Mechanical Equations on Bi-Para Conformal Geometry

Zeki Kasap *

Department of Elementary Education, Faculty of Education, Pamukkale University
20070 Denizli, Turkey.

Mehmet Tekkoyun †

Department of Mathematics, Pamukkale University
20070 Denizli, Turkey.

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Abstract

This study is an extented analogue to conformal geometry of the paper given by [14].

Also, the geometric and physical results related to bi-para-conformal-dynamical systems are also presented.

Keywords: Conformal geometry, bi-para Lagrangian, bi-para Hamiltonian.

MSC: 53C15, 70H03, 70H05.
1 Introduction

There are a great number of applications for differential geometry and mathematical physics. This applications can be use in many areas in this century. One of the most important applications of differential geometry is on geodesics. A geodesic is the shortest route between two points. Geodesics can be found with the help of the Euler-Lagrange and Hamilton equations. Also, the information about them can be seen in many mechanical and geometry books. It is well known that differential geometry provides a suitable field for studying Lagrangians and Hamiltonians of classical mechanics and field theory. So, the dynamic equations for moving bodies were obtained for Lagrangian and Hamiltonian mechanics by many authors and are illustrated as follows:

I. Lagrange Dynamics Equations [1] [2] [3]: Let \( M \) be an \( n \)-dimensional manifold and \( T M \) its tangent bundle with canonical projection \( \tau_M : TM \to M \). \( TM \) is called the phase space of velocities of the base manifold \( M \). Let \( L : TM \to R \) be a differentiable function on \( TM \) and is called the Lagrangian function. We consider closed 2-form on \( TM \):

\[
\Phi_L = -dd_J L. \tag{1}
\]

(if \( J^2 = -I \), \( J \) is a complex structure or if \( J^2 = I \), \( J \) is a para-complex structure and \( Tr(J) = 0 \)). Consider the equation

\[
i_\xi \Phi_L = dE_L. \tag{2}
\]

Where the semispray \( \xi \) is a vector field. We know that \( E_L = V(L) - L \) is an energy function and \( V = J(\xi) \) a Liouville vector field. Here \( dE_L \) denotes the differential of \( E_L \). It is well-known that (2) under a certain condition on \( \xi \) is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as Lagrange dynamical equation. The triple \((TM, \Phi_L, \xi)\) is known as Lagrangian system on the tangent bundle \( TM \). The operations run on (2) for any coordinate system.
(q^i(t), p_i(t)). Infinite dimension **Lagrangian’s equation** is obtained the form below:

\[
\frac{dq^i}{dt} = \dot{q}^i, \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i}, \quad i = 1, ..., n.
\]  

(3)

II. Hamilton Dynamics Equations [2][3]: Let M be the base manifold and its cotangent manifold T^*M. By a symplectic form we mean a 2-form \( \Phi \) on \( T^*M \) such that:

(i) \( \Phi \) is closed, that is, \( d\Phi = 0 \); (ii) for each \( z \in T^*M \), \( \Phi_z : T_z T^*M \times T_z T^*M \to \mathbb{R} \) is weakly nondegenerate. If \( \Phi_z \) in (ii) is nondegenerate, we speak of a **strong symplectic form**. If (ii) is dropped we refer to \( \Phi \) as a presymplectic form. Now let \( (T^*M, \Phi) \) us take as a symplectic manifold.

A vector field \( Z_H : T^*M \to TT^*M \) is called **Hamiltonian vector field** if there is a \( C^1 \) **Hamiltonian function** \( H : T^*M \to \mathbb{R} \) such that Hamilton dynamical equation is determined by

\[
i_{Z_H} \Phi = dH.
\]  

(4)

We say that \( Z_H \) is locally Hamiltonian vector field if \( \Phi \) is closed. Where \( \Phi \) shows the canonical symplectic form so that \( \Phi = -d\lambda, \lambda = J^*(\omega) \), such that \( J^* \) a dual of \( J \), \( \omega \) a 1-form on \( T^*M \). The triple \( (T^*M, \Phi, Z_H) \) is named **Hamiltonian system** which it is defined on the cotangent bundle \( T^*M \).

From the local expression of \( Z_H \) we see that \( (q^i(t), p_i(t)) \) is an integral curve of \( Z_H \) iff **Hamilton’s equations** are expressed as follows:

\[
\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.
\]  

(5)

Considering information the above, in a lot of articles and books, it is possible to show how differential geometric methods are applied in Lagrangian’s and Hamiltonian’s mechanics in the below. Some works in paracomplex geometry are used for mathematical models.

