AN ESTIMATE FROM BELOW FOR THE BUFFON NEEDLE PROBABILITY OF THE FOUR-CORNER CANTOR SET

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Abstract. Let \( C_n \) be the \( n \)-th generation in the construction of the middle-half Cantor set. The Cartesian square \( K_n = C_n \times C_n \) consists of \( 4^n \) squares of side-length \( 4^{-n} \). The chance that a long needle thrown at random in the unit square will meet \( K_n \) is essentially the average length of the projections of \( K_n \), also known as the Favard length of \( K_n \). A classical theorem of Besicovitch implies that the Favard length of \( K_n \) tends to zero. It is still an open problem to determine its exact rate of decay. Until recently, the only explicit upper bound was \( \exp(-c \log^* n) \), due to Peres and Solomyak. (\( \log^* n \) is the number of times one needs to take log to obtain a number less than 1 starting from \( n \)). In [11] the power estimate from above was obtained. The exponent in [11] was less than 1/6 but could have been slightly improved. On the other hand, a simple estimate shows that from below we have the estimate \( \frac{c}{n} \). Here we apply the idea from [4], [1] to show that the estimate from below can be in fact improved to \( c \log n/n \). This is in drastic difference from the case of random Cantor sets studied in [13].

1. Introduction

The four-corner Cantor set \( K \) is constructed by replacing the unit square by four sub-squares of side length 1/4 at its corners, and iterating this operation in a self-similar manner in each sub-square. More formally, consider the set \( C_n \) that is the union of \( 2^n \) segments:

\[
C_n = \bigcup_{a_j \in \{0,3\}, j=1,\ldots,n} \left[ \sum_{j=1}^{n} a_j 4^{-j}, \sum_{j=1}^{n} a_j 4^{-j} + 4^{-n} \right],
\]

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and let the middle half Cantor set be
\[ C := \bigcap_{n=1}^{\infty} C_n. \]

(It can also be written as \( C = \{ \sum_{n=1}^{\infty} a_n 4^{-n} : a_n \in \{0,3\}\}. \) The four corner Cantor set \( \mathcal{K} \) is the Cartesian square \( C \times C \).

Since the one-dimensional Hausdorff measure of \( \mathcal{K} \) satisfies \( 0 < \mathcal{H}^1(\mathcal{K}) < \infty \) and the projections of \( \mathcal{K} \) in two distinct directions have zero length, a theorem of Besicovitch (see [3, Theorem 6.13]) yields that the projection of \( \mathcal{K} \) to almost every line through the origin has zero length. This is equivalent to saying that the Favard length of \( \mathcal{K} \) equals zero. Recall (see [2, p. 357]) that the Favard length of a planar set \( E \) is defined by
\[
\text{Fav}(E) = \frac{1}{\pi} \int_{0}^{\pi} |\text{Proj} R_\theta E| \, d\theta, \tag{1.1}
\]
where \( \text{Proj} \) denotes the orthogonal projection from \( \mathbb{R}^2 \) to the horizontal axis, \( R_\theta \) is the counterclockwise rotation by angle \( \theta \), and \( |A| \) denotes the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \). The Favard length of a set \( E \) in the unit square has a probabilistic interpretation: up to a constant factor, it is the probability that the “Buffon’s needle,” a long line segment dropped at random, hits \( E \) (more precisely, suppose the needle’s length is infinite, pick its direction uniformly at random, and

**Figure 1.** \( \mathcal{K}_3 \), the third stage of the construction of \( \mathcal{K} \).
then locate the needle in a uniformly chosen position in that direction, at distance at most $\sqrt{2}$ from the center of the unit square).

The set $K_n = C_n^2$ is a union of $4^n$ squares with side length $4^{-n}$ (see Figure 1 for a picture of $K_3$). By the dominated convergence theorem, $\text{Fav}(K) = 0$ implies $\lim_{n \to \infty} \text{Fav}(K_n) = 0$. We are interested in good estimates for $\text{Fav}(K_n)$ as $n \to \infty$.

A lower bound $\text{Fav}(K_n) \geq \frac{c}{n}$ for some $c > 0$ follows from Mattila [8, 1.4]. Peres and Solomyak [13] proved that

$$\text{Fav}(K_n) \leq C \exp\left[-a \log_* n\right] \quad \text{for all } n \in \mathbb{N},$$

where

$$\log_* n = \min \left\{ k \geq 0 : \log \log \ldots \log n \leq 1 \right\}.$$

This result can be viewed as an attempt to make a quantitative statement out of a qualitative Besicovitch projection theorem [2], [15], using this canonical example of the Besicovitch irregular set.

