A Nice Representation for a Link Between Baskakov- and Szász–Mirakjan–Durrmeyer Operators and Their Kantorovich Variants

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Abstract. In this paper we consider a link between Baskakov–Durrmeyer type operators and corresponding Kantorovich type modifications of their classical variants. We prove a useful representation for Kantorovich variants of arbitrary order for integer values of the linking parameter which leads to a simple proof of convexity properties for the linking operators. This also solves an open problem mentioned in Baumann et al. (Results Math. 69(3):297–315, 2016). Another open problem is presented at the end of the paper.

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1. Introduction

In [11] Păltănea defined a non-trivial link between genuine Bernstein-Durrmeyer operators and classical Bernstein operators, depending on a positive real parameter $\rho$. For further results we refer to [2–5,9,12].

In [9] the authors of this paper considered the $k$th order Kantorovich modifications and proved for natural numbers $\rho$ a representation in terms of Bernstein basis functions (see [9, Theorem 2]).

In this paper we investigate linking operators acting on the unbounded interval $[0, \infty)$. Linking operators for the Szász–Mirakjan case were defined by Păltănea [12] and for Baskakov type operators by Heilmann and Raşa [8].
In what follows for $c \in \mathbb{R}$ we use the notations
\[ a^{c,j} := \prod_{l=0}^{j-1} (a + cl), \quad a^{c,j} := \prod_{l=0}^{j-1} (a - cl), \quad j \in \mathbb{N}; \quad a^{c,\bar{0}} = a^{c,\bar{1}} := 1 \]
which can be considered as a generalization of rising and falling factorials. This notation enables us to state the results for the different operators in a unified form.

In the following definitions of the operators we omit the parameter $c$ in the notations in order to reduce the necessary sub and superscripts.

Let $c \in \mathbb{R}, \ c \geq 0, \ n \in \mathbb{R}, \ n > c$. Furthermore let $\rho \in \mathbb{R}^+, \ j \in \mathbb{N}_0, \ x \in [0, \infty)$. Then the basis functions are given by
\[
p_{n,j}(x) = \begin{cases} 
  \frac{n^j}{j!} x^j e^{-nx}, & c = 0, \\
  \frac{\Gamma(j+1)}{j!} x^j (1 + cx)^{-(n+c)} & c > 0.
\end{cases}
\]

In the following definition we assume that $f : [0, \infty) \rightarrow \mathbb{R}$ is given in such a way that the corresponding integrals and series are convergent.

**Definition 1.** The operators of Baskakov-type are defined by
\[(B_{n,\infty} f)(x) = \sum_{j=0}^{\infty} p_{n,j}(x) f\left(\frac{j}{n}\right),\]
and the genuine Baskakov–Durumeyer type operators are denoted by
\[(B_{n,1} f)(x) = p_{n,0}(x) f(0) + \sum_{j=1}^{\infty} p_{n,j}(x)(n+c) \int_{0}^{\infty} p_{n+2c,j-1}(t) f(t) dt. \quad (1)\]
Depending on a parameter $\rho \in \mathbb{R}^+$ the linking operators are given by
\[(B_{n,\rho} f)(x) = p_{n,0}(x) f(0) + \sum_{j=1}^{\infty} p_{n,j}(x) \int_{0}^{\infty} \mu_{n,j,\rho}(t) f(t) dt, \quad (2)\]
where
\[
\mu_{n,j,\rho}(t) = \begin{cases} 
  \frac{(n\rho)^j}{\Gamma(j+1)} t^{j-1} e^{-n\rho t}, & c = 0, \\
  \frac{c^j}{B\left(j\rho, \frac{c}{\rho} + 1\right)} t^{j-1} (1 + ct)^{-(n+c)^\rho - 1} & c > 0,
\end{cases}
\]
with Euler’s Beta function
\[
B(x, y) = \int_{0}^{1} t^{x-1} (1 - t)^{y-1} dt = \int_{0}^{\infty} \frac{t^{x-1}}{(1 + t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0.
\]
For \( \rho \in \mathbb{R}^+ \) the operators \( B_{n,\rho} \) are well defined for functions \( f \in W_n^\rho \) having a finite limit \( f(0) = \lim_{x \to 0^+} f(x) \) where \( W_n^\rho \) denotes the space of functions \( f \in L_{1,\text{loc}}[0, \infty) \) satisfying certain growth conditions, i.e., there exist constants \( M > 0, 0 \leq q < n\rho + c \), such that a.e. on \( [0, \infty) \)

\[
|f(t)| \leq M e^{qt} \quad \text{for } c = 0,
\]

\[
|f(t)| \leq M t^{\frac{q}{c}} \quad \text{for } c > 0.
\]

