Chapter 1

LOW-FREQUENCY NOISE AS A SOURCE OF DEPHASING OF A QUBIT

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Abstract

With the growing efforts in isolating solid-state qubits from external decoherence sources, the material-inherent sources of noise start to play crucial role. One representative example is electron traps in the device material or substrate. Electrons can tunnel or hop between a charged and an empty trap, or between a trap and a gate electrode. A single trap typically produces telegraph noise and can hence be modeled as a bistable fluctuator. Since the distribution of hopping rates is exponentially broad, many traps produce flicker-noise with spectrum close to $1/f$. Here we develop a theory of decoherence of a qubit in the environment consisting of two-state fluctuators, which experience transitions between their states induced by interaction with thermal bath. Due to interaction with the qubit the fluctuators produce $1/f$-noise in the qubit’s eigenfrequency. We calculate the results of qubit manipulations - free induction and echo signals - in such environment. The main problem is that in many important cases the relevant random process is both non-Markovian and non-Gaussian. Consequently the results in general cannot be represented by pair correlation function of the qubit eigenfrequency fluctuations. Our calculations are based on analysis of the density matrix of the qubit using methods developed for stochastic differential equations. The proper generating functional is then averaged over different fluctuators using the so-called Holtsmark procedure. The analytical results are compared with simulations allowing checking accuracy of the averaging procedure and evaluating mesoscopic fluctuations. The results allow understanding some observed features of the echo decay in Josephson qubits.

Keywords: Qubits, Decoherence, $1/f$-noise

1. Introduction and model

The dynamics of quantum two-level systems has recently attracted special attention in connection with ideas of quantum computation. A crucial requirement is to the phase coherence in the presence
of noisy environment [1]. Solid state devices have many advantages for realization of quantum computation that has been confirmed by several successful experiments, for a review see, e. g., Ref. [2] and references therein. In solid-state realizations of quantum bits (qubits) the major intrinsic noise is due to material-specific fluctuations (substrate, etc). Concrete mechanisms of these fluctuations depend upon the realization. In particular, in the case of charge qubits the background charge fluctuations with 1/f spectrum are considered as most important [2]. They are usually attributed to random motion of charges either between localized impurity states, or between localized impurity states and metallic electrodes.

The conventional way to allow for the noisy environment is to describe it as a set of harmonics oscillators with a certain frequency spectrum. The resulting “spin-boson models” were extensively discussed in the literature, see for a review Refs. [3] and [4]. Applications of these models to concrete qubit implementations have been recently reviewed by Shnirman et al. in Ref. [5].

1.1 Spin-boson model

Conventionally, the quantum system which we will call the qubit is assumed to be coupled linearly to an oscillator bath with interaction Hamiltonian

$$\mathcal{H} = \sigma_z \hat{X}, \quad \hat{X} = \sum_j C_j (\hat{b}_j + \hat{b}_j^\dagger).$$

Here $\sigma_i$, $i = x, y, z$, are the Pauli matrices describing the qubit, while $\hat{b}_j$ and $\hat{b}_j^\dagger$ stand for bosons. The decoherence is then expressed in terms of the symmetric correlation function

$$S_X(\omega) \equiv \left\langle \left[ \hat{X}(t), \hat{X}(0) \right] \right\rangle +_\omega = 2J(\omega) \coth \frac{\omega}{2T},$$

where $J(\omega)$ the bath spectral density, $J(\omega) \equiv \pi \sum_j C_j^2 \delta(\omega - \omega_j)$. Here and below we put $\hbar = 1$ and $k_B = 1$. In the simplest case the decoherence can be characterized by [5]

$$\mathcal{K}(t) = -\ln \text{Tr} \left( e^{-i\hat{\Phi}(t)} e^{i\hat{\Phi}(0)} \hat{\rho}_b \right)$$

where $\hat{\rho}_b$ is the density matrix of the thermal bath, and the bath phase operator is defined as

$$\hat{\Phi}(t) \equiv i \sum_j (2C_j/\omega_j) e^{it\hat{H}_b} \left( \hat{b}_j^\dagger - \hat{b}_j \right) e^{-it\hat{H}_b},$$

$\hat{H}_b$ being the bath Hamiltonian. The quantity $\mathcal{K}(t)$ is conventionally expressed through the bath spectral density, $J(\omega)$, as [3]

$$\mathcal{K}(t) = \frac{8}{\pi} \int_0^\infty \frac{d\omega J(\omega)}{\omega^2} \left[ \sin \frac{\omega t}{2} \coth \frac{\omega}{2T} + \frac{i}{2} \sin \omega t \right] .$$

The most popular assumption about $J(\omega)$, namely $J(\omega) = (\alpha \pi \omega/\Theta(\omega_c - \omega))$, is “Ohmic dissipation”. Here $\alpha$ is a dimensionless coupling strength, while $\Theta(x)$ is the Heaviside unit step function.

Important features of this approach is that (i) the decoherence is determined solely by the pair correlation function $S_X(\omega)$ that assumes the noise to be Gaussian; (ii) $S_X(\omega)$ is related to the bath spectral density through the fluctuation-dissipation theorem, which assumes the system to be equilibrium. As long as these assumptions hold, the method provides powerful tools to analyze the decoherence.

Several attempts, see Ref. [5] and references therein, were made to extend the spin-boson model to the so-called “sub-Ohmic” case, in particular, to the case of 1/f-noise where $S_X(\omega) \propto |\omega|^{-1}$.

We believe that 1/f noise is a typical nonequilibrium phenomenon. It is due to the fact that some excitations of the environment relax so slowly that cannot reach the equilibrium during the measuring time. As a result, the fluctuation-dissipation theorem cannot be applied. Moreover, 1/f noise is not a stationary Markov process. Indeed, it is created by the fluctuators with exponential broad distribution of the relaxation rates and thus is not fully characterized by its pair correlation function.
Dephasing of a qubit by $1/f$-noise

Figure 1.1. Schematic sketch of the distribution of charged traps. They are located near the gate’s surface and produce oppositely charged images.

1.2 Spin-fluctuator model

Several attempts were made to study the role of non-Gaussian and non-Markovian nature of the $1/f$-noise for various examples of coherent quantum transport such as resonant tunneling [6], ballistic transport through a quantum point contact [7], Josephson effect [8], Andreev interferometer [9]. In connection to qubits, a similar model has been recently studied by Paladino et al. [10]. In this paper dynamical charged traps were considered as two-level systems (TLS) with exponentially-broad distribution of hopping rates, and the “free induction signal” of a qubit was numerically analyzed for a narrow distribution of the coupling constants, $v_i$, between the traps and the qubit. Quantum aspects of non-Markovian kinetics were addressed in Ref. [11].

The aim of the present work is to revisit this problem. In the following we will consider a spin-fluctuator model, similar to that considered in Ref. [10], which takes into account both nonequilibrium and non-Markovian effects. Analysis of this model shows that nonequilibrium effects are important. In particular, we will address the role of the distribution of coupling constants $v_i$ between the fluctuators and the qubit. The distribution of $v_i$ is probably broad for most possible devices and realistic situations because the fluctuators are located at different distances from the qubit. We will show that a broad distribution of the coupling constants leads to significant modifications of the decoherence dynamics.

