Spin dependent extension of Calogero-Sutherland model through anyon like representations of permutation operators

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Abstract

We consider a $A_{N-1}$ type of spin dependent Calogero-Sutherland model, containing an arbitrary representation of the permutation operators on the combined internal space of all particles, and find that such a model can be solved as easily as its standard $su(M)$ invariant counterpart through the diagonalisation of Dunkl operators. A class of novel representations of the permutation operator $P_{ij}$, which pick up nontrivial phase factors along with interchanging the spins of $i$-th and $j$-th particles, are subsequently constructed. These ‘anyon like’ representations interestingly lead to different variants of spin Calogero-Sutherland model with highly nonlocal interactions. We also explicitly derive some exact eigenfunctions as well as energy eigenvalues of these models and observe that the related degeneracy factors crucially depend on the choice of a few discrete parameters which characterise such anyon like representations.

PACS No. : 03.65.Fd, 75.10.Jm

Keywords : Calogero-Sutherland model, Permutation operator, anyon, Dunkl operator, Hecke algebra
1 Introduction

As it is well known, the Calogero-Sutherland (CS) model [1,2] and its spin dependent generalisations [3-8] fall into a very interesting class of quantum many-body systems with long ranged interactions, for which the complete excitation spectrum and various dynamical correlation functions can be calculated exactly. Moreover, in recent years, such integrable systems have found a lot of applications in apparently diverse subjects like fractional statistics in (1+1)-dimension [9-13], quantum Hall effect [14-16], the level statistics for disordered systems [17-19], matrix models [20,21], $W_\infty$ algebra [22-24], etc.

The dynamics of $A_{N-1}$ type spin CS model, associated with $N$ number of particles each having $M$ internal degrees of freedom and moving on a ring of length $L$, is governed by the $su(M)$ invariant Hamiltonian [7,8]

$$H = -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial}{\partial x_i} \right)^2 + \frac{\pi^2}{L^2} \sum_{i<j} \frac{\beta(\beta + P_{ij})}{\sin^2 \frac{\pi}{L}(x_i - x_j)}. \quad (1.1)$$

Here $\beta$ is a coupling constant and $P_{ij}$ is the permutation operator which interchanges the ‘spins’ of $i$-th and $j$-th particles. Thus, if the vector $|\vec{\alpha}\rangle = |\alpha_1 \alpha_2 \cdots \alpha_N\rangle$ represents a particular spin configuration of $N$ particles, with $\alpha_1, \alpha_2, \cdots, \alpha_N \in [1, M]$, then $P_{ij}$ will act on this vector as

$$P_{ij} |\alpha_1 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_N\rangle = |\gamma_1 \cdots \gamma_i \cdots \gamma_j \cdots \gamma_N\rangle, \quad (1.2)$$

where $\gamma_i = \alpha_j$, $\gamma_j = \alpha_i$, and $\gamma_k = \alpha_k$ when $k \neq i, j$. The Hamiltonian of the original spin independent CS model [2] can be recovered from (1.1) through the formal substitution $P_{ij} \to -1$. However it is worth noting that, in spite their much more complicated nature, the eigenstates of the spin CS model (1.1) can be obtained almost in the same way as its spin independent counterpart by diagonalising a set of simple differential operators known as Dunkl operators [7,8]. So it should be interesting to enquire whether there exist any other form of ‘permutation’ operator $P_{ij}$, than given by eqn.(1.2), which through substitution in (1.1) would generate a new quantum Hamiltonian that can be solved again through the diagonalisation of these Dunkl operators.
With the hope of making some progress to the above mentioned direction, in sec.2 of this article we briefly recapitulate the procedure of solving the standard $su(M)$ invariant CS model (1.1). In this context we curiously notice that the algebra of the permutation operators $P_{ij}$, rather than any of their particular representation like (1.2), plays an essential role in solving the model. Therefore, if one takes any other representation of $P_{ij}$ on the total internal space of the whole system and substitutes it to the expression (1.1), that would also yield a spin CS model which can be solved exactly in the same way as its standard $su(M)$ invariant counterpart. Next, in sec.3, we construct a new class of representations of the permutation operator $P_{ij}$, by considering a specific limit of some known braid group representations associated with the universal $\mathcal{R}$-matrix of $U_q(sl(M))$ quantum group. Such novel representations of the permutation operators, characterised by a set of discrete as well as continuous deformation parameters, interestingly lead to different types of exactly solvable spin CS models with highly nonlocal interactions which would break the $su(M)$ invariance. Subsequently, in sec.4, we focus our attention to some special cases of these models containing a small number of spin-$\frac{1}{2}$ particles and explicitly derive the related eigenvectors for several low-lying energy states. Sec.5 is the concluding section.

2 Solution of $su(M)$ invariant CS model

To solve the $su(M)$ invariant CS model (1.1), it is useful to make an ansatz for the corresponding wave function as [8]

$$\psi(x_1, \cdots, x_N; \alpha_1, \cdots, \alpha_N) = \Gamma^\beta \phi(x_1, \cdots, x_N; \alpha_1, \cdots, \alpha_N),$$

(2.1)

where $\Gamma = \prod_{i<j} \sin \frac{\pi}{L}(x_i - x_j)$ (here $\beta$ is assumed to be positive for avoiding singularity at $x_i \to x_j$). Next, by applying the canonical commutation relations $[\frac{\partial}{\partial x_j}, x_k] = \delta_{jk}$ and also making a change of coordinates like $z_j = e^{\frac{2\pi i}{L} x_j}$, one may easily find that

$$H\psi = H\Gamma^\beta \phi = \frac{2\pi^2}{L^2} \Gamma^\beta \mathcal{H} \phi,$$

(2.2)
where $H$ is the original Hamiltonian (1.1) and

$$
H = \sum_j \left( z_j \partial \frac{\partial}{\partial z_j} \right)^2 + \beta \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left( \frac{z_i}{\partial z_i} - \frac{z_j}{\partial z_j} \right) \\
- 2\beta \sum_{i<j}(1 + P_{ij}) \frac{z_i z_j}{(z_i - z_j)^2} + \frac{\beta^2}{12} N(N^2 - 1),
$$

(2.3)

Due to the ‘gauge transformation’ (2.2), the diagonalisation problem of $H$ is now reduced to the diagonalisation problem of effective Hamiltonian $H$. Thus, if $\phi$ is an eigenvector of $H$ with eigenvalue $\epsilon$, then $\psi$ would be the corresponding eigenvector of $H$ satisfying the relation

$$
H\psi = \frac{2\pi^2}{L^2} \epsilon \psi.
$$

(2.4)

To diagonalise $H$, however, it is convenient to introduce another operator $H^*$ which acts only on the coordinate degrees of freedom and may be given by

$$
H^* = \sum_j \left( z_j \partial \frac{\partial}{\partial z_j} \right)^2 + \beta \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left( \frac{z_i}{\partial z_i} - \frac{z_j}{\partial z_j} \right) \\
- 2\beta \sum_{i<j}(1 - K_{ij}) \frac{z_i z_j}{(z_i - z_j)^2} + \frac{\beta^2}{12} N(N^2 - 1),
$$

(2.5)

where $K_{ij}$s are the coordinate exchange operators defined through algebraic relations

$$
K_{ij}z_l = z_l K_{ij}, \quad K_{ij} \frac{\partial}{\partial z_i} = \frac{\partial}{\partial z_j} K_{ij}, \quad K_{ij}z_l = z_l K_{ij},
$$

(2.6a)

$$
K_{ij}^2 = 1, \quad K_{ij}K_{jl} = K_{il}K_{ij} = K_{jl}K_{il}, \quad [K_{ij}, K_{lm}] = 0,
$$

(2.6b)

$i, j, l, m$ being all different indices. It may be noted that the operator $H$ in eqn.(2.3) can be reproduced from the expression of $H^*$ in (2.5) through the formal substitution $K_{ij} \rightarrow -P_{ij}$. Due to such close connection between these two operators and also because $H^*$ satisfies some simple commutation relations like

$$
[H^*, K_{ij}] = [H^*, P_{ij}] = 0,
$$

(2.7)

one can easily construct an eigenfunction of $H$ from a given eigenfunction of $H^*$ in the following way.

