On Particles Collisions Near Rotating Black Holes

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Abstract. Scattering of particles with different masses and energy in the gravitational field of rotating black holes is considered as outside as inside the black hole. Expressions for scattering energy of particles in the centre of mass system are obtained. It is shown that scattering energy of particles in the centre of mass system can obtain very large values not only for extremal black holes but also for nonextremal ones if one takes into account multiple scattering. Numerical estimates for the time needed for the particle to get ultrarelativistic energy are given.

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1. INTRODUCTION

In our publications [1] we came to the conclusion that one can get very large energies in the centre of mass frame first mentioned in [2] for two particle collision of particles with equal masses close to the horizon of the rotating black hole (Active Galactic nucleus) if one considers multiple scattering. The effect of getting very large energy depends on the value of the angular momentum of one of the particles. The problem of the energy of collision of particles in vicinity of black holes of different types now is intensively studied by different authors [3,4,5]. Here we obtain similar formulas for particles of different masses in the field of Kerr’s black hole. To get very large energy one must have large time of rotating of the particle around the black hole coming closer and closer to the horizon. We give some quantitative estimates for the time needed for a particle to obtain ultrarelativistic energy outside the horizon. Then we investigate the case of scattering inside the horizon. The limiting formulas are obtained and it is shown that the collisions with infinite energy can not be realized even in the singularity.

The system of units $G = c = 1$ is used in the paper.

2. THE ENERGY OF COLLISIONS IN THE FIELD OF KERR’S BLACK HOLE

Let us consider particles falling on the rotating chargeless black hole. The Kerr’s metric of the rotating black hole in Boyer–Lindquist coordinates has the form

$$ds^2 = dt^2 - \frac{2Mr (dt - a \sin^2 \theta d\varphi)^2}{r^2 + a^2 \cos^2 \theta} - (a^2 \cos^2 \theta + r^2) \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - (r^2 + a^2) \sin^2 \theta d\varphi^2,$$  \hspace{1cm} (1)

where

$$\Delta = r^2 - 2Mr + a^2,$$  \hspace{1cm} (2)

$M$ is the mass of the black hole, $J = aM$ is angular momentum. In the case $a = 0$ the metric (1) describes the static chargeless black hole in Schwarzschild coordinates. The event horizon for the Kerr’s black hole corresponds to the value

$$r = r_H \equiv M + \sqrt{M^2 - a^2}. \hspace{1cm} (3)$$

The Cauchy horizon is

$$r = r_C \equiv M - \sqrt{M^2 - a^2}. \hspace{1cm} (4)$$

For equatorial ($\theta = \pi/2$) geodesics in Kerr’s met-
For the energy in the centre of mass frame of two colliding particles with rest masses $m_1$ and $m_2$ in arbitrary gravitational field. It can be obtained from

$$E_{\text{c.m.}}(0, 0, 0) = m_1 u_{i1} + m_2 u_{i2},$$

where $u^i = dx^i/ds$. Taking the squared (5) and due to $u^i u_i = 1$ one obtains

$$\frac{E^2_{\text{c.m.}}}{m_1 m_2} = \frac{m_1}{m_2} + \frac{m_2}{m_1} + u_{i1}(u_{i2}).$$

The scalar product does not depend on the choice of the coordinate frame so (9) is valid in an arbitrary coordinate system and for arbitrary gravitational field.

We denote $x = r/M$, $A = a/M$, $l_n = L_n/M$, $\Delta_s = \Delta^2 - 2x + A^2$ and

$$x_H = 1 + \sqrt{1 - A^2}, \quad x_C = 1 - \sqrt{1 - A^2}. \quad (10)$$

For the energy in the centre of mass frame of two colliding particles with specific energies $\varepsilon_1$, $\varepsilon_2$ and angular momenta $L_1$, $L_2$, which are moving in Kerr’s metric one obtains using (11), (5), (7), (9):

$$\frac{E^2_{\text{c.m.}}}{m_1 m_2} = \frac{m_1^2 + m_2^2}{2m_1 m_2} - \frac{\varepsilon_1 \varepsilon_2}{x \Delta_x}\left[l_2 (2x - 1) + 2 \varepsilon_1 \varepsilon_2 \left(2(x^2 - 1) + A^2(x + 1) - A(l_1^2 \varepsilon_1^2 + l_2^2 \varepsilon_2^2)\right) - \sqrt{2 \varepsilon_1^2 x^2 + 2 (l_1 - \varepsilon_1 A)^2 - l_1^2 x + (\varepsilon_1^2 - 1) x \Delta_x}\right.$$\n
$$\times \sqrt{2 \varepsilon_2^2 x^2 + 2 (l_2 - \varepsilon_2 A)^2 - l_2^2 x + (\varepsilon_2^2 - 1) x \Delta_x} \right]. \quad (11)$$

It corresponds to results in (2) and (7) for the case $\varepsilon_1 = \varepsilon_2$. Collisions of particles of equal masses with different specific energies were considered in (8).

