Noether-Lefschetz locus and generalisation of an example
due to Mumford

Ananyo Dan∗

September 23, 2014

Abstract

In this article we generalize the well-known example due to Mumford for a generically
non-reduced component of the Hilbert scheme of curves in \( \mathbb{P}^3 \) whose general element is
smooth. The example considers smooth curves in smooth cubic surfaces in \( \mathbb{P}^3 \). In this
article we give similar examples of generically non-reduced component of the Hilbert scheme
of curves in \( \mathbb{P}^3 \), for every integer \( d \geq 5 \), whose general element is a smooth curve contained
in a smooth degree \( d \) surface in \( \mathbb{P}^3 \) and not in any surface of smaller degree. The techniques
used are motivated by the study of Noether-Lefschetz locus.

1 Introduction

With Grothendieck’s construction of the Hilbert scheme one can give a scheme structure to
families of curves, which up to then were described only as algebraic varieties. In 1962, only
a few years after Grothendieck introduced the Hilbert scheme, Mumford [Mum62] showed that
there exists generically non-reduced (in the sense, the localization of the structure sheaf at every
point contains a non-trivial nilpotent element) irreducible components of the Hilbert scheme of
curves in \( \mathbb{P}^3 \) such that a general element is a smooth curve contained in a cubic surface in \( \mathbb{P}^3 \).
This example inspired the investigation of such components. Kleppe shows in [Kle81] that an
irreducible component \( L \) of the Hilbert scheme of curves parametrizing smooth curves contained
in a cubic surface in \( \mathbb{P}^3 \) is non-reduced if and only if for a general \( C \in L \) and a smooth cubic
surface \( X \) containing \( C \), \( h^1(O_X(-C)(3)) \neq 0 \). Using this condition he gives examples in [Kle85]

*The author has been supported by the DFG under Grant KL-2244/2 − 1

Humboldt Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, Berlin 10099.
e-mail: dan@mathematik.hu-berlin.de
Mathematics Subject Classification: 14C30, 14D07, 13D10
for each integer $d \geq 5$, non-reduced components of the Hilbert scheme of smooth space curves contained in a smooth degree $d$ surface in $\mathbb{P}^3$ but not in any surface (in $\mathbb{P}^3$) of smaller degree. The main tool used in this article comes from the study of Hodge loci, which contrasts previous approaches.

We recall briefly the main ideas used in [Kle81] to illustrate the difficulty in producing such examples. The most important observation is that for a smooth curve $C$ in a smooth cubic surface $X$ in $\mathbb{P}^3$ $H^1(N_C|X) = 0$ (since $H^0(N_C|X)^\vee \cong H^0(N_C^\vee|X \otimes K_C)$ where $K_C$ is the canonical divisor and $\deg(N_C^\vee|X \otimes K_C) = 2\rho_a(C) - 2 - C^2 \equiv -\deg(C) < 0$). Therefore, the natural morphism from $H^0(N_C|\mathbb{P}^3)$ to $H^0(N_X|\mathbb{P}^3 \otimes O_C)$ is surjective (use the normal short exact sequence). This means (using the normal pull-back morphism, say $\rho'$ from $H^0(N_X|\mathbb{P}^3)$ to $H^0(N_X|\mathbb{P}^3 \otimes O_C)$ and basic knowledge of flag Hilbert scheme) for any infinitesimal deformation of $X$ in $\mathbb{P}^3$, there exists a corresponding infinitesimal deformation of $C$ contained in this. Furthermore, if $\rho'$ is not surjective then there exists an infinitesimal deformation of $C$ not corresponding to any infinitesimal deformation of $X$. This condition is equivalent to $h^1(O_X(-C)(3)) \neq 0$ (use $N_X|\mathbb{P}^3 \cong O_X(3)$ and $H^1(O_X(3)) = 0$). An easy dimension count tells us that this is a necessary and sufficient condition for the corresponding irreducible component to be non-reduced (an important assumption used in this step is that a curve corresponding to a general point in this component is contained in a cubic surface in $\mathbb{P}^3$). For $d \geq 5$, $H^1(N_C|X)$ for a smooth curve $C$ in $X$ is never zero. So, it is not possible to duplicate this approach for $d \geq 5$, i.e., for finding non-reduced irreducible components of the Hilbert scheme of smooth curves contained in a smooth degree $d$ surface not contained in a surface of degree smaller than $d$.

