Differential-difference system related to toroidal Lie algebra

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Abstract

We present a novel differential-difference system in (2+1)-dimensional space-time (one discrete, two continuum), arisen from the Bogoyavlensky’s (2+1)-dimensional KdV hierarchy. Our method is based on the bilinear identity of the hierarchy, which is related to the vertex operator representation of the toroidal Lie algebra $sl_{tor}^2$.

1 Introduction and main results

Multi-dimensional generalization of classical soliton equations has been one of the most exciting topic in the field of integrable systems. Among other things, Calogero [1] proposed an interesting example that is a (2+1)-dimensional extension of the Korteweg-de Vries equation,

$$u_t = \frac{1}{4} u_{xxx} + uu_y + \frac{1}{2} u_x \int^x u_y dx. \quad (1)$$

Yu et al. [2] obtained multi-soliton solutions of the (2+1)-dimensional KdV equation (1) by using the Hirota’s bilinear method. Let us consider the following Hirota-type equations,

$$(D_x^4 - 4D_xD_y) \tau \cdot \tau = 0, \quad (2)$$
$$(D_yD_x^3 + 2D_yD_x - 6D_D) \tau \cdot \tau = 0, \quad (3)$$

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where we have used the $D$-operators of Hirota defined as
\[
D_x D_y \ldots f(x, y, \ldots) \cdot g(x, y, \ldots) = \frac{\partial_x \partial f(x + s, y + t, \ldots) g(x - s, y - t, \ldots)|_{s, t, \ldots = 0}}{\partial_x f(x, y, \ldots)}.
\]

We remark that we have introduced auxiliary variables $t'$ that is a hidden parameter in (1).

If we set $u = 2(\log \tau)_{xx}$ and use (2) to eliminate $\partial_{t'}$, then one can show that $u = u(x, y, t)$ solves (4).

Bogoyavlensky [3] showed that there is a hierarchy of higher-order integrable equations associated with (1). In the paper [4], Ikeda and Takasaki generalized the Bogoyavlensky’s hierarchy from the viewpoint of the Sato’s theory of the KP hierarchy [2, 4, 7, 8], and discussed the relationship to toroidal Lie algebras. We note that the relation between integrable hierarchy and toroidal algebras has been discussed also by Billig [9], Iohara, Saito and Wakimoto [10] by using vertex operator representations.

In the present paper, we propose the following differential-difference system that has the same symmetry:
\[
\begin{align*}
\partial_t u_k &= \Delta_k \left( \frac{\partial_x u_{k+1}}{1 - \exp(-u_{k+1} - u_k)} \right) \\
&- \left( \frac{\partial_x u_k}{1 - \exp(u_{k+1} + u_k)} - \frac{1 + \exp(u_{k+1} + u_k)}{1 - \exp(u_{k+1} + u_k)} v_k \right), \\
\Delta_{-k} v_k &= \frac{\partial_x u_{k+1}}{u_{k+1}} + \frac{\partial_x (u_{k+1} + u_k)}{1 - \exp(u_{k+1} + u_k)} \\
&+ \frac{\partial_x (u_k + u_{k-1})}{u_k} + \frac{\partial_x (u_k + u_{k-1})}{1 - \exp(u_k + u_{k-1})},
\end{align*}
\]

where $\Delta_{-k}$ denotes the backward difference operator $\Delta_{-k} \overset{\text{def}}{=} 1 - \exp(-\partial_k) (\Delta_{-k} u_k = u_k - u_{k-1})$. We also show that this system has soliton-type solutions.

## 2 Lie algebraic derivation of bilinear identity

Here we briefly review the Lie algebraic derivation of the bilinear identity of the Bogoyavlensky’s hierarchy [4], which is a generating function of Hirota-type differential equations. We remark that the Lie algebra considered in [4] is bigger than that of present article. Here we don’t include the derivations to $\mathfrak{sl}_2^{\text{tor}}$ since those are not needed for our purpose. Owing to this difference, the proof given below may be simpler than that of [4].