**Cruceanu, Fortuny and Geada** have presented paracomplex geometry which is related to algebra of paracomplex number and the study of the structures on differentiable manifolds called paracomplex structures. Furthermore, they have considered a compatible neutral pseudo-Riemannian metric,
the para-Hermitian and para-Kahler structures, and their variants \[4\]. Kaneyuki and Kozai have introduced a class of affine symmetric spaces, which are called para-Hermitian symmetric spaces, a paracomplex analogue Hermitian symmetric space \[5\]. In the study of para-Kahlerian manifolds, Tekkoyun has introduced paracomplex analogues of the Euler-Lagrange and Hamilton equations. Furthermore, the geometric results on the related mechanic systems have been presented \[6\]. Etayo and Santamaria studied connections attached to non-integrable almost biparacomplex manifolds. Manifolds endowed with three foliations pairwise transversal are called 3-webs. Similarly, they can be algebraically defined as biparacomplex or complex product manifolds, i.e., manifolds endowed with three tensor fields of type \((1,1)\), \(F, P\) and \(J = FoP\), where the two first are product and the third one is complex, and they mutually anti-commute. In this case, it is well known that there exists a unique torsion-free connection parallelizing the structure. A para-Kählerian manifold \(M\) is said to be endowed with an almost bi-para-Lagrangian structure (a bi-para-Lagrangian manifold) if \(M\) has two transversal Lagrangian distributions (involutive transversal Lagrangian distributions) \(D_1\) and \(D_2\) \[7\]. Carinena and Ibort obtained the Lax equations which are associated with a dynamical endowed with a bi-Lagrangian connection and a closed two-form \(\Omega\) parallel along dynamical field \(\Gamma\). The case of Lagrangian dynamical systems is analysed and the nonnoether constants of motion found by Hojman and Harleston are recovered as being associated to a reduced Lax equation. Completely integrable dynamical systems have been shown to be a particular case of these systems by their \[8\]. Gordejuela and Santamaria have proved that the canonical connection of a bi-Lagrangian manifold introduced which was by Hess is a Levi-Civita connection by showing that a bi-Lagrangian manifold (i.e. a symplectic manifold endowed with two transversal Lagrangian foliations) is endowed with a canonical semi-Riemannian metric \[9\]. Kanai has been concerned with closed \(C^\infty\) Riemannian manifolds of negative curvature whose geodesic flows have \(C^\infty\) stable and unstable foliations. In particular, we have shown that the geodesic flow of such a manifold is isomorphic to that of a certain closed Riemannian
manifold of constant negative curvature if the dimension of the manifold is greater than two and if the sectional curvature lies between $-\frac{9}{4}$ and $-1$ strictly [10]. Since they have shown fundamental physical properties in turbulence (conservation laws, wall laws, Kolmogorov energy spectrum,...), symmetries are used to analyse common turbulence models. A class of symmetry preserving turbulence models has been proposed. This class has been refined in such a way that the models respect the second law of thermodynamics. Moreover, an example of model belonging to the class has been numerically tested by Razafindralandy and Hamdouni [11].

A base-equation method has been implemented to actualize the hereditary algebra of the Korteweg-de Vries (KdV) hierarchy and the N-soliton manifold is reconstructed. The novelty of our approach is the fact that it can in a rather natural way, predict other nonlinear evolution equations which admit local conservation laws. Significantly enough, base functions themselves are found to provide a basis to regard the KdV-like equations as higher order degenerate bi-Lagrangian systems by Chakrabarti and Talukdar [12]. Bi-para-complex analogue of Lagrangian and Hamiltonian systems has been introduced on Lagrangian distributions by Tekkoyun and Sari. Additionally, the geometric and physical results related to bi-para-dynamical systems have also been presented by them [13]. Authors introduced generalized-quaternionic Kähler analogue of Lagrangian and Hamiltonian mechanical systems. Moreover, the geometrical-physical results which are related to generalized-quaternionic Kähler mechanical systems have also been also given by Tekkoyun and Yayli [14].

In the above studies; although para-complex mechanical systems were analyzed successfully in relatively broad area of science, they have not dealt with bi-para-complex conformal mechanical systems on the bi-Lagrangian conformal manifold. In this study, therefore, equations related to bi-para-conformal mechanical systems on the bi-Lagrangian conformal manifold used in obtaining geometric quantization have been presented.
2 Preliminaries

In this study, all the manifolds and geometric objects are $C^\infty$ and the Einstein summation convention ($\sum x_i = x_i$) is in use. Also, $A$, $F(TM)$, $\chi(TM)$ and $\Lambda^1(TM)$ denote the set of para-complex numbers, the set of para-complex functions on $TM$, the set of para-complex vector fields on $TM$ and the set of para-complex 1-forms on $TM$, respectively. The definitions and geometric structures on the differential manifold $M$ given in [4] may be extended to $TM$ as follows:

3 Conformal Geometry

A conformal map is a function which preserves angles. Conformal maps can be defined between domains in higher dimensional Euclidean spaces, and more generally on a Riemann or semi-Riemann manifold. A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics $g'$ and $g$ are equivalent if and only if

$$g' = \lambda^2 g$$

where $\lambda > 0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class [15]. Two Riemann metrics $g$ and $g'$ on $M$ are said to be equivalent if and only if

$$g' = e^{\lambda} g$$

where $\lambda$ is a smooth function on $M$. The equation given by (7) is called a Conformal Structure [16].