Let $\Pi_{ij}$s be the set of permutation operators which simultaneously interchange the spins as well as coordinates of two particles (i.e., $\Pi_{ij} = P_{ij}K_{ij}$) and $\Lambda$ be the
corresponding antisymmetric projection operator satisfying the relations

\[ \Pi_{ij} \Lambda = P_{ij} K_{ij} \Lambda = - \Lambda , \quad (2.8) \]

(or, equivalently, \( P_{ij} \Lambda = - K_{ij} \Lambda \)). Now, if \( \xi \equiv \xi(z_1, z_2, \ldots, z_N) \) is an eigenvector of \( \mathcal{H}^* \) with eigenvalue \( \epsilon \), then by using commutation relations (2.7) one can generate another eigenvector of \( \mathcal{H}^* \) with the same eigenvalue: \( \mathcal{H}^* \Lambda(\xi \rho) = \epsilon \Lambda(\xi \rho) \), where \( \rho \equiv \rho(\alpha_1, \alpha_2, \ldots, \alpha_N) \) is an arbitrary spin dependent function. However, due to property (2.8) of the antisymmetriser \( \Lambda \), it is evident that \( \Lambda(\xi \rho) \) may also be considered as an eigenfunction of the effective Hamiltonian \( \mathcal{H} \) with eigenvalue \( \epsilon \). As a result the eigenvectors of original Hamiltonian \( H \), defined through eqn.(2.4), can be written through the eigenfunctions of \( \mathcal{H}^* \) as

\[ \psi = \Gamma^\beta \Lambda(\xi \rho) . \quad (2.9) \]

Finally, for finding out the eigenfunctions of \( \mathcal{H}^* \), it may be observed that this operator can be expressed in a simple quadratic form like [8]

\[ \mathcal{H}^* = \sum_{i=1}^{N} d_i^2 , \quad (2.10) \]

where \( d_i \)s (\( i \in [1, N] \)) are the so called Dunkl operators [25-27]:

\[ d_i = z_i \frac{\partial}{\partial z_i} + \beta \left( i - \frac{N+1}{2} \right) - \beta \sum_{j>1} \frac{z_i}{z_i - z_j} (K_{ij} - 1) + \beta \sum_{j<i} \frac{z_j}{z_j - z_i} (K_{ij} - 1) , \quad (2.11) \]

which satisfy the relations

\[ [ d_i, d_j ] = 0 , \quad [ k_{i,i+1}, d_l ] = 0 , \quad k_{i,i+1}d_l - d_{i+1}k_{i,i+1} = -\beta , \quad (2.12) \]

when \( l \neq i, i + 1 \). So one should be able to construct the eigenvectors of \( \mathcal{H}^* \) by simultaneously diagonalising the mutually commuting operators \( d_i \). For this purpose, however, it is quite helpful to make the following ordering of the corresponding basis elements characterised by the monomials

\[ m_{\{\lambda_1, \lambda_2, \ldots, \lambda_N\}} = z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_N^{\lambda_n} , \quad (2.13) \]
where \( \{ \lambda_1, \ldots, \lambda_N \} \equiv [\lambda] \) is a sequence of non-negative integers with the homogeneity \( \lambda = \sum_{i=1}^{N} \lambda_i \). For this sequence \( [\lambda] \), one may now associate a partition \( [\lambda] \) where the entries \( \lambda_k \) are arranged in decreasing order. Next, an ordering among the partitions (which are obtained from all monomials with a given homogeneity \( \lambda \)) is defined by saying that \( [\lambda] \) is larger than \( [\mu] \) if the first nonvanishing difference \( \lambda_k - \mu_k \) is positive. This prescription naturally induces an ordering between any two monomials which belong to different partitions of homogeneity \( \lambda \). One can further order the monomials associated with the same partition by saying that \( [\lambda] \) is larger than \( [\lambda'] \) if the last nonvanishing difference \( \lambda_k - \lambda'_k \) is positive. Due to such global ordering it turns out that the operators \( d_i \) and \( H^* \) can be written through block-triangular matrices; each block representing the action of an operator on all monomials within a given homogeneity sector [7]. By using this important block-triangular property, for which the diagonal elements of a matrix can be identified with the corresponding eigenvalues, one finds that

\[
H^* \zeta_{\{\lambda_1, \lambda_2, \cdots, \lambda_N\}} = \sum_{i=1}^{N} \left[ \lambda_i - \beta \left( \frac{N + 1}{2} - i \right) \right]^2 \zeta_{\{\lambda_1, \lambda_2, \cdots, \lambda_N\}} \tag{2.14}
\]

where the eigenfunction \( \zeta_{\{\lambda_1, \lambda_2, \cdots, \lambda_N\}} \) is a suitable linear combination of \( m_{\{\lambda_1, \lambda_2, \cdots, \lambda_N\}} \) and other monomials of relatively lower orders.

Though it is rather difficult to write down the general form of \( \zeta_{\{\lambda_1, \lambda_2, \cdots, \lambda_N\}} \), one can find it out easily for the case of low-lying energy states associated with small number of particles. We shall explicitly derive a few of such \( \zeta_{\{\lambda_1, \lambda_2, \cdots, \lambda_N\}} \) in sec.4 of this article, and subsequently use them to generate the eigenstates of new spin CS models which are related to some ‘anyon like’ representations of the permutation operators.

## 3 Novel variants of spin CS model

In close analogy with the \( su(M) \) invariant CS model (1.1), we consider in the following another Hamiltonian \( \hat{H} \) which can differ from (1.1) only through the nature of
corresponding spin-spin interactions.

\[ \tilde{H} = -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial}{\partial x_i} \right)^2 + \frac{\pi^2}{L^2} \sum_{i<j} \frac{\beta(\beta + \tilde{P}_{ij})}{\sin^2 \frac{\pi}{L}(x_i - x_j)}, \tag{3.1} \]

where \( \tilde{P}_{ij} \)s are any possible set of ‘permutation’ operators which act on the combined internal space of \( N \) particles (i.e., on \( F \equiv C^M \otimes C^M \otimes \cdots \otimes C^M \)) and yield a representation of the algebraic relations

\[ P_{ij}^2 = 1, \quad P_{ij}P_{jl} = P_{il}P_{ij}, \quad [P_{ij}, P_{lm}] = 0, \tag{3.2} \]

\( i, j, l, m \) being all different indices.