We instead use results from the theory of Noether-Lefschetz locus to produce such examples. There are numerous examples of non-reduced components of the Noether-Lefschetz locus (see [Dan14, Theorems 6.16, 6.17]) which is the starting point for our study. Moreover, the tangent space at a point on the Noether-Lefschetz locus has an explicit description in terms of commutative algebra (see [Voi03, §6.2]). This suggest that using standard computer programming one can produce further examples of non-reduced components of the Noether-Lefschetz locus which would in turn give new examples of non-reduced irreducible components of the Hilbert scheme of smooth space curves. However, the second approach has not been explored in this article.

The first main result in this article gives a cohomological criterion for the existence of the
aforementioned components. But before we proceed, we need to recall the notion of Hodge locus. Consider, \(U_d \subseteq P(\mathbb{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)))\) the open subscheme parametrizing smooth projective hypersurfaces in \(\mathbb{P}^3\) of degree \(d\). Let \(X \overset{\pi}{\rightarrow} U_d\) be the corresponding universal family. For a given \(F \in U_d\), denote by \(X_F := \pi^{-1}(F)\). Let \(X \in U_d\) and \(U \subseteq U_d\) be a simply connected neighbourhood of \(X\) in \(U_d\) (under the analytic topology). Then \(\pi_{\mid \pi^{-1}(U)}\) induces a variation of Hodge structure \((\mathcal{H}, \nabla)\) on \(U\) where \(\mathcal{H} := R^2\pi_* \mathbb{Z} \otimes \mathcal{O}_U\) and \(\nabla\) is the Gauss-Manin connection. Note that \(\mathcal{H}\) defines a local system on \(U\) whose fiber over a point \(F \in U\) is \(\mathcal{H}^2(X_F, \mathbb{Z})\) where \(X_F := \pi^{-1}(F)\). Consider a non-zero element \(\gamma_0 \in \mathcal{H}^2(X_F, \mathbb{Z}) \cap \mathcal{H}^{1,1}(X_F, \mathbb{C})\) such that \(\gamma_0 \neq c_1(\mathcal{O}_{X_F}(k))\) for \(k \in \mathbb{Z}_{\geq 0}\). This defines a section \(\gamma \in (\mathcal{H} \otimes \mathbb{C})(U)\). Let \(\gamma\) be the image of \(\gamma\) in \(\mathcal{H}/F^2(\mathcal{H} \otimes \mathbb{C})\). The Hodge loci, denoted \(NL(\gamma)\) is then defined as

\[NL(\gamma) := \{G \in U | \gamma_G = 0\},\]

where \(\gamma_G\) denotes the value at \(G\) of the section \(\gamma\). See [Voi03, §5] for a detailed study of the subject. We then prove,

**Theorem 1.1.** Let \(X\) be a smooth degree \(d\) surface, \(\gamma \in H^{1,1}(X, \mathbb{Z})\) and \(C\) be a smooth semi-regular curve in \(X\) such that \(\gamma - [C]\) is a multiple of the class of the hyperplane section \(H_X\), where \([C]\) is the cohomology class of \(C\). Let \(P'\) be the Hilbert polynomial of \(C\). If \(h^1(\mathcal{O}_X(-C)(d)) = 0 = h^0(\mathcal{O}_X(-C)(d))\) and \(NL(\gamma)\) (closure taken in \(U_d\) under Zariski topology) is irreducible generically non-reduced then there is an irreducible generically non-reduced component of the Hilbert scheme \(\text{Hilb}_{P'}\) containing \(C\) and parametrizing smooth curves in \(\mathbb{P}^3\). Furthermore, given \(\gamma\) there always exists such a \(C\) (i.e., smooth, semi-regular and \(\gamma - [C]\) is a multiple of \(H_X\)) and \(C\) is not contained in a surface of degree less than \(d\).

See Theorems 3.3 and 4.1 for further details.

Combining this result with a result from Noether-Lefschtez locus proven in [Dan11] we conclude that

**Theorem 1.2.** For \(d \geq 5\) and \(m \gg d\), there exists a generically non-reduced irreducible component of the Hilbert scheme parametrizing smooth curves in \(\mathbb{P}^3\):

1. of degree \(md + 3\) and the arithmetic genus \(1 + (1/2)(md^2 + d(m^2 - 4m - 2) + 6m + 2)\),
2. a generic element in this component corresponds to a smooth curve contained in a smooth
degree $d$ surface in $\mathbb{P}^3$ but not in any surface of smaller degree

See Corollary 4.3 and Lemma 4.4 for a proof of the statement.

