RIESZ TRANSFORMS ON $Q$-TYPE SPACES WITH APPLICATION TO QUASI-GEOSTROPHIC EQUATION

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Abstract. By an equivalent characterization of Morrey space associated with the fractional heat semigroup, we establish a relation between the generalized $Q$-type spaces and Morrey spaces. By this relation, in this paper, we prove the boundedness of the singular integral operators on the $Q$-type spaces $Q^\beta_\alpha(\mathbb{R}^n)$. As an application, we get the well-posedness and regularity of the quasi-geostrophic equation with initial data in $Q^{\beta-1}_\alpha(\mathbb{R}^n)$.

1. INTRODUCTION

In this paper, we consider the boundedness of a class of singular integral operators on the $Q$-type space $Q^\beta_\alpha(\mathbb{R}^n)$. Here $Q^\beta_\alpha(\mathbb{R}^n)$ is a space defined as the set of all measurable functions with

$$
\sup_I (l(I))^{2\alpha-n+2\beta-2} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha-2\beta+2}} \, dx \, dy < \infty,
$$

where $\alpha \in (0, 1)$, $\beta \in (1/2, 1)$, the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$. This space is introduced in [18] to study the well-posedness of the generalized Naiver-Stokes equations. For $\beta = 1$, $Q^\beta_\alpha(\mathbb{R}^n)$ coincides with the classical space $Q_\alpha(\mathbb{R}^n)$ which is introduced in [13]. Furthermore, if $\alpha = 0$, $\beta = 1$, $Q^\beta_\alpha(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

As a new space between $W^{1,n}(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$, $Q_\alpha(\mathbb{R}^n)$ has been studied extensively by many authors since 1990s. In 1995, on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, R. Aulaskari, J. Xiao and R. Zhao first introduced a class of M"{o}bius invariant analytic function spaces, $Q_p(\mathbb{D})$, $p \in (0, 1)$. The class $Q_p(\mathbb{D})$, $p \in (0, 1)$ can be
Rewritten text: seen as subspaces and subsets of $BMOA$ and $UBC$ on $D$. Since then, many studies on $Q_p(D)$ and their characterization have been done. We refer the readers to [1], [2], [21] and [29] and the reference therein. In order to generalize $Q_p(D)$, $p \in (0, 1)$ to $\mathbb{R}^n$, in [13], M. Essen, S. Janson, L. Peng and J. Xiao introduced a class of Q-type spaces of several real variables, $Q_\alpha(\mathbb{R}^n)$, $\alpha \in (0, 1)$. Later, in [12], G. Dafni and J. Xiao established the Carleson measure characterization of $Q_\alpha(\mathbb{R}^n)$, $\alpha \in (0, 1)$. For more information of the spaces $Q_\alpha(\mathbb{R}^n)$ and their application, we refer to [28], [12] and [13]. For the generalization of $Q_\alpha(\mathbb{R}^n)$, we refer to [18] and [30].

It is easy to see that a function $f(x)$ belongs to $BMO(\mathbb{R}^n)$ if and only if

$$\sup_{l(I)\neq 0}(l(I))^{-2n} \int_I \int_I |f(x) - f(y)|^2 \, dx \, dy < \infty.$$ 

It can be also proved that if $\alpha \in (-\infty, 0)$ and $\beta = 1$, $Q_\alpha^\beta(\mathbb{R}^n) = BMO(\mathbb{R}^n)$. The similarity on the structure of $Q_\alpha^\beta(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$ shows that the two spaces share some common properties. It is well-known that the singular integral operators associated to the fractional heat semigroup $e^{-t(-\Delta)^\alpha}$ and establish a relation between $Q_\alpha^\beta(\mathbb{R}^n)$ and Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$. For $\beta = 1$ and $\alpha \in (0, 1)$, such relation was established by Z. Wu and C. Xie in [27]. In [28], J. Xiao gave another proof which is based on the Carleson measure characterization of $Q_\alpha, \alpha \in (0, 1)$ and Morrey spaces. Hence our result can be seen as a generalization of those in [27] and [28]. By this relation, the boundedness of $T$ on $Q_\alpha^\beta(\mathbb{R}^n)$ can be deduced by that on $L_{p,\lambda}(\mathbb{R}^n)$. See Section 3.

As an application, we consider the well-posedness and regularity of the quasi-geostrophic equations with initial data in $Q_\alpha^{\beta - 1}(\mathbb{R}^n)$. In recent years, Q-type spaces have been applied to the study of the fluid equations by several authors. For example, in [28], J. Xiao introduced a new critical space $Q_\alpha^{-1}(\mathbb{R}^n)$ which is derivatives of $Q_\alpha(\mathbb{R}^n)$, $\alpha \in (0, 1)$ and got the well-posedness of Naiver-Stokes equations with initial data in $Q_\alpha^{-1}(\mathbb{R}^n)$. When $\alpha = 0$, $Q_\alpha^{-1}(\mathbb{R}^n) = BMO^{-1}(\mathbb{R}^n)$, his result generalized the well-posedness obtained by Koch and Tataru in [17]. In [18], inspiring by [28] and the scaling invariance, we introduced a new Q-type space $Q_\alpha^\beta(\mathbb{R}^n)$ with $\alpha > 0$, $\max\{\frac{1}{2}, \alpha\} < \beta < 1$ such that $\alpha + \beta - 1 \geq 0$. We proved the well-posedness and regularity of the generalized Naiver-Stokes equations with some initial data in the space $Q_\alpha^{\beta - 1}(\mathbb{R}^n)$. For $\beta = 1$, our space $Q_\alpha^{\beta - 1}(\mathbb{R}^n)$ becomes $Q_\alpha^{-1}(\mathbb{R}^n)$ in [28]. So our result can be regarded as a generalization of those of [17] and [28].
In Section 4, we consider the two-dimensional subcritical quasi-geostrophic dissipative equations \((DQG)_\beta\) with small initial data in \(Q^{\beta,1}_{\alpha} (\mathbb{R}^n)\),

\[
\begin{cases}
\partial_t \theta + (-\Delta)^\beta u + (u \cdot \nabla) \theta = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}^+, \alpha > 0; \\
u = \nabla^\perp (-\Delta)^{-1/2} \theta; \\
\theta(0, x) = \theta_0 & \text{in } \mathbb{R}^2,
\end{cases}
\]

where \(\beta \in (\frac{1}{2}, 1)\), the scalar \(\theta\) represents the potential temperature, and \(u\) is the fluid velocity.

The equations \((DQG)_\beta\) are important models in the atmosphere and ocean fluid dynamics. It was proposed by P. Constantin and A. Majda, etc. that the equations \((DQG)_\beta\) can be regarded as low dimensional model equations for mathematical study of singularity in smooth solutions of unforced incompressible three dimensional fluid equations. See e.g. [10, 14, 15, 22, 23] and the references therein.

Owing to the importance in mathematical and geophysical fluid dynamics mentioned above, the equations \((DQG)_\beta\) have been intensively studied. Some important progress has been made. We refer the readers to [4, 5, 6, 7, 8, 11, 16, 25, 26] etc. for details.

In [19], F. Marchand and P. G. Lemarié-Rieusset get the well-posedness of the solutions to the equation \((DQG)_1\) with the initial data in \(BMO^{-1}(\mathbb{R}^2)\). However, because the space \(BMO^{-1}(\mathbb{R}^2)\) is invariant under the scaling: \(u_{0, \lambda}(x) = \lambda u_0(\lambda x)\), we see that under the fractional scaling associated to \(0 < \beta < 1\),

\[
\theta_{\lambda}(t, x) = \lambda^{2\beta-1} \theta(\lambda^{2\beta} t, \lambda x) \text{ and } \theta_{0, \lambda}(x) = \lambda^{2\beta-1} \theta_0(\lambda x),
\]

the space \(BMO^{-1}\) is not invariant.

The above observation implies that if we want to generalize the result in [19] to the general case \(\beta < 1\), we should choose a new space \(X^\beta\) which satisfies the following two properties. At first, the space \(X^\beta\) should be invariant under the scaling (1.2).

Secondly, \(BMO^{-1}\) is a “special” case of \(X^\beta\) for \(\beta = 1\).

It is proved in [18] that the space \(Q^{\beta, -1}_{\alpha} (\mathbb{R}^n)\) is exactly such a space. Therefore we could apply the approach in [18] to the equations \((DQG)_\beta\) and get the well-posedness and regularity of the solution to the equations \((DQG)_\beta\) with \(\beta > 1/2\).

It should be pointed out that the scope of \(\beta\) in the equations \((DQG)_\beta\) is depended upon the definition of \(Q^{\beta, -1}_{\alpha} (\mathbb{R}^n)\). In [18], we proved that the parameters \(\{\alpha, \beta\}\) should satisfy the condition: \(\max\{\alpha, \frac{1}{2}\} < \beta < 1\) and \(\alpha < \beta\) with \(\alpha + \beta - 1 \geq 0\). It is easy to see that \(\beta > \frac{1}{2}\).

