A cellular automaton for blocking queen games

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Abstract We show that the winning positions of a certain type of two-player game form interesting patterns which often defy analysis, yet can be computed by a cellular automaton. The game, known as Blocking Wythoff Nim, consists of moving a queen as in chess, but always towards (0, 0), and it may not be moved to any of \(k−1\) temporarily “blocked” positions specified on the previous turn by the other player. The game ends when a player wins by blocking all possible moves of the other player. The value of \(k\) is a parameter that defines the game, and the pattern of winning positions can be very sensitive to \(k\). As \(k\) becomes large, parts of the pattern of winning positions converge to recurring chaotic patterns that are independent of \(k\). The patterns for large \(k\) display an unprecedented amount of self-organization at many scales, and here we attempt to describe the self-organized structure that appears. This paper extends a previous study (Cook et al. in Cellular automata and discrete complex systems, AUTOMATA 2015, Lecture Notes in Computer Science, vol 9099, pp 71–84, 2015), containing further analysis and new insights into the long term behaviour and structures generated by our blocking queen cellular automaton.

Keywords Wythoff Nim · Blocking Wythoff Nim · Cellular automata · Self-organization

1 Blocking queen games (k-Blocking Wythoff Nim)

In the paper Larsson (2011), the game of \(k\)-Blocking Wythoff Nim was introduced, with rules as follows.

Formulation 1: As in Wythoff Nim (Wythoff 1907), two players alternate in removing counters from two heaps: any number may be removed from just one of the heaps, or the same number may be removed from both heaps. However, a player is allowed to reject the opponent’s move (so the opponent must go back and choose a different, non-rejected move), up to \(k−1\) times, where \(k\) is a parameter that is fixed for the game. The \(k\)th distinct attempted move must be allowed. Thus, if there are at least \(k\) winning moves among the options from a given position, then one of these winning moves can be played.

Formulation 2: There are \(k\) chess pieces on an infinite (single quadrant) chess board: one queen, and \(k−1\) pawns. On your turn you move the queen towards the origin. (The first player who cannot do this loses.) The queen cannot be moved to a position occupied by a pawn, but it can move over pawns to an empty position. After moving the queen, you complete your turn by moving the \(k\) pawns to wherever you like. The pawns serve to block up to \(k−1\) of the queen’s possible next moves.

Example game: Consider a game with \(k = 5\), where the queen is now at \(3, 3\) (yellow in Fig. 1). It is player A’s turn, and player B is blocking the four positions \{\(0,0\), \(1,1\), \(0,3\), \(3,0\)\} (dark brown and light olive). This leaves A with the options \{\(3,1\), \(3,2\), \(2,2\), \(2,3\), \(1,3\)\} (each is black or blue). Regardless of which of these A chooses, B will
then have at least five winning moves to choose from (ones marked yellow, or light, medium, or dark olive). These are winning moves because it is possible when moving there to block all possible moves of the other player immediately win. Therefore player B will win.

As shown in Fig. 1, there is a simple algorithm to compute the winning positions (for the previous player) for game \( k \). These are known as P-positions in combinatorial game theory, and we will refer to them as palace positions, the idea being that the queen wants to move to a palace: if you move her to a palace, you can win, while if you move her to a non-palace, your opponent can win. To win, you must always block (with pawns) all of the palaces your opponent might move to.

The idea is simply that a palace is built on any site that can see fewer than \( k \) other palaces when looking due north, west, or north-west. In this way, the pattern of palaces can be constructed, starting at \((0, 0)\) and going outward. For efficiency, a dynamic programming approach can be used, storing three numbers at each position, for the number of palaces visible in each of the three directions. With this technique, each line can be computed just from the information in the previous line, allowing significant savings in memory usage.

The case \( k = 1 \) corresponds to classical Wythoff Nim, solved in Wythoff (1907). In Larsson (2011), the game was solved for \( k = 2 \) and 3. When we say a game is solved we mean it is possible to give closed-form expressions for the P-positions, or at least that a winning move, if it exists, can be found in log-polynomial time in the heap sizes. For example, the set \{\([n \phi], [n \phi^2]\), \([n \phi^2], [n \phi]\)\}, where \( n \) runs over the nonnegative integers and \( \phi = \frac{1+\sqrt{5}}{2} \) is the golden ratio, provides a solution for classical Wythoff Nim. Combinatorial games with a blocking maneuver appear in Gavel and Strimling (2004), Holshouser and Reiter (2001), Holshouser and Reiter (NA) and Smith and Stânică (2002), and specifically for Wythoff Nim in Gurvich (2010), Hegarty and Larsson (2006), Larsson (2009) and Larsson (2015).

