1. Introduction

Let $\ell$ be an integer and $M_{\ell}^2$ denote the space of Siegel modular forms of weight $\ell$ and degree 2 on $\text{Sp}_2(\mathbb{Z})(\subseteq M_4(\mathbb{Z}))$ and by $S_{\ell}^2$ the subspace of cusp forms. These are holomorphic functions defined on the Siegel upperhalf space $H_2$ which consists of complex symmetric matrices $Z$ in $M_4(\mathbb{C})$ whose imaginary part is positive-definite. If we write such a $Z = \left( \begin{array}{cc} \tau & \epsilon \\ \epsilon' & \tau' \end{array} \right)$, then $F|_{z=0} = F \left( \left( \begin{array}{cc} \tau & 0 \\ 0 & \tau' \end{array} \right) \right)$ is a modular form in $\tau$ and $\tau'$ (see [3] for more details) with weight $\ell$, which we call the pullback of $F$ to $H \times H$.

The study of pullbacks of automorphic forms has a rich history, see eg., [1], [6], [7], [12]. In the context of Siegel modular forms, there are conjectures of Ikeda [7] relating such pullbacks to central critical values of $L$-functions for $GSp(2) \times GL(2) \times GL(2)$. Ichino’s beautiful result [6] studies this question for the Saito-Kurokawa (SK from now on) lifts of elliptic modular forms. Following the above notation, let us write $F|_{z=0} = \sum_{g_1,g_2} c_{g_1,g_2} g_1(\tau)g_2(\tau')$ (see also [12]), where $g_j$ runs over a Hecke basis of $S_{\ell}$, the space of elliptic cusp forms on $\text{SL}_2(\mathbb{Z})$. Then Ichino proves that if $F = F_g$ is the SK-lift of $g \in S_{2\ell-2}$ in the above, only the diagonal survives and $|c_{g_1,g_1}|$ is essentially given by the central value $L(1/2, \text{sym}^2 g_1 \times g)$.

It was moreover observed in [12] that comparison of the (normalised) norm of $F_g$ with the norm of its pullback provides a measure of the non-vanishing of the latter on average over the ‘projection’ of $F_g|_{z=0}$ along $g_1 \times g_1$, as $g_1 \in S_{2\ell-2}$ varies. By a formula (see (1.1) in [12]), this also provides a measure of density of $F$ along $F|_{z=0}$ (see [12] (1.13)). This is made more precise in the next paragraph. We now make a change of notation, and use $2k$ for the weight $2\ell-2$, in conformity to the above papers on the topic.

For an odd integer $k > 0$, let $g \in S_{2k}$ be a normalized Hecke eigenform for $\text{SL}_2(\mathbb{Z})$. Let $h \in S_{k+1/2}(\Gamma_0(4))$ be the Hecke eigenform associated to $g$ by the Shimura correspondence. Denote the Saito-Kurokawa lift of $g$ by $F_g \in S_{k+1}^2$. Let us define the quantity

\begin{equation}
N(F_g) := \frac{1}{v_1} \langle F_g|_{z=0}, F_g|_{z=0} \rangle / \frac{1}{v_2} \langle F_g, F_g \rangle,
\end{equation}

where $v_1 = \text{vol}(\text{SL}_2(\mathbb{Z}) \setminus H)$ and $v_2 = \text{vol}(\text{Sp}_2(\mathbb{Z}) \setminus H_2)$. Here $\langle F_g|_{z=0}, F_g|_{z=0} \rangle$ denotes the Petersson norm of $F_g|_{z=0}$ on $\text{SL}_2(\mathbb{Z}) \setminus H \times \text{SL}_2(\mathbb{Z}) \setminus H$ (see section 2 for more details).

\textbf{Abstract.} Using the amplification technique, we prove that ‘mass’ of the pullback of the Saito-Kurokawa lift of a Hecke eigenform $g \in S_{2k}$ is bounded by $k^{1/2} \pi^{k+1}$. This improves the previously known bound $k$ for this quantity.
Let $B_{k+1}$ denote the Hecke basis for $S_{k+1}$. Now Ichino’s formula \cite{6} immediately implies the following, as computed in \cite{12}:

\begin{equation}
N(F_g) = \frac{\pi^2}{15} \left( L(3/2, g) L(1, \text{sym}^2 g) \right)^{-1} \frac{12}{k} \sum_{f \in B_{k+1}} L\left( \frac{1}{2}, \text{sym}^2 f \times g \right).
\end{equation}

It is this quantity $N(F_g)$ that we are concerned with in this paper and we refer this quantity as the mass of the pullback of the Saito-Kurokawa lift. Let us recall that as a special case of conjectures of \cite{2}, Liu and Young in \cite{12} conjectured that $N(F_g) \sim 2$ as $k \to \infty$, and proved it on average over the family $g \in B_{2k}$ and $K \leq k \leq 2K$. In \cite{1} a stronger asymptotic formula was obtained by considering only the smaller family $g \in B_{2k}$. Their result says that there exists some $\eta > 0$ such that

\begin{equation}
\frac{12}{2k-1} \sum_{g \in B_{2k}} N(F_g) = 2 + O(k^{-\eta}).
\end{equation}

Now dropping all but one term, this asymptotic formula immediately gives $N(F_g) \ll k$. This bound is slightly better than the bound $N(F_g) \ll k \log k$ (cf. \cite{12}) that one gets by using the convexity bound for $L(1/2, \text{sym}^2 f \times g)$. In this paper we use the method of amplification (see \cite{9} for more details) to get a power-saving bound for an individual $N(F_g)$. We prove that

**Theorem 1.** For any $g \in B_{2k}$ and $\varepsilon > 0$, one has

\begin{equation}
N(F_g) \ll k^{1-\frac{1}{10}+\varepsilon},
\end{equation}

where the implied constant depends only on $\varepsilon$.

