End-periodic homeomorphisms and volumes of mapping tori

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Abstract
Given an irreducible, end-periodic homeomorphism \( f : S \to S \) of a surface with finitely many ends, all accumulated by genus, the mapping torus, \( M_f \), is the interior of a compact, irreducible, atoroidal 3-manifold \( \overline{M}_f \) with incompressible boundary. Our main result is an upper bound on the infimal hyperbolic volume of \( \overline{M}_f \) in terms of the translation length of \( f \) on the pants graph of \( S \). This builds on work of Brock and Agol in the finite-type setting. We also construct a broad class of examples of irreducible, end-periodic homeomorphisms and use them to show that our bound is asymptotically sharp.

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1 | INTRODUCTION

Infinite-type surfaces arise naturally as leaves of foliations of 3-manifolds. In the case of taut, depth-one foliations of closed, hyperbolic 3-manifolds, any noncompact leaf is a surface \( S \) of infinite genus with finitely many ends (all accumulated by genus). The leaf \( S \) is dense in an open submanifold which is the mapping torus \( M_f \) of an end-periodic homeomorphism \( f : S \to S \). As such, the study of homeomorphisms of infinite-type surfaces is intimately related to the study of foliated 3-manifolds. The goal of this paper is to relate the hyperbolic geometry of \( M_f \) to the geometry and dynamics of the monodromy \( f \) in a way that mirrors the connections between properties of pseudo-Anosov homeomorphisms of finite-type surfaces and the geometry of their mapping tori.
The motivating result from the finite-type setting is due to Brock [12], who related the translation length, $\tau(f)$, of a pseudo-Anosov homeomorphism $f$ on the pants graph with the hyperbolic volume of its mapping torus. In particular, he proved that there is some $K > 0$ so that

$$\frac{1}{K} \tau(f) \leq \text{Vol}(M_f) \leq K \tau(f),$$

where $K$ depends only on the surface. Agol [3] improved the upper bound on volume, giving an explicit value of the constant, and moreover proved that it is sharp.

For an end-periodic homeomorphism $f : S \to S$, the mapping torus $M_f$ is the interior of a compact, irreducible 3-manifold $M_f$ with incompressible boundary (Proposition 3.1). Handel and Miller proved a Nielsen–Thurston type classification theorem in the end-periodic setting, with the notion of irreducibility serving as an analogue of being pseudo-Anosov; see [17] or Subsection 2.2. We prove that when $f$ is irreducible, $M_f$ is atoroidal (Proposition 3.4), and further define a notion of strong irreducibility which ensures $M_f$ is also acylindrical (Lemma 3.5). Thurston’s Hyperbolization Theorem (see Theorem 2.20) then reinforces the analogy with the Nielsen–Thurston classification in the finite-type case, providing a convex hyperbolic metric on $M_f$ when $f$ is irreducible, and one with totally geodesic boundary when $f$ is strongly irreducible. Note that every 3-manifold with boundary admits a non-convex hyperbolic metric so it is important to make this distinction. We write $\text{Vol}(M_f)$ to denote the infimum of volumes of all such hyperbolic metrics on $M_f$. Our first theorem is an analogue of Brock’s upper bound on volume and our approach follows Agol’s proof in [3].

**Theorem 1.1.** For any irreducible, end-periodic homeomorphism $f : S \to S$ of a surface with finitely many ends, all accumulate by genus, we have $\text{Vol}(M_f) \leq V_{\text{oct}} \tau(f)$.

Here, $V_{\text{oct}}$ denotes the volume of a regular ideal octahedron, and $\tau(f)$ denotes the asymptotic translation length of $f$ on the pants graph $P(S)$. Throughout this paper, we assume that $S$ has finitely many ends, all accumulated by genus. Although $P(S)$ is disconnected for such surfaces, the end-periodic assumption implies that there is a nonempty subgraph, $P_f(S) \subset P(S)$, for which each component is invariant by a positive power of $f$, and hence $\tau(f)$ is well-defined (see Subsection 2.4 where we also discuss the path metric on the components of $P(S)$ we consider).

For any component $\Omega \subset P_f(S)$ which is invariant by $f$, we will construct a pants decomposition $P_\Omega$ of $\partial M_f$ for which $M_f - P_\Omega$ is acylindrical, and $P_\Omega$ depends only on the component $\Omega$; see Proposition 4.3. Denoting the volume of the complete hyperbolic metric with totally geodesic boundary on this manifold by $\text{Vol}(M_f - P_\Omega)$, which by a result of Storm [47] is equal to the infimum of volumes of all convex hyperbolic metrics on $M_f - P_\Omega$, and letting $\tau(f, \Omega)$ denote the translation distance of $f$ on $\Omega$, we prove the following.

**Theorem 1.2.** For any $f$-invariant component $\Omega \subset P_f(S)$,

$$\text{Vol}(M_f - P_\Omega) \leq V_{\text{oct}} \tau(f, \Omega).$$

The translation length of $f$ on various components of $P(S)$ is quite mysterious as the following corollary of Theorem 1.2 shows.
**Corollary 1.3.** For any irreducible, end-periodic homeomorphism $f : S \to S$ and $R > 0$, there exists an $f$-invariant component $\Omega \subset \mathcal{P}_f(S)$ such that $\tau(f, \Omega) \geq R$.

Theorem 1.2 also provides uniform lower bounds on asymptotic translation length. To state the result in this case, we let $V_{tet}$ denote the volume of a regular ideal tetrahedron in $\mathbb{H}^3$ (which is also the maximal volume of a tetrahedron in $\mathbb{H}^3$).

**Corollary 1.4.** Given an irreducible, end-periodic homeomorphism $f$, we have

$$\tau(f) \geq \frac{V_{tet} \xi(\partial \overline{M}_f)}{2V_{oct}}.$$

Here $\xi(\partial \overline{M}_f)$ is the complexity of the boundary, see Subsection 2.1. For a description of the lower bound which is more intrinsic to the surface $S$ and homeomorphism $f$, see Corollary 4.4.

We end by describing a fairly robust construction for examples of both strongly irreducible homeomorphisms and irreducible, but not strongly irreducible, homeomorphisms.

**Theorem 1.5.** Given any pure end-periodic homeomorphism $\phi : S \to S$ there exist both irreducible and strongly irreducible homeomorphisms $f, f' : S \to S$, respectively, so that $f$ and $f'$ agree with $\phi$ on the complement of a compact set.

The construction is sufficiently robust to prove the following. Note that when an end-periodic homeomorphism $f$ is strongly irreducible, $\overline{M}_f$ is acylindrical (see Proposition 3.1), and so the infimal volume is realized by $\text{Vol}(\overline{M}_f)$, which denotes the volume of the complete hyperbolic metric with totally geodesic boundary.

**Theorem 1.6.** There is a sequence of strongly irreducible end-periodic homeomorphisms $f_k : S \to S$ so that $\text{Vol}(\overline{M}_{f_k}) \to \infty$ and $\frac{\text{Vol}(\overline{M}_{f_k})}{\tau(f_k)} \to V_{oct}$ as $k \to \infty$. In fact, $|\text{Vol}(\overline{M}_{f_k}) - V_{oct}\tau(f_k)|$ is uniformly bounded, independent of $k$.

The examples in this theorem can also be taken to agree with any given end-periodic homeomorphism $\phi : S \to S$ outside some compact set.

**Outline of the paper**

In Section 2, we provide background on mapping class groups for finite-type and infinite-type surfaces, end-periodic homeomorphisms, pants graphs, decomposition spaces, and 3-manifolds. The mapping torus of an irreducible end-periodic homeomorphism is shown to be the interior of a compact, irreducible, atoroidal 3-manifold in Section 3. Furthermore, for a strongly irreducible end-periodic homeomorphism, we show that the compactified mapping torus is also acylindrical. In Section 4, we prove Theorem 1.2, Theorem 1.1, and Corollary 1.4 as applications of Proposition 4.3, which is proved in Section 5. Finally, we construct examples of irreducible (and strongly irreducible) end-periodic homeomorphisms and prove Theorem 1.6 and Corollary 1.3 in Subsection 6.1.
1.1 Related and future work

Brock in [11] proved that the pants graph is quasi-isometric to the Teichmüller space with its Weil–Petersson metric, which allows one to compare volumes of mapping tori with Weil–Petersson translation lengths. A direct proof of such a comparison was later given by Brock–Bromberg [13] and Kojima–McShane [30]. Pants distance has also been related to volumes of hyperbolic 3-manifolds in another setting by Cremaschi, Rodríguez-Migueles and Yarmola [19], see also [18, 43]. In forthcoming work, Landry, Minsky, and Taylor study stretch factors of end-periodic homeomorphisms arising from depth one foliations [31]. Finally, with Autumn Kent, the authors are currently investigating lower bounds on volumes in terms of translation lengths on the pants graph.

2 PRELIMINARIES

2.1 Surfaces and mapping class groups

Let \( Y \) be any surface. We will consider the mapping class group, \( \text{Map}(Y) \), of \( Y \), which is the group of orientation preserving homeomorphisms of \( Y \) up to isotopy. We do not distinguish the notation for a homeomorphism and the mapping class it defines, as the context will make clear the intended meaning.

Throughout this paper, we let \( S \) denote a boundaryless, infinite-type surface with \( 2 \leq n < \infty \) ends, all accumulated by genus (in particular, we assume \( S \) has no planar ends). The mapping class group of an infinite-type surface is an uncountably infinite group and is often called a big mapping class group. Important subgroups of big mapping class groups include the pure mapping class group, \( \text{PMap}(S) \), which is the subgroup of \( \text{Map}(S) \) consisting of mapping classes which fix each end of \( S \), as well as the compactly supported mapping class group, \( \text{Map}_c(S) \). For more on infinite-type surfaces and their associated big mapping class groups, see the survey article on these topics by Aramayona and Vlamis [7].

We define the complexity of a finite-type surface \( \Sigma = \Sigma_{g,n} \) of genus \( g \) and with \( n \) punctures to be \( \xi(\Sigma_{g,n}) = 3g - 3 + n \). We then let \( \overline{\Sigma} = \overline{\Sigma}_{g,n} \) denote the compact surface of genus \( g \) with \( n \) boundary components, which we declare to have the same complexity as \( \Sigma \). For example, we will call \( \overline{\Sigma}_{0,3} \) a pair of pants (compact) and \( \Sigma_{0,3} \) a thrice-punctured sphere (noncompact).

By a finite-type subsurface of \( S \), we mean a connected, finite-complexity, incompressible (that is, \( \pi_1 \)-injective) open subsurface \( \Sigma \subset S \) with \( \xi(\Sigma) \neq 0 \). Such a surface \( \Sigma \) is homeomorphic to the interior of a compact surface \( \overline{\Sigma} \) with boundary, and we assume that the inclusion extends to a locally injective map of \( \overline{\Sigma} \). We abuse notation and write \( \overline{\Sigma} \subset S \), even though it is only assumed to be embedded on its interior.

Let \( Y \) be a surface, either compact or noncompact, possibly of infinite type. By a curve in \( Y \), we mean the homotopy class of an essential (that is, non-null-homotopic and non-peripheral) simple closed curve in \( Y \). A line in \( Y \) is the proper homotopy class of a proper embedding of \( \mathbb{R} \) into \( Y \) which is essential (that is, not homotopic via a proper homotopy into an arbitrary neighborhood of an end). By proper homotopy we mean that the homotopy itself is a proper map this implies that it is also a homotopy through proper maps. An arc in \( Y \) is the relative homotopy class of an essential arc \( (I, \partial I) \to (Y, \partial Y) \) (that is, an embedding not homotopic into \( \partial Y \)) by a relative homotopy \( h_t : (I, \partial I) \to (Y, \partial Y) \). We often confuse curves, arcs, and lines with representatives of
their (relative/proper) homotopy classes. A multicurve, multiarc, or multiline in $Y$ is the union of a collection of curves, arcs, or lines, respectively, in $Y$, having pairwise disjoint representatives.

Given a curve $\alpha$ in $Y$, we let $T_\alpha$ denote the left Dehn twist about the curve $\alpha$. A pseudo-Anosov mapping class of a finite-type surface $Y$ is a mapping class with no periodic curves in $Y$. A partial pseudo-Anosov mapping class of any surface $Y$ is a mapping class that has a representative supported on a (finite-type) subsurface $\Sigma \subset Y$ which is pseudo-Anosov on $\Sigma$. We call $\Sigma$ the support of such a partial pseudo-Anosov. In general, we say that a mapping class is supported on a subsurface $\Sigma$ if it has a representative which is the identity outside of $\Sigma$. See [22] for more on mapping class groups of finite-type surfaces and the Nielsen–Thurston classification.

### 2.2 | End-periodic homeomorphisms

We refer the reader to [17, 24, 25, 27] for a more detailed discussion of the theory of end-periodic homeomorphisms, examples, and their relationship to depth-one foliations. We note that the exposition in these references discusses end-periodic homeomorphisms of more general types of surfaces than what we consider here.

**Definition 2.1** (End-periodic homeomorphisms). An end-periodic homeomorphism of $S$ is a homeomorphism $f$ of $S$ satisfying the following. There exists $m > 0$ such that for each end $E$ of $S$, there is a neighborhood $U_E$ of $E$, so that either

(i) $f^m(U_E) \subset U_E$ and the sets $\{f^{nm}(U_E)\}_{n>0}$ form a neighborhood basis of $E$; or

(ii) $f^{-m}(U_E) \subset U_E$ and the sets $\{f^{-nm}(U_E)\}_{n>0}$ form a neighborhood basis of $E$.

In the first case, $E$ is said to be an attracting end for $f$, and in the second case, $E$ is said to be a repelling end. We call such a neighborhood $U_E$ a nesting neighborhood for $E$, and note that by definition, we may always assume (as we will) that the closures of the nesting neighborhoods of the ends are pairwise disjoint.

There are many examples of end-periodic homeomorphisms given by Cantwell–Conlon in [16]. It is also straightforward to build more examples using ‘handle shifts’, which were introduced by Patel–Vlamis in [42].

**Definition 2.2.** Take a bi-infinite strip $\mathbb{R} \times [0, 1]$ and remove disks of radius $\frac{1}{4}$ centered at each point of $\mathbb{Z} \times \{\frac{1}{2}\}$. To the boundary of each disk (now removed) glue the boundary of a one-holed torus. Call this bi-infinite strip of handles $H$. Let $h$ be the homeomorphism of $H$ given by shifting each handle over by one while tapering to the identity in a neighborhood of the boundary (which consists of two copies of $\mathbb{R}$). We will call the pair $(H, h)$ a handle strip, as shown in Figure 1. Note that $H$ has two ends accumulated by genus: one toward which $h$ is shifting, called the attracting end, and one from which $h$ shifts away, called the repelling end. Now embed $H$ into any surface $S$ with at least one end accumulated by genus and extend $h$ by the identity to a homeomorphism, $\rho$, of $S$. The homeomorphism $\rho$ of $S$ is called a handle shift.

**Example 2.3.** Consider a finite number of handle strips $(H_1, h_1), \ldots, (H_k, h_k)$ embedded in a ladder surface $L$ (that is an infinite-type surface with exactly two ends, both accumulated by genus) so that the complement of $\bigcup_{i=1}^k H_i$ in $L$ consists of $k$ bi-infinite strips and so that the attracting
FIGURE 1 A handle strip composed of a bi-infinite strip of handles \( H \) and a shift map \( h \). The left end of the strip is repelling, while the right end is attracting.

FIGURE 2 Products of pairwise commuting handle shifts as end-periodic homeomorphisms

ends of \( H_i \) are sent into a single end of \( L \) and the repelling ends are sent into the other end of \( L \). See Figure 2 for the case of 2 handle strips. Let \( \rho_1, \ldots, \rho_k \) be the corresponding handle shifts. We obtain an end-periodic homeomorphism \( \rho \) by an isotopy of the product of \( \rho_1, \ldots, \rho_k \). This is shown for 2 handle strips in Figure 2. See Construction 2.10 for more examples constructed using handle shifts.

In the example of the ladder surface just described, the end-periodic homeomorphism generates a covering group acting with closed surface quotient (of genus 1 more than the number of handle shifts involved). We now explain the generalization of this behavior to arbitrary end-periodic homeomorphisms of infinite-type surfaces.

Given an end-periodic homeomorphism \( f : S \to S \), choose nesting neighborhoods for each end and let \( U_+ \) be the union of nesting neighborhoods of the attracting ends, and likewise \( U_- \) the union of nesting neighborhoods of repelling ends. Define

\[
U_+ = \bigcup_{n \geq 0} f^{-n}(U_+) \quad \text{and} \quad U_- = \bigcup_{n \geq 0} f^n(U_-)
\]
We call \( U_+ \) the positive escaping set for \( f \) and \( U_- \) the negative escaping set for \( f \). From Definition 2.1, any choice of nesting neighborhoods will give rise to the same sets \( U_\pm \). Our standing assumption that \( S \) has only finitely many ends, each accumulated by genus, together with the definition of nesting neighborhoods leads to the following properties of the escaping sets.

**Lemma 2.4.** Given an end-periodic homeomorphism \( f : S \rightarrow S \), the escaping sets \( U_+ \) and \( U_- \) are each \( f \)-invariant, and the infinite cyclic group \( \langle f \rangle \) acts freely, properly discontinuously, and cocompactly on each. Consequently, the quotients \( U_\pm \rightarrow S_\pm = U_\pm / \langle f \rangle \) are closed, orientable surfaces.

**Proof.** The \( f \)-invariance of each of \( U_+ \) and \( U_- \) is immediate from their definition.

Choose the nesting neighborhoods for the attracting ends so that their union \( U_+ \) has the property that \( f(U_+) \subset U_+ \). Then \( K_+ = U_+ - f(U_+) \) is a compact subset of \( U_+ \) and

\[
U_+ = \bigcup_{n \in \mathbb{Z}} f^n(K_+).
\]

From the fact that positive powers applied to nesting neighborhoods give a neighborhood basis of the ends in Definition 2.1, we see that the \( f^n \)-translates of \( K_+ \) form a locally finite cover of \( U_+ \). It follows that \( \langle f \rangle \) acts properly discontinuously and cocompactly on \( U_+ \). Since \( \langle f \rangle \) is torsion free, the action is free. The same proof is valid for the repelling ends. \( \square \)

Now note that \( U_+ \rightarrow S_+ \) is an infinite cyclic covering, but there may be multiple components of \( U_\pm \) that project to a single component of \( S_\pm \). Fixing any component \( U' \) of \( U_\pm \) defining an end \( E \) of \( S \) and projecting to a component \( S_{U'} \subset S_\pm \), then \( U' \rightarrow S_{U'} \) is a connected, infinite cyclic cover of the connected, closed, orientable surface \( S_{U'} \). As such, there is a nonseparating simple closed curve \( \alpha \subset S_{U'} \) that lifts to \( U' \) so that any lift \( \alpha_0 \subset U' \) separates \( U' \) into neighborhoods of the two ends of \( U' \). One of these is a nesting neighborhood \( U_E \) of an end \( E \) of \( S \). Moreover, if \( p_E \) is the period of \( E \), then \( f^{p_E}(U_E) \subset U_E \); indeed, \( \langle f^{p_E} \rangle \) generates the covering group of \( U' \rightarrow S_{U'} \), and \( \alpha_0 \cup f^{p_E}(\alpha_0) \) bounds a subsurface that serves as a fundamental domain. Note that

\[
U_{E_i} = f^i(U_E) \quad \text{for each} \quad i = 0, \ldots, p_E - 1
\]

are nesting neighborhoods for all the attracting ends \( E = E_0, \ldots, E_{p_E-1} \) in the \( f \)-orbit of \( E \) with the same properties that \( U_E \) has for \( E \).

When the nesting neighborhood of an attracting end \( E \) is chosen in this way, and the neighborhoods of the other ends in the \( f \)-orbit of \( E \) are given by (2.1), then we will call these \textit{good nesting neighborhoods} for the attracting ends. We similarly define good nesting neighborhoods for the repelling ends.

Using the unions, \( U_+ \) and \( U_- \), of good nesting neighborhoods for the attracting and repelling ends, respectively, we can choose a hyperbolic metric on \( S \) so that the boundaries of these neighborhoods are geodesics, and furthermore \( f^{\pm 1}|_{U_+} \) and \( f^{\pm 1}|_{U_-} \) are isometries onto their images; see [25].

**Corollary 2.5.** The number of components of \( S_+ \) and \( S_- \) is equal to the number of \( f \)-orbits of components of \( U_+ \) and \( U_- \), respectively. Moreover, \( \xi(S_+) = \xi(S_-) \).
Proof. The first statement is clear since the preimage of any component of \( S_\pm \) is an \( f \)-orbit of components of \( U_\pm \).

For the second claim, choose the unions \( U_+ \) and \( U_- \) of good nesting neighborhoods and a hyperbolic metric as above. Note that by our assumption on \( U_+ \) and \( U_- \), \( \overline{U_-} \cap \overline{U_+} = \emptyset \). In particular,

\[
\Sigma = S - (U_+ \cup U_-)
\]

is a subsurface with geodesic boundary, as is \( f(\Sigma) \). We will let \( K_{\pm} = \overline{U_{\pm}} - f^{\pm}(\overline{U_{\pm}}) \).

Now observe that \( S_+ \) admits a hyperbolic metric so that the restriction of the covering projection \( \overline{U_+} \to S_+ \) to \( K_+ \subset U_+ \) is a surjective local isometry, injective on the interior. Combining this with the Gauss–Bonnet theorem, we have

\[
\frac{4\pi}{3} \xi(S_+) = \text{Area}(S_+) = \text{Area}(K_+).
\]

Likewise, there is a hyperbolic metric on \( S_- \) so that

\[
\frac{4\pi}{3} \xi(S_-) = \text{Area}(S_-) = \text{Area}(K_-).
\]

On the other hand,

\[
\text{Area}(\Sigma \cup K_-) = \text{Area}(\Sigma) + \text{Area}(K_-), \tag{2.2}
\]

since \( \Sigma \) and \( K_- \) intersect along a finite union of simple closed geodesics, which has area 0. Similarly,

\[
\text{Area}(\Sigma \cup K_+) = \text{Area}(\Sigma) + \text{Area}(K_+). \tag{2.3}
\]

Now observe that \( f \) maps \( \Sigma \cup K_- \) to \( f(\Sigma) \cup f(K_-) = \Sigma \cup K_+ \), and so by Gauss-Bonnet the areas on the left-hand sides of (2.2) and (2.3) are equal. Thus, \( \text{Area}(K_-) = \text{Area}(K_+) \), and so \( \xi(S_+) = \xi(S_-) \), as required.

