Some new higher separation axioms via sets having non-empty interior

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Abstract: Some new higher separation axioms called \( \star \)-normal, \( s^\star \)-normal, and \( w^\star \)-normal are introduced by using a newly defined closure operator called \( \star \)-closure operator and interrelation among these variants of normality are established. It is also observed that the class of \( \kappa \)-normal spaces contains all these variants of normality.

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1. Introduction

Several forms of higher separation axioms exist in the topology literature in which closed sets or generalized closed sets are separated by open sets. These closed sets are complement of open sets (or generalized open sets). These forms of generalized open sets such as semi-open sets (\( \beta \)-open sets), \( \alpha \)-open sets (\( \nu \)-sets), feebly open sets, pre-open sets, etc. have been introduced and utilized to study general topology in the past. Interior of these closed sets (or generalized closed sets) need not be non-empty, whereas interior of a non-empty regularly closed (or canonically closed) set is always non-empty. In this paper, some higher separation axioms are introduced and studied by separating sets with non-empty interior which are obtained through a generalized closure operator called \( \star \)-closure operator. It is evident from Smyth (1995) that topological structures that are more general than the usual topology are worthy of study because they can provide a suitable framework for various approaches to digital topology. Cech (1966) introduced Cech closure operator which are obtained from the Kuratowski ones by omitting the requirement of idempotency. Liu (2010), studied...
representation of closure concepts via relations and established a one to one correspondence between quasi-discrete closures (Galton, 2003; Slapal, 2003) and reflexive relations. In this paper, the authors introduced some new higher separation axioms including \( \star \)-normal spaces and it is observed that this class of space is independent of the class of normal topological spaces. In contrary to normality, it is shown that the class of \( \star \)-normal spaces neither contains the class of compact Hausdorff spaces nor contains the class of metrizable spaces. Further, it is observed in Das (2013) that the class of \( \kappa \)-normal spaces contains several generalizations of normality such as densely normal spaces (Arhangel'skii, 1996), \( \Delta \)-normal spaces (Das, 2009), quasi normal spaces (Kalantan, 2008), \( \pi \)-normal spaces (Kalantan, 2008), and almost normal spaces (Singal & Arya, 1970). In this paper, we have shown that the important class of \( \kappa \)-normal spaces also contains the class of \( \star \)-normal spaces which is independent of normality. Throughout the present paper no separation axiom is assumed unless stated otherwise. Interior of a set \( A \) in \( X \) is either denoted by \( A^o \) or \( \text{int}_X(A) \). Similarly, closure of a set \( A \) in \( X \) is denoted by \( cl_X(A) \) or \( \overline{A} \).

2. Preliminaries

Definition 2.1  Let \( X \) be any topological space, then an operator \( \star : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) defined by \( \star(A) = cl(A^\star) \) is known as \( \star \)-closure operator.

It can be easily verified that \( \star \)-closure operator satisfies the following axioms:

1. Monotone i.e. \( A \subset B \implies \star(A) \subset \star(B) \) for every \( A, B \subset X \).
2. Idempotence i.e. \( \star(\star(A)) = \star(A) \) for every \( A \subset X \).
3. Join homomorphism i.e. \( \star(A \cup B) = \star(A) \cup \star(B) \) for every \( A, B \subset X \).

But \( \star \)-closure operator need not satisfy inflationary axiom and meet homomorphism, since there exist topological spaces in which \( \star(A) \) need not contain \( A \) and \( \star(A \cap B) \) need not be equal to \( \star(A) \cap \star(B) \) which are evident from the below-mentioned examples.

Example 2.2  Let \( X = \{a, b, c\} \) and \( \tau_X = \{\emptyset, X, \{a\}, \{b, c\}\} \). For \( A = \{a, b\} \), \( \star(A) = \{a\} \). Thus \( \star(A) \) does not contain \( A \).