Writing the right hand side (11) as $f(x) + (m_1^2 + m_2^2)/(2m_1 m_2)$, one obtains

$$\frac{E_{\text{c.m.}}(r)}{m_1 + m_2} = \sqrt{1 + \left(f(x) - 1\right) \frac{m_1 m_2}{(m_1 + m_2)^2}}. \quad (12)$$

This relation has maximal value for given $r$, specific particle energies $\varepsilon_1$, $\varepsilon_2$ and specific angular momenta $l_1$, $l_2$, if the particle masses are equal: $m_1 = m_2$.

To find the limit $r \to r_H$ for the black hole with a given angular momentum $A$ one must take in (11) $x = x_H + \alpha$ with $\alpha \to 0$ and do calculations up to the order $\alpha^2$. Taking into account $A^2 = x_H x_C$, $x_H + x_C = 2$, after resolution of uncertainties in the limit $\alpha \to 0$ one obtains

$$\frac{E_{\text{c.m.}}^2(r \to r_H)}{m_1 m_2} = \frac{m_1^2 + m_2^2}{2m_1 m_2} \frac{l_1 l_2}{4} + \frac{1}{8} \left[(l_1^2 + 4) \frac{l_2^2}{l_1 l_2 - l_2^2} + (l_1^2 + 4) \frac{l_1 l_2}{l_2^2 - l_1^2}\right], \quad (13)$$

where

$$l_{nH} = \frac{2 \varepsilon_n x_H}{A} = \frac{2 \varepsilon_n}{A} \left(1 + \sqrt{1 - A^2}\right). \quad (14)$$

is the limiting value of the angular momentum of the particle with specific energy $\varepsilon_n$ close to the horizon of the black hole. It can be obtained from the condition of positive derivative in (15) $dt/d\tau > 0$, i.e. going “forward” in time:

$$l_n < l_{nH} \left(1 + \frac{x_H + 1}{2}\right) + O(\alpha), \quad x = x_H + \alpha. \quad (15)$$

So close to the horizon one has the condition $l_n \leq l_{nH}$.

Note that for $l = l_H - \beta$ from (7) one gets

$$\left.\left(\frac{dr}{d\tau}\right)^2\right|_{r=r_H} = \frac{\beta^2 x_C}{x_H^2} > 0. \quad (16)$$

So there exists some region close to the horizon where one has particles moving with angular momentum arbitrary close to the limiting value $l = l_H$.

In another form (13) is

$$\frac{E_{\text{c.m.}}(r \to r_H)}{m_1 + m_2} = \left[1 + \frac{m_1 m_2}{(m_1 + m_2)^2}\right] \times \frac{(l_1 l_2 - l_2 l_1)^2 + 4(l_1 l_2 - l_2 l_1)^2}{4(l_1 - l_2)(l_2 - l_1)^2}\]. \quad (17)$$

In special case $\varepsilon_1 = \varepsilon_2$ (for example for nonrelativistic on infinity particles $\varepsilon_1 = \varepsilon_2 = 1$) formula (13) can be written as

$$\frac{E_{\text{c.m.}}(r \to r_H)}{m_1 + m_2} = \left[1 + \frac{m_1 m_2}{(m_1 + m_2)^2}\right] \times \frac{(4 + l_2^2) (l_1 - l_2)^2}{4(l_1 - l_2)(l_2 - l_1)^2}. \quad (18)$$
For the extremal black hole \( A = x_H = 1, \ l_H = 2\varepsilon, \) and the expression \( (19) \) is divergent when the dimensionless angular momentum of one of the falling into the black hole particles \( l = l_H = 2\varepsilon. \) The scattering energy in the centre of mass system is increasing without limit (for case \( \varepsilon = 1 \) it was established in [2]). For example, if \( l_1 = l_H \) then one obtains from Eq. \( (11) \)

\[
\frac{E_{\text{cm}}^2(x)}{2m_1m_2} \approx \frac{(2\varepsilon_2 - l_2)(2\varepsilon_1 - \sqrt{3\varepsilon_1^2 - 1})}{x - 1}, \quad x \to 1.
\]