In [24], the authors proved the global existence of the solutions of the subcritical quasi-geostrophic equations with small size initial data in the Besov norms spaces \(B^{1-2\beta, \infty}_{\infty}\). However our result cannot be deduced by the existence result in [24]. In addition, by the method in [18], we consider the regularity of the solutions to the equations \((DQG)_\beta\).
The organization of this paper is as follows. In Section 2 we state some preliminary knowledge, notation and terminology that will be used throughout this paper. In Section 3 we consider the boundedness of a class of singular integral operators on $Q^{\beta}_{\alpha}(\mathbb{R}^n)$. In Section 4 we give a well-posedness of the equations $(DQG)^{\beta}$ with the initial data in the spaces $Q^{\beta}_{\alpha} - 1(\mathbb{R}^n)$.

2. Preliminaries

In this paper the symbols $\mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ denote the sets of all complex numbers, integers and natural numbers, respectively. For $n \in \mathbb{N}$, $\mathbb{R}^n$ is the $n$–dimensional Euclidean space, with Euclidean norm denoted by $|x|$ and the Lebesgue measure denoted by $dx$. $\mathbb{R}^{n+1}_{+}$ is the upper half-space $\{ (t, x) \in \mathbb{R}^{n+1}_{+} : t > 0, x \in \mathbb{R}^n \}$ with Lebesgue measure denoted by $dtdx$.

A ball in $\mathbb{R}^n$ with center $x$ and radius $r$ will be denoted by $B = B(x, r)$; its Lebesgue measure is denoted by $|B|$. A cube in $\mathbb{R}^n$ will always mean a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. The sidelength of a cube $I$ will be denoted by $l(I)$. Similarly, its volume will be denoted by $|I|$.

The characteristic function of a set $A$ will be denoted by $1_A$. For $\Omega \subset \mathbb{R}^n$, the space $C^\infty_0(\Omega)$ consists of all smooth functions with compact support in $\Omega$. The Schwartz class of rapidly decreasing functions and its dual will be denoted by $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$, respectively. For a function $f \in S(\mathbb{R}^n)$, $\hat{f}$ means the Fourier transform of $f$.

The generalized $Q$–type spaces $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ are introduce as a substitute of the classical $Q^{\alpha}(\mathbb{R}^n)$ under the fractional dilation: $f_\lambda(x) = \lambda^{2\beta - 1} f(\lambda x)$, $0 < \beta < 1$. This space is defined as follows.

**Definition 2.1.** Let $-\infty < \alpha$ and $\max\{\alpha, 1/2\} < \beta < 1$. Then $f \in Q^{\beta}_{\alpha}(\mathbb{R}^n)$ if and only if

$$
sup_I (l(I))^{2\alpha - n + 2\beta - 2} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha - 2\beta + 2}} dxdy < \infty,$$

where the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

For $\beta = 1$ and $\alpha > -\infty$, the above space becomes $Q_{\alpha}(\mathbb{R}^n)$, which was introduced by M. Essen, S. Janson, L. Peng and J. Xiao in [13]. In 2004, G. Dafni and J. Xiao give the Carleson measure characterization of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ using a new type of tent spaces in [12]. Following the same idea, in order to study the $Q_{\alpha}$ initial data problem for the
generalized Naiver-Stokes equations, we consider the Carleson measure characterization of $Q^\beta_\alpha(\mathbb{R}^n)$ in [18]. Precisely, we get the following result.

Let $\phi(x)$ be a $C^\infty$ real-valued function on $\mathbb{R}^n$ satisfying the properties

\begin{equation}
\phi(x) \in L^1(\mathbb{R}^n), \quad |\phi(x)| \lesssim (1+|x|)^{-(n+1)}, \quad \int_{\mathbb{R}^n} \phi(x)dx = 0, \quad \phi_t(x) = t^{-n} \phi\left(\frac{x}{t}\right).
\end{equation}

In [18], we proved that $Q^\beta_\alpha(\mathbb{R}^n)$ has the following Carleson measure characterization.

**Theorem 2.2.** ([18, p. 2462]). Given $\phi$ be a function satisfying the above conditions (2.1). Let $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$. $f \in Q^\beta_\alpha(\mathbb{R}^n)$ if and only if

$$\sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2\alpha-n+2\beta-2} \int_0^r \int_{|y-x| < r} |f \ast \phi_t(y)|^2 t^{-(1+2(\alpha-\beta+1))} dtdy < \infty,$$

that is, $d\mu_{f, \phi, \alpha, \beta}(t, x) = |(f \ast \phi_t(x)|^2 t^{1-2(\alpha-\beta+1)} dtdx$ is a $1 - 2(\alpha + \beta - 1)/n -$ Carleson measure.

The main tool for the Carleson measure characterization of $Q^\beta_\alpha(\mathbb{R}^n)$ is the following fractional tent spaces.

**Definition 2.3.** For $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, we define $T^\infty_{\alpha, \beta}$ be the class of all Lebesgue measurable functions $f$ on $\mathbb{R}^{n+1}$ with

$$\|f\|_{T^\infty_{\alpha, \beta}} = \sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}} \int_{T(B)} |f(t, y)|^2 \frac{dtdy}{t^{1+2(\alpha-\beta+1)}} \right)^{1/2} < \infty.$$

In order to define the dual of $T^\infty_{\alpha, \beta}$, we need the following $T^1_{\alpha, \beta}$-atoms.

**Definition 2.4.** For $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, a function $a$ on $\mathbb{R}^{n+1}$ is said to be a $T^1_{\alpha, \beta}$-atom provided there exists a ball $B \subset \mathbb{R}^n$ such that $a$ is supported in the tent $T(B)$ and satisfies

$$\int_{T(B)} |a(t, y)|^2 \frac{dtdy}{t^{1-2(\alpha-\beta+1)}} \leq \frac{1}{|B|^{1-2(\alpha+\beta-1)/n}}.$$

We denote by $dA_{n-2(\alpha+\beta-1)}^\infty$ the $n-2(\alpha+\beta-1)$ dimensional Hausdorff capacity of a set $E$ and refer to [12] for the details of the Hausdorff capacity. For $x \in \mathbb{R}^n$, let $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+: |x - y| < t\}$ be the cone at $x$. Define the non-tangential maximal function $N(f)$ of a measurable function $f$ on $\mathbb{R}^{n+1}$ by

$$N(f)(x) := \sup_{(y, t) \in \Gamma(x)} |f(y, t)|.$$

The dual of $T^\infty_{\alpha, \beta}$ is defined as follows.
Definition 2.5. For $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, the space $T_{\alpha, \beta}^1$ consists of all measurable functions $f$ on $\mathbb{R}_+^{n+1}$ with

$$
\|f\|_{T_{\alpha, \beta}^1} = \inf_{\omega} \left( \int_{\mathbb{R}_+^{n+1}} |f(x, t)|^2 \omega^{-1}(x, t) \frac{dtx}{t^{1-2(\alpha-\beta+1)}} \right)^{1/2} < \infty,
$$

where the infimum is taken over all nonnegative Borel measurable functions $\omega$ on $\mathbb{R}_+^{n+1}$ with $\int_{\mathbb{R}^n} N\omega d\Lambda_{n-2(\alpha+\beta-1)} \leq 1$ and with the restriction that $\omega$ is allowed to vanish only where $f$ vanishes.

The above tent spaces and their dualities can be seen as the generalization of the usual one. For $\beta = 1$, $T_{\alpha, \beta}^\infty$ and $T_{\alpha, 1}^1$ coincide with $T_{\alpha}^\infty$ and $T_{\alpha}^1$, respectively which are introduced in [12]. For $\alpha = 0$ and $\beta = 1$, $T_{0, \beta}^\infty$ becomes the classical tent space $T_{\beta}^\infty$ in [9].

Let $\phi$ satisfy the conditions (2.1). For a function $F$ on $\mathbb{R}_+^{n+1}$, denote by $\Pi_{\phi}$ the operator

$$
(2.2) \quad \Pi_{\phi}(F) = \int_0^\infty F(\cdot, t) * \phi_t \frac{dt}{t}.
$$

In [18], we proved that $\Pi_{\phi}$ is a bounded and surjective operator from $T_{\alpha, \beta}^\infty$ to $Q_{\alpha, \beta}^\infty$.

Theorem 2.6. ([18, Theorem 3.20]). Consider the operator $\Pi_{\phi}$ defined by (2.2). The operator $\Pi_{\phi}$ is a bounded and surjective operator from $T_{\alpha, \beta}^\infty$ to $Q_{\alpha, \beta}^\infty(\mathbb{R}^n)$. More precisely, if $F \in T_{\alpha, \beta}^\infty$ then the righthand side of the above integral converges to a function $f \in Q_{\alpha, \beta}^\infty(\mathbb{R}^n)$, $\|f\|_{Q_{\alpha, \beta}^\infty} \lesssim \|F\|_{T_{\alpha, \beta}^\infty}$, and any $f \in Q_{\alpha, \beta}^\infty(\mathbb{R}^n)$ can be thus represented.

