First Law of Black Saturn Thermodynamics

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The physical process version and equilibrium state version of the first law of thermodynamics for a black object consisting of n-dimensional charged stationary axisymmetric black hole surrounded by a black ring, the so-called black Saturn, was derived. The general setting for our derivations is n-dimensional dilaton gravity with $p + 1$ strength form fields.

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I. INTRODUCTION

The idea that spacetime may have been more than four-dimensional manifold acquired popularity recently. One of the most promising approaches to unification of fundamental interactions of Nature is superstring/M-theory. Those unified theories are formulated in the spacetime of higher dimensions. Consequently, this approach triggers continuously growing interests in studying properties of black holes in higher dimensional theories. It happened that they reveal a variety of new and interesting properties. The uniqueness theorem for static n-dimensional black holes was quite well established [1]. But for stationary axisymmetric n-dimensional solutions the situation is far from obvious. In fact, it was shown [2] that even in five-dimensional spacetime a new type of black object emerged, the so-called black ring. This solution has $S^2 \times S^1$ topology of the event horizon and is equipped in the same mass and angular momentum as a spherical five-dimensional stationary axisymmetric black hole. However, if one assumes the topology of black hole event horizon as $S^3$ the uniqueness proof can be established (see Ref.[3] for the vacuum case and Ref.[4] for the stationary axisymmetric self-gravitating $\sigma$-model). For a review of a black ring story see [5] and references therein.

But it turned out that we can have more complicated black object, not only black hole or black ring. In Ref.[6] by means of the inverse scattering method an exact asymptotically flat five-dimensional solution describing black Saturn, i.e., a spherical black hole surrounded by a black ring, was derived. It was also revealed [7] that the configurations that approach maximal entropy in five-dimensional asymptotically flat vacuum gravity for fixed mass and angular momentum are black Saturns. Because of the growing interests in such black object our main aim will be to study the first law of mechanics for it. In our paper we shall look for the physical process version of the first law of black Saturn thermodynamics as well as equilibrium state version of it.

The physical process version of the first law of black object thermodynamics is realized by changing a stationary black hole or black ring by some infinitesimal physical process, e.g., by throwing matter into black object. If we assume that the final state of black object settles down to a stationary one, we can extract the changes of black object’s parameters and in this way obtain information about the first law of its mechanics. The physical process version of the first law of black hole thermodynamics was extensively studied in the context of Einstein and Einstein-Maxwell (EM) theory in Refs.[8,9] and in Einstein-Maxwell axion-dilaton (EMAD) gravity being the low-energy limit of the heterotic string theory in Ref.[10]. While the case Einstein gravity coupled to $(n-2)$-gauge form field strength was treated in Ref.[11]. The case of black rings in higher dimensional dilaton gravity containing $(p+1)$-form field strength was considered in [12], being the simplest generalization of five-dimensional one in which stationary black ring solution has been provided [13].

The other attitude to the problem of the first law of black hole thermodynamics is the so-called equilibrium state version. It was studied in the seminal paper of Bardeen, Carter and Hawking [14]. This attitude is based on taking into account the linear perturbations of a stationary electrovac black hole to another one. In Ref.[15] arbitrary asymptotically flat perturbations of a stationary black hole were considered, while the first law of black hole thermodynamics valid for an arbitrary diffeomorphism invariant Lagrangian with metric and matter fields possessing stationary and axisymmetric black hole solutions were given in Refs.[16]-[19]. The cases of higher curvature terms and higher derivative terms in the metric were considered in [20], while the situation when the Lagrangian is an arbitrary function of metric, Ricci tensor and a scalar field was elaborated in Ref.[21]. In Ref.[22] of a charged rotating black hole where fields were not smooth through the event horizon was treated. In Ref.[23], the authors using the notion of bifurcate Killing horizons and taking into account dipole charges were managed to find the first law of black hole
thermodynamics for black ring solutions. In the higher dimensional gravity containing \((p + 1)\)-form field strength and dilatons fields the first law of black ring mechanics choosing an arbitrary cross section of the event horizon to the future of the bifurcation surface was derived in Ref.\[26\]. On the other hand, the physical process version and the equilibrium state version of the first law of black ring thermodynamics in \(n\)-dimensional Einstein gravity with Chern-Simons term were derived in Ref.\[23\]. Using the covariant cohomological methods to the conserved charges for \(p\)-form gauge fields coupled to gravity the first law of thermodynamics was found in Ref.\[20\].

In our paper we shall consider the first law of black Saturn thermodynamics in generalized \(n\)-dimensional dilaton gravity with \(p+1\) strength forms being the simplest generalization of a five-dimensional theory which admits stationary black hole and black ring solutions. The model under considerations will be composed of axisymmetric stationary \(n\)-dimensional black hole surrounded by a black rings. Sec.II will be devoted to the physical process version of the first law of black Saturn thermodynamics. In Sec.III we shall consider the equilibrium state version of the first law of the black objects by choosing the arbitrary cross sections of event horizons of the adequate components of black Saturn to the future of their bifurcation surfaces. This method allows one to treat fields which are not necessary smooth through the adequate event horizons. In Sec.IV we concluded our investigations.

