Selberg Integral and $SU(N)$ AGT Conjecture

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Abstract

An intriguing coincidence between the partition function of super Yang-Mills theory and correlation functions of 2d Toda system has been heavily studied recently. While the partition function of gauge theory was explored by Nekrasov, the correlation functions of Toda equation have not been completely understood. In this paper, we study the latter in the form of Dotsenko-Fateev integral and reduce it in the form of Selberg integral of several Jack polynomials. We conjecture a formula for such Selberg average which satisfies some consistency conditions and show that it reproduces the $SU(N)$ version of AGT conjecture.

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1 Introduction

Two years ago Alday, Gaiotto and Tachikawa [1] presented an interesting observation that the partition functions of certain class of $\mathcal{N} = 2$ $SU(2)$ gauge theories [2,3] seem to coincide with the correlation function of 2D Liouville theory. After some translation rules of parameters, they confirmed a relation which may be written schematically as,

$$Z_{\mathcal{N}=2} = \langle V \cdots V \rangle_{\text{Liouville}}.$$ 

They conjectured that such correspondence exists for large class of $\mathcal{N} = 2$ gauge theories. Soon later, Wyllard [4] and others [5, 6] has presented a generalization to $SU(N)$ gauge theories.

This conjecture is illuminating in showing a correspondence between 4D Yang-Mills and 2D integrable models and will be fundamental in the understanding of the duality of gauge theories. It also will be relevant to understand strong coupling physics of multiple M5-branes. In this respect, it will be important to understand to which extent and how this conjecture holds. Especially, since the coincidence was found through the first few orders in the instanton expansion of $q = e^{\pi i \tau}$, the exact computation of conformal block is needed in the Liouville side.

Recently, A. Mironov et. al. [7,8] has embarked on an interesting step toward this direction. They used the Dotsenko-Fateev method [9] to calculate the conformal blocks (see [10,11] for earlier contributions). They analyzed the simplest example $SU(2)$, $N_f = 4$ and proved the AGT relation for a special choice of a parameter $\beta = -\epsilon_1/\epsilon_2 = 1$. The key step in their analysis is the reduction of the Dotsenko-Fateev (DF) formula to Selberg average with one or two Jack polynomial(s) which was computed explicitly by Kadell [12].

In this paper, we generalize this idea to $SU(N)$ case. We find that DF formula is reduced to $A_{N-1}$-type Selberg average of a product of $N$ Jack polynomials. While we do not manage to compute the integral, it is possible to guess the answer (3.32) at least for $\beta = 1$. As we will see, it is still nontrivial task to check if it reproduces the known results [13] and satisfies some consistency conditions that the integral should obey. With this conjectured formula, we can prove the $SU(N)$ version of AGT formula.

We organize the sections as follows. In §2 we briefly review the relevant results of Nekrasov’s formula and AGT conjecture. In §3 we derive the DF formula for the conformal block can reduced to Selberg integral. This part is a generalization of [7,8] from $SU(2)$ to $SU(N)$. In particular, we show how Selberg average of the product of $N$ Jack polynomials gives the DF formula. After presenting the known results [12,14], we give a conjecture for $N$ Jack average and examine the consistency conditions. In §4 we show that it reproduces the AGT conjecture properly.

Since this paper needs many technical detail, we have substantial amount of sections in the appendix. In appendix A, we summarize the notation for Young diagrams. In appendix B, we collect relevant materials on Jack polynomial which is essential in our computation. In appendix C we present the more general conjecture for Selberg integral for arbitrary $\beta$. While this formula needs modification, it satisfies various consistency condition nontrivially and may be useful in the future development. In appendix D we write the explicit computation of the check of consistency for $N$ Selberg integral. In appendix E, we give proofs of lemmas which are used to bring Selberg average into the form of Yang-Mills partition function.
2 A brief review of AGT conjecture and Nekrasov formula

Nekrasov’s partition function We first recall the partition function of $\mathcal{N} = 2$ super Yang-Mills theory \cite{2, 3}. With graviphoton deformation parameters $\epsilon_1, \epsilon_2$ which were introduced for the regularization, the partition function for $G = U(N_1) \times \cdots \times U(N_n)$ linear quiver gauge theory was obtained by the localization technique. It is schematically written as

$$Z_{\text{full}}(q; a, m; \epsilon) = Z_{\text{tree}}Z_{1\text{loop}}Z_{\text{inst}}, \quad Z_{\text{inst}}(q; a, m; \epsilon) = \sum_{\mathbf{Y}} q^{\mathbf{Y}} z(\mathbf{Y}, a, m), \quad (2.1)$$

where $\mathbf{Y} := (\mathbf{Y}^{(1)}, \cdots, \mathbf{Y}^{(n)})$, $q^{\mathbf{Y}} := \prod_{i=1}^{n} q_{z_i}^{[\mathbf{Y}^{(i)}]}$. The parameter $a$ (resp. $m$) represents the diagonalized VEV of vector multiplets (resp. mass of hypermultiplets) whereas $q_i = e^{\pi i \tau_i}$ is the instanton expansion parameter for $i$th gauge group $SU(N_i)$. The total partition function is decomposed into a product of the contributions of the perturbative parts $Z_{\text{tree}}$, $Z_{1\text{loop}}$ and non-perturbative instanton correction $Z_{\text{inst}}$. The latter is further decomposed into a sum of sets of Young diagrams. $\mathbf{Y}^{(i)} = (Y_{1}^{(i)}, \cdots, Y_{N_i}^{(i)})$ is a collection of $N_i$ Young diagram which parameterizes the fixed points of instanton moduli space for $i$th gauge group $U(N_i).

In this paper, we will mainly focus on the instanton part. The coefficient $z(\mathbf{Y}, a, m)$ is described as a product of the contributions of the gauge- and hyper multiplets which describes the system:

$$z(\mathbf{Y}, a, m) = \prod_{i=1}^{n} z_{\text{vect}}(a^{(i)}, \mathbf{Y}^{(i)}) \prod_{R} z_{R}(\mathbf{Y}, a, m), \quad (2.2)$$

where $R$ is the representation for each hypermultiplets:

$$z_{\text{bifund}}(a, \mathbf{Y}; b, \mathbf{W}; m) = \prod_{i=1}^{N_1} \prod_{s=1}^{N_2} G_{Y_i,W_s}(a_t - b_s - m) G_{W_s,Y_i}(b_s - a_t + m + 1 - \beta), \quad (2.3)$$

$$z_{\text{fund}}(a, \mathbf{Y}; m) = \prod_{s=1}^{N} f_{Y_s}(a_s - m - 1 + \beta), \quad (2.4)$$

$$z_{\text{adj}}(a, \mathbf{Y}; m) = z_{\text{fund}}(a, \mathbf{Y}, -1 + \beta - m), \quad (2.5)$$

$$z_{\text{adj}}(a, \mathbf{Y}; m) = z_{\text{bifund}}(a, \mathbf{Y}, a, \mathbf{Y}, \mathbf{m}), \quad (2.6)$$

$$z_{\text{vect}}(a, \mathbf{Y}) = 1/z_{\text{adj}}(a, \mathbf{Y}, 0). \quad (2.7)$$

In eq. (2.3), the hypermultiplet is supposed to transform as bifundamental associated with gauge group $U(N_1) \times U(N_2)$. Similarly, in eq. (2.4), the fundamental representation is associated with $U(N)$. The function $G$ in eq. (2.3) is a function with respect to the tableau $Y$’s arm-length and leg-length (see (A.1) for their definitions)

$$G_{A,B}(x) = \prod_{(i,j) \in A} \left(x + \beta(A_j' - i) + (B_i - j) + \beta\right), \quad (2.8)$$

and the function $f$ in (2.4) is defined as

$$f_{A}(z) = \prod_{(i,j) \in A} \left(z + \beta(i - 1) - (j - 1)\right). \quad (2.9)$$
Instead of considering general quiver gauge theories, we are mainly interested in the simplest case, \( G = SU(N) \), with \( N_f = 2N \) hypermultiplets in fundamental representation. In this specific example, the partition function is written as

\[
Z_{\text{full}}(q; a, \mu; \epsilon) = Z_{\text{tree}} Z_{1\text{loop}} Z_{\text{inst}}, \quad Z_{\text{inst}}(q; a, m; \epsilon) = \sum_{\vec{Y}} q^{\vec{Y}} N_{\vec{Y}}^{\text{inst}}(a, \mu),
\]

Here the product of the constants \( C_{\vec{Y}} \) should be proportional to either \( \omega_1 \) or \( \omega_{N-1} \) where \( \omega_i (i = 1, \cdots, N - 1) \) is the fundamental weight of \( A_{N-1} \).

\[\begin{align*}
N_{\vec{Y}}^{\text{inst}}(a, \mu) &= z_{\text{vec}}(\vec{Y}, a) \prod_{i=1}^{2N} z_{\text{fund}}(\vec{Y}, \mu_i) = \prod_{r,s=1}^{N} \prod_{k=1}^{2N} J_{\chi_k}(\mu_k + a_s) \prod_{l,s=1}^{N} g_{Y_s, Y_l}(a_{l} - a_s),
\end{align*}\]

with \( g_{AB}(x) := G_{AB}(x) G_{AB}(x+1-\beta) \). \( \mu_i (i = 1, \cdots, 2N) \) are mass parameters of hypermultiplets with fundamental representation.

**AGT conjecture** In [1] Alday, Gaiotto and Tachikawa pointed out that this partition function is identical to the correlation functions of Liouville theory when the gauge group is \( SU(2) \). It takes the form (here we give example of \( n \)-point function on sphere):

\[
\langle V_n(\infty)V_{n-1}(1)V_{n-2}(q_1) \cdots V_2(q_1 \cdots q_{n-3})V_1(0) \rangle = \sum_{\psi_1, \cdots, \psi_{n-3}} C_{V_1V_2V_1} \cdots C_{V_{n-3}V_{n-1}V_{n-1}} |F_{V_1V_2U_1 \cdots U_{n-3}V_{n-1}V_n}(z_1, \cdots, z_n)|^2.
\]

Here the product of the constants \( C_{V_1V_2U_1} \) etc. are from the 3-point functions. For Liouville case, it is given by DOZZ formula [15][20]. The function \( F \) carries the coordinate \( (q) \) dependence and reflects the contributions of the conformal descendants. It is called conformal block.

In order to give the identification of partition function with the correlator, we need some identification of parameters: \( a, m \leftrightarrow \alpha \) and the coordinate \( q \) in CFT is identified with the coupling constant \( q = e^{\pi i \tau} \) in Yang-Mills. Here \( \alpha \in \mathbb{R}^N \) is a parameter which appears in the exponential of the vertex operator \( V_\alpha = e^{i(\alpha, \phi)} \) inserted in the correlator.

With such identification, it is shown that \( Z_{\text{inst}} \) in the gauge theory written in a form [21] is identical to the conformal blocks, and the perturbative part \( Z_{1\text{loop}} \) corresponds to the (product of) three point functions [1][3].

To be more explicit, for the specific example of \( SU(N) \) gauge theory with \( N_f = 2N \) fundamental matter, the relevant Toda correlator is written in the form

\[
\langle V_{\alpha_1}(\infty)V_{\alpha_3}(1)V_{\alpha_2}(q)V_{\alpha_1}(0) \rangle,
\]

where the insertion of screening operators is necessary for the charge conservation. The conformal block of this correlation function is written in the form,

\[
F_{\alpha_1, \alpha_3, \alpha_2, \alpha_1}(q) = \sum_{\vec{Y}} q^{\vec{Y}} N_{\vec{Y}}^{\text{Toda}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4).
\]
AGT conjecture for $SU(N)$ [4,5] implies that partition function and the correlator are the same. In particular it implies,
\[ N_Y^{\text{inst}}(a, \mu) = N_Y^{\text{Toda}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4), \] (2.15)
if we identify the parameters,
\[ a = \alpha; \quad \mu = -\alpha_1 - (1 - \beta)\rho, \quad \tilde{\mu} = -\alpha_4 - (1 - \beta)\rho; \] (2.16)
where $\mu = (\mu_1, \cdots, \mu_N)$ and $\tilde{\mu} = (\mu_{N+1}, \cdots, \mu_{2N})$ are mass parameters of vector multiplets. $\alpha = \alpha_1 + \alpha_2 + \beta \sum_a N_a e_a + (1 - \beta) = -(\alpha_4 + \alpha_3 + \beta \sum \tilde{N}_a e_a + (1 - \beta))$ is the momentum which appears in the intermediate channel ($N_a$ and $\tilde{N}_a$ are the numbers of screening charges and $e_a$ is the simple root of $A_{N-1}$). Weyl vector $\rho = \sum_{i=1}^{N-1} \omega_i$ shows up to represent the corrections of the background charge. As explained, we choose $\alpha_2$ and $\alpha_3$ to be proportional to $\omega_1$.

