Continuous Symmetric Perturbations of Planar Power Law Forces

C. Azevêdo and P. Ontaneda

Abstract

We show the existence of periodic solutions for continuous symmetric perturbations of certain planar power law problems.

In this paper we study continuous symmetric perturbations of planar power law problems of the form $\ddot{r} = g(r, \mu)$, where the unperturbed problem is $\ddot{r} = g(r, 0) = -\frac{\kappa}{|r|^{\alpha+2}}r$, $\kappa > 0$, $0 \leq \alpha$. In particular, if $\alpha = 1$, the unperturbed problem is Kepler’s problem.

We prove the existence of periodic solutions of perturbed problems as above, close to a given circular orbit of the unperturbed problem. We have two cases. When $\alpha = 1$ (that is, the unperturbed problem is Kepler’s problem) we will require that the perturbed problems are symmetric with respect to the $x$ and $y$ axes. For $\alpha \neq 1$, we will require just one symmetry.

Here are the statements of our main results, for $\alpha = 1$ and $\alpha \neq 1$:

(Notation: $r(t, x, v, \mu)$ denotes the solution of $\ddot{r} = g(r, \mu)$ with initial conditions $r(0, x, v, \mu) = x$ and $\dot{r}(0, x, v, \mu) = v$.)

**Theorem 0.1** Let $C$ be a circle centered at the origin $(0, 0)$ of $\mathbb{R}^2$, $x_0 \in C \cap \{x\text{ -- axis}\}$ and let $U$ an open neighborhood of $C$ of the form $C \subset U \subset (\mathbb{R}^2 - \{(0,0)\})$.

Let $a > 0$ and $g : U \times (-a, a) \to \mathbb{R}^2$ continuous such that

(i) $g(r, 0) = -\frac{\kappa}{|r|^{\alpha+2}}r$, $\kappa > 0$,

(ii) $g(r, \mu)$ is $C^1$ in the variable $r \in U$, for each $\mu$,

(iii) for all $\mu$, $g$ is invariant (as a vector field) by the reflections

$$\varphi_1(x, y) = (-x, y), \quad \varphi_2(x, y) = (x, -y).$$

Then there is $\delta_0$, $0 < \delta_0 < a$, with the following property. For each $\mu \in (-\delta_0, \delta_0)$ there is a velocity $v_\mu$ such that the solution $r_{v_\mu, \mu}(t) := r(t, x_0, v_\mu, \mu)$ of $\ddot{r} = g(r, \mu)$ is periodic. Moreover, given $\eta > 0$ we can choose $\delta_0 > 0$ such that

(1) the traces of these solutions are simple closed curves, symmetric with respect to the $x$ and $y$ axes, and enclose the origin,

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\textit{(2)} all velocities $v_\mu$ are $\eta$-close, $\mu < \delta_0$.

\textbf{Theorem 0.2} Let $C$ be a circle centered at the origin $(0,0)$ of $\mathbb{R}^2$, $x_0 \in C \cap \{x - \text{axis}\}$ and $U$ an open neighborhood of $C$ of the form $C \subset U \subset (\mathbb{R}^2 - \{(0,0)\})$. Let $a > 0$ and $g : U \times (-a,a) \to \mathbb{R}^2$ continuous such that
(i) $g(r,0) = -\frac{\kappa}{|r|}r$, where $r = (x,y)$, $\kappa > 0$, and $0 \leq \alpha$, $\alpha \neq 1$, 
(ii) $g(r,\mu)$ is $C^1$ in the variable $r \in U$, for each $\mu$, 
(iii) for all $\mu$, $g$ is invariant (as a vector field) by the reflection
$$\varphi(x,y) = (x,-y).$$

Then there is $\delta_0$, $0 < \delta_0 < a$, with the following property. For each $\mu \in (-\delta_0,\delta_0)$ there is a velocity $v_\mu$ such that the solution $r_{v_\mu}(t) := r(t,x_0,v_\mu,\mu)$ of $\dot{r} = g(r,\mu)$ is periodic. Moreover, given $\eta > 0$ we can choose $\delta_0 > 0$ such that
(1) the traces of these solutions are simple closed curves, symmetric with respect to the $x$-axis, and enclose the origin, 
(2) all velocities $v_\mu$ are $\eta$-close, $\mu < \delta_0$.

Note that in both theorems $g$ is required to be defined just in a neighborhood of the circle $C$.

In general, we can not claim that $\mu \mapsto v_\mu$ is continuous, but it is possible to prove that the map $\mu \mapsto v_\mu$ satisfies a property a bit weaker than that of continuity. This is given in the addenda to the theorems above. These results were used in [5]. For more details see sections 2 and 3.

For $\alpha = 1$ a similar result is proved in [12] using Poincaré’s analytic continuation. Our result is more general in the sense that we do not demand the existence of any first integral. Also, we require $g$ to be just continuous. Still, the result in [12] requires a little less symmetry.

There are also some related results in the literature obtained using Calculus of Variations. These methods work best for $\alpha \geq 2$ (strong force), see [1], [6]. There are some results for $0 < \alpha < 2$, $\alpha \neq 1$, assuming more hypotheses, but in all cases the form of the perturbations is more restrictive and the domain must have a particular form, see [2], [3], [7]. There are also some results for $\alpha = 1$, which require less symmetry, see [2]. All these results hold, in general, for any dimension but demand more regularity and the perturbations are always potential. Note that our result holds also for non potential perturbations, where variational methods cannot be applied.

For $\alpha = 0$, we could not find related results obtained using Calculus of Variations. In [13], the results of [12] are extended to $\alpha = 0$, but in this case we demand less symmetry, and, as before, we do not demand the existence of any first integral and require $g$ to be just continuous.

It is interesting to note that the proofs of the theorems above, given in sections 2 and 3, do not use neither Calculus of Variations nor analytic continuation, but simple and elementary
geometric arguments. The results of this paper were motivated by (and needed in) the study of the fixed homogeneous circle problem [5].

This paper has three sections and two appendices. In the first section we give some preliminary results. The case \( \alpha = 1 \) is treated in section 2, and the case \( \alpha \neq 1 \) is treated in section 3. In the appendices we present proofs of some needed results for which we could not find an exact reference in the literature.

1 Preliminaries.

We begin presenting two lemmas that show that to find a periodic solution of a problem with symmetries it is enough to construct only a piece of a solution, with certain properties.

**Lemma 1.1** Let \( \Omega \subset \mathbb{R}^2 \) be an open set, with \( \varphi \Omega = \Omega \), where \( \varphi(x,y) = (x,-y) \). Let \( f : \Omega \to \mathbb{R}^2 \), be invariant by \( \varphi \), that is, \( f(\varphi p) = \varphi f(p) \), \( p \in \Omega \). If \( r : [0,\tau] \to \Omega \), \( \tau > 0 \), 
\[
\begin{align*}
    r(t) &= (x(t), y(t)), \\
    \dot{r}(t) &= (\dot{x}(t), \dot{y}(t)), \\
    \dot{r}(0), \dot{r}(\tau) &\text{ are vertical (that is,} \dot{x}(0) = \dot{x}(\tau) = 0) \\
\end{align*}
\]
is a solution of
\[
\mathbf{(1.1)}
\]
such that \( r(0), r(\tau) \in \{x\}-axis \), (that is, \( y(0) = y(\tau) = 0 \) and \( \dot{r}(0), \dot{r}(\tau) \) are vertical (that is, \( \dot{x}(0) = \dot{x}(\tau) = 0 \)), then the extension \( \bar{r} \) of \( r \), defined by:
\[
\bar{r}(t) = \begin{cases} 
    r(t - 2n\tau), & t \in [2n\tau, (2n + 1)\tau], \ n \in \mathbb{Z} \\
    \varphi r(2n\tau - t), & t \in [(2n-1)\tau, 2n\tau], \ n \in \mathbb{Z} 
\end{cases}
\]
is a periodic solution of \( \mathbf{(1.1)} \) with period \( 2\tau \). Moreover, the trace of \( \bar{r} \) is symmetric with respect to the \( x \)-axis.

