THE ASYMPTOTICAL BEHAVIOR OF SPECTRAL FUNCTION
OF ONE FAMILY OF DIFFERENTIAL OPERATORS.

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In $L_2[0, +\infty)$ consider operator $l$, defined by differential expression:

$$ly(x) = -y''(x) + q(x)y(x)$$

and boundary condition

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0.$$

Assume that $\alpha \in \mathbb{R}$, $q$ is a function continuous on $[0, +\infty)$, and having real values. Denote by $\varphi(x, \lambda)$ and $\theta(x, \lambda)$ the solutions to equation $ly = \lambda y$ with initial conditions

$$\varphi(0, \lambda) = \sin \alpha, \quad \varphi'(0, \lambda) = -\cos \alpha, \quad \theta(0, \lambda) = \cos \alpha, \quad \theta'(0, \lambda) = \sin \alpha.$$

Functions $\varphi, \theta$ and $\rho, m, \hat{f}$ (we will introduce them later) depend, naturally, on $\alpha$, but we will omit the argument $\alpha$, for not to overload the text with notations.

It is well known the Weyl theorem on representation of arbitrary function $f \in L_2[0, +\infty)$ as an integral along the spectrum of operator $L$.

**Theorem [1-3].** There exists such a non decreasing and bounded from below on $\mathbb{R}$ function $\rho$, for which the following statements take place:

1. \( \exists \hat{f}(\lambda) = \lim_{n \to \infty} \int_{-n}^{n} f(x) \varphi(x, \lambda) dx = \int_{0}^{\infty} f(x) \varphi(x, \lambda) dx, \)

   and the function $\hat{f}$ is considered as a limit in $L_2[0, +\infty)$ of "Fourier transform by measure $d\rho$" of $f$:

   $$f(x) = \lim_{n \to \infty} \int_{-n}^{n} \hat{f}(\lambda) \varphi(x, \lambda) d\rho(\lambda) = \int_{-\infty}^{\infty} \hat{f}(\lambda) \varphi(x, \lambda) d\rho(\lambda).$$

2. If $\sin \alpha \neq 0$, then for function $m(z)$ defined in half-plane $\text{Im } z > 0$ by the equality

   $$m(z) = -\text{ctg } \alpha + \int_{-\infty}^{+\infty} \frac{d\rho(\lambda)}{z - \lambda}, \quad (1)$$

   we have:

   $$\theta(x, z) + m(z) \varphi(x, z) \in L_2[0, +\infty) \quad \forall z : \text{Im } z > 0.$$
3. For an arbitrary Borel set \( E \subset \mathbb{R} \), not including the spectrum of operator \( L \), we have:
\[
\int_E d\rho(\lambda) = 0.
\]

The statement 3 of Theorem shows, that the measure \( d\rho \) is concentrated on the spectrum of operator \( L \). The function \( \rho(\lambda) \) is called the spectral function, and \( m(z) \) is called the function of Weyl-Titchmarsh of operator \( L \). It is easy to see from the representation (1) that the function \( m(z) \) is analytical in the upper half-plane and that if on certain interval \((a, b) \subset \mathbb{R}\) there exists
\[
\lim_{y \to +0} m(\lambda + iy) = m(\lambda) \in C(a, b),
\]
then \( \rho(\lambda) \) has on this interval continuous derivative, which is connected to \( m(\lambda) \) by the correspondence:
\[
\rho'(\lambda) = -\frac{\text{Im} m(\lambda)}{\pi}, \lambda \in (a, b).
\]

The investigation of properties of function \( p(\lambda) \), and its definition via the potential \( q \) and the number \( \alpha \), defining the operator \( L \), is, as a rule, a very difficult problem.

There are well known the general theorems of B.M.Levitan and V.A.Marchenko (see [4]—[6] on evaluation of \( \rho(\lambda) \) as \( \lambda \to \pm \infty \):
\[
\rho(\lambda) - \rho(-\infty) = o(\exp(-a \sqrt{|\lambda|})), \ (\lambda \to -\infty) \ \forall a > 0, \quad (3)
\]
\[
\rho(\lambda) = \frac{2\sqrt{\lambda}}{\pi \sin^2 \alpha} + \rho(-\infty) + \frac{\cos \alpha}{\sin^2 \alpha} + o(1) \ (\lambda \to +\infty). \quad (4)
\]

(it is assumed, that the norming \( \rho(0) = 0 \) is performed.)

The simplest case of \( q(x) \equiv 0 \) was studied by Titchmarsh in [7]. It turned out, that for \( \lambda > 0 \)
\[
\rho'(\lambda) = \frac{\sqrt{\lambda}}{\pi(\lambda \sin^2 \alpha + \cos^2 \alpha)},
\]
and for \( \lambda < 0 \) both of the cases \( \text{ctg} \alpha > 0 \) and \( \text{ctg} \alpha < 0 \) are strongly different:
\[
\rho'_\alpha(\lambda) \equiv 0, \ \lambda < 0, \ \text{ctg} \alpha < 0,
\]
\[
\rho'_\alpha(\lambda) = \frac{2\text{ctg} \alpha}{\sin^2 \alpha} \delta(\lambda - \lambda_0), \ \lambda < 0, \ \text{ctg} \alpha > 0,
\]
where \( \lambda_0 = -\text{ctg}^2 \alpha \), \( \delta \) is delta-function of Dirac.