### II. PHYSICAL PROCESS VERSION OF THE FIRST LAW OF BLACK SATURN MECHANICS

The general setting for our theory will be \(n\)-dimensional dilaton gravity with \(p+1\)-form field strength. This theory is the simplest generalization of the five-dimensional one with three form field strength and dilatons fields which contains stationary black hole and black ring solution. The Lagrangian for the theory under consideration is given by

\[
L = \epsilon \left( (n)R - \frac{1}{2} \nabla_{\mu} \varphi \nabla^{\mu} \varphi - \frac{1}{(p+1)!} e^{-\alpha \varphi} H_{\mu_1 \ldots \mu_{p+1}} H^{\mu_1 \ldots \mu_{p+1}} \right),
\]

where by \(\epsilon\) we denote the volume element, \(\varphi\) is the dilaton field and \(H_{\mu_1 \ldots \mu_{p+1}} = (p+1)! \nabla_{[\mu_1} B_{\mu_2 \ldots \mu_{p+1}]}\) is \((p+1)\)-form field strength. Let us assume that in the theory described by the relation \([1\)] the black Saturn solution is given. By the notion black Saturn we shall understand \(n\)-dimensional axisymmetric stationary black hole surrounded by a black rings. We begin with the physical version of the first law of black Saturn thermodynamics. In order to find this law one ought to get the explicit variations of mass and angular momenta. Evaluating the variations of the adequate fields, we achieve the following:

\[
\delta L = \epsilon \left( G_{\mu \nu} - T_{\mu \nu} \right) \delta g^{\mu \nu} - \epsilon \nabla_{\alpha} \left( e^{-\alpha \varphi} H^{\alpha \beta \gamma} \delta \eta^\alpha \right) - \epsilon \nabla_{\mu} \varphi \nabla^{\mu} \varphi - \frac{1}{(p+1)!} e^{-\alpha \varphi} H_{\mu_1 \ldots \mu_{p+1}} H^{\mu_1 \ldots \mu_{p+1}} \delta \varphi + d\Theta,
\]

where

\[
T_{\mu \nu} = \frac{1}{2} \nabla_{\mu} \varphi \nabla_{\nu} \varphi - \frac{1}{4} \delta_{\mu \nu} \nabla_{\alpha} \varphi \nabla^{\alpha} \varphi + \frac{1}{2(p+1)!} e^{-\alpha \varphi} \left( (p+1) H_{\mu_1 \ldots \mu_{p+1}} H_{\nu_1 \ldots \nu_{p+1}} - \frac{1}{2} g_{\mu \nu} H_{\mu_1 \ldots \mu_{p+1}} H^{\mu_1 \ldots \mu_{p+1}} \right).
\]

Having in mind the relation \([2\)] one achieves the symplectic \((n-1)\)-form \(\Theta_{j_1 \ldots j_{n-1}} [\eta_\alpha, \delta \eta_\alpha]\) in the following form:

\[
\Theta_{j_1 \ldots j_{n-1}} [\eta_\alpha, \delta \eta_\alpha] = \epsilon_{\mu j_1 \ldots j_{n-1}} \left[ \omega_\mu - e^{-\alpha \varphi} H^{\mu \nu_2 \ldots \nu_{p+1}} \delta B_{\nu_2 \ldots \nu_{p+1}} - \nabla^{m} \varphi \delta \varphi \right],
\]

where \(\omega_\mu = \nabla^\alpha g_{\mu \alpha} - \nabla_\mu \delta g_{\beta \alpha}^\beta\) and \(\eta_\alpha\) stands for the fields in the underlying theory while their variations are denoted by \(\delta \eta_\alpha\).

As in Refs.\[3, 23\], we identify variations of fields with a general coordinate transformations induced by an arbitrary Killing vector field \(\xi_\alpha\). In the next step, we calculate the Noether \((n-1)\)-form with respect to this above mentioned Killing vector, i.e., \(J_{j_1 \ldots j_{n-1}} = \epsilon_{m j_1 \ldots j_{n-1}} J^m [\eta_\alpha, \xi \eta_\alpha]\), namely

\[
J_{j_1 \ldots j_{n-1}} = d \left( Q^{GR} + Q^B \right)_{j_1 \ldots j_{n-1}} + 2 \epsilon_{m j_1 \ldots j_{n-1}} \left( G^m \eta - T^m \eta \right)_{j_1 \ldots j_{n-1}} \xi^\eta + p \left( \epsilon_{m j_1 \ldots j_{n-1}} \xi^d B_{d \alpha_2 \ldots \alpha_{p+1}} \nabla_{\alpha_2} \left( e^{-\alpha \varphi} H^{\alpha_2 \ldots \alpha_{p+1}} \right) \right).
\]
while \( Q_{j_1 \ldots j_{n-2}}^{GR} \) yields
\[
Q_{j_1 \ldots j_{n-2}}^{GR} = - \epsilon_{j_1 \ldots j_{n-2}ab} \nabla^a \xi^b, \tag{6}
\]
while \( Q_{j_1 \ldots j_{n-2}}^{B} \) has the following form:
\[
Q_{j_1 \ldots j_{n-2}}^{B} = \frac{p}{(p+1)!} \epsilon_{m\alpha j_1 \ldots j_{n-2}} \xi^d \ B_{\alpha_3 \ldots \alpha_{p+1}} \ e^{-\alpha \varphi} \ H^{m\alpha_3 \ldots \alpha_{p+1}}. \tag{7}
\]

