Badly approximable vectors in affine subspaces:
Jarník-type result

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Abstract. Consider irrational affine subspace $A \subset \mathbb{R}^d$ of dimension $a$. We prove that the set
$$\{\xi = (\xi_1, \ldots, \xi_d) \in A : \ q^{1/a} \cdot \max_{1 \leq i \leq d} ||q\xi_i|| \to \infty, \ q \to \infty\}$$
is an $\alpha$-winning set for every $\alpha \in (0, 1/2]$. This simple short communication may be considered as a supplement to our short paper [9].

1. Jarník’s result in simultaneous Diophantine approximations. All numbers in this paper are real. Notation $|| \cdot ||$ stands for the distance to the nearest integer. In 1938 V. Jarník (see [1], Satz 9 and [2], Statement E) proved the following result.

Theorem 1. (V. Jarník) Suppose that among numbers $\xi_1, \ldots, \xi_d$ there are at least two numbers which are linearly independent over $\mathbb{Z}$, together with 1. Then
$$\limsup_{t \to +\infty} \left( t \cdot \min_{q \in \mathbb{Z}, 1 \leq q \leq t} \max_{1 \leq i \leq d} ||q\xi_i|| \right) = +\infty.$$Nothing more can be said in a general situation. In 1926 A. Khintchine [3] proved the following result.

Theorem 2. (A. Khintchine) Let $\psi(t)$ increases to infinity as $t \to +\infty$. Then there exist two algebraically independent real numbers $\xi_1, \xi_2$ such that for all $t$ large enough one has
$$t \cdot \min_{q \in \mathbb{Z}, 1 \leq q \leq t} \max_{i=1,2} ||q\xi_i|| \leq \psi(t).$$A general form of such a result one can find in Jarník’s paper [2]. A corresponding lim sup result is due to J. Lesca [6]:

Theorem 3. (J. Lesca) Let $d \geq 2$. Let $\psi(t)$ be a positive continuous function in $t$ such that the function $t \mapsto \psi(t)/t$ is a decreasing function. Suppose that
$$\limsup_{t \to \infty} \psi(t) = +\infty.$$Then the set of all vectors $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, containing of algebraically independent elements, such that
$$t \cdot \min_{q \in \mathbb{Z}, 1 \leq q \leq t} \max_{1 \leq i \leq d} ||q\xi_i|| \leq \psi(t)$$for all $t$ large enough, being intersected with a given open set $G \subset \mathbb{R}^d$ is of cardinality continuum.

We would like to note that Jarník’s Theorem 1 as well as some other theorems by Khintchine and V. Jarník were discussed and generalized in author’s survey [7]. In particular in [7], Section 4.1 (see also [8]) one can find an improvement of Theorem 1 in terms of the best approximation vectors.

2. Affine subspaces. Let $\mathbb{R}^d$ be a Euclidean space with the coordinates $(x_1, \ldots, x_d)$, let $\mathbb{R}^{d+1}$ be a Euclidean space with the coordinates $(x_0, x_1, \ldots, x_d)$. Consider an affine subspace $A \subset \mathbb{R}^d$. Let $a = \dim A \geq 1$. Define the affine subspace $\mathcal{A} \subset \mathbb{R}^{d+1}$ in the following way:
$$\mathcal{A} = \{x = (1, x_1, \ldots, x_d) : (x_1, \ldots, x_d) \in A\}.$$
We define linear subspace \( \mathfrak{A} = \text{span} \mathcal{A} \), as the smallest linear subspace in \( \mathbb{R}^{d+1} \) containing \( \mathcal{A} \). So \( \dim \mathfrak{A} = a + 1 \).

Consider a sublattice \( \Gamma(A) = \mathfrak{A} \cap \mathbb{Z}^{d+1} \) of the integer lattice \( \mathbb{Z}^{d+1} \). We see that
\[
0 \leq \dim \Gamma(A) \leq a + 1.
\]
Of course here for a lattice \( \Gamma \subset \mathbb{Z}^{d+1} \) by \( \dim \Gamma \) we mean the dimension of the linear subspace \( \text{span} \Gamma \).

In the case \( \dim \Gamma(A) = a + 1 = \dim \mathfrak{A} \) we define \( A \) to be a completely rational affine subspace in \( \mathbb{R}^d \).

