Asymptotic expansion of 2-dimensional gradient graph with vanishing mean curvature at infinity

Zixiao Liu and Jiguang Bao*

February 14, 2022

Abstract

In this paper, we establish the asymptotic expansion at infinity of gradient graph in dimension 2 with vanishing mean curvature at infinity. This corresponds to our previous results in higher dimensions and generalizes the results for minimal gradient graph on exterior domain in dimension 2. Different from the strategies for higher dimensions, instead of the equivalence of Green’s function on unbounded domains, we apply a version of iteration methods from Bao–Li–Zhang [Calc. Var. PDE, 52(2015), pp. 39-63] that is refined by spherical harmonic expansions to provide a more explicit asymptotic behavior than known results.

Keywords: Monge–Ampère equation, Mean curvature equation, Asymptotic behavior, Spherical harmonic expansion.

MSC 2020: 35J60, 35C20, 35B20.

1 Introduction

We consider the asymptotic expansion at infinity of solutions to a family of mean curvature equations of gradient graph in dimension 2.

Let \((x, Du(x))\) denote the gradient graph of \(u\) in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\), where \(Du\) denotes the gradient of scalar function \(u\) and

\[ g_\tau = \sin \tau \delta_0 + \cos \tau g_0, \quad \tau \in \left[0, \frac{\pi}{2}\right] \]

is the linearly combined metric of standard Euclidean metric

\[ \delta_0 = \sum_{i=1}^{n} dx_i \otimes dx_i + \sum_{j=1}^{n} dy_j \otimes dy_j, \]

with the pseudo-Euclidean metric

\[ g_0 = \sum_{i=1}^{n} dx_i \otimes dy_i + \sum_{j=1}^{n} dy_j \otimes dx_j. \]

*Supported by the National Key Research and Development Program of China (No. 2020YFA0712904) and National Natural Science Foundation of China (No. 11871102 and No. 11631002).
As proved in [37], if \( u \in C^2(\mathbb{R}^n) \) is a classical solution of
\[
F_\tau(\lambda(D^2u)) = f(x),
\]
then \( Df(x) \) is the mean curvature of gradient graph \((x, Du(x))\) in \((\mathbb{R}^n \times \mathbb{R}^n, g_\tau)\). In (1.1), \( f(x) \) is a sufficiently regular function, \( \lambda(D^2u) = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) is the vector formed by \( n \) eigenvalues of Hessian matrix \( D^2u \) and
\[
F_\tau(\lambda) := \begin{cases}
\frac{1}{n} \sum_{i=1}^{n} \ln \lambda_i, & \tau = 0, \\
\frac{\sqrt{a^2+1}}{2b} \sum_{i=1}^{n} \ln \frac{\lambda_i + a - b}{\lambda_i + a + b}, & 0 < \tau < \frac{\pi}{4}, \\
-\sqrt{2} \sum_{i=1}^{n} \frac{1}{1 + \lambda_i}, & \tau = \frac{\pi}{4}, \\
\frac{\sqrt{a^2+1}}{b} \sum_{i=1}^{n} \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
\sum_{i=1}^{n} \arctan \lambda_i, & \tau = \frac{\pi}{2},
\end{cases}
\]
\[a = \cot \tau, b = \sqrt{|\cot^2 \tau - 1|}.
\]

When \( \tau = 0 \), Eq. (1.1) becomes the Monge–Ampère type equation
\[
\det D^2u = e^{nf(x)}.
\]
When \( \tau = \frac{\pi}{4} \), Eq. (1.1) can be translated into the inverse harmonic Hessian equation
\[
-\sqrt{2} \sum_{i=1}^{n} \frac{1}{\lambda_i} = f(x),
\]
which is a special form of the quadratic Hessian equations \( \sigma_k(\lambda) = f(x) \), where \( \sigma_k(\lambda) \) with \( k = 1, 2, \cdots, n \) denotes the \( k \)-th elementary symmetric function of \( \lambda(D^2u) \).

When \( \tau = \frac{\pi}{2} \), Eq. (1.1) becomes the Lagrangian mean curvature type equation
\[
\sum_{i=1}^{n} \arctan \lambda_i(D^2u) = f(x).
\]
(1.2)

Especially when \( f(x) \) is a constant, the Lagrangian mean curvature equation above is also known as the special Lagrangian equation.

For \( f(x) \) being a constant \( C_0 \), Warren [39] proved Bernstein-type results of (1.1) based on the results of Jörgens [26]–Calabi [11]–Pogorelov [35], Flanders [17] and Yuan [43, 44], which state that any classical solution with under suitable semi-convex conditions must be a quadratic. Especially when \( \tau = 0 \), there are different proofs and extensions of the Bernstein-type results of Monge–Ampère equations by Cheng–Yau [15], Caffarelli [6], Jost–Xin [27], Fu [18], Li–Xu–Simon–Jia [29], etc. When \( \tau = \frac{\pi}{4} \), there are Bernstein-type results of Hessian quotient equations
by Bao–Chen–Guan–Ji [1]. For generalizations of Bernstein-type results of Hessian and Hessian quotient equations, we refer to Chang–Yuan [12], Chen–Xiang [14], Li–Ren–Wang [31], Yuan [43], Du [16] etc. When \( \tau = \frac{\pi}{2} \), though the mean curvature relies only on \( Df \), Yuan [44] reveals the importance of the value of phase \( C_0 \). The phase (also known as the Lagrangian angle) \( \frac{\pi-2}{2} \) is called critical since the level set

\[
\left\{ \lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \arctan \lambda_i = C_0 \right\}
\]

is convex only when \( |C_0| \geq \frac{n-2}{2} \pi \). Another way to obtain a convexity/concavity structure is to restrict in the range of \( D^2 u > 0 \) as in Yuan [43]. For further relevant discussions, we refer to Warren–Yuan [40, 41], Wang–Yuan [38], Chen–Shankar–Yuan [13], Li–Li–Yuan [30], Bhattacharya–Shankar [4, 5], Bhattacharya [3] and the references therein.

For \( f(x) - C_0 \) having compact support and \( n \geq 3 \), there are exterior Bernstein type results of (1.1) by the authors [34], which state that any classical solution with the same semi-convex conditions must be asymptotic to quadratic polynomial at infinity, together with higher order expansions that give the precise gap between exterior minimal gradient graph and the entire case. For \( f(x) - C_0 \) having compact support and \( n = 2 \), the authors [32] proved similar exterior Bernstein type results of (1.1), which imply that exterior solutions are asymptotic to quadratic polynomial with additional \( \ln \)-term at infinity. When \( \tau = 0 \), such results were partially proved earlier for Monge–Ampère equations by Caffarelli–Li [9] and Hong [23]. When \( \tau = \frac{\pi}{2} \), such results were partially proved earlier for special Lagrangian equations by Li–Li–Yuan [30]. The refined asymptotic expansions in our earlier results [34, 32] are new even for \( \tau = 0 \) and \( \frac{\pi}{2} \) cases, it reveals that the gap between exterior minimal gradient graph and the entire case can be written into higher order errors.

For \( f(x) - C_0 \) vanishing at infinity, \( n \geq 3 \) and \( \tau \in [0, \frac{\pi}{4}] \), there are exterior Bernstein-type results of (1.1) by the authors [33], which provide both the asymptotic behavior and finer expansions of error terms. Especially when \( \tau = 0 \), the asymptotic behavior result of solutions to Monge–Ampère type equations were proved under stronger assumptions on \( f(x) \) by Bao–Li–Zhang [2].

Such asymptotic expansion for solutions of geometric curvature equations were earlier introduced in Han–Li–Teixeira [21], which refines the previous study on the Yamabe equation and the \( \sigma_k \)-Yamabe equations by Caffarelli–Gidas–Spruck [8], Korevaar–Mazzeo–Pacard–Schoen [28], Han–Li–Teixeira [22], etc. We would also like to mention that for the Monge–Ampère type equations, there are also classification results and asymptotic behavior analysis for \( f(x) - C_0 \) being a periodic function by Caffarelli–Li [10] or asymptotically periodic at infinity by Teixeira–Zhang [36], etc. Under additional assumptions on \( D^2 u \) at infinity, the asymptotic behavior results were obtained for general fully nonlinear elliptic equations by Jia [25].

In this paper, we consider the asymptotic behavior and further expansions or error terms at infinity of solutions to (1.1) with \( f(x) - C_0 \) vanishing at infinity and \( n = 2 \). Some special structures in \( n = 2 \) case enable us to deal with all \( \tau \in [0, \frac{\pi}{2}] \), which is different from \( n \geq 3 \) case in [33]. However, there are also disadvantages caused by \( n = 2 \), especially the lack of equivalence on the Green’s function on unbounded domain. The asymptotic behavior obtained here is a refinement of known results of the Monge–Ampère equations by Bao–Li–Zhang [2].
Consider classical solutions of
\[ F_\tau (\lambda (D^2 u)) = f(x) \text{ in } \mathbb{R}^2, \tag{1.3} \]
where \( f(x) \in C^m(\mathbb{R}^2) \) converge to some constant \( f(\infty) \) in the sense of
\[ \limsup_{|x| \to \infty} |x|^{\zeta + k} |D^k (f(x) - f(\infty))| < \infty, \quad \forall \ k = 0, 1, 2, \cdots, m \tag{1.4} \]
for some \( \zeta > 2 \) and \( m \geq 3 \).

From the definition of \( F_\tau \) operator, \( \lambda_i \) must satisfy
\[
\begin{cases} 
\lambda_i > 0, & \text{for } \tau = 0, \\
\frac{\lambda_i + a - b}{\lambda_i + a + b} > 0, & \text{for } \tau \in (0, \frac{\pi}{4}), \\
\lambda_i \neq -1, & \text{for } \tau = \frac{\pi}{4}, \\
\lambda_i + a + b \neq 0, & \text{for } \tau \in (\frac{\pi}{4}, \frac{\pi}{2}),
\end{cases}
\tag{1.5}
\]
for \( i = 1, 2 \). Thus we separate the solution into semi-convex and semi-concave cases. For simplicity, we consider the semi-convex case
\[
A > \begin{cases} 
0, & \tau = 0, \\
-(a - b)I, & \tau \in (0, \frac{\pi}{4}), \\
-I, & \tau = \frac{\pi}{4}, \\
-(a + b)I, & \tau \in (\frac{\pi}{4}, \frac{\pi}{2}), \\
-\infty, & \tau = \frac{\pi}{2},
\end{cases}
\tag{1.6}
\]
where \( I \) denotes the 2-by-2 identity matrix and the semi-concave case can be treated similarly.

Hereinafter, we assume \( D^2 u \) satisfy (1.5) in \( \mathbb{R}^2 \),
\[ f(\infty) \neq 0 \quad \text{for } \tau = \frac{\pi}{2} \quad \text{and} \quad f(\infty) \notin \partial \{ F_\tau (\lambda(A)) \mid A \text{ satisfies } (1.5) \}, \tag{1.6}\]
where the notation \( \partial \) denote the boundary of a set in \( \mathbb{R} \). We may assume further without loss of generality that \( f(\infty) > 0 \) for \( \tau = \frac{\pi}{2} \) case, otherwise consider \(-u\) instead.

