Sharp ill-posedness for the generalized Camassa-Holm equation in Besov spaces

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Abstract: In this paper, we consider the Cauchy problem for the generalized Camassa-Holm equation that includes the Camassa-Holm as well as the Novikov equation on the line. We present a new and unified method to prove the sharp ill-posedness for the generalized Camassa-Holm equation in $B^s_{p,∞}$ with $s > \max\{1 + 1/p, 3/2\}$ and $1 \leq p \leq ∞$ in the sense that the solution map to this equation starting from $u_0$ is discontinuous at $t = 0$ in the metric of $B^s_{p,∞}$. Our results cover and improve the previous work given in [22], solving an open problem left in [22].

Keywords: Generalized Camassa-Holm equation; Ill-posedness; Besov space.
MSC (2010): 35Q53, 37K10.

1 Introduction

In this paper, we consider the Cauchy problem for the generalized Camassa-Holm-Novikov (gCHN) equation which was proposed by Anco, Silva and Freire [1] as follows

\[
\begin{aligned}
  &m_t + u^k m_x + (k+1)u^{k-1}u_x m = 0, \quad (x, t, k) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{Z}^+, \\
  &m = u - u_{xx}, \\
  &u(0, x) = u_0(x),
\end{aligned}
\]

(1.1)

The gCHN is an evolution equation with $(k+1)$-order nonlinearities, and can be regarded as a subclass of the generalized Camassa-Holm (g-kbCH) equation considered in [15,17]

\[
m_t + u^k m_x + bu^{k-1}u_x m = 0, \quad k \in \mathbb{Z}^+, \quad b \in \mathbb{R}.
\]

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When \( k = 1 \), (1.1) reduces to the classical Camassa-Holm (CH) equation \([8–12]\)
\[
    u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx},
\]
(1.2)
which was originally derived as a bi-Hamiltonian system by Fokas and Fuchssteiner \([14]\) in the context of the KdV model and gained prominence after Camassa-Holm \([3]\) independently re-derived it as an approximation to the Euler equations of hydrodynamics. (1.2) is completely integrable \([3,7]\) with a bi-Hamiltonian structure \([6,14]\) and infinitely many conservation laws \([3,14]\). Also, it admits exact peaked soliton solutions (peakons) of the form \( ce^{-|x-ct|} \) with \( c > 0 \), which are orbitally stable \([13]\) and models wave breaking (i.e., the solution remains bounded, while its slope becomes unbounded in finite time \([5,9,10]\)).

When \( k = 2 \), (1.1) becomes the famous Novikov equation \([18,19,23–28,30]\)
\[
    u_t - u_{xxt} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx}.
\]
(1.3)
Home-Wang \([20]\) proved that the Novikov equation with cubic nonlinearity shares similar properties with the CH equation, such as a Lax pair in matrix form, a bi-Hamiltonian structure, infinitely many conserved quantities and peakon solutions given by the formula \( u(x,t) = \sqrt{c}e^{-|x-ct|} \).

Setting \( \Lambda^{-2} = (1 - \partial_x^2)^{-1} \), then we transform (1.1) equivalently into the following nonlinear transport type equation
\[
\begin{cases}
    u_t + u^k u_x = P(u) + Q(u), \\
    u(0,x) = u_0(x),
\end{cases}
\]
(1.4)
where
\[
    P(u) := -\partial_x\Lambda^{-2}\left(\frac{2k-1}{2}u^{k-1}u_x^2 + u^{k+1}\right) \quad \text{and} \quad Q(u) := -\frac{k-1}{2}\Lambda^{-2}(u^{k-2}u_x^3).
\]
(1.5)
As shown in \([1,15,17]\), the gCHN equation (1.1) admits a local conservation law, possesses single peakons of the form \( u(x,t) = c^{1/k}e^{-|x-ct|} \) as well as multi-peakon solutions and exhibits wave breaking phenomena (see \([29]\) and the references therein).