It is very interesting to consider quantitative analogs of Besicovitch theorem in general. The reader can find more of that in [15].

In [11] the following estimate from above was obtained

$$\text{Fav}(K_n) \leq \frac{C\tau}{n^\tau},$$

where $\tau$ was strictly less than $1/6$. This can be slightly improved, but it is still a long way till $\tau = 1$. Here we show, using the idea of [1], [4], that $\tau = 1$ is impossible.

**Theorem 1.** There exists $c > 0$ such that

$$\text{Fav}(K_n) \geq c \frac{\log n}{n} \quad \text{for all } n \in \mathbb{N}. \quad (1.2)$$

**Remark.** This result is somewhat surprising in light of the probabilistic result in [13]. There, the authors consider planar Cantor sets constructed randomly as follows. Starting from the unit square $U$, divide $U$ into four equal squares $U_1, U_2, U_3, U_4$. Similarly divide each of these into four squares $U_{j1}, U_{j2}, U_{j3}, U_{j4}$. For each $j$, randomly choose one square $U_{jk}$ (of side length $\frac{1}{16}$). The four chosen squares form the first level $\tilde{K}_1$. Repeat this process, always choosing the next generation randomly. The authors in [13] show that one expects

$$\frac{1}{Cn} \leq \text{Fav}(K_n) \leq \frac{C}{n}.$$
Proof. The proof is an immediate corollary of the idea of [4] if one applies the
duality between Cantor sets and Kakeya sets from [9]. As a “warm-up” we are
going to prove a much simpler estimate
\[ \text{Fav}(K_n) \geq \frac{C}{n} \quad \text{for all } n \in \mathbb{N}. \] (1.3)
This does not require [1], [4].

In what follows the square means only the Cantor square. Let \( L_\theta \) be the line
passing through the origin at an angle \( \theta \) with the x-axis. Let \( f_{n,\theta}(x) \) denote the
number of squares in \( K_n \) whose orthogonal projection onto the line \( L_\theta \) contains a
point \( x \) of this line. For each square \( Q \) of size \( 4^{-n} \) let \( \chi_{Q,\theta}(x) \) be the characteristic
function of the projection of \( Q \) onto \( L_\theta \). Then \( f_{n,\theta}(x) = \sum_{Q,\ell(Q)=4^{-n}} \chi_{Q,\theta}(x). \)
Therefore,
\[ \int \int f_{n,\theta}(x) dx \, d\theta \asymp 4^n \cdot 4^{-n} = 1. \] (1.4)

Let us denote the support of \( f_{n,\theta}(x) \) by \( E_{n,\theta} \), \( |E_{n,\theta}| \) being its length.
Knowing the first and second moment of \( f_{n,\theta}(x) \) we can estimate \( \int |E_{n,\theta}| \, d\theta \) by
using Cauchy inequality twice:
\[ 1 \asymp \int \int f_{n,\theta} dx \, d\theta \leq \int |E_{n,\theta}|^{\frac{1}{2}} \left( \int f_{n,\theta}^2(x) dx \right)^{\frac{1}{2}} d\theta \leq \]
\[ \left( \int |E_{n,\theta}| \, d\theta \right)^{\frac{1}{2}} \left( \int \int f_{n,\theta}^2(x) dx \, d\theta \right)^{\frac{1}{2}}. \]

Hence,
\[ \int |E_{n,\theta}| \, d\theta \geq c \frac{1}{\int \int f_{n,\theta}^2(x) dx \, d\theta}. \] (1.5)

Now
\[ \int \int f_{n,\theta}^2(x) dx \, d\theta = \sum_{Q,Q',\ell(Q)=\ell(Q')=4^{-n}} \int \int \chi_{Q,\theta}(x)\chi_{Q',\theta}(x) dx \, d\theta. \]
So for each pair \( P = (Q,Q') \), \( \ell(Q) = \ell(Q') = 4^{-n} \) (\( Q \) and \( Q' \) may coincide) we consider
\[ p_P := \int |\text{Proj}_\theta Q \cap \text{Proj}_\theta Q'| \, d\theta. \] (1.6)

