A Simple Modification of HLLEM Approximate Riemann Solver Applied to the Compressible Euler System at Low Mach Number

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Abstract. Due to its attractive properties in computing compressible flows, such as sharp capturing of discontinuities, satisfying entropy condition and positivity preservation, the HLLEM approximate Riemann solver has been widely applied for simulations of many compressible flow problems. However, when it comes to weakly compressible or incompressible flows, the HLLEM scheme cannot give physically correct solutions. In the current study, a simple low Mach number fix is applied to improve the accuracy and stability of HLLEM approximate Riemann solver in the low Mach limit. As a result, a modified HLLEM scheme called LM-HLLEM scheme is proposed. Numerical results demonstrate that the proposed LM-HLLEM scheme is able to compute various flow problems accurately and robustly ranging from compressible to low Mach incompressible flows.

Keywords. Low Mach number; Computational fluid dynamics; HLLEM scheme; Euler equations

1. Introduction

Godunov-type Riemann solvers have been widely used in computational fluid dynamics for their clear physical interpretation and robustness. However, many Godunov-type Riemann solvers fail to guarantee physically correct solution in most weakly compressible and incompressible cases[1]. In practical applications, most problems contain both incompressible flow regions with very low Mach number and compressible flow regions. Take the flow around a blunt body as example, there will be a shock in front of the blunt body while in the boundary layer region near the wall, the flow should be considered as incompressible. Therefore, improving Godunov-type schemes’ performance in incompressible flows is of great significance.

In order to overcome the low Mach limit of Godunov-type Riemann solvers, many scholars have done a series of researches. Turkel[2] proposed a preconditioning method, improved the accuracy and efficiency of solving the low Mach flows[3]. However, quite a number of test cases show that a cut-off problem is unavoidably for preconditioning method within the region where low Mach flows and high Mach flows both exist. And this problem limits its capability of accurate simulating in transition flow regions and low speed/high speed mixed flow regions. To cure the cut-off problem of Turkel’s preconditioning method, Li et al[4] developed a so called All-Speed-Roe scheme. Compared with other previous preconditioned schemes, Li’s new scheme is more robust and accurate. And asymptotic analysis is conducted in their subsequent works[5] to study the low Mach number behavior of this
scheme theoretically. Chen et al[6] proposed a low Mach fixed scheme, i.e. the EC-RoeM scheme. In their opinion, the left/right-moving acoustic waves can be rescaled into the scaling of convective velocity through a scaling Mach number function. Dellacherie[7] also has done a series of works to solve the low Mach number problems and introduced a low Mach correction. Based on this low Mach correction, Dellacherie proposed a low Mach Godunov scheme for the linear wave equation, and further extended it to nonlinear cases. Furthermore, he exhibited that the proposed method can be applied to other schemes and can be used to justify other existing low Mach schemes, like Rieper’s LMRoe[8] scheme, K. Oßwald’s L2Roe scheme[9], and other Godunov-type methods[10]. Based on the work of Dellacherie, Xie[11] proposed a simple and accurate all Mach HLLC-type scheme, i.e. AM-HLLC-P scheme. This new scheme is of high level of robustness against shock instability and retain the ability of sharp capturing of different kinds of discontinuities. With the introduction of an all Mach correction method, this scheme is further extended to the computation of flows at all Mach numbers.

In 1983, Harten, Lax and van Leer[12] proposed a uniform framework for constructing approximate Riemann solutions, namely the HLL scheme. Based on HLL scheme, the approximate Riemann solutions with different resolution for various discontinuity can be easily obtained. In [13], the authors indicated that under certain wavespeeds estimate, the HLL-type schemes are positively conservative and automatically satisfy the entropy condition. Furthermore, some researchers[13, 14] pointed out that there is a close connection among many upwind schemes, such as Roe scheme[15] and Osher scheme[16]. And these schemes can be represented uniformly by HLL-type scheme framework, therefore, the research on HLL-type schemes is of great significance. Based on HLL scheme, Einfeldt[13] obtained an approximate Riemann solver which can resolve various discontinuities through a method of linear distribution to modify the intermediate state, i.e. the HLLEM scheme. Since HLLEM scheme is based on HLL scheme, it also satisfies the entropy condition and is positivity. HLLEM scheme inherits many attractive properties of HLL scheme, and can resolve linear waves with minimal diffusion. It has been extended to many other applications like multi-phase flows[17], chemically reacting flows[18] and MHD equations[19]. However, when comes to low Mach flows, the HLLEM scheme also cannot guarantee a physically correct solution, like other Godunov-type schemes such as HLLC[20] scheme or Roe[15] scheme. And this low Mach number problem limits HLLEM’s performance for accurate simulation of compressible flows.

