Efficient quantum interpolation of smooth distributions

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We present an efficient method to interpolate smooth distributions with a quantum computer that aims to complement data uploading techniques. The quantum algorithm operates in four steps: i) a small sample of the distribution is uploaded onto a reduced set of qubits; ii) a Quantum Fourier Transform (QFT) is applied; iii) qubits encoding vanishing high frequencies are added to the register; and iv) an inverse QFT is operated on the total register. This leverages on the efficiency of the QFT and opens the door of quantum advantage to a relevant family of distributions. We demonstrate the power of the method with an efficient interpolation of a toy gaussian model, discussing uploading methods and precision. We then apply the QFT interpolation technique to quantum encoded images and present further advantages when processing data in superposition.

I. INTRODUCTION

Uploading a large amount of classical data onto a quantum state remains a bottleneck for quantum computation applications. Quantum states indeed offer a large Hilbert space to encode classical data, but uploading one element at a time makes a quantum approach inefficient, no matter its promises regarding processing power on the uploaded state. New strategies are needed to overcome the initial threshold of uploading data into a quantum state.

To this end, we present a method to interpolate smooth probability distributions over a larger space that can alleviate the data uploading effort, dramatically in some cases. The basic idea of this algorithm builds on classical resampling techniques that employ the Fourier Transform to interpolate band-limited signals. The original amplitude encoded distribution is first Fourier transformed, then complemented with high frequencies and, finally, an inverse Fourier transform over the larger space delivers the interpolated probability distribution.

To be precise, the relevant fact that makes this approach useful when interpolating classical distributions in a quantum computer is that the Quantum Fourier Transform (QFT) is efficient, that is, it only needs a polynomial number of gates as a function of the number of qubits involved.

This interpolation method can also be understood as a discrete-to-discrete instance of the ideas put forward in the Nyquist-Shannon sampling theorem. The sampling theorem states that all the information of a signal with finite band-width can be captured by samples obtained at a finite rate, known as Nyquist rate. We incorporate these ideas when discussing the accuracy of the proposed QFT resampling technique. The use of Fourier transforms is ubiquitous when reconstructing continuous signals from samples. The QFT can introduce efficient counterparts to techniques used in classical signal processing, and we present here an example in the context of data interpolation.

We will detail the QFT interpolation of classical probability distributions using a simple example of a gaussian toy model. We then apply it to resample images which are quantum encoded.

The Fourier interpolation technique has also been adapted with success in the context of Tensor Networks in Ref. [1].

II. INTERPOLATING A DISTRIBUTION

We showcase here the basic algorithm for interpolating an uploaded distribution to higher precision. The algorithm is detailed, then discussed in the context of enhancing quantum data uploading techniques. Afterwards, we discuss the accuracy of the interpolated distribution.

Given a quantum register $q$, with individual qubits $q_i$, from 0 to $n-1$, where a distribution $P$ has been amplitude encoded into a $2^n$ discrete version $P_m$. QFT interpolation proceeds as follows. Apply the QFT to the quantum register $q$. In Fourier space, the high-frequency modes of smooth distributions will be suppressed, or zero in the case of band-limited signals. Therefore, one can artificially pad the high frequency components with quantum states at zero amplitude and not fundamentally alter the composition of the original signal. This can be achieved on a quantum circuit by adding an ancillary qubit register $a$, with qubits $a_j$, from 0 to $m-1$, between original qubits $q_0$ and $q_1$ (the first and second most significant qubits) and then applying CNOT gates, controlled by $q_0$, targeting all qubits in the ancilla register $a$. Finally, an inverse QFT is applied to the entire quantum system.

The outcome of this circuit is the interpolation from the initial $2^n$ probability distribution to a larger $2^{n+m}$ space. Fig. 1 a) and c) showcase the interpolation of the original 4 qubit distribution to a larger 7 qubits space, and implemented using the quantum simulation library Quibo [2, 3]. In Fig. 1 b) we present the QFT interpolation algorithm on a quantum circuit.

The QFT interpolation algorithm for probability distributions is efficient. That being said, the cost of up-
loading the initial distribution into the small space needs to be taken into account. There are multiple proposals for uploading a probability distribution into a quantum register, some exact [4–9] and some training a quantum generator circuit [10–12]. If the cost of the initial uploading is already prohibitive this algorithm will only provide a marginal advantage. Still, this technique opens the door for uploading methods that are effective for a small number of qubits that can later be enhanced via QFT interpolation.