**Remark 1.1.** In condition (1.6), our major additional restriction is \( f(\infty) \neq 0 \) for \( \tau = \frac{\pi}{2} \). It corresponds to the critical phase in \( \mathbb{R}^2 \), which leads to a different phenomenon than supercritical case.

If \( \tau \in (0, \frac{\pi}{4}] \) and \( f(\infty) = \sup \{ F_\tau (A) \mid A \text{ satisfies } (1.5) \} = 0 \), then by the structure of \( F_\tau (\lambda) \), we have
\[
\lambda_1 (D^2 u(x)), \lambda_2 (D^2 u(x)) \to +\infty \quad \text{as } |x| \to \infty.
\]
These are not the asymptotic behavior under discussion and hence we rule out these situations by (1.6).

Let \( \text{Sym}(2) \) denote the set of 2-by-2 symmetric matrix and \( x^T \) denote the transpose of a vector \( x \in \mathbb{R}^2 \). We say a scalar function \( \varphi = O_l (|x|^{-k_1} (\ln |x|)^{k_2}) \) with \( l \in \mathbb{N}, k_1, k_2 \geq 0 \) if it satisfies
\[
|D^k \varphi| = O \left( |x|^{-k_1-k} (\ln |x|)^{k_2} \right) \quad \text{as } |x| \to \infty
\]
for all \( k = 0, 1, 2, \cdots, l \).

Our main result shows the following asymptotic behavior and expansion result at infinity.
\textbf{Theorem 1.2.} Let $u \in C^2(\mathbb{R}^2)$ be a classical solution of (1.3) with $D^2u$ satisfying (1.5) and $f \in C^m(\mathbb{R}^2)$ satisfy (1.4), (1.6) for some $\zeta > 2$ and $m \geq 3$. Assume further that
\begin{equation}
 u(x) \leq C (1 + |x|^2) \quad \text{in} \quad \mathbb{R}^2
\end{equation}
for some $C > 0$. Then there exist $A \in \text{Sym}(2)$ satisfying $F_r(\lambda(A)) = f(\infty)$ and (1.5), $b \in \mathbb{R}^2, c, d \in \mathbb{R}$ such that
\begin{equation}
 u(x) - \left( \frac{1}{2} x^T Ax + bx + c \right) - d \ln (x^T Px)
 = \begin{cases} 
 O_{m+1}(|x|^{2-\min(\zeta,3)}) , & \text{if } \zeta \neq 3, \\
 O_{m+1}(|x|^{-1}(\ln |x|)) , & \text{if } \zeta = 3,
\end{cases}
\end{equation}
as $|x| \to \infty$, where the matrix $P$ is given by
\begin{equation}
 P = (DF_r(\lambda(A)))^{-1} = \frac{1}{2} \left( \sin \tau A^2 + 2 \cos \tau A + \sin \tau I \right). 
\end{equation}
Furthermore, when $\zeta > 3$, there also exist $d_1, d_2 \in \mathbb{R}$ such that
\begin{equation}
 u(x) - \left( \frac{1}{2} x^T Ax + \beta x + c \right) - d \ln (x^T Px)
 = (x^T Px)^{-\frac{1}{2}} (d_1 \cos \theta + d_2 \sin \theta) + \begin{cases} 
 O_m(|x|^{2-\zeta}) , & \text{if } \zeta < 4, \\
 O_m(|x|^{-2}(\ln |x|)) , & \text{if } \zeta \geq 4,
\end{cases}
\end{equation}
as $|x| \to \infty$, where $\theta = \frac{p\frac{1}{2}x}{(x^T Px)^{\frac{1}{2}}}$.

\textbf{Remark 1.3.} For $\tau = 0$ case in Theorem 1.2 condition (1.7) is not necessary.

\textbf{Remark 1.4.} Theorem 1.2 generalizes the asymptotic expansion results of previous work [32] by the authors, where $f(x)$ being a constant since it corresponds to $\zeta = \infty$ and $m = \infty$ case in (1.8) and (1.10). But there are many differences in argumentation methods. By differentiating the equations we only obtain nonhomogeneous elliptic equations and inequalities on exterior domain. Furthermore, when $f(x) = \frac{b}{\sqrt{a^2+1}} f(\infty) + \frac{\pi}{2} \equiv 0$ and $\tau \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right)$, the equation can be translated into harmonic equations. But only if $f(x) \equiv 0$, it yields an additional perturbation term involving the second order derivatives of $u$. This leads to the difficult discussions as in (3.5). For a similar reason, $\tau = \frac{\pi}{4}$ case cannot be deduced into the Monge–Ampère equation $\det D^2v = 1$ or harmonic equations by a simple change of variable as in [32]. For these perturbed cases, we turn to study the algebraic form as in (3.6) and apply iteration methods instead of using the asymptotic behavior of solutions to the Monge–Ampère equations directly.

\textbf{Remark 1.5.} As in the discussions in [2][34][33] etc., $\zeta > 2$ in (1.4) is optimal in the sense that for $\zeta = 2$ we may construct radially symmetric solutions with $u = \frac{1}{2} x^T Ax + O((\ln |x|)^2)$ as $|x| \to \infty$. Furthermore, the asymptotic expansion (1.10) is optimal in the sense that the next order term in (1.10) may contain error terms like $|x|^{-2}\ln |x|$, which cannot be represented into $(x^T Px)^{-1}(d_3 \cos 2\theta + d_4 \sin 2\theta)$ for some $d_3, d_4 \in \mathbb{R}$.
Remark 1.6. By extension results as in Theorem 3.2 of [42], we may change the value of \( u \) and \( f \) on a dense subset without affecting the asymptotic behavior near infinity. Consequently by interior estimates as in Lemma 17.16 of [19], the regularity assumption on \( f \) can be relaxed to \( f \in C^0(\mathbb{R}^2) \) with \( D^m f \) exists outside a compact subset of \( \mathbb{R}^2 \) for some \( m \geq 3 \). Especially since \( m \geq 3 \), we may assume without loss of generality that \( u \in C^4(\mathbb{R}^2) \).

The paper is organized as follows. In section 2 we prove existence results for Poisson equations on exterior domain of \( \mathbb{R}^2 \). In section 3 we prove that \( u \) converge to a quadratic function \( \frac{1}{2}x^T Ax \) at infinity with a speed of \( O(|x|^{2-\epsilon}) \) for some \( \epsilon > 0 \), which is similar to the strategy used in [9, 2] etc. In section 4 we prove Theorem 1.2 by iteration and spherical harmonic decomposition, based on the results in sections 2 and 3.

2 Preliminary results on Poisson equations

In this section, we prove an existence result for Poisson equation on exterior domain.

Lemma 2.1. Let \( g \in C^\infty(\mathbb{R}^2) \) satisfy

\[
\|g(r\cdot)\|_{L^p(S^1)} \leq c_0 r^{-k_1} (\ln r)^{k_2} \quad \forall \ r > 1
\]  

(2.1)

for some \( c_0 > 0, k_1 > 0, k_2 \geq 0 \) and \( p \geq 2 \). Then there exists a smooth solution \( v \) of

\[
\Delta v = g \quad \text{in} \ \mathbb{R}^2 \setminus \overline{B_1}
\]  

(2.2)

such that

\[
|v(x)| \leq \begin{cases} 
C_0 |x|^{2-k_1} (\ln |x|)^{k_2}, & \text{if } k_1 \not\in \mathbb{N}_*, \\
C_0 |x|^{2-k_1} (\ln |x|)^{k_2+1}, & \text{if } k_1 \in \mathbb{N}_* \setminus \{2\}, \\
C_0 (\ln |x|)^{k_2+2}, & \text{if } k_1 = 2,
\end{cases}
\]  

(2.3)

for \( |x| > 1 \) for some \( C > 0 \).

For \( k_1 > 2 \) case, Lemma 2.1 is similar to the one proved earlier by the authors in [32]. The proof here is similar with minor modifications.

Proof of Lemma 2.1. Here we only provide detail proof for \( 0 < k_1 < 1 \) case, the rest parts follow with minor modifications on the choice of \( a_{k,m} \) and \( \epsilon > 0 \) below as in (2.7).

In polar coordinate we have

\[
\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2},
\]

where \( r := |x| \) represents the radial distance and \( \theta := \frac{x}{|x|} \) the angle. Let

\[
Y_1^{(0)}(\theta) \equiv \frac{1}{\sqrt{2\pi}}, \quad Y_1^{(k)}(\theta) = \frac{1}{\sqrt{\pi}} \cos k\theta \quad \text{and} \quad Y_2^{(k)}(\theta) = \frac{1}{\sqrt{\pi}} \sin k\theta,
\]

which forms a complete standard orthogonal basis of \( L^2(S^1) \). Decompose \( g \) and the wanted solution \( v \) into
\[ v(x) = a_{0,1}(r) + \sum_{k=1}^{+\infty} \sum_{m=1}^{2} a_{k,m}(r) Y_m^{(k)}(\theta), \quad g(x) = b_{0,1}(r) + \sum_{k=1}^{+\infty} \sum_{m=1}^{2} b_{k,m}(r) Y_m^{(k)}(\theta), \]

where

\[ a_{k,m}(r) := \int_{S^{n-1}} v(r \theta) \cdot Y_m^{(k)}(\theta) d\theta, \quad b_{k,m}(r) := \int_{S^{n-1}} g(r \theta) \cdot Y_m^{(k)}(\theta) d\theta. \]

By the linear independence of \( Y_m^{(k)}(\theta) \), (2.2) implies that

\[ a_{0,1}'(r) + \frac{1}{r} a_{0,1}'(r) = b_{0,1}(r) \quad \text{in } r > 1 \]

and for all \( k \in \mathbb{N}_+ \) with \( m = 1, 2 \),

\[ a_{k,m}''(r) + \frac{1}{r} a_{k,m}'(r) - \frac{k^2}{r^2} a_{k,m}(r) = b_{k,m}(r) \quad \text{in } r > 1. \]

By solving the ODE, there exist constants \( C_{k,m}^{(1)}, C_{k,m}^{(2)} \) such that for all \( r > 1 \),

\[ a_{k,m}(r) = C_{k,m}^{(1)} r^k + C_{k,m}^{(2)} r^{-k} \]

\[ -\frac{1}{2k} r^k \int_2^r \tau^{1-k} b_{k,m}(\tau) d\tau + \frac{1}{2k} r^{-k} \int_2^r \tau^{1+k} b_{k,m}(\tau) d\tau \quad (2.4) \]

for \( k \geq 1 \) and

\[ a_{0,1}(r) = C_{0,1}^{(1)} + C_{0,1}^{(2)} \ln r \]

\[ -\int_2^r \tau \ln \tau b_{0,1}(\tau) d\tau + \ln r \int_2^r \tau b_{0,1}(\tau) d\tau. \]

By (2.1),

\[ |b_{0,1}(r)|^2 + \sum_{k=1}^{+\infty} \sum_{m=1}^{2} |b_{k,m}(r)|^2 = ||g(r)||_{L^2(S^{n-1})}^2 \leq c_0^2 (2\pi)^{\frac{p-2}{p}} r^{p-2k} (\ln r)^{2k^2} \quad (2.5) \]

for all \( r > 1 \). Then by \( 0 < k_1 < 1 \), we have \( r^{1-k} b_{k,m}(r) \in L^1(2, +\infty) \) for all \( k \geq 2 \) and \( r^{k+1} b_{k,m}(r) \not\in L^1(2, +\infty) \) for all \( k \in \mathbb{N} \). We choose \( C_{k,m}^{(1)} \) and \( C_{k,m}^{(2)} \) in (2.4) such that

\[ a_{0,1}(r) := -\int_2^r \tau \ln \tau b_{0,1}(\tau) d\tau + \ln r \int_2^r \tau b_{0,1}(\tau) d\tau, \]

\[ a_{k,m}(r) := -\frac{1}{2} \int_2^r b_{k,m}(\tau) d\tau + \frac{1}{2} r^{-1} \int_2^r \tau^2 b_{k,m}(\tau) d\tau \]

for \( k = 1 \) and

\[ a_{k,m}(r) := -\frac{1}{2k} r^k \int_{+\infty}^r \tau^{1-k} b_{k,m}(\tau) d\tau + \frac{1}{2k} r^{-k} \int_2^r \tau^{1+k} b_{k,m}(\tau) d\tau \]
for all $k \geq 2$.

