Biharmonic Submanifolds in a Riemannian Manifold with Non-Positive Curvature

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Abstract. In this paper, we show that, for every biharmonic submanifold \((M, g)\) of a Riemannian manifold \((N, h)\) with non-positive sectional curvature, if \(\int_M |\eta|^2 v_g < \infty\), then \((M, g)\) is minimal in \((N, h)\), i.e., \(\eta \equiv 0\), where \(\eta\) is the mean curvature tensor field of \((M, g)\) in \((N, h)\). This result gives an affirmative answer under the condition \(\int_M |\eta|^2 v_g < \infty\) to the following generalized Chen’s conjecture: every biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature must be minimal. The conjecture turned out false in case of an incomplete Riemannian manifold \((M, g)\) by a counter example of Ou and Tang (in The generalized Chen’s conjecture on biharmonic sub-manifolds is false, a preprint, 2010).

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1. Introduction and Statement of Results

This paper is an extension of our previous paper [10] to biharmonic submanifolds of any co-dimension of a Riemannian manifold of non-positive curvature. Let us consider an isometric immersion \(\varphi : (M, g) \to (N, h)\) of a Riemannian manifold \((M, g)\) of dimension \(m\) into another Riemannian manifold \((N, h)\) of dimension \(n = m + p (p \geq 1)\). We have

\[\nabla^N_{\varphi_* X} \varphi_* Y = \varphi_* (\nabla_X Y) + B(X, Y),\]

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for vector fields $X$ and $Y$ on $M$, where $\nabla, \nabla^N$ are the Levi-Civita connections of $(M,g)$ and $(N,h)$, and $B : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)^\perp$ is the second fundamental form of the immersion $\varphi$ corresponding to the decomposition:

$$T_{\varphi(x)}N = d\varphi(T_xM) \oplus d\varphi(T_xM)^\perp \quad (x \in M),$$

respectively. Let $\eta$ be the mean curvature vector field along $\varphi$ defined by $\eta = \frac{1}{m} \sum_{i=1}^{m} B(e_i,e_i)$, where $\{e_i\}_{i=1}^{m}$ is a local orthonormal frame on $(M,g)$. Then, the generalized Chen’s conjecture (cf. [1–4,11–13]) is that:

For an isometric immersion $\varphi : (M,g) \to (N,h)$, assume that the sectional curvature of $(N,h)$ is non-positive. If $\varphi$ is biharmonic (cf. See Sect. 2), then, it is minimal, i.e., $\eta \equiv 0$.

In this paper, we will show

**Theorem 1.1.** Assume that $(M,g)$ is a complete Riemannian manifold of dimension $m$ and $(N,h)$ is a Riemannian manifold of dimension $m+p$ ($p \geq 1$) whose sectional curvature is non-positive. If $\varphi : (M,g) \to (N,h)$ is biharmonic and satisfies that $\int_M |\eta|^2 v_g < \infty$, then, $\varphi$ is minimal.

In our previous paper [10], we showed

**Theorem 1.2.** Assume that $(M,g)$ is complete and the Ricci tensor $\text{Ric}^N$ of $(N,h)$ satisfies that

$$\text{Ric}^N(\xi,\xi) \leq |A|^2. \quad (1.1)$$

If $\varphi : (M,g) \to (N,h)$ is biharmonic and satisfies that

$$\int_M H^2 v_g < \infty, \quad (1.2)$$

then, $\varphi$ has constant mean curvature, i.e., $H$ is constant.

Notice that, in Theorem 1.2 in case of codimension one, we only need the weaker assumption, non-positivity of the Ricci curvature of $(N,h)$ (cf. [13]). On the other hand, in Theorem 1.1, we should treat with a complete submanifold of an arbitrary co-dimension $p \geq 1$, and we need the stronger assumption non-positivity of the sectional curvature of $(N,h)$. In proving Theorem 1.1, the method of the proof of Theorem 1.2 [10] does not work anymore. We should turn our mind, and have a different and very simple proof. Finally, our Theorem 1.1 implies that the generalized Chen’s conjecture holds true under the assumption that $\int_M |\eta|^2 v_g$ is finite and $(M,g)$ is complete.
2. Preliminaries

2.1. Harmonic Maps and Biharmonic Maps

In this subsection, we prepare general materials about harmonic maps and biharmonic maps of a complete Riemannian manifold into another Riemannian manifold (cf. [5]).