Combining the asymptotic \cite{1.3} and the power-saving obtained from Theorem 1, we have the following corollary, showing the existence of SK-lifts with non-vanishing ‘mass’.

**Corollary 1.1.** We have

\begin{equation}
\# \{ g \in B_{2k} \mid N(F_g) \neq 0 \} \gg k^{1/210}.
\end{equation}

As for the proof of Theorem 1, we use the classical amplifier of Iwaniec-Sarnak \cite{9}. Now instead of inserting the amplifier in the sum $\sum_{g \in B_{2k}} N(F_g)$, we insert the amplifier in a modified sum $\sum_{g \in B_{2k}} S_g$, where $S_g = L(3/2, g) N(F_g)$ (cf. \cite{1.2}). Since $L(3/2, g)^{-1} \ll 1$, this doesn’t have any affect on the final bound for $N(F_g)$. Moreover, this modification helps in reducing the complexities of the further calculations. The proof of Theorem 1 follows a slightly different trajectory than that of the proof of the asymptotic \cite{1.3}. In particular, we have to be careful in keeping track of the dependence on the weight $2k$ throughout in a quantitative fashion.

Finally let us mention that one way of obtaining a better power saving seems to be an improvement in the error term that we obtain when we express $L(1, \text{sym}^2 f) (f \in B_k)$ as a Dirichlet polynomial (see lemma \cite{2.1}), which in turns relies on the results of Y.-K. Lau and J. Wu \cite{11}.

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2. Preliminaries

In this section we collect some necessary results and formulæ that will be used later in the paper. Throughout the article we follow the convention that \( f(x) \ll g(x) \) \( (g(x) \geq 0) \) means there exist constants \( M \) and \( N \) such that \( |f(x)| \leq M \cdot g(x) \) for \( x > N \). Moreover, \( \varepsilon \) will always denote an arbitrarily small positive constant, but not necessarily the same one from one occurrence to the next.

2.1. The norm \( \langle F_\ell |z=0, F_\ell |z=0 \rangle \): Let \( F_\ell \in S_k \) be defined as in section \([1]\). Then \( \langle F_\ell |z=0, F_\ell |z=0 \rangle \) as in the definition of \( N(F_\ell) \) in (1.1) is the product of Petersson inner products on \( \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \times \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \) and it is given by

\[
\langle F_\ell |z=0, F_\ell |z=0 \rangle = \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} |F_\ell \left( \begin{pmatrix} \tau' \\ \tau \end{pmatrix} \right) |^2 \text{Im}(\tau'+1)\text{Im}(\tau')^{k+1} d\mu(\tau) d\mu(\tau'),
\]

where \( d\mu(z) = y^{-2} dx dy \), if \( z = x + iy, y > 0 \).

2.2. Petersson trace formula: Let \( f \in B_k \) and \( \lambda_f(n) \) be the normalized Fourier coefficients of \( f \). Then we have

\[
\frac{2\pi^2}{k-1} \sum_{f \in B_k} \frac{\lambda_f(m)\lambda_f(n)}{L(1, \text{sym}^2 f)} = \delta_{mn} + 2\pi i^{-k} \sum_{c=1}^{\infty} S(m; n; c) J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right),
\]

where \( \delta_{mn} = 1 \) if \( m = n \) and 0 otherwise, \( S(m; n; c) \) is the Kloosterman sum and \( J_{k-1} \) is the Bessel function. For the Bessel function we have the best possible upper bounds given by (see \([10]\))

\[
|J_k(x)| \ll \min\{k^{-1/3}, |x|^{-1/3}\}
\]

for any real \( x \) and \( k \geq 0 \). We use the following bound for the rapid decay of the Bessel function near zero.

\[
J_k(x) \ll \frac{|x/2|^k}{\Gamma(k+1)}, \quad \text{for } x > 0.
\]

Let \( D_1 \) denote the off-diagonal term in (2.2). Then using (2.4) as above, one can truncate the \( c \) sum in \( D_1 \) at \( c \leq 100 \frac{4\pi \sqrt{mn}}{k} \) up to a very small error, say \( k^{-100} \). Then using (2.3) we can write

\[
D_1 = 2\pi i^{-k} \sum_{c \leq 100 \frac{4\pi \sqrt{mn}}{k}} \frac{S(m; n; c)}{c} J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right) + O(k^{-100}) \ll \frac{\sqrt{mn}}{k^{4/3}}.
\]

Let \( f \in S_k \) be a Hecke eigenform of weight \( k \) and \( A_f(m, n) \) the Fourier-Whittaker coefficients of the symmetric square lift of \( f \) (see \([4]\)). Then \( A_f(m, n) \) is given by

\[
A_f(m, n) = \sum_{d|\text{lcm}(m, n)} \mu(d) A_f(m/d, 1) A_f(n/d, 1), \quad \text{where} \quad A_f(r, 1) = \sum_{ab^2 = r} \lambda_f(a^2).
\]

2.3. The approximate functional equation: For \( f \in B_{k+1} \) and \( g \in B_{2k} \) we would use the approximate functional equation (cf. \([1]\) \([8]\))

\[
L(1/2, \text{sym}^2 f \times g) = 2 \sum_{n, m \geq 1} \frac{\lambda_g(n)A_f(m, n)}{n^{1/2}m} W(nm^2).
\]

Here \( W \) is a rapidly decaying smooth weight function. In fact \( W \) satisfies (see \([1]\))

\[
\chi^{(j)} W^{(j)}(x) \ll_{j, A} \left( 1 + \frac{x}{k^2} \right)^{-A}
\]