The broad distribution of the coupling constants, makes the model, which we will consider, essentially similar to the conventional models of the spectral diffusion in glasses. The concept of spectral diffusion was introduced by Klauder and Anderson [12] as early as in 1962 for the problem of spin resonance. They considered spins resonant to the external microwave field (spins $A$) which generate echo signals, and surrounding non-resonant spins (spins $B$). Due to interaction between spins $A$ and $B$, stochastic flip-flops of spins $B$ lead to a random walk of $A$-spins frequencies. This random walk is referred to as the spectral diffusion.

Black and Halperin [13] applied the concept of spectral diffusion to low-temperature physics of glasses. They used ideas of Ref. [12] to consider phonon echo and saturation of sound attenuation by two-level systems [14] in glasses. Important generalizations of these results were made by Hu and Walker [15], Mainard et al. [16] and Laikhtman [17].

Following these ideas, we assume that the qubit is a two-level system (TLS) surrounded by so-called fluctuators, which also are systems with two (meta)stable states. One can imagine several realizations of two-level fluctuators. Consider, e. g., structural dynamic defects, which usually accompany really quenched disorder. These defects, if not charged, behave as elastic dipoles, i. e., they interact with the qubit via deformational potential. The interaction strength in this case depends on the distance, $r$, between the fluctuator and the qubit as $r^{-3}$. Consequently, the distribution of the coupling constants is $P(v) \propto v^{-2}$. Charge traps near the gates also produce dipole electric fields, see Fig. 1.1. In this case $v = e^2 (a \cdot r)/r^3 \propto r^{-2}$. Assuming that $a \ll r$ (otherwise charge would tunnel directly to the qubit and
the device will not work) and integrating the fluctuators contributions along the 2D gate surface we again obtain \( P(v) \propto v^{-2} \). It is exactly the distribution of the coupling constants in glasses, where two-level systems (TLS) interact via dipole-dipole interaction [13].

It is crucial that due to their interaction with the environment the fluctuators switch between their states. This switching makes the fields acting upon the qubit time-dependent. The decoherence is caused by the time-dependence of the random field - a static field can only renormalize the qubit’s interlevel spacing. The dynamics of the fluctuators is relaxational, so they are rather “relaxators” than oscillators. The decoherence is thus determined by the fluctuators relaxation rates, which should be compared to the measurement time. At low temperatures the fluctuators are frozen in the ground states, and their dynamics is slow. Therefore, the fluctuators-induced decoherence significantly decreases with the temperature.

The paper is organized as follows. In Sec. 1.1.3 we reformulate the model for spectral diffusion in glasses for the case of a qubit and consider a simplified version of the theory. This approach is, strictly speaking, applicable only to the fluctuators with the small interlevel spacings, \( E \ll T \), and the numerical factors that follow from this approximation should not be trusted. However the resulting qualitative physical picture is believed to be correct, because the fluctuators with \( E \gtrsim T \) are frozen in their ground states, they thus do not fluctuate and do not contribute to the decoherence. The problem of a single neighboring fluctuator is considered in Sec. 1.2 followed by the discussion of the averaging over the ensemble of fluctuators in Sec. 1.3. In Sec. 1.5 we show that different fluctuators are responsible for the decoherence and for the flicker noise. Consequently, the decoherence cannot in general be expressed through the pair correlation function of random fields acting upon the qubit.

### 1.3 Detailed description

A qubit coupled to the environment will be modeled by the Hamiltonian

\[
\hat{\mathcal{H}} = \hat{\mathcal{H}}_q + \hat{\mathcal{H}}_{\text{man}} + \hat{\mathcal{H}}_{qF} + \hat{\mathcal{H}}_F ,
\]  

(1.6)

where \( \hat{\mathcal{H}}_q \) and \( \hat{\mathcal{H}}_F \) describe the qubit and the fluctuators separately. A completely isolated qubit is just a system that can be in one of two states and is characterized by the energies of these states and the tunneling probability between them. \( \hat{\mathcal{H}}_q \) can thus be written as the Hamiltonian of the qubit pseudospin in a static "magnetic field" \( \mathbf{B} = \{ B_x, B_z \} \), where \( B_z \) characterizes the splitting of the energies of the two states, and \( B_x \) is responsible for the tunneling. (Parallel shift of the qubit energies is, of course, irrelevant). One can diagonalize such a Hamiltonian, by simply choosing the direction of the new \( z \)-axis to be parallel to \( \mathbf{B} \) and write the rotated Hamiltonian \( \hat{\mathcal{H}}_q \) as

\[
\hat{\mathcal{H}}_q = -\frac{1}{2} B \sigma_z .
\]  

(1.7)

\( \hat{\mathcal{H}}_F \) in turn can be diagonalized and split into three parts

\[
\hat{\mathcal{H}}_F = \hat{\mathcal{H}}_{F}^{(0)} + \hat{\mathcal{H}}_{F-\text{env}} + \hat{\mathcal{H}}_{\text{env}} .
\]  

(1.8)

We model fluctuators by two-level tunneling systems, which Hamiltonians can also be diagonalized:

\[
\hat{\mathcal{H}}_{F}^{(0)} = \frac{1}{2} \sum_i E_i \tau^{(i)}_z
\]  

(1.9)

where the Pauli matrices \( \tau^{(i)} \) correspond to \( i \)-th fluctuator. The spacing between the two levels, \( E_i \) is formed by the diagonal splitting, \( \Delta_i \), and the tunneling overlap integral, \( \Lambda_i \)

\[
E_i = \sqrt{\Delta_i^2 + \Lambda_i^2} \equiv \Lambda_i / \sin \theta_i .
\]  

(1.10)

To account for the flip-flops of the fluctuators one needs to include environment. To be specific we model the environment by a bosonic bath. This applies not only to phonons, but also to electron-hole pairs
in conducting part of the system [18]. Neglecting the interactions of the fluctuators with the environment that do not cause the flip-flops we specify the environment-related parts of the Hamiltonian as

\[ \mathcal{H}_{\text{env}} = \sum_{\mu} \omega_\mu \left( \hat{b}_\mu^\dagger \hat{b}_\mu + 1/2 \right), \]

\[ \mathcal{H}_{\text{F-env}} = \sum_i \tau_x^{(i)} \sum_{\mu} \tau_z^{(i)} C_{i,\mu} \left( \hat{b}_\mu + \hat{b}_\mu^\dagger \right) \]

where \( \mu \) represents the boson quantum numbers (wave vector, etc.), and \( C_{i,\mu} \) are the constants of the coupling between the fluctuators and the bosons.

It is crucial to specify the interaction, \( \mathcal{H}_{qF} \), between the qubit and the fluctuators. Following Ref. [13], we assume that

\[ \mathcal{H}_{qF} = \sum_i v_i \sigma_z \tau_z^i, \quad v_i = g(r_i) A(n_i) \cos \theta_i. \]

Here \( \theta_i \) is determined by Eq. (1.10), \( n_i \) is the direction of elastic or electric dipole moment of \( i \)-th fluctuator, and \( r_i \) is the distance between the qubit and \( i \)-th fluctuator. The functions \( A(n_i) \) and \( g(r_i) \) are not universal.

Interaction between the fluctuators and the environment manifests itself through time-dependent random fields \( \xi_i(t) \) applied to the qubit. These low frequency, \( \omega \ll T \), fields can thus be treated classically. Accordingly, one can substitute \( \mathcal{H}_{qF} \) by the interaction of the qubit pseudospin with a random, time-dependent magnetic field, \( \chi_1(t) \), which is formed by independent contributions of surrounding fluctuators:

\[ \mathcal{H}_{qF} = \chi_1(t) \sigma_z, \quad \chi_1(t) = \sum_i v_i \xi_i(t), \]

so that \( v_i \), Eq. (1.13) determines the coupling strength of the \( i \)-th fluctuator with the qubit, while the random function \( \xi_i(t) \) characterizes the state of this fluctuator at each moment of time. Below we assume that fluctuator switching itself is an abrupt process that takes negligible time, and thus at any given \( t \) either \( \xi_i(t) = 0 \) or \( \xi_i(t) = 1 \).

