D-Instantons on the boundary

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Abstract

The Maldacena’s proposal has established an intriguing connection between string theory in $AdS$ spaces and gauge theory. In this paper we study the effects of adding $D(-1)$-branes to the system of $D3$- or $(D1 - D5)$-branes and we give arguments indicating that $D(-1)$-branes are necessary to describe four and two dimensional instantons.
1 Introduction

The description of supersymmetric gauge field theories by means of superstrings has been a challenging problem. In brane theory, gauge theory arises as an effective low energy description that is useful in some regions in the moduli space of vacua \[1\]. The underlying dynamics is always the same - brane volume dynamics in string theory. However, the physical interpretation of these objects has remained quite obscure. Recently, based on previous works on the structure of extremal black holes near the horizon \[2\] and absorption by these branes \[3\], Maldacena has proposed an exact correspondence between string theory on Anti-de-Sitter spaces (times a compact space) and superconformal quantum field theories living on the boundary of this space \[3\]. In particular, Type IIB theory on \(AdS_5 \times S^5\) should be dual to \(N = 4\) super Yang-Mills on the boundary.

This new AdS/CFT duality allows to describe many features of gauge theory by means of branes. In this sense, perturbative computation of correlation functions of local operator have been presented in \[5\][6], Wilson loops operators have been computed in \[7\][8] and the description of baryons by wrapping branes on the compact space has been given in \[9\] and \[10\].

In this paper we are interested in finding branes configurations which can lead to instantons solutions of \(d\) Yang-Mills field theory, the \(d\) dimensional space being the \(AdS_{d+1}\) boundary. If we construct systems of branes within the configuration of branes which causes the \(AdS\) background, in the AdS/CFT duality picture, the ‘small’ branes should behave as physical objects for the gauge field theory on the boundary. The natural \(D\)-brane objects to be identified with boundary instantons are the \(D\)-Instantons (\(D(-1)\)-branes). The world volume of these tiny objects is just a point. Therefore, as we will see, they hardly alter the basic structure of the space.

In section 2 of the paper we will remind that the \(AdS_5 \times S^5\) space appears as a supergravity solution for \(D3\)-branes in the low energy limit. Then, in view of the fundamental role of supergravity solutions of branes near the horizon, looking for solutions of composite branes (branes within branes) seems to be important. The description of smeared solutions of branes ending on branes have been extensively treated in literature \[11\] and some localized solutions have been also found for specific configurations of branes \[12\].

In section 3, we will focus on solutions of the \(D\)-Instanton within the
$N D3$-branes. We are interested in the configuration on the boundary and, therefore, in localized solutions. Here we will characterize the $D$–Instanton by means of a quasi conformally invariant harmonic function connected to the dilaton field and we will show how the presence of $D$–Instantons affects the structure of the space by developing throats which connect different vacua.

In section 4, we will describe the field theory on the branes, the coupling to the gravitational fields and the new features, self-duality and non-trivial instanton number, which the existence of $D(-1)$-branes induces on the gauge fields.

In section 5, we will approach the supergravity solution for $D$-Instantons in the bulk to the boundary. Here we will find that the system of $D(-1)$-branes stuck to the $N D3$-branes behaves as small $SU(2)$ Yang-Mills instantons. We will obtain the Yang-Mills solution corresponding to the system, the moduli space of both of them and the instanton measure.

Finally, in section 6, we will discuss supergravity solutions with lower supersymmetry ($D(-1)$-branes within a $D1$-$D5$ system) and its relation to $d = 2$ instantons.

We will conclude with a brief discussion of the results and new possibilities for a further research.

While this paper was being prepared for publication, we learned about a recent work on the same subject [13].

2 The AdS$_5 \times S^5$ background

D-branes can be identified with BPS-saturated, R-R charged $p$–brane solutions. The Supergravity solution carrying $Dp$ charge can be characterized in terms of a function $H_p(x_\perp)$ which is harmonic with respect to the directions transversal to the world volume [14]; i.e. it verifies the $10-(p+1)$ dimensional Laplace equation

