Effects of quantum deformation on the integer quantum Hall effect

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Abstract – In this work an application of the $\kappa$-deformed algebra in condensed-matter physics is presented. Starting by the $\kappa$-deformed Dirac equation we study the relativistic generalization of the $\kappa$-deformed Landau levels as well as the consequences of the deformation on the Hall conductivity. By comparing the $\kappa$-deformed Landau levels in the nonrelativistic regime with the energy levels of a two-dimensional electron gas (2DEG) in the presence of a normal magnetic field, upper bounds for the deformation parameter in different materials are established. An expression for the $\kappa$-deformed Hall conductivity of a 2DEG is obtained as well. The expression recovers the well-known result for the usual Hall conductivity in the limit $\varepsilon = \kappa^{-1} \to 0$. The deformation parameter breaks the Landau levels degeneracy and due to this, it is observed that deformation gives rise to new plateaus of conductivity in a such way that the plateaus widths of the $\kappa$-deformed Hall conductivity are less than the usual one. By studying the temperature dependence of the $\kappa$-deformed Hall conductivity, we show that an increase of the temperature causes the smearing of the plateaus and a diminution of the effect of the deformation, whilst an increase in the magnetic field enhances the effect of the deformation.

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The $\kappa$-Minkowski spacetime [1–4] is a usual framework for studying the effects of quantum deformation on the properties of physical systems in various contexts. A natural manner to access such properties, starts from the construction of the Dirac [5–8], Klein-Gordon and Schrödinger equations [9–12] within the quantum deformation framework. Some realizations in this direction have been reported with great interest [13,14]. In the relativistic context, the equations of motion are highly nonlinear and an analysis up to first order in the deformation parameter is the natural way to approach physical problems. Indeed, this consideration becomes more evident and acceptable when the problem is addressed in connection with some real physical system. This same idea can be applied in nonrelativistic context where physical systems in condensed matter can be studied. It is well known that the deformation parameter $\kappa$ can be interpreted as being the Planck mass $m_P$ or the quantum gravity scale [15] and it has implications to various properties of physical systems, for instance, in the quantization of a linear scalar field [16], in determining Landau levels [17], spin-(1/2) Aharonov-Bohm problem [18,19], Dirac oscillator [20,21], Casimir effect [22] and Dirac equation with anomalous magnetic moment interaction [23]. The investigation of physical systems on the $\kappa$-Minkowski spacetime constitutes a proper environment in which noncommutative theories may find applications. In the context of recent investigations, we can mention the study of the interference phenomena in quantum field theory [24], the study of modified diffusion equations defined on $\kappa$-spacetime to investigate the change
in the spectral dimension of a such space [25,26] and the analysis of bicovariant differential calculus [27,28].

In this letter, we study the effects of quantum deformation (through the parameter ε = κ⁻¹) on the Landau levels and on the integer quantum Hall effect [29]. As we shall show, starting by the κ-deformed Dirac equation we determine the κ-deformed Landau levels and obtain that the deformation parameter breaks the Landau levels degeneracy giving rise to new plateaus in the Hall conductivity. We also obtain upper bounds for the deformation parameter in different materials and discuss the effect of the temperature in the κ-deformed Hall conductivity. We would like to emphasize that in this work we are using the framework of quantum deformations as an effective theory, and not necessarily connecting it with quantum gravity.

One starts by considering a two-dimensional electron gas (2DEG) subject to an axial magnetic field parallel to the z-direction, B = Bẑ. Therefore, we are interested in the two-dimensional κ-deformed Dirac equation. Such an equation, obtained in [19] up to first order in the parameter ε, reads (in units such as $\hbar = c = 1$)

$$\left[ \beta \gamma \cdot \pi + \beta m^* + \frac{e}{2} \left( m^* \gamma \cdot \pi + \epsilon \sigma \cdot \mathbf{B} \right) \right] \psi = \mathcal{E} \psi, \quad (1)$$

where $\pi = p - eA$ is the generalized momentum, $m^*$ is the effective mass and $\psi$ is a two-component Dirac spinor. The $\kappa$ parameter corresponds to the two possible kinds of spinors in a two-dimensional space, which is related to the signature of the two-dimensional Dirac matrices [30]

