Advantage in two-way communication using non-classical states of light

Bohnishikha Ghosh, Amit Mukherjee, and S. Aravinda

1 Department of Physical Sciences, IISER Kolkata, Mohanpur 741246, India
2 Optics and Quantum Information Group, The Institute of Mathematical Sciences, HBNI, CIT Campus, Taramani, Chennai 600113, India
3 Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India

Maximal advantage in two-way communication via maximal violation of an inequality associated with the ‘Guess Your Neighbour’s Input’ game has been theoretically and experimentally established quite recently using a single-photon two-mode entangled state. We argue that such a maximal advantage can be achieved using single-mode non-classicality, embedded in a two-mode entangled state, called the generalized NOON state, irrespective of the average photon number of the state, which in principle can be made to be very small. The same state furnishes violation of Bell-type inequality without using a shared reference frame. It is shown that the usage of generalized NOON states can provide an advantage over the usage of single-photon two-mode entangled state under noisy apparatuses (beam splitters and photo detectors).

I. INTRODUCTION

Communicating messages efficiently is an important task in today’s technological world. As communication is a physical process, the fundamental nature of physical theories involved in the communication process dominates the efficiency of communication. The advent of quantum entanglement and nonlocality as a philosophical inquiry that severs it from the classical understanding, laid the foundation of quantum communication that provided an enormous advantage in communication over classical theories. Many other non-classical features like no-cloning, non-joint measurability, uncertainty and the non-classical nature of quantum optical states are responsible factors providing the advantage in many information theoretic and computational tasks.

The most dramatic way of departing from the classical understanding is the observation of non-locality in the observed statistics via the violation of Bell type inequalities. Most of the cases, the non-locality is associated with composite systems involving multiple particles. However, in the 90’s, Tan et al. first investigated the question concerning the non-locality of single particle within the restricted set of assumptions which could rule out certain class of local realistic models. Later, Hardy extended the protocol by Tan and co-authors, using single photon to rule out larger class of local realistic models. Hardy’s proposal met with many criticism and counter criticisms. Dunningham and Vedral demonstrated the existence of single photon entanglement even when the given state doesn’t violate any Bell-type inequality. The requirement of additional particles for sharing the reference frame could contribute to the violation of Bell type inequality making the nonlocality of single particle debatable. The debate is settled by Brask et al. by showing the violation of Bell’s inequality by the single photon entangled state without shared reference frame.

Although, as mentioned above, debates concerning non-locality and entanglement of a single particle still persist, these have provided many important applications in information theoretic tasks against classical one. Recently, by exploiting the superposition of single quantum particle in two spatially separated distant locations, advantage in communication was shown over the classical case in which the protocol is restricted by usage of single particle (single photon) and finite speed of propagation.

The common theme in both scenarios of non-locality and two-way communication is the usage of the single photon entangled state,

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B),$$

where $|i\rangle_A$ denotes the occupation number state of mode A, B, respectively. This prompts us to ask the reason behind the advantage of using single particle in the communication process. This question concerns both fundamental as well as practical importance. From a foundational perspective, we would like to investigate the relation between the optical non-classicality associated with the state and the communication advantage supplied by it. From a practical perspective, it is well known that single photons are difficult to produce. Parametric down conversion (PDC) sources, which are among the most widely used sources of single photons, possess some inherent drawbacks: photon heralding being a statistical process only allows for approximation to a deterministic single-photon source via PDC; multi-photon-pair emission (as opposed to single photons at the idler and the signal) limits the heralding rates and the fidelity of the generated single-photon states etc; the
spectral properties of the source may lead to a heralding of single photons in a mixture of frequency modes, thus reducing the purity of the heralded state \[ |\phi\rangle = \frac{1}{\sqrt{N}}(|\xi\rangle_A |0\rangle_B + |0\rangle_A |\xi\rangle_B), \]

where \(|\xi\rangle\) is a certain type of quantum optical non-classical state of the form

Here we show that the quantum optical non-classical state and the generalized NOON state are juxtaposed. The final section is devoted to an account of the one-way communication. In this section the effects of loss on the quantum optical non-classicality to quantum communication is formulated.

Using the non-classicality of quantum optical states. The authors of this work established that non-classicality is a prerequisite for entanglement by a lossless beam splitter. This very property of the beam splitter allows us to put non-classical states other than the single photon state to use in the existing protocol. Thus we can claim the quantum optical non-classicality to quantum computational advantage over classical computation [36–38].

In Section II the polytopic structure of the aforementioned tasks and the facet inequalities involved are briefly described. This is followed by detailed descriptions of these two tasks and the protocols involved in sections [I] and [IV] respectively. Section [V] describes certain methods to model loss in the devices used in the protocol for two-way communication. In this section the effects of loss on the GYNI inequalities furnished by the single photon entangled state and the generalized NOON state involving an even coherent state are juxtaposed. The final section summarizes our results and is indicative of possible future directions.

II. CLASSICAL BOUND : TWO-WAY COMMUNICATION AND NON-LOCALITY

In this section, we attempt to elucidate the settings in which the tasks of two-way communication as well as exhibition of non-locality are carried out.

First consider the two-way communication task. As a simplest scenario, consider two parties Alice and Bob, separated by a spatial distance \(d\), has to communicate their messages among themselves. The speed of message transmission is restricted as a classical signal can traverse the distance between the two parties only once within the given time window. The communication task is formulated as a game in which a referee provides inputs to the players Alice and Bob, and each player has to predict other parties input via his output process. For simple scenario, imagine both Alice and Bob has a two-input-two-output device, and Alice (Bob) inserts her input \(x(y)\) and obtains an output \(a\) (output \(b\) for Bob’s device). The task of the players is to predict other player’s input within the time interval \((r \leq \frac{d}{c})\), taken by a single particle to travel the distance \(d\) between the two parties.

Within the classical particle model, Alice (Bob) can encode her (his) message in the particle and send it to Bob (Alice). If Alice’s action precedes Bob’s, Alice can send (signal) her message to Bob but Bob can’t send his message to Alice. This situation is represented in a causal structure as \(A \prec B\) (Alice’s actions precedes Bob’s) and marginal probability of Alice’s outcome is unaffected by Bob’s action as

\[ P(a|x, y) = P(a|x, y'), \]

where \(P(a|x, y) = \sum_b P(a, b|x, y)\) and \(P(a|x, y') = \sum_b P(a, b|x, y')\). Similarly, we can represent a causal structure in which Bob’s action precedes Alice’s ones, denoted as \(B \prec A\), and we have

\[ P(b|x, y) = P(b|x', y). \]

Restricting the communication with single particle in a given time window in classical particle model, the correlations either belong to causal order \(A \prec B\) or \(B \prec A\) or any convex combination of these two. The only constraints on all the probabilities \(P(a, b|x, y)\) is to be a valid probability measures, so each probability has to satisfy the positivity condition \(P(a, b|x, y) \geq 0\), and the normalization condition, \(\sum_{a, b} P(a, b|x, y) = 1\). The set of all such correlations \(P(a, b|x, y)\) forms a polytope in 12 dimensional vector space. We call this polytope as the correlation polytope. The set of all deterministic correlations, corresponding to causal order \(A \prec B\) and \(B \prec A\), has to further satisfy the relations [4] and [I] forming a polytope called the one-way signaling polytope [40]. The total number of deterministic vertices in each of the one-way signaling polytopes \(A \prec B\) and \(B \prec A\) is 64 including the common 16 local deterministic vertices. The facets (represented by some inequalities) of the one-way
signaling polytope represents the maximally achievable classical result in maximal violation of this inequality. However, this doesn’t allow us to comment on whether they violate the other 15 inequalities or not.

