Relativistic Doppler Effect in Quantum Communication

ASHER PERES and DANIEL R. TERNO

Department of Physics, Technion—Israel Institute of Technology, 32000 Haifa, Israel

Abstract. When an electromagnetic signal propagates in vacuo, a polarization detector cannot be rigorously perpendicular to the wave vector because of diffraction effects. The vacuum behaves as a noisy channel, even if the detectors are perfect. The “noise” can however be reduced and nearly cancelled by a relative motion of the observer toward the source. The standard definition of a reduced density matrix fails for photon polarization, because the transversality condition behaves like a superselection rule. We can however define an effective reduced density matrix which corresponds to a restricted class of positive operator-valued measures. There are no pure photon qubits, and no exactly orthogonal qubit states.

1. Introduction

The long range propagation of polarized photons is an essential tool of quantum cryptography [1]. Usually, optical fibers are used, and the photons may be absorbed or depolarized due to imperfections. In some cases, such as communication with space stations, the photons must propagate in vacuo [2]. The beam then has a finite diffraction angle of order \( \lambda/a \), where \( a \) is the aperture size, and new deleterious effects appear. In particular a polarization detector cannot be rigorously perpendicular to the wave vector, and the transmission is never faithful, even with perfect detectors. Moreover, the “vacuum noise” depends on the relative motion of the observer with respect to the source.

The relativistic effects reported here are essentially different from those for massive particles [3] because massless particles have only two linearly independent polarization states. The properties that we discuss are kinematical, not dynamical. At the statistical level, it is not even necessary to involve quantum electrodynamics. Most formulas can be derived by elementary classical methods as shown below. It is only when we need to consider individual photons, for cryptographic applications, that quantum theory becomes essential.

This article consists of two parts. First we consider the propagation of a classical electromagnetic wave. The wave vector cannot be constant because of diffraction effects. A polarization detector cannot unambiguously distinguish orthogonal polarizations, even if the detector is perfect. The vacuum behaves as a noisy channel. We then show that this “noise” can be reduced and nearly cancelled by a relative motion of the observer toward the source.

In the second part of this paper, we investigate the transmission of a single photon. The
diffraction effects mentioned above lead to superselection rules which make it impossible to define a reduced density matrix for polarization. It is still possible to have “effective” density matrices; however, the latter depend not only on the preparation process, but also on the type of detection that is used by the observer.

2. Classical electromagnetic signals

Assume for simplicity that the electromagnetic signal is monochromatic. In a Fourier decomposition, the Cartesian components of the wave vector $k_\mu$ (with $\mu = 0, 1, 2, 3$) can be written in term of polar angles:

$$k_\mu = (1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),$$

where we use units such that $c = 1$ and $k_0 = 1$. Let us choose the z axis so that a well collimated beam has a large amplitude only for small $\theta$, and let us rotate the $x$ and $y$ axes so that a particular $k$ in which we are interested has $\phi = 0$ (we shall later return to arbitrary $k$ with $\phi \neq 0$). The Fourier transform of the electric field is perpendicular to

$$k = (\sin \theta, 0, \cos \theta).$$

If the emitter (“Alice”) selects a linear polarization angle $a$, the Fourier transform of $E$ is proportional to

$$E_k = (\cos a \cos \theta, \sin a, -\cos a \sin \theta).$$

The magnetic field Fourier transform is proportional to $B_k = k \times E_k$. The Poynting vector $P$ is parallel to $k$ and gives the energy flux, as usual. We shall henceforth omit the subscript $k$.

Suppose that the receiver (“Bob”) has an infinite flat detector parallel to the $xy$ plane. Then only the component

$$P_z \propto E_x^2 + E_y^2 = \cos^2 a \cos^2 \theta + \sin^2 a$$

is absorbed by the detector. Moreover, if Bob selects a polarization angle $b$ in the $xy$ plane, the flux detected by him will be proportional to

$$(E_x \cos b + E_y \sin b)^2 = (\cos a \cos b \cos \theta + \sin a \sin b)^2.$$ 

Under ideal conditions ($\theta = 0$), the maximal signal behaves as $\cos^2(a - b)$. For small but finite $\theta$, the fraction of the signal that is lost, when $a = b$, is $\theta^2 \cos^2 a$. More generally it is $\theta^2 \cos^2(a - \phi)$, where $\phi$ is the azimuthal angle of $k$, which was set to zero in the preceding calculation by a suitable choice of the $x$ axis.