$$\partial^2_{x_\perp} H_p(x_\perp) = 0. \quad (2.1)$$

Assuming that $H_p$ depends only on the radial coordinates $r = \sqrt{x_{p+1}^2 + \cdots + x_9^2}$,
we can solve (2.1) to get
\[ H_p(r) = 1 + \frac{Q_p}{r^{(7-p)}}. \] (2.2)
where the charge \( Q_p \) is related to the string tension \( T \equiv (2\pi\alpha)^{-1} \)
\[ Q_p = g(2\pi)^{(5-p)/2}(2\pi\alpha')^{(7-p)/2}(2\pi^{(7-p)/2}/\Gamma((7-p)/2))^{-1}. \] (2.3)
Then, the euclidean metric in string frame is
\[ ds^2_p = H_p^{-1/2}(dx_0^2 + ... + dx_p^2) + H_p^{1/2}(dx_{(p+1)}^2 + ... + dx_9^2) \] (2.4)
with the dilaton field \( \phi \) given by
\[ e^{2\phi} = H_p^{(3-p)/2}. \] (2.5)
The R-R gauge field strength associated with the \( p \)-brane can be also expressed in terms of the harmonic function \[ F^{(p+2)} = d\frac{1}{H_p} \wedge dx^0 \ldots \wedge dx^p, \] (2.6)
in case \( p \leq 3 \) (they carry electric charge) and its dual for \( p \geq 3 \) (they carry magnetic charge). Notice that the case \( p = 3 \) is self-dual. The flux of the dual field strengths on theirs \( S^{(8-p)} \) transversal spheres fix the value of the charges.

Now we consider the string background describing one 3-brane
\[ ds^2_3 = H_3^{-1/2}d\vec{x}^2 + H_3^{1/2}(dr^2 + r^2d\Omega_5^2), \quad H_3 = 1 + \frac{4\pi g\alpha'^2}{r^4} \] (2.7)
where \( \vec{x} = (x_0, ..., x_3) \) denotes the four dimensional world volume of the 3-brane and \( d\Omega_5^2 \) is the metric in \( S^5 \). There is no dilaton field\[ ] then the string frame and the Einstein frame are identical
\[ ds^2_E = ds^2_s. \] (2.8)
In case of \( N \) parallel 3-branes, the BPS condition of ‘No force’ implies
\[ H_3 = 1 + \frac{4N\pi g\alpha'^2}{r^4}. \] (2.9)
\[ ^1 \text{That corresponds to the fact that the theory on D3-branes is conformal.} \]
In this paper we will consider the low energy effective theory. In this limit
\[ \alpha' \to 0, \quad U \equiv \frac{r}{\alpha'} = \text{fixed} \quad (2.10) \]
the field theory on the 3-brane decouples from the bulk [4]. In this limit, the constant term in the harmonic function (2.9) can be neglected and the metric, in terms of the new variable \( U \), becomes
\[
\begin{align*}
\text{ds}^2_3 &= \alpha' \left[ \frac{u^2}{\sqrt{4\pi gN}} d\vec{x}^2 + \sqrt{4\pi gN} \frac{du^2}{u^2} + \sqrt{4\pi gNd\Omega_5^2} \right] \quad (2.11)
\end{align*}
\]
which describes the \( AdS_5 \times S^5 \) space and remains constant in \( \alpha' \) units. Notice that \( S^5 \) and \( AdS_5 \) have the same radius and, being spaces of opposite curvature, the total scalar curvature of the \( AdS_5 \times S^5 \) space is identically zero.

The supersymmetry group of euclidean \( AdS_5 \times S^5 \), \( SO(1,5) \times SO(6) \), and, as it has been shown in [6], the conformal compactification of \( AdS_5 \), on which \( SO(1,5) \) acts, is the sphere \( S^4 \). The supersymmetric group is the same as the superconformal group in four dimensions [16]. This fact led to Maldacena’s proposal [4]. According to it, when the effective coupling \( g_e = gN \) becomes large, the \( N = 4 \) superconformal theory on the boundary is governed by supergravity on \( AdS_5 \times S^5 \) where perturbation theory can be trusted.

Rescaling the \( u \to u\lambda^{-1} \) and \( \vec{x} \to \lambda \vec{x} \) variables by the factor \( \lambda^4 = 4\pi gN \), we obtain the metric
\[
\begin{align*}
\text{ds}^2_3 &= \alpha' \left[ \frac{u^2}{\sqrt{4\pi gN}} d\vec{x}^2 + \frac{du^2}{u^2} + d\Omega_5^2 \right] \quad (2.12)
\end{align*}
\]
and, using the inverse variable \( z = 1/u \), the metric
\[
\begin{align*}
\text{ds}^2_3 &= \alpha' \sqrt{4\pi gN} \left[ \frac{1}{z^2} (d\vec{x}^2 + dz^2) + d\Omega_5^2 \right] \quad (2.13)
\end{align*}
\]
which we will use, ignoring constant factors, from now on. In this representation of \( AdS_5 \) as the upper space \( z > 0 \), the boundary consist of a copy of \( R^4 \times S^5 \), at \( z = 0 \), and a single point at \( z = \infty \) which compactifies \( R^5 \times S^5 \) to \( S^4 \times S^5 \).
3 D-Instantons in $AdS_5 \times S^5$

The $p - (p + 4)$ system of branes is a BPS bound state which preserves 1/4 of the supersymmetries; i.e., we are dealing with a $N = 2$ SUSY theory. These $p - (p + 4)$ systems are marginally bound. This means that the total energy is the sum of the energies.

