AN INSIGHT INTO THE DESCRIPTION OF THE CRYSTAL STRUCTURE FOR MIRKOVIĆ-VILONEN POLYTOPES

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Abstract. We study the description of the crystal structure on the set of Mirković-Vilonen polytopes. Anderson and Mirković defined an operator and conjectured that it coincides with the Kashiwara operator. Kamnitzer proved the conjecture for type $A$ and gave an counterexample for type $C_2$. He also gave an explicit formula to calculate the Kashiwara operator for type $A$. In this paper we prove that a part of the AM conjecture still holds in general, answering an open question of Kamnitzer (2007). Moreover, we show that although the formula given by Kamnitzer does not hold in general, it is still valid in many cases regardless of the type. The main tool is the connection between MV polytopes and preprojective algebras developed by Baumann and Kamnitzer.

1. Introduction

Let $G$ be a complex connected reductive group and $G^\vee$ be its Langlands dual group. In the geometric Satake correspondence, the intersection cohomology of the affine Grassmanian associated with $G$ provides a geometric realization of the irreducible highest weight representations $V(\lambda)$ of $G^\vee$ and a family of subvarieties, called Mirković-Vilonen (MV) cycles, forms a basis of $V(\lambda)$ [15]. Attempting to understand combinatorial aspects of $V(\lambda)$ using MV cycles, Anderson defined MV polytopes as moment polytopes of MV cycles [1]. He proved that MV polytopes can be used to count weight multiplicities as well as tensor product multiplicities without giving a full combinatorial description of the polytopes. Later, Kamnitzer gave a complete combinatorial characterization of MV polytopes as pseudo-Weyl polytopes satisfying tropical Plücker relations [11] (see Section 2.2).

The set of MV polytopes, denoted by $\mathcal{MV}$, naturally inherits a crystal structure via an explicit bijection to Lusztig’s canonical basis [11] (see Theorem 2.1). Let $\tilde{f}_j$ be the Kashiwara operator. Kamnitzer studied the crystal structure on $\mathcal{MV}$ and gave a description of $\tilde{f}_j(P)$ for polytope $P \in \mathcal{MV}$ [10] (see Theorem 2.2). However, this description is non-explicit because it requires to solve many equations given by the tropical Plücker relations, which involves addition and taking minimum.

Anderson and Mirković proposed a conjecture to describe the crystal structure on $\mathcal{MV}$, which was called Anderson-Mirković (AM) conjecture in [10] (see Conjecture 3.1). More precisely, they defined a new operator $AM_j$ acting on $\mathcal{MV}$ and conjectured that it is the same as the Kashiwara operator $\tilde{f}_j$. By definition, for
any MV polytope $P$, $AM_j(P)$ is the smallest pseudo-Weyl polytope satisfying four conditions concerning $P$ (see Section 3.1). The description given by the operator $AM_j$ does not require the tropical Plücker relations and hence was considered more explicit than the one given by Kamnitzer.

In case of type $A$, Kamnitzer proved the AM conjecture and gave an explicit formula for the Kashiwara operators [10]. He also gave a counterexample of the conjecture for type $C_2$. An alternative proof for type $A$ was given by Saito [19] via the connection between MV polytopes and representations of quivers. Naito and Sagaki [18] proved modified versions of the AM conjecture for type $B$, $C$ and also gave explicit formulas for the Kashiwara operators using diagram automorphisms. As far as we know, up to now there is no result concerning type $D$, $E$ or other non-simply-laced types.

Our study was motivated by an interesting question raised by Kamnitzer in [10]. He found in his counterexample for type $C_2$ that $AM_j(P) \subset \tilde{f}_j(P)$. It means that $\tilde{f}_j(P)$ is indeed a pseudo-Weyl polytope satisfying the four conditions required by $AM_j$, but it is not the smallest one. Hence he asked:

**Question 1.1** ([10], Question 2). Is $AM_j(P)$ always contained in $\tilde{f}_j(P)$?

The aim of this paper is to give an affirmative answer to the above question. Namely, we proved that for any MV polytope $P$, $AM_j(P) \subseteq \tilde{f}_j(P)$ holds in general (see Theorem 3.4). In our proof we treat the simply-laced case and the non-simply-laced case separately. In the simply-laced case, Baumann and Kamnitzer has recently established a link between MV polytopes and representations of preprojective algebras [2]. Their results shed new light on the theory of MV polytopes and is the main tool for us to prove our theorem. In the non-simply-laced case, a diagram automorphism $\sigma$ on the Lie algebra $\mathfrak{g}$ induces a action on $\mathcal{MV}$ and the set $\tilde{\mathcal{MV}}$ for $\mathfrak{g}^\sigma$ is in bijection with the $\sigma$-invariant MV polytopes in $\mathcal{MV}$ ([18], [7]). Therefore, having the theorem proved for the simply-laced case, we can use diagram automorphisms to obtain the corresponding results for the non-simply-laced case.

We also show that although the explicit formula given by Kamnitzer in [10] for type $A$ does not hold in general, it is still true in many special cases (see Theorem 3.5 and 3.7). All our results are type-independent.

Note that it remains open whether the AM conjecture holds for all simply-laced cases as we did not find any counterexamples in type $D$ or $E$.

The paper is organized as follows. In Section 2 we recall the definition of MV polytopes and the crystal structure on the set of them. In Section 5 we introduce the AM conjecture and state our main results. We then recall the link between MV polytopes and representations of preprojective algebras in Section 4. The proofs of our results are presented in the last two sections. The simply-laced case is treated in Section 5 while in Section 6 we focus on the non-simply-laced case.

2. Mirković-Vilonen polytopes and the crystal structure

2.1. Notations. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank $n$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $C = (c_{ij})$ be the Cartan matrix. The vertices of the Dynkin diagram of $\mathfrak{g}$ is denoted by $I = \{1, \ldots, n\}$. Let $Q$ be the root system and $Q_+$ (resp. $Q_-$) be the positive (resp. negative) root lattice. For $1 \leq i \leq n$, denote by $\alpha_i$ (resp. $\beta_i$) the simple root (resp. simple coroot). Let $P$ be the weight lattice and $P_+$ be the set of dominant weights. The $i$-th fundamental
weight is denoted by $\varpi_i$. Let $P^*$ be the coweight lattice and $h_\mathbb{R} = P^* \otimes \mathbb{R}$ be the real form of $h$.

Let $W$ be the Weyl group of $\mathfrak{g}$. For any $w \in W$, the length of $w$ is denoted by $\ell(w)$. Let $w_0$ be the longest element in $W$ and $r = \ell(w_0)$. Denote by $s_1, \ldots, s_n$ the simple reflections. A Weyl group translate of a fundamental weight $w_0 \varpi_i$ is called a chamber weight. The collection of all chamber weights is denoted by $\Gamma = \{w_0 \varpi_i | w \in W, i \in I\}$.

The Bruhat order on $W$ is denoted by $\geq$. We will use the same notation for the usual partial order on $P^*$, namely $\mu \geq \nu$ if and only if $\mu - \nu \in \sum_{i \in I} \mathbb{N} h_i$. The twisted partial order $\geq_w$ for $w \in W$ is defined as $\mu \geq_w \nu$ if and only if $w^{-1} \mu \geq w^{-1} \nu$.

Let $\mathfrak{g}^\vee$ be the Langlands dual of $\mathfrak{g}$, namely the Lie algebra whose root system is dual to that of $\mathfrak{g}$. In particular, the weight lattice of $\mathfrak{g}^\vee$ is $P^*$.

### 2.2. Mirković-Vilonen polytopes

We recall the combinatorial definition of MV polytopes following [10].

For a collection of integers $M_*= (M_i)_{\gamma \in \Gamma}$ indexed by the set of chamber weights, the following inequalities are called edge inequalities:

\begin{equation}
M_{w_0 \varpi_i} + M_{w_0 \varpi_j} + \sum_{j \neq i} c_{ji} M_{w_0 \varpi_j} \leq 0, \quad \text{for all } i \in I, w \in W.
\end{equation}

Given $M_*$ satisfying the edge inequalities, we have the associated pseudo-Weyl polytope

$P(M_*) = \{ h \in h_\mathbb{R} | (\langle h, \gamma \rangle \geq M_{\gamma} \text{ for all } \gamma) \}$.

As shown in [11], there is a map $w \mapsto \mu_w$ from the Weyl group onto the set of vertices of the polytope $P(M_*)$ such that

$(\mu_w, w_0 \varpi_i) = M_{w_0 \varpi_i}$, \quad \text{for all } i \in I, w \in W.

The collection $\mu_* = (\mu_w)_{w \in W}$ is called the Gelfand-Goresky-MacPherson-Serganova (GGMS) datum of weight $(\mathfrak{g})$. We have

$P(M_*) = \text{conv}(\mu_*) = \{ h \in h_\mathbb{R} | h \geq_w \mu_w, \text{ for all } w \in W \}$.

Let $w \in W$ and $i, j \in I$ be such that $w s_i > w$, $w s_j > w$ and $i \neq j$. We say that $M_* = (M_\gamma)_{\gamma \in \Gamma}$ satisfies the tropical Plücker relation at $(w, i, j)$ if $c_{ij} = 0$ or $c_{ji} = c_{ji} = -1$ and

\begin{equation}
M_{w s_i \varpi_i} + M_{w s_j \varpi_j} = \min \{ M_{w \varpi_i} + M_{w s_i, s_j \varpi_j}, M_{w s_i, s_j \varpi_i} + M_{w s_j \varpi_j} \}.
\end{equation}

The relations for other possible values of $c_{ij}, c_{ji}$ are omitted as they will not be used in this paper. We refer to [11] for details.

We say that $M_*$ satisfies the tropical Plücker relations if it satisfies the tropical Plücker relation at each $(w, i, j)$.

The pseudo-Weyl polytope $P(M_*)$ is called a Mirković-Vilonen (MV) polytope of weight $(\mu_1, \mu_2)$ if $M_*$ satisfies the edge inequalities and $\mu_w = \mu_1$ and $\mu_{w_0} = \mu_2$. In this case, $M_*$ is called a Berenstein-Zelevinsky (BZ) datum of weight $(\mu_1, \mu_2)$.

The weight lattice $P$ acts on $h_\mathbb{R}$ by translation. Hence we have an action of $P$ on the set of MV polytopes. Namely for $\nu \in P$, $\nu + P(M_*) = P(M'_*)$ where $M'_* = M_* + \langle \nu, \gamma \rangle$. The orbit of an MV polytope of weight $(\mu_1, \mu_2)$ is called a stable MV polytope of weight $(\mu_1 - \mu_2)$. Let $\text{MV}$ denote the set of stable MV polytopes. In this paper, we always consider representatives in $\text{MV}$ with $\mu_{w_0} = 0$. 
2.3. The crystal structure. We recall the definition of abstract crystals following [13]. For more details on the theory of crystal bases we refer to [12].