It should be noticed that the standard permutation operator \( P_{ij} \), defined by eqn.\((1.2)\), is only a particular representation of \( P_{ij} \) satisfying the algebra \((3.2)\). Our aim is to construct here some other representations of \( P_{ij} \) on the vector space \( F \) and subsequently use such \( \tilde{P}_{ij} \) to generate new variants of spin CS model. However, before focussing our attention to those specific cases, let us investigate at present how a Hamiltonian like \((3.1)\), containing an arbitrary representation of permutation operators, can be solved exactly by using the techniques which have been already discussed in sec.2. For this purpose we assume that the form of the corresponding wave function \( \tilde{\psi} \) is again given by an ansatz like \((2.1)\) and make the ‘gauge transformation’: \( \tilde{H}\tilde{\psi} = \tilde{H}\Gamma^\beta\tilde{\phi} = \frac{2\pi^2}{L^2} \Gamma^\beta \tilde{H}\tilde{\phi} \), where the effective Hamiltonian \( \tilde{H} \) is now expressed as

\[ \tilde{H} = \sum_j \left( z_j \frac{\partial}{\partial z_j} \right)^2 + \beta \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right) \]

\[ -2\beta \sum_{i<j} (1 + \tilde{P}_{ij}) \frac{z_i z_j}{(z_i - z_j)^2} + \frac{\beta^2}{12} N(N^2 - 1). \tag{3.3} \]

Evidently, the above effective Hamiltonian \( \tilde{H} \) can also be reproduced from the operator \( \mathcal{H}^* \) in eqn.\((2.3)\), through the formal substitution: \( K_{ij} \rightarrow -\tilde{P}_{ij} \). So, for constructing the eigenvectors of \( \tilde{H} \) from that of \( \mathcal{H}^* \), we define a set of operators as \( \tilde{\Pi}_{ij} = K_{ij}\tilde{P}_{ij} \). Since both \( K_{ij} \) and \( \tilde{P}_{ij} \) satisfy an algebra like \((3.2)\), while acting on the coordinate and spin spaces respectively, the newly defined operators \( \tilde{\Pi}_{ij} \) would also produce a representation of the same permutation algebra on the full Hilbert space of \( N \) particles. Therefore, by
using only this permutation algebra, one can easily define a ‘generalised’ antisymmetric projection operator $\tilde{\Lambda}$ which will satisfy the relations

$$\tilde{\Pi}_{ij}\tilde{\Lambda} = K_{ij}\tilde{P}_{ij}\tilde{\Lambda} = -\tilde{\Lambda}.$$  \hspace{1cm} (3.4)

For example, such antisymmetric projection operators corresponding to the simplest $N = 2$ and $N = 3$ cases (denoted by $\tilde{\Lambda}_2$ and $\tilde{\Lambda}_3$ respectively) are given by

$$\tilde{\Lambda}_2 = 1 - \tilde{\Pi}_{12}, \hspace{0.5cm} \tilde{\Lambda}_3 = 1 - \tilde{\Pi}_{12} - \tilde{\Pi}_{23} - \tilde{\Pi}_{13} + \tilde{\Pi}_{23}\tilde{\Pi}_{12} + \tilde{\Pi}_{12}\tilde{\Pi}_{23}. \hspace{1cm} (3.5a,b)$$

Now, by exactly following the arguments of sec.2, it is straightforward to verify that $\tilde{\Lambda}(\xi\rho)$ will be an eigenvector of $\tilde{H}$ with eigenvalue $\epsilon$, when $\xi \equiv \xi(z_1, z_2, \cdots, z_N)$ is an eigenfunction of $H^*$ with same eigenvalue, and $\rho \equiv \rho(\alpha_1, \alpha_2, \cdots, \alpha_N)$ is an arbitrary spin dependent function. Finally, by using a relation like (2.2), the eigenfunction of spin CS model (3.1) corresponding to eigenvalue $\frac{2\pi \epsilon}{L^2}$ can also be expressed through the eigenfunction of $H^*$ as

$$\tilde{\psi} = \Gamma^\beta \tilde{\Lambda}(\xi\rho). \hspace{1cm} (3.6)$$

Thus from the above discussion it is clear that the spin CS Hamiltonian (3.1) can be solved much in the same way as its original counterpart (1.1) by introducing a generalised antisymmetric projection operator $\tilde{\Lambda}$. The key point in this approach is that for finding out the form of $\tilde{\Lambda}$, which satisfies equation (3.4) and consequently relates the eigenvectors of $H^*$ and $\tilde{H}$, one does not have to use any information about the representation $\tilde{P}_{ij}$ except that it satisfies the permutation algebra (3.2). So, each representation of $P_{ij}$ on the combined internal space of $N$ particles would generate an exactly solvable model whose eigenvectors can be obtained by using the equation (3.6). The standard representation (1.2) evidently reproduces the well known $su(N)$ invariant CS Hamiltonian (1.1) with eigenvectors given by eqn.(2.9).

Now, for constructing other possible representations of the permutation group ($S_N$) related algebra (3.2), we recall that it can be generated by $N - 1$ number of elements $P_{k,k+1}$ ($k \in [1, N - 1]$), which satisfy the relations

$$P_{k,k+1}P_{k+1,k+2}P_{k,k+1} = P_{k+1,k+2}P_{k,k+1}P_{k+1,k+2}, \hspace{0.5cm} [P_{k,k+1}, P_{l,l+1}] = 0, \hspace{0.5cm} P_{k,k+1}^2 = 1,$$
where \( |k-l| > 1 \). All other ‘non-nearest neighbour’ elements like \( P_{km} \) (with \( m-k > 1 \)) can be expressed through these generators as

\[
P_{km} = (P_{k,k+1}P_{k+1,k+2} \cdots P_{m-2,m-1}) P_{m-1,m} (P_{m-2,m-1} \cdots P_{k+1,k+2}P_{k,k+1}). \tag{3.8}
\]

It is worth observing in this context that the braid group \( B_N \) [28] also has \( N-1 \) number of generators \( b_k \) \((k \in [1, N-1])\) and the corresponding algebra looks very similar to the relations (3.7a,b):

\[
b_kb_{k+1}b_k = b_{k+1}b_kb_{k+1}, \quad [b_k, b_l] = 0 \tag{3.9}
\]

where \( |k-l| > 1 \). However, there is no analogue of the relation (3.7c) for the braid group generators. To ‘reduce’ this difference between the generators of \( S_N \) and \( B_N \), one may consider a specific class of braid group representations (BGRs) which satisfy the extra condition

\[
b_k^2 = (q - q^{-1}) b_k + 1, \tag{3.10}
\]

\( q \) being an arbitrary nonvanishing parameter. In fact, the equations (3.9) and (3.10) define together the Hecke algebra, which has interesting applications in many areas related to integrable models [28-30]. For the present purpose it is useful to notice that at the limit \( q \to 1 \), eqn.(3.10) becomes exactly equivalent to the relation (3.7c). Consequently, by taking this limit to some known representations of Hecke algebra and making the identification \( b_k \to P_{k,k+1} \), we might be able to construct new representations of algebra (3.7) satisfied by the permutation generators.

