Mathematical Proofs of Two Conjectures: 
The Four Color Problem and 
The Uniquely 4-colorable Planar Graph

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Version 2.0, 2010/05/20

Abstract

The famous four color theorem states that for all planar graphs, every vertex can be assigned one of 4 colors such that no two adjacent vertices receive the same color. Since Francis Guthrie first conjectured it in 1852, it was until 1976 with electronic computer that Appel and Haken first gave a proof by finding and verifying 1936 reducible unavoidable sets, and a simplified proof of Robertson, Sanders, Seymour and Thomas in 1997 only involved 633 reducible unavoidable sets. Both of proofs could not be realized effectively by hand. Different from related work mentioned above, an alternative method only involving 4 reducible unavoidable sets for proving the four color theorem is used in this paper, which takes form of mathematical proof rather than a computer-assisted proof and proves both the four color conjecture and the uniquely 4-colorable planar graph conjecture by mathematical method. Our work mainly includes the following five parts: the first part sets up some basic theory on the chromatic polynomial for 4-colorable planar graphs; the concept of uniquely near $k$-colorable planar graph is introduced in the second part, including some basic properties of this kind of graphs and a necessary and sufficient condition for uniquely near 4-colorable planar graphs; the third part builds generating system for maximal planar graphs, including some important results: for any uniquely 4-colorable planar graph, the graphs obtained by contracting 5-wheel operation have only two kinds of partitions of color group; the graphs obtained by contracting 6-wheel operation have two or four kind of partition of color group. Based on generating system and construction method for maximal planar graphs, the fourth part gives the upper bound of number of partition of color groups for uniquely 4-colorable graphs, which naturally solves the Friezin-Wilson-Fisk conjecture and the Jensen-Toft conjecture on the uniquely 4-colorable planar graphs, and gives the necessary and sufficient condition for the uniquely 4-colorable planar graph that this graph belongs to the recursive maximal planar graph. Based on the previous four parts, the last part gives the following result: for any maximal planar graph $G$, its chromatic polynomial $f(G, t)$ satisfies $f(G, 4) > 0$. Therefore, we give a brief mathematical proof of the four color theorem.
key words: the four color conjecture, the uniquely 4-colorable planar graph conjecture, maximal planar graph, recursive maximal planar graph (FWF graph), the contracting and extending operation for 4-colorable maximal planar graph, uniquely near $k$-colorable graph, chromatic polynomial.

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1 Introduction

In 1852, Francis Guthrie [31] proposed the four color conjecture [54, 43, 32, 7, 6, 64, 19]: four colors are sufficient to color any map drawn in the plane or
on a sphere so that no two regions with a common boundary line are colored with the same color, where two regions which have a point or a finite number of points in common are permitted to have the same color. It is also required that all regions be connected together. This problem can be converted to the vertex coloring problem for a graph easily. The specific method is as follows: Place a point, or vertex, in the middle of each country of a map M, and join two vertices with a line, or edge, whenever two countries have a common border to obtain a dual graph, which is planar. Therefore, the four color map problem can be converted into the 4-vertex-coloring problem for a planar graph equivalently.

The similar conjecture is shown as follows:

**Conjecture 1.1** (The Four Color Conjecture). For any planar graph \( G \), its vertices can be colored with 4 colors. Or for short, \( G \) is 4-colorable.

This conjecture was submitted to the London Mathematical Society in 1878 by Cayley, one of the most famous British mathematicians\[15, 16\]. From then on, it became the spotlight in the mathematical world. Nextly, Kempe \[45, 46\] and Tait \[62\] gave two different “proofs” for the four color conjecture in 1879 and 1880 respectively. Although two proofs were both incorrect, Kempe’s work laid the foundation for the computer-assisted proof\[3, 4, 5\] in 1976.

The concept of “configuration” was introduced by Kempe in his proof \[45, 46\]. He proved that in every normal map, the number of countries adjacent to a country may be two, three, four or five, but not all the countries are adjacent to six or more countries. In other words, the “configurations” composed of 2 to 5 countries are unavoidable. Each map contains at least one of these four kinds of configurations. Another concept implicitly given by Kempe was “reducibility”, which also came from his proof\[45, 46\]. He proved that if one country is adjacent to four countries in a colored map with 5 colors, then the number of countries in this map can be reduced. Since introducing the concepts of “configuration” and “reducibility”, several standard methods have been developed eventually to check the reducibility of configurations. So it is important for proofs of the four color theorem to find the unavoidable sets of reducible configurations. But the proof for larger reducible configuration needs to check large numbers of cases in detail, which is rather complicated. In a sense, the later computer-assisted proofs of the four color theorem were gradually developed from the above two concepts and related methods.

In 1890, Heawood\[42\] pointed out a fatal error in Kempe’s proof by constructing a counterexample. Nextly, he obtained the five color theorem by Kempe’s method: All planar graphs can be normally colored with no more than five colors. Kempe admitted Heawood’s counterexample, but none of them could correct the error. In the following 60 years, Heawood devoted himself exclusively to the research on the four color problem\[42\].

Another incorrect proof of four color problem\[62\] was given by Tait in 1880. His proof was based on the false assumption that each 3-connected planar graph is Hamiltonian, in which an important notion of 3-edge-coloring of a cubic plane graph was introduced. After 11 years, Petersen\[57\] pointed out the assumption of Tait’s proof was incorrect. The 3-edge-coloring of a 3-regular graph is called
Tait coloring. Tait proved that every 3-regular Hamiltonian graph contains Tait coloring, and thus believed that he had solved the four color problem based on the above false assumption. Although his error was found by Petersen [57], the explicit counterexample was not given until 1946 [68]. In 1968, Grinberg obtained a necessary condition for a planar graph to have a Hamiltonian cycle [39]. The result had been widely used to construct the non-Hamiltonian planar graph. It turns out that Tait’s work has an important impact on graph theory, particularly for edge coloring.

After Heawood’s work, the process for solving the four color problem slowed down. The subsequent efforts were further deepening, refinement and improvement with Kempe’s thought. The basic method was the smallest counterexample method, and the basic idea was to find unavoidable sets of reducible configurations.

The research for unavoidable sets started with Wernicke’s work [74] in 1904. Further, the concept of reducibility was explicitly introduced by Birkhoff [9] in 1913. Combining Kempe’s idea with a new idea of himself, Birkhoff proved some larger configurations are reducible. Nextly, in 1922 Franklin proved that the four color conjecture is true for maps with at most 25 regions [29]; The number was increased to 27 by Reynolds in 1926 [59], to 32 by Franklin in 1937 [30], to 35 by Winn in 1940 [76], to 39 by Ore and Stemple in 1970 [55], and finally to 95 by Mayer in 1976 [48]. Obviously, this method developed very slowly, and it helped little for the final solution to the four color problem.

If \( G \) is a maximal planar graph of order \( n \geq 3 \), then \( G \) is triangulated, that is, every face of a plane has a boundary of exactly three edges. A configuration of the maximal planar graph is the boundary cycle and inside part. For the four color conjecture, a set of configurations is unavoidable if any minimum counterexamples must contain one configuration in the set. A configuration is said to be reducible, if it is not contained in a planar graph which is triangulation of any minimum counterexamples.

On the research for finding unavoidable sets of reducible configurations, the great contribution was made by German mathematician Heesch [43], which became the foundation for the final solution with computer in 1976 [3, 4, 5]. Heesch put forward a more systematized method which could solve the four color problem effectively. It was given to the public by him in a seminar at Hamburg University and Keele University, which attracted Haken’s attention. He predicted that it might contain ten thousands of unavoidable sets of reducible configurations. But it was impossible to check the reducibility of all the configurations by hand. Moreover, he proposed another method “discharging” to prove the reducibility, which pushed forward the proof of the four color problem greatly. From 1960 to 1976, the research focused on finding unavoidable sets of reducible configurations by computer. The main contributors in the 1960s were Heesch, Haken, Durre and Shimamoto etc. And in the 1970s, Heesch, Allaire, Swart, Bernhart, Haken and Tutte etc. made great contributions on the four color problem. Among them, Shimamoto was frustrated by checking that one of his configurations was not D-reducible with computer [19].

The algorithm for checking “reducible obstacle” was proposed by Appel and
Haken, which could reduce computing time greatly. With the help of John Koch, they constructed successfully a complete proof which contained unavoidable sets of 1936 kinds of reducible configurations (later reduced to 1476 kinds) in 1976[3, 4, 5]. Each configuration needs to be checked one by one with computer. This proof was rechecked independently with different programs in different computers. Therefore, it took about 1200 hours with three computers to validate the four color conjecture, in which the number of judgments was about 10,000,000,000.

In 1997, a simplified proof[60, 61] with the similar idea as Appel and Haken was given by Robertson, Sander, Seymour and Thomas, in which only 633 configurations in the unavoidable sets were needed to check by computer. Additionally, a new computer-assisted proof[34] was given by Georges Gonthier with general purpose theorem proving software Coq in 2005.

However, these proofs were all computer-assisted and hard to check by hand. Thus, a computer-free method to solve the four color problem concisely is still concerned by the whole mathematical community.

In fact, vertex coloring problem is to classify all the vertices in a graph with different colors under two constraints: one is the adjacent vertices need to receive different colors; the other one is to minimum the number of colors, that is, the number of classes in the vertex-classification. Obviously, each color class corresponds to an independent set of a graph.

In this paper, let \( t \) be the number of colors used to color the graph \( G \), \( \chi(G) \) be the chromatic number of the graph \( G \), \( f(G, t) \) be the number of all possible colorings for the labeled graph \( G \) with \( t \) colors. Obviously, if \( t < \chi(G) \), the graph \( G \) can not be colored properly, so \( f(G, t) = 0 \); But if \( \chi(G) \leq t \), the number for this coloring must exist, that is \( f(G, t) > 0 \). For every given planar graph \( G \), when \( t = 4 \), the four color problem is equal to proving the inequality \( f(G, 4) > 0 \). This idea was proposed by Birkhoff to attack the four color problem in 1912[8, 9]. Later on, it was found that \( f(G, t) \) is a polynomial in the number \( t \), called the chromatic polynomial of the graph, which has become a fascinating branch in the field of graph theory until now[24]. But it was a pity that Birkhoff’s aim had not been reached. The best result given by Tutte[69] was that if \( t = \tau \sqrt{5} = 3.618... \) and \( \tau = \frac{1}{2}(1 + \sqrt{5}) \), then \( f(G, \tau \sqrt{5}) > 0 \). Although his result was indeed beautiful, it brushed past the four color problem, for the four color conjecture needs to prove \( f(G, 4) > 0 \).

An important tool used in this paper is the chromatic polynomial, which is the key to solve the four color conjecture. Some related researches are shown and discussed in Section 3 in order to make a systematic description.

Related theory and methods of the uniquely colorable graph, especially for the uniquely 4-colorable planar graph and the uniquely near \( k \)-colorable graph are the essential tool for proving the two conjectures in this paper. Similarly as the chromatic polynomial in Section 3, the advances of this field are introduced in Section 5.

Different from previous proofs, we give a novel method to prove the two conjectures in this paper, which combines the uniquely colorable graph with the chromatic polynomial. The main idea is built on the mathematical induction.
First, the Delete-One-Vertex recursion formulas of chromatic polynomials are built up on the vertices of degree 4 and 5 in a maximal planar graph, respectively. And the concept of uniquely near colorable graph is introduced, including basic characterizations of this kind of graphs. In order to attack the four color conjecture, the extending and contracting operations for maximal planar graphs have been proposed, and basic characterizations of uniquely 4-colorable planar graph have been described comprehensively. Furthermore, we give the necessary and sufficient condition for the uniquely 4-colorable graph, which naturally solves the Fröbenius-Wilson-Fisk Conjecture and the Jensen-Toft Conjecture on the uniquely 4-colorable graph. During the above-mentioned study, we prove that for any maximal planar graph $G$, its chromatic polynomial $f(G,t)$ satisfies $f(G,4) > 0$. Therefore, a short but strict mathematical proof is given here for the four color conjecture.

2 Relational definitions and notations

2.1 Basic notation

We begin with definitions of some useful terms we used in this paper, several of which may be non-standard. All graphs in this paper are restricted to the finite, simple and undirected ones.

In a given graph $G$, $V(G), E(G), d_G(u)$ and $\Gamma_G(u)$ denote the vertex-set, the edge-set, the vertex $u$’s degree and the vertices set adjacent to $u$ respectively, which can be written as $V, E, d(u)$ and $\Gamma(u)$ for short. The number of vertices is called order of the graph $G$ and the number of edges is called size of the graph $G$. Let $H$ denote a subgraph of $G$ if $V(H) \subseteq V(G), E(H) \subseteq E(G)$, and whenever $u, v \in V(H)$ are adjacent in the graph $G$, they are also adjacent in the graph $H$. $H$ is also called the vertex induced subgraph of $G$. An induced subgraph of $G$ with vertex set $V'$ is denoted by $G[V']$. Let $u, v$ be two different vertices in $V(G)$, the distance between $u$ and $v$ is the shortest path from $u$ to $v$, denoted by $d_G(u,v)$. The graph $G$ and $H$ are disjoint if they have no vertex in common. By starting with a disjoint union of $G$ and $H$, and adding edges to make each vertex of $G$ adjacent to each vertex of $H$, We can obtain the join $G \vee H$. The join $C_n \vee K_1$ which forms with a cycle and a single vertex, can be referred as a wheel with $n$ spokes, denoted by $W_n$ (the examples $W_2, W_3, W_4, W_5$ are shown in Figure 2.1). A graph is $k$-regular if all of its vertices have the same degree $k$. A 3-regular graph is usually called the cubic graph.
In order to identify the nonadjacent vertices $u$ and $v$, it is necessary to replace these two vertices with a single vertex, and make it adjacent to all the edges incident to $u$ or $v$. The contraction of an edge $e = uv$ is to replace the end-vertex $u$ and $v$ with a new single vertex, connect it with other vertices which were adjacent to $u$ or $v$. The resulting graph can be denoted by $G \circ e$ (or $G \circ \{u, v\}$).

### 2.2 Graph coloring

A vertex coloring of a graph $G$ is an assignment from color-set to its vertex-set such that no two distinct adjacent vertices have the same color. A $k$-vertex-coloring, or simply a $k$-coloring, of a graph $G$ is a mapping $f$ from the vertex set $V$ to the color sets $C(k) = \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ if vertex $x$ is adjacent to vertex $y$.

A graph $G$ is $k$-colorable if it has a $k$-coloring. The minimum $k$ for which a graph $G$ is $k$-colorable is called its chromatic number, denoted by $\chi(G)$. If $\chi(G) = k$, then the graph $G$ can be colored with $k$ colors, but not with $k - 1$ colors. Alternatively, each $k$-coloring $f$ of $G$ can be viewed as a partition $\{V_1, V_2, \ldots, V_k\}$ of $V$, where $V_i$ denotes the set of vertices assigned color $i$. So it can be written as $f = (V_1, V_2, \ldots, V_k)$. In other words, that is

$$V(G) = \bigcup_{i=1}^{k} V_i, V_i \neq \emptyset, V_i \cap V_j = \emptyset, i \neq j, i, j = 1, 2, \ldots, k \quad (2.1)$$

where $V_i$ is an independent set of $G$, $i = 1, 2, \ldots, k$. The set of all $k$-colorings of a graph $G$ can be denoted by $C_k(G)$. A $k$-colorable graph $G$ is called uniquely $k$-colorable if each $k$-coloring of $G$ induces the same partition of the vertex set $V$ into $k$-independent sets.

Similarly, an edge coloring of a graph $G$ [26, 27] is an assignment from color-set to its edge-set such that no two distinct adjacent edges have the same color. An $k$-edge-coloring of a graph $G$ is an edge-coloring with $k$ colors. A graph $G$ is $k$-edge-colorable if it has an $k$-edge-coloring. The edge chromatic number, $\chi'(G)$, of $G$ is the minimum number $k$ for which $G$ is $k$-edge-colorable. A graph $G$ is called uniquely $k$-edge-coloring if there is unique $k$-edge-coloring such that any other colorings are equivalent to it. Alternatively, a graph $G$ is uniquely $k$-edge-coloring if there is exactly one partition of the edge-set $E(G)$ into $k$ matchings.
In this paper, two isomorphic graphs $G$ and $H$ can be written as $G \cong H$. A graph $G$ is labeled if each vertices assign different labels, traditionally represented by integers. For a labeled graph $G$, two colorings are different if there is at least one vertex receives different colors. We use $f(G, t)$ to denote the number of $t$-colorings for a labeled graph $G$. It is called the chromatic polynomial of a graph $G$, which is introduced by Brikhoff to attack the four color problem in 1912[8]. More detailed researches can be found in the literatures [10, 65, 6, 69, 70, 71, 78, 77, 24].

2.3 Maximal planar graph

The maximal planar graph is a planar graph to which no new edges can be added without violating planarity. A triangulation is a planar graph in which every face is bounded by three edges (including infinite face). It can be easily proved that the maximal planar graph is equivalent to a triangulation. Thus, we can say that each maximal planar graph is a triangulation.

There exists one kind of uniquely 4-colorable planar graph $G$, which is called the recursive maximal planar graph. The graph $G$ can be obtained from $K_4$, embedding a 3-degree vertex in each triangular face continuously.

![Figure 2.2. Three recursive maximal planar graphs of minimal order.](image)

The set of all the recursive maximal planar graphs is denoted by $\Lambda$. All of this graphs with $n$ vertices can be written as $\Lambda_n$. Let $\gamma_n = |\Lambda_n|$. Obviously, $\gamma_4 = \gamma_5 = \gamma_6 = 1$. The corresponding recursive maximal planar graphs are shown in Figure 2.2.

Each face of any maximal planar graphs is a triangle. The vertex embedding operation in the triangular face $a - b - c$ is to add the vertex $u$ in this face, and make it adjacent to the vertices $a, b, c$ respectively, which can form a new maximal planar graph. This operation is denoted by $G + u$, shown in Figure 2.3, which is also called extending 3-wheel operation.

Another operation often used in the paper is to delete the 3-degree vertex, which is called contracting 3-wheel operation. Let $G'$ be a maximal planar graph, and $u$ be a 3-degree vertex. The operation for deleting 3-degree vertex $u$ needs to delete the vertex $u$ and all the edges incident to it. For example, the graph $G$ can be obtained from $G' = G + u$ by deleting the 3-degree vertex $u$ (shown in Figure 2.3).
The definitions and notations not mentioned can be found in Boudy’s new book [14].

3 Chromatic polynomial of a maximal planar graph

3.1 Introduction

Map coloring actually is a classification mode for all the countries in a map such that no two adjacent countries assigned the same color. If we convert a map into a planar graph by dual transform, then this problem will be changed into the vertex-coloring problem of a planar graph equivalently. Certainly, the vertex coloring problem of a graph is a mode of vertex partition, in which adjacent vertices must receive different colors. Accordingly, the basic scheme to attack the four color conjecture is the partition of color class for a graph. And this idea can be realized by the chromatic polynomial, which is a much useful mathematical tool. Although it is introduced for the labeled graph, which will generate a much larger number of colorings, the chromatic polynomial contains all the information on the partition of color class. Therefore, chromatic polynomial may be a preferable tool to prove the four color problem. Based on this tool, several scholars had made important contributions on attacking the four color conjecture, mainly including Birkhoff [8, 9, 10], Tutte [69, 70, 71], Read [58], Whitney [75]. Among them, the best result is that for any planar graph $G$, the chromatic polynomial $f(G, \tau \sqrt{5}) > 0, \tau \sqrt{5} = 3.618 \ldots$. However, it is a pity that his result brushed past the four color problem, for the four color conjecture needs to prove $f(G, 4) > 0$.

In order to prove the four color problem, the basic idea on the chromatic polynomial proposed in this paper is to study the recurrence equations for the chromatic polynomial of a maximal planar graph directly when color number $t = 4$. Especially, the result for recurrence equations when the minimal degree $\delta(G) = 5$ points out the direction to the mathematical proof of the four color problem. These recurrence equations are not only the foundation for the final proof of the four color conjecture, but also the base to prove the necessary and sufficient condition for uniquely 4-colorable planar graphs. With the guide of the
recurrence equation, we have to accomplish two tasks: one is the concept of the uniquely near $k$-colorable graph and the basic characterization for this graph. That is, the necessary and sufficient condition has been given for a uniquely near $k$-colorable graph. The other is the necessary and sufficient condition for the uniquely 4-colorable planar graph. This latter task was an unsolved conjecture proposed in 1977 [26, 28]. Although F. Thomas had given a computer-assisted proof [65] in 1998, his proof was not satisfied by mathematicians. In this paper, the mathematical proofs of these two conjectures will be given, which cost me 18 years’ attempt. Related results will be given in Section 3.3, 4 and 5 respectively.

3.2 Some related results on chromatic polynomial

In a given graph $G$, the number of $t$-colorings can be computed by $f(G,t)$. It was introduced by Birkhoff in 1912 [8]. He hoped it is helpful to solve the four color problem, which can be stated as follows:

**Theorem 3.1.** For any planar graph $G$ without loops, it is 4-colorable if and only if $f(G,4) > 0$.

So far, this attempt has not been realized, but the research on chromatic polynomial attracts many scholars’ interests. More detailed work on this field can be found from Birkhoff, Lewis, Read, Tutte and F.M.Dong [8, 9, 75, 58, 24]. In order to calculate the chromatic polynomial for a given graph, the basic tool is the Deletion-Contract Edge Formula.

Here the result graphs obtained by deleting and contracting edge $e$ are denoted by $G - e$ and $G \circ e$. In the contracting operation, the graphs are assumed to be simple graph (except in one case $W_2$, shown in Figure 2.1).

**Lemma 3.1.** [The Deletion-Contract Edge Formula] For a given graph $G$ and an edge $e$, then:

$$f(G, t) = f(G - e, t) - f(G \circ e, t) \quad (3.1)$$

**Lemma 3.2.** Let $G$ be the union of two subgraphs $G_1$ and $G_2$, whose intersection is a complete graph of order $k$, then

$$f(G, t) = f(G_1, t) \times f(G_2, t) \over t(t-1)\ldots(t-k-1) \quad (3.2)$$

Moreover, the recursive formula by vertex deletion [78] and the chromatic polynomial between graph and its complement [77] were given by us several years ago.