In this paper, a low Mach correction is introduced for HLLEM scheme to improve its ability for simulating weakly compressible and incompressible flows, and a LM-HLLEM scheme is developed. Inspired by the work of Dellacherie[7], this new scheme preserves the original form of HLLEM scheme to the maximum, thus it will be straightforward to introduce this low Mach correction to existing codes. Besides, LM-HLLEM scheme is more robust and accurate when compressible and weakly compressible phenomenon are coexisting within a flow region.

The paper is organized in the following way. In the second section, the Euler equations and the related discrete method is reviewed, the HLLEM scheme is also introduced in this section. In the third section, a low Mach correction is applied to HLLEM scheme. A few numerical tests are conducted in the fourth section. Finally, concluding remarks are made in the last section.

2. Euler equations and discrete method

2.1. Euler equations

Euler equations are the basic governing equations for inviscid flows, it can be written in the following integral form as,

\[
\frac{\partial}{\partial t} \int_{\Omega} U d\Omega + \int_{\Omega} F dS = 0, \quad (1)
\]
where \( \mathbf{U} \) and \( \mathbf{F} \) are conservative variable vector and inviscid flux vector respectively. \( \Omega \) is the control volume, \( \partial \Omega \) denotes the boundaries of the control volume, \( dS \) denotes the surface of the interface. The conservative variable vector and inviscid flux vector are defined as,

\[
\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \rho q \\ \rho u q + p n_x \\ \rho v q + p n_y \\ \rho H q \end{pmatrix},
\]

where \( \rho \) is density, \( p \) is static pressure, \( e \) is specific total energy, \( u \) and \( v \) are velocity vector components. \( q = u n_x + v n_y \), represents the directed velocity, where \( \mathbf{n} = (n_x, n_y)^T \) denotes the outward unit vector normal to the interface.

In this paper, a cell-centered finite-volume method is applied to system (1), assuming that flow variables are stored at the center of grid cell, therefore the time derivative of the conserved variable \( \mathbf{U} \) can be transformed as

\[
\frac{\partial \Omega}{\partial t} \int_{\Omega} \mathbf{U} d\Omega = \Omega \frac{\partial \mathbf{U}}{\partial t}.
\]

Combine equation (1) and equation (3), we can get the following expression,

\[
\frac{\partial \mathbf{U}}{\partial t} = \frac{1}{\Omega} \sum_{i} \Gamma_{ij} |\mathbf{F}_{ij}|.
\]

\[\text{Figure 1. Schematic diagram of control volume in a two-dimensional structured grid.}\]

The semidiscrete finite volume scheme over a control volume \( \Omega_i \), as shown in figure 1, can be written as

\[
\frac{d\mathbf{U}_i}{dt} = \frac{1}{|\Omega_i|} \sum_{j} \Gamma_{ij} |\mathbf{F}_{ij}|,
\]

where \( \mathbf{U}_i \) is cell average of \( \mathbf{U}_i \) on \( \Omega_i \), \( |\Omega_i| \) is the volume of \( \Omega_i \), \( \Gamma_{ij} \) denotes the common edge of two adjacent cell \( \Omega_i \) and \( \Omega_j \), \( |\Gamma_{ij}| \) is the length of this surface element, \( \mathbf{F}_{ij} \) is the numerical flux.

