EXACT NORM ESTIMATES FOR MULTIVARIATE DILATION OPERATORS BETWEEN TWO BILATERAL WEIGHT GRAND LEBESGUE SPACES

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Abstract.

We give in this short paper a sharp estimate for the norm of a multivariate dilation operator generated by multi-matrix (tensor) linear argument transformation (dilation operator) between two different weight Lebesgue-Riesz and Grand Lebesgue Spaces (GLS).

We consider also some examples and study the compactness of these operators.

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1. Introduction. Notations. Statement of problem.

A. Matrix dilation in Lebesgue-Riesz spaces.

Let here \( X = \mathbb{R}^d \) equipped with Borelian sigma-field and with Lebesgue measure \( dx \) and let \( f : \mathbb{R}^d \to \mathbb{R} \) be some (measurable) function belonging to the space \( L^p(\mathbb{R}^d), \ p \geq 1 \). The norm of a function \( f \) this space will be denoted as ordinary

\[
|f|_{L^p} = |f|_p \overset{def}{=} \left( \int_X |f(x)|^p \ dx \right)^{1/p}.
\]

Let also \( A \) be non degenerate: \( \det(A) \neq 0 \) linear (homogeneous) map (matrix) acting from \( \mathbb{R}^d \) to \( \mathbb{R}^d \).

Define a matrix dilation, or compression, in the terminology of an article \[33\], operator of a form

\[
V_A[f] = f(Ax).
\]
Obviously,
\[
|V_A[f]|_p^p = \int_{\mathbb{R}^d} |f(Ax)|^p \, dx = \int_{\mathbb{R}^d} |\det(A)|^{-1} |f(y)|^p \, dy = |\det(A)|^{-1} |f|_p^p,
\]
or equally
\[
|V_A[f]|_p = |\det(A)|^{-1/p} |f|_p.
\]

(1.2)

In particular, if \( d = 1 \) and following \( V_A[f](x) = f(A \cdot x) \), \( A = \text{const} \neq 0 \); then \( V_A[\cdot] \) is the classical dilation operator, and we have
\[
|V_A[f]|_p = |A|^{-1/p} |f|_p.
\]

(1.2a)

The last equalities (1.2) (and (1.2a)) may be rewritten as follows. Note that the fundamental function \( \phi(L_p(\mathbb{R}^d), \delta) \) of the \( L_p(\mathbb{R}^d) \) space has a form
\[
\phi(L_p(\mathbb{R}^d), \delta) = \delta^{|1/p}, \delta \in (0, \infty).
\]

At the same result is true for arbitrary Lebesgue-Riesz space \( L_p, 1 \leq p < \infty \) constructed over any measure space equipped with diffuse measure, see [1], chapters 1,2.

Recall that for arbitrary rearrangement invariant space \((S, || \cdot ||_S)\) (over \( R^d \) or arbitrary another measure space with diffuse measure) the fundamental function \( \phi(S, \delta) \) is defined as follows
\[
\phi(S, \delta) := ||I_D(\cdot)||_S,
\]
where \( I_D(\cdot) \) denotes an indicator function of a measurable set \( D \) and \( \text{mes}(D) = \delta, \delta \in [0, \infty] \).

So,
\[
|V_A[f]|_{L_p} = \phi(L_p, |\det A|^{-1}) \cdot |f|_{L_p}
\]

(1.3)
or equally
\[
||V_A[f]||_{L_p \to L_p} = \phi(L_p, |\det A|^{-1}).
\]

(1.3a)

**B. Matrix dilation in weighted Lebesgue-Riesz spaces.**

Let \(|x|, x \in \mathbb{R}^d\) be arbitrary (complete) norm in the whole space \( \mathbb{R}^d \), for instance, the classical Euclidean, and let \( ||A|| \) be correspondent matrix norm:
\[
||A|| \overset{\text{def}}{=} \sup_{0 \neq x \in \mathbb{R}^d} \left[ \frac{|Ax|}{|x|} \right],
\]
so that \(|Az| \leq ||A|| \cdot |z|, z \in \mathbb{R}^d\).

