MIRROR DUALITY VIA $G_2$ AND Spin(7) MANIFOLDS

SELMAN AKBULUT AND SEMA SALUR

Abstract. The main purpose of this paper is to give a construction of certain "mirror dual" Calabi-Yau submanifolds inside of a $G_2$ manifold. More specifically, we explain how to assign a $G_2$ manifold $(M, \varphi, \Lambda)$, with the calibration 3-form $\varphi$ and an oriented 2-plane field $\Lambda$, a pair of parametrized tangent bundle valued 2 and 3-forms of $M$. These forms can then be used to define different complex and symplectic structures on certain 6-dimensional subbundles of $T(M)$. When these bundles are integrated they give mirror CY manifolds. In a similar way, one can define mirror dual $G_2$ manifolds inside of a Spin(7) manifold $(N^8, \Psi)$. In case $N^8$ admits an oriented 3-plane field, by iterating this process we obtain Calabi-Yau submanifold pairs in $N$ whose complex and symplectic structures determine each other via the calibration form of the ambient $G_2$ (or Spin(7)) manifold.

1. Introduction

Let $(M^7, \varphi)$ be a $G_2$ manifold with the calibration 3-form $\varphi$. If $\varphi$ restricts to be the volume form of an oriented 3-dimensional submanifold $Y^3$, then $Y$ is called an associative submanifold of $M$. Associative submanifolds are very interesting objects as they behave very similarly to holomorphic curves of Calabi-Yau manifolds.

In [AS], we studied the deformations of associative submanifolds of $(M, \varphi)$ in order to construct Gromov-Witten like invariants. One of our main observations was that oriented 2-plane fields on $M$ always exist by a theorem of Thomas [1], and by using them one can split the tangent bundle $T(M) = E \oplus V$ as an orthogonal direct sum of an associative 3-plane bundle $E$ and a complex 4-plane bundle $V$. This allows us to define “complex associative submanifolds” of $M$, whose deformation equations may be reduced to the Seiberg-Witten equations, and hence we can assign local invariants to them, and assign various invariants to $(M, \varphi, \Lambda)$, where $\Lambda$ is an oriented 2-plane field on $M$. It turns out that these Seiberg-Witten equations on the submanifolds are restrictions of global equations on $M$.

In this paper, we explain how the geometric structures on $G_2$ manifolds with oriented 2-plane fields $(M, \varphi, \Lambda)$ provide complex and symplectic structures to certain 6-dimensional subbundles of $T(M)$. When these bundles integrated we obtain a pair of Calabi-Yau manifolds whose complex and symplectic structures are remarkably related to each other. We also study examples of Calabi-Yau manifolds which fit

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nicely in our mirror set-up. Later, we do similar constructions for Spin(7) manifolds with oriented 3-plane fields. We then explain how these structures lead to the definition of “dual $G_2$ manifolds” in a Spin(7) manifold, with their own dual Calabi-Yau submanifolds.

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2. Associative and Complex distributions of a $G_2$ manifold

Let us go through quickly over the basic definitions about $G_2$ manifolds. The main references are the two foundational papers [HL] and [BI], as well as [S], [B2], [BS], and [J]. We also need some properties introduced in [AS]. Now let $O = \mathbb{H} \oplus l\mathbb{H} = \mathbb{R}^8$ be the octonions which is an 8 dimensional division algebra generated by $<1, i, j, k, l, li, lj, lk>$, and let $imO = \mathbb{R}^7$ be the imaginary octonions with the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$, defined by $u \times v = im(v.u)$. The exceptional Lie group $G_2$ is the linear automorphisms of $imO$ preserving this cross product operation, it can also be defined in terms of the orthogonal 3-frames in $\mathbb{R}^7$:

$$G_2 = \{(u_1, u_2, u_3) \in (imO)^3 \mid \langle u_i, u_j \rangle = \delta_{ij}, \langle u_1 \times u_2, u_3 \rangle = 0 \}.$$  

Another very useful definition popularized in [BI] is the subgroup of $GL(7, \mathbb{R})$ which fixes a particular 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$. Denote $e^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7)$, then

$$G_2 = \{A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0 \}.$$  

(1)

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$  

Definition 1. A smooth 7-manifold $M^7$ has a $G_2$ structure if its tangent frame bundle reduces to a $G_2$ bundle. Equivalently, $M^7$ has a $G_2$ structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$ the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$ (pointwise condition). We call $(M, \varphi)$ a manifold with $G_2$ structure.

A $G_2$ structure $\varphi$ on $M^7$ gives an orientation $\mu \in \Omega^7(M)$ on $M$, and $\mu$ determines a metric $g = g_\varphi = \langle , \rangle$ on $M$, and a cross product structure $\times$ on the tangent bundle of $M$ as follows: Let $i_v = v \lrcorner$ be the interior product with a vector $v$, then

$$\langle u, v \rangle = [i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi]/6\mu.$$  

(2)

$$\varphi(u, v, w) = \langle u \times v, w \rangle.$$  

(3)
Definition 2. A manifold with $G_2$ structure $(M, \varphi)$ is called a $G_2$ manifold if the holonomy group of the Levi-Civita connection (of the metric $g_\varphi$) lies inside of $G_2$. Equivalently $(M, \varphi)$ is a $G_2$ manifold if $\varphi$ is parallel with respect to the metric $g_\varphi$, that is $\nabla_{g_\varphi}(\varphi) = 0$; which is equivalent to $d\varphi = 0$, $d(*_{g_\varphi}\varphi) = 0$. Also equivalently, at each point $x_0 \in M$ there is a chart $(U, x_0) \to (\mathbb{R}^7, 0)$ on which $\varphi$ equals to $\varphi_0$ up to second order term, i.e. on the image of $U \varphi(x) = \varphi_0 + O(|x|^2)$.

Remark 1. One important class of $G_2$ manifolds are the ones obtained from Calabi-Yau manifolds. Let $(X, \omega, \Omega)$ be a complex 3-dimensional Calabi-Yau manifold with Kähler form $\omega$ and a nowhere vanishing holomorphic 3-form $\Omega$, then $X^6 \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a $G_2$ manifold. In this case $\varphi = \text{Re} \Omega + \omega \wedge dt$. Similarly, $X^6 \times \mathbb{R}$ gives a noncompact $G_2$ manifold.

Definition 3. Let $(M, \varphi)$ be a $G_2$ manifold. A 4-dimensional submanifold $X \subset M$ is called coassociative if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called associative if $\varphi|_Y \equiv \text{vol}(Y)$; this condition is equivalent to the condition $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

\[
\langle \chi(u,v,w), z \rangle = *_{\varphi}(u,v,w,z)
\]

The equivalence of these conditions follows from the 'associator equality' of $[HL]$.

Definition 4. Let $(M, \varphi)$ be a $G_2$ manifold. Then $\psi \in \Omega^2(M, TM)$ is the tangent bundle valued 2-form defined by the identity:

\[
\langle \psi(u,v), w \rangle = \varphi(u,v,w) = \langle u \times v, w \rangle
\]

Now we have two useful properties from $[AS]$, the first property basically follows from definitions, the second property fortunately applies when the first property fails to give anything useful.

Lemma 1. ($[AS]$) To any 3-dimensional submanifold $Y^3 \subset (M, \varphi)$, $\chi$ assigns a normal vector field, which vanishes when $Y$ is associative.

Lemma 2. ($[AS]$) To any associative manifold $Y^3 \subset (M, \varphi)$ with a non-vanishing oriented 2-plane field, $\chi$ defines a complex structure on its normal bundle (notice in particular that any coassociative submanifold $X \subset M$ has an almost complex structure if its normal bundle has a non-vanishing section).

