EFFECTIVE EQUIDISTRIBUTION OF HOROCYCLE LIFTS

ILYA VINOGRAODOV

Abstract. We give a rate of equidistribution of lifts of horocycles from the space \( SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R}) \) to the space \( ASL(2, \mathbb{Z}) \backslash ASL(2, \mathbb{R}) \), making effective a theorem of Elkies and McMullen. This result constitutes an effective version of Ratner’s measure classification theorem for measures supported on general horocycle lifts. The method used relies on Weil’s resolution of the Riemann hypothesis for function fields in one variable and generalizes the approach of Strömbergsson to the case of linear lifts and that of Browning and the author to rational quadratic lifts.

1. Introduction

1.1. Background. In the theory of flows on homogeneous spaces, Ratner’s theorems on measure rigidity, topological rigidity, and orbit equidistribution \([18, 19]\) play a major role. Their applications go far beyond the realm of dynamical systems and include results in number theory and mathematical physics \([7, 22, 14, 15]\); thorough expositions and comprehensive references may be found in \([17]\).

In the last decade there has been an increased interest in obtaining effective versions of Ratner’s results, such as giving a rate of convergence of measures in the measure rigidity theorem. There are two general situations where effective results may be proved: when the group generating the flow is horospherical, or when it is “large” in an appropriate sense (cf. \([4, \text{Sec. 1.5.2}]\)). Recently, rates of convergence were obtained for several settings where the corresponding group is neither horospherical nor large. Green and Tao \([11]\) proved effective equidistribution of polynomial orbits on nilmanifolds, while Einsiedler, Margulis, and Venkatesh \([4]\) proved effective equidistribution for closed orbits of semisimple groups on general homogeneous spaces. Strömbergsson \([26]\) and Browning and the author \([2]\) gave rates for the convergence of measures on special horocycle lifts; the present paper further explores this direction by giving a rate of convergence for measures on general horocycle lifts.

1.2. Results. For \( x \in \mathbb{R} \) and \( y > 0 \), let

\[
n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}.
\]

(1.1)

It is a fundamental result in homogeneous dynamics that long closed horocycles \( \{n(x) : x \in [-\frac{1}{2}, \frac{1}{2}]\} \) on \( X = SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R}) \) equidistribute under the geodesic flow \( a(y) \) as \( y \to 0 \).
That is, for every bounded continuous \( f : X \to \mathbb{R} \),
\[
\lim_{y \to 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(n(x)a(y)) \, dx = \int_X f(g) \, d\mu_X(g),
\]
where \( \mu_X \) is the Haar probability measure on \( X \). This can be proved using thickening followed by applying the mixing property of \( a(y) \) \cite{12}, which is a general approach when the integral is taken over all unstable directions of a flow. The rate of convergence was given in \cite{20, 29} and is related to the zero-free region for the Riemann zeta function. It is proved that for \( f \in C_0^\infty(X) \),
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} f(n(x)a(y)) \, dx = \int_X f(g) \, d\mu_X(g) + o(y^{1/2}),
\]
where the error term depends on the error term in the Prime Number Theorem. In the present paper we establish a similar result for certain horocycle lifts.

Let \( G = \text{ASL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2 \), and set \( \Gamma = \text{ASL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2 \), which is a lattice in \( G \). We view elements of \( G \) as ordered pairs \((M, x)\) with \( M \in \text{SL}(2, \mathbb{R}) \) and \( x \in \mathbb{R}^2 \), and multiply them following the rule
\[
(M, x)(M', x') = (MM', xM' + x'),
\]
thinking of \( x, x' \) as row vectors in \( \mathbb{R}^2 \). Writing \( Y = \Gamma \backslash G \), we equip this homogeneous space with the Haar probability measure that we denote \( \mu_Y \). When no confusion can arise we shorten \((M, 0)\) to \( M \).

It is important to note the relationship between \( X \) and \( Y \), the latter being a bundle over the former with two-dimensional torus fiber. The space \( X \) parametrizes unimodular lattices in \( \mathbb{R}^2 \), while \( Y \) is the space of lattice translates in \( \mathbb{R}^2 \). Thus, each point in \( X \) corresponds to a lattice \( \Lambda \subset \mathbb{R}^2 \), and a choice of \( x \in \Lambda \backslash \mathbb{R}^2 \) determines the translated lattice \( \Lambda + x \subset \mathbb{R}^2 \), which corresponds to a point of \( Y \).

For a continuous function \( \xi = (\xi_1, \xi_2) : \mathbb{R} \to \mathbb{R}^2 \) define
\[
\tilde{n}(x) = (1, \xi(x))n(x);
\]
we call this a lift of a horocycle from \( X \) to \( Y \). Let \( \rho : \mathbb{R} \to \mathbb{R} \) be nonnegative, continuously differentiable, supported on a compact interval (without loss of generality, \( \text{supp} \rho \subset (-1, 0) \)), and of integral 1. It is natural to ask whether the lifted measures \( \nu_y \) defined by
\[
\int_{\mathbb{R}} f(\tilde{n}(x)a(y)) \rho(x) \, dx = \int_Y f(g) \, d\nu_y(g)
\]
have a weak-* limit as \( y \to 0 \). Using Ratner’s Theorem \cite{18}, Elkies and McMullen \cite{7} established a condition on \( \xi \) under which \( \nu_y \) converges to \( \mu_Y \). Let \( \Xi(x) = x\xi_1(x) + \xi_2(x) \). A horocycle lift is called rationally linear if for some \( \alpha, \beta \in \mathbb{Q} \),
\[
\text{Leb}\{x : \Xi(x) = \alpha x + \beta\} > 0.
\]

**Theorem 1** (\cite{7 Th. 2.2}). Suppose that a horocycle lift \( \tilde{n} \) is not rationally linear, that \( \Xi \) is Lipschitz, and that \( \xi_1 \) is continuous. Then, \( \nu_y \to \mu_Y \) in the weak-* topology as \( y \to 0 \).
In the present paper, we make convergence in this theorem effective, which requires an effective version of rational nonlinearity. We say $\Xi$ is $D$-nice for some $D \geq 2$ if $\Xi$ is twice continuously differentiable and there exist $x_0 \in \mathbb{R}$ and $C_1, C_2 > 0$ such that

\[(1.8) \quad C_1|x - x_0|^{D-2} \leq |\Xi''(x)| \leq C_2|x - x_0|^{D-2}\]

for every $x$ in the support of $\rho$. For such a lift, set $C = \max \left\{ C_1^{-1/2}, C_2^{1/2} \right\}$. We say that $\tilde{n}$ is $D$-nice if the corresponding $\Xi$ is $D$-nice.

**Theorem 2.** Fix a density function $\rho$ as before, and let $\tilde{n}$ be $D$-nice. Assume that $f$, $\rho$, and $\tilde{n}$ are such that all norms in (1.10) are finite. Then for every $\varepsilon > 0$ there exists a constant $C(\varepsilon, f, \rho, \tilde{n})$ such that

\[(1.9) \quad |\nu_y(f) - \mu_Y(f)| \leq C(\varepsilon, f, \rho, \tilde{n}) y^{\min\left\{ \frac{1}{16}, \frac{1}{2} \right\} - \varepsilon} \]

for all $y \in (0, 1)$. Moreover, we can take

\[(1.10) \quad C(\varepsilon, f, \rho, \tilde{n}) = A(\varepsilon, \eta) C\|f\|_{C^2_b} (C + \|\Xi\|_{C^1} + \|\xi_1\|_{L^\infty}) \|\rho\|_{W^{1,1}} \|\rho\|_{W^{2,1}}\]

for any $\eta \in (0, 1)$, and the function $A$ is universal.

