Hadronic vacuum polarization contribution to the muon $g - 2$ in holographic QCD

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We evaluate the leading-order hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon with two light flavors in minimal hard-wall and soft-wall holographic QCD models, as well as in simple generalizations thereof, and compare with the rather precise results available from dispersive and lattice approaches. While holographic QCD cannot be expected to shed light on the existing small discrepancies between the latter, this comparison in turn provides useful information on the holographic models, which have been used to evaluate hadronic light-by-light contributions where errors in data-driven and lattice approaches are more sizable. In particular, in the hard-wall model that has recently been used to implement the Melnikov-Vainshtein short-distance constraint on hadronic light-by-light contributions, a matching of the hadronic vacuum polarization to the data-driven approach points to the same correction of parameters that has been proposed recently in order to account for next-to-leading order effects.

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I. INTRODUCTION

Currently there is a 4.2σ discrepancy between the Standard Model prediction for the anomalous magnetic moment of the muon $a_\mu = (g - 2)_\mu/2$ as assembled in the White Paper (WP) [1] and the new experimental result, obtained by combining the BNL and the Fermilab E989 values, involving an experimental error of $41 \times 10^{-11}$, which is expected to be reduced further by additional data taking and future experiments. The uncertainty in the Standard Model prediction has a similar magnitude; it is almost entirely due to hadronic contributions from hadronic vacuum polarization (HVP) and from hadronic light-by-light scattering (HLbL), which according to [1] amount to

$$a_{\mu}^{\text{HVP,WP}} = (6845 \pm 40) \times 10^{-11}, \quad (1.1)$$
$$a_{\mu}^{\text{HLbL,WP}} = (92 \pm 19) \times 10^{-11}. \quad (1.2)$$

The data-driven computation of HVP thus claims an accuracy of 0.6%, whereas the much smaller HLbL contribution has about 20% uncertainty. To match the experimental progress, improvements in the theoretical predictions for both contributions are called for.

However, the data-driven HVP result has recently been questioned by a direct lattice QCD calculation [2] by the BMW collaboration which claims a similar error of 0.8%

$$a_{\mu}^{\text{HVP,BMW}} = (7075 \pm 55) \times 10^{-11} \quad (1.3)$$

but deviating from (1.1) by about 3% or 2.1σ. Taken at face value this would reduce the discrepancy between experiment and theory in the case of $a_\mu$ to about 1.5σ, while it may give rise to tensions with electroweak precision fits of the hadronic contribution to the running of the electromagnetic coupling [3–5].
Once this critical issue has been resolved, it will also be crucial to reduce the theoretical uncertainty in the HLbL contribution. In the latter, an important question has been the implementation of certain short-distance constraints in hadronic models [6–8], where recently holographic QCD has helped to shed light on the role of axial-vector mesons [9–12]. Holographic QCD also makes interesting quantitative predictions given the large spread of results in other hadronic models, which have led to a 100% uncertainty for the estimated contribution of axial-vector mesons in the WP. Being an approach which is based on the large color number \( N_c \) limit, it cannot be expected to help with the percent-level discrepancies in the highly constrained HVP contribution. However, given that it typically achieves an accuracy of 10-20%, it can provide potentially useful information in the case of HLbL. Investigating the performance in the case of HVP allows us to test the holographic QCD models with regard to their ability to describe photon-hadron interactions quantitatively.

In this paper we evaluate the leading-order HVP (LO-HVP) contribution of the minimal bottom-up holographic QCD models that have been employed in the study of the HLbL contribution, as well as simple generalizations thereof, and compare with relevant results obtained within the data-driven approach, in particular for the contributions of the lightest quark flavors, revisiting and extending the study of Hong, Kim, and Matsuzaki [13].

As shown in [14, 15], the leading order HVP contributions to the muon \( g - 2 \) is related to the hadronic vacuum polarization function through

\[
a_{\mu}^{\text{LO-HVP}} = 4\pi^2 \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dQ^2 f(Q^2) \Pi_{\text{had}}^{\text{em}}(Q^2), \tag{1.4}
\]

where \( Q^2 = -q^2 \) is the Euclidean momentum squared and

\[
f(Q^2) = \frac{m_\mu^2 Q^2 Z^3 (1 - Q^2 Z)}{1 + m_\mu^2 Q^2 Z^2}, \quad Z = -\frac{Q^2 - \sqrt{Q^4 + 4m_\mu^2 Q^2}}{2m_\mu^2 Q^2}. \tag{1.5}
\]

The hadronic vacuum polarization function needs to be renormalized such that \( \Pi_{\text{had}}^{\text{em}}(0) = 0 \). It is given by the vector-current correlator, defined by

\[
i \int d^4x e^{iqx} \langle 0 | T \{ J_v^{\mu}(x) J_{V}\bar{\mu}(0) \} | 0 \rangle = \delta^{ab} (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \Pi_V ( -q^2 ) \tag{1.6}
\]

in the flavor-symmetric case, via

\[
\Pi_{\text{em}}^{\text{had}} ( -q^2 ) = 2 \text{Tr} Q_{\text{em}}^2 \Pi_V ( -q^2 ), \tag{1.7}
\]

where \( Q_{\text{em}} = \text{diag}(\frac{2}{3}, -\frac{1}{3}, \ldots) \) is the quark charge matrix.

