Factorization for Hardy spaces and characterization for BMO spaces via commutators in the Bessel setting

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Abstract: Fix $\lambda > 0$. Consider the Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$ in the sense of Coifman and Weiss, where $\mathbb{R}_+: = (0, \infty)$ and $dm_\lambda := x^{2\lambda} dx$ with $dx$ the Lebesgue measure. Also consider the Bessel operators $\Delta_\lambda := -\frac{d^2}{dx^2} - \frac{2\lambda}{x} \frac{d}{dx}$, and $S_\lambda := -\frac{d^2}{dx^2} + \frac{\lambda^2 - \lambda}{x}$ on $\mathbb{R}_+$. The Hardy spaces $H^1_{\Delta_\lambda}$ and $H^1_{S_\lambda}$ associated with $\Delta_\lambda$ and $S_\lambda$ are defined via the Riesz transforms $R_{\Delta_\lambda} := \partial_x (\Delta_\lambda)^{-1/2}$ and $R_{S_\lambda} := x^\lambda \partial_x x^{-\lambda}(S_\lambda)^{-1/2}$, respectively. It is known that $H^1_{\Delta_\lambda}$ and $H^1(\mathbb{R}_+, dm_\lambda)$ coincide but they are different from $H^1_{S_\lambda}$. In this article, we prove the following: (a) a weak factorization of $H^1(\mathbb{R}_+, dm_\lambda)$ by using a bilinear form of the Riesz transform $R_{\Delta_\lambda}$, which implies the characterization of the BMO space associated to $\Delta_\lambda$ via the commutators related to $R_{\Delta_\lambda}$; (b) the BMO space associated to $S_\lambda$ can not be characterized by commutators related to $R_{S_\lambda}$, which implies that $H^1_{S_\lambda}$ does not have a weak factorization via a bilinear form of the Riesz transform $R_{S_\lambda}$.

Keywords: BMO; commutator; Hardy space; factorization; Bessel operator; Riesz transform.

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1 Introduction and statement of main results

Recall that the classical Hardy space $H^p$, $0 < p < \infty$, on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is defined as the space of holomorphic functions $f = u + iv$, i.e., those satisfying the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ in $\mathbb{D}$, such that

$$\|f\|_{H^p(\mathbb{D})} := \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}} < \infty.$$ 

It is well known that the product of two $H^2(\mathbb{D})$ functions belongs to Hardy space $H^1(\mathbb{D})$, but in fact the converse is also true, and is known as the Riesz factorization theorem: “A function $f$ is in $H^1(\mathbb{D})$ if and only if there exist $g, h \in H^2(\mathbb{D})$ with $f = g \cdot h$ and $\|f\|_{H^1(\mathbb{D})} = \|g\|_{H^2(\mathbb{D})}\|h\|_{H^2(\mathbb{D})}$.” This factorization result plays an important role in studying function theory and operator theory connected to the spaces $H^1(\mathbb{D})$, $H^2(\mathbb{D})$ and the space $BMOA(\mathbb{D})$ (analytic BMO).

The real-variable Hardy space theory on $n$-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 1$) plays an important role in harmonic analysis and has been systematically developed. We point out that two closely related characterizations for $H^1(\mathbb{R}^n)$ are that: (1) $H^1(\mathbb{R}^n)$ can be characterized in terms of Riesz transforms; (2) $H^1(\mathbb{R}^n)$ can be viewed as the boundary of the Hardy space $H^1(\mathbb{R}^n_{+1})$ consisting of systems of conjugate harmonic functions $F = (u_0, u_1, \ldots, u_n)$, which satisfy the generalized Cauchy–Riemann equations $\sum_{j=0}^n \partial u_j / \partial x_j = 0$ and $\partial u_k / \partial x_k = u_k / \partial x_j$ in $\mathbb{R}^{n+1}_+$. 


0 ≤ j, k ≤ n, see [12, 16]. However, the analogue of the Riesz factorization theorem, sometimes referred to as strong factorization, is not true for real-variable Hardy space $H^1(\mathbb{R}^n)$. Nevertheless, Coifman, Rochberg and Weiss [9] provided a suitable replacement that works in studying function theory and operator theory of $H^1(\mathbb{R}^n)$, the weak factorization via a bilinear form related to the Riesz transform (Hilbert transform in dimension 1).

The theory of the classical Hardy space is intimately connected to the Laplacian; changing the differential operator introduces new challenges and directions to explore. In 1965, Muckenhoupt and Stein in [15] introduced a notion of conjugacy associated with this Bessel operator $\Delta_{\lambda}$, which is defined by

$$\Delta_{\lambda} f(x) := -\frac{d^2}{dx^2} f(x) - \frac{2\lambda}{x} \frac{d}{dx} f(x), \quad x > 0.$$  

They developed a theory in the setting of $\Delta_{\lambda}$ which parallels the classical one associated to $\Delta$. Results on $L^p(\mathbb{R}_+, dm_{\lambda})$-boundedness of conjugate functions and fractional integrals associated with $\Delta_{\lambda}$ were obtained, where $p \in [1, \infty]$, $\mathbb{R}_+ := (0, \infty)$ and $dm_{\lambda}(x) := x^{2\lambda} dx$. Since then, many problems based on the Bessel context were studied; see, for example, [1, 2, 4, 5, 6, 14, 19, 21]. In particular, the properties and $L^p$ boundedness ($1 < p < \infty$) of Riesz transforms

$$R_{\Delta_{\lambda}} f := \partial_x (\Delta_{\lambda})^{-1/2} f$$

related to $\Delta_{\lambda}$ have been studied in [1, 2, 4, 15, 19]. The related Hardy space

$$H^1(\mathbb{R}_+, dm_{\lambda}) := \{ f \in L^1(\mathbb{R}_+, dm_{\lambda}) : R_{\Delta_{\lambda}} f \in L^1(\mathbb{R}_+, dm_{\lambda}) \}$$

with norm $\|f\|_{H^1(\mathbb{R}_+, dm_{\lambda})} := \|f\|_{L^1(\mathbb{R}_+, dm_{\lambda})} + \|R_{\Delta_{\lambda}} f\|_{L^1(\mathbb{R}_+, dm_{\lambda})}$ has been studied by Betancor et al. in [3]. For $f \in H^1(\mathbb{R}_+, dm_{\lambda})$, we have that the pair of functions

$$u(t, x) := P_t^{[\lambda]}(f)(x) \quad \text{and} \quad v(t, x) := Q_t^{[\lambda]}(f)(x), \quad t, x \in \mathbb{R}_+$$

satisfy the following generalized Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial t} \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} - \frac{2\lambda}{x} v \quad \text{in} \quad \mathbb{R}_+.$$  

Here $P_t^{[\lambda]}(f)$ is the Poisson integral of $f$ with the Poisson semigroup $P_t^{[\lambda]} := e^{-\sqrt{\lambda t}}$, and $Q_t^{[\lambda]}(f)$ is the conjugate Poisson integral, see Section 2 for precise definitions.

Following a different procedure in [15], the Riesz transform

$$R_{S_{\lambda}} f := A_{\lambda}(S_{\lambda})^{-1/2} f, \quad \text{where} \quad A_{\lambda} := x^\lambda \partial_x x^{-\lambda}$$

has also been studied by Betancor et al. [3], which is related to the other Bessel operator $S_{\lambda}$ defined by

$$S_{\lambda} f(x) := -\frac{d^2}{dx^2} f(x) + \frac{\lambda^2 - \lambda}{x^2} f(x), \quad x > 0. \quad (1.1)$$

Moreover, the corresponding Hardy space

$$H^1_{S_{\lambda}}(\mathbb{R}_+, dx) := \{ f \in L^1(\mathbb{R}_+, dx) : R_{S_{\lambda}} f \in L^1(\mathbb{R}_+, dx) \}$$

with norm $\|f\|_{H^1_{S_{\lambda}}(\mathbb{R}_+, dx)} = \|f\|_{L^1(\mathbb{R}_+, dx)} + \|R_{S_{\lambda}} f\|_{L^1(\mathbb{R}_+, dx)}$ was characterized. And the Poisson integral and the conjugate Poisson integral of the function $f \in H^1_{S_{\lambda}}(\mathbb{R}_+, dx)$ also satisfy a generalized Cauchy–Riemann equations.
A natural question is that: Do the Hardy spaces $H^1(\mathbb{R}_+, dm_\lambda)$ and $H^1_{S_\lambda}(\mathbb{R}_+, dx)$ have Riesz type factorization or weak factorization in terms of a bilinear form related to $R_{\Delta_\lambda}$ and $R_{S_\lambda}$, respectively?

The aim of this paper is twofold. We first build up a weak factorization for the Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$ in terms of a bilinear form related to $R_{\Delta_\lambda}$. Then we further prove that this weak factorization implies the characterization of the dual of $H^1(\mathbb{R}_+, dm_\lambda)$ via commutators related to $R_{\Delta_\lambda}$. Second, we point out that a weak factorization for the Hardy space $H^1_{S_\lambda}(\mathbb{R}_+, dx)$ in terms of a bilinear form related to $R_{S_\lambda}$ is not true, by proving that the dual of $H^1_{S_\lambda}(\mathbb{R}_+, dx)$ cannot be characterized by commutators related to $R_{S_\lambda}$.

To state our main results, we first recall some necessary notions and notation. Throughout this paper, for any $x, r \in \mathbb{R}_+$, $I(x, r) := (x - r, x + r) \cap \mathbb{R}_+$. From the definition of the measure $m_\lambda$ (i.e., $dm_\lambda(x) := x^{2\lambda} dx$), it is obvious that there exists a positive constant $C \in (1, \infty)$ such that for all $x, r \in \mathbb{R}_+$,

$$C^{-1}m_\lambda(I(x, r)) \leq x^{2\lambda}r + r^{2\lambda+1} \leq Cm_\lambda(I(x, r)). \quad (1.2)$$

Thus $(\mathbb{R}_+, \rho, dm_\lambda)$ is a space of homogeneous type in the sense of Coifman and Weiss [10], where $\rho(x, y) := |x - y|$ for all $x, y \in \mathbb{R}_+$.