We are interested in placing D-Instantons in the $AdS_5 \times S^5$ space. Therefore, we will construct supergravity solutions of $D(-1)$-branes within a collection of $N$ $D3$-branes in the decoupling limit. The solutions are required to preserve as much as possible the symmetries of this space. The localized solution should correspond to localized instantons in the four dimensional theory.

The supergravity background for such a system is represented by the following metric

$$ds^2_{(-1,3)} = H^{-1/2} \left[ \frac{1}{z^2} (d\vec{x}^2 + dz^2) + d\Omega_5^2 \right], \quad (3.1)$$

where we have taken off the prefactor which appears in (2.13), the 1- and 4-forms

$$F^{(1)} = dH^{-1}_1$$
$$F^{(4)} = d(z^4)(dx_0 \wedge ... \wedge dx_3), \quad (3.2)$$

and dilaton field

$$e^\phi = H_{-1}. \quad (3.3)$$

In this case the string and the Einstein metric do not coincide

$$ds^2_{\text{string}} = e^{\phi/2} ds_E \quad (3.4)$$

and, due to the relation (3.3), the Einstein metric still corresponds to $AdS_5 \times S^5$ space.

Assuming that the harmonic function $H_{-1}$ depends only on the coordinates of the $D3$-brane world volume $\vec{x} = (x_0, x_1, x_2, x_3)$ and on $z$, it must satisfy the Laplace equation in the ten-dimensional curved transverse space

$$\left[ \Delta_{||} + z^3 \frac{\partial}{\partial z} z^3 \frac{\partial}{\partial z} \right] H_{-1}(\vec{x}, z) = 0 \quad (3.5)$$

This condition preserves the flatness of the space.
where $\Delta_{||}$ represents the laplacian in the four dimensional space. This condition is invariant under translations in $x_i, n = 0, \ldots, 3$ and under the $SO(4) \times SO(6)$ symmetry group of $SO(1, 5)$. However, a conformal transformation that maps the point at infinity to the origin

$$z \leftrightarrow \frac{z}{z^2 + x^2}, \quad x_i \leftrightarrow \frac{x_i}{z^2 + x^2}, \quad i = 0, \ldots, 3$$

(3.6)
does change the harmonic function leaving invariant the laplacian. Therefore, for a given solution $H_{-1}(\vec{x}, z)$ of (3.5), its transformed function under (3.6), $H'_{-1} = H_{-1}(\frac{z}{z^2 + x^2}, \frac{x}{z^2 + x^2})$ is also a solution. Returning to the metric (3.4), it is straightforward to see that it exhibits this same behaviour; i.e., it is invariant under $SO(4) \times SO(6)$ and only the $H^{-1/2}_{-1}$ prefactor changes under (3.6).

In the representation of the bulk as the upper space $z > 0$ we have been using, the transformation (3.6) interchanges the two boundary regions. Now at infinity ($z^t = 0$) we have a copy of $R^4$ and the boundary at $z = 0$ ($z^t = \text{infinity}$) is just a point. Then, the compactified space is still $S^4 \times S^5$, but the normal vector has flipped. This change of orientation transforms the D-instanton into the anti D-instanton \[17\] and changes the sign of the flux of the 1-form defined in (3.2) on $S^4 \times S^5$. It will be clear later when we relate $D(-1)$-branes to Yang-Mills instantons that this operation precisely corresponds to coordinate inversion which sends the pseudoparticle with $q = Q$ into the antiparticle with $q = -Q$ \[18\].

Now we will construct specific supergravity solutions for the $D-$Instanton sitting on the $D3$-branes. That means $D(-1)-$brane configuration centered at the $u_0 = 0$.

A solution of (3.5) singular at a point on the boundary at $z = 0$ can be shown to be

$$e^\phi = \frac{cz^4}{((\vec{x} - \vec{x}(0))^2 + z^2)^4}$$

(3.7)

where the constant $c$ is related to the charge of the $D-$Instanton and $\vec{x}(0)$ is the position in $R^4$. Its transformed function under (3.6) will give us the solution singular at infinity

$$e^\phi = cz^4.$$ 

(3.8)

Then, the conformal transformation (3.6), transforms (3.7) into (3.8), leaving invariant the underlying $AdS_5 \times S^5$ metric. This fact allows us relate the behaviour of the system at the origin and at infinity. In the string frame
the configuration consists of two asymptotic $AdS_5 \times S^5$ spaces, one at
the singularity at the origin and the other at infinity, which are connected
by a throat [19] [20]. The space is geodesically complete, so in this sense is not
singular. That is analogous to classical instantons in field theories which join
two different vacua of the theory.

