Existence of weak solutions for non-stationary flows of fluids with shear thinning dependent viscosities under slip boundary conditions in half space

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Abstract

The author treats the system of motion for an incompressible non-Newtonian fluids of the stress tensor described by $p-$potential function subject to slip boundary conditions in $\mathbb{R}^3_+$. Making use of the Oseen-type approximation to this model and the $L^\infty$-truncation method, one can establish the existence theorem of weak solutions for $p-$potential flow with $p \in (\frac{2}{3}, 2]$ provided that large initial data.

Keywords: Non-Newtonian Fluid; slip boundary conditions; Oseen-type approximation; weak solution;
Mathematics Subject Classification(2000): 76D05, 35D05,54B15, 34A34.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open set. For any $T < \infty$, set $Q_T = \Omega \times (0, T)$. The motion of a homogeneous, incompressible fluid through $\Omega$ is governed by the following equations

$$\begin{cases}
\partial_t u - \text{div} S + (u \cdot \nabla)u + \nabla \pi = f, \quad \text{in } Q_T \\
\nabla \cdot u = 0, \quad \text{in } Q_T \\
u|_{t=0} = u_0(x), \quad \text{in } \Omega,
\end{cases}$$

(1)
where \( u \) is the velocity, \( \pi \) is pressure and \( f \) is the force, \( u_0 \) is initial velocity and \( S = (s_{ij})^{n}_{i,j=1} \) is stress tensor. The above system (1.1) has to be completed by boundary conditions except that \( \Omega \) is the whole space and by constitutive assumptions for the extra tensor. It is well known that the extra tensors are characterized by Stoke’s law \( S = \nu D(u) \), where \( D(u) \) is the symmetric velocity gradient, i.e.

\[
D_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).
\]

Assume that \( \nu \) is a constant, (1) is called incompressible Navier-Stokes equations.

However, there are phenomena that can be described by \( \nu = \nu(|D(u)|) \) with power-law ansatz, which presents certain non-Newtonian behavior of the fluid flows. These models are frequently used in engineering literature. We can refer the book by Bird, Armstrong and Hassager [15] and the survey paper due to Málak and Rajagopal [36]. Typical examples for the constitutive relations are

\[
S(D(u)) = \mu (\delta + |D(u)|) \frac{q-2}{q} D(u),
\]

with \( 1 < q < \infty \), \( \delta \geq 0 \), and \( \mu > 0 \).

These models (2) were studied initially by Ladyženska(see [30], [31] and [32]) and Lions(see [33]). There are many results improved in several directions by many mathematicians. For example, concerning the steady problem, there are several results proving existence of weak solution and interior regularity in bounded domain, we can refer [23, 26, 27, 11, 38] and references therein. The results of regularity up to boundary for the Dirichlet problems were obtained by T. Shinlkin [41] and H. Beirão da Veiga, et.al. [5, 6, 7, 8, 9, 10, 11, 20, 21].

For the unsteady Dirichlet boundary value problems, J. Málak, J. Nečas, and M. Růžička[35] firstly obtained the weak solution for \( p \geq 2 \). Later, L. Diening et.al. had generalized these results to \( p > \frac{2}{3} \) in [23] and \( p > \frac{6}{5} \) in [24]. There are also many papers dealing with regularity of for evolution Dirichlet boundary problems and we refer instance to [3, 4, 16, 8, 9, 10, 11]. For the study of these models with space periodic boundary conditions, we refer to the monograph [34] and papers [14, 22].

It should be emphasized that theoretical contributions mostly concern the no-slip boundary conditions and space periodic boundary conditions. However, many other boundary conditions are important for engineer experiment and computation science. Commonly used boundary conditions are Navier-
type boundary conditions, i.e.

\[ u \cdot n = 0, (S \cdot n) \tau - \alpha u \tau = 0, \text{ on } \partial \Omega \times (0, T), \]  

(3)

where \( \alpha \) is the smooth function which were introduced by Navier in \([39]\). Navier-stokes equations under Navier slip boundary conditions were studied by many mathematician,\([12, 13]\) and \([13]\). There also are some results for non-Newtonian fluid. For example, in \([5, 25]\), the authors investigated the regularity of steady flows with shear-dependent viscosity on the slip boundary conditions. M. Bulíček, J. Málek and K.R. Rajagopal obtained the weak solution for the evolutionary generalized Navier-stokes-like system of pressure and shear-dependent viscosity in \([17]\). Later, in \([18]\) they treated the fluid dependent on pressure, shear-rate and temperature with the Navier-type slip boundary conditions in the bounded domain.

In this paper, we study the system \((1)\) with stress tensor \( S \) induced by \( p \)-potential as in Definition 2.1 in the half space \( \mathbb{R}^3_+ \) with perfect slip boundary conditions on \( x_3 = 0 \), i.e.

\[ u_3 |_{x_3=0} = \frac{\partial u_i}{\partial x_3} |_{x_3=0} = 0 \text{ for } i = 1, 2. \]  

(4)

There have been some existence results in unbounded to study. For instance, Galdi and Grisanti obtained the existence and regularity of 2D exterior problems for steady flows with shear-thinning Ladyzhenskaya model in \([28]\). M. Pokorný in \([40]\) firstly obtained the weak solution of Cauchy problem for \( p \)-potential models approximated by the Bipolar non-Newtonian incompressible fluid. M. Bulíček, J.Málek, M. Majdoub and J. Málek studied the Cauchy problem of the fluid with pressure dependent viscosity in \([19]\). They established the existence theorem for unsteady flows approximated by means of higher differentiability models with the power \( p > \frac{9}{5} \).

After extend the initial velocity and force term by mirror method, you note that it is seem to transfer into the Cauchy problem as previous mentioned in \([40]\) and should have established the existence results. However, because of absent enough regularity and uniqueness, this Cauchy problem can not recover the system of \((1)\) with the boundary conditions \((4)\). As the boundary of \( \mathbb{R}^3_+ \) is concerned, the methods mentioned both in \([40]\) are \([19]\) are not applied without suitable boundary conditions for higher derivatives of the velocity.

We overcome these difficulties from the following two ways. Firstly, we can regularize the convection term \( u \cdot \nabla u \) in system \((1)\) and obtain an ap-
proximated system,
\[
\begin{aligned}
\begin{cases}
\partial_t u - \text{div}S(D(u)) + (u' \cdot \nabla)u + \nabla \pi = f, & \text{in } Q_T \\
\nabla \cdot u = 0, & \text{in } Q_T \\
u|_{t=0} = u_0(x), & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

subject to the slip boundary conditions (4), where \(u'\) is regularization of \(u\) in \(W^{2,1}\). Using Galerkin method, one has got the existence of (5) in any semiball in \(\mathbb{R}^3\). Letting the radius go to infinity, the Galerkin scheme converges to the solution of the approximated system. Second, we verify local Minty-Trick Theorem in [42] by means of the \(L^\infty\)-truncation method in [17], and establish the existence of weak solution for \(p\)-potential fluids (1) with \(p > \frac{8}{3}\) subject to the boundary conditions (1).

The paper is organized as follows. In the next section, after introduce the notations and definitions, we will present some auxiliary results and main theorem (see Theorem 2.8) in this paper. In section 3, we set an approximated system and the approximated solution. In section 4, we will prove the main result in this paper presenting the existence of weak solution to the system (1) with the slip boundary conditions (4) by \(L^\infty\)-truncation method.

## 2 Basic notations and statement of main result

In this section, we will introduce function spaces and the definitions of \(p\)-potential function and weak solution to system (1.1) with (1.5). We will also show the Korn-type inequalities for unbounded domain. Let \(M^{n \times n}\) be the vector space of all symmetric \(n \times n\) matrices \(\xi = (\xi_{ij})\). We equip \(M^{n \times n}\) with scalar product \(\xi : \eta = \sum_{i,j=1}^{n} \xi_{ij} \eta_{ij}\) and norm \(|\xi| = (\xi : \eta)^{\frac{1}{2}}\).

**Definition 2.1.** Let \(p > 1\) and let \(F : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}\) be a convex function, which is \(C^2\) on the \(\mathbb{R}_+ \cup \{0\}\), such that \(F(0) = 0, F'(0) = 0\). Assume that the induced function \(\Phi : M^{n \times n} \to \mathbb{R}_+ \cup \{0\}\), defined through \(\Phi(B) = F(|B|)\), satisfies

\[
\sum_{jklm} (\partial_{jklm} \Phi)(B) C_{jklm} \geq \gamma_1 (1 + |B|^2)^{\frac{p-2}{2}} |C|^2, \quad (6)
\]

\[
|\nabla_{n \times n} \Phi(B)| \leq \gamma_2 (1 + |B|^2)^{\frac{p-2}{2}} \quad (7)
\]
for all $B, C \in M^{n \times n}$ with constants $\gamma_1, \gamma_2 > 0$. Such a function $F$, resp. $\Phi$, is called a $p$-potential.