Birkhoff introduced the chromatic polynomial in order to solve the four color problem. So it is meaningful to study the chromatic polynomial of the maximal planar graph. Another beautiful work had been made by Tutte, as follows:

**Theorem 3.2.** [the Vertex-Elimination Formula] Let $G$ be a planar graph with a wheel graph $W_m$ as its subgraph. Then

$$f(G, \tau^2) = (-1)^m \tau^{1-m} f(G - \nu, \tau^2) \quad (3.3)$$
where $\tau^2 = \frac{3+\sqrt{5}}{2}$

**Theorem 3.3.** If $G$ is a maximal planar graph with $n$ vertices, then
\[ |f(G,t)| \leq \tau^{5-n}. \quad (3.4) \]

We now introduce three operations on graphs. Let $G$ be a planar graph with a 4-circuit $C = xyzlx$. There does not exist any vertices or edges inside the circuit besides a diagonal edge $e = xz$. Define $\theta_e$ as the “twisting” operation, which replaces the edge $xz$ with the edge $yl$. Another operation $\varphi_e(G)$ contracts the edge $xz$ to a single vertex $x'$, and deletes one member of each pair of double edges. The final operation $\Psi_e(G)$, similar as $\theta_e(G)$, contracts the edge $yl$ instead of the edge $xz$. These operations are shown in Figure 3.1.

![Figure 3.1. Three kinds of operations introduced for this planar graph.](image)

With Lemma 3.1, it is easy to prove that
\[ f(G,t) - f(\theta_e(G),t) = f(\Psi_e(G),t) - f(\varphi_e(G),t) \quad (3.5) \]
It holds for all the $t$. Let $t = \tau^2$. It can be obtained easily that

**Theorem 3.4.**
\[ f(G,\tau^2) + f(\theta_e(G),\tau^2) = \tau^{-3}\left\{f(\Psi_e(G),\tau^2) + f(\varphi_e(G),\tau^2)\right\} \quad (3.6) \]

Notice that $\tau = \frac{1+\sqrt{5}}{2}$. So $\sqrt{5} = \frac{5+\sqrt{5}}{2} = \tau + 2$. In this way, we can get

**Theorem 3.5. [Golden Identity.]** Let $G$ be a triangulation with $n$ vertices. Then
\[ f(G,\tau\sqrt{5}) = \sqrt{5} \times \tau^{3(K-3)} f^2(G,\tau^2) \quad (3.7) \]

**Theorem 3.6.** Let $G$ be a connected graph with at least one edge. Then $f(G,\tau+1)$ is non-zero.

Based on the work of Theorem [3.5] and Theorem [3.6], Tutte obtained an interesting result as follows:
\[ f(G,\tau\sqrt{5}) > 0 \quad (3.8) \]
It is clear that this result is much closer to the four color theorem, which is equivalent to \( f(G, 4) > 0 \). But \( \tau \sqrt{5} = 3.618 \ldots \), which is close to 4 but not 4. What a pity it is! One small false step will make a great difference.

Although Tutte’s result is perfect, once it was thought that it is impossible to attack the four color problem by chromatic polynomial. Nevertheless, our works below give new hopes to solving the four color problem by chromatic polynomial.

### 3.3 Delete-One-Vertex recursion formula for maximal planar graphs

Here we give some useful results for the chromatic polynomial of maximal planar graphs.

**Theorem 3.7.** (The chromatic polynomial for the vertex contraction with 4 degree vertex) Let \( G \) be a maximal planar graph, \( v \) a 4-degree vertex of \( G \), and \( \Gamma(v) = \{v_1, v_2, v_3, v_4\} \) (shown in Figure 3.2).

Then

\[
\begin{align*}
\chi(G, 4) &= \chi(G_1, 4) + \chi(G_2, 4) \\
\text{where } G_1 &= (G - v) \circ \{v_1, v_3\}, \text{ and } G_2 = (G - v) \circ \{v_2, v_4\}
\end{align*}
\]

![Figure 3.2. A maximal planar graph with a 4-degree vertex.](image)

*Proof. Let \( v \in V(G) \), \( d(v) = 4 \), \( \Gamma(v) = \{v_1, v_2, v_3, v_4\} \). Notice that the graph \( G \) is denoted by \( G[\Gamma(v)] \) in the following derivation. Now we first compute the chromatic polynomial of the graph \( G \) by the Lemma 3.1. In ease to understand, a method introduced by Zykov is used here [82], where the chromatic polynomials are represented by the corresponding graphical graphs which appear in the chromatic polynomials in \( t \). More details can be found in [43, 58].
By Lemma (3.2), the chromatic polynomial of the first subgraph is \( tf(G - v, t) \). Therefore,

\[
f(G, t) = (t - 2)
\]
When $t=4$, we can get that

$$f(G, 4) = (\quad - \quad) + (\quad - \quad) = (\quad - \quad) + (\quad - \quad) + v_2 \{v_1, v_3\} v_4 + v_1 \{v_2, v_4\} v_3$$

Notice that two graphs at the last line above denote $(G - v) \circ \{v_1, v_3\}$ and $(G - v) \circ \{v_2, v_4\}$ respectively, in which "\circ" represents the operation of vertex contraction in a graph. It is easily proved that they are both maximal planar graphs of order $n - 2$. Thus, we obtain that

$$f(G, t) = f((G - v) \circ \{v_1, v_3\}, t) + f((G - v) \circ \{v_2, v_4\}, t) = f(G_1, t) + f(G_2, t)$$
Theorem 3.8. (The chromatic polynomial for the vertex contraction with the 5-degree vertex) Let $G$ be a maximal planar graph, $v$ a 5-degree vertex of $G$, and $\Gamma(v) = \{v_1, v_2, v_3, v_4, v_5\}$ (shown in Figure 3.3).

Then

$$f(G, 4) = [f(G_1, t) - f(G_1 \cup \{v_1v_4, v_1v_3\}, t)] + [f(G_2, t) - f(G_2 \cup \{v_3v_1, v_3v_5\}, t)]$$

$$+ [f(G_3, t) - f(G_1 \cup \{v_4v_1\}, t)]$$

where $G_1 = (G - v) \circ \{v_2, v_5\}$, $G_2 = (G - v) \circ \{v_2, v_4\}$, $G_3 = (G - v) \circ \{v_3, v_5\}$.

Proof. Let $v \in V(G)$, $d(v) = 5$, $\Gamma(v) = \{v_1, v_2, v_3, v_4, v_5\}$, in which adjacent relations are shown in Figure 3.3. The chromatic polynomial of graph $G$ can be calculated by applying Lemma 3.2 repeatedly. If parallel edges appear in the process, reserve only one edge and use wheel graph $W_5$ to represent the chromatic polynomial of a maximal planar graph. In this way, we can obtain that

$$f(G, t) =$$

$$= $$

$$= $$

$$= $$

$$= $$
By Lemma (3.2), the chromatic polynomial of the first graph at the righthand of the last equation is $tf(G - v, t)$. Therefore, we can obtain that
when $t = 4$, the following equation holds:

\[
f(G, 4) = \frac{1}{4 - 1}
\]
\[ = \begin{array}{cc}
& \begin{array}{c}
\bullet v_1 \\
\bullet v_2 \\
\bullet v_3 \\
\bullet v_4 \\
\bullet v_5 \\
\end{array} \\
& \begin{array}{c}
\bullet v_1 \\
\bullet v_2 \\
\bullet v_3 \\
\bullet v_4 \\
\bullet v_5 \\
\end{array} \\
& \begin{array}{c}
\bullet v_1 \\
\bullet v_2 \\
\bullet v_3 \\
\bullet v_4 \\
\bullet v_5 \\
\end{array} \\
& \begin{array}{c}
\bullet v_1 \\
\bullet v_2 \\
\bullet v_3 \\
\bullet v_4 \\
\bullet v_5 \\
\end{array} \\
\end{array} \]
Notice that the fourth graph at the righthand of the last equation, denoted by $G'$, contains subgraph $K_5$ and so $f(G', 4) = 0$. Thus, we can obtain that
Actually, the first graph in the first bracket of the equation is $G_1 = (G - v) \circ \{v_2, v_3\}$; the first graph in the second bracket is $G_2 = (G - v) \circ \{v_2, v_4\}$; and the first graph in the third bracket is $G_3 = (G - v) \circ \{v_3, v_5\}$. This result is very important to the proof of the four color theorem, for all resulted values of each bracket are no less than zero. Obviously, the four color theorem can be proved if any bracket’s value is greater than zero. It may hold for every bracket’s values. Moreover, it is known that the graphs $G_1, G_2$ and $G_3$ is 4 colorable by induction, all of which are maximal planar graphs. Take the graph in the first bracket for example. In the second graph of the first bracket, the vertices $v_1, v_3, v_4$ and $v_2'$ (which is the new vertex from $v_2$ and $v_5$) can form a complete subgraph of order 4, so they have to be colored with different colors. Therefore, the four color theorem holds if there exists one kind of coloring in $C_4(G_1)$, which makes the vertex $v_1$ and $v_3$, or vertex $v_1$ and $v_4$ receive different colors. However, it is not an easy task for proving it, which will be given on the bases of Section 4, Section 5 and Section 6.

4 Uniquely near $k$-colorable graph

The $k$-colorable graph $G$ is called uniquely near $k$-colorable if $\exists V' \in V(G)$ so that $G[V']$ is uniquely $k$-colorable. In this section, basic characters of this kind of graphs will be described in detail, including a sufficient and necessary condition for a maximal planar graph to be uniquely near 4-colorable. This result is one of key theory to prove the four color conjecture by mathematics method, as shown in Section 7.

4.1 Some related concepts

Generally speaking, not every $k$-colorable graph is always an uniquely $k$-colorable graph. Thus, in each $k$-colorable graph $G$, does there certainly exist an induced subgraph which is uniquely $k$-colorable? The answer is negative. For instance, for every odd cycle $C_{2n+1}(n \geq 2)$, its chromatic number $\chi(C_{2n+1}) = 3$. But it
does not contain any uniquely 3-colorable induced subgraphs. Certainly, in a k-colorable graph \( G \), its vertex induced subgraph \( H \) may be uniquely k-colorable.

**Definition 4.1.** Let the graph \( G \) be \( k \)-colorable. It is called an **uniquely near \( k \)-colorable graph** if its induced subgraph \( G[V'] \) has an unique \( k \)-coloring, where \( V' \subset V(G) \).

Take the graph \( G \) shown in Figure 4.1(a) for example. It is easily known that it is a 4-colorable planar graph, but can not be uniquely 4-colorable. \( V' = \{1, 2, 3, 4, 5, 6, 7\} \) is a vertex subset in the graph \( G \). Thus, the subgraph \( G[V'] \) induced by \( V' \) has an unique 4-coloring. Thus, the graph \( G \) is an uniquely near 4-colorable planar graph.

In the following, the concept of **chromatic isomorphism** will be introduced by an example.

![Figure 4.1](image_url) An uniquely near 4-colorable graph.

The graph shown in the Figure 4.2(x) is a 3-colorable graph with 6 vertices. Its induced subgraph by \( V' = \{v_1, v_2, v_3, v_4\} \) has an unique 3-coloring. The color set is denoted by \( \{1, 2, 3\} \) (shown in Figure 4.2(z)).

For the subgraph shown in Figure 4.2(y), its unique partition of color class is \( \{v_1\}, \{v_2, v_4\}, \{v_3\} \). Let them receive different colors, denoted by 1, 2, 3 respectively. That is

\[
\{v_1\} \rightarrow 1, \{v_2, v_4\} \rightarrow 2, \{v_3\} \rightarrow 3 \quad (4.1)
\]

So we can obtain 3 kinds of colorings for the graph \( G \) (Shown in Figure 4.2(x)), denote by \( f_a, f_b, f_c \) respectively.

\[
f_a = \begin{pmatrix}
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 1 & 2 & 3 & 2 & 1 & 2
\end{pmatrix}
\]

\[
f_b = \begin{pmatrix}
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ 1 & 2 & 3 & 2 & 3 & 1
\end{pmatrix}
\]

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These 3 kinds of colorings are shown in Figure 4.2(a),(b),(c). With the change of colors in (4.1), the corresponding new colorings are generated, similar as $f_a, f_b, f_c$. This graph has 18 kinds of colorings, shown in Figure 4.2. In fact, it only has 3 kinds of different partitions of color class, shown in Figure 4.2(a),(b),(c). Other colorings can be obtained by permutation with these three colorings. Here we introduce the notion of chromatic isomorphism.
Figure 4.2. The graph $G$ with 6 vertices and its 18 kinds of 3-colorings.

**Definition 4.2.** Let $G$ be a $k$-colorable graph. $f_1, f_2 \in C_k(G)$. $f_1, f_2$ is called **chromatic isomorphic** if their partitions of color groups are the same. One representation is taken from every chromatic isomorphic set, and the set of all the representations is called the **chromatic isomorphic group**. The notation $C^0_k(G)$ denotes the chromatic isomorphic group formed by all the $k$-coloring of graph $G$. Actually, $C^0_k(G)$ is the set formed by all the partitions of $k$-color group, simplified by the graph $G$'s **partition set of $k$-color group**.

If two colorings of a graph are chromatic isomorphic, then they can be easily interconverted when their colors are adjusted properly. Therefore, these two colorings can be viewed as the same coloring. We just need to choose one of them. In Figure 4.2, it is easy to see that six colorings in every column have the same partitions of color class. Therefore, all the colorings in every column are chromatic isomorphic. So the graph $G$ in Figure 4.2 only has three kinds of different colorings, represented by three colorings in the first row.

It is easy to prove the following theorem:

**Theorem 4.1.** Let $G$ be a $k$-colorable graph, which is simple and undirected. The isomorphic class of this $k$-colorable graph $G$ is denoted by $C^0_k(G)$. The set of all the colorings in the graph $G$ is denoted by $C_k(G)$. Then,

$$|C_k(G)| = k!|C^0_k(G)|$$

(4.2)
Therefore, when we discuss some properties of all the $k$-colorings in $C_k(G)$, we only need to consider the isomorphic color class $C_k^i(G)$.

Let $G$ be a $k$-colorable graph. If there exist $k$ vertices $v_1, v_2, \ldots, v_k$ in $G$, for any coloring $f \in C_k(G)$, the values of $f(v_1), f(v_2), \ldots, f(v_k)$ are all different. It means that no two of these $k$ vertices belong to the same color class for all the partitions of color class. Without loss of generality, for each partition of $C_k^i(G)$,

$$V(G) = V_1 \cup V_2 \cup \ldots \cup V_k, V_i \cap V_j = \emptyset$$

where $i, j = 1, 2, \ldots, k$, and $V_i, i = 1, 2, \ldots, k$ denote independent sets. Let $v_i$ in the color class $i$, $i = 1, 2, \ldots, k$. In other word, for all the functions $f \in C_k^i(G)$,

$$f(v_i) = i, i = 1, 2, \ldots, k$$

(4.3)

The vertices $v_1, v_2, \ldots, v_k$ is called the color coordinates in the graph $G$, in which every vertex $v_i$ is called the $i$-th color axis, and a $k$-colorable graph $G$ is called the $k$-colorable coordinated graph. Let the graph $G$ be a $k$-colorable coordinated graph. The vertices in $C_k^i(G)$ which has the same color with color axis can form a new set, denoted by $V'_i$. That is, it includes all the vertices in the same color class as $v_i, i = 1, 2, \ldots, k$. More precisely,

$$V'_i = \bigcap_{f \in C_k^i} V_{if}, i = 1, 2, \ldots, k$$

(4.4)

where $V_{if}$ is the $i$-th color class under the coloring $f$, for any $f \in C_k^i(G)$.

Since that for all $f \in C_k^i(G)$, we have $v_i \in V_{if}(i = 1, 2, \ldots, k)$, so $V'_i \neq \emptyset$. Let

$$V' = V'_1 \cup V'_2 \cup \ldots \cup V'_k$$

(4.5)

$$V^c = V(G) - V'$$

(4.6)

**Definition 4.3.** The induced subset $V'_i$ obtained by $v_i$ according to Eq.(4.4) is called the color invariant group of $v_i$. The vertex set $V'$ defined by Eq.(4.5) is called the color invariant set concerning chromatic axes $v_1, v_2, \ldots, v_k$, simplified by the color invariant set of graph $G$. The vertex set $V_c$ is called the color variant set concerning $V'$, or the color invariant set of graph $G$.

**Definition 4.4.** Assume that $k$-colorable graph $G$ is a $k$-colorable coordinated graph concerning $v_1, v_2, \ldots, v_k$. Let $V'$ and $V^c$ denote color invariant set and color variant set defined by (4.5) and (4.6) respectively. Thus, the edge subset of graph $G$

$$E(V^c, V') = \{ uv, uv \in E(G), u \in V^c, v \in V' \}$$

(4.7)

is called edge cut between $V^c$ and $V'$. The subset of $V'$ incident with edge cut $E(V^c, V')$ is denoted by $V''$. Moreover, the induced subgraph $G[V'']$ is called the boundary of subgraph $G[V']$. 

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4.2 Basic characters for uniquely near $k$-colorable graphs

This section will mainly describe basic characters for uniquely near $k$-colorable graphs. Thus, we first discuss basic properties of this kind of graphs, including some useful lemmas. In order to understand basic characters of uniquely near $k$-colorable graphs, we give the definition of the standard form of uniquely near $k$-colorable graph.

4.2.1 Some useful lemmas

In graph $G$, vertex $v$ is adjacent to vertex subset $V'$ if it is adjacent to any one vertex in vertex subset $V'$.

**Lemma 4.1.** Let the graph $G$ be a $k$-colorable coordinated graph. Let the vertices $v_1, v_2, \ldots, v_k$ be the color axes in this graph. The definitions of $V'$ and $V^c$ are given in (4.5) and (4.6). Then, $\forall u \in V^c$, the vertex $u$ is adjacent to at most $k - 2$ vertices in different color classes.

*Proof.* If vertex $u$ is adjacent to other $k$ vertices $u_1, u_2, \ldots, u_k (u_i \in V'_i, i = 1, 2, \ldots, k)$, then it receives a color different from other vertices $u_1, u_2, \ldots, u_k$, which contradicts with the fact that the graph $G$ is $k$-colorable. Without loss of generality, assume vertex $u$ is adjacent to $u_1, u_2, \ldots, u_{k-1} (u_i \in V_i, i = 1, 2, \ldots, k-1)$. Then for all $f \in C_k^V(G), f(u) \notin \{1, 2, \ldots, k-1\}$. Since the graph $G$ is $k$-colorable, $f(u) = k$. So we can obtain $u \in V'_k$, which contradicts with the fact that $u \in V^c = V(G) - V'$. So the conclusion holds.

**Lemma 4.2.** Let the graph $G$ be a $k$-colorable coordinated graph. Let the vertices $v_1, v_2, \ldots, v_k$ be the color axes in this graph. The definitions of $V'$ and $V^c$ are given in (4.5) and (4.6). For all $u \in V^c$, let $u$ be adjacent to $u_1, u_2, \ldots, u_m$, where $u_i \in V'_i, 1 \leq i \leq m$. Then $G' = G + uv_{m+1}$ is $k$-colorable, where $v_{m+1} \in V'_{m+1}$.

*Proof.* Assume for all $f \in C_k(G), f(u) = m + 1$. Then $u$ belongs to $V'_{m+1}$, which contradicts with the fact that $u \in V^c$. So there exists $f_0 \in C_k(G)$, in which $f_0(u) \in \{m + 2, \ldots, k\}$. This shows that the graph $G' = G + uv_{m+1}$ is $k$-colorable.

**Lemma 4.3.** Let $G$ be an uniquely $k$-colorable graph with $n$ vertices, and the color classes are $V(G) = V_1 \cup V_2 \cup \ldots \cup V_k$. If there exists a color class with only one vertex in the unique partition of color class, denoted by $V_i = \{v\}$, and $d_G(v) = n - 1$, then $G - v$ is an uniquely $(k-1)$-colorable graph, whose unique partition of color class is $V(G - v) = V_2 \cup V_3 \cup \ldots \cup V_k$.

*Proof.* Assume $G - v$ is not uniquely $(k-1)$-colorable, namely besides the partition of $(k-1)$-color classes $V_2 \cup V_3 \cup \ldots \cup V_k$, there exists another different partition of $(k-1)$-color classes:

\[ U_2 \cup U_3 \cup \ldots \cup U_k. \]
Certainly, let $V_G \exists$ variant set respectively. If of the vertex $U$ different partitions of $k$-coloring classes, which contradicts with the fact that $G$ is uniquely $k$-colorable. In this way, $G - v$ is an unique $(k - 1)$-colorable graph, and its unique partition of color classes is $V(G - v) = V_2 \cup V_3 \cup \ldots V_k$.

Generally speaking, the color axes in this paper is given by $v_1, v_2, \ldots, v_k$, the corresponding color invariant groups are $V'_1, V'_2, \ldots, V'_k$, also called the coordinate system of graph $G$. Nextly, we will analyze the structure of color variant set $V^c$ in graph $G$.

4.2.2 The standard form

Here the standard form of uniquely near $k$-colorable graph is proposed in order to understand the basic structure of uniquely near $k$-colorable graph.

Let graph $G$ be a $k$-colorable coordinated graph. Let the vertices $v'_1, v'_2, \ldots, v'_k$ be the color axes in this graph. The $V'$ and $V^c$ are color invariant set and color variant set respectively. If $\exists u_1, u_2 \in V^c; \exists f_0 \in C^0_k(G)$, then $f_0(u_1) = f_0(u_2)$.

Certainly, $u_1$ is not adjacent to $u_2$. We can conclude the following properties of graph $G \circ \{u_1, u_2\}$:

1. $G \circ \{u_1, u_2\}$ is $k$-colorable, and if contracted vertex $u = \{u_1, u_2\}$ in $C^0_k(G \circ \{u_1, u_2\})$ is replaced by $\{u_1, u_2\}$, then $C^0_k(G \circ \{u_1, u_2\}) = C^0_k(G)$.

2. The vertex set $V'$ is color invariant group of $G \circ \{u_1, u_2\}$.

Furthermore, consider about any subset of $V^c$, $\{u_1, u_2, \ldots, u_m\}$, $m \geq 3$. If $\exists f_0 \in C^0_k(G)$ making $f_0(u_1) = f_0(u_2) = \ldots = f_0(u_m)$, then $u_1, u_2, \ldots, u_m$ belong to the same color group. Thus, the graph $G \circ \{u_1, u_2, \ldots, u_m\}$ contracted from vertex subset $\{u_1, u_2, \ldots, u_m\}$ has two properties as follows:

1. $G \circ \{u_1, u_2, \ldots, u_m\}$ is $k$-colorable. In $C^0_k(G \circ \{u_1, u_2, \ldots, u_m\})$, if the contracted vertex $u = \{u_1, u_2, \ldots, u_m\}$ is replaced by $\{u_1, u_2, \ldots, u_m\}$, then $C^0_k(G \circ \{u_1, u_2, \ldots, u_m\}) = C^0_k(G)$.

2. The vertex set $V'$ is color invariant group of $G \circ \{u_1, u_2, \ldots, u_m\}$.

Therefore, we can assume that $\forall f \in C^0_k(G), \forall u, v \in V(G)$, and $uv \notin E(G)$, $f(u) \neq f(v)$. Thus, we can also assume that

$$C^0_k(G) = C^0_k(G + uv)$$

Therefore, we can assume that $G[V^c]$ is a complete subgraph of graph $G$. Certainly, its order $|V^c| \leq k$.

