A hive model determination of multiplicity-free Schur function products and skew Schur functions

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Abstract

The hive model is a combinatorial device that may be used to determine Littlewood-Richardson coefficients and study their properties. It represents an alternative to the use of the Littlewood-Richardson rule. Here properties of hives are used to determine all possible multiplicity-free Schur function products and skew Schur function expansions. This confirms the results of Stembridge [11], Gutschwager [3] and Thomas and Yong [12], and sheds light on the combinatorial origin of the conditions for being multiplicity-free, as well as illustrating some of the key features and power of the hive model.

1 Introduction

Throughout this paper we will adopt the notation and terminology on Schur functions taken from the standard text by Macdonald [9]. The Schur functions $s_\lambda$, indexed by

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partitions \( \lambda \), form a \( \mathbb{Z} \)-basis of the ring of symmetric functions. This basis is orthonormal, and the corresponding bilinear form enables one to define skew Schur functions \( s_{\lambda/\mu} \), indexed by pairs of partitions \( \lambda \) and \( \mu \) with \( \mu \subseteq \lambda \). Each such skew Schur function \( s_{\lambda/\mu} \) can be expressed as a sum of Schur functions \( s_\nu \) with non-negative integer multiplicities given by the well-known Littlewood-Richardson coefficients, \( c_{\lambda \mu \nu} \). These same coefficients govern the decomposition of the product \( s_\mu s_\nu \) of two Schur functions as a sum of Schur functions \( s_\lambda \).

In [11], Stembridge classified the products of Schur functions that are multiplicity-free, that is those pairs of Schur functions for which every coefficient in the Schur function expansion of their product is 0 or 1, as in the following theorem:

**Theorem 1.1 (Stembridge [11])** The Schur function product \( s_\mu s_\nu \) is multiplicity-free if and only if one or more of the following is true:

- **P0** \( \mu \) or \( \nu \) is the zero partition 0;
- **P1** \( \mu \) or \( \nu \) is a one-line rectangle;
- **P2** \( \mu \) is a two-line rectangle and \( \nu \) is a fat hook (or vice versa);
- **P3** \( \mu \) is a rectangle and \( \nu \) is a near-rectangle (or vice versa);
- **P4** \( \mu \) and \( \nu \) are rectangles.

Here each partition \( \lambda \) is to be identified with the corresponding Young diagram \( F_\lambda \), and is said to be a rectangle if \( \lambda = (a^p) \) with \( a > 0 \) and \( p > 0 \), and a fat hook if \( \lambda = (a^p b^q) \) with \( a > b > 0 \) and \( p, q > 0 \). A rectangle \( (a^p) \) is a one-line rectangle if \( a = 1 \) or \( p = 1 \), and a two-line rectangle if \( a, p > 1 \) with \( a = 2 \) or \( p = 2 \). A fat hook \( (a^p b^q) \) is a near rectangle if any one or more of \( a - b, b, p \) or \( q \) is equal to 1.

Stembridge [11] also gave a corresponding theorem applicable to the case where the lengths of the partitions satisfy \( \ell(\mu), \ell(\nu) < n \) and the Schur function product is restricted to the ring, \( \Lambda_n \), of symmetric functions in a finite number of variables \( x_1, x_2, \ldots, x_n \). To do this he invoked, in the particular case \( m = \lambda_1 \), the use of the partition \( \lambda^* = (m - \lambda_n, m - \lambda_{n-1}, \ldots, m - \lambda_1) \), which may be said to be \( m^n \)-complementary to \( \lambda \).

More recently, Thomas and Yong [12] established a multiplicity-free result for the product of two Schubert classes \( \sigma_\mu \sigma_\nu \) in the cohomolgy ring \( H^*(Gr(m, \mathbb{C}^{m+n}), \mathbb{Z}) \) of the Grassmannian \( Gr(m, \mathbb{C}^{m+n}) \) of \( m \)-dimensional subspaces in \( \mathbb{C}^{m+n} \). The coefficients in this product of Schubert classes are just the usual Littlewood-Richardson coefficients, \( c^\lambda_{\mu \nu} \), but this time restricted to the case \( \lambda \subseteq m^n \). This allows, Thomas and Yong’s result to be recast
in terms of skew Schur functions. When this is done it coincides with the multiplicity-free result for skew Schur functions derived independently by Gutschwager [3].

Here it is convenient to state their common result just for basic skew Schur functions, that is for those cases $s_{\lambda/\mu}$ where the skew Young diagram $F^{\lambda/\mu}$ has neither empty rows nor empty columns. It need not be connected. As will be seen, every skew Schur function is equal, in a rather trivial way, to some basic skew Schur function. In stating the theorem it is also convenient to follow Thomas and Yong in letting the $m^n$-shortness of a partition $\lambda \subseteq m^n$ be the length of the shortest straight line segment of the path of length $m+n$ from the southwest to northeast corner of $F^{m^n}$ that separates $F^{\lambda}$ from the $\pi$-rotation of $F^{\lambda^*}$. With this definition, the result, jointly attributable both to Gutschwager and to Thomas and Yong, takes the form:

**Theorem 1.2 (Gutschwager [3], Thomas and Yong [12])** The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free if and only if one or more of the following is true:

- **R0** $\mu$ or $\lambda^*$ is the zero partition $0$;
- **R1** $\mu$ or $\lambda^*$ is a rectangle of $m^n$-shortness 1;
- **R2** $\mu$ is a rectangle of $m^n$-shortness 2 and $\lambda^*$ is a fat hook (or vice versa);
- **R3** $\mu$ is a rectangle and $\lambda^*$ is a fat hook of $m^n$-shortness 1 (or vice versa);
- **R4** $\mu$ and $\lambda^*$ are rectangles.

where $\lambda^*$ is the $m^n$-complement of $\lambda$ with $m = \lambda_1$ and $n = \lambda'_1$.

In deriving this result both Gutschwager [3] and Thomas and Yong [12] used the traditional model of Littlewood-Richardson coefficients [7,8]:

**Theorem 1.3 (Littlewood and Richardson [8])** The Littlewood-Richardson coefficient $c_{\mu\nu}^{\lambda}$ is equal to the number of semistandard Young tableaux of shape $\lambda/\mu$ and content $\nu$ whose reverse reading word is a lattice permutation.

Here, our aim is to rederive both Theorems [1.1] and [1.2] by means of a different combinatorial model, namely the hive model [2,5] for these coefficients:

**Theorem 1.4 (Knutson and Tao [5])** The Littlewood-Richardson coefficient $c_{\mu\nu}^{\lambda}$ is equal to the number of distinct integer LR-hives with boundary edge labels specified by the partitions $\lambda$, $\mu$ and $\nu$. 

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The hive model introduced in [5] arose as a reformulation of a convex polytope model [1]. It was described in more detail in [2], with an appendix providing a rather simple bijection between the tableaux of Theorem 1.3 and the hives of Theorem 1.4. For more information about the hive model, see also [4, 6].

There are a number of advantages to the hive model approach, including the fact that it allows a direct proof that all the cases enumerated in both Theorem 1.1 and Theorem 1.2 are indeed multiplicity-free. It also lends itself well to the uniform statement of Theorem 1.2, simultaneously covering both connected and disconnected cases. What is avoided in the hive model proof that all the indicated multiplicity-free cases are indeed multiplicity-free is any recourse to the non-trivial order filter, introduced by Stembridge and generalised by Gutschwager, that underlies two families of inequalities of the form $c_\lambda^{\mu
u} \geq c_\rho^{\sigma\tau}$ that are also heavily used by Thomas and Yong in the form of what they call Stembridge demolitions. Although we cannot avoid the use of such inequalities in dealing with all possible non-multiplicity-free cases, the hive model does allow us to be completely explicit about the route from the most general cases to those for which a multiplicity of at least two occurs. In doing so the hive model offers some insight into the origin of the breakdown of multiplicity-freeness for both products of Schur functions and expansions of skew Schur functions. This lies in the fact that within the appropriate LR-hives there always exists an elementary hexagon whose interior edge labels are not fixed. The precise nature of the somewhat numerous conditions for the breakdown of multiplicity-freeness can then be exposed in each case through consideration of a single hive diagram.

In the next section, we make some necessary definitions regarding partitions, skew Schur functions and Littlewood-Richardson coefficients. In Section 3 we define integer $n$-hives and the LR-hives whose enumeration for fixed boundary labels provides a model for evaluating Littlewood-Richardson coefficients. A sequence of lemmas regarding LR-hives and subhives are derived in Section 4. These are used in Sections 5 and 7, respectively, to prove that all the Schur function products and skew Schur functions listed in Theorems 1.1 and 1.2 are indeed multiplicity-free. The question of completeness of these lists is tackled in Sections 6 and 8, thereby completing the proof of both theorems. Some final remarks, including a corollary regarding multiplicity-free products of skew Schur functions, are offered in Section 9.

