Discrete Bakry-Émery curvature tensors and matrices of connection graphs

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Abstract

Connection graphs are natural extensions of Harary’s signed graphs. The Bakry-Émery curvature of connection graphs has been introduced by Liu, Münch and Peyerimhoff in order to establish Buser type eigenvalue estimates for connection Laplacians. In this paper, we reformulate the Bakry-Émery curvature of a vertex in a connection graph in terms of the smallest eigenvalue of a family of unitarily equivalent curvature matrices. We further interpret this family of curvature matrices as the matrix representations of a new defined curvature tensor with respect to different orthonormal basis of the tangent space at a vertex. This is a strong extension of previous works of Cushing-Kamtue-Liu-Peyerimhoff and Siconolfi on curvature matrices of graphs. Moreover, we study the Bakry-Émery curvature of Cartesian products of connection graphs, strengthening the previous result of Liu, Münch and Peyerimhoff. While results of a vertex with locally balanced structure cover previous works, various interesting phenomena of locally unbalanced connection structure have been clarified.

1 Introduction

In this paper, we study the discrete Bakry-Émery theory on connection graphs. Bakry-Émery Γ-calculus and curvature dimension inequalities [3, 4, 5, 6] provides an important extension of the lower Ricci curvature bounds on a Riemannian manifolds. In recent years, the discrete Bakry-Émery theory on graphs have attracted a lot of attention. The investigation of the discrete Bakry-Émery theory has lead to deep understandings on geometric and analytic properties of the underlying graphs [9, 11, 12, 14, 16, 18, 19, 20, 24, 26, 27, 29, 31, 33, 36, 40, 41]. Recently, Cushing et al. [14, Theorem 1.2] and Siconolfi [40, Theorem 27] discover independently that the Bakry-Émery curvature at each vertex of a graph can be reformulated as the smallest eigenvalue of a symmetric matrix, which is called a curvature matrix. The concept of curvature matrices leads to an interesting curvature flow on (mixed) weighted graphs based on the Bakry-Émery curvature [13].
Connection graphs are interesting extension of graphs, which serves as a simple model of discrete vector bundles. A connection graph \((G, \sigma)\) with \(d\)-dimensional connections is a graph \(G = (V, E)\) equipped with a connection function \(\sigma : E^{or} \to O(d)\) or \(U(d)\) satisfying \(\sigma_{xy} = \sigma_{yx}^{-1}\). Here, \(E^{or} := \{(x, y), (y, x) : \{x, y\} \in E\}\) is the set of oriented edges, one in each direction for each edge in \(E\), and \(\sigma_{xy} := \sigma((x, y))\). That is, we assign a linear transformation on each oriented edge.

Connection graphs constitute a particular class of gain graphs. A gain graph is a graph \(G\) coupled with a gain function \(\sigma : E^{or} \to H\) satisfying \(\sigma_{xy} = \sigma_{yx}^{-1}\), where \(H\) is an arbitrary group. Gain graphs are special cases of biased graphs \([45]\). Gain graphs have also been referred to as voltage graphs \([21]\), the motivation of which lies in Heawood map-coloring problem. With the special choices of \(H = O(d)\) or \(U(d)\), the function \(\sigma\) can be considered as a connection of the vector bundle on \(G\) \([32]\). We use the terminology of connection graph following the work \([10]\). Connection graphs are also referred to as metrized local systems in \([28]\), with a close relation to previous works \([8, 17]\). In the case of \(H = O(1) = \{\pm 1\}\), a connection graph is simply a signed graph introduced by Harary. Harary \([22, 23]\) introduced the concepts of balanced and anti-balanced signed graphs as models of social networks. For more historical review, we refer to \([38, \text{Section 3}]\) and \([31, \text{Section 2.1}]\).

Liu, Münch, Peyerimhoff \([35]\) extend the concept of Bakry-Émery curvature on graphs to the setting of connection graphs and derive higher order Buser-type eigenvalue estimates. They show their curvature of connections graphs has a characterization in terms of the heat semigroup for functions and the heat semigroup for vector fields of the underlying graphs \([32, \text{Theorem 3.2}]\).

It is natural to ask whether the reformulation of Bakry-Émery curvature as an eigenvalue problem in \([14]\) and \([40]\) can be extended to the setting of connection graphs.

We point out that the Bakry-Émery curvature of connection graphs and that of graphs are quite different in general. For example, a graph with bounded degrees and a positive lower Bakry-Émery curvature bound is finite \([31, 36]\). However, we find an infinite 3-regular connection graph with a positive curvature bound. (See Section \(8\).)

A key property used in both proofs of \([14]\) and \([40]\) is that the constant functions are eigenfunctions to the zero eigenvalue of the Laplacian and \(\Gamma, \Gamma_2^{or}\) operators in Bakry-Émery calculus. However, this is not true for the connection Laplacian and the associated \(\Gamma^{or}, \Gamma_2^{or}\) operators of \([35]\).

Building upon the key method of Schur complement in \([14]\), we overcome this difficulty by applying an additional trick of Schur complement in terms of pseudoinverses established by Albert \([1]\). It is highly nontrivial to check the necessary conditions of applying the results of Albert \([1]\) in our setting (see Proposition \(A.1\)). As a result, we reformulate the Bakry-Émery curvature at a vertex of a connection graph as the smallest eigenvalue of a curvature matrix. In fact, for each vertex, we find a family of unitarily equivalent curvature matrices (see Theorem \(3.2\) and Proposition \(3.5\)).

Inspired by the method of Siconolfi \([40]\), we can define at each vertex \(x\) a curvature tensor and a metric tensor (see Definition \(4.5\)), using the sesquilinear forms \(\Gamma_2^{or}\) and \(\Gamma^{or}\), and extensions of functions on the 1-sphere to functions on the 2-ball determined by the Schur complement. The tangent space \(T_x(G, \sigma)\) at \(x\) (see Definition \(4.3\)) is a linear space and becomes an inner product space when assigning the metric tensor. A curvature matrix is
then simply the matrix representation of the curvature tensor with respect to an orthonomal basis of the inner product space \((T_x(G, \sigma), g_x(\cdot, \cdot))\).

We further investigate the property of the Bakry-Émery curvature at a vertex in a connection graph as a function of the dimension parameter \(N \in (0, \infty]\). We show that this function is continuous, monotone non-decreasing and concave extending the previous approach on Bakry-Émery curvature function of graphs [16, 14]. Moreover, we prove that the curvature function is constant on \([N, \infty]\), if the multiplicity of the smallest eigenvalue of the \(N\)-dimensional curvature matrix is no smaller than the dimension \(d\) of the connection (see Theorem 5.1).

We derive decomposition formulas for the curvature matrices of the Cartesian product of two connection graphs, in terms of the curvature matrices of the corresponding vertices in the individual graphs (see Theorem 6.3 and Corollary 6.9). Employing those formulas, we establish several results on the Bakry-Émery curvature of the Cartesian product of connection graphs (Corollaries 6.10, 6.15 and 6.17), which strengthen previous results known in [35]. In particular, we have the following result. Let \((G, \sigma)\) and \((G', \sigma')\) be two connection graphs with 1-dimensional connections. Let \(x\) and \(x'\) be two vertices of \(G\) and \(G'\), respectively. Assume that one of \(x, x'\) has a balanced local connection structure. Then we show the curvature function of the Cartesian product at \((x, x')\) is given by the star product (see [16, Definition 7.1] or Definition 6.16) of the curvature functions of \(G\) at \(x\) and that of \(G'\) at \(x'\).

We study how the curvature changes under two kinds of operations to the local structure of a given vertex \(x\) in a connection graph: (i) Add an edge with connection between two neighbors of \(x\); (ii) Merge two vertices in the two sphere of \(x\) which have no common neighbors in the one sphere. While the curvature is always non-decreasing under the operation (ii), the change of curvature depends on the connection of the new added edge under operation (i).

Before we give a more detailed discussion of the results, we present a rough overview of each section:

- **Section 2** Preliminaries for connection graphs, Bakry-Émery curvature and Schur complement.
- **Section 3** Curvature matrices of connection graphs.
- **Section 4** Curvature tensors of connection graphs.
- **Section 5** Properties of curvature functions.
- **Section 6** Curvature matrices of Cartesian products.
- **Section 7** How the curvature changes under local operations.
- **Section 8** An infinite connection graph with positive Bakry-Émery curvature lower bound.
- **Appendix A** Explicit calculations of the matrices \(\Gamma^\sigma_2(x)\) and \(Q(x)\).

## 2 Preliminaries

In this section, we prepare notations and some preliminaries.
2.1 Connection graphs

Let $G = (V, w, \mu)$ be a weighted graph with a vertex set $V$, a vertex measure $\mu : V \to \mathbb{R}^+$, and an edge-weight function $w : V \times V \to \mathbb{R}^+ \cup \{0\}$ which satisfies $w_{xy} = w_{yx}$ and $w_{xx} = 0$ for all $x, y \in V$. The edge set $E \subset V \times V$ of $G$ is defined as $E := \{(x, y) : x, y \in V, w_{xy} \neq 0\}$. In particular, there are no multiple edges and self-loops in $E$. We say a vertex $x$ is adjacent to another vertex $y$, denoted by $x \sim y$, if $\{x, y\} \in E$. For any two vertices $x$ and $y$, the distance $d(x, y)$ is the length of the shortest path connecting them. We denote the ball centered at $x$ with radius $r$ by $B_r(x) = \{y \in V : d(x, y) \leq r\}$, and the $r$-sphere of $x$ by $S_r(x) = \{y \in V : d(x, y) = r\}$. A graph $G$ is locally finite if the 1-sphere $S_1(x)$ of every vertex $x$ contains only finitely many vertices. We restrict ourselves to locally finite weighted graphs in this article.

The degree $d_x$ of a vertex $x$ is defined as $d_x := \sum_{y \in V} w_{xy}$. We also use the notation of transition rate $p_{xy}$ from $x$ to $y$, where

$$p_{xy} := \frac{w_{xy}}{\mu_x}.$$  

For the case of $\mu_x = d_x$ for all $x \in V$, $p_{xy}$ is the transition probability from $x$ to $y$ of a random walk (a reversible Markov chain).

We denote by $E^{or} := \{(x, y), (y, x) : \{x, y\} \in E\}$ the set of oriented edges, one in each direction, for each edge in $E$. For an integer $d$, a connection graph $(G, \sigma)$ with $d$-dimensional connections is a weighted graph $G$ equipped with a connection function $\sigma : E^{or} \to O(d)$ or $U(d)$ satisfying $\sigma_{xy} = \sigma_{yx}^{-1}$, where we write $\sigma_{xy} := \sigma((x, y))$ for short.

The signature of a cycle $C = x_0 \sim x_1 \sim x_2 \sim \ldots x_n \sim x_0$ is defined as the conjugacy class of the product $\sigma_C = \sigma_{x_0x_1}\sigma_{x_1x_2}\ldots\sigma_{x_nx_0}$. A connection graph is called balanced if all of its cycles have signature (the conjugacy class of) $I_d$.

A switching function is a function $\tau : V \to U(d)$ or $O(d)$. Switching the connection function $\sigma$ by the function $\tau$ means replacing $\sigma$ by $\sigma^\tau$ where $\sigma^\tau_{xy} := \tau(x)^{-1}\sigma_{xy}\tau(y)$ for each edge $\{x, y\} \in E$. We say two connection functions $\sigma_1$ and $\sigma_2$ are switching equivalent if there exists a switching function $\tau$ such that $\sigma_1 = \sigma_2^\tau$. It is direct to check that switching equivalence is indeed an equivalent relation. A property or a quantity is called switching invariant if it is preserved by switching operations. For example, the signature of any cycle is switching invariant. Zaslavsky [44, Corollary 3.3 and Section 9] obtained the following characterization lemma of balancedness.

Lemma 2.1 (Zaslavsky’s switching lemma). A gain graph $(G, \sigma)$ is balanced if and only if $\sigma$ is switching equivalent to the all $I_d$ signature.

2.2 Bakry-Émery curvature

In this article, we study the Bakry-Émery curvature of connection graphs introduced in [35]. The definition of the curvature is motivated by the Bochner identity in Riemannian Geometry and Bakry-Émery $\Gamma$-calculus [4][1][5]. We give a brief review about it here. For any smooth function $f : M \to \mathbb{R}$ on a Riemannian manifold $M$, the following Bochner identity holds:

$$\frac{1}{2}\Delta|\text{grad } f|^2(x) = |\text{Hess } f|^2(x) + \langle \text{grad } \Delta f(x), \text{grad } f(x) \rangle + \text{Ric}(\text{grad } f, \text{grad } f)(x),$$
where $\text{Hess} f$ is the Hessian of $f$, $\Delta$ is the Laplace-Beltrami operator and $\text{Ric}$ stands for the Ricci curvature tensor. By Cauchy-Schwarz inequality, we have the first term on the right hand

$$|\text{Hess} f|^2(x) \geq \frac{1}{n}(\Delta f(x))^2.$$ 

If the Ricci curvature is lower bounded by $K$ at $x \in M$, i.e.,

$$\text{Ric}(v, v) \geq K\langle v, v \rangle,$$

for any $v \in T_x M$,

then the above identity yields

$$\frac{1}{2}\Delta|\text{grad} f|^2(x) - \langle \text{grad} \Delta f, \text{grad} f \rangle \geq \frac{1}{n}(\Delta f(x))^2 + K|\text{grad} f|^2(x). \quad (2.1)$$

Bakry-Émery \cite{5} introduced the following definition of $\Gamma$ and $\Gamma_2$: For any two smooth functions $f, g : M \to \mathbb{R}$,

$$2\Gamma(f, g) := \Delta(fg) - f\Delta(g) - g\Delta(f),$$

$$2\Gamma_2(f, g) := \Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Delta f).$$

Noticing that

$$\Gamma(f, g) = \langle \text{grad} f, \text{grad} g \rangle,$$

$$\Gamma_2(f, f) = \frac{1}{2}\Delta|\text{grad} f|^2 - \langle \text{grad} \Delta f, \text{grad} f \rangle,$$

the inequality \eqref{2.1} is reformulated as

$$\Gamma_2(f, f)(x) \geq \frac{1}{n}(\Delta f(x))^2 + K\Gamma(f, f)(x). \quad (2.2)$$

On an $n$-dimensional Riemannian manifold $M$, \eqref{2.2} holds at $x \in M$ for any smooth function $f$ if and only if the Ricci curvature is bounded from below by $K$ at $x$ \cite[pp.93-94]{3}.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For a function $f : V \to \mathbb{K}^d$ defined on a connection graph $(G, \sigma)$ with $d$-dimensional connections, the Laplace operator $\Delta^\sigma$ is defined as follows: for any vertex $x$,

$$\Delta^\sigma f(x) := \sum_{y, y \sim x} p_{xy}(\sigma_{xy} f(y) - f(x)).$$

This operator has been introduced as the connection Laplacian by Singer and Wu \cite{42}. The special choices of $H = O(d)$ or $U(d)$ ensure that operator $\Delta^\sigma$ is self-adjoint. For the case $H = U(1)$, this operator is known as discrete magnetic Laplacian \cite{39, 43}. For the case $H = O(1) = \{\pm 1\}$, it is known as the signed Laplace matrix \cite{40}. The Cheeger type eigenvalue estimates for the connection Laplacian have been discussed in \cite{2, 7, 33}.

The following definition of Bakry-Émery curvature of connection graphs based on the Laplacian $\Delta^\sigma$ has been introduced in \cite[Definition 3.2]{35}. We first define the corresponding operators $\Gamma^\sigma$ and $\Gamma_2^\sigma$: For any two functions $f, g : V \to \mathbb{K}^d$,

$$2\Gamma^\sigma(f, g) := \Delta(f^\top g) - f^\top \Delta^\sigma g - (\Delta^\sigma f)^\top g,$$

$$2\Gamma_2^\sigma(f, g) := \Delta(\Gamma^\sigma(f, g)) - \Gamma^\sigma(f, \Delta^\sigma g) - \Gamma^\sigma(\Delta^\sigma f, g).$$

We also write $\Gamma^\sigma(f) := \Gamma^\sigma(f, f)$ and $\Gamma_2^\sigma(f) = \Gamma_2^\sigma(f, f)$ for simplicity.
We denote by $\Delta$ the structure of $B$.

Observe that the curvature function $\Gamma: V \to \mathbb{K}^d$ has been derived in [35, Section 3.4]. Recall from [35, Proposition 3.1] that for any explicit expressions of the non-trivial blocks of these two matrices for combinatorial graphs.

The curvature function at any vertex of a connection graph $\Gamma: \sigma \rightarrow V$ has been derived in [35, Section 3.4]. Recall from [35, Proposition 3.1] that for any explicit expressions of the non-trivial blocks of these two matrices for combinatorial graphs.

The following switching invariance property was shown in [35, Proposition 3.5].

\[ \Gamma(x, \tau) = \Gamma(x, \tau f, \tau g), \quad \Gamma(x, \tau) = \Gamma(x, \tau f, \tau g), \quad (2.3) \]

and

\[ \Delta(x) = D(\tau)^{-1} \Delta f D(\tau), \quad (2.4) \]

where $D(\tau)$ stands for the diagonal matrix of the function $\tau$.

Observe that the curvature function $K_{x, \sigma, x}(\cdot)$ at a vertex $x$ only depends on the local structure of $B_2(x)$. Let us denote

\[ S_1(x) = \{y_1, \ldots, y_m\}, \quad \text{and} \quad S_2(x) = \{z_1, \ldots, z_n\}. \]

We denote by $\Delta(x)$ the following $(m + 1)d \times d$ matrix

\[ \Delta(x) = \left( -\frac{d}{\mu_x} I_d \quad p_{xy_1, \sigma y_1} \quad p_{xy_2, \sigma y_2} \quad \ldots \quad p_{xy_m, \sigma y_m} \right). \quad (2.5) \]

For any functions $f, g: V \to \mathbb{R}^d$, we denote the corresponding local $(m + 1)d$-vectors by

\[ \bar{f}(x) = (f(x)^T, f(y_1)^T, \ldots, f(y_m)^T), \quad \bar{g}(x) = (g(x)^T, g(y_1)^T, \ldots, g(y_m)^T). \]

Then we have

\[ (\Delta(x))^{-1} \Delta f(x) = \bar{f}(x)^T \Delta(x) \Delta(x)^{-1} \bar{g}(x). \]

That is the matrix $\Delta(x) \Delta(x)^{-1}$ is the Hermitian matrix corresponding to the symmetric sesquilinear form $(\Delta f(x)) \Delta f(x)$ at $x$. Let $\Gamma(x)$ and $\Gamma(x)$ be the Hermitian matrices corresponding to the symmetric sesquilinear forms $\Gamma(x)$ and $\Gamma(x)$ at $x$, respectively. The explicit expressions of the non-trivial blocks of these two matrices for combinatorial graphs has been derived in [35, Section 3.4]. Recall from [35, Proposition 3.1] that for any $f, g: V \to \mathbb{R}^d$

\[ 2\Gamma(f, g)(x) = \sum_{y \in V} p_{xy}(\sigma y f(y) - f(x))^\top (\sigma y g(y) - g(x)). \quad (2.6) \]
Therefore, the matrix $\Gamma^\sigma(x)$ has a non-trivial block of size $d|B_1(x)| \times d|B_1(x)|$:

$$
2\Gamma^\sigma(x) = \begin{pmatrix}
\sum_{i=1}^{m} p_{xy_i} I_d & -p_{xy_1} \sigma_{xy_1} & -p_{xy_2} \sigma_{xy_2} & \cdots & -p_{xy_m} \sigma_{xy_m} \\
-p_{xy_1} \sigma_{xy_1} & p_{xy_1} I_d & 0 & \cdots & 0 \\
-p_{xy_2} \sigma_{xy_2} & 0 & p_{xy_2} I_d & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-p_{xy_m} \sigma_{xy_m} & 0 & 0 & \cdots & p_{xy_m} I_d
\end{pmatrix}.
$$

(2.7)

The matrix $\Gamma^\sigma_2(x)$ has a non-trivial block of size $d|B_2(x)| \times d|B_2(x)|$:

$$
\Gamma^\sigma_2(x) = \begin{pmatrix}
\Gamma^\sigma_2(x)_{x,x} & \Gamma^\sigma_2(x)_{x,S_1} & \Gamma^\sigma_2(x)_{x,S_2} \\
\Gamma^\sigma_2(x)_{S_1,x} & \Gamma^\sigma_2(x)_{S_1,S_1} & \Gamma^\sigma_2(x)_{S_1,S_2} \\
\Gamma^\sigma_2(x)_{S_2,x} & \Gamma^\sigma_2(x)_{S_2,S_1} & \Gamma^\sigma_2(x)_{S_2,S_2}
\end{pmatrix},
$$

(2.8)

The explicit expression will be given in the Appendix. We only mention here that the block $(\Gamma^\sigma_2)_{S_2,S_2}$ is diagonal and positive definite.

Since the block $(\Gamma^\sigma_2)_{S_2,S_2}$ is diagonal, the matrix $\Gamma^\sigma_2(x)$, and hence the curvature function $K_{G,\sigma,x}(\cdot)$, is completely determined by the local connection structure of the incomplete 2-ball $B^\text{inc}_2(x)$ around $x$, which is obtained from the induced subgraph of $B_2(x)$ by removing the edges connecting vertices from $S_2(x)$ and assigning a restricted connection function of $\sigma$.

For the matrix $\Gamma^\sigma(x)$, we have the following observation.

**Proposition 2.4.** Let $(G, \sigma)$ be a connection graph with $d$-dimensional connections. At any vertex $x$, the eigenvalues of the matrix $\Gamma^\sigma(x)$ can be listed with multiplicity as

$$
0 = \lambda_1 = \cdots = \lambda_d < \lambda_{d+1} \leq \cdots \leq \lambda_{md}.
$$

**Proof.** It follows directly from (2.6) that

$$
2\Gamma^\sigma(f, f)(x) = \sum_{y \in V} p_{xy}(\sigma_{xy} f(y) - f(x))^2 \geq 0,
$$

where the equality holds if and only if $f(y) = \sigma_{xy}^{-1} f(x)$ for any $y \in S_1(x)$. This proves the proposition. \hfill \Box

### 2.3 Schur complement

A fundamental tool in our reformulation of the Bakry-Émery curvature as the smallest eigenvalue of a family of curvature matrices is the so-called Schur complement. Let $S$ be an Hermitian matrix

$$
S = \begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix},
$$

(2.9)

where $S_{11}$ and $S_{22}$ are square blocks. The Schur complement $S/S_{11}$ of $S_{11}$ in $S$ is defined as

$$
S/S_{11} := S_{22} - S_{21} S_{11}^\dagger S_{12},
$$

(2.10)

where $S_{11}^\dagger$ stands for the pseudoinverse of the matrix $S_{11}$. The following result has been proved in [1, Theorem 1(i)].
Theorem 2.5 (Albert). Let $S$ be an Hermitian matrix defined as above. Then $S$ is positive semidefinite (denoted by $S \succeq 0$) if and only if $S_{11}S_{11}^\dagger S_{12} = S_{12}$, $S_{11} \succeq 0$ and $S/S_{11} \succeq 0$.