2.2. The HLLEM scheme

The HLLEM scheme, proposed by Einfeldt[13], is a popular method to solve Riemann problem approximately, and the corresponding interface flux can be defined as,

\[
\mathbf{F}_{\text{HLLEM}} = \frac{S_R}{S_R - S_L} \mathbf{F}(\mathbf{U}_L)^+ - \frac{S_L}{S_R - S_L} \mathbf{F}(\mathbf{U}_R)^+ + \frac{S_R^+}{S_R^+-S_L^+} (\mathbf{U}_R - \mathbf{U}_L - \hat{\delta}_2 \hat{\delta}_2 \hat{R}_2 - \hat{\delta}_3 \hat{\delta}_3 \hat{R}_3),
\]

with
\[ S_L = \min(0, S_L), S_R = \max(0, S_R), \]  

the symbol \(^{\wedge}\) represents the Roe’s averaged values at the interface, \(S_L\) and \(S_R\) are the left and right wave speeds, and can be calculated from

\[ S_L = \min(q_L - a_L, \hat{q} - \hat{a}), S_R = \max(q_R + a_R, \hat{q} + \hat{a}), \]  

\(a\) represents the sound speed.

\[ \hat{\mathbf{R}}_2 = \begin{pmatrix} 1 & \hat{u} \\ \hat{v} & - \frac{\hat{v}^2 + \hat{u}^2}{2} \end{pmatrix}, \hat{\mathbf{R}}_3 = \begin{pmatrix} 0 \\ -n_y \\ n_x \\ -\hat{u}n_y + \hat{v}n_x \end{pmatrix}, \]  

and

\[ \hat{a}_2 = \Delta \rho - \frac{\Delta p}{\hat{a}^2}, \hat{\alpha}_3 = \hat{\rho}(-n_x \Delta u + n_y \Delta v), \]  

\[ \hat{\delta}_2 = \hat{\delta}_3 = \frac{\hat{a}}{\hat{a} + |\hat{q}|}, \]  

where \(\hat{\mathbf{R}}_2\) and \(\hat{\mathbf{R}}_3\) are the second and third right eigenvectors of Jacobian matrix \(\hat{\mathbf{A}}\), \(\hat{a}_2\) and \(\hat{\alpha}_3\) are the wave strengths. \(\hat{\delta}_2\) and \(\hat{\delta}_3\) are the anti-dissipation terms, which control the dissipation terms of linearly degenerate waves.

It is worth noting that when the anti-dissipation terms \(\hat{\delta}_2\) and \(\hat{\delta}_3\) are equal to 0, the HLLEM scheme will degenerate into HLL scheme.

3. **Modification on the HLLEM scheme at low Mach number**

As mentioned in the first section, the Godunov-type Riemann solvers cannot guarantee physically correct solutions at low Mach numbers. Dellacherie[7] has done a series of works, and proposed a low Mach correction for Godunov-type schemes. In this part, the low Mach correction will be briefly reviewed, and will be applied to HLLEM scheme.

3.1. **An all Mach correction**

Here we solve the compressible Euler equations through the finite-volume method, and the spatial convection term is discretized by a Godunov-type scheme, so the low Mach Godunov scheme can be constructed as

\[ \frac{d\mathbf{U}}{dt} + \frac{1}{|\Omega|} \sum_{\Gamma_{ij}} \left| \Gamma_{ij} \right| \mathbf{F}^{LM,X}_{ij} = 0. \]  

\(X\) represents a Godunov-type scheme, the numerical flux \(\mathbf{F}^{LM,X}_{ij}\) can be expressed as

\[ \mathbf{F}^{LM,X}_{ij} = \mathbf{F}^X + (1 - \theta_{ij}) \frac{\rho a_{ij}}{2} \begin{pmatrix} 0 \\ \Delta q_n_x \\ \Delta q_n_y \\ 0 \end{pmatrix}, \]  

where \(\mathbf{F}^X\) is the original flux of a unmodified scheme \(\mathbf{X}\), \(\theta_{ij}\) is scaling function, and \(\theta_{ij}\) is related to the local Mach number \(M_{ij}\).
\[ \theta_{ij} = \theta(M_{ij}), \theta(M) = \min(M, 1), \quad (14) \]

\[ M_{ij}, \rho_{ij} \text{ and } a_{ij} \text{ denote the calculate Mach number, density and sound speed at the interface of } i \text{ and } j \text{ respectively, which are calculated from the Roe's averaged values.} \]