We showcase one such case in App. A, where we use the QFT interpolation algorithm to enhance a probability distribution uploaded in the unary basis [13]. This uploading technique trades practical scalability for a device-friendly uploading method that is effective for a small number of amplitudes. After the original distribution is uploaded in unary using $2^n$ qubits, it is transformed into binary to $n$ qubits, where the full available $2^n$ Hilbert space is reclaimed via QFT interpolation.

In the following, we aim to establish the accuracy of the interpolated distribution when compared to an ideal uploading of the underlying distribution to the larger space.

We will bound the operational distinguishability between the interpolated quantum state and the ideal one by introducing their trace-distance,

$$\text{dist}_\text{Tr}(|\psi\rangle, |\phi\rangle) = \frac{1}{2} \| \langle \psi | - | \phi \rangle \|_1,$$

(1)

where $\|A\|_1 = \text{Tr} \sqrt{A^\dagger A}$. However, we first proceed by analyzing the $\ell_2$ norm difference between them. The reason for this choice is twofold. Quantum states are normalized under their $\ell_2$ norm, $\| |\psi\rangle \|_{\ell_2} = 1$ where $|\psi\rangle$ is any pure quantum state, and the QFT preserves this norm, that is $\| |\psi\rangle - |\phi\rangle \|_{\ell_2} = \| |\Psi\rangle - |\Phi\rangle \|_{\ell_2}$ with $|\Psi\rangle = QFT |\psi\rangle$ and $|\Phi\rangle = QFT^\dagger |\phi\rangle$. From their $\ell_2$ distance measure we can bound the trace-distance between the two states by employing the Fuchs-van-de-Graaf inequality [14], which is tight for pure states [15], when we restrict ourselves to states that amplitude-encode probability distributions, see App. B. From here on out, all norms are assumed to be $\ell_2$ unless stated otherwise.

We aim to upload a target quantum state $|\psi\rangle$ into $n+m$ qubits by using QFT interpolation on state $|\psi\rangle$, originally uploaded to $n$ qubits, which will be referred to as an $n$-qubit band-limited state. An $n$-qubit band-limited state refers to a quantum state with any number of qubits that only has $2^n$ non-zero Fourier components. More precisely, a state with initial distribution $P_n$ discretized over a span $\Delta x_n$, will only have non-zero Fourier components within the band $2n/\Delta x_n$.

Analogous to the Nyquist-Shannon sampling theorem [16, 17], if the target state $|\psi\rangle$ is $n$-qubit band-limited, the QFT interpolation technique is able to capture all the information of the underlying distribution, and the interpolation is perfect. However, that will not always be the case. Then, the best possible initial state $|\psi\rangle$ to interpolate will be the one minimizing $\| |\Psi\rangle - |\Phi\rangle \|^2 = \| |\Psi\rangle - |\Psi\rangle \|_1$. Classically, the best band-limited approximation is realized by a convolution of the original signal with the sinc function [18], analogous to applying a low-pass filter to the signal, so that the high frequencies are cut off while maintaining the band-limited frequencies intact. This, however, is not as straightforward with quantum states, as they have to maintain their $\ell_2$ norm throughout the process, and a low-pass filter is not a unitary operation. Therefore, see App. C for the detailed derivation, the optimal distance between states will be

$$\| |\Psi\rangle - |\Psi\rangle \|^2 = \frac{2\| |\Psi_{\text{out}}\rangle \|^2}{1 + \sqrt{1 - \| |\Psi_{\text{out}}\rangle \|^2}} \leq 2\| |\Psi_{\text{out}}\rangle \|^2,$$

(2)
where $|\Psi_{\text{out}}\rangle$ denotes the Fourier components of target state $|\psi\rangle$ outside of the $n$-qubit band-limit. The distance between the target state and the best possible interpolation will depend on the $L_2$ norm of the Fourier components that are filtered out. Using this result, we can bound the trace-distance between the target and interpolated state by

$$\text{dist}_{\text{Tr}}(|\psi\rangle, |\tilde{\psi}\rangle) \leq \sqrt{2}|||\Psi_{\text{out}}||\sqrt{1 - \frac{|||\Psi_{\text{out}}||^2}{2}}$$

$$\leq \sqrt{2}|||\Psi_{\text{out}}||.$$  \hspace{1cm} (3)