**Proof:** Since \( r(t), \ t \in [0,\tau] \), is a solution of \( \mathbf{(1.1)} \), we have that \( r(t - a) \) and \( r(a + \tau - t), t \in [a,a + \tau] \), are also solutions of \( \mathbf{(1.1)} \), for all \( a \in \mathbb{R} \). Since \( f \) is invariant by \( \varphi \), we have that \( \varphi r(a - t), t \in [a - \tau, a] \), is also a solution of \( \mathbf{(1.1)} \). Therefore, each part in the definition of \( \bar{r} \) is a solution of \( \mathbf{(1.1)} \).

A direct calculation shows that these parts, and its first derivatives, coincide at the endpoints of the intervals where they are defined. In this way, \( \bar{r} \) is a well defined solution of \( \mathbf{(1.1)} \). Moreover, \( \bar{r}(0) = \bar{r}(2\tau) \) and \( \dot{r}(0) = \dot{r}(2\tau) \), and the trace of \( \bar{r} \) is symmetric with respect to the \( x \)-axis (see figure below). \( \blacksquare \)

![Figure 1.1: The solution is symmetric with respect to the x-axis](image_url)
Analogously we have:

**Lemma 1.2** Let $\Omega \subset \mathbb{R}^2$ be an open set, with $\varphi_i \Omega = \Omega$, $i = 1, 2$, where $\varphi_1(x, y) = (-x, y)$ and $\varphi_2(x, y) = (x, -y)$. Let $f : \Omega \rightarrow \mathbb{R}^2$ be invariant by $\varphi_1$ and $\varphi_2$. If $r : [0, \tau] \rightarrow \Omega$, $\tau > 0$, is a solution of

$$\dot{r} = f(r) \quad (1.2)$$

such that $r(0) \in \{x\}$-axis, $r(\tau) \in \{y\}$-axis, $\dot{r}(0)$ is vertical and $\dot{r}(\tau)$ is horizontal, then the extension $\bar{r}$ of $r$, defined by:

$$\bar{r}(t) = \begin{cases} r(t - 4n\tau), & t \in [4n\tau, (4n + 1)\tau], \\ \varphi_1 r(4(n + 2)\tau - t), & t \in [(4n + 1)\tau, (4n + 2)\tau], \\ \varphi_2 \varphi_1 r(t - (4n + 2)\tau), & t \in [(4n + 2)\tau, (4n + 3)\tau], \\ \varphi_2 r(4n\tau - t), & t \in [(4n - 1)\tau, (4n)\tau], \end{cases}$$

with $n \in \mathbb{Z}$, is a periodic solution of (1.2) with period $4\tau$. Moreover, the trace of $\bar{r}$ is symmetric with respect to the $x$ and $y$ axes.

**Proof:** Analogous to the proof of the previous lemma (see figure 1.2). □

![Figure 1.2: The solution is symmetric with respect to the x and y axes](image)

We want to study now the transversality of solutions.

**Definition 1.3** Let $I$ be an interval, $\alpha : I \rightarrow \mathbb{R}^n$ of class $C^1$ and $H^{n-1}$ a hypersurface of $\mathbb{R}^n$. We say that $\alpha$ intersects $H^{n-1}$ transversally if $\alpha(\partial I) \cap H^{n-1} = \emptyset$, $\alpha(I) \cap \partial H^{n-1} = \emptyset$, and $\dot{\alpha}(t) \notin T_{\alpha(t)}H^{n-1}$ for all $t$ such that $\alpha(t) \in H^{n-1}$. Moreover, we say that $\alpha$ intersects $H^{n-1}$ transversally in a single point, if $\alpha$ intersects $H^{n-1}$ transversally and there is a unique $t$ such that $\alpha(t) \in H^{n-1}$.

The following proposition shows that the property “$\alpha$ intersects transversally in a single point” is open in the $C^1$ topology.

**Proposition 1.4** Let $E \subset \mathbb{R}^2$ be a closed segment and let $\alpha : [0, \bar{t}] \rightarrow \mathbb{R}^2$, $\bar{t} > 0$, be $C^1$, such that $\alpha$ intersects $E$ transversally in a single point.
Then there is $\epsilon > 0$ such that if $\beta : [0, \bar{t}] \to \mathbb{R}^2$ is $C^1$ and $\|\alpha - \beta\| < \epsilon$, $\|\dot{\alpha} - \dot{\beta}\| < \epsilon$, then $\beta$ intersects $E$ transversally in a single point. (See figure 1.3.)

Figure 1.3: $E$ is closed, $\alpha$ and $\beta$ are transversal

Remarks.

(1) The fact that transversal maps form an open set in the $C^1$ topology can be found in any differential topology textbook, but we could not find a reference for the “intersect in a single point” part of the statement of the proposition above. Therefore we present a proof of this proposition in an appendix.

(2) The condition of $E$ being closed is fundamental (see figure below, where $\alpha$ intersects transversally $E$ in a single point and $\beta$ is close to $\alpha$, but $\beta$ intersects $E$ in two points).

Figure 1.4: $E$ is not closed

The next proposition is essential in the proof of the theorems in sections 2 and 3. Before, we introduce some notation:

Let $U \times U_{\mu_0} \subset \mathbb{R}^2 \times \mathbb{R}$ be an open set, where $\mu_0 \in U_{\mu_0}$, and let $g : U \times U_{\mu_0} \to \mathbb{R}^2$ be continuous and $C^1$ in $r \in U$, for each $\mu$. For each $\mu \in U_{\mu_0}$, consider the ordinary differential equation

$$\dot{r} = g(r, \mu) \quad (1.3)$$

Denote by $r(t, x, v, \mu)$ a solution of (1.3), with initial conditions $r(0, x, v, \mu) = x$ and $\dot{r}(0, x, v, \mu) = v$. Let $x_0 \in U$, $v_0 \in \mathbb{R}^2$, $\mu_0 \in \mathbb{R}$ be fixed. We write $r_0(t) = r(t, x_0, v_0, \mu_0)$.

In this situation we have:

**Proposition 1.5** Let $[0, \bar{t}]$, $\bar{t} > 0$, be a time interval in which $r_0(t)$ intersects transversally the closed segment $E \subset \mathbb{R}^2$ in a single point.

Then there is $\delta > 0$ ($\delta$ depending on $(x_0, v_0, \mu_0, \bar{t})$) such that for $x, v, \mu$ satisfying $|x - x_0| < \delta$, $|v - v_0| < \delta$, $|\mu - \mu_0| < \delta$, the solution $r(t, x, v, \mu)$ of the perturbed problem (1.3) is defined in
[0, \tilde{t}] and, restricted to the interval [0, \tilde{t}], intersects E transversally in a single point. Moreover, the function \(0 < t(x, v, \mu) < \tilde{t}\), defined by \(r(t(x, v, \mu), x, v, \mu) \in E\), is continuous.

