On the Accelerating of Two-dimensional Smart Laplacian Smoothing on the GPU

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Abstract

This paper presents a GPU-accelerated implementation of two-dimensional Smart Laplacian smoothing. This implementation is developed under the guideline of our paradigm for accelerating Laplacian-based mesh smoothing \cite{13}. Two types of commonly used data layouts, Array-of-Structures (AoS) and Structure-of-Arrays (SoA) are used to represent triangular meshes in our implementation. Two iteration forms that have different choices of the swapping of intermediate data are also adopted. Furthermore, the feature CUDA Dynamic Parallelism (CDP) is employed to realize the nested parallelization in Smart Laplacian smoothing. Experimental results demonstrate that: (1) our implementation can achieve the speedups of up to 44x on the GPU GT640; (2) the data layout AoS can always obtain better efficiency than the SoA layout; (3) the form that needs to swap intermediate nodal coordinates is always slower than the one that does not swap data; (4) the version of our implementation with the use of the feature CDP is slightly faster than the version where the CDP is not adopted.

Keywords: GPU, CUDA, Mesh Smoothing, Laplacian Smoothing, Data Dependency

1 Introduction

The generation of computational mesh models plays a key role in numerical simulation. The quality of meshes strongly affects the computational efficiency and the accuracy of numerical results. In general, mesh models are needed to be optimized to improve the mesh quality after initially creating. There are typically two categories of mesh optimization methods: (1) the first one is mesh clear-up / modification, which changes the topology of meshes to improve the mesh quality \cite{21, 8, 5}; and (2) the other one is the mesh smoothing, which leaves the mesh connectivity unchanged but to relocate the position of mesh nodes to enhance the mesh quality \cite{21, 6, 13}. A number of mesh smoothing methods have been introduced and widely used in various applications. Among the mesh smoothing methods,
the Laplacian-based and the optimization-based methods are the two types of the most frequently used mesh smoothing approaches in practice [4, 17].

The basic idea behind the Laplacian mesh smoothing is quite simple. In each iteration, the new nodal position of each smoothed point is directly the geometric center of those of its adjacent/neighborhood nodes [9]. There are no other complex smoothing operations. Thus, the Laplacian mesh smoothing is computationally non-intensive, and frequently used in various applications.

However, when smoothing large meshes consisting of a large number of nodes and elements, the computational cost is still too high. To improve the efficiency, an effective strategy is to perform the mesh smoothing in parallel. For example, Mei, et al. [13] developed a generic paradigm for accelerating the Laplacian-based mesh smoothing on the GPU. Dahal and Newman [6] described efficient GPU-accelerated implementations of three triangular 2D mesh smoothing algorithms. In addition, D’Amato and Venere [7] presented an implementation of a non-Laplacian smoothing method on the GPU to optimize tetrahedral meshes in parallel. Other recent efforts attempting at parallelizing mesh smoothing include those work presented in [1, 2, 10].

In this paper, we present an efficient GPU-accelerated implementation of two-dimensional Smart Laplacian smoothing. The presented implementation is developed under the guideline of our previously proposed paradigm for accelerating Laplacian-based mesh smoothing [13]. Two types of commonly used data layouts, Array-of-Structures (AoS) and Structure-of-Arrays (SoA), are also used to represent triangular meshes in our implementation. Furthermore, the feature CUDA Dynamic Parallelism (CDP) is employed to realize the nested parallelization in Smart Laplacian Smoothing. The feature CDP is an extension to the CUDA programming model which enables a CUDA kernel to create and synchronize with new kernel(s) directly on the GPU [14]. We finally carry out several experimental tests to evaluate the performance of our GPU-accelerated implementation by comparing to the corresponding CPU implementation.

2 Background

2.1 Laplacian Smoothing

Laplacian smoothing is one of the most commonly used mesh smoothing algorithms [11]. The basic idea behind Laplacian smoothing is to relocate each node in a mesh to the geometric center of its neighboring nodes; see Figure 1.

Laplacian smoothing is computationally straightforward but does not always produce enhancements in mesh quality. In practical applications, inverted or even invalid elements in concave regions are probably created. To deal with the above problem, some variations such as the Weighted Laplacian smoothing [3, 19] and Constrained / Smart Laplacian smoothing [4, 20] have been designed.

The basic idea behind Smart Laplacian smoothing is also simple. For a node being smoothed such as the one \( v_0 \) presented in Figure 1, the mesh quality of its incident elements (those elements that share this node, e.g., the triangles \( T_1, T_2, T_3, T_4, \) and \( T_5 \) in Figure 1) is first evaluated; Then a newly smoothed position of this node is calculated according to the Laplacian smoothing operations. The quality of all the incident elements is evaluated again using the new nodal position. If the mesh quality is increased, then the new nodal position will be accepted; otherwise, the node will not be repositioned.