The new idea that we use in this paper is to look directly at the number of palaces that the queen can see from each position on the game board, and to focus on this palace number rather than just on the palace positions (P-positions). (The palace positions are exactly the locations with palace numbers less than \( k \).) In previous explorations, the palace number was in fact computed but then only used for finding the new palace positions, which when viewed on their own give the appearance of involving long-range information transfer. By observing the palace number directly, however, we can see that in almost all positions the palace number is surprisingly close to \( k \), and if we look at its deviation from \( k \) (the surplus number) then we are led to discover that these surplus numbers follow a local rule that is, surprisingly, independent of \( k \). The next section will present this local rule as a cellular automaton (CA). Other recent results connecting cellular automata to combinatorial games can be found in Fink (2012), Larsson (2013) and Larsson and Wästlund (2013).

The rest of the paper will present the rich structure visible in the patterns of surplus numbers for games with large \( k \). A surprising number of regions of self organization appear, largely independent of the particular value of \( k \), and some of the self-organized patterns are quite complex, involving multiple layers of self-organization. This is the first CA we are aware of that exhibits so many levels of self-organization. So far, these patterns offers many more questions than answers, so for now we will simply try to catalog our initial observations.
2 A cellular automaton perspective

It is clear that a local rule can propagate the palace number—it simply propagates the number of palaces visible to the north, northwest, and west (so actually it propagates three numbers whose sum is the palace number). When these sum to less than \( k \), a palace is placed at the given location and the running totals are increased by one. A visualization of these three running totals is given in the center and right columns of Fig. 16. In general, these running totals range between 0 and \( k \).

Using this dynamic programming approach, the information for each cell can be computed from just its parents and central grandparent, and is thus a CA (although not the main one we are presenting). However, such a CA needs a large number of states for the cells (we conjecture order \( O(k^2 \log k) \) states for the interesting parts) and each different value of \( k \) requires a different rule.

Surprisingly, there is also a local rule which is independent of \( k \), which can propagate a single number, namely the palace number minus the blocking number. We call this the surplus number. To get a picture such as in Fig. 1, we can apply this one rule to a simple initial condition that depends on \( k \). This single number is surprisingly close to zero (we conjecture within \( O(\log k) \)) everywhere except near the initial condition in the upper left, where it is on the order of \( k \). For example, in Fig. 1, except in the upper left, this number ranges only between \(-1\) (yellow) and \(+2\) (indigo). This small number of states strongly motivates understanding its operation in terms of being the emergent behavior of a cellular automaton.

Here in Sect. 2 we derive this CA rule, and in Sect. 3 we will see that the pictures generated by this rule are very complex yet surprisingly similar for different values of \( k \).

2.1 Definition of the CA

The CA we present is one-dimensional and operates on a diamond space-time grid as shown in Fig. 2, so that at each time step, each cell that is present at that time step derives from two parents at the previous time step, one being half a unit to the right, the other being half a unit to the left. On the diamond grid, there is no cell from the previous step that is in the same position as a cell on the current step. However, the three grandparents of a cell in the diamond grid do include a cell which is at the same spatial position as the grandchild cell.

Our CA rule is based not only on the states of the two parent cells, but also on the state of the central grandparent cell, as well as its parents and central grandparent, all shown in blue in Fig. 2. As such, this CA depends on the previous four time steps, i.e. it is a fourth-order CA. It is the first naturally occurring fourth-order CA that we are aware of. We say it is “naturally occurring” simply because we discovered these pictures by analyzing the blocking queen game, and only later realized that these pictures can also be computed by the fourth-order diamond-grid CA we present here.

The states in our CA are integers. In general they are close to 0, but the exact bounds depend on the initial conditions. The formula for computing a cell’s value, given its neighborhood, is described in Fig. 2.

2.2 The connection between the CA and the game

Since our definition of the CA appears to be completely different from the definition of the blocking queen game, we need to explain the correspondence.