One of the main ideas that we exploit is that the Hodge locus $NL(\gamma)$ of the cohomology class,
say $\gamma$ of a divisor $D$ on a surface $X$ is invariant if we translate it by a multiple of the hyperplane
section. By twisting the line bundle $\mathcal{O}_X(D)$ by some multiple of the hyperplane section, we
can conclude that a general curve in the resulting linear system is smooth and semi-regular in
the sense of Bloch (see [Blo72]). Then the Hodge locus and the flag Hilbert schemes are closely
related. In particular, if we denote by $P$ the Hilbert polynomial of this curve, there exists an
irreducible component $H_\gamma$ of the flag Hilbert scheme $Hilb_{P,Q}$ such that $pr_2(H_\gamma)_{\text{red}} \cong NL(\gamma)_{\text{red}}$
and $H_\gamma$ is non-reduced if and only if so is $NL(\gamma)$. The only point that needs to be checked is
that $pr_1(H_\gamma)$ is in fact an irreducible component of the Hilbert scheme of curves corresponding
to $P$. This is shown in Proposition 3.2.

Notation 1.3. We fix once and for all a few notations that will be used throughout this arti-
cle. By a surface or a curve we mean a scheme of pure dimension 2 or 1, respectively in $\mathbb{P}^3$.
For a Hilbert polynomial $P$ of a curves or a surface in $\mathbb{P}^3$, denote by $H_P$ the Hilbert scheme
parametrizing all subschemes in $\mathbb{P}^3$ with Hilbert polynomial $P$. Denote by $Q_d$ the Hilbert poly-
nomial of a degree $d$ surface in $\mathbb{P}^3$. For a pair of Hilbert polynomials $P, Q_d$, denote by $H_{P,Q_d}$ the
corresponding flag Hilbert scheme.

Acknowledgement: I would like to thank R. Kloosterman and K. R"{u}lling for reading the
article and helpful feedbacks.

2 Preliminaries

2.1. In this section we recall the basic definitions of Noether-Lefschetz locus. See [Voi02] §9, 10]
and [Voi03] §5, 6] for a detailed presentation of the subject.

Definition 2.2. Recall, for a fixed integer $d \geq 5$, the Noether-Lefschetz locus, denoted $NL_d$,
parametrizes the space of smooth degree $d$ surfaces in $\mathbb{P}^3$ with Picard number greater than 1.
Using the Lefschetz (1, 1)-theorem this is the parametrizing space of smooth degree $d$ surfaces with $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}) \neq \mathbb{Z}$.

**Notation 2.3.** Let $X \in U_d$ and $\mathcal{O}_X(1)$, the very ample line bundle on $X$ determined by the closed immersion $X \hookrightarrow \mathbb{P}^3$ arising (as in [Har77 II, Ex.2.14(b)]) from the graded homomorphism of graded rings $S \to S/(F_X)$, where $S = \Gamma_*(\mathcal{O}_{\mathbb{P}^3})$ and $F_X$ is the defining equations of $X$. Denote by $H_X$ the very ample line bundle $\mathcal{O}_X(1)$, $\mathcal{O}_X(1)$. Denote by $H^2(X, \mathbb{C})_{\text{prim}}$ the primitive cohomology. Given $\gamma \in H^2(X, \mathbb{C})$, denote by $\gamma_{\text{prim}}$ the image of $\gamma$ under the natural morphism from $H^2(X, \mathbb{C})$ to $H^2(X, \mathbb{C})_{\text{prim}}$. Since the very ample line bundle $H_X$ remains of type (1, 1) in the family $X$, we can therefore conclude that $\gamma \in H^{1,1}(X)$ remains of type (1, 1) if and only if $\gamma_{\text{prim}}$ remains of type (1, 1). In particular, $\text{NL}(\gamma) = \text{NL}(\gamma_{\text{prim}})$.

**2.4.** Note that, $\text{NL}_d$ is a countable union of subvarieties. Every irreducible component of $\text{NL}_d$ is locally of the form $\text{NL}(\gamma)$ for some $\gamma \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$, $X \in \text{NL}_d$ such that $\gamma_{\text{prim}} \neq 0$.

There is a natural analytic scheme structure on $\text{NL}(\gamma)$ (see [Voi03, Lemma 5.13]).

**Definition 2.5.** We now discuss the tangent space to the Hodge locus, $\text{NL}(\gamma)$. We know that the tangent space to $U$ at $X$, $T_XU$ is isomorphic to $H^0(N_X|_{\mathbb{P}^3})$. This is because $U$ is an open subscheme of the Hilbert scheme $H_{Q_d}$, the tangent space of which at the point $X$ is simply $H^0(N_X|_{\mathbb{P}^3})$. Given the variation of Hodge structure above, we have (by Griffith’s transversality) the differential map:

$$\nabla : H^{1,1}(X) \to \text{Hom}(T_XU, H^2(X, \mathcal{O}_X))$$

induced by the Gauss-Manin connection. Given $\gamma \in H^{1,1}(X)$ this induces a morphism, denoted $\nabla(\gamma)$ from $T_XU$ to $H^2(\mathcal{O}_X)$.