The constant $c$ are related to the electric charge $Q^{-1}$ of the $D-$Instanton.
This can be calculated as the flux of the 9–form,

$$F^9 = e^{2\phi(x,z)} * dH^{-1}_1(x,z) = -e^{\phi(x,z)} * d\phi(x,z), \quad (3.9)$$

dual to $F^1$, on the $S^4 \times S^5$ space

$$Q^{-1} = \frac{1}{\text{Vol}(\Sigma)\text{Vol}(S_5)} \int_{AdS_5 \times S^5} *d*d(e^\phi)$$

$$= \frac{c}{\text{Vol}(\Sigma)} \int_{x} d^4x \int_{0}^{\infty} dz \frac{1}{z^3} \frac{\partial_x}{\partial_x} \frac{z^4}{(x^2 + z^2)^4} \int_{x} d^4x \frac{1}{z^4} \frac{z^4}{(x^2 + z^2)^4} = \frac{c}{4}$$

(3.10)

where $\Sigma = \partial AdS_5$. A quantization condition relates the R-R electric charges
of $p-$branes to magnetic charges of $(p+6)$-branes[24][25]. The associated
R-R field strength $F^{(9)}$ is nine-dimensional and, therefore, the flux of its dual
has to be calculated on the $S^1$ sphere. For this reason its charge is quantized
and that gives a quantized charge for the $D(-1)-brane$. In the following we
will take $c = 1$ for one $D-$Instanton. And, as a BPS state, for $n$ instantons
we will have $c = n$.

Due to the linearity of the laplacian in (3.5) the multiinstanton solution
will be a superposition of solutions

$$e^\phi = \sum_i \frac{c_i z^4}{((x - x_i(0))^2 + z^2)^4} \quad (3.11)$$

as corresponding to a BPS state. At every of these singularities the space
will develop a throat.

Another set of solutions of (3.5) can be found by factorizing

$$H_{-1}(x,z) = F(x)G(z)$$

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as a product of two functions. Then, we have
\[ e^\phi = c \frac{z^4}{(x^2 + z^2)^4} \frac{(x^2 + z^2)^2}{x^2} \] (3.12)
singular at the origin, and its transformed under \((3.6)\) function
\[ e^\phi = cz^4 \frac{1}{x^2}, \] (3.13)
singular at the infinity. When gravity decouples from field theory, all the gravitational fields should behave as Green’s functions. Note that these last solutions are more regular on the boundary and they may not be a good description for the dilaton field.

Finally, we could also consider supergravity solutions for the \(D(-1)\)-brane smeared (as opposite to localized) in the \(D3\)-branes world volume. That can be achieved by integrating the \(x_i, (i = 0, ..., 3)\) coordinates (using the \(AdS_5 \times S^5\)) metric to obtain
\[ e^\phi = z^4, \] (3.14)
but this solution does tell us too much about the structure of the space because the transformation \((3.6)\) does not act longer.

4 Field Theory on the branes

We will describe in this section the action on the branes \((AdS_5 \times S^5)\) to which the superconformal field theory on the boundary is dual.

The bosonic part of the effective low energy field theory for a \(D3\)-brane brane, in the R-R sector, is given by the Born-Infeld action
\[ S = \int_4 e^{-\phi} \sqrt{det(G + F)} \] (4.15)
where \(G\) is the metric on the brane world-volume (which, in static gauge, includes also six scalar fields) and \(F\) is the gauge field strength. From this action is clear the coupling
\[ C_1 = \frac{1}{4} \int_4 e^{-\phi} F^2. \] (4.16)
The presence of a $D(-1)$-brane will induce a coupling of the gauge field to the axion field $A^{(0)}$, $F^{(1)} = dA^{(0)}$, which arises from a Chern-Simons term

$$C_2 = -\frac{1}{4} \int_4 A^{(0)} F \wedge F. \quad (4.17)$$

The fact that the instanton number $\frac{1}{8\pi^2} \int_4 A^{(0)} F \wedge F$ carries $(p - 4)$-brane charge was observed by Witten [27] for $5-$ and $9-$branes and as a general result by Douglas [26]. Then, the presence of a gauge field with non-trivial instanton number is necessary in order to induce the $D(-1)$-brane charge on a $D3$-brane. Moreover, we know that the $D(-1)$-$D3$ system is a BPS marginal configuration and, for correspondence to the properties of such a configuration, we must require the self-duality condition on this field. Therefore, the $D(-1)$-$D3$ action can be described as that of the $D3$-brane with an extra self dual gauge field [28].