Proof. Let $L \subset \mathbb{R}^7$ be an associative 3-plane, that is $\varphi_0|_L = \text{vol}(L)$. Then for every pair of orthonormal vectors $\{u, v\} \subset L$, the form $\chi$ defines a complex structure on the orthogonal 4-plane $L^\perp$, as follows: Define $j : L^\perp \to L^\perp$ by

\[
j(X) = \chi(u,v,X)
\]
This is well defined i.e. \( j(X) \in L^\perp \), because when \( w \in L \) we have:
\[
\langle \chi(u, v, X), w \rangle = *\varphi_0(u, v, X, w) = -*\varphi_0(u, v, w, X) = \langle \chi(u, v, w), X \rangle = 0
\]

Also \( j^2(X) = j(\chi(u, v, X)) = \chi(u, v, \chi(u, v, X)) = -X \). We can check the last equality by taking an orthonormal basis \( \{X_j\} \subset L^\perp \) and calculating
\[
\langle \chi(u, v, \chi(u, v, X_i)), X_j \rangle = *\varphi_0(u, v, \chi(u, v, X_i), X_j) = -*\varphi_0(u, v, X_j, \chi(u, v, X_i)) = -\delta_{ij}
\]

The last equality holds since the map \( j \) is orthogonal, and the orthogonality can be seen by polarizing the associator equality, and by noticing \( \varphi_0(u, v, X_i) = 0 \). Observe that the map \( j \) only depends on the oriented 2-plane \( \Lambda = \langle u, v \rangle \) generated by \( \{u, v\} \) (i.e. it only depends on the complex structure on \( \Lambda \)).

**Remark 2.** Notice that Lemma 1 gives an interesting flow on the 3-dimensional submanifolds of \( G_2 \) manifolds \( f : Y \hookrightarrow (M, \varphi) \) (call \( \chi \)-flow), described by:
\[
\frac{\partial}{\partial t} f = \chi(f_* \text{vol}(Y))
\]

For example, by [BS] the total space of the spinor bundle \( Q^7 \to S^3 \) (with \( \mathbb{C}^2 \) fibers) is a \( G_2 \) manifold, and the zero section \( S^3 \subset Q \) is an associative submanifold. We can imbed any homotopy 3-sphere \( \Sigma^3 \) into \( Q \) (homotopic to the zero-section). We conjecture that \( \chi \)-flow on \( \Sigma \subset Q \), takes \( \Sigma \) diffeomorphically onto the zero section. Note that, since any \( S^3 \) smoothly unknots in \( S^7 \) it is not possible to produce quick counterexamples by tying local knots; and an affirmative answer gives \( \Sigma \cong S^3 \).

![Figure 1](image-url)

Finally, we need some identities from [B2] (also see [K]) for \( (M^7, \varphi) \), which follow from local calculations by using the definition (1). For \( \beta \in \Omega^1(M) \) we have:
\[
(7) \quad |\varphi \wedge \beta|^2 = 4|\beta|^2, \quad \text{and} \quad |\ast \varphi \wedge \beta|^2 = 3|\beta|^2,
\]
\[
(8) \quad (\xi \lrcorner \varphi) \wedge \varphi = 2* (\xi \lrcorner \varphi), \quad \text{and} \quad * [*(\beta \wedge \varphi) \wedge \varphi] = 3\beta,
\]
\[
(9) \quad \beta \times (\beta \times u) = -|\beta|^2 u + \langle \beta, u \rangle \beta,
\]
where ∗ is the star operator. Let ξ be a vector field on any Riemannian manifold (M, g), and ξ# ∈ Ω^1(M) be its dual 1-form, i.e. ξ#(v) = ⟨ξ, v⟩. Then for α ∈ Ω^k(M):
\[
* (ξ \lrcorner α) = (-1)^{k+1} (ξ# \wedge *α).
\]

3. Mirror duality in G_2 manifolds

On a local chart of a G_2 manifold (M, ϕ), the form ϕ coincides with the form ϕ_0 ∈ Ω^3(\mathbb{R}^7) up to quadratic terms, we can express the corresponding tangent valued forms χ and ψ in terms of ϕ_0 in local coordinates. More generally, if e_1, ..., e_7 is any local orthonormal frame and e_1, ..., e_7 is the dual frame, from definitions we get:
\[
\begin{align*}
χ &= (e^{256} + e^{247} + e^{346} - e^{357})e_1 \\
&\quad + (-e^{156} - e^{147} - e^{345} + e^{367})e_2 \\
&\quad + (e^{157} - e^{146} + e^{245} + e^{267})e_3 \\
&\quad + (e^{127} + e^{136} - e^{235} - e^{567})e_4 \\
&\quad + (e^{126} - e^{137} + e^{234} + e^{467})e_5 \\
&\quad + (-e^{125} - e^{134} - e^{237} - e^{457})e_6 \\
&\quad + (-e^{124} + e^{135} + e^{236} + e^{456})e_7.
\end{align*}
\]
\[
\begin{align*}
ψ &= (e^{23} + e^{45} + e^{67})e_1 \\
&\quad + (e^{46} - e^{57} - e^{13})e_2 \\
&\quad + (e^{12} - e^{47} - e^{56})e_3 \\
&\quad + (e^{37} - e^{15} - e^{26})e_4 \\
&\quad + (e^{14} + e^{27} + e^{36})e_5 \\
&\quad + (e^{24} - e^{17} - e^{35})e_6 \\
&\quad + (e^{16} - e^{25} - e^{34})e_7.
\end{align*}
\]

The forms χ and ψ induce complex and symplectic structures on certain subbundles of T(M) as follows: Let ξ be a nonvanishing vector field of M. We can define a symplectic ω_ξ and a complex structure J_ξ on the 6-plane bundle V_ξ := ξ^⊥ by
\[
\begin{align*}
ω_ξ &= ⟨ψ, ξ⟩ \quad \text{and} \quad J_ξ(X) = X × ξ.
\end{align*}
\]

Now we can define
\[
\begin{align*}
\text{Re } Ω_ξ &= ϕ|_{V_ξ} \quad \text{and} \quad \text{Im } Ω_ξ = ⟨χ, ξ⟩.
\end{align*}
\]
In particular $\omega_\xi = \xi \lrcorner \varphi$, and $\Im \Omega_\xi = \xi \lrcorner \varphi$. Call $\Omega_\xi = \Re \Omega_\xi + i \Im \Omega_\xi$. The reason for defining these is to pin down a Calabi-Yau like structure on any $G_2$ manifold. In case $(M, \varphi) = CY \times S^1$ these quantities are related to the ones in Remark 1. Notice that when $\xi \in E$ then $J_\xi$ is an extension of $J$ of Lemma 2 from the 4-dimensional bundle $V$ to the 6-dimensional bundle $V_\xi$.

By choosing different directions, i.e. different $\xi$, one can find the corresponding complex and symplectic structures. In particular we will get two different complex structures if we choose $\xi$ in the associative subbundle $E$ (where $\varphi$ restricts to be 1), or if we choose $\xi$ in the complementary subbundle $V$, which we will call the coassociative subbundle. Note that $\varphi$ restricts to zero on the coassociative subbundle.

In local coordinates, it is a straightforward calculation that by choosing $\xi = e_i$ for any $i$, from equations (11) and (12), we can easily obtain the corresponding structures $\omega_\xi$, $J_\xi$, $\Omega_\xi$. For example, let us assume that $\{e_1, e_2, e_3\}$ is the local orthonormal basis for the associative bundle $E$, and $\{e_4, e_5, e_6, e_7\}$ is the local orthonormal basis for the coassociative bundle $V$. Then if we choose $\xi = e_3 = e_1 \times e_2$ then we get $\omega_\xi = e_1^2 - e_3^7 - e_5^6$ and $\Im \Omega_\xi = e_1^{157} - e_1^{245} + e_1^{267}$. On the other hand, if we choose $\xi = e_7$ then $\omega_\xi = e_1^6 - e_2^5 - e_3^4$ and $\Im \Omega_\xi = -e_1^{124} + e_1^{35} + e_1^{26} + e_1^{38}$ which will give various symplectic and complex structures on the bundle $V_\xi$.