Setting \( c = 0 \) in (1) leads to the Phillips operators [14], \( c > 0 \) was investigated in [16]. To the best of our knowledge \( c = 0 \) was first considered in [12].

In [8] (see also [1,13]) the authors considered the \( k \)-th order Kantorovich modifications of the operators \( B_{n,\rho} \), namely,

\[
B_{n,\rho}^{(k)} := D^k \circ B_{n,\rho} \circ I_k, \quad k \in \mathbb{N}_0,
\]

where \( D^k \) denotes the \( k \)-th order ordinary differential operator and

\[
I_0 f = f, \quad (I_k f)(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t) dt, \quad k \in \mathbb{N}.
\]

For \( k = 0 \) we omit the superscript \((k)\) as indicated by the definition above. These operators play an important role in the investigation of simultaneous approximation.

This general definition contains many known operators as special cases. For \( c = 0 \) we get the \( k \)-th order Kantorovich modification of linking operators considered in [13]. For \( \rho = 1, k \in \mathbb{N} \) we get the Baskakov–Durrmeyer type operators \( B_{n,1}^{(1)} \) (see [10] for \( c = 0 \) and [6, (1.3)] for \( c \geq 0 \), named \( M_{n+c} \) there) and the auxiliary operators \( B_{n,1}^{(k)} \) considered in [7, (3.5)], (named \( M_{n+c,k-1} \) there).

Concerning the limit of the operators \( B_{n,\rho}^{(k)} \) for \( \rho \to \infty \) some results are known which are cited here.

**Theorem 1** ([13, Theorem 4]). Let \( c = 0 \). Assume that \( f : [0, \infty) \to \mathbb{R} \) is integrable and there exist constants \( M > 0, q \geq 0 \) such that \( |f(t)| \leq M e^{qt} \) for \( t \in [0, \infty) \). Then for any \( b > 0 \) there is \( \rho_0 > 0 \), such that \( B_{n,\rho} f \) exists for all \( \rho \geq \rho_0 \) and we have

\[
\lim_{\rho \to \infty} (B_{n,\rho} f)(x) = (B_{n,\infty} f)(x), \quad \text{uniformly for } x \in [0, b].
\]

In [8] explicit representations for the images of polynomials for all operators \( B_{n,\rho}^{(k)} \) were calculated which led to the following result for \( c \geq 0 \).

**Theorem 2** ([8, Theorem 1, Theorem 2, Corollary 1]). For each polynomial \( p \) we have

\[
\lim_{\rho \to \infty} (B_{n,\rho}^{(k)} p)(x) = (B_{n,\infty}^{(k)} p)(x),
\]

uniformly on every compact subinterval of \([0, \infty)\).
A different function space was considered in [1] for the case \( c \geq 0 \).

**Theorem 3** ([1, Lemma 5, Corollary 3]). Let \( f \in C^2[0, \infty) \) with \( \|f''\|_\infty < \infty \). Then we have

\[
\lim_{\rho \to \infty} (B_{n, \rho} f)(x) = (B_{n, \infty} f)(x),
\]

uniformly on every compact subinterval of \([0, \infty)\).

