Universal Kounterterms in Lovelock AdS gravity

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Abstract

We show the universal form of the boundary term (Kounterterm series) which regularizes the Euclidean action and background-independent definition of conserved quantities for any Lovelock gravity theory with AdS asymptotics (including Einstein-Hilbert and Einstein-Gauss-Bonnet). We discuss on the connection of this procedure to the existence of topological invariants and Chern-Simons forms in the corresponding dimensions.

1 Introduction

Lovelock gravity [1] is the natural generalization of two basic features of General Relativity: general covariance and (at most) second-order field equations. It is also known to be free of ghosts when expanded on a flat space [2].

The introduction of (negative) cosmological constant leads to solutions that are not asymptotically flat. Therefore, the Euclidean bulk action and the conserved quantities evaluated on these solutions contain divergences that should be cured using a suitable regularization mechanism. The AdS/CFT correspondence provides a regularizing prescription for Einstein-Hilbert gravity coming from the holographic reconstruction of asymptotically AdS (AAdS) spacetimes in Fefferman-Graham coordinates [3, 4]. This procedure results into the addition to the bulk (plus the corresponding Gibbons-Hawking
term) of an intrinsic (Dirichlet) counterterms series $\mathcal{L}_{ct}$ [5, 6]. It is important to stress, however, that due to the increasing complexity of the algorithm to construct the counterterms in higher dimensions, holographic renormalization has been unable to provide a full expression for $\mathcal{L}_{ct}$ in any dimension. Moreover, as it can be seen in Einstein-Gauss-Bonnet gravity, the asymptotic resolution of the equations of motion in terms of a given data at the boundary turns out to be extremely difficult. These arguments motivate the search for another regularization scheme for Lovelock AdS gravity.

An alternative prescription for boundary terms which achieve the finiteness of both Euclidean action and conserved quantities in Einstein-Hilbert gravity with AdS asymptotics has been provided in [7, 8] as given polynomials in the extrinsic and intrinsic curvatures (Kounterterms). As the natural extension of what happens in Einstein-Gauss-Bonnet gravity [9], we show below that, whenever the effective cosmological constant is negative, the form of the Kounterterms is universal for all gravity theories of the Lovelock type [10].

## 2 Kounterterms in Einstein-Hilbert AdS gravity

AdS gravity action in $D = d + 1$ dimensions can be regularized using Kounterterms $B_d$, which are boundary terms with explicit dependence on the extrinsic curvature\footnote{Here, hatted curvatures refer to the ones of the bulk manifold. The cosmological constant is $\Lambda = -(D - 1)(D - 2)/2\ell^2$ in terms of the AdS radius $\ell$.}

$$I = \frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{-g} \left( \hat{R} - 2\Lambda \right) + c_d^{EH} \int_{\partial M} d^d x B_d$$

(1)

where $c_d^{EH}$ is a given coupling constant, fixed demanding a well-defined variational principle for AAdS spacetimes.

One of the simplest examples of the use of Kounterterms regularization is given by four-dimensional AdS gravity. Due to the fact that the Gauss-Bonnet term in four dimensions is a topological invariant, one has always the freedom to add it to the bulk action

$$I_4 = \int_M d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} \left( \hat{R} - 2\Lambda \right) + \alpha (\hat{R}^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu\rho\sigma} - 4\hat{R}^{\mu\nu} \hat{R}_{\mu\nu} + \hat{R}^2) \right],$$

(2)

with an arbitrary coupling constant $\alpha$. The presence of the Euler-Gauss-Bonnet term $\mathcal{E}_4$ does not modify the bulk dynamics.

In any gravity theory, the Wald’s entropy formula [11] amounts to taking derivatives of the bulk action with respect to the Riemann tensor and suitably projecting this quantity at the horizon. This method provides the correct value of the entropy for a large number of different gravity actions.
direct application of this prescription to Eq. (2) shifts the entropy by a constant $32\pi^2\alpha\chi(\Sigma_2)$, where $\chi(\Sigma_2)$ stands for the Euler characteristic of the two-dimensional transversal section.