2.2Iteration Forms

When calculating the smoothed coordinates of vertices according to the smoothing operation, there are typically two forms in terms of selecting the coordinates of neighboring nodes [13]. These two forms can be simply illustrated using the following formulations.
Form A:
\[ x_{i}^{q+1} = \frac{1}{N} \sum_{j=1}^{N} x_{j}^{q}, \]
where \( N \) is the number of neighboring nodes to node \( i \) and \( x_{i}^{q+1} \) is the new position for node \( i \) in the iteration pass \( (q+1) \).

Form B:
\[
\bar{x}_{i}^{q+1} = \frac{1}{N} \left( \sum_{j=1}^{N_{q}} x_{j}^{q} + \sum_{k=1}^{N_{q+1}} x_{k}^{q+1} \right), \quad \begin{cases} 
0 \leq N_{q} \leq N \\
0 \leq N_{q+1} \leq N \\
N_{q} + N_{q+1} = N 
\end{cases},
\]
where \( N \) is the number of neighboring nodes to node \( i \) and \( x_{i}^{q+1} \) is the new position for node \( i \) in the iteration pass \( (q+1) \). \( N_{q} \) and \( N_{q+1} \) are numbers of neighboring nodes derived from the iteration passes \( q \) and \( (q+1) \), respectively. Obviously, the Form A is a special case of the Form B where \( N_{q+1} = 0 \).

2.3 Data Layouts

In GPU-accelerated applications there are typically two commonly used data layouts, including the Array-of-Structures (AoS) and Structure-of-Arrays (SoA); see a simple illustration in Figure 2. Arranging data in AoS layout leads to coalescing issues as the data are interleaved. Arranging data as the SoA layout makes full use of the memory bandwidth even when individual elements of the structure are utilized. In practical applications, it is unable to determine which data layout can always achieve better efficiency. In general, the selecting of the proper data layout is application-specific.

```
struct PT {  
    float x;  
    float y;  
    bool flag; 
};
struct PT pts[N];
```
(a) AoS

```
struct PT {  
    float x[N];  
    float y[N];  
    bool flag[N]; 
};
struct PT pts;
```
(b) SoA

Figure 2: The data layouts: Array-of-Structures (AoS) and Structure-of-Arrays (SoA)
3 Our Implementation

3.1 Overview

Our implementation of the Smart Laplacian smoothing is mainly divided into four sub-procedures, including the initiating procedure, the finding of neighbors, the determining of constraints, and the iterative procedure to obtain the smoothed positions. The initiating procedure is to set several values for each vertex and calculate the mesh quality for each triangle. The finding of neighbors is to identify the neighboring vertices / nodes and to find the incident elements / triangles for each vertex. The determining of constraints is to identify which vertices are needed to be fixed or free. Finally, the iterating procedure is to iteratively calculate the positions of smoothed points.

3.2 Data Structures

We design two sets of mesh data structures according to the two layouts AoS and SoA, respectively. Corresponding implementations are also developed according to these mesh data structures. Due to the limit of the paper length, we only list the data structures represented with the SoA layout; see Listing 1.

```c
struct cuVert_SOA_ST {
    float *x, *y;        // Coordinates of a node
    int *nNeig, *neig;   // Number, and indices of neighboring nodes
    int *nLoca, *loca;   // Number, and indices of incident elements
    bool *bBoundary;     // Whether nodes are boundary vertices
    float *minQuality;   // Qualities of the worst incident elements
};
struct cuTrgl_SOA_ST {
    int *vIDs;           // Indices of vertices of triangles
    float *Quality;      // Qualities of triangles
};
struct cuMesh_SOA_ST {
    int nVert, nTrgl;   // Number of vertices and triangles
    cuVert_SOA_ST verts; cuTrgl_SOA_ST trgls;
};
```

Listing 1: Mesh data structures represented with the SoA layout

3.3 Implementation Details

3.3.1 The Initiating Procedure

This procedure is to (1) calculate the mesh quality metric of each triangle for the first time, and (2) initiate several values to prepare for the subsequent calculations. We adopt the mesh quality metric, $\alpha$, proposed by Lee and Lo [12] to measure the quality of triangular elements.