We define the extra stress $S$ induced by $F$, resp. $\Phi$, by

$$S(B) = \frac{\partial \Phi(B)}{\partial B} = F'(|B|) \frac{B}{|B|}$$

for all $B \in M^{n \times n} \setminus \{0\}$. From (6), (7) and $F'(0) = 0$, it is easy to know that $S$ can be continuously extended by $S(0) = 0$.

As in [22] and [34], one can obtain from (6) and (7) the following properties of $S$.

**Theorem 2.2.** There exist constants $c_1, c_2 > 0$ independent of $\gamma_1, \gamma_2$ such that for all $B, C \in M^{n \times n}$ there holds

$$S(0) = 0,$$

$$\sum_{i,j} S_{ij}(B)B_{ij} \geq c_1\gamma_1(1 + |B|^2)^{\frac{p-2}{2}}|B|^2,$$

$$|S(B)| \leq c_2\gamma_2(1 + |B|^2)^{\frac{p-2}{2}}|B|.$$

Let $\Omega$ be an open set in $\mathbb{R}^3$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing continuous convex function that vanishes at zero. We define the Orlicz space $L^\phi(\Omega)$ which is the Banach space by

$L^\phi(\Omega) = \left\{ \text{v is a measurable function : } \int_\Omega \phi(|v|)dx < \infty \right\}$,

equipped with the norm

$$\|v\|_{L^\phi} = \inf \left\{ \lambda > 0 : \int_\Omega \phi \left( \frac{|v|}{\lambda} \right) dx \leq 1 \right\}.$$

The Sobolev-Orlicz space $W^{k,\phi}(\Omega)$ is also Banach space by

$W^{k,\phi}(\Omega) = \{ v \in L^\phi(\Omega) : D^\alpha v \in L^\phi(\Omega), \ \forall \alpha, \ |\alpha| \leq k \}$

with the norm

$$\|v\|_{W^{k,\phi}} = \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^\phi}.$$

Note that if $\phi(s) = s^p$ with some $p \in [1, \infty)$, we write $L^p(\Omega) = L^\phi(\Omega)$ which corresponding to the standard one for Lebesgue space. $L^\infty(\Omega)$ and $W^{k,p}(\Omega)$
are defined by usual ways respectively. Let $D(\Omega)$ be the set of functions $v(x) \in C_0^{\infty}(\Omega)$. we defined

$$D_{k,\phi}(\Omega) = \frac{D(\Omega)}{\|v\|_{D_{k,\phi}}} \quad \text{with} \quad \|v\|_{D_{k,\phi}} = \|\nabla^k v\|_{L^\phi}. $$

Similarly, $D_{k,p}(\Omega)$ with $p \in [1, \infty)$ denotes the space $D_{k,\phi}(\Omega)$ with $\phi(s) = s^p$.

When $\Omega = \mathbb{R}^n_+$, set

$$D_{1,\phi}^1(\mathbb{R}^3_+) = \{v \in D(\mathbb{R}^3_+) : v(x) \to 0 \text{ as } |x| \to \infty \text{ and } v \cdot \hat{n}|_{x_3=0} = 0\}$$

$$H = \{v \in L^2(\mathbb{R}^3_+) : \text{div}v = 0 \text{ and } v \cdot \hat{n}|_{x_3=0} = 0\}$$

$$H(\Omega) = \{v \in L^2(\Omega) : \text{div}v = 0 \text{ and } v \cdot \hat{n}|_{\partial \Omega} = 0\}$$

In the rest of the paper, the functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are defined as $\phi(s) = s^2(1 + s^2)^{\frac{p-2}{2}}$ and $\psi = \min(s^\delta, s^{\frac{3}{3-p}})$.

We list some auxiliary lemmas needed in the following text. Some results need to deal with Orlicz spaces, we will frequently use the following results in Orlicz space.

**Lemma 2.3.** Let $p \in (1, 3)$. There exists a constant $C > 0$, such that

$$\|E u\|_{D^1,\phi(\mathbb{R}^3)} \leq C \|u\|_{D_1,\phi(\mathbb{R}^3_+)}$$

for all $u \in D_{1,\phi}^1(\mathbb{R}^3_+)$, where $E$ is the extension operator.

*Proof.* It is easy to obtain this conclusion by the idea in [2].

**Lemma 2.4.** Let $p \in (1, 3)$. There exists a constant $C > 0$, such that

$$\|v\|_{L^\psi} \leq C \|\nabla v\|_{L^\phi},$$

for each $v \in D_{1,\phi}^1(\mathbb{R}^3_+)$.  

*Proof.* From Lemma 2.3 and Lemma 3.1 in [19], we can obtain this inequality.

Combine Lemma 2.3 and Lemma 3.2 in [19], one can conclude the following Korn’s inequality.

**Lemma 2.5.** Let $1 < p < 2$. There is a constant $C > 0$ such that for all vector value functions $v \in D_{1,\phi}^1(\mathbb{R}^3_+)$ there holds

$$\|\nabla v\|_{L^\phi} \leq C \|D(v)\|_{L^\delta}.$$ 

As a consequence of Lemma 2.3, Lemma 2.4 and Lemma 2.5, it is easy to conclude the following interpolation result.
Corollary 2.6. Assume that \( q \in [2, \frac{5p}{3}] \) and \( u \) satisfies
\[
\|u\|_{L^\infty(0,T;L^2(\mathbb{R}_+^3))} + \|D(u)\|_{L^p(0,T;L^3(\mathbb{R}_+^3))} \leq K.
\]
Then there exists a constant \( C \) independent of \( u \) and \( K \) such that
\[
\|u\|_{L^q(0,T;L^q(\mathbb{R}_+^3))} \leq CK.
\]

Next, we consider the following Laplace equation
\[
\begin{dcases}
\Delta u = f, & \text{in } \mathbb{R}_+^3; \\
\frac{\partial u}{\partial n} = 0, & \text{on } \{x_3 = 0\}, \\
u(x) \to 0, & \text{as } |x| \to \infty
\end{dcases}
\]
then there exists a unique solution given by for any \( x \in \mathbb{R}_+^3 \)
\[
u(x) = \int_{\mathbb{R}_+^3} \left( \frac{1}{|x-y|} - \frac{1}{|x-y^*|} \right) f dy
\]
with \( y^* = (y_1, y_2, -y_3) \) for \( y = (y_1, y_2, y_3) \in \mathbb{R}_+^3 \).

Using the Calderón-Zygmund singular integral operator theory, we conclude that for all \( q \in (1, \infty) \)
\[
\begin{align*}
\|\nabla^2 u\|_{L^q} & \leq K\|f\|_{L^q}, \\
\|\nabla u\|_{L^q} & \leq K\|v\|_{L^q} \text{ with } f = \text{div}v, \\
\|u\|_{L^q} & \leq K\|S\|_{L^q} \text{ with } f = \text{div}\text{div}S.
\end{align*}
\]

For any given \( R > 0 \), set \( \Omega = \{x \in \mathbb{R}_+^3 : |x| \leq R\} \). Denote by \( \partial \Omega_1 = \{x \in \mathbb{R}_+^3 : |x| = R\} \) and \( \partial \Omega_2 = \partial \Omega \cap \{x_3 = 0\} \). The following proposition shows that there exists a basis in \( W^{2,2} \) with the boundary conditions (4).

Proposition 2.7. The eigenvalue problem
\[
\begin{dcases}
-\Delta u + \nabla p = \lambda u, & \text{in } \Omega \\
\nabla \cdot u = 0, & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega_1, \\
\quad u_3 = 0, \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0 & \text{on } \partial \Omega_2,
\end{dcases}
\]
\( \lambda \in \mathbb{R} \), \( u \in D^{1,2}(\Omega) \cap H(\Omega) \) admits a denumerable positive eigenvalue \( \{\lambda_i\} \) clustering at infinity. Moreover, the corresponding eigenfunctions \( \{\omega_i\} \) are in \( W^{2,2}(\Omega) \), and associate pressure fields \( p_i \in W^{1,2}(\Omega) \). Finally, \( \{\omega_i\} \) are orthogonal and complete in \( W^{2,2}(\Omega) \cap H(\Omega) \). Furthermore, for any \( \Omega' \subset \subset \Omega \), \( \omega_i \in W^{3,2}(\Omega' \cup \partial \Omega_2) \).
Proof. We can infer this conclusion from the estimates of Theorem 3.2 in [37] and the Rellich theorem. Then it is easy to obtain the estimates of regularity.