(19)

Note that the small value of \( r - r_H \) for the radial coordinate of the point of the collision of particles with high energy in the centre of mass frame does not mean small distance because the metrical coefficient \( g_{rr} = -r^2/\Delta \to \infty. \)

If \( A = 1 \) and \( l = l_H = 2\varepsilon \) then from \( (7) \) one gets

\[
\left( \frac{dr}{d\tau} \right)^2 = \frac{(x - 1)^2}{x^3} \left( 2\varepsilon^2 + (\varepsilon^2 - 1)x \right).
\]

(20)

For \( \varepsilon \geq 1 \) expression \( (20) \) is nonnegative and the particle with such angular momentum falling from infinity can achieve the event horizon.

In Refs. [1] [7] we are shown that in order to get the unboundedly growing energy one must have the time interval (as coordinate as proper time) from the beginning of the falling inside the black hole to the moment of collision also growing infinitely. Give some quantitative estimates.

In case of the extremal rotating black hole \( A = 1 \) and the limiting angular momentum \( l_1 = 2\varepsilon_1 \) from \( (5), (7) \) one obtains

\[
\frac{dt}{dx} = \frac{-M\varepsilon_1\sqrt{2}(x^2 + 2 + x)}{(x - 1)^2\sqrt{\varepsilon_1^2(x + 2) - x}}.
\]

(21)

So the time of movement in the vicinity of events horizon up to the point of collision with radial coordinate \( x_f \to x_H = 1 \)

\[
\Delta t \sim \frac{4M\varepsilon_1}{(x_f - 1)\sqrt{3\varepsilon_1^2 - 1}}.
\]

(22)

Taking into account \( (19) \) one obtains time of movement before collision with a given value of the energy \( E \) in the centre of mass frame

\[
\Delta t \sim \frac{E^2}{m_1m_2} \frac{2M\varepsilon_1}{(2\varepsilon_2 - l_2)\sqrt{3\varepsilon_1^2 - x} - (2\varepsilon_1 - \sqrt{3\varepsilon_1^2 - x})}.
\]

(23)

For \( \varepsilon_1 = \varepsilon_2 = 1, \ l_2 = 0 \) one gets

\[
\Delta t \sim \frac{E^2}{m_1m_2} \frac{M}{2(\sqrt{2} - 1)} \approx 6 \cdot 10^{-6} \frac{M}{M_\odot} \frac{E^2}{m_1m_2} s,
\]

(24)

where \( M_\odot \) is the mass of the Sun.

So to have the collision of two protons with the energy of the order of the Grand Unification one must wait for the black hole of the star mass the time \( \approx 10^{24} \) s, which is larger than the age of the Universe \( \approx 10^{18} \) s. However for the collision with the energy \( 10^8 \) larger than that of the LHC one must wait only \( \approx 10^8 \) s.

Can one get the unlimited high energy of this scattering energy for the case of nonextremal black hole?

3. THE ENERGY OF COLLISIONS FOR NONEXTREMAL BLACK HOLE

For a particle falling on the black hole from infinity one must have \( \varepsilon \geq 1 \). In this section we consider the case \( \varepsilon = 1 \), when the particles falling into the black hole are nonrelativistic at infinity. Formula \( (8) \) leads to limitations on the possible values of the angular momentum of falling particles: the massive particle free falling in the black hole with dimensionless angular momentum \( A \) to achieve the horizon of the black hole must have angular momentum from the interval

\[
-2 \left( 1 + \sqrt{1 + A} \right) = l_L \leq l \leq l_R = 2 \left( 1 + \sqrt{1 - A} \right).
\]

(25)

Putting the limiting values of angular momenta \( l_L, l_R \) into the formula \( (16) \) one obtains the maximal values of the collision energy of particles freely falling from infinity

\[
E_{\text{c.m.}} \Big( r \to r_H \Big) = \frac{E_{\text{c.m.}}(r \to r_H)}{m_1 + m_2} = \sqrt{1 + \frac{2m_1m_2}{(m_1 + m_2)^2} \left( 2 + \sqrt{1 + A} + \sqrt{1 - A^2} \right)^2}
\]

\[
= \sqrt{1 + \frac{2m_1m_2}{(m_1 + m_2)^2} \left( 2 + \sqrt{1 - A^2} \right)^2}
\]