In the following, recall that the terms ‘curve' and 'line' actually mean a (proper) homotopy class of such.

**Definition 2.6** (Irreducibility/Strong irreducibility). An end-periodic homeomorphism, \( f : S \to S \), is **irreducible** if it has

1. no **periodic curves**;
2. no **AR-periodic lines**, that is, a line \( \ell \) with one end in an attracting end of \( S \) and the other in a repelling end of \( S \) so that there exists \( k \in \mathbb{Z} \) with \( f^k(\ell) = \ell \); and
3. no **reducing curves**, that is, a curve \( \gamma \) so that there exists \( m, n \in \mathbb{Z} \) with \( m < n \) and such that \( f^n(\gamma) \) is contained in a nesting neighborhood of an attracting end and \( f^m(\gamma) \) is contained in a nesting neighborhood of a repelling end.

An end-periodic homeomorphism \( f \) is **strongly irreducible** if it is irreducible and contains no periodic lines, that is, a line \( \ell \) in \( S \) such that \( f^k(\ell) = \ell \) for some \( k \in \mathbb{Z} \) (with no constraints on the ends of the line).
Remark. Note that the language used above to define periodic and AR-periodic lines differs from that used in the literature. See, for example, [17, 24, 25].

In their unpublished work, Handel and Miller developed an analogue of the Nielsen–Thurston classification/reduction theory for end-periodic homeomorphisms. In that theory, irreducible end-periodic homeomorphisms play the role of pseudo-Anosov homeomorphisms due to the analogous properties their invariant laminations enjoy. See Cantwell–Conlon–Fenley [17] for a detailed exposition and expansion of this theory. The concept of strong irreducibility is new, and arose in the current paper from geometric considerations of the mapping torus; see Proposition 3.1.

In his thesis [23], Fenley produces the irreducible end-periodic homeomorphism shown in Figure 3. His proof of irreducibility relies on building and analyzing the corresponding weighted train tracks. We will provide a general construction of both irreducible and strongly irreducible end-periodic homeomorphisms in Subsection 6.1. However, our proofs of irreducibility rely on subsurface projection and curve complex geometry rather than weighted train tracks.

2.3 End behavior

Given a homeomorphism $f : S \to S$ (or its associated mapping class $f \in \text{Map}(S)$), its ‘end behavior’ describes the way in which $f$ acts on neighborhoods of the ends. It is convenient to make this precise in two different ways as follows.

First, we will say that two elements $f, f' \in \text{Map}(S)$ have the same fine end behavior if $f^{-1}f' \in \text{Map}_c(S) = \text{PMap}_c(S)$: that is, if $f^{-1}f'$ is compactly supported. Thus, for any end of $S$, $f$ and $f'$, agree on some neighborhood of that end. In Subsection 6.1, we will show that for any pure, end-periodic homeomorphism $f : S \to S$, there are both irreducible and strongly irreducible end-periodic homeomorphisms with the same fine end behavior as $f$; see Theorem 1.5.

To construct examples in this level of generality, it is convenient to have a notion of ‘coarse end behavior’. This coarse end behavior will also provide a convenient description of the surfaces $S_\pm$ associated to an end-periodic homeomorphism as described above. To formalize this, we will first introduce a few more definitions and state a result of Aramayona–Patel–Vlamis [5].

Let $H^{sep}_1(S, \mathbb{Z})$ be the subgroup of $H_1(S, \mathbb{Z})$ spanned by homology classes of separating curves, and let $\tilde{H}^{sep}_1(S, \mathbb{Z}) \leq H^1(S, \mathbb{Z})$ be the subgroup naturally identified with $\text{Hom}(H^{sep}_1(S, \mathbb{Z}), \mathbb{Z})$ by the isomorphism $H^1(S, \mathbb{Z}) \to \text{Hom}(H_1(S, \mathbb{Z}), \mathbb{Z})$ arising from the Universal Coefficient Theorem.
Given an end $E$ of $S$, let $\gamma$ be an oriented curve that separates $E$ from the other ends (with orientation such that the component of $S - \gamma$ to the right of $\gamma$ is a neighborhood of $E$). Recall that we require that $S$ is boundaryless and has at least two ends accumulated by genus. Then, $\gamma$ defines a nonzero element $[\gamma] \in H^1_{sep}(S, \mathbb{Z})$, canonically associated to $E$. When convenient, we denote this class $v_E = [\gamma] \in H^1_{sep}(S, \mathbb{Z})$. Given $f \in \text{PMap}(S)$, $[\gamma] = [f(\gamma)] = f_\ast([\gamma])$ and Aramayona–Patel–Vlamis define an integer $\varphi_{[\gamma]}(f)$ which can be viewed as the ‘signed genus’ between $\gamma$ and $f(\gamma)$. If, for example, $\gamma$ and $f(\gamma)$ are disjoint, then $|\varphi_{[\gamma]}(f)|$ is the genus of the subsurface bounded by $\gamma$ and $f(\gamma)$ (with a negative sign if $f(\gamma)$ is to the left of $\gamma$); see [5, section 3] for a more precise description. Furthermore, they show that $\varphi_{[\gamma]} : \text{PMap}(S) \to \mathbb{Z}$ is a well-defined homomorphism that depends only on the homology class $[\gamma]$; see [5, Proposition 3.3].

According to [5, Proposition 4.4], this induces a surjective homomorphism

$$
\Phi^* : \text{PMap}(S) \to H^1_{sep}(S, \mathbb{Z}),
$$
by $\Phi^*(f)([\gamma]) = \varphi_{[\gamma]}(f)$. In other words, $\Phi^*(f)$ describes ‘how much genus has been shifted’ on each end by $f$. It follows from [5, Theorem 5] and its proof that $\Phi^*$ is a surjective homomorphism whose kernel is precisely $\text{Map}_c(S)$, the closure of the compactly supported mapping class subgroup. We define the coarse end behavior of $f \in \text{PMap}(S)$ to be the cohomology class $\Phi^*(f) \in H^1_{sep}(S, \mathbb{Z})$.

**Remark.** We note that $\Phi^*(f) = \Phi^*(f')$ if and only if $f^{-1}f' \in \text{Map}_c(S)$ (thanks to Ghaswala for pointing this out). In particular, if $f$ and $f'$ have the same fine end behavior, then they have the same coarse end behavior (since $\text{Map}_c(S) < \text{Map}_c(S)$). Aramayona, Patel, and Vlamis construct a section of $\Phi^*$ onto an abelian group generated by commuting handle shifts, thus giving $\text{PMap}(S)$ the structure of a semidirect product; see [5, Theorem 3, Corollary 6]. Recall that we require that $S$ is boundaryless and has at least two ends accumulated by genus. Although we will use handle shifts in Construction 2.10, we will be using a different collection of handle shifts. Since we will not appeal to this structure, we do not discuss it further.

**Lemma 2.7.** Suppose $f : S \to S$ is a pure, end-periodic homeomorphism, and let $S_\pm$ be the surfaces obtained as quotients of the positive and negative escaping sets. If $E$ is an end of $S$ with associated homology class $v_E \in H^1_{sep}(S, \mathbb{Z})$, then the genus of the corresponding component of $S_\pm$ is $1 + |\Phi^*(f)(v_E)|$.

**Proof.** Since $f$ is pure, we may choose a simple closed curve $\gamma$ such that $[\gamma] = v_E$, so that $\gamma$ cuts off a neighborhood $U$ of, say, an attracting end $E$ and $f(\overline{U}) \subset U$. In particular, $\gamma$ and $f(\gamma)$ are disjoint and bound a subsurface of genus $|\Phi^*(f)(v_E)|$. This subsurface serves as a fundamental domain for the action of $(f)$ on the corresponding component of $U_+$, and hence the genus of the component of $S_+$ is one more than $|\Phi^*(f)(v_E)|$, as required. The case of a repelling end is similar.

Given any $f \in \text{PMap}(S)$, let $E_1, \ldots, E_n$ denote the ends of $S$, $v_i = v_{E_i}$, for $i = 1, \ldots, n$, and set

$$
|\Phi^*(f)| = \sum_{i=1}^n |\Phi^*(f)(v_i)|. \quad (2.4)
$$

With this definition, we have the following.
Corollary 2.8. If \( f : S \to S \) is an end-periodic homeomorphism with associated quotient surfaces \( S_\pm \) from the escaping sets, then

\[
\xi(S_+) = \xi(S_-) = \frac{3}{2} |\Phi^*(f)|.
\]

Proof. By the Lemma 2.7, the genus of the component of \( S_\pm \) corresponding to \( E_i \) is \(|\Phi^*(f)(v_i)| + 1\). Therefore,

\[
\xi(S_+) + \xi(S_-) = \sum_{i=1}^{n} (3(1 + |\Phi^*(f)(v_i)|) - 3) = 3 \sum_{i=1}^{n} |\Phi^*(f)(v_i)|.
\]

By Corollary 2.5, \( \xi(S_+) = \xi(S_-) = \frac{1}{2}(\xi(S_+) + \xi(S_-)) \), completing the proof.

To construct end-periodic homeomorphisms with a specified fine end behavior, the next corollary shows that it suffices to construct an end-periodic homeomorphism with the same coarse end behavior.

Corollary 2.9. Let \( f_1, f_2 : S \to S \) be pure, end-periodic homeomorphisms with the same coarse end behavior. Then, there exists a mapping class \( f'_1 \) with the same fine end behavior as \( f_2 \) such that \( f_1 \) is conjugate to \( f'_1 \) by an element of \( \text{PMap}(S) \).

Proof. For each \( i = 1, 2 \), choose good nesting neighborhoods \( U^i_\pm \) for \( f_i \) so that the neighborhoods \( U^i_E \) of the end \( E \) is bounded by a single curve \( v^i_E \) that projects to a nonseparating curve in the quotient surface \( S^i_\pm \). We further assume that the neighborhood are chosen so that there is an orientation preserving homeomorphism

\[
h_0 : S - (U^1_+ \cup U^1_-) \to S - (U^2_+ \cup U^2_-).
\]

The existence of such a homeomorphism follows from the classification of surfaces, since we can increase the genus of either \( S - (U^i_+ \cup U^i_-) \), \( i = 1, 2 \) by choosing smaller good nesting neighborhoods. Moreover, we may assume \( h_0(v^1_E) = v^2_E \) for each end \( E \). Since \( f_1 \) and \( f_2 \) have the same coarse end behavior, by Lemma 2.7, for each end \( E \), the image \( S^1_E \subset S^1_\pm \) of \( U^1_E \) is a closed surface of the same genus as the image \( S^2_E \subset S^2_\pm \) of \( U^2_E \). By the classification of surfaces again there is an orientation preserving homeomorphism \( S^1_E \to S^2_E \) that sends the image of \( v^1_E \) to the image of \( v^2_E \). This homeomorphism lifts to a homeomorphism \( h_E : U^1_E \to U^2_E \) which is equivariant with respect to the restriction of the semigroup generated by \( f_1 \) and \( f_2 \) if \( E \) is an attracting end, and equivariant with respect to \( f_1^{-1} \) and \( f_2^{-1} \) if \( E \) is a repelling end. That is, on \( U_E \) we have we have that \( f_2 = h_E f_1 h_E^{-1} \) or \( f_2^{-1} = h_E f_1^{-1} h_E^{-1} \) if \( E \) is attracting or repelling, respectively. Observe that in the repelling case, on \( f_2^{-1}(U_E) \) we have \( f_2 = h_E f_1^{-1} h_E^{-1} \). Because all of these homeomorphisms are orientation preserving, adjusting \( h_0 \) if necessary, we may assume that \( h_E \) agrees with \( h_0 \) on \( U^1_\pm \). We may therefore glue these together into a single pure homeomorphism \( h : S \to S \). Setting \( f'_1 = h f_1^{-1} h^{-1} \) we see that \( f'_1 \) and \( f_2 \) agree on

\[
U^2_+ \cup f_2^{-1}(U^2_-),
\]

the complement of which has compact closure. Thus, \( f'_1 \) and \( f_2 \) have the same fine end behavior.
Note that for \( v_1, \ldots, v_n \) as above (for example, in (2.4)), we have \( \sum_i v_i = 0 \) in \( H^1_{\text{sep}}(S, \mathbb{Z}) \), and any \( n-1 \) elements of this set forms a basis. Given any \( \omega \in H^1_{\text{sep}}(S, \mathbb{Z}) \), we thus have

\[
\sum_{i=1}^n \omega(v_i) = 0,
\]

and moreover, for any integers \( w_1, \ldots, w_n \) with \( \sum w_i = 0 \), there exists a unique \( \omega \in H^1_{\text{sep}}(S, \mathbb{Z}) \) so that \( \omega(v_i) = w_i \) for all \( i = 1, \ldots, n \). Thus, specifying the coarse end behavior is equivalent to specifying such a collection of integers \( w_1, \ldots, w_n \) with sum zero.

For an end-periodic homeomorphism on a surface with \( n \) ends, the integers \( w_1, \ldots, w_n \) describing the coarse end behavior (as above, summing to zero) are all nonzero. Conversely, for any prescribed such coarse end behavior, in the following construction, we explain how to build examples of end-periodic homeomorphisms with that coarse end behavior. Moreover, in Subsection 6.1, we will show that many of the end-periodic homeomorphisms arising from Construction 2.10 are irreducible and strongly irreducible.

**Construction 2.10.** As above, let \( v_1, \ldots, v_n \) be separating curves bounding pairwise disjoint neighborhoods \( U_1, \ldots, U_n \) of the ends \( E_1, \ldots, E_n \) of \( S \), respectively, and fix any nonzero integers \( w_1, \ldots, w_n \) with \( \sum w_i = 0 \). Next, properly embed pairwise-disjoint handle strips \( \{(H_j, h_j)\}_{j=1}^m \) into \( S \) so that for each \( w_i > 0 \), there are exactly \( w_i \) ends of the handle strips sent to \( E_i \), all of which are attracting, while if \( w_i < 0 \), then there are \(-w_i\) ends of the handle strips sent to \( E_i \), all of which are repelling.

We assume the embeddings are such that each \( v_i \) meets each of the handle strips exiting the end \( E_i \) in a single separating arc in the handle strip like the red arc in the top of Figure 1, and that in \( U_i \) the complement of the handle strips is a union of properly embedded half strips, \([0, 1] \times [0, \infty) \subset \overline{U}_i\); see Figure 4. As in Example 2.3, the associated handle shifts \( \rho_1, \ldots, \rho_m \) pairwise-commute, and the map \( \rho = \Pi_1^m \rho_i \) is isotopic to an end-periodic homeomorphism with good nesting neighborhoods \( U_1, \ldots, U_n \). By construction, \( \Phi^*(\rho)(v_i) = w_i \). In Section 6, we will construct examples of irreducible and strongly irreducible end periodic homeomorphisms with the same fine end behavior as \( \rho \) by composing \( \rho \) with a compactly supported homeomorphism.

**Remark.** The notion of coarse end behavior can be extended to an arbitrary element \( f \in \text{Map}(S) \) by recording the permutation of the ends, together with the cohomology class \( \Phi^*(f^m) \), where
\( m \geq 1 \) is the order of the permutation. The purpose of the discussion of end behaviors is to provide a framework for illustrating the generality of the examples described above and elaborated upon in Subsection 6.1. Expanding to non-pure end-periodic homeomorphisms only serves to complicate the notation and obfuscate the ideas. For this reason, we do not elaborate on, or further pursue end behaviors for non-pure elements.

2.4 Graphs associated to surfaces

Graphs built from topological data associated to curves and arcs on a surface have become an essential tool in studying surfaces and their mapping class groups. Most notable among these is the curve graph, \( C(Y) \), of a surface \( Y \), whose vertices are curves on \( Y \) with edges connecting any two vertices that have disjoint representatives on \( Y \).

When \( Y \) has finite type, a germainal result of Masur–Minsky [35] with numerous implications is that \( C(Y) \) is infinite diameter and hyperbolic. Unfortunately, it is not hard to see that \( C(Y) \) has diameter 2 when \( Y \) has infinite type. The actions of big mapping class groups on various other graphs of curves, simplicial complexes, and metric spaces have been studied extensively. See, for just a few examples, Aramayona–Valdez [6], Bavard [9], Durham–Fanoni–Vlamis [21], Lanier–Loving [32], and Mann–Rafi [33].

Another useful graph is the arc and curve graph, \( AC(Y) \): the graph whose vertices are curves and arcs on \( Y \), and whose edges correspond to pairs of curves and/or arcs having disjoint representatives. When \( Y \) is an annulus, the vertices of \( AC(Y) \) are also arcs (which recall means homotopy classes of essential arcs), though in this setting we require the homotopies to \( \text{fix} \) the endpoints (otherwise there would be only one vertex). In this case, edges connect vertices with representatives having disjoint interiors.

Given \( \Sigma \subset Y \) a non-annular, essential subsurface, the \textit{subsurface projection}

\[
\pi_\Sigma : C(Y) \longrightarrow P(AC(\Sigma))
\]

takes a vertex \( \alpha \in C(Y) \) to the collection of curves and arcs in \( \Sigma \) obtained by taking the union of all distinct homotopy classes occurring in the intersection of \( \alpha \) with \( \Sigma \), after \( \Sigma \) and \( \alpha \) are put in minimal position (we send points in an edge to the union of the images of the endpoints of the edge). Note that our definition differs slightly from the projections defined by Masur and Minsky [36].

When \( \Sigma \) is an annulus, we first consider the cover \( Z_\Sigma \) of \( Y \) corresponding to \( \pi_1 \Sigma \). This cover is naturally the interior of a compact annulus, \( \overline{Z_\Sigma} \), (obtained by adding the ideal boundary) and we let \( \pi_\Sigma(\alpha) \) be the union of the closures of those components of the preimage of \( \alpha \) in \( Z_\Sigma \) that limit to points on both boundary components of \( \overline{Z_\Sigma} \).

We define the \( \Sigma \)-\textit{subsurface distance} as

\[
d_\Sigma(\alpha, \beta) = \text{diam}_{AC(\overline{\Sigma})}(\pi_\Sigma(\alpha) \cup \pi_\Sigma(\beta)),
\]

whenever \( \pi_\Sigma(\alpha), \pi_\Sigma(\beta) \neq \emptyset \). Observe that in this case we have

\[
d_\Sigma(\alpha, \beta) \geq 2 \Leftrightarrow i(\alpha, \beta) \neq 0. \quad (2.5)
\]
If \( \pi_S(\alpha), \pi_S(\beta), \pi_S(\gamma) \neq \emptyset \), then this distance satisfies the triangle inequality

\[
d_{\Sigma}(\alpha, \beta) \leq d_{\Sigma}(\alpha, \gamma) + d_{\Sigma}(\gamma, \beta).
\]

The graph of central interest in this paper is the pants graph of an infinite-type surface \( S \), which we now describe. First, a pants decomposition on \( S \) is a multicurve in \( S \) that cuts \( S \) into pairs of pants. An elementary move on a pants decomposition \( P \) replaces a single curve in \( P \) with a different one intersecting it a minimal number of times, producing a new pants decomposition \( P' \). If \( P' \) is obtained from \( P \) by an elementary move and \( Q = P \cap P' \) is the maximal multicurve in common, then \( S - Q \) has one component \( \Sigma \) which has complexity one. There are two types of elementary moves, depending on the homeomorphism type of \( \Sigma \), and these are illustrated in Figure 5.

**Definition 2.11.** The pants graph, \( P(S) \), is the graph whose vertices are (isotopy classes of) pants decompositions on \( S \), with edges between pants decompositions that differ by an elementary move. We define a path metric on (the components of) \( P(S) \) so that an edge corresponding to an elementary move that occurs on a one-holed torus has length 1, and an edge corresponding to an elementary move that occurs on a four-holed sphere has length 2. The group \( \text{Map}(S) \) still acts isometrically on \( P(S) \) with this metric.

Brock proved in [12] that for finite-type surfaces \( Y, P(Y) \) is a coarse model for the Teichmüller space of \( Y \) with the Weil–Petersson metric. For infinite-type surfaces, the pants graph is not connected, and thus the distance between pants decompositions may be infinite. See [10] for an alternative topology on this graph and other facts about it.

**Definition 2.12.** Given any end-periodic homeomorphism \( f : S \to S \), we define the asymptotic translation distance of \( f \) on \( P(S) \) to be

\[
\tau(f) = \inf_{P \in P(S)} \liminf_{n \to \infty} \frac{d(P, f^n(P))}{n},
\]

where this infimum is over all pants decompositions \( P \in P(S) \). Observe that \( \tau(f^n) = n \tau(f) \) for all \( n > 0 \).
For a pants decomposition $P$ of $S$, it may be the case that for all $n > 0$, $P$ and $f^n(P)$ are not in the same component, and thus $d(P, f^n(P)) = \infty$ for all $n > 0$. If this is the case, then the same is necessarily true for all $P'$ in the same component as $P$. For the purposes of studying $\tau(f)$, such components are useless, and thus we let $P_f(S)$ be the subgraph of $P(S)$ spanned by all $f$-asymptotic pants decompositions; that is, those pants decompositions $P$ for which $P$ and $f^n(P)$ differ by a finite number of elementary moves for some $n > 0$ (and are thus a finite distance in $P(S)$). So, in the definition of $\tau(f)$, it suffices to consider only pants decompositions $P$ taken from $P_f(S)$. Note that, since $P_f(S)$ is nonempty, $\tau(f)$ is finite.

**Remark.** The behavior of end periodic homeomorphisms on the pants graph is reminiscent of the work of Funar and Kapoudjian in [26]. Indeed, their asymptotic mapping class group of infinite genus is a subgroup of the mapping class group of the blooming Cantor tree that preserves a certain component of its pants graph. Thanks to Ghaswala for pointing out this reference.

Given a component $\Omega \subset P_f(S)$, we define $\tau(f, \Omega)$ by the same formula as $\tau(f)$, except the infimum is taken over all $P \in \Omega$ (rather than over all $P \in P(S)$). As such, $\tau(f, \Omega) \geq \tau(f)$.

### 2.5 Decomposition spaces

Our proof of Theorem 1.1 will utilize some decomposition theory and a result of Armentrout [8].

Let $M$ be a 3-manifold. A decomposition $G$ of $M$ is a partition of $M$ into disjoint sets. Given such a decomposition, there is a natural map $f : M \rightarrow M/G$ obtained by collapsing each element $X \in G$ to a point and endowing $M/G$ with the quotient topology. If $\pi : M \rightarrow N$ is any quotient map, then there is a naturally associated decomposition of $M$ which comes from the preimages of points in $N$ under $\pi$. As such, a quotient map may also be referred to as a decomposition. For more details about decompositions; see [20].