Let also \( \alpha = \text{const} > 0 \) and \( \mu_\alpha(\cdot) \) be weight measure defined on the Borelian sets on \( \mathbb{R}^d \) :
\[
\mu_\alpha(D) \overset{\text{def}}{=} \int_D |x|^{\alpha} \, dx.
\]
The correspondent for the space $L_{p,\alpha}$ norm for the function $f : \mathbb{R}^d \to \mathbb{R}$ will be denoted by $|f|_{p,\alpha}$:

$$|f|_{p,\alpha}^p := \int_{\mathbb{R}^d} |f(x)|^p \ |x|^\alpha \ dx.$$ 

We deduce

$$|V_A[f]|_{p,\alpha}^p = \int_{\mathbb{R}^d} |f(Ax)|^p \ |x|^\alpha \ dx = |\det A|^{-1} \int_{\mathbb{R}^d} |f(y)|^p \ |A^{-1}y|^\alpha \ dy \leq$$

$$|\det A|^{-1} \ |A^{-1}|^\alpha \int_{\mathbb{R}^d} |f(y)|^p \ |y|^\alpha \ dy = |\det A|^{-1} \ |A^{-1}|^\alpha \ |f|_{p,\alpha}^p;$$

or equally

$$|V_A[f]|_{p,\alpha} \leq |\det A|^{-1/p} \ |A|^{-\alpha/p} \ |f|_{p,\alpha},$$

or equally

$$||V_A||_{(L_{p,\alpha} \to L_{p,\alpha})} \leq |\det A|^{-1/p} \ |A|^{-\alpha/p}. \quad (1.4)$$

Equivalent form:

$$||V_A||_{(L_{p,\alpha} \to L_{p,\alpha})} \leq \phi \left( L_{p,\alpha}, |\det(A)|^{-1} \ |A|^{-\alpha} \right). \quad (1.5)$$

If for example the matrix $A$ is diagonal: $A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$, and following $|\det A| = \prod_k |\lambda_k|$, then the relation (1.4) can be transformed as an equality:

$$||V_A||_{(L_{p,\alpha} \to L_{p,\alpha})} = |\det A|^{-1+\alpha}/p, \ \det A \neq 0. \quad (1.6)$$

In particular, if $\lambda_k = \lambda = \text{const} \neq 0, \ k = 1, 2, \ldots, d$ then

$$||V_A||_{(L_{p,\alpha} \to L_{p,\alpha})} = |\lambda|^{-d(1+\alpha)/p}. \quad (1.7)$$

Our purpose in this short article is investigation of these operators in Grand Lebesgue Spaces (GLS), as well as in its multivariate anisotropic version AGLS.

We set ourselves a goal to derive the sharp estimates for the norms of these estimates, similar to the estimates (1.2), (1.7).

Analogous statement of this problem for another Banach spaces, namely: Lorentz, Orlicz, Marcinkiewicz etc. closed relatively as a rule with computation of the so-called Boyd’s and Shimogaki etc. indices and as a consequence with application in turn in the theory of operators interpolation, theory of Fourier series in the one-dimensional case $d = 1$ see, e.g. in monograph and articles [1], chapter 7; [23], [26], [28], [31], [34], [33] etc.

The immediate predecessor of offered article is the authors preprint [31], where are obtained an upper bounds for the norms of dilation operators acting in these spaces in the one-dimensional case.

Another operators acting in GLS spaces: Hardy, Riesz, Fourier, maximal, potential, composition etc. are investigated, e.g. in [9], [12], [13], [19], [20].
2. Grand Lebesgue Spaces (GLS).

We recall first of all here for reader convenience some definitions and facts from the theory of GLS spaces.

Recently, see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11] etc. appear the so-called Grand Lebesgue Spaces GLS

\[ G(\psi) = G = G(\psi; a; b); \quad a; b = \text{const}; \quad a \geq 1, \quad b \leq \infty \]

spaces consisting on all the measurable functions \( f : X \to \mathbb{R} \) with finite norms

\[ ||f||G(\psi) \overset{df}{=} \sup_{p \in (a,b)} \left[ |f|^p \psi(p) \right]. \]  

(2.1)

Here \( \psi = \psi(p), \ p \in (a, b) \) is some continuous positive on the open interval \( (a; b) \) function such that

\[ \inf_{p \in (a,b)} \psi(p) > 0. \]

We define formally \( \psi(p) = +\infty, \ p \notin [a, b] \).

We will denote

\[ \text{supp}(\psi) \overset{df}{=} (a; b). \]

The set of all such a functions with support \( \text{supp}(\psi) = (A, B) \) will be denoted by \( \Psi(A, B) \).

These spaces are non-trivial, are rearrangement invariant; and are used, for example, in the theory of Probability, theory of Partial Differential Equations, Functional Analysis, theory of Fourier series, Martingales, Mathematical Statistics, theory of Approximation etc.

They does not coincide in general case with classical rearrangement invariant spaces: Lorentz, Orlicz, Marcinkiewicz etc., see [8], [10], [9], [22].

Notice that the classical Lebesgue - Riesz spaces \( L_p \) are extremal case of Grand Lebesgue Spaces; the exponential Orlicz spaces are the particular cases of Grand Lebesgue Spaces, see [11], [12].