For a completely rational affine subspace \( A \) by \( d(A) \) we denote the fundamental \((a + 1)\)-dimensional volume of the lattice \( \Gamma(A) \).

We see from Dirichlet principle that for any completely rational affine subspace \( A \) of dimension \( a \) there exists a positive constant \( \gamma = \gamma(A) \) such that for any \( \xi = (\xi_1, ..., \xi_d) \in A \) the inequality
\[
\max_{1 \leq i \leq d} ||q\xi_i|| \leq \frac{\gamma}{q^{1/a}}
\]
has infinitely many solutions in positive integers \( q \).

One can easily see that for any affine subspace \( A \) of dimension \( a \) the set
\[
\Omega = \{ \xi = (\xi_1, ..., \xi_d) \in A : \inf_{q \in \mathbb{Z}_+} q^{1/a} \cdot \max_{1 \leq i \leq d} ||q\xi_i|| > 0 \}
\]
is an 1/2-winning set in \( A \). Here we do not want to discuss the definitions \((\alpha, \beta)\)-games and \((\alpha, \beta)\)-winning set or \(\alpha\)-winning set. This definitions were given in W.M. Schmidt’s paper [10]. All the definitions and basic properties of winning sets one can find in the book [11], Chapter 3. In particular, every \(\alpha\)-winning set in \( A \) has full Hausdorff dimension. A countable intersection of \(\alpha\)-winning sets inn \( A \) is also an \(\alpha\)-winning set.

In the case when \( A \) is not a completely rational subspace the result about winning property of the set \( \Omega \) admits a small improvement. This improvement is related to Jarník’s result cited behind.

**Theorem 4.** Let
\[
0 < \alpha < 1, \ 0 < \beta < 1, \ \gamma = 1 + \alpha\beta - 2\alpha > 0.
\]
Suppose that \( \dim \Gamma(A) < a \). Then the set
\[
\Omega^* = \{ \xi = (\xi_1, ..., \xi_d) \in A : q^{1/a} \cdot \max_{1 \leq i \leq d} ||q\xi_i|| \to \infty, \ q \to \infty \}
\]
is \((\alpha, \beta)\)-winning set in \( A \). In particular, it is an \(\alpha\)-winning set for every \(\alpha \in (0, 1/2] \).

Here we should note that certain results concerning badly approximable vectors in affine subspaces one can find in [11, 5, 9, 12].

3. **Lemmata.** Consider the set of all \((a + 1)\)-dimensional complete sublattices of the integer lattice \( \mathbb{Z}^{d+1} \). It is a countable set. One can easily see that for any positive \( H \) there exist not more than a finite number of such sublattices \( \Gamma \) with the fundamental volume \( \det \Gamma \leq H \). Hence we can order the set \( \{V_\nu\}_{\nu=1}^\infty \) of all \( a \)-dimensional affine subspaces in \( \mathbb{R}^d \) in such a way that values \( d_\nu = d(V_\nu) = \det \Gamma(V_\nu) \) form an increasing sequence:
\[
1 = d_1 \leq d_2 \leq \cdots \leq d_\nu \leq d_{\nu+1} \leq \cdots.
\]
We see that
\[
d_\nu \to \infty, \ \nu \to \infty. \quad (2)
\]
Some of consecutive values of \( d_\nu \) may be equal. We define a sequence \( d_{\nu_k} \) of all different elements from the sequence \( \{d_\nu\} \):
\[
1 = d_{\nu_1} = \cdots = d_{\nu_{\nu-1}} < d_{\nu_2} = \cdots = d_{\nu_{\nu-1}} < d_{\nu_3} = \cdots < d_{\nu_k} = d_{\nu_{k+1}} = \cdots = d_{\nu_{k-1}} < d_{\nu_k} = \cdots
\]
(of course \( \nu_1 = 1 \)). For \( V_j \) we define the affine subspace \( \mathcal{V}_j \subset \mathbb{R}^{d+1} \) as
\[
\mathcal{V}_j = \{ x = (1, x_1, \ldots, x_d) : (x_1, \ldots, x_d) \in V_j \}
\]
and consider linear subspace \( \mathfrak{W}_j = \text{span} \mathcal{V}_j \).

