Hyperfine structure and \((e^-, e^+)-\)pair annihilation in the muonium-positronium MuPs and positronium hydrides.

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Abstract

The hyperfine structure of the ground states in a number of positronium hydrides (TPs, DPs, \(^1\)HPs) and MuPs \((\mu^+e^-_2e^+)\) is determined with the use of highly accurate variational wave functions. We also evaluate the probabilities of various processes in the MuPs system, including the \((e^-, e^+)-\)pair annihilation and its conversion into the charge conjugate system \(\mu^+e^-_2e^+ \rightarrow \mu^-e^-e^+_2\).
I. INTRODUCTION.

In this work we consider the bound states in the positronium hydrides $\infty$HPs, TPs, DPs, $^1$HPs and MuPs ($\mu^+e^-e^+$). The last system is the main interest in this study. Each of these neutral systems contain one heavy positively charged particle, i.e. hydrogen nucleus or $\mu^+$, two electrons $e^-$ and one positron $e^+$. Below, such four-body systems are designated as $A^+e^-e^+$ systems, where the notation $A$ designates a heavy particle with $m_A \gg m_e$. In atomic units $h = 1, m_e = 1, e = 1$ the Hamiltonian of the four-body $A^+e^-e^+$ system is written in the form (in atomic units):

$$H = -\frac{1}{2m_A}\Delta_1 - \frac{1}{2}\Delta_2 - \frac{1}{2}\Delta_3 - \frac{1}{2}\Delta_4 + \frac{1}{r_{12}} - \frac{1}{r_{13}} - \frac{1}{r_{14}} - \frac{1}{r_{23}} - \frac{1}{r_{24}} + \frac{1}{r_{34}}$$

(1)

where the notation $1$ (also $A$) designates the heaviest particle $A^+$, the notation $2$ (or $+$) means the positron, while $3$ (or $-$) and $4$ (or $-$) stand for electrons. This system of notations will be used everywhere below in our study.

Our first goal is to determine the wave functions which correspond to the bound states of the Hamiltonian, Eq.(1). In other words, we need to find all negative eigenvalues $E$ and unit-norm functions $\Psi$ for which the corresponding Schrödinger equation $H\Psi = E\Psi$ is obeyed. It is clear that the total energies and other bound state properties of the $A^+e^-e^+$ systems must be analytical functions of the inverse masses of heavy particle $\frac{1}{m_A}$. Note that some state in the $A^+e^-e^+$ system is stable if its total energy (in atomic units) is less than the corresponding threshold value

$$E_{th}(m_A) = -\frac{0.5}{1 + m_A^{-1}} - 0.25 = -\frac{(3m_A + 1)}{4(m_A + 1)} \geq -\frac{3}{4} = -0.75a.u.$$  

(2)

As follows from the results of numerical calculations for all positronium hydrides the total energy of the ground state is bounded between $\approx -0.78631730(15)$ a.u. (MuPs) and $\approx -0.789196770(3)$ a.u. ($\infty$HPs). This indicates clearly that each of these positronium hydrides is a weakly bound four-body system. In fact, it was shown long ago that each of these hydrides has only one bound (ground) $S$-state $1$, where $S$ designates a state with $L = 0$ and $L$ is the total orbital angular momentum of the four-body system. Moreover, it is bound if (and only if) the two electrons form the singlet pair, i.e. the total electron spin equals zero.

In general, the positronium hydrides are of interest for astrophysics $2, 3$. Almost 20 years ago the $^1$HPs hydride was created in the laboratory during collisions between
Theoretically, the positronium hydride $^{\infty}$HPs has extensively been investigated in earlier studies [5] - [10], [11], [12] (all references on HPs before 1998 can be found in [13]). The bound muonium-positronium MuPs has never been observed in the laboratory.

In this work our main attention will be given to some properties of the muonium-positronium system (or MuPs, for short), but we also evaluate the probabilities of some processes within it. The hyperfine structure of the MuPs system is discussed in Section II. Section III contains numerical evaluations of different annihilation probabilities for muonium-positronium. In Section IV we consider the annihilation rates of the electron-positron pairs in other positronium hydrides. In Section V we discuss a possibility to observe the conversion of MuPs into its charge conjugate system $\mu^+e^-e^+\rightarrow\mu^-e^+e^+$. Concluding remarks can be found in Section VI.