\( Q_{j_1 \ldots j_{n-1}} \equiv (Q^{GR} + Q^{B})_{j_1 \ldots j_{n-1}} \) constitutes the Noether charge for the \( n \)-dimensional dilaton gravity with \( p+1 \)-form strength fields. One has in mind that \( \mathcal{J}[\xi] = dQ[\xi] + \xi^\alpha \mathcal{C}_\alpha \), where \( \mathcal{C}_\alpha \) is an \((n-1)\)-form constructed from dynamical fields, i.e., from \( g_{\mu \nu} \), \((p+1)\)-form field \( H^{\mu_1 \ldots \mu_{p+1}} \) and dilaton fields. \( \mathcal{C}_\alpha \) reduces to the following form:
\[
C_{\alpha j_1 \ldots j_{n-1}} = 2 \epsilon_{m j_1 \ldots j_{n-1}} \left[ G^m_{\alpha} - T^m_{\alpha}(B, \varphi) \right] + p \ \epsilon_{m j_1 \ldots j_{n-1}} \ \nabla_{\alpha_2} \left( e^{-\alpha \varphi} H^{m\alpha_2 \ldots \alpha_{p+1}} \right) B_{\alpha_3 \ldots \alpha_{p+1}}. \tag{8}
\]

When \( \mathcal{C}_\alpha = 0 \) one gets the source-free Eqs. of motion but on the other hand, we get the following:
\[
G_{\mu \nu} - T_{\mu \nu}(B, \varphi) = T_{\mu \nu}(\text{matter}), \tag{9}
\]
\[
\nabla_{\mu_1} \left( e^{-\alpha \varphi} H^{\mu_1 \ldots \mu_{p+1}} \right) = j^{\mu_2 \ldots \mu_{p+1}}(\text{matter}). \tag{10}
\]

Let us assume further that \( (g_{\mu \nu}, B_{\alpha_1 \ldots \alpha_{p+1}}) \) are solutions of source-free equations of motion and \( (\delta g_{\mu \nu}, \delta B_{\alpha_1 \ldots \alpha_{p+1}}, \delta \varphi) \) are the linearized perturbations satisfying Eqs. of motion with sources \( \delta T_{\mu \nu}(\text{matter}) \) and \( \delta j^{\mu_1 \ldots \mu_{p+1}}(\text{matter}) \), then we reach to the relation of the form as follows:
\[
\delta C^\alpha_{j_1 \ldots j_{n-1}} = 2 \epsilon_{m j_1 \ldots j_{n-1}} \left[ \delta T^m_{\alpha}(\text{matter}) + p \ B_{\alpha_3 \ldots \alpha_{p+1}} \ \delta j^{\alpha_3 \ldots \alpha_{p+1}}(\text{matter}) \right]. \tag{11}
\]

The fact that the Killing vector field \( \xi^\alpha \) describes also a symmetry of the background matter field, it provides the formula for a conserved quantity connected with \( \xi^\alpha \). It follows directly
\[
\delta H_{\xi} = -2 \int_{\Sigma(i)} \epsilon_{m j_1 \ldots j_{n-1}} \left[ \delta T^m_{\alpha}(\text{matter}) \xi^\alpha + p \ \xi^\alpha \ B_{\alpha_3 \ldots \alpha_{p+1}} \ \delta j^{\alpha_3 \ldots \alpha_{p+1}}(\text{matter}) \right] \tag{12}
\]
\[
+ \int_{\partial \Sigma(i)} \left[ \delta Q(t) - t \cdot \Theta \right].
\]