We now state our first main result on the weak factorization (via Riesz transform $R_{\Delta_\lambda}$ and its adjoint operator $\widetilde{R}_{\Delta_\lambda}$) of the Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$.

**Theorem 1.1.** Let $p \in (1, \infty)$ and $p'$ be the conjugate of $p$. For any $f \in H^1(\mathbb{R}_+, dm_\lambda)$, there exist numbers $\{\alpha_j^k\}_{k, j}$, functions $\{g_j^k\}_{k, j} \subset L^p(\mathbb{R}_+, dm_\lambda)$ and $\{h_j^k\}_{k, j} \subset L^{p'}(\mathbb{R}_+, dm_\lambda)$ such that

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \quad (1.3)$$

in $H^1(\mathbb{R}_+, dm_\lambda)$, where the operator $\Pi$ is defined as follows: for $g \in L^p(\mathbb{R}_+, dm_\lambda)$ and $h \in L^{p'}(\mathbb{R}_+, dm_\lambda)$,

$$\Pi(g, h) := gR_{\Delta_\lambda}h - h\widetilde{R}_{\Delta_\lambda}g. \quad (1.4)$$

Moreover, there exists a positive constant $C$ independent of $f$ such that

$$C^{-1}\|f\|_{H^1(\mathbb{R}_+, dm_\lambda)} \leq \inf \left\{ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \left\| g_j^k \right\|_{L^p(\mathbb{R}_+, dm_\lambda)} \left\| h_j^k \right\|_{L^{p'}(\mathbb{R}_+, dm_\lambda)} : f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_j^k \Pi(g_j^k, h_j^k) \right\} \leq C\|f\|_{H^1(\mathbb{R}_+, dm_\lambda)}. \quad (1.5)$$

Our second main result provides a characterization of the BMO space $\text{BMO}(\mathbb{R}_+, dm_\lambda)$, which is the dual of $H^1(\mathbb{R}_+, dm_\lambda)$, in terms of the commutators adapted to the Riesz transform $R_{\Delta_\lambda}$.

Recall the definition of the BMO space associated with the Bessel operator, which is the dual space of $H^1(\mathbb{R}_+, dm_\lambda)$.

**Definition 1.2** ([21]). A function $f \in L^1_{\text{loc}}(\mathbb{R}_+, dm_\lambda)$ belongs to the space $\text{BMO}(\mathbb{R}_+, dm_\lambda)$ if

$$\sup_{x, r \in (0, \infty)} \frac{1}{m_\lambda(I(x, r))} \int_{I(x, r)} \left| f(y) - \frac{1}{m_\lambda(I(x, r))} \int_{I(x, r)} f(z) \, dm_\lambda(z) \right| \, dm_\lambda(y) < \infty.$$
Suppose \( b \in L^1_{\text{loc}}(\mathbb{R}_+, dm_\lambda) \) and \( f \in L^p(\mathbb{R}_+, dm_\lambda) \). Let \([b, R_\Delta]\) be the commutator defined by
\[
[b, R_\Delta]f(x) := b(x)R_\Delta f(x) - R_\Delta(bf)(x).
\]

**Theorem 1.3.** Let \( b \in \cup_{q>1} L^q_{\text{loc}}(\mathbb{R}_+, dm_\lambda) \) and \( p \in (1, \infty) \).

1. If \( b \in \text{BMO}(\mathbb{R}_+, dm_\lambda) \), then the commutator \([b, R_\Delta]\) is bounded on \( L^p(\mathbb{R}_+, dm_\lambda) \) with the operator norm
\[
\| [b, R_\Delta] \|_{L^p(\mathbb{R}_+, dm_\lambda) \to L^p(\mathbb{R}_+, dm_\lambda)} \leq C\| b \|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)}.
\]

2. If \([b, R_\Delta]\) is bounded on \( L^p(\mathbb{R}_+, dm_\lambda) \), then \( b \in \text{BMO}(\mathbb{R}_+, dm_\lambda) \) and
\[
\| b \|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} \leq C \| [b, R_\Delta] \|_{L^p(\mathbb{R}_+, dm_\lambda) \to L^p(\mathbb{R}_+, dm_\lambda)}.
\]

We will provide the proof of Theorems 1.1 and 1.3 in the following structure: we first provide the proof of (1) in Theorem 1.3, which plays the key role to prove Theorem 1.1. Then, (2) in Theorem 1.3 follows directly from the weak factorization of \( H^1(\mathbb{R}_+, dm_\lambda) \) in Theorem 1.1 via duality.

The next main result that we provide is that the BMO space \( \text{BMO}_{\lambda}(\mathbb{R}_+, dx) \) associated with \( S_\lambda \), which is the dual of \( H^1_{S_\lambda}(\mathbb{R}_+, dx) \), can not be characterized by the commutators with respect to the Riesz transform \( R_\lambda \).

**Theorem 1.4.** There exists a locally integrable function \( b \notin \text{BMO}_{\lambda}(\mathbb{R}_+, dx) \), such that for \( 1 < p < \infty \), the commutator \([b, R_\lambda]\) is bounded on \( L^p(\mathbb{R}_+, dx) \) with the operator norm
\[
\| [b, R_\lambda] \|_{L^p(\mathbb{R}_+, dx) \to L^p(\mathbb{R}_+, dx)} \leq C_b,
\]

where \( C_b \) is a positive constant related to the function \( b \).

As a consequence, we have the following argument.

**Corollary 1.5.** A weak factorization for \( H^1_{S_\lambda}(\mathbb{R}_+, dx) \) in the form of Theorem 1.1 with respect to a bilinear form related to \( R_\lambda \) is not true.

An outline of the paper is as follows. In Section 2 we recall the Hardy spaces associated with \( \Delta_\lambda \) and \( S_\lambda \). Also we collect some fundamental estimates of the kernel of the Riesz transforms \( R_\Delta \) and \( R_\lambda \), especially the size estimate and the H"older’s regularity in Proposition 2.2 (Proposition 2.5), and the kernel lower bounds in Proposition 2.3 for \( R_\Delta \). In Section 3 we prove Theorems 1.1 and 1.3. We note that our auxiliary result Lemma 3.1 is an important ingredient of the proof of Theorem 1.1, an earlier analogue of which appears in [13] but with different proof. In Section 4 we prove Theorem 1.4 by providing a specific example of the locally integrable function \( b \) from the classical BMO space \( \text{BMO}(\mathbb{R}_+, dx) \) but \( b \notin \text{BMO}_{\lambda}(\mathbb{R}_+, dx) \). Then we prove Corollary 1.5.

Throughout the paper, we denote by \( C \) and \( \tilde{C} \) positive constants which are independent of the main parameters, but they may vary from line to line. For every \( p \in (1, \infty) \), we denote by \( p' \) the conjugate of \( p \), i.e., \( \frac{1}{p} + \frac{1}{p'} = 1 \). If \( f \leq Cg \), we then write \( f \lesssim g \) or \( g \gtrsim f \); and if \( f \lesssim g \lesssim f \), we write \( f \sim g \). For any \( k \in \mathbb{R}_+ \) and \( I := I(x, r) \) for some \( x, r \in (0, \infty) \), \( kI := I(x, kr) \).
2 Hardy and BMO spaces, Riesz transforms associated with $\Delta_\lambda$ and $S_\lambda$

In this section we recall the Hardy and BMO spaces, and some important properties of and Riesz transforms related to the Bessel operator $\Delta_\lambda$ and $S_\lambda$ from [15, 3, 4, 5].

We now recall the atomic characterization of the Hardy spaces $H^1(\mathbb{R}_+, dm_\lambda)$ in [3].

**Definition 2.1** ([3]). A function $f$ is called a $(1, \infty)\Delta_\lambda$-atom if there exists an open bounded interval $I \subset \mathbb{R}_+$ such that $\text{supp}(f) \subset I$, $\|f\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} \leq [m_{\lambda}(I)]^{-1}$ and $\int_0^\infty a(x) \, dm_\lambda(x) = 0$.

We point out that from [3], the Hardy space $H^1(\mathbb{R}_+, dm_\lambda)$ can also be characterized via atomic decomposition. That is, an $L^1(\mathbb{R}_+, dm_\lambda)$ function $f \in H^1(\mathbb{R}_+, dm_\lambda)$ if and only if

$$f = \sum_{j=1}^\infty \alpha_j a_j \quad \text{in} \quad L^1(\mathbb{R}_+, dm_\lambda),$$

where for every $j$, $a_j$ is a $(1, \infty)\Delta_\lambda$-atom and $\alpha_j \in \mathbb{R}$ satisfying that $\sum_{j=1}^\infty |\alpha_j| < \infty$. Moreover,

$$\|f\|_{H^1(\mathbb{R}_+, dm_\lambda)} \approx \inf \left\{ \sum_{j=1}^\infty |\alpha_j| \right\},$$

where the infimum is taken over all the decompositions of $f$ as above.

We also note that $H^1(\mathbb{R}_+, dm_\lambda)$ can also be characterized in terms of the radial maximal function associated with the Hankel convolution of a class of functions, including the Poisson semigroup and the heat semigroup as special cases. It is also proved in [3] that $H^1(\mathbb{R}_+, dm_\lambda)$ is the one associated with the space of homogeneous type $(\mathbb{R}_+, \rho, dm_\lambda)$ defined by Coifman and Weiss in [10].

