Tame topology and non-integrability of dynamical systems

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Abstract
In this paper we study the general concept of integrability in the broad sense within the frame of differential Galois theory. We concentrate on the gradient systems which are not integrable. In spite of it, if we consider them as the real dynamical systems, they have trajectories with finiteness properties of o-minimal type.

1 Introduction

From the time of publication of fundamental papers by Sergey Lvovich Ziglin [24, 25], the interest in the relation of Galoisian approach to non-integrability and complex behaviour of dynamical systems has generated a series of studies on the algebraic aspects of “chaos theory”. In the second half of the XX century several interesting papers were published

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by R. Churchill, J.J. Morales-Ruiz, J.M. Peris, D.L. Rod and B.D. Sleeman [6], [16], [20]. Quite recently K. Yagasaki and S. Yamanaka studied Hamiltonian systems with two degrees of freedom and established a deep connection between the existence of transverse heteroclinic orbits and the non-integrability [22, 23]. In general, the relation of Galoisian obstruction to integrability and chaotic dynamics is quite complicated. In this paper we give a curious example of a non-integrable gradient system which is not only non-chaotic but also is such that over \( \mathbb{R} \) all its trajectories have tame topology in the sense of o-minimal geometry (cf. [10] and [11]).

2 Preliminaries

2.1 Integrability in the broad sense

In this section we follow the notations of V.I. Arnold [2]. Let \( \dot{x} = X_H \) be a Hamiltonian system and \( F : M^{2n} \to \mathbb{R} \) be a first integral of it (i.e., \( \{H, F\} = 0 \)). It is possible to choose canonical coordinates \( x_1, \ldots, x_n, y_1, \ldots, y_n \) in a neighborhood of a point \( x_0 \in M \) with \( dF(x_0) \neq 0 \), such that \( F(x, y) = y_1 \). The Hamiltonian function \( H \) does not depend on \( x_1 \) in these coordinates, therefore putting \( F = y_1 = \text{const} \) gives us for \( j \geq 2 \)

\[
\dot{x}_j = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial x_j}
\]

which is a Hamiltonian system with \( n - 1 \) degrees of freedom.

Therefore, having one first integral enabled us to lower the degree of freedom of the system by one. Now to generalize this, assume that we have \( n - 1 \) first integrals that are in involution, it can be shown that the degree of freedom is reduced to one and the system will be “integrable by quadratures”. This generalization can be stated as follows

**Theorem 2.1.1.** Let \( M \) be a 2n-dimensional symplectic manifold. A Hamiltonian vector field \( X_H \) on \( M \) is integrable in the sense of Liouville, if there exist \( n \) first integrals \( F_j : M \to \mathbb{R}, 1 \leq j \leq n \), such that at every point of \( M \) satisfy the following

1. \( dF_1, \ldots, dF_n \) are linearly independent over a dense open set \( U \subset M \);
2. \( \{F_i, F_j\} = 0 \) for all \( i, j \).

The involutivity condition (i.e., the second property) implies that the Hamiltonian system has an abelian Lie algebra \( \mathcal{F}_a \) of functionally independent first integrals \( F_i \) with respect to the Poisson brackets, and an abelian Lie algebra \( \mathcal{S}_a \) of symmetries which
preserve the first integrals. O. Bogoyavlenskij [4] observed that these first integrals may be non-involutive and the symmetries may be non-symplectic, therefore, the two Lie algebras are no longer identified and gave the following definition.

**Definition 2.1.2.** The dynamical system \( \dot{x} = X \) on a smooth manifold \( M^n \) is called integrable in the broad sense if there exist

1. \( k \) functionally independent first integrals \( F_1, ..., F_k, \ 1 \leq k < n \),
2. an abelian \((n - k)\)-dimensional Lie algebra \( S \) of symmetries (i.e., vector fields \( Y_i \) such that \( Y_i(F_j) = 0, \ \forall i, j \)), that are linearly independent at the every point of \( M \).