Note that we directed \( \chi_1(t) \) along \( z \)-axis, i.e., neglected possible transitions between the qubit eigenstates induced by the random fields. This can be justified by the low frequency of this field. (From the practical point of view a qubit that switches frequently by the external noise does not make a decent device.)

To manipulate the qubit one should be able to apply ac “magnetic field”, \( F(t) \). In general this field not only causes the transitions between the qubit eigenstates \( (F_z) \), but also modulates the qubit level spacing \( B \) in time \( (F_x) \). The latter effect is parasitic and should be reduced. Here we simply neglect this modulation and assume that \( F \) is parallel to the \( x \)-axis \( (F_x = F) \). Accordingly

\[ \mathcal{H}_{\text{man}} = \left(1/2\right) F(t) \cdot \sigma_x. \]

For the manipulation to be resonant, the frequency \( \Omega \) of the external field \( F \) should be close to \( B \).

Combining Eqs. (1.7), (1.9), (1.14), and (1.15) we substitute the initial Hamiltonian (1.6) by:

\[ \mathcal{H} = \frac{1}{2} \left[ E_0 + \chi(t) \right] \sigma_z + \frac{1}{2} F(t) \sigma_x + \frac{1}{2} \sum_i E_i \tau_z^i. \]

Here \( E_0 \) determines the original \( (t = 0) \) value of the eigenfrequency of the qubit, while \( \chi(t) \) is the random modulation of this eigenfrequency caused by the flips of the fluctuators:

\[ E_0 = B + \chi_1(0), \quad \chi(t) = \chi_1(t) - \chi_1(0). \]
finite transition rates between the fluctuator’s states. The transition rates can be calculated in the second order of the perturbation theory for the fluctuator-phonon/electron interaction [19, 18]. As a result, the ratio, \( \gamma^+ / \gamma^- \), of the rates for the upper, \( \gamma^+ \), and the lower, \( \gamma^- \), states in the equilibrium is determined by the energy spacing between these two states: \( \gamma^+_i / \gamma^-_i = \exp (E_i / T) \). Accordingly, these rates can be parameterized as

\[
\gamma^+_i = (1/2) \gamma_i(T, E_i) \left[ 1 \pm \tanh(E_i / T) \right] \sin^2 \theta_i
\]

(1.18)

where \( \theta_i \) is given by Eq. (1.10).

The dependence of \( \gamma_0(E_i) \) on \( E_i \) is determined by the concrete relaxation mechanism: for phonons [19] \( \gamma_0(E_i) \propto E_i^3 \), while for electrons [18] \( \gamma_0(E_i) \propto E_i \). Note that the average value of \( \xi_i(t) \) depends only on \( E_i \) and \( T \):

\[
\langle \xi(t) \rangle_{t \to \infty} = \gamma^- / (\gamma^+ + \gamma^-) = [1 + \exp (E_i / T)]^{-1}.
\]

(1.19)

There can also be a direct interaction of the qubit with the bosonic bath. One can introduce the transition rate \( \gamma_q(T) \) due to this interaction in the same way as \( \gamma_i(T) \).

Below we will often use a simplified version of the theory assuming that the only fluctuators that contribute to dephasing are those with \( E_i \ll T \). In this approximation

\[
\gamma^+_i = (1/2) \gamma_0(T) \sin^2 \theta_i,
\]

(1.20)

\( \gamma_0 \) is thus the maximal fluctuator relaxation rate at a given temperature \( T \). This assumption significantly simplifies the formulas and produces, as one can show [17, 20], correct order-of-magnitude results.

The model formulated above differs from the spin-boson models by statistics of the random force \( X(t) \). It allows one to describe the qubit decoherence in a simple way without loosing track of the essential physical picture.

### 1.4 Qubit manipulations

We parameterize the qubit’s density matrix as

\[
\hat{\rho} = \begin{pmatrix}
  n & -i f e^{\Omega t} \\
  i f^* e^{-\Omega t} & 1 - n
\end{pmatrix},
\]

(1.21)

and commute it with the Hamiltonian, Eq. (1.16), using the resonance approximation. (The frequency, \( \Omega \) of the applied field \( F \) is assumed to be close to the qubit eigenfrequency, \( E_{0i} \).) Including also the inherent qubit relaxation (\( \gamma_q \)) we obtain the following equations of motion:

\[
\frac{\partial n}{\partial t} = -2\gamma_q(n - n_0) - F \text{Re} f,
\]

(1.22)

\[
\frac{\partial f}{\partial t} = i [E_0 + \mathcal{X}(t) - \Omega] f - \gamma_q f + F \left( n - \frac{1}{2} \right).
\]

(1.23)

These equations have been obtained in Refs. [21, 22] for the problem of spectral diffusion in glasses, \( F \) in Eq. (1.16) playing the role of the Rabi frequency of the resonant pair.

The equation set (1.22) belongs to the class of stochastic differential equations. In the following we use the methods [23–29] developed for these equations to study the qubit response to various types of the manipulation.

Currently the experimentally observable signal is an accumulated result of numerous repetitions of the same sequence of inputs (e. g., pulses of the external field). To be compared with such measurements, solutions of Eq. (1.21) should be averaged over realizations of the stochastic dynamics of the fluctuators. Provided that the time intervals between the sequences of inputs are much longer than the single-shot measuring time we can separate the averaging over the initial states of the fluctuators, \( \xi_i(0) \), from the averaging over their stochastic dynamics, \( \mathcal{X}(t) \).
In the absence of the external ac field \( n(t) \) should approach its equilibrium value given by the Fermi function, \( n_0(E_0) = [1 + \exp(E_0/T)]^{-1} \), while the off-diagonal matrix element of the density matrix should tend to zero. If the external ac field is switched off at \( t = 0 \) then the solution of Eq. (1.21) has the form

\[
\begin{align*}
n(t) &= n_0(E_0) + [n(0) - n_0(E_0)]e^{-2\gamma_0 t}, \\
f(t) &= f(0) e^{-\gamma_0 t+i(E_0-\Omega)t+i\int_0^t \chi(t') dt'}.
\end{align*}
\]

We need to average this signal, known as the free induction signal over both \( E_0 \) and \( \chi(t) \). If the time \( t \) after the pulse is short enough, then the fluctuators remain in their original states and \( \chi(t) \) can be neglected. At \( t = 0 \) the system of the fluctuators is supposed to be in the equilibrium, and thus probability \( \xi_i = 1 \) is \( n_0(E_i) \). Substituting \( E_0 \) from Eq. (1.17) we find that \( \exp[i(E_0 - \Omega)t] \) averaged over the initial realization equals to

\[
\left< e^{i(E_0-\Omega)t} \right>_{\xi} = e^{i(B-\Omega)t} \prod_j [1 - n_0(E_j) + n_0(E_j)e^{i\nu_j t}] .
\]

It follows from Eq. (1.26) that the observed free induction signal involves the oscillations with frequencies that differ from \( B - \Omega \) by various combinations of \( \nu_j \). As a result, in the presence of a large number of the fluctuators the free inductance signal decays in time even when \( \chi(t) = 0 \), i.e. when the fluctuators do not switch during one experimental run. However, this decay has little to do with the decoherence due to irreversible processes.