4 Bi-Para-Complex Geometry

Let $M$ be a differentiable manifold. An almost bi-para-complex structure on $M$ is denoted by two tensor fields $F$ and $P$ of type (1,1) giving $F^2 = P^2 = 1$, $F \circ P + P \circ F = 0$ [7]. It is seen that
\(P \circ F\) is an almost complex structure. If the matrix-structure defined by the almost bi-para-complex structure is integrable then for every point \(p \in M\) there exists an open neighbourhood \(U\) of \(p\) and local coordinates \((U; x^i, y^j)\) such that

\[
F(\partial/\partial x^i) = \partial/\partial y^i, F(\partial/\partial y^i) = \partial/\partial x^i, \quad (8)
\]

\[
P(\partial/\partial x^i) = \partial/\partial x^i, \quad P(\partial/\partial y^i) = -\partial/\partial y^i, \quad \forall i = 1, n
\]

[17]. The existence of these kind of local coordinates on \(M\) permits to construct holomorphic local coordinates, \((U; z^k)\), \(z^k = x^k + iy^k\), \(i^2 = -1\), \(k = 1, n\), or para-holomorphic local coordinates, \((U; z^k)\), \(z^k = x^k + jy^k\), \(k = 1, n\), \(j^2 = 1\) [18] [19]. \((M, g, J)\) is a para-Kählerian manifold that always has two transversal distributions defined by the eigen-spaces associated to \(+1\) and \(-1\) eigenvalues of \(J\).

Besides, the mentioned distributions are involutive Lagrangian distributions if somebody thinks of the symplectic form \(\Phi\) defined by

\[
\Phi(X, Y) = g(JX, Y), \forall X, Y \in \chi(M).
\]

Consider \(x^i, y^i\) to be a real coordinate system on a neighborhood \(U\) of any point \(p\) of \(M\). Also let \(\{\partial/\partial x^i, \partial/\partial y^i\}\) and \(\{dx^i, dy^i\}\) be natural bases over \(R\) of the tangent space \(T_p(M)\) and the cotangent space \(T^*_p(M)\) of \(M\), respectively. Then the below equalities may be written by

\[
J(\partial/\partial x^i) = \partial/\partial y^i, \quad \text{j}(\partial/\partial y^i) = \partial/\partial x^i. \quad (9)
\]

Let \(z^i = x^i + j y^i\), \(j^2 = 1\), also be a para-complex local coordinate system on \(M\). So the vector fields will be shown:

\[
\partial/\partial z^i = \frac{1}{2}\{\partial/\partial x^i - j \partial/\partial y^i\}, \quad \partial/\partial \bar{z}^i = \frac{1}{2}\{\partial/\partial x^i + j \partial/\partial y^i\}.
\]

which represent the bases of \(M\). Also, the dual covector fields are

\[
dz^i = dx^i + jdy^i, \quad d\bar{z}^i = dx^i - jdy^i.
\]
which represent the cobases of $M$. Then the following expression can be written

$$J(\frac{\partial}{\partial z^i}) = -j \frac{\partial}{\partial z^i}, \quad J(\frac{\partial}{\partial \bar{z}^i}) = j \frac{\partial}{\partial \bar{z}^i}. \quad (12)$$

The dual endomorphism $J^*$ of $T^*_p(M)$ at any point $p$ of the manifold $M$ satisfies that $J^{*2} = I$, and is denoted by

$$J^*(dz^i) = -jd\bar{z}^i, \quad J^*(d\bar{z}^i) = jdz^i. \quad (13)$$

Let $V^A$ be a commutative group $(V, +)$ endowed with a structure of unitary module over the ring $A$. Let $V^R$ denote the group $(V, +)$ endowed with the structure of real vector space inherited from the restriction of scalars to $R$. The vector space $V^R$ will then be called the real vector space associated to $V^A$. Setting

$$J(u) = ju, \quad P^+(u) = e^+u, \quad P^-(u) = e^-u, \quad u \in V^A, \quad (14)$$

the equalities

$$J^2 = 1_V, \quad P^{*2} = P^+, \quad P^{-2} = P^-, \quad P^+ \circ P^- = P^- \circ P^+ = 0$$

$$P^+ + P^- = 1_V, \quad P^+ - P^- = J,$$

$$P^+ = (1/2)(1_V - J), \quad P^- = (1/2)(1_V + J),$$

$$j^2 = 1, \quad e^{*2} = e^+, \quad e^{-2} = e^-, \quad e^+ \circ e^- = e^- \circ e^+ = 0,$$