For \( \rho = 1 \) and \( \rho = \infty \) nice explicit representations are known, i.e.,

\[
(B_n^1 f)(x) = \frac{n^{c,k}}{n^{c,k-1}} \sum_{j=0}^{\infty} p_{n+c,k,j}(x) \int_0^\infty p_{n-c(k-2),j+k-1}(t)f(t)dt,
\]

\[
B_n^{(k)} (f; x) = \frac{n^{c,k}}{n^{k-1}} \sum_{j=0}^{\infty} p_{n+c,k,j}(x) \Delta_{\frac{k}{n}} I_k \left(f; \frac{j}{n}\right),
\]

where the forward difference of order \( k \) with the step \( h \) for a function \( g \) is given by \( \Delta_{\frac{k}{n}} g(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g(x+ih) \). By using Peano’s representation theorem for divided differences (see, e.g., [15, p. 137]) this can also be written as

\[
B_n^{(k)} (f; x) = \frac{n^{c,k}}{n^{k-1}} \sum_{j=0}^{\infty} p_{n+c,k,j}(x) \int_0^\infty N_{n,k,j}(t)f(t)dt,
\]

where \( N_{n,k,j} \) denotes the B-spline of order \( k \) to the equidistant knots \( \frac{j}{n}, \ldots, \frac{j+k}{n} \), defined by

\[
N_{n,1,j}(t) = \begin{cases} 1, & \frac{j}{n} \leq t < \frac{j+1}{n}, \\ 0, & \text{otherwise}, \end{cases}
\]

\[
N_{n,k,j}(t) = \frac{n}{k-1} \left\{ \binom{k}{j} N_{n,k-1,j}(t) + \left( \frac{j+k}{n} - t \right) N_{n,k-1,j+1}(t) \right\}.
\]

Our goal was to find useful representations also for \( \rho \neq 1, \infty \) for the general case \( k \in \mathbb{N} \).

Throughout this paper we will use the following well-known formula for the basis functions.

\[
p'_{n,j}(x) = n \left[ p_{n+c,j-1}(x) - p_{n+c,j}(x) \right].
\]

**2. The Representation Theorem**

First we treat the case \( k = 1 \). In other words we prove an explicit representation for a non-trivial link between Baskakov–Durrmeyer type and Baskakov-Kantorovich operators. Let \( f \in W^2_\rho \).

We start with the definition of \( B_{n, \rho}^{(1)} = D \circ B_{n, \rho} \circ I \) and consider

\[
\omega_{n,j,\rho}(t) = \begin{cases} \int_t^\infty \mu_{n,1,\rho}(u)du, & j = 0, \\ \int_0^t (\mu_{n,j,\rho}(u) - \mu_{n,j+1,\rho}(u)) \, du, & 1 \leq j. \end{cases}
\]
Thus
\[
\omega'_{n,j,\rho}(t) = \begin{cases} 
-\mu_{n,1,\rho}(t), & j = 0, \\
\mu_{n,j,\rho}(t) - \mu_{n,j+1,\rho}(t), & 1 \leq j.
\end{cases}
\]

Note that
\[
\omega_{n,j,\rho}(0) = 0, \ j \in \mathbb{N} \text{ and } \lim_{t \to \infty} \omega_{n,j,\rho}(t) = 0, \ j \in \mathbb{N}_0. \tag{6}
\]

By applying (4), an appropriate index transform and the definition of \(\omega_{n,j,\rho}\) we derive
\[
B_{n,\rho}^{(1)}(f; x) = \sum_{j=1}^{\infty} p'_{n,j}(x) \int_{0}^{\infty} \mu_{n,j,\rho}(t) I_1(f; t) dt
\]
\[
= n \sum_{j=1}^{\infty} p_{n+c,j-1}(x) \int_{0}^{\infty} \mu_{n,j,\rho}(t) I_1(f; t) dt
\]
\[
- n \sum_{j=1}^{\infty} p_{n+c,j}(x) \int_{0}^{\infty} \mu_{n,j,\rho}(t) I_1(f; t) dt
\]
\[
= np_{n+c,0}(x) \int_{0}^{\infty} \mu_{n,1,\rho}(t) I_1(f; t) dt
\]
\[
+ n \sum_{j=1}^{\infty} p_{n+c,j}(x) \int_{0}^{\infty} [\mu_{n,j+1,\rho}(t) - \mu_{n,j,\rho}(t)] I_1(f; t) dt
\]
\[
= -n \sum_{j=0}^{\infty} p_{n+c,j}(x) \int_{0}^{\infty} \omega'_{n,j,\rho}(t) I_1(f; t) dt.
\]