3.1. A useful example.

Let us take a Calabi-Yau 6-torus $T^6 = T^3 \times T^3$, where $\{e_1, e_2, e_3\}$ is the basis for one $T^3$ and $\{e_4, e_5, e_6\}$ the basis for the other (terms expressed with a slight abuse of notation). We can take the product $M = T^6 \times S^1$ as the corresponding $G_2$ manifold with the calibration 3-form $\varphi = e_1^{123} + e_1^{145} + e_1^{167} + e_1^{246} - e_1^{257} - e_1^{347} - e_1^{356}$, and with the decomposition $T(M) = E \oplus V$, where $E = \{e_1, e_2, e_3\}$ and $V = \{e_4, e_5, e_6, e_7\}$. Now, if we choose $\xi = e_7$, then $V_\xi = \langle e_1, \ldots, e_6 \rangle$ and the symplectic form is $\omega_\xi = e_1^6 - e_2^5 - e_3^4$, and the complex structure is

$$J_\xi = \begin{pmatrix}
 e_1 & \mapsto & -e_6 \\
 e_2 & \mapsto & e_5 \\
 e_3 & \mapsto & e_4
\end{pmatrix}$$

and the complex valued $(3, 0)$ form is $\Omega_\xi = (e_1^2 + ie_6^2) \wedge (e_2^2 - ie_5^2) \wedge (e_3^2 - ie_4^2)$; note that this is just $\Omega_\xi = (e_1 - i\xi(e^1)) \wedge (e_2 - i\xi(e^2)) \wedge (e_3 - i\xi(e^3))$.

On the other hand, if we choose $\xi' = e_3$ then $V_{\xi'} = \langle e_1, \ldots, \hat{e}_3, \ldots, e_7 \rangle$ and the symplectic form is $\omega_{\xi'} = e_1^7 - e_2^4 - e_3^5$ and the complex structure is

$$J_{\xi'} = \begin{pmatrix}
 e_1 & \mapsto & -e_2 \\
 e_4 & \mapsto & e_7 \\
 e_5 & \mapsto & e_6
\end{pmatrix}$$
Also $\Omega_\xi = (e^1 + ie^2) \wedge (e^4 - ie^7) \wedge (e^5 - ie^6)$, as above this can be expressed more tidily as $\Omega_\xi' = (e^1 - iJ_\xi(e^1)) \wedge (e^4 - iJ_\xi'(e^4)) \wedge (e^5 - iJ_\xi'(e^5))$. In the expressions of $J$’s the basis of associative bundle $E$ is indicated by bold face letters to indicate the differing complex structures on $T^6$. To sum up: If we choose $\xi$ from the coassociative bundle $V$ we get the complex structure which decomposes the 6-torus as $T^3 \times T^3$. On the other hand if we choose $\xi$ from the associative bundle $E$ then the induced complex structure on the 6-torus corresponds to the decomposition as $T^2 \times T^4$. This is the phenomenon known as “mirror duality”. Here these two $SU(3)$ and $SU(2)$ structures are different but they come from the same $\varphi$ hence they are dual. These examples suggests the following definition of “mirror duality” in $G_2$ manifolds:

**Definition 5.** Two Calabi-Yau manifolds are mirror pairs of each other, if their complex structures are induced from the same calibration 3-form in a $G_2$ manifold. Furthermore we call them strong mirror pairs if their normal vector fields $\xi$ and $\xi'$ are homotopic to each other through nonvanishing vector fields.

**Remark 3.** In the above example of $CY \times S^1$, where $CY = T^6$, the calibration form $\varphi = \text{Re } \Omega + \omega \wedge dt$ gives Lagrangian tori fibration in $X_\xi$ and complex tori fibration in $X_{\xi'}$. They are different manifestations of $\varphi$ residing on one higher dimensional $G_2$ manifold $M^7$. In the next section this correspondence will be made precise.

In Section 4.2 we will discuss a more general notion of mirror Calabi-Yau manifold pairs, when they sit in different $G_2$ manifolds, which are themselves mirror duals of each other in a $\text{Spin}(7)$ manifold.

### 3.2. General setting.

Let $(M^7, \varphi, \Lambda)$ be a manifold with a $G_2$ structure and a non-vanishing oriented 2-plane field. As suggested in [AS] we can view $(M^7, \varphi)$ as an analog of a symplectic manifold, and the 2-plane field $\Lambda$ as an analog of a complex structure taming $\varphi$. This is because $\Lambda$ along with $\varphi$ gives the associative/complex bundle splitting $T(M) = E_{\varphi, \Lambda} \oplus V_{\varphi, \Lambda}$. Now, the next object is a choice of a non-vanishing unit vector field $\xi \in \Omega^0(M, TM)$, which gives a codimension one distribution $V_\xi := \xi^\perp$ on $M$, which is equipped with the structures $(V_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ as given by (11) and (12).

Let $\xi^\#$ be the dual 1-form of $\xi$. Let $e_\xi^\#$ and $i_\xi = \xi^\#^\perp$ denote the exterior and interior product operations on differential forms. Clearly $e_\xi^\#^\perp \circ i_\xi + i_\xi \circ e_\xi^\#^\perp = id$.

(13) \[ \varphi = e_\xi^\# \circ i_\xi(\varphi) + i_\xi \circ e_\xi^\#(\varphi) = \omega_\xi \wedge \xi^\# + \text{Re } \Omega_\xi. \]

This is just the decomposing of the form $\varphi$ with respect to $\xi \oplus \xi^\perp$. Recall that the condition that the distribution $V_\xi$ be integrable (the involutive condition which implies $\xi^\perp$ comes from a foliation) is given by:

(14) \[ d\xi^\# \wedge \xi^\# = 0. \]
Even when $V_\xi$ is not integrable, by [Th] it is homotopic to a foliation. Assume $X_\xi$ be a page of this foliation; for simplicity assume this 6-dimensional manifold is smooth.

It is clear from definitions that $J_\xi$ is an almost complex structure on $X_\xi$, because from (2) we can write

$$\omega_\xi^3 = (\xi \bullet \varphi)^3 = \xi \bullet [(\xi \bullet \varphi) \wedge (\xi \bullet \varphi) \wedge \varphi] = \xi \bullet (6|\xi|^2\mu) = 6\mu_\xi$$

where $\mu_\xi = \mu|_{V_\xi}$ is the induced orientation form on $V_\xi$.

**Lemma 3.** $J_\xi$ is compatible with $\omega_\xi$, and it is metric invariant.

**Proof.** Let $u, v \in V_\xi$

$$\omega_\xi(J_\xi(u), v) = \omega_\xi(u \times \xi, v) = \langle \psi(u \times \xi, v), \xi \rangle = \varphi(u \times \xi, v, \xi) \quad \text{by (5)}$$

$$= -\varphi(\xi, \xi \times u, v) = -\langle \xi \times (\xi \times u), v \rangle \quad \text{by (3)}$$

$$= -\langle -|\xi|^2u + (\xi, u)\xi, v \rangle = |\xi|^2\langle u, v \rangle - \langle \xi, u \rangle \langle \xi, v \rangle \quad \text{by (9)}$$

$$= \langle u, v \rangle.$$

By plugging in $J_\xi(u), J_\xi(v)$ for $u, v$: $\langle J_\xi(u), J_\xi(v) \rangle = -\omega_\xi(u, J_\xi(v)) = \langle u, v \rangle \quad \Box$

**Lemma 4.** $\Omega_\xi$ is a non-vanishing $(3, 0)$ form.

**Proof.** By a local calculation as in Section 3.1 we see that $\Omega_\xi$ is a $(3, 0)$ form, and is non-vanishing because $\Omega_\xi \wedge \overline{\Omega_\xi} = 8\text{vol}(X_\xi)$, i.e.

