Dynamics controlled by additive noise

G.D. Lythe
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Cambridge CB3 9EW, UK

Abstract

Analysis is presented of a system whose dynamics are dramatically simplified by tiny amounts of additive noise. The dynamics divide naturally into two phases. In the slower phase, trajectories are close to an invariant manifold; this allows small random disturbances to exert a controlling influence. A map is derived which provides an accurate description of the trajectories.

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1. Introduction

In certain systems described by differential equations, tiny amounts of added noise have a controlling influence on the dynamics. The influence of noise is a simplifying one: the larger the noise level the more regular trajectories become. Sensitivity to noise arises because solutions of the differential equations spend most of their time near a slow invariant manifold. Noise with magnitude $\epsilon$ controls the dynamics if $\epsilon > \sqrt{\mu}$, where $1/\mu$ is the timescale for dynamics near the invariant manifold.

The type of behaviour described here is found, for example, in differential equations modelling the resonant interaction of wave modes [1], turbulent flows [2], pulsating laser oscillations [3] and plane Poiseuille flow [4]. In this work, the following system, derived as a model for the shear instability of tall convection cells [5], is analysed:

\[
\begin{align*}
\dot{x} &= \mu x - y^2 + z^2, \\
\dot{y} &= y(x - 1 + \delta) + \delta z, \\
\dot{z} &= z(-x - 1 + \delta) + \delta y;
\end{align*}
\]

\[0 < \mu < 1, \quad 0 \leq \delta < 0.5. \tag{1}\]

The dynamics in the case $\delta = 0$ were analysed by Hughes and Proctor in terms of a one-dimensional map [5]. In this work a map is derived which describes the solutions when $\delta$ is non-zero and the system is subject to white noise with magnitude $\epsilon$ such that $\epsilon \ll \sqrt{\mu} \ll 1$. The dynamics are organised about the line $y = z = 0$ (the unstable manifold of the fixed point at the origin) and, to lowest order in $\mu$ and $\delta^2$, can be described exclusively in terms of the parameter $\mu |\ln \epsilon|$.

In the simplest case, Figure 1(a), the solutions consist of long periods (slow phases) during which $y$ and $z$ are $O(\epsilon)$ and $|x|$ slowly increases, occasionally interrupted by fast phases during which $|x|$ decreases. The solution is then fully specified in terms of the probability distribution of $x_{\text{max}}$, the turning point of $x$ at the end of a slow phase. An accurate expression for this distribution is obtained by modelling the slow phase with the stochastic differential equation

\[dy = yg(x)dt + \epsilon dw \tag{2}\]

where $w$ is the Wiener process [6] and $x$ is taken to be a function of time satisfying $\dot{x} = \mu x$. The function $g(x)$, which approximates $x(t) - 1 + \delta + \frac{\dot{x}}{y}(t)$, begins negative and passes through 0 in the course of a slow phase. The ratio $r(t) = \frac{\dot{x}}{y}$ is obtained by assuming that it is slaved to the instantaneous value of $x$. Because $g(x)$ is
Figure 1. Noise-dependent solutions. The four trajectories shown are solutions of (1), differing only in the level of added noise, $\epsilon$. The greater the noise level, the simpler the solution. Sensitivity to noise arises because the solutions spend most of their time near $y = z = 0$ with $|x|$ slowly increasing. ($\delta = 0.3$ and $\mu = 0.01$.) Fixed points are marked with crosses.

slowly-varying, (2) is a dynamic bifurcation problem in which trajectories remain near $y = z = 0$ for a time $\mathcal{O}(\frac{1}{\sqrt{\mu}})$ after $g(x)$ becomes positive. Note that it is only by the action of noise that $y$ changes sign. The condition $\epsilon \ll \sqrt{\mu}$ is necessary to justify the neglect, for long periods, of the quadratic terms in the equation for $\dot{x}$. The analysis, based on separating slow and fast phases, is accurate when $\mu \ll 1$.

The solution of (2) specifies the mean and standard deviation of $y$ as a function of $x$. It is thus possible to calculate, as a function of $x$, the probability that $y^2 - z^2 > \mu x$; taking the derivative with respect to $x$ of this probability gives the distribution
of \( x_{\text{max}} \), which has a non-Gaussian form \([7]\). The maximum of the distribution is at \( x_{\text{max}} = \hat{x} \) where, to lowest order in \( \mu \), \( \hat{x} \) satisfies

\[
F(\hat{x}) - F(\alpha) = \mu |\ln \epsilon| \tag{3}
\]

where \( \frac{d}{dx}F(x) = g(x)/x \) and \( g(\alpha) = 0 \). The standard deviation of the distribution of \( x_{\text{max}} \) is

\[
\sigma_{x_{\text{max}}} = \frac{\pi}{2\sqrt{2}} \frac{\mu \hat{x}}{g(\hat{x})}. \tag{4}
\]

Note that the width of the distribution is \( \mathcal{O}(\mu) \), not \( \mathcal{O}(\epsilon) \). The probability distribution of \( x_{\text{max}} \) is affected only slightly by the introduction of a finite correlation time ('colour') in the noise \([8]\). On the other hand, multiplicative noise or mean-zero deterministic perturbations have a much less dramatic effect on trajectories than additive Gaussian noise.