**Proof:** Let \(\epsilon > 0\) be as in proposition 1.3 (taking \(\alpha = r_0\)). Since solutions of ODE depend continuously on the initial data (e.g. see [11], p.34), there exist \(\delta > 0\) (\(\delta\) depending on \((x_0, v_0, \mu_0, \tilde{t})\)) such that if \((x, v, \mu)\) satisfies:

\[
\|x - x_0\| < \delta, \quad \|v - v_0\| < \delta, \quad |\mu - \mu_0| < \delta, \quad (1.4)
\]

then the solution \(r = r(t, x, v, \mu)\) of the problem (1.3) is defined on \([0, \tilde{t}]\) and satisfies:

\[
\|r - r_0\| = \sup_{t \in I} \|r(t, x, v, \mu) - r_0(t, x_0, v_0, \mu_0)\| < \epsilon;
\]

\[
\|\dot{r} - \dot{r}_0\| = \sup_{t \in I} \|\dot{r}(t, x, v, \mu) - \dot{r}_0(t, x_0, v_0, \mu_0)\| < \epsilon.
\]

By proposition 1.3, we have that \(r([0, \tilde{t}])\) intersects \(E\) transversally in a single point. Also, by this same proposition, we have that the function \(t(x, v, \mu)\) is well defined for any \((x, v, \mu)\) satisfying (1.3). Write \((x, v, \mu) = u\). Since the function \(t\) is bounded, to prove its continuity it is enough to show that, if \(u_n \to u\) and \(t(u_n) \to b\), then \(b = t(u)\).

By definition of \(t\), we have that \(r(t(u_n), u_n) \in E\). Since \(r\) is continuous we have that \(\lim_{n \to +\infty} r(t(u_n), u_n) = r(b, u)\); hence \(r(b, u) \in E\) because \(E\) is closed. But \(r(t(u), u) \in E\) and, by proposition 1.3, \(t(u)\) is unique. This implies that \(b = t(u)\). This proves the proposition. \(\blacksquare\)

# 2 Perturbations of Kepler’s problem.

In this section we prove the existence of periodic solutions of perturbed problems with symmetries, close to a circular solution of Kepler’s problem (the unperturbed problem).

As before, \(r(t, x, v, \mu)\) denotes a solution of \(\ddot{r} = g(r, \mu)\), with initial conditions \(r(0, x, v, \mu) = x\) and \(\dot{r}(0, x, v, \mu) = v\).

In this section we prove theorem 0.1:

**Theorem 0.1** Let \(C\) be a circle centered at the origin \((0, 0)\) of \(\mathbb{R}^2\), \(x_0 \in C \cap \{x-axis\}\) and let \(U\) an open neighborhood of \(C\) of the form \(C \subset U \subset (\mathbb{R}^2 - \{(0,0)\})\).

Let \(a > 0\) and \(g : U \times (-a, a) \to \mathbb{R}^2\) continuous such that

(i) \(g(r, 0) = -\kappa \nabla r\), \(\kappa > 0\),

(ii) \(g(r, \mu)\) is \(C^1\) in the variable \(r \in U\), for each \(\mu\),

(iii) for all \(\mu\), \(g\) is invariant (as a vector field) by the reflections

\[
\varphi_1(x, y) = (-x, y), \quad \varphi_2(x, y) = (x, -y).
\]

Then there is \(\delta_0, 0 < \delta_0 < a\), with the following property. For each \(\mu \in (-\delta_0, \delta_0)\) there is a velocity \(v_\mu\) such that the solution \(r_{v_\mu}(t) := r(t, x_0, v_\mu, \mu)\) of \(\ddot{r} = g(r, \mu)\) is periodic. Moreover, given \(\eta > 0\) we can choose \(\delta_0 > 0\) such that
the traces of these solutions are simple closed curves, symmetric with respect to the x and y axes, and enclose the origin.

(2) all velocities $v_\mu$ are $\eta$-close, $\mu < \delta_0$.

Remarks 2.1

(1) It can easily be deduced from proposition 1.4 and (2) of the theorem above that we can choose $\delta_0$ in the theorem such that the periodic solutions intersect transversally the x-axis in exactly two points. Follows that these points are $x_0$ and $-x_0$. This fact is used in [5].

(2) Recall that a simple closed curve $S$, in the plane $\mathbb{R}^2$, encloses a point $p \not\in S$, if $p$ belongs to the bounded component of $\mathbb{R}^2 \setminus S$.

(3) $\delta_0$ of theorem 0.1 depends only on $g$, $U$ and the radius of $C$.

(4) From the proof of the theorem (or from the fact that the solutions $r_{v_\mu}$, $\mu$ are symmetric) follows that $v_\mu$ is vertical, that is, $v_\mu = (0, v_\mu)$ or, equivalently, orthogonal to the $x-$axis.

(5) In general, we cannot claim that $\mu \mapsto v_\mu$ is continuous, but it is possible to prove that the map $\mu \mapsto v_\mu$ satisfies a property a bit weaker than that of continuity: there is a correspondence $\mu \mapsto V_\mu \subset \mathbb{R}^2$, $V_\mu \neq \emptyset$, such that $r_{v_\mu}$ is periodic, for all $v \in V_\mu$. Moreover, the set $\mathcal{V} = \bigcup_{\mu \in [0, \delta_0]} V_\mu = \{(v, \mu); v \in V_\mu\}$ is compact and connected. (See the figure below.)

If $\mu \mapsto v_\mu$ is already continuous, we can take $V_\mu = \{v_\mu\}$. In this particular case $\mathcal{V} = \bigcup_{\mu \in [0, \delta_0]} V_\mu = \{(v, \mu); v \in V_\mu\}$ is the graph of $\mu \mapsto v_\mu$, $\mu \in [0, \delta_0]$, which is certainly compact and connected.

We make then the following addendum to theorem 0.1

Addendum (to theorem 0.1) We can choose $\delta_0 > 0$ in theorem 0.1 such that there is a compact connected $\mathcal{V} \subset \mathbb{R}^2 \times \mathbb{R}$ with the following properties:

1) $V_\mu := \mathcal{V} \cap \left(\mathbb{R}^2 \times \{\mu\}\right) \neq \emptyset$, for all $\mu \in [0, \delta_0]$,

2) $r_{v_\mu}$ is periodic, for $(v, \mu) \in \mathcal{V}$.

Moreover, the trace of $r_{v_\mu}$ is a simple closed curve symmetric with respect to the x and y axes, and encloses the origin.

Proof of the Theorem 0.1 First, recall that every circle centered at the origin is the trace of a periodic solution of Kepler’s problem. This solution has constant angular speed $w$, with
\[ w = \sqrt{Ra}^{3/2}, \text{ where } a \text{ is the radius of the circular solution.} \]

Let \( r_0(t) \) be the circular solution of Kepler’s problem in the \((x, y)\)-plane whose trace is \( C \). We can assume that \( r_0(0) = x_0 \) lies in the positive \( x \)-axis and \( \dot{r}_0(0) = v_0 \) has the same direction as the positive \( y \)-axis, that is, \( v_0 = (0, v_0), v_0 > 0 \).

Choose \( \bar{t} > 0 \) such that \( r_0(t) = r(t, x_0, v_0, 0) \), restricted to the interval \([0, \bar{t}]\), intersects the \( y \)-axis transversally in a single point (see figure below).

