ALGEBRAIC $K$-THEORY AND LOCALLY CONVEX ALGEBRAS

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1. Introduction

The category of locally convex algebras is very vast. Besides natural Fréchet-algebras arising from differential calculus on manifolds it also contains algebras defined purely algebraically (any complex algebra with a countable basis is a locally convex algebra in a natural way with the fine topology). The category is therefore very flexible and allows constructions mixing analytic and algebraic structures. It seems to cover nearly all examples of (usually noncommutative) algebras associated with differential geometric objects, such as algebras of differential or pseudodifferential operators, algebras of differential forms, deformation algebras etc., etc. In connection with index theory and noncommutative geometry it therefore seems highly desirable to have at our disposal tools such as a computable version of topological $K$-theory - preferably in bivariant form as it had been developed for $C^*$-algebras by Kasparov and others - and associated invariants, also in connection with the various classes of “infinitesimals” used by Connes [4].

In [9], the first named author had developed such a theory for the smaller category consisting of all locally convex algebras that can be topologized by a family of submultiplicative seminorms (i.e. the category of projective limits of Banach algebras). This theory has all the desirable properties and covers already many of the important examples. However, there are still very natural examples of locally convex algebras that do not fit into this category. Particularly relevant examples are given by algebras of differential operators and in particular, in the simplest case, by the so-called Weyl algebra.

In [7] a bivariant $K$-theory $kk^{alg}$ on the category of all locally convex algebras was then indeed developed. This bivariant homology theory again has very good formal properties, in particular the usual properties of (differentiable) homotopy invariance, long exact sequences associated with extensions and stability under tensoring by the algebra $K$ of rapidly decreasing matrices. Therefore the $kk^{alg}$-invariants can be computed for many interesting examples of locally convex algebras as modules over the graded coefficient ring $R_* = kk^{alg}(\mathbb{C}, \mathbb{C})$. However, the determination of $R_*$ and more generally, a $K$-theoretic description of $kk^*_{0}(\mathbb{C}, A)$ remained open in [7] (it was shown though that $R_0$ admits a non-trivial unital homomorphism into $\mathbb{C}$).

In the present paper we show that the problem of determining the coefficient ring can be overcome simply by stabilizing by the Schatten ideal $L_p$ rather than by $K$. In fact for the $L_p$-stabilized theory $kk^{L_p}$ we show that $kk^{L_p}_{0}(\mathbb{C}, A) = K_0(A \hat{\otimes} L_p)$ for $p > 1$ and thus, in particular, that $kk^{L_p}_{*}(\mathbb{C}, \mathbb{C}) = \mathbb{Z}[u, u^{-1}]$, with a generator $u$ in degree 2 (see Corollary 6.2.3). Stabilization by $L_p$ is a very natural operation in Connes’ setting for noncommutative geometry. Similar results hold for stabilization with more general symmetrically normed multiplicative Banach ideals (see definition 2.2.1) including the dual $L_{1+}$ of the Macaev ideal.

The key to our result is a locally convex version of an old homotopy invariance theorem proved by Higson in [15] on the basis of previous work by Kasparov and one of us. One ingredient in our proof of this result is the development of a setup in which the technology of abstract Kasparov modules can be applied algebraically (and thus in particular to locally convex algebras).
Technically, we prove smooth homotopy-invariance of $M_2$-stable and split-exact functors (see definition 3.1.1), i.e. in particular of negative algebraic $K$-theory, for “weakly $J$-stable” locally convex algebras, $J$ being a harmonic Banach ideal (see definition 2.2.2). All algebras of the form $B \otimes J$, where $B$ is an arbitrary locally convex algebra, are weakly $J$-stable. The result implies that, for such an algebra $A$, the canonical evaluation maps from the algebra $A[0,1]$ of smooth $A$-valued functions on $[0,1]$ induce the same map $K_i(A[0,1]) \to K_i(A)$ for each non-positive $i \in \mathbb{Z}$. In particular, since $A[t] \subset A[0,1]$, we conclude that $A$ is $K_0$-regular. We mention that the original version of Higson’s theorem has been used in the category of $C^*$-algebras in a similar spirit by Suslin-Wodzicki in their proof of the Karoubi conjecture [17], see also [16] for a nice exposition of this result and of related ideas. Our conclusions are not obvious, even in the case $i = 0$, since we do not make any assumption concerning openness of the group of invertibles, stability under functional calculus etc. Indeed, as it turns out, it is wrong for $i = 1$, contradicting a claim in [14] concerning $K_1(L_p)$. We include a computation of $K_1(J)$ for any harmonic Banach ideal.

For higher algebraic $K$-theory, smooth homotopy invariance fails in general for weakly $J$-stable algebras. This fact is, from our point of view, due to the lack of excision. Cortiñas (see [3]) has given a long exact sequence relating the algebraic and topological $K$-theory of such algebras. Our results imply the existence of a homomorphism of graded rings $kk_{alg}(\mathbb{C}, \mathbb{C}) = R_* \to \mathbb{Z}[u, u^{-1}]$. It seems that a proof of the assertion that this homomorphism is an isomorphism could be derived from the results claimed in [18]. However, we were not able to verify the arguments in [18].

There is independent partially published work by Wodzicki. There are hints to a deep relationship between the results announced in [21], see also [11] pp.3, and implications of our results. It should be interesting to pursue this further.

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\section*{2. $J$-STABLE BIVARIANT $K$-THEORY}

\subsection*{2.1. Definitions.}

By a locally convex algebra we mean an algebra over $\mathbb{C}$ equipped with a complete locally convex topology such that the multiplication $A \times A \to A$ is (jointly) continuous.

This means that, for every continuous seminorm $\alpha$ on $A$, there is another continuous seminorm $\alpha'$ such that

$$\alpha(xy) \leq \alpha'(x)\alpha'(y)$$

for all $x, y \in A$. Equivalently, the multiplication map induces a continuous linear map $A \otimes A \to A$ from the completed projective tensor product $A \hat{\otimes} A$. All homomorphisms between locally convex algebras will be assumed to be continuous.

Every Banach algebra or projective limit of Banach algebras obviously is a locally convex algebra. But so is every algebra over $\mathbb{C}$ with a countable basis if we equip it with the “fine” locally convex topology, see e.g. [10]. The fine topology on a complex vector space $V$ is given by the family of all seminorms on $V$. 
Let \([a, b]\) be an interval in \(\mathbb{R}\). We denote by \(C[a, b]\) the algebra of complex-valued \(C^\infty\)-functions \(f\) on \([a, b]\), all of whose derivatives vanish in \(a\) and in \(b\) (while \(f\) itself may take arbitrary values in \(a\) and \(b\)). Also the subalgebras \(C(a, b), C(a, b)\) and \(C(a, b)\) of \(C[a, b]\), which, by definition consist of functions \(f\), that vanish in \(a\), in \(b\), or in \(a\) and \(b\), respectively, will play an important role. The topology on these algebras is the usual Fréchet topology.

Given two complete locally convex spaces \(V\) and \(W\), we denote by \(V \hat{\otimes} W\) their completed projective tensor product (see [20], [10]). We note that \(C[a, b]\) is nuclear in the sense of Grothendieck [20] and that, for any complete locally convex space \(C\), the space \(C\) is isomorphic to the space of \(C^\infty\)-functions on \([a, b]\) with values in \(V\), whose derivatives vanish in both endpoints, [20], § 51.

Given a locally convex algebra \(A\), we write \(A[a, b], A[a, b]\) and \(A(a, b)\) for the locally convex algebras \(A \hat{\otimes} C[a, b], A \hat{\otimes} C[a, b]\) and \(A \hat{\otimes} C[a, b]\) (their elements are \(A\)-valued \(C^\infty\)-functions whose derivatives vanish at the endpoints). The algebra \(A(0, 1)\) is called the suspension of \(A\) and denoted by \(\Sigma A\).

We denote by \(M_n\) the algebra of \(n \times n\)-matrices over \(\mathbb{C}\) and abbreviate, as usual, \(A \otimes M_n\) by \(M_nA\). Given continuous homomorphisms \(\phi_1, \ldots, \phi_n : A \to B\), we denote by \(\phi_1 \oplus \cdots \oplus \phi_n : A \to M_n(B)\) the diagonal sum of the \(\phi_i\).

We also consider the algebra

\[
M_\infty = \lim_{k \to \infty} M_k(\mathbb{C}).
\]

It is a locally convex algebra with the fine topology (which is also the inductive limit topology in the representation as inductive limit of the algebras \(M_k(\mathbb{C})\)). Note that, if \(A\) is a finitely generated algebra, every homomorphism from \(A\) to \(M_\infty\) factors through \(M_k\) for some \(k \in \mathbb{N}\). Finally, given a locally convex algebra \(A\), we denote by \(A^+\) its unitization (as a locally convex space \(A^+\) is the direct sum \(A \oplus \mathbb{C}\)).

### 2.2. Banach Ideals

Let \(H\) be an infinite dimensional separable Hilbert space. Denote by \(\mathcal{B}(H)\) the algebra of bounded operators on \(H\). Given \(p \in [1, \infty)\), the Schatten ideal \(\mathcal{L}_p \subset \mathcal{B}(H)\) is defined as

\[
\mathcal{L}_p = \{x \in \mathcal{B}(H) \mid \text{Tr} |x|^p < \infty\}
\]

Equivalently, a compact operator \(x\) is in \(\mathcal{L}_p\) if the sequence \((\mu_n)\) of its singular values is in \(\ell^p(\mathbb{N})\). \(\mathcal{L}_p\) is a Banach algebra with the norm

\[
\|x\|_p = (\text{Tr} |x|^p)^{1/p}.
\]

We denote by \(\mathcal{L}_\infty\) the algebra of all compact operators on \(H\).

**Definition 2.2.1.** A symmetrically normed, multiplicative Banach ideal \(\mathcal{J}\) assigns to each infinite-dimensional separable Hilbert space \(H\) a normed ideal \(\mathcal{J}(H)\) in the algebra \(\mathcal{B}(H)\) of bounded operators such that

- the norm \(\|\cdot\|_{\mathcal{J}}\) on \(\mathcal{J}(H)\) is complete,
- \(\|ABC\|_{\mathcal{J}} \leq \|A\|_{\mathcal{J}}\|B\|_{\mathcal{J}}\|C\|\) for all \(A, C \in \mathcal{B}(H)\) and \(B \in \mathcal{J}(H)\).
• there is a natural continuous homomorphism \( \mathcal{J}(H) \otimes \mathcal{J}(H) \to \mathcal{J}(H \otimes H) \), compatible with the homomorphism \( \mathcal{B}(H) \otimes \mathcal{B}(H) \to \mathcal{B}(H \otimes H) \), and
• the assignment \( H \mapsto \mathcal{J}(H) \) is functorial under unitary transformations.

By abuse of notation, in the following we also denote by \( \mathcal{J} \) the algebra \( \mathcal{J}(H) \) (which depends on \( H \) only up to unitary isomorphism).

We will abbreviate the term symmetrically normed, multiplicative Banach ideal to Banach ideal, since we are considering only such. The main examples we have in mind are of course the Schatten ideals \( L^p \) and the ideal \( L^\infty \), but also the Macaev ideal and its dual, which we denote by \( L^1+ \). The Banach ideal \( L^1+ \) is of major importance in Connes’ non-commutative geometry. It consists of all compact operators with singular values \( (\mu_n) \), satisfying \( \sum_{i=1}^{N} \mu_n = O(\log(N)) \) (see [4], pp.439). Although our main result does not apply to \( L^p \) it still applies to \( L^1+ \).

The only non-trivial Banach ideal which is closed in operator norm in \( \mathcal{B}(H) \), is the ideal \( L^\infty \). For our purposes, in particular for the proof of theorem 4.2.1 we need to consider a Banach ideal as a closed ideal in some algebra related to \( \mathcal{B}(H) \). This can be achieved using the following construction.

Given a Banach ideal \( \mathcal{J} \), we construct the algebra \( \mathcal{B}(H) \rtimes \mathcal{J}(H) \). Topologically, this algebra is defined to be \( \mathcal{B}(H) \oplus \mathcal{J}(H) \) with \( l_1 \)-norm and the multiplication is such that as many summands as possible are put in \( \mathcal{J}(H) \). Since \( \mathcal{J}(H) \) is symmetrically normed, \( \mathcal{B}(H) \rtimes \mathcal{J}(H) \) is clearly a Banach algebra. Note that \( \mathcal{J}(H) \) is a closed ideal in \( \mathcal{B}(H) \rtimes \mathcal{J}(H) \) and that there is a split extension (see definition 3.1.1)

\[
0 \to \mathcal{J}(H) \to \mathcal{B}(H) \rtimes \mathcal{J}(H) \to \mathcal{B}(H) \to 0.
\]

**Definition 2.2.2.** A Banach ideal \( \mathcal{J} \) is called harmonic if \( \mathcal{J}(H) \) contains a compact operator with singular values given by the harmonic series for one (and hence each) Hilbert space \( H \). If the Hilbert \( H \) space has a standard basis \( \{e_i, i \in \mathbb{N}\} \), we denote by \( \omega \) the operator which multiplies the \( n \)-th basis vector \( e_n \) by \( 1/n \). \( \mathcal{J} \) is harmonic if and only if \( \omega \in \mathcal{J}(l^2(\mathbb{N})) \).