Example 2.3  Let \( X \) be the real line with usual topology and let \( A = [0, 1] \) and \( B = [1, 2] \). Then \( \star(A) \cap \star(B) = \{1\} \) and \( \star(A \cap B) = \emptyset \). Thus \( \star(A \cap B) \neq \star(A) \cap \star(B) \).

From the above example it is clear that \( \star(A) \) is non-inflationary, whereas \( \star(A) \) satisfies any one of the following cases:

- (Case I:)(1) \( A = \star(A) \).
- (Case II:)(2) \( A \subseteq \star(A) \).
- (Case III:)(3) \( \star(A) \subseteq A \).

For any topological space if a set satisfies Case I, then the set is regularly closed (Kuratowski, 1958), if it satisfies Case II, then the set is semi-open (Levine, 1963) (semi-open sets are referred as \( \beta \)-sets in Njastad 1965) and sets satisfying Case III are known as pre-closed sets (Mashhour, Abd El-Monsef, & El-Deeb, 1982).

Definition 2.4  (Mashhour et al., 1982) A subset \( A \) of a topological space \( X \) is said to be pre-closed if \( \star(A) \subseteq A \).

Definition 2.5  A subset \( A \) of topological space \( X \) is said to be \( \star \)-closed if \( \text{int}(A) \neq \emptyset \) and \( \star(A) \subseteq A \).

Complement of a \( \star \)-closed set is said to be \( \star \)-open.
**Definition 2.6** Let $X$ be a topological space and $A$ be a subset of $X$. Then a point $x$ in $X$ is said to be $\star$-limit point of $A$ if for every open neighborhood $U_x$ around $x$ there exists an non-empty open set $V$ in $X$ such that $V \subseteq A$ and $V \cap U_x$ is not empty.

**Remark 2.7** Let $X$ be a topological space and $A \subseteq X$ then $A$ is $\star$-closed iff $A$ contains all its $\star$-limit points. Every $\star$-limit point of a set $A$ is a limit point of the set but the converse need not to be true.

**Example 2.8** Let $X = \mathbb{R}$ with co-finite topology and $A = [0, 1]$, here $\{2\}$ is a limit point of $A$ but not a $\star$-limit point of $A$ because open sets around $\{2\}$ does not intersect any open set inside $A$.

From the above definition it is obvious that every $\star$-closed set is pre-closed but the converse need not to be true. Further every closed set is pre-closed but $\star$-closed sets are independent of closed sets.

The following implications are obvious from the definitions.

$$
\text{clopen} \longrightarrow \text{regularly closed} \longrightarrow \star\text{-closed} \\
\text{closed} \longrightarrow \text{pre-closed}
$$

**Example 2.9** A set which is pre-closed but not $\star$-closed. Let $X = \{a, b, c\}$ and $\tau_X = \{X, \emptyset, \{b, c\}\}$ be the topology on $X$, then $A = \{a, b\}$ is a pre-closed set which is not $\star$-closed in $X$ as $\text{int}(A) = \emptyset$.

**Example 2.10** A set which is closed but not $\star$-closed.

Let $X = \{a, b, c\}$ and $\tau_X = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then $A = \{b\}$ is closed but not $\star$-closed in $X$ as $\text{int}(A) = \emptyset$.

**Example 2.11** A set which is $\star$-closed but not closed.

Let $X = \{a, b, c\}$ and $\tau_X = \{X, \emptyset, \{a\}, \{b, c\}\}$ be the topology on $X$, then $A = \{a, b\}$ is not closed in $X$ but is $\star$-closed as $\text{int}(A) \neq \emptyset$ and $\star(A) = \{a\} \subseteq A$.

**Proposition 2.12** A subset $U$ of $X$ is $\star$-open iff $U$ is pre-open and $\overline{U} \neq X$.

**Proposition 2.13** Let $X$ be a topological space and $A$ and $B$ be two $\star$-closed sets in $X$ then $A \cup B$ is $\star$-closed.