We focus on this “identity” in the following.

3 Correlation functions of Toda theory and Selberg Formula

In this section, we give a brief review on Dotsenko-Fateev integral representation of the correlation function of Toda theory. We will focus on the four point functions. We show, by generalizing the argument of [7], that the integral reduces to the product of Selberg average of $N$ Jack polynomials for $SU(N)$ Toda theory. Finally, we present our conjecture on Selberg average which will lead to $SU(N)$ AGT conjecture.

3.1 $W_N$ algebra and Dotsenko-Fateev integral

The correlator in $SU(N)$ Toda field theory is given as the conformal block for $W_N$ algebra which consists of the operator algebra chiral operators $W^{(s)}(z)$ with spin $s = 2, \cdots, N$. It has a free boson representation [24]. Let $\phi(z) = (\phi_1(z), \cdots, \phi_N(z))$ be free bosons which satisfies the operator-product expansion: $\phi_j(z)\phi_k(0) \sim \delta^{jk} \log(z)$.

\[ R_N = : \prod_{m=1}^{N} (Q \frac{d}{dz} - i(h_m, \partial_z \phi)) : = \sum_k W^{(k)}(z) \left( Q \frac{d}{dz} \right)^{N-k}, \] (3.1)

$h_m$ are vectors in $\mathbb{R}^N$ and defined by $(h_j)_k = \delta_{jk} - \frac{1}{N}$. Since it satisfies $\sum_{m=1}^{N} (h_j)_m = 0$, a component of $\phi$ is decoupled. The definition (3.1) gives $W^{(0)}(z) = 1$ and $W^{(1)}(z) = 0$. The Virasoro generator is

\[ W^{(2)}(z) = \frac{1}{2} : (\partial_z \phi)^2 : -i Q(\rho, \partial_z^2 \phi), \quad \rho = \sum_{i=1}^{N-1} \omega_i = \left( \frac{N-1}{2}, \frac{N-3}{2}, \cdots, -\frac{N-1}{2} \right), \] (3.2)

which has the central charge $c = (N-1)(1 + N(N+1)Q^2)$.

The primary operator of $W_N$ algebra is given as the vertex operators:

\[ V_{\alpha}^a(z) = : e^{(\alpha, \phi(z))} :, \] (3.3)
which has the OPE with the $W_N$ generators as
\begin{equation}
W_k(z) V_\alpha(0) = \frac{w_k(\alpha)}{z^k} V_\alpha(0) + O(z^{-k+1}),
\end{equation}
with
\begin{align}
w_2(\alpha) &= \Delta(\alpha) = \frac{1}{2} (\alpha, \alpha) + iQ(\rho, \alpha), \quad \text{(3.5)} \\
w_k(\alpha) &= (-1)^k \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \prod_{m=1}^k (Q(k - m) + i(h_{i_m}, \alpha)) . \quad \text{(3.6)}
\end{align}

In order to derive nonvanishing correlation function of the form $\langle V_{\tilde{a}_1}(z_1) \cdots V_{\tilde{a}_M}(z_M) \rangle$, we have freedom to insert screening operators,
\begin{equation}
Q_j^{(\pm)} = \int \frac{dz}{2\pi i} V_j^{(\pm)}(z) = \int \frac{dz}{2\pi i} : e^{\alpha_\pm (e_j, \phi(z))} : . \quad \text{(3.7)}
\end{equation}
By the requirement of conformal invariance, $w_2(\alpha) = 1$, we need to put $w_2(\alpha_\pm e_j) = 1$. By writing $Q = ib - i/b$, the two solutions are $\alpha_+ = b$, $\alpha_- = -1/b$.

For the computation of four point functions $\langle V_{\tilde{a}_1}(\infty)V_{\tilde{a}_2}(1)V_{\tilde{a}_3}(q)V_{\tilde{a}_4}(0) \rangle$ we insert $N_a$ screening currents integrated along $[0, q]$ and $\tilde{N}_a$ currents integrated $[1, \infty]$. This is a useful prescription to see the connection with the Selberg formula [8]. For simplicity, we assume we need only the screening operators $Q^{(\pm)}$ in the correlator. It gives the Dotsenko-Fateev integral [9] for the four point functions,
\begin{equation}
Z_{\text{DF}}(q) = \left\langle \left( : e^{(\alpha_1, \phi(0))} :: e^{(\alpha_2, \phi(q))} : e^{(\alpha_3, \phi(1))} :: e^{(\alpha_4, \phi(\infty))} : \right) \prod_{a=1}^{N-1} \left( \int_0^q : e^{b(e_a, \phi(z))} : dz \right) \right( \int_1^\infty : e^{b(e_a, \phi(z))} : dz \right) \right\rangle_{\tilde{N}_a} . \quad \text{(3.8)}
\end{equation}
For the charge conservation, this correlator has nonvanishing norm only when
\begin{equation}
\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4 + b \sum_a (N_a + \tilde{N}_a) e_a + 2iQ\rho = 0. \quad \text{(3.9)}
\end{equation}
We apply Wick’s theorem to evaluate the correlator
\begin{equation}
\left\langle \left( : e^{(\tilde{\alpha}_1, \phi(z_1))} : \cdots : e^{(\tilde{\alpha}_n, \phi(z_n))} : \right) \right\rangle = \prod_{1 \leq i < j \leq n} (z_j - z_i)^{(\tilde{\alpha}_i, \tilde{\alpha}_j)} , \quad \text{(3.10)}
\end{equation}
where $e_a$ are the simple roots of $SU(N)$, and $(\cdot\cdot\cdot)$ the bilinear symmetric form on the space dual to the Cartan subalgebra. To be consistent with the parameters introduced in the last section, defining $\tilde{\alpha}_i = \alpha_i/b$, $\beta = b^2$, [3.8] becomes
\begin{equation}
Z_{\text{DF}}(q) = q^{(\alpha_1, \alpha_2)/\beta} (1 - q)^{(\alpha_2, \alpha_3)/\beta} \prod_{a=1}^{N-1} \prod_{I=1}^{N_a} \int_0^q dz_{I}^{(a)} \prod_{J=1}^{N_a+\tilde{N}_a} \int_1^\infty dz_{J}^{(a)} \prod_{i \leq j}^{N_a+\tilde{N}_a} (z_{j}^{(a)} - z_{i}^{(a)})^{2\beta} \times \prod_{i=1}^{N_a+\tilde{N}_a} (z_{i}^{(a)} - q)^{(\alpha_1, e_a)} (z_{i}^{(a)} - 1)^{(\alpha_3, e_a)} \prod_{a=1}^{N-2} N_a + \tilde{N}_a \prod_{a=1}^{N_a+1} \prod_{i=1}^{N_a} (z_{i}^{(a+1)} - z_{i}^{(a)})^{-\beta} . \quad \text{(3.11)}
\end{equation}
We note that we do not include the 3-point functions in the correlator. Thus this expression should be compared with the instanton contribution of Yang-Mills partition functions in AGT conjecture [10].
3.2 Reduction to Selberg integral

In 1944 Selberg find a proof of a noteworthy multiple integral which now plays the role as one of the most fundamental hypergeometric integrals [7]. Here we consider its $A_{N-1}$ extension [13] ($A_{N-1}$ Selberg integral):

$$S_{\vec{u},\vec{v},\beta} = \int dx \prod_{a=1}^{N-1} \left| \Delta(x^{(a)}) \right|^{2\beta} \prod_{i=1}^{N_a} (x_i^{(a)})^{u_a} (1 - x_i^{(a)})^{v_a} \prod_{a=1}^{N-2} \Delta(x^{(a)}, x^{(a+1)})^{-\beta},$$  \hspace{1cm} (3.12)

where $\int dx := \int dx(1) \ldots \int dx(N-1)$. As indicated, the integral contains parameters $\vec{u} = (u_1, \cdots, u_{N-1})$, $\vec{v} = (v_1, \cdots, v_{N-1})$ and $\beta$. Similarly, $A_{N-1}$ Selberg average is the integration with the Selberg integration kernel,

$$\langle f \rangle_{\vec{u},\vec{v},\beta} = \frac{1}{S_{\vec{u},\vec{v},\beta}} \int dx \prod_{a=1}^{N-1} \left| \Delta(x^{(a)}) \right|^{2\beta} \prod_{i=1}^{N_a} (x_i^{(a)})^{u_a} (1 - x_i^{(a)})^{v_a} \times \prod_{a=1}^{N-2} \Delta(x^{(a)}, x^{(a+1)})^{-\beta} f(x).$$  \hspace{1cm} (3.13)

In this subsection, we rewrite the Dotsenko-Fateev integral in the form of $A_{N-1}$ Selberg average for the product of $N$ Jack polynomials (see appendix [B] for a summary of relevant material and [25,26] for further mathematical details). In physics literature, Jack polynomial is the eigenfunction of quantum Calogero-Sutherland model and relevant to the representation theory of $W_N$ algebra. See for example [27,28]. The appearance of the product of $N$ Jack polynomials reminds us of another line of recent developments [29,30] for the computation of conformal block where the convenient basis for the Hilbert space is expressed in terms of Jack polynomial. In particular for $\beta = 1$, it is expressed as product of $N$ Schur polynomial. While the mathematical origin of the appearance of Jack polynomial is different, there should be a good hint to be learned from each other.

**Proposition 1** The integral (3.11) can be written in the following form (up to $U(1)$ factor),

$$Z_{DF}(q) = \sum_{\vec{Y}} q^{\vec{Y}} \left[ \prod_{a=1}^{N} j_{Y_a}^{(\beta)} \left( -r_k^{(a)} - v_a^+ / \beta \right) \right] + \left[ \prod_{a=1}^{N} j_{Y_a}^{(\beta)} \left( r_k^{(a)} + v_a^- / \beta \right) \right].$$  \hspace{1cm} (3.14)

Here we have to explain some notations. $\vec{Y}$ is a collection of $N$ Young diagrams, $j_{Y}^{(\beta)}$ is normalized Jack symmetric polynomial. We introduced new parameters $v_{a+}$ and $u_{a+}$ by

$$v_{a+} = (\alpha_2, e_a), \quad u_{a+} = (\alpha_1, e_a), \quad u_{a-} = (\alpha_4, e_a),$$

where we use a relation

$$u_{a+} + u_{a-} + v_{a+} + v_{a-} + \beta \sum_b C_{ab}(N_b + \tilde{N}_b) = 2\beta - 2$$  \hspace{1cm} (3.16)

implied by Eq. (3.9) to define $u_{a-}$. The Selberg average $\langle \cdots \rangle_{\pm}$ is taken with respect to these parameters, $\langle \cdots \rangle_{\pm} := \langle \cdots \rangle_{\vec{u}_{\pm},\vec{v}_{\pm},\beta}$. $r_k^{(a)}$ and $\tilde{r}_k^{(a)}$ is related to the integration variables $x_i^{(a)}$ and $y_i^{(a)}$ through

$$r_k^{(a)} := \tilde{p}_k^{(a)} - p_k^{(a-1)}, \quad \tilde{r}_k^{(a)} := \tilde{p}_k^{(a)} - p_k^{(a-1)}, \quad \tilde{p}_k := \sum_i (y_i^{(a)})^k,$$

with $p_k^{(0)} = p_k^{(N)} = \tilde{p}_k^{(0)} = \tilde{p}_k^{(N)} = 0$. Finally $v_a' := -\sum_{s=1}^{a-1} v_{s-}$, and $v_{(N-a)+}' := \sum_{s=1}^{a} v_{(N-s)+}$. 


In particular, when $N = 2$, the above reduce to (notice that $v_{1-} = v_{2+} = 0$)

$$Z_{DF}(q) = \sum_{A,B} q^{|A|+|B|} \left\langle j_A^{(\beta)} (-p_k - \frac{v_+}{\beta}) j_B^{(\beta)} (p_k) \right\rangle + \left\langle j_A^{(\beta)} (\tilde{p}_k) j_B^{(\beta)} (-\tilde{p}_k - \frac{v_-}{\beta}) \right\rangle,$$  \hspace{1cm} (3.18)

which was used in [7]. The proposition is a generalization of their result.