Note that \( \mathcal{L}_p \) for \( p > 1 \) and \( \mathcal{L}^1_+ \) are harmonic, whereas \( \mathcal{L}^1 \) is clearly not.

Let \( \mathcal{J} \) be a harmonic Banach ideal. We have the following chain of inclusions of sub-algebras of \( \mathcal{B}(l^2(\mathbb{N})) \)

\[
M_\infty \subset \mathcal{K} \subset \mathcal{L}_1 \subset \mathcal{J} \subset \mathcal{L}_\infty
\]

in which only the three right-most algebras are Banach ideals. Clearly, every smooth compact operator is trace class. The fact, that \( \mathcal{L}_1 \) is contained in any harmonic Banach ideal is due to the following chain of inequalities

\[
\|A\|_{\mathcal{L}_1} \geq \sum_{i=1}^{k} \mu_i \geq k\mu_k.
\]

This implies, given any \( A \in \mathcal{L}_1 \), that there is a bounded operator \( B \in \mathcal{B}(l^2(\mathbb{N})) \) such that \( UB\omega U^* = |A| \), for some partial isometry \( U \), hence \( A \in \mathcal{J} \).
2.3. Bivariant \( kk \)-theory. The bivariant homology theory \( kk^{\text{alg}} \) associates with any pair \( A, B \) of locally convex algebras abelian groups \( kk^{\text{alg}}_n(A, B), n \in \mathbb{Z} \), see section 6.1 for a definition and [7] for complete proofs. We list some important properties of \( kk^{\text{alg}} \).

**Theorem 2.3.1.** (a) Every continuous homomorphism \( \alpha : A \to B \) determines an element \( kk(\alpha) \) in \( kk^{\text{alg}}_0(A, B) \). Given two homomorphisms \( \alpha \) and \( \beta \), we have \( kk(\alpha \circ \beta) = kk(\beta)kk(\alpha) \).

(b) Every extension (see section 3.1 for a definition)

\[
E : 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0
\]

determines canonically an element \( kk(E) \) in \( kk^{\text{alg}}_{-1}(B, I) \). The class of the cone extension

\[
0 \to \Sigma A \to A(0,1] \to A \to 0
\]
is the identity element in \( kk^{\text{alg}}_0(A, A) = kk^{\text{alg}}_{-1}(A, \Sigma A) \).

If

\[
(E) : 0 \to A_1 \to A_2 \to A_3 \to 0
\]

\[
\downarrow \alpha \quad \downarrow \beta
\]

\[
(E') : 0 \to B_1 \to B_2 \to B_3 \to 0
\]
is a morphism of extensions (a commutative diagram where the rows are extensions), then \( kk(E)kk(\alpha) = kk(\beta)kk(E') \).

(c) \( kk^{\text{alg}} \) satisfies Bott periodicity: \( kk^{\text{alg}}_{n-2}(A, B) \cong kk^{\text{alg}}_n(A, B) \).

(d) Let \( D \) be any locally convex algebra. Every extension

\[
E : 0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0
\]
induces exact sequences in \( kk^{\text{alg}}(D, \cdot) \) and \( kk^{\text{alg}}(\cdot, D) \) of the following form:

\[
\begin{align*}
kk^{\text{alg}}_0(D, I) & \xrightarrow{-kk(i)} kk^{\text{alg}}_0(D, A) \xrightarrow{-kk(q)} kk^{\text{alg}}_0(D, B) \\
kk^{\text{alg}}_1(D, B) & \xrightarrow{-kk(q)} kk^{\text{alg}}_1(D, A) \xleftarrow{-kk(i)} kk^{\text{alg}}_1(D, I)
\end{align*}
\]

and

\[
\begin{align*}
kk^{\text{alg}}_0(I, D) & \xleftarrow{kk(i)} kk^{\text{alg}}_0(A, D) \xleftarrow{kk(q)} kk^{\text{alg}}_0(B, D) \\
kk^{\text{alg}}_1(B, D) & \xrightarrow{kk(q)} kk^{\text{alg}}_1(A, D) \xleftarrow{kk(i)} kk^{\text{alg}}_1(I, D)
\end{align*}
\]

The vertical arrows in (1) and (2) are (up to a sign) given by right and left multiplication, respectively, by the class \( kk(E) \).

(e) For each locally convex algebra \( D \), there is a multiplicative transformation \( \tau_D : kk^*\text{alg}(A, B) \to kk^*\text{alg}(A \otimes D, B \otimes D) \) such that \( \tau_D(kk(\alpha)) = kk(\alpha \otimes \text{id}_D) \), for any homomorphism \( \alpha : A \to B \).

For the proof see [7].
Definition 2.3.2. Let $A$ and $B$ be locally convex algebras and $\mathcal{J}$ be a Banach ideal. We define

$$kk_n^\mathcal{J}(A, B) = kk_{n}^{\text{alg}}(A, B \hat{\otimes} \mathcal{J})$$

A priori, the definition depends on an additional variable, the Hilbert space. To see the independence, we only need to know that the action of the unitary group of the Hilbert space $H$ on $\mathcal{J}(H)$ is trivial after applying $H \mapsto kk_{n}^{\text{alg}}(A, B \otimes \mathcal{J}(H))$. This is easy to see and a particular case of Lemma 2.3.3.

Having the independence of the Hilbert space, we can use the natural map $\mathcal{J}(H) \hat{\otimes} \mathcal{J}(H) \to \mathcal{J}(H \otimes H)$ and its obvious associativity property in combination with property 2.3.1(e), to define an associative product $kk_n^\mathcal{J}(A, B) \times kk_m^\mathcal{J}(B, C) \to kk_{n+m}^\mathcal{J}(A, C)$.

The properties listed in 2.3.1 extend to the bivariant homology theory $kk^\mathcal{J}$, since the projective tensor product is exact on extensions with continuous split.

The following theorem, which is of independent interest, will show that for the ideals $L_p$ ($p \in [1, \infty]$ or $p = 1^+$), the theories $kk^{L_p}$ are all naturally isomorphic.

Theorem 2.3.3. Let $A$ and $B$ be locally convex algebras. We assume that there is a continuous homomorphism $\alpha : A \to B$ and a continuous map $\beta : B \hat{\otimes} B \to A$, such that

- $\beta \circ (\alpha \otimes \alpha) : A \otimes A \to A$ is the multiplication on $A$ and
- $\alpha \circ \beta : B \otimes B \to B$ is the multiplication on $B$.

Under these conditions, $[\alpha] \in kk_{\text{alg}}(A, B)$ is invertible.

Proof. Consider the locally convex vector space $\Sigma A \oplus B$. Let $t \mapsto \phi_t$ be a smooth homeomorphism from $[0, 1]$ to itself with vanishing derivatives at the endpoints and satisfying $\phi_0 = 0$. Endowed with the multiplication

$$(a_t, b) \cdot (a'_t, b') = (a_t a'_t + \beta(\alpha(a_t) \otimes \phi_t b' + \phi_t b \otimes \alpha(a'_t) + \phi_t b \otimes \phi_t b'), bb')$$

it is a locally convex algebra which we denote by $L(\alpha, \beta)$. There is a natural homomorphism $A(0, 1) \to L(\alpha, \beta)$ given by

$$A(0, 1) \ni a_t \mapsto (a_t - \phi_t a_1, \alpha(a_1)).$$

Similarly, there is a natural map $L(\alpha, \beta) \to B(0, 1]$ given by the assignment

$L(\alpha, \beta) \ni (a_t, b) \mapsto \alpha(a_t) + \phi_t b \in B(0, 1]$.

Consider the following diagram of extensions.

$$
\begin{array}{cccc}
0 & \to & \Sigma A & \to & A(0, 1] & \to & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Sigma A & \to & L(\alpha, \beta) & \to & B & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Sigma B & \to & B(0, 1] & \to & B & \to & 0 \\
\end{array}
$$

By 2.3.1(b), the commutativity of the diagram implies that the class in $kk_{\text{alg}}^0(A, B)$ determined by the extension in the middle is left and right inverse to $kk(\alpha) \in kk_{\text{alg}}(A, B)$. \qed
Remark 2.3.4. The argument in the proof of 2.3.3 shows at the same time that \( \Sigma \alpha \) induces an isomorphism \( E(\Sigma A) \cong E(\Sigma B) \) for any half-exact (see 4.1.1) functor \( E \) on the category of locally convex algebras which is diffotopy invariant.

Corollary 2.3.5. For \( 1 \leq q < \infty \) or \( q = 1 + \frac{1}{2p} \), the natural map \( \text{kk}^{\ell q}_n(A, B) \to \text{kk}^{\ell q}_n(A, B) \) defines an isomorphism for all \( A, B \) and \( n \).

Proof. Note that the multiplication \( \mathcal{L}_{2p} \otimes \mathcal{L}_{2p} \to \mathcal{L}_{2p} \) factors continuously through \( \mathcal{L}_p \) by Hölders inequality. By 2.3.3 the inclusion map \( \alpha : A \otimes \mathcal{L}_p \to B \otimes \mathcal{L}_q \) induces an isomorphism in \( \text{kk}^{\ell q} \) whenever \( p \leq q \leq 2p \) and, by iteration, whenever \( p \leq q \).

3. \( M_2 \)-stable and split-exact functors

3.1. Split exactness. We consider functors from the category of locally convex algebras with continuous homomorphisms, which we denote by LCA, to the category of abelian groups, which we denote by Ab. There is a natural forgetful functor from the category of locally convex algebras to the category of locally convex vector spaces. A sequence

\[ 0 \to A \to B \to C \to 0 \]

of locally convex algebras is called an extension, if it is a split-extension in the category of locally convex vector spaces, i.e. the middle term is a topological direct sum of kernel and co-kernel. An extension is called split-extension if there is a continuous splitting \( C \to B \) which at the same time is a homomorphism.

Definition 3.1.1. Let \( E : \text{LCA} \to \text{Ab} \) be a functor.

(a) The functor \( E \) is called split-exact, if any split extension of locally convex algebras is mapped to a split extension of abelian groups, i.e. if, for every extension \( 0 \to I \to A \to B \to 0 \) of locally convex algebras with a homomorphism splitting \( B \to A \), the induced sequence \( 0 \to E(I) \to E(A) \to E(B) \to 0 \) is exact (and then automatically also split).

(b) The functor \( E \) is called \( M_2 \)-stable, if for any locally convex algebra \( A \), the natural inclusion map \( j_2 : A \to A \otimes M_2 \) which embeds \( A \) in the upper left corner induces an isomorphism under \( E \).

The most important example of a split-exact and \( M_2 \)-stable functor is of course the algebraic \( K \)-theory functor \( K_0 \), but there are other examples which are of independent interest. Cortiñas has heavily used the diffotopy invariance result (Theorem 4.2.1), which we will derive for all weakly \( J \)-stable, split-exact and \( M_2 \)-stable functors, for the functors \( HP_* \), \( K_*^{\text{inf}} \) and \( KH_* \) in order to obtain results about the structure of higher algebraic \( K \)-theory of weakly \( J \)-stable algebras, see [5] and the definitions in section [4].

Proposition 3.1.2. Let \( E \) be a split exact functor and let \( \phi, \psi : A \to B \) be homomorphisms with \( \phi(x)\psi(y) = \psi(x)\phi(y) = 0 \) for all \( x, y \in A \). We have that the linear map \( \phi + \psi : A \to B \) is a homomorphism and that \( E(\phi + \psi) = E(\phi) + E(\psi) \).
We will show that every split exact and $M_2$-stable functor admits a pairing with abstract Kasparov modules. Kasparov modules are a convenient way of encoding the extended functoriality of $M_2$-stable and split-exact functors. The underlying construction is the one of quasi-homomorphisms, which were introduced in [8]. Next, we give a brief introduction to the basics of quasi-homomorphisms.

3.2. Quasi-homomorphisms. Let $\alpha$ and $\bar{\alpha}$ be two homomorphisms $A \to D$ between locally convex algebras. Assume that $B$ is a closed subalgebra of $D$ such that $\alpha(x) - \bar{\alpha}(x) \in B$ and $\alpha(x)B \subset B$, $B\bar{\alpha}(x) \subset B$ for all $x \in A$. We call such a pair $(\alpha, \bar{\alpha})$ a quasi-homomorphism from $A$ to $B$ relative to $D$ and denote it by $(\alpha, \bar{\alpha}) : A \to B$.