**Definition 2.13** (Upper semicontinuous). A decomposition $G$ of $M$ is said to be upper semicontinuous if each element $X \in G$ is compact, and if for each $X \in G$ and each open subset $U \subseteq M$ with $X$ contained in $U$, there exists a further open set $V \subseteq M$ containing $X$ such that for each $X' \in G$ with $X' \cap V \neq \emptyset$, we have that $X' \subseteq U$. A quotient map $\pi : M \rightarrow N$ is said to be upper semicontinuous if the associated decomposition space is upper semicontinuous.

The following result gives alternative characterizations of an upper semicontinuous decomposition; see [20, Proposition I.1.1].

**Proposition 2.14.** Let $G$ be a decomposition of a 3-manifold $M$. The following are equivalent.

1. The decomposition $G$ is upper semicontinuous.
2. For each open set $U$ in $M$, let $U^*$ denote the union of all sets in $G$ which are contained in $U$. Then, the set $U^*$ is open in $M$.
3. The associated quotient map $f : M \rightarrow M/G$ is closed.

**Definition 2.15** (Cellular decomposition). A subset $X$ of a 3-manifold $M$ is said to be cellular if there exists a sequence $\{B_i\}$ of 3-cells in $M$ such that for each $i \geq 1$, $B_{i+1} \subset \text{Int}(B_i)$, and $X = \cap_{i=1}^{\infty} B_i$. Equivalently, a subset $X$ of a 3-manifold $M$ is cellular if and only if $X$ is compact and has arbitrarily
small neighborhoods which are homeomorphic to Euclidean 3-space. A decomposition $G$ of a 3-manifold $M$ is a **cellular decomposition** if $G$ is an upper semicontinuous decomposition of $M$ and if each element $X \in G$ is a cellular subset of $M$.

The following result of Armentrout [8, Theorem 2] gives a sufficient condition for when the quotient space is homeomorphic to the original space. We apply this result to the double of $\overline{M_f}$ in the proof of Proposition 4.3.

**Theorem 2.16.** Suppose $M$ is a 3-manifold and $G$ is a cellular decomposition of $M$ such that $M/G$ is a 3-manifold. Then, $M \to M/G$ is homotopic to a homeomorphism.

### 2.6 Surfaces in 3-manifolds and foliations

Suppose $\overline{M}$ is a compact, orientable, 3-manifold with (possibly empty) boundary. We say $\overline{M}$ is **irreducible** if every smoothly embedded 2-sphere in $\overline{M}$ bounds a 3-ball in $\overline{M}$. An **incompressible surface** in $\overline{M}$ is the image of a compact, connected, orientable surface by an embedding $(\overline{\Sigma}, \partial \overline{\Sigma}) \to (\overline{M}, \overline{\partial M})$ such that $\overline{\Sigma}$ is not a sphere or disk, and such that the induced map on fundamental groups is injective. We often write $\overline{\Sigma} \subset \overline{M}$ for an incompressible surface, though it will always come from an embedding of pairs.

An incompressible annulus $(A, \partial A) \subset (\overline{M}, \overline{\partial M})$ is **boundary parallel** if there is an embedded solid torus $V \subset \overline{M}$ so that $\partial V$ is the union of $A$ with an annulus $A' = V \cap \overline{\partial M}$ intersecting each other along their common boundary $\partial A = \partial A'$. If $\overline{M}$ is irreducible, then by [51, Lemma 5.3], an incompressible annulus is boundary parallel if and only if the inclusion is homotopic by a homotopy of pairs $h_t : (A, \partial A) \to (\overline{M}, \overline{\partial M})$ to a map of $A$ into $\overline{\partial M}$. An **essential annulus** in $\overline{M}$ is an incompressible annulus which is not boundary parallel. If no essential annulus exists, we say that $\overline{M}$ is **acylindrical**. The manifold $\overline{M}$ is **atoroidal** if for every incompressible torus, the inclusion is homotopic into $\overline{\partial M}$.

Let $\mathcal{F}$ be a transversely oriented, codimension-one foliation of a compact 3-manifold $\overline{M}$ for which the boundary is a union of leaves. We say such a foliation is **taut** if for every leaf $\lambda$ of $\mathcal{F}$ there is a closed curve or embedded arc $(\gamma_\lambda, \partial \gamma_\lambda) \subset (\overline{M}, \overline{\partial M})$ which is transverse to $\mathcal{F}$ and intersects $\lambda$.

We will need the following ‘normal form’ theorem for surfaces in taut foliated 3-manifolds due to Roussarie [44, Théorème 1] (see also [15, Theorem 9.5.5]). A strengthening of an absolute version of this theorem is due to Thurston [48] (see also [28]), but we will use the following relative version.

**Theorem 2.17.** Suppose $\mathcal{F}$ is a transversely oriented, taut foliation of a 3-manifold with boundary, $\overline{M}$, for which $\overline{\partial M}$ is a union of leaves. Suppose $\overline{\Sigma} \subset \overline{M}$ is a properly embedded, incompressible torus or annulus. After an isotopy of the embedding which is the identity on the boundary (in the annulus case) we have:

1. $\overline{\Sigma}$ is a torus and is either transverse to $\mathcal{F}$ or is a leaf of $\mathcal{F}$; or
2. $\overline{\Sigma}$ is an annulus and is transverse to $\mathcal{F}$, except at finitely many circle tangencies occurring in the interior of $\overline{M}$.

### 2.7 Dehn filling

Given a 3-manifold $\overline{M}$ with a torus boundary component $T$, we can **Dehn fill** $\overline{M}$ along $T$ by gluing a solid torus $D \times S^1$ to $\overline{M}$ by a homeomorphism $\varphi : D \times S^1 \to T$. The homeomorphism type of the
Dehn filling is determined by the isotopy class of $\beta \subset T$ to which $\partial D \times \{x\}$ is identified. We call $\beta$ the \textit{Dehn filling coefficient}, and write $M(\beta)$ to denote the $\beta$-Dehn filled manifold. Choosing a basis $\mu, \lambda$ for $\pi_1(T) \cong \mathbb{Z}^2$, the isotopy class of $\beta$ on $T$ is given by $\beta = \pm (p\lambda + q\mu)$, with $p$ and $q$ relatively prime. So, we may also refer to this Dehn filling coefficient as $(p, q)$, writing $M(\beta) = M(p, q)$.

Additionally, we allow a Dehn filling coefficient of $'\infty'$, which means that we do not glue in a solid torus at all (that is, we leave the boundary component unfilled). Any sequence of distinct slopes is said to converge to $\infty$.

If $M$ is a 3-manifold with $k$ torus boundary components and $\beta = (\beta_1, \ldots, \beta_k)$ is a $k$-tuple of Dehn filling coefficients (some or all of which may be $\infty$), we may perform $\beta_i$-Dehn filling on each boundary component, which we denote by $M(\beta_1, \ldots, \beta_k)$ or simply $M(\beta)$. Finally, we also consider Dehn filling on torus ends of a noncompact manifold. These are ends which have neighborhoods homeomorphic to $T^2 \times (0, 1)$, where $T^2$ is a torus. We perform Dehn filling on such ends by first deleting a product open neighborhood of the end and then Dehn filling the resultant boundary torus.

A link in a 3-manifold, $L \subset M$, is the image of an embedding of a disjoint union of circles (a link with one component is a knot). Given a link $L \subset M$, we let $M - L$ denote the exterior of $L$ in $M$ obtained by removing an open tubular neighborhood of $L$ from $M$.

It is straightforward to see that $(1, n)$-Dehn filling the exterior of a knot which is a simple closed curve in a surface fiber of a fibered 3-manifold (for an appropriate basis of the fundamental group of the torus boundary) has the same effect as changing the monodromy by the $n^{th}$ power of a Dehn twist; see, for example, [45].

**Proposition 2.18.** Let $\gamma$ be a curve in $Y$ and $K$ the associated knot in the mapping torus $M_f$ for $f \in \text{Map}(Y)$. Then $(M_f - K)(1, n) \cong M_f \circ T^n\gamma$.

More generally, if $L$ is a link consisting of curves $\gamma_1, \ldots, \gamma_k$ with $\gamma_i$ lying on fiber $Y \times \{\frac{i}{k+1}\} \subset Y \times (0, 1) \subset M_f$, for each $i$, then

$$(M_f - L)((1, n_1), \ldots, (1, n_k)) \cong M_f \circ T^{n_k}\gamma_k \circ \cdots \circ T^{n_1}\gamma_1.$$  

**2.8 \quad Pared 3-manifolds**

The following definitions provide the appropriate topological framework for stating Thurston's Hyperbolization Theorem (see Theorem 2.20), which ensures that the manifolds we are interested in admit hyperbolic structures. Given a 3-manifold $\overline{M}$ and closed subset $\mathfrak{p} \subset \partial \overline{M}$, we say that $(\overline{M}, \mathfrak{p})$ is a \textit{pared} 3-manifold if:

1. $\overline{M}$ is compact, connected, orientable, irreducible and $\partial \overline{M} \neq \emptyset$;
2. $\pi_1(\overline{M})$ is not virtually abelian;
3. $\mathfrak{p}$ consists of incompressible annuli and tori, and contains all tori in $\partial \overline{M}$;
4. any incompressible annulus $(S^1 \times I, S^1 \times \partial I) \to (\overline{M}, \mathfrak{p})$ is properly homotopic into $\mathfrak{p}$ (that is, homotopic as a map of pairs); and
5. any non-cyclic abelian subgroup of $\pi_1(\overline{M})$ is conjugate into the fundamental group of some component of $\mathfrak{p}$.

A pared manifold $(\overline{M}, \mathfrak{p})$ has \textit{incompressible boundary} if $\partial \overline{M} - \mathfrak{p}$ is incompressible (when $\mathfrak{p} = \emptyset$, this condition simply means that $\partial \overline{M}$ is incompressible). An essential annulus in $(\overline{M}, \mathfrak{p})$ is an
incompressible annulus \((S^1 \times I, S^1 \times \partial I) \to (\overline{M}, \partial \overline{M} - \mathfrak{p})\) which is not homotopic into \(\partial \overline{M}\). One says that a pared manifold \((\overline{M}, \mathfrak{p})\) is \textit{acylindrical} if it has incompressible boundary and contains no essential annulus. If \(\mathfrak{p} = \emptyset\), this agrees with the definition of acylindrical given above.

Observe that if \((\overline{M}, \mathfrak{p})\) is a pared manifold with incompressible boundary and \(\mathfrak{p}\) meets each boundary component of \(\overline{M}\) of genus at least 2 in annuli whose core curves define a pants decomposition of these boundary components, then \((\overline{M}, \mathfrak{p})\) is acylindrical. To see this, observe that if there were an incompressible annulus with boundary in \(\partial \overline{M} - \mathfrak{p}\), then since this surface is a union of thrice-punctured spheres, the boundary of the annulus may be isotoped into \(\mathfrak{p}\), and hence the entire annulus is properly homotopic relative to the boundary into \(\mathfrak{p}\) by (4).

Suppose \(\overline{M}\) is a compact, connected, orientable, irreducible 3-manifold with incompressible, nonempty boundary. If all components of \(\partial \overline{M}\) have genus at least 2, then \((\overline{M}, \emptyset)\) is a pared manifold if \(\overline{M}\) is atoroidal: conditions (1)–(2) are immediate, (3)-(4) are vacuous, and (5) follows from the Torus Theorem; see, for example, [29, Corollary IV.4.3]. Similarly, if \(\mathfrak{p} \neq \emptyset\) and contains all tori in \(\partial \overline{M}\), then \((\overline{M}, \mathfrak{p})\) is pared if \(\overline{M}\) is atoroidal and not Seifert fibered, see [29, Theorem IV.4.1].

Remark. It is sometimes convenient to replace some or all of the annuli in \(\mathfrak{p}\) by their core curves, so that \(\mathfrak{p}\) is a union of tori, annuli, and simple closed curves. Note that up to homeomorphism, this does not affect the manifold \(\overline{M} - \mathfrak{p}\) or its boundary \(\partial \overline{M} - \mathfrak{p}\).

A ‘pants block’ is a pared manifold built as a quotient of \(\Sigma \times I\), where \(\Sigma\) is a complexity-one surface (so \(\Sigma \cong \Sigma_{1,1}\) or \(\Sigma \cong \Sigma_{0,4}\)) and \(I\) an interval, say \(I = [0, 1]\), together with some additional decoration. We make this precise with the following.

**Definition 2.19** (Pants block). A \textbf{pants block} is a pair \((\overline{M}, \partial_1 \overline{M})\), where \(\overline{M}\) is a handlebody of genus 2 or 3, and \(\partial_1 \overline{M} \subset \partial \overline{M}\) is union of simple closed curves defining a pants decomposition of the boundary of \(\overline{M}\). We further assume that \((\overline{M}, \partial_1 \overline{M})\) admits a quotient map of pairs

\[
\varphi : (\overline{\Sigma} \times I, \alpha_0 \cup \alpha_1 \cup (\partial \overline{\Sigma}) \times I) \to (\overline{M}, \partial_1 \overline{M})
\]

such that

(a) \(\overline{\Sigma}\) is a compact, complexity-one surface;
(b) \(\alpha_i \subset \overline{\Sigma} \times \{i\}\), for \(i = 0, 1\) are curves of the form \(\alpha_i = \alpha^i \times \{i\}\), with \(\alpha^0, \alpha^1 \subset \overline{\Sigma}\) curves intersecting minimally (thus intersecting once in \(\overline{\Sigma}_{1,1}\) and twice in \(\overline{\Sigma}_{0,4}\)); and
(c) \(\varphi\) is 1-to-1, except on \(\partial \overline{\Sigma} \times I\), where \(\varphi^{-1}(\varphi(x, t)) = \{x\} \times I\), for all \((x, t) \in \partial \overline{\Sigma} \times I\).

We sometimes refer to a pants block simply as a \textbf{block}.

Up to homeomorphism (ignoring the pared locus), there are exactly two distinct pants blocks, as shown in Figure 6. It is straightforward to see that pants blocks are acylindrical pared manifolds using the product structure, since

\[
B - \partial_1 B \cong \overline{\Sigma} \times I - (\alpha_0 \cup \alpha_1 \cup (\partial \overline{\Sigma} \times I)).
\]

It will be convenient to keep track of where the components of \(\partial_1 B\) came from. As such, we use the following notation for the components of \(\partial_1 B\):

\[
\partial^-_1 B = \varphi(\alpha_0) \quad \partial^+_1 B = \varphi(\alpha_1) \quad \partial^0_1 B = \varphi(\partial \overline{\Sigma} \times I).
\]
FIGURE 6  Pants blocks corresponding to the one-holed torus and four-holed sphere. The curve $\alpha_0$ is shown in blue, the curve $\alpha_1$ is shown in red, and the image (under $\varphi$) of $\partial \Sigma$ is shown in grey. The meridian disks for the handlebodies are drawn for illustrative purposes only.

We also write $\partial_2 B = \partial B - \partial_1 B$, which is a disjoint union of thrice-punctured spheres, $\Sigma_{0,3}$, (two for $\Sigma_{1,1}$ and four for $\Sigma_{0,4}$), and set

$$\partial^-_2 B = \partial_2 B \cap \varphi(\Sigma \times \{0\}) \text{ and } \partial^+_2 B = \partial_2 B \cap \varphi(\Sigma \times \{1\}).$$

We often denote a pants block simply as $B$, suppressing the decoration and the quotient map, though they are in fact part of the structure. When we want to distinguish between the two types of pants blocks built from $\Sigma_{0,4}$ and $\Sigma_{1,1}$, we write $B^S$ and $B^T$, respectively.

**Remark.** We note that $(B, \partial_1 B)$ is a pared 3-manifold: everything except condition (4) in the definition is immediate. Condition (4) follows from an identification of the fundamental group as a free group, in which all components of $\partial_1 B$ are distinct homotopy classes. The product structure implies $\partial_2 B$ is incompressible, and by the comments above, $(B, \partial_1 B)$ is acylindrical.

### 2.9 Geometry of hyperbolic 3-manifolds

By a **convex hyperbolic 3-manifold**, we mean a 3-manifold which is the quotient of a closed, convex subset of $\mathbb{H}^3$ by a discrete, torsion-free subgroup $\Gamma < \text{PSL}_2(\mathbb{C})$. A **convex hyperbolic metric** on a 3-manifold, $M$, is a metric which makes $M$ isometric to a convex hyperbolic 3-manifold. A special case of Thurston’s Geometrization Theorem for Haken manifolds is the following; see [37, 40, 50].

**Theorem 2.20.** Suppose $(\overline{M}, \mathcal{P})$ is a pared 3-manifold and $\partial \overline{M} - \mathcal{P} \neq \emptyset$ is incompressible. Then $\overline{M} - \mathcal{P}$ admits a convex hyperbolic metric $\sigma$. If $(\overline{M}, \mathcal{P})$ is additionally assumed to be acylindrical and non-degenerate, then there is a unique (up to isometry) convex hyperbolic structure $\sigma_{\text{min}}$ on $\overline{M} - \mathcal{P}$ for which $\partial \overline{M} - \mathcal{P}$ is totally geodesic.
The non-degeneracy assumption on \((\overline{M}, p)\) rules out the case that the structure degenerates to a lower dimensional manifold. Because of acylindricity, this simply means that \((\overline{M}, p)\) is not a pair of pants times an interval, with \(p\) being the three annuli which are the boundary curves of the pants times the interval. We also note that \(p\) is allowed to be empty in the theorem above, though \(\partial \overline{M}\) is not. The uniqueness of \(\sigma_{\min}\) in Theorem 2.20 is a consequence of Mostow–Prasad rigidity by doubling \(\overline{M} - p\) over the boundary.

We will write \(\text{Vol}(\overline{M} - p, \sigma)\) for the total volume of the manifold \(\overline{M} - p\) with respect to the metric \(\sigma\). In the special case that \((\overline{M}, p)\) is acylindrical, we will write \(\text{Vol}(\overline{M} - p)\) to mean \(\text{Vol}(\overline{M} - p, \sigma_{\min})\), where \(\sigma_{\min}\) is the metric given by Theorem 2.20.

A pants block \((B, \partial_1 B)\) is an acylindrical, pared manifold, as described in the remark above; thus Theorem 2.20 gives a unique hyperbolic structure on \(\widetilde{B} = B - \partial_1 B\) with totally geodesic boundary. In fact, this can be constructed explicitly from regular ideal octahedra: one octahedron in the case of \(\overline{M}^T\) and two in the case of \(\overline{M}^S\); see, for example, Agol [3, Lemma 2.3].

**Proposition 2.21.** The manifold \(\widetilde{B}\) has a convex hyperbolic metric \(\sigma_{\overline{s}}\) with totally geodesic boundary consisting of a union of thrice-punctured spheres. The metric \(\sigma_{\overline{s}}\) is obtained by gluing together half the faces of a regular ideal octahedron for \(B = B^T\) and two regular ideal octahedra for \(B = B^S\). Consequently,

\[
\text{Vol}(\widetilde{B}^T, \sigma_{\overline{s}}) = V_{\text{oct}} \quad \text{and} \quad \text{Vol}(\widetilde{B^S}, \sigma_{\overline{s}}) = 2V_{\text{oct}},
\]

where \(V_{\text{oct}}\) is the volume of a regular ideal octahedron.

The notation \(\sigma_{\min}\) is justified by the following theorem due to Storm [47, Theorem 3.1]; see also Storm [46] and Agol–Storm–Thurston [4].

**Theorem 2.22.** Given a pared manifold \((\overline{M}, p)\) which is acylindrical and non-degenerate with \(\partial \overline{M} - p \neq \emptyset\) and given any convex hyperbolic metric \(\sigma\) on \(\overline{M} - p\), we have \(\text{Vol}(\overline{M} - p, \sigma_{\min}) \leq \text{Vol}(\overline{M} - p, \sigma)\) with equality precisely when \(\sigma_{\min} = \sigma\).

Suppose \((\overline{M}, p)\) is a pared 3-manifold and \(\partial \overline{M} - p \neq \emptyset\) is incompressible. In this situation, we define

\[
\text{Vol}(\overline{M} - p) = \inf\{\text{Vol}(\overline{M} - p, \sigma) \mid \sigma \text{ a convex hyperbolic metric on } \overline{M} - p\}.
\]

(2.6)

If \(p = \emptyset\), we also write \(\text{Vol}(\overline{M}) = \text{Vol}(\overline{M} - \emptyset)\).

By Theorem 2.22, \(\text{Vol}(\overline{M} - p)\) is realized as \(\text{Vol}(\overline{M} - p, \sigma_{\min})\) when \((\overline{M}, p)\) is acylindrical. The main fact we will need about the general case is the second half of the result of Storm [47, Theorem 3.1] referenced above.

**Theorem 2.23.** If \((\overline{M}, p)\) is a pared 3-manifold with \(\partial \overline{M} - p \neq \emptyset\) incompressible, then \(\text{Vol}(\overline{M} - p)\) is equal to half of the simplicial volume of the double of \(\overline{M}\) over \(\partial \overline{M} - p\). In particular, for any degree \(k < \infty\) covering, \((\overline{M'}, p') \rightarrow (\overline{M}, p)\), one has \(\text{Vol}(\overline{M'} - p') = k \text{Vol}(\overline{M} - p)\).

It will also be useful to relate the infimal volume of a pared manifold to the infimal volume of the manifold after forgetting the paring locus; see Storm [47, Corollary 1.2, Corollary 2.10, and Theorem 3.2].
Theorem 2.24. Suppose \((\overline{M}, \mathcal{P})\) is a pared acylindrical 3-manifold and that \(\partial \overline{M}\) is incompressible, nonempty, and all components have genus at least 2. Then \(\text{Vol}(\overline{M}) \leq \text{Vol}(\overline{M} - \mathcal{P})\).

Since \((\overline{M}, \mathcal{P})\) is pared and \(\partial \overline{M}\) contains no tori we see that \((\overline{M}, \emptyset)\) is also pared and hence admits a convex hyperbolic structure by Theorem 2.20 so \(\text{Vol}(\overline{M})\) is well-defined.

Storm’s proof of Theorem 2.24 exploits the behavior of volume (and simplicial volume) under Dehn filling. The behavior of the hyperbolic and simplicial volumes is part of Thurston’s Dehn Surgery Theorem; see Thurston [49, Theorems 5.8.2, 6.5.4, and 6.5.6], as well as [41, Theorem 1A] and [34, 15.4.1]. The version of Thurston’s theorem, which we state here, follows by doubling over the totally geodesic boundary and appealing to [38]. The notion of a sequence of slopes going to \(\infty\) is natural from the perspective of Thurston’s Hyperbolic Dehn Surgery Theorem.

