The Bessel function expression of characteristic function

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Abstract

In this paper, we give a unified method to derive the classical characteristic functions of all elliptical and related distributions in terms of Bessel functions. The approach is based on the stochastic representation of elliptical random variable and the characteristic function of uniform distribution on the unit sphere surface in $\mathbb{R}^n$. In particular, we present the simple closed form of characteristic functions for commonly used distributions such as multivariate $t$, Pearson Type II, Pearson Type VII, Kotz type and Bessel distributions. Some extensions are also investigated.

Keywords: Characteristic functions; Bessel representation; Elliptical distributions; Location-scale mixture of elliptical distributions; Reciprocal formula; Skew-elliptical distributions

1 Introduction

The characteristic function (CF) of a random vector $\mathbf{X}$ is the function $\psi_{\mathbf{X}} : \mathbb{R}^n \to \mathbb{C}$ defined as $\psi_{\mathbf{X}}(\mathbf{t}) = E(e^{i\mathbf{t}'\mathbf{X}}), \ \mathbf{t} \in \mathbb{R}^n$. Characteristic functions play an important role in probability and statistics. In recent years, there has been a growing interest in the multivariate elliptical distributions and related multivariate distributions such that the skew-elliptical distributions,

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location-scale mixture of elliptical distributions, and so on. The characteristic functions of many multivariate distributions have been derived by many authors. For instance, a general treatment for multivariate elliptical distributions has been given in Fang, Kotz and Ng (1990). Arellano-Valle and Genton (2005) studied the moment generating function for fundamental skew-symmetric distributions, Arellano-Valle and Azzalini (2006) studied the moment generating function for unified skew-normal distributions. Kim and Genton (2011) concentrated on the characteristic functions of scale mixtures of multivariate skew-normal distributions which include some well-known distributions, for example, the skew-normal and skew-t. Vilca, Balakrishnan and Zeller (2014) investigated the multivariate skew-normal generalized hyperbolic distribution and derived its moment generating functions. Some recent results show that the characteristic functions are very useful in the study of stochastic orderings and deriving the moments for multivariate skew-elliptical distributions, see, e.g., Yin (2021), Jamali et al. (2020), Pu et al. (2021), and Yin and Balakrishnan (2024), and so on.

In the existing literature, most of characteristic functions are derived based on contour integrals which unfamiliar to statistics students, or by solving ordinary differential equations, some expressions are too complex so as to limit the application in practice, and even some expressions are wrong. The purpose of this short note is to complement the existing literature with a new simple and direct derivation for the multivariate elliptical distributions and their location-scale mixtures, which include some well-known distributions such as the multivariate normal, multivariate \( t \), multivariate Kotz-type, multivariate Pearson type VII, multivariate stable law, Cauchy, logistic, Laplace and so on. The advantage of this method is that the calculation process is simple, the contour integrals are avoided, and the expressions are more concise.

The paper is organized as follows. In Section 2, we recall some important concepts of elliptical symmetric distributions, several special functions including generalized hypergeometric series and Bessel functions and characteristic function of the uniform distribution on the unit sphere surface in \( \mathbb{R}^n \). Section 3 is the main results, in which we present a unified expression for the characteristic functions of multivariate elliptical distributions in terms of Bessel functions, and provides simple closed form of characteristic functions for commonly used distributions. Some extensions are given in Section 4. Finally, Section 5 provides the conclusions.
2 Preliminaries

In this section, we first present a brief overview of elliptical distributions, some special functions such as the generalized hypergeometric series and the Bessel functions, and an alternative expression for characteristic function of the uniform distribution on the unit sphere surface in $\mathbb{R}^n$.

2.1 Elliptical distributions

The class of elliptical distributions provides a generalization of the multivariate normal distributions. This class includes various distributions such as symmetric Kotz type distribution, symmetric multivariate Pearson type VII and symmetric multivariate stable law, multivariate Student-$t$, Cauchy, logistic, Laplace and so on. They are useful to modeling multivariate random phenomena which have heavier tails than the normal as well as having some skewness. Such a rich class of distributions can be used to model multivariate regression problems with skew-elliptical error structure.

An $n \times 1$ random vector $X = (X_1, X_2, \cdots, X_n)'$ is said to have an elliptical distribution if its characteristic function is $e^{it'\mu} \phi(t'\Sigma t)$ for all $t \in \mathbb{R}^n$, where $\phi$ is called the characteristic generator satisfying $\phi(0) = 1$, $\mu$ (n-dimensional vector) is its location parameter and $\Sigma$ ($n \times n$ positive semi-definite matrix) is its dispersion matrix. We shall write $X \sim ELL_n(\mu, \Sigma, \phi)$. It is well known that $X$ admits the stochastic representation

$$X = \mu + RA'U^{(n)}, \quad (2.1)$$

where $A$ is a square matrix such that $A'A = \Sigma$, $U^{(n)}$ is uniformly distributed on the unit sphere surface in $\mathbb{R}^n$, $R \geq 0$ is the random variable with $R \sim F$ in $[0, \infty)$ called the generating variate and $F$ is called the generating distribution function, $R$ and $U^{(n)}$ are independent. In general, an elliptically distributed random vector $X \sim ELL_n(\mu, \Sigma, \phi)$ does not necessarily possess a density. However, if density of $X$ exists it must be of the form

$$f(x) = c_n|\Sigma|^{-\frac{1}{2}}g((x - \mu)^T\Sigma^{-1}(x - \mu)), \quad x \in \mathbb{R}^n, \quad (2.2)$$

for some non-negative function $g$ satisfying the condition

$$\int_0^\infty z^{n-1}g(z)dz < \infty,$$
and a normalizing constant $c_n$ given by

$$c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \left(\int_0^\infty z^{\frac{n}{2}-1} g(z) dz\right)^{-1}. \tag{2.3}$$

The function $g$ is called the density generator. One sometimes writes $X \sim ELL_n(\mu, \Sigma, g)$ for the $n$-dimensional elliptical distributions generated from the function $g$. In this case $R$ in (2.1) has the pdf given by

$$h_R(v) = c_n \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} v^{n-1} g(v^2), \quad v \geq 0. \tag{2.4}$$

A comprehensive review of the properties and characterizations of elliptical distributions can be found in Cambanis et al. (1981), Fang, Kotz and Ng (1990); see also some recent papers, Zuo, Yin and Balakrishnan (2021), Wang and Yin (2021) and Yin, Wang and Sha (2022).