However, an arbitrary coupling of the Euler term is inconsistent from the point of view of the variational principle. Indeed, the theory has a well-posed variational principle for AAdS spacetimes which satisfy the condition

$$\hat{R}^{\alpha\lambda} + \frac{1}{\ell^2} \delta^{\alpha\lambda}_\mu = 0$$

only if the coupling constant is chosen as $\alpha = \ell^2/(64\pi G)$ [12]. This simple argument produces the on-shell cancelation of divergences in the conserved quantities defined through the Noether theorem.

On the other hand, for an arbitrary value of $\alpha$, the Gauss-Bonnet invariant introduces additional divergent terms in the Euclidean action, e.g., for Schwarzschild-AdS black holes. The problem of a finite Euclidean action might be regarded as disconnected from the variational principle. This is what makes remarkable the fact that fixing $\alpha$ as above also provides a mechanism to regularize the Euclidean action for asymptotically AdS spacetimes and to reproduce the correct black hole thermodynamics [7].

Then, we see that the ambiguity present in Eq. (2) is removed by considering either the variational principle or the regularization problem in what might be called topological regularization of AdS gravity. However, the entropy is still modified by a constant which, in the case of topological static black holes is given by $\text{vol}(\Sigma^k_2)\ell^2 k/4G$, in terms of the volume of the two-dimensional cross-section $\Sigma^k_2$, which can be spherical, locally-flat or hyperbolic (for $k = +1, 0, -1$, respectively). Most of the thermodynamic quantities are insensitive to an additive constant in the entropy, but for $k = -1$ the entropy itself becomes negative for physical black holes ($r_+ < \ell$).

Negative entropy for hyperbolic black holes can be avoided considering, instead, the boundary term $B_3$ that is locally equivalent to the Gauss-Bonnet term, which is dictated by the four-dimensional Euler theorem

$$\int_{M_4} d^4x \ E_4 = 32\pi^2\chi(M_4) + \int_{\partial M_4} d^3x \ B_3,$$  

where $\chi(M_4)$ is the Euler characteristic of the four-dimensional manifold.

In Gauss-normal coordinates to describe the spacetime

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2(\rho)d\rho^2 + h_{ij}(x, \rho)dx^i dx^j,$$

the boundary $\partial M$ is defined by setting $\rho = \text{const}$. The term $B_3$ is a well-known object known as second Chern form, which is constructed using the second fundamental form (closely related to the extrinsic curvature), and whose explicit form in the coordinate frame (5) is

$$B_3 = 4\sqrt{-h} \delta^{[j_1 j_2 j_3]}_{[i_1 i_2 i_3]} K^i_{j_1} \left( \frac{1}{2} R^{i j_2 i_3}_{j_1 j_2} (h) - \frac{1}{3} K^{i j_2}_{j_1 j_3} K^{j_3}_{j_1} \right).$$

(6)
The above expression is a compact way of writing down a polynomial in the intrinsic curvature $R_{ij}^{kl}(h)$ associated to the boundary metric $h_{ij}$ and the extrinsic curvature $K_{ij} = -\frac{1}{2N} \frac{\partial h_{ij}}{\partial \rho}$, using the totally antisymmetrized Kronecker delta. Because the AdS action can be regularized using Eq.(6) that does not depend only on intrinsic tensors on $\partial M$, what we have is an alternative counterterm series [7].