As we shall show, the holographic results can deviate by up to 50% from the data-driven result, even after accounting for the fact that the holographic QCD results, being essentially a large-\( N_c \) approximation, can only account for a subset of multi-hadron intermediate states. However, the simplest HW model that can simultaneously fit the rho meson mass and the pion decay constant as well as the leading-order short-distance behavior of the vector correlator is performing quite reasonably. Interestingly, the amount of correction expected from next-to-leading order effects in the large-momentum domain, where perturbative corrections to the asymptotic behavior proportional to \( \alpha_s / \pi \) should play a role, turn out to be consistent with the corrections proposed recently by two of us in the case of the HLbL contribution [11].

In the next section, we shall review the minimal holographic QCD models included in our study to the extent necessary for evaluating the LO-HVP contribution to \( a_{\mu} \) in Sect. III. Sect. IV summarizes our conclusions.
II. MINIMAL HOLOGRAPHIC QCD MODELS

In this work we shall limit ourselves to holographic QCD models with a minimal set of adjustable parameters with anti-de Sitter background geometry and simple generalizations thereof.

A. Hard-wall models

In hard-wall (HW) AdS/QCD models, a five-dimensional anti-de Sitter (AdS) background geometry is chosen. In terms of a holographic radial coordinate $z$ where the conformal boundary is at $z = 0$, the line element is given by (using a mostly-minus metric convention)

$$ds^2 = g_{MN} dx^M dx^N = \frac{1}{z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu - dz^2 \right), \quad 0 < z \leq z_0. \quad (2.1)$$

Conformal invariance is broken by a hard cutoff at $z = z_0$, where suitable boundary conditions for bulk fields dual to the quantum operators of the four-dimensional (large-$N_c$) gauge theory are imposed.

The fields dual to left and right quark bilinears $\bar{\psi} \gamma^\mu T^a P_L, P_R \psi$ are five-dimensional $U(N_f)_{L,R}$ gauge fields $A^{(L,R)}$, where chiral symmetry breaking can be implemented through spontaneous symmetry breaking by a bifundamental scalar $X$ [16, 17] or through different boundary conditions [18] on vector and axial-vector fields, $V = \frac{1}{2} [A^{(L)} + A^{(R)}]$ and $A = \frac{1}{2} [A^{(L)} - A^{(R)}]$, or both [19].

The five-dimensional action of models with a bifundamental scalar $X$ introduced first by Erlich et al. [16] (termed HW1 in the following) reads

$$S = \int d^5x \sqrt{g} \text{Tr} \left\{ |DX|^2 + 3|X|^2 - \frac{1}{4g_5^2} (F_L^2 + F_R^2) \right\}, \quad (2.2)$$

whereas the model of Hirn and Sanz (termed HW2) has only the Yang-Mills part.

In both cases, the field equations for transverse vector fields $\partial_\mu V^\mu = 0$ are given by

$$\partial_z \left( \frac{1}{z} \partial_z V^a_\mu(q, z) \right) + \frac{q^2}{z} V^a_\mu(q, z) = 0, \quad (2.3)$$

where we have Fourier transformed with respect to the spacetime coordinates of the boundary theory. Splitting the vector field further as $V^a_\mu(q, z) = V(q, z) v^a_\mu$, the on-shell action, given by the boundary term

$$S = -\frac{1}{2g_5^3} \int d^4x \left( \frac{1}{z^2} V^a_\mu \partial_z V^{\mu a} \right) \bigg|_{z=\epsilon}, \quad (2.4)$$

is interpreted as generating functional for QCD flavor currents. In both HW1 and HW2 models, boundary conditions on $V$ are such that there is no contribution from $z = z_0$, while the conformal boundary at $z = 0$ needs regularization by a finite cutoff $z = \epsilon$ when imposing $V(q, \epsilon) = 1$.

Thus we find

$$\Pi (q^2) = -\frac{1}{g_5^2q^2} \frac{\partial_z V(q, z)}{z} \bigg|_{z=\epsilon}. \quad (2.5)$$
The equations of motion (2.3) can be solved in terms of Bessel functions. Using the boundary conditions $V(q, \epsilon) = 1$ and $\partial_z V(q, z)|_{z=z_0} = 0$ yields

$$V(q, z) = \frac{z J_1(qz) Y_0(qz_0) - z Y_1(qz) J_0(qz_0)}{\epsilon J_1(\epsilon \epsilon) Y_0(qz_0) - \epsilon Y_1(\epsilon \epsilon) J_0(qz_0)}|_{\epsilon \to 0}. \quad (2.6)$$

Plugging this result into equation (2.5) yields

$$\Pi_V (-q^2) = -\frac{1}{g^2} \frac{1}{q \epsilon} J_0(qz_0) Y_0(q \epsilon) - Y_0(qz_0) J_0(q \epsilon) \left. \frac{\partial_z V(q, z)}{z}|_{z=z_0} \right| \bigg|_{\epsilon \to 0}, \quad (2.7)$$

and expanding this at $\epsilon \to 0$ gives

$$\Pi_V (-q^2) = \frac{1}{g^2} \left[ \pi \frac{y_0(qz_0)}{2 J_0(qz_0)} + \gamma - \log 2 + \log q \epsilon + O(\epsilon^2) \right], \quad (2.8)$$

where $\gamma$ is the Euler-Mascheroni constant. This expression is divergent for $\epsilon \to 0$ and has to be renormalized. Adding a counterterm

$$S_{\text{c.t.}}(\mu) = \int d^4 x \left( \frac{1}{2 g^2} \ln \epsilon \mu \right) \text{tr} [F_{\mu\nu}(x, \epsilon)]^2 \quad (2.9)$$