**Remark 2.14** Intersection of two $\star$-closed sets need not be $\star$-closed in $X$. Let $X$ be the real line with usual topology and let $A = [0, 1]$ and $B = [1, 2]$. Then $A$ and $B$ are regularly closed, thus $\star$-closed but $A \cap B = \{1\}$ is not $\star$-closed.

**Remark 2.15** Union of two $\star$-open sets need not be $\star$-open. Let $X$ be the real line with usual topology and let $A = (-\infty, 0)$, $B = (-1, \infty)$. Then $A, B$ are both $\star$-open whereas $A \cup B$ is not $\star$-open as $A \cup B = X$.

3. Some higher separation axioms

**Definition 3.1** A topological space $X$ is said to be

1. $\star$-normal if every pair of disjoint $\star$-closed sets can be separated by disjoint open sets.
2. strongly $\star$-normal ($s^\star$-normal) if every pair of disjoint pre-closed sets can be separated by disjoint open sets.
(3) weakly $\star$-normal ($w^*\text{-normal}$) if every pair of disjoint closed sets whose interior is non-empty can be separated by disjoint open sets.

Recall that a space $X$ is completely normal if every pair of separated sets can be separated by disjoint open sets and a space $X$ is $\kappa$-normal (Stechepin, 1972) (mildly normal (Singal & Singal, 1973)) if every pair of disjoint regularly closed sets can be separated by disjoint open sets.

**Lemma 3.2** A topological space $X$ is $\star$-normal iff for every $\star$-closed set $A$ contained in a $\star$-open set $U$ there exists an open set $V$ such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Since every regularly closed set is $\star$-closed and closure of every open set is regularly closed, the proof of the following theorem is analogous to the proof of Urysohn’s lemma.

**Theorem 3.3** A topological space $X$ is $\star$-normal iff for every pair of disjoint $\star$-closed sets there exists a continuous function $f : X \to [0, 1)$ such that $f(A) = 0$ and $f(B) = 1$.

**Proof.** Let $X$ be a $\star$-normal space and $A$, $B$ be two disjoint $\star$-closed sets in $X$. Then $A \subseteq X - B = U_1$ (say). Since $U_1$ is a $\star$-open set containing the $\star$-closed set $A$, by Lemma 3.2, there exists an open set $U_2$ such that $A \subseteq U_1 \subset \overline{U}_1 \subset U_2$. Since $\overline{U}_2$ is closure of an open set, $\overline{U}_2$ is regularly closed. Thus $U_2$ is a $\star$-closed set contained in a $\star$-open set $U_1$. Since $U_1$ is a open set which satisfies the condition of being a $\star$-open set i.e $\overline{U}_1 \neq X$ and $U_1 \subset \overline{U}_1$. Thus again applying Lemma 3.2, there exist open sets $U_3$, $U_4$ such that $A \subset U_3 \subset \overline{U}_3 \subset U_4$ and $\overline{U}_4 \subset U_2 \subset \overline{U}_2 \subset U_1$. Thus by continuing this process we obtain for each dyadic rational $r$, an open set $U_r$ satisfying $r \geq s$ implies $U_s \subset U_r$. Let us define a mapping $f : X \to [0, 1)$ by

$$f(x) = \begin{cases} 
1 & \text{if } x \notin U_r \\
\inf \{x : x \in U_r\} & \text{if } x \text{ belongs to some } U_r.
\end{cases}$$

Clearly $f(A) = 0$ and $f(B) = 1$. Continuity of $f$ can be proved as in the Urysohn’s Lemma. Conversely, let $A$ and $B$ be two disjoint $\star$-closed subsets of $X$ for which a continuous function $f : X \to [0, 1)$ exists satisfying $f(A) = 0$ and $f(B) = 1$. Thus $f^{-1}(0, \frac{1}{2})$ and $f^{-1}(\frac{1}{2}, 1)$ are two disjoint open sets separating $A$ and $B$, respectively. Hence $X$ is $\star$-normal.