Much more informative for studies of genuine decoherence are echo experiments, when the system is subject to two (or three) short ac pulses with different durations \( \tau_1 \) and \( \tau_2 \), the time interval between them being \( \tau_{12} \) (or \( \tau_{12} \) and \( \tau_{13} \), respectively). Considering echo, we assume that the external pulses are short enough for both relaxation and spectral diffusion during each of the pulses to be neglected. The echo decay is known to be proportional to the "phase-memory functional" [30]

\[
\Psi[\beta(t'), t] = \left< \exp\left( i \int_0^t \beta(t')\chi(t') dt' \right) \right>_{\xi_i} .
\]

Here for 2-pulse echo \( t = 2\tau_{12} \) and

\[
\beta(t') = \begin{cases} 
0 & \text{for} \; t' \leq 0, \\
1 & \text{for} \; 0 < t' \leq \tau_{12}, \\
-1 & \text{for} \; \tau_{12} < t'. 
\end{cases}
\]

In the case of the 3-pulse echo one would put \( t = \tau_{12} + \tau_{13} \) and

\[
\beta(t') = \begin{cases} 
0 & \text{for} \; t' \leq 0, \\
1 & \text{for} \; 0 < t' \leq \tau_{12}, \\
0 & \text{for} \; \tau_{12} < t' \leq \tau_{13}, \\
-1 & \text{for} \; \tau_{13} < t'. 
\end{cases}
\]

The functional (1.27) can be used to describe the decoherence of the free induction signal, substituting \( \beta(t') = \Theta(t') \). In this case, however, one should understand \( \chi(t') \) as \( \sum_i \nu_i \xi_i(t') \) rather than use Eq. (1.17). For the echo experiments \( t \) in Eq. (1.27) is chosen in such a way that the integral of \( \beta(t') \) from zero to \( t \) vanishes. As a result, the time-independent part of \( \chi \), i.e., dispersion of \( \xi_i(0) \) becomes irrelevant. Below we evaluate the phase-memory functional (1.27) for the free induction as well as for schemes of the measurement.

2. Results for a single fluctuator

2.1 Random telegraph noise

The process, which is described by a random function, \( \xi(t) \), that acquires only two values: either \( \xi = 0 \) or \( \xi = 1 \) is known as random telegraph process. In this section we assume that \( E_i \gg T \) and thus
the two states of each fluctuator are statistically equivalent, i.e., the time-average value of \( \xi(t) \) equals to
\[
\langle \xi(t) \rangle_{t \to \infty} = 1/2.
\]

To evaluate the memory functional (1.27) we first introduce auxiliary random telegraph processes defined as
\[
z_{\pm}(t) = \pm(-1)^{n(t,t)},
\]
where \( n(t_1, t_2) \) is a random sequence of integers describing number of "flips" during the period \((t_1, t_2)\), so that \( n(t, t) = 0 \). The fact that \( z_{\pm}^2(t) = 1 \) substantially simplifies the calculations. The 'flips" of a given fluctuator induced by its interaction with the bosonic bath should not be correlated with each other. Accordingly, \( n(t_1, t_2) \) obeys the Poisson distribution, i.e., the probability, \( \mathcal{P}_n(t_1, t_2) \), that \( n(t_1, t_2) = n \) equals to
\[
\mathcal{P}_n(t_1, t_2) = \frac{\langle n(t_1, t_2) \rangle^n}{n!} e^{-\langle n(t_1, t_2) \rangle}, \quad \langle n(t_1, t_2) \rangle = \gamma |t_1 - t_2|,
\]
where \( \gamma \) has a meaning of the average frequency of "flips". From Eqs. (1.31) and (1.30) it follows that
\[
\langle z_{\pm}(t) \rangle = \sum_{n=0}^{\infty} (-1)^n \mathcal{P}_n = \pm e^{-2\gamma|t|},
\]
\[
\langle z_{\pm}(t_1) z_{\pm}(t_2) \rangle = \langle (-1)^{n(t_2, t_1)} \rangle = e^{-2\gamma(t_1 - t_2)}, \quad t_1 \geq t_2.
\]

It is convenient to describe different measurement schemes by making use of the generating functionals
\[
\psi_{\pm}[\beta, t] = \exp \left[ \frac{iv}{2} \int_0^t \beta(t') z_{\pm}(t') dt' \right] \bigg|_{z_{\pm}(t)} \equiv \psi_{\pm}(\beta, t),
\]
where \( \beta(t') \) is the same function as in Eqs. (1.27), (1.28), and (1.29), while the constant \( v \) will later play the role of the qubit-fluctuator coupling constant. To evaluate the functionals (1.32) consider a set of the correlation functions
\[
M_n(t_1, t_2, \ldots, t_n) \equiv \langle z_{\pm}(t_1) z_{\pm}(t_2) \cdots z_{\pm}(t_n) \rangle, \quad t_1 \geq t_2 \geq \ldots \geq t_n.
\]

It is convenient to use a recursive formula
\[
M_n(t_1, t_2, \ldots, t_n) = e^{-2\gamma(t_1 - t_2)} M_{n-2}(t_3, \ldots, t_n),
\]
which follows directly from Eqs. (1.31) and (1.30). Combining Eq. (1.33) with the Taylor expansion of Eq. (1.32) we obtain an exact integral equation for \( \psi_{\pm}[\beta, t] \):
\[
\psi_{\pm}(\beta, t) = 1 \pm i(v/2) \int_0^t dt_1 e^{-2\gamma t_1} \beta(t_1)
\]
\[
- (v^2/4) \int_0^t dt_1 \int_0^t dt_2 e^{-2\gamma(t_1 - t_2)} \beta(t_1) \beta(t_2) \psi[\beta, t_2].
\]

One can evaluate second time-derivative of both sides of Eq. (1.34) and transform this integral equation into a second order differential equation [25]
\[
\frac{d^2 \psi_{\pm}}{dt^2} + \left[ 2\gamma - \frac{d \ln \beta(t)}{dt} \right] \frac{d \psi_{\pm}}{dt} + \frac{v^2}{4} \psi_{\pm} = 0
\]
with initial conditions
\[
\psi_{\pm}(0) = 1, \quad \frac{d \psi_{\pm}}{dt} \bigg|_{t=0} = \pm \frac{iv}{2} \beta(t = -0).
\]
2.2 Generating functional

In the limit $E_i \ll T$ the random functions $\xi_i(t)$ can be expressed through $z_+(t)$ or by $z_-(t)$ with equal probabilities. Using Eq. (1.17) we thus can rewrite the memory functional (1.27) in terms of the functionals $\psi_\pm$, Eq. (1.32):

$$\psi[\beta, t] = \frac{1}{2} \sum_\pm e^{\pm i\theta/2} \int_0^t \beta(t') dt' \psi_\pm[\beta(t), t].$$  

(1.37)

From Eqs. (1.37, 1.35) follows the differential equation for $\psi[\beta, t]$:

$$\frac{d^2 \psi}{dt^2} + \left[2\gamma - \frac{d \ln \beta(t)}{dt} - iv\beta(t)\right] \frac{d \psi}{dt} - i\gamma \psi = 0$$

(1.38)

with initial conditions

$$\psi(0) = 1, \quad \left. \frac{d \psi}{dt} \right|_{t=0} = i\beta(-0) \frac{\nu}{2}.$$  

(1.39)

Below we use this equation to analyze decay of the free induction and echo signals. Note that Eq. (1.38) is the $E_i/T \to 0$ limit of equation derived in Ref. [25] for arbitrary $E_i/T$.

