OPEN STRINGS IN CONSTANT ELECTRIC AND MAGNETIC FIELDS

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ABSTRACT

Various properties of open strings in external constant E.M. fields are reviewed. In particular, the charged-particle pair production rate in an external electric field is evaluated, and shown to reduce to Schwinger’s formula in the limit of low-intensity fields. Open strings in external magnetic fields are shown to undergo an infinite number of phase transitions as the strength of the field increases.

1. Open Strings in a Constant Electric-Field Background

In the presence of an external constant electric field, quantum field theory predicts a nonvanishing probability for the creation of charged particle pairs. The rate of particle pair production can be estimated using simple semiclassical arguments. Namely, one represents a virtual charged particle pair in an external electric field of strength $E$ by a potential well of depth $2m$, where $m$ is the mass of the charged particle, superimposed to a linear potential

$$V = 0, \quad x < 0, \quad V = 2m - eEx, \quad x \geq 0.$$  \hspace{1cm} (1)

By denoting with $e$ the charge of the particle and using the standard semi-classical approximation one would get a rate $w \sim \exp(-O(1)m^2/|eE|)$. This estimate is quite accurate. Indeed, the exact rate for a particle of spin $s$ minimally coupled to an external electric field was found long ago by Schwinger$^1$:

$$w = \frac{2s + 1}{8\pi^3} \sum_{k=1}^{\infty} (-1)^{(2s+1)(k+1)} (eE/k)^2 \exp(-k^2m^2/|eE|).$$  \hspace{1cm} (2)

A natural question arising at this point is whether it is possible to find an exact formula for the pair-production rate in string theory. The answer to this question is in the affirmative, in the case of the open bosonic and supersymmetric strings$^2$.

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$^\dagger$ On leave of absence from INFN, Sez. di Pisa, Pisa, Italy.
Open bosonic strings in an external constant e.m. field are exactly soluble, to lowest order in the string coupling constant. Their world-sheet action reads, in the gauge \( A_\mu = (1/2) F_\nu \sigma X^\nu \) and for a state of total electric charge \( e_1 + e_2 \)

\[
S = -\frac{1}{2\pi} \int d\sigma d\tau \partial_a X^\mu \partial^a X_\mu + \frac{1}{2} e_1 \int d\tau F_{\mu\nu} X^\nu \partial_\tau X^\mu |_{\sigma=0} + \frac{1}{2} e_1 \int d\tau F_{\mu\nu} X^\nu \partial_\tau X^\mu |_{\sigma=\pi}.
\]

(3)

Here we set \( \alpha' = 1/2 \). The world-sheet coordinates are \( \sigma \in [0, \pi] \) and the proper time \( \tau \). The 26-dimensional target-space metric is \( \eta_{\mu\nu} = (-1, +1, ..., +1) \). The e.m. field couples only through boundary terms, thus the equations of motion of all coordinates are the same as in the free-string case

\[
\partial_a \partial^a X^\mu = 0.
\]

(4)

The only effect of the e.m. field is to modify the boundary conditions obeyed by the \( X^\mu \). Notice that the target-space metric remains flat to lowest order in the string coupling constant \( g = e^{\phi} \). The back reaction due to the presence of a non-zero stress-energy tensor is in fact of order \( g \) with respect to the tree-level metric. When the external field is purely electrical one can choose a coordinate system in which only the component \( F_{01} \) of \( F_{\mu\nu} \) is non-vanishing. By introducing standard light-cone coordinates \( X^\pm = (X^0 \pm X^1)/\sqrt{2} \) the boundary conditions read

\[
\partial_\sigma X^\pm = \pm \beta_1 \partial_\tau X^\pm, \quad \sigma = 0,
\]

\[
\partial_\sigma X^\pm = \mp \beta_2 \partial_\tau X^\pm, \quad \sigma = \pi.
\]

(5)

In Eq. (3) \( \beta_i \equiv \pi e_i E, \ E = F_{01} \). The boundary conditions and mode expansion for the transverse coordinates \( X^i \ i = 2, ..., 25 \) are the same as for the free open string. Equations (4,5) imply instead the following mode expansion for the coordinates \( X^\pm \)

\[
X^\pm = x^+ + ia_0^+ \phi_0^+ + i \sum_{n=1}^{\infty} a_n^+ \phi_n^+(\sigma, \tau) - h.c.,
\]

\[
\phi_n^\pm = (n \mp i\epsilon)^{-1/2} e^{-i(n\mp i\epsilon)\tau} \cos[(n \mp i\epsilon)\sigma \pm \arctan(\beta_1)],
\]

\[
(a_0^\pm) = \pm ia_0^\pm, \quad \epsilon = \frac{1}{\pi}(\arctan(\beta_1 \mp \arctan(\beta_2)).
\]