Choose \( \delta, 0 < \delta < 1, \delta \) sufficiently small (as in proposition 1.5 with \( \bar{t} \) as above and \( E = y \)-axis), and define \( V_{\bar{t}} := \{ \sigma v_0; \sigma \in (1 - \delta, 1 + \delta) \} \subset \mathbb{R}^2 \). Let \( l : V_{\bar{t}} \times (-\delta, \delta) \to \mathbb{R}, l(v, \mu) = \dot{y}(t(v, \mu), x_0, v, \mu) \), where \( \dot{y}(t(v, \mu), x_0, v, \mu) \) is the second coordinate of \( \dot{r}(t(v, \mu), x_0, v, \mu) = (\dot{x}(t(v, \mu), x_0, v, \mu), \dot{y}(t(v, \mu), x_0, v, \mu)) \) and \( t(v, \mu) \) is the time at which the solution intersects the \( y \)-axis. Here \( t(v, \mu) := t(x_0, v, \mu) \) is as in proposition 1.5 (this is why we chose \( \bar{t} \) and wanted \( \delta \) sufficiently small). Note that, since \( \delta < 1 \), \( v \in V_{\bar{t}} \) has the same direction as \( v_0 \).

Observe that \( l(v, \mu) = 0 \) if and only if \( r_{v, \mu} \), restricted to \([0, \bar{t}]\), intersects orthogonally the \( y \)-axis (and in a single point).

**Claim 2.1** Let \( \eta > 0 \). Then there is \( \delta_0, 0 < \delta_0 < \delta \), and \( v_+, v_- \in V_{\bar{t}} \) such that \( |v_0 - v_-| < \eta, |v_0 - v_+| < \eta, |v_-| < |v_0| < |v_+| \), and for all \( \mu \in (-\delta_0, \delta_0) \), we have \( l(v_+, \mu) > 0 \) and \( l(v_-, \mu) < 0 \).

**Proof of the claim:** For \( \mu = 0 \) we have Kepler’s problem. Hence there is \( v_+ \) and \( v_- \) belonging to \( V_{\bar{t}} \), such that \( |v_-| < |v_0| < |v_+| \) and \( |v_0 - v_-| < \eta, |v_0 - v_+| < \eta \), with \( l(v_+, 0) > 0 \) and \( l(v_-, 0) < 0 \) (see figure below).
Since \( l \) is continuous, for \( \epsilon = \frac{1}{2} \min\{l(v_+, 0), |l(v_-, 0)|\} > 0 \), there is \( \delta_0 > 0 \) so that \( 0 < \delta_0 < \delta \) and \( |l(v_-, 0) - l(v_+, \mu)| < \epsilon, |l(v_+, 0) - l(v_+, \mu)| < \epsilon, \) for \( \mu \in (-\delta_0, \delta_0) \). It follows that \( l(v_+, \mu) > 0 \), and \( l(v_-, \mu) < 0 \), for \( \mu \in (-\delta_0, \delta_0) \). This proves the claim.

By the claim above (choose any \( \eta > 0 \)) and by the intermediate value theorem, there is \( \delta_0 > 0 \) such that for each \( \mu \in (-\delta_0, \delta_0) \) there exists \( v_\mu \in V_\delta \) satisfying \( l(v_\mu, \mu) = 0 \). Therefore, for each \( \mu \), the solution of the perturbed problem \( r_\mu(t) := r_{v_\mu, \mu}(t) = r(t, x_0, v_\mu, \mu) \) intersects orthogonally the \( y \)-axis at \( t = t(v_\mu, \mu) \).

Finally, lemma 1.2 implies that \( r_\mu(t) \) can be extended to a periodic solution, with period \( 4t(v_\mu, \mu) \), whose trace is symmetric with respect to the \( x \) and \( y \) axes.

To show that we can choose \( \delta_0 \), such that the trace of \( r_\mu(t), \mu \in (-\delta_0, \delta_0) \), is a simple closed curve, let \( \tilde{r}_\mu(t) \) be the map \( r_\mu(t) \) considered as a map with domain \( S^1 = \{ z \in \mathbb{C}; |z| = 1 \} \) (that is, \( \tilde{r}_\mu(e^{i\theta}) = r_\mu(\frac{\tau}{2\pi} + \theta), \tau = 4t(v_\mu, \mu) \)). Note that \( \tilde{r}_\mu \) and \( r_\mu \) have the same trace.

From the continuous dependence of the solutions with respect to initial conditions, and from the fact that \( t(v_\mu, \mu) \) is continuous, it is straightforward to verify that, choosing \( \delta_0 \) and \( \eta \) sufficiently small, we have that, for \( \mu \in (-\delta_0, \delta_0) \), \( \tilde{r}_\mu \) is close to \( \tilde{r}_0 \) in the space of maps from \( S^1 \) to \( \mathbb{R}^2 \) (with the \( C^1 \) topology).

Since \( \tilde{r}_0 \) is a embedding, and the space of embeddings is open in the \( C^1 \) topology, we have that we can choose \( \delta_0 \) and \( \eta \) sufficiently small such that, for \( \mu \in (-\delta_0, \delta_0) \), \( \tilde{r}_\mu \) is also a embedding. This implies that the trace of \( \tilde{r}_\mu \) is homeomorphic to \( S^1 \), that is, it is a simple closed curve in \( \mathbb{R}^2 \).

To show that we can choose \( \delta_0 \) sufficiently small such that the trace of \( r_\mu(t), \mu \in (-\delta_0, \delta_0) \), encloses the origin, recall first that the trace of \( r_0 \) encloses the origin. This means that \( (0, 0) \) is in the bounded component of \( \mathbb{R}^2 - \text{(trace } r_0) = \mathbb{R}^2 - C \). Equivalently, \( \tilde{r}_0 : S^1 \to \mathbb{R}^2 - \{(0, 0)\} \) is not homotopy trivial.

Choosing \( \delta_0 \) and \( \eta \) sufficiently small we have that \( \tilde{r}_\mu \) is close to \( \tilde{r}_0, \mu \in (-\delta_0, \delta_0) \). Therefore \( \tilde{r}_\mu : S^1 \to \mathbb{R}^2 - \{(0, 0)\} \) is homotopic to \( \tilde{r}_0 \) in \( \mathbb{R}^2 - \{(0, 0)\} \). Hence \( \tilde{r}_\mu \) is not homotopy trivial either. It follows that the trace of \( r_\mu \) encloses the origin, \( \mu \in (-\delta_0, \delta_0) \). □

For the proof of the addendum we will use the following lemma. We present a proof of this lemma in an appendix.

**Lemma 2.2** Let \([a, b], [c, d] \subset \mathbb{R}\) be closed intervals and \( f : [a, b] \times [c, d] \to \mathbb{R} \) be continuous, such that

\[
\begin{align*}
  f(a, y) &< 0, \quad y \in [c, d], \\
  f(b, y) &> 0, \quad y \in [c, d].
\end{align*}
\]

Then there is \( \mathcal{W} \subset f^{-1}(0) \) connected and compact such that \( \mathcal{W}_0 := \mathcal{W} \cap ([a, b] \times \{y\}) \neq \emptyset \), for all \( y \in [c, d] \).

9
Proof of the Addendum: We use the notation from the proof of the theorem.

Recall that the solutions $r_{\mu}(t) = r(t, x_0, v_{\mu}, \mu)$ are such that $l(v_{\mu}, \mu) = 0$. Moreover, if $l(v, \mu) = 0$, then $(v, \mu)$ determines a periodic solution $r_{v, \mu}$ of $\dot{r} = g(r, \mu)$.

Recall also that, by the claim above, there is $v^+, v^-, \delta_0 > 0$ such that $l(v^+, \mu) > 0$, $l(v^-, \mu) < 0$, for $\mu \in (-\delta_0, \delta_0)$.