$$e^+ + e^- = 1, \quad e^+ - e^- = j,$$

$$e^- = (1/2)(1 - j), \quad e^+ = (1/2)(1 + j).$$

can be found. Also, we calculated which

$$P^\mp \left(\frac{\partial}{\partial z^i}\right) = -e^\mp \frac{\partial}{\partial z^i}, \quad P^\mp \left(\frac{\partial}{\partial \bar{z}^i}\right) = e^\mp \frac{\partial}{\partial \bar{z}^i},$$

$$P^{*\mp} (dz^i) = -e^{*\mp}d\bar{z}^i, \quad P^{*\mp} (d\bar{z}^i) = e^{*\mp}dz^i. \quad (16)$$

If the conformal manifold $(M, g, J = P^+ - P^-)$ satisfies the following conditions simultaneously then the conformal manifold is an almost para-conformal Hermitian manifold. The first expression can be given as follows:

$$g(X, Y) + g(X, Y) = 0 \iff g(X, Y) = 0, \quad \forall X, Y \in \chi(D_1), \quad (17)$$
since $P^+$ and $P^-$ are the projections over $D_1$ and $D_2$ respectively. Then $(P^+ - P^-)(X) = P^+ X - P^- X = P^+ X = X$, $(P^+ - P^-)(Y) = P^+ Y - P^- Y = P^+ Y = Y$. Similarly the second expression can be shown as follows:

$$g(X, Y) + g(X, Y) = 0 \iff g(X, Y) = 0, \quad \forall X, Y \in \chi(D_2). \quad (18)$$

Let $X = X_1 + X_2, Y = Y_1 + Y_2$ be vector fields on $M$ such that $X_1, Y_1 \in D_1$ and $X_2, Y_2 \in D_2$. Then

$$g(JX, Y) = g(JX_1 + JX_2, Y_1 + Y_2) = g(X_1 - X_2, Y_1 + Y_2)$$

$$= g(X_1, Y_1) - g(X_2, Y_1) + g(X_1, Y_2) - g(X_2, Y_2)$$

$$= -g(X_2, Y_1) + g(X_1, Y_2),$$

$$g(X, JY) = g(X_1 + X_2, JY_1 + JY_2) = g(X_1 + X_2, Y_1 - Y_2)$$

$$= g(X_1, Y_1) + g(X_2, Y_1) - g(X_1, Y_2) - g(X_2, Y_2)$$

$$= g(X_2, Y_1) - g(X_1, Y_2),$$

and hence

$$g(JX, Y) + g(X, JY) = -g(X_2, Y_1) + g(X_1, Y_2) + g(X_2, Y_1) - g(X_1, Y_2) = 0,$$

for all vector fields $X, Y$ on $M$. If the conditions (17) and (15) are true then $D_1$ and $D_2$ are Lagrangian distributions in terms of the 2-form $\Phi(X, Y) = g(JX, Y)$. Therefore, if the almost para-complex structure $J$ is integrable then $(M, g, J)$ is a para-conformal Kählerian manifold, or equivalently, $(M, \Phi, D_1, D_2)$ is a bi-Lagrangian conformal manifold. [20, 21, 22]. Where $W_{\pm}$ is a conformal para-complex structure to be similar to an integrable almost (para)-complex $P^\pm$ given in (16). Similarly $W^\pm_{\mp}$ are the dual of $W_{\pm}$ structures. So, we adapt the following equations using (17):

$$W^\pm \left( \frac{\partial}{\partial z_i} \right) = -e^\pm e^{\lambda} \frac{\partial}{\partial z_i}, \quad W^\mp \left( \frac{\partial}{\partial z_i} \right) = e^{\mp} e^{-\lambda} \frac{\partial}{\partial z_i},$$

$$W^{\mp*} (dz_i) = -e^\mp e^{\lambda} dz_i \quad \text{and} \quad W^{\pm*} (dz_i) = e^{\mp} e^{-\lambda} dz_i. \quad (20)$$