to the action gives rise to an additional term

$$\Pi_{V}^{\text{c.t.}} = -\frac{1}{g^2} \log(\mu \epsilon), \quad (2.10)$$

which cancels the divergent part of (2.8). The resulting renormalized vacuum polarization is

$$\Pi_{V}^{\text{ren}} (-q^2) = \lim_{\epsilon \to 0} \left[ \Pi_V (-q^2) + \Pi_{V}^{\text{c.t.}} (-q^2) \right] = \frac{1}{g^2} \left[ \frac{1}{2} \frac{y_0(qz_0)}{J_0(qz_0)} + \gamma + \ln \frac{q}{2 \mu} \right], \quad (2.11)$$

where $\mu$ can be chosen such that $\Pi_{V}^{\text{ren}}(0) = 0$ holds, as required for $\alpha$ in (1.4) to be identified with the standard fine structure constant in the Thomson limit. For Euclidean momenta this yields

$$\Pi_{V}^{\text{ren}} (Q^2) = \frac{1}{g^2} \left[ K_0(Qz_0) \frac{Qz_0}{I_0(Qz_0)} + \ln \frac{Qz_0}{2} + \gamma \right]. \quad (2.12)$$

As can be seen from (2.11), in the time-like domain the vacuum polarization function has an infinite series of poles at $q^2 = m_n^2$ with $m_n$ given determined by $J_0(m_n z_0) = 0$, corresponding to an infinite tower of (stable) vector mesons (as expected in a large-$N_c$ limit). The latter are described by normalizable solutions of

$$\partial_z \left( \frac{1}{z} \partial_z \psi_n \right) + \frac{m_n^2}{z} \psi_n = 0, \quad (2.13)$$

with boundary conditions $\psi_n'(z_0) = 0$, $\psi_n(0) = 0$, and are explicitly given by

$$\psi_n(z) = \frac{\sqrt{2z} J_1(m_n z)}{z_0 J_1(m_n z_0)}. \quad (2.14)$$
The unrenormalized vector current correlator can then be represented as

$$\Pi_V(q^2) = \sum_{n=1}^{\infty} \frac{F_n^2}{(q^2 - m_n^2)m_n^2}$$

(2.15)

with decay constants $F_n$, defined as $\langle 0| J^{a\mu}_V(0)| V_n^b \rangle = F_n \delta^{ab} \varepsilon^\mu$, given by

$$F_n = \lim_{\epsilon \to 0} \frac{1}{g_5} \frac{\psi'_n(\epsilon)}{\epsilon} = \frac{1}{g_5} \frac{\sqrt{2m_n}}{z_0 J_1(m_n z_0)}.$$  

(2.16)

1. Parameters of the HW1 model

In the chiral limit, the HW1 model has only three free parameters, $g_5$, $z_0$, and the chiral condensate described by $X(z_0)$. In the application to HLbL contributions [9], those were matched to the pion decay constant $f_\pi$, the $\rho$ meson mass, and $g_5$ was set such that the short-distance constraint on $\Pi_V$ from QCD [20, 21]

$$\Pi_V(Q^2) = \frac{N_c}{24\pi^2} \left( 1 + \frac{\alpha_s}{\pi} \right) \log \left( \frac{Q^2}{\mu^2} \right) - \frac{\alpha_s}{24\pi} \frac{N_c}{3} \frac{\langle G^2 \rangle}{Q^4} + \frac{14N_c}{27} \frac{\pi \alpha_s \langle q\bar{q} \rangle^2}{Q^6}$$

(2.17)

is satisfied to leading order. This determines

$$\frac{1}{g_5^2} = \frac{N_c}{12\pi^2},$$

(2.18)

which holds true in other bottom-up models with bulk geometry that is at least asymptotically AdS.

The HW cutoff $z_0$ directly determines the mass of the lightest vector meson, which we choose as $m_\rho = 775$ MeV, corresponding to $z_0 = 3.103$ GeV$^{-1}$.

(2.19)

This remains unchanged when finite quark masses are introduced in the HW1 model; the latter modify, however, axial-vector meson masses and vector meson masses with open flavor quantum numbers [23].

2. Parameters in the HW2 model

In the inherently chiral HW2 model due to Hirn and Sanz [18], chiral symmetry breaking is implemented without a symmetry breaking bifundamental scalar $X$, through different boundary conditions for vector and axial-vector fields.

When $g_5$ and $z_0$ are chosen as above, the pion decay constant can no longer be fitted to phenomenological values. In the application to HLbL contributions for the muon anomalous magnetic moment [9, 10, 24, 25], which are dominated by the coupling of pions to two photons, $f_\pi$ is fixed first, leaving the choice of matching either the infrared parameter $m_\rho$.

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through $z_0$ or the ultraviolet behavior through $g_5$. In [9, 25], the former option, with physical values for $f_\pi$ and $m_\rho$, is referred to as HW2 model. This matches the short-distance constraints on transition form factors and also the Melnikov-Vainshtein short-distance constraint [6] only at the level of 61%. Conversely, the leading term in (2.17) is too large at the level of 164%. Matching these constraints at the expense of a much too heavy rho meson (987 MeV) was called HW2(UV-fit).²

The symmetry breaking boundary conditions can in fact also be used in conjunction with the symmetry breaking scalar $X$. This possibility has been proposed in [19] and also explored in [11] for HLbL contributions to $a_\mu$, where it was referred to as HW3. In the vector sector, however, this coincides with the HW1 model so that our HVP results for the latter also pertain to the HW3 case.