Now, we state the main result as follows. If \( q \in (1, \frac{5p}{6}] \), let \( U^* \) be the dual space of

\[
U = L^p(0, T; D^{1,\phi}(\mathbb{R}^3_+)) \cap L^{q'}(0, T; D^{1,q'}(\mathbb{R}^3_+)).
\]

Theorem 2.8. Let \( \frac{8}{5} < p \leq 2 \), \( \phi(s) = s^2(1 + s^2)^{\frac{p-2}{2}} \) and \( S(D(u)) \) be induced by the \( p \)-potential of Definition 2.1. Let \( f \in L^2(0, T; H) \) or \( f \in U^* \), and \( u_0 \in H \) with \( \nabla \cdot u_0 = 0 \) in the sense of distribution. Then there exists a vector function

\[
u \in L^\infty(0, T; H) \cap L^p(0, T; D^{1,\phi}(\mathbb{R}^3_+))\]

with \( u_t \in U^* \) satisfies the following identity

\[
\int_0^T \langle u_t, \psi \rangle dt + \int_{Q_T} (S(x, t, D(u)) - u \otimes u) : D(\psi) dx dt = \int_{Q_T} f \cdot \psi dx dt
\]

holds for all \( \psi \in U \) with \( \text{div}\psi = 0 \), \( \psi_3|_{x_3=0} = 0 \), where \( Q_T = \mathbb{R}^3_+ \times (0, T) \).

3 Construction of the approximation solutions

In this section, for brevity, assume that \( f = 0 \), we will set the approximation system and hunt for the solutions. Given \( u \in L^2(\Omega) \), where \( \Omega \) is an open proper subset of \( \mathbb{R}^3 \) with smooth boundary, define \( u^\epsilon \) by

\[
u^\epsilon = J_\epsilon u := PE'F_\epsilon E u
\]

where the symbol \( E \) is an extension operator from \( L^2(\Omega) \) to \( L^2(\mathbb{R}^3) \), \( F_\epsilon \) is the standard mollifier operator, \( E' \) is the restriction from \( L^2(\mathbb{R}^3) \) to \( L^2(\Omega) \), and \( P \) is the Leray projector from \( L^2(\Omega) \) to \( H(\Omega) \) [29]. Hence it is easy to see that

\[
u^\epsilon \rightharpoonup u \quad \text{in} \quad D^{1,\phi}(\Omega) \cap H(\Omega) \quad \text{as} \quad \epsilon \to 0,
\]

\[
\|u^\epsilon\|_{W^{2,2}} \leq \frac{C}{\epsilon^2}\|u\|_{L^2}
\]

where \( C \) is independent of \( \epsilon \).
Now we investigate the following approximation system
\[
\begin{aligned}
\partial_t u - \text{div} S(D(u)) + (u^\varepsilon \cdot \nabla) u + \nabla \pi &= 0, & \text{in } Q_T \\
\nabla \cdot u &= 0, & \text{in } Q_T \\
u|_{t=0} &= u_0(x), & \text{in } \mathbb{R}^3_+ , \\
u_3|_{x_3=0} &= \frac{\partial u}{\partial x_3} \bigg|_{x_3=0} = 0 \ (i = 1, 2) , \\
u(x) &\to 0, \text{ as } |x| \to \infty.
\end{aligned}
\]

where \( u^\varepsilon \) is given in \([13]\).

**Theorem 3.1.** Let \( u_0 \in H \) with boundary conditions \([4]\) in the distribution sense and \( S(D(u)) \) be induced by a \( p \)-potential from Definition 2.1 with \( \frac{3}{2} < p < 2 \). Then there exists a weak solution \( u \in \mathbb{L}^p(0, T; D^{1,\phi}(\mathbb{R}^n_+)) \cap L^\infty(0, T; H) \) to the system \((21)\) in the following sense
\[
\int_0^T \langle u_t, \psi \rangle dt + \int_{Q_T} (S(x, t, D(u)) - u^\varepsilon \otimes u) : D(\psi) \, dx \, dt = 0 \quad (22)
\]
holds for all \( \psi \in U \) with \( \text{div} \phi = 0, \ \psi_3|_{x_3=0} = 0 \), where \( Q_T = \mathbb{R}^3_+ \times (0, T) \).

**Proof.** The proof is based on standard arguments for approximation of a solution by the Galerkin method. Let \( \{\Omega_R\} \) be a sequence of bounded domains expanding in \( \mathbb{R}^3_+ \), such that \( \Omega_R = \{x \in \mathbb{R}^3_+: |x| \leq R\} \) with \( R \to \infty \), and
\[
\mathbb{R}^3_+ = \bigcup_{R=1}^{\infty} \Omega_R.
\]

Now we consider the following problem in \( \Omega_R \),
\[
\begin{aligned}
\partial_t u_R - \text{div} S(D(u_R)) + (u^R \cdot \nabla) u_R + \nabla \pi &= 0, & \text{in } \Omega_R \times [0, T] \\
\nabla \cdot u_R &= 0, & \text{in } \Omega_R \times [0, T] \\
u_R|_{t=0} &= u_{0,R}(x), & \text{in } \Omega_R, \\
u_R|_{x=R} &= u_{R,3}|_{x_3=0} = \frac{\partial u_{R,i}}{\partial x_3} \bigg|_{x_3=0} = 0 \ (i = 1, 2) .
\end{aligned}
\]

where both \( u^R \) given by \([13]\) and \( u_{0,R} = PE'u_0 \) with respect to the domain \( \Omega_R \). For simplicity we omit the subscript \( R \) for \( u_R(x, t) \). In \( \Omega_R \) we consider the eigenfunctions \( \{\omega_i(x)\}_{i \in \mathbb{N}} \) of the problem \([16]\), and put
\[
u_m(x, t) = \sum_{k=1}^{m} c_{k,m}(t) \omega_k(x).
\]
Denote $\omega^k(x) = J, \omega_k(x)$ for all $k = 1, 2, \cdots$, and one yields $c_{k,m}(t)$ solves the following system of ordinary differential equations

$$\left( \partial_t u_m, \omega_k \right) + \left( S(D(u_m)) - u_m^e \otimes u_m, D(\omega_k) \right) = 0, \ k = 1, \cdots, m.$$  \hfill (24)

Due to the continuity of $S$, the local-in-time existence follows from Caratheodory theory. The global-in-time existence will be established by the following a priori estimates.

Multiply the equations (24) by $c_{k,m}$, then sum over $k$ and integrate on $(0, t)$. It is easy to obtain

$$\|u_m\|^2_{L^2(\Omega_R)} + 2 \int_0^t (S(D(u_m)), D(u_m)) = \|u_0\|^2_{L^2(\Omega_R)}. \hfill (25)$$

Hence

$$\sup_{0 \leq t \leq T} \|u_m(t)\|^2_{L^2(\Omega_R)} + 2 \int_0^t \|u_m(\tau)\|^p_{D^{1,p}(\Omega_R)} d\tau \leq C(R, u_0). \hfill (26)$$

From (24), (26) and $\{\omega_i(x)\}_{i \in \mathbb{N}}$ is dense in $D^{1,p}(\Omega_R) \cap H(\Omega_R)$, one infers that

$$\|u_m'\|_{L^p'((0,T) ; (D^{1,p}(\Omega_R) \cap H(\Omega_R))^*)} \leq C(R, u_0). \hfill (27)$$

By Aubin-Lions Lemma, it follows, as $m \to \infty$

\begin{align*}
    & u_m' \rightharpoonup u', \quad \text{weakly in } L^p'(0, T; (D^{1,p}(\Omega_R) \cap H(\Omega_R))^*); \quad (28) \\
    & u_m \rightharpoonup^* u, \quad \text{weakly}^{*} \text{ in } L^\infty(0, T; H(\Omega_R)); \quad (29) \\
    & u_m \rightharpoonup u, \quad \text{weakly in } L^p(0, T; D^{1,p}(\Omega_R)); \quad (30) \\
    & u_m \rightharpoonup u, \quad \text{strongly in } L^q(0, T; L^q(\Omega_R)) \ q \in \left[1, \frac{5p}{3}\right); \quad (31) \\
    & S(D(u_m)) \rightharpoonup \tilde{S}^R, \quad \text{weakly in } L^p(\Omega_R \times (0, T)). \quad (32)
\end{align*}

It is easy to see that

$$\left\langle u_t, \omega_k \right\rangle + \left( \tilde{S}^R, D(\omega_k) \right) - \left( u^e \cdot \nabla \omega_k, u \right) = 0. \hfill (33)$$

Multiplying both sides of (33) by $c_{k,m}$ and summing over $k$ we find

$$\left\langle u_t, u_m \right\rangle + \left( \tilde{S}^R, D(u_m) \right) - \left( u^e \cdot \nabla u_m, u \right) = 0. \hfill (34)$$