However, many applications do not have access to the the filtered distribution, as only the low-resolution values of the original distribution are available. Then, the interpolated state will suffer from aliasing effects, where the Fourier components of subsampled distributions are mixed with its own high-frequency modes due to the periodic nature of the discrete Fourier transform. In other words, the frequency components outside the $n$-qubit band limit, $|\Psi_{\text{out}}\rangle$, are added to the low frequency components, which we will call $|\Psi_{\text{in}}\rangle$, creating artifacts in the interpolated distribution.

The accuracy of the QFT interpolation algorithm will be worse in this direct approach, but we can still provide analytical bounds on the distinguishability due to the effects of aliasing. Now the distance between the target and interpolated state will be, refer to App. D for details,

$$|||\Psi\rangle - |\tilde{\Psi}\rangle||^2 = \frac{2|||\Psi_{\text{out}}||^2}{N} - \frac{(N - 1)^2}{N}$$

$$\leq 2|||\Psi_{\text{out}}||^2,$$  \hspace{1cm} (4)

where $N \geq 1$ is the normalization factor of the target state $|\tilde{\Psi}\rangle$ due to the effects of aliasing. Therefore, the upper bound on the trace-distance under aliasing effects remains the same as Eq. 3.

We have shown how the QFT interpolation algorithm approaches the target quantum state for a probability distribution. The smaller the norm of the high-frequency components, the better the interpolation will be. Moreover, when the distribution is $n$-qubit band-limited, that is $|||\Psi_{\text{out}}|| = 0$, it captures all the information of the underlying function and can be interpolated with as many qubits as needed. Even for non-band-limited distributions, we show in Fig. 1 that with an initial uploading of 4 qubits, a Gaussian distribution can be interpolated with high fidelity to an exponentially larger space.

### III. RESAMPLING IMAGES

Interpolation methods, referred to also as resampling, are very common techniques in image processing. Moreover, natural images tend to have suppressed high-frequency components. This is used in some of the most prevalent image processing techniques [19], where Fourier or Fourier-like [20] transforms are one of the main components. The field of quantum image processing has experienced a lot of progress during the recent years [21–25] due to the efficiency of such transforms in quantum circuits. Therefore, resampling quantum encoded images using the QFT interpolation algorithm seems a natural step forward to generalize the technique to more dimensions.

Encoding images into a quantum state is a costly task. Most algorithms that attempt to upload a classical image into a quantum state [22, 23] inevitably scale with the pixel count of the image. This encoding techniques quickly become prohibitive when large images need to be uploaded. The QFT interpolation algorithm can alleviate part of this complexity, by applying QFT interpolation to resample an image in an efficient way after an initial uploading has taken place. This method also supports other uploading methods that rely on machine learning or quantum sensing techniques.

The QFT interpolation algorithm can be extended to two, and up to any, dimensions. The image data that we want to process requires the following amplitude encoding. For two dimensions, values labelled by $(x, y)$ will be encoded into the amplitude of the quantum state $|x\rangle|y\rangle$ in the computational basis of registers $q_x$ and $q_y$. If this encoding is realized, applying the QFT interpolation protocol to both $q_x$ and $q_y$ registers separately (and simultaneously) achieves the 2-dimensional discrete Fourier transform over the data.

**Grayscale images.** The quantum resources needed to encode and interpolate a grayscale image with $2^n \times 2^n$ pixels using the proposed technique is $2n$ qubits, plus constant ancilla depending on the uploading method, and $2m$ ancillary qubits used to enlarge the image to $2^{n+m} \times 2^{n+m}$ pixels. An example of an interpolated image using this method is displayed on Fig. 2 (left). A $512 \times 512$ image (top left) has been enlarged four times in both axis. This implementation requires $2 \times 9 + 2 \times 2 = 22$ qubits, omitting encoding ancillas, to transform the original image to the enlarged space and has been simulated using the quantum simulation library Qibo using images available in the image processing library scikit-image [26].