Summarizing the values of the parameters for the three different fits we have, for $N_c = 3$,

\begin{align*}
\text{HW1} & : \quad g_5 = 2\pi, \quad z_0 = 3.103 \text{ GeV}^{-1}; \\
\text{HW2} & : \quad g_5 = 4.932, \quad z_0 = 3.103 \text{ GeV}^{-1}; \\
\text{HW2 (UV-fit)} & : \quad g_5 = 2\pi, \quad z_0 = 2.4359 \text{ GeV}^{-1}.
\end{align*}

(2.20)

B. Soft-wall model (SW)

A shortcoming of the HW models is that the masses of highly excited vector mesons do not rise as $m_n^2 \sim \sigma n$ as expected from linear confinement with string constant $\sigma$, but instead like $m_n^2 \sim n^2$.

The so-called soft-wall model introduced in [26] (a precursor of which appeared in [27]) achieves a strictly linear dependence of $m_n^2$ on $n$ by introducing a dilaton $\Phi(z) = \kappa^2 z^2$ as an additional background field in the five-dimensional Lagrangian [26, 28]

\begin{equation}
S = \int d^5x e^{-\Phi(z)} \sqrt{g} \text{Tr} \left\{ |DX|^2 + 3|X|^2 - \frac{1}{2g_5^2} (F_L^2 + F_R^2) \right\},
\end{equation}

(2.21)

while keeping the AdS metric (2.1). The $z$-coordinate is, however, now unbounded from above, $z \in [0, \infty)$.

Solving the correspondingly modified field equation

\begin{equation}
\partial_z \left( \frac{e^{-\Phi}}{z} \partial_z V^a_\mu \right) + \frac{q^2 e^{-\Phi}}{z} V^a_\mu = 0
\end{equation}

(2.22)

with the boundary conditions $V(q, \epsilon) = 1$ in the limit $\epsilon \to 0$ and $\lim_{z \to \infty} V(q, z) = 0$ gives [29]

\begin{equation}
V(q, z) = \Gamma \left( 1 - \frac{q^2}{4\kappa^2} \right) U \left( \frac{-q^2}{4\kappa^2}, 0, (\kappa z)^2 \right),
\end{equation}

(2.23)

with the confluent hypergeometric function of second kind $U$ (also known as Tricomi function). Plugging this into

\begin{equation}
\Pi_V (-q^2) = -\frac{e^{-\Phi} \partial_z V(q, z)}{g_5^2 q^2 z} \bigg|_{z=\epsilon} = -\frac{1}{g_5^2 q^2 z} \partial_z V(q, z) \bigg|_{z=\epsilon},
\end{equation}

(2.24)

² These two choices correspond essentially to the parameters in “Set 1” and “Set 2” of [10], where only the Hirn-Sanz model was considered.
switching to the Euclidean momentum $Q^2 = -q^2$ and expanding in a series for small $\epsilon$ gives

$$\Pi_{V}(Q^2) = \frac{1}{2g_5^2} \left[ \psi \left( \frac{Q^2}{4\kappa^2} + 1 \right) + \ln \left( \kappa^2 \epsilon^2 \right) + 2\gamma \right] + \mathcal{O}(\epsilon), \quad (2.25)$$

where $\psi(z) = d \ln \Gamma(z)/dz$. Renormalizing as above and using $\psi(0) = -\gamma$ we find

$$\Pi_{V}^{\text{ren}}(Q^2) = \frac{1}{2g_5^2} \left[ \psi \left( \frac{Q^2}{4\kappa^2} + 1 \right) + \gamma \right], \quad (2.26)$$

with $g_5$ fixed as in the HW1 model.

From the poles of the digamma function $\psi$ one can see that the spectrum of vector mesons is now given by

$$m_n^2 = 4\kappa^2 n, \quad n = 1, 2, 3, \ldots . \quad (2.27)$$

Alternatively, the hadronic vacuum polarization can be calculated as above from the normalizable solutions given by [26, 29]

$$\psi_n(z) = \sqrt{\frac{2}{n}} \kappa^2 z^2 L_n^1(\kappa^2 z^2), \quad (2.28)$$

where the $L_n^1$ are the generalized Laguerre polynomials of order one, leading to decay constants

$$F_n = \frac{1}{g_5} \psi_n''(0) = \frac{1}{g_5} \kappa^2 \sqrt{8n}. \quad (2.29)$$

C. Interpolating models

The phenomenological study of pion form factors for HW and SW models in [28] came to the conclusion that the HW1 model generally performed better than the SW model. In [28, 30], a simple interpolating model was proposed that combines features of HW and SW models, whereas in [31–33] a more sophisticated version including a dynamical tachyon for chiral symmetry breaking was developed, which achieves a similar behavior.

1. Semi-hard wall model (SHW)

In the model proposed by Kwee and Lebed in [28, 30], a semi-hard wall was set up by replacing the dilaton background of the SW model $e^{-\Phi} = e^{-\kappa^2 z^2}$ with a background given by

$$e^{-\Phi(z)} = \frac{e^{\lambda^2 z_0^2} - 1}{e^{\lambda^2 z_0^2} + e^{\lambda^2 z^2} - 2}. \quad (2.30)$$

The HW model is recovered by $\lambda z_0 \to \infty$ at fixed $z_0$, whereas at large $z$ the dilaton behaves as $\Phi(z) \sim \lambda^2 z^2$.