For $a_{0,1}(r)$, we notice that there are cancellation properties as below. By (2.5) we have

$$|a_{0,1}(r)| = \left| \int_{2}^{r} \ln \frac{r}{\tau} b_{0,1}(\tau) d\tau \right| \leq Cc_{0} \int_{2}^{r} \ln \frac{r}{\tau} (\ln \tau)^{k_{2}} d\tau \leq Cc_{0} \left( \ln r \int_{2}^{r} (\ln \tau)^{k_{2}} d\tau - \int_{2}^{r} (\ln \tau)^{k_{2}+1} d\tau \right). \tag{2.6}$$

By a direct computation, for any $0 < k_{1} < 2$ we have

$$\int_{2}^{r} (\ln \tau)^{k_{2}+1} d\tau = \frac{1}{2 - k_{1}} \left( r^{2-k_{1}} (\ln r)^{k_{2}+1} - 2^{2-k_{1}} (\ln 2)^{k_{2}+1} - (k_{2} + 1) \int_{2}^{r} (\ln \tau)^{k_{2}} d\tau \right).$$

Consequently

$$\ln r \int_{2}^{r} (\ln \tau)^{k_{2}} d\tau - \int_{2}^{r} (\ln \tau)^{k_{2}+1} d\tau = \frac{1}{2 - k_{1}} \left( r^{2-k_{1}} (\ln r)^{k_{2}+1} - 2^{2-k_{1}} (\ln 2)^{k_{2}+1} - (k_{2} + 1) \int_{2}^{r} (\ln \tau)^{k_{2}} d\tau \right) - \frac{1}{2 - k_{1}} \left( r^{2-k_{1}} (\ln r)^{k_{2}+1} - 2^{2-k_{1}} (\ln 2)^{k_{2}+1} - (k_{2} + 1) \int_{2}^{r} (\ln \tau)^{k_{2}} d\tau \right) \leq \frac{k_{2} + 1}{2 - k_{1}} \left( 2^{2-k_{1}} (\ln 2)^{k_{2}+1} - (k_{2} + 1) \int_{2}^{r} (\ln \tau)^{k_{2}} d\tau \right) \leq C \int_{1}^{\infty} (\ln \tau)^{k_{2}} d\tau$$

for some $C > 0$ for all $r > 2$. This yields

$$|a_{0,1}(r)| \leq Cc_{0} r^{2-k_{1}} (\ln r)^{k_{2}}, \quad \forall r > 2.$$ 

Similar argument also holds for $k_{1} > 2$ case with the integrate range changed from $(2, r)$ into $(r, +\infty)$. But for $k_{1} = 2$, from (2.6) we have no cancellation and it yields

$$|a_{0,1}(r)| \leq Cc_{0} r^{2-k_{1}} (\ln r)^{k_{2}+1}, \quad \forall r > 2.$$ 

For $0 < k_{1} < 1$ case, we pick $0 < \epsilon := \frac{1}{2} \min\{1, 2 - 2k_{1}, 2k_{1}\}$ such that

$$\begin{cases} 
3 - 2k_{1} - \epsilon > -1, \\
1 - 2k_{1} - \epsilon > -1, \\
3 - 2k - 2k_{1} + \epsilon < -1, \quad \forall k \geq 2.
\end{cases}$$

Then by (2.5) and Hölder inequality, we have
\[
a_{0,1}(r) + \sum_{k=1}^{\infty} \sum_{m=1}^{2} a_{k,m}(r) \\
\leq C_0 r^2 4^{-2k_1} (\ln r)^{2k_2} + 2 \sum_{k=1}^{\infty} \sum_{m=1}^{2} r^{-2k} \left| \int_{2}^{r} \tau^{1+k} b_{k,m}(\tau) d\tau \right|^2 \\
+ 2 \sum_{m=1}^{2} \tau^2 \left| \int_{2}^{r} b_{1,m}(\tau) d\tau \right|^2 \\
+ 2 \sum_{k=2}^{\infty} \sum_{m=1}^{2} \tau^{2k} \left| \int_{\infty}^{r} \tau^{1-k} b_{k,m}(\tau) d\tau \right|^2 \\
\leq C_0 r^2 4^{-2k_1} (\ln r)^{2k_2} \\
+ 2 \sum_{k=1}^{\infty} \sum_{m=1}^{2} r^{-2k} \left| \int_{2}^{r} \tau^{3+2k-2k_1-\epsilon} (\ln \tau)^{2k_2} d\tau \cdot \int_{2}^{r} \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1-\epsilon}} \right| \\
+ 2 \sum_{m=1}^{2} \tau^2 \int_{2}^{r} \tau^{1-2k_1-\epsilon} (\ln \tau)^{2k_2} d\tau \cdot \int_{2}^{r} \tau^{2k_1} (\ln \tau)^{-2k_2} b_{1,m}^2(\tau) \frac{d\tau}{\tau^{1-\epsilon}} \\
+ 2 \sum_{k=2}^{\infty} \sum_{m=1}^{2} \tau^{2k} \left| \int_{r}^{+\infty} \tau^{3-2k-2k_1+\epsilon} (\ln \tau)^{2k_2} d\tau \cdot \int_{r}^{+\infty} \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1+\epsilon}} \right| \\
\leq C_0 r^2 4^{-2k_1} (\ln r)^{2k_2} + C r^{4-2k_1-\epsilon} (\ln r)^{2k_2} \int_{r}^{+\infty} \sum_{k=1}^{\infty} \sum_{m=1}^{2} \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1+\epsilon}} \\
+ C r^{4-2k_1+\epsilon} (\ln r)^{2k_2} \int_{r}^{+\infty} \sum_{k=2}^{\infty} \sum_{m=1}^{2} \tau^{2k_1} (\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1+\epsilon}} \\
\leq C_0 r^2 4^{-2k_1} (\ln r)^{2k_2}
\]

for some \( C > 0 \) relying only on \( k_1, k_2 \) and \( p \).

For \( 1 < k_1 < 2 \) case, we only need to change \( a_{1,m} \) with \( m = 1, 2 \) into

\[
a_{1,m} = -\frac{1}{2} r \int_{+\infty}^{r} \tau b_{1,m}(\tau) d\tau + \frac{1}{2} \tau^{-1} \int_{2}^{r} \tau^2 b_{1,m}(\tau) d\tau,
\]

and \( 0 < \epsilon := \frac{1}{2} \min\{1, 4 - 2k_1, 2k_1 - 2\} \). The estimates on \( a_{0,1}^2(r) + \sum_{k=1}^{\infty} \sum_{m=1}^{2} a_{k,m}^2(r) \) follow similarly.

For \( k_1 = 1 \) case, we choose \( \epsilon := \frac{1}{2} \) and we use the following estimates of \( a_{1,m} \).

\[
a_{1,m}^2(r) \leq C_0 r^2 \int_{2}^{r} \tau^{-1} (\ln \tau)^{2k_2} d\tau \cdot \int_{2}^{r} \tau^2 (\ln \tau)^{-2k_2} b_{1,m}^2(\tau) \frac{d\tau}{\tau} \\
+ C r^{2-\epsilon} \int_{2}^{r} \tau^{3} (\ln \tau)^{2k_2} d\tau \cdot \int_{2}^{r} \tau^2 (\ln \tau)^{-2k_2} b_{1,m}^2(\tau) \frac{d\tau}{\tau} \\
\leq C_0 r^2 (\ln r)^{2k_2+2}.
\]

The rest parts of estimate follow similarly.

For \( k_1 = 2 \) case, we choose \( \epsilon := \frac{1}{2} \) and change \( a_{1,m} \) with \( m = 1, 2 \) into (2.7). In this case, the estimates of \( a_{0,1} \) shall be
\begin{align*}
    u_{0,1}^2(r) & \leq Cc_0^2 \int_2^r \tau^{-1}(\ln \tau)^{2k_2} d\tau \cdot \int_2^r \tau^4 (\ln \tau)^{-2k_2} \frac{d\tau}{\tau} \\
    & \quad + Cc_0^2 (\ln r)^2 \int_2^r \tau^{-1}(\ln \tau)^{2k_2} d\tau \cdot \int_2^r \tau^4 (\ln \tau)^{-2k_2} v_{0,1}(\tau) \frac{d\tau}{\tau} \\
    & \leq Cc_0^2 (\ln r)^{2k_2+4}.
\end{align*}

The rest parts of estimate follow similarly.

This proves that \( v(r) \) is well-defined, is a solution of (2.2) in distribution sense [20] and satisfies

\[
    \|v(r)\|_{L^2(\mathbb{S}^1)}^2 \leq \left\{ \begin{array}{ll}
        Cc_0^2 r^{4-2k_1} (\ln r)^{2k_2}, & \text{if } k_1 \not\in \mathbb{N}_*, \\
        Cc_0^2 r^{4-2k_1} (\ln r)^{2k_2+2}, & \text{if } k_1 \in \mathbb{N}_* \setminus \{2\}, \\
        Cc_0^2 (\ln r)^{2k_2+4}, & \text{if } k_1 = 2,
    \end{array} \right.
\]

By interior regularity theory of elliptic differential equations, \( v \) is smooth [19]. Then the pointwise decay rate at infinity follows from re-scaling method and weak Harnack inequality as Theorem 8.17 of [19] etc. (see also Lemma 3.1 of [32]).

**Remark 2.2.** By Hölder inequality, the constant \( C \) relying on \( p \) in (2.3) remains finite when \( p = \infty \) in (2.1). Furthermore, by approximation method, the results hold for more general right hand side term other than smooth functions.