Let a function \( f : X \to \mathbb{R} \) be such that

\[ \exists(a, b) : \quad 1 \leq a < b \leq \infty \Rightarrow \forall p \in (a, b) \ |f|^p < \infty. \]

Then the function \( \psi = \psi(p) \) may be naturally defined by the following way:

\[ \psi_f(p) := |f|^p, \ p \in (a, b). \]

Recall now that the fundamental function \( \phi(G\tau, \delta), \ 0 \leq \delta \leq \mu(X) \) for the Grand Lebesgue Space \( G\tau \) may be calculated by the formula

\[ \phi(G\tau, \delta) = \sup_{p \in (a,b)} \left[ \frac{\delta^{1/p}}{\tau(p)} \right]. \]

This notion play a very important role in the theory of operators, Fourier analysis etc., see [1]. The detail investigation of the fundamental function for GLS is done in [9], [11].
3. Main result: the norm of matrix dilation operator on GLS.

Let the measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) be such that there exists a function \( \psi(\cdot) \) from the set \( \Psi(a, b), 1 \leq a < b \leq \infty \) for which \( f \in G\Psi(a, b) \). For instance, the function \( \psi(\cdot) \) can be picked as a natural function for the function \( f(p) := |f|_p \), if of course the last function is finite for all the values \( p \) from some non-trivial interval \( (a, b) \).

Suppose the function \( \psi(\cdot) \) has a form

\[
\psi(p) = \frac{\nu(p)}{\zeta(p)}, \quad p \in (a, b),
\]

where both the functions \( \nu(\cdot), \zeta(\cdot) \) belong also to the same set \( G\Psi(a, b) \).

**Theorem 3.1.** Assume (in the notations of the first two sections) \( \det(A) \neq 0 \). Then

\[
||V_A[f]||_{G\nu} \leq \phi(G\zeta, |\det(A)|^{-1}) ||f||_{G\psi},
\]

and the last estimate is in general case non-improvable.

**Proof.** We can and will suppose without loss of generality \( ||f||_{G\psi} = 1 \). Then

\[
|f|_p \leq \psi(p), \quad p \in (a, b).
\]

We use the inequality (1.2):

\[
|V_A[f]|_p = |\det(A)|^{-1/p} |f|_p \leq |\det(A)|^{-1/p} \psi(p),
\]

or equally

\[
\frac{|V_A[f]|_p}{\nu(p)} \leq \frac{|\det(A)|^{-1/p}}{\zeta(p)}. \tag{3.2}
\]

Let us take the supremum over \( p \in (A, B) \) from both the sides of the last inequality (3.2), taking into account the direct definitions of norm and fundamental function for Grand Lebesgue Spaces:

\[
||V_A[f]||_{G\nu} \leq \phi\left(G\zeta, |\det(A)|^{-1}\right) ||f||_{G\psi}. \tag{3.3}
\]

Inequality (3.3) becomes an equality if for example \( \psi(p) = |f|_p \), i.e. the function \( \psi(p) \) is the natural function for the source function \( f(\cdot) \).

This completes the proof of theorem 3.1.

4. Weight estimates for GLS.

The notion of Grand Lebesgue Spaces may be easy generalized on the arbitrary measure space with sigma-finite measure, see \([2]-[11]\) etc. Denote for instance

\[
||f||_{G\psi_\alpha} \overset{\text{def}}{=} \sup_{p \in (a, b)} \left[ \frac{|f|_{p, \alpha}}{\psi(p)} \right], \tag{4.0}
\]

where as before \( \alpha = \text{const} > 0, \psi(\cdot) \in \Psi(a, b), 1 \leq a < b \leq \infty \).
Assume again in the notations of the first two sections \( x \in R^d, \det(A) \neq 0 \). Let also \( \psi(\cdot), \zeta(\cdot) \) be two functions from the set \( \Psi(a,b) \). Introduce the new function \( \nu = \nu(p) \) by an equality
\[
\nu(p) := \psi(p) \cdot \zeta(p); \quad (4.1)
\]
obviously, the function \( \nu = \nu(p) \) belongs also to the set \( \Psi(a,b) \).

**Theorem 4.1.**
\[
||V_A[f]||G_{\nu,\alpha} \leq \phi(G\zeta, |\det(A)|^{-1} ||A||^{-\alpha}) \ ||f||G_{\psi,\alpha}, \quad (4.2)
\]
and the last estimate is also in general case non-improvable, for example when the matrix \( A \) is diagonal and the function \( \psi(\cdot) \) is natural for the function \( f : \psi(p) = |f|_p < \infty, \ p \in (a,b) \).

