K3 Surfaces with Nine Cusps

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Abstract

By a K3-surface with nine cusps I mean a surface with nine isolated double points $A_2$, but otherwise smooth, such that its minimal desingularisation is a K3-surface. It is shown, that such a surface admits a cyclic triple cover branched precisely over the cusps. This parallels the theorem of Nikulin, that a K3-surface with 16 nodes is a Kummer quotient of a complex torus.

Contents

1 Introduction 1
2 Cyclic triple covers of $K3$-surfaces 2
3 The lattice generated by the nine cusps 3
4 The code of the nine cusps 4
5 The double cover branched over the dual cubic 6
6 References 8

1 Introduction

If $E_1, ..., E_{16}$ are 16 disjoint, smooth curves on a $K3$-surface $X$ then the divisor $\sum_{i=1}^{16} E_i$ is divisible by 2 in $Pic(X)$. This was observed by V.V. Nikulin [N]. Equivalently: If $\tilde{X}$ is the surface obtained from $X$ by blowing down the 16 rational curves to nodes $e_i \in X$, there is a double cover $A \to \tilde{X}$, with $A$ a complex torus, branched exactly over the 16 nodes $e_i$. The surface $\tilde{X}$ is the Kummer surface of the complex torus $A$.

The aim of this note is to prove an analog of Nikulin’s theorem in the case of nine cusps (double points $A_2$) instead of 16 nodes (double points $A_1$):
Theorem. Let \( E_i, E'_i, i = 1, \ldots, 9 \) be 18 smooth rational curves on a K3-surface \( X \) with
\[
E_i, E'_i = 1, \quad E_i.E'_j = E_i.E'_j = 0 \text{ for } i \neq j,
\]
then there are integers \( a_i, a'_i = 1, 2, a_i \neq a'_i \), such that the divisor \( \sum_{i=1}^{9}(a_i E_i + a'_i E'_i) \) is divisible by 3 in Pic(\( X \)). Equivalently: If \( \bar{X} \) is the surface obtained from \( X \) by blowing down the nine pairs of rational curves to cusps \( e_i \in X \), then there is a cyclic cover \( A \to \bar{X} \) of order three, with \( A \) a complex torus, branched exactly over the nine cusps \( e_i \).

The proof I give here essentially parallels Nikulin’s proof in \([N]\).

In the case of Nikulin’s theorem of course each complex torus \( A \) of dimension two appears (the covering involution is the map \( a \mapsto -a \)). But complex tori of dimension two admitting an automorphism of order three with nine fix-points are rarer. If the K3-surface \( X \) is algebraic, then its Picard number is \( \geq 19 \). So in this case the surface \( X \) and the covering surface \( A \) can depend on at most one parameter.

Examples of abelian surfaces with an automorphism of order three are given in \([BH]\): Each selfproduct \( A = C \times C \), with \( C \) an elliptic curve, admits the automorphism
\[
(x, y) \mapsto (-x, x - y).
\]

It is shown in \([BH]\) that the quotient \( \bar{X} \) then is a double cover of the plane \( \mathbb{P}_2 \), branched over the sextic \( C^* \) dual to a plane cubic \( C \subset \mathbb{P}_2 \), a copy of the elliptic curve \( C \). The nine cusps of \( \bar{X} \) of course lie over the nine cusps of \( C^* \).

By deformation theory of K3-surfaces, one may convince oneself, that there are also non-algebraic K3-surfaces with nine cusps.

Convention: Throughout this note the base field for algebraic varieties is \( \mathbb{C} \).

## 2 Cyclic triple covers of K3-surfaces

By a configuration of type \( A_2 \) on a smooth surface I mean a pair \( E, E' \) of smooth rational curves with \( E^2 = (E')^2 = -2, \quad E.E' = 1 \). Such a pair can be contracted to a double point \( A_2 \) (a cusp).

**Lemma 1.** Let \( X \) be a K3-surface carrying \( p \) disjoint configurations of type \( A_2 \), and \( \bar{X} \) the surface obtained from \( X \) by contracting them to cusps. If there is a smooth complex surface \( Y \) and a triple cover \( Y \to \bar{X} \) branched (of order three) precisely over the \( p \) cusps, then either \( p = 6 \) with \( Y \) a K3-surface or \( p = 9 \) with \( Y \) a torus.