Consider, now, the nonlocality scenario involving two parties Alice and Bob with each of the parties having an access to device having two inputs and two outputs. In nonlocality scenario, the correlations has to satisfy the following no-signaling conditions

\[
P(a|xy) = P(a|x'y'),
\]

\[
P(b|xy) = P(b|x'y).
\]

The set of correlations satisfying no-signaling conditions forms a no-signaling (NS) polytope in 8 dimensions with 8 nonlocal vertices and 16 local deterministic vertices. The facets of the local polytope is the CHSH inequality.

The point of difference between the geometric structure of the correlation polytope and the no-signaling polytope is worth considering. The NS polytope is a subset of the correlation polytope and the dimension of NS polytope is 8 while the dimension of the correlation polytope is 12, thus making them two different geometrical entities. These geometric considerations as well as the different physical scenarios concerning the two-way communication and nonlocality tasks, it is not trivial to compare these two tasks. This is the rationale behind considering the same state as a resource for both the tasks and studying the advantage in both the scenarios.

### III. TWO-WAY COMMUNICATION

We now describe the method of using some generalized NOON states for carrying out the task of two-way communication. Although, we focus on generalized NOON states involving even coherent states, in Appendix 3 we show that generalized NOON states involving single mode squeezed states also violates inequality maximally.

\[
\frac{1}{4} \sum_{x,y,a,b} \delta_{y,a \oplus \beta_1 b \oplus \beta_0} \delta_{x,b \oplus \alpha_1 a \oplus \alpha_0} P(a, b|x, y) \leq \frac{1}{2}.
\]

where inputs and outputs are represented as \( x, y, a, b \in \{0, 1\} \) and \( \forall \alpha_0, \alpha_1, \beta_0, \beta_1 \in \{0, 1\} \). We have worked with one of these 16 inequalities. We denote the inequality of our interest by \( \mathcal{J} \),

\[
\mathcal{J} = \frac{1}{4} (P(0, 0|0, 0) + P(0, 0|1, 1) + P(1, 1|0, 1) + P(1, 1|1, 0)) \leq \frac{1}{2}.
\]

#### A. Communication protocol

At the outset, it is necessary to elucidate the protocol described in [1] and [2].

**Preparation:** Alice and Bob receive an even coherent state prepared in the superposition of their locations, i.e an even-coherent NOON state:

\[
|\phi_{A,B} \rangle = \frac{1}{\sqrt{N_e}} |\alpha_e \rangle_A |0 \rangle_B + |0 \rangle_A |\alpha_e \rangle_B,
\]

where \( |\alpha_e \rangle = N_e |\alpha \rangle + |\alpha \rangle \), i.e a Cat state. Alice receives mode A and Bob receives mode B of the aforementioned state. Here,

\[
N_e = \frac{1}{\sqrt{2[1 + \exp(-2|\alpha|^2)]}}.
\]

The average photon number of the input state (c.f Appendix A) can be tuned to a desirable value, in principle, this can be made as small as possible. Although we have concentrated on even-coherent NOON states, it is to be noted that the protocol remains valid for odd-coherent NOON states as well.

**Encoding:** Alice and Bob encode their respective inputs as phases on their local states by operating on the polarization degree of freedom. Naturally, this doesn’t work when Alice or Bob receive \( |0 \rangle \).

**Beam splitter operation:** The encoded even-coherent NOON state is made to pass through a 50:50 beam splitter (BS). In the current section, we assume that the beam splitter is lossless. A possible treatment of lossy beam splitters is provided in Section V.

**Detection:** The output of the BS is detected in the local laboratories of Alice and Bob. The success of the protocol is based on the simultaneous detection within the time window \( \tau \) of odd or even number of photons at both Alice and Bob’s ends. This shall
be further elucidated in the subsequent section. In the current section, we assume that the detectors at the two ends are lossless. This assumption is relaxed in Section [V].

B. Execution of the protocol

\( \hat{a} \) and \( \hat{a}^\dagger \) are single mode annihilation and creation operators satisfying the commutation relation \( [\hat{a}, \hat{a}^\dagger] = 1 \). \( |n\rangle \) is an eigen state of number operator \( \hat{a} = \hat{a}^\dagger \) having eigen value \( n \). The annihilation and creation operators acting on the number basis can be given as \( \hat{a} |n\rangle = \sqrt{n} |n-1\rangle \), \( \hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \). The set of eigen states \( |n\rangle \) forms an orthonormal complete set \( \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{I} \). A coherent state \( |\alpha\rangle \) is an eigen state of \( \hat{a} \) with an eigen-value \( \alpha \), \( \hat{a} |\alpha\rangle = \alpha |\alpha\rangle \).

In number basis \( |\alpha_e\rangle \) is represented as

\[
|\alpha_e\rangle = \frac{1}{\cosh |\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} |2n\rangle
\]  

(9)

Even and odd (\( \propto (|\alpha\rangle \pm |\alpha\rangle) \)) coherent states and their respective preparations have been studied extensively. The reader may refer to [41] in order to understand experimental methods of preparing the aforementioned state.

The protocol described in Section [III A] can be carried out as described below. Let \( |\phi\rangle \) (c.f Eq. (8)) be the state shared by Alice and Bob. Here the first mode is with Alice while second mode is with Bob. We describe the production of the state \( |\phi\rangle \) using a 50:50 beam splitter (BS), assuming that we have the means to produce Cat states at our disposal. Let us define the action of 50:50 BS on input modes as

\[
\hat{a}^\dagger \rightarrow \hat{a}^\dagger + \hat{b}^\dagger \sqrt{2}, \quad \hat{b}^\dagger \rightarrow \hat{a}^\dagger - \hat{b}^\dagger \sqrt{2}.
\]  

(10)

This action on the input modes can be represented by

\[
|\phi\rangle = \frac{e^{-|\alpha|^2}}{1 + (-1)^x + y e^{-|\alpha|^2}} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} [(-1)^x |2n, 0\rangle + (-1)^y |0, 2n\rangle]
\]  

(16)

The average photon number of this state can be tuned to be quite small in principle, and is approximately \( \frac{|\alpha|^2}{4} \) when \( |\alpha| \) is small (c.f Appendix A). After encoding their

the following \( 2 \times 2 \) unitary matrix.

\[
BS = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]  

(11)

Consider the action of this BS on \( |\alpha\rangle |0\rangle \)

\[
|\alpha\rangle |0\rangle \equiv \exp (a\hat{a}^\dagger - \alpha^* \hat{a}) \otimes |0\rangle |0\rangle
\]

\[
\rightarrow \exp \left( \alpha \hat{a}^\dagger \hat{b}^\dagger - \alpha^* (\frac{\hat{a} + \hat{b}}{\sqrt{2}}) \right) |0\rangle |0\rangle
\]

\[
= |\alpha \rangle |\alpha \rangle
\]  

(12)

