JACOB’S LADDERS AND SOME NONLINEAR INTEGRAL EQUATIONS CONNECTED WITH THE POISSON-LOBACHEVSKY INTEGRAL

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ABSTRACT. We obtain some new properties of the signal generated by the Riemann zeta-function in this paper. Namely, we show the connection between the function \( \zeta \left( \frac{1}{2} + it \right) \) and a nonlinear integral equation related to the Poisson-Lobachevsky integral.

1. THE RESULT

1.1. Let us remind the Poisson parametric integral

\[
\int_0^\pi \ln(1 + a^2 - 2a \cos x)dx = \begin{cases} 0 & , \quad a \in (0, 1), \\ 2\pi \ln a & , \quad a > 1. \end{cases}
\]

It is well-known that Lobachevsky applied his new geometric system to the calculations of the wide set of integrals. One of these integrals (see [2], eq. (111)) leads, after a small transformation, right to the integral (1.1).

Remark. The integral (111) of [2] was used by Lobachevsky to answer the unfair criticism of Ostrogradsky.

1.2. In this paper we obtain some new properties of the signal

\[
Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right)
\]

generated by the Riemann zeta-function, where

\[
\vartheta(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + i \frac{t}{2} \right) = \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O \left( \frac{1}{t} \right),
\]

namely, the nonlinear integral equation connected with the function \( \zeta \left( \frac{1}{2} + it \right) \) and the Poisson-Lobachevsky integral (1.1).

Next let us remind that

\[
\dot{Z}^2(t) = \frac{d\varphi_1(t)}{dt}, \quad \varphi_1(t) = \frac{1}{2} \varphi(t),
\]

where

\[
\dot{Z}^2(t) = \frac{Z^2(t)}{2\Phi'_{\varphi} [\varphi(t)]} = \frac{|\zeta \left( \frac{1}{2} + it \right)|^2}{1 + O \left( \frac{\ln \ln t}{\ln t} \right) \ln t}
\]

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Corollary 1.

2.1. It is clear that (1.3) is followed by:

the nonlinear integral equation

these methods in the case of the nonlinear integral equation (1.3)?

2

Remark

Theorem

1.3. The following theorem holds true.

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Jacob’s ladder . . .

Remark

This paper is a continuation of the series [3] - [22].

(1.3)

(there is an infinite set of Jacob’s ladders).

1.3. The following theorem holds true.

Theorem. Every Jacob’s ladder \( \varphi_1(t) = \frac{1}{2} \varphi(t) \), where \( \varphi(t) \) is the exact solution of the nonlinear integral equation

\[
\int_0^{\mu[\varphi(T)]} Z^2(t)e^{-\frac{2}{\pi T}t}dt = \int_0^T Z^2(t)dt,
\]

is the asymptotic solution of the following nonlinear integral equation

\[
\int_{x^{-1}(T + \pi)}^{x^{-1}(T + \pi_a)} \ln[1 + a^2 - 2a \cos(x(t) - T)] \left| \zeta \left( \frac{i}{2} + it \right) \right|^2 dt = 1,
\]

where \( x(t) = x(t; a) \), and

\[
\tau_a = \arccos \frac{a}{2}, \quad a \in (0, 1),
\]

i.e. the following asymptotic formula

\[
\int_{x^{-1}(T + \pi)}^{x^{-1}(T + \pi_a)} \ln[1 + a^2 - 2a \cos(x(t) - T)] \left| \zeta \left( \frac{i}{2} + it \right) \right|^2 dt = 1 + O \left( \frac{\ln \ln T}{\ln T} \right),
\]

as \( T \to \infty \) holds true for every Jacob’s ladder and for every fixed \( a \in (0, 1) \).

Remark 2. There are the fixed point methods and other methods of the functional analysis used to study the nonlinear equations. What can be obtained by using these methods in the case of the nonlinear integral equation (1.3)?

This paper is a continuation of the series [3] - [22].

2. Corollaries and remarks

2.1. It is clear that (1.3) is followed by:

Corollary 1.

\[
\int_{x^{-1}(T + \pi)}^{x^{-1}(T + \pi_a)} \ln[1 + a^2 - 2a \cos(x(t) - T)] \left| \zeta \left( \frac{i}{2} + it \right) \right|^2 dt = 0,
\]

i.e. we have homogenous nonlinear integral equation.