For the free induction signal $\beta(t > 0) = 1$ and $\beta(-0) = 0$. For this $\beta$-function Eq. (1.38) with initial conditions (1.39) yields the following phase memory functional

$$\psi_{pm}(t) = e^{-\gamma t} \sum_{q=\pm 1} \sum_{p=\pm 1} p \left(1 - q \frac{i \nu}{2\gamma} + p \mu\right) e^{-(iqv/2 + p\gamma\mu)t}.$$  

(1.40)

where $\mu = 1 - (v/2\gamma)^2$. This solution is the $E_i/T \to 0$ limit of the result obtained in Ref. [17]. At short times, $\gamma t \ll 1$, Eq. (1.40) can be approximated as

$$\psi_{pm}(t) = 1 - \gamma \left(t - \frac{\sin \nu t}{\nu}\right).$$  

(1.41)

However $\psi_{pm}$ given by Eq. (1.40) does not describe the free induction decay, $\exp \left[iv \int_0^t \xi(t') dt' \right]$, even within our simplified model. We have to consider

$$\psi_{fi}(t) = \exp \left(i\frac{\nu t}{2}\right) \frac{\psi_+ + \psi_-}{2}$$

(1.42)

rather than $\psi_{pm}(t)$, because the latter neglects the dispersion of $E_0$, (1.17), which is due to the term $v\xi(0)$. From Eqs. (1.35) and (1.36) it follows that

$$\psi_{fi}(t) = e^{i(\nu/2 - \gamma)t} \left(\mu^{-1} \cosh \gamma \mu t + \sinh \gamma \mu t\right).$$  

(1.43)

This expression is the $E_i/T \to 0$ limit of the result obtained in Ref. [10]. Comparison of the free induction decay (1.43) with decay of the phase memory is presented in Fig. 1.2. This difference is especially important for the case of $1/f$-noise, when many fluctuators contribute.

The calculation of the echo decay can be done in a similar way. The results for 2- and 3-pulse echo are, respectively (cf. with Ref. [17]):

$$\psi_{e2}(2\tau_2) = \frac{e^{-2\gamma\tau_2}}{2|\mu|^2} \sum_\pm \left[(1 + \mu_2^2)(1 \pm \mu_1) e^{\pm 2i\mu_1 \gamma \tau_2} \right.$$  

$$\left.- (1 - \mu_2^2)(1 \mp i \mu_2) e^{\mp 2i\mu_2 \gamma \tau_2}\right],$$  

(1.44)

$$\psi_{e3}(\tau_2 + \tau_3) = \psi_{e2}(2\tau_2) + \frac{1}{2} \left(\frac{v}{2|\mu|}\right)^2 \cos 2\mu_2 \gamma \tau_2$$  

$$\left.- \cos 2\mu_1 \gamma \tau_2\right) \left(1 - e^{-2\gamma(\tau_3 - \tau_2)}\right) e^{-2\gamma \tau_2}.$$  

(1.45)
Here $\tau_{13}$ is the time between first and third pulse, $\mu_1 + i \mu_2 = \sqrt{1 - (v/2\gamma)^2}$. The function $\psi_{12}(2\tau_{12})$ is shown in Fig. 1.3. Note that at $v > 2\gamma$ the time dependence of the echo signal shows steps similar to what was experimentally observed for the charge echo [2]. The expressions for the echo signal have a simple form at $v \gg \gamma$. In particular, the expression for the two-pulse echo acquires the form

$$
\psi_{e2}(2\tau_{12}) = e^{-2\gamma\tau_{12}} \left[ 1 + (2\gamma/v) \sin v\tau_{12} \right].
$$

Consequently, the plateaus occur at $v\tau_{12} = k\pi/2 - \arctan(2\gamma/v)$ and their heights exponentially decay with the number $k$.

3. Summation over many fluctuators

To average over a set of the fluctuators we assume that dynamics of different fluctuators are not correlated, i.e., $\langle \xi_i(t) \xi_j(t') \rangle = 0$, unless $i = j$. Under this condition the generating functional is a product of the partial functionals, $\psi^{(i)}$. Hence, the generating functional can be presented as

$$
\Psi[\beta(t), t] = \prod_i \psi^{(i)}(t) = e^{\sum_i \ln \psi^{(i)}(t)} \equiv e^{-K(t)}.
$$

Since logarithm of the product is a self-averaged quantity, at large number of fluctuators, $N \gg 1$, one can replace the sum $\sum_i \ln \psi^{(i)}(t)$ by $N \langle \ln \psi \rangle_F$, where $\langle \cdots \rangle_F$ denotes average over the fluctuators interlevel spacings $E_i$, their interaction strength, $v_j$, and tunneling parameters, $\sin \theta_j$. Furthermore, for $N \gg 1$ one can employ the Holtsmark procedure [31], i.e., to replace $\langle \ln \psi \rangle_F$ by $\langle \psi - 1 \rangle_F$, assuming that each of $\psi^{(i)}$ is close to 1. Thus, for large $N$ the approximate expression for $K(t)$ is

$$
K(t) \approx \sum_i \langle 1 - \psi^i(t) \rangle = N \langle 1 - \psi(t) \rangle_F.
$$

To evaluate $K(t)$ one has to specify the distributions of the parameters $E_i$, $v_i$, and $\theta_i$ that characterize the fluctuators. Taking into account only the fluctuators with $E_i \lesssim T$ we can write the number of fluctuators per unit volume as $P_0 T$. It is natural to assume that the density of states, $P_0$ is a $T$-independent constant as it is in structural glasses [14].

The conventional estimation of the distribution function of relaxation rates is based on the fact that the tunneling splitting $\Lambda$ depends exponentially on the distance in real space between the positions of the
two-state fluctuator (on the distance between the charge trap and the gate), as well as on the height of the barrier between the two states. Assuming that parameters like distances and barrier heights are distributed uniformly, one concludes that $\Lambda$-distribution is $\propto \Lambda^{-1}$. Since $\gamma$ is parameterized according to Eq. (1.20), this implies $\mathcal{P}(\theta) = 1/\sin \theta$. As a result, cf. with Ref. [14],

$$\mathcal{P}(E, \theta) = P_0 / \sin \theta .$$  \hfill (1.48)

The coupling constants, $v_i$, determined by Eq. (1.13), contain $\cos \theta_i$ and thus are statistically correlated with $\theta_i$. It is convenient to introduce an uncorrelated random coupling parameter, $u_i$ as

$$u_i = g(r_i)A(n_i), \quad v_i = u_i \cos \theta_i .$$  \hfill (1.49)

It is safe to assume that direction, $n_i$, of a fluctuator is correlated neither with its distance from the qubit, $r_i$, nor with the tunneling parameter $\theta_i$ and thus to replace $A(n)$ by its angle average, $\langle |A(n)| \rangle_n$. It is likely that the coupling decays as power of the distance, $r$: $g(r) \propto g/r^b$. If the fluctuators are located near a $d$-dimensional surface, then

$$\mathcal{P}(u, \theta) = \frac{\eta^{d/b}}{\sin \theta} u^{-d/b-1}, \quad \eta = \frac{\bar{g}}{r_T^b}, \quad r_T = \frac{a_d}{(P_0 T)^{1/d}} .$$  \hfill (1.50)

Here $a_d$ is a dimensionless constant depending on the dimensionality $d$ while $r_T$ is a typical distance between the fluctuators with $E_i \lesssim T$. In the following we will for simplicity assume that

$$r_{\min} \ll r_T \ll r_{\max} ,$$  \hfill (1.51)

where $r_{\min}$ ($r_{\max}$) are distances between the qubit and the nearest (most remote) fluctuator. Under this condition $\eta \propto T^{b/d}$ is the typical constant of the qubit - fluctuator coupling. As soon as the inequality (1.51) is violated the decoherence starts to depend explicitly on either $r_{\min}$ or $r_{\max}$, i. e., become sensitive to mesoscopic details of the device. This case will be analyzed elsewhere. In the following we assume that $d = b$, as it is for charged traps located near the gate electrode, see Fig. 1.1.

