DOA estimation based on fractional low-order multi-sensor time-frequency analysis in heavy tailed noise

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Abstract. In order to improve the performance of Direction-of-arrival (DOA) estimation in heavy-tailed noise, a novel DOA estimation method based on fractional low-order multi-sensor time-frequency distribution is proposed. We first use the high-resolution of the fractional low-order multi-sensor time-frequency analysis (FLOM-MTFA) to obtain the time-frequency matrices of the array signal. Then, the high-energy points of the fractional low-order multi-sensor time-frequency matrices is selected to improve the MSNR. We finally compute the averaged FLOM-MTFA matrix and exploit estimation signal parameter via rotational invariance techniques to estimate DOA. Theoretical analysis and simulation results show that the proposed method can effectively estimate DOA in heavy tailed noise.

1. Introduction

Direction-of-arrival (DOA) based on antenna array is estimated to be one of the important research directions in array signal processing. The main purpose of DOA estimation is to estimate and extract the direction of arrival of the space target signal by using the measured data received by the sensor array arranged in a certain way in space [1]. DOA estimate has broad application prospects in military and civilian fields such as radar, passive sonar, biomedicine, radio astronomy, and seismic survey [2].

At present, many effective algorithms have been proposed for the DOA estimation in Gaussian noise. These algorithms can be roughly divided into two categories, methods based on subspace analysis [3]-[4] and algorithms based on sparse reconstruction [5]-[6]. The methods based on subspace analysis are represented by the multiple signal classification (Multiple signal Classification, MUSIC) algorithm [3] and the subspace rotation invariance (Estimation Signal Parameter via Rotational Invariance Techniques, ESPRIT) algorithm [4]. The MUSIC algorithm decomposes the received data covariance matrix into a signal subspace and a noise subspace, and uses the orthogonality of the two subspaces to solve the spatial spectrum to achieve super-resolution DOA estimation. The ESPRIT algorithm uses the rotation-invariant characteristics of the signal subspace of the covariance matrix of the received signal, and obtains the low-rank subspace of all signal sources through correlation operations, thereby completing DOA estimation. Although algorithms based on subspace analysis can obtain super-resolution estimation performance, they also have some problems. On the one hand, the second-order statistics of the observed signal need to be obtained in the solution process. When the number of sampling snapshots is relatively small, the covariance of the actual received data is a biased estimate, which makes it impossible to effectively estimate the signal source. On the other hand, when the signal source is a coherent signal source, the rank of the received data covariance is reduced, the signal subspace penetrates into the noise subspace, and a large number of eigenvalues equal to the...
signal source cannot be obtained, which makes DOA estimation fail. The algorithm based on sparse reconstruction is proposed based on sparse representation and signal reconstruction theory. Among them, the sparse reconstruction algorithm based on $l_1$ norm minimization has received widespread attention [5]. This type of algorithm performs singular value decomposition on the received data matrix and extracts most of the power of the signal into the measurement matrix of lower dimensions, thereby achieving good reconstruction of the sparse solution of DOA estimation. Compared with algorithms based on subspace analysis, algorithms based on sparse reconstruction have better robustness, can effectively process coherent signal sources, and allow the use of fewer snapshots for DOA estimation. However, the performance of the DOA estimation algorithm based on sparse reconstruction is easily affected by the regularization parameter. When the constraint parameters are not properly selected, it will cause deviations in the signal reconstruction results, and reduce the accuracy of DOA estimation.

In recent years, there have also been some methods for DOA estimation under heavy tailed noise, mainly including DOA estimation algorithms based on fractional lower-order moments (FLOM) [7]-[8], DOA estimation algorithm based on correlation entropy [9]-[10] and DOA estimation algorithm based on sparse recovery [11]-[12]. The method based on FLOS realizes DOA estimation by constructing a class-covariance matrix based on FLOM and using subspace analysis methods (such as MUSIC, ESPRIT). This method can not only realize DOA estimation in tail noise, but also obtain good performance under Gaussian noise. However, algorithms based on fractional low-order statistics are sub-optimal and require larger sampling snapshots. The DOA estimation algorithm based on correlation entropy uses correlation entropy to suppress the impulsive characteristics of heavy tailed noise, and builds a pseudo-covariance matrix based on correlation entropy to complete DOA estimation. This method can obtain good estimation performance under both heavy tailed noise and Gaussian noise, but the algorithm has high computational complexity. The DOA estimation algorithm based on sparse recovery treats the abnormal points in the heavy tailed noise as a sparse vector, uses compressed sensing algorithms to reconstruct the sparse signal and sparse noise at the same time, and uses the signal vector and sparse noise vector to form a new sparse vector to recover joint sparse vectors, and then identify the sparse components of the signal, and then determine the DOA information of the signal. This algorithm can realize DOA estimation in heavy tailing noise, but when the impulsive abnormal points of heavy tailed noise increase, it will cause the increase of sparse elements, causing its estimation performance to decrease.