(6)

Notice that for small electric fields, that is for fields \( e_i E \ll \alpha'^{-1}, \epsilon \approx 2\alpha'(e_1 + e_2)E \). The main effect of the non-vanishing electric field is an imaginary shift in the free-string frequencies \( n \rightarrow n \mp i\epsilon \). Moreover, similarly to the case of a particle in an external magnetic field, the center-of-mass coordinates \( x^\pm \) do not commute, but obey instead the equation

\[
[x^+, x^-] = \frac{-i\pi}{\beta_1 + \beta_2}.
\]

(7)

By using the above mode expansion one easily finds the Virasoro operators \( L_n \). They can be written as a sum of a “transverse” component \( L_n^\perp \), involving only
the coordinates $X^2, \ldots, X^{25}$, and identical to the free-string one, and a “longitudinal” one, $L^\parallel$. The component $L^\parallel_0$ reads, for $\epsilon > 0$,

$$L^\parallel_0 = - \sum_{n=0}^{\infty} (n + i\epsilon)(a_n^+)^*a_n - \sum_{n=1}^{\infty} (n - i\epsilon)(a_n^-)^*a_n^+ + \frac{1}{2}i\epsilon(1 - i\epsilon).$$  \hspace{2cm} (8)

Besides the imaginary shift in frequencies, the main difference with the free-string case is the shift in the vacuum energy. This shift can be determined either by spectral flow (with a twist $\theta = i\epsilon)^2$, or by imposing that the Virasoro algebra closes in the standard form $[L_m, L_n] = (n - m)L_{n+m} + 1/12 cm(m^2 - 1)^3$.

The vacuum to vacuum transition amplitude $\langle 0 | \exp(-iTH) | 0 \rangle$ is given in terms of the free-energy density $F$ by a standard field-theory argument as $\exp(-iTVF)$. Thus the total pair-production rate $w$, equal to the probability of vacuum decay, is $w = -2\Im F$.

In open string theory the free-energy density $F$ is given, at one loop, by a sum of four terms arising from the four different geometries open strings can propagate on: the torus, Klein bottle, annulus, and Moebius strip

$$-iTVF = \frac{1}{2} T + \frac{1}{2} K + \frac{1}{2} A + \frac{1}{2} M$$ \hspace{2cm} (9)

The first two geometries correspond to closed string states: they do not depend on the end-point charges $e_1$ and $e_2$, since the torus and the Klein bottle have no boundaries, thus they do not give rise to terms contributing to $\Im F$.

The annulus contribution reads

$$A = \frac{1}{2} \lim_{\delta \to 0} \int_\delta^\infty dtt^{-1}\Tr e^{-\pi t(L_0 - 1)} \approx \frac{1}{2} \Tr \log(L_0 - 1).$$ \hspace{2cm} (10)

The trace in Eq. (10) is taken over all variables (oscillators, momenta etc.) and over all possible charge sectors of the open string. These charge sectors, in a consistent model, are determined by the embedding of $U(1)_{e.m.}$ into the open string gauge group.

The integration variable $t$ is the (real) modular parameter of the annulus.

The M"{o}bius strip has only one boundary, and thus contributes only to sectors where $e_1 = e_2$. Its contribution to the free-energy is

$$M = \pm \frac{1}{2} \lim_{\delta \to 0} \int_\delta^\infty dtt^{-1}\Tr \mathcal{P} e^{-\pi t(L_0 - 1)}. \hspace{2cm} (11)$$

The operator $\mathcal{P}$ implements the change of orientation of the open string $\sigma \to \pi - \sigma$. The M"{o}bius-strip contribution, together with the Klein-bottle one, is needed in order to cancel the small-$t$ divergences arising already in the free-string annulus integral. Only after annulus, Klein-bottle and M"{o}bius-strip contributions are added, and only for a specific choice of the gauge group, do the free-string small-$t$ divergencies cancel. The correct gauge group turns out to be $SO(8192)$ for the bosonic string, and $SO(32)$ for the open superstring.\end{document}
Cancellation of divergencies in the presence of an external field requires to take into proper account the back reaction on the target-space metric induced by the presence of the background field. However, since we are only interested in the imaginary part of the free-energy, which receives no contribution form the small-$t$ region of integration, we can safely ignore this complication.