$$\frac{1}{2i} \Omega_\xi \wedge \overline{\Omega_\xi} = \text{Im} \Omega_\xi \wedge \text{Re} \Omega_\xi = (\xi \bullet \varphi) \wedge [\xi \bullet (\xi^# \wedge \varphi)]$$

$$= -\xi \bullet [(\xi \bullet \varphi) \wedge (\xi^# \wedge \varphi)]$$

$$= \xi \bullet [(\xi \bullet \varphi) \wedge (\xi^# \wedge \varphi)] \quad \text{by (10)}$$

$$= |\xi^# \wedge \varphi|^2 \xi \cdot \text{vol}(M)$$

$$= 4|\xi^#|^2 \langle \ast \xi^# \rangle = 4 \text{vol}(X_\xi). \quad \text{by (7)} \Box$$

We can easily calculate $\ast \text{Re} \Omega_\xi = -\text{Im} \Omega_\xi \wedge \xi^#$ and $\ast \text{Im} \Omega_\xi = \text{Re} \Omega_\xi \wedge \xi^#$. In particular if $\ast$ is the star operator of $X_\xi$ (for example by (15) $\ast \omega_\xi = \omega_\xi^2/2$), then

$$\ast \text{Re} \Omega_\xi = \text{Im} \Omega_\xi.$$

Notice that $\omega_\xi$ is a symplectic structure on $X_\xi$ whenever $d\varphi = 0$ and $L_\xi(\varphi)|_{V_\xi} = 0$, where $L_\xi$ denotes the Lie derivative along $\xi$. This is because $\omega_\xi = \xi \bullet \varphi$ and:

$$d\omega_\xi = L_\xi(\varphi) - \xi \bullet d\varphi = L_\xi(\varphi).$$

Also $d^*\varphi = 0 \implies d^*\omega_\xi = 0$, without any condition on the vector field $\xi$, since

$$\ast \varphi = \ast \omega_\xi - \text{Im} \Omega_\xi \wedge \xi^#,$$
and hence $d(*\omega_\xi) = d(*\varphi|_X) = 0$. Also $d\varphi = 0 \implies d(Re \Omega_\xi) = d(\varphi|_X) = 0$.

Furthermore, $d^*\varphi = 0$ and $L_\xi(*\varphi)|_V = 0 \implies d(Im \Omega_\xi) = 0$; this is because $Im \Omega_\xi = \xi_{\varphi} (*\varphi)$, where $*$ is the star operator on $(M, \varphi)$. Also, $J_\xi$ is integrable when $d\Omega = 0$ (e.g. [Hi]). By using the following definition, we can sum up all the conclusions of the above discussion as Theorem 5 below.

**Definition 6.** $(X^6, \omega, \Omega, J)$ is called an almost Calabi-Yau manifold, if $X$ is a Riemannian manifold with a non-degenerate 2-form $\omega$ (i.e. $\omega^3 = 6\text{vol}(X)$) which is co-closed, and $J$ is a metric invariant almost complex structure which is compatible with $\omega$, and $\Omega$ is a non-vanishing $(3,0)$ form with $Re \Omega$ closed. Furthermore, when $\omega$ and $Im \Omega$ are closed, we call this a Calabi-Yau manifold.

**Theorem 5.** Let $(M, \varphi)$ be a $G_2$ manifold, and $\xi$ be a unit vector field which comes from a codimension one foliation on $M$, then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is an almost Calabi-Yau manifold with $\varphi|_X = Re \Omega_\xi$ and $*\varphi|_X = \omega_\xi$. Furthermore, if $L_\xi(\varphi)|_X = 0$ then $d\omega_\xi = 0$, and if $L_\xi(*\varphi)|_X = 0$ then $J_\xi$ is integrable; when both of these conditions are satisfied then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is a Calabi-Yau manifold.

**Remark 4.** If $\xi$ and $\xi'$ are sections of $V$ and $E$ respectively, then from [M] the condition $L_\xi(*\varphi)|_{X_\xi} = 0$ (complex geometry of $X_\xi$) implies that deforming associative submanifolds of $X_\xi$ along $\xi$ in $M$ keeps them associative; and $L_{\xi'}(\varphi)|_{X_{\xi'}} = 0$ (symplectic geometry of $X_{\xi'}$) implies that deforming coassociative submanifolds of $X_{\xi'}$ along $\xi'$ in $M$ keeps them coassociative (e.g. for an example see Example 1).

Notice that both complex and symplectic structure of the CY-manifold $X_\xi$ in Theorem 3 is determined by $\varphi$ when they exist. Recall that (c.f. [V]) elements $\Omega \in H^{3,0}(X_\xi, \mathbb{C})$ along with topology of $X_\xi$ (i.e. the intersection form of $H^3(X_\xi, \mathbb{Z})$) parametrize complex structures on $X_\xi$ as follows: We compute the third betti number $b_3(M) = 2h^{2,1} + 2$ since

$$H^3(X_\xi, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} = 2(\mathbb{C} \oplus H^{2,1}).$$

Let $\{A^i, B_j\}$ be a symplectic basis of $H_3(X, \mathbb{Z})$, $i = 1, ..., h^{2,1} + 1$, then

$$X_i = \int_{A^i} \Omega \tag{18}$$

give complex numbers which are local homegenous coordinates of the moduli space of complex structures on $X_\xi$, which is a $h^{2,1}$ dimensional space (there is an extra parameter here since $\Omega$ is defined up to scale multiplication).

As we have seen in the example of the Section 3.1, the choice of $\xi$ can give rise to quite different complex structures on $X_\xi$ (e.g. $SU(2)$ and $SU(3)$ structures). For example, assume $\xi \in \Omega^0(M, V)$ and $\xi' \in \Omega^0(M, E)$ be unit vector fields, such that the the codimension one plane fields $\xi^\perp$ and $\xi'^\perp$ come from foliations. Let $X_\xi$ and $X_{\xi'}$ be pages of the corresponding foliations. By our definition $X_\xi$ and $X_{\xi'}$ are
mirror duals of each other. Decomposition $T(M) = E \oplus V$ gives rise to splittings $TX_\xi = E \oplus \bar{E}$, and $TX_{\xi'} = C \oplus V$, where $\bar{E} = \xi^\perp(V) \subset V$ is a 3-dimensional subbundle, and $C = (\xi')^\perp(E) \subset E$ is a 2-dimensional subbundle. Furthermore, $E$ is Lagrangian in $TX_\xi$ i.e. $J_\xi(E) = \bar{E}$, and $C, V$ are complex in $TX_{\xi'}$ i.e. $J_{\xi'}(C) = C$ and $J_{\xi'}(V) = V$. Also notice that, $\text{Re} \Omega_\xi$ is a calibration form of $E$, and $\omega_{\xi'}$ is a calibration form of $C$. In particular, $\langle \Omega_\xi, E \rangle = 1$ and $\langle \omega_{\xi'} \wedge \xi'^\#, E \rangle = 0$; and $\langle \Omega_{\xi'}, E \rangle = 0$ and $\langle \omega_{\xi'} \wedge (\xi')^\#, E \rangle = 1$.

If $X_\xi$ and $X_{\xi'}$ are strong duals of each other, we can find a homotopy of non-vanishing unit vector fields $\xi_t$ ($0 \leq t \leq 1$) starting with $\xi \in V$ ending with $\xi' \in E$. This gives a 7-plane distribution $\Xi = \xi_t^\perp \oplus \bar{\partial}_t$ on $M \times [0,1]$ with integral submanifolds $X_\xi \times [0,\epsilon)$ and $X_{\xi'} \times (1-\epsilon,1]$ on a neighborhood of the boundary. Then by [Th] and [Th1] we can homotop $\Xi$ to a foliation extending the foliation on the boundary (possibly by taking $\epsilon$ smaller). Let $Q^7 \subset M \times [0,1]$ be the smooth manifold given by this foliation, with $\partial Q = X_\xi \cup X_{\xi'}$, where $X_\xi \subset M \times \{0\}$ and $X_{\xi'} \subset M \times \{1\}$.