Proof. First of all, \( Y \) is kähler: Indeed, \( X \) is kähler by \([S]\). Blow up \( X \) in the \( p \) points, where the \( p \) pairs of curves in the \( A_2 \) configurations meet. The resulting surface \( \bar{X} \) is kähler by \([B, \text{ Theoreme II 6}]\). Pull back the covering \( Y \to \bar{X} \) to a covering \( \tilde{Y} \to \bar{X} \). Here \( \tilde{Y} \) is kähler, since it is a smooth surface in some \( \mathbb{P}_1 \)-bundle over \( \bar{X} \), which is kähler by \([B, \text{ Theoreme principal II}]\). The surface \( Y \) is obtained from \( \tilde{Y} \) by blowing down (-1)-curves, so it is kähler too by \([F]\).

The canonical bundle of \( Y \) admits a section with zeros at most in \( p \) points. So there are no such zeros and \( K_Y \) is trivial.

By the classification of surfaces \([BPV, \text{ p.188}]\) the covering surface \( Y \) therefore either is K3 with \( e(Y) = 24 \) or a torus with \( e(Y) = 0 \). The Euler number of \( Y \) is computed in terms of \( p \) as
\[
e(Y) = 3 \cdot e(\bar{X}) - 2 \cdot p = 3 \cdot (24 - 2p) - 2 \cdot p = 72 - 8 \cdot p.
\]
The possibilities are \( p = 6 \) with \( Y \) a K3-surface and \( p = 9 \) with \( Y \) a complex torus.

Consider the \( 2p \) rational curves \( E_i, E_i' \subset X \) forming the \( p \) configurations of type \( A_2 \). The cyclic cover, lifted to \( X \), is branched along all of these curves of order three. So there must be a divisor
\[
\sum_{i=1}^{p} a_i \cdot E_i + a_i' \cdot E_i' , \quad a_i, a_i' = 1 \text{ or } 2
\]
divisible by three in \( \text{Pic}(X) \). This implies that all intersection numbers
\[
(a_i E_i + a_i' E_i'). E_i = -2 \cdot a_i + a_i'
\]
\[
(a_i E_i + a_i' E_i'). E_i' = a_i - 2 \cdot a_i'
\]
are divisible by three. Hence
\[
a_i = 1 \iff a_i' = 2 \quad \text{and} \quad a_i = 2 \iff a_i' = 1.
\]

3 The lattice generated by the nine cusps

Here let \( X \) be a K3-surface and \( L = H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22} \) its lattice provided with the (unimodular) intersection form. Assume that on \( X \) there are nine disjoint \( A_2 \)-configurations \( E_i, E_i' \), \( i = 1, \ldots, 9 \). Following [N] we denote by \( I \subset L \) the sublattice spanned (over \( \mathbb{Z} \)) by the 18 classes \( [E_i], [E_i'] \).

Let me denote by \( \overline{I} \subset L \) the primitive sublattice spanned by these classes over \( \mathbb{Q} \) and let me put
\[Q := \overline{I}/I.\]

To study \( Q \) we split \( I \) in two sublattices by the base change
\[E_i, E_i' \quad \text{replaced by} \quad E_i, F_i := 2E_i + E_i' \text{ for } i = 1, \ldots, 9.\]

The essential point is that the intersection numbers
\[E_i.F_j = -3 \cdot \delta_{i,j}, \quad F_i.F_j = -6 \cdot \delta_{i,j}\]
are divisible by 3.

Lemma 2. If a class
\[
\sum_{i=1}^{9} \epsilon_i[E_i] + \varphi_i[F_i], \quad \epsilon_i, \varphi_i \in \mathbb{Q},
\]
belongs to \( \overline{I} \) then
\[\epsilon_i \in \mathbb{Z}, \quad 3 \cdot \varphi_i \in \mathbb{Z}.\]

In particular the order of the finite group \( Q \) is \( |Q| = 3^n \) for some \( n \geq 0 \).

Proof. We just intersect the class with \( E_k \) and \( E_k' \) to find
\[
(\sum_{i=1}^{9} \epsilon_i[E_i] + \varphi_i[F_i]). E_k = -2 \cdot \epsilon_k - 3 \cdot \varphi_k \in \mathbb{Z}
\]
\[
(\sum_{i=1}^{9} \epsilon_i[E_i] + \varphi_i[F_i]). E'_k = \epsilon_k \in \mathbb{Z}.
\]
This implies the assertion.