The annulus contribution to the free-energy can be evaluated straightforwardly, since the Virasoro operators are sums of free oscillators

$$A = \sum_{e_1, e_2 \in Q} \lim_{\delta \to 0} V T \frac{\beta_1 + \beta_2}{(2\pi)^2} \int_\delta^\infty dtt^{-1} \int \frac{d^{24}p}{(2\pi)^{24}} e^{-\pi tp^2/2} T^\perp T^\parallel. \quad (12)$$

In this equation $Q$ denotes the set of boundary charges determined by the embedding of $U(1)_{e.m.}$ into the open-string gauge group. $T^\perp$ and $T^\parallel$ arise from taking the trace over transverse and longitudinal oscillators, respectively. The integration over transverse momenta is standard. The normalization factor in front of the integral is the only place where the $\beta_i$ enter explicitly, instead of $\epsilon$, and it is determined by noticing that the commutation relation (7) implies that the $x^\pm$ phase-space integration measure is

$$\frac{|\beta_1 + \beta_2|}{2\pi^2} dx^+ dx^-.$$

(13)

Obviously $T^\perp$ has the same form as for the free open string, since it involves only transverse oscillators

$$T^\perp = \eta(it/2)^{-24}\eta(it/2)^2. \quad (14)$$

Notice the contribution $\eta^2$ coming from the coordinate-reparametrization ghosts.

$T^\parallel$ depends on the external electric field and reads

$$T^\parallel = e^{-\pi t(\epsilon^2/2-1/12)} \left[ 2i \sin(\pi |\epsilon| t/2) \prod_{n=1}^\infty \left| 1 - e^{-\pi t(n+ie)} \right|^2 \right]^{-1}. \quad (15)$$

This quantity is imaginary, as imposed by equation (9). The free-energy $F$ though, has a nonzero imaginary part, arising from the integration in $t$. Indeed, $T^\parallel$ has the form

$$T^\parallel = [2i \sin(\pi |\epsilon| t/2)][-1] f(t), \quad (16)$$

with $f(t)$ regular for $t > 0$, and thus it has simple poles for positive $t$, located at $t = 2k/|\epsilon|$, $k$ integer. When integrating in $t$ one has to deform the contour of integration so as to avoid these poles. The correct prescription on the contour is, as expected, the one allowing for the analytic continuation $t \to it$. Thus, $\Im F$ reduces to a sum over residues at the poles, according to Cauchy’s theorem

$$A = \sum_{e_1, e_2 \in Q} V T \frac{\beta_1 + \beta_2}{(2\pi)^{26} \epsilon} \sum_{k=1}^\infty (-1)^{k+1} e^{-\pi |\epsilon| k (|\epsilon|/k)^{13}} \eta(ik/|\epsilon|)^{-24}. \quad (17)$$

This formula can be recast in the following simple form

$$A = \sum_{e_1, e_2 \in Q} V T \frac{\beta_1 + \beta_2}{2\epsilon} \sum_{k=1}^\infty (-1)^{k+1} Z(2k/|\epsilon|), \quad (18)$$
where $Z(t)$ is the free-string partition function on the annulus. Equation (18) holds in any compactification that does not involve the $X^\pm$ coordinates. Notice also that the term under the $Q$-summation sign in Eq. (18) gives the complete pair-production rate for open-string states with $e_1 \neq e_2$, since in this case $F$ receives no contribution from the Möbius strip. To compare Eq. (18) with Schwinger’s result (2) we must compactify the bosonic string to four dimensions and recall that four distinct open-string sectors contribute to the production rate of a particle pair of given charge $e = e_1 + e_2$. These are the sectors with boundary charges ($e_1, e_2$), ($e_2, e_1$), ($-e_1, -e_2$), and ($-e_2, -e_1$). By denoting with $M_S$ the mass of an open-string state $S$ and after a simple computation one finds

$$w = \sum_S \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\beta_1 + \beta_2}{8\pi^4 \epsilon} \left( \frac{|\epsilon|}{k} \right)^2 e^{-k\pi(M_S^2 + \epsilon^2)/|\epsilon|}.$$  (19)

When $\beta_i \ll 1$, $\epsilon \to (e_1 + e_2)E$, and Eq. (19) reduces to Eq. (2).