## 5 Conformal Bi-Para Euler-Lagrangians

Here, conformal bi-para-Euler-Lagrange equations and a conformal bi-para-mechanical system will be obtained under the consideration of the basis $\{e^+, e^-\}$ on the bi-Lagrangian conformal manifold.
$(M, \Phi, D_1, D_2)$. Let $(W^+, W^-)$ be an almost bi-para-complex conformal structure on $(M, \Phi, D_1, D_2)$, and $(z^i, \bar{z}^i)$ be its para-complex coordinates. Let the vector field $\xi$ be a semispray given by

$$
\xi = e^+(\xi^{i+} \partial_{z^i} + \bar{\xi}^{i+} \partial_{\bar{z}^i}) + e^-(\xi^{i-} \partial_{z^i} + \bar{\xi}^{i-} \partial_{\bar{z}^i});
$$

$$z^i = z^{i+}e^+ + z^{i-}e^-; \quad \bar{z}^i = \bar{z}^{i+}e^+ + \bar{z}^{i-}e^-; \quad \xi^{i+}e^+ + \xi^{i-}e^-;
$$

$$
(21)
$$

where the dot indicates the derivative with respect to time $t$. The vector field denoted by $V = (W^+ - W^-)(\xi)$ and given by

$$
(W^+ - W^-)(\xi) = e^+( -e^\lambda \xi^{i+} \partial_{z^i} + e^{-\lambda \bar{\xi}^{i+} \partial_{\bar{z}^i}}) - e^-( -e^\lambda \xi^{i-} \partial_{z^i} + e^{-\lambda \bar{\xi}^{i-} \partial_{\bar{z}^i}})
$$

is called conformal bi-para Liouville vector field on the bi-Lagrangian conformal manifold. The maps given by $T, P : M \to A$ such that $T = \frac{1}{2}m_i(\bar{z}^i)^2 = \frac{1}{2}m_i(z^i)^2$, $P = m_i gh$ are called the kinetic energy and the potential energy of the system, respectively. Here $m_i, g$ and $h$ stand for mass of a mechanical system having $n_i$ particle, the gravity acceleration and distance to the origin of a mechanical system on the bi-Lagrangian conformal manifold,

respectively. Then $L : M \to A$ is a map that satisfies the conditions;

i) $L = T - P$ is a conformal bi-para Lagrangian function,

ii) the function given by $E_L = V(L) - L$ is a conformal bi-para energy function.

The operator $i_{(W^+ - W^-)}$ induced by $W^+ - W^-$ and shown by

$$
i_{(W^+ - W^-)}(Z_1, Z_2, ..., Z_r) = \sum_{i=1}^r \omega(Z_1, ..., (W^+ - W^-)Z_i, ..., Z_r)
$$

is said to be vertical derivation, where $\omega \in \wedge^r M$, $Z_i \in \chi(M)$. The vertical differentiation $d_{(P^+ - P^-)}$ is defined by

$$
d_{(W^+ - W^-)} = [i_{(W^+ - W^-)}, d] = i_{(W^+ - W^-)}d - di_{(W^+ - W^-)}
$$

where $d$ is the usual exterior derivation. For an almost para-complex structure $W^+ - W^-$, the closed
para-conformal Kählerian form is the closed 2-form given by \( \Phi_L = -dd_{(W^+ - W^-)} L \) such that

\[
d_{(W^+ - W^-)} L = e^+ \left( -e^\lambda \frac{\partial L}{\partial z^i} dz^i + e^{-\lambda} \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i \right) - e^- \left( -e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i + e^{-\lambda} \frac{\partial L}{\partial z^i} dz^i \right) : \mathcal{F}(M) \rightarrow \Lambda^1 M \quad (25)
\]

Let \( \xi \) be the second order differential equations given by equation (21) and

\[
i_\xi \Phi_L = \Phi_L(\xi)
\]

\[
= -e^+ \xi^+ e^\lambda \frac{\partial \lambda}{\partial z^i} dz^i + e^+ \xi^0 e^\lambda \frac{\partial L}{\partial z^i} dz^i - e^- \xi^+ e^\lambda \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + e^- \xi^0 e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i
\]

\[
+ e^+ \xi^+ e^{-\lambda} \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + e^+ \xi^0 e^{-\lambda} \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i - e^- \xi^+ e^{-\lambda} \frac{\partial \lambda}{\partial z^i} dz^i + e^- \xi^0 e^{-\lambda} \frac{\partial L}{\partial z^i} dz^i
\]

\[
+ e^- \xi^0 e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i + e^- \xi^0 e^\lambda \frac{\partial L}{\partial z^i} dz^i - e^- \xi^0 e^{-\lambda} \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + e^- \xi^0 e^{-\lambda} \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i
\]

\[
= e^+ \xi^+ e^\lambda \frac{\partial \lambda}{\partial z^i} dz^i + e^+ \xi^0 e^\lambda \frac{\partial L}{\partial z^i} dz^i - e^- \xi^+ e^\lambda \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + e^- \xi^0 e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i
\]

\[
+ e^+ \xi^0 e^{-\lambda} \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + e^+ \xi^0 e^{-\lambda} \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i - e^- \xi^0 e^{-\lambda} \frac{\partial \lambda}{\partial z^i} dz^i + e^- \xi^0 e^{-\lambda} \frac{\partial L}{\partial z^i} dz^i
\]