Proof. The proof of [1, Theorem 1(i)] was given for real symmetric matrices. The argument there works for Hermitian matrices straightforwardly. For our later purpose, we present a slightly different proof in terms of the corresponding sesquilinear forms.

If it holds that

$$S_{11} \succeq 0, \text{ and } S_{11}S_{11}^\dagger S_{12} = S_{12},$$

then we have the following identity: For any vectors $v_1, v_2$ whose dimension coincides with the size of $S_{11}, S_{12}$, respectively,

$$(v_1^\top \ v_2^\top) \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2^\top (S_{22} - S_{21}S_{11}S_{12})v_2 + \left( S_{11}^{1/2}S_{12}v_2 + (S_{11}^{1/2})^\top v_1 \right)^\top \left( (S_{11}^{1/2})^\dagger S_{12}v_2 + S_{11}^{1/2}v_1 \right)$$

$$= v_2^\top (S/S_{11})v_2 + \left( S_{11}^{1/2}S_{12}v_2 + S_{11}^{1/2}v_1 \right)^2.$$  \tag{2.12}

In the above, we use the identity $S_{11}^{1/2}(S_{11}^{1/2})^\dagger S_{12} = S_{12}$ which is a consequence of the condition $S_{11}(S_{11})^\dagger S_{12} = S_{12}$ and $S_{11} = S_{11}^{1/2}S_{11}^{1/2}$.

Under the condition (2.11), we derive from (2.12) directly that $S/S_{11} \succeq 0$ implies $S \succeq 0$.

Conversely, we suppose $S \succeq 0$. Then there exists a matrix $H$ such that $S = HH^\top$. Let us write

$$H = \begin{pmatrix} X \\ Y \end{pmatrix},$$

where the number of rows in $X$ (in $Y$) coincides with the size of $S_{11}$ (of $S_{12}$). Thus we have $S_{11} = XX^\top$, $S_{12} = S_{21} = XY^\top$, and $S_{22} = YY^\top$. Hence, we derive

$$S_{11}S_{11}^\dagger S_{12} = XX^\top(XX^\top)^\dagger S_{12} = XX^\dagger(XX^\top) = XX^\top = S_{12},$$

where we use the properties that $XX^\top(XX^\top)^\dagger = X^\dagger$ and $XX^\dagger X = X$. That is, the property $S \succeq 0$ implies the condition (2.11), and, by applying (2.12), $S/S_{11} \succeq 0$. \hfill \Box

Remark 2.6. In case that $S_{11}$ is positive definite (denoted by $S_{11} \succ 0$), the above Theorem simply reads that $S \succeq 0$ if and only if the Schur complement $S/S_{11} \succeq 0$.

For our later purpose, we prove the following technical lemma.

Lemma 2.7. Let $S$ be an Hermitian matrix defined in (2.7) such that $S_{11} \succeq 0$ and $S_{11}S_{11}^\dagger S_{12} = S_{12}$. Let $v_1, v_2$ be vectors whose dimensions coincide with the sizes of $S_{11}, S_{12}$, respectively. Then the norm square term in (2.12)

$$\left| (S_{11}^{1/2})^\dagger S_{12}v_2 + S_{11}^{1/2}v_1 \right|^2$$

...
vanishes if and only if it holds that

$$S \left( \begin{array}{c} v_1 \\ v_2 \\ \end{array} \right) = \left( \begin{array}{cc} S_{11} & S_{12} \\ S_{21} & S_{22} \\ \end{array} \right) \left( \begin{array}{c} v_1 \\ v_2 \\ \end{array} \right) = \left( \begin{array}{c} 0 \\ * \\ \end{array} \right) .$$  \tag{2.13}

Proof. By the fact that $$X^\dagger X X^\dagger = X^\dagger$$ for any matrix $$X$$, we have

$$S_{11}^\dagger S_{12} v_2 + S_{11}^\dagger S_{11}^\dagger v_1 = 0$$

if and only if

$$S_{11}^\dagger S_{12}^\dagger S_{12} v_2 + S_{11}^\dagger S_{11}^\dagger S_{12} = 0 .$$

The latter one can be reformulated as $$S_{11} S_{11}^\dagger = S_{12} .$$

3 Curvature matrices of connection graphs

By Definition 2.2, the $$N$$-Bakry-Émery curvature $$K_{G,\sigma,x}(N)$$ of the vertex $$x$$ of a signed graph $$(G, \sigma)$$ is the solution of the following semidefinite programming problem:

$$\begin{align*}
\text{maximize} & \quad K \\
\text{subject to} & \quad \Gamma_2^\sigma(x) - \frac{1}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\dagger - K \Gamma^\sigma(x) \succeq 0 .
\end{align*}$$  \tag{3.1}

Since the block $$\Gamma_2^\sigma(x) S_2, S_2 > 0$$, we derive from Theorem 2.5 that the inequality (3.1) if and only if

$$Q(x) - \frac{1}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\dagger - K \Gamma^\sigma(x) \succeq 0 ,$$  \tag{3.2}

where

$$Q(x) := Q(\Gamma_2^\sigma(x)) := \Gamma_2^\sigma(x) S_1, S_1 - \Gamma_2^\sigma(x) S_1, S_2 \Gamma_2^\sigma(x) S_2, S_2$$

is the Schur complement of $$\Gamma_2^\sigma(x) S_2, S_2$$ in the matrix $$\Gamma_2^\sigma(x)$$. Notice that $$Q(x)$$ is Hermitian.

Let $$B$$ be a $$(m+1)d \times (m+1)d$$ nonsingular matrix

$$B = \left( \begin{array}{c} b_0^\top \\ b_1^\top \\ \vdots \\ b_m^\top \end{array} \right) ,$$  \tag{3.4}

where, for $$i = 0, 1, \ldots, m$$, we have

$$b_i^\top = \left( \begin{array}{c} b_{i1}^\top \\ \vdots \\ b_{id}^\top \end{array} \right)$$

with $$b_{ij} \in \mathbb{R}^{(m+1)d}$$, $$j = 1, \ldots, d$$, such that

$$B(2\Gamma^\sigma(x)) B^\top = \left( \begin{array}{cc} 0_d & 0_{d \times md} \\ 0_{md \times d} & I_{md} \end{array} \right) .$$  \tag{3.5}

Here, we use the notation $$I_{md}$$ for a $$md \times md$$ identity matrix, and $$0_d$$ for a $$d \times d$$ zero matrix.
Recalling from Proposition 2.4 that the zero eigenvalue of $2\Gamma^\sigma(x)$ has multiplicity $d$ and all other eigenvalues are positive, such a matrix $B$ satisfying (3.5) exists. Indeed, such a matrix is not unique.

We read from (3.5) that
$$b_0^\top 2\Gamma^\sigma(x)b_0 = 0.$$ Since $2\Gamma^\sigma(x)$ is a positive semidefinite Hermitian matrix, we obtain that $b_0$ is a matrix which is composed of $d$ linearly independent eigenvectors $b_0, i = 1, \ldots, d$ corresponding to the zero eigenvalue of the matrix $2\Gamma^\sigma(x)$. By (2.6), any $f: V \rightarrow \mathbb{K}^d$ such that $\Gamma^\sigma(f,f)(x) = 0$ satisfies
$$\sigma_{xy}f(y) = f(x), \quad \text{for } y \in S_1(x).$$

Hence, we derive that $b_0^\top = E_d p_0^\top$ for some nonsingular $d \times d$ matrix $E_d$, where
$$p_0^\top := \begin{pmatrix} I_d & \sigma_{x1} & \cdots & \sigma_{xm} \end{pmatrix}.$$ (3.6)

The blocks $b_i, i = 1, \ldots, m$ can be constructed from the nonzero eigenvalues and corresponding eigenvectors of $2\Gamma^\sigma(x)$. They can also be given by simply setting $b_i = p_i, i = 1, \ldots, m$ with
$$p_1^\top := \begin{pmatrix} 0_d & \frac{1}{\sqrt{p_{x1}}} I_d & 0_d & \cdots & 0_d \end{pmatrix}, \ldots, p_m^\top := \begin{pmatrix} 0_d & 0_d & 0_d & \cdots & \frac{1}{\sqrt{p_{xm}}} I_d \end{pmatrix}.$$ (3.7)

The fact that $b_0$ is composed of eigenvectors to the zero eigenvalue of $2\Gamma^\sigma(x)$ implies that
$$b_0^\top \Delta^\sigma(x) = 0_d.$$ (3.8)

The inequality (3.3) holds if and only if
$$2BQ(x)B^\top - \frac{2}{N} (B\Delta^\sigma(x))(B\Delta^\sigma(x))^\top - K \begin{pmatrix} 0_d & 0_{d \times md} \\ 0_{md \times d} & I_{md} \end{pmatrix} \succeq 0.$$ (3.9)

Due to (3.8), we can denote
$$B\Delta^\sigma(x) := \begin{pmatrix} 0_d \\ v_0 \end{pmatrix},$$ (3.10)
where $v_0(x) = v_0(G, \sigma, x, B)$ is a $md \times d$ matrix. Then, we have
$$B\Delta^\sigma(x)\overline{B\Delta^\sigma(x)}^\top = \begin{pmatrix} 0_d & 0_{d \times md} \\ 0_{md \times d} & v_0 v_0^\top \end{pmatrix}.$$ (3.11)

We further introduce the notation
$$2BQ(x)B^\top = \begin{pmatrix} a & \omega^\top \\ \overset{\omega}{\omega} & (2BQ(x)B^\top)_{11} \end{pmatrix},$$ where
$$a := a(G, \sigma, x, B) = 2b_0^\top Q(x)\overline{b_0},$$ (3.12)
and
$$\omega^\top := \omega(G, \sigma, x, B)^\top = \begin{pmatrix} 2b_0^\top Q(x)\overline{b_1} & \cdots & 2b_0^\top Q(x)\overline{b_m} \end{pmatrix}$$ (3.13)
and
$$(2BQ(x)B^\top)_{11} = \begin{pmatrix} \overline{b_1} \\ \vdots \\ \overline{b_m} \end{pmatrix} 2Q(x) \begin{pmatrix} \overline{b_1} & \cdots & \overline{b_m} \end{pmatrix}.$$ (3.14)
is the $md \times md$ matrix obtained from $2BQ(x)\overline{B}^T$ via removing the first $d$ columns and the first $d$ rows. Therefore, (3.9) can be reformulated as

$$\begin{pmatrix} a & \omega^T \\ \omega & (2BQ(x)\overline{B}^T)_{11} - \frac{2}{N}v_0\overline{v}_0^T - KI_{md\times md} \end{pmatrix} \succeq 0. \quad (3.15)$$

Indeed, we have $a \succeq 0$ and $aa^\dagger \omega^T = \omega^T$ (Proposition A.1 in the Appendix). Therefore, we apply Theorem 2.5 again to derive that (3.15) holds if and only if

$$(2BQ(x)\overline{B}^T)_{11} - \frac{2}{N}v_0\overline{v}_0^T - KI_{md\times md} - \omega a^\dagger \omega^T \succeq 0. \quad (3.16)$$

That is, the $N$-Bakry-Émery curvature $K_{G,\sigma,x}(N)$ is equal to the smallest eigenvalue of the matrix

$$(2BQ(x)\overline{B}^T)_{11} - \omega a^\dagger \omega^T - \frac{2}{N}v_0\overline{v}_0^T.$$

In conclusion, we introduce the following concept of curvature matrices and reformulate the curvature as an eigenvalue problem.

**Definition 3.1** (Curvature matrices). Let $x$ be a vertex in a connection graph $(G, \sigma)$ with the matrix $Q(x)$ defined in (3.3). Let $B$ be a nonsingular matrix satisfying (3.5). We define the curvature matrix $A_\infty(G, \sigma, x, B)$ of $x$ corresponding to the matrix $B$ as the $md \times md$ matrix below:

$$A_\infty(G, \sigma, x, B) := (2BQ(x)\overline{B}^T)_{11} - \omega a^\dagger \omega^T, \quad (3.17)$$

where $a$ and $\omega$ are defined by (3.12) and (3.13). Given $N \in (0, \infty]$, we define

$$A_N(G, \sigma, x, B) := A_\infty(G, \sigma, x, B) - \frac{2}{N}v_0\overline{v}_0^T, \quad (3.18)$$

where $v_0$ is given in (3.10).

**Theorem 3.2.** Let $x$ be a vertex in a connection graph $(G, \sigma)$. Let $B$ be a nonsingular matrix satisfying (3.7). Then we have for any $N \in (0, \infty]$:

$$K_{G,\sigma,x}(N) = \lambda_{\min}(A_N(G, \sigma, x, B)). \quad (3.19)$$

We first observe that the curvature matrices have the following property under switching the connections.

**Proposition 3.3.** Let $x$ be a vertex in a connection graph $(G, \sigma)$. Let $\tau : V \rightarrow U(d)$ be a switching function. Then the matrix $A_N(G, \sigma, x, B)$, with $B$ being a nonsingular matrix satisfying (3.7) and $N \in (0, \infty]$, has the following property

$$A_N(G, \sigma, x, \tau_{B_1(x)}\overline{B}_1(x)) \overline{B}_1(x)) = \tau_{S_1(x)}^TA_N(G, \sigma, x, B)\tau_{S_1(x)},$$

where

$$\tau_{S_1(x)} := \begin{pmatrix} \tau(y_1) \\
\vdots \\
\tau(y_m) \end{pmatrix} \text{ and } \tau_{B_1(x)} := \begin{pmatrix} \tau(x) \\
\tau_{S_1(x)} \end{pmatrix}. \quad (3.11)$$
We first prepare the following lemma for the Schur complement.

**Lemma 3.4.** Consider a block matrix

\[ S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

with \( D \) being an invertible square matrix. Let

\[ U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \]

be an invertible matrix with the blocks \( U_1, U_2 \) of the same sizes as \( A, D \) respectively. Then the Schur complement of the bottom right block in \( USU^{-1} \) equals

\[ U_1(A - BD^{-1}C)U_1^{-1}. \]

**Proof.** It is directly to check that

\[ USU^{-1} = \begin{pmatrix} U_1AU_1^{-1} & U_1BU_2^{-1} \\ U_2CU_1^{-1} & U_2DU_2^{-1} \end{pmatrix}. \]

By definition, the Schur complement of the bottom right block in \( USU^{-1} \) is

\[ U_1AU_1^{-1} - (U_1BU_2^{-1})(U_2DU_2^{-1})^{-1}(U_2CU_1^{-1}) = U_1(A - BD^{-1}C)U_1^{-1}. \]

\[ \square \]

**Proof of Proposition 3.3.** It is direct to check for any \( f, g : V \to \mathbb{C}^d \) that

\[ \Gamma^\sigma (f, g) = \Gamma^\sigma (\tau f, \tau g), \text{ and } \Gamma_2^\sigma (f, g) = \Gamma_2^\sigma (\tau f, \tau g). \]

Therefore, the matrices \( \Gamma^\sigma(x) \) and \( \Gamma_2^\sigma(x) \) change under the switching by \( \tau : V \to U(d) \) as follows:

\[ \Gamma^\sigma(x) = \tau_{B_1(x)}^\top \Gamma^\sigma(x) \tau_{B_1(x)} \text{ and } \Gamma_2^\sigma(x) = \tau_{B_2(x)}^\top \Gamma_2^\sigma(x) \tau_{B_2(x)}, \tag{3.20} \]

where

\[ \tau_{B_2(x)} = \begin{pmatrix} \tau_{B_1(x)} \\ \tau(z_1) \\ \vdots \\ \tau(z_n) \end{pmatrix}. \]

Let us denote by \( B^\tau := \tau_{B_1(x)}^\top B\tau_{B_1(x)}. \) By the first identity in (3.20), we check that

\[ B^\tau 2\Gamma^\sigma(x) B^\tau = \begin{pmatrix} 0_{d \times d} & 0_{d \times md} \\ 0_{md \times d} & I_{md \times md} \end{pmatrix}. \]

By Lemma 3.4 and the second identity in (3.20), we obtain

\[ Q(\Gamma_2^\sigma(x)) = \tau_{B_1(x)}^\top Q(\Gamma_2^\sigma(x)) \tau_{B_1(x)}, \tag{3.21} \]

which further implies that

\[ B^\tau 2Q(\Gamma_2^\sigma(x)) B^\tau = \tau_{B_1(x)}^\top B2Q(\Gamma_2^\sigma(x)) B\tau_{B_1(x)}. \tag{3.22} \]
If the local connection structure \( B_2^{inc}(x) \) is balanced, then the first \( d \) rows and first \( d \) columns of the matrices \( 2Q(\Gamma_2^\sigma(x)), 2Q(\Gamma_2^\sigma(x)) \) vanish, and, hence, the curvature matrices satisfy
\[
A_\infty(G, \sigma^\tau, x, B^\tau) = 2Q(\Gamma_2^\sigma(x))_1^1 = \tau_{S_1(x)}^\top A_\infty(G, \sigma, x, B)_{\overline{S_1(x)}}.
\]

If, otherwise, the local connection structure \( B_2^{inc}(x) \) is unbalanced, we apply Lemma 3.4 to conclude that
\[
A_\infty(G, \sigma^\tau, x, B^\tau) = \tau_{S_1(x)}^\top A_\infty(G, \sigma, x, B)_{\overline{S_1(x)}}. \tag{3.23}
\]

By definition, we calculate straightforwardly that
\[
\Delta^\sigma(x) = \tau_{B_1(x)}^\top \Delta^\sigma(x)_{\overline{B_1(x)}}. \tag{3.24}
\]

This leads to
\[
\Delta^\sigma(x) \Delta^\sigma(x)^\top = \tau_{B_1(x)}^\top \Delta^\sigma(x)_{\overline{B_1(x)}} \tau_{B_1(x)}. \tag{3.25}
\]

We derive that
\[
B^\tau \Delta^\sigma(x)_{\overline{B_1(x)}} \Delta^\sigma(x)^\top_{\overline{B_1(x)}} = \tau_{B_1(x)}^\top B \Delta^\sigma(x)_{\overline{B_1(x)}} \Delta^\sigma(x)^\top_{\overline{B_1(x)}}. \tag{3.26}
\]

Recall that the first \( d \) rows and first \( d \) columns of both sides vanish, we have
\[
(B^\tau \Delta^\sigma(x)_{\overline{B_1(x)}} \Delta^\sigma(x)^\top_{\overline{B_1(x)}})_1^1 = \tau_{S_1(x)}^\top (B \Delta^\sigma(x)_{\overline{B_1(x)}} \Delta^\sigma(x)^\top_{\overline{B_1(x)}})_1^1 \tau_{S_1(x)}. \tag{3.27}
\]

That is,
\[
\nu_0(\sigma^\tau)_{\overline{B_1(x)}} \nu_0(\sigma)_{\overline{B_1(x)}} = \tau_{S_1(x)}^\top \nu_0(\sigma)_{\overline{B_1(x)}} \nu_0(\sigma)_{\overline{B_1(x)}} \tau_{S_1(x)}. \tag{3.28}
\]

Combining (3.23) and (3.26) completes the proof.

Next, we show that different choices of the matrix \( B \) lead to unitarily equivalent curvature matrices.

**Proposition 3.5.** Let \( x \) be a vertex in a connection graph \((G, \sigma)\). Let \( B_1, B_2 \) be two nonsingular matrices satisfying (3.24). Then the two curvature matrices \( A_N(G, \sigma, x, B_1) \) and \( A_N(G, \sigma, x, B_2) \) are unitarily equivalent. Indeed, we have
\[
A_N(G, \sigma, x, B_1) = (B_1 B_2^{-1})_1^1 A_N(G, \sigma, x, B_2) (B_1 B_2^{-1})_1^1. \tag{3.29}
\]

We prepare some lemmas.

**Lemma 3.6.** Let \( t_0, b_0, b_1, \ldots, b_m \) be \((m + 1)d \times d\) matrices such that both
\[
B_1 := \begin{pmatrix}
  t_0^\top \\
  b_1^\top \\
  \vdots \\
  b_m^\top
\end{pmatrix}, \quad \text{and} \quad B_2 := \begin{pmatrix}
  b_0^\top \\
  b_1^\top \\
  \vdots \\
  b_m^\top
\end{pmatrix}
\]

are nonsingular and satisfy (3.25). Then we have
\[
A_N(G, \sigma, x, B_1) = A_N(G, \sigma, x, B_2). \tag{3.30}
\]
Proof. Recall that
\[ A_N(G, \sigma, x, B_2) = (2B_2Q(x)\overline{B_2^T})_1 - \overline{\omega a^T} - \frac{2}{N}(B_2\Delta\sigma(x)\overline{B_2\Delta\sigma(x)^T})_1. \]
Due to (3.11), (3.14), we have the two terms
\[ (2B_2Q(x)\overline{B_2^T})_1 = (2B_1Q(x)\overline{B_1^T})_1 \]
and
\[ (B_2\Delta\sigma(x)\overline{B_2\Delta\sigma(x)^T})_1 = (B_1\Delta\sigma(x)\overline{B_1\Delta\sigma(x)^T})_1 \]
are independent of the choice of \( b_0 \).
Recall that \( \overline{b_0} \) and \( \overline{t_0} \) are both composed of linearly independent eigenvectors corresponding to the zero eigenvalue of \( 2\Gamma^\sigma(x) \). Hence, there exists a nonsingular \( d \times d \) matrix \( E_d \) such that \( b_0^T = E_d t_0^T \). Then, we derive that
\[
\overline{\omega a^T} = \begin{pmatrix}
\overline{b_1^T 2Q(x)\overline{t_0}} \\
\vdots \\
\overline{b_m^T 2Q(x)\overline{t_0}}
\end{pmatrix} E_d^T \left[ E_d t_0^T 2Q(x)\overline{t_0} E_d^T \right]^\dagger E_d \left( t_0^T 2Q(x)\overline{b_1} \cdots t_0^T 2Q(x)\overline{b_m} \right)
\]
\[
= \begin{pmatrix}
\overline{b_1^T 2Q(x)\overline{t_0}} \\
\vdots \\
\overline{b_m^T 2Q(x)\overline{t_0}}
\end{pmatrix} \left( t_0^T 2Q(x)\overline{b_1} \cdots t_0^T 2Q(x)\overline{b_m} \right)^\dagger \left( t_0^T 2Q(x)\overline{b_1} \cdots t_0^T 2Q(x)\overline{b_m} \right).
\]
That is, (3.29) holds true in this case too.

Lemma 3.7. Let \( B_1, B_2 \) be two nonsingular matrices satisfying (3.2). Then the \( md \times md \) matrix \( (B_1B_2^{-1})_1 \) is unitary.