Similar to Lemma 3.2 in [34], by interior estimate, we have the following

**Lemma 2.3.** Let \( g \in C^\infty(\mathbb{R}^2) \) satisfy

\[
    g = O_l(|x|^{-k_1} (\ln |x|)^{k_2}) \quad \text{as } |x| \to \infty
\]

for some \( k_1 > 0, k_2 \geq 0 \) and \( l - 1 \in \mathbb{N} \). Then

\[
    v_g = \left\{ \begin{array}{ll}
        O_{l+1}(|x|^{-k_1} (\ln |x|)^{k_2}), & \text{if } k_1 \not\in \mathbb{N}_*, \\
        O_{l+1}(|x|^{-k_1} (\ln |x|)^{k_2+1}), & \text{if } k_1 \in \mathbb{N}_* \setminus \{2\}, \\
        O_{l+1}(|x|^{-k_1} (\ln |x|)^{k_2+2}), & \text{if } k_1 = 2,
    \end{array} \right. \quad \text{as } |x| \to \infty,
\]

where \( v_g \) denotes the solution found in Lemma 2.1.

### 3 Quadratic part of \( u \) at infinity

In this section, we prove a weaker asymptotic behavior than (1.8), which concerns only on the quadratic part of \( u \) at infinity.

**Lemma 3.1.** Let \( u, f \) be as in Theorem 1.2 Then there exist \( A \in \text{Sym}(2) \) satisfying \( F_\tau(\lambda(A)) = f(\infty) \) and \( \epsilon > 0 \) such that

\[
    \left| u(x) - \frac{1}{2} x^T A x \right| = O_2(|x|^{2-\epsilon})
\]

as \( |x| \to \infty \).
For \( \tau = 0 \), the results in Lemma 3.1 were proved earlier in Bao–Li–Zhang [2]. More rigorously, in the proof of Theorem 1.2 in [2] (see also Theorem 2.2 of [33]), we have the following result for Monge–Ampère type equations, which holds for general \( n \geq 2 \).

**Theorem 3.2.** Let \( u \in C^0(\mathbb{R}^n) \) be a convex viscosity solution of

\[
\det D^2 u = \psi(x) \quad \text{in} \quad \mathbb{R}^n
\]

with \( u(0) = \min_{\mathbb{R}^n} u = 0 \), where \( 0 < \psi \in C^0(\mathbb{R}^n) \) and

\[
\psi^{\frac{1}{n}} - 1 \in L^n(\mathbb{R}^n).
\]

Then there exists a linear transform \( T \) satisfying \( \det T = 1 \) such that \( v := u \circ T \) satisfies

\[
\left| v - \frac{1}{2}|x|^2 \right| \leq C|x|^{2-\varepsilon}, \quad \forall \ |x| \geq 1
\]

for some \( C > 0 \) and \( \varepsilon > 0 \).

For \( \tau \in (0, \frac{\pi}{4}] \) cases, the results follow from Legendre transform and the asymptotic behavior for the Monge–Ampère type equations and Poisson equations. More explicitly, as proved for general \( n \geq 2 \) cases in Theorem 2.1 and Remark 2.6 of [33], we have the following result.

**Theorem 3.3.** Let \( u \in C^2(\mathbb{R}^2) \) be a classical solution of (1.3) with \( \tau \in [0, \frac{\pi}{4}] \) and \( f \in C^1(\mathbb{R}^2) \) satisfy

\[
|x|^\zeta |f(x) - f(\infty)| + |x|^{1+\zeta'} |Df(x)| \leq C, \quad \forall \ |x| > 1
\]

for some \( C > 0 \) with \( \zeta > 1, \zeta' > 0 \) for \( \tau \in [0, \frac{\pi}{4}] \) and \( \zeta, \zeta' > 0 \) for \( \tau = \frac{\pi}{4} \). Let \( D^2 u \) and \( f(\infty) \) satisfy (1.5) and (1.6) respectively. For \( \tau \in (0, \frac{\pi}{4}] \), we assume further that \( u \) satisfies (1.7). Then there exist \( \epsilon > 0, A \in \text{Sym}(2) \) satisfying \( F_\tau(\lambda(A)) = f(\infty) \) and (1.5) and \( C > 0 \) such that

\[
||D^2 u||_{C^0(\mathbb{R}^2)} \leq C \quad \text{and} \quad |D^2 u(x) - A| \leq \frac{C}{|x|^{\epsilon}}, \quad \forall \ |x| > 1.
\]

**Proof of Lemma 3.1** with \( 0 \leq \tau \leq \frac{\pi}{4} \). In Lemma 3.1 we have \( f \in C^m(\mathbb{R}^2) \) satisfies (1.4) for some \( \zeta > 2 \) and \( m \geq 3 \). Especially \( f \in C^1(\mathbb{R}^2) \) satisfies condition (3.2) with \( \zeta' = \zeta > 2 \). By Theorem 3.3 we have \( ||D^2 u||_{C^0(\mathbb{R}^2)} \) is bounded for all \( 0 < \alpha < 1 \) and \( D^2 u(x) \) converge to matrix \( A \) at Hólder speed \( |x|^{1-\epsilon} \). By Newton–Leibnitz formula, since \( Du, u \) are bounded on \( \partial B_1 \), for any \( |x| > 1 \) we let \( e := \frac{2}{|x|} \in \partial B_1 \) to obtain

\[
\left| Du(x) - Du(e) - \int_0^{1} Ads \cdot e \right| = \left| \int_0^{|x|-1} D^2 u((s+1)e) - Ads \cdot e \right| \leq C|x|^{1-\epsilon}.
\]

Consequently there exists \( C > 0 \) such that

\[
|Du(x) - Ax| \leq C|x|^{1-\epsilon}, \quad \forall \ |x| > 1.
\]

By Newton–Leibnitz formula again we have

11
\[ u(x) - u(c) - \int_0^{|x|^{-1}} x^T A \cdot eds = \int_0^{|x|^{-1}} (Du((s + 1)e) - Ax) \cdot eds \leq C|x|^{2-\epsilon}, \quad \forall |x| > 1 \]

for some \( C > 0 \). This finishes the proof of Lemma 3.1 with \( \tau \in [0, \frac{\pi}{4}] \).

**Proof of Lemma 3.1 with \( \frac{\pi}{4} < \tau < \frac{\pi}{2} \).** By semi-convex condition (1.5), we have \( \lambda_i > -(a + b) \) for \( i = 1, 2 \) and consequently by a direct computation as in [24, 39],

\[ \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b} = \arctan \frac{\lambda_i + a}{b} - \frac{\pi}{4}. \]

Consequently Eq. (1.3) with \( \tau \in (\frac{\pi}{4}, \frac{\pi}{2}) \) and semi-convex condition \( D^2u > -(a + b)I \) becomes

\[ \arctan \frac{\lambda_1(D^2u) + a}{b} + \arctan \frac{\lambda_2(D^2u) + a}{b} = \frac{b}{\sqrt{a^2 + 1}} f(x) + \frac{\pi}{2} \text{ in } \mathbb{R}^2. \]

Let \( v := \frac{1}{b}(u + \frac{2}{b}|x|^2) \), then \( v \) satisfies (1.7) for some new \( C > 0 \) and the Lagrangian mean curvature equation

\[ \arctan \lambda_1(D^2v) + \arctan \lambda_2(D^2v) = \frac{b}{\sqrt{a^2 + 1}} f(x) + \frac{\pi}{2} \text{ in } \mathbb{R}^2. \quad (3.3) \]

If \( \frac{b}{\sqrt{a^2 + 1}} f(\infty) \neq -\frac{\pi}{2} \), we may assume without loss of generality that \( \frac{b}{\sqrt{a^2 + 1}} f(\infty) + \frac{\pi}{2} > 0 \), otherwise we consider the equation satisfied by \(-v\) as replacement. Then the desired result (3.1) follows from \( \tau = \frac{\pi}{2} \) case, which will be proved below.

It remains to prove for \( \frac{b}{\sqrt{a^2 + 1}} f(\infty) + \frac{\pi}{2} = 0 \) case. For any sufficiently small \( \delta > 0 \), there exists \( R_0 > 0 \) such that

\[ \frac{b}{\sqrt{a^2 + 1}} f(x) + \frac{\pi}{2} < \delta, \quad \forall |x| > R_0. \]

Since \( D^2v > -I \), together with the continuity of \( D^2u \) in \( \mathbb{R}^2 \), we have \( D^2v \) bounded on entire \( \mathbb{R}^2 \). Consequently \( D^2u \) is bounded on entire \( \mathbb{R}^2 \). For any sufficiently large \( |x| > 1 \), we set

\[ R := |x| \quad \text{and} \quad u_R(y) := \frac{4}{R^2} u(x + \frac{R}{2} y) \text{ in } B_1. \]

Then \( u_R \) is a classical solution of

\[ F_\tau(\lambda(D^2u_R(y))) = f(x + \frac{R}{2} y) =: f_R(y) \text{ in } B_1. \]

Since \( D^2u_R \) are uniformly (to \( R \)) bounded, the equations above are uniformly elliptic. Consequently, by the definition of \( u_R \), we have \( ||u_R||_{C^0(B_1)} \) are also uniformly bounded. By interior Hölder estimates for second derivatives as in Theorem 17.11 of [19], we have

\[ ||D^2u_R||_{C^0(B_{\frac{1}{2}})} \leq C \quad (3.4) \]
for some $\alpha, C > 0$ uniform to sufficiently large $R$. Now we turn to the algebraic form of (3.3) i.e.,

$$
\Delta v = (1 - \det D^2 v) \cdot \tan \left( \frac{b}{\sqrt{a^2 + 1}} f(x) + \frac{\pi}{2} \right) =: g(x)
$$