Theorem 2.25. Let \((M, \sigma)\) be a finite volume hyperbolic 3-manifold with totally geodesic boundary and \(k\) torus cusps. Suppose \(\beta^n = (\beta^n_1, \ldots, \beta^n_k)\) are a sequence of Dehn filling coefficients so that for each \(i\), either \(\beta^n_i = \infty\) for all \(n\), or \(\beta^n_i \to \infty\). Then for \(n\) sufficiently large, \(M(\beta^n)\) admits a complete hyperbolic structure \(\sigma(\beta^n)\) with totally geodesic boundary and

\[
\text{Vol}(M(\beta^n), \sigma(\beta^n)) \to \text{Vol}(M, \sigma).
\]

Moreover,

\[
\text{Vol}(M(\beta^n), \sigma(\beta^n)) \leq \text{Vol}(M, \sigma)
\]

with equality if and only if \(\beta^n = (\infty, \ldots, \infty)\).

3 | COMPACTIFIED MAPPING TORI

Given an end-periodic homeomorphism \(f : S \to S\), we let

\[
M_f = S \times [0, 1]/(x, 1) \sim (f(x), 0)
\]
de note the mapping torus of \(f\). We will write \((\psi_s)\) to denote the suspension flow on \(M_f\), defined as the descent to \(M_f\) of the local flow on \(S \times [0, 1]\) given by \(\psi_s(x, t) = (x, t + s)\), when \(0 \leq t \leq t + s \leq 1\). The goal of this section is to prove the following result.

Proposition 3.1. Let \(f : S \to S\) be an irreducible, end-periodic homeomorphism of a surface with finitely many ends, all accumulated by genus. Then the mapping torus, \(M_f\), is the interior of a compact, irreducible, atoroidal 3-manifold, \(\overline{M}_f\), with incompressible boundary. If \(f\) is strongly irreducible, then \(\overline{M}_f\) is also acylindrical.

The fact that \(M_f\) is the interior of a compact, irreducible manifold with boundary is well-known: the boundary points are obtained by adding ‘endpoints at infinity’ to each positive (respectively, negative) ray of the suspension flow that passes through \(U_+ \times \{0\}\) (respectively, \(U_- \times \{0\}\)); see [25]. As we will make explicit use of this compactification in our discussion, we discuss this in more detail.
Suppose \( f \) is any end-periodic homeomorphism of \( S \). We begin by describing the compactification of \( M_f \) in a way that is well-suited to our discussion. As a starting point, we identify the infinite cyclic cover of \( M_f \) dual to the fibration over \( S^1 \) by

\[ p : S \times (-\infty, \infty) \to M_f \]

together with the action of the covering group isomorphic to \( \mathbb{Z} \), generated by the homeomorphism \( F \) given by

\[ F(x, t) = (f(x), t - 1). \]

This is a covering action with quotient \( M_f \). Indeed, \( S \times [0, 1] \) is a fundamental domain, and the only points that are nontrivially identified are of the form \((x, 1) \sim F(x, 1) = (f(x), 0)\).

Recall that \( U_+, U_- \subset S \) are the positive and negative escaping sets, respectively, as defined in Subsection 2.2. We define a partial compactification of \( S \times (-\infty, \infty) \) inside \( S \times [-\infty, \infty] \) by

\[ \tilde{M}_\infty = S \times (-\infty, \infty) \cup U_+ \times \{\infty\} \cup U_- \times \{-\infty\}. \]

Since \( U_+ \) and \( U_- \) are \( f \)-invariant, we may extend the action of \( \langle F \rangle \) by defining

\[ F(x, \pm \infty) = (f(x), \pm \infty) \]

whenever \( x \in U_\pm \). By Lemma 2.4, \( \langle f \rangle \mid_{U_\pm} \) acts cocompactly, with quotient closed (not necessarily connected) surfaces \( S_+ = U_+ / \langle f \rangle \) and \( S_- = U_- / \langle f \rangle \).

**Lemma 3.2.** Suppose \( f \) is end-periodic and \( \langle F \rangle \) acts on \( S \times (-\infty, \infty) \) as above. Then the action of \( \langle F \rangle \cong \mathbb{Z} \) extends to a properly discontinuous, cocompact action on \( \tilde{M}_\infty \) by \( F(x, \pm \infty) = (f(x), \pm \infty) \). The quotient, \( M_f \), is thus a compact manifold with \( \partial \tilde{M}_f \cong S_+ \cup S_- \).

**Proof.** Recall that

\[ U_\pm = \bigcup_{k=1}^{\infty} f^{\mp k} U_\pm \]

where \( U_\pm \) are good nesting neighborhoods. Since \( U_\pm \) is invariant by \( f \), it follows that the action of \( F \) extends to an action by the given formula.

By considering an appropriate open covering of \( \tilde{M}_\infty \), we note that for any compact set \( K \subset \tilde{M}_\infty \), there is some \( n \in \mathbb{N} \) and a decomposition \( K = K_+ \cup K_- \cup K_0 \) where

\[ K_+ \subset (f^{-n}(U_+) - f^n(U_+)) \times [0, \infty], \]

\[ K_- \subset (f^n(U_-) - f^{-n}(U_-)) \times [-\infty, 0], \]

and

\[ K_0 \subset (S - (f^n(U_+) \cup f^{-n}(U_-))) \times [-n, n]. \]
Note that only finitely many $\langle F \rangle$-translates of each of $K_0, K_+, \text{ or } K_-$ can intersect themselves (the case of $K_0$ by inspection of the second coordinate, and $K_\pm$ by inspection of the first coordinate). Therefore, we must show that only finitely many $\langle F \rangle$-translates of any one of these sets can nontrivially intersect another one. And for this, it suffices to consider the $\langle F \rangle$-translates of $K_0$ with either $K_+$ or $K_-$ and the $\langle F \rangle$-translates of $K_+$ with $K_-$. 

In the first case, we observe that for any integer $m > n$, $F^m(K_0) \cap K_+ = \emptyset$, by considering the second coordinates. On the other hand, for any integer $m < -2n$, inspection of the first coordinate shows that $F^m(K_0) \cap K_+ = \emptyset$. Thus, $F^m(K_0) \cap K_+ = \emptyset$ for all but at most $3n + 1$ values of $m \in \mathbb{Z}$. Similar reasoning applies to show that at most $3n + 1$ $\langle F \rangle$-translates of $K_0$ nontrivially intersect $K_-$. 

Next observe that for all integers $m < 0$, $F^m(K_+) \cap K_- = \emptyset$, by considering the second coordinates. Finally, note that for all $k > n$, $f^k(U_+) \cap f^n(U_-) = \emptyset$, and so if $m > 2n$, then $F^m(K_+) \cap K_- = \emptyset$. Consequently, there are at most $2n + 1$ values of $m \in \mathbb{Z}$ so that $F^m(K_+) \cap K_- \neq \emptyset$. Therefore, the action is properly discontinuous, and hence $\bar{M}_f = \bar{M}_\infty / \langle F \rangle$ is a 3-manifold with boundary $S_\pm$.

All that remains is to prove that the action is cocompact. For convenience, define

$$C_n = S - (f^n(U_+) \cup f^n(U_-)) \quad \text{and} \quad \Delta_\pm = \bar{U}_\pm - f^\pm 1(U_\pm).$$

We observe that $C_n$ is compact for all $n > 0$, and $\Delta_\pm$ is a compact fundamental domain for $\langle f \rangle$ on $U_\pm$.

Now we suppose $\{z_k\}_{k=1}^{\infty} \subset \bar{M}_f$ is any sequence, and show that after passing to a subsequence, it converges. Since $\partial \bar{M}_f = S_+ \cup S_-$ is compact, without loss of generality we may assume $z_k \in M_f$, for all $k$.

For each $k \geq 1$, let $(x_k, t_k)$ be the unique point in $p^{-1}(z_k) \cap S \times [0,1)$. Since each $C_n$ is compact, we may assume that for all $n$, only finitely many $x_k$ lie in $C_n$. After passing to a subsequence, we may assume that for any $n$, all but finitely many $x_k$ lie in $f^n(U_+)$ or in $f^{-n}(U_-)$. Assuming the first case (the second is similar) and passing to a further subsequence, we can choose an increasing sequence $\{n_k\}_{k=1}^{\infty}$ so that $x_k \in f^{n_k}(\Delta_+)$. Now observe that

$$F^{-n_k}(x_k, t_k) = (f^{-n_k}(x_k), t_k + n_k) \in \Delta_+ \times [0, \infty).$$

The set on the right is compact, and so the sequence converges. It follows that $\{z_k\}$ converges, as required. \[\square\]

Note that the foliation of $\bar{M}_\infty$ by fibers of the projection $\bar{M}_\infty \to [-\infty, \infty]$ is invariant by $\langle F \rangle$, and so descends to a transversely oriented foliation $\mathcal{F}$ of $\bar{M}_f$ for which $\partial \bar{M}_f$ is a union of (compact) leaves.

**Lemma 3.3.** Suppose $f$ is an end-periodic homeomorphism of $S$ and $\mathcal{F}$ the foliation on $\bar{M}_f$ just defined. Then, $\mathcal{F}$ is a taut foliation.

**Proof.** Any transversal of the foliation of $\bar{M}_\infty$ connecting a point of $U_- \times \{-\infty\}$ to a point of $U_+ \times \{\infty\}$, projects to a transversal of $\mathcal{F}$ on $\bar{M}_f$. Doing this for each component of $U_+$ and $U_-$, produces enough transversals to intersect every leaf of $\mathcal{F}$, as required. \[\square\]
We let $D\overline{M}_f$ be the double of $\overline{M}_f$ over its boundary. This induces a foliation $\mathcal{F}_D$ on $D\overline{M}_f$ which is also taut.

**Proposition 3.4.** Suppose $f$ is an irreducible, end-periodic homeomorphism. Then $\overline{M}_f$ is a compact, irreducible, atoroidal 3-manifold with incompressible boundary.

**Proof.** By Lemma 3.2, we know that $\overline{M}_f$ is a compact 3-manifold with boundary. Since $M_f$ is a mapping torus, it follows that $M_f$ (and thus $\overline{M}_f$) is irreducible. Since each component of $\partial \overline{M}_f$ is covered by a component of the $\pi_1$-injective subsurface $U_\pm \times \{\pm \infty\} \subset \tilde{M}_\infty$, it follows that $\partial \overline{M}_f$ is incompressible.

To prove that $\overline{M}_f$ is atoroidal, we suppose that $T \subset \overline{M}_f$ is an incompressible torus and derive a contradiction. By Theorem 2.17, after an isotopy of the inclusion, we can assume that $T$ is transverse to $\mathcal{F}$ (since $\mathcal{F}$ has no torus leaves).

Cutting $M_f$ open along a fiber, we obtain a product $S \times [0,1]$ and the torus is cut into a union of annuli connecting a finite, disjoint union of curves in $S \times \{0\}$ to a finite disjoint union of curves in $S \times \{1\}$ (the torus cannot be disjoint from any fiber since $S \times [0,1]$ contains no incompressible tori). Since $M_f$ is obtained by gluing $S \times \{1\}$ to $S \times \{0\}$ by $f$, this produces a finite $f$-invariant union of curves, and thus an $f$-periodic curve. This is a contradiction to the irreducibility assumption, and thus $\overline{M}_f$ is atoroidal. □

Suppose $f$ is an irreducible, end-periodic homeomorphism. Since the boundary components of $\overline{M}_f$ are closed surfaces of genus at least 2, Proposition 3.4 and Theorem 2.20 imply that $\overline{M}_f$ admits a convex hyperbolic metric, since $(\overline{M}_f, \emptyset)$ is pared; see the discussion following Theorem 2.20.

**Lemma 3.5.** Suppose $f$ is an irreducible, end-periodic homeomorphism and $(A, \partial A) \subset (\overline{M}_f, \partial \overline{M}_f)$ is an essential annulus. Then both boundary components of $A$ are in $S_+$ or both are in $S_-$, and their preimages in $\tilde{M}_\infty$ are lines in the surface $U_+ \times \{\infty\}$ or $U_- \times \{-\infty\}$, respectively.

If $f$ is strongly irreducible, then $\overline{M}_f$ is acylindrical.

**Proof.** An application of Theorem 2.17 implies that, after an isotopy of the inclusion which is the identity on $\partial A$, we may assume $A$ is transverse to $\mathcal{F}$, except possibly at finitely many circle tangencies occurring in $M_f \cap A$. In particular, $\mathcal{F}$ induces a foliation $\mathcal{F}_A$ on $A$ whose leaves are components of intersection with the leaves of $\mathcal{F}$ (consequently, each component of $\partial A$ is a leaf of $\mathcal{F}_A$).

If there is a circle leaf $\alpha$ of $\mathcal{F}_A$ in the interior of $A$ (this happens, for example, if there is a circle tangency) then $\alpha$ is also a curve in some fiber $S$ of $M_f$. Since $\alpha$, viewed as a loop on $A$, generates $\pi_1(A)$ and lies on $S$, it follows that the inclusion of $A$ lifts to an embedding into $\tilde{M}_\infty$. In particular, the preimage of $A$ in this covering space is a disjoint union of copies of $A$ permuted by the covering transformation $F$.

Suppose one boundary component of $A$ lies on $S_-$ and the other on $S_+$. Then any lift of $A$ has one boundary component in $U_- \times \{-\infty\}$ and the other in $U_+ \times \{\infty\}$. This means there is an $f$-reducing curve which corresponds to $(S \times \{0\}) \cap A$, contradicting irreducibility.

Therefore, both boundary components of $A$ lie in $S_+$ or both lie in $S_-$. In $\tilde{M}_\infty \subset S \times [-\infty, \infty]$, any lift of $A$ is therefore boundary parallel, implying the same for $A$ in $\overline{M}_f$, contradicting the fact that $A$ is essential.
From the contradictions above, we conclude that $A$ has no closed leaves except its boundary curves. In this case, all leaves of $\mathcal{F}_A$ spiral in toward the boundary components of $A$. Let $\bar{A} \subset \bar{M}_\infty$ be any component of the preimage of $A$. Note that the foliation $\mathcal{F}_A$ lifts to a foliation $\bar{\mathcal{F}}_A$ on $\bar{A}$ whose non-boundary leaves are intersections with fibers $S \times \{t\}$.

Claim 3.6. $\bar{A}$ is the universal covering of $A$ and some power of $F$ preserves $\bar{A}$ and generates the covering group.

Proof. Either some positive power of $F$ preserves $\bar{A}$, or no positive power does. In the former case, the smallest such positive power generates an infinite order covering action on $\bar{A}$ with quotient $A$. In this case, $\bar{A}$ must be the universal cover and we are done. If there is no positive power that preserves $\bar{A}$, then $\bar{A}$ must be a lift of $A$. In this case, the boundary of $\bar{A}$ is a union of curves in $\mathcal{U}_+ \times \{\infty\} \cup \mathcal{U}_- \times \{-\infty\}$. Since the non-boundary leaves of $\bar{\mathcal{F}}_A$ limit onto the boundary curves of $\bar{A}$, we see that some fiber $S \times \{t\}$ contains a path (the leaf) that accumulates on $\mathcal{U}_\pm \times \{\pm \infty\}$, which is impossible. This contradiction means we are in the first case that $\bar{A}$ is the universal cover, as required.

Let $F^m$ be the smallest positive power of $F$ that preserves $\bar{A}$ and thus generates the covering group of $\bar{A} \to A$. Let $\ell' \subset (S \times \{t\}) \cap \bar{A}$ be a leaf of $\bar{\mathcal{F}}_A$ and consider the strip in $\bar{A}$ between $\ell'$ and $F^m(\ell')$, which is contained in $S \times [t, t + m]$. This is a union of leaves of $\bar{\mathcal{F}}_A$ which are slices of a product structure on the strip. Note that in the covering to $A$ (or $\bar{M}_f$), the restriction to each leaf in the strip injects to a leaf of $\mathcal{F}_A$ and spirals in toward $\partial A$. In particular, the two ends of the strip must exit ends of $S \times [t, t + m]$ (since the only sequences in $S \times [t, t + m]$ which when projected to $\bar{M}_f$ accumulate on $\partial \bar{M}_f$ are those that exit the ends). Since $F(x, s) = (f(x), s - 1)$, it follows that after projecting out the $\mathbb{R}$-direction, $\ell'$ projects to a line $\ell_0$ in $S$ and the strip projects to a proper homotopy from $\ell_0$ to $f^m(\ell_0)$. That is, $\ell_0$ is an $f$-periodic line. We note that $\ell_0$ is in fact an essential line. For, if not, then it would bound an infinite monogon (topological half-plane) in $S$, and hence each leaf of $\bar{\mathcal{F}}_A$ would also bound a monogon in the surface slice $S \times \{s\}$. The union of these would project to the interior of a solid torus in $\bar{M}_f$ whose boundary is the union of $A$ and an annulus in $\partial \bar{M}_f$, contradicting the fact that $A$ is essential. If $f$ is strongly irreducible, this is a contradiction. Therefore, in this case, there is no essential annulus and $\bar{M}_f$ is acylindrical.

In general, the boundary of $\bar{A}$ consists of two lines in $\mathcal{U}_\pm \times \{\pm \infty\}$. If one of these lines lies in $\mathcal{U}_- \times \{-\infty\}$ and the other lies in $\mathcal{U}_+ \times \{\infty\}$, then $\ell_0$ is an AR-periodic line with one end in an attracting end of $S$ and the other in a repelling end, a contradiction. It follows that both components of the boundary of $\bar{A}$ are contained in $\mathcal{U}_+ \times \{\infty\}$ or both are contained in $\mathcal{U}_- \times \{-\infty\}$. This proves the first claim of the lemma, and since we have already proved the second claim, we are done.

Proof of Proposition 3.1. This is immediate from Proposition 3.4 and Lemma 3.5.

Finally, we note that the suspension flow $(\psi_s)$ on $M_f$ can be reparameterized to a local flow so that the flow lines that exit every compact set limit to a point on $\partial \bar{M}_f$ in finite time. We do this so that it extends to a local flow $(\bar{\psi}_s)$ on $\bar{M}_f$. This lifts to a local flow on $\bar{M}_\infty \subset S \times [-\infty, \infty]$ and has flow lines given by $\{x\} \times [-\infty, \infty]$, with either or both endpoints missing if $x$ is not in one or both of $\mathcal{U}_\pm \times \{\pm \infty\}$. Moreover, we can assume that $(\bar{\psi}_s)$ on $\bar{M}_f$, and its negative on the reflected copy of $\bar{M}_f$ in the double $D \bar{M}_f$, glue together to a well-defined flow $(\psi_s^D)$ on $D \bar{M}_f$. 

\[ \text{Proof of Proposition 3.1. This is immediate from Proposition 3.4 and Lemma 3.5.} \]
4 | A QUOTIENT FROM A PATH OF PANTS DECOMPOSITIONS

In this section, we begin by describing a method for decomposing 3-manifolds that will be useful in proving bounds on volumes (see the remark below for other instances of such decompositions in the literature). We will use this idea to prove Theorems 1.1 and 1.2 as well as the corollaries described in the introduction. In our situation, the decomposition will arise naturally from a quotient map with various properties; see Proposition 4.3.

4.1 | Block decompositions

First, recall from Definition 2.19 that a pants block is a compact 3-manifold, $B$, with pared boundary $\partial_1 B \subset \partial B$, where $\partial_1 B$ is a disjoint union of three or six embedded circles, depending on whether the block comes from $\Sigma_{1,1}$ or $\Sigma_{0,4}$, respectively. We view $\partial_1 B$ as a stratified space (for bookkeeping purposes only), with strata being $\partial_1 B$, $\partial_2 B = \partial B - \partial_1 B$, and the interior $\text{Int}(B)$. We also recall that $\hat{B} = B - \partial_1 B$ admits a complete hyperbolic structure with totally geodesic, thrice-punctured sphere boundary components.

Suppose $N$ is a compact 3-manifold, possibly with boundary, and $L \subset N$ a link. This is the image of a disjoint union of finitely many circles by a (smooth) embedding. We require each component of $L$ to be contained in the interior of $N$ or the boundary of $N$ (that is, if a component intersects the boundary, then it is contained in it).

**Definition 4.1.** A block map to $N$ relative to $L$ is a map of pairs $(B, \partial_1 B) \to (N, L)$ which is injective on each component of each stratum of $B$, and so that the image of each component of $\partial_2 B$ is either disjoint from $\partial N$ or contained in it (but distinct components may have the same image). We also require that the image of $B$ meets $L$ precisely in the image of $\partial_1 B$. Given a block map, it is convenient to identify the block with its image $B \subset N$, with the map itself implicit, thus allowing us to identify the components of each stratum in $N$ (some may be equal).

A block decomposition of $N$ relative to $L$ (or a block decomposition of $(N, L)$) is a finite set of block maps $\{B_i\}_{i=1}^n$ to $N$ relative to $L$, so that (after identifying the blocks with their images) we have

1. the interiors of the $B_i$ are pairwise disjoint;
2. the union of the blocks is all of $N$; and
3. whenever two blocks intersect, they do so in a union of components of strata.

Thus, we can view $N$ as being built from a finite number of blocks. The blocks are glued together in pairs along closures of three-punctured spheres in their boundaries. The link $L$ is the union of the images of the 1-boundaries, $\bigcup \partial_1 B_i$.

Note that if $(N, L)$ admits a block decomposition, then $L \cap \partial N$ defines a pants decomposition of $\partial N$ (which may be empty).

**Remark.** Using block decompositions to build 3-manifolds is very similar to the construction of Minsky’s ‘model manifold’ defined by Minsky [39] and used by Brock–Canary–Minsky [14] in the proof the Ending Lamination Conjecture. Our particular definition includes as a special case the construction Agol used in [3] for mapping tori of finite-type surfaces.
An immediate consequence of Proposition 2.21 is the following result. Recall that $V_{\text{oct}}$ is the volume of the regular ideal octahedron in hyperbolic 3-space.

**Corollary 4.2.** Suppose $L \subset N$ is a link in a compact 3-manifold $N$ with a block decomposition of $N$ relative to $L$ that consists of $n_T$ blocks of type $B^T$ and $n_S$ blocks of type $B^S$. Then, $N - L$ admits a complete hyperbolic metric with totally geodesic thrice-punctured sphere boundary components and volume $V_{\text{oct}}(n_T + 2n_S)$.