Let us pass to the limit for $m \to \infty$ in to (34). By these properties (28), (29) and (32), we know that as $m \to \infty$

\begin{align*}
    & \int_0^T \left\langle u_t, u_m \right\rangle \to \int_0^T \left\langle u_t, u \right\rangle = \|u\|^2_{L^2(\Omega_R)} - \|u_0\|^2_{L^2(\Omega_R)}, \\
    & \int_0^T \left( \tilde{S}^R, D(u_m) \right) \to \int_0^T \left( \tilde{S}^R, D(u) \right).
\end{align*}
Since

$$(u^\epsilon \cdot \nabla u_m, u) - (u^\epsilon \cdot \nabla u, u) = \int_{\Omega_R} (u^\epsilon \otimes u) \cdot \nabla (u_m - u) \, dx$$

From (30) and (31), we know that $u^\epsilon \otimes u \in L^2(\Omega_R \times (0, T))$. Hence $\int_0^T (u^\epsilon \cdot \nabla u_m, u) \to 0$ as $m \to \infty$, since $(u^\epsilon \cdot \nabla u, u) = 0$. Subtracting (34) by (24) and passing to limit as $m \to \infty$, we get

$$\lim_{m \to \infty} \int_0^T (S(D(u_m)), D(u_m)) \, dt = \int_0^T (\tilde{S}^R, D(u)) \, dt. \quad (35)$$

Choose $\psi = 1$ in $\Omega_R \times [0, T]$ in Appendix [42], then by Minty-Trick, it shows that

$$\tilde{S}^R = S(D(u)) \text{ a.e. in } \Omega_R \times (0, T).$$

Next we must consider the limit as $R$ tends to $\infty$. Let $u^R$ be the Galerkin approximation solution for the equation (23), therefore, from (25), we obtain

$$\|u^R\|_{L^\infty(0, T; H)} + 2 c_1 \gamma_1 \int_0^T \int_{\Omega_R} \phi(u^R(x, \tau)) \, dx \, d\tau \leq \|u_0\|_2^2 := K, \quad (36)$$

From above (36), there exists $C(K)$ independent of $R$ and $\epsilon$, we know that

$$\int_0^T \int_{|D(u)| \geq 1} |D(u^R)|^p \leq C(K) \quad \text{and} \quad \int_0^T \int_{|D(u^R)| \leq 1} |D(u^R)|^2 \leq C(K). \quad (37)$$

From (11), we can observe that

$$\int_0^T \int_{\Omega_R} |S(D(u^R))|^r \leq c_2 \gamma_2 \int_0^T \int_{\Omega_R} (1 + |D(u^R)|^2)^{\frac{r-2p}{2}} |D(u^R)|^r$$

$$\leq c_2 \gamma_2 \left( \int_0^T \int_{|D(u^R)| \geq 1} + \int_0^T \int_{|D(u^R)| \leq 1} \right) (1 + |D(u^R)|^2)^{\frac{r-2p}{2}} |D(u^R)|^r$$

$$:= c_2 \gamma_2 (J_1 + J_2). \quad (38)$$

Since
Hence, it follows that, for any $2 \leq p \in \mathbb{R}$ one infers that
\[
\begin{align*}
\text{It suffices to study the more estimates for } u \{ \text{mates from the above estimates, thus we start at the Galerkin approximation}\}
\end{align*}
\]
\[
\begin{align*}
J_1 &= \int_0^T \int_{|D(u_R)| \geq 1} \left(1 + |D(u_R)|^2\right)^{\frac{(n-2)r}{2}} |D(u_R)|^r \leq \int_0^T \int_{|D(u_R)| \geq 1} |D(u_R)|^{(n-1)r},
\end{align*}
\]
or
\[
\begin{align*}
J_2 &= \int_0^T \int_{|D(u_R)| \leq 1} \left(1 + |D(u_R)|^2\right)^{\frac{(n-2)r}{2}} |D(u_R)|^r \leq \int_0^T \int_{|D(u_R)| \leq 1} |D(u_R)|^r,
\end{align*}
\]

thus take $r = p' = \frac{p}{p-1} > 2$ in (38), by (39), (41) and $r = 2$ in (38), by (40), (41) respectively, we have
\[
\begin{align*}
\int_0^T \int_{\Omega_R} |S(D(u_R))|^{p'} \, dx \, dt &\leq C(K), \\
\int_0^T \int_{\Omega_R} |S(D(u_R))|^2 \, dx \, dt &\leq C(K).
\end{align*}
\]

Hence, it follows that, for any $2 \leq s \leq p'$, by the interpolation,
\[
\| S(D(u_R)) \|_{L^s(Q_T)} \leq C(K).
\]

One infers that
\[
\| (u^R)' \|_{L^2(0,T;D^{1,2}(\Omega_R) \cap H(\Omega_R))^{\ast}} \leq C(\varepsilon, K)
\]

We can extend the $u^R$ to zero outside of $\Omega_R$. From above estimates (36), (41), (42) and Aubin-Lions Lemma, we can extract a subsequence of $\{u^R\}$, denoting it by $\{u^{R_k}\}_{k \in \mathbb{N}}$, such that as $R_k \to \infty$,

\[
\begin{align*}
(u^{R_k})' &\to u', \quad \text{weakly in } L^{p'}(0, T; (D^{1,2}(\mathbb{R}_+^3) \cap H)^{\ast}); \\
u^{R_k} &\rightharpoonup u, \quad \text{weakly}^{\ast} \text{ in } L^{\infty}(0, T; H); \\
u^{R_k} &\to u, \quad \text{weakly in } L^p(0, T; D^{1,\phi}(\mathbb{R}_+^3)); \\
u^{R_k} &\to u, \quad \text{strongly in } L^q(0, T; L^q_{loc}(\mathbb{R}_+^3)) \text{ for } q \in [1, \frac{5p}{3}); \\
S(D(u^{R_k})) &\rightharpoonup G, \quad \text{weakly in } L^{p'}(\mathbb{R}_+^3 \times (0, T)).
\end{align*}
\]

It suffices to study the more estimates for $u^R$ such that $D(u^{R_k}) \to D(u)$ a.e. in $\mathbb{R}_+^3 \times (0, T)$, as $R_k \to \infty$. Since we can not obtain more regularity estimates from the above estimates, thus we start at the Galerkin approximation solutions $\{u^R_m\}$ for given domain $\Omega_R$, and still denote it by $u_m$ as above.
From Proposition 2.7 let us consider \( \eta \) smooth cut-off functions such that \( \eta(x) = 1 \) for \( |x| \leq \frac{R}{2} \) and \( \eta(x) = 0 \) for \( |x| \geq R \), for any \( x \in \mathbb{R}^3 \), \( |\nabla^k \eta| \leq \frac{C}{R^k} \) and \( \left| \frac{\nabla \eta}{\eta} \right| \leq \frac{C}{R} \). We observe that

\[
\text{curl}(\eta^2 \text{curl}\ u_m) = 2\eta \nabla \eta \times \text{curl}\ u_m + \eta^2 \Delta u_m, \quad \text{curl}(\eta^2 \text{curl}\ u_m)\big|_{x=\hat{R}} = 0,
\]

\[
\text{curl}(\eta^2 \text{curl}\ u_m) \cdot \hat{n}|_{x=0} = \eta \nabla \eta \cdot (\text{curl}\ u_m \times \hat{n})|_{x=0} + \eta^2 \Delta u_{m,3} = 0.
\]

Therefore, take \(-\text{curl}(\eta^2 \text{curl}\ u_m)\) as a test function in (24), it infers that

\[
(\partial_t u_m, -\text{curl}(\eta^2 \text{curl}\ u_m)) + (S(D(u_m)) - u_m^c \otimes u_m, D(-\text{curl}(\eta^2 \text{curl}\ u_m))) = 0.
\]

We will give the estimates one by one, \((\partial_t u_m, -\text{curl}(\eta^2 \text{curl}\ u_m)) = \frac{1}{2} \|\eta \text{curl}\ u_m\|^2_2\)

From the boundary conditions (4) and integrate by parts, we have

\[
|\langle u_m^c \otimes u_m, D(\text{curl}(\eta^2 \text{curl}\ u_m))\rangle| = \left| \int_{\Omega_R} \text{curl}(u_m^c \cdot \nabla u_m) \cdot \eta^2 \text{curl}\ u_m \, dx \right|
\]

\[
= \left| \int_{\Omega_R} \eta^2 \text{curl}\ u_m^c \cdot \nabla u_m \cdot \text{curl}\ u_m \, dx + \int_{\Omega_R} \eta^2 u_m^c \cdot \nabla \text{curl}\ u_m \cdot \text{curl}\ u_m \, dx \right|
\]

\[
\leq \|\nabla u_m^c\|_6 \|\nabla u_m\|_3 \|\eta \text{curl}\ u_m\|_2 + \left\| \frac{\nabla \eta}{\eta} \right\|_{\infty} \|u_m^c\|_{\infty} \|\eta \text{curl}\ u_m\|_2^2
\]

\[
\leq \frac{1}{e^2} \|u_0\|_2 \|\nabla u_m\|_3 \|\eta \text{curl}\ u_m\|_2 + \frac{C}{Re^2} \|u_0\|_2 \|\eta \text{curl}\ u_m\|_2^2.
\]