**Proof** is alike to one in theorem 3.1. Indeed, let \( ||f||G_{\psi,\alpha} = 1 \). We the inequality (1.4):
\[
|V_A[f]|_{p,\alpha} \leq |\det A|^{-1/p} ||A||^{-\alpha/p} |f|_{p,\alpha} \leq |\det A|^{-1/p} ||A||^{-\alpha/p} \psi(p),
\]
or equally
\[
\frac{|V_A[f]|_{p,\alpha}}{\nu(p)} \leq \frac{|\det A|^{-1/p} ||A||^{-\alpha/p}}{\zeta(p)}. \quad (4.3)
\]
It remains to take the supremum from both the sides of the last inequality (4.3) over \( p : p \in (a,b) \):
\[
||V_A[f]||G_{\nu,\alpha} \leq \phi(G\zeta, |\det(A)|^{-1} ||A||^{-\alpha}) = \phi(G\zeta, |\det(A)|^{-1} ||A||^{-\alpha}) \ ||f||G_{\psi,\alpha},
\]
Q.E.D.

5. **Main result: dilation operators in mixed (anisotropic) Lebesgue spaces.**

We recall here the definition of the so-called anisotropic (mixed in Bochner’s sense) Lebesgue (Lebesgue-Riesz) spaces; see the source work [24]. More detail information about this spaces see in the classical books of Besov O.V., Ilin V.P., Nikol’skii S.M. [23], chapter 16,17; Leoni G. [27], chapter 11; using for us theory of operators interpolation in this spaces see in [23], chapter 17,18. 

Let \((X_j, A_j, \mu_j), j = 1, 2, \ldots, l\) be measurable spaces with sigma-finite non-trivial measures \( \mu_j \); in the considered in this report case \( X_j = R^{m_j} \).

Set
\[
X = R^d = \bigotimes_{j=1}^l X_j,
\]
evidently \( d = \sum_j m_j \).

Let also
\[
p = \vec{p} = (p_1, p_2, \ldots, p_l) \quad (5.1)
\]
be \( l \)-dimensional numerical vector such that \( 1 \leq p_j \leq \infty \).
Recall that the anisotropic Lebesgue space $L_{\vec{p}}$ consists of all the total measurable real valued function $f = f(x_1, x_2, \ldots, x_l) = f(\vec{x})$

$$f : \otimes_{j=1}^l X_j \to \mathbb{R}$$

with finite norm $|f|_{\vec{p}} \overset{def}{=} \left( \int_{X_1} \mu_1(dx_1) \left( \int_{X_{l-1}} \mu_{l-1}(dx_{l-1}) \ldots \left( \int_{X_1} |f(\vec{x})|^{p_1} \mu_1(dx_1) \right)^{p_2/p_1} \right)^{p_3/p_2} \ldots \right)^{1/p_l}$.

Note that in general case $|f|_{p_1,p_2} \neq |f|_{p_2,p_1}$, but $|f|_{p,p} = |f|_p$.

Observe also that if $f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2)$ (condition of factorization), then $|f|_{p_1,p_2} = |g_1|_{p_1} \cdot |g_2|_{p_2}$, (formula of factorization).

**Definition 5.1.** Let $D$ be Borelian subset of the whole space $R^d$ and

$$q = \vec{q} = (q_1, q_2, \ldots, q_d)$$

be $d$ – dimensional numerical vector such that $1 \leq q_j < \infty$.

We define as before as a capacity of a fundamental function the following expression

$$\phi_{\vec{q}}(D) \overset{def}{=} ||I_D(\cdot)||_{\vec{q}}.$$  (5.3)

**Remark 5.1.** Note that in general case $d \geq 2$ the value $\phi_{\vec{q}}(D)$ does not dependent only on the volume, i.e. on the measure of the set $D$.

**Remark 5.2.** This notion of fundamental function, or in other words, non-linear volume, may be easy generalized on the arbitrary multivariate, for instance, over the Euclidean space $R^d$, Banach functional space $(L, || \cdot ||_L)$ as follows

$$\phi_L(D) \overset{def}{=} ||I_D(\cdot)||_L.$$  (5.4)

if there exists.

This function $D \to \phi_L(D)$ is obviously non-negative and sub-additive:

$$\phi_L(\bigcup_{k=1}^n D_k) \leq \sum_{k=1}^n \phi_L(D_k), \ 1 \leq n \leq \infty.$$  

If the space $L$ coincides with the classical $L_1$ ones, then the function $D \to \phi_L(D)$ is ordinary sigma additive Lebesgue measure. In the case when $L = L_{p,p,p,\ldots,p}$ we return to the fundamental function for $L_p(R^d)$ space.