\[
+ e^- \xi^0 e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i + e^- \xi^0 e^\lambda \frac{\partial L}{\partial z^i} dz^i - e^- \xi^0 e^{-\lambda} \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + e^- \xi^0 e^{-\lambda} \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i
\]

\[
- e^- \xi^0 e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i + e^- \xi^0 e^\lambda \frac{\partial L}{\partial z^i} dz^i - e^- \xi^0 e^{-\lambda} \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + e^- \xi^0 e^{-\lambda} \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i
\]

\[
= e^+ \left( -\xi^+ e^\lambda \frac{\partial L}{\partial z^i} + \xi^0 e^{-\lambda} \frac{\partial L}{\partial z^i} \right) - e^- \left( -\xi^+ e^\lambda \frac{\partial L}{\partial \bar{z}^i} + \xi^0 e^{-\lambda} \frac{\partial L}{\partial \bar{z}^i} \right) - L \quad (27)
\]

Since the closed conformal para-Kählerian form \( \Phi_L \) on \( M \) is in a para-symplectic structure, it is found that

\[
E_L = e^+ \left( -\xi^+ e^\lambda \frac{\partial L}{\partial z^i} + \xi^0 e^{-\lambda} \frac{\partial L}{\partial z^i} \right) - e^- \left( -\xi^+ e^\lambda \frac{\partial L}{\partial \bar{z}^i} + \xi^0 e^{-\lambda} \frac{\partial L}{\partial \bar{z}^i} \right) - L \quad (27)
\]

and thus

\[
dE_L = e^+ \left[ -\xi^+ e^\lambda \frac{\partial \lambda}{\partial z^i} dz^i - \xi^+ e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i - \xi^0 e^- e^\lambda \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + \xi^0 e^- e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i \right]
\]

\[
- e^- \left[ -\xi^+ e^\lambda \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i - \xi^+ e^- e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i - \xi^0 e^\lambda \frac{\partial \lambda}{\partial z^i} dz^i + \xi^0 e^- e^\lambda \frac{\partial L}{\partial z^i} dz^i \right] - \frac{\partial \lambda}{\partial z^i} dz^i
\]

\[
+ e^+ \left[ -\xi^+ e^\lambda \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i - \xi^+ e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i - \xi^0 e^- e^\lambda \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i + \xi^0 e^- e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i \right]
\]

\[
- e^- \left[ -\xi^+ e^\lambda \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i - \xi^+ e^- e^\lambda \frac{\partial L}{\partial \bar{z}^i} d\bar{z}^i - \xi^0 e^\lambda \frac{\partial \lambda}{\partial z^i} dz^i + \xi^0 e^- e^\lambda \frac{\partial L}{\partial z^i} dz^i \right] - \frac{\partial \lambda}{\partial \bar{z}^i} d\bar{z}^i \quad (28)
\]
Use of equation (2) gives:

\begin{equation}
\begin{aligned}
e^+ \xi^i + e^\lambda \frac{\partial L}{\partial z^i}; & \quad e^+ \xi^i + e^\lambda \frac{\partial L}{\partial z^i} - e^- \xi^i + e^\lambda \frac{\partial L}{\partial z^i} - e^- \xi^i + e^\lambda \frac{\partial L}{\partial z^i} \\
+ e^+ \xi^i + e^\lambda \frac{\partial L}{\partial z^i} + e^+ \xi^i + e^\lambda \frac{\partial L}{\partial z^i} - e^- \xi^i + e^\lambda \frac{\partial L}{\partial z^i} - e^- \xi^i + e^\lambda \frac{\partial L}{\partial z^i} + \frac{\partial L}{\partial z^i} = 0.
\end{aligned}
\end{equation}

If a curve denoted by \( \alpha : A \rightarrow M \) is considered to be an integral curve of \( \xi, \xi(L) = \frac{\partial L}{\partial t} \), then the following equation is obtained:

\begin{equation}
\begin{aligned}
(e^+ - e^-) e^\lambda \left[ e^+ \xi^i + \frac{\partial L}{\partial z^i} + e^+ \xi^i + \frac{\partial L}{\partial z^i} + e^- \xi^i - \frac{\partial L}{\partial z^i} + e^- \xi^i - \frac{\partial L}{\partial z^i} \right] \left( \frac{\partial L}{\partial z^i} \right) \\
+ (e^+ - e^-) e^\lambda \left[ e^+ \xi^i + \frac{\partial L}{\partial z^i} + e^+ \xi^i + \frac{\partial L}{\partial z^i} + e^- \xi^i - \frac{\partial L}{\partial z^i} + e^- \xi^i - \frac{\partial L}{\partial z^i} \right] \left( \frac{\partial L}{\partial z^i} \right) + \frac{\partial L}{\partial z^i} = 0,
\end{aligned}
\end{equation}

or

\begin{equation}
\begin{aligned}
(e^+ - e^-) e^\lambda \left( \frac{\partial L}{\partial z^i} \right) + (e^+ - e^-) e^\lambda \xi \left( \frac{\partial L}{\partial z^i} \right) + \frac{\partial L}{\partial z^i} = 0,
\end{aligned}
\end{equation}