Let us make an order on pairs. We call a pair \( P \) a \( k \)-pair if \( Q, Q' \) are in a \( 4^{-k} \)
square, but not in any \( 4^{-k-1} \)-square, \( k = 0, 1, \ldots, n \). We have \( 4^k \) of \( 4^{-k} \)-squares, so
we have \( \asymp 4^k \cdot (4^{n-k})^2 \) \( k \)-pairs. For each \( k \)-pair \( P \) we obviously have
\[ p_P \leq C 4^{-n} 4^{k-n}. \]
Putting this together we get
\[
\int \int f_{n, \theta}^2(x) dx \, d\theta = \sum_P p_P = \sum_{k=0}^n \sum_{P \text{ is a } k\text{-pair}} p_P
\]
\[
\leq C \sum_{k=0}^{n-1} \sum_{P \text{ is a } k\text{-pair}} 4^{-n} 4^{k-n} \leq C \sum_{k=0}^{n-1} 4^{k} (4^{n-k})^2 4^{-n} 4^{k-n} \leq C n .
\]
This estimate and (1.5) give us
\[
\int |E_{n, \theta}| \, d\theta \geq \frac{c}{n}.
\]

To prove (1.2) one needs to count pairs in a much more interesting way, which one gets from [1].

First we consider axis $0X$, where 0 is the origin and the axis has angle $\arctan \frac{1}{2}$ with the horizontal axis. We also need $0Y$, the orthogonal axis. Project original unit square on $0X$. We obtain the segment $I_0 := [0, L]$, $L = \sqrt{2} \cos (\frac{\pi}{4} - \arctan \frac{1}{2})$ on $0X$. Notice that projections of Cantor squares of size $4^{-k}$, $k = 0, ..., n$, generate the 4-adic structure on $I_0 = [0, L]$. Segments of these 4-adic structure will be called $I_\sigma$, where $\sigma$ is the word of length at most $n$ in the alphabet of $\{0, 1, 2, 3\}$.

We have $4^n$ points that are the projections of the centers of $4^n$ squares $Q$ of size $4^{-n}$. We will call this set $S$, and use the notation $s$ (maybe with indices) for elements of $S$. Each $s$ recovers its $Q_s$ uniquely. Let $y_s$ be the $0Y$ coordinate of the center of $Q_s$. Note that each $s$ is the center of an interval $I_\sigma$, and that the projections of all cubes $Q$ onto this axis are disjoint. This is an important feature of the argument.

Along with the usual Euclidean distance $|s_1 - s_2|$ between the points $s_1, s_2 \in S$, we have another very simple distance which will play the crucial role in proving (1.2). Namely,
\[
d(s_1, s_2) := \min \{|I_\sigma|, s_1 \in I_\sigma, s_2 \in I_\sigma\}.
\]
This is just the usual 4-adic distance scaled by $L$. Of course $|s_1 - s_2| \leq d(s_1, s_2)$.

For $j = 0, 1, ..., \log n$, $k \in [-n + j, 0]$, we call pair $P$ a $(j, k)$-pair, if
\[
\frac{|s_1 - s_2|}{|y_{s_1} - y_{s_2}|} \asymp 4^{-j}, |s_1 - s_2| \asymp 4^{-k-j}.
\]
Now the pair $P = (Q, Q')$ of squares of size $4^{-n}$ is just a pair $(s_1, s_2), s_i \in S$. 

For every \((j, k)\)-pair \(P = (s_1, s_2)\) one immediately has
\[
\|p\|_P \leq C \frac{1}{4^n} \cdot \frac{4^{-n}}{|y_{s_1} - y_{s_2}|}.
\] (1.7)

where \(p\) is as in \((1.6)\). Now we want to estimate the number \(A_{j,k}\) of all \((j, k)\)-pairs. If \((s_1, s_2)\) is a \((j, k)\)-pair, then
\[
|s_1 - s_2| \asymp 4^{-k-j}
\]

But also
\[
4^j |s_1 - s_2| \leq C |y_{s_1} - y_{s_2}|,
\]

and
\[
|y_{s_1} - y_{s_2}| \leq C' d(s_1, s_2).\] (1.8)

The last inequality is obvious but it is the most crucial for the proof!

This is because we just obtained \(d(s_1, s_2) \geq c 4^{-k}\). How many 4-adic intervals are such that \(d(s_1, s_2) \geq c 4^{-k} \geq 4^{-k-a}\) \((a\) is absolute), and \(|s_1 - s_2| \asymp 4^{-k-j}\)? Corresponding two 4-adic intervals of size \(4^{-n}\) should be both in \(C 4^{-k-j}\)-neighborhood of the 4-adic points of \(1, 2, 3, \ldots, k, k+1, \ldots, k+a\)-generations. We have \(4, 4^2, \ldots, 4^{k+a}\) such points correspondingly.

Therefore,
\[
A_{j,k} \leq C \sum_{m=0}^{k+a} 4^m \left( \frac{4^{-k-j}}{4^n} \right)^2 = C 4^{2n-k-2j}
\]

Another way to count the number of \((j, k)\) pairs is as follows.