In the sequel for \( \xi = (x_1, \ldots, x_d) \in A \) we consider \( a \)-dimensional ball
\[
B(\xi, \rho) = \{ \xi' = (\xi'_1, \ldots, \xi'_d) \in A : \max_{1 \leq i \leq d} |\xi_i - \xi'_i| \leq \rho \}
\]
and \( d \)-dimensional ball
\[
\hat{B}(\xi, \rho) = \{ \xi' = (\xi'_1, \ldots, \xi'_d) \in \mathbb{R}^d : \max_{1 \leq i \leq d} |\xi_i - \xi'_i| \leq \rho \}.
\]
Obviously
\[
B(\xi, \rho) = \hat{B}(\xi, \rho) \cap A.
\]

**Lemma 1.** Suppose that \( U, V \subset \mathbb{R}^d \) are two affine subspaces. Put \( L = U \cap V \) and suppose that \( \dim U > \dim L \). Suppose that affine subspace \( L' \subset U \) has dimension \( \dim L' = \dim U - 1 \), and \( L' \cap L = \emptyset \). Define \( \hat{U} \subset U \) to be a half-subspace with the boundary \( L' \) and such that \( \hat{U} \cap L = \emptyset \). Then \( \text{dist}(\hat{U}, V) > 0 \).

Proof. In affine subspace \( \text{aff}(U \cup V) \) of dimension \( w = \dim U + \dim V - \dim L \) there exists an affine subspace \( L'' \supset L' \) with dimension \( \dim L'' = w - 1 \) such that \( L'' \cap V = \emptyset \). So \( \text{dist}(L'', U) > 0 \). The subspace \( L'' \) divides \( \text{aff}(U \cup V) \) into two parts, and lemma follows. \( \square \)

**Corollary.** Consider two affine subspaces \( A, V \subset \mathbb{R}^d \). Suppose that for \( \xi \in A \) the ball \( B(\xi, \rho) \subset A \) satisfies the property
\[
\text{dist}(B(\xi, \rho), A \cap V) \geq \varepsilon > 0.
\]
Then there exists positive \( \delta = \delta(A, V, \xi, \varepsilon) \) such that for any \( \xi' \in B(\xi, \rho) \) one has
\[
\hat{B}(\xi', \delta) \cap V = \emptyset.
\]

Proof. From the conditions of our Corollary we see that \( \dim (A \cap V) < \dim A \). So we can take a subspace \( L' \) of dimension \( \dim L' = \dim A - 1 \) which separates the ball \( B(\xi, \rho) \) from the subspace \( A \cap V \) in \( A \). Now we use Lemma 1. \( \square \)

**Lemma 2.** Let \( \rho > 0 \) and \( \xi \in A \). Consider a ball \( \hat{B}(\xi, \rho) \subset \mathbb{R}^d \) such that
\[
\hat{B}(\xi, \rho) \cap \mathfrak{W}_j = \emptyset, \quad 1 \leq j \leq n.
\]
Define \( k = k(n) \) from the condition
\[
\nu_k \leq n < \nu_{k+1}.
\]
Put
\[
\kappa = \kappa_{d, \xi} = (2\sqrt{d})^a \times \sqrt{1 + (|\xi_1| + 1)^2 + \ldots + (|\xi_d| + 1)^2}, \quad \sigma = \sigma_{a, d, \xi} = \frac{1}{\kappa_{d, \xi}(a+1)!}
\]
and
\[
T = (\sigma d \nu_n \rho^{-a})^{\frac{1}{a+1}}.
\]
Then the set of all rational points \( \left( \frac{b_1}{q}, \ldots, \frac{b_d}{q} \right) \in \hat{B}(\xi, \rho) \) with \( q \leq T \) lie in a certain \((a-1)\)-dimensional affine subspace.
Proof. We may suppose that the set of rational points from \( \hat{B}(\xi, \rho) \) with \( q \leq T \) consists of more than \( a \) points (otherwise there is nothing to prove). We take arbitrary \( a + 1 \) points 
\[
\left( \frac{b_{1,j}}{q_j}, \ldots, \frac{b_{d,j}}{q_j} \right) \in \hat{B}(\xi, \rho), \quad 1 \leq q_j \leq T, \quad \gcd(q_j, b_{1,j}, \ldots, b_{d,j}) = 1, \quad 1 \leq j \leq a + 1
\]
and prove that primitive integer vectors
\[
\mathbf{b}_j = (q_j, b_{1,j}, \ldots, b_{d,j}), \quad 1 \leq j \leq a + 1
\]
are linearly dependent. Then the lemma will be proved.