Lemma 2 shows in particular

\[ \bar{I} = E + \bar{F} \]

with \( E \subset I \) the lattice spanned by the classes \([E_i]\), with \( F \subset I \) the lattice spanned by the classes \([F_i]\), and \( \bar{F} \) the primitive sublattice of \( L \) spanned over \( \mathbb{Q} \) by \( F \).

**Lemma 3.** The order \(|Q|\) is \(3^n\) with \(n \geq 3\).

Proof. Choose a system of \(n\) generators for \(Q\). They are the residues of \(n\) classes \(q_1, \ldots, q_9 \in \bar{F}\). The set of these \(n\) classes can be extended to a \(\mathbb{Z}\)-basis

\[ q_1, \ldots, q_n, f_{n+1}, \ldots, f_9 \]

of \(\bar{F}\). So, if \(n \leq 2\), the lattice \(\bar{F}\) has a \(\mathbb{Z}\)-basis \(q_1, q_2, f_3, \ldots, f_9\) with \(f_3, \ldots, f_9\) integral linear combinations of the classes \([F_1], \ldots, [F_9]\). We extend this basis to a \(\mathbb{Z}\)-basis of \(\bar{I}\) with the classes \(e_1 = [E_1], \ldots, e_9 = [E_9]\), and to a basis of \(L\) with some classes \(t_{19}, \ldots, t_{22}\). In the basis

\[ f_3, \ldots, f_9, e_1, \ldots, e_9, q_1, q_2, t_{19}, \ldots, t_{22} \]

the intersection matrix is

\[
\begin{array}{ccc}
7 & 9 & 6 \\
(f_i, f_j) & (f_i, e_j) & * \\
(f_i, e_j) & (e_i, e_j) & * \\
* & * & *
\end{array}
\]

Each summand in the Leibniz expansion of the determinant contains at least ten factors

\[ f_i f_j, \ f_i e_j \text{ or } e_i e_j. \]

At most nine of them can be \(e_i e_j\). At least one of them must be a factor \(f_i f_j\) or \(f_i e_j\) divisible by 3. This shows that the determinant of the \(22 \times 22\) intersection matrix is divisible by 3, a contradiction with unimodularity.

4 The code of the nine cusps

Each class in \(\bar{I}\) is of the form

\[ \sum_{i=1}^{9} \epsilon_i \cdot E_i + \varphi_i \cdot F_i, \quad \epsilon_i \in \mathbb{Z}, \ \varphi_i \in \frac{1}{3} \mathbb{Z}. \]

By sending

\[ \varphi_i \mapsto \varphi_i \mod \mathbb{Z} \]

we identify \(Q\) with an \(\mathbb{F}_3\) sub-vector space of \(\mathbb{F}_3^9\). By lemma 3 the sub-vector space \(Q \subset \mathbb{F}_3^9\) has dimension \(\geq 3\). In this section we want to identify this sub-vector space.

In analogy with coding theory, we call each vector \(q = (q_i)_{i=1, \ldots, 9} \in Q\) a word, and the number of its non-zero coefficients its length \(|q|\). By lemma 1 all vectors \(q \in Q\) have length \(|q| = 0, 6\) or \(9\). As \(\dim_{\mathbb{F}_3}(Q) \geq 3\), the space \(Q\) contains at least \(3^3 - 3 = 24\) words of length 6.
We say that two words \( q, q' \) overlap in \( r \) places, if there are precisely \( r \) ciphers \( i \) such that both coefficients \( q_i \) and \( q'_i \) are nonzero. It is clear that any two nonzero words of length six overlap in at least three places. If they overlap in six places, they are linearly dependent: In fact, if \( q + q' \neq 0 \), we have \( q_i = q'_i \) for at least one \( i \). Then \( q + 2q' \) has length \( \leq 5 \), hence \( q + 2q' = 0 \).