When the two boundary charges are equal, only two open-string sectors contribute to the production rate of a given particle pair: $(e, e)$ and $(-e, -e)$. The annulus amplitude in Eq. (17), therefore, would reduce to one-half of Schwinger’s result. This discrepancy is resolved by taking into account the contribution of the Möbius strip to equation (9).

For a given state $S$ this contribution is equal to annulus one, up to a sign. This sign is positive or negative according to whether the state $S$ is even or odd under $P$ (defined in Eq. (11)). The sum of the Möbius and annulus contribution thus reduces to Schwinger’s formula for all states which are not eliminated from the physical spectrum by the Möbius projection.

An interesting property of Eqs. (17,18) is that they diverge for a (finite) critical value of the electric field, namely when the force applied by the electric field on either of the boundary charges equals the string tension

$$\min_i |e_i E| = (2\pi\alpha')^{-1}.  \quad (20)$$

This instability arises already at the classical level. It is interesting to notice that when $(e_1 + e_2) \to 0$, but $e_1 \neq 0$, the annulus partition function $A$ does not reduce to the free-string one. In fact, by setting $\beta_1 = -\beta_2 + \Delta$, and taking the limit $\Delta \to 0$, one finds $\epsilon = \Delta/(1 - \beta_1^2) \pi + O(\Delta^2)$. Using Eqs. (12,14,15) one then finds that the neutral-string annulus amplitude $A_{\text{neutral}}$ and the free-string one $A_{\text{free}}$ are related by

$$A_{\text{neutral}} = (1 - \beta_1^2)A_{\text{free}}.  \quad (21)$$

This formula agrees with ref.3.

Open superstrings in an external field can be solved exactly at lowest order in the string coupling constant by using the same procedure already outlined in the bosonic-string case. The world-sheet action of the critical 10-dimensional superstring reads

$$S = S_{\text{bosonic}} + \frac{i}{2\pi} \int d\sigma d\tau \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu + \ldots$$
\[ + \frac{i}{4} e_1 \int d\tau F_{\mu\nu} \bar{\psi}^\mu \rho^\nu \psi^\nu \bigg|_{\sigma=0} + \frac{i}{4} e_2 \int d\tau F_{\mu\nu} \bar{\psi}^\mu \rho^0 \psi^\nu \bigg|_{\sigma=\pi}. \tag{22} \]

The \( \rho^i \) are the 2-dimensional Dirac matrices.

The appropriate boundary conditions on the 2-dimensional bosons \( X^\pm \) are given in Eq. (5). The boundary conditions on their fermionic partners \( \psi^\pm \) are

\[
(1 \mp \beta_1) \psi^\pm_R = (1 \pm \beta_1) \psi^\pm_L, \quad \sigma = 0, \\
(1 \pm \beta_2) \psi^\pm_R = -(1 \mp \beta_2) \psi^\pm_L, \quad \sigma = \pi. \tag{23} \]

In this equation the subindices \( R, L \) denote the 2-dim. handedness of the fermions. The variable \( a \) is equal to 0 when the fermions are given anti-periodic (Neveu-Schwarz) boundary conditions and to 1 when they are given periodic (Ramond) boundary conditions. The transverse fermionic coordinates obey standard free-superstring boundary conditions.

All fermionic coordinates obey free equations of motion which, together with Eq. (23), completely determine the fermion mode expansion. Canonical quantization gives rise to the Virasoro operators \( L_n = L_n^F + L_n^B \). The bosonic contribution to these operators is denoted by \( L_n^B \) and is the same as in the bosonic open string. The fermionic contribution \( L_n^F \) can be further decomposed into a sum of two terms: \( L_n^{F,\perp} \), containing only transverse oscillators, and identical to the corresponding operator for a free open superstring, and \( L_n^{F,\|} \), depending only on the fermions \( \psi^\pm \). By introducing the canonical anti-commuting operators \( d^\alpha_\mu \) obeying \( \{d^\alpha_\mu, d^\beta_\nu\} = \eta^{\mu\nu} \delta^\alpha_\beta + m \), one finds, in particular,

\[ L_0^{F,\|} = \sum_{n \in \mathbb{Z} + 1/2 + a/2} -(n + i\epsilon) : d^{-}_n d^+_n : + \frac{a}{8} \frac{i\epsilon}{2} (a - i\epsilon), \quad \epsilon > 0. \tag{24} \]

The shift in the vacuum energy can be determined by spectral flow, once a normal-ordering prescription for \( d^\pm_0 \) is given, or by closure of the Virasoro algebra in exact analogy with the bosonic case. Notice that the shift in the Ramond-sector vacuum energy exactly cancels between bosons and fermions (cfr. Eq. (8)).