In order to effectively improve the performance of DOA estimation in heavy tailed noise, we combine the fractional low-order moment with high-precision time-frequency analysis methods, and propose a DOA estimator based on FLOM-MTFA (Fractional low-order multi-sensor time-frequency analysis). This method introduces fractional low-order moments into multi-sensor time-frequency analysis, constructs a fractional low-order time-frequency matrix, and then reconstructs the class-covariance matrix based on FLOM-MTFA and finally uses the ESPRIT algorithm to estimate DOA. Simulation experiments show that under heavy tailed noise, the DOA estimation performance of the proposed method is better than traditional methods and it is robust to different noise characteristic indexes.

2. Array model of signal

Considering that $M$ sensors with a spatial distance of half wavelength $d = \frac{\lambda}{2}$ form a uniform linear array (ULA), and the sensor numbers are 1 to $N$ respectively. There are $P$ ($P < M$) narrowband signals $\{s_p(t)\}_{p=1}^P$ with center wavelength $\lambda$ in the far field at $\{\theta_p\}_{p=1}^P$, that is, the DOAs of the $P$ signals are $\{\theta_1, \theta_2, \cdots, \theta_P\}$. Thus, the received data vector formed by the output of $M$ sensors can be expressed as
\[ X(t) = Y(t) + W(t) = AS(t) + W(t) \]  
(1)

where 
\[ S(t) = [s_1(t), s_2(t), \ldots, s_p(t)]^T \]

is the \( P \times 1 \) dimensional signal vector, and
\[ W(t) = [w_1(t), w_2(t), \ldots, w_M(t)]^T \]

is the \( M \times 1 \) dimensional noise vector. 
\[ A = [a(\theta_1), a(\theta_2), \ldots, a(\theta_P)] \]

is the \( P \times P \) dimensional array steering matrix, and the ideal array steering vector corresponding to the \( p \)-th signal is specifically expressed as
\[ a(\theta_p) = \begin{bmatrix} 1, e^{j2\pi \sin(\theta_p)/\lambda}, \ldots, e^{j2\pi(M-1)\sin(\theta_p)/\lambda} \end{bmatrix} \]  
(2)

In this paper, we consider alpha stable distribution noise to simulate heavy tailed noise. Since the alpha stable distribution does not have a closed-form probability density function, it is usually characterized by a characteristic function, and its characteristic function is
\[ \phi(u) = \exp\left[iu\gamma|u|^\alpha [1 + j\beta \text{sgn}(u)\omega(u, \alpha)]\right] \]  
(3)

\[ \omega(u, \alpha) = \begin{cases} \tan(\pi\alpha / 2) & \alpha \neq 1 \\ (2 / \pi)\log|u| & \alpha = 1 \end{cases} \]  
(4)

\[ \text{sgn}(u) = \begin{cases} 1 & u > 0 \\ 0 & u = 0 \\ -1 & u < 0 \end{cases} \]  
(5)

where \( \alpha \) is the characteristic index, \( \gamma \) is the dispersion coefficient, and \( \beta \) determines the degree of skew of the distribution. Since there is no finite second moment in \( \alpha \) stable distribution noise, its variance is meaningless. The mixed signal-to-noise ratio \( \text{MSNR} = 10\log(\sigma_s^2 / \gamma) \) (dB) is defined in the article to characterize the relationship between signal power and noise power, where \( \sigma_s^2 \) represents the variance of the signal, and \( \gamma \) represents the dispersion coefficient of heavy tailed noise.

3. DOA estimation method based on flom-ntfa

In this section, we first introduce the fractional low-order multi-sensor time-frequency analysis and analyze its characteristics, and then propose DOA estimation based on the fractional low-order multi-sensor time-frequency analysis.

3.1. Fractional low-order multi-sensor time-frequency analysis

If signal \( X(t) \) is the vector signal received by the array, its fractional low-order correlation matrix can be expressed as
\[ R_{\alpha}(t, \tau) = E\left\{ K_{\alpha}(t, \tau) \right\} \]
\[ = E\left\{ X\left( t + \frac{\tau}{2} \right)^{\alpha} \right\} \left\{ X\left( t - \frac{\tau}{2} \right)^{\alpha} \right\} \]  
(6)

where \( E\{\} \) represents the expected operation, \( (\cdot)^H \) represents the Hermitian transpose, \( \chi^{(b)} = [\chi_{b-1}^{(b)}, \ldots, \chi] \), and \( K_{\alpha}(t, \tau) \) can be expressed as
\[ K_{\omega}(t,\tau) = \begin{bmatrix} x(t+\tau/2)^\omega_i \cdot x(t-\tau/2)^\omega_j \end{bmatrix} \]

\[= \begin{bmatrix} K_{\omega_1}(t,\tau) & K_{\omega_2}(t,\tau) & \cdots & K_{\omega_M}(t,\tau) \\ K_{\omega_M}(t,\tau) & K_{\omega_2}(t,\tau) & \cdots & K_{\omega_1}(t,\tau) \\ \vdots & \vdots & \ddots & \vdots \\ K_{\omega_M}(t,\tau) & K_{\omega_2}(t,\tau) & \cdots & K_{\omega_1}(t,\tau) \end{bmatrix} \]