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\[
\chi^{(j)} W^{(j)}(x) \ll_{j, A} \left( 1 + \frac{x}{k^2} \right)^{-A}
\]
for any $j, A \geq 0$. For our case we take

\begin{equation}
W(x) = \frac{1}{2\pi i} \int_{(1)} \frac{\Lambda_k(s) \left( \cos \frac{\pi s}{10A} \right)^{-60A}}{s^x} ds,
\end{equation}

where (by invoking [11] p. 2626), by replacing $k$ with $k + 1$, $\kappa$ with $k$, so that we are in the case $\kappa < k$)

\begin{equation}
\Lambda_k(s) = (2\pi)^{-3s} \Gamma(s + 2k - \frac{1}{2}) \Gamma(s + k - \frac{1}{2}) \Gamma(s + \frac{1}{2}).
\end{equation}

We also require the following set-up in our proof. As in [1] we define

\begin{equation}
M_f(r) := \frac{12}{2k - 1} \sum_{g \in B_{2k}} \lambda_g(r) \frac{L(1/2, \text{sym}^2 f \times g)}{L(1, \text{sym}^2 g)}.
\end{equation}

Using Deligne’s bound, positivity and Theorem 1.4 in [11] we have

\begin{equation}
M_f(r) \ll r^\kappa, M_f(1) \ll r^\kappa.
\end{equation}

Also applying Petersson formula (2.2) to (2.11) we get $M_f(r) = M_f^{(1)}(r) + M_f^{(2)}(r)$, where $M_f^{(1)}(r)$ is the diagonal contribution and $M_f^{(2)}(r)$ is the off-diagonal contribution. We have the following

\begin{equation}
M_f^{(1)}(r) = \frac{2}{\zeta(2)} \sum_{m} A(m, r) \frac{W(m^2)}{r^{1/2} m}.
\end{equation}

For the off-diagonal term we have

\begin{equation}
M_f^{(2)}(r) = \frac{4\pi r^k}{\zeta(2)} \sum_{n, m, c \geq 1} \frac{A(n, m)}{m^{1/2} n} W(mn^2) S(m, r, c) \frac{1}{c} J_{2k-1} \left( \frac{4\pi \sqrt{mr}}{c} \right).
\end{equation}

Now using the rapid decay of $W$ as in (2.3), we can truncate the $m-$sum at $m \leq k^{2+\varepsilon}n^{-2}$ up to a negligible error, say $k^{-100}$. If one wants to bound $M_f^{(2)}(r)$, then by the insertion of smooth partitions of unity for the $m$ and $c$ sums, it is enough to bound the quantity

\begin{equation}
M_f^{(2)}(r, M, C) = \sum_{n, c} \Omega_1(c/C) \sum_{d(c)} e\left( \frac{dr}{c} \right) \sum_{m} A(n, m) e\left( \frac{dm}{c} \right) \Omega_2\left( \frac{m}{M} \right) J_{2k-1} \left( \frac{4\pi \sqrt{mr}}{c} \right)
\end{equation}

for

\begin{equation}
M \leq \frac{k^{2+\varepsilon}}{n^2}, \quad C \leq 100 \frac{Mr}{k}.
\end{equation}

The truncation over $c$ comes from the rapid decay of the Bessel function near 0 (see (2.4)). Here $\Omega_1$ and $\Omega_2$ are fixed, smooth, compactly supported weight functions. From (2.16) we immediately get that

\begin{equation}
k^2 r^{-1} \leq M \leq k^{2+\varepsilon}, \quad cn \ll r^{1/2} k^\varepsilon.
\end{equation}

Also using the Voronoï formula we can write (see [11] and [13])

\begin{equation}
M_f^{(2)}(r, M, C) = \sum_{n, c} \Omega_1(c/C) \sum_{m_1 \leq m_2} \sum_{m_1 m_2} A(m_1, m_2) \frac{S(n, \pm m_2, c/m_1)}{m_1 m_2} e\left( \frac{dr}{c} \right) \Omega_2\left( \frac{m_2 m_1^2}{c^2 n} \right).
\end{equation}

We also have (see lemma 5.1 in [11])

\begin{equation}
\Psi^\pm(x) \ll A, \quad k^\frac{x c^2}{r^{1/2} (1 + \frac{c x^3}{k^\frac{1}{2} M^{1/2} r^{3/2}})^{-A}}, \quad \text{for } x \geq \frac{1}{c^2 n},
\end{equation}
which follows from a very careful estimation of certain oscillatory integrals (see \[1\]) and

\[
|\sum_{d|c} e^{\frac{dr}{c}} S(nd, \pm m_2, c/m_1)| \leq c \tau(c)(c,n).
\]

We also make use of the useful observation that almost all \(L(1, \text{sym}^2 f)\) for \(f \in B_{k+1}\) can be approximated by a convergent Dirichlet series with rapidly decaying weight function (see lemma 6.1 in \[1\]). More precisely:

**Lemma 2.1.** Given \(\delta_1, \delta_2 > 0\), there is a \(\delta_3\) such that

\[
L(1, \text{sym}^2 f) = \sum_{d_1,d_2} \lambda_f(d_1^2) \exp \left(-\frac{d_1d_2^2}{k\delta_1}\right) + O(k^{-\delta_3})
\]

for all but \(O(k^{\delta_3})\) cusp forms \(f \in B_{k+1}\).

We also note from the proof of lemma 2.1 in \[1\] that one can take, \(\delta_3 < \delta_1 \delta_2 / 62\).