Next, from (2.1), in the case \( a = \frac{1}{2}, \ b > 1 \) we obtain

Corollary 2.

\[
\int_{x^{-1}(T + \pi)}^{x^{-1}(T + \pi_a)} \ln[1 + b^2 - 2b \cos(x(t) - T)] \left| \zeta \left( \frac{i}{2} + it \right) \right|^2 dt =
\]

\[
= 2 \ln b \int_{x^{-1}(T + \pi)}^{x^{-1}(T + \pi_a)} \left| \zeta \left( \frac{i}{2} + it \right) \right|^2 dt,
\]
Remark 3. If we apply the usual way in the case (1.1), \( a \geq 2 \), we obtain the following theorem: every Jacob’s ladder \( \varphi_1(t) = \frac{1}{2} \varphi(t) \), where \( \varphi(t) \) is the exact solution of the nonlinear integral equation

\[
\int_0^{\mu(x(T))} Z^2(t)e^{-\frac{t}{\varphi_1(t)}}\,dt = \int_T^{T_0} Z^2(t)\,dt,
\]

is the asymptotic solution of the following nonlinear integral equation

\[
\int_{x^{-1}(T+\pi)}^{x^{-1}(T)} \ln[1 + a^2 - 2a \cos(x(t) - T)] \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \,dt = 2\pi \ln a \ln T, \quad a \geq 2,
\]

where \( \ln[1 + a^2 - 2a \cos(x(t) - T)] \geq 0 \), i.e. the following asymptotic formula

\[
\int_{x^{-1}(T+\pi)}^{x^{-1}(T)} \ln[1 + a^2 - 2a \cos(\varphi_1(t) - T)] \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \,dt \sim 2\pi \ln a \ln T, \quad T \to \infty
\]

holds true.

2.2. On Riemann hypothesis the following Littlewood’s estimate

\[
0 < \gamma' - \gamma < \frac{A}{\ln \ln \gamma}, \quad \gamma \to \infty
\]

holds true (see [1]), where \( \gamma, \gamma' \) stand for consecutive zeroes of the function \( \zeta \left( \frac{1}{2} + it \right) \). Since

\[
\frac{1}{\gamma' - \gamma} > \frac{1}{A} \ln \ln \gamma > 2, \quad \gamma > \gamma_0,
\]

then we obtain from (2.1)

**Corollary 3.** On Riemann hypothesis

\[
\int_{x^{-1}(\gamma+\pi)}^{x^{-1}(\gamma)} \ln \left[ 1 + \frac{1}{(\gamma' - \gamma)^2} - \frac{2}{\gamma' - \gamma} \cos(\varphi_1(t) - \gamma) \right] \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \,dt \sim -2\pi \ln(\gamma' - \gamma) \ln \gamma, \quad \gamma > \max\{\gamma_0, T_0[\varphi_1]\}.
\]

Next, we obtain

**Corollary 4.**

\[
\int_{x^{-1}(p+\pi)}^{x^{-1}(p)} \ln[1 + (p' - p)^2 - 2(p' - p) \cos(\varphi_1(t) - p)] \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \,dt \sim 2\pi \ln(p' - p) \ln p, \quad p \geq T_0[\varphi_1],
\]

where \( p, p'; \quad p < p' \) denote the consecutive prime numbers.
3. Proof of the Theorem

3.1. Let us remind that the following lemma holds true (see [8], (2.5); [9], (3.3)): for every integrable function (in the Lebesgue sense) \(f(x), \ x \in [\varphi_1(T), \varphi_1(T + U)]\) we have

\[
\int_T^{T+U} f[\varphi_1(t)] \tilde{Z}^2(t) dt = \int_{\varphi_1(T)}^{\varphi_1(T+U)} f(x) dx, \ U \in \left(0, \frac{T}{\ln T}\right)
\]

where

\[
t - \varphi_1(t) \sim (1 - c)\pi(t),
\]

c is the Euler’s constant and \(\pi(t)\) is the prime-counting function. In the case

\[
\hat{T} = \varphi_1^{-1}(T), \ \hat{T} + \hat{U} = \varphi_1^{-1}(T + U)
\]

we obtain from (3.1)

\[
\int_{\varphi_1^{-1}(T)}^{\varphi_1^{-1}(T+U)} f[\varphi_1(t)] \tilde{Z}^2(t) dt = \int_T^{T+U} f(x) dx.
\]