Integration by parts and observing (6) leads to
\[
B_{n,\rho}^{(1)}(f; x) = n \sum_{j=0}^{\infty} p_{n+c,j}(x) \int_{0}^{\infty} \omega_{n,j,\rho}(t) f(t) dt. \tag{7}
\]

**Lemma 1.** Let \(\rho \in \mathbb{N}\). Then
\[
\omega_{n,j,\rho}(t) = \sum_{i=0}^{\rho-1} p_{n\rho+c,i+j\rho}(t).
\]

**Proof.** First we treat the case \(c = 0\).

For \(l \in \mathbb{N}\), integration by parts leads to
\[
\int u^{l-1} e^{-n\rho u} du = -(l\rho - 1)! \sum_{i=0}^{l-1} \frac{1}{(n\rho)^{i+1}} \cdot \frac{1}{(l\rho - 1 - i)!} u^{l-1-i} e^{-n\rho u} + C
\]
\[
= -(l\rho - 1)! \sum_{i=0}^{l-1} \frac{1}{(n\rho)^{i+1}} \cdot \frac{1}{i!} u^{i} e^{-n\rho u} + C.
\]
Thus for \( j = 0 \)
\[
\omega_{n,0,\rho}(t) = \frac{(n\rho)^\rho}{(\rho - 1)!} \int_0^\infty u^{\rho-1} e^{-n\rho u} du
\]
\[
= -\frac{(n\rho)^\rho}{(\rho - 1)!} \sum_{i=0}^{\rho-1} \frac{1}{i!} \frac{1}{i^i} u^i e^{-n\rho u} \bigg|_0^\infty
\]
\[
= \sum_{i=0}^{\rho-1} (n\rho)^i \frac{1}{i!} t^i e^{-n\rho t}
\]
\[
= \sum_{i=0}^{\rho-1} p_{n\rho,i}(t).
\]

For \( j \in \mathbb{N} \) we get
\[
\omega_{n,j,\rho}(t) = \frac{(n\rho)^j\rho}{(j\rho - 1)!} \int_0^t u^{j\rho-1} e^{-n\rho u} du - \frac{(n\rho)^{(j+1)}\rho}{((j+1)\rho - 1)!} \int_0^t u^{(j+1)\rho-1} e^{-n\rho u} du
\]
\[
= -\frac{(n\rho)^j\rho}{(j\rho - 1)!} \cdot (j\rho - 1)! \sum_{i=0}^{j\rho-1} \frac{1}{(n\rho)^i} \cdot \frac{1}{i!} u^i e^{-n\rho u} \bigg|_0^t
\]
\[
+ \frac{(n\rho)^{(j+1)}\rho}{((j+1)\rho - 1)!} \cdot ((j+1)\rho - 1)! \sum_{i=0}^{(j+1)\rho-1} \frac{1}{(n\rho)^i} \cdot \frac{1}{i!} u^i e^{-n\rho u} \bigg|_0^t
\]
\[
= \sum_{i=j\rho}^{(j+1)\rho-1} (n\rho)^i \frac{1}{i!} t^i e^{-n\rho t}
\]
\[
= \sum_{i=0}^{\rho-1} p_{n\rho,i+j\rho}(t).
\]

Now we consider \( c > 0 \).

For \( l \in \mathbb{N} \), integration by parts leads to
\[
\int u^{l\rho-1}(1 + cu)^{-(\frac{n}{c} + l)\rho+1} du
\]
\[
= -\frac{(lp - 1)!}{\Gamma(\frac{n}{c} + l)\rho + 1} \sum_{i=0}^{l\rho-1} \frac{\Gamma(\frac{n}{c} + l)\rho - i}{(lp - 1 - i)!} \frac{1}{i!} c^i u^{l\rho-1-i}(1 + cu)^{-(\frac{n}{c} + l)\rho+i} + C
\]
\[
= -\frac{(lp - 1)!}{\Gamma(\frac{n}{c} + l)\rho + 1} \sum_{i=0}^{l\rho-1} \frac{\Gamma(\frac{n}{c} \rho + 1 + i)}{i!c^{l\rho-i}} u^i(1 + cu)^{-(\frac{n}{c} \rho+1+i)} + C.
\]
Thus for $j = 0$