## 2 Skew Schur functions and Littlewood-Richardson coefficients

Let $n$ be a fixed positive integer, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of weight $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and length $\ell(\lambda) \leq n$. The parts of $\lambda$ are non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ with $\lambda_i > 0$ for all $i \leq \ell(\lambda)$ and $\lambda_i = 0$ for all $i > \ell(\lambda)$. Such
a partition $\lambda$ specifies a Young diagram $F^\lambda$ consisting of $|\lambda|$ boxes whose row lengths are the parts $\lambda_i$ of $\lambda$ and whose column lengths are the parts $\lambda'_j$ of the conjugate partition $\lambda'$. Schematically, we have:

$$F^\lambda = \lambda_1 \lambda_2 \lambda_3 \lambda_4 = \lambda'_1 \lambda'_2 \lambda'_3 \lambda'_4.$$  

It is sometimes convenient to write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ in the form $\lambda = (a^p, b^q, \ldots)$ if $\lambda_i = a$ for $1 \leq i \leq p$, $\lambda_i = b$ for $p < i \leq p + q$, etc., with $a > b > \cdots > 0$ and $p, q, \ldots > 0$. In addition, for any pair of partitions $\lambda$ and $\mu$ we define $\lambda + \mu$ to be the partition obtained by adding corresponding parts of $\lambda$ and $\mu$, and $\lambda \cup \mu$ is the partition obtained by arranging all the parts of $\lambda$ and $\mu$ in weakly decreasing order.

We write $\mu \subseteq \lambda$ if all the boxes of $F^\mu$ are contained in $F^\lambda$, that is to say $\mu_i \leq \lambda_i$ for all $i$, or equivalently, $\mu'_j \leq \lambda'_j$ for all $j$. In such a case the corresponding skew diagram $F^{\lambda/\mu}$ is the diagram obtained by deleting from $F^\lambda$ all the boxes of $F^\mu$. Schematically, we have:

$$F^\lambda = \mu_1 \mu_2 \mu_3 \mu_4 = F^{\lambda/\mu} = \lambda_1 - \mu_1 \lambda_2 - \mu_2 \lambda_3 - \mu_3 \lambda_4 - \mu_4.$$  

Just as to each partition $\lambda$ there corresponds a Schur function $s_\lambda$, so to each pair of partitions $\lambda$ and $\mu$ with $\mu \subseteq \lambda$ there corresponds a skew Schur function $s_{\lambda/\mu}$ [7,9]. This may be defined by noting first that there exists a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the ring of symmetric functions $\Lambda$ such that $\langle s_\mu, s_\nu \rangle = \delta_{\mu\nu}$, and then defining $s_{\lambda/\mu}$ by the relations [9]

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle,$$  \hspace{1cm} (2.1)

for all partitions $\nu$.

Since the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ arise as the multiplicities in the expansion of the Schur function product

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda,$$  \hspace{1cm} (2.2)  

it follows from (2.1) that they must also arise as the multiplicities in the skew Schur function expansion

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$  \hspace{1cm} (2.3)

The Littlewood-Richardson rule implies that $c^\lambda_{\mu\nu}$ can only be non-zero if

$$|\lambda| = |\mu| + |\nu| \quad \text{and} \quad \ell(\mu), \ell(\nu) \leq \ell(\lambda) \leq \ell(\mu) + \ell(\nu).$$  \hspace{1cm} (2.4)
Although it is by no means obvious from the Littlewood-Richardson rule, the Littlewood-
Richardson coefficients satisfy a number of symmetry properties, including:

\[ c_{\lambda \mu \nu} = c_{\lambda \nu \mu} \quad \text{and} \quad c_{\lambda \mu' \nu'} = c_{\lambda \mu \nu}. \] \hspace{1cm} (2.5)

Moreover, for all partitions \( \lambda, \mu \) and \( \nu \) and all non-negative integers \( a, b \) and \( c \) with \( a = b + c \), we have

\[ c_{\lambda+\langle 1^a \rangle} \geq c_{\mu+\langle 1^b \rangle, \nu+\langle 1^c \rangle} \quad \text{and} \quad c_{\lambda \cup \langle a \rangle} \geq c_{\mu \cup \langle b \rangle, \nu \cup \langle c \rangle}. \] \hspace{1cm} (2.6)

These inequalities, which are related by conjugacy, have been derived in [3], as a generalisation of the \( c = 0 \) case given in [11]. Although a direct proof of both inequalities may be based on the hive model, we do not present the proof here.

It is useful to note [10] that

\[ s_\lambda = s_{\lambda*} \quad \text{and} \quad s_{\lambda/\mu} = s_{(\lambda/\mu)*}, \] \hspace{1cm} (2.7)

where \( F_{\lambda*} \) and \( F_{(\lambda/\mu)*} \) are obtained by rotating \( F_\lambda \) and \( F_{\lambda/\mu} \), respectively, through \( \pi \) radians.

**Example 2.1** If \( \lambda = (432) \) and \( \mu = (2) \), then the \( \pi \)-rotations of \( F_\lambda \) and \( F_{\lambda/\mu} \) take the form:

\[ F_{\lambda} \quad \Rightarrow \quad F_{\lambda*} \quad \text{and} \quad F_{\lambda/\mu} \quad \Rightarrow \quad F_{(\lambda/\mu)*}. \]

so that we have \( s_{432} = s_{444/21} \) and \( s_{432/2} = s_{442/21} \).

As far as \( m^n \)-complements are concerned

\[ s_{m^n/\lambda} = \begin{cases} s_{\lambda*} & \text{if } \lambda \subseteq m^n; \\ 0 & \text{otherwise}, \end{cases} \] \hspace{1cm} (2.8)

where \( \lambda_k^* = m - \lambda_{n-k+1} \) for \( k = 1, 2, \ldots, n \).

An important consequence of this rather trivial observation is that for \( \mu \subseteq \lambda \subseteq m^n \) we have

\[ (s_{\lambda/\mu}, s_\nu) = (s_\lambda, s_\mu s_\nu) = (s_{m^n/\lambda^*}, s_\mu s_\nu) = (s_{m^n/\nu}, s_{\lambda^*} s_\mu) = (s_{\nu^*}, s_{\lambda^*} s_\mu) \] \hspace{1cm} (2.9)

with the result non-zero only if \( \nu \subseteq m^n \). It is this identity which enables us to conclude that the skew Schur function \( s_{\lambda/\mu} \) is multiplicity-free if and only if the product of Schubert classes \( \sigma_{\lambda^*} \sigma_\mu \) is multiplicity-free.
**Example 2.2** By way of illustration, if \( m = 9 \), \( n = 5 \), \( \lambda = (99666) \) and \( \mu = (552) \) then \( \lambda^* = (333) \) and \( F_{\lambda/\mu} \) and its \( m^n \)-complement take the form:

\[
F_{\lambda/\mu} = \begin{array}{cccccccc}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]

\[
F_{\mu} \cup F_{\lambda^{*}} = \begin{array}{cccccccc}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]

In this example it can be seen that the sequence of straight line segments of the two paths from the southwest to northeast corners of the \( m^n = 9^5 \) rectangle that border the inner and outer boundaries of \( F_{\lambda/\mu} \) are given by \( s_{in} = (2, 2, 1, 3, 2, 4) \) and \( s_{out} = (6, 3, 3, 2) \), respectively, where the terminology of [3] has been adopted. The shortness of \( \mu \) and \( \lambda^* \) as defined in [12] are just the smallest components, 1 and 2 respectively, of these two sequences.

A further useful fact about skew Schur functions is that

\[
s_{\lambda/\mu} = s_{\hat{\lambda}/\hat{\mu}},
\]

(2.10)

where \( F_{\hat{\lambda}/\hat{\mu}} \) is the skew Young diagram obtained from \( F_{\lambda/\mu} \) by deleting any empty rows, that is those for which \( \lambda_i = \mu_i \), and any empty columns, that is those for which \( \lambda'_j = \mu'_j \). The skew Schur function \( s_{\hat{\lambda}/\hat{\mu}} \) is said to be basic. This identity therefore allows each skew Schur function to be expressed as a basic skew Schur function. It should be noted that if \( s_{\lambda/\mu} \) is itself basic, then \( \mu_i < \lambda_i \) and \( \mu_i \leq \lambda_{i+1} \) for \( i = 1, 2, \ldots, \ell(\lambda) - 1 \), with \( \ell(\mu) < \ell(\lambda) \). If we just have \( \mu_i < \lambda_i \) for all \( i = 1, 2, \ldots, \ell(\lambda) \), then we say that \( s_{\lambda/\mu} \) is row-basic.

**Example 2.3** In the case \( \lambda = (985333) \) and \( \mu = (755321) \) the construction of \( F_{\hat{\lambda}/\hat{\mu}} \) from \( F_{\lambda/\mu} \) is illustrated by:

\[
F_{\lambda/\mu} = \begin{array}{cccccccc}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]

\[
F_{\hat{\lambda}/\hat{\mu}} = \begin{array}{cccccccc}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]

Hence \( \hat{\lambda} = (6522) \) and \( \hat{\mu} = (421) \), so that \( s_{985333/755321} = s_{6522/421} \), with \( s_{6522/421} \) basic, but in this instance not connected.

If \( F_{\lambda/\mu} \) is not connected, but consists of two components \( F^\theta \) and \( F^\phi \) that have no edge in common, and may themselves be either Young diagrams or skew Young diagrams, then

\[
s_{\lambda/\mu} = s_{\theta} s_{\phi}.
\]

(2.11)

**Example 2.4** In the case \( \lambda/\mu = 65221/421 \) we have the disconnected diagram

\[
F_{\lambda/\mu} = \begin{array}{cccccccc}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]

\[
F^\theta = \begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\]

\[
F^\phi = \begin{array}{cccc}
\ast & \ast \\
\end{array}
\]

from which it can be seen that \( \theta = 221/1 \) and \( \phi = 43/2 \). Hence \( s_{65221/421} = s_{221/1} s_{43/2} \).
3 The hive model

An $n$-hive is an array of numbers $a_{ij}$, with $0 \leq i, j, i+j \leq n$, placed at the vertices of an equilateral triangular graph. Typically, for $n = 4$ their arrangement is as shown below. Such an $n$-hive is said to be an integer hive if all of its entries are non-negative integers.