Since \( S(D(u)) \) is deduced by a \( p \)-potential \( \Phi \) and satisfies the property (4), hence

\[
(S(D(u_m))), D(-\text{curl}(\eta^2 \text{curl}\ u_m))) = -\int_{\Omega_R} (S(D(u_m)) : \text{curl}(\eta^2 \text{curl}\ D(u_m))) \, dx
\]

\[
= \int_{\Omega_R} (S(D(u_m)) : \text{curl}(2\text{sym}(\eta \nabla \eta \otimes \text{curl}\ u_m))) \, dx
\]

\[
= \int_{\Omega_R} (\eta^2 \text{curl}\ S(D(u_m)) : \text{curl}\ D(u_m)) \, dx - \int_{x=0} (\eta^2 S(D(u_m)) \times \hat{n} : \text{curl}\ D(u_m)) \, d\sigma
\]

\[
- \int_{\Omega_R} (S(D(u_m)) : \text{curl}(2\text{sym}(\eta \nabla \eta \otimes \text{curl}\ u_m))) \, dx\]

\[
\geq c_1 \gamma_1 \int_{\Omega_R} \eta^2 (1 + |D(u_m)|^2)^{-\gamma_2} \left| \text{curl}\ D(u_m) \right|^2 \, dx + I_1 + I_2,
\]

where \( I_1 = -\int_{x=0} (\eta^2 S(D(u_m)) \times \hat{n} : \text{curl}\ D(u_m)) \, d\sigma \), \( I_2 = -\int_{\Omega_R} (S(D(u_m)) : \text{curl}(2\text{sym}(\eta \nabla \eta \otimes \text{curl}\ u_m))) \, dx \), hereafter the symbol \( \text{sym}(B) \) is the symmetric part of the matrix \( B \), and denote \( I_\eta(u) = \int_{\Omega_R} \eta^2 (1 + |D(u)|^2)^{-\gamma_2} \left| \text{curl}\ D(u) \right|^2 \, dx \).
Since the boundary conditions and incompressibility imply that
\[ D_{jk}(u_m) \partial_3 D_{jk}(u_m) = 0 \]
for \( j, k = 1, 2, 3 \) on \( x_3 = 0 \), thus from the formula \( S(D(u)) = F'(|D(u)|) \frac{D(u)}{|D(u)|} \), we have
\[
I_1 = - \int_{x_3=0} \eta^2 (S(D(u_m))) \times \hat{n} : \text{curl}(D(u_m)) d\sigma
\]
\[
= \sum_{i,j,k} \int_{x_3=0} \eta^2 F'(|D(u)|) \left( \frac{D_{jk}(u)}{|D(u)|} \right) n_i \varepsilon_{ij s} \partial_i D_{jk}(u_m) \varepsilon_{ij s} d\sigma
\]
\[
= \sum_{i,j,k} \int_{x_3=0} \eta^2 F'(|D(u)|) \left( \frac{D_{jk}(u)}{|D(u)|} \right) \partial_3 D_{jk}(u_m) = 0
\]
(54)
and from the construction of cut-off function \( \eta \), one obtains
\[
|I_2| \leq \left| \int_{\Omega_R} (S(D(u_m))) : \text{curl}(2\text{sym}(\eta \nabla \eta) \otimes \text{curl}u_m)) dx \right|
\]
\[
+ \int_{\Omega_R} (S(D(u_m))) : 2\text{sym}(\eta \nabla \eta) \otimes \Delta u_m) dx \right|
\]
\[
\leq \frac{C}{R^2} \int_{\Omega_R} \left( 1 + |D(u_m)|^2 \right)^{\frac{p-p=2}{2}} |D(u_m)|^2 dx + \frac{C}{R} \int_{\Omega} \eta |S(D(u_m)) \Delta u_m|
\]
\[
\leq \frac{C}{R^2} + \frac{1}{R} \int_{\Omega_R} \eta \left( 1 + |D(u_m)|^2 \right)^{\frac{p-2}{2}} |D(u_m)||\Delta u_m|
\]
\[
\leq \frac{C}{R^2} + \frac{1}{R} I_{\eta}(u_m)^\frac{1}{2} \left( \int_{\Omega_R} \left( 1 + |D(u_m)|^2 \right)^{\frac{p-2}{2}} |D(u_m)|^2 dx \right)^{\frac{1}{2}} \leq \frac{C(\delta)}{R^2} + \delta I_{\eta}(u_m)
\]
(55)
where the constants \( C \) are independent of \( R, m \). The following three interpolation inequalities will be used
\[
\| \cdot \|_3 \leq \| \cdot \|_2^{\alpha} \cdot \| D(u) \|_{\beta}^{1-\alpha} \quad \alpha = \frac{2(p-1)}{3p-2}, \quad \beta = \frac{p-1}{2}, \quad \beta = \frac{p-1}{2}
\]
\[
\| \cdot \|_3 \leq \| \cdot \|_p^{\beta} \cdot \| D(u) \|_{3p}^{1-\beta} \quad \beta = \frac{p-1}{2}
\]
\[
I_{\eta}(u) \geq C \|D(u)\|_{3p, \geq 1}^p
\]
(56)
(57)
(58)
where \( \|v\|_{3, \leq 1} := \int_{|v| \leq 1} |v|^4 dx \) and \( \|v\|_{3, \geq 1} := \int_{|v| \geq 1} |v|^4 dx \). Therefore, from
where \( \lambda \) with \( \lambda \) conclude that for any bounded domain \( \Omega \)

Subtly modify the proofs of Lemma 6.2 and Lemma 6.3 in [19], we can

exist a constant \( C \) estimates (37), we deduce that there is a constant \( C \)

where \( \sigma \) provided that \( W \) into \( C \)

which finishes the proof.

\[ (59) \]

Subtly modify the proofs of Lemma 6.2 and Lemma 6.3 in [19], we can conclude that for any bounded domain \( \Omega' \) of \( \mathbb{R}^3_+ \), as \( R \) large enough, there exist a constant \( C \) independent of \( R, m \).

\[ \frac{d}{dt} \| \eta \mathbf{curl} u_m \|_2^2 + C I_\eta(u_m) \leq C \left( \| \mathbf{curl} u_m \|_2 \right) \]

with \( \lambda = \frac{2p}{3p-3} \), where \( C \) only depends on \( \epsilon \) and \( \| u_0 \|_2 \). Dividing above inequality by \( (\| \eta \mathbf{curl} u_m \|_2 + 1)^\lambda \), integrating with respect to time and using the estimates (37), we deduce that there is a constant \( C \) that depends on \( \epsilon \) and \( \| u_0 \|_2 \), does not depend on \( m, R \) such that

\[ \int_0^T I_\eta(u_m)(\| \eta \mathbf{curl} u_m \|_2^2 + 1)^{-\lambda} dt \leq C \left( 1 + \frac{1}{R^2} \right). \]  

Subtly modify the proofs of Lemma 6.2 and Lemma 6.3 in [19], we can conclude that for any bounded domain \( \Omega' \) of \( \mathbb{R}^3_+ \), as \( R \) large enough, there exist a constant \( C \) independent of \( R, m \) and \( 0 < \gamma < 1 \) such that

\[ \int_0^T \left( \int_{\Omega'} \left| \nabla^2 u_m \right|^p dx \right)^\gamma dt \leq C. \]  

Applying the standard interpolation, (36), (61) and the Young inequality, we conclude that there exists a constant \( C \) is independent of \( R, m \) such that

\[ \int_0^T \| u_m \|_{W^{1, p} \Omega'}^p \]

provided that \( \sigma \) small enough. Whence let \( m \rightarrow \infty \) we have got

\[ \int_0^T \| u_R \|_{W^{1, +\sigma, p} \Omega'}^p \leq C \]

where \( C \) does not depend on \( R \). Since the compact embedding \( W^{1, +\sigma, p} \Omega' \)

into \( W^{1, p} \Omega' \), Thus we obtain there exists a subsequence \( \{ u_{R_k} \} \) such that

\[ D(u_{R_k}) \rightarrow D(u) \text{ a.e. in } \mathbb{R}^3_+ \times (0, T), \text{ as } R_k \rightarrow \infty, \]

which finishes the proof. \( \square \)
The proof of main result

We have constructed the approximation solution $u_\epsilon$ for the problem (1), and now we will show the approximation solutions converge to the solution of the original problem (1) in this section. Denote $Q_T = \mathbb{R}_+^3 \times (0, T)$.

To finish the proof of theorem 2.8 we need to consider the estimates of the pressure as follows.