**Multi-layer images.** We highlight here a genuinely quantum speed-up that arises when working with multi-layer data. By properly encoding the multiple layers, we can process all the quantum data with a single call to the QFT interpolation algorithm. Starting from the same encoding technique used for a single image, each layer $l_i$ of the image will be labelled by state $|l_i\rangle$ of an new label quantum register $q_l$, with $\lceil \log(l) \rceil$ qubits, where $l$ is the total number of layers. That is, the value of pixel $(x, y)$ of layer $l_i$ will be encoded in the amplitude of the quantum state $|x\rangle|y\rangle|l_i\rangle$.

With a quantum image encoded this way, applying the QFT interpolation algorithm in the same way as one would for gray-scale images, that is acting on quantum registers $q_x$ and $q_y$ only, will perform the interpolation
Figure 2. Example of the QFT resampling technique on images. (left) Grayscale image enlarged four times using the two dimensional QFT resampling technique. (right) RGB image enlarged four times using the two dimensional QFT resampling technique, ancillary qubits are used to label each color layer. Both quantum circuits require the same gate complexity regardless of the layers of the image.

to all layers of the image in superposition, at no extra quantum cost. Since every pixel value is now entangled with its label, the amplitude interference that makes the QFT possible (and efficient in the quantum setting) will only act on the amplitudes of pixel states that share the same label state. We showcase an example in Fig. 2 (right), where we interpolate an RGB image using two qubits to keep track of the color channel. Since the pixel dimensions are the same as Fig. 2 (left), the depth of the quantum circuit required for interpolation is the same for both instances.

The speed-up provided by acting on all layers in superposition is made more apparent the more layers, or data instances sharing the same shape, that are encoded in the proposed way. Additionally, further quantum advantage can be achieved when implementing quantum transformations to all subsets within an image at the same time. Beyond that, this can be extended to other types of data uploaded in superposition. If the initial cost of uploading data using a label register is overcome, any quantum processing using a transformation such as the QFT can be applied to all members of the superposition in singular cost.

IV. CONCLUSION

We have proposed a quantum technique to interpolate distributions using the efficient Quantum Fourier Transform. We show that the QFT interpolation technique achieves favourable results on data with negligible high-frequency components.

This technique can be used to enhance current distribution uploading algorithms by focusing on smaller scale and accurate uploading techniques that can then be efficiently resampled via the QFT interpolation algorithm. It can also be extended to any uploading technique that deals with band-limited quantum signals, even without a classical counterpart. Additionally, this interpolation technique can be used on genuinely quantum states, but the interpolation will be extended along computational basis states which will fail to capture correlations that go beyond that.

We have also showcased the power of processing quantum data in superposition. Here, encoding data into a quantum superposition by using label ancillas allows for the implementation of quantum transformations in parallel. This yields a substantial speed-up when the same transformation is required in different data sets.

We would like to further highlight the implementation of efficient quantum transformations in order to provide quantum advantages to algorithms that might not initially rely on them. After all, at the core of the efficiency of Shor’s factorization algorithm [27] is the use of the QFT. Looking into areas where these types of transformations are extensively used may result in further avenues for quantum advantage. Image and signal processing are fields that have evolved around such transformations, further work might borrow from such well understood fields in order to enhance quantum algorithms in different ways, or explore interpolation techniques with more advanced transformations.

The code used to simulate the quantum circuits presented is available online [28].

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Appendix A: Enhancing a unary uploading

We illustrate how the QFT resampling algorithm can enhance uploading techniques using the amplitude distribution in the unary basis presented in Ref. [13]. This proposal requires only nearest-neighbor connectivity and employs gates that are well suited for implementation on near-term quantum devices. The distribution is uploaded in the unary basis where only one qubit is in state $|1\rangle$ while the others remain at $|0\rangle$, i.e., $|10000\rangle, |01000\rangle, \ldots, |00001\rangle$. This, however, reduces the Hilbert space available for computation and requires linear depth with the number of amplitudes needed to upload, limiting its usability. While this basis helps with gate application and control of the device, one would want to exploit the exponential Hilbert space that qubits support.

In order to reclaim the lost Hilbert space using QFT interpolation we propose the following. Upload a probability distribution into the unary basis over a $2^n$ qubit total. Then, perform a unitary change of basis from unary to binary basis [29], as detailed in Alg. 1 and illustrated with a small example in Fig. 3 for $n = 3$. After the change of basis, the quantum state contains $n$ qubits storing the superposition and $2^n-n$ clean ancillas at state $|0\rangle$. This allows the implementation of the QFT interpolation algorithm using the clean ancilla register to encode the high frequency components. Now the full extent of the available Hilbert space is used to encode the interpolated probability distribution. Shown in Fig. 4 are the simulation results of a 16 qubit total QFT interpolation with unary uploading.