Next we recall the Poisson integral, the conjugate Poisson integral and the properties of the Riesz transforms. As in [3], let $\{P_t^\lambda\}_{t > 0}$ be the Poisson semigroup $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t > 0}$ defined by

$$P_t^\lambda f(x) := \int_0^\infty P_t^\lambda(x, y) f(y) y^{2\lambda} \, dy,$$

where

$$P_t^\lambda(x, y) = \int_0^\infty e^{-t(xz)}^{-\lambda+1/2} J_{\lambda-1/2}(xz)(yz)^{-\lambda+1/2} J_{\lambda-1/2}(yz)^{2\lambda} \, dz$$

and $J_\nu$ is the Bessel function of the first kind and order $\nu$. Weinstein [20] established the following formula for $P_t^\lambda(x, y)$: $t$, $x$, $y \in \mathbb{R}_+$,

$$P_t^\lambda(x, y) = \frac{2\lambda t}{\pi} \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} \, d\theta.$$

The $\Delta_\lambda$-conjugate of the Poisson integral of $f$ is defined by

$$Q_t^\lambda f(x) := \int_0^\infty Q_t^\lambda(x, y) f(y) y^{2\lambda} \, dy,$$

where

$$Q_t^\lambda(x, y) = -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} \, d\theta.$$

From this, we deduce that for any $x$, $y \in \mathbb{R}_+$,

$$R_{\Delta_\lambda}(x, y) = \partial_x \int_0^\infty P_t^\lambda(x, y) \, dt = -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} \, d\theta = \lim_{t \to 0} Q_t^\lambda f(x). \quad (2.1)$$
We note that, as indicated in [3], this Riesz transform $R_{\Delta}$ is a Calderón–Zygmund operator (see also [5]). For the convenience of the readers we provide all the details of the verification here.

**Proposition 2.2.** The kernel $R_{\Delta}(x, y)$ satisfies the following conditions:

i) for every $x, y \in \mathbb{R}_+$ with $x \neq y$,

$$|R_{\Delta}(x, y)| \lesssim \frac{1}{m_{\lambda}(I(x, |x - y|))};$$  \hfill (2.2)

ii) for every $x, x_0, y \in \mathbb{R}_+$ with $|x_0 - x| < |x_0 - y|/2$,

$$|R_{\Delta}(y, x_0) - R_{\Delta}(y, x)| + |R_{\Delta}(x_0, y) - R_{\Delta}(x, y)| \lesssim \frac{|x_0 - x|}{|x_0 - y|} \frac{1}{m_{\lambda}(I(x_0, |x_0 - y|))}. \hfill (2.3)$$

**Proof.** Recall that

$$R_{\Delta}(x, y) = \frac{-2\lambda}{\pi} \int_0^\pi \frac{(x - y\cos \theta)(\sin \theta)^{2\lambda - 1}}{(x^2 + y^2 - 2xy \cos \theta)^{\lambda + 1}} d\theta.$$ 

We first verify (2.2). Suppose $x, y \in \mathbb{R}_+$ with $x \neq y$. We now consider the following two cases.

Case 1, $x \leq 2|x - y|$. Note that $|x - y\cos \theta| \leq |x^2 + y^2 - 2xy \cos \theta|^{1/2}$. Combining the fact that

$$\int_0^\pi (\sin \theta)^{2\lambda - 1} d\theta = 2 \int_0^\pi (\sin \theta)^{2\lambda - 1} d\theta = \frac{\Gamma(\lambda)\sqrt{\pi}}{\Gamma(\lambda + 1/2)} \hfill (2.4)$$

and the property (1.2) of the measure $dm_{\lambda}$, we obtain that

$$|R_{\Delta}(x, y)| \lesssim \int_0^\pi \frac{(\sin \theta)^{2\lambda - 1}}{|x - y|^{2\lambda + 1}} d\theta \lesssim \frac{1}{|x - y|^{2\lambda + 1}} \sim \frac{1}{m_{\lambda}(I(x, |x - y|))}.$$ 

Case 2, $x > 2|x - y|$. Note that in this case, $x/2 \leq y \leq 3x/2$. Thus, by noting that $1 - \cos \theta \geq 2\left(\frac{\theta}{\pi}\right)^2$ for $\theta \in [0, \pi]$, we have

$$|R_{\Delta}(x, y)| \lesssim \int_0^\pi \frac{(\sin \theta)^{2\lambda - 1}}{|(x - y)^2 + 2xy(1 - \cos \theta)|^{\lambda + 1/2}} d\theta \lesssim \frac{1}{|x - y|^{2\lambda + 1}} \left\{ \begin{array}{l} \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda - 1}}{(\pi/2)^{\lambda + 1/2}} d\theta \\ \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda - 1}}{(\pi/2)^{\lambda + 1/2}} d\theta \end{array} \right\} \leq \frac{1}{|x - y|^{2\lambda + 1}} \frac{1}{m_{\lambda}(I(x, |x - y|))}.$$ 

Combining the estimates in Cases 1 and 2 we obtain (2.2).

We turn to (2.3). We point out that it suffices to prove that when $|x_0 - x| < |x_0 - y|/2$,

$$|R_{\Delta}(x_0, y) - R_{\Delta}(x, y)| \lesssim \frac{|x_0 - x|}{|x_0 - y|} \frac{1}{m_{\lambda}(I(x_0, |x_0 - y|))}. \hfill (2.5)$$
then the estimate for \(|R_{\Delta \lambda}(y, x_0) - R_{\Delta \lambda}(y, x)|\) follows similarly.

By the Mean Value Theorem, there exists \(\xi := tx_0 + (1 - t)x\) for some \(t \in (0, 1)\) such that

\[ |R_{\Delta \lambda}(x_0, y) - R_{\Delta \lambda}(x, y)| = |x_0 - x||\partial_x R_{\Delta \lambda}(\xi, y)|. \]

Observe that

\[
|\partial_x R_{\Delta \lambda}(\xi, y)| \lesssim \left| \int_0^\pi \frac{(\sin \theta)^{2\lambda - 1}}{(\xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda + 1}} d\theta \right| + \left| \int_0^\pi \frac{(\xi - y \cos \theta)^2 (\sin \theta)^{2\lambda - 1}}{(\xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda + 2}} d\theta \right|
\]

\[
\lesssim \int_0^\pi \frac{(\sin \theta)^{2\lambda - 1}}{(\xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda + 1}} d\theta
\]

\[
\lesssim \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda - 1}}{(\xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda + 1}} d\theta.
\]

To show (2.5), it suffices to prove that

\[
\int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda - 1}}{(\xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda + 1}} d\theta \lesssim \frac{1}{|x_0 - y|} m_\lambda(I(x_0, |x_0 - y|)).
\]

To see this, we first point out that from the definition of \(\xi\),

\[
\frac{1}{2} |x_0 - y| \leq |\xi - y| \leq \frac{3}{2} |x_0 - y|. \tag{2.6}
\]

Then, we consider the following two cases.

Case 1, \(x_0 \leq 2|x_0 - y|\). It follows that

\[
\int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda - 1}}{(\xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda + 1}} d\theta \lesssim \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda - 1}}{(\xi - y)^{2\lambda + 2}} d\theta \lesssim \frac{1}{(x_0 - y)^{2\lambda + 2}};
\]

Case 2, \(x_0 > 2|x_0 - y|\). Note that in this case \(\xi \sim y \sim x_0\). By (2.6), we also have that

\[
\int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda - 1}}{(\xi^2 + y^2 - 2\xi y \cos \theta)^{\lambda + 1}} d\theta \lesssim \int_0^{\pi/2} \frac{\theta^{2\lambda - 1}}{[(\xi - y)^2 + \frac{4}{\pi^2} \xi y \theta^2]^{\lambda + 1}} d\theta
\]

\[
\lesssim \frac{1}{\xi y^{\lambda}(\xi - y)^2} \int_{\xi y^2}^\infty \frac{1}{(1 + \beta^{2\lambda - 1})^{\lambda + 1}} d\beta
\]

\[
\sim \frac{1}{x_0^{\lambda}(x_0 - y)^2}.
\]

Combining the two cases above, we get that (2.5) holds.

Next we recall the following estimates of the kernel \(R_{\Delta \lambda}(x, y)\) of the Riesz transform \(R_{\Delta \lambda}\), which will be used in the sequel.

**Proposition 2.3.** The Riesz kernel \(R_{\Delta \lambda}(x, y)\) satisfies:

1) There exist \(K_1 > 2\) large enough and a positive constant \(C_{K_1, \lambda}\) such that for any \(x, y \in \mathbb{R}_+\) with \(y > K_1 x\),

\[
R_{\Delta \lambda}(x, y) \geq C_{K_1, \lambda} \frac{x}{y^{2\lambda + 2}}. \tag{2.7}
\]
We estimate $I$ by considering the following three cases.

**ii)** There exist $K_2 \in (0, 1)$ small enough and a positive constant $C_{K_2, \lambda}$ such that for any $x, y \in \mathbb{R}_+$ with $y < K_2x$,

$$R_{\Delta_\lambda}(x, y) \leq -C_{K_2, \lambda} \frac{1}{x^{2\lambda+1}}. \quad (2.8)$$

**iii)** There exist $K_3 \in (1/2, 2)$ such that $|K_3 - 1|$ small enough and a positive constant $C_{K_3, \lambda}$ such that for any $x, y \in \mathbb{R}_+$ with $0 < |1 - y/x| < |K_3 - 1|$,

$$|R_{\Delta_\lambda}(x, y) + \frac{1}{\pi} \frac{1}{x^\lambda y^\lambda} \frac{1}{x - y}| \leq C_{K_3, \lambda} \frac{1}{x^{2\lambda+1}} \left( \log_+ \frac{\sqrt{xy}}{|x - y|} + 1 \right).$$

We point out that an earlier version of these three properties can be deduced from [4, p.711], see also [3, p.207]. See also the first version of these estimates in [15, p.87]. However, to obtain our main results in Theorems 1.1 and 1.3, we provide the current version of these kernel estimates in (2.8).

Moreover, from (iii) in Proposition 2.3 we can deduce the following inequality which will be used in the sequel.

**Remark 2.4.** There exist $K_3 \in (0, 1/2)$ small enough and a positive constant $C_{K_3, \lambda}$ such that for any $x, y \in \mathbb{R}_+$ with $0 < y/x - 1 < K_3$,

$$R_{\Delta_\lambda}(x, y) \geq \frac{1}{\pi} \frac{1}{x^\lambda y^\lambda} \frac{1}{y - x} - C_{K_3, \lambda} \frac{1}{x^{2\lambda+1}} \left( 1 + \log_+ \frac{\sqrt{xy}}{|x - y|} \right) \geq C_{K_3, \lambda} \frac{1}{x^\lambda y^\lambda} \frac{1}{y - x}.$$

**Proof of Proposition 2.3.** For any fixed $x, y \in \mathbb{R}_+$ with $x \neq y$, write $y = kx$. Then $k \in ((0, 1) \cup (1, \infty))$. By (2.1), we denote that $R_{\Delta_\lambda}(x, kx) = -\frac{2\lambda}{\pi} \frac{1}{x^{2\lambda+1}} I$, where

$$I = \int_0^\pi \frac{(1 - k \cos \theta)(\sin \theta)^{2\lambda-1}}{(1 + k^2 - 2k \cos \theta)^{\lambda+1}} \, d\theta.$$

We estimate $I$ by considering the following three cases.