Then the following equations are found:

\begin{equation}
\begin{aligned}
(e^+ - e^-) \frac{\partial}{\partial t} \left( e^\lambda \frac{\partial L}{\partial z^i} \right) + \frac{\partial L}{\partial z^i} = 0, \quad (e^+ - e^-) \frac{\partial}{\partial t} \left( e^\lambda \frac{\partial L}{\partial z^i} \right) - \frac{\partial L}{\partial z^i} = 0.
\end{aligned}
\end{equation}

Thus equations (32) are seen to be conformal bi-para Euler-Lagrange equations on the distributions \( D_1 \) and \( D_2 \), and then the triple \( (M, \Phi_L, \xi) \) is seen to be a conformal bi-para mechanical system with taking into account the basis \( \{e^+, e^-\} \) on the bi-Lagrangian conformal manifold \( (M, \Phi, D_1, D_2) \).
6 Conformal Bi-Para Hamiltonians

In the part, conformal bi-para Hamilton equations and a conformal bi-para Hamiltonian mechanical system on the bi-Lagrangian conformal manifold \((M, \Phi, D_1, D_2)\) will be derived. Let \((z_i, \bar{z}_i)\) be its para-complex coordinates. Let \(\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\}\) and \(\{dz_i, d\bar{z}_i\}\), be bases and cobases of \(T_p(M)\) and \(T_p^*(M)\) of \(M\), respectively. Let us assume that an almost bi-para-complex conformal structure, a bi-para-conformal Liouville form and a bi-para-complex conformal 1-form on the distributions \(D_1\) and \(D_2\) are shown by \(W^*+ - W^*-\), \(\lambda\) and \(\omega\), respectively. Then, we using [23] and (20):

\[
\omega = \frac{1}{2}[(z_i dz_i + \bar{z}_i d\bar{z}_i)e^+ + (e^{2\lambda} z_i dz_i + e^{2\lambda} \bar{z}_i d\bar{z}_i)e^-],
\]

\[
\lambda = (W^{*+} - W^{*-})(\omega) = \frac{1}{2}[(-e^+ e^\lambda z_i d\bar{z}_i + e^+ e^\lambda \bar{z}_i dz_i)] - \frac{1}{2}(-e^- e^\lambda z_i d\bar{z}_i + e^- e^\lambda \bar{z}_i dz_i].
\]

(33)

It is well known that if \(\Phi\) is a closed para-Kählerian form on the bi-Lagrangian conformal manifold, then \(\Phi\) is also a para-symplectic structure on the bi-Lagrangian conformal manifold. Given a bi-para-conformal Hamiltonian vector field \(Z_H\) fixed with the bi-para-conformal Hamiltonian energy \(H\) that is

\[
Z_H = (Z_i \frac{\partial}{\partial z_i} + \bar{Z}_i \frac{\partial}{\partial \bar{z}_i})e^+ + (Z_i \frac{\partial}{\partial \bar{z}_i} + \bar{Z}_i \frac{\partial}{\partial z_i})e^-.
\]

(34)

Then closed 2-form is

\[
\Phi = -d\lambda = e^+ \bar{Z}_i dz_i - e^- Z_i d\bar{z}_i - \frac{1}{2} \left[ e^+ e^\lambda \frac{\partial \lambda}{\partial z_i} Z_i z_i dz_i + e^+ e^\lambda \frac{\partial \lambda}{\partial \bar{z}_i} \bar{Z}_i \bar{z}_i d\bar{z}_i - e^- e^\lambda \frac{\partial \lambda}{\partial z_i} Z_i \bar{z}_i d\bar{z}_i - e^- e^\lambda \frac{\partial \lambda}{\partial \bar{z}_i} \bar{Z}_i z_i dz_i \right]\]

\[
= -e^+ Z_i d\bar{z}_i + e^- Z_i dz_i - \frac{1}{2} \left[ -e^+ e^\lambda \frac{\partial \lambda}{\partial z_i} Z_i z_i d\bar{z}_i - e^+ e^\lambda \frac{\partial \lambda}{\partial \bar{z}_i} \bar{Z}_i \bar{z}_i dz_i + e^- e^\lambda \frac{\partial \lambda}{\partial z_i} Z_i \bar{z}_i d\bar{z}_i + e^- e^\lambda \frac{\partial \lambda}{\partial \bar{z}_i} \bar{Z}_i z_i dz_i \right].
\]