The dependences of the generating functional $\psi(\beta, t|u, \gamma)$ on the coupling constants $u$ and transition rates $\gamma$ of the fluctuators are determined by Eq. (1.27). Substituting equations (1.20) and (1.49) into (1.27) and the result - into (1.47) one obtains $K(t)$ in the form:

$$K(t) = \int \frac{du}{u^2} \frac{d\theta}{\sin \theta} \left\{ 1 - \psi(\beta, t|u \cos \theta, \gamma_0 \sin^2 \theta) \right\} .$$  \hfill (1.52)

This expression allows one to calculate the dephasing rate in the case of many surrounding fluctuators for various qubit manipulations.

To start with consider the phase memory functional with $\beta(t') = \Theta(t')$, i. e., free induction signal will $\xi_i(0) = 0$. At small times, $t \ll \gamma_0^{-1}$, one can use Eq. (1.41) for $\psi_{pm}(t)$. The integration over $u$ yields

$$\int_0^{\infty} \frac{du}{u^2} [1 - \psi_{pm}(t)] = \frac{\pi}{4} t^2 \gamma_0 \sin^2 \theta .$$

Performing the following integral over $\theta$ one obtains $K_{pm} \propto \eta \gamma_0 t^2$.

It is slightly trickier to estimate $K_{pm}(t)$ at large times, $\gamma_0 t \gg 1$. One can show that the decoherence in this limit is due to the fluctuators, which coupling with the qubit is atypically weak: $u \sim t^{-1} \ll \gamma_0$. As a result, in the leading in $1/(\gamma_0 t)$ approximation $\psi_{pm}(t) = \cos ut/2$. This asymptotics can be interpreted in the following way. At $t \gg 1/\gamma_0$ a typical fluctuator had flipped many times and its contribution to the qubit phase, which is proportional to

$$\left\langle \int_0^t [\xi(t') - \xi(0)] \, dt' \right\rangle \propto t .$$
and does not depend on \( \gamma \) and, hence on \( \theta \). Therefore the integral over \( \theta \) in (1.52) diverges logarithmically. The proper cut-off is determined by the condition \( \gamma t \approx 1 \), i.e., is the value of \( \theta = \theta_{\text{min}}(t) \), which allows approximately one flip during the time \( t \). Using Eq. (1.20) we estimate \( \theta_{\text{min}}(t) \) as \( (\gamma_0 t)^{-1/2} \). This cut-off reflects the fact that the fluctuators with too low tunneling rates do not change their states during the measurement time \( t \).

The estimate for \( K_{pm}(t) \) can be then summarized as (cf. with Ref. [17]),

\[
K_{pm}(t) \approx \eta \cdot \begin{cases} 
\gamma_0 t^2 & \text{for } \gamma_0 t \ll 1; \\
t \ln \gamma_0 t & \text{for } \gamma_0 t \gg 1.
\end{cases} 
\] (1.53)

Now we can define the dephasing time \( \tau_\varphi \) by the condition \( K(\tau_\varphi) = 1 \). Using Eq. (1.53), we get

\[
\tau_\varphi = \max \left\{ \eta^{-1} \ln^{-1}(\gamma_0/\eta), (\eta \gamma_0)^{-1/2} \right\}. 
\] (1.54)

The echo decay can be calculated in a similar way, cf. with Ref. [17]:

\[
K_{e2}(2\tau_{12}) \sim \eta \tau \cdot \min \left\{ 1, \gamma_0 \tau_{12} \right\}, 
\] (1.55)

\[
K_{e3}(\tau_{12} + \tau_{13}) \sim \eta \gamma_{12} \min \left\{ 1, \ln \tau_{13}/\tau_{12} \right\}. 
\] (1.56)

The dephasing time for the two-pulse echo decay is then

\[
\tau_\varphi = \max \left\{ \eta^{-1}, (\eta \gamma_0)^{-1/2} \right\}. 
\] (1.57)

Let us discuss the physical meaning of the results (1.53)–(1.57). If there is no flips of the fluctuators, then the contribution of a given fluctuator to the total phase gain during the observation time is \( \xi(0) \int_0^t \beta(t') \, dt' \). During the time interval \( t \ll \gamma_0^{-1} \) each fluctuator can flip only once. If it flips at time \( t_1 \), the accumulated relative phase is \( \pm 2 \int_{t_1}^t \beta(t') \, dt' \). To obtain the total phase gain one has to average over all possible moments of flips:

\[
|\delta \varphi(t)| \sim \gamma_0 \int_0^t dt_1 \left| \int_{t_1}^t \beta(t') \, dt' \right|. 
\]

Since \( v \propto r^{-3} \), nearest neighbors are important, and typical value of \( v \) is \( \eta \). Thus we immediately obtain \( K(t) \sim \eta \gamma_0 t^2 \). It can be shown [13] that in this case the random process is Markovian. Consequently in this case the situation can be characterized by a pair conditional probability \( K(E, t|E_0) \) to find the spacing \( E \) at time \( t \) under condition that at \( t = 0 \) it was \( E_0 \). It has the Lorentzian form,

\[
K(E, t|E_0) = \frac{1}{\pi} \frac{\Gamma(t)}{(E - E_0)^2 + \Gamma^2(t)}, 
\] (1.58)

where \( \Gamma(x) \sim \eta \gamma_0 t \). This time dependence can be easily understood in the following way. The nearest region of \( r \), where at least one fluctuator flips during the time interval \( t \) gives the maximal contribution. The size of this region can be estimated from the condition \( P_0 T r^3 \gamma_0 t \approx 1 \) that yields \( r^{-3} \approx P_0 T \gamma_0 t \). The corresponding change in the qubit’s interlevel spacing is then given by the interaction strength at this distance, \( \eta \gamma_0 t \).