Case (a) $k > 2$. In this case, we write

$$I = \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(1 + k^2 - 2k \cos \theta)^{\lambda+1}} \, d\theta$$

$$+ \left[ \left( \int_0^{\pi/2} - \int_0^{\pi/2} \frac{-k \cos \theta}{(1 + k^2 - 2k \cos \theta)^{\lambda+1}} (\sin \theta)^{2\lambda-1} \, d\theta \right) \right]$$

$$= \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(1 + k^2 - 2k \cos \theta)^{\lambda+1}} \, d\theta$$

$$- k \int_0^{\pi/2} \left[ \frac{1}{(1 + k^2 - 2k \cos \theta)^{\lambda+1}} - \frac{1}{(1 + k^2 + 2k \cos \theta)^{\lambda+1}} \right] \cos \theta (\sin \theta)^{2\lambda-1} \, d\theta$$

$$=: I_1 - I_2.$$

As $k > 2$, from (2.4), we deduce that

$$I_1 \leq \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(k - 1)^{2\lambda+2}} \, d\theta = \frac{\Gamma(\lambda) \sqrt{\pi}}{\Gamma(\lambda + 1/2)} \frac{1}{(k - 1)^{2\lambda+2}}.$$
On the other hand, by the Mean Value Theorem, there exists $t \in (-1, 1)$ depending on $\theta$ and $\lambda$, such that

$$I_2 = 4k^2(\lambda + 1) \int_0^{\pi/2} \frac{(\cos \theta)^2}{(1 + k^2 - 2tk \cos \theta)^{\lambda+2}} (\sin \theta)^{2\lambda-1} \, d\theta$$

$$\geq 4k^2(\lambda + 1) \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda-1} - (\sin \theta)^{2\lambda+1}}{(1 + k^2 + 2k)^{\lambda+2}} \, d\theta$$

$$= \frac{\Gamma(\lambda)\sqrt{\pi}}{\Gamma(\lambda + 1/2)} \frac{k^2}{(k + 1)^{2\lambda+4}} \frac{\lambda + 1}{\lambda + 1/2},$$

where the last equality follows from the second equality in (2.4). Let $a_1 \in \left(1, \frac{\lambda+1}{\lambda+\frac{1}{2}}\right)$. Observe that there exists $K_1$ such that when $k > K_1$,

$$\frac{k^2}{(k + 1)^{2\lambda+4}} \frac{\lambda + 1}{\lambda + 1/2} > a_1 \frac{1}{(k - 1)^{2\lambda+2}}.$$

Thus when $k > K_1$,

$$R_{\Delta_\lambda}(x, kx) = \frac{2\lambda}{\pi} \frac{1}{x^{2\lambda+1}} (I_2 - I_1) \geq \frac{2\lambda}{\pi} \frac{1}{x^{2\lambda+1}} \frac{\Gamma(\lambda)\sqrt{\pi}}{\Gamma(\lambda + 1/2)} \frac{a_1 - 1}{(k - 1)^{2\lambda+2}} \geq \frac{x}{(kx)^{2\lambda+2}}.$$

This implies (2.7).

Case (b) $k \in (0, 1)$. Similar to the argument in Case (a), we have that

$$I_1 \geq \frac{\Gamma(\lambda)\sqrt{\pi}}{\Gamma(\lambda + 1/2)} \frac{1}{(k + 1)^{2\lambda+2}}$$

and

$$I_2 \leq \frac{\Gamma(\lambda)\sqrt{\pi}}{\Gamma(\lambda + 1/2)} \frac{k^2}{(k + 1)^{2\lambda+4}} \frac{\lambda + 1}{\lambda + 1/2}.$$

Then for some fixed $a_2 \in (0, 1)$, there exists $K_2 \in (0, 1)$ such that when $0 < k < K_2$,

$$\frac{k^2}{(k - 1)^{2\lambda+4}} \frac{\lambda + 1}{\lambda + 1/2} < a_2 \frac{1}{(k + 1)^{2\lambda+2}}.$$

Thus when $0 < k < K_2$, there exists $C_{K_2, \lambda}$ such that

$$R_{\Delta_\lambda}(x, kx) = \frac{2\lambda}{\pi} \frac{1}{x^{2\lambda+1}} (I_2 - I_1) \leq \frac{2\lambda}{\pi} \frac{1}{x^{2\lambda+1}} \frac{\Gamma(\lambda)\sqrt{\pi}}{\Gamma(\lambda + 1/2)} \frac{a_2 - 1}{(k + 1)^{2\lambda+2}} \leq -C_{K_2, \lambda} \frac{x}{x^{2\lambda+1}}.$$

This implies (2.8).

Case (c) $k \in (1/2, 2)$. In this case, we write

$$I = \left(\int_0^{\pi/2} + \int_{\pi/2}^\pi\right) \frac{(1 - k) + k(1 - \cos \theta)}{(1 + k^2 - 2k \cos \theta)^{\lambda+1}} (\sin \theta)^{2\lambda-1} \, d\theta =: J_1 + J_2.$$

To estimate $J_1$, using Taylor’s Theorem and the Mean Value Theorem, we see that for $\theta \in [0, \pi/2]$, there exists $t_1, t_2, t_3 \in (0, 1)$ such that

$$(\sin \theta)^{2\lambda-1} = \theta^{2\lambda-1} - \frac{t_1(2\lambda - 1)}{6}\theta^{2\lambda+1},$$

(2.9)
From (2.9), (2.10) and (2.11), it follows that

\[1 - \cos \theta = \frac{\theta^2}{2} - \frac{t_2}{4!} \theta^4, \quad (2.10)\]

and

\[
\left[ (1 - k)^2 + k \theta^2 - \frac{k t_2}{12} \theta^4 \right]^{-\lambda - 1} = \left[ (1 - k)^2 + k \theta^2 \right]^{-\lambda - 1} + \frac{k t_2 (\lambda + 1)}{12} \theta^4 \left[ (1 - k)^2 + k \theta^2 - \frac{k t_2}{12} \theta^4 \right]^{-\lambda - 2}. \quad (2.11)
\]

From (2.9), (2.10) and (2.11), it follows that

\[
J_1 = \int_0^{\pi/2} \frac{(1 - k) (\frac{k}{2} \theta^2 - \frac{t_2 k}{12} \theta^4) (\theta^{2\lambda - 1} - \frac{t_1 (2\lambda - 1)}{6} \theta^{2\lambda + 1})}{(1 - k)^2 + k \theta^2 - \frac{k t_2}{12} \theta^4} \lambda + 1 \, d\theta
\]

\[= \int_0^{\pi/2} \frac{(1 - k) \theta^{2\lambda - 1}}{(1 - k)^2 + k \theta^2 + \lambda + 1} d\theta
\]

\[+ \int_0^{\pi/2} \frac{(\frac{k}{2} - \frac{2\lambda - 1}{6} (1 - k) t_1) \theta^{2\lambda + 1} - (\frac{2\lambda - 1}{12} t_1 k + \frac{t_2 k}{24} \theta^{2\lambda + 3} + \frac{2\lambda - 1}{144} t_1 t_2 k \theta^{2\lambda + 5}}{(1 - k)^2 + k \theta^2 + \lambda + 1} d\theta
\]

\[+ \frac{k}{12} (\lambda + 1) \int_0^{\pi/2} \frac{t_2 (1 - k) \theta^{2\lambda + 3} + \left( \frac{k}{2} - \frac{2\lambda - 1}{6} (1 - k) t_1 \right) \theta^{2\lambda + 5}}{(1 - k)^2 + k \theta^2 + \lambda + 1} d\theta
\]

\[= \int_0^{\pi/2} \frac{1}{(1 - k)^2 + k \theta^2 - \frac{k t_2}{12} \theta^4} \lambda + 2 d\theta
\]

\[= J_{11} + J_{12} + J_{13}.
\]

Observe that

\[
\int_0^{\infty} \frac{\beta^{2\lambda - 1}}{(1 + \beta^2)^{\lambda + 1}} d\beta = \frac{1}{2} B(\lambda, 1) = \frac{1}{2\lambda},
\]

where \(B(p, q)\) is the Beta function. Then we have that

\[
J_{11} = \frac{1 - k}{1 - k^{2\lambda + 2}} \int_0^{\pi/2} \frac{\theta^{2\lambda - 1}}{1 + (\sqrt{k} \theta^2)^{2\lambda + 1}} d\theta
\]

\[= \frac{1}{k \lambda} \left[ \frac{1}{2\lambda} - \int_0^{\infty} \frac{\beta^{2\lambda - 1}}{(1 + \beta^2)^{\lambda + 1}} d\beta \right].
\]

By this and the fact that

\[0 < \int_0^{\infty} \frac{\beta^{2\lambda - 1}}{(1 + \beta^2)^{\lambda + 1}} d\beta < \frac{2}{\pi^2} \frac{(k - 1)^2}{k},\]

we see that \(J_{11} - \frac{1}{2\lambda} \frac{1}{k^{2\lambda + 1}} \to 0, \ k \to 1.\)

Similarly, we have that

\[|J_{12}| \lesssim \int_0^{\pi/2} \frac{\theta^{2\lambda + 1} + \theta^{2\lambda + 3} + \theta^{2\lambda + 5}}{(1 - k)^2 + k \theta^2 + 1} d\theta
\]
Proposition 3.1, this BMO $H(1.1)$ is bounded on $R$ where $BMO(\cdot)$ and regularity properties, as proved in [4, Proposition 4.1].

Combining the estimates of $J_{11}, J_{12}, J_{13}$ and $J_2$, we finish the proof of Proposition 2.3.

