Confinement and soliton solutions in the \( SL(3) \) Toda model
 coupled to matter fields

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Abstract

We consider an integrable conformally invariant two dimensional model associated to the affine Kac-Moody algebra \( \hat{\mathfrak{sl}}_3(\mathbb{C}) \). It possesses four scalar fields and six Dirac spinors. The theory does not possess a local Lagrangian since the spinor equations of motion present interaction terms which are bilinear in the spinors. There exists a submodel presenting an equivalence between a \( U(1) \) vector current and a topological current, which leads to a confinement of the spinors inside the solitons. We calculate the one-soliton and two-soliton solutions using a procedure which is a hybrid of the dressing and Hirota methods. The soliton masses and time delays due to the soliton interactions are also calculated. We give a computer program to calculate the soliton solutions.
1 Introduction

The non-perturbative aspects of Lorentz invariant field theories are related in one or another way to classical soliton solutions. The best known example is provided by the strong coupling sector of non-abelian gauge theories. It is believed that the fundamental particles, namely gluons and quarks, get confined and form bound states which should correspond to the known spectrum of hadrons. The most popular mechanism to explain such confinement is based on a condensation of magnetic monopoles producing a vacuum which is a magnetic superconductor. The quarks and gluons get then confined by the dual Meissner effect. The monopoles responsible for that mechanism are not excitations of the fundamental fields appearing in the Lagrangian. They are classical soliton solutions carrying a topological charge. There are many ideas based on Montonen-Olive duality conjecture [1] where the monopoles are interpreted as the fundamental particles of a dual Lagrangian describing the strong coupling sector of gauge theories.

The development of exact methods for studying solitons and non-perturbative aspects of field theories is important for many reasons. Besides its relevance for gauge theories, it finds applications in many areas of condensed matter physics, non-linear phenomena and fluid dynamics. In many cases, the symmetries involved allow the reduction of the number of dimensions, and integrable non-linear models in low dimensions play an important role. The soliton theory in two dimensions has reached a high degree of development. It is known how to construct practically all the soliton solutions of two dimensional models using some very basic methods. Such theories have a zero curvature representation with the potentials taking values on an affine Kac-Moody algebra. In addition, it should present some “vacuum” solutions such that the zero curvature potentials belong to an abelian subalgebra when evaluated on them. The soliton solutions are then constructed via the dressing method [2] where the dressing transformation are generated by the operators which diagonalize the adjoint action of the “oscillators” of that abelian subalgebra [3]. Some of the structures of two-dimensional integrable models can in fact be generalized to theories in higher dimensions [4].

In this paper we consider a two-dimensional field theory involving four scalar fields and six Dirac spinors. The interactions among the fields present some peculiar properties. There are the usual coupling of bilinears in the spinors to exponentials of the scalars. However, some of the equations of motion for the spinors present terms which are bilinear in the spinors themselves. That fact makes it difficult to believe that one can find a local Lagrangian for the theory. In spite of that fact, the model presents a lot of symmetries. It is conformally invariant, possesses local gauge symmetries as well as vector and axial conserved currents bilinear in the spinors. In addition, it presents three species of one-soliton solutions carrying non-trivial topological charges. One of the most striking properties of the model is that it presents a confinement mechanism of the spinors inside the solitons. For a special submodel there exists an equivalence between a $U(1)$ vector conserved current, bilinear in the spinors, and a topological currents depending only on the first derivative of some scalars. The equivalence is such that the charge density associated to that $U(1)$ current can only exist on those regions where the space derivative of the scalars are non-vanishing. If one then consider excitations of the fields around a given soliton solution, one observes the charge density can exists only inside the solitons, since the scalars tend to constants outside them. Therefore outside the solitons, the spinors can live in “white” states carrying no $U(1)$ charge. That resembles very much what one expects to happen with the confinement mechanism of QCD, even though
our model presents a bag model like confinement instead of the dual Meissner effect.

The model possesses a zero curvature representation based on the $\mathfrak{sl}_3(\mathbb{C})$ affine kac-Moody algebra. It constitutes a particular example of the so-called affine Toda models coupled to matter fields which have been introduced in paper [4]. The corresponding model associated to $\mathfrak{sl}_2(\mathbb{C})$ has been studied in paper [6] where it was shown, using bosonization methods, that the equivalence between the currents holds true at the quantum level and so the confinement mechanism does take place in the quantum theory.

The objective of the present paper is to calculate exactly the three species of one-soliton solutions as well as the corresponding six two-soliton solutions describing the scattering between any two of them. We also evaluate the time-delays for the interactions among the solitons. The character of the interactions, i.e. the relative strength and if it is attractive or repulsive, is determined by the scalar product of the roots associated to the solution. The solutions are constructed using a mixture of the dressing [2] and Hirota [7] methods. The dressing method in association to the vertex operator representations of the Kac-Moody algebra is quite good in finding the spectrum of solitons, to construct the so-called tau-functions [3], and to provide an ansatz for the solution. The Hirota method, which does not provide us any hint of the correct tau-functions, is then used to actually evaluate the explicit expression of the soliton solutions. We have implemented an algorithm in MATHEMATICA to execute the procedures of the Hirota method. In this way we escape from two difficulties of the methods. The Hirota method needs the tau-functions and an ansatz for them, but it does not tell us how to construct those. That is solved through the dressing method, which in its turn needs the evaluation of matrix elements in the vertex operator representation. We avoid that calculation through the Hirota expansion method.

The paper is organized as follows. In Section 2 we describe in detail the model, its symmetries and the zero curvature representation. In Section 3 we discuss the dressing and Hirota methods, construct the tau-functions and the corresponding Hirota’s equations. In Section 4 we discuss how to use the infinitely degenerate vacua to introduce non-trivial topological charges. Although we work with a $\mathfrak{sl}_3(\mathbb{C})$ Kac-Moody algebra, the topological charges take values in the weight lattices of three $\mathfrak{sl}_2(\mathbb{C})$ subalgebras. In Section 5 we show how to use the breaking of the conformal symmetry to evaluate the masses of the solitons just from their asymptotic behaviour. In Section 6 we show how to evaluate the time delays associated to the interactions of the solitons in a two-soliton solution. In Sections 7 and 8 the one-soliton and two-soliton solution respectively are calculated. The appendix A contains the results about Kac-Moody needed for the calculations performed in the paper, and in appendix B we present the algorithm for MATHEMATICA used to evaluate the soliton solutions.

As a matter of conventions, we use for the space-time coordinates in the two-dimensional Minkowski space the notations $x^0 = t$ and $x^1 = x$. The light front coordinates are defined as $x^\pm = x^0 \pm x^1$. Hence, we have $\partial_\pm = \frac{1}{2} (\partial_0 \pm \partial_1)$, and $\partial^2 = \partial_0^2 - \partial_1^2 = 4 \partial_+ \partial_-$. For the $\gamma$-matrices we use the following representation:

$$
\gamma_0 = -i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0 \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$
The affine Toda models coupled to matter fields have been introduced in paper [5]. In the present paper we consider the model associated to the affine Kac–Mody algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, which equations of motion are [6]

\[
\partial^2 \varphi_a = m_a^2 \bar{\psi}^a V_a \left[ \frac{1 + \gamma_5}{2} - e^3 \eta \frac{1 - \gamma_5}{2} \right] \psi^a + m_3^2 \bar{\psi} \psi_3 \left[ \frac{1 + \gamma_5}{2} - e^3 \eta \frac{1 - \gamma_5}{2} \right] \psi^3, \quad a = 1, 2, \tag{2.1}
\]

\[
\partial^2 \eta = 0, \tag{2.2}
\]

\[
i \gamma^\mu \partial_\mu \psi^i = m_i V_i \left[ \frac{1 + \gamma_5}{2} + e^3 \eta \frac{1 - \gamma_5}{2} \right] \psi^i + U^i, \quad i = 1, 2, 3, \tag{2.3}
\]

\[
i \gamma^\mu \partial_\mu \tilde{\psi}^i = m_i V_i^{-1} \left[ e^3 \eta \frac{1 + \gamma_5}{2} + \frac{1 - \gamma_5}{2} \right] \tilde{\psi}^i + \tilde{U}^i, \quad i = 1, 2, 3, \tag{2.4}
\]

where $m_3 = m_1 + m_2$, $\psi^i$ and $\tilde{\psi}^i$, $i = 1, 2, 3$, are independent Dirac spinors, and $\tilde{\psi}^i = (\psi^i)^T \gamma_0$. Notice that we can not take $\tilde{\psi}^i$ to be the complex conjugate of $\psi^i$, since that is incompatible with the equations of motion. In addition, we have

\[
V_1 = e^{i \gamma_5 (2 \varphi_1 - \varphi_2)} e^{\gamma_5 \eta}, \quad V_2 = e^{i \gamma_5 (2 \varphi_2 - \varphi_1)} e^{\gamma_5 \eta}, \quad V_3 = e^{i \gamma_5 (\varphi_1 + \varphi_2)} e^{2 \gamma_5 \eta}. \tag{2.6}
\]

Denoting the two component spinors as

\[
\psi^i = \begin{pmatrix} \psi^i_R \\ \psi^i_L \end{pmatrix}, \quad U^i = \begin{pmatrix} U^i_R \\ U^i_L \end{pmatrix}, \tag{2.7}
\]

and similarly for $\tilde{\psi}^i$ and $\tilde{U}^i$, we see that

\[
U^1_R = U^2_R = U^3_R = 0, \quad \tilde{U}^1_L = \tilde{U}^2_L = \tilde{U}^3_R = 0, \tag{2.8}
\]

and

\[
U^1_L = -\frac{m_2 m_3}{m_1} e^\eta \left[ \psi^1_R \tilde{\psi}^1_L e^{i(\varphi_2 - \varphi_1)} + \psi^3_R \tilde{\psi}^2_L e^{-i(\varphi_1 + \varphi_2)} \right], \tag{2.9}
\]

\[
U^2_L = \frac{m_1 m_3}{m_2} e^\eta \left[ \psi^2_R \tilde{\psi}^1_L e^{i(\varphi_2 - \varphi_1)} - \psi^3_R \tilde{\psi}^2_L e^{-i(\varphi_1 + \varphi_2)} \right], \tag{2.10}
\]

\[
U^3_R = \frac{m_1 m_2}{m_3} e^\eta \left[ \psi^1_L \tilde{\psi}^1_R e^{i(\varphi_2 - \varphi_1)} + \psi^2_L \tilde{\psi}^2_R e^{-i(\varphi_1 + \varphi_2)} \right], \tag{2.11}
\]

\[
\tilde{U}^1_R = -\frac{m_1 m_3}{m_2} e^\eta \left[ \psi^1_R \tilde{\psi}^3_L e^{i(\varphi_2 - \varphi_1)} + \psi^2_R \tilde{\psi}^3_L e^{-i(\varphi_1 + \varphi_2)} \right], \tag{2.12}
\]

\[
\tilde{U}^2_R = \frac{m_1 m_2}{m_3} e^\eta \left[ \psi^1_L \tilde{\psi}^3_R e^{i(\varphi_2 - \varphi_1)} - \psi^2_L \tilde{\psi}^3_R e^{-i(\varphi_1 + \varphi_2)} \right], \tag{2.13}
\]

\[
\tilde{U}^3_L = \frac{m_1 m_2}{m_3} e^\eta \left[ \psi^1_R \tilde{\psi}^2_L e^{i(\varphi_2 - \varphi_1)} - \psi^2_R \tilde{\psi}^2_L e^{-i(\varphi_1 + \varphi_2)} \right]. \tag{2.14}
\]

\[\text{1The spinors have been rescaled by a factor } \frac{1}{2\sqrt{m_i/1}} \text{ with respect to those in } [3], \text{ and the scalars } \varphi_a \text{ by a factor } i.\]
The model under consideration admits a representation in terms of the Lax-Zakharov-Shabat zero curvature condition

$$
\partial_+ A_– - \partial_- A_+ + [A_+, A_-] = 0,
$$

(2.15)

where the potentials are given by

$$
A_+ = -BF^+ B^{-1}, \quad A_- = -\partial_ B B^{-1} + F^-.
$$

(2.16)

An important ingredient in the definition of the potentials is the use of the principal gradation of $g$ (see the definition in the appendix A). The four scalar fields of the theory live in the subgroup $B$ obtained by exponentiating the zero grade subalgebra $g_0$ given in (A.24), and we parametrize it as

$$
B = e^{\sum_{\alpha=1}^5 \varphi_\alpha H^0_\alpha + \eta Q_{\text{psal}} + \nuC}.
$$

(2.17)

The model possesses six Dirac spinor fields $\psi^i$, and $\tilde{\psi}^i$, $i = 1, 2, 3$, which live in the subspaces with non-zero grades. i.e. we have

$$
F^\pm \equiv E^\pm_{\pm3} + F^\pm_1 + F^\pm_2,
$$

(2.18)

where $E_{\pm3}$ are fixed elements of the subspaces $g_{\pm3}$, and the mappings $F^\pm_1$ and $F^\pm_2$ take values in the subspaces $g_{\pm1}$ and $g_{\pm2}$ respectively. We choose the elements $E_{\pm3}$ as

$$
E_{\pm3} = \frac{1}{6} \left[ (2m_1 + m_2) H^\pm_{11} + (2m_2 + m_1) H^\pm_{21} \right],
$$

(2.19)

and use the following parametrisation for $F^\pm_1$ and $F^\pm_2$:

$$
F^+_2 = \frac{1}{2} \left[ m_3 \psi^3_R E^0_{\alpha_3} + m_1 \tilde{\psi}^1_R E^1_{\alpha_1} + m_2 \tilde{\psi}^2_R E^1_{\alpha_2} \right],
$$

(2.20)

$$
F^+_1 = \frac{1}{2} \left[ m_1 \psi^1_R E^0_{\alpha_1} + m_2 \psi^2_R E^0_{\alpha_2} + m_3 \tilde{\psi}^3_R E^1_{\alpha_3} \right],
$$

(2.21)

$$
F^-_2 = \frac{1}{2} \left[ m_1 \psi^3_L E^1_{\alpha_1} + m_2 \psi^2_L E^{1}_{\alpha_2} - m_3 \tilde{\psi}^3_L E^0_{\alpha_3} \right],
$$

(2.22)

$$
F^-_1 = \frac{1}{2} \left[ m_3 \psi^3_L E^0_{\alpha_3} - m_1 \tilde{\psi}^1_L E^1_{\alpha_1} + m_2 \tilde{\psi}^2_L E^1_{\alpha_2} \right].
$$

(2.23)

The model described by equations (2.1)-(2.3) is invariant under the conformal transformations

$$
x_+ \to f(x_+), \quad x_- \to g(x_-),
$$

(2.24)

with the fields transforming as

$$
\varphi_a(x_+, x_-) \to \hat{\varphi}_a(\hat{x}_+, \hat{x}_-) = \varphi_a(x_+, x_-),
$$

(2.25)

$$
e^{-\nu(x_+, x_-)} \to e^{-\hat{\nu}(\hat{x}_+, \hat{x}_-)} = (f')^\delta (g')^\delta e^{-\hat{\nu}(\hat{x}_+, \hat{x}_-)},
$$

(2.26)

$$
e^{-\eta(x_+, x_-)} \to e^{-\hat{\eta}(\hat{x}_+, \hat{x}_-)} = (f')^{1/3} (g')^{1/3} e^{-\hat{\eta}(\hat{x}_+, \hat{x}_-)},
$$

(2.27)

$$
\psi^i(x_+, x_-) \to \hat{\psi}^i(\hat{x}_+, \hat{x}_-) = e^{\frac{1}{2}(1+\gamma_5) \log(f') \frac{h(\alpha)}{4}} e^{\frac{1}{2}(1-\gamma_5) \log(g') \frac{h(\alpha)}{4}} \psi^i(x_+, x_-),
$$

(2.28)

$$
\tilde{\psi}^i(x_+, x_-) \to \hat{\tilde{\psi}}^i(\hat{x}_+, \hat{x}_-) = e^{\frac{1}{2}(1+\gamma_5) \log(f') \frac{h(\alpha)}{4}} e^{\frac{1}{2}(1-\gamma_5) \log(g') \frac{h(\alpha)}{4}} \tilde{\psi}^i(x_+, x_-),
$$

(2.29)
where \( h(\alpha_i) \) is the height of the root, i.e. \( h(\alpha_1) = h(\alpha_2) = 1 \) and \( h(\alpha_3) = 2 \), and where the conformal weight \( \delta \), associated to \( e^{-\nu} \), is arbitrary. The two dimensional Lorentz transformations \( x_+ \rightarrow \lambda x_+ \) and \( x_- \rightarrow x_-/\lambda \) are arbitrary. The two dimensional Lorentz transformations \( x_+ \rightarrow \lambda x_+ \) and \( x_- \rightarrow x_-/\lambda \) are contained in \((2.24)\), as one observes by choosing \( f(x_+) = \lambda x_+ \) and \( g(x_-) = x_-/\lambda \).