Proof. By direct calculation, we have
\[
\begin{pmatrix}
0_d \\
0_{md \times d} \\
I_{md}
\end{pmatrix} = B_1 2\Gamma^\sigma(x)B_1^T = (B_1B_2^{-1})_1 B_2 2\Gamma^\sigma(x)B_2^T \overline{B_1B_2^{-1}}^T
\]
\[
=(B_1B_2^{-1})_1 \begin{pmatrix}
0_d \\
0_{md \times d} \\
I_{md}
\end{pmatrix} \overline{B_1B_2^{-1}}^T.
\]
Therefore, we have \( (B_1B_2^{-1})_1 \overline{B_1B_2^{-1}}^T_1 = I_{md} \).

Lemma 3.8. For \( i = 1, 2 \), let
\[
B_i = \begin{pmatrix}
b_{0,i}^T \\
b_{1,i}^T \\
\vdots \\
b_{m,i}^T
\end{pmatrix}
\]
be a nonsingular matrix satisfying (3.2). Then we have
\[
B_1B_2^{-1} = \begin{pmatrix}
E_d \\
\eta \\
(B_1B_2^{-1})_1
\end{pmatrix},
\]
for some \( md \times d \) matrix \( \eta \), where \( E_d \) is the unique \( d \times d \) matrix with \( b_{0,1}^T = E_d b_{0,2}^T \).
Proof. By (3.5), we derive
\[
B_1B_2^{-1}\begin{pmatrix}
0_d & 0 \\
0 & I_{md}
\end{pmatrix} = B_12\Gamma^\sigma(x)\overline{B_2}^\top = \begin{pmatrix}
0_d & 0 \\
\begin{pmatrix} b_{1,1}^	op \\
\vdots \\
b_{m,1}^	op 
\end{pmatrix} & 2\Gamma^\sigma(x)\begin{pmatrix} \overline{b_{1,2}} & \cdots & \overline{b_{m,2}} \end{pmatrix}
\end{pmatrix}.
\] (3.31)

In the second equality above, we use the fact \(2\Gamma^\sigma(x)\overline{b_{0,i}} = 0, \ i = 1, 2\). Let us write
\[
B_1B_2^{-1} = \begin{pmatrix} *_{1} & *_{2} \\
\eta & (B_1B_2^{-1})_1 
\end{pmatrix}.
\]
Then we have
\[
B_1B_2^{-1}\begin{pmatrix}
0_d & 0 \\
0 & I_{md}
\end{pmatrix} = \begin{pmatrix}
0_d & 0 \\
0 & (B_1B_2^{-1})_1 
\end{pmatrix}.
\]
Hence, we solve from (3.31) that \(*_{2} = 0\). Furthermore, we solve from
\[
B_1 = \begin{pmatrix} *_{1} & 0 \\
\eta & (B_1B_2^{-1})_1 
\end{pmatrix}B_2
\]
that \(b_{0,1} = b_{0,2}^\top\). Therefore, we have \(*_{1} = E_d\).

Corollary 3.9. Let \(B_1, B_2, B_3\) be three nonsingular matrices satisfying (3.5). Then, we have
\[
(B_1B_2^{-1})_1(B_2B_3^{-1})_1 = (B_1B_3^{-1})_1
\] (3.32)

Proof. This follows directly from Lemma 3.8 and the identity below:
\[
\begin{pmatrix} E_d & 0 \\
\eta & C \end{pmatrix} \begin{pmatrix} E_d' & 0 \\
\eta' & C' \end{pmatrix} = \begin{pmatrix} * & 0 \\
* & CC' \end{pmatrix}.
\]

Now, we are prepared for the proof of Proposition 3.5.

Proof of Proposition 3.5. Let us denote by \(C := B_1B_2^{-1}\). By Lemma 3.8 we have
\[
C = \begin{pmatrix} I_d & 0 \\
\eta & C_1 \end{pmatrix},
\]
where \(\eta\) is a \(md \times d\) matrix and \(C_1\) is the matrix obtained from \(C\) by deleting the first \(d\) rows and first \(d\) columns.

Now we compare the two curvature matrices
\[
A_\infty(G, \sigma, x, B_i) := (B_i2Q_{x}B_i^\top)_1 - \omega_i^\top a_i^\top \omega_i, \ i = 1, 2,
\]
where \(a_i\) and \(\omega_i\) are defined by (3.12) and (3.13) using \(B_i\), for \(i = 1, 2\).

By Lemma 3.6 we can assume \(a_1 = a_2\) without loss of generality. Let us write \(a := a_1 = a_2\) for short.
We start with
\[ B_12Q(x)B_1^\top = CB_22Q(x)B_2^\top C^\top \]
\[ = \begin{pmatrix} I_d & 0 \\ \eta & C_1 \end{pmatrix} \begin{pmatrix} a & \omega_2^\top \\ \omega_2 & (B_22Q(x)B_2^\top)_1 \end{pmatrix} \begin{pmatrix} I_d & \eta^\top \\ 0 & C_1^\top \end{pmatrix} \]
\[ = \begin{pmatrix} a & \eta a + C_1\omega_2 & \eta a\eta^\top + C_1\omega_2\eta^\top + \eta\omega_2^\top C_1^\top + C_1(B_22Q(x)B_2^\top)_1C_1^\top \end{pmatrix}. \]

Suppose the local connection structure $B_2^{inc}(x)$ is unbalanced. It is direct to check that the Schur complement of $a$ in $B2Q(x)B^\top$ satisfies
\[
A_\infty(G, \sigma, x, B_1) = (B_12Q(x)B_1^\top)_1 - \omega_1 a^\top \omega_1
\]
\[= \eta a\eta^\top + C_1\omega_2\eta^\top + \eta\omega_2^\top C_1^\top + C_1(B_22Q(x)B_2^\top)_1C_1^\top - (\eta a + C_1\omega_2) a^\top (a\eta^\top + \omega_2^\top C_1^\top)
\]
\[= C_1 \left( (B_22QxB_2^\top)_1 - \omega_2 a^\top \omega_2 \right) C_1^\top
\]
\[= C_1 A_\infty(G, \sigma, x, P)C_1^\top. \]

For the case that the local connection structure $B_2^{inc}(x)$ is balanced, we have $a = 0_d$ and $\omega_1 = \omega_2 = 0_{md \times d}$. Hence, $A_\infty(G, \sigma, x, B_1) = C_1 A_\infty(G, \sigma, x, B_2)C_1^\top$ still holds true.

Moreover, we derive that
\[
\begin{pmatrix} 0 \\ v_0(B_1) \end{pmatrix} = B_1\Delta^\sigma(x) = CB_2\Delta^\sigma(x) = \begin{pmatrix} I_d & 0 \\ \eta & C_1 \end{pmatrix} \begin{pmatrix} 0 \\ v_0(B_2) \end{pmatrix} = \begin{pmatrix} 0 \\ C_1v_0(B_2) \end{pmatrix}.
\]

Then, we obtain $v_0(B_1) = C_1v_0(B_2)$, and, hence
\[v_0(B_1)v_0(B_1)^\top = C_1v_0(B_2)v_0(B_2)^\top C_1^\top.
\]

This completes proof. \hfill \Box

**Remark 3.10.** We can choose the following matrix $B_0$ which satisfies (3.5) as a canonical choice in calculating the curvature matrices:
\[ B_0 := \begin{pmatrix} p_0^\top \\ p_1 \\ \vdots \\ p_m^\top \end{pmatrix} = \begin{pmatrix} I_d & \sigma_{xy1} \cdots \sigma_{xym} \\ \sqrt{p_{y1}}I_d & \cdots & \sqrt{p_{ym}}I_d \end{pmatrix} \tag{3.33} \]

It is direct to check that $B_0$ is a nonsingular matrix satisfying (3.5). Moreover, we have that
\[v_0(B_0) = \begin{pmatrix} \sqrt{p_{y1}}\sigma_{xy1} \\ \vdots \\ \sqrt{p_{ym}}\sigma_{xym} \end{pmatrix}. \tag{3.34} \]

**Remark 3.11.** Indeed, the matrix $a$ vanishes if the local connection structure $B_2^{inc}(x)$ is balanced (Proposition A.1 in the Appendix). Whenever each edge is assigned with the connection $1 \in O(1)$, our curvature matrix $A_N(G, \sigma, x, B_0)$ coincides with the curvature matrix for Bakry-Émery curvature of graph Laplacians introduced in [14] and [40].
Example 3.12. Here we consider an example of an unbalanced $U(2)$-connection graph shown in Figure 1 with \((\sigma_1)_{23} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}):

\[ \begin{array}{cccc}
  I_2 & 2 & I_2 \\
  I_2 & (\sigma_1)_{23} & I_2 \\
  1 & 3 & 4
\end{array} \]

Figure 1: An unbalanced connection graph \((G_1, \sigma_1)\).

Let us derive the curvature matrix \(A_{\infty}(G_1, \sigma_1, 1, B_0)\) by Definition 3.1 and (3.33). The matrices \(2\Gamma_{\sigma_1}(1)\) and \(2\Gamma_{\sigma_1}^{\sigma_1}(1)\) are

\[
2\Gamma_{\sigma_1}(1) = \begin{pmatrix}
  2 & 0 & -1 & 0 & -1 & 0 \\
  0 & 2 & 0 & -1 & 0 & -1 \\
  -1 & 0 & 1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 1 & 0 & 0 \\
  -1 & 0 & 0 & 0 & 1 & 0 \\
  0 & -1 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
2\Gamma_{\sigma_1}^{\sigma_1}(1) = \frac{1}{2} \begin{pmatrix}
  10 & 0 & -7 & -i & -7 & -i & 2 & 0 \\
  0 & 10 & i & -7 & i & -7 & 0 & 2 \\
  -7 & -i & 10 & 0 & 2 & 4i & -2 & 0 \\
  -7 & -i & 2 & 4i & 10 & 0 & -2 & 0 \\
  -i & -7 & -4i & 2 & 0 & 10 & 0 & -2 \\
  i & -7 & 4i & 2 & 0 & -2 & 0 & 2 \\
  2 & -2 & 0 & -2 & 0 & 2 & 2 & 0 \\
  0 & 2 & 0 & -2 & 0 & -2 & 0 & 2
\end{pmatrix}.
\]

Then, we have the Schur complement of \(2(\Gamma_{\sigma_1}^{\sigma_1}(1))_{S_2(1), S_2(1)}\) in \(2\Gamma_{\sigma_1}^{\sigma_1}(1)\) is

\[
2Q(1) = \frac{1}{2} \begin{pmatrix}
  8 & 0 & -5 & -i & -5 & -i \\
  0 & 8 & i & -5 & i & -5 \\
  -5 & -i & 8 & 0 & 0 & 4i \\
  i & -5 & 0 & 8 & -4i & 0 \\
  -5 & -i & 0 & 4i & 8 & 0 \\
  i & -5 & -4i & 0 & 0 & 8
\end{pmatrix}.
\]

With the canonical choice of

\[
B_0 = \begin{pmatrix}
  1 & 0 & 1 & 0 & 1 \\
  0 & 1 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
we have the matrix
\[
B_0 Q(1) B_0^\top = \frac{1}{2}
\begin{bmatrix}
4 & 4i & 3 & 3i & 3 & 3i \\
-4i & 4 & -3i & 3 & -3i & 3 \\
3 & 3i & 8 & 0 & 0 & 4i \\
-3i & 3 & 0 & 8 & -4i & 0 \\
3 & 3i & 0 & 4i & 8 & 0 \\
-3i & 3 & -4i & 0 & 8 & 0
\end{bmatrix}.
\]

Then, the curvature matrix is the Schur complement of \( \frac{1}{2} \begin{pmatrix} 4 & 4i \\ -4i & 4 \end{pmatrix} \) in \( B_0 Q(1) B_0^\top \)
\[
A_\infty(G_1, \sigma_1, 1, B_0) = \frac{1}{8}
\begin{pmatrix}
23 & -9i & -9 & 7i \\
9i & 23 & -7i & -9 \\
-9 & 7i & 23 & -9i \\
-7i & -9 & 9i & 23
\end{pmatrix}.
\]

The eigenvalues of \( A_\infty(G_1, \sigma_1, 1, B_0) \) are \( \{3/2, 2, 2, 6\} \). Therefore, the curvature
\[
K_{G_1, \sigma_1, 1}(\infty) = \frac{3}{2}.
\]

4 Curvature tensors of connection graphs

In order to motivate the definition of discrete Bakry-Émery curvature tensors of connection graphs, we first present an alternative proof of Theorem 3.2. This alternative proof is more close to the work by Siconolfi [40] in spirit.

**Lemma 4.1.** Let \( x \) be a vertex in a connection graph \((G, \sigma)\). Then we have for any \( N \in (0, \infty] \)
\[
K_{G, \sigma, x}(N) = \inf_{f: V \to \mathbb{K}^d \text{ with } \Gamma_\sigma(f)(x) \neq 0} \frac{\Gamma_\sigma^2(f)(x) - (1/N)\Delta^\sigma f(x)^2}{\Gamma_\sigma(f)(x)}.
\]

**Proof.** It is enough to show for any \( f: V \to \mathbb{K}^d \) with \( \Gamma_\sigma(f)(x) = 0 \) that
\[
\Gamma_\sigma^2(f)(x) - (1/N)\Delta^\sigma f(x)^2 \geq 0.
\]

By Remark 3.10 it holds for any \( f \) with \( \Gamma_\sigma(f)(x) = 0 \) that \( f(y_i) = \sigma_{xy_i}^{-1} f(x) \) for any \( i = 1, \ldots, m \). Then we have \( \Delta^\sigma f(x) = 0 \). Next, we show \( \Gamma_\sigma^2(f)(x) \geq 0 \) for such an \( f \).

Let us denote by \( b_0^\top := (f(x)^\top \sigma_{xy_1} \cdots f(x)^\top \sigma_{xy_m}) \).

By the identity (2.12), we estimate
\[
2\Gamma_\sigma^2(f)(x) \geq b_0^\top 2Q(x)b_0.
\]

If \( f(x) = 0 \), then we have \( 2\Gamma_\sigma^2(f)(x) \geq 0 \). Otherwise, we have \( f(x) \neq 0 \). Let \( b_0, \ldots, b_d \) be a basis for the eigenspace of \( \Gamma_\sigma(x) \) to its zero eigenvalue. This basis provides
\[
b_0^\top = \begin{pmatrix} b_0^1 \\ \vdots \\ b_0^d \end{pmatrix}.
\]
Recall that we have the matrix \( a = b_0^T 2Q(x)b_0 \geq 0 \) (see Proposition 4.1 in the Appendix). Observe that \( b_0^T 2Q(x)b_{01} \) is a diagonal entry of the matrix \( a \), and hence, is nonnegative. Therefore, \( \Gamma^2(x) \geq 0 \). \( \blacksquare \)

An alternative proof of Theorem 2.2: For a given function \( f : V \to \mathbb{R}^d \), we denote by

\[
f_0 := f(x), \quad f_1 := \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_m) \end{pmatrix}, \quad \text{and} \quad f_2 := \begin{pmatrix} f(z_1) \\ \vdots \\ f(z_n) \end{pmatrix}.
\]

Applying Lemma 4.1, we derive

\[
K_{G,\sigma,x}(N) = \inf_{f:V \to \mathbb{R}^d, T^\sigma(f)(x) \neq 0} \Gamma^\sigma(f)(x) \frac{\Gamma^2(f)(x) - (1/N)|\Delta^\sigma f(x)|^2}{\Gamma^\sigma(f)(x)}
\]

\[
= \inf_{f:V \to \mathbb{R}^d, T^\sigma(f)(x) \neq 0} \left( f_0^T f_1^T \right) \left( 2\Gamma^2(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^T \right) \left( f_0 \right) \left( f_1 \right)
\]

\[
= \inf_{f:V \to \mathbb{R}^d, T^\sigma(f)(x) \neq 0} \left( f_0^T f_1^T \right) \left( 2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^T \right) \left( f_0 \right) \left( f_1 \right) + |T(f_0, f_1, f_2)|^2
\]

\[
= \inf_{f_0, f_1} \inf_{f_2} \left( f_0^T f_1^T \right) \left( 2Q - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^T \right) \left( f_0 \right) \left( f_1 \right) + |T(f_0, f_1, f_2)|^2,
\]

where \( T(f_0, f_1, f_2) = \left( (2\Gamma^2(x)_{S_2,S_2})^{-1/2} (2\Gamma^2(x)_{S_2,B_1}) \right) \left( f_0 \right) \left( f_1 \right) + (2\Gamma^2(x)_{S_2,S_2})^{1/2} f_2 \in \mathbb{R}^{nd} \) is the vector determined by (2.12). Applying Lemma 2.7 yields that

\[
\inf_{f_2} |T(f_0, f_1, f_2)|^2 = 0.
\]

We continue the calculation in (4.3) to derive

\[
K_{G,\sigma,x}(N) = \inf_{f_0, f_1} \left( f_0^T f_1^T \right) \left( 2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^T \right) \left( f_0 \right) \left( f_1 \right)
\]

\[
= \inf_{f_0, f_1} \left( f_0^T f_1^T \right) B^{-1} B \left( 2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^T \right) B^{-1} B^{-1}^T \left( f_0 \right) \left( f_1 \right)
\]

\[
= \inf_{f_0, f_1} \left( f_0^T f_1^T \right) B^{-1} B \left( 2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^T \right) B^{-1} B^{-1}^T \left( f_0 \right) \left( f_1 \right),
\]

where \( B \) is the nonsingular matrix satisfying (3.5). We denote by

\[
\left( \begin{array}{c} f_{B,0} \\ f_{B,1} \end{array} \right) := \left( B^{-1}\right)^T \left( f_0 \right) \left( f_1 \right).
\]
Then we derive from (4.5) that
\[ K_{G, \sigma, x}(N) = \inf_{f_{B,0}} \left( f_{B,0}^\top f_{B,1} \right) B \left( 2Q(x) - \frac{2}{N} \Delta \sigma(x) \Delta \sigma(x)^\top \right) B^\top \left( \frac{f_{B,0}}{f_{B,1}} \right) \]
\[ = \inf_{f_{B,0} \neq 0} f_{B,1} A_N(G, \sigma, x, B) f_{B,0} + |T_B(f_{B,0}, f_{B,1})|^2. \]  
(4.7)

where \( T_B(f_{B,0}, f_{B,1}) = a^{1/2} f_{B,1} + (a^{1/2})^\top \omega \) is the vector determined by the identity (2.12). Applying Lemma 2.7 yields that
\[ \inf_{f_{B,0}} |T_B(f_{B,0}, f_{B,1})|^2 = 0. \]  
(4.8)

Now we continue the calculation in (4.7) as follows
\[ K_{G, \sigma, x}(N) = \inf_{f_{B,1} \neq 0} f_{B,1} A_N(G, \sigma, x, B) f_{B,1} + \left| f_{B,1} \right|^2 \]
\[ = \lambda_{\min}(A_N(G, \sigma, x, B)). \]  
(4.9)

We have the following key observations about the vectors achieving the infimums in (4.4) and (4.8):

**Observation 1**: By Lemma 2.7, the infimum in (4.4) is attained by \( f_2 \) satisfying
\[ 2\Gamma_2^\sigma(x) \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Explicitly, the infimum is attained by
\[ f_2 = - (2\Gamma_2^\sigma(x) s_2 s_1) \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} \sum_{y} p_y p_{y_2} \sigma_{y_1y_2} \sigma_{y_1y_2}^\top \\ \sum_{y} p_y p_{y_2} \sigma_{y_1y_2} \sigma_{y_1y_2}^\top \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}. \]  
(4.10)

Notice that, in the above we make use of the fact that \( \Gamma_2^\sigma(x) s_2 s_1 \) is a real matrix. Indeed, it is a positive diagonal matrix.

**Observation 2**: The infimum in (4.8) is attained by a vector \( f_{B,0} \) satisfying
\[ B \left( 2Q(x) - \frac{2}{N} \Delta \sigma(x) \Delta \sigma(x)^\top \right) B^\top \left( \frac{f_{B,0}}{f_{B,1}} \right) = 0. \]  
(4.11)
Let us denote the submatrix of $B$ composed of the first $d$ rows by $b_0^\top$. Then combining (4.6) and (4.11) leads to

$$0 = b_0^\top \left(2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Sigma^\sigma(x)^\top \right) \left(\frac{f_0}{f_1} \right) = b_0^\top (2Q(x)) \left(\frac{f_0}{f_1} \right).$$

(4.12)

Recall that there exists a nonsingular $d \times d$ matrix $E_d$ such that

$$b_0^\top = E_d p_0^\top := E_d \begin{pmatrix} \sigma_{xy_1} & \cdots & \sigma_{xy_m} \end{pmatrix}.$$

Then we derive from (4.12) that

$$p_0^\top 2Q(x) \left(\frac{f_0}{f_1} \right) = 0.$$

(4.13)

We point out that the matrix $p_0^\top 2Q(x)$ is a zero matrix when the local connection structure $B_2^{i nc}(x)$ is balanced (see (A.16) and (A.17) in the Appendix). In that case, the equation (4.13) holds true for any $f_0, f_1$.

Based the above two observations, we introduce the following linear maps.

**Definition 4.2.** Let $x$ be a given vertex in a connection graph $(G, \sigma)$. We define a linear map $\Psi_x : \mathbb{K}^{(m+1)d} \to \mathbb{K}^{(m+n+1)d}$ via

$$\Psi_x(w) = \begin{pmatrix} w \cr -2\Gamma_2'(x) S_2 S_2^{-1} 2\Gamma_2'(x) S_2 B_1 w \end{pmatrix}$$

(4.14)

for any $w \in \mathbb{K}^{(m+1)d}$.

**Definition 4.3.** Let $x$ be a given vertex in a connection graph $(G, \sigma)$. Any two functions $f, f' : B_1(x) \to \mathbb{K}^d$ are defined to be equivalent if

$$\begin{pmatrix} \sigma_{xy_1} f(y_1) - f(x) \\ \vdots \\ \sigma_{xy_m} f(y_m) - f(x) \end{pmatrix} = \begin{pmatrix} \sigma_{xy_1} f'(y_1) - f'(x) \\ \vdots \\ \sigma_{xy_m} f'(y_m) - f'(x) \end{pmatrix}.$$

We denote by $[f]$ the equivalent class of the function $f : B_1(x) \to \mathbb{K}^d$. We can define the following linear operations: For any equivalent classes $[f], [f']$ and $\lambda \in \mathbb{K}$,

$$[f] + [f'] = [f + f'], \quad \lambda[f] = [\lambda f].$$

We define the tangent space $T_x(G, \sigma)$ at the vertex $x$ as the linear space of the equivalent classes $[f]$ of any function $f : B_1(x) \to \mathbb{K}^d$.