**Proof:** Let us derive the proposition in the rest of this subsection. Following the procedure in [11] for $SU(2)$, we rename the integration variables in (3.11) $Z_I =: qx_I$, $1 \leq I \leq N_a$ and $z_J =: \frac{1}{y_J}$, $N_a + 1 \leq J \leq N_a + \tilde{N}_a$. Then Eq.(3.11) is rewritten as a double average

$$\left\langle \prod_{a=1}^{N-1} \prod_{i=1}^{N_a} (1 - qx_i^{(a)} y_j^{(b)})^{v_{a+}} \prod_{j=1}^{\tilde{N}_b} (1 - qy_j^{(a)} y_j^{(b)})^{v_{a-}} \right\rangle_{a,b=1,i,j}^{N-1,\tilde{N}_b} = \prod_{i=1}^{\tilde{N}_b} (1 - qy_j^{(a)} y_j^{(b)})^{C_{ab}} \left\langle \prod_{a=1}^{N_a} \prod_{j=1}^{\tilde{N}_b} (1 - qx_i^{(a)} y_j^{(b)})^{C_{ab}} \right\rangle_{a,b=1,i,j}^{N-1,\tilde{N}_b}, \hspace{1cm} (3.19)$$

where $C_{ab}$ is $A_{N-1}$ Cartan matrix,

$$C_{ab} = \begin{cases} 2 & a = b \\ -1 & a = b \pm 1 \\ 0 & |a - b| > 1 , \end{cases}$$

and the Selberg average $(\cdots)_+$ (resp. $(\cdots)_-$) is taken over the variables $x_i^{(a)}$ (resp. $y_i^{(a)}$) with parameters $\tilde{u}_+, \tilde{v}_+$ (resp. $\tilde{u}_-, \tilde{v}_-$).

We change the second product in the integral (3.19) into exponential form

$$\prod_{a,b=1}^{N} \prod_{i=1}^{N_a} \prod_{j=1}^{\tilde{N}_b} (1 - qx_i^{(a)} y_j^{(b)})^{C_{ab}} = \exp \left\{ -\beta \sum_{a,b=1}^{N_a} \sum_{i,j} \ln(1 - qx_i^{(a)} y_j^{(b)}) \right\}$$

$$= \exp \left\{ -\beta \sum_{a,b=1}^{N_a} \sum_{k=1}^{\infty} \frac{q^k}{k} \frac{1}{p_k^{(a)} p_k^{(b)}} \right\}$$

$$= \exp \left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N-1} \frac{1}{p_k^{(a)} p_k^{(a-1)}} - \sum_{a=1}^{\tilde{N}_b} \frac{1}{p_k^{(a)} p_k^{(a+1)}} \right\}$$

$$= \exp \left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N-1} r_k^{(a)} \right\}.$$ \hspace{1cm} (3.20)

In the second line, we performed Taylor expansion and rewrite the variables $x, y$ by $p_k^{(a)}$ and $\tilde{p}_k^{(b)}$. In the last line, we rewrite $p_k, \tilde{p}_k$ by $r_k^{(a)}$. In the new variables,

$$\prod_{a=1}^{N-1} \prod_{i=1}^{N_a} (1 - qx_i^{(a)})^{v_{a-}} = \exp \left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N-1} \frac{1}{p_k^{(a)} v_{a-}} \right\} \equiv \exp \left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N} \frac{1}{r_k^{(a)} v_{a-}} \right\}.$$ \hspace{1cm} (3.21)

\footnote{The $U(1)$ prefactors are omitted for its irrelevance to the Nekrasov function.}
In the second equivalence we change the basis from \( p_k^{(a)} \) to \( r_k^{(a)} \). The coefficients \( v'_a \) are determined from \( v_{a-} \) with an additional condition \( v'_a := 0 \) which is somewhat arbitrary. Similarly,

\[
\prod_{a=1}^{N-1} \prod_{\alpha=1}^{N} (1 - q y_j^{(a)})^{v_{a+}} = \exp \left\{ - \beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N} \tilde{r}_k^{(a)} \frac{v'_a}{\beta} \right\}. \tag{3.22}
\]

This time we define \( v'_a \) from another condition \( v'_N = 0 \) for the convenience of later arguments.

Combining the above factors together, the integrand in (3.19) takes the form

\[
\exp \left\{ - \beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N} \left[ (r_k^{(a)} + v'_a) \tilde{r}_k^{(a)} + \frac{v'_a - v_{a-}}{\beta} \right] \right\}
\]

\[
= \prod_{a=1}^{N} (1 - q)^{v'_a + v'_{a-}/\beta} \sum_{\vec{y}} \prod_{a=1}^{N} q^{\vec{y}} j_{\lambda_a} \left( -r_k^{(a)} - \frac{v'_a + v'_{a-}}{\beta} \right) j_{\lambda_a} \left( r_k^{(a)} + \frac{v'_a - v_{a-}}{\beta} \right), \tag{3.23}
\]

where we have made use of the Cauchy-Stanley identity (B.7) for the Jack polynomial in the second line

\[
\exp(\beta \sum_{k=1}^{\infty} \frac{1}{k} p_k p'_k) = \sum_{R} j_R^{(\beta)}(p) j_{\lambda_R}^{(\beta)}(p'). \tag{3.24}
\]

So the conformal blocks (3.5) finally becomes

\[
\prod_{a=1}^{N} (1 - q)^{v'_a + v'_{a-}/\beta} \sum_{\vec{y}} \left( \prod_{a=1}^{N} \left( j_{\lambda_a} \left( -r_k^{(a)} - \frac{v'_a + v'_{a-}}{\beta} \right) \right) + \prod_{a=1}^{N} \left( j_{\lambda_a} \left( r_k^{(a)} + \frac{v'_a - v_{a-}}{\beta} \right) \right) \right) . \tag{3.25}
\]

Absorbing the prefactor into the \( U(1) \) part of the product, we arrive at (3.14). QED

### 3.3 Known results and a conjecture on Selberg average

The Dotzenko-Fateev integral is now reduced to the evaluation of Selberg average of \( N \) Jack polynomials. Let us first summarize the known results on Selberg average in the literature.

**SU(2) case:** The relevant Selberg averages for one and two Jack polynomials were obtained by Kadell [12],

\[
\left\langle J_Y^{(\beta)}(p) \right\rangle_{u,v,\beta}^{SU(2)} = \frac{[N \beta]_Y [u + N \beta + 1 - \beta]_Y}{\prod_{(i,j) \in Y} \left( (Y_j - i) + (Y_i - j) + \beta \right) [u + v + 2N \beta + 2 - 2\beta]_Y}, \tag{3.26}
\]

\[
\left\langle J_A^{(\beta)}(p + w) J_B^{(\beta)}(p) \right\rangle_{u,v,\beta}^{SU(2)} = \frac{[v + N \beta + 1 - \beta]_A [u + N \beta + 1 - \beta]_B}{[N \beta]_A [u + v + N \beta + 2 - 2\beta]_B} \times \prod_{i < j} \left( A_i - A_j + (j - i) \beta \right) \prod_{i < j} \left( B_i - B_j + (j - i) \beta \right) \beta
\]

\[
\times \frac{\prod_{i,j} (u + v + 2N + 2 - (1 + i + j) \beta)}{\prod_{i,j} (u + v + 2N + 2 + A_i + B_j - (1 + i + j) \beta)} \beta \times \prod_{i < j} \frac{u + v + 2N + 2 - (1 + i + j) \beta}{u + v + 2N + 2 - (1 + i + j) \beta} \beta, \tag{3.27}
\]
where we have used the following notation
\[
[x]_A = \prod_{(i,j) \in A} (x - \beta(i - 1) + j - 1) = (-1)^{|A|} f_A(-x),
\]
and Pochhammer symbol
\[
(x)_k = \frac{\Gamma(x + k)}{\Gamma(x)} = x(x+1)\ldots(x+k-1).
\]

\(J_Y^{(\beta)}\), the Jack polynomial, is related to normalized one \(j_Y^{(\beta)}\) as \(\text{(B.7)}\). Inclusion of a shift \(w\) of the argument for the two Jack case was conjectured in \(\text{[7]}\). Together with the identity \(j_A^{(\beta)}(-p/\beta) = (-1)^{|A|} j_A^{(1/\beta)}(p)\) and an identification of parameter \(w = (v+1 - \beta)/\beta\), these are sufficient to evaluate \(\text{(3.14)}\) for \(SU(2)\) case \(\text{[7]}\).

\(SU(n+1)\) case: The one-Jack Selberg integral for \(SU(n+1)\) could be calculated by the formula offered by Warnaar \(\text{[13]}\). To perform the integral, we need to restrict the parameter \(v\) as,
\[
v_2 = \ldots = v_n = 0, \quad \text{and} \quad v_1 = v.
\]
As already explained, in Toda field theory, this condition is necessary to solve conformal Ward identity for the W-algebra \(\text{[4][22]}\). The formula by Warnaar is,
\[
\left\langle J_B^{(\beta)}(p_k^{(n)}) \right\rangle_{SU(n+1)}^{\bar{a}, \bar{r}, \beta} = \prod_{1 \leq i < j \leq N_n} ((j - i + 1) \beta)_{B_i - B_j} \times \prod_{a=1}^n \prod_{i=1}^{N_n} (u_{n-a+1} + \ldots + u_n + a + (N_n - a - i + 1) \beta)_{B_i}.
\]
To evaluate \(\text{(3.14)}\), we need Selberg average of \((n+1)\) Jack polynomials. While we do not perform the integration so far, we find a formula for \(\beta = 1\) which reproduces known results and satisfies consistency conditions\(\footnote{Actually we could guess a formula for general \(\beta\) (see appendix \([C]\)) which reproduces the known results. While the formula looks quite reasonable, it does not pass one of the consistency checks. It seems that some modifications up to the terms proportional to \(1 - \beta\) are needed.}\). As explained in appendix \(\text{[B]}\) the Jack polynomial for \(\beta = 1\) is called Schur polynomial and we write \(J_Y^{(\beta)}|_{\beta = 1} = \chi_Y\).

**Conjecture** We propose the following formula of Selberg average for \(n+1\) Schur polynomials,
\[
\left\langle \chi_{Y_1}(v_1^{(r)}) \ldots \chi_{Y_r}(v_r^{(r)}) \right\rangle_{SU(n+1)}^{\bar{a}, \bar{r}, \beta} = \prod_{s=1}^n \left\{ (-1)^{|Y_s|} \times \left[ \frac{v_s + N_s - N_s - 1}{N_s + N_s - 1} \right]_{Y_s'} \times \prod_{1 \leq i < j \leq N_s} (j-i+1)_{Y_s', Y_s''} \times \prod_{1 \leq i \leq j \leq N_s} (j-i)_{Y_s', Y_s''} \times \prod_{1 \leq i < j \leq n+1} \right\}
\]
\[
\times \prod_{1 \leq i \leq N_s} \prod_{j=1}^{N_s-1} \left[ \frac{v_i + u_t \ldots + u_{s-1} + N_t - N_t - 1}{v_i + u_t \ldots + u_{s-1} + N_t - N_t - 1 - (i+j)} \times \frac{v_i + u_t \ldots + u_{s-1} + N_t - N_t - 1 - (i+j)}{v_i + u_t \ldots + u_{s-1} + N_t - N_t - 1 - (i+j)} \right]^{Y_s'},
\]
\[
\times \prod_{1 \leq i \leq N_s} \prod_{j=1}^{N_s-1} \left[ \frac{v_i + u_t \ldots + u_{s-1} + N_t - N_t - 1 - N_s + N_s - 1}{v_i + u_t \ldots + u_{s-1} + N_t - N_t - 1 - N_s + N_s - 1 + 1 + Y_t + Y_s - (i+j)} \right]^{Y_s''}.
\]
\(\text{(3.32)}\)
\[ v_r := \sum_{n=r}^N v_n = v \delta_{r1} \] after imposing the constraint \((3.30)\).

As we wrote, this formula seems reasonable since

- It reproduces the AGT relation as we will see in the next section.
- It is reduced to the known results for \(\beta = 1\) with the help of \((3.31)\).

(a) For \(Y_1 \cdots = Y_n = \emptyset\), and \(Y_{n+1} = B\), the above reduce to the \(A_n\) one Jack integral \((3.31)\).

(b) For \(n = 1\), \(Y_1 = A\) and \(Y_2 = B\), it coincides with the \(A_1\) two Jack integral \((3.27)\).

(c) For \(n = 2\), \(Y_1 = R\), \(Y_2 = \emptyset\), and \(Y_3 = B\), the above is consistent with the \(A_2\) two Jack integral \((C.5)\) given by Warnaar \[14\].

(d) For \(N_n = 0\), \(u_n = 0\) and \(Y_{n+1} = \emptyset\), the above reduces to the formula for \(A_{n-1}\).