Nextly, we consider about the bound between $V^c$ and $V'$. Based on lemma 4.1, we can assume that each vertex of $V^c$ is adjacent to $k - 2$ color groups of $V'$, and adjacent to all the vertices in each color groups. In summary, we can give the standard form of uniquely near $k$-colorable graph as follow:

**The standard form:** Let graph $G$ be a $k$-colorable coordinated graph. Let the vertices $v'_1, v'_2, \ldots, v'_k$ be the color axes in this graph. Let $V'$ and $V^c$ be color invariant set and color variant set respectively. The set $V^c$ satisfies the
following properties: $|V^c| \leq k$; $G[V^c]$ is a complete subgraph; $\forall u \in V^c$, vertex $u$ is adjacent to each vertex of $k - 2$ color groups in $V^c$.

4.3 Basic characters of uniquely near 4-colorable maximal graphs

Suppose that $G$ is a $k$-colorable graph, $f \in C_k(G)$, $u \in V(G)$. The chromatic neighborhood of vertex $u$ on coloring $f$ is the color set $\{f(v) : v \in \Gamma(u)\}$, also denoted by $C(f, \Gamma(u))$.

In this subsection, we will give a necessary and sufficient condition of uniquely near 4-colorable maximal planar graph after studying the characteristic of maximal planar graph. Firstly, we should introduce an important result obtained by Chartrand and Geller in 1969 as follow:

**Lemma 4.4.** If $G$ is a uniquely 4-colorable planar graph, then $G$ is a maximal planar graph.

**Theorem 4.2.** The necessary and sufficient condition for a 4-colorable graph $G$ to be uniquely near 4-colorable is as follow:

There exist 4 vertices $v_1, v_2, v_3, v_4$ in the graph $G$, for all colorings $f \in C_k(G)$, $f(v_i) \neq f(v_j)$, $i \neq j$, $i, j = 1, 2, 3, 4$.

**Proof.** Clearly, the necessity condition holds.

In following we mainly prove the sufficient condition, that is to prove: If there exist 4 vertices $v_1, v_2, v_3, v_4$ in $G$ such that $\forall f \in C_k(G)$, $f(v_1), f(v_2), f(v_3), f(v_4)$ are different from each other, then a vertices induced graph of $G$ containing vertices $v_1, v_2, v_3, v_4$ is uniquely 4-colorable.

Let the color invariant group induced by vertex $v_i$ be $V'_i$, $i = 1, 2, 3, 4$; $V' = V'_1 \cup V'_2 \cup V'_3 \cup V'_4$ be the color invariant set of graph $G$; $V^c = V(G) - V'$ be the color variant set of graph $G$; $G[V'']$ be the bound of $G[V']$. Our aim is to prove that $G[V']$ is a uniquely 4-colorable graph.

When $|V^c| = 0$, $G[V']$ is clearly a uniquely 4-colorable graph. Therefore, we can assume $|V^c| \geq 1$.

The definition of $G[V']$’s boundary is referred to definition 4.4. The proof of this theorem’s sufficient condition is achieved mainly by discussing the boundary $G[V'']$ of $G[V']$. Without loss of the generality, we can always assume that the induced subgraph $G[V^c]$ is connected in the following. Because when $G[V^c]$ is not connected, we only need to consider one of the connected components and the proof works in the same way. On the basis of above assumption, it is easy to prove that $G[V'']$ is a cycle, so it can be denoted by $G[V''] = C'_m = \{u_1, u_2, \ldots, u_m\}$, where the vertex subscript of $C'_m$ is arranged in a clockwise direction (see figure 4.3). In terms of the length $m$ of cycle $C'_m$, we consider the following two cases: $m \geq 4$ and $m = 3$.

Case 1. $m \geq 4$.

Under this condition, clearly $G[V']$ is not a maximal planar graph, consequently, it is not a uniquely 4-colorable planar graph according to lemma 4.6.
Here, we divide into several subcases in accordance to prove that this situation, \( m \geq 4 \), cannot happen.

Subcase 1.1. \( \exists f \in C^0_4(G) \) such that \( f(u_1) = f(u_3) \) (setting red), as shown in figure 4.3(a),(b).

In this case, we assign color blue to vertex \( u_2 \). So the vertex \( u_4 \) can be colored in three possible ways: blue (see figure 4.3(a)), and green (or yellow) (see figure 4.3(b)), respectively.

Now consider the coloring of graph in figure 4.3(a), in graph \( G[V'] \) we let \( G[V']^{r-y} \) denote the induced graph by these vertices that colored by red and yellow and \( G[V']^{b-g} \) denote the induced graph by those vertices that colored by blue and green.

Consider vertices \( u_1 \) and \( u_3 \). If they are not in the same component in \( G[V']^{r-y} \), then we exchange the two colors assigned to vertices in the component containing vertex \( u_3 \) and make the colors assigned to other vertices of \( G \) unchanged. Obviously, the resulted color group partition is a new 4-coloring of \( G \). This is in contradiction to the fact that \( V' \) is the color invariant set of \( G \).

If vertices \( u_1 \) and \( u_3 \) are in the same component in \( G[V']^{r-y} \), then vertices \( u_2 \) and \( u_4 \) surely are not in the same component in \( G[V']^{b-g} \), since otherwise the red-yellow chain from vertex \( u_1 \) to \( u_3 \), denoted by \( l_1 \), and the blue-green chain from vertex \( u_2 \) to \( u_4 \), denoted by \( l_2 \), must intersect at one vertex \( w \). However, the color assigned to \( w \) can not be any one of red, yellow, blue, green, yielding a contradiction. In this case, it is easy to prove that \( V' \) is not the color invariant set of \( G \) similarly.

For the coloring of graph in figure 4.3(b), by applying the above Kempe-chain method similarly we can also prove that \( V' \) is not the color invariant set of graph \( G \).
Subcase 1.2. \( \exists f \in C^0(G) \) such that \( f(u_1) = f(u_3), f(u_1) \neq f(u_3) \) and \( f(u_2) \neq f(u_3) \). Without loss of the generality, let \( f(u_1) = \text{red}, f(u_2) = \text{blue} \) and \( f(u_3) = \text{yellow} \) respectively, so the color assigned to vertex \( u_4 \) may be in following three possible ways.

1.2.1. \( f(u_4) = \text{green} \) (shown in figure 4.3(c)). For this situation, consider the red-yellow chain from vertex \( u_1 \) to \( u_3 \) and the blue-green chain from vertex \( u_2 \) to \( u_4 \), and the proof of this situation is similar to subcase 1.1.

1.2.2. \( f(u_4) = \text{blue} \). This case is also similar to subcase 1.1, so omitted here.

1.2.3. \( f(u_4) = \text{red} \) (shown in figure 4.3(d)). This situation must have \( m \geq 5 \), for otherwise, if \( m = 4 \), vertex \( u_1 \) is adjacent to vertex \( u_4 \), yielding a contradiction. Accordingly, consider vertex \( u_5 \in V(C'_m) \). If \( f(u_5) = \text{yellow} \), this case is similar to subcase 1.1, so omitted; If \( f(u_4) = \text{green} \) or \( f(u_4) = \text{blue} \), then similar to subcase 1.1, we can also prove that \( V' \) is not the color invariant set of graph \( G \) by considering the Kempe red-yellow chain from vertex \( u_1 \) to \( u_4 \) and the Kempe blue-green chain from \( u_2 \) to \( u_5 \) respectively.

In conclusion, we have proved case 1: When \( m \geq 4 \), \( G[V'] \) is not the uniquely 4-colorable maxima planar graph, but this case cannot happen at all.

Case 2. \( m = 3 \), namely \( G[V'''] = C'_3 = \{u_1, u_2, u_3\} \).

For this case, consider the subgraphs \( G[V'] \) and \( G[V' \cup \{u_1, u_2, u_3\}] \) of \( G \), both of them are maximal planar graph and satisfy

\[
G = G[V'] \bigcup G[V' \cup \{u_1, u_2, u_3\}], G[V'] \bigcap G[V' \cup \{u_1, u_2, u_3\}] \cong K_3.
\]

Therefore, \( G \) is the so-called Quasi-separable graph. Since graph \( G \) is a 4-colorable maximal planar graph, accordingly, \( G[V'] \) and \( G[V' \cup \{u_1, u_2, u_3\}] \) also are 4-colorable. Suppose \( f_1 \) and \( f_2 \) are two different 4-colorings of \( G[V'] \), they are easily extended into two different 4-colorings of \( G \) by the boundary \( C'_3 = \{u_1, u_2, u_3\} \), yielding a contradiction to the fact that \( V' \) is the color invariant group of \( G \). So the conclusion holds in this case.
5 Operational generating system of maximal planar graphs

With respect to research on many problems about coloring of maximal planar graphs, it is naturally an important task to clearly understand the structure of maximal planar graphs. In this field, some relevant results have been given in studying on methods of computer assisted proof of the four color conjecture [4-5,61]. Some scholars have designed some constructive algorithms to investigate special properties of maximal planar graphs, but no proof that can be directly used in this paper has come out.

Here a novel operational generating system of maximal planar graphs is given. This system consists of two parts of operating objects and basic operators, where the operating objects are maximal planar graphs, and with totally four pairs of the basic operators, such as: extending 2-wheel operation and its inverse operation—contracting 2-wheel operation; extending 3-wheel operation and its inverse operation—contracting 3-wheel operation; extending 4-wheel operation and its inverse operation—contracting 4-wheel operation; extending 5-wheel operation and its inverse operation—contracting 5-wheel operation. The basic function of this system is to use $K_3$ as basic operating object and apply the four kinds of basic operators to generate any a given maximal planar graph. With this method, we can construct all the maximal planar graphs with minimal degree $\delta \geq 4$. In addition, we propose extending and contracting operations based on any 4-coloring, and give the extending and contracting operations under some constrains, especially get some important results: for any uniquely 4-colorable planar graph, there are two kinds of partitions of color group in the graph obtained by contracting 5-wheel operation; there are two or four kinds of partitions of color group in the graph obtained by contracting 6-wheel operation.

5.1 The basic operational generating system of maximal planar graphs

In this section, we mainly give the definition of the basic operators and some correlated properties which are used in the basic operational generating system of maximal planar graphs without any condition on their colorings.

The extending 2-wheel operation means: firstly, add a new edge between two adjacent vertices, which shall generate 2-parallel edges; secondly, add a new vertex in the face of the 2-parallel edges and make the new vertex adjacent to the two vertices on the 2-circle.

The contracting 2-wheel operation means: firstly, delete the wheel center and the two edges incident to it; secondly, delete one of the parallel edges that is the two edges of the 2-wheel circle.

In the second section of this paper, we have introduced the extending 3-wheel operation on maximal planar graphs: add a new vertex on a certain face of the maximal planar graph, and then add three edges which makes the new vertex adjacent to the three vertices of the face; we have also introduced the
contracting 3-wheel operation: delete a certain 3-degree vertex and the edges incident to it.

The so-called contracting 4-wheel operation is: in a maximal planar graph, delete a certain 4-degree vertex and the edges incident to it, and then do the contracting operation to a pair of the vertices adjacent to the deleted vertex. About the extending 4-wheel operation, it is the inverse operation of the contracting 4-wheel operation. The detailed definition is given as follows.

\[ \begin{align*}
\text{Figure 5.1. the schematic diagram of the extending 4-wheel operation.}
\end{align*} \]

Let \( G \) be a maximal planar graph. Let \( x, y, z \in V(G) \), and \( x - u - y \) is a length 2 path. The so-called extending 4-wheel operation with respect to the path \( x - u - y \) is that, replace the \( x - u - y \) path by a new face with four vertices \( x, u, u', y \), that is, split the vertex \( u \) into two vertices \( u \) and \( u' \), and split the edge \( xu \) and \( uy \) into two edges \( xu, xu' \) and \( uy, uy' \) respectively. This process is shown by the first to the fourth graphs in figure 5.1. Then the four vertices \( x, u', y, u \) form a length 4 circle in the new graph. And then, add a new vertex \( v \) in the face of the length 4 circle \( x - u' - y - u - x \), and make \( v \) adjacent to vertices \( x, u', y, u \) respectively, which will generate edges \( vx, vu', vy \) and \( vu \). The resulted new graph is denoted by \( G * xuy \), and is called the graph generated by extending 4-wheel operation. This is shown by the fifth graph in figure 5.1.

We call the graph shown in figure 5.2 \textbf{funnel-graph}. The degree-1 vertex is called the \textbf{funnel top}; the degree-3 vertex is called \textbf{funnel stem}; the two degree-2 vertices are called \textbf{funnel bottom}. 

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On a maximal planar graph, the contracting 5-wheel operation and the extending 5-wheel operation is similar to the contracting 4-wheel operation and the extending 4-wheel operation, except that the contracting 5-wheel operation is performed on the funnel-subgraph of maximal planar graph, while the extending 4-wheel operation is performed on the length-2 path in maximal planar graph. Here we only give a graphical illustrative definition, shown in figure 5.3, to the contracting 5-wheel operation and the extending 5-wheel operation.

In this paper, for \( i = 2, 3, 4, 5 \), we use \( \zeta_i^-(G) \) to denote the resulted graph from the contracting \( i \)-wheel operation on maximal planar graph \( G \), and use \( \zeta_i^+(G) \) to denote the resulted graph from the extending \( i \)-wheel operation on maximal planar graph \( G \).

The following two propositions are easy to prove.

**Proposition 5.1.** Let \( G \) be a maximal planar graph, then \( \zeta_i^-(G) \) and \( \zeta_i^+(G) \) \((i = 2, 3, 4, 5)\) are maximal planar graph.

**Proposition 5.2.** Let \( G \) be an \( n \)-vertex maximal planar graph, then the resulted graphs \( \zeta_2^-(G) \) or \( \zeta_3^-(G) \) from contracting 2-wheel operation or contracting 3-wheel operation are \((n - 1)\)-vertex maximal planar graph respectively; and the resulted graphs \( \zeta_4^-(G) \) or \( \zeta_5^-(G) \) from contracting 4-wheel operation or contracting 3-wheel operation are \((n - 2)\)-vertex maximal planar graph respectively, namely

\[
|\zeta_4^-(G)| = |\zeta_5^-(G)| = |G| - 2 = n - 2.
\]

**Theorem 5.1.** Let \( G \) be an \( n \)-vertex maximal planar graph. Then by conducting repeatedly contracting 2-wheel operation, contracting 3-wheel operation, contracting 4-wheel operation and contracting 5-wheel operation, the graph can be contracted to a 3-vertex complete graph \( K_3 \).

**Proof.** When \( n = 4 \), for there is only one maximal planar graph \( K_4 \), the conclusion holds obviously. Suppose that the conclusion holds when number of vertices is no more than \( n \), which means that for any graph with no more than \( n \) vertices, it can be contracted to a 3-vertex complete graph \( K_3 \) by conducting repeatedly contracting 2-wheel operation, contracting 3-wheel operation, contracting 4-wheel operation and contracting 5-wheel operation.
Now we consider the situation when the number of the vertices is \( n + 1 \). For any \((n+1)\)-vertex graph \( G \), if \( G \) has any degree-2 or degree-3 vertex, then by deleting the degree-2 or degree-3 vertex and the incident edges, we get an \( n \)-vertex graph \( \zeta(G) \) or \( \zeta^{-}(G) \), then by the induction hypothesis, the conclusion holds. If \( \delta(G) = 4 \) or \( \delta(G) = 5 \), then choose some degree-4 or degree-5 vertex, by doing a contracting 4-wheel operation or a contracting 5-wheel operation, we get graph \( \zeta^{-4}(G) \) or \( \zeta^{-5}(G) \), both of them are \((n-2)\)-vertex graph \( K_3 \), by doing repeatedly contracting 2-wheel operation, contracting 3-wheel operation, contracting 4-wheel operation and contracting 5-wheel operation.

![Figure 5.3. The schematic diagram of extending 5-wheel operation.](image)

Through theorem 5.1, we know that every \( n \)-vertex maximal planar graph can be contracted to \( K_3 \) by 4 basic contracting operation. Of course, through the reverse direction of contracting \( k \)-wheel operation of graph \( G \), from \( K_3 \), do the corresponding extending \( k \)-wheel operation, we can get the original graph \( G \). From this, we can conclude that

**Corollary 5.2.** For any two maximal planar graph \( G \) and \( G' \) whose numbers of vertices are larger than 5, we can get \( G' \) from \( G \) by doing the above 4 pairs of contracting and extending operations.

We use \( \Psi = \{ \zeta_2^{-}, \zeta_2^{+}, \zeta_3^{-}, \zeta_3^{+}, \zeta_4^{-}, \zeta_4^{+}, \zeta_5^{-}, \zeta_5^{+} \} \) to denote the 4 basic pairs of contracting operations and extending operations; use \( S(G) = (K_3, \Psi) \) to denote the operational generating system of maximal planar graphs. Any maximal planar graph can be generated by this system.

### 5.2 The generation method of maximal planar graph

Through the method mentioned in the last section, we give the construction method and steps of the entire \( n \)-vertex maximal planar graph whose \( \delta(G) = 4 \). Especially, we construct all the maximal planar graph whose number of vertices is 6 to 11 and \( \delta(G) = 4 \).

Let \( \text{Aug}(G) \) denote the automorphism group of graph \( G \), \( x - u - y \) and \( x' - u' - y' \) are two different paths of graph \( G \). \( x - u - y \) and \( x' - u' - y' \) are called equal, if there exist a \( \sigma \) in \( \text{Aug}(G) \), which makes \( \sigma(x) = x' \), \( \sigma(u) = u' \), \( \sigma(z) = z' \). Otherwise, these two paths are unequal.
5.2.1 the construction method and steps of $n$-vertex maximal planar graph

Step 1. Generated from the $(n - 2)$-vertex maximal planar graph by doing extending 4-wheel operation and extending 5-wheel operation.

The detailed method is: for any $(n - 2)$-vertex maximal planar graph, firstly choose the length-2 unequal paths. As shown by figure 5.4, in the 7-vertex maximal planar graph $G_7$ whose $\delta(G) = 4$, there are 4 different length-2 paths, they are: 444 type, 445 type, 454 type and 545 type respectively, where 444 type means that the degree sequence of the length-2 path is 444, and the other types are similar to this. Then for every length-2 path selected, do extending 4-wheel operation. For example, for the 7-vertex maximal planar graph $G_7$ whose $\delta(G) = 4$, the two 9-vertex maximal planar graphs resulted from doing extending 4-wheel operation on 444 type and 454 type length-2 paths respectively are isomorphic. On 545 type length-2 path, the degree sequence of the 9-vertex maximal planar graph resulted from doing extending 4-wheel operation is 444444477; the degree sequence of the 9-vertex maximal planar graph resulted from doing extending 5-wheel operation is 4444555556. The above cases are shown by figure 5.4 respectively.

Step 2: Generated from $(n - 3)$-vertex maximal planar graph by doing the combination of extending 2-wheel operation and extending 4-wheel operation; or the combination of extending 3-wheel operation and extending 5-wheel operation.

For example, for the 9-vertex maximal planar graph, it can be only generated from 6-vertex maximal planar graph. Due to the 6-vertex maximal planar graph whose $\delta(G) = 4$ is only regular dodecahedron (as the first graph in figure 5.4), it generates a 9-vertex maximal planar graph $G$ by doing the combination of extending 2-wheel operation and extending 4-wheel operation, its degree sequence is 444444666 and $\delta(G) = 4$; it generates a 9-vertex maximal planar graph whose degree sequence is 444555555 and $\delta(G) = 4$ by doing the combination of extending 3-wheel and extending 5-wheel operation.

We have in fact given constructions of all five 9-vertex maximal planar graphs whose $\delta(G) = 4$ by above examples. Note that 445 type can not generate any 9-vertex maximal planar graph whose $\delta(G) = 4$; all other graphs resulted from doing extending 5-wheel operation are isomorphic to one of the five graphs.
Figure 5.4 the schematic diagram of the generation procedure of maximal planar graph whose number of vertices is 7 to 9 and $\delta(G) = 4$.

5.2.2 All the maximal planar graph whose number of vertices is 6 to 11 and $\delta(G) = 4$

In order to prove the main result of this section, we need to investigate all maximal planar graphs whose number of vertices is 6 to 11 and $\delta(G) = 4$. 
Table 5.1 gives the number of maximal planar graphs in different orders and $\delta(G) = 4$. And by above generation methods, we accordingly construct these maximal planar graphs, as shown in figure 5.4 figure 5.8 respectively.

Table 5.1. The count chart of the maximal planar graph satisfied and the order of the graph is 6 to 11

| Order | 6 | 7 | 8 | 9 | 10 | 11 |
|-------|---|---|---|---|----|----|
| Graphs | 1 | 1 | 2 | 5 | 13 | 36 |

Figure 5.5. The maximal planar graph whose order is 6 and 7, $\delta(G) = 4$.

Figure 5.6. All two maximal planar graph whose order is 8 and $\delta(G) = 4$. 
Figure 5.7 all five maximal planar graph whose order is 9 and $\delta(G) = 4$. 
Figure 5.8 all 13 maximal planar graph whose order is 10 and $\delta(G) = 4$. 
Figure 5.9 all 36 maximal planar graph whose order is 11 and $\delta(G) = 4$. 

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5.3 The extending \( k \)-wheel operation and contracting \( k \)-wheel operation based on coloring

Based on the last two sections, in this section, we introduce the contracting \( k \)-wheel operation based on coloring \( f \) of the maximal planar graph and the relevant inverse operations– the extending \( k \)-wheel operation based on coloring \( f \), where \( k = 2, 3, 4, 5 \), and give the related basic properties.

The contracting 2-wheel operation and the extending 2-wheel operation based on 4-coloring graphs are almost the same as the contracting 2-wheel operation and the extending 2-wheel operation without considering coloring, only except the issue of assigning colors to the central vertex of wheel, so no more discussion here, see section 5.1.

Let \( G \) be a 4-colorable maximal planar graph. If \( v \) is a 3-degree vertex of \( G \), and \( \Gamma(v) = \{ v_1, v_2, v_3 \} \), then \( \forall f \in C_4(G) \), the contracting 3-wheel operation based on coloring \( f \) of 3-degree vertex \( v \), means that delete vertex \( v \) from graph \( G \). Naturally, the resulted graph from contracting 3-wheel operation is \( G - v \), and it is still a 4-colorable maximal planar graph. And the extending 3-wheel operation on face \( v_1 - v_2 - v_3 \) with respect to coloring \( f \in C_4(G - v) \) of graph \( G - v \) is that, add a new vertex \( v \) on that face, let \( v \) adjacent to \( v_1, v_2, v_3 \) respectively, and assign to \( v \) a color different from \( f(v_1), f(v_2), f(v_3) \).