We now recall the Hardy and BMO spaces associated with $S_\lambda$. Betancor et al. [3] introduced an equivalent definition of the Hardy spaces $H^1_{S_\lambda}(\mathbb{R}_+, dx)$ associated with $S_\lambda$ via the maximal function via Poisson semigroups, i.e.

$$H^1_{S_\lambda}(\mathbb{R}_+, dx) := \left\{ f \in L^1(\mathbb{R}_+, dx) : \sup_{t > 0} |e^{-t\sqrt{S_\lambda}}(f)| \in L^1(\mathbb{R}_+, dx) \right\}$$

with norm

$$\|f\|_{H^1_{S_\lambda}(\mathbb{R}_+, dx)} = \|f\|_{L^1(\mathbb{R}_+, dx)} + \sup_{t > 0} |e^{-t\sqrt{S_\lambda}}(f)|\|L^1(\mathbb{R}_+, dx)\|.$$  

Moreover, they proved that $H^1_{S_\lambda}(\mathbb{R}_+, dx)$ is equivalent to the Hardy space $H^1(\mathbb{R}_+, dx)$ (see Theorem 3.1 and Proposition 3.9 in [3]), where $H^1(\mathbb{R}_+, dx)$ is defined in [8] as follows

$$H^1(\mathbb{R}_+, dx) = \left\{ f \in L^1(\mathbb{R}_+, dx) : f_0 \in H^1(\mathbb{R}) \right\}$$

with the norm defined by $\|f\|_{H^1(\mathbb{R}_+, dx)} := \|f_0\|_{H^1(\mathbb{R})}$. Here $f_0(x) := f(x)$ if $x \in \mathbb{R}_+$, $f_0(x) := -f(-x)$ if $x \in \mathbb{R}_-$, which is also called the odd extension of $f$ on $\mathbb{R}_+$. We point out that the atoms in $H^1(\mathbb{R}_+, dx)$ may not have cancellation property. It follows from Chang et al. [8] that the dual space of $H^1(\mathbb{R}_+, dx)$ is $BMO_2(\mathbb{R}, dx)$. We further point out that as proved in [11, Proposition 3.1], this $BMO_2(\mathbb{R}, dx)$ is equivalent to $BMO_o(\mathbb{R}_+, dx)$, which is defined as

$$BMO_o(\mathbb{R}_+, dx) = \left\{ f \in L^1_{loc}(\mathbb{R}_+, dx) : f_0 \in BMO(\mathbb{R}) \right\}.$$  

where $BMO(\mathbb{R})$ is the standard BMO space on $\mathbb{R}$ introduced by John–Nirenberg. Thus, we have that

$$BMO_{S_\lambda}(\mathbb{R}_+, dx) = BMO_o(\mathbb{R}_+, dx).$$

We finally note that the Riesz transform $R_{S_\lambda}$ related to Bessel operator $S_\lambda$ (defined as in (1.1)) is bounded on $L^2(\mathbb{R}_+, dx)$, and the kernel $R_{S_\lambda}(x, y)$ of $R_{S_\lambda}$ satisfies the following size and regularity properties, as proved in [4, Proposition 4.1].

$$\lesssim \frac{1}{(1 - k)^{2\lambda + 2}} \int_0^{\pi/2} \frac{\theta^{2\lambda + 1}}{[1 + \left(\frac{\sqrt{k}}{|k - 1|}|\theta|^2\right)^{\lambda + 1}} d\theta + \int_0^{\pi/2} \frac{\theta^{2\lambda + 3} + \theta^{2\lambda + 5}}{[(1 - k)^2 + k^2]^\lambda + 1} d\theta$$

$$\lesssim \int_0^{\pi/2} \frac{\theta^{2\lambda + 1}}{(1 + \beta^2)^{\lambda + 1}} d\beta + \int_0^{\pi/2} (\theta + \theta^3) d\theta$$

$$\lesssim \int_0^1 \beta^{2\lambda + 1} d\beta + \int_1^{\pi/2} \beta^{-1} d\beta + 1$$

$$\lesssim \log_+ \frac{\sqrt{k}}{|k - 1|} + 1,$$

and

$$|J_{13}| \lesssim \int_0^{\pi/2} \frac{\theta^{2\lambda + 3} + \theta^{2\lambda + 5} + \theta^{2\lambda + 9} + \theta^{2\lambda + 9}}{[(1 - k)^2 + k^2/4]^{\lambda + 2}} d\theta$$

$$\lesssim \frac{1}{(1 - k)^{2\lambda + 4}} \int_0^{\pi/2} \frac{\theta^{2\lambda + 3}}{[1 + (\frac{\sqrt{k}}{|k - 1|})^2 |\theta|^2]^{\lambda + 2}} d\theta + 1 \lesssim \log_+ \frac{\sqrt{k}}{|k - 1|} + 1.$$
Proposition 2.5 ([4]). There exists \( C > 0 \) such that for every \( x, y \in \mathbb{R}_+ \) with \( x \neq y \),

\[
\begin{align*}
(i) \quad |R_{S_\lambda}(x, y)| & \leq \frac{C}{|x - y|}; \\
(ii) \quad \left| \frac{\partial}{\partial x} R_{S_\lambda}(x, y) \right| + \left| \frac{\partial}{\partial y} R_{S_\lambda}(x, y) \right| & \leq \frac{C}{|x - y|^2}.
\end{align*}
\]

3 Hardy space factorization and BMO space characterization in the setting of \( \Delta_\lambda \)

In this section we provide the details of the proof of Theorems 1.1 and 1.3, in the following structure: we first provide the proof of (1) in Theorem 1.3, which plays the key role for the proof of Theorem 1.1. Then the proof of (2) in Theorem 1.3 follows from Theorem 1.1.

Proof of (1) of Theorem 1.3. We first prove the upper bound, i.e., for \( b \in \text{BMO}(\mathbb{R}_+, d\mu_\lambda) \) and \( p \in (1, \infty) \), there exists a positive constant \( C \) such that for any \( f \in L^p(\mathbb{R}_+, d\mu_\lambda) \),

\[
\left\| \left[ b, R_{\Delta_\lambda} \right] f \right\|_{L^p(\mathbb{R}_+, d\mu_\lambda)} \leq C \left\| f \right\|_{L^p(\mathbb{R}_+, d\mu_\lambda)}.
\]

(3.1)

Note that (3.1) follows from [7, Theorem 2.5] since the Riesz transform \( R_{\Delta_\lambda} \) is a Calderón–Zygmund operator as indicated in Proposition 2.2.

We now prove Theorem 1.1 based on (1) of Theorem 1.3. To begin with, we now provide an auxiliary lemma for the Hardy space \( H^1(\mathbb{R}_+, d\mu_\lambda) \) associated with the Bessel operator, which plays an important role in the proof of our main result.

Lemma 3.1. Let \( f \) be a function satisfying the following estimates:

\[
\begin{align*}
(i) \quad \int_0^\infty f(x)x^{2\lambda} \, dx &= 0; \\
(ii) \quad \text{there exist intervals } I(x_1, r) \text{ and } I(x_2, r) \text{ for some } x_1, x_2, r \in \mathbb{R}_+ \text{ and positive constants } D_1, D_2 \text{ such that } & \quad |f(x)| \leq D_1 \chi_{I(x_1, r)}(x) + D_2 \chi_{I(x_2, r)}(x); \\
(iii) \quad |x_1 - x_2| & \geq 4r.
\end{align*}
\]

Then there exists a positive constant \( C \) independent of \( x_1, x_2, r, D_1, D_2 \), such that

\[
\left\| f \right\|_{H^1(\mathbb{R}_+, d\mu_\lambda)} \leq C \log_2 \left( \frac{|x_1 - x_2|}{r} \right) \left[ D_1 m_\lambda(x_1, r) + D_2 m_\lambda(x_2, r) \right].
\]

Proof. Assume that \( f := f_1 + f_2 \), where \( \text{supp} f_i \subset I(x_i, r) \) for \( i = 1, 2 \). We will show that \( f \) has the following \((1, \infty)_{\Delta_\lambda}\)-atomic decomposition

\[
f = \sum_{i=1}^2 \sum_{j=1}^{J_0+1} a_i^j a_i^j,
\]

(3.2)

where \( J_0 \) is the smallest integer larger than \( \log_2 \left( \frac{|x_1 - x_2|}{r} \right) \), for each \( j \), \( a_i^j \) is a \((1, \infty)_{\Delta_\lambda}\)-atom and \( a_i^j \) a real number satisfying that

\[
|a_i^j| \lesssim D_i m_\lambda(I(x_i, r)).
\]

(3.3)
To this end, we write
\[ f = \sum_{i=1}^{\frac{n}{2}} [f_i - \bar{\alpha}_i^1 \chi_{I(x_i, 2r)}] + \sum_{i=1}^{\frac{n}{2}} \bar{\alpha}_i^1 \chi_{I(x_i, 2r)} =: f_1^1 + f_2^1 + \sum_{i=1}^{\frac{n}{2}} \bar{\alpha}_i^1 \chi_{I(x_i, 2r)}, \]
where
\[ \bar{\alpha}_i^1 := \frac{1}{m(\lambda(I(x_i, 2r)))} \int_{I(x_i, r)} f(x) x^2 \lambda \, dx. \]
Let
\[ \alpha_i^1 := \|f_i^1\|_{L^{\infty}(\mathbb{R}_+, dm_\lambda)} m(\lambda(I(x_i, 2r))) \]
and \( \alpha_i^2 := f_i^1 / \alpha_i^1. \) Then we see that \( \alpha_i^1 \) is a \( (1, \infty)_{\Delta_\lambda} \)-atom supported on \( I(x_i, 2r) \) and \( \alpha_i^1 \)

\[ \sum \bar{\alpha}_i^1 \chi_{I(x_i, 2r)} \]

where for \( j \in \{1, 2, \ldots, J_0\} \),
\[ f = \sum_{i=1}^{J_0} \left[ \sum_{j=1}^{J_0} f_{ij} \right] + \sum_{i=1}^{J_0} \bar{\alpha}_i^{J_0} \chi_{I(x_i, 2^{J_0} r)} + \sum_{i=1}^{J_0} \sum_{j=1}^{J_0} \alpha_i^j a_i^j \] + \sum_{i=1}^{J_0} \bar{\alpha}_i^{J_0} \chi_{I(x_i, 2^{J_0} r)},

where for \( j \in \{2, 3, \ldots, J_0\} \),
\[ \bar{\alpha}_i^j := \frac{1}{m(\lambda(I(x_i, 2^j r)))} \int_{I(x_i, r)} f(x) x^2 \lambda \, dx, \]
\[ f_i^j := \bar{\alpha}_i^j \chi_{I(x_i, 2^{j-1} r) - \bar{\alpha}_i^j \chi_{I(x_i, 2^j r)},} \]
\[ \alpha_i^j := \|f_i^j\|_{L^{\infty}(\mathbb{R}_+, dm_\lambda)} m(\lambda(I(x_i, 2^j r))) \text{ and } a_i^j := f_i^j / \alpha_i^j. \]
Moreover, for each \( i \) and \( j \), \( a_i^j \) is a \( (1, \infty)_{\Delta_\lambda} \)-atom and \( |a_i^j| \leq D_i m(\lambda(I(x_i, r))). \)