A \( g \)-crystal is a set \( B \) with maps \( \text{wt} : B \to P, \varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\} \) and \( \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\} \) for all \( i \in I \), satisfying the following axioms:

1. For any \( b \in B \), \( \varepsilon_i(b) = \varepsilon_i(b + (h_i, \text{wt}(b))) \).
2. If \( \tilde{e}_i(b) \in B \), then \( \text{wt}((\tilde{e}_i(b)) = \text{wt}(b) + \alpha_i, \varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1 \) and \( \varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1 \).
3. If \( \tilde{f}_i(b) \in B \), then \( \text{wt}(\tilde{f}_i(b)) = \text{wt}(b) - \alpha_i, \varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1 \) and \( \varphi_i(\tilde{f}_i(b)) = \varphi_i(b) - 1 \).
4. For \( b, b' \in B \), we have \( b' = \tilde{e}_i(b) \) if and only if \( b = \tilde{f}_i(b') \).
5. For \( b \in B \), if \( \varphi(b) = -\infty \), then \( \varepsilon_i(b) = \tilde{f}_i(b) = 0 \).

Let \( B \) be Lusztig’s canonical basis for the positive part of the quantized enveloping algebra \( U_q^+(g) \) [15]. It is known that \( B \) carries a \( g \)-crystal structure.

Let \( i = (i_1, \ldots, i_r) \) be a reduced word of \( w_0 \). For \( 1 \leq k \leq r \), let \( w_k = s_i \cdots s_i \). The \( i \)-Lusztig datum of an MV polytope \( P(M_\bullet) \) is \( n_\bullet = (n_1, \ldots, n_r) \in \mathbb{N}^r \) defined by

\[
n_k = -M_{w_k} - 1 - M_{w_1} - 1 - \sum_{j \neq i_k} a_{ij} M_{w_j}.
\]

The meaning of the \( i \)-Lusztig datum of an MV polytope is the following [10]: the reduced word \( i \) determines a path \( e = w_0 = w_{i_r} \cdots w_{i_1} \) through the 1-skeleton of the polytope. The integers \( (n_1, \ldots, n_r) \) are just the lengths of the edges along the path.

Let \( B^\vee \) be the canonical basis of \( U_q^+(g) \). For any reduced word \( i \) of \( w_0 \), Lusztig showed that there is a bijection \( \phi_i : B^\vee \to \mathbb{N}^r \), known as Lusztig’s parametrization of the canonical basis \( B^\vee \). \( \phi_i(b) \) is called the \( i \)-Lusztig datum of \( b \).

**Theorem 2.1** ([11]). There is a coweight-preserving bijection \( b \mapsto P(b) \) between the canonical basis \( B^\vee \) and the set of MV polytopes \( \mathcal{MV} \). Under this bijection, the \( i \)-Lusztig datum of \( b \) equals that of \( P(b) \).

Thus the set \( \mathcal{MV} \) inherits a crystal structure from the canonical basis. The action of the Kashiwara operator \( \tilde{f}_j \) (\( j \in I \)) can be described as follows:

**Theorem 2.2** ([10]). Let \( P = P(M_\bullet) \) be an MV polytope and assume that \( \tilde{f}_j P = P(M'_\bullet) \). Then \( M'_\bullet \) is uniquely determined by \( M'_{\varpi_j} = M_{\varpi_j} - 1 \) and \( M'_\gamma = M_\gamma \) for \( \gamma = w \varpi_j \) with \( s_j w < w \).

As pointed out in [10], this description is non-explicit because the rest of \( M'_\gamma \) are determined by the tropical Plücker relations. To calculate them one needs to recursively solve a number of equations involving + and taking minimum. Hence it is interesting to seek a more explicit description or even an explicit formula to calculate the rest of \( M'_\gamma \).

3. The Anderson-Mirković conjecture and main results

3.1. The Anderson-Mirković conjecture. For \( j \in I \) we define \( W_j^- = \{ w \in W | s_j w < w \} \), \( W_j^+ = \{ w \in W | s_j w > w \} \). Let \( \Gamma_j = \bigcup_{i \in I} W_j^- \varpi_i \) and \( \Gamma_j = \Gamma \setminus \Gamma_j \).

Let \( P = P(M_\bullet) \) be an MV polytope with GGMS datum \( \mu_\bullet \) and fix \( j \in I \). Let \( c_j(P) = M_{\varpi_j} - M_{s_j \varpi_j} - 1 \). We define a map \( r_j : \mathfrak{h}_R \to \mathfrak{h}_R \) by \( r_j(\alpha) = s_j(\alpha) + c_j h_j \).

Note that \( r_j(\mu_{s_j}) = \mu_e - h_j \). The \( j \)-th AM operator \( AM_j : \mathcal{MV} \to \mathcal{MV} \) is defined
as following: $P' = AM_j(P) = \text{conv}(\mu_j)$ is the smallest pseudo-Weyl polytope such that

(i). $\mu_{\mu_w} = \mu_w$ for all $w \in W_j^-$,
(ii). $\mu_e = h_j$,
(iii). $P'$ contains $\mu_w$ for all $w \in W_j^+$,
(iv). if $w \in W_j^-$ is such that $\langle \mu_w, \alpha_j \rangle \geq c_j(P)$, then $P'$ contains $r_j(\mu_w)$.

**Conjecture 3.1** (Anderson-Mirković). *For any MV polytope $P$ and any $j \in I$, $AM_j(P)$ is an MV polytope. Moreover $AM_j(P) = \tilde{f}_j(P)$.*

If the conjecture holds, we have a more explicit description of the action of Kashiwara operators in the sense that the AM operator only requires the smallest pseudo-Weyl polytope without involving tropical Plücker relations.

Kamnitzer has proved the following result:

**Proposition 3.2** ([10], Proposition 5.3). *Let $P = P(M_\bullet)$ be an MV polytope and let $M_\bullet$ be defined as

\[
M_\bullet = \begin{cases} 
M_{\gamma}, & \text{if } \gamma \in \Gamma_j \\
\min\{M_{\gamma}, M_{s_j \gamma} + c_j(P)\langle h_j, \gamma \rangle\}, & \text{if } \gamma \in \Gamma_j.
\end{cases}
\]

If $M_\bullet$ satisfies the edge inequalities, then $P(M_\bullet) = AM_j(P)$.***

**Remark 3.3.** Let $\tilde{f}_j(P) = P(M_\bullet)$. Suppose that $M_\bullet$ is defined as (3.1). Then $M_\bullet$ automatically satisfies the edge inequalities since $P(M_\bullet)$ is already an MV polytope. Thus by the above proposition, Conjecture 3.1 holds.

In the case of type $A$, Kamnitzer proved Conjecture 3.1 by showing that (3.1) indeed holds for $\tilde{f}_j(P) = P(M_\bullet)$ [10]. He also gave a counterexample of the conjecture for type $C_2$. Naito and Sagaki [18] proved modified versions of Conjecture 3.1 for type $B$ and $C$. So far we have not seen any results concerning type $D$, $E$ or other non-simply-laced types.

3.2. **Main results.** In this subsection we state our main results. The proofs will be given in the next two sections.

Our first result answers an open question raised by Kamnitzer in 2007 (Question 1.1).

**Theorem 3.4.** *For any MV polytope $P$ and any $j \in I$, we have $AM_j(P) \subseteq \tilde{f}_j(P)$.***

That is, the MV polytope $\tilde{f}_j(P)$ satisfies the condition (i)-(iv) in the definition of AM operators. For type $D$ and $E$, it remains open whether it is the smallest pseudo-Weyl polytope satisfying those conditions.

The next result shows that for particular $\gamma \in \Gamma_j$, the BZ datum of $\tilde{f}_j(P)$ is indeed given by (3.1).

**Theorem 3.5.** *Let $P = P(M_\bullet)$ be an MV polytope and $j \in I$. Let $\tilde{f}_j(P) = P(M_\bullet)$. If $\langle h_j, \gamma \rangle = 1$, then $M'_\gamma = \min\{M_{\gamma}, M_{s_j \gamma} + c_j(P)\}$.***

**Remark 3.6.** Note that in the case of type $A$, all $\gamma \in \Gamma_j$ satisfies the condition $\langle h_j, \gamma \rangle = 1$. Hence by proving this theorem we obtain a new proof of Conjecture 3.1 for type $A$.**
For $\gamma \in \Gamma$, such that $\langle h_j, \gamma \rangle > 0$, $M'_j$ could not be characterized by (3.1) even in the simply-laced case. We will give an example for type $D_4$ in Section 5.4 where $M'_\gamma$ is indeed not given by (3.1) but the AM conjecture still holds.

**Theorem 3.7.** Let $P = P(M_\bullet)$ be an MV polytope and $j \in I$. Set
\[
m = m(j, P) = M_{w_j} - M_{s_j w_j} - M_{s_i w_i}.
\]
Then $m \geq 0$ and for any positive integer $k > m$, $AM_j(\bar{f}^k_j(P)) = \bar{f}^{k+1}_j(P)$.

Moreover, let $P^{(k)} = P(M^{(k)}_\bullet) = \bar{f}^k_j(P)$ and $P^{(k+1)} = P(M^{(k+1)}_\bullet) = \bar{f}^{k+1}_j(P)$, we have
\[
(3.2) \quad M^{(k+1)}_\gamma = \begin{cases} 
M^{(k)}_\gamma, & \text{if } \gamma \in \Gamma^j \\
M^{(k)}_\gamma + c_j(P^{(k)})(h_j, \gamma), & \text{if } \gamma \in \Gamma_j.
\end{cases}
\]

Recall that for any MV polytope $P$, the set $\{\bar{f}^k_j(P) | k \in \mathbb{Z}\}$ is called a $j$-string in the crystal graph (see [13]). In any $j$-string, there exists a unique polytope $P_0$ such that $\varepsilon_j(P_0) = 0$. We call $P_0$ a $j$-extremal polytope. The above theorem indicates that for each $j$-string, $AM_j = \bar{f}_j$ holds for all MV polytopes $P$ lying far enough from the $j$-extremal polytope and the BZ-datum of $\bar{f}_j(P)$ can be explicitly calculated. So for each $j$-string, there are infinitely many MV polytopes for which $AM_j = \bar{f}_j$ holds.

### 3.3. A characterization of AM operators.

The original definition of AM operators uses the GGMS datum $\mu_\bullet$. In this subsection we give a characterization of AM operators using the BZ datum $M_\bullet$, which will be more convenient for us to prove our results.