**Claim 1.** Any two linearly independent vectors \( q, q' \) of length six overlap in three or in four places.

Proof. We have to exclude, that \( q \) and \( q' \) overlap in five places. Assume to the contrary that they do. By rescaling the basis vectors of \( \mathbb{F}^3 \) we may assume

\[
q = (1, 1, 1, 1, 1, 0, 0, 0)
\]

and

\[
q' = (0, q'_2, q'_3, q'_4, q'_5, q'_6, q'_7, 0, 0), \quad q'_i = 1 \text{ or } 2.
\]

Since \( q + q' \) again is a word of length six, w.l.o.g.

\[
q' = (0, 2, 1, 1, 1, q'_7, 0, 0).
\]

Then

\[
q + 2q' = (1, 2, 0, 0, 0, 0, 2q'_7, 0, 0) \notin Q,
\]

contradiction. \( \square \)

Now, let me call the nine ciphers 1, ..., 9 'points' and those triplets \( \{i, j, k\} \) of ciphers 'lines', for which there is a word \( q \) of length six with \( q_i = q_j = q_k = 0 \). As there are at least 24 words of length six, there are at least twelve lines. As two linearly independent words of length six overlap in four or three places, two different lines intersect in one point, or not at all (parallel lines). This allows to count the number of lines:

**Claim 2.** There are precisely 12 lines, and therefore the dimension of \( Q \) is \( n = 3 \).

Proof. Through each point, there are at most four distinct lines. So there are at most \( 9 \times 4 = 36 \) incidences of lines with points. As on each line there are three points, we have indeed at most \( 36/3 = 12 \) lines. \( \square \)

This proof shows in particular, that through each point there are exactly four lines, or in other words: Each pair of points lies on a (uniquely determined) line.

**Claim 3.** For each line there are precisely two parallel lines. These two parallel lines do not intersect.

Proof. Each line \( L \) meets \( 3 \times 3 = 9 \) other lines, hence there are two lines \( L', L'' \) parallel to it. If \( L' \) and \( L'' \) would meet in a point, then through this point we would have five lines: the two lines \( L' \) and \( L'' \) and the three lines joining this point with the three points on \( L \). \( \square \)

**Claim 4.** The code \( Q \) contains a word of length nine.

Proof. Take three parallel lines \( L, L', L'' \) and two words \( q, q' \) vanishing on the lines \( L, L' \) respectively. These two words \( q \) and \( q' \) overlap in precisely three places (the points of \( L'' \)). After replacing \( q \) by \( 2 \cdot q \) if necessary, we may assume \( q_i = q'_i \) for one \( i \in L'' \). Then \( q - q' \) is a word of length six, i.e., \( q_i = q'_i \) for all \( i \in L'' \). So \( q + q' \) is a word of length nine. \( \square \)
Claim 4 proves the theorem from the introduction: The existence of a word of length nine shows that there is a linear combination

\[ D := \sum_{i=1}^{9} \varphi_i F_i \in Pic(X) \quad \text{with} \quad 0 < \varphi_i < 1, 3 \varphi_i \in \mathbb{Z}. \]

The divisor

\[ 3 \cdot D - \sum_{\varphi_i=2} 3 \cdot E'_i \]

contains all curves \( E_i \) and \( E'_i, i = 1, \ldots, 9 \), with multiplicity 1 or 2, and it is divisible by 3.

5 The double cover branched over the dual cubic

A smooth cubic \( C \subset \mathbb{P}_2 \) has nine flexes. On the dual cubic \( C^* \subset \mathbb{P}_2^* \) they yield nine cusps. So the double cover \( \bar{X} \to \mathbb{P}_2^* \) is an example of a K3-surface with nine cusps. Here I want to understand the 3-torsion property on \( \bar{X} \) in terms of plane projective geometry, independently of the theory in the preceding sections and of [BH].

The nine flexes of \( C \) in a natural way have the structure of an affine plane over \( \mathbb{F}_3 \). In fact, if \( C \) is given in Hesse normal form

\[ x_0^3 + x_1^3 + x_2^3 + 3\lambda x_0 x_1 x_2 = 0, \]

a transitive action of the vector space \( \mathbb{F}_3^2 \) on the curve \( C \) and thus on the set of its flexes is induced by the symmetries

\[ \sigma : x_i \mapsto x_{i+1}, \quad \tau : x_i \mapsto \omega^i \cdot x_i, \quad \omega \text{ a primitive third root of unity}. \]

Of course the 'lines' used in the preceding section must be the lines in this affine plane. This section will give a proof.