![Figure 2.](image)

We can define $\Phi \in \Omega^2(M \times [0,1])$ with $\Phi|_{X_\xi} = \Omega_\xi$ and $\Phi|_{X_{\xi'}} = \xi_t \star \omega_{\xi'}$

$\Phi = \Phi(\varphi, \Lambda, t) = \langle \omega_{\xi_t} \wedge \xi_t', E \rangle \xi_t'' \star \omega_{\xi_t} + \langle \text{Re} \Omega_{\xi_t}, E \rangle \Omega_{\xi_t}$

where $\xi_t'' = J_{\xi \times \xi'}(\xi_t) = \xi_t \times (\xi \times \xi')$ (hence $\xi_0'' = -\xi'$ and $\xi_1'' = \xi$). This can be viewed as a correspondence between the complex structure of $X_\xi$ and the symplectic structure of $X_{\xi'}$. In general, the manifold pairs $X_\alpha$ and $X_\beta$ (as constructed in Theorem 5) determine each others almost Calabi-Yau structures via $\varphi$ provided they are defined.

**Proposition 6.** Let $\{\alpha, \beta\}$ be orthonormal vector fields on $(M, \varphi)$. Then on $X_\alpha$ the following hold

(i) $\text{Re} \Omega_\alpha = \omega_\beta \wedge \beta'^\# + \text{Re} \Omega_\beta$
(ii) $\text{Im} \Omega_\alpha = \alpha \lrcorner \left( \ast \omega_\beta \right) - \left( \alpha \lrcorner \text{Im} \Omega_\beta \right) \wedge \beta^\#$

(iii) $\omega_\alpha = \alpha \lrcorner \text{Re} \Omega_\beta + \left( \alpha \lrcorner \omega_\beta \right) \wedge \beta^\#$

Proof. Since $\text{Re} \Omega_\alpha = \varphi|_{X_\alpha}$ (i) follows. Since $\text{Im} \Omega_\alpha = \alpha \lrcorner \varphi$ following gives (ii)

\[
\begin{align*}
\alpha \lrcorner \left( \ast \omega_\beta \right) &= \alpha \lrcorner \left[ \beta \lrcorner \left( \beta \lrcorner \varphi \right) \right] \\
&= \alpha \lrcorner \beta \lrcorner \left( \beta^\# \wedge \ast \varphi \right) \\
&= \alpha \lrcorner \varphi + \beta^\# \wedge \left( \alpha \lrcorner \beta \lrcorner \ast \varphi \right) \\
&= \alpha \lrcorner \varphi + \left( \alpha \lrcorner \text{Im} \Omega_\beta \right) \wedge \beta^#
\end{align*}
\]

(iii) follows from the following computation

\[
\begin{align*}
\alpha \lrcorner \text{Re} \Omega_\beta &= \alpha \lrcorner \beta \lrcorner \left( \beta^\# \wedge \varphi \right) = \alpha \lrcorner \varphi + \beta^\# \wedge \left( \alpha \lrcorner \beta \lrcorner \ast \varphi \right) = \alpha \lrcorner \varphi - \left( \alpha \lrcorner \omega_\beta \right) \wedge \beta^#
\end{align*}
\]

Notice that even though the identities of Proposition 6 hold only after restricting the right hand side to $X_\alpha$, all the individual terms are defined everywhere on $(M, \varphi)$. Also, from the construction, $X_\alpha$ and $X_\beta$ inherit vector fields $\beta$ and $\alpha$, respectively.

Corollary 7. Let $\{\alpha, \beta\}$ be orthonormal vector fields on $(M, \varphi)$. Then there are $A_{\alpha\beta} \in \Omega^3(M)$, and $W_{\alpha\beta} \in \Omega^2(M)$ satisfying

\[
\begin{align*}
(a) \quad \varphi|_{X_\alpha} &= \text{Re} \Omega_\alpha \quad \text{and} \quad \varphi|_{X_\beta} = \text{Re} \Omega_\beta \\
(b) \quad A_{\alpha\beta}|_{X_\alpha} &= \text{Im} \Omega_\alpha \quad \text{and} \quad A_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner \left( \ast \omega_\beta \right) \\
(c) \quad W_{\alpha\beta}|_{X_\alpha} &= \omega_\alpha \quad \text{and} \quad W_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner \text{Re} \Omega_\beta
\end{align*}
\]

For example, when $\varphi$ varies through metric preserving $G_2$ structures [B2], (hence fixing the orthogonal frame $\{\xi, \xi'\}$), it induces variations of $\omega$ one side, and $\Omega$ on the other side.

Remark 5. By using Proposition 6, in the previous torus example of 3.1 one can show a natural correspondence between the groups $H^{2,1}(X_\xi)$ and $H^{1,1}(X_{\xi'})$. Even though $\mathbb{T}^7$ is a trivial example of a $G_2$ manifold, it is an important special case since the $G_2$ manifolds of Joyce are obtained by smoothing quotients of $\mathbb{T}^7$ by finite group actions. We believe this process turns the subtori $X_{\xi}$’s into Borcea-Voisin manifolds with a similar correspondence of their cohomology groups.

For the discussion of the previous paragraph to work, we need a non-vanishing vector field $\xi$ in $T(M) = E \oplus V$, moving from $V$ to $E$. The bundle $E$ always has a non-zero section, in fact it has a non-vanishing orthonormal 3-frame field; but $V$ may not have a non-zero section. Nevertheless the bundle $V \rightarrow M$ does have a non-vanishing section in the complement of a 3-manifold $Y \subset M$, which is a transverse self intersection of the zero section. In [AS], Seiberg-Witten equations
of such 3-manifolds were related to associative deformations. So we can use these partial sections and, as a consequence, $X_\xi$ and $X_{\xi'}$ may not be closed manifolds. The following is a useful example:

**Example 1.** Let $X_1$, $X_2$ be two Calabi-Yau manifolds, where $X_1$ is the cotangent bundle of $S^3$ and $X_2$ is the $O(-1) \oplus O(-1)$ bundle of $S^2$. They are conjectured to be the mirror duals of each other by physicists (c.f. [Ma]). By using the approach of this paper, we identify them as 6-dimensional submanifolds of a $G_2$ manifold. Let’s choose $M = \wedge^2_+ (S^4)$; this is a $G_2$ manifold by Bryant-Salamon [BS].

Let $\pi: \wedge^2_+ (S^4) \to S^4$ be the bundle projection. The sphere bundle of $\pi$ (which is also $\overline{\mathbb{C}P}^3$) is the so-called twistor bundle, let us denote it by $\pi_1: Z(S^4) \to S^4$. It is known that the normal bundle of each fiber $\pi_1^{-1}(p) \cong S^2$ in $Z(S^4)$ can be identified by $O(-1) \oplus O(-1) [S]$. Now we take $E$ to be the bundle of vertical tangent vectors of $\pi$, and $V = \pi^*(TS^4)$, lifted by connection distribution. Let $\xi$ be the pull-back of the vector field on $S^4$ with two zeros (flowing from north pole $n$ to south pole $s$), and let $\xi'$ be the radial vector field of $E$. Clearly $X_\xi = T^* S^3$ and $X_{\xi'} = O(-1) \oplus O(-1)$.

Note that $\xi$ is non-vanishing in the complement of $\pi_1^{-1}\{n, s\}$, whereas $\xi'$ is non-vanishing in the complement of the zero section of $\pi$. Clearly on the set where they are both defined, $\xi$ and $\xi'$ are homotopic through nonvanishing vector fields $\xi_t$. This would define a cobordism between the complements of the zero sections of the bundles $T^* S^3$ and $O(-1) \oplus O(-1)$, if the distributions $\xi_t^\perp$ were involutive.