Let \( G \) be a 4-colorable maximal planar graph. Let \( v \) be a 4-degree vertex of \( G \), and \( \Gamma(v) = \{ v_1, v_2, v_3, v_4 \} \). When \( f \in C_4(G) \), it must be that \( f(v_1) = f(v_3) \), or \( f(v_2) = f(v_4) \). From now on, we always assume that \( f(v_1) = f(v_3) \), as shown in figure 5.10(a). Then contracting 4-wheel operation based on coloring \( f \) of 4-degree vertex \( v \) means that, delete vertex \( v \) from graph \( G \), and do contracting operation on vertex pair \( \{ v_1, v_3 \} \), as shown in graph 5.10(b). Extending 4-wheel operation based on coloring \( f \) means that, in a 4-colorable maximal planar graph, do following operations on a length-2 path \( x - u - y \): firstly, do similar to the extending 4-wheel operation without considering of coloring (see figure 5.10); secondly, assign to the new central vertex \( v \) of wheel a different color from \( f(x), f(u), f(y) \).

![Figure 5.10](image_url)
Let $v$ be a 5-degree vertex of 4-colorable maximal planar graph, and $\Gamma(v) = \{v_1, v_2, v_3, v_4, v_5\}$. When $f \in C_4^5(G)$, without lose of generality, let $f(v_1) = f(v_3), f(v_2) = f(v_5)$, as shown in figure 5.11(a). Then the **contracting 5-wheel operation based on coloring** $f$ means that, delete vertex $v$ from graph $G$, and do contracting operations on vertex pair $\{v_2, v_5\}$ or $\{v_1, v_3\}$, here we choose to contract vertex pair $\{v_2, v_5\}$, as shown in figure 5.11(b).

**Figure 5.11.** the schematic diagram of contracting 5-wheel operation.

So far, we have given the contracting and the extending operations of $W_3, W_4, W_5$, under the condition of a given 4-coloring $f$ of a maximal planar graph. It is easy to see that, when $3 \leq k \leq 5$, the extending $k$-wheel operation and the contracting $k$-wheel operation are in one-one correspondence. But when $k \geq 6$, the similar extending operation and contracting operation are not in one-one correspondence, but in one-many correspondence. Below we first give 4-coloring extending 6-wheel operation and contracting 6-wheel operation, then define extending $k$-wheel operation and contracting $k$-wheel operation.
Figure 5.12. the schematic diagram of extending 6-wheel operation and contracting 6-wheel operation.

**Definition 5.1.** If $v$ is a 6-degree vertex of $G$, and $\Gamma(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. When $f \in C_4^6(G)$, for $W_6 = G[\Gamma(v)]$, there might be 4 kinds of colorings as shown in figure 5.12, and on every kind of coloring, the definition of the relevant extending 6-wheel operation and contracting 6-wheel operation is clear from the shown graphs.

**Definition 5.2.** Let graph $G$ be 4-colorable, and $f$ be one of the 4-coloring. Let $v$ be a degree $k \geq 3$ vertex of $G$, and $\Gamma(v) = \{v_1, v_2, \ldots, v_k\}$. The so-called **contracting $k$-wheel operation** of wheel $W_k = G[\Gamma(v)]$ in graph $G$ is that, delete the $k$-degree vertex $v$, then merge those neighbors of $k$-degree vertex $v$ with the same color into one vertex, and if the merged graph is maximal planar,
then the contracting vertex operation finished; otherwise, check that if there are two vertices with the same color, and if it is the case, then merge these two vertices, repeat this process until the merged graph is maximal planar. According to the above definition of contracting $k$-wheel operation, doing the extending operations step by step reversely is the so-called extending $k$-wheel operation.

We use $\zeta_k^-(G,v)$ to denote the contracting $k$-wheel operation on $k$-degree vertex $v$ of the maximal planar graph $G$, in case of no confusion, it can be abbreviated as $\zeta_k^-(G)$. Use $\zeta_k^+(G,v)$ to denote the extending $k$-wheel operation on maximal planar graph by the above three methods, in case of no confusion, abbreviated as $\zeta_k^+(G)$, where the added degree-$k$ vertex is denoted by $v$.

**Theorem 5.3.** Let $G$ be a unique 4-colorable maximal planar graph. Let $v$ be a 5-degree vertex of graph $G$, and the neighborhood structure of $v$ and the uniquely 4-coloring of the subgraph are shown in figure 5.11(a). Then there are exact two 4-color group partitions of the resulted graph $\zeta_v^-(G)$ from contracting 5-wheel operation on $G$, namely

$$|C_4^4(\zeta_v^0(G))| = 2$$

(5.1)

**Proof.** Let the four-color set be $\{1 = \text{red}, 2 = \text{blue}, 3 = \text{yellow}, 4 = \text{green}\}$, and let $G[i,j]$ denote the induced subgraph from the subset of vertices with colors $i$ and $j$, where $i \neq j$ and $i, j = 1, 2, 3, 4$. Then $G[i,j]$ must be a tree, for otherwise, it is easy to get contradiction to the fact that $G$ is uniquely 4-colorable. Consider $\zeta_v^-(G)$ in figure 5.11(b), it is resulted from deleting the vertex $v$ with green color (color 4), and contracting two non-adjacent red vertices (color 1). Therefore, in $\zeta_v^-(G)$, there must exist a circle with alternative red and blue vertices, as shown in figure 5.12. Now, we consider the coloring of $\zeta_v^-(G)$ in more details.

![Figure 5.13. The schematic diagram of the proof of theorem 5.5.](image)
• $\zeta_-(G)[3, 4]$ has two connected components, and each of them is a tree;

• $\zeta_-(G)[2, 4]$ is a tree, resulted by deleting leaf $v$ from tree $G[2, 4]$;

• $\zeta_-(G)[2, 3]$ is a tree, exactly the same as tree $G[2, 3]$;

• $\zeta_-(G)[1, 4]$ is a tree, resulted by deleting 2-degree vertex $v$ from tree $G[1, 4]$ and contracting $v_2$ and $v_5$;

• $\zeta_-(G)[1, 3]$ is a tree, resulted by contracting $v_2$ and $v_5$ from tree $G[1, 3]$;

• $\zeta_-(G)[1, 2]$ is connected and there exists exactly one circle $C$, either vertex $v_1$ or $v_2$ is inside the circle $C$, as shown in figure 5.13 (a), (b). Without lose of generality, assume that vertex $v_1$ is inside circle $C$. Obviously, circle $C$ divides the $G[3, 4]$ into two components from vertex $v$; the two components are $\zeta_-(G)[3, 4]$. Now let $V^y$ and $V^g$ denote the vertex sets colored yellow and green outside the circle respectively; and $V^3$ and $V^4$ denote the vertex sets colored yellow and green inside the circle respectively.

Then because $G$ is uniquely 4-colorable, we have

$$C_4^0(\zeta_-(G)) = \{\{V^1, V^2, V^3 \cup V^y, V^4 \cup V^g\}, \{V^1, V^2, V^3 \cup V^g, V^4 \cup V^y\}\}.$$

**Theorem 5.4.** Let graph $G$ is a uniquely 4-colorable maximal planar graph. Let $v$ be a vertex of degree 6, and the neighborhoods' structure of $v$ is shown in Figure 5.14. Let $f$ be the uniquely 4-coloring. Then there are 3 different colorings of $f$ in $\Gamma(v)$, and all the possible colorings on 6-circle are shown in Figure 5.12(b), (c), (d). Let $\zeta_-(G)$ be the graph resulted by contracting 6-wheel operation on graph $G$, then the 6-wheel subgraph itself in graph $G$ can be contracted into only three possible kinds of structures : line type (as shown in Figure 5.12(b) left), star type (as shown in Figure 5.12(c) right), up and down triangles type (as shown in Figure 5.12(d) right), so we have the following conclusions:

1. If after contracting 6-wheel operation based on $f$, the 6-wheel subgraph is contracted into line type, then $|C_4^0(\zeta_-(G))| = 2$.

2. If after contracting 6-wheel operation based on $f$, the 6-wheel subgraph is contracted into star type, then $|C_4^0(\zeta_-(G))| = 2$.

3. If after contracting 6-wheel operation based on $f$, the 6-wheel subgraph is contracted into up and down triangles type, then $|C_4^0(\zeta_-(G))| = 4$. 

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Figure 5.14. 6-wheel induction subgraph.

Proof. Let the four-color set of graph $G$ is $\{1 – \text{red}, 2 – \text{blue}, 3 – \text{yellow}, 4 – \text{green}\}$, and let the definition of $G[i,j]$ is the same as in theorem 5.5, where $i \neq j$ and $i, j = 1, 2, 3, 4$. Similarly, $G[i,j]$ is a tree.

1. If after the contracting 6-wheel operation based on $f$, the 6-wheel subgraph is contracted into line type, then consider the contracting 6-wheel operation of the first and the second graphs in Figure 5.12(b), it is the graph resulted by deleting the red vertex (color 1) and merging two non-adjacent blue (color 2) vertices and two non-adjacent yellow (color 3) vertices. Therefore, in $\zeta_v(G)$, there must be a circle with alternative blue and yellow vertices, as shown in figure 5.15(a),(b). Similar to the proof of theorem 5.3, we have $|C_4^0(\zeta_v(G))| = 2$.

2. If after the contracting 6-wheel operation based on $f$, the 6-wheel subgraph is contracted into star type, then consider the contracting 6-wheel operation of the second and the third graphs in Figure 5.12(c), it is the graph resulted by deleting the red vertex (color 1) and merging three non-adjacent yellow (color 3) vertices. Therefore, in $\zeta_v(G)$, there must be a circle with alternative green and yellow vertices, as shown in figure 5.15(c),(d). Similar to the proof of theorem 5.3, we have $|C_4^0(\zeta_v(G))| = 2$.

3. If after the contracting 6-wheel operation based on $f$, the 6-wheel subgraph is contracted into up and down triangles type, then consider the contracting 6-wheel operation of the second and the third graphs in Figure 5.12(d), it is the graph resulted by deleting the red vertex (color 1) and merging two non-adjacent green (color 2) vertices. Therefore, in $\zeta_v(G)$, there must be a circle with alternative green and yellow vertices and a circle with alternative green and blue vertices, as shown in figure 5.15(e). Similar to the proof of theorem 5.3, we have $|C_4^0(\zeta_v(G))| = 4$. 
Let graph $G$ be a 4-colorable maximal planar graph, pick a coloring $f^*$ of $C^4_0(G)$, we use $\zeta_v^-(f^*, G)$ to denote the graph resulted by doing contracting wheel operation on vertex $v$ of graph $G$ based on coloring $f^*$. And for graph $\zeta_v^-(f^*, G)$, it is resulted by doing contracting 6-wheel operation on vertex $v$ based on coloring $f^*$. Because of this, we introduce the recover of extending $k$-wheel operation on vertex $v$ based on coloring $f^*$, in short, recover-extending $k$-wheel operation, that is, recover the structure of graph $\zeta_v^-(f^*, G)$ back to the original graph $G$, and the coloring of $G$ is $f^*$. More precisely, as shown in figure 5.11, by doing contracting 5-wheel operation on the graph shown in figure 5.11(a), we get the graph shown in figure 5.11(b); in graph of figure 5.11(b), do extending 5-wheel operation on the funnel subgraph induced by vertex subset \{v_1, v_2 = \{v_2, v_5\}, v_3, v_4\}, and the color of vertex $v$ is green.

Obviously, we have the following fact:

**Proposition 5.3.** In a maximal planar graph $G$, the graph resulted by doing contracting $k$-wheel operation $m$ times based on coloring $f^*$ is $G'$. Then do it reversely, by $m$ times recover-extending $k$-wheel operations based on coloring $f^*$, we get the original graph $G$.
6 A necessary and sufficient condition for the uniquely 4-colorable planar graphs—the solution of FWF and JT conjectures

This section, as a key to this paper, describes the characters for the uniquely 4-colorable planar graph. A sufficient and necessary condition for uniquely 4-colorable graphs is given, namely the graph $G$ is a recursive maximal planar graph. This work not only solves the Fiorini-Wilson-Fisk conjecture and the equivalent open conjecture proposed by Jensen-Toft on uniquely 4-colorable graph, but also is the foundation to our proof of the four color theorem in next section.

6.1 Advances on the uniquely 4-colorable planar graph

6.1.1 General results about unique vertex coloring

The definition of uniquely colorable graph came from one paper of Cartwright and Harary in 1968 \[20\], as well as a 1967 paper of Gleason and Cartwright \[35\]. A graph is uniquely colorable if there is exactly one partition of vertex set into color classes. Cartwright and Harary established some sufficient conditions for a labeled graph to have an unique coloring. They also showed that if graph $G$ has an unique coloring with $t$ colors, then $t = \chi(G)$, where $\chi(G)$ is the chromatic number of $G$, the smallest positive integer $s$ for which there is a coloring of $G$ using exactly $s$ colors. Since that, many useful work has been made in this field.

Some useful and basic properties on uniquely $k$-colorable graph are given by Harary, Hedetniemi and Robinson in 1969 \[40\], which are shown as follows:

1. If the graph $G$ is uniquely vertex colorable, then for all $i \neq j, i, j \in 1, 2, \ldots, k$, the graph $G_{i,j}$ is connected, where $G_{i,j}$ is the induced subgraph of two color class $V_i$ and $V_j$.

2. Every $k$-chromatic graph of order $n$ is $(k-1)$-connected, and it has at least $(k-1)n - \frac{1}{2}k(k-1)$ edges.

3. For all $k \geq 3$, there exists an uniquely $k$-colorable graph which does not contain any subgraphs isomorphic to $K_k$.

4. If $G$ is uniquely $k$-colorable, $H$ is the homomorphic image of $G$ such that $\chi(H) = k$, then $H$ is uniquely $k$-colorable.

Naturally, we may propose the problem whether there exists a uniquely $k$-colorable graph without $K_3$, for $k \geq 3$. Until now this problem have not been solved.

In 1972, Nešetřil studied the properties of critical uniquely colorable graphs \[52, 53\]. Investigations into edge-critical uniquely colorable graphs were carried out by Müller \[50, 51\], Aksionov \[2\] and Melnikov & Steinberg \[49\].
In 1973, Wang and Artzy [73] obtained a useful conclusion that “For \( k \geq 3 \), if there exists a uniquely \( k \)-colorable graph which does not contain any subgraph isomorphic to \( K_3 \), the number of vertices should be strictly greater than \( k^2 + k - 1 \).”

In 1974, Osterweil [56] constructed a kind of uniquely 3-colorable graphs, which is called 6−cliquerings. Moreover, he demonstrated how these techniques used to produce uniquely \( k \)-colorable graphs \( k > 3 \).

In 1976, Bollobas and Sauer [13] proved that “For all \( k \geq 2, g \geq 3 \), there exists an uniquely \( k \)-colorable graph with the girth of at least \( g \).” The concept of the critically uniquely \( k \)-colorable graph was introduced by Tutte and Erdos. They obtained the result that there is a critically uniquely \( k \)-colorable graph with at least \( n \) vertices. Bollobas’s result [12] was generalized by Dmitriev [22] in 1980.

In 1982, Dmitriev [23] gave some useful properties of uniquely colorable graph, such as the minimum degree, the number of edges.

In 1990, S.J Xu [80] proved that “if \( G \) is a uniquely \( k \)-colorable graph with order \( n \) and size \( m \), then \( m \geq (k-1)n - \frac{1}{2}k(k-1) \), which is the best possible bound”. And he gave a famous conjecture that ”If \( G \) is a uniquely \( k \)-colorable graph with order \( n \) and size \( (k-1)n - \frac{1}{2}k(k-1) \)”, then \( G \) contains a \( K_k \) as its subgraph. At that time, Chao and Chen [17] showed that ”For each integer \( n \geq 12 \), there exists a uniquely 3-colorable graph of order \( n \) without any triangles”.

In 2001, Akbari, Mirrokni and Sadjad negated Xu’s conjecture [1]. They gave that “There exists a \( K_3 \)-free uniquely 3-colorable graph \( G \) with 24 vertices and \( SH(G) = 45 \)” edges, where \( SH(G) = (k-1)n - \frac{1}{2}k(k-1) \).

### 6.1.2 Uniquely edge colorable graphs

Uniquely edge colorable graphs are introduced by Greenwell and Kronk [38] in 1973. They proposed an influential conjecture that “If \( G \) is a uniquely edge-3-colorable cubic graph, then \( G \) is a planar graph with a triangle”. In 1975, Fiorini [25] independently studied the problem on uniquely edge colorable graphs. He got some similar results like Greenwell and Kronk. After that, this class of graphs was discussed detailedly by Thomason [66, 67], Fiorini and Wilson [26], C.Q Zhang [81], Goldwasser&C.Q Zhang [36, 37] and Kriesell [47].

### 6.1.3 Uniquely colorable planar graphs

In 1969, Chartrand and Geller [18] started to investigate uniquely colorable planar graphs. They showed that “Uniquely 3-colorable planar graphs with at least four vertices contain at least two triangles, uniquely 4-colorable planar graphs are maximal planar graphs, and uniquely 5-colorable planar graphs do not exist”.

For convenience, a useful theorem is listed as follows:
Theorem 6.1. If $G$ is a uniquely 4-colorable planar graph, then it is a maximal planar graph. (This result is the same as the Lemma 4.5, for completeness of this section, we repeat it here.)

Naturally, some problems on maximal planar graphs are proposed, such as the basic characters of uniquely 4-colorable planar graphs, the necessary and sufficient condition of uniquely 4-colorable planar graphs. These problems are the primary concerns about this class of graphs. Many scholars have made important contributions in this field\[11, 21, 79, 80\].

In 1977, Fiorini and Wilson \[26\] gave the following conjecture:

Conjecture 6.1. Every 3-edge-colorable cubic planar graph with at least 4 vertices contains a triangle.

In the same year, an equivalent form was proposed as an unsolved problem by Fisk\[28\], so this conjecture is called the Fiorini-Wilson-Fisk Conjecture. More work on this problem can be found in the literature\[28, 29, 30, 31, 32, 34\]. Especially, Thomas gave a computer-assisted proof of this conjecture\[65\], where the method was similar as that of the four color problem. However, a strict and exact mathematical proof can not be found until now.

Uniquely 4-colorable planar graphs had attracted the interests of Jensen and Toft, whose early work had mentioned the characters of uniquely 4-colorable planar graphs. In 1995, they gave a structural characterization of uniquely 3- and 4-colorable planar graphs \[44\].

From 1969, Chartrand and Geller \[18\] made important works on unique coloring of planar graph, especially for the unique colorings of 3-colorable, 4-colorable and 5-colorable graphs. Some important results were obtained as follows:

Theorem 6.2. Let $G$ be a planar graph whose chromatic number $\chi(G) = 3$. If $G$ contains a triangle $T$ such that for any vertex $v$ in $G$ there is a triangle sequence $T, T_1, T_2, \ldots, T_m$, in which adjacent triangles have a common edge and $v \in T_m$, then $G$ can be uniquely 3-colorable.

Theorem 6.3. If a graph $G$ is a uniquely 3-colorable planar graph with at least 4 vertices, then it contains at least two triangles.

Until now, the basic characters of uniquely 3-colorable graph is still unclearly. The work on uniquely 5-colorable graph is shown as follows:

Theorem 6.4. There exists no uniquely 5-colorable planar graphs.

However, we did not know more about whether all the maximal planar graphs are uniquely 4-colorable, and what is the basic character of uniquely 4-colorable graphs.

In 1995, Jensen and Toft gave the following conjecture\[44\]:

Conjecture 6.2. Every uniquely 4-colorable planar graph is a recursive maximal planar graph. That is, all the uniquely 4-colorable planar graphs of order $n$ belong to $\Lambda_n$, denoted by $G \in \Lambda_n$. $\Lambda_n$ denotes the set of recursive maximum planar graphs with $n$ vertices.
The recursive maximum planar graph can be obtained from $K_4$, embedding new vertices of degree 3 repeatedly in triangular faces. More details can be found in next section.

In fact, an equivalent form was proposed by Fiorini and Wilson in 1977 [26].

**Conjecture 6.3.** Every uniquely 3-edge colorable cubic planar graph with at least four vertices contains a triangle.

In the same year, this conjecture was also proposed by Fisk [28]. Therefore, this conjecture is called the Fiorini-Wilson-Fisk conjecture (or simply the FWF conjecture). Easy to prove, these two conjectures are equivalent to each other.

In 1998, Böhm, Stiebitz and Voigt [11] proved that the minimum counterexample for conjecture 6.3 was 5-connected.

The FWF conjecture was proved by Thomas in 1998 by a computer-assisted method similar as in the proof of four color conjecture in 1997. In this paper, the structure of a recursive maximal planar graph is studied in depth. The idea of counting by partition of color class is used to prove that the necessary and sufficient condition of a uniquely 4-colorable planar graph $G$ is that it is a recursive maximal planar graph. In this way, we can obtain a short logical and mathematical proof for the Jensen-Toft conjecture, and also solve the FWF conjecture naturally.

### 6.2 Recursive maximal planar graph (FWF graph)

In this section, we will prove that the necessary and sufficient condition for a uniquely 4-colorable planar graph $G$ is being a recursive maximal planar graph. The foundation to attack this conjecture is to analyze the properties of recursive maximal planar graphs. This kind of graph is also called the FWF graph. Here we introduce one class of graphs called the (2,2)-FWF graph to prove this conjecture. Below, we give some related properties of the FWF graph, especially for the (2,2)-FWF graph.

#### 6.2.1 Basic properties

**Theorem 6.5.** If $G$ is a FWF graph of order $n$, then it has at least two vertices of degree 3. And when $n \geq 5$, any two vertices of degree 3 are not adjacent to each other.

*Proof.* Here we prove it by induction on the number of vertices. When $n = 4, 5$, $\gamma_4 = \gamma_5 = 1$; And the corresponding graphs are shown in the Figure 2.2.

So the theorem holds obviously. Assume that it holds when the number of vertices is $n$. That is, for any FWF graph $G$ with $n$ vertices, it has at least two 3-degree vertices, and all the vertices of degree 3 are not adjacent each other. Now we consider the case that the number of vertices is $n + 1$.

If a graph $G \in \Lambda_{n+1}$, then $G$ is constructed by adding a 3-degree vertex $v$ in any triangular face of a FWF graph with $n$ vertices. If $G$ has only one vertex of
degree 3, let $\Gamma_G(v) = \{v_1, v_2, v_3\}$, then $G - v$ should be a FWF graph of order $n$. It has at least two vertices of degree 3, and all of them are not adjacent to each other. This comes to a contradiction. Since two vertices of degree 3 in $G - v$ must be any two in $\{v_1, v_2, v_3\}$, but they are adjacent to each other. It means that there exists at least one vertex $u \notin \{v_1, v_2, v_3\}$ in $G - v$. The vertex $u$ is a 3-degree vertex. Naturally, $v$ is another 3-degree vertex, which is not adjacent to $v$. So this theorem holds.