**Lemma 3.8.** For $\gamma \in \Gamma$, we have $\gamma \in \Gamma^j \iff \langle h_j, \gamma \rangle \leq 0$, $\gamma \in \Gamma_j \iff \langle h_j, \gamma \rangle > 0$.

**Proof.** If $\gamma \in \Gamma^j$, then by definition there exist $w \in W^-_j$ and $i \in I$ such that $\gamma = w\varepsilon_i$. Thus $\langle h_j, \gamma \rangle = \langle w^{-1}(h_j), \varepsilon_i \rangle \leq 0$. Since $s_j w < w$, we know that $w^{-1}(h_j) \in Q^-$. Hence $\langle h_j, \gamma \rangle \leq 0$. Similarly, if $\gamma \in \Gamma_j$, we have $\langle h_j, \gamma \rangle \geq 0$.

Now we only need to prove that $\langle h_j, \gamma \rangle = 0$ implies $\gamma \in \Gamma_j$. Assume that $\gamma = w\varepsilon_i$ for some $w \in W$ and $i \in I$. If $w \in W_j^-$ we are done. If not, let $w' = s_j w$ then $\gamma = w'\varepsilon_i$ and $w' \in W_j^-$.

The following lemma is in fact a generalization of [10] Proposition 5.3 and the proof is similar. Nevertheless, for readers’ convenience, we present a complete proof here.

**Lemma 3.9.** Let $P = P(M_\bullet)$ be an MV polytope, $j \in I$ and $P' = P(M'_\bullet)$ be the smallest pseudo-Weyl polytope satisfying the following three conditions:
\begin{itemize}
  \item[(a)] $M'_\gamma = M_\gamma$, for all $\gamma \in \Gamma^j$;
  \item[(b)] $M'_\varepsilon_i = M_{\varepsilon_i}$ for $i \neq j$ and $M'_\varepsilon_j = M_{\varepsilon_j} - 1$;
  \item[(c)] $M'_\gamma \leq \min\{M_\gamma, M_{s_j \gamma} + c_j(h_j, \gamma)\}$, for all $\gamma \in \Gamma_j$.
\end{itemize}

Then $P' = AM_j(P)$.

**Proof.** First we need to prove that $P'$ satisfies the conditions (i) to (iv) in the definition of AM operators.

It is easy to see that (a) and (b) are equivalent to (i) and (ii) respectively. By (c) we know that $M'_\gamma \leq M_\gamma$ for all $\gamma \in \Gamma_j$. Together with (a) we have $M'_\gamma \leq M_\gamma$ for all $\gamma \in \Gamma$, which by definition implies (iii).
Now it remains to prove (iv). Let \( w \in W_j^- \) such that \( \langle \mu_w, \alpha_j \rangle \geq c_j \). We need to show that \( r(\mu_w) \in P' \), i.e. \( \langle r(\mu_w), \gamma \rangle \geq M'_\gamma \) for all \( \gamma \in \Gamma \).

First we consider \( \gamma = s_j v \varpi_i \) for any \( v \in W_j^- \) and \( i \in I \). By definition of MV polytopes, we have \( \langle \mu_w, v \varpi_i \rangle \geq M_{v \varpi_i} = \langle \mu_v, v \varpi_i \rangle \). From this inequality we can deduce that \( \langle r_j(\mu_w), \gamma \rangle \geq \langle r_j(\mu_v), \gamma \rangle \). We claim that \( \langle r_j(\mu_v), \gamma \rangle \geq M'_\gamma \). Hence \( \langle r_j(\mu_w), \gamma \rangle \geq M'_\gamma \).

To prove the claim, note that

\[
\langle r(\mu_w), \gamma \rangle = \langle s_j(\mu_w) + c_j h_j, s_j w \varpi_i \rangle = M_{w \varpi_i} + c_j \langle h_j, \gamma \rangle.
\]

It is clear that \( \gamma \in \Gamma_j \) as \( v \in W_j^- \). By our assumption (c) we have \( M'_\gamma \leq M_{w \varpi_i} + c_j \langle h_i, \gamma \rangle \).

Next we consider \( \gamma = v \varpi_i \) for any \( v \in W_j^- \) and \( i \in I \). Since \( v \in W_j^- \), we know that \( \langle h_j, \gamma \rangle \leq 0 \). Let \( d_j = \langle \mu_w, \alpha_j \rangle - c_j \). Then \( d_j \geq 0 \) and \( r(\mu_w) = \mu_w - d_j h_j \). We have

\[
\langle r(\mu_w), \gamma \rangle = \langle \mu_w, \gamma \rangle - d_j \langle h_j, \gamma \rangle \geq \langle \mu_w, \gamma \rangle.
\]

Note that \( \langle \mu_w, \gamma \rangle \geq M_\gamma \). Also we know that \( M_\gamma = M'_\gamma \) since \( \gamma = v \varpi_i \in \Gamma_j \). Hence \( \langle \mu_w, \gamma \rangle \geq M'_\gamma \).

Since \( W_j^- \cup s_j W_j^- = W \) we see that \( \langle r(\mu_w), \gamma \rangle \geq M'_\gamma \) for all \( \gamma \in \Gamma \). We have proved that condition (iv) is satisfied.

Now, let \( P'' = P(M'_\bullet) \) be a pseudo-Weyl polytope satisfying (i) to (iv), we need to prove \( P' \subseteq P'' \), for which we only need to show that \( P'' \) satisfies (a), (b) and (c).

We only need to prove (c). Since \( P'' \) satisfies (iii), we have that \( M''_\gamma \leq M_\gamma \) for all \( \gamma \in \Gamma_j \). It remains to prove \( M''_\gamma \leq M_{s_j \gamma} + c_j \langle h_j, \gamma \rangle \) for all \( \gamma \in \Gamma_j \).

There exists \( w \in W_j^- \) such that \( \gamma = s_j w \varpi_i \) since \( \gamma \in \Gamma_j \). If \( \langle \mu_w, \alpha_j \rangle \geq c_j \), then \( r_j(\mu_w) \in P'' \) by (iv). We have

\[
M''_\gamma \leq \langle r_j(\mu_w), \gamma \rangle = M_{s_j \gamma} + c_j \langle h_j, \gamma \rangle.
\]

If \( \langle \mu_w, \alpha_j \rangle < c_j \), we have

\[
M''_\gamma < M_\gamma \leq \langle \mu_w, \gamma \rangle < \langle r_j(\mu_w), \gamma \rangle = M_{s_j \gamma} + c_j \langle h_j, \gamma \rangle
\]

as desired. \( \square \)

**Remark 3.10.** Assume that \( P = P(M_\bullet) \) is an MV polytope and \( \tilde{f}_j(P) = P(M'_\bullet) \). By Theorem 3.2 we know that \( M'_\bullet \) satisfies the condition (a) and (b) in the above lemma. Therefore, to prove Theorem 3.4 it is enough to show that \( M'_\bullet \) satisfies the condition (c).

### 4. MV polytopes and preprojective algebras

In this section we briefly recall the relationship between MV polytopes and representations of preprojective algebras developed by Baumann and Kamnitzer [2]. It is the main tool to prove our results in the simply-laced case. We will also need some knowledge on the geometric realization of crystals via nilpotent varieties [14], which are certain varieties of modules over preprojective algebras.
4.1. **Preprojective algebras and their modules.** Let $Q$ be a quiver whose underlying graph is the Dynkin diagram of $\mathfrak{g}$. Let $\Omega$ be the set of arrows in $Q$ and $s,t: \Omega \to I$ be the two maps indicating the source and target of an arrow. We define a new quiver $\overline{Q}$ from $Q$ by adding to each arrow $a \in \Omega$ an opposite arrow $a^*$. Set $\Omega^* = \{a^*|a \in \Omega\}$. So the set of arrows in $\overline{Q}$ is $H = \Omega \cup \Omega^*$. For each $a \in \Omega$, set $(a^*)^* = a$. We also define a map $\epsilon : H \to \{\pm 1\}$ by assigning $\epsilon(a) = 1$ if $a \in \Omega$ and $\epsilon(a) = -1$ if $a \in \Omega^*$. The **preprojective algebra** $\Lambda(Q)$ is defined as the path algebra $\mathbb{C}\overline{Q}$ modulo the two-sided ideal generated by $\sum_{a \in H} \epsilon(a)aa^*$ (see [17] for details).

A $\Lambda$-module $X$ consists of an $I$-graded $\mathbb{C}$-vector space $X = \oplus_{i \in I} X_i$ and a collection of linear maps $X_a : X_{s(a)} \to X_{t(a)}$ for each arrow $a \in H$ satisfying the following relations:

$$\sum_{a \in H, t(a) = i} \epsilon(a)X_aX_a^* = 0,$$

for each $i \in I$.

In our case ($\mathfrak{g}$ is a simple Lie algebra) it is known that $\Lambda$ is a finite-dimensional algebra. Denote by $\text{mod}\Lambda$ the category of finite-dimensional $\Lambda$-modules. For any $\Lambda$-module $X$, we call $\dim X = (\dim X_i)_{i \in I} \in \mathbb{N}^\mathbb{N}$ the dimension vector of $X$. It will also be convenient for us to treat $\dim X = \sum_{i \in I} (\dim X_i)\alpha_i$ as an element in the positive root lattice. Let $(-,-): \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be the standard symmetric bilinear form determined by $\langle \alpha_i, \alpha_k \rangle$ for any $i, k \in I$.

The following lemma is very useful:

**Lemma 4.1 ([3]).** For any $\Lambda$-modules $X, Y$, the following holds:

(4.1) $\dim \text{Ext}^1_{\Lambda}(X,Y) = \dim \text{Hom}_{\Lambda}(X,Y) + \dim \text{Hom}_{\Lambda}(Y,X) - (\dim X, \dim Y)$.

In particular, we have $\dim \text{Ext}^1_{\Lambda}(X,Y) = \dim \text{Ext}^1_{\Lambda}(Y,X)$.

For any $i \in I$, let $S_i$ be the (one-dimensional) simple $\Lambda$-module concentrated at $i$, $I_i$ (resp. $P_i$) be the injective hull (resp. projective cover) of $S_i$.

For any $\Lambda$-module $X$, the $i$-socle (resp. $i$-top) of $X$ is the $S_i$-isotypic component of the socle (resp. top) of $X$, denoted by $\text{soc}_iX$ (resp. $\text{top}_iX$). For a sequence $(j_1, \ldots, j_t)$ of indices with $1 \leq j_p \leq n$ for all $p$, there is a unique chain

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_t \subseteq X$$

of submodules of $X$ such that $X_{j_p}/X_{j_{p-1}} = \text{soc}_{j_p}(X/X_{j_{p-1}})$. Define $\text{soc}_{(j_1, \ldots, j_t)}(X)$ to be the submodule $X_{j_t}$ in the above chain.

