Allpass Feedback Delay Networks

Sebastian J. Schlecht, Senior Member, IEEE

Abstract—In the 1960s, Schroeder and Logan introduced delay-based allpass filters, which are still popular due to their computational efficiency and versatile applicability in artificial reverberation, decorrelation, and dispersive system design. In this work, we extend the theory of allpass systems to any arbitrary connection of delay lines, namely feedback delay networks (FDNs). We present a complete characterization of uniallpass FDNs, i.e., FDNs, which are allpass for an arbitrary choice of delays. Further, we develop a solution to the completion problem, i.e., given an FDN feedback matrix to determine the remaining gain parameters such that the FDN is allpass. Particularly useful for the completion problem are feedback matrices, which yield a homogeneous decay of all system modes. Finally, we apply the uniallpass characterization to previous FDN designs, namely, Schroeder’s series allpass and Gardner’s nested allpass for single-input, single-output systems, and, Poletti’s unitary reverberator for multi-input, multi-output systems and demonstrate the significant extension of the design space.

Index Terms—Filter Design; Allpass Filter; Feedback Delay Networks; SISO; MIMO; Delay State Space

I. INTRODUCTION

Allpass filters are unique as they preserve the signal’s energy and only alter the signal phase [1]. Schroeder and Logan introduced delay-based allpass filters in the 1960s [2] to create “colorless” artificial reverberation. A decade later, Gerzon generalized the delay-based filters to feedback delay networks (FDNs) [3] and the single-input, single-output (SISO) allpass structure to multi-input, multi-output (MIMO) allpass networks [4].

An FDN essentially consists of a set of delay lines interconnected via a feedback matrix [3], see Fig. 1. FDNs can have single or multiple input and output channels distributed by the input, output, and direct gains. Further, FDNs have well-established system properties such as losslessness and stability [5] [6], decay control [7] [8], impulse response density [9] [10], and, modal distribution [11]. Compared to general high-order allpass filters [12], FDNs are sparse filters, which are less flexible, but more computationally efficient. SISO allpass FDNs can be combined from simple allpass filters in series [2] [13] or by nesting [14] to create more complex structures while retaining the allpass characteristic. Rocchesso and Smith also suggested an almost allpass FDN with equal delays in [5] Th. 2. MIMO allpass filters can be similarly generated from simple unitary building blocks [4] [15] or by generalizing the allpass lattice structure [16].

Both SISO and MIMO allpass FDNs were applied to a wide range of roles including: 1) increasing the echo density as preprocessing to an artificial reverberator [2] [17]; 2) increasing echo density of in the feedback loop of reverberators [18] [20], 3) decorrelation for widening the auditory image of a sound source [21] [23]; 4) as reverberator in electro-acoustic reverberation enhancement systems [16] [19] [24] [25]; 5) linear dynamic range reduction [26] [27]; and 6) dispersive system design [28] [29]. In the broader context of control theory, allpass FDNs are strongly related to Schur diagonal stability [30], e.g., stability properties of asynchronous networks.

In this work, we extend the theory of allpass FDNs for both SISO and MIMO. In particular, we study uniallpass FDNs, i.e., FDNs, which are allpass for arbitrary delay lengths. While not all allpass FDNs are uniallpass, the more straightforward design criterion significantly extends practical filter structures.

The feedback matrix determines many filter properties of the FDN. Thus, it is often desirable to first design the feedback matrix and subsequently choose the input, output, and direct gains such that the resulting FDN is allpass. We refer to this procedure as the completion problem. We call feedback matrices, which have a solution to the completion problem as being allpass admissible. A particularly useful class of feedback matrices are lossless mixing matrices in conjunction with diagonal delay-proportional absorption matrices. They result in homogeneous decay of the impulse response, i.e., all system eigenvalues have the same magnitude [7]. The main contributions of this work are:

- Necessary and sufficient conditions for SISO and MIMO FDNs to be uniallpass (Section III).
- Characterization of admissible feedback matrices in uniallpass FDN (Section IV-B).
- Completion algorithms for uniallpass SISO and MIMO FDN (Section IV-D).

The term uniallpass is introduced here with similar motivation as unilossless feedback matrices in [6] which yields lossless FDNs regardless of delay lengths.
• Characterization of uniallpass FDNs with homogeneous decay (Section [V])
• Embedding of previous designs in the proposed characterization (Section [VI]).

This work extends the design space of delay-based allpass filters from a handful of known structures to a freely parametrizable extensive class. In particular, the solution of the completion problem allows to combine feedback matrix design with the allpass property and potentially improves application designs mentioned above.

The remaining manuscript is structured as follows. Section [II] introduces FDN and allpass prior art and reviews a characteristic of admissible feedback matrices and presents a completion algorithm. Section [III] derives a solution for uniallpass FDN designs. Section [IV] presents a characterization of allpass state space systems. Section [V] derives a solution for uniallpass FDN designs.

II. PROBLEM STATEMENT AND PRIOR ART

In the following, we state the problem formulation of this work and review the prior art.