3. **Proof of Theorem \[1\]**

Recall that \(S_g = L(3/2, g) N(F_g)\). We start we the following sum with the amplifier given by \(|\sum_{n \leq N} \alpha_n \lambda_g(n)\|^2\)

\[
S_A = \frac{12}{2k-1} \sum_{g \in B_{2k}} \left| \sum_{n \leq N} \alpha_n \lambda_g(n) \right|^2 S_g.
\]

Expanding the sum (3.1) and using the Hecke relation \(\lambda_g(m) \lambda_g(n) = \sum_{d|\text{lcm}(m,n)} \lambda_g(mn/d^2)\), we have

\[
S_A = \frac{12}{2k-1} \sum_{g \in B_{2k}} \sum_{n_1,n_2 \leq N} \alpha_{n_1} \overline{\alpha_{n_2}} \sum_{d|\text{lcm}(n_1,n_2)} \lambda_g(n_1n_2/d^2) S_g.
\]

Now substituting for \(S_g\) and using the approximate functional equation (see \[2\]) and the Petersson formula (see \[2\]) for the sum over \(g\) we get

\[
S_A = \frac{\pi^2}{15k} \sum_{n_1,n_2 \leq N} \alpha_{n_1} \overline{\alpha_{n_2}} \sum_{f \in B_{k+1}} \sum_{d|\text{lcm}(n_1,n_2)} \left( \mathcal{M}_f^{(1)}(n_1n_2/d^2) + \mathcal{M}_f^{(2)}(n_1n_2/d^2) \right),
\]

where the quantities \(\mathcal{M}^{(i)}(r)\) for \(i = 1, 2\) are as in section \[2\]. To apply the Petersson formula for the sum over \(f\), it is convenient to introduce the quantity \(L(1, \text{sym}^2 f)\) in the above sum. Thus we write

\[
S_A = \frac{\pi^2}{15k} \sum_{d|N} \sum_{n_1,n_2 \leq N/d} \alpha_{dn_1} \overline{\alpha_{dn_2}} \sum_{f \in B_{k+1}} \frac{L(1, \text{sym}^2 f)}{L(1, \text{sym}^2 f)} \left( \mathcal{M}_f^{(1)}(n_1n_2) + \mathcal{M}_f^{(2)}(n_1n_2) \right).
\]

Making a change of variables and rearranging the summation we get

\[
S_A = \frac{\pi^2}{15k} \sum_{d|N} \sum_{n_1,n_2 \leq N/d} \alpha_{dn_1} \overline{\alpha_{dn_2}} \sum_{f \in B_{k+1}} \frac{L(1, \text{sym}^2 f)}{L(1, \text{sym}^2 f)} \left( \mathcal{M}_f^{(1)}(n_1n_2) + \mathcal{M}_f^{(2)}(n_1n_2) \right).
\]
Now we use lemma 2.1 to get
\[
S_A = \frac{\pi^2}{15} \cdot \frac{12}{k} \sum_{d \leq N \frac{n_1, n_2 \leq N}{d}} \alpha_{dn_1} \alpha_{dn_2} \sum_{f \in B_{k+1}} \frac{1}{L(1, \text{sym}^2 f)} \sum_{d_1, d_2} \frac{\lambda_f(d_1^2)}{d_1 d_2} \exp \left( - \frac{d_1 d_2^2}{k \delta_i} \right)
\]
(3.3)
× \left( M_f^{(1)}(n_1 n_2) + M_f^{(2)}(n_1 n_2) \right) + O \left( \left( N^E \sum_{n_1, n_2 \leq N} |\alpha_{d_1} \alpha_{d_2}| \right) (k^{-\delta_i+\epsilon} + k^{\delta_i-1+\epsilon}) \right).

As in [11], the error term comes from two sources: the error in lemma 2.1 and the forms $f \in B_{k+1}$ for which (2.21) doesn’t hold. In the former case we use that the sum over $d_1, d_2$ is bounded by $\log(k)$ (from the bound of [5] and lemma 2.1); in the latter case we estimate trivially using (2.12).

Denote by $S_A^{(1)}$ and $S_A^{(2)}$ the terms corresponding to $M_f^{(1)}$ and $M_f^{(2)}$ in (3.3) respectively. We first bound $S_A^{(1)}$.

3.1. The term $S_A^{(1)}$. Expanding out $M_f^{(1)}$ from (2.13) we have
\[
M_f^{(1)}(n_1 n_2) = \frac{2}{\zeta(2)} \sum_{a_1, a_2, b_1, b_2, d_1, d_2 \mid (a_1^2, a_2^2)} \mu \left( \frac{n_1 n_2}{a_1 b_1^2}, \frac{a_2 b_2^2}{a_1 b_1^2} \right) \frac{a_2 b_2^2 \lambda_f \left( \frac{a_2 b_2^2}{a_1 b_1^2} \right)}{(n_1 n_2)^{3/2} a_1 b_1} W \left( \frac{n_1 n_2^2 a_1^2 b_1^4}{a_2^2 b_2^4} \right). \]

3.1.1. The diagonal: We apply the Petersson formula (2.2) for the sum over $f$ in $S_A^{(1)}$ and denote by $S_A^{(1)}$ the corresponding diagonal term. Then we have
\[
S_A^{(1)} = \frac{2\pi^2}{15 \zeta(2)^2} \sum_{d \leq N \frac{n_1, n_2 \leq N}{d}} \alpha_{dn_1} \alpha_{dn_2} \sum_{a_1, a_2, b_1, b_2, d_1, d_2 \mid (a_1^2, a_2^2)} \mu \left( \frac{n_1 n_2}{a_1 b_1^2}, \frac{a_2 b_2^2}{a_1 b_1^2} \right) \frac{a_2 b_2^2 \lambda_f \left( \frac{a_2 b_2^2}{a_1 b_1^2} \right)}{(n_1 n_2)^{3/2} a_1 b_1} W \left( \frac{n_1 n_2^2 a_1^2 b_1^4}{a_2^2 b_2^4} \right) \exp \left( - \frac{d_1 d_2^2}{k \delta_i} \right).
\]