Since \( E_{+3} \) commutes with the Cartan subalgebra generators \( H_1 \) and \( H_2 \), it follows that equations \((2.1)-(2.5)\) are also invariant under the transformations

\[
\begin{align*}
B(x^+, x^-) & \rightarrow h_L(x^-) B(x^+, x^-) h_R(x^+), \\
F^+_m(x^+, x^-) & \rightarrow h^{-1}_R(x^+) F^+_m(x^+, x^-) h_R(x^+), \\
F^-_m(x^+, x^-) & \rightarrow h_L(x^-) F^-_m(x^+, x^-) h^{-1}_L(x^-),
\end{align*}
\]

where

\[
h_L(x^-) = e^{i(\xi_L^1(x^-) H_1 + \xi_L^2(x^-) H_2)}, \quad h_R(x^+) = e^{i(\xi_R^1(x^+) H_1 + \xi_R^2(x^+) H_2)}. \tag{2.33}
\]

That implies that the fields transform as

\[
\begin{align*}
\varphi_a & \rightarrow \varphi_a + \xi^a_L, \quad \eta \rightarrow \eta, \quad \tilde{\nu} \rightarrow \tilde{\nu}, \\
\psi^1 & \rightarrow e^{i(1+\gamma_1)(-2\xi^1_R + \xi^1_L)} e^{i(1-\gamma_3)(2\xi^1_L - \xi^1_R)} \psi^1, \\
\psi^2 & \rightarrow e^{i(1+\gamma_3)(\xi^1_R - 2\xi^1_L)} e^{i(1-\gamma_3)(-\xi^1_L + 2\xi^1_R)} \psi^2, \\
\psi^3 & \rightarrow e^{i(1+\gamma_3)(-\xi^1_L - \xi^3_L)} e^{i(1-\gamma_3)(\xi^1_L + \xi^3_L)} \psi^3.
\end{align*}
\]

and \( \tilde{\psi}^i \) transforms in the same way as \( \psi^i \) with \( \xi^a_{R,L} \) replaced by \( -\xi^a_{R,L} \).

Notice that, by taking \( \xi^a_R = -\xi^a_L = -\frac{1}{2} \theta^a \), with constant \( \theta^a \), we get a \( GL_1(\mathbb{C}) \times GL_1(\mathbb{C}) \) global symmetry, with the fields transforming as

\[
\varphi_a \rightarrow \varphi_a, \quad \eta \rightarrow \eta, \quad \tilde{\nu} \rightarrow \tilde{\nu}. \tag{2.38}
\]

and

\[
\psi^1 \rightarrow e^{i(2\theta^1 - \theta^2)} \psi^1, \quad \psi^2 \rightarrow e^{i(2\theta^2 - \theta^1)} \psi^2, \quad \psi^3 \rightarrow e^{i(\theta^1 + \theta^2)} \psi^3 \tag{2.39}
\]

with \( \tilde{\psi}^i \) transforming in the same way as \( \psi^i \) with \( \theta^a \) replaced by \( -\theta^a \).

On the other hand, if we take \( \xi^a_R = \xi^a_L = -\frac{1}{2} \chi^a \), with constant \( \chi^a \), we get a \( GL_1(\mathbb{C}) \times GL_1(\mathbb{C}) \) chiral symmetry,

\[
\varphi_a \rightarrow \varphi_a - \chi^a, \quad \eta \rightarrow \eta, \quad \tilde{\nu} \rightarrow \tilde{\nu}. \tag{2.40}
\]

and

\[
\psi^1 \rightarrow e^{i(\gamma_3(2\chi^1 - \chi^3) - \gamma_1)} \psi^1, \quad \psi^2 \rightarrow e^{i(\gamma_3(2\chi^2 - \chi^1))} \psi^2, \quad \psi^3 \rightarrow e^{i(\gamma_1 + \gamma_3)} \psi^3 \tag{2.41}
\]

with \( \tilde{\psi}^i \) transforming in the same way as \( \psi^i \) with \( \chi^a \) replaced by \( -\chi^a \).

If our theory had a Lagrangian it would follow from the Noether theorem and the above \( GL_1(\mathbb{C}) \times GL_1(\mathbb{C}) \) symmetries that it would possess two vector and two chiral conserved currents. However, a careful analysis of the equations of motion \((2.1)-(2.3)\) reveals that only half of such currents exist. Indeed, one can check that, as a consequence of \((2.1)-(2.3)\), the vector current

\[
J_\mu = i \sum_{i=1}^{3} m_i^2 \bar{\psi}^i \gamma_\mu \psi^i. \tag{2.42}
\]
and the chiral current

\begin{equation}
J_5^\mu = i \sum_{i=1}^{3} m_i^2 \bar{\psi}^i \gamma_\mu \gamma_5 \psi^i + 2 \partial_\mu (m_1 \varphi_1 + m_2 \varphi_2) \tag{2.43}
\end{equation}

are conserved

\begin{equation}
\partial^\mu J_\mu = 0, \quad \partial^\mu J_5^\mu = 0. \tag{2.44}
\end{equation}

The existence of these two conserved currents implies that the quantities

\begin{equation}
J = i \sum_{i=1}^{3} m_i^2 \bar{\psi}^i \gamma_\mu \gamma_5 \psi^i + 2 \partial_\mu (m_1 \varphi_1 + m_2 \varphi_2) \tag{2.45}
\end{equation}

and \( \bar{J} = i \sum_{i=1}^{3} m_i^2 \bar{\psi}^i \gamma_\mu \gamma_5 \psi^i - 2 \partial_\mu (m_1 \varphi_1 + m_2 \varphi_2) \tag{2.46} \)

satisfy

\begin{equation}
\partial^- \bar{J}^0 = 0, \quad \partial^+ \bar{J}^0 = 0. \tag{2.47}
\end{equation}

Indeed, the currents \( J^\mu_{\pm} = J_\mu \pm J_5^\mu \) are obviously conserved. Their light front components have the form

\begin{equation}
J_+(^+) = 2 i \sum_{i=1}^{3} m_i^2 \bar{\psi}^i \gamma_\mu \gamma_5 \psi^i + 2 \partial_\mu (m_1 \varphi_1 + m_2 \varphi_2), \quad J_-^{(+)} = 2 \partial_\mu (m_1 \varphi_1 + m_2 \varphi_2), \tag{2.48}
\end{equation}

\begin{equation}
J_+^{(-)} = -2 \partial_\mu (m_1 \varphi_1 + m_2 \varphi_2), \quad J_-^{(-)} = 2 i \sum_{i=1}^{3} m_i^2 \bar{\psi}^i \gamma_\mu \gamma_5 \psi^i - 2 \partial_\mu (m_1 \varphi_1 + m_2 \varphi_2). \tag{2.49}
\end{equation}

and the conservation laws

\begin{equation}
\partial_+ J_-^{(\pm)} + \partial_- J_+^{(\pm)} = 0 \tag{2.50}
\end{equation}

lead to relations \( \text{(2.47)} \).

One can now perform a reduction of the theory by imposing the constraints

\begin{equation}
J = 0, \quad \bar{J} = 0, \tag{2.51}
\end{equation}

which are equivalent to \( \bar{J} = 0 \)

\begin{equation}
\sum_{i=1}^{3} m_i^2 \bar{\psi}^i \gamma_\mu \gamma_5 \psi^i = -2 \epsilon^{\mu\nu} \partial_\nu (m_1 \varphi_1 + m_2 \varphi_2) \tag{2.52}
\end{equation}

Consequently, in the submodel defined by \( \text{(2.51)} \), the vector current \( \text{(2.42)} \) is proportional to a topological current. The relation \( \text{(2.52)} \) has very important consequences for the physical properties of the theory. The time component of it implies that the charge density associated to the current \( \text{(2.42)} \) is proportional to the space derivative of the \( \varphi \) fields, i.e.

\begin{equation}
J^0 = i \sum_{i=1}^{3} m_i^2 \bar{\psi}^i \gamma_0 \psi^i = -2 \partial_\varphi (m_1 \varphi_1 + m_2 \varphi_2). \tag{2.53}
\end{equation}

\^2\text{We use the normalisation } \epsilon^{01} = 1.
Consequently, there can exist charges only on those regions where the space derivative of the scalar fields are non-vanishing. As we will see below the soliton solutions of this theory are localized in space, in the sense that the scalar fields are not constant only in a region with a size determined by the soliton masses. Therefore, in the quantum theory, if we look for excitations around a soliton solution, the spinor field can live outside the solitons in configurations with charge zero only. We then have a confining mechanism which resembles that of the bag model for QCD.

3 Dressing transformations and the Hirota method

We construct the soliton solutions using a combination of the so called dressing transformation method \[2\] and the Hirota method \[7\]. Let us start with the discussions of the former one.

The fact that the potentials \(A_\pm\) in (2.16) satisfy the zero curvature (2.15) implies that they can be represented as
\[
A_\mu = -\partial_\mu T T^{-1},
\]
where \(T\) is a group element obtained by exponentiating the affine Kac-Moody algebra \(\hat{\mathfrak{sl}}_3(C)\) (see appendix \[A\]). In addition, the potentials \(A_\pm\) satisfy the so called grading condition of the Leznov–Saveliev method \[8\]. This condition states that the potentials \(A_\pm\) has some limitations on their decomposition over grading subspaces. In the case under consideration the potential \(A_-\) has only components belonging to the subspaces \(g_0, g_{-1}, g_{-2}\) and \(g_{-3}\), and the potential \(A_+\) has only components from the subspaces \(g_{+1}, g_{+2}\) and \(g_{+3}\). It can be shown \[8\] that potentials satisfying the grading condition can be represented in form (2.16). Here \(E_{\pm 3}\) are arbitrary mappings taking values in \(g_{\pm 3}\) and satisfying the conditions
\[
\partial_+ E_{-3} = 0, \quad \partial_- E_{+3} = 0.
\]
Choosing them in form (2.19) one fixes the class of equations under consideration. The dressing transformation method allows to construct gauge transformations which do not violate the grading condition and the form of \(E_{\pm 3}\). Therefore one gets new solutions to the equations of motion starting from some initial solution to these equations.

The concrete procedure to obtain desired gauge transformation is as follows. Consider a constant group elements \(\rho\) such that the element
\[
\Sigma = T \rho T^{-1}
\]
admits the generalized Gauss decomposition
\[
\Sigma = \Sigma_- \Sigma_0 \Sigma_+,
\]
where \(\Sigma_-\), \(\Sigma_0\), and \(\Sigma_+\) take values in the subgroups corresponding respectively to the subalgebras \(g_{<0}\), \(g_0\) and \(g_{>0}\) defined in appendix \[A\]. If decomposition (3.4) exists, we can introduce a new group element \(T^\rho\) related to \(T\) in two different ways:
\[
T^\rho = \Sigma_+ T = \Sigma_0^{-1} \Sigma_-^{-1} T \rho.
\]
We now define the flat connection
\[
A_\mu^\rho = -\partial_\mu T^\rho T^{-1} = \Theta_\pm A_\mu \Theta_\pm^{-1} - \partial_\mu \Theta_\pm \Theta_\pm^{-1}
\]
with 
\[ \Theta_+ = \Sigma_+, \quad \Theta_- = \Sigma_0^{-1} \Sigma_1^{-1} \equiv \Sigma_0^{-1} \tilde{\Theta}_-. \] (3.7)

We see that \( A_\mu^\rho \) and \( A_\mu \) are related by two gauge transformations; one with a group element involving only non-negative grade generators and the other only non-positive grade generators. That fact implies that \( A_\mu^\rho \) satisfies the grading condition. Moreover, it can be shown that the gauge transformations under consideration preserve the form of \( E_{\pm 3} \). Consequently, the dressing method defines transformations on the space of solutions of the model defined by the zero curvature condition \((2.15)\). Such transformations are parametrised by the constant group element \( \rho \).

A quite general procedure to construct soliton solutions in integrable theories, using the dressing transformation method, is described in paper [3]. It constitutes a generalization to practically any two-dimensional integrable hierarchy, of the so-called “solitonic specialization” in the context of the Leznov–Saveliev solution for Toda type models [9, 10, 8]. The idea is to start from a “vacuum” solution such that the connections \( A_{\pm} \), when evaluated on it, belong to an algebra of oscillators, i. e. an abelian (up to central terms) subalgebra of the Kac–Moody algebra. One then looks for the eigenvectors \( V_i \), in \( g \), of such oscillators. The solitons belong to the orbits of solutions obtained by the dressing transformation performed by elements of the form \( \rho = e^{V_1} e^{V_2} \ldots e^{V_n} \).

Notice that
\[ \tilde{\nu} = -\frac{1}{12} \sum_{i=1}^{3} m_i^2 x^+ x^-, \quad \varphi_a = \eta = \psi^i = \tilde{\psi}^i = 0, \quad a = 1, 2, \quad i = 1, 2, 3, \] (3.8)
is a “vacuum” solution of the equations of motion \((2.1)\)–\((2.3)\). The potentials \((2.16)\) evaluated on it become
\[ A_+ = -E_{+3}, \quad A_- = E_{-3} + \frac{1}{12} \sum_{i=1}^{3} m_i^2 x^+ C \] (3.9)
and so we can take the initial group element \( T \) in the form
\[ T = e^{x^+ E_{+3}} e^{-x^- E_{-3}}. \] (3.10)

Here we have used the equality
\[ [ E_{+3}, E_{-3} ] = \frac{1}{12} \sum_{i=1}^{3} m_i^2 C \] (3.11)
which follows from \((2.19)\) and \((A.12)\).

Notice that \( E_{\pm 3} \) belong to the “algebra of oscillators” generated by the elements \( H_a^n \) (see relation \((A.12)\)). That is the so-called homogeneous Heisenberg subalgebra \( \hat{sl}_3(\mathbb{C}) \) of the affine Kac–Moody algebra \( \hat{sl}_3(\mathbb{C}) \).