The representations of braid group (and also of Hecke algebra at some special cases), in turn, can be derived in a systematic way by using the universal \( \mathcal{R} \)-matrix associated with various quantum groups [31-33]. A class of such BGRs, operating on the tensor product space \( \mathcal{F} \), are given by

\[
b_k = \sum_{\sigma=1}^{M} \epsilon_\sigma(q) e^k_{\sigma\sigma} \otimes e^{k+1}_{\sigma\sigma} + \sum_{\sigma \neq \gamma} e^{i\phi_{\gamma\sigma}} e^k_{\sigma\gamma} \otimes e^{k+1}_{\gamma\gamma} + (q - q^{-1}) \sum_{\sigma < \gamma} e^k_{\gamma\gamma} \otimes e^{k+1}_{\sigma\sigma}, \tag{3.11}
\]
where \( e^k_{\sigma \gamma} \) are the basis operators on the \( k \)-th vector space with elements \( (e^k_{\sigma \gamma})_{\tau \delta} = \delta_{\sigma \tau} \delta_{\gamma \delta} \), \( \phi_{\gamma \sigma} \) are number of independent antisymmetric deformation parameters: \( \phi_{\gamma \sigma} = -\phi_{\sigma \gamma} \), and each of the \( \epsilon_{\sigma}(q) \) can be freely taken as either \( q \) or \( -q^{-1} \) for any value of \( \sigma \). So for every possible choice of the set of parameters \( q, \phi_{\sigma \gamma} \) and \( \epsilon_{\sigma}(q) \), eqn. (3.11) will give us a distinct braid group representation. Though the derivation of relation (3.11) is not relevant for our purpose, it is worth noting that in the special case when all \( \epsilon_{\sigma}(q) \)s take the same value (i.e., all of them are either \( q \) or \( -q^{-1} \)), the corresponding BGRs can be obtained from the fundamental representation of the universal \( \mathcal{R} \)-matrix associated with \( U_q(sl(M)) \) quantum group, for generic values of the parameter \( q \) [31-33]. On the other hand if \( \epsilon_{\sigma}(q) \)s do not take the same value for all \( \sigma \), the corresponding ‘nonstandard’ BGRs are found to be connected with the universal \( \mathcal{R} \)-matrix of \( U_q(sl(M)) \) quantum group when \( q \) is a root of unity [34-36]. Furthermore, the parameters \( \phi_{\sigma \gamma} \) and \( \epsilon_{\sigma}(q) \) have appeared previously in the context of multi-parameter dependent quantisation of \( GL(M) \) group [37] and the asymmetric vertex model studied by Perk and Schultz [38]. However, one may also directly check that the BGRs given by eqn. (3.11) obey both the relations (3.9) and (3.10), and therefore, can be considered as some representations of the Hecke algebra. Consequently, by taking the \( q \to 1 \) limit of the expression (3.11), we get a class of representations of the permutation algebra (3.7) as

\[
\hat{P}_{k,k+1} = \sum_{\sigma=1}^{M} \epsilon_{\sigma} e^k_{\sigma \sigma} \otimes e^{k+1}_{\sigma \sigma} + \sum_{\sigma \neq \gamma} e^{i\phi_{\gamma \sigma}} e^k_{\sigma \gamma} \otimes e^{k+1}_{\gamma \sigma}, \tag{3.12}
\]

where \( \epsilon_{\sigma} \) can be freely chosen to be either 1 or \( -1 \) for each value of \( \sigma \). By inserting the above expression of \( \hat{P}_{k,k+1} \) to eqn. (3.8), one can easily construct the representations of ‘non-nearest neighbour’ permutation elements. It might be noted that, the parameters \( \epsilon_{\sigma} \) in eqn. (3.12) have some apparent similarity with the grading parameters which appear in the supersymmetric exchange operator and related CS model [39]. However, in contrast to the case of ref.39, our expression (3.12) defines some representations of permutation algebra (3.7) on the usual vector space \( \mathcal{F} \) which does not carry any \( Z_2 \) grading.
It is evident that when $\epsilon_\sigma = 1, \phi_{\gamma \sigma} = 0$ for all values of $\sigma, \gamma$, the expression (3.12) coincides with the standard form of permutation operator (1.2). Similarly, the case $\epsilon_\sigma = -1, \phi_{\gamma \sigma} = \pi$ for all $\sigma, \gamma$ reproduces again the representation (1.2) up to an overall sign factor. However to get an insight to other situations, let us consider first the simplest case of two spin-$\frac{1}{2}$ particles (i.e, $N = M = 2$), where we have only one permutation operator $\tilde{P}_{12}$. By using the relation (3.12), one can explicitly write down the action of this operator on the associated vector as

$$
\tilde{P}_{12}|11\rangle = \epsilon_1|11\rangle, \quad \tilde{P}_{12}|12\rangle = e^{i\theta}|21\rangle, \quad \tilde{P}_{12}|21\rangle = e^{-i\theta}|12\rangle, \quad \tilde{P}_{12}|22\rangle = \epsilon_2|22\rangle, \quad (3.13)
$$

where $\theta = \phi_{12}$. It is curious to notice that, somewhat similar to the case of anyons, the above representation of $\tilde{P}_{12}$ not only interchanges the spin of two particles but also picks up some spin-dependent phase factors. Consequently, when substituted in the Hamiltonian (3.1), such representation of permutation operator would break the $su(2)$ symmetry which is present in the original spin-$\frac{1}{2}$ CS model.

More interesting things happen if one considers the motion of three spin-$\frac{1}{2}$ particles (i.e. $N = 3, M = 2$). In this case we naturally have three permutation operators $\tilde{P}_{12}$, $\tilde{P}_{23}$ and $\tilde{P}_{13}$, which act on the direct product of three spin spaces. However, the forms of $\tilde{P}_{12}$ and $\tilde{P}_{23}$ would again be given by equations like (3.13) while these operators act on the direct product of two spin spaces where they are nontrivial. To find out the action of the remaining element $\tilde{P}_{13}$, we have to simultaneously use the above mentioned forms of $\tilde{P}_{12}$, $\tilde{P}_{23}$ and the relation $\mathcal{P}_{13} = \mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{12}$ (which is derivable from (3.8)):

$$
\tilde{P}_{13}|1\alpha_21\rangle = \epsilon_1|1\alpha_21\rangle, \quad \tilde{P}_{13}|1\alpha_22\rangle = f(\alpha_2)|2\alpha_21\rangle, \quad (3.14a, b)
$$

$$
\tilde{P}_{13}|2\alpha_21\rangle = g(\alpha_2)|1\alpha_22\rangle, \quad \tilde{P}_{13}|2\alpha_22\rangle = \epsilon_2|2\alpha_22\rangle, \quad (3.14c, d)
$$

where $\alpha_2 = 1, 2$ denotes the spin orientation of 2nd particle and $f(1) = g(1)^{-1} = \epsilon_1 e^{2i\theta}$, $f(2) = g(2)^{-1} = \epsilon_2 e^{2i\theta}$. It is rather surprising to notice that, for $\epsilon_1 \neq \epsilon_2$, the phase factors in the expressions (3.14b,c) not only depend on the spin orientations of 1st and 3rd particle but also on that of the intermediate 2nd particle. Therefore, the operator $\tilde{P}_{13}$ no longer acts like identity on the 2nd internal space and generates a
three-body interaction when substituted to the CS Hamiltonian (3.1). It is worth noting that the form of \( \bar{P}_{12} \) in eqn.(3.13) is quite similar to the supersymmetric exchange operators [6,39] associated with two species of particles, where \( \epsilon_1 \) and \( \epsilon_2 \) play the role of corresponding grading parameters. However, even in the supersymmetric case the phase factors associated with a permutation operator like \( P_{13} \) can depend only on the gradings of spin components in the 1st and 3rd internal space, and are completely independent of the spin orientation in the 2nd internal space. So the fact that the operator \( P_{13} \) given by eqn.(3.14) induces a three-body interaction is rather unique to the present situation.

