A detailed discussion of superfield supergravity prepotential perturbations in the superspace of the $AdS_5/CFT_4$ correspondence

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Abstract

This paper presents a detailed discussion of the issue of supergravity perturbations around the flat five dimensional superspace required for manifest superspace formulations of the supergravity side of the $AdS_5/CFT_4$ Correspondence.

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1 Introduction

The importance of deeply understanding the superspace geometry of five dimensions, has received attention during the last few years [1], [2], [3], motivated mainly by the postulate of the AdS/CFT correspondence [4]. In this respect, the study of supergravity theories represents an unavoidable issue [5], [6], [7], even more keeping in mind the existence of the supergravity side of the $AdS_5/CFT_4$ correspondence. Indeed the works of [6, 7] present complete nonlinear descriptions of such superspaces based on a particular choice of compensators. It has long been known [8], that the superspace geometry changes when different compensators are introduced. So one of the goals of the current study is to begin the process of looking at what features of the work of [6, 7] are universal (i.e. independent of compensator choice).

When the superspace approach is used, the conventional representation for Grassmann variables for $SUSY\ D = 5, \mathcal{N} = 1$ (often denominated $\mathcal{N} = 2$) considers these variables obeying a pseudo-Majorana reality condition, then the spinor coordinates are dotted with an $SU(2)$ index. Thus the conventional approach first doubles the number of fermionic coordinates, by the introduction of the $SU(2)$ index, then halves this number by the imposition of the pseudo-Majorana reality condition. However, as noted previously [1], there is no fundamental principle that demands the use of Majorana-symplectic spinors for describing the fermionic coordinate. Indeed, it was demonstrated in [1] that complex 4-components spinors provide an adequate basis for describing such a fermionic coordinate. Building on this previous work, in this paper it will be shown that it is possible to develop successfully a geometrical approach to five dimensional $\mathcal{N} = 1$ supergravity theory, using this unconventional representation for Grassmann variables.

The geometrical approach to supergravity involves calculating fields strengths to determine the form of the torsions and curvatures of the theory. With this information in hand, we can set constraints such that the super spin connections and the vector-supervector component of the inverse supervierbein become dependent variables of the theory. Once this is accomplished, the route to deducing the prepotentials of the supergravity theory are opened. On the other hand, there is an alternative approach which is based in the torsion superfield $C_{AB}^C$ associated to the superspace derivative $E_A$, ...
namely, the super-anholonomy \([E_A, E_B] = C_{AB}^C E_C\). Using the superspace derivative \(\nabla_A\) to calculate the super (anti)-commutator \([\nabla_A, \nabla_B]\), we will be able to write all super-torsion components in terms of the anholonomy and the spinorial connection. Through a choice of suitable constraints on superspace through some super-torsion components, we will be able to write the spin connection superfield in terms of the anholonomy, eliminating by this way the spin connection as independent fields. Once this is accomplished, the linearized theory is considered through perturbations around the flat superspace. In this linearized regime, all super holonomy components can be obtained in terms of semi-prepotentials. Hence the torsion and curvature of the theory are determined in terms of these semi-prepotentials.

2 Superspace geometry: the unconventional representation

Let us start considering the supercoordinate \(Z^A = (x^m, \theta^\mu, \bar{\theta}^\mu)\), where the bosonic and fermionic coordinates are given respectively by \(x^m\) and \(\theta^\mu\), where \(m = 0, \ldots, 4\) and \(\mu = 1, \ldots, 4\). As already mentioned, unlike the conventional representation for Grassmann variables for \(SUSY\ D = 5, \mathcal{N} = 1\), the unconventional representation for the Grassmann variables \((\theta^\mu, \bar{\theta}^\mu)\) given in [1] where there is not an \(SU(2)\) index appended to the spinor coordinates of the superspace, will be used.

Under this unconventional spinorial representation for \(SUSY\ D = 5, \mathcal{N} = 1\), the spinorial supercovariant derivatives is given by (for all details concerning this algebra, see [I])

\[
D_\mu = \partial_\mu + \frac{1}{2}(\gamma^m)_{\mu\nu} C^\nu_\sigma \bar{\theta}^\sigma \partial_m, \quad \bar{D}_\mu = \bar{\partial}_\mu - \frac{1}{2}(\gamma^m)_{\mu\sigma} \theta^\sigma \partial_m, \quad (1)
\]

which satisfies the algebra

\[
\{D_\mu, \bar{D}_\nu\} = (\gamma^m)_{\mu\nu} \partial_m, \quad \{D_\mu, D_\nu\} = \{\bar{D}_\mu, \bar{D}_\nu\} = 0. \quad (2)
\]

In order to construct the \(SUGRA\ D = 5, \mathcal{N} = 1\) version associated with this representation, it is necessary to consider the supervector derivative \(\nabla_A\), a
superspace supergravity covariant derivative, which is covariant under general supercoordinate and superlocal Lorentz groups, given by

\[ \nabla_A = E_A + \Upsilon_A; \quad \Upsilon_A \equiv \frac{1}{2}\omega_{Ac}^d M_d^c + \Gamma_A Z, \quad A = (a, \alpha, \bar{\alpha}) \] (3)

This supergravity superderivative is written in terms of a superderivative \( E_A \), a spin super connection \( \omega_{Ac}^d \), the Lorentz generator \( M_d^c \), a central charge super connection \( \Gamma_A \) and the central charge generator \( Z \). It can be seen there is an absence of any \( SU(2) \) connection, which is characteristic of the unconventional fermionic representation considered here.

We should also mention one other possibility (though we will not study it in this work). Since the bosonic dimension is five, it follows that the field strength of a 5-vector gauge field is thus a two-form. Hence in superspace there must be a super two-form field strength (as appears in (2.5) below). However, by Hodge duality, there should be expected to be a formulation of supergravity here where the central charge connection \( \Gamma_A \) can be set to zero and instead there is introduced a super two-form \( \Gamma_{AB} \).

The superspace derivative \( E_A \) is given through the super vielbein \( E_A^M \) by

\[ E_A = E_A^M D_M, \] (4)

with \( D_M \) being the supervector (this is the flat situation) which components are \((\partial_m, D_\mu, \bar{D}_\mu)\) satisfying the algebra (2). The supertorsion \( T_{AB}^C \), curvature superfield \( R_{ABc}^d \) and central charge superfield strength are given through the algebra

\[ [\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + \frac{1}{2} R_{ABc}^d M_d^c + F_{AB} Z. \] (5)

Again, there is no curvature associated with \( SU(2) \) generators.

The anholonomy superfield \( C_{AB}^C \) associated to the superspace derivative \( E_A \) is given by

\[ [E_A, E_B] = C_{AB}^C E_C. \] (6)

This superfield structure will play a fundamental role in our analysis.