Another type of consistency conditions is also considered. For the simplest case, we start from multiplying a trivial zero factor

\[ v + (p^{(1)}_1 - v_1) + (p^{(1)}_1 - p^{(2)}_1) + \cdots + (p^{(n-1)}_1 - p^{(n)}_1) + p^{(n)}_1 = 0 \]

in the integrand of \((3.32)\). We then apply to each term a property of Schur polynomial,

\[ p_1 \chi_R(p_k) = \sum_{\tilde{R}} \chi_{\tilde{R}}(p_k), \quad (3.33) \]

where the summation is over all possible Young diagrams which can be obtained from \(R\) by adding one cell. This gives rise to a consistency condition for any combination \((Y_1, \cdots, Y_{n+1})\);

\[ v \left( \chi_{Y_1}(-p^{(1)}_k - v'_1) \cdots \chi_{Y_r}(p^{(r-1)}_k - p^{(r)}_k - v'_r) \cdots \chi_{Y_{n+1}}(p^{(n)}_k) \right)^{SU(n+1)}_{v'_{r,\tilde{u}^i,\beta=1}} \]

\[ + \sum_{r=1}^{n+1} \sum_{Y_r} \left( \chi_{Y_1}(-p^{(1)}_k - v'_1) \cdots \chi_{Y_r}(p^{(r-1)}_k - p^{(r)}_k - v'_r) \cdots \chi_{Y_{n+1}}(p^{(n)}_k) \right)^{SU(n+1)}_{v'_{r,\tilde{u}^i,\beta=1}} = 0. \quad (3.34) \]

While this looks trivial, the cancellation becomes rather nontrivial. We give a detailed computation for the simpler cases, \(n = 2\) \((SU(3))\) with \(Y_1, Y_2, Y_3\) being rectangle Young diagrams, in appendix \(D\).

We may write easily some generalizations of \((3.33)\) such as,

\[ \chi_{[n]}(p_k) \chi_R(p_k) = \sum_{\tilde{R}} \chi_{\tilde{R}}(p_k), \quad (3.35) \]

where \(\tilde{R}/R\) is \([n]\). We hope that such series of consistency conditions may serve as a proof of the formula \((3.32)\) in the future.

### 4 AGT conjecture from Selberg integral

In the following, we present a ‘proof’ of AGT conjecture for \(SU(n+1)\) case by using the postulated formulae for Selberg average in \((3.3)\). It is a generalization of the proof for \(SU(2)\) case in \([7,8]\). As we
already mentioned, what we need to see is the coincidence of partition function,

\[ Z_{\text{inst}}(q) = Z_{\text{DF}}(q), \tag{4.1} \]

up to \( U(1) \) factor but we would like to see the stronger condition, namely the coefficient \( N^{\text{Inst}} \) in the instanton partition function \( 2.11 \) with the similar coefficient \( N^{\text{Toda}} \) in \( 3.14 \)

\[ N^{\text{Inst}}_Y = N^{\text{Toda}}_Y. \tag{4.2} \]

We show that this stronger identity holds at \( \beta = 1 \).

We note that both coefficients have the factorized form:

\[ N^{\text{Inst}}_Y \equiv N^{\text{Inst}}_{Y^+} N^{\text{Inst}}_{Y^-}, \quad N^{\text{Toda}}_Y \equiv N^{\text{Toda}}_{Y^+} N^{\text{Toda}}_{Y^-}, \tag{4.3} \]

with

\[
N^{\text{Inst}}_{Y^+} \equiv \prod_{s=1}^{n+1} \prod_{k=1}^{n+1} f_{Y_s}(\mu_k + a_s) \prod_{t,s=1}^{n+1} \int \frac{(-1)^{|Y_s|} \sqrt{G_{Y_s,Y_s(0)} G_{Y_s,Y_s(1-\beta)}}}{G_{Y_s,Y_s(1-\beta)}} \tag{4.4}
\]

and

\[
N^{\text{Toda}}_Y \equiv \prod_{a=1}^{n+1} J_{Y_a}^{(\beta)}(-t^{(a)} - \frac{v_a^+}{\beta}) = \prod_{a=1}^{n+1} \sqrt{G_{Y_a,Y_a(0)} G_{Y_a,Y_a(1-\beta)}} \prod_{a=1}^{n+1} J_{Y_a}^{(\beta)}(-t^{(a)} - \frac{v_a^+}{\beta}) \pm. \tag{4.5}
\]

We remind that \( t^{(a)} \equiv p^{(a)}(a-1) \), \( v_a^- = -\sum_{s=1}^{a-1} v_{s-} \) and \( v_{(N-a)+}^{(a)} = \sum_{s=1}^{a} v_{(N-s)+} \). Therefore, the problem left is to figure out whether the \( (n+1) \)-Jack Selberg integral has the same form with its Nekrasov counterpart for \( \beta = 1 \),

\[ N^{\text{Toda}}_{Y^\pm} = N^{\text{Inst}}_{Y^\pm}. \tag{4.6} \]

### 4.1 Special case: \( \vec{Y} = (\emptyset, \ldots, \emptyset, B) \), arbitrary \( \beta \)

In the following, we prove \( 4.6 \) for ‘+’ part. Proof for ‘−’ is similar. We will omit the lower index “+” in \( v_{a+} \) and \( u_{a+} \) as long as there are no misunderstanding.

We start from the simplest case, when \( Y_1 = \cdots = Y_n = \emptyset \), \( Y_{n+1} = B \). In this case, the Selberg integral is already proved by Warnaar for arbitrary \( \beta \). So our proof for this case is exact and holds without the restriction of \( \beta \).

In the instanton part, we have,

\[
N^{\text{Inst}}_{(\emptyset, \ldots, B)} = \frac{(-1)^{|B|} \prod_{k=1}^{n+1} f_B(\mu_k + a_{n+1})}{\sqrt{G_{B,B(0)} G_{B,B(1-\beta)}}} \prod_{m=1}^{n} G_{B,\emptyset(a_{n+1}-a_m)}. \tag{4.7}
\]
On the other hand, the one-Jack Selberg integral is given in (3.31)

\[
N_{\text{Toda}}^{(\emptyset, \ldots, \emptyset, B)} = \left\langle \frac{j_B^{(\beta)}(p_k)}{SU(n+1)} \right\rangle_+^N \times \frac{G_{B,B}(0)}{G_{B,B}(1 - \beta)} \times \prod_{1 \leq i < j \leq N} \frac{((j - i + 1)\beta)_{B_i-B_j}}{((j - i)\beta)_{B_i-B_j}} \times \prod_{a=1}^{n} \prod_{i=1}^{N_n} \frac{(u_{n-a+1} + \cdots + u_n + a + (N_n - a - i + 1)\beta)_{B_i}}{(v_{n-a+1} + \cdots + u_n + a + 1 + (N_n + N_n-a+1 - N_n-a - a - i)\beta)_{B_i}}.
\]

(4.8)

To see the equivalence, first we note that the function \(f_B(x)\) in \(N_{\text{inst}}\) is linked to the notation \([x]_B\) by (3.28). Then we need to rewrite \(G_{AB}\) in terms of \((x)_B\) in (4.8). For this purpose, we need the following lemmas which will be proved in appendix:

**Lemma 1**

\[
\prod_{1 \leq i < j \leq N} \frac{((j - i + 1)\beta)_{B_i-B_j}}{((j - i)\beta)_{B_i-B_j}} = \frac{[N\beta]_B}{G_{B,B}(0)}
\]

(4.9)

**Lemma 2**

\[
\prod_{i=1}^{N} (x - i\beta)_{B_i} = [x - \beta]_B
\]

(4.10)

**Lemma 3**

\[
[x]_B = (-1)^{|B|}G_{B,\emptyset}(-x + 1 - \beta)
\]

(4.11)

With the help of these formulae, we arrive at the results

\[
N_{\text{Toda}}^{(\emptyset, \ldots, \emptyset, B)} = \left\langle \frac{j_B^{(\beta)}(p_k)}{SU(n+1)} \right\rangle_+^N \times \frac{[N\beta]_B}{G_{B,B}(0)G_{B,B}(1 - \beta)} \times \prod_{a=1}^{n} \frac{(-1)^{|B|}u_{n-a+1} + \cdots + u_n + N_n\beta + a - a\beta}_{B} \times \frac{(-1)^{|B|}u_{n-a+1} + \cdots + u_n + N_n\beta + N_n-a+1\beta - N_n-a\beta + a - a\beta)}{G_{B,\emptyset}(-(-v_{n-a+1} + u_{n-a+1} + \cdots + u_n + N_n\beta + N_n-a+1\beta - N_n-a\beta + a - a\beta)).}
\]

(4.12)

This is equivalent to (4.7), with the identifications of parameters (where we have omitted the lower index”+”
\[ \mu_{n+1} + a_{n+1} = -N_n \beta, \]
\[ \vdots \]
\[ \mu_a + a_{n+1} = -(u_s + \cdots + u_n + N_n \beta + (n - s)(1 - \beta)) \]
\[ \vdots \]
\[ \mu_1 + a_{n+1} = -(u_1 + \cdots + u_n + N_n \beta + n(1 - \beta)) \]
\[ a_n - a_{n+1} = v_n + u_n + 2N_n \beta - N_{n-1} \beta + 1 - \beta, \]
\[ \vdots \]
\[ a_s - a_{n+1} = v_s + u_s + \cdots + u_n + N_n \beta + N_s \beta - N_{s-1} \beta + (n - s)(1 - \beta), \]
\[ \vdots \]
\[ a_1 - a_{n+1} = v_1 + u_1 + \cdots + u_n + N_n \beta + N_1 \beta + n(1 - \beta), \]

with the restriction \( v_2 = \cdots = v_n = 0 \) and \( v_1 = v \). While this looks complicated, it is simplified in the vector notation in \( \mathbb{R}^{n+1} \),

\[ a = a_1 + a_2 + \beta \sum_a N_a e_a + (1 - \beta) \rho, \quad \mu = -a_1 - (1 - \beta) \rho, \]

where \( a = \sum_{i=1}^{N+1} a_i h_i \) and \( \mu = \sum_{i=1}^{N+1} \mu_i h_i \). We note that \( a \) thus written can be identified with the momentum of the vertex in the intermediate channel. This gives \([2.16]\). Eq. \((4.14)\) is the desired identification of parameters in \( SU(N+1) \) AGT conjecture \([4, 5]\). We note that this holds for arbitrary \( \beta \).

### 4.2 General case: arbitrary \( Y^J, \beta = 1 \)

By interpolation method, we have derived that the \((N+1)\)-Schur Selberg integral has the form of \((3.32)\):

\[ N_{Y^J}^{\text{Toda}} = \left( \chi_{\mathbb{Y}}(p_k^{(1)} - (v_1 + \cdots + v_n)) \cdots \chi_{\mathbb{Y}}(p_k^{(r-1)} - \frac{v_r + \cdots + v_n}{\beta}) \chi_{\mathbb{Y}^{n+1}}(p_k^{(n)}) \right)^{SU(n+1)} \]

\[ = \prod_{i=1}^{n} \left\{ (-1)^{Y_i^J} \right\} \times \frac{[v_s + N_s - N_{s-1}]_{Y_s}}{[N_s + N_{s-1}]_{Y_s}} \times \prod_{1 \leq i < j \leq N_s + N_{s-1}} \frac{(j - i + 1)_{Y_i^J - Y_j^J}}{(j - i)_{Y_i^J - Y_j^J}} \times \prod_{1 \leq i < j \leq N_s} \frac{(j - i + 1)_{Y_{(i+1)^J} - Y_{(j+1)^J}}}{(j - i)_{Y_{(i+1)^J} - Y_{(j+1)^J}}} \times \prod_{1 \leq i < j \leq N_s} \frac{[v_i + u_t + \cdots + u_{s-1} + N_t - N_{i-1}]_{Y_i^J}}{[v_i - v_s + u_t + \cdots + u_{s-1} + N_t - N_{i-1} - N_s]_{Y_i^J}} \times \frac{[-v_s + u_t + \cdots + u_{s-1} - N_s + N_{s-1}]_{Y_s}}{[v_t - v_s + u_t + \cdots + u_{s-1} - N_{t-1} - N_s + N_{s-1}]_{Y_s}} \times \prod_{i=1}^{N_t} \prod_{j=1}^{N_{s-1}} \frac{v_t - v_s + u_t + \cdots + u_{s-1} + N_t - N_{i-1} - N_s + N_{s-1} + 1 - (i + j)}{v_t - v_s + u_t + \cdots + u_{s-1} + N_t - N_{i-1} - N_s + N_{s-1} + 1 + Y_{i^J}^t + Y_{j^J}^s - (i + j)}. \]

Then with the lemmas \((4.9)\) to \((4.11)\) introduced in the last section and a new assistant (which only holds at \( \beta = 1 \))

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*There is some degree of freedom to choose the possible identifications.