Neighbouring entries define two distinct types of triangles and neighbouring triangles define three distinct types of rhombus:

In each rhombus, with the labelling as shown above, the hive condition takes the form:

$$b + c \geq a + d. \quad (3.12)$$

In what follows we make use of edge labels more often than vertex labels. Each edge in the hive is labelled by means of the difference, $\epsilon = q - p$, between the labels, $p$ and $q$, of the two vertices connected by this edge, with $q$ always to the right of $p$. Thus in both triangles $T1$ and $T2$, we have $\alpha = b - a$, $\beta = c - b$ and $\gamma = c - a$, so that in each case

$$\alpha + \beta = \gamma. \quad (3.13)$$

Similarly, in the case of all three of the above rhombi, $R1$, $R2$ and $R3$, we have

$$\alpha + \delta = \beta + \gamma, \quad (3.14)$$

and the hive conditions take the form:

$$\alpha \geq \gamma \quad \text{and} \quad \beta \geq \delta. \quad (3.15)$$
where, of course, either one of these conditions implies the other, and
\[ \alpha = \gamma \text{ if and only if } \beta = \delta. \]  
\hspace{2cm} (3.16)

We are now in a position to define LR-hives:

**Definition 3.1** Let \( n \) be a positive integer, and let \( \lambda, \mu \) and \( \nu \) be any partitions for which \( \ell(\lambda), \ell(\mu), \ell(\nu) \leq n \) and \( |\mu| + |\nu| = |\lambda| \). An LR-hive is any integer \( n \)-hive with its vertex labels satisfying the hive conditions (3.12) and its boundary vertex labels given by \( a_{00} = 0 \), \( a_{0,i} = \nu_1 + \cdots + \nu_i \), \( a_{j,n-j} = |\nu| + \mu_1 + \cdots + \mu_j \) and \( a_{k,0} = \lambda_1 + \cdots + \lambda_k \) for \( i, j, k = 1, 2, \ldots, n \).

Equivalently, its edge labels satisfy (3.13), (3.14) and (3.15) for all constituent triangles of type \( T_1 \) and \( T_2 \), and rhombi of type \( R_1 \), \( R_2 \) and \( R_3 \), and its boundary edge labels are given by \( \lambda_i, \mu_j \) and \( \nu_k \) for \( i, j, k = 1, 2, \ldots, n \). Schematically, we have:

The labelling has been given first in terms of vertex labels and then in terms of edge labels. The right hand edge labelling scheme is the one that we will adopt for all subsequent LR-hives, and it is precisely this type of hive whose enumeration determines the Littlewood-Richardson coefficient \( c_{\mu \nu}^\lambda \) as in Theorem 1.4.

### 4 Some properties of LR-hives and subhives

All edge labels in any integer hive are, of course, integers. In the case of an LR-hive, these integer labels are necessarily non-negative and along any straight line parallel to a boundary they weakly decrease in one particular direction. This is clearly true on the boundary, since the labels are all parts of partitions. It is also true of all interior edge labels as may be seen from the following pair of diagrams of an arbitrary LR-hive:
In the first of these diagrams the hive conditions (3.15) applied to the rhombi of type $R_1$, $R_1$, $R_2$ that constitute the corridors between the interior edges with labels $x$, $y$, $z$ and the boundary edges with labels $a$, $b$, $c$, respectively, imply that $x \geq a \geq 0$, $y \geq b \geq 0$, $z \geq c \geq 0$. Thus all interior edge labels are non-negative, as required. In the second diagram, the same hive conditions applied to rhombi of type $R_1$ and $R_2$ imply that $a \geq x$ and $x \geq b$, so that $a \geq b$. This weakly decreasing condition may readily be extended to cover all edge labels on the straight line containing $a$ and $b$. Analogous results apply to edge labels along any straight line parallel to one or other of the three hive boundaries.

We now establish some properties of subhives of any given LR-hive. These properties will play an important role in the proof of our two main theorems.

**Lemma 4.1** Each of the following diagrams represents a subhive of an LR-hive. The subhive may be oriented in any manner within the full hive. If each edge signified by a solid line is assigned some fixed label, then these labels are sufficient to determine the labels of all the remaining edges signified by dashed lines.

![Diagram](image)

**Proof:** In each case the repeated use of the triangle condition (3.13) is sufficient to fix all the unassigned edge labels. Case (i) is covered by the fact that (3.13) fixes any one edge label of an elementary triangle in terms of the other two. Applying this to each of the three elementary subtriangles of (ii) then fixes the labels of the three interior edges. In the cases (iii) and (iv) one can successively determine the labels on all the dashed line edges by the application of (3.13) to each elementary triangle taken in turn from left to right along each of these two diagrams.

**Lemma 4.2** Each of the following diagrams represents a subhive of an LR-hive, which for illustrative purposes has been given a specific orientation. If each edge signified by a
solid line is assigned the labels \(a, b, c, \ldots\) as indicated, then these labels are sufficient to fix the labels of all the remaining edges signified by dashed lines, including those dashed line edge labels that have been indicated.

![Diagram](image)

**Proof:** Here it is helpful to consider the following subhive:

![Subhive](image)

Within this diagram the repeated application of the rhombus conditions (3.15) to the sequence of rhombi of type \(R3\) between \(a\) and \(x\), and then a sequence of type \(R1\) between \(x\) and \(b\), shows that \(a \geq x \geq b\). It follows that if \(a = b\) then \(x = a\). Applying this to the isosceles trapezium and the equilateral triangle shows that all the horizontal edges of both diagrams must have label \(a\). Then each of the subhives is seen to consist of a sequence of thin strip subhives of the case (iv), dealt with in Lemma 4.1. It follows that all the edge labels are fixed not only in each thin strip, but also in both the isosceles trapezium and the equilateral triangle. With the orientation as shown, the triangle condition (3.13) and equal edge rhombus condition (3.16) imply that if the edge labels on the left hand boundaries are given by \(b, c, \ldots\) then those on the right hand boundary must be \(a - b, a - c, \ldots\). For different orientations \(a - b, a - c, \ldots\) must be replaced by either \(a + b, a + c, \ldots\) or \(b - a, c - a, \ldots\).

**Lemma 4.3** In each of the following diagrams the elementary hexagon represents a subhive of an LR-hive. Each edge signified by a solid line within the LR-hive is assigned some fixed label. Then these labels are sufficient to determine the labels of all the remaining edges signified by dashed lines if either (i) any interior edge label of the hexagon is fixed, or (ii) any boundary edge label of the hexagon is 0, or (iii) any two neighbouring edge labels are equal on any of the six lines constituting the two triangles bounding the hexagon.
Proof: The three diagrams exemplify the three possibilities referred to in the lemma. In the first of these we can apply case (i) of Lemma 4.1 to each of the six elementary triangles constituting the hexagon. These may be taken in turn, say anticlockwise beginning with one involving the fixed edge label, signified by $a$ in the illustrative example. In case (ii) the hive condition 3.15 gives $b \geq z$, so that for $b = 0$ we have $z = 0$, since all edge labels of a LR-hive are non-negative. Having fixed one interior edge label of the hexagon, the remainder follow as in case (i). In case (iii) the application of the hive conditions (3.15) gives $c \geq x \geq d \geq y \geq e$, so that if $c = d$ we have $x = d$, and if $d = e$ we have $y = d$. In either case we have fixed one interior edge label of the hexagon and the remainder are then fixed as in case (i). This completes the proof, since all other examples of these three cases can be treated in exactly the same way.

Although this lemma does not exhaust the list of conditions that fix all interior edge labels of a hexagonal subhive of an LR-hive, avoiding these conditions turns out to be a crucial first step in the construction of examples for which the interior edge labels of a hexagon are not fixed. The existence of such a situation will then be shown to characterise those Schur function products and skew Schur functions that are not multiplicity-free.

5 Multiplicity-free products

In order to provide a hive-based proof of Stembridge’s Theorem 1.1 we first prove:

Lemma 5.1 All the Schur function products $s_\mu s_\nu$ listed under cases P0–P4 of Theorem 1.1 are multiplicity-free.

Proof: In order that all terms in the product $s_\mu s_\nu$ are accounted for, we choose $n = \ell(\mu) + \ell(\nu)$. It then suffices to show that for any fixed $\lambda$ there exists at most one LR-hive with boundary edge labels specified by the parts of $\lambda$, $\mu$ and $\nu$. This is accomplished by first parametrising the pair $\mu$ and $\nu$, and then showing that for each fixed, but unknown $\lambda$, the hive conditions (3.13)–(3.15) serve to fix all the interior edge labels. Without the necessity of testing all possible hive conditions, this implies that for each $\lambda$ there exists at most one LR-hive with the required boundary edge labels, and hence that $s_\mu s_\nu$ is multiplicity-free.
We consider the five cases in turn.