The idea is that the states of this CA correspond to palace numbers minus \( k \), which are generally integers close to zero. One can easily prove a bound of \( 3k + 1 \) for the number of states needed for the game with blocking number \( k \), since each row, column, and diagonal can have at most \( k \) palaces, so palace numbers are always in the range \([0, 3k]\). However, in practice the number of states needed (after the initial ramping-up region near the origin)

\[
g = a - b - c + e + f + p. \]

These green squares correspond to the blue cells and show the palace compensation terms. For any blue cell containing a negative value (and therefore a palace, see Sect. 2.2), the corresponding palace compensation term must be added. In the formula, \( p \) represents the total contribution of these palace compensation terms. Note that location \( d \) only affects \( g \) via its palace compensation term, so only its sign matters.
appears to be far smaller, more like log \( k \). For example, when \( k = 500 \), only eight states are needed, ranging between \(-4\) and \(3\).

Surprisingly, this single CA is capable of computing the pattern of palace numbers regardless of the value of \( k \). Different values of \( k \) simply require different initial conditions: The initial condition for a given value of \( k \) is simple: Every site in the quadrant opposite the game quadrant should be \( k \), and every site in the other two quadrants should be 0.

**Theorem 1** The \( k \)-Blocking Wythoff Nim position \((x, y)\) is a P-position if and only if the CA given in Fig. 2 gives a negative value at that position, when the CA is started from an initial condition defined by

\[
CA(x, y) = \begin{cases} 
  k & x < 0 \text{ and } y < 0 \\
  0 & x < 0 \text{ and } y \geq 0 \\
  0 & x \geq 0 \text{ and } y < 0 
\end{cases}
\]

**Proof** First we will consider the case of no P-positions occurring within the CA neighborhood, so compensation terms can be ignored.

As shown in Fig. 2, we will let \( v_1 \) be the number of P-positions directly above \( a \) and \( c \), and similarly for \( v_2 \) and \( v_3 \), as well as for the diagonals \( d_i \) and horizontal rows \( h_i \).

This gives us \( a = v_1 + d_2 + h_1 - k \), and so on: when adding the \( k \)-value to each cell this represents the sum of the numbers of P-positions in the three directions.

We would like to express \( g \) in terms of the other values. Notice that

\[
a + e + f = \sum_{i=1}^{3} v_i + \sum_{i=1}^{3} d_i + \sum_{i=1}^{3} h_i - 3k = b + c + g
\]

and therefore \( a + e + f = b + c + g \), allowing us to express \( g \) in terms of the other values as \( g = a - b - e + f \).

All that remains is to take any P-positions in the CA neighborhood into account, so as to understand the compensation terms.

If there is a P-position at \( a \), then \( b, c, d, \) and \( g \) (i.e. the positions in a line to the right, down, or right-down) will all be one higher than they were before taking that palace into account. Since the equation \( a + e + f = b + c + g \) was true when ignoring the palace at \( a \), it becomes wrong when the palace at \( a \) produces its effect of incrementing \( b, c, d \), and \( g \), because that makes the right hand side go up by three while the left hand side is untouched. To compensate for this, we can add a term \( p_a \) to the left hand side, which is three if there is a palace at \( a \), and 0 otherwise.

Similarly, if \( b \) is a P-position, then this increments \( d, e, \) and \( f \), so to compensate, we will need to subtract 2 from the left hand side of \( a + e + f = b + c + g \). Hence \( p_b = -2 \) if there is a palace at \( b \), and otherwise \( p_b = 0 \).

We can see that we are computing exactly the compensation terms shown in the green squares of Fig. 2. Once we include all the compensation terms, i.e. \( p = \sum p_i \), the formula for \( g \) becomes correct even in the presence of local P-positions, and it corresponds exactly to the rule given in Fig. 2.

The initial condition can be confirmed to produce (via the CA rule) the correct values in the first two rows and columns of the game quadrant, and from that point onwards the reasoning given above shows that the correct palace numbers, and therefore the correct P-positions, are being computed by the CA rule. \(\square\)

### 2.3 Notes on reversibility

The reversed version of this CA computes \( a \), given \( b, c, d, e, f, \) and \( g \). This is done with the equation \( a = g - f - e + c + b - p \), which is equivalent to the equation in the caption of Fig. 2. However, the palace compensation term \( p \) can depend on \( a \), so this equation has not fully isolated \( a \) on the left hand side. (The forward direction did not have this problem, since \( p \) does not depend on \( g \).)

Writing \( p = p_a + p_{bcdef} \) to separate the palace compensation term due to \( a \) from the other palace compensation terms, we get \( a + p_a = g - f - e + c + b - p_{bcdef} \). Since \( p_a \) is 3 when \( a \) is negative, and 0 otherwise, this equation always yields either one or two solutions for \( a \). If the right hand side is 3 or more, then \( a \) must be equal to it. If the right hand side is negative, then \( a \) must be 3 less than it. And if the right hand side is 0, 1, or 2, then \( a \) can either be equal to it or be 3 less than it—we are free to choose. The reversed rule is non-deterministic, but it can always find a compatible value. In other words, there is no “Garden of Eden” pattern for this rule, if we assume that all integers are permissible states.