**Remark 5.3.** Assume that the set $D$ is direct (Cartesian) product of the (measurable) sets $F_j$:

$$D = \otimes_{j=1}^q F_j, \ F_j \subset R^{m_j}.$$  

Since the indicator function $I_D$ is factorable, we deduce by means of the formula of factorization
As a consequence: in this case the value \( \phi_q(D) \) dependent only on the "individual" volumes \( \{\mu_j(F_j)\} \).

Let us consider the following important example. Indeed, we claim to compute the fundamental function of an ellipsoid relative the norm in anisotropic Lebesgue spaces.

Some additional notations. \( \vec{1} = (1,1,\ldots,1) \); and for the \( d \)-dimensional vector \( \vec{p} = p = (p_1,p_2,\ldots,p_{d-2},p_{d-1},p_d) \), where \( d \geq 2 \), we define its right-hand side truncation

\[
\vec{p}(d) = p(d) \overset{\text{def}}{=} (p_1,p_2,\ldots,p_{d-2},p_{d-1}).
\]

Let \( a = \vec{a} = (a_1,a_2,\ldots,a_d) \) be numerical \( d \)-dimensional vector with positive entries \( a_i > 0 \); \( d = 1,2,\ldots \). Define the ellipses (ellipsoids)

\[
E_a = E_{\vec{a}} = \left\{ x = (x_1,x_2,\ldots,x_d) : \sum_{i=1}^{d} \frac{x_i^2}{a_i^2} \leq 1 \right\},
\]

\[
E_a(R) = E_{\vec{a}}(R) = \left\{ x = (x_1,x_2,\ldots,x_d) : \sum_{i=1}^{d} \frac{x_i^2}{a_i^2} \leq R^2 \right\} = E_{R\vec{a}},
\]

so that the ordinary Euclidean centered unit \( d \)-dimensional ball \( B \) is equal to the ellipsoid \( B = E_{1,1,\ldots,1} \) and the ordinary Euclidean centered ball \( B(R) \) with radii \( R \) is equal to the ellipsoid \( B(R) = E_{1,1,\ldots,1}(R) \).

Denote also for simplicity

\[
\theta(\vec{p},\vec{a}) = \theta^{(d)}(\vec{p},\vec{a}) = \phi_{\vec{p}}(E_{\vec{a}}), \quad \theta(\vec{p}) = \theta^{(d)}(\vec{p}) \overset{\text{def}}{=} \theta(\vec{p},\vec{1});
\]

then obviously

\[
\theta^{(d)}(\vec{p},\vec{a}) = \theta^{(d)}(\vec{p}) \cdot \prod_{i=1}^{d} a_i^{1/p_i} = \theta(\vec{p},\vec{1}) \prod_{i=1}^{d} a_i^{1/p_i}
\]

and

\[
\theta(\vec{p},\vec{a}; R) := \phi_{\vec{p}}(E_{\vec{a}}(R)) = \theta^{(d)}(\vec{p}) \cdot \prod_{i=1}^{d} a_i^{1/p_i} \cdot R^{\sum_{i=1}^{d} 1/p_i}.
\]

We derive after some computations \( \theta^{(1)}(p_1) = 2^{1/p_1} \),

\[
\theta^{(2)}(p_1,p_2) = \theta^{(1)}(p_1) \cdot B^{1/p_2}(1/2,1+p_2/(2p_1)),
\]

where \( B(\alpha, \beta) \) denotes usually beta function.

Moreover, we can deduce the following recurrent relation

\[
\frac{\theta^{(d+1)}(\vec{p})}{\theta^{(d)}(\vec{p}(d+1))} = Z_{d+1}(p_1,p_2,\ldots,p_d,p_{d+1}) = Z_{d+1},
\]

where \( \vec{p} \in R^{d+1}, \ Z_1(p_1) = 2^{1/p_1} \).
\[ Z_{d+1}(p_1, p_2, \ldots, p_d, p_{d+1}) \overset{\text{def}}{=} B^{1/p_{d+1}} \left( \frac{1}{2}, 1 + \frac{p_{d+1}}{2} \left( \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_d} \right) \right). \] (5.9)

Therefore
\[ \theta^{(d)}(p_1, p_2, \ldots, p_d) = Z_1(p_1) \cdot Z_2(p_1, p_2) \cdot \ldots \cdot Z_d(p_1, p_2, \ldots, p_d) = \]
\[ \overset{\text{def}}{=} W_d(p_1, p_2, \ldots, p_d) = W_d(\vec{p}). \] (5.10)

For instance,
\[ \theta^{(3)}(p, q, r) = 2^{1/p} B^{1/q}(1/2, 1 + q/(2p)) B^{1/r}(1/2, 1 + (r/2)(1/p + 1/q)), \]
\[ \theta^{(4)}(p, q, r, s) = \theta^{(3)}(p, q, r) \cdot B^{1/s}(1/2, 1 + (s/2)(1/p + 1/q + 1/r)) \]
etc.