(3.5)

in $\mathbb{R}^2$. By approximation and Lemma 2.1 there exists a solution $v_g$ solving (3.5) on $\mathbb{R}^2 \setminus B_1$ with estimate

$$
|v_g(x)| \leq C|x|^{2-\zeta}(|x|), \quad \forall \, |x| > 1
$$

for some $C > 0$. Let $v_R(y) := v_g(x + R y)$ in $B_1$, where $R = |x| > 2$, then

$$
\Delta v_R = g(x + R y) =: g_R(y) \quad \text{in } B_1.
$$

By conditions (1.4) and (3.4), we have

$$
||v_R||_{C^0(B_1)} \leq CR^{-\zeta} \ln R \quad \text{and} \quad ||g_R||_{C^\alpha(B_{\frac{1}{2}})} \leq CR^{\alpha-\zeta}
$$

for some $C > 0$ uniform to $R > 2$. By interior Schauder estimates, we have

$$
||v_R||_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq CR^{\zeta}
$$

for some $0 < \alpha, \zeta < 1$ and $C > 0$. Consequently there exists $C > 0$ such that

$$
v_g = O_2(|x|^{2-\zeta}) \quad \text{as } |x| \to \infty.
$$

Since $v - v_g$ is harmonic on $\mathbb{R}^2 \setminus B_1$ with bounded Hessian matrix, by spherical harmonic expansion as in (2.4), there exists $A_v \in \text{Sym}(2), \beta_v \in \mathbb{R}^2$ and $c_v, d_v \in \mathbb{R}$ such that

$$
v - v_g = \frac{1}{2} x^T A_v x + \beta_v \cdot x + d_v \ln |x| + c_v + O_2(|x|^{-1})
$$

for all $l \in \mathbb{N}$ as $|x| \to \infty$. Combining the two asymptotic behavior above, we have

$$
u(x) = bv(x) - \frac{a}{2}|x|^2
$$

$$
= \frac{1}{2} x^T (bA_v - aI)x + b\beta_v \cdot x + bd_v \ln |x| + bc_v + bv_g + O_2(|x|^{-1})
$$

$$
= \frac{1}{2} x^T (bA_v - aI)x + O_2(|x|^{2-\zeta})
$$

as $|x| \to \infty$. This finishes the proof of (3.1). \hfill \Box

**Proof of Lemma 3.1 with $\tau = \frac{\pi}{2}$.** Consider the algebraic form of Eq. (1.3) with $\tau = \frac{\pi}{2}$ i.e.,

$$
\cos f(x) \cdot \Delta u + \sin f(x) \det D^2 u = \sin f(x)
$$

in $\mathbb{R}^2$. By condition (1.6), $\cot f(\infty) \neq 0$ and consequently we have

$$
\det D^2 u + \cot f(\infty) \cdot \Delta u = 1 + (\cot f(\infty) - \cot f(x)) \Delta u
$$

in $\mathbb{R}^2$. Change of variable by setting
\[ v(x) := u(x) + \frac{\cot f(\infty)}{2} |x|^2, \]

which satisfies

\[
\det D^2 v = (\lambda_1(D^2 u) + \cot f(\infty)) \cdot (\lambda_2(D^2 u) + \cot f(\infty)) = 1 + \cot^2 f(\infty) + (\cot f(\infty) - \cot f(x)) \Delta u \]

\[ =: g(x) \quad (3.6) \]

in \( \mathbb{R}^2 \). To obtain the desired results, we shall obtain the asymptotic behavior of \( g(x) \) at infinity and apply Theorem 3.2.

**Step 1:** We prove the boundedness of \( D^2 u \) by interior Hessian estimate.

Since \( f(\infty) \in (0, \pi) \), for any sufficiently small \( 0 < \delta < f(\infty) \), there exists \( R_0 > 0 \) such that

\[ |f(x) - f(\infty)| < \delta, \quad \forall \ |x| > R_0. \]

By Eq. (1.3) with \( \tau = \frac{\pi}{2} \), for all \( i = 1, 2 \), we have

\[ \arctan \lambda_i + \frac{\pi}{2} > \arctan \lambda_1 + \arctan \lambda_2 > f(\infty) - \delta > 0, \quad \forall \ |x| > R_0. \]

Consequently by the monotonicity of \( \arctan \) function, we have

\[ D^2 u > -\cot(f(\infty) - \delta)I \quad \forall \ |x| > R_0. \]

By (1.7) and the quadratic growth condition from above, there exists \( C > 0 \) such that

\[ |u(x)| \leq C(1 + |x|^2), \quad \forall \ x \in \mathbb{R}^2. \]

(3.7)

For sufficiently large \( |x| > 2R_0 \), we set

\[ R := |x| > 2R_0 \quad \text{and} \quad u_R(y) := \frac{4}{R^2} u(x + \frac{R}{2} y), \quad y \in B_1, \]

(3.8)

where \( B_r(x) \) denote the ball centered at \( x \) with radius \( r \) and \( B_r := B_r(0) \). Then \( u_R \in C^4(B_1) \) satisfies

\[ \arctan \lambda_1(D^2 u_R) + \arctan \lambda_2(D^2 u_R) = f(x + \frac{R}{2} y) =: f_R(y) \quad \text{in} \ B_1. \]

(3.9)

By a direct computation, (3.7) implies that there exists a constant \( C \) uniform to \( R > 2R_0 \) such that

\[ \sup_{B_1} |u_R| \leq \frac{4}{R^2} \sup_{B_{\frac{R}{2}}(x)} |u| \leq C. \]

Together with condition (1.4) on \( f \) with \( \zeta > 2 \) and \( m \geq 3 \), we have

\[
||f_R||_{C^1,B_1} = \sup_{B_1} (|f_R| + |Df_R|) + \sup_{y,z \in B_1 \ y \neq z} \frac{|Df_R(y) - Df_R(z)|}{|y - z|}
\leq \sup_{B_1} (|f_R| + |Df_R| + |D^2f_R|)
\leq CR^{-\zeta},
\]

14
for some constant $C > 0$ uniform to $R > 2R_0$. Furthermore, since

$$\sup_{B_1} |f_R(y) - f(\infty)| \to 0 \quad \text{as } R \to \infty.$$  

By $f(\infty) > 0$, there exists $\delta > 0$ uniform to sufficiently large $R$ such that $|f_R| \geq \delta > 0$.

Now we introduce the following interior Hessian estimates for Lagrangian mean curvature equations as in Theorems 1.1 and 1.2 by Bhattacharya [3].

**Theorem 3.4.** Let $u$ be a $C^4$ solution of (1.2) in $B_r \subset \mathbb{R}^n$, where $f \in C^{1,1}(B_r)$ and

$$|f| \geq \frac{n-2}{2} \pi + \delta \quad (3.10)$$

for some $\delta > 0$. Then we have

$$|D^2u(0)| \leq C_1 \exp \left( \frac{C_2}{r^{2n-2}} \left( \text{osc}_{B_1(0)} u + 1 \right)^{2n-2} \right),$$

where $C_1, C_2$ are positive constants depending on $\|f\|_{C^{1,1}(B_r)}, n$, and $\delta$.

Applying Theorem 3.4 to the equation satisfies by $u_R$ in $B_1$, there exists $C > 0$ uniform to $R > 2R_0$ such that

$$|D^2u_R(0)| \leq C_1 \exp \left( C_2 \left( \text{osc}_{B_1(0)} u_R + 1 \right)^2 \right) \leq C.$$

Consequently $D^2u$ is bounded on entire $\mathbb{R}^2$.

**Step 2:** Now we compute the asymptotic behavior of $g(x)$ at infinity.

By the definition of $g(x)$ in Eq. (3.6) and condition (1.4), since $D^2u$ is bounded on entire $\mathbb{R}^2$, we have

$$g(x) - (1 + \cot^2 f(\infty)) = (\cot f(\infty) - \cot f(x)) \Delta u \leq \frac{C}{\sin^2 f(\infty)} |f(x) - f(\infty)| \leq C|x|^{-\zeta}$$

for some constant $C > 0$. Consequently there exists $C > 0$ such that

$$\int_{\mathbb{R}^2 \setminus B_1} |g^2(x) - (1 + \cot^2 f(\infty))|^\frac{1}{2} \, dx \leq C \int_{\mathbb{R}^2 \setminus B_1} |g(x) - (1 + \cot^2 f(\infty))|^2 \, dx \leq C \int_{\mathbb{R}^2 \setminus B_1} |x|^{-2\zeta} \, dx < \infty.$$

**Step 3:** Obtain the asymptotic behavior of $v$.

Since $1 + \cot^2 f(\infty) > 0$, by the continuity of $D^2v$ and $n = 2$, we have either $D^2v > 0$ or $D^2v < 0$ for sufficiently large $|x| > 2R_0$. We may assume without loss of generality that
$D^2v > 0$, otherwise we consider $-v$ instead. Thus by extension result as Theorem 3.2 of \cite{42}, we can apply Theorem 3.2 (see also Corollary 2.3 in \cite{33}) after re-scaling $\tilde{v} := \frac{1}{(1+\cot^2 f(\infty))^{1/2}}v$, there exist $A_v \in \text{Sym}(2)$ satisfying $\det A_v = 1 + \cot^2 f(\infty)$ such that

$$\left| v - \frac{1}{2}x^T A_v x \right| \leq C|x|^{2-\epsilon} \quad \forall |x| > 1$$

for some $C > 0$ and $\epsilon > 0$. By the definition of $v$, we have

$$\left| u - \frac{1}{2}x^T (A_v - \cot f(\infty)I) x \right| \leq C|x|^{2-\epsilon} \quad \forall |x| > 1.$$ 

By taking $A := A_v - \cot f(\infty)I$, it is easy to verify that

$$\cos f(\infty) \text{trace}(A) + \sin f(\infty) \det A = \sin f(\infty),$$

and hence $\text{arctan} \lambda_1(A) + \text{arctan} \lambda_2(A) = f(\infty)$ and the first part of the desired result in Lemma 3.1 follows immediately.

**Step 4:** We prove interior gradient and Hessian estimates by scaling.

Let $u_R, f_R$ be as in (3.8) and (3.9). From the results in Step 1, $D^2u$ is bounded on entire $\mathbb{R}^2$. Consequently equations (3.9) are uniformly (to $R > 2R_0$) elliptic. By interior Hölder estimates for second derivatives as in Theorem 17.11 of \cite{19}, we have

$$[D^2u_R]_{C^0(B_{1/2})} \leq C$$

for some $\alpha, C > 0$, where $\alpha$ relies only on $\|D^2u_R\|_{C^0(B_{1})}$ and $C$ relies only on $\|u_R\|_{C^2(B_{1})}$ and $\|f_R\|_{C^1(B_{1})}$. Consequently

$$\|D^2u_R\|_{C^0(B_{1/2})} < C$$

for some $C > 0$ for all $R > 2R_0$. We would like to mention that $n = 2$ is necessary to apply interior estimates as Theorem 17.11 in \cite{19}. For higher dimensions, it is generally required that the operator has a concavity structure (see for instance Theorem 17.14 in \cite{19} and Theorem 8.1 in \cite{7}).

To obtain the desired result, it remains to prove the gradient and Hessian estimate on the difference between $u$ and $\frac{1}{2}x^T Ax$. Let

$$w(x) := u(x) - \frac{1}{2}x^T Ax \quad \text{and} \quad w_R(y) := \frac{4}{R^2}w(x + \frac{R}{2}y) \quad \forall y \in B_{1}. \quad (3.11)$$

From the results in Steps 1-4, there exists $C > 0$ uniform to all $R > 2R_0$ such that

$$\|u_R\|_{L^\infty(B_{1})} \leq C \quad \text{and} \quad \|w_R\|_{L^\infty(B_{1})} \leq CR^{-\epsilon}.$$ 

Applying Newton–Leibnitz formula between (3.9) and $\text{arctan} \lambda_1(A) + \text{arctan} \lambda_2(A) = f(\infty)$, we have

$$a_{ij}^R D_{ij} w_R = f_R(y) - f(\infty) \quad \text{in } B_{1}, \quad (3.12)$$

where
\[ a_{ij}^R(y) = \int_0^1 D_{M_{ij}} F_\tau(A + tD^2 w_R(y)) dt, \]

are uniformly elliptic and having uniformly bounded (to \( R > 2R_0 \)) \( C^0 \) norm. Hereinafter, we let \([a_{ij}]\) denote the 2-by-2 matrix with the \( i, j \)-position being \( a_{ij} \) and \( D_{M_{ij}} F_\tau(M) \) denote the value of partial derivative of \( F_\tau(\lambda(M)) \) with respect to \( M_{ij} \) variable at \( M = [M_{ij}] \). By \( (1.4) \), there exists uniform \( C > 0 \) such that

\[ \| f_{R-1} \|_{L^\infty(B_1)} + \sum_{k=1}^{m-1} \| D^k f_{R} \|_{C^\alpha(B_1)} \leq CR^{-\zeta}. \]  

(3.13)

By interior Schauder estimates as Theorem 6.2 in [19],

\[ \| w_R \|_{C^{2,\alpha}(B_2^\frac{1}{2})} \leq C \left( \| w_R \|_{L^\infty(B_1)} + \| f_{R-1} \|_{C^\alpha(B_1)} \right) \leq CR^{-\min\{\epsilon, \zeta\}}. \]

This finishes the proof of Lemma 3.1 by choosing \( \epsilon \) as the minimum of \( \epsilon \) and \( \zeta \).