4.2. **Geometric crystals.** For any $\nu = \sum_{i \in I} \nu_i \alpha_i \in Q_+$, let $\Lambda(\nu)$ be the variety of all finite-dimensional $\Lambda$-modules with dimension vector $\nu$. It is an affine algebraic variety and the group $G(\nu) = \prod_{i \in I} \text{GL}(\mathbb{C}^{\nu_i})$ acts on it by conjugation. Denote by $\text{Irr} \Lambda(\nu)$ the set of irreducible components of $\Lambda(\nu)$. Let $\mathcal{B} = \bigsqcup_{\nu} \text{Irr} \Lambda(\nu)$.

For any $Z \in \text{Irr} \Lambda(\nu)$, we define the weight of $Z$ to be $\text{wt}(Z) = -\nu \in Q_-$. For any $i \in I$ and $c \in \mathbb{N}$, define

$$\Lambda(\nu)_{i,c} = \{X \in \Lambda(\nu)| \dim \text{top}_i X = c\}.$$

They are locally closed subsets and $\Lambda(\nu) = \bigsqcup_{c \in \mathbb{N}} \Lambda(\nu)_{i,c}$. Thus for any $Z \in \text{Irr} \Lambda(\nu)$, there is a unique $c$ such that $Z \cap \Lambda(\nu)_{i,c}$ is open dense in $Z$. We then define two maps $\epsilon_i(Z) = c$ and $\varphi_i(Z) = \epsilon_i(Z) + \langle h_i, \text{wt}(Z) \rangle$.

It is also known that the map $Z \mapsto Z \cap \Lambda(\nu)_{i,c}$ gives a bijection

$$\{Z \in \text{Irr} \Lambda(\nu)|\epsilon_i(Z) = c\} \longleftrightarrow \text{Irr} \Lambda(\nu)_{i,c}.$$
Denote by $\Omega(\nu, i, c)$ the set of triples $(X, Y, f)$ where $X \in \Lambda(\nu)_{i,0}$, $Y \in \Lambda(\nu + c\alpha_i)_{i+c}$ and $f : X \to Y$ is an injective morphism of $\Lambda$-modules. We then have the following diagram

$$
\Lambda(\nu)_{i,0} \xrightarrow{p} \Omega(\nu, i, c) \xrightarrow{q} \Lambda(\nu + c\alpha_i)_{i,c},
$$

where $p$ and $q$ are the first and second projections. Then $p$ is a locally trivial fibration with a smooth and connected fibre and $q$ is a principal $G(\nu)$-bundle. Hence the above diagram induces the following bijection

$$
\{ Z \in \text{Irr} \Lambda(\nu) | \varepsilon_i(Z) = 0 \} \xrightarrow{\pi_i^c} \{ Z \in \text{Irr} \Lambda(\nu + c\alpha_i) | \varepsilon_i(Z) = c \}
$$

Then we define operators $\overline{e}_i, \overline{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ as follows:

$$
\overline{e}_i(Z) = \overline{f}_i^{-1}c^\max(Z), \quad \overline{f}_i(Z) = \overline{f}_i^{c+1}c^\max(Z), \text{ for any } Z \text{ with } \varepsilon_i(Z) = c.
$$

**Theorem 4.2** ([13]). Equipped with the maps $\omega_t, \varepsilon_t, \varphi_t, \overline{e}_t, \overline{f}_t$ defined above, the set $\mathcal{B} = \bigsqcup_{\nu \in Q_\Lambda} \text{Irr} \Lambda(\nu)$ is a crystal and isomorphic to the crystal structure on the canonical basis $\mathcal{B}$.

### 4.3. MV polytopes via preprojective algebras

Let $\gamma$ be a chamber weight. Then there exist $i \in I$ and $w \in W$ such that $w$ has a reduced expression $w = s_{ki}s_{k_2} \cdots s_{k_s}$ with $k_s = i$ and $\gamma = w\varpi_i$. We define a $\Lambda$-module

$$
N(-w\varpi_i) = \text{soc}_{\mathcal{I}(k_s, k_{s-1}, \ldots, k_1)}(I_i).
$$

This is the unique (up to isomorphism) $\Lambda$-submodule of $I_i$ with dimension vector $\varpi_i - w\varpi_i$.

**Remark 4.3.** This definition is due to Geiß, Leclerc and Schröer [6] while the notation $N(-w\varpi_i)$ is after Baumann and Kamnitzer who alternatively defined this module using Nakajima’s quiver and reflection functors [2]. A detailed proof of the equivalence between the two definitions can be found in [8]. The choice of the definition and the notation will be convenient for us to prove our results.

Note that $-w\varpi_i$ is also a chamber weight. The following useful lemma can be easily deduced using definitions and [6] Proposition 9.6.

**Lemma 4.4.** (1). For any $\gamma \in \Gamma_j$, $N(\gamma)$ has trivial $j$-top.

(2). For any $\gamma \in \Gamma_j$, $\dim_{\text{top}_j} N(\gamma) = \langle h_j, \gamma \rangle$. And we have the following short exact sequence:

$$
0 \to N(s_j\gamma) \to N(\gamma) \to S_j^{\oplus(h_j, \gamma)} \to 0.
$$

For any $\gamma \in \Gamma$, define an operator $D_\gamma := \dim_{\text{Hom}_\Lambda}(N(\gamma), -)$. It is clear that for any $\nu \in \mathbb{N}^n$, $D_\gamma$ is a constructible function on $\Lambda(\nu)$. Thus for each irreducible component of $\Lambda(\nu)$, it takes a constant value on a dense open subset. For each $Z \in \text{Irr} \Lambda(\nu)$, denote by $D_\gamma(Z)$ the generic value on $Z$. The following theorem tells us how to construct MV polytopes using representations of preprojective algebras.

**Theorem 4.5** ([2]). (i). For any $\gamma \in \Gamma$ and $Z \in \text{Irr} \Lambda(\nu)$, set $M_\gamma = -D_\gamma(Z)$. Then $M_\gamma$ satisfies the edge inequalities and tropical Plücker relations. Hence $P(M_\gamma)$ is an MV polytope.

(ii). The map $\text{Pol} : \mathcal{B} \to \text{MV}$ defined by $Z \mapsto P((-D_\gamma(Z))_{\gamma \in \Gamma})$ is an crystal isomorphism.
5. **Proofs of the Results: The simply-laced case**

In this section we assume that \( \mathfrak{g} \) is of simply-laced type. The following notations will be fixed throughout this section: Let \( P = \bar{P}(\mathfrak{m}_{\bullet}) \) be an MV polytope and \( P' = \bar{f}_j(P(\mathfrak{m}_{\bullet})) = P(\mathfrak{m}_{\bullet}) \) for \( j \in I \). Let \( Z = \text{Pol}^{-1}(P) \). Then \( Z' = \bar{f}_j(Z) = \text{Pol}^{-1}(\bar{f}_j(P)) \).

### 5.1. The Proof of Theorem 3.4 in the simply-laced case

By the definition of \( \bar{f}_j \) on \( B \), there exist general points \( T \) in \( Z \) and \( T' \) in \( Z' \) with the following short exact sequence:

\[
0 \rightarrow T \rightarrow T' \rightarrow S_j \rightarrow 0.
\]

Recall that \( c_j(P) = M_{\varpi_j} - M_{s_j\varpi_j} - 1 \). We first show that \( c_j(P) \) is closely related to the map \( \varphi_j \) in the crystal structure:

**Lemma 5.1.** For any \( j \in I \), we have \( c_j(P) = \varphi_j(Z) - 1 \).

**Proof.** Since \( N(\varpi_j) = P_j \) (see [2], Proposition 3.6), we have

\[
D_{\varpi_j}(Z) = \dim \text{Hom}_A(P_j, T) = \dim T_j.
\]

By Lemma 4.4, we have the following short exact sequence

\[
0 \rightarrow N(s_j\varpi_j) \rightarrow N(\varpi_j) \rightarrow S_j \rightarrow 0.
\]

Applying \( \text{Hom}_A(-, T) \), we have the following long exact sequence

\[
0 \rightarrow \text{Hom}_A(S_j, T) \rightarrow \text{Hom}_A(P_j, T) \rightarrow \text{Hom}_A(N(s_j\varpi_j), T) \rightarrow \text{Ext}^1_{\Lambda}(S_j, T) \rightarrow 0.
\]

Comparing dimensions of modules in the above sequence, we deduce that

\[
D_{s_j\varpi_j}(Z) = \dim \text{Ext}^1_{\Lambda}(S_j, T) + \dim \text{Hom}_A(P_j, T) - \dim \text{Hom}_A(S_j, T).
\]

Now using (4.1), we have

\[
D_{s_j\varpi_j}(Z) = \dim \text{Hom}_A(T, S_j) - (\dim S_j, \dim T) + \dim T_j
\]

\[
= \varepsilon_j(Z) + \langle h_j, \text{wt}(Z) \rangle + \dim T_j
\]

Using (5.1) and (5.2), we can deduce that

\[
c_j(P) = M_{\varpi_j} - M_{s_j\varpi_j} - 1 = -D_{\varpi_j}(Z) + D_{s_j\varpi_j}(Z) - 1
\]

\[
= \varepsilon_j(Z) + \langle h_j, \text{wt}(Z) \rangle - 1 = \varphi_j(Z) - 1.
\]

\[\square\]

**Lemma 5.2.** For any \( \gamma \in \Gamma \), we have \( D_{\gamma}(Z') \geq D_{\gamma}(Z) \).

**Proof.** Since we have the exact sequence

\[
0 \rightarrow T \rightarrow T' \rightarrow S_j \rightarrow 0,
\]

we know that \( \text{Hom}_A(N(\gamma), T) \rightarrow \text{Hom}_A(N(\gamma), T') \).

\[\square\]

**Lemma 5.3.** For \( \gamma \in \Gamma_j \), we have \( D_{\gamma}(Z') \geq D_{s_j\gamma}(Z) - c_j(\langle h_j, \gamma \rangle) \).