As in the case of black hole or black ring first law of mechanics, let us choose \( \xi^\alpha_{(i)} \) to be an asymptotic time translation \( t^\alpha \) for each black object. Just we draw the conclusion that the variation of the ADM mass for each object will be given as follows:
\[
\alpha \ \delta M_{(i)} = -2 \int_{\Sigma(i)} \epsilon_{m j_1 \ldots j_{n-1}} \left[ \delta T^m_{\alpha}(\text{matter}) t^\alpha + p \ t^\alpha B_{\alpha_3 \ldots \alpha_{p+1}} \ \delta j^{\alpha_3 \ldots \alpha_{p+1}}(\text{matter}) \right] \tag{13}
\]
\[
+ \int_{\partial \Sigma(i)} \left[ \delta Q(t) - t \cdot \Theta \right],
\]
where \( \alpha = \frac{a}{\Sigma} \) and \( i \) stands for the case of black hole or black ring. Further on, we take into account the Killing vector fields \( \psi_{(i)} \) which are responsible for the rotation in the adequate directions for black hole, and \( \phi_{(i)} \) which are connected with the rotation of black rings. It provides finally the following relations for angular momenta respectively for black hole and black rings. The variations of angular momenta for \( n \)-dimensional black hole imply
\[
\delta J^{BH}_{(i)} = 2 \int_{\Sigma(i)} \epsilon_{m j_1 \ldots j_{n-1}} \left[ \delta T^m_{\alpha}(\text{matter}) \psi^\alpha_{(i)} + p \ \psi^\alpha_{(i)} B_{\alpha_3 \ldots \alpha_{p+1}} \ \delta j^{\alpha_3 \ldots \alpha_{p+1}}(\text{matter}) \right] \tag{14}
\]
\[
- \int_{\partial \Sigma(i)} \left[ \delta Q(\psi_{(i)}) - \psi_{(i)} \cdot \Theta \right],
\]
while the variations of angular momenta for surrounding black rings yield
\[
\delta J^{BR}_{(i)} = 2 \int_{\Sigma(i)} \epsilon_{m j_1 \ldots j_{n-1}} \left[ \delta T^m_{\alpha}(\text{matter}) \phi^\alpha_{(i)} + p \ \phi^\alpha_{(i)} B_{\alpha_3 \ldots \alpha_{p+1}} \ \delta j^{\alpha_3 \ldots \alpha_{p+1}}(\text{matter}) \right] \tag{15}
\]
\[
- \int_{\partial \Sigma(i)} \left[ \delta Q(\phi_{(i)}) - \phi_{(i)} \cdot \Theta \right].
\]
To consider the physical process version of the first law of black Saturn thermodynamics, let us assume that \((g_{\mu\nu}, B_{a_1\ldots a_p}, \varphi)\) are solutions to the source free Einstein equations with \((p + 1)\) form fields and scalar dilaton fields. Moreover, let \(\xi^\mu_{(BH)}\) denotes the event horizon Killing vector field connected with black hole

\[
\xi^\mu_{(BH)} = t^\mu + \sum_i \Omega_i \xi^{\mu(i)},
\]

and \(\xi^\mu_{(BR(i))}\) denotes the event horizon Killing vector field for \(i = th\) black ring

\[
\xi^\mu_{(BR(i))} = t^\mu + \sum_m \omega_i \phi^{\mu(m)},
\]

Let us perturb the black Saturn by dropping into it some matter. Furthermore, suppose that the black Saturn will be not destroyed during this process and it settles down to a stationary solution \([9]\). Our next task will be to find changes of masses and angular momenta of the black objects under consideration according to relations \((13)-(15)\). Changes of the horizons' area will be computed using the \(n\)-dimensional Raychaudhuri equation. In addition, we shall assume that \(\Sigma_{0(i)}\) is an asymptotically flat hypersurface which terminating on the \(i = th\) event horizon of the considered black objects, i.e., black hole or on \(a = th\) black ring surrounded the higher dimensional black hole under consideration. Then, one takes into account the initial data on \(\Sigma_{0(i)}\) for a linearized perturbations of \((\delta g_{\mu\nu}, \delta B_{a_1\ldots a_p}, \delta \varphi)\) with \(\delta T^m_{\mu\nu}(\text{matter})\) and \(\delta j^{\alpha_2\ldots\alpha_{p+1}}(\text{matter})\). We require that \(\delta T^m_{\mu\nu}(\text{matter})\) and \(\delta j^{\alpha_2\ldots\alpha_{p+1}}(\text{matter})\) disappear at infinity and the initial data for \((\delta g_{\mu\nu}, \delta B_{a_1\ldots a_p}, \delta \varphi)\) vanish in the vicinity of each black object horizon \(H_{(BH)}\) and \(H_{BR(i)}\) on the adequate hypersurfaces \(\Sigma_{0(i)}\). The above conditions provide that for the initial time \(\Sigma_{0(i)}\), each of the black object is unperturbed. On its own, it provides the perturbations vanish near the internal boundary \(\partial\Sigma_{0(i)}\), one gets from relations \((13)\) and \((14)\) that the following is fulfilled:

\[
\alpha \left( \delta M_{BH} + \sum_a \delta M_{BR(a)} \right) - \sum_i \Omega_i \delta J^{(i)}_{BH} - \sum_a \sum_i \omega_i \delta J^{(i)}_{BR(a)} = \]

\[
- 2 \int_{\Sigma_{0(i)}} \epsilon_{m_1\ldots m_{\frac{\alpha}{2} - 1}} \left[ \delta T^k_m \xi^k_i \right] + p \xi_i \delta j^{\alpha_2\ldots\alpha_{p+1}}(\text{matter})
\]

\[
- 2 \sum_a \int_{\Sigma_{0(i)}} \epsilon_{m_1\ldots m_{\frac{\alpha}{2} - 1}} \delta T^k_m \xi^k_i + p \xi_i \delta j^{\alpha_2\ldots\alpha_{p+1}}(\text{matter})
\]

