Sudden violation of the CHSH inequality in a two-qubit system

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Received 29 March 2010
Accepted for publication 16 April 2010
Published 30 September 2010
Online at stacks.iop.org/PhysScr/T140/014055

Abstract

In this paper, I study the dynamics of the violation of the CHSH inequality for two qubits interacting with a common zero-temperature non-Markovian environment. I demonstrate sudden violation of the inequality for two qubits initially prepared in a factorized state. Due to the strong coupling between the qubits and the reservoir, the dynamics is characterized by numerous sharp revivals. Furthermore, I focus on a more realistic physical system in which the spontaneous emission for the qubits is taken into account. When including spontaneous emission even for small decay parameters, revivals in the violation are heavily damped out. If the decay rates exceed a certain threshold, the inequality turns out to be always satisfied.

PACS numbers: 03.67.Bg, 03.65.Ud, 03.65.Yz, 42.50.−p

(Some figures in this article are in colour only in the electronic version.)

1. Introduction

The interaction of a quantum system with an environment is generally seen as a source of decoherence, leading to the loss of quantum properties and the appearance of the classical world [1]. In fact, due to the creation of correlation with the environment, the evolution of the quantum system is not unitary any more. If the system is prepared in an entangled state, such an entanglement is very likely to be lost [2]. However, environments can also create correlations among quantum systems. For example, the interaction of a bipartite system with a common reservoir creates correlations between the parts [3], irrespective of the existence of a direct coupling between them [4, 5]. In a shared environment highly entangled long-living states or even sub-radiant decoherence-free states can exist [6]. Many studies have demonstrated the existence of non-zero stationary entanglement in systems of two qubits or two harmonic oscillators sharing the same reservoir [7]. The ability to create correlations persists if the common reservoir has a non-zero temperature [8], and is increased by memory effects in non-Markovian reservoirs [9–11].

In the last decade, a lot of effort has been made to understand the dynamics of entanglement and quantum correlations in systems that are good candidates for applications in quantum information theory and technology. However, not all the protocols in quantum information theory rely on entanglement. On the contrary, in many cases, entanglement is simply not enough and non-local properties, expressed by the violation of a Bell inequality, are required.

In this paper I study the non-local properties, namely the violation of the CHSH inequality, of a system of two qubits interacting with a common non-Markovian reservoir. I consider a factorized state of the qubits with one excitation, and study, as a function of time and of the qubit–cavity coupling, the sudden violation of CHSH inequality. The conditions for which the violation is maximum are identified. Afterwards I add another dynamical ingredient and study how the dynamics of the CHSH violation gets modified when the spontaneous emission of the qubits is included.

2. The model

The system under investigation comprises two qubits interacting with a common zero-temperature bosonic reservoir. The Hamiltonian of the system in rotating-wave approximation, and in units of $\hbar$, is $H = H_0 + H_{\text{int}}$,

$$H_0 = \omega_0 \left( \sigma_+^{(1)} \sigma_-^{(1)} + \sigma_+^{(2)} \sigma_-^{(2)} \right) + \sum_k \omega_k a_k \dagger a_k,$$  

$$H_{\text{int}} = (\alpha_1 \sigma_+^{(1)} + \alpha_2 \sigma_+^{(2)}) \sum_k g_k a_k + \text{h.c.},$$

where $\sigma_+^{(i)}$, $\sigma_-^{(i)}$ are the $\sigma_+$ and $\sigma_-$ Pauli operators of the $i$th qubit, $a_k$ and $a_k \dagger$ are the boson annihilation and creation operators, respectively, and $\omega_0$, $\omega_k$, $\alpha_1$, $\alpha_2$, $g_k$ are appropriate coupling strengths.
where $\sigma_\pm^{(1)}$ and $\sigma_\pm^{(2)}$ are the Pauli raising and lowering operators for qubits 1 and 2, respectively, $\omega_0$ is the Bohr frequency of the two identical qubits, $\alpha_1$ and $\alpha_2$ are dimensionless environment–qubit coupling constants, $\alpha_1^*$ and $\alpha_2^*$, and $g_0$ and $g_0$ are the annihilation and creation operators, and the frequency and coupling constants of the field mode $k$, respectively.

In the following, I assume that the two qubits interact resonantly with a lossy cavity, so I choose a Lorentzian spectral distribution to describe the properties of the environment,

$$J(\omega) = \frac{W^2}{\pi} \frac{\lambda}{(\omega - \omega_0)^2 + \lambda^2}, \quad \text{(3)}$$

where $\lambda$ is the width of the spectral distribution describing the cavity losses, and $W$ in the limit of ideal cavity (when $\lambda \to 0$) is proportional to the vacuum Rabi frequency $R$ through $W = R/\alpha_T$ with $\alpha_T = (\alpha_1^2 + \alpha_2^2)^{1/2}$ being the collective coupling constant.