We observe that the $\varepsilon$-loss in the error term can be replaced by a logarithmic loss with a slightly more tedious computation as in [2]. The overall error would then be a constant times $y^{\min\left\{ \frac{1}{16}, \frac{1}{2} \right\} \log^x (2 + 1/y)$ for some $x > 0$. The norms used to define $C(\varepsilon, f, \rho, \tilde{n})$ are rigorously defined in (3.13), (3.14).

1.3. **Discussion.** This paper stands in the series of works that prove effective equidistribution results in the setting of a sequence on measures supported on the unstable manifold of a diagonal flow. The first and by now classical is [20]; it gives the optimal rate of equidistribution of long closed horocycles on quotients of the $SL(2, \mathbb{R})$ by using Eisenstein series to relate this question to the zero-free region of the Riemann zeta function. The case of non-uniform measure on the horocycle was treated by Strömbergsson [24]; this work also proves effective equidistribution for horocycle pieces of optimal intermediate length (length of piece can be nearly as short as the square root of the length of the horocycle). Horocycle lifts to $Y$ were first studied in [7] where an ineffective equidistribution theorem for general lifts is proved. Strömbergsson [26] used number-theoretic techniques similar to those employed in the present paper to give a rate for equidistribution of linear irrational lifts (in our notation, these correspond to $\xi$ being a constant that is not in $\mathbb{Q}^2$ and our results do not apply as $\Xi'' = 0$). The rate in this setup depends on the Diophantine properties of $\xi$. The method of Strömbergsson was further developed in [2] to treat the case of rational quadratic lifts, with the case $\xi(x) = (x/2, -x^2/4)$ being the most interesting. The treatment of this particular lift yields a rate for the convergence of the gap distribution of the sequence $\sqrt{n} \mod 1$. 
The present paper completes the effectivization of equidistribution theorems for lifts. The powers of $y$ (up to $y^3$) that appear in error terms in the aforementioned theorems are

\begin{align}
(1.11) & \quad y^{1/2}, \text{ Sarnak [20]}, \\
(1.12) & \quad y^{1/4}, \text{ Strömbergsson [26]}, \text{ assuming best Diophantine condition}, \\
(1.13) & \quad y^{1/4}, \text{ Browning, V. [2]}, \\
(1.14) & \quad y^{1/16}, \text{ present work, assuming best lift}.
\end{align}

We also remark that $y^{3/4}$ in (1.11) would be equivalent to the Riemann hypothesis. The novelty of the present paper is that it does not make use of quadratic niceties of [2] or linear simplicity of [26], allowing for the treatment of general lifts. The key result is Proposition 5, which establishes cancellations in a certain Kloosterman-like exponential sum (2.1) for all values of the indices involved.

In addition to the theorems mentioned above, we must also mention the recent result of Ubis [28], who used the “Fourier method” on $\mathbb{R}^d$ to prove effective equidistribution of certain manifolds on $(\Gamma \setminus \text{SL}(2, \mathbb{R}))^d$. Fix $\tilde{a}(y) = (a(y), \ldots, a(y)) \in \text{SL}(2, \mathbb{R})^d$ and consider its $d$-dimensional unstable manifold. Then, given a submanifold that is “totally curved” and has positive codimension, Ubis gives a rate of equidistribution of this submanifold under the action of $\tilde{a}(y)$. Although the method of this paper is different from that of the present work, the setup is quite similar, which gives hope that other equidistribution statements of this flavor (for example, results of Shah on equidistribution of curves [23, 21, 22]) will be effectivized in the near future.

Related results on effective equidistribution for SL(2, $\mathbb{R}$) include papers of Tanis and Vishe [27] and Flaminio, Forni, and Tanis [9] on period integrals, both building on the work on the seminal paper of Flaminio and Forni [8] on invariant distributions for the horocycle flow. Effective equidistribution of “relatively large” orbits is proven by Einsiedler, Margulis, and Venkatesh [11].

Another direction of refining convergence in Ratner’s theorem is extending weak-* convergence to unbounded test functions, known as the problem of convergence of moments. This question was answered affirmatively in certain situations, relating to theta functions and their application to values of inhomogeneous quadratic forms [13]; the pair correlation function of the sequence $\sqrt{n}$ modulo 1 [10]; directions of Euclidean lattice points [5]; directions of hyperbolic lattice points [16].

The question of convergence of moments is open for the main result of this paper, Theorem 2 as is the question of the rate of equidistribution of the unipotent flow $\{\tilde{n}(x) : x \in \mathbb{R}\}$ with $\xi(x) = (x/2, -x^2/4)$. We hope to return to these questions in future work.

1.4. Plan of paper. Section 2 contains an application of number-theoretic techniques to control a special exponential sum. In Section 3 we single out the main term from the integral in the statement of Theorem 2; we then bound the error term in Section 4.

1.5. Notation. Given functions $f, g : S \to \mathbb{R}$, with $g$ positive, we will write $f \ll g$ if there exists a constant $c$ such that $|f(s)| \leq cg(s)$ for all $s \in S$. 

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2. **Special exponential sums**

In this section we make a detailed examination of the exponential sum

\[ S_c(k, l, n) = \sum_{\substack{(c,d) = 1 \\ 0 < d < c}} e \left( \frac{ld - kd}{c} - nc\Xi \left( -\frac{d}{c} \right) \right). \tag{2.1} \]

We distinguish two cases that require different treatment, according as \( l = 0 \) or \( l \neq 0 \).

Consider first the case \( l = 0 \); the cancellations in the sum \( S \) must come from analytic properties of \( \Xi \). Writing

\[ S = \sum_{h|c} \mu(h) \sum_{d=1}^{c/h} e \left( -nc\Xi \left( -\frac{d}{c} \right) - \frac{kd}{c} \right) = \sum_{h|c} \mu(h) S \left( \frac{c}{h} \right), \tag{2.2} \]

we massage the inner sum over \( d \). We have

\[ S(x) = \sum_{d=1}^{x} e \left( -hnx\Xi \left( -\frac{d}{x} \right) - \frac{kd}{x} \right) = \sum_{d=1}^{x} e(w(d)). \tag{2.3} \]

Since \( w''(d) = -\frac{hn\Xi''}{x} \left( -\frac{d}{x} \right) \), our assumption on \( \Xi \) implies that \( |w''(d)| \sim c_1, c_2, \frac{hn}{x} \left| -\frac{d}{x} - z_0 \right|^{D-2} \). Let \( \delta > 0 \). Using the van der Corput estimate (cf. [10, p. 8]) when \( \left| -\frac{d}{x} - z_0 \right| > \delta \) and the trivial estimate otherwise, we get the bound

\[ |S(x)| \ll \delta x + C_2^{1/2} h^{1/2} x^{1/2} \delta^{D-2} + \frac{x^{1/2}}{C_1^{1/2} h^{1/2} n^{1/2}} \delta^{D-2}. \tag{2.4} \]

