A toy model for the dynamical discrepancies on galactic scales

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ABSTRACT
In this study a simple toy model solution to the missing gravity problem on galactic scales is reverse engineered from galactic data via imposing broad assumptions. It is shown that the toy model solution can be written in terms of baryonic quantities, is highly similar to pseudo-isothermal dark matter on galactic scales and can accommodate the same observations. In this way, the toy model solution is similar to MOND modified gravity in the Bekenstein–Milgrom formulation. However, it differs in the similarity to pseudo-isothermal dark matter and in the functional form. In loose terms, it is shown that pseudo-isothermal dark matter can be written in terms of baryonic quantities. The required form suggests that it may be worth looking into a mechanism that can increase the magnitude of the post-Newtonian correction from general relativity for low accelerations.

Key words: dark matter – gravitation – galaxies: general.

1 INTRODUCTION
Since the 1930s astronomical observations have made it increasingly clear that the dynamical behaviour of the Universe is poorly understood. These observations include the virialization of the Coma cluster (Zwicky 1933), rotation curves (Bosma 1978; Rubin, Thonnard & Ford 1980), lensing of galaxy clusters (Mellier, Ford & Soucail 1989), observation of cluster mergers (Clowe et al. 2006), and large-scale structure surveys (Dodelson & Ligori 2006). The discrepancy between the observed and predicted dynamics can be remedied by introducing a significant, as of yet unobserved mass component (dark matter) or modifying the theory responsible for the predicted dynamics – with the former being the vastly more explored option.

Analyses of rotation curve data have historically been the source of much debate (e.g. de Blok & McGaugh 1998; Sanders 2003; Sanders & Noordermeer 2007; McGaugh, Lelli & Schombert 2016; Petersen & Frandsen 2017; Frandsen & Petersen 2018; Kroupa et al. 2018; Li et al. 2018; McLaugh et al. 2018; Rodrigues et al. 2018a, b; Chang & Zhou 2019; Petersen 2019; Sanders 2019; Tian & Ko 2019) since Tully & Fisher (1977) inferred a close relation between the visible baryonic matter and behaviour of galaxies at large radii – supposedly dominated by dark matter. The debate spawned the idea of modified classical dynamics, introduced by Milgrom (1983), as an alternative to the dark matter hypothesis. Milgrom coined the non-relativistic modification of gravitational dynamics, relevant on galactic scales, modified Newtonian dynamics (MOND). He defined two classes, namely MOND modified inertia and MOND modified gravity (Milgrom 1983). The former model class was reinvigorated after McGaugh et al. (2016) and Lelli et al. (2017) found in their analysis of rotation curve data from the SPARC data base (Lelli, McGaugh & Schombert 2016), that data from it globally follow an analytical relation (dubbed the RAR) that carries the features of MOND modified inertia. The results of McGaugh et al. (2016) and Lelli et al. (2017) even inspired dark matter model builders to reproduce the RAR, with the unique characteristics of MOND modified inertia, within the dark matter hypothesis e.g. Chashchina, Foot & Silagadze (2017), Edmonds et al. (2017), Dai & Lu (2017), Cai, Liu & Wang (2018), and Berezhiani, Famaey & Khoury (2018). Subsequently, the discussion surrounding MOND modified inertia has continued, with Li et al. (2018) coming out with evidence in support and Petersen & Frandsen (2017), Frandsen & Petersen (2018), and Petersen (2019) the contrary.

As the solution to the missing gravity problem continues to elude physicists, good ideas continue to grow in importance and new inspirations become of increasingly greater value. In this article, a toy model is derived via reverse engineering from galactic rotation curve data. The purpose of the toy model is to inspire dark matter and modified gravity model builders by highlighting a curious connection between the toy model and the baryonic mass components on galactic scales and analysing what this could mean.