A. MIMO Feedback Delay Network

The MIMO FDN is given in the discrete-time-domain by the difference equation in delay state space form [5], see Fig. [1]:

\[ \begin{align*}
    y(n) &= C s(n) + D x(n), \\
    s(n+m) &= A s(n) + B x(n),
\end{align*} \tag{1} \]

where \( y(n) \) and \( y(n) \) are the \( N_{in} \times 1 \) input and \( N_{out} \times 1 \) output vectors at time sample \( n \), respectively. The FDN dimension \( N \) is the number of delay lines. The FDN consists of the \( N \times N \) feedback matrix \( A \), the \( N \times N_{in} \) input gain matrix \( B \), the \( N_{out} \times N \) output gain matrix \( C \) and the \( N_{out} \times N_{in} \) direct gain matrix \( D \). The lengths of the \( N \) delay lines in samples are given by the vector \( m = [m_1, \ldots, m_N] \). The \( N \times 1 \) vector \( s(n) \) denotes the delay-line outputs at time \( n \). The vector argument notation \( s(n+m) \) abbreviates the vector \( s_i(n+m_1), \ldots, s_i(n+m_N) \). Although, large parts of the derivations are general, we mainly focus on our results on FDNs with equal input and output channels, i.e., \( N_{in} = N_{out} \). We refer to an FDN where the number of delay lines is equal to the input and output channels as full MIMO, i.e., \( N_{in} = N_{out} = N \). A SISO FDN has \( N_{in} = N_{out} = 1 \), which is emphasized by notating vectors and scalars \( b, c \) and \( d \) instead of matrices \( B, C \) and \( D \).

The \( N_{out} \times N_{in} \) transfer function matrix of an FDN in the z-domain [5] corresponding to (1) is

\[ H(z) = C(D_m z^{-1} - A)^{-1} B + D, \tag{2} \]

where \( D_m(z) = \text{diag}([z^{-m_1}, z^{-m_2}, \ldots, z^{-m_N}]) \) is the diagonal \( N \times N \) delay matrix [7]. The system order is given by the sum of all delay units, i.e., \( \mathfrak{N} = \sum_{i=1}^{N} m_i \) [3]. For commonly used delays \( m \), the system order is much larger than the FDN size, i.e., \( \mathfrak{N} \gg N \).

The transfer function matrix [3] can be stated as a rational polynomial [5, 20], i.e.,

\[ H(z) = \frac{Q_{m,A,B,C,D}(z)}{p_{m,A}(z)}, \tag{3} \]

where the denominator is a scalar-valued polynomial

\[ p_{m,A}(z) = \det(P(z)), \tag{4} \]

where \( \det \) denotes the determinant and the loop transfer function is

\[ P(z) = D_m(z^{-1}) - A. \tag{5} \]

The numerator is a matrix-valued expression with

\[ Q_{m,A,B,C,D}(z) = D \det(P(z)) + C \text{adj}(P(z)) B, \tag{6} \]

where \( \text{adj}(A) \) denotes the adjugate of \( A \) [11]. The FDN system poles \( \lambda_i \), where \( 1 \leq i \leq \mathfrak{N} \), are the roots of the generalized characteristic polynomial (GCP) \( p_{m,A}(z) \) in (4). Thus, the system poles \( \lambda_i \) are fully characterized by the delays \( m \) and the feedback matrix \( A \).

B. Allpass Property

A transfer function matrix \( H(z) \) with real coefficients is allpass if

\[ H(z)(H(z^{-1})^\top) = I, \tag{7} \]

where \( I \) denotes an identity matrix of appropriate size and \( ^\top \) denotes the transpose operation. If \( N_{in} = N_{out} \), a MIMO system is allpass if \( \det H(z) \) is allpass [31, p. 772], i.e.,

\[ |\det H(z)| = 1 \quad \text{for any } \omega. \tag{8} \]

In particular, \( H(z) \) is unitary for \( z \) on the unit circle.

For allpass filters, the coefficients of the denominator polynomial are in reversed order and possibly with reversed signs of the denominator coefficients [11]. Thus, for an allpass FDN in (5), we have

\[ \det H(z) = \pm z^{-\mathfrak{N}} p_{m,A}(z^{-1}). \tag{9} \]

In the following, we present a classic result for allpass state space systems.

C. Allpass State Space Systems

For a moment, we consider that all delays are single time steps, i.e., \( m = 1 \), where \( 1 \) denotes a vector or matrix of ones with appropriate size. The time-domain recursion in (1) reduces to the standard state space realization of a linear time-invariant (LTI) filter. We state a classic sufficient and necessary condition for state space systems to be allpass [32].

**Theorem 1.** Assume that the \( N_{out} \times N_{in} \) transfer function has a realization \( H(z) = C(zI - A)^{-1} B + D \). There exists a solution of the equation

\[ \begin{bmatrix}
    A & B \\
    C & D
  \end{bmatrix}
\begin{bmatrix}
    P \\
    0
  \end{bmatrix}
= \begin{bmatrix}
    A^\top & C^\top \\
    0 & I
  \end{bmatrix}
\begin{bmatrix}
    P \\
    0
  \end{bmatrix}, \tag{10}
\]

where \( P = P^\top \). If and only if \( H(z) \) is an allpass function.

In the Section [III] we present an extension of this theorem for allpass FDNs.
D. Principal Minors and Diagonal Similarity

To demonstrate system properties of an FDN independent from delays \( \bm{m} \), we have earlier developed a representation of \( p_{m,\bm{A}}(z) \) based on the principal minors of \( \bm{A} \) \[6, 20\]. This representation is also useful to derive the uniallpass property of FDNs.