We design a simple CUDA kernel to calculate the qualities of all triangular elements, where each thread is responsible for computing the quality of a single triangle. The element quality is stored in the component `float *Quality` in the structure `cuTrgl_SOA_ST`.

We also design another simple kernel to initiate the values of the components `nNeig`, `nLoca`, and `bBoundary` as 0, 0, and `false`, respectively. Each thread within this kernel / grid is responsible for initiating the values for only one points. The above set values mean that: currently there are no found adjacent points and incident triangles for any vertex. In addition, any vertex of the mesh is initially set to be free point that can be relocated in smoothing.
3.3.2 The Finding of Neighbors

The finding of neighbors is to find: (1) the adjacent / neighboring nodes and (2) the incident triangles for each vertex. As described in our previous work [13], the finding of neighbors can be quite easily performed according to the topology of the mesh. More specifically, for a triangle the second and the third vertices must be the adjacent points of the first vertex; similarly, the first and the third vertices are definitely the adjacent points of the second vertex. And obviously, each triangle is the incident triangle of its three vertices.

As also explained in our previous work [13], due to the data dependencies, we allocate only one thread block and a single thread within the thread block to perform the finding of neighbors. The only one thread takes the complete responsibilities to finding the adjacent points and incident triangles for all vertices. In fact, this procedure carried out on the GPU is the same as the corresponding version performed on the CPU.

3.3.3 The Determining of Constraints

The determining of constraints is to identify which vertices of a mesh can be relocated in smoothing and which cannot be. For planar polygonal meshes, the constraints include the boundary vertices and other specifically-defined points such as some feature points. In our implementation, we only consider the boundary vertices as constraints.

We also adopt the method introduced in [13] to determine the boundary vertices. More specifically, by taking advantage of the indices of the adjacent points, it is quite easy to check whether or not a vertex is a boundary one. The basic idea behind determining the boundary vertices is straightforward: if all the neighbors of a vertex, e.g., the node $v_0$ in Figure 1, have been recorded twice, then the vertex is internal; otherwise, it is a boundary vertex.

We develop a specific kernel to carry out this determine of boundary vertices. Each thread within the thread grid is invoked to check whether a points is a boundary one, i.e., to check whether all the neighbors / adjacent points of a vertex has been recorded twice. If not, then update the corresponding flag value bool bBoundary from being false to true.

3.3.4 The Iterating Procedure

The final and key procedure is to iteratively calculate the smoothed positions of all vertices. The main feature of this procedure is that: only one thread block is allocated because of the data dependencies. In Smart Laplacian smoothing, the calculating of the positions of all smoothed points in the $(i + 1)^{th}$ iteration depends on the positions of all smoothed points in the $i^{th}$ iteration. In other words, there exist data dependencies between the $(i + 1)^{th}$ iteration and the $i^{th}$ iteration.

Due to the data dependencies existing in different passes of iterations, only one thread block is allocated. Each thread within the thread block is invoked to calculate the smoothed positions of $(n + \text{BLOCK\_SIZE} - 1) / \text{BLOCK\_SIZE}$ vertices in one iterative step, where $n$ is the number of all vertices and \text{BLOCK\_SIZE} denotes the number of threads within the only one thread block. The barrier of synchronization __syncthreads() is used to guarantee all threads within the only one block finishing calculating one pass of all smoothed positions.

In Smart Laplacian smoothing it is “Smart” to determine whether or not a non-constrained vertex should be relocated. This “Smart” determination is typically carried out by comparing the mesh quality of the original mesh before relocating the vertex and the mesh quality after the relocating. In other words, when using the new nodal coordinates after relocating, if the mesh quality of the local mesh (i.e., all the incident triangles) of the vertex being smoothed definitely improves, for example, if the min value of the mesh quality metric of all the incident triangles is increased, then this vertex should be
repositioned; otherwise, the position of the vertex will be left unchanged. Therefore, after obtaining the new position of a smoothed vertex, it is needed to temporarily re-evaluate the mesh quality of the local mesh, and then compare the mesh qualities.

After completely obtaining all the smoothed positions in one iteration step, the mesh quality of all triangles is needed to be updated using all of the new nodal coordinates. The qualities of all triangles updated in \(i^{th}\) iteration will be used to “Smart” calculate the new smoothed positions in the \((i + 1)^{th}\) iteration. We implement the above updating of mesh quality in each iteration step by with or without the use of the feature, CUDA Dynamic Parallelism (CDP).

“GPU-” versions: In this version, the feature CDP is not adopted. Alternatively, in this kernel of iterating, each thread within the only thread block is responsible for: (1) calculating the quality of all of its incident triangles, and (2) finding the min value of the qualities of the incident triangles. The min quality is then stored in the component float \text{minQuality} of the data structure \text{cuVertSOAST}.