### 3 Self-organization

This section consists mainly of empirical observations, corresponding to conjectures rather than proven theorems. We believe that these observations are nonetheless interesting enough to warrant their dissemination, and we hope that researchers will be able to prove these and/or other facts about this automaton’s behavior in the future.

The top row of Fig. 3 shows the palace number patterns for games 100 and 1000. The patterns are strikingly similar, given that the value of \( k \) differs by an order of magnitude. The pattern for game 1000 has the appearance of being “the same, but ten times bigger” than the pattern for game 100. The middle and lower rows of Fig. 3 zoom in on
subregions where the two patterns are in fact identical, without any scaling factor.

3.1 Terminology

As an aid to our discussion of these complex images, we will give names to the prominent features in them.

We can see that this system self-organizes itself into 11 regions with 14 borders and 6 junctions. Ignoring duplicates due to the mirror symmetry, there are 7 regions, 7 borders, and 4 junction points.

Let us first consider the regions, shown in Fig. 4. The region at the upper left (the game’s terminal region, and the CA’s starting region), in the shape of a dented triangle, is the hood (Fig. 4). The triangular regions adjacent to the hood, with a periodic interior, (visible in all panels of Fig. 3), are the épaulettes (Fig. 4). The rhomboid region that emanates from between the pair of épaulettes is the fabric (Fig. 4). The solid black regions at the top and at the left constitute the outer space (Fig. 4). Between the outer space and the épaulettes we find the arms which extend indefinitely (Fig. 4). Extending next to the arms, and of
similar width, we have the warps (Fig. 4). Each warp contains a number of threads (strings of yellow dots, clearly visible in the top left panel of Fig. 3) which come out of the fabric. Between the warps lies the inner sector (Fig. 4), and the blue stripes in the warps and in the inner sector are the weft.

Next, let us consider the junction points. The hood, épaulettes, and fabric all meet at the nose. The hood, épaulette, arm, and outer space all meet at the shoulder. The warp, fabric, épaulette, and arm all meet at the armpit, which is often a hotspot of highly positive palace numbers. Finally, the fabric, warps, and inner sector meet at the prism (located just before \((5k/3, 5k/3)\)). The inner sector often contains slightly higher palace numbers than the warps, especially near the main diagonal, giving the impression of light being emitted from the prism, as in Fig. 5.

Next, let us consider the borders. The hood and épaulette meet cleanly at the casing, which extends from the nose to the shoulder. The hood contains all the states from \(-k\) to 0, but after the casing, the CA uses very few states. The épaulette and arm meet at the hem, which extends from the shoulder to the armpit. The épaulette and fabric meet at the rift, which has no features of its own near the nose, but as it goes towards the armpit for large \(k\), it slowly widens into a narrow, relatively thread-free space. The fabric and warp meet at the fray, where threads almost parallel to the fray unravel from the fabric, and threads in the other direction exit the fabric and start merging to form the thicker threads of the warp. There is no clear boundary distinguishing the warp from the inner sector, rather, the warp simply runs out of threads, as can be seen in Fig. 5. The warp also meets the arm cleanly, at the inside of the arm, as can be seen on the left side of Fig. 6. At the boundary between the warp and the arm, we can interpret the yellow nature of the arm as due to it being packed full of threads (indeed, it could be viewed as a giant thread), while the warp simply has a much lower density of threads than the arm. Threads that bend into the inner sector, and stop being parallel to the rest of the warp, are sometimes called beams Larsson (2012).

The often-occurring slightly-separated periodic part of the arm, bordering the outer space, is the skin, clearly visible in Figs. 9, 10, and 11.