We will show that $(\alpha, \bar{\alpha})$ induces a homomorphism $E(\alpha, \bar{\alpha}) : E(A) \to E(B)$ in the following way. Define $\alpha' : A \to A \oplus D$ by $\alpha'(x) = (x, \alpha(x))$, $\bar{\alpha}' = (x, \bar{\alpha}(x))$ and denote by $D'$ the subalgebra of $D \oplus A$ generated by all elements $\alpha'(x)$, $x \in A$ and by $0 \oplus B$. We obtain an extension with two splitting homomorphisms $\alpha'$ and $\bar{\alpha}'$:

$$0 \to B \to D' \to A \to 0$$

where the map $D' \to A$ by definition maps $(x, \alpha(x))$ to $x$ and $(0, b)$ to $0$. The map $E(\alpha, \bar{\alpha})$ is defined to be $E(\alpha') - E(\bar{\alpha}') : E(A) \to E(B) \subset E(D')$ (this uses split-exactness). Note that $E(\alpha, \bar{\alpha})$ is independent of $D$ in the sense that we can enlarge $D$ without changing $E(\alpha, \bar{\alpha})$ as long as $B$ maintains the properties above.

**Proposition 3.2.1.** The assignment $(\alpha, \bar{\alpha}) \to E(\alpha, \bar{\alpha})$ has the following properties:

(a) $E(\bar{\alpha}, \alpha) = -E(\alpha, \bar{\alpha})$

(b) If the linear map $\varphi = \alpha - \bar{\alpha}$ is a homomorphism and satisfies $\varphi(x)\bar{\alpha}(y) = \bar{\alpha}(x)\varphi(y) = 0$ for all $x, y \in A$, then $E(\alpha, \bar{\alpha}) = E(\varphi)$.

**Proof.** (a) This is obvious from the definition. (b) This follows from proposition 3.1.2 and the fact that $\varphi + \bar{\alpha} = \alpha$. □

**Definition 3.2.2.** We say that a functor $E : \text{LCA} \to \text{Ab}$ is

- invariant under inner automorphisms, if $E(\text{Ad}U) = \text{id}_{E(A)}$ for any invertible element $U$ in a unital algebra $A$,

- invariant under idealizing automorphisms, if the following holds:

  Whenever $A$ is isomorphic to a subalgebra of a unital algebra $D$ and $U$ is an invertible element in $D$ such that $AU, UA, AU^{-1}, U^{-1}A \subset A$, then $E(\text{Ad}U) = 1_{E(A)}$ for the automorphism $\text{Ad}U : A \to A$.

**Lemma 3.2.3.** (a) Every split-exact functor that is invariant under inner automorphisms is also invariant under idealizing automorphisms.

(b) Every $M_2$-stable functor is invariant under inner automorphisms.

(c) Every split-exact and $M_2$-stable functor is invariant under idealizing automorphisms.

(d) Assume that $E$ is invariant under inner automorphisms and let $(\alpha, \bar{\alpha}) : A \to B$ be a quasi-homomorphism relative to $D$. If there is an invertible element $U$ in $D$ such that $BU, UB, BU^{-1}, U^{-1}B \subset B$ and such that moreover $U\alpha(x) - \alpha(x) \in B$, $\bar{\alpha}(x) = U\alpha(x)U^{-1}$ for all $x \in A$, then $E(\alpha, \bar{\alpha}) = 0$. 

Proof. (a) Let $A$ be a locally convex algebra and $U \in D$ be an idealizing element of $A \subset D$. Let $B$ be the sub-algebra of $D \oplus \mathbb{C}[t, t^{-1}]$ which is generated by $A \oplus 0$ and $U \oplus t$ (we view $\mathbb{C}[t, t^{-1}]$ as a locally convex algebra with the fine topology). There is a natural split extension

$$0 \to A \to B \to \mathbb{C}[t, t^{-1}] \to 0.$$ 

The inner automorphism determined by $U \oplus t$ induces the identity on $E(B)$ and thus on $E(A)$.

(b) Let $U$ induce an inner automorphism of the algebra $A$. Consider the inner automorphism of $\hat{A} \otimes M_2$ induced by $U \oplus 1$. Since, by an easy argument, the inclusions of $A$ into the upper left corner and the inclusion into the lower right corner of $\hat{A} \otimes M_2$ determine the same isomorphism under $E$, this implies that $E$ is invariant under idealizing automorphisms.

(c) Combine (b) and (a).

(d) Note that $W = (1, U)$ defines an idealizing automorphism of the algebra $D'$ that appears in the split-extension defining the map $E(\alpha, \bar{\alpha})$. By (c), we know that $E(\bar{\alpha}') = E(\text{Ad}W) \circ E(\alpha') = E(\alpha')$. Hence, we get $E(\alpha, \bar{\alpha}) = 0$. □

3.3. Kasparov modules.

**Definition 3.3.1.** Let $A, I$ and $D$ be locally convex algebras. Assume that $D$ is unital and contains $I$ as a closed ideal. An abstract Kasparov $(A, I)$-module relative to $D$ is a triple $(\varphi, U, P)$ where

- $\varphi$ is a continuous homomorphism from $A$ into $D$.
- $U$ is an invertible element and $P$ is an idempotent element in $D$ such that the following commutators are in $I$ for all $x \in A$:

$$[U, \varphi(x)], [P, \varphi(x)], [U, P].$$

**Remark 3.3.2.** (a) The condition that $I$ is a closed ideal in $D$ could be weakened to demanding that the inclusion $I \to D$ and the homomorphisms $A \otimes I \to I$ and, $I \otimes A \to I$ which are given by mapping $A$ to $D$ and multiplying in $D$ are continuous for the projective topology on the tensor product. Furthermore, just as for quasihomomorphisms, the morphisms that we construct from a Kasparov module will be independent of the algebra $D$ in the sense that $D$ can be enlarged as long as the conditions on the commutators and the ideal are maintained.

(b) In Definition 3.3.1 one could replace the condition that $[U, P] \in I$ by the condition that $[U, P]\varphi(x)$ and $\varphi(x)[U, P]$ are in $I$ for all $x \in A$.

In the sequel, given an idempotent $P$ in a unital algebra, we will denote by $P^\perp$ the idempotent $1 - P$. Given an abstract Kasparov module $(\varphi, U, P)$, we define invertible matrices $W_P$ and $U_P$ in $M_2(D)$ as follows

$$W_P = \begin{pmatrix} P & P^\perp \\ -P^\perp & P \end{pmatrix},$$
and
\[ U_P = W_P \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} W_P^{-1} = \begin{pmatrix} PUP + P^\perp & -PUP^\perp \\ -P^\perp UP & P + P^\perp UP^\perp \end{pmatrix}. \]

We obtain in this way a quasihomomorphism \(((\varphi \oplus 0), \text{Ad} U_P \circ (\varphi \oplus 0)) : A \to M_2(I)\).

**Definition 3.3.3.** Let \( E \) be a functor that is split exact and \( M_2 \)-stable and let \((\varphi, U, P)\) be an abstract \((A, I)\)-Kasparov module. We denote by \( E(\varphi, U, P) \) the map \( E(A) \to E(M_2 I) \cong E(I) \) associated with the quasihomomorphism \(((\varphi \oplus 0), \text{Ad} U_P \circ (\varphi \oplus 0)) : A \to M_2 I \) and the natural inclusion \( j_2 : I \to M_2 I \).

**Lemma 3.3.4.** Let \( E \) be split exact and \( M_2 \)-stable. We consider abstract \((A, I)\)-Kasparov modules.

(a) \( E(\varphi, UU', P) = E(\varphi, U, P) + E(\varphi, U', P) \) whenever all terms are defined.
(b) If \( PP' = P'P = 0 \), then \( E(\varphi, U, P + P') = E(\varphi, U, P) + E(\varphi, U, P') \).
(c) If \( P \) commutes with all \( \varphi(x) \), \( \varphi'(x) \) and if \( P\varphi(x) = P\varphi'(x) \) for all \( x \), then \( E(\varphi, U, P) = E(\varphi', U, P) \).
(d) If \( \varphi(x)U - \varphi(x) \in I \) for all \( x \in A \), then \( E(\varphi, U, P) = 0 \).
(e) If \( PU - P'U' \in I \), then \( E(\varphi, U, P) = E(\varphi, U', P') \).

**Proof.** (a) We have \((UU')_P = U_P U'_P\). This implies
\[
E(\varphi, UU', P) = E(\varphi \oplus 0, Ad((UU')_P) \circ (\varphi \oplus 0)) = E(\varphi \oplus 0, Ad(U_P) \circ (\varphi \oplus 0)) + E(Ad(U_P)) \circ (\varphi \oplus 0), Ad(U_P U'_P) \circ (\varphi \oplus 0)) = E(\varphi, U, P) + E(Ad(U_P)) \circ E(\varphi, U', P) = E(\varphi, U, P) + E(\varphi, U', P)
\]
The last equality follows, since \( Ad(U_P) \) is an idealizing automorphism of \( M_2(I) \) and we have shown the invariance under such in Lemma 3.2.3(c).

(b) One easily checks that \( 1 - (U_{P+P'})^{-1}U_P U_{P'} \in M_2(I) \). The assertion then follows from (a) combined with (d).

(c) A computation using the definition of a Kasparov module and \( M_2 \)-stability shows that \( E(P^\perp \varphi, U, P) = E(P^\perp \varphi \oplus 0, P^\perp \varphi \oplus 0) = 0 \).

This observation implies that
\[
E(\varphi, U, P) = E(P\varphi, U, P) = E(P\varphi', U, P) = E(\varphi', U, P).
\]
(d) The element \( U_P \) satisfies the hypotheses of lemma 3.2.3(d) with respect to \( \varphi \oplus 0 \).

(e) We have the following chain of congruences mod the ideal \( I \):
\[
P \equiv U^{-1}U'P' = (U^{-1}U'P')P' \equiv PP' \equiv P(PUU'^{-1}) = PUU'^{-1} \equiv P'
\]
(\( \equiv \) denotes congruence mod \( I \)).

The explicit form of the matrix \( U_P \) is as follows:
\[
U_P = \begin{pmatrix} PUP + P^\perp & -PUP^\perp \\ -P^\perp UP & P + P^\perp UP^\perp \end{pmatrix}
\]
and similarly for \( U'_P \). Therefore the matrices
\[
U'^{-1}_P U_P (1 \oplus 0)
\]
Moreover, \( V \) and \( W \) are in \( M_2(I^+) \). These two matrices produce an algebraic equivalence in \( M_2(I^+) \) between the idempotents \( 1 \oplus 0 \) and \( U_{p_1} U_p (1 \oplus 0) U_{p_1}^{-1} U_{p}^{-1} \). Using the standard technique (e.g. [2] prop. 4.3.1) there exists an invertible element \( V \) in \( M_4(I^+) \) such that

\[
V(U_{p_1}^{-1} U_p \oplus 1 \oplus 1)(1 \oplus 0 \oplus 0 \oplus 0) = (1 \oplus 0 \oplus 0 \oplus 0)
\]

and

\[
(1 \oplus 0 \oplus 0 \oplus 0)(U_{p_1} U_p^{-1} \oplus 1 \oplus 1)V^{-1} = (1 \oplus 0 \oplus 0 \oplus 0).
\]

Moreover, \( V \) can be chosen such that \( V \equiv 1 \oplus 1 \oplus 1 \oplus 1 \). This last observation, together with \( M_2 \)-stability, implies the third equality in the following computation.

\[
0 = E(\varphi \oplus 0 \oplus 0 \oplus 0, \varphi \oplus 0 \oplus 0 \oplus 0)
= E(\varphi \oplus 0 \oplus 0 \oplus 0, \Ad(V) \circ \Ad(U_{p_1}^{-1} U_p \oplus 1 \oplus 1) \circ (\varphi \oplus 0 \oplus 0 \oplus 0))
= E(\varphi \oplus 0, \Ad(U_{p_1}^{-1} U_p) \circ (\varphi \oplus 0))
= E(\varphi \oplus 0, \Ad(U_{p_1}^{-1}) (\varphi \oplus 0) + E(\Ad(U_{p_1}^{-1})) \circ E(\Ad(U_{p_1}^{-1}) \circ (\varphi \oplus 0))
= E(\varphi, U^{-1}, P') + E(\varphi, U, P)
\]

The last equality is due to the observation that \( \Ad(U_{p_1}^{-1}) \) is an idealizing automorphism of \( M_2(I) \). This proves the assertion

\[
E(\varphi, U', P') = E(\varphi, U, P),
\]

using (a) and (d).

3.4. Diffotopy invariance via Kasparov modules. In this section we construct explicit Kasparov modules which induce the evaluation maps, stabilized by the smooth compact operators. They are used in the proof of the diffotopy invariance theorem (see Theorem 1.2.1).

We let \( C^\infty(S^1) \) act in the usual way on \( \ell^2(\mathbb{Z}) \) and denote by \( P \) the Hardy projection from \( \ell^2(\mathbb{Z}) \) onto the subspace \( \ell^2(\mathbb{N}) \).

We denote by \( D \) the subalgebra of \( B(\ell^2(\mathbb{Z})) \) generated algebraically by \( C^\infty(S^1) \) together with \( P \) and all smooth compact operators. There is a natural isomorphism of \( C^\infty(S^1) \) with the Schwartz space of rapidly decreasing \( \mathbb{Z} \)-sequences \( s \subset \ell^2(\mathbb{Z}) \) defined by the action on the basis vector \( e_0 \in \ell^2(\mathbb{Z}) \). A vector is called smooth if it belongs to the image of this embedding.