Using Mellin inversion we can write
\[
S_A^{(1)} = \frac{2\pi^2}{15 \zeta(2)^2} \int \left( \int \int \zeta(2+4u) \zeta(2+2v) \zeta(1+u+\frac{v}{2}) W(u) \Gamma(v) k^{\delta_i+v} B_N(u, v) \frac{du}{2\pi i} \frac{dv}{2\pi i} \right)
\]
where
\[
B_N(u, v) = \sum_{1 \leq d \leq N \frac{n_1, n_2 \leq N}{d}} \alpha_{dn_1} \alpha_{dn_2} \sum_{a_1, a_2, b_1, b_2, d_1, d_2 \mid (a_1^2, a_2^2)} \mu \left( \frac{n_1 n_2}{a_1 b_1^2}, \frac{a_2 b_2^2}{a_1 b_1^2} \right) \frac{(n_1 n_2)^{3/2+2u} a_1^2 b_1^{-2u}}{d_1^2 a_2^{-2u}}
\]
and from (2.9),
\[
W(u) = \frac{\Lambda_k \left( \frac{1}{2} + u \right)}{\Lambda_k \left( \frac{1}{2} \right)} \left( \cos \frac{\pi u}{104} \right)^{-604} \cdot \frac{1}{u}. \tag{3.4}
\]

We further denote by $|B_N(u, v)|$ the sum as in $B_N(u, v)$, but with all terms replaced by their absolute values. First we move the line of integration w.r.t $u$ to $-\delta$, for some $1/2 < \delta < 1$ and encounter the pole of $\tilde{W}$ at $u = 0$ and the poles of $\zeta$ at $u = -v/2$ and $u = -1/4$. Then the integral over $u$ equals
\[
R(v) := \zeta(2) \zeta(1+\frac{v}{2}) B_N(0, v) + \zeta(2-2v) \tilde{W} \left( \frac{-v}{2} \right) B_N \left( \frac{-v}{2}, v \right) + \zeta(\frac{3}{4}) + \frac{v}{2} \tilde{W} \left( \frac{-1}{4} \right) B_N \left( \frac{-1}{4}, v \right).
\]

\[
+ \int_{(-\delta)} \zeta(2+4u) \zeta(1+u+\frac{v}{2}) \tilde{W}(u) B_N(u, v) \frac{du}{2\pi i}.
\]
Let us call by \( R_1(v), R_2(v) \) the functions of \( v \) appearing on the first and second line in the above expression for \( R(v) \). Next we move the line of integration w.r.t \( v \) to \( \varepsilon > 0 \) and cross the pole of \( \zeta \) at \( v = 1/2 \). The contribution to the residue only comes from the last two terms in \( R_1(v) \). Then the contribution from \( R_1 \) to \( S_A^{(1)} \) becomes (a sum of four terms):

\[
R_1 := (|B_N(0, \varepsilon)| + |B_N(-\frac{\varepsilon}{2}, \varepsilon)|)|k\varepsilon| + \frac{1}{4} \left| B_N(-\frac{1}{4}, \varepsilon) \right| |k^{\delta_i/2-1/2}| + \frac{1}{4} \left| B_N(-\frac{1}{4}, \varepsilon) \right| k^{-1/2+\varepsilon}.
\]

Since \( R_2(v) \) is entire, its contribution to \( S_A^{(1)} \) is just the integral over the two new lines of integrations, namely

\[
R_2 := |B_N(-\delta, \varepsilon)| |k^{-\delta_i+\varepsilon}|
\]

Noting the following bounds: \( |B_N(0, \varepsilon)| \leq |B_N(0, 0)|, |B_N(-\frac{1}{4}, \frac{1}{2})| \leq |B_N(-\frac{1}{4}, \varepsilon)| \leq |B_N(-\frac{1}{4}, 0)| \) and \( |B_N(-\delta, \varepsilon)| \leq |B_N(-\delta, 0)| \), we have

\[
S_A^{(1)} \ll (|B_N(0, 0)| + |B_N(-\frac{\varepsilon}{2}, \varepsilon)|)|k\varepsilon| + \frac{1}{4} \left| B_N(-\frac{1}{4}, 0) \right| (|k^{\delta_i/2-1/2}| + |B_N(-\delta, 0)| k^{-\delta_i+\varepsilon}).
\]

3.1.2. The off-diagonal: Denote the off-diagonal terms of \( S_A^{(1)} \) by \( S_A^{(12)} \), then we have

\[
S_A^{(12)} = 2\pi i^{-k} \frac{2\pi^2}{15 \zeta(2)^2} \sum_{d \leq N} \sum_{n_1, n_2 \leq N/d} \alpha_{dn_1} \alpha_{dn_2} \sum_{a_1, a_2} \sum_{b_1, b_2} \sum_{a_1, a_2} \frac{\mu(\frac{n_1 n_2}{a_1 b_1}) a_2 b_2}{c(n_1 n_2)^{3/2} a_1 b_1 d_1 d_2} \times S\left(\frac{a_1^2 a_2^2}{d_1^2}, \frac{a_1^2 a_2^2}{d_2^2}, c\right) W\left(\frac{a_1^2 a_2^2}{d_1^2}, \frac{a_1^2 a_2^2}{d_2^2}\right) \exp\left(-\frac{d_1 d_2^2}{k^{\delta_i}}\right) J_k\left(\frac{4\pi a_1 a_2 d_1}{\epsilon d_2}\right).
\]

We can truncate the sum over \( c \) at \( c \leq 100 \frac{\delta_i a_1 a_2 d_1}{d_2} \) by using the rapid decay of Bessel function near 0 (see (2.4)). Next we use the bound (2.8) for \( W \) with \( j = 0 \) and \( A = \frac{1}{2} + \frac{\varepsilon}{2} \) and the trivial bounds \(|S(*, *, c)| \leq c, J_k(x) \ll k^{-1/3} \) (see (2.3)) to see that

\[
S_A^{(12)} \ll \left( \sum_{d \leq N} \sum_{n_1, n_2 \leq N/d} |\alpha_{dn_1} \alpha_{dn_2}| \sum_{a_1, a_2, b_1, b_2} \frac{|\mu(\frac{n_1 n_2}{a_1 b_1})| a_2 b_2}{(n_1 n_2)^{5/2+\varepsilon}} \right) k^{-1/3+\delta_i+\varepsilon}.
\]