II. THE HYPERFINE STRUCTURE OF THE GROUND STATE IN MUONIUM-POSITRONIUM.

The hyperfine structure (i.e. the appropriate shift of the energy level and its splitting) is determined by the spin-spin interaction between particles. The general expression for the hyperfine interaction of a number of particles with non-zero spin values can be written in the form

$$H_{HF} = -\sum_{(ij)} a_{ij}(s_i \cdot s_j)$$

where in the case of $A^+e^-e^+$ system the sum is calculated for all six pairs of particles $(ij)$. However, as mentioned above in the ground state of the MuPs system the two electrons are always in the singlet state, i.e. their total spin equals zero. Also, in this work we are interested in the hyperfine structure splitting only. In such a case Eq.(3) can be re-written to the form

$$H_{HF} = -a(I_A \cdot s_+) - b(s_+ \cdot S_-) - c(I_A \cdot S_-)$$

where $S_-$ is the total electron spin (i.e. $S_- = s_1 + s_2$ in our current notations), $s_+$ is the positron spin and $I_A$ is the spin of the $A-$particle.

In the MuPs system both electrons are in the singlet state, i.e. $S_- = 0$. Therefore, from
Eq. (4) one finds $H_{HF} = -a(I_\mu \cdot s_+)$, where the coupling constant $a$ is written in the form

$$a = \frac{8\pi \alpha^2}{3} \mu_B^2 \frac{g_\mu}{m_\mu} \frac{g_+}{m_e} \langle \delta_{\mu^+e^+} \rangle = \frac{2\pi \alpha^2}{3} \frac{g_\mu}{m_\mu} g_+ \langle \delta_{\mu^+e^+} \rangle$$

(5)

where $\langle \delta_{\mu^+e^+} \rangle$ is the expectation value of the muon-positron delta-function $\delta_{\mu^+e^+} = \delta(r_{\mu^+e^+}) = \delta(r_{\mu^+} - r_{e^+})$ determined for the ground state of MuPs and expressed in atomic units. Also, In Eq.(5) the factor $\alpha = 7.2973525679 \cdot 10^{-3}$ is the fine structure constant and $\mu_B = \frac{e \hbar}{2m_e}$ is the Bohr magneton which equals $\frac{1}{2}$ in the atomic units ($e = 1, \hbar = 1$ and $m_e = 1$). The value of $\mu_B$ in SI units is $\approx 9.27401543 \cdot 10^{-24} J \cdot T^{-1} [14]$. In our calculations we have used the following values for the muon mass $m_\mu$ and for the factors $g_+$ and $g_\mu [14, 15]$:

$$m_\mu = 206.768264m_e, \quad g_+ = -2.0023193043718, \quad g_\mu = -2.0023318396$$

(6)

where $m_e$ is the electron/positron mass at rest. With these numerical values Eq.(5) takes the form

$$a = 14229.1255 \cdot \langle \delta_{\mu^+e^+} \rangle$$

(7)

For MuPs the diagonalization of the $H_{HF}$ operator yields the two energies: $\epsilon(J = 0) = \frac{3}{4}a$ and $\epsilon(J = 1) = -\frac{1}{4}a$, where $a > 0$. The notation $J$ denotes the total spin of the muon-positron pair. From our numerical calculations we have found that in the ground state of MuPs the numerical value of muon-positron delta-function is $\langle \delta_{\mu^+e^+} \rangle \approx 1.613451 \cdot 10^{-3}$. From here one finds that the energy difference between $\epsilon(J = 0) = \frac{3}{4}a$ and $\epsilon(J = 1) = -\frac{1}{4}a$ levels equals $a \approx 22.985 \, MHz$. The uncertainty in this value can be evaluated as $\approx 10 \, kHz$. To convert the atomic units into $MHz$ we have used the conversion factor $6.57968392061 \cdot 10^9 MHz/a.u$. The value $22.958 \, MHz$ must be compared with the total ground state energy (non-relativistic) obtained for the MuPs system $E = -0.7683171715 \, a.u. \approx 5.0552841393 \cdot 10^9 MHz$. Analogous calculations of the hyperfine structure splitting can be performed for all positronium hydrides mentioned above (see Section IV below).