Correspondingly
\[ \theta(\vec{p}, \vec{a}; R) := \phi_{\vec{p}}(E_{\vec{a}(R)}) = W_d(\vec{p}) \cdot \prod_{i=1}^{d} a_i^{1/p_i} \cdot R^{d+1}. \] (5.11)

**Remark 5.4.** It is easily to verify that when \( \vec{a} = \vec{p} = \vec{1} \), the expression (5.11) gives the Euclidean volume of \( d \) – dimensional ball with radii \( R \).

**Remark 5.5.** Obviously, the expression for \( \theta(\vec{p}, \vec{a}; R) \) does not dependent from the center of our ellipsoid. Namely, the fundamental function of the non-centered ellipsoid of the form
\[ \tilde{E}_a = \left\{ x = (x_1, x_2, \ldots, x_d) : \sum_{i=1}^{d} \frac{(x_i - x_0^i)^2}{a_i^2} \leq R^2 \right\} \]
is at the same as in the case when \( x_0^i = 0 \).

**Remark 5.6.** It is no hard to calculate the fundamental function for parallelepiped
\[ Q = Q(x_1^0, x_2^0, \ldots, x_d^0; x_1^0 + \delta_1, x_2^0 + \delta_2, \ldots, x_d^0 + \delta_d) = \]
\[ \{ \vec{x} : x_j^0 \leq x_j \leq x_j^0 + \delta_j \}, \delta_j \in (0, \infty). \]
We conclude using the fact that the indicator function \( I_Q(x) \) is factorable
\[ \phi_{\vec{p}}(Q) = \prod_{j=1}^{d} \delta_j^{1/p_j}. \]

**A. Anisotropic Lebesgue-Riesz spaces.**

Let us return to the announced problem of calculation of the norm of multivariate dilation operator in the mixed (anisotropic) Lebesgue spaces. Namely,
\[ R^d = \otimes_{j=1}^l R^{m_j}, \quad \sum_{j=1}^l m_j = d, \]

\[ p = \bar{p} = (p_1, p_2, \ldots, p_l), \quad p_l \geq 1. \]

Let also \( \vec{A} = A = (A_1, A_2, \ldots, A_l) \), where \( A_j : R^{m_j} \to R^{m_j} \) are linear non-degenerate \( \det(A_j) \neq 0 \) operators (matrices), so that \( \vec{A} \) is matrix tensor.

Let also \( f(\vec{x}) = f(x_1, x_2, \ldots, x_l), \ x_j \in R^{m_j} \) be measurable function from the anisotropic space \( L_{\bar{p}} \). We consider the following tensor dilation operator

\[ V_{\vec{A}}[f] \overset{\text{def}}{=} f(A_1 x_1, A_2 x_2, \ldots, A_l x_l). \] (5.12)

Denote

\[ \Lambda_p(\vec{A}) = \prod_{j=1}^l \left| \det A_j \right|^{-1/p_j}. \] (5.13)

**Proposition 5.1.**

\[ |V_{\vec{A}}[f]|_{\bar{p}} \leq \Lambda_p(\vec{A}) |f|_{\bar{p}.} \] (5.14)

where the equality is attained if for example, the function \( f(\cdot) \) is factorable:

\[ f(x) = \prod_{j=1}^l g_j(x_j), \ g_j(\cdot) \in L_{p_j}(R^{m_j}). \] (5.15)

**B. Anisotropic Grand Lebesgue-Riesz spaces.**

Let \( Q \) be convex (bounded or not) subset of the set \( \otimes_{j=1}^l [1, \infty] \). Let \( \psi = \psi(\vec{p}) \) be continuous in an interior \( Q^0 \) of the set \( Q \) strictly positive function such that

\[ \inf_{\vec{p} \in Q^0} \psi(\vec{p}) > 0; \quad \inf_{\vec{p} \not\in Q^0} \psi(\vec{p}) = \infty. \]

We denote the set all of such a functions by \( \Psi(Q) \).