A principal minor \( \det \bm{A}(I) \) of a matrix \( \bm{A} \) is the determinant of a submatrix \( \bm{A}(I) \) with equal row and column indices \( I \subset \langle N \rangle \). The set of all indices is denoted by \( \langle N \rangle = \{1, 2, \ldots, N\} \) and \( I^c \) is the relative complement in \( \langle N \rangle \), i.e., \( I^c = \langle N \rangle \setminus I \). \(|I|\) indicates the cardinality of set \( I \).

For a given feedback matrix \( \bm{A} \) and delays \( \bm{m} \), the generalized characteristic polynomial \( p_{m,\bm{A}}(z) \) is given by

\[
p_{m,\bm{A}}(z) = \sum_{k=0}^{\infty} c_k z^k
\]

where \( c_k = \left\{ \begin{array}{ll}
\sum_{I \subseteq \langle N \rangle} (-1)^{N-|I|} \det \bm{A}(I^c), & \text{for } I_k \neq \emptyset \\
0, & \text{otherwise}
\end{array} \right. \]

The principal minors of invertible matrices \( \bm{A} \) are related by Jacoby’s identity \[33\], i.e.,

\[
\det \bm{A}^{-1}(I) = \frac{\det \bm{A}(I^c)}{\det \bm{A}} \quad \text{for any } I \subset \langle N \rangle.
\]

Diagonally similar matrices \( \bm{A} \) and \( \bm{B} \), i.e., there exists nonsingular diagonal matrix \( \bm{E} \) with \( \bm{E} \bm{A} \bm{E}^{-1} = \bm{B} \), have the same principal minors \[34\]. The converse is not true in general \[34\], however, if \( \bm{A} \) and \( \bm{B} \) have the same principal minors, then they are diagonally similar \[6, \text{Th. 8}\].

In the following section, we derive the analogue of Theorem \[1\] for uniallpass FDNs with arbitrary delays \( \bm{m} \).

III. UNIALPASS FEEDBACK DELAY NETWORKS

The central question of the present work is which system parameters constitute an allpass transfer function \( \bm{H}(z) \) in \( \mathcal{H} \).

We start with a matrix equation for a given feedback system with system matrices \( \bm{A}, \bm{B}, \bm{C}, \) and \( \bm{D} \) for arbitrary delays \( \bm{m} \).

There exists a solution of the equation

\[
\begin{bmatrix}
\bm{A} & \bm{B} \\
\bm{C} & \bm{D}
\end{bmatrix}
\begin{bmatrix}
\bm{P} & \bm{0} \\
\bm{0} & \bm{I}
\end{bmatrix}
\begin{bmatrix}
\bm{A}^T & \bm{C}^T \\
\bm{B}^T & \bm{D}^T
\end{bmatrix}
= \begin{bmatrix}
\bm{P} & \bm{0} \\
\bm{0} & \bm{I}
\end{bmatrix}
\]

where \( \bm{P} \) is diagonal, if and only if \( \bm{H}(z) \) is uniallpass, i.e., allpass for any \( \bm{m} \).

While in \( \mathcal{H} \), \( \bm{P} \) is diagonal for uniallpass FDN, \( \bm{P} \) is not necessarily diagonal for specific delays \( \bm{m} \). For instance, allpass FDNs with equal delays \( \bm{m} = k \bm{1} \) with \( k \in \mathbb{N} \), \( \bm{P} \) is only necessarily symmetric as in Th. \[1\]. For longer delays \( \bm{m} \), it can become quickly impractical to determine the allpass property for specific \( \bm{m} \) such that the uniallpass property is more useful albeit slightly restrictive. However, based on observations of unilessors matrices, we conjecture for many \( \bm{m} \) that \( \bm{P} \) tends to be close to diagonal \[6\].

In the following subsections, we derive central aspects of Theorem \[2\].

A. System Matrix

First, we establish a convenient notation based on system matrices with \( N_{in} = N_{out} \), i.e.,

\[
\bm{V} = \begin{bmatrix}
\bm{A} & \bm{B} \\
\bm{C} & \bm{D}
\end{bmatrix}
\]

which is of size \( N_{\bm{V}} \times N_{\bm{V}} \), where \( N_{\bm{V}} = N_{out} = N_{in} \).

The Schur complement of the invertible block \( \bm{D} \) in \( \bm{V} \) is a matrix defined by

\[
\frac{\bm{V}}{\bm{D}} = \bm{A} - \bm{BD}^{-1} \bm{C}
\]

and equivalently the Schur complement of the invertible block \( \bm{A} \) is

\[
\frac{\bm{V}}{\bm{A}} = \bm{D} - \bm{CA}^{-1} \bm{B}
\]

If \( \bm{A}, \bm{D}, \frac{\bm{V}}{\bm{D}}, \) and \( \frac{\bm{V}}{\bm{A}} \) are invertible, the block-wise inverse of the system matrix \( \mathcal{H} \) is

\[
\frac{\bm{V}}{\bm{D}} = \left[ \begin{array}{cc}
\frac{\bm{D}}{\bm{V}} & \frac{\bm{D}}{\bm{V}} \frac{\bm{B}}{\bm{V}}(\frac{\bm{A}}{\bm{V}})^{-1} \\
\frac{\bm{B}}{\bm{V}} & \frac{\bm{D}}{\bm{V}} \frac{\bm{B}}{\bm{V}}(\frac{\bm{A}}{\bm{V}})^{-1}
\end{array} \right]
\]

Further, the inverse of the Schur complements are related by

\[
\frac{\bm{V}}{\bm{D}} = \frac{\bm{A}}{\bm{D}} + \frac{\bm{A}}{\bm{D}} \frac{\bm{B}}{\bm{V}}(\frac{\bm{A}}{\bm{V}})^{-1} \frac{\bm{C}}{\bm{D}}
\]

B. Delay-Independent Allpass Condition

The main challenge in Theorem \[2\] is that we want the allpass property to be independent of the choice of the delays \( \bm{m} \). Thus, we derive an allpass criterion which only depends on the system matrix \( \bm{V} \).