![Fig. 4](image) Names of the seven self-organized regions of blocking queen games (for large \(k\)), and of some of the borders between them, and of the points where these borders meet

![Fig. 5](image) (left) A comparison between \(k = 497\) and \(k = 500\). Positions where the two images differ are masked in white. All other colors show places where the two images match, meaning that if the palace number at position \((x, y)\) is \(p\) in the image for \(k = 497\), then the palace number at position \((x + 1, y + 1)\) is \(p + 3\) in the image for \(k = 500\). P-positions are shown in yellow (and brown, in the hood), and N-positions are shown in black and blue. Note that the hood, épaulettes, and fabric match perfectly. (right) A comparison between \(k = 499\) and \(k = 500\). In this comparison, “matching” means that if the palace number at position \((x, y)\) is \(p\) in the image for \(k = 499\), then the palace number at position \((x + 1, y)\) is \(p + 1\) in the image for \(k = 500\). Note that the épaulette and arm match perfectly, as does half of the hood. Both comparisons show a 1000 × 1000 region.
The periods (and all integral linear combinations of those). Only the warp. If viewing this document electronically, zoom in for detail. When the rift hits the hem. This starts the fray, which separates the meta-glider behavior in the fabric from the merging threads of the warp. If nobody moves into the hood, thereby winning the game. When players are not making mistakes, the game ends when (and only when) somebody moves into the hood, thereby winning the game.

The fray, warp, central sector, armpits, and prism are all very sensitive to $k$, but all the other regions are not, with the exception of the fabric and the rift, which are sensitive only to $k \mod 3$. The fabric, fray, warp, prism, and central sector are all full of weft, which generally defies analysis until it becomes periodic. The left-hand warp does not contain vertical weft stripes, and the right-hand warp does not contain horizontal weft stripes, except within threads. Often the centermost beams will communicate with each other via the weft, and this process can usually be analyzed to calculate the slopes of these beams, which are generally quadratic irrationals.

This is the greatest complexity of self-organization that we have seen for a system that has no structured input.

3.2 Structure within the regions

The hood and the épaulettes have a very regular structure. The palace numbers in the hood increase steadily in each row and column, increasing by one when the greater coordinate is incremented, and by two when the lesser coordinate is incremented. Thus the hood contains all palace numbers from 0, at the upper left corner, to $k$, where the hood meets the épaulettes at the casing, a line with slope 2 (before the nose) and 1/2 (after the nose) that connects the nose at $(k/3, k/3)$ to the shoulders at $(0, k)$ and $(k, 0)$. The hood is exactly the region where every position is a winning move, because all possible further moves can be blocked, thereby immediately winning the game. When players are not making mistakes, the game ends when (and only when) somebody moves into the hood, thereby winning the game.

The palace numbers in the épaulettes form a two-dimensional periodic pattern, with periods $(5, 1)$ and $(-1, 2)^1$ (and all integral linear combinations of those). Only the palace numbers $k-1, k$, and $k+1$ appear in the épaulettes. In the épaulette’s periodic region of size 11, $k-1$ (a P-position) appears 5 times, and $k$ and $k+1$ (both N-positions (no palace)) each appear 3 times. This pattern can be verified rigorously at least in the region outside the hood and below $(k, k/3)$, and by the local definition of the CA, one can see that the pattern develops beyond this bound (for large $k$).

The rift develops a periodicity of $(81, 31)$, which is also a periodicity of the épaulette, as is the hem’s early period, $(9, 23)$, and later period, $(41, 106)$. It seems likely that for very large $k$, even larger periods may develop.

The arms, shown in the bottom row of Fig. 3, have a random appearance, although they often contain temporary black and yellow stripes at one of the two angles parallel to their sides. Despite this initial appearance of disorder, the arms have many interesting properties, discussed below in Sect. 3.4.

The fabric exhibits further self-organization. The larger black regions visible in the middle row of Fig. 3 form a rough grid, and in much larger pictures ($k \gg 1000$) the grid morphs into a larger-scale grid, which is at a slightly different angle and has cells about 3.5 times larger. Regions of the small-grid pattern appear to travel through the large grid like meta-gliders, visible in the top right of Fig. 6. These grid cells are separated by threads of P-positions, which are able to split and merge to form smaller and larger threads, and sometimes seem to disappear.

Threads typically have a measurable (vertical, horizontal, and/or diagonal) thickness, which is added when they merge, as happens frequently just after the fray, as in Fig. 6. For example, in the left-hand warp the threads have integer vertical thicknesses, that is, each thread has a fixed number of P-positions that occur in every column. Furthermore, this fixed number is always a Fibonacci number.

3.3 Region prefix properties

If we look around the nose, we see one of three pictures, depending on the value of $k \mod 3$, because this determines the precise pixel arrangement of the casing at the nose. These three shapes are shown in Fig. 7. Since there are only three possibilities for the hood boundary shape at the nose, and the CA rule can then be used to produce the épaulettes and fabric without knowing $k$, we see that despite the chaotic nature of the fabric, there are in fact only three fabric patterns that can be produced. Figs. 3 and 8 both show examples of how the full fabric area matches between different games with $k$ congruent modulo 3.