### 4.2 A block decomposition from a path in $P(S)$

Now assume $f$ is an irreducible, end-periodic homeomorphism of $S$. Choose a pants decomposition $P$ in any $f$-invariant component $\Omega \subset P_f(S)$ and a path $P = P_0, P_1, \ldots, P_n = f^{-1}(P)$ in $\Omega$. We fix representatives of the pants decompositions and of the isotopy class of $f$ (all denoted by the same names) so that $f(P_n) = P_0$. We also assume that our representatives are such that $P_k$ and $P_{k+1}$ agree on the complement of a complexity 1 (open) subsurface $Z_k$. We further assume that on $Z_k$, $P_k$ and $P_{k+1}$ meet in simple closed curves $\alpha_k^-, \alpha_k^+$, respectively, that intersect each other minimally: once or twice depending on the homeomorphism type of $Z_k$. We let $n_T$ and $n_S$ denote the number of surfaces $Z_k$ homeomorphic to $\Sigma_{1,1}$ and $\Sigma_{0,4}$, respectively, with $n_S + n_T = n$. As usual, $\overline{Z}_k \subset S$ is the compactified subsurface (which may only be embedded on its interior; see Subsection 2.1).

For each $0 \leq k \leq n$, we also view $P_k, Z_k \subset S$ as a subset of the slice

$$P_k, Z_k \subset S \times \left\{ \frac{k}{n} \right\} \subset S \times [0,1]$$

via the inclusion in the first factor. Gluing $S \times \{1\}$ to $S \times \{0\}$ via $f$, we produce $M_f$. Each of $P_k$ and $Z_k$ are well-defined subsets of $M_f \subset \overline{M}_f$, where we identify $P_n$ with $P_0$ via the gluing map $f$, and similarly identify $Z_n$ with $Z_0$. Define $L = \bigcup_k P_k \subset M_f$.

For each $0 \leq k < n$, set

$$W_k = Z_k \times \left[ \frac{k}{n}, \frac{k+1}{n} \right], \quad \overline{W}_k = \overline{Z}_k \times \left[ \frac{k}{n}, \frac{k+1}{n} \right], \quad W = \bigsqcup_{k=0}^{n-1} W_k, \quad \text{and} \quad \overline{W} = \bigsqcup_{k=0}^{n-1} \overline{W}_k.$$

The set $W$ and each $W_k$ is naturally an open subset of $M_f$, and by another abuse of notation we also view $\overline{W}$ and each $\overline{W}_k$ as subsets of $M_f$ (again, only the interior is actually embedded).

Each $\overline{W}_k$ admits a natural map to a block $\phi_k : \overline{W}_k \to B_k$ (by definition of a block). Here, $\alpha_k^-, \alpha_k^+$, and $\partial \overline{Z}_k \times \left[ \frac{k}{n}, \frac{k+1}{n} \right]$ map to $\partial_1^- B_k, \partial_1^+ B_k$, and $\partial_2^\pm B_k$, respectively. We write $\partial_2 \overline{W}_k$ and $\partial_2^\pm \overline{W}_k$ for the unions of 3-punctured spheres mapping to $\partial_2 B_k$ and $\partial_2^\pm B_k$, respectively. We identify $\partial_2 \overline{W}_k$, and $\partial_2^\pm \overline{W}_k$ with their images in $M_f$ when convenient. We also write $\partial_1 \overline{W}_k = \partial_1 \overline{Z}_k \times \left[ \frac{k}{n}, \frac{k+1}{n} \right]$ to denote vertical boundary annuli which project to $\partial_1^\pm B_k$.

To state the main proposition of this section, recall from Corollary 2.8 that $\xi(\partial \overline{M}_f) = \xi(S_+) + \xi(S_-) = 3|\Phi^*(f)|$.

**Proposition 4.3.** Given a sequence of elementary moves $P = P_0, \ldots, P_n = f^{-1}(P) \subset \Omega \subset P_f(S)$, together with the notation and assumptions above, there is a manifold $\overline{M}_f$ and a quotient map $h : \overline{M}_f \to \tilde{M}_f$ with the following properties:
(1) $h$ restricts to a homeomorphism from $\partial \widetilde{M}_f \to \partial \hat{M}_f$;
(2) $h$ is homotopic (rel boundary) to a homeomorphism $H : \overline{M}_f \to \hat{M}_f$;
(3) $L = h(L)$ is a link with $n + \frac{3}{2} |\Phi^*(f)|$ components;
(4) $h|_{\overline{W}_k} : \overline{W}_k \to \hat{M}_f$ is the composition of $\phi_k$ and a block map $B_k \subset \hat{M}_f$ relative to $L$. Moreover, 
   \{B_k\}_{k=0}^{n-1}$ is a block decomposition of $(\hat{M}_f, L)$;
(5) the pants decomposition $P_{\Omega} = h^{-1}(L \cap \partial \hat{M}_f) = H^{-1}(L \cap \partial \hat{M}_f) \subset \partial \overline{M}_f$ depends only on the
   component $\Omega \subset \ell(S)$ containing $P$, up to isotopy; and
(6) $(\overline{M}_f, P_{\Omega})$ is a pared acylindrical manifold.

The pants decomposition $P_{\Omega}$ of $\partial \overline{M}_f$ is the one alluded to in the introduction, and so by Proposition
4.3 (6) together with Theorem 2.20, $\overline{M}_f - P_{\Omega}$ admits a complete hyperbolic structure, $\sigma(f, \Omega)$, with
totally geodesic, thrice-punctured sphere boundary components. So, by Theorem 2.22,
\[ \text{Vol}(\overline{M}_f - P_{\Omega}) = \text{Vol}(\overline{M}_f - P_{\Omega}) = \text{Vol}(\overline{M}_f - P_{\Omega}, \sigma(f, \Omega)). \]

### 4.3 Applications of Proposition 4.3

We will prove the proposition in the next section, but first we use it to deduce the first two
theorems from the introduction and their corollaries.

**Theorem 1.2.** For any $f$-invariant component $\Omega \subset P_f(S),$
\[ \text{Vol}(\overline{M}_f - P_{\Omega}) \leq V_{\text{oct}}(\tau(f, \Omega)). \]

**Proof.** Observe that for any $N > 0$, $\overline{M}_{fN}$ is an $N$-fold cover of $\overline{M}_f$. We will apply Proposition 4.3
to $\overline{M}_{fN}$ for an appropriate choice of $N$, which we now specify. Fix any $f$-invariant component
$\Omega \subset P_f(S)$ and let $\varepsilon > 0$. Let $N > 0$ and $P = P_0, \ldots, P_n = f^{-N}(P) \subset \Omega$ be a sequence of elementary
moves so that
\[ |\tau(f, \Omega) - \frac{n_T + 2n_S}{N}| < \varepsilon, \]
where $n = n_S + n_T$, as above. Recall that the path metric we defined on (the components of) $P(S)$ has edges of length 1 corresponding to an elementary move that occurs on a one-holed
torus has length 1, and edges of length 2 corresponding to an elementary move that occurs on
a four-holed sphere.

For the path $P_0, \ldots, P_n$ above, let $P^{N}_{\Omega} \subset \overline{M}_{fN}$ be the pants decomposition from part (5). Observe
that by replacing the sequence $P_0, \ldots, P_n$ with a concatenation of $P = P_0 = P'_{0}, P'_1, \ldots, P'_k = f^{-1}(P)$
and its image under $f^{-1}, f^{-2}, \ldots, f^{-N+1}$, we get the same pants decomposition $P^{N}_{\Omega}$ on $\partial \overline{M}_{fN}$, by
Proposition 4.3 (5), but this necessarily lifts the pants decomposition $P_{\Omega}$ of $\partial \hat{M}_f$. Since $\text{Vol}(\overline{M}_{fN} - P^{N}_{\Omega})$ is the volume of the metric with totally geodesic boundary on $\overline{M}_{fN} - P^{N}_{\Omega}$, it follows that
\[ \text{Vol}(\overline{M}_{fN} - P^{N}_{\Omega}) = NV_{\text{oct}}(\overline{M}_f - P_{\Omega}). \]
Therefore, if we can prove
\[ \text{Vol}(\overline{M}_{fN} - P^{N}_{\Omega}) \leq V_{\text{oct}}(n_T + 2n_S) \quad (4.1) \]
we will have
\[
\text{Vol}(\overline{M}_f - P_{\Omega}) = \frac{\text{Vol}(M_{f^N} - P^N_{\Omega})}{N} \leq V_{\text{oct}} \frac{n_T + 2n_S}{N} \leq V_{\text{oct}}(\tau(f, \Omega) + \varepsilon),
\]
and letting $\varepsilon \to 0$ will then complete the proof.

To prove the inequality (4.1), we apply Proposition 4.3 (4) to the path $P_0, \ldots, P_n$ and $M_{f^N}$, producing a block decomposition of $\overline{M}_f$ relative to $\hat{L}$. By Corollary 4.2, the drilled manifold $\overline{M}_f - \hat{L}$ also admits a complete hyperbolic metric $\sigma$ with totally geodesic thrice-punctured sphere boundary components having volume $\text{Vol}(\overline{M}_f - \hat{L}, \sigma) = V_{\text{oct}}(n_T + 2n_S)$. We now observe that $\overline{M}_f - P_{\Omega}$ is homeomorphic to a Dehn filling on $\overline{M}_f - \hat{L}$ of some of the components of $\hat{L}$. By Proposition 4.3(6) and Theorem 2.20, there is a convex hyperbolic metric $\sigma(f_N, \Omega)$ on $\overline{M}_f - P^N_{\Omega}$ with totally geodesic boundary. Appealing to Theorem 2.25, we see
\[
\text{Vol}(\overline{M}_f - P^N_{\Omega}) \leq \text{Vol}(\overline{M}_f - \hat{L}) = V_{\text{oct}}(n_T + 2n_S).
\]
This proves (4.1) and so completes the proof of the theorem. \qed

From Theorem 1.2, we obtain the first theorem.

**Theorem 1.1.** For any irreducible, end-periodic homeomorphism $f : S \to S$ of a surface with finitely many ends, all accumulate by genus, we have $\text{Vol}(\overline{M}_f) \leq V_{\text{oct}}\tau(f)$.

**Proof.** Given $\varepsilon > 0$, suppose $\Omega \subset P_f(S)$ is a component such that
\[
|\tau(f) - \tau(f, \Omega)| < \varepsilon
\]
and let $N > 0$ be such that $f^N$ preserves $\Omega$. Then,
\[
\frac{\tau(f^N, \Omega)}{N} = \tau(f, \Omega) \leq \tau(f) + \varepsilon.
\]
According to Theorem 1.2, we have
\[
\text{Vol}(\overline{M}_f - P_{\Omega}) \leq V_{\text{oct}}\tau(f^N, \Omega),
\]
where $P_{\Omega}$ is the pants decomposition of $\partial \overline{M}_f$ from Proposition 4.3(5).

On the other hand, by Theorems 2.22 and 2.24,
\[
\text{Vol}(\overline{M}_f^N) \leq \text{Vol}(\overline{M}_f - P_{\Omega}) = \text{Vol}(\overline{M}_f - P_{\Omega}),
\]
and so by Theorem 2.23 and the inequalities above we have
\[
\text{Vol}(\overline{M}_f) = \frac{\text{Vol}(\overline{M}_f^N)}{N} \leq \frac{V_{\text{oct}}\tau(f^N, \Omega)}{N} \leq V_{\text{oct}}(\tau(f) + \varepsilon).
\]
Letting $\varepsilon \to 0$ completes the proof. \qed

Next, we prove the lower bound on translation length stated in the introduction. Recall that $V_{\text{tet}}$ is the volume of a regular ideal tetrahedron in $\mathbb{H}^3$. 
Corollary 1.4. Given an irreducible, end-periodic homeomorphism \( f \), we have
\[
\tau(f) \geq \frac{V_{\text{tet}} \xi(\partial \tilde{M}_f)}{2V_{\text{oct}}}.
\]

Proof. First observe that \( \tau(f) = \inf \tau(f, \Omega) \), where the infimum is taken over all components \( \Omega \subset \mathcal{P}(S) \), so it suffices to prove that for any component \( \Omega \), \( \tau(f, \Omega) \) is bounded below by the quantity on the right in the corollary. We need only consider components \( \Omega \subset \mathcal{P}_f(S) \), so we fix such a component. Let \( N > 0 \) be such that \( f^N(\Omega) = \Omega \), and let \( P_\Omega \subset \partial \tilde{M}_{f^N} \) be the pants decomposition given by Proposition 4.3(5). By Proposition 4.3(6) together with Theorem 2.20, \( \tilde{M}_{f^N} - P_\Omega \) admits a convex hyperbolic structure \( \sigma_{\text{min}} \) with totally geodesic boundary so that
\[
\text{Vol}(\tilde{M}_{f^N} - P_\Omega) = \text{Vol}(\tilde{M}_{f^N} - P_\Omega).
\]
The number of pants curves in \( P_\Omega \) is \( \xi(\partial \tilde{M}_{f^N}) = N \xi(\partial \tilde{M}_f) \), and so the double of \( \tilde{M}_{f^N} - P_\Omega \) admits a complete hyperbolic structure, obtained by doubling \( \sigma_{\text{min}} \), having
\[
\xi(\partial \tilde{M}_{f^N}) = N \xi(\partial \tilde{M}_f)
\]
cusps. According to Adams [2], the volume of this hyperbolic structure on the double of \( \tilde{M}_{f^N} \) is at least \( V_{\text{tet}} \) times the number of cusps, and thus
\[
\tau(f, \Omega) = \frac{\tau(f^N, \Omega)}{N} \geq \frac{V_{\text{tet}} \xi(\partial \tilde{M}_{f^N} - P_\Omega)}{N V_{\text{oct}}} \geq \frac{V_{\text{tet}} N \xi(\partial \tilde{M}_f)}{2N V_{\text{oct}}} = \frac{V_{\text{tet}} \xi(\partial \tilde{M}_f)}{2V_{\text{oct}}}.
\]

The following provides an alternate description of the lower bound in Corollary 1.4 in terms of the coarse end behavior \( \Phi^*(f) \) of \( f \) as discussed in Subsection 2.3.

Corollary 4.4. Given an irreducible, end-periodic homeomorphism \( f \) with coarse end behavior \( \Phi^*(f) \), we have
\[
\tau(f) \geq \frac{3V_{\text{tet}} |\Phi^*(f)|}{2V_{\text{oct}}}.
\]

Proof. Since \( \partial \tilde{M}_f = S_+ \cup S_- \), the result is immediate from Corollaries 2.8 and 1.4.

5 | PROOF OF PROPOSITION 4.3

The bulk of the proof of Proposition 4.3 involves defining the quotient map \( h : \tilde{M}_f \to \hat{M}_f \), analyzing the structure of the sets of the associated decomposition, and then appealing to Theorem 2.16. Since the hypotheses of that theorem require a closed manifold, we will double \( \tilde{M}_f \) over \( \partial \tilde{M}_f \).

We now define a quotient map \( h : \tilde{M}_f \to \hat{M}_f \) in terms of a decomposition of \( \tilde{M}_f \). The elements in the decomposition are either:

(1) singletons in \( W \); or
(2) maximal components of a flow line of the local flow \( (\tilde{\psi}_s) \) (defined at the end of Section 3) intersected with \( \tilde{M}_f - W \).
We note that there may, a priori, be a flow line that entirely misses $W$. We will see that this does not happen, and as Corollary 5.5 shows, every component of intersection of a flow line with $\overline{M}_f - W$ is a compact arc. We first verify this for points in $L$ and then for points outside. Along the way, we verify that the image of $L$ under $h$ is the appropriate union of circles from Proposition 4.3 (3).

For each component $\alpha \subset L$, we say that $\alpha$ flips forward if $\psi_{1/n}(\alpha) \not\subset L$. Note that $\alpha$ flips forward if and only if it is equal to $\alpha_k^- \subset Z_k \subset S \times \{\frac{k}{n}\}$ for some $0 \leq k < n$. Thus, there are precisely $n$ components of $L$ that flip forward. We say that $\alpha$ flips forward after $k \geq 0$ steps if either $k = 0$ and $\alpha$ flips forward, or if $k > 0$ and $\psi_{j/n}(\alpha)$ does not flip forward for any $0 \leq j < k$, but $\psi_{k/n}(\alpha)$ flips forward. We similarly say that a component $\alpha \subset L$ flips backward if $\psi_{-1/n}(\alpha) \not\subset L$ and note that there are exactly $n$ components of $L$ that flip backward. Flipping backward after $k \geq 0$ steps is defined in the analogous manner. Briefly, we will also say that $\alpha$ eventually flips forward/backward, if it does so after $k$ steps, for some $k \geq 0$.

For any component $\alpha \subset L$ and (possibly infinite) interval $I \subseteq \mathbb{R}$ containing $0$, we let

$$A(\alpha, I) = \bigcup_{t \in I} \psi_t(\alpha).$$

**Lemma 5.1.** Every component $\alpha \subset L$ eventually flips forward or backward.

**Proof.** We suppose $\alpha$ does not eventually flip forward or backward and arrive at a contradiction. Indeed, under this assumption we have $A(\alpha, \mathbb{R})$ is a bi-infinite annulus that meets $S = S \times \{0\}$ in an $f$-invariant subset of $P_0$. This means that there is either a periodic simple closed curve in $P_0$ or else the $f$-orbit of a curve in $P_0$ contains components in both $U_+$ and $U_-$. Either conclusion contradicts the irreducibility of $f$. □

For the next lemma, we let $n = n_S + n_T$, as in Subsection 4.2.

**Lemma 5.2.** The intersection of $\overline{M}_f - W$ with the union of $(\overline{\psi})$ flow lines through $L$ consists of $n + \frac{1}{2}|\Phi^*(f)|$ (possibly degenerate) compact annuli. There are $3|\Phi^*(f)|$ boundary components of these annuli on $\partial \overline{M}_f$ forming a pants decomposition of $\partial \overline{M}_f$, and $n$ boundary components which are curves of the form $\alpha_{k}^\pm$ on some $\overline{W}_k$.

By a ‘degenerate annulus’, we mean an annulus $S^1 \times [a, b]$, where $a = b$, which is therefore just a circle. In this case, the two boundary components of the annulus consist of the circle, and this has the form $\alpha_{k}^+ = \alpha_{k+1}^-$. 

**Proof.** Fix a component $\alpha \subset L$. By Lemma 5.1, $\alpha$ either flips forward after $k \geq 0$ steps, backward after $k' \geq 0$ steps, or both (for some $k$ and $k'$). If it flips forward after $k$ steps and does not eventually flip backward, then $A(\alpha, (−\infty, \frac{k}{n}])$ is a half-open annulus with its boundary component given by $\alpha_j^- = \psi_{k/n}(\alpha) \subset \overline{W}_j$, for some $j$. The open end limits to a curve in $S_+ \subset \partial \overline{M}_f$.

Similarly, if $\alpha$ flips backward after $k' \geq 0$ steps, but does not eventually flip forward, then $A(\alpha, [\frac{k'}{n}, \infty))$ is a half-open annulus with boundary component $\alpha_j^+ = \psi_{−k'/n}(\alpha) \subset \overline{W}_j$, for some $j$. The open end limits to a curve in $S_-.$

If $\alpha$ both flips forward after $k \geq 0$ steps and backward after $k' \geq 0$ steps, then $A(\alpha, [−\frac{k'}{n}, \frac{k}{n}])$ is a compact annulus with one boundary component $\alpha_j^- = \psi_{k/n}(\alpha) \subset \overline{W}_j$ and the other $\alpha_{j'}^+ = \psi_{−k'/n}(\alpha) \subset \overline{W}_{j'}$, for some $j, j'$. 


The annuli of the first two types compactify in $\overline{M}_f$ to have boundary components on $\partial \overline{M}_f$. Since $(\overline{\psi}_s)$ is the local flow extending the reparameterization of $(\psi_s)$, the union of all resulting compact annuli (from all three cases) are exactly the union of the flow lines through $L$ intersected with $\overline{M}_f - W$.

We can count the number of annuli by counting the number of boundary components of the annuli and dividing by 2. There are exactly two boundary components on $W_k$, and this accounts for $2n$ boundary components. The remaining boundary components form a multicurve in $\partial \overline{M}_f$. If this is a pants decomposition, then we have $\xi(\partial \overline{M}_f) = 3|\Phi^*(f)|$ such boundary components, and hence the number of annuli is $n + \frac{3}{2}|\Phi^*(f)|$, as required.

To see that the boundary components on $S_+$ form a pants decomposition, we argue as follows (the case of $S_-$ is similar). Choose a good nesting neighborhood $U_+$ of the attracting end with the property that $P_j \cap U_+ = P_k \cap U_+$, for all $j, k$, and so that $f(U_+) \subset U_+$. Note that this means that no component of $P_k \cap U_+ \subset S \times \{\frac{k}{n}\}$ eventually flips forward. Every point of $U_+$ flows forward to $S_+$ (since $U_+ \subset \partial_+^e$) and this flow defines a surjective local diffeomorphism $U_+ \to S_+$; more precisely, this is one end of the infinite cyclic covering $U_+ \to S_+$. Because $P_k$ defines a pants decomposition of the end $U_+$, this projects by the flow to a pants decomposition of $S_+$, completing the proof. □

Let $A_1, \ldots, A_m$ be the annuli from Lemma 5.2, with $m = n + \frac{3}{2}|\Phi^*(f)|$, and let $A$ be the union of these annuli. Note that the product structure on $A_j \cong S^1 \times I$ obtained from the flow has the property that $\{\ast\} \times I$ is an element of the decomposition defining the quotient $\hat{M}_f$. Furthermore, every component of $L$ has the form $S^1 \times \{\ast\} \subset A_j$, for some $j$. Therefore, we have the following.

**Corollary 5.3.** We have $\hat{L} = h(L) = h(A)$ and the restriction $h|_A : A \to \hat{M}_f$ factors as

$$A \to \bigsqcup_{j=1}^m S^1 \to \hat{M}_f,$$

where the first map projects out the interval directions of the annuli, and the second map is injective.

Thus far we have described the restriction of $h$ to both $W$ (on which $h$ is injective) and $A$. The next lemma describes the structure of the rest of $\overline{M}_f$.