Similarly we have the following transformation by acting BS,

\[
|0\rangle |\alpha\rangle \rightarrow |\frac{\alpha}{\sqrt{2}}\rangle |\frac{\alpha}{\sqrt{2}}\rangle, \quad |\alpha\rangle |0\rangle \rightarrow |\frac{-\alpha}{\sqrt{2}}\rangle |\frac{\alpha}{\sqrt{2}}\rangle
\]

\[
|0\rangle |\alpha\rangle \rightarrow |\frac{-\alpha}{\sqrt{2}}\rangle |\frac{\alpha}{\sqrt{2}}\rangle
\]  

(13)

Thus the action of BS on \( |\phi\rangle \) leads to the following state

\[
\frac{1}{\sqrt{N(\alpha_e)}}(|\alpha_e\rangle |0\rangle + |0\rangle |\alpha_e\rangle + |\alpha\rangle |\alpha\rangle + |\alpha\rangle |\alpha\rangle)
\]

\[
\equiv \frac{1}{\sqrt{N(\alpha_e)}}(|\alpha_e\rangle |0\rangle + |0\rangle |\alpha_e\rangle)
\]

\[
\rightarrow N'|\left(\frac{\alpha}{\sqrt{2}e}\right) |\left(\frac{\alpha}{\sqrt{2}e}\right)\rangle \]  

(14)

By reversible application of BS on \( N'|\left(\frac{\alpha}{\sqrt{2}e}\right) |\left(\frac{\alpha}{\sqrt{2}e}\right)\rangle \) we get

\[
|\phi\rangle = \frac{1}{\sqrt{N(\alpha_e)}} (|\alpha_e\rangle |0\rangle + |0\rangle |\alpha_e\rangle)
\]  

(15)

where \( N(\alpha_e) = \frac{2(1+\exp\{-|\alpha|^2\})}{1+\exp\{-2|\alpha|^2\}} \)

Alice and Bob encodes their respective inputs into the phase of polarization degree of freedom of their respective modes resulting \( |\phi'\rangle = N(x,y)((-1)^x |\alpha_e\rangle |0\rangle + (-1)^y |0\rangle |\alpha_e\rangle) \) which is represented in number basis upon normalization as

inputs, they send the encoded state to BS. The action of BS on \( |\phi'\rangle \) results in the following state
\[ |\phi'_{\text{out}}(x, y)\rangle = \frac{e^{-|\alpha|^2}}{1 + (-1)^{x+y}e^{-|\alpha|^2}} \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \frac{\alpha^{2n}}{\sqrt{(2n)!}} \frac{1}{2^n} \left( \frac{2n}{k} \right)^{\frac{1}{2}} \left[ (-1)^x + (-1)^{y+2n-k} \right] |k, 2n-k\rangle. \] (17)

Given inputs \( x, y \in \{0, 1\} \) we have,

\[ |\phi'_{\text{out}}(x = 0, y = 0)\rangle = -|\phi'_{\text{out}}(x = 1, y = 1)\rangle = \frac{2e^{-|\alpha|^2}}{1 + e^{-|\alpha|^2}} \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \frac{\alpha^{2n}}{\sqrt{(2n)!}} \frac{1}{2^n} \left( \frac{2n}{2k} \right)^{\frac{1}{2}} |2k, 2n-k\rangle. \] (18)

\[ |\phi'_{\text{out}}(x = 0, y = 1)\rangle = -|\phi'_{\text{out}}(x = 1, y = 0)\rangle = \frac{2e^{-|\alpha|^2}}{1 - e^{-|\alpha|^2}} \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \frac{\alpha^{2n}}{\sqrt{(2n)!}} \frac{1}{2^n} \left( \frac{2n}{2k-1} \right)^{\frac{1}{2}} |2k-1, 2n-2k+1\rangle. \] (19)

The output state of the BS is directed to Alice and Bob’s labs for detection. In our protocol the measurement outcome \( a, b \in \{0, 1\} \) is associated with photon number parity resolving detectors:

\[ P_{\text{even}} = \sum_{l=0}^{\infty} |2l\rangle\langle 2l|, \quad P_{\text{odd}} = \sum_{l=0}^{\infty} |2l+1\rangle\langle 2l+1|. \] (20)

The outputs \( a, b = 0 \) correspond to detection of photon number with even parity at both Alice and Bob’s lab. Similarly the outputs \( a, b = 1 \) correspond to detection of photon number with odd parity at both Alice and Bob’s lab. It is clear from Eq. (18) and Eq. (19), that \( P(00|0\rangle|0) = P(11|0\rangle|1) = P(11|1\rangle|0) = P(00|1\rangle|1) = 1 \). The outcomes \( a, b \) satisfy the functional relation \( a = x \oplus y \) and \( b = x \oplus y \) with \( x, y \) being the inputs, thus facilitating Alice and Bob to deterministically predict the input of the other party, consequently furnishing a maximal violation of Inequality (6). The very same protocol can be used to show that the state \( \mathcal{N}|\alpha\rangle|0\rangle + |0\rangle|\alpha\rangle \) doesn’t furnish any violation of inequality (6) by noting that this state, on passing through the BS is separable (c.f. Equations [12] and [13]).

### IV. NON-LOCALITY

Consider a two party scenario in which both Alice and Bob have two choice of dichotomic measurements \( M_{0/1}^A \) and \( M_{0/1}^B \), with \( \pm 1 \) outcome respectively. The correlations that satisfy the local realistic condition is bounded by Bell-type inequality given as [22] [33]

\[ \mathcal{I} = \frac{1}{4} \sum_u \sum_v (-1)^{u\cdot v} \xi(v) \leq 1. \] (21)

Here \( u, v \in \{0, 1\}^2 \) and \( \xi(v) = \langle M_{v_1}^A M_{v_2}^B \rangle \) is the corresponding correlation function.

In Ref. [18], the authors demonstrated the non-local nature of the single photon state (11) by showing the violation of inequality (21). Here we show that the inequality (21) is being violated by \( |\phi\rangle \).

Consider the measurements as follows: \( M_{0}^A = M_{0}^B = 2|0\rangle\langle 0| - I, M_{1}^A = 2|\beta_1\rangle\langle \beta_1| - I \) and \( M_{1}^B = 2|\beta_2\rangle\langle \beta_2| - I \), where \( |\beta_k\rangle \) are coherent states. We denote \( \beta_1 = re^{i\phi_1} \) and \( \beta_2 = re^{i\phi_2} \), where \( j = -1 \). The correlators \( \langle M_{l}^A M_{m}^B \rangle, l, m \in \{0, 1\} \) are given in the Appendix (C). For \( r = 0.1 \), inequality (21) is violated by the even-coherent NOON for all the values of \( \phi_k \in [0, 2\pi] \) and \( |\alpha| \) and the violation is much greater than the violation achieved by single photon state (11). Contrariwise, the state \( \mathcal{N}|\alpha\rangle|0\rangle + |0\rangle|\alpha\rangle \) for all values of \( |\alpha| \) shows no violation of the aforesaid inequality. These results are summarized in Figures (1) and (2). In these figures, the plane plotted in yellow ochre marks the \( z = 1 \) plane, i.e. the classical bound of inequality (21).

