2D non-perturbative modeling of oscillations in rapidly rotating stars

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We present and discuss results of a recently developed two dimensional non-perturbative method to compute accurate adiabatic oscillation modes of rapidly rotating stars. The 2D calculations fully take into account the centrifugal distortion of the star while the non-perturbative method includes the full influence of the Coriolis acceleration. These characteristics allow us to compute oscillation modes of rapid rotators - from high order p-modes in δ Scuti stars, to low order p- and g-modes in β Cephei or Be stars.

1 Introduction

It is a well known fact that rotation plays a key role in stellar evolution. From early stages of stellar formation to the final steps of evolution, rotation generates various dynamical processes such as meridional circulation and rotationally induced turbulence which drive chemical element mixing and transport of angular momentum (for example Maeder 2009). These processes are not fully understood and are still poorly modeled, but asteroseismology can provide important constraints provided the effects of rotation on stellar pulsation are better understood. In rotating stars, the centrifugal acceleration breaks the spherical symmetry - causing distortion - and the resonant cavity of the modes is modified. The Coriolis acceleration enters the equation of motion and affects the motion of the waves and the frequencies of the normal modes. For slow rotators, the effects of rotation on oscillation frequencies have been extensively investigated with perturbative methods (see Dziembowski & Goode 1992, Gough & Thompson 1990, Saio 1981, Soufi et al. 1998, and references therein). In this approach, the angular rotation velocity Ω is considered as small compared to the oscillation frequencies, thereby allowing their expansion as a power series in Ω. Perturbation methods cease to be valid whenever the rotation frequency is no longer negligible in front of the break-up frequency (√GM/R3) or the oscillation frequency. Then for moderate to rapid rotors a non-perturbative treatment is necessary. In the non-perturbative approach, the pulsation equations are projected onto the spherical harmonic basis. The effects of the Coriolis acceleration and the stellar distortion cause a coupling between the different spectral components, and the eigenvalue problem must be solved directly by a two-dimensional method. Such an approach has been applied to g and r-modes for uniformly rotating stars under the Cowling approximation (Lee & Saio 1987), for acoustic modes in uniformly rotating polytropes (Lignières et al. 2006, Reese et al. 2006), in uniformly rotating ZAMS models (Lovekin & Deupree 2008), and in differentially rotating ZAMS models with a conservative rotation law (Lovekin et al. 2009, Reese et al. 2009). We present here a 2D non-perturbative code which allows us to calculate adiabatic oscillations for the whole frequency range from high order g-modes to high order p-modes. Particular care has been taken so as to be able to compute pulsations for all types of stellar models. In the oscillation code, no hypotheses have been made on the fluid microphysics – non polytropic, non barotropic – the rotation profile is free – not necessarily conservative – it can be differential in radius and in latitude. The paper is organized as follows: in the next section, the formalism is explained. In section 3, we describe the numerical method used to solve the eigenfunction problem. A conclusion and perspectives follow.

2 The formalism

For the computation of pulsations, the stellar structure is reduced to its dynamical behavior. We compute oscillation modes as the adiabatic response of the structure to small perturbations – i.e. of the density, pressure, gravitational potential and velocity field – using the eulerian formalism. The velocity field of the equilibrium structure is only due to
solenoidal rotation:

$$\mathbf{v}_0 = \mathbf{\Omega} \times \mathbf{r} \quad (1)$$

where $\mathbf{\Omega} = \Omega(r, \theta) \cos(\theta) \mathbf{e}_r - \Omega(r, \theta) \sin(\theta) \mathbf{e}_\theta \quad (2)$

$\mathbf{e}_r$ and $\mathbf{e}_\theta$ being the classical spherical basis vectors. Then the equations describing the oscillations of a self rotating fluid are the perturbed equations of motion:

$$\rho_0 \left( \frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}_0 \cdot \nabla) v' + (v' \cdot \nabla) v_0 \right) + \rho' (\mathbf{v}_0 \cdot \nabla) v_0 = -\nabla p' - \rho' \nabla \Phi_0 - \rho_0 \nabla \Phi' \quad (3)$$