**Lemma 5.4.** Every component $X$ of $\overline{M}_f - (A \cup W)$ is homeomorphic to a product of a 3-punctured sphere with a (possibly degenerate) interval $I \subset [-\infty, \infty]$ with $0 \in \partial I$. The 3-punctured sphere $Y$ is a component of $\bigcup \partial_2 \overline{W}_j$, and the homeomorphism $Y \times I \to X$ is given by $(x, t) \mapsto \psi_t(x)$. Furthermore, $Y \times \partial I$ consists of two (possibly equal) 3-punctured sphere components of

$$\bigcup_{j=0}^{n-1} \partial_2 \overline{W}_j \cup (\partial \overline{M}_f - A).$$

Note that in the statement of this lemma, we may have to make sense of $\psi_{-\infty}(x)$ for some $x$, which is defined to be the point $\lim_{s \to -\infty} \psi_s(x)$ in $S_+ \subset \partial \overline{M}_f$, when this limit exists. We make a similar convention for $\psi_{-\infty}(x)$. It might seem more natural to use the local flow, $(\overline{\psi}_s)$, but the parameterization of $(\overline{\psi}_s)$ is not well-behaved with respect to the $S$-fibers. In particular, we cannot produce the nice homeomorphism described in the statement.
Proof. Consider any 3-punctured sphere component $Y \subset \partial^+ W_j$ for any $j$. This is the interior of a (compact) pair of pants we denote $\overline{Y} \subset \overline{Z}_j \times \{ \frac{j+1}{n} \}$, and the boundary consists of three (not necessarily distinct) pants curves $\alpha_1, \alpha_2, \alpha_3$ in $P_{j+1}$ (with $j + 1$ taken modulo $n$). If one of these pants curves flips forward after $k$ steps, then taking $I = [0, \frac{k}{n}]$, the map $Y \times I \to M_f$, given by the formula in the lemma maps to $M_f - (A \cup W)$, with $Y \times \{ \frac{k}{n} \}$ mapping to a 3-punctured sphere component of $\partial^+ W_i$, for some $i$. If none of these curves eventually flip forward, then we take $I = [0, \infty]$ and the formula provides the homeomorphism, this time sending $Y \times \{ \infty \}$ to a 3-punctured sphere component of $S_+ - A$.

We can argue similarly for a 3-punctured sphere component $Y \subset \partial^- W_j$ for any $j$, this time producing an interval $I$ of the form $[-\frac{k}{n}, 0]$ if one of the boundary pants curves flips backward after $k$ steps or $[-\infty, 0]$ if it does not eventually flip backward.

Each of the products produced above is a component of $\overline{M}_f - (A \cup W)$, and thus it suffices to show that this accounts for all components of this complement. To see this, we simply note that any point $x$ in such a component $X$ flows forward (remaining inside $X$) for time less than $\frac{1}{n}$ until it reaches some fiber $S \times \{ \frac{j}{n} \}$. It meets this in a 3-punctured sphere component of the complement of $P_j$, and (each of) the boundary pants curves must eventually flip forward or backward. The first time one of these flips forward or backward specifies a length of time we need to flow $x$ (inside $X$) until it hits $\bigcup \partial^+_2 W_i$. Therefore, $X$ is one of the components already discussed, completing the proof.

Corollary 5.5. Every flow line of $(\bar{\psi}_s)$ has nontrivial intersection with $W$. Consequently, every set in the decomposition is either a point or compact arc of a flow line with at most one endpoint on $\partial \overline{M}_f$.

We are now ready for the proof of Proposition 4.3. Most of the proof involves analyzing the decomposition defining the quotient $h : \overline{M}_f \to \hat{M}_f$ using the description of the decomposition elements from the lemmas/corollary above. From this analysis, we will show that $\hat{M}_f$ is a 3-manifold with boundary and that $h$ restricts to a homeomorphism $\partial \overline{M}_f \to \partial \hat{M}_f$. Proving that $h$ is homotopic to a homeomorphism involves analyzing the associated quotient of the double $D \overline{M}_f$ and applying Theorem 2.16. The remaining items follow fairly quickly after this is done.

Proof of Proposition 4.3. We begin by proving that $\hat{M}_f$ is a 3-manifold with boundary. The first step is to find a neighborhood of each point in $\hat{M}_f$ homeomorphic to a ball or half-ball in $\mathbb{R}^3$. For this, observe that the restriction of $h$ to $W$ is a homeomorphism onto its image, since $W$ is an open set on which $h$ is injective. So, all points in $h(W)$ have neighborhoods homeomorphic to a ball in $\mathbb{R}^3$. Next, observe that by Lemma 5.4, each component $X$ of $\overline{M}_f - (A \cup W)$ is homeomorphic to a product $X \cong Y \times I$, with $\{y\} \times I = \bigcup_{s \in I} \psi_s(y)$ an arc of a flow line. The restriction of $h$ to $X$ factors as the composition of the projection $X \to Y$ and an inclusion $Y \to \hat{M}_f$, $X \to Y \hookrightarrow \hat{M}_f$.

If the interval $I$ does not contain either $\infty$ or $-\infty$, then any point in the image of $Y$ in $\hat{M}_f$ has a neighborhood homeomorphic to a ball in $\mathbb{R}^3$, constructed from half-balls in $\overline{W}_j$ and $\overline{W}_i$. If $I$ contains $\infty$, then it cannot contain $-\infty$, and $Y \times \{0\} \subset X$ is a component of some $\partial^+_2 W_j$ while $Y \times \{\infty\}$ is a component of $S_+ - A$, and in this case every point in the image of $Y$ in $\hat{M}_f$ has a
neighbourhood homeomorphic to a half-ball in $\mathbb{R}^3$, coming from a half-ball in $\overline{W}_j$. If $I$ contains $-\infty$, we similarly find a half-ball neighborhood.

The only remaining points of $\hat{M}_f$ are those in the image of $A$. For these, we consider each component annulus $A_j \subset A$ and build neighborhoods from all the $W_k$ it meets. There are different cases depending on whether the annulus $A_j$ meets $\partial M_f$ or not.

First, suppose $A_j \cap \partial M_f = \emptyset$. See Figure 7 for an illustration of the construction of the neighborhood of a point in the image of $A_j$. Then we can write

$$A_j = \bigcup_{s \in [0,1]} \psi_s(\alpha_k^+),$$

for some some $t > 0$ and integer $k$ so that $\psi_t(\alpha_k^+) = \alpha_{\ell}^-$, for some integer $\ell$. Observe that $k$ is such that $\alpha_k^+ \subset \mathbb{Z}_k \times \{\frac{k+1}{n}\}$ and does not flip backward (cf. proof of Lemma 5.2). The annulus may also meet various pieces $\overline{W}_{k_1}, \ldots, \overline{W}_{k_j}$ along the vertical annuli $\partial^v \overline{W}_{k_1}, \ldots, \partial^v \overline{W}_{k_j}$. Any point in the image of $A_j$ is given by the decomposition element of the form $\{s\} \times I \subset S^1 \times I \cong A_j$, and a neighborhood of this point is obtained from neighborhoods in $\overline{W}_k$ and $\overline{W}_\ell$ on the ‘top and bottom’, together with neighborhoods ‘along the sides’ from the images of $\overline{W}_{k_1}, \ldots, \overline{W}_{k_j}$.

When the annulus $A_j$ meets $S_+$ or $S_-$, we have a similar picture, except in the former case the top reaches $S_+$ instead of some $\overline{W}_\ell$, and in the latter, the bottom reaches $S_-$ instead of ‘starting’ on some $\overline{W}_k$. In this setting, we get half-ball neighborhoods (since the top/bottom are missing).

We have found neighborhoods of every point in $\hat{M}_f$ homeomorphic to either an open ball or half-ball in $\mathbb{R}^3$. From the construction of these neighborhoods, it is straightforward to show that any two distinct points in $\hat{M}_f$ have disjoint neighborhoods, and that there is a countable basis for the topology. Since $\hat{M}_f$ is the image of the compact space $\overline{M}_f$, it is also compact. Thus, $\hat{M}_f$ is a compact 3-manifold with boundary.

Since the points with neighborhoods homeomorphic to half-balls are precisely those in the image of $\partial \overline{M}_f$, we have $\partial \hat{M}_f = h(\partial \overline{M}_f)$. To see that the restriction of $h$ to $\partial \overline{M}_f$ is injective,
observe that by Corollary 5.5, any element of the decomposition meets $\partial \tilde{M}_f$ in at most one point. This proves item (1). According to Corollary 5.3, $\tilde{L} = h(L) = h(A)$ is the continuous image of a disjoint union of $n + \frac{3}{2} |\Phi^+(f)|$ circles. Since $\tilde{M}_f$ is Hausdorff, the injection is an embedding, proving item (3). Moreover, since $\mathcal{A}$ meets $\partial \tilde{M}_f$ in a pants decomposition, $\tilde{L}$ meets $\partial \tilde{M}_f$ in a pants decomposition which we denote by $P_\Omega$, where $\Omega$ is the component of $\mathcal{P}(S)$ containing $P$.

Next, we want to see that $h|_{\tilde{W}_k} : \tilde{W}_k \to \tilde{M}_f$ factors through a block map relative to $L = h(L)$, and that the maps $h|_{\tilde{W}_k}$ define a block decomposition. This follows from (a) the fact that $h|_{\tilde{W}_k}$ is injective on the interior of product (by definition of the decomposition), (b) $h$ collapses $\partial^i \tilde{W}_k$ to circles contained in $\tilde{L}$ since these annuli are sub-annuli of $A$, (c) $h$ is injective on each pair of pants $Y \subset \partial^2 \tilde{W}_k$ by Lemma 5.4, and (d) $h$ is injective on each curve $x^\pm_k$ since this is either the bottom or top boundary component of an annulus in $\mathcal{A}$.

In fact, Corollary 5.3 and Lemma 5.4 tell us that $\tilde{M}_f$ is built by gluing the blocks together along pairs of pants and unions of circle components of 1-boundaries $\partial B_k$ contained in $\tilde{L}$ (by definition of the decomposition space, the interiors of the blocks are disjoint). This gives a block decomposition relative to $\tilde{L}$, proving item (4).

Next we prove item (2), namely that $h$ is homotopic to a homeomorphism $H : \tilde{M}_f \to \tilde{M}_f$. For this, we consider the doubles $D\tilde{M}_f$ and $D\tilde{M}_f$, and let $Dh : D\tilde{M}_f \to D\tilde{M}_f$ be the associated quotient map, which, restricted to each copy of $\tilde{M}_f$, maps via $h$ to one of the copies of $\tilde{M}_f$. Note that because $D\tilde{M}_f$ is a 3-manifold, hence Hausdorff, and $Dh$ is a quotient map from the compact space $D\tilde{M}_f$, it follows that $Dh$ is a closed map. By Proposition 2.14, the decomposition is upper semicontinuous. Furthermore, by Corollary 5.5, each element of the decomposition defining the quotient map $h$ is an arc (or point) meeting $\partial \tilde{M}_f$ in at most one point, hence each element of the decomposition defining the quotient map $Dh$ is also an arc or point. These are in fact arcs of flow lines of the extension of $(\tilde{\psi}_s)$ to $D\tilde{M}_f$ and so it is easy to find a sequence of neighborhoods, each homeomorphic to a cell, whose intersection is the arc/point. Therefore, the decomposition defining $Dh$ is cellular, hence $Dh$ is cellular, and so by Theorem 2.16, $Dh$ is homotopic to a homeomorphism.

Now we note that the inclusion of $\tilde{M}_f$ to $D\tilde{M}_f$ induces an injection $\pi_1 \tilde{M}_f$, and since $Dh_*$ is an isomorphism, it follows that $h_* : \pi_1 \tilde{M}_f \to \pi_1 \tilde{M}_f$ is also injective. By a result of Waldhausen [51, Theorem 6.1], $h$ is homotopic to a homeomorphism; in fact, this is through a homotopy $h_t$ with $h_t|_{\tilde{M}_f} = h_0|_{\tilde{M}_f}$ for all $t$. This completes the proof of item (2).

By construction, $\tilde{L} \cap \partial \tilde{M}_f = h(P_\Omega)$ and this is equal to $H(P_\Omega)$, since $h|_{\tilde{M}_f} = H|_{\tilde{M}_f}$. Suppose we replace the path $P = P_0, \ldots, P_n = f^{-1}(P)$ in the construction with a path between $P'$ and $f^{-1}(P')$ in the same component $\Omega \subset \mathcal{P}(S)$. Since $P = P'$ outside a compact set of $S$ (and the same is true for all pants in the path from $P'$ to $f^{-1}(P'))$, one sees that the annuli $\mathcal{A}'$ constructed analogously to $A$ meet $\partial \tilde{M}_f$ in the same pants decomposition. Therefore, $P_\Omega$ does indeed depend only on the component $\Omega$. This proves item (5).

All that remains is to prove item (6), stating that $(\tilde{M}_f, P_\Omega)$ is a pared acylindrical manifold. All the conditions in the definition of pared manifold are immediate from the preceding results, except condition (4), stating that an incompressible annulus $(A, \partial A) \subset (\tilde{M}_f, N(P_\Omega))$ with boundary in a neighborhood $N(P_\Omega)$ of $P_\Omega$ must be homotopic into $N(P_\Omega)$, via a homotopy of pairs. Consider such an annulus $(A, \partial A) \subset (\tilde{M}_f, N(P_\Omega))$.

Since the preimage $\tilde{P}_\Omega$ of $P_\Omega$ in $\tilde{M}_\infty \subset S \times [-\infty, \infty]$ is a pants decomposition of $\partial \tilde{M}_\infty = U_\pm \times \{\pm \infty\}$, $A$ lifts to an annulus $\tilde{M}_\infty$ with boundary in $N(\tilde{P}_\Omega)$. By Lemma 3.5, $A$ is not essential, and hence must be boundary parallel. Since $N(P_\Omega)$ is a neighborhood of a pants decomposition, being
boundary parallel gives a homotopy of pairs taking $A$ into $N(P\Omega)$, proving that $(\overline{M}_f, P\Omega)$ is a pared manifold.

Now observe that $\overline{M}_f$ has incompressible boundary, and hence so does $(\overline{M}_f, P\Omega)$. Since the paring locus $P\Omega$ is a pants decomposition of the boundary, $(\overline{M}_f, P\Omega)$ is acylindrical, as explained in the discussion after the definition of pared manifolds. This completes the proof of item (6) of the proposition, and we are done.

6 | SHARPNESS OF THE UPPER BOUND

In this section, we describe a construction of end-periodic homeomorphisms and provide sufficient conditions which ensure that they are either strongly irreducible or irreducible (but not strongly irreducible). The construction allows us to prescribe the end behavior (Theorem 1.5), and also produce sequences of examples showing that the bound on volume given in Theorem 1.1 is asymptotically sharp (Theorem 1.6). The proof of Theorem 1.5 will follow immediately from Proposition 6.3 together with Corollary 2.9, while the proof of Theorem 1.1 will appear at the end of Subsection 6.3.

6.1 | Examples of irreducible, end-periodic homeomorphisms

Recall that $S$ is a boundaryless infinite-type surface with $n$ ends, each accumulated by genus, with $2 \leq n < \infty$. We will construct a general family of examples of irreducible and strongly irreducible end-periodic homeomorphisms of $S$ with arbitrary coarse end behavior. All examples will be of the form $f = \rho h$, where $\rho$ is as in Construction 2.10 and $h$ is supported on a compact subsurface $C$. Whether the resulting homeomorphism $f$ is irreducible or strongly irreducible will depend on the choice of subsurface $C$.

Definition 6.1 (Fully separating/partially separating). We say a subsurface $C$ is fully separating if $C$ separates every end of $S$, in the sense that the end space of each component of $S - C$ is a singleton. We say a subsurface $C$ is partially separating (with respect to a given end-periodic homeomorphism) if $C$ separates the collection of attracting ends from the collection of repelling ends.

Note that a fully separating subsurface is necessarily partially separating with respect to any end-periodic homeomorphism, but the converse is not necessarily true. For any partially separating subsurface $C$, the boundary $\partial C$ decomposes as a disjoint union $\partial C = \partial_+ C \cup \partial_- C$ according whether the ends cut off are attracting or repelling. Examples and a non-example are illustrated in Figure 8.

Convention 6.2. As in Construction 2.10, we fix any coarse end behavior of $S$ and realize it by an end-periodic homeomorphism (isotopic to) $\rho = \Pi^{m}_1 \rho_i$, where $\{\rho_i\}_1^m$ are pairwise-commuting handle shifts. We continue with the notation and assumptions from that construction, letting $v_1, \ldots, v_n$ be separating curves cutting off good nesting neighborhoods of the ends, and let $U_+$ and $U_-$ denote the resulting union of good nesting neighborhoods of the attracting and repelling ends,
Consider an end-periodic homeomorphism of the 5-ended surface shown here with 3-repelling ends on the left and 2-attracted ends on the right. Both the finite-type subsurface bounded by the red curves and the finite-type subsurface bounded by the green curves are partially separating (upper left). The finite-type subsurface bounded by the blue curves is fully separating (upper right). The finite-type surface bounded by the pink curves is neither fully nor partially separating (bottom).

respectively. In addition, we now let $C \subset S$ be a compact subsurface which is partially separating with respect to $\rho$ and disjoint from $U_+$ and $U_-$. We further assume, as we may, that

1. $S - (U_+ \cup U_- \cup C)$ is either empty or a union of planar subsurfaces;
2. $\partial C$ meets each of the handle strips in arcs isotopic to the arcs of intersection with the curves $v_1, \ldots, v_n$; and
3. $C$ intersects each handle strip in a subsurface with one boundary component and genus at least 2.

See Figure 9 for more explanation. When $C$ is fully separating, we observe that item (1) implies that $S - C = U_+ \cup U_-$. Fix a component $\eta$ of $\partial_- C$ and fix a component $\alpha$ of $\partial_+ C$. By items (2) and (3), we have $i(\rho(\eta), \rho^{-1}(\alpha)) = 0$. For this fixed $\eta$, we choose $h \in \text{Map}(S)$ which is supported on $C$ and for which $d_C(\rho(\eta), h(\rho(\eta))) \geq 9$ (see Subsection 2.4 for discussion of subsurface distance). We also let $B \subset B' \subset \mathcal{A}(C)$ denote the balls of radius two and three about $\rho(\eta)$, respectively. Since $i(\rho^{-1}(\alpha), \rho(\eta)) = 0$, we have $d_C(\rho^{-1}(\alpha), \rho(\eta)) = 1$ and so $\rho^{-1}(\alpha) \in B$. We will consider the end-periodic homeomorphism $f = \rho h$. By construction, $U_+$ and $U_-$ are still good nesting neighborhoods of the attracting and repelling ends of $f$, respectively.

**Proposition 6.3.** If $C$ is fully separating, then $f = \rho h$ is strongly irreducible. If $C$ is partially separating with respect to $\rho$, but not fully separating, then $f = \rho h$ is irreducible but not strongly irreducible.

Assuming the lemmas and corollary proved in the remainder of Subsection 6.1, we now provide a straightforward proof of Proposition 6.3.
A partially separating subsurface $C$ (shown in blue) for an end-periodic homeomorphism $\rho$ with repelling ends on the left and attracting ends on the right. The intersection of each handle strip with $C$ is a subsurface of genus 3 with a single boundary component (shown in pink). The boundary of $C$ intersects the strips in arcs that represent the same isotopy class as the intersection with the curves $v_1, \ldots, v_5$ defining good nesting neighborhoods of the ends.

**Proof.** Let $f = \rho h$ and $C = \text{supp}(h)$ be as in Convention 6.2. Suppose that $C$ is partially separating. Then by Corollary 6.7, Lemma 6.8, and Lemma 6.9, $f$ is irreducible. Furthermore, if $C$ is fully separating then there are no periodic lines by Lemma 6.9. Hence, $f$ is strongly irreducible in this case.

Note that in the case that $C$ is partially, but not fully separating, there exist at least two ends of the same type (that is, either both positive or both negative) that are contained in the same complementary component of $C$. This implies that there is a bi-infinite line $\ell$ passing between these two ends, which is disjoint from $C$ and which lives in the complement of the handle strips that comprise $\rho$. Since $\ell$ does not intersect the support of $h$ (namely, $C$) or any handle strips, then it is an AR-periodic line passing between two ends of the same type and so $f$ is irreducible, but not strongly irreducible.

The following lemmas will be used to show that there are no periodic lines or reducing curves.

**Lemma 6.4.** Let $\eta$ and $\alpha$ be as in Convention 6.2, and let $\gamma$ be any curve with $\pi_C(\gamma) \neq \emptyset$. If $d_C(\gamma, \rho(\eta)) \geq 2$, then $\pi_C(\rho^{-1}(\gamma)) \neq \emptyset$. If $d_C(\gamma, \rho^{-1}(\alpha)) \geq 2$, then $\pi_C(\rho(\gamma)) \neq \emptyset$.

**Proof.** If $d_C(\gamma, \rho(\eta)) \geq 2$, then $i(\gamma, \rho(\eta)) \neq 0$; see (2.5). Therefore, $i(\rho^{-1}(\gamma), \eta) \neq 0$, and hence $\pi_C(\rho^{-1}(\gamma)) \neq \emptyset$. A similar argument proves the second claim.

**Lemma 6.5.** Let $\alpha, \eta,$ and $B$ be as in Convention 6.2. Then for all $k \geq 1$, we have

(i) $\pi_C(f^{-k}(\alpha)) \subset h^{-1}B$;
(ii) $\pi_C(f^k(\eta)) \subset B$.

Consequently, $f^k(\eta)$ and $f^{-k}(\alpha)$ fill $C$. 

\[ FIGURE \\text{9} \quad \text{A partially separating subsurface} \ C \ (\text{shown in blue}) \text{ for an end-periodic homeomorphism} \ \rho \ \text{with repelling ends on the left and attracting ends on the right. The intersection of each handle strip with} \ C \ \text{is a subsurface of genus 3 with a single boundary component (shown in pink). The boundary of} \ C \ \text{intersects the strips in arcs that represent the same isotopy class as the intersection with the curves} \ v_1, \ldots, v_5 \ \text{defining good nesting neighborhoods of the ends.} \]
Proof. We will split the proof into two pieces, first showing that (i) holds and then (ii). In each case, the proof is by induction.

**Proof of (i).** As noted in Convention 6.2, \( \rho^{-1}(\alpha) \subset B \), and since \( C = \text{supp}(h) \) we have \( f^{-1}(\alpha) = h^{-1}\rho^{-1}(\alpha) \subset h^{-1}B \) and our base case holds.

Assume, by induction, that there exists \( m \geq 1 \) such that \( \pi_C(f^{-m}(\alpha)) \subset h^{-1}B \). Since \( d_C(h^{-1}\rho(\eta), \rho(\eta)) \geq 7 \), we have \( d_C(f^{-m}(\alpha), \rho(\eta)) \geq 6 \), and so Lemma 6.4 implies that \( \pi_C(\rho^{-1}f^{-m}(\alpha)) \neq \emptyset \). Since \( i(\eta, f^m(\alpha)) = 0 \) (because \( i(\alpha, f^m(\alpha)) = i(\alpha, \rho^m(\alpha)) = 0 \) by construction), we also have that \( i(\rho^{-1}f^{-m}(\alpha), \rho^{-1}(\alpha)) = 0 \). This implies that \( d_C(\rho^{-1}f^{-m}(\alpha), \rho^{-1}(\alpha)) = 1 \).