“CDP-” versions: In this version, the feature CDP is used. We design a child kernel specifically for updating the quality of all triangles by using new nodal coordinates. Within this child kernel, each thread is invoked to evaluate the quality of only one triangle. We also develop another child kernel for finding the min quality of the incident triangles for all vertices. Each thread in this kernel is responsible for finding the min values of the quality of the incident triangles. Note that the above two child kernels are only invoked once in the parent kernel of iterating.

4 Results

To evaluate the performance of our GPU implementations, we perform the experimental tests on the GeForce GT640 (GDDR5) graphics cards with CUDA 6.5. The CPU experiments are performed on Windows 7 SP1 with a dual Intel i5 3.2 GHz processor and 8GB of RAM memory. We perform the experimental tests for both the GPU- and CDP- versions of our implementation. Each version of our implementation is tested in two cases where the iteration Form A and Form B are adopted.

The test data includes 5 planar triangular meshes which are composed of 1, 5, 10, 50, and 100K vertices, respectively. The original unsmoothed triangular meshes are generated according to the standard Delaunay triangulation algorithm. First, five sets of uniformly distributed points in 2D are randomly generated using the generator provided by Qi, et al. [15]; and then Delaunay meshes are created for these sets of discrete points using the library Triangle [16].

We evaluate our implementation on the single precision. For the GPU-version, the running time and corresponding speedups achieved when using the Form A and Form B are listed in Table 1 and Table 2, respectively. Similarly, For the CDP-version, the running time and corresponding speedups obtained when iterating in the Form A and Form B are presented in Table 3 and Table 4, respectively. We find that the highest speedup is up to 44.76x; see Table 3.

5 Discussion

We have investigated the related work involving the accelerating of mesh smoothing on the GPU, and have found that currently there are only several recent efforts [13, 7, 6]. Among the above efforts, the work presented in [7] focused on smoothing tetrahedral meshes, while in [13] and [6] the work aimed at developing the GPU-accelerated mesh smoothing for planar meshes.

The GPU-accelerated implementations of 2D Laplacian mesh smoothing were both introduced in [13] and [6]. However, the above implementations are only developed for the original Laplacian smoothing, rather than the Smart Laplacian smoothing. To the best of the authors’ knowledge, the work presented in this paper is the first attempt at accelerating Smart Laplacian smoothing on the GPU.
### Table 1: Performance of the GPU-version developed in the iteration Form A

| Size | Running time (/ms) | Speedup |
|------|--------------------|---------|
|      | CPU | GPU-AoS | GPU-SoA | CPU | GPU-AoS | GPU-SoA |
| 1K   | 218 | 16.0    | 18.8    | 13.63 | 11.60   |
| 5K   | 1217| 72.2    | 86.0    | 16.86 | 14.15   |
| 10K  | 2012| 117.6   | 138.1   | 17.11 | 14.57   |
| 50K  | 14383| 530.5  | 618.0   | 27.11 | 23.27   |
| 100K | 33836| 802.8   | 932.8   | 42.15 | 36.27   |

### Table 2: Performance of the GPU-version developed in the iteration Form B

| Size | Running time (/ms) | Speedup |
|------|--------------------|---------|
|      | CPU | GPU-AoS | GPU-SoA | CPU | GPU-AoS | GPU-SoA |
| 1K   | 171 | 14.9    | 17.7    | 11.48 | 9.66    |
| 5K   | 1139| 52.4    | 76.6    | 21.74 | 14.87   |
| 10K  | 1888| 79.3    | 115.0   | 23.81 | 16.42   |
| 50K  | 11918| 424.5  | 513.1   | 28.08 | 23.23   |
| 100K | 26021| 776.1   | 896.3   | 33.53 | 29.03   |

### Table 3: Performance of the CDP-version developed in the iteration Form A

| Size | Running time (/ms) | Speedup |
|------|--------------------|---------|
|      | CPU | GPU-AoS | GPU-SoA | CPU | GPU-AoS | GPU-SoA |
| 1K   | 218 | 14.1    | 16.5    | 15.46 | 13.21   |
| 5K   | 1217| 59.2    | 66.8    | 20.56 | 18.22   |
| 10K  | 2012| 96.1    | 107.7   | 20.94 | 18.68   |
| 50K  | 14383| 477.7  | 532.4   | 30.11 | 27.02   |
| 100K | 33836| 755.9   | 832.5   | 44.76 | 40.64   |