The above mentioned feature of ‘non-nearest neighbour’ permutation operator \( P_{13} \) becomes even more prominent when, by using the relations (3.12) and (3.8), one constructs the action of a general element \( \bar{P}_{kl} \) \((l - k > 1)\) on the related vector space \( \mathcal{F} \) as

\[
\bar{P}_{kl} |\alpha_1\alpha_2 \cdots \alpha_k \cdots \alpha_l \cdots \alpha_N\rangle = \exp \left\{ i \phi_{\alpha_k\alpha_l} + i \sum_{\tau=1}^{M} n_\tau (\phi_{\tau\alpha_l} - \phi_{\tau\alpha_k}) \right\} |\gamma_1\gamma_2 \cdots \gamma_k \cdots \gamma_l \cdots \gamma_N\rangle, (3.15)
\]

where \( \gamma_k = \alpha_l \), \( \gamma_l = \alpha_k \) and \( \gamma_j = \alpha_j \) for \( j \neq k, l \); we have used the notation \( e^{i\phi_{\sigma\sigma}} = \epsilon_\sigma \), and assumed that the particular spin orientation \( \alpha_p = \tau \) occurs \( n_\tau \) number of times when the index \( p \) in \( \alpha_p \) runs from \( k + 1 \) to \( l - 1 \). Thus, the phase factor associated with the element \( \bar{P}_{kl} \) now depends on the spin configuration of \( l - k + 1 \) number of particles: \( \alpha_k, \alpha_{k+1}, \cdots, \alpha_l \). So, if we substitute this ‘anyon like’ representation of permutation operators to the spin CS Hamiltonian (3.1), it will lead to an exactly solvable model with highly nonlocal interactions. It is fascinating to observe that such solvable models with nonlocal interactions are quite similar to the quantum spin chains with open boundary conditions; since, from eqn.(3.15) it may be seen that the spin dependent interaction between 1st and \( N \)-th particle, i.e. \( P_{1N} \), would be much more complicated in form than all other nearest neighbour spin interactions like \( P_{12}, P_{23} \) etc. This more complicated nature of operator \( P_{1N} \) is probably connected with the existence of some nonperiodic or twisted boundary condition on the CS model. However, it should be
noted that the symmetry properties of these spin CS models are completely different
from that of the $BC_N$ type CS models [40-42], which are well known for their releven-
tce in one dimensional physics with boundary. For example, the particles in $BC_N$ type CS
model can interact even with their ‘mirror images’ and also with an impurity located
at the origin. So these interaction terms, depending in particular on the summation
of coordinates like $(x_i + x_j)$, break the translational invariance of the related system.
On the other hand the CS Hamiltonians given by eqn.(3.1), which depend only on
the difference of particle coordinates, would remain translational invariant even in the
presence of new types of spin-spin interactions.

4 Explicit solutions of different spin CS models

In the previous section we have seen that the eigenfunctions of CS Hamiltonian (3.1),
associated with an arbitrary representation of the permutation operators, can be con-
structed by diagonalising the Dunkl operators and subsequently using the general re-
lation (3.6). Then we have also found some concrete examples of such exactly solvable
spin CS model, by inserting the anyon like representations (3.12) and (3.15) to the
Hamiltonian (3.1). In the following we like is to explicitly derive a few of the re-
lated eigenfunctions, by restricting to systems which contain a small number of spin-$\frac{1}{2}$
particles.

For this purpose we first consider the simplest spin CS model, which contain two
spin-$\frac{1}{2}$ particles moving on a circle. The operator $\tilde{P}_{12}$, given by eqn.(3.13), would
represent the spin dependent interaction between these two particles. Now, by using
eqn.(2.11), one may explicitly write down the related Dunkl operators as

$$d_1 = z_1 \frac{\partial}{\partial z_1} - \frac{\beta}{2} - \beta \frac{z_1}{z_1 - z_2} (K_{12} - 1), \quad d_2 = z_2 \frac{\partial}{\partial z_2} + \frac{\beta}{2} + \beta \frac{z_1}{z_1 - z_2} (K_{12} - 1).$$  (4.1)

So, according to our discussion in sec.2, the trivial monomial with homogeneity zero
(i.e. $\xi = 1$) would be an eigenvector of the operators $d_1, d_2$ and will also correspond
to the ground state of operator $H^* = d_1^2 + d_2^2$. By using eqn.(4.1), it is rather easy to see that the eigenvalues of the operators $d_1$, $d_2$ and $H^*$ in this case will be given by $-\beta/2$, $\beta/2$ and $\beta^2/2$ respectively. Therefore, by applying the relation (3.4), the ground state of spin CS Hamiltonian (3.1) associated with energy eigenvalue $\frac{2\pi^2}{L^2}\epsilon = \frac{\pi^2\beta^2}{L^2}$ can be found as

$$\tilde{\psi} = \sin^{\beta} \left( \frac{\pi}{L}(x_1 - x_2) \right) \left( 1 - \tilde{P}_{12} \right) \rho(\alpha_1, \alpha_2),$$  \hspace{1cm} (4.2)

where $\rho(\alpha_1, \alpha_2)$ is an arbitrary spin dependent function. However, for spin-$\frac{1}{2}$ case the function $\rho(\alpha_1, \alpha_2)$ can be chosen in four different ways: $|11\rangle$, $|12\rangle$, $|21\rangle$ and $|22\rangle$. By inserting these forms of $\rho$ to eqn.(4.2) and also using eqn.(3.13), we get three degenerate eigenfunctions like

$$\tilde{\psi}_1 = (1 - \epsilon_1) \Gamma^{\beta}_{(2)} |11\rangle, \hspace{0.5cm} \tilde{\psi}_2 = (1 - \epsilon_2) \Gamma^{\beta}_{(2)} |22\rangle, \hspace{0.5cm} \tilde{\psi}_3 = \Gamma^{\beta}_{(2)} \left( |12\rangle - e^{i\theta} |21\rangle \right),$$ \hspace{1cm} (4.3a, b, c)

where $\Gamma^{\beta}_{(2)} = \sin^{\beta} \left[ \frac{\pi}{L}(x_1 - x_2) \right]$. Notice that the choice of $\rho$ as $|12\rangle$ or $|21\rangle$ would lead to the same wave function $\tilde{\psi}_3$ up to a multiplicative constant.

Next, for constructing the first excited states of the above 2-body problem, let us consider the monomials of homogeneity one. Evidently, $z_2$ and $z_1$ are two such monomials belonging to the partition $(1+0)$ and $z_2$ is of higher order than $z_1$ according to the convention discussed in sec.2. It is not difficult to check that for this simple case the action of Dunkl operators (4.1) on $z_2$, $z_1$ will generate two simultaneously diagonalisable and triangular matrices, whose eigenvectors (i.e. $z'_2$ and $z_1$) would satisfy the relations

$$d_1z'_2 = -\frac{\beta}{2} z'_2, \hspace{0.5cm} d_2z'_2 = \left( 1 + \frac{\beta}{2} \right) z'_2, \hspace{0.5cm} d_1z_1 = \left( 1 + \frac{\beta}{2} \right) z_1, \hspace{0.5cm} d_2z_1 = -\frac{\beta}{2} z_1,$$ \hspace{1cm} (4.4)

where $z'_2 = (1 + \beta)z_2 + \beta z_1$. Consequently, $\epsilon = \beta^2/2 + \beta + 1$ will be the eigenvalue of the operator $H^*$ for both of the eigenstates $z'_2$ and $z_1$. So, by using (3.6), we find that the first excited states of spin CS Hamiltonian (3.1) are given by the expressions

$$\tilde{\psi}' = \Gamma^{\beta}_{(2)} \left( 1 - K_{12}\tilde{P}_{12} \right) \rho(\alpha_1, \alpha_2)z_1, \hspace{0.5cm} \tilde{\psi}'' = \Gamma^{\beta}_{(2)} \left( 1 - K_{12}\tilde{P}_{12} \right) \rho(\alpha_1, \alpha_2)z'_2,$$ \hspace{1cm} (4.5a, b)
where, as before, $\rho(\alpha_1, \alpha_2)$ is an arbitrary spin dependent function. If we insert four possible choice of $\rho$ to eqn.(4.5a), that would lead to four degenerate wave functions like

\[
\tilde{\psi}'_1 = \Gamma_{(2)}^\beta (z_1 - \epsilon_1 z_2) |11\rangle, \quad \tilde{\psi}'_2 = \Gamma_{(2)}^\beta (z_1 - \epsilon_2 z_2) |22\rangle, \\
\tilde{\psi}'_3 = \Gamma_{(2)}^\beta (z_1 - z_2) \left( |12\rangle + e^{i\theta} |21\rangle \right), \quad \tilde{\psi}'_4 = \Gamma_{(2)}^\beta (z_1 + z_2) \left( |12\rangle - e^{i\theta} |21\rangle \right),
\]