The 2-toroidal Lie algebra $\mathfrak{sl}_2^{\text{tor}}$ [11, 12] is the universal central extension of the double loop algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$, while the affine Lie algebra $\tilde{\mathfrak{sl}}_2$ is the central extension of $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$. Let $A$ be the ring of Laurent polynomials of two variables $s$ and $t$. As a vector space, $\mathfrak{sl}_2^{\text{tor}}$ is isomorphic to $\mathfrak{sl}_2 \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}] \oplus \Omega_A/dA$, where $\Omega_A$ denotes the module of Kähler differentials of $A$ defined with the canonical derivation $d : A \to \Omega_A$. We define the Lie algebra structure of $\mathfrak{sl}_2^{\text{tor}}$ by
\[
[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x|y)(da)b \quad (x, y \in \mathfrak{sl}_2, \ a, b \in A),
\]
\[
[\mathfrak{sl}_2^{\text{tor}}, \Omega_A/dA] = 0,
\]
where $(x|y)$ denotes the Killing form and $\bar{\tau} : \Omega_A \to \Omega_A/dA$ the canonical projection.
In terms of generating series $X_m(z)$ ($X = E, F, H, m \in \mathbb{Z}$), $K^s_m(z)$ and $K^t_m(z)$, defined by

$$X_m(z) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} X \otimes s^n t^m z^{-n-1},$$

$$K^s_m(z) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} s^n t^m d \log s \cdot z^{-n},$$

$$K^t_m(z) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} s^n t^m d \log t \cdot z^{-n-1},$$

the relation (6) can be expressed as

$$X_n(z) Y_n(w) = \frac{1}{z - w}[X, Y]_{m+n}(w) + \frac{1}{(z - w)^2} (X|Y) K^s_{m+n}(w) + \frac{m}{z - w} (X|Y) K^t_{m+n}(w) + \text{regular as } z \to w.$$  \hspace{1cm} (8)

There exists a class of representations of $\mathfrak{sl}_{2}^{\text{tor}}$, which comes directly from that of $\hat{\mathfrak{sl}}_{2}$. We consider the space of polynomials,

$$F_y \overset{\text{def}}{=} \mathbb{C}[y_j, j \in \mathbb{Z}] \otimes \mathbb{C} [\exp(\pm y_0)],$$

and define the generating series $\varphi(z)$ and $V_m(y; z)$ by

$$\varphi(z) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} n y_n z^{n-1}, \quad V_m(y; z) \overset{\text{def}}{=} \exp \left[ m \sum_{n \in \mathbb{Z}} y_n z^n \right].$$

**Proposition 1.** (cf. [10, 13]) Let $(V, \pi)$ be a representation of $\hat{\mathfrak{sl}}_{2}$ such that $d \log s \mapsto c \cdot \text{id}_V$ for $c \in \mathbb{C}$. Then we can define the representation $\pi^{\text{tor}}$ of $\mathfrak{sl}_{2}^{\text{tor}}$ on $V \otimes F_y$ such that

$$X_m(z) \mapsto X^\pi(z) \otimes V_m(z),$$

$$K^s_m(z) \mapsto c \cdot \text{id}_V \otimes V_m(z),$$

$$K^t_m(z) \mapsto c \cdot \text{id}_V \otimes \varphi(z) V_m(z),$$

where $X = E, F, H, m \in \mathbb{Z}$ and $X^\pi(z) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} \pi(X \otimes s^n) z^{-n-1}$.