Apart from the above loss in intensity, the angle mismatch may introduce errors. Consider again the case $\phi = 0$. When Alice emits a wave that is linearly polarized along the $x$ or $y$ axes, Bob’s intensity pattern varies as $A_x^2 \cos^2 b$ and $A_y^2 \sin^2 b$, respectively. These intensity distributions would be unambiguously recognized as pertaining to the corresponding linear polarizations. However, for a general linear polarization state that Alice may send ($0 \neq a \neq \pi/2$) the intensity will be distributed according to Eq. (5). As a result, the angle $a'$ that Bob will ascribe to it would be related to the actual polarization direction by $\tan a' = \tan a / \cos \theta$. 

2
In a real experiment, the angles $\theta$ and $\phi$ are distributed in a continuous way around the $z$ axis (exactly how depends on the properties of the laser) and one has to take a suitable average over them. Since the definition of polarization explicitly depends on the direction of $k$, taking the average over many values of $k$ leads to an impure polarization and therefore may cause not only an attenuation of the beam, but also identification errors.

Let us now consider the effect of a motion of Bob relative to Alice, with a constant velocity $v = (0, 0, v)$. The Lorentz transformation of $k_\mu$ in Eq. (1) yields new components

$$k'_0 = \gamma (1 - v \cos \theta) \quad \text{and} \quad k'_z = \gamma (\cos \theta - v),$$

where $\gamma = (1 - v^2)^{-1/2}$. Considering again a single Fourier component, we have, instead of the unit vector $k$, a new unit vector

$$k' = \left( \frac{\sin \theta}{\gamma (1 - v \cos \theta)}, 0, \frac{\cos \theta - v}{1 - v \cos \theta} \right).$$

In other words, there is a new tilt angle $\theta'$ given by

$$\sin \theta' = \sin \theta / \gamma (1 - v \cos \theta).$$

For small $\theta$, such that $\theta^2 \ll |v|$, we have

$$\theta' = \theta \sqrt{\frac{1 + v}{1 - v}}.$$  

The square root is the familiar relativistic Doppler factor. For large negative $v$, the diffraction angle becomes arbitrarily small, and side-ways losses can be reduced to zero.

It is noteworthy that the same Doppler factor was obtained by Jarett and Cover [4] who considered only the relativistic transformations of bit rate and noise intensity, without any specific physical model. This remarkable agreement shows that information theory should properly be considered as a branch of physics.

3. Quantized electromagnetic signals

We now turn to the quantum language in order to discuss applications to secure communication. The ideal scenario is that Alice sends isolated photons (one particle Fock states). In a more realistic setup, the transmission is by means of weak coherent pulses containing on the average less than one photon each.

A basis of the one-photon space is spanned by states of definite momentum and helicity,

$$|k, \epsilon_k^\pm \rangle \equiv |k\rangle \otimes |\epsilon_k^\pm \rangle,$$

where helicity states $|\epsilon_k^\pm \rangle$ are explicitly defined in Eq. (15) below. The momentum basis is normalized by

$$\langle q | k \rangle = (2\pi)^3 (2k^0) \delta^{(3)}(q - k).$$

Polarization states that correspond to different momenta belong to distinct Hilbert spaces and cannot be superposed (an expression such as $|\epsilon_k^+ \rangle + |\epsilon_q^+ \rangle$ is meaningless if $k \neq q$). The complete basis (10) does not violate this superselection rule, owing to the
orthogonality of the momentum basis. Therefore, a generic one-photon state is given by a wave packet [5]

$$|\Psi\rangle = \int d\mu(k) f(k)|k, \alpha(k)\rangle,$$

(12)

where the polarization state $|\alpha(k)\rangle$ corresponds to the 3-vector

$$\alpha(k) = \alpha_+(k)\epsilon^+_k + \alpha_-(k)\epsilon^-_k,$$

(13)

$|\alpha_+|^2 + |\alpha_-|^2 = 1$, and the explicit form of $\epsilon^\pm_k$ is given below. The Lorentz-invariant measure is

$$d\mu(k) = \frac{d^3k}{(2\pi)^3 2k_0},$$

(14)

and normalized states satisfy $\int d\mu(k)|f(k)|^2 = 1$. Since diffraction angles and frequency spreads are usually small [2, 5], $f(k)$ significantly differs from zero only in the vicinity of a certain momentum that we shall denote by $k_A$.