(4.6)

which naturally share the same energy eigenvalue $\frac{2\pi^2}{L^2} \epsilon = \frac{\pi^2}{L^2} (\beta^2 + 2\beta + 2)$. However, if we substitute the four different forms of $\rho(\alpha_1, \alpha_2)$ to eqn.(4.5b), that will only reproduce the above four wave functions.

Now we like to analyse the effect of different permutation operators on the above constructed ground state and first excited state wave functions, by tuning the discrete parameters $\epsilon_1$, $\epsilon_2$ as well as the continuous parameter $\theta$. For this purpose, first we insert $\epsilon_1 = \epsilon_2 = 1$ to the eqns.(4.3), (4.6) and curiously notice that some of the wave functions appearing in eqn.(4.3) would become trivial for these values of discrete parameters. So we write down the explicit form of the remaining nontrivial wave functions as

\[
\psi^{(1)}_0 = \Gamma_{(2)}^\beta \left( |12\rangle - e^{i\theta} |21\rangle \right), \quad \psi^{(1)}_1 = \Gamma_{(2)}^\beta (z_1 - z_2) |11\rangle, \quad \psi^{(2)}_1 = \Gamma_{(2)}^\beta (z_1 - z_2) |22\rangle, \\
\psi^{(3)}_1 = \Gamma_{(2)}^\beta (z_1 - z_2) \left( |12\rangle + e^{i\theta} |21\rangle \right), \quad \psi^{(4)}_1 = \Gamma_{(2)}^\beta (z_1 + z_2) \left( |12\rangle - e^{i\theta} |21\rangle \right),
\]

(4.7)

which shows the existence of a nondegenerate ground state $\psi^{(1)}_0$ with energy $\pi^2 \beta^2 / L^2$ and a four-fold degenerate first excited state $\psi^{(i)}_1$ with energy $\pi^2 (\beta^2 + 2\beta + 2) / L^2$. It was mentioned earlier that for the values $\epsilon_1 = \epsilon_2 = 1$ and $\theta = 0$, the permutation operator (3.13) coincides with its standard counterpart and yields the $su(2)$ invariant CS model (1.1). Consequently, the ground state and first excited states of this $su(2)$ invariant model can be easily reproduced by simply putting $\theta = 0$ in the expression (4.7).

Next we take the values of our discrete paramaters as $\epsilon_1 = -\epsilon_2 = 1$, which is related to a new variant of spin CS model. Again, by inserting these values to (4.3) and (4.6), one may find out the corresponding nontrivial wavefunctions as

\[
\psi^{(1)}_0 = 2 \Gamma_{(2)}^\beta |22\rangle, \quad \psi^{(2)}_0 = \Gamma_{(2)}^\beta \left( |12\rangle - e^{i\theta} |21\rangle \right),
\]
\[ \psi_1^{(1)} = \Gamma^{\beta}_{(2)} (z_1 - z_2)|11\rangle, \quad \psi_1^{(2)} = \Gamma^{\beta}_{(2)} (z_1 + z_2)|22\rangle, \quad \psi_1^{(3)}, \quad \psi_1^{(4)}, \quad (4.8) \]

where the forms of \( \psi_1^{(3)} \) and \( \psi_1^{(4)} \) are identical to their previous forms which appeared in eqn. (4.7). So for these values of discrete parameters and the related nonstandard spin CS model, one gets a doubly degenerate ground state along with a four-fold degenerate first excited state. Similarly, for the values of discrete parameters as \( \epsilon_1 = -\epsilon_2 = -1 \), it is easy to see that the associated wave functions would be given by

\[ \psi_0^{(1)} = 2\Gamma^{\beta}_{(2)}|11\rangle, \quad \psi_0^{(2)} = \Gamma^{\beta}_{(2)} (|12\rangle - e^{i\theta}|21\rangle), \]
\[ \psi_1^{(1)} = \Gamma^{\beta}_{(2)} (z_1 + z_2)|11\rangle, \quad \psi_1^{(2)} = \Gamma^{\beta}_{(2)} (z_1 - z_2)|22\rangle, \quad \psi_1^{(3)}, \quad \psi_1^{(4)}, \quad (4.9) \]

which again shows a doubly degenerate ground state and a four-fold degenerate first excited state. However, it is worth noting that the forms of present eigenfunctions \( \psi_0^{(1)}, \psi_1^{(1)} \) and \( \psi_1^{(2)} \) are quite different from their respective forms in eqn. (4.8) associated with \( \epsilon_1 = -\epsilon_2 = 1 \) sector. Finally, one may also find out the wave functions corresponding to the case \( \epsilon_1 = \epsilon_2 = -1 \) as

\[ \psi_0^{(1)} = 2\Gamma^{\beta}_{(2)}|11\rangle, \quad \psi_0^{(2)} = 2\Gamma^{\beta}_{(2)}|22\rangle, \quad \psi_0^{(3)} = \Gamma^{\beta}_{(2)} (|12\rangle - e^{i\theta}|21\rangle), \]
\[ \psi_1^{(1)} = \Gamma^{\beta}_{(2)} (z_1 + z_2)|11\rangle, \quad \psi_1^{(2)} = \Gamma^{\beta}_{(2)} (z_1 + z_2)|22\rangle, \quad \psi_1^{(3)}, \quad \psi_1^{(4)}. \quad (4.10) \]

So, in this sector, one interestingly gets a triply degenerate ground state along with a four-fold degenerate first excited state.

Thus from the nature of above construction it is clear that, the ground state energy and the first excited state energy of 2-particle spin CS model do not depend on the choice of parameters \( \epsilon_1, \epsilon_2 \) and \( \theta \) in the related Hamiltonian. However, the values of discrete parameters \( \epsilon_1 \) and \( \epsilon_2 \) can affect the degeneracy of the ground state in a very significant way. While the standard choice \( \epsilon_1 = \epsilon_2 = 1 \) yields a nondegenerate ground state, other possible choice of these two discrete parameters would give us a doubly or triply degenerate ground state. On the other hand, this degeneracy factor does not change at all with the variation of continuous parameter \( \theta \). So, only the explicit form of these ground state wave functions, and not their degeneracy factor, would depend
on the value of $\theta$. Furthermore it turns out that, in contrast to the case of ground state, the first excited state always remain four-fold degenerate for any possible choice of the parameters $\epsilon_1$, $\epsilon_2$ and $\theta$.