2 FORMULATING THE MISSING GRAVITY PROBLEM FOR ROTATION CURVES
In relation to rotation curves, the missing gravity problem can be neatly formulated analytically via Newtonian mechanics as follows: solving the Poisson equation outside a given mass distribution (i.e. solving the Laplace equation; \( \nabla^2 \Phi = 0 \)) in cylindrical coordinates yields the potentials

\[
\Phi_k = c(k)e^{-|z|}J_0(kr),
\]

where \( c(k) \) is a constant of integration.
where \( J_0(kr) \) is the Bessel function of the first kind of zeroth order and the coordinate system is arranged such that the \( z \)-axis is perpendicular to the galactic plane. These solutions are valid outside a given mass distribution and are such not mass specific. The mass specific solution is developed with \( \Phi_i \) as basis functions. This leads to the total (abbreviated ‘tot’) centripetal acceleration

\[
g_{\text{tot}} = - \int dk S(k) \nabla \Phi_i, \tag{2}
\]

where \( S(k) \) is a weight function to be determined by the mass distribution. Assuming the mass is distributed in a plane (Binney \\& Tremaine 2008),

\[
\Sigma(r) = - \frac{1}{2\pi G_N} \int dk S(k) c(k) J_0(kr). \tag{3}
\]

The procedure in Newtonian mechanics is then to assume that \( \Sigma(r) \) is the mass distribution of the baryonic matter only. By doing so \( S(k)c(k) \), can be determined via Hankel transformation. The existence of dark matter or some modification of Newtonian gravity can be inferred when it is concluded that the Newtonian model does not agree well with data at large radii. In particular, \( v_\text{rot}(r) \sim \text{const} \) is required by observations (Bosma 1978; Rubin et al. 1980) while \( v_N(r) \sim r^{-\frac{3}{2}} \) at large radii from Newtonian mechanics. Hence, it is clear that, in this formalism, \( \Sigma(r) \) cannot only be the visible (baryonic) mass. \( \Sigma(r) \) should be a sum of the baryonic mass as well as some additional mass term, \( \Sigma_m \), where ‘m’ signifies ‘missing’ mass. This missing mass term can be interpreted as either arising from dark matter or from some modification of Newtonian gravity.

The total surface density can then be written as

\[
\Sigma_{\text{tot}}(r) = \Sigma_N(r) + \Sigma_m(r) \Rightarrow g_{\text{tot}} = g_N + g_m. \tag{4}
\]

### 3 THE TOY MODEL

The toy model consists of an analytical expression for the radial component (Notation: \( g_{\text{rot}} = \nabla |\mathbf{r} - \mathbf{d}| = \frac{\partial}{\partial |\mathbf{r} - \mathbf{d}|} \)) of \( g_m \) derived via reverse engineering from galactic rotation curve data. The reverse engineering process itself consists of three broad assumptions motivated by the data:

(i) \( v_{\text{rot}}(r) \sim \text{const} \) at large radii (Bosma 1978; Rubin et al. 1980).
(ii) \( g_{\text{rot}} \) should not diverge at \( r \rightarrow 0 \).
(iii) \( v_{\text{rot}} \sim v_N \) at small radii (Petersen 2019), corresponding also to the maximum disc approximation (e.g. van Albada et al. 1985).

All three assumptions are rather uncontroversial, albeit the third one less so. The third assumption represents the maximum disc approximation or equivalently the cored dark matter profiles. The slight controversy of this assumption does not persist in the existing galaxies which uphold this assumption, but in the assumption that all galaxies do. Petersen (2019) recently motivated this assumption in an analysis of the gas dominated galaxies of the SPARC data base and indicated that this might in fact be the case in all galaxies.