For \( \sum_{i=1}^{2} \bar{\alpha}_i^{J_0} \chi_{I(x_i, 2^{J_0} r)}, \) we set
\[ \bar{\alpha}_i^{J_0} := \frac{1}{m(\lambda(I(x_i, 2^{J_0} + 1) r)))} \int_{I(x_i, r)} f(x) x^2 \lambda \, dx \]
\[ = - \frac{1}{m(\lambda(I(x_i, 2^{J_0} + 1) r)))} \int_{I(x_i, r)} f(x) x^2 \lambda \, dx. \]
Then
\[
\sum_{i=1}^{2} \alpha_i^{j_0} \chi_{I(x_i, 2^{j_0}r)} = \left[ \alpha_1^{j_0} \chi_{I(x_1, 2^{j_0}r)} - \alpha_1^{j_0} \chi_{I(\bar{x}_1 + x_2, 2^{j_0+1}r)} \right] + \left[ \alpha_2^{j_0} \chi_{I(\bar{x}_2 + x_2, 2^{j_0+1}r)} + \alpha_2^{j_0} \chi_{I(x_2, 2^{j_0}r)} \right]
\]
\[= \sum_{i=1}^{2} f_i^{j_0+1}. \]

For \(i = 1, 2\), let
\[
\alpha_i^{j_0+1} := \left\| f_i^{j_0+1} \right\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} m_\lambda \left( I \left( \frac{x_1 + x_2}{2}, 2^{j_0+1}r \right) \right) \quad \text{and} \quad a_i^{j_0+1} := f_i^{j_0+1} / \alpha_i^{j_0+1}.
\]

Then we see that \(a_i^{j_0+1}\) is a \((1, \infty)_{\Delta_\lambda}\)-atom and \(\alpha_i^{j_0+1}\) satisfies (3.3). Thus, we have (3.2) holds, which implies that \(f \in H^1(\mathbb{R}_+, dm_\lambda)\) and
\[
\left\| f \right\|_{H^1(\mathbb{R}_+, dm_\lambda)} \leq 2 \sum_{i=1}^{2} \sum_{j=1}^{j_0+1} \left| \alpha_i^j \right| \lesssim \log \frac{|x_1 - x_2|}{r} \sum_{i=1}^{2} D_i m_\lambda(I(x_i, r)).
\]

This finishes the proof of Lemma 3.1.

\[ \square \]

**Remark 3.2.** From the proof of the Lemma 3.1, we see that this result holds for general Hardy space \(H^1(X, d, \mu)\) on the spaces of homogeneous type \((X, d, \mu)\) in the sense of Coifman and Weiss [10].

Next we provide the following estimate of the bilinear operator \(\Pi\), which is defined in (1.4).

**Proposition 3.3.** Let \(p \in (1, \infty)\). There exists a positive constant \(C\) such that for any \(g \in L^p(\mathbb{R}_+, dm_\lambda)\) and \(h \in L^{p'}(\mathbb{R}_+, dm_\lambda)\),
\[
\left\| \Pi(g, h) \right\|_{H^1(\mathbb{R}_+, dm_\lambda)} \leq C \left\| g \right\|_{L^p(\mathbb{R}_+, dm_\lambda)} \left\| h \right\|_{L^{p'}(\mathbb{R}_+, dm_\lambda)}.
\]

**Proof.** Since \(R_{\Delta_\lambda}\) and \(\bar{R}_{\Delta_\lambda}\) are both bounded on \(L^r(\mathbb{R}_+, dm_\lambda)\) for any \(r \in (1, \infty)\), we see that \(\Pi(g, h) \in L^1(\mathbb{R}_+, dm_\lambda)\) for any \(g \in L^p(\mathbb{R}_+, dm_\lambda)\) and \(h \in L^{p'}(\mathbb{R}_+, dm_\lambda)\) and
\[
\int_0^\infty \Pi(g, h)(x)x^{2\lambda} dx = 0.
\]

Moreover, from (1) of Theorem 1.3, it follows that for every \(f \in BMO(\mathbb{R}_+, dm_\lambda)\), \(g \in L^p(\mathbb{R}_+, dm_\lambda)\) and \(h \in L^{p'}(\mathbb{R}_+, dm_\lambda)\),
\[
\left| \int_0^\infty f(x) \Pi(g, h)(x)x^{2\lambda} dx \right| = \left| \int_0^\infty g(x)[f, R_{\Delta_\lambda}]h(x)x^{2\lambda} dx \right| \lesssim \left\| h \right\|_{L^{p'}(\mathbb{R}_+, dm_\lambda)} \left\| g \right\|_{L^p(\mathbb{R}_+, dm_\lambda)} \left\| f \right\|_{BMO(\mathbb{R}_+, dm_\lambda)}.
\]

Therefore, the proof of Proposition 3.3 is completed. \[ \square \]

The following proposition will lead to an iterative argument to prove the lower bound appearing in Theorem 1.1.
**Proposition 3.4.** Let $p \in (1, \infty)$. For every $\epsilon > 0$, there exist positive constants $M$ and $C$ such that for every $(1, \infty)_{\lambda}$-atom $a$, there exist $g \in L^p(\mathbb{R}_+, dm_\lambda)$ and $h \in L^{p'}(\mathbb{R}_+, dm_\lambda)$ satisfying that
\[
\|a - \Pi(g, h)\|_{H^1(\mathbb{R}_+, dm_\lambda)} < \epsilon
\]
and
\[
\|g\|_{L^p(\mathbb{R}_+, dm_\lambda)} \|h\|_{L^{p'}(\mathbb{R}_+, dm_\lambda)} \leq CM^{2\lambda+1}.
\]

*Proof.* Assume that $a$ is a $(1, \infty)_{\lambda}$-atom with $\text{supp } a \subset I(x_0, r)$. Observe that if $r > x_0$, then $I(x_0, r) = (x_0 - r, x_0 + r) \cap \mathbb{R}_+ = I(x_0 - \frac{r}{2}, x_0 + \frac{r}{2})$. Therefore, without loss of generality, we may assume that $r \leq x_0$. Let $K_2$ and $K_3$ be the constants appeared in (ii) of Proposition 2.3 and Remark 2.4 respectively, and $K_0 > \max\{\frac{1}{K_2}, \frac{1}{K_3}\}$ large enough. For any $\epsilon > 0$, let $M$ be a positive constant large enough such that $M \geq 100K_0$ and $\frac{\log_2 M}{M} < \epsilon$.

We now consider the following two cases.

Case (a): $x_0 \leq 2Mr$. In this case, let $x_0 := x_0 + 2MK_0r$. Then
\[
(1 + K_0)x_0 \leq y_0 \leq (1 + 2MK_0)x_0.
\]
Define
\[
g(x) := \chi_{I(y_0, r)}(x) \quad \text{and} \quad h(x) := \frac{a(x)}{R_{\lambda}g(x_0)}.
\]
By the fact that $y/x_0 > K_2^{-1}$ for any $y \in I(y_0, r)$ and Proposition 2.3 ii), we see that
\[
\left| \widetilde{R}_{\lambda} \gamma g(x_0) \right| = \left| \int_{y_0 - r}^{y_0 + r} R_{\lambda}(y, x_0)y^{2\lambda} dy \right| \geq \int_{y_0 - r}^{y_0 + r} \frac{1}{y} dy \sim \frac{r}{y_0} \sim \frac{1}{M}.
\]
Moreover, from the definitions of $g$ and $h$, it follows that
\[
\|g\|_{L^p(\mathbb{R}_+, dm_\lambda)} \|h\|_{L^{p'}(\mathbb{R}_+, dm_\lambda)} \leq \frac{1}{|R_{\lambda}g(x_0)|} \left[ m_\lambda(I(y_0, r)) \right]^{\frac{1}{p}} \left[ m_\lambda(I(x_0, r)) \right]^{-\frac{1}{p}} \lesssim M \left( \frac{y_0^{2\lambda}r}{2} \right)^{1/p} \left( \frac{x_0^{2\lambda}r}{2} \right)^{-1/p} \lesssim M^{2\lambda+1}.
\]

By the definition of the operator $\Pi$ as in (1.4), we write
\[
a(x) - \Pi(g, h)(x) = a(x) \frac{\widetilde{R}_{\lambda}g(x_0)}{R_{\lambda}g(x_0)} - g(x)R_{\lambda}h(x) =: W_1(x) + W_2(x).
\]
Then it is obvious that $\text{supp } W_1 \subset I(x_0, r)$ and $\text{supp } W_2 \subset I(y_0, r)$. Moreover, let
\[
C_1 := \frac{1}{m_\lambda(I(x_0, r))} \frac{m_\lambda(I(y_0, r))}{m_\lambda(I(x_0, |x_0 - y_0|))} \quad \text{and} \quad C_2 := \frac{1}{m_\lambda(I(x_0, |x_0 - y_0|))}.
\]
From the cancellation property $\int_0^\infty a(y)y^{2\lambda} dy = 0$, the Hölder’s regularity of the Riesz kernel $R_{\lambda}(x, y)$ in (2.3), and the fact that $|y - x| \sim |x_0 - y_0|$ for $y \in I(x_0, r)$ and $x \in I(y_0, r)$, we have
\[
|W_2(x)| = \chi_{I(y_0, r)}(x) |R_{\lambda}h(x)| \lesssim M \chi_{I(y_0, r)}(x) \left| \int_{I(x_0, r)} [R_{\lambda}(x, y) - R_{\lambda}(x, x_0)] a(y)y^{2\lambda} dy \right|
\]
\[ \lesssim M_{I(y_0, r)}(x) \frac{r}{|x_0 - y_0|} \frac{1}{m_\lambda(I(x_0, |x_0 - y_0|))} \]
\[ \lesssim C_2 M_{I(y_0, r)}(x). \]