![FIG. 1. (Colour online) On the left, the single-photon state (Eq. 1) shows violation of inequality (21) for a subset of \( \phi_1, \phi_2 \in [0, 2\pi] \). On the right, the even-coherent NOON state shows a greater violation of inequality (21) for all values of \( \phi_k \) in [0, 2\pi]. Here \( r = 0.1 \) and \( |\alpha| = 0.1 \).](image-url)

### V. LOSSY APPARATUS

We now describe some models of loss in the devices used in the protocol. The model of the lossy beam splitter described below is quite specific. There exist other models of lossy beam splitters.
A. Lossy beam splitters

It is widely known [44,45] that the unitary action of the lossless beam splitter on the input modes at frequency \( \omega \) can be summarised as follows.

\[
\begin{pmatrix}
\hat{b}_1(\omega) \\
\hat{b}_2(\omega)
\end{pmatrix} = T(\omega)
\begin{pmatrix}
\hat{a}_1(\omega) \\
\hat{a}_2(\omega)
\end{pmatrix},
\]

(22)

where \( \hat{a}_i(\omega) \) and \( \hat{b}_i(\omega) \) are the continuum annihilation operators of the input and output modes respectively. Here

\[
T(\omega) = \begin{pmatrix} t(\omega) & r(\omega) \\ r^*(\omega) & -t^*(\omega) \end{pmatrix}
\]

(23)

is a 2 \times 2 unitary matrix satisfying \( |t(\omega)|^2 + |r(\omega)|^2 = 1 \) and \( t(\omega)r^*(\omega) + r(\omega)t^*(\omega) = 0 \). In presence of losses in the system this unitarity doesn’t hold and \( |t(\omega)|^2 + |r(\omega)|^2 \leq 1 \), with the equality holding in the lossless case only. Hence in lossy systems the reflectivity and the transmittivity do not add up to unity [46].

The imaginary part the dielectric permittivity of matter contributes to the losses in the system. The dielectric permittivity can be considered to be approximately real if the losses in the chosen frequency interval are sufficiently small. In that case, the input-output relations of the beam splitter correspond to unitary transformations between input- and output-mode operators, as described previously. However, when the effects of absorption of radiation and consequent loss are taken into account, the relations between the input and output mode operators are no longer unitary. The sources of noise must be taken into account so as to preserve the canonical commutation relations for the outgoing fields. A Kramers-Kronig consistent quantization scheme of the electromagnetic field in dispersive and absorbing inhomogeneous media has been provided in ref. [46]. This work, has in turn been used to derive the unitary transformation that relates the output quantum state to the input quantum state by Knöll et al [47]. For the sake of completeness, we shall briefly describe this formalism. The idea is to define a U(4) matrix such that it transforms four input modes, two of the incoming field and two of the device, to four output modes, of which two are of the outgoing field and the other two are of the device. This approach thus allows for mode and energy conservation of the entire system whose constituents are the field and the device. Let us define the four dimensional input and output vector operators.

\[
\hat{\alpha}(\omega) = \begin{pmatrix}
\hat{a}_1(\omega) \\
\hat{a}_2(\omega) \\
\hat{g}_1(\omega) \\
\hat{g}_2(\omega)
\end{pmatrix},
\]

(24)

Here \( \hat{a}_j(\omega) \) \( (j \in \{1,2\}) \) are the amplitude operators of the incoming damped wave at frequency \( \omega \), \( \hat{g}_j(\omega) \) describe device excitations by playing the role of the noise operators. Similarly, we may write down the four-dimensional output vector operator as follows.

\[
\hat{\beta}(\omega) = \begin{pmatrix}
\hat{b}_1(\omega) \\
\hat{b}_2(\omega) \\
\hat{h}_1(\omega) \\
\hat{h}_2(\omega)
\end{pmatrix}.
\]

(25)

Here \( \hat{b}_j(\omega) \) and \( \hat{h}_j(\omega) \) \( (j \in \{1,2\}) \) are the corresponding output mode operators of the outgoing damped wave and the device, respectively, at frequency \( \omega \). Now we relate \( \hat{\beta}(\omega) \) and \( \hat{\alpha}(\omega) \) to each other as

\[
\hat{\beta}(\omega) = \Lambda(\omega)\hat{\alpha}(\omega),
\]

(26)

\[
\Lambda(\omega)\Lambda^\dagger(\omega) = I,
\]

(27)

where \( \Lambda(\omega) \in U(4) \), is given as follows.

\[
\Lambda(\omega) = \begin{pmatrix}
T(\omega) & \Lambda(\omega) \\
-S(\omega)C^{-1}(\omega)T(\omega) & C(\omega)S^{-1}(\omega)\Lambda(\omega)
\end{pmatrix},
\]

(28)

where the matrices \( T(\omega) \) and \( \Lambda(\omega) \) are \( 2 \times 2 \) matrices that represent the transmission and absorption respectively. They satisfy the following relation.

\[
T(\omega)T^\dagger(\omega) + \Lambda(\omega)\Lambda^\dagger(\omega) = I
\]

(29)

It must also be noted that \( C(\omega) \) and \( S(\omega) \) in equation [28], are commuting positive Hermitian matrices given by the following equations:

\[
C(\omega) = \sqrt{T(\omega)T^\dagger(\omega)},
\]

(30)

\[
S(\omega) = \sqrt{\Lambda(\omega)\Lambda^\dagger(\omega)}.
\]

(31)
The effects of absorption on the output field modes can be summarized by the following equation. Evidently, the dependence on the input device modes arise due to the presence of absorption.

\[
\hat{b}(\omega) = \mathbf{T}(\omega)\hat{a}(\omega) + \mathbf{A}(\omega)\hat{g}(\omega),
\]

where \(\hat{b}(\omega)\), \(\hat{a}(\omega)\), and \(\hat{g}(\omega)\) are column vectors of the output field operators, the input field operators, and the input device operators respectively. For our purpose of describing a 50:50 lossy beam splitter, we have chosen \(\mathbf{T}\) and \(\mathbf{A}\) to be frequency independent and of the following form.

\[
\mathbf{T} = \sqrt{\frac{\eta}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

where \(\eta \in [0, 1]\). The reader is requested to refer to Appendix D for the form of \(\mathbf{A}\) used in our calculations. Let us now write the four mode initial state by considering the modes of the device to be vacuum.

\[
|\phi\rangle = \frac{e^{-|\alpha|^2}}{1 + (-1)^{x+y}e^{-|\alpha|^2}} \sum_{n=0}^{\infty} \frac{a^{2n}}{\sqrt{(2n)!}} \{(-1)^x |2n, 0, 0, 0\rangle + (-1)^y |0, 2n, 0, 0\rangle\}.
\]

On passing through the lossy beam splitter, the following output state is obtained.

\[
|\phi'_{\text{out}}\rangle = A(x, y, \alpha) \sum_{n=0}^{\infty} \frac{(\alpha^n)}{2^n} \sum_{k_1=0}^{2n} \sum_{k_2=0}^{2n-k_1} \sum_{k_3=0}^{2n-k_1-k_2} \frac{((-1)^{x+k_1+k_2} + (-1)^{y+k_2+k_3})}{\sqrt{k_1!k_2!k_3!(2n-k_1-k_2-k_3)!}} |k_1, k_2, k_3, 2n-k_1-k_2-k_3\rangle,
\]

where \(A(x, y, \alpha) = \frac{e^{-|\alpha|^2}}{1 + (-1)^{x+y}e^{-|\alpha|^2}}\). Thus, on tracing over the device modes of \(\phi'_{\text{out}} = |\phi'_{\text{out}}\rangle \langle \phi'_{\text{out}}|\) (c.f. Appendix D) and on performing the projective measurements described in Eq. (20), we obtain the probabilities \(P(00|00) = P(00|11)\) and \(P(11|01) = P(11|10)\). It is to be noted that the detectors at both ends are assumed to be ideal and hence projective measurements are performed. If this assumption is relaxed, finding the corresponding probabilities may become all the more cumbersome. Thus, by this method, the left hand side of the GYNI inequality (4) can be obtained as a function of \(\eta\).