![Figure 3.](image)

Here the change of complex structures $X_{\xi'} \sim X_\xi$ happens as follows. Let $S^3_\lambda \to S^2$ be the Hopf map with fibers consisting of circles of radius $\lambda$, clearly $S^3_\infty = S^2 \times \mathbb{R}$

$$(\mathbb{C}^2 - 0) \times S^2 \to (S^3_\lambda \times \mathbb{R}) \times S^2 \xrightarrow{\lambda \to \infty} S^3_\infty \times S^3_\infty$$

where the complex structure on $S^3_\lambda \times S^3_\lambda$ is the obvious one, induced from exchanging the factors. In general if we allow the vector fields $\xi$ and $\xi'$ be homotopic through
vector fields $\xi_t$ possibly with zeros, or the family $\xi_t^\perp$ not remain involutive the cobordism between $X_\xi$ and $X_{\xi'}$ will have singularities.

**Remark 6.** If we apply the construction of Example 1 to the total space of the spinor bundle $Q \to S^3$ (see Remark 2), the two dual 6-manifolds we get are $S^2 \times \mathbb{R}^4$ and $S^3 \times \mathbb{R}^3$.

There is also a concept of mirror-dual $G_2$ manifolds in a Spin(7) manifold, hence we can talk about mirror dual CY manifolds coming from two different mirror dual $G_2$ submanifolds of a Spin(7) manifold. This is the subject of the next section.

### 4. Mirror duality in $\text{Spin}(7)$ Manifolds

Similar to Calabi-Yau case there is a notion of mirror duality between $G_2$ manifolds [AG], [AV], [GYZ], [SV]. In this section we will give a definition of mirror $G_2$ pairs, and an example which shows that associative and co-associative geometries in mirror $G_2$ pairs are induced from the same calibration 4-form in a $\text{Spin}(7)$ manifold, and hence these geometries dual to each other. Let us first recall the basic definitions and properties of $\text{Spin}(7)$ geometries. The main references in this subject are [HL] and [Ti].

**Definition 7.** An 8-dimensional Riemannian manifold $(N, \Psi)$ is called a $\text{Spin}(7)$ manifold if the holonomy group of its metric connection lies in $\text{Spin}(7) \subset \text{GL}(8)$.

Equivalently, a $\text{Spin}(7)$ manifold is an 8-dimensional Riemannian manifold with a triple cross product $\times$ on its tangent bundle, and a closed 4-form $\Psi \in \Omega^4(N)$ with $\Psi(u,v,w,z) = (u \times v \times w, z)$.

**Definition 8.** A 4-dimensional submanifold $X$ of a $\text{Spin}(7)$ manifold $(N, \Psi)$ is called Cayley if $\Psi|_X \equiv \text{vol}(X)$.

Analogous to the $G_2$ case, we introduce a tangent bundle valued 3-form, which is just the triple cross product of $N$.

**Definition 9.** Let $(N, \Psi)$ be a $\text{Spin}(7)$ manifold. Then $\Upsilon \in \Omega^3(N, TN)$ is the tangent bundle valued 3-form defined by the identity:

$$\langle \Upsilon(u,v,w), z \rangle = \Psi(u,v,w,z) = \langle u \times v \times w, z \rangle.$$

$\text{Spin}(7)$ manifolds can be constructed from $G_2$ manifolds. Let $(M, \varphi)$ be a $G_2$ manifold with a 3-form $\varphi$, then $M \times S^1$ (or $M \times \mathbb{R}$) has holonomy group $G_2 \subset \text{Spin}(7)$, hence is a $\text{Spin}(7)$ manifold. In this case $\Psi = \varphi \wedge dt + ^*_{7}\varphi$, where $^*_{7}$ is the star operator of $M$.  

Now we will repeat a similar construction for a $\text{Spin}(7)$ manifold $(N, \Psi)$, which we did for $G_2$ manifolds. Here we make an assumption that $T(M)$ admits a non-vanishing 3-frame field $\Lambda = \langle u, v, w \rangle$, then we decompose $T(M) = K \oplus D$, where
$K = \langle u, v, w, u \times v \times w \rangle$ is the bundle of Cayley 4-planes (where $\Psi$ restricts to be 1) and $D$ is the complementary subbundle (note that this is also a bundle of Cayley 4-planes since the form $\Psi$ is self dual). In the $G_2$ case, existence of an analogous decomposition of the tangent bundle followed from [T] (in this case we can just restrict to a submanifold which a 3-frame field exists). On a chart in $N$ let $e_1, \ldots, e_8$ be an orthonormal frame and $e^1, \ldots, e^8$ be the dual coframe, then the calibration 4-form is given as (c.f. [HL])

$$
\Psi = e^{1234} + (e^{12} - e^{34}) \wedge (e^{56} - e^{78}) + (e^{13} + e^{24}) \wedge (e^{57} + e^{68}) + (e^{14} - e^{23}) \wedge (e^{58} - e^{67}) + e^{5678}
$$

which is a self dual 4-form, and the corresponding tangent bundle valued 3-form is

$$
\Upsilon = (e^{234} + e^{256} - e^{278} + e^{357} + e^{368} + e^{458} - e^{467})e_1
+ (-e^{134} - e^{156} + e^{178} + e^{457} + e^{468} - e^{358} + e^{367})e_2
+ (e^{124} - e^{456} + e^{478} - e^{157} - e^{168} + e^{258} - e^{267})e_3
+ (-e^{123} + e^{356} - e^{378} - e^{257} - e^{268} - e^{158} + e^{167})e_4
+ (e^{126} - e^{346} + e^{137} + e^{247} + e^{148} - e^{238} + e^{678})e_5
+ (-e^{125} + e^{345} + e^{138} + e^{248} - e^{147} + e^{378} - e^{578})e_6
+ (-e^{128} + e^{348} - e^{135} - e^{245} + e^{146} - e^{236} + e^{568})e_7
+ (e^{127} - e^{347} - e^{136} - e^{246} - e^{145} + e^{356} - e^{567})e_8.
$$

This time we show that the form $\Upsilon$ induce $G_2$ structures on certain subbundles of $T(N)$. Let $\gamma$ be a nowhere vanishing vector field of $N$. We define a $G_2$ structure $\varphi_\gamma$ on the 7-plane bundle $V_\gamma := \gamma^\perp$ by (where $*_8$ is the star operator on $N^8$)

$$
\varphi_\gamma := \langle \Upsilon, \gamma \rangle = \gamma^\perp \Psi = *_8(\Psi \wedge \gamma^\#).
$$

Assuming that $V_\gamma$ comes from a foliation, we let $M_\gamma$ be an integral submanifold of $V_\gamma$. We have $d\varphi_\gamma = 0$, provided $\mathcal{L}_\gamma(\Psi)|_{V_\gamma} = 0$. On the other hand, we always have $d(*\varphi_\gamma) = 0$ on $M_\gamma$. To see this, we use

$$
\Psi = \varphi_\gamma \wedge \gamma^\# + *\gamma \varphi_\gamma
$$

where $*\gamma$ is the star operator on $M_\gamma$, and use $d\Psi = 0$ and the foliation condition $d\gamma^\# \wedge \gamma^\# = 0$, and the identity $\theta|_{M_\gamma} = \gamma^\perp [\theta \wedge \gamma^\#]$ for forms $\theta$. In order to state the next theorem, we need an definition:

**Definition 10.** A manifold with $G_2$ structure $(M, \varphi)$ is called an almost $G_2$-manifold if $\varphi$ is co-closed.
Theorem 8. Let $(N^8, \Psi)$ be a Spin$(7)$ manifold, and $\gamma$ be a unit vector field which comes from a foliation, then $(M_\gamma, \varphi_\gamma)$ is an almost $G_2$ manifold. Furthermore if $\mathcal{L}_\gamma(\Psi)|_{M_\gamma} = 0$ then $(M_\gamma, \varphi_\gamma)$ is a $G_2$ manifold.

Proof. Follows by checking Definition 1 and by the discussion above.

The following theorem says the induced $G_2$ structures on $M_\alpha, M_\beta$ determine each other via $\Psi$; more specifically $\varphi_\alpha$ and $\varphi_\beta$ are restrictions of a global 3-form of $N$.