(7)

where \[ K_{\omega_i}(t,\tau) = \begin{bmatrix} x_i(t+\tau/2)^\omega_i \cdot x_i(t-\tau/2)^\omega_j \end{bmatrix}, \]

\[i, j = 1, 2, \ldots, M, x^{(0)} = |\alpha x|, \]

this operation only changes the amplitude of the signal, without changing the frequency and phase information. When \(0 < b < \alpha\) is satisfied, \(x^{(0)}\) has a finite variance.

According to (6) and (7), we can obtain

\[ R_{\omega}(t,\tau) = \text{AR}_{\omega}(t,\tau)A^H + \sigma^2 \delta(\tau)I_M \]

(8)

where \(R_{\omega}(t,\tau)\) represents the score correlation matrix of the transmitted signal, the correlation matrix of noise is \(R_{\omega}(t,\tau) = \sigma^2 \delta(\tau)I_M\) and \(I_M\) is the unit matrix of \(M \times M\) dimension.

According to the correlation matrix \(R_{\omega}(t,\tau)\) of signal \(x(t)\), its multi-sensor time-frequency distribution can be expressed as

\[ P_{\omega}(t,f) = \mathcal{F}_{t,f} \{Q(t,\tau) \ast R_{\omega}(t,\tau)\} \]

(9)

where \(\mathcal{F}_{t,f}\) represents the Fourier transform, \(Q(t,\tau)\) represents the delay kernel function, and \(Q(t,\tau) = \mathcal{F}_{t,f}^{-1}\{h(t,f)\}\), where \(h(t,f)\) is the smoothing window of variables \(t\) and \(f\).

Substituting (8) into (9), we can get

\[ P_{\omega}(t,f) = \mathcal{F}_{t,f} \{Q(t,\tau) \ast [\text{AR}_{\omega}(t,\tau)A^H + \sigma^2 I_M]\} \]

\[= \mathcal{F}_{t,f} \{Q(t,\tau) \ast [\text{AR}_{\omega}(t,\tau)A^H + \sigma^2 I_M] \ast Q(t,\tau)\} \]

\[= \mathcal{F}_{t,f} \{Q(t,\tau) \ast [\text{AR}_{\omega}(t,\tau)A^H + \sigma^2 I_M] \ast [Q(t,\tau) \ast Q(t,\tau)]\} \]

(10)

where \(P_{\omega}(t,f) = \mathcal{F}_{t,f} \{Q(t,\tau) \ast R_{\omega}(t,\tau)\}\) and \(\sigma^2 = \sigma^2 \cdot q(0,0), q(\nu,\tau) = \mathcal{F}_{t,f} \{Q(t,\tau)\}\)

3.2. DOA estimation method based on FLOM-MTFA

According to the analysis in Section 3.1, FLOM-MTFA has fractional low-order characteristics, which can effectively suppress heavy tailed noise. In addition, FLOM-MTFA has higher time-frequency resolution, which can effectively improve DOA estimation performance. The DOA estimation method based on FLOM-MTFA can be as follows.

First, select the high-energy time-frequency coordinates in \(P_{\omega}(t,f)\). Assuming that the number of array elements is \(M\), FLOM-MTFA can be expressed as:

\[ P_{\omega}(t,f) = \begin{bmatrix} P_{\omega_1,1}(t,f) & P_{\omega_2,1}(t,f) & \cdots & P_{\omega_M,1}(t,f) \\ P_{\omega_1,2}(t,f) & P_{\omega_2,2}(t,f) & \cdots & P_{\omega_M,2}(t,f) \\ \vdots & \vdots & \ddots & \vdots \\ P_{\omega_1,M}(t,f) & P_{\omega_2,M}(t,f) & \cdots & P_{\omega_M,M}(t,f) \end{bmatrix} \]

(11)

In order to improve the anti-noise performance of the multi-sensor time-frequency distribution, the autocorrelation time-frequency distribution in \(P_{\omega}(t,f)\) is selected for time-frequency distribution fusion, which is:
The threshold screening method is used to select the high-energy time-frequency coordinates in \( P_{\text{mix}}(t,f) \), and the high-energy time-frequency coordinate set \( C_r \) is obtained. If \( |P_{\text{mix}}(t,f)| > \vartheta \), then \((t_i,f_i)\) represents high-energy time-frequency coordinates, where the threshold \( \vartheta \) is generally set to \( \vartheta = 0.05 \cdot \max\{P_{\text{mix}}\} \).