**Proof.** By Lemma 4.3, we have the following short exact sequence:

\[
0 \rightarrow N(s_j\gamma) \rightarrow N(\gamma) \rightarrow S_j^{\oplus(\langle h_j, \gamma \rangle)} \rightarrow 0.
\]

Applying \( \text{Hom}_A(-, T') \), we have the following long exact sequence

\[
0 \rightarrow \text{Hom}_A(S_j^{\oplus(\langle h_j, \gamma \rangle)}, T') \rightarrow \text{Hom}_A(N(\gamma), T') \rightarrow \text{Ext}^1_{\Lambda}(S_j^{\oplus(\langle h_j, \gamma \rangle)}, T').
\]

\[\square\]
Hence the following inequality holds:
\[
\dim \text{Hom}_\Lambda(N(\gamma), T') + \dim \text{Ext}^1_\Lambda(S_j^{\oplus(h_j, \gamma)}, T') \\
\geq \dim \text{Hom}_\Lambda(S_j^{\oplus(h_j, \gamma)}, T') + \dim \text{Hom}_\Lambda(N(s_j \gamma), T').
\]
Using (\ref{1}), we deduce that
\[
D_\gamma(Z') \geq D_{s_j \gamma}(Z') + \langle h_j, \gamma \rangle(\dim S_j, \dim T') - \dim \text{Hom}_\Lambda(T', S_j^{\oplus(h_j, \gamma)}).
\]
By Lemma \ref{5.2} (b), we have \(D_{s_j \gamma}(Z) = D_{s_j \gamma}(Z')\). It is clear that \(\dim T' = \dim T + \alpha_j, \dim \text{top}_j(T') = \varepsilon_j(Z') = \varepsilon_j(Z) + 1\). We then deduce that
\[
D_\gamma(Z') \geq D_{s_j \gamma}(Z) - \langle h_j, \text{wt}(Z) - \alpha_j \rangle(\dim S_j, \dim T') - \varepsilon_j(Z)
\]
\[
= D_{s_j \gamma}(Z) - \langle h_j, \text{wt}(Z) \rangle - 2 + \varepsilon_j(Z) + 1 \langle h_j, \gamma \rangle
\]
\[
= D_{s_j \gamma}(Z) - (\varphi_j(Z) - 1) \langle h_j, \gamma \rangle.
\]

By Lemma \ref{5.1}, we have \(D_\gamma(Z') \geq D_{s_j \gamma}(Z) - c_j(P) \langle h_j, \gamma \rangle\). \qed

The following corollary follows straightforward from the proof above.

**Corollary 5.4.** \(D_\gamma(Z') = D_{s_j \gamma}(Z) - c_j(P) \langle h_j, \gamma \rangle\) if and only if the last map in the sequence \(\ref{5.4}\) is surjective.

Combining Lemma \ref{5.2} and Lemma \ref{5.3}, we know that for any \(\gamma \in \Gamma_j\),
\[
D_\gamma(Z') \geq \max\{D_\gamma(Z), D_{s_j \gamma}(Z) - c_j(P) \langle h_j, \gamma \rangle\}.
\]
Recall that \(M_\gamma = -D_\gamma(Z)\). The above inequality is
\[
M'_j \leq \min\{M_\gamma, M_{s_j \gamma} + c_j(P) \langle h_j, \gamma \rangle\}.
\]
Hence the condition (c) in Lemma \ref{3.9} is satisfied for \(P(M'_j) = \tilde{f}_j(P)\). By Remark \ref{3.10}, we have completed the proof of Theorem \ref{3.3} in the simply-laced case.

5.2. The proof of Theorem \ref{3.5} in the simply-laced case. Since \(\langle h_j, \gamma \rangle = 1\), we know that \(\gamma \in \Gamma_j\). By Lemma \ref{5.2} and \ref{5.3}, we already have
\[
D_\gamma(Z') \geq \max\{D_\gamma(Z), D_{s_j \gamma}(Z) - c_j(P)\}.
\]
We only need to prove that the equality holds in the above formula. Applying \(
\text{Hom}_\Lambda(-, T)\) and \(
\text{Hom}_\Lambda(-, T')\) to the sequence \(\ref{5.3}\), we have the following commutative diagram whose rows and columns are exact:

\[
\begin{array}{c}
0 \rightarrow \text{Hom}_\Lambda(S_j, T) \rightarrow \text{Hom}_\Lambda(S_j, T') \\
\downarrow \quad \quad \downarrow \\
0 \rightarrow \text{Hom}_\Lambda(N(\gamma), T) \rightarrow \text{Hom}_\Lambda(N(\gamma), T') \\
\downarrow \quad \quad \downarrow \\
\text{Hom}_\Lambda(N(s_j \gamma), T) \rightarrow \text{Hom}_\Lambda(N(s_j \gamma), T') \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\text{Ext}^1_\Lambda(S_j, T) \rightarrow \text{Ext}^1_\Lambda(S_j, T') \rightarrow \text{Ext}^1_\Lambda(S_j, S_j) = 0
\end{array}
\]

\[
\begin{array}{cc}
\alpha & \beta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\phi_j & \psi_j \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]

\[
\begin{array}{cc}
\gamma & \delta \\
\downarrow & \downarrow \\
\end{array}
\]
If $\beta$ is surjective, we have $D_\gamma(Z') = D_{s,\gamma}(Z) - c_j(P)$ by Corollary 5.3.

Now assume that $\beta$ is not surjective, then $\alpha$ is not surjective either. We claim that this implies $D_\gamma(Z') = D_\gamma(Z)$, namely

$$\dim \text{Hom}_\Lambda(N(\gamma), T) = \dim \text{Hom}_\Lambda(N(\gamma), T').$$

Suppose in the contrary that $\dim \text{Hom}_\Lambda(N(\gamma), T) < \dim \text{Hom}_\Lambda(N(\gamma), T')$. Hence there exists $f \in \text{Hom}_\Lambda(N(\gamma), T')$ such that the following diagram is commutative:

$$\begin{array}{ccc} N(\gamma) & \xrightarrow{p_1} & T'/T \simeq S_j \\
 f & \downarrow & \\
 T' & \xrightarrow{p_2} & S_j \end{array}$$

Hence there exits $g \in \text{Hom}_\Lambda(N(s_j\gamma), T)$ making the following commutative diagram whose rows are exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N(s_j\gamma) & \xrightarrow{g} & N(\gamma) & \xrightarrow{p_1} & S_j & \longrightarrow & 0 \\
 & & \uparrow & \downarrow f & \downarrow & \downarrow \text{id} & \downarrow & \downarrow & \\
 0 & \longrightarrow & T & \longrightarrow & T' & \xrightarrow{p_2} & S_j & \longrightarrow & 0 \\
\end{array}$$

Now let us consider any $\Lambda$-module $X$ which is an extension of $T$ by $S_j$. By the definition of $f_j$ on the geometric crystal $B$, we know that $X \in f_j(Z)$. Since $T'$ is a general point in $f_j(Z)$, using the well-known fact that the function $\dim \text{Hom}_\Lambda(N(\gamma), -) : f_j(Z) \rightarrow Z$ is upper semi-continuous (see for example [4]), we have

$$\dim \text{Hom}_\Lambda(N(\gamma), X) \geq \dim \text{Hom}_\Lambda(N(\gamma), T') > \dim \text{Hom}_\Lambda(N(\gamma), T).$$

The previous arguments for $T'$ can be applied to $X$. So there exists the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N(s_j\gamma) & \longrightarrow & N(\gamma) & \longrightarrow & S_j & \longrightarrow & 0 \\
 & & \uparrow & \downarrow \text{id} & \downarrow & \downarrow & \downarrow & \downarrow & \\
 0 & \longrightarrow & T & \longrightarrow & X & \longrightarrow & S_j & \longrightarrow & 0 \\
\end{array}$$

This implies that the map $\alpha$ is surjective, which contradicts to our assumption.

To summarize, we have proved the fact that $D_\gamma(Z')$ equals either $D_\gamma(Z)$ or $D_{s,\gamma}(Z) - c_j(P)$. Together with [5.5], it completes the proof of Theorem 3.7 in the simply-laced case.

5.3. The proof of Theorem 3.7 in the simply-laced case. We keep the notations in the previous subsections.

**Lemma 5.5.** $\dim \text{Ext}_\Lambda^1(T, S_j) = D_{-s_j\varpi_j}(Z) - D_{\varpi_j}(Z) + D_{s_j\varpi_j}(Z)$.

**Proof.** We already know from the proof of Lemma 5.1 that

$$D_{\varpi_j}(Z) = \dim T_j,$$

$$D_{s_j\varpi_j}(Z) = \varepsilon_j(Z) + \langle h_j, \text{wt}(Z) \rangle + \dim T_j.$$

By definition $N(-s_j\varpi_j) = S_j$. Hence $D_{-s_j\varpi_j}(Z) = \dim \text{Hom}_\Lambda(S_j, T)$. 
Now using (4.1), we deduce that

\[ \dim \text{Ext}^1_{\Lambda}(T, S_j) = \dim \text{Hom}_{\Lambda}(T, S_j) + \dim \text{Hom}_{\Lambda}(S_j, T) - (\dim T, \dim S_j) \]

\[ = \varepsilon_j(Z) + \dim \text{Hom}_{\Lambda}(S_j, T) + \langle h_j, \text{wt}(Z) \rangle \]

\[ = D_{\varepsilon_j}(Z) - D_{\omega_j}(Z) + D_{\omega_j}(T). \]

Hence the proof is completed. \(\square\)

By the above lemma, we have

\[ m(j, P) = \dim \text{Ext}^1_{\Lambda}(T, S_j) \geq 0. \]

Now let \( n \geq m(j, P) \) be a positive integer, let \( T(n) \) be a general point in \( Z(n) = \tilde{f}_j(P(n)) \) and \( T(n+1) \) be a general point in \( Z(n+1) = \tilde{f}_j(P(n+1)) \) such that the short exact sequence holds:

\[ 0 \rightarrow T(n) \rightarrow T(n+1) \rightarrow S_j \rightarrow 0. \]

By Theorem 3.4, which we have already proved in the simply-laced case, and Remark 3.3, we see that (3.2) implies \( AM_j(M_k) = \tilde{f}_j(P(k)) \). Thus to prove the remaining part of Theorem 3.7, we only need to prove:

\[ (5.6) \quad D_{\gamma}(Z(n+1)) = D_{s_{\gamma}}(Z(n)) - c_j(P(n))(h_j, \gamma), \text{ for any } \gamma \in \Gamma. \]

Since \( n \geq m(j, P) \), we have \( \text{Ext}^1_{\Lambda}(T(n+1), S_j) = 0 \). We then use the same argument as in the proof of Lemma 5.3. The last map in sequence (5.4) is now just a zero map. By Corollary 5.4, (5.6) holds as desired.

### 5.4. An example for type D

Let \( M''_\gamma = (M''_\gamma)_{\gamma \in \Gamma} \) be given by (3.1). In this subsection we give an example of type \( D_4 \) in which \( M''_\gamma \neq M''_\nu \) but \( \tilde{f}_j(P) = AM_j(P) \) still holds.