During a time interval \( t \gg \gamma_0^{-1} \) a substantial contribution comes from the fluctuators with \( \gamma_0^{-1} \ll \gamma^{-1} \ll t \), which experience many flip-flops. Since a fluctuator having \( \xi = 1 \) can flip only to the state with \( \xi = 0 \) the contributions of successive hops are not statistically independent. The density of most important fluctuators is of the order \( P_0 T r^3 \ln \gamma_0 t \), and the substantial region of \( r \) is determined by the relation \( P_0 T r^3 \ln \gamma_0 t \sim 1 \). As a result \( \Gamma(t) \sim \eta \ln \gamma_0 t \). This dependence holds until \( t \lesssim \gamma_{\text{min}}^{-1} \), where \( \gamma_{\text{min}} \) is the minimal relaxation rate in the system. At \( t \gtrsim \gamma_{\text{min}}^{-1} \) the quantity \( \Gamma(t) \) saturates at the value of the order of \( \Gamma_{\infty} \approx \eta \ln(\gamma_0/\gamma_{\text{min}}) \). The random process in this case is non-Markovian and cannot be fully characterized by the pair correlation function (1.58).
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The above estimates do not describe decay of the free induction signal due to beats between contributions of different fluctuators. For $t \ll \gamma_0^{-1}$ this decay can be evaluated in the same way as influence of the static inhomogeneous broadening [12, 17]. Averaging Eq. (1.26) we get $K_f(t) = \Gamma_\infty t$. At large time, $t \gg \gamma_0^{-1}$, one can assume that the probability to find a fluctuator in a given state is just the equilibrium one. Thus

$$\psi_f(t) \approx 1 - n_0 + n_0 e^{i(1-n_0)vt} = 1 - n_0 \left(1 - e^{i(1-n_0)vt}\right),$$

Averaging this solution over the hopping rates and positions of the fluctuators we arrive at the same result. Consequently, the free induction signal decays much more rapidly than the phase memory.

The important point is that at large observation time, $t \gg \gamma_0^{-1}$, there are optimal fluctuators responsible for decoherence. The distance $r_{\text{opt}}(T)$, between the optimal fluctuators and the qubit is determined by the condition

$$v(r_{\text{opt}}) \approx \gamma_0(T).$$

(1.59)

If the coupling decays as $1/r^b$ and the fluctuators are distributed in a $d$-dimensional space, then $r^{d-1} dr \to \mathcal{P}(v) \propto v^{-(1+d/b)}$. From this one concludes that at $d \leq b$ the decoherence is controlled by optimal fluctuators located at the distance $r_{\text{opt}}$ provided they exist. At $d > b$ the decoherence at large time is determined by most remote fluctuators with $v = v_{\text{min}}$. If $d \leq b$, but the closest fluctuator has $v_{\text{max}} \ll \gamma_0$, then it is the quantity $v_{\text{max}}$ that determines the decoherence. In both last cases $K(t) \propto t^2$, and one can apply the results of Ref. [10], substituting for $v$ either $v_{\text{min}}$ or $v_{\text{max}}$. Since $r_{\text{opt}}$ depends on the temperature there can exist a specific mesoscopic behavior of the decoherence rate. A similar mesoscopic behavior of the decoherence has been discussed for a microwave-irradiated Andreev interferometer [9].

4. Simulations

The procedure outlined above leaves several questions unanswered. First, how many experimental runs one needs to obtain the ensemble-averaged result for a single fluctuator? Second, when the contributions of several fluctuators can be described by averages over the fluctuators’ parameters?

The second question is the most delicate. The situation with a qubit interacting with environment in fact differs from that of a resonant two-level system in spin or phonon echo experiments. In the first case the experiment is conducted using a single qubit surrounded by a set of fluctuators with fixed locations, while in the second case many resonant TLSs participate the absorption. Consequently, one can assume that each TLS has its own environment and calculate the properties averaged over positions and transition rates of the surrounding fluctuators. How many surrounding fluctuators one needs to replace the set of fluctuators with fixed locations (and transition rates) by an averaged fluctuating medium?

4.1 One fluctuator

To answer these questions we have made a series of simulations. First, using Eq. (1.27) we have calculated the two-pulse echo decay for a qubit with a single neighboring fluctuator. The random switching between the $\xi = 0$ and $\xi = 1$ states of this fluctuator was modeled by the Poisson process with time constant $\gamma$. The result of simulations for $u/2\gamma = 5$ are shown in Fig. 1.4. The left panel presents the average over 100 random runs, while the right one shows the average over 1000 runs. Ultimate averaging over infinite number of runs would give the analytical result (1.44) shown by the curve 1 in Fig. 1.3. One can see that averaging over 1000 runs is sufficient to reproduce the analytical result with good accuracy, in particular, to observe the characteristic plateau around. Note that the plateau is qualitatively similar to experimentally observed for the charge echo in Josephson qubits. [2] Note also that with only 100 runs made, the dispersion of the signal indicated by the error bars is huge. Therefore, experimentally one needs at least many hundred runs to obtain reliable averages.
4.2 Check of the Holtsmark procedure

To check the validity of the summation over different fluctuators using the Holtsmark procedure, we perform simulations for many fluctuators. The fluctuators are assumed to be uniformly distributed in space at distances smaller than some $r_{\text{max}}$. Then, the normalized distribution function of the coupling constants and relaxation rates, $P(u, \theta)$ can be specified as

$$P(u, \theta) = \frac{N_{u_{\text{min}}}/u^2}{[\sin \theta \ln(\tan \theta_{\text{min}}/2)]^{-1}} \quad (1.60)$$

with $u \in \{u_{\text{min}}, \infty\}$, while $\theta \in \{\theta_{\text{min}}, \pi/2\}$. Here small $\theta$ correspond to slow fluctuators, $\gamma = \gamma_0 \sin^2 \theta$.

The quantity $\eta (1.50)$ that characterizes the fluctuator density is given now as $\eta = N_{u_{\text{min}}} / \ln(\tan \theta_{\text{min}}/2)$. The results of simulations for small times are shown in Fig. 1.5. For simplicity, in these simulations the transition rate was assumed to be the same, $\gamma = \gamma_0$, for all fluctuators. In order to check the analytical result (1.53) for the free induction signal, it is convenient to plot $K(t)/\eta t$ versus $2\gamma t$. One can see that the predicted asymptotic behavior works rather well for $\gamma t \ll 1$. By substituting (1.44) into (1.52) it can be easily shown that the two-pulse echo signal has a similar asymptotic for small $t$, $K \propto t^2$, only the coefficient is twice that for the free induction signal. This result is also perfectly reproduced by the simulations, which justifies the use of the Holtsmark procedure for $\gamma t \ll 1$, i.e. when $K$ is small. The results for large times are shown in Fig. 1.6. Here it was important to take into account scatter in values of $\gamma$ because behavior at large $t$ is controlled by numerous fluctuators that flip very seldom, i.e. have small $\gamma$. One observes that analytically predicted behavior of the phase memory for the free induction signal, $K \approx t \ln t$ at $t \gg \gamma_0^{-1}$ is fully confirmed by the simulations. The analytical result was obtained by making a rough cutoff of slow fluctuators at $\theta = (\gamma_0 t)^{-1/2}$ that led to $K_{\text{pm}}(t) \approx \eta t \ln \gamma_0 t$, see Eq. 1.53. From simulations we can see that a more accurate expression at large times is $K_{\text{pm}}(t) \approx (\eta t/2) \ln 2\gamma_0 t$, which corresponds to the straight line in Fig. 1.6.

4.3 Many fluctuators with fixed locations

The curves presented in Figs. 1.5 and 1.6 were calculated by averaging over many random sets of fluctuators. This allowed us to make a reliable check of the analytical results based on the Holtsmark procedure. The next step is to check whether it is appropriate to average over the fluctuators positions though in a real system the fluctuators’ parameters are fixed. For this purpose we compare the results for three different sets of fixed fluctuators with different coupling constants distributed again according to Eq. (1.60), however with fixed $\theta$. For each set of fluctuators, we find the average signal over 1000 runs and show it in Fig. 1.7. The results strongly depend on the fluctuator density $\eta = N_{u_{\text{min}}}$. For $\eta = 300$, the right panel, different sets of fluctuators lead to similar behavior of $K(t)$, rather close to the “expected”
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Figure 1.5. Short-time phase memory decay of two-pulse echo (1) and free induction (2) for $u_{\text{min}} = 2\gamma$. Lines (3) and (4), correspond to analytically calculated slopes: $\pi/4$ and $\pi/8$, respectively. The results are averaged over 25000 realizations of random telegraph noise in $N = 10$ fluctuators randomly distributed in space.