The first step will be to calculate the super (anti)-commutator (5) using the superspace derivative (3). By this way we will be able to write all supertorsion components in terms of the anholonomy and the spinorial connection.
Then choosing suitable constraints on superspace through some super-torsion components, we will be able to write the spin connection superfield in terms of the anholonomy, eliminating thus the spin connection as independent fields. Finally eliminated the spin connection, the next step will be to obtain a specific form for all components of the anholonomy superfield. To carry out this, it will be necessary to provide an specific structure to the super vielbein $E^M_A$. This structure will be based in all “fundamental” geometric objects which appear in $D = 5 \mathcal{N} = 1$ SUSY. These fundamental objects are the following spinor metric and gamma matrices

$$\eta_{\alpha\beta}; \quad (\gamma^a)_\alpha^\beta; \quad (\sigma^{ab})_\alpha^\beta.$$ (7)

This will be explained in detail later.

Let’s start now with the first step, which is to calculate the super (anti)-commutator (5) using the superspace derivative (3) and then identify all super-torsion components. The super (anti)-commutator can be written as

$$[\nabla_A, \nabla_B] = [E_A, E_B] + [\Upsilon_A, E_B] + [\Upsilon_A, \Upsilon_B],$$ (8)

using the anholonomy definition we have

$$[\nabla_A, \nabla_B] = C_{AB}^C E_C + [E_A, \Upsilon_B] + [\Upsilon_A, E_B] + [\Upsilon_A, \Upsilon_B].$$ (9)

Hence finally we have the explicit form of the algebra, given by (see appendix to details)

$$[\nabla_a, \nabla_b] = C_{ab}^C \nabla_C - \omega_{[ab]}^c \nabla_c$$

$$+ \frac{1}{2} \left[ -C_{ab}^C \omega_C^{ef} + E_a \omega_b^{ef} - E_b \omega_a^{ef} - \omega_{[a}^{ec} \omega_{bc]}^{f} \right] M_{fe}$$

$$+ \left[ -C_{ab}^C \Gamma_C + E_a \Gamma_b - E_b \Gamma_a \right] Z.$$ (10)

$$\{\nabla_\alpha, \nabla_\beta\} = C_{\alpha\beta}^C \nabla_C + \frac{3}{4} \left[ \omega_\alpha^{cd} (\sigma_{dc})_\beta^\gamma + \omega_\beta^{cd} (\sigma_{dc})_\alpha^\gamma \right] \nabla_\gamma$$

$$+ \left[ -\frac{1}{2} C_{\alpha\beta}^C \omega_C^d + \frac{1}{2} E_a \omega_\beta^d - \frac{1}{2} E_\beta \omega_a^d + \omega_\alpha^b \omega_\beta^d - \omega_\alpha^b \omega_\beta^d \right] M_d^c$$

$$+ \left[ -C_{\alpha\beta}^C \Gamma_C + E_\alpha \Gamma_\beta + E_\beta \Gamma_\alpha \right] Z,$$ (11)
\[ \{ \nabla_\alpha, \nabla_\beta \} = C_{\alpha\beta}^{\gamma} \nabla_\gamma + \frac{1}{4} \bar{\omega}_{\betacd} \sigma_{\alpha}^{\gamma} \nabla_\gamma + \frac{1}{4} \omega_{\alpha}^{\gamma} \sigma_{\betacd}^{\gamma} \nabla_\gamma \\
+ \left[ -\frac{1}{2} C_{\alpha\beta}^{\gamma} \omega_{\alpha}^{\gamma} \sigma_{\betacd}^{\gamma} + \frac{1}{2} E_{\alpha} \bar{\omega}_{\beta} \sigma_{\alpha}^{\gamma} \sigma_{\betacd}^{\gamma} + \frac{1}{2} \omega_{\alpha}^{\gamma} b^{c} \bar{\omega}_{\beta} \right] M_{dc} \\
+ \left[ -C_{\alpha\beta}^{\gamma} \Gamma_{\alpha}^{\gamma} C_{\alpha\beta}^{\gamma} + E_{\alpha} \Gamma_{\beta} + E_{\beta} \Gamma_{\alpha} \right] Z, \]

(12)

\[ \nabla_\alpha, \nabla_\beta = \nabla_\alpha - \omega_{\alpha} \nabla_\beta - \frac{i}{4} \omega_{\alpha} \sigma_{\betacd}^{\gamma} \nabla_\gamma \\
+ \left[ -C_{\alpha\beta}^{\gamma} \omega_{\alpha}^{\gamma} \sigma_{\betacd}^{\gamma} + E_{\alpha} \omega_{\beta} \sigma_{\alpha}^{\gamma} - \omega_{\alpha}^{\gamma} \sigma_{\betacd}^{\gamma} - \omega_{\alpha}^{\gamma} \sigma_{\betacd}^{\gamma} \omega_{\alpha}^{\gamma} \sigma_{\betacd}^{\gamma} \right] M_{dc} \\
+ \left[ -C_{\alpha\beta}^{\gamma} \Gamma_{\alpha}^{\gamma} + E_{\alpha} \Gamma_{\beta} + E_{\beta} \Gamma_{\alpha} \right] Z. \]

(13)

Comparing the algebra (10)-(13) from (5), the super torsion components can be identified in terms of the super anholonomy and super spin connection components, as it is shown below

\[ T_{ab}^{c} = C_{ab}^{c} + \omega_{ab}^{c} - \omega_{ab}^{c}; \quad T_{ab}^{\gamma} = C_{ab}^{\gamma}; \quad T_{ab}^{\bar{\gamma}} = C_{ab}^{\bar{\gamma}}. \]

\[ T_{\alpha\beta}^{c} = C_{\alpha\beta}^{c}; \quad T_{\alpha\beta}^{\gamma} = C_{\alpha\beta}^{\gamma} + \frac{i}{4} \omega_{\beta}^{\gamma} \sigma_{\alpha}^{\gamma} + \frac{i}{4} \omega_{\alpha}^{\gamma} \sigma_{\beta}^{\gamma}; \quad T_{\alpha\beta}^{\bar{\gamma}} = C_{\alpha\beta}^{\bar{\gamma}}. \]

\[ T_{ab}^{c} = C_{ab}^{c} - \omega_{ab}^{c}; \quad T_{ab}^{\gamma} = C_{ab}^{\gamma} + \frac{i}{4} \omega_{bc}^{\gamma} \sigma_{ab}^{\gamma}; \quad T_{ab}^{\bar{\gamma}} = C_{ab}^{\bar{\gamma}}. \quad (14) \]

In order to eliminate the spin connections as independent fields, it is necessary to impose some restrictions on the torsion superfield. To accomplish this, the following suitable constraints are considered, through which we are able to write the spin connection in terms of the anholonomy:

\[ T_{ab}^{c} = 0 \Rightarrow \omega_{abc} = \frac{1}{2} (C_{abc} - C_{acb} - C_{bca}); \]

(15)

\[ T_{ab}^{c} = 0 \Rightarrow \omega_{abc} = C_{abc}, \]

(16)
leaving thus the spin connections as dependent fields. It is worth noticing that keeping in mind general relativity (torsion free theory) as a low energy limit of \(SUGRA\), the constraint (15) seems a “natural” choice. On the other hand, to keep the flat supergeometry (\(SUSY\)), represented by the algebra shown in (2), as a particular solution of this curve supergeometry (\(SUGRA\)), it is necessary that the superspace satisfies the following restriction

\[ T_{\alpha\beta}^c = (\gamma^c)_{\alpha\beta} \]  