Since $v^+ \in V_\delta$, we can choose $\epsilon_+ > 0$ such that $v^+ = (1+\epsilon_+)v_0$ (because $\|v_0\| < \|v^+\|$ and $v_+$ and $v_0$ have the same direction). Analogously, we can choose $\epsilon_- < 0$ such that $v^- = (1+\epsilon_-)v_0$.

Define $f : [\epsilon_-, \epsilon_+] \times [0, \delta_0/2] \to \mathbb{R}$, $f(s, \mu) = l((1+s)v_0, \mu)$. Since $f(\epsilon_+, \mu) > 0$, $f(\epsilon_-, \mu) < 0$, $\mu \in [0, \delta_0/2]$, the lemma above tells us that there is a connected compact $W \subset f^{-1}(0)$, such that $W_\mu := W \cap ([\epsilon_-, \epsilon_+] \times \{\mu\}) \neq \emptyset$ for $\mu \in [0, \delta_0/2]$. Therefore, for $(v, \mu) \in V := \left\{ ((1+s)v_0, \mu) ; (s, \mu) \in W \right\}$, we have that $l(v, \mu) = 0$ and $V$ satisfies the conditions of the addendum to the theorem. $\blacksquare$

Remark 2.3 For the proof of the existence of figure eight periodic solutions in [5] we needed that the solutions $r_{v_{\mu}, \mu}$ given in theorem 0.1 above satisfy the following property: $r_{v_{\mu}, \mu}$ intersects the $x$-axis in exactly two points. (It follows that these points are $x_0$ and $-x_0$.) To verify that the solutions $r_{v_{\mu}, \mu}$ satisfy this property, it is enough to apply proposition 1.4 to the solutions $\tilde{r}_{\mu}(e^{i\theta}) = r_{\mu}(\frac{\pi}{2\mu} \theta)$ twice (see the end of the proof of theorem 0.1): once to $\{ e^{i\theta} ; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \}$ and then to $\{ e^{i\theta} ; \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \}$.

3 Other Perturbations.

In this section we prove the existence of periodic solutions of perturbed problems. These solutions are close to a given circular solution of the unperturbed problem $\tilde{r} = -\frac{\kappa}{\rho^2} \tilde{r}$, with $0 \leq \alpha$, $\alpha \neq 1$ and $\kappa > 0$. We will require the perturbed problems to be symmetric with respect to the $x$-axis.

Before proving the main result of this section let us recall some general facts about central force problems (see [3], p. 70-81, [4], p. 30-35, or [4], p. 33-41).

Let $U : (0, +\infty) \to \mathbb{R}$, $U \in C^\infty$. Consider the planar problem

$$\ddot{r} = -\nabla U(r) \tag{3.5}$$

where $r = (x, y)$ and $U(r) = U(r)$, $r = |r| = \sqrt{x^2 + y^2}$. Using polar coordinates $(r, \varphi)$, the
Hamiltonian and the angular momentum can be written in the following form

\[
\begin{align*}
H(r, \dot{r}) &= \frac{1}{2} \dot{r}^2 + \frac{K^2}{2r^2} + U(r), \\
K &= r^2 \dot{\varphi}.
\end{align*}
\] (3.6)

Let \( r(t) \) be a solution of this problem.

**Remarks.**

(1) The following claims are equivalent:

\[
\begin{align*}
i) \quad & \dot{r}(t_0) = 0, \\
ii) \quad & r(t_0) \perp \dot{r}(t_0).
\end{align*}
\]

(2) Suppose that there are \( t_1 < t_2 \) such that \( \dot{r}(t_1) = \dot{r}(t_2) = 0 \), \( r(t_1) \neq r(t_2) \) and \( \dot{r}(t) \neq 0 \), \( t \in (t_1, t_2) \). Then we have two possibilities:

a) \( \dot{r}(t) > 0 \), for all \( t \in (t_1, t_2) \). It follows that

\[
\begin{align*}
r(t_1) &= r_{\min} \\
r(t_2) &= r_{\max}
\end{align*}
\]

b) \( \dot{r}(t) < 0 \), for all \( t \in (t_1, t_2) \). It follows that

\[
\begin{align*}
r(t_1) &= r_{\max} \\
r(t_2) &= r_{\min}
\end{align*}
\]

Certainly, if \( \dot{r} \equiv 0 \) then \( r_{\max} = r_{\min} = r(t) \) for all \( t \), and the solution \( r(t) \) is circular. The points where \( r = r_{\min} \) are called **pericenters**, and the points where \( r = r_{\max} \) are called **apocenters**.

Therefore, if the orbit \( r(t) \) is not circular and has, at least, one pericenter and one apocenter then the orbit describes a curve that goes from an apocenter to a pericenter or from a pericenter to an apocenter successively (see figure below). Moreover, \( r(t) \) is defined for all \( t \) and the movement happens in the interior of an annulus defined by the circles with radii \( r_{\min} \) and \( r_{\max} \). Consequently, if we know \( r(t) \) between a pericenter and an apocenter (or vice-versa), we know the whole function \( r(t) \) (see fig. 3.8). The proof of these facts uses the symmetry of the problem and is similar to the proofs of lemmas [11] and [12] (see [4], [8]).

![Figure 3.7: Orbit of r(t)](image-url)
Note that if \( r(t) \) has an apocenter and a pericenter then \(|\dot{\varphi}(t)| = \frac{|K|}{|r(t)|} \geq \frac{|K|}{r_{\max}} > 0\). Hence, since \( r(t) \) is defined for all \( t \), \( \lim_{t \to +\infty} |\varphi(t)| = +\infty \), that is, \( r(t) \) "goes around" the origin infinitely many times.

(3) The angle between a pericenter and a successive apocenter is given by

\[
\Phi = \int_{r_{\min}}^{r_{\max}} \frac{K/r^2 \, dr}{\sqrt{2(E - U_ff(r))}},
\]

where \( U_ff = \frac{K^2}{2r^2} + U(r) \) is the effective potential and \( K \) is the angular momentum.

By the symmetry of the potential, we see that \( \Phi \) does not depend on which successive \( r_{\max}, r_{\min} \) we choose.

Let \( \{r_n(t)\} \) be a sequence of solutions that approaches a circular solution \( r_0(t) \), with radius \( r_0 \). Suppose that \( r_n(t) \) has an apocenter and a pericenter. Then \( \Phi_n \to \pi \sqrt{\frac{U'(r_0)}{2U'(r_0) + r_0U''(r_0)}} \), where \( \Phi_n \) is the angle between a pericenter and a successive apocenter of \( r_n \) (see [4], p.37). Here \( \{r_n(t)\} \) approaches \( r_0(t) \)” means that \( (r_n(0), \dot{r}_n(0)) \to (r_0(0), \dot{r}_0(0)) \).

(4) Let \( r_0(t) \) be a circular solution of (3.5) with \( r_0(0) = x_0 \neq 0 \) and \( \dot{r}_0(0) = v_0 \neq 0 \). Assume \( x_0 \perp v_0 \). For \( \epsilon > 0 \) let \( r_\epsilon(t) \) denote a solution of (3.5) with \( r_\epsilon(0) = x_0 \) and \( \dot{r}_\epsilon(0) = (1 + \epsilon)v_0 \). Write \( a = |x_0| \) and assume that \( U'(a) > 0 \).

Claim: If \( \epsilon > 0 \) we have that \( x_0 \) is a pericenter of \( r_\epsilon(t) \); similarly, if \( \epsilon < 0 \) we have that \( x_0 \) is an apocenter of \( r_\epsilon(t) \).