*Check the appendix for the proof.
Lemma 4

\[
\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \frac{(x + 1 - (i + j)\beta)}{(x + 1 + A_i' + B_j - (i + j)\beta)_{\beta}} = \frac{(-1)^{B} [x - N_2\beta + 1 - \beta]_{A'} [x - N_1\beta + 1 - \beta]_{B}}{G_{A,B}(x)G_{B,A}(-x)} \tag{4.16}
\]

Equation (4.15) transforms to

\[
\left\langle \chi_Y \left(-p_k^{(1)} - (v_1 + \cdots + v_n)\right) \cdots \chi_Y \left(p_k^{(r-1)} - p_k - \frac{v_r + \cdots + v_n}{\beta}\right) \cdots \chi_{Y_{n+1}}(p_k^{(n)})\right\rangle_{SU(n+1)}^{A,B}
\]

\[
= \prod_{s=1}^{n} \left\{ (-1)^{Y_s} \times \frac{[v_s + N_s - N_{s-1}]_Y}{G_{Y_s,Y_s}(0)} \right\} \times \frac{[N_n]_{Y_{n+1}}}{G_{Y_{n+1},Y_{n+1}}(0)} \times \prod_{1 \leq t < s \leq n+1} \frac{1}{G_{Y_t,Y_s}\left(\frac{[v_t - v_s + u_t + \cdots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1}]}{y_t}\right)_{Y_s}} \times \frac{(-1)^{Y_s}}{G_{Y_s,Y_t}\left(-\left(\frac{v_t - v_s + u_t + \cdots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1}}{y_t}\right)\right)} \tag{4.17}
\]

Further notice that for \( \beta = 1 \),

\[
[x]_{A'} = (-1)^{|A|} [-x]_A = f_A(x), \quad G_{A',A'}(x) = G_{A,A}(x) \tag{4.18}
\]

is equivalent to its Nekrasov counterpart (4.1) \( N_{Y_{\text{inst}}}^{\text{inst}} \) at \( \beta = 1 \) with the identifications (4.13) and the following (where we have again omitted the lower index”+” in \( v_{a+} \) and \( u_{a+} \))

\[
a_t - a_s = v_t - v_s + u_t + \cdots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1} ,
\]
\[
\mu_s + a_t = v_t + u_t + \cdots + u_{s-1} + N_t - N_{t-1} ,
\]
\[
\mu_t + a_s = v_s - u_t - \cdots - u_{s-1} + N_s - N_{s-1} ,
\]
\[
\mu_s + a_s = v_s + N_s - N_{s-1} ,
\]

where \( 1 \leq t < s \leq n \). The above are of course in accordance with (4.13) and (4.14). This implies AGT relation for \( SU(n+1) \) at \( \beta = 1 \).

5 Conclusion and further prospects

In this paper, we conjectures some formulae for \( A_n \) Selberg average with \( n + 1 \) Jack polynomials and proves AGT relation for \( SU(n+1) \) based on this conjecture. For the particular combination of Young diagram, namely \( \bar{Y} = (\emptyset, \cdots, \emptyset, B) \), our proof is exact since the corresponding Selberg average is already proved. For this particular case, the proof is exact for arbitrary \( \beta \). Our proof is based on a few lemmas and some of which seem not very straightforward.

The obvious problem is that our formulae for Selberg average are not based on the explicit evaluation but determined only by consistency. So, we need substantial work in the future to prove them. One idea
may be to use the recursion formula of $W_{1+\infty}$ algebra [31]. This idea looks natural since Schur polynomial has simple transformation law with $W_{1+\infty}$ transformation. This should work at least for $\beta = 1$.

The difficulty of the proof for $\beta \neq 1$ case has different origin. For $\beta = 1$, we need to compare the factors of factorized form of $N^\text{Toda}(\vec{Y})$ or $N^\text{inst}(\vec{Y})$ for each $\vec{Y}$. On the other hand, for $\beta \neq 1$, each factor does not coincide but we need to compare the sum $\sum_{|\vec{Y}|=m} N^{\cdots}(\vec{Y})$ for arbitrary $m = 1, 2, 3, \cdots$ in both side. This will be certainly more difficult to prove it. We hope to say something meaningful in such direction, possibly with the help of the relation with the integrable models.

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## A Young diagrams

![Young diagrams](image)

**Figure 1:** example of Young tableaux

Young diagrams are very useful in representing conjugacy classes in group theory. The above is a Young diagram $Y$ of $(8,6,5,5,4,2,1)$. The $i$th column is named as $Y_i$. $h = Y_1$ is the height of $Y$, while $m = Y'_1$ is called the length of $Y$, where $Y'$ stands for the transposed Young diagram.

The arm-length and leg-length of the cell $(i, j)$ in the tableaux $Y$ are denoted by $\text{Arm}_Y(i, j)$ and $\text{Leg}_Y(i, j)$ defined separately as

$$\text{Arm}_Y(i, j) = Y'_j - i, \quad \text{Leg}_Y(i, j) = Y_i - j. \quad (A.1)$$

For the cell $(i, j) = (3, 2)$, the arm-length and leg-length are 5 and 4, respectively.
B  Jack polynomials

Jack polynomials $J_Y^{(β)}[z_1, ⋮, z_M]$ are a kind of symmetric polynomials of variables $z_1, ⋮, z_M$ labeled by a Young diagram $Y$. Detailed properties of Jack polynomial is given in [25]. they are characterized by the fact that they are the eigenfunctions of Calogero-Sutherland Hamiltonian written in the form,

$$\mathcal{H} = \sum_{i=1}^{M} D_i^2 + \beta \sum_{i<h} \frac{z_i + z_j}{z_i - z_j} (D_i - D_j), \quad D_i := z_i \frac{\partial}{\partial z_i}. \quad (B.1)$$

Sometimes they are written as functions of power sum $p_k(z) = \sum_i z_i^k$. In the text, we write the Jack polynomial in terms of them, $J_Y^{(β)}(p_1, p_2, ⋯) \equiv J_Y^{(β)}(p_k) := J_Y^{(β)}[z_1, ⋮, z_M]$. The explicit form of low level ones are listed below;

$$J_Y^{(β)}(p_k) = p_1,$$
$$J_Y^{(β)}[2](p_k) = \frac{p_2 + \beta p_1^2}{\beta + 1}, \quad J_Y^{(β)}[11](p_k) = \frac{1}{2}(p_1^2 - p_2), \quad (B.2)$$
$$J_Y^{(β)}[3](p_k) = \frac{2p_3 + 3\beta p_1 p_2 + \beta^2 p_1^3}{(\beta + 1)(\beta + 2)}, \quad J_Y^{(β)}[21](p_k) = \frac{(1-\beta)p_1 p_2 - p_3 + \beta p_1^2}{(\beta + 1)(\beta + 2)}, \quad J_Y^{(β)}[111](p_k) = \frac{1}{6}p_1^3 - \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3. \quad (B.3)$$

Jack polynomials are orthogonal with each other $(J_Y^{(β_1)}, J_Y^{(β_2)}) \propto \delta_{Y_1 Y_2}$. There are two inner products defined for the symmetric polynomial which has such property. One is defined in terms of products of power sum,

$$\langle p_1^{k_1} ⋯ p_n^{k_n}, p_1^{l_1} ⋯ p_n^{l_n} \rangle_β = \delta_{k^i l^i} - \sum_{i=1}^n i! k_i l_i \cdot \quad (B.3)$$

We write the norm for this inner product as $\langle J_Y^{(β)}, J_Y^{(β)} \rangle = ||J_Y^{(β)}||^2$. The explicit form of the norm is given in the literature [25][34]

$$||J_A^{(β)}||^2 = \frac{Q_Y}{P_Y}, \quad (B.4)$$

with $P_Y$ and $Q_Y$ given by

$$P_Y = \prod_{(i,j) \in Y} \left( \beta (Y_j' - i) + (Y_i - j) + \beta \right) = G_{Y,Y}(0), \quad (B.5)$$

$$Q_Y = \prod_{(i,j) \in Y} \left( \beta (Y_j' - i) + (Y_i - j) + 1 \right) = G_{Y,Y}(1-\beta). \quad (B.6)$$

In this paper, we denote the normalized Jack polynomials as,

$$J_Y^{(β)}(p) := \frac{J_Y^{(β)}(p)}{||J_Y^{(β)}||} = \sqrt{\frac{G_{Y,Y}(0)}{G_{Y,Y}(1-\beta)}} J_Y^{(β)}(p). \quad (B.7)$$

Especially, at $β = 1$, Jack polynomials reduce to Schur polynomials $\chi_Y$:

$$J_Y^{(β)}|_{β = 1} = J_Y^{(β)}|_{β = 1} = \chi_Y. \quad (B.8)$$
The relation between Jack polynomial and Toda theory is that Jack polynomial is characterized as the null states of W-algebra, as discussed, for example, in [27]. In particular, the Calogero-Sutherland Hamiltonian (B.1) is written in terms of Virasoro and W-generators (see, for example, eq.(52) of [27]).

The relevance of Jack polynomial in Selberg integral is through the Cauchy-Riemann relations,

$$\prod_{i,j}(1 - x_i y_j)^{-\beta} = \sum_Y J^{(\beta)}_Y(x) J^{(\beta)}_Y(y) ||J_Y||^{-2}, \quad \prod_{i,j}(1 + x_i y_j) = \sum_Y J^{(1/\beta)}_Y(x) J^{(\beta)}_Y(y).$$  \hfill (B.9)

The first property was essentially used in the text.

C  Formula for general $\beta$

Here we write a formula of $A_n$ Selberg average for product of $n + 1$ Jack polynomials which generalizes (3.32). While some modifications on the terms proportional to $1 - \beta$ are required to meet the constraints (3.31), it survives other constraints which are quite nontrivial. We write this formula since it may give a useful hints in the future development, though some modifications are necessary.

The formula for $n + 1$ Jack polynomials should be close to the following,

$$\left< J^{(\beta)}_{Y_1} (-p^{(1)}_k - \frac{v_1 + \cdots + v_t}{\beta}) \cdots J^{(\beta)}_{Y_t} (p^{(t-1)}_k - \frac{v_e + \cdots + v_n}{\beta}) \cdots J^{(\beta)}_{Y_{n+1}} (p^{(n)}_k) \right>_{SU(n+1)}$$

$$= \prod_{s=1}^{n} \left< (-1)^{|Y_s|} [v_s + N_s \beta - N_s - 1 \beta] Y_s \prod_{1 \leq i < j \leq N} (j - i + 1) \beta Y_{s_i} Y_{s_j} \right> \times \prod_{1 \leq i < j \leq N_n} \frac{(j - i + 1) \beta Y_{s_i} Y_{s_j}}{(j - i) \beta Y_{s_i} Y_{s_j}}$$

$$\times \prod_{i=1}^{N_n} \prod_{j=1}^{N_n+1} \left( v_{i} - v_{i} + u_{i} + \cdots + u_{s_i - 1} - N_i \beta - N_i - 1 \beta + (s - t + 1)(1 - \beta) \right) Y_{s'_i} Y_{s'_j} - \left( -s + t + 1 \right) \beta$$

$$\times \prod_{s=1}^{n} \left[ u_s + \cdots + u_n + N_n \beta - (n - s + 1)(1 - \beta) + (n - s) + 2 \right] Y_{s_n} + \left( v_s + v_{s+1} + \cdots + v_{s+n} + N_{s+n} \beta - N_{s+n} - 1 \beta + (n - s + 1)(1 - \beta) + 1 - (i + j) \beta \right)$$

$$\times \prod_{i=1}^{N_n} \prod_{j=1}^{N_n} \left[ v_s + v_{s+1} + \cdots + v_{s+n} + N_{s+n} \beta - N_{s+n} + 1 \beta - (n - s + 1)(1 - \beta) + 1 - (i + j) \beta \right].$$

(C.1)

It satisfies consistency conditions with the known results:

(a) For $Y_1 = \cdots = Y_n = 0$, and $Y_{n+1} = B$, with the help of (E.31) the above reduce to the $A_n$ one Jack integral (3.31). The proof of this statement is obvious.
\( A \) restrictions is only claimed by Warnaar’s two Jack integral given by Warnaar \([14]\) as below

\[
\langle J_A^{(\beta)}(-p_k - \frac{v_1}{\beta}) J_B^{(\beta)}(p_k) \rangle_{u,v,\beta}^{SU(2)} = (-1)^{|A|} \times \frac{[v + N\beta_A|u + N\beta + 1 - \beta|_B]}{[N\beta_A|u + v + N\beta + 2 - 2\beta|_B} \times \prod_{1 \leq i < j \leq N} (A_i^j - A_j^i + (j - i)\beta)_{\beta} \prod_{1 \leq i < j \leq N} (B_i - B_j + (j - i)\beta)_{\beta} \prod_{i,j=1}^{N} (u + v + 2N\beta + A_i^j + B_j + 1 - \beta + 1 - (i + j)\beta)_{\beta},
\]

which is consistent with the \( A_1 \) two Jack integral \([3,27]\) by considering

\[
J_A^{(\beta)}(-p/\beta) = (-1)^{|A|} J_A^{(1/\beta)}(p).
\]

