CLASSIFICATION OF POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH HARDY TERM

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Abstract. We study the elliptic equation \( \Delta u + \mu/|x|^2 + K(|x|)u^p = 0 \) in \( \mathbb{R}^n \setminus \{0\} \), where \( n \geq 1 \) and \( p > 1 \). In particular, when \( K(|x|) = |x|^l \), a classification of radially symmetric solutions is presented in terms of \( \mu \) and \( l \). Moreover, we explain the separation structure for the equation, and study the stability of positive radial solutions as steady states.

1. Introduction. We study the elliptic equation with Hardy term

\[ \Delta u + \frac{\mu}{|x|^2} u + K(|x|)u^p = 0, \quad (1.1) \]

where \( n \geq 1 \), \( p > 1 \), and \( K \) is a non-negative radially symmetric continuous function on \( \mathbb{R}^n \setminus \{0\} \). Here, \( \mu \) is a real parameter. In order to have a positive local solution near 0, we need additional conditions on \( \mu \) and \( K \). When \( n > 2 \) and \( \mu = 0 \), the equation

\[ \Delta u + K(|x|)u^p = 0 \quad (1.2) \]

with \( u(0) = \alpha > 0 \), has a positive local radial solution \( u \in C^2(0, \varepsilon) \cap C[0, \varepsilon) \) for small \( \varepsilon > 0 \) under the condition

\[ \int_0^r K(r) \, dr < \infty \quad (1.3) \]

where \( r = |x| \). By \( u_\alpha(r) \) we denote the unique local solution with \( u_\alpha(0) = \alpha > 0 \). Considering the Kelvin transform \( v \) of \( u \),

\[ v(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}), \]

we see that \( v \) satisfies

\[ \Delta v + \frac{1}{|x|^{n+2-p(n-2)}} K(\frac{x}{|x|^2}) v^p = 0 \]

and (1.3) corresponds to

\[ \int_0^\infty r^{n-1-(n-2)p} K(r) \, dr < \infty. \quad (1.4) \]

When \( K = |x|^l \) with \( l > -2 \), we observe the dual structure for \( p \) between \( \frac{n+2+2l}{n-2} \) and \( \frac{n+l}{n-2} \) in the following sense that the solution structure for \( p \geq \frac{n+2+2l}{n-2} \) reflects the solution structure for \( \frac{n+l'}{n-2} < p \leq \frac{n+2+2l'}{n-2} \) for \( l' = p(n-2) - (n+2+l) \).

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The second transformation
\[ u(|x|) = \left( \frac{n-2}{n-2-2\sigma} \right)^{\frac{1}{2-\sigma}} |y|^\sigma w(|y|), \quad |x| = |y|^{\frac{n-2-2\sigma}{n-2}}, \]
for \( \sigma < \frac{n-2}{2} \), leads to the equation
\[
\Delta w + \frac{\mu}{|x|^2} w + |y|^{(p-\frac{n+2}{n-2})\sigma} K(|y|^{\frac{n-2-2\sigma}{n-2}}) w^p = 0, \quad (1.5)
\]
where \( \mu = \sigma(n-2-\sigma) \). Setting \( \tilde{K}(|y|) = |y|^{(p-\frac{n+2}{n-2})\sigma} K(|y|^{\frac{n-2-2\sigma}{n-2}}) \), we observe that (1.3) corresponds to
\[
\int_0^{s^{1-(p-1)\sigma}} \tilde{K}(s) ds < \infty. \quad (1.6)
\]
Under condition (1.6), we see that (1.5) possesses a local radial solution \( w \) such that \( w(r) = O(r^{-\sigma}) \) at 0.

This paper is organized as follows. Classification of radial solutions for Lane-Emden equation is reviewed in Section 2. In Section 3 we study the existence of regular and singular solutions of (1.1). Moreover, we explain the separation structure for (1.1) and study the stability. In Section 4 we provide the graphs of the regions where separation phenomena occur.