In recent years, the issue of well-posedness in different spaces for the g-kbCH equation has been a fascinating object of research due to its abundant physical and mathematical properties and a series of achievements have been made in the study of the g-kbCH equation. For general \( k \), using a Galerkin-type approximation scheme, Himonas-Holliman \([17]\) established the local well-posedness of the g-kbCH equation in the Sobolev space \( H^s(\mathbb{R} \text{ or } \mathbb{T}) \). Zhao-Li-Yan \([31]\) extended the above well-posedness result to the Besov space \( B_{p,r}^s(\mathbb{R}) \) with \( s > \max\{1 + \frac{1}{p}, \frac{3}{2}\} \) and \( 1 \leq p, r \leq \infty \). However, for \( r = \infty \), they established the continuity of the data-to-solution map in a weaker topology. Subsequently, Chen-Li-Yan \([4]\) solved the critical case for \( (s,p,r) = (\frac{3}{2},2,1) \). Guo, Liu, Molinet and Yin \([16]\) established the ill-posedness of the Camassa-Holm equation in the critical Sobolev space \( H^{3/2}(\mathbb{R} \text{ or } \mathbb{T}) \) and even in the Besov space \( B_{p,r}^{1+1/p}(\mathbb{R} \text{ or } \mathbb{T}) \) with \( p \in [1,\infty], r \in (1,\infty] \) by proving the norm inflation. In our recent paper \([22]\), we proved the solution map to the Camassa-Holm equation starting from \( u_0 \) is discontinuous at \( t = 0 \) in the metric of \( B_{p,\infty}^s(\mathbb{R}) \), which implies the ill-posedness for this equation in \( B_{p,\infty}^s(\mathbb{R}) \). More precisely, we established...
Theorem 1.1 (See [22]) Let $s > 2 + \max \{3/2, 1 + 1/p\}$ with $1 \leq p \leq \infty$. There exits $u_0 \in B_{p,\infty}^s(\mathbb{R})$ and a positive constant $\varepsilon_0$ such that the data-to-solution map $u_0 \mapsto S_t(u_0)$ of the Cauchy problem (1.4)-(1.5) with $k = 1$ satisfies

$$\limsup_{t \to 0^+} \|S_t(u_0) - u_0\|_{B_{p,\infty}^s} \geq \varepsilon_0.$$ 

In addition, for the Novikov equations, we proved that Theorem 1.1 holds for only $p = 2$ in [22], while for the case $p \neq 2$ the difficulty lies mainly in the construction of initial data $u_0$ due to the appearance of $u_0^2 \partial_x u_0$. Naturally, the method in [22] seems to be invalid when proving the ill-posedness for (1.4)-(1.5) with $k \geq 3$ in $B_{p,\infty}^s(\mathbb{R})$ ($p \neq 2$) since the construction of initial data makes the computation of $u_0^k \partial_x u_0$ more difficult.

In this present paper, we shall develop a new and unified method to study the ill-posedness problem for (1.4)-(1.5) with general $k \in \mathbb{Z}^+$. Our main aim is to prove the solution map to the Cauchy problem (1.4)-(1.5) starting from $u_0$ is discontinuous at $t = 0$ in the metric of $B_{p,\infty}^s(\mathbb{R})$, which implies the ill-posedness for this equation in $B_{p,\infty}^s(\mathbb{R})$. Furthermore, we expect that the sharp index pair $(s, p)$ satisfies that $s > \max \{3/2, 1 + 1/p\}$ with $1 \leq p \leq \infty$. Now let us state our main result of this paper.

**Theorem 1.2** Let $k \in \mathbb{Z}^+$ be fixed. Assume that

$$s > \max \left\{ \frac{3}{2}, 1 + \frac{1}{p} \right\} \quad \text{with} \quad 1 \leq p \leq \infty. \quad (1.6)$$

There exits $u_0 \in B_{p,\infty}^s(\mathbb{R})$ and a positive constant $\varepsilon_0$ such that the data-to-solution map $u_0 \mapsto S_t(u_0)$ of the Cauchy problem (1.4)-(1.5) satisfies

$$\limsup_{t \to 0^+} \|S_t(u_0) - u_0\|_{B_{p,\infty}^s} \geq \varepsilon_0.$$ 

**Remark 1.1** As mentioned above, system (1.4)-(1.5) unifies the Camassa-Holm and Novikov equations. In [22], we only obtained the ill-posedness for the Novikov equation in $B_{2,\infty}^s$ with $s > \frac{7}{2}$ due to the technical difficulty. Thus Theorem 1.2 covers our recent results on both the Camassa-Holm and Novikov equations in [22]. Also, our method here is new and makes the proof simpler.