3. BOUNDEDNESS OF THE SINGULAR INTEGRAL OPERATORSON Q-TYPE SPACES $Q_{\alpha}^\beta$

In this section, we will prove a class of singular integral operators are bounded on Q-type spaces $Q_{\alpha}^\beta(\mathbb{R}^n)$. Our method is based on the characterizations of $Q_{\alpha}^\beta(\mathbb{R}^n)$ and the Morrey space $L_{2, \lambda}$ associated to the fractional heat semigroup $e^{-t(-\Delta)^{\beta}}$. Before we state the main results in this section, we give a relation between $Q_{\alpha}^\beta(\mathbb{R}^n)$, a class of conformally invariant Sobolev spaces and the fractional BMO type space $BMO^\beta(\mathbb{R}^n)$.

Definition 3.1. Let $\beta \in (1/2, 1)$. Then $f \in BMO^\beta(\mathbb{R}^n)$ if and only if

$$
\sup_I \left( (l(I))^{4\beta-4-2n} \int_I \int_I |f(x) - f(y)|^2 dxdy \right)^{1/2} < \infty,
$$

where the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$. 
In [28], J.Xiao proved that $Q_α(\mathbb{R}^n)$ is a space between the Sobolev space $W^{1,n}(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. In this section we prove that a similar relation holds for $Q_α^2(\mathbb{R}^n)$ and $BMO^β(\mathbb{R}^n)$. For this purpose, we introduce a conformally invariant Sobolev space $CIS_β(\mathbb{R}^n)$.

**Definition 3.2.** Let $β \in (1/2,1)$ and $f \in C^1(\mathbb{R}^n)$. $f \in CIS_β(\mathbb{R}^n)$ if

$$\|f\|_{CIS_β} = \sup_I \left( |I|^{\frac{4β-2-n}{n}} \int_I |\nabla f(x)|^2 dx \right)^{1/2} < \infty,$$

where the supremum is taken over all cubes $I$ with the edge length $l(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

**Theorem 3.3.** Let $n \geq 2$ and $\max\{α,1/2\} < β < 1$ with $α + β - 1 \geq 0$. If

$$E_β(\mathbb{R}^n) = \left\{ f \in C^1(\mathbb{R}^n) : \|f\|_{E_β} = \left( \int_{\mathbb{R}^n} |\nabla f(x)|^{\frac{2n}{n-1}} dx \right)^{\frac{n}{2}} \right\},$$

then

$$E_β(\mathbb{R}^n) \subseteq CIS_β(\mathbb{R}^n) \subseteq Q_α^2(\mathbb{R}^n) \subseteq BMO^β(\mathbb{R}^n).$$

**Proof.** If $n \geq 2$, by Hölder’s inequality, we have for any cube $I \subset \mathbb{R}^n$,

$$\int_I |\nabla f(x)|^2 dx \leq \left( \int_I |\nabla f(x)|^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} |I|^{\frac{4β-2-n}{n}}.$$

This implies $E_β(\mathbb{R}^n) \subseteq CIS_β(\mathbb{R}^n)$.

Now we prove $CIS_β(\mathbb{R}^n) \subseteq Q_α^2(\mathbb{R}^n)$. For a cube $I \subset \mathbb{R}^n$, denote by $cI$ the cube with volume being $c^n |I|$ and the center of $I$. For $f \in CIS_β(\mathbb{R}^n)$, we have

$$|f(z + y) - f(y)| \leq \int_0^1 |\nabla f(y + tz)| |z| dt.$$

Hence we can get

$$\left( \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{n+2α-2β+2}} dxdy \right)^{1/2}$$

$$= \left( \int_I \int_I \left( \frac{|f(x) - f(y)|}{|x - y|} \right)^2 \frac{1}{|x - y|^{n+2α-2β}} dxdy \right)^{1/2}$$

$$\leq \left( \int_I \int_{|x - y| < \sqrt{n} |I|^{1/n}} \frac{|f(x) - f(y)|^2}{|x - y|} |x - y|^{2β-n-2α} dxdy \right)^{1/2}$$

$$\leq \left( \int_I \int_{|z| < \sqrt{n} |I|^{1/n}} \frac{|f(z + y) - f(y)|^2}{|z|} |z|^{2β-n-2α} dzdy \right)^{1/2}$$

Riesz Transform on $Q$-type Space
We see that if \( H \) and \( e \) are such that

\[
\begin{align*}
\text{Theorem 3.3.} & \quad \text{we know that} \\
\text{By Definition 2.1,} & \quad \text{we have} \\
\text{Relation between} & \quad L_{\beta,n}(\mathbb{R}^n) \text{ is a special case of} \quad Q_{\alpha}(\mathbb{R}^n). \\
\end{align*}
\]

Because

\[
\int_{|z|<\sqrt{m}I} |z|^{2\beta-2\alpha-n} \, dz \leq \int_{|z|<\sqrt{m}I} |z|^{2\beta-2\alpha-1} \, dz \leq C |I|^{\frac{2\beta-2\alpha}{n}},
\]

we have

\[
\left( \int_I \int_I |x-y|^{n+2\alpha-2\beta-2} \, dx \, dy \right)^{1/2} \leq C \int_0^1 \left[ \int_{(1+\sqrt{m})I} |\nabla f(\omega)|^2 |I|^{\frac{2\alpha-2\beta}{n}} \, d\omega \right]^{1/2} \, dt \\
= C |I|^{\frac{\beta-\alpha}{n}} \left( \int_{(1+\sqrt{m})I} |\nabla f(\omega)|^2 \, d\omega \right)^{1/2}.
\]

Hence we get

\[
\left( |I|^{\frac{2\alpha-n+2\beta-2}{n}} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha-2\beta+2}} \, dx \, dy \right)^{1/2} \\
\leq |I|^{\frac{2\alpha-n+2\beta-2}{2n}} |I|^{\frac{\beta-\alpha}{2n}} \left( \int_{(1+\sqrt{m})I} |\nabla f(\omega)|^2 \, d\omega \right)^{1/2} \\
\leq |I|^{\frac{4\beta-n-2}{2n}} \left( \int_{(1+\sqrt{m})I} |\nabla f(\omega)|^2 \, d\omega \right)^{1/2}.
\]

By Definition 2.1, we know that \( CIS_{\beta}(\mathbb{R}^n) \subseteq Q_{\alpha}(\mathbb{R}^n) \). This completes the proof of Theorem 3.3. \( \blacksquare \)

Recall that Morrey space \( L_{p,\lambda}(\mathbb{R}^n) \) is defined as follows.

\[
\| f \|_{L_{p,\lambda}} = \sup_I \left( \frac{1}{(I(I))^{-\lambda}} \int_I |f(x) - f_I|^p \, dx \right)^{1/p} < \infty.
\]

We see that if \( \lambda = n \), \( L_{p,\lambda}(\mathbb{R}^n) = BMO(\mathbb{R}^n) \) by John-Nirenberg inequality. Owing to \( BMO(\mathbb{R}^n) \) is a special case of \( Q_{\alpha}(\mathbb{R}^n) \), it is natural to ask if there exists a general relation between \( L_{p,\lambda}(\mathbb{R}^n) \) and \( Q_{\alpha}(\mathbb{R}^n) \). In [28], by a characterization of \( L_{p,\lambda}(\mathbb{R}^n) \)
associated to the semigroup $e^{-t(-\Delta)}$, J. Xiao established such a relation. Precisely he proved that for $\alpha \in (0, 1)$, $Q_\alpha(R^n) = (-\Delta)^{-\frac{\alpha}{2}}L_{2,n-2\alpha}(R^n)$.