Let \( g \) be of type \( D_4 \) with Dynkin diagram

```
1
\downarrow
2
\downarrow
3
4
```

Let \( j = 1 \) and we choose a particular \( \gamma_0 = -s_1s_2s_3s_2s_2 \in \Gamma \). It is easy to check that \( \langle \gamma_0, h_1 \rangle = 2 \).

The modules \( N(\gamma_0) \) and \( N(s_1\gamma_0) \) can be visualized as follows:

```
N(\gamma_0)  1
  \downarrow
2
  \downarrow
3
4
1
2

N(s_1\gamma_0)  2
  \downarrow
3
\uparrow
4
2
```

The variety \( \Lambda(2\alpha_2) \) contains just one point, namely \( T = S_2 \oplus S_2 \). So \( Z = \{ T \} \) is the unique irreducible component. Let \( P = P(M'_\bullet) = \text{Pol}(Z) \) and \( P' = P(M''_\bullet) = \tilde{f}_1(P) \). Denote by \( X \) the unique extension (up to isomorphism) of \( S_1 \) by \( S_2 \). Then
$T' = X \oplus S_2$ is a general point in $Z' = \tilde{f}_1(Z)$. Then we have

\begin{align*}
M_{\gamma_0} &= -\dim \text{Hom}_A(N(\gamma_0), T) = 0, \\
M_{s_1\gamma_0} &= -\dim \text{Hom}_A(N(s_1\gamma_0), T) = -2, \\
c_1(P) &= c_1(Z) + \langle h_1, \text{wt}(Z) \rangle - 1 = 0 - \langle h_1, 2\alpha_2 \rangle - 1 = 1.
\end{align*}

So we have $M_{s_1\gamma_0} + c_1(P)\langle h_1, \gamma_0 \rangle = 0$. Hence

\begin{equation*}
M''_{\gamma_0} = \min\{M_{\gamma_0}, M_{s_1\gamma_0} + c_1(P)\langle h_1, \gamma_0 \rangle\} = 0.
\end{equation*}

However, $M'_{\gamma_0} = -\dim \text{Hom}_A(N(\gamma_0), T') = -1 \neq M''_{\gamma_0}$.

Next we will show that $AM_0(P) = \tilde{f}_1(P)$.

Let $\text{AM}_0(P) = P(M_\sigma)$. By Lemma 6.1 and Theorem 6.3 we know that $M'_\gamma \leq M''_\gamma$ for all $\gamma \in \Gamma$. And we need to prove $M'_\gamma = M''_\gamma$.

In fact we have $M'_\gamma = M''_\gamma = M''_\gamma$ for all $\gamma \neq \gamma_0$. The reason is that $\gamma_0$ is the only chamber weight such that $\langle \gamma, h_1 \rangle = 2$. To check this, one can find out all $\gamma \in \Gamma$ by exhausting all the possible modules $N(\gamma)$. Then by Theorem 6.3 $M'_\gamma = M''_\gamma$ for all $\gamma \neq \gamma_0$.

Note that $M_\gamma = -1$ and $M''_{\gamma_0} = 0$. So we have $M_{\gamma_0} = M''_{\gamma_0}$ or $M_{\gamma_0} = M'_{\gamma_0}$. Now we claim that $M''_\gamma$ does not satisfy the edge inequalities, which implies $M_{\gamma_0} = M''_{\gamma_0}$ because $M_\gamma$ satisfies the edge inequalities.

Let $w = s_1s_2s_3s_2$. So $\gamma_0 = -w\varpi_2$. Note that $w_0\varpi_k = -\varpi_k$ for all $k \in I$ in case of $D_4$. We have $\gamma_0 = \varpi_0 - \varpi_2$. Using the fact $w_0s_2 = s_2w_0$, we have $-w\varpi_2 = w\varpi_0s_2\varpi_2$. Therefore, the left hand side of the edge inequality (2.1) applied for $M''_\gamma$ with $w\varpi_0 \in W$ and $i = 2$ is the following:

\begin{align*}
M''_{-w\varpi_2} + M''_{-w_0s_2\varpi_2} + \sum_{k \in \{1,3,4\}} c_{k2}M''_{-w_0\varpi_k}
&= M''_{-\gamma_0} + M''_{-s_1s_2\varpi_2} - M''_{-s_1\varpi_1} - M''_{-s_1s_2s_3\varpi_3} - M''_{-s_1s_2s_3s_2\varpi_4}
\end{align*}

We already know that $M''_{-\gamma_0} = 0$. It is easy to see that $M''_{-s_1\varpi_1} = 0$, $M''_{-s_1s_2\varpi_2} = M''_{-s_1s_2s_3\varpi_3} = M''_{-s_1s_2s_3s_2\varpi_4} = -1$. Hence

\begin{align*}
M''_{-w\varpi_2} + M''_{-w_0s_2\varpi_2} + \sum_{k \in \{1,3,4\}} c_{k2}M''_{-w_0\varpi_k}
&= 1 > 0,
\end{align*}

which verifies that $M''_\gamma$ does not satisfy the edge inequalities.

6. Proofs of the results: The non-simply-laced case

Relying on the results in the simply-laced case, we will use diagram automorphisms to prove our results in the non-simply-laced case.

6.1. Diagram automorphisms. We assume that $\mathfrak{g}$ is a simple Lie algebra of simply-laced type and keep the notations in Section 2. Let $\sigma$ be an automorphism of the Dynkin diagram, i.e. $\sigma : I \to I$ is a bijection such that $c_{ij} = c_{\sigma(i)\sigma(j)}$. Let $k$ be the order of $\sigma$. It is well known that $\sigma$ induces a Lie algebra automorphism of $\mathfrak{g}$ by assigning the Chevalley generators $e_i$, $f_i$ and $h_i$ to $e_{\sigma(i)}$, $f_{\sigma(i)}$ and $h_{\sigma(i)}$ respectively. It is called a diagram automorphism of $\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h}$ is stable under $\sigma$, which induces $\sigma \in \text{GL}(\mathfrak{h}^*)$ by $\langle h, \sigma(\lambda) \rangle = \langle \sigma(h), \lambda \rangle$. Define

$$
\mathfrak{g}^\sigma := \{ x \in \mathfrak{g} | \sigma(x) = x \}, \quad \mathfrak{h}^\sigma := \{ h \in \mathfrak{h} | \sigma(h) = h \}.
$$
Then it is well-known that $g^\sigma$ is also a simple Lie algebra (see for example [9] Section 8.3). The Cartan subalgebra of $g^\sigma$ is $h^\sigma$. Moreover, $\sigma$ induces a group automorphism of the Weyl group $W$ by $\sigma(s_i) = s_{\sigma(i)}$. Set $W^\sigma := \{w \in W|\sigma(w) = w\}$. $h^\sigma$ is stable under the action of $W^\sigma$.

Throughout this section, we will only consider diagram automorphisms $\sigma$ illustrated in Figure 6.1 where the dashed arrows indicate the action of $\sigma$ on $I$ and vertices not connected with any arrows are invariant under $\sigma$. Then the fixed point subalgebra $g^\sigma$ is a simple Lie algebra of type $C_l$, $B_l$, $G_2$ and $F_4$ respectively. Note that we have covered all non-simply-laced types.

From now on we set $\hat{g} = g^\sigma$. Let $\hat{C} = (\hat{c}_{ij})$ be the Cartan matrix of $\hat{g}$ with index set $\hat{I} = \{1, 2, \ldots, l\}$. It is convenient to treat $\hat{I}$ as a subset of $I = \{1, 2, \ldots, n\}$. Let $\hat{W}$ be the Weyl group associated with $\hat{g}$ with $\hat{s}_i (i \in \hat{I})$ the simple reflections. Let $\hat{\pi}_i (i \in \hat{I})$ be the fundamental weights for $\hat{g}$ and set

$$\hat{\Gamma} = \{\hat{w}\hat{\pi}_i|\hat{w} \in \hat{W}, i \in \hat{I}\}.$$ 

Let $k$ be the order of $\sigma$. From Figure 6.1 we see that $k = 3$ if $g$ is of type $D_4$ and $\sigma$ is given by $\sigma(1) = 3, \sigma(3) = 4, \sigma(4) = 1$ and $k = 2$ in all remaining cases.

Let $k_i$ be the number of elements in the $\sigma$-orbit of $i \in I$. Again from Figure 6.1 we can see that $k_i \in \{1, 2, 3\}$ for any $i$ and the $\sigma$-orbit of $i \in I$ is isomorphic to $A_1 \times \cdots \times A_1$ ($k_i$ copies). It is well known that there exists a group isomorphism
\[ \Theta : \hat{W} \simeq W^\sigma \] such that \( \Theta(\hat{s}_i) = s_i^\sigma \) for all \( i \in \hat{I} \) (see for example [18]), where

\[ s_i^\sigma = \prod_{t=0}^{k_i-1} s_{\sigma^t(i)}. \]

Note that the product is independent on the order of \( s_{\sigma^t(i)} \) for different \( t \).

The following basic results will be used later.

**Lemma 6.1.** (a). The Chevalley generators of \( \Theta : \hat{g} \rightarrow g^\sigma \) with the Kashiwara operators was proved in [18] for \( g^\sigma \).

(b). The fundamental weight \( \hat{\varpi}_i \) is the restriction of \( \varpi_i \) to the subspace \( \mathfrak{h}^\sigma \).

(c). Let \( \hat{w} \in \hat{W} \) and \( w = \Theta(\hat{w}) \in W^\sigma \). For any \( i \in \hat{I} \), we have

\[ \hat{s}_i \hat{w} < \hat{w} \iff s_i^\sigma w < w \iff s_\sigma(w) < w, \ldots, s_{\sigma^{k_i-1}(i)}w < w. \]

(d). For any \( \hat{w} \in \hat{W} \) and \( h \in \mathfrak{h}^\sigma \), we have \( \hat{w} h = \Theta(\hat{w}) h \).

**Proof.** (a) and (b) are well-known results. (c) and (d) can be found in [18] Remark 2.3.2(3). \( \square \)

6.2. **Diagram automorphisms and MV polytopes.** In this subsection we describe the relations between \( \sigma \)-invariant MV polytopes for \( g \) and MV polytopes for \( \hat{g} \).

For details we refer to [7] and [18].