The Cauchy problem for the Degasperis-Procesi equation reads as

$$\begin{cases} 
\partial_t u + uu_x = -\frac{3}{2} \partial_x (1 - \partial_x^2)^{-1}(u^2), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}. 
\end{cases} \quad (1.7)$$

**Remark 1.2** We should mention that the $H^1$ norm of solutions to the g-kbCH is conserved if and only if $b = k + 1$, which naturally excludes the Degasperis-Procesi equation for the case $k = 1$ and $b = 3$. However, following the procedure in the proof of Theorem 1.2 with suitable modification, we can prove Theorem 1.2 holds for the Degasperis-Procesi equation.
2 Preliminaries

**Notation** The notation $A \lesssim B$ (resp., $A \gtrsim B$) means that there exists a harmless positive constant $c$ such that $A \leq cB$ (resp., $A \geq cB$). Given a Banach space $X$, we denote its norm by $\| \cdot \|_X$. For $I \subset \mathbb{R}$, we denote by $\mathcal{C}(I; X)$ the set of continuous functions on $I$ with values in $X$. Sometimes we will denote $L^p(0, T; X)$ by $L^p_T X$.

Next, we will recall some facts about the Littlewood-Paley decomposition, the nonhomogeneous Besov spaces and some of their useful properties (For more details, see [2]). Let $\mathcal{B} := \{ \xi \in \mathbb{R} : |\xi| \leq \frac{1}{2} \}$ and $\mathcal{C} := \{ \xi \in \mathbb{R} : \frac{2}{3} \leq |\xi| \leq \frac{3}{2} \}$. There exist two radial functions $\chi \in C^\infty_c(B)$ and $\varphi \in C^\infty_c(C)$ both taking values in $[0, 1]$ such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1 \quad \forall \xi \in \mathbb{R}^d.$$ 

**Definition 2.1 (See [2])** For every $u \in S'(\mathbb{R})$, the Littlewood-Paley dyadic blocks $\Delta_j$ are defined as follows

$$\Delta_j u = \begin{cases} 0, & \text{if } j \leq -2; \\ \chi(D) u = \mathcal{F}^{-1}(\chi \mathcal{F} u), & \text{if } j = -1; \\ \varphi(2^{-j} D) u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u), & \text{if } j \geq 0. \end{cases}$$

**Definition 2.2 (See [2])** Let $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$. The nonhomogeneous Besov space $B^s_{p, r}(\mathbb{R})$ is defined by

$$B^s_{p, r}(\mathbb{R}) := \left\{ f \in S'(\mathbb{R}) : \| f \|_{B^s_{p, r}(\mathbb{R})} < \infty \right\},$$

where

$$\| f \|_{B^s_{p, r}(\mathbb{R})} = \left\{ \begin{array}{ll} \left( \sum_{j \geq -1} 2^{sjr} \| \Delta_j f \|_{L^p(\mathbb{R})} \right)^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty, \\ \sup_{j \geq -1} 2^{sj \| \Delta_j f \|_{L^p(\mathbb{R})}}, & \text{if } r = \infty. \end{array} \right.$$ 

**Remark 2.1** It should be emphasized that the fact $B^s_{p, \infty}(\mathbb{R}) \hookrightarrow B^t_{p, \infty}(\mathbb{R})$ with $s > t$ will be often used implicitly.

Finally, we give some important properties which will be also often used throughout the paper.

**Lemma 2.1 (See [2])** Let $(p, r) \in [1, \infty]^2$ and $s > \max \{ 1 + \frac{1}{p}, \frac{2}{3} \}$. Then we have

$$\| uv \|_{B^s_{p, r}(\mathbb{R})} \leq C \| u \|_{B^{s-2}_{p, r}(\mathbb{R})} \| v \|_{B^s_{p, r}(\mathbb{R})}.$$

**Lemma 2.2 (See [2])** For $(p, r) \in [1, \infty]^2$, $B^{s-1}_{p, r}(\mathbb{R})$ with $s > 1 + \frac{1}{p}$ is an algebra. Moreover, for any $u, v \in B^{s-1}_{p, r}(\mathbb{R})$ with $s > 1 + \frac{1}{p}$, we have

$$\| uv \|_{B^{s-1}_{p, r}(\mathbb{R})} \leq C \| u \|_{B^{s-1}_{p, r}(\mathbb{R})} \| v \|_{B^{s-1}_{p, r}(\mathbb{R})}.$$
Remark 2.2 Let \((p, r) \in [1, \infty]^2\) and \(s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}\), using Lemmas 2.1-2.2, we have