3.2. The term \( S_A^{(2)} \). Now we proceed to estimate \( S_A^{(2)} \). From the arguments in section 2 it is enough to bound

\[
S_A^{(2)}(M, C) := \frac{12}{k} \sum_{d \leq N} \sum_{n_1, n_2 \leq N/d} \alpha_{dn_1} \alpha_{dn_2} \sum_{j \in B_{k+1}} \frac{1}{L(1, \text{sym}^2 f)} \sum_{d_1, d_2} \lambda_j(d_1) \lambda_j(d_2) \exp\left(-\frac{d_1 d_2^2}{k^{\delta_i}}\right) A^{(2)}(n_1 n_2, M, C).
\]

We use (2.19) and (2.20) in the above equation and also note that the using the exponential decay, the sum over \( d_1 \) can be truncated at \( d_1 \leq k^{\delta_i+\varepsilon} \) with a very small error. Thus we are left to bound

\[
S_A^{(2)}(M, C) \ll \sum_{d \leq N} \sum_{n_1, n_2 \leq N/d} \alpha_{dn_1} \alpha_{dn_2} \left( \sum_{d_1 \leq k^{\delta_i+\varepsilon}} \sum_{n_1, n_2} \sum_{c \leq \varepsilon \leq 2C} T(d_1, n_1 n_2, n, c, M) + O(k^{-100}) \right),
\]

where
where

\[ T(d_1, n_2 n_1, n, c, M) = k^\varepsilon \sum_{m_2 m_1^2 \leq k^\varepsilon M^{1/2}(n_1 n_2)^{3/2} n} \frac{m_1 \tau(c)(c,n)}{(n_1 n_2)^{1/2} n^2 d_1 M^{1/2}} \left| \frac{12}{k} \sum_{f \in B_{k+1}} \frac{\lambda_f(d_1^2) A(m_2, m_1)}{L(1, \text{sym}^2 f)} \right| \]

\[ \ll k^\varepsilon \sum_{a, l_1, l_2, m_1, m_2} \frac{a^3 l_1^2 m_2 m_1^2 \leq k^\varepsilon M^{1/2}(n_1 n_2)^{3/2} n}{a^3 l_1^2 m_1^2 c} \frac{m_1 \tau(c)(c,n)}{(n_1 n_2)^{1/2} n^2 d_1 M^{1/2}} \sum_{h | (m_1^2, m_2^2)} \left| \frac{12}{k} \sum_{f \in B_{k+1}} \frac{\lambda_f(d_1^2) \lambda_f(m_2^2 m_1^2 / h^2)}{L(1, \text{sym}^2 f)} \right|. \]

We use (6.4) and the Hecke relations to arrive at the previous step.

Now we apply the Petersson formula and using the rapid decay of Bessel function near 0 for the off-diagonal term (see (6.3)) we get with the same conditions on the variables as above that

\[ T(d_1, n_2 n_1, n, c, M) \ll k^\varepsilon \sum_{a, l_1, l_2, m_1, m_2} \frac{a^3 l_1^2 m_2 m_1^2 \leq k^\varepsilon M^{1/2}(n_1 n_2)^{3/2} n}{a^3 l_1^2 m_1^2 c} \left( \delta_{d_1 h = m_2, m_1} + O\left( \frac{d_1 m_1 m_2}{h k^{4/3}} \right) \right). \]

Since the sum over \( l_2 \) is free, it is \( \ll \left( \frac{k^\varepsilon M^{1/2}(n_1 n_2)^{3/2} n}{a^3 l_1^2 m_1^2 c} \right)^{1/2} \). Thus

\[ T(d_1, n_2 n_1, n, c, M) \ll k^\varepsilon \sum_{a, l_1, l_2, m_1, m_2} \frac{a^3 l_1^2 m_2 m_1^2 \leq k^\varepsilon M^{1/2}(n_1 n_2)^{3/2} n}{a^3 l_1^2 m_1^2 c} \left( \delta_{d_1 h = m_1, m_2} + O\left( \frac{d_1 m_1 m_2}{h k^{4/3}} \right) \right). \]

Now the sum over \( l_1 \) is \( \ll \left( \frac{k^\varepsilon M^{1/2}(n_1 n_2)^{3/2} n}{a^3 l_1^2 m_1^2 c} \right)^{1/4} \) and the sum over \( a \) is \( \ll \left( \frac{k^\varepsilon M^{1/2}(n_1 n_2)^{3/2} n}{a^3 l_1^2 m_1^2 c} \right)^{-1/12} \). Thus

\[ T(d_1, n_2 n_1, n, c, M) \ll k^\varepsilon \frac{(n_1 n_2)^{3/2} \tau(c)(c,m)}{n^{1/2} d_1 M^{1/2}} \sum_{m_1, m_2} \frac{1}{m_1^{2/3} m_2^{1/3}} \sum_{h | (m_1^2, m_2^2)} \left( \delta_{d_1 h = m_1, m_2} + O\left( \frac{d_1 m_1 m_2}{h k^{4/3}} \right) \right). \]

Now we evaluate the two inside sums separately. Let \( T_1 \) and \( T_2 \) correspond to the diagonal and off-diagonal terms respectively in the above sum. Then with the same conditions on the variables as above, we have

\[ T_1 := \sum_{m_1, m_2} \frac{1}{m_2^{2/3} m_1^{1/3}} \sum_{h} \delta_{d_1 h = m_1 m_2}. \]

Making the following change of variable, \( m_2^2 = d_1^2 h^2 / m_1^2 \) we get

\[ T_1 = \sum_{h, m_1} \frac{m_1^{1/3}}{h^{2/3} d_1^{2/3}} \ll \sum_{m_1} \frac{m_1^{1/3+\varepsilon}}{d_1^{2/3}} \ll \frac{\sigma_{1/3+\varepsilon}(c)}{d_1^{2/3}}. \]