All integer vectors (6) belong to the cylinder
\[
C = C_\xi(T, \rho) = \{ x = (x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1} : 0 \leq x_0 \leq T, \quad \max_{1 \leq j \leq d} |x_0 \xi_j - x_j| \leq \rho T. \}
\]
Suppose that they are linearly independent. Then \( \mathcal{L} = \text{span}(\mathbf{b}_1, \ldots, \mathbf{b}_{a+1}) \) is an \((a + 1)\)-dimensional completely rational linear subspace. By \( D \) we denote the fundamental \((a + 1)\)-dimensional volume of the lattice \( \mathcal{L} \cap \mathbb{Z}^{d+1} \). From (3) we see that \( \mathcal{L} \neq \mathcal{V}_j, \quad 1 \leq j \leq n. \)

From (4) we see that
\[
D \geq d_{\nu_n}. \quad (7)
\]
Now we consider the section \( \mathcal{L} \cap C \) which is an \((a + 1)\)-dimensional convex polytope. As it is inside \( C \), its \((a + 1)\)-dimensional measure is less than
\[
(2\sqrt{d\rho T})^a \times T \sqrt{1 + (|\xi_1| + 1)^2 + \ldots + (|\xi_d| + 1)^2} = \kappa \rho^a T^{a+1} = \kappa \sigma d_{\nu_n}.
\]
But the section \( \mathcal{L} \cap C \) consist of \( a + 1 \) independent points from the lattice \( \mathcal{L} \cap \mathbb{Z}^{d+1} \). For the fundamental volume of this lattice we have lower bound (7). That is why
\[
\frac{d_{\nu_n}}{(a + 1)!} \leq \frac{D}{(a + 1)!} < \kappa \sigma d_{\nu_n} = \frac{d_{\nu_n}}{(a + 1)!}.
\]
This is a contradiction. Lemma is proved. \( \square \)

**Lemma 3.** (W.M. Schmidt’s escaping lemma, Lemma 1B, [11], Chapter 3) Let \( t \) be such that
\[
(\alpha \beta)^t < \frac{\gamma}{2}.
\]
Suppose a ball \( B_j \subset A \) with the radius \( \rho_j \) occurs in the game (as a Black ball). Suppose \( V \) is an \((d - 1)\)-dimensional affine subspace passing through the center of the ball \( B_j \). Then White can play in such a way that the ball \( B_{j+t} \) is contained in the halfspace \( \Pi \) such that the boundary of \( \Pi \) is parallel to the subspace \( V \) and the distance between \( \Pi \) and \( V \) is equal to \( \frac{\rho_j \gamma}{2} \).

**Corollary.** Suppose a ball \( B_j \subset A \) with the radius \( \rho_j \) occurs in the game (as a Black ball). Suppose that \( V, V' \subset A \) are two proper affine subspaces of \( A \). Then White can play in such a way that the distance from the ball \( B_{j+2t} \) to each of subspaces \( V, V' \) is greater than \( \frac{\rho_{j+t} \gamma}{2} \) (here \( \rho_{j+t} \) is the radius of the ball \( B_{j+t} \)).

**4. Proof of Theorem 4.** Suppose that \( t = t(\alpha, \beta) \) satisfies the condition of Lemma 3. Put \( j_k = 2tk \) and \( R_0 = 1 \). Suppose that the first Black ball \( B_0 \subset A \) with the radius \( \rho_0 \) lies inside the
Hence we apply Lemma 2 to see that all rational points \( \xi \in B_{j_r} \) one has

\[
\max_{1 \leq i \leq d} \|q \xi_i\| \geq \frac{(\alpha \beta)^t \gamma \rho_0}{2} R^{-(a+1)t} \cdot q, \quad \forall q < R_r
\]

with a certain \( R_r \) which we define later in the inductive step.

We shall prove it by induction in \( r \).

The base of induction is obvious.