The Anisotropic Grand Lebesgue Spaces \( AGLS = AGLS(\psi) \) space consists by definition on all the measurable functions

\[ f : \otimes_{j=1}^l R^{m_j}(= R^d) \to R \]

with finite (mixed) norm

\[ ||f||_{AGLS} = \sup_{\vec{p} \in Q^0} \left[ \frac{|f|_{\bar{p}}}{\psi(\vec{p})} \right]. \] (5.16)

These spaces appear (and investigated) at first (perhaps) in the articles [13], [32]; therein are described also some possible its applications.

As before, the (multivariate) fundamental function \( \phi_{AGLS}(D), \ D \subset R^d \) in these space can be defined as follows:
\[
\phi_{AGLS\psi}(D) \overset{def}{=} ||I_D||AGLS\psi.
\]

Assume again that the set \(D\) is direct (Cartesian) product of the (measurable) sets \(F_j:\)

\[
D = \bigotimes_{j=1}^{g} F_j, \quad F_j \subset R^{m_j}.
\]

Assume in addition that the function \(\psi(\vec{p})\) is factorable:

\[
\psi(\vec{p}) = \prod_{j=1}^{l} \psi_j(\vec{p}_j)
\]

and that the domain \(G\) is also factorable:

\[
G = \bigotimes_{j=1}^{g} G_j, \quad G_j \subset R^{m_j}.
\]

Since the indicator function \(I_D\) is also factorable, we deduce by means of the formula of factorization

\[
\phi_{AGLS\psi}(D) = \prod_{j=1}^{g} \phi_{AGLS\psi_j}(F_j).
\]

In order to formulate (and prove) the main result of our report, we need to introduce some new preliminary notations.

\[
\delta_j := |\det A_j|^{-1/m_j}, \quad j = 1, 2, \ldots, l;
\]

\[
K_j^{m_j} := \bigotimes_{s=1}^{m_j} [0, \delta_j], \quad K = K(\vec{m}, \vec{p}) := \bigotimes_{j=1}^{l} K_j^{m_j}.
\]

so that the set \(K_j^{m_j}\) is a cube of the volume (measure) \(|\det A_j|^{-1}\) in the correspondent space \(R^{m_j}\).

Let further the functions \(\psi = \psi(\vec{p}), \quad \zeta = \zeta(\vec{p}), \quad \vec{p} \in D\) be two functions from certain non-trivial domain \(\Psi(D)\). Define a new functions from this class \(\Psi(D)\)

\[
\nu(\vec{p}) = \psi(\vec{p}) \cdot \zeta(\vec{p}).
\]

**Theorem 5.1.**

\[
||V_A[f]||G\nu \leq \phi(G\zeta, K(\vec{m}, \vec{p}))) ||f||G\psi,
\]

and the last estimate is also in general case non-improvable, for example when the matrix tensor \(\vec{A}\) is diagonal and the function \(\psi(\cdot)\) is natural for the factorable function \(f : \psi(\vec{p}) = |f|_{\vec{p}} < \infty, \quad p \in D.\)

**Proof.** Of course, we can and will suppose without loss of generality \(f \in G\psi\); in opposite case it is nothing to prove. Moreover, it is reasonable to assume \(||f||G\psi = 1.\) Then

\[
|f|_{\vec{p}} \leq \psi(\vec{p}), \quad p \in D.
\]

We use the inequality (5.14):
\[ |V^*_A[f]|_\vec{p} \leq \Lambda_{\vec{p}}(\vec{A}) \|f\|_{\vec{p}} \leq \Lambda_{\vec{p}}(\vec{A}) \psi(\vec{p}), \quad p \in D, \]

or equally

\[
\frac{|V^*_A[f]|_{\vec{p}}}{\nu(\vec{p})} \leq \frac{\Lambda(\vec{A})}{\zeta(p)}. \tag{5.20}
\]

Let us take the supremum over \( p \in D \) from both the sides of the last inequality, taking into account the direct definitions of norm and fundamental function for Anisotropic Grand Lebesgue Spaces:

\[ \|V_A[f]\|_{G\nu} \leq \phi(G\zeta, K(\vec{m}, \vec{p})) = \phi(G\zeta, K(\vec{m}, \vec{p})) \|f\|_{G\psi}, \tag{5.21} \]

Q.E.D.

6. Concluding remark. Examples and counterexamples.

A. An example.

Let us show that the condition \( \det(A) \neq 0 \) in the theorem 3.1 (and in another ones) is essential. Let \( f = f(x,y), (x,y) \in \mathbb{R}^2 \) be measurable non-negative factorable function

\[ f(x,y) = g(x) \cdot g(y), \]

where \( g(\cdot) \in L_p(R) \), \( \lim_{y \to 0} g(y) = \infty \), and consider the linear degenerate operator \( A \) (matrix \( 2 \times 2 \)) such that \( A(x,y) = (x,0) \), so that

\[ f(A \cdot (x,y)) := f(x,0). \]

So, the operator \( A \) is the coordinate projections.