**Theorem 6.6.** (1) There exists no maximal planar graph exactly having two adjacent vertices of degree 3.

(2) There exists no maximal planar graph exactly having three vertices of degree 3, and each two of them are adjacent.

**Proof.** Assume that $G$ is a maximal planar graph, where $u, v \in V(G)$, $d(u) = d(v) = 3$, and $uv \in E(G)$. Since vertex $u$ is also a 3-degree vertex, $\Gamma(u) = \{v, x, y\}$. Notice that $G$ is a maximal planar graph, vertex $u$ must be in the triangular face which consists of the vertices $v$, $x$, and $y$. In other words, vertex $v$ is adjacent to vertices $x$ and $y$. These four vertices can form a subgraph $K_4$ (shown in Figure 6.1). Since $G$ is a maximal planar graph, if there exist any other vertex, then it can form a triangle with $u$ or $v$. It contradicts the fact that $d(u) = d(v) = 3$. Otherwise, if there exists no other vertices, then $G$ is isomorphism to $K_4$ with four vertices of degree 3. Therefore, there exists no maximal planar graph with two adjacent vertices of degree 3.

![Figure 6.1. The schematic for the proof of Theorem 6.6](image)

Similarly, we can conclude that there exists no maximal planar graph having exactly three vertices of degree 3, and each two of them are adjacent. Let $u, v, x \in V(G)$, $d(u) = d(v) = d(x) = 3$, and $uv, ux, vx \in E(G)$. There exist three vertices in $\Gamma(u)$, two of which are $v$ and $x$. Let $y$ denote the other vertex adjacent to $u$. So $\Gamma(u) = \{v, x, y\}$. Since a face can be constructed with three vertices $v$, $x$, and $y$, each two of them are adjacent. They can form an induced subgraph $K_4$ as shown in Figure 6.1. Since $G$ is a maximal planar graph, if there exists any other vertex, it can form triangles with vertices $u$, $v$, or $x$. It contradicts with the fact that $d(u) = d(v) = 3$. Otherwise, $G \cong K_4$. Obviously, $K_4$ contains four vertices of degree 3. Thus, there exists no maximal planar graph with three vertices of degree 3, any two of which are adjacent.

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Theorem 6.7. Let $G$ be a maximal planar graph having only one vertex of degree 3. A subgraph without any 3-degree vertex can be obtained by deleting 3-degree vertices repeatedly.

Proof. Let $v$ be a unique vertex of degree 3 in graph $G$, and $\Gamma_G(v) = \{u_1, u_2, u_3\}$. Thus, these three vertices can form a triangle, each two of which are adjacent. Let $G_1$ denote $G - v$. So it is also a maximal planar graph. There are four cases may exist as follows:

1. $\delta(G_1) \geq 4$;
2. There exists only one 3-degree vertex;
3. There exist two 3-degree vertices;
4. There exist three 3-degree vertices.

For case (1), the theorem holds naturally. For case (3) and (4), we know they do not exist by Theorem 6.6. So we just need to consider about case (2). In this case, there exists a 3-degree vertex in subgraph $G_1$, denoted by $v_1$. Let $G_2 = G_1 - v_1$. Like the method mentioned above, if $\delta(G_2) \geq 4$, then the theorem holds. Otherwise, the graph $G_2$ must contain a 3-degree vertex. In this way, we can get $\delta(G_m) \geq 4$ in finite steps $m$. Otherwise, $G_m \cong K_4$ if $G_m$ contains only four vertices. It means the graph $G$ is a FWF graph. But it contains only one 3-degree vertex. It is contradicted with Theorem 6.5. Therefore, this theorem holds.

6.3 The (2,2)-FWF graph

In this section, we introduce and study the (2,2)-FWF graph, which is a special kind of the FWF graph. The FWF graph is called the (2,2) -FWF graph if it contains only two vertices of degree 3, and the distance between them is 2. To understand the structure of a (2,2)-FWF graph, the complete graph $K_4$ is divided into three regions, and then its vertices are labeled correspondingly. As is shown in Figure 6.2, the triangle is called the outside triangle when its vertices are labeled by 1, 2, 3, and the vertex $u$ (also labeled by 4) is called the central vertex. Here we define that the vertex 1–4 are colored by yellow, green, blue and red respectively. The four vertices and their corresponding colorings are called the basic axes in the color coordinate of a (2,2)-FWF graphs. Four color axes are 1 (yellow), 2 (green), 3 (blue) and $u$ (red). Obviously, there exists no (2,2)-FWF graph of order 4; and there is only one (2,2)-FWF graph with 5 vertices under isomorphism, which can be obtained by embedding a 3-degree vertex in the region I, II or III of the graph $K_4$(shown in Figure 6.2).
Figure 6.2. The basic framework of the color coordinates.

Without loss of generality, new vertices are added in the region $II$. Thus, the vertex is colored by yellow (Figure 6.3 (a)); the non-isomorphic $(2,2)$-FWF graphs of 6-order can be obtained by embedding a 3-degree vertex in any region of the $(2,2)$-FWF graph of 5-order. It is easy to prove that this kind of graphs with 6 vertices obtained by embedding new vertices in any face is isomorphic. Therefore, the $(2,2)$-FWF graphs with 6 vertices are also unique. In general, we make an agreement that the 6th vertex is embedded in the face composed of the vertices 2, 4, 5 (i.e. the sub-region $I$ of the region $II$), which is colored by blue. (Figure 6.3 (b)). Apparently, the agreement makes that new vertices are added in the region $I$ and $II$, but not in the region $III$ of the $(2,2)$-FWF graphs of higher order.

Figure 6.3. Two $(2,2)$-FWF graphs. (a) a graph of order 5; (b) a graph of order 6.

Based on the definitions above, we discuss about the classification of $(2,2)$-FWF graphs. Two methods are introduced as follow.

The first is based on the region where the 3-degree vertices are embedded: (1) The $(2,2)$-FWF graphs are obtained by successively embedding the 3-degree vertices only in the region $II$. The graphs belong to this type are shown in Figure 6.4; (2) They are obtained by successively and randomly embedding the
3-degree vertices in the region \( I \) and \( II \), as shown in Figure 6.5. An easily known fact about planar graphs is:

**Proposition 6.1.** [54] *Any face in the (maximal) planar graph can become the infinite outside face.*

That is, the (2,2)-FWF graphs mentioned above are obtained by embedding 3-degree vertices randomly in the region \( I \) and \( II \). We can transform any one 3-degree vertex in the region \( I \) or \( II \) to the outside triangular face by proposition 6.8, which is equivalent to the first classification. It means that this kind of (2,2)-FWF graphs are obtained by successively embedding 3-degree vertices only in the region \( II \). Therefore, we only consider this kind of graphs in the following sections.

**Figure 6.4.** The (2,2)-FWF graphs obtained by embedding 3-degree vertices only in the region \( II \),

(a) the adjacent type; (b) the non-adjacent type.

**Figure 6.5.** The (2,2)-FWF graphs obtained by embedding the 3-degree vertices in the region \( I \) and \( II \) randomly.

The second classification is based on whether there exists a common edge between the two triangular surfaces of two 3-degree vertices. It is called the adjacent type if there is a common edge; otherwise, the non-adjacent type. As shown in Figure 6.4, the first graph belongs to the adjacent type, while the last two graphs belong to the non-adjacent type.
**Theorem 6.8.** $\gamma_5 = \gamma_6 = 1, \gamma_n = |\Lambda_n| = 2^{n-7} + 1, n \geq 7$

*Proof.* $\gamma_5 = \gamma_6 = 1$ is known by above paragraphs. According to the agreement made above, graph 6.6(a) gives the corresponding vertices coloring for the (2, 2)-recursive maximal planar graph of order 6. Under such agreement, the vertex 7 naturally has 3 possible choices: ① embedding the new vertex 7 on the face of vertices 2, 4, 6 and assigning yellow color to it, as shown in figure 6.6(b); ② embedding the new vertex 7 on the face of vertices 4, 5, 6 and assigning green color to it, as shown in figure 6.6(c); ③ embedding the new vertex 7 on the face of vertices 2, 5, 6 and assigning red color to it, as shown in figure 6.6(d). It is easy to prove that, the two graphs shown in figure 6.6(b) and 6.6(c) are not isomorphic, because there is a 5-degree vertex in figure 6.6(c), but not in figure 6.6(b). At the same time, it is also easy to prove that, the two graphs shown in figure 6.6(c) and figure 6.6(d) are isomorphic and the isomorphic mapping between them is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 5 & 6 & 2 & 4 & 3 & 1 \end{pmatrix}$$

So, there only exist two (2, 2)-FWF graphs with order 7 which are not isomorphic to each other.
From now on, we make an agreement that vertex 4 is selected as a central vertex among two 3-degree vertices, as shown in 6.6(c). The (2, 2)-FWF graphs with order 8 are constructed from two (2, 2)-FWF graphs with order 7. Similar to the proof of (2, 2)-FWF graphs with order 7, we can obtain two (2, 2)-FWF graphs on the basis of figure 6.6(b): one is embedding the vertex 8 on the face of vertices 4, 6, 7 and assigning green color to it, as shown in figure 6.7(a); another is embedding the vertex 8 on the face of vertices 2, 4, 7 and assigning blue color to it, as shown in figure 6.7(b). Concerning the graph shown in figure 6.6(c), it can generate two (2, 2)-FWF graphs with order 8: one is embedding the vertex 8 on the face of vertices 4, 6, 7 and assigning yellow color to it, as shown in figure 6.7(c); another is embedding the vertex 8 on the face of vertices 4, 5, 7 and assigning blue color to it, as shown in figure 6.7(d). Because there is a 6-degree vertex in the neighbors of 3-degree vertex 8 in graph of figure 6.7(d) but not in graph of figure 6.7(c), these two graphs are not isomorphic and $\gamma_8 = 4$.

Moreover, by similar method as above, it is easy to prove that the number of (2, 2)-FWF graphs which are not isomorphic to each other in $\Lambda_n$ is only $2^{n-7}+1$ when $n \geq 8$. 
The following theorem can be deduced easily from the proof process of Theorem 6.8:

**Theorem 6.9.** (1) Every (2,2)-FWF graph $G$ of order $n$ has one and only one $(n - 1)$-degree vertex, it is called the central vertex of the graph $G$, denoted by $u$. And in any partitions of color group in the (2,2)-FWF graph $G$, only the central vertex is colored by red; (2) For the adjacent type of the (2,2)-FWF graphs obtained by embedding the 3-degree vertices only in the region $II$, not only its central vertex is colored by red, but also the vertex 2 of the color axis is colored by green.

### 6.4 The color sequence of a (2,2)-FWF graph

Without loss of generality, we can always assume that the (2,2)-FWF graph $G$ is obtained by embedding the 3-degree vertices only in the region $II$ in the following contents. Thus, a (2,2)-FWF graph $G$ can be uniquely represented by its color sequence. The specific method is shown as follows:
Let \( V(G) = \{1, 2, 3, 4 = u, 5, \ldots, n\} \), where vertex 1 = \( x \) indicates the first fixed 3-degree vertex, while the vertex \( n = y \) indicates the second 3-degree vertex; the vertices 1 = \( x \), 2, 3, and 4 = \( u \) indicate the 1st, 2nd, 3rd, 4th color axis respectively; while the vertex 4 = \( u \) is the central vertex; the vertex \( n − 1 \) signifies the 3-degree vertex of the subgraph \( G_{n-1} = G − n \); the vertex \( n − 2 \) denotes the 3-degree vertex of the subgraph \( G_{n-1} − (n − 1) \); the rest may be deduced by analogy. The sequence \( c_1c_2\ldots c_n \) is used to indicate the corresponding color sequence of the sequence \( (1, 2, 3, 4 = u, 5, \ldots, n) \), and the parameter \( c_i \) is the color of the vertex \( i \) in the \((2,2)\)-FWF graph \( G \). So we can obtain that

\[
c_i \in \{1 = y(yellow), 2 = g(green), 3 = b(blue), 4 = r(red)\}
\]

According to the definition of the \((2,2)\)-FWF graph, we can know that this representation can also determine the structure of a graph uniquely. This structure starts from \( K_4 \) (shown in Figure 6.2), and selects a triangular face embedded the vertices according to the coloring of each vertex.

**Example 6.1.** For the color sequence \( c_1c_2c_3c_4c_5c_6c_7c_8c_9 = ygbrgybgyg \), its corresponding \((2,2)\)-FWF graph can be analyzed easily, shown in Figure 6.8.

![Figure 6.8. A color sequence \( c_1c_2c_3c_4c_5c_6c_7c_8c_9 = ygbrgybgyg \) and its corresponding \((2,2)\)-FWF graph.](image)

For the color sequence of a \((2,2)\)-FWF graph, we can obtain the following theorem:

**Theorem 6.10.** Let \( c_1c_2\ldots c_n \) be the color sequence of a \((2,2)\)-FWF graph. With the agreement in section 6.3, the colors of the first six vertices in this sequence is determined, namely \( c_1 = y, c_2 = g, c_3 = b, c_4 = r, c_5 = y, c_6 = b \); if \( G \) belongs to the adjacent type, then \( c_7 = y \); otherwise, \( c_7 = g \).
6.5 The coloring of the induced graph obtained by extending 4-wheel operation

In this section, we just discuss the vertex coloring problem of the induced graph from a (2,2)-FWF graph by extending 4-wheel operation. We know that a given (2,2)-FWF graph is uniquely 4-colorable, and according to the definition of the color coordinate in the section 6.4, every vertex can be also colored determinately.

**Definition 6.1.** Let \( G \) be a (2,2)-FWF graph, and \( x - u - y \) is a path of length 2 in \( G \). A natural 4-coloring of the induce graph \( G \ast xuy \) means that vertex \( v \) has color different from that of the vertices \( x, y \) and \( u \), and the color of vertex \( u' \) is the same as vertex \( u \), and other vertices have colors unchanged. Obviously, it is a 4-coloring of the graph \( G \ast xuy \), which is called **natural 4-coloring**.

Naturally, one question is proposed whether the induced graph \( G \ast xuy \) obtained by extending 4-wheel operation graph is uniquely 4-colorable. This question is discussed as a key problem in this section. The answer is negative, that is, \(|C_4^0(G \ast xuy)| > 1\). Here the definition of the color neighborhood is introduced as follows:

**Definition 6.2.** Let \( G \) denote a \( k \)-colorable graph, and \( f \in C_k(G), u \in V(G) \). The **color neighborhood** of vertex \( u \) on coloring \( f \) is a set which contains the colors of all vertices adjacent to vertex \( u \), denoted as \( C(f, \Gamma(u)) \).

**Theorem 6.11.** Let \( G \) be a (2,2)-FWF graph with \( n \) vertices. The vertices \( x, y \) are two vertices of degree 3 and the vertex \( u \) is the central vertex. Then, the induced graph \( G \ast xuy \) is not uniquely 4-colorable.

**Proof.** Obviously, if vertex \( x \) and \( y \) are in the same color class, since the vertices \( u \) and \( u' \) are colored by red, the vertex \( v \) in the graph \( G \) has two possible colors to choose. Hence, the graph \( G \ast xuy \) is not uniquely 4-colorable. So we only need to consider the case that \( x \) and \( y \) belong to different color classes.

According to the classification in the subsection 6.4, all the (2,2)-FWF graphs can be classified into two types: the adjacent type and non-adjacent type both only in the region II. We treat them respectively as follows:

**Case 1 :** the adjacent type only in the region II.

For the graph \( G \) belong to the adjacent type only in the region II, based on the theorem 6.9, we know that vertex \( x, 2, 3 \) are coordinate axes \( 1, 2, 3 \), colored by yellow, green and blue respectively; and the central vertex \( u \) is colored by red. Since all the 3-degree vertices can only be embedded in the subregion I of the region II, the vertex \( y \) can be colored by yellow or blue, illustrated by Figure 6.9(a). But when the vertex \( y \) is colored by yellow, which is the same with vertex \( x \), this case is not needed to consider about. So we only discuss the case that the vertex \( y \) is colored by blue. With the definition of extending 4-wheel operation, for (2,2)-FWF graph \( G \) belongs to the adjacent type only in region II, we can do an extending 4-wheel operation on the path \( x - u - y \), result in a graph \( G \ast xuy \).
Figure 6.9. A graph belong to adjacent type only in region $II$ and two colorings of its induced graph by extending 4-wheel operation.

We can easily obtain two colorings of this graph: One is the natural 4-coloring given by definition 6.1, in which the vertex $u'$ is colored by red and the vertex $v$ embedded newly is colored by green. And the colorings of other vertices remain unchanged. Obviously, it is a coloring of the graph $G \ast xuy$, illustrated in figure 6.9(b). Besides, by the discussion above, in the natural 4-coloring of $G \ast xuy$, in the color neighborhood of the central vertex $u$, the green color class has only one vertex $v$; and in the color neighborhood of the vertex $u'$, the yellow color class has only one vertex $x$; the vertices $u$ and $u'$ both have red colors in the graph $G \ast xuy$. Therefore, in the natural 4-coloring of $G \ast xuy$, let the vertices $u, u', x, y$ and $v$ be colored by green, yellow, red, red and blue respectively; other vertices remain unchanged. Clearly, this is a 4-coloring of the graph $G \ast xuy$, for there is only one vertex $v$ of green color in the color neighborhood of the vertex $u$. Then change the color of vertex $u$ from red to green, only vertex $v$ has the same green color. Similarly, there is only one vertex $x$ of yellow color in the color neighborhood of the vertex $u'$, so change the
color of vertex \( u' \) from red to yellow, only vertex \( x \) has the same yellow color. Then these two non-adjacent vertices \( x \) and \( y \) are colored by red, the vertex \( v \) is colored by blue instead of green, and other vertices remain unchanged. It is another 4-coloring of the graph \( G * xuy \), as shown in Figure 6.9(c). These two colorings are different apparently. Hence, the case 1 is proved.

**Case 2:** Non-adjacent type only in region II.

According to Theorem 6.10, the first six vertices of all the \((2,2)\)-FWF graphs are colored in the same way, illustrated as follows:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
y & g & b & r & y & b & \ldots \\
\end{pmatrix}
\]

(6.1)

The color coordinate axes vertex 1(or vertex \( x \)), 2, 3 and 4(or vertex \( u \)) have yellow, green, blue, and red colors respectively. Vertex 1 has degree 3 and adjacent vertices of the red central vertex 4, the green vertex 2 and the blue vertex 3. Therefore, vertex 1 has degree 5 in the graph \( G * xuy \), and in the natural coloring has adjacent vertices of the green vertex 2, the blue vertex 3, the red vertex 4(or vertex \( u \)), the red vertex \( u' \) and the blue vertex \( v \) respectively.

Since the graph \( G \) belongs to non-adjacent type, so the vertices 7, 8, \ldots, \( n \) must be added in the triangular face formed by vertices 4, 5 and 6, as shown in the Figure 6.6(c) and Figure 6.11(a). According to Theorem 6.10, the 7th vertex only can be colored by green, illustrated as follows:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
y & g & b & r & y & b & g & \ldots \\
\end{pmatrix}
\]

(6.2)

Then, we can know that the green vertex 2, which is also a coordinate axis, is a 5-degree vertex. The neighborhoods of vertex 2 are vertex 1(yellow), vertex 3(blue), vertex 5(yellow), vertex \( u' \) (red) and vertex 6(blue). They are shown as follows.

\[
C(f, \Gamma(1)) = \{g\{2\}, b\{3, v\}, r\{u, u'\}\}
\]

(6.3)

\[
C(f, \Gamma(2)) = \{y\{1, 5\}, b\{3, 6\}, r\{u'\}\}
\]

(6.4)

Now we take the representative graph in Figure 6.10(a) for example, give a computing process to get a new 4-coloring from the natural 4-coloring (Figure 6.10(b)) of the induced graph \( G * xuy \) by extending 4-wheel operation.

Firstly, in the natural 4-coloring of \( G * xuy \), the vertex 1 is colored by green instead of yellow. By formula (6.3), the two end-vertices are both colored by green, which is a unique edge with so-called pseudo color. Other vertices are colored properly, illustrated by Figure 6.10(c).

Secondly, the vertex 2 is colored by red instead of green. Thus, the edge (1,2) is no longer with pseudo color. But the edge \((u', 2)\) becomes pseudo color, for these two end-vertices are colored by red. Other vertices are all proper, illustrated by Figure 6.10(d).

Thirdly, the vertex \( u' \) is colored yellow instead of red. Thus, the pseudo color edge \((u', 2)\) becomes proper. There may be several vertices colored by yellow in
the color neighborhood of vertex $u'$, which can form a set $C_4(u', \text{yellow})$. Therefore, this step generates several pseudo color edges whose number is $|C_4(u', \text{yellow})|$. Obviously, other edges are all proper, illustrated by Figure 6.10(e).

Fourthly, all the vertices in $C_4(u', \text{yellow})$ are changed to red color. Since in the 4-coloring of the third step, only two vertices $u$ and 2 are colored by red. Obviously, $C_4(u', \text{yellow}) \subset \Gamma(u')$. So all the vertices in $C_4(u', \text{yellow})$ are not adjacent to the vertex $u$. In the vertex set adjacent to vertex 2 after the third step, the vertices 1, 3 and 6 are colored by green, blue and blue. Although the vertex 5 is colored by yellow, it is a 5-degree vertex and not in $C_4(u', \text{yellow})$. Therefore, the edges between vertex 2 and all the red vertices in $C_4(u', \text{yellow})$ are not pseudo. Moreover, the vertices in $C_4(u', \text{yellow})$ form an independent set of the graph. So they can not generate pseudo color edges by themselves. This step is illustrated by Figure 6.10(f).

Thus, based on the natural 4-coloring of the graph $G * xuy$, we can obtain a new coloring different from the natural 4-coloring. It means that the induced graph constructed from non-adjacent $(2,2)$-FWF graph by extending 4-wheel operation is not uniquely 4-colorable.

To sum up the case 1 and 2, this theorem is proved.

(a): A representative $(2,2)$-FWF $G$. 
(b): The natural 4-coloring of the induced graph $G * xuy$ by extending 4-wheel operation.

(c): The coloring of the induced graph $G * xuy$ when vertex 1 is colored by green instead of yellow, which generates a pseudo color edge $\{1,2\}$. 
(d): The coloring of the induced graph $G * xuy$ based on the step (c) when vertex 2 is colored by red instead of green, which generates a pseudo color edge \{u',2\}.

(e): The coloring of the induced graph $G * xuy$ based on the step (d) when
vertex $u'$ is colored by yellow instead of red, which has several pseudo color edges $\{\{u', e''\}, u'' \in C_4(u', yellow)\}$.

(f): Another new coloring of the induced graph $G * xuy$ based on the step(e) when the vertices in the set $C_4(u', yellow)$ are colored by red.

Figure 6.10. A graph belong to adjacent type only in region II and two colorings of its induced graph by extending 4-wheel operation.