$$\omega_{n,0,\rho}(t) = \frac{c^{\rho}}{B(\rho, \frac{n}{c} \rho + 1)} \int_0^\infty u^{\rho-1}(1 + cu)^{-\left(\frac{n}{c} + 1\right)\rho-1} du$$

$$= -\frac{c^{\rho}}{B(\rho, \frac{n}{c} \rho + 1)} \frac{(\rho - 1)!}{\Gamma((\frac{n}{c} + 1)\rho + 1)} \sum_{i=0}^{\rho-1} \frac{\Gamma(\frac{n}{c} \rho + 1 + i)}{i! c^{\rho-1}} u^i(1 + cu)^{-\left(\frac{n}{c} + 1\right)\rho-1} \left|_t^\infty\right.$$  

$$= \sum_{i=0}^{\rho-1} \frac{(np + c)^{c,i}}{i!} t^i(1 + ct)^{-\left(\frac{n}{c} + 1\right)i}$$

$$= \sum_{i=0}^{\rho-1} p_{n,\rho + c,i}(t).$$

For $j \in \mathbb{N}$ we get

$$\omega_{n,j,\rho}(t) = \frac{c^{j\rho}}{B(j\rho, \frac{n}{c} \rho + 1)} \int_0^t u^{j\rho-1}(1 + cu)^{-\left(\frac{n}{c} + j\right)\rho-1} du$$

$$= -\frac{c^{j\rho}}{B(j\rho, \frac{n}{c} \rho + 1)} \frac{(j\rho - 1)!}{\Gamma((\frac{n}{c} + j)\rho + 1)} \sum_{i=0}^{j\rho-1} \frac{\Gamma(\frac{n}{c} \rho + 1 + i)}{i! c^{j\rho-1}} u^i(1 + cu)^{-\left(\frac{n}{c} + 1\right)\rho-1} \left|_0^t\right.$$  

$$= -\frac{c^{j\rho}}{B((j + 1)\rho, \frac{n}{c} \rho + 1)} \frac{(j + 1)\rho - 1)!}{\Gamma((\frac{n}{c} + j + 1)\rho + 1)} \sum_{i=0}^{(j + 1)\rho-1} \frac{\Gamma(\frac{n}{c} \rho + 1 + i)}{i! c^{(j + 1)\rho-1}} u^i(1 + cu)^{-\left(\frac{n}{c} + 1\right)\rho-1} \left|_0^t\right.$$  

$$= -\sum_{i=0}^{j\rho-1} \frac{(np + c)^{c,i}}{i!} t^i(1 + ct)^{-\left(\frac{n}{c} + 1\right)i} + \sum_{i=0}^{(j + 1)\rho-1} \frac{(np + c)^{c,i}}{i!} t^i(1 + ct)^{-\left(\frac{n}{c} + 1\right)i}$$

$$= \sum_{i=0}^{\rho-1} p_{n,\rho + c,j\rho,i}(t).$$

With Lemma 1 and (7) we have now the desired representation of $B_{n,\rho}^{(1)}(f; x)$ in terms of the basis functions, i.e.,

$$B_{n,\rho}^{(1)}(f; x) = n \sum_{j=0}^\infty p_{n+c,j}(x) \int_0^\infty \left\{ \sum_{i=0}^{\rho-1} p_{n,\rho + c,i+j\rho}(t) \right\} f(t) dt.$$  

(8)
So, for the kernel function

\[ K_{n,j,\rho}(t) = \rho - 1 \sum_{i=0}^{\rho-1} p_{n,\rho+c,i+j\rho}(t) \]

we have \( K_{n,j,1}(t) = p_{n+c,j}(t) \) and we conjecture (see also Sect. 5) that

\[
\lim_{\rho \to \infty} K_{n,j,\rho}(t) = \begin{cases} 1, & t \in \left[ \frac{j}{n}, \frac{j+1}{n} \right], \\ 0, & t \in [0, \infty) \setminus \left[ \frac{j}{n}, \frac{j+1}{n} \right]. \end{cases}
\]