Using the CA, starting from an infinite casing pattern, we can make any of the three fabric patterns in an infinitely large version. The fabric patterns that we see in practice are simply prefixes of one of these three infinite fabrics.

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1. The periods $(5, 1)$ and $(-1, 2)$ are for the épaulette below and to the left of the main diagonal which is of course symmetric with the épaulette above and to the right of the main diagonal.
The arms similarly can be formed by the CA rule from the shoulder, and in this case there is only one possible pattern. Fig. 8 shows how this pattern remains very stable as \( k \) increases.

### 3.4 Properties of the skin

Where the arm borders the outer space, the arm grows a periodic skin (as can be seen in Fig. 9) which is slightly separated from the rest of the arm, emitting a single vertical (in the case of the upper arm) line once per period. This skin consists of solidly packed P-positions, with a vertical thickness of \( f_{2n} \) (or \( f_{2n} + 1 \) in the column where the line is emitted), a diagonal thickness of \( f_{2n+1} \), a horizontal thickness of \( f_{2n+2} \), a horizontal period of \( f_{2n+3} \), and a vertical period of \( f_{2n+1} \), where \( f_{2n} \) is the \( 2n \)th Fibonacci number.
The pattern in the skin, of palace numbers \( k - 1 \) (yellow) and \( k - 2 \) (olive), is given by the pattern shown in Fig. 10.

Initially a thin periodic skin is produced that continues for some time before it changes to form a thicker skin with a longer period, this thickening process is repeated until eventually there is no more thickening and the skin remains periodic. The skin originally emerges at the top of the arm with \( n = 1 \), and after about 200 pixels (horizontally, about 80 vertically) it changes to the form with \( n = 2 \), then after about 7700 pixels it changes to the form with \( n = 3 \), and by around 743,000 pixels it changes to \( n = 4 \). These four skin forms can be seen in Fig. 11. We conjecture that for large \( k \) (or in the “infinite arm” described in Sect. 3.3), the skin will continue to thicken in this manner, with \( n \) increasing by one each time, approaching a slope of \( \phi^2 \). In the other direction, one can even see the \( n = 0 \) stage of this pattern as part of the épaulette at the start of the arm for the first 10 pixels or so, although it is not clearly visible due to the lack of black pixels between this skin and the rest of the arm.

The boundary between the skin (which has palace numbers \( k - 2 \) and \( k - 1 \)) and the outer space (which has constant palace number \( k \)) consists of steps of width 2 or 3.

**Fig. 10** The function \( s(i,j) = |\phi(i+j)| - |\phi(i)| - |\phi(j)| \) with five highlighted skin-patterns. The smallest pattern (a single dotted line) interacts with the épaulette, so it does not correspond perfectly with the arm’s skin. (And of course, épaulettes do not have skin.) The cells in the stripes correspond to \(-2\) if olive and \(-1\) if yellow, in the updates of the CA. For example the \( g\)-value of the cell \((9,2)\) can be computed via the formula \( g = a - b - c + e + f + p \) as \( g = -1 + 2 + 2 - 2 + (3 - 2 - 2 - 1 + 1 + 1) = -1 \). Notice that the total palace compensation number is 0. Skin period 5, 13, 34 and 89 appears at game 5, 15, 81 (about x-coordinate 13500), and about game 490 respectively.

**Fig. 11** Skin patterns appearing for \( k = 508 \), for comparison with Fig. 10. Two periods of the respective patterns are included except for the lower left picture of the shoulder, which shows the full transformation from hood to skin of period \((5,2)\), including three periods of period \((2,1)\) skin.
Each time the skin thickens, the pattern of steps expands according to the rule \( \{2 \rightarrow 23, 3 \rightarrow 233\} \), starting from the pattern of just 2 for the skin \( n = 0 \). The skin pattern of positions of palace numbers \( k - 2 \) and \( k - 1 \) can be computed from this pattern of steps and from the assumption that there are \( k - f_{2n+3} \) palaces to the left of the skin in each row. Skin positions that can see \( f_{2n+3} - 1 \) other skin positions are yellow, and positions that can see \( f_{2n+3} - 2 \) other skin positions are olive, where each position is said to be able to see other positions that are directly above it, or to the left of it, or on the diagonal to the upper left.

This pattern of yellow and olive in the skin is rotationally symmetric within each period (strictly between the columns that produce the vertical lines), with yellow and olive being swapped when the pattern is rotated \( 180^\circ \), since the sum of visible positions in the original and rotated version is simply the sum of the vertical, horizontal, and diagonal widths of the skin \( (-3, \text{since the position in question cannot see itself}) \).