3.2. We obtain from (1.1), \(0 < a < 1\)

\[
\int_0^\pi \ln(1 + a^2 - 2a \cos \tau) d\tau = -\int_0^\pi \ln(1 + a^2 - 2a \cos \tau) d\tau,
\]

where

\[
\tau_a = \arccos \frac{a}{2},
\]

and

\[
\ln(1 + a^2 - 2a \cos \tau) > 0, \ \tau \in (\tau_a, \pi),
\]

\[
\ln(1 + a^2 - 2a \cos \tau) < 0, \ \tau \in (0, \tau_a).
\]

Putting

\[
f(t) = \ln[1 + a^2 - 2a \cos(t - T)], \ U = \pi; \ t = \tau + T
\]

in (3.3), we obtain (see (3.3), (3.4))

\[
\int_{\varphi_1^{-1}(T+\tau_a)}^{\varphi_1^{-1}(T+\tau_a)} \ln[1 + a^2 - 2a \cos(\varphi_1(t) - T)] \tilde{Z}^2(t) dt =
\]

\[
= -\int_{\varphi_1^{-1}(T+\tau_a)}^{\varphi_1^{-1}(T+\tau_a)} \ln[1 + a^2 - 2a \cos(\varphi_1(t) - T)] \tilde{Z}^2(t) dt
\]

where

\[
\int_T^{T+\pi} \ln[1 + a^2 - 2a \cos(t - T)] dt = \int_0^\pi \ln(1 + a^2 - 2a \cos \tau) d\tau.
\]
3.3. We obtain by using the mean-value theorem for the integrals in \((3.7)\) (see (1.2), (3.6))

\[
\int_{\varphi_{1}^{-1}(T+\pi)}^{\varphi_{2}^{-1}(T+\tau_{a})} \ln[1 + a^{2} - 2a\cos(\varphi_{1}(t) - T)] \tilde{Z}_{2}^{2}(t) dt = \\
\frac{1}{\ln t_{1}} \ln t_{1} \int_{\varphi_{1}^{-1}(T+\tau_{a})}^{\varphi_{1}^{-1}(T+\pi)} \ln[1 + a^{2} - 2a\cos(\varphi_{1}(t) - T)] \zeta \left(\frac{1}{2} + it\right)^{2} dt,
\]

where \(t_{1}, t_{2} \in (\varphi_{1}^{-1}(T), \varphi_{2}^{-1}(T+\pi))\), and

\[
(3.9) \quad t_{1} = \varphi_{1}^{-1}(T_{1}), \ t_{2} = \varphi_{1}^{-1}(T_{2}), \ T_{1}, T_{2} \in (T, T+\pi).
\]

3.4. Next, we obtain from \((3.2)\) by \((3.9)\) \((t_{1} \to \infty \iff T \to \infty)\)

\[
(3.10) \quad t_{1} - T_{1} = \mathcal{O}\left(\frac{t_{1}}{\ln t_{1}}\right) \Rightarrow 1 - \frac{T_{1}}{t_{1}} = \mathcal{O}\left(\frac{1}{\ln t_{1}}\right) \to 0, \ T \to \infty,
\]

i.e.

\[
(3.11) \quad t_{1} \sim T_{1} \sim T, \ t_{2} \sim T_{2} \sim T,
\]

and (see \((3.10)\), \((3.11)\); \(0 < T_{1} - T, T_{2} - T < \pi)\)

\[
(3.12) \quad t_{1} - T = \mathcal{O}\left(\frac{T}{\ln T}\right), \ t_{2} - T = \mathcal{O}\left(\frac{T}{\ln T}\right).
\]

Now,

\[
(3.13) \quad \ln t_{1} = \ln T + \mathcal{O}\left(\frac{t_{1} - T}{T}\right) = \ln T + \mathcal{O}\left(\frac{1}{\ln T}\right),
\]

and similarly,

\[
(3.14) \quad \ln t_{2} = \ln T + \mathcal{O}\left(\frac{1}{\ln T}\right).
\]

Then the formula \((1.3)\) follows from \((3.7)\) by \((3.8)\), \((3.10)\) and \((3.14)\).

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Jacob’s ladder and the asymptotic formula for the integral of the fifth order expression of the function ζ(t).