Lorentz transformations of quantum states are most easily computed by referring to some standard momentum, which for photons is $p^\nu = (1, 0, 0, 1)$. Accordingly, standard right and left circular polarization vectors are $\epsilon^+_p = (1, \pm i, 0)/\sqrt{2}$. If we are interested in linear polarization, all we have to do is to use Eq. (13) with $\alpha_+ = (\alpha_-)^*$, so that $\alpha(k)$ is real. In general, $\alpha(k)$ corresponds to elliptic polarization.

Under a Lorentz transformation $\Lambda$, these states become $|k_{\Lambda}, \alpha(k_{\Lambda})\rangle$, where $k_{\Lambda}$ is the spatial part of a four-vector $k_{\Lambda} = \Lambda k$, and the new polarization vector can be obtained by an appropriate rotation that is described below. For each $k$ a polarization basis is labeled by the helicity vectors,

$$\epsilon^\pm_k = R(\hat{k})\epsilon^\pm_p,$$

(15)

and the corresponding quantum states are just $|k, \epsilon^\pm_k\rangle$. As usual, $\hat{k}$ denotes the unit 3-vector in the direction of $k$. The matrix that rotates the standard direction $(0, 0, 1)$ to $\hat{k} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is

$$R(\hat{k}) = \begin{pmatrix} \cos \theta \cos \phi & -\sin \phi & \cos \phi \sin \theta \\ \cos \theta \sin \phi & \cos \phi & \sin \phi \sin \theta \\ -\sin \theta & 0 & \cos \theta \end{pmatrix},$$

(16)

and likewise for $\hat{k}_{\Lambda}$.

Under a general Lorentz transformation, be it a rotation or a boost, helicity is preserved, but the states (and corresponding geometric vectors) acquire helicity-dependent phases,

$$\alpha_+\epsilon^+_k + \alpha_-\epsilon^-_k \rightarrow \alpha_+ e^{i\xi(\Lambda, k)}\epsilon^+_k + \alpha_- e^{-i\xi(\Lambda, k)}\epsilon^-_k,$$

(17)

where the explicit expressions for $\xi(\Lambda, \hat{k})$ are given in [6] The superselection rule that was mentioned above makes it impossible to define a reduced density matrix in the usual way. We can however define an “effective” reduced density matrix for polarization, as follows. The labelling of polarization states by Euclidean vectors $\epsilon^\pm_k$, and the fact that photons
are spin-1 particles, suggest the use of a $3 \times 3$ matrix with entries labelled $x$, $y$ and $z$. Classically, they correspond to different directions of the electric field. For example, when $\mathbf{k} = k_\ell \hat{z}$, only $\rho_{xx}$, $\rho_{xy}$, $\rho_{yy}$ are non-zero.

For a generic photon state $|\Psi\rangle$, let us try to construct a reduced density matrix $\rho_{xx}$ that gives the expectation value of an operator representing the polarization in the $x$ direction, irrespective of the particle’s momentum. To have a momentum-independent polarization is to tacitly admit longitudinal photons. Therefore, in terms of real transversal photons of a momentum $\mathbf{k}$, this problem suggests to find the direction that is perpendicular to $\mathbf{k}$ and closest to an arbitrary unit vector $\hat{x}$. That is, we are looking for a unit Euclidean complex vector $\mathbf{e}_x(\mathbf{k})$ such that $\mathbf{k} \cdot \mathbf{e}_x(\mathbf{k}) = 0$ and $\hat{x} \cdot \mathbf{e}_x(\mathbf{k}) = \max$.