The optimal choice for \( \delta \) is \( x^{-1/D} \), giving the bound \( C h^{1/2} n^{1/2} x^{1-1/D} \) for \( S(x) \), where \( C = \max \left\{ C_2^{1/2}, C_1^{-1/2} \right\} \). The contribution of the case \( l = 0 \) is thus

\[ S \ll C \sum_{h|c} |\mu(h)| h^{1/2} n^{1/2} \left( \frac{c}{h} \right)^{1-1/D}, \tag{2.5} \]

\[ \ll C n^{1/2} c^{1-1/D} \sum_{h|c} |\mu(h)| h^{1/D+1/2-1}, \tag{2.6} \]

\[ \ll \varepsilon C n^{1/2} c^{1-1/D+\varepsilon}. \tag{2.7} \]

Consider now the case \( l \neq 0 \). We adopt the convention that the range of summation includes only those values of the indices for which the summands are defined, allowing us to
drop the coprimality condition. We begin by applying the Weyl-van der Corput inequality \[10\] eq. (2.3.5) for some \( H \in [1, c] \) to be chosen later, which gives

\[
S^2 \ll \frac{c^2}{H} + \frac{c}{H} \sum_{1 \leq h \leq H} \left| \sum_{d=1}^{c} e \left( \frac{l(d+h-d)}{c} - nc \left( \Xi \left( -\frac{d}{c} \right) - \Xi \left( -\frac{d}{c} \right) \right) \right) \right|.
\]

Writing

\[
a_d = e \left( \frac{l(d+h-d)}{c} \right), \quad b_d = e \left( -nc \left( \Xi \left( -\frac{d}{c} \right) - \Xi \left( -\frac{d}{c} \right) \right) \right),
\]

we set

\[
T = \sum_{d=1}^{c} a_d b_d;
\]

here \( T = T(c, h, l, n) \) depends on \( c \), \( h \), \( l \), and \( n \); and we follow the convention that the terms with \( a_d \) undefined are assumed to be zero. Now we seek to get cancellations in the sum \( T \).

Summing by parts, we can write

\[
T = b_c \sum_{q=1}^{c} a_d + \sum_{d=1}^{c-1} \sum_{q=1}^{d} a_q (b_d - b_{d+1}),
\]

provided \( c \geq 2 \) (when \( c = 1 \), the bound \( T \ll 1 \) is satisfactory). Set \( A_d = \sum_{q=1}^{d} a_q \) and \( B_d = b_d - b_{d+1} \); we need to bound \( A_d \) and \( B_d \). For the first, we use smoothing to write the sum as a complete sum modulo \( c \) followed by standard estimates for exponential sums; for the second, we rely on Taylor’s theorem and smoothness of \( \Xi \).

Let \( \delta \in (0, 1) \) be a number we will choose later depending on \( c \), and let \( I_\alpha : [0, 1] \to \{0, 1\} \) be the indicator of \([0, \alpha]\) for \( \alpha \in [0, 1] \). Let \( \psi : \mathbb{R} \to \mathbb{R} \) be smooth, of integral 1, supported on \([-1, 1]\). Then, \( \psi_\delta(x) = \frac{1}{\delta} \psi \left( \frac{x}{\delta} \right) \) is smooth, of integral 1, supported on \([-\delta, \delta]\). Set

\[
I_\alpha(x) = \begin{cases} 
I_{\alpha+\delta} * \psi_\delta(x), & \alpha + \delta \leq 1 \\
I_1(x), & \alpha + \delta > 1.
\end{cases}
\]

Using the notation where \( e_c(\cdot) = e(\frac{\cdot}{c}) \), we can write

\[
A_d = \sum_{q=1}^{c} e_c(l(q + h - \overline{q})) I_{d/c}(q/c)
\]

\[
= \sum_{q=1}^{c} e_c(l(q + h - \overline{q})) I_{d/c}(q/c) + O(\delta c).
\]

We introduce quantities

\[
c_{k,d}^\delta = \int_{0}^{1} \overline{I_{d/c}^\delta(x)} e(-kx) dx, \quad U_c(h, k, l) = \sum_{q=1}^{c} e_c(l(q + h - \overline{q}) + kq).
\]
Then, we can write
\begin{equation}
A_d = \sum_{k \in \mathbb{Z}} c_{k,d}U_c(h, k, l) + O(\delta c).
\end{equation}

Now the sum $U_c(h, k, l)$ can be treated using results of Bombieri \cite{Bombieri1975} for $c$ a prime, generalized by Cochrane and Zheng \cite{CochraneZheng1997} for $c$ a prime power. We begin by recording the easy multiplicative property
\begin{equation}
U_{q_1q_2}(h, k, l) = U_{q_1}(h, k\bar{q}_2, l\bar{q}_2)U_{q_2}(h, k\bar{q}_1, l\bar{q}_1)
\end{equation}
whenever $q_1, q_2 \in \mathbb{N}$ are coprime and $\bar{q}_1, \bar{q}_2 \in \mathbb{Z}$ satisfy $q_1\bar{q}_1 + q_2\bar{q}_2 = 1$. This renders it sufficient to study $U_{p^m}(h, k, l)$ for a prime power $p^m$. We may write $U_{p^m}(h, k, l)$ in the form
\begin{equation}
\sum_{q \mod p^m}^* e_{p^m}\left(\frac{f_1(q)}{f_2(q)}\right),
\end{equation}
where $f_1(q) = kq^2(q + h) - hl$ and $f_2(q) = q(q + h)$. The symbol $\sum^*$ emphasizes the fact that $q$ is only taken over values for which $q \nmid f_2(q)$, in which scenario $f_1(q)/f_2(q)$ means $f_1(q)\overline{f_2(q)}$. We proceed by establishing the following result, which is far from optimal, but sufficient for our needs.

**Lemma 3.** Let $p$ be a prime and $m \in \mathbb{N}$. Then we have
\begin{equation}
U_{p^m}(h, k, l) \ll \begin{cases}
p^{1/2}(p, (k, hl))^{1/2}, & m = 1, 
p^{2m/3}(p^m, (k, lh))^{m/3}, & m > 1.
\end{cases}
\end{equation}

**Proof.** When $m = 1$, the result follows from \cite{CochraneZheng1997} eq. (1.2)], which is a restatement of Bombieri’s result \cite{Bombieri1975}.

When $m > 1$, we use \cite{CochraneZheng1997} Cor. 3.2]. In their notation, we have $d(f_1) = 3$, $d(f_2) = 2$, $d(f) = 5$, $d^*(f) = 3$,
\begin{equation}
d_p(f) = \begin{cases}
0, & p \mid k, p \mid hl,
2, & p \mid k, p \nmid hl,
1, & p \nmid k, p \mid hl,
5, & p \nmid k, p \nmid hl.
\end{cases}
\end{equation}

In the last three cases, $d^*_p(f) = 2, 1, 3$, respectively, so that, by \cite{CochraneZheng1997} Cor. 3.2], $U_{p^m}(h, k, l) \ll p^{2m/3}$, which is satisfactory. In the first case, we choose $t \in \mathbb{N}$ so that $p^t \mid (k, hl)$. If $t \geq m$, the trivial bound on $U_{p^m}(h, k, l)$ is satisfactory. If $t < m$, we write $t = t_1 + t_2$, where $p^{t_1} \mid h$ and $p^{t_2} \mid l$,
\begin{equation}
U_{p^m}(h, k, l) = p^tU_{p^{m-t}}(hp^{-t_1}, kp^{-t}, lp^{-t_2}) \ll p^tp^{2(m-t)/3} = p^{2m/3+t/3},
\end{equation}
which is satisfactory. \(\square\)