(c) For \( n = 2, Y_1 = R, Y_2 = \emptyset, \) and \( Y_3 = B, \) with the help of \([E.31]\) the above reduce to

\[
\langle J_R^{(\beta)}(-p_k) J_B^{(\beta)}(p_k) \rangle_{u,v,\beta}^{SU(3)} = (-1)^{|R|} \times \prod_{1 \leq i < j \leq N_1} (\frac{(j - i + 1)\beta}{R_i^j - R_j^i}) \times \prod_{1 \leq i < j \leq N_2} (\frac{(j - i + 1)\beta}{B_i - B_j}) \times \frac{1}{[v_1 - v_2 + 1 + 2N_1\beta - N_2\beta + 2(1 - \beta)]_{R}} \times \frac{1}{[v_1 + u_1 + 1 + N_1\beta + 2 - 2\beta]_{R}} \times \frac{N_1 N_2}{1} \times \prod_{i,j=1}^{N_1 N_2} \frac{[v_1 + u_1 + 1 + u_2 + N_2\beta + N_1\beta + 2(1 - \beta) + 1 - (i + j)\beta]_{\beta}}{[v_1 + u_1 + 1 + u_2 + N_2\beta + 3(1 - \beta)]_B} \times \frac{[v_1 + N_1\beta]_R}{[N_1\beta]_R} \times \frac{1}{[v_2 + u_2 + 2N_2\beta - N_1\beta + 2(1 - \beta)]_B} \times \frac{1}{[u_2 + N_2\beta + (1 - \beta)]_B} \times \frac{1}{1}.
\]

Notice the shift in \( j_R \)'s argument, and the restrictions \( v_2 = 0, \quad v_1 = v, \quad v_1 + v_2 = \beta - 1 \) (this last restrictions is only claimed by Warnaar’s \( A_2 \) two Jack integral), the above is consistent with the \( A_2 \) two Jack integral given by Warnaar \([14]\) as below

\[
\langle J_R^{(\beta)}(p_k) J_B^{(\beta)}(p_k) \rangle_{u,v,\beta}^{SU(3)} = \prod_{1 \leq i < j \leq N_1} (\frac{(j - i + 1)\beta}{R_i^j - R_j^i}) \prod_{1 \leq i < j \leq N_2} (\frac{(j - i + 1)\beta}{B_i - B_j}) \times \frac{[u_1 + N_1\beta + 1 - \beta]_R}{[v_1 + u_1 + 1 + 2N_1\beta - N_2\beta + 2 - 2\beta]_R} \times \frac{[u_2 + N_2\beta + 1 - \beta]_B}{[v_2 + u_2 + 2N_2\beta - N_1\beta + 2 - 2\beta]_B} \times \prod_{i,j=1}^{N_1 N_2} \frac{(u_1 + u_2 + N_1\beta + N_2\beta + R_i + B_j + 1 - \beta + 1 - (i + j)\beta)_{\beta}}{[u_1 + u_2 + N_1\beta + N_2\beta + R_i^j + B_j + 1 - \beta + 1 - (i + j)\beta]_{\beta}}.
\]
(d) For $N_n = 0$ (so that $u_n = v_n = 0$, and $Y_{n+1} = \emptyset$), the above reduce to

$$
\left\langle J_{Y_1}(p_1^{(1)} - \frac{v_1 + \cdots + v_{(n-1)}}{\beta}) \cdots J_{Y_r}(p_r^{(r-1)} - \frac{v_r + \cdots + v_{(n-1)}}{\beta}) \cdots J_{Y_{n+1}}(p_{n+1}^{(n)}) \right\rangle_{SU(n+1)}^{A_{n+1}^{-1}}
$$

$$
= \prod_{s=1}^{n-1} \prod_{1 \leq i < j \leq N_{i+1} + N_s} \left\langle \frac{\prod_{j=1}^{i-1} Y_i}{\prod_{j=1}^{i-1} Y_s} \right\rangle^{(j-i)\beta} Y_{i+1}^{(s)} Y_{i+1}^{(s)} \times \prod_{1 \leq i < j \leq N_{n-1}} \left\langle \frac{\prod_{j=1}^{i-1} Y_i}{\prod_{j=1}^{i-1} Y_s} \right\rangle^{(j-i)\beta} Y_{n}^{(s)} Y_{n}^{(s)}
$$

$$
\times \prod_{1 \leq i < j \leq N_{n-1}} \left\langle \frac{\prod_{j=1}^{i-1} Y_i}{\prod_{j=1}^{i-1} Y_s} \right\rangle^{(j-i)\beta} Y_{n}^{(s)} Y_{n}^{(s)}
$$

This is just the expression of

$$
\left\langle J_{Y_1}(p_1^{(1)} - \frac{v_1 + \cdots + v_{(n-1)}}{\beta}) \cdots J_{Y_r}(p_r^{(r-1)} - \frac{v_r + \cdots + v_{(n-1)}}{\beta}) \cdots J_{Y_{n+1}}(p_{n+1}^{(n)}) \right\rangle_{SU(n+1)}^{A_{n+1}^{-1}}
$$

D Proof of consistency relations

Here we present the detailed computation of the second sets of consistency conditions (3.34) in the text. When $n = 2$ ($SU(3)$ case), making use of (4.18), and setting $Y_1 = R$, $Y_2 = A$, $Y_3 = B$, the conjecture (3.32) becomes

$$
\left\langle \chi_R(-p_1^{(1)} - v) \chi_A(p_1^{(1)} - p_2^{(2)}) \chi_B(p_2^{(2)}) \right\rangle_{SU(3)}^{A_{3}^{-1}}
$$

$$
= \frac{\left[ -v - N_1 \right]_R}{G_R(0)} \times \frac{\left[ N_1 - N_2 \right]_A}{G_A(0)} \times \frac{\left[ N_2 \right]_B}{G_B(0)}
$$

$$
\times \frac{\left[ -v - u_1 - N_1 \right]_R}{\left[ -v - u_1 - N_1 + N_2 \right]_R} \times \frac{\left[ u_1 + N_1 - N_2 \right]_A}{\left[ u_1 + u_1 + N_1 - N_2 \right]_A} \times \frac{\left[ u_1 + u_2 + N_2 \right]_B}{\left[ u_1 + u_2 + N_2 \right]_B} \times \frac{\left[ u_2 + N_2 \right]_B}{\left[ u_2 - N_1 + N_2 \right]_B}
$$

$$
\times \prod_{j=1}^{N_1} \prod_{i=1}^{N_1} v + u_1 + 2N_1 - N_2 + 1 + (i+j)
$$

$$
\times \prod_{j=1}^{N_2} \prod_{i=1}^{N_2} v + u_1 + u_2 + N_1 - N_2 + 1 + (i+j)
$$

$$
\times \prod_{j=1}^{N_2} \prod_{i=1}^{N_2} u_2 - N_1 + 2N_2 + 1 + (i+j)
$$

(D.1)
where we have switched the name of $i$ and $j$ in the last three lines.

For simplicity, we consider the case with $R, A, B$ being rectangle Young diagrams, when (3.33) reduce to

$$p_1\chi_A(p_k) = \chi_{\hat{A}}(p_k) + \chi_{\breve{A}}(p_k),$$

as illustrated in Figure 2.

![Figure 2: The white cells stands for $A$, with length $r_A$ and height $s_A$. the left is the diagram of $\hat{A}$, with an extra grey cell compared to $A$; the right is the diagram of $\breve{A}$, with an extra black cell compared to $A$. $A_i = s_A$, $A'_j = r_A$, $\hat{A}_1 = s_A + 1$, and $\breve{A}_1' = r_A + 1$.]

Now at $\beta = 1$, there are

$$[x]_A = \prod_{i=1}^{r_A} \prod_{j=1}^{s_A} (x - i - j), \quad G_{A,A}(0) = \prod_{i=1}^{r_A} \prod_{j=1}^{s_A} (r_A + s_A - i - j + 1).$$

Furthermore with the information given in Figure 2, we find several lemmas shown below

$$\frac{[x]_{\hat{A}}}{[x]_A} = x + s_A, \quad \frac{[x]_{\breve{A}}}{[x]_A} = x - r_A,$$

$$\frac{G_{A,A}(0)}{G_{\hat{A},\hat{A}}(0)} = \prod_{j=1}^{s_A} \frac{r_A + s_A - j}{r_A + s_A - j + 1} = \frac{r_A}{r_A + s_A}, \quad \frac{G_{A,A}(0)}{G_{\breve{A},\breve{A}}(0)} = \prod_{i=1}^{r_A} \frac{r_A + s_A - i}{r_A + s_A - i + 1} = \frac{s_A}{r_A + s_A},$$

$$\prod_{j=1}^{N_1} \prod_{i=1}^{N_2} \frac{x + 1 + A'_j + B_i - (i + j)}{x + 1 + A'_j + B_i - (i + j)} = \prod_{i=1}^{r_B} \frac{x + s_B - s_A - i}{x + s_B - s_A - i + 1} \times \prod_{i=r_B+1}^{r_B + N_2} \frac{x - s_A - i}{x - s_A - i + 1}.$$
\[
\frac{N_1 N_2}{\prod_{j=1}^{N_2} x + 1 + A_j' + B_i - (i + j)} = \prod_{i=1}^{N_2} \frac{x + 1 + r_A + B_i - i - 1}{x + 1 + r_A + 1 + B_i - i - 1} = \prod_{i=1}^{r_B} \frac{x + s_B + r_A - i}{x + s_B + r_A - i + 1} \times \prod_{i=r_B+1}^{N_2} \frac{x + r_A - i}{x + r_A - i + 1} \tag{D.7}
\]

\[
\frac{N_1 N_2}{\prod_{j=1}^{N_2} x + 1 + A_j' + B_i - (i + j)} = \prod_{j=1}^{N_1} \frac{x + 1 + A_j' + s_B - 1 - j}{x + 1 + A_j' + s_B + 1 - 1 - j} = \prod_{j=1}^{s_A} \frac{x + r_A + s_B - j}{x + r_A + s_B - j + 1} \times \prod_{j=s_A+1}^{N_1} \frac{x + s_B - j}{x + s_B - j + 1} \tag{D.8}
\]

and

\[
\frac{N_1 N_2}{\prod_{j=1}^{N_2} x + 1 + A_j' + B_i - (i + j)} = \prod_{j=1}^{N_1} \frac{x + 1 + A_j' + 0 - (r_B + 1) - j}{x + 1 + A_j' + 1 - (r_B + 1) - j} = \prod_{j=1}^{s_A} \frac{x + r_A - r_B - j}{x + r_A - r_B - j + 1} \times \prod_{j=s_A+1}^{N_1} \frac{x - r_B - j}{x - r_B - j + 1} \tag{D.9}
\]

With the help of the above lemmas, we can calculate that

\[
\begin{align*}
\left< \chi_R(-p_{k}^{(1)} - v)\chi_A(p_{k}^{(1)} - p_{k}^{(2)}\chi_B(p_{k}^{(2)})) \right> \\
\left< \chi_R(-p_{k}^{(1)} - v)\chi_A(p_{k}^{(1)} - p_{k}^{(2)}\chi_B(p_{k}^{(2)})) \right> \\
= \frac{[v - u_1 - N_1]_R}{[v - u_1 - N_1]_R} \times \frac{G_{R,R}(0)}{G_{R,R}(0)} \times \frac{[v - u_1 - N_1]_R}{[v - u_1 - N_1]_R} \times \frac{[v - u_1 - N_1 + N_2]_R}{[v - u_1 - N_1 + N_2]_R} \\
\times \frac{N_1 N_2}{v + u_1 + 2N_1 - N_2 + 1 + R_j' + A_i - (i + j)} \\
\times \frac{N_1 N_2}{v + u_1 + 2N_1 - N_2 + 1 + R_j' + A_i - (i + j)} \\
\times \frac{N_1 N_2}{v + u_1 + u_2 + N_1 + N_2 + 1 + R_j' + B_i - (i + j)} \tag{D.10}
\end{align*}
\]

\[
= (-v - N_1 + s_R) \times \frac{r_R}{r_R + s_R} \times \frac{v - u_1 - N_1 + s_R}{v - u_1 - N_1 + s_R} \times \\
\frac{v + u_1 + 2N_1 - N_2 + s_A - s_R - r_A}{v + u_1 + 2N_1 - N_2 + s_A - s_R - r_A} \times \\
\frac{v + u_1 + u_2 + N_1 + N_2 + s_B - s_R - r_B}{v + u_1 + u_2 + N_1 + N_2 + s_B - s_R - r_B} \times \\
\frac{v + u_1 + u_2 + N_1 + N_2 + s_B - s_R}{v + u_1 + u_2 + N_1 + N_2 + s_B - s_R} .
\]
Likewise, we have

\[
\frac{\langle \chi_R(-p_k^{(1)} - v) \chi_A(p_k^{(1)} - p_k^{(2)}) \chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v) \chi_A(p_k^{(1)} - p_k^{(2)}) \chi_B(p_k^{(2)}) \rangle} = (-v - N_1 - r_R) \times \frac{s_R}{r_R + s_R} \times \frac{-v - u_1 - N_1 - r_R}{-v - u_1 - N_1 + N_2 - r_R} \times \\
\times \frac{v + u_1 + 2N_1 - N_2 + s_A + r_R - r_A}{v + u_1 + 2N_1 - N_2 + r_R - r_A} \times \frac{v + u_1 + N_1 - N_2 + r_R}{v + u_1 + 2N_1 - N_2 + r_R - r_A} \times \\
\times \frac{v + u_1 + u_2 + N_1 + N_2 + s_B + r_B - r_R}{v + u_1 + u_2 + N_1 + N_2 + r_B - r_R} \times \frac{v + u_1 + u_2 + N_1 + r_R}{v + u_1 + u_2 + N_1 + N_2 + r_B - r_R},
\]