Momentum-independent polarization states thus consist of physical (transversal) and unphysical (longitudinal) parts with a polarization vector $\mathbf{e}^\ell = \mathbf{k}$. For example, a polarization state along the $x$-axis is

$$|\hat{x}\rangle = x_+(\mathbf{k})|\epsilon^+_k\rangle + x_-(\mathbf{k})|\epsilon^-_k\rangle + x_\ell(\mathbf{k})|\epsilon^\ell_k\rangle,$$

where $x_\pm(\mathbf{k}) = \mathbf{e}_k^\pm \cdot \hat{x}$, and $x_\ell(\mathbf{k}) = \hat{x} \cdot \hat{k} = \sin \theta \cos \phi$. It follows that $|x_+|^2 + |x_-|^2 + |x_\ell|^2 = 1$, and

$$\mathbf{e}_x(\mathbf{k}) = \frac{x_+(\mathbf{k})\epsilon^+_k + x_-(\mathbf{k})\epsilon^-_k}{\sqrt{x_+^2 + x_-^2}}.$$  

Note that $\langle \hat{x}|\hat{y}\rangle = \hat{x} \cdot \hat{y} = 0$, whence

$$|\hat{x}\rangle \langle \hat{x}| + |\hat{y}\rangle \langle \hat{y}| + |\hat{z}\rangle \langle \hat{z}| = \mathbb{1}.$$  

To the direction $\hat{x}$ corresponds a projection operator

$$P_{xx} = |\hat{x}\rangle \langle \hat{x}| \otimes \mathbb{1}_p = |\hat{x}\rangle \langle \hat{x}| \otimes \int d\mu(\mathbf{k})|\mathbf{k}\rangle \langle \mathbf{k}|,$$

where $\mathbb{1}_p$ is the unit operator in momentum space. The action of $P_{xx}$ on $|\Psi\rangle$ follows from Eq. (18) and $\langle \epsilon^+_k|\epsilon^-_k\rangle = 0$. Only the transversal part of $|\hat{x}\rangle$ appears in the expectation value:

$$\langle \Psi|P_{xx}|\Psi\rangle = \int d\mu(\mathbf{k})|f(\mathbf{k})|^2|x_+(\mathbf{k})\alpha^+_\mathbf{k}(\mathbf{k}) + x_-(\mathbf{k})\alpha^-_\mathbf{k}(\mathbf{k})|^2.$$  

Define the transversal part of $|\hat{x}\rangle$:

$$|b_x(\mathbf{k})\rangle \equiv (|\epsilon^+_k\rangle \langle \epsilon^+_k| + |\epsilon^-_k\rangle \langle \epsilon^-_k|)|\hat{x}\rangle = x_+(\mathbf{k})|\epsilon^+_k\rangle + x_-(\mathbf{k})|\epsilon^-_k\rangle.$$  

Likewise define $|b_y(\mathbf{k})\rangle$ and $|b_z(\mathbf{k})\rangle$. These three state vectors are neither of unit length nor mutually orthogonal. For $\mathbf{k} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ we have

$$|b_x(\mathbf{k})\rangle = \left[(\cos \theta \cos \phi + i \sin \phi)|\epsilon^+_k\rangle + (\cos \theta \cos \phi - i \sin \phi)|\epsilon^-_k\rangle\right]/\sqrt{2},$$

$$c(\theta, \phi)|\mathbf{k}, \mathbf{e}_x(\mathbf{k})\rangle,$$

where $\mathbf{e}_x(\mathbf{k})$ is given by Eq. (19), and $c(\theta, \phi) = \sqrt{x_+^2 + x_-^2}$. 

5
Finally, a POVM element $E_{xx}$ which is the physical part of $P_{xx}$, namely is equivalent to $P_{xx}$ for physical states (without longitudinal photons) is

$$E_{xx} = \int d\mu(k)|k, b_x(k)\rangle\langle k, b_x(k)|,$$

and likewise for other directions. The operators $E_{xx}$, $E_{yy}$ and $E_{zz}$ indeed form a POVM in the space of physical states, owing to Eq. (20). It then follows from Eq. (23) and similar definitions for other directions that, for any $k$,

$$\langle b_x(k)|b_x(k)\rangle + \langle b_y(k)|b_y(k)\rangle + \langle b_z(k)|b_z(k)\rangle = \mathbb{1}_{\perp k},$$

where $\mathbb{1}_{\perp k}$ is the identity operator in the subspace of polarizations orthogonal to $k$.