**Proposition 4.1.** Let $\frac{3}{5} < p \leq 2$, $\phi(s) = s^2(1 + s^2)\frac{p-2}{2}$ and $S(D(u))$ be induced by the $p-$potential of Definition 2.1. Assume that a vector function $u \in L^\infty(0, T; H) \cap L^p(0, T; D^{1,\phi}(\mathbb{R}_+^3))$ with $u_t \in U^*$ satisfies the following identity (22) for any $\psi \in U$ with divergence free and $\psi_3|_{x_3=0} = 0$.

Then there exists a unique pressure $\pi$ solves the following problem

$$\int_0^T \langle u_t, \phi \rangle dt + \int_{Q_T} (S(D(u)) - u^\epsilon \otimes u : D(\phi)) + \pi \nabla \cdot \phi dx dt = 0$$

for any $\varphi \in C^\infty(Q_T)$ with $\varphi_3|_{x_3=0} = 0$. Here $\pi = \pi_1 + \pi_2$ with

$$\|\pi_1\|_p \leq C(r)\|u^\epsilon \otimes u\|_r, \ \forall r \in (2, \frac{5p}{3}], \ \|\pi_2\|_s \leq C(s)\|D(u)\|_{L^s}, \ \forall s \in [2, p'].$$

Moreover, if there is a sequence $\{u_n\}_{n=1}^\infty$ satisfying

$$u_n \rightharpoonup u \text{ weakly in } L^r(Q_T) \text{ for all } r \in [2, \frac{5p}{3}],$$

$$u_n \rightarrow u \text{ a.e. in } Q_T,$$

$$\nabla u_n \rightarrow \nabla u \text{ a.e. in } Q_T,$$

$$\|u_n\|_{D^{1,\phi}} \leq C \text{ uniformly w.r.t. } n,$$

then there is a subsequence $\{\pi_n\}$ solving (62) such that $(\pi_n = \pi_1^n + \pi_2^n)$

$$\pi_1^n \rightharpoonup \pi_1 \text{ weakly in } L^q(Q_T) \text{ for all } q \in (1, \frac{5p}{6}],$$

$$\pi_2^n \rightharpoonup \pi_2 \text{ weakly in } L^s(Q_T) \text{ for all } s \in [2, p'],$$

and the pair $(u, \pi)$ solves (62).

**Proof.** The proof follows almost the procedure showed in [19], therefore, we can check the result for the smooth vector functions $u \in D(\mathbb{R}_+^3)$ with divergence free and the boundary conditions (4). Hence, we solve the pressure
\[ \pi = \pi_1 + \pi_2 \] to the problem (62) determined by the following two equations

\[
\begin{cases}
- \Delta \pi_1 = \text{div} \text{div}(u' \otimes u), & \text{in } \mathbb{R}_+^3; \\
\frac{\partial \pi_1}{\partial \hat{n}} = \text{div}(u' \otimes u) \cdot \hat{n}, & \text{on } \{x_3 = 0\}, \\
\pi_1(x) \to 0, & \text{as } |x| \to \infty
\end{cases}
\] (64)

and

\[
\begin{cases}
\Delta \pi_2 = \text{div} \text{div}(S(D(u))), & \text{in } \mathbb{R}_+^3; \\
\frac{\partial \pi_2}{\partial \hat{n}} = \text{div}(S(D(u)) \cdot \hat{n}, & \text{on } \{x_3 = 0\}, \\
\pi_2(x) \to 0, & \text{as } |x| \to \infty
\end{cases}
\] (65)

One can simplify that the boundary conditions of equations both in (64) and (65). In fact, \( \text{div}(u' \otimes u) \cdot \hat{n} = 0 \) as \( u_3 = 0 \) on \( x_3 = 0 \). By calculation, we observe that on \( x_3 = 0 \),

\[ \text{div}(S(D(u)) \cdot \hat{n} = G'(|D(u)|) \partial_3(|D(u)|^2) + G(|D(u)|) \Delta u_3 \]

where \( G(|D(u)|) = \frac{F'(|D(u)|)}{|D(u)|} \). From the (4) and incompressibility, we know that \( \Delta u_3 = \partial_3(|D(u)|^2) = 0 \) on \( x_3 = 0 \). Therefore, \( \pi_1 \) and \( \pi_2 \) can be determined by (12) with \( f = u' \otimes u \) and \( f = S(D(u)) \) respectively. The estimates (63) can be obtained by (15). The remained proof is the same procedure of Proposition 4.1 in [19].

**The proof of Theorem 2.8.** From (36), let \( R \to \infty \), we obtain

\[ \|u'\|_{L^\infty(0,T;H)} + 2C \int_0^T \int_{\mathbb{R}_+^3} (\phi(D(u_\epsilon)) + |u_\epsilon|^r) \, dx \, dt \leq K, \] (66)

for all \( r \in [2, \frac{5p}{3}] \) and \( C,K \) are independent of \( \epsilon \). From (43), we know that

\[ \|S(D(u_\epsilon))\|_{L^s(Q_T)} \leq C(K), \ s \in [2,p']. \] (67)

From Proposition 4.1, it shows that

\[ \int_0^T \|\pi_\epsilon^1\|^q_q \leq C(q,K) \text{ and } \int_0^T \|\pi_\epsilon^2\|^s_s \leq C(s,K) \] (68)

for all \( q \in (1, \frac{5p}{6}) \) and all \( s \in [2,p'] \) hold.

By (22), (43) and (68), one obtains

\[ \|u'\|_{L^r(0,T;D^1,q'((\mathbb{R}_+^3)^*)) \cap L^s(0,T;D^1,s'((\mathbb{R}_+^3)^*))} \leq K. \] (69)
Summing these estimates (66), (67), (69) and Aubin-Lions Lemma, we conclude that there exist a sequence of \( \{ \epsilon_k \} \) with \( \epsilon_k \to 0 \) as \( k \to \infty \), functions \( u \in L^p(0, T; D^{1,\phi}(\mathbb{R}^3_+)) \cap L^\infty(0, T; H) \) and \( \tilde{S} \in L^s(Q_T) \), such that as \( k \to \infty \)

\[
\begin{align*}
  u_{\epsilon_k} & \rightharpoonup^* u, \text{ weakly* in } L^\infty(0, T; H) \quad (70) \\
  u_{\epsilon_k} & \to u, \text{ weakly in } L^p(0, T; D^{1,\phi}(\mathbb{R}^3_+)), \quad (71) \\
  u_{\epsilon_k}' & \to u', \text{ weakly in } L^s(0, T; (D^{1,s'}(\mathbb{R}^3_+))^*) \cap L^q(0, T; (D^{1,q'}(\mathbb{R}^3_+))^*), \quad (72) \\
  S(x, t, D(u_{\epsilon_k})) & \to \tilde{S} \text{ weakly in } L^s(Q_T), \ s \in [2, p'] \quad (73) \\
  u_{\epsilon_k}' \otimes u_{\epsilon_k} & \to u \otimes u, \text{ weakly in } L^q(Q_T), \ q \in (1, \frac{5p}{6}] \quad (74) \\
  \pi_{\epsilon_k}^1 & \to \pi^1 \text{ weakly in } L^q(Q_T), \ q \in (1, \frac{5p}{6}] \quad (75) \\
  \pi_{\epsilon_k}^2 & \to \pi^2 \text{ weakly in } L^s(Q_T), \ q \in [2, p'] \quad (76) \\
\end{align*}
\]

Then the identity

\[
\begin{align*}
  \int_0^T \langle u_t, \varphi \rangle dt + \int_{Q_T} (\tilde{S} - u \otimes u : D(\varphi)) + \pi \nabla \cdot \varphi dxdtdt = 0 \quad (78)
\end{align*}
\]

with \( \pi = \pi^1 + \pi^2 \) holds for any \( \varphi \in (L^p(0, T; D^{1,\phi}(\mathbb{R}^3_+)) \).

For convenience, sometimes we denote \( u_k = u_{\epsilon_k}, \ S_k = S(x, t, D(u_{\epsilon_k})). \) To end this proof, we must prove that \( \tilde{S} = S(x, t, D(u)) \) a.e. in \( \mathbb{R}^3_+ \times [0, T] \). As in the [34], it suffices to prove that as \( k \to \infty \)

\[
\int_{G \times [\delta, T-\delta]} (S(x, t, D(u_k)) - S(x, t, D(u))) : (D(u_k) - D(u)) dxdtdt \to 0 \quad (79)
\]

for all bounded compact set \( G \subset \mathbb{R}^3_+ \) and any \( 0 < \delta < \frac{T}{2} \).
If (79) holds, for any positive cutoff function \( \Psi \in C^\infty_0(\mathbb{R}^3_+ \times (0, T)) \), then

\[
\int_{Q_T} (\tilde{S} - S(x, t, D(v))) : (D(u) - D(v))\Psi \, dx \, dt,
\]

\[
= \int_{Q_T} (\tilde{S} - S(x, t, D(u_k))) : (D(u) - D(v))\Psi \, dx \, dt
\]

\[
- \int_{Q_T} (S(x, t, D(u_k)) - S(x, t, D(u))) : (D(u_k) - D(u))\Psi \, dx \, dt
\]

\[
+ \int_{Q_T} (S(x, t, D(u_k)) - S(x, t, D(v))) : (D(u_k) - D(v))\Psi \, dx \, dt
\]

\[
- \int_{Q_T} S(D(u))(D(u_k) - D(u))\Psi \, dx \, dt,
\]

\[
:= I_1 + I_2 + I_3 + I_4.
\]

By (71) and (73), we know \( I_i \to 0 (i = 1, 2, 4) \) as \( k \to \infty \). From (9), it shows that \( I_3 \geq 0 \).