In [30] two sets of parameters were considered, involving $\lambda z_0 = 2.1$ and 1, where the first choice was found to give a good agreement with the pion form factor $F_\pi(Q^2)$ comparable to that of the HW1 model, albeit at the cost of not matching the pion decay constant very well; the second choice led to a similar prediction for $F_\pi(Q^2)$ as the SW model.
In the following we shall choose the two free parameters $\lambda$ and $z_0$ such that in addition to the mass of the lightest rho meson this model reproduces the mass of $\rho(1450)$, which [34] lists as $1465 \pm 25$ MeV. This leads to

$$\lambda z_0 = 1.697, \quad z_0 = 2.9738 \text{ GeV}^{-1}, \quad (2.31)$$

which is right in between the two parameter sets explored in [30].

With (2.30) closed analytical results are no longer available and one has to resort to numerical solutions of the equations of motion for the vector modes. In contrast to the SW model, $m_n^2$ becomes a linear function of $n$ only for large $n$, where the spacing is determined by $m_{n+1}^2 - m_n^2 \sim 4\lambda^2 n$ ($\approx 1.3 \text{ GeV}^2$ for our choice (2.31)). The decay constants are again given by $F_n = \psi''(0)/g_5$ with $g_5$ still determined by (2.18). Table I lists the results for the first 8 modes.

The full subtracted self energy function with $\Pi_{V}^{\text{ren}}(0) = 0$ can then be calculated through

$$\Pi_{V}^{\text{ren}} (-q^2) = \sum_{n=1}^{m} \frac{q^2 F_n^2}{(q^2 - m_n^2) m_n^4} + \mathcal{O} \left( \frac{q^2}{m_{m+1}^2} \right). \quad (2.32)$$

2. Tachyon condensation model (TC)

Finally we consider a more sophisticated but still relatively simple bottom-up model, developed in [31–33], where chiral symmetry breaking is implemented by a brane-antibrane effective action with an open string tachyon mode as proposed by Sen [35]. This is based on a pair of Dirac-Born-Infeld type Lagrangians augmented by a tachyon potential $V(|T|)$, which in the case of a single flavor reads

$$S = \int d^4 x dz V(|T|) \left( \sqrt{\det A^{(L)}} + \sqrt{\det A^{(R)}} \right), \quad (2.33)$$

where

$$A_{MN}^{(L,R)} = g_{MN} + \frac{2\pi\alpha'}{g_V} F_{MN}^{(L,R)} + \pi\alpha' \lambda [(D_M T)^* (D_N T) + (D_N T)^* (D_M T)], \quad (2.34)$$

and

$$D_M T = \left( \partial_M + i A_M^{(L)} - i A_M^{(R)} \right) T, \quad T = \tau e^{i\theta}, \quad (2.35)$$

with a Gaussian potential

$$V = K e^{-\frac{1}{2} \mu^2 + \tau^2}. \quad (2.36)$$

The background geometry is derived from a six-dimensional AdS soliton, which has been proposed by Kuperstein and Sonnenschein as a holographic model of four-dimensional Yang-Mills theory [36]. It is given by

$$ds_5^2 = g_{tt} dt^2 - g_{zz} dz^2 - g_{xx} dx_3^2 - \frac{R^2}{z^2} \left[ dx_{1,3}^2 - f_A^{-1} dz^2 \right], \quad (2.37)$$

where $f_A = 1 - (z/z_A)^5$.

Similarly to the case of the bifundamental scalar $X$ in the HW and SW models, the scalar $T$ mediates chiral symmetry breaking by its possible vacuum solutions. In order to match
the scaling dimension of a quark bilinear, the AdS/CFT correspondence requires to set the mass of the field $\tau$ to

$$m^2 R^2 = -\frac{R^2 \mu^2}{2\pi \alpha' \lambda} = -3,$$

which leads to the differential equation for its profile $\tau(z)$

$$\tau'' - \frac{4\mu^2 z f_\Lambda}{3} \tau'^3 + \left( -\frac{3}{z} + \frac{f'_\Lambda}{2f_\Lambda} \right) \tau' + \left( \frac{3}{z^2 f_\Lambda} + \mu^2 \tau'^2 \right) \tau = 0$$

and a UV asymptotic behavior parametrized by two constants $c_1$ and $c_3$

$$\tau = c_1 z + \frac{\mu^2}{6} c_3 z^3 \log z + c_3 z^3 + O(z^5).$$

Choosing the source parameter $c_1$ corresponding to the quark mass, the parameter $c_3$ is tuned such that the tachyon diverges exactly at $z = z_\Lambda$.

The parameter $\mu$ in the tachyon potential does not have a physical meaning as it can be absorbed in the definition of $\tau$; in the following it will be set to $\mu^2 = \pi$. In [32] a fit of light unflavored mesons (composed of $u$ and $d$ quarks) gave

$$z_\Lambda^{-1} = 522\text{MeV}, \quad c_1 = 0.0125 z_\Lambda^{-1}.$$  \hspace{1cm} (2.41)

We will refer to this as TC (fit 1). In [33] the parameters are chosen as

$$z_\Lambda^{-1} = 549\text{MeV}, \quad c_1 = 0.0094 z_\Lambda^{-1},$$

which gives a slightly higher mass for the lightest rho meson [referred to as TC (fit 2) in the following].

