Model-independent Reconstruction of the Cosmological Scale Factor as a Function of Lookback Time

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Abstract

We present a model-independent method of reconstructing scale factor against lookback time from the Observational Hubble parameter Data (OHD). The reconstruction method is independent of dynamical models and is only based on the Friedmann–Robertson–Walker metric. We also calculate the propagation of error in the reconstruction process. The reconstruction data errors mainly come from trapezoidal rule approximation and the uncertainty from OHD. Furthermore, the model discrimination ability of original OHD and reconstructed $a$–$t$ data is discussed under a dimensionless method. The $a$–$t$ data can present the differences between cosmology models more clearly than $H$–$z$ data by comparing their coefficients of variations. Finally, we add 50 simulated $H(z)$ data to estimate the influence of future observation. More Hubble measurements in the future will help constrain cosmological parameters more accurately.

Unified Astronomy Thesaurus concepts: Cosmological models (337); Cosmological parameters (339); Stellar distance (1595); Astronomy data analysis (1858)

1. Introduction

The cosmological scale factor $a(t)$ is one of the most fundamental quantities that describe the smooth background universe, although it is not observable. A whole relation between scale factor $a$ and cosmic time $t$ almost contains all information about cosmological kinematics, such as the expansion history, the Hubble parameter (expansion rate) $H = \dot{a}/a$ ($\dot{a} \equiv da/dt$), and the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$. Riemann & Mead (2014) introduced a model-independent method of developing a scale factor–lookback time data from Type Ia supernovae (SNe Ia) and radio-galaxy data, i.e., the Hubble diagram of modulus data against redshift. They used the scale factor plot as a better way to find the transition redshift of the universe at which the universe transitions from decelerating to accelerating.

The assumption that cosmic curvature equals zero was made in the reconstruction process from Type Ia supernovae to scale factor–lookback time data (Daly & Djorgovski 2004). We reconstruct the scale factor against cosmic time from the Observational Hubble parameter Data (OHD). The expression of the lookback time contains an integral of Hubble parameter, so using OHD to reconstruct cosmic time is model independent and does not require any assumptions. We use the trapezoidal rule for approximating integrals of Hubble parameters. The reconstruction data errors come from the trapezoidal rule and OHD’s errors.

The scale factor $a$, cosmic time $t$ data has a more basic status than Hubble parameter $H_e$ redshift $z$ data, since the former directly appears in the Friedmann–Robertson–Walker metric. Although the error propagation from OHD to reconstructed data would increase errors, we expect that the $a$–$t$ data have a higher sensitivity to model discrimination. We calculate the $H$–$z$ and its variance as well as $a$–$t$ for several models such as Phantom, $\Lambda$CDM, and Chevallier–Polarski–Linder (CPL) parameterization. The result shows the $a$–$t$ plot has better model discrimination than the $H$–$z$ plot by comparing the coefficient of variation, although it cannot be seen directly from the figure.

The primary purpose of this paper is to present a model-independent approach to reconstruct the scale factor against cosmic lookback time data from OHD. The paper is organized as follows. We introduce the reconstruction method in Section 2. The error propagation is also calculated there. The reconstructed $a$–$t$ data are shown in Section 3. Section 4 discusses the model discrimination ability of original $H$–$z$ and reconstructed $a$–$t$ data. In the Section 5, we use simulated $H(z)$ data to forecast the improvement effects of future $H(z)$ observation on this reconstruction.

2. Reconstruction

2.1. Dimensionless Cosmic Time $\tau$

The Friedmann–Robert–Walker metric is

$$ds^2 = c^2dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

where $t$ is cosmological time. The age of the universe corresponding to redshift $z$ is

$$t(z) = t_{1f} \int_z^{\infty} \frac{dz'}{(1 + z')E(z')}$$

where Hubble time $t_{1f} \equiv 1/H_0$, $H_0$ is the Hubble constant, and $E(z) = H(z)/H_0$.