On the other hand, using (2.3) and \(|y - x_0| \sim |x_0 - y_0|\) for \(y \in I(y_0, r)\),
\[
|W_1(x)| \leq M_{I(x_0, r)}(x) \|a\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} \int_{I(y_0, r)} \frac{r}{|x_0 - y|} \frac{1}{m_\lambda(I(x_0, |x_0 - y|))} y^{2\lambda} dy
\[
\lesssim M_{I(x_0, r)}(x) \frac{1}{m_\lambda(I(x_0, r))} \frac{1}{|x_0 - y_0|} \lesssim C_1 M_{I(x_0, r)}(x).
\]

Moreover, note that
\[
\int_0^\infty [a(x) - \Pi(g, h)(x)] x^{2\lambda} dx = 0.
\]

Hence, the function \(f(x) = a(x) - \Pi(g, h)(x)\) satisfies all conditions in Lemma 3.1. Now from Lemma 3.1, we have that
\[
\|a - \Pi(g, h)\|_{H^1(\mathbb{R}_+, dm_\lambda)} \lesssim \log_2 \left( \frac{|x_0 - y_0|}{r} \right) \left[ C_1 m_\lambda(I(x_0, r)) + C_2 m_\lambda(I(y_0, r)) \right]
\[
\lesssim \log_2 \left( \frac{|x_0 - y_0|}{r} \right) \frac{r}{|x_0 - y_0|}
\]
\[
\lesssim \frac{\log_2 M}{M} < \epsilon. \quad (3.5)
\]

Case (b): \(x_0 > 2Mr\). In this case, let \(y_0 := x_0 - Mr/K_0\). Then \(\frac{2K_0 - 1}{2K_0} x_0 < y_0 < x_0\). Let \(g\) and \(h\) be as in Case (a). For every \(y \in I(y_0, r)\), from the facts that \(K_0 > \max\{\frac{1}{K_2}, \frac{1}{K_3}\} + 1\) and \(M \geq 100K_0\), we have
\[
0 < \frac{x_0}{y_0} - 1 < \tilde{K}_3.
\]

By Remark 2.4 and \(y \sim y_0 \sim x_0\) for any \(y \in I(y_0, r)\), we conclude that
\[
\left| \tilde{R}_{\lambda, g}(x_0) \right| \gtrsim \left| \int_{y_0 - r}^{y_0 + r} \frac{1}{x_0 y_0} \frac{1}{x_0 - y} y^{2\lambda} dy \right| \sim \left| \int_{y_0 - r}^{y_0 + r} \frac{1}{x_0 - y_0} y^{2\lambda} dy \right| \sim \frac{1}{M}. \quad (3.6)
\]

Moreover,
\[
\|g\|_{L^p(\mathbb{R}_+, dm_\lambda)} \|h\|_{L^{p'}(\mathbb{R}_+, dm_\lambda)} \lesssim M \left( \frac{m_\lambda(I(y_0, r))}{m_\lambda(I(x_0, r))} \right)^{1/p} \sim M.
\]

Let \(W_1, W_2, C_1\) and \(C_2\) be as in Case (i). Then similarly, we have that
\[
|W_2(x)| \lesssim M_{I(y_0, r)}(x) \int_{I(x_0, r)} |R_{\lambda, g}(x, y) - R_{\lambda, g}(x, x_0)| a(y) y^{2\lambda} dy
\[
\lesssim M_{I(y_0, r)}(x) \frac{r}{|x_0 - y_0|} \frac{1}{m_\lambda(I(y_0, |x_0 - y_0|))} \sim C_2 M_{I(y_0, r)}(x),
\]

and
\[
|W_1(x)| \leq M_{I(x_0, r)}(x) \|a\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} \int_{I(y_0, r)} \frac{r}{|x_0 - y|} \frac{1}{m_\lambda(I(x_0, |x_0 - y|))} y^{2\lambda} dy
\]
proof of Proposition 3.4. Then (3.5) follows from Lemma 3.1 in this case, which together with Case (a) completes the proof of Proposition 3.4.

We now use the above proposition in an iterative fashion to deduce the first main result Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 3.3, we have that for any \( g \in L^p(\mathbb{R}^+, dm_\lambda) \) and \( h \in L^{p'}(\mathbb{R}^+, dm_\lambda) \),

\[
\|\Pi(g, h)\|_{H^1(\mathbb{R}^+, dm_\lambda)} \lesssim \|g\|_{L^p(\mathbb{R}^+, dm_\lambda)} \|h\|_{L^{p'}(\mathbb{R}^+, dm_\lambda)}.
\]

From this, for any \( f \in H^1(\mathbb{R}^+, dm_\lambda) \) having the representation (1.3) with

\[
\sum_{k=1}^\infty \sum_{j=1}^\infty \|\alpha_j^k\|_p \left\|g_j^k\right\|_{L^p(\mathbb{R}^+, dm_\lambda)} \left\|h_j^k\right\|_{L^{p'}(\mathbb{R}^+, dm_\lambda)} < \infty,
\]

it follows that

\[
\|f\|_{H^1(\mathbb{R}^+, dm_\lambda)} \lesssim \inf \left\{ \sum_{k=1}^\infty \sum_{j=1}^\infty \|\alpha_j^k\|_p \left\|g_j^k\right\|_{L^p(\mathbb{R}^+, dm_\lambda)} \left\|h_j^k\right\|_{L^{p'}(\mathbb{R}^+, dm_\lambda)} : f = \sum_{k=1}^\infty \sum_{j=1}^\infty \alpha_j^k \Pi(g_j^k, h_j^k) \right\}.
\]

To see the converse, let \( f \in H^1(\mathbb{R}^+, dm_\lambda) \). We will show that \( f \) has a representation as in (1.3) with

\[
\inf \left\{ \sum_{k=1}^\infty \sum_{j=1}^\infty \|\alpha_j^k\|_p \left\|g_j^k\right\|_{L^p(\mathbb{R}^+, dm_\lambda)} \left\|h_j^k\right\|_{L^{p'}(\mathbb{R}^+, dm_\lambda)} \right\} \lesssim \|f\|_{H^1(\mathbb{R}^+, dm_\lambda)}.
\]

(3.7)

To this end, assume that \( f \) has the following atomic representation \( f = \sum_{j=1}^\infty \alpha_j^1 a_j^1 \) with \( \sum_{j=1}^\infty |\alpha_j^1| \leq C_3 \|f\|_{H^1(\mathbb{R}^+, dm_\lambda)} \) for certain constant \( C_3 \in (1, \infty) \). We show that for any \( \epsilon \in (0, 1/C_3) \) and any \( K \in \mathbb{N} \), \( f \) has the following representation

\[
f = \sum_{k=1}^K \sum_{j=1}^\infty \alpha_j^k \Pi(g_j^k, h_j^k) + E_K,
\]

(3.8)

where, \( M \) is as in Proposition 3.4, \( g_j^k \in L^p(\mathbb{R}^+, dm_\lambda) \), \( h_j^k \in L^{p'}(\mathbb{R}^+, dm_\lambda) \) for each \( k \) and \( j \), \( \{\alpha_j^k\}_j \in \ell^1 \) for each \( k \) and \( E_K \in H^1(\mathbb{R}^+, dm_\lambda) \) satisfying that

\[
\left\|g_j^k\right\|_{L^p(\mathbb{R}^+, dm_\lambda)} \left\|h_j^k\right\|_{L^{p'}(\mathbb{R}^+, dm_\lambda)} \lesssim M^{\frac{2k}{p}+1},
\]

(3.9)

\[
\sum_{j=1}^\infty |\alpha_j^k| \leq \epsilon^{k-1} C_3^k \|f\|_{H^1(\mathbb{R}^+, dm_\lambda)}
\]

(3.10)
and

\[ \| E_K \|_{H^1(\mathbb{R}^+, dm_\lambda)} \leq (\epsilon C_3)^K \| f \|_{H^1(\mathbb{R}^+, dm_\lambda)}. \]  

(3.11)

In fact, for given \( \epsilon \) and \( a_j \), by Proposition 3.4, there exist \( g_j^1 \in L^p(\mathbb{R}^+, dm_\lambda) \) and \( h_j^1 \in L^{p'}(\mathbb{R}^+, dm_\lambda) \) with

\[ \| g_j^1 \|_{L^p(\mathbb{R}^+, dm_\lambda)} \| h_j^1 \|_{L^{p'}(\mathbb{R}^+, dm_\lambda)} \lesssim M^{2\frac{1}{p} + 1} \]

and

\[ \| a_j^1 - \Pi (g_j^1, h_j^1) \|_{H^1(\mathbb{R}^+, dm_\lambda)} < \epsilon. \]

Now we write

\[ f = \sum_{j=1}^{\infty} a_j^1 a_j^2 = \sum_{j=1}^{\infty} a_j^1 \Pi (g_j^1, h_j^1) + \sum_{j=1}^{\infty} a_j^1 [a_j^1 - \Pi (g_j^1, h_j^1)] =: M_1 + E_1. \]

Observe that

\[ \| E_1 \|_{H^1(\mathbb{R}^+, dm_\lambda)} \leq \sum_{j=1}^{\infty} |a_j^1| \| a_j^1 - \Pi (g_j^1, h_j^1) \|_{H^1(\mathbb{R}^+, dm_\lambda)} \leq \epsilon C_3 \| f \|_{H^1(\mathbb{R}^+, dm_\lambda)}. \]