Proposition 9. Let $(N, \Psi)$ be a Spin$(7)$ manifold, and $\{\alpha, \beta\}$ be an orthonormal vector fields on $N$. Then the following holds on $M_\alpha$

$$\varphi_\alpha = -\alpha \lrcorner (\varphi_\beta \wedge \beta \# + \ast_7 \varphi_\beta)$$

Proof. The proof follows from the definitions, and by expressing $\varphi_\alpha$ and $\varphi_\beta$ in terms of $\beta \#$ and $\alpha \#$ by the formula (13).

As in the $G_2$ case, by choosing different $\gamma$'s, one can find various different $G_2$ manifolds $M_\gamma$ with interesting structures. Most interestingly, we will get certain “dual” $M_\gamma$’s by choosing $\gamma$ in $K$ or in $D$. This will shed light on more general version of mirror symmetry of Calabi-Yau manifolds. First we will discuss an example.

4.1. An example.

Let $T^8 = T^4 \times T^4$ be the Spin$(7)$ 8-torus, where $\{e_1, e_2, e_3, e_4\}$ is the basis for the Cayley $T^4$ and $\{e_5, e_6, e_7, e_8\}$ is the basis for the complementary $T^4$. We can take the corresponding calibration 4-form (20) above, and take the decomposition $T(N) = K \oplus D$, where $\{e_1, e_2, e_3, e_4\}$ is the orthonormal basis for the Cayley bundle $K$, and $\{e_5, e_6, e_7, e_8\}$ is the local orthonormal basis for the complementary bundle $D$. Then if we choose $\gamma = e_4 = e_1 \times e_2 \times e_3$ then we get

$$\varphi_{\gamma} = -e^{123} + e^{356} - e^{378} - e^{257} - e^{268} - e^{158} + e^{167}$$

On the other hand, if we choose $\gamma' = e_5$ then we get

$$\varphi_{\gamma'} = e^{126} - e^{346} + e^{137} + e^{247} + e^{148} - e^{238} + e^{678}$$

which give different $G_2$ structures on the 7 toris $M_\gamma$ and $M_\gamma'$.

Note that if we choose $\gamma$ from the Cayley bundle $K$, we get the $G_2$ structure on the 7-torus $M_\gamma$ which reduces the Cayley 4-torus $T^4 = T^3 \times S^1$ (where $\gamma$ is tangent to $S^1$ direction) to an associative 3-torus $T^3 \subset M_\gamma$ with respect to this $G_2$ structure. On the other hand if we choose $\gamma'$ from the complementary bundle $D$, then the Cayley 4-torus $T^4$ will be a coassociative submanifold of the 7-torus $M_\gamma$ with the corresponding $G_2$ structure. Hence associative and coassociative geometries are dual to each other as they are induced from the same calibration 4-form $\Psi$ on a
Spin(7) manifold. This suggests the following definition of the “mirror duality” for $G_2$ manifolds.

**Definition 11.** Two 7-manifolds with $G_2$ structures are mirror pairs, if their $G_2$-structures are induced from the same calibration 4-form in a Spin(7) manifold. Furthermore they are strong duals if their normal vector fields are homotopic.

**Remark 7.** For example, by [BS] the total space of an $\mathbb{R}^4$ bundle over $S^4$ has a Spin(7) structure. By applying the process of Example 1, we obtain mirror pairs $M_\gamma$ and $M_{\gamma'}$ to be $S^3 \times \mathbb{R}^4$ and $\mathbb{R}^4 \times S^3$ with dual $G_2$ structures.

### 4.2. Dual Calabi-Yau’s inside of Spin(7).

Let $(N^8, \Psi)$ be a Spin(7) manifold, and let $\{\alpha, \beta\}$ be an orthonormal 2-frame field in $N$, each coming from a foliation. Let $(M_\alpha, \varphi_\alpha)$ and $(M_\beta, \varphi_\beta)$ be the $G_2$ manifolds given by Theorem 8. Similarly by Theorem 5, the vector fields $\beta$ in $M_\alpha$, and $\alpha$ in $M_\beta$ give almost Calabi-Yau’s $X_{\alpha\beta} \subset M_\alpha$ and $X_{\beta\alpha} \subset M_\beta$. Let us denote $X_{\alpha\beta} = (X_{\alpha\beta}, \omega_{\alpha\beta}, \Omega_{\alpha\beta}, J_{\alpha\beta})$ likewise $X_{\beta\alpha} = (X_{\beta\alpha}, \omega_{\beta\alpha}, \Omega_{\beta\alpha}, J_{\beta\alpha})$. Then we have

**Proposition 10.** The following relations hold:

(i) $J_{\alpha\beta}(u) = u \times \beta \times \alpha$

(ii) $\omega_{\alpha\beta} = \beta \llcorner \alpha \llcorner \Psi$

(iii) $\Re \Omega_{\alpha\beta} = \alpha \llcorner \Psi|_{X_{\alpha\beta}}$

(iv) $\Im \Omega_{\alpha\beta} = \beta \llcorner \Psi|_{X_{\alpha\beta}}$

**Proof.** (i), (ii), and (iii) follow from definitions, and the formula $X \times Y = (X \llcorner Y \llcorner \varphi)^\#$.

\[
\begin{align*}
\Im \Omega_{\alpha\beta} &= \beta \llcorner \ast_7 \varphi_\alpha = \beta \llcorner \ast_7 (\alpha \llcorner \Psi) \\
&= \beta \llcorner [\alpha \llcorner \ast_8 (\alpha \llcorner \Psi)] \\
&= \beta \llcorner [\alpha \llcorner (\alpha^\# \wedge \Psi)] \quad \text{by (10)} \\
&= \beta \llcorner [\Psi - \alpha^\# \wedge (\alpha \llcorner \Psi)] \\
&= \beta \llcorner [\Psi - \alpha^\# \wedge (\beta \llcorner \alpha \llcorner \psi)]
\end{align*}
\]

Left hand side is already defined on $X_{\alpha\beta}$, by restricting to $X_{\alpha\beta}$ we get (iii). \(\square\)

**Corollary 11.** When $X_{\alpha\beta}$ and $X_{\beta\alpha}$ coincide, they are oppositely oriented manifolds and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, and $\Re \Omega_{\alpha\beta} = -\Im \Omega_{\beta\alpha}$ (as forms on $X_{\alpha\beta}$).

Now let $\{\alpha, \beta, \gamma\}$ be an orthonormal 3-frame field in $(N^8, \Psi)$, and let $(M_\alpha, \varphi_\alpha)$, $(M_\beta, \varphi_\beta)$, and $(M_\gamma, \varphi_\gamma)$ be the corresponding almost $G_2$ manifolds. As before, the
orthonormal vector fields \( \{ \gamma, \beta \} \) in \( M_\alpha \) and \( \{ \gamma, \alpha \} \) in \( M_\beta \) give rise to corresponding almost Calabi-Yau’s \( X_{\alpha, \gamma}, X_{\alpha, \beta} \) in \( M_\alpha \), and \( X_{\beta, \gamma}, X_{\beta, \alpha} \) in \( M_\beta \).

In this way \((N^8, \Psi)\) gives rise to 4 Calabi-Yau descendents. By Corollary 11, \( X_{\alpha \beta} \) and \( X_{\beta \alpha} \) are different geometrically; they may not even be the same as smooth manifolds, but for simplicity we may consider it to be the same smooth manifold obtained from the triple intersection of the three \( G_2 \) manifolds.