The FDN is allpass if and only if the determinant of the transfer function \( \det \bm{H}(z) \) is allpass, see \[8\]. Applying the matrix determinant lemma \[33\] in \( \mathcal{H} \) and using the Schur complement notation \[15\], we have

\[
\det \bm{H}(z) = \frac{\det \bm{D}_m(z^{-1}) - \bm{A} + \bm{BD}^{-1} \bm{C}}{\det \bm{D}_m(z^{-1}) - \bm{A}}
\]

\[
= \frac{p_{m,\bm{V}/\bm{D}}(z) \det \bm{D}}{p_{m,\bm{A}}(z)}.
\]

According to \[6\], for \( \det \bm{H}(z) \) to be allpass, the coefficients of denominator and numerator of \( \mathcal{H} \) are in reversed order, i.e.,

\[
\det \bm{D} \det \frac{\bm{V}}{\bm{D}} = \pm z^{-m} p_{m,\bm{A}}(z^{-1})
\]

For the special case \( \bm{m} = [1, 2, \ldots, N^{-1}] \), \[21\] holds if and only if

\[
\det \bm{D} \det \frac{\bm{V}}{\bm{D}}(I) = \pm \det \bm{A}(I^c) \quad \forall I \subset \langle N \rangle
\]
are directly related to the principal minors of $A$. For arbitrary delays $m$, (22) is sufficient for (21) to hold as the coefficients in (11) are merely summations for $|I_k| > 1$. In other words, an FDN is uniallpass if and only if (22) is satisfied.

C. Proof of Theorem 2

Proof. First, we assume $H(z)$ is uniallpass with realization $A, B, C,$ and $D$. As an uniallpass FDN is allpass for any $m$, it is allpass also for $m = 1$ and therefore $V$ satisfies (10) in Theorem 1 for some symmetric $P$. Thus, the system matrix $V^\top$ is similar to the inverse system matrix $V$, i.e.,

$$
\begin{bmatrix}
P & 0 \\
0 & I
\end{bmatrix}
V^\top
\begin{bmatrix}
P^{-1} & 0 \\
0 & I
\end{bmatrix} = V^{-1} .
$$

(23)

Thus, $\det V^\top = \det V^{-1}$ and consequently

$$
\det V = \pm 1 .
$$

(24)

From Jacoby’s identity (12) with $I_N = \langle N \rangle$ in $\langle N_N \rangle$,

$$
\det V(I_N)/\det V = \det(V)^{-1}(I_N)
$$

(25)

$$
\det D = \pm \det A .
$$

(26)

The lower right block in (23) yields

$$
D^+ = (V/A)^{-1} .
$$

(27)

As the FDN is uniallpass, also (22) holds. In other words, the principal minors of $A^1$ and $V/D$ are equal. Therefore, $A^1$ and $V/D$ are diagonally similar and $P$ is diagonal [6, 34].

For the opposite direction let us assume, there exists diagonal matrix $P$ satisfying (23). Thus, $A^\top$ is diagonally similar to $(V/D)^{-1}$. Therefore, $A^\top$ (and also $A$) and $(V/D)^{-1}$ have equal principal minors. Further, $A^{-1}$ and $V/D$ have equal principal minors. Thus with (12) and (26), we have

$$
\det V/D(I) = \det A^{-1}(I)
$$

$$
= \det A(I^c)/\det A
$$

$$
= \pm \det A(I^c)/\det D
$$

(28)

for all $I \subset \langle N \rangle$ as in (22). Therefore, $H(z)$ is allpass.

D. Discussion

Allpass FDNs are strongly related to unilossless matrices, i.e., feedback matrices $A$ such that all FDN poles $\lambda_i$ are on the unit circle. In [6, Th. 1], an irreducible $X$ is unilossless if and only if there exists a non-singular diagonal matrix $E$ such that

$$
E X^\top E^{-1} = X^{-1} .
$$

(29)

However, compared to (23), the lower-right block of $E$ related to the input and output part is not necessarily $I$.

For any uniallpass FDNs, we have $\det D = \pm \det A$, see (26). Thus, like in Schroeder allpass structures [22], there is an inherent relation between the direct component and the decay rate of the response.

In the following section, we present methods to design uniallpass FDNs based on a desired feedback matrix $A$.

IV. UNIALPASS FDN COMPLETION

Uniallpass FDNs can be generated by a simple procedure for $N_{\text{in}}$ input and output channels and $N$ delay lines. First, generate an orthogonal system matrix $V$ of size $N_V \times N_V$ with $N_V = N + N_{\text{in}}$. Optionally, apply a similarity transform with a non-singular diagonal matrix $\text{diag}(P, I)$. However, note that the similarity transform does not alter the transfer function, but may change computational properties. Lastly, divide the system matrix $V$ into the submatrices $A, B, C,$ and $D$ according to (14). However, this procedure does not allow to specify directly the feedback matrix $A$ and the resulting filter properties.