In order to draw a comparison between the effect of the lossy beam splitter (assuming the aforementioned model) on the state \(|\psi\rangle = \frac{(-1)^x|11\rangle + (-1)^y|00\rangle}{\sqrt{2}}\), has been studied. The measurements used in this case are consistent with ref. [1, 2], i.e, the existence of a single photon at Alice or Bob’s end, after the completion of the protocol. It turns out that the LHS of the GYNI inequality (4) is given by \(\eta\) in this case. These results are summarized in Fig. (3).

a. Alternate description of loss

Off late, a development [3] proposes methods to realize arbitrary linear transformations allowing for both loss and gain. A lossy beam splitter, though no longer a unitary operation, is certainly a linear transformation. The input and output modes can be augmented with ancilla modes (device modes) and singular value decomposition of the full network can be performed which allows each component to be further decomposed into a series of elementary operations. Fig. (4) represents the results of the following steps pictorially.

\[
\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{g}_1 \\ \hat{g}_2 \end{pmatrix} \rightarrow \begin{pmatrix} \hat{b}_1' \\ \hat{b}_2' \\ \hat{h}_1' \\ \hat{h}_2' \end{pmatrix} \rightarrow \begin{pmatrix} \hat{b}_1 = \sqrt{1-\eta} \hat{b}'_1 + \sqrt{1-\eta} \hat{h}'_1 \\ \hat{b}_2 = \sqrt{1-\eta} \hat{b}'_2 - \sqrt{1-\eta} \hat{h}'_2 \\ \hat{h}_1 = -\sqrt{1-\eta} \hat{b}'_1 + \sqrt{1-\eta} \hat{h}'_1 \\ \hat{h}_2 = -\sqrt{1-\eta} \hat{b}'_2 - \sqrt{1-\eta} \hat{h}'_2 \end{pmatrix}
\]

(37)

It can be verified that the four-mode output is identical to that which results from the action of Eq. (D6) on the
four-mode input. Thus, the alternate description can also be used to describe the mixing between the input modes of the field and those of the device and results precisely in the aforementioned model.

![Figure 3](image_url)

**FIG. 3.** (Colour online) The blue plane marks the classical bound of the GYNI inequality. The green plane in the plane that gives the LHS of the GYNI inequality \( J \) for the single photon state. The surface in yellow is the GYNI inequality given by the even-coherent NOON state, plotted as a function of \( \eta \) and \( |\alpha|^2 \). It is to be noted that the yellow curve shows violation of the GYNI inequality for quantum efficiency \( \eta \) of the beam splitter, less than 0.5 and small values of \( |\alpha|^2 \) and hence of the average photon number.

**B. Loss at the detection end**

At the detection end, in the absence of loss, it suffices to study the output using projectors on the even and the odd subspace, as given in Eq. (20). However, in the presence of loss, projective measurements have to be replaced by POVMs, taking the factors that contribute to the loss into consideration. A model for photon detection with loss and saturation has been given in ref. [49]. Suppose a detector can resolve up to \( N \) photons, this detector can then be modelled by a positive operator-valued measure (POVM) with \( N + 1 \) outcomes: \( \{ \Pi_0, \Pi_1, ..., \Pi_N \} \), satisfying completeness, \( \sum_{i=0}^{N} \Pi_i = I \), and positivity, \( \Pi_i \geq 0 \), for all \( i = 0, ..., N \). For a perfect detector with no loss (unit quantum efficiency \( \kappa \)) and no saturation, \( \{ \Pi_i = |i\rangle\langle i|, i = 1, 2, ... \} \). Let us define the \( m \)-th POVM effect of a lossy detector with quantum efficiency \( \kappa \) is

\[
\Pi_m = \sum_{n=m}^{\infty} w_{m,n}(\kappa) |n\rangle\langle n|,
\]

(38)

with

\[
w_{m,n}(\kappa) = \kappa^m (1 - \kappa)^{n-m} \binom{n}{m},
\]

(39)

for \( m = 0, 1, 2, ... \). Note in Eq. (38) that when \( m \) photons arrive at the detector, there exists a non-zero probability, if \( \kappa \) is not unity, for the detector to count more than \( m \) photons. Thus, \( \Pi_0 = \sum_{n=0}^{\infty} (1 - \eta)^n |n\rangle\langle n| \). This implies that even in the absence of incoming photons, an inefficient detector might detect some photons. This precisely is what the term ‘dark count’ refers to. The effect of saturation prevents the detector from resolving between \( N \) and \( N + 1 \) or more photons. Thus we have for the \( N \)-th outcome

\[
\Pi_N = I - \sum_{m=0}^{N-1} \Pi_m
\]

(40)

When \( N = 1 \), this POVM reduces to \( \{ \Pi_0, \Pi_1 \} \), where \( \Pi_1 = I - \Pi_0 \).

Recall that for the even-coherent NOON state, the measurements defined in Eq. (20) have been course grained from all possible outcomes of lossless, photon number resolving detectors. We use a similar approach to group the aforementioned effects into two broad effects, which in the limit of \( \kappa \) going to 1 and \( N \) going to infinity, would reduce to Eq. (20). We do this in two case: when the saturation is even and when the saturation is odd.
Case 1: When the detector can resolve up to $2N$ photons

We define the following POVM for our state:

$$\Pi_e^{(2N)} = \sum_{m=0}^{N-1} \Pi_{2m} + \frac{1}{2} (I - \sum_{m=0}^{2N-1} \Pi_m),$$  \hspace{1cm} (41)

$$\Pi_o^{(2N)} = \sum_{m=0}^{N-1} \Pi_{2m+1} + \frac{1}{2} (I - \sum_{m=0}^{2N-1} \Pi_m).$$  \hspace{1cm} (42)

Thus, our POVM is $\{\Pi_e^{(2N)}, \Pi_o^{(2N)}\}$. The corresponding GYNI inequality and its violation can then be studied in terms of functions of $|\alpha|^2$, $N$, and $\kappa$. In order to compare with the effect of lossy detector on the single-photon entangled state, we use the POVM: $\{\Pi_0 = \Pi_0 + \frac{1}{2} (I - \sum_{m=0}^{1} \Pi_{m}), \Pi_1 = \frac{1}{2} (I - \sum_{m=0}^{1} \Pi_{m})\}$, where $\Pi_1 = \sum_{n=0}^{\infty} n \kappa |n\rangle \langle n|$ and $\Pi_0$ is identical to what has been defined earlier. In case of the single-photon state the left hand side of the GYNI inequality is simply $\kappa$, similar to what is observed in the case of the lossy beam splitter.