### 2.2 Special functions

The following special functions and mathematical results will be useful in our analyses. The details can be found in Gradshteyn and Ryzhik (2007) and Slater (1960). The series

$$1_F2(\alpha_1; \beta_1, \beta_2; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k}{(\beta_1)_k(\beta_2)_k} \frac{z^k}{k!},$$

is called a generalized hypergeometric series of order $(1, 2)$, where $(\alpha)_k$ and $(\beta)_k$ represent Pochhammer symbols and $(x)_k = x(x+1) \cdots (x+k-1)$, and $(x)_0 = 1$. The series given by

$$2F1(\alpha, \beta; \gamma; z) \equiv F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \frac{z^k}{k!},$$

is called a generalized hypergeometric series of order $(2, 1)$. The series

$$1F1(\alpha; \gamma; z) \equiv F(\alpha, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k} \frac{z^k}{k!},$$
is called a confluent hypergeometric function or Kummer’s function. The series
\[ {}_0F_1(\gamma; z) = \sum_{k=0}^{\infty} \frac{1}{(\gamma)_k} \frac{z^k}{k!}, \]
is called a a generalized hypergeometric series of order \((0, 1)\). Bessel function of the first kind, \(J_\nu(x)\), is defined as
\[ J_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!\Gamma(\nu + k + 1)} \left( \frac{x}{2} \right)^{\nu + 2k}, \]
or
\[ J_\nu(x) = \frac{x^\nu}{2^{\nu}\Gamma(\nu + 1)} {}_0F_1 \left( \nu + 1; -\frac{x^2}{4} \right). \]
Modified Bessel function of the first kind, \(I_\nu(x)\), is defined as
\[ I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu + k + 1)} \left( \frac{x}{2} \right)^{\nu + 2k}. \]
Modified Bessel function of the second kind (also called the MacDonal function) \(K_\nu(x)\) with order \(\nu\) may be defined by the following integral:
\[ K_\nu(x) = \left( \frac{2}{x} \right)^\nu \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}} \int_0^{\infty} \frac{\cos(xu)}{(1 + u^2)^{\nu + \frac{1}{2}}} du, x > 0, \nu > -\frac{1}{2}. \]
I and K have asymptotic properties: For \(\nu > 0\), we have
\[ x^{-\nu}I_\nu(x) \to \frac{1}{2^{\nu}\Gamma(1 + \nu)}, \quad x^{\nu}K_\nu(x) \to 2^{\nu-1}\Gamma(\nu), \text{ as } x \to 0. \]

2.3 Characteristic function of \(U^{(n)}\)
Let \(U^{(n)}\) be uniformly distributed on the unit sphere surface in \(\mathbb{R}^n\) which has been introduced in Section 1. Let \(\Omega_n(||t||^2), t \in \mathbb{R}^n\) be the characteristic function of \(U^{(n)}\), where \(||t||^2 = t't\). The well-known three equivalent forms of \(\Omega_n(||t||^2)\) can be found in Fang et al. (1990, p.70). Another result for characteristic function of \(U^{(n)}\) is due to Schoenberg (1938) which will be useful for the rest of the paper. Here we present a simple proof by using an integral formula for \(J_\nu\).
Lemma 2.1. (Schoenberg (1938)). The characteristic function of $U(n)$ can be expressed as

$$\Omega_n(||t||^2) = \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{||t||}\right)^{n/2} J_{n/2}(||t||), \ t \in \mathbb{R}^n,$$  \hspace{1cm} (2.5)

where $J_\nu$ is the Bessel function of the first kind of order $\nu$.

**Proof** It follows from the proof of Theorem 3.1 in Fang et al. (1990) that

$$\Omega_n(||t||^2) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sqrt{\pi} \int_{-1}^{1} e^{||t||u} (1 - u^2)^{n/2-1} du.$$

By using the formula

$$\int_{-1}^{1} e^{iux} (1 - x^2)^{\nu-1} dx = \sqrt{\pi} \Gamma(\nu) \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} J_{\nu-\frac{1}{2}}(\mu),$$

we arrive at the result.

**Remark 2.1.** Using the relationship of $J_\nu$ and $\, _0F_1$ leads the following well known fact (see, e.g. Fang et al. (1990), P.69):

$$\Omega_n(||t||^2) = \, _0F_1\left(\frac{n}{2}; -\frac{||t||^2}{4}\right), \ t \in \mathbb{R}^n.$$

### 3 Main results

In this section, we derive the CFs of elliptical distributions. Moreover, we give some important special cases. The following result provided a concrete form of the characteristic generator function in terms of the Bessel functions for any elliptical distribution with density generator $g$.