2. Lane-Emden equation. For last several decades, the Lane-Emden equation
\[
\Delta u + |x|^lu^p = 0, \quad (2.1)
\]
where \( l > -2 \), has been studied extensively. We refer the readers to [2, 5, 6, 8, 9, 11, 13, 14, 15]. We review some well-known results for (2.1). For \( 1 < p \leq \frac{n+l}{n-2} \), (2.1) has no positive supersolution in any exterior domain while the other case has a positive supersolution. On the other hand, if \( \frac{n+l}{n-2} < p < \frac{n+2+2l}{n-2} \), then (2.1) has no positive entire solution for \( -2 < l \leq 0 \) and no positive entire radial solution for \( l > 0 \). When \( p \geq \frac{n+2+2l}{n-2} \), for every \( \alpha > 0 \), there exists a entire radial solution \( u_\alpha \) with \( u_\alpha(0) = \alpha \), and (2.1) has the scale invariance \( u_\alpha(r) = \alpha u_1(\alpha^{1/m}r) \), where \( m = \frac{n+2+2l}{p-1} \). When \( p = \frac{n+2+2l}{n-2} \), all entire radial solutions are
\[
u_e(x) = \frac{[(n+l)(n-2)\epsilon(\alpha-2)/2(2+l)]^{2/(n-2)}}{(\epsilon + |x|^{2(l+1)})^{(n-2)/(2(l+1))}}
\]
for \( \epsilon > 0 \). If \( -2 < l \leq 0 \), every positive solution is radially symmetric (up to translation for \( l = 0 \)). When \( p > \frac{n+2+2l}{n-2} \), every local solution \( u_\alpha \) remains positive entirely and \( r^m u_\alpha(r) \rightarrow L \) as \( r \rightarrow \infty \) where \( L := [m(n-2-m)]^{1/2} \). Furthermore, if \( p \geq p_c(n,l) := \frac{(n-2)^2-2(l+2)(n+1)+2l+2}{(n-2)(n-10-4l)} \sqrt{(n+l)^2-(n-2)^2} \) for \( n > 10 + 4l \), any two positive entire radial solution do not intersect each other. This property is closely related with the stability of solutions.

By the Kelvin transform, we observe that there exist infinitely many singular solutions for \( \frac{n+l}{n-2} < p < \frac{n+2+2l}{n-2} \). When \( p > \frac{n+2+2l}{n-2} \), \( r^{-m}L \) is a unique singular solution, which is defined for \( p > \frac{n+l}{n-2} \). When \( p = \frac{n+2+2l}{n-2} \), there are singular solutions with different asymptotic behavior. More precisely, for each \( 0 < d_1 < L \), there exists a singular solution \( u_\alpha(r) \) such that \( 0 < d_1 := \min \left\{ r^m u_\alpha(r) < L < d_2 := \max r^m u_\alpha(r) \right\} \left[ \frac{n-2}{(n+l)(n-2)} \right]^{1/(p+1)} \) where \( L^{p-1}(d^2_2 - d^2_1)/2 = (d^{p+1}_2 - d^{p+1}_1)/(p+1) \), and \( r^m u_\alpha(r) \) is periodic in \( t = \log r \).
3. Schrödinger equation with Hardy term. We consider the elliptic equation with Hardy term
\[ \Delta u + \frac{\mu}{|x|^2} u + |x|^l u^p = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \tag{3.1} \]
where \( \mu < \left(\frac{n-2}{2}\right)^2 \) and \( p > 1 \). Set
\[ \nu = \nu_\pm := \frac{n - 2 \pm \sqrt{(n - 2)^2 - 4\mu}}{2}, \]
the solutions of the quadratic equation, \( \nu(n - 2 - \nu) = \mu \).