\[
= \int_{H_{BR(i)}} \gamma^k \delta j^{\alpha_2\ldots\alpha_{p+1}}(\text{matter}) + \sum_a \int_{H_{BR(a)}} \gamma^k \delta j^{\alpha_2\ldots\alpha_{p+1}}(\text{matter})
\]

where \(\epsilon_{m_1\ldots m_{\frac{\alpha}{2} - 1}} = n^\delta \epsilon_{m_1\ldots m_{\frac{\alpha}{2} - 1}}\) while \(n^\delta\) is a future directed unit normal to the adequate hypersurface \(\Sigma_{0(i)}\) for each black object. On the other hand, each \(k^i\) is tangent vector to the affinely parametrized null geodesics generators of the adequate black object event horizon. Due to the fact of the conservation of each current \(\gamma^i\) and the assumption that all of the matter falls into each of the considered black object we replace in Eq.\((18)\) \(n^\delta\) by the vector \(k^i\). Let us recall relation which will be useful in finding the integrals over the event horizons of the adequate black object

\[
p! \left( \xi \right) \delta j^{\alpha_2\ldots\alpha_{p+1}}(\text{matter}) = \xi^d H_{d\alpha_2\ldots\alpha_{p+1}} \delta j^{\alpha_2\ldots\alpha_{p+1}}(\text{matter})
\]

Having in mind that in the stationary background \(\theta\) expansion and \(\sigma_{ij}\) shear vanish and using \(n\)-dimensional Raychaudhuri lead us to the fact that \(R_{a_2\ldots a_p+1}^d k_{(i)}^d\) \(\xi_{(i)} = 0\). This in turn implies the following for each black object:

\[
\frac{1}{2} \left[ k^\mu_{(i)} \nabla_\mu \varphi \right] k_{(i)}^\nu \nabla_\nu \varphi + \frac{1}{2p!} e^{-\alpha \varphi} H_{\mu_2\ldots \mu_p+1} H_{\nu}^{\nu_2\ldots \nu_p+1} k^\mu_{(i)} k^\nu_{(i)} \nabla_\mu \varphi \nabla_\nu \varphi \mid_{\Sigma_{0(i)}} = 0.
\]

Using the fact that \(\mathcal{L}_{k^\mu} \varphi = 0\), it is easily seen that, \(H_{\mu_2\ldots \mu_p+1} k^\mu_{(i)} = 0\). Because of the fact that \(H_{\mu_2\ldots \mu_p+1} k^\mu_{(i)} = 0\), then by asymmetry of \(H_{\mu_2\ldots \mu_p+1}\) it follows that \(H_{\mu_2\ldots \mu_p+1} \sim \delta \mu_2(\ldots \delta \mu_{p+1})\). The pull-back of \(H_{\mu_2\ldots \mu_p+1} k^\mu_{(i)}\) to the adequate event horizon is equal to zero. Thus, \(\xi^{(BH)}_{(i)} B_{d\alpha_2\ldots\alpha_{p+1}}\) is a closed \(p\)-form on the event horizon of the considered black hole.
For the case of \( n \)-dimensional stationary axisymmetric black hole we get the following:

\[
-2 \int_{\mathcal{H}(BH)} \epsilon_{m_{j_1}...j_{n-1}p} \Phi_{(BH)}^{k} B_{k\alpha_3...\alpha_{p+1}} \delta j^{m_{\alpha 3}...\alpha_{p+1}}(\text{matter}) =
\]

\[
- \Phi_{(BH)} \int_{\mathcal{H}(BH)} \epsilon_{m_{j_1}...j_{n-1}} 2p \delta j^{m_{\alpha 3}...\alpha_{p+1}}(\text{matter}) = \Phi_{(BH)} \delta Q(BH),
\]

where \( \delta Q(BH) \) denotes the net flux of charge flowing into the considered black hole.

For the case of black rings surrounded the black hole the situation is a little bit more complicated. Namely, due to the Hodge theorem (see e.g., [27]) it may be rewritten as a sum of an exact and harmonic form. An exact one does not contribute to the above expression because of the fact that equations of motion must be satisfied. The only contribution originates from the harmonic part of \( \xi_{(BR(a))}^d H_{d\alpha_2...\alpha_{p+1}} \) and \( \delta q_{(BR(a))} \) is the variation of a local charge [23]. Summing it all up, we conclude that the following is fulfilled:

\[
\alpha \left( \delta M_{BH} + \sum_a \delta M_{BR(a)} \right) - \sum_i \Omega_{(i)} \delta J_{BH}^{(i)} - \sum_a \sum_i \omega_{(i)(a)} \delta J_{BR(a)}^{(i)} =
\]

\[
\Phi_{(BH)} \delta Q_{(BH)} + \sum_a \Phi_{(BR(a))} \delta q_{(BR(a))}
\]

\[
= 2 \int_{\mathcal{H}(BH)} \delta T^\mu_{\nu} \xi_{(BH)}^\mu k_{\alpha(BH)} + 2 \sum_a \int_{\mathcal{H}(BR(a))} \delta T^\mu_{\nu} \xi_{(BR(a))}^\mu k_{\alpha(BR(a))}
\]