(17)

Finally to ensure the existence of (anti)chiral scalar superfields in supergravity, we have to impose a generalization of \(D_\alpha \tilde{\chi} = 0\) in curved superspace. This is accomplished through \(\nabla\alpha \tilde{\chi} = 0\), which means

\[ \{\nabla_\alpha, \nabla_\beta\} \tilde{\chi} = 0. \]  

(18)

Hence

\[ \{\nabla_\alpha, \nabla_\beta\} \tilde{\chi} = T_{\alpha\beta} C \nabla_C \tilde{\chi} + \frac{1}{2} R_{\alpha\beta c d} M_d \tilde{\chi} + F_{\alpha\beta} Z \tilde{\chi} = 0. \]  

(19)

Therefore we have an additional set of constraints, the so called representation preserving constraints, given by

\[ T_{\alpha\beta}^c = 0; \quad T_{\alpha\beta}^\gamma = 0; \quad T_{\alpha\beta}^\bar{\gamma} = 0. \]  

(20)

The constraints shown in (15)-(17) are the so called conventional constraints, which essentially allow us to eliminate the spin superconnection as independent field and to keep \(SUSY\) as a particular solution. These two set of constraint, together with the representation preserving constraint shown in (20), are called the conformal constraints of the theory, whose corresponding supergeometry is the conformal supergravity.

3 Perturbation around the flat superspace

The supergravity theory we are building up is represented by the algebra (10)-(13), which explicitly gives the field strengths and curvature of the theory. After the spin connection is eliminated as independent field, this algebra essentially depends on the anholonomy. Then the following logical step
is to find a specific form for all components of the anholonomy superfield
in terms of simpler functions. When this is accomplished, the constructed
SUGRA theory will be described by these functions, which will contain all
the basic physical information. To carry out this, we need to provide a spec-
cific structure to the super vielbein $E^M_A$ using all “fundamental” geometric
objects which appear in $D = 5 \mathcal{N} = 1$ SUSY, namely, those given by (7).

First of all let us consider

$$E_A = E^M_A D_M,$$  \hspace{1cm} (21)

and let us start considering its vectorial component, which is writing as

$$E_a = E^a_M D_M = E^m_a \partial_m + E^{\mu}_a D_{\mu} + \bar{E}^{\mu}_a \bar{D}_{\mu},$$ \hspace{1cm} (22)

now expanding $E^m_a$ around the flat solution $\delta^m_a$ we have

$$E^m_a = \delta^m_a + H^m_a,$$ \hspace{1cm} (23)

hence finally we obtain the perturbative version of (22)

$$E_a = \partial_a + H^m_a \partial_m + H^\mu_a D_{\mu} + \bar{H}^{\mu}_a \bar{D}_{\mu}.$$ \hspace{1cm} (24)

It is not complicated to realize that the fields $H^m_a, H^\mu_a$ and its conjugated
$\bar{H}^{\mu}_a$ cannot be expressed in terms of the fundamental geometric objects given
by (7), thus in some sense they are considered as fundamental objects of the
theory. Indeed these field are identified as the graviton $H^m_a$ and the gravitino
$H^\mu_a$ with its “conjugated” $\bar{H}^{\mu}_a$.

On the other hand, the spinorial component $E_\alpha$ of the superfield $E_A$ is
written as

$$E_\alpha = E^\alpha_M D_M = E^m_\alpha \partial_m + E^\mu_\alpha D_{\mu} + \bar{E}^\mu_\alpha \bar{D}_{\mu},$$ \hspace{1cm} (25)

again expanding $E^\mu_\alpha$ around the flat solution $\delta^\mu_\alpha$ we have

$$E^\mu_\alpha = \delta^\mu_\alpha + H^\mu_\alpha,$$ \hspace{1cm} (26)

hence we obtain the perturbative version of (25)

$$E_\alpha = D_\alpha + H^\mu_\alpha D_{\mu} + \bar{H}^{\mu}_\alpha \bar{D}_{\mu} + H^m_\alpha \partial_m.$$ \hspace{1cm} (27)
The fields $H_\alpha^\mu$ and $\bar{H}_\alpha^\mu$ can be expressed as a linear combination of the fundamental objects (7) by

\[ H_\alpha^\mu = \delta_\mu^\alpha \psi_1^1 + i (\gamma^a)_\alpha^\mu \psi_1^1 + \frac{1}{4} (\sigma^{ab})_\alpha^\mu \psi_1^{ab}, \]

and

\[ \bar{H}_\alpha^\mu = \delta_\mu^\alpha \psi_2^2 + i (\gamma^a)_\alpha^\mu \psi_2^2 + \frac{1}{4} (\sigma^{ab})_\alpha^\mu \psi_2^{ab}. \]

The coefficients $\psi$'s are the so called semi-prepotentials of the theory. We will see later that it is possible to obtain an explicit form to some semi-prepotentials in terms of $H$'s fields by using the constraints on supertorsion components.

Using (28) and (29) in (27) we obtain the spinorial components of the superspace derivative in terms of the semi-prepotentials $\psi$'s and the fields $H_\alpha^m$

\[ E_\alpha = D_\alpha + \left[ \delta_\alpha^\mu \psi_1^1 + i (\gamma^a)_\alpha^\mu \psi_1^1 + \frac{1}{4} (\sigma^{ab})_\alpha^\mu \psi_1^{ab} \right] D_\mu 
\]

\[ + \left[ \delta_\alpha^\mu \psi_2^2 + i (\gamma^a)_\alpha^\mu \psi_2^2 + \frac{1}{4} (\sigma^{ab})_\alpha^\mu \psi_2^{ab} \right] \bar{D}_\mu + H_\alpha^m \partial_m, \]

hence

\[ \bar{E}_\alpha = \bar{D}_\alpha + \left[ \delta_\alpha^\mu (\psi_2)^* - i (\gamma^a)_\alpha^\mu (\psi_2)^* - \frac{1}{4} (\sigma^{ab})_\alpha^\mu (\psi_2^{ab})^* \right] D_\mu 
\]

\[ + \left[ \delta_\alpha^\mu (\psi_1)^* - i (\gamma^a)_\alpha^\mu (\psi_1)^* - \frac{1}{4} (\sigma^{ab})_\alpha^\mu (\psi_1^{ab})^* \right] \bar{D}_\mu + \bar{H}_\alpha^m \partial_m; \]