3.2. Modification for HLLEM scheme

In this section, the low Mach correction in equation (13) is applied to HLLEM scheme to improve its performance in the simulation of low Mach number flows. Here X scheme is HLLEM scheme. Following the low Mach correction in equation (13), the modified numerical fluxes can be written as

\[ F^{\text{LM-HLLEM}}_j = F^{\text{HLLEM}}_j + (1 - \theta_{ij}) \frac{\rho_{ij} a_{ij}}{2} \begin{pmatrix} 0 \\ \Delta qn_x \\ \Delta qn_y \\ 0 \end{pmatrix}, \quad (15) \]

where \( \theta_{ij} \) is defined in equation (14), and \( \left( \right)_{ij} \) represents Roe’s averaged values.

Besides, we shall introduce a new definition of the local Mach number \( M_{ij} \), namely

\[ M_{ij} = \max \left( \frac{\sqrt{u_i^2 + v_i^2}}{a_i}, \frac{\sqrt{u_j^2 + v_j^2}}{a_j} \right), \quad (16) \]

and this modification is proposed by Thornber et al[21].

Compared with other alternatives, Thornber’s modification of local Mach number has little influence on stability of numerical simulation near the sonic point or at high speed flows[22].

4. Numerical experiments

In this section, several numerical test cases will be conducted to verify the effectiveness of Dellacherie’s low Mach number correction applied in HLLEM scheme.

4.1. Modified Sod shock tube

This test case is a modified version of the classical Sod shock tube problem[23], which is very useful to evaluate the entropy satisfaction property of numerical schemes. The solution of this problem consists of a right shock wave, a right travelling contact wave and a left sonic rarefaction wave. In this paper, this case is used to verify whether the LM-HLLEM scheme satisfies the entropy condition. The initial condition is set as

\[ (\rho_L, u_L, p_L) = (1.0, 0.75, 1.0), \quad (\rho_R, u_R, p_R) = (0.125, 0.0, 0.1), \]

besides, the position of the discontinuity is set as \( x = 0.3 \). The computational domain is \( x \in [0,1] \), and is evenly discretized with 400 cells. Nonreflecting boundary conditions are used at the boundaries. The time step is determined by local CFL condition, and is set as \( CFL = 0.9 \).

The numerical solutions of different solvers at \( t = 0.2s \) are shown in figure 2. The results from the LM-HLLEM solver are almost identical to that of the HLLEM scheme, and agree well with the exact solution. As shown in figure 2, this newly developed all Mach number scheme can capture shock wave, rarefaction wave and contact discontinuity sharply. In the left rarefaction wave region, no so-called rarefaction shock generated near the sonic point, which indicates that this LM-HLLEM scheme satisfies the entropy condition. On the contrary, the Roe scheme does not satisfy the entropy condition, so there is an obvious nonphysical rarefaction shock near the sonic point.
4.2. Gresho vortex

The Gresho vortex test is used to demonstrate the accuracy of LM-HLLEM scheme in low Mach incompressible flow regions. Gresho vortex is a rotating flow in which centrifugal force is balanced by pressure gradient. The simulation region is set as \( \Omega = [0,1] \times [0,1] \), the initial background state set as

\[
\rho_0 = 1.0, \quad u_0 = (u_0, 0)^T, \quad p_0 = 1.0, \quad a_0 = \sqrt{\gamma p_0 \rho_0} = \sqrt{\gamma}, \quad u_0 = M_0 a_0,
\]

where \( M_0 \) is the global Mach number. A vortex of \( R = 0.4 \) is set at position \((x_0, y_0) = (0.5, 0.5)\) at the initial time. The initial condition is defined as

\[
u_r(r) = u_0 \begin{cases} 
\frac{2r}{R} & 0 \leq r < \frac{R}{2} \\
2 \left(1 - \frac{r}{R}\right) & \frac{R}{2} \leq r < R \\
0 & R \leq r
\end{cases}
\]  