$$s = \frac{i}{2} \text{Tr}(\gamma_0 \gamma_1 \gamma_2) = \pm 1, \quad (2)$$

and can be used to characterize the two possible spin states "up" and "down" [31]. We take the representation for the two-dimensional Dirac matrices as $\beta = \gamma_0 = \sigma_1$, $\gamma_1 = i \sigma_2$, $\gamma_2 = -i \sigma_1$, where $\left( \sigma_1, \sigma_2, \sigma_3 \right)$ are the standard Pauli matrices. The vector potential associated with the magnetic field is conveniently taken in the Landau gauge as $\mathbf{A} = (-By, 0, 0)$. Thus, adopting the fermionic decomposition

$$\psi = \left( \psi_1 \psi_2 \right) = \left( f(y) \ g(y) \right) e^{ip_x x}, \quad (3)$$

where $p_x \in \mathbb{R}$ is the eigenvalue of $\hat{p}_x$ operator, we obtain a set of two coupled first-order differential equations, namely,

$$\begin{align*}
\left( 1 + \frac{m^* \epsilon}{2} \right) \left( p_x + eBy - s \frac{d}{dy} \right) g(y) &= (\mathcal{E} - \mathcal{E}_+) f(y), \\
\left( 1 - \frac{m^* \epsilon}{2} \right) \left( p_x + eBy + s \frac{d}{dy} \right) f(y) &= (\mathcal{E} - \mathcal{E}_-) g(y),
\end{align*} \quad (4,5)$$

with $\mathcal{E}_\pm = \pm (m^* + \epsilon \kappa e B)/(2 \epsilon)$.

The second-order equation implied by (4) and (5) is seen to be

$$\left[ -\frac{d^2}{dy^2} + (eBy + p_x)^2 \right] f(y) = (\mathcal{E}^2 - m^*^2) f(y) + (1 - m^* \epsilon) \kappa e B f(y). \quad (6)$$

We note then that eq. (6) can be compared to the problem of an one-dimensional simple harmonic oscillator. Therefore, the κ-deformed relativistic Landau levels are

$$\mathcal{E}^{(c)}_{n,s} = \pm \sqrt{m^*^2 + (2n + 1 - s)eB + m^* \epsilon \kappa e B}, \quad (7)$$

with $n = 0, 1, 2, \ldots$. The above expression coincides with the result found in [17] (for the case $p_z = 0$) and shows that the energy eigenvalues are modified by the deformation parameter. Moreover, the limit of $\epsilon \to 0$ leads us to the usual relativistic Landau levels [32], as it should be.

Let us now investigate the influence of the κ-deformed algebra in the Hall conductivity of a 2DEG. Thus, taking the nonrelativistic limit of eq. (6) by setting $\mathcal{E}^{(c)} = m^* + E^{(c)}$, with $m^* \gg E^{(c)}$, and restoring Planck’s constant, the κ-deformed nonrelativistic Landau levels are seen to be

$$E^{(c)}_{n,s} = \left( n + \frac{1}{2} \right) \frac{\epsilon \kappa e B}{m^*} - \left( 1 - m^* \epsilon \right) \frac{\epsilon \kappa e B}{2m^*}. \quad (8)$$

In the usual case (i.e., for $\epsilon = 0$), we note that the inclusion of the electron spin causes a Zeeman splitting due to the term $\epsilon \kappa e B/2m^*$, thus doubling the number of Landau levels. However, for electron with $g = 2$ these levels are
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degenerate as a consequence of supersymmetric characteristic of the Hall Hamiltonian [35]. On the other hand, as we shall see, the deformation breaks this degeneracy. A schematic representation of the Landau levels in both cases is depicted in fig. 1.

We shall now use eq. (8) to impose an upper bound on the deformation parameter. In ref. [19], by using a relation between the deformation parameter and the anomalous magnetic moment of the free electron, an upper bound for the product $m^*\epsilon$ was determined as being smaller than $\alpha/2\pi \approx 0.00116$, where $\alpha$ is the fine-structure constant. In this manner, using the electron mass $m_e = 5.11 \times 10^5$ eV, this leads us to $\epsilon < 2.27 \times 10^{-9}$ (eV)$^{-1}$. On the other hand, it is well known that in the realm of condensed-matter physics, an effective mass and an effective anomalous magnetic moment can assume different values depending on the environment and external applied magnetic fields [36]. For instance, in electrons confined to the GaAs layer of a GaAs/AlxGa1-xAs heterojunction (which constitute a 2DEG) the effective mass and an effective anomalous magnetic moment of the free electron, an upper bound for the product $m^*\epsilon$ was determined as being smaller than $\epsilon < 2.27 \times 10^{-9}$ (eV)$^{-1}$.