Notice that $T_x(G, \sigma)$ is isomorphic to $\mathbb{K}^{md}$ as linear spaces. Indeed the isomorphic map is given by

$$\Pi : T_x(G, \sigma) \to \mathbb{K}^{md},$$

such that for any $f : B_1(x) \to \mathbb{K}^d$ we have

$$\Pi([f]) := \begin{pmatrix} \sigma_{xy_1} f(y_1) - f(x) \\ \vdots \\ \sigma_{xy_m} f(y_m) - f(x) \end{pmatrix}.$$
Proposition 4.4. Let $x$ be a given vertex in a connection graph $(G, \sigma)$. Then there exists a linear map

$$\phi_x : T_x(G, \sigma) \to \mathbb{K}^d,$$

such that

$$p_0^\top 2Q(x) \begin{pmatrix} \sigma_{xy_1}(v_1 + \phi_x(v)) \\ \vdots \\ \sigma_{xy_m}(v_m + \phi_x(v)) \end{pmatrix} = 0 \quad (4.15)$$

holds true for any $v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in T_x(G, \sigma)$ with each $v_i \in \mathbb{K}^d$.

Proof. We derive from (4.15) that

$$0 = p_0^\top 2Q(x) \begin{pmatrix} 0 \\ \sigma_{xy_1} \bar{v}_1 \\ \vdots \\ \sigma_{xy_m} \bar{v}_m \end{pmatrix} + p_0^\top 2Q(x)p_0\bar{\phi}_x(v). \quad (4.16)$$

By Proposition A.1, we have $a := p_0^\top 2Q(x)p_0 \succeq 0$. Hence, there exists a nonsingular matrix $P$ such that

$$a = P \text{diag}(1, \ldots, 1, 0, \ldots, 0)P^\top.$$

Then, we have

$$\text{diag}(1, \ldots, 1, 0, \ldots, 0)P^\top \bar{\phi}_x(v) = -P^{-1}p_0^\top 2Q(x) \begin{pmatrix} 0 \\ \sigma_{xy_1} \bar{v}_1 \\ \vdots \\ \sigma_{xy_m} \bar{v}_m \end{pmatrix}. \quad (4.17)$$

By (A.19), there exists $d \times d$ matrices $X_{ij,k}$ and $X_{ij}$, $i, j = 1, \ldots, m$, $k = 1, \ldots, n$ such that

$$a = \sum_{i,j,k} X_{ij,k}X_{ij,k}^\top + \sum_{i,j} X_{ij}X_{ij}^\top.$$

Observing from (A.16) and (A.17) that there exists $d \times d$ matrices $Y_{ij,k}$, $Y'_{ij,k}$ and $Y_{ij}$, $Y'_{ij}$ such that

$$p_0^\top 2Q(x) = \begin{pmatrix} (p_0^\top 2Q(x))_0 & (p_0^\top 2Q(x))_1 & \cdots & (p_0^\top 2Q(x))_m \end{pmatrix}$$

with

$$(p_0^\top 2Q(x))_i = \sum_{j,k} X_{ij,k}Y_{ij,k} + \sum_j X_{ij}Y_{ij}, \quad i = 1, \ldots, m$$

and

$$(p_0^\top 2Q(x))_0 = \sum_{i,j,k} X_{ij,k}Y'_{ij,k} + \sum_{i,j} X_{ij}Y'_{ij}.$$
Then we read from
\[
\text{diag}(1, \ldots, 1, 0, \ldots, 0) = P^{-1}a \left( P^\top \right)^{-1} = \sum_{ij,k} (P^{-1}X_{ij,k}) P^{-1}X_{ij,k}^\top + \sum_{ij} (P^{-1}X_{ij}) P^{-1}X_{ij}^\top
\]
that the rows of \( P^{-1}X_{ij,k} \) and \( P^{-1}X_{i,j} \) corresponding to the zero rows in \( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \) vanish. Hence, the corresponding rows in the matrix \( P^{-1}p_0^\top 2Q(x) \) also vanish. Therefore, the equation (4.17) has solutions and the solution of the following linear equation provides a particular solution of (4.17):
\[
\overline{P^\top \phi_x(v)} = - P^{-1}p_0^\top 2Q(x) \begin{pmatrix} 0 \\ \sigma_{xy_1}^{\top} v_1 \\ \vdots \\ \sigma_{xym}^{\top} v_m \end{pmatrix}.
\]
That is, we can set
\[
\overline{\phi_x(v)} = \left( P^\top \right)^{-1} p_0^\top 2Q(x) \begin{pmatrix} 0 \\ \sigma_{xy_1}^{\top} v_1 \\ \vdots \\ \sigma_{xym}^{\top} v_m \end{pmatrix},
\]
which is a linear map from \( T_x(G, \sigma) \) to \( \mathbb{K}^d \).

Next, we define discrete Bakry-Émery Ricci curvature tensor and the metric tensor.

**Definition 4.5** (Curvature tensor and metric tensor). Let \( x \) be a given vertex in a connection graph \( (G, \sigma) \). For any linear map \( \phi_x : T_x(G, \sigma) \to \mathbb{K}^d \) satisfying (4.15), we define a discrete Bakry-Émery Ricci curvature tensor \( \text{Ric}_x^\sigma \) as a \((0,2)\)-tensor, i.e., a sesquilinear map
\[
\text{Ric}_x^\sigma : T_x(G, \sigma) \times T_x(G, \sigma) \to \mathbb{K},
\]
by
\[
\text{Ric}_x^\sigma(v_1, v_2) := \Psi_x \circ \Phi_x(v_1)^\top 2\Gamma_0^\sigma(x) \Psi_x \circ \Phi_x(v_2), \tag{4.18}
\]
for any \( v_1, v_2 \in T_x(G, \sigma) \), where \( \Phi_x : T_x(G, \sigma) \to \mathbb{K}^{(m+1)d} \) is the linear map defined as
\[
\Phi_x(v) = \begin{pmatrix} \phi_x(v) \\ \sigma_{xy_1}^{-1} (v_1 + \phi_x(v)) \\ \vdots \\ \sigma_{xym}^{-1} (v_m + \phi_x(v)) \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \in T_x(G, \sigma). \tag{4.19}
\]

For \( N \in (0, \infty] \), we define the discrete \( N \)-Bakry-Émery Ricci curvature tensor by
\[
\text{Ric}_N^\sigma(v_1, v_2) := \text{Ric}_x^\sigma(v_1, v_2) - \frac{2}{N} \Phi_x(v_1)^\top \Delta^\sigma(x) \Delta^\sigma(x)^\top \Phi_x(v_2). \tag{4.20}
\]
We define a metric tensor \( g_x \) as a \((0,2)\)-tensor, i.e., a sesquilinear map
\[
g_x : T_x(G, \sigma) \times T_x(G, \sigma) \to \mathbb{K},
\]
by
\[
g_x(v_1, v_2) := \Phi_x(v_1)^\top 2\Gamma^\sigma(x) \Phi_x(v_2). \tag{4.21}
\]
for any \( v_1, v_2 \in T_x(G, \sigma) \).
Remark 4.6. Notice that the metric tensor is independent of the choices of the linear map $\phi_x$. Indeed, we have

$$g_x(v_1, v_2) = \sum_{i=1}^{m} p_{x_i} v_i v_i^\top,$$

for any $v_i = (v_{i1}, \ldots, v_{im})^\top \in T_x(G, \sigma)$, $i = 1, 2$. The metric tensor $g_x$ provides an inner product to the linear space $T_x(G, \sigma)$.

Proposition 4.7. Let $x$ be a given vertex in a connection graph $(G, \sigma)$ with a balanced local connection structure $B_2^{\text{inc}}(x)$. Then the definition of the tensor $\text{Ric}_N^x$ is independent of the choices of the linear map $\phi_x$ satisfying (4.15).

Proof. In the case of a balanced local connection structure $B_2^{\text{inc}}(x)$, the signature of any 3-cycle $x \sim y_i \sim y_j \sim x$ or any 4-cycle $x \sim y_i \sim z_k \sim y_j \sim x$ is (the conjugacy class of) $I_d$.

By Appendix (A.18) and (A.16)-(A.17), the coefficient matrices in (4.16) are

$$a = 0, \quad (p_0 \Delta_2^\sigma(t)) = (0 \ 0 \cdots 0).$$

And the equation (4.16) becomes

$$0_d \phi_x(v) = 0_d.$$

Therefore, the vector $\phi_x(v)$ can be arbitrarily chosen.

For any vector $v \in \mathbb{K}^{md}$, we have by (4.19) that $\Phi_x(v) = u_1 + u_2$ with

$$u_1 := \begin{pmatrix} 0 \\ \sigma^{-1}_{xy_1} v_1 \\ \vdots \\ \sigma^{-1}_{xy_m} v_m \end{pmatrix}, \quad u_2 := \begin{pmatrix} \phi_x(v) \\ \sigma^{-1}_{xy_1} \phi_x(v) \\ \vdots \\ \sigma^{-1}_{xy_1} \phi_x(v) \end{pmatrix} = p_0 \phi_x(v).$$

By (2.5), we observe that

$$u_2^\top \Delta^\sigma(x) = \phi_x(v)^\top p_0 \Delta^\sigma(x) = \mathbf{0}_{1 \times d}.$$ 

Since $(p_0 \Delta_2^\sigma(t)) = 0_{d \times md}$, we have that

$$u_2^\top 2Q(x) = \phi_x(v)^\top (p_0 \Delta_2^\sigma(t)) = \mathbf{0}_{1 \times md}.$$ 

Therefore, we derive by definition that

$$\text{Ric}_N^x(v, v) = \Phi_x(v)^\top (2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\top ) \Phi_x(v)$$

$$= (u_1 + u_2)^\top (2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\top ) (u_1 + u_2)$$

$$= u_1^\top (2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\top ) u_1.$$ 

In conclusion, the definition of the tensor $\text{Ric}_N^x$ is independent of the choices of $\phi_x$ in the locally balanced case. \hfill \Box

Proposition 4.8. Let $x$ be a given vertex in a $U(1)$-connection graph $(G, \sigma)$ with an unbalanced local connection structure $B_2^{\text{inc}}(x)$. Then the linear map $\phi_x$ satisfying (4.15) is unique.

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Proof. For the case of $U(1)$-connection, the signature of each edge is a complex number of norm one, i.e. $e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Recall from (A.19) that the coefficient $a := p_j^\dagger 2Q(x)\Phi_0$ of $\phi_\sigma(v)$ in (4.10) has the form

$$a = \sum_{i,j,k} X_{ij,k} X_{ij,k} + \sum_{i,j} X_{ij} X_{ij},$$

where $X_{ij,k} = C_{ij,k} (1 - e^{i\theta_{ij,k}})$ and $X_{ij} = C_{ij} (1 - e^{i\theta_{ij}}), \forall i, j = 1, \ldots, m, \forall k = 1, \ldots, n$ and $C_{ij,k}, C_{ij}$ are positive coefficients.

If the local connection structure of the vertex $x$ is unbalanced, then there exists some $\{ij, k\}$ or $\{ij\}$ such that

$$e^{i\theta_{ij,k}} = \cos(\theta_{ij,k}) + i \sin(\theta_{ij,k}) \neq 1, \quad \text{or} \quad e^{i\theta_{ij}} = \cos(\theta_{ij}) + i \sin(\theta_{ij}) \neq 1,$$

and

$$1 - e^{i\theta_{ij,k}} = (1 - \cos(\theta_{ij,k})) - i \sin(\theta_{ij,k}) \neq 0, \quad \text{or} \quad 1 - e^{i\theta_{ij}} = (1 - \cos(\theta_{ij})) - i \sin(\theta_{ij}) \neq 0.$$  

We derive that either $\Re(X_{ij,k}) > 0$ and $X_{ij,k} X_{ij,k} > 0$, or $\Re(X_{ij}) > 0$ and $X_{ij} X_{ij} > 0$. Hence, $a > 0$ and $\phi_\sigma(v)$ can be uniquely determined by the equation (4.10). In other words, the linear map $\phi_\sigma$ is unique in the case of $U(1)$-connection.

Theorem 4.9. Let $x$ be a vertex in a connection graph $(G, \sigma)$. Then we have for any $N \in (0, \infty)$$$

$$K_{G, \sigma, x}(N) = \inf_{v \in T_x(G, \sigma) \setminus \{0\}} \frac{\text{Ric}_N^\sigma(v, v)}{g_x(v, v)},$$

(4.22)

Proof. Recall from Lemma 6.1 that

$$K_{G, \sigma, x}(N) = \inf_{f : V \to \mathbb{K}^d} \frac{\Gamma_2^\sigma(f(x)) - (1/N)\Delta^\sigma f(x)^2}{\Gamma^\sigma(f)(x)},$$

By the calculations in the alternative proof of Theorem 3.2 it is enough to take the infimum above over a sub-class of functions:

$$\{f : V \to \mathbb{K}^d : \Gamma^\sigma(f)(x) \neq 0, \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = \Psi_x \circ \Phi_x(f_1)\},$$

where we adopt the notations $f_0, f_1, f_2$ given in (1.2). By Definition 4.5 we obtain

$$K_{G, \sigma, x}(N) = \inf_{f_1 \in \mathbb{K}^{md}} \frac{\text{Ric}_N^\sigma(f_1, f_1)}{g_x(f_1, f_1)}.$$  

Since $g_x(f_1, f_1) \neq 0$ if and only if $f_1 \neq 0$. That is, we arrive at (4.22).  

Notice that $\mathbb{K}^{md} = \{(v^1, \ldots, v^{md})^\top : v_i \in \mathbb{K}, i = 1, \ldots, md\}$ is the space of $md$-tuples. It can also be considered as the linear space spanned by the basis $e_1, \ldots, e_{md}$, where $e_i = (0, \ldots, \epsilon_{i-1}, 1, \ldots, 0)$ with the only 1 appear in the $i$-th position. Given any orthonormal basis of the inner product space $(\mathbb{K}^{md}, g_x(\cdot, \cdot))$, we have a unique matrix corresponding to the curvature tensor $\text{Ric}_N^\sigma$. 

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Proposition 4.10. For any nonsingular matrix $B$ satisfying (3.3), there exists an orthonormal basis $e_1, \ldots, e_{md}$ of the inner product space $(\mathbb{K}^{md}, g_x(\cdot, \cdot))$, such that for any $v = \sum_{i=1}^{md} v^i e_i = \sum_{i=1}^{md} v^B_i e_i$,

$$\text{Ric}_N^x(v, v) = (v^B_B \cdots v^B_{md}) A_N(G, \sigma, x, B) \begin{pmatrix} v^B_B \\ \vdots \\ v^B_{md} \end{pmatrix}. \quad (4.23)$$

Proof. For any $v = \sum_{i=1}^{md} v^i e_i$ and the given $B$, we define

$$v_B := \begin{pmatrix} v^B_1 \\ \vdots \\ v^B_{md} \end{pmatrix} = \begin{pmatrix} 0_{md \times d} & I_{md} \end{pmatrix} (B^{-1})^\top \Phi_x(v), \quad (4.24)$$

where $\Phi_x$ is the linear map given in (4.19). Then we check directly that

$$g_x(v, v) = \Phi_x(v)^\top B^{-1} B 2I^\sigma_x(x) B^\top B^{-1}^\top \Phi_x(v)$$

$$= \Phi_x(v)^\top B^{-1} \begin{pmatrix} 0_{d \times d} & I_{md} \end{pmatrix} B^{-1}^\top \Phi_x(v)$$

$$= v_B^\top v_B. \quad (4.25)$$

Let $\xi_B : \mathbb{K}^{md} \to \mathbb{K}^{md}$ be the linear map defined by $\xi_B(v) := v_B$ for any $v \in \mathbb{K}^{md}$. If $\xi_B(v) = v_B = 0$, we derive from (4.25) that $g_x(v, v) = 0$, and hence, $v = 0$ since $g_x$ is an inner product. That is, the linear map $\xi_B$ is injective. Since $\xi_B$ is a linear map between two linear spaces of the same dimension, it is surjective. Thus, there exists a nonsingular matrix $M_B$ such that

$$\begin{pmatrix} v^B_1 \\ \vdots \\ v^B_{md} \end{pmatrix} = M_B \begin{pmatrix} v^1 \\ \vdots \\ v^{md} \end{pmatrix}. \quad (4.26)$$

Then

$$\begin{pmatrix} e_1 \\ \vdots \\ e_{md} \end{pmatrix} := (M_B^{-1})^\top \begin{pmatrix} e_1 \\ \vdots \\ e_{md} \end{pmatrix}$$

is a basis of $\mathbb{K}^{md}$ such that $v = \sum_{i=1}^{md} v^i e_i = \sum_{i=1}^{md} v^B_i e_i$. Due to (4.25), the basis $e_1, \ldots, e_{md}$ is an orthonormal basis of the inner product space $(\mathbb{K}^{md}, g_x(\cdot, \cdot))$.

Finally we check by definition

$$\text{Ric}_N^x(v, v) = \Psi_x \circ \Phi_x(v)^\top (2\Gamma^\sigma_x(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\top) \Psi_x \circ \Phi_x(v)$$

$$= \Phi_x(v)^\top (2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\top) \Phi_x(v)$$

$$= \Phi_x(v)^\top B^{-1} B (2Q(x) - \frac{2}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\top) B^\top B^{-1}^\top \Phi_x(v)$$

$$= v_B^\top A_N(G, \sigma, x, B) v_B.$$ 

That is, (4.23) holds true. \qed

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Remark 4.11. In the identity (4.23), the tensor $\text{Ric}^N_\phi(v, v)$ and the linear map $\xi_B$ sending $v$ to $v_B$ depend on the choices of $\phi_x$, and the matrix $A_N(G, \sigma, x, B)$ is independent of the choices of $\phi_x$.

In view of Proposition 4.10, we give an alternative proof of Proposition 3.5.

An alternative proof of Proposition 3.5. For any two nonsingular matrices $B_1$ and $B_2$ satisfying (3.5), the curvature matrices $A_N(G, \sigma, x, B_1)$ and $A_N(G, \sigma, x, B_2)$ are the matrices of the discrete Bakry-Émery Ricci curvature tensor with respect to different orthonormal basis of the inner product space $(\mathbb{K}^{md}, g_x(\cdot, \cdot))$. Hence, they are unitarily equivalent.

Next, we derive the unitary matrix explicitly. For any $v \in \mathbb{K}^{md}$, let $v_{B_1}$ and $v_{B_2}$ be defined as in (4.24) with respect to $B_1$ and $B_2$. Then, we calculate

\[
v_{B_2} = (0_{md \times d} \quad I_{md}) (B_2^{-1})^\top \Phi_x(v) = (0_{md \times d} \quad I_{md}) (B_1 B_2^{-1})^\top (B_1^{-1})^\top \Phi_x(v) = (B_1 B_2^{-1})^\top v_{B_1}.
\]

In the third equality above, we apply Lemma 3.8. Proposition 4.10 tells that

\[
\text{Ric}^N_\phi(v, v) = v_{B_1}^\top A_N(G, \sigma, x, B_1) \overline{v_{B_1}} = v_{B_2}^\top A_N(G, \sigma, x, B_2) \overline{v_{B_2}}.
\]

This leads to $A_N(G, \sigma, x, B_1) = (B_1 B_2^{-1})_1 A_N(G, \sigma, x, B_2)(B_1 B_2^{-1})_1^\top$. \hfill \Box

Remark 4.12. From the above proof, we see that the matrix $(B_1 B_2^{-1})_1^{-1}$ is actually the transformation matrix between two orthonormal basis of the inner product space $(\mathbb{K}^{md}, g_x(\cdot, \cdot))$. Therefore, it is quite natural to see $(B_1 B_2^{-1})_1$ is unitary (Lemma 3.7) and the transitive property in Corollary 3.9 holds true.

5 Properties of curvature functions

In this section, we study the properties of curvature functions with the help of curvature matrices.

Theorem 5.1. Let $(G, \sigma)$ be a connection graph and $x$ be a given vertex. Then the curvature function $K_{G, \sigma, x} : (0, \infty) \to \mathbb{R}$ satisfying the following properties:

(i) It is continuous, monotone non-decreasing and concave with

\[
\lim_{N \to 0} K_{G, \sigma, x}(N) = -\infty \quad \text{and} \quad \lim_{N \to \infty} K_{G, \sigma, x}(N) < \infty.
\]

(ii) If there exist $0 < N_1 < N_2 \leq \infty$ such that $K_{G, \sigma, x}(N_1) = K_{G, \sigma, x}(N_2)$, then the function $K_{G, \sigma, x}$ is constant on the interval $[N_1, \infty]$.

(iii) If the multiplicity of $\lambda_{\text{min}}(A_{N_0}(G, \sigma, x))$ is larger than $d$ for some $N_0 \in (0, \infty)$, then the function $K_{G, \sigma, x}$ is constant on the interval $[N_0, \infty]$. 

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Proof. The continuity of the curvature function is due to the fact that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. (see the proof of Proposition 3.1.)

For two Hermitian matrices $M_1$ and $M_2$, we recall that

$$
\lambda_{\min}(M_1 + M_2) = \inf_{v \neq 0} \frac{v^\top (M_1 + M_2) v}{v^\top v} \geq \inf_{v \neq 0} \frac{v^\top M_1 v}{v^\top v} + \inf_{v \neq 0} \frac{v^\top M_2 v}{v^\top v} = \lambda_{\min}(M_1) + \lambda_{\min}(M_2), \quad (5.1)
$$

where the inequality holds with equality if and only if $M_1$ and $M_2$ share an eigenvector corresponding to $\lambda_{\min}(M_1)$ and $\lambda_{\min}(M_2)$.

By Proposition 3.5 and Theorem 3.2, we consider the minimal eigenvalue of the matrix $A_N(G, \sigma, x, B_0)$ with respect to the canonical choice $B_0$ given in (3.33). Observe the following facts from (3.34) that $v_0(B_0)v_0(B_0)^\top = \sum_y p_{xy}v_0(B_0)$.

Therefore, the matrix $v_0(B_0)v_0(B_0)^\top$ has eigenvalues $\sum_y p_{xy}$ with multiplicity $d$ and eigenvalues $0$ with multiplicity $(m-1)d$.

The monotonicity of the curvature function follows from (5.1) as below. For any $0 < N_1 < N_2 \leq \infty$,

$$
\lambda_{\min}(A_{N_2}) \geq \lambda_{\min}(A_{N_1}) + \lambda_{\min}\left(\left(\frac{2}{N_1} - \frac{2}{N_2}\right) v_0(B_0)v_0(B_0)^\top\right) = \lambda_{\min}(A_{N_1}), \quad (5.2)
$$

where we write $A_{N_i} = A_{N_i}(G, \sigma, x, B_0)$ for short.

