{p-1} = 0, \) that can be confirmed by Lemma 5.1.

Now assume \( c_p > 0. \) Imitating the proof of Lemma 5.2 one obtains

\[ |f_j u_p|^2 = |(e_{k-2p+1} \cdots e_2)(c_p) c_0 \cdot f_{k-2p+2} u_{p-1}|^2. \]

Next we calculate

\[ q^{-c_p} t_{k-2p+1} f_{k-2p+1} (e_{k-2p+1} \cdots e_2)(c_p) c_0 \cdot f_{k-2p+2} u_{p-1} \]

\[ = q^{-c_p} \sum_j (e_{c_p-j}) f_{k-2p+1} (e_{k-2p+1} \cdots e_2)(c_p) c_0 \cdot f_{k-2p+2} u_{p-1}. \]
Since \(f_j^{(e_p-j)} \cdots e_2^{(e_p)} e_0^{(e_p)} f_{k-2p+1}^{(e_p)} f_{k-2p+2u_p-1}^{(e_p)} f_{k-2p+1}^{(e_p)} = l - 1 + c_p - j\), the summand of the above expression survives only when \(j = c_p, c_p - 1\). Noting \([c_{p-1}^p] = 0\), we obtain

\[|f_ju_p|^2 = |(e_0^{(e_p)} \cdots e_2^{(e_p)}) e_0^{(e_p)} f_{k-2p+1}^{(e_p)} f_{k-2p+2u_p-1}|^2.\]

Calculating similarly, one gets \(|f_ju_p-1|^2 = |e_0^{(e_p)} f_{k-2p+2u_p-1}|^2\). After removing \(e_0^{(e_p)}\), we arrive at

\[|f_ju_p|^2 = q^{e_p}(2l-1-e_p) \left[\frac{2l-1}{c_p}\right] |f_2 \cdots f_{k-2p+2u_p-1}|^2.\]

Calculating similarly, we obtain

\[|f_2 \cdots f_{k-2p+2u_p-1}|^2 = |f_{k-2p+2u_p-1}|^2 = q^{e_p-1}[c_{p-1}]n_{p-1}^2.\]

(3) The proof goes parallel to that of (2). When \(c_p = 0\),

\[|f_ju_p|^2 = |f_{k-2p+1}u_p|^2 = 0\]

from Lemma 5.1 Assume \(c_p > 0\). One obtains

\[|f_ju_p|^2 = |f_1 f_2 \cdots f_{k-2p+1}u_p|^2.\]

Noting that \(f_iu_p = 0\) for \(i = 1, \ldots, k - 2p + 1\), we have

\[f_1 f_2 \cdots f_{k-2p+1}u_{p-1/2} = f_1 f_2 \cdots f_{k-2p+1}e_0^{(e_p)} f_{k-2p+1}^{(e_p)} e_2^{(e_p)} e_0^{(e_p)} u_p - 1 = \alpha \cdot f_1 f_2 \cdots f_{k-2p+1}^{(e_p)} e_2^{(e_p)} e_0^{(e_p)} u_p - 1 = 0.\]

Here \(\alpha\) is a product of \(q\)-integers.

The proof is complete. \(\square\)

Now we are in a position to prove Proposition 5.2. (1) is a simple consequence of Lemma 5.2. (2) when \(j \geq k + 1\) is settled by Lemma 5.3. (1). To show when \(j \leq k\) note that

\[|e_ju_k| = u_k, q^{-l-j} f_j e_j u_k = q^{2\beta j} |f_ju_k|^2 + q^{\beta j}[\beta_j]n_k^2,\]

where

\[\beta_j = -\langle h_j, wt u_k \rangle = \begin{cases} \frac{c_{p-1}}{2} + 1 - \frac{c_{p+1}}{2} & \text{if } j \equiv k \ (2), \\ 0 & \text{if } j \not\equiv k \ (2). \end{cases}\]

Thus we are left to evaluate \(|f_ju_k|^2\). Examining the proof of Lemma 5.2 carefully, one notices that the same recursion formula is valid when \(m > p\), namely, one has

\[|f_ju_m|^2 = q^{e_m(2l-e_m)} \left[\frac{2l}{c_m}\right] |f_ju_{m-1}|^2\]

for \(m > p\).

The formula for \(|f_ju_k|^2\) is obtained from this, Lemma 5.3 (2) or (3) and Lemma 5.2. Calculating explicitly we obtain (2).

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