A diagram automorphism \( \sigma \) induces an action on the set \( \mathcal{MV} \) of MV polytopes as follows. Let \( P = P(\mu_*) = P(M_*) \) be an MV polytope for \( g \). \( \sigma(P) := P(\mu'_*) \), where \( \mu'_w = \sigma^{-1}(\mu_{\sigma(w)}) \) for \( w \in W \). Assuming that the BZ datum of \( \sigma(P) \) is \( M'_* \), we have \( M'_* = M_{\sigma(\gamma)} \) for any \( \gamma \in \Gamma \). We then define

\[ \mathcal{MV}^\sigma = \{ P \in \mathcal{MV} | \sigma(P) = P \}. \]

We know that an MV polytope \( P \in \mathcal{MV}^\sigma \) if and only if \( \sigma(\mu_w) = \mu_{\sigma(w)} \) for all \( w \in W \), if and only if \( M_* = M_{\sigma(\gamma)} \) for all \( \gamma \in \Gamma \).

Let \( \hat{\mathcal{MV}} \) be the set of MV polytopes for \( \hat{g} \). A polytope in \( \hat{\mathcal{MV}} \) will be denoted by \( \hat{P} = \hat{P}(\hat{M}_*) = \hat{P}(\hat{\mu}_*) \). The Kashiwara operators on \( \hat{\mathcal{MV}} \) are denoted by \( \hat{f}_j \) for any \( j \in \hat{I} \).

Let \( P = P(\mu_*) \) be an MV polytope for \( g \), we define a map \( \Phi : \mathcal{MV}^\sigma \rightarrow \hat{\mathcal{MV}} \) as follows: Define \( (\hat{\mu}_*)_{\hat{w}} = (\hat{\mu}_{\Theta(\hat{w})})_{\hat{w}} \in \mathfrak{h}^\sigma \cap \mathbb{R} \) given by \( \hat{\mu}_{\hat{w}} = \mu_{\Theta(\hat{w})} \). Then define \( \Phi(P(\mu_*)) = \hat{P}(\Phi(\mu_*)) \).

Now we define operators \( \hat{f}_i^\sigma \), \( i \in \hat{I} \) on \( \mathcal{MV}^\sigma \) as

\[ \hat{f}_i^\sigma = \prod_{t=0}^{k_i-1} \hat{f}_{\sigma^t(i)}. \]

Note that this definition does not depend on the order of \( \hat{f}_{\sigma^t(i)} \) for different \( t \).

**Theorem 6.2.** The map \( \Phi : \mathcal{MV}^\sigma \rightarrow \hat{\mathcal{MV}} \) is a bijection such that \( \Phi \circ \hat{f}_j^\sigma = \hat{f}_j \circ \Phi \) for all \( j \in \hat{I} \).

**Proof.** It was proved in [7] that \( \Phi \) is a bijection. The property that \( \Phi \) commutes with the Kashiwara operators was proved in [18] for \( g \) of type \( A \). But for type \( D \) and \( E \) it can be proved in the same way. In fact, one only need to notice that Lemma 2.6.1, Lemma 2.7.2, Proposition 2.7.3 and Proposition 2.7.4 in [18] all hold in general. \( \square \)
6.3. Proof of Theorem 3.4: The non-simply-laced case. Let \( \hat{P} = \hat{P}(\tilde{M}_\bullet) \) be an MV polytope for \( \mathfrak{g} \). Fix \( j \in \hat{I} \) and let \( \hat{P}' = \hat{f}_j(\hat{P}) = \hat{P}(\tilde{M}_\bullet) \). Let \( P = P(M_\bullet) = \Phi^{-1}(\hat{P}) \) and \( P' = P(M'_\bullet) = \Phi^{-1}(\hat{P}') \). By Theorem 6.2 we have \( P' = \hat{f}_j(P) \).

**Lemma 6.3.** For any \( i \in \hat{I} \) and \( \hat{w} \in \hat{W} \), we have \( \tilde{M}_\hat{w}_{\hat{\varnothing}_i} = M_{\Theta(\hat{w})_{\hat{\varnothing}_i}}. \)

*Proof.* See [18, Remark 2.5.2]. \( \square \)

**Lemma 6.4.** For any \( i \in \hat{I} \), we have \( c_i(\hat{P}) = c_i(P) \).

*Proof.* By definition, \( c_i(\hat{P}) = \tilde{M}_\hat{w}_i - \tilde{M}_{\hat{w}_i} - 1 \). We have that \( \tilde{M}_\hat{w}_i = M_{\varnothing_i} \) and \( \tilde{M}_{\hat{w}_i} = M_{\hat{\varnothing}_i} \) (Lemma 6.3). By definition, we have \( s_i^\sigma = s_is_{\sigma(i)} \cdots s_{\sigma^{k_i-1}(i)} \). Note that \( \sigma(i), \ldots, \sigma^{k_i-1}(i) \) are not connected with \( i \) in the Dynkin graph. Hence we have \( M_{\hat{\varnothing}_i} = M_{\varnothing_i} \). Therefore we have \( c_i(\hat{P}) = M_{\varnothing_i} - M_{\varnothing_i} - 1 = c_i(P) \). \( \square \)

**Lemma 6.5.** For any \( \hat{\gamma} \in \hat{\Gamma}_j \), we have

\[
M'_\gamma \leq \min \{ \tilde{M}_\hat{\gamma}, \tilde{M}_{\hat{\sigma}_j \hat{\gamma}} + c_j(\hat{P}) (\langle h_j, \hat{\gamma} \rangle) \}.
\]

*Proof.* Since \( \hat{\gamma} \in \hat{\Gamma}_j \), we have \( \hat{\gamma} = \hat{w}_{\hat{\varnothing}_i} \) where \( \hat{s}_j \hat{w} > \hat{w} \). Let \( w = \Theta(\hat{w}) \) and \( \gamma = w_{\varnothing_i} \). Hence \( \langle h_j^\sigma, \gamma \rangle = \langle \hat{h}_j, \hat{\gamma} \rangle > 0 \). By Lemma 6.1 (c), we deduce that \( \langle h_{\sigma(j)}, \gamma \rangle \geq 0 \) for any \( 0 \leq t \leq k_j - 1 \).

By Lemma 6.1 and 6.3, we can deduce that (6.1) is equivalent to

\[
M'_\gamma \leq \min \{ M_\gamma, M_{s_j \gamma} + c_j(P) (\langle h_j^\sigma, \gamma \rangle) \}.
\]

Hence we can return to the simply-laced case and only need to prove (6.2). In the following we only consider the case \( k_j = 2 \). The case \( k_j = 3 \) can be treated in the same way.

Recall that \( f_j^\sigma = f_j f_{\sigma(j)} \). So \( P' = f_j^\sigma(P) = f_j f_{\sigma(j)}(P) \). Let \( P^{(1)} = P(M^{(1)}_\bullet) = f_{\sigma(j)}(P) \). Then \( P' = f_j(P^{(1)}) \).

Since we have proved Theorem 3.4 in the simply-laced case, we have

\[
M'_\gamma = M^{(1)}_\gamma \quad \text{if} \quad \langle h_j, \gamma \rangle = 0,
\]

\[
M'_\gamma \leq \min \{ M^{(1)}_\gamma, M^{(1)}_{s_j \gamma} + c_j(P^{(1)}) (\langle h_j, \gamma \rangle) \} \quad \text{if} \quad \langle h_j, \gamma \rangle > 0.
\]

Recall \( h_j^\sigma = h_j + h_{\sigma(j)} \) and note that \( \langle h_j^\sigma, \gamma \rangle > 0 \). We need to consider the following three cases:

Case 1: \( \langle h_j, \gamma \rangle = 0 \) and \( \langle h_{\sigma(j)}, \gamma \rangle > 0 \). Now we have \( M'_\gamma = M^{(1)}_\gamma \). Again by Theorem 3.4 in the simply-laced case, we have

\[
M^{(1)}_\gamma \leq \min \{ M_\gamma, M_{s_j \gamma} + c_{\sigma(j)}(P) (\langle h_{\sigma(j)}, \gamma \rangle) \}.
\]

We claim that \( c_{\sigma(j)}(P) = c_j(P) \). In fact, \( P \in MV^\sigma \) implies that \( M_\gamma = M_{\sigma(\gamma)} \) for all \( \gamma \in \Gamma \). Hence we have

\[
c_{\sigma(j)}(P) = M_{\varnothing_{\sigma(j)}} - M_{s_{\sigma(j)} \varnothing_{\sigma(j)}} - 1 = M_{\varnothing_{j}} - M_{\varnothing_{j} \varnothing_{j}} - 1 = c_j(P).
\]

Since \( \langle h_j, \gamma \rangle = 0 \), it is easy to see that \( s_{\sigma(j)} \gamma = s_j^\sigma \gamma, \langle h_{\sigma(j)}, \gamma \rangle = \langle h_j^\sigma, \gamma \rangle \). Therefore (6.2) holds in this case.

Case 2: \( \langle h_j, \gamma \rangle > 0 \) and \( \langle h_{\sigma(j)}, \gamma \rangle = 0 \). Now we have

\[
M'_\gamma \leq \min \{ M^{(1)}_\gamma, M^{(1)}_{s_j \gamma} + c_j(P^{(1)}) (\langle h_j, \gamma \rangle) \}
\]
We claim that \( c_j(P^{(1)}) = c_j(P) \). In fact, since \( P^{(1)} = f_{\sigma(j)}(P) \) and \( \varpi_j, s_j \varpi_j \in \Gamma_{\sigma(j)} \), we have
\[
c_j(P^{(1)}) = M_{\varpi_j}^{(1)} - M_s^{(1)} = 1 = M_{\varpi_j} - M_s - 1 = c_j(P).
\]

Since \( \langle h_{\sigma(j)}, \gamma \rangle = 0 \), we have \( \langle h_{\sigma(j)}, s_j \gamma \rangle = 0 \) and \( \langle h_j, \gamma \rangle = \langle h_j^\sigma, \gamma \rangle \). Hence \( M_{\gamma}^{(1)} = M_{\gamma}, M_{s_j \gamma}^{(1)} = M_{s_j \gamma} \). Therefore (6.2) holds in this case.

Case 3: \( \langle h_j, \gamma \rangle > 0 \) and \( \langle h_{\sigma(j)}, \gamma \rangle > 0 \). Again by the theorem in the simply-laced case, we have
\[
\begin{align*}
M_{\gamma}^{(1)} &\leq \min\{M_{\gamma}, M_{s_{\sigma(j)}} \gamma + c_{\sigma(j)}(P) \langle h_{\sigma(j)}, \gamma \rangle \}, \\
M_{s_j \gamma}^{(1)} &\leq \min\{M_{s_j \gamma}, M_{s_{\sigma(j)} s_j \gamma} + c_{\sigma(j)}(P) \langle h_{\sigma(j)}, s_j \gamma \rangle \}.
\end{align*}
\]

We can deduce that
\[
(6.3) 
M'_{\gamma} \leq \min\{M_{\gamma}, M_{s_{\sigma(j)} s_j \gamma} + c_{\sigma(j)}(P) \langle h_{\sigma(j)}, s_j \gamma \rangle + c_j(P^{(1)}) \langle h_j, \gamma \rangle \}.
\]

As in the previous two cases we have \( c_j(P^{(1)}) = c_j(P) \) and \( c_{\sigma(j)}(P) = c_j(P) \). Note that \( \langle h_{\sigma(j)}, s_j \gamma \rangle = \langle h_{\sigma(j)}, \gamma \rangle \). So (6.3) can be simplified as
\[
(6.4) 
M'_{\gamma} \leq \min\{M_{\gamma}, M_{s_{\sigma(j)} s_j \gamma} + c_j(P) \langle h_{\sigma(j)}, h_j, \gamma \rangle \},
\]

which is the same as (6.2).