In this section, we present procedures related to the completion problem, i.e., determining $B, C,$ and $D$ given $A$ such that $V$ is uniallpass. The following subsections are: [IV-A] determining $P$ given uniallpass $V$; [IV-B] characterize admissible feedback matrices $A$; [IV-C] completion where $P = I$; and [IV-D] completion for any diagonal $P$.

A. Determining Diagonal Similarity

Given a uniallpass FDN with system matrix $V$. The diagonal similarity matrix $P$ in (13) can be computed by solving the discrete-time Lyapunov equation (30)

$$
P - APA^\top = BB^\top .
$$

(30)

We give an alternative solution, which is helpful for the further development below. The system matrix $V$ satisfies (23), thus $V$ is diagonally similar to an orthogonal matrix. We review here, key aspects of Engel and Schneider’s algorithm to determine the diagonal similarity [36].

A system matrix $V$ is diagonally similar to an orthogonal matrix if and only if $V^{-1} \otimes V^\top$ is diagonally similar to a $\{0, 1\}$-matrix $J$, i.e., $J \in \{0, 1\}^{N_V \times N_V}$. Operation $\otimes$ denotes an element-wise division also called Hadamard quotient, i.e.,

$$
(A \otimes B)_{ij} = \begin{cases}
a_{ij}/b_{ij} & \text{for } b_{ij} \neq 0 \\
0 & \text{otherwise.}
\end{cases}
$$

(31)

Thus with (17), the similarity transform $P$ can be readily retrieved from

$$
P^{-1}JP = (V/D)^{-1} \otimes A^\top .
$$

(32)

For fully connected matrices $A$ and $(V/D)^{-1}$, i.e., having only non-zero elements, $J$ contains only ones. Then, (32) can be simply solved by a singular value decomposition. For non-fully connected $A$ and $(V/D)^{-1}$, the computation is performed on the spanning tree of the adjacency graph of $A$, for more details see [36].

B. Admissible Feedback Matrix

In the following, we characterize the feedback matrix $A$ of uniallpass FDNs with system matrix $V$. First, we assume that $V$ is orthogonal. The following theorem by Fiedler [37] gives sufficient and necessary conditions for such $A$.

Theorem 3 (Fiedler [37], Theorem 2.2). Every $N \times N$ submatrix of an orthogonal $N_V \times N_V$ matrix has at least
2N − NV = N − Nio singular values equal to one and Nio singular values less than one.

Conversely, if A is a N × N matrix that has N − k singular values equal to one and the remaining k singular values less than one, then for every NV ≥ N + k there exists an orthogonal NV × NV matrix containing A as a submatrix, and for no NV smaller than N + k does such matrix exist.

In particular for the SISO case with Nio = 1, A has exactly one singular value less than one and the other singular values are one. In the full MIMO case, i.e., Nio = N, A has all singular values less than one. Thus, any admissible feedback matrix A of a unialpass FDN is diagonally similar to a matrix with singular values as described above. There are various techniques to generate matrices with prescribed eigenvalues and singular values [38, 39]. Note, that for a stable FDN, the moduli of the eigenvalues of A are less than one [5].

C. Orthogonal Completion

We give a simple method for completing an orthogonal unialpass system. Given an N × N submatrix A of an NV × NV orthogonal matrix V, i.e., VVT = I. Therefore, P = I in (13). The block matrices in (13) for VV† = I and V†V = I yield then

\[ I − AA^T = BB^T, \]  
\[ I − A^T A = C^T C, \]  
\[ −BD = AC^T. \]  

The equations can be solved with a singular value decomposition, e.g., B is the rank-Nio decomposition of I − AA†.

Particularly in the full MIMO case, any matrix A with all singular values less than one can be completed to a unialpass FDN. As demonstrated in the Section VI this result is a large extension to prior designs.

D. General Completion

Here, we complete a feedback matrix A, which is part of any (not necessarily orthogonal) unialpass FDN. The first part of the procedure is general, where as the latter part focuses on the SISO case. From (33) and (17), we have

\[ D = (V/A)^{-1} = (D − CA^T B)^{-1} \]  

and further

\[ −A^T BD = PC^T, \]  
\[ −DCA^{-1} = B^T P^{-1}. \]  

Therefore, (18) is

\[ (V/D)^{-1} = A^T + PC^T D^{-1} B^T P^{-1}. \]  

Given the system matrix V of a unialpass FDN, thus, V† and V are diagonally similar and the Hadamard quotient V† ⊗ V† is diagonally similar to a \{0, 1\}-matrix. Thus,

\[ Q = \left( A^T + PC^T D^{-1} B^T P^{-1} \right) \odot A^T \]  

is diagonally similar to a \{0, 1\}-matrix J. In particular, the diagonal elements of Q are ones, and therefore

\[ (A)_{ii} = (A^{-1})_{ii} + \left( C^T D^{-1} B^T \right)_{ii}. \]  