The right-hand side of Eq. (22), may be found by the same procedure as described in Refs. [9, 10, 11]. Namely, considering \( n \)-dimensional Raychauduri Eq. and using the fact that the null generators of the event horizon of the perturbed black ring coincide with the null generators of the unperturbed stationary black ring, lead to the conclusion that

\[
\kappa_{(i)} \delta A_{(i)} = \int_{\mathcal{H}(i)} \delta T^\mu_{\nu}(\text{matter}) \xi_{(i)}^\nu k_{\mu(i)},
\]

where \( \kappa_{(i)} \) is the surface gravity of the adequate black object.

In the light of what has been shown, according to Eqs. (22) and (23) we arrived at the desired expression for physical process version of the first law of black Saturn mechanics in Einstein gravity with additional \((p+1)\)-form field strength and dilaton fields. It yields

\[
\alpha \left( \delta M_{BH} + \sum_a \delta M_{BR(a)} \right) - \sum_i \Omega_{(i)} \delta J_{BH}^{(i)} - \sum_a \sum_i \omega_{(i)(a)} \delta J_{BR(a)}^{(i)} =
\]

\[
\Phi_{(BH)} \delta Q_{(BH)} + \sum_a \Phi_{(BR(a))} \delta q_{(BR(a))} = 2\kappa_{(BH)} \delta A_{(BH)} + 2 \sum_a \kappa_{(BR(a))} \delta A_{(BR(a))}.
\]

One should have in mind that a proof of physical process version of the first law of thermodynamics for \( n \)-dimensional black Saturn also provides support for cosmic censorship [6].

III. EQUILIBRIUM STATE VERSION OF THE FIRST LAW OF BLACK SATURN MECHANICS

Now, we would like to extend the above analysis and to pay more attention to the first law of black Saturn dynamics by choosing an arbitrary cross section of the adequate event horizon of each black object to the future of the bifurcation sphere. In Ref. [23] it was shown that this attitude helped one to treat fields which were not necessarily smooth through the event horizon, having in mind the only requirement that the pull-back of these fields in the future of the bifurcation surface be smooth. To begin with, let us consider the spacetime with asymptotic conditions at infinity and equipped with the Killing vector fields \( \xi_{(i)} \), which introduces an asymptotic symmetry. It was revealed in Ref. [23] that there exist conserved quantities \( H_{\xi_{(i)}} \), which imply

\[
\delta H_{\xi_{(i)}} = \int_{\infty} \left( \delta Q(\xi_{(i)}) - \xi_{(i)} \Theta \right).
\]
is the variation which has no effect on \( \xi_n \) because of the fact that the Killing vector field is treated as a fixed background and it ought not to be varied in expression (25). In our case in Eq. (25) the symbol \((i)\) stands for black hole or each of for each of the black rings surrounded it.

In our considerations we were bound to the case of stationary axisymmetric \( n \)-dimensional black hole and \( a \) black rings surrounded the black hole. The Killing vector fields will be given by Eqs. (16) and (17). In order to derive equilibrium state version of the first law of black Saturn mechanics let us consider asymptotically hypersurfaces \( \Sigma_i \) ending on the part of the event horizons \( \mathcal{H}_i \) of each black object building black Saturn to the future of the bifurcation surfaces. The cross sections of the black object horizons will constitute the inner boundaries of the hypersurfaces \( \Sigma_i \). It will be denoted by \( S_{\mathcal{H}_i} \), where \( i \) stands respectively for black hole or \( a \)-th black ring. In our considerations we shall compare variations between two neighbouring states of the considered black objects. There is a freedom which points can be chosen to correspond when one compares two slightly different solutions. In our case we choose this freedom [14] to make \( S_{\mathcal{H}_i} \) the same of the two solutions (freedom of the general coordinate transformation) as well as we consider the case when the null vector remains normal to \( S_{\mathcal{H}_i} \). Of course, the stationarity and axisymmetry of the solution will be preserved which in turn causes that \( \delta \alpha^i, \delta \psi(i) \) and \( \delta \phi(i) \) will be equal to zero. It provides that the variation of the Killing vector field \( \xi_n(i) \) is of the form, respectively for black hole \( \delta \xi_n(BH) = \sum_i \delta \Omega(i) \psi(i) \) and for \( a \)-th black ring \( \delta \xi(a)_{BR(a)} = \sum_i \delta \omega(i)_{(a)} \phi(i) \).