**P0.** This case is trivial since $s_\mu s_0 = s_\mu$ and $s_0 s_\nu = s_\nu$ for all $\mu$ and $\nu$, respectively.

**P1.** Thanks to the symmetry properties (2.5), we need only consider the case for which $\mu = (a)$ with $a > 0$ and $\nu$ fixed but arbitrary. Then for any $\lambda$, the corresponding LR-hive takes the form in $M_1$. Applying Lemma 4.2 to the triangle $BCD$ fixes all its edge labels, including those on $BD$. Case (iv) of Lemma 4.1 then serves to fix all the edge labels of $ABDE$. Thus all edge labels of the complete LR-hive are fixed, so that each product of type $P1$ is multiplicity-free.

**P2.** Thanks once again to the symmetry properties (2.5), we need only consider the case for which $\mu = (a^2)$ with $a > 0$ and $\nu = (b^p c^q)$ with $b > c > 0$, $p, q > 0$ and $n = p + q + 2$. If $p, q > 2$ then, for any $\lambda$ with $\ell(\lambda) \leq n$, the corresponding LR-hives take the form in $M_2$. Lemma 4.2 implies that all the edge labels of $BCD$, $ABJI$ and $DEF$ are fixed. Since those on $DF$ must both be $a$, Lemma 4.2 implies that all the edge labels of $DFG$ are also fixed. This is then sufficient to determine all edge labels of $FIKG$, thanks again to Lemma 4.2. This only leaves the triangle $IJK$ of side length 2 to be considered. Its boundary edges labels are all known, so that, by virtue of case (ii) of Lemma 4.1 its interior edge labels are also fixed. This fixes all edge labels in the complete LR-hive, and the corresponding product is multiplicity-free.

If either $q = 2$ or $p = 2$, then the previous diagram must be modified as shown below. The argument then proceeds exactly as before with the trapeziums $ABJI$ and $FIKG$ replaced by the triangles $AJI$ and $FIK$, as appropriate.

On the other hand if either $p = 1$ or $q = 1$, then $\nu$ is a near rectangle, and this is a situation covered by case **P3.**

**P3.** Again thanks to the symmetry properties (2.5) it is sufficient to consider the two cases (i) $\mu = (a^p b)$ and $\nu = (c^q)$ and (ii) $\mu = (a b^p)$ and $\nu = (c^q)$ with $a > b > 0$, $c, p, q > 0$.
and \( n = p + q + 1 \). In each subcase (i) and (ii) there are three possibilities, depending on the relative size of \( p \) and \( q \). For any \( \lambda \), with \( \ell(\lambda) \leq n \), the LR-hives in the subcase (i) may take one or other of the following forms:

\[
\begin{array}{c}
A & J & B & C \\
E & & & D \\
F & & & I \\
K & c & c & a \\
\end{array}
\]

\[
\begin{array}{c}
A & J & B & C \\
E & a & a & I \\
F & b & b & b \\
D & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c}
A & K & J & B & C \\
E & a & a & a & I \\
F & b & b & b & b \\
D & 0 & 0 & 0 & 0 \\
\end{array}
\]

Considering the first diagram, successive applications of Lemma 4.2 fix all the edge labels of the triangles \( D E F \) and \( B C D \), as well as those of the trapezium \( A J K F \). Since the edge labels of \( F I \) are all \( a \), all the edge labels of \( F I K \) are then fixed by virtue of Lemma 4.2. This means that the boundary edge labels of the thin strip \( B D I J \) are all known, so that thanks to case (iii) of Lemma 4.1 all the remaining interior edge labels are also fixed. This completes the edge labelling of the complete hive. A similar argument applies to the other two diagrams.

Similarly, for the subcase (ii), we have the following types of LR-hive, and the argument goes through precisely as before.

\[
\begin{array}{c}
A & J & B & C \\
E & a & a & I \\
F & b & b & b \\
D & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c}
A & J & B & C \\
E & b & b & b \\
F & a & a & a \\
D & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c}
A & K & J & B & C \\
E & a & a & a & I \\
F & b & b & b & b \\
D & 0 & 0 & 0 & 0 \\
\end{array}
\]

It follows that the case \( P3 \) is also multiplicity-free.

**P4.** For this case, let \( \mu = (a^p) \) and \( \nu = (b^q) \) with \( a, b, p, q > 0 \) and \( n = p + q \). There are three subcases corresponding to \( p < q \), \( p = q \) and \( p > q \). For each of these, for any \( \lambda \) with \( \ell(\lambda) \leq n \), the corresponding LR-hives take the form:

\[
\begin{array}{c}
A & J & B & C \\
E & a & a & a \\
F & b & b & b \\
G & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c}
A & J & B & C \\
E & b & b & b \\
F & a & a & a \\
G & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c}
A & K & J & B & C \\
E & a & a & a & a \\
F & b & b & b & b \\
G & 0 & 0 & 0 & 0 \\
\end{array}
\]

In the first of these LR-hives, thanks to Lemma 4.2 all the edge labels of the triangles \( D E F \) and \( A B F \), as well as those of the trapezium \( B C D G \), are fixed. Since the edge labels on \( F D \) are all \( a \), Lemma 4.2 fixes all the edge labels of \( D F G \), thereby completing the edge labelling of the complete hive. Thus at most one LR-hive exists. A similar argument applies to the other two diagrams, and this case **P4** is also multiplicity-free.

This completes the proof of Lemma 5.1.
6 Completeness of the list in Stembridge’s theorem

To complete the proof of Stembridge’s Theorem 1.1 it is necessary to show that all cases other than those of $P_0$–$P_4$ are not multiplicity-free. This can be done by following Stembridge’s argument based on the use of just the second part of (2.6). In the context of the hive model, we do this by first considering three further cases, for which we shall show that there exists at least one partition $\lambda$ such that $c_{\lambda}^{\mu \nu} > 1$.

Lemma 6.1 The product $s_\mu s_\nu$ is not multiplicity-free in each of the following cases:

- **Q1** $\mu = (ab)$ and $\nu = (cd)$ with $a > b > 0$, $c > d > 0$;
- **Q2** $\mu = (abc)$ and $\nu = (d^2)$ with $a > b > c > 0$ and $d > 1$;
- **Q3** $\mu = (a^2b^2)$ and $\nu = (c^3)$ with $a > b + 1$, $b > 1$ and $c > 2$.

Proof: For $\lambda = (xyz)$, $(wxyz)$ and $(uvxyzw)$ the corresponding hives take the form shown below, with $p = a + d - w$ in $H_2$, and $p = a + c - v$, $q = a + c - u$ and $r = a + b + c - u$ in $H_3$:

In each case the solid lines divide the hive into portions for which the edge labels are determined, including all the dashed line interior edges. In each case it will be observed that we are left with a hexagon on which the six boundary edge labels are necessarily fixed from a knowledge of $\lambda$, $\mu$ and $\nu$.

Now, for each given $\mu$ and $\nu$ we will identify one particular partition $\lambda$ for which there exists exactly two distinct labellings of the interior edges of the hexagon that satisfy all the hive conditions. It will then follow that $c_{\lambda}^{\mu \nu} = 2$, so that $s_\mu s_\nu$ is not multiplicity-free.

We consider each case in turn.

**Q1.** If we take $\lambda = (a + c - 1, b + d, 1)$, then with the stated conditions, there are exactly two LR-hives $H_1$ corresponding to this given $\lambda$, as the following figures show:
Thus $s_{ab}s_{cd}$ is not multiplicity-free.

**Q2.** For $\lambda = (a+d-1, b+d-1, c+1, 1)$, with the stated conditions, we can complete the labelling of the interior edges of two LR 3-subhives of $H2$, as illustrated in the following figures:

The edge labelings of these pairs of LR 3-hives serve to complete the interior edge labelling of the corresponding pairs of LR 4-hives in which they are embedded. The existence of two LR-hives corresponding to the given $\lambda$ shows that $s_{ab}s_{cd}$ is not multiplicity-free.

**Q3.** For $\lambda = (a + c - 1, a + c - 2, b + c - 1, b + 1, 2, 1)$, with the stated conditions, we can complete the labelling of the interior edges of two LR 3-subhives of $H3$ as illustrated in the following figures:

The edge labelings of these pairs of LR 3-hives serve to complete the interior edge labelling of the corresponding pairs of LR 6-hives in which they are embedded. The existence of two LR-hives corresponding to the given $\lambda$ shows that $s_{a^2b^2}s_{cd}$ is not multiplicity-free.

This completes the proof of Lemma 6.1.

**Note** It should be pointed out that the conditions on $a, b, c, d$ as stated in Lemma 6.1 arise naturally. In $H1$, to avoid being multiplicity-free, Lemma 4.3 implies that $a > b > 0$ and $c > d > 0$, as stated for case Q1. In $H2$, Lemma 4.3 implies that we require $a > b$, $b > c > 0$ and $d > p > 0$ so that $a > b > c > 0$ and $d > 1$ as stated for case Q2. In
Proof of Theorem \ref{thm:main}. Suppose $Q3$ gives $a > r > b$, $b > b - w > 0$ and $c > p > q$. Hence $a > b + 1$, $b > 1$ and $w > 0$. Then, in order to avoid fixing an interior edge of the hexagon by means of the hive condition \eqref{eq:three} we must also have $p > w$, so that $c > p > w > 0$. Hence $c > 2$, as required to complete the conditions listed for case Q3.