In case we have a decomposition \( T(N) = K \oplus D \) of the tangent bundle of \((N^8, \Psi)\) by Cayley plus its orthogonal bundles (Section 4); we can choose our frame special and obtain interesting CY-manifolds. For example, if we choose \( \alpha \in \Omega^0(M, K) \) and \( \beta, \gamma \in \Omega^0(N, D) \) we get one set of complex structures, whose types are indicated by the first row of the following diagram. On the other hand, if we choose all \( \{ \alpha, \beta, \gamma \} \) lie entirely in \( K \) or \( D \) we get another set of complex structures, as indicated by the second row of the diagram.

\[
\begin{array}{cccc}
(N^8, \Psi) & (M_\alpha, \varphi_\alpha) & (M_\beta, \varphi_\beta) \\
X_{\alpha \gamma} & X_{\alpha \beta} & X_{\beta \gamma} & X_{\beta \alpha} \\
SU(3) & SU(3) & SU(2) & SU(3) \\
SU(2) & SU(2) & SU(2) & SU(2)
\end{array}
\]

Here all the corresponding symplectic and the holomorphic forms of the resulting Calabi-Yau’s come from restriction of global forms induced by \( \Psi \). The following gives relations between the complex/symplectic structures of these induced CY-manifolds; i.e. the structures \( X_{\alpha \gamma}, X_{\beta \gamma} \) and \( X_{\alpha \beta} \) satisfy a certain triality relation.

**Proposition 12.** We have the following relations;

(i) \( Re \Omega_{\alpha \gamma} = \alpha \lrcorner (\ast_6 \omega_{\beta \gamma}) + \omega_{\alpha \beta} \wedge \beta \# \)

(ii) \( Im \Omega_{\alpha \gamma} = \omega_{\beta \gamma} \wedge \beta \# - \gamma \lrcorner \ast_6 (\omega_{\alpha \beta}) \)

(iii) \( \omega_{\alpha \gamma} = \alpha \lrcorner Im \Omega_{\beta \gamma} + (\gamma \lrcorner \omega_{\alpha \beta}) \wedge \beta \# \)

First we need to prove a lemma;

**Lemma 13.** The following relations hold:

\[
\begin{align*}
\alpha \lrcorner \ast_6 (\omega_{\beta \gamma}) &= \alpha \lrcorner \Psi + \gamma \# \wedge (\alpha \lrcorner \gamma \lrcorner \Psi) + \beta \# \wedge (\alpha \lrcorner \beta \lrcorner \Psi) - \gamma \# \wedge \beta \# \wedge (\alpha \lrcorner \gamma \lrcorner \beta \lrcorner \Psi). \\
Im \Omega_{\gamma \beta} &= -\gamma \lrcorner \Psi - \beta \# \wedge (\gamma \lrcorner \beta \lrcorner \Psi). \\
Re \Omega_{\gamma \beta} &= \alpha \lrcorner \Psi - \gamma \# \wedge (\gamma \lrcorner \alpha \lrcorner \Psi).
\end{align*}
\]
Proof.

\[ \alpha \downarrow \ast_6 (\omega_{\beta\gamma}) = \alpha \downarrow \gamma \downarrow \beta \downarrow \ast_8 (\gamma \downarrow \beta \downarrow \Psi) \]
\[ = -\alpha \downarrow \gamma \downarrow \beta \downarrow \ast_6 (\gamma \# \land \beta \# \land \Psi) \]
\[ = -\alpha \downarrow \gamma \downarrow \beta \downarrow [-\gamma \# \land \Psi + \gamma \# \land (\beta \downarrow \Psi)] \]
\[ = \alpha \downarrow \Psi + \gamma \# \land (\alpha \downarrow \gamma \downarrow \Psi) + \beta \# \land (\alpha \downarrow \beta \downarrow \Psi) \]
\[ -\gamma \# \land \beta \# \land (\alpha \downarrow \gamma \downarrow \beta \downarrow \Psi). \]

\[ \text{Im} \Omega_{\beta\gamma} = \gamma \downarrow \ast_7 (\beta \downarrow \Psi) \]
\[ = \gamma \downarrow \beta \downarrow \ast_8 (\beta \downarrow \Psi) \]
\[ = -\gamma \downarrow \beta \downarrow (\beta \# \land \Psi) \]
\[ = -\gamma \downarrow \Psi - \beta \# \land (\gamma \downarrow \beta \downarrow \Psi). \]

\[ \text{Re} \Omega_{\alpha\gamma} = (\alpha \downarrow \Psi)|_{X \alpha\gamma} = \gamma \downarrow [\gamma \# \land (\alpha \downarrow \Psi)] = \alpha \downarrow \Psi - \gamma \# \land (\gamma \downarrow \alpha \downarrow \Psi). \]

Proof of Proposition 12. We calculate the following by using Lemma 13:

\[ \alpha \downarrow \ast_6 (\omega_{\beta\gamma}) + \omega_{\alpha\beta} \land \beta \# = \alpha \downarrow \Psi \land (\alpha \downarrow \gamma \downarrow \Psi) + \beta \# \land (\alpha \downarrow \beta \downarrow \Psi) \]
\[ -\gamma \# \land \beta \# \land (\alpha \downarrow \gamma \downarrow \beta \downarrow \Psi) + (\beta \downarrow \alpha \downarrow \Psi) \land \beta \# \]
\[ = \alpha \downarrow \Psi - \gamma \# \land (\gamma \downarrow \alpha \downarrow \Psi). \]

Since we are restricting to \( X_{\alpha\gamma} \) we can throw away terms containing \( \gamma \# \) and get (i). We prove (ii) similarly:

\[ \text{Im} \Omega_{\alpha\gamma} = (\gamma \downarrow \ast_7 \varphi_{\alpha}) \]
\[ = \gamma \downarrow \alpha \downarrow \ast_8 (\alpha \downarrow \Psi) \]
\[ = -\gamma \downarrow \Psi + \gamma \downarrow [\alpha \# \land (\alpha \downarrow \Psi)] \]
\[ = -\gamma \downarrow \Psi - \alpha \# \land (\gamma \downarrow \alpha \downarrow \Psi). \]

\[ \omega_{\beta\gamma} \land \beta \# = \gamma \downarrow (\ast_6 \omega_{\alpha\beta}) + (\gamma \downarrow \beta \downarrow \Psi) \land \beta \# \]
\[ = -\gamma \downarrow \Psi - \alpha \# \land (\gamma \downarrow \beta \downarrow \Psi) - \alpha \# \land (\gamma \downarrow \alpha \downarrow \Psi) \]
\[ + \beta \# \land (\gamma \downarrow \alpha \downarrow \Psi). \]

Here, we used Lemma 13 with different indices \((\alpha, \beta, \gamma) \mapsto (\gamma, \alpha, \beta)\), and since we are restricting to \( X_{\alpha\gamma} \) we threw away terms containing \( \alpha \# \). Finally, (iii) follows by plugging in Lemma 13 to definitions.

Following says that Calabi-Yau structures of \( X_{\alpha\gamma} \) and \( X_{\beta\gamma} \) determine each other via \( \Psi \). Proposition 14 is basically a consequence of Proposition 6 and Corollary 11.
Proposition 14. We have the following relations

(i) \( \text{Re} \Omega_{\alpha\gamma} = \alpha \cdot (\ast_6 \omega_{\beta\gamma}) - (\alpha \cdot \text{Re} \Omega_{\beta\gamma}) \land \beta^\# \)

(ii) \( \text{Im} \Omega_{\alpha\gamma} = \omega_{\beta\gamma} \land \beta^\# + \text{Im} \Omega_{\beta\gamma} \)

(iii) \( \omega_{\alpha\gamma} = \alpha \cdot \text{Im} \Omega_{\beta\gamma} + (\alpha \cdot \omega_{\beta\gamma}) \land \beta^\# \)

Proof. All follow from the definitions and Lemma 11 (and by ignoring \( \alpha^\# \) terms).

Remark 8. After this paper was written we learned that the results similar to Theorem 5 already appeared in [AW], [ASa], [C], [CS] (where they use more restricted vector fields \( \xi \)), and we also found out that the idea of studying induced hypersurface structures, from manifolds with exceptional holonomy goes back to earlier works of Calabi and Gray.

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MI, 48824

E-mail address: akbulut@math.msu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NY, 14627

E-mail address: salur@math.rochester.edu