The annulus amplitude now reads

\[ \mathcal{A} = \sum_{e_1, e_2 \in \Phi} \sum_{a, b} \lim_{\delta \to 0} VT \frac{\beta_1 + \beta_2}{(2\pi)^2} C \left[ \begin{array}{c} a \\ b \end{array} \right] \left[ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] \int_\delta^\infty dt t^{-1} \text{Tr} a \left[ (-1)^{bf} e^{-\pi t(L_0 - 1)} \right]. \tag{25} \]

\( F \) is the fermion-number operator commuting with all world-sheet bosons and anti-commuting with the world-sheet fermions. The values of the parameters \( a \) and \( b \) are 0 or 1, and \( a \) is defined as before to be 0 in the Neveu-Schwarz sector and 1 in the Ramond sector. The weights in Eq. (25) are chosen so as to implement space-time supersymmetry in 10 dimensions

\[ C \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = -C \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = -C \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] = \pm C \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = \frac{1}{2}. \tag{26} \]
the trace in Eq. (23) factorizes into the product of a purely bosonic contribution, which is evaluated using Eqs. (12, 14, 15), and a fermionic one, which reads

\[
\text{Tr}_a (-1)^b F e^{-\pi t L_0^F} = T_{F}^{\text{ghost}} T_{F}^{\perp} T_{F}^{\parallel},
\]

\[
T_{F}^{\text{ghost}} = \text{Tr}_a (-1)^b F e^{-\pi t L_0^F}^{\text{ghost}} = \eta(it/2)/\Theta \left[ \begin{array}{c} a \\ b \end{array} \right] (0|it/2), \quad a + b \neq 2,
\]

\[
T_{F}^{\perp} = \text{Tr}_a (-1)^b F e^{-\pi t L_0^F}^{\perp} = \left\{ \Theta \left[ \begin{array}{c} a \\ b \end{array} \right] (0|it/2)/\eta(it/2) \right\}^3,
\]

\[
T_{F}^{\parallel} = \text{Tr}_a (-1)^b F e^{-\pi t L_0^F}^{\parallel} = \Theta \left[ \begin{array}{c} a - 2i|\epsilon| \\ b \end{array} \right] (0|it/2)/\eta(it/2).
\]

The only term depending on the external electric field in Eq. (27) is, as expected, \( T_{F}^{\parallel} \).

Here the conventions on theta functions with characteristics are as in \(^2\). The contribution \( T_{F}^{\text{ghost}} \), arising from the world-sheet supersymmetry ghosts, takes a special form in the presence of a fermionic zero-mode (when \( a + b = 2 \)). We need not to consider it because the contribution of the \( a + b = 2 \) sector to \( \mathcal{A} \) gets cancelled by \( T_{F}^{\perp} \), since the theta function with \( a = b = 1 \) vanishes identically.

It is immediate to notice that the poles in the \( t \)-integration arise only from the trace over bosons. Thus, the annulus amplitude can be evaluated exactly as in the bosonic case. The residue at the pole \( t = 2k/|\epsilon| \) is given by a bosonic term times the fermionic term

\[
T_{F}(t = 2k/|\epsilon|) = (-1)^k e^{\pi k |\epsilon|} \left\{ \Theta \left[ \begin{array}{c} a \\ b \end{array} \right] (0|ik/|\epsilon|)/\eta(ik/|\epsilon|) \right\}^4. \quad (28)
\]