The eigenvectors of \( E_{\pm 3} \) are
\[ V_{\pm \alpha_i}(z) = \sum_{n=-\infty}^{\infty} z^{-n} E_{\pm \alpha_i}^n \quad i = 1, 2, 3, \] (3.12)

\(^{3}\)Recall that the group element \( T \) entering \((3.1)\) is defined up to the right multiplication by a constant element of the group.
with \( z \) being an arbitrary parameter. Indeed, one has

\[
[E_{+3}, V_{\pm\alpha_i}(z)] = \pm z^2 m_i V_{\pm\alpha_i}(z), \quad [E_{-3}, V_{\pm\alpha_i}(z)] = \pm \frac{1}{2} z \ m_i V_{\pm\alpha_i}(z).
\] (3.13)

Notice that \( V_{\alpha_i}(z) \) and \( V_{-\alpha_i}(-z) \) have the same eigenvalues. It turns out that the soliton solutions are obtained by taking the constant group element \( \rho \) to be products of exponentials of the operators

\[
V_{\alpha_i}(a_{\alpha_i}^\pm, z) = a_{\alpha_i}^+ V_{\alpha_i}(z) + a_{\alpha_i}^- V_{-\alpha_i}(-z).
\] (3.14)

We now perform the dressing transformation (3.6) starting from the vacuum potential (3.9). The transformed potential \( A_\mu^\nu \) has the structure given by relation (2.13) with \( B \) and \( F^\pm \) given by (2.17) and (2.18), therefore we have the equality

\[
b(E_{+3} + F_1^+ + F_2^+) b^{-1} = \Theta_\pm E_{+3} \Theta_\pm^{-1} + \partial_\pm \Theta_\pm^{-1},
\] (3.15)

where

\[
b = e^{i(\varphi_1 H_1 + \varphi_2 H_2)},
\] (3.16)

and the equality

\[
-i \partial_-(\varphi_1 H_1 + \varphi_2 H_2) - \partial_\nu \tilde{C} + E_{-3} + F_1^- + F_2^-
= \Theta_\pm E_{-3} \Theta_\pm^{-1} - \partial_\pm \Theta_\pm^{-1} + \frac{1}{12} \sum_{i=1}^3 m_i^2 x^+ \tilde{C}.
\] (3.17)

These equalities relate the new fields and the parameters of the dressing transformation. Notice that the \( \eta \) field appears in (2.16) multiplying the grading operator \( Q_{\text{ppal}} \), and that involves the operator \( D \) (see (A.20)). Since \( D \) is not the result of any commutator it follows that the dressing transformation method does not excite the field \( \eta \) if one starts from a solution where it vanishes.

In order to construct the solution solutions we have to split (3.15) and (3.17) into the grading subspaces. For instance, taking grade 3 component of (3.15) for \( \Theta_- \), and the grade 0 one of (3.17) for \( \Theta_- \) too, gives

\[
\Sigma_{0}^{-1} = b e^{\left(\tilde{\nu} + \frac{i}{12} \sum_{i=1}^3 m_i^2 x^+ x^-\right) C}.
\] (3.18)

Then from (3.4) and (3.10) one gets for

\[
\Sigma = e^{x^+ E_{+3} - x^- E_{-3}} | \rho e^{x^- E_{-3} - x^+ E_{+3}} = \tilde{\Theta}_-^{-1} b^{-1} \rho e^{-\left(\tilde{\nu} + \frac{i}{12} \sum_{i=1}^3 m_i^2 x^+ x^-\right) C}.
\] (3.19)

The idea now is to consider matrix elements of both sides of (3.19) in states of the fundamental representations of \( g \) to obtain the explicit space-time dependence of the fields evaluated on the solutions determined by the constant group element \( \rho \). For instance, one obtains from (3.19) that

\[
e^{-i\varphi_1} = \frac{\tau_1}{\tau_0}, \quad e^{-i\varphi_2} = \frac{\tau_2}{\tau_0}, \quad e^{-\left(\tilde{\nu} + \frac{i}{12} \sum_{i=1}^3 m_i^2 x^+ x^-\right)} = \tau_0.
\] (3.20)

with

\[
\tau_j = \langle \lambda_j | \Sigma | \lambda_j \rangle, \quad j = 0, 1, 2.
\] (3.21)
where \(|\lambda_j\rangle\) are the highest weight states of the fundamental representations of \(\mathfrak{g}\), satisfying (A.28) and (A.29), and where we have used that, as a consequence of (A.29) and (A.33),

\[
\Theta_+ |\lambda_j\rangle = |\lambda_j\rangle, \quad \langle \lambda_j | \tilde{\Theta}^{-1} = \langle \lambda_j |.
\]  

(3.22)

In order to obtain the expressions for the spinor fields we represent the mappings \(\Theta_+\) and \(\tilde{\Theta}^-\) as

\[
\Theta_+ = e^{\sum_{m>0} t^{(+m)}}, \quad \tilde{\Theta}_- = e^{\sum_{m>0} t^{(-m)}},
\]

(3.23)

where the mappings \(t^{(+m)}\) and \(t^{(-m)}\) take values in the subspaces \(\mathfrak{g}_{+m}\) and \(\mathfrak{g}_{-m}\), respectively. According to (A.29) and (A.33) for the highest weight vectors of the fundamental representations we have

\[
t^{(+m)} |\lambda_j\rangle = 0, \quad \langle \lambda_j | t^{(-m)} = 0, \quad m > 0, \quad j = 0, 1, 2.
\]

(3.24)

Considering the components of grades 1 and 2 of (3.15) for \(\Theta_-\), implies that

\[
F^+_2 = [t^{(-1)}, E_{+3}], \quad F^+_1 = [t^{(-2)}, E_{+3}] + \frac{1}{2} [t^{(-1)}, [t^{(-1)}, E_{+3}]].
\]

(3.25)

Analogously, considering the components of grades \(-1\) and \(-2\) of (3.15) for \(\Theta_+\), one gets

\[
F^-_2 = [t^{(1)}, E_{-3}]; \quad F^-_1 = [t^{(2)}, E_{-3}] + \frac{1}{2} [t^{(1)}, [t^{(1)}, E_{-3}]].
\]

(3.26)

Now, multiplying both sides of (3.19) from the left by generators of grades 1 and 2 and taking the expectation value on the states \(|\lambda_j\rangle\), one can relate the components of \(t^{(-1)}\) and \(t^{(-2)}\) to the matrix elements of \(\Sigma\). The peculiarity here is that to extract \(t^{(-2)}\) one needs to consider two matrix elements for each one of its components. By adding and subtracting the corresponding relations coming from (3.19), one obtains the components of \(t^{(-2)}\) in terms of matrix elements of \(\Sigma\), and a quadratic relation involving the components of \(t^{(-1)}\). A similar thing happens when we multiply both sides of (3.19) from the right by generators of grades \(-1\) and \(-2\) to relate \(t^{(+1)}\) and \(t^{(+2)}\) to matrix elements of \(\Sigma\). Once we have obtained the relations between \(t^{(+1)}\) and \(t^{(+2)}\) to matrix elements of \(\Sigma\), we can use (3.22), (3.24) and (2.20)–(2.23) to obtain the explicit space-time dependence of spinor fields on the solutions determined by \(\rho\). The results are

\[
\psi^1_R = \frac{m_3 t^{R_{(1)}}}{m_1 t_2} - \frac{m_2 t^{R_{(0)}}}{m_1 t_0}, \quad \psi^1_L = -\frac{\tau^3_{L_1}}{\tau_1}, \quad \psi^2_R = \frac{m_3 t^{R_{(1)}}}{m_2 t_1} - \frac{m_1 t^{R_{(0)}}}{m_2 t_0}, \quad \psi^2_L = -\frac{\tau^3_{L_2}}{\tau_2},
\]

(3.27)

\[
\psi^3_R = \frac{\tau^3_R}{\tau_0}, \quad \psi^3_L = \frac{m_2 t^{L_{(1)}}}{m_3 t_1} + \frac{m_1 t^{L_{(2)}}}{m_3 t_2},
\]

(3.28)

\[
\bar{\psi}^1_R = -\frac{\tau^3_{L_1}}{\tau_1}, \quad \bar{\psi}^1_L = \frac{m_2 t^{L_{(1)}}}{m_1 t_0} - \frac{m_1 t^{L_{(2)}}}{m_1 t_2}, \quad \bar{\psi}^2_R = -\frac{\tau^3_{L_2}}{\tau_2}, \quad \bar{\psi}^2_L = \frac{m_1 t^{L_{(1)}}}{m_2 t_0} - \frac{m_3 t^{L_{(2)}}}{m_2 t_1},
\]

(3.29)

\[
\bar{\psi}^3_R = \frac{m_2 t^{R_{(1)}}}{m_3 t_1} - \frac{m_1 t^{R_{(2)}}}{m_3 t_2}, \quad \bar{\psi}^3_L = -\frac{\tau^3_{L_3}}{\tau_0}.
\]

(3.30)
where \( \tau_j, j = 0, 1, 2, \) are defined in (3.21), and where we have denoted

\[
\begin{align*}
\tilde{\tau}_{R,2}^{(0)} &= \langle \alpha | E_1 \Sigma | \alpha \rangle, \\
\tau_{R,(0)}^{(0)} &= \langle \lambda_0 | E_1 \Sigma | \lambda_0 \rangle, \\
\tilde{\tau}_{L}^{(0)} &= \langle \lambda_1 | E_1 \Sigma | \lambda_1 \rangle, \\
\tau_{R,(0)}^{(1)} &= \langle \lambda_0 | E_1 \Sigma | \lambda_0 \rangle, \\
\tilde{\tau}_{R,(1)}^{(1)} &= \langle \lambda_1 | E_1 \Sigma | \lambda_1 \rangle,
\end{align*}
\]

(3.33)

As we mentioned above in the process of obtaining the components of \( t^{(\pm)} \) in terms of the matrix elements of \( \Sigma \) one gets quadratic relations involving the components of \( t^{(\pm)} \). These relations are given by

\[
\begin{align*}
\tilde{\tau}_{R}^{(2)} \tilde{\tau}_{R}^{(0)} + \tilde{\tau}_{R}^{(0)} \tau_{R,(0)}^{(2)} - \tau_{R,(0)}^{(2)} \tau_{R,(0)}^{(0)} &= 0, \\
\tilde{\tau}_{R}^{(0)} \tau_{R,(0)}^{(3)} - \tau_{R,(0)}^{(3)} \tau_{R,(0)}^{(0)} + \tau_{R,(0)}^{(0)} \tau_{R,(0)}^{(2)} &= 0, \\
\tilde{\tau}_{L}^{(2)} \tilde{\tau}_{L}^{(1)} + \tilde{\tau}_{L}^{(1)} \tau_{L,(1)}^{(2)} - \tau_{L,(1)}^{(2)} \tau_{L,(1)}^{(1)} &= 0, \\
\tilde{\tau}_{L}^{(0)} \tau_{L,(0)}^{(3)} - \tau_{L,(0)}^{(3)} \tau_{L,(0)}^{(0)} + \tau_{L,(0)}^{(0)} \tau_{L,(0)}^{(2)} &= 0.
\end{align*}
\]

(3.42)

According to the “solitonic specialization” [3, 4, 10, 11], the soliton solutions are obtained by taking \( \rho \) to be the product of exponentials of the operators \( (3.13) \). The one-soliton solutions are obtained by taking

\[
\rho = \rho_i = \exp[V_a(a_{\alpha i}^-, z)], \quad i = 1, 2, 3.
\]

(3.45)

Therefore, there are three species of solitons. Similarly, the two-soliton solutions are obtained by taking

\[
\rho = \rho_i \rho_j = \exp[V_a(a_{\alpha i}^+, z_1)] \exp[V_{\alpha j}(b_{\alpha j}^+, z_2)],
\]

(3.46)

and so there are six of such type of solutions. Using (3.13) one then gets from (3.19) that for the one-soliton solutions

\[
\Sigma = \Sigma_i = e^{x_+ E_{i+3}} e^{-x^+ E_{i+3}} e^{x^+ E_{i+3}} = \exp e^{g_i(z)} V_{\alpha_i}(a_{\alpha_i}^+, z),
\]

(3.47)

and for the two-soliton solutions

\[
\Sigma = \Sigma_{ij} = \Sigma_i \Sigma_j = e^{x_+ E_{i+3}} e^{-x^+ E_{i+3}} e^{x^+ E_{i+3}} e^{-x^+ E_{i+3}}
\]

\[
= \exp e^{g_i(z_1)} V_{\alpha_i}(a_{\alpha_i}^+, z_1) \exp e^{g_j(z_2)} V_{\alpha_j}(b_{\alpha_j}^+, z_2),
\]

(3.48)

where

\[
\Gamma_i(z) = \frac{1}{2} m_i \left( \frac{z}{x^+ - \frac{x^-}{z}} \right) = \gamma (x - vt)
\]

(3.49)

with

\[
\gamma = \frac{1}{2} m_i \left( z + \frac{1}{z} \right) = (\text{sign } z) \frac{m_i}{\sqrt{1 - v^2}} = \epsilon m_i \cosh \theta; \quad v = \frac{z^2 - 1}{z^2 + 1} = \tanh \theta.
\]

(3.50)
where we have introduced the rapidity $\theta$ as $z = \epsilon e^{i\theta}$, with $\epsilon = \pm 1$. Therefore, if $z$ is real we have $|v| < 1$, where the speed of light has been normalized to unity. A generalization to the $n$-soliton case is evident.

In order to obtain the final expressions for one-soliton and two-soliton solutions one has to evaluate the matrix elements in (3.21) and (3.33)–(3.41) for the group elements (3.47) and (3.48). That calculation can be performed using vertex operator representations for $g^{12, 11}$. We have to use in fact the homogeneous vertex operator realization of the three fundamental representations of $g$. It turns out that such representations are integrable in the sense that the step operators of $g$ are nilpotent $^{11}$. Indeed, one can verify that the operators (3.12) satisfy $^{12, 11} \hspace{1cm} V_{\alpha_{i}}(z_{1}) V_{\alpha_{i}}(z_{2}) \to 0 \hspace{0.5cm} \text{as} \hspace{0.5cm} z_{1} \to z_{2}$. (3.51)

That implies that the matrix elements in (3.21) and (3.33)–(3.41) truncate when one expands the exponentials in (3.47) and (3.48), and consequently they become polynomials in the $\Gamma_{i}$'s. In this sense, those matrix elements are the Hirota tau-functions $^{7}$ for the model described by equations (2.1)–(2.5). Note that due to (3.51) the maximal power of each $\epsilon^{\Gamma_{i}}$ in the expansion of a tau-function over them is equal to 2. Therefore, a general tau-function for an $n$-soliton solution has the form

$$
\tau = \sum_{p_{1}, \ldots, p_{n}=0}^{2} c_{p_{1}, \ldots, p_{n}} \exp \left[ p_{1} \Gamma_{i_{1}}(z_{1}) + \cdots + p_{n} \Gamma_{i_{n}}(z_{n}) \right].
$$

(3.52)

The connection between dressing transformations, solitonic specialization and the Hirota method has been discussed in $^{3}$ for any hierarchy of integrable models possessing a zero curvature representation in terms of an affine Kac–Moody algebra. It is a quite general result that explains a lot of the structures known in soliton theory. In addition, on the practical side it allows one to use the good features of each method.

The Hirota method is quite powerful to actually evaluate the explicit expression for the soliton solutions. Besides, being a recursive method it allows a simple implementation on a computer program for algebraic manipulation like Mathematica. However, one of its difficulty is to find the relation between the tau-functions and the fields that lead to the truncation of the Hirota expansion. But the dressing transformation method together with the solitonic specialization does exactly that.

The dressing method on the other hand, requires the evaluation of matrix elements in vertex operator representations of the Kac-Moody algebra, which in the case of higher soliton solutions become extremely tedious. One can take advantage of those features to speed up calculations.