Now using (24), (30) and (31) in (6) and keeping linear terms, we are able to express all the anholonomy components in terms of the graviton, gravitino, the semi-prepotential fields $\psi$'s, and the fields $H_\alpha^m$. Hence we have

\[ C_{ab}^c = \partial_a H_b^c - \partial_b H_a^c; \]

\[ C_{ab}^\gamma = \partial_a H_b^\gamma - \partial_b H_a^\gamma; \]

\[ C_{ab}^{\bar{\gamma}} = \partial_a \bar{H}_b^{\bar{\gamma}} - \partial_b \bar{H}_a^{\bar{\gamma}}. \]

\[ C_{\alpha\beta}^c = \left[ \delta_\alpha^\gamma \psi_2^2 + i (\gamma^a)_\alpha^\gamma \psi_2^2 + \frac{1}{4} (\sigma^{ab})_\alpha^\gamma \psi_2^{ab} \right] (\gamma^c)_{\beta}\gamma + D_\alpha H_\beta^c + (\alpha \leftrightarrow \beta); \]
\begin{align}
C_{\alpha\beta}^{\gamma} &= D_\alpha \left[ \delta_\beta^\gamma \psi^1 + \iota (\gamma^a)_\beta^\gamma \psi_a^1 + \frac{1}{4} (\sigma^{ab})_\beta^\gamma \psi_{ab}^1 \right] + (\alpha \leftrightarrow \beta); \quad (36) \\
C_{\alpha\beta}^{\gamma} &= D_\alpha \left[ \delta_\beta^\gamma \psi^2 + \iota (\gamma^a)_\beta^\gamma \psi_a^2 + \frac{1}{4} (\sigma^{ab})_\beta^\gamma \psi_{ab}^2 \right] + (\alpha \leftrightarrow \beta); \quad (37)
\end{align}

\begin{align}
C_{\alpha\beta}^c &= \eta_{\alpha\beta} \left[ \eta^{mc} \left( (\psi_m^1) - \psi_m^1 \right) + \frac{1}{4} \eta^{\mu\nu} (D_\mu \bar{H}_\nu^c + \bar{D}_\nu H_\mu^c) \right] \\
&\quad + (\gamma^a)_{\alpha\beta} \left[ \delta_\alpha^c (1 + \psi^1 + (\psi^1)^*) - H_a^c + \frac{1}{4} \eta^{\mu\nu} (\psi^1_{mn} - (\psi^1_{mn})^*) + X_a^c \right] \\
&\quad + (\sigma^{ab})_{\alpha\beta} \left[ \frac{1}{2} \eta^{\mu\nu} \eta^{\beta\epsilon} (\psi^1_{\epsilon m} + (\psi^1_{\epsilon m})^*) - \frac{1}{8} \epsilon^{mncab} (\psi^1_{mn} + (\psi^1_{mn})^*) + X_{cab} \right]; \quad (38)
\end{align}

\begin{align}
X_a^c &\equiv - \frac{1}{4} (\gamma^a)_{\alpha\beta} (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H_\alpha^c); \quad (39) \\
X_{cab} &\equiv - \frac{1}{8} (\sigma^{ab})_{\alpha\beta} (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H_\alpha^c); \quad (40)
\end{align}

\begin{align}
C_{\alpha\beta}^{\gamma} &= D_\alpha \left[ \delta_\beta^\gamma (\psi^2)^* - \iota (\gamma^a)_\beta^\gamma (\psi_a^2)^* - \frac{1}{4} (\sigma^{ab})_\beta^\gamma (\psi_{ab}^2)^* \right] \\
&\quad + D_\beta \left[ \delta_\alpha^\gamma \psi^1 + \iota (\gamma^a)_\alpha^\gamma \psi_a^1 + \frac{1}{4} (\sigma^{ab})_\alpha^\gamma \psi_{ab}^1 \right] - (\gamma^a)_{\alpha\beta} H_a^\gamma; \quad (41)
\end{align}

\begin{align}
C_{\alpha\beta}^{\gamma} &= D_\alpha \left[ \delta_\beta^\gamma (\psi^1)^* - \iota (\gamma^a)_\beta^\gamma (\psi_a^1)^* - \frac{1}{4} (\sigma^{ab})_\beta^\gamma (\psi_{ab}^1)^* \right] \\
&\quad + D_\beta \left[ \delta_\alpha^\gamma \psi^2 + \iota (\gamma^a)_\alpha^\gamma \psi_a^2 + \frac{1}{4} (\sigma^{ab})_\alpha^\gamma \psi_{ab}^2 \right] - (\gamma^a)_{\alpha\beta} \bar{H}_a^\gamma; \quad (42)
\end{align}

\begin{align}
C_{ab}^c &= \bar{H}_b^\gamma (\gamma^c)_{\alpha\gamma} + D_a H_b^c - \partial_b H_\alpha^c; \quad (43) \\
C_{ab}^{\gamma} &= D_a H_b^\gamma - \partial_b \left[ \delta_\alpha^\gamma \psi^1 + \iota (\gamma^a)_\alpha^\gamma \psi_a^1 + \frac{1}{4} (\sigma^{ac})_\alpha^\gamma \psi_{ac}^1 \right]; \quad (44) \\
C_{ab}^{\gamma} &= D_a \bar{H}_b^\gamma - \partial_b \left[ \delta_\alpha^\gamma \psi^2 + \iota (\gamma^a)_\alpha^\gamma \psi_a^2 + \frac{1}{4} (\sigma^{ac})_\alpha^\gamma \psi_{ac}^2 \right]; \quad (45)
\end{align}

With all the components of the anholonomy written in terms of the semi-prepotential fields \(\psi\)’s and fields \(H\)’s, the next step will be to use some suitable constraint to write the \(\psi\)’s fields in terms of \(H\)’s fields. We will see that a direct consequence of keeping \(SUSY\) as a particular solution allow
us to determinate the semi-prepotentials $\psi_a^1$ and $\psi_a^{1b}$ in terms of $H_a^b$ and its
conjugate, and that the existence of (anti)chiral scalar superfield allow us to
determinate the semi-prepotentials $\psi_a^1$ and $\psi_a^{1b}$ in terms of $H_a^b$. Let us start
considering $T_{\alpha\beta}^c = (\gamma^c)_{\alpha\beta}$, a "rigid constraint" given in (17), which by (14)
means
\[
C_{\alpha\beta}^c = (\gamma^c)_{\alpha\beta}.
\] (46)