Assumption 1 can be accommodated by requiring \( g_{\text{rot}} \sim r^{-1} \) at large radii. In order to also uphold assumption 2, a first ansatz could be

\[
g_{\text{rot}}(r) = 1 - e^{-\frac{z}{r_0}}, \tag{5}
\]

where \( r_0 \) is the scale length of the missing mass and \( g_{\text{rot}} \) is evaluated at \( z = 0 \) since this denotes the galactic plane to which the rotation curve data refer. Equation (5) has \( \lim_{r \to 0}(g_{\text{rot}}) \neq 0 \), which violates assumption 3. A re-evaluated ansatz could therefore be

\[
g_{\text{rot},i}(r) = \Phi_m^0 \left[ 1 - e^{-\frac{z}{r}} - e^{-\frac{z}{r_m}} \right], \tag{6}
\]

where \( \Phi_m^0 \) is a constant of proportionality that is in general a function of scale length and mass. Equation (6) is not unique in that there exist other parametrizations that uphold the three assumptions. What makes this particular parametrization interesting is its relation to the baryonic matter. To uncover this relation, assume that \( g_{\text{rot}} \) exists in the same function space as \( g_{N,r} \). This is equivalent to taking

\[
S(k)c(k) = S_N(k) + S_m(k) \Rightarrow g_{\text{rot}} = - \int_{\infty}^{\infty} dk S_m(k) J_1(kr)e^{-|z|}. \tag{7}
\]

where \( S_N \) is the weight function obtained from the baryonic mass distribution using Newtonian dynamics and \( J_1(kr) \) is the Bessel function of first kind of first order. Taking \( z = 0 \) in equation (7), using equation (6) and Hankel transforming reveals

\[
S_m(k) = -\Phi_m^0 \int_{0}^{\infty} dr' r' J_1(kr') \left[ \frac{1 - e^{-\frac{z}{r'}}}{r'} - \frac{e^{-\frac{z}{r_m}}}{r_m} \right]
= - \frac{1}{(1 + (kr_m)^2)^{\frac{3}{2}}} \frac{\Phi_m^0}{k}. \tag{8}
\]

Compare this to the weight function for an exponential baryonic disc (Binney \\& Tremaine 2008):

\[
S_\text{b}(k) = - \frac{G_N m_d}{(1 + (kr_d)^2)^{\frac{3}{2}}}, \tag{9}
\]

where \( r_d \) and \( m_d \) are the scale length and mass of the baryonic disc, respectively. The difference in powers of \( k \) between equations (8) and (9) can be captured by an indefinite integral over \( z \).

\[
g_{\text{rot},i} = g_{N,r} - \frac{\Phi_m^0}{G_N m_d} \int d|z| \partial_i \Phi_d(r, z, r_m). \tag{10}
\]

In order for assumption 1 to appear from data, both the magnitude of \( g_{\text{rot}} \) and the involved scale lengths must – to some degree – be fine tuned towards the baryonic ones. Hence, as an ansatz it is reasonable to let \( r_m \rightarrow r_d \) such that

\[
\frac{1}{r_c} \Rightarrow \frac{\Phi_m^0}{G_N m_d}. \tag{12}
\]

By modelling the baryonic disc and gas as exponential discs (Binney \\& Tremaine 2008), a sum can be introduced into equation (11) in a straightforward manner. In cases where there is no baryonic bulge, then

\[
\sum_i \int d|z| \partial_i \Phi_d = \int d|z| g_{N,r,i}.
\]

A priori there is no reason why the missing mass term should not depend on the baryonic bulge. However, introducing a baryonic bulge is consistent with assumptions 1–3. The only possible impact of a bulge is in removing a bulge component can be consistently added allowing equation (11) to be written

\[
g_{\text{rot},i} = g_{N,r,i} - \frac{1}{r_c} \int d|z| g_{N,r,i}. \tag{13}
\]
where
\[ g_{N,r} = \sum_i \partial_r \Phi_{d}^i + \partial_t \Phi_b \]
with \( \lim_{|z|\to\infty} (\int |z|g_{N,r}) = 0 \) and \( \lim_{|z|\to0} (\int |z|\partial_t \Phi_b) = 0 \) imposed.