The dynamics of two qubits interacting with a common Lorentzian-structured reservoir has been studied in [9] for the case of one excitation and in [11] for a generic state of two identically coupled qubits. Since here I consider the dynamics of a factorized state with one excitation of the form $|\Psi(t)\rangle = c_1(t)|10\rangle_E + c_2(t)|01\rangle_E + \sum_k c_k(t)|00\rangle_k E_k$, I am going to use the model of [9]. There the dynamics of the qubits is expressed on the basis of super-radiant and sub-radiant states

$$|\psi_-\rangle = r_2|10\rangle - r_1|01\rangle, \quad \text{(4)}$$

$$|\psi_+\rangle = r_1|10\rangle + r_2|01\rangle, \quad \text{(5)}$$

where the relative coupling strengths $r_1 = \alpha_1/\alpha_T$ and $r_2 = \alpha_2/\alpha_T$ have been introduced ($r_1^2 + r_2^2 = 1$).

Considering the dynamics in such a basis is particularly convenient since the sub-radiant state does not evolve in time. Thus, the evolution of the amplitudes of the first and second qubits is just

$$c_1(t) = r_2\beta_+ + r_1\beta_+ E(t), \quad \text{(6)}$$

$$c_2(t) = -r_1\beta_+ + r_2\beta_+ E(t), \quad \text{(7)}$$

with $\beta_\pm = \langle \psi_\pm | \psi_0 \rangle$ and

$$E(t) = e^{-\lambda t/2} \left[ \cosh(\Omega t/2) + \frac{\lambda}{\Omega} \sinh(\Omega t/2) \right], \quad \text{(8)}$$

where $\Omega = \sqrt{\lambda^2 - 4R^2}$.

In order to evaluate the time evolution of the violation of the CHSH Bell inequality, I use equation [12], which allows one to express the maximum of the Bell function (by an appropriate choice of angles) as a function of the two-qubit density matrix elements. In the $|11\rangle$, $|10\rangle$, $|01\rangle$, $|00\rangle$ basis of the qubits, the maximum of the Bell function reads as

$$B = 2 \max_{i,j} |u_i + u_j|^2, \quad \text{(9)}$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig1.png}
\caption{Dynamics of the CHSH inequality violation in a common Lorentzian structured reservoir as a function of scaled time and relative coupling strength for two atoms prepared in a factorized state $\Psi_0 = |10\rangle$. No spontaneous emission by the qubits is included.}
\end{figure}

with

$$u_1 = (\rho_{11} + \rho_{44} - \rho_{22} - \rho_{33})^2, \quad u_2 = 4 \left( |\rho_{23}|^2 + |\rho_{14}|^2 \right), \quad u_3 = 4 \left( |\rho_{24}|^2 - |\rho_{13}|^2 \right), \quad \text{(10)}$$

where $\rho_{22} = |c_1(t)|^2$, $\rho_{33} = |c_2(t)|^2$, $\rho_{23} = c_1(t)c_2^*(t)$ and $\rho_{44} = 1 - |c_1(t)|^2 - |c_2(t)|^2$. For the particular initial state considered, I have $\rho_{11} = 0$ and $\rho_{14} = 0$.

3. CHSH violation dynamics

Here I investigate the CHSH inequality violation for two qubits prepared initially in the state

$$|\psi(0)\rangle = |10\rangle. \quad \text{(11)}$$

I consider a cavity characterized by a high but experimentally feasible quality factor (the width is $\lambda = 10^{-1} \gamma_0$, with $\gamma_0$ being the Markovian decay rate of the qubits), and strong coupling conditions. I set the parameter $S = R/\lambda = 10$, indicating the strength of the coupling [9].

In figure 1, I plot the CHSH violation as a function of scaled time ($\tau = \lambda t$) and of the relative coupling strength $r_1$. During the evolution, the system passes through highly entangled states, strongly violating the CHSH inequality. Cycles of birth and death of non-locality follow one after the other, until for long times the inequality is finally satisfied. The sudden appearance of the violation is a consequence of the shared reservoir, providing an indirect coupling between the qubits. Such a reservoir-mediated interaction certainly plays a role also in the revivals of non-locality, which are also caused by the memory effects of the non-Markovian reservoir.