The concavity of the curvature function can also be derived from (5.1). Indeed, for any $0 < N_1 < N_2 \leq \infty$ and $\gamma \in (0, 1)$, we have

$$
\lambda_{\min}(A_{\gamma N_1 + (1-\gamma)N_2}) = \lambda_{\min}\left(\gamma A_{N_1} + (1-\gamma)A_{N_2} + \left(\frac{2\gamma}{N_1} + \frac{2(1-\gamma)}{N_2} - \frac{2}{\gamma N_1 + (1-\gamma)N_2}\right) v_0(B_0)v_0(B_0)^\top\right) \\
\geq \gamma \lambda_{\min}(A_{N_1}) + (1-\gamma) \lambda_{\min}(A_{N_2}).
$$

We further calculate that

$$
\lim_{N \to \infty} K_{G, \sigma, x}(N) = K_{G, \sigma, x}(\infty) = \lambda_{\min}(A_\infty) < \infty,
$$

and

$$
\lim_{N \to 0} K_{G, \sigma, x}(N) = \lim_{N \to 0} \lambda_{\min}(A_N) \leq \lim_{N \to 0} \left(\lambda_{\max}(A_\infty) + \lambda_{\min}\left(-\frac{2}{N} v_0(B_0)v_0(B_0)^\top\right)\right) \\
= \lambda_{\max}(A_\infty) - \lim_{N \to 0} \frac{2}{N} \sum_y p_{xy} = -\infty.
$$

This completes the proof of the property (i).
For the property (ii), the assumption that $\mathcal{K}_{G,\sigma,x}(N_1) = \mathcal{K}_{G,\sigma,x}(N_2)$ forces the inequality holds true with equality in (5.2). By (5.1), this means that the matrices $A_{N_1}$ and $v_0(B_0)v_0(B_0)^\top$ share an eigenvector corresponding to their minimal eigenvalues. This further implies that $\lambda_{\min}(A_N) = \lambda_{\min}(A_{N_1})$ for any $N \in [N_1, \infty]$.

Next, we show the property (iii). Let us denote by $v_0(B_0) = (v_0 \cdots v_{d})$ with each $v_i \in \mathbb{K}^{m \times d}$. By definition, the vectors $v_i$, $i = 1, \ldots, d$ are pairwise orthogonal.

We claim that there exists a nonzero vector $w$ in the minimal eigenspace $E_{\min}(A_{N_0})$ which is orthogonal to the space span$\{v_0, \ldots, v_d\}$. Indeed, such a vector $w$ can be constructed as follows. Pick $d + 1$ linearly independent vectors $\xi_j$, $j = 1, \ldots, d + 1$ in the space $E_{\min}(A_{N_0})$. Then there exist numbers $a_j \in \mathbb{K}$, $i = 1, \ldots, d$, $j = 1, \ldots, d + 1$ and vectors $w_j$, $j = 1, \ldots, d + 1$ orthogonal to the space span$\{v_0, \ldots, v_d\}$ such that

$$\xi_j = \sum_{i=1}^{d} a_j^i v_i + w_j, \quad j = 1, \ldots, d + 1.$$  

If there exists a $1 \leq j_0 \leq d + 1$ with $\sum_{i=1}^{d} a_j^i v_i = 0$, we can set $w = w_{j_0}$. Otherwise, the vectors $\sum_{i=1}^{d} a_j^i v_i$, $j = 1, \ldots, d + 1$ are $d + 1$ nonzero vectors in the $d$-dimensional space span$\{v_0, \ldots, v_d\}$, and hence are linearly dependent. There exist $c_1, \ldots, c_{d+1} \in \mathbb{K}$ such that $\sum_{j=1}^{d+1} c_j \sum_{i=1}^{d} a_j^i v_i = 0$. Then we can set $w = \sum_{j=1}^{d+1} c_j \xi_j = \sum_{j=1}^{d+1} c_j w_j$.

Notice that such a vector $w$ is an eigenvector corresponding to both $\lambda_{\min}(A_{N_0})$ and $\lambda_{\min}(v_0(B_0)v_0(B_0)^\top)$. Therefore, the curvature $\mathcal{K}_{G,\sigma,x}(N) = \lambda_{\min}(A_N)$ is constant for $N \in [N_0, \infty]$ by (5.1).

**Remark 5.2.** The above proof follows closely the method in [14, Section 3]. In fact, for the case of $d = 1$, i.e., for connection graphs with $O(1)$ or $U(1)$ connections, one can employ the same proof for [14, Theorem 1.3] to show the following property holds true: There exists a unique threshold $N_1 \in (0, \infty]$ such that $\mathcal{K}_{G,\sigma,x}$ is analytic, **strictly** monotone increasing and **strictly** concave on $(0, N_1)$ and constant on $[N_1, \infty]$.

For connection graphs with $d$-dimensional connections, $d \geq 2$, the situation is still very mysterious:

1. Is the multiplicity condition in Theorem 5.1 (iii) necessary?
2. Is the multiplicity of $\lambda_{\min}(A_N)$ monotonically increasing w.r.t. $N$?
3. Does there exist a threshold $N_1 \in (0, \infty]$ such that the curvature function $\mathcal{K}_{G,\sigma,x}$ is analytic on $(0, N_1)$ and constant on $[N_1, \infty]$?

### 6 Curvature matrices of Cartesian products

In this section, we study the curvature matrices of Cartesian products of connection graphs.

**Definition 6.1.** Let $(G, \sigma)$ be a connection graph. The signature group $\Sigma$ is defined to be the group generated by the elements of the set $\{\sigma_{xy} : (x, y) \in E^{\text{or}}\}$.  

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**Definition 6.2.** Let \((G, \sigma)\) and \((G', \sigma')\) be two connection graphs with \(G = (V, w, \mu), G' = (V', w', \mu')\) and \(\sigma : E_{G}^{\text{or}} \to U(\mathbb{d}), \sigma' : (E')_{G'}^{\text{or}} \to U(\mathbb{d})\). Let \(\alpha, \beta\) be two positive real numbers. We define the Cartesian product of this two connection graphs as a connection graph

\[
(G, \sigma) \times_{\alpha, \beta} (G', \sigma') := (G \times_{\alpha, \beta} G', \sigma^\times).
\]

Here, \(G \times_{\alpha, \beta} G'\) is the Cartesian product of the two graphs \(G\) and \(G'\) assigned with the following edge weights and vertex measure: For \(x, y \in V\), and \(x', y' \in V'\),

\[
w_{(x,x')(y,y')} = \alpha w_{xy} \mu_{x'},
\]

\[
w_{(x,x')(x,y')} = \beta w'_{x'y'} \mu_x,
\]

\[
\mu(x,x') = \mu_x \mu_{x'}.
\]

The connection \(\sigma^\times\) of each oriented edge in the Cartesian product of \(G\) and \(G'\) is given as below:

\[
\sigma^\times_{(x,x')(y,y')} = \sigma_{xy},
\]

\[
\sigma^\times_{(x,x')(x,y')} = \sigma'_{x'y'}.
\]

Here we use the same notation \(w, \mu\) for the product graph and the graph \(G\). The associated graphs can be determined by the vertices input.

**Theorem 6.3.** Let \((G, \sigma)\) and \((G', \sigma')\) be two \(U(\mathbb{d})\)-connection graphs. Suppose that the two signature groups \(\Sigma\) and \(\Sigma'\) commute. For any \(x \in V\) and \(x' \in V'\), the curvature matrices with respect to the canonical choice \(B_0\) of matrix \(B\) in \((6.33)\) satisfy

\[
A^\infty(x,x') = \begin{pmatrix} \alpha A^\infty(x) & \beta A^\infty(x') \end{pmatrix} + R(x,x'),
\]

where

\[
R(x,x') := \begin{pmatrix} \alpha^3 \omega (\alpha^2 a - (\alpha^2 a + \beta^2 a')^\dagger) & -\alpha^3 \beta^2 \omega (\alpha^2 a + \beta^2 a')^\dagger (\omega')^\dagger \\ -\alpha^3 \beta^2 \omega (\alpha^2 a + \beta^2 a')^\dagger \omega^\dagger & \beta^3 \omega ((\beta^2 a')^\dagger - (\alpha^2 a + \beta^2 a')^\dagger) (\omega')^\dagger \end{pmatrix}
\]

is positive semidefinite. Here, \(a, a', \omega, \omega'\) are given by \((6.22), (6.23)\), and we use the notation \(A^\infty(x) = A^\infty(G, \sigma, x, B_0)\) for short. In particular, if one of the two local connection structures \(B_2^{\text{inc}}(x)\) is balanced, we have \(R(x,x') = 0\).

Before the proof of Theorem 6.3 we first establish several lemmas.

**Lemma 6.4.** Consider two connection graphs \((G, \sigma)\) and \((G', \sigma')\). Suppose that the two signature groups \(\Sigma\) and \(\Sigma'\) commute. Let \(\tau : E_{G}^{\text{or}} \to \Sigma\) and \(\tau' : (E')_{G'}^{\text{or}} \to \Sigma'\) be two switching functions of \((G, \sigma)\) and \((G', \sigma')\), respectively. Then

\[
(G, \sigma^\tau) \times_{\alpha, \beta} (G', \sigma'^{\tau'}) = (G \times_{\alpha, \beta} G', (\sigma^\times)^{\tau^\times}),
\]

where \(\tau^\times(x,x') := \tau(x)\tau'(x')\) for any \(x \in V\) and \(x' \in V'\). That is, the Cartesian product \((G, \sigma)\) and \((G', \sigma')\) is switching equivalent to the Cartesian product of \((G, \sigma^\tau)\) and \((G', \sigma'^{\tau'})\).
Proof. For any \(x, y \in V\) with \(\{x, y\} \in E\) and any \(x' \in V'\), we calculate that
\[
\tau^x(x, x')\sigma^x_{(x,x')(y,x')}\tau^x(y, x')^{-1} = \tau(x)\tau'(x')\sigma_{xy}^x\tau'(x')^{-1}\sigma_{xy} \tau(y)^{-1} = \tau(x)\sigma_{xy} \tau(y)^{-1},
\]
where we use the community of \(\Sigma\) and \(\Sigma'\) in the second equality. Similarly, we have, for any \(x \in V\) and any \(x', y' \in V'\) with \(\{x', y'\} \in E'\), that
\[
\tau^x(x, x')\sigma^x_{(x,x')(x,y')}\tau^x(x, y')^{-1} = \tau(x)\tau'(x')\sigma_{x'y'}^x\tau'(x')^{-1}\tau(x)^{-1} = \tau'(x')\sigma_{x'y'} \tau'(y')^{-1}.
\]
This completes the proof. \(\square\)

**Lemma 6.5.** Let \((G, \sigma)\) and \((G', \sigma')\) be two connection graphs. Consider two vertices \(x \in V\) and \(x' \in V'\). Suppose that
\[
\sigma_{xy} = I_d, \quad \sigma_{x'y'} = I_d, \quad \text{for any } y \in S_1(x), y' \in S_1(x').
\]
Then we have the following direct sum decomposition for the \(Q(x, x') := Q(\Gamma^x_2(x, x'))\)
\[
Q(x, x') = \begin{pmatrix}
\alpha^2 Q(x)_{x,x} + \beta^2 Q(x')_{x',x'} & \alpha^2 Q(x)_{x,S_1(x)} & \beta^2 Q(x')_{x',S_1(x')} \\
\alpha^2 Q(x)_{S_1(x),x} & \alpha^2 Q(x)_{S_1(x),x} & 0 \\
\beta^2 Q(x')_{S_1(x'),x'} & 0 & \beta^2 Q(x')_{S_1(x'),S_1(x')} \end{pmatrix}.
\]
(6.4)

In particular, we have
\[
Q(x, x')_1 = \alpha^2 Q(x)_1 \oplus \beta^2 Q(x')_1.
\]

Proof. It is obtained by a straightforward computation using definitions. The details are presented in the Appendix. \(\square\)

**Corollary 6.6.** Let \((G, \sigma)\) and \((G', \sigma')\) be two connection graphs. If the two signature groups \(\Sigma\) and \(\Sigma'\) commute, then the matrix \(Q(x, x')\) satisfies (6.4) for any \(x \in V\) and \(x' \in V'\).

**Proof.** Let \(\tau, \tau'\) be two switching functions such that the connections \(\sigma^\tau\) and \(\sigma'^{\tau'}\) satisfy (A.21). Applying Lemma 6.3 and Lemma 6.5 yields the decomposition (6.4) for the matrix \(Q(\Gamma^x_2(\sigma^\tau)(x, x'))\). Due to (A.21), we have
\[
Q(\Gamma^x_2(\sigma^\tau)(x, x')) = (\tau^x_{\Gamma^x_2(x,x')})^\top Q(\Gamma^x_2(\sigma^\tau)(x, x')) \tau^x_{\Gamma^x_2(x,x')}.
\]
(6.5)

By community of \(\Sigma\) and \(\Sigma'\) and applying (A.21) again, we can derive the decomposition (6.4) for the matrix \(Q(x, x') = Q(\Gamma^x_2(\sigma^\tau)(x, x'))\). \(\square\)

**Remark 6.7.** If both the local connection structures \(B_2^{inc}(x)\) and \(B_2^{inc}(x')\) are balanced, then the first \(d\) rows and \(d\) columns of the matrix \(Q(x, x')\) vanish. That is, we have
\[
\begin{pmatrix}
0_d & 0 & 0 \\
0 & \alpha^2 Q(x)_1 & 0 \\
0 & 0 & \beta^2 Q(x')_1 \\
\end{pmatrix}.
\]

We need further the following lemma.
Lemma 6.8. For two positive semidefinite Hermitian matrices $A$ and $B$, there exists a nonsingular matrix $P$ such that $PA^TP$ and $PB^TP$ are both diagonal matrices.

Proof. Before the proof, we prepare two facts about positive semidefinite matrices.

- Every diagonal entry of a positive semidefinite matrix is non-negative.
- If a diagonal entry is 0, then all entries in the same row and same column are 0.

For the first fact, since any principal minor of a positive semidefinite matrix is non-negative, especially, every diagonal entry is a principal minor of size $1 \times 1$, hence is non-negative.

For the second one, since a positive semidefinite matrix $D$ can be written as

$$D = XX^T$$

for some square matrix $X$. We denote $X$ as

$$X = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{pmatrix},$$

where each $x_i^T$ is a row vector. Then the identity (6.6) can be rewritten as

$$\begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} = D.$$

Notice that the diagonal entries are exactly

$$\begin{pmatrix} x_1^T x_1 \\ x_2^T x_2 \\ \vdots \\ x_n^T x_n \end{pmatrix} = \begin{pmatrix} d_{11} \\ d_{22} \\ \vdots \\ d_{nn} \end{pmatrix}.$$

If these is a zero diagonal entry, without loss of generality, assume that $d_{ii} = 0$, i.e. $x_i^T x_i = 0$, we can get $x_i^T = (0 \ 0 \ \cdots \ 0)$ and all entries in the $i$-th row and $i$-th column of $D = XX^T$ are zero.

Now we prove this lemma. Since the matrices $A$ and $B$ are both positive semidefinite, then $A + B$ is positive semidefinite. There exists a nonsingular matrix $P_1$ such that $P_1(A + B)^{\top}P_1 = \text{diag}(I_r, 0)$, here $r$ is the rank of $A + B$. Then

$$P_1A^{\top}P_1 = \text{diag}(S_1, 0), \quad P_1B^{\top}P_1 = \text{diag}(S_2, 0),$$

where $S_1$ and $S_2$ are positive semidefinite and satisfy $S_1 + S_2 = I_r$. Hence there exists a unitary matrix $T$ such that the matrix $TS_1^{\top}$ is a diagonal matrix. Besides, the matrix $TS_2^{\top} = I_r - TS_1^{\top}$ is also a diagonal matrix. Let $P = \text{diag}(T, I_{n-r})P_1$, we have that $PA^{\top}$ and $PB^{\top}$ are both diagonal matrices. \qed
Proof of Theorem 6.3. First observe that for any \( x, y \in V \) and any \( x', y' \in V' \)
\[
p(x, x')(y, y') = \alpha p_{xy}, \quad \text{and} \quad p(x, x')(x', y') = \beta p_{x'y'}.
\]
By Lemma 6.8, we have the canonical choice of matrix \( B \) at \( (x, x') \) in the Cartesian product is
\[
B_0 = \begin{pmatrix}
I_d & \sigma_{xy} & \cdots & \sigma_{xy_m} & \sigma'_{x'y_i} & \cdots & \sigma'_{x'y_{m'}} \\
\frac{1}{\sqrt{\alpha p_{xy}}} I_d & I_d & \cdots & I_d & \frac{1}{\sqrt{\beta p_{x'y_i}}} I_d & \cdots & \frac{1}{\sqrt{\beta p_{x'y_{m'}}}} I_d
\end{pmatrix}.
\]
Using Corollary 6.9 we derive that
\[
B_0 B_0^\top \begin{pmatrix}
\alpha^2 a + \beta^2 a' \\
\alpha^2 \omega^\top \\
\beta^2 \omega'^\top
\end{pmatrix}
= \begin{pmatrix}
\alpha B_0 2Q(x') B_0^\top \\
\alpha (B_0 2Q(x) B_0^\top)\begin{pmatrix}I_d \end{pmatrix}
\end{pmatrix}.
\]
Here, we use the same notations \( B_0, Q \) in \( B_0 2Q(x) B_0^\top \), \( B_0 2Q(x') B_0^\top \), \( B_0 2Q(x, x') B_0^\top \)
for the graphs \( (G, \sigma) \), \( (G', \sigma') \) and their Cartesian product graph. Those matrices are
distinguished by the vertex input and are determined in the corresponding graphs.

By definition, we have the curvature matrix
\[
A_\infty(x, x') = \begin{pmatrix}
\alpha (B_0 2Q(x) B_0^\top)\begin{pmatrix}I_d \end{pmatrix} \\
\beta (B_0 2Q(x') B_0^\top)\begin{pmatrix}I_d \end{pmatrix}
\end{pmatrix}.
\]
Next, we show that \( R(x, x') \) is positive semidefinite. If one of the two local connection
dependences \( B_2^{\text{inc}}(x) \) and \( B_2^{\text{inc}}(x') \) is balanced, we have one of \( \omega \) and \( \omega' \) vanishes
(Proposition A.1 in the Appendix), and hence \( R(x, x') = 0 \). In the following, we assume
that both the two local connection structures \( B_2^{\text{inc}}(x) \) and \( B_2^{\text{inc}}(x') \) are unbalanced. In this case, both \( a \) and \( a' \)
in \( \text{(6.1)} \) are positive semidefinite (Proposition A.1 in the Appendix).
By Lemma 6.8, there exists a nonsingular matrix \( P \) such that
\[
P a P^\top = \text{diag}(\mu_1, \ldots, \mu_d), \quad \text{for some} \ \mu_i \in \mathbb{R}_{\geq 0}, \ i = 1, \ldots, d.
\]
and 

\[ P \alpha^T P^T = \text{diag}(\lambda_1, \ldots, \lambda_d), \text{ for some } \lambda_i \in \mathbb{R}_{\geq 0}, i = 1, \ldots, d. \]

Then, we obtain

\[ a^\dagger = P^T \text{diag}(\mu_1^\dagger, \ldots, \mu_d^\dagger)P, \quad (a')^\dagger = P^T \text{diag}(\lambda_1^\dagger, \ldots, \lambda_d^\dagger)P \]  \hspace{1cm} (6.7)

and

\[ (\alpha^2 a + \beta a')^\dagger = P^T \text{diag}((\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger, \ldots, (\alpha^2 \mu_d + \beta^2 \lambda_d)^\dagger)P, \]  \hspace{1cm} (6.8)

where we use the notation \(^\dagger\) for

\[ \lambda^\dagger = \begin{cases} 
\frac{1}{\lambda}, & \text{if } \lambda \neq 0, \\
0, & \text{otherwise}
\end{cases} \]

Let us write \( R = R(x, x') \) for short. Notice that we have

\[ R = \begin{pmatrix} \beta^3 \omega & \beta^3 \omega^T \\
\beta^3 \omega & \beta^3 \omega^T \end{pmatrix} \begin{pmatrix} (\alpha^2 a)^\dagger - (\alpha^2 a + \beta^2 a')^\dagger & -(\alpha^2 a + \beta^2 a')^\dagger \\
-(\alpha^2 a + \beta^2 a')^\dagger & (\beta^2 a')^\dagger - (\alpha^2 a + \beta^2 a')^\dagger \end{pmatrix} \begin{pmatrix} \alpha^2 \omega^T \\
\beta^3 \omega^T \end{pmatrix}, \]  \hspace{1cm} (6.7)

Employing (6.7) and (6.8), we can reformulate \( R \) as below:

\[ R = \tilde{W}^T \begin{pmatrix} \text{diag}((\alpha^2 \mu_1)^\dagger - (\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger) & -\text{diag}((\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger) \\
-\text{diag}((\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger) & \text{diag}((\beta^2 \lambda_1)^\dagger - (\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger) \end{pmatrix} W, \]  \hspace{1cm} (6.9)

where we use the notation \( \text{diag}(\lambda_i) := \text{diag}(\lambda_1, \ldots, \lambda_d) \) and

\[ W := \begin{pmatrix} \alpha^2 \omega^T \\
\beta^3 \omega^T \end{pmatrix}. \]

We further introduce the notations

\[ D_1 := \text{diag} \left( \sqrt{(\alpha^2 \mu_1)^\dagger - (\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger} \right), \quad \text{and} \quad D_2 := \text{diag} \left( \sqrt{(\beta^2 \lambda_1)^\dagger - (\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger} \right). \]

Observe that the middle matrix in (6.9) can be decomposed as

\[ \begin{pmatrix} \text{diag}((\alpha^2 \mu_1)^\dagger - (\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger) & -\text{diag}((\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger) \\
-\text{diag}((\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger) & \text{diag}((\beta^2 \lambda_1)^\dagger - (\alpha^2 \mu_1 + \beta^2 \lambda_1)^\dagger) \end{pmatrix} = \begin{pmatrix} D_1 & -D_2 \\
-D_2 & D_1 \end{pmatrix}, \]

and, hence, is positive semidefinite. This implies that

\[ R = \tilde{W}^T \begin{pmatrix} D_1 & -D_2 \\
-D_2 & D_1 \end{pmatrix} W \]  \hspace{1cm} (6.10)

is positive semidefinite. \( \square \)

**Corollary 6.9.** Let \((G, \sigma)\) and \((G', \sigma')\) be two \(U(d)\)-connection graphs. Suppose that the two signature groups \( \Sigma \) and \( \Sigma' \) commute. For any \( x \in V, x' \in V' \) and any \( N, N' \in (0, \infty] \), the matrices \( A_N(x), A_{N'}(x') \) and \( A_{N+N'}(x, x') \) with respect to the canonical choice \( B_0 \) of matrix \( B \) in (6.5) satisfy

\[ A_{N+N'}(x, x') = \begin{pmatrix} \alpha A_N(x) \\
\beta A_{N'}(x') \end{pmatrix} + R(x, x') + J(x, x'), \]  \hspace{1cm} (6.11)
where $R(x, x')$ is given in (6.3) and

$$J(x, x') := \frac{2}{N + N'} \begin{pmatrix} \alpha N' \langle v_0(x), v_0(x) \rangle^\top - \sqrt{\alpha \beta} \langle v_0(x), v_0(x') \rangle^\top \\ \beta N' \langle v_0(x), v_0(x') \rangle^\top - \sqrt{\alpha \beta} \langle v_0(x), v_0(x) \rangle^\top \end{pmatrix},$$

(6.12)

with $v_0$ defined in (3.34). Moreover, $J(x, x')$ is positive semidefinite.

