Nonexistence of self-similar singularities in the viscous magnetohydrodynamics with zero resistivity

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Abstract

We are concerned on the possibility of finite time singularity in a partially viscous magnetohydrodynamic equations in $\mathbb{R}^n$, $n = 2, 3$, namely the MHD with positive viscosity and zero resistivity. In the special case of zero magnetic field the system reduces to the Navier-Stokes equations in $\mathbb{R}^n$. In this paper we exclude the scenario of finite time singularity in the form of self-similarity, under suitable integrability conditions on the velocity and the magnetic field. We also prove the nonexistence of asymptotically self-similar singularity. This provides us information on the behavior of solutions near possible singularity of general type as described in Corollary 1.1 below.

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1 Introduction

The equations of magnetohydrodynamics (MHD) with zero resistivity in $\mathbb{R}^n$, $n = 2, 3$, are the following.

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = \nu \Delta v - \nabla (p + \frac{1}{2} |b|^2) + (b \cdot \nabla) b, \quad (1.1)$$

$$\frac{\partial b}{\partial t} + (v \cdot \nabla) b = (b \cdot \nabla) v, \quad (1.2)$$

$$\text{div } v = \text{div } b = 0, \quad (1.3)$$

$$v(x, 0) = v_0(x), \quad b(x, 0) = b_0(x), \quad (1.4)$$

where $v = (v_1, \cdots, v_n)$, $v_j = v_j(x, t)$, $j = 1, \cdots, n$, is the velocity of the flow, $b = (b_1, \cdots, b_n)$, $b_j = b_j(x, t)$, is the magnetic field, $p = p(x, t)$ is the scalar pressure, $\nu > 0$ is the viscosity of the fluid, and $v_0$, $b_0$ are the given initial velocity and magnetic fields, satisfying $\text{div } v_0 = \text{div } b_0 = 0$, respectively.

The system (1.1)-(1.4) describes the macroscopic behavior of electrically conducting incompressible fluids with extremely high conductivity. In the original (fully viscous) equations of magnetohydrodynamics, besides the viscosity term, $\nu \Delta v$, in (1.1) we have the resistivity term, $\eta \Delta b$, in the right hand side of (1.2), where $\eta$ is the resistivity constant, which is inversely proportional to the electrical conductivity constant, $\sigma$. In the extremely high electrical conductivity cases, which occur frequently in the cosmical and geophysical problems we ignore the resistivity term to have our system (1.1)-(1.4) (see e.g. [4]). We are concerned here the mathematical question of the global well-posedness/finite time singularity of the system (1.1)-(1.4). The proof of local well-posedness of the Cauchy problem is rather standard (actually the necessary essential estimates are derived in the proof of Lemma 2.1 below), and similar to the case of fully viscous MHD is done in [17]. The question of spontaneous apparition of singularity from a local classical solution is a challenging open problem in the mathematical fluid mechanics. The situation is similar to the both of the cases of ideal MHD and fully viscous MHD. We just refer some of the studies on the finite time blow-up problem in the ideal MHD([11 6 7 10 8] and references therein). In order to discuss the self-similar singularity of the system (1.1)-(1.4) we first observe that it has the following scaling property: If $(v, b, p)$ is a solution of (1.1)-(1.4) corresponding to the initial data $(v_0, b_0)$, then for any $\lambda > 0$ the functions

$$v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t), \quad b^\lambda(x, t) = \lambda b(\lambda x, \lambda^2 t),$$
and
\[ p^\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t) \]
are also solutions with the initial data \( v_0^\lambda(x) = \lambda v_0(\lambda x), \ b_0^\lambda(x) = \lambda b_0(\lambda x). \)

In view of the above scaling property the self-similar blowing up solution \((v(x,t), b(x, t))\) of the system (1.1)-(1.4), if it exists, should be of the form,

\[ v(x,t) = \frac{1}{\sqrt{T_* - t}} V \left( \frac{x}{\sqrt{T_* - t}} \right), \quad (1.5) \]
\[ b(x, t) = \frac{1}{\sqrt{T_* - t}} B \left( \frac{x}{\sqrt{T_* - t}} \right), \quad (1.6) \]
\[ p(x, t) = \frac{1}{T_* - t} P \left( \frac{x}{\sqrt{T_* - t}} \right) \quad (1.7) \]
for \( t \) close to the possible blow-up time \( T_* \). If we substitute (1.5)-(1.7) into (1.1)-(1.4), then we find that \((V, B, P)\) should be a solution of the stationary system:

\[ \frac{1}{2} V + \frac{1}{2} (y \cdot \nabla) V + (V \cdot \nabla) V = \nu \Delta V + (B \cdot \nabla) B \]
\[ -\nabla(P + \frac{1}{2} |B|^2), \quad (1.8) \]
\[ \frac{1}{2} B + \frac{1}{2} (y \cdot \nabla) B + (V \cdot \nabla) B = (B \cdot \nabla) V, \quad (1.9) \]
\[ \text{div} V = \text{div} B = 0. \quad (1.10) \]