**Lemma 2.6 ([Voi03, Lemma 5.16]).** The tangent space at $X$ to $\text{NL}(\gamma)$ is equal to $\ker(\nabla(\gamma))$.

Another important notion that will be used in this article is that of semi-regularity. We recall first the definition.

**Definition 2.7.** Let $X$ be a surface and $C \subset X$, a curve in $X$. Since $X$ is smooth, $C$ is local complete intersection in $X$. Denote by $i$ the closed immersion of $C$ into $X$. This gives rise to the short exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{N}_{C|X} \to 0$$

(1)
where \( \mathcal{N}_{C|X} \) is the normal sheaf of \( C \) in \( X \). The \textit{semi-regularity map} \( \pi \) is the boundary map from \( H^1(\mathcal{N}_{C|X}) \) to \( H^2(\mathcal{O}_X) \). We say that \( C \) is \textit{semi-regular} if \( \pi \) is injective.

**Theorem 2.8** ([Dan14, Theorem 4.8]). Let \( X \) be a smooth degree \( d \) surface and \( C \) be a curve in \( X \). Let \( \gamma = [C] \in H^{1,1}(X, \mathbb{Z}) \), the cohomology class of \( C \). Denote by \( P \) the Hilbert polynomial of \( C \). We have the following commutative diagram

\[
\begin{array}{cccccc}
T_{(C,X)}H_{P,Q,d} & H^0(X, \mathcal{N}_X|\mathbb{P}^3) & \nabla(\gamma) & H^2(X, \mathcal{O}_X) \\
\downarrow & \rho_C & \uparrow & \pi_C \\
0 & H^0(C, \mathcal{N}_{C|X}) & \phi_C & H^0(C, \mathcal{N}_{C|\mathbb{P}^3}) & \beta_C & H^0(C, \mathcal{N}_X|\mathbb{P}^3 \otimes \mathcal{O}_C) & \delta_C & H^1(C, \mathcal{N}_{C|X})
\end{array}
\]

where the horizontal exact sequence comes from the normal short exact sequence

\[
0 \to \mathcal{N}_{C|X} \to \mathcal{N}_{C|\mathbb{P}^3} \to \mathcal{N}_X|\mathbb{P}^3 \otimes \mathcal{O}_C \to 0,
\]

\( \pi_C \) is the semi-regularity map and \( \rho_C \) is the natural pull-back morphism.

**Corollary 2.9.** Let \( X \) be a smooth degree \( d \) surface in \( \mathbb{P}^3 \), \( C \subset X \) a semi-regular curve satisfying \( H^1(\mathcal{O}_X(-C)(d)) = 0 = H^0(\mathcal{O}_X(-C)(d)) \). Then,

\[
\dim T_X(\text{NL}([C])) = h^0(\mathcal{N}_{C|\mathbb{P}^3}) - h^0(\mathcal{N}_{C|X}),
\]

where \([C]\) is the cohomology class of \( C \).

**Proof.** Notations as in the diagram in Theorem 2.8. Let \( \gamma = [C] \). Since \( C \) is semi-regular, \( \pi_C \) is injective. It follows directly from the above theorem that,

\[
T_X(\text{NL}(\gamma)) = \ker(\nabla(\gamma)) = \ker(\delta_C \circ \rho_C) = \rho_C^{-1}(\text{Im } \beta_C).
\]

Using the long exact sequence associated to

\[
0 \to \mathcal{O}_X(-C)(d) \to \mathcal{O}_X(d) \to i_* \mathcal{O}_C(d) \to 0,
\]
Recall, the following theorem which describes the relation between the Hodge locus to the cohomology class of a semi-regular curve $C$ and deformation of a surface $X$ containing $C$ such that $C$ remains a curve under deformation. a curve.

**Theorem 2.10** ([Dan14, Theorem 5.7]). Let $X$ be a surface, $C$ be a semi-regular curve in $X$ and $\gamma \in H^{1,1}(X,\mathbb{Z})$ be the class of the curve $C$. For any irreducible component $L'$ of $\overline{NL(\gamma)}$ (the closure is taken in the Zariski topology on $U_d$) there exists an irreducible component $H'$ of $H_{P,Q,d}^{red}$ containing the pair $(C,X)$ such that $pr_2(H')$ coincides with the associated reduced scheme $L'_\text{red}$, where $pr_2$ is the second projection map from $H_{P,Q,d}$ to $H_{Q,d}$.