In Fig. 10 we see\(^2\) that the observed pattern is the same as the one studied by Kari and Szabados (2015), obtained by the function \( s(i,j) = [\phi(i+j)] - [\phi(i)] - [\phi(j)] \in \{0, 1\} \), \((i,j) \in \mathbb{N}^2\). For example, the second-largest yellow-olive part of Fig. 10, between the lines \( j = 13 - 13i/34 \) and \( j = 21 - 13i/34 \), is identical to the skin shown in Fig. 9.

This pattern has a three-way symmetry and self-similar nature, which can be seen more clearly by shifting the coordinates into a hexagonal grid, as shown in Fig. 12.

We find that this pattern has many interesting properties. It can be generated by drawing three sets of parallel lines, as shown in Fig. 12, where each set of parallel lines is spaced according to the infinite Fibonacci word, where 0 (the more common symbol) indicates an inter-line spacing of \( \phi \), and 1 indicates a spacing of 1. After drawing two of the three sets of lines, the third set is constrained by requiring that its lines pass within \( (\phi-1)/2 \) of each intersection point of the first two sets of lines. This results in many small triangles of side length \( (\phi-1)/2 \). These small triangles topologically form a hexagonal grid, and if we color them according to whether the small triangle is horizontal-side-up or horizontal-side-down, then we again obtain the pattern shown in Fig. 12.

Another interesting property is indicated in Fig. 13, namely, if we consider six grid locations at the vertices of a grid-aligned hexagon whose side lengths alternate between two lengths, then the colors of those six positions must be symmetric across one of the three possible lines of reflection, as shown in Fig. 13. This property often allows the immediate determination of the color at positions far beyond the edge of a region whose colors are known.

The pattern can be seen to have a self-similar nature, leading to the natural question of whether there is a recursive rule for growing such a pattern from smaller versions of itself. The inflation rule shown in Fig. 14 provides such a rule. We can see that when following this

\(^2\) We thank Michal Szabados for pointing this out to us.
rule, the size of the pattern increases at each step by the golden ratio \( \phi \).

### 3.5 Properties of the arm and warp

The arms and warp eventually become periodic, starting with the skin, and then the arm, and then the threads of the warp. Since information is sent in the form of blue stripes from the skin into the arm, as well as through the arm and warp, we see that the outer parts of the skin, arm, and warp must become periodic before the inner parts can become periodic. Since information does not flow outward very easily, the outer parts are often able to maintain their periodicity even while the inner parts have not yet become periodic. The boundary between the periodic part and the transient part can move slowly inward or outward. For example, it must move outward for the skin to change to a higher period. If it crosses the arm and weft, then they will remain periodic forever. We do not know if this boundary moves like a random walk or whether it behaves in a more predictable way.

Because the information starts flowing inwards from the skin, we find that the horizontal periods in the arm are multiples of those in the skin. That is, if \( f_n \) is the horizontal period in the skin then \( x f_n \) is the horizontal period in the arm, where \( x \in \mathbb{N} \). Similarly, we find that when the layers in the warp (a warp layer is the area between consecutive threads) below the arm become periodic, each layer has a horizontal period that is a multiple of the horizontal period in the layer above it, with the top layer having the same horizontal period as the arm. There seems to be no obvious method for predicting when the arm will become periodic and what length the period will have when it does. In general the number of steps before the arm stabilizes and the length of the arm period both increase as the game number \( k \) increases. However this is not a hard and fast rule; we found many games with blocking numbers \( k \) that have an arm period of \( 5-4-5 \), or \( 5-3-3-5 \), or \( 5-3-2-3-5 \). The expansion only works when the edges of two abutting triangles both expand into the same pattern. If the original pattern of triangles does not satisfy this matching requirement, then the expansion is undefined. The patterns shown here all satisfy this matching requirement, and for any pattern satisfying this requirement, the expansion of the pattern will also satisfy the requirement. The last expansion shown is an example, for a small patch of four triangles, showing how the expanded patterns fit together. The two-color pattern in Fig. 13 is given by the two possible configurations of markings (60° rotated versions of each other) appearing at the vertices in this triangular grid.
thickness of threads in the warp. This means that predicting periodicity in the warp is even more challenging than predicting periodicity in the arm. If the slopes of a pair of threads that border the same layer differ, then the periodic diagonal and vertical blue lines traveling through that layer can arrive at the other thread that borders that layer at a number of different positions relative to each other, and this gives rise to larger periods in subsequent inner layers. For this reason looking at games whose blocking numbers are close to each other is less helpful for estimating periodicity in the warp. For example, while the first layer of the warp for games 346, 347 and 348 all have a horizontal period of 34, the horizontal periods differ in the second layer with a period of 68 for game 347 and a period of 34 for games 346 and 348.