We list some elementary properties concerning the action of the smooth algebras that we consider on \( \ell^2(\mathbb{Z}) \).

**Lemma 3.4.1.**

(a) There is an isomorphism of topological vector spaces \( s \hat\otimes s \rightarrow \mathcal{K} \) extending the map \( (\lambda_i)_{i \in \mathbb{Z}} \otimes (\mu_j)_{j \in \mathbb{Z}} \mapsto (\lambda_i \mu_j)_{i,j \in \mathbb{Z}} \). In particular, any finite-rank operator built out of smooth vectors is a smooth compact operator.

(b) The action of \( \mathcal{K} \cong s \hat\otimes s \) on the Hilbert space \( \ell^2(\mathbb{Z}) \) is induced by \( a \otimes b(\xi) = a(b|\xi) \) where \( \langle \cdot | \cdot \rangle \) denotes the canonical bilinear form on \( \ell^2(\mathbb{Z}) \). In particular the image \( k(\ell^2(\mathbb{Z})) \) of any \( k \in \mathcal{K} \) is contained in \( s \). Any eigenvector for an eigenvalue \( \neq 0 \) of an element \( k \in \mathcal{K} \) is smooth.
(c) The algebra \( K \) (resp. its unitization \( K^+ \)) is a subalgebra of the \( C^* \)-algebra \( \mathcal{L}_\infty \) of compact operators (resp. of its unitization \( \mathcal{L}_\infty^+ \)) which is closed under functional calculus by holomorphic functions for arbitrary elements (in particular the spectrum of an element in \( K \) is the same as in \( \mathcal{L}_\infty \)), and under functional calculus by \( C^\infty \)-functions for self-adjoint and normal elements.

(d) The algebra \( D \) preserves the space of smooth vectors.

Proof. The only non-trivial assertion is (c). Let \( N \) be the unbounded operator in \( H = l^2(\mathbb{Z}) \) defined by \( N(e_i) = |1 + i|e_i \) on the domain \( \mathcal{D} \) consisting of all (finite) linear combinations of vectors \( e_i \) in the standard orthonormal basis for \( H \). It is trivial to check that \( K \) can be identified with the set of operators \( a \) in \( \mathcal{B}(H) \) for which \( N^i a N^j \) extends from \( \mathcal{D} \) to a bounded operator on \( H \) for all \( i, j \in \mathbb{N} \).

Define norms \( \eta_k(a) = \sum_{i+j=k} ||N^i a N^j|| \) where || \( \cdot || \) denotes the operator norm. Then, for \( a, b \in K \) we have:

\[
\eta_k(ab) = \sum_{i+j=k} ||N^i ab N^j|| \leq \sum_{i+j=k} ||N^i a|| \cdot ||b N^j|| \leq \sum_{i+j=k} \eta_k(a) \eta_j(b)
\]

Thus, the family \( (\eta_j) \) defines a differential seminorm in the sense of \( \mathcal{B} \). The topology on \( K \) is described by the family of sub-multiplicative norms \( || \cdot ||_k \) defined by

\[
||a||_k = \sum_{0 \leq j \leq k} \eta_j(a)
\]

The results in \( \mathcal{B} \) then immediately imply that \( K \) is a subalgebra of the algebra \( \mathcal{L}_\infty \) of compact operators on \( H \) which is closed under functional calculus by holomorphic functions for arbitrary elements (see \( \mathcal{B}, \) 3.12) and by functional calculus by \( C^\infty \)-functions for self-adjoint or normal elements (see \( \mathcal{B}, \) 6.4).

□

Linearly, \( D \) splits into \( K \oplus C^\infty(S^1) \oplus C^\infty(S^1) \) via the linear isomorphism

\[
K \oplus C^\infty(S^1) \oplus C^\infty(S^1) \ni (k, f, g) \mapsto k + f + g P \in D.
\]

We topologize \( D \) by norms \( || \cdot ||_k \oplus P_k \oplus P_k \), where \( || \cdot ||_k \) is of the type described in Lemma \( \mathcal{B}, \) 3.4.1 (c) and \( P_k \) is the usual sub-multiplicative norm on \( C^\infty(S^1) \) defined by

\[
P_k(f) = \sum_{0 \leq j \leq k} \frac{1}{j!} ||f(j)||
\]

An easy computation shows that these norms are sub-multiplicative. This shows that \( D \) is a locally convex algebra and contains the algebra \( K \) of smooth compact operators as a closed ideal.

We identify \( C^\infty(S^1) \) with the algebra of smooth periodic functions on the interval \([0, 4]\) and divide this interval into four succeeding intervals \( I_1, I_2, I_3, I_4 \) of length 1.

We define a homomorphism \( \varphi : \mathbb{C}[0, 1] \rightarrow C^\infty(S^1) \) in the following way:

\[
\varphi(f) \equiv f(0) \quad \text{on} \quad I_1 \quad \quad \varphi(f) = f \quad \text{on} \quad I_2
\]

\[
\varphi(f) \equiv f(1) \quad \text{on} \quad I_3 \quad \quad \varphi(f) = \tilde{f} \quad \text{on} \quad I_4
\]

where we identify the interval \([0, 1]\) with each of the intervals \( I_k \) and where we put \( \tilde{f}(t) = f(1 - t) \), for \( t \in [0, 1] \).
Let now $u_t$, $t \in [0,1]$ be a smooth family of unitary elements in $C^\infty(S^1)$ such that each $u_t$ has winding number 1 and such that $u_0 \equiv 1$ outside a closed interval $I'_1$ contained in the interior of $I_1$ and $u_1 \equiv 1$ outside an interval $I'_2$ contained in the interior of $I_3$ (there is an obvious explicit family with these properties).

Since $[P, \varphi(x)] \in \mathcal{K}$ and $[P, u_t] \in \mathcal{K}$ for each $x$ and $t$, $(\varphi, u_t, P)$ defines an abstract Kasparov module for each $t$.

**Lemma 3.4.2.** Let $E$ be an $M_2$-stable and split exact functor. Then

$$E(\varphi, u_0, P) = E(j \circ \text{ev}_0) \quad E(\varphi, u_1, P) = E(j \circ \text{ev}_1)$$

where $\text{ev}_1 : \mathbb{C}[0,1] \to \mathbb{C}$ are the evaluation maps and $j : \mathbb{C} \to \mathcal{K}$ is the natural inclusion.

**Proof.** Let $h$ be a function in $C^\infty(S^1)$ such that $h \equiv 1$ on $I'_1$, $h \equiv 0$ outside $I_1$ and $0 \leq h \leq 1$ and put

$$Q = \begin{pmatrix} h & (h(1-h))^\frac{1}{2} \\ (h(1-h))^\frac{1}{2} & 1-h \end{pmatrix}$$

We consider $Q$ as an idempotent in $M_2(D)$. It commutes exactly with $\varphi(f) \oplus \text{ev}_0(f), f \in \mathbb{C}[0,1]$ and we have

$$Q(\varphi(f) \oplus \text{ev}_0(f)) = Q(\text{ev}_0(f) \oplus \text{ev}_0(f))$$

and

$$(u_0 \oplus 1)Q^\perp = Q^\perp(u_0 \oplus 1) = Q^\perp.$$ We have, using the last identity and Lemma 3.3.4(e), the following equalities:

$$E(\varphi, u_0, P) = E(\varphi \oplus \text{ev}_0, u_0 \oplus 1, P \oplus P)$$
$$= E((\varphi \oplus \text{ev}_0)Q, u_0 \oplus 1, P \oplus P)$$
$$= E((\text{ev}_0 \oplus \text{ev}_0)Q, u_0 \oplus 1, P \oplus P)$$
$$= E(\text{ev}_0 \oplus \text{ev}_0, u_0 \oplus 1, P \oplus P)$$
$$= E(\text{ev}_0, u_0, P)$$

An analogous argument shows that $E(\varphi, u_1, P) = E(\text{ev}_1, u_1, P)$.

Now, note that $u_0$ is connected to the standard unitary $z = \{t \mapsto e^{2\pi it}\}$ by a continuous path $u_t$ in $C^\infty(S^1)$ with $u_0 = z$ and $u_1 = u_0$. Since $P$ commutes with all elements in $C^\infty(S^1)$ modulo compact operators we see that $Pw_tP$ describes a continuous path of operators, which are Fredholm on $\ell^2(\mathbb{N})$. The operator $PzP$ is clearly of index $-1$, then so is $v = Pu_0P$. Applying functional calculus by the holomorphic function, which is equal to $z^{-1/2}$ in a neighborhood of 1 and equal to 0 in a neighborhood of 0, to $v^*v$, there is a positive element $h$ in the algebra generated by $PKP$ and $P$, which differs from $P$ only by an element in $\mathcal{K}$ such that $vh$ is a partial isometry still of index $-1$ (in fact with the same kernel and cokernel as $v$).

Choose a partial isometry $k$ in $\mathcal{B}(\ell^2(\mathbb{N}))$ with support ker $v$ and range contained in coker $v$. Since the range and support of $k$ consist of smooth vectors, $k$ is in $\mathcal{K}$. Thus $a = vh + k$ is an isometry of index -1 on $\ell^2(\mathbb{N}) \subset \ell^2(\mathbb{Z})$ which differs from $v$ only by an element in $\mathcal{K}$.

In the same way one gets a coisometry $b$ of index 1 on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{N})$ and equal to $P^\perp u_0 P^\perp$ modulo $\mathcal{K}$. Adding a partial isometry of rank 1 in $\mathcal{K}$, that connects ker $b$
and coker $a$, to $a + b$ one obtains a unitary $u$ in $B(l^2(\mathbb{Z}))$ that differs from $u_0$ only by an element in $\mathcal{K}$ such that $uP = a$. 
Using Lemma 3.3.4 (e) and the explicit form of the quasihomomorphism defined by the abstract Kasparov module $(ev_0, u, P)$, we have

$$E(ev_0, u_0, P) = E(ev_0, u, P) = E(ev_0, \text{Ad}(a + P^\perp)ev_0)$$

However, by the choice of $a$, we have $ev_0 = \text{Ad}(a + P^\perp)ev_0 + j \circ ev_0$, so that $E(ev_0, u_0, P) = E(j_0 \circ ev_0)$ by 3.2.1 (b). The argument for $E(ev_1, u_1, P)$ is exactly symmetric. □

### 4. Diffotopy invariance theorem

#### 4.1. Half-exact functors and diffotopy invariance

We continue with some definitions.

**Definition 4.1.1.** Let $E : \text{LCA} \rightarrow \text{Ab}$ be a functor from the category of locally convex algebras to the category of abelian groups. Let $\mathcal{J}$ be a Banach ideal. We say that

- $E$ is diffotopy invariant, if the maps $ev_t : E(A[0, 1]) \rightarrow E(A)$ induced by the different evaluation maps for $t \in [0, 1]$ are all the same (it is easy to see that this is the case if and only if the map induced by evaluation at $t = 0$ is an isomorphism).
- $E$ is half-exact, if, for every extension $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ of locally convex algebras, the induced short sequence $E(I) \rightarrow E(A) \rightarrow E(B)$ is exact.
- $E$ is $\mathcal{J}$-stable, if the map $E(A) \rightarrow E(A \hat{\otimes} \mathcal{J})$ induced by the natural inclusion $j : A \rightarrow A \hat{\otimes} \mathcal{J}$, defined by any minimal idempotent in $\mathcal{J}$, is an isomorphism for each locally convex algebra $A$ (note that any Banach ideal contains minimal idempotents).
- $E$ is weakly $\mathcal{J}$-stable, if there is a natural map $E(A \hat{\otimes} \mathcal{J}) \rightarrow E(A)$, such that the composition $E(A) \rightarrow E(A \hat{\otimes} \mathcal{J}) \rightarrow E(A)$ with the map induced by the inclusion map $A \rightarrow A \hat{\otimes} \mathcal{J}$ defined by a minimal idempotent in $\mathcal{J}$ is the identity for each locally convex algebra $A$.

**Remark 4.1.2.** We are using weak $\mathcal{J}$-stability only for Banach ideals $\mathcal{J}$, but note that weak stability is hereditary for certain subalgebras of $\mathcal{J}$. In particular, if $\mathcal{J}$ contains all smooth compact operators (for instance if $\mathcal{J}$ is harmonic, see end of section 2.2), then any weakly $\mathcal{J}$-stable functor is also weakly $\mathcal{K}$-stable.

**Definition 4.1.3.** Two homomorphisms $\alpha, \beta : A \rightarrow B$ between locally convex algebras are called diffotopic if there is a homomorphism $\varphi : A \rightarrow B[0, 1]$ such that

$$\alpha = ev_0 \circ \varphi \quad \beta = ev_1 \circ \varphi.$$

**Remark 4.1.4.** In [9] and elsewhere it was stated that the condition defining diffotopy in 4.1.3 was equivalent to the fact that there exists a family of homomorphisms $\varphi_t : A \rightarrow B$, $t \in [0, 1]$ such that, for each $x$ in $A$, the function $t \mapsto \varphi_t(x)$ is in $B[0, 1]$ and such that $\alpha = \varphi_0$, $\beta = \varphi_1$. The equivalence between the two conditions holds only if $A$ and $B$ are Fréchet, i.e. metrizable. A counterexample in the general
case, due to Frerick and Shkarin, was communicated to us by L.Frerick. Since only the condition given in the definition above was used in all the proofs, this does not affect any of the results in [9] or [10].