Using (38) in the expression (46), we obtain
\[
(\gamma^c)_{\alpha\beta} = \eta_{\alpha\beta} \left[ \eta^{mc} ((\psi^1_m)^* - \psi^1_m) + \frac{1}{4} \eta^{\mu\nu} (D_\mu \bar{H}_\nu^c + \bar{D}_\nu H^c_\mu) \right]
+ (\gamma^a_{\alpha\beta}) \left[ \delta_c^a c (1 + \psi^1 + (\psi^1)^*) - H^c_a + \frac{i}{4} \eta^{\epsilon \mu \delta} \eta^{n} (\psi^1_{mn} - (\psi^1_{mn})^*) + X_a^c \right]
+ (\sigma_{ab})_{\alpha\beta} \left[ \frac{1}{2} \eta^{m[a} \eta^{b]c} (\psi^1_m + (\psi^1_m)^*) - \frac{1}{8} \epsilon^{mncab} (\psi^1_{mn} + (\psi^1_{mn})^*) + X^{cab} \right],
\] (47)
showing thus that the rigid constraint leads to the following three independent
equations
\[
\eta^{mc} ((\psi^1_m)^* - \psi^1_m) + \frac{1}{4} \eta^{\mu\nu} (D_\mu \bar{H}_\nu^c + \bar{D}_\nu H^c_\mu) = 0,
\] (48)
\[
\delta_c^a c (\psi^1 + (\psi^1)^*) - H^c_a + \frac{i}{4} \eta^{\epsilon \mu \delta} \eta^{n} (\psi^1_{mn} - (\psi^1_{mn})^*) + X_a^c = 0,
\] (49)
\[
\frac{1}{2} \eta^{m[a} \eta^{b]c} (\psi^1_m + (\psi^1_m)^*) - \frac{1}{8} \epsilon^{mncab} (\psi^1_{mn} + (\psi^1_{mn})^*) + X^{cab} = 0.
\] (50)
From the equations (48) we have
\[
\psi^1_a - (\psi^1_a)^* = -\frac{1}{16} \eta_{ac} \eta^{\alpha\beta} (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H^c_\alpha),
\] (51)
and from the equation (49) we obtain
\[
\psi^1 + (\psi^1)^* = \frac{1}{3} [H^a_\alpha + \frac{1}{4} (\gamma^a_{\alpha\beta}) (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H^c_\alpha)]
\] (52)
and
\[
\psi^1_{ab} - (\psi^1_{ab})^* = -\eta_{c[a} H^c_{\beta]} - \frac{1}{16} \eta_{c[a} (\gamma^c_{\beta]} (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H^c_\alpha).
\] (53)
From (50) it is found the following two expressions
\[
\psi^1_a - (\psi^1_a)^* = \frac{1}{16} (\sigma_{ac}) \eta^{\alpha\beta} (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H^c_\alpha);
\] (54)
\[\psi_{ab}^1 + (\psi_{ab}^1)^* = \frac{1}{12} \epsilon_{abckl}(\sigma^{kl})^{\alpha\beta}(D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H_\alpha^c). \quad (55)\]

Thus from (51) and (54) we obtain
\[\psi_a^1 = \frac{1}{8} \left[ -\eta_{ac} \eta^{\alpha\beta} + \frac{1}{4}(\sigma_{ac})^{\alpha\beta} \right] (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H_\alpha^c), \quad (56)\]

and from (53) and (55) we have
\[\psi_{ab}^1 = -\frac{1}{2} \eta_{c[a} H_{b]}^c + \frac{1}{8} \left[ -\eta_{c[a}(\gamma_{b])^{\alpha\beta} + \frac{1}{3} \epsilon_{abckl}(\sigma^{kl})^{\alpha\beta} \right] (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H_\alpha^c), \quad (57)\]

thus the graviton may be expressed in terms of the semi-prepotential \(\psi_{ab}^1\) as
\[H_a^d = -\eta^{bd} \psi_{ab}^1 + \frac{1}{8} \epsilon^{bd} \left[ -\eta_{c[a}(\gamma_{b])^{\alpha\beta} + \frac{1}{3} \epsilon_{abckl}(\sigma^{kl})^{\alpha\beta} \right] (D_\alpha \bar{H}_\beta^c + \bar{D}_\beta H_\alpha^c). \quad (58)\]

In order to obtain \(\psi_a^2\) and \(\psi_{ab}^2\), the "chiral constraint" \(T_{a\beta}^c = 0\) shown in (20) is used, which leads to
\[C_{\alpha\beta}^c = 0. \quad (59)\]

Thus using the expression (35) in the condition (59), we finally obtain
\[\psi_a^2 = \frac{1}{16} (\sigma_{ca})^{\alpha\beta} D_\alpha H_\beta^c; \quad (60)\]
\[\psi_{ab}^2 = -\frac{1}{12} \epsilon_{abde} (\sigma^{de})^{\alpha\beta} D_\alpha H_\beta^c. \quad (61)\]

The remaining two constraints associate to the chiral representation, that is, \(T_{a\beta}^{\gamma} = 0\) and \(T_{a\beta}^{\bar{\gamma}} = 0\), lead respectively to
\[C_{\alpha\beta}^{\gamma} = \frac{1}{4} \omega_{ac\alpha}(\sigma^{cd})_{\beta}^{\gamma} + \frac{1}{4} \omega_{\beta cd}(\sigma^{cd})_{\alpha}^{\gamma}; \quad (62)\]
\[C_{\alpha\beta}^{\bar{\gamma}} = 0, \quad (63)\]

where the spin connection component \(\omega_{abc}\) is given by
\[\omega_{ab}^c = \bar{H}_b^\gamma (\gamma^c)_{\alpha\gamma} + D_a H_b^c - \partial_b H_\alpha^c. \quad (64)\]

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Using (64) in (62) we have

\[ 20D_\alpha \psi^1 = i(\sigma_{ab} \gamma^c \sigma^{ab})^\beta_\alpha D_\beta \psi^1_c + \frac{1}{4}(\sigma_{ab} \sigma^{cd} \sigma^{ab}) D_\beta \psi^1_{cd} + \frac{1}{4}(\sigma_{ab} \sigma_d \sigma^{ab})^\beta_\alpha \left[ H_c \gamma^d (\gamma^c)_{\alpha \gamma} + D_\alpha H_d - \partial_\alpha H_c \right], \quad (65) \]

and from (37) in (63) we obtain

\[ 20D_\alpha \psi^2 = i(\sigma_{ab} \gamma^c \sigma^{ab})^\beta_\alpha D_\beta \psi^2_c + \frac{1}{4}(\sigma_{ab} \sigma^{cd} \sigma^{ab}) D_\beta \psi^2_{cd}. \quad (66) \]

4 The Bianchi identities

So far we have impose some restrictions on superspace through some constraints on the supertorsion components. It was necessary to impose the constraints (15) and (16) to leave the spin connections as dependent fields of the anholonomy, the constraint (17) to keep rigid supersymmetry as a particular solution, and (20) to ensure the existence of (anti)chiral scalar superfields in supergravity. When all these constraint are imposed, the geometry of the superspace is restricted, in consequence the Bianchi identities, which can be written by

\[ \left[ \nabla_A, [\nabla_B, \nabla_C] \right] + (-1)^{A(B+C)} \left[ \nabla_B, [\nabla_C, \nabla_A] \right] + (-1)^{C(A+B)} \left[ \nabla_C, [\nabla_A, \nabla_B] \right] = 0, \quad (67) \]

now contain non trivial information. This information can be read by the following three equations

\[ \nabla_A T_{BC}^D F + (-1)^{A(B+C+D)} T_{BC}^D T_{AD}^F + (-1)^{A(B+C)} \frac{1}{2} R_{BC}^{cd} \Phi_{dcA}^F \]
\[ + (-1)^{A(B+C)} \nabla_B T_{CA}^F + (-1)^{C(B+A)+BD} T_{CA}^D T_{BD}^F + (-1)^{C(A+B)} \frac{1}{2} R_{CA}^{cd} \Phi_{dcB}^F \]
\[ + (-1)^{C(A+B)} \nabla_C T_{AB}^F + (-1)^{CD} T_{AB}^D T_{CD}^F + \frac{1}{2} R_{AB}^{cd} \Phi_{dcC}^F = 0; \quad (68) \]

\[ (-1)^{A(B+C+D)} T_{BC}^D R_{ADc}^d + \nabla_A R_{BCc}^d \]
\[ + (-1)^{C(A+B)+BD} T_{CA}^D R_{BDe}^d + \nabla_B R_{CAe}^d \]
\[ + (-1)^{CD} T_{AB}^D R_{CDe}^d + \nabla_C R_{ABe}^d = 0; \quad (69) \]
\(-1\)^{(A+B+C+D)} T_{BC}^D F_{AD} + \nabla_A F_{BC} \\
\quad + (-1)^{(C(A+B)+BD) T_{CA}^D F_{BD} + (-1)^A(B+C) \nabla_B F_{CA}} \\
\quad + (-1)^{C D T_{AB}^D F_{CD} + (-1)^C(A+B) \nabla_B F_{CA}} = 0; \quad (70)

where
\[
\Phi_{abC}^D = \left( \begin{array}{ccc} \Phi_{abc}^d & 0 & \Phi_{abcd}^d \\ 0 & \delta^{d}_{\gamma} & 0 \\ \eta_{c[a} & 0 & \sigma_{ab] \gamma} \end{array} \right) .
\]