Since \( E_1 \in H^1(\mathbb{R}^+, dm_\lambda) \), for the given \( C_3 \), there exist a sequence of atoms \( \{a_j^2\}_j \) and numbers \( \{\alpha_j^2\}_j \) such that \( E_2 = \sum_{j=1}^{\infty} \alpha_j^2 a_j^2 \) and

\[ \sum_{j=1}^{\infty} |\alpha_j^2| \leq C_3 \| E_1 \|_{H^1(\mathbb{R}^+, dm_\lambda)} \leq \epsilon C_3 \| f \|_{H^1(\mathbb{R}^+, dm_\lambda)}. \]

Another application of Proposition 3.4 implies that there exist functions \( g_j^2 \in L^p(\mathbb{R}^+, dm_\lambda) \) and \( h_j^2 \in L^{p'}(\mathbb{R}^+, dm_\lambda) \) with

\[ \| g_j^2 \|_{L^p(\mathbb{R}^+, dm_\lambda)} \| h_j^2 \|_{L^{p'}(\mathbb{R}^+, dm_\lambda)} \lesssim M^{\frac{2}{p} + 1} \text{ and } \| a_j^2 - \Pi (g_j^2, h_j^2) \|_{H^1(\mathbb{R}^+, dm_\lambda)} < \epsilon. \]

Thus, we have

\[ E_1 = \sum_{j=1}^{\infty} a_j^2 a_j^2 = \sum_{j=1}^{\infty} a_j^2 \Pi (g_j^2, h_j^2) + \sum_{j=1}^{\infty} a_j^2 [a_j^2 - \Pi (g_j^2, h_j^2)] =: M_2 + E_2. \]

Moreover,

\[ \| E_2 \|_{H^1(\mathbb{R}^+, dm_\lambda)} \leq \sum_{j=1}^{\infty} |\alpha_j^2| \| a_j^2 - \Pi (g_j^2, h_j^2) \|_{H^1(\mathbb{R}^+, dm_\lambda)} \leq \epsilon \sum_{j=1}^{\infty} |\alpha_j^2| \leq (\epsilon C_3)^2 \| f \|_{H^1(\mathbb{R}^+, dm_\lambda)}. \]

Now we conclude that

\[ f = \sum_{j=1}^{\infty} \alpha_j^1 a_j^1 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \alpha_j^k \Pi (g_j^k, h_j^k) + E_2, \]

Continuing in this way, we deduce that for any \( K \in \mathbb{N} \), \( f \) has the representation (3.8) satisfying (3.9), (3.10) and (3.11). Thus letting \( K \to \infty \), we see that (1.3) holds. Moreover, since \( \epsilon C_3 < 1 \), we have that

\[ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |\alpha_j^k| \leq \sum_{k=1}^{\infty} \epsilon^{-1} (\epsilon C_3)^k \| f \|_{H^1(\mathbb{R}^+, dm_\lambda)} \lesssim \| f \|_{H^1(\mathbb{R}^+, dm_\lambda)}, \]

which implies (3.7) and hence, completes the proof of Theorem 1.1.
Next we turn to the proof of (2) of Theorem 1.3.

**Proof of (2) of Theorem 1.3.** Assume that $[b, R_{\Delta_{\lambda}}]$ is bounded on $L^{p'}(\mathbb{R}_+, dm_{\lambda})$ for a given $p' \in (1, \infty)$ and

$$f \in (H^1(\mathbb{R}_+, dm_{\lambda}) \cap L^\infty_c(\mathbb{R}_+, dm_{\lambda})),$$

where $L^\infty_c(\mathbb{R}_+, dm_{\lambda})$ is the subspace of $L^\infty(\mathbb{R}_+, dm_{\lambda})$ consisting of functions with compact supports in $\mathbb{R}_+$. From Theorem 1.1, we deduce that

$$\langle b, f \rangle = \sum_{k=1}^\infty \sum_{j=1}^\infty \alpha_j^k \langle b, \Pi \left(g_j^k, h_j^k\right) \rangle = \sum_{k=1}^\infty \sum_{j=1}^\infty \alpha_j^k \langle g_j^k, [b, R_{\Delta_{\lambda}}]h_j^k \rangle.$$ 

This implies that

$$\langle b, f \rangle \leq \sum_{k=1}^\infty \sum_{j=1}^\infty |\alpha_j^k| \left\| g_j^k \right\|_{L^p(\mathbb{R}_+, dm_{\lambda})} \left\| [b, R_{\Delta_{\lambda}}]h_j^k \right\|_{L^{p'}(\mathbb{R}_+, dm_{\lambda})}$$

$$\leq \left\| [b, R_{\Delta_{\lambda}}] \right\|_{L^{p'}(\mathbb{R}_+, dm_{\lambda}) \to L^{p'}(\mathbb{R}_+, dm_{\lambda})} \sum_{k=1}^\infty \sum_{j=1}^\infty |\alpha_j^k| \left\| g_j^k \right\|_{L^p(\mathbb{R}_+, dm_{\lambda})} \left\| h_j^k \right\|_{L^{p'}(\mathbb{R}_+, dm_{\lambda})}$$

$$\leq \left\| [b, R_{\Delta_{\lambda}}] \right\|_{L^{p'}(\mathbb{R}_+, dm_{\lambda}) \to L^{p'}(\mathbb{R}_+, dm_{\lambda})} \left\| f \right\|_{H^1(\mathbb{R}_+, dm_{\lambda})}.$$ 

Then by the fact that $H^1(\mathbb{R}_+, dm_{\lambda}) \cap L^\infty_c(\mathbb{R}_+, dm_{\lambda})$ is dense in $H^1(\mathbb{R}_+, dm_{\lambda})$ and the duality between $H^1(\mathbb{R}_+, dm_{\lambda})$ and BMO$(\mathbb{R}_+, dm_{\lambda})$, we finish the proof of Theorem 1.3. \qed

## 4 Proof of Theorems 1.4 and Corollary 1.5

In this section, we give the proofs of Theorem 1.4 and Corollary 1.5.

**Proof of Theorem 1.4.** From (i) and (ii) in Proposition 2.5, we get that $R_{S_{\lambda}}$ falls into the scope of classical Calderón–Zygmund operators (see for example [17]). Hence, by the result of Coifman et al. [9], we have that: For $1 < p < \infty$, if $b$ lies in the classical BMO space BMO$(\mathbb{R}_+, dx)$ in the sense of John–Nirenberg, then the commutator $[b, R_{S_{\lambda}}]$ is bounded on $L^p(\mathbb{R}_+, dx)$ with the operator norm

$$\left\| [b, R_{S_{\lambda}}] \right\|_{L^p(\mathbb{R}_+, dx) \to L^p(\mathbb{R}_+, dx)} \leq C \left\| b \right\|_{\text{BMO}(\mathbb{R}_+, dx)}, \quad (4.1)$$

However, the BMO space associated with the Bessel operator $S_{\lambda}$ is the BMO$_o(\mathbb{R}_+, dx)$ as defined in (2.12), which is the dual of the Hardy space $H^1_{S_{\lambda}}(\mathbb{R}_+, dx)$ associated with $S_{\lambda}$. As indicated in [11],

$$\text{BMO}_o(\mathbb{R}_+, dx) \subsetneq \text{BMO}(\mathbb{R}_+, dx).$$

Hence, we now choose

$$b_0(x) = \log(x), \quad x > 0.$$ 

Then it is obvious that this function $b_0 \in \text{BMO}(\mathbb{R}_+, dx)$ but $b_0 \not\in \text{BMO}_o(\mathbb{R}_+, dx)$. Hence

$$\left\| [b_0, R_{S_{\lambda}}] \right\|_{L^p(\mathbb{R}_+, dx) \to L^p(\mathbb{R}_+, dx)} \leq C \left\| b_0 \right\|_{\text{BMO}(\mathbb{R}_+, dx)}$$

but

$$\left\| b_0 \right\|_{\text{BMO}_o(\mathbb{R}_+, dx)} = \infty.$$ 

\qed
Proof of Corollary 1.5. Suppose $H^1_{S_{\lambda}}(\mathbb{R}_+, dx)$ has a weak factorization in the following form: for certain $p \in (1, \infty)$ and $f \in H^1_{S_{\lambda}}(\mathbb{R}_+, dx)$, there exist numbers $\{\alpha^k_j\}_{k,j}$, functions $\{g^k_j\}_{k,j} \subset L^p(\mathbb{R}_+, dx)$ and $\{h^k_j\}_{k,j} \subset L^{p'}(\mathbb{R}_+, dx)$ such that

$$f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha^k_j \Pi_{S_{\lambda}} \left( g^k_j, h^k_j \right)$$

in $H^1_{S_{\lambda}}(\mathbb{R}_+, dx)$, where the operator $\Pi_{S_{\lambda}}$ is defined as follows: for $g \in L^p(\mathbb{R}_+, dx)$ and $h \in L^{p'}(\mathbb{R}_+, dx)$,

$$\Pi_{S_{\lambda}}(g,h) := gR_{S_{\lambda}}h - h\widetilde{R}_{S_{\lambda}}g.$$  \hfill (4.2)

We let $b_0(x) = \log(x)$, $x > 0$. Then for the index $p$ above, from the inequality (4.1) we have

$$\| [b_0, R_{S_{\lambda}}] \|_{L^p(\mathbb{R}_+, dx) \rightarrow L^p(\mathbb{R}_+, dx)} \leq C \| b_0 \|_{\text{BMO}(\mathbb{R}_+, dx)} < \infty.$$

Now following the same proof of (2) in Theorem 1.3 and the duality of $H^1_{S_{\lambda}}(\mathbb{R}_+, dx)$ with $\text{BMO}_o(\mathbb{R}_+, dx)$, we obtain directly that

$$\|b_0\|_{\text{BMO}_o(\mathbb{R}_+, dx)} \leq C \| [b_0, R_{S_{\lambda}}] \|_{L^p(\mathbb{R}_+, dx) \rightarrow L^p(\mathbb{R}_+, dx)},$$

which contradicts with the fact that

$$\|b_0\|_{\text{BMO}_o(\mathbb{R}_+, dx)} = \infty.$$

\[\square\]

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