**Theorem 3.1.** Suppose $X \sim ELL_n(\mu, \Sigma, g)$. Then, the characteristic function of $X$ is given by

$$\psi_X(t) = e^{it'\mu} \phi(t'\Sigma t), \ t \in \mathbb{R}^n$$ \hspace{1cm} (3.1)

for some real valued function $\phi$ defined as

$$\phi(u^2) = c_n(2\pi)^{\frac{n}{2}} u^{-\frac{n+2}{2}} \int_0^{\infty} r^{\frac{n}{2}} J_{n/2}(ru) g(r^2) dr, \ u \geq 0,$$ \hspace{1cm} (3.2)

where $c_n$ is the normalizing constant given by (2.3).
Proof Without loss of generality we assume that $\mu = 0, \Sigma = I_n$. For $t \in \mathbb{R}^n$, by using the stochastic representation (2.1) and (2.4), we obtain
\[
\phi(t^t) = E(e^{iR^tU^{(n)}}) = \int_0^\infty E(e^{iR^tU^{(n)}})P(R \in dr) \\
= \int_0^\infty \Omega_n(r^2||t||^2)h_R(r)dr \\
= \int_0^\infty \Omega_n(r^2||t||^2)\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} r^{n-1}g(r^2)dr.
\]
Therefore, utilizing (2.5) we have
\[
\phi(t^t) = c_n(2\pi)^\frac{n}{2}||t||^{-\frac{n-2}{2}} \int_0^\infty r^\frac{n}{2} J_{n-2}(r||t||)g(r^2)dr,
\]
and we arrive at (3.2). This completes the proof.

The following theorem discusses the inverse transformation problem which was put forward by one of reviewers.

**Theorem 3.2.** Suppose $X \sim ELL_n(\mu, \Sigma, \phi)$ has a density. Then, the density generator of $X$ is given by
\[
g(r) = \frac{r^{-\frac{n-2}{2}}}{c_n(2\pi)^\frac{n}{2}} \int_0^\infty u^{\frac{n}{2}} J_{n-2}(\sqrt{ru})\phi(u^2)du, \quad r \geq 0,
\]
where $c_n$ is the normalizing constant given by (2.3).

**Proof** Using the Hankel transform pair (see, Davies (2002), P.227), the integral transform (3.2) has the reciprocal formula
\[
r^{\frac{n}{2}-1}g(r^2) = \frac{1}{c_n(2\pi)^\frac{n}{2}} \int_0^\infty u^{\frac{n}{2}} J_{n-2}(ru)\phi(u^2)du, \quad r \geq 0,
\]
from which we get (3.3) immediately, ending the proof.

In the following, we shall using Theorem 3.1 to derive the CFs of uniform distribution in the unit sphere, multinormal, multivariate $t$, Pearson Type II, Pearson Type VII, Kotz type and multivariate Bessel. Of course, most of results are not new, the proposed method offers simplicity and ease of use, making it potentially valuable from a pedagogical perspective.
3.1 Uniform distribution in the unit sphere

Let \( V^{(n)} \) be distributed according to the uniform distribution in the unit sphere in \( \mathbb{R}^n \) (see Fang et al. (1990), P.74). \( V^{(n)} \) has the stochastic representation \( V^{(n)} = R U^{(n)} \). The density of \( R \) is given by

\[
f(r) = nr^{n-1}1_{(0 \leq r \leq 1)},
\]

and the density generator of \( V^{(n)} \) is given by

\[
g(r) = 1, \quad 0 \leq r \leq 1.
\]

Then

\[
c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \left( \int_0^1 z^{\frac{n}{2}-1} dz \right)^{-1} = \frac{n \Gamma\left(\frac{n}{2}\right)}{2 \pi^{\frac{n}{2}}}.
\]

By using (3.1) and (3.2) we find the characteristic function of \( V^{(n)} \) given by

\[
\psi_{V^{(n)}}(t) = 2^{\frac{n-2}{2}} n \Gamma\left(\frac{n}{2}\right) ||t||^{-\frac{n-2}{2}} \int_0^{||t||} r^{\frac{n}{2}} J_{n-2}(r||t||) dr
\]

\[
= 2^{\frac{n-2}{2}} n \Gamma\left(\frac{n}{2}\right) ||t||^{-\frac{n}{2}} J_{\frac{n}{2}}(||t||), \quad t \in \mathbb{R}^n,
\]

where we have used the integral (Gradshteyn and Ryzhik (2007, p.676)):

\[
\int_0^1 x^{\nu+1} J_\nu(ax) dx = \frac{1}{a} J_{\nu+1}(a).
\]

Equivalently,

\[
\psi_{V^{(n)}}(t) = {}_0F_1\left(\frac{n}{2} + 1; -\frac{||t||^2}{4}\right), \quad t \in \mathbb{R}^n.
\]

(3.5)

**Remark 3.1.** The characteristic function of \( V^{(n)} \) is given in Fang et al. (1990), P.75) by

\[
\phi(t't) = \frac{2}{B\left(\frac{1}{2}, \frac{n+1}{2}\right)} \int_0^\infty \cos(t'tx)(1 - x^2)^{-\frac{n+1}{2}} dx.
\]

We remark that the \( \phi \) above does not satisfies \( \phi(0) = 1 \). Actually,

\[
\phi(t't) = \frac{2}{B\left(\frac{1}{2}, \frac{n+1}{2}\right)} \int_0^1 \cos(||t||x)(1 - x^2)^{-\frac{n+1}{2}} dx.
\]

(3.6)
By using formula 8 in Gradshteyn and Ryzhik (2007, p.442)), which states that, for \(a > 0, u > 0, \Re(u) > -\frac{1}{2}\),

\[
\int_0^u (u^2 - x^2)^{\nu - \frac{1}{2}} \cos(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{a}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu}(au).
\]

we can prove that (3.3) and (3.5) are equivalent.