The upper limit of the integral of $t(z)$ is infinity, whereas observation data only exist in low redshift ($0 \leq z \leq 2.4$ for the data we use). To solve this problem, we turn to the lookback
time $t_{L}$, which is the time measured back from the present epoch $t_0$ to any earlier time $t(z)$. It can be written as

$$ t_{L}(z) = t_0 - t(z) = t_0 - \int_{0}^{z} \frac{dz'}{(1 + z')E(z')} \, dz'. $$

The “direction” of $t_{L}$ is opposite from $t(z)$. For simplicity, we define dimensionless cosmic time $\tau$ by Hubble time $t_H$ as:

$$ \tau = 1 - \frac{t_{L}}{t_H} = 1 - \int_{0}^{z} \frac{dz'}{(1 + z')E(z')} \approx 0, \quad \text{in general.} $$

The $\tau$ in the Big Bang is not zero. In the $\Lambda$CDM model, $\tau$ $(z \rightarrow +\infty)$ depends on the cosmological density parameters by the expansion rate $E(z)$. But according to current observation constraint values, $\tau|_{z \rightarrow \infty} \approx 0$ (Planck Collaboration et al. 2020), see Figure 1.

The relation between scale factor $a$ and dimensionless cosmic time $\tau$ can be expressed as

$$ \tau = 1 + \int_{1}^{a} \frac{1}{a' \cdot E(a')} \, da', $$

where we use $a=a_0/(1+z)$ and set $a_0=1$.

### 2.2. Reconstruction Method

The OHD contains

$$\{z_i, H_i, \Delta H_i\}, \quad i = 1, ..., N$$

where $\Delta H_i$ means the error of OHD. $\Delta$ is used to represent data error.
where the Hubble constant \( H_0 = 67.4 \pm 0.5 \text{ km s}^{-1} \text{ Mpc}^{-1} \), which comes from Planck 2018 data (Planck Collaboration et al. 2020).

The process from \( d\tau/da \) to \( \tau \) causes most errors since we use trapezoidal rule, which approximates an integral. The trapezoidal method has its own proper error, and \( d\tau/da \) data errors also contribute,

\[
s_n = \int_{a_1}^{a} \left( \frac{d\tau}{da} \right) da \\
\approx \left( \frac{d\tau}{da} \right)_{1} (1 - a_{1}) \\
+ \frac{1}{2} \sum_{i=2}^{n} \left[ \left( \frac{d\tau}{da} \right)_{i-1} + \left( \frac{d\tau}{da} \right)_{i} \right] (a_{i-1} - a_{i}) \tag{20}
\]

and

\[
\Delta s_n = \Delta s_{n, \text{trape}} + \Delta s_{n, \text{data}}. \tag{21}
\]

Errors of \( s \) that correspond to errors of \( d\tau/da \) are easily calculated:

\[
\Delta s_{n, \text{data}} = \Delta \left( \frac{d\tau}{da} \right)_{1} (1 - a_{1}) \\
+ \frac{1}{2} \sum_{i=2}^{n} \left[ \Delta \left( \frac{d\tau}{da} \right)_{i-1} + \Delta \left( \frac{d\tau}{da} \right)_{i} \right] (a_{i-1} - a_{i}). \tag{22}
\]

We estimate trapezoidal rule’s proper error by:

\[
\left| \int_{a}^{b} f(x) dx - \frac{b - a}{2} [f(a) + f(b)] \right| \leq \frac{1}{4} M (b - a)^2, \tag{23}
\]

where the constant \( M \) satisfies \( |f'(x)| \leq M, \forall x \).