Write \( r_\epsilon = |r_\epsilon| \) and \( v_\epsilon = |v_\epsilon| \), \( \epsilon \in \mathbb{R} \). Then \( v_\epsilon = (1 + \epsilon)v_0 \). Write also \( K_\epsilon \) for the angular momentum of \( r_\epsilon \). A simple calculation shows that \( K_\epsilon = (1+\epsilon)av_0 \). Since \( r_0 \) is a circular solution...
we have \( v_0^2 = aU'(a) \). Thus \( K^2 = (1 + \epsilon)^2 a^3 U'(a) \).

Since \( r_\epsilon \) is a solution of (3.6), we have that \( r_\epsilon \) satisfies (just differentiate the first equation):

\[
\ddot{r}_\epsilon (0) = \frac{K^2}{a^3} - U'(a) = \frac{(1 + \epsilon)^2 a^3 U'(a)}{a^3} - U'(a) = \epsilon (2 + \epsilon) U'(a)
\]

It follows that the sign of \( \dot{r}_\epsilon (0) \) is equal to the sign of \( \epsilon \). Therefore, since \( r_\epsilon (0) = a, \dot{r}_\epsilon (0) = 0 \) we have that, if \( \epsilon > 0 \), \( r_\epsilon (t) > a \) for \( t \neq 0 \) close to 0; hence \( x_0 \) is a pericenter of \( r_\epsilon (t) \). Similarly for \( \epsilon < 0 \). This proves the claim.

(5) If we consider (3.5) of the form

\[
\ddot{r} = -\frac{\kappa}{|r|^{\alpha+2}} r, \quad \text{where } r = (x, y), \kappa > 0, \text{ with } 0 \leq \alpha,
\]

the potential \( U \) of the problem is given by:

\[
U(r) = \begin{cases} 
-\frac{\gamma}{r^{\alpha}}, & \text{with } \alpha \gamma = \kappa, \quad 0 < \alpha, \\
\kappa \ln r, & \alpha = 0.
\end{cases} \quad (3.7)
\]

Observe that, if \( r_0(t) \) is a circular solution of an attractive central problem, with potential \( U \), we have that, for \( r_0(0) = x_0 \) and \( \dot{r}_0(0) = v_0 \):

\[
|v_0| = \sqrt{pU'(p)}, \quad \text{with } p = |x_0|.
\]

For \( U \) as in (3.7), if \( 0 < \alpha \), \( |v_0| = \frac{\sqrt{\alpha \gamma}}{\sqrt{p}} \) and if \( \alpha = 0 \), \( |v_0| = \sqrt{\kappa} \). Note that, in this last case, \( |v_0| \) is independent of \( x_0 \).

**Theorem 0.2** Let \( C \) be a circle centered at the origin \((0, 0)\) of \( \mathbb{R}^2 \), \( x_0 \in C \cap \{x - \text{axis}\} \) and \( U \) an open neighborhood of \( C \) of the form \( C \subset U \subset (\mathbb{R}^2 - \{(0, 0)\}) \).

Let \( a > 0 \) and \( g : U \times (-a, a) \to \mathbb{R}^2 \) continuous such that

(i) \( g(r, 0) = -\frac{\kappa}{|r|^\alpha} r \), where \( r = (x, y) \), \( \kappa > 0 \), and \( 0 \leq \alpha \), \( \alpha \neq 1 \),

(ii) \( g(r, \mu) \) is \( C^1 \) in the variable \( r \in U \), for each \( \mu \),

(iii) for all \( \mu \), \( g \) is invariant (as a vector field) by the reflection

\[
\varphi(x, y) = (x, -y).
\]

Then there is \( \delta_0, 0 < \delta_0 < a \), with the following property. For each \( \mu \in (-\delta_0, \delta_0) \) there is a velocity \( v_\mu \) such that the solution \( r_{v_\mu}(t) := r(t, x_0, v_\mu, \mu) \) of \( \dot{r} = g(r, \mu) \) is periodic. Moreover, given \( \eta > 0 \) we can choose \( \delta_0 > 0 \) such that

(1) the traces of these solutions are simple closed curves, symmetric with respect to the \( x \)-axis, and enclose the origin,
(2) all velocities $v_\mu$ are $\eta$-close, $\mu < \delta_0$.

The Remarks 2.1 also hold in this case. Thus we also give an addendum to theorem 0.2:

**Addendum (to theorem 0.2)** We can choose $\delta_0 > 0$ in theorem 0.2 such that there is a compact connected set $V \subset \mathbb{R}^2 \times \mathbb{R}$ with the following properties:

1) $V_\mu := V \cap \left( \mathbb{R}^2 \times \{\mu\} \right) \neq \emptyset$, for all $\mu \in [0, \delta_0]$.

2) $r_{V,\mu}$ is periodic, for $(v, \mu) \in V$.

Moreover, the trace of $r_{V,\mu}$ is a simple closed curve symmetric with respect to the $x$-axis, and encloses the origin.

The proof of theorem 0.2 is similar to the proof of theorem 0.1 (Kepler’s perturbation problem). We just need to study the behavior of the solutions near a circular solution of the unperturbed problem (which we knew in the case of Kepler’s problem: they are all ellipses).

Let $r_0(t) = (x_0(t), y_0(t))$ be a circular solution of the planar problem

$$\ddot{r} = -\frac{\kappa}{\|r\|^{\alpha+2}} r, \quad 0 \leq \alpha, \quad \alpha \neq 1, \quad \kappa > 0 \quad (3.8)$$

Suppose that $r_0(0) = x_0$ and $\dot{r}_0(0) = v_0$, $v_0$ with the same direction as the positive $y$-axis and $x_0$ in the positive $x$-axis.

Let $r_\epsilon(t) = (x_\epsilon(t), y_\epsilon(t))$ be a solution of (3.8) with the same initial position $x_0$ and with initial velocity $v_\epsilon = (1 + \epsilon)v_0$.

Write $r_\epsilon = \|r_\epsilon\|$, $v_\epsilon = \|v_\epsilon\|$ and $v_0 = \|v_0\|$. Let $\varphi_\epsilon$ be such that $(r_\epsilon, \varphi_\epsilon)$ are the polar coordinates of $r_\epsilon$, with $\varphi_\epsilon(0) = 0$.

Since $\dot{r}_\epsilon(0)$ is orthogonal to $r_\epsilon(0)$, we have $\dot{r}_\epsilon(0) = 0$ (see remark (1) above) and $r_\epsilon(0)$ is a maximum or minimum of $r_\epsilon(t)$; hence $r_\epsilon(0)$ is an apocenter or pericenter of the solution $r_\epsilon(t)$. Note that remark (4) above implies that $r_\epsilon(0) = x_0$ is a pericenter of $r_\epsilon$, for $\epsilon > 0$ and an apocenter of $r_\epsilon$, for $\epsilon < 0$.

It follows from proposition 1.5 that, for $\epsilon$ close to 0, there is a minimum $t_\epsilon^* > 0$ such that $r_\epsilon$ intersects the negative $x$-axis (transversally) in $r_\epsilon(t_\epsilon^*)$. Equivalently, $\varphi_\epsilon(t_\epsilon^*) = \pi$. The next lemma tells us when $v_\epsilon(t_\epsilon^*)$ points to the left or to the right; i.e. when the sign of $\dot{x}_\epsilon(t_\epsilon^*)$ is positive or negative.