The function $\tau(z)$ we obtain as the solution of equation (2.39) is plotted in figure 1. The value we obtain for the other constant is

$$c_3 \approx 0.37 z_\Lambda^{-3}.$$  \hspace{1cm} (2.43)

Given $\tau(z)$, one can proceed by expanding the action (2.33) up to quadratic order in the fields as in the other models above.
For the vector gauge fields $V_\mu = [A_\mu^{(L)} + A_\mu^{(R)}]/2$ and corresponding field strength $V_{\mu\nu}$, the action up to quadratic order reads

$$S_V = \frac{(2\pi\alpha')^2}{g_V^4} \mathcal{K} \int d^4x dze^{-\frac{1}{4}\mu^2 z^2} \left[ \frac{1}{2} \bar{g}_{zz} V_\mu V^{\mu} + g_{xx} \bar{g}_{zz} \partial_z V_\mu \partial_z V^{\mu} \right], \quad (2.44)$$

where we have defined $\bar{g}_{zz} = g_{zz} + 2\pi\alpha' \lambda (\partial_z \langle \tau \rangle)^2$. This leads to the mode equations

$$-\frac{1}{e^{-\frac{1}{4}\mu^2 z^2} \bar{g}_{zz}^2} \partial_z \left( e^{-\frac{1}{2}\mu^2 z^2} g_{xx} \bar{g}_{zz}^{-1} \partial_z \psi_n(z) \right) = m_n^2 \psi_n(z). \quad (2.45)$$

As in the SHW model, solutions have to be obtained by relying on numerical calculations, which is best done by transforming the differential equations to Liouville normal form (see the Appendix of [33]). The vector correlator, which is given by

$$\Pi_V(-q^2) = -2 \frac{(2\pi\alpha')^2 \mathcal{K} R \partial_z V(q, z)}{q^2} \bigg|_{z=\epsilon}. \quad (2.46)$$

for a solution with boundary condition $V(q, 0) = 1$, can then be calculated alternatively by summing a sufficiently large number of modes according to (2.32). The behavior at large $q^2$ can be shown [33] to be of a form similar to the SW model. Matching to the leading-order term of the OPE result leads to

$$\frac{2 (2\pi\alpha')^2 \mathcal{K} R}{g_V^4} = \frac{N_c}{12\pi^2}. \quad (2.47)$$

Like the SHW model, the TC model has a linear dependence of $m_n^2$ on $n$ only for large $n$, where it can be shown [33] that the spacing is given by $m_{n+1}^2 - m_n^2 \sim 6z_{\Lambda}^{-2}$ ($\approx 1.6\text{GeV}^2$ and $1.8\text{GeV}^2$ for (2.41) and (2.42), respectively).

III. NUMERICAL RESULTS

A. Masses and decay constants

In Table I we list our results for the masses and decay constants of the vector mesons in the various models. In the HW1, SW, SHW models we have fixed $m_1 = 775\text{ MeV}$ while setting $g_5$ such that the asymptotic behavior of the vector correlator is matched. The extra parameter in the SHW model was used to additionally match $m_2 = 1465\text{ MeV}$, while in the TC model we have considered the two sets presented in [32] and [33]. The simpler HW2 model, which instead has fewer parameters, is considered in the two versions used in [9, 10] for evaluating the HLbL contribution to the muon $g-2$: an IR fit where $m_\rho$ and $f_\pi$ is matched but short-distance constraints are only satisfied at the level of 61%, and a UV-fit where $f_\pi$ and the short-distance behavior are correct but $m_\rho$ too heavy by 27%.

In the HW models, the masses of excited rho mesons rise very quickly, asymptotically like $m_n^2 \sim n^2$, whereas the SW, SHW, and TC models have $m_n^2 \sim n$ as required by linear confinement. While in the HW models the first excited rho meson has a mass significantly higher than the experimental value $m_2 = 1465\text{ MeV}$, in the simple SW this value is 25%
too low, this can be remedied in the SHW by our choice for its parameters, and also by the overall fits [32, 33] in the TC model. In the SHW and TC models, also $m_3$ and $m_4$ are well compatible with the next states on the radial Regge trajectory, which in [37] were assumed to be $\rho(1900)$ and $\rho(2150)$. In Fig. 2 we plot the increments of the masses squared for the first 12 modes in the models with an asymptotically linear behavior, which shows that the simple SW model has a much smaller value ($m_{n+1}^2 - m_n^2 \approx m_\rho^2 \approx 0.601$ GeV$^2$) than the SHW and TC models. The latter are in fact closer to the observed slope of radial Regge trajectories, which in [37] was determined as $1.38(4)$ GeV$^2$.

Regarding decay constants, sufficient experimental information is only available for the lightest rho meson through $\Gamma(\rho^0 \to e^+e^-) = 7.04(6)$ keV [34], which yields [38]

$$F_{\rho^0}^2 = 3m_{\rho}^2\Gamma(\rho^0 \to e^+e^-)/4\pi a^2 = (348(1) \text{ MeV})^2.$$  \hfill (3.1)

The largest deviations from this result, of about 25% and 20%, respectively, are found in the SW model and in the HW2(UV-fit) case, whereas the HW1 model are merely 5% too low. Note that $F_n \propto g_5^{-2}$, which implies that one could match the experimental result by a corresponding adjustment of $g_5$.

### B. $N_f = 2$ LO-HVP contribution to $a_\mu$

In Table II we finally give the results for the leading-order HVP contribution to the anomalous magnetic moment of the muon with two light flavors, $a_{\mu(N_f=2)}^{\text{LO-HVP}}$, by using equation (1.4). As mentioned above, for the HW and SW models we can use closed form expressions for $\Pi_V$. For the SHW and TC models we rely on numerical results and the expansion (2.32). The corresponding integrands in the master formula (1.4) are displayed in Fig. 3.

With the exception of the model HW2 (with fitted $m_\rho$ and $f_\pi$), where the asymptotic behavior of $\Pi_V$ is about a factor 1.6 larger than the OPE result, the holographic results are much smaller than the results obtained for the $N_f = 2$ contributions in dispersive and lattice approaches (the latter are about [1] $590 \times 10^{-10}$ and [2] $640 \times 10^{-10}$, respectively).