### 3.2 Multinormal

The random vector \(X = (X_1, X_2, \ldots, X_n)'\) is said to have a multivariate normal distribution \(N_n(\mu, \Sigma)\) if its density generator is of the form \(g_n(u) = \exp\left(-\frac{u^2}{2}\right)\) and the normalizing constant is given by \(c_n = (2\pi)^{-\frac{n}{2}}\). Applying (3.2) to \(c_n\) and \(g_n\) we get

\[
\phi(u^2) = u^{-\frac{n+2}{2}} \int_0^\infty r^{\frac{n}{2}-1} J_{n-2}(ru) \exp\left(-\frac{u^2}{2}\right) dr. \tag{3.7}
\]

By the following formula (Gradshteyn and Ryzhik (2007, p.706))

\[
\int_0^\infty x^\mu e^{-ax^2} J_\nu(\beta x) dx = \frac{\beta^\nu \Gamma\left(\frac{\nu}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)}{\left(2\nu + 1\right)^{\frac{\nu}{2}\mu + \nu + \frac{1}{2}} \Gamma\left(\nu + 1\right)} 1_F\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}; 1; \nu + 1, -\frac{\beta^2}{4a}\right),
\]

one gets that

\[
\phi(u^2) = 1_F\left(\frac{n}{2}; \frac{n}{2}, -\frac{u^2}{2}\right) = e^{-\frac{1}{2}u^2}.
\]

Thus, we get the following well known characteristic function of \(N_n(\mu, \Sigma)\):

\[
\psi_X(t) = e^{it'\mu} \exp\left(-\frac{1}{2}t'\Sigma t\right), \quad t \in \mathbb{R}^n.
\]

### 3.3 Multivariate \(t\)

The \(n\)-dimensional random vector \(X\) is said to have a multivariate generalized \(t\) distribution if its probability density function is given by (see Fang et al. (1990))

\[
f(x) = \frac{c_n}{\sqrt{|\Sigma|}} \left(1 + \frac{(x - \mu)'\Sigma^{-1}(x - \mu)}{s}\right)^{-\frac{n+m}{2}}, \quad x \in \mathbb{R}^n, s > 0, m \in \mathbb{N}_+.
\]
If \( m = s \), then \( X \) follows the multivariate \( t \) distribution. Its characteristic function is given by

\[
\psi_X(t) = e^{i\mu'\mu} \phi(t'\Sigma t), \quad t \in \mathbb{R}^n, \tag{3.8}
\]

where

\[
\phi(u^2) = \frac{2^{\frac{n}{2}} \Gamma(\frac{n}{2})}{\int_{0}^{\infty} z^{\frac{n}{2} - 1} g(z)dz} u^{-\frac{n-2}{2}} \int_{0}^{\infty} r^n J_{\frac{n-2}{2}}(ru)g(r^2)dr. \tag{3.9}
\]

Here,

\[
g(u) = \left(1 + \frac{u}{s}\right)^{-\frac{n+m}{2}}.
\]

It follows the formula (see Gradshteyn and Ryzhik (2007, p.678))

\[
\int_{0}^{\infty} J_\nu(bx) x^{\nu+1} dx = \frac{a^{\nu-\mu}b^\mu}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ab), \tag{3.10}
\]

with \( \nu = \frac{n-2}{2}, x = r, b = ||t||, a = \sqrt{s} \), we get

\[
\int_{0}^{\infty} r^\frac{n}{2} J_{\frac{n-2}{2}}(r||t||)g(||t||^2)dr = \frac{s^{\frac{m+n}{2}} ||t||^{\frac{m+n-3}{2}}}{2^{\frac{m+n-2}{2}} \Gamma(\frac{m+n}{2})} K_{-\frac{m}{2}}(\sqrt{s}||t||),
\]

where \( K \) is the modified Bessel function of the second kind. It is easy to verify that

\[
\int_{0}^{\infty} z^{\frac{n}{2} - 1} g(z)dz = s^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{m}{2})}{\Gamma(\frac{n+m}{2})}. \tag{3.11}
\]

Substituting (3.9) and (3.10) into (3.8) and note that \( K_\nu = K_{-\nu} \), we find that

\[
\phi(u^2) = \frac{||t||^{\frac{n}{2}} s^\frac{m}{2}}{2^{\frac{m+n}{2}} \Gamma(\frac{m}{2})} K_{\frac{m}{2}}(\sqrt{su}), \quad u \geq 0, \tag{3.12}
\]

which has been found by Joarder and Ali (1996), Joarder and Alam (1995) obtained the characteristic function of the elliptic \( t \)-distribution using the conditional expectation technique, Song et al. (2014) got the same result by using the fact that the multivariate/generalized \( t \) distributions can be expressed as a normal variance-mean mixture. For the special case of \( n = 1, m = s \), see Gaunt (2021). In particular, when \( m = 1, s = 1 \), by using

\[
K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x},
\]

one gets the characteristic function of multivariate Cauchy given by

\[
\psi_X(t) = e^{i\mu'\mu} \exp(-\sqrt{t'\Sigma t}), \quad t \in \mathbb{R}^n. \tag{3.13}
\]
3.4 Pearson Type II

The $n$-dimensional random vector $X$ is said to have a symmetric multivariate Pearson type II distribution with parameters $m > -1$, $\mu \in \mathbb{R}^n$, $\Sigma : n \times n$ with $\Sigma > 0$ if its probability density function is given by

$$f(x) = \frac{\Gamma\left(\frac{n}{2} + m + 1\right)}{\pi^{\frac{n}{2}}\Gamma(m + 1)\sqrt{|\Sigma|}} (1 - (x - \mu)^T\Sigma^{-1}(x - \mu))^m,$$

where $0 \leq (x - \mu)^T\Sigma^{-1}(x - \mu) \leq 1$.

This distribution was introduced by Kotz (1975) and will be denoted by $\text{MPII}_n(\mu, \Sigma)$. A closed form of the characteristic function of the multivariate Pearson type II distribution has been obtained by Joarder (1997).