In the following, we select the parts of $\sigma$ and $\tau$ from those of $\mu$ and $\nu$, respectively, so that $\mu = \sigma \cup \zeta$ and $\nu = \tau \cup \xi$ for some $\zeta$ and $\xi$. If the choice is made in such a way that $s_\mu s_\nu$ is not multiplicity-free, then there exists at least one $\rho$ such that $c^\rho_{\sigma\tau} \geq 2$. Now let $\lambda = \rho \cup \eta$ where $\eta$ is formed by pairing up the parts of $\zeta$ and $\xi$ in any convenient way so that each $\eta_k = \zeta_i + \xi_j$ for some $i$ and $j$. It then follows from the repeated application of the second part of \eqref{eq:four} with $a = \eta_k$, $b = \zeta_i$ and $c = \xi_j$ that $c^\lambda_{\mu\nu} \geq c^\rho_{\sigma\tau} \geq 2$. Thus $s_\mu s_\nu$ is not multiplicity-free.

The cases $s = 0$ or $t = 0$ are covered by case P0, and are multiplicity-free. If $s \geq 2$ and $t \geq 2$ we select $\{a, b\} \subseteq \{a_1, a_2, \ldots, a_s\}$ and $\{c, d\} \subseteq \{b_1, b_2, \ldots, b_t\}$ in such a way that $\sigma = (ab)$ and $\tau = (cd)$ are a pair of partitions of the type covered by case Q1.

If $s \geq 3$ the case $t \geq 2$ has already been dealt with. We can therefore take $t = 1$ so that $\nu = (b^q)$. If $q = 1$ or $b = 1$ then $s_\mu s_\nu$ is multiplicity-free since the situation is covered by case P1, which were shown to be multiplicity-free before. For $q > 1$ and $b > 1$, it is possible to select from the distinct parts of $\mu$ and $\nu$ those parts that constitute partitions $\sigma = (abc)$ and $\tau = (d^2)$ appropriate to the non-multiplicity-free case Q2.

If $s = 2$ and $t = 1$, suppose $\mu = (a^p b^q)$ and $\nu = (c^r)$. If $c = 1$ or $r = 1$, then the situation is covered by case P1; if $c = 2$ or $r = 2$, then the situation is covered by case P2; if $a = b + 1$ or $b = 1$ or $p = 1$ or $q = 1$, then the situation is covered by case P3. For all these cases $s_\mu s_\nu$ is multiplicity-free. Thus we consider the case where $a > b + 1$, $b > 1$, $p, q > 1$ and $c, r > 2$. In this case we can always select $\sigma = (a^2 b^2)$ and $\tau = (c^3)$, and this is covered by the non-multiplicity-free case Q3.

Any case with $s = 1$ and $t \geq 2$ is related by the first symmetry condition of \eqref{eq:two} to a case with $t = 1$ and $s \geq 2$ that has already been dealt with, so only the case $s = 1$ and $t = 1$ is left. This case appears in the list under P4, and is multiplicity-free.

This completes the proof of Theorem \ref{thm:main}.
7 Multiplicity-free skew Schur functions

In this section we prove:

**Lemma 7.1** All the basic skew Schur functions $s_{\lambda/\mu}$ listed under cases $R0$–$R4$ of Theorem 1.2 are multiplicity-free.

**Proof:** For each case the strategy is to parametrise the pair $\lambda$ and $\mu$ and then, for each fixed but unknown $\nu$, to use the hive conditions (3.13)–(3.15) to show that all the interior edge labels are fixed. Without the necessity of testing all possible hive conditions, this implies that for each $\nu$ there exists at most one LR-hive with the required boundary edge labels, and hence that $s_{\lambda/\mu}$ is multiplicity-free.

Since $s_{\lambda/\mu}$ is basic, the required LR-hives are integer $n$-hives, with $n = \ell(\lambda)$ and all edge labels positive along the boundary specified by $\lambda$ and at least one edge label 0 along the boundary specified by $\mu$. We consider the four cases in turn.

**R0.** There are two subcases, $S0$: $\mu = 0$, and $S0^\pi$: $\lambda = (m^n)$ for some positive integer $m$, as illustrated by:

Lemma 4.2 implies immediately that in each case there exists a single LR-hive. In the case of $S0$ the equal edge labels 0 suffice to show that $\nu = \lambda$, while in the case of $S0^\pi$ the equal edge labels $m$ fix the parts of $\nu$ to be $\nu_k = m - \mu_{n-k+1}$ for $k = 1, 2, \ldots, n$.

**R1.** There are two major subcases which we designate by $S1$ and $S1^\pi$ in which $\mu$ and $\lambda^*$, respectively, are rectangles of $m^n$-shortness 1. Each has four subcases, as illustrated by:
These skew Young diagrams for $s_{\lambda/\mu}$ have been arranged so that those of type $S1^\pi$ are just the $\pi$-rotations of their left-hand neighbour of type $S1$. Moreover, the right-hand block of four are just the conjugates of the left-hand block. Thanks to the rotation symmetry (2.7) and the conjugate symmetry (2.6), it is therefore only necessary to consider two cases, which we choose to be $S1^\pi(a)$ and $S1^\pi(b')$.

$S1^\pi(a)$. Suppose $\lambda = (ab^{n-1})$ and $\mu$ is arbitrary, then the corresponding Young diagram and LR-hives take the form:

For given $\mu$ and $b$, Lemma 4.2 implies that all the edge labels of $ABC$ are fixed, including those on $AB$. It then follows from case (iv) of Lemma 4.1 that, for any given $\nu$, all the edge labels of $ABDE$ are also fixed. Thus all the hive edge labels are fixed, and $s_{\lambda/\mu}$ must be multiplicity-free, as required.

$S1^\pi(b')$. Suppose $\lambda = (a^{n-1}b)$ and $\mu$ is arbitrary. Then this case is exemplified by:

That $s_{\lambda/\mu}$ is multiplicity-free then follows from an argument entirely analogous to that used for $S1^\pi(a)$.
This completes the argument that each $s_{\lambda/\mu}$ of $R1$ is multiplicity-free.

**R2.** The two major subcases, $\mu$ a rectangle of $m^n$-shortness 2 and $\lambda^*$ a fat hook, and vice versa, we designate by $S2$ and $S2^\pi$, respectively. They each possess four subcases, as illustrated by:

![Diagram of skew Young diagrams]

Once again, these skew Young diagrams have been arranged so that those of type $S2^\pi$ are just the $\pi$-rotations of those of type $S2$, and the right-hand block of four is just the conjugate of the left-hand block of four. Thanks to the rotation symmetry (2.7) and the conjugate symmetry (2.5), it is therefore only necessary to consider two cases, which we choose to be $S2(a)$ and $S2(b')$.

**S2(a).** In this case $\lambda = (a'b^sc^t)$ and $\mu = (d^p0^q)$ with $r + s + t = p + q = n$ and $q = 2$, as illustrated in the following figure:

![Diagram of skew Young diagrams]

First, all the edge labels in subhives $ACD$, $IKGD$ and $CEF$ can be determined uniquely by Lemma 4.2 with all the edge labels on $CF$ being $c$, by Lemma 4.2 again, the labels in $BCF$ can be determined uniquely and then all the edge labels in $KFBJ$ can be determined uniquely. Since the edge labels on the boundary of $GKJ$ are known, and this triangle has side length two, then all its interior edge labels are fixed by those on the boundary by using case (ii) of Lemma 4.1 Hence all edge labels are fixed and this case is also multiplicity-free.
S2(b'). In this case $\lambda = (a'b'c')$ and $\mu = (d'0'q')$ with $r + s + t = p + q = n$ and $p = 2$, as illustrated in the following figure:

By Lemma 4.2 all the edge labels in ABC, IKH and CJKE can be determined uniquely, and the edge labels on JF and FC are all b and c, respectively. It follows, that all the edge labels in the subhives BCF and HJFG can also be determined by Lemmas 4.2 again. This only leaves the interior labels of BGF undetermined, which can be determined immediately by case (ii) of Lemma 4.1. Then all the edge labels in the complete hive are determined, and once again this case is multiplicity-free.

The symmetry conditions (2.7) and (2.5) then establish the fact that all case R2 examples are multiplicity-free.

R3. The two major subcases, $\mu$ a rectangle and $\lambda^*$ a fat hook of $m^n$-shortness 1, and vice versa, we designate by S3 and S3$^\pi$, respectively. They each possess six subcases, as illustrated by:

Once again, these skew Young diagrams have been arranged so that those of type S3$^\pi$ are just the $\pi$-rotations of those of type S3. This time the right-hand block of six is just the
conjugate of the left-hand block of six. Thanks to the rotation symmetry (2.7) and the conjugate symmetry (2.5), it is therefore only necessary to consider three cases, which we choose to be $S_3(a)$, $S_3(b)$ and $S_3(c)$.

$S_3(a)$. Suppose $\lambda = (ab^s c^t)$, $\mu = (d^p 0^q)$ with $s, t, p, q > 0$ and $1 + s + t = p + q = n$. The case $p = 1$ and $p = s + t$ have been covered in $S_1(b')$ and $S_1(a)$. This leaves three cases to discuss: $1 < p < 1 + s$, $p = 1 + s$, and $1 + s < p < s + t$, as the following figure shows:

We only consider the first case $1 < p < 1 + s$. The argument for the others is similar.

First, by Lemma 4.2 the edge labels in $AIF$ and $ADEC$ can be determined, and the edge labels on $AD$ are all equal to $c$. Again Lemma 4.2 implies that all edge labels in $ADF$ are fixed. Thus all edge labels on $GE$ are determined. The use, yet again, of Lemma 4.2 suffices to fix all edge labels in $HEG$. Finally, the edge labels of the region $FJHG$ are fixed by virtue of case (iv) of Lemma 4.1. Thus we have determined all the edge labels of this hive. Hence $s_{\lambda/\mu}$ is multiplicity-free.