### 3 General criteria for non-reducedness

3.1. In this section we give criterion in terms of the vanishing of certain cohomology groups under which there exists irreducible, *generically non-reduced* components of the Hilbert scheme of curves in $\mathbb{P}^3$ parametrizing *smooth* curves contained in a smooth degree $d$ surface but not in a surface of lower degree. We later use these criteria to produce several examples.

**Proposition 3.2.** Let $P_0$ be the Hilbert polynomial of a curve $C$ in $\mathbb{P}^3$. Assume that there exists an integer $d$ and a smooth degree $d$ surface, say $X$ containing $C$, such that $h^1(\mathcal{O}_X(-C)(d)) = 0 = h^0(\mathcal{O}_X(-C)(d))$. Let $L$ be an irreducible component of $H_{P_0}$ containing $C$. Then, for a general element $D \in L$, $h^0(\mathcal{I}_D(d)) > 0$ i.e., $D$ is contained in a smooth degree $d$ surface.

**Proof.** Denote by $i$ the natural closed immersion of $C$ into $X$. It suffices to prove that $h^0(\mathcal{O}_C(d)) < h^0(\mathcal{O}_{\mathbb{P}^3}(d))$. Then by upper semi-continuity, $h^0(\mathcal{O}_D(d)) < h^0(\mathcal{O}_{\mathbb{P}^3}(d))$ as $D$ varies over an open neighbourhood of $C$ in the Hilbert scheme $H_{P_0}$. For $j : D \hookrightarrow \mathbb{P}^3$, the closed immersion, the short exact sequence

$$0 \rightarrow \mathcal{I}_D(d) \rightarrow \mathcal{O}_{\mathbb{P}^3}(d) \rightarrow j_* \mathcal{O}_D(d) \rightarrow 0$$
implies that $h^0(\mathcal{O}_{\mathbb{P}^3}(d)) \leq h^0(\mathcal{I}_D(d)) + h^0(\mathcal{O}_D(d))$. Hence, we have $h^0(\mathcal{I}_D(d)) > 0$.

Using the short exact sequence

$$0 \to \mathcal{O}_X(-C)(d) \to \mathcal{O}_X(d) \to i_* \mathcal{O}_C(d) \to 0$$

we have, $h^0(\mathcal{O}_X(d)) = h^0(\mathcal{O}_C(d))$ because $h^0(\mathcal{O}_X(-C)(d)) = 0 = h^1(\mathcal{O}_X(-C)(d))$ by assumption. It then suffices to prove $h^0(\mathcal{O}_X(d)) < h^0(\mathcal{O}_{\mathbb{P}^3}(d))$. But this follows from the short exact sequence,

$$0 \to \mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}^3}(d) \to \mathcal{O}_X(d) \to 0$$

and the fact that $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-d)$. The proposition then follows.

Using this result we can show the following theorem:

**Theorem 3.3.** Let $X$ be a smooth degree $d$ surface, $\gamma \in H^{1,1}(X, \mathbb{Z})$ and $C$ be a smooth semi-regular curve in $X$. Assume that, $\gamma - [C]$ is a multiple of the class of the hyperplane section $H_X$, where $[C]$ is the cohomology class of $C$. Let $P'$ be the Hilbert polynomial of $C$. If $h^1(\mathcal{O}_X(-C)(d)) = 0 = h^0(\mathcal{O}_X(-C)(d))$ and $\overline{\text{NL}(\gamma)}$ is an irreducible generically non-reduced component of $\text{NL}_d$ then there is an irreducible generically non-reduced component of $H_{P'}$ containing $C$ and parametrizing smooth curves in $\mathbb{P}^3$.

**Proof.** Since $\gamma - [C]$ is a multiple of the hyperplane section $H_X$, $\overline{\text{NL}(\gamma)}$ is (scheme-theoretically) isomorphic to $\overline{\text{NL}([C])}$. Hence, $\overline{\text{NL}([C])}$ is generically non-reduced.

Since $C$ is semi-regular, Theorem 2.10 implies that there exists an irreducible component $H_\gamma$ of $H_{P',Q_{d,\text{red}}}$ such that $\text{pr}_2(H_\gamma)$ is isomorphic to $\text{NL}([C])_{\text{red}}$. Denote by $L_\gamma := \text{pr}_1(H_\gamma)$. Notice that the fiber over $C \in L_\gamma$ to the morphism $\text{pr}_1 : H_\gamma \to L_\gamma$ is isomorphic to $\mathbb{P}(I_0(C))$. Since $h^0(\mathcal{I}_C(d)) - 1 = h^0(\mathcal{O}_X(-C)(d)) = 0$, we have that $\text{pr}_1 : H_{P',Q_{d,\text{red}}} \to H_{P',\text{red}}$ is an isomorphism onto its image on an open neighbourhood of $C$. Proposition 3.2 implies that there exists an open neighbourhood $U \subset H_{P'}$ containing $C$ which is in the image of $\text{pr}_1$, hence $\text{pr}_1^{-1}(U)$ is isomorphic to $U$. Since $H_\gamma$ is an irreducible component of $H_{P',Q_{d,\text{red}}}$, $L_\gamma$ is an irreducible component of $H_{P',\text{red}}$ and