The linearised continuity equation:

$$\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) \rho' + \nabla \cdot (\rho_0 v') = 0 \quad (4)$$

The Poisson equation for the perturbed gravitational potential:

$$\nabla^2 \Phi' = 4\pi G \rho' \quad (5)$$

and the perturbed adiabatic relation:

$$\left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) \left( \frac{p'}{\rho_0} - \frac{p'}{\Gamma_1 p_0} \right) + v' \cdot \left( \nabla \ln \rho_0 - \frac{1}{\Gamma_1} \nabla \ln p_0 \right) = 0 \quad (6)$$

By adding an auxiliary equation for the derivative of the gravitational potential $d\Phi' = d\Phi' / \partial \zeta$, the problem is reduced to a set of first order differential and algebraic equations. Together with specific boundary conditions, we get an eigenvalue problem, the eigenvalues of which are the oscillation frequencies, and the eigenfunctions of which are the eulerian perturbations of density, pressure, gravitational potential, its radial derivative, and the three spatial components of the perturbed velocity field.

### 2.1 Spheroidal geometry

Due to the distorted shape caused by rotation, we chose to use a spheroidal coordinate system. This system (found by Bonazzola et al. 1998) is convenient for setting up proper boundary conditions. As done in Reese et al. (2006), $\zeta$ is defined as the radial coordinate, and is related to the spherical $r$ coordinate by:

- In the stellar interior $\zeta \in [0; 1]$, domain $V$:

  $$r(\zeta, \theta) = (1 - \epsilon) \zeta + \frac{5 \zeta^3 - 3 \zeta^5}{2} (R_s(\theta) - 1 + \epsilon) \quad (7)$$

- In the outer domain $V_2$, $\zeta \in [1; 2]$:

  $$r(\zeta, \theta) = 2 + (1 - \epsilon) \zeta$$

  $$+ \left( 2 \zeta^3 - 9 \zeta^2 + 12 \zeta - 4 \right) (R_s(\theta) - 1 + \epsilon) \quad (8)$$

where $\theta$ is the colatitude, $\epsilon = 1 - R_{ps} / R_{eq}$ the flatness and $R_s(\theta)$ the radius at the surface. With this mapping, the surface of the star is given by $\zeta = 1$, which is very convenient for avoiding discontinuities at the stellar surface. At the center, surfaces of constant $\zeta$ tend to be spherical, so that central regularity conditions are simplified. In the outer region, iso-$\zeta$ surfaces regain a spherical shape at $\zeta = 2$.

![Fig. 1](https://www.an-journal.org) Coordinate system used in computing the pulsation modes. The domain $V$ corresponds to the star itself. $V_2$ encompasses the star, its outer limit being a sphere of radius $r = 2$ (twice the equatorial radius).

### 2.2 Boundary conditions

In order to complete the eigenvalue problem defined by the four equations Eq.(3), Eq.(4), Eq.(5), Eq.(6), it is necessary to specify a number of boundary conditions.

At the center of the star, where the mapping is almost spherical, the requirement is that the velocity field components are regular. This condition is easily expressed for functions expanded over the spherical harmonic basis. Concerning the scalar quantities, we ensure that the radial component associated with the spherical harmonic $Y^m_\ell$ behaves like $r'$. By setting $\Phi' \Phi' \Phi' / \partial r = 0$, which is easily expressed on the iso-$\zeta$ surface $\zeta = 1$.

It is also necessary to impose a condition on the gravitational potential so as to ensure that it goes to zero at infinity. With the mapping described in the former section, we can safely impose the classical condition on the different harmonic components of $\Phi'$ on the outer border of $V_2$, at $\zeta = 2$.