To use the Hirota method we have to find the equations satisfied by tau-functions. They are determined by substituting the relations between the fields and tau-functions (3.20) and (3.27)–(3.32) into the equations of motion (2.1)–(2.5). Notice that, except for $\eta$, we have 15 fields in the model (2.1)–(2.5) (three scalars and six two-component spinors). However, in (3.20) and (3.27)–(3.32) we have introduced 21 tau-functions. The six equations missing are the algebraic relations (3.42)–(3.44) provided by the dressing method. Substituting (3.20) and (3.27)–(3.32) into (2.1)–(2.5) one gets that the equations for $\varphi_{a}$ and $\tilde{\nu}$ lead to the following three Hirota equations

$$
4 \tau_{1} \tau_{2} (\tau_{0} \partial_{+} \partial_{-} \tau_{0} - \partial_{-} \tau_{0} \partial_{+} \tau_{0}) = m_{1}^{2} \tau_{0} \tau_{1} \tilde{\tau}_{R}^{a_{1}} \tau_{L}^{a_{1}} + m_{2}^{2} \tau_{0} \tau_{2} \tilde{\tau}_{R}^{a_{2}} \tau_{L}^{a_{2}}
$$

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The components of the equations of motion (2.1)–(2.5) for the quadratic terms
\[
\begin{align*}
- m_2^2 \tau_2^2 r_{L(1)}^0 & \tilde{\tau}_{R(1)}^3 - m_1 m_2 \tau_1 \tau_2 r_{L(2)}^0 \tilde{\tau}_{R(1)}^3, \\
- m_1 m_2 \tau_1 \tau_2 r_{L(1)}^0 & \tilde{\tau}_{R(2)}^3 - m_1 \tau_1 \tau_2 r_{L(2)}^0 \tilde{\tau}_{R(2)}^3, \\
4 \tau_0 \tau_2 (\tau_1 \partial_+ \partial_- \tau_1 - \partial_- \tau_1 \partial_+ \tau_1) &= m_2^2 \tau_1 \tau_2 \tilde{\tau}_{R}^0 \tilde{\tau}_{L}^0 + m_3^2 \tau_0 \tau_1 \tau_2 \tilde{\tau}_{L}^0 \tilde{\tau}_{R}^0, \\
- m_2^3 \tau_0 \tau_2 \tilde{\tau}_{L(1)}^0 \tilde{\tau}_{R(0)}^1 & + m_2^2 \tau_2 \tilde{\tau}_{L(0)}^0 \tilde{\tau}_{R(0)}^0 + m_2^2 \tau_2 \tilde{\tau}_{L(0)}^1 \tilde{\tau}_{R(0)}^1, \\
+ m_2^2 \tau_0 \tau_2 \tilde{\tau}_{L(2)}^0 & + m_2 \tau_1 \tau_2 \tilde{\tau}_{L(2)}^0 \tilde{\tau}_{R(2)}^0, \\
4 \tau_0 \tau_1 (\tau_2 \partial_+ \partial_- \tau_2 - \partial_- \tau_2 \partial_+ \tau_2) &= m_1^2 \tau_1 \tau_2 \tilde{\tau}_{R}^1 \tilde{\tau}_{L}^0 + m_3^2 \tau_0 \tau_2 \tau_3 \tilde{\tau}_{L}^0 \tilde{\tau}_{R}^0, \\
- m_1 m_3 \tau_0 \tau_1 \tau_2 \tilde{\tau}_{L(1)}^0 \tilde{\tau}_{R(0)}^1 & + m_1^2 \tau_2 \tilde{\tau}_{L(0)}^0 \tilde{\tau}_{R(0)}^1 + m_1 \tau_1 \tau_2 \tilde{\tau}_{L(0)}^1 \tilde{\tau}_{R(0)}^1, \\
+ m_2 \tau_0 \tau_2 \tilde{\tau}_{L(2)}^1 & + m_2 \tau_2 \tilde{\tau}_{L(2)}^0 \tilde{\tau}_{R(2)}^1, \\
(3.53) \end{align*}
\]

The components of the equations of motion (2.1)–(2.3) for \(\psi\)'s and \(\tilde{\psi}\)'s that do not involve the quadratic terms \(U^i\) and \(\tilde{U}^i\) (see (2.3)) lead to the Hirota equations
\[
\begin{align*}
2 (\tau_1 \partial_+ \tau_{L(1)}^0 - \tau_{R(1)}^0 \partial_+ \tau_1) &= m_3 \tau_0 \tau_{R(2)}^0 - m_2 \tau_1 \tau_{R(0)}^0, \\
(3.56) \\
2 (\tau_2 \partial_+ \tau_{L(2)}^0 - \tau_{R(2)}^0 \partial_+ \tau_2) &= m_3 \tau_0 \tau_{R(1)}^0 - m_1 \tau_1 \tau_{R(0)}^0, \\
(3.57) \\
2 (\tau_0 \partial_- \tau_{R(1)}^0 - \tau_{L(1)}^0 \partial_- \tau_0) &= m_2 \tau_2 \tau_{L(1)}^0 + m_1 \tau_1 \tau_{L(2)}^0, \\
(3.58) \\
2 (\tau_1 \partial_- \tau_{R(0)}^0 - \tau_{L(2)}^0 \partial_- \tau_1) &= m_3 \tau_0 \tau_{L(2)}^0 + m_2 \tau_2 \tau_{L(0)}^0, \\
(3.59) \\
2 (\tau_2 \partial_- \tau_{R(0)}^0 - \tau_{L(0)}^0 \partial_- \tau_2) &= m_3 \tau_0 \tau_{L(1)}^0 + m_1 \tau_1 \tau_{L(0)}^0, \\
(3.60) \\
2 (\tau_0 \partial_+ \tau_{L(1)}^0 - \tau_{L(0)}^0 \partial_+ \tau_0) &= -m_2 \tau_2 \tau_{R(1)}^0 - m_1 \tau_1 \tau_{R(2)}^0, \\
(3.61) 
\end{align*}
\]

and the other components of the equations (2.1)–(2.3) for \(\psi\)'s and \(\tilde{\psi}\)'s involving the quadratic terms \(U^i\) and \(\tilde{U}^i\) lead to
\[
\begin{align*}
2 m_1 \tau_1^2 (\tau_2 \partial_+ \tau_{L(2)}^0 - \tau_{R(2)}^0 \partial_+ \tau_2) + 2 m_2 \tau_2^2 (\tau_1 \partial_+ \tau_{L(1)}^0 - \tau_{R(1)}^0 \partial_+ \tau_1) &= m_2^2 \tau_1 \tau_{L(2)}^0 \tilde{\tau}_{R(0)}^0 - m_2^2 \tau_0 \tau_1 \tau_2 \tilde{\tau}_{L(0)}^0 \tilde{\tau}_{R(0)}^0 - m_2 \tau_2 \tau_1 \tau_2 \tilde{\tau}_{L(2)}^0 \tilde{\tau}_{R(0)}^0, \\
(3.62) \\
2 m_2 \tau_2^2 (\tau_0 \partial_- \tau_{R(0)}^0 - \tau_{L(0)}^0 \partial_- \tau_0) + 2 m_3 \tau_0^2 (\tau_{R(2)}^0 \partial_- \tau_2 - \tau_2 \partial_- \tau_{R(2)}^0) &= m_3^2 \tau_0 \tau_{L(0)}^0 \tilde{\tau}_{R(2)}^0 - m_3^2 \tau_0 \tau_1 \tau_2 \tilde{\tau}_{L(2)}^0 \tilde{\tau}_{R(0)}^0 + m_3 \tau_1 \tau_2 \tilde{\tau}_{L(0)}^0 \tilde{\tau}_{R(2)}^0, \\
- m_2^2 \tau_0 \tau_2 \tilde{\tau}_{L(1)}^0 & + m_1 m_3 \tau_0 \tau_1 \tau_2 \tilde{\tau}_{L(1)}^0 \tilde{\tau}_{R(0)}^0, \\
- m_2 \tau_2 \tilde{\tau}_{L(0)}^0 & + m_2 \tau_2 \tilde{\tau}_{L(2)}^0 \tilde{\tau}_{R(0)}^0, \\
2 m_1 \tau_1^2 (\tau_0 \partial_- \tau_{R(0)}^0 - \tau_{L(0)}^0 \partial_- \tau_0) - 2 m_3 \tau_0^2 (\tau_{R(1)}^0 \partial_- \tau_1 - \tau_{R(1)}^0 \partial_- \tau_1) &= m_1 \tau_1 \tau_{R(1)}^0 \tilde{\tau}_{L(2)}^0 + m_1 m_3 \tau_0 \tau_1 \tau_2 \tilde{\tau}_{L(2)}^0 \tilde{\tau}_{R(0)}^0, \\
- m_1 m_2 \tau_1 \tau_2 \tilde{\tau}_{L(1)}^0 & + m_1 m_2 \tau_1 \tau_2 \tilde{\tau}_{L(2)}^0 \tilde{\tau}_{R(0)}^1, \\
2 m_2 \tau_2^2 (\tau_0 \partial_+ \tau_{L(0)}^0 - \tau_{R(0)}^0 \partial_+ \tau_0) + m_3 \tau_0^2 (\tau_{R(0)}^0 \partial_+ \tau_2 - \tau_2 \partial_+ \tau_{R(0)}^0) &= m_2 \tau_2 \tau_{L(0)}^0 \tilde{\tau}_{R(2)}^0 + m_2 \tau_2 \tau_{L(0)}^0 \tilde{\tau}_{R(2)}^0, \\
+ m_2 \tau_2 \tilde{\tau}_{L(2)}^0 & + m_2 \tau_2 \tilde{\tau}_{L(0)}^0 \tilde{\tau}_{R(2)}^0, \\
2 m_1 \tau_1^2 (\tau_0 \partial_+ \tau_{L(0)}^0 - \tau_{R(0)}^0 \partial_+ \tau_0) + m_3 \tau_0^2 (\tau_{R(0)}^0 \partial_+ \tau_1 - \tau_{R(0)}^0 \partial_+ \tau_1) &= m_1 \tau_1 \tau_{L(0)}^0 \tilde{\tau}_{R(1)}^0 + m_1 m_3 \tau_0 \tau_2 \tilde{\tau}_{L(1)}^0 \tilde{\tau}_{R(0)}^0, \\
- m_1 m_2 \tau_1 \tau_2 \tilde{\tau}_{L(1)}^0 & - m_1 m_2 \tau_1 \tau_2 \tilde{\tau}_{L(0)}^0 \tilde{\tau}_{R(2)}^0, \\
- m_1 m_2 \tau_1 \tau_2 \tilde{\tau}_{L(1)}^0 & - m_1 m_2 \tau_1 \tau_2 \tilde{\tau}_{L(0)}^0 \tilde{\tau}_{R(2)}^0, \\
(3.64) \end{align*}
\]

13
\[2 m_1 \tau_1^2 (\tau_2 \partial_+ \tilde{\tau}_{R,(2)}^\alpha - \tilde{\tau}_{R,(2)}^\alpha \partial_+ \tau_2) + m_2 \tau_2^2 (\tau_1 \partial_- \tilde{\tau}_{R,(1)}^\alpha - \tilde{\tau}_{R,(1)}^\alpha \partial_- \tau_1)\]
\[= m_3^2 \tau_1 \tau_2 \tilde{\tau}_{L}^\alpha + m_2 m_3 \tau_0 \tau_2 \tilde{\tau}_{L,(2)}^\alpha - m_2^2 \tau_2 \tilde{\tau}_{L,(0)}^\alpha \]
\[- m_1 m_3 \tau_1 \tau_2 \tilde{\tau}_{L,(1)}^\alpha + m_1 \tau_1 \tilde{\tau}_{R,(0)}^\alpha \tilde{\tau}_{L,(0)}^\alpha.\]  

(3.67)

The equivalence between the vector and topological currents (2.32) in terms of the tau-
functions become

\[2 m_1 \tau_2 (\tau_1 \partial_+ \tau_0 - \tau_0 \partial_+ \tau_1) + 2 m_2 \tau_1 (\tau_2 \partial_+ \tau_0 - \tau_0 \partial_+ \tau_2)\]
\[= m_1 m_2 \tau_2 \tau_{R,(0)}^\alpha \tilde{\tau}_{R}^\alpha - m_1 m_3 \tau_0 \tau_{R,(2)}^\alpha \tilde{\tau}_{R}^\alpha + m_1 m_2 \tau_1 \tau_{R,(0)}^\alpha \tilde{\tau}_{R}^\alpha\]
\[= m_2 m_3 \tau_0 \tau_{L,(1)}^\alpha \tilde{\tau}_{L}^\alpha + m_1 m_3 \tau_1 \tau_{L,(2)}^\alpha \tilde{\tau}_{L}^\alpha + m_1 m_2 \tau_2 \tau_{L,(0)}^\alpha \tilde{\tau}_{L}^\alpha\]
\[- m_1 m_3 \tau_0 \tau_{L,(1)}^\alpha \tilde{\tau}_{L,(0)}^\alpha + m_1 \tau_1 \tilde{\tau}_{R,(0)}^\alpha \tilde{\tau}_{L,(0)}^\alpha.\]  

(3.68)

(3.69)

Resuming we can say that the soliton solution can then be obtained by evaluating the matrix elements (3.21) and (3.33)–(3.41) in the homogeneous vertex operator realization of the three fundamental representations of the affine Kac-Moody algebra \(\mathfrak{sl}_3(\mathbb{C})\). Alternatively one can obtain the same solutions by using the Hirota method. In such case one uses the ansatz (3.52) for the tau-function provided by the dressing method. The coefficients can be determined recursively using a program for algebraic manipulation. In appendix we give the code of a MATHEMATICA program implementing the Hirota method. The explicit soliton solutions are given in Sections 7 and 8.

4 Topological charge

In \((1 + 1)\) dimensions the topological current is defined as \(J_\mu^{\text{top}} \sim \epsilon_{\mu\nu} \partial^\nu \Phi\), where \(\Phi\) is some Lorentz scalar quantity. The reason is that if \(\Phi\) does not present singularities, the space-time derivatives acting on it commute and therefore \(J_\mu^{\text{top}}\) is conserved independently of the equations of motion, i.e. \(\partial^\mu J_\mu^{\text{top}} = 0\). The topological charge is then \(Q^{\text{top}} = \int dx J_0^{\text{top}} \sim \Phi(+\infty) - \Phi(-\infty)\). For finite energy solutions the fields have to approach a vacuum configuration for \(x \to \pm \infty\), and therefore the introduction of a topological current only makes sense in theories with degenerate vacua. Our theory (2.1)–(2.5) has four scalars fields from which we can build topological charges. However, \(\eta\) is a free field and its vacuum configuration do not lead to interesting charges. The field \(\tilde{\nu}\) is not suitable either because one needs \(\tilde{\nu} \to -\frac{1}{12} \sum_{i=1}^3 m_i^2 x^+ x^-\) as \(x \to \pm \infty\). Therefore, we are left with the two scalars \(\varphi_a\), \(a = 1, 2\). Then, following the approach adopted in the abelian affine Toda models we introduce the quantity

\[\varphi = \sum_{a=1}^2 \frac{2 \alpha_a}{\alpha_a^2} \varphi_a,\]  

(4.1)

where \(\alpha_a\), \(a = 1, 2\), are the simple roots of \(\mathfrak{sl}_3(\mathbb{C})\) (see appendix 1). We then have that \(\varphi_a = (\varphi|\lambda_a)\), where \(\lambda_a\) are the fundamental weights of \(\mathfrak{sl}_3(\mathbb{C})\) defined by the relation

\[2 \frac{(\alpha_a|\lambda_b)}{(\alpha_a|\alpha_a)} = \delta_{ab}.\]  

(4.2)
One observes from (2.9)–(2.6) that the fields $\varphi_a$ enter into the equations of motion through the combinations $e^{\pm i\langle \varphi|\alpha_j\rangle}$, $j = 1, 2, 3$, which are invariant under the transformations

$$\varphi \rightarrow \varphi + 2\pi \mu$$

(4.3)

where $\mu$ is any vector on the root space satisfying the condition $\langle \alpha_i|\mu \rangle \in \mathbb{Z}$. Such vectors are the so-called weights of $\mathfrak{sl}_3(\mathbb{C})$ and they constitute an infinite discrete lattice called the weight lattice [13]. In the case of the abelian affine Toda models the vacua of the theory is determined by such weight lattice. However, in the model described by equations (2.1)–(2.5) one notices that for any constants $\varphi_a^{(0)}$ and $\eta^{(0)}$

$$\psi^i = \tilde{\psi}^i = 0, \quad \eta = \eta^{(0)}, \quad \varphi_a = \varphi_a^{(0)}, \quad \nu = -\frac{1}{12} \sum_{i=1}^{3} m_i^2 x^+ x^-$$

(4.4)

is a vacuum configuration. Consequently, the vacuum configurations of the $\varphi_a$ fields are not determined by the weight lattice of $\mathfrak{sl}_3(\mathbb{C})$. What we will see below is that the topological charges of the one-soliton solutions of (2.1)–(2.3) lie in fact on the weight lattice of the three $\mathfrak{sl}_2(\mathbb{C})$ subalgebras associated to the three positive roots of $\mathfrak{sl}_3(\mathbb{C})$.