J. Moser, ‘Jacob’s ladders and the multiplicative asymptotic formula for short and microscopic parts of the Hardy-Littlewood integral’, (2009), arXiv:0907.0501

J. Moser, ‘Jacob’s ladders and the asymptotic approximate solutions of a nonlinear differential equation’, (2010), arXiv:1005.2052.

J. Moser, ‘Jacob’s ladders and the oscillations of the function ζ(1/2 + it) in the main part of its mean-value; law of the almost exact equality of the corresponding areas’, (2010), arXiv:1006.5158.

J. Moser, ‘Jacob’s ladders and the nonlocal interaction of the function Z(t) with the function Z^2(t) on the distance ∼ (1 - c)π(t) for a collection of disconnected sets’, (2010), arXiv:1006.5158.

J. Moser, ‘Jacob’s ladders and the nonlocal interaction of the function Z^2(t) on the distance ∼ (1 - c)π(t) for a collection of disconnected sets’, (2010), arXiv:1007.5147.

J. Moser, ‘Jacob’s ladders and some new consequences from A. Sleberg’s formula’, (2010), arXiv:1010.0868.

J. Moser, ‘Jacob’s ladders and new orthogonal systems generated by Jacobi polynomials’, (2010), arXiv:1010.3540.

J. Moser, ‘Jacob’s ladders, Bessel’s functions and the asymptotic solutions of a new class of nonlinear integral equations’, (2010), arXiv:1011.3359.

J. Moser, ‘Jacob’s ladders, Gram’s sequence and some nonlinear integral equations connected with the functions J_n(x) and |ζ(1/2 + it)|^4’, (2010), arXiv:1011.6198.

### References

[1] J.E. Littlewood, ‘Two notes on the Riemann zeta-function’, Proc. Cam. Phil. Soc., 22 (1924), 234-242.
[2] N.I. Lobachevsky, ‘Collected papers’, part III, M. - L., 1959 (in russian).
[3] J. Moser, ‘Jacob’s ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral’, Math. Notes 2010, 88, pp. 414-422, arXiv:0901.3973.
[4] J. Moser, ‘Jacob’s ladders and the tangent law for short parts of the Hardy-Littlewood integral’, (2009), arXiv:0906.0059.
[5] J. Moser, ‘Jacob’s ladders and the multiplicative asymptotic formula for short and microscopic parts of the Hardy-Littlewood integral’, (2009), arXiv:0907.0501.
[6] J. Moser, ‘Jacob’s ladders and the quantization of the Hardy-Littlewood integral’, (2009), arXiv:0909.3928.
[7] J. Moser, ‘Jacob’s ladders and the first asymptotic formula for the expression of the sixth order argument of the function ζ(t)’, (2009), arXiv:0911.1246.
[8] J. Moser, ‘Jacob’s ladders and the first asymptotic formula for the expression of the fifth order argument of the function ζ(t)’, (2009), arXiv:0912.0130.
[9] J. Moser, ‘Jacob’s ladders, the iterations of Jacob’s ladder and asymptotic formulae for the integrals of the products Z^2[φ^2(t)]Z^2[φ^n-1(t)] · · · Z^2[φ^0(t)] for arbitrary fixed n ∈ N^0 (2010), arXiv:1001.1632.
[10] J. Moser, ‘Jacob’s ladders and the asymptotic formula for the integral of the eighth order expression |ζ(1/2 + it)|^4 |ζ(1/2 + it)|^4’, (2010), arXiv:1001.2114.
[11] J. Moser, ‘Jacob’s ladders and the asymptotically approximate solutions of a nonlinear diophantine equation’, (2010), arXiv:1001.3019.
[12] J. Moser, ‘Jacob’s ladders and the asymptotic formula for short and microscopic parts of the Hardy-Littlewood integral of the function |ζ(1/2 + it)|^4’, (2010), arXiv:1001.4007.
[13] J. Moser, ‘Jacob’s ladders and the nonlocal interaction of the function ζ(1/2 + it) with arg ζ(1/2 + it) on the distance ∼ (1 - c)π(t)’, (2010), arXiv:1004.0169.
[14] J. Moser, ‘Jacob’s ladders and the Z^2 - transformation of polynomials in ln φ^1(t)’, (2010), arXiv:1005.2052.
[15] J. Moser, ‘Jacob’s ladders and the oscillations of the function |ζ(1/2 + it)|^2 around the main part of its mean-value; law of the almost exact equality of the corresponding areas’, (2010), arXiv:1006.4316.