Notice that this term contributes an extra \((-1)^k\) to the sum over residues when \( a = 1 \), that is for space-time fermions, in analogy with Schwinger’s formula Eq. (2). After a few manipulations, analogous to the ones performed in the bosonic case, one arrives at the following formula for the vacuum decay rate when \( 10 - D \) dimensions are compactified

\[
w = \frac{1}{(2\pi)^{D-1}} \sum_S \frac{\beta_1 + \beta^2}{\pi \epsilon} \sum_{k=1}^{\infty} (-1)^{(k+1)(a_S+1)}(|\epsilon|/k)^{D/2} e^{-\pi k M^2_S/|\epsilon|}. \quad (29)
\]

Here the variable \( S \) labels the string states surviving the GSO and Möbius projections, \( a_S = 0 \) for states in the Neveu-Schwarz sector and \( a_S = 1 \) for Ramond-sector states. In \( D = 4 \) and for small fields this expression reduces to Eq. (2). Notice that in Eq. (29) the production rates for each particle species diverge at \( \beta_i = 1 \). This behavior is different from the bosonic-string one, where the production rates vanished species by species, but their sum diverged, thanks to the stronger divergence of the statistical sum over all string states.

Finally, one may notice that equation (5) interpolates between free-string (Neumann) boundary conditions, at \( \beta_i = 0 \), and reflecting boundary conditions at
\( \beta_i = 1 \). This latter case corresponds to the critical value of the electric field given in Eq. (20). At \( \beta_i = 1 \), thus, the world-sheet dynamics of the bosonic string becomes formally equivalent to that of a black-hole solution of 2-dimensional gravity coupled to 24 free massless scalars \( ^7 \) (see also K. Schoutens’s and E. Verlinde’s contributions to these proceedings).

2. Open Strings in a Constant Magnetic-Field Background

Open strings in an external purely magnetic field exhibit new interesting features already at tree-level \( ^8 \). Let us examine at first the bosonic string. In this case by choosing a coordinate system in which the only non-vanishing component of the magnetic field is \( H \equiv H_{12} \), the Virasoro operator \( L_0 \) reads

\[
L_0 = (2b_0^\dagger b_0 + 1) \frac{|h|}{2} - \frac{1}{2} h^2 - |h| \sum_{k=1}^{\infty} (a_k^\dagger a_k - b_k^\dagger b_k) + L_0^{free}.
\]  (30)

Here \( L_0^{free} \) denotes the free-string Virasoro operator of all the transverse coordinates \( X^0, X^3, \ldots, X^{(D-1)} \), and we introduced the complexified creation and annihilation operators

\[
\sqrt{2} a_k = \alpha_k^1 + i \text{ sign } (h) \alpha_k^2, \quad \sqrt{2} b_k = \alpha_k^1 - i \text{ sign } (h) \alpha_k^2.
\]  (31)

Equation (30) differs from the free string expression by terms involving

\[
h = \frac{1}{\pi} (\arctan 2 \alpha' e_1 \pi H + \arctan 2 \alpha' e_2 \pi H).
\]  (32)

Notice in particular the presence of a magnetic-dipole coupling, non-linear in the magnetic-field strength

\[
|h| \sum_{k=1}^{\infty} (a_k^\dagger a_k - b_k^\dagger b_k) = h S_{12},
\]  (33)

where \( S_{\mu\nu} \) is the target-space spin operator. In particular, in four dimensions, \( S_{12} \) is the spin component along the magnetic field. Since, in the low-field limit \( (\alpha' e H \ll 1) \) \( h \rightarrow 2\alpha'(e_1 + e_2) H \), it is apparent that open string states have all a gyromagnetic ratio \( g = 2^{9,10} \). Moreover, in \( D = 4 \) and in the low-field limit, Eq. (30) itself reduces to the well-known field-theoretical formula giving the energy levels of a particle of mass \( M \), spin \( \vec{S} \), and gyromagnetic ratio \( g \) in an external magnetic field (see for instance \( ^{11} \))

\[
E^2 = (2n + 1) e H - g_s e \vec{H} \cdot \vec{S} + p_3^2 + M^2.
\]  (34)