Assume that \( l > \nu_-(p - 1) - 2 \). If \( |x| \) is replaced by \( \bar{K}(|x|) \), then this condition corresponds to (1.6) which is necessary to have local radial solutions satisfying \( u(r) = O(r^{-\nu_-}) \) at 0 where \( r = |x| \). We regard this type as regular solution.

Let \( L^{p-1} = m(n - 2 - m) \) with \( m = \frac{2 + l}{p - 1} \), and set \( L = [m(n - 2 - m)]^{\frac{1}{p-1}} \) when \( L > 0 \), i.e., \( p(n - 2) > n + l \). Note that (i) \( m > \nu_- \) (see the condition of \( l \)); (ii) if \( m \leq \frac{n-2}{2} \) (or \( p(n - 2) \geq n + 2l \)), then \( L^{p-1} > \mu \).

3.1. Nonexistence. We consider the nonexistence of positive solutions.

**Theorem 3.1.** If \( \mu \geq L^{p-1} \), then (3.1) has no positive solution.

**Proof.** Let \( \bar{u} \) be the spherical average of \( u \) and \( V(t) = r^m \bar{u}, t = \log r \). Then, \( \bar{u} \) and \( V \) satisfy that
\[ \Delta \bar{u} + \frac{\mu}{|x|^2} \bar{u} + |x|^l \bar{u}^p \leq 0 \]
by Jensen’s inequality and
\[ V'' + aV' + V^p - (L^{p-1} - \mu)V \leq 0, \tag{3.2} \]
where \( a = n - 2 - 2m \).

**Case 1:** \( a \leq 0 \).

Since \( V'(t) \leq e^{-a(t-T)}V'(T) \) on \([T, t]\) for \( T \) large, we see that \( V' \geq 0 \) near \(+\infty\).

Multiplying (3.2) by \( e^{at} \) and integrating over \([T, t]\) for \( T \) large, we have
\[ 0 \leq e^{at}V'(t) \leq e^{at}V'(T) - \int_T^t e^{at}V^p dt. \]
Hence, we have
\[ \int_t^\infty e^{at}V^p dt \leq e^{at}V'(t). \]
If \( a = 0 \), then \( V'(t) = \infty \), a contradiction. If \( a < 0 \), we have \(-\frac{1}{a}V^p(t) \leq V'(t) \) since \( V \) is increasing. Then, integrating over \([T, t]\), we have
\[ -\frac{1}{a}(t - T) \leq \frac{1}{p-1} \left( \frac{1}{V^{p-1}(T)} - \frac{1}{V^{p-1}(t)} \right), \]
a contradiction for \( t \) large.

**Case 2:** \( a > 0 \).

Let \( W(t) = V(\bar{t}) \). Then \( W \) satisfies that \( W'' - aW' + W^p \leq 0 \) which has no positive solution near \(+\infty\) as observed in Case 1. Hence, (3.2) has no positive solution near \(-\infty\). \( \square \)

In particular, we have the nonexistence if \( \mu \geq \left(\frac{n-2}{2}\right)^2 =: \bar{\mu} \), the Hardy constant.

In fact, \( L^{p-1} = \bar{\mu} \) is the maximum at \( p = \frac{n+2+2l}{n-2} \).

For any \( p > 1 \), the nonexistence holds when
• $n = 1$ and $\mu \geq \frac{1}{4}$;
• $n = 2$ and $\mu \geq 0$.

Now, we mention the following two well-known nonexistence results. By making use of the second transformation from Lane-Emden equation, we have the nonexistence of regular radial solutions.

**Theorem 3.2.** If $\mu < L^{p-1}$ and $p(n-2) < n+2+2l$, then (3.1) has no regular radial solution.

The second condition is restated for $n = 1, 2$ as follows:
• $n = 1$ and $p > -2l - 3$ (or $l > -\frac{1}{2}(p-1) - 2$);
• $n = 2$ and $l > -2$;
• $n \geq 3$ and $p < \frac{n+2+2l}{n-2}$.