Following Xiao’s idea in [28], we will prove that a similar result holds for the space $Q_\alpha(R^n)$. At first we prove an equivalent characterization of $L_{2,n-2\alpha}(R^n)$ via the semigroup $e^{-t(-\Delta)^\alpha}$. Here $e^{-t(-\Delta)^\alpha}$ denotes the convolution operator defined by Fourier transform:

$$
e^{-t(-\Delta)^\alpha}f(\xi) = e^{-t|\xi|^{2\alpha}}\widehat{f}(\xi).$$

**Lemma 3.4.** Given $\gamma \in (0, 1)$. Let $f$ be a measurable complex-valued function on $\mathbb{R}^n$. Then $f \in L_{2,n-\gamma}(\mathbb{R}^n)$ if and only if

$$\sup_{x \in \mathbb{R}^n, r \in (0, \infty)} r^{2\gamma-n} \int_0^r \int_{|y-x|<r} \left| \nabla e^{-t(-\Delta)^\alpha}f(y) \right|^2 \, t \, dy \, dt < \infty.$$

**Proof.** Take $(\psi_0)_t(x) = t\nabla e^{-t^2(-\Delta)^\alpha}(x, 0)$ with the Fourier symbol $(\widehat{\psi_0})_t(x)(\xi) = t|\xi|^{2\alpha}e^{-t|\xi|^{2\alpha}}$. For a ball $B = \{y \in \mathbb{R}^n : |y-x|<r\}$, the mean of $f$ on $2B$ is defined by $f_{2B} = \frac{1}{|2B|} \int_{2B} f(x) \, dx$. We split $f$ into $f = f_1 + f_2 + f_3$, where $f_1 = (f - f_{2B})1_{2B}$, $f_2 = (f - f_{2B})1_{(2B)^C}$ and $f_3 = f_{2B}$. Because

$$\int (\psi_0)_t(x) \, dx = \int t\nabla e^{-t^2(-\Delta)^\alpha}(x, 0) \, dx = 0,$$

we have

$$t\nabla e^{-t^2(-\Delta)^\alpha}f(y) = (\psi_0)_t * f(y) = (\psi_0)_t * f_1(y) + (\psi_0)_t * f_2(y).$$

It is easy to see that

$$\int_0^r \int_B |(\psi_0)_t * f_1(y)|^2 \frac{dy \, dt}{t} \lesssim \int_0^r \int_{\mathbb{R}^n} |(\psi_0)_t * f_1(y)|^2 \frac{dy \, dt}{t} = \left\| \left( \int_0^\infty |(\psi_0)_t * f_1(y)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2(dy)}.$$

Because $(\psi_0)_t = \nabla e^{-t(-\Delta)^\alpha}$, we have $\int (\psi_0)_1(x) \, dx = 1$ and $(\psi_0)_1$ belongs to the Schwartz class $S$. Also the function

$$G(f) = \left( \int_0^\infty |(\psi_0)_t * f_1(y)|^2 \frac{dt}{t} \right)^{1/2}$$

is a Littlewood-Paley $g$-function. So we can get

$$\int_0^r \int_B |(\psi_0)_t * f_1(y)|^2 \frac{dy \, dt}{t} \lesssim \int_{2B} |f(y) - f_{2B}|^2 \, dy \lesssim r^{-n-2\gamma} \|f\|^2_{L_{2,n-2\gamma}}.$$
Now we estimate the term associated with $f_2(y)$. Because

$$| (\psi_0)_t * f_2(y) | \leq \int_{\mathbb{R}^n} t | \nabla e^{-t\Delta} (y-z) f_2(z) | dz$$

$$\leq \int_{\mathbb{R}^n \setminus 2B} t | \nabla e^{-t\Delta} (y-z) | f(z) - f_{2B} | dz$$

$$\lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{t | f(z) - f_{2B} |}{(1 + t^{-\frac{1}{2\beta}} | z - y |)^{n+1}} dz,$$

where in the last inequality we have used the following estimate:

$$| \nabla e^{-t\Delta} (x,y) | \lesssim \frac{1}{t^{\frac{1}{2\beta}}} \frac{1}{(1 + t^{-\frac{1}{2\beta}} | x - y |)^{n+1}}.$$

Set $B_k = B(x,2^k)$. For every $(t,y) \in (0,r) \times B(x,r)$, we have $0 < t < r$ and $|x-y| < r$. If $z \in B_{k+1} \setminus B_k$, we have $|x-y| < |x-z|/2$ and

$$| (\psi_0)_t * f_2(y) | \lesssim \int_{\mathbb{R}^n \setminus 2B} \frac{t | f(z) - f_{2B} |}{(t + |x-z|)^{n+1}} dz$$

$$\lesssim t \sum_{k=1}^{\infty} \frac{(2^k+1)^n}{(2^k r)^{n+1}} \left( \frac{1}{(2^{k+1}r)^n} \int_{2^{k+1}B} | f(z) - f_{2B} |^2 dz \right)^{1/2}$$

$$\lesssim t \sum_{k=1}^{\infty} \frac{1}{2^k r} \left( \frac{1}{(2^{k+1}r)^n} \int_{2^{k+1}B} | f(z) - f_{2^{k+1}B} |^2 dz \right)^{1/2}$$

$$+ \sum_{k=1}^{\infty} \frac{1}{2^k r} | f_{2^{k+1}B} - f_{2B} |$$

$$= t(S_1 + S_2).$$

For $S_1$, we have

$$S_1 = t \sum_{k=1}^{\infty} \frac{1}{2^k r} \left( \frac{(2^{k+1}r)^{n-2\gamma}}{(2^k r)^n} \frac{1}{(2^{k+1}r)^n} \int_{2^{k+1}B} | f(z) - f_{2^{k+1}B} |^2 dz \right)^{1/2}$$

$$\lesssim t \sum_{k=1}^{\infty} \frac{1}{2^k r} r^{-\gamma} \| f \|_{L^{2,n-2\gamma}}$$

$$\lesssim tr^{-1-\gamma} \| f \|_{L^{2,n-2\gamma}}.$$

For $S_2$, we have

$$S_2 \lesssim t \sum_{k=1}^{\infty} \frac{1}{2^k r} \left[ | f_{2B} - f_{4B} | + \cdots + | f_{2^kB} - f_{2^{k+1}B} | \right].$$
For any $j$ with $2 \leq j \leq k$, it is easy to see that

$$|f_{2jB} - f_{2j+1B}| \lesssim \frac{1}{|2jB|} \int_{2jB} |f(z) - f_{2j+1B}|dz$$

$$\lesssim \left( \frac{1}{|2jB|} \int_{2jB} |f(z) - f_{2j+1B}|^2dz \right)^{1/2}$$

$$\lesssim r^{-\gamma} \|f\|_{L^2_{2,n-2\gamma}}.$$

Then we have

$$S_2 \lesssim \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot r^{-\gamma} \|f\|_{L^2_{2,n-2\gamma}} \lesssim tr^{-1-\gamma} \|f\|_{L^2_{2,n-2\gamma}}.$$

Therefore, we can get

$$\int_0^r \int_B |(\psi_0)_{t + f_2(y)}|^2 {}^t t^{-1}dtdydt \lesssim \int_0^r \int_B t^2 r^{-2\gamma-2} \|f\|_{L^2_{2,n-2\gamma}}^2 dydt$$

$$\lesssim \|f\|_{L^2_{2,n-2\gamma}}^2 r^{-2\gamma-2} |B| \int_0^r tdt$$

$$\lesssim r^{\gamma-2} \|f\|_{L^2_{2,n-2\gamma}}^2.$$

For the converse, let $S(I) = \{(t, x) \in \mathbb{R}^{n+1}_+, 0 < t < l(I), x \in I\}$ if $f$ such that

$$\sup_I [l(I)]^{2\gamma-n} \int_{S(I)} \left| t \nabla e^{-t^{2\beta}(-\Delta)^{\beta}} f(y) \right|^2 \frac{dydt}{t}$$

$$= \sup_I [l(I)]^{2\gamma-n} \int_{S(I)} \left| \nabla e^{-t^{2\beta}(-\Delta)^{\beta}} f(y) \right|^2 tdydt < \infty.$$

Denote

$$\Pi_{\psi_0} F(x) = \int_{\mathbb{R}^{n+1}_+} F(t, y)(\psi_0)t(x - y) \frac{dydt}{t}.$$

We will prove that if

$$\|F\|_{C_\gamma} = \sup_I \left( [l(I)]^{2\gamma-n} \int_{S(I)} |F(t, y)|^2 \frac{dydt}{t} \right)^{1/2} < \infty,$$

then for any cube $J \subset \mathbb{R}^n$,

$$\int_J |\Pi_{\psi_0} F(x) - (\Pi_{\psi_0} F)_J|^2 dx \lesssim [l(J)]^{n-2\gamma} \|F\|_{L^2_{2,n-2\gamma}}^2.$$

For this purpose, we split $F$ into $F = F_1 + F_2 = F|_{S(2J)} + F|_{\mathbb{R}^{n+1}\setminus S(2J)}$ and get
\[ \int_J |\Pi_{\psi_0} F_1(x)|^2 \, dx \leq \int_J |\Pi_{\psi_0} F_1(x)|^2 \, dx \]
\[ \leq \int_{S(2J)} |F(t, y)|^2 \frac{dy dt}{t} \]
\[ \lesssim [l(J)]^{n-2\gamma} \|F\|_{l_2}^2. \]

Now we estimate the term associated with \( F_2 \). We have
\[ \int_J |\Pi_{\psi_0} F_1(x)|^2 \, dx = \int_J \left( \int_{R^{n+1}} (\psi_0)_y(x-y) F_2(t, y) t^{-1} \, dy dt \right)^2 \, dx \]
\[ \lesssim \int_J \left( \int_{R^{n+1} \setminus S(2J)} |(\psi_0)_y(x-y)||F_2(t, y)| \frac{dy dt}{t} \right)^2 \, dx \]
\[ = \int_J \left( \sum_{k=1}^{\infty} \int_{S(2^{k+1}J) \setminus S(2^kJ)} |(\psi_0)_y(x-y)||F_2(t, y)| \frac{dy dt}{t} \right)^2 \, dx. \]