For the trajectory shown in Figure 1(a), successive values of \( x_{\text{max}} \) are independent and the specification of the probability distribution of \( x_{\text{max}} \) provides an essentially complete description of the solution of (1). More generally the value \( \hat{x} \) is an upper limit on \( x_{\text{max}} \). This still has a profound, and simplifying, influence on the dynamics.

A different approximation scheme is appropriate for the fast phase. The simplest is to assume that the brief excursions away from the vicinity of \( y = z = 0 \) are semicircular, ie that \( x_0 \) is related to the previous value of \( x_{\text{max}} \) by

\[
x_{\text{max}} - 1 + \delta = 1 - \delta - x_0, \tag{5}
\]

where \( x_0 \) is the minimum value of \( x \) in a cycle (and thus the starting point for a slow phase). Corrections to (5), necessary for an accurate description of the dynamics in the cases shown in Figure 1(c) and 1(d), will be introduced in the next section. They are not necessary in the case shown in Figure 1(a) because, in every slow phase, \(|y|\) and \(|z|\) descend to \( \mathcal{O}(\epsilon) \) before \( g(x) \) passes through 0 and so \( x_{\text{max}} \) is in fact independent of the previous \( x_0 \). The trajectory shown in Figure 1(b) differs from 1(a) in that \( x \) changes sign in the course of each fast phase. Because of the invariance of (1) under the transformation

\[
x \mapsto -x, \quad \text{and} \quad y \leftrightarrow z, \tag{6}
\]

the analysis of each slow phase can proceed for \( x < 0 \) as before, with the roles of \( y \) and \( z \) reversed and \( x \) replaced by \( |x| \). Thus the statistics of \( |x_{\text{max}}| \) are the
same as in Figure 1(a), and successive values of $x_{\text{max}}$ have opposite signs. (At the same parameter values as Figure 1(a), there is a stable noisily periodic orbit with $x$ always negative.)

The change from trajectories in which $x$ always keeps the same sign (Figure 1(a)) to those in which $x$ changes sign (Figure 1(b)) occurs when $\hat{x}$ is sufficiently large, i.e., when $\mu|\ln \epsilon|$ is sufficiently large. The next qualitative change that occurs for larger $\mu|\ln \epsilon|$ (larger $\mu$ or smaller $\epsilon$) is a period-doubling bifurcation (Figure 1(c)). This occurs when the value of $|x_0|$ that follows $\hat{x}$ approaches $1 - \delta$ so that $y$ and $z$ can remain above the noise level throughout a slow phase. The condition for this bifurcation also translates into a condition on $\mu|\ln \epsilon|$. After the period-doubling bifurcation, the parameter $\mu|\ln \epsilon|$ can still be used as a bifurcation parameter but trajectories can no longer simply be described in terms of one probability distribution. A more sophisticated analysis, in terms of a map, is appropriate. This is developed in the next section.

2. Description by an analytical map

Starting from a given value of $x_{\text{max}}$, there is a fast phase which carries the system to a value of $x$, $x_0$, at which a slow phase begins. This slow phase carries the system to a new value of $x_{\text{max}}$. The object is to describe analytically solutions of (1) in terms of a one-dimensional map of successive values of $x_{\text{max}}$ (values of $x$ at the end of a slow phase at which $\dot{x} = 0$). The strategy is to treat the slow phase, in which noise may have a controlling influence, separately from the fast phase, in which noise is unimportant but the nonlinearities are more troublesome.

Analysis of the fast phase is based on the function $h(t)$ where $h = (x - 1 + \delta)^2 + y^2 - z^2$. This satisfies $\dot{h} = 2\mu(x - 1 + \delta)x + 4xz^2$. (Typically $z < \delta$ during the fast phase.) By integrating $\dot{h}$ along a semicircular trajectory, the following approximation was obtained:

$$\left(|x'| - 1 + \delta\right)^2 = \left(|x_{\text{max}}| - 1 + \delta\right)^2 + \left(|x_{\text{max}}| - 1 + \delta\right)(6\mu + 4\delta^2). \quad (7)$$