Algorithm 1: Unary to binary encoding.

| Line | Description |
|------|-------------|
| 1    | Unary2Binary($n$) |
| 2    | $c \leftarrow$ Circuit($2^n$) |
| 3    | Ensure $q = 0$ |
| 4    | for $i \leftarrow 0$ to $n - 1$ do |
| 5    | $qq \leftarrow 2^{n-i-1}$ |
| 6    | $c$.add(CNOT($q$, $qq$)) |
| 7    | for $j \leftarrow 1$ to $qq - 1$ do |
| 8    | $c$.add(CNOT($q+j$, $q$)) |
| 9    | for $j \leftarrow 1$ to $qq - 1$ do |
| 10   | $c$.add(SWAP($q+j$, $q+j+qq$).controlled_by($q$)) |
| 11   | $q \leftarrow q + qq$ |
| 12   | $c$.add(X($2^n - 1$)) |
| 13   | return |

Binary basis in qubits $\{2^n - 2^i\}$ with $i \leftarrow n$ to 1

Figure 4. Comparison of a Gaussian probability distribution uploaded using the unary basis to 16 qubits $P_4$ that is enlarged into $P_{16}$ using the QFT interpolation scheme to span the full $2^{16}$ Hilbert space.

Appendix B: Trace-distance for pure states encoding real probability distributions

The trace-distance for pure quantum states is defined as

$$\operatorname{dist}_T(|\psi\rangle, |\phi\rangle) = \frac{1}{2} \|\psi\rangle \langle \psi\| - |\phi\rangle \langle \phi\|_1.$$ (B1)

At the same time, for pure quantum states, the Fuchs-van-de-Graaf inequality is tight, meaning that

$$\|\psi\rangle \langle \psi\| - |\phi\rangle \langle \phi\|_1 = 2\sqrt{1 - (|\langle \psi\|\phi\rangle|^2}.$$ (B2)

Recall then, that the $\ell_2$ norm of the same pure states can be decomposed as

$$\|\psi\rangle - |\phi\rangle\|^2 = \|\psi\|^2 + \|\phi\|^2 - 2 \Re(\langle \psi|\phi\rangle ).$$ (B3)

In particular, when the pure states being considered are amplitude encodings of probability distributions, that is,
all amplitudes are real valued, the $\ell_2$ norm can be simplified as

$$|||\psi| - |\phi|||^2 = 2(1 - |\langle \psi | \phi \rangle|).$$

(B4)

Therefore one can substitute the value

$$|\langle \psi | \phi \rangle| = \left(1 - \frac{|||\psi| - |\phi|||^2}{2}\right)$$

(B5)

in Eq. B2 in order to recover the trace-distance from the $\ell_2$ distance in this particular case. Precisely,

$$\text{dist}_\text{tr}(|\psi\rangle, |\phi\rangle) = |||\psi| - |\phi||| \sqrt{1 - \frac{|||\psi| - |\phi|||^2}{4}}.$$  

(B6)

Appendix C: Distance for band-filtered pure states

The bra-ket notation will be dropped in the derivation section for clarity.

The band-limited version of target state $\Psi$ can be defined as

$$\tilde{\Psi} = \frac{\Psi_{\text{in}}}{||\Psi_{\text{in}}||},$$

(C1)

where $\Psi_{\text{in}}$ are the components of the target $\Psi$ that fall within the $n$-qubit band-limited space. Alternatively, $\Psi_{\text{out}}$ are the components outside of the $n$-qubit band-limit, and $\Psi_{\text{in}} + \Psi_{\text{out}} = \Psi$.