\( \square \)

4 Asymptotic behavior and expansions of \( u \) at infinity

In this section, we prove the asymptotic behavior and expansions of solution \( u \) at infinity following the line of iteration method as by Bao–Li–Zhang [2] with an improvement from spherical harmonic expansion.

Lemma 4.1. Let \( u, f \) be as in Theorem 1.2 and

\[ w(x) := u(x) - \frac{1}{2} x^T Ax, \]

where \( A \in \text{Sym}(2) \) is from Lemma 3.1 Then there exist \( C, \alpha, \epsilon' > 0 \) such that

\[
\begin{cases}
|D^k w(x)| \leq C|x|^{2-k-\epsilon'}, \\
|D^{m+1} w(x_1) - D^{m+1} w(x_2)| \leq C|x_1|^{1-m-\epsilon'-\alpha},
\end{cases}
\]  

(4.1)

for all \( |x| > 2, k = 0, \ldots, m + 1 \) and \( |x_1| > 2, x_2 \in B_{|x_1|/2}(x_1) \).

Proof. For sufficiently large \( |x| > 1 \), we set \( R, u_R, f_R, w_R \) as in the proof of Lemma 3.1 i.e., \( (3.8), (3.9) \) and \( (3.11) \). As proved in Lemma 3.1 together with \( u \in C^2(\mathbb{R}^2) \) we have

\[ |D^k w(x)| \leq C|x|^{2-k-\epsilon}, \quad \forall \, |x| > 1 \]

for some \( C > 0 \) and \( \epsilon > 0 \) from Lemma 3.1 for \( k = 0, 1, 2 \). It remains to prove the higher order derivatives following the Step 4 in the proof of Lemma 3.1.

In fact, we consider the scaled equation

\[ F_\tau(\lambda(D^2 u_R)) = f_R(y) \quad \text{in} \, B_1. \]  

(4.2)
Since $D^2 u_R$ are uniformly (to $R$) bounded, $F_r(\lambda(D^2 u_R))$ are uniformly elliptic. By Theorem 17.11 in [19], there exist $\alpha, C, R_0 > 0$ such that

$$||D^2 u_R||_{C^\alpha(B_{\frac{3}{2}})} \leq C, \quad \forall \, R > 2. \quad (4.3)$$

Apply Newton–Leibnitz formula between (4.2) and $F_r(\lambda(A)) = f(\infty)$ to obtain linearized equation (3.12). By the boundedness of $D^2 u_R$ and estimate (4.3), the coefficients $a^{ij}_l(y)$ are uniformly elliptic and having finite $C^\alpha$-norm for some $\alpha > 0$ uniform to $R > 2$. Together with (3.13), by interior Schauder estimates we have

$$||w_R||_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C \left(||w_R||_{L^\infty(B_{\frac{3}{2}})} + ||f_R - f(\infty)||_{C^\alpha(B_{\frac{3}{2}})}\right) \leq CR^{-\min\{\epsilon, \zeta\}}$$

for some $C > 0$ uniform to all $R > 2$.

It remains to obtain higher order derivative estimates. For any $e \in \partial B_1$, we act partial derivative $D_e$ to both sides of (4.2) and obtain

$$\tilde{a}^{ij}_l(D_e w_R) = D_e f_R(y) \quad \text{in} \, B_1, \quad \text{where} \quad \tilde{a}^{ij}_l(y) := D_{Mj} f_r(D^2 u_R(y)).$$

By the boundedness of $D^2 u_R$ and estimate (4.3), the coefficients $\tilde{a}^{ij}_l$ are uniformly elliptic with uniformly bounded $C^\alpha$-norm to all $R > 2$. By interior Schauder estimate again and the arbitrariness of $e \in \partial B_1$, we have

$$||w_R||_{C^{3,\alpha}(\overline{B}_{1/3})} \leq C \left(||w_R||_{L^\infty(\overline{B}_{2/3})} + ||D f_1, R||_{C^\alpha(\overline{B}_{2/3})}\right) \leq CR^{-\min\{\epsilon, \zeta\}}.$$

By taking further derivatives of Eq. (4.2), higher order derivatives follow from Schauder estimate and this finishes the proof of (4.1).

Next we prove an iteration type lemma that improves the estimates in Lemma 4.1. The result follows as in Lemma 2.2 in Bao–Li–Zhang [2] with minor modifications.

**Lemma 4.2.** Let $u \in C^2(\mathbb{R}^2)$ be a classical solution of

$$F(D^2 u) = f(x) \quad \text{in} \, \mathbb{R}^2. \quad (4.4)$$

Suppose $F$ is smooth up to the boundary of the range of $D^2 u$ and is uniformly elliptic. Suppose further that $f \in C^m(\mathbb{R}^2)$ satisfies (1.4) for some $\zeta > 2, m \geq 3$. If there exists a quadratic polynomial $v$ satisfying $F(D^2 v) = f(\infty)$ and $w := u - v$ satisfies

$$\left\{ \begin{array}{l} |D^k w(x)| \leq C|x|^{2-\epsilon-k} (\ln |x|)^{p_0}, \\
\frac{|D^{m+1} w(x_1) - D^{m+1} w(x_2)|}{|x_1 - x_2|^m} \leq C|x_1|^{1-m-\epsilon-\alpha} (\ln |x|)^{p_0} \end{array} \right. \quad (4.5)$$

for some $0 < \epsilon < \frac{1}{2}$ and $p_0 \in \mathbb{N}$ for all $|x| > 2, k = 0, \ldots, m + 1$ and $|x_1| > 2, x_2 \in B_{|x_1|/2}(x_1)$. Then

$$\left\{ \begin{array}{l} |D^k w(x)| \leq C|x|^{2-2\epsilon-k} (\ln |x|)^{2p_0}, \\
\frac{|D^{m+1} w(x_1) - D^{m+1} w(x_2)|}{|x_1 - x_2|^m} \leq C|x_1|^{1-m-2\epsilon-\alpha} (\ln |x|)^{2p_0}, \end{array} \right. \quad (4.6)$$

for all $|x| > 2, k = 0, \ldots, m + 1$ and $|x_1| > 2, x_2 \in B_{|x_1|/2}(x_1)$. 

18
Proof. Acting partial derivative $D_k$ to both sides of Eq. (4.4), we have
\begin{equation}
  a_{ij}(x)D_{ij}(D_ku(x)) = D_kf(x), \quad \text{where} \quad a_{ij}(x) = D_{Mij}F(D^2u(x)).
\end{equation}
By the assumptions on $F$ and $u$, $a_{ij}(x)$ are uniformly elliptic coefficients. By the first formula of (4.5), since $m \geq 3$, there exists $C > 0$ such that
\begin{equation}
  |a_{ij}(x) - D_{Mij}F(A)| \leq C|x|^{-\epsilon}(\ln |x|)^{p_0}, \quad |Da_{ij}(x)| \leq C|x|^{-1-\epsilon}(\ln |x|)^{p_0}
\end{equation}
for all $|x| > 2$. For the given $\alpha \in (0, 1)$, together with the second formula in (4.5) we have
\begin{equation}
  \frac{|Da_{ij}(x_1) - Da_{ij}(x_2)|}{|x_1 - x_2|^\alpha} \leq C|x_1|^{-1-\epsilon-\alpha}(\ln |x_1|)^{p_0}, \quad |x_1| > 2, \quad x_2 \in B_{|x_1|/2}(x_1).
\end{equation}
For any $l = 1, 2$, we act partial derivative $D_l$ to both sides of (4.7). Let $h_1 := D_{kl}u$, then we have
\begin{equation}
  D_{Mij,Mqr}F(D^2u)D_{ijk}uD_{qr}u + D_{Mij}F(D^2u)D_{ij}h_1 = D_{kl}f(x),
\end{equation}
i.e.,
\begin{equation}
  a_{ij}(\infty)D_{ij}h_1 = f_2(x) \quad \text{in} \quad \mathbb{R}^2,
\end{equation}
where
\begin{equation}
  f_2(x) := D_{kl}f - D_{Mij,Mqr}F(D^2u)D_{ijk}uD_{qr}u + (a_{ij}(\infty) - a_{ij}(x))D_{ij}h_1.
\end{equation}
Notice that $[a_{ij}(\infty)] = [D_{Mij}F(A)]$ is a positive symmetric matrix, we set $Q := [a_{ij}(\infty)]^{1/2}$ and $\tilde{h}_1(x) := h_1(Qx)$. Since trace is invariant under cyclic permutations,
\begin{equation}
  \Delta \tilde{h}_1(x) = f_2(Qx) =: \tilde{f}_2(x)
\end{equation}
in $Q^{-1}(\mathbb{R}^2 \setminus B_1)$. Since $Q$ is invertible, by $\zeta > 2 > 2\epsilon$ and (4.5), we have
\begin{equation}
  |\tilde{f}_2(x)| \leq |D_{kl}f(Qx)| + C|D_{ijk}u(Qx)| \cdot |D_{qr}u(Qx)| + C|a_{ij}(\infty) - a_{ij}(Qx)|
  \leq C|x|^{-2-\epsilon} + C|x|^{-2-2\epsilon}(\ln |x|)^{2p_0}
  \leq C|x|^{-2-2\epsilon}(\ln |x|)^{2p_0}
\end{equation}
in $x \in Q^{-1}(\mathbb{R}^2 \setminus B_1)$. By Lemmas 2.1 and 2.3 and $0 < 2\epsilon < 1$, there exists a function $\tilde{h}_2$ satisfying (4.8) on an exterior domain with estimate
\begin{equation}
  |D^k\tilde{h}_2(x)| \leq C|x|^{-2\epsilon-k}(\ln |x|)^{2p_0}, \quad \forall k = 0, 1.
\end{equation}
By the definition of $\tilde{h}_1$ and the first line of (4.5), $\tilde{h}_1 - \tilde{h}_2$ is harmonic on exterior domain of $\mathbb{R}^2$ with vanishing speed
\begin{equation}
  \tilde{h}_1 - \tilde{h}_2 - \delta_{kl} = O(|x|^{-\epsilon}(\ln |x|)^{p_0}) \quad \text{as} \quad |x| \to \infty.
\end{equation}
By spherical harmonic expansion as in (2.4), see also formula (2.23) in [2], there exists $C > 0$ such that

$$|\tilde{h}_1 - \tilde{h}_2 - \delta_{kl}| \leq C|x|^{-1}$$

for all $|x| > 2$. By taking $h_2(x) := \tilde{h}_2(Q^{-1}x)$, there exists $C > 0$ such that

$$|h_1(x) - h_2(x) - A_{kl}| \leq C|x|^{-1}, \quad \forall \ |x| > 2$$

and hence

$$|D^k w(x)| \leq C|x|^{2-2e^{-k}(\ln |x|)^{2p_0}}, \quad \forall \ |x| > 2.$$ 

By taking higher order derivatives as in the proof of Lemma 4.1, this finishes the proof of the first formula in (4.6).