### 5.1 Impact of Data Layouts (AoS and SoA)

We have observed that: the version of our implementation that is developed using the data layout AoS is slightly faster than that is developed using the layout SoA. This behavior has previously been observed in our previous work [13]. As also explained in our previous work, the better performance obtained by the GPU version based upon the AoS data structures is due to the use of the aligned global memory accesses. Therefore, in practical applications, we recommend the developers to use the data layout AoS to implement the Smart Laplacian smoothing. However, it is should be also noted that: the design of the AoS format mesh data structures is much more complex than that of the SoA format.
Table 4: Performance of the CDP-version developed in the iteration Form B

| Size | Running time (/ms) | Speedup |
|------|--------------------|---------|
|      | CPU    | GPU-AoS | GPU-SoA | GPU-AoS | GPU-SoA |
| 1K   | 171    | 13.5    | 15.4    | 12.67   | 11.10   |
| 5K   | 1139   | 50.0    | 58.1    | 22.78   | 19.60   |
| 10K  | 1888   | 74.8    | 82.8    | 25.24   | 22.80   |
| 50K  | 11918  | 404.4   | 424.8   | 29.47   | 28.06   |
| 100K | 26021  | 750.2   | 801.7   | 34.69   | 32.46   |

5.2 Impact of Iteration Forms

By comparing the absolute running time of two variations developed using the two iteration forms, i.e., the Form A and the Form B, we have found that: the variation with the Form B is a little faster than the variation with the Form A in the cases whenever which data layour (AoS or SoA) is adopted or whether the feature CDP is used; see Figure 3 for a groups of performance comparison.

The causes to the above results have been also described in our previous work [13]. The first cause is that: the Form A needs more calculations due to swapping intermediate nodal coordinates during iterating. The other cause is that: the convergence speed of the Form A is much lower than that of the Form B; and thus the Form A needs much more iterations for converging. Therefore, the Form B is suggested in practical applications.

![Figure 3](image-url)

Figure 3: Comparison of the running time of two variations implemented in the Form A and Form B

5.3 Performance of the Use of CUDA Dynamic Parallelism

We have observed that: the “CDP” version is slightly faster than the “GPU” version. This is perhaps leaded by the following causes. In the GPU version, only a thread block is allocated; and after obtaining the new positions of interior points, the quality of local mesh for each point is needed to be re-evaluated. Within this only one thread block, a single thread takes the responsibilities for re-calculating the quality of local meshes for several points, rather than only one point. In this case, the power of massively parallel computing is not fully exploited. The second reason is that: when re-evaluating the quality of the local mesh for a single point, each of the incident elements is needed to evaluate it quality once.
Due to the fact that a triangle has three vertices, it is thus needed to evaluate its quality three times. Obviously, there exists redundancies in the calculating of the quality of triangles.

Figure 4: Comparison of the speedups of the GPU-version and CDP-version of our implementation

In the CDP version, we first design a specific kernel to re-calculate the quality of all triangles after obtaining the new positions in each iteration. Each thread within this kernel is responsible for computing the quality metric of only one triangle. We also develop another kernel to find the worst triangle with the lowest quality of the local mesh for each point. For that the quality of each of the incident elements (i.e., triangles) have been calculated, it is quite easy to find the worst triangle that has the lowest quality among the incident elements. In this kernel, each thread is designed to find the worst incident element for only one point. Obviously, in this CDP version, the power of the massively parallel computing in the above two child kernels / grids can be efficiently exploited. However, the computing capability of the parent kernel is not fully exploited since only one thread (typically the first thread within the thread grid) is needed to invoke the above two child kernels. This is probably the reason why the CDP version is slightly rather than significantly faster than the GPU version.

6 Conclusion

We have presented an efficient GPU-accelerated implementation of the Smart Laplacian smoothing for optimizing planar triangular meshes. We have developed our implementation by using two data layouts, Array-of-Structures (AoS) and Structure-of-Arrays (SoA), and employing two iteration forms (Form A and Form B). The feature CDP is also adopted to realize the nested parallelization to iteratively determine the smoothed vertices' positions. We have evaluated the performance of our implementation using five randomly created triangular meshes on the GPU GT640. Experimental results have indicated that: our implementation can achieve the speedups of up to 44x. We have also found that: the data layout AoS can obtain better efficiency than the SoA layout. It has been demonstrated that: the Form A that needs to swap intermediate nodal coordinates is always slower than the Form B that does not swap data. We have also observed that: the version of our implementation with the use of the feature CDP is slightly faster than the version where the CDP is not adopted.

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