It is well known that every metrizable space is normal and every compact Hausdorff space is normal, but the following example establishes that these facts need not hold for $\star$-normal spaces.

**Example 3.4** A metrizable compact Hausdorff space which is not $\star$-normal. Let $X$ be the real line with usual topology and $Y = [-2, 2]$. Here $A = [-2, -1] \cup (-1/n : n \in \mathbb{Z})$ and $B = [1, 2] \cup (1/n : n \in \mathbb{Z})$ are two disjoint $\star$-closed sets which cannot be separated by disjoint open sets. Thus $X$ is metrizable but not $\star$-normal and the subspace topology on $Y$ is compact Hausdorff but not $\star$-normal.

**Lemma 3.5** A topological space $X$ is $s^\star$-normal iff for every pre-closed set $A$ contained in a pre-open set $U$ there exists an open set $V$ such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

**Proof.** Let $A$ be a pre-closed set in $X$ and $U$ be a pre-open set containing $A$. Then $A$ and $X - U$ are disjoint pre-closed sets in $X$ and so by $s^\star$-normality of $X$, there exist two disjoint open sets $V$ and $W$ such that $A \subseteq V$ and $X - U \subset W$. Now $V \cap W = \phi$. Thus $A \subseteq V \subseteq X - W \subseteq U$ implies $A \subseteq V \subseteq \overline{V} \subseteq U$. Conversely, let $A$ and $B$ be two disjoint pre-closed sets in $X$. Since $X - B$ is pre-open in $X$ and $A \subseteq X - B = U$, by hypothesis there exists an open set $V$ such that $A \subseteq V \subseteq \overline{V} \subseteq U$. Here $W$ and $X - W$ are two disjoint open sets in $X$ containing $A$ and $B$, respectively. Thus the space is $s^\star$-normal.

Since every $\star$-closed set is pre-closed, the proof of the following theorem is analogous to the proof of Theorem 3.3.
Theorem 3.6 A topological space $X$ is $s^\star$-normal iff for every pair of disjoint pre-closed sets $A$ and $B$ there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

Lemma 3.7 A topological space $X$ is $w^\star$-normal iff for every closed set $A$ with a non-empty interior contained in an open set $U$ satisfying $U \neq X$, there exists an open set $V$ such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Proof Let $X$ be a $w^\star$-normal space. Let $A$ be a closed set with $\text{int} A \neq \emptyset$ and $U$ be an open set containing $A$ satisfying $\overline{U} \neq X$. Here $A$ and $B = \overline{X} - U$ are two disjoint closed sets whose interiors are non-empty since $X - \overline{U} \subseteq X - U = B$. So by $w^\star$-normality of $X$ there exist two disjoint open sets $V$ and $W$ such that $A \subseteq V$ and $X - U \subseteq W$. Now $V \cap W = \emptyset$. Thus $A \subseteq V \subseteq X - W \subseteq U$ which implies $A \subseteq V \subseteq \overline{V} \subseteq U$. The converse part is obvious. 

The proof of the following theorem is analogous to the proof of Urysohn’s type lemma.

Theorem 3.8 A topological space is $w^\star$-normal iff for every pair of disjoint closed sets $A$, $B$ with non-empty interiors there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

The following implications are obvious from the definitions defined in the present paper and notions defined in Arhangel’skii (1996), Das (2009, 2013), Kalantan (2008), Singal and Arya (1970) but none of these is reversible.

Example 3.9 A space which is $\star$-normal but not completely normal.