Here $x'$ is written instead of $x_0$ because, if $x'$ and $x_{\text{max}}$ have opposite signs, a new slow phase does not begin immediately – notice the looping back in Figure 1(d). Given $x_{\text{max}}$, to generate $x_0$ (the starting value for the next slow phase) further iterates of (7), $x'', x''', \ldots$ are generated. (These are approximations for successive turning points of $x$.) When two successive values have the same sign, the procedure ends and the system is deemed to have reached $x_0$. Analysis of the fast phase is simpler when $\delta$ is 0 or is extremely small [5] because looping back of trajectories does not occur.
The slow phase starts with \( x = x_0 \). If \( |x_0| \) is sufficiently far from \( 1 - \delta \) that \( |y| \) and \( |z| \) descend to the noise level before \( g(t) \) passes through 0, the probability distribution of \( x_{\text{max}} \) can be described very accurately by (3) and (4). On the other hand, if the minimum value of \( |y| \) in a slow phase is above the noise level then the relationship between \( x_{\text{max}} \) and the previous value of \( x_0 \) is

\[
F(x_0) = F(x_{\text{max}}). \tag{8}
\]

It is possible to derive an expression which has both (3) and (8) as limits [7]. The changeover between régimes is very rapid as a function of \( x_0 \), occurring when the minimum value of \( y \) that would be found in a slow phase without noise (typically \( e^{-1/\mu} \)) equals the noise level \( \epsilon \). In practice, therefore, \( x_{\text{max}} \) is generated by taking the minimum of the two values given by (3) and (8) – the effect of noise, if any, is always to reduce the average value of \( x_{\text{max}} \). The map thus obtained is compared with that resulting from numerical solution of (1) in Figure 2, where a rather low noise level is chosen to exhibit some of the structure of the map. For larger values of \( \epsilon \), the cutoff value of \( x_{\text{max}} \), given by (3), means that only a few humps are seen.

The bifurcation structure of multi-modal maps is extremely rich, but the smearing in \( x \) due to (4) means that, for non-vanishing \( \mu \) only the first few bifurcations are distinguished. Finite \( \mu \) also has the effect of making the map inaccurate in the region \(|x_{\text{max}}| - 1 + \delta| < \sqrt{\mu}\), where the distinction between fast and slow phases breaks down.

The numerical solutions exhibited in this work were produced using a second order stochastic algorithm [9]. Adding very small noise usually presents no difficulties because the noise-sensitive variables are themselves small, but for extremely small noise levels, it may be necessary to use \( \ln y \) and \( \ln z \) as variables rather than \( y \) and \( z \).

3. Relevance to shear in convection

When a horizontal layer of fluid is heated from below, convection begins if the temperature difference across the layer exceeds a threshold value. Just above this threshold, the motion often consists of rolls [10]. If these rolls are tall and thin they can exist for long periods but are typically subject to shear (a net sideways movement) which can temporarily destroy the roll pattern. In this situation, the motion of the fluid is in two space dimensions and can therefore be described in terms of a scalar field known as the stream function. The stream function in this case can be written as a sum of three important modes (representing ‘convection’,
Figure 2. Map of successive values of $x_{\text{max}}$. If $x_1$ is the turning point of $x$ at the end of a slow phase then the next such turning point is $x_2$. The dots are numerical results from simulation of (1), with $\epsilon = 10^{-80}$ noise added independently to each variable, and the solid lines calculated as described in the text. ($\mu = 0.01$, $\delta = 0.3$.) The dotted line is $x_1 = x_2$. The most important effect of noise is to set an upper limit, (in this case 3.6) on $x_{\text{max}}$. This reduces the map to a finite number of humps. (In numerical simulations of trajectories, a value of $x$ at which $\dot{x} = 0$ is deemed to be at the end of a slow phase if, in addition, $\dot{y}/y > 0$ and $\dot{z}/z > 0$. This has the effect of restricting the maps to $|x_{\text{max}}| > \sqrt{1 - 2\delta}$.)

'shear' and 'tilt') plus other slaved modes [11]. Substitution of this form for the stream function into the partial differential equations for Boussinesq convection results in a third order system of ordinary differential equations which successfully models the behaviour of the solutions of the partial differential equations when the rolls are tall and the Prandtl number is small. The system (1) is produced from this by rescaling the three variables so that the coefficients of all quadratic terms are unity [12].

In the model system (1), the variable $x$ is proportional to the amplitude of the convective roll pattern, which persists for long periods and is occasionally destabilised by shear before reestablishing itself. The variables $y$ and $z$ are linear combinations of 'shear' and 'tilt', $\mu$ is proportional to the amount by which the temperature difference across the layer exceeds the critical value, and $\delta$ is a function of the
Prandtl number. (A fifth order set of ordinary differential equations is appropriate when the Prandtl number is not small and there is a magnetic field present [11].)

4. Conclusion

Dynamical systems whose solutions return repeatedly to the neighbourhood of a slow invariant manifold can be controlled by added noise with magnitude \( \epsilon \) if \( \epsilon < e^{-1/\mu} \), where \( \frac{1}{\mu} \) is the timescale for dynamics near the manifold. It is possible to construct analytically a one-dimensional map when \( \epsilon \ll \sqrt{\mu} \ll 1 \). Noise affects the dynamics by setting an upper limit on the size of excursions away from the invariant manifold. The upper limit is a function of \( \mu|\ln \epsilon| \) and the width of the appropriate probability distribution is proportional to \( \mu \) (not to \( \epsilon \)).

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