$$\dim \overline{\text{NL}([C])} + h^0(\mathcal{O}_X(C)) - 1 = \dim H_\gamma = \dim L_\gamma,$$
for a general pair \((C, X) \in H\gamma\), where the first equality follows from the fiber dimension theorem applied to the surjective projection map \(pr_2 : H_{\gamma} \to \text{NL}(\gamma)\).

Now, Corollary 2.9 implies that \(\dim T_X(\text{NL}(C)) = h^0(N_{C|\mathbb{P}^3}) - h^0(N_{C|X})\). Since \(h^0(N_{C|X}) = h^0(O_X(C)) - 1\) (see (1)),

\[
\dim T_X(\text{NL}(C)) - \dim \text{NL}(C) = h^0(N_{C|\mathbb{P}^3}) - \dim L_{\gamma}.
\]

This implies \(h^0(N_{C|\mathbb{P}^3}) > \dim L_{\gamma}\) for a general \(C \in L_{\gamma}\) because \(\text{NL}(C)\) is generically non-reduced. Since \(L_{\gamma}\) is an irreducible component of \((H_{P'})_{\text{red}}\), the corresponding component of \(H_{P'}\) is generically non-reduced. Since \(C\) is smooth, a curve corresponding to a general closed point on \(L_{\gamma}\) is smooth (see [Har77, Ex. III.10.2]). This completes the proof of the theorem.

4 Generalisation of the example of Mumford

We first show how to go from a curve in \(\mathbb{P}^3\) to a curve satisfying the conditions in Theorem 5.3. This in turn gives us a clue to produce examples of non-reduced components of Hilbert scheme parametrizing smooth curves. Using a result from the previous chapter, we give several examples. In particular, we prove Theorem 1.2.

**Theorem 4.1.** Let \(d \geq 5\) be an integer, \(X\) a smooth degree \(d\) surface and \(\gamma \in H^{1,1}(X, \mathbb{Z})\). Suppose that \(\gamma\) is the class of a (not necessarily effective) divisor \(C\) of the form \(\sum_i a_i C_i\). Assume that \(\text{NL}(\gamma)\) is an irreducible generically non-reduced component of \(\text{NL}_d\). Then, for \(m \gg 0\), there exists a smooth curve \(C'\) in the linear system corresponding to \(O_X(C)(m)\) satisfying: If \(P'\) is the Hilbert polynomial of \(C'\), there exists an irreducible generically non-reduced component of the Hilbert scheme \(H_{P'}\) containing \(C'\) such that a generic curve on this component is smooth and not contained in a surface of degree less than \(d\).

**Proof.** Using Serre’s vanishing theorem we have \(H^i(O_X(C)(m - 1 - i)) = 0\) for \(m \gg 0\) and \(i \geq 1\). Hence, \(O_X(C)(m - 1)\) is globally generated. Then, [Har77, Ex. II. 7.5(d)] states that \(O_X(C)(m)\) is very ample. Bertini’s theorem implies that a general curve \(C'\) in the linear system corresponding to \(O_X(C)(m)\) is smooth, semi-regular for \(m \gg 0\).

A lemma of Enriques-Severi-Zariski [Har77, Corollary III.7.7], tells us that for \(m \gg d\), we
have $H^1(\mathcal{O}_X(-C')(d)) = H^1(\mathcal{O}_X(-C)(d-m)) = 0$. Furthermore, $\deg(\mathcal{O}_X(-C)(d-m)) < 0$ for $m \gg 0$ implying that for such values of $m$, $H^0(\mathcal{O}_X(-C')(d)) = H^0(\mathcal{O}_X(-C)(d-m)) = 0$.

Denote by $P'$ the Hilbert polynomial of $C'$. Then, Theorem 3.3 implies that there exists an irreducible generically non-reduced component, say $L'$ of $H_{P'}$ containing $C'$ and parametrizing smooth curves.