Since \( d_C(\rho(\eta), \rho^{-1}(\alpha)) = 1 \), the triangle inequality guarantees that \( \pi_C(\rho^{-1}f^{-m}(\alpha)) \subset B \). Thus, as \( C = \text{supp}(h) \), we have

\[
h^{-1}\pi_C(\rho^{-1}f^{-m}(\alpha)) = \pi_C(h^{-1}\rho^{-1}f^{-m}(\alpha)) = \pi_C(f^{-m-1}(\alpha)) \subset h^{-1}B,
\]

as desired. \( \square \)

**Proof of (ii).** Note that \( f(\eta) = \rho(\eta) \), since \( \eta \subset \partial_- C \). So, we immediately have that \( \pi_C(f(\eta)) \subset B \) and our base case holds.

Now, assume that for some \( m \geq 1 \), \( \pi_C(f^m(\eta)) \subset B \). Therefore, we have that \( h\pi_C(f^m(\eta)) = \pi_C(hf^m(\eta)) \subset hB \). By Convention 6.2, \( d_C(\rho(\eta), \rho^{-1}(\alpha)) = 1 \) and \( d_C(h(\rho(\eta)), \rho(\eta)) \geq 9 \), and thus \( d_C(hf^m(\eta), \rho^{-1}(\alpha)) \geq 6 \). Therefore, Lemma 6.4 implies that \( \pi_C(f^{m+1}(\eta)) = \pi_C(\rho hf^m(\eta)) \neq \emptyset \).

Since \( i(\eta, f^m(\eta)) = 0 \), we must also have that \( i(f(\eta), f^{m+1}(\eta)) = 0 \). Thus, \( \pi_C(f^{m+1}(\eta)) \subset B \). \( \square \)

So, we have shown by induction that for all \( k \geq 1 \), \( \pi_C(f^{-k}(\alpha)) \subset h^{-1}B' \) and \( \pi_C(f^k(\eta)) \subset B' \). Since \( d_C(h\rho(\eta), \rho(\eta)) \geq 9 \), we have \( d_C(f^k(\eta), f^{-k}(\alpha)) \geq 5 \), and thus \( f^k(\eta) \) and \( f^{-k}(\alpha) \) fill \( C \) for all \( k \geq 1 \).

**Lemma 6.6.** Let \( \gamma^+ \subset \overline{U}_+ \) and \( \gamma^- \subset \overline{U}_- \) be curves such that \( \pi_C(\rho^{-1}(\gamma^+)) \neq \emptyset \) and \( \pi_C(\rho(\gamma^-)) \neq \emptyset \). Let \( \alpha, \eta \), and \( B' \) be as in Convention 6.2. Then for all \( k \geq 1 \), we have

(i) \( \pi_C(f^{-k}(\gamma^+)) \subset h^{-1}B' \); and

(ii) \( \pi_C(f^k(\gamma^-)) \subset B' \).

Consequently, \( f^k(\gamma^-) \) and \( f^{-k'}(\gamma^+) \) fill \( C \) for all \( k, k' \geq 1 \).

**Proof.** As in Lemma 6.5, we will first show that (i) holds, and then prove (ii). In both cases, the proof is by induction.

**Proof of (i).** As noted in Convention 6.2, we have \( d_C(\rho(\eta), \rho^{-1}(\alpha)) = 1 \), and since \( i(\rho^{-1}(\gamma^+), \rho^{-1}(\alpha)) = 0 \), it follows that \( \pi_C(\rho^{-1}(\gamma^+)) \subset B \subset B' \). Therefore, we have \( h^{-1}\pi_C(\rho^{-1}(\gamma^+)) = \pi_C(h^{-1}(f^{-m}(\gamma^+))) \subset h^{-1}B' \), proving the base case.

Assume by induction that for some \( m \geq 1 \), \( \pi_C(f^{-m}(\gamma^+)) \subset h^{-1}B' \). Since \( d_C(h^{-1}(\rho(\eta)), \rho(\eta)) \geq 7 \), and Lemma 6.4 implies that \( \pi_C(\rho^{-1}f^{-m}(\gamma^+)) \neq \emptyset \). As in that proof we have

\[
h^{-1}\pi_C(\rho^{-1}f^{-m}(\gamma^+)) = \pi_C(h^{-1}\rho^{-1}f^{-m}(\gamma^+)) = \pi_C(f^{-m-1}(\gamma^+)) \neq \emptyset.
\]

Since \( i(\gamma^+, \alpha) = 0 \), this implies that \( i(f^{-(m+1)}(\gamma^+), f^{-(m+1)}(\alpha)) = 0 \). Then as \( \pi_C(f^{-(m+1)}(\alpha)) \subset h^{-1}B \), Lemma 6.5 implies \( \pi_C(f^{-(m+1)}(\gamma^+)) \subset h^{-1}B' \), as desired. \( \square \)
Proof of (ii). Since \( i(\gamma^-, \eta) = 0 \), we have \( i(f(\gamma^-), f(\eta)) = 0 \). Because \( h \) has no effect on \( \partial C \), we have \( f(\eta) = \rho h(\eta) = \rho(\eta) \), and Lemma 6.5 implies that \( h(\pi_C(\rho(\gamma^-))) = \pi_C(f(\gamma^-)) \subset B \subset B' \), proving the base case.

Assume by induction that for some \( m \geq 1 \), \( \pi_C(f^m(\gamma^-)) \subset B' \). Then since \( d_C(h^{-1}(f^m(\gamma^-)), \rho(\eta)) \leq 4 \). The assumption that \( d_C(h\rho(\eta), \rho(\eta)) \geq 9 \), then implies \( d_C(h f^m(\gamma^-), \rho^{-1}(\alpha)) \geq 5 \). From Lemma 6.4 we deduce that \( \pi_C(f^m(\gamma^-)) \subset B' \), proving the base case.

Hence, we have shown by induction that for all \( k \geq 1 \), \( \pi_C(f^{-k}(\gamma^+) \subset h^{-1}B' \) and \( \pi_C(f^k(\gamma^-)) \subset B' \), for any curves \( \gamma^+ \subset U_+ \) and \( \gamma^- \subset U_- \) which project nontrivially to \( C \) after one handle shift. Since \( d_C(h(\rho(\gamma^-)), \rho(\gamma^-)) \geq 9 \), it follows that \( d_C(h^{-k}(\gamma^+), f^{-k}(\gamma^-)) \geq 3 \), and we can conclude that \( f^k(\gamma^-) \) and \( f^{-k}(\gamma^+) \) fill \( C \) for all \( k, k' \geq 1 \).

**Corollary 6.7.** There is no curve in \( U_+ \) which escapes into \( U_- \) under backward iteration of the map \( f \), and there is no curve in \( U_- \) which escapes into \( U_+ \) under forward iteration of the map \( f \).

**Proof.** Let \( \beta^- \subset U_- \) and \( \beta^+ \subset U_+ \) be any curves. Then there exists \( q, r \geq 0 \) so that \( f^q(\beta^-) \subset U_- \) and \( \pi_C(f^q(\beta^-)) \neq \emptyset \), and \( f^{-r}(\beta^+) \subset U_+ \) and \( \pi_C(f^{-r}(\beta^+)) \neq \emptyset \). Note that the curves \( \gamma^+ = f^q(\beta^-) \) and \( \gamma^- = f^{-r}(\beta^+) \) satisfy the hypotheses of Lemma 6.6. Thus, for all \( k > \max\{q, r\} \), \( f^k(\beta^-) \) and \( f^{-k}(\beta^+) \) project nontrivially to \( C \). Consequently, \( \beta^- \) cannot escape into \( U_+ \) under forward iteration of \( f \) and \( \beta^+ \) cannot escape into \( U_- \) under backward iteration of \( f \).

**Lemma 6.8.** Let \( C \) be either partially or fully separating and let \( f = \rho h \) be as in Convention 6.2. Then, there are no curves that are periodic under the homeomorphism \( f \). That is, for curves \( \delta \) on \( S \), \( f^m(\delta) \neq \delta \) for all \( m \neq 0 \).

**Proof.** Suppose \( \delta \) is a simple closed curve in \( S \) which is periodic under \( f \).

Note that if any part of \( \delta \) essentially intersects \( U_+ \) or \( U_- \), then \( \delta \) cannot be periodic because it will always move further into \( U_+ \) under positive powers of \( f \) or will always move further into \( U_- \) under negative powers of \( f \). Hence, if \( \delta \) is periodic under \( f \), then \( \delta \subset S - (U_+ \cup U_-) \).

There are now two cases to consider. Either \( \delta \) essentially intersects \( C \), or \( \delta \) is contained entirely in \( S - (U_+ \cup U_- \cup C) \). If \( \delta \) is contained entirely in \( S - (U_+ \cup U_- \cup C) \), then by the assumptions on \( U_+ \), \( U_- \), and \( C \), this means that \( \delta \) must essentially intersects some handle strip. Since the components of the complement of the handle strips in \( S - (U_+ \cup U_- \cup C) \) are contractible (being homeomorphic to closed disks, minus a finite set of points on the boundary). However, if \( \delta \) essentially intersects a handle strip outside of \( C \), then this portion of \( \delta \) will also be carried further out into an end along this handle strip under either forward or backward iteration of \( f \).

So, we assume that \( \delta \) essentially intersects \( C \) but is disjoint from \( U_+ \) and \( U_- \), and that \( m > 0 \) is such that \( f^m(\delta) = \delta \). Let \( \gamma^+ \subset U_+ \) and \( \gamma^- \subset U_- \) be such that \( \pi_C(h^{-1}(\gamma^+)) \neq \emptyset \) and \( \pi_C(h(\gamma^-)) \neq \emptyset \), as in Lemma 6.6. Note that since \( i(\delta, \gamma^+) = 0 \), this implies that \( i(f^k(\delta), f^k(\gamma^+)) = 0 \) for all \( k \in \mathbb{Z} \).

But, Lemma 6.6 states that \( f^k(\gamma^-) \) and \( f^{-k}(\gamma^-) \) fill \( C \) for all \( k > 0 \). In particular, since \( f^m(\delta) = \delta = f^{-m}(\delta) \) essentially intersects \( C \), we must have \( i(f^m(\delta), f^m(\gamma^+)) = 0 \) or \( i(f^{-m}(\delta), f^{-m}(\gamma^-)) \neq 0 \), which is a contradiction.
Lemma 6.9. If $C$ is partially separating, then $f$ has no AR-periodic lines. If $C$ is fully separating, then $f$ has no periodic lines.

Proof. We will first consider the case where $C$ is partially separating. Suppose $f$ has an AR-periodic line $\ell$ with period $m$, so that $f^{mp}(\ell) = \ell$ for all $p \in \mathbb{Z}$. As $\ell$ cannot fill either $U_+$ or $U_-$, we can find curves $\beta^+ \subset U_+$ and $\beta^- \subset U_-$ which are each disjoint from $\ell$. There exist integers $q, r \geq 0$ such that $f^q(\beta^-) = \rho^q(\beta^-) \subset \overline{U}_-$ but $\pi_C(\rho^q f^{-r}(\beta^-)) \neq \emptyset$ and $f^{-r}(\beta^+) = \rho^{-r}(\beta^+) \subset \overline{U}_+$ but $\pi_C(\rho^{-1} f^{-r}(\beta^+)) \neq \emptyset$.

The curves $\gamma^+ = f^q(\beta^-)$ and $\gamma^- = f^{-r}(\beta^+)$ satisfy the hypotheses of Lemma 6.6, and thus $f^k(\gamma^+) = f^{q+k}(\beta^-)$ and $f^{-k'}(\gamma^-) = f^{-r-k'}(\beta^+)$ fill $C$ for all $k, k' \geq 1$. Note that since $i(\ell, \beta^\pm) = 0$, this implies that $i(f^k(\ell), f^k(\beta^\pm)) = 0$ for all $k \in \mathbb{Z}$. Since $f^{mp}(\ell) = \ell$ for all $p \in \mathbb{Z}$, this implies that $i(\ell, f^{mp}(\beta^\pm)) = 0$ for all $p \in \mathbb{Z}$. But, for $p$ large enough so that $mp > \max\{q, r\}$, Lemma 6.6 implies that $f^{mp}(\beta^-)$ and $f^{-mp}(\beta^+)$ fill $C$, contradicting the fact that $\ell$ was disjoint from $\beta^-$ and $\beta^+$.

We now consider the case where $C$ is fully separating and suppose that $f$ has a periodic line $\ell$ with period $m$. If $\ell$ passes between two ends of $S$ of the same type (that is, both attracting or both repelling), then in order to be invariant, $\ell$ must intersect $C$ (or else it would get shifted farther out the end under forward or backward iteration of $f$). If $\ell$ passes between two ends of $S$ of distinct types (that is, one is attracting and one is repelling), then $\ell$ must also intersect $C$ since $C$ is fully separating. As $\ell$ can only intersect $U_-$ and $U_+$ in a finite number of arcs and rays, we can find curves $\beta^- \subset U_-$ and $\beta^+ \subset U_+$ as before which are disjoint from $\ell$. But, this gives rise to the same contradiction as in the case when $C$ was partially separating, since $f^{pm}(\beta^-)$ and $f^{-pm}(\beta^+)$ must fill $C$ for sufficiently large $p$ by Lemma 6.6.

6.2 Sharpness of the upper bound

In this subsection, we will use some of the examples of irreducible end-periodic homeomorphisms we have constructed to show that Theorem 1.1 is asymptotically sharp, in the sense made precise by the following theorem from the introduction. Recall that $V_{\text{oct}}$ is the volume of a regular ideal octahedron.

Theorem 1.6. There is a sequence of strongly irreducible end-periodic homeomorphisms $f_k : S \to S$ so that $\text{Vol}(\overline{M}_{f_k}) \to \infty$ and $\frac{\text{Vol}(\overline{M}_{f_k})}{\tau(f_k)} \to V_{\text{oct}}$ as $k \to \infty$. In fact, $|\text{Vol}(\overline{M}_{f_k}) - V_{\text{oct}} \tau(f_k)|$ is uniformly bounded, independent of $k$.

We will only be considering strongly irreducible end-periodic homeomorphisms $f = \rho h$, as constructed in Subsection 6.1. Thus, by Proposition 6.3, we will require our subsurface $C = \text{supp}(h)$ to be fully separating.

Convention 6.10. We make the same assumptions on $S, \alpha, \eta, \rho, h, C, U_+, B,$ and $B'$ from Convention 6.2, in addition to the following. We assume that $C$ is fully separating, that there is a 4-times punctured sphere $\Sigma = \Sigma_{0,4}$ embedded in the subsurface $C \cap \rho(C) \subset C \subset S$, and that each of the four boundary curves in $\Sigma$ as well as some curve $\gamma_0 \subset \Sigma$ have all distinct topological types in $S$: that is, there is no homeomorphism of $S$ taking any one of these curves to any other one. For example, the curves can be chosen to all be separating curves which each cut off a subsurface of distinct
topological type. In particular, no two boundary curves of $\Sigma$ or $\gamma_0$ are homotopic. For example, the conditions on $\Sigma$ hold when $\Sigma$ is embedded in $C \cap \rho(C)$ so that $S - \Sigma$ consists of four pairwise non-homeomorphic components and $\gamma_0$ cuts off a compact subsurface of $S$ not homeomorphic to any of the components of $S - \Sigma$.

We also fix a pants decomposition $P$ of $S$ which is $\rho$-invariant on $U_+$ and $U_-$, and which contains the four boundary curves of $\Sigma$, as well as $\partial C_+$, $\partial C_-$ and $\rho(\partial C_-)$.

**Lemma 6.11.** With the assumptions in Convention 6.10 (and hence also Convention 6.2), for all curves $\gamma \subset C \cap \rho(C)$, we have that for all $k \geq 1$:

(i) $\pi_C(f^{-k}(\gamma)) \subset h^{-1}B'$; and

(ii) $\pi_C(f^k(\gamma)) \subset B'$.

**Proof.** The proof will proceed in a similar manner as that of Lemma 6.6, with proofs of both (i) and (ii) by induction.

**Proof of (i).** Since $i(\alpha, \gamma) = 0$, we have that $i(\rho^{-1}(\alpha), \rho^{-1}(\gamma)) = 0$. In addition, $\rho^{-1}(\gamma) \subset C$ since $\rho^{-1}(C \cap \rho(C)) \subset C$. As $d_C(\rho^{-1}(\alpha), \rho(\eta)) = 1$, we have $h^{-1}\pi_C(\rho^{-1}(\gamma)) = \pi_C(f^{-1}(\gamma)) \subset h^{-1}B'$, proving the base case.

So, assume by induction that for some $m \geq 1$, $\pi_C(f^{-m}(\gamma)) \subset h^{-1}B'$. As $d_C(h^{-1}\rho(\eta), \rho(\eta)) \geq 9$, we have $d_C(f^{-m}(\gamma), \rho(\eta)) \geq 6$, and hence Lemma 6.4 implies that $\pi_C(\rho^{-1}f^{-m}(\gamma)) \neq \emptyset$. As before, we have that $h^{-1}\pi_C(\rho^{-1}f^{-m}(\gamma)) = \pi_C(f^{-m-1}(\gamma)) \neq \emptyset$. Since $i(\gamma, \alpha) = 0$, we have that $i(f^{-m-1}(\alpha), f^{-m-1}(\gamma)) = 0$. By Lemma 6.5, we know that $\pi_C(f^{-m-1}(\alpha)) \subset h^{-1}B$, and so we must have that $\pi_C(f^{-m-1}(\gamma)) \subset h^{-1}B'$, as desired.  

**Proof of (ii).** Since $\gamma \subset C \cap \rho(C)$, we have $i(\gamma, \eta) = 0$, and hence $i(f(\gamma), f(\eta)) = 0$. As $f(\eta) = \rho h(\eta) = \rho(\eta)$, we have $\pi_C(f(\gamma)) \subset B \subset B'$, proving the base case.

Now, assume by induction that for some $m \geq 1$, $\pi_C(f^m(\gamma)) \subset B'$. Then since $d_C(h\rho(\eta), \rho(\eta)) \geq 9$, we have $d_C(h f^m(\gamma), \rho(\eta)) \geq 6$, and hence Lemma 6.4 implies $\pi_C(f^{m+1}(\gamma)) = \pi_C(\rho h f^m(\gamma)) \neq \emptyset$. Since $i(\eta, \eta) = 0$, it follows that we have $i(f^{m+1}(\gamma), f^{m+1}(\eta)) = 0$. As $\pi_C(f^{m+1}(\eta)) \subset B$ by Lemma 6.5, we therefore have that $\pi_C(f^{m+1}(\gamma)) \subset B'$, as desired.

Having proved (i) and (ii), this completes the proof.

Let $L \subset \overline{M}_f$ be the link defined by

$$L = \partial \Sigma \cup \gamma_0 \subset S \times \{1\} \subset M_f \subset \overline{M}_f.$$  

Write $\overline{M}_0 = \overline{M}_f - L$ for the complement of $L$ in $\overline{M}_f$. Note that $\partial \overline{M}_0 = \partial \overline{M}_f$, but $\overline{M}_0$ has five ends with neighborhoods homeomorphic to $T^2 \times (0, \infty)$.

**Lemma 6.12.** The manifold $\overline{M}_0$ admits a convex hyperbolic structure such that $\partial \overline{M}_0$ is totally geodesic.

**Proof.** Instead of simply removing $L$ from $\overline{M}_f$, we instead remove an open tubular neighborhood of $L$ to produce a compact 3-manifold whose boundary consists of $\partial \overline{M}_f$ together with five tori. Declaring the union of these five tori to be the paring locus, we have that the resulting manifold,
together with the tori, satisfies all conditions of being a pared 3-manifold with incompressible boundary, except possibly conditions (4) and (5). We can prove these, and simultaneously prove that the pared manifold is acylindrical, by proving that the double, $D\overline{M}_0$, is atoroidal (see Subsection 2.8). If we do this, then we may apply Theorem 2.20 to deduce that $\overline{M}_0$ admits a convex hyperbolic structure with totally geodesic boundary.

So, suppose to the contrary that there is an embedded, torus $T$ in $D\overline{M}_0$. There are two cases to consider: (a) $T$ is contained in (one copy of) $\overline{M}_0$; and (b) $T$ essentially intersects $\partial \overline{M}_0$ in $D\overline{M}_0$.

First, suppose that $T$ is contained entirely inside of (one copy of) $\overline{M}_0$, and let $S' = S - L \subset \overline{M}_0$. We assume that $T \cap S'$ (this can always be done by a small perturbation) such that $\Gamma = T \cap S'$ has the minimal number of components among all tori isotopic to $T$ which are transverse to $S'$. We claim that $T - \Gamma$ is a union of annuli. Suppose to the contrary, that $T - \Gamma$ is not a union of annuli. Then, since the Euler characteristic of $T$ is 0, there must be some disk $D$ in $T - \Gamma$. Note that $\partial D$ is an inessential curve in $S'$. Otherwise, $\partial D$ bounds an essential curve in $S'$ and $D$ is a compressing disk contradicting the incompressibility of $S'$. Since $\partial D$ is not essential in $S'$, then it bounds a disk in $S'$ which together with $D$ bounds a 3-ball in $\overline{M}_0$. The 3-ball can be used to define an isotopy removing the intersection of $T$ with $S'$ forming $\partial D$, which contradicts the minimality of $\Gamma$. Therefore, $T - \Gamma$ consists entirely of annuli.

Let $M_0 \subset \overline{M}_0$ denote the interior, so that $M_0 = M_f - (L \times \{1\})$. Setting

$$N = S \times [0, 1] - (L \times \{1\} \cup f(L) \times \{0\}),$$

then $M_0$ is obtained from $N$ by gluing

$$\partial_1 N = (S - L) \times \{1\} \subset S \times \{1\}$$

via the homeomorphism $f$ to

$$\partial_0 N = (S - f(L)) \times \{0\} \subset S \times \{0\}.$$
Let $A_1 \subset A$ be the component with one boundary component $\partial_- A_1 = f(\partial_+ A) \subset \partial_0 N$. The other boundary component $\partial_+ A_1$ is not equal to $f(\partial_+ A)$, for if it were, then $T$ would be a peripheral torus, contradicting the assumption that $T$ is essential. If $\partial_+ A_1 \subset \partial_0 N$, then $A_1$ is boundary parallel in $S \times [0, 1]$ and so $\partial_1 A_1$ is the boundary of an annulus $G' \subset S \times \{1\}$. This must also contain a component $f(\alpha_i) \subset f(L)$ (or else we could have reduced the number of components of $\Gamma$). Note that both $f(\alpha_i)$ and $f(\alpha_j)$ are isotopic in $S \times \{0\}$ to $f(\partial_+ A)$ and hence are isotopic to each other. It follows that $i = j$ by our choice of $L$, and hence $f(G)$ is properly contained in $G'$. Since $f(\partial_- A)$ is also the boundary component of some annulus $A' \subset A$ which is disjoint from $A_1$, it must be that the other boundary component of $A'$ is contained in $G' - f(G)$. This annulus $A'$ is also boundary parallel in $S \times [0, 1]$, but note that the annulus in $S \times \{0\}$ with the same boundary curves as $A'$ is entirely contained in $G' - f(G)$, which can contain no components of $f(L)$. Thus, $A'$ is boundary parallel in $N$, providing an isotopy of $T$ to reduce the number of components of intersection with $S'$, again a contradiction. This case is illustrated in the diagram on the left of Figure 10.