**Proof.** Using the operator product expansion for $\hat{\mathfrak{sl}}_{2}$,

$$X(z) Y(w) = \frac{1}{z - w} [X, Y](w) + \frac{1}{(z - w)^2} (X|Y) K + \text{regular as } z \to w,$$

and the property $V_m(z) V_n(z) = V_{m+n}(z)$, it is straightforward to show that $X_m(z)$ above satisfies (8). The remaining relations can be checked by direct calculations. □
To see the relationship to soliton theory, we shall consider the representation of $\tilde{sl}_2$ on the fermionic Fock space $[7, 8]$. Let $\psi_j, \psi_j^*$ ($j \in \mathbb{Z}$) be free fermions with the canonical anti-commutation relation. In terms of the generating series defined as

$$\psi(\lambda) = \sum_{n \in \mathbb{Z}} \psi_n \lambda^n, \quad \psi^*(\lambda) = \sum_{n \in \mathbb{Z}} \psi^*_n \lambda^{-n},$$

the canonical anti-commutation relation is written as

$$[\psi_{\lambda}(\lambda), \psi^*_{\mu}(\mu)]_+ = \delta(\lambda/\mu), \quad [\psi_{\lambda}(\lambda), \psi_{\mu}(\mu)]_+ = [\psi^*_{\lambda}(\lambda), \psi^*_{\mu}(\mu)]_+ = 0,$$

where $\delta(\lambda) \equiv \sum_{n \in \mathbb{Z}} \lambda^n$ is the formal delta-function.

Consider the fermionic Fock space $\mathcal{F}$ with the vacuum vector $|\text{vac}\rangle$ satisfying

$$\psi_j |\text{vac}\rangle = 0 \text{ for } j < 0, \quad \psi_j^* |\text{vac}\rangle = 0 \text{ for } j \geq 0,$$

and the dual Fock space $\mathcal{F}^*$ with the dual vacuum vector $\langle \text{vac}|$ satisfying

$$\langle \text{vac}|\psi_j = 0 \text{ for } j \geq 0, \quad \langle \text{vac}|\psi_j^* = 0 \text{ for } j < 0, \quad \langle \text{vac}|\text{vac}\rangle = 1.$$

As mentioned in $[7, 8]$, a level-1 representation of $\tilde{sl}_2$ is given by the elements,

$$:\psi(\lambda)\psi^*(-\lambda) : = \sum_{j,n \in \mathbb{Z}} (-1)^j :\psi_{j+n}\psi_j^* : \lambda^n,$$

where $:\cdot:\cdot$ denotes the fermionic normal ordering, $:\psi_i\psi_j^* : \equiv \psi_i\psi_j^* - \langle \text{vac}|\psi_i\psi_j^*|\text{vac}\rangle$. Applying Proposition $[7]$, we can construct a representation of $\mathfrak{sl}_2^\text{tor}$ on the space $\mathcal{F}_y \equiv \mathcal{F} \otimes F_y$ with the vacuum vector $|\text{vac}\rangle^\text{tor} \equiv |\text{vac}\rangle \otimes 1$.

We now introduce the following operator acting on $\mathcal{F}_y \otimes \mathcal{F}_y^*$:

$$\Omega^\text{tor} \equiv \sum_{m \in \mathbb{Z}} \oint \frac{d\lambda}{2\pi i\lambda} \psi(\lambda)V_m(\lambda; y) \otimes \psi^*(\lambda)V_{-m}(\lambda; y^*).$$

Using the anti-commutation relation ($[7]$) and the relation $V_m(y; \lambda)V_n(y; \lambda) = V_{m+n}(y; \lambda)$, we can obtain the following identity by direct calculations:

$$[\Omega^\text{tor}, \psi(p)\psi^*(p)V_n(y; p) \otimes 1 + 1 \otimes \psi(p)\psi^*(p)V_n(y^*; p)] = 0,$$

which means the action of $\mathfrak{sl}_2^\text{tor}$ on $\mathcal{F}_y \otimes \mathcal{F}_y^*$ commutes with $\Omega^\text{tor}$. Then it is straightforward to show that

$$\Omega^\text{tor} (g|\text{vac}\rangle^\text{tor} \otimes g|\text{vac}\rangle^\text{tor}) = 0 \quad (10)$$

for $g = \exp(X), X \in \mathfrak{sl}_2^\text{tor}$.