Because \((\psi_0)_t\) satisfies the estimate
\[ |(\psi_0)_t(x-y)| \lesssim \frac{t}{t^{n+1} (1 + t^{-1}|x-y|)^{n+1}}, \]
we have
\[ \int_J |\Pi_{\psi_0} F_2(x)|^2 \, dx \lesssim \int_J \left( \sum_{k=1}^{\infty} \int_{S(2^{k+1}J) \setminus S(2^kJ)} \frac{t}{[l + 2^k l(J)]^{n+1}} |F_2(t, y)| \frac{dy dt}{t} \right)^2 \, dx \]
\[ \lesssim \int_J \left( \sum_{k=1}^{\infty} 2^{k(l(J))} t^{-(n+1)} \int_{S(2^{k+1}J) \setminus S(2^kJ)} |F_2(t, y)| \, dy dt \right)^2 \, dx \]
\[ \lesssim \|F\|_{l_2}^2 [l(J)]^{n-2\gamma}. \]

Therefore, we get
\[ \int_J |\Pi_{\psi_0} F(x) - (\Pi_{\psi_0} F)_J|^2 \, dx \leq \int_J |\Pi_{\psi_0} F(x)|^2 \, dx \]
\[ \lesssim \int_J |\Pi_{\psi_0} F_1(x)|^2 \, dx + \int_J |\Pi_{\psi_0} F_2(x)|^2 \, dx \]
\[ \lesssim \|F\|_{l_2}^2 [l(J)]^{n-2\gamma}. \]

Because
\[ \Pi_{\psi_0} F(x) = \int (\psi_0)_z * (\psi_0)_t * f \frac{dt}{t}, \]
by Calderón’s reproducing formula, we have \( \Pi_{\psi_0} F(x) = f(x) \), that is, \( f(x) = \Pi_{\psi_0} F(x) \in \mathcal{L}_{2-n-2\gamma} \). This completes the proof of Lemma 3.4. \( \blacksquare \)
Theorem 3.5. For $\alpha > 0$, $\max\{\alpha, 2\beta\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$, we have

$$Q^\beta_\alpha(\mathbb{R}^n) = (-\Delta)^{-\frac{\alpha-\beta+1}{2}} L_{2, n-2(\alpha+\beta-1)}(\mathbb{R}^n).$$

Proof. For $f \in L_{2, n-2(\alpha+\beta-1)}$, let $F(t, y) = t^{\alpha-\beta+1} t \nabla e^{-t^{2\beta}(-\Delta)^{\beta}} f(y)$. By Lemma 3.4, we have

$$r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |F(t, y)|^2 \frac{dydt}{t^{1+2(\alpha-\beta+1)}} \lesssim r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |t^{\alpha-\beta+1} t \nabla e^{-t^{2\beta}(-\Delta)^{\beta}} f(y)|^2 \frac{dydt}{t^{1+2(\alpha-\beta+1)}} \lesssim r^{2(\alpha+\beta-1)-n} \int_0^r \int_{|y-x|<r} |t \nabla e^{-t^{2\beta}(-\Delta)^{\beta}} f(y)|^2 \frac{dydt}{t} \lesssim \|f\|_{L_{2, n-2(\alpha+\beta-1)}}.$$ 

This implies $F \in T_{\alpha, \beta}^\infty$. By Theorem 2.6, $\Pi_{\psi_0}$ is bounded from $T_{\alpha, \beta}^\infty$ to $Q^\beta_\alpha(\mathbb{R}^n)$. Therefore we have

$$\|f\|_{Q^\beta_\alpha} = \|\Pi_{\psi_0} F\|_{Q^\beta_\alpha} \lesssim \|F\|_{T_{\alpha, \beta}^\infty}.$$ 

Because $\widehat{F}(t, \xi) = t^{\alpha-\beta+2} |\xi| e^{-t^{2\beta} |\xi|^2 \widehat{f}(\xi)}$, we have

$$\Pi_{\psi_0} \widehat{F}(\xi) = \int_0^\infty \widehat{F}(t, \xi) (\psi_0)(t) \frac{dt}{t} = \int_0^\infty t^{\alpha-\beta+2} |\xi| e^{-t^{2\beta} |\xi|^2 \widehat{f}(\xi)} \frac{dt}{t} = |\xi|^2 \widehat{f}(\xi) \int_0^\infty t^{\alpha-\beta+2} e^{-t^{2\beta} |\xi|^2} dt.$$ 

Set $t^{2\beta} = s$ and $|\xi|^{2\beta} s = u$. We can get

$$\Pi_{\psi_0} \widehat{F}(\xi) = \int_0^\infty s^{-\frac{\alpha+1}{2\beta}} e^{-2s |\xi|^{2\beta}} s^{\frac{1}{2\beta} - 1} ds \widehat{f}(\xi) |\xi|^2 = \widehat{f}(\xi) |\xi|^2 \int_0^\infty (u |\xi|^{-2\beta})^{-\frac{\alpha-\beta+3}{2\beta} - 1} e^{-u |\xi|^{-2\beta}} du = \widehat{f}(\xi) |\xi|^2 |\xi|^{-\frac{\alpha+1}{\beta} + 2\beta - 2\beta} \int_0^\infty u^{-\frac{\alpha+1}{2\beta} - 1} e^{-2u} du.$$ 

Because $\frac{1}{2} < \beta < 1$ and $0 < \alpha < \beta$, the integral $\int_0^\infty u^{-\frac{\alpha+1}{2\beta} - 1} e^{-2u} du < \infty$. We denote it by $C_{\alpha, \beta}$ and get

$$\Pi_{\psi_0} \widehat{F}(\xi) = C_{\alpha, \beta} \widehat{f}(\xi) |\xi|^{-\frac{\alpha+1}{\beta} + 2\beta - 2\beta}.$$ 

By the inverse Fourier transform, we have
\[
\Pi_{\psi_0} F(x) = C_{\alpha,\beta}(-\Delta)^{-\frac{\alpha-\beta+1}{2}} f(x).
\]
Conversely, suppose \( g \in Q_\beta^\alpha(\mathbb{R}^n) \). Set \( G(t, y) = t^{1-(\alpha-\beta+1)} \nabla e^{-t^{2\beta}(-\Delta)^\beta} g(y) \). We have, by the equivalent characterization of \( Q_\beta^\alpha(\mathbb{R}^n) \) (see [18] for details),
\[
\left( [l(I)]^{2(\alpha+\beta-1)} \int_{S(I)} \left| t^{1-2(\alpha-\beta+1)} \nabla e^{-t^{2\beta}(-\Delta)^\beta} g(y) \right|^2 dy dt \right)^{1/2} \leq \int_{\mathbb{R}^n} |t^{1+2(\alpha-\beta+1)}| dy dt.
\]
Hence we get
\[
\hat{f}(\xi) = \Pi_{\psi_0} \hat{G}(t, \xi) = \int_0^\infty \left| t^{1-2(\alpha-\beta+1)} \nabla e^{-t^{2\beta}(-\Delta)^\beta} g(y) \right|^2 dy dt = C_{\alpha,\beta} |\xi|^{1+(\alpha-\beta)} \hat{g}(\xi) = C_{\alpha,\beta}((-\Delta)^{-\frac{\alpha-\beta+1}{2}} g)(\xi).
\]
Then \( f(x) = C_{\alpha,\beta}(-\Delta)^{-\frac{\alpha-\beta+1}{2}} g \). This completes the proof of this theorem.

Based on the above theorem, we can deduce the boundedness of the convolution singular integral operators on \( Q_\beta^\alpha(\mathbb{R}^n) \) directly and state this result as the following theorem.

**Theorem 3.6.** Let \( T \) be a singular operator defined by
\[
Tf(x) = \int_{\mathbb{R}^n} K(x - y) f(y) dy,
\]
where the kernel \( K(x) \) satisfies
\[
|\partial_\xi^\gamma K(x)| \leq A_\gamma |x|^{-n-\gamma}, \quad (\gamma > 0).
\]
Or equivalently, let \( \hat{T}\hat{f}(\xi) = m(\xi) \hat{f}(\xi) \), where the symbol \( m(\xi) \) satisfies
\[
|\partial_\xi^\gamma m(\xi)| \leq A_\gamma |\xi|^{-\gamma}
\]
for all \( \gamma \). Suppose \( \alpha > 0 \), \( \max\{\alpha, \frac{1}{2}\} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \). We have \( T \) is bounded on the \( Q \)-type spaces \( Q_\beta^\alpha(\mathbb{R}^n) \).
Proof. It is well-known that the singular integral operator \( T \) is bounded on the Morrey space \( L_{2, n-2(\alpha+\beta-1)}(\mathbb{R}^n) \). Moreover as a convolution operator, \( T \) can commutate with the fractional Laplace operator \((−Δ)^{-\frac{(\alpha−\beta+1)}{2}}\). By Theorem 3.5, we complete the proof of this theorem.