Let $X = \{a, b, c, d, e\}$ and $r_x$ be the topology generated by the subbasic open sets $\{\{a\}, \{b, c\}, \{d, e\}, \{\}\}$. This space is not completely normal because for separated sets $\{a\}$ and $\{c\}$ there does not exist disjoint open sets separating them. In this space the $\star$-closed sets are $\{\{a\}, \{b\}, \{b, c\}, \{d, e\}, \{a, b, c, d\}, \{b, d\}, \{a, c, d, e\}\}$ and so every pair of disjoint $\star$-closed sets can be separated by disjoint open sets. Hence $X$ is $\star$-normal.

Example 3.10 A space which is $\star$-normal but not $s^\star$-normal.

Let $X$ be the space as defined in Example 3.9. This space is not $s^\star$-normal because for disjoint pre-closed sets $\{a\}$ and $\{c, d, e\}$ there does not exist disjoint open sets separating them.

Example 3.11 A space which is $w^\star$-normal but not $\star$-normal.

Let $X = \{a, b, c, d\}$ and $r_x$ be the topology generated by the subbasic open sets $\{\{a\}, \{d\}, \{b, c, d\}, \{a, b, c\}\}$. The space is not $\star$-normal because for disjoint $\star$-closed sets $\{a, b\}$ and $\{c, d\}$ there does not exist disjoint open sets separating them. But every pair of disjoint closed sets can be separated by disjoint open sets. Thus the space is normal and so $w^\star$-normal.
Example 3.12 A space which is $w^*$-normal but not normal.

Example 3.9 is $w^*$-normal but not normal because for disjoint closed sets (a) and (c) there does not exist disjoint open sets separating them.

Example 3.13 A space which is normal but not $s^*$-normal.

Example 3.11 is normal but not $s^*$-normal because for disjoint pre-closed sets (a, b) and (c, d) there does not exist disjoint open sets separating them.

Example 3.14 A space which is $\kappa$-normal but not $w^*$-normal.

Let $X = \{a, b, c, d, e, f\}$ and $r$ be the topology generated by the subbasic open sets $\{(a), (a, b, c), (a, b, c, d), (e), (c, d, e), (b, c, d, e), (f)\}$. Here regularly closed sets are $\{(a), (b, c, d, e), (e, f), (a, b, c, d), (a, b, c, d, f), (e), (a, f), (b, c, d, e), (a, b, c, d, e), (f)\}$. This space is $\kappa$-normal as every pair of disjoint regularly closed sets can be separated by disjoint open sets but is not $w^*$-normal because for disjoint closed sets having non-empty interior (a, b) and (d, e) there does not exist disjoint open sets separating them.

Example 3.15 A space which is completely normal but neither $\neq$-normal nor $s^*$-normal.

The space defined in Example 3.11 is completely normal but is not $\neq$-normal because for disjoint $\neq$-closed sets (a, b) and (c, d) there does not exist disjoint open sets separating them and is not $s^*$-normal because for disjoint pre-closed sets (a, b) and (c, d) there does not exist disjoint open sets separating them.

From Example 3.4, it is clear that every compact Hausdorff spaces need not be $\neq$-normal. Thus it is natural to ask: Which compact Hausdorff spaces are $\neq$-normal? It is established in the following result that in the class of door spaces every compact Hausdorff space is $\neq$-normal.

Recall that a subset $A$ of a space $X$ is locally closed if $A$ is open in its closure in $X$. A space $X$ is a \textit{submaximal space} (Bourbaki, 1961) if every subset of $X$ is locally closed. A space $X$ is a \textit{door space} (Arhangel’skii & Collins, 1995) if every subset of $X$ is open or closed. Clearly, every door space is submaximal.

Lemma 3.16 In a door space every $\neq$-closed set is closed.

Proof Let $X$ be a door space and $A$ be a $\neq$-closed subset of $X$. Since $X$ is a door space, $A$ is either open or closed. If $A$ is open then $A = A\overline{\neq}$ as $A$ is $\neq$-closed. Thus $A$ is in either case closed. □

Theorem 3.17 In a door space every normal space is $\neq$-normal.