In Liouville integrability the dynamical system is Hamiltonian and these two conditions are unified in the single condition of involutivity of the first integrals. In general, a Hamiltonian system that is integrable in the Liouville sense is also integrable in the broad sense but the converse is not always true, therefore integrability in the broad sense is a generalization of the Liouville integrability. A. Maciejewski and M. Przybylska in [15] called the integrability in the broad sense, “B-integrability”.

A dynamical system on an \( n \)-dimensional manifold \( N \) using the cotangent lift can be included in the \( 2n \)-dimensional cotangent bundle \( T^*N \). Let \( \phi : N \longrightarrow N \) be a diffeomorphism, then we have the following

\[
\begin{array}{ccc}
T^*N & \xrightarrow{\tilde{\phi}} & T^*N \\
\downarrow & & \downarrow \\
N & \xrightarrow{\phi} & N
\end{array}
\]

where \( \tilde{\phi} \) is called the cotangent lift of \( \phi \). If \( X \) is a vector field on the manifold \( N \), using \( X \) one can define the dynamical system

\[ \dot{x}_i = X_i, \ i = 1, ..., n. \quad (2) \]

Furthermore, let us suppose a function \( f : T^*N \longrightarrow \mathbb{R} \) defined as follows

\[ f(p) := \langle p, X(x) \rangle, \ p \in T^*_xN. \quad (3) \]

it is easy to show that the Hamiltonian vector field \( X_f \) associated to \( f \) is the following
\[ X_f = \left( X^i, -\frac{\partial f}{\partial x^i} \right). \]  

(4)

and we say that \( X_f \) is the cotangent lift of \( X \). Let \( X \) be a vector field on \( N \) and its cotangent lift be denoted by \( Y \).

**Theorem 2.1.3.** If the vector field \( X \) is (meromorphically) integrable in the broad sense, then \( Y \) is (meromorphically) Liouville integrable.

**Proof.** See Ayoul and Zung [3]. \[ \square \]

In [24, 25] S.L. Ziglin started a new approach to integrability using the monodromy group of the variational equation. Later Morales and Ramis [17, 18], took another approach to the integrability using the differential Galois group of the variational equation and further developed the theory. Morales and Ramis also stated that all the previous Galoisian approaches to the Ziglin theory had a common inconvenience, which is they restrict to the case where the variational equations belong to the Fuchsian class (i.e., their singularities must be regular) and this restriction is no longer necessary (for more details see the excellent survey [7]). We now state the theorem by Morales and Ramis which has attracted many researchers to the subject.

**Theorem 2.1.4.** Assume that a complex analytical Hamiltonian system is meromorphically completely integrable in a neighborhood of the integral curve \( x = x(t) \) (i.e., the solution of the Hamiltonian system). Then the identity component of the differential Galois group of the variational equations along the integral curve is Abelian.

As a remark, if the variational equation has irregular singularity at infinity, then Theorem 2.1.4 only provides obstruction to the existence of rational first integrals.

Later in [19] Morales, Ramis and Simó proved this result for higher variational equations. In [3] Ayoul and Zung proved that the differential Galois group of the variational equations (of any order) of the original system is isomorphic to its counterpart for the cotangent lifted system. Therefore, the condition that the system is integrable in Theorem 2.1.4 can be generalized to being integrable in the broad sense and the group being the differential Galois group of variational equations of any order (see also [15] Theorem 13).

**Theorem 2.1.5.** Assume that a dynamical system is meromorphically integrable in the broad sense in a neighborhood of the integral curve \( x = x(t) \) (i.e., the solution of the dynamical system). Then the identity component of the differential Galois group of the variational equations (of any order) along the integral curve is Abelian.
2.2 Pfaffian manifolds

In this section we gather together some facts from the Khovanskii’s theory of fewnomials [12], which we shall use to the study of gradient trajectories. We assume that the ambient space is \( \mathbb{R}^n \) and we consider real analytic foliations of codimension 1 (cf. [8] Section I.4).