- for the terms \(P(u)\) and \(P(v)\), there holds
  \[
  \|P(u) - P(v)\|_{B^{s-1}_{p,r}} \lesssim \|u - v\|_{B^{s-1}_{p,r}} (\|u\|_{B^{s}_{p,r}} + \|v\|_{B^{s}_{p,r}})^{k-1} (\|u\|_{B^{s}_{p,r}} + \|v\|_{B^{s}_{p,r}}),
  \]

  \[(2.8)\]

- for the terms \(Q(u)\) and \(Q(v)\) (notice that \(Q(u) = 0\) for \(k = 1\)), there holds
  \[
  \|Q(u) - Q(v)\|_{B^{s-1}_{p,r}} \lesssim \|u - v\|_{B^{s-1}_{p,r}} (\|u\|_{B^{s}_{p,r}} + \|v\|_{B^{s}_{p,r}})^{k-2} (\|u\|_{B^{s}_{p,r}}^2 + \|v\|_{B^{s}_{p,r}}^2).
  \]

  \[(2.9)\]

Lemma 2.3 (See [2]) For \(1 \leq p \leq \infty\) and \(s > 0\). There exists a constant \(C\), depending continuously on \(p\) and \(s\), we have

\[
\|2^js \left[\Delta_j, v\right] \partial_x f\|_{L^p} \|_{L^\infty} \leq C \left(\|\partial_x v\|_{L^\infty} \|f\|_{B^{s}_{p,\infty}} + \|\partial_x f\|_{L^\infty} \|\partial_x v\|_{B^{s-1}_{p,\infty}}\right),
\]

where we denote the standard commutator \([\Delta_j, v] \partial_x f = \Delta_j (v \partial_x f) - v \Delta_j \partial_x f\).

3 Proof of Theorem 1.2

3.1 Construction of Initial Data

We need to introduce smooth, radial cut-off functions to localize the frequency region. Precisely, let \(\hat{\phi} \in C_0^\infty(\mathbb{R})\) be an even, real-valued and non-negative function on \(\mathbb{R}\) and satisfy

\[
\hat{\phi}(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{4}, \\ 0, & \text{if } |\xi| \geq \frac{1}{2}. \end{cases}
\]

Lemma 3.1 Define the function \(f_n(x)\) by

\[
f_n(x) = \phi(x) \cos \left(\frac{17}{12} 2^n x\right) \quad \text{with} \quad n \gg 1.
\]

Then we have

\[
\Delta_j(f_n) = \begin{cases} f_n, & \text{if } j = n, \\ 0, & \text{if } j \neq n. \end{cases}
\]

Proof. See [21].

Lemma 3.2 Define the initial data \(u_0(x)\) as

\[
u_0(x) := \sum_{n=0}^{\infty} 2^{-ns} \phi(x) \cos \left(\frac{17}{12} 2^n x\right).
\]

Then for any \(s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}\) and \(k \in \mathbb{Z}^+\), we have for some \(n\) large enough

\[
\|u_0\|_{B^{s}_{p,\infty}} \leq C, \\
\|u_0^k \partial_x \Delta_n u_0\|_{L^p} \geq c 2^{n(1-s)},
\]