Conversely, if \((V, B, P)\) is a smooth solution of the system (1.8)-(1.10), then the triple of functions \((v, b, p)\) defined by (1.5)-(1.7) is a smooth solution of (1.1)-(1.4), which blows up at \( t = T_* \). The search for self-similar singularities of the form, (1.5)-(1.7) was suggested first by Leray for the 3D Navier-Stokes equations in [14], and its nonexistence was first proved by Nečas, Ružička and Šverák in [15] under the condition of \( V \in L^3(\mathbb{R}^3) \cap H^1_{\text{loc}}(\mathbb{R}^3) \), the result of which was generalized later by Tsai to the case \( L^p(\mathbb{R}^3) \cap H^1_{\text{loc}}(\mathbb{R}^3) \) with \( p > 3 \) in [19]. Their proofs crucially depend on the maximum principle of the Leray system,

\[ \frac{1}{2} V + \frac{1}{2} (y \cdot \nabla) V + (V \cdot \nabla) V = -\nabla P + \nu \Delta V, \quad \text{div} V = 0, \]

which corresponds to a special case \( (B = 0) \) in (1.8)-(1.10). The corresponding maximum principle for (1.8)-(1.10), however, cannot be obtained by applying similar method used in [15, 19] (The situation is similar even if we have
‘special’ resistivity term $\nu \Delta B$ to the right hand side of (1.9)). Due to this fact there are difficulties in extending the nonexistence results for the self-similar singularity of the 3D Navier-Stokes system to our system (1.1)-(1.4). Recently, the author of this paper developed new method to prove nonexistence of the self-similar singularity of the 3D Euler system under suitable integrability condition on the vorticity (2). Here we first combine the argument in (2) together with the results by [15, 19] to obtain the nonexistence of self-similar blowing up solutions, the precise statement of which is in the following theorem.

**Theorem 1.1** Suppose there exists $T_*>0$ such that we have a representation of a solution $(v,b)$ to (1.1)-(1.4) by (1.5)-(1.6) for all $t \in (0,T_*)$ with $(V,B)$ satisfying the following conditions:

(i) $(V,B) \in [C^1(\mathbb{R}^n)]^2$, $\nabla V \in L^\infty(\mathbb{R}^n)$, and $\text{div} \ V = \text{div} \ B = 0$.

(ii) In the case $n = 3$, there exists $q_1 \in [3,\infty)$ such that $V \in L^{q_1}(\mathbb{R}^3)$.

In the case $n = 2$, $V \in L^2(\mathbb{R}^2)$.

(iii) There exists $q_2 > 0$ such that $B \in L^q(\mathbb{R}^n)$ for all $q \in (0,q_2)$.

Then, $V = B = 0$.

**Remark 1.1** In order to illustrate the integrability condition for $B$ in (iii) above we make the following observations: If a function $f(x)$ on $\mathbb{R}^n$ satisfies

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^k)|f(x)| < \infty \quad \forall k \in \mathbb{N},$$

then $f \in L^p(\mathbb{R}^n)$ for all $p \in (0,\infty)$. Indeed, given $p \in (0,\infty)$, we choose $k = \left[\frac{n+1}{p}\right]$. Then, we have

$$\int_{\mathbb{R}^n} |f(x)|^p dx \leq \int_{\mathbb{R}^n} \left(\frac{C}{1 + |x|^k}\right)^p dx \leq C(n,p) \int_0^\infty \frac{r^{n-1}}{(1 + r^{n+1})} dr < \infty.$$ 

Under different type of decay conditions on $(V,B)$ from the above theorem, we could also have similar nonexistence result as follows.

**Theorem 1.2** Suppose there exists $T_*>0$ such that we have a representation of a solution $(v,b)$ to (1.1)-(1.4) by (1.5)-(1.6) for all $t \in (0,T_*)$ with $(V,B)$ satisfying the following conditions:
\( (i) \) \((V, B) \in [H^m(\mathbb{R}^n)]^2, \ m > n/2 + 1 \)

\( (ii) \) \( \|\nabla V\|_{L^\infty} + \|\nabla B\|_{L^\infty} < \eta \), where \( \eta \) is a sufficiently small constant to be determined in Lemma 2.1 in the next section.

Then, \( V = B = 0 \).