Specially, taking \( T = R_j, j = 1, 2, \cdots, n \) as the Riesz transforms, we have the following corollary.

Corollary 3.7. Suppose \( \alpha > 0, \max\alpha, \frac{1}{2} < \beta < 1 \) with \( \alpha + \beta - 1 \geq 0 \). For \( j = 1, 2, \cdots, n \), the Riesz transforms \( R_j = \partial_j(-Δ)^{-1/2} \) are bounded on the \( Q-\)type spaces \( Q^β_α(\mathbb{R}^n) \).

Remark 3.8. There exists another method to prove Theorem 3.6. In fact we can get the boundedness of \( T \) on \( Q^β_α(\mathbb{R}^n) \) directly by its characterization associated to \( e^{-t(-Δ)^β} \). In Section 4, this method can be applied to study the well-posedness of the equations \((DQG)_β\) with the initial data in \( Q^β_α(\mathbb{R}^n) \). See Lemma 4.5.

4. WELL-POSEDNESS AND REGULARITY OF QUASI-GEOSTROPHIC EQUATION

In this section, we study the well-posedness and regularity of quasi-geostrophic equation with initial data in the space \( Q^β_α(\mathbb{R}^2) \). We introduce the definition of \( X^β_α(\mathbb{R}^n) \).

Definition 4.1. The space \( X^β_α(\mathbb{R}^n) \) consists of the functions which are locally integrable on \((0, \infty) \times \mathbb{R}^2\) such that \( \sup_{t>0} t^{-\frac{1}{\beta}} \|f(t, \cdot)\|_{\mathbb{B}^0_{\alpha, 1}} < \infty \) and

\[
\sup_{x \in \mathbb{R}^2, r > 0} r^{2-\alpha-2n+2\beta-2} \int_0^r \int_{|y-x_0| < r} |f(t, y)|^2 + |R_1 f(t, y)|^2 + |R_2 f(t, y)|^2 \frac{dydt}{t^{\alpha/\beta}} < \infty,
\]

where \( R_j, j = 1, 2 \) denote the Riesz transforms in \( \mathbb{R}^2 \).

For the quasi-geostrophic dissipative equations

\[
\begin{align*}
\partial_t \theta &= -(-Δ)^β + \partial_1(\theta R_2 \theta) - \partial_2(\theta R_1 \theta), \\
\theta(0, x) &= \theta_0(x),
\end{align*}
\]

where \( \beta \in (\frac{1}{2}, 1) \). The solution to equations (4.1) can be represented as

\[
u(t, x) = e^{-t(-Δ)^β} u_0 + B(u, u),
\]

where the bilinear form \( B(u, v) \) is defined by

\[
B(u, v) = \int_0^t e^{-(t-s)(-Δ)^β} (\partial_1(v R_2 u) - \partial_2(v R_1 u)) ds.
\]

In order to prove the well-posedness, we need the following preliminary lemmas. For their proofs, we refer the readers to Lemma 4.8 and Lemma 4.9 in [18].
Lemma 4.2. ([18, Lemma 4.8]). Given $\alpha \in (0, 1)$. For a fixed $T \in (0, \infty]$ and a function $f(t, x)$ on $\mathbb{R}^{1+n}$, let $A(t) = \int_0^t e^{-(t-s)(-\triangle)\beta}(-\triangle)^{\beta}f(s, x)ds$. Then

$$
\int_0^T \|A(t, \cdot)\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} < \int_0^T \|f(t, \cdot)\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}}.
$$

Lemma 4.3. ([18, Lemma 4.9]). For $\beta \in (1/2, 1)$ and $N(t, x)$ defined on $(0, 1) \times \mathbb{R}^n$, let $A(N)$ be the quantity

$$
A(\alpha, \beta, N) = \sup_{x \in \mathbb{R}^n, r \in (0, 1)} r^{2\alpha-n+2\beta-2} \int_0^{2\beta} \int_{|y-x|<r} |f(t, x)| \frac{dxdt}{t^{\alpha/\beta}}.
$$

Then for each $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ there exists a constant $b(k)$ such that the following inequality holds:

$$
\int_0^1 \left\| (-\triangle)^{\frac{k+1}{2}} e^{-t(-\triangle)^{\beta}} \int_0^t N(s, \cdot)ds \right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \leq b(k) A(\alpha, \beta, N) \int_0^1 \int_{\mathbb{R}^n} |N(s, x)| \frac{dxds}{s^{\alpha/\beta}}.
$$

Remark 4.4. Similarly when $k = 0$, we can prove the following inequality:

$$
\int_0^1 \left\| (-\triangle)^{\frac{1}{2}} e^{-t(-\triangle)^{\beta}} \int_0^t N(s, \cdot)ds \right\|_{L^2}^2 \frac{dt}{t^{\alpha/\beta}} \leq A(\alpha, \beta, N) \int_0^1 \int_{\mathbb{R}^n} |N(s, x)| \frac{dxds}{s^{\alpha/\beta}}.
$$

Lemma 4.5. Assume $\alpha > 0$ and $\max\{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$. Let $R_j, j = 1, 2$ be the Riesz transforms. Then for any $x_0 \in \mathbb{R}^n$,

$$
\left( \sup_{r>0} r^{2\alpha-n+2\beta-2} \int_0^{2\beta} \int_{|y-x_0|<r} |R_j f(t, y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)^{1/2} \leq \left( \sup_{x \in \mathbb{R}^n, r>0} r^{2\alpha-n+2\beta-2} \int_0^{2\beta} \int_{|y-x_0|<r} |f(t, y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)^{1/2}.
$$

Proof. We split $f(t, y)$ into

$$
f(t, y) = f_0(t, y) + \sum_{k=1}^{\infty} f_k(t, y),
$$

where $f_0(t, y) = f(t, y) \chi_{B(x_0, 2r)}(y)$ and $f_k(t, y) \chi_{B(x_0, 2^{k+1}r) \setminus B(x_0, 2^kr)}(y)$. We have
Riesz Transform on $Q$-type Space

$$\left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |R_j f(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)$$

$$\leq \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |R_j f_0(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)$$

$$+ \sum_{k=1}^{\infty} \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |R_j f_k(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)$$

$$=: M_0 + \sum_{k=1}^{\infty} M_k.$$

By the $L^2$ boundedness of Riesz transforms $R_j$, $j = 1, 2$, we have

$$M_0 \lesssim \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |f(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)$$

$$\lesssim C \sup_{x \in \mathbb{R}^n, r>0} \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |f(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right).$$

Now we estimate the terms $M_k$. We only need to estimate the integral as follows.

$$I = \int_{|y-x_0|<r} |R_j f_k(t,y)|^2 dy.$$

As a singular integral operator,

$$R_j g(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x_j - y_j|^{n+1}} g(y) dy.$$

By Hölder’s inequality, we can get

$$I = \int_{|y-x_0|<r} \left| \int_{2^k r \leq |z-x_0| < 2^{k+1} r} \frac{y_j - z_j}{y - z |^{n+1}} f(t,z) dz \right|^2 dy$$

$$\lesssim \int_{|y-x_0|<r} \left( \frac{1}{(2^k r)^n} \int_{|z-x_0|<2^{k+1} r} |f(t,z)|^2 dz \right) dy$$

$$\lesssim \frac{1}{2^{kn}} \int_{|z-x_0|<2^{k+1} r} |f(t,z)|^2 dz.$$

So we have

$$M_k = \left( r^{2\alpha-n+2\beta-2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} |R_j f_k(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}} \right)^{1/2}.$$
Therefore we can get
\[ \frac{1}{2} \sum_{k=1}^{\infty} M_k \]
\[ \lesssim \left[ 1 + \sum_{k=1}^{\infty} 2^{-k(\alpha+\beta-1)} \right] \sup_{x_0 \in \mathbb{R}^n, r > 0} \left( \int_0^2 \int_{|x-x_0| < r} |f(t, z)|^2 dy dt \right)^{1/2} \]
This completes the proof of Lemma 4.5.

Now we give the main result of this section.

**Theorem 4.6.** (Well-posedness).

(i) The subcritical quasi-geostrophic equation (4.1) has a unique small global mild solution in $(X_\alpha^\beta(\mathbb{R}^2))^2$ for all initial data $b_0$ with $\nabla \cdot b = 0$ and $\|b_0\|_{Q_{\alpha,-1}^\beta}$ being small.

(ii) For any $T \in (0, \infty)$, there is an $\varepsilon > 0$ such that the quasi-geostrophic equation (4.1) has a unique small mild solution in $(X_\alpha^\beta(\mathbb{R}^2))^2$ on $(0, T) \times \mathbb{R}^2$ when the initial data $b_0$ satisfies $\nabla \cdot b_0 = 0$ and $\|b_0\|_{Q_{\alpha,-1}^\beta}$ \(\leq \varepsilon\). In particular, for all $b_0 \in (VQ_{\alpha,-1}^\beta)^2$ with $\nabla \cdot b_0 = 0$, there exists a unique small local mild solution in $(X_\alpha^\beta(\mathbb{R}^2))^2$ on $(0, T) \times \mathbb{R}^2$.