To complete the construction of the density matrix, we introduce additional directions. Following a standard practice of state reconstruction [7], we consider $P_{x+z,x+z}$, $P_{x-iz,x-iz}$ and similar combinations. For example,

$$P_{x+z,x+z} = {\mathcal{F}}(|\hat{x}\rangle + |\hat{z}\rangle)(|\hat{x}\rangle + |\hat{z}\rangle) \otimes \mathbb{1}_p$$

The diagonal elements of the new polarization density matrix are defined as

$$\rho_{mm} = \langle \Psi|E_{mm}|\Psi\rangle, \quad m = x, y, z,$$

and the off-diagonal elements are recovered by combinations such as

$$\rho_{xz} = \langle \Psi(|\hat{x}\rangle|\hat{z}\rangle \otimes \mathbb{1}_p)|\Psi\rangle = \langle \Psi|E_{x+z,x+z} + E_{x-iz,x-iz} - E_{xx} - E_{zz}|\Psi\rangle,$$

where we denote $|\hat{x}\rangle|\hat{z}\rangle \otimes \mathbb{1}_p$ as $P_{xz}$, and by $E_{xz}$ its “physical” part. We then get a simple expression for the reduced density matrix corresponding to the polarization state $|\alpha(k)\rangle$:

$$\rho_{mn} = \langle \Psi|E_{mn}|\Psi\rangle = \int d\mu(k)|f(k)|^2\langle \alpha(k)|b_m(k)\rangle\langle b_n(k)|\alpha(k)\rangle, \quad m, n, = x, y, z.$$  

Our basis states $|k, e_k\rangle$ are direct products of momentum and polarization. Owing to the transversality requirement $e(k) \cdot k = 0$, they remain direct products under Lorentz transformations. All the other states have their polarization and momentum degrees of freedom entangled. As a result, if one is restricted to polarization measurements as described by the above POVM, there do not exist two orthogonal polarization states. In general, any measurement procedure with finite momentum sensitivity will lead to the errors in identification. This can be seen as follows.

Let two states $|\Phi\rangle$ and $|\Psi\rangle$ be of the form in Eq. (12). Their reduced polarization density matrices, $\rho_\Phi$ and $\rho_\Psi$, respectively, are calculated using Eq. (31). Since the states are entangled, the von Neumann entropies of the reduced density matrices, $S = -\text{tr}(\rho \ln \rho)$, are positive [8]. Therefore, both matrices are at least of rank two. Since the overall dimension is 3, it follows that $\text{tr}(\rho_\Phi\rho_\Psi) > 0$ and these states are not perfectly distinguishable. An immediate corollary is that photon polarization states cannot be cloned perfectly. This is because no-cloning theorem, in its various versions [9], forbids an exact copying of unknown non-orthogonal states.

To quantify the distinguishability of a pair of quantum states, we shall use the simplest criterion, namely the probability of error $P_E$, defined as follows: an observer receives a
single copy of one of the two known states and performs any operation permitted by quantum theory, in order to decide which state was supplied. The probability of a wrong answer for an optimal measurement is [10]

\[ P_E(\rho_1, \rho_2) = \frac{1}{2} + \frac{1}{4} \text{tr}|\rho_1 - \rho_2|, \]  

where, for any operator \( O \), the operator \( |O| \) is defined as \( \sqrt{O^\dagger O} \). As shown below, the distinguishability of polarization density matrices depends on the observers’ motion. First we present some general considerations and then illustrate them with a simple example.

Let us take the \( z \)-axis to coincide with the average direction of propagation so that the mean photon momentum is \( k_A \hat{z} \). Typically, the spread in momentum is small, but not necessarily equal in all directions. Usually the intensity profile of laser beams has cylindrical symmetry, and we may assume that \( \Delta x \sim \Delta y \sim \Delta r \) where the index \( r \) means radial. We may also assume that \( \Delta r \gg \Delta z \), since the momentum spread along the average propagation direction does not necessarily increase the mixness of the reduced density matrix, as we shall see below. We then have

\[ f(k) \propto f_1[(k_z - k_A)/\Delta_z] f_2(k_r/\Delta_r). \]

We approximate

\[ \theta \approx \tan \theta \equiv k_r/k_z \approx k_r/k_A. \]