We write $c = uv$, where $u$ is square-free and $v$ is square-full. That is, $p \mid u$ implies $p^2 \nmid u$ and $p \mid v$ implies $p^2 \mid v$. Using the multiplicativity property (2.17), we may apply Lemma 3 for different primes to arrive at the following result.
Lemma 4. Let $c \in \mathbb{N}$ and let $h, k, l \in \mathbb{Z}$. Then for every $\varepsilon > 0$ we have

\begin{equation}
U_c(h, k, l) \ll_{\varepsilon} c^2 u^{1/2}(u, (k, hl))^{1/2} v^{2/3}(v, (k, hl))^{1/3}.
\end{equation}

We substitute this bound into (2.10) together with the bound for Fourier coefficients

\begin{equation}
c_{k,d}^\delta \ll_{\gamma, \psi} \frac{1}{1 + k} \cdot \left(\frac{1}{k\delta + 1}\right)^\gamma
\end{equation}

for $\gamma \geq 0$. Choosing $\gamma = \varepsilon$, we get

\begin{equation}
Ad \ll_{\varepsilon} \sum_{k \in \mathbb{Z}} c^{2} u^{1/2}(u, (k, hl))^{1/2} v^{2/3}(v, (k, hl))^{1/3} \frac{1}{1 + k} \cdot \left(\frac{1}{k\delta + 1}\right)^\varepsilon + O(\delta c)
\end{equation}

(2.25)

\begin{equation}
\ll \sum_{k \in \mathbb{Z}} c^{2} u^{1/2}(u, hl)^{1/2} v^{2/3}(v, hl)^{1/3} \frac{(c, k)^{1/2}}{1 + k} \cdot \left(\frac{1}{k\delta + 1}\right)^\varepsilon + O(\delta c).
\end{equation}

Now we observe that

\begin{equation}
\sum_{k=1}^{K} \frac{(c, k)^{1/2}}{s^{1/2}} \ll \sum_{s | c} s^{1/2} \sum_{k \leq K} \frac{1}{s} \ll \sum_{s | c} s^{1/2} \frac{K}{s}
\end{equation}

(2.26)

\begin{equation}
\ll K \sum_{s | c} s^{-1/2} \ll K \tau(c) \ll_{\varepsilon} K c^{\varepsilon}.
\end{equation}

Summing by parts, we conclude that

\begin{equation}
Ad \ll_{\varepsilon} c^2 u^{1/2}(u, hl)^{1/2} v^{2/3}(v, hl)^{1/3} \delta^{-\varepsilon} + c\delta \ll_{\varepsilon} c^2 u^{1/2}(u, hl)^{1/2} v^{2/3}(v, hl)^{1/3},
\end{equation}

choosing $\delta = c^{-1/2}$. Combining this deduction with $B_d \ll \frac{2H}{c} \sup |\Xi''|$ (and the trivial bound for $b_c$ in the boundary term of (2.11)), we get

\begin{equation}
T \ll c^2 u^{1/2}(u, hl)^{1/2} v^{2/3}(v, hl)^{1/3} + \sum_{d=1}^{c^{-1}} c^2 u^{1/2}(u, hl)^{1/2} v^{2/3}(v, hl)^{1/3} \frac{nH}{c} \sup |\Xi''|
\end{equation}

(2.29)

\begin{equation}
\ll (1 + \sup |\Xi''|) nH c^2 u^{1/2}(u, hl)^{1/2} v^{2/3}(v, hl)^{1/3}.
\end{equation}

We finally get

\begin{equation}
|S|^2 \ll \frac{c^2}{H} + \frac{c}{H} \sum_{1 \leq h \leq H} (1 + \sup |\Xi''|) nH c^2 u^{1/2}(u, hl)^{1/2} v^{2/3}(v, hl)^{1/3}
\end{equation}

(2.31)

\begin{equation}
\ll \frac{c^2}{H} + c^{1+\varepsilon} H n u^{1/2}(u, l)^{1/2} v^{2/3}(v, l)^{1/3} (1 + \sup |\Xi''|),
\end{equation}

using (2.26) with $h$ in place of $k$. The optimal choice for $H$ is $[c^{1/4}]$, so that

\begin{equation}
S \ll_{\varepsilon} (1 + \sup |\Xi''|)^{1/2} c^{5/8+\varepsilon} u^{1/4} v^{1/3} n^{1/2} (u, l)^{1/4} (v, l)^{1/6}.
\end{equation}

(2.32)

Note that we are not concerned with the value of $\varepsilon$, and thus don’t distinguish between $\varepsilon$ and $\varepsilon/2$. We have thus proved the following proposition.
**Proposition 5.** Let \( S = S_c(k, l, n) \) be the sum defined in (2.1). Write \( c = uv \) with \( u \) square-free and \( v \) square-full. Then, we have

\[
(2.34) \quad S_c(k, 0, n) \ll_{\varepsilon} C n^{1/2} c^{1-1/D+\varepsilon}
\]

\[
(2.35) \quad S_c(k, l, n) \ll_{\varepsilon} (1 + \sup |\zeta'|^{1/2}) c^{5/8+\varepsilon} u^{1/4} v^{1/3} n^{1/2} (u, l)^{1/4} (v, l)^{1/6}.
\]

### 3. Fourier decomposition

In this section we develop the tools necessary to prove Theorem 2 and decompose \( f \) into a Fourier series on the torus. We proceed exactly as in [26, 2]. To begin with we note that

\[
(3.1) \quad f((1, \xi) M) = f((1, \xi + n) M)
\]

for \( n \in \mathbb{Z}^2 \). So for \( M \) fixed, \( f \) is a well defined function on \( \mathbb{R}^2/\mathbb{Z}^2 \) and we can expand it into a Fourier series as

\[
(3.2) \quad f((1, \xi) M) = \sum_{m \in \mathbb{Z}^2} \hat{f}(M, m) e(m \xi),
\]

where

\[
(3.3) \quad \hat{f}(M, m) = \int_{\mathbb{T}^2} f((1, \xi') M) e(-m \xi') d\xi'.
\]

Note that

\[
(3.4) \quad \hat{f}(TM, m) = \hat{f}(M, m(T^{-1})^t),
\]

for \( T \in \text{SL}(2, \mathbb{Z}) \). Set \( \tilde{f}_n(M) = \hat{f}(M, (n, 0)) \). These functions of \( M \in \text{SL}(2, \mathbb{R}) \) are left-invariant under the group \((\frac{1}{0} \frac{Z}{1})\) by (3.4).

Now it follows from (3.4) that

\[
(3.5) \quad \tilde{f}_n \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} M \right) = \hat{f} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} M, (n, 0) \right) = \hat{f} \left( M, (n, 0) \left( \begin{array}{cc} d & -c \\ -b & a \end{array} \right) \right)
\]

\[
(3.6) \quad = \hat{f}(M, (nd, -nc)).
\]

Therefore we can rewrite (3.2) with \( \xi = (\xi_1(x), \xi_2(x)) \) and \( M = (\frac{\sqrt{x}}{\sqrt{y}}, \frac{\sqrt{y}}{\sqrt{1}}) \) as

\[
(3.7) \quad f((1, \xi) M) = f_0(M) + \sum_{n \neq 1} \sum_{(c, d) = 1} \tilde{f}_n \left( \begin{pmatrix} * & * \\ c & d \end{pmatrix} M \right) e(n (d\xi_1(x) - c\xi_2(x))),
\]

where \( \begin{pmatrix} * & * \\ c & d \end{pmatrix} \) is any matrix in \( \text{SL}(2, \mathbb{Z}) \) with \( c \) and \( d \) in the second row as specified.