By (3.2) and using the following integral (Gradshteyn and Ryzhik (2007, p.679))

$$\int_0^1 x^{\nu+1}(1-x^2)^\mu J_\nu(bx)dx = 2^\mu \Gamma(\mu + 1)b^{-(\mu+1)}J_{\nu+\mu+1}(b),$$

with $\mu = m$ and $\nu = \frac{n-2}{2}$, we get the characteristic function of multivariate Pearson type II distribution:

$$\psi_X(t) = e^{it^T\mu}2^{\frac{n}{2}+m}\Gamma\left(\frac{n}{2} + m + 1\right) (||\Sigma^{\frac{1}{2}}t||)^{-\frac{n}{2}-m}J_{\frac{n}{2}+m}(||\Sigma^{\frac{1}{2}}t||), \quad t \in \mathbb{R}^n.$$

(3.14)

Using of the following relation between the Bessel function of the first kind and the generalized hypergeometric function (Slater (1960), 1.8.5)

$$\binom{b}{1}\binom{-x^2/4}{1} = \left(\frac{x}{2}\right)^{-b}\Gamma(b + 1)J_b(x),$$

(3.15)

we get the following equivalent form of (4.10):

$$\psi_X(t) = e^{it^T\mu}\binom{n}{0}\binom{\frac{n}{2} + m + 1, -\frac{t^T\Sigma t}{4}}{1}, \quad t \in \mathbb{R}^n.$$

(3.16)

The last expression has been obtained by Li (1994).

Remark 3.2. Result (3.15) simplified the expression (2.2)-(2.4) in Sutr (1986), and also revised Joarder (1997, (2.1)) in which the following wrong identity was used:

$$\binom{b}{1}\binom{-x^2/4}{1} = \left(\frac{x}{2}\right)^{-b}\Gamma(b + 1)J_b(x).$$

(3.17)
3.5 Pearson Type VII

Let \( X \) be an \( n \)-dimensional vector distributed according to a symmetric multivariate Pearson type VII distribution with density (cf. Fang et al. (1990)):

\[
f(x) = c_n \left(1 + \frac{1}{m}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)^{-N},
\]
where \( N > \frac{n}{2}, m > 0 \) are parameters, \( c_n \) is the normalizing constant given by

\[
c_n = \frac{\Gamma(N)}{(m\pi)^{n/2}\Gamma(N - n/2)}.
\]

This subclass includes a number of important distributions such as the multivariate \( t \)-distribution for \( N = \frac{n+m}{2} \) and the multivariate Cauchy distribution for \( m = 1 \) and \( N = \frac{n+1}{2} \).

Using the following integral formula (see Gradshteyn and Ryzhik (2007, p.678))

\[
\int_0^\infty J_\nu(bx)x^{\mu+1}dx = \frac{a^{\nu-\mu}b^\mu}{2\mu\Gamma(\mu+1)}K_{\nu-\mu}(ab),
\]
we get

\[
\int_0^\infty r^{\frac{N}{2}}J_{\frac{N}{2}}\left(r||t||\right)\left(\frac{m+r^2}{m}\right)^{-N}dr = \frac{m^{\frac{N}{2}+\frac{N}{2}}||t||^{N-1}}{2^{N-1}\Gamma(N)}K_{\frac{N}{2}-N}(\sqrt{m||t||}).
\]

It is easy to verify that

\[
\int_0^\infty r^{\frac{N}{2}+1}\left(1 + \frac{r}{m}\right)^{-N}dr = m^{\frac{N}{2}}\frac{\Gamma(\frac{N}{2})\Gamma(N - \frac{n}{2})}{\Gamma(N)}.
\]

Substituting (3.17) and (3.18) into (3.2) we obtain

\[
\psi(||t||^2) = e^{it^T \mu \phi(t^T \Sigma t)}, \; t \in \mathbb{R}^n,
\]
and

\[
\phi(u^2) = \frac{2^{\frac{N}{2}-1+N+1}}{\Gamma(N - \frac{n}{2})}m^{\frac{N}{2}+\frac{N}{2}}u^{N-\frac{n}{2}}K_{\frac{N}{2}-N}(\sqrt{mu})
\]
\[
= \frac{2^{\frac{N}{2}-N+1}}{\Gamma(N - \frac{n}{2})}m^{\frac{N}{2}+\frac{N}{2}}u^{N-\frac{n}{2}}K_{\frac{N}{2}-N}(\sqrt{mu}).
\]
In particular, when \( N = (n + m)/2 \) and \( m = s \), we recover the result (3.11).

The characteristic generator of \( X \) is obtained in Fang et al. (1990, (3.30)):

\[
\phi(u^2) = \frac{2\Gamma(N - (n - 1)/2)}{\sqrt{\pi}\Gamma(N - n/2)} \int_0^\infty \cos(\sqrt{mt})(1 + t^2)^{-N+(n-1)/2} dt,
\]

which is equivalent to (3.19). In fact, by using the following integral (see Gradshteyn and Ryzhik (2007, p.442))

\[
\int_0^\infty (\beta^2 + x^2)^{\nu-1/2} \cos(ax)dx = \frac{1}{\sqrt{\pi}} \cos(\pi\nu) \left( \frac{2\beta}{a} \right) ^\nu \Gamma(\nu + \frac{1}{2}) K_{\nu}(a\beta),
\]

where \( a > 0, \Re(\beta) > 0, \Re(\nu) < \frac{1}{2} \), we can rewritten above \( \phi \) as

\[
\phi(u^2) = \frac{2\Gamma(N - (n - 1)/2)}{\sqrt{\pi}\Gamma(N - n/2)} \frac{1}{\sqrt{\pi}} \cos \left( \pi \left( \frac{n}{2} - N \right) \right) \left( \frac{2}{\sqrt{mu}} \right)^{n/2-N} \times \Gamma \left( \frac{n}{2} - N + \frac{1}{2} \right) K_{N-1/2}(\sqrt{mu})
\]

\[
= \frac{2^{\frac{s}{2}-N+1}}{\Gamma(N - \frac{n}{2})} \left( \frac{1}{\sqrt{mu}} \right)^\frac{n}{2}-N K_{N-\frac{n}{2}}(\sqrt{mu}),
\]

where we have used the following formula for gamma function:

\[
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, x \in (0,1),
\]

as desired.