In this work we study the case of \( \text{ctg} \alpha > 0 \) and analyse the behavior of the derivative of spectral function on \((-\infty, 0)\) for potentials of special type.

The following problem of common form is interesting for our study:

Let \( Q \in C(0, +\infty) \). Denominate as \( \rho_\alpha(\varepsilon, Q, \lambda) \) the spectral function of the operator, defined by differential expression:
\[
-\frac{d^2}{dx^2} y(x) + \varepsilon Q(x)y(x)
\]
and boundary condition
\[
y(0) \cos \alpha + y'(0) \sin \alpha = 0, \ \text{ctg} \alpha > 0.
\]
Is it true that on \((-\infty, 0)\) the family of distributions \(\rho'_\alpha(\varepsilon, Q, \lambda)\) converges as \(\varepsilon \to 0\) (in any "reasonable" sense) to the derivative of spectral function \((4)\) of non-perturbed operator \(-y''\)? The positive solution of this problem would allow us to write for functions \(f \in L_2[0, +\infty)\) the "approximate spectral expansion" of the following kind:

\[
f(x) = \int_{\lambda_0+\beta}^{+\infty} + \int_{\lambda_0-\beta}^{-\beta} \hat{f}(\lambda) \varphi_\alpha(x, \lambda) d\rho_\alpha(\varepsilon, Q, \lambda)
\]

\((\beta = \beta(\varepsilon) \downarrow 0)\), omitting the integral on the complement to small neighborhood of the spectrum of non-perturbed operator. As we know, such investigations have not been performed before.

Titchmarsh in \([7]\) have expressed through Gankel functions of the first type the Weyl-Titchmarsh function \(m(\lambda, \varepsilon)\) of operator

\[
yy'' - \varepsilon xy, \ v(0) \cos \alpha + y'(0) \sin \alpha = 0, \ \ctg \alpha > 0.
\]

He proved, that for all \(\lambda \in \mathbb{R}\)

\[
m(\lambda, \varepsilon) = \frac{H^{(1)}_{1/3}(A) \sin \alpha - \sqrt{A} H^{(1)}_{-2/3}(A) \cos \alpha}{H^{(1)}_{1/3}(A) \cos \alpha + \sqrt{A} H^{(1)}_{-2/3}(A) \sin \alpha},
\]

where \(A = \frac{2\lambda^{3/2}}{\varepsilon}, \lambda^{3/2} = -i|\lambda|^{3/2}\) with \(\lambda < 0, H^{(1)}_{\rho}\) are Hankel functions of the first type.

Using the representation \((6)\) it was proved in \([6]\) , that in certain vicinity of the point \(\lambda_0\) the functions \(m(\lambda, \varepsilon)\) have only one pole \(\lambda_0(\varepsilon)\) with the asymptotics

\[
\lambda_0(\varepsilon) = \lambda_0 - \frac{\varepsilon}{2} \tan \alpha + O(\varepsilon^2), \ (\varepsilon \to +0),
\]

and its imaginary part satisfies the inequality:

\[
-\exp\left(-\frac{\tan^3 \alpha}{\varepsilon}\right) \leq \Im \lambda_0(\varepsilon) < 0.
\]

But the behavior of \(\rho'(\lambda, \varepsilon)\) as \(\lambda \in (-\infty, 0)\) and \(\varepsilon \to +0\) was not studied by Titchmarsh. We could solve this problem in certain sence. In spite of the existence of explicit formula for \(\rho'(\lambda, \varepsilon)\) (taking into account \((2)\) we have \(\rho'(\lambda, \varepsilon) = -\frac{\Im m(\lambda, \varepsilon)}{2}\), due to the fact that from \((6)\) it follows, that the function \(m(\lambda, \varepsilon)\) is analytical in closed upper half-plane except one point \(0\) this was not easyly to realize. The matter is that in putting in \((6)\) known asymptotic series for \(H^{(1)}_{1/3}(A)\) and \(H^{(1)}_{-2/3}(A)\) as \(A \to -i\infty\) (along negative part of imaginary axis), then for \(m(\lambda, \varepsilon)\) we obtain the asymptotic series, all members of which are real. Therefore, mentioned series does not give information concerning \(\rho'(\lambda, \varepsilon)\).