Integrating (3.7) over \( x \), we obtain

\[
(3.8) \quad \int_{\mathbb{R}} f(\tilde{n}(x)a(y)) \rho(x) dx = M(y) + E(y),
\]

where

\[
(3.9) \quad M(y) = \int_{\mathbb{R}} f_0 \left( \begin{pmatrix} \sqrt{y} \\ 0 \end{pmatrix} \frac{x}{\sqrt{y}} \right) \rho(x) dx
\]
and

\[ E(y) = \sum_{n \geq 1} \int_{\mathbb{R}} e(n (d\xi_1(x) - c\xi_2(x))) \tilde{f}_n \left( \begin{pmatrix} * & * \\ c & d \end{pmatrix} \left( \frac{x}{\sqrt{y}} \quad \frac{x}{\sqrt{y}} \right) \right) \rho(x) dx. \]

The main term in this expression is \( M(y) \) and, as is well-known (cf. [8, 25]), we have

\[ M(y) = \int_X f \, d\mu \int_{\mathbb{R}} \rho(x) dx + O_{\varepsilon}(\|f\|_{C^4_b}\|\rho\|_{W^{1,1}y^{1/2-\varepsilon}}) \]

for every \( \varepsilon > 0 \), where the Sobolev norm of \( \rho \) is defined in (3.14). This statement is nothing more than effective equidistribution of horocycles under the geodesic flow on \( \text{SL}(2, \mathbb{Z})/\text{SL}(2, \mathbb{R}) \).

We need not seek the best error term for this problem, since there will be larger contributions to the error term in Theorem 2.

It remains to estimate \( E(y) \) as \( y \to 0 \), which we do in Section 4.

We end this section with several technical results that will help us to estimate \( E(y) \). First, however, we give a precise definition of \( \| \cdot \|_{C^m_b} \) and \( \| \cdot \|_{W^{k,p}} \) for functions on \( G \) and hence also on \( X \). Following [26], we let \( \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}^2 \) be the Lie algebra of \( G \) and fix

\[ X_1 = ((0 \ 1), 0), \quad X_2 = ((0 \ 0), 0), \quad X_3 = ((1 \ 0 \ -1), 0), \quad X_4 = ((\pi / \sqrt{y}, 0), (1, 0)), \quad X_5 = ((0 \ 0), (0, 1)) \]

(3.12)

to be a basis of \( \mathfrak{g} \). Every element of the universal enveloping algebra \( U(\mathfrak{g}) \) corresponds to a left-invariant differential operator on functions on \( X \). We define

\[ \|f\|_{C^m_b} = \sum_{\deg D \leq m} \|Df\|_{L^\infty}, \]

where the sum runs over monomials in \( X_1, \ldots, X_5 \) of degree at most \( m \). We also Sobolev norms of functions on \( \mathbb{R} \).

for \( 1 \leq p < \infty \) and a positive integer \( k \), set

\[ \|\rho\|_{W^{k,p}} = \sum_{s=0}^{k} \|\rho^{(s)}\|_{L^p} = \sum_{s=0}^{k} \left( \int_{\mathbb{R}} |\rho^{(s)}(x)|^p dx \right)^{1/p}. \]

The following result is [26, Lemma 4.2].

**Lemma 6.** Let \( m \geq 0 \) and \( n > 0 \) be integers. Then

\[ \tilde{f}_n \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \ll_{m, n} \frac{\|f\|_{C^m_b}}{n^m(c^2 + d^2)^{m/2}}, \quad \forall \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \in \text{SL}(2, \mathbb{R}). \]

(3.15)

Passing to Iwasawa coordinates in \( \text{SL}(2, \mathbb{R}) \), we write

\[ \tilde{f}_n(u, v, \theta) = \tilde{f}_n \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \left( \frac{1}{\sqrt{v}} \quad 0 \quad 0 \quad 1/\sqrt{v} \right) \left( \begin{pmatrix} \cos \theta \ - \sin \theta \\ \sin \theta \ - \cos \theta \end{pmatrix} \right) \right). \]

(3.16)

for \( u \in \mathbb{R}, v > 0 \) and \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \). The following is [26, Lemma 4.4].

**Lemma 7.** Let \( m, k_1, k_2, k_3 \geq 0 \) and \( n > 0 \) be integers, and let \( k = k_1 + k_2 + k_3 \). Then

\[ \partial_u^{k_1} \partial_v^{k_2} \partial_\theta^{k_3} \tilde{f}_n(u, v, \theta) \ll_{m, k} \|f\|_{C^m_b} n^{-m} v^{m/2-k_1-k_2}. \]

(3.17)
As a consequence of Lemma 6, we get the bound

\[ \tilde{f}_n \left( u, \frac{\sin \theta}{c^2 y}, \theta \right) \ll_m \| f \|_{C^m} \min \left\{ 1, \left( \frac{|\sin \theta|}{nc \sqrt{y}} \right)^m \right\} \]

for every integer \( m \geq 0 \). We also note that for \( a, A, B > 0 \) and \( B - A > -1 \) we have

\[ \int_{-\pi}^{\pi} \frac{d\theta}{|\sin \theta|^A} \min \left\{ 1, \left( \frac{|\sin \theta|}{a} \right)^B \right\} \ll \min \left\{ a^{-B}, a^{-A+1} \right\}. \]

4. Error Terms

The purpose of this section is to estimate \( E(y) \) in (3.10). We begin with the case \( c = 0 \). Then \( d = \pm 1 \) by coprimality, and \[26\] eq. (25) yields

\[ E_{c=0}(y) = \int_{\mathbb{R}} \tilde{f}_n \left( \frac{x}{\sqrt{y}}, \frac{x}{\sqrt{y}} \right) \rho(x) dx \ll \| f \|_{C^2_{\text{y}}} \frac{y}{n^2}. \]

After summing over \( n \), the contribution from this term is clearly much smaller than that claimed in Theorem 2.

The remaining contribution (\( c \neq 0 \)) to the error term \( E(y) \) in (3.10) is

\[ E_{c \neq 0}(y) = \sum_{c \neq 0 \atop (c,d)=1} \int_{\mathbb{R}} e \left( n(d\xi_1(x) - c\xi_2(x)) \right) \tilde{f}_n \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} x/\sqrt{y} \\ 1/\sqrt{y} \end{array} \right) \right) \rho(x) dx. \]

Now we proceed to the change of variables, following \[26\] Lemma 6.1. Writing the argument of \( \tilde{f}_n \) in Iwasawa coordinates (3.16), we get

\[ \int_{\mathbb{R}} e \left( n(d\xi_1(x) - c\xi_2(x)) \right) \tilde{f}_n \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} x/\sqrt{y} \\ 1/\sqrt{y} \end{array} \right) \right) \rho(x) dx = \int_{0}^{\pi} g(\theta) d\theta, \]

for \( c > 0 \), where

\[ g(\theta) = e \left( n \left( d\xi_1 \left( -\frac{d}{c} + y \tan \theta \right) - c\xi_2 \left( -\frac{d}{c} + y \tan \theta \right) \right) \right) \]

\[ \times \tilde{f}_n \left( \frac{a}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) \rho \left( -\frac{d}{c} + y \tan \theta \right) \frac{y}{\sin^2 \theta}. \]