Proof The proof is obvious by Lemma 3.16. □

Corollary 3.18 In a door space, every compact Hausdorff space is $\neq$-normal.

Since every door space is submaximal it is natural to ask the following question:

Question 3.19 Does there exist a submaximal normal space which is not $\neq$-normal?

4. $\neq$-Closure operator in lower separation axioms

It is well known that in a $T_1$-space every singleton set is closed. Also it is observed in Kohli and Das (2002) that a space is Hausdorff if and only if every singleton set is $\theta$-closed. But in contrary to this the following result establishes that if in a space every singleton is $\neq$-closed then the space is discrete, thus satisfies all separation axioms.
THEOREM 4.1 A topological space $X$ is discrete if and only if every singleton in $X$ is $\star$-closed.

Proof Let $X$ be a discrete space and let $x \in X$. Since $X$ is discrete, every singleton is open. Thus $\text{int}\{x\} \neq \emptyset$ and $\star\{x\} \subseteq \{x\}$. Hence $\{x\}$ is not only $\star$-closed but also regularly closed. Conversely, let every singleton in $X$ is $\star$-closed. Then interior of every of singleton set $\{x\}$ in $X$ is non-empty and so $\text{int}\{x\} = \{x\}$. Hence $X$ is discrete. \hfill $\Box$

Definition 4.2 A topological space $X$ is said to be a $T_1^*$ space if for every distinct points $x$ and $y$ there exist $\star$-open sets $U$ and $V$ such that $x \in U$, $y \not\in U$ and $y \in V$, $x \not\in V$.

Definition 4.3 A topological space $X$ is said to be a $T_2^*$ space if for every distinct points $x$ and $y$ there exist disjoint $\star$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

From the definitions it is obvious that every $T_2^*$ space is $T_1^*$.

THEOREM 4.4 Every $T_2$ space is $T_2^*$.

Proof Let $X$ be a $T_2$ space and let $x, y$ be two distinct points in $X$. Then there exist two disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$. Since $\overline{U} \cap V = \emptyset$, here $\overline{U} \neq X$, $U \subset \overline{U}$. Similarly, since $U \cap \overline{V} = \emptyset$, we have $\overline{V} \neq X$ and $V \subset \overline{V}$. Thus, by Proposition 2.12, $U$ and $V$ are two disjoint $\star$-open sets in $X$ separating $x$ and $y$, respectively. Hence $X$ is $T_2^*$.

From definitions and above observation the following implications are obvious.

$T_2 \implies T_2^* \implies T_1^*$

Example 4.5 A $T_2^*$ space which is not $T_2$.

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Here the $\star$-open sets are $\{b\}, \{a\}, \{c\}, \{b, c\}$. Hence the space is $T_2^*$ but is not $T_2$ even not $T_1$ as distinct points $\{a\}$ and $\{c\}$ cannot be separated by open sets.

Remark 4.6 From the above example it is clear that $T_1^*$ spaces need not imply $T_2$-ness. Thus it is natural to ask : Does $T_1$-ness imply $T_1^*$-ness? The following example provide answer to this question in negative. Thus the notions $T_1$ and $T_1^*$ are independent of each other.

Example 4.7 A $T_1^*$ space which is not $T_1^*$.

Let $X$ be an infinite set with co-finite topology.

Clearly the space is $T_1$. But the space is not $T_1^*$ as there does not exists any $\star$-open set in $X$. Suppose there exists a $\star$-open set $U \neq X$ in $X$ containing $x$ then there is a point $y \not\in U$. Thus $X - U$ is a $\star$-closed set containing $y$. Since $X - U$ is $\star$-closed, $\text{int}(X - U) \neq \emptyset$. Thus $X - U$ contains infinitely many points and so $x \not\in (X - U)$ will become a limit point of $(X - U)$. Hence $(X - U)$ cannot be $\star$-closed and so $U$ cannot be $\star$-open.