(D.11)

\[
\frac{\langle \chi_R(-p_k^{(1)} - v) \chi_A(p_k^{(1)} - p_k^{(2)}) \chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v) \chi_A(p_k^{(1)} - p_k^{(2)}) \chi_B(p_k^{(2)}) \rangle} = (N_1 - N_2 + s_A) \times \frac{r_A}{r_A + s_A} \times \frac{u_1 + N_1 - N_2 + s_A}{v + u_1 + N_1 - N_2 + s_A} \times \\
\times \frac{v + u_1 + 2N_1 - N_2 + r_R + s_A - s_R}{v + u_1 + 2N_1 - N_2 + r_R + s_A} \times \frac{v + u_1 + N_1 - N_2 + s_A - s_R}{v + u_1 + 2N_1 - N_2 + r_R - r_A} \times \\
\times \frac{u_2 - N_1 + 2N_2 + s_B - s_A - r_B}{u_2 - N_1 + 2N_2 + s_B - s_A} \times \frac{u_2 - N_1 + 2N_2 - s_A - r_B}{u_2 - N_1 + 2N_2 - s_A - r_B},
\]

(D.12)

\[
\frac{\langle \chi_R(-p_k^{(1)} - v) \chi_A(p_k^{(1)} - p_k^{(2)}) \chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v) \chi_A(p_k^{(1)} - p_k^{(2)}) \chi_B(p_k^{(2)}) \rangle} = (N_1 - N_2 - r_A) \times \frac{s_A}{r_A + s_A} \times \frac{u_1 + N_1 - N_2 - r_A}{v + u_1 + N_1 - N_2 - r_A} \times \\
\times \frac{v + u_1 + 2N_1 - N_2 + r_R - r_A - s_R}{v + u_1 + 2N_1 - N_2 + r_R - r_A} \times \frac{v + u_1 + N_1 - N_2 - r_A - s_R}{v + u_1 + 2N_1 - N_2 - r_A - s_R} \times \\
\times \frac{u_2 - N_1 + 2N_2 + s_B + r_A - r_B}{u_2 - N_1 + 2N_2 + s_B + r_A} \times \frac{u_2 - N_1 + 2N_2 + r_A - r_B}{u_2 - N_1 + 2N_2 + r_A - r_B},
\]

(D.13)

\[
\frac{\langle \chi_R(-p_k^{(1)} - v) \chi_A(p_k^{(1)} - p_k^{(2)}) \chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v) \chi_A(p_k^{(1)} - p_k^{(2)}) \chi_B(p_k^{(2)}) \rangle} = (N_2 + s_B) \times \frac{r_B}{r_B + s_B} \times \frac{u_1 + u_2 + N_2 + s_B}{u_2 - N_1 + N_2 + s_B} \times \frac{u_2 + N_2 + s_B}{u_2 - N_1 + N_2 + s_B} \times \\
\times \frac{v + u_1 + u_2 + N_1 + N_2 + r_R + s_B}{v + u_1 + u_2 + N_1 + N_2 + r_R + s_B} \times \frac{v + u_1 + u_2 + N_2 + s_B}{v + u_1 + u_2 + N_2 + r_R - s_R} \times \\
\times \frac{u_2 - N_1 + N_2 + r_A + s_B - s_A}{u_2 - N_1 + 2N_2 + r_A + s_B} \times \frac{u_2 - N_1 + N_2 + s_B}{u_2 - N_1 + 2N_2 + s_B - s_A},
\]

(D.14)
and
\[
\frac{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle} = (N_2 - r_B) \times \frac{s_B}{r_B + s_B} \times \frac{u_1 + u_2 + N_2 - r_B}{v + u_1 + u_2 + N_2 - r_B} \times \frac{u_2 + N_2 - r_B}{u_2 - N_1 + N_2 - r_B} \times \frac{u_2 + N_2 - r_B}{u_2 - N_1 + N_2 - r_B} \times \frac{u_2 - N_1 + N_2 - r_B}{u_2 - N_1 + N_2 - r_B - s_R} \times \frac{u_2 - N_1 + 2N_2 + r_A - r_B - s_R}{u_2 - N_1 + 2N_2 - r_B - s_A}.
\]

(D.15)

Summing \( v \) and the above six expressions together, we obtain
\[
v + \frac{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle} + \frac{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle} + \frac{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle} + \frac{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle} + \frac{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle} + \frac{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle}{\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \rangle} = 0.
\]

(D.16)

This reproduces (3.34), which serves as a quite nontrivial check of our conjecture (3.32).

E Proof of the lemmas

E.1 Proof of Lemma 1

Lemma 1
\[
\prod_{1 \leq i < j \leq N} \frac{(j - i + 1)^{\beta}_{B_i - B_j}}{(j - i)^{\beta}_{B_i - B_j}} = \frac{[N\beta]_B}{G_{B,B}(0)}.
\]

(E.1)

Proof: Since \((x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}\), we obtain
\[
\frac{(j - i + 1)^{\beta}_{B_i - B_j}}{(j - i)^{\beta}_{B_i - B_j}} = \frac{\Gamma((j - i + 1)\beta + B_i - B_j)}{\Gamma((j - i + 1)\beta)} \times \frac{\Gamma((j - i)\beta)}{\Gamma((j - i + 1)\beta + B_i - B_j)} = \frac{(B_i - B_j + (j - i)\beta)_{\beta}}{(j - i)^{\beta}_{B_i - B_j}}.
\]

(E.2)

So we only need to prove the following
\[
\prod_{1 \leq i < j \leq N} \frac{(B_i - B_j + (j - i)\beta)_{\beta}}{(j - i)^{\beta}_{B_i - B_j}} = \frac{\prod_{(i,j) \in B} (N\beta - \beta(i - 1) + j - 1)}{\prod_{(i,j) \in B} \beta(B'_j - i) + (B_i - j) + \beta} = \frac{[N\beta]_B}{G_{B,B}(0)}.
\]

(E.3)
Suppose the length of $B$ to be $m$, The left hand side can be expressed as

$$\prod_{1 \leq i < j \leq N} \frac{(B_i - B_j + (j - i)\beta)_{\beta}}{(j - i)\beta} = \prod_{i=1}^{m} \prod_{j=m+1}^{N} \frac{(B_i + (j - i)\beta)_{\beta}}{(j - i)\beta} \times \prod_{i=1}^{m-1} \prod_{j=i+1}^{m} \frac{(B_i - B_j + (j - i)\beta)_{\beta}}{(j - i)\beta}, \quad (E.4)$$

where

$$\prod_{i=1}^{m} \prod_{j=m+1}^{N} \frac{(B_i + (j - i)\beta)_{\beta}}{(j - i)\beta} = \prod_{i=1}^{m} \prod_{j=m+1}^{N} \frac{(1 + (j - i)\beta)_{\beta} (2 + (j - i)\beta)_{\beta} \cdots (B_i + (j - i)\beta)_{\beta}}{(1 + (j - i)\beta)_{\beta} \cdots (B_i - 1 + (j - i)\beta)_{\beta}} \quad (E.5)$$

$$= \prod_{i=1}^{m} \prod_{j=m+1}^{N} \frac{k - 1 + (j - i + 1)\beta}{k - 1 + (j - i)\beta} \quad (E.6)$$

So what is left is to prove the following equation:

$$\prod_{i=1}^{m-1} \prod_{j=i+1}^{m} \frac{(B_i - B_j + (j - i)\beta)_{\beta}}{(j - i)\beta} = \prod_{i=1}^{B_i} \prod_{j=1}^{N} \frac{(m\beta - \beta(i - 1) + j - 1)}{\beta(B_j' - i) + (B_i - j) + \beta}. \quad (E.7)$$

Notice when $1 \leq j \leq B_m$, we have $B_j' = m$,

$$\prod_{j=1}^{B_m} \frac{(m\beta - \beta(m - 1) + j - 1)}{\beta(m - m) + (B_m - j) + \beta} = 1. \quad (E.8)$$

Thus the sufficient condition of (E.6) is

$$\prod_{j=i+1}^{m} \frac{(B_i - B_j + (j - i)\beta)_{\beta}}{(j - i)\beta} = \prod_{j=1}^{B_i} \frac{(m\beta - \beta(i - 1) + j - 1)}{\beta(B_j' - i) + (B_i - j) + \beta}, \quad (E.9)$$

which becomes our new goal.

In Figure 3 we have

$$B_j' = \begin{cases} 
  m_1 & B_{m_2} + 1 \leq j \leq B_{m_1} \\
  m_2 & B_{m_3} + 1 \leq j \leq B_{m_2} \\
  \vdots & \vdots \\
  m_n & 1 \leq j \leq B_{m_n}
\end{cases}$$
and if \( m_{t-1} + 1 \leq i \leq m_t \), we have \( B_i = B_{m_t} \). Besides, we define \( B_{m_{n+1}} = 0 \).

Now the denominator on the right hand side of (E.8) is

\[
R_1 = \prod_{j=1}^{B_i} \left[ \beta(B'_j - i) + (B_i - j) + \beta \right] = \prod_{k=t}^{n} \prod_{j=B_{m_k+1}}^{B_{m_k}} \left[ (B_i - j) + \beta(m_k - i + 1) \right], \tag{E.9}
\]

and the left hand side of (E.8) is

\[
L = \prod_{j=m_{t+1}+1}^{m} \frac{(B_i - B_j + (j - i)\beta)}{(j - i)\beta} \frac{1 + (j - i + 1)\beta}{1 + (j - i)\beta} \cdots \frac{B_i - B_j - 1 + (j - i + 1)\beta}{B_i - B_j - 1 + (j - i)\beta}, \tag{E.10}
\]

Name the term in the last line to be \( H \), we see \( H = 1 \) unless \( B_j \neq B_{j+1} \), (i.e., primary rows \( j = m_k \)). And notice that \( B_{m_k+1} = B_{m_k+1} \), we can count only over the primary rows.
As a result, we find

\[ H = \prod_{k=t+1}^{n-1} \frac{1}{(B_i - B_{mk+1} - 1 + (m_k - i + 1)\beta)} \frac{1}{(B_i - B_{mk+1} + (m_k - i + 1)\beta)} \cdots \frac{1}{(B_i - B_{mk} + (m_k - i + 1)\beta)} \]

\[ = \prod_{k=t+1}^{n-1} \prod_{j=B_{mk+1}+1}^{B_{mk}} \frac{1}{(B_i - j) + \beta(m_k - i + 1)} . \]

(E.11)

Combine the above three equations, we obtain

\[ R_1 \times L = \frac{[(m - i + 1)\beta][1 + (m - i + 1)\beta] \cdots [(B_i - B_m - 1 + (m - i + 1)\beta)]}{[(m_t - i + 1)\beta][(1 + (m_t - i + 1)\beta)] \cdots [(B_i - B_{mt+1} - 1 + (m_t - i + 1)\beta)]} \times \]

\[ \times \prod_{j=1}^{B_m} [(B_i - j) + \beta(m - i + 1)] \times \prod_{j=B_{mt+1}+1}^{B_i} [(B_i - j) + \beta(m_t - i + 1)] \]

\[ = \prod_{j=1}^{B_i} [(B_i - j) + \beta(m - i + 1)] = \prod_{j=1}^{B_i} [(m\beta - \beta(i - 1) + j - 1)] . \]

(E.12)

This is equivalent to (E.8), thus complete the proof of lemma 1.