The generalization to higher even dimensions ($D = 2n$) considers also the regularizing effect of the Euler term in the corresponding dimension, which is not longer quadratic in the curvature for $D > 4$ [13]. The boundary formulation equivalent to the bulk topological invariants is given by the $n$-th Chern form $B_{2n-1}$ which appears as the correction due to the boundary in the $2n$-dimensional Euler theorem

$$B_{2n-1} = 2n\sqrt{-h} \int_0^1 dt \delta^{[i_1 \cdots i_{2n-1}]} R_{j_1}^{i_1} \left( \frac{1}{2} R_{j_2 j_3}^{i_2 i_3} - t^2 K_{j_2 j_3}^{i_2 i_3} \right) \times \cdots \times \left( \frac{1}{2} R_{j_2n-j_2n-1}^{i_2n-i_{2n-1}} - t^2 K_{j_2n-j_2n-1}^{i_2n-i_{2n-1}} \right).$$  (7)

Then, the action (1) becomes finite if we adjust the coefficient of the Kounterterms as

$$\ell_{2n-1}^{EH} = \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{n (2n - 2)!} .$$  (8)

In the even-dimensional case one can also understand the Kounterterms as a transgression form (a gauge-invariant extension of a Chern-Simons density) for the Lorentz group $SO(2n-1,1)$ [14].

The absence of topological invariants of the Euler class in odd dimensions can make quite difficult the problem of finding a suitable Kounterterms series in this case. However, one is able to get some insight from the simplest case and the fact that three-dimensional AdS gravity can be formulated as a Chern-Simons density for the AdS group. Indeed, using a single copy of the Chern-Simons term for the group $SO(2,2)$, ones reproduces Einstein-Hilbert bulk action plus a boundary term

$$I = Tr(AdA + \frac{2}{3} A^3) = \frac{1}{16\pi G} \int_M d^3 x \sqrt{-g} \left( \hat{R} - 2\Lambda \right) - \frac{1}{16\pi G} \int_{\partial M} d^{d+1} x \sqrt{-h} K ,$$  (9)

which turns out to be half of the Gibbons-Hawking term. A well-posed action principle is achieved by imposing a given boundary condition on the extrinsic curvature, which is consistent to fixing the conformal data for the metric at the boundary [15]. In doing so, a single extrinsic boundary term is enough for the variational and the regularization problems. It can also be shown that this prescription is equivalent to the intrinsic regularization developed by Balasubramanian and Kraus [5] up to a two-dimensional topological invariant.

The generalization to higher odd-dimensional Einstein-Hilbert AdS gravity of the extrinsic regularization mentioned above is far from straightforward. However, an interesting hint is given by
higher dimensional Chern-Simons forms for the AdS group, which produce a bulk gravity action of the Lovelock type. By simply restoring Lorentz covariance at the boundary, one obtains a regularizing boundary term constructed with the second fundamental form [16, 17].

The Kounterterm series in $D = 2n + 1$ dimensions is written in a compact form thanks to a double parametric integration

$$B_{2n} = 2n \sqrt{-h} \int_0^1 dt \int_0^t ds \delta^{[j_1 \cdots j_{2n}]} (K_{j_1}^{i_1} \delta_{j_2}^{i_2} \left( \frac{1}{2} R_{j_3 j_4}^{i_3 i_4} - t^2 K_{j_3 j_4}^{i_3 i_4} + \frac{s^2}{l^2} \delta_{j_3}^{i_3} \delta_{j_4}^{i_4} \right) \times \cdots \times \left( \frac{1}{2} R_{j_2n-1 j_2n}^{i_2n} - t^2 K_{j_2n-1 j_2n}^{i_2n} + \frac{s^2}{l^2} \delta_{j_2n-1}^{i_2n} \delta_{j_2n}^{i_2n} \right) ) \right),$$

that, when written as a polynomial of the extrinsic and intrinsic curvatures, produces the relative coefficients of the terms in the expansion. The variational principle for AAdS spacetimes uniquely determine the weight factor of $B_{2n}$ as

$$c_{2n}^{EH} = \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{2^{2n-2} (n-1)!^2} ;$$

value that could have also been calculated by means of the cancelation of the highest-order divergences in the evaluation of the Euclidean action. Remarkably, all the subleading divergent contributions are also eliminated by the suitable choice of a single coupling constant [8].