**Proof.** By the Picard contraction principle we only need to prove the bilinear form $B(u, v)$ is bounded on $X_\alpha^\beta$. We split the proof into two parts.

**Part I.** $B_{\infty,0}^{0,1}$—boundedness. The proof of this part has been given in [19]. For completeness, we give the details. We have
\[ \|B(u, v)\|_{B_{\infty,0}^{0,1}} \lesssim \int_0^t \|e^{-(t-s)}(-\Delta)^{\beta/2} (\partial_1 (gR_2 f) - \partial_2 (gR_1 f))\|_{B_{\infty,0}^{0,1}} ds \]
\[ \lesssim \int_0^t (t-s)^{\frac{1}{2\gamma}} \frac{C_\beta}{s^{1+1/\gamma}} \|u\|_{B_{\infty,0}^{\alpha,1}} \|v\|_{B_{\infty,0}^{0,1}} ds \]
\[ \lesssim \|u\|_{X_\alpha^\beta} \|v\|_{X_\alpha^\beta} \int_0^t \frac{ds}{(t-s)^{1+1/\gamma}}. \]
Because when $\frac{1}{2} < \beta < 1$,
\[
\int_0^{t/2} \frac{1}{(t-s)^{1+(1-\beta)/2}} ds \lesssim t^{1-\beta-1}
\]
and
\[
\int_{t/2}^{t} \frac{1}{(t-s)^{1+(1-\beta)/2}} ds \lesssim t^{-2+\frac{1}{\beta}} \int_{t/2}^{t} \frac{1}{(t-s)^{1/\beta}} ds \lesssim t^{1-\beta-1}.
\]
Then we can get
\[
t^{1-\frac{1}{\beta}} \|B(u, v)\|_{\dot{B}_{\infty,1}^0} \lesssim \|u\|_{X_{\alpha}^\beta} \|v\|_{X_{\alpha}^\beta},
\]
where in the above estimates we have used the fact that $\|R_j f\|_{\dot{B}_{\infty,1}^0} \lesssim \|f\|_{\dot{B}_{\infty,1}^0}$ for $f \in \dot{B}_{\infty,1}^0$. In fact by Bernstein’s inequality, we have
\[
\sum_l \|\Delta_l R_j f\|_{L^{\infty}} = \sum_l \|\partial_j (-\Delta)^{-1/2} \Delta_l f\|_{L^{\infty}} \lesssim \sum_l 2^l \|(-\Delta)^{-1/2} \Delta_l f\|_{L^{\infty}} \lesssim \sum_l 2^l 2^{-l} \|\Delta_l f\|_{L^{\infty}} \lesssim \|f\|_{\dot{B}_{\infty,1}^0}.
\]
On the other hand, by Young’s inequality, we have
\[
t^{1-\frac{1}{\beta}} \|e^{-t(-\Delta)^\beta} u_0\|_{\dot{B}_{\infty,1}^0} \lesssim \|u_0\|_{\dot{B}_{\infty,1}^{1-2\beta,\infty}} \lesssim \|u_0\|_{Q_{\alpha}^{\beta,-1}}.
\]

**Part II.** \(L^2\)-boundedness. This part contributes to the operation of \(B(u, v)\) on the Carleson part of \(X_{\alpha}^\beta\). We split again the estimate into two steps.

**Step I.** We want to prove the following estimate:
\[
r^{2\alpha - 2 + 2\beta - 2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |B(u, v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_{\alpha}^{\beta}} \|v\|_{X_{\alpha}^{\beta}}.
\]
By symmetry, we only need to deal with the term
\[
\int_0^t e^{-(t-s)(-\Delta)^\beta} [\partial_1 (v R_1 u)] ds = B_1(u, v) + B_2(u, v) + B_3(u, v),
\]
where
\[
B_1(u, v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} \partial_1 [(1 - 1_{r,x}) v R_1 u] ds,
\]
\[
B_2(u, v) = (-\Delta)^{-1/2} \partial_1 \int_0^t e^{-(t-s)(-\Delta)^\beta} (-\Delta)(y) (-\Delta)^{1/2} (1 - e^{-s(-\Delta)^\beta})(1_{r,x}) v R_1 u) ds
\]
and
\[ B_3(u, v) = (-\Delta)^{-1/2} \partial_t \int_0^t (1_{r,x}) v R_1 u ds. \]

For \( B_1 \), it can be proved that the fractional heat kernel satisfies the following estimate (\([20]\)):
\[
|\nabla e^{-t(-\Delta)}(x, y)| \lesssim \frac{1}{t^{n/2}} \left( \frac{1}{1 + \frac{|x-y|}{10t^{2/3}}} \right)^{n+1} \lesssim \frac{1}{(t^{2/3} + |x-y|)^{n+1}}.
\]

For \( 0 < t < r^{2\beta} \), taking \( n = 2 \) in (4.5), we have
\[
|B_1(u, v)(t, x)| \lesssim \int_0^t \int_{|z-s| \geq 10r} |R_1 u(s, z)||v(s, z)| ds dz ds
\]
\[
\lesssim \left( \int_0^{r^{2\beta}} \int_{|z-s| \geq 10r} |R_1 u(s, z)|^2 ds dz ds \right)^{1/2} \left( \int_0^{r^{2\beta}} \int_{|z-s| \geq 10r} |v(s, z)|^2 ds dz ds \right)^{1/2}
\]
\[
:= I_1 \times I_2.
\]

For \( I_1 \), we have
\[
I_1 \lesssim \left( \sum_{k=3}^{\infty} \frac{1}{(2^k r)^3} \int_0^{r^{2\beta}} \int_{|z-s| \leq 2^{k+1} r} |R_1 u(s, x)|^2 ds dx \right)^{1/2}
\]
\[
\lesssim \left( \sum_{k=3}^{\infty} \frac{1}{(2^k r)^3} (2^k r)^{2\alpha+2\beta-2} (2^k r)^{2-2\beta} \int_0^{r^{2\beta}} \int_{|z-s| \leq 2^{k+1} r} |R_1 u(s, x)|^2 ds dx \right)^{1/2}
\]
\[
\lesssim \|u\|_{X_\alpha^{\beta}} \left( \sum_{k=3}^{\infty} \frac{1}{(2^k r)^{2\beta-1}} \right)^{1/2}
\]
\[
\lesssim \left( \frac{1}{r^{2\beta-1}} \right)^{1/2} \|u\|_{X_\alpha^{\beta}}.
\]

Similarly, we can get \( I_2 \lesssim \left( \frac{1}{r^{2\beta-1}} \right)^{1/2} \|v\|_{X_\alpha^{\beta}} \) and \( |B_1(u, v)| \lesssim \frac{1}{r^{2\beta-1}} \|u\|_{X_\alpha^{\beta}} \|v\|_{X_\alpha^{\beta}} \).

Then we have
\[
\int_0^{r^{2\beta}} \int_{|x-y| < r} |B_1(u, v)|^2 dy dt \lesssim \frac{1}{r^{4\beta-2}} \int_0^{r^{2\beta}} dt \left( \frac{1}{t^{2\beta}} \right)^2 \|u\|_{X_\alpha^{\beta}}^2 \|v\|_{X_\alpha^{\beta}}^2
\]
\[
\lesssim \frac{1}{r^{4\beta-2}} r^{2\beta-2\alpha} \|u\|_{X_\alpha^{\beta}}^2 \|v\|_{X_\alpha^{\beta}}^2
\]
\[
\lesssim r^{2-2\alpha-2\beta+2} \|u\|_{X_\alpha^{\beta}}^2 \|v\|_{X_\alpha^{\beta}}^2,
\]

where in the second inequality we have used the fact \( 0 < \alpha < \beta \). That is to say
For $B_2$, by the $L^2$-boundedness of Riesz transform, we have

\[
\int_0^{r^{2\beta}} \int_{|x-y|<r} |B_2(u, v)(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim \|u\|_{\dot{X}^\alpha_\beta}^2 \|v\|_{\dot{X}^\alpha_\beta}^2.
\]

On the other hand, we have, by H"older’s inequality, we get

\[
\int_0^{r^{2\beta}} \int_{|x-y|<r} |B_2(u, v)(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim \|u\|_{\dot{X}^\alpha_\beta}^2 \|v\|_{\dot{X}^\alpha_\beta}^2.
\]

For $B_3$, by the $L^2$–boundedness of Riesz transform, we have

\[
\int_0^{r^{2\beta}} \int_{|x-y|<r} |B_3(u, v)(t, y)|^2 \frac{dy dt}{t^{\alpha/\beta}} \lesssim r^{2-2\alpha-2\beta+2} \|u\|_{\dot{X}^\alpha_\beta}^2 \|v\|_{\dot{X}^\alpha_\beta}^2.
\]