### 3.6 Kotz type

Let \( X \) be distributed according to a symmetric Kotz type distribution with density

\[
f(x) = c_n|\Sigma|^{-\frac{1}{2}}[(x - \mu)^T \Sigma^{-1} (x - \mu)]^{N-1} \exp \left( -r[(x - \mu)^T \Sigma^{-1} (x - \mu)]^{s} \right),
\]

for \( x \in \mathbb{R}^n \), where \( \mu \in \mathbb{R}^n \), \( \Sigma \) is a positive \( n \times n \) matrix, \( r, s > 0, 2N + n > 2 \) are parameters and \( c_n \) is a normalizing constant:

\[
c_n = \frac{s \Gamma(\frac{N}{2})}{\pi^{\frac{n}{2}} \Gamma(2N + n - 2)/2s)}^{r(2N+n-2)/2s}.
\]
The density generator takes the form
\[ g(l) = l^{N-1} e^{-rl}, \quad l > 0. \]

We denote it by \( X \sim KTD_n(\mu, \Sigma, N, r, s) \). When \( N = s = 1 \) and \( r = \frac{1}{2} \), the distribution reduces to a multivariate normal distribution and when \( N = 1 \) and \( r = \frac{1}{2} \) the distribution reduces to a multivariate power exponential distribution. A series form of the characteristic function of the symmetric Kotz type distribution has been obtained by Iyengar and Tong (1989), Li (1994). Here we give an integral form in terms of the Bessel functions. In doing so, by (3.1) and (3.2) the characteristic function of \( X \) is given by
\[ \psi_X(t) = e^{i\mathbf{t}'\mu} \phi(t'\Sigma t), \quad t \in \mathbb{R}^n, \quad (3.22) \]
where
\[
\phi(u^2) = c_n(2\pi)^{\frac{n}{2}} u^{-\frac{n-2}{2}} \int_0^\infty l^\frac{n}{2} + 2(N-1) J_{n-2}(lu)e^{-rl^2s} dl
= \frac{2^n s \Gamma((\frac{n}{2}) \Gamma(2N+n-2)/2s)}{\Gamma(2N+n-2)/2s} u^{-\frac{n-2}{2}}
\times \int_0^\infty l^\frac{n}{2} + 2(N-1) J_{n-2}(lu)e^{-rl^2s} dl, \quad u \geq 0. \quad (3.23)
\]
In particular, when \( s = 1 \), using the following integral (see Gradshteyn and Ryzhik (2007, p.706))
\[
\int_0^\infty x^\mu e^{-\alpha x^2} J_\nu(\beta x) dx = \frac{\beta^\nu \Gamma(\frac{1}{2} (\nu + \mu + 1))}{2^{\nu+1} \Gamma(\nu + \mu + 1) \Gamma(\nu+1)} 1_F(\frac{1}{2} (\nu + \mu + 1); \nu + 1; -\frac{\beta^2}{\alpha}),
\]
we get
\[ \phi(u^2) = 1_F\left(\frac{n}{2} + N - 1; \frac{n}{2} - \frac{u^2}{4r}\right), \quad u \geq 0, \quad (3.24) \]
which is the same form as of Li (1994); when \( s = \frac{1}{2} \), using the integral (see Gradshteyn and Ryzhik (2007, p.702))
\[
\int_0^\infty e^{-ax} J_\nu(\beta x)x^{\nu+1} dx = \frac{2\alpha(2\beta)^\nu \Gamma(\nu + \frac{3}{2})}{\sqrt{\pi} (\alpha^2 + \beta^2)^{\nu+\frac{1}{2}}},
\]
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we get

$$
\phi(u^2) = \frac{2^{n-1} \Gamma(n) \Gamma(n+1) r^{2N+n-1}}{\sqrt{\pi} \Gamma(2N+n+2)(r^2 + u^2)^{n+1}} \quad u \geq 0.
$$

(3.25)

When \( n = 2 \) and \( N = 1 \), (3.24) reduces to the simple form (see Nadarajah (2003))

$$
\phi(u^2) = \frac{r^3}{(r^2 + u^2)^{\frac{n}{2}}} \quad u \geq 0.
$$

(3.26)

### 3.7 Multivariate Bessel

An \( n \times 1 \) random vector \( \mathbf{X} \) is said to have a symmetric multivariate Bessel distribution if the density has the form (see, e.g. Fang et al. (1990))

$$
f(\mathbf{x}) = C_n |\Sigma|^{-\frac{n}{2}} g((\mathbf{x} - \mathbf{\mu})^T \Sigma^{-1} (\mathbf{x} - \mathbf{\mu})) \quad \mathbf{x} \in \mathbb{R}^n,
$$

where

$$
g(t) = \left( \frac{\sqrt{t}}{\beta} \right)^a K_a \left( \frac{\sqrt{t}}{\beta} \right), \quad a > -\frac{n}{2}, \beta > 0,
$$

$$
C_n^{-1} = 2^{a+n-1} \frac{\beta^{n+a}}{\Gamma \left( a + \frac{n}{2} \right)}.
$$

It can be shown that

$$
\int_0^\infty r^{\frac{n}{2}} J_{n-2}(ru) r^a K_a \left( \frac{r}{\beta} \right) dr = 2^{\frac{n}{2}+a-1} u^{\frac{n}{2}-1} \beta^{-a} \frac{\Gamma \left( \frac{n}{2} + a \right)}{(u^2 + \beta^{-2})^{\frac{n}{2}+a}},
$$

from which and using (3.2) we get that

$$
\phi(u^2) = \frac{1}{(1 + u^2 \beta^2)^{\frac{n}{2}+a}} \quad u \geq 0.
$$

So the characteristic function of \( \mathbf{X} \) is given by

$$
\psi_{\mathbf{X}}(\mathbf{t}) = \frac{e^{i\mathbf{t}^T \mathbf{\mu}}}{(1 + \beta^2 \mathbf{t}^T \Sigma \mathbf{t})^{\frac{n}{2} + a}}, \quad \mathbf{t} \in \mathbb{R}^n.
$$