Table 1: Upper bound for the deformation parameter. Values used for $m^*$ and $g^*$ are from refs. [33,34].

| Material | $m^*$ (eV)$^{-1}$ | $g^*$ |
|----------|-----------------|-------|
| InSb     | 0.016           | -51.3 |
| GaAs     | 0.067           | -0.44 |
| CdTe     | 0.110           | -1.64 |

Table 2: Upper bound for the deformation parameter. Values used for $m^*$ and $g^*$ are from refs. [33,34].

| Material | $m^*$ (eV)$^{-1}$ | $g^*$ |
|----------|-----------------|-------|
| InSb     | 0.016           | -51.3 |
| GaAs     | 0.067           | -0.44 |
| CdTe     | 0.110           | -1.64 |


In fig. 2, we show the $\kappa$-deformed and usual DOS.

The discrete spectrum is schematically depicted by delta peaks. As we can observe, the deformation parameter $\epsilon$ modifies the energy spacing by lifting the degeneracy of the Landau levels with different $s$ values, once that for $\epsilon = 0$ states with quantum numbers $n$ and $s = -1$ have the same energy as states with quantum numbers $n + 1$ and $s = +1$.

We shall now show how the lifting the degeneracy due to the deformation $\epsilon$, affects the quantum Hall effect by studying the plateaus of conductivity. In [38], it was shown that in the linear response approximation and when the Fermi energy is within the energy gap, the Hall conductivity at zero temperature is given by

$$\sigma_{xy}(\epsilon_F, 0) = \frac{\epsilon}{A} \frac{\partial N}{\partial B}$$

(10)

where $A$ is the area of the surface and $N$ is the number of states below the Fermi energy $\epsilon_F$. Thus, using eq. (9) we obtain

$$N^{(c)}(E) = A \int_{-\infty}^{\epsilon_F} D^{(c)}(E) dE,$$

$$= \frac{A|\psi|}{2\pi\hbar} \left\{ \frac{m^*\epsilon_F}{eB} + \frac{1}{2} - \frac{(1 - m^*\epsilon)s}{2} \right\},$$

(11)

where $|x|$ is the floor function. As the integer part in the above expression is a constant for a fixed value of the Fermi energy, the $\kappa$-deformed Hall conductivity is seen to be

$$\frac{\sigma_{xy}^{(\kappa)}(\epsilon_F, 0)}{\sigma_0} = - \left[ \frac{m^*\epsilon_F}{eB} + \frac{1}{2} - \frac{(1 - m^*\epsilon)s}{2} \right],$$

(12)

where $\sigma_0 = e^2/h$ is the quantum of conductivity. We note that the above expression recovers the well-known result for the usual Hall conductivity in the limit $\epsilon \to 0$. In fig. 3, we plot the $\kappa$-deformed and usual Hall conductivity as a function of the magnetic field. We can observe the presence of new plateaus (dashed lines) in the deformed case when compared with the usual one (solid line). These new plateaus are a direct consequence of degeneracy breaking of the Landau levels with different $s$ values caused by the deformation (see fig. 2).

In fig. 4, we show the $\kappa$-deformed Hall conductivity for three different values of the Fermi energy. We can observe...
that increasing the Fermi energy the plateaus are shifted to higher magnetic fields. Moreover, the plateaus widths increase with the increasing of the Fermi energy. A plateau transition in the Hall conductivity occurs when a Landau level crosses the Fermi energy, then \( n_0 = n + 1 = 1, 2, \ldots \) counts the number of Landau levels occupied below the Fermi energy, i.e.,

\[
\epsilon_F = \frac{\hbar e B}{m^*} \left( n_0 - \frac{1}{2} \right) + \frac{\hbar e B (1 - m^* \epsilon) s}{2},
\]

in a such way that \( \sigma_{xy}^{(c)} = -n_0 \sigma_0 \) on the plateaus. Thus, considering eq. (13), we can determine the plateaus widths. From fig. 3 and fig. 4, we can observe the presence of two plateaus widths. The small ones are essentially due to the deformation parameter. Their widths are seen to be

\[
\Delta B_1^{(c)} = \frac{m^* \epsilon F}{\hbar e} \frac{m^* \epsilon}{2}.
\]

Therefore, if \( \epsilon = 0 \), \( \Delta B_1^{(c)} = 0 \), and these plateaus disappear. Thus, confirming that these plateaus are exclusively associated with the deformation. The widths of the larger ones are given by

\[
\Delta B_2^{(c)} = \frac{m^* \epsilon F}{\hbar e} \frac{2 - (1 - 2m^* \epsilon)}{2}.
\]

This latter result for the widths of the \( \kappa \)-deformed plateaus should be compared with the usual ones (which can be obtained from (15) by setting \( \epsilon = 0 \)). Therefore, we observe that the plateaus widths for the \( \kappa \)-deformed system are less than the usual ones. Moreover, \( \Delta B_2^{(c)} \) is an increasing function of \( \epsilon \) whilst \( \Delta B_2^{(c)} \) is a decreasing function of \( \epsilon \).