III. ELECTRON-POSITRON ANNIHILATION IN MUONIUM-POSITRONIUM.

The muonium-positronium system is not a stable four-body system. Its instability is mainly related with the $(e^-, e^+)$-pair annihilation. In some works such an annihilation is called the positron annihilation. The life-time of MuPs against positron annihilation is
\[ \approx 2.247 \cdot 10^{-10} \text{ sec} \] (see below). Another possible decay channel arises from the instability of the \( \mu^+ \) -muon. It usually decays into one positron, one electron neutrino and one muon antineutrino (see Section IV below). The corresponding life-time is \[ \approx 2.19703 \pm 4 \cdot 10^{-10} \text{ sec} \] which is approximately 15 times longer than the life-time of MuPs against three-photon annihilation. Muonium-positronium conversion in MuPs is also possible (see discussion in the fourth Section). In this Section we consider annihilation of the \((e^-, e^+)\)-pair in the ground state of the MuPs system.

First, consider the two- and three-photon annihilation rates. As is well known from Quantum Electrodynamics (see, e.g., [16]) an isolated electron-positron pair or Ps \((e^- e^+)\), which is in the singlet \( ^1S \) state, annihilates with the emission of two, four, six and any even number of photons. The largest annihilation rate is for two-photon annihilation:

\[
\Gamma_{2\gamma}(\text{Ps}, ^1S) = 4\pi\alpha^4c_0^{-1}\left[1 - \frac{\alpha}{\pi}\left(5 - \frac{\pi^2}{4}\right)\right] \langle \delta(r_{+-}) \rangle = 4 \times 50.17280269804 \cdot 10^9 \cdot \langle \delta_{+-} \rangle \text{ sec}^{-1} \tag{8}
\]

where the notation \( \delta(r_{+-}) = \delta_{+-} \) is the two-body electron-positron delta-function and \( \langle \delta_{+-} \rangle \) is its expectation value determined for the singlet \( ^1S \) state of electron-positron pair. In this formula and everywhere below we shall use the following numerical values for speed of light \( c = 2.99792458 \cdot 10^8 \text{ m} \cdot \text{sec}^{-1} \) and for Bohr radius \( a_0 = 0.5291772108 \cdot 10^{-10} \text{ m} \) [14]. Note that our expression for \( \Gamma_{2\gamma} \), Eq.(8), also includes the lowest order radiative correction [17]. Analogously, an isolated electron-positron pair, which is in the triplet \( ^3S \) state, annihilates with the emission of three, five, seven and any odd number of photons. The largest annihilation rate is for three-photon annihilation:

\[
\Gamma_{3\gamma}(\text{Ps}, ^3S) = \frac{16(\pi^2 - 9)}{9} \alpha^5c_0^{-1}\langle \delta(r_{+-}) \rangle = \frac{4}{3} \times 1.35927229774 \cdot 10^8 \langle \delta_{+-} \rangle \text{ sec}^{-1} \tag{9}
\]

In an arbitrary atom, ion or molecule which contain the bound positron we have a number of electron-positron pairs which are generally in mixed spin states and this is the case in the MuPs system. This means that we cannot predict the actual spin state of these electron-positron pairs. In such cases it is assumed that each of the four possible spin states of the electron-positron pair has equal probability, which implies the probability of \( \frac{1}{4} \) to be in its singlet state and the probability of \( \frac{3}{4} \) to be in its triplet state [18]. The total probability of the two-photon annihilation of \((e^-, e^+)\)-pair which is in a mixed spin state equals the product of \( \Gamma_{2\gamma}(\text{Ps}, ^1S) \) (Eq.(8)), the factor \( \frac{1}{4} \), and the number of electron-positron pairs \( n \).
In this case one finds the formulae presented above for the MuPs system \((n = 2)\)

\[
\Gamma_2(\text{MuPs}) = n\pi\alpha^4c\alpha_0^{-1}\left[1 - \frac{\alpha}{\pi}\left(5 - \frac{\pi^2}{4}\right)\right]\langle\delta(r_{+-})\rangle = 100.3456053781 \cdot 10^9 \langle\delta_{+-}\rangle \sec^{-1} \tag{10}
\]