### 3 Kounterterms in Einstein-Gauss-Bonnet AdS gravity

In higher dimensions than four, the Gauss-Bonnet term is not longer topological and thus, it contributes to the equations of motion. In fact, the field equations derived from the action

$$I = \frac{1}{16\pi G} \int_M d^{d+1} x \sqrt{-g} \left[ \hat{R} - 2\Lambda + \alpha (\hat{R}^{\mu \nu \sigma \rho} \hat{R}_{\mu \nu \sigma \rho} - 4 \hat{R}^{\mu \nu} \hat{R}_{\mu \nu} + \hat{R}^2) \right] + c_d^{EGB} \int_M d^d x B_d,$$

tell us that there are vacuum solutions which correspond to constant-curvature spacetimes with an effective AdS radius $\ell_{\text{eff}}$ modified by the Gauss-Bonnet coupling as

$$\frac{1}{\ell_{\text{eff}}^2} = \frac{1 \pm \sqrt{1 - 4(D - 3)(D - 4)/\alpha / \ell^2}}{2(D - 3)(D - 4)\alpha} .$$

It was shown in Ref.[9] that the regularization prescription given above for Einstein-Hilbert gravity with AdS asymptotics is still valid for the action (12). This means that the form of the Kounterterms in even and odd dimensions (Eqs. (7) and (10), respectively) is preserved, as long as we change the
AdS radius from $\ell$ to the effective one $\ell_{\text{eff}}$. The corresponding coupling constants are modified by the variational principle as

$$c_{EGB}^{2n-1} = \frac{1}{16\pi G} \frac{(-1)^n \ell_{\text{eff}}^{2n-2}}{n (2n-2)!} \left( 1 - \frac{2\alpha}{\ell_{\text{eff}}^2} (D - 2)(D - 3) \right),$$

for $D = 2n$ and

$$c_{EGB}^{2n} = \frac{1}{16\pi G} \frac{(-1)^n \ell_{\text{eff}}^{2n-2}}{2^{2n-2} n (n-1)!^2} \left( 1 - \frac{2\alpha}{\ell_{\text{eff}}^2} (D - 2)(D - 3) \right),$$

for $D = 2n + 1$.

For Einstein-Gauss-Bonnet gravity, the Dirichlet counterterms have been found only in five dimensions, which are constructed requiring general covariance and not using holographic renormalization techniques for AAdS spacetimes [18]. In turn, Kounterterms method provides a prescription which leads to a finite value for both conserved charges and Euclidean action in a background-independent fashion for all dimensions. A covariant formula for the vacuum energy for AAdS spacetimes has been derived, whose evaluation for topological static black holes generalize the corresponding result in Einstein-Hilbert gravity [6].

### 4 Kounterterms in Lovelock AdS gravity

In $D = d + 1$ dimensions, we will consider a regularized Lovelock action of the form

$$I = \frac{1}{16\pi G} \int_M \sum_{p=0}^{\lfloor (D-1)/2 \rfloor} \alpha_p L_p + c_d \int_{\partial M} d^d x \, B_d,$$

where $G$ is the gravitational constant in $D$ dimensions, $L_p$ corresponds to the dimensional continuation of the Euler term in $2p$ dimensions

$$L_p = \frac{1}{2p} \sqrt{-g} \delta_{[\mu_1 \ldots \mu_{2p}]} \hat{R}_{\nu_1 \nu_2}^{\mu_1 \mu_2} \ldots \hat{R}_{\nu_{2p-1} \nu_{2p}}^{\mu_{2p-1} \mu_{2p}} d^D x,$$

and $\{\alpha_p\}$ is a set of arbitrary coefficients.