Evidently, \( f(\cdot, \cdot) \in L_p(R^2) \), but

\[ f(A(x,y)) = f(x,0) \notin L_p(R^2). \]

Note that the coordinate projections on matrix weighted \( L_p \) - spaces, for instance, Hilbert’s transform is investigated in the recent article [29].

B. Particular case: Exponential Orlicz spaces.

It is known, see [8], [10], [9], [22], that the so-called exponential Orlicz’s spaces, for example, the Orlicz’s spaces with correspondent Young function

\[ \Phi(u) = \Phi^{(\lambda)}(u) = e^{Cu} - 1, \quad \lambda, C = \text{const} > 0 \]

are particular case of Grand Lebesgue Space with correspondent \( \psi = \psi_\Phi \) – function, for example,

\[ \psi_\Phi^{(\lambda)}(p) := \psi^{(\lambda)}(p) = p^{1/\lambda}, \quad p \geq 1. \]

Therefore, the theorem 3.1 may be applied to these Orlicz spaces.

See also [15], [16], [18] etc.
The correspondent fundamental function $\phi(G\psi^{(\lambda)}, \delta)$ is calculated and estimated in [11].

C. Periodical case.

At the same results may be derived in the case when $X = (-\pi, \pi)^d$ (case of torus), where the algebraic operations are understood mod $(2\pi)$; or more generally when

$$X = (-\pi, \pi)^{d_1} \otimes R^{d_2}, \; d_1, d_2 = 1, 2, \ldots.$$ 

D. Example to the third section.

Let us consider the $G\psi$ - space $\tilde{G} = \tilde{G}(a, h; \alpha, \beta)$ over real line $R^1$ with the following $\psi$ - function

$$\tilde{\psi} = \psi(a; \alpha, \beta; p) = (p - a)^{-\alpha}, \; p \in (a, h);$$

$$\tilde{\psi}(p) = \psi(a; \alpha, \beta; p) = p^\beta, \; p \in (h, \infty),$$

where

$$\alpha, \beta = \text{const} > 0, \; a = \text{const} \geq 1,$$

the value $h = h(a, \alpha, \beta) > a$ is the unique positive solution of an equation

$$(h - a)^{-\alpha} = h^\beta,$$

so that the function $p \to \tilde{\psi}(p), \; p \in (a, \infty)$ is continuous.

This space does not coincides with the known rearrangement invariant spaces: Lorentz, Orlicz, Marcinkiewicz etc., see [8], [10], [11], [9], [22].

The correspondent fundamental function $\phi(\delta) = \phi((a; \alpha, \beta, \delta)$ obeys a following asymptotical behavior:

$$\tilde{\phi}(\delta) \sim \beta^\beta |\ln \delta|^{-\beta}, \; \delta \to 0+;$$

$$\tilde{\phi}(\delta) \sim \left(\frac{a^2 \alpha}{e}\right)^\alpha \delta^{1/\alpha} (\ln \delta)^{-\alpha}, \; \delta \to \infty,$$

see [11].

It remains to apply the inequality (3.3).

E. Possible generalisations.

It may be investigated analogously to the 5th section the multivariate weight case as well as the affine linear non-centered transform of the form

$$f \to f(A \cdot x + b).$$

F. About compactness of dilation operator.
Of course, in general case the dilation operator $V_A[\cdot]$ acting from one GLS to suitable ones, is non-compact, even in the very simple case $A = I$ – unit operator: $V_A[f] = f$.

Let us consider an opposite case. We borrow the notations, conditions and proposition of theorem 3.1.

Let also $\psi_1(\cdot), \psi_2(\cdot)$ be two function from the set $\Psi(a,b)$, $1 \leq a < b \leq \infty$. We recall the following relation definition:

$$\psi_1 \ll \psi_2 \Leftrightarrow \lim_{\psi_2(p) \to \infty} \frac{\psi_1(p)}{\psi_2(p)} = 0,$$

see [9], [11]. It is proved in these articles that in this case the GLS space $G\psi_1$ is compact embedded in the space $G\psi_2$.

We deduce as a slight corollary: let the new function $\theta = \theta(p)$ from this set $\Psi(a,b)$ be such that $\theta(\cdot) \ll \nu(\cdot)$. Then we derive under conditions of theorem 3.1 that the operator $V_A[\cdot]$ acting from the space $G\psi_1$ into the space $G\theta$ is compact.

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