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### Table I. Vector meson masses $m_n$ and decay constants $F_n^{1/2}$ in MeV.

| $n$ | $m_n$ (MeV) | $F_n^{1/2}$ (MeV) | $m_n$ (MeV) | $F_n^{1/2}$ (MeV) | $m_n$ (MeV) | $F_n^{1/2}$ (MeV) | $m_n$ (MeV) | $F_n^{1/2}$ (MeV) |
|-----|-------------|-----------------|-------------|-----------------|-------------|-----------------|-------------|-----------------|
| 1   | 775         | 329.1           | 775         | 314.0           | 775         | 314.0           | 775         | 314.0           |
| 2   | 1779        | 615.8           | 1779        | 485.8           | 1782        | 1453            | 1806        | 438.9           |
| 3   | 2789        | 863.3           | 2789        | 498.7           | 1903        | 498.7           | 1806        | 488.7           |
| 4   | 3800        | 1089            | 3800        | 2230            | 2158        | 2269            | 538.4       | 564.9           |
| 5   | 4812        | 1300            | 4812        | 2511            | 2466        | 2593            | 577.3       | 605.4           |
| 6   | 5824        | 1500            | 5824        | 2762            | 2744        | 2885            | 610.4       | 639.6           |
| 7   | 6836        | 1692            | 6836        | 2991            | 2999        | 3153            | 639.4       | 669.7           |
| 8   | 7848        | 1876            | 7849        | 3203            | 3236        | 3402            | 665.6       | 696.6           |

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3 In [13] $\Pi_V$ in the HW1 model was calculated by truncating the infinite sum in (2.15) at $n = 4$, which produces a result that is about 1% lower than the full contribution. With the slightly different choice of $m_1 = 775.49$ MeV of [13], we would obtain $a_{\mu(N_f=2)}^{\text{LO-HVP}} = 476.4 \times 10^{-10}$, while the truncated result given in [13] is $a_{\mu(N_f=2)}^{\text{LO-HVP}} = 470.5 \times 10^{-10}$. 

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FIG. 2. Plots of $m_{n+1}^2 - m_n^2$ in GeV$^2$ in models where $m_n^2 \sim n$ for large $n$: the SW model (light green), the SHW model (darker green), and the TC model (fit 1 in red and fit 2 in orange). The gray line is the phenomenological value given in [37] as 1.38(4) GeV$^2$, with the gray shaded region representing its uncertainty.

FIG. 3. Plot of the integrand $f(Q^2)\Pi(Q^2) = f(Q^2)4\pi^2\Pi_{\text{em}}^{\text{had}}(Q^2)$ in (1.4) for the HW1 model (blue), SW model (light green), SHW model (darker green), and the TC model (red for fit 1 and orange for fit 2). The HW2 model is only shown in the version UV-fit (purple), where the UV behavior is reproduced at the expense of an unrealistically heavy rho meson; when $m_\rho$ and $f_\pi$ are fitted, the integrand peaks at about 0.39 and the UV behavior is about a factor 1.6 too large.

However, the holographic QCD models should be viewed as (more or less crude) large-$N_c$ approximations. HVP contributions with multi-hadron states such as four pions correspond to higher-order contributions in the large-$N_c$ expansion so that it appears more reasonable to compare only with contributions from intermediate states corresponding to the $\rho$ and $\omega$ channels, which are dominated by two and three-pion states as well as $\pi^0\gamma$. Table III lists their contributions to $a_\mu$ according to [39] and [40], which combined give approximately

$$a_{\mu}^{\text{LO-HVP}}(\pi\pi,\pi\pi\pi,\pi\gamma) = 557(3) \times 10^{-10}. \quad (3.2)$$
TABLE II. Values of $a_{\mu}^{\text{LO-HVP}}$ with $N_f = 2$ for the different hQCD models in multiples of $10^{-10}$.

The last column gives the ratio of these results over $a_{\mu}^{\text{LO-HVP}}(\pi\pi,\pi\pi\pi,\pi\gamma)$ as obtained in the dispersive approach (3.2).

| Model          | $a_{\mu}^{\text{LO-HVP}}$ | Mismatch |
|----------------|-----------------------------|----------|
| HW1            | 476.9                       | 0.86     |
| HW2(IR|UV-fit) | 773.9|304.0 | 1.39|0.55 |
| SW             | 276.4                       | 0.50     |
| SHW            | 415.4                       | 0.75     |
| TC (fit 1|2)    | 442.3|403.6 | 0.79|0.72 |

While the HW2 model in its two versions brackets this result, with rather large deviations on either side, the HW1 model, where $m_\rho$ and $f_\pi$ as well as the short-distance behavior can be fitted simultaneously, is only a factor 0.86 smaller, and thus comes closest of all models considered here.

The smallest result, at only 50%, is obtained with the SW model. As we have seen above, the SW model with a strictly linear dependence of $m_n^2$ on $n$ underestimates the masses of all excited rho mesons. While this should tend to overestimate $a_{\mu}^{\text{LO-HVP}}$, the decay constant squared of the ground-state rho meson is only at 30% of its experimental value, which is thus responsible for the strong attenuation. However, the simple modification (2.30) in the SHW model, which leads to a much improved mass spectrum, also brings the decay constants closer to realistic values, yielding a result for $a_{\mu}^{\text{LO-HVP}}$ that is only 25% below (3.2). The more sophisticated TC model turns out to be comparable, coming somewhat closer with parameters of fit 1.