4 COMPARING THE TOY MODEL TO ROTATION CURVE DATA

In comparing the toy model to data, the geometry in \((g_{N,r}, g_{N,\rho z})\)-space \((g_2)\) will be investigated. \(g_2\)-space – used by Milgrom (1983) to write the now famous mass discrepancy acceleration relation (MDAR) – presents several advantages compared to more conventional spaces (e.g. the classical \(v(r)\) plane) in that it highlights the details of the solution to the missing gravity problem. For example, the difference in the \(v(r)\) plane for a Navarro-Frenk-White (NFW) dark matter density profile and a pseudo-isothermal dark matter density profile, both fitted to data, can be difficult to see. In \(g_2\)-space however, the difference is very explicit and clear (Fig. 2).

There exist various different models of the baryonic matter in rotationally supported galaxies. The most common are (Sofue 2013):

(i) Exponential disc and de Vaucouleurs bulge with surface mass densities
\[
\Sigma_d(r) = \Sigma_0 e^{-r/\tilde{r}_d},
\]
\[
\Sigma_b(r) = \Sigma_0 e^{-\left(\frac{r}{\tilde{r}_b}\right)^{4/3}},
\]
where \(\kappa = 7.6695\) (Binney & Tremaine 2008).

(ii) Miyamoto–Nagai disc and Plummer bulge with potentials
\[
\Phi_d(r, z) = -\frac{G N m_d}{\sqrt{r^2 + (d_d + z)^2}},
\]
\[
\Phi_b(r, z) = -\frac{G N m_b}{\sqrt{r^2 + z^2 + \tilde{r}_b}},
\]
where both potentials are in cylindrical coordinates and \(d_d, \tilde{r}_b\) are scale lengths.

In relation to the geometry in \(g_2\)-space, it is interesting to note the differences introduced by the bulge profiles. Even though the centripetal acceleration of the de Vaucouleurs bulge, \(g_{N,\rho z}^{\text{dV}}\), vanishes for both \(r \to 0\) and \(r \to \infty\), the maximum of \(g_{N,\rho z}^{\text{dV}}\) is located at incredibly small radius (few pc). This radius is well beyond conventionally sampled radii (Lelli et al. 2016) and so for the sampled radii a de Vaucouleurs-like bulge will cause \(g_{N,\rho z}\) to increase towards small radii and thus force the \(g_2\)-space geometry to be single valued (see Fig. 1). Similarly to the de Vaucouleurs bulge, the centripetal acceleration from the Plummer bulge, \(g_{N,\rho z}^{\text{Pl}}\), vanishes for both \(r \to 0\) and \(r \to \infty\). However, contrary to the de Vaucouleurs bulge, the maximum of \(g_{N,\rho z}^{\text{Pl}}\) is located at radii often sampled by data (few kpc). This means the \(g_2\)-space in this case is double valued in the range of sampled radii (see Fig. 1). Without bulges, the exponential disc and the Miyamoto–Nagai disc gives rise to similar, \(\sim\)elliptical geometries in \(g_2\)-space – with the latter a bit more ‘squeezed’ than the former (see Fig. 1).

In relation to the \(g_2\)-space geometry seen in the SPARC data base (Lelli et al. 2016; Frandsen & Petersen 2018; Petersen 2019), it is interesting to note that the majority of galaxies\(^1\) (121/152) have no bulge. Hence, the overall \(g_2\)-space geometry is expected to be dominated by the disc geometries shown in Fig. 1 – this is indeed what is found in Petersen & Frandsen (2017) and Petersen (2019) and, to a lesser extent, in Frandsen & Petersen (2018). In Frandsen & Petersen (2018) and Petersen (2019) it is shown that galaxies can in general be grouped according to whether data for the \(g_2\)-space geometry curve leftward \((r_{\text{tot}} < r_N)\), rightward \((r_{\text{tot}} > r_N)\), or nowhere \((r_N = r_{\text{tot}})\) for decreasing radius, where \(r_N\) and \(r_{\text{tot}}\) denote the radii corresponding to the maxima in \(g_N\) and \(g_{N,\rho z}\), respectively. Frandsen & Petersen (2018) show that larger subsets curve rightward (73/152 galaxies) and nowhere (46/152 galaxies) whereas a smaller subset curves leftward (33/152 galaxies), leading to an overall geometry ~ nowhere with a slight rightward inclination. Petersen (2019) show that gas-dominated galaxies – which to leading order have \(g_2\)-space geometries independent of the mass-to-light ratios – overall display a pronounced rightward geometry, indicating that leftward (and possibly nowhere) galaxies could be an artefact of an underlying radial dependence of the mass-to-light ratios (e.g. as suggested by Kroupa et al. 2018) not accounted for in Frandsen & Petersen (2018). Equation (13) relies on this assumption as it cannot accommodate leftward or nowhere geometries for which data are sampled at \(r < r_N\) (see Fig. 1). Via de Vaucouleurs-like bulges, equation (13) can accommodate nowhere geometries which are not sampled at \(r < r_N\). The toy model can be adjusted to accommodate all types of geometries by introducing more complexity, but this makes the connection with the baryonic