$S_3(b)$. Suppose $\lambda = (a^r bc^t)$, $\mu = (d^p 0^q)$ with $r, t, p, q > 0$ and $r + 1 + t = p + q = n$. Since $p = 1$ and $p = r + t$ have been covered in $S_1(b')$ and $S_1(a)$, there are four subcases $1 < p < r$, $p = r$, $p = r + 1$ and $r + 1 < p < r + t$, as illustrated below:
By way of example, we consider the fourth subcase. The others may be dealt with similarly. By Lemma 4.2, all the edge labels in subhives BMN, AKG and DCGE are fixed, and the edge labels on DE are all c. Then by Lemma 4.2 once again, the edge labels in DEKF can be determined. Finally, thanks to Lemma 4.1 the edge labels of the thin strip MDFN are completely determined. We can therefore conclude that $s_{\lambda/\mu}$ is again multiplicity-free.

**S3(c).** Suppose $\lambda = (a^rb^sc)$ and $\mu = (d^pq^0)$ with $r, s, p, q > 0$ and $r + s + 1 = p + q = n$. Since $p = 1$ and $p = r + s$ have been covered in S1(b') and S1(a) respectively, there are three cases to consider: $1 < p < r$, $p = r$ and $r < p < r + s$. We choose to illustrate just the case $r < p < r + s$:

By Lemma 4.2, the edge labels in ABC and BGH are fixed by the hive boundary edge labels and the edge labels 0 along CE force all the edge labels on FJ to also be 0. Thanks to Lemma 4.2 it follows that the edge labels in HJFK are fixed, with those on HK all equal to b. Then Lemma 4.2 fixes all the edge labels in LHK. This leaves the thin strips BLFC and CFJE each of which may be dealt with through the use of Lemma 4.1. This fixes all the edge labels in the complete hive. Once again $s_{\lambda/\mu}$ is multiplicity-free. Similar arguments cover all the other subcases.

Hence, by the symmetry conditions (2.7) and (2.5), it follows that all the skew Schur functions of case R3 are also multiplicity-free.

**R4.** Here both $\mu$ and $\lambda^*$ are rectangles. The latter implies that $\lambda$ itself is a fat hook. In this situation, which we designate by S4, We can set $\lambda = (a^rb^s)$, $\mu = (c^pq^0)$ with $r, s, p, q > 0$ and $r + s = p + q = n$. There are three subcases to cover: $p < r$, $p = r$ and $p > r$. These are illustrated by:
In the first of these, by Lemma 4.2, the edge labels in ABC, BEH and CFED can be determined and the edge labels on DE are all equal to b. Since, in addition, the edge labels on BD are known, all the edge labels in BDE are fixed by Lemma 4.2. Thus we have determined all the edge labels of this hive. A similar argument applies in the other two subcases. Thus in all three subcases $s_{\lambda/\mu}$ is multiplicity-free.

This completes the proof of Lemma 7.1 that all the cases listed in Theorem 1.2 are multiplicity-free, as claimed.

8 Completeness of the list in the main theorem

It remains to show that the list of multiplicity-free skew Schur functions given in Theorem 1.2 is exhaustive. To this end we first consider three further cases, for which we shall show that there exists at least one partition $\nu$ such that $c_{\mu\nu}^\lambda > 1$.

Lemma 8.1 The skew Schur functions $s_{\lambda/\mu}$ are not multiplicity-free in each of the following cases:

T1 $\lambda = (abc)$ and $\mu = (de)$ with $a > b > c > 0$, $d > e > 0$, $a > d$ and $b > e$;

T2 $\lambda = (abcd)$ and $\mu = (e^2)$ with $a > b > c > d > 0$ and $b > e > 1$;

T3 $\lambda = (a^2b^2c^2)$ and $\mu = (d^3)$ with $a > b + 1$, $b > c + 1$, $c > 1$ and $b > d > 2$.

Proof:

For $\nu = (xyz)$, $(xyzw)$ and $(wxyzuv)$ the corresponding LR-hives take the form shown below, with $p = c - d + w$ in $K2$, and $p = a + b - w$, $q = b + c - d - v$, $r = d - c + u$ and $s = d - c + v$ in $K3$: 

![Diagram](image-url)
The solid lines divide the hive into portions for which the edge labels are determined, including all the dashed line interior edges. In each case, we are left with a hexagon on which the six boundary edge labels are necessarily fixed from a knowledge of \( \lambda, \mu \) and \( \nu \).

A priori the skew Schur functions \( s_{\lambda/\mu} \) identified here need not be basic. They are row-basic since the stated conditions ensure that \( \lambda_i > \mu_i \) for all \( i \), so that no row of \( F_{\lambda/\mu} \) is empty. However, some columns may be empty. If so, then these may be deleted to give \( F_{\hat{\lambda}/\hat{\mu}} \), with \( s_{\hat{\lambda}/\hat{\mu}} \) basic. Since the pair \( \hat{\lambda} \) and \( \hat{\mu} \) belong to the same case \( T_1, T_2 \) or \( T_3 \) as the original pair \( \lambda \) and \( \mu \), and \( s_{\lambda/\mu} = s_{\hat{\lambda}/\hat{\mu}} \), in accordance with (2.10), we can, without loss of generality, confine our attention to those cases for which \( s_{\lambda/\mu} \) is itself basic.

Then, for each given pair \( \lambda \) and \( \mu \), such that \( s_{\lambda/\mu} \) is basic, we will identify one particular partition \( \nu \) for which there are precisely two distinct labellings of the interior edge labels of the hexagon that satisfy all the hive conditions. It will then follow that \( c_{\mu/\nu} = 2 \), so that \( s_{\lambda/\mu} \) is not multiplicity-free. We consider each case in turn.

**T1.** Here, with \( \lambda = (abc) \) and \( \mu = (de) \) and \( s_{\lambda/\mu} \) basic, there are just two overlapping subcases to consider, one with \( c + 1 \geq d \) and the other with \( d \geq c + 1 \). In each subcase, we offer an appropriate partition \( \nu = (xyz) \) for which there exist two LR-hives:

**T1(i)** \[ a > b \geq c + 1 \geq d \geq e + 1 > 1 \quad \nu = (a - 1, b - e, c - d + 1) ; \]

**T1(ii)** \[ a > b \geq d \geq c + 1 \geq e + 1 > 1 \quad \nu = (a - 1, b + c - d - e + 1, 0) . \]

For these \( \nu \) the corresponding pairs of LR-hives are given explicitly by:
Thus there are precisely two LR-hives corresponding to the given \( \nu \), so that \( s_{abc/de} \) is not multiplicity-free. It might be noted that the case \( a = 3, b = 2, c = 1, d = 2 \) and \( e = 1 \) belongs to both of the above subcases, and that in each case \( \nu = (21) \). This case corresponds to the multiplicity 2 appearing in the well known expansion \( s_{321/21} = s_3 + 2s_{21} + s_{111} \).

\textbf{T2.} For \( \nu = (xyzw) \) figure \( K2 \) shows the preliminary constraints on interior edge labels that arise from fixing the boundary edge labels.

Since \( s_{\lambda/\mu} \) is basic, there are two subcases to deal with, as tabulated below:

\begin{align*}
\textbf{T2}(i) & \quad a > b > c \geq d + 1 \geq e > 1 \quad \nu = (a - 1, b - 1, c - e + 1, d - e + 1); \\
\textbf{T2}(ii) & \quad a > b > c \geq e \geq d + 1 > 1 \quad \nu = (a - 1, b + d - e, c - e + 1, 0).
\end{align*}

For these \( \nu \) we are able to complete the labelling of the interior edges of two LR 3-subhives of the required pair of LR 4-hives, as illustrated in the following figures:

The edge labellings of these pairs of LR 3-hives serve to complete the interior edge labelling of the corresponding pairs of LR 4-hives in which they are embedded. The
existence of precisely two LR-hives corresponding to the given \( \nu \), shows that \( s_{abcd/e} \) is not multiplicity-free. Once again it might be noted that the case \( a = 4, b = 3, c = 2, d = 1 \) and \( e = 2 \) belongs to both subcases, and that in each case \( \nu = (321) \). This corresponds to the multiplicity 2 appearing in the expansion \( s_{4321/2^2} = s_{42} + s_{41^2} + s_{32} + 2s_{321} + s_{31^3} + s_{2^3} + s_{2^2 1^2} \).

**T3.** Setting \( \nu = (wxyzuv) \), figure K3 illustrates the impact of the specification of boundary edge labels on the interior edges.

There are just two subcases to deal with, and in each of these we consider \( \nu = (w, x, y, z, u, v) \) as tabulated below:

- **T3(i)** \( a - 1 > b > c + 1 \geq d \geq 2 \) \( \nu = (a - 1, a - 2, b - 1, b - d + 1, c - d + 2, c - d + 1) \);
- **T3(ii)** \( a - 1 > b > d \geq c + 1 \geq 2 \) \( \nu = (a - 1, a + c - d - 1, b + c - d, b - d + 1, 1, 0) \).

In each subcase, we are then able to complete the labelling of the interior edges of two LR 3-subhives, as illustrated in the following figures, where \( f = d - c + 1 \) in the pair of subhives \( T3(ii) \).