**Remark 1.** Observe that the kernel \( K_{n,j,\rho} \) can also be written in terms of classical Baskakov type operators applied to the characteristic functions \( \chi_{\left[ \frac{j\rho}{n}, \frac{(j+1)\rho}{n} \right]}(t) \), i.e.,

\[
B_{n,\rho+c,\infty}(\chi_{\left[ \frac{j\rho}{n}, \frac{(j+1)\rho}{n} \right]}(t); t) = \sum_{i=0}^{\infty} p_{n,\rho+c,i}(t) \chi_{\left[ \frac{j\rho}{n}, \frac{(j+1)\rho}{n} \right]} \left( \frac{i}{n\rho+c} \right)
\]

\[
= \sum_{i=j\rho}^{(j+1)\rho-1} p_{n,\rho+c,i}(t)
\]

\[
= \sum_{i=0}^{\rho-1} p_{n,\rho+c,i+j\rho}(t).
\]

**3. Representation for the \( k \)-th Order Kantorovich Modification**

In this section we generalize the representation of the operators to \( k \in \mathbb{N} \).

**Theorem 4.** Let \( n, k \in \mathbb{N} \), \( n - k \geq 1 \), \( \rho \in \mathbb{N} \) and \( f \in W_{n}^\rho \). Then we have the representation

\[
B^{(k)}_{n,\rho}(f; x) = \frac{n^c_k}{(n\rho)^c_{k-1}} \sum_{j=0}^{\infty} p_{n+k\rho,\rho+j}(x)
\]

\[
\times \int_0^{\rho-1} \cdots \int_0^{\rho-1} p_{n-k\rho+k-2,\rho+i_1+\cdots+i_k+k-1}(t)f(t)dt.
\]

**Proof.** We prove the theorem by induction.

For \( k = 1 \) see (8).

\( k \Rightarrow k+1 \): From the definition of the operators \( B^{(k+1)}_{n,\rho} \) we get

\[
B^{(k+1)}_{n,\rho}(f; x) = D^1 \circ B^{(k)}_{n,\rho} \circ I_1(f; x)
\]

\[
= \frac{n^c_k}{(n\rho)^c_{k-1}} \sum_{j=0}^{\infty} p'_{n+k\rho,\rho+j}(x)
\]

\[
\times \int_0^{\rho-1} \cdots \int_0^{\rho-1} p_{n-k\rho+k-2,\rho+i_1+\cdots+i_k+k-1}(t)I_1(f; t)dt.
\]
By using (4) and an appropriate index transform we derive

\[
B_{n,\rho}^{(k+1)}(f; x) = \frac{n^{c,k+1}}{(n\rho)^{c,k-1}} \sum_{j=0}^{\infty} p_{n+(k+1)c,j}(x) 
\]

\[
\times \int_0^\infty \sum_{i_1=0}^{\rho-1} \cdots \sum_{i_k=0}^{\rho-1} \sum_{i_{k+1}=0}^{\rho-1} \left[p_{n\rho-(k-2)c,j,\rho+i_1+\cdots+i_k+k-1}(t) \right. 

\left. - p_{n\rho-(k-2)c,j,\rho+i_1+\cdots+i_k+k-1}(t) \right] I_1(f; t) dt. 
\]

(9)

Now we rewrite the difference of the basis functions in the integral and use again (4), i.e.,

\[
p_{n\rho-(k-2)c,j,\rho+i_1+\cdots+i_k+k-1}(t) - p_{n\rho-(k-2)c,j,\rho+i_1+\cdots+i_k+k-1}(t) 
\]

\[
= \sum_{i_{k+1}=0}^{\rho-1} \left[p_{n\rho-(k-2)c,j,\rho+i_1+\cdots+i_k+k+1}(t) - p_{n\rho-(k-2)c,j,\rho+i_1+\cdots+i_k+k+1}(t) \right] 
\]

\[
= - \frac{1}{n\rho - (k-1)c} \sum_{i_{k+1}=0}^{\rho-1} p'_{n\rho-(k-1)c,j,\rho+i_1+\cdots+i_k+k+1}(t). 
\]