Now consider the second sum

\[ T_2 := \sum_{m_1, m_2} \frac{1}{m_2^{2/3} m_1^{1/3}} \sum_{h | (m_1^2, m_2^2)} \frac{d_1 m_1 m_2}{h k^{4/3}} = k^{-4/3} d_1 \sum_{m_1, m_2} \frac{m_2^{2/3} m_1^{1/3}}{h}. \]
In both of the above sums $m_2 m_1^2 \leq k^e M^{1/2} (n_1 n_2)^{3/2}$. Put $m_1^2 = hm_3$. Then
\[
T_2 = k^{-4/3} d_1 \sum_{h,m_1,m_2 \mid h m_3 m_2 \leq k^e M^{1/2} (n_1 n_2)^{3/2} n} \frac{m_1^{1/3} m_2^{1/3}}{h^{2/3}}.
\]

Now using the fact that $hm_3|c^2$ we find that the sums over $h$ and $m_3$ are $\ll c^e \sigma_{1/3} (c^2)$. The remaining sum over $m_2$ is then $\ll k^e M^{2/3} (n_1 n_2)^2 n^{4/3}$. This implies $T_2 \ll d_1 k^{-4/3 + e} c^{2e} \sigma_{1/3} (c^2) M^{2/3} (n_1 n_2)^2 n^{4/3}$. Thus we have
\[
T(d_1, n_2 n_1, n, c, M) \ll k^{e} \frac{(n_1 n_2)^{3/4} \sigma(c, n)}{n^{1/6} d_1 M^{1/6}} \left( \sigma_{1/3 + e} (c) \frac{1}{d_1^{2/3}} + d_1 k^{-4/3 + e} c^{2e} \sigma_{1/3} (c^2) M^{2/3} (n_1 n_2)^2 n^{4/3} \right).
\]

Using that $\sigma_{1/3} (c) \ll c^{1/3}$, $(c, n) \leq c$ and that $M \leq k^{2 + e}$ from (2.17), we get
\[
(3.10) \quad T(d_1, n_1 n_2, n, c, M) \ll (n_1 n_2)^{11/4 + e} c^{5/3 + e} n^{1/6} k^{-1/3 + e}.
\]

Finally, using (3.10) in (3.9) we use $c \ll (n_1 n_2)^{1/2} k^{e} n^{-1}$ (from (2.17)) and note that sum over $n$ is $\ll (n_1 n_2)^{-3/4} k^{e}$, we get
\[
(3.11) \quad S_A^{(2)} (M, C) \ll \left( \sum_{n_1, n_2 \leq N} |\alpha_{n_1} \alpha_{n_2}| \right) k^{-1/3 + \delta_1 + e}.
\]

Putting everything together from (3.7), (3.8) and (3.11), we have
\[
(3.12) \quad S_A \ll (|B_N(0, 0)| + |B_N(-\frac{\varepsilon}{2}, 0)|) k^{e} + |B_N(-\frac{1}{4}, 0)| k^{\delta_1/2 - 1/2} + |B_N(-\delta, 0)| k^{-2\delta + e} + \sum_{n_1, n_2 \leq N} |\alpha_{n_1} \alpha_{n_2}| \left( N^{20/3 + e} k^{-1/3 + \delta_1 + e} + N^{e} k^{-\delta_1 + e} + N^{e} k^{\delta_2 - 1 + e} \right).
\]

where $\delta_1, \delta_2 > 0$ are arbitrary, $1/2 < \delta < 1$ and $\delta_3 < \delta_1 \delta_2 / 62$.

3.3. **Choice of the amplifier:** For a fixed $g_0$ in the sum (3.1) we choose the $\alpha_n$s following Iwaniec-Sarnak ([9]) as below
\[
(3.13) \quad \alpha_n = \begin{cases} \lambda_{g_0} (p), & \text{if } n = p \leq N^{1/2}; \\ -1, & \text{if } n = p^2 \leq N; \\ 0, & \text{otherwise}. \end{cases}
\]

Substituting in (3.1) and using the Hecke relation $\lambda_{g_0} (p^2) - \lambda_{g_0} (p^2) = 1$, we find that
\[
(3.14) \quad \frac{12}{2k - 1} S_{g_0} \left| \sum_{p \leq N^{1/2}} 1 \right| ^2 \leq S_A.
\]

We proceed to bound the quantities $|B_N(0, 0)|$, $|B_N(-\delta, 0)|$, $|B_N(-\varepsilon / 2, \varepsilon)|$, $|B_N(-1/4, 0)|$ and $S_A^{(12)}$ in (3.12) with the choice of $\alpha_n$s as in (3.13).

3.4. **The estimation of $|B_N(\ast, \ast)|$.**
3.4.1. Estimation of $|B_N(0,0)|$: Since the $\alpha_\nu$'s are supported on primes and prime squares we can write $|B_N(0,0)|$ as

$$
|B_N(0,0)| \leq \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1p_2)^{3/2}} B_{p_1, p_2} + 2 \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1)^{3/2}p_2^2} B_{p_1, p_2} + \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1p_2)^3} B_{p_1^2, p_2^2},
$$

where

$$
B_{n_1, n_2} := \sum_{d|n_1n_2} \sum_{a_2b_2} \mu\left(\frac{n_1n_2}{a_2b_2}\right) \sigma_0(a_2^2) b_2^2.
$$

We have $B_{p_1, p_2} \ll 1$ if $p_1 \neq p_2$ and is $\ll p^2$, if $p_1 = p_2 = p$; $B_{p_1, p_2^2} \ll p_2^2$ and $B_{p_1^2, p_2^2} \ll p_1^2 p_2^2$. Thus

$$
|B_N(0,0)| \ll \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1p_2)^{3/2}} + \sum_{p \leq N^{1/2}} \frac{1}{p} + 2 \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1)^{3/2}p_2^2} + \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1p_2)^3} \ll \log \log N.
$$

Here we use that $\sum_{p \leq x} p^{-1} \ll \log \log x$.