The maximum of \( (d\tau/da) \) reconstruction data is 1.5, and we take \( M = 10 \) for an estimate of trapezoidal rule error:

\[
\Delta s_{1, \text{trape}} = \frac{1}{4} M (1 - a_{1})^2 \tag{24}
\]

\[
\Delta s_{n, \text{trape}} = \Delta s_{1} + \sum_{i=2}^{n} \frac{1}{4} M (a_{i-1} - a_{i})^2 \tag{25}
\]

\[
\Delta \tau_i = \Delta s_i, \quad i = 1, \ldots, n. \tag{26}
\]

### 3. Reconstruction Results

The 43 OHD are shown in Figure 2 and Table 1, which contains 31 Hubble measurements from the method of cosmic chronometers and 12 from the radial BAO method. The reconstructed \( a - d\tau/da \) and \( a - \tau \) data are shown in Figure 3 and Table 2. The error in the early universe (low \( \tau \)) is enormous. According to Equation (22) and (25), the error is accumulated in calculating high-\( z \) or low-\( \tau \) data. The main sources of error are the accumulation effect of trapezoidal rule and errors of Hubble measurements. In order to reduce the error from trapezoidal rule, more future OHD data is needed to increase the number of approximate trapezoids to have a more accurate result.

To test the validation of reconstruction data, we use Markov Chain Monte Carlo (MCMC) method to fit the matter density parameter \( \Omega_m \) in \( \Lambda \)CDM model:

\[
E(z) = \sqrt{\Omega_m(1 + z)^3 + \Omega_\Lambda} \tag{27}
\]

\[
E(z) = \sqrt{\Omega_m(1 + z)^3 + \Omega_\Lambda} \tag{28}
\]

\[
\Omega_m + \Omega_\Lambda = 1 \tag{29}
\]

where we ignore radiation \( (\Omega_k) \) and consider a flat model \( (\Omega_k = 0) \).

According to Equation (4) and \( a = 1/(1 + z) \), the relation between \( \tau \) and \( a \) in \( \Lambda \)CDM is:

\[
\tau(a) = 1 + \int_{1}^{a} \frac{da'}{a' \cdot E(a')} \tag{30}
\]

where the only free parameter is \( \Omega_m \).

The result of direct fitting from OHD data is

\[
\Omega_m = 0.290 \pm 0.008. \tag{31}
\]

which is close to the result from Planck Collaboration et al. (2020) \( \Omega_m = 0.315 \pm 0.007 \) with similar uncertainty, but they are not consistent.

The fitting result from \( a-\tau \) data is:

\[
\Omega_m = 0.372 \pm 0.087. \tag{32}
\]

This result implies an universe with \( \Lambda \)-dominated accelerating expansion, which is consistent with SNe Ia and BAO results. Compared with OHD fit of \( \Lambda \)CDM result \( \Omega_m = 0.290 \pm 0.008 \), this exercise shows the reconstructed \( a-\tau \)’s parameter is consistent with the Planck result, which verifies the feasibility of the reconstruction method. However, it also shows its limitation that the trapezoidal rule introduces a much larger uncertainty.

### 4. Model Discrimination

To test the \( H-\tau \) and \( a-\tau \) plot’s ability for model discrimination, we find the models that are in good agreement with the results obtained from the observations, such as the Planck result (Planck Collaboration et al. 2020), Supernovae Ia (SNe Ia), Baryon Acoustic Oscillations (BAO), and Cosmic Microwave Background (CMB) (Wang et al. 2016). A nonparametric
smoothing method, the Gaussian process, is also used to get an independent result. The models we choose to distinguish are Phantom, ΛCDM, CPL parameterization.

The observational constraints on the parameters are $\theta_i \pm \Delta \theta_i$, $i = 1,...,n$, the estimated error in a model quantity $f(\theta_1,...,\theta_n)$ is

\[
\Delta f(\theta_1,...,\theta_n) \approx \left( \sum_{i=1}^{n} \left( \frac{\partial f}{\partial \theta_i} \Delta \theta_i \right)^2 \right)^{1/2}.
\]  (33)

4.1. Phantom model

In the Phantom model, the Hubble parameter $H$ about redshift $z$ is given by:

\[
H_{\text{Phantom}}(z) = H_0 [\Omega_m(1+z)^3 + (1 - \Omega_m)(1+z)^{-3(1+w)}]^{1/2}.
\]  (34)