Together with (9) and integration by parts this leads to

\[
B_{n,\rho}^{(k+1)}(f; x) 
\]

\[
= \frac{n^{c,k+1}}{(n\rho)^{c,k-1}} \sum_{j=0}^{\infty} p_{n+(k+1)c,j}(x) 
\]

\[
\times \int_0^\infty \sum_{i_1=0}^{\rho-1} \cdots \sum_{i_k=0}^{\rho-1} \sum_{i_{k+1}=0}^{\rho-1} -p'_{n\rho-(k-1)c,j,\rho+i_1+\cdots+i_k+k+1}(t) I_1(f; t) dt 
\]

\[
= \frac{n^{c,k+1}}{(n\rho)^{c,k-1}} \sum_{j=0}^{\infty} p_{n+(k+1)c,j}(x) 
\]

\[
\times \int_0^\infty \sum_{i_1=0}^{\rho-1} \cdots \sum_{i_k=0}^{\rho-1} \sum_{i_{k+1}=0}^{\rho-1} p_{n\rho-(k-1)c,j,\rho+i_1+\cdots+i_k+k+1}(t) f(t) dt. 
\]

\[
\square
\]

4. Convexity of the Linking Operators

In [13, Theorem 6] Păltănea considered convexity properties of the operators \(B_{n,\rho}\) in case \(c = 0\), i.e., the case of Szász–Mirakjan linking operators. In a long and tricky proof he showed that \((B_{n,\rho}f)^{(r)} \geq 0\) for each function \(f \in W^r_n \cap C^r[0, \infty), f^{(r)} \geq 0\).

By using Păltănea’s method this can be generalized to \(B_{n,\rho}^{(k)}, c = 0, k \in \mathbb{N}\) (see [1, Theorem 3]). From the representation in Theorem 4 we know that
all the operators $B^{(k)}_{n,\rho}$, $\rho \in \mathbb{N}$, are positive and the convexity properties for $B^{(k)}_{n,\rho}$, $\rho \in \mathbb{N}$ now follow as a corollary. For $c > 0$ this solves an open problem mentioned in [1].

**Corollary 1.** Let $f \in W^\rho_n$ with $f^{(r)}(x) \geq 0$, $r \in \mathbb{N}_0$, for all $x \in [0, \infty)$. Then

$$D^r \left( B^{(k)}_{n,\rho}(f; x) \right) \geq 0$$

for each $k \in \mathbb{N}$, $x \in [0, \infty)$.

**Proof.**

$$D^r B^{(k)}_{n,\rho}f = D^r D^k B_{n,\rho} I_k f = D^{k+r} B_{n,\rho} I_{k+r} f^{(r)} = B^{(k+r)}_{n,\rho} f^{(r)} \geq 0,$$

as $f^{(r)} \geq 0$. □

5. Remarks on Convergence to B-Splines

In [9] the authors considered the topic for the linking Bernstein operators with a different approach. In [3, Theorem 2.3] Gonska and Păltănea proved the uniform convergence on $[0, 1]$ of the linking Bernstein operators to the classical Bernstein operators for every function $f \in C[0, 1]$. With explicit representations for the images of monomials and density arguments this can be extended to the $k$-th order Kantorovich modifications (see [9, Introduction]). From the

![Figure 1](image-url)  

**Figure 1.** The dotted line belongs to $\rho = 10$, the dashed line to $\rho = 30$ and the solid line to $\rho = 150$
representation of the $k$-th order Kantorovich modifications of the linking Bernstein operator and the classical Bernstein operator (see [9, Theorem 2, (2)]) one can derive immediately that
\[
\lim_{\rho \to \infty} \frac{1}{\rho^{k-1}} \sum_{i_1=0}^{\rho-1} \cdots \sum_{i_k=0}^{\rho-1} \tilde{p}_{n \rho + k - 2, j \rho + i_1 + \cdots + i_k + k - 1}(t) = N_{n, k, j}(t)
\]
for $t \in [0, 1]$ with the Bernstein basis functions \( \tilde{p}_{n, j}(t) = \binom{n}{j} t^j (1 - t)^{n-j} \).