In particular, when \( a = \frac{1}{2} \) and \( \beta = 1 \), we get the characteristic function of the Type-II multivariate \( \mathcal{L} \) distribution (see, Shi et al. (2022)):

$$
\psi_{\mathbf{X}}(\mathbf{t}) = \frac{e^{i\mathbf{t}^T \mathbf{\mu}}}{(1 + \mathbf{t}^T \Sigma \mathbf{t})^{\frac{n+1}{2}}}, \quad \mathbf{t} \in \mathbb{R}^n.
$$
3.8 Multivariate Logistic

An elliptical vector $X$ belongs to the family of multivariate logistic distributions if its density generator has the form

$$g(u) = \frac{e^{-u}}{(1 + e^{-u})^2}.$$ 

A generalized elliptically symmetric logistic distribution including the multivariate logistic distribution can be found in Yin et al. (2022). By using (3.1) and (3.2) we find the characteristic function of $X$ given by

$$\psi_X(t) = e^{it^t\mu} \phi(t^t \Sigma t), \ t \in \mathbb{R}^n$$

where

$$\phi(u^2) = c_n (2\pi)^{n/2} u^{-n/2} \int_0^\infty r^{n/2} J_{n/2}(ru) \frac{e^{-r^2}}{(1 + e^{-r^2})^2} dr, \ u \geq 0.$$ 

Here, $c_n$ is the normalizing constant given by

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left( \int_0^\infty x^{n/2-1} \frac{e^{-x}}{(1 + e^{-x})^2} dx \right)^{-1}.$$ 

The characteristic generator of $X$ is given in Li (1994) by

$$\phi(x) = \sum_{k=0}^\infty \left( \frac{x}{4\pi} \right)^k \frac{c_n}{k!c_{n+2k}}.$$ 

3.9 Multivariate exponential power

An elliptical vector $X$ is said to have a multivariate exponential power distribution if its density generator has the form

$$g(u) = \exp(-ru^s), \ r > 0, s > 0.$$ 

When $r = s = 1$, this family of distributions clearly reduces to the multivariate normal family. If $s = 1/2, r = \sqrt{2}$, we have the family of double exponential or Laplace distributions. By using (3.1) and (3.2) we find the characteristic function of $X$ given by

$$\psi_X(t) = e^{it^t\mu} \phi(t^t \Sigma t), \ t \in \mathbb{R}^n$$
where
\[ \phi(u^2) = c_n(2\pi)^{n/2} u^{-n/2} \int_0^\infty r^{n/2} J_{n/2}^2(ru) \exp(-ru^2) dr, \quad u \geq 0. \]

Here, \( c_n \) is the normalizing constant given by
\[
c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left( \int_0^\infty x^{n/2-1} \exp(-rx^2) dx \right)^{-1} = \frac{s \Gamma(n/2)}{(2\pi)^{n/2} \Gamma(n/2s)} r^{n/2s}. \]

4 Some extensions

In this section, we consider CFs of the location-scale mixture of elliptical distributions and CFs of multivariate generalized skew-elliptical distribution, which can be as the generalizations of elliptical distributions.

4.1 CFs of location-scale mixture of elliptical distributions

An \( n \)-dimensional random variable \( X \) is said to have a location-scale mixture of elliptical distributions with the parameters \( \mu, \gamma \) and \( \Sigma \), if
\[
X = \mu + V \gamma + \sqrt{V} \Sigma^{1/2} Z, \tag{4.1}
\]
where \( Z \sim ELL_n(0, I_n, g) \), \( V \) is a nonnegative, scalar-valued random variable with the distribution \( F \), \( Z \) and \( V \) are independent, \( \mu, \gamma \in \mathbb{R}^n \), \( \Sigma \in \mathbb{R}^{n \times n} \) with \( \Sigma > 0 \), and \( \Sigma^{1/2} \) is the square root of \( \Sigma \). Here \( 0 \) is an \( n \times 1 \) vector of zeros, and \( I_n \) is \( n \times n \) identity matrix.

Note that when \( Z \sim KTD_n(0, I_n, N, \frac{1}{2}, s) \) we have the variance-mean mixture of the Kotz-type distribution introduced by Arslan (2009); When \( Z \sim N_n(0, I_n) \) we get the multivariate normal variance-mean mixture distribution (see, e.g., McNeil et al. (2005)).

**Theorem 4.1.** The characteristic function of \( X \) defined by (4.1) has the form
\[
\psi_X(t) = c_n(2\pi)^{n/2} e^{it^\prime \mu} \int_0^\infty e^{ivt^\prime \gamma} \phi(vt^\prime \Sigma t) dF(v), \quad t \in \mathbb{R}^n, \tag{4.2}
\]
where \( c_n \) is defined by (2.3) and \( \phi \) is defined by (3.2).
Proof By using (3.1) and (3.2), the characteristic function of $X$ can be written as

$$
\psi_X(t) = e^{it\mu}E\{E(e^{it'(V\gamma+\sqrt{V}\Sigma^\frac{1}{2}})|V)\}
$$

$$
= e^{it\mu}E\{e^{it'\gamma}E(e^{it'\sqrt{V}\Sigma^\frac{1}{2}})|V)\}
$$

$$
= e^{it\mu} \int_0^\infty e^{it'\gamma}E(e^{it'\sqrt{V}\Sigma^\frac{1}{2}})P(V \in dv)
$$

$$
= e^{it\mu} \int_0^\infty e^{it'\gamma}\phi(vt'\Sigma)P(V \in dv)
$$

$$
= e^{it\mu} \int_0^\infty e^{it'\gamma}\phi(vt'\Sigma)dF(v).
$$

This ends the proof of Theorem 4.1.