### 2.3 Change of variable

In order to have solutions with a good behavior at the surface of the star, we choose to use $\pi' = p'/\rho_0$ rather than the pressure perturbation directly.

### 3 Numerical method

In order to isolate the radial, spheroidal and toroidal components of the fluid’s motion, we take the radial component of the equation of motion (Eq.3), its divergence and the radial part of its curl. We then get a set of 7 equations and 7 unknowns which are: the perturbed velocity vector, $(\pi')$, the density perturbation $(\rho')$, the perturbation of the gravitational potential $(\Phi')$ and its derivative $(d\Phi'/d\zeta)$, along with the related boundary conditions.
3.1 Projection onto the spherical harmonics

To solve this eigenvalue problem, we develop all the variables onto a series of spherical harmonics, following Rieutord [1987].

For the velocity field perturbation:

\[ v'_\ell(\zeta, \theta, \phi) = i \sum_{l' \geq |m|} u_{l'\ell}(\zeta) Y_{l'm}(\theta, \phi) \quad (9) \]

\[ v'_\ell(\zeta, \theta, \phi) = i \sum_{l' \geq |m|} \left( v'_{l'\ell}(\zeta) \frac{\partial Y_{l'm}(\theta, \phi)}{\partial \theta} + w'_{l'\ell}(\zeta) \frac{m}{\sin \theta} \frac{\partial Y_{l'm}(\theta, \phi)}{\partial \phi} \right) \]

\[ v'_\ell(\zeta, \theta, \phi) = -i \sum_{l' \geq |m|} \left( v'_{l'\ell}(\zeta) \frac{m}{\sin \theta} Y_{l'm}(\theta, \phi) + w'_{l'\ell}(\zeta) \frac{\partial Y_{l'm}(\theta, \phi)}{\partial \theta} \right) \]

where \( l'_{2\rho} = l_{2} + (-1)^\rho \). For any scalar variable \((\pi', \rho', \Phi', d\Phi')\):

\[ f'(\zeta, \theta, \phi) = \sum_{l' \geq |m|} f'_{l'\ell}(\zeta) Y_{l'm}(\theta, \phi) \quad (10) \]

where \( Y_{l'm} \) is the spherical harmonic of degree \( l \) and azimuthal order \( m \), and \( f'_{l'\ell}(\zeta) \) the radial functions that are to be determined. We include these spectral developments Eq.(9) and Eq.(10) into the equations system Eq.(6) to Eq.(9). We get a linear partial differential equations system in terms of the variables \( u'_{\ell}, v'_{\ell}, w'_{\ell}, \pi'_{\ell}, \rho'_{\ell}, \Phi'_{\ell} \) and \( d\Phi'_{\ell} \) which can formally be written as:

\[ \sum_{l \geq |m|} \mathbf{E}(Y_{l'm}(\theta, \phi)) = 0 \quad (11) \]

In the general case – where rotation breaks the spherical symmetry – the problem is not separable in \( \zeta \) and \( \theta \). As a result, a coupled equations system is obtained by projecting Eq.(11) onto the spherical harmonics basis:

\[ \forall \ell \geq |m|, \sum_{l' \geq |m|} \int \sin \theta \, d\theta \, d\phi \frac{4\pi}{4\pi} \mathbf{E}(Y_{l'm}(\theta, \phi)) Y_{l'm}^* = 0 \quad (12) \]

where \( Y_{l'm}^* \) is the complex conjugate of \( Y_{l'm} \). Finally, to get a finite and equal number of equations and variables, we truncate the series at the \( 2M^{th} \) term. Given that the equilibrium model is symmetrical with respect to the equator, half of the terms are dropped in the projection, and the system only couples terms with the same symmetry, i.e. parity. The selection rules operating here are:

\[ \ell = |m| + 2(j - 1) + p, \quad j \in [1 : M] \]

\[ p = 0 \text{ if } m \text{ and } \ell \text{ are of the same parity,} \]

\[ p = 1 \text{ otherwise.} \]

We then are able to solve the eigenvalue problem separately for the even and odd eigenfunctions, and for a given azimuthal order.