Case 2: When the detector can resolve up to $2N - 1$ photons

The structure of the POVM in this case follows almost immediately from the previous one:

$$\Pi_e^{(2N-1)} = \sum_{m=0}^{N-1} \Pi_{2m} + \frac{1}{2} (I - \sum_{m=0}^{2N-2} \Pi_m),$$  \hspace{1cm} (43)

$$\Pi_o^{(2N-1)} = \sum_{m=0}^{N-2} \Pi_{2m+1} + \frac{1}{2} (I - \sum_{m=0}^{2N-2} \Pi_m).$$  \hspace{1cm} (44)

The POVM is $\{\Pi_e^{(2N-1)}, \Pi_o^{(2N-1)}\}$. The corresponding GYNI inequality and its violation can, once again, be studied in terms of functions of $|\alpha|^2$, $N$, and $\kappa$. In order to compare the effect of the lossy detector with odd saturation on the single-photon entangled state with that on the even-coherent NOON state, we can quite simply define the POVM to be $\{\Pi_0, \Pi_1\}$, where $\Pi_1 = I - \Pi_0$. Using this POVM on the single-photon state the left hand side of the GYNI inequality is once again found to be $\kappa$. The results are summarized in Fig. 5. It is to be noted that the violation of the GJNI inequality furnished by the even-coherent NOON state persists when the efficiency $\kappa$ is close to unity, even if the saturation number is small.

VI. CONCLUSION

Understanding the relation between an optical non-classicality and information theoretic tasks is an important way of revealing the non-classical nature of these states, and this greatly enhances our understanding of the quantum world that differs from that of classical. In this work we have used the concept of generalized NOON states, a single mode non-classical state embedded in two mode entangled state, and use it to show the maximal advantage in case of two way communication and that it gives rise to Bell-inequality violation. These two tasks are fundamentally different in the implementation form as well as in its polytopic characterization resisting to compare these two tasks. By using the same state in both the tasks reveals underpinning nature of single mode non-classicality which is responsible for the advantage of these two tasks. Perfection is too impossible to reach in an experimental scenarios. We show that the two-way communication task using the even-coherent NOON state under noisy beam splitter can outperform (in the regime of low average photon number and low efficiency)
the usage of single photon two mode entangled state. In case of the application of the lossy detector, the results in both states are comparable (maximal violation) when the efficiency is close to unity and saturation and the average photon number of the even-coherent NOON state are high. The fact that the even-coherent NOON state, in the regime of high quantum efficiency, ends up exhibiting some violation (not maximal) even when its average photon number is low, irrespective of the saturation number, is noteworthy.

We would now like to direct the reader’s attention to some open questions. We have investigated the performance of even-coherent NOON state under noisy BS and noisy detectors separately. Taking into account the presence of noise at each step simultaneously is likely to put a better comparison between generalized NOON state and the single photon two mode entangled state. The quantitative relation between non-classicality measures and performance in the two-way communication task would enhance our understanding of the relation between the non-classicality and an information theoretic task. In our work we have used two single mode non-classical states namely even coherent states and squeezed states. This prompts us to ask whether it is possible to obtain a similar advantage using other non-classical states. Since single mode squeezed states supply an advantage in two-way communication, any single mode non-classical pure Gaussian state is likely to supply a similar advantage. We expect non-Gaussian states to behave the same way.

Recent progress in single electron sources [50–52], and an experimental demonstration of single electron entanglement and non-locality [53] motivate to ask of the possibility of extending the questions addressed in our work to two- or multi-way communication using entangled states of single massive particles.

Appendix A: Average photon number of the chosen state

The average photon number $\bar{N}$ of the state $|\phi'\rangle$ (Eq. 16) obtained from $\langle\phi'| \hat{N}_A \otimes I + I \otimes \hat{N}_B |\phi'\rangle$ as a function of $\alpha$.

$$\bar{N} = \frac{|\alpha|^2}{(1 + e^{-|\alpha|^2})^2} \tag{A1}$$

It is evident from the above equation that the average photon number of $|\phi'\rangle$ is less than 1 iff $|\alpha|^2$ is less than 1.

Appendix B: Generalized NOON states involving squeezed states

In Ref. [12], the authors show the violation of causal Inequality [5] by using single-photon two-mode entangled state. Let's begin with some preliminary notions about the squeezed vacuum states. The squeezed vacuum state $|\xi\rangle = S(\xi)|0\rangle$ is obtained by operating the squeezing operator $S(\xi)$ on the vacuum state $|0\rangle$, where $\xi = re^{i\theta}$, $r$ is the squeezing parameter such that $0 \leq r < \infty$ and $0 \leq \theta \leq 2\pi$. The squeezed vacuum state in terms of photon number state is expressed as follows:

$$|\xi_{sq}\rangle \equiv S(\xi)|0\rangle = \frac{1}{\cosh r} \sum_{m=0}^{\infty} (-1)^m \frac{\sqrt{(2m)!}}{2^m m!} e^{im\theta} (\tanh r)^m |2m\rangle \tag{B2}$$

The important point to specify here is that the average photon number of squeezed vacuum state is $\langle \bar{N} \rangle_{\xi_{sq}} = \langle \xi_{sq}|a^\dagger a|\xi_{sq}\rangle = (\sinh r)^2$. Thus, although we are using the squeezed state, the average photon number can, in principle, be made comparable to lowered below 1.

Let us now apply the protocol given in Section IIIA to the squeezed vacuum NOON state $|\psi\rangle = \frac{1}{\sqrt{N(\xi)}} (|\xi_{sq}\rangle |0\rangle + |0\rangle |\xi_{sq}\rangle)$, where $N(\xi)$ is the normalization factor. After encoding the information of the inputs, as explained in the Section IIIB, the state will be

$$|\psi'\rangle = \frac{1}{\sqrt{N_{xy}(\xi)}} (-1)^x |\xi_{sq}\rangle |0\rangle + (-1)^y |0\rangle |\xi_{sq}\rangle \tag{B3}$$

where $N_{xy}(\xi)$ is the modified normalization factor (input dependent). Alternately, it may be expressed in the number basis as.

VII. ACKNOWLEDGEMENTS

We, the authors, would like to express our gratitude towards Sibasish Ghosh for his valuable inputs and for having discussed at length with us. BG and SA acknowledge the Institute of Mathematical Sciences, Chennai for providing them a visit during which this work was conceived.
\[ |\psi'\rangle = \frac{1}{\sqrt{N_{xy}(\xi)}} \frac{1}{\sqrt{\cosh r}} \sum_{m=0}^{\infty} \frac{(-\tanh r)^m}{2^m m!} \sqrt{(2m)!} \left[ (-1)^x |2m, 0\rangle + (-1)^y |0, 2m\rangle \right]. \] \hspace{1cm} (B4)

Acting the BS on the state (B4) and for given inputs \( x, y \in \{0, 1\} \), the output states can be given as

\[ |\psi_{out}'(x = 0, y = 0)\rangle = -|\psi_{out}'(x = 1, y = 1)\rangle = \frac{1}{\sqrt{2(\cosh r + 1)}} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(-\tanh r)^m}{2^{2m-1} m!} \sqrt{(2m)!} \left( \frac{2m}{2k} \right)^{\frac{1}{2}} |2k, 2(m-k)\rangle, \] \hspace{1cm} (B5)

\[ |\psi_{out}'(x = 0, y = 1)\rangle = -|\psi_{out}'(x = 1, y = 0)\rangle = \frac{1}{\sqrt{2(\cosh r + 1)}} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(-\tanh r)^m}{2^{2m-1} m!} \sqrt{(2m)!} \left( \frac{2m}{2k-1} \right)^{\frac{1}{2}} |2k-1, 2(m-k)+1\rangle. \] \hspace{1cm} (B6)

By using the measurements given in Eq. (20) and from Eq. (26) the associated correlations take values as \( P(00|00) = P(11|01) = P(11|10) = P(00|11) = 1 \) and violates GYNI maximally.