E.2 Proof of Lemma 2

Lemma 2

\[ \prod_{i=1}^{N} (x - i\beta)_{B_i} = \left[ x - \beta \right]_B \]

(E.13)

Proof: Use (3.29), we find

\[ \prod_{i=1}^{N} (x - i\beta)_{B_i} \]

\[ = \prod_{i=1}^{N} \frac{\Gamma(x - i\beta + B_i)}{\Gamma(x - i\beta)} = \prod_{i=1}^{m} \frac{\Gamma(x - i\beta + B_i)}{\Gamma(x - i\beta)} = \prod_{i=1}^{m} (x - i\beta)(x - i\beta + 1) \cdots (x - i\beta + B_i - 1) = \]

\[ = \prod_{i=1}^{m} \prod_{j=1}^{B_i} (x - i\beta + j - 1) = \prod_{(i,j) \in B} (x - \beta - \beta(i - 1) + j - 1) = \left[ x - \beta \right]_B , \]

where \( m \) is the length of \( B \).

E.3 Proof of Lemma 3

Lemma 3

\[ [x]_B = (-1)^{|B|} G_{B,\beta}(-x + 1 - \beta) \]

(E.15)
Proof:

\[ [x]_B = \prod_{j=1}^{B_1} \prod_{i=1}^{B_j'} (x - \beta(i - 1) + j - 1) = \prod_{j=1}^{B_1} \prod_{i=1}^{B_j'} (x - \beta(B_j' - i) + j - 1) = (-1)^{|B|} G_{B,\emptyset}(-x + 1 - \beta). \] (E.16)

The second equivalence is based on the fact that when \( j \) is fixed, both \( i - 1 \) and \( B_j' - i \) count from 0 to \( B_j' - 1 \).

E.4 Proof of Lemma 4

Lemma 4

\[
\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \frac{(x + 1 - (i + j)\beta)_{|\beta}}{(x + 1 + A_i' + B_j - (i + j))_{|\beta}} = \frac{(-1)^{|B|}[x - N_2 \beta + 1 - \beta]_{|A'}[x - N_1 \beta + 1 - \beta]_{|B}}{G_{A,B}(x)G_{B,A}(-x)}
\] (E.17)

Actually this lemma holds only for \( \beta = 1 \). For this value, the equation becomes,

\[
\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \frac{x + 1 - (i + j)}{x + 1 + A_i' + B_j - (i + j)} = \prod_{(i,j) \in A} \frac{x - N_2 + i - j}{x + A_i' + B_j - i - j + 1} \prod_{(i,j) \in B} \frac{x - N_1 + i - j}{x - B_j' - A_i + i + j - 1}.
\] (E.18)

We have switched the name of \( i \) and \( j \) on the left hand side.

Proof: **Step 1:** Proof for \( B = \emptyset \).

The left hand side of (E.18) is,

\[
L_0 = \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \frac{x + 1 - (i + j)}{x + 1 + A_i' + B_j - (i + j)} = \prod_{i=1}^{N_2} \prod_{j=1}^{h} \frac{x + 1 - (i + j)}{x + 1 + A_j' - (i + j)} = \prod_{i=1}^{N_2} \prod_{j=1}^{h} \frac{x + 1 - (i + j)}{x + 1 + A_j'} = \prod_{i=1}^{N_2} \prod_{j=1}^{h} \frac{x + 1 - (i + j)}{x + 1 + A_j'},
\] (E.19)

where \( h \) is the height of \( A \).

On the other hand, the right hand side of (E.18) becomes,

\[
R_0 = \prod_{(i,j) \in A} \frac{x - N_2 + i - j}{x + (A_i' - i) + j + 1} = \prod_{(i,j) \in A} \frac{x - N_2 + i - j}{x + i - j}.
\] (E.20)

We see \( L_0 = R_0 \), the equation (E.18) holds with \( B = \emptyset \).

**Step 2:** Induction for other cases. Suppose (E.18) is valid for \( B \). As shown in Figure 4 let us construct \( C \) which has only one cell difference from \( B \): \( C_m = B_m + 1, B_{B_m+1}' = m - 1, C_{B_m+1}' = m \), with \( m \) the length of \( B \). (Notice that the special case \( B_m = 0 \) means \( C_m \) starts from a new column, thus we can build any diagram from zero).
Figure 4: Construction of $C$. The white cells stands for $B$, while $C$ has one extra cell (marked in black) than $B$ in the last column.

so we just need to prove that

$$
\prod_{i=1}^{N_2} \prod_{j=1}^{N_1} \frac{x + 1 - (i + j)}{x + 1 + A_j' + C_i - (i + j)} = \prod_{(i,j) \in A} \frac{x - N_2 + i - j}{x + A_j' + C_i - i - j + 1} \prod_{(i,j) \in C} \frac{x - N_1 - i + j}{x - C_j' - A_i + i + j - 1}.
$$

(E.21)

The left hand side of (E.21) is

$$
L = \prod_{i=1}^{N_2} \prod_{j=1}^{N_1} \frac{x + 1 - (i + j)}{x + 1 + A_j' + C_i - (i + j)}
$$

$$
= \prod_{(i,j) \in A} \frac{x - N_2 + i - j}{x + A_j' + C_i - i - j + 1} \prod_{j=1}^{N_1} \frac{x + 1 + A_j' + B_m - (m + j)}{x + 1 + A_j' + B_m + 1 - (m + j)}.
$$

(E.22)

The first term on the right hand side of (E.21) is

$$
R_1 = \prod_{(i,j) \in A} \frac{x - N_2 + i - j}{x + A_j' + B_i - i - j + 1} \prod_{j=1}^{N_1} \frac{x + 1 + A_j' + B_m - (m + j)}{x + 1 + A_j' + B_m + 1 - (m + j)}.
$$

(E.23)

And the second term becomes

$$
R_2 = \prod_{(i,j) \in C} \frac{x - N_1 - i + j}{x - C_j' - A_i + i + j - 1}
$$

$$
= \prod_{(i,j) \in B} \frac{x - N_1 - i + j}{x - C_j' - A_i + i + j - 1} \times \frac{x - N_1 - m + B_m + 1}{x - A_m + B_m}
$$

$$
= \frac{x - N_1 - m + B_m + 1}{x - A_m + B_m} \times \prod_{(i,j) \in B} \frac{x - N_1 - i + j}{x - B_j' - A_i + i + j - 1} \prod_{i=1}^{m-1} \frac{x - m - A_i + i + B_m + 1}{x - m - A_i + i + B_m}.
$$

(E.24)
Since we have assumed the equation (E.18) is correct for $B$, we only need to prove

\[
\prod_{j=1}^{A_m} \frac{x + 1 + A_j' + B_m - (m + j)}{x + 1 + A_j' + B_m + 1 - (m + j)} = \prod_{j=1}^{A_m} \frac{x + 1 + A_j' + B_m - (m + j)}{x + 1 + A_j' + B_m + 1 - (m + j)} 
\]

\[
L' = \prod_{j=A_m+1}^{N_1} \frac{x + A_j' - j + B_m - m + 1}{x + A_j' - j + B_m - m + 2} = \prod_{j=h+1}^{N_1} \frac{x + A_j' - j + B_m - m + 1}{x + A_j' - j + B_m - m + 2} 
\]  

(E.27)

which is equivalent to

\[
\prod_{j=A_m+1}^{N_1} \frac{x + A_j' - j + B_m - m + 1}{x + A_j' - j + B_m - m + 2} = \frac{x - N_1 - m + B_m + 1}{x - A_m + B_m} \times \frac{x - m - A_i + i + B_m + 1}{x - m - A_i + i + B_m} 
\]

(E.26)

The left hand side of the above transforms to

\[
L' = \prod_{j=h+1}^{N_1} \frac{x + A_j' - j + B_m - m + 1}{x + A_j' - j + B_m - m + 2} 
\]

Here $h$ is again the height of $A$. Name the second term of the last line as $L_1'$,

\[
L_1' = \prod_{j=A_m+1}^{h} \frac{x + A_j' - j + B_m - m + 1}{x + A_j' - j + B_m - m + 2} 
\]

(E.28)

This time we call the last term of the last line as $L_3$. 
The second term of the right hand side of (E.26) has the form

\[ R'_2 = \prod_{i=1}^{m-1} \frac{x - m - A_i + i + B_m + 1}{x - m - A_i + i + B_m} \]

\[ = \prod_{i=1}^{m-1} \frac{x - m - A_i + i + B_m + 1}{x - m + i + B_m + 1} \frac{x - m + i + B_m + 1}{x - m - A_i + i + B_m} \]

\[ = \prod_{i=1}^{m-1} \left( \prod_{j=1}^{A_i} \frac{x - m + i + B_m - j + 1}{x - m + i + B_m - j} \prod_{j=0}^{A_i} \frac{x - m + i + B_m - j + 1}{x - m + i + B_m - j} \right) \]

\[ = \prod_{i=1}^{m-1} \frac{x - m + i + B_m + 1}{x - m + i + B_m} \times \prod_{i=1}^{m-1} \prod_{j=1}^{A_i} \left( \frac{x - m + i + B_m - j + 1}{x - m + i + B_m - j} \frac{x - m + i + B_m - j + 1}{x - m + i + B_m - j} \right) \quad (E.29) \]

\[ \prod_{i=1}^{m-1} \prod_{j=1}^{A_i} \left( \prod_{j=1}^{A_i} \frac{x - m + i + B_m - j + 1}{x - m + i + B_m - j} \frac{x - m + i + B_m - j + 1}{x - m + i + B_m - j} \right) \]

Figure 5: \( \prod_{i=1}^{m-1} \prod_{j=1}^{A_i} \) is represented by the area marked by grey and black, while \( \prod_{j=A_m+1}^{h} \prod_{i=1}^{A'_i} \) is represented only by the black cells. Their difference, the grey cells, stands for \( \prod_{i=1}^{m-1} \prod_{j=1}^{A'_i} \), which leads to the following equation.

\[ \prod_{i=1}^{m-1} \prod_{j=1}^{A'_i} \]

so we find (see Figure 5)

\[ \frac{R'_2}{L_3} = \prod_{i=1}^{m-1} \frac{x - m + i + B_m + 1}{x - m + i + B_m} \times \prod_{i=1}^{m-1} \prod_{j=1}^{A_m} \left( \frac{x - m + i + B_m - j + 1}{x - m + i + B_m - j} \frac{x - m + i + B_m - j + 1}{x - m + i + B_m - j} \right) \]

\[ = \prod_{i=1}^{m-1} \frac{x - m - A_m + i + B_m + 1}{x - m - A_m + i + B_m} \quad (E.30) \]

Combine (E.27), (E.28) and (E.30), it is straightforward to find that (E.26) is tenable, thus complete the proof.
E.5 Proof of Lemma 5

Lemma 5

\[
\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \frac{(x + 1 - (i + j)\beta) \beta}{(x + 1 + B_j - (i + j)\beta) \beta} = \frac{[x - N_1\beta + 1 - \beta]_B}{[x + 1 - \beta]_B}, \tag{E.31}
\]

\[
\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \frac{(x + 1 - (i + j)\beta) \beta}{(x + 1 + A'_j - (i + j)\beta) \beta} = \frac{[x - N_2\beta + 1 - \beta]_{A'}}{[x + 1 - \beta]_{A'}}.
\]

These are actually the special case of Lemma 4, but hold for arbitrary \(\beta\).

Proof: For the first statement, we have

\[
L = \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \frac{(x + 1 - (i + j)\beta) \beta}{(x + 1 + B_j - (i + j)\beta) \beta}
= \prod_{i=1}^{N_1} \prod_{j=1}^{m} \frac{(x + 1 - (i + j)\beta)(x + 2 - (i + j)\beta) \ldots (x - (i + j - 1)\beta)}{(x + 1 + B_j - (i + j)\beta)(x + 2 + B_j - (i + j)\beta) \ldots (x + B_j - (i + j - 1)\beta)}
= \prod_{i=1}^{N_1} \prod_{j=1}^{m} \frac{B_j}{x + k - (i + j - 1)\beta} = \prod_{j=1}^{m} \prod_{k=1}^{B_j} \frac{x - N_1\beta + k - j\beta}{x + k - j\beta}
= \prod_{(i,j) \in B} \frac{x - N_1\beta - i\beta + j}{x - i\beta + j} = \frac{[x - N_1\beta + 1 - \beta]_B}{[x + 1 - \beta]_B} = R,
\]

where \(m\) is the length of \(B\).

The second statement can be proved in totally the same way.

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