In the case of three-photon annihilation the \(\Gamma_3(\text{MuPs})\) annihilation rate equals the product of \(\Gamma_3(\text{Ps,}^3S)\) (Eq.(9)), the factor \(\frac{3}{4}\), and the number of electron-positron pairs \(n\), i.e.

\[
\Gamma_3(\text{MuPs}) = n\frac{4(\pi^2 - 9)}{3}\alpha^5c\alpha_0^{-1}\langle\delta(r_{+-})\rangle = 2.718545954 \cdot 10^8 \langle\delta_{+-}\rangle \sec^{-1} , \tag{11}
\]

where \(\langle\delta_{+-}\rangle\) is the expectation value of the electron-positron delta-function determined for the ground state in the MuPs system.

Now, let us discuss the four- and five-photon annihilation of the electron-positron pairs in the MuPs system. It was shown in [19] that the rates of the four- and two-photon annihilation in para-positronium (i.e. in the \((e^-,e^+)\)-pair in its singlet state) are related to each other by the following approximate equation

\[
\Gamma_{4\gamma}(\text{Ps},^1S) \approx 0.274 \left(\frac{\alpha}{\pi}\right)^2 \Gamma_{2\gamma}(\text{Ps},^1S) \tag{12}
\]

By multiplying the both sides of this equation by the factor \(\frac{1}{4}\) and the total number of electron-positron pairs (in MuPs \(n = 2\)) one finds an analogous expression for the MuPs system

\[
\Gamma_{4\gamma}(\text{MuPs}) \approx 0.274 \left(\frac{\alpha}{\pi}\right)^2 \Gamma_{2\gamma}(\text{MuPs}) \tag{13}
\]

where \(\Gamma_{4\gamma}(\text{MuPs})\) and \(\Gamma_{2\gamma}(\text{MuPs})\) are the corresponding annihilation rates of the MuPs system. For the two-photon annihilation rate \(\Gamma_{2\gamma}\) in Eq.\,(13) one can use the explicit expression Eq.\,(10). Note that in Eq.\,(13) the formula for the \(\Gamma_{2\gamma}\) rate must be used which does not contain the lowest order radiative correction. But, for approximate evaluations we can ignore such a small difference in \(\Gamma_{2\gamma}\). For the five-photon annihilation rate in the MuPs system one analogously finds the following result

\[
\Gamma_{5\gamma}(\text{MuPs}) \approx 0.177 \left(\frac{\alpha}{\pi}\right)^2 \Gamma_{3\gamma}(\text{MuPs}) \tag{14}
\]

This result is based on the formula from Ref.\,[19]. The numerical values of the \(\Gamma_{2\gamma},\Gamma_{3\gamma},\Gamma_{4\gamma}\) and \(\Gamma_{5\gamma}\) annihilation rates computed with the use of these formulas are: \(\Gamma_{2\gamma} \approx 2.4522354(30) \cdot 10^9 \sec^{-1},\ \Gamma_{3\gamma} \approx 6.643554(10) \cdot 10^6 \sec^{-1},\ \Gamma_{4\gamma} \approx 3.6253(1) \cdot 10^3 \sec^{-1}\) and \(\Gamma_{5\gamma} \approx 6.1723(1) \sec^{-1}\). They also can be found in Table I. The four- and five-photon annihilation rates (i.e.
\( \Gamma_4 \) and \( \Gamma_5 \) have never been evaluated (accurately) in earlier studies. Table I also contains the numerical values of \( \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5 \) annihilation rates determined for the positronium hydrides \( \infty \)HPs, TPs, DPs and \( 1 \)HPs. The numerical values of these \( n \)-photon annihilation rates allow one to estimate the total annihilation rate \( \Gamma \approx \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 \approx \Gamma_2 + \Gamma_3 \) for each of the positronium hydrides and MuPs.