So far we have discussed the conductivity at zero temperature. The expression for the temperature dependence of the Hall conductivity for an ideal 2DEG, but now taking into account the two possible \( s \) values and the deformation, reads [39–41]

\[
\frac{\sigma_{xy}^{(s)}(\mu, T)}{\sigma_0} = \sum_{n=0}^{\infty} (n + 1) \left[ \sigma_F \left( E_{n+s}^{(c)} \right) \right] - \sum_s n_F \left( E_{n+1,s}^{(c)} \right),
\]

where \( n_F(E) = 1/\exp [(E - \mu)/(k_B T)] + 1 \) is the Fermi distribution function, \( \mu \) is the chemical potential, \( T \) is the temperature and \( k_B \) is the Boltzmann constant. As shown in fig. 5, an increase of the temperature causes the smearing of the plateaus. This smearing is even more important for the small plateaus which are generated by the deformation. Figure 5 also suggests that increasing the temperature, the curve of the \( \kappa \)-deformed Hall conductivity comes close to the curve of the usual one. In order to confirm this, let us define

\[
\Delta \sigma = \frac{\sigma_{xy}^{(c)}(\mu, T) - \sigma_{xy}(\mu, T)}{\sigma_0},
\]

as the difference between the \( \kappa \)-deformed and usual Hall conductivities. This quantity is depicted in fig. 6 and we
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Fig. 6: (Color online) The difference between the κ-deformed and usual Hall conductivity $\Delta \sigma$ vs. the magnetic field $B$ for different temperatures. We use $\mu = 4.57 \times 10^{-13}$ ergs.

Fig. 7: (Color online) The κ-deformed Hall conductivity $-\sigma_{xy}^{(2)}/\sigma_0$ vs. the chemical potential $\mu$ for three different temperatures.

can indeed conclude that increasing the temperature, the κ-deformed Hall conductivity approaches the usual one. Therefore, increasing the temperature results in the weakening of the effect of the deformation. Moreover, by increasing the magnetic field, $\Delta \sigma$ increase. These two behaviors can be understood in the following way. The plateaus generated by the deformation are associated with the breaking of the degeneracy of the states. By increasing the temperature, the states tend to become degenerated again. On the other hand, the increasing of the magnetic field does the opposite, increasing the energy difference between states with different spin values.

For a fixed value of the external magnetic field, expression (16) allows us to study the dependence of the Hall conductivity with the chemical potential $\mu$ as well. This dependence is depicted in fig. 7 and we can observe that an increase in the chemical potential increases the conductivity. Moreover, as discussed above for the dependence with the magnetic field, an increase of the temperature results in smearing of the plateaus.

In conclusion, we have studied the effects of the κ-deformed algebra on the relativistic Landau levels and its consequences on the Hall conductivity. The κ-deformed relativistic and nonrelativistic Landau levels were determined, and by comparing the latter with the Landau levels of a 2DEG in a normal magnetic field, we have determined an upper bound for the deformation parameter in different materials. The values of upper bounds encountered show that the deformation parameter is dependent on the material and may be four orders of magnitude larger than that found for free electrons. It has been shown the presence of new plateaus in the κ-deformed Hall conductivity when compared with the usual one. These new plateaus stem from the fact that the deformation parameter breaks the degeneracy of the Landau levels. As a consequence of the presence of these new plateaus, the widths of the κ-deformed Hall plateaus are less than the usual ones. It was also shown that an increase of the Fermi energy causes the shifting of the plateaus to higher magnetic fields. Finally, an increase of the temperature causes the smearing of the plateaus and decreases the effect of the deformation, whilst an increase in the magnetic field does the opposite, enhancing the effect of the deformation. Given the relation between the deformation parameter and the effective anomalous magnetic moment, we hope that some future experiment may be able to detect the effect of the deformation on the Landau levels and consequently on the Hall conductivity.

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