It is easy to check that diffotopy is an equivalence relation. If $\alpha$ and $\beta$ are diffotopic and $E$ is a diffotopy invariant functor, then obviously $E(\alpha) = E(\beta)$.

The next lemma illustrates the important consequences of the combination of the properties of diffotopy invariance and half-exactness. Note that the algebraic $K$-theory functor $K_0$ is half-exact, in fact, it satisfies excision (see [1]). Using the formally similar property of split-exactness, we are going to conclude that the $J$-stabilized version of the algebraic $K$-theory functor is diffotopy invariant. However, split-exactness does not imply half-exactness in general, nor does the reverse implication hold. For some of the notation, the reader is referred to section 6.1.

**Lemma 4.1.5.** Let $E : \text{LCA} \to \text{Ab}$ be a functor from the category of locally convex algebras to the category of abelian groups which is diffotopy invariant and half-exact. Then

(a) $E$ has long exact sequences, i.e. for each extension $0 \to I \to A \to B \to 0$ of locally convex algebras there is a long exact sequence (infinite to the left) of the form

$$\cdots \to E(\Sigma A) \to E(\Sigma B) \to E(I) \to E(A) \to E(B)$$

(b) There is a natural isomorphism $E(JA) \cong E(\Sigma A)$

**Proof.** (a) is well known and follows from a standard argument using mapping cones. (b) Apply the long exact sequence from (a) to the extension $0 \to JA \to TA \to A \to 0$ and use the fact that $E(TA) = E(\Sigma TA) = 0$ by diffotopy invariance of $E$ and the fact that $TA$ is smoothly contractible. □

### 4.2. Main Theorem

In this section we state and prove the diffotopy invariance theorem.

**Theorem 4.2.1.** Every functor from the category of locally convex algebras to the category of abelian groups which is split exact, $M_2$-stable and weakly $J$-stable for some harmonic Banach ideal, is diffotopy invariant.

**Proof.** Given a weakly $J$-stable, $M_2$-stable and split-exact functor $E : \text{LCA} \to \text{Ab}$, these properties are inherited by the functor $E^A : \text{LCA} \to \text{Ab}$ which assigns $B \mapsto E(A \otimes B)$, for any given locally convex algebra $A$. This implies that w.l.o.g. it is enough to show that the two evaluation maps from $E(\mathbb{C}[0, 1])$ to $E(\mathbb{C})$ are equal. Using the result of Lemma 3.4.2 and the injectivity of $E(j) : E(\mathbb{C}) \to E(\mathcal{K})$, which followed from weak $J$-stability and remark 4.1.2, it suffices to show that we have an equality of $(\mathbb{C}[0, 1], \mathcal{K})$-Kasparov modules

$$E(\varphi, u_0, P) = E(\varphi, u_1, P).$$

Using the weak $J$-stability again we conclude that it also suffices to show that the following $(\mathbb{C}[0, 1], \mathcal{K} \hat{\otimes} J)$-Kasparov modules are equal:
\[ E(\varphi \otimes 1, u_0 \oplus 1 \oplus \ldots, P \otimes 1) = E(\varphi \otimes 1, u_1 \oplus 1 \oplus \ldots, P \otimes 1). \]

Note that \( u_0 u_0^{-1} = e^{ih} \) for some self-adjoint element \( h \in C^\infty(S^1) \). Denote by \( Z \) the unitary in \((C^\infty(S^1) \otimes J)^+ \subset (D \otimes J)^+\) given by
\[ Z = e^{ih} \oplus 1 \oplus 1 \oplus \ldots. \]

where \( D \) is as in 3.3.

We have to show that \( E(\varphi \otimes 1, Z, P \otimes 1) = 0 \). The element \( Z \) is the product of elements
\[ X_1 = e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus \ldots \]

and
\[ X_2 = 1 \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus \ldots. \]

Here, the term with \( \pm \frac{ih}{2n} \) appears precisely \( 2^n \) times. The elements \( X_i \) lie in \((C^\infty(S^1) \otimes J)^+\) since they are exponentials of elements \( ih \otimes x_i \) with \( x_i \leq 2\omega \in J \). This is the only place, where we use the assumption that \( J \) is harmonic in an essential way.

Note that the elements \( X_i \) commute with \((C^\infty(S^1) \otimes 1)\) and with \( P \otimes 1 \) modulo \( K \otimes J \) and hence define abstract Kasparov \((C[0,1], K \otimes J)\)-modules relative to \((D \otimes J)^+\).

Note further that \( X_1 = W_1, X_2 = W_2 \) with
\[ W_1 = e^{ih} \oplus e^{-ih} \oplus 1 \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus \ldots \]

and
\[ W_2 = e^{ih} \oplus 1 \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus e^{ih} \oplus e^{-ih} \oplus 1 \oplus \ldots. \]

Clearly, \( W_2 = (1 \otimes Y)W_1^{-1}(1 \otimes Y^{-1}) \) for some permutation matrix \( Y \in B(H) \).

We see that \([\varphi(C[0,1]) \otimes 1, 1 \otimes Y] = [P \otimes 1, 1 \otimes Y] = 0\) so that abstract Kasparov \((C[0,1], K \otimes J)\)-modules relative to \( D \otimes (B(H) \ltimes J) \) are defined, since \( K \otimes J \) is a closed ideal in \( D \otimes (B(H) \ltimes J) \). (For a definition of \( B(H) \ltimes J \) see section 2.2) By lemma 8.3.4(a) we see that \( E(\varphi \otimes 1, X_1, P \otimes 1) = E(\varphi \otimes 1, W_1 \otimes Y W_1^{-1}(1 \otimes Y)^{-1}, P \otimes 1) = 0 \).

A similar reasoning applies to \( X_2 \). This finishes the proof. \( \square \)

5. \( K_0 \) of a stable algebra.

5.1. Stabilized functors. In this section we consider split-exact and \( M_2 \)-stable functors \( E' : \text{LCA} \to \text{Ab} \), defined on the category of locally convex algebras.

Definition 5.1.1. Let \( A \) be a locally convex algebra and let \( E' \) be a \( M_2 \)-stable and split-exact functor. Let \( J \) be a Banach ideal. The algebra \( A \) is called weakly \( J \)-stable with respect to the functor \( E' \) if the functor \( B \mapsto E'(B \otimes A) \) is weakly \( J \)-stable (see definition 4.1.1).

Proposition 5.1.2 shows that \( J \) is weakly \( J \)-stable, so that there is always one obvious weakly \( J \)-stable algebra. Moreover, if \( B \) is weakly \( J \)-stable and \( A \) is any locally convex algebra, the \( A \otimes B \) is weakly \( J \)-stable.

Let now \( J \) be a harmonic Banach ideal and \( A \) be a weakly \( J \)-stable algebra with respect to the functor \( E' \). In this section we show, as a corollary of theorem 4.2.1 that the associated \( A \)-stabilized functor \( E = E'(? \otimes A) : \text{LCA} \to \text{Ab} \) satisfies
diffotopy invariance. See the remark after definition 3.1.1 for important examples of split-exact and $M_2$-stable functors to which such a result could be applied.

**Proposition 5.1.2.** The functor $E : \text{LCA} \to \text{Ab}$ which assigns $A \mapsto E'(A \otimes \mathcal{J})$ is weakly $\mathcal{J}$-stable in the sense of definition 4.1.1, i.e. the algebra $\mathcal{J}$ is weakly $\mathcal{J}$-stable with respect to any $M_2$-stable and split exact functor.

*Proof.* The natural map $\theta : \mathcal{J}(H) \hat{\otimes} \mathcal{J}(H) \to \mathcal{J}(H \hat{\otimes} H)$ induces a natural map $\theta_A : E(A \otimes \mathcal{J}) \to E(A)$ for every locally convex algebra $A$. We want to show that $\theta_A \circ j_A = \text{id}_{E(A)}$ for the natural map $j_A : E(A) \to E(A \otimes \mathcal{J})$ induced by the inclusion $j : \mathbb{C} \to \mathcal{J}$, i.e. $j_A = E'(j \otimes \text{id}_{\mathcal{J}})$.

There is an isometry $V$ in $\mathcal{B}(H \otimes H)$ such that $\theta \circ (j \otimes \text{id}_{\mathcal{J}})(x) = \text{Ad}(V) = V x V^*$ for $x \in \mathcal{J}$. Choose a second isometry $V'$ in $\mathcal{B}(H \otimes H)$ such that $VV^* + V'V'^* = 1$. Denote by $O_2$ the algebra generated algebraically by $1 \otimes V, 1 \otimes V^*$ and $1 \otimes V', 1 \otimes V'^*$ in $A \hat{\otimes} \mathcal{B}(H \otimes H)$ and by $D$ the algebra generated by $A \hat{\otimes} \mathcal{J}$ together with $1 \otimes V, 1 \otimes V^*$ and $1 \otimes V', 1 \otimes V'^*$ inside $A \hat{\otimes} \mathcal{B}(H \otimes H)$. We have a split extension

$$0 \to A \hat{\otimes} \mathcal{J} \to D \to O_2 \to 0$$

It is easy to see that there is a unitary $U$ in $M_2(O_2)$ and hence in $M_2(D)$ such that $U x U^* = (1 \otimes V) x (1 \otimes V^*)$ for $x$ in the subalgebra

$$
\begin{pmatrix}
A \hat{\otimes} \mathcal{J} & 0 \\
0 & 0
\end{pmatrix}
$$

of $M_2(A \hat{\otimes} \mathcal{J})$. Thus, by $M_2$ stability of $E$ it is clear that $E(\text{Ad}(1 \otimes V)) = \text{id}$ on $E(D)$. On the other hand, from the split extension above, we see that $E(D) = E(A \hat{\otimes} \mathcal{J}) \oplus E(O_2)$ so that the restriction of $\text{Ad}(1 \otimes V)$ induces the identity on $E(A \hat{\otimes} \mathcal{J})$. $\square$

**Corollary 5.1.3.** Let $E' : \text{LCA} \to \text{Ab}$ be an $M_2$-stable and split exact functor and let $A$ be a weakly $\mathcal{J}$-stable algebra with respect to the functor $E'$, for a fixed harmonic Banach ideal $\mathcal{J}$. Under these circumstances, the functor $E = E'(? \otimes A)$ is diffotopy invariant.

*Proof.* This follows now from 4.2.1 $\square$

### 5.2. Algebraic $K$-theory

We continue by applying Corollary 5.1.3 to the algebraic $K$-theory functor $K_0$ and identify algebraic $K$-theory of stable algebras with a suitable group of homotopy classes of maps.

Consider the algebra $Q \mathbb{C} = \mathbb{C} * \mathbb{C}$ and denote by $e, \bar{e}$ the two generators $e = \nu_1(1), \bar{e} = \nu_2(1)$. The argument in [3], 3.1, shows that $E(Q \mathbb{C} \otimes A) \cong E(A) \oplus E(A)$ and that $E(q \mathbb{C} \hat{\otimes} A) \cong E(A)$ for each $M_2$-stable and diffotopy invariant functor $E$. With the classical description of $K_0$ in [1], the generator of $K_0(q \mathbb{C} \otimes \mathcal{J})$ is given by the difference of equivalence classes of the idempotent elements $p$ and $\bar{p}$ in $M_2((q \mathbb{C} \otimes \mathcal{J})^+)$:

$$p = W \begin{pmatrix} \bar{e} & 0 \\ 0 & e \end{pmatrix} W^{-1} \quad \text{where} \quad W = \begin{pmatrix} \bar{e} & e \\ -\bar{e} & \bar{e} \end{pmatrix}$$
and,
\[
\bar{p} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}.
\]
Note that \( p - \bar{p} \in M_2(q\mathbb{C} \otimes \mathcal{J}) \) and that therefore \([p] - [\bar{p}] \in K_0(q\mathbb{C} \otimes \mathcal{J})\).

**Proposition 5.2.1.** For each locally convex algebra \( A \), one has natural isomorphisms
\[
K_0(A \otimes \mathcal{J}) \cong \langle q\mathbb{C}, A \otimes \mathcal{J} \otimes M_\infty \rangle \cong \lim_{\longrightarrow \, n} \langle q\mathbb{C}, A \otimes \mathcal{J} \otimes M_n \rangle.
\]

*Proof.* First of all, note that the righthand side is an abelian semigroup by block sum. The second isomorphism follows from the fact that \( q\mathbb{C} \) is finitely generated and from properties of the fine topology on \( M_\infty \).

We construct a map \( \phi : \langle q\mathbb{C}, A \otimes \mathcal{J} \otimes M_\infty \rangle \to K_0(A \otimes \mathcal{J}) \). It sends a diffotopy class \( \langle h \rangle \in \langle q\mathbb{C}, A \otimes \mathcal{J} \otimes M_\infty \rangle \) to the difference of equivalence classes of projections \([h(p)] - [h(\bar{p})]\) in \( K_0((A \otimes \mathcal{J})^+) \). The difference lies in the direct summand \( K_0(A \hat{\otimes} \mathcal{J}) \).