It is easy to similarly derive the wave functions and their degeneracy factors related to the higher excitations of two spin-$\frac{1}{2}$ particles. However, in the following, we like to focus our attention to the CS model containing three spin-$\frac{1}{2}$ particles and explore whether such a system exhibits any new interesting feature. In this case, we have to simultaneously diagonalise three Dunkl operators $d_1$, $d_2$ and $d_3$, which can be explicitly written by using the relation (2.11). It is easy to check that, the trivial monomial with homogeneity zero ($\xi = 1$) would be the simplest eigenstate of $d_1$, $d_2$ and $d_3$ with eigenvalues $-\beta$, 0, and $\beta$ respectively. Moreover, due to the relation (2.10), $\epsilon = 2\beta^2$ will be the eigenvalue for the operator $H^*$ corresponding to this eigenstate. So, by using eqn.(3.6), one can find out the ground state of CS model (3.1) as

$$\tilde{\psi} = \Gamma^\beta_3 \tilde{\Lambda}_3 \rho(\alpha_1, \alpha_2, \alpha_3)$$

where $\Gamma^\beta_3 = \sin^\beta \left[ \frac{\pi}{L} (x_1 - x_2) \right] \sin^\beta \left[ \frac{\pi}{L} (x_2 - x_3) \right]$, and the ‘generalised’ antisymmetric projection operator $\tilde{\Lambda}_3$ is given by the expression (3.5b) which at present contains the representations of permutation operators like (3.13) and (3.14). Furthermore, by taking the arbitrary spin-dependent function $\rho(\alpha_1, \alpha_2, \alpha_3)$ in the above equation in eight possible ways: $|111\rangle$, $|112\rangle$, $|121\rangle$, $|122\rangle$ $|211\rangle$, $|212\rangle$, $|221\rangle$ and $|222\rangle$, we obtain four distinct eigenfunctions given by

$$\tilde{\psi}_1 = (1 - \epsilon_1) \Gamma^\beta_3 |111\rangle, \quad \tilde{\psi}_2 = (1 - \epsilon_1) \Gamma^\beta_3 \left( |112\rangle - e^{i\theta} |121\rangle + e^{2i\theta} |211\rangle \right),$$
$$\tilde{\psi}_3 = (1 - \epsilon_2) \Gamma^\beta_3 |222\rangle, \quad \tilde{\psi}_4 = (1 - \epsilon_2) \Gamma^\beta_3 \left( |122\rangle - e^{i\theta} |212\rangle + e^{2i\theta} |221\rangle \right).$$

It is evident that $\frac{2\pi^2}{L^2} \epsilon = \frac{4\pi^2\beta^2}{L^2}$ would be the energy eigenvalue for all of these degenerate states.

Next we consider the monomials $z_3$, $z_2$ and $z_1$, which correspond to the partition $(1 + 0 + 0)$ of homogeneity one sector. In this case one can again simultaneously diagonalise the triangular matrix representations which are generated by the action of
Dunkl operators on these three monomials and obtain the related eigenvectors as
\[ z_3' = \beta (z_1 + z_2) + (1 + \beta)z_3 , \quad z_2' = \beta z_1 + (1 + 2\beta)z_2 , \quad z_1' = z_1 . \tag{4.13} \]

So, with the help of eqn.(3.6), we find that the first excited states of spin CS model (3.1) would be given by the expressions: \( \tilde{\psi}' = \Gamma^{\beta}_{(3)}\tilde{\Lambda}(3) (\rho z_1'), \tilde{\psi}'' = \Gamma^{\beta}_{(3)}\tilde{\Lambda}(3) (\rho z_2'), \) and \( \tilde{\psi}''' = \Gamma^{\beta}_{(3)}\tilde{\Lambda}(3) (\rho z_3'). \) By inserting the previously mentioned eight possible forms of the arbitrary function \( \rho(\alpha_1, \alpha_2, \alpha_3) \) to \( \tilde{\psi}' \), we obtain a set of six distinct and degenerate eigenfunctions like
\[ \tilde{\psi}'_1 = (1 - \epsilon_1) \Gamma^{\beta}_{(3)} (z_1 + z_2 + z_3) |111\rangle , \quad \tilde{\psi}'_2 = (1 - \epsilon_2) \Gamma^{\beta}_{(3)} (z_1 + z_2 + z_3) |222\rangle , \]
\[ \tilde{\psi}'_3 = (1 - \epsilon_1) \Gamma^{\beta}_{(3)} \left\{ z_1 |211\rangle - e^{-i\theta} z_2 |121\rangle + e^{-2i\theta} z_3 |112\rangle \right\} , \]
\[ \tilde{\psi}'_4 = (1 - \epsilon_2) \Gamma^{\beta}_{(3)} \left\{ z_1 |122\rangle - e^{i\theta} z_2 |212\rangle + e^{2i\theta} z_3 |221\rangle \right\} , \]
\[ \tilde{\psi}'_5 = \Gamma^{\beta}_{(3)} \left\{ (z_1 - \epsilon_1 z_2) |112\rangle - e^{i\theta} (z_1 - \epsilon_1 z_3) |121\rangle + e^{2i\theta} (z_2 - \epsilon_1 z_3) |211\rangle \right\} , \]
\[ \tilde{\psi}'_6 = \Gamma^{\beta}_{(3)} \left\{ (z_1 - \epsilon_2 z_2) |221\rangle - e^{-i\theta} (z_1 - \epsilon_2 z_3) |212\rangle + e^{-2i\theta} (z_2 - \epsilon_2 z_3) |122\rangle \right\} . \tag{4.14} \]

It may be noted that if one substitutes the eight possible forms of arbitrary function \( \rho \) to \( \tilde{\psi}'' \) or \( \tilde{\psi}''' \), that will only reproduce the above set of six wave functions. Moreover, by using eqns.(2.10) and (2.11), it is easy to check that \( \epsilon = 3\beta^2 + 2\beta + 1 \) would be the eigenvalue of operator \( \mathcal{H}^* \) for all \( z_i' \) in eqn.(4.13). Consequently, the degenerate wave functions appearing in eqn.(4.14) will share the same energy eigenvalue \( 2\pi^2 (3\beta^2 + 2\beta + 1) L^2 \).

Now, similar to the case of two particles, let us analyse again the effect of different permutation operators on the above constructed ground state and first excited state wave functions associated with three spin-\(\frac{1}{2}\) particles. In this context it is interesting to observe that, all four wave-functions in eqn.(4.12) would vanish identically for the choice of two discrete parameters as \( \epsilon_1 = \epsilon_2 = 1 \). This observation is also consistent with the simple fact that the usual antisymmetrisation of more than two spin-\(\frac{1}{2}\) particles always yields the trivial result. So, to obtain the related ground state, it is necessary...
to consider the monomials of homogeneity one instead of homogeneity zero. Therefore, we insert the values $\epsilon_1 = \epsilon_2 = 1$ to eqn.(4.14) and find that there exist two nontrivial wave functions which may be explicitly written as

$$
\psi_0^{(1)} = \Gamma_{(3)}^\beta \left\{ (z_1 - z_2)|112\rangle - e^{i\theta} (z_1 - z_3)|121\rangle + e^{2i\theta} (z_2 - z_3)|211\rangle \right\}, \\
\psi_0^{(2)} = \Gamma_{(3)}^\beta \left\{ (z_1 - z_2)|221\rangle - e^{-i\theta} (z_1 - z_3)|212\rangle + e^{-2i\theta} (z_2 - z_3)|122\rangle \right\}.
$$

(4.15)

Thus, for these values of discrete parameters, one gets a doubly degenerate ground state with energy eigenvalue $2\pi^2 (3\beta^2 + 2\beta + 1)/L^2$. It may be noticed that the above equation will also reproduce the ground state of usual $su(2)$ invariant spin CS model (1.1), after the substitution $\theta = 0$.