3.2 Radial behavior

In order to ensure a proper regular behavior at the center of the star, and to avoid numerical convergence problems, we scale the radial components of the spectral decomposition by the appropriate powers of \( \zeta \). If we suppose a priori the regularity of the scalar variables \( f_\ell (\tilde{\zeta}, \rho_\ell, \Phi_\ell) \), following Nikiforov and Uvarov (1983), these radial functions satisfy:

\[ f_\ell(\zeta) \sim \zeta^k \text{ when } \zeta \to 0 \quad (14) \]

For the velocity field, the same treatment gives:

\[ u'_\ell \sim \zeta^{k-1}, \quad v'_\ell \sim \zeta^{k-1} \quad \text{and} \quad w'_\ell \sim \zeta^k \quad (15) \]

This leads to the following scaling:

\[ f'_\ell = \zeta^{k-1} \tilde{f}' \quad \Phi'_\ell = \zeta^k \Phi' \quad \rho'_\ell = \zeta^k \rho' \quad d\Phi'_\ell = \zeta^{k-1}d\Phi' \]

\[ u'_\ell = \zeta^{k-1} \tilde{u}' \quad v'_\ell = \zeta^{k-1} \tilde{v}' \quad w'_\ell = \zeta^k \tilde{w}' \quad (16) \]

3.3 Final eigenvalue system

In the set of equations, 4 are differential equations for the variables \( u', \pi', \Phi' \) and \( d\Phi' \), with respect to the radial coordinate \( \zeta \), and 3 are not:

\[ \frac{dy_1}{d\zeta} = (A_{11} + \delta\sigma A_{12})y_1 + (A_{21} + \delta\sigma A_{22})y_2 \quad (17) \]

\[ 0 = (B_{11} + \delta\sigma B_{12})y_1 + (B_{21} + \delta\sigma B_{22})y_2 \quad (18) \]

where \( \sigma = \sigma_0 + \delta\sigma \), \( y_1 \) and \( y_2 \) are the column vectors – with 4 \( M \) and 3 \( M \) components respectively – containing the unknown coefficients of the spectral decomposition:

\[ y_1 = (\tilde{\pi}'_{1\ell}, ..., \tilde{\pi}'_{1\ell_m}, \tilde{\Phi}'_{1\ell_m}, ..., \tilde{\Phi}'_{1\ell_m}, \tilde{\rho}'_{1\ell_m}, ..., \tilde{\rho}'_{1\ell_m}) \]

\[ y_2 = (\tilde{v}'_{1\ell}, ..., \tilde{v}'_{1\ell_m}, \tilde{w}'_{1\ell}, ..., \tilde{w}'_{1\ell_m}) \]

where

\[ \ell_1 = |m| + p \text{ and } \ell_M = |m| + 2(M - 1) + p \quad (19) \]

Using (18), a matrix inversion allows to express \( y_2 \) as a linear function of \( y_1 \). The system is then reduced to a differential system of 4 independent variables:

\[ \Rightarrow \frac{dy_1}{d\zeta} = (A + \delta\sigma A_{1\ell_0})y_1 \quad (20) \]

3.4 Radial resolution: finite differences

Considering two consecutive layers \( i \) and \( i+1 \), such that \( \zeta(i+1) - \zeta(i) = h \), a Taylor development of any function \( y \) gives Scuflaire et al. [2008]:

\[ y(i) + \frac{h}{2} \frac{dy}{d\zeta}(i) + \frac{h^2}{12} \frac{d^2y}{d\zeta^2}(i) = y(i+1) - \frac{h}{2} \frac{dy}{d\zeta}(i+1) + \frac{h^2}{12} \frac{d^2y}{d\zeta^2}(i+1) + o(h^3) \quad (21) \]