The two-, three-, four- and five-photon annihilations are the leading annihilation processes in MuPs and other positronium hydrides. In some applications, however, the one-photon and zero-photon annihilations may also play an important role. For the zero-photon annihilation rate \( \Gamma_0 \) we shall use the following expression (found in [20])

\[
\Gamma_0 = \xi \cdot \frac{147 \sqrt{3 \pi^3}}{2} \cdot \alpha^{12} a_0^{-1} \cdot \langle \delta_{++--} \rangle = 5.0991890 \cdot 10^{-4} \cdot \xi \cdot \langle \delta_{++--} \rangle \sec^{-1}
\]

(15)

where \( \langle \delta_{++--} \rangle \) is the expectation value of the four-particle delta-function in the ground state of muonium-positronium (MuPs). Its numerical value is the probability to find all four particles at one spatial point with spatial radius \( \alpha a_0 \). The unknown (dimensionless) factor \( \xi \) has the numerical value close to unity. The expectation value of the four-particle delta-function determined in our calculations is \( \approx 1.785222 \cdot 10^{-4} \) (in a.u.). From here one finds that \( \Gamma_0(MuPs) \approx 9.10318(10) \cdot 10^{-8} \xi \sec^{-1} \). For approximate evaluations we can assume that the factor \( \xi \) equals unity. In this case one finds that \( \Gamma_0(MuPs) \approx 9.10318(10) \cdot 10^{-8} \sec^{-1} \).

Now, consider the one-photon annihilation of the electron-positron pair in MuPs (this can proceed with the emission of one fast electron). The probability of such a process is given by the formula (its rigorous derivation can be found in [21])

\[
\Gamma_1^{(1)} = \frac{64 \pi^2}{27} \cdot \alpha^{8} a_0^{-1} \cdot \langle \delta_{++--} \rangle = 1.066420947 \cdot 10^3 \cdot \langle \delta_{++--} \rangle \sec^{-1}
\]

(16)

where \( \langle \delta_{++--} \rangle = \langle \delta(r_{++})\delta(r_{--}) \rangle \) is the expectation value of the triple electron-positron delta-function in the ground state of the MuPs system. Its numerical value is the probability to find all three corresponding particles at one spatial point with spatial radius \( \alpha a_0 \). Our best numerical treatment to-date gives \( \langle \delta_{++--} \rangle \approx 3.631815 \cdot 10^{-4} \) resulting in \( \Gamma_1^{(1)} \approx 3.87063(10) \cdot 10^{-1} \sec^{-1} \) for the MuPs ground state.

In addition to this one-photon annihilation in MuPs another one-photon annihilation of the \((e^-, e^+)\)-pair is possible. In [13] such an annihilation was called the second one-photon annihilation. The corresponding annihilation rate is designated as \( \Gamma_1^{(2)} \). In this
case the probability of one-photon annihilation is \( \sim \langle \delta_{\mu^+} \rangle \) and one (of two) annihilation \( \gamma \)-quanta is absorbed by the heavy \( \mu^+ \) muon. The muon takes all photon’s energy (i.e. \( \approx 0.51099906 \) MeV) and its momentum. The Lorentz \( \gamma \)-factor of the final/accelerated muon is \( \approx 1.00483633 \), i.e. the acceleration of the final \( \mu^+ \) muon produced by the absorbed \( \gamma \)-quantum is very small. It follows from here that the final \( \mu^+e^- \) system (or muonium) can be found either in its ground \( 1S(L = 0) \)−state (\( P_g \geq 97\% \)), or in the excited \( 2S(L = 0) \)−state (\( P_e \leq 1\% \)), or in the unbound state (\( P_u \leq 1\% \)). Formally, the rate of the second one-photon annihilation in the MuPs system can be evaluated from the approximate equality \( \Gamma_{1\gamma}^{(2)} \approx \Gamma_{1\gamma}^{(1)} \). To obtain the more accurate value of \( \Gamma_{1\gamma}^{(2)} \) one needs undertake an extensive QED consideration.

IV. HYPERFINE STRUCTURE AND ELECTRON-POSITRON ANNIHILATION IN POSITRONIUM HYDRIDES.

The formulas presented above can also be used to compute the hyperfine structure and evaluate the probabilities of electron-positron annihilation in positronium hydrides. In this Section we restrict ourselves to the analysis of the two systems: \( \infty \)HPs and \( 1 \)HPs. The total energies obtained for the ground state of these two systems in our calculations are -0.789 196 764 445 a.u. and -0.788 870 709 151 a.u., respectively. The proton mass used in these calculations is \( M_p = 1836.15267261 \) \( m_e \). For masses of deuterium and tritium nuclei we have used the values \( M_d = 3670.4829652 \) \( m_e \) and \( M_t = 5496.92158 \) \( m_e \).