The remaining procedure is only for the SISO case, which is emphasized by notating vectors and scalars b, c and d instead of matrices. From the unialpass property, we have d = ± det A. We restate (40)

\[ Q = \left( A^{-1} + \frac{c^T \tilde{b}}{d} \right) \odot A^T, \]  

where \( \tilde{c} = cP \), \( \tilde{b} = P^{-1}b \). We can also rewrite (41) for the SISO case, i.e.,

\[ (A)_{ii} = (A^{-1})_{ii} + \left( c^T d^{-1} b^T \right)_{ii}. \]  

More concisely, we can write

\[ d\alpha = c^T \circ b = \tilde{c}^T \circ \tilde{b}, \]  

where \( \alpha_{\circ} = (A)_{ii} - (A^{-1})_{ii} \), and \( \circ \) denotes the element-wise product, also called Hadamard product. By inspecting the individual matrix entries for \( 1 \leq i, j \leq N \)

\[ \left( \tilde{c}_i \tilde{b}_j \right) \left( \tilde{b}_i \tilde{c}_j \right) = \tilde{c}_i \tilde{c}_j \tilde{b}_i \tilde{b}_j = \left( \tilde{c}_i \tilde{b}_i \right) \left( \tilde{b}_i \tilde{c}_i \right), \]  

we derive an important identity

\[ d\alpha = \tilde{c}^T \circ \tilde{b} = \left( \tilde{c}^T \circ \tilde{b} \right) d^2 = \alpha \alpha^T. \]  

Because Q is diagonally similar to a \{0, 1\}-matrix J, we have

\[ Q \circ Q^T = J. \]  

We use this identity in the following to determine the input and output gains. By substituting (42) and (46) in Q ∘ QT, we derive

\[ Q \circ Q^T \circ A \circ A^T = A^{-1} \circ A^T + \]  
\[ A^{-1} \circ \tilde{b} \circ c^T \circ c^T \circ \tilde{b} = F, \]  

By substituting (47) into (48) and by sorting the terms we can write more concisely,

\[ A^{-1} \circ \tilde{b} \circ c^T \circ \tilde{b} = F, \]  

where

\[ F = d(J \circ A \circ A^T - A^{-1} \circ A^T - \alpha \alpha^T). \]  

By Hadamard multiplying the equation with \( \tilde{b} \circ c \) and substituting (46), we get

\[ A^{-1} \circ \tilde{b} \circ c^2 - F \circ \tilde{b} \circ c + A^{-1} \circ d^2 \alpha \alpha^T = 0, \]  

where \( \cdot^2 \) denotes the element-wise square. Each matrix entry in (51) is a quadratic equation and can be solved independently. From the two possible solutions for each matrix entry, one is selected such that the solution matrix is of rank 1. From (57),

\[ - P \tilde{b} \tilde{d} = - \tilde{b} \tilde{d} = A \tilde{c}^T \]  

(52)
such that
\[ \text{diag}(P) = -\left( \mathbf{Ac}^\top \right) \otimes \left( \tilde{b}d \right) \] (53)
we can recover \( P \) and therefore \( b \) and \( c \) from \( \tilde{b} \) and \( \tilde{c} \). This concludes the completion algorithms for SISO uniallpass FDNs. In the following section, we study the completion of a special class of feedback matrices.

V. HOMOGENEOUS DECAY ALLPASS FDN

A. Homogeneous Decay

A typical requirement in artificial reverberation and audio decorrelation is that all modes decay at the same rate, i.e., all system eigenvalues have the same magnitude, i.e., \( |\lambda_i| = \gamma \) for \( 1 \leq i \leq N \). We refer to this property as homogeneous decay. In FDNs, this can be achieved by delay-proportional absorption in combination with a lossless matrix \[7\]. Thus, the feedback matrix is
\[ A = UT \] (54)
with unilossless matrix \( U \), diagonal matrix \( \Gamma \) with \[6\]
\[ \Gamma_{ii} = \gamma^{m_i} \text{ for } 1 \leq i \leq N. \] (55)
For \( \gamma < 1 \), the singular values of \( A \) are then \( \Gamma_1, \ldots, \Gamma_N \) and the eigenvalues of \( A \) have moduli less than 1. From Section IV-C any such feedback matrix can be completed into a full MIMO uniallpass FDN. Note that this is a significant extension to Poletti’s design \[16\] as shown below in Section VI.

In \[54\], \( U \) can be a unilossless triangular matrix, i.e., with a diagonal of ones \[6\]. In Section VI we revisit this structure for series allpasses. In the following, we focus on the more intricate case of orthogonal \( U \).