(17)

And the initial pressure field is given by

\[
p(r) = p_0 + u_0^2 \begin{cases} 
\frac{2r^2}{R^2} + 2 - \log 16 & 0 \leq r < \frac{R}{2} \\
2 \frac{r^2}{R^2} - \frac{8r}{R} + 4 \log \left(\frac{r}{R}\right) + 6 & \frac{R}{2} \leq r < R, \\
0 & R \leq r
\end{cases}
\]  

(18)

where the radius can be expressed as

\[r = \sqrt{(x-x_0)^2 + (y-y_0)^2}.\]  

(19)

Periodic boundary conditions are applied to the left and right boundaries of simulating region. The far field boundary conditions are applied to upper and lower boundaries. The computational region is evenly dispersed into a \( 100 \times 100 \) Cartesian grid, this test case is run for the Mach number \( M_0 = 10^{-2} \), and the CFL-number is 0.9.

A dimensionless pressure field is defined to characterize the order of pressure fluctuation in incompressible limit,
\[ P_N(x, y) = \frac{p(x, y) - p_{\min}}{p_{\max} - p_{\min}}. \] (20)

Since Gresho vortex is a steady solution of the incompressible Euler equation, the dimensionless pressure field should not change over time, therefore this test case is very suitable for evaluating the performance of numerical schemes in low Mach flow fields.

![Figure 3](image1.png)

(a) Initial State  
(b) Roe Scheme  
(c) HLLEM Scheme  
(d) LM-HLLEM Scheme

**Figure 3.** The normalized pressure contour map of the Gresho vortex after one period. (a) Initial State; (b) Roe Scheme; (c) HLLEM Scheme; (d) LM-HLLEM Scheme.

Figure 3 shows the dimensionless pressure field of Gresho vortex after one period under the condition \( M_0 = 10^{-2} \), it can be clearly observed that the original Riemann solver, like Roe scheme and HLLEM scheme, contains excessive numerical dissipation in the low Mach number region, so the structure of vortex is completely dissipated after one period, which is greatly different from the initial solution. On the contrary, this newly developed low Mach number scheme maintains the vortex structure well and is in good agreement with the initial solution, showing its low dissipation property. Figure 4 shows the dimensionless pressure distribution curves along the center line of Gresho vortex calculated in different schemes. As shown, the LM-HLLEM scheme is more accurate than original HLLEM scheme, and shows good agreement with the exact solution. This test case verifies the accuracy of LM-HLLEM scheme in low Mach number cases.
4.3. Low Mach number inviscid flow around the NACA0012 airfoil

The inviscid steady flow around the NACA 0012 airfoil is a classical test case to evaluate the performance of numerical schemes in low Mach number flows. An O-type mesh is applied, which contains 241 nodes in the circumferential direction and 121 nodes in the normal direction. The angle of attack is set as $0^\circ$, and the inlet Mach number is set as $M_0 = 0.1, 0.01, 0.001$. Each case is executed for 50000 time steps with LU-SGS approach, and the CFL-number is 200. The residuals (L2-norm of density) dropped by at least five orders of magnitude. A normalized pressure field is defined by equation (20) to represent the order of pressure fluctuations in the incompressible limit.

As shown in figure 5, the unmodified HLLEM scheme cannot guarantee a physically correct solution in low Mach flow, while the LM-HLLEM scheme manages to converge to a solution that approaches incompressible flow.

5. Concluding remarks

In this paper, we proposed a low Mach number scheme called LM-HLLEM scheme. Three test cases were conducted to evaluate its performance in compressible and incompressible flows. The results of test cases shown that the LM-HLLEM scheme can capture shocks as sharply as the original HLLEM scheme, and it can be applied to the simulations of weakly compressible and incompressible flows.
Figure 5. The normalized pressure fields for inviscid flows around NACA0012 airfoil.

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