6.6 A necessary and sufficient condition for the uniquely 4-colorable planar graph

On the basis of the previous sections, we now characterize the uniquely vertex-4-colorable planar graphs, and give a positive proof of the Jensen-Toft conjecture, and naturally also a positive proof of the FWF conjecture.

Theorem 6.12. Let $G$ be a 4-colorable maximal planar graph with minimum degree $\delta(G) = 4$, then $G$ is not uniquely 4-colorable.

Proof. By induction on the number of vertices $n$. The minimum graph which satisfies the precondition is the regular octahedron, as shown in the Figure 6.11. It is 3-colorable, so can not be uniquely 4-colorable.
Assume that the conclusion holds for any graph with \( n \) vertices \( (n \geq 6) \), we need to consider the case when the number of vertices is \( n + 1 \). Let \( G \) be a maximal planar graph with \( n + 1 \) vertices, which is 4-colorable and \( \delta(G) \geq 4 \). Two cases are to be considered as follows:

Let \( v \in V(G) \), \( d(v) = 4 \) and \( \Gamma(v) = \{v_1, v_2, v_3, v_4\} \). Notice that \( G[\Gamma(v)] \) denotes the graph \( G \) in our deduction below. Here we analyze the chromatic polynomial of the graph \( G \).

By Theorem 3.7, we can obtain that

\[
f(G, 4) = f((G - v) \circ \{v_1, v_3\}, 4) + f((G - v) \circ \{v_2, v_4\})
\]

Let \( G_1 = (G - v) \circ \{v_1, v_3\}, G_2 = (G - v) \circ \{v_2, v_4\} \). The graph \( G \) is 4-colorable, so

\[
f(G, 4) = f(G_1, 4) + f(G_2, 4) > 0
\]

Because of \( f(G_i, 4) \geq 0 \), \( i = 1, 2 \), there are two possible cases to be considered, which are listed as follows:

**Case 1:** \( f(G_i, 4) > 0 \), \( i = 1, 2 \).

In this case, without loss of generality, assume that for all colorings in \( C_4(G) \), either vertex \( v_1 \) and \( v_3 \) or vertex \( v_2 \) and \( v_4 \) are in different colors, since otherwise \( G \) is not uniquely 4-colorable. There are two different partitions of color class for \( C_4(G) \), one is \( \{v_2, \ldots, \}, \{v_4, \ldots, \}, \{v_1, v_3, \ldots, \} \) and \( \{v, \ldots, \} \) with respect to graph \( G_1 \); the other one is \( \{v_1, \ldots, \}, \{v_3, \ldots, \}, \{v_2, v_4, \ldots, \} \) and \( \{v, \ldots, \} \) with respect to graph \( G_2 \). Obviously, these two kinds of partitions are different. Hence, the theorem holds.

**Case 2:** \( f(G_1, 4) > 0 \), \( f(G_2, 4) = 0 \); or \( f(G_2, 4) > 0 \), \( f(G_1, 4) = 0 \).

For all \( f \in C_4(G) \), the condition \( f(G_2, 4) = 0 \) holds only if \( f(v_2) \neq f(v_4) \). Here only the graph \( G_1 \) needs to be considered. We analyze the degrees of all the vertices in \( G_1 \) and \( G \). They are shown as follows:

\[
d_{G_1}(v_2) = d_G(v_2) - 2;
\]
\[
d_{G_1}(v_4) = d_G(v_4) - 2;
\]
\[
d_G(v_2) = d_G(v_1) + d_G(v_3) - 4;
\]
\[ d_{G_1}(v_i) = d_G(v_i), i = 5, 6, \ldots, n; \]

Since \( \delta(G) \geq 4 \), \( d_{G_1}(v_2') = d_G(v_1) + d_G(v_3) - 4 \geq 4 \); for two vertices \( v_2, v_4 \)
in \( G_1 \), owing to \( d_G(v_2) \geq 4 \) and \( d_G(v_4) \geq 4 \), we can obtain that

\[ 2 = 4 - 2 \leq d_G(v_2) - 2 = d_{G_1}(v_2), 2 = 4 - 2 \leq d_G(v_4) - 2 = d_{G_1}(v_4). \]

Hence, there exist 4 subcases as follows:

**Subcase 2.1:** \( d_{G_1}(v_2) \geq 4, d_{G_1}(v_4) \geq 4; \)

**Subcase 2.2:** \( d_{G_1}(v_2) = 2 \) or \( d_{G_1}(v_4) = 2; \)

**Subcase 2.3:** \( d_{G_1}(v_2) = 3, d_{G_1}(v_4) \geq 4 \) or \( d_{G_1}(v_2) \geq 4, d_{G_1}(v_4) = 3; \)

**Subcase 2.4:** \( d_{G_1}(v_2) = 3, d_{G_1}(v_4) = 3; \)

Now we analyze these four subcases mentioned above respectively.

**Subcase 2.1:** Since \( d_{G_1}(v_2) = d_G(v_2) - 2 \geq 4, d_{G_1}(v_4) = d_G(v_4) - 2 \geq 4, \)
The graph \( G_1 \) is a maximal planar graph with \( n-2 \) vertices, in which the degrees of every vertex are equal to or larger than 4. Since \( n-2 < n \), by the induction hypothesis, the graph \( G_1 \) is not uniquely 4-colorable. It can be easily proved that the following process can obtain a 4-coloring of \( G \) from each 4-coloring of \( G_1 \) in a reverse order \((d) \rightarrow (e) \rightarrow (b)\) (shown in Figure 6.10). The process starts from adding a new vertex \( v \) to the graph \( G_1 \), then put it adjacent to \( v_2, v_4 \) and \( v_2' \), color the vertex \( v \) with a color (say red) different from these of \( v_2, v_4 \) and \( v_2' \). Then, split the vertex \( v_2' \) into two vertices \( v_1 \) and \( v_3 \) as shown by Figure 6.10(b), and make them adjacent to the vertex \( v \), thus we obtain a coloring of the graph \( G \). Obviously, different colorings of the \( G_1 \) lead to different colorings of the graph \( G \), this proves that the theorem holds for graphs of order \( n+1 \) in this subcase.

**Subcase 2.2:** If any one of the degrees \( d_{G_1}(v_2) = d_G(v_2) - 2 \) and \( d_{G_1}(v_4) = d_G(v_4) - 2 \) is equal to 2, the graph \( G_1 \) is not uniquely colorable, the process is the same as the reverse order \((d) \rightarrow (e) \rightarrow (b)\) in Figure 6.10. Thus, the theorem holds for this subcase.

**Subcase 2.3:** If exact one of the degrees \( d_{G_1}(v_2) = d_G(v_2) - 2 \) and \( d_{G_1}(v_4) = d_G(v_4) - 2 \) is three, by theorem 6.7, by deleting 3-degree vertices repeatedly we can always obtain a subgraph \( G_m \) without 3-degree vertices. Hence, \( \delta(G_m) \geq 4 \). By the induction hypothesis, we use the similar methods as subcases 2.1 and 2.2. Thus, the graph \( G_m \) is not uniquely 4-colorable.

**Subcase 2.4:** If both of the degrees \( d_{G_1}(v_2) = d_G(v_2) - 2 \) and \( d_{G_1}(v_4) = d_G(v_4) - 2 \) are equal to three, then these two vertices \( v_2 \) and \( v_4 \) are not adjacent to each other. When the graph \( G_1 \) is not a FWF graph, the 3-degree vertices are deleted repeatedly until we can obtain a graph where all the vertices are of degrees no less than 4. By the induction hypothesis, the resulted subgraph is not uniquely 4-colorable. We can use a similar method as in above subcases to prove that the graph \( G \) is not uniquely 4-colorable. But when the graph \( G_1 \) is a FWF graph, it contains exact two 3-degree vertices, and the distance between them is 2. Hence, the graph \( G_1 \) is a \((2,2)\)-FWF graph, and the graph \( G \) is the induced graph from the graph \( G_1 \) by the extending 4-wheel operation on the path of \( v_2 - v_2' - v_4 \). So it is not uniquely 4-colorable by the theorem 6.12.

To sum up the above four subcases, this theorem holds.
Let the graph $G$ be a 4-colorable maximal planar graph with order 4 according to Theorem 6.12. A natural question is whether there are more 4-colorings in $C^i_4(G)$, $i = 3, 4, 5$.

Lemma 6.1. Let the graph $G$ be a 4-colorable maximal planar graph, then

$$|C^i_4(G)| \geq |C^i_4(\zeta^+_4(G))|, \quad i = 3, 4, 5.$$

The proof of this lemma is easy, so omitted here.

Theorem 6.13. Let $G$ be a 4-colorable maximal planar graph with the minimum degree $\delta(G) = 4$. If there is no three 4-degree vertices which are not adjacent to each other in $G$ and $|V(G)| \geq 10$, then

$$|C^0_4(G)| \geq 5 \quad (6.3)$$

Proof. By induction on $n$, the number of vertices containing in maximal planar graph $G$ which satisfies the conditions of this theorem.

Note that the minimum order of the maximal planar graph whose $\delta(G) = 4$ is 6, namely the dodecahedral, it is 3-colorable. When the number of vertices is $n = 10$, the number of colorings of all the maximal planar graphs whose $\delta(G) = 4$ satisfy formula (6.3), see appendix I. Note that the numbers of 4-degree vertices in these graphs are 2, 3, 4, 5 and 6 respectively. For maximal planar graphs with only one 4-degree vertex, their the minimum order is 13. The proof is as follows.

From figure 5.5-5.9 we know that, when $6 \leq n \leq 11$, there is no maximal planar graphs with only one 4-degree vertex and $\delta(G) = 4$. When $n = 12$, there is no maximal planar graphs with only one 4-degree vertex and $\delta(G) = 4$. The proof is as follow:

For a maximal planar graph $G$ of order 12, by the Euler’s formula, $\sum_{i=1}^{12} d(v_i) = 2m = 6 \times 12 - 12 = 60$, where $d(v_i)$ denotes the degree of vertex $v_i$ and $m$ denotes the number of edges contained in $G$. If there is only a 4-degree vertex in $G$, it will contain ten 5-degree vertices and one 6-degree vertex. Suppose $v, u \in V(G)$ and $d(v) = 6, d(u) = 4$, $\Gamma(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. If $u \in \{v_1, v_2, v_3, v_4, v_5, v_6\}$, without loss of generality, we may let $u = v_1$, then assume that $\Gamma(v_1) = \{v_2, v_6, v, x\}$, then both vertices $v_2$ and $v_6$ are 6-degree vertex because $v_2 - v_6 - v - x - v_2$ and $u - v_2 - v_3 - v_4 - v_5 - v_6 - u$ form 4-cycle and 6-cycle respectively. In addition, all of vertices in $G$ have degree 5 except vertices $u, v$, so we can assume $\Gamma(v_2) = \{x, u, v, v_3, y\}$, $\Gamma(v_6) = \{x, u, v, v_5, z\}$, where $y, z \notin \{v_1, v_2, v_3, v_4, v_5, v_6\}$. In this way, when make all of the vertices $v_1, v_2, v_3, v_4, v_5, v_6$ be 5-degree, accordingly, it will destroy the planarity of $G$ or yield some 3- or 4-degree vertices. Similarly, if $u \notin \{v_1, v_2, v_3, v_4, v_5, v_6\}$, then there is no such maximal planar graph either. Therefor, if a maximal planar $G$ has only one 4-vertex and $\delta(G) \geq 4$, then the order of $G$ must be greater than 12.

We have constructed a maximal planar graph $G$ with order 13 such that it just contains a 4-degree vertex and satisfies $\delta(G) = 4$. Further more, we have
proved that there is only one such graph and given all of the 4-color group partitions of it, see appendix II.

Suppose that the conclusion holds when the number of vertices \( n \geq 11 \), namely \( |C^0_4(G)| \geq 5 \) for every maximal planar graph \( G \) of order \( n \geq 11 \) and \( \delta G = 4 \).

Now, we consider the graphs \( G \) with order \( n+1 \) satisfying above conditions.

Let \( v \) be a 4-degree vertex of \( G \). We can obtain graph \( \zeta_v^{-}(G) \) by contracting 4-wheel operation about wheel central vertex \( v \) or the combined operation of contracting 4-wheel and 3-wheel or the combined operation of contracting 4-wheel and 2-wheel. The graphs after contracting are always denoted by \( \zeta_v^{-}(G) \).

In graph \( \zeta_v^{-}(G) \), the 4-degree vertices have following two cases.

6.12: The graphic about illustrating \( \delta(\zeta_v^{-}(G)) \geq 5 \).

Firstly, \( \zeta_v^{-}(G) \) has no 4-degree vertex.

Two possibilities lead to this case. One is that there is only one 4-degree vertex in graph \( G \), and the degrees of two vertices in \( \Gamma(v) \) not contracted are no less than 7. Another is that graph \( G \) has many 4-degree vertices which form a path, as shown in figure 6.12, two adjacent degree-4 vertices and three adjacent degree-4 vertices structures respectively. Here, we only consider the situation of two adjacent degree-4 vertices: the degree-4 vertex \( u_2 \) is adjacent to degree-4 vertex \( v \), and \( \Gamma(v,u_2) = \{x,u_1,y,w\} \). When \( d(w) \geq 7 \) and \( d(u_1) \geq 7 \), the degree of vertex \( u_2 \) is 2 in graph \( \zeta_v^{-}(G) \). Thus, we can obtain a maximal planar graph with no 4-degree vertices when we carry out the contracting 2-wheel operation concerning vertex \( u_2 \) on the basis of graph \( \zeta_v^{-}(G) \).

According to the theorem 6.12, there are at least two different 4-colorings for graph \( G \), thus we can conclude that there are also at least two different 4-colorings for graph \( \zeta_v^{-}(G) \). Being aware of that \( \zeta_v^{-}(G) \) is a maximal planar graph with minimum degree 5, that is, a maximal planar graph with minimum degree 5 is not uniquely 4-colorable, this result is just what we want in this section. So no more discussion to this case here.

Secondly, \( \zeta_v^{-}(G) \) has 4-degree vertices.

Many possibilities can lead to this case, we will not enumerate them one by one here.
Note that there will not be three 4-degree vertices not adjacent to each other after doing all kinds of contracting operation, so by the induction hypothesis, there are at least five different 4-colorings for graph $\zeta_v(G)$. Reversely, doing the recover-extending 4-wheel operations (possibly recover-extending 3- and 2-wheel operations) accordingly will obtain the original graph $G$. Note that no extending 5-wheel operation has been done in each recover-extending wheel operation, so from lemma 6.1, the conclusion holds, namely $|C^0_4(G)| \geq 5$.

In summary of above two cases, the conclusion holds when the number of vertices is $n + 1$. The proof is finished.

**Theorem 6.14.** Let $G$ be a 4-colorable maximal planar graph with the minimum degree $\delta(G) = 5$, then $G$ is not uniquely 4-colorable.

*Proof.* By reduction to absurdity. Suppose $G$ is a unique 4-colorable planar graph, and $f^* \in C^0_4(G)$ is the unique 4-coloring of $G$. Firstly, we prove the following fact:

**Fact.** For any maximal planar graph $G$ with the minimum degree 5, there must be a 5-degree vertex or a 6-degree vertex $v \in V(G)$ and corresponding contracting 5-wheel or 6-wheel operation on the basis of $f^*$, such that the resulted graph $\zeta_v(G)$ must has 4-degree vertices, namely $\delta(\zeta_v(G)) = 4$.

This fact mainly stem from Franklin’s work: In 1922, Franklin proved a set of unavoidable configurations as shown in figure 6.13[29], where the unavoidable configurations concerning 2-wheel, 3-wheel and 4-wheel are omitted here. That means, for any maximal planar graph $G$ with minimum degree 5, $G$ must has one of the configurations shown in figure 6.13.

![Unavoidable Sets](https://via.placeholder.com/150)

**6.13:** The unavoidable sets of maximal planar graph with minimum degree 5.

To $x, y, z$, $\bullet$ denotes 5-degree vertices, $\circ$ denotes 6-degree vertices.

To the first configuration (shown in figure 6.13(a)), since vertices $x, y$ or $z$ all have degree 5, it is easy to prove that no matter what coloring is $f^*$ on this configuration, there always be a contracting 5-wheel operation or the combined operation of contracting 5-wheel and 3-wheel such that the resulted graph $\zeta_v(G)$ must has degree-4 vertices, that is $\delta(\zeta_v(G)) = 4$. By theorem 5.3, we have $|C^0_4(\zeta_v(G))| = 2$. However, by theorem 6.13, we have $|C^0_4(\zeta_v(G))| \geq 5$, 

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a contradiction! Therefore, we have proved that the maximal planar graphs containing three 5-degree vertices which are adjacent to each other and with the minimum degree 5 are not uniquely 4-colorable.

To the second configuration shown in figure 6.13(b), where vertices $x, y$ are 5-degree vertex and vertex $z$ is 6-degree vertex, $f^*$ has following two 4-coloring types for this configuration:

The first 4-coloring type is that 4-degree vertices are generated from contracting 5-wheel operation. Specifically, $\zeta^-(G)$ resulted from contracting 5-wheel operation has degree-4 vertices, namely $\delta(\zeta^-(G)) = 4$. The possible colorings of this type have 6 possibilities as shown in figure 6.14.

6.14: The 4-coloring types yielding 4-degree vertices after carrying out contracting 5-wheel operation for 5-5-6 configuration.

6.15: The 4-coloring types unyielding 4-degree vertices after carrying out contracting 5-wheel operation for 5-5-6 configuration.
The second 4-coloring type is that no 4-degree vertex is generated from the contracting 5-wheel operation. Such 4-coloring type only has one possibility as show in figure 6.15(a). From this graph we have two 4-colorings as shown in figure 6.15(b),(c). To graphs in figure 6.15(b),(c), when doing contracting 6-wheel operation on two yellow vertices, there will be two 4-degree vertices (x and y) in resulted graph $\zeta^{-}(G)$. And to graph in figure 6.15(b), if two blue vertices are contracted, there will be one 4-degree vertex (y) in $\zeta^{-}(G)$. By theorem 5.4, we have $|C_{4}^{0}(\zeta^{-}(G))| = 4$. However, by theorem 6.13, we also have $|C_{4}^{0}(\zeta^{-}(G))| \geq 5$, a contradiction! Therefore, we have proved that the maximal planar graphs containing subgraph shown in figure 6.13(b) and with minimum degree 5 are not unique 4-colorable.

To the third configuration as shown in figure 6.13(c), where vertex x is 5-degree vertex and vertices y, z are 6-degree vertex, $f^{*}$ has following two 4-coloring types for this configuration:

The first 4-coloring type is that 4-degree vertices are generated from contracting 5-wheel operation. Specifically, $\zeta^{-}(G)$ resulted from contracting 5-wheel operation has degree-4 vertices, namely $\delta(\zeta^{-}(G)) = 4$. The possible colorings of this type have 2 possibilities as shown in figure 6.16(a),(b).

The second 4-coloring type is that no 4-degree vertex is generated from the contracting 5-wheel operation. Such 4-coloring type only has one possibility as show in figure 6.16(c). When coloring other five uncolored vertices in figure 6.16(c), we will have 13 kinds of 4-colorings, as shown in figure 6.17.
Analyzing the above 13 kinds 4-colorings, it is easy to know that the following results.

Based on the 1st, 2nd, 3rd, 6th, 7th, 8th colorings in figure 6.17, we do the contracting 6-wheel operation concerning the blue vertex $y$ and contract the two vertices colored by green and the two vertices colored by red in $\Gamma(y)$. That is the so-called line type contraction and the vertex $z$ (colored by yellow) in $\zeta_v(G)$ becomes a 4-degree vertex.

6.17: The 4-coloring types unyielding 4-degree vertices after carrying out contracting 5-wheel operation for 5-6-6 configuration.
Based on the 4th, 5th, 9th, 10th colorings in figure 6.17, we do the contracting 6-wheel operation concerning the blue vertex $y$ and contract the two vertices colored by red in $\Gamma(y)$. That is the so-called up and down triangle contraction and the 5-degree vertex $x$ (colored by green) in $\zeta_v(G)$ becomes a 4-degree vertex.

Based on the 11th, 12th, 13th colorings in figure 6.17, we do the contracting 6-wheel operation concerning the blue vertex $y$ and contract the three vertices colored by green in $\Gamma(y)$. That is the so-called star type contraction and the vertex $z$ (colored by yellow) in $\zeta_v(G)$ becomes a 4-degree vertex.

Since the graphs after contracting 5-wheel or 6-wheel operation must contain 4-degree vertices, so by similar proof to previous two cases, we know that the conclusion holds in this case, that is, the maximal planar graphs with minimum degree 5 containing subgraph shown in figure 6.13(c) are not uniquely 4-colorable.

In summary of above cases, this theorem is proved.

**Theorem 6.15.** Let $G$ be a 4-colorable maximal planar graph, then $G$ is uniquely 4-colorable if and only if $G$ is a FWF graph.

**Proof.** If $G$ is a FWF graph, then obviously $G$ is uniquely 4-colorable. So we only need prove the if-only part. By theorem 6.12 and 6.14, if $G$ is a uniquely 4-colorable maximal planar graph, then it must be $\delta(G) = 3$. Now, repeatedly do the contracting 3-wheel operation until the resulted graph $G'$ has no 3-degree vertex, that is $\delta(G') \geq 4$. Note that for graph $G'$ resulted from $G$ by contracting 3-wheel operation, we have $|C_4^0(G')| = |C_4^0(G)|$. Then by theorem 6.12 and theorem 6.13, we have $|C_4^0(G')| \geq 2$. So we have $|C_4^0(G)| \geq 2$, that is, $G$ is not a uniquely 4-colorable, a contradiction! So we know that after repeatedly contracting 3-wheel operations until $|V(G')| = 4$, it must be that $G' \cong K_4$, so $G$ is a FWF graph. The if-only part is finished and the theorem is proved.

### 7 The mathematical proof of the four color theorem

Based on works in section 3, 4, 5 and 6, we give the mathematical proof of four color theorem in this section.

**Theorem 7.1.** Let $G$ be a maximal planar graph. Then, $f(G, 4) > 0$.

**Proof.** By induction on the number of vertices $n$.

When $n = 3, 4, 5$, this theorem holds obviously.

Assume that this theorem holds when the graph order is no less than 5 and no larger than $n - 1$. We consider the case that the order of graphs is $n$. We only consider the simple maximal planar graph. For any maximal planar graph $G$, it satisfies $3 \leq \delta(G) \leq 5$. So we need to consider these three cases based on the minimum degree.