4.2 CFs of scale mixture of the uniform distributions

A random vector $X \sim S_n(g)$ is said to have a scale mixture of the uniform (SMU) representation, if it can be represented as $X = WV^{(n)}$, where $V^{(n)}$ is the random vector distributed uniformly inside a unit sphere in $\mathbb{R}^n$, $W$ is a positive random variable and is distributed independently of $V^{(n)}$. The SMU class is more general than the normal scale-mixing class and the SMU property is characterised by Theorem 2 in Fung and Seneta (2008). $X = WV^{(n)}$ if and only if $X$ is star unimodal which is equivalent to $g'(x) \leq 0$ for $x > 0$. In this case, $W$ has a density given by

$$
f_W(w) = -\frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}w^n g'(w^2), \ w > 0. \quad (4.3)
$$

Examples of star unimodal elliptical distributions are multivariate normal, multivariate exponential power, multivariate $t$, multivariate Cauchy, Stable laws and Pearson type VII distributions, and so on.

The characteristic function of $X$ is given by

$$
\psi_X(t) = E(E(e^{it'V^{(n)}}|W))
$$

$$
= \int_0^\infty E(e^{it'V^{(n)}})P(W \in dw)
$$

$$
= -(2\pi)^{\frac{n}{2}}||t||^{-\frac{n}{2}} \int_0^\infty w^{\frac{n}{2}}J_{\frac{n}{2}}(w||t||)g'(w^2)dw, \ t \in \mathbb{R}^n,
$$

where we have used (3.3) and (4.3).
4.3 CFs of multivariate generalized skew-elliptical distribution

An $n$-dimensional random variable $Y$ is said to have a multivariate generalized skew-elliptical (GSE) distribution, if its density has the following form (cf. Genton and Loperfido, 2005)

$$f(x) = 2|\Sigma|^{-\frac{1}{2}}c_n g((x - \mu)^T \Sigma^{-1}(x - \mu))\pi(\Sigma^{-\frac{1}{2}}(x - \mu)), \ x \in \mathbb{R}^n, \quad (4.4)$$

where $\Sigma^{-\frac{1}{2}}$ is the inverse of $\Sigma^{\frac{1}{2}}$, $|\Sigma|^{-\frac{1}{2}}c_n g((x - \mu)^T \Sigma^{-1}(x - \mu))$ is a probability density function of $n$-variate elliptical distribution $X \sim \text{ELL}_n(\mu, \Sigma, g)$, $c_n$ is the normalizing constant given by (2.3), $\pi$ is the skewing function which satisfies $0 \leq \pi(y) \leq 1$ and $\pi(-y) + \pi(y) = 1$ for $y \in \mathbb{R}^n$.

If $\pi(y) = F_g^{\ast}(\alpha' y)$, where $F_g^{\ast}(\cdot)$ is a univariate cdf of standard elliptical distribution with density generator $g^{\ast}$, one gets the skew-elliptical distribution introduced in Vernic (2006); If $\pi(y) = H(\alpha' \Sigma^{\frac{1}{2}} y)$, where $H(\cdot)$ is the cdf of a distribution symmetric around 0, one gets the skew-elliptical distribution introduced in Azzalini and Dalla-Valle (1996). The skew-elliptical distributions include the more familiar skew-normal, skew-$t$ and skew-Cauchy distributions.

The characteristic function of (4.4), in the special case of $\mu = 0$ and $\Sigma = I_n$, has been derived by Shushi (2016) as follows:

$$\psi_X(t) = 2\phi(t' t)k_n(t), \ t \in \mathbb{R}^n, \quad (4.5)$$

where $\phi$ is the characteristic generator of $X$, $k_n$ is a function that satisfies $0 \leq k_n(t) \leq 1$ and $k_n(-t) + k_n(t) = 1$ for $t \in \mathbb{R}^n$.

In general case, $\psi_X(t)$ has the form (see, Shushi (2018)):

$$\psi_X(t) = 2e^{it' \mu}\phi(t' \Sigma t)a_n(t), \ t \in \mathbb{R}^n, \quad (4.6)$$

where

$$a_n(t) = \frac{c_n}{\phi(t' \Sigma t)} \int_{\mathbb{R}^n} \cos(t' y)g(y' \Sigma^{-1} y)\pi(y)dy.$$ 

Here, $a_n$ satisfies $0 \leq a_n(t) \leq 1$ and $a_n(-t) + a_n(t) = 1$ for $t \in \mathbb{R}^n$.\"
Applying Theorem 3.1, the characteristic generator $\phi$ in (4.5) and (4.6) is given by

$$
\phi(u^2) = c_n(2\pi)^{n/2} u^{-n/2} \int_0^\infty r^{\frac{n}{2}} J_{n/2}(ru) g(r^2) dr, \quad u \geq 0,
$$

where $c_n$ is the normalizing constant given by (2.3).

5 Conclusions

In this paper, we have given a unified argument for the characteristic functions of all elliptical and related distributions in terms of Bessel functions which avoid the computation of the contour integration. The approach was based on the stochastic representation of elliptical random variable and the characteristic function of uniform distribution on the unit sphere surface in $\mathbb{R}^n$. In particular, we present the simple closed form of characteristic functions for commonly used distributions such as multivariate $t$, Pearson Type II, Pearson Type VII, Kotz type and Bessel distributions. The results can be easily generalized to the case having location and scale parameters and to the skewed multivariate cases. We remark that the obtained results can be also expressed in terms of the confluent hypergeometric functions.

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