I note that sudden violation of the CHSH inequality appears up to a certain value of the relative strength of the coupling. In particular for the chosen coupling conditions no violation occurs when $r_1 > 0.6$. For stronger coupling conditions the region of violation in the $r_1$ space of parameters
widens; nevertheless it never reaches \(1/\sqrt{2}\). In fact for \(r_1 = r_2 = 1/\sqrt{2}\) the qubits are symmetrically coupled to the cavity, and entanglement and non-local properties in general cannot be created out of the factorized state in equation (11).

4. Including spontaneous emission

In this section, I study how the dynamical violation of the CHSH inequality is modified when including spontaneous emission for the two qubits. In [13], I have studied the entanglement time evolution of two entangled qubits interacting with the same Lorentzian structured reservoir (leaky cavity) and emitting independently outside the cavity due to spontaneous emission. By means of equations (1) and (2) of [13], I have solved the dynamics in the case of qubits having the same transition frequency, equally and resonantly coupled with the cavity. Here I generalize that master equation to include the case of different couplings between the qubits and the cavity. So the master equation describing the dynamics of our system (with some renaming of parameters) becomes

\[
\frac{\partial \hat{\rho}}{\partial t} = -i[H, \hat{\rho}] - \lambda(\sigma_+^1 a \hat{\rho} + \hat{\rho} a^\dagger) - 2\sigma_-^1 \hat{\rho} \sigma_+^1 - \frac{\gamma_1}{2} (\sigma_+^1 \hat{\rho} \sigma_-^1 - 2 \sigma_-^1 \hat{\rho} \sigma_+^1) - \frac{\gamma_2}{2} (\sigma_+^2 \hat{\rho} \sigma_-^2 - 2 \sigma_-^2 \hat{\rho} \sigma_+^2),
\]

with

\[
H = \alpha_1 W[r_1 \sigma_+^1 + r_2 \sigma_+^2] + \text{h.c.},
\]

where \(\sigma_+^1\) and \(\sigma_+^2\), \(\alpha_1\), \(r_1\) and \(r_2\), \(\lambda\) and \(W\) have been defined above, while \(a\) and \(a^\dagger\) are the annihilation and creation operators for the cavity mode and \(\gamma_{1/2}\) are the spontaneous emission rates for the two qubits. For the sake of simplicity I assume \(\gamma_1 = \gamma_2 = \gamma_3\).

As in the previous section, I consider the sudden appearance of CHSH violation for two qubits prepared in the state \(|\psi(0)\rangle = |10\rangle\). I study the dynamics for the same strong coupling conditions \(S = R/\lambda = 10\), in particular for the same width of the spectral distribution, and vacuum Rabi frequency. The two-qubit spontaneous emission decay rates are set equal to \(\gamma_3/\gamma_0 = 1/50\); therefore they are 50 times smaller than the vacuum Rabi frequency \(\gamma_0\). In figure 2, I show the CHSH violation as a function of time and of the relative coupling parameter \(r_1\). The effect of spontaneous emission is apparent: the long series of sharp revivals is dramatically damped out. For the chosen spontaneous emission parameter only one small revival is present. The amount of violation of the CHSH inequality reduces with increasing decay rate, and beyond a certain threshold value the CHSH inequality is always satisfied. For the coupling conditions chosen, such a threshold parameter is \(\gamma_3 = 1/9\gamma_0\).

5. Conclusion

I have investigated the time evolution of the violation of the CHSH inequality for a system of two qubits interacting with a common Lorentzian structured reservoir. The reservoir-induced correlations between the qubits drive the evolution of an initial factorized state as \(|\psi(0)\rangle = |10\rangle\) towards highly entangled states, strongly violating the CHSH inequality. After cycles of birth and death of non-locality, progressively damped out, the CHSH inequality becomes always satisfied. As a second step I consider a more realistic system in which spontaneous emission of the two qubits is taken into account. The time evolution of the CHSH violation well describes the dramatic changes that the introduction of spontaneous emission in this model brings to the dynamics of the system. Birth of non-local properties appears only if the spontaneous emission decay rate is below a certain threshold; even in that case oscillations are strongly damped and the inequality becomes permanently satisfied much earlier.

The dynamics of the CHSH violation was studied for a system of two non-interacting initially entangled qubits in two independent non-Markovian reservoirs [12, 14]. There, death and revivals of the violation have been observed due to the memory effects of the reservoirs. In that case, however, correlations between the qubits cannot be created starting from a factorized state; as a consequence, an initially separable state cannot evolve into an entangled one, possibly violating the CHSH inequality.

Acknowledgments

I thank S Maniscalco and J Pöllö for enlightening discussions and G Compagno and his group in Palermo for their kind hospitality and useful suggestions. I acknowledge the M Ehrnrooth Foundation for financial support.

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