We have examined the ‘periodicity’ in the arm for the first few hundred games and we end this section with some observations. For games with very low blocking numbers (i.e. <34) the arm is still quite thin and for these numbers we see a few different period lengths for the arm [e.g. periods of (65, 26), (143, 56), (338, 132), and (247, 96)]. For games from 34 to 80 the behaviour of the arm seems more predictable and with the exception of games 57–63 which have an arm period of (26, 10) all other games between 34 and 80 have an arm period of (13, 5). So from 34 to 80 the period length of the arm is either the same as that of the skin at (13, 5) or double it, (26, 10). In games 81 to about 500 we get a thicker skin that has a period of (34, 13) and this gives rise to arms and warps with periods that are multiples of this skin period. We checked games 81 to about 500 up to about 80,000, looking for arm periods with lengths that are small multiples of (34, 13). Interestingly we found that the smaller arm period of 1 × (34, 13) only appeared for games after 344 with games 81–344 having arm periods that were large multiples of (34, 13). As mentioned earlier in this section, the periodicity that propagates from the arm into the warp gives periods in the warp that are multiples of those in the arm and so these arm periods have a huge influence on periodicity in the warp and on the slope of the threads that separate the different layers of the warp.

Given that information flows inwards, the periodicity in arm and warp are centrally important to determining the behaviour in the inner sector and the slopes of the threads that bound it. We are currently completing work Cook et al. (NA) that gives a detailed analysis of the inner sector and relates its behavior to a type of generalised tag system Cook (2004), Post (1943).

3.6 A note on generic information propagation

Figure 16 illustrates properties which are more easily seen when decomposing the palace numbers into the diagonal and horizontal components used in the definition of our CA.

First, we see that the behavior is affected by the blocking number mod 3, for example, it takes an increase from \( k \) to \( k + 3 \) to develop another diagonal color in a central sector; also the horizontal palace numbers gradually shift colors right-to-left, again with a period of three. This holds for both diagonal and horizontal decomposition.

Secondly, information is ‘filtered’ by the threads and beams as described in the figure caption, and this can be seen (with more difficulty) by inspecting the standard plot of each game which appears as the leftmost image in each row. We see that the inner sectors have different characteristics that depend on how information travels. Horizontal information is symmetric with vertical information about the central diagonal and so it is sufficient for us to note only the presence of horizontal and diagonal information. From the top row of Fig. 16 we see that in the inner sector of game 306 information travels horizontally and not diagonally. From
In the second row of Fig. 16 we see that in the inner sector of game 307 information travels only diagonally, and so there is no information propagation in the inner sector. From the third row of Fig. 16 we see that in the inner sector of game 308 information travels both diagonally and horizontally. From the forth row of Fig. 16 we see that in the inner sector of game 309 information travels both diagonally and horizontally, and the lines that travels towards the innermost beam from the warp are both diagonal and horizontal.

4 Conjectures and questions

Most of the observations in Sect. 3 may be viewed as open problems. Here we list a few.

- The arm’s skin (as observed for small n) has a vertical thickness of $f_{2n}$ (except where a line is emitted), a diagonal thickness of $f_{2n+1}$, a horizontal thickness of $f_{2n+2}$, a horizontal period of $f_{2n+3}$, and a vertical period of $f_{2n+1}$, where $n$ is the thickness level (here $f_i$ is the $i$th Fibonacci number). Can this pattern be explained, and does it continue?
- Are the armpits the unique regions from which the queen views the most palaces (for large $k$)?
- Do the innermost threads (the beams in the inner sector) always have slopes corresponding to algebraic numbers? If there is only one innermost upper thread, is the slope a root of a second degree polynomial with rational coefficients? Is the number of beams (with irrational slope) bounded (ranging over $k$)?
Why is it that the threads of the warp merge in such a way that their thickness is always a Fibonacci number?

Is it true that the threads (in the arm and warp) bound the number of palace positions that the queen sees looking just diagonally, and this number is at most one out of two numbers?

Why can no information propagate vertically beyond the inner sector?

Why do games whose blocking numbers differ by a large Lucas number have similar inner sectors?

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