B. SISO FDN

We construct homogeneous decay uniallpass FDNs for SISO. We substitute \[54\] into \[30\]
\[ P - UTPTU^\top = bb^\top. \] (56)
We right-multiply with \( U \) and substitute \( R = \Gamma^2 P \) and \( \tilde{b} = U^\top b \) such that
\[ PU - UR = bb^\top, \] (57)
which is called a displacement equation \[40\]. In the following, we denote the diagonal entries of a diagonal matrix \( P \) with a single index, e.g., \( P_i = P_i \). The solution of the displacement equation \[57\] is the Cauchy-like matrix \[40\]
\[ U = bb^\top \odot K \]
\[ = \text{diag}(b)K \text{ diag} \left( \tilde{b} \right), \] (58)
where the \( N \times N \) Cauchy matrix \( K \) has elements
\[ K_{ij} = \frac{1}{P_i - R_j}. \] (59)
Then, the inverse of the Cauchy matrix is given by \[41\]
\[ K^{-1} = \text{diag}(\alpha)K^\top \text{ diag}(\beta), \] (60)
where the elements of \( N \times 1 \) vectors \( \alpha \) and \( \beta \) are
\[ \alpha_i = \frac{A(R_i)}{B(R_i)} \text{ and } \beta_i = \frac{B(P_i)}{A(P_i)} \] (61)
and
\[ A(x) = \prod_{k=1}^{N}(x - P_k) \text{ and } B(x) = \prod_{k=1}^{N}(x - R_k), \] (62)
where \( \cdot \) denotes the derivative with respect to \( x \). Thus, the diagonal elements of \( P \) and \( R \) are the zeros of the polynomials \( A(x) \) and \( B(x) \). Thus, taking the inverse in \[58\] and substituting \[60\], yields
\[ U^{-1} = \text{diag} \left( \tilde{b} \right)^{-1}K^{-1}\text{diag} \left( b \right)^{-1} \]
\[ = \text{diag} \left( \tilde{b} \right)^{-1}\text{diag}(\alpha)K^\top \text{diag}(\beta)\text{diag}(b)^{-1}. \] (63)
Because \( U^{-1} = U^\top \), we have
\[ \text{diag} \left( \tilde{b} \right)^2 = \text{diag}(\alpha) \text{ and } \text{diag}(b)^2 = \text{diag}(\beta). \] (64)
Therefore, \( \alpha \) and \( \beta \) need to be positive. And the unitary matrix is given by
\[ U_{ij} = \frac{\sqrt{\beta_i \alpha_j}}{P_i - R_j}. \] (65)

C. Admissible Parameters

Firstly, we give a sufficient condition for \( P \) and \( R \) to be admissible, i.e., \( \alpha \) and \( \beta \) in \[64\] are positive. Secondly, for a given decay gains \( \Gamma \), we determine similarity matrix \( P \) such that \( P \) and \( \Gamma^2 P \) are admissible. The choice of \( P \) is effectively a parametrization of \( U \) in \[65\] such that a uniallpass FDN exists with \( A = UT \).

We show that following choice of \( P \) and \( R \) is admissible, i.e.,
\[ R_1 < P_1 < R_2 < P_2 < \cdots < R_N < P_N. \] (66)
Because of \[62\], we say that the zeros of \( A(x) \) and \( B(x) \) are strictly interleaved.

With Rolle’s theorem, the zeros of the derivatives \( A'(x) \) and \( B'(x) \) are strictly interleaving the zeros of \( A(x) \) and \( B(x) \), respectively \[42\]. Thus, with \[66\], we have that
\[ \text{sign} A'(P_i) = \text{sign} B'(R_i) = (-1)^{N-i}, \] (67)
where sign denotes the sign operator. Similarly, because of \[66\], we have
\[ \text{sign} A(R_i) = (-1)^{N+1-i} \text{ and } \text{sign} B(P_i) = (-1)^{N-i}. \] (68)
Therefore, with \[61\], we have
\[ \text{sign} \alpha_i = \frac{(-1)^{N+1-i}}{(-1)^{N-i}} = 1 \text{ and } \text{sign} \beta_i = \frac{(-1)^{N-i}}{(-1)^{N-1-i}} = 1 \]
such that \( P \) and \( R \) in \[66\] yield an admissible solution to \[64\].

Thus, for a given decay gain \( \Gamma \), we choose \( P \) such that \( P \) strictly interleaves \( R = \Gamma^2 \). With \[66\], we have
\[ 0 < \frac{P_{i-1}}{P_i} < \Gamma_i^2 \text{ for } 2 \leq i \leq N \] (69)
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The Schroeder series allpass of $N$ allpasses with matrix blocks $A, b, c, d$ as in (71).

The similarity matrix $P$ in (13) is a diagonal matrix with diagonal elements

$$P_{ii} = \frac{1}{g_i^2 - 1}. \quad (72)$$

Fig. 2(a) depicts the system matrix $V$ of the Schroeder series allpass for $N = 6$. The feedback matrix $A$ is triangular with gains $g_1, \ldots, g_N$ on the main diagonal. The remaining gains $b, c, d$ are determined by the gains $g_i$ as well. Therefore, there exists $A = U\Gamma$ with triangular unilossless $U$ and $\Gamma = \text{diag}([g_1, \ldots, g_N])$ such that the Schroeder series allpass can have homogeneous decay, see (54). At the same time, the series allpass is a highly limited structure with a particular feedback matrix.

B. SISO - Nested Allpass

The nested allpass as proposed by Gardner [14] is a recursive nesting of Schroeder allpasses, i.e.,

$$H_{\text{Gardner}} = H_N(z), \quad (73)$$

where $H_1(z) = \frac{g_1 + z^{-m_1}}{1 + g_1 z^{-m_1}}$ and for $k > 1$

$$H_k(z) = \frac{g_k + z^{-m_k} H_{k-1}(z)}{1 + g_k z^{-m_k} H_{k-1}(z)}. \quad (74)$$