Thus all models which reproduce $m_\rho$, $f_\pi$ and $F_\rho$ reasonably well also do not deviate too strongly from the dispersive result for $a_{\mu}^{\text{LO-HVP}}$, but uniformly underestimate it. Since the latter is proportional to $g_5^{-2}$, this suggests that its value, obtained from matching the leading-order term in the vector correlator (2.17), should be corrected to account for the next-to-leading order term, which is indeed positive. Exactly such a correction was proposed by two of us in the evaluation of the HLbL contribution within the (massive) HW1 and HW3 models [11, 12], where it has the effect of reducing the holographic HLbL result, as this brings the asymptotic behavior of transition form factors down by amounts that are roughly consistent with perturbative corrections to the leading-order pQCD results at moderately high $Q^2$ values [41, 42]. At the same time, the coefficient of the logarithm in the asymptotic expression (2.17) is increased by a similar amount, which is consistent with the next-to-leading order terms in this expression.
In the case of the HW1 model, $F_{\rho 0}$ can be matched by reducing $g_{\rho 0}^2$ by a factor 0.9. This happens to bring the HW1 result for the $\pi^0$ pole contribution to the HLbL part of $a_\mu$ into perfect agreement with the dispersive result \[11\]: 
$$a_{\mu(HW1,3)}^{\pi^0} = (6.17 \ldots 6.39) \times 10^{-10}$$
while [43] $a_{\mu(\text{disp.})}^{\pi^0} = 6.26^{+30}_{-25} \times 10^{-10}$.

With $F_{\rho 0}$ matched, the HW1 result for $a_{\mu}^{\text{LO-HVP}}$ becomes correspondingly larger, namely $533.2 \times 10^{-10}$, which is less than 5% smaller than the dispersive result (3.2).\[^4\]

### IV. CONCLUSION

By considering a number of simple bottom-up holographic-QCD models we have found that their quantitative predictions are too spread out to be of help with the task of determining the HVP contribution to the anomalous magnetic moment of the muon, which is currently afflicted by the largest uncertainty with regard to the ongoing efforts of testing the Standard Model by a new round of experiments. However, a comparison of the holographic results for the LO-HVP contribution with the existing data-driven results at or below percent accuracy allows us to assess the various holographic models with regard to their ability to account for the relevant interactions between hadrons and photons. This is useful because holographic QCD can provide interesting estimates for HLbL contributions, where conventional approaches have uncertainties that are comparable with or larger than expected errors in the large-$N_c$ limit that holographic QCD is based upon.\[^5\]

In particular, we have considered the holographic SW and HW models that have been used previously for estimating the HLbL contributions of pseudoscalars and axial-vector mesons (see the recent review \[12\]), and we have also explored two simple extensions that aim at interpolating between the HW and SW models, while keeping their respective advantages.

We have found that the original HW1 model [16] turned out to come closest to the phenomenological value of the rho meson decay constant as well as to the value for $a_{\mu}^{\text{LO-HVP}}$ obtained in dispersive approaches. The somewhat simpler HW2 model, which was used in holographic calculations of the axial-vector contribution in two versions which either fit IR or UV constraints, brackets the latter with rather large deviations in both directions. The SW model turns out to give the worst fit, but already the simple improvement of a semi-hard wall as proposed in \[28, 30\] reduces the deviation considerably; the more sophisticated TC model achieves roughly the same with the parameters considered previously in \[31–33\].

In the HW1 model, the LO-HVP result is simply proportional to the coupling $g_{\rho}^{-2}$ determining the asymptotic behavior of the vector correlator. Reducing $g_{\rho}^2$ by a factor 0.9 or 0.85 has been proposed in \[11, 12\] as a simple way to account for next-to-leading order QCD effects for the large-$Q^2$ behavior of transition form factors. In the case of the rho meson decay constant, a factor of 0.9 leads to a perfect fit with the phenomenological value and a result for $a_{\mu}^{\text{LO-HVP}}$ that is only 5% too small. As shown already in \[11, 12\], the same reduction of $g_5^2$ brings about a perfect agreement of the pion pole contribution in the HW1 model with the data-driven result of \[43\]. We interpret this as a support for the predictions for pseudoscalar and axial-vector meson contributions obtained by two of us in various versions of the HW1 and HW3 model \[11, 12\], where a theoretical error was formed by taking the unchanged results of these models as upper bound and those with $g_5^2$ reduced by a factor of

\[^4\] Interestingly, in \[22\] it has been argued that inclusion of the effects of a gluon condensate within a modified HW1 model leads to an increase of about 6% of the holographic value for $a_{\mu}^{\text{LO-HVP}}$.

\[^5\] For example, the contribution of axial vector mesons is currently assigned a 100% uncertainty in \[1\].
as lower bound. The corresponding values with the factor 0.9 could then be regarded as the best guess within these models.\footnote{We do not reproduce these numbers here. They can be easily obtained by applying the correction factors given in Table III of \cite{11} to the results in Table II therein.}

As an outlook we would like to refer to the many possible improvements that can be considered for bottom-up holographic QCD models. In \cite{22} it has already been shown that incorporating the effects of a gluon condensate within a modified HW1 model leads to an increase of about 6\% of the holographic value for $a_{\mu}^{\text{LO-HVP}}$, bringing it very close to the data-driven result. It would be interesting to study even more extensions such as models that relax the assumption $N_f \ll N_c$ of the 't Hooft limit \cite{44}.

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