**Case 1:** $\delta(G) = 3$;
Let \( v \in V(G) \), \( d(v) = 3 \), and \( G_1 = G[\Gamma(v)] \), \( G_2 = G - v \). Then we can obtain that
\[
G_1 \cap G_2 = G[\Gamma(v)] \cong K_3
\] (6.1)
Notice that \( G_1 = G[\Gamma(v)] \cong K_4 \), we can obtain the following result by Lemma 3.2:
\[
f(G, t) = f(G_1 \cup G_2, t) = \frac{f(G_1, t) \times f(G_2, t)}{f(K_3, t)} = (t - 3)f(G_2, t)
\] (6.2)
By the hypothesis of induction, \( f(G_2, 4) > 0 \). Thus, \( f(G, 4) = f(G_2, 4) > 0 \).
Hence, the theorem holds when \( \delta(G) = 3 \).

**Case 2:** \( \delta(G) = 4 \);
Let \( v \in V(G) \), \( d(v) = 4 \), \( \Gamma(v) = \{v_1, v_2, v_3, v_4\} \). This case is shown in Figure 3.2. Notice that we use \( G[\Gamma(v)] \) to denote \( G \). By Theorem 3.7, we can obtain that \( f(G, 4) = f(G_1, 4) + f(G_2, 4) \), where \( G_1 = (G - v) \circ \{v_1, v_3\} \) and \( G_2 = (G - v) \circ \{v_2, v_4\} \). It is easy to prove that the graph \( G_1 \) and the graph \( G_2 \) are both maximal planar graphs with \( n - 2 \) vertices. By the induction’s hypothesis, we can obtain:
\[
f(G_1, 4) = f((G - v) \circ \{v_1, v_3\}, 4) > 0
\] (6.3)
\[
f(G_2, 4) = f((G - v) \circ \{v_2, v_4\}, 4) > 0
\] (6.4)
Therefore,
\[
f(G, 4) = f(G_1, 4) + f(G_2, 4) > 0
\] (6.5)
Hence, the theorem holds when \( \delta(G) = 4 \).

**Case 3:** \( \delta(G) = 5 \);
Let \( v \in V(G) \), \( d(v) = 5 \), \( \Gamma(v) = \{v_1, v_2, v_3, v_4, v_5\} \). The adjacent relation is shown in Figure 3.3. By Theorem 3.8, we can obtain
\[
f(G, 4) = [f(G_1, 4) - f(G_1 \cup \{v_1 v_4, v_1 v_3\}, 4)]
\]
\[
+ [f(G_2, 4) - f(G_2 \cup \{v_3 v_1, v_3 v_5\}, 4)]
\] (6.6)
\[
+ [f(G_3, 4) - f(G_1 \cup \{v_4 v_1\}, 4)]
\]
where \( G_1 = (G - v) \circ \{v_2, v_5\} \), \( G_2 = (G - v) \circ \{v_2, v_4\} \) and \( G_3 = (G - v) \circ \{v_3, v_5\} \).
Now we consider about the graph \( G_1 = (G - v) \circ \{v_2, v_4\} \). It is easy to prove the following conclusions:
(1) \( G_1 \) is a maximal planar graph with \( n - 2 \) vertices.
(2) The degrees of the vertex in \( G_1 \) and \( G \) satisfy the following equations:
\[
d_{G_1}(v_1) = d_G(v) - 2;
\]
\[
d_{G_1}(v_2) = d_G(v_2) + d_G(v_5) - 3;
\]
\[
d_{G_1}(v_3) = d_G(v_3) - 1;
\]
\[
d_{G_1}(v_4) = d_G(v_4) - 1.
\]
Therefore, $G_1$ can be 4-colorable by the hypothesis of induction.

Notice that the graph $G_1 \cup \{v_1v_4, v_1v_3\}$ is obtained by only adding two edges $v_1v_3, v_1v_4$ in the graph $G_1$. So if we can prove $\exists f \in C_4(G_1)$ such that $f(v_1) = f(v_3)$ or $f(v_1) = f(v_4)$, then we can obtain

$$f(G_1, 4) > f(G_1 \cup \{v_1v_4, v_1v_3\}, 4) \quad (6.7)$$

Then,

$$f(G, 4) = [f(G_1, t) - f(G_1 \cup \{v_1v_4, v_1v_3\}, t)]$$

$$+ [f(G_2, t) - f(G_2 \cup \{v_3v_1, v_3v_5\}, t)]$$

$$+ [f(G_3, t) - f(G_1 \cup \{v_3v_1\}, t)]$$

where $f(G_2) - f(G_2 \cup \{v_3v_1, v_3v_5\}, t) \geq 0, f(G_3, t) - f(G_1 \cup \{v_4v_1\}, t) \geq 0$. But $f(G_1, t) - f(G_1 \cup \{v_1v_4, v_1v_3\}, t) > 0$. Therefore, $f(G, 4) > 0$.

Now we need to prove that such a coloring $f$ exists.

Assume that $\forall f \in C_4(G_1)$, the values of $f(v_1), f(v_2), f(v_3), f(v_4)$ are different from each other. Let $f(v_i) = i, (i = 1, 2, 3, 4)$. By Theorem 4.2, we can know that there exists a vertex subset $V'$ which contains vertices $v_1, v_2, v_3$ and $v_4$ in the graph $G_1$. And its induced subgraph is uniquely 4-colorable. If the graph $G_1$ is uniquely 4-colorable, that is, $G'_1 = G_1[V'] = G_1$. By Theorem 6.15, we can know that graph $G_1$ is a recursive maximal planar graph. From Theorem 6.5, it at least contains two 3-degree vertices. But for $\delta(G) = 5$ and the analysis above, the graph $G_1$ contains at most one 3-degree vertex($v_1$). It becomes a contradiction. It indicates that it is not correct for the hypothesis that any two values of $f(v_1), f(v_2), f(v_3), f(v_4)$ are all different. Notice that vertex $v_2$ is adjacent to the vertices $v_1, v_3, v_4$ in the graph $G_1$. Thus, $\exists f \in C_4(G_1)$ such that $f(v_1) = f(v_3)$ or $f(v_1) = f(v_4)$.

If the graph $G_1$ is not uniquely 4-colorable, the induced subgraph $G'_1 = G_1[V']$ by $G_1$ is a proper subset of $G_1$. Notice that $G'_1 = G_1[V']$ is an uniquely 4-colorable maximal planar graph. So it is also a FWF graph. Thus, the boundary of $G'_1 = G_1[V']$ in $G_1$ is a triangle(a triangle including the infinite face). Let $u_1, u_2, u_3$ be three vertices of this triangle. Then, $v_1, v_3, v_4 \in \{u_1, u_2, u_3\}$ (shown in Figure 7.1).

Next we will prove that $G_1[V']$ is not a FWF graph actually.

**Case 1:** If $G_1[V']$ at most contains only one 3-degree vertex, it is certainly not a FWF graph;

**Case 2:** There exists at most one 3-degree vertex in $\{u_1, u_2, u_3\}$. Let $u_1$ be the 3-degree vertex. Then, there may exist any vertices of degree 3, denoted by $v_1$. $d(v_1) = 3$. But in the graph $G'_1[V'] - v_1$, there exists only one 3-degree vertex $u_1$. Therefore, $G_1[V'] - v_1$ is not a FWF graph. And the graph $G_1[V']$ is not either.

**Case 3:** There exist at least two vertices of degree 3 in $\{u_1, u_2, u_3\}$ in $G_1[V']$. But this case does not exist by Theorem 6.5.
To sum up the case 1, 2 and 3, we know that $G_1[V']$ is not a recursive maximal planar graph. It means that the hypothesis that all the values of $f(v_1), f(v_2), f(v_3)$ and $f(v_4)$ are different is not correct. From the proof above, $\exists f \in C_4(G_1)$, the equations $f(v_1) = f(v_3)$ or $f(v_1) = f(v_4)$ holds.

In the same way, we can prove that the value of the second bracket is larger than zero. Obviously, the difference of two chromatic polynomials in the third bracket is no less than zero. So the conclusion holds when $\delta(G) = 5$. Therefore, the four color theorem is proved completely.

8 Conclusion

In this paper, with the idea of the color class partition as main thread, with the vertex contraction formula of the chromatic polynomials(Theorem 3.7 and 3.8) as basic tool, we first mathematically prove the conjecture of uniquely 4-colorable planar graphs, namely a necessary and sufficient condition for a 4-colorable maximal planar graph $G$ to be uniquely 4-colorable is that $G$ is a recursive maximal planar graph. This conjecture was put forward independently by several scholars in 1977[26, 28]. In order to prove the four color conjecture, the concept of uniquely near $k$-colorable graphs is proposed, the basic characters of these graphs are studied, and a necessary and sufficient condition for a maximal planar graph to be uniquely near 4-colorable is obtained that there exist 4 vertices belong to different color classes in any 4-coloring partition. The
mathematical proof of four color conjecture is based on these three aspects: Theorem 3.7 and Theorem 3.8; the necessary and sufficient condition of uniquely 4-colorable planar graph; and the necessary and sufficient condition of uniquely near 4-colorable maximal planar graph.

In fact, in this paper we prove that the configuration $W_5$ is reducible, in other words, we find four reducible unavoidable configurations $W_2$, $W_3$, $W_4$ and $W_5$ as shown by Figure 2.1. Apparently, they constitute a minimum reducible unavoidable configuration set.

The four color conjecture was put forward in 1852 (some investigations were even earlier, in about 1840), so far 157 years have passed. Although the computer-assisted proof was given in 1976 and in fact was admitted by some mathematicians, especially by Tutte, a more brief proof with mathematical and logical reasoning is expected by most or even all of mathematicians. The work in this paper should fulfill this expectation.

Through the proof of the four color conjecture in this paper, we can obtain a useful “perception” that to attack any mathematical problem and even non-mathematical problem, the best way might be to start from the very original point of it. As mentioned before, the graph coloring problem is just to classify the vertices in a graph, subject to constraints of number of colors and that adjacent vertices have different colors. In fact, the original proof idea in this paper is firmly around the most fundamental concept that all these information is contained in chromatic polynomials. With this thought, can the Total-coloring Conjecture of graphs be solved?

Acknowledgements

The first draft of this paper was written in 1991, when I worked in the Mathematics Department of Shaanxi Normal University, my Alma Mater. I reported my work to several professors there, including Prof. Guojun Wang, Prof. Xiansun Wei, Prof. Zhongqiang Yang, Prof. Taihe Fan, Prof. Baolin Guo and Prof. Wanmin Zhang, etc. All of them gave me some useful advices, especially Prof. Zhongqiang Yang pointed out a fatal error, which was a necessary and sufficient condition for uniquely 4-colorable planar graphs. In other words, there were some errors in the proof of the FWF conjecture. To overcome it, I have since spent more than eighteen years on this problem until August 6th, 2009. I would like to thank my teachers Prof. Xinmin Wang, Prof. Hongke Du for their encouragement, caring and support on my work in 1991, especially Prof. Hongke Du, who has been encouraging and directing me on it since then. Here I would like to thank them all deeply.

After the draft was completed in Nov. 9th, 2009, Prof. Jianfang Wang, Prof. Liang Sun discussed with me directly. They pointed out that there was one obvious leak on the proof of “If a maximal planar graph has a minimum degree of 5, it is not uniquely 4-colorable. Here I also would like to thank them deeply. At the same time, many experts on this field have looked over my paper,
such as Prof. Zhongfu Zhang, Prof. Bin Yao, Prof. Xiangen Chen, Prof. Hui Chen, Prof. Muchun Li, Prof. Huiying Qiang, student Zepeng Li and many international experts such as Prof. Jensen. All of them have pointed out the same leak as Prof. Liang Sun and Prof. Jianfang Wang’s. Here I would like to thank them all deeply.

During the drafting of first version and second version of this paper, my students Fang Xi, Mei Chen, Enqiang Zhu, Jingming Liu, Zhen Chen, Yufang Huang, Ziqi Wei and Dongming Zhao spent a lot of time on this paper, from translation, drawing, typewriting to proofreading, especially for Enqiang Zhu and Fang Xi. During the accomplishment of second version of my paper, Enqiang Zhu paid lots of attention on it. He discussed with me about some problems in my proof, completed the drawing of numbers of figures and English translation; Fang Xi was in charge of the overall work of final draft, who had maximal workload and often worked very late into the night. Moreover, Prof. Chunling Quan validated all the graphs in Figure 5.7, 5.8 and 5.9 by electronic computer. Here I would like to thank them all deeply.

I would like to thank my colleges Prof. Daoheng Yu for his continuous encouragement and Prof. Tian Liu for his helps to improve our English presentation.

Here, I also would like to thank my tutors Prof. Ziguo Wang, Prof. Yinghuo Wang (Fellow of Chinese Academy of Engineering) and Prof. Zheng Bao (Fellow of Chinese Academy of Sciences) for their advice and help through years.

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9 Appendix I

This appendix provides all the colorings of all the maximal planar graphs $G$ of orders from 6 to 10 and with minimum degree $\delta(G) = 4$.

Table 1: Lower Bound of number of partition of color groups of all the maximal planar graphs whose orders are from 6 to 10 and contain no three adjacent vertices of degree 4.

| Order | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|----|
| Lower Bound | 4 | 5 | 3 | 6 | 5 |

1. There is only one maximal planar graph of order 6 whose minimal degree is 4. It has 4 different colorings, whose partitions of color group are shown as follow:

\[
\{\{v_1\}\{v_2, v_6\}\{v_3, v_4\}\{v_5\}\}, \{\{v_1, v_6\}\{v_2\}\{v_3, v_4\}\{v_5\}\}
\]
\[
\{\{v_1, v_5\}\{v_2, v_6\}\{v_3\}\{v_4\}\}, \{\{v_1, v_5\}\{v_2, v_6\}\{v_3\}\{v_4\}\}
\]

The following figures are shown for its drawing and 4 different colorings:
2. There is only one maximal planar graph of order 7 whose minimal degree is 4. It has 5 different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
&\{\{v_1, v_7\}\{v_2, v_5\}\{v_3, v_4\}\{v_6\}\}, \{\{v_1, v_7\}\{v_2, v_6\}\{v_3, v_4\}\{v_5\}\}, \\
&\{\{v_1, v_7\}\{v_2, v_5\}\{v_3\}\{v_4, v_6\}\}, \{\{v_1, v_7\}\{v_2, v_6\}\{v_3, v_5\}\{v_4\}\}, \\
&\{\{v_1, v_7\}\{v_2\}\{v_3, v_5\}\{v_4, v_6\}\}\end{align*}
\]
3. There are two kinds of maximal planar graphs of order 8 whose minimal degree is 4, which are shown as follows:

3.1 Its degree sequence is $4444466$, and it has 12 kinds of different colorings, whose partitions of color group are shown as follow:

- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_4, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5\}\{v_3, v_4, v_6, v_7\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_4, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_4, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_4, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_4, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_6\}\}$
- $\{\{v_1, v_8\}\{v_2, v_5, v_7\}\{v_3, v_6\}\}$

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3.2 Its degree sequence is $4445555$, and it has 3 kinds of different colorings, whose partitions of color group are shown as follow:
4. There are five kinds of maximal planar graphs of order 9 whose minimal degree is 4.

4.1 Its degree sequence is 44444666, and it is 3-colorable, whose unique partitions of color group are shown as follow:
4.2 Its degree sequence is 44455556, and it has 6 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
\{\{v_1, v_7, v_8\}\{v_2, v_6\}\{v_4, v_5, v_9\}\{v_3\}\}, & \quad \{\{v_1, v_7, v_8\}\{v_2\}\{v_4, v_5, v_9\}\{v_3, v_6\}\}\} \\
\{\{v_1, v_8\}\{v_2, v_6\}\{v_4, v_5, v_9\}\{v_3, v_7\}\}, & \quad \{\{v_1, v_7\}\{v_2, v_8\}\{v_3, v_6\}\{v_4, v_5, v_9\}\}\} \\
\{\{v_1, v_7, v_8\}\{v_2, v_6\}\{v_3, v_4\}\{v_5, v_9\}\}, & \quad \{\{v_1, v_7, v_8\}\{v_2, v_5\}\{v_3, v_6\}\{v_4, v_9\}\}\} \\
\end{align*}
\]

4.3 Its degree sequence is 4445555555, and it has 2 kinds of different colorings, whose partitions of color group are shown as follow:
4.4 Its degree sequence is 4444444477, and it has 17 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\{\{v_1, v_9\}\{v_2, v_8\}\{v_3, v_4\}\{v_5, v_6, v_9\}\}, \{\{v_1, v_9\}\{v_2, v_4\}\{v_3, v_7\}\{v_5, v_6, v_9\}\}
\]
4.5 Its degree sequence is 444445566, and it has 7 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
\{\{v_1, v_5, v_9\}\{v_2, v_8\}\{v_3, v_6\}\{v_4, v_7\}\}, & \{\{v_1, v_5, v_9\}\{v_2, v_7\}\{v_3, v_4\}\{v_6, v_8\}\} \\
\{\{v_1, v_5, v_9\}\{v_2, v_4\}\{v_3, v_7\}\{v_6, v_8\}\}, & \{\{v_1, v_5, v_9\}\{v_3, v_4, v_7\}\{v_6, v_8\}\} \\
\{\{v_1, v_5, v_9\}\{v_2, v_4, v_7\}\{v_3\}\{v_6, v_8\}\}, & \{\{v_1, v_5, v_9\}\{v_2, v_7\}\{v_3, v_4, v_8\}\} \\
\{\{v_1, v_5, v_9\}\{v_2, v_4, v_7\}\{v_6\}\{v_3, v_8\}\}, & \{\{v_1, v_5, v_9\}\{v_2, v_7\}\{v_3, v_4, v_8\}\} \\
\{\{v_1, v_5, v_9\}\{v_2, v_4, v_7\}\{v_8\}\{v_3, v_6\}\} & \\
\end{align*}
\]
5. There are 13 kinds of maximal planar graphs of order 10 whose minimal degree is 4.

5.1 Its degree sequence is $444455567$, and it has 7 kinds of different colorings, whose partitions of color group are shown as follow:

$$\{\{v_1, v_9\}\{v_2, v_4\}\{v_5, v_8, v_{10}\}\{v_3, v_6, v_7\}\}$$
$$\{\{v_1, v_9\}\{v_2, v_6, v_7\}\{v_5, v_8, v_{10}\}\{v_3, v_4\}\}$$
$$\{\{v_1, v_6, v_7\}\{v_2, v_5, v_{10}\}\{v_3, v_8\}\{v_4, v_9\}\}$$
$$\{\{v_1, v_6\}\{v_2, v_5, v_{10}\}\{v_3, v_7, v_8\}\{v_4, v_9\}\}$$
$$\{\{v_1, v_7\}\{v_2, v_6, v_{10}\}\{v_3, v_5, v_8\}\{v_4, v_9\}\}$$
$$\{\{v_1, v_5\}\{v_2, v_6, v_{10}\}\{v_3, v_7, v_8\}\{v_4, v_9\}\}$$
5.2 Its degree sequence is 444445577, and it has 10 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
&\{\{v_1, v_7, v_{10}\}\{v_2, v_4\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_9\}\{v_3, v_5\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\} \{\{v_1, v_7, v_{10}\}\{v_2, v_4, v_7\}\{v_3, v_5, v_9\}\{v_6, v_8\}\}
\end{align*}
\]
5.3 Its degree sequence is $444444488$, whose unique partitions of color group are shown as follow:
5.4 Its degree sequence is 444455666, and it has 14 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
&\{\{v_1, v_5, v_9\}\{v_2, v_4, v_6\}\{v_3, v_7, v_8\}\{v_{10}\}\}\{\{v_1, v_5, v_9\}\{v_2, v_4\}\{v_3, v_7, v_8\}\{v_{10}\}\} \\
&\{\{v_1, v_5, v_9\}\{v_2, v_4, v_7\}\{v_3, v_8\}\{v_{10}\}\}\{\{v_1, v_5, v_9\}\{v_2, v_6\}\{v_3, v_7, v_8\}\{v_{10}\}\} \\
&\{\{v_1, v_5, v_9\}\{v_2, v_6, v_{10}\}\{v_3, v_7, v_8\}\{v_{10}\}\}\{\{v_1, v_5, v_9\}\{v_2, v_{10}\}\{v_3, v_7, v_8\}\{v_{10}\}\} \\
&\{\{v_1, v_5, v_9\}\{v_2, v_{10}\}\{v_3, v_4, v_8\}\{v_{10}\}\}\{\{v_1, v_5, v_9\}\{v_2, v_{10}\}\{v_3, v_4, v_8\}\{v_{10}\}\} \\
&\{\{v_1, v_5, v_9\}\{v_2, v_{10}\}\{v_3, v_4, v_7\}\{v_{10}\}\}\{\{v_1, v_5, v_9\}\{v_2, v_{10}\}\{v_3, v_4, v_8\}\{v_{10}\}\} \\
&\{\{v_1, v_5, v_9\}\{v_2, v_{10}\}\{v_3, v_4, v_7\}\{v_{10}\}\}\{\{v_1, v_5, v_9\}\{v_2, v_{10}\}\{v_3, v_4, v_8\}\{v_{10}\}\}
\end{align*}
\]
5.5 Its degree sequence is 4445555666, and it has 6 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
\{\{v_1, v_6, v_8\}\{v_2, v_9\}\{v_3, v_5, v_7\}\{v_4, v_{10}\}\} & \{\{v_1, v_6, v_8\}\{v_2, v_7, v_9\}\{v_3, v_5\}\{v_4, v_{10}\}\} \\
\{\{v_1, v_6, v_8\}\{v_2, v_4\}\{v_3, v_5, v_7\}\{v_9, v_{10}\}\} & \{\{v_1, v_6, v_8\}\{v_2, v_7\}\{v_3, v_4\}\{v_5, v_9, v_{10}\}\} \\
\{\{v_1, v_6, v_8\}\{v_2, v_7, v_9\}\{v_3, v_4\}\{v_5, v_{10}\}\} & \{\{v_1, v_6, v_8\}\{v_2, v_4\}\{v_3, v_7\}\{v_5, v_9, v_{10}\}\} \\
\end{align*}
\]
5.6 Its degree sequence is 444446666, and it is uniquely 3-colorable and has 28 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
&\{\{v_1, v_3\}\{v_2, v_5, v_9\}\{v_4, v_6, v_7, v_9\}\{\{v_1\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\{v_6, v_7, v_9\}}
\end{align*}
\]

\[
\begin{align*}
&\{\{v_1, v_5\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\{v_6, v_7\}\}, \{\{v_1, v_{10}\}\{v_2, v_5\}\{v_3, v_4, v_8\}\{v_6, v_7, v_9\}}
\end{align*}
\]

\[
\begin{align*}
&\{\{v_1, v_7\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\{v_6, v_9\}\}, \{\{v_1, v_{10}\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\{v_6\}\}
\end{align*}
\]

\[
\begin{align*}
&\{\{v_1, v_7, v_9\}\{v_2, v_5\}\{v_3, v_4, v_8\}\{v_6, v_{10}\}\}, \{\{v_1, v_{10}\}\{v_2, v_7\}\{v_3, v_4, v_8\}\{v_5, v_9\}\}
\end{align*}
\]

\[
\begin{align*}
&\{\{v_1, v_6, v_9\}\{v_2, v_7\}\{v_3, v_4, v_8\}\{v_5, v_{10}\}\}, \{\{v_1, v_6, v_7, v_9\}\{v_2\}\{v_3, v_4, v_8\}\{v_5, v_{10}\}\}
\end{align*}
\]
$$\{\{v_1, v_6, v_7, v_9\}\{v_2, v_{10}\}\{v_3, v_4, v_8\}\{v_5\}\}, \{\{v_1, v_6, v_7\}\{v_2, v_{10}\}\{v_3, v_4, v_8\}\{v_5, v_9\}\}$$
$$\{\{v_1, v_6, v_7, v_9\}\{v_2, v_5, v_{10}\}\{v_3, v_4\}\{v_8\}\}, \{\{v_1, v_6, v_7\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\{v_9\}\}$$
$$\{\{v_1, v_6, v_7, v_9\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\}, \{\{v_1, v_6, v_7, v_9\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\{v_9\}\}$$
$$\{\{v_1, v_6, v_9\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\{v_7\}, \{\{v_1, v_6\}\{v_2, v_5, v_{10}\}\{v_3, v_4, v_8\}\{v_7, v_9\}\}$$
$$\{\{v_1, v_6, v_{10}\}\{v_2, v_5\}\{v_3, v_4, v_8\}\{v_7, v_9\}\}, \{\{v_1, v_6, v_7, v_9\}\{v_2, v_5, v_{10}\}\{v_4, v_8\}\{v_3\}\}$$
$$\{\{v_1, v_6, v_7, v_9\}\{v_2, v_5\}\{v_3, v_8\}\{v_4, v_{10}\}\}, \{\{v_1, v_6, v_7, v_9\}\{v_2, v_5, v_{10}\}\{v_3, v_8\}\{v_4\}\}$$
$$\{\{v_1, v_6, v_7\}\{v_2, v_5, v_{10}\}\{v_3, v_8\}\{v_4\}\}, \{\{v_1, v_6, v_9\}\{v_2, v_5, v_{10}\}\{v_3, v_8\}\{v_4, v_7\}\}$$
$$\{\{v_1, v_6\}\{v_2, v_5, v_{10}\}\{v_3, v_8\}\{v_4, v_7, v_9\}\}, \{\{v_1, v_6, v_{10}\}\{v_2, v_5\}\{v_3, v_8\}\{v_4, v_7, v_9\}\}$$
$$\{\{v_1, v_6\}\{v_2, v_5, v_{10}\}\{v_3, v_8\}\{v_4, v_7, v_9\}\}, \{\{v_1, v_7, v_9\}\{v_2, v_5, v_{10}\}\{v_3, v_6\}\{v_4, v_8\}\}$$
5.7 Its degree sequence is 4445555556, and it has 4 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
\{v_1, v_6, v_{10}\} \{v_2, v_8\} \{v_3, v_4, v_5, v_7, v_9\}, \\
\{v_1, v_6\} \{v_2, v_4, v_{10}\} \{v_3, v_8\} \{v_5, v_7, v_9\} \\
\{v_1, v_{10}\} \{v_2, v_6\} \{v_3, v_4, v_8\} \{v_5, v_7, v_9\}, \\
\{v_1, v_6\} \{v_2, v_{10}\} \{v_3, v_4, v_8\} \{v_5, v_7, v_9\}
\end{align*}
\]