**Lemma 3.1** The sign of $\dot{x}_\epsilon(t_\epsilon^*)$ is given by the following table:
0 ≤ α < 1 & 1 < α

| ε > 0 | \( \dot{x}_e(t_\epsilon^*) > 0 \) | \( \dot{x}_e(t_\epsilon^*) < 0 \) |
| ε < 0 | \( \dot{x}_e(t_\epsilon^*) < 0 \) | \( \dot{x}_e(t_\epsilon^*) > 0 \) |

Table 1: Sign of \( \dot{x}_e(t_\epsilon^*) \)

Proof: Let us assume first that 0 ≤ α < 1. In this case, it is known that, for small \( \epsilon \), \( r_\epsilon \) has an apocenter and a pericenter (see [8]).

Let \( \Phi(\epsilon) \) be the angle between a pericenter and a successive apocenter of the solution \( r_\epsilon(t) \). Since 0 ≤ α < 1, we have that \( \frac{\pi}{2} < \lim_{\epsilon \to 0} \Phi(\epsilon) = \frac{\pi}{\sqrt{2-\alpha}} < \pi \) (see remark (3) above). This, together with remark (4) above, imply that, for \( \epsilon \) sufficiently close to 0, we have: (see fig. 3.10)

(1) if \( \epsilon > 0 \), \( r_\epsilon(t_\epsilon^*) \) is between an apocenter and a pericenter (in that exact order, where we consider one “first” if it happens at an earlier time). If follows that \( r_\epsilon \) is a decreasing function in this interval; hence \( \dot{r}_\epsilon(t_\epsilon^*) < 0 \). Since \( x_\epsilon = r_\epsilon \cos \phi_\epsilon \) and \( \phi_\epsilon(t_\epsilon^*) = \pi \) we have that \( \dot{x}_\epsilon(t_\epsilon^*) = \dot{r}_\epsilon(t_\epsilon^*) \cos \pi > 0 \).

(2) If \( \epsilon < 0 \), \( r_\epsilon(t_\epsilon^*) \) is between a pericenter and an apocenter. Then \( r_\epsilon \) is an increasing function in this interval; hence \( \dot{r}_\epsilon(t_\epsilon^*) > 0 \). As before, it follows that \( \dot{x}_\epsilon(t_\epsilon^*) < 0 \).

Now, let us assume that 1 < α. By the form of the graph of the effective potential in this case (see [8]) we have that, for \( t > 0 \), \( r_\epsilon(t) \) is either an increasing function or a decreasing function. By remark (4) above, \( r_\epsilon(t) \) is an increasing function, for \( \epsilon > 0 \) and \( t > 0 \), and \( r_\epsilon(t) \) is a decreasing function, for \( \epsilon < 0 \) and \( t > 0 \). An argument similar to the one used in cases (1) and (2) above completes the proof of the lemma.

Proof of theorem 0.2 As before, \( r(t, x, v, \mu) \) denotes a solution of \( \ddot{r} = g(r, \mu) \), with initial conditions \( r(0, x, v, \mu) = x \) and \( \dot{r}(0, x, v, \mu) = v \).

We proceed in a similar fashion as in the final part of the proof of theorem 0.1.
Choose $\bar{t} > 0$ such that $r_0(t) = r(t, x_0, v_0, 0)$, restricted to the interval $[0, \bar{t}]$, intersects (transversally) the negative $x$-axis at a single point.

Let $\delta > 0$, $\delta$ sufficiently small (as in proposition 1.3) with $\bar{t}$ as above and $E = \{(x, 0); x \leq 0\}$, and define $V_\delta = \{\sigma v_0; \sigma \in (1 - \delta, 1 + \delta)\} \subset \mathbb{R}^2$. Define the function $\hat{l} : V_\delta \times (-\delta, \delta) \to \mathbb{R}$, $\hat{l}(v, \mu) = \hat{x}(t(v, \mu), x_0, v, \mu)$, where $\hat{x}(t(v, \mu), x_0, v, \mu)$ is the first coordinate of the velocity vector $\dot{r}(t(v, \mu), x_0, v, \mu) = (\dot{x}(t(v, \mu), x_0, v, \mu), \dot{y}(t(v, \mu), x_0, v, \mu))$ and $t(v, \mu) > 0$ is the minimal time at which the solution intersects the negative $x$-axis. Here $t(v, \mu) := t(x_0, v, \mu)$ is as in proposition 1.5.

As in claim 2.1, we can verify, using lemma 3.1, that there are $\delta_0$, $0 < \delta_0 < \delta$, and $v_-, v_+ \in V_\delta$, $|v_-| < |v_0| < |v_+|$, such that, for all $\mu \in (-\delta_0, \delta_0)$, we have:

1. If $0 \leq \alpha < 1$, then $\bar{l}(v_-, \mu) > 0$ and $\bar{l}(v_+, \mu) > 0$.
2. If $1 < \alpha$, then $\bar{l}(v_+, \mu) < 0$ and $\bar{l}(v_-, \mu) > 0$.

In any case, by the intermediate value theorem, there is $\delta_0 > 0$ such that, for all $\mu \in (-\delta_0, \delta_0)$, there exists $v_\mu \in V_\delta$ such that $\bar{l}(v_\mu, \mu) = 0$. Therefore, for each $\mu \in (-\delta_0, \delta_0)$, the solution of the perturbed problem $r_\mu(t) := r_{v_\mu}(t) = r(t, x_0, v_\mu, \mu)$, intersects orthogonally the $x$-axis at $t = t(v_\mu, \mu)$.

Finally, lemma 1.4 implies that $r_\mu(t)$ can be extended to a periodic solution, with period $2\pi(v_\mu, \mu)$, whose trace is symmetric with respect to the $x$-axis. The rest of the proof is exactly the same as the proof of theorem 0.1.

The proof of the addendum to theorem 0.2 is similar as the proof of the addendum to theorem 0.1.

Remark 3.2 For the proof of the existence of figure eight periodic solutions in 5 we need also that the solutions $r_{v_\mu}$ given by theorem 0.2 above, satisfy also the following property: $r_{v_\mu}$ intersects the $x$-axis in exactly two points. One of these points is $x_0 = (x_0, 0)$ and the other is $(x_0, 0)$, and we can take $x_0 > 0$ and $x_0' < 0$. To verify that the solutions $r_{v_\mu}$ satisfy this property, it is enough to apply twice proposition 1.3 to the solutions $r_\mu(e^{i\theta}) = r_\mu(\frac{\pi}{2\alpha} \theta)$. Once to $\{e^{i\theta}; -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ and then to $\{e^{i\theta}; \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$.

A Proof of Proposition 1.4

Consider $E$ contained in the $y$-axis. Let $\pi_1(x, y) = x$ and let $t_0$ be the unique point in the intersection of $a$ with $E$. Note that $\alpha(t) \in E$ if and only if $\pi_1(\alpha(t)) = 0$. We assume $\pi_1(\alpha(t)) > 0$.

Since $\dot{\alpha}$ is continuous, there is an interval $[a, b] \subset (0, \bar{t})$, $t_0 \in (a, b)$, such that $\pi_1(\dot{\alpha}(t)) > 0$, for all $t \in [a, b]$. Also, we can suppose that $\text{diam}(\alpha[a, b]) < \frac{1}{3} \text{dist}(\alpha(t_0), \partial E) = \gamma$. Hence the
function $\pi_1 \alpha(t)$ is an increasing function on $[a, b]$. Then $\pi_1 \alpha(a) < 0$ and $\pi_1 \alpha(b) > 0$, because $\pi_1 \alpha(t_0) = 0$.