Note that if \(j = 0\), there would be \(4^{2(n-k)} 4^k\) such pairs, since there are \(4^k\) intervals of length \(4^{-k}\), and each contains \(4^{2(n-k)}\) pairs of intervals of length \(4^{-n}\). Increasing \(j\) by 1 decreases the number of pairs by a factor of \(\frac{1}{4^n}\). One can see this by noting that if a pair \(s_1, s_2\), satisfies these conditions, then the 4-adic expansions of \(s_1\) and \(s_2\) are almost uniquely determined for \(j\) digits. Hence
\[
A_{j,k} \leq C 4^{2n-k-2j}.
\]

Using this and \((1.7)\) we get
\[
\sum_{p \in (j,k)-\text{pairs}} \|p\| \leq C 4^{2n-k-2j} \frac{4^{-2n}}{4^{-k}} \asymp 4^{-2j}.
\]

The union of all \((j, k)\)-pairs over all \(k\) is called: \(\mathcal{P}'_j\).
So fix \( j \), and get
\[
\sum_{p \in \mathcal{P}'_j} p P = 0 \sum_{k = -n+j}^0 \sum_{p \in \{j,k\}-\text{pairs}} p P \leq C \frac{n}{4^j}. \quad (1.9)
\]

Now let \( J_j := [c_1 4^{-j}, c_2 4^{-j}] \), where \( c_1 \) is sufficiently small and \( c_2 \) is sufficiently large. These are intervals of \textit{angles} \( \theta \) with respect to the axis \( 0X \), where zero angle means we are line parallel to the axis \( 0X \).

Here is a crucial geometric observation:

If \( P = (Q, Q') \), \( Q \neq Q' \) is so that
\[
\text{Proj}_\theta Q \cap \text{Proj}_\theta Q' \neq \emptyset, \theta \in J_j \quad \text{then} \quad P \in \mathcal{P}'_j. \quad (1.10)
\]

Let us throw into \( \mathcal{P}'_j \) also all \( (Q, Q) \) pairs. The resulting collection is called \( j \)-pairs: \( \mathcal{P}_j \). As
\[
\int_{J_j} |E_{n,\theta}| d\theta \geq c \left( \int_{J_j} f_{n,\theta} dx d\theta \right)^2 \int_{J_j} \int f_{n,\theta} dx d\theta,
\]
\[
\int_{J_j} \int f^2_{n,\theta} dx d\theta \leq \sum_{p \in \mathcal{P}_j} p P \leq C \frac{n}{4^j} + \int_{J_j} \int \sum_{p = (Q, Q), \ell(Q) = 4^{-n}} \chi_{Q,\theta}(x) dx d\theta \leq C \frac{n}{4^j} + C 4^{-j} 4^n 4^{-n} \leq C \frac{n}{4^j}, \quad (1.11)
\]
and
\[
\int_{J_j} \int f_{n,\theta} dx d\theta \leq C |J_j| \cdot 4^n \cdot 4^{-n} \asymp 4^{-j},
\]
we combine this to obtain
\[
\int_{J_j} |E_{n,\theta}| d\theta \geq c 4^{-2j} \frac{4^j}{n} = \frac{c}{n}. \quad (1.12)
\]

\textbf{Remark.} Notice that if \( (1.11) \) stops to be valid if \( j > \log_4 n + \text{Const} \). This explains why we did not get a better estimate from below than that in the Theorem.

Summing \( (1.12) \) over \( j = 0, \ldots, \log n \) we obtain \( (1.2) \). Theorem is completely proved.
2. Median value of $|E_{n,\theta}|$

**Question 1.** What is the median value of $|E_{n,\theta}|$?

Let us call this median value $M_n$. We can prove the following simple theorem, which immediately implies (1.3) of course.

**Theorem 2.** $M_n \geq \frac{c}{n}$.

**Proof.** We are going to prove

$$\int \frac{1}{|E_{n,\theta}|} d\theta \leq C n. \quad (2.1)$$

If one uses Tchebyshev’s inequality this immediately gives $M_n \geq \frac{c}{n}$.