We shall define the topological current and charge as

$$J_{\mu}^{\text{top}} = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial^{\nu} \varphi, \quad Q^{\text{top}} = \int dx \ J_{0}^{\text{top}} = \frac{1}{2\pi} [\varphi(+\infty) - \varphi(-\infty)]$$

(4.5)

In terms of the tau-functions introduced in (3.20) one can express the topological charge as

$$Q^{\text{top}} = \frac{i}{2\pi} \sum_{a=1}^{2} \frac{2 \alpha_a}{\alpha_a^2} \ln \left| \frac{\tau_a}{\tau_0} \right|^{\pm\infty}_{-\infty}.$$ 

(4.6)

The relation (3.20) implies that $-i \varphi_a = \ln |\tau_a/\tau_0| + i \arg(\tau_a/\tau_0)$. Therefore, in order to have real solutions for the fields $\varphi_a$, one needs $|\tau_a| = |\tau_0|$. Given the solutions for the tau-functions, the relation (3.20) determines $\varphi_a$ only modulo $2\pi$. Therefore, the topological charges are defined up to an integer linear combination of the simple roots $\alpha_a$ (recall that we use the normalization $\alpha_a^2 = 2$). When evaluating the charges we should then use the prescription that the arguments of the tau-functions lie between $\pi$ and $-\pi$.

Notice that the topological current in (2.52), which is equivalent to the spinor vector current is the projection of (4.5) onto the vector $-4\pi (m_1 \lambda_1 + m_2 \lambda_2)$.

### 5 Masses of fundamental particles and solitons

The model described by equations (2.1)–(2.3) is conformally invariant. It means, in particular, that if one has a solution of the equations with some finite mass, then by a scale transformation it can be continuously transformed to a solution with any other mass. Hence to have stable solutions we have to break the conformal invariance. The simplest way to do this is to freeze the field $\eta$. This fixes also the masses of the fundamental fields. Indeed, taking, for example, $\eta = 0$, and considering the linear part of the equations (2.1)–(2.5) one observes that the masses of the fundamental particles are (see section 7 of [5])

$$m_{\psi^i} = m_{\tilde{\psi}^i} = m_i, \quad m_{\varphi_a} = m_\nu = 0.$$ 

(5.1)
The mass of a soliton is defined to be its rest frame energy. The energy is measured by the Hamiltonian or the energy-momentum tensor. The model under consideration does not possess a Lagrangian and so we cannot calculate the canonical energy-momentum tensor. However, in reference [5] it was shown that our model belongs to a class of theories obtained by the Hamiltonian reduction from the two-loop WZNW model [14, 15]. The energy-momentum tensor $L_{\mu\nu}$, of the two-loop WZNW model is of the Sugawara form, i.e. quadratic in the conserved currents. We take the energy-momentum tensor of our model to be the reduced form of $L_{\mu\nu}$ under the Hamiltonian reduction mentioned above. Denote that by $L_{\mu\nu}^{\text{red}}$. Such tensor is conserved and traceless, i.e.

$$\partial^\mu L_{\mu\nu}^{\text{red}} = 0, \quad L_{\mu}^{\text{red}\mu} = 0.$$ (5.2)

However, according to the arguments given in [16, 5] the masses measured by such tensor must vanish, since it corresponds to a conformally invariant field theory. On the other hand, the conformal symmetry can be broken by freezing the $\eta$ field to a constant value. We define the energy momentum tensor of such theory with a broken conformal symmetry by

$$\Theta_{\mu\nu}^{\text{broken}} \equiv \Theta_{\mu\nu} |_{\eta=0},$$ (5.3)

where

$$\Theta_{\mu\nu} = L_{\mu\nu}^{\text{red}} + S_{\mu\nu}$$ (5.4)

with

$$S_{\mu\nu} = \kappa (\partial_\mu \partial_\nu - g_{\mu\nu} \partial^2) \left[ \tilde{\nu} + \frac{i}{3} (\varphi_1 + \varphi_2) \right]$$ (5.5)

and $\kappa$ being the coupling constant of the two-loop WZNW model [3, 14, 15]. The tensor $\Theta_{\mu\nu}$ is conserved but not traceless. We define the soliton masses to be the space integral of $\Theta_{00}^{\text{broken}}$ on the soliton rest frame.

Therefore, using (5.4) and (5.3), we obtain the following expression for the mass $M$ of the soliton moving with the velocity $v$:

$$\frac{M}{\sqrt{1 - v^2}} = -\int_{-\infty}^{\infty} dx \left( \Theta_{00}^{\text{broken}} - E_{\text{vac}} \right)$$

$$= -\kappa \partial_x \left[ \tilde{\nu} + \frac{i}{3} (\varphi_1 + \varphi_2) + \frac{x^+x^-}{12} \sum_{i=1}^{3} m_i^2 \right] \bigg|_{-\infty}^{+\infty} = \frac{\kappa}{3} \partial_x \sum_{j=0}^{2} \ln \tau_j \bigg|_{-\infty}^{+\infty}. \quad (5.6)$$

Here we have used relation (3.20), and the fact that the integral of $L_{\mu\nu}^{\text{red}}$ vanishes, as discussed above, due to the conformal symmetry. The term $E_{\text{vac}}$ is due to the nonzero vacuum energy density, associated to the fact that the vacuum configuration of the field $\tilde{\nu}$ is $-(1/12) \sum_{i=1}^{3} m_i^2 x^+x^-$. Thus, the soliton mass is determined by the asymptotic behaviour of the solution.

6 Time delays

A soliton is a classical solution that travels with a constant speed without dispersion and keeps its form under scattering on other solitons. The effect of scattering is only a phase
shift or a displacement in the soliton position. Now we will show that the solitons we have been working with are true solitons in this sense and calculate the so-called time delays of the scattering of two solitons, using the procedure elaborated in papers \[17, 6\].

Consider two solitons that in the distant past are well apart, then collide near \( t = 0 \) and then separate again in the distant future. Therefore, except for the region where the scattering occurs, the solitons are free and so travel with constant velocities. Let us write the trajectories of one of the solitons before and after the collision as

\[
x = vt + x(I) \quad \text{and} \quad x = vt + x(F),
\]

since the velocity is the same. The lateral displacement at fixed time is measured by

\[
\Delta(x) = x(F) - x(I),
\]

and the time delay is defined as

\[
\Delta(t) = t(F) - t(I) = -\frac{\Delta(x)}{v}
\]

with the intercepts of the trajectories with the time axis being given by \( t(F) = -x(F)/v \) and \( t(I) = -x(I)/v \). The lateral displacement and the time delay are not Lorentz invariant, and so we consider the invariant

\[
E\Delta(x) = -p\Delta(t)
\]

with \( E \) and \( p = vE \) being the energy and momentum of the soliton respectively. Since \( E \) is positive it follows that \( \Delta(x) \) has the same sign in any reference frame, with only its value being frame dependent. The time delay on the other hand may change the sign under Lorentz transformations. One can show that the lateral displacements and time delays for the two solitons participating in the collision have to satisfy \[17, 6\]

\[
E_1\Delta_1(x) + E_2\Delta_2(x) = 0, \quad p_1\Delta_1(t) + p_2\Delta_2(t) = 0
\]

where the index \( i \) labels the quantities associated to the \( i \)-th particle. Notice therefore that, since the energies are positive, the lateral displacements have opposite signs. Clearly, in the center of momentum frame where \( p_1 + p_2 = 0 \), the time delays are equal, i.e. \( \Delta_1^{cm}(t) = \Delta_2^{cm}(t) \). Therefore, from (6.4) we have that \( E_1^{cm}\Delta_1^{cm}(x)/p_1 = -E_2^{cm}\Delta_2^{cm}(x)/p_1 = -\Delta_1^{cm}(t) = -\Delta_2^{cm}(t) \). Consequently, since \( \Delta(x) \) has the same sign in any frame, it follows that, if the first particle moves to the right faster then the second particle (so that \( p_1 \) is positive in the center of momentum frame), then \( -\Delta_1(x) \), \( \Delta_2(x) \), \( \Delta_1^{cm}(t) \) and \( \Delta_2^{cm}(t) \) all have the same sign. The physical interpretation of that sign is related to the character (attractive or repulsive) of the interaction forces. Indeed, if the force is attractive then the first particle will accelerate as it approaches the second particle and then decelerate. That means that \( \Delta_1(x) \) is positive and so the common sign negative. Therefore, attractive forces lead to a negative time delay in the center of momentum frame, and clearly repulsive forces lead to a positive time delay. These considerations assume that the two particles pass through each other, and there is no reflection. However, when the masses of the two particles are equal there is the possibility of occurring reflection.

In order to evaluate the time delay we choose the first particle to move faster to the right than the second particle, i.e. \( v_1 > v_2 \), with \( v_1 > 0 \), and therefore from (3.50) \( |z_1| > |z_2| \) or
\[ \theta_1 > \theta_2, \] where we have denoted \( z_a = e_a e^{\theta_a} \) with \( e_a = \pm 1 \). Let us track the first soliton in time \(|7|\), i.e. hold \( x - v_1 t \) fixed as time varies. One gets

\[ x - v_2 t = x - v_1 t + (v_1 - v_2) t. \] (6.6)

We then get from (3.49) that, if \( \epsilon_2 = 1, \) \( e^{\Gamma(z_2)} \to 0 \) as \( t \to -\infty \), and \( e^{\Gamma(z_2)} \to \infty \) as \( t \to \infty \) \((\Gamma \) stands for one of the three \( \Gamma_i \) of (3.49), corresponding to the second soliton). For the case \( \epsilon_2 = -1 \) the limits get interchanged.

As we will see in section \( \| \), taking \( \epsilon_2 = 1 \), all the two-soliton solutions become, in the limit \( t \to -\infty \), a one-soliton solution. Now, in the limit \( t \to \infty \), the two-solitons also become a one-soliton but with the replacement

\[ e^{\Gamma(z_1)} \to \left( \frac{z_1 - \omega z_2}{z_1 + \omega z_2} \right)^n e^{\Gamma(z_1)}, \] (6.7)

where \( n \) is either 1 or 2, and \( \omega = \pm 1 \), depending upon the type of two-soliton solution under consideration. Therefore, the relevant effect of the scattering on the solutions is a lateral displacement of the first soliton given by

\[ \gamma_1 (x - v_1 t) \to \gamma_1 \left[ x - v_1 t + \frac{1}{\gamma_1} \ln \left( \frac{z_1 - \omega z_2}{z_1 + \omega z_2} \right)^n \right]. \] (6.8)

If one takes \( \epsilon_2 = -1 \) instead, one observes that the direction of the arrow in (6.8) reverses. Therefore, using (6.3) one sees that the lateral displacement for the first soliton is given by

\[ \Delta_1(x) = -\frac{\epsilon_2}{\gamma_1} \ln \left( \frac{z_1 - \omega z_2}{z_1 + \omega z_2} \right)^n = -\frac{\epsilon_1 \epsilon_2}{m \cosh \theta_1} \ln \left( \frac{e^{(\theta_1 - \theta_2)/2} - \epsilon_1 \epsilon_2 \omega e^{-(\theta_1 - \theta_2)/2}}{e^{(\theta_1 - \theta_2)/2} + \epsilon_1 \epsilon_2 \omega e^{-(\theta_1 - \theta_2)/2}} \right)^n, \] (6.9)

where \( m \) stands for one of the masses \( m_i, i = 1, 2, 3 \), corresponding to the first soliton. Since \( \epsilon_1 \epsilon_2 \omega = \pm 1 \), one observes that \( \Delta_1(x) \) can in fact be written as

\[ \Delta_1(x) = -\frac{\omega n}{m \cosh \theta_1} \ln \left[ \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \right]. \] (6.10)

Using (6.3) and (6.10), one gets that the time delay is given by (assuming \( v_1 > v_2 \), with \( v_1 > 0 \))

\[ \Delta_1(t) = \frac{\omega n}{m \sinh \theta_1} \ln \left[ \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \right]. \] (6.11)

Notice that we have taken \( \theta_1 > \theta_2 \) and so the hyperbolic tangent can vary from 0 to 1 and therefore its logarithm is always negative, and so from (6.10) one sees that \( \Delta_1(x) \) for \( v_1 > v_2 \), with \( v_1 > 0 \), has the same sign as \( \omega \). Therefore, from the above considerations we conclude that the forces between the solitons are attractive for \( \omega = 1 \), and repulsive for \( \omega = -1 \). In section \( \| \) we discuss the details of each type of two-soliton solution.

### 7 One-soliton solutions

As we have discussed in section \( \| \) the one-soliton solutions are obtained by taking the constant group element \( \rho \), parametrising the dressing transformations, as in (3.45). Therefore, we have
a species of solitons for each positive root of \( \mathfrak{sl}_3(\mathbb{C}) \). We give below the results obtained by either evaluating the matrix elements (3.21) and (3.33)-(3.41), or by using the Hirota method (see program in appendix B). As we show below the one-soliton solution associated to a given root \( \alpha_i \) excite only the tau-functions (or fields) associated to the same root. Therefore, each species of solitons belongs to one of the three \( \mathfrak{sl}_2(\mathbb{C}) \) subalgebras associated to the positive roots of \( \mathfrak{sl}_3(\mathbb{C}) \).

We have checked that the three one-soliton and six two-soliton solutions constructed in this paper satisfy the equivalence between the vector and topological currents given by (2.52), or equivalently (3.68) and (3.69). Therefore, they are solutions of the submodel defined by the constraints (2.51), and that presents the confinement of the spinor charge inside the solitons.

### 7.1 One soliton of species \( \alpha_1 \)

In this case we have \( \Sigma = \Sigma_1 = \exp \left[ e^{F_1(z)} V_{\alpha_1} (a^\pm_{\alpha_1}, z) \right] \). The only non-vanishing tau-functions are given by

\[
\begin{align*}
\tau_0 &= 1 - \frac{a^-_{\alpha_1} a^+_{\alpha_1}}{4} e^{2F_1(z)}, \\
\tau_1 &= 1 + \frac{a^-_{\alpha_1} a^+_{\alpha_1}}{4} e^{2F_1(z)}, \\
\tau_2 &= 1 - \frac{a^-_{\alpha_1} a^+_{\alpha_1}}{4} e^{2F_1(z)},
\end{align*}
\]

(7.1)

\[
\begin{align*}
\tau^{01}_{R,(0)} &= a^+_{\alpha_1} z e^{F_1(z)}, \\
\tau^{01}_{R,(2)} &= a^+_{\alpha_1} z e^{F_1(z)}, \\
\tau^{01}_{L} &= a^+_{\alpha_1} e^{F_1(z)}.
\end{align*}
\]

(7.2)

As discussed below (4.6), in order to have \( \varphi \) real one needs \(|\tau_0| = |\tau_0|\), and consequently \( a^-_{\alpha_1} a^+_{\alpha_1} \) has to be pure imaginary. The topological current (4.6) is then

\[
Q^{\text{top}} = -\frac{1}{2} \alpha_1 \text{ (sign } z) \]

(7.4)

and so it is (up to a sign) the fundamental weight of the \( \mathfrak{sl}_2(\mathbb{C}) \) subalgebra generated by \( H_1 \) and \( E_{\pm \alpha_1} \). Using (5.6) and (3.50), one obtains that the mass of the soliton is \( M^{(\alpha_1)}_{\text{sol}} = 2k \).