It remains to prove the Hölder semi-norm part. In fact for sufficiently large $|x|$, we set $R := |x|$ and

$$h_{2,R}(y) := \tilde{h}_2 \left(x + \frac{R}{4}y\right), \quad f_{2,R}(y) = \frac{R^2}{16} \tilde{f}_2 \left(x + \frac{R}{4}y\right), \quad |y| \leq 2.$$ 

By condition (1.4) on $f(x)$ and (4.5), for any large $|x_1|$ and $x_2 \in B_{|x_1|/2}(x_1)$ with $x_1 \neq x_2$, we have

$$\left|\frac{\tilde{f}_2(x_1) - \tilde{f}_2(x_2)}{|x_1 - x_2|^\alpha}\right| \leq \left|\tilde{D}_{kl} f (Qx_1) - \tilde{D}_{kl} f (Qx_2)\right| |x_1 - x_2|^\alpha$$

$$+ \left| (a_{ij}(\infty) - a_{ij}(Qx_1)) D_{ij} h_1 (Qx_1) - (a_{ij}(\infty) - a_{ij}(Qx_2)) D_{ij} h_1 (Qx_2) \right| |x_1 - x_2|^\alpha$$

$$+ \left| F_{M_{ij,M_{qr}}} (D^2 u) D_{ijk} u D_{qrt} u (Qx_1) - F_{M_{ij,M_{qr}}} (D^2 u) D_{ijk} u D_{qrt} u (Qx_2) \right| |x_1 - x_2|^\alpha$$

$$\leq C|x_1|^{-\epsilon - 2 - \alpha} + C|x_1|^{-1 - \epsilon} (\ln |x_1|)^{p_0} \cdot |x_1|^{-1 - \epsilon - \alpha} (\ln |x_1|)^{p_0}$$

$$\leq C|x_1|^{-2 - 2\epsilon - \alpha} (\ln |x_1|)^{2p_0}.$$ 

Thus by a direct computation, there exists $C > 0$ uniform to all $R > 1$ such that

$$\|f_{2,R}\|_{C^\alpha(B_R)} \leq CR^{-2\epsilon} (\ln |x|)^{2p_0}.$$ 

By the interior Schauder estimates of Poisson equation, we have

$$\|h_{2,R}\|_{C^{2,\alpha}(B_{R})} \leq C \left(\|h_{2,R}\|_{L^\infty(B_2)} + \|f_{2,R}\|_{C^\alpha(B_2)}\right) \leq CR^{-2\epsilon} (\ln R)^{2p_0}$$

By the definition of $h_{2,R}$ and the non-degeneracy of $Q$ we have the second formula in (4.6).

For higher order derivatives, the results follow from taking further derivatives to both sides of the equation and apply interior Schauder estimates as in the proof of Lemma 4.1.
Now we are able to prove the asymptotic behavior at infinity by iteration. More explicitly, we have the following results.

**Proposition 4.3.** Let \( u, f \) be as in Theorem 1.2 then there exists \( A \in \text{Sym}(2) \) satisfying \( F_t(\lambda(A)) = f(\infty) \) and (1.5), \( b \in \mathbb{R}^2 \) and \( d, c, R, 1 \in \mathbb{R} \) such that (1.8) holds as \( |x| \to \infty \), where \( Q \) is given by (1.9). Furthermore, when \( \zeta > 3 \), then there exist \( d_1, d_2 \) such that (1.10) holds.

**Proof.** By Lemmas 3.1 and 4.1 there exist \( \alpha, \epsilon' > 0 \) such that (4.1) holds. Let \( p_1 \in \mathbb{N} \) be the positive integer such that

\[
2^{p_1} \epsilon' < 1 \quad \text{and} \quad 1 < 2^{p_1 + 1} \epsilon' < 2.
\]

(If necessary, we may choose \( \epsilon' \) smaller to make both inequalities hold.) Let \( \epsilon_1 := 2^{p_1} \epsilon' \). Applying Lemma 4.2 \( p_1 \) times, we have

\[
\begin{align*}
&\left| D^k w(x) \right| \leq C |x|^{2-\epsilon_1-k}, \\
&\left| D^{m+1} w(x_1)-D^{m+1} w(x_2) \right| \leq C |x_1|^{1-m-\epsilon_1-\alpha},
\end{align*}

(4.9)
\]

for all \( |x| > 2, k = 0, \ldots, m + 1 \) and \( |x_1| > 2, x_2 \in B_{|x_1|/2} (x_1) \).

Now we consider the linearized equation again. Applying Newton–Leibnitz formula between Eq. (1.3) and \( F_t(\lambda(A)) = f(\infty) \), we have

\[
\tilde{a}_{ij}(\infty)D_{ij} w = f(x) - f(\infty) + (\tilde{a}_{ij}(\infty) - \tilde{a}_{ij}(x))D_{ij} w =: f_3(x)
\]

in \( \mathbb{R}^2 \), where the coefficients are uniformly elliptic and

\[
\tilde{a}_{ij}(x) = \int_0^1 D_{M_{ij}} F_t(A + tD^2 w(x)) dt \to \tilde{a}_{ij}(\infty) = D_{M_{ij}} F_t(A)
\]

as \( |x| \to \infty \). Let \( Q := [D_{M_{ij}} F(A)]^{1 \over 2} \) and \( \tilde{w}(x) := w(Qx). \) By the invariance of trace under cyclic permutations again, we have

\[
\Delta \tilde{w} = f_3(Qx) =: \tilde{f}_3(x).
\]

(4.10)

By the definition of \( \tilde{a}_{ij}(x) \), condition (1.4) on \( f \) and (4.9) we have

\[
|\tilde{f}_3(x)| \leq C |x|^{-2\epsilon_1}
\]

for some \( C, R_1 > 0 \) for all \( |x| > 2R_1 \). By Lemmas 2.1 and 2.3, there exist a function \( \tilde{h}_3 \) solving (4.10) in \( \mathbb{R}^2 \setminus \overline{B_{R_1}} \) with estimate

\[
|\tilde{h}_3(x)| \leq C |x|^{2-2\epsilon_1}
\]

for some \( C > 0 \) for all \( |x| > 2R_1 \). Thus \( \tilde{w} - \tilde{h}_3 \) is harmonic on an exterior domain of \( \mathbb{R}^2 \) with \( \tilde{w} - \tilde{h}_3 = O(|x|^{2-\epsilon_1}) \) as \( |x| \to \infty \). By spherical harmonic expansion as in (2.4) or the proof of (2.31) in [2], there exist \( \tilde{b} \in \mathbb{R}^2 \) and \( \tilde{d}_1, \tilde{d}_2 \in \mathbb{R} \) such that

\[
\tilde{w}(x) - \tilde{h}_3(x) = \tilde{b} \cdot x + \tilde{d}_1 \ln |x| + \tilde{d}_2 + O(|x|^{-1})
\]
as $|x| \to \infty$ and consequently
\[
|\bar{w}(x) - \bar{b} \cdot x| \leq |\bar{h}_3(x)| + |\bar{d}_1 \ln |x| + \bar{d}_2 + O(\ln |x|) + O(|x|) = O(|x|^2) + |\bar{d}_1 \ln |x| + \bar{d}_2 + O(|x|^{-1})| = O(|x|^{2-2\epsilon_1}) = o(|x|)
\]
as $|x| \to \infty$.

Let
\[
\bar{w}_1(x) := \bar{w}(x) - \bar{b} \cdot x.
\]
By interior estimates as used in Lemma 3.1, we have
\[
|D^k \bar{w}_1(x)| \leq C|x|^{2-2\epsilon_1-k},
\]
for some $C > 0$ for all $k = 0, \ldots, m+1$ and $|x| > 2R_1$. As in the process in obtaining (4.10), \(\bar{w}_1\) satisfies
\[
\Delta \bar{w}_1 = \bar{f}_4(x) = O(|x|^{-\zeta}) + O(|x|^{-4\epsilon_1}). \quad (4.11)
\]
Since $\zeta > 2$ and $4\epsilon_1 \in (2, 4)$, Lemmas 2.1 and 2.3 there exists a function $\bar{h}_4$ solving (4.11) in $\mathbb{R}^2 \setminus B_{R_1}$ with estimate
\[
|\bar{h}_4(x)| \leq \begin{cases} 
C|x|^{2-\zeta} (\ln |x|) + C|x|^{2-4\epsilon_1}, & \zeta \notin \mathbb{N}_*, \\
C|x|^{2-\zeta} + C|x|^{2-4\epsilon_1}, & \zeta \in \mathbb{N}_*, 
\end{cases}
\]
for some $C > 0$ for all $|x| > 2R_1$. Thus $\bar{w}_1 - \bar{h}_4$ is harmonic in $|x| > 2R_1$ with $|\bar{w}_1 - \bar{h}_4| = O(|x|^{2-2\epsilon_1})$. Since $2-2\epsilon_1 < 1$, by spherical harmonic expansion, there exist $\bar{d}, \bar{d}_3 \in \mathbb{R}$ such that
\[
\bar{w}_1(x) - \bar{h}_4(x) = \bar{d} \ln |x| + \bar{d}_3 + O(|x|^{-1})
\]
as $|x| \to \infty$ and consequently,
\[
|\bar{w}_1(x) - \bar{d} \ln |x|| \leq |\bar{h}_4(x)| + \bar{d}_3 + O(|x|^{-1}) = \begin{cases} 
O(|x|^{2-\zeta}), & \text{if } \zeta < 4\epsilon_1, \text{ and } \zeta \neq 3 \\
O(|x|^{2-\zeta} (\ln |x|)), & \text{if } \zeta < 4\epsilon_1, \text{ and } \zeta = 3 \\
O(|x|^{2-4\epsilon_1}), & \text{if } \zeta \geq 4\epsilon_1.
\end{cases} \quad (4.12)
\]
Again, we follow the process in obtaining (4.10).