Let me in this section denote by a line in the set of flexes, a line in the sense of the affine structure just mentioned.

The flexes are cut out on \( C \) by the coordinate triangle \( x_0 x_1 x_2 = 0 \). Two parallel lines are formed e.g. by the triplet of flexes \((0 : 1 : -\omega^k)\) and the triplet \((1 : 0 : -\omega^k)\). (All pairs of parallel lines are equivalent to this one, so let us restrict our attention to this pair.) The inflectional tangents there are

\[ -\lambda \omega^k \cdot x_0 + x_1 + \omega^{2k} \cdot x_2 = 0 \quad \text{and} \quad x_0 - \lambda \omega^k \cdot x_1 + \omega^{2k} \cdot x_2 = 0. \]

The essential remark is, that they touch a nondegenerate conic, which in dual coordinates \((\xi_0 : \xi_1 : \xi_2)\) has the equation

\[ \xi_0 \cdot \xi_1 + \lambda \xi_2^2 = 0. \]

(Of course, here we have to exclude \( \lambda = 0 \), the case of the Fermat cubic, where these triplets of inflectional tangents are concurrent.) This implies that the corresponding six cusps on the dual cubic \( C^* \) in \( \mathbb{P}_2^* \) are cut out by the nondegenerate conic, whose equation was just given. This conic intersects \( C^* \) in each cusp with multiplicity 2, so does not touch the tangent of the cusp.
Clearly, the inverse image of this conic on $\bar{X}$ decomposes into two smooth rational curves $\bar{R}, \bar{R}' \subset \bar{X}$ passing through our six distinguished cusps. Denote by $R, R'$ the proper transforms of these curves on the smooth surface $X$. A computation in local coordinates shows, that each curve $R$ or $R'$ meets just one of the two rational curves $E_i, E_i'$ from the $A_2$-configuration over each of the six distinguished cusps. Let me call $E_i$ those curves which meet $R$, and $E_i'$ the curves intersecting $R'$, $i = 1, ..., 6$.

**Lemma 4.** For general choice of $\lambda$, the K3 surface $X$ has Picard number 19.

By [PS, §8] there is only a countable set of K3-surfaces with Picard number 20. So, all we have to show is that the structure of $X$ indeed varies with the elliptic curve $C$. In fact, a copy of $C$ (the proper transform of the branch locus) lies on $X$, where it passes through the intersection points in $E_i \cap E_i'$, $i = 1, ..., 6$. So, if the structure of $X$ would not vary with $C$, we would have on $X$ more than countably many elliptic pencils, a contradiction.

This implies that $NS(X)$ is generated (over $\mathbb{Q}$) by the classes of $E_i, E_i'$, $i = 1, ..., 9$, and by the pullback $[H]$ of the class of a line on $\mathbb{P}^2$. Now put

$$R - R' \sim \sum_{i=1}^{9} (n_i \cdot [E_i] + n_i' \cdot [E_i']) + n \cdot H, \quad n, n_i, n_i' \in \mathbb{Q}.$$  

From

$$(R - R').H = (R - R').E_i = (R - R').E_i' = 0 \quad \text{for } i = 7, 8, 9$$

we conclude $n = n_i = n_i' = 0$ for $i = 7, 8, 9$. The other intersection numbers are ($i = 1, ..., 6$)

$$
\begin{align*}
1 & \quad = \quad (R - R').E_i \quad = \quad -2n_i + n_i' \\
-1 & \quad = \quad (R - R').E_i' \quad = \quad n_i - 2n_i'
\end{align*}
$$

This implies

$$n_i' = -n_i = \frac{1}{3}.$$  

We have shown that the class

$$\frac{1}{3} \sum_{i=1}^{6} (E_i' - E_i) = R - R'$$

is integral. This is equivalent to the fact that the classes

$$\sum_{i=1}^{6} (2 \cdot E_i + E_i') \quad \text{and} \quad \sum_{i=1}^{6} (E_i + 2 \cdot E_i')$$

are divisible by 3.

Finally, we remark: For a 3-divisible set of six cusps on $\bar{X}$ the pattern, in which the curves $E_i$ and $E_i'$ organize themselves into unprimed and primed ones, is given by their intersections with $R$ or $R'$.  

7
6 References

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