We can develop a 5th order finite differences scheme that only involves two consecutive layers \( \zeta(i) \) and \( \zeta(i+1) \). We then obtain a global eigenvalue system:

\[ A \delta Y = \delta\sigma A \delta\sigma Y \quad (22) \]
Where $AA$ and $AA_{\delta \sigma}$ are block diagonal matrices. The block $i$ couples the layers $i$ and $i+1$ and:

$$Y = \begin{pmatrix} y_1(1) \\ \vdots \\ y_1(N) \end{pmatrix} \quad \text{(layers 1, \ldots, N)} \quad (23)$$

### 3.5 Inverse iteration algorithm

In order to solve Eq. (22) we use a generalization of the inverse iteration method (see [Dupret 2001]). Starting from a first estimate of the eigenvector $Y_0$ and eigenfrequency correction $\delta \sigma_0$ (we take $\delta \sigma_0 = 0$, i.e. $\sigma = \sigma_0$), we compute the next step in the iteration using the formula:

$$Y_{k+1} = AA^{-1} AA_{\delta \sigma} Y_k \quad (24)$$

provided $AA^{-1} AA_{\delta \sigma}$ is diagonalizable. Solving the eigenvalue problem is then equivalent to solving the linear system:

$$AA Y_{k+1} = AA_{\delta \sigma} Y_k \quad (25)$$

To do so, we perform a LU factorization of the matrix $AA = LU$, where $L$ is a lower triangular matrix and $U$ an upper triangular one. Therefore, the LU factorization needs to be made only once, and the only thing to be done afterwards is to solve at each step of the inverse iteration the two triangular systems:

$$L X = AA_{\delta \sigma} Y_k \quad (26)$$

$$U Y_{k+1} = X$$

To avoid ill-conditioning problems, we adopt a special kind of pivoting strategy where the pivoting is done alternatively on the columns and on the lines of the matrix. Thanks to the chosen finite difference resolution (Sect. 3.4), $AA$ and $AA_{\delta \sigma}$ are block diagonal matrices, therefore, $L$ and $U$ are also block diagonal matrices, where the blocks are triangular. The non-zero elements are all located inside the blocks and, during the algorithm, the permutations between lines and the permutations between columns will keep the non-zero elements in the same block. This allows us to keep the narrow band shape for the system, and reduces the memory needed and the computational time. Once the eigenvector $Y$ is computed with sufficiently high precision, the eigenvalue can be calculated using a generalization of the Rayleigh ratio:

$$\delta \sigma = \frac{Y^* AA_{\delta \sigma}^* AA Y}{Y^* AA_{\delta \sigma} AA_{\delta \sigma} Y} \quad (27)$$

where $Y^*$ and $AA_{\delta \sigma}^*$ are the hermitian conjugate of $Y$ and $AA_{\delta \sigma}$. The value of $\delta \sigma$ given by Eq. (27) minimizes:

$$S^2 = || (AA - \delta \sigma AA_{\delta \sigma}) Y ||^2 \quad (28)$$

The solution is then obtained when a certain iteration criterion is reached, which depends on the precision we want on the oscillation frequency.

### 4 Conclusion and perspectives

We presented here a new code that performs 2D non-perturbative calculations of adiabatic oscillations for all kinds of stellar structures – stratified, differentially rotating. We used a numerical method which efficiently saves computational time and memory, and would make the code available for seismic interpretation. It is now under a series of test – comparison with perturbative methods for evolved stellar models, and comparison with non-perturbative computations for polystrophic models. With this new tools, we aim at modeling oscillations of 2D stratified models of stars. A first step will be to take as the equilibrium model a 1D evolved stellar model where the whole effect of rotation on the microphysics have been taken into account, and use a self-consistent method (used in [Roxburgh 2006], for example) in order to compute the distortion due centrifugal acceleration.

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