![T3(i)](image)

![T3(ii)](image)

The edge labellings of these pairs of LR 3-hives serve to complete the interior edge labelling of the corresponding pairs of LR 6-hives in which they are embedded. The existence of precisely two LR-hives corresponding to the given \( \nu \), then suffices to show that \( s_{a^2b^2c^2/d^3} \) is not multiplicity-free. It might be noted that the two subcases coincide when \( a = 6, b = 4, c = 2 \) and \( d = 3 \), in which case \( \nu = (54321) \). This corresponds to the multiplicity 2 occurring in the decomposition \( s_{642^2 2^2 3^3} = s_{6531} + s_{6521^2} + s_{6432} + s_{6431^2} + s_{6421} + s_{6421^3} + s_{6331} + s_{6332} + s_{6321^2} + s_{5341} + s_{532^3} + s_{532^3} + s_{532^2} + s_{5321^2} + s_{5321} + s_{542^2} + s_{542^2} + s_{542^2} + s_{542^2} + 2s_{54321} + s_{5431^3} + s_{5432} + s_{542^2} + s_{542^2} + s_{533^2} + s_{533^2} + s_{532^2} + s_{532^2} + s_{532^2} + s_{532^2} + s_{4^32} + s_{4^32} + s_{4^32} + s_{4^32} \).

This completes the proof of Lemma 8.1.
Note: It should be pointed out that the conditions on \( a, b, c, d, e \) as stated in Lemma 8.1 arise naturally. In \( K1 \), to avoid being multiplicity-free, part (iii) of Lemma 4.3 implies that \( a > b > c \) and \( d > e > 0 \), while part (ii) implies \( c > 0 \). The remaining conditions \( a > d \) and \( b > e \) of case \( T1 \) arise from the fact that (3.12) and (3.13) imply \( a > d + e \geq d \) and \( b > e \) with an interior edge label of the hexagon fixed to be \( a \) or \( b \) if either \( a = d \) or \( b = e \), respectively. In \( K2 \), part (iii) of Lemma 4.3 implies that we require \( a > b > c \), \( d > e \) and \( e > p = e - d + w > 0 \) so that \( e > 1 \) and \( d > w \) with \( w \geq 0 \). The remaining condition \( b > e \) of case \( T2 \) arises because (3.12) and (3.13) imply \( b \geq e + w \geq e \) and setting \( b = e \) would fix an interior edge label of the hexagon. Finally in \( K3 \), to avoid being multiplicity-free, Lemma 4.3 gives \( a > p > b, b > q > c \) and \( d > r = d - c + u > s = d - c + v \), so that \( a > b + 1, b > c + 1 \) and \( c > u > v \), with the last implying \( c > 1 \). In addition, (3.13) gives \( b = q + s \), so that \( b > q \) implies \( s > 0 \). Then our condition \( d > r > s \) gives \( d > 2 \), as required. The condition \( q > c \) further implies \( b > d + v \), which yields the final condition \( b > d \) of case \( T3 \).

Definition 8.2 The skew Schur functions \( s_{\lambda/\mu} \) are not multiplicity-free in each of the following cases:

\[ \text{U1} \]

(i) \( \lambda = (a^2bc), \mu = (de) \) with \( a > b > c > 0, d > e > 0 \) and \( a > d \),
(ii) \( \lambda = (abc^2), \mu = (d^2e) \) with \( a > b > c > 0, d > e > 0, b > d \) and \( c > e \);

\[ \text{U2} \]

(i) \( \lambda = (a^2bcd), \mu = (e^2) \) with \( a > b > c > d > 0 \) and \( a > e + 1 > 2 \),
(ii) \( \lambda = (abcd^2), \mu = (e^3) \) with \( a > b > c > d > 0, c > e > 1 \) and \( d > 1 \);

\[ \text{U3} \]

(i) \( \lambda = (a^3b^2c^2) \) and \( \mu = (d^3) \) with \( a > b + 1, b > c + 1, c > 1 \) and \( a > d + 2 > 4 \),
(ii) \( \lambda = (a^2b^2c^3) \) and \( \mu = (d^4) \) with \( a > b + 1, b > c + 1, c > 2 \) and \( b > d + 1 > 3 \).

Proof: Once again we note that under the stated conditions \( s_{\lambda/\mu} \) is necessarily row-basic, but may not be basic. However, in each case we can obtain \( F_{\lambda/\hat{\mu}} \) from \( F_{\lambda/\mu} \) by the deletion of empty columns. This deletion procedure is such that the pair \( \lambda \) and \( \hat{\mu} \) necessarily belong to the same case, \( \text{U1}(i) - \text{U3}(ii) \), as the original pair \( \lambda \) and \( \mu \). Since \( s_{\lambda/\mu} = s_{\lambda/\hat{\mu}} \), it follows once again, that without loss of generality, we can confine attention to those \( s_{\lambda/\mu} \) that are basic. We consider each such case in turn.

\[ \text{U1}(i) \]

Since \( s_{\lambda/\mu} \) is basic, we have \( b \geq e \). If \( b > e \) then the pair \( \sigma = (abc) \) and \( \tau = (de) \) are such that \( s_{\sigma/\tau} \) is row-basic. It follows from case \( T1 \) of Lemma 8.1 that there exists at least one \( \rho \) such that \( c_{\sigma/\rho} \geq 2 \). Then by the second part of (2.6) \( c_{\mu/\nu} \geq c_{\sigma/\rho} \geq 2 \), with \( \nu = \rho \cup \{a\} \).

If \( b = e \), we begin with \( \sigma = (a - 1, b, c) \) and \( \tau = (d - 1, b - 1) \). Case \( T1 \) of Lemma 8.1 applies to this pair, so there exists at least one \( \rho \) such that \( c_{\sigma/\rho} \geq 2 \). Then by the second
part of (2.6) \( c'_{\tau'\rho'} \geq c'_{\tau\rho} \geq 2 \) with \( \tau' = \tau \cup \{a - 1\} \), \( \tau'' = \tau \cup \{0\} \) and \( \rho' = \rho \cup \{a - 1\} \). At last by the first part of (2.6), we have \( c'_{\mu\nu} \geq c'_{\tau'\rho'} \geq 2 \) with \( \nu = \rho' \).

**U1**(ii). Since \( s_{\lambda/\mu} \) is basic, we have \( c \geq d \). With \( \sigma = (abc) \) and \( \tau = (de) \) it follows from case **T1** of Lemma 8.1 that there exists at least one \( \rho \) such that \( c'_{\tau\rho} \geq 2 \). Then by the second part of (2.6) \( c'_{\tau'\rho'} \geq c'_{\tau\rho} \geq 2 \) with \( \nu = \rho \cup \{c - d\} \).

**U2**(i). Since \( s_{\lambda/\mu} \) is basic, we have \( b \geq e \). If \( b > e \), let \( \sigma = (abcd) \) and \( \tau = (e^2) \). Then by case **T2** of Lemma 8.1 there exists at least one \( \rho \) such that \( c'_{\tau\rho} \geq 2 \). The second part of (2.6) then implies \( c'_{\mu\nu} \geq c'_{\tau'\rho'} \geq 2 \) with \( \nu = \rho \cup \{a\} \).

If \( b = e \), let \( \sigma = (a - 1, b, c, d) \) and \( \tau = ((b - 1)^2) \). By case **T2** of Lemma 8.1 there exists at least one \( \rho \) such that \( c'_{\tau\rho} \geq 2 \). Then by the second part of (2.6) \( c'_{\tau'\rho'} \geq c'_{\tau\rho} \geq 2 \) with \( \nu = \rho \cup \{d - e\} \).

**U2**(ii). Since \( s_{\lambda/\mu} \) is basic, we have \( d \geq e \). Let \( \sigma = (abcd) \) and \( \tau = (e^3) \), so that by case **T2** of Lemma 8.1 there exists at least one \( \rho \) such that \( c'_{\tau\rho} \geq 2 \). Then by the second part of (2.6) \( c'_{\mu\nu} \geq c'_{\tau'\rho'} \geq 2 \) with \( \nu = \rho \cup \{a\} \).

If \( b = d \), let \( \sigma = ((a - 1)^2b^2c^2) \) and \( \tau = ((b - 1)^3) \). By case **T3** of Lemma 8.1 there exists at least one \( \rho \) such that \( c'_{\tau\rho} \geq 2 \). Then by the second part of (2.6) \( c'_{\mu\nu} \geq c'_{\tau'\rho'} \geq 2 \) with \( \nu = \rho \cup \{a\} \).

**U3**(i). Since \( s_{\lambda/\mu} \) is basic, we have \( b \geq d \). If \( b > d \), let \( \sigma = (a^2b^2c^2) \) and \( \tau = (d^3) \). By case **T3** of Lemma 8.1 there exists at least one \( \rho \) such that \( c'_{\tau\rho} \geq 2 \). Then by the second part of (2.6) \( c'_{\mu\nu} \geq c'_{\tau'\rho'} \geq 2 \) with \( \nu = \rho \cup \{c - d\} \).

**U3**(ii). Since \( s_{\lambda/\mu} \) is basic, we have \( c \geq d \). Let \( \sigma = (a^2b^2c^2) \) and \( \tau = (d^3) \). By case **T3** of Lemma 8.1 there exists at least one \( \rho \) such that \( c'_{\tau\rho} \geq 2 \). Then by the second part of (2.6) \( c'_{\mu\nu} \geq c'_{\tau'\rho'} \geq 2 \) with \( \nu = \rho \cup \{c - d\} \).