The dimensionless time $\tau$ about scale factor $a$ is:

\[
\tau_{\text{Phantom}}(a) = 1 + \int_1^{a} da' \left[ \frac{\Omega_m}{a'^2} + \frac{1-\Omega_m}{a'^{3(1+w)}} \right]^{1/2}.
\]  (35)

We follow Rani et al. (2017)’s scheme: fix $\omega = -2$ and take $\Omega_m = 0.315 \pm 0.007$ from the Planck result (Planck Collaboration et al. 2020). Note that Phantom model with $\omega = -2$ is actually far from Planck constraints. It may not be a good choice to describe our real universe, but here we add this model in the hope that more different models can be compared to illustrate and test the model discrimination ability of $a-\tau$ model.

4.2. ΛCDM Model

ΛCDM model is most widely accepted and is in very good agreement with observation. Ignoring the curve $(\Omega_k)$ and radiation term $(\Omega_R)$, the Hubble parameter is written as:

\[
H_{\Lambda\text{CDM}}(z) = H_0 [\Omega_m(1+z)^3 + (1 - \Omega_m)]^{1/2}.
\]  (36)
The dimensionless time is:
\[ \tau_{\Lambda CDM}(a) = 1 + \int_1^a \frac{da'}{a'[\Omega_m + 1 - \Omega_m]^{1/2}}. \] (37)

We take \( \Omega_m = 0.315 \pm 0.007 \) (Planck Collaboration et al. 2020).

4.3. Chevallier–Polarski–Linder (CPL) Model

CPL is one of the most popular parameterizations of the dark energy equation of state (Chevallier & Polarski 2001; Linder 2003). It has two equation of state parameters \( \omega_0, \omega_1 \):

\[ \omega_{\text{CPL}} = \omega_0 + \omega_1 \frac{z}{1 + z} = \omega_0 + \omega_1 (1 - a). \] (38)

The corresponding Hubble parameter and cosmic time are:

\[ H_{\text{CPL}}(z) = H_0 (1 + z)^3 + (1 - \Omega_m) (1 + z)^{3(1 + \omega_0 + \omega_1)} \times \exp \left( \frac{3 \omega_1}{1 + z} \right)^{1/2}. \] (39)

\[ \tau_{\text{CPL}}(a) = 1 + \int_1^a \frac{da'}{a'[\Omega_m + 1 - \Omega_m]^{1/2}} \exp \left( -3 \omega_1 (1 - a') \right)^{1/2}. \] (40)

We use \( \Omega_m = 0.300 \pm 0.014, \omega_0 \) obtained from the joint

\[ \omega_1 = -0.082 \pm 0.134, \omega_1 \]

analysis of Supernovae Ia (SNe Ia), BAO, and CMB data (Wang et al. 2016).

The collect results of the \( H-z \) plot and \( a-\tau \) plot are shown in Figures 4 and 5. In the \( H-z \) plot, the curves of Phantom, \( \Lambda \)CDM, and CPL models are entangled together, and their error bands overlap, which means \( H-z \) data cannot distinguish these models well. Whereas in the \( a-\tau \) plot, the error bands should have separated more clearly in theory due to the integral effect, but we cannot see it clearly from the figure. A parameterized comparision method is needed.

Here we use the coefficient of variation (CV) to measure the dispersion of different models in a standard and dimensionless way. It is defined as the ratio of the standard deviation \( \sigma \) to the average value \( \mu \):

\[ CV = \frac{\sigma}{\mu}. \] (41)

The calculation result is shown in Table 3. According to the \( H-z \) plot data, the differences of CV values between models are \( \leq 0.01 \), which only account for 0.2% or less. From the CV data of the \( H-z \) plot, we can see that the differences between models are in tenths or single digits, which exceeds 100% of some CV
values. Therefore, the dispersion degree between models improves by reconstructing \( a - \tau \) data. We believe the data indicate that the \( a - \tau \) data’s model selection ability is better.