**Proof.** The decomposition formula (6.11) is a direct consequence of Theorem 6.3 and the definition of $A_N(x) = A_\infty(x) - \frac{2}{N} v_0(x) v_0(x)^\top$. It remains to show that $J(x, x')$ is positive semidefinite. This follows from the following observation:

$$(N + N') J(x, x') = 2 \begin{pmatrix} v_0(x) \\ v_0(x') \end{pmatrix} \begin{pmatrix} \sqrt{\alpha \frac{N'}{N} I_d} & - \sqrt{\alpha \beta \frac{N'}{N} I_d} \\ - \sqrt{\beta \frac{N'}{N} I_d} & \sqrt{\alpha \beta \frac{N'}{N} I_d} \end{pmatrix} \begin{pmatrix} v_0(x)^\top \\ v_0(x')^\top \end{pmatrix}.$$  (6.13)

\[ \square \]

**Corollary 6.10.** Let $(G, \sigma)$ and $(G', \sigma')$ be two connection graphs. Suppose that the two signature groups $\Sigma$ and $\Sigma'$ commute. For any $x \in V$, $x' \in V'$, and $N, N' \in (0, \infty)$, we have

$$K_{G \times_{\alpha, \beta} G', \sigma \times, (x, x')}(N + N') \geq \min\{\alpha K_{G, \sigma, x}(N), \beta K_{G', \sigma', x'}(N')\}.$$  (6.14)

Moreover, if one of the two local connection structures $B_2^{inc}(x)$ and $B_2^{inc}(x')$ is balanced, we have

$$K_{G \times_{\alpha, \beta} G', \sigma \times, (x, x')}(\infty) = \min\{\alpha K_{G, \sigma, x}(\infty), \beta K_{G', \sigma', x'}(\infty)\}.$$  (6.15)

**Proof.** By Corollary 6.9 we have the decomposition formula (6.11). Then, we derive from the fact that the matrices $R(x, x')$ and $J(x, x')$ are positive semidefinite and Theorem 3.2 that

$$K_{G \times_{\alpha, \beta} G', \sigma \times, (x, x')}(N + N') \geq \min\{\alpha K_{G, \sigma, x}(N), \beta K_{G', \sigma', x'}(N')\}.$$  

If one of the two local connection structures $B_2^{inc}(x)$ and $B_2^{inc}(x')$ is balanced, we have

$$A_\infty(x, x') = \begin{pmatrix} \alpha A_\infty(x) \\ \beta A_\infty(x') \end{pmatrix}.$$  

Applying Theorem 3.2 tells us that

$$K_{G \times_{\alpha, \beta} G', \sigma \times, (x, x')}(\infty) = \min\{\alpha K_{G, \sigma, x}(\infty), \beta K_{G', \sigma', x'}(\infty)\}.$$  

\[ \square \]

**Remark 6.11.** For two connection graphs $(G, \sigma)$ and $(G', \sigma')$ with connections of (possibly) different dimensions, we can define the Cartesian product of them as follows. Suppose that

$$\sigma : E^o \rightarrow O(d_1) \text{ or } U(d_1), \text{ and } \sigma' : (E')^o \rightarrow O(d_2) \text{ or } U(d_2)$$

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for two positive integers $d_1$ and $d_2$. The Cartesian product of $(G, \sigma)$ and $(G', \sigma')$ is the connection graph $(G \times_{\alpha, \beta} G', \sigma^\otimes)$ with a $d_1 d_2$-dimensional connection $\sigma^\otimes$. The connection $\sigma^\otimes$ of each oriented edge in the Cartesian product of $G$ and $G'$ is given as below:

$$\sigma^\otimes_{(x,x')(y,x')} = \sigma_{xy} \otimes I_{d_2},$$

$$\sigma^\otimes_{(x,x')(x,y')} = I_{d_1} \otimes \sigma'_{x'y'}.$$

Notice that $(G \times_{\alpha, \beta} G', \sigma^\otimes)$ coincides with the Cartesian product of $(G, \sigma \otimes I_{d_2})$ and $(G', I_{d_1} \otimes \sigma')$ in the sense of Definition 6.2. The signature groups of $(G, \sigma \otimes I_{d_2})$ and $(G', I_{d_1} \otimes \sigma')$ always commute. Moreover, we have [35, Corollary 3.4]

$$K_{G,\sigma \otimes I_{d_2},x} = K_{G,\sigma,x}, \quad K_{G',I_{d_1} \otimes \sigma',x'} = K_{G',\sigma',x'}. $$

Therefore, the two estimates in Corollary 6.10 holds true for $(G \times_{\alpha, \beta} G', \sigma^\otimes)$.

Below we give an example of two unbalanced connection graphs with non-commuting signature groups $\Sigma$ and $\Sigma'$, for which the curvature of the Cartesian product is smaller than the minimum of the two individual curvatures. This shows that the condition on the community of signature groups $\Sigma$ and $\Sigma'$ in Corollary 6.10 is necessary.

**Example 6.12.** Let $(G, \sigma)$ and $(G', \sigma')$ be given in Figure 2 and Figure 3 with

$$\sigma_{AC} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \sigma'_{23} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

respectively. We calculate that

$$K_{G,\sigma,A}(\infty) = \frac{1}{2}, \quad \text{and} \quad K_{G',\sigma',1} = \frac{3}{2}.$$

It is clearly that $\sigma_{AC} \sigma'_{23} \neq \sigma'_{23} \sigma_{AC}$. We can check the curvature $K_{G \times G', \sigma \times_{(\alpha,1)}(A,1)}(\infty) < 0$ (with $\alpha = \beta = 1$). (Indeed, the matrix $4\Gamma^\times_2(A,1)$ has a negative eigenvalue $-0.7660 < 0$.)

Next, we present an example of two locally unbalanced connection graphs, for which the curvature of the Cartesian product equals the minimum of the two individual curvatures. That is, the condition for the equality (6.15) in Corollary 6.10, i.e., one of the two graphs is locally balanced, is sufficient but not necessary.

**Example 6.13.** Consider again the graphs $G$ and $G'$ depicted in Figure 2 and Figure 3 respectively. We assign $O(1)$-connections $\mathbf{\sigma}$ and $\mathbf{\sigma}'$ to $G$ and $G'$ by replacing each $I_2$ in $\sigma$ and $\sigma'$ by $+1$ and assigning $\mathbf{\sigma}_{AC} = -1$ and $\mathbf{\sigma}'_{23} = -1$. 
We calculate the curvature
\[ K_{G,\overline{\sigma},A}(\infty) = \frac{1}{2}, \quad \text{and} \quad K_{G',\overline{\sigma}',1}(\infty) = \frac{3}{2}. \]

The local connection structure of \((A, 1)\) of the Cartesian product of \((G, \overline{\sigma})\) and \((G', \overline{\sigma}')\) is depicted in Figure 4 in which edges with signature +1 (signature −1) are drawn as solid lines (dashed lines).

![Figure 4: The local connection structure of \((A, 1)\) in \((G, \overline{\sigma}) \times (G', \overline{\sigma}')\).](image)

We can check that the curvature at the vertex \((A, 1)\) is \(\frac{1}{2}\), which is equal to the minimum of \(K_{G,\overline{\sigma},A}(\infty)\) and \(K_{G',\overline{\sigma}',1}(\infty)\).

In the following, we present an example of two signed graphs, for which the curvature of the Cartesian product lies strictly in between the two individual curvatures.

**Example 6.14.** Consider the signed graph \((G, \overline{\sigma})\) given in Example 6.13, and the signed graph \((G_2, \overline{\sigma}_2)\) given in Figure 5. Notice that we only label the negative edges in the figures for simplicity.

![Figure 5: The signed graph \((G_2, \overline{\sigma}_2)\).](image)

Then the curvature matrix at the vertex 1 is
\[
A_{\infty}(G_2, \overline{\sigma}_2, 1) = \frac{1}{4} \begin{pmatrix} 5 & 3 & 0 \\ 3 & 1 & 4 \\ 0 & 4 & 6 \end{pmatrix}.
\]

Thus, the curvature is \(K_{G_2,\overline{\sigma}_2,1}(\infty) \approx -0.5502\). Recall that the curvature at the vertex \(A\) is \(K_{G,\overline{\sigma},A} = \frac{1}{2}\).
The local structure of the Cartesian product at the vertex \((1, A)\) is shown in Figure 6. Again, for clarity, the edges with signature +1 (signature −1) are drawn as solid lines (dashed lines).

![Figure 6: The local connection structure of \((1, A)\) in \((G_2, \sigma_2) \times (G, \sigma)\).](image)

The curvature matrix at the vertex \((1, A)\) is

\[
A_\infty(1, A) = \frac{1}{8} \begin{pmatrix}
19 & -15 & -9 & -9 & 0 \\
-15 & 19 & 9 & 9 & 0 \\
-9 & 9 & 19 & 15 & 0 \\
-9 & 9 & 15 & 11 & 8 \\
0 & 0 & 0 & 8 & 12
\end{pmatrix}.
\]

Hence the curvature is about −0.454, which satisfies

\[ -0.5502 \approx \min\{K_{G_2, \sigma_2, 1}, K_{G, \sigma, A}\} < -0.454 < \max\{K_{G_2, \sigma_2, 1}, K_{G, \sigma, A}\} = \frac{1}{2}. \]

Next, we restrict to the case that the connections lies in \(O(1)\) or \(U(1)\).

**Corollary 6.15.** Let \((G, \sigma)\) and \((G', \sigma')\) be two connection graphs with 1-dimensional connections. For \(x \in V\) and \(x' \in V'\), we have

\[
\min\{\alpha K_{G, \sigma, x}(\infty), \beta K_{G', \sigma', x'}(\infty)\} \leq K_{G \times_\alpha, G', \sigma \times_\lambda, x, x'}(\infty) \leq \max\{\alpha K_{G, \sigma, x}(\infty), \beta K_{G', \sigma', x'}(\infty)\}.
\]

**Proof.** Since the signature groups are always commute in the case of \(d = 1\), the lower bound estimate follows directly from Corollary 6.10. Next, we show the upper bound estimate. Let \(u\) and \(u'\) be the eigenvectors with unit norm corresponding to \(\lambda_{\min}(A_\infty(x))\) and \(\lambda_{\min}(A_\infty(x'))\), respectively. Denote by \(c\) and \(c'\) two numbers to be determined. Applying Theorem 6.3, we estimate

\[
\lambda_{\min}(A_\infty(x, x')) \leq \frac{\left(\frac{cu}{|c|^2} \frac{c'u'}{|c'|^2}\right) \left(\frac{\alpha A_\infty(x)}{\beta A_\infty(x')}\right) \left(\frac{cu}{|c|^2} \frac{c'u'}{|c'|^2}\right) + \left(\frac{cu}{|c|^2} \frac{c'u'}{|c'|^2}\right) R \left(\frac{cu}{|c|^2} \frac{c'u'}{|c'|^2}\right)}{|c|^2 + |c'|^2}.
\]

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Applying (6.10), we derive
\[
\left( \begin{array}{c}
\alpha u^T \\
\beta u^T \\
\end{array} \right) R \left( \begin{array}{c}
cu \\
c'u' \\
\end{array} \right) = \left( \begin{array}{c}
\alpha u^T \\
\beta u^T \\
\end{array} \right) \left( \begin{array}{c}
\frac{D_1}{D_1 - D_2} \\
\frac{D_2}{D_1 - D_2} \\
\end{array} \right) W \left( \begin{array}{c}
cu \\
c'u' \\
\end{array} \right) = \zeta^T \zeta,
\]
where we use
\[
\zeta := (D_1 - D_2) W \left( \begin{array}{c}
cu \\
c'u' \\
\end{array} \right) = \frac{\alpha}{\beta} D_1 P \omega^T u - \frac{\alpha}{\beta} D_2 P \omega^T.
\]
Notice that \( \zeta \) is a number due to \( d = 1 \). We choose \( c \) and \( c' \) such that \( \zeta = 0 \).

Then we estimate
\[
\lambda_{\min}(A_\infty(x, x')) \leq \frac{1}{|c|^2 + |c'|^2} \left( |c|^2 \alpha \lambda_{\min}(A_\infty(x)) + |c'|^2 \beta \lambda_{\min}(A_\infty(x')) \right)
\]
\[
\leq \max \{ \alpha \lambda_{\min}(A_\infty(x)), \beta \lambda_{\min}(A_\infty(x')) \}.
\]
This completes the proof. \( \Box \)

Let us recall the following definition of star product of functions from [16].

**Definition 6.16.** [16] Definition 7.1 Let \( f_1, f_2 : [0, \infty) \to \mathbb{R} \) be continuous and monotone increasing functions with \( \lim_{t \to 0} f_i(t) = -\infty \), \( i = 1, 2 \). Then the star product of \( f_1 \) and \( f_2 \) is defined as a function \( f_1 \ast f_2 : [0, \infty) \to \mathbb{R} \) given by
\[
f_1 \ast f_2(t) := f_1(t_1) = f_2(t_2),
\]
where \( t_1 + t_2 = t \) such that \( f_1(t_1) = f_2(t_2) \).

Notice that the star product is commutative and associative [16] Propositions 7.5 and 7.6.

**Corollary 6.17.** Let \((G, \sigma)\) and \((G', \sigma')\) be two connection graphs with 1-dimensional connections. Let \( x \in V \), \( x' \in V' \) be two vertices. Assume that one of the two local connection structures \( B_2^{inc}(x) \) and \( B_2^{inc}(x') \) is balanced. For any \( N, N' \in (0, \infty] \), we have
\[
\min \{ \alpha K_{G, \sigma, x}(N), \beta K_{G', \sigma', x'}(N') \} \leq K_{(G_1 \times \cdots \times G_n, \sigma \times \cdots \times \sigma')(x, x')} \leq \max \{ \alpha K_{G, \sigma, x}(N), \beta K_{G', \sigma', x'}(N') \}.
\]

Consequently, the curvature functions in this case satisfies
\[
K_{(G_1 \times \cdots \times G_n, \sigma \times \cdots \times \sigma')(x, x')} = (\alpha K_{G, \sigma, x}) \ast (\beta K_{G', \sigma', x'}).
\]

**Proof.** The lower bounds follows from Corollary 6.10 since the signature groups commute. Due to one of the two local connection structures \( B_2^{inc}(x) \) and \( B_2^{inc}(x') \) being balanced, we have \( R(x, x') = 0 \) in (6.11). Let \( u \) and \( u' \) be the eigenvectors with unit norm corresponding to \( \lambda_{\min}(A_N(x)) \) and \( \lambda_{\min}(A_N(x')) \), respectively. Let \( c \) and \( c' \) be two numbers such that
\[
\left( \begin{array}{c}
\alpha u^T \\
\beta u^T \\
\end{array} \right) J(x, x') \left( \begin{array}{c}
cu \\
c'u' \\
\end{array} \right) = 0.
\]

Due to the decomposition formula (6.13) for \( J(x, x') \), the existence of \( c \) and \( c' \) can be shown by a similar argument as in the proof of Corollary 6.14. Then we estimate
\[
\lambda_{\min}(A_{N+N}(x, x')) \leq \frac{1}{|c|^2 + |c'|^2} \left( |c|^2 \alpha \lambda_{\min}(A_N(x)) + |c'|^2 \beta \lambda_{\min}(A_N(x')) \right)
\]
\[
\leq \max \{ \alpha \lambda_{\min}(A_N(x)), \beta \lambda_{\min}(A_N(x')) \}.
\]
This proves the curvature upper bound estimate. The identity of curvature functions follows by applying Theorem 6.11 (i) and [16] Proposition 7.3. \( \Box \)
7 How the curvature changes under local operations

In this section, we study how the curvature at a given vertex changes under operations on the local connection structure $B^\_2$. We discuss two operations:

(i) Add a new spherical edge in $S_1(x)$;

(ii) Merge two vertices in $S_2(x)$ which have no common neighbors.

**Theorem 7.1.** Let $(G, \sigma)$ be a connection graph and $x \in V$ be a given vertex. Assume that $x$ is $S_1$-in regular, i.e., $p^-(y) = p_{yx}$ is independent of $y \in S_1(x)$. Suppose $y_1, y_2 \in S_1(x)$ are non-adjacent, i.e., $w_{y_1y_2} = 0$. Denote by $\tilde{G}$ the weighted graph obtained from $G$ by assigning a positive $\tilde{w}_{y_1y_2} > 0$ and keeping other edges weights. The new connection $\tilde{\sigma}$ coincide with $\sigma$ except that

$$\tilde{\sigma}_{y_1y_2} := \sigma_{y_1x}\sigma_{xy_2}, \quad \text{and} \quad \sigma_{y_2y_1} := \sigma_{y_1y_2}^{-1}. \quad (7.1)$$

That is, we add a spherical edge in $S_1(x)$ which produce a balanced triangle. Then, we have for any $N \in (0, \infty]$,

$$K_{\tilde{G}, \tilde{\sigma}, x}(N) \geq K_{G, \sigma, x}(N).$$

**Proof.** First, observe that the matrices $\Delta^\sigma(x)$ and $\Gamma^\sigma(x)$ stay put under the operation of adding spherical edges. We only need to consider how the matrix $\Gamma^\sigma_2(x)$ changes. By Proposition 3.3, the curvature is unchanged under switching the connections. We can switch all the connections of the edges $\{(x, y_i) : i = 1, \ldots, m\}$ to be $I_d$ by a switching function $\tau$ with $\tau(x) = I_d$ and $\tau(y_i) = \sigma_{xy_i}^{-1}, i = 1, \ldots, m$. By the Appendix, the difference matrix is

$$4\Gamma^\sigma_2(x) - 4\Gamma^\sigma_2(x) = \begin{pmatrix}
0 & c_2\sigma_{y_2y_1} - c_1I_d & \sigma_{y_1y_2} - c_1I_d & \ldots & 0 \\
c_2\sigma_{y_2y_1} - c_1I_d & 3c_1I_d + c_2I_d & -2(c_1 + c_2)\sigma_{y_1y_2} & \ldots & 0 \\
c_1\sigma_{y_1y_2} - c_2I_d & -2(c_1 + c_2)\sigma_{y_1y_2} & 3c_2I_d + c_1I_d & \ldots & 0 \\
\ldots & \ldots & \ldots & \ddots & \ldots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix},$$

where we use the notations $c_1 := p_{xy_1}\tilde{p}_{y_1y_2}$ and $c_2 := p_{xy_2}\tilde{p}_{y_2y_1}$. By the $S_1(x)$-in regularity at $x$, we have

$$c_1 = p_{xy_1}\tilde{p}_{y_1y_2} = \frac{w_{xy_1}\tilde{w}_{y_1y_2}}{p_{yx}\mu(x)} = \frac{\tilde{w}_{y_1y_2}}{p_{yx}\mu(x)} = \frac{\tilde{w}_{y_1y_2}}{\mu(x)} = c_2.$$ 

By our assumption (7.1), the switching function $\tau$ will switch $\sigma_{y_1y_2}$ to be $I_d$ too. Then, the difference matrix becomes

$$4\Gamma^\sigma_2(x) - 4\Gamma^\sigma_2(x) = c_1 \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 4I_d & -4I_d & \ldots & 0 \\
0 & -4I_d & 4I_d & \ldots & 0 \\
\ldots & \ldots & \ldots & \ddots & \ldots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix},$$

which is clearly positive semidefinite. \hfill \box
If, instead, we add a new spherical edge between \( y_1 \) and \( y_2 \) with a connection \( \tilde{\sigma}_{y_1y_2} \) which produces an unbalanced triangle, the difference matrix, after switching all connections \( \sigma_{xy_i}, \ i = 1, \ldots, m \) to be \( I_d \), is

\[
4\Gamma_2^a(x) - 4\Gamma_2^b(x) = c_1 \begin{pmatrix}
0 & \sigma_{y_2y_1} - I_d & \sigma_{y_1y_2} - I_d & \ldots & 0 \\
\sigma_{y_2y_1} - I_d & 4I_d & 4I_d & \ldots & 0 \\
\sigma_{y_1y_2} - I_d & 4I_d & 4I_d & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix},
\] (7.2)

which is indefinite. Indeed, the unbalancedness of the new triangle is equivalent to the fact that \( \tilde{\sigma}_{y_1y_2} \neq I_d \) in the above matrix. That is, there exists \( v \in \mathbb{K}^d \) such that \( \tilde{\sigma}_{y_1y_2} v = \lambda v \) where \( \lambda \in \mathbb{K} \) satisfying \( \text{Re}(\lambda) < 1 \). Then, we calculate for \( c \in \mathbb{R} \)

\[
(cv^\top v^\top) \begin{pmatrix}
0 & \sigma_{y_2y_1} - I_d \\
\sigma_{y_2y_1} - I_d & 4I_d \\
\end{pmatrix} \begin{pmatrix}
cv^\top \\
v^\top \\
\end{pmatrix} = 2 (c(\text{Re}(\lambda) - 1) + 2) v^\top v,
\]

which is negative for large enough \( c \). This implies that the matrix (7.2) is indefinite.

Below, we give three examples of adding a spherical edge which produces an unbalanced triangle, for which the curvature increases, stays put and decreases, respectively. The connection graphs in those examples are signed graphs, i.e., graphs with \( O(1) \)-connections. For simplicity, we only label the edges with \(-1\) in the figures and the unlabeled edges have the connection \(+1\).

**Example 7.2.** Let \((G_3,\sigma_3)\) be the signed graph given in Figure 7.

![Figure 7: The signed graph \((G_3,\sigma_3)\).](image)

By definition, the curvature matrix at the vertex 1 is

\[
A_\infty(G_3,\sigma_3,1) = \frac{1}{5} \begin{pmatrix}
3 & 6 & 12 \\
6 & 7 & 4 \\
12 & 4 & 13 \\
\end{pmatrix}.
\]

Hence the curvature is \( K_{G_3,\sigma_3,1}(\infty) \approx -0.569 \).