In view of Remark 6.10, the proof of Theorem 3.4 is now completed.

6.4. Proof of Theorem 3.5 in the non-simply-laced case. We keep the notations in the previous subsection. Assuming that \( \langle h_j, \hat{\gamma} \rangle = \langle h_j^\sigma, \hat{\gamma} \rangle = 1 \), we need to prove
\[
(6.5) 
\hat{M}_{\hat{\gamma}} = \min\{\hat{M}_{\hat{\gamma}}, \hat{M}_{s_{\hat{\gamma}}} + c_j(\tilde{P}) \}.
\]

Since \( \langle h_j, \hat{\gamma} \rangle = 1 \), we know that \( \hat{\gamma} \in \hat{\Gamma}_j \). Hence \( \hat{\gamma} = \hat{w} \varpi_i \), where \( \hat{s}_j \hat{w} > \hat{w} \). Let \( w = \Theta(w) \) and \( \gamma = w \varpi_i \). By the same reason as in the proof of Lemma 6.3, (6.5) is equivalent to
\[
(6.6) 
M_{\gamma}^{(1)} = \min\{M_{\gamma}, M_{s_{\gamma}} + c_j(P) \}.
\]

By Lemma 6.1 (c) we know that \( \langle h_{\sigma^t(j)}, \gamma \rangle \geq 0 \) for any \( 0 \leq t \leq k_j - 1 \). Since \( h_j^\sigma = \sum_{t=0}^{k_j-1} h_{\sigma^t(j)} \) and \( \langle h_j^\sigma, \hat{\gamma} \rangle = 1 \), we deduce that there exist a unique \( t_1 \) such that \( \langle h_{\sigma_1(j)}, \gamma \rangle = 1 \) and for all other \( t, \langle h_{\sigma^t(j)}, \gamma \rangle = 0 \).

Now we are again in the simply-laced case. In the following we assume \( k_j = 2 \). The case \( k_j = 3 \) can be proved similarly.

Recall that \( P' = f_j^\sigma(P) = f_j f_{\sigma(j)}(P) \). As in the last subsection, Let \( P^{(1)} = P(M_{\gamma}^{(1)}) = f_{\sigma(j)}(P) \). So we have \( P' = f_j(P^{(1)}) \).

Since we have proved the theorem in the simply-laced case, we know that
\[
M_{\gamma}^{(1)} = \begin{cases}
M_{\gamma}^{(1)} & \text{if } \langle h_j, \gamma \rangle = 0, \\
\min\{M_{\gamma}^{(1)}, M_{s_{\gamma}} + c_j(P^{(1)}) \} & \text{if } \langle h_j, \gamma \rangle = 1.
\end{cases}
\]

If \( \langle h_j, \gamma \rangle = 0 \), then \( \langle h_{\sigma(j)}, \gamma \rangle = 1 \). Again by the theorem in the simply-laced case, we have
\[
M_{\gamma}^{(3)} = \min\{M_{\gamma}, M_{s_{\sigma(j)} \gamma} + c_{\sigma(j)}(P) \}.
\]

Note that \( s_{\sigma(j)} \gamma = s_{\gamma}^{s_{\sigma(j)} \gamma} \) because \( s_j \gamma = \gamma \). So \( M_{s_{\sigma(j)} \gamma} = M_{\gamma}^{s_{\sigma(j)} \gamma} \). In the previous subsection we have proved that \( c_{\sigma(j)}(P) = c_j(P) \). Thus (6.6) is proved.
If \( \langle h_j, \gamma \rangle = 1 \), then \( \langle h_{\sigma(j)}, \gamma \rangle = 0 \) and \( \langle h_{\sigma(j)}, s_j \gamma \rangle = 0 \). In this case we have
\[
M_{\gamma}^{(1)} = M_{\gamma}, \quad M_{\hat{\gamma}}^{(1)} = M_{\hat{\gamma}} = M_{\gamma}^{(1)} \gamma \quad \text{and} \quad c_j(P^{(1)}) = c_j(P) \quad \text{(proved in the previous subsection). Hence } (6.6) \text{ still holds.}
\]

6.5. Proof of Theorem 3.7 in the non-simply-laced case. Let \( \tilde{P} = \tilde{P}(\tilde{M}_* \delta) \) be an MV polytope for \( \hat{g} \) and \( j \in \hat{I} \).

Let \( k = m(\tilde{P}, \tilde{I}) \). By Lemma 6.3 we know that \( \tilde{M}_{s_j} = M_{s_j} \), \( \tilde{M}_{s_j} = M_{s_j} \) and \( \tilde{M}_{s_j} = M_{s_j} \). Since \( s_j^j = s_j \), we have
\[
m(j, \tilde{P}) = m(j, P) = M_{s_j} - M_{s_j} - M_{s_j}.
\]
which is positive by the theorem in the simply-laced case.

Now assume that \( k \) is an integer such that \( k > m(j, \tilde{P}) \). Let \( \tilde{P}(k) = \tilde{P}(\tilde{M}_* \delta) = \tilde{f}_j(k, \tilde{P}) \) and \( \tilde{P}(k+1) = \tilde{P}(\tilde{M}_* \delta) = \tilde{f}_j(k+1, \tilde{P}) \). We are going to prove
\[
\tilde{M}^{(k+1)} = \left\{ \begin{array}{ll}
\tilde{M}^{(k)} & \text{if } \tilde{f} \in \tilde{G} \\
\tilde{M}_{\gamma}^{(k)} + c_j(P^{(k)}) \langle h_j, \gamma \rangle & \text{if } \tilde{f} \in \tilde{G}.
\end{array} \right.
\]

We only need to prove for the case \( \tilde{f} \in \tilde{G} \). We have \( \tilde{f} = \tilde{w} \tilde{w}_j \), where \( \tilde{d}_j \tilde{w} > \tilde{w} \).

Let \( \tilde{w} = \Theta(w) \) and \( \gamma = w \tilde{w}_j \). Then, as in the previous sections, we just need to prove
\[
m(\sigma(j), \tilde{P}) = m(j, P).
\]

In the following we assume \( k_j = 2 \). The case \( k_j = 3 \) is similar. Let \( P^{(k+1)} = P(M_{\sigma(j)} \delta) = \Phi^{-1}(\tilde{P}(k+1)) = \tilde{f}_j^{k+1}(P) \). Let \( P^{(k)} = P(M_{\sigma(j)} \delta) = \Phi^{-1}(\tilde{P}(k)) = \tilde{f}_j^{k}(P) \). Let \( P'' = P(M_{\sigma(j)} \delta) = \tilde{f}_j^{k}(P) \). Let \( \tilde{P} = \tilde{P}(\tilde{M}_* \delta) = \tilde{f}_j^{k}(P) \), hence \( P(k) = \tilde{f}_j^{k}(\tilde{P}) \).

We claim that \( m(\sigma(j), \tilde{P}) = m(j, P) \). In fact, we have
\[
\langle h_j, \tilde{w}_{\sigma(j)} \rangle = \langle h_j, -s_{\sigma(j)} \tilde{w}_{\sigma(j)} \rangle = \langle h_j, s_{\sigma(j)} \tilde{w}_{\sigma(j)} \rangle = 0.
\]

Noting that \( P = \hat{f}_j^{k}(P) \), we have \( \tilde{M}_{s_{\sigma(j)}} = M_{s_{\sigma(j)}} \), \( \tilde{M}_{s_{\sigma(j)}} = M_{s_{\sigma(j)}} \) and \( M_{s_{\sigma(j)}} \tilde{w}_{\sigma(j)} = M_{s_{\sigma(j)}} \tilde{w}_{\sigma(j)} \). Thus we deduce that
\[
m(\sigma(j), \tilde{P}) = M_{\tilde{w}_{\sigma(j)}} - M_{s_{\sigma(j)}} \tilde{w}_{\sigma(j)} - M_{s_{\sigma(j)}} \tilde{w}_{\sigma(j)} = M_{\tilde{w}_{\sigma(j)}} - M_{s_{\sigma(j)}} \tilde{w}_{\sigma(j)} - M_{s_{\sigma(j)}} \tilde{w}_{\sigma(j)} = m(\sigma(j), P) = m(j, P),
\]
where the last equality holds because \( P \in M\mathcal{Y}^{\sigma} \).

Applying the theorem in the simply-laced case for \( \tilde{P} \) and \( \sigma(j) \), we have
\[
M''_\gamma = \left\{ \begin{array}{ll}
M^{(k)}_\gamma & \text{if } \gamma \in \Gamma^\sigma(j), \\
M^{(k)}_{s_{\sigma(j)} \gamma} + c_{\sigma(j)}(P^{(k)}) \langle h_{\sigma(j)}, \gamma \rangle & \text{if } \gamma \in \Gamma^\sigma(j).
\end{array} \right.
\]

Let \( \tilde{P} = P(\tilde{M}_* \delta) = \tilde{f}_j^{k+1}(P) \). So \( P' = \tilde{f}_j^{k}(P) \).
As above we can prove that $m(j, \tilde{P}) = m(\sigma(j), P) = m(j, P)$. Then we apply the theorem in the simply-laced case for $\tilde{P}$ and $j$ and deduce that

\begin{equation}
M_{\gamma}^{(k+1)} = \begin{cases} 
M''_{\gamma} & \text{if } \gamma \in \Gamma_j, \\
M''_{\sigma_j}(P'')(h_j, \gamma) & \text{if } \gamma \in \Gamma_j.
\end{cases}
\end{equation}

To prove (6.10), we only need to combine (6.11) and (6.12) and there are three cases to be considered. Case 1: $\gamma \in \Gamma_{\sigma(j)}$ and $\gamma \in \Gamma_j$. Case 2. $\gamma \in \Gamma_{\sigma(j)}$ and $\gamma \in \Gamma_j$. The remaining part is completely similar to the last part in the proof of Lemma 6.5 and hence we omit it.

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