Definition 4.8 A topological space $X$ is said to be $\star$-regular if for every closed set $A$ and a point $x \not\in A$ there exist disjoint $\star$-open sets $U$ and $V$ such that $x \in U$ and $A \subset V$.

THEOREM 4.9 Every regular space is $\star$-regular.

Proof Let $X$ be a regular space and $A$ be a closed set with $x \not\in A$. By regularity of $X$ there exist disjoint open sets $U$ and $V$ such that $x \in U$, $A \subset V$. Since $\overline{U} \cap V = \emptyset$, we have $\overline{U} \neq X$ and $U \subset \overline{U}$. Thus $U$ is a $\star$-open set. Similarly, $V \neq X$ and $V \subset \overline{V}$. So $U$ and $V$ are disjoint $\star$-open sets separating $x$ and $A$, respectively. Hence $X$ is $\star$-regular. \hfill $\Box$
5. Subspaces and preservation under mapping

Let X be a topological space and Y be its subspace then the following example shows that every \( \star \)-closed subset A of Y need not be \( \star \star \)-closed in X.

Example 5.1 Let \( X = \{a, b, c, d\} \) and \( r_x = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, c, d\}\} \). Let \( Y = \{a, b, c\} \), then A = \( \{a, b\} \) is \( \star \)-closed in Y but not \( \star \star \)-closed in X.

Proposition 5.2 Let X be a topological space and Y be its \( \star \star \)-closed subspace, then every \( \star \star \)-closed subset of Y is \( \star \)-closed in X.

Proof Let A be a \( \star \)-closed set in Y which implies A contains all its \( \star \)-limit points i.e there does not exists any point x outside A such that every open set in Y around x intersects an open set inside A. Since Y is \( \star \star \)-closed in X then there does not exists any point y in X \( \setminus Y \) such that every open set in X around y intersects any open set inside Y. To prove A is \( \star \)-closed in X, let \( z \notin A \). Here two cases arises either \( z \in Y \) or \( z \in X \setminus Y \). In both the cases there exists an open set in X around z which does not intersect any open set in A. Thus z is not a \( \star \)-limit point of Y which implies A is \( \star \)-closed in X.

Theorem 5.3 A \( \star \star \)-closed subspace of a \( \star \star \)-normal space is \( \star \star \)-normal.

Proof Let X be a \( \star \star \)-normal space and Y be its \( \star \star \)-closed subspace. To show that Y is \( \star \star \)-normal, let A and B be two disjoint \( \star \star \)-closed sets in Y. Since, Y is \( \star \star \)-closed subspace of X then A and B are also two disjoint \( \star \)-closed sets in X. So by \( \star \star \)-normality of X there exist two disjoint open sets say U and V in X such that A \( \subseteq U \) and B \( \subseteq V \). Clearly, \( U \cap Y \) and \( V \cap Y \) are two disjoint open sets in Y such that A \( \subseteq U \cap Y \) and B \( \subseteq V \cap Y \). Thus Y is a \( \star \star \)-normal space.

Let X be a topological space and Y be its subspace then every closed subset A of Y with \( \text{int}_Y(A) \neq \emptyset \) need not be closed in X is shown in the following example.

Example 5.4 Let \( X = \{a, b, c, d\} \) with \( r_x = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, c, d\}\} \) and \( Y = \{a, b, c\} \). Then A = \( \{a, b\} \) is closed in Y with \( \text{int}_Y(A) \neq \emptyset \) but not closed in X.

Theorem 5.5 Let X be a topological space and Y be its closed subspace with \( \text{int}_Y(Y) \neq \emptyset \), then every closed subset A of Y with \( \text{int}_Y(A) \neq \emptyset \) is closed in X with \( \text{int}_X(A) \neq \emptyset \).