We add a negative edge between the vertices 3 and 4 and obtain the signed graph \((\tilde{G}_3,\tilde{\sigma}_3)\) shown in Figure 8. Then we have the curvature matrix

\[
A_\infty(\tilde{G}_3,\tilde{\sigma}_3,1) = \frac{1}{9} \begin{pmatrix}
31 & 6 & 4 \\
6 & 9 & 6 \\
4 & 6 & 31 \\
\end{pmatrix}.
\]
and, therefore, the curvature $K_{	ilde{G}_3, \tilde{\sigma}_3, 1}(\infty) \approx 0.36 > K_{G_3, \sigma_3, 1}(\infty)$.  

**Example 7.3.** Consider the signed graph $(G_4, \sigma_4)$ given in Figure 9. The signed graph $(\tilde{G}_4, \tilde{\sigma}_4)$ in Figure 10 is obtained by adding a negative edge between the vertices 2 and 3. We check directly that $K_{G_4, \sigma_4, 1}(\infty) = 0 = K_{\tilde{G}_4, \tilde{\sigma}_4, 1}(\infty)$. That is, the curvature stays put.

![Figure 9: The signed graph $(G_4, \sigma_4)$](image1)

![Figure 10: The signed graph $(\tilde{G}_4, \tilde{\sigma}_4)$](image2)

**Example 7.4.** Consider the signed graph $(G_5, \sigma_5)$ given in Figure 11. We have the curvature $K_{G_5, \sigma_5, 1}(\infty) = 2$. We add a negative edge between the vertices 2 and 3. The curvature becomes $K_{\tilde{G}_5, \tilde{\sigma}_5, 1}(\infty) = \frac{3}{2} < K_{G_5, \sigma_5, 1}(\infty)$. In the following, we study the operation of merging two vertices in the 2-sphere $S_2(x)$ which have no common neighbors. We define the operation of merging two vertices $z_k$, $z_\ell$ with no common neighbors in $(G, \sigma)$ as follows. Derive from $(G, \sigma)$ a new connection graph $(G', \sigma')$:

- Identify the two vertices $z_k$ and $z_\ell$ as a new vertex $z$, i.e., set $V' := (V \setminus \{z_k, z_\ell\}) \cup \{z\}$.
- We further set $w'_{uz} := w_{uz_k} + w_{uz_\ell}$, $w'_{uv} = w_{uv}$, $\mu'(z) = \mu(z_k) + \mu(z_\ell)$ and $\mu'(u) = \mu(u)$ for any $u, v \in V \setminus \{z_k, z_\ell\}$. The connection $\sigma'$ is given by
  
  $$(\sigma'_{uz})^{-1} = \sigma'_{uz} := \sigma_{uz_k}, \text{ for any } u \in V \setminus \{z_k, z_\ell\} \text{ with } w_{uz_k} \neq 0,$$
  
  $$(\sigma'_{zu})^{-1} = \sigma'_{zu} := \sigma_{uz_\ell}, \text{ for any } u \in V \setminus \{z_k, z_\ell\} \text{ with } w_{uz_\ell} \neq 0,$$
  
  and $\sigma'_{uv} = \sigma_{uv}$ for any $u, v \in V \setminus \{z_k, z_\ell\}$ with $\omega'_{uv} \neq 0$.

**Theorem 7.5.** Let $(G, \sigma)$ be a connection graph and $x$ be a given vertex. Let $z_k, z_\ell$ be two vertices in $S_2(x)$ without common neighbors and $(G', \sigma')$ be the new connection graph obtained by merging the two vertices $z_k$ and $z_\ell$. Then, we have for any $N \in (0, \infty]$ 

$$K_{G', \sigma', x}(N) \geq K_{G, \sigma, x}(N).$$
Proof. Without loss of generality, we assume that the two vertices to be merged are indexed as \( z_n \) and \( z_{n-1} \) in \( S_2(x) \). Recall we denote \( S_2(x) = \{ z_1, \ldots, z_n \} \), where \( n \) is the number of vertices in the 2-sphere \( S_2(x) \). Observe that the merging operation keeps the matrices \( \Delta^\sigma(x) \) and \( \Gamma^\sigma(x) \) unchanged.

Notice that the new matrix \( \Gamma'^\sigma(x) \) has a smaller size than the matrix \( \Gamma^\sigma(x) \). Indeed, we have by Appendix

\[
\Gamma'^\sigma(x) = \left( C' \right) \Gamma^\sigma(x) \left( C' \right)^\top,
\]

where

\[
C' = \begin{pmatrix}
I_{d(|B_2(x)|-2)} & 0 & 0 \\
0 & I_d & I_d
\end{pmatrix},
\]

is a matrix of the size \( d(|B_2(x)| -1) \times d(|B_2(x)|) \).

It is trivial to check that

\[
(C')\Delta^\sigma(x) \Delta^\sigma(x)^\top (C')^\top = \Delta^\sigma(x) \Delta^\sigma(x)^\top;
\]

\[
(C')\Gamma^\sigma(x) (C')^\top = \Gamma^\sigma(x),
\]

where we extend the sizes of \( \Delta^\sigma(x) \Delta^\sigma(x)^\top \) and \( \Gamma^\sigma(x) \) to be \( d|B_2(x)| \times d|B_2(x)| \) by zeros.

Therefore, we have for any \( N \in (0, \infty) \),

\[
\Gamma'^\sigma_2(x) - \frac{1}{N} \Delta^\sigma'(x) \Delta^\sigma'(x)^\top - K_{G,\sigma,x}(N) \Gamma^\sigma(x) = C' \left( \Gamma'_2(x) - \frac{1}{N} \Delta^\sigma(x) \Delta^\sigma(x)^\top - K_{G,\sigma,x}(N) \Gamma^\sigma(x) \right) (C')^\top \geq 0.
\]

That is, we have \( K_{G',\sigma',x}(N) \geq K_{G,\sigma,x}(N) \).

Remark 7.6. Notice that the operation of merging two vertices in \( S_2(x) \) which have no common neighbors produces a new 4-cycle. In contrast to the 3-cycle case in Theorem 7.1, the curvature never decrease no matter this new 4-cycle is balanced or not.

8 An infinite connection graph with positive Bakry-Émery curvature lower bound

To conclude, we give an interesting example that is very different from the case without connections. As we know, in that situation, if the curvature is positively lower bounded on a locally finite graph with bounded vertex degree, we then can dig out that it is a finite graph \cite{30}. The following example shows that there exists an infinite, 3-regular unbalanced signed graph, whose local curvature is strictly positive everywhere.

Example 8.1. Consider the infinite signed graph in Figure 13. Notice that the local connection structures at every vertex are identical. The local connection structure at 1 is depicted in Figure 14.
The matrices $4\Gamma_2^\sigma(1)$ and $4\Gamma^\sigma(1)$ are as follows:

$$4\Gamma^\sigma(1) = \begin{pmatrix}
8 & 2 & -2 & 2 \\
2 & 2 & 0 & 0 \\
-2 & 0 & 2 & 0 \\
-2 & 0 & 0 & 2 \\
2 & 0 & 0 & 2
\end{pmatrix};$$

$$4\Gamma_2^\sigma(1) = \begin{pmatrix}
28 & 11 & -12 & -12 & 11 & 1 & -2 & -2 & -2 & 1 \\
11 & 11 & -6 & -2 & 2 & 2 & -2 & 0 & 0 \\
-12 & -6 & 12 & 6 & -2 & 0 & 2 & 0 & 0 \\
-12 & -2 & 6 & 12 & -6 & 0 & 0 & 2 & 0 \\
11 & 2 & -2 & -6 & 11 & 0 & 0 & -2 & 2 \\
1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-2 & -2 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\
-2 & 0 & 0 & 2 & -2 & 0 & 0 & 2 & 0 \\
1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Using the canonical choice

$$B_0 = \begin{pmatrix}
1 & -1 & 1 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

we obtain the curvature matrix

$$A_\infty(1, B_0) = \frac{1}{4} \begin{pmatrix}
7 & -2 & 2 & 1 \\
-2 & 8 & 0 & 2 \\
2 & 0 & 8 & -2 \\
1 & 2 & -2 & 7
\end{pmatrix}.$$
Therefore, the curvature $\mathcal{K}_1(\infty) = \lambda_{\text{min}}(A_{\infty}(1, B_0)) = \frac{7-\sqrt{17}}{4} > 0$.

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A  Explicit calculations of the matrices $\Gamma_2^\sigma(x)$ and $Q(x)$

In this Appendix, we present the calculations about the matrices $\Gamma_2^\sigma(x)$ and the Schur complement $Q(x)$ for a given vertex $x$ in a connection graph $(G, \sigma)$. Notice that for any $\sigma_{xy}$ of an oriented edge from $x$ to $y$, the inverse $\sigma_{xy}^{-1} = \overline{\sigma_{xy}}$.

We first present the explicit expression of the matrix $\Gamma_2^\sigma(x)$ given in (2.8). We use the following notations:

$$S_1(x) = \{y_1, \ldots, y_m\}, \text{ and } S_2(x) = \{z_1, \ldots, z_n\}.$$

We use the indices $i, j = 1, \ldots, m$ and $k, \ell = 1, \ldots, n$.

The matrix $\Gamma_2^\sigma(x)$ is an Hermitian matrix such that

$$4\Gamma_2^\sigma(x)_{x,x} = 3 \sum_{i=1}^{m} p_{xy_i} p_{y_i,x} I_d + \left( \frac{d_x}{\mu_x} \right)^2 I_d,$$

$$4\Gamma_2^\sigma(x)_{x,y_i} = \sum_{j=1, j \neq i}^{m} p_{xy_j} p_{y_j,y_i} \overline{\sigma_{xy_j} \sigma_{y_j,y_i}} - \left( 2 p_{y_i,x} + \frac{d_y}{\mu_y} + \frac{d_x}{\mu_x} \right) p_{xy_i} \overline{\sigma_{xy_i}}, \forall i,$$

$$4\Gamma_2^\sigma(x)_{x,z_k} = \sum_{i=1}^{m} p_{xy_i} p_{y_i,z_k} \overline{\sigma_{xy_i} \sigma_{y_i,z_k}}, \forall k,$$

$$4\Gamma_2^\sigma(x)_{y_i,y_j} = \sum_{j=1, j \neq i}^{m} p_{xy_j} p_{y_j,y_i} I_d + \left( 2 p_{xy_i} + \frac{3d_y}{\mu_y} - \frac{d_x}{\mu_x} \right) p_{xy_i} I_d, \forall i,$$

$$4\Gamma_2^\sigma(x)_{y_i,y_j} = -2 \left( p_{xy_i} p_{y_i,y_j} + p_{xy_j} p_{y_j,y_i} \overline{\sigma_{y_j,y_i}} + 2 p_{xy_j} p_{xy_j} \overline{\sigma_{xy_j} \sigma_{y_j,y_i}} \right), \forall i, \forall j \neq i,$$

$$4\Gamma_2^\sigma(x)_{y_i,z_k} = -2 p_{xy_i} p_{y_i,z_k} \overline{\sigma_{y_i,z_k}}, \forall i, k,$$

$$4\Gamma_2^\sigma(x)_{z_k,z_k} = \sum_{i=1}^{m} p_{xy_i} p_{y_i,z_k} I_d, \forall k,$$

$$4\Gamma_2^\sigma(x)_{z_k,z_\ell} = 0, \forall k, \forall \ell \neq k.$$

In particular, we have the block

$$4\Gamma_2^\sigma(x)_{S_2,S_2} = \begin{pmatrix} \sum_{i=1}^{m} p_{xy_i} p_{y_i,z_1} I_d & \cdots & \sum_{i=1}^{m} p_{xy_i} p_{y_i,z_n} I_d \end{pmatrix}. \quad (A.9)$$
That is, the block \( \Gamma_2^n(x)_{S_2,S_2} \) is real, diagonal, and invertible.

Next, we present the explicit expression for the Schur complement

\[
Q(x) = \Gamma_2^n(x)/\Gamma_2^n(x)_{S_2,S_2} = \Gamma_2^n(x)_{B_1,B_1} - \Gamma_2^n(x)_{B_1,S_2} \Gamma_2^n(x)_{S_2,S_2}^{-1} \Gamma_2^n(x)_{S_2,B_1}.
\]

The matrix \( Q(x) \) is an Hermitian matrix such that

\[
4Q(x)_{x,x} = 3 \sum_{i=1}^{m} p_{xy} p_{yx} I_d + \left( \frac{d_x}{\mu_x} \right)^2 I_d - n \left( \sum_{i=1}^{m} p_{xy} p_{yx} \gamma_{xy,y_i} \sum_{i=1}^{m} p_{xy} p_{yx} \sigma_{xy,y_i} \right) \sum_{i=1}^{m} p_{xy} p_{yx} \sigma_{xy,y_i}^\top, \quad (A.10)
\]

\[
4Q(x)_{x,y_i} = \sum_{j=1,j\neq i}^{m} p_{xy} p_{yx} \gamma_{xy,y_j} \left( 2p_{xy} x + \frac{d_y}{\mu_y} + \frac{d_x}{\mu_x} \right) p_{xy} \sigma_{xy,y_i} + 2 \sum_{k=1}^{n} \left( \sum_{i=1}^{m} p_{xy} p_{yx} \sigma_{xy,y_i} \gamma_{xy,y_k} \right) \sum_{i=1}^{m} p_{xy} p_{yx} \sigma_{xy,y_i}^\top, \quad \forall i, (A.11)
\]

\[
4Q(x)_{y_i,y_i} = \sum_{j=1,j\neq i}^{m} p_{xy} p_{yx} I_d + \left( 2p_{xy} + \frac{3d_y}{\mu_y} - \frac{d_x}{\mu_x} \right) p_{xy} I_d - 4 \sum_{k=1}^{n} \sum_{j=1}^{m} \frac{p_{xy}^2 p_{yx}^2}{p_{xy} p_{yx}} I_d, \quad \forall i, (A.12)
\]

\[
4Q(x)_{y_i,y_j} = -2(\gamma_{xy,y_j} + \gamma_{xy,y_j} p_{xy} p_{yx} \gamma_{xy,y_j}) \sigma_{xy,y_i} + 2p_{xy} p_{yx} \sigma_{xy,y_i} \gamma_{xy,y_j} - 4 \sum_{k=1}^{n} \sum_{h=1}^{m} p_{xy} p_{yx} \gamma_{xy,y_i} p_{xy} p_{yx} \sigma_{xy,y_j}^\top, \quad \forall i, \forall j \neq i. (A.13)
\]

Next, we show the following key property.

**Proposition A.1.** Let \( x \) be a given vertex in a connection graph \((G, \sigma)\). For any nonsingular matrix \( B \) satisfying (3.2), let \( a := a(G, \sigma, x, B) \) and \( \omega^\top := \omega^\top(G, \sigma, x, B) \) be given in (3.12) and (3.13), respectively. Then we have

\[
a \succeq 0, \text{ and } aa^\top \omega^\top = \omega^\top.
\]

Moreover, we have \( a = 0, \omega = 0 \) when the local connection structure \( B^{loc}_2(x) \) is balanced.

We first prepare the following key lemma for the proof of Proposition A.1.

**Lemma A.2.** Suppose \( A = \sum_{i=1}^{m} X_i X_i^\top, B = \sum_{j=1}^{\ell} X_j Y_j \), where \( X_i, Y_j i = 1, \ldots, m, j = 1, \ldots, \ell \) are \( n \times n \) square matrices, and \( m \geq \ell \). Then we have

\[
AA^\top B = B.
\]

**Proof.** Notice that \( A \succeq 0 \). Thus there exists a nonsingular matrix \( P \) such that

\[
A = P \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix} P^\top, \text{ with } \lambda_k \succeq 0, \ k = 1, \ldots, n. \quad (A.14)
\]
Denote for any real number \( \lambda \) that
\[
\lambda^\dagger = \begin{cases} 
\frac{1}{\lambda}, & \text{if } \lambda \neq 0; \\
0, & \text{otherwise}
\end{cases}
\]
Then we calculate
\[
AA^\dagger B = P \left( \begin{array}{ccc}
\lambda_1^\dagger & \cdots & \lambda_n^\dagger \\
\vdots & \ddots & \vdots \\
\lambda_1 & \cdots & \lambda_n
\end{array} \right) P^{-1} \sum_{j=1}^{\ell} X_j Y_j
\]
\[
= P \sum_{j=1}^{\ell} \left( \begin{array}{ccc}
\lambda_1^\dagger & \cdots & \lambda_n^\dagger \\
\vdots & \ddots & \vdots \\
\lambda_1 & \cdots & \lambda_n
\end{array} \right) P^{-1} X_j Y_j.
\]
We claim that
\[
\left( \begin{array}{ccc}
\lambda_1^\dagger \\
\vdots \\
\lambda_n^\dagger
\end{array} \right) P^{-1} X_i = P^{-1} X_i, \ \forall i = 1, \ldots, m. \quad (A.15)
\]
Indeed, we derive from (A.14) that
\[
P^{-1} A(P^T)^{-1} = \left( \begin{array}{ccc}
\lambda_1 & \cdots & \lambda_n \\
\vdots & \ddots & \vdots \\
\lambda_1 & \cdots & \lambda_n
\end{array} \right) = \sum_{i=1}^{m} (P^{-1} X_i)(P^{-1} X_i)^\top.
\]
Denoting by
\[
P^{-1} X_i = \begin{pmatrix} [P^{-1} X_i]_1 \\ \vdots \\ [P^{-1} X_i]_n \end{pmatrix},
\]
we obtain \( \sum_{i=1}^{m} [P^{-1} X_i]_k(P^{-1} X_i)_k = \lambda_k, \ k = 1, \ldots, n. \) Whenever \( \lambda_k = 0 \) for some \( k \), we have
\[
\sum_{i=1}^{m} |[P^{-1} X_i]_k|^2 = 0 \Rightarrow [P^{-1} X_i]_k = 0, \ \forall i = 1, \ldots, m.
\]
Therefore,
\[
\lambda_k \lambda_k^\dagger [P^{-1} X_i]_k = \begin{cases} 
[P^{-1} X_i]_k, & \text{if } \lambda_k > 0; \\
0, & \lambda_k = 0
\end{cases} = [P^{-1} X_i]_k.
\]
That is, (A.15) holds true.
Applying (A.15), we continue our calculation as
\[
AA^\dagger B = P \sum_{j=1}^{\ell} (P^{-1} X_j)Y_j = \sum_{j=1}^{\ell} X_j Y_j = B.
\]
This finishes the proof.
Proof of Proposition A.1. Consider the nonsingular matrix

\[ B_0 = \begin{pmatrix} P_0^\top \\ P_1 \\ \vdots \\ P_m \end{pmatrix} \]

satisfying (3.3) with \( p_0^\top = (I_d \quad \sigma_{xy_1} \quad \cdots \quad \sigma_{xy_m}) \) and \( p_i, i = 1, \ldots, m \) given in (3.4). We first calculate the \( d \times (m + 1)d \) matrix \( p_0^\top Q(x) \) and denote it as

\[ p_0^\top Q(x) = ( (p_0^\top Q(x))_0 \quad (p_0^\top Q(x))_1 \quad \cdots \quad (p_0^\top Q(x))_m ) . \]

The \( d \times d \) blocks are given in the following:

\[
(p_0^\top Q(x))_0 = \sum_{k=1}^n \left( \sum_{i=1}^m p_{xy_i} p_{y_i z_k} \sigma_{xy_i} \sigma_{y_i z_k} \right) \left( \sum_{i=1}^m p_{xy_i} p_{y_i z_k} \sigma_{xy_i} \sigma_{y_i z_k}^\top \right) - \sum_{i=1}^m \sum_{i=1}^n p_{xy_i} p_{y_i z_k} I_d \\
+ \sum_{i=1}^m \sum_{j \neq i} p_{xy_j} p_{y_j y_j} \sigma_{y_j y_j} \sigma_{y_j y_j}^\top - \sum_{i=1}^m \sum_{j \neq i} p_{xy_i} p_{y_j y_j} I_d = n \sum_{i=1}^m p_{xy_i} p_{y_i z_k} I_d \\
+ \sum_{i=1}^m \sum_{j \neq i} p_{xy_i} p_{y_i y_j} \sigma_{y_i y_j} \sigma_{y_i y_j}^\top - \sum_{i=1}^m \sum_{j \neq i} p_{xy_i} p_{y_j y_j} I_d \\
= - \sum_{i=1}^m \sum_{j \neq i} p_{xy_i} p_{y_j y_j} (I_d - \sigma_{y_i y_j} \sigma_{y_i y_j}^\top) \\
- \sum_{i=1}^m \sum_{j \neq i} p_{xy_i} p_{y_j y_j} (I_d - \sigma_{y_i y_j} \sigma_{y_i y_j}^\top) \\
- \sum_{i=1}^m \sum_{j \neq i} p_{xy_i} p_{y_j y_j} (I_d - \sigma_{y_i y_j} \sigma_{y_i y_j}^\top) \\
= - \sum_{i=1}^m \sum_{j \neq i} p_{xy_i} p_{y_j y_j} \sigma_{y_i y_j} \sigma_{y_i y_j}^\top,
\]

(A.16)
and, for each \(i = 1, \ldots, m\),

\[
(p_0^T 4Q(x))_i = \sum_{j \neq i} p_{xy_i} p_{y_j y_i}(\sigma_{xy_i} - \sigma_{y_j y_i}) + 2 \sum_{j \neq i} p_{xy_i} p_{y_j y_i}(\sigma_{xy_i} - \sigma_{y_j y_i})
\]

\[
+ 2 \sum_{k=1}^{n} p_{xy_i} p_{y_j z_k} \left( \sum_{j=1}^{m} \frac{p_{xy_j} p_{y_j y_i} \sigma_{xy_j y_j y_j}}{p_{xy_j} p_{y_j y_i}} \sigma_{y_j y_j} \right)
\]

\[
= \sum_{j \neq i} p_{xy_i} p_{y_j y_i}(I_d - \sigma_{y_j y_j} \sigma_{xy_i}) + 2 \sum_{j \neq i} p_{xy_i} p_{y_j y_i}(I_d - \sigma_{y_j y_j} \sigma_{xy_i})
\]

\[
+ 2 \sum_{k=1}^{n} p_{xy_i} p_{y_j y_i} \left( I_d - \frac{p_{xy_j} p_{y_j y_i} \sigma_{xy_j y_j y_j}}{p_{xy_j} p_{y_j y_i}} \sigma_{y_j y_j} \right)
\]

\[
= \sum_{j \neq i} p_{xy_i} p_{y_j y_i}(I_d - \sigma_{y_j y_j} \sigma_{xy_i}) + 2 \sum_{j \neq i} p_{xy_i} p_{y_j y_i}(I_d - \sigma_{y_j y_j} \sigma_{xy_i})
\]

\[
+ 2 \sum_{k=1}^{n} p_{xy_i} p_{y_j y_i} \left( I_d - \frac{p_{xy_j} p_{y_j y_i} \sigma_{xy_j y_j y_j}}{p_{xy_j} p_{y_j y_i}} \sigma_{y_j y_j} \right)
\]

\[
= \sum_{j \neq i} p_{xy_i} p_{y_j y_i}(I_d - \sigma_{y_j y_j} \sigma_{xy_i}) + 2 \sum_{j \neq i} p_{xy_i} p_{y_j y_i}(I_d - \sigma_{y_j y_j} \sigma_{xy_i})
\]

\[
+ 2 \sum_{k=1}^{n} p_{xy_i} p_{y_j y_i} \left( \sum_{j=1}^{m} \frac{p_{xy_j} p_{y_j y_i} \sigma_{xy_j y_j y_j}}{p_{xy_j} p_{y_j y_i}} \sigma_{y_j y_j} \right)
\]

\[
\text{(A.17)}
\]