From the relation $F^{(1)} = de^{-\phi}$ between the gravitational field strenght and the dilaton, it follows that the couplings in (4.16) and (4.17) are similar. Then, in the limit where the field theory on the brane decouples from the bulk, the properties of self-duality and non trivial instanton number will remain.

5 $D-$Instantons localized on the boundary and small Instantons

The $AdS/CFT$ correspondence tells us that in the limit where gravity decouples, the theory on the branes should be dual to the super Yang-Mills theory on the boundary of $AdS_5 \times S^5$. Therefore, the system of $D-$Instantons sitting on $D3$-branes should describe $YM$ instantons on the boundary. Here we have a configuration space of Super Yang-Mills gauge fields and, as we have discussed in the preceding section, among them there exist self-dual configuration with non-trivial instanton number; i.e., Yang-Mills instantons. Then, by studying the new gravitational fields introduced by the $D-$Instantons we will try to find out some properties of Yang-Mills instantons. In this sense, both configurations are dual.

In the following we will focus on the first set of supergravity solutions (3.7), (3.8) and (3.11) for one instanton, though some of the arguments could
be extended to the second one. We can see that the solution approaches on the boundary to
\[ e^\phi = \frac{z^4}{((\vec{x} - \vec{x}(0))^2 + z^2)^4}, \quad z \to 0. \]  
(5.18)
This kind of singularities in the dilaton have been related to Yang-Mills instanton in the limit of small scale size \[22\] in the context of heterotic strings. The parameters which describe our D-Instanton solution (5.18) on the boundary are its position given by the four coordinates \( \vec{x}(0) \), and the parameter \( z \) which can be understood as an \( UV \) regularization of the Yang-Mills boundary theory \[21\]. Then, we can identify the position of the \( D- \)Instanton with that of the Yang-Mills instanton on the boundary and the regulator \( z \) with the size of the small instanton \[22\]. Note that we have placed the \( D- \)Instanton at the point where the collection of \( D3 \)-branes were sitting originally \( u_0 = 0 \) and, interpreting \( u \) as a scale of energies, that means a point in the IR. So, we can see that, in the holographic spirit, infrared effects in the bulk theory have been reflected as ultraviolet effects on the boundary theory.

Following with the identifications, the electric R-R charge carried by the \( D- \)Instanton which flow through the throat might represent in the dual picture the instanton number. Working on a regulated boundary \[7\]-\[9\] and from
\[ n \sim Q^{-1} \sim \lim_{z \to 0} \int \frac{d^4x}{z^4} \frac{1}{(x^2 + z^2)^4}, \]  
(5.19)
where we have ignore constant factors, we obtain
\[ Tr\{F \wedge F\} \sim \frac{z^4}{(x^2 + z^2)^4} \]  
(5.20)
which is the correct expression for Yang-Mills instantons of size \( z \). Then, thinking of gauge fields \( A_i \) on \( S^3 \subset S^4 \) we arrive to the known pseudoparticle solution
\[ A_i = \frac{z^2 x^2}{(x^2 + z^2)^{1/2}} g^{-1} \partial_i g, \]  
(5.21)
where \( g = (x_3 - i \vec{x} \vec{\sigma})/(x_i x^i)^{1/2} \) is the imbedding of \( S^3 \) into the group manifold of \( SU(2) \). We see that the size of this instanton shrinks to zero as the regulator parameter \( z \) goes to zero.
We will discuss next the moduli space of $D$–Instantons in order to compare it to that of $SU(2)$ instantons on $S^4$.

We have characterized the $D$–Instanton by the harmonic function $H_{-1}(\vec{x}, z)$ solution of the Laplacian equation (3.5) in $AdS_5 \times S^5$ space. The solutions we have found are invariant under rotation $SO(6)$ in $S^5$ and rotations $SO(5)$ in $S^4$, but there exist a $SO(1, 5)$ (3.6) transformation which transforms it into other. Then, the moduli space of our supergravity solution is

$$\frac{SO(1, 5)}{SO(5)} \times \frac{SO(6)}{SO(6)}$$

(5.22)

which coincides with the moduli space of $SU(2)$ instantons on $S^4$ of instanton number one [29]. Let us note that $AdS_5 = SO(1, 5)/SO(5)$. Then, the moduli space of one instanton solution is $AdS_5$ and coordinates $\vec{x}, z$ in $AdS_5$ are coordinates of instanton moduli space. Therefore, the natural measure on this moduli space is $d\mu = \sqrt{g} dx_0...dx_4 dz$ in $AdS_5$ space. Using the $AdS_5$ metric

$$ds^2 = \frac{1}{z^2}(dz^2 + d\vec{x}^2),$$

(5.23)

the measure on the moduli space can be expressed as

$$d\mu = \frac{d^4x \ dz}{z^5}$$

(5.24)

which is the well known instanton measure [23].