### 7.2 One soliton of species \( \alpha_2 \)

In this case we have \( \Sigma = \Sigma_2 = \exp \left[ e^{F_2(z)} V_{\alpha_2} (a^\pm_{\alpha_2}, z) \right] \). The non-vanishing tau-functions are given by

\[
\begin{align*}
\tau_0 &= 1 - \frac{a^-_{\alpha_2} a^+_{\alpha_2}}{4} e^{2F_2(z)}, \\
\tau_1 &= 1 - \frac{a^-_{\alpha_2} a^+_{\alpha_2}}{4} e^{2F_2(z)}, \\
\tau_2 &= 1 + \frac{a^-_{\alpha_2} a^+_{\alpha_2}}{4} e^{2F_2(z)},
\end{align*}
\]

(7.5)

\[
\begin{align*}
\tau^{02}_{R,(0)} &= a^+_{\alpha_2} z e^{F_2(z)}, \\
\tau^{02}_{R,(1)} &= a^+_{\alpha_2} z e^{F_2(z)}, \\
\tau^{02}_{L} &= a^+_{\alpha_2} e^{F_2(z)}.
\end{align*}
\]

(7.6)

Similarly as above, in order to have \( \varphi \) real we need \( a^-_{\alpha_2} a^+_{\alpha_2} \) be pure imaginary. The topological current (4.6) is then

\[
Q^{\text{top}} = -\frac{1}{2} \alpha_2 \text{ (sign } z) \]

(7.8)

and so it is (up to a sign) the fundamental weight of the \( \mathfrak{sl}_2(\mathbb{C}) \) subalgebra generated by \( H_2 \) and \( E_{\pm \alpha_2} \). The mass of the soliton is \( M^{(\alpha_2)}_{\text{sol}} = 2k \).
In this case we have $\Sigma = \Sigma_3 = \exp \left[ e^{R_3(z)} V_{a_3}(a_{a_3}, z) \right]$. The non-vanishing tau-functions are given by
\begin{align}
\tau_0 &= 1 - \frac{a_{a_3}^{-1}}{4} e^{2 R_3(z)}, & \tau_1 &= 1 + \frac{a_{a_3}^{+1}}{4} e^{2 R_3(z)}, & \tau_2 &= 1 + \frac{a_{a_3}^{+1}}{4} e^{2 R_3(z)}, \\
\tau_R^{a_3} &= a_{a_3}^{+1} z e^{R_3(z)}, & \tau_{L(1)}^{a_3} &= -a_{a_3}^{+1} e^{R_3(z)}, & \tau_{L(2)}^{a_3} &= -a_{a_3}^{+1} e^{R_3(z)}, \\
\tau_{R(1)}^{a_3} &= a_{a_3}^{+1} e^{R_3(z)}, & \tau_{R(2)}^{a_3} &= a_{a_3}^{+1} e^{R_3(z)}, & \tau_{L}^{a_3} &= -\frac{a_{a_3}^{+1}}{z} e^{R_3(z)}.
\end{align}

In order to have $\varphi_a$ real we need $a_{a_3}^{-1} a_{a_3}^{+1}$ be pure imaginary. The topological current (4.4) is then
\begin{equation}
Q_{top} = -\frac{1}{2} \alpha_3 (\text{sign } z)
\end{equation}
and so it is (up to a sign) the fundamental weight of the $\mathfrak{sl}_2(\mathbb{C})$ subalgebra generated by $H_1 + H_2$ and $E_{\pm a_3}$. The mass of the soliton is $M_{\text{sol}}^{(a_3)} = 2 \kappa m_3$.

8 Two-soliton solutions

We now present the six two-soliton solutions corresponding to the pairs we can form with the three species of solitons constructed in section 4. The solutions are constructed by performing the dressing transformations with the constant group element $\rho$ leading to the elements $\Sigma_{ij}$ given in (3.48). Instead of evaluating the matrix elements in (3.21) and (3.33)–(3.41) using the homogeneous vertex operator realization of the three fundamental representations of the $\mathfrak{g}$, we calculated the solutions using the Hirota method with ansatz (5.52). The evaluation of the corresponding coefficients was performed using the Mathematica program described in appendix 4.

8.1 Two solitons of species $\alpha_1/\alpha_1$

Here we give the two-soliton solution obtained by taking the group element $\Sigma = \Sigma_{11} = \exp \left[ e^{R_1(z_1)} V_{a_1}(a_{a_1}, z_1) \right] \exp \left[ e^{R_1(z_2)} V_{a_1}(b_{a_1}, z_2) \right]$. The non-vanishing tau-functions are
\begin{align}
\tau_0 &= 1 - \frac{a_{a_1}^{-1}}{4} e^{2 R_1(z_1)} - \frac{b_{a_1}^{-1}}{4} e^{2 R_1(z_2)} - \frac{(a_{a_1}^{+1} a_{a_1}^{-1} + a_{a_1}^{-1} b_{a_1}^{+1}) z_1 z_2}{4 (z_1 + z_2)^2} e^{R_1(z_1)+R_1(z_2)} \\
&\quad + \frac{a_{a_1}^{+1} b_{a_1}^{-1} b_{a_1}^{+1} (z_1 - z_2)^4}{16 (z_1 + z_2)^4} e^{2 R_1(z_1)+2 R_1(z_2)}, \\
\tau_1 &= 1 + \frac{a_{a_1}^{+1}}{4} e^{2 R_1(z_1)} + \frac{b_{a_1}^{-1}}{4} e^{2 R_1(z_2)} + \frac{(a_{a_1}^{+1} b_{a_1}^{-1} z_1^2 + a_{a_1}^{-1} b_{a_1}^{+1} z_2^2)}{4 (z_1 + z_2)^2} e^{R_1(z_1)+R_1(z_2)} \\
&\quad + \frac{a_{a_1}^{-1} b_{a_1}^{+1} b_{a_1}^{-1} (z_1 - z_2)^4}{16 (z_1 + z_2)^4} e^{2 R_1(z_1)+2 R_1(z_2)}, \\
\tau_2 &= 1 - \frac{a_{a_1}^{-1}}{4} e^{2 R_1(z_1)} - \frac{b_{a_1}^{+1}}{4} e^{2 R_1(z_2)} - \frac{(a_{a_1}^{+1} b_{a_1}^{-1} + a_{a_1}^{-1} b_{a_1}^{+1}) z_1 z_2}{4 (z_1 + z_2)^2} e^{R_1(z_1)+R_1(z_2)} \\
&\quad + \frac{a_{a_1}^{+1} b_{a_1}^{-1} b_{a_1}^{+1} (z_1 - z_2)^4}{16 (z_1 + z_2)^4} e^{2 R_1(z_1)+2 R_1(z_2)},
\end{align}
\[ \tau_{R}^{a_1} = a_{a_1} e^{f_1(z_1)} + b_{a_1} e^{f_1(z_2)} + \frac{a_{a_1} a_{a_1} b_{a_1}^+ (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{2f_1(z_1) + f_1(z_2)} + \frac{a_{a_1} b_{a_1} b_{a_1}^+ (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{f_1(z_1) + 2f_1(z_2)}, \]  
\[ \tau_{L}^{a_1} = a_{a_1} e^{f_1(z_1)} + b_{a_1} e^{f_1(z_2)} + \frac{a_{a_1} a_{a_1} b_{a_1}^+ (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{2f_1(z_1) + f_1(z_2)} + \frac{a_{a_1} b_{a_1} b_{a_1}^+ (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{f_1(z_1) + 2f_1(z_2)}, \]  
\[ \tau_{L,(2)}^{a_1} = -\frac{a_{a_1}}{z_1} e^{f_1(z_1)} - \frac{b_{a_1}}{z_2} e^{f_1(z_2)} + \frac{a_{a_1} a_{a_1} b_{a_1}^+ (z_1 - z_2)^2}{4 z_1 (z_1 + z_2)^2} e^{f_1(z_1) + 2f_1(z_2)} + \frac{a_{a_1} b_{a_1} b_{a_1}^+ (z_1 - z_2)^2}{4 z_2 (z_1 + z_2)^2} e^{2f_1(z_1) + f_1(z_2)}, \]  
\[ \tau_{L,(0)}^{a_1} = -\frac{a_{a_1}}{z_1} e^{f_1(z_1)} - \frac{b_{a_1}}{z_2} e^{f_1(z_2)} + \frac{a_{a_1} a_{a_1} b_{a_1}^+ (z_1 - z_2)^2}{4 z_1 (z_1 + z_2)^2} e^{f_1(z_1) + 2f_1(z_2)} + \frac{a_{a_1} b_{a_1} b_{a_1}^+ (z_1 - z_2)^2}{4 z_2 (z_1 + z_2)^2} e^{2f_1(z_1) + f_1(z_2)}, \]  
\[ \tau_{R,(0)}^{a_1} = a_{a_1} z_1 e^{f_1(z_1)} + b_{a_1} z_2 e^{f_1(z_2)} - \frac{a_{a_1} a_{a_1} b_{a_1}^+ z_1 (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{f_1(z_1) + 2f_1(z_2)} - \frac{a_{a_1} b_{a_1} b_{a_1}^+ z_2 (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{2f_1(z_1) + f_1(z_2)}, \]  
\[ \tau_{R,(2)}^{a_1} = a_{a_1} z_1 e^{f_1(z_1)} + b_{a_1} z_2 e^{f_1(z_2)} - \frac{a_{a_1} a_{a_1} b_{a_1}^+ z_1 (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{f_1(z_1) + 2f_1(z_2)} - \frac{a_{a_1} b_{a_1} b_{a_1}^+ z_2 (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{2f_1(z_1) + f_1(z_2)}. \]  

One can check that by taking the limit \(e^{f_1(z_2)} \to 0\), such solution becomes the one-soliton solution \((7.4)\)–\((7.3)\). Now, considering the ratios of tau-functions appearing in relations \((3.20)\) and \((3.27)\)–\((3.32)\) one observes that in the limit \(e^{f_1(z_2)} \to 0\), such solution becomes the one-soliton solution \((7.1)\)–\((7.3)\) again but with the replacement

\[ e^{f_1(z_2)} \to \left( \frac{z_1 - z_2}{z_1 + z_2} \right)^2 e^{f_1(z_1)} \]  

and with the ratio \(\tau_1/\tau_0\) changing sign. Therefore, comparing with \((6.7)\), we observe that the lateral displacement and time delay for this two-soliton solution is given by \((3.10)\) and \((6.11)\), respectively, with \(n = 2\) and \(\omega = 1\). Consequently, the solitons of species \(a_1\) experience an attractive force.
8.2 Two solitons of species $\alpha_2/\alpha_2$

Here we give the two-soliton solutions obtained by taking the group element $\Sigma = \Sigma_{22} = \exp \left[ e^{\Gamma_{2(z_1)} V_{\alpha_2} (a_{\alpha_2}^\pm, z_1)} \right] \exp \left[ e^{\Gamma_{2(z_2)} V_{\alpha_2} (b_{\alpha_2}^\pm, z_2)} \right]$. The non-vanishing tau-functions are

\[
\tau_0 = 1 - \frac{a_{\alpha_2}^+ a_{\alpha_2}^-}{4} e^{2 \Gamma_{2(z_1)}} - \frac{b_{\alpha_2}^- b_{\alpha_2}^+}{4} e^{2 \Gamma_{2(z_2)}} - \frac{(a_{\alpha_2}^+ b_{\alpha_2}^- + a_{\alpha_2}^- b_{\alpha_2}^+)}{(z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + \Gamma_{2(z_2)}}
+ \frac{a_{\alpha_2}^- a_{\alpha_2}^+ b_{\alpha_2}^- b_{\alpha_2}^+ (z_1 - z_2)^4}{16 (z_1 + z_2)^4} e^{2 \Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}},
\]

(8.11)

\[
\tau_1 = 1 - \frac{a_{\alpha_2}^+ a_{\alpha_2}^-}{4} e^{2 \Gamma_{2(z_1)}} - \frac{b_{\alpha_2}^- b_{\alpha_2}^+}{4} e^{2 \Gamma_{2(z_2)}} - \frac{(a_{\alpha_2}^+ b_{\alpha_2}^- + a_{\alpha_2}^- b_{\alpha_2}^+)}{(z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + \Gamma_{2(z_2)}}
+ \frac{a_{\alpha_2}^- a_{\alpha_2}^+ b_{\alpha_2}^- b_{\alpha_2}^+ (z_1 - z_2)^4}{16 (z_1 + z_2)^4} e^{2 \Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}},
\]

(8.12)

\[
\tau_2 = 1 + \frac{a_{\alpha_2}^+ a_{\alpha_2}^-}{4} e^{2 \Gamma_{2(z_1)}} + \frac{b_{\alpha_2}^- b_{\alpha_2}^+}{4} e^{2 \Gamma_{2(z_2)}} + \frac{(a_{\alpha_2}^+ b_{\alpha_2}^- + a_{\alpha_2}^- b_{\alpha_2}^+)}{(z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + \Gamma_{2(z_2)}}
+ \frac{a_{\alpha_2}^- a_{\alpha_2}^+ b_{\alpha_2}^- b_{\alpha_2}^+ (z_1 - z_2)^4}{16 (z_1 + z_2)^4} e^{2 \Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}},
\]

(8.13)

\[
\tau_{R}^{\alpha_2} = a_{\alpha_2}^- e^{\Gamma_{2(z_1)}} + b_{\alpha_2}^- e^{\Gamma_{2(z_2)}} + \frac{a_{\alpha_2}^- a_{\alpha_2}^+ b_{\alpha_2}^- b_{\alpha_2}^+ (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}}
+ \frac{a_{\alpha_2}^- b_{\alpha_2}^- b_{\alpha_2}^+ (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{2 \Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}},
\]

(8.14)

\[
\tau_{L}^{\alpha_2} = a_{\alpha_2}^+ e^{\Gamma_{2(z_1)}} + b_{\alpha_2}^+ e^{\Gamma_{2(z_2)}} + \frac{a_{\alpha_2}^+ a_{\alpha_2}^+ b_{\alpha_2}^+ b_{\alpha_2}^- (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{2 \Gamma_{2(z_1)} + \Gamma_{2(z_2)}}
+ \frac{a_{\alpha_2}^+ b_{\alpha_2}^- b_{\alpha_2}^+ (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + \Gamma_{2(z_2)}},
\]

(8.15)

\[
\tau_{L,(1)}^{\alpha_2} = \frac{-a_{\alpha_2}^- a_{\alpha_2}^+}{z_1} e^{\Gamma_{2(z_1)}} - \frac{b_{\alpha_2}^-}{z_2} e^{\Gamma_{2(z_2)}} + \frac{a_{\alpha_2}^- b_{\alpha_2}^- b_{\alpha_2}^+ (z_1 - z_2)^2}{4 z_1 (z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}}
+ \frac{a_{\alpha_2}^- a_{\alpha_2}^- b_{\alpha_2}^- (z_1 - z_2)^2}{4 z_2 (z_1 + z_2)^2} e^{2 \Gamma_{2(z_1)} + \Gamma_{2(z_2)}},
\]

(8.16)

\[
\tau_{L,(0)}^{\alpha_2} = \frac{-a_{\alpha_2}^- a_{\alpha_2}^+}{z_1} e^{\Gamma_{2(z_1)}} - \frac{b_{\alpha_2}^-}{z_2} e^{\Gamma_{2(z_2)}} + \frac{a_{\alpha_2}^- b_{\alpha_2}^- b_{\alpha_2}^+ (z_1 - z_2)^2}{4 z_1 (z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}}
+ \frac{a_{\alpha_2}^- a_{\alpha_2}^- b_{\alpha_2}^- (z_1 - z_2)^2}{4 z_2 (z_1 + z_2)^2} e^{2 \Gamma_{2(z_1)} + \Gamma_{2(z_2)}},
\]

(8.17)

\[
\tau_{R,(0)}^{\alpha_2} = a_{\alpha_2}^+ z_1 e^{\Gamma_{2(z_1)}} + b_{\alpha_2}^+ z_2 e^{\Gamma_{2(z_2)}} - \frac{a_{\alpha_2}^+ b_{\alpha_2}^+ b_{\alpha_2}^- z_1 (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}}
- \frac{a_{\alpha_2}^+ a_{\alpha_2}^+ b_{\alpha_2}^+ (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{2 \Gamma_{2(z_1)} + \Gamma_{2(z_2)}},
\]

(8.18)

\[
\tau_{R,(1)}^{\alpha_2} = a_{\alpha_2}^+ z_1 e^{\Gamma_{2(z_1)}} + b_{\alpha_2}^+ z_2 e^{\Gamma_{2(z_2)}} - \frac{a_{\alpha_2}^+ b_{\alpha_2}^- b_{\alpha_2}^+ z_1 (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{\Gamma_{2(z_1)} + 2 \Gamma_{2(z_2)}}
- \frac{a_{\alpha_2}^- a_{\alpha_2}^+ b_{\alpha_2}^- (z_1 - z_2)^2}{4 (z_1 + z_2)^2} e^{2 \Gamma_{2(z_1)} + \Gamma_{2(z_2)}}.
\]

(8.19)
One can check that by taking the limit \(e^{\Gamma_2(z_2)} \to 0\), such solution becomes the one-soliton solution (7.3)–(7.7). Now, considering the ratios of tau-functions appearing in the relations (3.20) and (3.27)–(3.32) one observes that in the limit \(e^{\Gamma_2(z_2)} \to \infty\), such solution becomes the one-soliton solution (7.3)–(7.4) again, but with the replacement

\[ e^{\Gamma_2(z_1)} \to \left(\frac{z_1 - z_2}{z_1 + z_2}\right)^2 e^{\Gamma_2(z_1)} \]  

(8.20)

and with the ratio \(\tau_2/\tau_0\) changing sign. Therefore, comparing with (5.7), we observe that the lateral displacement and time delay for this two-soliton solution is given by (3.10) and (6.11), respectively, with \(n = 2\) and \(\omega = 1\). Consequently, the solitons of species \(\alpha_2\) experience an attractive force.