To rewrite (10) into bosonic language, we prepare two lemmas:
Lemma 1 ("Boson-Fermion correspondence"). [4] [5] For any \(|\nu| \in \mathcal{F}\), we have the following formulas,

\[
\langle \text{vac}| \psi_0^* \exp(H(x))\psi(\lambda)|\nu \rangle = \exp(\xi(x, \lambda))\langle \text{vac}| \exp(H(x - [\lambda^{-1}])|\nu \rangle, \\
\langle \text{vac}| \psi_{-1} \exp(H(x))\psi^*(\lambda)|\nu \rangle = \lambda \exp(-\xi(x, \lambda))\langle \text{vac}| \exp(H(x + [\lambda^{-1}])|\nu \rangle, 
\]

where we have used the following notation,

\[
x = (x_1, x_3, \ldots), \quad H(x) = \sum_{n=1}^{\infty} \sum_{j \in \mathbb{Z}} x_n \psi_j \psi_{n+j}^*, \\
\xi(x, \lambda) \equiv \sum_{n=1}^{\infty} x_n \lambda^n, \quad [\lambda^{-1}] \equiv (1/\lambda, 1/2\lambda, 1/3\lambda, \ldots).
\]

Lemma 2. [4] [7] Let \(P(n) = \sum_{j=0}^{\infty} n^j P_j\), where \(P_j \in \text{Diff}(z)\) are differential operators that may not depend on \(z\). If

\[
\sum_{n \in \mathbb{Z}} z^n P(n) g(z) = 0
\]

for some formal series \(g(z) = \sum_j g_j z^j\), then

\[
P(\epsilon - z \partial_z) g(z)|_{z=1} = 0
\]

as a polynomial in \(\epsilon\).

Define the \(\tau\)-function as

\[
\tau(x, y) \equiv \langle \text{tor}| \exp(H(x))g(\text{tor})\rangle.
\]

From the relation (10), together with Lemma 1 and Lemma 2, we have the following bilinear identity:

\[
\oint \frac{d\lambda}{2\pi i} \exp(\xi(x - x', \lambda))\tau(x - [\lambda^{-1}], y_0 + \eta(b, \lambda^2), y - b) \\
\times \tau(x' + [\lambda^{-1}], y_0 - \eta(b, \lambda^2), y' + b) = 0, \quad (11)
\]

where \(y = (y_2, y_4, \ldots)\) and \(\eta(b, \lambda^2) \equiv \sum_{n=1}^{\infty} b_{2n} \lambda^{2n}\).

Expanding (11), we can obtain Hirota-type differential equations including (2), (3) \((x_1 = x, x_3 = t', y_0 = y, y_2 = t)\). In this sense, the bilinear identity (11) is a generating function of Hirota-type differential equations of the Bogoyavlensky’s hierarchy. The \(N\)-soliton solution of (11) is obtained as follows [3]:

\[
\tau_N(x, y) = \sum_{l=0}^{N} \sum_{j_1 < \ldots < j_l} c_{j_1 \ldots j_l} \prod_{m=1}^{l} a_{j_m} \exp(\eta_{j_m}(x, y)), \quad (12)
\]

\[
\eta_j(x, y) \equiv \sum_{n=1}^{\infty} 2p_j^{2n-1} x_{2n-1} + \sum_{n=1}^{\infty} r_j p_j^{2n} y_{2n},
\]

\[
c_{j_1 \ldots j_l} \equiv \prod_{1 \leq m < n \leq l} \frac{(p_{j_m} - p_{j_n})^2}{(p_{j_m} + p_{j_n})^2}.
\]
3 Derivation of the differential-difference system

We now apply the Miwa transformation \([8, 14]\),

\[
\begin{align*}
x' & = l[\alpha] + m[\beta] + n[\gamma], \\
x & = (l + 1)[\alpha] + (m + 1)[\beta] + (n + 1)[\gamma],
\end{align*}
\]

(13)

to the bilinear identity (11). Here we have used the notation \(l[\alpha] = (la, la^2/2, la^3/3, \ldots)\).