Since
\[
F_{\tau}(\lambda(D^2(\frac{1}{2}x^T A x + d \ln |x|))) = f(\infty) + O(|x|^{-4})
\]
as $|x| \to \infty$, we set
\[
\bar{w}_2(x) := \bar{w}_1(x) - \bar{d} \ln |x|,
\]
which satisfies
\[ \Delta \tilde{w}_2 = O(|x|^{-\zeta}) + O(|x|^{-4}) + \begin{cases} O(|x|^{-2\zeta}), & \text{if } \zeta < 4\epsilon_1, \text{ and } \zeta \neq 3 \\ O(|x|^{-2\zeta}(\ln|x|)^2), & \text{if } \zeta < 4\epsilon_1, \text{ and } \zeta = 3 \\ O(|x|^{-8\epsilon_1}), & \text{if } \zeta \geq 4\epsilon_1. \end{cases} \]

Since \(8\epsilon_1 \in (4, 8)\), we have

\[ \Delta \tilde{w}_2 = O(|x|^{-\zeta}) + O(|x|^{-4}). \]

By Lemmas 2.1 and 2.3, we have a solution \(\tilde{h}_5\) on exterior domain with estimate

\[ |\tilde{h}_5(x)| \leq \begin{cases} C|x|^{2-\zeta} + C|x|^{-2}(\ln|x|), & \text{if } \zeta \notin \mathbb{N}_+, \\ C|x|^{2-\zeta}(\ln|x|) + C|x|^{-2}(\ln|x|), & \text{if } \zeta \in \mathbb{N}_+, \end{cases} \]

for some \(C > 0\) for all |\(x| > 2R_1\). Together with (4.12), by spherical harmonic expansion we have \(\tilde{c} \in \mathbb{R}\) such that

\[ \tilde{w}_2(x) - \tilde{h}_5(x) = \tilde{c} + O(|x|^{-1}). \quad (4.13) \]

Rotating back by \(Q^{-1}\) matrix, since \(P = Q^{-2}\), we have \(\beta \in \mathbb{R}^2, c, d \in \mathbb{R}\) such that

\[ \left| u(x) - \left( \frac{1}{2} x^T Ax + \beta x + d \ln(x^T P x) + c \right) \right| \leq C|\tilde{w}_2 - \tilde{c}| \leq C|\tilde{h}_5(Q^{-1} x)| + C|x|^{-1} \leq \begin{cases} C|x|^{2-\zeta} + C|x|^{-1}, & \text{if } \zeta \neq 3, \\ C|x|^{2-\zeta}(\ln|x|) + C|x|^{-1}, & \text{if } \zeta = 3, \end{cases} \]

for some \(C > 0\) for sufficiently large \(|x|\). Estimates for higher order derivatives follow similarly as in Lemma 4.1. The second equality in (1.9) can be obtained in (1.4) of [32]. This finishes the proof of (1.8).

It remains to prove that when \(\zeta > 3\), we have (1.10) at infinity. In fact from (4.13), we iterate once more by setting \(\tilde{w}_3 := \tilde{w}_2(x) - \tilde{c}\), which satisfies

\[ \Delta \tilde{w}_3 = O(|x|^{-\zeta}) + O(|x|^{-4}) \text{ as } |x| \to \infty. \]

By Lemmas 2.1 and 2.3, we have a solution \(\tilde{h}_6\) on exterior domain with estimate

\[ |\tilde{h}_6(x)| \leq \begin{cases} C|x|^{2-\zeta} + C|x|^{-2}(\ln|x|), & \text{if } \zeta \notin \mathbb{N}_+, \\ C|x|^{2-\zeta}(\ln|x|) + C|x|^{-2}(\ln|x|), & \text{if } \zeta \in \mathbb{N}_+, \end{cases} \]

for some \(C > 0\) in \(|x| > 2R_1\). Since \(\tilde{w}_3 - \tilde{h}_6\) is harmonic on an exterior domain of \(\mathbb{R}^2\) and satisfies \(|\tilde{w}_3 - \tilde{h}_6| = O(|x|^{-1})\) as \(|x| \to \infty\), by spherical harmonic expansion there exists \(d_4, d_5 \in \mathbb{R}\) such that

\[ \tilde{w}_3 - \tilde{h}_6 = d_4 \cos \theta |x|^{-1} + d_5 \sin \theta |x|^{-1} + O(|x|^{-2}) \]

as \(|x| \to \infty\), where \(\theta = \frac{x}{|x|}\) here. By rotating back through \(Q^{-1}\), we have the 0-order estimates in (1.10) since \(P = Q^{-2}\). For higher order derivatives, the result follows from interior estimate as in [33].
References

[1] Jiguang Bao, Jingyi Chen, Bo Guan, and Min Ji. Liouville property and regularity of a Hessian quotient equation. *Amer. J. Math.*, 125(2):301–316, 2003.

[2] Jiguang Bao, Haigang Li, and Lei Zhang. Monge-Ampère equation on exterior domains. *Calc. Var. Partial Differential Equations*, 52(1-2):39–63, 2015.

[3] Arunima Bhattacharya. Hessian estimates for Lagrangian mean curvature equation. *Calc. Var. Partial Differential Equations*, 60(6):Paper No. 224, 2021.

[4] Arunima Bhattacharya and Ravi Shankar. Optimal regularity for Lagrangian mean curvature type equations. *arXiv. 2009.04613*, 2020.

[5] Arunima Bhattacharya and Ravi Shankar. Regularity for convex viscosity solutions of Lagrangian mean curvature equation. *arXiv. 2006.02030*, 2020.

[6] Luis Caffarelli. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math.* (2), 131(1):135–150, 1990.

[7] Luis Caffarelli and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

[8] Luis Caffarelli, Basilis Gidas, and Joel Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42(3):271–297, 1989.

[9] Luis Caffarelli and Yanyan Li. An extension to a theorem of Jörgens, Calabi, and Pogorelov. *Comm. Pure Appl. Math.*, 56(5):549–583, 2003.

[10] Luis Caffarelli and Yanyan Li. A Liouville theorem for solutions of the Monge-Ampère equation with periodic data. *Comm. Pure Appl. Math.*, 56(5):549–583, 2003.

[11] Eugenio Calabi. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. *Michigan Math. J.*, 5:105–126, 1958.

[12] Sun-Yung Alice Chang and Yu Yuan. A Liouville problem for the sigma-2 equation. *Discrete Contin. Dyn. Syst.*, 28(2):659–664, 2010.

[13] Jingyi Chen, Ravi Shankar, and Yu Yuan. Regularity for convex viscosity solutions of special Lagrangian equation. *arXiv. 1911.05452*, 2019.

[14] Li Chen and Ni Xiang. Rigidity theorems for the entire solutions of 2-Hessian equation. *J. Differential Equations*, 267(9):5202–5219, 2019.

[15] Shiu Yuen Cheng and Shing-Tung Yau. Complete affine hypersurfaces. I. The completeness of affine metrics. *Comm. Pure Appl. Math.*, 39(6):839–866, 1986.

[16] Shi-Zhong Du. Necessary and sufficient conditions to Bernstein theorem of a Hessian equation. *arXiv. 2106.06211*, 2021.

[17] Harley Flanders. On certain functions with positive definite Hessian. *Ann. of Math. (2),* 71(1):135–150, 1960.

[18] Lei Fu. An analogue of Bernstein’s theorem. *Houston J. Math.*, 24(3):415–419, 1998.

[19] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[20] Matthias Günther. Conformal normal coordinates. *Ann. Global Anal. Geom.*, 11(2):173–184, 1993.

[21] Qing Han, Xiaoxiao Li, and Yichao Li. Asymptotic expansions of solutions of the Yamabe equation and the $\sigma_k$-Yamabe equation near isolated singular points. *Comm. Pure Appl. Math.*, 74(9):1915–1970, 2021.

[22] Zheng-Chao Han, Yanyan Li, and Eduardo V. Teixeira. Asymptotic behavior of solutions to the $\sigma_k$-Yamabe equation near isolated singularities. *Invent. Math.*, 182(3):635–684, 2010.

[23] Guanghao Hong. A Remark on Monge-Ampère equation over exterior domains. *arXiv. 2007.12479*, 2020.

[24] Rongli Huang and Zhizhang Wang. On the entire self-shrinking solutions to Lagrangian mean curvature flow. *Calc. Var. Partial Differential Equations*, 41(3-4):321–339, 2011.

[25] Xiaobiao Jia. Asymptotic behavior of solutions of fully nonlinear equations over exterior domains.
Konrad Jörgens. Über die Lösungen der Differentialgleichung $rt - s^2 = 1$. Math. Ann., 127:130–134, 1954.

Jürgen Jost and Yuan Long Xin. Some aspects of the global geometry of entire space-like submanifolds. volume 40, pages 233–245. 2001. Dedicated to Shing-Shen Chern on his 90th birthday.

Nick Korevaar, Rafe Mazzeo, Frank Pacard, and Richard Schoen. Refined asymptotics for constant scalar curvature metrics with isolated singularities. Invent. Math., 135(2):233–272, 1999.

An-Min Li, Ruiwei Xu, Udo Simon, and Fang Jia. Affine Bernstein problems and Monge-Ampère equations. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

Dongsheng Li, Zhisu Li, and Yu Yuan. A Bernstein problem for special Lagrangian equations in exterior domains. Adv. Math., 361:106927, 29, 2020.

Ming Li, Changyu Ren, and Zhizhang Wang. An interior estimate for convex solutions and a rigidity theorem. J. Funct. Anal., 270(7):2691–2714, 2016.

Zixiao Liu and Jiguang Bao. Asymptotic expansion and optimal symmetry of minimal gradient graph equations in dimension 2. Communications in Contemporary Mathematics, 0(0):2150110, 0.

Zixiao Liu and Jiguang Bao. Asymptotic expansion at infinity of solutions of Monge-Ampère type equations. Nonlinear Anal., 212:Paper No. 112450, 17, 2021.

Zixiao Liu and Jiguang Bao. Asymptotic Expansion at Infinity of Solutions of Special Lagrangian Equations. J. Geom. Anal., 32(3):90, 2022.

Aleksei Vasil’evich Pogorelov. On the improper convex affine hyperspheres. Geometriae Dedicata, 1(1):33–46, 1972.

Eduardo V. Teixeira and Lei Zhang. Global Monge-Ampère equation with asymptotically periodic data. Indiana Univ. Math. J., 65(2):399–422, 2016.

Chong Wang, Rongli Huang, and Jiguang Bao. On the second boundary value problem for Lagrangian mean curvature equation. arXiv:1808.01139, 2018.

Dake Wang and Yu Yuan. Hessian estimates for special Lagrangian equations with critical and supercritical phases in general dimensions. Amer. J. Math., 136(2):481–499, 2014.

Micah Warren. Calibrations associated to Monge-Ampère equations. Trans. Amer. Math. Soc., 362(8):3947–3962, 2010.

Micah Warren and Yu Yuan. Hessian estimates for the sigma-2 equation in dimension 3. Communications on Pure and Applied Mathematics, 62(3):305–321, 2009.

Micah Warren and Yu Yuan. Hessian and gradient estimates for three dimensional special Lagrangian equations with large phase. Amer. J. Math., 132(3):751–770, 2010.

Min Yan. Extension of convex function. J. Convex Anal., 21(4):965–987, 2014.

Yu Yuan. A Bernstein problem for special Lagrangian equations. Invent. Math., 150(1):117–125, 2002.

Yu Yuan. Global solutions to special Lagrangian equations. Proc. Amer. Math. Soc., 134(5):1355–1358, 2006.

Z. Liu & J. Bao
School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875, China
Email: liuzixiao@mail.bnu.edu.cn, jgbao@bnu.edu.cn