It is well known [9] that it is sufficient to analyze the Bianchi identities (68) and (70), since all equations contained in (69) are identically satisfied when (68) and (70) hold. Hence using the constraints (15), (16), (17) and (20) in the Bianchi identities (68) and (70) we will be able to obtain the curvature and field strength superfield components in terms of the smaller set of superfields of the theory.

5 Symmetries and semi-prepotentials

In order to obtain some information on \(\psi's\), let’s see the behaviour of them under the scale, U(1) and Lorentz Symmetry, which are represented respectively by
\[
E_a \rightarrow E_a' = e^{f_0} E_a; \quad E_\alpha \rightarrow E_\alpha' = e^{\frac{1}{2} f_0} E_\alpha; \quad \bar{E}_\alpha \rightarrow \bar{E}_\alpha' = e^{\frac{1}{2} f_0} \bar{E}_\alpha \quad (71)
\]
\[
E_a \rightarrow E_a' = e^{\frac{1}{2} f_0} E_a; \quad E_\alpha \rightarrow E_\alpha' = e^{-\frac{1}{2} f_0} E_\alpha; \quad \bar{E}_\alpha \rightarrow \bar{E}_\alpha' = e^{-\frac{1}{2} f_0} \bar{E}_\alpha \quad (72)
\]
\[
E_a \rightarrow E_a' = \Lambda_a^b E_b; \quad E_\alpha \rightarrow E_\alpha' = e^{\frac{1}{2} \Lambda_{ab}(\sigma_{ab})_{\alpha}^\beta} E_\beta; \quad \bar{E}_\alpha \rightarrow \bar{E}_\alpha' = e^{\frac{1}{2} \Lambda_{ab}(\sigma_{ab})_{\alpha}^\beta} \bar{E}_\beta \quad (73)
\]

Let us begin considering the scale transformation
\[
E_a \rightarrow E_a' = e^{f_0} E_a, \\
E_\alpha \rightarrow E_\alpha' = e^{\frac{1}{2} f_0} E_\alpha, \\
\bar{E}_\alpha \rightarrow \bar{E}_\alpha' = e^{\frac{1}{2} f_0} \bar{E}_\alpha. \quad (74)
\]

Considering the infinitesimal version of (74) and the perturbative expression of \((E_a, E_\alpha, \bar{E}_\alpha)\) around the flat solution, we have
\[
E_a \rightarrow E_a' = (1 + f_0)(\partial_a + H_a^m \partial_m + H_a^\mu D_\mu + \bar{H}_a^\mu \bar{D}_\mu), \quad (75)
\]
\[ E_\alpha \rightarrow E'_\alpha = (1 + \frac{1}{2} f_0) E_\alpha, \]  
(76)

\[ \bar{E}_\alpha \rightarrow \bar{E}'_\alpha = (1 + \frac{1}{2} f_0) \bar{E}_\alpha. \]  
(77)

The Eq. (75) can be written as

\[ E_a \rightarrow E'_a = \partial_a + (f_0 \delta_a^m + H_a^m) \partial_m + H_a^\mu D_\mu + \bar{H}_a^\mu \bar{D}_\mu, \]  
(78)

showing that there is a shift on \( H_a^m \) due to the scale transformation, as shown below

\[ E_a \rightarrow E'_a = e^{f_0} E_a \Rightarrow H_a^m \rightarrow H_a^m + f_0 \delta_a^m. \]  
(79)

On the other hand, the Eqs. (76) and (77) can be written as

\[ E_\alpha \rightarrow E'_\alpha = E_\alpha + \frac{1}{2} f_0 \delta_\alpha^\mu D_\mu \]  
(80)

\[ \bar{E}_\alpha \rightarrow \bar{E}'_\alpha = \bar{E}_\alpha + \frac{1}{2} f_0 \delta_\alpha^\mu \bar{D}_\mu. \]  
(81)

Now using the explicit form of \( E_\alpha \) and \( \bar{E}_\alpha \) given in (30) and (31), we have

\[ E_\alpha \rightarrow E'_\alpha = e^{\frac{1}{2} f_0} E_\alpha \Rightarrow \psi^1 \rightarrow \psi^1 + \frac{1}{2} f_0, \]  
(82)

\[ \bar{E}_\alpha \rightarrow \bar{E}'_\alpha = e^{\frac{1}{2} f_0} \bar{E}_\alpha \Rightarrow (\psi^1)^* \rightarrow (\psi^1)^* + \frac{1}{2} f_0, \]  
(83)

showing thus that the scale transformation prodeces a shift on the real parte of the semi-prepotential \( \psi^1 \). This can be seen more clearly through the following useful decomposition

\[ \psi^1 = \frac{1}{2} (\hat{\psi}^1 + i \tilde{\psi}^1), \]  
(84)

where \( \hat{\psi} \) and \( \tilde{\psi} \) are real functions. Hence we have

\[ E_\alpha \rightarrow E'_\alpha = e^{\frac{1}{2} f_0} E_\alpha \Rightarrow \frac{1}{2} (\hat{\psi}^1 + i \tilde{\psi}^1) \rightarrow \frac{1}{2} (\hat{\psi}^1 + i \tilde{\psi}^1) + \frac{1}{2} f_0, \]  
(85)

\[ \bar{E}_\alpha \rightarrow \bar{E}'_\alpha = e^{\frac{1}{2} f_0} \bar{E}_\alpha \Rightarrow \frac{1}{2} (\hat{\psi}^1 - i \tilde{\psi}^1) \rightarrow \frac{1}{2} (\hat{\psi}^1 - i \tilde{\psi}^1) + \frac{1}{2} f_0, \]  
(86)
hence
\[ \hat{\psi}^1 \rightarrow \hat{\psi}^1 + f_0; \quad \nu \tilde{\psi}^1 \rightarrow \nu \tilde{\psi}^1, \]  
(87)
showing thus that under the scale transformation there is a shift on \( \hat{\psi}^1 \), leaving invariant \( \tilde{\psi}^1 \) in the expression (84).