To arrive at this formula, it suffices to recall that the wave equation of a string state \( \Psi \) is

\[
(L_0 - 1)\Psi \equiv \alpha'[E^2 - (p^0)^2]\Psi = 0, \tag{35}
\]

and to notice that the eigenvalues \( n \) of \( b_0^\dagger b_0 \) are the Landau levels. The Virasoro operator \( L_0 \) possesses the usual tachionic mode of the free bosonic string, but it also shows additional tachionic modes when \( H \neq 0 \). By direct examination of Eq. (30)
one may see that only the highest-helicity states belonging to the first ("parent") Regge trajectory can become tachyonic. By denoting with $|0\rangle$ the Fock vacuum for the string oscillators $a_k$, $b_k$ and $\alpha_k$, these states read

$$\Psi(m) = \left(a_1^\dagger\right)^m |0\rangle. \tag{36}$$

The value of $\alpha' E^2$ on them is

$$\left(\alpha' E^2\right)\Psi(m) = \left[\frac{|h|}{2} - \frac{h^2}{2} + (1 - |h|)m - 1\right] \Psi(m). \tag{37}$$

For fixed value of the magnetic field, and thus of $|h|$, $\alpha' E^2$ is negative on all states with

$$m < \frac{1 + |h|(|h| - 1)/2}{1 - |h|}. \tag{38}$$

In particular, the highest helicity of the open-string massless vector ($m = 1$) is always tachyonic when $H \neq 0$. This instability is well-known in Yang-Mills theory$^{12}$. The $m = 0$ instability of open bosonic strings was also noticed in ref.$^3$. There is an infinite number of states given by Eq. (37), correspondingly, as the intensity of the external magnetic field increases, open strings may undergo an infinite number of phase transitions. As $H$ approaches infinity $h \to 1$. In this limit all states $\Psi(m)$ would become tachyonic, in the standard superstring vacuum, with a common (negative) square energy $\alpha' E^2 = -1$.

The meaning of these phase transitions is that the vacuum where the VEVs of the $\Psi(m)$ vanish is unstable, when the magnetic field passes the threshold value given by Eq. (38): in the stable vacuum $\Psi(m)$-condensates should appear (cfr.$^{12,13}$).

The previous analysis can be straightforwardly extended to the open superstring.

In the superstring case one must introduce fermionic oscillators, either in the Ramond or Neveu-Schwarz sector, besides the bosonic ones. The fermionic oscillators can be complexified in the same way as the bosonic coordinates

$$\sqrt{2}d_k = d_k^1 + i \text{sign}(h)d_k^2, \quad \sqrt{2}\tilde{d}_k = d_k^1 - i \text{sign}(h)d_k^2 \tag{39}$$

Using the same notations as before one finds that the Virasoro operator $L_0$ takes the following form in the Ramond sector

$$L_0 = (2n + 1)\frac{|h|}{2} + |h|d_0^\dagger d_0 - \frac{|h|}{2} - |h| \sum_{k=1}^{\infty} \left(d_k^\dagger d_k - \tilde{d}_k^\dagger \tilde{d}_k + a_k^\dagger a_k - b_k^\dagger b_k\right) + L_0^{free}. \tag{40}$$

Recalling that the wave equation in the Ramond sector takes the form $L_0 \Psi = 0$ one easily finds that all Ramond states have positive square energy. This result follows from the inequality $h \leq 1$. This inequality is due to the fact that in string theory the magnetic-dipole coupling is non-linear (see Eq. (32)). The field-theoretical formula (34) with $g = 2$ would instead give instabilities for any spin larger than 1/2.
The Virasoro operator in the Neveu-Schwarz sector reads

\[ L_0 = (2n + 1) \frac{|h|}{2} - |h| \sum_{k=1/2}^{\infty} \left( d_k^* d_k - \tilde{d}_k^* \tilde{d}_k + a_k^* a_k - b_k^* b_k \right) + L_0^{\text{free}}. \]  

The Neveu-Schwarz wave equation is \((L_0 - 1/2) \Psi = 0\), this implies that there exist Neveu-Schwarz physical states (i.e. states surviving the GSO projection) with \(\alpha'E^2 < 0\), when \(H\) is sufficiently large. A brief analysis of Eq. (11) shows that these states have the form

\[ (a_1^m d_{1/2}^0)_{\text{NS}}, \]  

and that they become tachionic when

\[ m < \frac{|h|}{2(1 - |h|)}. \]  

They are, again, highest-helicity states belonging to the parent Regge trajectory of the Neveu-Schwarz sector. When \(h \to 1\) the negative square energy of all these states becomes \(\alpha'E^2 = -1/2\).

An interesting and difficult problem still to be solved is to find an ansatz for the scalar potential of the fields given in Eqs. (36,42). Knowledge of this potential would allow a detailed study of high magnetic field phase transitions in string theory.

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