By local Minty Trick theorem (see the appendix of [42]), we know \( \tilde{S} = S(x, t, D(u)) \) a.e. in \( \mathbb{R}^3_+ \times (0, T) \).

Next, we will prove that (79) by some \( L^\infty \)-truncation method. As in [26] or [17], Let

\[
g^k = \phi(\nabla u_k) + \phi(\nabla u) + (|S_k| + |S(x, t, D(u))|)(|D(u_k) + D(u)|)
\]

The following lemma shows that the properties of \( g^k \) on \( \mathbb{R}^3_+ \times (0, T) \), its proof can be found in [26] for the steady case and in [17] for the unsteady case.

**Lemma 4.2.** \( \eta > 0 \), there exists \( L \leq \frac{\eta}{K} \) and there are a subsequence \( \{u_k\}_{k=1}^\infty \) and sets \( E^k = \{(x, t) \in \mathbb{R}^3_+ \times (0, T) : L^2 \leq |u_k - u| \leq L\} \) such that

\[
\int_{E^k} g^k \, dx \, dt \leq \eta.
\]

Set \( Q^k = \{(x, t) \in \mathbb{R}^3_+ \times (0, T) : |u_k - u| \leq L\} \) and \( \psi^k = (u_k - u) \left( 1 - \min \left( \frac{|u_k - u|}{L}, 1 \right) \right) \). One can prove that the following proposition that shows the required properties of \( \psi^k \).

**Proposition 4.3.** (1) \( \psi^k \in L^p(0, T; D^{1,\phi}(\mathbb{R}^3_+)) \cap L^\infty(0, T; H) \) and

\[
\|\psi^k\|_{L^\infty(\mathbb{R}^3_+ \times (0, T))} \leq L;
\]

(2) \( \psi^k \to 0 \) in \( L^p(0, T; D^{1,\phi}(\mathbb{R}^3_+)) \);

(3) \( \psi^k \to 0 \) in \( L^s(0, T; L^{s,\text{loc}}(\mathbb{R}^3_+)) \) for all \( 1 \leq s < \infty \);
For any bounded domain with Lipschitz boundary $G$, we have the following estimates

$$|\text{div}\psi^k| \leq \frac{1}{T} (u^k - u) \cdot \nabla|u^k - u| \chi_{Q^k}; \text{ and}$$

$$|\text{div}\psi^k|_{L^p(0,T;L^q(\mathbb{R}^3_+))} \leq C\eta, \quad |\nabla\psi^k|_{L^p(0,T;L^q(\mathbb{R}^3_+))} \leq C\eta,$$

where $C$ is independent of $k$ and $\chi_{Q^k}$ denotes the characteristic function of the set $Q^k$.

Proof. It is easy to see that (1) and (2) hold. By the simple calculation, we know that (4) holds. Now we check that (3). Indeed, since $D^{1,0}(\mathbb{R}^3_+) \hookrightarrow L^p(G)$ is compact for all $G \subset \subset \mathbb{R}^3_+$, it follows that $\psi^k \to 0$ in $L^p(0, T; L^p(G))$ and there exists a subsequence $\psi^{k'} \to 0$ a.e. in $G \times (0, T) := G_T$. Therefore, for $p < s < \infty$ with

$$\int_{G_T} |\psi^{k'}|^s \, dx \, dt \leq \|\psi^{k'}\|_{L^s(\mathbb{R}^3_+ \times (0, T))}^s \int_{G_T} |\psi^{k'}|^p \, dx \, dt \leq L \|\psi^{k'}\|_{L^p(G_T)}^p \leq \eta.$$

For $1 \leq s \leq p$, we have

$$\int_{G_T^s} |\psi^{k'}|^s \, dx \, dt \leq \left( \int_{G_T} |\psi^{k'}|^p \, dx \, dt \right)^{\frac{s}{p}} \left| G_T^s \right|^{1 - \frac{s}{p}} \leq K \left| G_T^s \right|^{1 - \frac{s}{p}}.$$

Let $|G_T^s| < \delta$ small enough, by Vitali’s theorem we have $\psi^{k'} \to 0$ in $L^s(0, T; L^s_{\text{loc}}(\mathbb{R}^3_+))$ for all $1 \leq s \leq p$. Hence, we proved (3). 

Let

$$z^k(x, t) = \int_{\mathbb{R}^3_+} N(x, y) \text{div}\psi^k(y, t) \, dy \forall x \in \mathbb{R}^3_+,$$

then we have the following estimates

$$\|\nabla z^k\|_{L^s(G_T)} \leq C(s, G, T) \|\psi^k\|_{L^s(G_T)};$$

$$\|\nabla^2 z^k\|_{L^p(0,T;L^q(\mathbb{R}^3_+))} \leq C(p) \|\text{div}\psi^k\|_{L^p(0,T;L^q(\mathbb{R}^3_+))} \leq C\eta.$$

(81)

For any bounded domain with Lipschitz boundary $G \subset \mathbb{R}^3_+$, we can choose another bounded set $G'$ with $G \subset \subset G' \subset \subset \mathbb{R}^3_+$, and any positive number $\delta > 0$. Define smooth functions $\tau \in C^\infty_0 \left( \frac{\delta}{2}, T - \frac{\delta}{2} \right)$ and $\zeta \in C^\infty_0 \left( \mathbb{R}^3_+ \right)$ such that $0 \leq \tau \leq 1$ in $\left( \frac{\delta}{2}, T - \frac{\delta}{2} \right)$ and $\tau \equiv 1$ in $\left[ \delta, T - \delta \right]$, $0 \leq \zeta \leq 1$ in $G'$ and $\zeta \equiv 1$ in $G$. Let $\varphi^k = \tau \zeta (\psi^k - \nabla z^k)$, then $\varphi^k \in (L^p(0, T; D^{1,0}_n(\mathbb{R}^3_+)))$. 

20
Take \( \varphi^k \) as a test function in (78), we follows
\[
\int_Q S(x, t, D(u_k)) : \nabla \varphi^k \, dx \, dt \\
= \int_Q \tilde{S} : \nabla \varphi^k \, dx \, dt - \int_0^T \langle u'_k - u', \varphi^k \rangle \, dt \\
- \int_Q \nabla \cdot (u'_k \otimes u_k - u \otimes u) \cdot \varphi^k \, dx \, dt \\
+ \int_Q (\pi^k - \pi) (\nabla \cdot \varphi^k) \, dx \, dt
\]
\[
:= D_1 + D_2 + D_3 + D_4.
\]

From proposition 4.3, we can know that \( D_1 \to 0 \) as \( k \to \infty \). Since
\[
\int_{G'_T} |\nabla \cdot (u'_k \otimes u_k)| \sigma \, dx \, dt = \int_{G'_T} |(u'_k \cdot \nabla) u_k| \sigma \, dx \, dt \\
\leq \left( \int_{G'_T} |u'_k| \gamma \, dx \, dt \right)^{\frac{1}{\gamma}} \left( \int_{G'_T} |\nabla u_k| \sigma \, dx \, dt \right)^{\frac{1}{\gamma}}.
\]

From (66), let \( \sigma \gamma' = \frac{5p}{3}, \sigma \gamma = p \) for \( G' \) is bounded, where \( \frac{1}{\gamma} + \frac{1}{\gamma'} = 1 \), then \( \sigma = \frac{5}{8}p > 1 \) provided that \( p > \frac{8}{5} \). Therefore, there is a constant \( C \) independent of \( k \) such that
\[
D_3 \leq \int_{Q_T} |\nabla \cdot (u'_k \otimes u_k - u \otimes u)(\psi^k - \nabla z^k)| \, dx \, dt \\
\leq C\|\psi^k - \nabla z^k\|_{L^\sigma(G'_T)} = o(1) \text{ as } k \to \infty.
\]

As \( \nabla \cdot \varphi^k = \tau \nabla \zeta \cdot (\psi^k - \nabla z^k) \), thus
\[
D_4 \leq \|\pi^1_k - \pi^1\|_{L^q(G'_T)} \|
abla \cdot \varphi^k\|_{L^{\sigma'}(G'_T)} + \|\pi^2_k - \pi^2\|_{L^{\sigma'}(G'_T)} \|
abla \cdot \varphi^k\|_{L^{\sigma'}(G'_T)}.
\]
thus from Proposition 4.3 (3), and (81), it shows that
\[
\|
abla \cdot \varphi^k\|_{L^{\sigma'}(G'_T)} = \|\tau \nabla \zeta \cdot (\psi^k - \nabla z^k)\|_{L^{\sigma'}(G'_T)} \leq C\|\psi^k - \nabla z^k\|_{L^{\sigma'}(G'_T)} \to 0 \text{ as } k \to \infty.
\]