The absolute value of the hyperfine structure splitting in the \( 1 \)HPs system is \( \approx 3.61 \) MHz. This value represents the energy splitting between the triplet and singlet states of hyperfine structure. As expected the absolute value of hyperfine structure splitting in the \( 1 \)HPs system is significantly smaller (\( \approx 6 \) times smaller) than such a splitting in the MuPs system (see Section II). In the DPs and TPs hydrides the hyperfine structure splittings are also relatively small.

Annihilation rates for the \( \infty \)HPs system are: \( \Gamma_{2\gamma} \approx 2.4568468(30) \cdot 10^9 \) sec\(^{-1} \), \( \Gamma_{3\gamma} \approx 6.656047(10) \cdot 10^6 \) sec\(^{-1} \), \( \Gamma_{4\gamma} \approx 3.6321(1) \cdot 10^3 \) sec\(^{-1} \) and \( \Gamma_{5\gamma} \approx 6.3565(1) \) sec\(^{-1} \), respectively. The first one-photon annihilation rate in the \( \infty \)HPs system is \( \Gamma_{1\gamma}^{(1)} \approx 3.93809(10) \cdot 10^{-1} \) sec\(^{-1} \), while the zero-photon annihilation rate is \( \Gamma_{0\gamma} \approx \xi_0 \cdot 9.4502(1) \cdot 10^{-8} \) sec\(^{-1} \), where \( \xi_0 \) is an unknown numerical factor (\( \xi_0 \approx 1 \)). For the \( 1 \)HPs system these annihilation rates are:
\[ \Gamma_2 \approx 2.4562527(30) \cdot 10^9 \text{ sec}^{-1}, \quad \Gamma_3 \approx 6.654438(10) \cdot 10^6 \text{ sec}^{-1}, \quad \Gamma_4 \approx 3.6312(1) \cdot 10^3 \text{ sec}^{-1} \]
and \[ \Gamma_5 \approx 6.3550(1) \text{ sec}^{-1}, \]
respectively. The first one-photon annihilation rate is \[ \Gamma_1^{(1)} \approx 3.93025(10) \cdot 10^{-1} \text{ sec}^{-1} \] and zero-photon annihilation rate is \[ \Gamma_0 \approx \xi_1 \cdot 9.4289(1) \cdot 10^{-8} \text{ sec}^{-1} \], where \( \xi_1 \) is an unknown numerical factor. No attempt was made to evaluate the second one-photon annihilation rate \( \Gamma_1^{(2)} \) accurately in this study. Annihilation rates for the DPs and TPs hydrides can be found in Table I.

It should be mentioned that our current expressions for annihilation rates \( \Gamma_2, \Gamma_3, \Gamma_4, \) and \( \Gamma_5 \) which are used above for MuPs and other positronium hydrides have been derived from a rigorous consideration based on Quantum Electrodynamics whereas the annihilation rates determined in Ref.\[13\] are based on very approximate relations.

V. MUONIUM-POSITRONIUM CONVERSION.

In the four-body MuPs system there is a possibility to observe a very interesting process of muonium (or \( \mu^+e^- \)) conversion into the charge conjugate system \( \mu^-e^+ \) \[22\] - \[26\]. The muonium-antimuonium conversion has attracted significant theoretical and experimental attention for many years (see, e.g., \[22\] - \[26\], \[27\], \[28\] and references therein). In atomic physics such a process corresponds to a spontaneous conversion of the incident atom into its anti-atom. In the four-body MuPs system this process is even more interesting, since during such a conversion the original \( \mu^+e^- \) system is transformed into the charge conjugate four-body system \( \mu^-e^+_2e^- \) (or \( \text{MuPs} \)) in which the heaviest particle has the negative charge. The newly arising \( \mu^-e^+_2e^- \) system contains two positrons \( e^+ \) and one electron \( e^- \). Very likely that the newly arising system \( \text{MuPs} \) can be in the same (atomic) bound state as the original system MuPs.