Let us consider now $M$ $D$–Instantons. The moduli space of instanton with charge $M$ and gauge group $SU(N)$ has different components. Each of this components describes how the $M$ instantons have been placed in the $SU(2)$ factors of $SU(N)$. That seems closely related, in the dual picture of $D$–Instantons, to the way in which the $M$ $D(-1)$-branes have been attached to the $N$ $D3$-branes. Then, as an example, the symmetric component of the moduli space of $M$ instantons would correspond to the $M$ $D(-1)$-branes stuck to different $M$ $D3$-branes (symmetrized).
6 Topological defects in two dimensions

Let us describe now another system of branes where the presence of D-Instantons within leads to topological defects in the $d=2$ gauge field on the boundary. We will consider a collection of $N_5$ $D5$-branes with world volume coordinates $x_i, (i = 0, 1, ..., 5)$ wrapping on a compact manifold $M_4$ and $N_1$ $D$-strings parallel to the first collection in $x_0, x_1$ coordinates and smeared in the $M_4$ coordinates. All of them are sitting at the point $x_6 = x_7 = x_8 = x_9 = 0$ of the transverse space. As a BPS bound state, the number of supersymmetries broken by such a state is $1/4$.

The exact string background describing this configuration is represented by the conformal sigma-model with the following metric

$$ds^2 = (H_1 H_5)^{-1/2}(dx_0^2 + dx_1^2) + H_1^{1/2} H_5^{-1/2}(dx_2^2 + ... + dx_5^2)$$

(6.25)

where the harmonic functions $H_1$ and $H_5$, depending only on the transverse radial coordinate $r = \sqrt{x_0^2 + ... + x_5^2}$, are

$$H_1 = 1 + \frac{g \alpha'}{v r^2}, \quad H_5 = 1 + \frac{g \alpha'}{r^2},$$

(6.26)

non-trivial seven and three strength field forms (2.6) and dilaton field

$$e^{2 \phi} = H_1 / H_5.$$  

(6.27)

In the decoupling limit [4]

$$\alpha' \to 0, \quad u = \frac{r}{\alpha'} = \text{fixed}, \quad v = \frac{V_4}{(2\pi)^4 \alpha'^2} = \text{fixed}, \quad g_6 = \frac{g}{\sqrt{v}}$$

(6.28)

with $V_4$ being the $M_4$ volume, we find the low energy metric

$$ds^2 = \alpha' g_6 \sqrt{N_1 N_5} \left[ u^2(dx_0^2 + dx_1^2) + \frac{du^2}{u^2} + d\Omega_3^2 + \beta(N_1, N_5)(dx_2^2 + ... + dx_5^2) \right],$$

(6.29)

where we have used the rescaling of section 2 with $\lambda^4 = g_6^2 N_1 N_5$, $d\Omega_3^2$ is the metric in the unit three sphere and $\beta(N_1, N_5) = (\alpha' g_6 N_5 v^{1/2})^{-1}$. This metric
describes the $AdS_3 \times S^3 \times M_4$ space, where the $M_4$ factor is a compact hiperkäler manifold ( $T^4$ or $K^3$ ) depending on the charges of the branes. Note that in this limit the dilaton field is a constant $e^{2\phi} = N_1/(N_5 v)$. Any of our further results will not depend on the moduli space $M_4$ or on constant factors of (3.29), then we will consider the seven dimensional metric, wich in terms of the inverse variable $z = 1/u$ reads

$$ds^2 = \left[ \frac{1}{z^2}(dx_0^2 + dx_1^2 + dz^2) + d\Omega_3^2 \right].$$  

(6.30)

In this representation, the $AdS_3 \times S^3$ space is the upper space $z > 0$, its supersymmetric group is $SO(1,3) \times SO(4)$ and the conformal compactified boundary consist of the plane $R^2$ at $z = 0$ and a point at infinity. The AdS/CFT picture tell us that the type $IIB$ string theory on $AdS_3 \times S^3$ in the limit of large $N$ is dual to the $N = 2$ superconformal Yang-Mills theory on the boundary.

Now, as in section 3, we can add $D$-Instantons to the system of branes. This new collection of branes breaks $1/2$ of the supersymmetries. In this case the $p - (p + 2)$ BPS system of branes is truly bound. The supergravity solution for the system is

$$ds^2 = H^{-1/2} \left[ \frac{1}{z^2}(dx_0^2 + dx_1^2 + dz^2) + d\Omega_3^2 \right],$$  

(6.31)

with $H_{-1}$ satisfying the laplace condiction

$$\left[ \Delta_{\parallel} + z \frac{\partial}{\partial z} \frac{1}{z} \frac{\partial}{\partial z} \right] H_{-1}(\vec{x}, z) = 0,$$  

(6.32)

where $\Delta_{\parallel}$ represents the laplacian in the plane. We have also the $F^{(1)}$ strength given in (3.2) and the dilaton field of (3.3) which leaves the metric in the Einstein frame invariant.