where \(C\) and \(c\) are some positive constants.
Proof. By the definition of Besov space and the support of \( \phi(2^{-j} \cdot) \), we have
\[
\|u_0\|_{B^s_{p,\infty}} \leq C.
\]
Using Lemma 3.1 yields
\[
\Delta_n u_0(x) = 2^{-ns} \phi(x) \cos \left( \frac{17}{12} 2^n x \right),
\]
equivalently,
\[
\partial_x \Delta_n u_0 = 2^{-ns} \phi'(x) \cos \left( \frac{17}{12} 2^n x \right) - \frac{17}{12} 2^n 2^{-ns} \phi(x) \sin \left( \frac{17}{12} 2^n x \right).
\]
Thus, we have
\[
u_0^k \partial_x \Delta_n u_0 = 2^{-ns} u_0^k(x) \phi'(x) \cos \left( \frac{17}{12} 2^n x \right) - \frac{17}{12} 2^n 2^{-ns} u_0^k(x) \phi(x) \sin \left( \frac{17}{12} 2^n x \right).
\]
Since \( u_0^k(x) \) is a real-valued and continuous function on \( \mathbb{R} \), then there exists some \( \delta > 0 \) such that
\[
|u_0^k(x)| \geq \frac{1}{2} |u_0^k(0)| = \frac{1}{2} (\phi(0) \sum_{n=0}^{\infty} 2^{-ns})^k = \frac{2^{sk} \phi^k(0)}{2^{2^s - 1} k} \text{ for any } x \in B_\delta(0). \tag{3.10}
\]
Thus we have from (3.10)
\[
\|u_0^k \partial_x \Delta_n u_0\|_{L^p} \geq C 2^n 2^{-ns} \|\phi(x) \sin \left( \frac{17}{12} 2^n x \right)\|_{L^p(B_\delta(0))} - C 2^{-ns} \|\phi'(x) \phi^k(x) \cos \left( \frac{17}{12} 2^n x \right)\|_{L^p}
\geq (c2^n - C) 2^{-ns}.
\]
Thus we have
\[
\|u_0^k \partial_x \Delta_n u_0\|_{L^p} \geq (c2^n - C) 2^{-ns}.
\]
We choose \( n \) large enough such that \( C < \frac{c}{2} 2^n \) and then finish the proof of Lemma 3.2.

3.2 Error Estimates

Proposition 3.1 Assume that \( \|u_0\|_{B^s_{p,\infty}} \lesssim 1 \). Let \( u \in L^\infty_T B^s_{p,\infty} \) be the solution of the Cauchy problem (1.4), then under the assumptions of Theorem 1.2, we have
\[
\|S_t(u_0) - u_0\|_{B^{s-1}_{p,\infty}} \lesssim t. \tag{3.11}
\]
Furthermore, there holds
\[
\|w\|_{B^s_{p,\infty}} \lesssim t^2, \tag{3.12}
\]
here and in what follows we denote
\[
w := S_t(u_0) - u_0 - tv_0 \quad \text{with} \quad v_0 := P(u_0) + Q(u_0) - u_0^k \partial_x u_0.
\]
Proof. For simplicity, we denote \( u(t) := S_t(u_0) \) here and in what follows. Due to the fact 
\[ B^s_{p,\infty} \hookrightarrow \text{Lip}, \]
we know that there exists a positive time \( T = T(\|u_0\|_{B^s_{p,\infty}}) \) such that
\[ \|u(t)\|_{L^\infty B^s_{p,\infty}} \leq C\|u_0\|_{B^s_{p,\infty}} \leq C. \]
Using the Mean Value Theorem and Remark 2.2 with \( v = 0 \), we obtain from (1.4) that
\[
\|u(t) - u_0\|_{B^{s-1}_{p,\infty}} \leq \int_0^t \|\partial_\tau u\|_{B^{s-1}_{p,\infty}} d\tau \\
\leq \int_0^t (\|P(u)\|_{B^{s-1}_{p,\infty}} + \int_0^t \|Q(u)\|_{B^{s-1}_{p,\infty}}) d\tau + \int_0^t \|u^k \partial_x u\|_{B^{s-1}_{p,\infty}} d\tau \\
\lesssim \int_0^t \|u\|^k_{L^\infty B^s_{p,\infty}} \\
\lesssim \int_0^t \|u_0\|^k_{B^s_{p,\infty}} \\
\lesssim t. \tag{3.13}
\]
By the Mean Value Theorem and (1.4), then we obtain that
\[
\|w\|_{B^{s-2}_{p,\infty}} \leq \int_0^t \|\partial_\tau u - v_0\|_{B^{s-2}_{p,\infty}} d\tau \\
\lesssim \int_0^t (\|P(u) - P(u_0)\|_{B^{s-2}_{p,\infty}} + \|Q(u) - Q(u_0)\|_{B^{s-2}_{p,\infty}}) d\tau \\
+ \int_0^t \|u^k \partial_x u - u_0^k \partial_x u_0\|_{B^{s-2}_{p,\infty}} d\tau. \tag{3.14}
\]
Using Remark 2.2 again yields
\[
\int_0^t (\|P(u) - P(u_0)\|_{B^{s-2}_{p,\infty}} + \|Q(u) - Q(u_0)\|_{B^{s-2}_{p,\infty}}) d\tau \lesssim t^2, \tag{3.15}
\]
Notice that the simple fact
\[ a^{k+1} - b^{k+1} = (k + 1)(a - b)\xi^k, \quad \xi \text{ is between } a \text{ and } b, \]
we obtain from Lemma 2.2 and (3.13) that
\[
\|u^k \partial_x u(\tau) - u_0^k \partial_x u_0\|_{B^{s-2}_{p,\infty}} \lesssim \|u^{k+1}(\tau) - u_0^{k+1}\|_{B^{s-1}_{p,\infty}} \\
\lesssim \|u(\tau) - u_0\|_{B^{s-1}_{p,\infty}} \|u_0\|_{B^{s-1}_{p,\infty}}^k \\
\lesssim \tau. \tag{3.16}
\]
Inserting (3.15) and (3.16) into (3.14) yields
\[ \|w\|_{B^{s-2}_{p,\infty}} \lesssim t^2. \]
Thus, we finish the proof of Proposition 3.1.
Now we present the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Notice that