The map is well-defined by the preceding corollary.

This map is surjective since all generators of \( K_{0}^{alg}(A \hat{\otimes} \mathcal{J}) \) and their negatives are hit and since the map is clearly a map of abelian semigroups. In order to prove injectivity we use Lemma 7 in \([9]\) which we reprove for sake of completeness.

**Lemma 5.2.2.** Let \( \varphi : q\mathbb{C} \to M_2(q\mathbb{C}) \) be the restriction of the homomorphism \( Q\mathbb{C} \to M_2((Q\mathbb{C})^+) \), which sends \( e \) to \( p \) and \( \bar{e} \) to \( \bar{p} \). Then \( \varphi \) is diffotopic to the inclusion map \( \iota : q\mathbb{C} \to M_2(q\mathbb{C}) \).

*Proof.* Let \( \gamma_t : q\mathbb{C} \to M_2(q\mathbb{C}) \), \( t \in [0, \pi/2] \) be the restriction of the homomorphism
\[
\gamma_t : Q\mathbb{C} \to M_2((Q\mathbb{C})^+) \text{ which is defined by}
\]
\[
\gamma_t'(e) = W_t \begin{pmatrix}
\bar{e}^\perp & 0 \\
0 & e
\end{pmatrix} W_{-t}
\]
\[
\gamma_t'(\bar{e}) = W_t \begin{pmatrix}
\bar{e}^\perp & 0 \\
0 & \bar{e}
\end{pmatrix} W_{-t}
\]
where
\[
W_t = \begin{pmatrix}
\bar{e}^\perp & 0 \\
0 & \bar{e}^\perp
\end{pmatrix} + \begin{pmatrix}
\bar{e} \cos t & \bar{e} \sin t \\
\bar{e} \sin t & \bar{e} \cos t
\end{pmatrix}
\]
For each \( t \) the difference \( \gamma_t'(e) - \gamma_t'(\bar{e}) \) lies in the ideal \( M_2(q\mathbb{C}) \). Therefore \( \gamma_t \) defines a diffotopy, which connects \( \varphi \) with \( \iota \).

To prove injectivity we now use Lemma 5.2.2. Assume that \( \eta_1, \eta_2 : q\mathbb{C} \to A \hat{\otimes} \mathcal{J} \hat{\otimes} M_n \) are homomorphisms, such that \([\eta_1'(p)] = [\eta_2'(p)]\) in \( K_0(A \hat{\otimes} \mathcal{J}) \) (where \( \eta_i \) denotes the induced map \( M_2(q\mathbb{C}^+) \to (A \hat{\otimes} \mathcal{J})^+ \otimes M_{2n} \)). This means that the there exists a projector \( q \in (A \hat{\otimes} \mathcal{J})^+ \hat{\otimes} M_k \) and an invertible element \( u \in (A \hat{\otimes} \mathcal{J})^+ \hat{\otimes} M_{2n+k} \) such that \( u(\eta_1'(p) + q)u^{-1} = \eta_2'(p) + q \). This element \( u \) can even be chosen to be connected to 1 by a differentiable family \( u_t, t \in [1, 2] \), such that \( 1 - u_t \in A \hat{\otimes} \mathcal{J} \hat{\otimes} M_{2n+k} \) for all \( t \).
Consider the homomorphisms \( \zeta_1, \zeta_2 : qC \to A \otimes J \otimes M_{2n+k} \) defined as restrictions of the maps from \( QC \) that map \( e \) to \( \eta'_1(p) \oplus q \) and \( \eta'_2(p) \oplus q \), respectively, and \( \bar{e} \) to \( \bar{p} \oplus q \). Note that \( \zeta_1 = M_2(\eta_1) \circ \phi \).

According to Lemma 5.2.2, \( \zeta_1 = M_2(\eta_1) \circ \varphi \) is diffeotopic to \( \eta_1 = M_2(\eta_1) \circ \iota \) and similarly \( \zeta_2 \) is diffeotopic to \( \eta_2 \). On the other hand, the family \( \zeta_t, t \in [1, 2] \), of homomorphisms \( qC \to A \otimes J \otimes M_{2n+k} \), obtained as restrictions of the maps from \( QC \), which map \( e \) to \( u_t(\eta'_1(p) \oplus q)u_t^{-1} \) and \( \bar{e} \) to \( \eta'_1(\bar{p}) \oplus q \), defines a diffeotopy connecting \( \zeta_1 \) to \( \zeta_2 \). \( \square \)

6. Determination of \( kk^J(C, A) \)

6.1. Bivariant \( kk \)-theory revisited. We now finally have to use the explicit definition of \( kk^{alg} \). To this end we recall some constructions and notation from [10].

Let \( V \) be a complete locally convex space. Consider the algebraic tensor algebra \( T_{alg} V = V \oplus V \otimes V \oplus V^{\otimes 3} + \cdots \)

with the usual product given by concatenation of tensors. There is a canonical linear map \( \sigma : V \to T_{alg} V \) mapping \( V \) into the first direct summand. We equip \( T_{alg} V \) with the locally convex topology given by the family of all seminorms of the form \( \alpha \circ \varphi \), where \( \varphi \) is any homomorphism from \( T_{alg} V \) into a locally convex algebra \( B \) such that \( \varphi \circ \sigma \) is continuous on \( V \), and \( \alpha \) is a continuous seminorm on \( B \). We further denote by \( TV \) the completion of \( T_{alg} V \) with respect to this locally convex structure.

For any locally convex algebra \( A \) we have the natural extension

\[
0 \to JA \to TA \xrightarrow{\pi} A \to 0.
\]

Here \( \pi \) maps a tensor \( x_1 \otimes x_2 \otimes \ldots \otimes x_n \) to \( x_1x_2 \ldots x_n \in A \) and \( JA \) is defined as Ker \( \pi \). This extension is (uni)versal in the sense that, given any extension \( 0 \to I \to E \to B \to 0 \) of a locally convex algebra \( B \), admitting a continuous linear splitting, and any continuous homomorphism \( \alpha : A \to B \), there is a morphism of extensions

\[
0 \to JA \to TA \xrightarrow{\alpha} A \to 0
\]

\[
0 \to I \xrightarrow{\gamma} E \xrightarrow{\tau} B \to 0
\]

The map \( \tau : TA \to E \) is obtained by choosing a continuous linear splitting \( s : B \to E \) in the given extension and mapping \( x_1 \otimes x_2 \otimes \ldots \otimes x_n \) to \( s'(x_1)s'(x_2)\ldots s'(x_n) \in E \), where \( s' := s \circ \alpha \). Then \( \gamma \) is the restriction of \( \tau \).

Choosing \( 0 \to JB \to TB \to B \to 0 \) in place of the second extension \( 0 \to I \to E \to B \to 0 \), we see that \( A \to JA \) is a functor, i.e. any homomorphism \( \alpha : A \to B \) induces a homomorphism \( J(\alpha) : JA \to JB \).

Using the universal property of \( TA \), one can associate a classifying map with any linearly split extension of locally convex algebras of the form

\[
0 \to I \to E_1 \to E_2 \to \ldots \to E_n \to A \to 0
\]

We consider such an extension as a complex, denoting the arrows (boundary maps) by \( \pi_i \), and we say that it is linearly split if there is a continuous linear map \( s \) of
degree -1 such that $s\pi + \pi s = \text{id}$. Every such splitting $s$ induces a commutative diagram of the form

$$0 \to I \to E_1 \to E_2 \to \cdots \to E_n \to A \to 0$$

$$0 \to J^n A \to T(J^{n-1}A) \to T(J^{n-2}A) \to \cdots \to TA \to A \to 0$$

The leftmost vertical arrow in this diagram is the classifying map for this $n$-step extension. It depends on $s$ only up to difftopy. Recall then from [10] the following definitions.

**Definition 6.1.1.** Let $A$ and $B$ be locally convex algebras. For any continuous homomorphism $\varphi : A \to B$, we denote by $\langle \varphi \rangle$ the equivalence class of $\varphi$ with respect to difftopy and we set

$$\langle A, B \rangle = \{ \langle \varphi \rangle \mid \varphi \text{ is a continuous homomorphism } A \to B \}$$

Given $n \in \mathbb{Z}$ we set

$$kk_n^{alg}(A, B) = \lim_{\to k} \langle J^{k-n} A, \Sigma^k B \hat{\otimes} K \rangle$$

where the inductive limit is with respect to the natural maps

$$\langle J^k A, K \hat{\otimes} B(0, 1)^k \rangle \to \langle J^{k+1} A, K \hat{\otimes} B(0, 1)^{k+1} \rangle$$

mapping the difftopy class of $\alpha$ to the difftopy class of $\alpha'$, where $\alpha'$ is defined by the commutative diagram

$$0 \to J^{k+1} A \to TJ^k A \to J^k A \to 0$$

$$0 \to B'(0, 1)^{k+1} \to B'(0, 1)^k[0, 1) \to B'(0, 1)^k \to 0$$

where $B' = K \hat{\otimes} B$.

Thus, by definition of $kk^J$, we have

$$kk_n^{alg}(A, B) = \lim_{\to k} \langle J^{k-n} A, \Sigma^k B \hat{\otimes} J \hat{\otimes} K \rangle$$

In the sequel we will also use stabilization by the finite matrix algebras $M_n$.

**Proposition 6.1.2.** The natural maps, induced by the inclusion $M_n \to K$,

$$\langle J^k A, \Sigma^k B \hat{\otimes} J \hat{\otimes} M_n \rangle \to \langle J^k A, \Sigma^k B \hat{\otimes} J \hat{\otimes} K \rangle$$

define an isomorphism $\alpha$ of abelian groups:

$$\lim_{\to k} \lim_{\to n} \langle J^k A, \Sigma^k B \hat{\otimes} J \hat{\otimes} M_n \rangle \cong kk^J_0(A, B).$$

**Proof.** Let $V$ be an isometry in $B(H)$ and $j_n : J \to M_n(J)$ the inclusion map (into the upper left corner). A standard argument shows that $j \circ \text{Ad} V : J \to J \hat{\otimes} M_n$ is difftopic to $j$.

Denote by $\theta : J \hat{\otimes} J \to J$ the natural tensor product of operators and by $\varphi : K \to J$
the natural inclusion. We claim that the inverse \( \beta \) to the map \( \alpha \) above is induced by

\[
j_n \circ \theta \circ (\text{id}_J \otimes \varphi) : J \hat{\otimes} K \to J \otimes M_n
\]

The identity \( \beta \circ \alpha = 1 \) follows from the fact that the composition

\[
J \cong J \otimes M_n \to J \hat{\otimes} K \xrightarrow{\text{id}_J \hat{\otimes} \varphi} J \hat{\otimes} J \xrightarrow{\theta} J
\]

is of the form \( \text{Ad} V \) as above.

To show that \( \alpha \circ \beta = 1 \), consider the inclusions \( \iota_l, \iota_r : K \to H \hat{\otimes} K \) into the left, resp. right, factor using the standard rank 1 projector onto the first basis vector in \( H \). By \([10], 2.2.1\), \( \iota_l \) and \( \iota_r \) are diffotopic and in fact both diffotopic to the natural isomorphism \( K \to K \hat{\otimes} K \).

We want to show that the composition of the following maps

\[
J \hat{\otimes} K \xrightarrow{\text{id}_J \hat{\otimes} \iota_l} J \hat{\otimes} K \hat{\otimes} K \xrightarrow{\text{id}_J \hat{\otimes} \phi \otimes \iota_l} J \hat{\otimes} J \hat{\otimes} K \xrightarrow{\theta \otimes \text{id}_K} J \hat{\otimes} K
\]

is diffotopic to the identity. This composition can also be factored as follows

\[
J \hat{\otimes} K \xrightarrow{\text{id}_J \hat{\otimes} \iota_l} J \hat{\otimes} K \hat{\otimes} K \xrightarrow{\text{id}_J \hat{\otimes} \phi \otimes \iota_l} J \hat{\otimes} J \hat{\otimes} K \xrightarrow{\theta \otimes \text{id}_K} J \hat{\otimes} K.
\]

Replacing in this composition \( \iota_l \) by the diffotopic map \( \iota_r \) we again obtain a map of \( J \hat{\otimes} K \to J \hat{\otimes} K \) of the form \( \text{Ad} V \otimes \text{id}_K \) which is diffotopic to \( \text{id} \) (using the fact that \( j_2 : K \to K \otimes M_2 \) is diffotopic to the natural isomorphism \( K \cong K \otimes M_2 \)).

6.2. Main theorem. Our main result in this section is the following computation. Its proof requires some preparation and is given in section 6.3.

**Theorem 6.2.1.** For every locally convex algebra \( A \) and for every harmonic Banach ideal \( J \) one has

\[
\text{kk}^J_0(A, A) = \text{K}_0(A \hat{\otimes} J).
\]

**Remark 6.2.2.** In particular this shows that \( \text{K}_0(A \hat{\otimes} L_p) \) does not depend on \( p \) for \( 1 < p < \infty \) (see also \([14], 4.1\)).