In pictorial language, polarization planes for different momenta are tilted by angles up to \( \sim \Delta_r/k_A \), so that we expect an error probability of the order \( \Delta_r^2/k_A^2 \). In the density matrix \( \rho_{mn} \) all the elements of the form \( \rho_{mz} \) should vanish when \( \Delta z \to 0 \). Moreover, if \( \Delta z \to 0 \), the non-vanishing \( xy \) block goes to the usual (monochromatic) polarization density matrix,

\[ \rho_{\text{pure}} = \begin{pmatrix} |\alpha|^2 & \beta & 0 \\ \beta^* & 1 - |\alpha|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

As an example we consider two states which, if the momentum spread could be ignored, would be \( |k_A \hat{z}, \epsilon_{k_A}^\pm \rangle \). To simplify the calculations we assume a Gaussian distribution:

\[ f(k) = N e^{-(k_z - k_A)^2/2\Delta_z^2} e^{-k_r^2/2\Delta_r^2}, \]

where \( N \) is a normalization factor and \( \Delta_z \ll \Delta_r \). Moreover, we take the polarization components to be \( \epsilon_{k}^\pm \equiv R(\hat{k})\epsilon_{p}^\pm \). That means we have to analyze the states

\[ |\Psi_{\pm} \rangle = \int d\mu(k) f(k)|\epsilon_{k}^\pm, k\rangle, \]

where \( f(k) \) is given above.

We expand \( R(\hat{k}) \) up to second order in \( \theta \). Reduced density matrices are calculated by techniques similar to those for massive particles [3], using rotational symmetry around the \( z \)-axis and normalization requirements. The leading order in \( \Omega \equiv \Delta_r/k_A \) gives

\[ \rho_+ = \frac{1}{2}(1 - \frac{1}{2}\Omega^2) \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\Omega^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

\[ \rho_- = \frac{1}{2}(1 + \frac{1}{2}\Omega^2) \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{2}\Omega^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]
and $\rho_-=\rho_+^\ast$. At the same level of precision,
\[ P_E(\rho_+,\rho_-) = \Delta^2/4k_A^2. \] (39)

It is interesting to note that an optimal strategy for distinguishing between these two states is a polarization measurement in the $xy$-plane. Then the effective $2 \times 2$ density matrices are perfectly distinguishable, but there is a probability $\Omega^2/2$ that no photon be detected at all. The above result was valid due to the special form of the states that we chose. Potential errors in the upper $2 \times 2$ blocks were averaged out in the integration over $\phi$.

We now calculate Bob’s reduced density matrix. We again assume that Bob moves along the $z$-axis with a velocity $v$. Recall that reduced density matrices have no transformation law [3], (only the complete density matrix has one) except in the limiting case of sharp momenta. The only way to calculate Bob’s reduced density matrix is to transform the complete state, and only then take a partial trace. According to Eqs. (31) and (37), reduced density matrices in both frames are given by the expression
\[ (\rho_{\pm})_{mn} = \int d\mu(k)|f(k)|^2\langle R(\hat{k})\epsilon^\pm_p |b_m(k)\rangle\langle b_n(k)|R(\hat{k})\epsilon^\pm_p \rangle. \] (40)

Note that pure boosts preserve the orientation of the coordinate axes in 3-space, and therefore do not affect the indices of $\rho_{mn}$. The measure $\mu(k)$ is Lorentz-invariant and $\epsilon^\pm_p$ are constant by definition. Since $f$ is a scalar function, it transforms as $f'(k) = f(k\Lambda^{-1})$, where primes indicate Bob’s frame, as in Eqs. (6–9). This is the only frame dependent expression in (40), since the phase factors $e^{\pm i\xi}$ cancel out for the helicity eigenstates. Therefore, there are two equivalent methods to calculate Bob’s polarization density matrix. One is to change the argument of $f(k)$ to $k\Lambda^{-1}$, and another is to change the argument of the rotation matrix $R(\hat{k})$ to $\hat{k}\Lambda$. Using the second option, we obtain
\[ (\rho'^{\pm})_{mn} = \int d\mu(k)|f(k)|^2\langle R(\hat{k}\Lambda)\epsilon^\pm_p |b_m(k)\rangle\langle b_n(k)|R(\hat{k}\Lambda)\epsilon^\pm_p \rangle. \] (41)