Next we take the values of discrete parameters as $\epsilon_1 = -\epsilon_2 = 1$, which would lead to a nonstandard type of spin CS model. By putting these values of $\epsilon_1$ and $\epsilon_2$ to eqns.(4.12) and (4.14) respectively, it is straightforward to find that for such nonstandard spin CS model there exist a doubly degenerate ground state with energy $4\pi^2 \beta^2 /L^2$:

$$
\psi_0^{(1)} = 2 \Gamma_{(3)}^\beta |222\rangle, \quad \psi_0^{(2)} = 2 \Gamma_{(3)}^\beta \left( |122\rangle - e^{i\theta} |212\rangle + e^{2i\theta} |221\rangle \right),
$$

(4.16)

and a four-fold degenerate first excited state with energy $2\pi^2 (3\beta^2 + 2\beta + 1)/L^2$:

$$
\psi_1^{(1)} = 2 \Gamma_{(3)}^\beta (z_1 + z_2 + z_3) |222\rangle, \quad \psi_1^{(2)} = 2 \Gamma_{(3)}^\beta \left\{ z_1 |122\rangle - e^{i\theta} z_2 |212\rangle + e^{2i\theta} z_3 |221\rangle \right\}, \\
\psi_1^{(3)} = \Gamma_{(3)}^\beta \left\{ (z_1 - z_2)|112\rangle - e^{i\theta} (z_1 - z_3)|121\rangle + e^{2i\theta} (z_2 - z_3)|211\rangle \right\}, \\
\psi_1^{(4)} = \Gamma_{(3)}^\beta \left\{ (z_1 + z_2)|221\rangle - e^{-i\theta} (z_1 + z_3)|212\rangle + e^{-2i\theta} (z_2 + z_3)|122\rangle \right\}.
$$

(4.17)

By using eqns.(1.12) and (1.14), one can similarly obtain the ground states and first excited states of spin CS models associated with other values of the two discrete parameters. In particular, it is easy to see that the sector $\epsilon_1 = -\epsilon_2 = -1$ would again lead to a doubly degenerate ground state and a four-fold degenerate first excited state. On the other hand, the remaining sector $\epsilon_1 = \epsilon_2 = -1$ would give us a four-fold degenerate ground state and a six-fold degenerate first excited state.

It is rather interesting to notice that the energy of ground states appearing in eqn.(1.16) is actually lower than that of the previous ground states (1.15). Consequently, the choice of two discrete parameters as $\epsilon_1 = -\epsilon_2 = \pm 1$ and $\epsilon_1 = \epsilon_2 = -1$,
would provide us nonstandard variants of spin CS model whose ground state energy is lower than that of the usual spin CS model \([1,1]\) associated with the \(\epsilon_1 = \epsilon_2 = 1\) sector. Furthermore, the ground state energy of usual spin CS model turns out to be exactly same with the first excited state energy of all other nonstandard variants of this model. Thus we curiously find that for the CS model containing three spin-\(\frac{1}{2}\) particles, it is not only possible to change the degeneracy factor of the ground state, but also its energy level, by tuning two discrete parameters which appear in the anyon like representations of permutation operators.

5 Concluding Remarks

Here we carefully analyse the method of constructing solutions of spin dependent Calogero-Sutherland (CS) model and observe that the algebra of the permutation operators, rather than any of their particular representation, plays an important role in this context. Moreover we consider a \(A_{N-1}\) type of spin CS model, containing an arbitrary representation of permutation operators \(BB\) on the combined internal space \((\mathcal{F})\) of all particles, and find that such a model can be solved almost in the same way as its standard \(su(M)\) invariant counterpart by introducing a ‘generalised’ antisymmetric projection operator.

Next, with the aim of constructing new variants of spin CS model, we search for some explicit representations of permutation operators on the space \(\mathcal{F}\). Here we interestingly notice that a class of known representations of the Hecke algebra, characterised by a deformation parameter \(q\), reduces to such representations at the limit \(q \to 1\). These representations of permutation operator \(P_{kl}\) \((k < l)\) not only interchange the spins of \(k\)-th and \(l\)-th particles, but also pick up nontrivial phase factors depending on the spin configuration of all particles indexed by \(k, \ k + 1, \ldots, \ l\). Moreover these ‘anyon like’ representations are found to be dependent on \(\frac{M(M-1)}{2}\) number of continuously variable antisymmetric parameters \(\phi_{\gamma\sigma}\), as well as \(M\) number of discrete parameters \(\epsilon_\sigma\) which
can be freely chosen to be 1 or $-1$. At the special case $\phi_{\gamma\sigma} = 0$ and $\epsilon_{\sigma} = 1$ for all values of $\gamma$, $\sigma$, they coincide with the usual representation of permutation operator, which only interchanges the spins of two particles and leads to the standard $su(M)$ invariant CS model. However, other possible values of the parameters $\epsilon_{\sigma}$ and $\phi_{\gamma\sigma}$ would generate novel variants of spin CS Hamiltonian, containing highly nonlocal type of spin dependent interactions, which violate the $su(M)$ invariance.

Subsequently, we explicitly derive a few low-lying energy states of the above mentioned spin CS models, by restricting to systems which contain a small number of spin-$\frac{1}{2}$ particles. For the case of two spin-$\frac{1}{2}$ particles, we find that there exists a non-degenerate ground state associated with the choice of discrete parameters $\epsilon_1 = \epsilon_2 = 1$. However, other possible values of the parameters $\epsilon_1$ and $\epsilon_2$ curiously yield 2-fold or 3-fold degenerate ground state with the same energy level. Thus it turns out that, the choice of discrete parameters in the representations of permutation operators can affect the degeneracy of the related ground states in a significant way. On the other hand, this degeneracy factor is found to be insensitive to the value of the continuous parameter $\phi_{12}$. More interesting things happen if one considers a system with three spin-$\frac{1}{2}$ particles. In this case we find that, both the degeneracy factor of the ground state as well as its energy level crucially depend on the choice of two discrete parameters $\epsilon_1$ and $\epsilon_2$. In fact, the ground state energy associated with the sector $\epsilon_1 = \epsilon_2 = 1$ exactly coincides with the energy of first excited states associated with $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_1 = -\epsilon_2 = \pm 1$ sectors.

The approach presented here for constructing novel types of spin CS models might have some further implications in several directions. As it is well known, the $su(M)$ invariant Haldane-Shastry model is related to a ‘frozen’ limit of the spin CS model [7]. So it should be encouraging to explore whether the Hamiltonian of this exactly solvable Haldane-Shastry model can also be modified through our anyon like representations of permutation operators. Moreover, it might be fruitful to investigate about various dynamical correlation functions and thermodynamic quantities of such new models in
connection with the fractional statistics. Another relevant problem is to establish the integrability of different spin CS models which are discussed in this article and find out the algebra of corresponding conserved quantities. By investigating along this line we have observed very recently that [43] a multi-parameter dependent extension of $Y(gl_N)$ Yangian [44,45], as well as its ‘nonstandard’ variants, curiously play the role of symmetry algebra for these CS models. It may be hoped that the representation theory of such extended Yangian algebra would give us some valuable insight about the degeneracy factors of the related quantum states.

Acknowledgments

It is a pleasure to thank Profs. Anjan Kundu, Sunil Mukhi and B.S. Shastry for many illuminating discussions. The author is also grateful to the Referee for his constructive suggestions which helped to improve this paper considerably.
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