Let us consider now the \( U(1) \) transformation
\[ E_\alpha \rightarrow E'_\alpha = e^{i \frac{1}{2} f} E_\alpha, \]
\[ \bar{E}_\alpha \rightarrow \bar{E}'_\alpha = e^{-i \frac{1}{2} f} \bar{E}_\alpha. \]  
(88)
Considering the infinitesimal version of (88) and the perturbative expression of \((E_\alpha, \bar{E}_\alpha)\) around the flat solution, we obtain
\[ E_\alpha \rightarrow E'_\alpha = E_\alpha + \frac{1}{2} f \delta_\alpha^\mu D_\mu, \]  
(89)
\[ \bar{E}_\alpha \rightarrow \bar{E}'_\alpha = \bar{E}_\alpha - \frac{1}{2} f \delta_\alpha^\mu \bar{D}_\mu. \]  
(90)
Now using the explicit form of \( E_\alpha \) and \( \bar{E}_\alpha \) given in (30) and (31), we have
\[ E_\alpha \rightarrow E'_\alpha = \frac{1}{2} (\hat{\psi}^1 + i \tilde{\psi}^1) \rightarrow \frac{1}{2} (\hat{\psi}^1 + i \tilde{\psi}^1) + \frac{1}{2} f, \]  
(91)
\[ \bar{E}_\alpha \rightarrow \bar{E}'_\alpha = \frac{1}{2} (\hat{\psi}^1 - i \tilde{\psi}^1) \rightarrow \frac{1}{2} (\hat{\psi}^1 - i \tilde{\psi}^1) - \frac{1}{2} f. \]  
(92)
Again, as in the previous case, there is a shift on the semi-prepotential \( \psi^1 \). In this case the \( U(1) \) symmetry produces a shift on the imaginary part of the semi-prepotential \( \psi^1 \). This can be seen clearly using the decomposition shown in Eq. (84) as following
\[ E_\alpha \rightarrow E'_\alpha = e^{i \frac{1}{2} f} E_\alpha \Rightarrow \psi^1 \rightarrow \psi^1 + \frac{1}{2} f, \]  
(93)
\[ \bar{E}_\alpha \rightarrow \bar{E}'_\alpha = e^{-i \frac{1}{2} f} \bar{E}_\alpha \Rightarrow (\psi^1)^* \rightarrow (\psi^1)^* - \frac{1}{2} f. \]  
(94)
thus
\[ \hat{\psi}^1 \rightarrow \hat{\psi}^1; \quad \nu \tilde{\psi}^1 \rightarrow \nu \tilde{\psi}^1 + f, \]  
(95)

hence the \( U(1) \) transformation produces a shift on \( \tilde{\psi}^1 \), leaving invariant \( \hat{\psi}^1 \) in the expression (84) for the semi-prepotential \( \psi^1 \).
Now let’s consider the Lorentz transformation on the vector component $E_a$

$$E_a \rightarrow E'_a = \Lambda^b_a E_b.$$  \hfill (96)

Now we consider the infinitesimal Lorentz transformation

$$\Lambda^b_a = \delta^b_a + \epsilon^b_a; \quad \epsilon_{ab} = -\epsilon_{ba}$$ \hfill (97)

acting on the perturbative expression of $E_a$ around the flat solution

$$E_a \rightarrow E'_a = (\delta^b_a + \epsilon^b_a)(\partial_b + H^m_b \partial_m + H^\mu_b D_\mu + H^\mu_b D_\mu)$$ \hfill (98)

$$E_a \rightarrow E'_a = \partial_a + (\epsilon^m_a + H^m_a)\partial_m + H^\mu_a D_\mu + H^\mu_a D_\mu$$ \hfill (99)

Hence

$$E_a \rightarrow E'_a = \Lambda^b_a E_b \Rightarrow H^m_a \rightarrow H^m_a + \epsilon^m_a,$$ \hfill (100)

showing thus that the Lorentz transformation produces a shift on the graviton.

Now considering the Lorentz transformation on the spinorial components

$$E_\alpha \rightarrow E'_\alpha = e^{i\frac{1}{8} \Lambda^{ab}(\sigma_{ab})_\alpha^\beta} E_\beta,$$ \hfill (101)

$$\bar{E}_\alpha \rightarrow \bar{E}'_\alpha = e^{-i\frac{1}{8} \Lambda^{ab}(\sigma_{ab})_\alpha^\beta} \bar{E}_\beta.$$ \hfill (102)

Considering the infinitesimal transformation we have

$$E_\alpha \rightarrow E'_\alpha = [\delta^\alpha_\beta + \frac{1}{8} \Lambda^{ab}(\sigma_{ab})_\alpha^\beta] E_\beta,$$ \hfill (103)

$$\bar{E}_\alpha \rightarrow \bar{E}'_\alpha = [\delta^\alpha_\beta - \frac{1}{8} \Lambda^{ab}(\sigma_{ab})_\alpha^\beta] \bar{E}_\beta,$$ \hfill (104)

and using the perturbation around the flat solution, we obtain

$$E_\alpha \rightarrow E'_\alpha = E_\alpha + \frac{1}{8} \Lambda^{ab}(\sigma_{ab})_\alpha^\mu D_\mu,$$ \hfill (105)

$$\bar{E}_\alpha \rightarrow \bar{E}'_\alpha = \bar{E}_\alpha - \frac{1}{8} \Lambda^{ab}(\sigma_{ab})_\alpha^\mu \bar{D}_\mu.$$ \hfill (106)

Now using the explicit form of $E_\alpha$ and $\bar{E}_\alpha$ given in (30) and (31), we have

$$E_\alpha \rightarrow E'_\alpha = e^{i\frac{1}{8} \Lambda^{ab}(\sigma_{ab})_\alpha^\beta} E_\beta \Rightarrow \psi^{1}_{ab} \rightarrow \psi^{1}_{ab} + \frac{i}{2} \Lambda_{ab},$$ \hfill (107)

$$\bar{E}_\alpha \rightarrow \bar{E}'_\alpha = e^{i\frac{1}{8} \Lambda^{ab}(\sigma_{ab})_\alpha^\beta} \bar{E}_\beta \Rightarrow (\psi^{1}_{ab})^* \rightarrow (\psi^{1}_{ab})^* - \frac{i}{2} \Lambda_{ab},$$ \hfill (108)

showing thus that the Lorentz transformation produces a shift on the imaginary part of the pre-potential $\psi^{1}_{ab}$. 

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6 Conclusions

Using the notion of superspace, a geometrical approach to five dimensional $\mathcal{N} = 1$ supergravity theory was discussed in detail. There was not used the conventional representation for Grassmann variables, based in spinors obeying a pseudo-Majorana reality condition. Instead, the unconventional representation for the Grassmann variables $(\theta^\mu, \bar{\theta}^{\dot{\mu}})$ given in [1] for a $SUSY$ $D = 5$, $\mathcal{N} = 1$ representation, was successfully extended for a supergravity theory, dispensing with the use of a $SU(2)$ index to the spinor coordinates of the superspace.

The components of the torsion and curvature superfield were found through the super (anti)-commutator of the superspace supergravity covariant derivative, finding these superfields as function of both the anholonomy and the spin connection. Imposing suitable constraints on superspace through some super-torsion components, the spin connection was written in terms of the anholonomy, eliminating thus the spin connection as an independent field.