Following the arguments of section 3, we find the solution

$$e^\phi = c \frac{z^2}{((\vec{x} - \vec{x}(0))^2 + z^2)^2},$$  

(6.33)

invariant under $SO(3) \times SO(4)$. Its transformed function under (3.6) is the solution singular at infinity

$$e^\phi = cz^2.$$  

(6.34)
In this case, the $D-$Instanton connects two $AdS_3 \times S^3$ spaces by a throat located at $(\vec{x}(0), z = 0)$ and at infinity. The charge which flows through the throat is given by

$$Q^{-1} = \frac{1}{Vol(\Sigma)Vol(S^3)} \int_{AdS_3 \times S^3} *d*d(e^\phi)$$

$$= \frac{c}{Vol(\Sigma)} \int d^2x \int_0^\infty dz \frac{\partial_x \partial_z}{(x^2 + z^2)^2} \frac{z^2}{2}$$

$$= \frac{c}{2Vol(\Sigma)} \lim_{z \to 0} \int d^2x \frac{1}{z^2} \frac{z^2}{(x^2 + z^2)^2} = \frac{c}{2}.$$

(6.35)

With regard to the field theory on the branes, the presence of $D(-1)$-branes induces a non trivial monopole (or two dimensional instanton) number

$$-\frac{1}{2} \int_2 A^{(0)} F$$

(6.36)

which couples to the axion field.

Now, after having described the supergravity solution, we are able to repeat the discussion of section 5 in order to indentify the solution on the boundary

$$e^\phi = \frac{z^2}{((\vec{x} - \vec{x}(0))^2 + z^2)^2}, \quad z \to 0.$$

(6.37)

with two dimensional gauge instantons.

Here we find again the same kind of singularities in the dilaton field related to small instantons. The parameter $z$ plays the role of an $UV$ regulator for the instanton size. This is another example of the $IR-UV$ connection. The infrared regulator $1/z_0 = u_0 = 0$ we have used to place the $D(-1)$-brane in the bulk has been translated to an $UV$ regulator on the boundary.

The correct expression for the two dimensional instantons can be obtained from the identification of the monopole number with the charge of the $D-$Instanton

$$\int d^2x F \sim \lim_{z \to 0} \int d^2x \frac{1}{z^2} \frac{z^2}{(x^2 + z^2)^2}$$

(6.38)

$^3$ A detailed discussion about gauge field description of $2d$ instantons can be found in Polyakov’s book [23] or in NSVZ’s review [31].
which leads to the two dimensional gauge field solution

\[ A_{\bar{w}} = \frac{z^2 |w|^2}{(|w|^2 + z^2)^2} g^{-1} \partial_{\bar{w}} g, \quad A_w = -\bar{A}_{\bar{w}}, \quad (6.39) \]

\( w, \bar{w} \) being complex coordinates on the plane and

\[ g = \frac{1}{1 + |w|^2} \begin{pmatrix} 1 & -w \\ \bar{w} & 1 \end{pmatrix} \quad (6.40) \]

the embedding of the plane into the group manifold of \( SU(2) \).

Finally, we observe that the moduli space of one \( D \)-Instanton in \( AdS_3 \times S^3 \), characterized by the harmonic function \( H_{-1} \),

\[ \frac{SO(1,3)}{SO(3)} \times \frac{SO(4)}{SO(4)} \quad (6.41) \]

agrees with that of the two dimensional instantons on \( S^2 \) of 2\( d \) instanton number one. Again, the coordinates of the \( AdS \) space \( AdS_3 = SO(1,3)/SO(3) \) are the coordinates of the instanton moduli space. Then, from the \( AdS_3 \) metric

\[ ds^2 = \frac{1}{z^2} (dz^2 + d\bar{x}_0^2 + dx_1^2), \quad (6.42) \]

we obtain the measure on the instanton moduli space

\[ d\mu = \frac{d^2x \; dz}{z^3}. \quad (6.43) \]

It is easy to see that this measure is a right 2\( d \) instanton measure by analyzing the simplest example of instanton in \( CP(O(3)) \) model \[23\]. This instanton is given by analytic functions \( \frac{a}{z-b} \) and its moduli space by the two complex coordinates \( a \) and \( b \). The one-instanton contribution is required to be dimensionless and, by translation invariance, to depend on \( |a - b| \). Therefore, it must be

\[ d\mu = \frac{d^2a \; d^2b}{|a - b|^4} = \frac{d^2x \; d\rho}{\rho^3} \quad (6.44) \]

and, as \( \rho = |a - b| \) in this model is the size of instanton \( z \), this measure coincides with \(6.43\).