It remains to prove that for a general $C_g \in L'$, there does not exist a smooth surface of smaller degree containing it. This is equivalent to saying that $H^0(\mathcal{I}_{C_g}(k)) = 0$ for all $k < d$. By the upper-semicontinuity theorem, it therefore suffices to show that $H^0(\mathcal{I}_{C'}(k)) = 0$ for all $k < d$. Since $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-d)$, $H^0(\mathcal{I}_X(k)) = 0$ for $k < d$. Since $\deg(\mathcal{O}_X(-C')(k)) = \deg(\mathcal{O}_X(-C)(-m+k)) < 0$, $H^0(\mathcal{O}_X(-C')(k)) = 0$ for $k < d$. So, the short exact sequence,

$$0 \to \mathcal{I}_X(k) \to \mathcal{I}_{C'}(k) \to \mathcal{O}_X(-C')(k) \to 0$$


tells us $H^0(\mathcal{I}_{C'}(k)) = 0$ for $k < d$. This completes the proof of the theorem. \hfill \square

We now recall some examples of non-reduced components of the Noether-Lefschetz locus.

**Theorem 4.2** ([Dan14, Theorem 6.17]). Let $d \geq 5$ and $C$ be a divisor in a smooth degree $d$ surface, say $X$, of the form $2l_1+l_2$, where $l_1,l_2$ are coplanar lines. Let $\gamma$ be the cohomology class of $C$ in $H^{1,1}(X,\mathbb{Z})$. Then, $\overline{NL(\gamma)}$ is a generically non-reduced component of the Noether-Lefschetz locus.

**Corollary 4.3.** Let $d \geq 5$ be an integer, $X$ a smooth degree $d$ surface containing two coplanar lines $l_1,l_2$. Let $C$ be a divisor in $X$ of the form $2l_1+l_2$ and $C'$ be a general element in the linear system $|C + mH_X|$ for $m \gg 0$. If $P'$ is the Hilbert polynomial of $C'$, there exists an irreducible generically non-reduced component of the Hilbert scheme $H_{P'}$ containing $C'$ such that a generic curve on this component is smooth and not contained in a surface of degree less than $d$.

**Proof.** Let $\gamma'$ be the cohomology class of $C$. Theorem 4.2 states that $\overline{NL(\gamma')}$ is an irreducible generically non-reduced component of $NL_d$. Then, Theorem 4.1 implies the corollary. \hfill \square

The following lemma tells us the degree and the arithmetic genus of the curve $C'$ as in Corollary 4.3:

---

10
Lemma 4.4. Let $C'$ be as in Corollary 4.3. Then, $\text{deg}(C') = md + 3$ and the arithmetic genus, $\rho_a(C') = 1 + (1/2)(md^2 + d(m^2 - 4m - 2) + 6m + 2)$.

Proof. Clearly, $\text{deg}(C') = md + 3$. We prove the formula for the arithmetic genus. Using the adjunction formula,

$$\rho_a(C') = 1 + (1/2)(C'^2 + (d - 4) \text{deg}(C'))$$

$$= 1 + (1/2)(C'^2 + m^2d + 2m \text{deg}(C) + (d - 4)(md + 3))$$

$$= 1 + (1/2)(4l_1^2 + l_2^2 + 4 + md^2 + d(m^2 - 4m + 3) - 12 + 6m)$$

$$= 1 + (1/2)(4(2 - d) + (2 - d) + md^2 + d(m^2 - 4m + 3) - 8 + 6m)$$

$$= 1 + (1/2)(md^2 + d(m^2 - 4m - 2) + 6m + 2)$$

This proves the lemma.

5 Additional remarks

Remark 5.1. Like in many cases, the $m$ specified in Corollary 4.3 can be easily computed. The proof of Theorem 4.1 and hence Corollary 4.3 suggest that we simply need $C'$ such that $H^0(O_X(-C')(d)) = 0 = H^1(O_X(-C')(d))$ and $H^1(O_X(C')) = 0$, which are the main conditions used in Theorem 3.3. We write this in the following corollary:

Corollary 5.2. For any $m \geq 2d - 3$ the conclusion of Corollary 4.3 holds true.

Proof. Note that it suffices to find the Castelnuovo-Mumford regularity of $O_X(C)$. Indeed, if it is equal to $t$ then simply take $m \geq t + 4$ and see that $H^1(O_X(C)(m)) = 0$,

$$0 = H^1(O_X(C)(t))^\vee = H^1(O_X(C)(m - 4))^\vee \overset{\text{SD}}{=} H^1(O_X(-C)(d - m)) = H^1(O_X(-C')(d)).$$

and, $0 = H^2(O_X(C)(t))^\vee = H^0(O_X(-C)(d - m)) = H^0(O_X(-C')(d))$.