A boost along the $z$-axis preserves $k_r$ and $\phi$. On the other hand, from Eq. (7) it follows that
\[ k_z' \approx k_A \sqrt{\frac{1-v}{1+v}}. \] (42)

Thus, at leading order in $\theta$ we have
\[ \theta' \approx \sqrt{\frac{1+v}{1-v}} k_r \approx \frac{1+v}{1-v} \theta, \] (43)

which is substituted into $R(\hat{k}_A)$. Since everything else in the integral remains the same, the effect of relative motion is given by a substitution
\[ \Omega \rightarrow \sqrt{\frac{1+v}{1-v}} \Omega. \] (44)

It follows that
\[ P'_E = \frac{1+v}{1-v} P_E, \] (45)
which may be either larger or smaller than $P_E$. As expected, we obtain for one-photon states the same Doppler effect as in the preceding classical calculations.

Although reduced polarization density matrices have no general transformation rule, the above results, as well as the analysis of massive particles [3], show that such rules can be derived for particular classes of experimental procedures. We can then ask how these effective transformation rules $\rho' = T[\rho]$ fit into the framework of general state transformations. A general state transformation $T$ is usually required to be completely positive (CP), namely [11, 12],

$$T[\rho] = \sum_i M_i \rho M_i^\dagger,$$

where the $M_i$ are bounded operators. It can be proved that distinguishability, as expressed by natural measures like $P_E$, cannot be improved by any CP transformation [10]. It is also known that the CP requirement may fail if there is a prior entanglement of $\rho$ with another system [13]. Since from [3] and Eq. (45) it follows that in our case distinguishability can be improved, we conclude that these transformations are not completely positive. The reason is that the Lorentz transformation acts not only on the “interesting” polarization variables, but also on the “hidden” momentum variables that we elected to ignore and to trace out.

This technicality has one important consequence. In quantum information theory quantum channels are described as completely positive maps [14, 15, 16] that act on qubit states. Qubits themselves are realized as particles’ discrete degrees of freedom. If relativistic motion is important, then not only does the vacuum behave as a noisy quantum channel, but the very representation of a channel by a CP map fails.

Acknowledgments

Work by AP was supported by the Gerard Swope Fund and the Fund for Encouragement of Research. DRT was supported by a grant from the Technion Graduate School.
References

[1] Gisin, N., Ribordy, G., Tittel, W., and Zbinden, H., 2002 Rev. Mod. Phys. 74, 145.

[2] Buttler, W. T., Hughes, R.J., Lamoreaux, S. K., Morgan, G. L., Nordholt, J. E., and Peterson, C. G., 2000, Phys. Rev. Lett. 84, 5652.

[3] Peres, A., Scudo, P. F., and Terno, D. R., 2002, Phys. Rev. Lett. 88, 230402.

[4] Jarett, K. and Cover, T., 1981, IEEE Trans. Info. Theory, IT-27, 152.

[5] Mandel, L. and Wolf, E., 1995, Optical Coherence and Quantum Optics (Cambridge: Cambridge University Press).

[6] Lindner, N. H., Peres, A. and Terno, D. R., 2003, e-print hep-th/0304017.

[7] Chuang, I. L. and Nielsen, M. A., 1997, J. Modern Optics 44, 2455.

[8] Peres, A., 1995, Quantum Theory: Concepts and Methods (Dordrecht: Kluwer).

[9] Peres, A., 2002, e-print quant-ph/0205076.

[10] Fuchs, C. A. and van de Graaf, J., 1999, IEEE Trans. Info. Theory, IT-45, 1216.

[11] Davies, E. B., 1976, Quantum Dynamics of Open Systems (New York: Academic Press), ch. 8.

[12] Kraus, K., 1983, States, Effects and Operations (Berlin: Springer).

[13] Pechukas, P., 1994, Phys. Rev. Lett. 73, 1060.

[14] Holevo, A. S., 1999, Russ. Math. Surveys 53, 1295.

[15] Amosov, G. G., Holevo, A. S., and Werner, R. F., 2000, Probl. Info. Transmission 36, 305.

[16] King, C. and Ruskai, M. B., 2001, IEEE Trans. Info. Theory, IT-47, 192.