---

Appendix C: Bell correlators

As mentioned in Section 4, the Bell correlators found by performing displacement measurements on the state \( |\phi\rangle \) (Eq. 15).

\[ \langle M_0^A M_0^B \rangle = N[6e^{-|\alpha|^2} - (1 + e^{-2|\alpha|^2})] \] \hspace{1cm} (C1)

\[ \langle M_0^A M_1^B \rangle = N[2(4e^{-|\alpha|^2} - (1 + e^{-2|\alpha|^2})) (2e^{-|\beta|^2} - 1) + 2(e^{-|\alpha+\beta|^2} + e^{-|\alpha-\beta|^2}) + 2e^{-|\alpha|^2 + |\beta|^2}] \cos (2 \text{Im}(\alpha \beta^*)) - (1 + e^{-2|\alpha|^2}) + 4e^{-|\alpha|^2} (2e^{\beta^2} (\cosh (\alpha^2) + \cosh (\alpha \beta) - 1) \] \hspace{1cm} (C2)

\[ \langle M_1^A M_0^B \rangle = N[2(4e^{-|\alpha|^2} - (1 + e^{-2|\alpha|^2})) (2e^{-|\beta|^2} - 1) + 2(e^{-|\alpha+\beta|^2} + e^{-|\alpha-\beta|^2}) + 2e^{-|\alpha|^2 + |\beta|^2}] \cos (2 \text{Im}(\alpha \beta^*)) - (1 + e^{-2|\alpha|^2}) + 4e^{-|\alpha|^2} (2e^{\beta^2} (\cosh (\alpha^2) + \cosh (\alpha \beta) - 1) \] \hspace{1cm} (C3)

\[ \langle M_1^A M_1^B \rangle = N[2(e^{-|\beta|^2} - 1) (e^{-|\alpha+\beta|^2} + e^{-|\alpha-\beta|^2} + 2e^{-(|\alpha|^2 + |\beta|^2)} \cos (2 \text{Im}(\alpha \beta^*)) - (1 + e^{-2|\alpha|^2}) + 2(e^{-|\beta|^2} - 1) (e^{-|\alpha+\beta|^2} + e^{-|\alpha-\beta|^2} + 2e^{-(|\alpha|^2 + |\beta|^2)} \cos (2 \text{Im}(\alpha \beta^*)) - (1 + e^{-2|\alpha|^2}) + 4e^{-|\alpha|^2} [2(e^{-|\beta|^2} (\cosh (\beta^2) - 1) (2e^{-|\beta|^2} \cosh (\alpha^2) - 1)]) + (2e^{-|\beta|^2} \cosh (\alpha \beta^2) - 1)] \] \hspace{1cm} (C4)

where

\[ N = \frac{1}{(1 + e^{-|\alpha|^2})^2} \]

Appendix D: Action of the beam splitter on the four-mode Fock states

[47] for further details. Let

\[ \tilde{\rho}_{in} = |\psi_{in}\rangle \langle \psi_{in}|, \] \hspace{1cm} (D1)

with

\[ |\psi_{in}\rangle = |n_1, n_2, n_3, n_4\rangle = \prod_{\nu=1}^{4} \frac{\hat{\alpha}^{|n_\nu}}{\sqrt{n_\nu!}} |0\rangle, \] \hspace{1cm} (D2)
be the density operator of the system in the case when \(n_1\) and \(n_2\) quanta are the field mode excitations and \(n_3\) and \(n_4\) quanta are device mode excitations. From the application of Eq. (28) on \(\hat{\rho}_{in}\), we obtain
\[
\hat{\rho}_{out} = |\psi_{out}\rangle\langle\psi_{out}|, \tag{D3}
\]
with
\[
|\psi_{out}\rangle = \frac{1}{\sqrt{n_\nu}} \left(\sum_{\mu=1}^{4} \Lambda_{\mu\nu} \hat{a}_{\mu}^\dagger\right)^{n_\nu} |0\rangle \tag{D4}
\]
The term in brackets raised to the exponent \(n_\nu\) can be expanded using multinomial expansion. The density operator of the outgoing field modes is then obtained by tracing out over the device modes.
\[
\hat{\rho}_{out}^{(F)} = T r^{(D)}\{|\psi_{out}\rangle\langle\psi_{out}|\}. \tag{D5}
\]
Based on our choices of \(T\) and \(A\) (Eqs. 33-34), the following is the form of \(A\) used in our calculations.
\[
A = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\eta} & \sqrt{1-\eta} \\ \sqrt{1-\eta} & \sqrt{\eta} \end{pmatrix}. \tag{D6}
\]

[1] Flavio Del Santo and Borivoje Dakić, “Two-way communication with a single quantum particle,” Phys. Rev. Lett. 120, 060503 (2018).
[2] Francesco Massa, Amir Moqanaki, Flavio del Santo, Borivoje Dakic, and Philip Walther, “Experimental two-way communication with one photon,” in CLEO Pacific Rim Conference 2018 (Optical Society of America, 2018) p. F1D.4.
[3] A. Einstein, B. Podolsky, and N. Rosen, “Can quantum-mechanical description of physical reality be considered complete?” Phys. Rev. 47, 777–780 (1935).
[4] J. Bell, “On the Einstein-Podolsky-Rosen paradox,” Physics 1, 195 (1964).
[5] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner, “Bell nonlocality,” Rev. Mod. Phys. 86, 419–478 (2014).
[6] Harry Buhrman, Richard Cleve, Serge Massar, and Ronald de Wolf, “Nonlocality and communication complexity,” Rev. Mod. Phys. 82, 665–698 (2010).
[7] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters, “Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels,” Phys. Rev. Lett. 70, 1895–1899 (1993).
[8] Charles H. Bennett and Stephen J. Wiesner, “Communication via one- and two-particle operators on einstein-podolsky-rosen states,” Phys. Rev. Lett. 69, 2881–2884 (1992).
[9] L. Mandel and E. Wolf, Optical Coherence and Quantum Optics (Cambridge University Press, 1995).
[10] Rodney Loudon, The Theory of Light (Oxford University Press, 2000).
[11] S. M. Tan, D. F. Walls, and M. J. Collett, “Nonlocality of a single photon,” Phys. Rev. Lett. 66, 252–255 (1991).
[12] Lucien Hardy, “Nonlocality of a single photon revisited,” Phys. Rev. Lett. 73, 2279–2283 (1994).
[13] D. M. Greenberger, M. A. Horne, and A. Zeilinger, “Nonlocality of a single photon?” Phys. Rev. Lett. 75, 2064–2064 (1995).
[14] Lev Vaidman, “Nonlocality of a single photon revisited again,” Phys. Rev. Lett. 75, 2063–2063 (1995).
[15] Lucien Hardy, “Hardy replies;,” Phys. Rev. Lett. 75, 2065–2066 (1995).
[16] Jacob Dunningham and Vlatko Vedral, “Nonlocality of a single particle,” Phys. Rev. Lett. 99, 180404 (2007).
[17] S. J. van Enk, “Single-particle entanglement,” Phys. Rev. A 72, 064306 (2005).
[18] Jonatan Bohr Brask, Rafael Chaves, and Nicolas Brunner, “Testing nonlocality of a single photon without a shared reference frame,” Phys. Rev. A 88, 012111 (2013).
[19] Nicolas Sangouard, Christoph Simon, Hugues de Riedmatten, and Nicolas Gisin, “Quantum repeaters based on atomic ensembles and linear optics,” Rev. Mod. Phys. 83, 33–80 (2011).
[20] Nicolas Sangouard and Hugo Zbinden, “What are single photons good for?” Journal of Modern Optics 59, 1458–1464 (2012).
[21] A. I. Lvovsky, H. Hansen, T. Aichele, O. Benson, J. Mlynek, and S. Schiller, “Quantum state reconstruction of the single-photon fock state,” Phys. Rev. Lett. 87, 050402 (2001).
[22] C. K. Hong, Z. Y. Ou, and L. Mandel, “Measurement of subpicosecond time intervals between two photons by interference,” Phys. Rev. Lett. 59, 2044–2046 (1987).
[23] T.B. Pittman, B.C. Jacobs, and J.D. Franson, “Heralding single photons from pulsed parametric down-conversion,” Optics Communications 246, 545 – 550 (2005).
[24] Yu-Ping Huang, Joseph B. Altepeter, and Prem Kumar, “Optimized heralding schemes for single photons,” Phys. Rev. A 84, 033844 (2011).
[25] Wojciech Wasilewski, Czeslaw Radzewicz, Robert Frankowski, and Konrad Banaszek, “Statistics of multiphoton events in spontaneous parametric down-conversion,” Phys. Rev. A 78, 033831 (2008).
[26] Daryl Achilles, Christine Silberhorn, and Ian A. Walmsley, “Direct, loss-tolerant characterization of nonclassical photon statistics,” Phys. Rev. Lett. 97, 043602 (2006).
[27] Wolfgang Maurer, Malte Avenhaus, Wolfram Helwig, and Christine Silberhorn, “How colors influence numbers: Photon statistics of parametric down-conversion,” Phys. Rev. A 80, 053815 (2009).
[28] Malcolm N. O’Sullivan, Kam Wai Clifford Chan, Vasudevan Lakshminarayanan, and Robert W. Boyd, “Conditional preparation of states containing a definite number of photons,” Phys. Rev. A 77, 023804 (2008).
[29] P Sekatski, N Sangouard, F Bussières, C Clausen, N Gisin, and H Zbinden, “Detector imperfections in photon-pair source characterization,” Journal of Physics B: Atomic, Molecular and Optical Physics 45, 124016 (2012).