Consider the short exact sequence,

$$0 \to O_X(l_1 + l_2) \to O_X(2l_1 + l_2) \to O_{l_1} \otimes O_X(2l_1 + l_2) \to 0$$
arising by tensoring with $\mathcal{O}_X(l_1 + l_2)$,

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(l_1) \to \mathcal{O}_{l_1} \otimes \mathcal{O}_X(l_1) \to 0$$

where $l_1, l_2$ as in Corollary 4.3. The exactness after tensor product follows from that $\mathcal{O}_X(l_1 + l_2)$ is locally free $\mathcal{O}_X$-module, hence flat.

We are going to compute the Castelnuovo Mumford regularity of $\mathcal{O}_X(l_1 + l_2)$ and $\mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2)$. We have,

**Lemma 5.3.** The sheaf $\mathcal{O}_X(l_1 + l_2)$ is $d - 4$-regular.

**Lemma 5.4.** The sheaf $\mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2)$ is $2d - 7$-regular.

This would imply that $\mathcal{O}_X(2l_1 + l_2)$ is $t$-regular for $t = \max\{2d - 7, d - 4\} = 2d - 7$, where the last equality follows from $d \geq 5$.

**Proof of Lemma 5.3.** Consider the short exact sequence,

$$0 \to \mathcal{O}_X(-l_1 - l_2) \to \mathcal{O}_X \to \mathcal{O}_{l_1 + l_2} \to 0.$$

[Har77, Ex. III.5] implies that for all $k \in \mathbb{Z}$, the induced map $H^0(\mathcal{O}_X(k)) \to H^0(\mathcal{O}_{l_1 + l_2}(k))$ is surjective and $H^1(\mathcal{O}_X(k)) = 0$. So,

$$0 = H^1(\mathcal{O}_X(-l_1 - l_2)(k)) \overset{\text{SD}}{=} H^1(\mathcal{O}_X(l_1 + l_2)(d - 4 - k))^\vee, \forall k \in \mathbb{Z}.$$

In other words, $H^1(\mathcal{O}_X(l_1 + l_2)(-k + d - 4)) = 0$ for all $k \in \mathbb{Z}$. Now, $H^2(\mathcal{O}_X(l_1 + l_2)(k)) \overset{\text{SD}}{=} H^0(\mathcal{O}_X(-l_1 - l_2)(-k + d - 4))$ is zero if the degree of $\mathcal{O}_X(-l_1 - l_2)(-k + d - 4)$ is less than zero, which happens if $k > d - 6$. This proves the lemma.

**Proof of Lemma 5.4.** Using Serre duality, we can conclude

$$H^1(\mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2)(k))^\vee = H^0(\mathcal{O}_{l_1} \otimes \mathcal{O}_X(-2l_1 - l_2)(-k)(-2)).$$

Now, $\deg(\mathcal{O}_{l_1} \otimes \mathcal{O}_X(-2l_1 - l_2)(-k)(-2)) = l_1(-2l_1 - l_2 - (k + 2))H_X = -2(2 - d) - 1 - (k + 2) =$
$2d - 7 - k$ where the second last equality follows from $l_1^2 = 2 - d$ which can be computed using the adjunction formula. Therefore, for $k > 2d - 7$, $H^1(\mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2)(k)) = 0$. This proves the lemma.

\begin{flushright}
\qed
\end{flushright}

\section*{References}

[Blö72] S. Bloch. Semi-regularity and de-Rham cohomology. \textit{Inventiones Math.}, 17:51–66, 1972.

[Dan14] A. Dan. Non-reduced components of the Noether-Lefschetz locus. \textit{arXiv:1407.8491}, 2014.

[Har77] R. Hartshorne. \textit{Algebraic Geometry}. Graduate text in Mathematics-52. Springer-Verlag, 1977.

[Kle81] J. O. Kleppe. Hilbert flag scheme, its properties and its connection with the Hilbert scheme: applications to curves in 3-space. \textit{Mathematisk Institut, Universitetet i Oslo}, 1981.

[Kle85] J. O. Kleppe. \textit{Non-reduced components of the Hilbert scheme of smooth space curves}. Lecture Notes in Math.-1266. Springer-Verlag, 1985. Proceedings Rocca di Papa, pp. 181-207.

[Mum62] D. Mumford. Further pathologies in algebraic geometry. \textit{Amer. J. Math.}, 84:642–648, 1962.

[Voi02] C. Voisin. \textit{Hodge Theory and Complex Algebraic Geometry-I}. Cambridge studies in advanced mathematics-76. Cambridge University press, 2002.

[Voi03] C. Voisin. \textit{Hodge Theory and Complex Algebraic Geometry-II}. Cambridge studies in advanced mathematics-77. Cambridge University press, 2003.