For the convenience of the reader, let us recall the definition of a Pfaffian hypersurface and its topological properties.

**Definition 2.2.1.** A Pfaffian hypersurface of \( \mathbb{R}^n \) is a triple \((V, \mathcal{F}, M)\), where

1. \( M \) is an open semianalytic set of \( \mathbb{R}^n \),
2. \( \mathcal{F} \) is a foliation of codimension 1 defined in an open neighbourhood of the closure of \( M \),
3. \( V \) is a leaf of \( \mathcal{F}_M \), i.e., is a connected maximal integral submanifold of the restricted foliation,
4. \( \text{Sing} \mathcal{F} \cap M = \emptyset \).

A Pfaffian hypersurface is called *separating* if \( M \setminus V \) has two connected components and \( V \) is their common boundary in \( M \). A Pfaffian hypersurface has the *Rolle property* (i.e., is *Rollian*) if each analytic path \( \gamma : [0, 1] \rightarrow M \), such that \( \gamma(0), \gamma(1) \) are points in \( V \), has at least one point \( \gamma(t) \) such that the tangent vector \( \gamma'(t) \) is tangent at the foliation \( \mathcal{F} \) (i.e., if at this point \( \mathcal{F} \) is determined by a local generator \( \omega \) then \( \gamma'(t) \in \ker \omega \)). By the theorem of Khovanskii-Rolle, a separating hypersurface is Rollian but the converse is not true.

In the context of Pfaffian geometry, Khovanskii’s finiteness theorem has the following form (cf. [8] Theorem 1).

**Theorem 2.2.2.** Let \( M \) be an open semianalytic set in \( \mathbb{R}^n \) and \( X \subset M \) semianalytic and bounded in \( \mathbb{R}^n \). For each finite collection of Pfaffian hypersurfaces \((V_1, \mathcal{F}_1, M), \ldots, (V_p, \mathcal{F}_p, M)\) which have the Rolle property for the paths in \( X \), there exists a number \( b_0 \in \mathbb{N} \) (depending only on \( M, X, \mathcal{F}_1, \ldots, \mathcal{F}_p \)) such that the number of connected components of \( X \cap V_1 \cap \cdots \cap V_p \) is smaller than \( b_0 \).

**Remark 2.2.3.** Separating Pfaffian hypersurfaces are interesting examples of “tame spaces” as predicted by Grothendieck in his sketch of a programme [11]. Separating Pfaffian hypersurfaces in the real plane \( \mathbb{R}^2 \) are called the *Pfaffian curves*. Pfaffian curves have similar properties to the semianalytic arcs, therefore, their topology can be described within the theory of o-minimal structures (cf. [10],[21]).
3 Gradient trajectories in the plane

3.1 Non-integrability of planar vector fields

In a very interesting paper [1], P.B. Acosta-Humánez, J.T. Lázaro, J.J. Morales-Ruiz and C.Pantazi tackled non-integrability of complex planar vector fields of the form $X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$ with $P, Q \in \mathbb{C}[x, y]$ (cf. [1] Section I, Theorem B). In fact they have elaborated an effective algorithm in order to check the necessary conditions for the integrability of polynomial vector fields in the plane. Their method is illustrated with several families of examples, however, in the presented families there are no examples with scalar potentials, i.e., gradient vector fields. On the other hand, gradient trajectories are intensively studied in the range of problems related with the stability of critical points of the gradient systems (see e.g. [13] Chapter I, Section 8). Let us note the common conviction that gradient trajectories have quite “moderated” topology. More precisely, K.Kurdyka, T.Mostowski and A. Parusiński in their landmark paper [14], proved the gradient conjecture of R. Thom. Simultaneously, they have stated an even more general conjecture called the finiteness conjecture for the gradient systems. This conjecture in the case of plane is easily proved by the elementary methods of subanalytic geometry (see [14], Proposition 2.1). Let us consider the following gradient dynamical system

$$\dot{x} = \text{grad} \, F, \quad \text{where} \quad F \in \mathbb{R}[x, y]. \quad (5)$$

Taking into account that for $n = 2$ (i.e., in the real plane) any subanalytic set is actually semianalytic, it is easy to see that gradient trajectories of system (5) are finite unions of Pfaffian curves (cf. Remark 2.2.3).