Proof Let Y be a closed subspace of X with \( \text{int}_Y(Y) \neq \emptyset \) and A be closed in Y with \( \text{int}_Y(A) \neq \emptyset \). Let \( \text{int}_Y(A) = U \) then there exists an open set V in X such that U = V \( \cap Y \). Since Y closed in X and \( \text{int}Y \neq \emptyset \). Let \( \text{int}Y = W. \) Now \( W \cap V \) is an open set in A so \( \text{int}_X(A) \) is non-empty and A is closed in X. Therefore A is closed in X with \( \text{int}_X(A) \neq \emptyset \).

Theorem 5.6 Let X be a \( w^* \)-normal space and Y be a closed subspace of X with \( \text{int}_Y(Y) \neq \emptyset \), then Y is \( w^* \)-normal.

Proof Let X be a \( w^* \)-normal space and Y be a closed subspace of X with \( \text{int}_Y(Y) \neq \emptyset \). To show that Y is \( w^* \)-normal, let A and B be two disjoint closed sets in Y with \( \text{int}_Y(A) \neq \emptyset \) and \( \text{int}_Y(B) \neq \emptyset \). Since Y is closed subspace of X and \( \text{int}_Y(Y) \neq \emptyset \), A and B are also two disjoint closed sets in X and \( \text{int}_X(A) \neq \emptyset \), \( \text{int}_X(B) \neq \emptyset \). So by \( w^* \)-normality of X there exist two disjoint open sets say U and V in X such that A \( \subseteq U \) and B \( \subseteq V \). Clearly, \( U \cap Y \) and \( V \cap Y \) are two disjoint open sets in Y such that A \( \subseteq U \cap Y \) and B \( \subseteq V \cap Y \). Thus, Y is a \( w^* \)-normal space.

Let X be a topological space and Y be its subspace then every pre-closed subset A of Y need not be pre-closed in X is evident from Example 5.4, in which A is preclosed in Y but not pre-closed in X.

Theorem 5.7 Let X be a topological space and Y be its pre-closed subspace, then every pre-closed subset of Y is pre-closed in X.
Theorem 5.8 A pre-closed subspace of a $s^*$-normal space is $s^*$-normal.

Proof Let $X$ be a $s^*$-normal space and $Y$ be its pre-closed subspace. To show that $Y$ is $s^*$-normal, let $A$ and $B$ be two disjoint pre-closed sets in $Y$. Since $Y$ is pre-closed subspace of $X$, $A$ and $B$ are also two disjoint pre-closed sets in $X$. So by $s^*$-normality of $X$, there exist two disjoint open sets say $U$ and $V$ in $X$ such that $A \subseteq U$ and $B \subseteq V$. Clearly, $U \cap Y$ and $V \cap Y$ are two disjoint open sets in $Y$ such that $A \subseteq U \cap Y$ and $B \subseteq V \cap Y$. Thus $Y$ is a $s^*$-normal space.

Theorem 5.9 A continuous clopen image of a $*$-normal space is $*$-normal.

Proof Let $X$ be a $*$-normal space and $f: X \rightarrow Y$ be a continuous, closed and onto map. To show that $Y$ is $*$-normal, let $A$ and $B$ be two disjoint $*$-closed sets in $Y$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $*$-closed sets in $X$. Since $X$ is $*$-normal, there are two disjoint open sets $U$ and $V$ in $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Also, since $f$ is closed map, $f(U \setminus A)$ and $f(V \setminus B)$ are closed in $Y$. Clearly $S = Y \setminus f(U \setminus A)$ and $T = Y \setminus f(V \setminus B)$ are open sets in $Y$ such that $A \subseteq S$ and $B \subseteq T$. Also $S \cap T = (Y \setminus f(U \setminus A)) \cap (Y \setminus f(V \setminus B)) = Y \setminus f(U \setminus A \cup V \setminus B) = Y \setminus f(X) = Y \setminus \emptyset$ which implies $S \cap T = \emptyset$. This proves that $Y$ is $*$-normal.