When $\mu \geq 0$ and $l \leq 0$, the radial symmetry of regular solutions leads to the following assertion.

**Theorem 3.3.** If $0 \leq \mu < L^{p-1}$ and $p(n-2) < n+2+2l$ with $l \leq 0$, then (3.1) has no regular solution.

It is natural question to ask whether (3.1) has nonradial solution for all $\mu < 0$. In [10], Jin, Li and Xu gave a partial answer: If $l = 0$, $\mu < -\frac{n-2}{4}$ and $p = \frac{n+2}{n-2}$, then (3.1) has nonradial solutions. However, it is still open for $-\frac{n-2}{4} \leq \mu < 0$.

### 3.2. Regular solution

We consider the existence of solutions of the equation

$$u'' + \frac{n-1}{r}u' + \frac{\mu}{r^2}u + ru^p = 0, \quad \lim_{r \to 0} r^{\nu_+}u(r) = \alpha > 0. \quad (3.3)$$

When $n > 2$ and $l > \nu_-(p-1) - 2$, (3.3) has a unique local solution $u_{\alpha} \in C^2(0, \delta)$ for $\delta > 0$ small.

**Theorem 3.4.** Let $\mu < L^{p-1}$ and $p(n-2) \geq n+2+2l$. Then, (3.3) has one-parameter family of regular solutions.

First, we apply the Fowler transform to $u$. Namely, $V(t) = r^m u(r)$, $t = \log r$. Then, $V$ satisfies

$$V'' + aV' - L^{p-1}V + V^p = 0, \quad (3.4)$$

where $a = n - 2 - 2m$ and $L := (L^{p-1} - \mu)^{\frac{1}{p-1}}$. By making use of the inverse Fowler transform, we observe that (3.3) for $n = 1, 2$ also possesses one-parameter family of solutions under the assumptions. In particular, we have the critical cases.

• For $n = 1$, assume $m^2 + m + \mu < 0$ and $1 < p \leq -2l - 3 < -2\nu_-(p-1) + 4$.

The critical problem is

$$u'' + \frac{\mu}{r^2}u + \frac{1}{r^{2\nu_+}}u^p = 0, \quad \lim_{r \to 0} r^{\nu_-}u(r) = \alpha > 0.$$

We regard the solutions as even functions in $\mathbb{R}\setminus\{0\}$.

• For $n = 2$, assume $m^2 + \mu < 0$ and $l \leq -2$. The critical problem is

$$u'' + \frac{\mu}{r^2}u + \frac{1}{r^{2l+2}}u^p = 0, \quad \lim_{r \to 0} r^{-\sqrt{\delta^2}}u(r) = \alpha > 0.$$

• For $n \geq 3$, assume $\mu < L^{p-1}$ and $p \geq \frac{n+2+2l}{n-2}$ for supercritical case.

The asymptotic behavior of solutions has two types: the first is fast decay for the critical case; the second is slow decay for the supercritical case.

**Theorem 3.5.** Let $\mu < L^{p-1}$ and $p(n-2) \geq n+2+2l$. 

\[ n = 1 \quad \mu \geq \frac{1}{4}; \quad n = 2 \quad \mu \geq 0. \]
(i) If \( p(n - 2) = n + 2 + 2l \), then \( \lim_{r \to \infty} r^{\mu} u_\alpha(r) = c > 0 \). If \( n > 2 \), for some \( \epsilon > 0 \),

\[
\bar{u}_\epsilon(x) = \frac{[2(n+l)(\bar{\mu}-\mu)]^{\frac{1}{p}}}{{\sqrt{\mu}} x^{\frac{2}{p}} - \sqrt{1-\epsilon} \sqrt{\mu} \left( \epsilon + \frac{2(n+l-2\sqrt{\mu})}{\sqrt{\mu}} \right) x^{(n-2)/(2+l)}}
\]

are the solutions.