In Riemannian gravity, the equation of motion for a generic Lovelock gravity is obtained varying with respect to the metric and takes the form

$$E^\nu_\mu = \sum_{p=0}^{\lfloor (D-1)/2 \rfloor} \frac{\alpha_p}{2p} \delta_{[\mu_1 \ldots \mu_{2p}]} \hat{R}_{\nu_1 \nu_2}^{\mu_1 \mu_2} \ldots \hat{R}_{\nu_{2p-1} \nu_{2p}}^{\mu_{2p-1} \mu_{2p}} = 0.$$
For a given set of coefficients \( \{\alpha_p\} \), the vacua of a Lovelock theory are defined as the maximally symmetric spacetimes that are globally of constant curvature. We will assume that all the corresponding cosmological constants are real and negative, with different effective AdS radii \( \ell_{\text{eff}} \) given by the solutions to the equation

\[
\frac{1}{[(D-1)/2]} \sum_{p=0}^{\infty} \frac{\alpha_p}{(D-2p-1)!} (-\ell_{\text{eff}}^{-2})^p = 0. \tag{19}
\]

If we consider spacetimes which approach asymptotically to any of these constant-curvature vacuum solutions, we can prove that the Kounterterms regularization is universal, as the explicit expression for the series keeps the form of Eqs. (7) and (10) (in even and odd dimensions, respectively) for any Lovelock AdS theory [10]. Once again, we only have to pass from the original AdS radius \( \ell \) to the effective one \( \ell_{\text{eff}} \) given above. The finiteness of the Euclidean action and Noether charges is achieved if and only if the coupling constant are consistently adjusted as

\[
c_{2n-1} = -\frac{1}{16 \pi n G} \sum_{p=1}^{n-1} \frac{(-1)^{n-p} p \alpha_p}{(2n-2p)!} \ell_{\text{eff}}^{2(n-p)}, \tag{20}
\]

and

\[
c_{2n} = -\frac{1}{16 \pi G} \frac{(2n-1)!!}{2^{n-1} n!} \sum_{p=1}^{n} \frac{(-1)^{n-p} p \alpha_p}{(2n-2p+1)!} \ell_{\text{eff}}^{2(n-p)}, \tag{21}
\]

for even and odd (bulk) dimensions, respectively. As the Kounterterms provides a method to get rid of all the divergences in a background-independent way for Lovelock AdS gravity, it is natural the appearance of a vacuum energy in \( D = 2n + 1 \) dimensions for globally AdS spacetime, whose form generalizes the result of Einstein-Hilbert and Einstein-Gauss-Bonnet theories.

### 5 Conclusions

We have argued that the regularization of any gravity theory of the Lovelock type can be carried out using boundary terms which are a given polynomial in the extrinsic and intrinsic curvatures (Kounterterms), whose form is universal. Indeed, the only difference appears at the level of the dimensionality, because even and odd dimensions must be treated separately. In this context, this fact is simply related to the existence of topological invariants of the Euler class in even dimensions. This difference is not so surprising, because in standard holographic renormalization there appear technical differences between these two cases (existence of vacuum energy and Weyl anomaly).

In the Dirichlet problem for gravity, one has to supplement the bulk action with the Gibbons-Hawking term such that the variations of the extrinsic curvature are canceled out. In Lovelock gravity, the Myers procedure leads to generalized Gibbons-Hawking terms for the same purpose [19].
In this way, one understands why standard counterterms can be only constructed up with covariant quantities of the metric $h_{ij}$. However, the holographic interpretation of a regularized stress tensor associates it to a metric $g_{(0)ij}$ at $\partial M$ which, on the contrary to $h_{ij}$, is regular at the boundary.

In the Fefferman-Graham frame, the leading order of the expansion in the extrinsic curvature $K_{ij}$ is the same as the one of the boundary metric $h_{ij}$. This seems to explain why fixing the extrinsic curvature instead of the metric one can still produce a finite action principle for AdS gravity [15].

A direct comparison between intrinsic counterterms and the Kounterterms prescription has been given in Ref. [20] for two particular Lovelock theories which feature a symmetry-enhancement from Lorentz to AdS group, and where the first problem is exactly solvable [21]. This indicates that an explicit comparison of both procedures might be possible also in Einstein-Hilbert and Einstein-Gauss-Bonnet theories with AdS asymptotics.

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