Let \((M, g)\) be an \(m\)-dimensional complete Riemannian manifold, and the target space \((N, h)\) is an \(n\)-dimensional Riemannian manifold. For every \(C^\infty\) map \(\varphi\) of \(M\) into \(N\). Let \(\Gamma(\varphi^{-1}TN)\) be the space of \(C^\infty\) sections of the induced bundle \(\varphi^{-1}TN\) of the tangent bundle \(TN\) by \(\varphi\). The tension field \(\tau(\varphi)\) is defined globally on \(M\) by

\[
\tau(\varphi) = \sum_{i=1}^{m} B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN),
\]

(2.1)

where the second fundamental form \(B(\varphi)\) is defined by

\[
B(\varphi)(X, Y) = \nabla^N_{\varphi_*(X)}\varphi_*(Y) - \varphi_*(\nabla_X Y)
\]

for \(X, Y \in \mathfrak{X}(M)\). Then, a \(C^\infty\) map \(\varphi : (M, g) \rightarrow (N, h)\) is harmonic if \(\tau(\varphi) = 0\). The bitension field \(\tau_2(\varphi)\) is defined globally on \(M\) by

\[
\tau_2(\varphi) = J(\tau(\varphi)) = \overline{\Delta}\tau(\varphi) - \mathcal{R}(\tau(\varphi)),
\]

(2.2)

where

\[
J(V) := \overline{\Delta}V - \mathcal{R}(V),
\]

\[
\overline{\Delta}V := \nabla^* \nabla V = -\sum_{i=1}^{m} \{\nabla_{e_i} (\nabla e_i V) - \nabla_{\nabla e_i e_i V}\},
\]

\[
\mathcal{R}(V) := \sum_{i=1}^{m} R^N(V, \varphi_*(e_i)) \varphi_*(e_i).
\]

(2.3)

Here, \(\nabla\) is the induced connection on the induced bundle \(\varphi^{-1}TN\), and \(R^N\) is the curvature tensor of \((N, h)\) (cf. [6]) given by

\[
R^N(U, V)W = [\nabla^N_U, \nabla^N_V]W - \nabla^N_{[U, V]}W \quad (U, V, W \in \mathfrak{X}(N)).
\]

A \(C^\infty\) map \(\varphi : (M, g) \rightarrow (N, h)\) is called to be biharmonic (cf. [2, 5, 7]) if

\[
\tau_2(\varphi) = 0.
\]

2.2. Setting of Isometric Immersions

In this subsection, we prepare fundamental materials of general facts on isometric immersions (cf. [8, 9]). Let \(\varphi\) be an isometric immersion of an \(m\)-dimensional Riemannian into an \((m + p)\)-dimensional Riemannian manifold \((N, h)\). Then, the induced bundle \(\varphi^{-1}TN\) of the tangent bundle \(TN\) of \(N\) by \(\varphi\) is decomposed into the direct sum:

\[
\varphi^{-1}TN = \tau M \oplus \nu M,
\]

(2.4)
where \( \varphi^{-1}TN = \bigcup_{x \in M} T_{\varphi(x)}N \), \( \tau M = d\varphi(TM) = \bigcup_{x \in M} d\varphi(T_x M) \), and \( \nu M = \bigcup_{x \in M} d\varphi(T_x M)^\perp \) is the normal bundle. For the induced connection \( \nabla \) on \( \varphi^{-1}TN \) of the Levi-Civita connection \( \nabla^N \) of \( (N, h) \) by \( \varphi \), \( \nabla_X(d\varphi(Y)) \) is decomposed corresponding to (2.4) as

\[
\nabla_X(d\varphi(Y)) = d\varphi(\nabla_X Y) + B(X, Y)
\]

for all \( C^\infty \) vector fields \( X \) and \( Y \) on \( M \). Here, \( \nabla \) is the Levi-Civita connection of \( (M, g) \) and \( B(X, Y) \) is the second fundamental form of the immersion \( \varphi : (M, g) \to (N, h) \).