**Remark 1.2** The above theorem implies the ‘stability of the null solution’ of the stationary system (1.8)-(1.10). Namely, there exists \( \eta > 0 \) such that if \((V, B)\) is a solution to (1.8)-(1.10) and belongs to a ball \(B(0, \eta) = \{ X = (V, B) \in H^m(\mathbb{R}^n); \|\nabla X\|_{L^\infty} < \eta \}\), where \( m > n/2 + 1 \), then \((V, B) = (0, 0)\).

Next, we consider more refined scenario of ‘asymptotically self-similar singularity’, which means that the local in time smooth solution evolves into a self-similar profile as the possible singularity time is approached. A similar notion was considered previously by Giga and Kohn in the context of the nonlinear scalar heat equation in [9]. Recently, the author of this paper ([3]) considered it in the context of 3D Euler and the 3D Navier-Stokes equations (see also [11]), and excluded its scenario. We apply the idea developed in [3] to exclude asymptotically self-similar singularity of our system (1.1)-(1.4).

**Theorem 1.3** Let \((v, b) \in [C([0,T); H^m(\mathbb{R}^n))]^2, m > n/2 + 1, \) be a classical solutions to (1.1)-(1.4). Suppose there exist functions \( \bar{V}, \bar{B} \) satisfying the conditions (i)-(iii) for \( V, B \) in Theorem 1.1 such that the following boundedness and the convergence hold true:

\[
\sup_{0 < t < T} (T-t)^{1-m/2} \left\| v(\cdot, t) - \frac{1}{\sqrt{T-t}} \bar{V} \left( \frac{\cdot}{\sqrt{T-t}} \right) \right\|_{L^1} + \sup_{0 < t < T} (T-t)^{1-m/2} \left\| b(\cdot, t) - \frac{1}{\sqrt{T-t}} \bar{B} \left( \frac{\cdot}{\sqrt{T-t}} \right) \right\|_{L^1} < \infty,
\]

\[
\lim_{t \to T} (T-t)^{1-m/2} \left\| \nabla v(\cdot, t) - \frac{1}{\sqrt{T-t}} \nabla \bar{V} \left( \frac{\cdot}{\sqrt{T-t}} \right) \right\|_{L^\infty} + \lim_{t \to T} (T-t)^{1-m/2} \left\| \nabla b(\cdot, t) - \frac{1}{T-t} \nabla \bar{B} \left( \frac{\cdot}{\sqrt{T-t}} \right) \right\|_{L^\infty} = 0.
\]
Then, $\tilde{V} = \tilde{B} = 0$, and $(v, b)$ can be extended to a solution of (1.1)-(1.4) in $[0, T + \delta] \times \mathbb{R}^n$, and belongs to $C([0, T + \delta]; H^m(\mathbb{R}^n))$ for some $\delta > 0$.

Remark 1.3 Unlike to the cases of the Euler equations (3), the convergence of (1.12) is not in the critical Besov space norms for the quantities of vorticities and current densities, but in the L^p norm for the gradients of velocities and magnetic fields. Actually due to the non-symmetry of the viscosity terms (the term $\nu \Delta v$ for the velocity evolution equations (1.1), and zero for the magnetic field evolution equations (1.2)) we cannot obtain critical Besov type of norm estimates in the procedure of proof of the above theorem (see the proof in the next section below).

As an immediate corollary of Theorem 1.3 we have the following information of the behaviors of solution near possible singularity, which is not necessarily of the self-similar type.

Corollary 1.1 Let $(v, b) \in [C([0, T_*); H^m(\mathbb{R}^n))]^2, m > n/2 + 1$, be a classical solutions to (1.1)-(1.4), which blows up at $T$. We expand the solution of the form:

$$v(x, t) = \frac{1}{\sqrt{T - t}} \tilde{V} \left( \frac{x}{\sqrt{T - t}} \right) + \bar{v}(x, t),$$

$$b(x, t) = \frac{1}{\sqrt{T - t}} \tilde{B} \left( \frac{x}{\sqrt{T - t}} \right) + \bar{b}(x, t),$$

where $(\tilde{V}, \tilde{B})$ satisfies the conditions (i)-(iii) for $(V, B)$ in Theorem 1.1. Then, either

$$\limsup_{t \nearrow T} \left[ (T - t)^{-\frac{1}{2m}} (\|\bar{v}(t)\|_{L^1} + \|\bar{b}(t)\|_{L^1}) \right] = \infty,$$  

or there exists $\varepsilon_0 > 0$ such that

$$\limsup_{t \nearrow T} \left[ (T - t)(\|\nabla \bar{v}(t)\|_{L^\infty} + \|\nabla \bar{b}(t)\|_{L^\infty}) \right] > \varepsilon_0.$$  