For $B_3(u, v)$, we have
Then we have
\[
\int_0^{r^{2\beta}} \int_{|y-x|<r} |B_3(u, v)(t, y)| \frac{dydt}{t^{\alpha/\beta}} = \int_0^{r^{2\beta}} \int_{|y-x|<r} (-\Delta)^{-1/2} \partial_t (-\Delta)^{1/2} e^{-t(-\Delta)^{\beta}} \left( \int_0^t (1_{r,x}) v R_1 udh \right) \frac{dydt}{t^{\alpha/\beta}}
\]
\[
\lesssim \int_0^{r^{2\beta}} \left\| (-\Delta)^{1/2} e^{-t(-\Delta)^{\beta}} \left( \int_0^t (1_{r,x}) v R_1 udh \right) \right\| \frac{dt}{t^{\alpha/\beta}}
\]
\[
\lesssim r^{2-2\alpha+6\beta-2} \left( \int_0^t \||M(r^{2\beta} s, r \cdot)||_{L^1} \frac{ds}{s^{\alpha/\beta}} \right) C(\alpha, \beta, f)
\]
\[
\lesssim r^{2-2\alpha+6\beta-2} r^{2-4\beta} r^{2-4\beta} \|u\|_{X_\alpha^2} \|v\|_{X_\alpha^2}
\]
\[
\lesssim r^{2-2\alpha-2\beta+2} \|u\|_{X_\alpha^2} \|v\|_{X_\alpha^2}.
\]

**Step II.** For $j = 1, 2$, we want to prove
\[
(4.6) \quad r^{2\alpha-2+\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |A_i B(u, v)| \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^2} \|v\|_{X_\alpha^2},
\]
where $R_j$ are the Riesz transforms $\partial_j (-\Delta)^{-1/2}$. Similar to Step I, we can split $B(u, v)$ into $B_i(u, v), i = 1, 2, 3$. We denote by $A_i, i = 1, 2, 3$

\[
(4.7) \quad A_i := r^{2\alpha-2+\beta-2} \int_0^{r^{2\beta}} \int_{|x-y|<r} |A_i B(u, v)| \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X_\alpha^2} \|v\|_{X_\alpha^2}.
\]

In order to estimate the term $A_1$, we need the following lemma.

**Lemma 4.7.** For $\beta > 0$, if we denote by $K_j^{\beta}$ the kernel of the operator $e^{-t(-\Delta)^{\beta}} R_j$, we have
\[
(1 + |x|)^{n+|\alpha|} \partial^\alpha e^{-t(-\Delta)^{\beta}} R_j \in L^\infty.
\]

**Proof.** By the Fourier transform, we have $K_j^{\beta} = F^{-1} \left( \frac{\xi_j}{|\xi|} e^{-|\xi|^2} \right)$, where $F^{-1}$ denotes the inverse Fourier transform. Because
\[
\left[ \partial^\alpha K_j^{\beta}(x) \right](\xi) = \frac{\xi_j}{|\xi|} |\xi|^{\alpha} e^{-|\xi|^2} \in L^1,
\]
we have
\[
|\partial^\alpha K_j^{\beta}(x)| \leq \int_{\mathbb{R}^2} \left| \frac{\xi_j}{|\xi|} |\xi|^{\alpha} e^{-|\xi|^2} \right| d\xi \leq C.
\]
Then $\partial^\alpha K_j^{\beta}(x) \in L^\infty$. If $|x| \leq 1$, we have
\[
(1 + |x|)^{n+|\alpha|} |K_j^{\beta}(x)| \lesssim C |K_j^{\beta}(x)| \lesssim C.
\]
If \(|x| > 1\), by Littlewood-Paley decomposition and write

\[ K_j^\beta(x) = (Id - S_0)K_j^\beta + \sum_{l < 0} \Delta_l K_j^\beta, \]

where \((Id - S_0)K_j^\beta \in S(\mathbb{R}^n)\) and \(\Delta_l K_j^\beta = 2^{2l}\omega_{j,l}(2^{l}x)\) where \(\omega_{j,l}(\xi) = \psi(\xi)\xi_j e^{-|2^l \xi|^{2\beta}}\in L^1\). Then \(\omega_{j,l}(x)_{(l < 0)}\) are a bounded set in \(S(\mathbb{R}^n)\). So we have

\[
(1 + 2|\xi|)^{N\delta(2+|\alpha|)}|\partial^\alpha \Delta_l K_j^\beta(x)| \lesssim C_N
\]

and

\[
|\partial^\alpha S_0 K_j^\beta(x)| \lesssim C \sum_{2^{|x|} \leq 1} 2^{l(2+|\alpha|)} + \sum_{2^{|x|} > 1} 2^{l(2+|\alpha|) - N}|x|^{-N}
\]

\[
\lesssim C|x|^{-(2+|\alpha|)}.
\]

This completes the proof of Lemma 4.7.

Now we complete the proof of Theorem 4.6. In Lemma 4.7, we take \(\alpha = 1\) and get

\[
\left| \partial_x R_j e^{-t(-\Delta)^{\beta}}(x,y) \right| \lesssim \frac{1}{(t^{\frac{2\beta}{\alpha}} + |x-y|)^{n+1}}.
\]

Similar to the proof in Part I, we can get

\[
A_1 := r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}} \int_{|x-y| < r} |R_j B_1(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X^\alpha_\beta} \|v\|_{X^\alpha_\beta}.
\]

By Lemma 4.5, we know

\[
r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}} \int_{|y-x| < r} |R_j f(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}}
\]

\[
\lesssim \sup_{r > 0, x \in \mathbb{R}^n} r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}} \int_{|y-x| < r} |f(t,y)|^2 \frac{dydt}{t^{\alpha/\beta}}.
\]

By the above estimate, we have

\[
A_i := r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}} \int_{|x-y| < r} |R_j B_i(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}}
\]

\[
\lesssim r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}} \int_{|x-y| < r} |B_i(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}},
\]

where \(i = 2, 3\). Following the estimate to \(B_i\), \(i = 2, 3\), we can get

\[
A_i := r^{2\alpha-2+2\beta-2}\int_0^{r^{2\beta}} \int_{|x-y| < r} |R_j B_i(u,v)|^2 \frac{dydt}{t^{\alpha/\beta}} \lesssim \|u\|_{X^\alpha_\beta} \|v\|_{X^\alpha_\beta}.
\]
This completes the proof of Theorem 4.6.

Following the method applied in Section 5 of [18], we can easily get the regularity of the solution to the quasi-geostrophic equations (4.1). So we only state the result and omit the details of the proof. For convenience of the study, we introduce a class of spaces $X^\beta,k_\alpha$ as follows.

**Definition 4.8.** For a nonnegative integer $k$ and $\beta \in (1/2, 1]$, we introduce the space $X^\beta,k_\alpha$ which is equipped with the following norm:

$$\|u\|_{X^\beta,k_\alpha} = \|u\|_{N^\beta,k_\alpha,\infty} + \|u\|_{N^\beta,k_\alpha,C},$$

where

$$\|u\|_{N^\beta,k_\alpha,\infty} = \sup_{\alpha_1 + \ldots + \alpha_n = k} \sup_t \int_0^t \frac{2^{\beta-1+k}}{t^{\beta}} \|\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u(\cdot, t)\|_{B^\beta_{\infty,1}},$$

$$\|u\|_{N^\beta,k_\alpha,C} = \sup_{\alpha_1 + \ldots + \alpha_n = k} \sup_{x_0, r} \left( \frac{r^{2\alpha-2n+2\beta-2}}{2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} \left| \frac{k}{\pi^\beta t^{\alpha/\beta}} \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u(t, y) \right|^2 \frac{dy dt}{t^{\alpha/\beta}} \right)^{1/2},$$

$$+ \sum_{j=1}^{2^k} \sup_{\alpha_1 + \ldots + \alpha_n = k} \sup_{x_0, r} \left( \frac{r^{2\alpha-2n+2\beta-2}}{2} \int_0^{r^{2\beta}} \int_{|y-x_0|<r} \left| R^k_j \frac{k}{\pi^\beta t^{\alpha/\beta}} \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u(t, y) \right|^2 \frac{dy dt}{t^{\alpha/\beta}} \right)^{1/2}.$$

Now we state the regularity result.

**Theorem 4.9.** Let $\alpha > 0$ and $\max \{\alpha, 1/2\} < \beta < 1$ with $\alpha + \beta - 1 \geq 0$. There exists an $\varepsilon = \varepsilon(n)$ such that if $\|u_0\|_{Q^{2\beta-1}_\infty} < \varepsilon$, the solution $u$ to equations (4.1) verifies:

$$t^{\frac{k}{\pi^\beta}} \nabla^k u \in X^\beta_{\alpha,0}$$

for any $k \geq 0$.

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