Formally, the muonium-positronium conversion is not prohibited by any conservation law. However, it is very hard to observe such a conversion under actual experimental conditions. Mainly, this is related to the very short life-time of the incident MuPs system. The positively charged muon \( \mu^+ \) is an unstable particle which decays as follows:

\[ \mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu \quad (17) \]

where \( \nu_e \) and \( \bar{\nu}_\mu \) are the electron neutrino and muon antineutrino, respectively. The muon mean life-time \( \tau_\mu \) is \( \approx (2.19703\pm4\cdot10^{-5})\cdot10^{-6} \text{ sec} \). That part of the Fermi theory Lagrangian
\[ \mathcal{L}_F \text{ which corresponds to the muon decay Eq.(17) is} \]
\[ \mathcal{L}_W = -\frac{1}{\sqrt{2}} G_F \left[ \bar{\psi}_{\nu_\mu} \gamma_\lambda (1 + \gamma_5) \psi_\mu \right] \left[ \bar{\psi}_e \gamma_\lambda (1 + \gamma_5) \psi_{\nu_e} \right] . \tag{18} \]

where \( G_F \) is the Fermi coupling constant, while \( \psi_\mu, \psi_e, \psi_{\nu_e} \) and \( \psi_{\nu_\mu} \) are the wave functions of the muon, electron, electron neutrino and muon antineutrino, respectively. Also in this equation \( \gamma_\lambda \) and \( \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_0 \) are the corresponding Dirac (4 \( \times \) 4) matrices.

In general, the Fermi theory Lagrangian \( \mathcal{L}_F \) must also include the bare quantum-electrodynamic Lagrangian \( \mathcal{L}_{QED} \) and bare quantum chromodynamic Lagrangian \( \mathcal{L}_{QCD} \) which is responsible for strong interactions, i.e. \( \mathcal{L}_F = \mathcal{L}_W + \mathcal{L}_{QED} + \mathcal{L}_{QCD} \). The \( \mathcal{L}_{QCD} \) Lagrangian is not of interest for our present purposes. The quantum-electrodynamic Lagrangian \( \mathcal{L}_{QED} \) is of the form
\[ \mathcal{L}_{QED} = -\sum_f \bar{\psi}_f \left( \tau_\lambda p_\lambda + m_f \right) \psi_f \psi_f - \frac{1}{4} \left( \partial_\kappa A_\lambda - \partial_\lambda A_\kappa \right)^2 + e \sum_f Q_f \left( \bar{\psi}_f \gamma_\lambda \psi_f \right) A_\lambda , \tag{19} \]

where the explicitly shown sums are over all fermion species \( f \) (in the present case \( f = \mu^+, e^+ \)), with rest mass \( m_f \) and electric charge \( Q_f \) (in the units of \( e \)). The notation \( A_\lambda \) stands for the four components of the electromagnetic field \( A_\lambda = (A_0, -\mathbf{A}) \). The Greek letters \( \kappa \) and \( \lambda \) designate four-dimensional indices, taking on the values 0, 1, 2, 3. In these equations and Eq.(21) below the sum is assumed over any repeated Greek index and the summation sign will not be used in such cases.

The analytical expression for the decay rate \( \Gamma_\mu \) of positive muon \( \mu^+ \) follows from the Fermi \( V - A \) theory \cite{29}
\[ \Gamma_\mu = \frac{1}{\tau_\mu} = \frac{G_F^2 \cdot m_\mu^5}{192\pi^3} (1 + \Delta q) , \tag{20} \]

where \( G_F \) is the Fermi coupling constant, \( m_\mu \) is the muon rest mass and \( \Delta q \) is the corresponding relativistic correction \cite{29}. The current value of the Fermi constant \( G_F \) is \( (1.16637 \pm 0.00001) \cdot 10^{-5} \text{ GeV}^{-2} \) \cite{29} (see also \cite{30}, \cite{31} and references therein).