2 Proof of the theorems

Proof of Theorem 1.1 We assume classical solution $(v, b)$ of the form (1.5)-(1.6). We will show that this assumption leads to $V = B = 0$. By
consistency with the initial condition, \( b_0(x) = \frac{1}{\sqrt{f_*}} B(\frac{x}{\sqrt{f_*}}) \), we can rewrite the representation (1.6) in the form,

\[
b(x, t) = \left( 1 - \frac{t}{T_*} \right)^{-\frac{1}{2}} b_0 \left( \left( 1 - \frac{t}{T_*} \right)^{-\frac{1}{2}} x \right) \quad \forall t \in [0, T_\ast).
\] (2.17)

Let \( a \mapsto X(a, t) \) be the particle trajectory mapping, defined by the ordinary differential equations,

\[
\frac{\partial X(a, t)}{\partial t} = v(X(a, t), t) ; \quad X(a, 0) = a.
\]

We set \( A(x, t) := X^{-1}(x, t) \), which is called the back-to-label map, satisfying

\[
A(X(a, t), t) = a, \quad X(A(x, t), t) = x.
\] (2.18)

We note that for our smoothness condition (i) decay condition on the velocity (ii) the existence of \( A(\cdot, t) \) is guaranteed at least for \( t \) close to \( T_* \) (see \[5\]), which is enough for our purpose in the proof. Taking dot product (1.2) by \( b \), we obtain

\[
\frac{\partial |b|}{\partial t} + (v \cdot \nabla)|b| = \alpha |b|,
\] (2.19)

where \( \alpha(x, t) \) is defined as

\[
\alpha(x, t) = \begin{cases} 
\sum_{i,j=1}^{n} S_{ij}(x, t)\xi_i(x, t)\xi_j(x, t) & \text{if } b(x, t) \neq 0 \\
0 & \text{if } b(x, t) = 0
\end{cases}
\]

with

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \quad \text{and} \quad \xi(x, t) = \frac{b(x, t)}{|b(x, t)|}.
\]

In terms of the particle trajectory mapping we can rewrite (2.19) as

\[
\frac{\partial}{\partial t} |b(X(a, t), t)| = \alpha(X(a, t), t) |b(X(a, t), t)|.
\] (2.20)

Integrating (2.20) along the particle trajectories \( \{X(a, t)\} \), we have

\[
|b(X(a, t), t)| = |b_0(a)| \exp \left[ \int_0^t \alpha(X(a, s), s) ds \right].
\] (2.21)
Taking into account the simple estimates
\[-\|\nabla v(\cdot, t)\|_{L^\infty} \leq \alpha(x, t) \quad \forall x \in \mathbb{R}^n,\]
we obtain from (2.21) that
\[|b_0(a)| \exp \left[ - \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq |b(X(a, t), t)|,\]
which, in terms of the back-to-label map, can be rewritten as
\[|b_0(A(x, t))| \exp \left[ - \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq |b(x, t)|. \tag{2.22}\]
Combining this with the self-similar representation formula in (2.17), we have
\[|b_0(A(x, t))| \exp \left[ - \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq \left(1 - \frac{t}{T_*}\right)^{-\frac{n}{2q} - \frac{1}{2}} |b_0\left(\left(1 - \frac{t}{T_*}\right)^{-\frac{1}{2}} x\right)| \tag{2.23}\]
Given \(q \in (0, q_2)\), computing \(L^q(\mathbb{R}^n)\) norm of each side of (2.23), we obtain
\[\|b_0\|_{L^q} \exp \left[ - \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq \|b_0\|_{L^q} \left(1 - \frac{t}{T_*}\right)^{-\frac{n}{2q} - \frac{1}{2}} \tag{2.24}\]
where we used the fact \(\det(\nabla A(x, t)) \equiv 1\). Now, suppose \(B \neq 0\), which is equivalent to assuming that \(b_0 \neq 0\), then we divide (2.24) by \(\|b_0\|_{L^q}\) to have
\[\exp \left[ - \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds \right] \leq \left(1 - \frac{t}{T_*}\right)^{-\frac{n}{2q} - \frac{1}{2}} \tag{2.25}\]
Passing \(q \searrow 0\) in (2.25), we deduce that
\[\int_0^t \|\nabla v(\cdot, s)\|_{L^\infty} ds = \infty \quad \forall t \in (0, T_*)\]
This contradicts with the assumption that the flow is smooth on \((0, T_*),\)
i.e. \(v \in C^1([0, T_*); C^1(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n))\), which is implied by the by the explicit representation formula (1.5)-(1.6), combined with the assumption (i). Hence we need to have \(B = 0\). Setting \(B = 0\) in the system (1.1)-(1.4),
it reduces to the incompressible Navier-Stokes system in \( \mathbb{R}^n \). When \( n = 3 \) we apply Nečas-Ružička-Šverák’s result in \([15]\) for \( q_1 = 3 \) and Tsai’s result in \([19]\) for \( q_1 \in (3, \infty) \) respectively. Then, we obtain \( V = 0 \). In the case \( n = 2 \) we recall that in the 2D Navier-Stokes equations for the initial data \( v_0(\cdot) = \frac{1}{\sqrt{T^*}}V(\frac{\cdot}{\sqrt{T^*}}) \in L^2(\mathbb{R}^2) \) the solution \( v \) belongs to \( C^\infty((0, \infty) \times \mathbb{R}^2) \)(see e.g. \([18]\)), and hence we need to have \( V = 0 \). □