### 8.3 Two solitons of species \(\alpha_3/\alpha_3\)

Here we give the two-soliton solutions obtained by taking the group element \(\Sigma = \Sigma_{33} = \exp \left[ e^{\Gamma_3(z_1)} V_{\alpha_3} (a_{\alpha_3}, z_1) \right] \exp \left[ e^{\Gamma_3(z_2)} V_{\alpha_3} (b_{\alpha_3}, z_2) \right] \). The non-vanishing tau-functions are

\[ \tau_0 = 1 - \frac{a_{\alpha_3} a_{\alpha_3}^+ a_{\alpha_3}^+}{4} e^{2 \Gamma_3(z_1)} + \frac{b_{\alpha_3} b_{\alpha_3}^+}{4} e^{2 \Gamma_3(z_2)} - \frac{1}{16} \left( \frac{(a_{\alpha_3} b_{\alpha_3}^- + a_{\alpha_3}^- b_{\alpha_3}^+) z_1 z_2}{(z_1 + z_2)^2} e^{\Gamma_3(z_1) + \Gamma_3(z_2)} \right), \]  

(8.21)

\[ \tau_1 = 1 + \frac{a_{\alpha_3} a_{\alpha_3}^- a_{\alpha_3}^+}{4} e^{2 \Gamma_3(z_1)} + \frac{b_{\alpha_3} b_{\alpha_3}^+}{4} e^{2 \Gamma_3(z_2)} + \frac{1}{16} \left( \frac{(a_{\alpha_3} b_{\alpha_3}^+ z_1^2 + a_{\alpha_3}^- b_{\alpha_3}^+ z_2^2)}{(z_1 + z_2)^2} e^{\Gamma_3(z_1) + \Gamma_3(z_2)} \right), \]  

(8.22)

\[ \tau_2 = 1 + \frac{a_{\alpha_3} a_{\alpha_3}^- a_{\alpha_3}^+}{4} e^{2 \Gamma_3(z_1)} + \frac{b_{\alpha_3} b_{\alpha_3}^+}{4} e^{2 \Gamma_3(z_2)} + \frac{1}{16} \left( \frac{(a_{\alpha_3} b_{\alpha_3}^+ z_1^2 - a_{\alpha_3}^- b_{\alpha_3}^+ z_2^2)}{(z_1 + z_2)^2} e^{\Gamma_3(z_1) + \Gamma_3(z_2)} \right), \]  

(8.23)

\[ \tau_{R}^{\alpha_3} = a_{\alpha_3}^+ z_1 e^{\Gamma_3(z_1) z_1} + b_{\alpha_3}^+ z_2 e^{\Gamma_3(z_2)} - \frac{a_{\alpha_3} b_{\alpha_3}^- b_{\alpha_3}^+}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} e^{\Gamma_3(z_1) + \Gamma_3(z_2)} \]  

(8.24)

\[ \tau_{L}^{\alpha_3} = -\frac{a_{\alpha_3}^-}{z_1} e^{\Gamma_3(z_1)} - \frac{b_{\alpha_3}^-}{z_2} e^{\Gamma_3(z_2)} + \frac{a_{\alpha_3}^- b_{\alpha_3}^- b_{\alpha_3}^+}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} e^{\Gamma_3(z_1) + \Gamma_3(z_2)} \]  

(8.25)

\[ \tau_{L,(1)}^{\alpha_3} = -a_{\alpha_3}^+ e^{\Gamma_3(z_1)} - b_{\alpha_3}^+ e^{\Gamma_3(z_2)} - \frac{a_{\alpha_3}^- a_{\alpha_3}^+ b_{\alpha_3}^+}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} e^{\Gamma_3(z_1) + \Gamma_3(z_2)} \]  

(8.26)
\[
\tau^\alpha_{L,(2)} = -a^+_\alpha a^+_\alpha e^{R_3(z_1)} - b^+_\alpha a^+_\alpha e^{R_3(z_2)} - \frac{a^-_\alpha a^-_\alpha b^+_\alpha (z_1 - z_2)^2}{4(z_1 + z_2)^2} e^{2R_3(z_1) + R_3(z_2)}
- \frac{a^-_\alpha b^-_\alpha b^+_\alpha (z_1 - z_2)^2}{4(z_1 + z_2)^2} e^{R_3(z_1) + 2R_3(z_2)},
\]
\(\text{Eq. (8.27)}\)

\[
\tau^\alpha_{R,(1)} = a^-_\alpha e^{R_3(z_1)} + b^-_\alpha e^{R_3(z_2)} + \frac{a^-_\alpha a^-_\alpha b^-_\alpha (z_1 - z_2)^2}{4(z_1 + z_2)^2} e^{2R_3(z_1) + R_3(z_2)}
+ \frac{a^-_\alpha b^-_\alpha b^+_\alpha (z_1 - z_2)^2}{4(z_1 + z_2)^2} e^{R_3(z_1) + 2R_3(z_2)},
\]
\(\text{Eq. (8.28)}\)

\[
\tau^\alpha_{R,(2)} = a^-_\alpha e^{R_3(z_1)} + b^-_\alpha e^{R_3(z_2)} + \frac{a^-_\alpha a^-_\alpha b^-_\alpha (z_1 - z_2)^2}{4(z_1 + z_2)^2} e^{2R_3(z_1) + R_3(z_2)}
+ \frac{a^-_\alpha b^-_\alpha b^+_\alpha (z_1 - z_2)^2}{4(z_1 + z_2)^2} e^{R_3(z_1) + 2R_3(z_2)},
\]
\(\text{Eq. (8.29)}\)

One can check that by taking the limit \(e^{R_3(z_2)} \to 0\), such solution become the one-soliton solution (7.3)–(7.11). Now, considering the ratios of tau-functions appearing in the relations (8.31) and (8.32) one observes that in the limit \(e^{R_3(z_2)} \to \infty\), such solution becomes the one-soliton solution (8.3)–(7.11) again, but with the replacement

\[
e^{R_3(z_1)} \to \left(\frac{z_1 - z_2}{z_1 + z_2}\right)^2 e^{R_3(z_1)}
\]
\(\text{Eq. (8.30)}\)

and with the ratios \(\tau_1/\tau_0\) and \(\tau_2/\tau_0\) changing signs. Therefore, comparing with (6.7), we observe that the lateral displacement and time delay for this two-soliton solution is given by (8.10) and (6.11), respectively, with \(n = 2\) and \(\omega = 1\). Consequently, the solitons of species \(\alpha_3\) experience an attractive force.

### 8.4 Two solitons of species \(\alpha_1/\alpha_2\)

Here we give the two-soliton solutions obtained by taking the group element \(\Sigma = \Sigma_{12} = \text{exp}[e^{R_3(z_1)}V_{a_1}(a^\pm_{\alpha_1}, z_1)] \text{exp}[e^{R_3(z_2)}V_{a_2}(a^\pm_{\alpha_2}, z_2)]\). The non-vanishing tau-functions are

\[
\tau_0 = 1 - \frac{a^-_{\alpha_1} a^-_{\alpha_1}}{4} e^{2R_1(z_1)} - \frac{a^-_{\alpha_2} a^+_{\alpha_2}}{4} e^{2R_2(z_2)}
+ \frac{a^-_{\alpha_1} a^+_{\alpha_2} a^+_{\alpha_2} (z_1 + z_2)^2}{16(z_1 - z_2)^2} e^{2R_1(z_1) + 2R_2(z_2)},
\]
\(\text{Eq. (8.31)}\)

\[
\tau_1 = 1 + \frac{a^-_{\alpha_1} a^+_{\alpha_1}}{4} e^{2R_1(z_1)} - \frac{a^-_{\alpha_2} a^+_{\alpha_2}}{4} e^{2R_2(z_2)}
- \frac{a^-_{\alpha_1} a^+_{\alpha_2} a^+_{\alpha_2} (z_1 + z_2)^2}{16(z_1 - z_2)^2} e^{2R_1(z_1) + 2R_2(z_2)},
\]
\(\text{Eq. (8.32)}\)

\[
\tau_2 = 1 - \frac{a^-_{\alpha_1} a^+_{\alpha_1}}{4} e^{2R_1(z_1)} + \frac{a^-_{\alpha_2} a^+_{\alpha_2}}{4} e^{2R_2(z_2)}
- \frac{a^-_{\alpha_1} a^+_{\alpha_2} a^+_{\alpha_2} (z_1 + z_2)^2}{16(z_1 - z_2)^2} e^{2R_1(z_1) + 2R_2(z_2)},
\]
\(\text{Eq. (8.33)}\)

\[
\tau^\alpha_{R} = a^-_{\alpha_1} e^{R_1(z_1)} - \frac{a^-_{\alpha_1} a^-_{\alpha_2} a^+_{\alpha_2} (z_1 + z_2)}{4(z_1 - z_2)} e^{R_1(z_1) + 2R_2(z_2)},
\]
\(\text{Eq. (8.34)}\)
\[ \tau_{R}^{a_2} = a_{a_2} e^{r_2(z_2)} + \frac{a_{a_1} a_{a_2}^+ a_{a_1}^+ (z_1 + z_2)}{4 (z_1 - z_2)} e^{2 r_1(z_1) + r_2(z_2)}, \] (8.35)

\[ \tau_{R}^{a_3} = a_{a_1} a_{a_2}^+ z_1 z_2 e^{r_1(z_1) + r_2(z_2)}, \] (8.36)

\[ \tau_{L}^{a_1} = a_{a_1} e^{r_1(z_1)} + \frac{a_{a_2} a_{a_1} a_{a_2}^+ (z_1 + z_2)}{4 (z_1 - z_2)} e^{r_1(z_1) + 2 r_2(z_2)}, \] (8.37)

\[ \tau_{L}^{a_2} = a_{a_2} e^{r_2(z_2)} - \frac{a_{a_1} a_{a_2}^+ a_{a_2}^* (z_1 + z_2)}{4 (z_1 - z_2)} e^{2 r_1(z_1) + r_2(z_2)}, \] (8.38)

\[ \tau_{L}^{a_3} = \frac{a_{a_1} a_{a_2}^* e^{r_1(z_1) + r_2(z_2)}}, \] (8.39)

\[ \tau_{L,(2)}^{a_1} = \frac{-a_{a_1} e^{r_1(z_1)} + \frac{a_{a_2} a_{a_1} a_{a_2}^+ (z_1 + z_2)}{4 z_1 (z_1 - z_2)} e^{r_1(z_1) + 2 r_2(z_2)}}, \] (8.40)

\[ \tau_{L,(0)}^{a_1} = \frac{-a_{a_1} e^{r_1(z_1)} + \frac{a_{a_2} a_{a_1} a_{a_2}^+ (z_1 + z_2)}{4 z_1 (z_1 - z_2)} e^{r_1(z_1) + 2 r_2(z_2)}}, \] (8.41)

\[ \tau_{L,(1)}^{a_2} = \frac{-a_{a_2} e^{r_2(z_2)} + \frac{a_{a_2} a_{a_1} a_{a_2}^+ (z_1 + z_2)}{4 z_1 (z_1 - z_2)} e^{2 r_1(z_1) + r_2(z_2)}}, \] (8.42)

\[ \tau_{L,(0)}^{a_2} = \frac{-a_{a_2} e^{r_2(z_2)} + \frac{a_{a_2} a_{a_1} a_{a_2}^+ (z_1 + z_2)}{4 z_1 (z_1 - z_2)} e^{2 r_1(z_1) + r_2(z_2)}}, \] (8.43)

\[ \tau_{L,(1)}^{a_3} = \frac{-a_{a_1} a_{a_2}^+ z_1 e^{r_1(z_1) + r_2(z_2)}}, \] (8.44)

\[ \tau_{L,(2)}^{a_3} = \frac{-a_{a_1} a_{a_2}^+ z_2 e^{r_1(z_1) + r_2(z_2)}}, \] (8.45)

\[ \tau_{R,(0)}^{a_1} = a_{a_1} z_1 e^{r_1(z_1)} - \frac{a_{a_1} a_{a_2}^+ a_{a_2}^+ z_1 (z_1 + z_2)}{4 (z_1 - z_2)} e^{r_1(z_1) + 2 r_2(z_2)}, \] (8.46)

\[ \tau_{R,(2)}^{a_1} = a_{a_1} z_1 e^{r_1(z_1)} + \frac{a_{a_1} a_{a_2}^+ a_{a_2}^+ z_1 (z_1 + z_2)}{4 (z_1 - z_2)} e^{r_1(z_1) + 2 r_2(z_2)}, \] (8.47)

\[ \tau_{R,(0)}^{a_2} = a_{a_2} z_2 e^{r_2(z_2)} + \frac{a_{a_1} a_{a_1} a_{a_2}^+ z_2 (z_1 + z_2)}{4 (z_1 - z_2)} e^{2 r_1(z_1) + r_2(z_2)}, \] (8.48)

\[ \tau_{R,(1)}^{a_2} = a_{a_2} z_2 e^{r_2(z_2)} + \frac{a_{a_1} a_{a_1} a_{a_2}^+ z_2 (z_1 + z_2)}{4 (z_1 - z_2)} e^{2 r_1(z_1) + r_2(z_2)}, \] (8.49)

\[ \tau_{R,(1)}^{a_3} = a_{a_1} a_{a_2}^+ z_2 e^{r_1(z_1) + r_2(z_2)}, \] (8.50)

\[ \tau_{R,(2)}^{a_3} = -a_{a_1} a_{a_2}^+ z_1 e^{r_1(z_1) + r_2(z_2)}, \] (8.51)

One can check that by taking the limit \( e^{r_2(z_2)} \to 0 \), such solution becomes the one-soliton solution \((7.4)\)–\((7.3)\). Now, considering the ratios of tau-functions appearing in the relations \((3.20)\) and \((3.27)\)–\((3.32)\), one observes that in the limit \( e^{r_2(z_2)} \to \infty \), such solution becomes the one-soliton solution \((7.4)\)–\((7.3)\) again, but with the replacement

\[ e^{r_1(z_1)} \to \left( \frac{z_1 + z_2}{z_1 - z_2} \right) e^{r_1(z_1)} \] (8.52)
and with the ratios \( \tau_2/\tau_0, \tau_1^\alpha/\tau_1, \bar{\tau}_L^{(2)}/\tau_2 \), and \( \bar{\tau}_L^{(0)}/\tau_0 \) changing signs. If we track the soliton \( \alpha_2 \) instead of \( \alpha_1 \), we get the same results (interchanging \( \Gamma_1 \leftrightarrow \Gamma_2 \)), but the ratios of tau-functions changing signs are \( \tau_1/\tau_0, \tau_2^\alpha/\tau_2, \bar{\tau}_R^{(2)}/\tau_0 \), and \( \bar{\tau}_R^{(2)}/\tau_1 \). Therefore, comparing with (6.7), we observe that the lateral displacement and time delay for this two-soliton solution is given by (6.10) and (6.11), respectively, with \( n = 1 \) and \( \omega = -1 \). Consequently, the solitons of species \( \alpha_1 \) and \( \alpha_2 \) experience a repulsive force.