Suppose that the ball \( B_{j_{r-1}} = B(\xi_{j_{r-1}}, \rho_{j_{r-1}}) \in A, \xi_{j_{r-1}} = (\xi_{j_{r-1},1}, \ldots, \xi_{j_{r-1},d}) \) which occurs as a Black ball satisfies the condition specified. Note that \( \rho_{j_{r-1}} = \rho_0 (\alpha \beta)^{j_{r-1}}. \) Consider the ball \( \hat{B}_{j_{r-1}} = \hat{B}(\xi_{j_{r-1}}, 2\rho_{j_{r-1}}) \in \mathbb{R}^d. \) Define \( k_r \) as the maximal \( k \) such that \( \hat{B}_{j_{r-1}} \cap \mathcal{M}_j = \emptyset, 1 \leq j \leq \nu_k. \)

Then we apply Lemma 2 to see that all rational points \( \left( \frac{b_1}{q}, \ldots, \frac{b_d}{q} \right) \in \hat{B}_{j_{r-1}} \) with

\[
q \leq \left( \sigma_{a,d, \xi_{j_{r-1}}}(2\rho_0)^{-a} \right)^{\frac{1}{a+1}} \left( \frac{1}{\alpha \beta} \right)^{\frac{2at(r-1)}{a+1}} d_{\nu_k r_{j_{r-1}}}
\]

lie in a certain \((a-1)\)-dimensional affine subspace. We denote this subspace by \( V'_r. \) As \( \max_{1 \leq i \leq d} |\xi_{j_{r-1},i}| \leq W \) we see that

\[
\sigma_{a,d, \xi_{j_{r-1}}} \geq \Sigma_{a,d,W} = \frac{1}{(2\sqrt{d})^a \sqrt{1 + (W + 1)^2 d (a + 1)!}}
\]

We put

\[
R_r = \left( \Sigma_{a,d,W}(2\rho_0)^{-a} \right)^{\frac{1}{a+1}} \left( \frac{1}{\alpha \beta} \right)^{\frac{2at(r-1)}{a+1}} d_{\nu_k r_{j_{r-1}}}. \tag{9}
\]

By Corollary to Lemma 3 White can play in such a way that

\[
\text{dist}(B_{j_r}, V_r) \geq \frac{\gamma \rho_{j_{r-1}+t}}{2} \tag{10}
\]

and

\[
\text{dist}(B_{j_r}, V'_r) \geq \frac{\gamma \rho_{j_{r-1}+t}}{2} \tag{11}
\]

So the inductive step is described and we must show that (9) is valid. But it is clear from (11) that for any \( \xi \in B_{j_r} \) one has

\[
\max_{1 \leq i \leq d} \|q \xi_i\| \geq \frac{1}{2} \gamma \rho_{j_{r-1}+t} q = \frac{\gamma \rho_0}{2} (\alpha \beta)^{(2t-1)r} q, \quad \forall q < R_r. \tag{12}
\]

Moreover by Corollary to Lemma 1 from (10) we see that

\[
k_r \to +\infty, \quad r \to +\infty.
\]

Hence

\[
d_{\nu_k r} \to +\infty, \quad r \to +\infty. \tag{13}
\]

Consider a point \( \xi \in \bigcap_j B_j. \) For positive integer \( q \) define \( r \) from the condition

\[
R_{r-1} \leq q < R_r.
\]

Then we make use of \( \xi \in B_{j_r}. \) From the inequality \( q \geq R_{r-1} \) and (9) we see that

\[
\alpha \beta \geq \omega_1 q^{-\frac{a+1}{2at} d_{\nu_k r_{j_{r-1}}}^{-1}}.
\]
where $\omega_1 = \omega_1(a, d, W, \alpha, \beta, t) > 0$. We substitute this estimate into (12) to see that

$$\max_{1 \leq i \leq d} ||q\xi_i|| \geq \omega_2 q^{-1/a} d_{\nu_{k-2}}^{1/a}, \quad R_{r-1} \leq q < R_r,$$

with positive $\omega_2 = \omega_2(a, d, W, \alpha, \beta, t)$. From (13) for $\xi \in \bigcap_j B_j$ we deduce that

$$q^{1/a} \cdot \max_{1 \leq i \leq d} ||q\xi_i|| \to +\infty, \quad q \to \infty.$$

So White can enforce Black to reach a point $\xi$ with the desired properties. Theorem 4 is proved. □

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