**Example 3.1.1.** Consider the planar gradient field

$$\dot{x} = \text{grad} \, F, \quad \text{where} \quad F(x, y) = \frac{1}{3} x^3 + \frac{1}{2} x^2 + (x + y)^2 y^2 + \frac{1}{4} y^4$$

and its associated foliation

$$\frac{dy}{dx} = \frac{\partial F/\partial y}{\partial F/\partial x} = \frac{2(x + y)y^2 + 2(x + y)^2 y + y^3}{x^2 + x + 2(x + y)y^2} \quad (6)$$

Following [1] Section 3, we will show that the linearized second variational equation $LVE_2$ of (6) along a particular solution has non-commutative identity component $G_2^0$ of its Galois group $G_2$. 

6
First let us observe that the straight line $\Gamma = \{y = 0\}$ is an invariant curve of (6) and we have the second linearized variational equation along $\Gamma$

$$
\begin{align*}
\chi_1' &= 2 \frac{2x}{x+1} \chi_1 \\
\chi_2' &= \frac{2x}{x+1} \chi_2 + \frac{12}{x+1} \chi_1
\end{align*}
$$

(7)

In the notations of [1] Section 3 we have

$$
\begin{align*}
\beta_1(x) &= \frac{2x}{x+1} \ , \ \beta_2(x) = \frac{12}{x+1} \\
\omega &= e^{2x} / (x+1)^2 , \ \theta_1 = \int \frac{12e^{2x}}{(x+1)^3}
\end{align*}
$$

In the notation of Maple 2019, we obtain

$$
\theta_1 = -(12x + 18)e^{2x} / (x+1)^2 - 24e^{-2}Ei_1(-2x - 2),
$$

(8)

where $Ei_1(z)$ denotes the special function “exponential integral”. In order to apply Proposition 3.1 from [1] Section 3, we have to check the hypotheses $(H_1)$ and $(H_2)$. $(H_1)$ is clearly fulfilled. Concerning $(H_2)$ let us suppose that $\theta_1$ is a rational function in $x$ and $\omega$. Then, since from the expression for $\theta_1$ in (8), we obtain

$$
Ei_1(-2x - 2) = -\frac{1}{24e^{-2}} (\theta_1 + (12x + 18)\omega),
$$

$Ei_1(-2x - 2)$ would be rational in $x$ and $\omega$, a contradiction, since $Ei_1(-2x - 2)$ is not an elementary function.

4 Final remarks and conclusions

As R.C. Churchill and D.L. Rod observed in their joint note [6], an important consequence of chaos in a Hamiltonian system is its non-integrability. On the other hand the study of chaotic dynamical systems from the algebraic point of view is still in the infancy and as shows e.g Example 3.1.1 non-integrability in the Arnold-Liouvillean sense may be unrelated with the complexity of the topology of solutions (cf [12], Introduction). The Morales-Ramis theory in its classical form [17], [18] deals with the problems of non-integrability in the complex case mainly because it is based on the Picard-Vessiot
theory with the additional assumption that the field of constants is algebraically closed. However from the time of discovery of so called “real Picard-Vessiot fields” [9] it seems very interesting to develop an analogous theory over the field of real numbers. In such a case it would be possible to study “tame topology” of the solutions following the theory of fewnomials. In fact in the case of linear systems the relation of real Liouville extensions with Khovanskii-Gel’fond theory is presented in [8].

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