**Corollary 6.2.3.** Let \( J \) be a harmonic Banach ideal. The coefficient ring \( \text{kk}^J_0(\mathbb{C}, \mathbb{C}) \) is isomorphic to \( \mathbb{Z}[u, u^{-1}] \).

**Proof.** By properties of \( \text{kk}^\text{alg} \) we are reduced to a computation of \( \text{kk}^J_0(\mathbb{C}, \mathbb{C}) \) and \( \text{kk}^J_0(\mathbb{C}, \Sigma) \). By Theorem 6.2.1 these groups are isomorphic to the algebraic \( K \)-groups \( \text{K}_0(J) \) and \( \text{K}_0(\Sigma J) \). Both algebras appearing are smooth sub-algebras of \( C^\ast \)-algebras whose \( K \)-theory is well-known, i.e.

\[
\text{K}_0(J) = \text{K}_0(\mathcal{L}_\infty) = \mathbb{Z}
\]

and

\[
\text{K}_0(\Sigma J) = \text{K}_0(\mathcal{C}_0(\mathbb{R}, \mathcal{L}_\infty)) = 0.
\]

This finishes the proof of the corollary.

**Corollary 6.2.4.** Let \( W \) be the Weyl algebra, i.e the unital algebra with two generators \( x \) and \( y \) satisfying the relation \( xy - yx = 1 \) (with the fine topology). Then for every harmonic Banach ideal \( J \) we have \( \text{kk}^J_0(\mathbb{C}, W) = \mathbb{Z} \) and \( \text{kk}^J_0(\mathbb{C}, W) = 0 \).

**Proof.** This follows from the result in \([10], 12.4\) in combination with 6.2.3 above.
Similarly, the \( J \)-stable \( K \)-theory of many other algebras can now be computed as an abelian group rather than just as a module over the coefficient ring.

In order to organize the notation in the following computations, we introduce a category \( H_J \). The objects are locally convex algebras and the morphisms between two locally convex algebras are given by

\[
[A, B] = \lim_{\rightarrow n} \langle A, B \otimes J \otimes M_n \rangle.
\]

Given two diffotopy classes of continuous homomorphisms \( \phi : A \to B \otimes J \otimes M_n \) and \( \psi : B \to C \otimes J \otimes M_m \), their composition in \( H_J \) is defined to be the diffotopy class of

\[
A \xrightarrow{\phi} B \otimes J \otimes M_n \xrightarrow{\psi \otimes \text{id}} B \otimes J \otimes M_n \otimes J \otimes M_m \xrightarrow{\text{id} \otimes \theta} C \otimes J \otimes M_{nm}.
\]

It is clear that composition is associative at the level of diffotopy classes. Moreover, note that endo-functors like \( J(\_); \Sigma \_ \) descend to endo-functors on \( H_J \), since, for example, there is a natural map \( J(A \otimes J \otimes M_n) \to J(A) \otimes J \otimes M_n \).

In particular, using the new notation, we have

\[
kk^J_i(A, B) = \lim_{\rightarrow k} [J^{k-i} A, \Sigma^k B]
\]

by proposition 6.1.2 and

\[
K_0(A \otimes J) = [qC, A]
\]

by proposition 5.2.1

**Proposition 6.2.5.** For all locally convex algebras \( A \) and \( B \) one has

\[
kk^J_i(A, B) = \lim_{\rightarrow k} [J^{2k-i} A, B]
\]

This identity was noted in [10], remark 8.4. In order to give an explicit proof, we have to introduce some notation. Throughout, we are working in the category \( H_J \).

Denote by \( \rho_A : JA \to \Sigma A \) the classifying map of the cone extension of \( A \). Denote by \( \varepsilon'_A : J \Sigma A \to A \) the classifying map of the Toeplitz extension tensored with \( A \). We define \( \varepsilon_A = \varepsilon'_A \circ J(\rho_A) : J^2 A \to A \). (The map \( \varepsilon_A \) has the important interpretation as the classifying map for the 2-step extension given by the Yoneda product of the Toeplitz extension and the cone extension.)

Furthermore, note that

\[
J^2(\varepsilon_A) = \varepsilon_{J^2(A)}
\]

by Korollar 3.1.1 in [9], or by Lemma 4.6 in [10], for all algebras \( A \). For \( \phi : A \to B \), we also have the identity

\[
\phi \circ \varepsilon_A = \varepsilon_B \circ J^2(\phi).
\]

We define inductively \( \varepsilon_A^n : J^n \Sigma^n A \to A \) by setting \( \varepsilon_A^n = \varepsilon_A^{n-1} \circ J^{n-1}(\varepsilon_{\Sigma^{n-1} A}) \) and \( \rho_A^n : J^n A \to \Sigma^n A \) by setting \( \rho_A^n = \rho_{\Sigma^n A} \circ J^{n-1}(\rho_A^{n-1}) \). We also define \( \varepsilon_A^n : J^{2n} A \to A \) by setting \( \varepsilon_A^n = \varepsilon_A^{n-1} \circ \varepsilon_{J^{2n-2} A} \). Note that, in the definition of \( \varepsilon_A^n \), all other choices lead to the same definition of \( \varepsilon_A^n \), since \( J^2(\varepsilon_A) = \varepsilon_{J^2 A} \).
Note that \( \varepsilon_A^n \circ J^n(\rho_A^n) = \varepsilon_A^n : J^{2n}A \to A \), as the following induction argument shows.

\[
\varepsilon_A^n \circ J^n(\rho_A^n) = \varepsilon_A^{n-1} \circ J^{n-1}(\varepsilon_A^{n-1} \circ J^n(\rho_{\Sigma^{n-1}A} \circ J(\rho_A^{n-1}))) \\
= \varepsilon_A^{n-1} \circ J^{n-1}(\varepsilon_A^{n-1} \circ J(\rho_{\Sigma^{n-1}A} \circ J^2(\rho_A^{n-1}))) \\
= \varepsilon_A^{n-1} \circ J^{n-1}(\varepsilon_A^{n-1} \circ J(\rho_A^{n-1} \circ \varepsilon_{J^{n-1}A})) \\
= \varepsilon_A^{n-1} \circ J^{n-1}(\rho_A^{n-1} \circ \varepsilon_{J^{n-2}A}).
\]

With this notation, the abelian group \( kk^J(A,B) \) is defined as the direct limit of a system of abelian groups \( [J^nA, \Sigma^nB] \) via a stabilization map

\[
[J^nA, \Sigma^nB] \ni \psi \mapsto \rho_{\Sigma^nB} \circ J(\psi) \in [J^{n+1}, \Sigma^{n+1}B].
\]

The \( k \)-th stabilization map is given by

\[
[J^nA, \Sigma^nB] \ni \phi \mapsto \rho_{\Sigma^nB}^k \circ J^k(\phi) \in [J^{n+k}A, \Sigma^{n+k}B],
\]

which, again, follows from an easy induction argument.

The right hand side is defined as the direct limit of abelian semi-groups \( [J^{2n}A, B] \) via the stabilization map

\[
[J^{2n}A, B] \ni \psi \mapsto \varepsilon_B \circ J^2(\psi) = \psi \circ \varepsilon_{J^{2n}A} \in [J^{2n+2}A, B].
\]

**Proof.** First of all, we define a map \( [J^nA, \Sigma^nB] \to [J^{2n}A, B] \) by \( \phi \mapsto \varepsilon_B^n \circ J^n(\phi) \in [J^{2n}, B] \). This assignment induces a map of directed systems since it is compatible with the stabilization maps. Indeed, \( \rho_{\Sigma^nB} \circ J(\phi) \) is mapped to \( \varepsilon_B^{n+1} \circ J^{n+1}(\rho_{\Sigma^nB} \circ J(\phi)) \) and the following computation shows that this is the desired result.

\[
\varepsilon_B^{n+1} \circ J^{n+1}(\rho_{\Sigma^nB} \circ J(\phi)) = \varepsilon_B^n \circ J^n(\varepsilon_B^n(\rho_{\Sigma^nB} \circ J(\rho_{\Sigma^nB} \circ J^2(\phi)))) \\
= \varepsilon_B^n \circ J^n(\varepsilon_B^n(J(\rho_{\Sigma^nB} \circ J^2(\phi)))) \\
= \varepsilon_B^n \circ J^n(\varepsilon_B^n(\rho_{\Sigma^nB} \circ J^2(\phi))) \\
= \varepsilon_B^n \circ J^n(\rho_{\Sigma^nB} \circ \varepsilon_{J^2A}) \\
= \varepsilon_B^n \circ J^n(\rho_{\Sigma^nB} \circ \varepsilon_{J^{2n}A}).
\]

This implies that there is a well-defined map

\[
\sigma_{A,B} : \lim_k [J^kA, \Sigma^kB] \to \lim_k [J^{2k}A, B].
\]

Consider a class \( \phi : J^nA \to \Sigma^nB \) which is represented by a homomorphism \( \phi : J^nA \to \Sigma^nB \). If the composition \( \varepsilon_B^m \circ J^n(\phi) \circ \varepsilon_{J^{2m}A} : J^{2m+2n}A \to B \) is homotopic to zero for some \( m \in \mathbb{N} \), then also \( \varepsilon_B^m \circ [J^n(\phi)] \circ [\varepsilon_{J^{2m}A}] = 0 \) as element in \( \lim_k [J^kJ^{2m+2n}A, \Sigma^kB] \).

Note that, for any \( A \) and \( B \), the functor \( J \) maps \( \lim_k [J^kA, \Sigma^kB] \) isomorphically onto \( \lim_k [J^kJA, \Sigma^k JB] \). Furthermore, \( [\varepsilon_A] \) and \( [\varepsilon_A] \) are invertible for all \( A \). This together implies that \( [\phi] = 0 \) as a class in \( \lim_k [J^{k+n}A, \Sigma^{k+n}B] \) and hence that \( \sigma_{A,B} \) is injective.
We still have to show that $\sigma_{A,B}$ is surjective. Consider a class $[\phi] \in \lim_{k \to} [J^{2k}A, B]$ which is represented by a homomorphism $\phi : J^{2n}A \to B$. The element $\phi$ gives rise to a class in $\lim_{k \to} [J^k J^{2n}A, \Sigma^k B]$. The natural map

$$
\lim_{k \to} [J^k A, \Sigma^k B] \to \lim_{k \to} [J^k J^{2n}A, \Sigma^k B]
$$

which is induced by the assignment

$$
[J^k A, \Sigma^k B] \ni \psi \mapsto \psi \circ J^k(\varepsilon^n_A) \in [J^k J^{2n}A, \Sigma^k B]
$$

is an isomorphism. I.e. there is some $k \in \mathbb{N}$ and $\eta : J^k A \to \Sigma^k B$ such that

$$
\eta \circ J^k(\varepsilon^n_A) = \rho_B^k \circ J^k(\phi).
$$

We claim that $[\eta] \in \lim_{k \to} [J^k A, \Sigma^k B]$ does the job. Indeed, it is mapped to

$$
\varepsilon_B^k \circ J^k(\eta) : J^{2k}A \to B.\text{ Stabilizing yields}
$$

$$
\varepsilon_B^k \circ J^k(\eta) \circ \varepsilon_A^{2k} = \varepsilon_B^k \circ J^k(\eta) \circ J^{2k}(\varepsilon_A^n) = \varepsilon_B^k \circ J^k(\rho_B^k) \circ J^{2k}(\phi) = \varepsilon_B^k \circ J^{2k}(\phi) = \phi \circ \varepsilon_A^{2k}.
$$

This shows the surjectivity of $\sigma_{A,B}$. It is another easy check to show, that the assignment is compatible with the several composition products. \(\square\)

6.3. Proof of the main theorem. We now proceed by proving Theorem 6.2.1. By proposition 5.2.1 it suffices to shows that there is a natural isomorphism

$$
[q\mathbb{C}, A] \cong \lim_{k \to} [J^{2k}\mathbb{C}, A].
$$

The existence of a natural map will become apparent in the sequel of the proof.

Proof. There is a natural map $\alpha_1 : J^2(\mathbb{C}) \to q\mathbb{C}$ given as the classifying map of a 2-step extension which comes as the Yoneda-product of the Toeplitz extension and a canonical extension

$$
0 \to \Sigma q\mathbb{C} \to E \to \mathbb{C} \to 0,
$$

where $E = \{f : [0, 1] \to Q\mathbb{C} | f(0) \in \mathbb{C} \ast 0, f(1) \in \mathbb{C} \ast 0, f(t) - f(t') \in q\mathbb{C}, \forall t \in [0, 1]\}$. Denote by $\delta : q\mathbb{C} \to \mathbb{C}$ the restriction of $id \ast 0 : \mathbb{C} \ast \mathbb{C} \to \mathbb{C}$. It follows from the definition of $\varepsilon_{\mathbb{C}} : J^2(\mathbb{C}) \to \mathbb{C}$ that $\delta \circ \alpha_1 = \varepsilon_{\mathbb{C}}$.