5.8 Its degree sequence is 4445555555, and it has 8 kinds of different colorings, whose partitions of color group are shown as follow:
\[
\{\{v_1, v_6, v_9\}, \{v_2, v_7\}, \{v_3, v_4, v_8\}, \{v_5, v_{10}\}\}, \{\{v_1, v_6, v_9\}, \{v_2, v_{10}\}, \{v_3, v_4, v_8\}, \{v_5, v_7\}\}
\{\{v_1, v_6, v_9\}, \{v_2, v_4, v_{10}\}, \{v_3, v_8\}, \{v_5, v_7\}\}, \{\{v_1, v_6, v_9\}, \{v_2, v_4, v_{10}\}, \{v_3, v_8\}, \{v_5, v_7\}\}
\{\{v_1, v_6\}, \{v_2, v_3\}, \{v_4, v_{10}\}, \{v_5, v_7, v_9\}\}, \{\{v_1, v_6\}, \{v_2, v_3\}, \{v_4, v_{10}\}, \{v_5, v_7, v_9\}\}
\}
5.9 Its degree sequence is 444445667, and it has 20 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
&\{\{v_1, v_7, v_{10}\}\{v_2, v_5, v_9\}\{v_3, v_4, v_8\}\{v_6\}\}, \{\{v_1, v_7\}\{v_2, v_5, v_9\}\{v_3, v_4, v_8\}\{v_6, v_{10}\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_5\}\{v_3, v_4, v_8\}\{v_6, v_9\}\}, \{\{v_1, v_7, v_{10}\}\{v_2, v_5, v_9\}\{v_3, v_4, v_8\}\{v_6, v_8\}\} \\
&\{\{v_1, v_{10}\}\{v_2, v_5, v_9\}\{v_3, v_4, v_8\}\{v_6, v_9\}\}, \{\{v_1, v_6, v_{10}\}\{v_2, v_5, v_9\}\{v_3, v_7\}\{v_4, v_8\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_6, v_9\}\{v_3, v_4, v_8\}\{v_5\}\}, \{\{v_1, v_7\}\{v_2, v_6, v_9\}\{v_3, v_4, v_8\}\{v_5, v_{10}\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_6\}\{v_3, v_4, v_8\}\{v_5, v_9\}\}, \{\{v_1, v_7, v_{10}\}\{v_2, v_6, v_9\}\{v_3, v_4\}\{v_5, v_8\}\} \\
&\{\{v_1, v_{10}\}\{v_2, v_5\}\{v_3, v_4, v_8\}\{v_6, v_9\}\}, \{\{v_1, v_7, v_{10}\}\{v_2, v_6, v_9\}\{v_3, v_4, v_8\}\{v_4, v_6\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_5\}\{v_3, v_4, v_9\}\{v_6, v_8\}\}, \{\{v_1, v_7\}\{v_2, v_5, v_9\}\{v_3, v_5, v_8\}\{v_4, v_6, v_{10}\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2\}\{v_3, v_5, v_8\}\{v_4, v_6, v_9\}\}, \{\{v_1, v_7, v_{10}\}\{v_2, v_5\}\{v_3, v_5, v_8\}\{v_4, v_6, v_9\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_9\}\{v_3, v_5, v_8\}\{v_4, v_6, v_9\}\}, \{\{v_1, v_7\}\{v_2, v_5\}\{v_3, v_5, v_8\}\{v_4, v_6, v_{10}\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_6\}\{v_3, v_5, v_8\}\{v_4, v_9\}\}, \{\{v_1, v_7, v_{10}\}\{v_2, v_6, v_9\}\{v_3, v_5, v_8\}\{v_4\}\} \\
&\{\{v_1, v_7, v_{10}\}\{v_2, v_6, v_9\}\{v_3, v_5, v_8\}\{v_4, v_9\}\}, \{\{v_1, v_7, v_{10}\}\{v_2, v_6, v_9\}\{v_3, v_5, v_8\}\{v_4, v_{10}\}\}
\end{align*}
\]
5.10 Its degree sequence is 444555566, and it has 14 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
\{\{v_1, v_5, v_{10}\}, \{v_2, v_4, v_8\}\} & \cup \{\{v_3, v_6, v_7\}, \{v_9\}\}, \\
\{\{v_1, v_6, v_{10}\}, \{v_2, v_4, v_8\}\} & \cup \{\{v_3, v_5, v_7\}, \{v_9\}\}, \\
\{\{v_1, v_5, v_{10}\}, \{v_2, v_4, v_8\}\} & \cup \{\{v_3, v_6, v_7\}, \{v_9\}\}, \\
\{\{v_1, v_6, v_{10}\}, \{v_2, v_4, v_8\}\} & \cup \{\{v_3, v_5, v_7\}, \{v_9\}\}, \\
\{\{v_1, v_5, v_{10}\}, \{v_2, v_4, v_8\}\} & \cup \{\{v_3, v_6, v_7\}, \{v_9\}\}, \\
\{\{v_1, v_6, v_{10}\}, \{v_2, v_4, v_8\}\} & \cup \{\{v_3, v_5, v_7\}, \{v_9\}\}, \\
\{\{v_1, v_5, v_{10}\}, \{v_2, v_4, v_8\}\} & \cup \{\{v_3, v_6, v_7\}, \{v_9\}\}, \\
\{\{v_1, v_6, v_{10}\}, \{v_2, v_4, v_8\}\} & \cup \{\{v_3, v_5, v_7\}, \{v_9\}\}.
\end{align*}
\]
\{v_1, v_5, v_9\}, \{v_2, v_9\}, \{v_3, v_8\}, \{v_6, v_7\}, \{v_1, v_{10}\}, \{v_2, v_5, v_9\}, \{v_3, v_4, v_8\}, \{v_6, v_7\}\}
\{v_1, v_6, v_{10}\}, \{v_2, v_5, v_9\}, \{v_3, v_4, v_8\}, \{v_7, v_9\}, \{v_1, v_5, v_{10}\}, \{v_2, v_6\}, \{v_3, v_4, v_8\}, \{v_7, v_9\}\}
\{v_1, v_6, v_{10}\}, \{v_2, v_5\}, \{v_3, v_4, v_8\}, \{v_7, v_9\}, \{v_1, v_8\}, \{v_2, v_5, v_9\}, \{v_3, v_6, v_7\}, \{v_4, v_{10}\}\}
5.11 Its degree sequence is 444555566, and it has 13 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
&\{v_1, v_6, v_{10}\}\{v_2, v_5, v_8\}\{v_3, v_4\}\{v_7, v_9\}, \\
&\{v_1, v_6, v_{10}\}\{v_2, v_5, v_8\}\{v_3, v_4, v_9\}\{v_7\}, \\
&\{v_1, v_7, v_9\}\{v_2, v_5, v_8\}\{v_3, v_4\}\{v_6, v_{10}\}, \\
&\{v_1, v_7, v_9\}\{v_2, v_5, v_8\}\{v_3, v_4, v_9\}\{v_6, v_{10}\}, \\
&\{v_1, v_6, v_{10}\}\{v_2, v_7\}\{v_3, v_4, v_8\}\{v_5, v_9\}, \\
&\{v_1, v_6, v_{10}\}\{v_2, v_7\}\{v_3, v_4, v_8\}\{v_5, v_9\}, \\
&\{v_1, v_6, v_{10}\}\{v_2, v_5\}\{v_3, v_8\}\{v_4, v_7, v_9\}, \\
&\{v_1, v_6, v_{10}\}\{v_2, v_5, v_8\}\{v_3, v_4, v_7, v_9\}.
\end{align*}
\]
\[ \{\{v_1, v_6, v_{10}\}\{v_2, v_5, v_8\}\{v_3, v_9\}\{v_4, v_7\}\}, \{\{v_1, v_7, v_9\}\{v_2, v_5, v_8\}\{v_3, v_6\}\{v_4, v_{10}\}\}, \{\{v_1, v_{10}\}\{v_2, v_5, v_8\}\{v_3, v_6\}\{v_4, v_7, v_9\}\} \]
5.12 Its degree sequence is 444555566, and it has 5 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
\{\{v_1, v_5, v_{10}\}\{v_2, v_4\}\{v_3, v_6\}\{v_7, v_9\}\}, \{\{v_1, v_5, v_9\}\{v_2, v_7\}\{v_3, v_4, v_8\}\{v_6, v_{10}\}\} \\
\{\{v_1, v_5, v_{10}\}\{v_2, v_4, v_8\}\{v_3, v_9\}\{v_6, v_7\}\}, \{\{v_1, v_5, v_{10}\}\{v_2, v_8\}\{v_3, v_4, v_9\}\{v_6, v_7\}\} \\
\{\{v_1, v_{10}\}\{v_2, v_4, v_8\}\{v_3, v_5, v_9\}\{v_6, v_7\}\}
\end{align*}
\]
5.13 Its degree sequence is 4444455577, and it has 15 kinds of different colorings, whose partitions of color group are shown as follow:

\[
\begin{align*}
\{\{v_1, v_6, v_8\}\{v_3, v_4\}\{v_2, v_5, v_9\}\{v_7, v_{10}\}\}, \{\{v_1, v_6\}\{v_3, v_4, v_8\}\{v_2, v_5, v_9\}\{v_7, v_{10}\}\} \\
\{\{v_1, v_6, v_{10}\}\{v_3, v_4, v_8\}\{v_2, v_5, v_9\}\{v_7, v_9\}\}, \{\{v_1, v_6, v_9\}\{v_3, v_4, v_8\}\{v_2, v_5, v_7\}\{v_{10}\}\} \\
\{\{v_1, v_7, v_{10}\}\{v_3, v_4, v_8\}\{v_2, v_5, v_9\}\{v_6, v_9\}\}, \{\{v_1, v_7, v_{10}\}\{v_3, v_4, v_8\}\{v_2, v_5, v_9\}\{v_6, v_{10}\}\} \\
\{\{v_1, v_7, v_{10}\}\{v_3, v_4\}\{v_2, v_5, v_9\}\{v_6, v_9\}\}, \{\{v_1, v_7, v_{10}\}\{v_3, v_4\}\{v_2, v_5, v_9\}\{v_6, v_{10}\}\} \\
\{\{v_1, v_6\}\{v_3, v_4, v_8\}\{v_2, v_5, v_9\}\{v_7, v_{10}\}\}, \{\{v_1, v_6, v_{10}\}\{v_3, v_4, v_8\}\{v_2, v_5, v_9\}\{v_7, v_{10}\}\} \\
\{\{v_1, v_6, v_9\}\{v_3, v_4, v_8\}\{v_2, v_5, v_{10}\}\}, \{\{v_1, v_6, v_{10}\}\{v_3, v_4, v_8\}\{v_2, v_5, v_{10}\}\} \\
\{\{v_1, v_6, v_9\}\{v_3, v_4, v_8\}\{v_2, v_5, v_{10}\}\}, \{\{v_1, v_6, v_{10}\}\{v_3, v_4, v_8\}\{v_2, v_5, v_{10}\}\} \\
\end{align*}
\]
10 Appendix II

This appendix proves that there is a unique maximal planar graph of order 13 with only one 4-degree vertex and all other vertices with degrees bigger than 4, and gives construction and all the 4-colorings of this kind of graph.

**Lemma 10.1.** For a triangle $uxy$, if the maximal planar graph $G$ obtained by adding vertices in this triangle face has all the vertices' degree bigger than 4, then at least 9 vertices are needed.

**Proof.** Since the maximal planar graph with minimum degree 5 has at least 12, this lemma holds.

**Lemma 10.2.** For a triangle $uxy$, if the maximal planar graph $G$ obtained by adding vertices in this triangle face has only one 4-degree vertex, and other vertices' degree is bigger than 4, then at least 10 vertices are needed.

**Proof.** By Euler's formula, the maximal planar graph with only one degree-4 vertex and all other vertices having degree bigger than 4 has order at least 11. However, there does not exist any this kind of planar graph of order 11 and 12. Thus this lemma holds.

**Lemma 10.3.** For a quadrilateral $uxyz$, if the graph $G$ obtained by adding vertices in this quadrilateral face has only one 4-degree vertex, and other vertices with degree bigger than 4, then at least 5 vertices are needed.
Proof. Obviously, this lemma holds.

By Euler’s formula, there are at most two degree sequences of a maximal planar graph with only one degree-4 vertex and order of 13, as follow: 4555555555566, 4555555555557.

For first kind of graph, it contains one vertex of degree 4, ten vertices of degree 5 and two vertices of degree 6. Nextly, we consider about this kind of maximal planar graph as following cases:

Case 1: there are two adjacent vertices of degree 6.

Let \( u, v \) be two vertices of degree 6 in graph \( G \), and \( \Gamma(u) = \{u_1, u_2, u_3, u_4, u_5, v\} \), as shown in Figure 1. There is a 6 circle \( u_1u_2u_3u_4u_5vu_1 \) in this graph, and vertex \( v \) is not adjacent to \( u_2 \) and \( u_3 \). Otherwise, by lemma 10.1 and lemma 10.2, the order of graph \( G \) is bigger than 13 or there are at least two vertices of degree 4, or exists degree-3 vertex. Moreover, if \( v \) is adjacent to \( u_3 \), vertices \( vu_3u_2u_3 \) can form a quadrilateral. In order to assure this quadrilateral at most has one vertex of degree 4, 5 vertices are at least added to this quadrilateral. Therefore, there is one more vertex in quadrilateral \( vu_5u_4u_3 \). Thus there exists either degree-3 vertex or two vertices \( u_4, u_5 \) of degree 4.

![Figure 1](image1.png)

According to discussion above, there are two adjacent vertices \( u, v \) in graph \( G \), as shown in Figure 2. \( \Gamma(u) = \{u_1, u_2, u_3, u_4, u_5, v\} \), \( \Gamma(v) = \{u_1, u, u_5, v_1, v_2, v_3\} \). Besides all the vertices in the following Figure 2, there are three more vertices in graph \( G \), as \( x, y, z \). Obviously, if we add vertices \( x, y, z \) in the face of \( xyzu_1u_2u_3u_4u_5v_1v_2v_3 \), it is impossible to have other vertex of degree-4 in this face. Thus, there exists no two adjacent vertices of degree 6.

![Figure 2](image2.png)
Case 2: The distance of two vertices of degree 6 is two in graph $G$. Assume there are two vertices $u, v$ of degree 6 in graph $G$, one vertex $w$ of degree 4.

Case 2.1 vertex $w$ is not adjacent to vertices $u, v$, as shown in Figure 3. Obviously, vertex $w$ does not belong to the vertices in Figure 3. And if we add this vertex $w$ in face $u_1u_2u_3u_4u_5v_1v_2v_3v_4$, all of vertices $u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4$ don’t have degree of 5.

![Figure 3.](image)

Case 2.2 vertex $w$ is adjacent to one vertex in $u, v$, as shown in Figure 3. Assume it be vertex $v$, no matter vertex $w$ is any one of $v_1, v_2, v_3, v_4$, if we add one new vertex, it is impossible to make other vertices having degree 5 except vertices $u, v, x$.

Case 2.3 vertex $w$ is adjacent to vertices $u, v$, as shown in Figure 4.

![Figure 4.](image)

In summary, there is only one maximal planar graph whose degree sequence of 455555555556, as shown in Figure 4.

Nextly, we consider about the case that degree sequence of 455555555557. There is only one vertex of degree 4, eleven vertices of degree 5 and one vertex degree of 7. Now we analyze all the cases of this structure of maximal planar graph. Let $u$ be a vertex of degree 7 in graph $G$, there must exist 7-wheel structure in Figure 5.

If there are not any vertices of degree 4 in $u_1, u_2, u_3, u_4, u_5, u_6, u_7$, according to lemma 1,2 and3, there does not contain any string in circle $u_1u_2u_3u_4u_5u_6u_7u_1$. Thus, if the order of graph $G$ is equal to 13, it is impossible to assure all the
vertices having degree of 5. Thus, there must exist any one vertex of degree 4 in $u_1, u_2, u_3, u_4, u_5, u_6, u_7$. Assume $d(u_1) = 4$, as shown in Figure 6. Consider the vertices in circle $u_8 u_2 u_3 u_4 u_5 u_6 u_7 u_8$, all of vertices in this circle does not have degree of 5. Therefore, there does not exist any maximal planar graph whose degree sequence of 455555555557.

If the distance of two vertices of degree 6 is 3 in graph $G$, let $u, v$ be this two vertices of degree 6. Thus, there is not any common element in $\Gamma(u)$ and $\Gamma(v)$. So $|V(G)| \geq |T(u)| + |\Gamma(v)| + 2 = 14$, which contracts with the fact that the order of graph $G$ is 13. In summary, there is only one maximal planar graph of order 13 whose minimal degree is 4, shown in Figure 4.

Next, we give all the 14 kinds of 4-colorings of the graph in Figure 4, whose partitions of color group are shown as follow:

$$\begin{align*}
\{\{v_1, v_7, v_8, v_{12}\}\{v_4, v_6, v_{13}\}\{v_3, v_9, v_{11}\}\{v_2, v_5, v_{10}\}\}
\{\{v_1, v_7, v_8, v_{12}\}\{v_4, v_{10}, v_{13}\}\{v_2, v_3, v_{11}\}\{v_5, v_6, v_9\}\}
\{\{v_1, v_7, v_8, v_{12}\}\{v_{10}, v_{13}\}\{v_3, v_9, v_{11}\}\{v_2, v_5, v_6\}\}
\{\{v_1, v_7, v_8, v_{12}\}\{v_4, v_5, v_{10}\}\{v_2, v_3, v_{11}\}\{v_6, v_9, v_{13}\}\}
\{\{v_1, v_7, v_8, v_{12}\}\{v_4, v_5, v_{10}\}\{v_3, v_9, v_{13}\}\{v_2, v_6, v_{11}\}\}
\{\{v_1, v_7, v_8, v_{12}\}\{v_{10}, v_{11}\}\{v_2, v_5, v_{10}\}\{v_3, v_9, v_{13}\}\}
\{\{v_1, v_6, v_9, v_{11}\}\{v_4, v_6, v_{12}\}\{v_2, v_5, v_{10}\}\{v_3, v_9, v_{13}\}\}
\{\{v_1, v_6, v_9, v_{11}\}\{v_4, v_8, v_{13}\}\{v_2, v_3, v_{10}\}\{v_5, v_7, v_{12}\}\}
\{\{v_1, v_6, v_9, v_{11}\}\{v_4, v_5, v_{12}\}\{v_2, v_7, v_8\}\{v_3, v_{10}, v_{13}\}\}
\{\{v_1, v_9, v_{11}\}\{v_4, v_5, v_{12}\}\{v_2, v_7, v_8\}\{v_3, v_{10}, v_{13}\}\}
\{\{v_1, v_8, v_9\}\{v_4, v_5, v_{12}\}\{v_3, v_{10}, v_{13}\}\{v_2, v_7, v_{11}\}\}
\{\{v_1, v_6, v_9, v_{11}\}\{v_4, v_8, v_{12}\}\{v_2, v_5, v_7\}\{v_3, v_{10}, v_{13}\}\}
\end{align*}$$

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\{\{v_1, v_6, v_9, v_{11}\}\{v_4, v_5, v_{12}\}\{v_2, v_3, v_{10}\}\{v_7, v_8, v_{13}\}\}

\{\{v_1, v_9, v_{11}\}\{v_4, v_5, v_6, v_{12}\}\{v_2, v_3, v_{10}\}\{v_7, v_8, v_{13}\}\}

\{\{v_1, v_7, v_{11}\}\{v_4, v_5, v_6, v_{12}\}\{v_2, v_3, v_{10}\}\{v_8, v_9, v_{13}\}\}