Let us assume further that \((g_{\mu\nu}, B \alpha_1...\alpha_p, \phi)\) are solutions of the equations of motion and their variations \((\delta g_{\mu\nu}, \delta B \alpha_1...\alpha_p, \delta \phi)\) are the linearized perturbations satisfying Eqs. of motion. Furthermore, one requires that the pull-back of \( B \alpha_1...\alpha_p \) to the future of the bifurcation surface be smooth, but not necessarily smooth on it [22]. Moreover, we require that \( B \alpha_1...\alpha_p \) and \( \delta B \alpha_1...\alpha_p \) fall off sufficiently rapid at infinity. Then, those fields do not contribute to the canonical energy and canonical momenta. For our black object we have the following relation:

\[
\alpha \left( \delta M_{BH} + \sum_a \delta M_{BR(a)} \right) - \sum_i \Omega(i) \delta J(i)_{BH} - \sum_a \sum_i \omega(i)_{(a)} \delta J(i)_{BR(a)} = \int_{S_{\mathcal{H}(BH)}} \left( \delta Q(\xi(BH)) - \xi(BH) \Theta \right) + \sum_a \int_{S_{\mathcal{H}(BR(a))}} \left( \delta Q(\xi(BR(a))) - \xi(BR(a)) \Theta \right). \tag{26}
\]

First, we calculate the integral over the symplectic \((n-1)\)-form bounded with the dilaton field. We express \( \epsilon_{\mu \alpha_1...\alpha_j} \) by the volume element on \( S_{\mathcal{H}(i)} \) and by the vector \( N(i) \), the ingoing future directed null normal to \( S_{\mathcal{H}(i)} \), which is normalized to \( N(i)(i) \xi_n(i) = -1 \). It gives the expression for each black object written as

\[
\int_{S_{\mathcal{H}(i)}} \xi(i)_{j1...j_{n-1}} = \int_{S_{\mathcal{H}(i)}} \epsilon_{j1...j_{n-2}N(i)(i) \xi_n(i) \xi_{\mu(i)} \nabla^\mu \phi} = 0, \tag{27}
\]

where we used the fact that \( \mathcal{L}_\xi \phi = 0 \).

The same arguments as quoted in the previous section lead us to the following:

\[
\int_{S_{\mathcal{H}(BH)}} Q_{j1...j_{n-2}(\xi(BH))} + \sum_a \int_{S_{\mathcal{H}(BR(a))}} Q_{j1...j_{n-2}(\xi(BR(a)))} = \Phi(BH) Q(BH) + \sum_a \Phi(BR(a)) q(BR(a)). \tag{28}
\]

The variation \( \delta \) of \( Q_{j1...j_{n-2}(\xi)} \) implies

\[
\delta \int_{S_{\mathcal{H}(BH)}} Q_{j1...j_{n-2}(\xi(BH))} + \sum_a \delta \int_{S_{\mathcal{H}(BR(a))}} Q_{j1...j_{n-2}(\xi(BR(a)))} \tag{29}
\]

\[
= \delta(\Phi(BH) Q(BH)) + \sum_a \delta(\Phi(BR(a)) q(BR(a)))
\]

\[
- \frac{p}{(p+1)!} \int_{S_{\mathcal{H}(BH)}} \sum_i \delta \Omega(i) \psi(i) B_{\alpha_3...\alpha_{p+1} \epsilon_{mkj1...j_{n-2}}} e^{-\alpha \phi} H^{mk\alpha_3...\alpha_{p+1}}
\]

\[
- \frac{p}{(p+1)!} \sum_a \int_{S_{\mathcal{H}(BR(a))}} \sum_i \delta \omega(i)_{(a)} \psi(i) B_{\alpha_3...\alpha_{p+1} \epsilon_{mdj1...j_{n-2}}} e^{-\alpha \phi} H^{md\alpha_3...\alpha_{p+1}}.
\]

It can be easily verify that this relation and Eq. (29) enables us to write

\[
\delta \Phi(BH) Q(BH) + \sum_a \delta \Phi(BR(a)) q(BR(a)) \tag{30}
\]
Taking into account symplectic \((n-1)\)-form for the potential \(B_{\nu_1...\nu_p}\) and the fact that on each event horizon of black object one has \(H_{\mu_2...\mu_{p+1}}\xi^{(i)}_{(i)} \sim \xi_{(i)} \cdots \xi_{(i)}\) and expressing \(\epsilon_{\mu_2...\mu_{n-2}}\) in the same form as in the above case, one arrives at the following:

\[
\int_{S_n(BH)} \xi^{(i)}_{(i)}(BH) \Theta^{B}_{j_1...j_{n-1}} = \sum_a \int_{S_n(BR(a))} \xi^{(i)}_{(i)}(BR(a)) \Theta^{B}_{j_1...j_{n-1}} + \sum_a \int_{S_n(BR(a))} \Theta^{B}_{j_1...j_{n-1}} \xi^{(i)}_{(i)}(BR(a))
\]

(31)

According to Eq. (30) and (31) one can conclude that the following is fulfilled:

\[
\bar{\delta} \int_{S_n(BH)} Q^{GR}_{j_1...j_{n-2}}(\xi(BH)) - \xi^{(i)}_{(i)}(BH) \Theta^{B}_{j_1...j_{n-1}} + \sum_a \bar{\delta} \int_{S_n(BR(a))} Q^{GR}_{j_1...j_{n-2}}(\xi(BR(a))) - \xi^{(i)}_{(i)}(BR(a)) \Theta^{B}_{j_1...j_{n-1}} = \Phi(BH) \delta Q(BH) + \sum_a \Phi(BR(a)) \delta Q(BR(a)).
\]