Let \( \{\xi_1, \cdots, \xi_p\} \) be a local unit normal vector fields along \( \varphi \) that are orthogonal at each point, and let us decompose \( B(X, Y) \) as

\[
B(X, Y) = \sum_{i=1}^{p} b^i(X, Y) \xi_i,
\]

where \( b^i(X, Y)(i = 1, \cdots, p) \) are the \( p \) second fundamental forms of \( \varphi \). For every \( \xi \in \Gamma(\nu M) \), \( \nabla_X \xi \), denoted also by \( \nabla_X^N \xi \) is decomposed correspondingly to (2.4) into

\[
\nabla^N_X \xi = -A_\xi(X) + \nabla^\perp_X \xi,
\]

where \( \nabla^\perp \) is called the normal connection of \( \nu M \). The linear operator \( A_\xi \) of \( \Gamma(TM) \) into itself, called the shape operator with respect to \( \xi \), satisfies that

\[
\langle A_\xi(X), Y \rangle = \langle B(X, Y), \xi \rangle
\]

for all \( C^\infty \) vector fields \( X \) and \( Y \) on \( M \). Here, we denote the Riemannian metrics \( g \) and \( h \) simply by \( \langle \cdot, \cdot \rangle \).

We denote the tension field \( \tau(\varphi) \) of an isometric immersion \( \varphi : (M, g) \to (N, h) \) as

\[
\tau(\varphi) = \text{Trace}_g(\nabla d\varphi) = \sum_{i=1}^{m} B(e_i, e_i)
\]

\[
= \sum_{k=1}^{p} (\text{Trace}_g b^k) \xi_k
\]

\[
= m \sum_{k=1}^{p} H_k \xi_k
\]

\[
= m \eta,
\]

where \( \nabla \) is the induced connection on \( TM \otimes \varphi^{-1}TN \), \( H_k := \frac{1}{m} \text{Trace}_g b^k = \frac{1}{m} \text{Trace}_g (A_{\xi_k})(k = 1, \cdots, p) \), and \( \eta := \sum_{k=1}^{p} H_k \xi_k \) is the mean curvature vector field of \( \varphi \). Let us recall that \( \varphi : (M, g) \to (N, h) \) is minimal if \( \eta \equiv 0 \).
3. Proof of Main Theorem

Assume that \( \varphi : (M, g) \rightarrow (N, h) \) is a biharmonic immersion. Then, since (2.9): \( \tau(\varphi) = m \eta \), the biharmonic map equation

\[
\tau_2(\varphi) = \Delta(\tau(\varphi)) - R(\tau(\varphi)) = 0 \tag{3.1}
\]

is equivalent to that

\[
\Delta \eta - \sum_{i=1}^{m} R^N(\eta, d\varphi(e_i))d\varphi(e_i) = 0. \tag{3.2}
\]

Take any point \( x_0 \) in \( M \), and for every \( r > 0 \), let us consider the following cut-off function \( \lambda \) on \( M \):

\[
\begin{cases}
0 \leq \lambda(x) \leq 1 & (x \in M), \\
\lambda(x) = 1 & (x \in B_r(x_0)), \\
\lambda(x) = 0 & (x \notin B_{2r}(x_0)) \\
|\nabla \lambda| \leq \frac{2}{r} & (\text{on } M),
\end{cases}
\]

where \( B_r(x_0) := \{ x \in M : d(x, x_0) < r \} \) and \( d \) is the distance of \((M, g)\). In both sides of (3.2), taking inner product with \( \lambda^2 \eta \), and integrate them over \( M \), we have

\[
\int_M \langle \Delta \eta, \lambda^2 \eta \rangle v_g = \int_M \sum_{i=1}^{m} \langle R^N(\eta, d\varphi(e_i))d\varphi(e_i), \eta \rangle \lambda^2 v_g. \tag{3.3}
\]

Since the sectional curvature of \((N, h)\) is non-positive, \( h(R^N(u, v)v, u) \leq 0 \) for all tangent vectors \( u \) and \( v \) at \( T_yN(y \in N) \), the right hand side of (3.3) is non-positive, i.e.,

\[
\int_M \langle \Delta \eta, \lambda^2 \eta \rangle v_g \leq 0. \tag{3.4}
\]