We define natural maps $\alpha_n = \alpha_1 \circ \varepsilon_{J^{2n}}^{n-1} \in [J^{2n}\mathbb{C}, q\mathbb{C}]$.

Since $A \mapsto K_0(A \otimes \mathcal{J})$ is a half-exact and dffeotopy invariant functor by Theorem 4.2.1 we conclude by Lemma 4.1.3 that there are natural isomorphisms $K_0(J^{2n}C \otimes \mathcal{J}) \cong K_0(\Sigma^{2n}\mathcal{J})$. However, the righthand side of the last equation is isomorphic to $K_0(C(\mathbb{R}^{2n})) = \mathbb{Z}$ and contains a canonical generator. Denote the canonical generators of $K_0(J^{2n}C \otimes \mathcal{J})$ by $\beta_n \in [q\mathbb{C}, J^{2n}(\mathbb{C})]$. The following identities are immediate, once we have the alternative description of $K_0(\mathbb{C} \otimes \mathcal{J})$, given in proposition 5.2.1.
\[ \alpha_n \circ \beta_n = \text{id}_{\mathbb{C}} \]
\[ \varepsilon_{j_n-k\mathbb{C}}^k \circ \beta_n = \beta_{n-k} \]
\[ \varepsilon_{\mathbb{C}}^n \circ \beta_n = \delta \]

It is clear from the identities above that the maps \( \alpha^* : \lim_{n \in \mathbb{N}} [J^{2n}\mathbb{C}, A] \to [q\mathbb{C}, A] \) and \( \beta^* : [q\mathbb{C}, A] \to \lim_{n \in \mathbb{N}} [J^{2n}\mathbb{C}, A] \), which are induced from \( \alpha_n \) and \( \beta_n \) by precomposition, are well defined. Furthermore, \( \beta^* \circ \alpha^* \) is equal to the identity. We now show that \( \alpha^* \circ \beta^* \) is also equal to the identity.

\[ \beta_n \circ \alpha_n \circ \varepsilon_{j2n\mathbb{C}}^n = \varepsilon_{j2n\mathbb{C}}^n \circ J^{2n}(\beta_n \circ \alpha_n) = J^{2n}(\varepsilon_{\mathbb{C}}^n) \circ J^{2n}(\beta_n \circ \alpha_n) = J^{2n}(\delta \circ \alpha_n) = J^{2n}(\delta \circ \alpha_1 \circ \varepsilon_{j2\mathbb{C}}^{n-1}) = J^{2n}(\varepsilon_{\mathbb{C}}^n) = \varepsilon_{j2n\mathbb{C}}^n \]

\[ \square \]

7. Computation of \( K_1(\mathcal{J}) \) for harmonic Banach ideals

In this section we want to give a computation of the algebraic \( K \)-theory group \( K_1(\mathcal{J}) \) for a harmonic Banach ideal \( \mathcal{J} \) (see definition [2.2.2]). The result contradicts an old result in [14] prop. 4.1. The error in the proof of proposition 4.1 in [14] was brought to our attention by Valqui and Cortiñas. This concrete computation fits nicely with the far more general structure theorem about higher algebraic \( K \)-theory of locally convex algebras stabilized by harmonic Banach ideals, which was obtained by Cortiñas in [5] using our diffotopy invariance theorem for weakly \( \mathcal{J} \)-stable, \( M_2 \)-stable, split-exact functors.

Let \( A \) be a locally convex algebra. In this section \( A^+ \) denotes the unitization by \( \mathbb{Z} \) rather than by \( \mathbb{C} \). A priori, this difference matters and one has to be careful not to use the complex unitization.

The abelian group \( K_1(\mathcal{J}) \) is defined as \( \ker (K_1(\mathcal{J}^+) \to K_1(\mathbb{Z})) \). Denote by \( \mathcal{J}^2 \) the (algebraic) square of \( \mathcal{J} \), i.e. the image of the algebraic tensor product \( \mathcal{J} \otimes \mathcal{J} \) under the multiplication map. We consider the map \( \mathcal{J}^+ \to \mathcal{J}^+/\mathcal{J}^2 \). Since \( \mathcal{J}^+/\mathcal{J}^2 \) is abelian, we have a naturally defined determinant map

\[ \bigcup_{n \in \mathbb{N}} \text{Gl}_n(\mathcal{J}^+/\mathcal{J}^2) \to (\mathcal{J}^+/\mathcal{J}^2)^\times = \mathbb{Z}^\times \ltimes \mathcal{J}/\mathcal{J}^2. \]

We get induced maps

\[ \text{det} : K_1(\mathcal{J}) \to K_1(\mathcal{J}/\mathcal{J}^2) \to \ker (\mathbb{Z}^\times \ltimes \mathcal{J}/\mathcal{J}^2 \to \mathbb{Z}^\times) = \mathcal{J}/\mathcal{J}^2 \]

(we use here the identification of the additive group \( \mathcal{J}/\mathcal{J}^2 \) with the multiplicative group \( \{1 + a \mid a \in \mathcal{J}/\mathcal{J}^2\} \)).

We want to show that \( \text{det} : K_1(\mathcal{J}) \to \mathcal{J}/\mathcal{J}^2 \) is an isomorphism. Note, that it is obviously surjective, since \( \text{det}(e^a) = a + \mathcal{J}^2 \) for any \( a \in \mathcal{J} \).
Theorem 7.1. Let $J$ be a harmonic Banach ideal. The natural determinant map yields an isomorphism

$$K_1(J) \cong J/J^2.$$ 

The preceding result implies that our difftopy result does not extend to higher algebraic $K$-theory. The first topological $K$-theory of the Schatten ideals is well-known to be zero. The proof of the preceding theorem is given after stating and proving a lemma.

Note that, every class $[z] \in K_1(J)$ is represented by an element $z \in Gl_n(J^+)$ which maps to the identity under the canonical evaluation onto $Gl_n(\mathbb{Z})$. Using the next lemma, we are able to show injectivity of the determinant map.

Lemma 7.2. (a) Every invertible element in $M_n(J^+)$ which maps to 1 $\in \mathbb{Z}$ is a product of exponentials of elements in $M_n(J)$.

An invertible element in $M_n(J^2)^+$ which is connected by a norm continuous path of elements, which are invertible in $M_n(J^2)^+$, to 1 $\in M_n(J^2)^+$ is a product of exponentials of elements in $M_n(J^2)$.

(b) Let $g \in M_n(J^2)$. The element $e^g$ is invertible in $M_n(J)^+$ and $[e^g] = 0$ in $K_1(J)$.

(c) Let $g, h \in M_n(J)$. We have that $[e^{g+h}] = [e^g] + [e^h]$ in $K_1(J)$.

(d) Let $g \in M_n(J)$. If $tr(g) \in J^2$, then $[e^g] = 0$ in $K_1(J)$.

Proof. Since $M_n(J)$ is isomorphic to $J$ and $(M_nJ)^2 = M_nJ^2$ we can restrict our reasoning, for a proof of (a),(b) and (c), to the case $n = 1$.

(a) If $a \in J^+$ is close to one, then the logarithmic series converges to an element in $J$. An easy calculation shows that the logarithm lies in $J^2$ if $a \in (J^2)^+$ (it can be written as a product of $a$ by an element in $J^+$). The assertion follows by standard arguments using compactness and the fact that the group of invertible elements in $J^+$ which maps to 1 $\in \mathbb{Z}$ is connected.

(b) The proof of this lemma follows the idea in [14], 4.1, using a scheme going back to [15]. As in the proof of the homotopy invariance theorem we consider certain invertible elements which we want to represent by commutators. Consider $M_2(J)$ as acting on $H \oplus H$ and choose an isometry between the second copy of $H$ in this direct sum and $\oplus_{n\in\mathbb{N}}H$. Under this identification, we define

$$X_1 = e^g \oplus e^{-g/2} \oplus e^{-g/2} \oplus e^{g/4} \oplus \ldots$$

and

$$X_2 = 1 \oplus e^{g/2} \oplus e^{g/2} \oplus e^{-g/4} \oplus \ldots$$

(again, with $e^{\pm g/2^n}$-term appearing $2^n$ times) can be considered as elements in $M_2(J)$, since $J$ is harmonic (compare to the proof of Theorem 1.2). Clearly $X_1X_2 = e^g \oplus 1$. In order to show that $[e^g] = 0$, it suffices to show $[e^g \oplus 1] = 0$ in $K_1(J)$. We want to show that $X_1$ and $X_2$ are products of commutators. This implies that $e^g \oplus 1$ is also a product of commutators and hence finishes the proof.

We first concentrate on $X_1$. As before we construct matrices

$$W_1 = e^{g/2} \oplus e^{-g/2} \oplus 1 \oplus e^{g/8} \ldots$$
and

\[ W_2 = e^{9/2} \oplus 1 \oplus e^{-9/2} \oplus e^{9/8} \ldots \]

which satisfy \( W_1 W_2 = X_1 \). There is an explicit isomorphism \( M_2(\mathcal{J})^+ \cong M_3(\mathcal{J})^+ \) (preserving the ideal \( \mathcal{J}^2 \)) such that the matrix \( W_1 \) is mapped to a matrix of the form \( \gamma \oplus \gamma^{-1} \oplus 1 \). The element \( h = 1 - \gamma \) is in \( \mathcal{J}^2 \) (since the exponential series was already convergent in \( (\mathcal{J}^2)^+ \)) and therefore decomposes into \( \alpha \beta \) with \( \alpha, \beta \in \mathcal{J} \). To see this assume that \( h = \sum_{i \leq n} a_i b_i \) with \( a_i \) and \( b_i \) in \( \mathcal{J} \). Then there are \( A \) and \( B \) in \( M_n \mathcal{J} \) such that \( h \oplus 0 = AB \). Identify then \( \mathcal{J} \) with \( M_n(\mathcal{J}) \) using \( \text{Ad} V \) for a suitable isometry \( V \).

By the proof of Vaserstein’s lemma (or by direct computation), we have that

\[
\gamma \oplus \gamma^{-1} \oplus 1 = \begin{bmatrix}
\begin{pmatrix}
\gamma & 0 & 0 \\
0 & 1 & 0 \\
-\gamma \beta & 0 & 1
\end{pmatrix} & \begin{pmatrix}
\gamma^{-1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} & \begin{pmatrix}
1 & 0 & 0 \\
0 & \gamma^{-1} & 0 \\
0 & 0 & \gamma \beta
\end{pmatrix}
\end{bmatrix}
\]

where \([\cdot, \cdot]\) denotes a multiplicative commutator. Thus, \( W_1 \) is a product of two multiplicative commutators in \( M_2(\mathcal{J})^+ \). A similar reasoning applies to \( W_2 \) and matrices occurring in a similar decomposition of \( X_2 \). Thus \( [e^g] = 0 \) in \( K_1(\mathcal{J}) \).

\( (c) \) Since the leading terms in the series expansion of \( e^{g+h}e^{-g}e^{-h} - 1 \) vanish, this element can be written as a sum of four terms where each term is a product of \( g^2, gh, hg, h^2 \) respectively, by an element in \( \mathcal{J}^+ \). Therefore \( e^{g+h}e^{-g}e^{-h} \) as well as its inverse lie in \( (\mathcal{J}^2)^+ \). By \( (a) \) \( e^{g+h}e^{-g}e^{-h} \) is a product of exponentials of elements in \( (\mathcal{J}^2)^+ \). Using \( (b) \), this implies the claim.

\( (d) \) A matrix with trace in \( \mathcal{J}^2 \) is a finite sum of

- off-diagonal matrices with one entry in \( \mathcal{J} \),
- matrices of the form \( a \otimes e_{ij} - a \otimes e_{jj} \) with \( a \in \mathcal{J} \) and
- a matrix \( a \otimes e_{11} \) with \( a \in \mathcal{J}^2 \).

The classes in \( K_1(\mathcal{J}) \) of their exponentials are zero, whence the claim by iterated application of \( (c) \).

\[ \square \]

We now proceed with the proof of the theorem.

**Proof.** Let \( A \) be an invertible element in \( M_n(\mathcal{J}^+) \) which maps to the identity in \( M_n(\mathbb{Z}) \) and with determinant zero. By \( (a) \) of the preceding lemma it is of the form \( e^{h_1} \cdots e^{h_k} \) and \( \det A = h_1 + \cdots + h_k + \mathcal{J}^2 = \mathcal{J}^2 \). The matrix \( e^{h_1} \oplus \cdots \oplus e^{h_n} \in M_{nk}(\mathcal{J})^+ \) has the same class in \( K_1(\mathcal{J}) \) and the trace of its logarithm is just a lift of its determinant \( \det A \) to \( \mathcal{J} \) and hence in \( \mathcal{J}^2 \). By \( (d) \) of the preceding lemma the class in \( K_1(\mathcal{J}) \) is zero. This shows injectivity of the determinant map. Surjectivity was obvious, hence the assertion. \[ \square \]

**Remark 7.0.1.** It is of course always true that \( K_1(\mathcal{J}) \) maps surjectively onto \( \mathcal{J}/[\mathcal{J}, \mathcal{J}] \), \( \mathcal{J} \) being a Banach ideal.
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