For each black object one has

\[
\int_{S_n(i)} Q^{GR}_{j_1...j_{n-2}}(\xi(i)) = 2\kappa(i) A(i),
\]

(33)

where \(A(i) = \int_{S_n(i)} \xi_{j_1...j_{n-2}}\) is the area of the black object horizon. It now follows that:

\[
\bar{\delta} \int_{S_n(BH)} Q^{GR}_{j_1...j_{n-2}}(\xi(BH)) + \sum_a \bar{\delta} \int_{S_n(BR(a))} Q^{GR}_{j_1...j_{n-2}}(\xi(BR(a))) = 2\delta \left(\kappa(BH)A(BH)\right) + 2 \sum_a \left(\kappa(BR(a))A(BR(a))\right) + \sum_i \delta\Omega(i) J^{(i)}(BH) + 2 \sum_a \sum_i \delta\omega(i)(a) J^{(i)}(BR(a)),
\]

(34)

where \(J^{(i)}(BH) = \frac{1}{2} \int_{S_n(BH)} \xi_{j_1...j_{n-2}} a^b \nabla^a \psi(i)^{b}\) and \(J^{(i)}(BR(a)) = \frac{1}{2} \int_{S_n(BR(a))} \xi_{j_1...j_{n-2}} a^b \nabla^a \phi(i)^{b}\) is the angular momentum connected with the Killing vector fields responsible for the rotations in the adequate directions. Having in mind calculations conducted in Ref. [14] it could be verified that the following integral is satisfied:

\[
\int_{S_n(BH)} \xi^{(i)}_{(i)}(BH) \Theta^{GR}_{j_1...j_{n-2}}(\xi(BH)) + \sum_a \int_{S_n(BR(a))} \xi^{(i)}_{(i)}(BR(a)) \Theta^{GR}_{j_1...j_{n-2}}(\xi(BR(a))) = 2A(BH) \delta\kappa(BH) + 2 \sum_i \delta\Omega(i) J^{(i)}(BH) + 2 \sum_a A(BR(a)) \delta\kappa(BR(a)) + 2 \sum_a \sum_i \delta\omega(i)(a) J^{(i)}(BR(a)).
\]

(35)
The above yields the conclusion that

\[ \delta \int_{S^n(BH)} Q_{j_1 \ldots j_{n-2}}^{GR}(\xi(BH)) - \xi_{(j_1)BH}^{j_1} \Theta_{j_1 \ldots j_{n-1}}^{GR} \]

\[ + \sum_a \left[ \delta \int_{S_a(BR(a))} Q_{j_1 \ldots j_{n-2}}^{GR}(\xi(BR(a))) - \xi_{(j_1)BR(a)}^{j_1} \Theta_{j_1 \ldots j_{n-1}}^{GR} \right] \]

\[ = 2\kappa_{(BH)} \delta A_{(BH)} + 2 \sum_a \kappa_{(BR(a))} \delta A_{(BR(a))}. \]

As a direct consequence of relations (32) and (39), we find that we have obtained the first law of black Saturn mechanics in Einstein n-dimensional gravity with additional \((p + 1)\)-form field strength and dilaton fields. It may be written as

\[ \alpha \left( \delta M_{BH} + \sum_a \delta M_{BR}^{(a)} \right) - \sum_i \Omega_{(i)} \delta J_{BH}^{(i)} - \sum_i \omega_{i(a)} \delta J_{BR(a)}^{(i)} \]

\[ + \Phi_{(BH)} \delta Q_{BH} + \sum_a \Phi_{(BR(a))} \delta q_{BR(a)} = 2\kappa_{(BH)} \delta A_{(BH)} + 2 \sum_a \kappa_{(BR(a))} \delta A_{(BR(a))}. \]

### IV. CONCLUSIONS

In our paper we considered the first law of black Saturn thermodynamics in a higher dimensional theory being the generalization of five-dimensional one with three-form field strength and dilaton field which contains stationary black hole and black ring solutions. Black Saturn under consideration is composed with stationary \(n\)-dimensional black hole and black rings surrounded this black hole. We provide both physical process version and equilibrium state version of the first law of black Saturn mechanics. During the physical process version we change infinitesimally black object under considerations by throwing matter into them. Assuming that black Saturn will be not destroyed in the process of it we find the changes of the ADM masses, angular momenta as well as areas of the considered black objects.

Considering equilibrium state version of the first law of black Saturn dynamics we chose arbitrary cross sections of each black object event horizons to the future of bifurcation surfaces, contrary to the previous derivations which are bounded to the considerations of bifurcation surfaces as the boundaries of hypersurfaces extending to spatial infinity. Our attitude enables one to treat fields which are not necessary smooth through each event horizon of the adequate black object.

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