Therefore, the difference between the target state and the initial state used in the interpolation is

$$||\Psi - \tilde{\Psi}||^2 = ||\Psi_{\text{out}}||^2 + \left(1 - \frac{1}{||\Psi_{\text{in}}||}ight)^2 ||\Psi_{\text{in}}||^2.$$  

(C2)

Developing the square and using the relation $||\Psi_{\text{in}}||^2 + ||\Psi_{\text{out}}||^2 = 1$ results in

$$||\Psi - \tilde{\Psi}||^2 = 2 + 2 \frac{||\Psi_{\text{out}}||^2 - 1}{\sqrt{1 - ||\Psi_{\text{out}}||^2}}$$

(C3)

that will only depend on the norm of the high frequency components. By combining the terms into

$$||\Psi - \tilde{\Psi}||^2 = 2 \left(\frac{1}{\sqrt{1 - ||\Psi_{\text{out}}||^2}} + 1 + ||\Psi_{\text{out}}||^2\right)^2$$

(C4)

and multiplying both terms in the fraction by $\sqrt{1 - ||\Psi_{\text{out}}||^2}$ we reach the simplified form shown in the main text,

$$||\Psi - \tilde{\Psi}||^2 = \frac{2||\Psi_{\text{out}}||^2}{\left(1 + \sqrt{1 - ||\Psi_{\text{out}}||^2}\right)^2}.$$  

(C5)

Appendix D: Distance for aliased pure states

The derivation of the $\ell_2$ distance between the target and aliased quantum state will proceed similarly to the band-limited one, with more detail on the normalization constant $N$.

The difference now can be written as

$$||\Psi - \tilde{\Psi}||^2 = ||\Psi - \frac{1}{N} (\Psi_{\text{in}} + \Phi)||^2,$$  

(D1)

where $N = ||\Psi_{\text{in}} + \Phi||$ is the normalization factor of the interpolated state $\tilde{\Psi}$ and $\Phi$ is the aliasing effect of the off band-limit components which satisfies $||\Phi|| = ||\Psi_{\text{out}}||$. Notice that $N \geq 1$, since

$$||\Psi_{\text{in}} + \Phi||^2 \geq ||\Psi_{\text{in}}||^2 + ||\Phi||^2 = ||\Psi_{\text{in}}||^2 + ||\Psi_{\text{out}}||^2 = 1,$$  

(D2)

due to $\Psi_{\text{in}}$ and $\Psi_{\text{out}}$ composing to reconstruct $\Psi$, which is a normalized quantum state.

The norm outside the band limit remains the same, leaving

$$||\Psi - \tilde{\Psi}||^2 = ||\Psi_{\text{out}}||^2 + \left(1 - \frac{1}{N}\right)||\Psi_{\text{in}} - \frac{1}{N} \Phi||^2.$$  

(D3)

The second term can be decomposed, making the distance then

$$||\Psi - \tilde{\Psi}||^2 = ||\Psi_{\text{out}}||^2 + \left(1 - \frac{1}{N}\right)^2 ||\Psi_{\text{in}}||^2 + \frac{1}{N^2} ||\Phi||^2 - 2 \left(1 - \frac{1}{N}\right) \frac{1}{N} \text{Re}(\langle \Psi_{\text{in}} | \Phi \rangle).$$  

(D4)

In order to get rid of the last term, we recall that the norm $N$ of the aliased state is

$$N^2 = ||\Psi_{\text{in}} + \Phi||^2 = ||\Psi_{\text{in}}||^2 + ||\Phi||^2 + 2 \text{Re}(\langle \Psi_{\text{in}} | \Phi \rangle),$$  

(D5)

and since $||\Psi_{\text{in}}||^2 + ||\Phi||^2 = ||\Psi_{\text{in}}||^2 + ||\Psi_{\text{out}}||^2 = 1$, we can substitute

$$2 \text{Re}(\langle \Psi_{\text{in}} | \Phi \rangle) = N^2 - 1$$  

(D6)

into Eq. D4 to reach

$$||\Psi - \tilde{\Psi}||^2 = ||\Psi_{\text{out}}||^2 + \left(1 - \frac{1}{N}\right)^2 \left(1 - ||\Psi_{\text{out}}||^2\right) + \frac{1}{N^2} ||\Psi_{\text{out}}||^2 - \left(1 - \frac{1}{N}\right) \frac{1}{N} (N^2 - 1) \frac{1}{N}.$$  

(D7)

This equation simplifies to the form shown in the main text,

$$||\Psi - \tilde{\Psi}||^2 = \frac{2||\Psi_{\text{out}}||^2}{N} - \frac{(N-1)^2}{N}.$$  

(D8)