We have the same integral with limits \(-\pi\) and \( 0 \) if \( c < 0 \). Combining terms with positive and negative \( c \) gives

\[ E_{c \neq 0}(y) = \sum_{c \neq 0 \atop (c,d)=1} \sum_{n \geq 1} \int_{-\pi}^{\pi} g(\theta) d\theta. \]

Let

\[ \tilde{g}(\theta) = \rho \left( -\frac{d}{c} \right) \tilde{f}_n \left( \frac{a}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) e \left( -nc \Xi \left( -\frac{d}{c} \right) \right) \frac{y}{\sin^2 \theta}, \]

where \( \Xi(z) = z\xi_1(z) + \xi_2(z) \).
Lemma 8. For every $\varepsilon > 0$, we have

\begin{equation}
\sum_{c, n \geq 1} \sum_{d \in \mathbb{Z}} \int_{\theta = -\pi}^{\pi} (g(\theta) - \tilde{g}(\theta)) d\theta \ll_{\varepsilon} \|f\|_{C_b^4} (1 + \|\Xi\|_{C_1} + \|\xi_1\|_{L^\infty}) \|\rho\|_{W^{1,1}} \frac{1}{y^{1/2 - \varepsilon}}
\end{equation}

for $0 < y < 1$.

Proof. We write

\begin{equation}
g(\theta) - \tilde{g}(\theta) = \tilde{f}_n \left( \frac{d}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) \frac{y}{\sin^2 \theta} \times \left[ e \left( n \left( d\xi_1 \left( -\frac{d}{c} + y \cot \theta \right) - c\xi_2 \left( -\frac{d}{c} + y \cot \theta \right) \right) \rho \left( -\frac{d}{c} + y \cot \theta \right) - e \left( nc\Xi \left( -\frac{d}{c} \right) \rho \left( -\frac{d}{c} \right) \right) \right] \right.
\end{equation}

\begin{equation}
= \tilde{f}_n \left( \frac{d}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) \frac{y}{\sin^2 \theta} e \left( -nc\Xi \left( -\frac{d}{c} \right) \right) \times \left[ e \left( O(\|\Xi\|_{C_1} + \|\xi_1\|_{L^\infty}) ncy \cot \theta \right) \rho \left( -\frac{d}{c} + y \cot \theta \right) - \rho \left( -\frac{d}{c} \right) \right].
\end{equation}

When $ncy |\cot \theta| < 1$, we use Taylor expansion of the exponential; in the complementary case we bound it trivially. The contribution of the first option comes in two parts since $e(z) = 1 + O(z)$. The first part is controlled by using (3.18) with $m = 2$ and (3.19) with $A = 2$ and $B = 2$, together with elementary inequalities. We have the bound

\begin{equation}
\sum_{c, d, n} \int_{\theta = -\pi}^{\pi} \tilde{f}_n \left( \frac{d}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) \frac{y}{\sin^2 \theta} e \left( -nc\Xi \left( -\frac{d}{c} \right) \right) \times \left[ \rho \left( -\frac{d}{c} + y \cot \theta \right) - \rho \left( -\frac{d}{c} \right) \right] \mathbb{1}_{ncy |\cot \theta| < 1}
\end{equation}

\begin{equation}
\ll \|f\|_{C_b^2} \int_{-\pi}^{\pi} \frac{y d\theta}{\sin^2 \theta} \min \left\{ 1, \left( \frac{\sin \theta}{nc\sqrt{y}} \right)^2 \right\} \times \sum_{(c, d) = 1} |\rho \left( -\frac{d}{c} \right) - \rho \left( -\frac{d}{c} + y \cot \theta \right) | \mathbb{1}_{ncy |\cot \theta| < 1}
\end{equation}

\begin{equation}
\ll \|f\|_{C_b^2} \int_{-\pi}^{\pi} \frac{y d\theta}{\sin^2 \theta} \min \left\{ 1, \left( \frac{\sin \theta}{nc\sqrt{y}} \right)^2 \right\} \times \sum_{h | c} |\mu(h)| \sum_{d \in \mathbb{Z}} \int_{-\frac{\rho(h)}{c} + y \cot \theta}^{\rho(h)} |\rho'(t)| dt \mathbb{1}_{y |\cot \theta| < \frac{1}{nc}}.
\end{equation}
Now we use the condition $y|\text{ctg} \theta| < \frac{1}{nc}$ to recast the sum in $d$ and the integral in $t$ to a single integral over the real line:

\begin{align}
&\ll \| f \|^2_{c_b^2} \int_{-\pi}^{\pi} \sum_{n,c} \frac{yd\theta}{\sin^2 \theta} \min \left\{ 1, \left( \frac{\sin \theta}{nc\sqrt{y}} \right)^2 \right\} \sum_{h|c} |\mu(h)| \int_{-\infty}^{\infty} |\rho'(t)| \, dt \\
&\ll \varepsilon \| f \|^2_{c_b} \| \rho \|_{W^{1,1}} \int_{-\pi}^{\pi} \sum_{n,c} \frac{yd\theta}{\sin^2 \theta} \min \left\{ 1, \left( \frac{\sin \theta}{nc\sqrt{y}} \right)^2 \right\} c^\varepsilon
\end{align}

At this step we take advantage of (3.19) with $A = 2 = B$, giving the bound

\begin{align}
&\ll \| f \|^2_{c_b^2} \| \rho \|_{W^{1,1}} \sum_{n,c} y c^\varepsilon \min \left\{ (nc\sqrt{y})^{-2}, (nc\sqrt{y})^{-1} \right\} \\
&\ll \| f \|^2_{c_b^2} \| \rho \|_{W^{1,1}} \sum_{k=1}^{\infty} y k^\varepsilon \min \left\{ (k\sqrt{y})^{-2}, (k\sqrt{y})^{-1} \right\} \\
&\ll \| f \|^2_{c_b^2} \| \rho \|_{W^{1,1}, y^{1/2-\varepsilon}}
\end{align}

The second part is controlled by using (3.18) with $m = 4$ followed by (3.19) with $A = 3$ and $B = 4$, together with elementary inequalities. The bound in this case is

\begin{align}
&\sum_{c,d,n} \int_{\theta = -\pi}^{\pi} \bar{f}_n \left( \frac{d}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) \frac{yd\theta}{\sin^2 \theta} c (-nc\Xi (-\frac{d}{c})) \\
&\times \left[ O((\| \Xi \|_{c_b^2} + \| \xi_1 \|_{L^\infty})ncy|\text{ctg} \theta|\rho (-\frac{d}{c} + y \text{ctg} \theta)) \right] 1_{ncy|\text{ctg} \theta| < 1} \\
&\ll \| f \|^2_{c_b^2} (\| \Xi \|_{c_b^2} + \| \xi_1 \|_{L^\infty}) \int_{-\pi}^{\pi} \frac{yd\theta}{\sin^2 \theta} \sum_{n,c} \min \left\{ 1, \left( \frac{\sin \theta}{nc\sqrt{y}} \right)^4 \right\} ncy|\text{ctg} \theta| \\
&\times \sum_{h|c} |\mu(h)| \sum_{d \in \mathbb{Z}} \rho \left( -\frac{dh}{c} + y \text{ctg} \theta \right) 1_{ncy|\text{ctg} \theta| < 1} \\
&\ll \| f \|^2_{c_b^2} (\| \Xi \|_{c_b^2} + \| \xi_1 \|_{L^\infty}) \int_{-\pi}^{\pi} \frac{ncy^2 d\theta}{\sin \theta^3} \sum_{n,c} \min \left\{ 1, \left( \frac{\sin \theta}{nc\sqrt{y}} \right)^4 \right\} \\
&\times \sum_{h|c} |\mu(h)| \left( \frac{c}{h} \int_{-\infty}^{\infty} \rho(t) \, dt + \int_{-\infty}^{\infty} |\rho'(t)| \, dt \right).
\end{align}
Now the last line is at most a constant times $c^{1+\epsilon}\|\rho\|_{W^{1,1}}$, since $\rho$ is of integral 1 and is supported on an interval of length 1. We get