\(^1\)Here the galaxy selection criteria of Frandsen & Petersen (2018) and Petersen (2019) are applied, bringing the number of galaxies down to 152 from the 175 listed in the data base.
matter less pronounced. Since the need for additional complexity is still under debate, further considerations in this direction are beyond the scope of our toy model.

5 COMPARING THE TOY MODEL TO DARK MATTER MODELS

Another point to make is in comparing the g2-space geometry of equation (13) to that of common dark matter models. Here, the pseudo-isothermal and NFW density profiles for dark matter will be considered.

\[
\rho_{\text{iso}}(r, z) = \frac{\rho_0}{1 + \left(\frac{\sqrt{r^2 + z^2}}{R_0}\right)^2}, \\
\rho_{\text{NFW}}(r, z) = \frac{\rho_1}{\sqrt{\frac{r^2 + z^2}{R_1}} \left(1 + \frac{\sqrt{r^2 + z^2}}{R_1}\right)^2}.
\]

(17)

Fig. 2 illustrates the g2-space geometry of equation (13) and the dark matter distributions of equation (17). It is clear that the primary differences between the models are present at small radii – as discussed in Petersen & Frandsen (2017), Frandsen & Petersen (2018), Petersen (2019), and Tian & Ko (2019). In the absence of a bulge, the solution to the missing gravity problem dictates the geometry at small radii. Finer details of the solution are clearly visible, as exemplified by the difference in geometry between the NFW and pseudo-isothermal dark matter geometries (without bulges). Given how sensitive the g2-space geometry is to the finer details of the solution to the missing gravity problem, it is remarkable how similar the geometry of equation (13) is to that of pseudo-isothermal dark matter.

6 SUMMARY AND DISCUSSION

In this study, it has been shown how a toy model solution to the missing gravity problem (equation 6) can be reverse engineered from galactic rotation curve data. It has also been shown that this solution can be written in terms of the Newtonian centripetal acceleration from the baryonic matter only (equation 13). The details of the toy model have been discussed in the context of its g2-space geometry and the results of Frandsen & Petersen (2018), Petersen & Frandsen (2017), and Petersen (2019) obtained from the SPARC data base (Lelli et al. 2016). The g2-space geometry highlights the finer details of the proposed solution to the missing gravity problem and as such provides an appropriate platform to discuss the details of a solution based on rotation curve data. The toy model is found to accommodate the most common g2-space geometries of the SPARC data base, but not geometries which are consistent with some form of cuspy dark matter (nowhere geometries sampled at \( r < r_N \) or leftward geometries). However, as recently proposed by Petersen (2019) such geometries could be an artefact from a radial dependence of the mass-to-light ratios not accounted for in Frandsen & Petersen (2018). In Petersen (2019) it is found that gas dominated galaxies follow a pronounced rightward geometry, consistent with some form of cored dark matter. Accepting this possibility, the toy model is consistent with all geometries present in the SPARC data base.