By the bounded estimates for \( \pi^i_k (i = 1, 2) \) in (68), we claim that
\[
D_4 = o(1) \text{ as } k \to \infty.
\]

To estimate the term \( D_2 \), we set \( w = u_k - u \in L^p(0, T; D^{1,\phi}(\mathbb{R}^3)) \). As the argument of the footnotes on page 79 in [17], for \( 1 < p < 6 \), there exists a
sequence \( \{w_n\} \subset C^\infty(0, T; D(\mathbb{R}^3_+)) \) with \( \text{div} w_n = 0 \), satisfying \( w_n' \to w' \) in \( L^p(0, T; W^{-1, p'}(G')) \) and \( w_n \to w \) in \( L^p(0, T; D^{1, \phi}(\mathbb{R}^3_+)) \). One has

\[
D_2 = \int_0^T \langle (\tau u_k)' - (\tau u)'_k, \zeta(\psi_k' - \nabla z^k)' \rangle dt
= \lim_{n \to \infty} \int_0^T \langle (w_n)'_k, \tau \zeta(w_n (1 - \min(\frac{|w_n|}{L}, 1)) - \nabla z^k) \rangle dt = D_{21} + D_{22}.
\]

We can refer [17], then it is easy to obtain that \( D_{21} \leq C |G'_T| L^2 \).

Since \( \text{div} w_n = 0 \), from (81) and the estimates (69) of \( u'_k \), we can obtain that \( D_{22} = o(1) \).

So far, we can conclude that

\[
\int_0^T \int_{\mathbb{R}^3_+} S(x, t, D(u_k)) : D(\varphi^k) dx dt \leq o(1) + C\eta, \text{ as } k \to \infty.
\]

It follows that

\[
\int_0^T \int_{\mathbb{R}^3_+} S(x, t, D(u_k)) : D(\psi^k) \tau \zeta dx dt
\leq \int_0^T \int_{\mathbb{R}^3_+} S(x, t, D(u_k)) : D(\nabla z^k) \tau \zeta dx dt
- \int_0^T \int_{\mathbb{R}^3_+} S(x, t, D(u_k)) : (\text{sym}(\psi^k - \nabla z^k) \otimes \nabla \zeta) \tau dx dt + o(1) + C\eta.
\]

From (81)

\[
\left| \int_0^T \int_{\mathbb{R}^3_+} S(x, t, D(u_k)) : D(\nabla z^k) \tau \zeta dx dt \right| \leq K\|\nabla z^k\|_{L^p(Q)} \leq C\eta,
\]

and

\[
\left| - \int_0^T \int_{\mathbb{R}^3_+} S(x, t, D(u_k)) : ((\psi^k - \nabla z^k) \otimes \nabla \zeta) \tau dx dt \right| \leq CK\|\psi^k - \nabla z^k\|_{L^p(G'_T)} = o(1).
\]

Therefore, we have

\[
\int_0^T \int_{\mathbb{R}^3_+} S(x, t, D(u_k)) : D(\psi^k) \tau \zeta dx dt \leq C\eta + o(1) \text{ as } k \to \infty,
\]

where \( C \) is independent of \( k, \eta \).
On the other hand,

\[
\int_0^T \int_{\mathbb{R}^3} S(x, t, D(u_k)) : D(\psi^k) \tau \zeta dx dt \\
= \int_{Q^k} S(x, t, D(u_k)) : D(u_k - u) \left(1 - \min \left(\frac{|u_k - u|}{L}, 1\right)\right) \tau \zeta dx dt \\
+ \int_{Q^k} S(x, t, D(u_k)) : \text{sym} \left(\frac{u_k - u}{L} \otimes \nabla |u_k - u|\right) \tau \eta dx dt \\
= \int_{Q^k} (S(x, t, D(u_k)) - S(x, t, D(u))) : D(u_k - u) \tau \zeta dx dt \\
+ \int_{Q^k} S(x, t, D(u_k)) : D(u_k - u) \tau \zeta dx dt \\
- \int_{Q^k} S(x, t, D(u_k)) : D(u_k - u) \min \left(\frac{|u_k - u|}{L}, 1\right) \tau \zeta dx dt \\
+ \int_{Q^k} S(x, t, D(u_k)) : \text{sym} \left(\frac{u_k - u}{L} \otimes \nabla |u_k - u|\right) \tau \eta dx dt \\
:= J_1 + J_2 + J_3 + J_4.
\]

Clearly, since \( D(u_k) \to D(u) \) in \( L^p(0, T; L^\phi(\mathbb{R}^3)) \), thus \( J_2 \to 0 \) as \( k \to \infty \). As in (4) of Proposition 4.3 we can compute that

\[
|J_3| + |J_4| \leq \int_{E^k} |S(x, t, D(u_k)) : D(u_k - u)| dx dt \\
+ L \int_{Q^k \setminus E^k} |S(x, t, D(u_k)) : D(u_k - u)| dx dt \\
\leq \int_{E^k} g^k dx dt + L \int_{Q^k \setminus E^k} g^k dx dt \leq \eta + KL \\
\leq C\eta.
\]

Consequently,

\[
\int_{Q^k} (S(x, t, D(u_k)) - S(x, t, D(u))) : D(u_k - u) \tau \zeta dx dt \leq o(1) + C\eta, \text{ as } k \to \infty.
\]

As above, we know that \( u_k \to u \) a.e. in \( Q_T \). Hence, choose a subsequence which still denote \( \{u_k\} \) such that \( |G_T' \setminus Q^k| \leq 2^{-k} \), for all \( k \in \mathbb{N} \). Thus there exists \( k_0 \in \mathbb{N} \) such that \( 2^{-k_0} < \eta \), and one can find

\[
\sum_{k=k_0+1}^\infty |G_T' \setminus Q^k| \leq 2^{-k_0} < \eta.
\]

23
Setting $M = \bigcup_{k=k_0+1}^{\infty} (G_T^r \setminus Q^k)$, we easily obtain
\[
\int_M (S(x,t,D(u_k)) - S(x,t,D(u))) : D(u_k - u) \tau \zeta \, dx \, dt \leq C \eta.
\] (85)
Whence (84) and (85) infer that
\[
\int_{G_T^r} (S(x,t,D(u_k)) - S(x,t,D(u))) : D(u_k - u) \tau \zeta \, dx \, dt \leq o(1) + C \eta, \quad \text{as } k \to \infty.
\]

From (9) it implies
\[
\int_{G_{st}} (S(x,t,D(u_k)) - S(x,t,D(u))) : D(u_k - u) \, dx \, dt \to 0, \quad \text{as } k \to \infty.
\]
This proves the main theorem.

**Remark 4.4.** We can assume that $f \in L^p(0,T; (D^1_\phi(\mathbb{R}_x^3))^*) \cap L^2(Q_T)$. Then all estimates (66)-(69) also depend on $\|f\|_{L^p(0,T; (D^1_\phi(\mathbb{R}_x^3))^*) \cap L^2(Q_T)}$. In the proof of (82), we must estimate the term $\int_0^T \langle f, \varphi^k \rangle \, dt$. Indeed it is easy to obtain it as follows;
\[
\int_0^T \langle f, \varphi^k \rangle \, dt \leq \int_0^T \langle f, \tau \zeta (u_k - u) \rangle \, dt - \int_0^T \langle f, \tau \zeta (u_k - u) \min\left(\frac{u_k - u}{L}, 1\right) \rangle \, dt - \int_0^T \langle f, \tau \zeta \nabla \zeta^k \rangle \, dt.
\]
The first term on the right vanishes as $k \to \infty$, while the second term is estimated analogously as $J_3$ and $J_4$. Finally, the third term is small thanks to (81). Therefore, we obtain
\[
\int_0^T \langle f, \varphi^k \rangle \, dt \leq o(1) + c \eta.
\]
It shows that this term can not change the statement of the main theorem.

## 5 Conclusion

In this work, we focus on studying the existence of weak solutions for partial differential equations of fluids with shear thinning dependent viscosities. It is easy to obtain this issue about fluids with shear thick case without Orlicz space frame. We have generalized the previous results in the case of unbounded domain until $p > \frac{8}{5}$. 24
Acknowledgements

The work of author is a part of Projects 11571279, and 11201411 supported by National Natural Science Foundation of China and a part of Project GJJ151036 supported by Education Department of Jiangxi Province and partly supported by Youth Innovation Group of Applied Mathematics in Yichun University (2012TD006).

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