On the other hand, the right hand side coincides with

\[
\int_M \langle \nabla^2 \eta, \nabla(\lambda^2 \eta) \rangle v_g = \int_M \sum_{i=1}^{m} \langle \nabla_{e_i} \eta, \nabla_{e_i}(\lambda^2 \eta) \rangle v_g \\
= \int_M \lambda^2 \sum_{i=1}^{m} |\nabla_{e_i} \eta|^2 v_g \\
+ 2 \int_M \sum_{i=1}^{m} \lambda (e_i \lambda) \langle \nabla_{e_i} \eta, \eta \rangle v_g, \tag{3.5}
\]

since \( \nabla_{e_i}(\lambda^2 \eta) = \lambda^2 \nabla_{e_i} \eta + 2\lambda(e_i \lambda) \eta \). Therefore, we have

\[
\int_M \lambda^2 \sum_{i=1}^{m} |\nabla_{e_i} \eta|^2 v_g \leq -2 \int_M \sum_{i=1}^{m} \langle \lambda \nabla_{e_i} \eta, (e_i \lambda) \eta \rangle v_g. \tag{3.6}
\]
Now apply with $V := \lambda \nabla e_i \eta$, and $W := (e_i \lambda) \eta$, to Young’s inequality: for all $V, W \in \Gamma(\varphi^{-1}TN)$ and $\epsilon > 0$,

$$\pm 2 \langle V, W \rangle \leq \epsilon |V|^2 + \frac{1}{\epsilon} |W|^2,$$

the right hand side of (3.6) is smaller than or equal to

$$\epsilon \int_M \lambda^2 \sum_{i=1}^m |\nabla e_i \eta|^2 v_g + \frac{1}{\epsilon} \int_M |\eta|^2 \sum_{i=1}^m |e_i \lambda|^2 v_g. \quad (3.7)$$

By taking $\epsilon = \frac{1}{2}$, we obtain

$$\int_M \lambda^2 \sum_{i=1}^m |\nabla e_i \eta|^2 v_g \leq \frac{1}{2} \int_M \lambda^2 \sum_{i=1}^m |\nabla e_i \eta|^2 v_g + 2 \int_M |\eta|^2 \sum_{i=1}^m |e_i \lambda|^2 v_g.$$

Thus, we have

$$\int_M \lambda^2 \sum_{i=1}^m |\nabla e_i \eta|^2 v_g \leq 4 \int_M |\eta|^2 \sum_{i=1}^m |e_i \lambda|^2 v_g \leq \frac{16}{r^2} \int_M |\eta|^2 v_g < \infty. \quad (3.8)$$

Since $(M, g)$ is complete, we can tend $r$ to infinity, and then the left hand side goes to $\int_M \sum_{i=1}^m |\nabla e_i \eta|^2 v_g$, we obtain

$$\int_M \sum_{i=1}^m |\nabla e_i \eta|^2 v_g \leq 0. \quad (3.9)$$

Thus, we have $\nabla_X \eta = 0$ for all vector field $X$ on $M$.

Then, we can conclude that $\eta \equiv 0$. For, applying (2.7):

$$\nabla_X \xi_k = -A_{\xi_k}(X) + \nabla_X^{\perp} \xi_k,$$

to $\eta = \sum_{k=1}^p H_k \xi_k$, we have

$$0 = \nabla_X \eta = -A_{\eta}(X) + \nabla_X^{\perp} \eta, \quad (3.10)$$

which implies that, for all vector field $X$ on $M$,

$$\begin{aligned}
A_{\eta}(X) &= 0, \\
\nabla_X^{\perp} \eta &= 0.
\end{aligned} \quad (3.11)$$

by comparing the tangential and normal components. Then, by the first equation of (3.11), we have

$$\langle B(X, Y), \eta \rangle = \langle A_{\eta}(X), Y \rangle = 0, \quad (3.12)$$

for all vector fields $X$ and $Y$ on $M$. This implies that $\eta \equiv 0$ since $\eta = \frac{1}{m} \sum_{i=1}^m B(e_i, e_i)$. \qed
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