(ii) If \( p(n - 2) > n + 2 + 2l \), then \( u_\alpha \) satisfies

\[
\lim_{r \to \infty} r^{\mu} u_\alpha(r) = L.
\]

3.3. Singular solution. Now, we look for solutions which are not regular. We call this type of solutions singular solutions.

**Theorem 3.6.** Let \( \mu < L^{-1} \) and \( p(n - 2) \geq n + 2 + 2l \).

(i) If \( p(n - 2) = n + 2 + 2l \), then there are two types: the first has the self-similar singularity, \( r^{-m} L \); the second is of periodic type. Precisely, for each \( 0 < d_1 < L < d_2 \) with \( \frac{d_2^2 - d_1^2}{2} L^{p-1} = \frac{d_1^{p+1} - d_2^{p+1}}{p+1} \),

\[
0 < d_1 = \min r^{m} u_\alpha(r) < L < d_2 = \max r^{m} u_\alpha(r) < \bar{v},
\]

where \( \bar{v} = \left[ \frac{(n+l)(n-2) - n+l}{2} \right]^{-\frac{1}{p+1}} \) for \( n > 2 \); \( \bar{v} = \left[ -\frac{n+1}{2} \right]^{-\frac{1}{p+1}} \) for \( n = 2 \);

\[\bar{v} = \left[ -\frac{l+1}{2} \right]^{-\frac{1}{p+1}} \] for \( n = 1 \), and \( r^{-m} u_\alpha(r) \) is periodic in \( t = \log r \).

(ii) If \( p(n - 2) > n + 2 + 2l \), then \( r^{-m} L \) is the unique singular radial solution.

If \( l \leq -2 \), then each problem for \( p > 1 \) is a supercritical case. Hence, for \( l = -2 \),

\( L = (-\mu)^{-\frac{1}{m+1}} \) is the unique singular radial solution while for \( \nu_-(p-1) - 2 < l < -2 \), \( r^{-m} L \) is the unique singular radial solution.

3.4. Separation. We investigate the separation structure of positive solutions of (3.3). For the separation structure of (1.2), see [3, 4, 16].

**Theorem 3.7.** Let \( a = n - 2 - 2m \) and \( p(n - 2) \geq n + 2 + 2l \). Assume \( L^{p-1} > \mu \geq L^{p-1} - \frac{a^2}{\pi(p+1)^2} \). Then, any two positive solutions of (3.3) do not intersect.

We omit the proof of Theorem 3.7 since the arguments are the same as in the proofs of Theorem 3.2 and Theorem 1.2 in [4]. Another way is to make use of the second transformation in the introduction. Importantly, (2.1) and (3.1) are closely related via (3.4). In order to clarify the relationship among parameters, we analyze each case and explain the conditions. For \( p(n - 2) \geq n + 2 + 2l \), there exist \( \mu_-(n, p, l) < \mu_+(n, p, l) < \bar{\mu} \) such that the separation occurs for \( 0 < \mu_- \leq \mu \leq \mu_+ \).

**Case I-A-1:** \( l > -2 \) and \( n \leq 10 + 4l \).

Observe that

\[
\lim_{p \to -\frac{n+2+2l}{n-2}} \mu_\pm = \bar{\mu}, \quad \lim_{p \to \infty} \mu_\pm = 0.
\]

For given \( 0 < \mu < \bar{\mu} \), there exist \( p_+ > p_- > \frac{n+2+2l}{n-2} \) such that separation happens for \( p_- \leq p < p_+ = \frac{\nu_+ + 2}{\nu_-} + 1 \). Moreover, \( p_\pm \) is decreasing in \((0, \bar{\mu})\).