The significance of these results is that it allows us to prove the main theorem.

**Proof of Theorem 1.2** Let \( \lambda \) and \( \mu \) be such that \( s_{\lambda/\mu} \) is basic, with \( \lambda \) and \( \mu \) having \( s \) and \( t \) distinct non-zero parts, respectively, with \( s \geq 1 \) and \( t \geq 0 \). Then for \( s > 0 \) and \( t > 0 \) we let \( \lambda = (a_1^{p_1}, a_2^{p_2}, \ldots, a_s^{p_s}) \) and \( \mu = (b_1^{q_1}, b_2^{q_2}, \ldots, b_t^{q_t}) \), with \( a_1 > a_2 > \cdots > a_s > 0, b_1 > b_2 > \cdots > b_t > 0, \ell(\lambda) = p_1 + p_2 + \cdots + p_s = n \) and \( \ell(\mu) = q_1 + q_2 + \cdots + q_t < n \), where \( p_i, q_j > 0 \) for \( i = 1, 2, \ldots, s \) and \( j = 1, 2, \ldots, t \).

First we recall that the results of Section 7 imply that \( s_{\lambda/\mu} \) is multiplicity-free in each of the cases S0–S8. Then we consider all possible values of \( s \) and \( t \) in turn.

In the following, we select the parts of \( \sigma \) and \( \tau \) from those of \( \lambda \) and \( \mu \), respectively, in such a way that if \( \sigma_i = \lambda_j \) then \( \tau_i = \mu_j \). Since \( s_{\lambda/\mu} \) is basic, this guarantees that \( s_{\sigma/\tau} \)
is at least row-basic. If \( s_{\sigma/\tau} \) is not multiplicity-free, there exists at least one \( \rho \) such that \( c_{\tau/\rho}^s \geq 2 \). Setting \( \lambda = \sigma \cup \xi \) and \( \mu = \tau \cup \xi \), we now define \( \eta \) to be the partition such that each \( \eta_k = \zeta_l - \xi_l \) for some \( l \), and let \( \nu = \rho \cup \eta \). It then follows from the repeated application of the second part of (2.6) with \( a = \zeta_l, b = \xi_l \) and \( c = \eta_k \), that \( c_{\lambda/\mu}^s \geq c_{\tau/\rho}^s \geq 2 \). Thus \( s_{\lambda/\mu} \) is not multiplicity-free.

The case \( t = 0 \) is covered for all \( s \) by \( S_0 \), and is multiplicity-free. If \( s \geq 3 \) and \( t \geq 2 \), we select the \( \sigma \) and \( \tau \) according to the relations between the various \( p_i \) and \( q_i \). Three situations may arise: (i) If \( \ell(\mu) \leq p_1 \) then we can always select \( \sigma = (a^2bc) \) and \( \tau = (de) \) which is covered by case \( U_1(i) \) of Lemma 8.2. (ii) If \( p_1 < \ell(\mu) \) and \( q_1 < n - p_s \) then we can select \( \sigma = (abc) \) and \( \tau = (de) \) which is covered by case \( T_1 \) of Lemma 8.1. (iii) Finally, if \( p_1 < \ell(\mu) \) and \( q_1 \geq n - p_s \) then we can select \( \sigma = (abc^2) \) and \( \tau = (d^2e) \) which is covered by case \( U_1(ii) \) of Lemma 8.2.

If \( s \geq 4 \) the case \( t \geq 2 \) has already been dealt with. We can therefore take \( t = 1 \) so that \( \mu = (e^q) \) with \( 1 \leq q < n \). If \( q = 1 \) or \( e = 1 \) or \( e = a - 1 \) or \( q = n - 1 \) then \( s_{\lambda/\mu} \) is multiplicity-free since the situation is covered by case \( S_1 \), which was dealt with in Section 7. For \( a > e + 1 > 2 \) and \( 1 < q < n - 1 \), we need only consider the following subcases: (i) If \( 2 \leq q \leq p_1 \) we can select \( \sigma = (a^2bcd) \) and \( \tau = (e^2) \) which is covered by case \( U_2(i) \) of Lemma 8.2. (ii) If \( p_1 < q < \ell(\lambda) - p_s \) we can select \( \sigma = (abcd) \) and \( \tau = (e^2) \) which is covered by case \( T_2 \) of Lemma 8.1. (iii) If \( \ell(\mu) - p_s \leq q \leq n - 2 \) we can select \( \sigma = (abcd^2) \) and \( \tau = (e^3) \) which is covered by case \( U_2(ii) \) of Lemma 8.2.

The case \( s = 1 \) is covered for all \( t \) by \( S_0 \), and is multiplicity-free. Similarly, the case \( s = 2 \) and \( t = 1 \) is covered by \( S_4 \), and is multiplicity-free. By virtue of the rotation symmetry (2.7), the case \( s = 2 \) and \( t = 2 \) is identical with that of \( s = 3 \) and \( t = 1 \), which is still to be fully covered. On the other hand the cases \( s = 2 \) and \( t \geq 3 \) are identical with their images under rotation, for which \( s \geq 4 \) and \( t = 1 \), that have just been covered.

This just leaves the case \( s = 3 \) and \( t = 1 \). This is dealt with by noting that if \( \lambda = (a^{p_1}b^{p_2}c^{p_3}) \) and \( \mu = (d^q) \) are such that they are not covered by one or other of the multiplicity-free cases listed under \( S_1, S_2 \) or \( S_3 \), that is \( p_i \geq 2 \) for \( i = 1, 2, 3, a > b + 1, b > c + 1, c > 1, d > 2, a > d + 2 \) and \( 3 \leq q \leq n - 3 \), then they are covered by one of the following three subcases: (i) If \( 3 \leq q \leq p_1 \) we select \( \sigma = (a^3b^2c^2) \) and \( \tau = (d^3) \) which is covered by case \( U_3(i) \) of Lemma 8.2. (ii) If \( p_1 < q < p_1 + p_2 \) we can select \( \sigma = (a^2b^2c^3) \) and \( \tau = (d^3) \) which is covered by case \( T_3 \) of Lemma 8.1. (iii) If \( p_1 + p_2 \leq q \leq n - 3 \) we can select \( \sigma = (a^2b^2c^3) \) and \( \tau = (d^4) \) which is covered by case \( U_3(ii) \) of Lemma 8.2.

This completes the proof of Theorem 1.2.
9 Final remarks

We have shown that the hive model is well suited to the derivation of the two Theorems 1.1 and 1.2 on multiplicity-free Schur function products and skew Schur functions, respectively. The use of LR-hives has allowed a direct proof that all the cases enumerated in both theorems are indeed multiplicity-free. In addition it has enabled us to demonstrate that the breakdown of multiplicity-freeness always has a common origin, in the sense that it can be traced back to the existence of a vertex in the relevant LR-hives that is surrounded by an elementary hexagon, none of whose interior edge labels is fixed either by the criteria of Lemma 4.3 or any other means.

The proof offered here of the skew Schur function theorem, unlike that of Gutschwa-ger [3], is quite independent of Stembridge’s product of Schur functions theorem. In fact, since the hive model proof has covered simultaneously both connected and disconnected cases, it is possible to recover from Theorem 1.2 not only Theorem 1.1 but also the following:

**Corollary 9.1** The product $s_\theta s_\phi$ of any two basic skew Schur functions is multiplicity-free if and only if one or more of the following is true:

- **V1** $\theta$ is a one-line rectangle and $\phi$ or $\phi^\pi$ is a partition (or vice versa);
- **V2** $\theta$ is a two-line rectangle and $\phi$ or $\phi^\pi$ is a fat hook (or vice versa);
- **V3** $\theta$ is a rectangle and $\phi$ or $\phi^\pi$ is a near-rectangle (or vice versa);
- **V4** $\theta$ and $\phi$ are rectangles.

**Proof:** Thanks to (2.11) every product $s_\theta s_\phi$ of two basic skew Schur functions can be expressed as a single basic skew Schur function $s_{\lambda/\mu}$ where $F^\lambda/\mu$ is constructed, as in Example 2.4, by joining $F^\theta$ and $F^\phi$ corner to corner. Then one applies Theorem 1.2 to all those cases for which $F^\lambda/\mu$ is of the required disconnected form. The cases $R0$ are always connected, while each multiplicity-free disconnected case of Theorem 1.2 gives rise through the identity $s_{\lambda/\mu} = s_\theta s_\phi$ to a corresponding case in this corollary, and vice versa, as follows: $R1 \leftrightarrow V1$; $R2 \leftrightarrow V2$; $R3 \leftrightarrow V3$ and $R4 \leftrightarrow V4$. It follows that every multiplicity-free skew Schur function product is of one or other of the types $V1$–$V4$. All other cases are not multiplicity-free.

By exactly the same argument, one can recover Theorem 1.1 as a further corollary by restricting attention to those cases for which both $\theta$ and $\phi$ are partitions. The correspondence between the multiplicity-free disconnected cases of Theorem 1.2 of the required form and those of Theorem 1.1 is given by: $R1 \leftrightarrow P1$; $R2 \leftrightarrow P2$; $R3 \leftrightarrow P3$ and $R4 \leftrightarrow P4$. All other cases are not multiplicity-free.
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