Lastly, the g2-space geometry of the toy model has been compared to those of NFW and pseudo-isothermal dark matter. This comparison shows a striking similarity between the toy model and isothermal dark matter to such a degree that one might consider the two loosely degenerate in terms of g2-space geometries. Extending...
this line of thought and collecting the results, the toy model shows that pseudo-isothermal dark matter can loosely be written in terms of Newtonian quantities derived from the baryonic matter (as far as the g2-space geometry is considered). This curious connection is an important take away from this study and should be viewed as a constraint for model builders. The toy model also suggest a low-acceleration departure from the predictions of general relativity, as also predicted by e.g. MOND modified gravity (Brada & Milgrom 1995). However, whereas MOND modified gravity relies on an interpolation function, the toy model does not. A curious note is that the functional form of the toy model (equation 13) is reminiscent of what is obtained from the post-Newtonian correction to general relativity, indicating that a mechanism to increase the magnitude of the post-Newtonian correction from general relativity for low accelerations may be worth considering.

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REFERENCES

Berezhiani L., Famaey B., Khoury J., 2018, J. Cosmol. Astropart. Phys., 1809, 021
Binney J., Tremaine S., 2008, Galactic Dynamics, 2nd edn., Princeton University Press, Princeton, New Jersey
Bosma A., 1978, PhD thesis, Groningen Univ.
Brada R., Milgrom M., 1995, MNRAS, 276, 453
Cai R.-G., Liu T.-B., Wang S.-J., 2018, Phys. Rev. D, 97, 023027
Chang Z., Zhou Y., MNRAS, 486, 1658
Chashchina O., Foot R., Silagadze Z., 2017, Phys. Rev. D, 95, 023009
Clowe D., Bradac M., Gonzalez A. H., Markevitch M., Randall S. W., Jones C., Zaritsky D., 2006, ApJ, 648, L109
Dai D.-C., Lu C., 2017, Phys. Rev. D, 96, 124016
de Blok W. J. G., McGaugh S. S., 1998, ApJ, 508, 132
Dodson S., Liguori M., 2006, Phys. Rev. Lett., 97, 231301
Edmonds D., Farrah D., Minic D., Ng Y. J., Takeuchi T., 2017, Int. J. Mod. Phys. D, 27, 1830001
Frandsen M. T., Petersen J., 2018, preprint (arXiv:1805.10706)
Kroupa P. et al., 2018, Nat. Astron., 2, 925
Lelli F., McGaugh S. S., Schombert J. M., 2016, AJ, 152, 157
Lelli F., McGaugh S. S., Schombert J. M., Pawlowski M. S., 2017, ApJ, 836, 152
Li P., Lelli F., Mcgaugh S., Schombert J., 2018, A&A, 615, A3
McGaugh S. S., Lelli F., Schombert J. M., 2016, Phys. Rev. Lett., 117, 201101
McGaugh S. S., Li P., Lelli F., Schombert J. M., 2018, Nat. Astron., 2, 924
Mellier Y., Ford B., Soucail G., 1989, Gravitational Lensing: Proceedings of a Workshop Held in Toulouse. Springer, Heidelberg, Berlin
Milgrom M., 1983, ApJ, 270, 365
Petersen J., 2019, preprint (arXiv:1906.09798)
Petersen J., Frandsen M. T., 2017, preprint (arXiv:1710.03096)
Rodrigues D. C., Marra V., del Popolo A., Davari Z., 2018a, Nat. Astron., 2, 668
Rodrigues D. C., Marra V., Del Popolo A., Davari Z., 2018b, Nat. Astron., 2, 927
Rubin V. C., Thomann N., Ford W. K., Jr., 1980, ApJ, 238, 471
Sanders R. H., 2003, MNRAS, 342, 901
Sanders R. H., 2019, MNRAS, 485, 513
Sanders R. H., Noordermeer E., 2007, MNRAS, 379, 702
Sofue Y., Planets, Stars and Stellar Systems, 5, 985
Tian Y., Ko C.-M., 2019, MNRAS, 488, L41
Tully R. B., Fisher J. R., 1977, A&A, 54, 661
van Albada T. S., Bahcall J. N., Begeman K., Sancisi R., 1985, ApJ, 295, 305
Zwicky F., 1933, Helv. Phys. Acta, 6, 110

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