3.4.2. Estimation of $|B_N(-\delta,0)|$: We have

$$
|B_N(-\delta,0)| \leq \sum_{n_1, n_2 \leq N} \frac{\alpha_{n_1} \overline{\alpha_{n_2}}}{(n_1n_2)^{3/2-2\delta}} B_{n_1, n_2},
$$

where $B_{n_1, n_2}$ is as in the estimation for $|B_N(0,0)|$. Now evaluating similarly as in the case of $|B_N(0,0)|$, we have

$$
|B_N(-\delta,0)| \ll \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1p_2)^{3/2-2\delta}} + \sum_{p \leq N^{1/2}} \frac{1}{p^{1-4\delta}} + 2 \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1)^{3/2-2\delta}p_2^{4\delta}} + \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1p_2)^{1-4\delta}} \ll N^{-\frac{1}{2}+4\delta}.
$$

Here we use the fact that, for $s \neq 1$, $\sum_{p \leq x} \frac{1}{p^s} \leq \sum_{n \leq x} \frac{1}{n^s} \asymp x^{1-s}$.

3.4.3. Estimation of $|B_N(-\varepsilon/2,\varepsilon)|$: We have $|B_N(-\varepsilon/2,\varepsilon)|$ is

$$
\leq \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1p_2)^{3/2-2\delta}} B_{p_1, p_2} + 2 \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1)^{3/2-2\delta}p_2^{4\delta}} B_{p_1, p_2} + \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1p_2)^{1-2\delta}} B_{p_1^2, p_2^2}.
$$

We have $B_{p_1, p_2} \leq (p_1p_2)^{1-\varepsilon}$, $B_{p_1, p_2} \leq (p_1p_2)^{1-\varepsilon}$ and $B_{p_1^2, p_2^2} \leq (p_1^2p_2^2)^{1-\varepsilon}$. Thus we have

$$
|B_N(-\varepsilon/2,\varepsilon)| \ll \log \log N.
$$

3.4.4. Estimation of $|B_N(-\frac{1}{4},0)|$: We have

$$
|B_N(-\frac{1}{4},0)| \leq \sum_{n_1, n_2 \leq N} \frac{\alpha_{n_1} \overline{\alpha_{n_2}}}{n_1n_2} \sum_{d|n_1n_2} \sum_{a_2b_2} \mu\left(\frac{n_1n_2}{a_2b_2}\right) \sigma_0(a_2^2) b_2.
$$

Following the similar calculations as in the case of $|B_N(0,0)|$, we find that

$$
|B_N(-\frac{1}{4},0)| \ll \log \log N.
$$
3.5. Estimation of $S_A^{(12)}$: From (3.8) we have

$$S_A^{(12)} \ll \left( \sum_{n_1, n_2 \leq N} \frac{\left| \alpha_{n_1} \alpha_{n_2} \right|}{(n_1 n_2)^{5/2} + \varepsilon} \sum_{d(n_1 n_2)} d^{\delta + \varepsilon} \sum_{a_2, b_2 \leq B} |\mu\left(\frac{n_1 n_2}{a_2 b_2}\right)| a_2^{2+\varepsilon} b_2^{4+2\varepsilon} \right) k^{-1/3+\delta_1+\varepsilon}. $$

We have that the inside summation is

$$\ll \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1 p_2)^{5/2+\varepsilon}} B'_{p_1, p_2} + 2 \sum_{p_1, p_2 \leq N^{1/2}} (p_1 p_2)^{3/2+\varepsilon} B'_{p_1, p_2} + \sum_{p_1, p_2 \leq N^{1/2}} \frac{1}{(p_1 p_2)^{5/2+\varepsilon}} B'_{p_1, p_2}$$

and $B'_{p_1, p_2} \ll (p_1 p_2)^{2+\varepsilon}$, $B'_{p_1^2, p_2^2} \ll (p_1 p_2^2)^{2+\varepsilon}$ and $B'_{p_1^2, p_2^2} \ll (p_1^2 p_2)^{2+\varepsilon}$. Thus

$$S_A^{(12)} \ll \log \log N \cdot k^{-1/3+\delta_1+\varepsilon}.$$ 

3.6. Completion of Theorem [1] Now substituting in (3.14) with $N = k^\eta$ and using the fact that $\left| \sum_{p \leq N^{1/2}} 1 \right|^2 \sim N/(\log N)^2$ and $\sum_{n_1, n_2 \leq N} |\alpha_{n_1} \alpha_{n_2}| \ll \frac{N}{(\log N)^2}$, we have for any $1/2 < \delta < 1$

$$\frac{12}{2k-1} S_{R_0} \ll k^{-\eta+\varepsilon} + k^{-\delta_1+\varepsilon} + k^{-1+\delta_2+\varepsilon} + k^{-\frac{2n}{7}+4\eta+2\delta+\varepsilon} + k^{-\frac{20n}{21}+2\delta+\varepsilon} \ll k^{1+\frac{1}{210}}.$$ 

We make the following choice: $\delta_3 = \frac{\delta_2}{\delta_1} - \varepsilon$ and note that since $1/2 < \delta < 1$, the fourth term in (3.15) is irrelevant. Then we equate all the exponents of $k$ in (3.15). A simple calculation shows that $\delta_1 = 27/91$ gives the answer. Also for this choice of $\delta_1$ we get $\eta \approx \frac{1}{210}$, $\delta_2 \approx \frac{209}{210}$ and $\frac{1}{210} > \delta_3 > \frac{1}{210}$. Thus we have

$$S_{R_0} \ll k^{1-\frac{1}{210}+\varepsilon}. \qed$$

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