**Case I-A-2:** \( l > -2 \) and \( n > 10 + 4l \).

In this case, \( \mu_- \) satisfies that

\[
\mu_- \geq -\frac{(2n + l - 2)(n - 10 - 4l)^2}{108(l + 2)} = \mu_\ast
\]

For given \( 0 < \mu < \bar{\mu} \), there exist \( p_+ > p_- > \frac{n+2+2l}{n-2} \) such that separation happens for \( p_- \leq p < p_+ = \frac{\nu_+ + 2}{\nu_-} + 1 \). Moreover, \( p_\pm \) is decreasing in \((0, \bar{\mu})\).
for $p \geq \frac{n+2+2l}{n-2}$. Moreover, $\mu_- = \mu_+$ only when $p = \frac{n+2+2l}{n-10-4l} = p_*(m = \frac{n-10-4l}{6})$.

Note that $\mu_- = -\frac{(n-1)(n-10)^2}{108}$ when $p = \frac{n+2}{n-10}$ and $l = 0$. In other words, for given $0 < \mu < \bar{\mu}$, there exist $p_+ > p_- > \frac{n+2+2l}{n-2}$ such that separation happens for $p_- \leq p < p_+= \frac{l+2}{\nu_-} + 1$, while for $\mu = 0$, $p \geq p_c$ and for $\mu_* \leq \mu < 0$, $p_- \leq p \leq p_+$. Here, $p_-$ is decreasing in $\mu$, and $p_+$ is decreasing only in $(0, \bar{\mu})$. Note that $p_-(0) = p_c$ and $p_+$ is increasing in $[\mu_*, 0)$, $p_+(0) = \infty$, and

$$\lim_{\mu \to \mu_*^-} p_\pm = \frac{n + 2 + 2l}{n - 2}, \quad \lim_{\mu \to \mu_*^+} p_\pm = p_*.$$

**Case I-B-1:** $l = -2$.

It is easy to see that $0 > \mu \geq -\frac{\bar{\mu}}{p-1}$ and $1 < p < p_+ = -\frac{\bar{\mu}}{\mu} + 1$.

**Case I-B-2:** $\sigma(p - 1) - 2 < l < -2$.

Observe that

$$\lim_{\mu \to 1^-} \mu_\pm = -\infty, \quad \lim_{\mu \to \infty} \mu_\pm = \infty.$$

For given $-\infty < \mu < 0$, $\frac{l+2}{\nu_-} + 1 = p_- < p \leq p_+.$

**Case II-A:** $n = 2$ and $l < -2$.

Observe similarly that

$$\lim_{\mu \to 1} \mu_\pm = -\infty, \quad \lim_{\mu \to \infty} \mu_\pm = \infty.$$

For given $-\infty < \mu < 0$, $\frac{l+2}{\nu_-} + 1 = p_- < p \leq p_+.$

**Case II-B:** $n = 1$ and $l < -2$.

There is an upper bound of $p$ for this case. Observe that $\mu_- \leq \mu < \mu_+ \leq \bar{\mu} = \frac{1}{4}$ and

$$\lim_{\mu \to 1} \mu_\pm = -\infty, \quad \lim_{\mu \to \infty} \mu_\pm = \infty.$$

For given $-\infty < \mu < 1$, $\frac{l+2}{\nu_-} + 1 = p_- < p \leq p_+.$

$$\lim_{\mu \to p_\pm} p_\pm = -2l - 3, \quad \lim_{\mu \to \infty} p_\pm = 1.$$

In the last section, we describe the above conditions as the regions surrounded by the graphs in the coordinate $(\mu, p)$-plane.

3.5. **Stability.** We study stability by adopting the arguments in [12]. Note that if $\mu \geq L^{p-1} - \frac{a^2}{4(p-1)}$, then $\bar{\mu} = (\frac{n-2}{2})^2 \geq pL^{p-1} + \mu = pL^{p-1} - (p - 1)\mu$.