We have obtained the following representation for function \(\rho'(\lambda, \varepsilon)\), which takes place for any \(\lambda < 0\) and \(\varepsilon > 0\):

\[
\rho'(\lambda, \varepsilon) = \frac{\pi^{-1} \tau(\beta_{1,1}(a) \beta_{2,2}(a) + \beta_{1,2}(a) \beta_{2,1}(a))}{(\beta_{1,1}(a) \cos \alpha - \beta_{1,2}(a) \tau \sin \alpha)^2 + (\beta_{2,1}(a) \cos \alpha + \beta_{2,2}(a) \tau \sin \alpha)^2},
\]

\((7)\).
where $\tau = \sqrt{-\lambda}$, $a = |A| = \frac{2x^2}{\Omega}$,

$$
\beta_{1,1}(a) = 1 + \frac{3}{2} \frac{B_{1/3}(a)}{\Omega_{1/3}(a)}, \quad \beta_{1,2}(a) = \frac{\Omega_{2/3}(a)}{\Omega_{1/3}(a)} + \frac{3}{2} \frac{B_{2/3}(a)}{\Omega_{1/3}(a)},
$$

$$
\beta_{2,1}(a) = \frac{B_{1/3}(a)}{2\Omega_{1/3}(a)}, \quad \beta_{2,2}(a) = \frac{B_{2/3}(a)}{2\Omega_{1/3}(a)},
$$

$$
\Omega_p(a) = \frac{1}{\pi} \int_0^\pi \exp(a \cos t) \cos(pt)dt,
$$

$$
B_p(a) = \frac{1}{\pi} \int_0^{+\infty} \exp(-a \sin t) \sin(pt)dt,
$$

$$
B^p(a) = \frac{1}{\pi} \int_0^{+\infty} \exp(-a \sin t) \sin(pt)dt.
$$

The representation (7) allows us to prove the following theorems 1-3:

**Theorem 1.** For any $\varepsilon > 0$ the following asymptotics takes place:

$$
\rho'(\lambda, \varepsilon) \sim \frac{\exp\left(-\frac{4|\lambda|^{3/2}}{3\varepsilon}\right)}{\pi \sqrt{-\lambda} \sin^2 \alpha}, \quad (\lambda \to -\infty).
$$

**Theorem 2.** There exist positive constants $c_1$ and $c_2$, effectively dependent on $\alpha$, such that with $\varepsilon \in (0, c_1)$ take place evaluations:

$$
\rho'(\lambda, \varepsilon) = O(|\lambda|^{-1/2} \exp\left(-\frac{4|\lambda|^{3/2}}{3\varepsilon}\right)), \quad \lambda \in (-\infty, -2 \cot^2 \alpha),
$$

$$
\rho'(\lambda, \varepsilon) = O(|\lambda|^{1/2} \exp\left(-\frac{4|\lambda|^{3/2}}{3\varepsilon}\right)), \quad \lambda \in (-\frac{1}{2} \cot^2 \alpha, -c_2 \varepsilon^{2/3}),
$$

$$
\rho'(\lambda, \varepsilon) = O(\varepsilon^{1/3}), \quad \lambda \in [-c_2 \varepsilon^{2/3}, 0).
$$

Constants in symbols $O$ depend only on $\alpha$ and are effective.

Denote $T(\lambda, \varepsilon) = \beta_{1,1}(a) \cos \alpha - \beta_{1,2}(a) \sin \alpha$. The answer to the question on the behavior of $\rho'(\lambda, \varepsilon)$ on the "critical" segment $I = [-2 \cot^2 \alpha, -\frac{1}{4} \cot^2 \alpha]$ gives the following

**Theorem 3.** There exists such a positive constant $c_3$, depending effectively on $\alpha$, that for any $\varepsilon \in (0, c_3)$ the following statements take place:

1) Function $T(\lambda, \varepsilon)$ has on segment $I$ unique zero $\lambda_1(\varepsilon)$ with asymptotics:

$$
\lambda_1(\varepsilon) = \lambda_0 - \frac{\varepsilon}{2} \cot \alpha + O(\varepsilon^2).
$$

2) For any $d > 0, \lambda \in I \setminus I_{d, \varepsilon}$ ($I_{d, \varepsilon} = [\lambda_1(\varepsilon) - d, \lambda_1(\varepsilon) + d]$) the evaluation takes place:

$$
\rho'(\lambda, \varepsilon) = d^{-2} \exp\left(-\frac{4|\lambda|^{3/2}}{3\varepsilon}\right).
$$

3) If $\exp\left(-\frac{\cot^2 \alpha}{3\varepsilon}\right) \leq d \leq \varepsilon^2$, then

$$
\int_{I_{d, \varepsilon}} d\rho(\lambda) = \frac{2 \cot \alpha}{\sin^2 \alpha} + O(\varepsilon).
$$
Corollary. In space $C^*(I)$ the family of functions $\rho'(\lambda, \varepsilon)$ converges weakly to
$$\frac{2\text{ctg} \alpha}{\min_{\lambda} \delta(\lambda - \lambda_0)}$$ as $\varepsilon \to +0$.

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