In order to prove Theorem 1.2 and Theorem 1.3 we establish the following continuation principle for local classical solution of (1.1)-(1.4).

**Lemma 2.1** Let \((v, b) \in [C([0, T); H^m(\mathbb{R}^n))]^2, m > n/2 + 1, \) be a classical solution to (1.1)-(1.4). There exists an absolute constant \( \eta > 0 \) such that if

\[
\sup_{0 \leq t < T} (T - t) \left\{ \| \nabla v(t) \|_{L^\infty} + \| \nabla b(t) \|_{L^\infty} \right\} < \eta,
\]

(2.26)

then the solution \((v(x, t), b(x, t))\) can be extended to be functions on \([0, T + \delta] \times \mathbb{R}^n\) and belongs to \( C([0, T + \delta]; H^m(\mathbb{R}^n))\) for some \( \delta > 0 \).

**Proof** Let \( \alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \) be a standard multi-index with \(|\alpha| = \alpha_1 + \cdots + \alpha_n\). We take operation \( D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \) on (1.1), and take \( L^2(\mathbb{R}^n) \) inner product it with \( D^\alpha, \) summing over \(|\alpha| \leq m \) after integration by parts. Then, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| v \|_{H^m}^2 + \nu \| \nabla v \|_{H^m}^2 = \sum_{|\alpha| \leq m} (D^\alpha(v \cdot \nabla)v - (v \cdot \nabla)D^\alpha v, D^\alpha v)_{L^2} + \sum_{|\alpha| \leq m} (D^\alpha(b \cdot \nabla)b - (b \cdot \nabla)D^\alpha b, D^\alpha v)_{L^2} + \sum_{|\alpha| \leq m} (b \cdot \nabla)D^\alpha b, D^\alpha v)_{L^2},
\]

(2.27)

where we used the facts,

\[
((v \cdot \nabla)D^\alpha v, D^\alpha v)_{L^2} = \frac{1}{2} \int_{\mathbb{R}^n} (v \cdot \nabla) |D^\alpha v|^2 dx = -\frac{1}{2} \int_{\mathbb{R}^n} (\text{div } v) |D^\alpha v|^2 dx = 0,
\]