Figure 3a shows an instance of the nested allpass for $N = 2$. The corresponding state space realization is

$$A_{ij} = \begin{cases} -g_i g_j & \text{for } i = j \\ 1 & \text{for } i = j - 1 \\ 0 & \text{for } i < j - 1 \\ 1 - g_j^2 \prod_{k=1}^{i-1} (1 - g_k^2) & \text{for } i > j \end{cases}, \quad (75a)$$

$$b_i = \begin{cases} 1 & \text{for } i = N \\ 0 & \text{otherwise} \end{cases}, \quad (75b)$$

$$c_i = \hat{g}_i \prod_{k=i}^N (1 - g_k^2), \quad (75c)$$

$$d = g_N, \quad (75d)$$

where $g_i$ and $m_i$ denote the feedforward-feedback gains and delay lengths, respectively. Fig. 2a shows an instance for $N = 2$. The corresponding state space realization is

$$A_{ij} = \begin{cases} -g_i & \text{for } i = j \\ 0 & \text{for } i < j \end{cases}, \quad (71a)$$

$$b_i = \prod_{k=1}^{i-1} g_k, \quad (71b)$$

$$c_i = (1 - g_i^2) \prod_{k=i+1}^N g_k, \quad (71c)$$

$$d = \prod_{k=1}^N g_k, \quad (71d)$$

and the similarity transform $P$ in (13) is a diagonal matrix with diagonal elements

$$P_{ii} = \frac{1}{g_i^2 - 1}. \quad (72)$$

**VI. Application**

In this section, we show that three well-known delay-based allpass structures are uniallpass FDNs: Schroeder’s series allpass [43], Gardner’s nested allpasses [14], and Poletti’s unitary reverberator [16]. Reviewing these previous designs also reveals their limited design space and demonstrates the significant extension introduced by Theorem 2. We conclude this section by presenting a complete numerical example of a SISO uniallpass FDN with homogeneous decay.

A. SISO - Series Schroeder Allpass

The Schroeder series allpass of $N$ feedforward-feedback delay allpasses is

$$H_{\text{Schroeder}}(z) = \prod_{i=1}^N \frac{g_i + z^{-m_i}}{1 + g_i z^{-m_i}}, \quad (70)$$

and $P_1$ and $\Gamma_1 < 1$ are unconstraint. Note, that $\Gamma$ does not need to be sorted in any way. As we have not constraint the decay gains $\Gamma$, we have shown that there exists SISO uniallpass FDNs with homogeneous decays for any delay $m$ and any decay rate $0 < \gamma < 1$. The similarity matrix $P$ acts as an additional design parameter within the constraints of (69).
where $\hat{g}_1 = 1$ and $\hat{g}_j = g_{j-1}$ for $2 \leq j \leq N$. The similarity transform $P$ in (13) is a diagonal matrix with diagonal elements

$$P_{ii} = \frac{-1}{\prod_{k=i}^{N} 1 - g_k}. \quad (76)$$

Fig. 3b depicts the system matrix $V$ of the nested allpasses for $N = 6$. The feedback matrix $A$ is Hessenberg and all gains including $b$, $c$, and $d$ are determined by the gains $g_i$. Series allpasses are strongly related to nested allpasses as they share the same parameter space, however, differ in the structure. Interestingly, the feedback matrix of nested allpasses induce a much more complex decay pattern than the series allpass counterpart.

**C. MIMO - Poletti Reverberator**

The MIMO reverberator proposed by Poletti is a direct multichannel generalization of the Schroeder allpass structure in lattice form, see Fig. 4a. The loop gain $\gamma$ controls the decay rate of the response tail such that

$$H_{\text{Poletti}}(z) = (\gamma I + UD_m(z))(I + \gamma UD_m(z))^{-1}. \quad (77)$$

The state space realization is

$$A = -\gamma U, \quad (78a)$$
$$B = (1 + \gamma)I, \quad (78b)$$
$$C = (1 - \gamma)U, \quad (78c)$$
$$D = \gamma I, \quad (78d)$$

and the similarity matrix in (13) is

$$P = \frac{1 + \gamma}{\sqrt{1 - \gamma^2}}I. \quad (79)$$

Fig. 4b depicts the system matrix $V$ of Poletti’s allpass for $N = 4$ and $N_o = 4$. While the direct and input gains, $D$ and $B$, respectively, are scaled identity matrices, the feedback matrix $A$ and output gains $C$ are scaled versions of the unitary matrix $U$. Interestingly, Poletti’s allpass has homogeneous decay only for equal delays, which is mostly an undesirable parameter choice.

**D. SISO Homogeneous Decay Uniallpass FDN**

We give a numerical example of a SISO allpass FDN with homogeneous decay following the procedure in Section 4. Let $N = 6$, $\gamma = 0.99$ and $m = [13, 22, 1, 10, 5, 3]$. Then with (55), we have

$$\Gamma = \text{diag}([0.878 \ 0.802 \ 0.990 \ 0.904 \ 0.951 \ 0.970])$$
presented a full characterization of uniallpass FDNs, which are allpass feedback delay networks (FDNs). In particular, we developed a novel characterization for uniallpass FDNs, which are allpass for any choice of delay lengths. Further, we introduced the uniallpass completion, i.e., completing a given feedback matrix to a uniallpass FDN. While the full MIMO case is relatively simple, also a solution to the SISO case was presented. Further, we solved the completion problem for a particular class of feedback matrices, which yields homogeneous decay of the impulse response. We reviewed three previous allpass FDN designs within this novel characterization and an additional numerical example for homogeneous decay uniallpass FDNs.

Future research questions should address application-specific designs of uniallpass FDNs, for instance, in audio signal processing, where additional constraints are required. Further research is also needed for the design of frequency-dependent FDNs with the allpass property.

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