Figure 1.6. Long-time phase memory decay of the free induction for $u_{\text{min}} = 0.02\gamma$. Solid line corresponds to $(1/2)\ln(2\gamma_0 t)$. The results are averaged over 5000 realizations of random telegraph noise in $N = 10$ fluctuators randomly distributed in space.

Figure 1.7. Two-pulse echo decay for three different fluctuators sets $u_{\text{min}} = 2\gamma$, $N = 10$ (left panel), and 300 (right panel). The behavior obtained by averaging over 5000 different sets. For larger densities the reproducibility is even better. However for $\eta = 10$, the left panel, each set of fluctuators is characterized by a very specific type of the signal. The function $K(t)$ for a given set usually differs dramatically from the “expected” behavior that we obtained by averaging over 5000 sets. For such a low fluctuator density, it is hopeless to fit the experimental data with our analytical formulas obtained by averaging over fluctuator ensemble, like Eq.(1.53). Note that small error bars on the plot mean only that the measured signal is reproducible if it is averaged over 1000 experimental runs. However, even a slightly different arrangement of fluctuators in the gate can produce a very different signal.

If the fluctuator density is low, or, in other words if we are in the mesoscopic limit, the signal should be essentially determined by one, the most important fluctuator. To check this we have calculated the echo signal in the presence of $N$ fluctuators, and compared it with the signal in the absence of the strongest...
fluctuator. The results are shown in Fig. 1.8. One can see that one needs really many fluctuators to avoid strong mesoscopic fluctuations. More detailed studies of mesoscopic fluctuations are planned for future.

5. Comparison with the noise in the random frequency deviation

The conventional way is to express the environment-induced decoherence through the noise spectrum, $S_X(\omega) = 2 \int_0^\infty dt e^{i\omega t} \langle \langle X(t)X(0) \rangle \rangle_F$. Using Eq. (1.17) we get

$$S_X(\omega) = 2 \cos^2 \theta \left( \sum_i \frac{2\gamma_i}{\omega^2 + (2\gamma_i)^2} \cdot v_i^2 \right) F.$$ 

Let us start averaging over fluctuators by integration over $\theta_i$. Since we are interested in small frequencies, we can replace $\sin \theta \to \theta, \cos \theta \to 1$ and replace the upper limit $\pi/2$ of the integration by infinity. In this way we get $S_X(\omega) = \pi \eta u_{\text{max}}/2\omega$. Here we have taken into account that the summation over the fluctuator strength, $u$, is divergent at the upper limit corresponding to the minimal distance, $r_{\text{min}}$, between the fluctuator and the qubit. Thus, the closest fluctuators are most important. We observe that our model leads to $1/f$ noise in the random force acting upon the qubit. However, the noise is mainly determined by the nearest fluctuators, while decoherence (at long times) is dominated by the fluctuators at the distance $r_{\text{opt}}$ given by Eq. (1.59). Since $r_{\text{opt}} \gg r_{\text{min}}$, the decoherence cannot, in general, be expressed only through $S_X(\omega)$.

Now we can compare our result given by Eq. (1.53). From Eq. (1.5) one obtains (cf. with Ref. [5])

$$K(t) = (1/2)\eta u_{\text{max}} t^2 |\ln \omega_{ir} t|$$

where $\omega_{ir}$ is the so-called intrinsic infrared cutoff frequency for the $1/f$ noise. [3] It is clear that the results differ significantly. Even in the case when $\gamma_0 t \ll 1$ when the random process is Markovian, results differ both by order of magnitude and temperature dependence. The reason for this discrepancy is that dephasing and $1/f$-noise are determined by different sets of fluctuators.

6. Applicability range of the model

Let us discuss the applicability range for the used approach. Firstly, random fields acting on the qubit were assumed to be classical. This is correct provided the typical hopping rate of a fluctuator, $\gamma_0$, is much less than its typical interlevel spacing, $E \sim T$. This is the case, indeed, for fluctuator interaction with both with phonons and electrons because of weak coupling. Secondly, we did not discuss the mechanism of decoherence due to direct translation of excitation from the qubit to fluctuators. The most important
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part of such interaction can be written as

$$\mathcal{H}_{tr} = \sum_i U_i \left( \sigma_+^{(i)} \tau_-^{(i)} + \sigma_+^{(i)} \tau_-^{(i)} \right),$$

where $\sigma_\pm = \sigma_x \pm i \sigma_y$, and $\tau_\pm = \tau_x \pm i \tau_y$. One can expect that the coupling constant $U_i$ is of the same order of magnitude as $v_i$, i.e. $\sim g/r^3$. Assuming the constant fluctuator density of states $P_0$ we can estimate the typical energy defect for translation of an excitation to the fluctuator at the distance $r$ as $(\delta E) \sim (Rr^3)^{-1}$. The effect of the above off-diagonal interaction can be then estimated as $U/(\delta E) \approx P_0 g \approx \eta/T$. This ratio should be small within the applicability range of our theory. Indeed, qubit will not be useful if its characteristic decay rate, $\tau_\phi \approx \eta$, exceeds its interlevel spacing $E_0$, which should be, in turn, much less than the temperature. However, at very low temperatures and at not too long observation times the above processes could be important [32].

Another issue, which has not been analyzed, is the decoherence near the degeneracy points where $\cos \theta_q \to 0$. These points are of specific importance since linear coupling between the qubit and the fluctuators vanishes. The conventional way, see, e.g., Ref. [33]) is to introduce the model coupling as $V_2 = \lambda X^2(t) \sigma_z$. We believe that the model still needs a careful derivation.

7. Conclusions

The simple model discussed above reproduces essential features of the decoherence of a qubit by “slow” dynamical defects – fluctuators. This model is valid for qubits of different types. The main physical picture is very similar to that of the spectral diffusion in glasses.

The phase memory decay is due to irreversible processes in the fluctuator system. In the case on ensemble-averaged measurements it can be directly determined from the echo-type experiments. In the experiments of the free-induction type the decay of the signal is due both to the finite phase memory and to the beats between different values of the qubit eigenfrequency.

The effective rate of the phase memory decay depends on the relation between the typical interaction strength, $\eta(T)$, and the typical fluctuator relaxation rate $\gamma_0(T)$. The first quantity is just a typical deviation of the qubit eigenfrequency produced by a thermal fluctuator (with the inter-level spacing of the order of temperature $T$) located at a typical distance. The second quantity is the maximal flip-flop rate for the thermal fluctuators.

The estimates for the phase memory time in the limiting cases $\eta \gg \gamma_0$ and $\eta \ll \gamma_0$ are summarized in Eq. (1.54). In the first limiting case during the decoherence time only few fluctuators flip. Consequently, the decoherence is governed by Markovian processes. In the opposite limiting case, typical fluctuators experience many flip-flops during the decoherence time. the subsequent deviations in the qubit frequency being statistically-dependent. As a result, the decoherence process is essentially non-Markovian.

The details of the decoherence depend strongly on the concrete type of the fluctuators, namely on the distribution of their flip-flop rates, on the range of their effective field acting upon the qubit, and on the distribution of the fluctuators in real space.

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