and

\[
(D^\alpha v, D^\alpha \nabla (p + \frac{1}{2} |v|^2))_{L^2} = -(D^\alpha (\text{div } v), D^\alpha (p + \frac{1}{2} |v|^2))_{L^2} = 0.
\]
Applying the well-known commutator estimate ([13]),
\[
\sum_{|\alpha| \leq m} \| D^\alpha (fg) - f D^\alpha g \|_{L^2} \leq C (\| \nabla f \|_{H^{m-1}} + \| f \|_{H^m} \| g \|_{L^\infty}),
\]
to the terms of the right hand side of (2.27), we have
\[
\frac{1}{2} \frac{d}{dt} \| v \|_{H^m}^2 + \nu \| \nabla v \|_{H^m}^2 \leq C' \| \nabla v \|_{L^\infty} \| v \|_{H^m}^2 + \sum_{|\alpha| \leq m} ((b \cdot \nabla) D^\alpha b, D^\alpha v)_{L^2}. \tag{2.28}
\]
Similarly, starting from (1.2), we can deduce
\[
\frac{1}{2} \frac{d}{dt} \| b \|_{H^m}^2 = - \sum_{|\alpha| \leq m} (D^\alpha (v \cdot \nabla) b - (v \cdot \nabla) D^\alpha b, D^\alpha b)_{L^2}
\]
\[
+ \sum_{|\alpha| \leq m} (D^\alpha (b \cdot \nabla) v - (v \cdot \nabla) D^\alpha v, D^\alpha b)_{L^2} + \sum_{|\alpha| \leq m} ((b \cdot \nabla) D^\alpha v, D^\alpha b)_{L^2}
\]
\[
\leq C' \| \nabla v \|_{L^\infty} \| b \|_{H^m}^2 + C' \| \nabla b \|_{L^\infty} \| v \|_{H^m} \| b \|_{H^m} + \sum_{|\alpha| \leq m} ((b \cdot \nabla) D^\alpha v, D^\alpha b)_{L^2}. \tag{2.29}
\]
We observe that
\[
\sum_{|\alpha| \leq m} ((b \cdot \nabla) D^\alpha b, D^\alpha v)_{L^2} = - \sum_{|\alpha| \leq m} ((b \cdot \nabla) D^\alpha v, D^\alpha b)_{L^2},
\]
which is obvious by the integration by part. Thus, adding (2.28) to (2.29), we obtain
\[
\frac{1}{2} \frac{d}{dt} (\| v \|_{H^m}^2 + \| b \|_{H^m}^2) + \nu \| \nabla v \|_{H^m}^2 \leq C (\| \nabla v \|_{L^\infty} + \| \nabla b \|_{L^\infty}) (\| v \|_{H^m}^2 + \| b \|_{H^m}^2), \tag{2.30}
\]
where we used the inequality, \( ab \leq \frac{1}{2} (a^2 + b^2). \) From (2.30) we first derive the inequality
\[
\| v(t) \|_{H^m}^2 + \| b(t) \|_{H^m}^2 + \nu \int_{t_0}^t \| \nabla v(s) \|_{H^m}^2 ds
\]
\[
\leq (\| v(t_0) \|_{H^m}^2 + \| b(t_0) \|_{H^m}^2) \exp \left[ \int_{t_0}^t (\| \nabla v(s) \|_{L^\infty} + \| \nabla b(s) \|_{L^\infty}) ds \right] \tag{2.31}
\]
for all $0 \leq t_0 < t$, which implies the continuation principle that if
\[
\int_{t_0}^{T} (\|\nabla v(s)\|_{L^\infty} + \|\nabla b(s)\|_{L^\infty}) ds < \infty,
\]
then $\|v(T)\|_{H^m} + \|b(T)\|_{H^m} < \infty$, and we can continue our classical solution $(v(t), b(t)) \in [H^m(\mathbb{R}^n)]^2$ up to $[t_0, T + \delta]$ so that $(v, b) \in [C([0, T + \delta]; H^m(\mathbb{R}^n))]^2$ for some $\delta > 0$. Next, using the estimate (2.30), we derive
\[
\frac{d}{dt} \left\{ (T - t)(\|v\|_{H^m}^2 + \|b\|_{H^m}^2) \right\} + \nu(T - t)\|\nabla v\|_{H^m}^2 + (\|v\|_{H^m}^2 + \|b\|_{H^m}^2) \leq C_0(T - t)(\|\nabla v\|_{L^\infty} + \|\nabla b\|_{L^\infty})(\|v\|_{H^m}^2 + \|b\|_{H^m}^2) \tag{2.32}
\]
for a constant $C_0 = C_0(m, n)$. We suppose
\[
\sup_{0 < t < T} (T - t)(\|\nabla v(t)\|_{L^\infty} + \|\nabla b(t)\|_{L^\infty}) \leq \frac{1}{2C_0}.
\]
Then,
\[
\frac{d}{dt} \left\{ (T - t)(\|v\|_{H^m}^2 + \|b\|_{H^m}^2) \right\} + \nu(T - t)\|\nabla v\|_{H^m}^2 + \frac{1}{2}(\|v\|_{H^m}^2 + \|b\|_{H^m}^2) \leq 0,
\]
and integrating this over $[t_0, T]$, we have
\[
\sup_{t_0 < t < T} (T - t)(\|v\|_{H^m}^2 + \|b\|_{H^m}^2) + \nu \int_{t_0}^{T} (T - t)\|\nabla v(t)\|_{H^m}^2 dt + \frac{1}{2} \int_{t_0}^{T} (\|v(t)\|_{H^m}^2 + \|b(t)\|_{H^m}^2) dt \leq (T - t_0)(\|v(t_0)\|_{H^m}^2 + \|b(t_0)\|_{H^m}^2).
\tag{2.33}
\]
Since $H^m(\mathbb{R}^n) \hookrightarrow Lip(\mathbb{R}^n)$ for $m > n/2 + 1$, we have
\[
\int_{t_0}^{T} (\|\nabla v(t)\|_{L^\infty} + \|\nabla b(t)\|_{L^\infty}) dt \leq C \int_{t_0}^{T} (\|v(t)\|_{H^m} + \|b(t)\|_{H^m}) dt \leq C \sqrt{T - t_0} \left[ \int_{t_0}^{T} (\|v(t)\|_{H^m} + \|b(t)\|_{H^m}^2) dt \right]^{\frac{1}{2}} < \infty.
\]
Applying the continuation principle derived above, we can continue our local solution as described in the theorem. □
Proof of Theorem 1.2 We just observe that
\[(T - t)\|\nabla v(t)\|_{L^\infty} = \|V\|_{L^\infty}, \quad (T - t)\|\nabla b(t)\|_{L^\infty} = \|\nabla B\|_{L^\infty}\]
for all \(t \in (t_0, T)\). Hence, our smallness condition, \(\|\nabla V\|_{L^\infty} + \|\nabla B\|_{L^\infty} < \eta\),
leads to
\[
\sup_{t_0 < t < T} (T - t) \{\|\nabla v(t)\|_{L^\infty} + \|\nabla b(t)\|_{L^\infty}\} < \eta.
\]
Applying Lemma 2.1, for initial time at \(t = t_0\), we conclude that \((v, b) \in [C([t_0, T); H^m(\mathbb{R}^n)])^2\) cannot have singularity at \(t = t_0\), hence we need to have \(V = B = 0\). □