We calculate from (A.10), (A.13) and (A.16)-(A.17) that

\[
2a(B_0) := \sqrt{n} 4Q(x) = p_0^T 4Q(x)
\]

\[
= \sum_{i=1}^{m} \sum_{k=1}^{n} p_{xy_i} p_{y_j z_k} I_d - \sum_{k=1}^{n} \left( \sum_{j=1}^{m} \frac{p_{xy_j} p_{y_j y_i} \sigma_{xy_j y_j y_j}}{p_{xy_j} p_{y_j y_i}} \sigma_{y_j y_j} \right)
\]

\[
+ 2 \sum_{i=1}^{n} \sum_{j \neq i} p_{xy_i} p_{y_j y_i} I_d - \sum_{i=1}^{n} \sum_{j \neq i} p_{xy_i} p_{y_j y_i} \sigma_{xy_i y_j y_j y_j} + \sigma_{xy_i y_j y_j}
\]

\[
= \sum_{k=1}^{m} \sum_{i=1}^{n} \sum_{j \neq i} p_{xy_i} p_{y_j z_k} p_{xy_j y_i} \left( I_d - \sigma_{xy_i y_j y_j} \sigma_{y_j y_j} \right) \left( I_d - \sigma_{xy_i y_j y_j} \sigma_{y_j y_j} \right)
\]

\[
+ \sum_{i=1}^{m} \sum_{j \neq i} p_{xy_i} p_{y_j y_i} \left( I_d - \sigma_{xy_i y_j y_j} \sigma_{y_j y_j} \right) \left( I_d - \sigma_{xy_i y_j y_j} \sigma_{y_j y_j} \right),
\]

\[
\text{(A.18)}
\]

and

\[
\omega^T(B_0) := \omega^T(G, \sigma, x, B_0) = (\omega_1^T \omega_2^T \cdots \omega_m^T),
\]

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where each term \( \omega_i \) is given by

\[
2 \omega_i^T(B_0) = p_0^T 4Q(x) p_0 = \left( \sum_{j \neq i} \left( \frac{1}{\sqrt{P_{xy}^{ij}}} p_{xy}^{ij} p_{yji}^{ji} + 2 \sqrt{P_{xy}^{ij}} p_{yji}^{ji} \right) (I_d - \sigma_{xyj}^i \sigma_{yji}^j \sigma_{xy}^{ji}) \right) \sigma_{xy}^{ji} + 2 \sum_{k=1}^n \sqrt{P_{xy}^{ik} p_{yki}^{ik}} (\sum_{j \neq i} \sum_{r=1}^m P_{xy}^{ij} p_{yji}^{ji} (I_d - \sigma_{xyj}^i \sigma_{yji}^j \sigma_{xy}^{ji} \sigma_{xyz}^{ji}) \sigma_{xyz}^{ji}) \sigma_{xy}^{ji}, \quad \forall i.
\]

Since the matrices \( I_d - \sigma_{xyj}^i \sigma_{yji}^j \sigma_{xy}^{ji} \) and \( I_d - \sigma_{xyj}^i \sigma_{yji}^j \sigma_{xy}^{ji} \sigma_{xyz}^{ji} \sigma_{xy}^{ji} \) are conjugate transformations of each other, and the same holds true for matrices \( I_d - \sigma_{xyj}^i \sigma_{yji}^j \sigma_{xy}^{ji} \sigma_{xyz}^{ji} \) and \( I_d - \sigma_{xyj}^i \sigma_{yji}^j \sigma_{xy}^{ji} \sigma_{xyz}^{ji} \), we have that \( 2a(B_0) \) has the form as in Lemma A.2:

\[
a(B_0) = \sum_{i,j,k} X_{ij,k} X_{ij,k}^T + \sum_{i,j} X_{ij} X_{ij}^T,
\]

where

\[
X_{ij,k} = \sqrt{P_{xy}^{ij} p_{yki}^{ik}} (I_d - \sigma_{xyj}^i \sigma_{yji}^j \sigma_{xy}^{ji} \sigma_{xyz}^{ji} \sigma_{xy}^{ji}), \quad \forall i, j = 1, \ldots, m, \forall k = 1, \ldots, n
\]

and

\[
X_{ij} = \sqrt{P_{xy}^{ij}} (I_d - \sigma_{xyj}^i \sigma_{yji}^j \sigma_{xy}^{ji}), \quad \forall i, j = 1, \ldots, m.
\]

In particular, we have \( a(B_0) \geq 0 \) and \( a(B_0) = 0 \) if the local connection structure \( B_2^{inc}(x) \) is balanced. The blocks \( 2 \omega_i^T(B_0) \) can be formulated into the following form

\[
2 \omega_i^T(B_0) = \sum_{j,k} X_{ij,k} Y_{ij,k} + \sum_j X_{ij} Y_{ij}
\]

for some suitable choices of matrices \( Y_{ij,k} \) and \( Y_{ij} \), \( i, j = 1, \ldots, m, k = 1, \ldots, n \). We observe particularly that \( \omega(B_0) = 0 \) if the local connection structure \( B_2^{inc}(x) \) is balanced. Applying Lemma A.2 to \( 2a(B_0) \) and \( 2 \omega_i^T(B_0) \) yields

\[
a(B_0) a(B_0) \omega_i^T(B_0) = 2 \omega_i^T(B_0), \quad \forall i = 1, \ldots, m.
\]

Therefore, we have the equality as follows

\[
a(B_0) a(B_0) \omega(B_0)^T = \omega(B_0)^T.
\]

For a general nonsingular matrix \( B \) satisfying (3.3) with the form (3.3), we divide each \( b_i^T \) into \((m + 1)\) blocks of the size \( d \times d \) denoted as:

\[
b_i^T = (b_{i,0}^T \quad b_{i,1}^T \quad \cdots \quad b_{i,m}^T), \quad \forall i = 1, \ldots, m.
\]

Then we have

\[
2a(B) := 2a(G, \sigma, x, B) = b_0^T 4Q(x) b_0 = E_d p_0^T 4Q(x) p_0 E_d^T = E_d 2a(B_0) E_d^T = \sum_{i,j,k} (E_d X_{ij,k}) (E_d X_{ij,k})^T + \sum_{i,j} (E_d X_{ij}) (E_d X_{ij})^T.
\]

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In particular, we observe that $a(B) \succeq 0$ and $a(B) = 0$ if and only if $a(B_0) = 0$. Moreover, we have the blocks

$$2\omega_i^\top(B) = b_0^\top 4Q(x) \overline{b_0} = ((b_0^\top 4Q(x))_0 \ (b_0^\top 4Q(x))_1 \ \cdots \ (b_0^\top 4Q(x))_m) \begin{pmatrix} b_{i,0} \\ b_{i,1} \\ \vdots \\ b_{i,m} \end{pmatrix}, \ \forall \ i. \quad (A.20)$$

Since each block

$$(b_0^\top 4Q(x))_i = (Edp_0^\top 4Q(x))_i = Ed(p_0^\top 4Q(x))_i, \ \forall \ i,$$

where $(p_0^\top 4Q(x))_i$ has the form $\sum_{j,k} X_{ij,k} Y'_{ij,k} + \sum_j X_{ij} Y'_{ij}$, we have

$$(b_0^\top 4Q(x))_i = \sum_{j,k} (EdX_{ij,k}) Y'_{ij,k} + \sum_j (EdX_{ij}) Y'_{ij}, \ \forall \ i.$$  

Combining the above identities and (A.20), we obtain

$$2\omega_i^\top(B) = \sum_{j,k} (EdX_{ij,k}) \tilde{Y}_{ij,k} + \sum_j (EdX_{ij}) \tilde{Y}_{ij}, \ \forall \ i$$

for some suitable $\tilde{Y}_{ij,k}$s and $\tilde{Y}_{ij}$s. Therefore, we can apply Lemma A.2 to derive

$$a(B)a(B)^\top(B) = \omega^\top(B)$$

for any $B$. We also mention that $\omega(B) = 0$ if the local connection structure $B^{inc}_2(x)$ is balanced. This completes the proof.

Below we consider the Cartesian product of two connection graphs $(G, \sigma)$ and $(G', \sigma')$. For two vertices $x \in V$ and $x' \in V'$, we use the following notations:

$$S_1(x) = \{y_1, y_2, \ldots, y_m\}, \ S_2(x) = \{z_1, z_2, \ldots, z_n\}$$

and

$$S_1(x') = \{y'_1, y'_2, \ldots, y'_m\}, \ S_2(x') = \{z'_1, z'_2, \ldots, z'_n\}.$$  

Next, we give the proof of Lemma 6.5

**Proof of Lemma 6.5** Consider two vertices $x$ and $x'$ in the connection graphs $(G, \sigma)$ and $(G', \sigma')$, respectively. Recall we have the following assumption:

$$\sigma_{xy} = I_d, \ \text{and} \ \sigma'_{x'y'} = I_d, \ \text{for any} \ y \in S_1(x), y' \in S_1(x'). \quad (A.21)$$
Applying the formulas (A.1)-(A.8), we derive the Hermitian matrix $4\Gamma_2^\times(x, x')$ as follows:

$$4\Gamma_2^\times(x, x')(x, x')(x, x') = \alpha^2 4\Gamma_2^\sigma(x, x) + \beta^2 4\Gamma_2^\sigma(x', x', x') + 2\alpha\beta \sum_{i=1}^{m'} \sum_{j=1}^{m'} p_{xy_j} p_{x'y'_j I_d}, \quad (A.22)$$

$$4\Gamma_2^\times(x, x')(x, x')(y, x') = \alpha^2 4\Gamma_2^\sigma(x, y) - 2\alpha\beta p_{xy_j}(\sum_{j=1}^{m'} p_{x'y'_j I_d}), \quad \forall i = 1, \ldots, m, \quad (A.23)$$

$$4\Gamma_2^\times(x, x')(x, x')(x, y') = \beta^2 4\Gamma_2^\sigma(x') - 2\alpha\beta p_{x'y'_j}(\sum_{j=1}^{m'} p_{xy_j} I_d), \quad \forall i = 1, \ldots, m', \quad (A.24)$$

Every $d \times d$ block in (A.22)-(A.27) is in the rows of $4\Gamma_2^\times(x, x')$ corresponding to the vertex $(x, x')$.

$$4\Gamma_2^\times(x, x')(y, x')(y, x') = \alpha^2 4\Gamma_2^\sigma(x, y), \quad \forall i, j = 1, \ldots, m, \text{ with } j \neq i, \quad (A.28)$$

$$4\Gamma_2^\times(x, x')(y, x')(x, y') = 2\alpha\beta p_{xy_j} p_{x'y'_j I_d}, \quad \forall i = 1, \ldots, m, \quad \forall j = 1, \ldots, m', \quad (A.29)$$

$$4\Gamma_2^\times(x, x')(x, y')(x, x') = \alpha^2 4\Gamma_2^\sigma(x, x), \quad \forall i = 1, \ldots, m, \quad \forall j = 1, \ldots, n, \quad (A.30)$$

Every $d \times d$ block in (A.28)-(A.34) is in the rows of $4\Gamma_2^\times(x, x')$ corresponding to the vertex set $\{(y_1, x'), \ldots, (y_m, x')\}$.

$$4\Gamma_2^\times(x, x')(x, y')(x, y') = \beta^2 4\Gamma_2^\sigma(x') + 2\alpha\beta p_{x'y'_j}(\sum_{j=1}^{m'} p_{xy_j} I_d), \quad \forall i = 1, \ldots, m', \quad (A.35)$$

$$4\Gamma_2^\times(x, x')(x, y')(y, y') = \beta^2 4\Gamma_2^\sigma(x'), \quad \forall i, j = 1, \ldots, m', \text{ with } i \neq j, \quad (A.36)$$

$$4\Gamma_2^\times(x, x')(y, y')(x, x') = 0, \quad \forall i = 1, \ldots, m', \quad \forall k = 1, \ldots, n, \quad (A.37)$$

$$4\Gamma_2^\times(x, x')(y, y')(y, y') = 0, \quad \forall i = 1, \ldots, m', \quad \forall j = 1, \ldots, m', \quad (A.38)$$

Every $d \times d$ block in (A.35)-(A.40) is in the rows of $4\Gamma_2^\times(x, x')$ corresponding to the vertex.
set \(\{(x, y'_1), \ldots, (x, y'_{m'})\}\).

\[
4\Gamma_2^\times(x, x')_{(z_k, x')(z_k, x')} = \alpha^2 4\Gamma_2^\times(x, z_k, z_k), \quad \forall k = 1, \ldots, n,
\]

\[
4\Gamma_2^\times(x, x')_{(z_k, x')(z_k, x')} = 0, \quad \forall k, \ell = 1, \ldots, n \text{ with } k \neq \ell,
\]

\[
4\Gamma_2^\times(x, x')_{(z_k, x')(y_i, y'_j)} = 0, \quad \forall k = 1, \ldots, n, \quad \forall i = 1, \ldots, m, \quad \forall j = 1, \ldots, m',
\]

\[
4\Gamma_2^\times(x, x')_{(z_k, x')(x, x')} = 0, \quad \forall k = 1, \ldots, n, \quad \forall \ell = 1, \ldots, n',
\]

(A.41)\(\quad\)

(A.42)\(\quad\)

(A.43)\(\quad\)

(A.44)\(\quad\)

Every \(d \times d\) block in \((A.41)-(A.44)\) is in the rows of \(4\Gamma_2^\times(x, x')\) corresponding to the vertex set \(\{(z_1, x'), \ldots, (z_n, x')\}\).

\[
4\Gamma_2^\times(x, x')_{(y_i, y'_j)(y_i, y'_j)} = 2\alpha\beta p_{xy, p_{x'y'_j}^I}I_d, \quad \forall i = 1, \ldots, m, \quad \forall j = 1, \ldots, m',
\]

\[
4\Gamma_2^\times(x, x')_{(y_i, y'_j)(y_i, y'_j)} = 0, \quad \forall i, r = 1, \ldots, m, \quad \forall j, s = 1, \ldots, m', \quad \text{with } i \neq r \text{ or } j \neq s,
\]

\[
4\Gamma_2^\times(x, x')_{(y_i, y'_j)(z_k, y'_k)} = 0, \quad \forall i = 1, \ldots, m, \quad \forall j = 1, \ldots, m', \quad \forall k = 1, \ldots, n',
\]

(A.45)\(\quad\)

(A.46)\(\quad\)

(A.47)\(\quad\)

Every \(d \times d\) block in \((A.45)-(A.47)\) is in the rows of \(4\Gamma_2^\times(x, x')\) corresponding to the vertex set \(\{(y_1, y'_1), \ldots, (y_m, y'_m)\}\).

\[
4\Gamma_2^\times(x, x')_{(x, z'_k)(x, z'_k)} = \beta^2 4\Gamma_2^\times(x', z'_k, z'_k), \quad \forall k = 1, \ldots, n',
\]

\[
4\Gamma_2^\times(x, x')_{(x, z'_k)(x, z'_k)} = 0, \quad \forall k, \ell = 1, \ldots, n', \quad \text{with } k \neq \ell.
\]

(A.48)\(\quad\)

(A.49)\(\quad\)

Every \(d \times d\) block in \((A.48)-(A.49)\) is in the rows of \(4\Gamma_2^\times(x, x')\) corresponding to the vertex set \(\{(x, z'_1), \ldots, (x, z'_n')\}\).

Then we calculate the matrix \(N := 4\Gamma_2^\times(x, x')B_1, S_2(4\Gamma_2^\times(x, x')S_2, S_2)^{-1}4\Gamma_2^\times(x, x')S_2, B_1\) :

\[
N(x, x')_{(x, x')(x, x')} = \sum_{k=1}^{n} \left( \frac{\sum_{i=1}^{m} p_{xy, p_y, z_k p_{x, y'_j}^I}}{\sum_{i=1}^{m} p_{xy, p_y, z_k}} \right) \left( \sum_{i=1}^{m} p_{xy, y'_j, z_k} \sigma_{y_j, y'_j} \right),
\]

\[
+ \sum_{k=1}^{n'} \left( \frac{\sum_{i=1}^{m'} p_{x', y'_j, p_{y'_j, z'_k}^I} \sigma_{x', y'_j, z'_k} \sigma_{x', y'_j, z'_k}^I}{\sum_{i=1}^{m'} p_{x', y'_j, p_{y'_j, z'_k}}^I} \right).
\]

(A.50)\(\quad\)

\[
N(x, y')_{(x, x')(y, y')} = -2\alpha\beta p_{xy, y'_j} \sum_{j=1}^{m'} p_{x, y'_j} \left( \sum_{j=1}^{m} p_{xy, y'_j, z_k} \sigma_{y_j, y'_j} \right),
\]

\[
\forall i = 1, \ldots, m,
\]

(A.51)\(\quad\)

\[
N(x, y')_{(x, x')(y, y')} = -2\beta p_{x', y'_j} \sum_{j=1}^{m'} p_{x', y'_j} \left( \sum_{j=1}^{m} p_{x', y'_j, z'_k} \sigma_{y'_j, y'_j} \right),
\]

\[
\forall i = 1, \ldots, m',
\]

(A.52)\(\quad\)

Every \(d \times d\) block in \((A.50)-(A.52)\) is in the rows of \(N(x, x')\) corresponding to the vertex

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\[(x, x'). \]

\[N(x, x')_{(y_i, x')_{(x, x')}} = \frac{N(x, x')_{(x, x')_{(y_i, x')}}}{\top}, \quad \forall \ i = 1, \ldots, m, \quad (A.53)\]

\[N(x, x')_{(y_i, x')_{(y_i, x')}} = 2\alpha\beta p_{xy}\sum_{j=1}^{m'} p_{x'y'_j} I_d + 4\alpha^2 \sum_{k=1}^{n} \frac{\sum_{j=1}^{m} p_{xy} p_{yi} z_k p_{x'y'} p_{yi} z_k}{\sum_{j=1}^{m} p_{xy} p_{yi} z_k}, \quad \forall \ i = 1, \ldots, m, \quad (A.54)\]

\[N(x, x')_{(y_i, x')_{(y_j, x')}} = 4\alpha^2 \sum_{k=1}^{n} \frac{\sum_{j=1}^{m} p_{xy} p_{yi} z_k p_{x'y'} p_{yi} z_k}{\sum_{j=1}^{m} p_{xy} p_{yi} z_k}, \quad \forall \ i, j = 1, \ldots, m, \quad \text{with } j \neq i, \quad (A.55)\]

\[N(x, x')_{(y_i, x')_{(x, x')}} = 2\alpha\beta p_{xy} p_{x'y'_i} I_d, \quad \forall \ i = 1, \ldots, m, \quad \forall \ j = 1, \ldots, m', \quad (A.56)\]

Every \(d \times d\) block in \(A.53-A.56\) is in the rows of \(N(x, x')\) corresponding to the vertex set \(\{(y_1, x'), \ldots, (y_m, x')\}\).

\[N(x, x')_{(x, y'_i)_{(x, x')}} = \frac{N(x, x')_{(x, x')_{(y_i, x')}}}{\top}, \quad \forall \ i = 1, \ldots, m', \quad (A.57)\]

\[N(x, x')_{(x, y'_i)_{(y_i, x')}} = \frac{N(x, x')_{(y_i, x')_{(y_i, x')}}}{\top}, \quad \forall \ i = 1, \ldots, m, \quad \forall \ j = 1, \ldots, m', \quad (A.58)\]

\[N(x, x')_{(x, y'_i)_{(y_j, x')}} = 2\alpha\beta p_{x'y'_i} \sum_{j=1}^{m'} p_{x'y'_j} I_d + 4\beta^2 \sum_{k=1}^{n'} \frac{(p_{x'y'_i} p_{x'y'_j})^2}{\sum_{k=1}^{n'} p_{x'y'_i} p_{x'y'_j}}, \quad \forall \ i = 1, \ldots, m', \quad (A.59)\]

\[N(x, x')_{(x, y'_i)_{(x, y'_j)}} = 4\beta^2 \sum_{k=1}^{n'} \frac{p_{x'y'_i} p_{x'y'_j} p_{x'y'_j} p_{x'y'_j} \sum_{k=1}^{n'} p_{x'y'_i} p_{x'y'_j}}{\sum_{k=1}^{m'} p_{x'y'_i} p_{x'y'_j}}, \quad \forall \ i, j = 1, \ldots, m', \quad \text{with } j \neq i. \quad (A.60)\]

Every \(d \times d\) block in \(A.57-A.60\) is in the rows of \(N(x, x')\) corresponding to the vertex set \(\{(x, y'_1), \ldots, (x, y'_{m'})\}\).

Finally we derive the matrix \(Q(x, x'):\)

\[Q(x, x')_{(x, x')_{(x, x')}} = Q(x, x) + Q(x')_{x', x'}, \]

\[Q(x, x')_{(x, x')_{(y_i, x')}} = Q(x, y_i), \quad \forall \ i = 1, \ldots, m, \]

\[Q(x, x')_{(x, y'_i)_{(x, x')}} = Q(x')_{x', y'_i}, \quad \forall \ i = 1, \ldots, m', \]

\[Q(x, x')_{(y_i, x')_{(x, y'_i)}} = \frac{Q(x, x')_{(x, y'_i)_{(y_i, x')}}}{\top}, \quad \forall \ i = 1, \ldots, m, \]

\[Q(x, x')_{(x, y'_i)_{(x, x')}} = \frac{Q(x, x')_{(x, x')_{(y_i, x')}}}{\top}, \quad \forall \ i = 1, \ldots, m', \]

\[Q(x, x')_{(y_i, y'_j)_{(y_i, x')}} = Q(x)_{y_i, y_j}, \quad \forall \ i = 1, \ldots, m, \]

\[Q(x, x')_{(x, y'_i)_{(y_j, x')}} = Q(x)_{y'_i, y'_j}, \quad \forall \ i = 1, \ldots, m', \]

\[Q(x, x')_{(x, y'_i)_{(x, y'_j)}} = Q(x')_{y'_i, y'_j}, \quad \forall \ i, j = 1, \ldots, m', \quad \text{with } j \neq i. \]

That is, we obtain the identity \(I.4\). \hfill \(\blacksquare\)
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