[30] Andreas Christ and Christine Silberhorn, “Limits on the deterministic creation of pure single-photon states using parametric down-conversion,” Phys. Rev. A 85, 023829 (2012).

[31] Pieter Kok, Hwang Lee, and Jonathan P. Dowling, “Creation of large-photon-number path entanglement conditioned on photodetection,” Phys. Rev. A 65, 052104 (2002).

[32] Christopher C. Gerry, “Non-classical properties of even and odd coherent states,” Journal of Modern Optics 40, 1053–1071 (1993), https://doi.org/10.1080/09500349314551131.

[33] V. Bužek, A. Vidiella-Barranco, and P. L. Knight, “Superpositions of coherent states: Squeezing and dissipation,” Phys. Rev. A 45, 6570–6585 (1992).

[34] V V Dodonov, “Nonclassical states in quantum optics: a squeezed review of the first 75 years,” Journal of Optics B: Quantum and Semiclassical Optics 4, R1–R33 (2002).

[35] M. S. Kim, W. Son, V. Bužek, and P. L. Knight, “Entanglement by a beam splitter: Nonclassicality as a prerequisite for entanglement,” Phys. Rev. A 65, 032323 (2002).

[36] Saleh Rahimi-Keshari, Timothy C. Ralph, and Carlton M. Caves, “Sufficient conditions for efficient classical simulation of quantum optics,” Phys. Rev. X 6, 021039 (2016).

[37] Victor Veitch, Christopher Ferrie, David Gross, and Joseph Emerson, “Negative quasi-probability as a resource for quantum computation,” New Journal of Physics 14, 113011 (2012).

[38] Victor Veitch, Nathan Wiebe, Christopher Ferrie, and Joseph Emerson, “Efficient simulation scheme for a class of quantum optics experiments with non-negative wigner representation,” New Journal of Physics 15, 013037 (2013).

[39] Scott Aaronson and Alex Arkhipov, “The computational complexity of linear optics,” in Proceedings of the forty-third annual ACM symposium on Theory of computing (ACM, 2011) pp. 333–342.

[40] Cyril Branciard, Mateus Arajo, Adrien Feix, Fabio Costa, and Časlav Brukner, “The simplest causal inequalities and their violation,” New Journal of Physics 18, 013008 (2016).

[41] H. Le Jeannic et al., “Remote preparation of continuous-variable qubits using loss-tolerant hybrid entanglement of light,” Optica 5, 1012–1015 (2018).

[42] R. F. Werner and M. M. Wolf, “All-multipartite bell-correlation inequalities for two dichotomic observables per site,” Phys. Rev. A 64, 032112 (2001).

[43] Marek Žukowski and Časlav Brukner, “Bell’s theorem for general n-qubit states,” Phys. Rev. Lett. 88, 210401 (2002).

[44] U. Leonhardt, Measuring the Quantum State of Light, by Ulf Leonhardt, pp. 204. ISBN 0521497302. Cambridge, UK: Cambridge University Press, July 1997. (1997) p. 204.

[45] Stephen M. Barnett, John Jeffers, Alessandra Gatti, and Rodney Loudon, “Quantum optics of lossy beam splitters,” Physical Review A 57, 2134–2145 (1998).

[46] T. Grüner and D.-G. Welsch, “Quantum-optical input-output relations for dispersive and lossy multilayer dielectric plates,” Phys. Rev. A. 54, 1661 (1996).

[47] L. Knöll, S. Scheel, E. Schmidt, D.-G. Welsch, and A.V. Chizov, “Quantum-state transformation by dispersive and absorbing four-port devices,” Phys. Rev. A 59, 4716 (1999).

[48] N. Tischler, C. Rockstuhl, and K. Slowik, “Quantum optical realization of arbitrary linear transformations allowing for loss and gain,” Phys. Rev. X 8, 021017 (2018).

[49] Si-Hui Tan, Leonid A. Krivitsky, and Berthold-Georg Englert, “Measuring quantum correlations using lossy photon-number-resolving detectors with saturation,” Journal of Modern Optics 63, 276 (2016).

[50] Erwann Bocquillon, Vincent Freulon, J-M Berroir, Pascal Degiovanni, Bernard Plaçais, A Cavanna, Yong Jin, and Gwendal Fève, “Coherence and indistinguishability of single electrons emitted by independent sources,” Science 339, 1054–1057 (2013).

[51] Izhar Neder, Nissim Ofek, Y Chung, M Heiblum, D Mahalu, and V Umansky, “Interference between two indistinguishable electrons from independent sources,” Nature 448, 333 (2007).

[52] Yang Ji, Yunchul Chung, D Sprinzak, M Heiblum, D Mahalu, and Hadas Shtrikman, “An electronic mach–zehnder interferometer,” Nature 422, 415 (2003).

[53] David Dasenbrook, Joseph Bowles, Jonatan Bohr Brask, Patrick P Hofer, Christian Flindt, and Nicolas Brunner, “Single-electron entanglement and nonlocality,” New Journal of Physics 18, 043036 (2016).