Proof of Theorem 1.3 We change variables from the physical ones \((x, t) \in \mathbb{R}^n \times [0, T)\) to the ‘self-similar variables’ \((y, s) \in \mathbb{R}^n \times [0, \infty)\) as follows:
\[
y = \frac{x}{\sqrt{T - t}}, \quad s = \frac{1}{2} \log \left(\frac{T}{T - t}\right).
\]
Based on this change of variables, we transform the functions \((v, p) \mapsto (V, P)\) according to
\[
v(x, t) = \frac{1}{\sqrt{T - t}} V(y, s), \quad (2.34)
\]
\[
b(x, t) = \frac{1}{\sqrt{T - t}} B(y, s), \quad (2.35)
\]
\[
p(x, t) = \frac{1}{\sqrt{T - t}} P(y, s). \tag{2.36}
\]
Substituting \((v, b, p)\) into \((1.1)-(1.4)\), we obtain the following equivalent evolution equations for \((V, P)\),
\[
\begin{cases}
\frac{1}{2} V_s + \frac{1}{2} V + \frac{1}{2} (y \cdot \nabla) V + (V \cdot \nabla) V = \nu \Delta V + (B \cdot \nabla) B - \nabla (P + \frac{1}{2} |V|^2), \\
\frac{1}{2} B_s + \frac{1}{2} B + \frac{1}{2} (y \cdot \nabla) B + (V \cdot \nabla) B = (B \cdot \nabla) V, \\
\text{div} V = \text{div} B = 0, \\
V(y, 0) = V_0(y) = \sqrt{T} v_0(\sqrt{T} y), \quad B(y, 0) = B_0(y) = \sqrt{T} b_0(\sqrt{T} y).
\end{cases}
\tag{2.37}
\]
In terms of \((V, B)\) the conditions (1.11) and (1.12) are translated into
\[
\sup_{0 < s < \infty} \left( \|V(\cdot, s) - \bar{V}(\cdot)\|_{L^1} + \|B(\cdot, s) - \bar{B}(\cdot)\|_{L^1} \right) < \infty,
\]
and
\[
\lim_{s \to \infty} \|\nabla V(\cdot, s) - \nabla \bar{V}(\cdot)\|_{L^\infty} = \lim_{s \to \infty} \|\nabla B(\cdot, s) - \nabla \bar{B}(\cdot)\|_{L^\infty} = 0,
\]
respectively, from which, thanks to the standard interpolation, we can have
\[
\lim_{s \to \infty} \|V(\cdot, s) - \bar{V}(\cdot)\|_{H^1(B_R)} = \lim_{s \to \infty} \|B(\cdot, s) - \bar{B}(\cdot)\|_{H^1(B_R)} = 0 \tag{2.38}
\]
for all \(0 < R < \infty\), where \(B_R = \{x \in \mathbb{R}^n \mid |x| < R\}\). Similarly to [11], we consider scalar test functions \(\xi \in C^1_c(0, 1)\) with \(\int_0^1 \xi(s) ds \neq 0\), \(\psi \in C^1_c(\mathbb{R}^n)\) and the vector test function \(\phi = (\phi_1, \cdots, \phi_n) \in C^1_c(\mathbb{R}^n)\) with \(\text{div} \phi = 0\). We multiply the first equation of (2.37) by \(\xi(s - k)\phi(y)\), and integrate it over \(\mathbb{R}^n \times [k, k + 1]\), and then we integrate by part for the terms including the time derivative and the pressure term to obtain
\[
\begin{align*}
- \int_0^1 \int_{\mathbb{R}^n} \xi_s(s)\phi(y) \cdot V(y, s + k) dy ds \\
+ \int_0^1 \int_{\mathbb{R}^n} \xi(s)\phi(y) \cdot [V + (y \cdot \nabla)V + 2(V \cdot \nabla) V](y, s + k) dy ds \\
- 2 \int_0^1 \int_{\mathbb{R}^n} \xi(s)\phi(y) \cdot (B \cdot \nabla)B(y, s + k) dy ds \\
+ 2\nu \int_0^1 \int_{\mathbb{R}^n} \xi(s)\nabla \phi(y) \cdot \nabla V(y, s + k) dy ds = 0, \tag{2.39}
\end{align*}
\]
and
\[
\begin{align*}
- \int_0^1 \int_{\mathbb{R}^n} \xi_s(s)\psi(y)B(y, s + k) dy ds \\
+ \int_0^1 \int_{\mathbb{R}^n} \xi(s)\psi(y)\left[ B + (y \cdot \nabla)B + 2(V \cdot \nabla)B \right](y, s + k) dy ds \\
- 2 \int_0^1 \int_{\mathbb{R}^n} \xi(s)\psi(y)(B \cdot \nabla)V(y, s + k) dy ds = 0. \tag{2.40}
\end{align*}
\]
Passing to the limit \(k \to \infty\) in (2.39)-(2.40), using the convergence (2.38), \(\int_0^1 \xi_s(s) ds = 0\) and \(\int_0^1 \xi(s) ds \neq 0\), we find that \(\bar{V}, \bar{B} \in C^1(\mathbb{R}^n)\) satisfies
\[
\int_{\mathbb{R}^n} [\bar{V} + (y \cdot \nabla)\bar{V} + 2(V \cdot \nabla)\bar{V} - 2(B \cdot \nabla)\bar{B}] \cdot \phi dy + 2\nu \int_{\mathbb{R}^n} \nabla \bar{V} \cdot \nabla \phi dy = 0,
\]
13
\[ \int_{\mathbb{R}^n} [\bar{B} + (y \cdot \nabla) \bar{B} + 2(\bar{V} \cdot \nabla) \bar{B} - 2(\bar{B} \cdot \nabla) \bar{V}] \psi \, dy = 0, \]