\begin{align}
(4.21) & \ll \|f\|_{C^1_b} (\|\Xi\|_{C^1_b} + \|\xi_1\|_{L^\infty}) \|\rho\|_{W^{1,1}} \sum_{n,c} y^2 nc^{2+\epsilon} \min\{ (nc\sqrt{y})^{-2}, (nc\sqrt{y})^{-4} \} \\
(4.22) & \ll \|f\|_{C^1_b} (\|\Xi\|_{C^1_b} + \|\xi_1\|_{L^\infty}) \|\rho\|_{W^{1,1}} \sum_{k=1}^\infty y^2 k^{2+\epsilon} \min\{ (k\sqrt{y})^{-2}, (k\sqrt{y})^{-4} \} \\
(4.23) & \ll \|f\|_{C^1_b} (\|\Xi\|_{C^1_b} + \|\xi_1\|_{L^\infty}) \|\rho\|_{W^{1,1}} y^{1/2-\epsilon}.
\end{align}

Now we peruse the second option, $ncy|\cot\theta| \geq 1$. Again, we distinguish two subcases, $1 \geq ncy \geq |\tan\theta|$ and $1 < ncy \geq |\tan\theta|$. The first subcase is dealt with using (3.18) with $m = 3$ followed by elementary estimates for the integral in $\theta$:

\begin{align}
(4.24) & \sum_{c,d,n} \int_{\theta=-\pi}^{\pi} \left| f_n \left( \frac{d}{c}, \frac{-\sin 2\theta}{2cy^2}, \frac{\sin^2 \theta}{c^2 y^2}, \theta \right) \right| y \frac{d\theta}{\sin^2 \theta} \\
& \times \left[ \rho \left( -\frac{d}{c} + y \cot \theta \right) + \rho \left( -\frac{d}{c} \right) \right] 1_{1 \geq ncy \geq |\tan\theta|} \\
(4.25) & \ll \|f\|_{C^1_b} \int_{-\pi}^{\pi} \sum_{n,c} y \frac{d\theta}{\sin^2 \theta} \min\left\{ 1, \left( \frac{|\sin \theta|}{nc\sqrt{y}} \right)^3 \right\} \\
& \times \sum_{(c,d)=1} \left[ \rho \left( -\frac{d}{c} + y \cot \theta \right) + \rho \left( -\frac{d}{c} \right) \right] 1_{1 \geq ncy \geq |\tan\theta|} \\
(4.26) & \ll \|f\|_{C^1_b} \int_{-\pi}^{\pi} \sum_{n,c} y \frac{d\theta}{\sin^2 \theta} \min\left\{ 1, \left( \frac{|\sin \theta|}{nc\sqrt{y}} \right)^3 \right\} c^{1+\epsilon} \|\rho\|_{W^{1,1}} 1_{1 \geq ncy \geq |\tan\theta|}.
\end{align}

Using the same reasoning as before convert the sum over $d$ into an integral, we arrive at the bound

\begin{align}
(4.27) & \ll \|f\|_{C^1_b} \|\rho\|_{W^{1,1}} \sum_{n,c} y c^{1+\epsilon} \int_{0 \leq \theta \leq ncy} \frac{\theta d\theta}{n^3 c^3 y^{3/2}} 1_{1 \geq ncy} \\
(4.28) & \ll \|f\|_{C^1_b} \|\rho\|_{W^{1,1}} \sum_{n,c} y^{-1/2} n^{-1} c^{\epsilon} y^2 1_{1 \geq ncy} \\
(4.29) & \ll \|f\|_{C^1_b} \|\rho\|_{W^{1,1}} y^{1/2-\epsilon}.
\end{align}
In the second subcase, we only keep the condition \( ncy > 1 \) to get the bound

\[
\sum_{c,d,n,y} \int_{-\pi}^{\pi} \left| \tilde{f}_n \left( \frac{d}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) \right| \frac{y \, d\theta}{\sin^2 \theta} \left[ \rho \left( -\frac{d}{c} + y \cot \theta \right) + \rho \left( -\frac{d}{c} \right) \right] \, 1_{1 < ncy} \leq \|f\|_{C^3_b} \|\rho\|_{W^{1,1}} \sum_{n,c} \int_{-\pi}^{\pi} y \, d\theta \sin^2 \theta. \tag{4.30}
\]

We need to analyze

\[
\tilde{E}_{c \neq 0}(y) = \sum_{c,n \geq 1} \int_{\theta = -\pi}^{\pi} \sum_{(c,d) = 1 \atop d \in \mathbb{Z}} \tilde{f}_n \left( \frac{d}{c} - \frac{\sin 2\theta}{2c^2 y}, \frac{\sin^2 \theta}{c^2 y}, \theta \right) e \left( -nc\Xi \left( -\frac{d}{c} \right) \right) \rho \left( -\frac{d}{c} \right) \frac{y \, d\theta}{\sin^2 \theta}. \tag{4.35}
\]

Define Fourier coefficients

\[
b_l^{(n,c)}(\theta) = \int_0^1 \tilde{f}_n \left( u, \frac{\sin^2 \theta}{c^2 y}, \theta \right) e(-lu) \, du, \tag{4.36}
\]

\[
a_k^{(n,c)} = \int_0^1 \rho(u)e(-ku) \, du. \tag{4.37}
\]

Then, we can use the fact that \( \rho \) is supported within \((-1,0]\) to write

\[
\tilde{E}_{c \neq 0}(y) = \sum_{c,n \geq 1 \atop k,l \in \mathbb{Z}} \int_{\theta = -\pi}^{\pi} \sum_{(c,d) = 1 \atop 0 < d < c} b_l^{(n,c)}(\theta) a_k^{(n,c)} e \left( \frac{l\bar{d}}{c} - \frac{l \sin 2\theta}{2c^2 y} \right) \times e \left( -nc\Xi \left( -\frac{d}{c} \right) \right) \frac{y \, d\theta}{\sin^2 \theta} \tag{4.38}
\]