In general, the branching ratio of the muonium conversion is determined by the ratio \( R_g \) of the conversion \( G_C \) and Fermi coupling constants \( G_F \), i.e. \( R_g = \frac{G_C}{G_F} \). The conversion constant \( G_C \) appears in the effective Lagrangian \( \mathcal{L}_C \) for the \( \mu^+ e^- \rightarrow \mu^- e^+ \) conversion
\[ \mathcal{L}_C = \frac{1}{\sqrt{2}} G_C \left[ \bar{\psi}_\mu \gamma_\lambda (1 - \gamma_5) \psi_e \right] \left[ \bar{\psi}_e \gamma_\lambda (1 - \gamma_5) \psi_\mu \right] + h.c. \tag{21} \]
where ‘h.c.’ means the hermitian conjugate expression. The theoretically predicted ratio 

\[ R_g = \frac{G_C}{G_F} \]

is relatively small (see, e.g., [27]). The total probability of the \( \mu^+e^- \rightarrow \mu^-e^+ \) conversion can be approximately represented in the form [27], [28]

\[ P_c \approx 2.6 \cdot 10^{-5} \cdot \left( \frac{G_C}{G_F} \right)^2 \approx 2.34 \cdot 10^{-10} \quad , \]

where we used the most recent experimental value of the \( R_g \) ratio \( R_g \approx 0.0030 \) [32]. In fact, in [32] it was found that \( R_g < 0.0030 \). The results of other experiments in which \( R_g \) has been measured at different energies can be found in [33] (\( R_g \leq 0.14 \)) and in [34] (\( R_g \leq 0.008 \)). The evaluation which follows from Eq.(22) indicates that we can observe muonium conversion only in two MuPs systems of each 100 millions created in experiments.

VI. CONCLUSION.

The hyperfine structure splitting and annihilation of the electron-positron pairs in the ground bound state of muonium-positronium MuPs has been studied. Its is shown that the hyperfine splitting between singlet \( J = 0 \) and triplet \( J = 1 \) spin states in MuPs is \( \approx 22.958(10) \) MHz. We also consider the annihilation of electron-positron pairs in the MuPs system. The largest two-photon annihilation rate is \( \Gamma_{2\gamma} \approx 2.4522354(30) \cdot 10^9 \) sec\(^{-1}\). The numerical values of the three-, four- and five-photon annihilations are \( \Gamma_{3\gamma} \approx 6.643554(10) \cdot 10^6 \) sec\(^{-1}\), \( \Gamma_{4\gamma} \approx 3.6253(1) \cdot 10^3 \) sec\(^{-1}\) and \( \Gamma_{5\gamma} \approx 6.3446(1) \) sec\(^{-1}\), respectively. These values are accurate and based on the results of rigorous QED analysis, rather than on approximate relations used in our earlier work [13]. The rates of zero- and one-photon annihilations have been also determined for the MuPs system: \( \Gamma_{0\gamma}(\text{MuPs}) \approx \xi \cdot 9.1032(1) \cdot 10^{-8} \) sec\(^{-1}\) and \( \Gamma_{1\gamma}^{(1)}(\text{MuPs}) \approx 3.8706(1) \cdot 10^{-1} \) sec\(^{-1}\). The second one-photon annihilation rate \( \Gamma_{1\gamma}^{(2)}(\text{MuPs}) \) has not been evaluated in this study. The expression for zero-photon annihilation rate \( \Gamma_{0\gamma}(\text{MuPs}) \) also contains an unknown numerical factor \( \xi \) which must be derived from Quantum Electrodynamics.

Analogous annihilation rates have been evaluated for other positronium hydrides \( ^\infty \text{HPs}, \text{TPs}, \text{DPs and} \; ^1\text{HPs} \). Note that in our current computations we have used the variational expansions based on six-dimensional gaussoids [35] in which all non-linear parameters have been varied. The most recent version of this method includes a number of substantial improvements made in the optimization of the non-linear parameters and in overall accuracy.
and numerical stability of our procedure \cite{36,37}. Finally, we improved the results of previous studies performed for positronium hydrides. We also discuss the possibility to observe the muonium-antimuonium conversion in MuPs. It is shown that such a conversion transforms the incident four-body system $\mu^+ e^- e^+ e^- \rightarrow$ into its charge conjugate system $\mu^- e^- e^+ e^-$. It is expected that the new system $\mu^- e^- e^+ e^-$ will remain in the same bound state (the ground $1^1S_e$-state).

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