(26)

For \( A = 1 - \epsilon \) with \( \epsilon \to 0 \) formula \( (26) \) gives:

\[
E_{\text{c.m.}}^{\max} = 2 \left( 2^{1/4} + 2^{1/4} \right) \frac{m_1m_2}{\epsilon^{1/4}} + O(\epsilon^{1/4}).
\]

(27)

So even for values close to the extremal \( A = 1 \) of the rotating black hole \( E_{\text{c.m.}}^{\max}/\sqrt{m_1m_2} \) can be not very large as mentioned in [2] for the case \( m_1 = m_2 \). So for \( A_{\max} = 0.998 \) considered as the maximal possible dimensionless angular momentum of the astrophysical black holes (see [10]), from \( (26) \) one obtains

\[
E_{\text{c.m.}}^{\max}/\sqrt{m_1m_2} \approx 18.97.
\]

Does it mean that in real processes of particle scattering in the vicinity of the rotating nonextremal black holes the scattering energy is limited so that no Grand Unification or even Planckian energies can
be obtained? Let us show that the answer is no! If one takes into account the possibility of multiple scattering so that the particle falling from infinity on the black hole with some fixed angular momentum changes its momentum in the result of interaction with particles in the accreting disc and after this is again scattering close to the horizon then the scattering energy can be unlimited.

From (7) one can obtain the permitted interval in \( r \) for particles with \( \varepsilon = 1 \) and angular momentum \( l = l_H - \delta \). To do this one must put the left hand side of (7) to zero and find the root. In the second order in \( \delta \) close to the horizon one obtains

\[
l = l_H - \delta \Rightarrow x < x_5 \approx x_H + \frac{\delta^2 x_5^2}{4 x_H \sqrt{1 - A^2}}.
\]

The effective potential for the case \( \varepsilon = 1 \) defined by the right hand side of (7)

\[
V_{\text{eff}}(x, l) = -\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 = -\frac{1}{x} + \frac{l^2}{2 x^2} - \frac{(A - l)^2}{x^3}
\]

(29)

(see, for example, Fig. 1) leads to the following behavour of the particle. If the particle goes from infinity to the black hole it can achieve the horizon if the inequality (20) is valid. However the scattering energy in the centre of mass frame given by (20) is not large. But if the particle is going not from the infinity but from some distance defined by (28) then due to the form of the potential it can have values of \( l = l_H - \delta \) large than \( l_R \) and fall on the horizon. If the particle falling from infinity with \( l \leq l_R \) arrives to the region defined by (28) and here it interacts with other particles of the accretion disc or it decays into a lighter particle which gets an increased angular momentum \( l_1 = l_H - \delta \), then due to (13) the scattering energy in the centre of mass system is

\[
E_{\text{c.m.}} \approx \frac{1}{\sqrt{\delta}} \sqrt{\frac{2 m_1 m_2 (l_H - l_2)}{1 - \sqrt{1 - A^2}}} \quad (30)
\]

and it increases without limit for \( \delta \to 0 \). For \( A_{\text{max}} = 0.998 \) and \( l_2 = l_L \), \( E_{\text{c.m.}} \approx 3.85 \text{m}/\sqrt{\delta} \).

Note that for rapidly rotating black holes \( A = 1 - \varepsilon \) the difference between \( l_H \) and \( l_R \) is not large

\[
l_H - l_R = 2 \sqrt{1 - \frac{A}{A}} \left( \sqrt{1 - A} + \sqrt{1 + A - A} \right) \approx 2(\sqrt{2} - 1) \sqrt{\varepsilon}, \quad \varepsilon \to 0 .
\]

(31)

For \( A_{\text{max}} = 0.998, l_H - l_R \approx 0.04 \) so the possibility of getting small additional angular momentum in interaction close to the horizon seems much probable. The probability of multiple scattering in the accretion disc depends on its particle density and is large for large density.