As the situation concerning the convergence of the operators in the Baskakov-type case is more complicated (see Theorems 1, 2 and 3) we can only conjecture that
\[
\lim_{\rho \to \infty} \frac{1}{\rho^{k-1}} \sum_{i_1=0}^{\rho-1} \cdots \sum_{i_k=0}^{\rho-1} p_{n \rho - c(k-2), j \rho + i_1 + \cdots + i_k + k - 1}(t) = N_{n, k, j}(t)
\]
for $t \in [0, \infty)$.

We fortify our conjecture by the following illustrations (see Fig. 1) where we chose $k = 1, c = 1, n = 5$ and $j = 1$.

References

[1] Baumann, K., Heilmann, M., Raşa, I.: Further results for $k$th order Kantorovich modification of linking Baskakov type operators. Results Math. 69(3), 297–315 (2016)

[2] Gonska, H., Pălănea, R.: Quantitative convergence theorems for a class of Bernstein–Durrmeyer operators preserving linear functions. Ukr. Math. J. 62(7), 1061–1072 (2010)

[3] Gonska, H., Pălănea, R.: Simultaneous approximation by a class of Bernstein–Durrmeyer operators preserving linear functions. Czechoslov. Math. J. 60(135), 783–799 (2010)

[4] Gonska, H., Raşa, I., Stanila, E.-D.: The eigenstructure of operators linking the Bernstein and the genuine Bernstein–Durrmeyer operators. Mediterr. J. Math. 11, 561–576 (2014)

[5] Gonska, H., Raşa, I., Stanila, E.-D.: Lagrange-type operators associated with $U^n_{\rho}$. Publ. Inst. Math. (Beograd) (N.S.) 96(110), 159168 (2014)

[6] Heilmann, M.: Direct and converse results for operators of Baskakov–Durrmeyer operators. Approx. Theory Appl. 5(1), 105–127 (1989)

[7] Heilmann, M.: Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren. Habilitationsschrift Universität Dortmund (1992)

[8] Heilmann, M., Raşa, I.: $k$-th order Kantorovich modification of linking Baskakov type operators. In: Agrawal, P.N. et al. (eds.) Recent Trends in Mathematical Analysis and its Applications, Proceedings of the Conference ICRTMAA 2014, Rorkee, India, December 2014, Proceedings in Mathematics and Statistics 143, pp. 229–242 (2015)
[9] Heilmann, M., Raşa, I.: A nice representation for a link between Bernstein-Durrmeyer and Kantorovich operators. In: Giri, D. et al. (eds.) Proceedings of ICMC 2017, Haldia, India, January 2017, Springer Proceedings in Mathematics and Computing, pp. 312–320 (2017)

[10] Mazhar, S.M., Totik, V.: Approximation by modified Szász operators. Acta Sci. Math. 49, 257–269 (1985)

[11] Păltănea, R.: A class of Durrmeyer type operators preserving linear functions. Ann. Tiberiu Popoviciu Sem. Funct. Eq. Approx. Conv. (Cluj-Napoca) 5, 109–117 (2007)

[12] Păltănea, R.: Modified Szász–Mirakjan operators of integral form. Carpath. J. Math. 24(3), 378–385 (2008)

[13] Păltănea, R.: Simultaneous approximation by a class of Szász–Mirakjan operators. J. Appl. Funct. Anal. 9(3–4), 356–368 (2014)

[14] Phillips, R.S.: An inversion formula for Laplace transforms and semi-groups of linear operators. Ann. Math. 59(2), 325–356 (1954)

[15] Schumaker, L.L.: Spline Functions: Basic Theory. Cambridge University Press, Cambridge (2007)

[16] Wagner, M.: Quasi-Interpolanten zu genuinen Baskakov–Durrmeyer–Typ Operatoren. Dissertation Universität Wuppertal (2013)

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