for all vector test function \( \phi \in C^1_0(\mathbb{R}^n) \) with \( \text{div} \, \phi = 0 \), and scalar test function \( \psi \in C^1_0(\mathbb{R}^n) \). Hence, there exists a scalar function \( \bar{P}' \), which can be written without loss of generality as \( \bar{P}' = \bar{P} + \frac{1}{2} |\bar{B}|^2 \) for another scalar function \( \bar{P} \), such that

\[ \bar{V} + (y \cdot \nabla) \bar{V} + 2(\bar{V} \cdot \nabla) \bar{V} = 2\nu \Delta \bar{V} + 2(\bar{B} \cdot \nabla) \bar{B} - 2\nu (\bar{V} + \frac{1}{2} |\bar{B}|^2), \]

(2.41)

and

\[ \bar{B} + (y \cdot \nabla) \bar{B} + 2(\bar{V} \cdot \nabla) \bar{B} = 2(\bar{B} \cdot \nabla) \bar{V}. \]

(2.42)

On the other hand, we can pass \( s \to \infty \) directly in the incompressibility equations for \( V \) and \( B \) in (2.37) to have

\[ \text{div} \, \bar{V} = \text{div} \, \bar{B} = 0. \]

(2.43)

The equations (2.41)-(2.43) show that \( (\bar{V}, \bar{B}) \) is a classical solution of (1.8)-(1.10). Since, by hypothesis, \( (\bar{V}, \bar{B}) \) satisfies the condition (i)-(iii) of Theorem 1.1, we can deduce \( \bar{V} = \bar{B} = 0 \) by that theorem. Hence, (??) leads to

\[ \lim_{s \to \infty} \|\nabla V(s)\|_{L^\infty} = \lim_{s \to \infty} \|\nabla B(s)\|_{L^\infty} = 0. \]

Thus, for \( \eta > 0 \) given in Lemma 2.1, there exists \( s_1 > 0 \) such that

\[ \|\nabla V(s_1)\|_{L^\infty} + \|\nabla B(s_1)\|_{L^\infty} < \eta. \]

Let us set \( t_1 = T[1 - e^{2s_1}] \). Going back to the original physical variables, we have

\[ (T - t_1)\|\nabla v(t_1)\|_{L^\infty} + (T - t_1)\|\nabla b(t_1)\|_{L^\infty} < \eta. \]

Applying Lemma 2.1, we conclude the proof. \( \square \)

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