Then, construct the time-frequency analysis matrix and calculate the spatial spectrum function. Construct the time-frequency analysis matrix through the selected high-energy time-frequency coordinates, as follows:

\[
G_n(t,f) = \frac{1}{C_n} \sum_{i=1}^{C_n} P_n(t_i,f_i) \tag{13}
\]

The eigenvalue decomposition is used to get the eigenvector \( \{\hat{V}_1, \ldots, \hat{V}_M\} \). Then, the signal subspace \( \mathbf{V}_s \) can be expressed as

\[
\mathbf{V}_s = \begin{bmatrix} \mathbf{V}_u & \text{first row} \\ \text{last row} & \mathbf{V}_d \end{bmatrix} \tag{14}
\]

Finally, DOA estimates can be obtained by

\[
\theta_i = \arccos\left(\frac{j \lambda \log(\psi_i)}{2\pi d}\right) \tag{15}
\]

where \( \psi_i \) is the eigenvalues of the matrix \( \mathbf{V} \), and \( \mathbf{V}_d = \mathbf{V}_s \mathbf{\psi} \). Note that the matrix \( \mathbf{\psi} \) can be obtained by using a least square sense to solve \( \mathbf{V}_d = \mathbf{V}_s \mathbf{\psi} \).

4. Simulation signal analysis

In this section, we sample Monte Carlo simulation experiments to verify the effectiveness of the proposed algorithm in a tail noise environment. The simulation experiment uses a ULA composed of 8 antenna array elements \( M = 8 \), where the interval between adjacent antenna elements is \( d = \lambda/2 \), assuming there are 8 far-field narrowband statistically independent target signal sources, the incident angles are \( \theta_1 = 10^\circ \), \( \theta_2 = 30^\circ \), \( \theta_3 = 50^\circ \) respectively. All simulation results are subjected to 1000 Monte Carlo simulations. The root mean square error is used as the standard to measure the performance of the algorithm. The normalized root mean square error is defined as

\[
\text{NRMSE} = \sqrt{\frac{1}{ZM} \sum_{z=1}^{Z} \sum_{n=1}^{N} \left( \hat{\theta}_{n,z} - \theta_n \right)^2} \tag{16}
\]

Among them, \( Z \) represents the number of Monte Carlo experiments, \( \theta_n \) represents the actual incident angle of the \( n\)-th signal, and \( \hat{\theta}_{n,z} \) represents the estimated value of the incident angle of the \( n\)-th signal in the \( z\)-th experiment.

In order to verify the effectiveness of the proposed method based on FLOM-MTFA, the proposed method is compared with the method based on FLOM-SCM (fractional lower order sample covariance matrix). Set the mixed signal-to-noise ratio is increased from -10dB to 10dB at 2dB intervals, and 1000 Monte Carlo experiments are performed under each MSNR. The NRMSE curves of FLOM-MTFA and FLOM-SCM algorithm are shown in Figure 1.
Figure 1. DOA estimation performance under different mixed signal-to-noise ratios.

Figure 1 shows that the proposed algorithm can effectively adapt to the heavy tailed noise, and the performance of the proposed algorithm is significantly better than the method based on FLOM-SCM. Especially, the performance of FLOM-MTFA is advantages under the condition of high MSNR. In addition, it can be seen from Figure 1 that with the increase of MSNR, the NRMSE of the DOA estimation of the proposed FLOM-MTFA gradually decreases. When MSNR is greater than 5dB, the NRMSE of the proposed algorithm is close to -8dB.

In order to verify the influence of the noise characteristic index on the performance of the algorithm, the NRMSE curves of FLOM-MTFA is plotted under different characteristic indexes in Figure 2. The MSNR is set to 6dB and the range of the characteristic index is set from 1 to 2.

Figure 2. DOA estimation performance under different noise characteristic indices.

It can be seen from Figure 2 that, under different values of characteristic index, the proposed method is verified. In addition, with the value of characteristic index gradually increases, the root mean square error also gradually decreases. For $\alpha = 2.0$, that is, in the case of Gaussian noise, the DOA estimation method proposed in this paper also has good estimation performance.

5. Conclusion
This paper uses the alpha stable distribution to describe the heavy tailed noise, and proposes a new DOA estimation method based on fractional low-order multi-sensor time-frequency analysis. We introduce fractional low-order statistics and multi-sensor time-frequency analysis into array signal processing, and construct a time-frequency distribution matrix through fractional low-order multi-sensor time-frequency analysis. Then, the DOA is estimated by using the ESPRIT method. The simulation results show that the estimation performance of this method is obviously better than that of the existing methods, and the method also has good estimation performance for different characteristic indexes.
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