\[
\leq \sum_{c,n \geq 1 \atop k,l \in \mathbb{Z}} \int_{\theta = -\pi}^{\pi} \left| b_l^{(n,c)}(\theta) a_k^{(n,c)} \right| \sum_{(c,d) = 1 \atop 0 \leq d < c} e \left( \frac{l\bar{d} - kd}{c} - nc\Xi \left( -\frac{d}{c} \right) \right) \left| \frac{y \, d\theta}{\sin^2 \theta} \right. \tag{4.39}
\]
Our objective is to get savings for the sum over $d$ and use the bounds
\begin{equation}
\begin{aligned}
&b_l^{(n,c)}(\theta) \ll \left\| f \right\|_{C^m_b} \min \left\{ \left( \frac{\sin \theta}{nc \sqrt{y}} \right)^m, 1 \right\} \\
&\quad \quad \quad l^{-2} \left\| f \right\|_{C^{m+2}_b} n^{-4} \min \left\{ \left( \frac{\sin \theta}{nc \sqrt{y}} \right)^{m-4}, 1 \right\} 
\end{aligned}
\end{equation}
for any $m \geq 4$, and
\begin{equation}
a_k \ll \eta \left( 1 + |k| \right)^{-1-\eta} \left\| \rho \right\|_{W^{1,1}}^{1-\eta} \left\| \rho \right\|_{W^{2,1}}^\eta, \text{ for } \eta \in (0, 1).
\end{equation}
The bound (4.40) is taken from \cite[Lemma 4.1]{2}, while the bound (4.41) follows from (4.37) and integration by parts. We use the first bound with $m = 2$ and $m = 6$, and note that
\begin{equation}
\int_{-\pi}^{\pi} \min \left\{ \left( \frac{|\sin \theta|}{a} \right)^2, 1 \right\} \frac{d\theta}{\sin^2 \theta} \ll \frac{1}{a(1+a)},
\end{equation}
for $a > 0$, following (3.18). This inequality will be applied with $a = nc \sqrt{y}$.

We write
\begin{equation}
S_c(k, l, n) = S = \sum_{(c,d)=1, 0 < d < c} e \left( \frac{ld - kd}{c} - nc \Xi \left( -\frac{d}{c} \right) \right).
\end{equation}
Cancellations in the exponential sum are proved in Section 2 where Proposition 5 is established, distinguishing two cases, when $l = 0$ and when $l \neq 0$. Combining contributions of these two cases, we control $E_{c \neq 0}(y)$ by
\begin{equation}
E_{c \neq 0}(y) \ll C \sum_{c, n \geq 1} \left| a_k \right| \int_{-\pi}^{\pi} \left| b_0(\theta) \right| \frac{yd\theta}{\sin^2 \theta} n^{1/2} c^{-1-1/D+\varepsilon}
\end{equation}
\begin{equation}
+ \left( 1 + C_2^{1/2} \right) \sum_{c, n \geq 1} \left| a_k \right| \int_{-\pi}^{\pi} \left| b_l(\theta) \right| \frac{yd\theta}{\sin^2 \theta} n^{1/2} v^{1/2} n^{1/2} (u, l)^{1/2} (v, l)^{1/2}
\end{equation}
\begin{equation}
= E_{l=0}(y) + E_{l \neq 0}(y).
\end{equation}
For $E_{l=0}(y)$ we use the first bound from (4.40) with $m = 2$, and (4.41) followed by (4.42). After bringing out factors of $F = C \left\| f \right\|_{C^2_b} \left\| \rho \right\|_{W^{1,1}}^{1-\eta} \left\| \rho \right\|_{W^{2,1}}^\eta$, we get
\begin{equation}
E_{l=0} \ll \eta \varepsilon F \sum_{c, n, k} \left( 1 + |k| \right)^{-1-\eta} \frac{n^{1/2} c^{-1-1/D+\varepsilon}}{nc \sqrt{y} (1 + nc \sqrt{y})}
\end{equation}
\begin{equation}
\ll F \sum_{c, n} \frac{\sqrt{y} c^\varepsilon}{\sqrt{nc^{1/D} (1 + cn \sqrt{y})}}
\end{equation}
\begin{equation}
\ll F \sum_{n=1}^{\infty} \sqrt{n} \left[ \sum_{c \geq n^{1/4}} \frac{1}{c^{1+\beta-\varepsilon} n^{\sqrt{y}}} + \sum_{c \geq 1} \frac{1}{c^{1+\beta-\varepsilon} n^{\sqrt{y}}} + \sum_{c \leq n^{1/4}} \frac{1}{c^{1+\beta-\varepsilon}} \right].
\end{equation}
Bounding sums over $c$ gives

$$(4.49) \quad E_{l=0} \ll F \left[ \sum_{n=1}^{\infty} \sqrt{\frac{y}{n}} \frac{n^{1/D-\varepsilon}}{y^{1/n}} \mathbb{1}_{n \sqrt{y} \leq 1} + \sum_{n=1}^{\infty} \sqrt{\frac{y}{n}} \frac{n^{1/n}}{n \sqrt{y}^{1+n \sqrt{y}}} \mathbb{1}_{n \sqrt{y} \geq 1} \right]$$

$$(4.50) \quad \ll F \left[ \sum_{n \leq \sqrt{y}} n^{-3/2} y^{1/2n-\varepsilon} + \sum_{n > \sqrt{y}} n^{-3/2} + \sum_{n \leq \sqrt{y}} n^{-5/2} y^{1/n} \right]$$

$$(4.51) \quad \ll F y^{1/2n-\varepsilon}.$$ 

When $l \neq 0$, we use $$(4.41), (4.40)$$ with $m = 6$, and $(4.42)$. Abbreviating $H = \|f\|_{C_b^8} \|\rho\|_{W^{1,1}}$, we have

$$(4.52) \quad E_{l \neq 0} \ll H \sum_{c,l,n} \frac{l^{-2} n^{-4} y}{\sqrt{y} (1 + nc \sqrt{y})} (n^{1/2} c^{5/8+\varepsilon} u^{1/4} v^{1/3} (uv, l)^{1/4})$$

$$(4.53) \quad \ll H \sum_{c,n} \frac{n^{-4} y}{\sqrt{y} (1 + nc \sqrt{y})} (n^{1/2} c^{5/8+\varepsilon} u^{1/4} v^{1/3})$$

Now we divide the sum over $c = uv$ into dyadic intervals $[2^{j-1}, 2^j)$, $j \in \mathbb{N}$. This gives

$$(4.54) \quad E_{l \neq 0} \ll H \sum_{n,j} n^{-5} y^{1/2} \sum_{v \leq 2^j} \sum_{u \leq 2^j/v} \frac{u^{-1/8+\varepsilon} v^{-1/24+\varepsilon}}{1 + nc \sqrt{y}}$$

$$\ll H \sum_{n,j} n^{-5} y^{1/2} \sum_{v \leq 2^j} \frac{v^{-1/24+\varepsilon}}{1 + n 2^{j-1} \sqrt{y}} \sum_{u \leq 2^j/v} u^{-1/8+\varepsilon}$$

$$\ll H \sum_{n,j} n^{-5} y^{1/2} \frac{2^{7j/8+\varepsilon}}{1 + n 2^{j-1} \sqrt{y}} \sum_{v \leq 2^j} v^{-1/24+\varepsilon} v^{-7/8+\varepsilon}.$$ 

The sum over $v$ is convergent as square-full numbers have square-root density. The remaining sum gives the bound

$$(4.57) \quad H y^{1/16-\varepsilon},$$

as needed.

The total error contribution is

$$(4.58) \quad H y^{1/16-\varepsilon} + F y^{1/2n-\varepsilon} + \|f\|_{C_b^8} y^{1/2-\varepsilon} (\|\xi_1\|_{L^\infty} + \|\Xi\|_{C_b^1} + 1) \|\rho\|_{W^{1,1}},$$

coming from $$(4.57), (4.51),$$ and $(4.7)$, which is majorized by the expression in the statement of the theorem.
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Department of Mathematics, Princeton University, Princeton, NJ 08544, United States

E-mail address: ivinogra@math.princeton.edu