Let $\varepsilon_1 = \min\{\dist(\alpha(t), E); t \in [0, \bar{t}] \setminus (a, b)\}$, $\varepsilon_2 = \min\{\pi_1 \dot{\alpha}(t); t \in [a, b]\}$ and consider $\varepsilon = \frac{1}{2} \min\{\varepsilon_1, \varepsilon_2, \gamma\}$. (Note that $\varepsilon_1$, $\varepsilon_2$ and $\gamma$ are positive.)

Let $\beta : [0, \bar{t}] \to \mathbb{R}^2$ be of class $C^1$, such that $\|\alpha - \beta\| < \varepsilon$, $\|\dot{\alpha} - \dot{\beta}\| < \varepsilon$. We have the following claims:

**Claim 1:** $\beta(t) \notin E$, $t \in [0, \bar{t}] \setminus (a, b)$.

In fact, for all $x \in E$, we have $\dist(x, \beta) \geq \dist(x, \alpha) - \dist(\alpha, \beta) \geq \dist(E, \alpha) - \dist(\alpha, \beta)$, and since $t \in [0, \bar{t}] \setminus (a, b)$, we have $\dist(x, \beta) \geq \varepsilon_1 - \frac{\varepsilon_1}{2} = \frac{\varepsilon_1}{2}$. Since this holds for all $x \in E$, we have $\dist(E, \beta) \geq \frac{\varepsilon_1}{2} > 0$.

**Claim 2:** There exists a single $t_1 \in (a, b)$ such that $\beta(t_1) \in E$. Moreover, $\beta(t_1)$ is an interior point of $E$ and $\dot{\beta}(t_1) \notin E$.

First we verify that $\pi_1 \beta(t)$ is an increasing function on $[a, b]$ and that $\pi_1 \beta(a) < 0$ and $\pi_1 \beta(b) > 0$. In fact, since $|\dot{\alpha}(t) - \dot{\beta}(t)| < \frac{\varepsilon_1}{2}$, we have that $|\pi_1 \dot{\alpha}(t) - \pi_1 \dot{\beta}(t)| < \frac{\varepsilon_1}{2}$, then

$$\pi_1 \beta(t) > \pi_1 \alpha(t) - \frac{\varepsilon_1}{2} > 0,$$ for $t \in [a, b]$.

Also, $|\beta(t) - \alpha(t)| < \frac{\varepsilon_1}{2}$, then $|\pi_1 \beta(t) - \pi_1 \alpha(t)| < \frac{\varepsilon_1}{2}$. For $t = a$, we have:

$$\pi_1 \beta(a) < \pi_1 \alpha(a) + \frac{\varepsilon_1}{2} < \pi_1 \alpha(a) + \frac{\|\pi_1 \alpha(a)\|}{2} < 0.$$

In the same way, we verify that $\pi_1 \beta(b) > 0$. We have then that there exists a single $t_1 \in (a, b)$ such that $\pi_1 \beta(t_1) = 0$, that is, $\beta(t_1) \in \{y\}$-axis, not necessarily in $E$. On the other hand,

$$|\beta(t_1) - \alpha(t_0)| \leq |\beta(t_1) - \alpha(t_1)| + |\alpha(t_1) - \alpha(t_0)| \leq \varepsilon + \gamma \leq 2\gamma = \frac{2}{3} \dist(\alpha(t_0), \partial E).$$

that is, there is a single $t_1 \in (a, b)$ with $\beta(t_1)$ in the interior int $E$ of $E$. Since $\pi_1 \dot{\beta}(t_1) \neq 0$, we have that $\dot{\beta}(t_1) \notin E$. This proves claim 2.

From claims 1 and 2, we conclude that there is a single $t_1 \in [0, \bar{t}]$ with $\beta(t_1) \in E$. Moreover, $\beta(t_1) \in \text{int } E$, $\dot{\beta}(t_1) \notin E$ and $t_1 \in (0, \bar{t})$. This proves the proposition. ■

**B Proof of lemma 2.2:**

Here we present a proof of lemma 2.2. The proof is a direct application of the naturality of the Mayer-Vietoris sequence. The referee pointed out to us that a proof using Leray-Schauder degree theory can be found in [10].

**Proof of the lemma 2.2:** Let $Z = [a, b] \times [c, d]$, $X = f^{-1}(0)$, $A = \{a\} \times [c, d]$, $B = \{b\} \times [c, d]$, $Y_0 = [a, b] \times \{c\}$, $Y_1 = [a, b] \times \{d\}$. We want to prove that there is a compact connected
\( \mathcal{W} \subset X \), with \( \mathcal{W} \cap Y_i \neq \emptyset \), \( i = 0, 1 \). If this is not the case, a result in general topology asserts that there are disjoint compact sets \( X_0, X_1 \subset X \) with \( Y_i \cap X \subset X_i \) and \( X \) is the Mayer-Vietoris sequence of \( (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \) \( (H_* \text{ denotes singular homology with } \mathbb{Z}_2 \text{ coefficients}) \). We identify \((1, 0) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) with the class in \( H_0(\mathcal{A} \cup \mathcal{B}) \) determined by \( A \), and \((0, 1) \) with the class determined by \( B \). Write \( x = (1, 1) \in H_0(\mathcal{A} \cup \mathcal{B}) \). Since \( \mathcal{A}, \mathcal{B} \) belong to the same path-connected component of \( Z - X_i \) \( (\text{because } Y_i \subset Z - X_i, i \pmod{2}) \), we have that \( i_i (x) = 0 \in H_0(Z - X_i) \), where \( i_i : H_0(\mathcal{A} \cup \mathcal{B}) \to H_0(Z - X_i) \) is induced by the inclusion. Note that, since \( f(A) \subset (-\infty, 0) \), \( f(B) \subset (0, +\infty) \), \( i(x) \neq 0 \in H_0(Z - X) \), where \( i : H_0(\mathcal{A} \cup \mathcal{B}) \to H_0(Z - X) \) is induced by the inclusion. Consider the following diagram of Mayer-Vietoris sequences:

\[
\begin{array}{ccccccccc}
& & H_1(\mathcal{A} \cup \mathcal{B}) & \longrightarrow & H_0(\mathcal{A} \cup \mathcal{B}) & \longrightarrow & H_0(A \cup B) & \longrightarrow & 0 \\
& & \downarrow & \phi & \downarrow & \theta & \downarrow & \psi & \downarrow & \iota_1 \oplus \iota_2 \\
& & H_1(Z) & \longrightarrow & H_0(Z - X) & \longrightarrow & H_0(Z - X_0) & \longrightarrow & H_0(Z - X_1) & \longrightarrow & 0
\end{array}
\]

The first line is the Mayer-Vietoris sequence of \( \mathcal{A} \cup \mathcal{B} = (\mathcal{A} \cup \mathcal{B}) \cup (\mathcal{A} \cup \mathcal{B}) \), and the second line is the Mayer-Vietoris sequence of \( Z = (Z - X_0) \cup (Z - X_1) \). Note that \( (Z - X_0) \cap (Z - X_1) = Z - (X_0 \cup X_1) = Z - X \). The vertical maps are induced by inclusions. Since \( \phi(x) = (x, x) \), we have \( (\iota_1 \oplus \iota_2) \phi(x) = (\iota_1(x), \iota_2(x)) = (0, 0) \). Then \( \psi \iota(x) = 0 \) and we have \( \iota(x) \in \text{Im} \theta \). But \( H_1(Z) = 0 \). Consequently, \( \iota(x) = 0 \), a contradiction. \( \blacksquare \)

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