\[ S_t(u_0) - u_0 = tv_0 + w(t, u_0) \quad \text{and} \quad v_0 = P(u_0) + Q(u_0) - u_0^k \partial_x u_0, \]

by the triangle inequality and Propositions 3.1, we deduce that

\[
\| S_t(u_0) - u_0 \|_{L^p_{B_{p, \infty}}} \geq 2^{ns} \| \Delta_n(S_t(u_0) - u_0) \|_{L^p} \\
= 2^{ns} \| \Delta_n(tv_0 + w(t, u_0)) \|_{L^p} \\
\geq t2^{ns} \| \Delta_n(v_0) \|_{L^p} - 2^{2n-2} \| \Delta_n(w(t, u_0)) \|_{L^p} \\
\geq t2^{ns} \| \Delta_n(u_0^k \partial_x u_0) \|_{L^p} - t2^{ns} \| \Delta_n(P(u_0) + Q(u_0)) \|_{L^p} \\
- C2^n \| w(t, u_0) \|_{L^p_{B_{p, \infty}}} \\
\geq t2^{ns} \| u_0^k \partial_x u_0 \|_{L^p} - t2^{ns} \| [\Delta_n, u_0^k \partial_x u_0] \|_{L^p} \\
- t \| P(u_0) + Q(u_0) \|_{L^p_{B_{p, \infty}}} - C2^{2n} t^2 \\
\geq t2^{ns} \| u_0^k \partial_x u_0 \|_{L^p} - Ct \| 2^{ns} \| [\Delta_n, u_0^k \partial_x u_0] \|_{L^p} \\
- t \| P(u_0) + Q(u_0) \|_{L^p_{B_{p, \infty}}} - C2^{2n} t^2. \tag{3.17}
\]

By Lemmas 2.2-2.3, one has

\[
\| P(u_0) + Q(u_0) \|_{L^p_{B_{p, \infty}}} \lesssim \| u_0^{-1} (\partial_x u_0)^2 + u_0^{k+1} \|_{L^p_{B_{p, \infty}}^{k+1}} + \| u_0^{-2} (\partial_x u_0)^3 \|_{L^p_{B_{p, \infty}}^{k+1}} \lesssim 1
\]

and

\[
\| 2^{ns} \| [\Delta_n, u_0^k \partial_x u_0] \|_{L^p} \|_{L^p} \lesssim \| \partial_x(u_0^k) \|_{L^p} \| u_0 \|_{L^p_{B_{p, \infty}}} + \| \partial_x u_0 \|_{L^p} \| \partial_x(u_0^k) \|_{L^p_{B_{p, \infty}}} \lesssim 1.
\]

Gathering all the above estimates and Lemma 3.2 together with (3.17), we obtain

\[
\| S_t(u_0) - u_0 \|_{L^p_{B_{p, \infty}}} \geq ct2^n - Ct - C2^{2n} t^2.
\]

Taking large \( n \) such that \( c2^n \geq 2C \), we have

\[
\| S_t(u_0) - u_0 \|_{L^p_{B_{p, \infty}}} \geq \frac{c}{2} t2^n - C2^{2n} t^2.
\]

Thus, picking \( t2^n \approx \varepsilon \) with small \( \varepsilon \), we have

\[
\| S_t(u_0) - u_0 \|_{L^p_{B_{p, \infty}}} \geq \frac{c}{2} \varepsilon - C\varepsilon^2 \geq c_1 \varepsilon.
\]

This completes the proof of Theorem 1.2.

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**Conflict of interest** The authors declare that they have no conflict of interest.
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