We first consider the case \(b = (b_2, b_4, \ldots) = 0\). In this case, the bilinear identity (11) is reduced to that of the ordinary KP hierarchy. Thus we have the Hirota-Miwa equation (or the discrete KP equation),

\[
\begin{align*}
\alpha(\beta - \gamma) & \tau(l + 1, m, n; y)\tau(l, m + 1, n + 1; y) \\
+ \beta(\gamma - \alpha) & \tau(l, m + 1, n; y)\tau(l + 1, m, n + 1; y) \\
+ \gamma(\alpha - \beta) & \tau(l, m, n + 1; y)\tau(l + 1, m + 1, n; y) = 0
\end{align*}
\]

(14)

where \(\tau(l, m, n; y)\) denotes

\[
\tau(l, m, n; y) = \tau(x = l[\alpha] + m[\beta] + n[\gamma], y).
\]

We then consider the time-evolutions with respect to \(y_0, y_2\). Collecting the coefficient of \(b_2\) in the bilinear identity (11), we have

\[
\oint \frac{d\lambda}{2\pi i} \exp(\xi(x - x', \lambda)) \left( D_{y_2} - \lambda^2 D_{y_0} \right) \times \tau(x' + [\lambda^{-1}], y) \cdot \tau(x - [\lambda^{-1}], y) = 0.
\]

Applying the Miwa transformation (13), we obtain

\[
\begin{align*}
\alpha^2 & \beta \gamma(\beta - \gamma)D_{y_2}\tau(l + 1, m, n; y) \cdot \tau(l, m + 1, n + 1; y) \\
& + \alpha \beta^2(\gamma - \alpha)D_{y_2}\tau(l, m + 1, n; y) \cdot \tau(l + 1, m, n + 1; y) \\
& + \alpha \beta \gamma^2(\alpha - \beta)D_{y_2}\tau(l, m, n + 1; y) \cdot \tau(l + 1, m + 1, n; y) \\
& = \beta \gamma(\beta - \gamma)D_{y_0}\tau(l + 1, m, n; y) \cdot \tau(l, m + 1, n + 1; y) \\
& + \alpha \beta(\gamma - \alpha)D_{y_0}\tau(l, m + 1, n; y) \cdot \tau(l + 1, m, n + 1; y) \\
& + \alpha \beta(\alpha - \beta)D_{y_0}\tau(l, m, n + 1; y) \cdot \tau(l + 1, m + 1, n; y) \\
& - (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) \\
& \times D_{y_0}\tau(l, m, n; y) \cdot \tau(l + 1, m + 1, n + 1; y)
\end{align*}
\]

(15)

We further impose the condition \(\beta = \gamma\). Then the \(\tau\)-function \(\tau(l, m, n; y)\) depends only on \(k \overset{\text{def}}{=} m - n, l\) and \(y\). In this sense, we rewrite

\[
\tau(l, m, n; y) \rightarrow \tau(l, k; y).
\]

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Under this condition, the equations (14) and (15) are reduced to

\[2\alpha \tau(l+1,k;y)\tau(l,k;y)\]
\[=-(\alpha+\beta)\tau(l,k+1;y)\tau(l+1,k-1;y)\]
\[=-(\alpha-\beta)\tau(l,k-1;y)\tau(l+1,k+1;y) = 0,\]  