4. COLLISION OF PARTICLES INSIDE KERR BLACK HOLE

As one can see from formula (11) the infinite value of the collision energy in the centre of mass system can be obtained inside the horizon of the black hole on the Cauchy horizon (1). Indeed, the zeroes of the denominator in (11): \( x = x_H, x = x_C, x = 0 \).

Let us find the expression for the collision energy for \( x \to x_C \). Denote

\[
l_C = \frac{2 \varepsilon x_C}{A} = \frac{2 \varepsilon}{A} \left( 1 - \sqrt{1 - A^2} \right), \quad l_{nC} = \frac{2 \varepsilon n x_C}{A} .
\]

(32)

Note that for \( \varepsilon = 1 \) the Cauchy horizon can be crossed by the free falling from the infinity particle under the same conditions on the angular momentum (25) as in case of the event horizon and \( l_L < l_C \leq l_R \leq l_H \).

To find the limit \( r \to r_C \) for the black hole with a given angular momentum \( A \) one must take in (11) \( x = x_C + \alpha \) and do calculations with \( \alpha \to 0 \). The limiting energy has three different expressions depending on the values of angular momenta. If

\[
(l_1 - l_{1C})(l_2 - l_{2C}) > 0 ,
\]

then

\[
\frac{E_{\text{c.m.}}^2(r \to r_C)}{2 m_1 m_2} = \frac{m_1^2 + m_2^2}{2 m_1 m_2} \frac{l_{1C} l_{2C}}{4} + \frac{1}{8} \left( l_{1C}^2 + 4 \right) \frac{l_{2C} - l_{1}}{l_{1C} - l_{1}} + (l_{2C} + 4) \frac{l_{1C} - l_{1}}{l_{2C} - l_{1}} ,
\]

(34)
This formula is similar to (13) if everywhere $H \leftrightarrow C$. If

$$(l_1 - l_{1C})(l_2 - l_{2C}) = 0,$$  \hspace{1cm} (35)

for example, $l_1 = l_{1C}$, then

$$\frac{E_{c.m.}}{\sqrt{m_1 m_2}} \approx \sqrt{\frac{4(l_2 - l_{2C})^2(x_H^2 x_C + x_H)}{x_C(x_H - x_C)(x - x_C)}}, \ x \to x_C.$$  \hspace{1cm} (36)

If

$$(l_1 - l_{1C})(l_2 - l_{2C}) < 0,$$  \hspace{1cm} (37)

then

$$\frac{E_{c.m.}}{2\sqrt{m_1 m_2}} \approx \sqrt{\frac{x_H(l_1 - l_{1C})(l_2 - l_{2C})}{x_C(x_H - x_C)(x - x_C)}}, \ x \to x_C.$$  \hspace{1cm} (38)

It seems that the limits of (36) and (38) are infinite for all values of angular momenta $l_1, l_2$ and (37). This could be interpreted as instability of the internal Kerr’s solution [11]. However, from Eq. (3) we can see

$$\frac{dt}{d\tau}(x \to x_C + 0) \approx \left\{ \begin{array}{ll} +\infty, & \text{if } l > l_C, \\ -\infty, & \text{if } l < l_C. \end{array} \right.$$  \hspace{1cm} (39)

That is why the collisions with infinite energy cannot be realized (see also [11]).

For the particle falling to singularity in the equatorial plane of the Kerr’s black hole with $A \neq 0$

$$\frac{dt}{d\tau}(x \to 0) \approx \left\{ \begin{array}{ll} +\infty, & \text{if } l < \varepsilon A, \\ -\infty, & \text{if } l > \varepsilon A. \end{array} \right.$$  \hspace{1cm} (40)

In case $l = \varepsilon A$, $A \neq 0$ the righthand side of (4) for massive particle is negative for $x \to 0$ and falling to singularity for such particle is impossible. So for particles colliding in the vicinity of singularity one has $(l_1 - \varepsilon_1 A)(l_2 - \varepsilon_2 A) > 0$. Then from (11) one gets

$$\frac{E_{c.m.}(r \to 0)}{m_1 + m_2} = \sqrt{1 + \frac{m_1 m_2}{(m_1 + m_2)^2} \frac{(l_1 - l_2 + (\varepsilon_2 - \varepsilon_1)A)^2}{(l_1 - \varepsilon_1 A)(l_2 - \varepsilon_2 A)}}.$$  \hspace{1cm} (41)

That is why collision of particles with infinite energy in the centre of mass frame is impossible even in singularity.

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