(35)
And then it follows

\[ i_{Z_H} \Phi = \Phi(Z_H) \]
\[ = \bar{Z}_i e^+ \left[ 1 - \frac{1}{2} e^\lambda \left( z_i \frac{\partial \lambda}{\partial z_i} + \bar{z}_i \frac{\partial \lambda}{\partial \bar{z}_i} \right) \right] dz_i - Z_i e^+ \left[ 1 + \frac{1}{2} e^\lambda \left( z_i \frac{\partial \lambda}{\partial z_i} + \bar{z}_i \frac{\partial \lambda}{\partial \bar{z}_i} \right) \right] d\bar{z}_i \]
\[ - \bar{Z}_i e^- \left[ 1 - \frac{1}{2} e^\lambda \left( z_i \frac{\partial \lambda}{\partial z_i} + \bar{z}_i \frac{\partial \lambda}{\partial \bar{z}_i} \right) \right] dz_i + Z_i e^- \left[ 1 + \frac{1}{2} e^\lambda \left( z_i \frac{\partial \lambda}{\partial z_i} + \bar{z}_i \frac{\partial \lambda}{\partial \bar{z}_i} \right) \right] d\bar{z}_i \]  

(36)

On the other hand, the differential of the bi-para-conformal Hamiltonian energy \( H \) is calculated as follows:

\[ dH = \left( \frac{\partial H}{\partial z_i} dz_i + \frac{\partial H}{\partial \bar{z}_i} d\bar{z}_i \right) e^+ + \left( \frac{\partial H}{\partial \bar{z}_i} dz_i + \frac{\partial H}{\partial z_i} d\bar{z}_i \right) e^- . \]  

(37)

By means of equation (4), using equation (36) and (37), the conformal bi-para Hamiltonian vector field is seen to be

\[ Z_H = (Z_i \frac{\partial}{\partial z_i} + \bar{Z}_i \frac{\partial}{\partial \bar{z}_i}) e^+ + (Z_i \frac{\partial}{\partial \bar{z}_i} + \bar{Z}_i \frac{\partial}{\partial z_i}) e^- . \]  

(38)

If a curve \( \alpha : I \subset A \to M \) is an integral curve of the conformal bi-para Hamiltonian vector field \( Z_H \), i.e., \( Z_H(\alpha(t)) = \dot{\alpha}(t) \), \( t \in I \). In the local coordinates, we get \( \alpha(t) = (z_i(t), \bar{z}_i(t)) \) and

\[ \dot{\alpha}(t) = \left( \frac{dz_i}{dt} \frac{\partial}{\partial z_i} + \frac{d\bar{z}_i}{dt} \frac{\partial}{\partial \bar{z}_i} \right) e^+ + \left( \frac{dz_i}{dt} \frac{\partial}{\partial \bar{z}_i} + \frac{d\bar{z}_i}{dt} \frac{\partial}{\partial z_i} \right) e^- . \]  

(39)

Taking equations (38), (39), the following equations are found

\[ \frac{dz_i}{dt} = - (e^+ - e^-) \left[ 1 + \frac{1}{2} e^\lambda \left( z_i \frac{\partial \lambda}{\partial z_i} + \bar{z}_i \frac{\partial \lambda}{\partial \bar{z}_i} \right) \right] \frac{\partial H}{\partial \bar{z}_i} , \quad \frac{d\bar{z}_i}{dt} = (e^+ - e^-) \left[ 1 + \frac{1}{2} e^\lambda \left( z_i \frac{\partial \lambda}{\partial z_i} + \bar{z}_i \frac{\partial \lambda}{\partial \bar{z}_i} \right) \right] \frac{\partial H}{\partial z_i} . \]  

(40)

Hence, equations (40) are seen to be conformal bi-para Hamilton equations on the bi-Lagrangian conformal manifold \((M, \Phi, D_1, D_2)\), and then the triple \((M, \Phi, Z_H)\) is seen to be a conformal bi-para Hamiltonian mechanical system with the use of basis \(\{e^+, e^-\}\) on \((M, \Phi, D_1, D_2)\).

7 Conclusion

It is seen in the above, formalisms of Lagrangian and Hamiltonian mechanics had intrinsically been described by taking into account the basis \(\{e^+, e^-\}\) on the bi-Lagrangian conformal manifold \((M, \Phi, D_1, D_2)\).
Conformal bi-para Lagrangian and bi-para Hamiltonian models have arisen to be very important tools since they present a simple method to describe the model for bi-para-conformal mechanical systems. So, the equations derived in (32) and (40) are only considered to be a first step to realize how bi-para-complex conformal geometry has been used in solving problems in different physically spaces. For further research, bi-para-complex conformal Lagrangian and Hamiltonian vector fields derived here are suggested to deal with problems in different fields of physics. In the literature, the equations, which explains the linear orbits of the objects, were presented. This study explained the non-linear orbits of the objects in the space by the help of revised equations using Weyl theorem.

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