5. Future Data Simulation

In this section, we simulate a future OHD to study its influence on the \( a - \tau \) method. We use the simulation method from Ma & Zhang (2011):

\[
H_{\text{sim}}(z) = H_{\text{fid}}(z) + \Delta H
\]

(42)

where \( \Delta H \) is the deviation between simulation value and fiducial value.

For the data simulation, we have to first choose a model. We choose the \( \Lambda \)CDM model as our fiducial model:

\[
H_{\text{fid}}(z) = H_0 \sqrt{\Omega_m(1 + z)^3 + \Omega_\Lambda}.
\]

(43)

The parameters come from Planck Collaboration et al. (2020), which are

\[
H_0 = 67.4 \pm 0.5,
\]

(44)

\[
\Omega_m = 0.315 \pm 0.007.
\]

(45)

From the spatially flat \( \Lambda \)CDM model, \( \Omega_\Lambda \) can be calculated by \( \Omega_m + \Omega_\Lambda = 1 \). Then, we can obtain a set of fiducial values from Equation (43).

Next, we need to estimate the uncertainties of future OHD. The 43 Hubble measurements’ uncertainties are shown in Figure 6. After removing one value with an obvious deviation, we draw two outlines to represent the general trend of the uncertainties. Two bound lines are expressed as one upper line \( \sigma_+ = 8.19z + 34.31 \) and one lower line \( \sigma_- = 2.16z + 2.25 \). This simulation method believes the future measurements will also conform to this trend, and their value will fall within this error strip between \( \sigma_- \) and \( \sigma_+ \).

\[
\sigma_+ = \frac{\sigma_0(z)}{4}
\]

(46)

This random uncertainty \( \sigma(z) \) can be used to determine the deviation \( \Delta H \) by a Gaussian distribution \( N(0, \sigma(z)) \). Then 50 Hubble parameters are simulated using the method described above and the result is shown in Figure 7. Using the previous reconstructed method in Section 2 to constrain the matter density parameter \( \Omega_m \), the corresponding \( a - \tau \) plot is shown in Figure 8. When we only apply the original 43 observed data points, the fitting curve describes red points well with the result of \( \Omega_m = 0.372 \pm 0.087 \), but it can be seen that there is still a definite gap between the fitting curve and the \( \Lambda \)CDM model curve. When we apply total data, including 43 observed data and 50 simulated data, the error of data points becomes larger due to the integral effect, while the result’s trend is closer to the \( \Lambda \)CDM model with a result of \( \Omega_m = 0.303 \pm 0.047 \). These are consistent with our expectations. The \( a - \tau \) plot result should be biased toward \( \Lambda \)CDM.
model after adding 50 data simulated by the ΛCDM model to the sample. In addition, the error of Ω_m decreased from 0.087 to 0.047 after adding more OHD. More data points are conducive to the accuracy of the study. In other words, as more Hubble measurements are added in the future, the a–τ plot can magnify the differences between models based on the integral effect so that the a–τ plot has a better model discriminating ability than the H–z plot. Note that the differences are not significant enough to be observed directly from the graph. We use the coefficient of variation to compare numerically.

We simulated 50 H(z) data with a fiducial ΛCDM model based on Ma & Zhang (2011) and reconstruct total a–τ data to forecast the improvement effects of future H(z) observations. On the one hand, due to the integration characteristics, errors are also accumulated. Therefore, when the sample takes all data, the error bar increases. On the other hand, the a–τ plot shows a more accurate result of constraining cosmological parameters. If there are more Hubble measurements in the future, the a–τ method can better present the model to help people find a better model to describe our present universe.

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Figure 8. The reconstructed a–τ data from Hubble parameters. Red points are 43 reconstruction results from observed data. Black points are 90 reconstruction results from total OHD, which include 43 observed data points and 50 simulated data points. The blue curve represents the ΛCDM model from Planck Collaboration et al. (2020). The red curve is the best ΛCDM fitting with original 43 OHD and the black curve is the fitting with total data. Note that a|τ=0 = 1, since the definition of τ (Equation (4)) implies that τ|τ=0 = 1 − H/Hi.