**Theorem 3.8.** Let $\mu < L^{p-1}$ and $p(n-2) > n + 2 + 2l$. Assume $\bar{\mu} \geq pL^{p-1} + \mu$. Then, every radial regular steady state $u$ satisfies

$$|x|^{2l}u^{p-1}(x) \leq L^{p-1}(=L^{p-1} - \mu)$$

and $u$ is linearly stable in the sense that the linearized operator $-\Delta - \frac{\mu}{|x|^2} - p|x|^l u^{p-1}$ has no negative spectrum.

**Proof.** Suppose the inequality. Then,

$$-p|x|^l u^{p-1}(x) - \frac{\mu}{|x|^2} \geq (-pL^{p-1} + (p - 1)\mu) \frac{1}{|x|^2}$$

and by Hardy’s inequality,

$$\int |\nabla \phi|^2 - \frac{\mu}{|x|^2} \phi^2 - p|x|^l u^{p-1}\phi^2 \geq 0$$
for \( \phi \in H^1(\mathbb{R}^n) \). Let \( L^{p-1} = Q(m) - \mu \) and \( Q(m) = L^{p-1} \). It follows from (3.4) that \( Q(m - \partial_t)V - \mu V = V \). Since \( V \) and \( L^p \) are bounded, we have
\[
Q(m - \partial_t)V - \mu V - L^p \geq p(Q(m) - \mu)(V - L).
\]
Then, we obtain
\[
[pQ(m) - Q(m - \partial_t) - (p-1)\mu]W = 0,
\]
where \( W = V - L \). Now, we claim that the characteristic polynomial
\[
P(\lambda) = pQ(m) - Q(m - \lambda) - (p-1)\mu
\]
has two negative roots, \( \lambda_1, \lambda_2 \). The product of the two roots is positive, \( P(0) = (p-1)(Q(m) - \mu) > 0 \), and
\[
P(m - \frac{n-2}{2}) = pQ(m) - \bar{\mu} - (p-1)\mu \leq 0.
\]
Since \( m < \frac{n-2}{2} \), we arrive at the conclusion that \( P(\lambda) \) has two negative roots.

Note that \( V(t), V'(t) \to 0 \) as \( t \to -\infty \). Let \( Y = W' - \lambda_1W \). Then, \( Y' - \lambda_2Y \leq 0 \). Since \( e^{-\lambda_2t}Y \) is decreasing and zero at \( t = -\infty \), we see that \( W' - \lambda_1W \leq 0 \) and \( W \) is also zero at \( t = -\infty \). Therefore, \( W = V - L \leq 0 \), i.e., (3.5) holds.

**Theorem 3.9.** Let \( \mu < L^{p-1} \) and \( p(n-2) > n+2+2 \). Assume \( \bar{\mu} < pL^{p-1} + \mu \). Then, two positive solutions of (3.3) intersect infinitely many times.

It is interesting to analyze the instability of solutions. The operator \(-\Delta + V \) with \( V = -\frac{\mu}{|x|^2} - p|x|^l|u|^{p-1} \) has a negative eigenvalue, if \( V \) is bounded. See [12, Lemma 5.1]. For the case that \( V \) is unbounded, see [7, the appendix].

4. **Separation region.** We consider six cases according to the dimension and the range of \( l \). In the following plots, separation phenomena occur in grey parts.

- **Case 1:** \( n \geq 3 \).
  - **Case 1-A:** \( l > -2 \).

\[
\text{Case I-A-1: } n \leq 10 + 4l \quad \text{Case I-A-2: } n > 10 + 4l
\]

**Figure 1.** \( n \geq 3 \) and \( l > -2 \).
Case I-B: $\sigma(p-1) - 2 < l \leq -2$.

Case I-B-1: $l = -2$

Case I-B-2: $\sigma(p-1) - 2 < l < -2$

Figure 2. $n \geq 3$ and $\sigma(p-1) - 2 < l \leq -2$.

Case II: $n = 1, 2$ and $l < -2$.

Case II-A: $n = 2$

Case II-B: $n = 1$

Figure 3. $n = 1, 2$ and $l < -2$.

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