Taking a perturbation around the flat superspace, the components of the superspace derivative were found as the sum of the “rigid” (SUSY) part and perturbative terms. These perturbative terms, arising from the super vielbein components, were written in terms of functions when the vectorial component of the superspace derivative was considered. These functions were a two vectorial component superfield and its supersymmetric partner, namely, the graviton and gravitino at quantum level. On the other hand, when the spinorial component of the superspace derivative was considered, it was possible to write the perturbative terms as a linear combination of fundamental geometric objects of SUSY $\mathcal{N} = 1$, introducing thus the semi-prepotential of the theory.

Using the perturbative version (linearized theory) of the superspace derivative, it was possible to find all components of the super anholonomy in terms of the simpler set of superfields, namely, the graviton, the gravitino and semi-prepotentials. Demanding consistence with rigid SUSY and the existence of (anti)chiral scalar superfields in supergravity, some suitable constraints on superspace were imposed, then all semi-prepotentials were written in terms of the smaller set of superfields of the theory, leaving the two scalar semi-
prepotentials $\psi^1$ and $\psi^2$ superfields. Using the Bianchi identities, three set of equations written in terms of superfields were found. Two of these set containing enough information to determinate the curvature and field strength superfield components in terms of the smaller set of superfields of the theory.

It was explained in detail the behaviour of the semi-prepotentials under the action of the scale, $U(1)$ and Lorentz Symmetry. It was found that the scale transformation produces a shift on the real part of the scalar semi-prepotential $\psi^1$, leaving invariant the scalar semi-prepotential $\psi^2$. A similar behaviour was found when the $U(1)$ transformation was considered, where only the semi-prepotential $\psi^1$ was affected, producing a shift on its imaginary part, leaving invariant its real sector. Finally, when the Lorentz transformation was considered, it was found that there is a shift on the imaginary part of the semi-prepotential $\psi^1_{ab}$, leaving invariant its real part as well as the semi-prepotential $\psi^2_{ab}$.

As mentioned before, this work represents a first stage towards the identification of universal features of the work of [6, 7], that is, the identification of all relevant elements independent of compensator choice, and therefore, in principle, independent of any chosen superspace geometry.

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**Appendix**

The action of the generator $M^b_a$ on spinors $\Psi_\alpha$ and vectors $X_a$ is given by

$$[M_{ab}, \Psi_\alpha] = \frac{i}{2} (\sigma_{ab})_\alpha^\gamma \Psi_\gamma; \quad [M_{ab}, X_c] = \eta_{ca} X_b - \eta_{cb} X_a,$$

(109)
hence
\[
[M_{ab}, \Phi_{acd}] = \frac{i}{2} (\sigma_{ab})_{\alpha}^{\gamma} \Phi_{\gamma cd} + \eta_{ca} \Phi_{abd} - \eta_{cb} \Phi_{aad} + \eta_{da} \Phi_{acb} - \eta_{db} \Phi_{aeb}. \quad (110)
\]

For instance let us consider
\[
\{\nabla_\alpha, \nabla_\beta\} = C_{\alpha\beta}^{\gamma} E_C + \{E_\alpha, \Upsilon_\beta\} + \{\Upsilon_\alpha, E_\beta\} + \{\Upsilon_\alpha, \Upsilon_\beta\}, \quad (111)
\]
thus computing each anticommutator
\[
\{E_\alpha, \Upsilon_\beta\} = \frac{1}{2} (E_\alpha \omega_{\beta c}) M_{d c} + \frac{i}{4} \omega_{\beta}^{cd} (\sigma_{de})_{\alpha}^{\gamma} E_\gamma + E_\alpha \Gamma_\beta Z, \quad (112)
\]
\[
\{\Upsilon_\alpha, \Upsilon_\beta\} = \frac{1}{4} \{\omega_{ab}^{c} M_{c b}, \omega_{\beta d}^{e} M_{e d}\} + \frac{i}{4} \omega_{ab}^{c} (\sigma_{c}^{b})_{\beta}^{\gamma} \Gamma_\gamma Z + \frac{i}{4} \omega_{\beta d}^{e} (\sigma_{e}^{d})_{\alpha}^{\gamma} \Gamma_\gamma Z. \quad (113)
\]

Using (112) and (113) in (111) we have
\[
\{\nabla_\alpha, \nabla_\beta\} = C_{\alpha\beta}^{\gamma} E_C + \frac{1}{2} (E_\alpha \omega_{\beta c}) M_{d c} + \frac{i}{4} \omega_{\beta}^{cd} (\sigma_{de})_{\alpha}^{\gamma} E_\gamma + E_\alpha \Gamma_\beta Z + \frac{1}{2} (E_\beta \omega_{\alpha d}) M_{d c} + \frac{i}{4} \omega_{\alpha}^{cd} (\sigma_{de})_{\beta}^{\gamma} E_\gamma + E_\beta \Gamma_\alpha Z + \frac{1}{4} \{\omega_{ab}^{c} M_{c b}, \omega_{\beta d}^{e} M_{e d}\} + \frac{i}{4} \omega_{ab}^{c} (\sigma_{c}^{b})_{\beta}^{\gamma} \Gamma_\gamma Z + \frac{i}{4} \omega_{\beta d}^{e} (\sigma_{e}^{d})_{\alpha}^{\gamma} \Gamma_\gamma Z, \quad (114)
\]
with
\[
\Sigma_{\alpha\beta} = \{\omega_{ab}^{c} M_{c b}, \omega_{\beta d}^{e} M_{e d}\}
\]
\[
= \omega_{ab}^{ba} \omega_{\beta c}^{de} [M_{ab}, M_{cd}] + \omega_{ab}^{ba} [M_{ab}, \omega_{\beta cd}] M_{dc} + \omega_{\beta}^{ba} [M_{ab}, \omega_{abcd}] M_{dc}
\]
\[
= 4 \omega_{a}^{b} \omega_{\beta d}^{e} M_{dc} + \frac{i}{2} \left[ (\sigma_{ab})_{\alpha}^{\gamma} \omega_{\beta}^{ba} + (\sigma_{ab})_{\beta}^{\gamma} \omega_{\alpha}^{ba} \right] \omega_{\gamma cd} M_{dc} \quad (115)
\]
and
\[
E_C = \nabla C - \frac{1}{2} \omega_{Cc}^{d} M_{d c} - \Gamma C Z \quad (116)
\]
\[ \{ \nabla_\alpha, \nabla_\beta \} = C_\alpha \beta C C_\gamma + \frac{i}{4} \left[ \omega_\alpha^{cd} (\sigma_{dc})_\beta^\gamma + \omega_\beta^{cd} (\sigma_{dc})_\alpha^\gamma \right] \nabla_\gamma \\
+ \left[ -\frac{1}{2} C_\alpha \beta C \omega^d_{Cc} + \frac{1}{2} E_\alpha \omega^d_{\beta c} \right. \\
\left. + \frac{1}{2} E_\beta \omega^d_{\alpha c} + \omega^b_{\alpha c} \omega^d_{\beta b} \right] M^c_d \\
+ \left[ -C_\alpha \beta C \Gamma^C C + E_\alpha \Gamma \beta + E_\beta \Gamma \alpha \right] Z. \]

(117)

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