(16)

\[\alpha\beta^2 D_{y_2} (2\alpha \tau(l+1,k;y)\cdot \tau(l,k;y)\]
\[=-(\alpha+\beta)\tau(l,k+1;y)\cdot \tau(l+1,k-1;y)\]
\[=-(\alpha-\beta)\tau(l,k-1;y)\cdot \tau(l+1,k+1;y)) \]
\[=\alpha(\alpha+\beta)\tau(l,k+1;y)\cdot \tau(l+1,k-1;y)\]
\[=-\alpha(\alpha-\beta)\tau(l,k-1;y)\cdot \tau(l+1,k+1;y)) .\]  

(17)

Furthermore, we can construct N-soliton solution by applying (13) to (12):

\[\tau_N(l,k;y_0,y_2) = \sum_{l=0}^{N} \sum_{j_1<...<j_l} c_{j_1...j_l} \prod_{m=1}^{l} \phi_{j_m}(l,k;y_0,y_2),\]  

(18)

\[\phi_j(l,k;y_0,y_2) \overset{\text{def}}{=} a_j \exp(r_j y_0 + r_j p_j^2 y_2) \left(\frac{1+p_j^2}{1-p_j^2}\right)^l \left(\frac{1+p_j^2}{1-p_j^2}\right)^k ,\]

where \(c_{j_1...j_l}\) is the same as the continuum one (12). We remark that the N-soliton \(\tau\)-function can be written as Wronskian determinant. Using the determinant expression, we can show that both (16) and (17) are reduced to the Plücker relations.

Introducing the variables as

\[\partial_t \overset{\text{def}}{=} \frac{2\alpha}{\alpha^2}\partial_{y_0} - 2\partial_{y_2}, \quad \partial_x \overset{\text{def}}{=} \partial_{y_2} - \frac{1}{\beta^2}\partial_{y_0},\]

\[u_k(t,x) \overset{\text{def}}{=} \log \left[ \left(\frac{\beta-\alpha}{\beta+\alpha}\right)^{1/2} \frac{\tau(l+1,k+1)\tau(l,k)}{\tau(l,k+1)\tau(l+1,k)} \right] ,\]

\[v_k(t,x) \overset{\text{def}}{=} \partial_x \log \frac{\tau(l,k+2)}{\tau(l,k)} ,\]

we have the differential-difference equations (1) and (3), which have the N-soliton solution corresponding to the \(\tau\)-function (18).

4 Concluding remarks

In this paper, we have introduced the differential-difference system (1) and (3), which is related to toroidal Lie algebra \(\mathfrak{sl}_\text{tor}^\alpha\). Since the symmetry of the toroidal Lie algebra allows us to introduce extra parameters of wave numbers in the soliton solution (i.e., \(r_j\) in (12) and (18)), it might be possible to construct some interesting solutions. In particular, we can obtain a class of traveling-wave solutions that has the shape of the character “V” (Figure 4), which is a special case of the two-soliton solutions. The existence of the V-soliton type solution is one of the features of this class of equations.

We note that there exist solutions of the same shape for the (2+1)-dimensional KdV equation (4), and for a (2+1)-dimensional generalization of the nonlinear Schrödinger
(NLS) equation \[\text{[13]}\] that also has the symmetry of the toroidal Lie algebra $\mathfrak{sl}_{2}^{tor}$ \[\text{[17]}\]. We also remark that Oikawa et al. \[\text{[14]}\] discussed the propagation of the V-soliton in a two-layer fluid, which is governed by a equation similar to the (2+1)-dimensional NLS equation.

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Figure 1: Special case of the two-soliton solutions ("V-soliton") with $p_1 = p_2$ ($p_1 = p_2 = 0.3$, $r_1 = 0.15$, $r_2 = -0.1$, $a_1 = a_2 = 1$, $\alpha = 0.8$, $\beta = 0.5$).
Figure 2: Two-soliton solutions with $p_1 \neq p_2$ ($p_1 = 0.3$, $p_2 = 0.23$, $r_1 = 0.15$, $r_2 = -0.1$, $a_1 = a_2 = 1$, $\alpha = 0.8$, $\beta = 0.5$).