To prove (2.1) we use [9]. Let us fix a small positive $\varepsilon$, and let $\mu_n$ be an equidistributed measure on $C_n$. Let $\text{Proj}_\theta$ stand (as always) for the orthogonal projection onto line $L_\theta$. Notice that given two points $z, \zeta \in C$ we have

$$\varepsilon |z - \zeta| \cong |\{\theta : |\text{Proj}_\theta(z) - \text{Proj}_\theta(\zeta)| \leq \varepsilon\}|.$$

Using this we write

$$\int \int \frac{\varepsilon}{|z - \zeta|} d\mu_n(z) d\mu_n(\zeta) \cong \int \int |\{\theta : |\text{Proj}_\theta(z) - \text{Proj}_\theta(\zeta)| \leq \varepsilon\}| d\mu_n(z) d\mu_n(\zeta)$$

Introduce

$$\Phi_{\varepsilon}(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \varepsilon \\ 0, & \text{otherwise} \end{cases}$$

Then we repeat

$$\int \int \frac{\varepsilon}{|z - \zeta|} d\mu_n(z) d\mu_n(\zeta) \cong \int \int \Phi_{\varepsilon}(|\text{Proj}_\theta(z) - \text{Proj}_\theta(\zeta)|) d\mu_n(z) d\mu_n(\zeta) d\theta =$$

$$\int \int \Phi_{\varepsilon}(|x - y|) d\mu_{n,\theta}(x) d\mu_{n,\theta}(y) d\theta,$$

where $d\mu_{n,\theta}$ is the projection of the measure $\mu_n$ on the line $L_\theta$. In our old notation

$$d\mu_{n,\theta} = f_{n,\theta}(x) dx. \quad (2.2)$$

Of course

$$\int \Phi_{\varepsilon}(|x - y|) d\mu_{n,\theta}(y) = \mu_{n,\theta}(B(x, \varepsilon)),$$

and finally we get

$$\int \int \frac{1}{|z - \zeta|} d\mu_n(z) d\mu_n(\zeta) \geq c \int \int \frac{\mu_{n,\theta}(B(x, \varepsilon))}{\varepsilon} d\mu_{n,\theta}(x) d\theta. \quad (2.3)$$
The left hand side is $\leq Cn$. One can see this by noting that for each square $Q$ of side length $4^{-n}$ in $\mathcal{K}_n$, and for each $k = 0, 1, \ldots, n$, there are $4^{n-k}$ squares $Q^i$ at distance $4^{-k}$.

In (2.3) we now use Fatou’s lemma:

$$\int \liminf_{\varepsilon \to 0} \frac{\mu_{n,\theta}(B(x, \varepsilon))}{\varepsilon} d\mu_{n,\theta}(x) d\theta \leq Cn.$$  \hfill (2.4)

Recalling (2.2) we obtain

$$\int \int_{E_{n,\theta}} f_{n,\theta}(x)^2 dx d\theta \leq Cn.$$  \hfill (2.5)

Recalling (1.4) we can rewrite it as

$$\int \frac{\int_{E_{n,\theta}} f_{n,\theta}(x)^2 dx}{(\int_{E_{n,\theta}} f_{n,\theta}(x) dx)^2} d\theta \leq Cn.$$  \hfill (2.6)

By Cauchy inequality

$$\frac{1}{|E_{n,\theta}|} \leq \frac{\int_{E_{n,\theta}} f_{n,\theta}(x)^2 dx}{(\int_{E_{n,\theta}} f_{n,\theta}(x) dx)^2}.$$  \hfill (2.6)

Combine this and (2.6) and obtain the desired estimate

$$\int \frac{1}{|E_{n,\theta}|} d\theta \leq Cn.$$  \hfill (2.6)

Inequality (2.1) and, therefore, Theorem 2 are completely proved.

\[ \square \]

3. Sierpiński’s Cantor set

Consider now another Cantor set, which, by analogy with Sierpiński’s gasket, we call Sierpiński’s Cantor set $\mathcal{S}$. We take an equilateral triangle with side length 1, leave 3 triangles of size $1/3$ at each corner, and then continue this for $n$ generations. On step $n$ we get $3^n$ equilateral tringles of size $3^{-n}$. Call this union of triangles $\mathcal{S}_n$. Its intersection is $\mathcal{S}$,

$$0 < H^1(\mathcal{S}) < \infty,$$

and this is a Besicovitch irregular set, so, by Besicovitch projection theorem (see [10])

$$\zeta_n := \int |S_{n,\theta}| d\theta \to 0, \ n \to \infty.$$  \hfill (2.6)

Question 2. What is the order of magnitude of $\zeta_n$?

This is the same question, which we had for 4-corner Cantor set.
Absolutely the same reasoning as above proves

**Theorem 3.**

\[ \zeta_n = \int |S_{n, \theta}| d\theta \geq \frac{c \log n}{n}. \]

In fact, projection of the triangles on the base side generate 3-adic lattice on the base side. Then we notice that (1.8) and (1.10) hold now as well. The proof is the same after these observations.

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