Therefore, the annulus $A_1$ is necessarily vertical and $\partial_+ A_1 \subset \partial_1 N$. There is thus an annulus $A_2 \subset A$ with $\partial_- A_2 = f(\partial_+ A_1) \subset \partial_0 N$. Either $A_2$ is boundary parallel in $S \times [0, 1]$, or $\partial_+ A_2 \subset \partial_1 N$ and there is an annulus $A_3 \subset A$ with $\partial_- A_3 = f(\partial_+ A_2)$. Continuing in this way, ‘tracing around the torus $T$’ we obtain a sequence of annuli $A_1, A_2, \ldots, A_m$ which are all vertical, but for which $\partial A_{m+1} \subset \partial_0 N$. Then $A_{m+1}$ is boundary parallel in $S \times [0, 1]$, and as before, there must be some curve $f(\alpha_i) \subset f(L)$ isotopic in $S \times \{0\}$ to $\partial_- A_{m+1}$. This means that $\alpha_i \subset L$ is isotopic to $\partial_+ A_m$ in $S \times \{1\}$.

The annuli $A_1, \ldots, A_m$ glue together in $M_0$ to give an annulus from $\partial_+ A$ to $\partial_+ A_m$. Since $\partial_+ A$ is isotopic to $\alpha_i$ in $S$ and $\partial_+ A_m$ is isotopic to $\alpha_j$ in $S$, inside $M_f$ we can flow this glued-up annulus forward to produce an isotopy from $f^m(\alpha_i)$ to $\alpha_j$. Since $f$ is irreducible, $f^m(\alpha_i)$ cannot be isotopic to $\alpha_i$, and hence $i \neq j$. On the other hand, $f^m$ is a homeomorphism from $S$ to itself sending $\alpha_i$ to $\alpha_j$ (up to isotopy), but this contradicts the fact that all components of $L$ in $S$ have different homeomorphism types. This contradiction shows that there is no essential torus in $\overline{M}_0$. An illustration of this case is shown in the diagram on the right of Figure 10.

So, suppose now there is an embedded, non-peripheral torus $T$ in $D\overline{M}_0$ which essentially intersects a higher genus boundary component of $\overline{M}_0$ in $D\overline{M}_0$. An argument similar to the first case implies that we may assume that $T$ meets $\partial\overline{M}_0$ minimally and transversely, so that each component of $T - \partial\overline{M}_0$ compactifies to an essential annulus in one of the copies of $\overline{M}_0$ in $D\overline{M}_0$. Let $A \subset \overline{M}_0$ be any such annulus.

Note that $A$ meets $\partial\overline{M}_0 = \partial\overline{M}_f$ in two essential curves, and so $A$ is incompressible in $\overline{M}_f$. Since $\overline{M}_f$ is acylindrical and since $\partial A$ are essential curves in $\overline{M}_f$, $A$ must be boundary parallel in $\overline{M}_f$. So,
there must be some annulus $A' \subset \partial M_f$, so that $A \cup A'$ must bound a solid torus in $\overline{M}_f$. Since $A$ is essential in $\overline{M}_0$, there must be some curve $\alpha_i \in L$, which is contained in this solid torus, but this solid torus gives a homotopy of $\alpha_i$ into $\partial \overline{M}_f$. This contradicts Lemma 6.11. Therefore, $\overline{M}_0$ admits a complete hyperbolic metric such that the boundary is totally geodesic. □

We will use the following result of Adams [1, Theorem 3.1] to conclude that the convex hyperbolic metric on $\overline{M}_0$ from the previous lemma makes the thrice-punctured spheres coming from $\Sigma - L$ totally geodesic in $\overline{M}_0$.

**Theorem 6.13.** A properly embedded incompressible thrice-punctured sphere in a hyperbolic 3-manifold is isotopic to a totally geodesic properly embedded thrice-punctured sphere.

After an isotopy, we can assume that $\Sigma - L$ consists of pairwise disjoint, totally geodesic thrice-punctured spheres in $\overline{M}_0$. Next, we cut open along these surfaces to obtain four totally geodesic thrice-punctured sphere boundary components. We will make use of them in the following proof of Theorem 1.6 which concludes this section.

**Proof of Theorem 1.6.** Fix a four-holed sphere $\Sigma$ and a pants decomposition $P$ as in Convention 6.10. As before, $L$ denotes the union of the four boundary curves of $\Sigma$ together with the pants curve $\gamma_0$ in $\Sigma$, and $\overline{M}_0 := \overline{M}_f - (L \times \{1\})$. By Lemma 6.12, $\overline{M}_f$ admits a convex hyperbolic metric with totally geodesic boundary. By Theorem 6.13, we can cut along the two totally geodesic thrice-punctured spheres inside $\overline{M}_0$ which yields a new 3-manifold, $\overline{M}_0'$ with four more boundary components, each of which is a thrice-punctured sphere.

Now isometrically glue $2k$ copies of the four-holed sphere block, $\widetilde{B}$, (with its complete hyperbolic metric with totally geodesic boundary) into $\overline{M}_0'$, stacked vertically so that the bottom of the first block is glued to the bottom new boundary component and so that the top of the last block is glued to the top new boundary component. We call this new manifold $\overline{M}_k$, and note that

$$\text{Vol}(\overline{M}_k) = \text{Vol}(\overline{M}_0) + 4kV_{\text{oct}}.$$  

Next observe that we may alternatively view $\overline{M}_k$ as being obtained from $\overline{M}_f$ by removing a link $L_k \subset \overline{M}_f$, each component of which is contained in fiber of $\overline{M}_f$. More precisely, we can take $L_k$ to consist of $5 + 2k$ curves, consisting of $(\partial \Sigma \cup \gamma_0) \times \{\frac{1}{4}\}$ together with $2k$ curves of the form $\beta_i \times \{\frac{1}{4} + \frac{1}{4k}\}$, where $\beta_i \subset \Sigma$ (the choice of interval $[\frac{3}{4}, \frac{5}{4}]$ is arbitrary: up to isotopy, we could of course take any interval in $[0,1]$). For any integer $s$, we let $\overline{M}_{k,s}$ denote the result of performing $(1, s)$-Dehn filling on each torus cusp of $\overline{M}_k$. For $s = 0$, this gives us back the original mapping torus, $\overline{M}_{k,0} \cong \overline{M}_f$.

Suppose that $P$ and $f^{-1}(P)$ differ by $r$ pants moves, and suppose that the pants move which occurs on $\Sigma$ occurs at the $m$th vertex in the path between $P$ and $f^{-1}(P)$ in $\mathcal{P}(S)$. As in Section 4, we may view this move as occurring in the fiber $S \times \{\frac{m}{r}\}$. Gluing in the $2k$ pants blocks and performing $(1,0)$-Dehn filling to get $\overline{M}_{k,0} \cong \overline{M}_f$ corresponds to choosing a new path between $P$ and $f^{-1}(P)$ which now has length $r + 2k$ (we have essentially inserted $2k$ redundant pants moves that occur inside $\Sigma$). So, we may perform an isotopy so that the initial pants move on $\Sigma$ now occurs in the fiber $S \times \{\frac{m}{N+2k}\}$, and so that the $i$th pants block (after filling) has one boundary component in the fiber $S \times \{\frac{m+i-1}{N+2k}\}$ and the other boundary component in the fiber $S \times \{\frac{m+i}{N+2k}\}$. 
By Proposition 2.18, $\overline{M}_{k,s} \cong \overline{M}_{fD_k,s}$, where $D_{k,s}$ is a product of powers of Dehn twists about the curves in $L_k$ (projected into $\Sigma \subset S$). Letting $\overline{D}_{k,s}$ denote the double of $\overline{M}_k$ over the boundary, for each $s > 0$ we can view the double $\overline{D}_{k,s}$ of $\overline{M}_{k,s}$ over its boundary as distinct Dehn fillings of $\overline{D}_k$. By Theorem 2.25, for $s$ sufficiently large, $\overline{D}_{k,s}$ is hyperbolic, and as $s \to \infty$, $\text{Vol}(\overline{D}_{k,s}) \to \text{Vol}(\overline{D}_k)$. By Mostow rigidity, the involution interchanging the two sides of the double is isotopic to an isometry with respect to the hyperbolic metric on $\overline{D}_{k,s}$ and fixes $\partial \overline{M}_{k,s}$. Consequently, the hyperbolic metric on $\overline{M}_{k,s}$ also has totally geodesic boundary and

$$\text{Vol}(\overline{M}_{k,s}) \to \text{Vol}(\overline{M}_k) = \text{Vol}(\overline{M}_0) + 4kV_{\text{oct}}.$$  

For each $k > 0$, let $s_k$ be such that

$$\text{Vol}(\overline{M}_{k,s_k}) \geq 4kV_{\text{oct}}.$$  

We note that the end-periodic monodromy $f_k = fD_{k,s_k} = \rho h_{D_{k,s_k}}$ for $\overline{M}_{k,s_k}$ is strongly irreducible. This is because our only requirement on the map $h$ in Subsection 6.1 was that $d_{C}(h,\rho(\eta),\rho(\eta)) \geq 9$ in $AC(C)$. As $L_k$ is disjoint from $\rho(\eta)$, this implies $D_{k,s_k}$ must preserve $\rho(\eta)$. Hence, $hD_{k,s_k}\rho(\eta) = h\rho(\eta)$, and therefore we have $d_{C}(hD_{k,s_k}\rho(\eta),\rho(\eta)) \geq 9$.

By Theorem 1.1, we have that $\text{Vol}(\overline{M}_{f_k}) \leq V_{\text{oct}}\tau(f_k) \leq V_{\text{oct}}(n_T + 2n_S + 4k)$, where $n_S$ and $n_T$ are the number of pants moves on four-punctured spheres and tori, respectively, in the original path between $P$ and $f^{-1}(P)$ (note that all $2k$ pants move from the blocks occur on four-punctured spheres). On the other hand, $\text{Vol}(\overline{M}_{f_k}) \geq 4kV_{\text{oct}}$, and therefore

$$|\text{Vol}(\overline{M}_{f_k}) - V_{\text{oct}}\tau(f_k)| \leq V_{\text{oct}}(n_T + 2n_S).$$  

Since $\text{Vol}(\overline{M}_{f_k}) \geq 4kV_{\text{oct}} \to \infty$ as $k \to \infty$, this provides the required sequence $\{f_k\}$.  

### 6.3 Large component translation

Here we prove the following corollary from the introduction.

**Corollary 1.3.** For any irreducible, end-periodic homeomorphism $f : S \to S$ and $R > 0$, there exists an $f$-invariant component $\Omega \subset P_f(S)$ such that $\tau(f, \Omega) \geq R$.

**Proof.** By Theorem 1.2, it suffices to find a sequence of $f$-invariant components $\Omega_k \subset P_f(S)$ so that

$$\text{Vol}(\overline{M}_f - P_{\Omega_k}) \to \infty$$  

as $k \to \infty$. For this, it is useful to observe that any pants decomposition $P$ of $\partial \overline{M}_f$ arises as $P_0$ for some $f$-invariant component $\Omega \subset P_f(S)$ if and only if the preimage of $P$ in $U_\pm \times \{\pm \infty\}$ are pants decompositions. This is because $\Omega$ being $f$-invariant is equivalent to requiring that any $P_0 \in \Omega$ has the property that $P_0$ and $f(P_0)$ differ by a finite number of pants moves. This is equivalent to $P_0$ being $f$-invariant when restricted to some nesting neighborhoods, and since these nesting neighborhoods form ‘half’ the infinite cyclic covers $U_\pm \to S_\pm$, this is equivalent to $P \subset S_\pm$ lifting to pants decompositions on $U_\pm$. 


Now choose any pants decomposition $P$ in an $f$–invariant component $\Omega \subset P_f(S)$ and consider $P_\Omega \subset \partial \overline{M}_f$. Let $\gamma_0 \subset P_\Omega$ be any component such that $\partial \overline{M}_f - (P_\Omega - \gamma_0)$ contains a four-punctured sphere component $\Sigma$. Consider the convex hyperbolic metric on $\overline{M}_f - P_\Omega$ with totally geodesic boundary consisting of thrice-punctured spheres, and as in the proof of Theorem 1.6, isometrically glue on $2k$ copies of $\hat{B}$ stacked vertically so that the bottom of the first block is glued to $\Sigma - \gamma_0$. The resulting manifold $\overline{M}_k$ is homeomorphic to $\overline{M}_f - (P_\Omega \cup L_k)$, where $L_k \subset M_f \subset \overline{M}_f - P_\Omega$ is a link in the interior. After an isotopy, we can assume that the curves in $L_k$ are of the form $\beta \times \{x_\beta\}$ with respect to some product structure on a collar neighborhood $N \cong \partial \overline{M}_f \times [0, 1]$ of $\partial \overline{M}_f$, with $\beta \subset \Sigma$ and $x_\beta \in (0, 1)$ for all $\beta$.

Similar to Proposition 2.18, performing $(1, s)$-Dehn filling on each torus cusp of $\overline{M}_k$ simply changes the product structure on the collar neighborhood of $\partial \overline{M}_f$, and consequently changes the pants decomposition on the boundary by a composition of powers of Dehn twists in the curves $L_k$. We can therefore denote the result of this Dehn filling as $\overline{M}_f - P_{k,s}$ for some pants decomposition $P_{k,s}$ of $\partial \overline{M}_f$. On the other hand, since all the Dehn twists occur on curves in $\Sigma$, and the entire subsurface $\Sigma$ lifts to $U_{\pm}$, it follows that $P_{k,s}$ lifts to pants decompositions on $U_{\pm}$. Therefore, $P_{k,s} = P_{\Omega_{k,s}}$ for some $f$–invariant component $\Omega_{k,s} \subset P_f(S)$. As in the proof of Theorem 1.6, appealing to Theorem 2.25, we have

$$\text{Vol}(\overline{M}_f - P_{\Omega_{k,s}}) \rightarrow \text{Vol}(\overline{M}_f - P_\Omega) + 4kV_{\text{oct}}.$$ 

We may therefore choose $s_k$ sufficiently large so that

$$\text{Vol}(\overline{M}_f - P_{\Omega_{k,s_k}}) \geq 4kV_{\text{oct}}.$$ 

The term on the right tends to $\infty$, and thus setting $\Omega_k = \Omega_{k,s_k}$ completes the proof. \hfill $\Box$

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**References**

1. C. C. Adams, *Thrice-punctured spheres in hyperbolic 3-manifolds*, Trans. Amer. Math. Soc. 287 (1985), no. 2, 645–656.
2. C. C. Adams, *Volumes of N-cusped hyperbolic 3-manifolds*, J. Lond. Math. Soc. (2) 38 (1988), no. 3, 555–565.
3. I. Agol, *Small 3-manifolds of large genus*, Geom. Dedicata 102 (2003), 53–64.
4. I. Agol, P. A. Storm, and W. P. Thurston, *Lower bounds on volumes of hyperbolic Haken 3-manifolds*, J. Amer. Math. Soc. 20 (2007), no. 4, 1053–1077. (With an appendix by Nathan Dunfield.)
5. J. Aramayona, P. Patel, and N. G. Vlamis, *The first integral cohomology of pure mapping class groups*, Int. Math. Res. Not. **2020** (2020), no. 22, 8973–8996.

6. J. Aramayona and F. Valdez, *On the geometry of graphs associated to infinite-type surfaces*, Math. Z. **289** (2018), no. 1–2, 309–322.

7. J. Aramayona and N. G. Vlamis, *Big mapping class groups: An overview*, *In the tradition of Thurston: geometry and topology*, Ken’ichi Ohshika and Athanase Papadopoulos (eds.), Springer, Cham, 2020, pp. 459–496.

8. S. Armentrout, *Cellular decompositions of 3-manifolds that yield 3-manifolds*, Bull. Amer. Math. Soc. **75** (1969), 453–456.

9. J. Bavard, *Hyperbolicité du graphe des rayons et quasi-morphismes sur un gros groupe modulaire*, Geom. Topol. **20** (2016), no. 1, 491–535.

10. B. Branman, *Spaces of pants decompositions for surfaces of infinite type*, to appear in Groups Geom. Dyn.

11. J. F. Brock, *The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores*, J. Amer. Math. Soc. **16** (2003), no. 3, 495–535.

12. J. F. Brock, *Weil-Petersson translation distance and volumes of mapping tori*, Comm. Anal. Geom. **11** (2003), no. 5, 987–999.

13. J. F. Brock and K. W. Bromberg, *Inflexibility, Weil-Petersson distance, and volumes of fibered 3-manifolds*, Math. Res. Lett. **23** (2016), no. 3, 649–674.

14. J. F. Brock, R. D. Canary, and Y. N. Minsky, *The classification of Kleinian surface groups, II: The ending lamination conjecture*, Ann. of Math. (2) **176** (2012), no. 1, 1–149.

15. A. Candel and L. Conlon, *Foliations. II*, Graduate Studies in Mathematics, vol. 60, American Mathematical Society, Providence, RI, 2003.

16. J. Cantwell and L. Conlon, *Examples of endperiodic automorphisms*, arXiv:1008.2549, 2016.

17. J. Cantwell, L. Conlon, and S. R. Fenley, *Endperiodic automorphisms of surfaces and foliations*, Ergodic Theory Dynam. Systems **41** (2021), no. 1, 66–212.

18. T. Cremaschi and J. A. Rodriguez-Migueles, *Hyperbolicity of link complements in Seifert-fibered spaces*, Algebr. Geom. Topol. **20** (2020), no. 7, 3561–3588.

19. T. Cremaschi, J. A. Rodriguez-Migueles, and A. Yarmola, *On volumes and filling collections of multicurves*, J. Topol. **15** (2022), no. 3, 1107–1153.

20. R. J. Daverman, *Decompositions of manifolds*, Pure and Applied Mathematics, vol. 124, Academic Press, Inc., Orlando, FL, 1986.

21. M. G. Durham, F. Fanoni, and N. G. Vlamis, *Graphs of curves on infinite-type surfaces with mapping class group actions*, Ann. Inst. Fourier (Grenoble) **68** (2018), no. 6, 2581–2612.

22. B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.

23. S. R. Fenley, *Depth-one foliations in hyperbolic 3-manifolds*, Ph.D. thesis, Princeton University, 1989.

24. S. R. Fenley, *Asymptotic properties of depth one foliations in hyperbolic 3-manifolds*, J. Differential Geom. **36** (1992), no. 2, 269–313.

25. S. R. Fenley, *End periodic surface homeomorphisms and 3-manifolds*, Math. Z. **224** (1997), no. 1, 1–24.

26. L. Funar and C. Kapoudjian, *An infinite genus mapping class group and stable cohomology*, Comm. Math. Phys. **287** (2009), no. 3, 784–804.

27. D. Gabai, *Foliations and genera of links*, Topology **23** (1984), no. 4, 381–394.

28. J. Hass, *Minimal surfaces in foliated manifolds*, Comment. Math. Helv. **61** (1986), no. 1, 1–32.

29. W. H. Jaco and P. B. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220, viii+192.

30. S. Kojima and G. McShane, *Normalized entropy versus volume for pseudo-Anosovs*, Geom. Topol. **22** (2018), no. 4, 2403–2426.

31. M. Landry, Y. Minsky, and S. Taylor, *Flows, growth rates, and the veering polynomial*, Glasnik Matematicki **57** (2022), no. 1, 119–128.

32. J. Lanier and M. Loving, *Curve graphs of surfaces with finite-invariance index 1*, Preprint, 2021.

33. K. Mann and K. Rafi, *Large scale geometry of big mapping class groups*, to appear in Geom. Topol.

34. B. Martelli, *An introduction to geometric topology*, arXiv:1610.02592, 2016.

35. H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149.
36. H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, Geom. Funct. Anal. **10** (2000), no. 4, 902–974.
37. C. McMullen, *Riemann surfaces and the geometrization of 3-manifolds*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 2, 207–216.
38. W. Menasco and A. W. Reid, *Totally geodesic surfaces in hyperbolic link complements*, Topology ’90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, de Gruyter, Berlin, 1992, pp. 215–226.
39. Y. Minsky, *The classification of Kleinian surface groups. I. Models and bounds*, Ann. of Math. (2) **171** (2010), no. 1, 1–107.
40. J. W. Morgan, *On Thurston’s uniformization theorem for three-dimensional manifolds*, The Smith conjecture (New York, 1979), Pure Appl. Math., vol. 112, Academic Press, Orlando, FL, 1984, pp. 37–125.
41. W. D. Neumann and D. Zagier, *Volumes of hyperbolic three-manifolds*, Topology **24** (1985), no. 3, 307–332.
42. P. Patel and N. G. Vlamis, *Algebraic and topological properties of big mapping class groups*, Algebr. Geom. Topol. **18** (2018), no. 7, 4099–4142.
43. J. A. Rodriguez-Migueles, *A lower bound for the volumes of complements of periodic geodesics*, J. Lond. Math. Soc. (2) **102** (2020), no. 2, 695–721.
44. R. Roussarie, *Plongements dans les variétés feuilletées et classification de feuilletages sans holonomie*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), no. 43, 101–141.
45. J. R. Stallings, *Constructions of fibred knots and links*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, RI, 1978, pp. 55–60.
46. P. A. Storm, *Minimal volume Alexandrov spaces*, J. Differential Geom. **61** (2002), no. 2, 195–225.
47. P. A. Storm, *Hyperbolic convex cores and simplicial volume*, Duke Math. J. **140** (2007), no. 2, 281–319.
48. W. P. Thurston, *Foliations of 3-manifolds that are circle bundles*, Ph.D. thesis, UC Berkeley, 1972.
49. W. P. Thurston, *Geometry and topology of 3-manifolds*, Lecture notes, Princeton University, Princeton, NJ, 1978.
50. W. P. Thurston, *Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds*, Ann. of Math. (2) **124** (1986), no. 2, 203–246.
51. F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.