In case we consider \( M \) instantons the components of the moduli space of the 2\( d \) instantons might be related to the different \( D(-1) - D1 \) bound states.
7 Conclusion

In this paper we have analyzed the effects of $D$-Instantons on the boundary of $AdS$ spaces. Though they hardly change the properties of the space, throats joining different vacua are developed in their presence. It has been also remarked that the existence of $D$-Instantons does not disturb the metric in the Einstein frame.

As predicted by the AdS/CFT correspondence, we have shown that these $D$-Instantons behave as Yang-Mills instantons, in case of $D3$-branes, and $2d$ instantons, in case of $(D1 - D5)$-branes, in the dual picture on the boundary and we have given exact expressions for the corresponding gauge instantons in four and two dimensions.

We have also studied the moduli space of the supergravity solutions finding a total correspondence to the moduli space of gauge instantons in the case of instanton number one and we have shown that the natural measure on $AdS$ spaces is exactly the measure of the partition function in instantonic backgrounds. We have also discussed possible conjectures about the multiinstanton measure.

It would be interesting to go further on this subject. In particular, a better understanding of the moduli space of $M$ $D$-Instantons and its correspondence to the $ADHM$ description of Yang-Mills instantons. Another direction for future research could be the calculation of expectation values in instantonic backgrounds by using branes technology.

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References

[1] J. Polchinski, [hep-th/9611050]
[2] G. Gibbons, Nucl. Phys. B207 (1982); R. Kallosh and A. Peet, Phys. Rev. D46 (1992) 5223; S. Ferrara, G. Gibbons and R. Kallosh, Nucl. Phys. B500 (1997); G. Gibbons and P. Townsend, Phys. Rev. Lett. 71 (1993) 5223.

[3] I.R. Klebanov, Nucl. Phys. B496 (1997) 231; S.S. Gubser, I.R. Klebanov and A. A. Tseytlin, Nucl. Phys. B499 (1997) 217; S.S. Gubser and I.R. Klebanov, Phys. Lett. B413 (1997) 41.

[4] J. Maldacena, hep-th/9711200.

[5] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, hep-th 9802109.

[6] E. Witten, hep-th/9802150.

[7] J. Maldacena, hep-th/9803002.

[8] S.J. Rey and J. Yee, hep-th/9803001.

[9] E. Witten, hep-th/9805112.

[10] D.J. Gross and H. Ooguri, hep-th/9805129.

[11] G. Papadopoulos and P. Townsend, Phys. Lett. B380 (1996) 273; A.A. Tseytlin, Nucl. Phys. B475 (1996) 149; A.A. Tseytlin, Class. Quant. Grav. 14 (1997) 2085.

[12] N. Itzhaki, A.A. Tseytlin and S. Yankielowicz, hep-th/9803103.

[13] C.S. Chu, P.M. Ho and Y.Y. Wu, hep-th/9806103.

[14] G. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197.

[15] E. Bergshoeff and M. de Roo, Phys. Lett. B380 (1996) 265.

[16] R. Haag, J. Lopuszanski and M. Sohnius, Nucl. Phys. B88 (1985) 257.

[17] C.G. Callan and J. Maldacena, hep-th/9708147.

[18] R. Jackiw and C. Rebbi, Phys. Rew. D14 (1976) 517.

[19] G.W. Gibbons, M.B. Green and M.J. Perry, Phys. Lett. 370B (1996) 37.
[20] E. Bergshoeff and K. Behrndt, hep-th/9803090.
[21] L. Susskind and E. Witten, hep-th/9805114.
[22] C. G. Callan, J. Harvey and A. Strominger, Nucl. Phys. 367(1991) 60
[23] A.M. Polyakov in *Gauge Fields and Strings*, Harwood Academic Publisher, 1987.
[24] R.I. Nepomechie, Phys. Rev. D31(1985) 1921.
[25] C. Teitelboim, Phys. Lett. B167(1986) 63; Phys. Lett. 167B(1986) 69.
[26] M.R. Douglas, hep-th/9512077.
[27] E. Witten, Nucl. Phys. B460(1996) 541.
[28] I. Chepelev and A.A. Tseytlin, hep-th/9704127.
[29] R. Jackiw, C. Nohl and C. Rebbi, Phys. Rew. D 1642.
[30] G. Horowitz, J. Maldacena and A. Strominger, Phys. Lett. B383(1996) 151.
[31] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Phys. Rep. 116(1984) 103.