**8.5 Two solitons of species \( \alpha_1/\alpha_3 \)**

Here we give the two-soliton solutions obtained by taking the group element \( \Sigma = \Sigma_{13} = \exp \left[ e^{\Gamma_1(z_1)} V_{\alpha_1}(a_{\alpha_1}^\pm, z_1) \right] \exp \left[ e^{\Gamma_3(z_2)} V_{\alpha_3}(a_{\alpha_3}^\pm, z_2) \right] \). The non-vanishing tau-functions are

\[
\tau_0 = 1 - \frac{a^-_{\alpha_1} a^+_{\alpha_1}}{4} e^{2\Gamma_1(z_1)} - \frac{a^-_{\alpha_3} a^+_{\alpha_3}}{4} e^{2\Gamma_3(z_2)} + \frac{a^-_{\alpha_1} a^-_{\alpha_3} a^+_{\alpha_1} a^+_{\alpha_3}}{16} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} e^{2\Gamma_1(z_1) + 2\Gamma_3(z_2)},
\]  

(8.53)

\[
\tau_1 = 1 + \frac{a^-_{\alpha_1} a^+_{\alpha_1}}{4} e^{2\Gamma_1(z_1)} + \frac{a^-_{\alpha_3} a^+_{\alpha_3}}{4} e^{2\Gamma_3(z_2)} + \frac{a^-_{\alpha_1} a^-_{\alpha_3} a^+_{\alpha_1} a^+_{\alpha_3}}{16} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} e^{2\Gamma_1(z_1) + 2\Gamma_3(z_2)},
\]  

(8.54)

\[
\tau_2 = 1 - \frac{a^-_{\alpha_1} a^+_{\alpha_1}}{4} e^{2\Gamma_1(z_1)} + \frac{a^-_{\alpha_3} a^+_{\alpha_3}}{4} e^{2\Gamma_3(z_2)} - \frac{a^-_{\alpha_1} a^-_{\alpha_3} a^+_{\alpha_1} a^+_{\alpha_3}}{16} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} e^{2\Gamma_1(z_1) + 2\Gamma_3(z_2)},
\]  

(8.55)

\[
\bar{\tau}_R^{\alpha_1} = a^-_{\alpha_1} e^{\Gamma_1(z_1)} + \frac{a^-_{\alpha_1} a^-_{\alpha_1} a^+_{\alpha_1} a^+_{\alpha_3}}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} e^{2\Gamma_1(z_1) + 2\Gamma_3(z_2)},
\]  

(8.56)

\[
\bar{\tau}_R^{\alpha_2} = -\frac{a^-_{\alpha_3} a^+_{\alpha_1} z_1}{z_1 + z_2} e^{\Gamma_1(z_1) + \Gamma_3(z_2)},
\]  

(8.57)

\[
\bar{\tau}_R^{\alpha_3} = a^+_{\alpha_3} z_2 e^{\Gamma_3(z_2)} + \frac{a^-_{\alpha_1} a^+_{\alpha_1} a^+_{\alpha_3} z_2}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} e^{2\Gamma_1(z_1) + \Gamma_3(z_2)},
\]  

(8.58)

\[
\tau_L^{\alpha_1} = a^+_{\alpha_1} e^{\Gamma_1(z_1)} - \frac{a^-_{\alpha_1} a^+_{\alpha_1} a^+_{\alpha_3} (z_1 - z_2)^2}{4(z_1 + z_2)} e^{2\Gamma_1(z_1) + \Gamma_3(z_2)},
\]  

(8.59)

\[
\tau_L^{\alpha_2} = -\frac{a^-_{\alpha_1} a^+_{\alpha_3} z_2}{z_1 + z_2} e^{\Gamma_1(z_1) + \Gamma_3(z_2)},
\]  

(8.60)

\[
\tau_L^{\alpha_3} = -\frac{a^-_{\alpha_3} e^{\Gamma_3(z_2)}}{z_2} + \frac{a^-_{\alpha_1} a^-_{\alpha_1} a^+_{\alpha_3} (z_1 - z_2)^2}{4} e^{2\Gamma_1(z_1) + \Gamma_3(z_2)},
\]  

(8.61)

\[
\bar{\tau}_L^{\alpha_1} = -\frac{a^-_{\alpha_1} e^{\Gamma_1(z_1)}}{z_1} + \frac{a^-_{\alpha_1} a^-_{\alpha_3} a^+_{\alpha_3} (z_1 - z_2)^2}{4} e^{\Gamma_1(z_1) + 2\Gamma_3(z_2)},
\]  

(8.62)

\[
\bar{\tau}_L^{\alpha_2} = -\frac{a^-_{\alpha_1} e^{\Gamma_1(z_1)}}{z_1} - \frac{a^-_{\alpha_1} a^-_{\alpha_3} a^+_{\alpha_3} (z_1 - z_2)^2}{4} e^{\Gamma_1(z_1) + 2\Gamma_3(z_2)},
\]  

(8.63)

\[
\bar{\tau}_L^{\alpha_3} = -\frac{a^-_{\alpha_1} a^+_{\alpha_1} z_1}{z_2} e^{\Gamma_1(z_1) + \Gamma_3(z_2)},
\]  

(8.64)
One can check that by taking the limit $e^{F_3(z_2)} \to 0$, such solution become the one-soliton solution \((7.1), (7.3)\). Now, considering the ratios of tau-functions appearing in the relations \((8.21)\) and \((8.27), (8.32)\) one observes that in the limit $e^{F_3(z_2)} \to \infty$, such solution becomes the one-soliton solution \((7.1)\)–\((7.3)\) again, but with the replacement

$$ e^{F_1(z_1)} \to \left( \frac{z_1 - z_2}{z_1 + z_2} \right) e^{F_1(z_1)} $$

and with the ratios $\tau_1/\tau_0$, $\tau_2/\tau_0$, $\tilde{\tau}_L/\tau_1$, $\tilde{\tau}_L(0)/\tau_0$, and $\tilde{\tau}_L(0)/\tau_1$ changing signs. If we track the soliton $\alpha_3$ instead of $\alpha_1$, we get the same results (interchanging $\Gamma_1 \leftrightarrow \Gamma_3$), but the ratios of tau-functions changing signs are $\tau_1/\tau_0$, $\tilde{\tau}_R/\tau_0$, $\tilde{\tau}_R(1)/\tau_1$, and $\tilde{\tau}_R(2)/\tau_2$. Therefore, comparing with \((8.7)\), we observe that the lateral displacement and time delay for this two-soliton solution is given by \((8.10)\) and \((8.11)\), respectively, with $n = 1$ and $\omega = 1$. Consequently, the solitons of species $\alpha_1$ and $\alpha_3$ experience an attractive force.

### 8.6 Two solitons of species $\alpha_2/\alpha_3$

Here we give the two-soliton solutions obtained by taking the group element $\Sigma = \Sigma_{23} = \exp \left[ e^{F_2(z_1)} V_{\alpha_2}(a_{\alpha_2}^\pm, z_1) \right] \exp \left[ e^{F_3(z_2)} V_{\alpha_3}(a_{\alpha_3}^\pm, z_2) \right]$. The non-vanishing tau-functions are

$$ \tau_0 = 1 - \frac{a_{\alpha_2}^{-} a_{\alpha_2}^{+}}{4} e^{2 F_2(z_1)} - \frac{a_{\alpha_3}^{-} a_{\alpha_3}^{+}}{4} e^{2 F_3(z_2)} $$

$$ + \frac{a_{\alpha_2}^{-} a_{\alpha_3}^{-} a_{\alpha_2}^{+} a_{\alpha_3}^{+} (z_1 - z_2)^2}{16 (z_1 + z_2)^2} e^{2 F_2(z_1)+2 F_3(z_2)}, $$

\[(8.75)\]
\[ \tau_1 = 1 - \frac{a_{a_2}^+ a_{a_2}^+ a_{a_3} a_{a_3}^+}{4} e^{2 R_2(1)} + a_{a_3} a_{a_3}^+ e^{2 R_3(2)} - \frac{a_{a_2}^- a_{a_2}^- a_{a_3}^+ a_{a_3}^+ (z_1 - z_2)^2}{16 (z_1 + z_2)^2} e^{2 R_2(1) + 2 R_3(2)}, \]  
\[ \tau_2 = 1 + \frac{a_{a_2}^- a_{a_2}^- a_{a_3} a_{a_3}^+}{4} e^{2 R_2(1)} + \frac{a_{a_2}^- a_{a_2}^- a_{a_3} a_{a_3}^+}{4} e^{2 R_3(2)} + \frac{a_{a_2}^- a_{a_2}^+ a_{a_3} a_{a_3}^+}{16 (z_1 + z_2)^2} e^{2 R_2(1) + 2 R_3(2)}, \]  
\[ \tilde{\tau}_R^{\alpha_1} = \frac{a_{a_3}^- a_{a_3}^+ z_1}{z_1 + z_2} e^{R_2(1) + R_3(2)}, \]  
\[ \tilde{\tau}_R^{\alpha_2} = a_{a_2}^- e^{R_2(1)} + a_{a_2}^- a_{a_3} a_{a_3}^+ (z_1 - z_2) e^{R_2(1) + 2 R_3(2)}, \]  
\[ \tilde{\tau}_R^{\alpha_3} = a_{a_3}^+ z_2 e^{R_3(2)} + \frac{a_{a_2}^+ a_{a_2} a_{a_3} a_{a_3}^+ (z_1 - z_2) z_2}{4 (z_1 + z_2)} e^{2 R_2(1) + R_3(2)}, \]  
\[ \tilde{\tau}_L^{\alpha_1} = \frac{a_{a_2}^- a_{a_3}^+ z_2}{z_1 + z_2} e^{R_2(1) + R_3(2)}, \]  
\[ \tilde{\tau}_L^{\alpha_2} = a_{a_2}^+ e^{R_2(1)} - \frac{a_{a_2}^+ a_{a_3} a_{a_3}^+ (z_1 - z_2)}{4 (z_1 + z_2)} e^{R_2(1) + 2 R_3(2)}, \]  
\[ \tilde{\tau}_L^{\alpha_3} = -a_{a_3}^- e^{R_3(2)} + \frac{a_{a_2}^- a_{a_3} a_{a_3}^+ (z_1 - z_2) z_2}{4 z_2 (z_1 + z_2)} e^{2 R_2(1) + R_3(2)}, \]  
\[ \tilde{\tau}_{L,(2)}^{\alpha_1} = -\frac{a_{a_3}^- a_{a_3}^+ z_1}{z_2 (z_1 + z_2)} e^{R_2(1) + R_3(2)}, \]  
\[ \tilde{\tau}_{L,(0)}^{\alpha_1} = \frac{a_{a_3}^- a_{a_3}^+ z_1}{z_1 + z_2} e^{R_2(1) + R_3(2)}, \]  
\[ \tilde{\tau}_{L,(1)}^{\alpha_2} = -\frac{a_{a_2}^- e^{R_2(1)}}{z_1} + \frac{a_{a_2}^- a_{a_3} a_{a_3}^+ (z_1 - z_2)}{4 z_1 (z_1 + z_2)} e^{R_2(1) + 2 R_3(2)}, \]  
\[ \tilde{\tau}_{L,(0)}^{\alpha_2} = -\frac{a_{a_2}^- e^{R_2(1)}}{z_1} - \frac{a_{a_2}^- a_{a_3} a_{a_3}^+ (z_1 - z_2)}{4 z_1 (z_1 + z_2)} e^{R_2(1) + 2 R_3(2)}, \]  
\[ \tau_{L,(1)}^{\alpha_3} = -a_{a_3}^+ e^{R_3(2)} + \frac{a_{a_2}^+ a_{a_3} a_{a_3}^+ (z_1 - z_2)}{4 (z_1 + z_2)} e^{2 R_2(1) + R_3(2)}, \]  
\[ \tau_{L,(2)}^{\alpha_3} = -a_{a_3}^+ e^{R_3(2)} - \frac{a_{a_2}^+ a_{a_3} a_{a_3}^+ (z_1 - z_2)}{4 (z_1 + z_2)} e^{2 R_2(1) + R_3(2)} e^{2 R_2(1) + 2 R_3(2)}, \]  
\[ \tau_{R,(0)}^{\alpha_1} = -\frac{a_{a_2}^- a_{a_3}^+ z_1 z_2}{z_1 + z_2} e^{R_2(1) + R_3(2)}, \]  
\[ \tau_{R,(2)}^{\alpha_1} = a_{a_2}^- a_{a_2}^+ z_2 e^{R_2(1) + R_3(2)}, \]  
\[ \tau_{R,(0)}^{\alpha_2} = a_{a_2}^+ z_1 e^{R_2(1)} - \frac{a_{a_2}^- a_{a_3} a_{a_3}^+ z_1 (z_1 - z_2)}{4 (z_1 + z_2)} e^{R_2(1) + 2 R_3(2)}, \]  
\[ \tau_{R,(1)}^{\alpha_2} = a_{a_2}^+ z_1 e^{R_2(1)} + \frac{a_{a_2}^+ a_{a_3} a_{a_3}^+ z_1 (z_1 - z_2)}{4 (z_1 + z_2)} e^{R_2(1) + 2 R_3(2)}, \]
\[
\tilde{\tau}_{R,(1)}^{\alpha_3} = a_{\alpha_3}^{-} e^{\Gamma_3(z_2)} + \frac{a_{\alpha_2}^{-} a_{\alpha_3}^{-} a_{\alpha_2}^{+} (z_1 - z_2)}{4 (z_1 + z_2)} e^{2 \Gamma_2(z_1) + \Gamma_3(z_2)},
\]
(8.94)

\[
\tilde{\tau}_{R,(2)}^{\alpha_3} = a_{\alpha_3}^{-} e^{\Gamma_3(z_2)} - \frac{a_{\alpha_2}^{-} a_{\alpha_3}^{-} a_{\alpha_2}^{+} (z_1 - z_2)}{4 (z_1 + z_2)} e^{2 \Gamma_2(z_1) + \Gamma_3(z_2)}.
\]
(8.95)

One can check that by taking the limit \( e^{\Gamma_3(z_2)} \to 0 \), such solution become the one-soliton solution (7.5)–(7.7). Now, considering the ratios of tau-functions appearing in the relations (3.20) and (3.27)–(3.32) one observes that in the limit \( e^{\Gamma_3(z_2)} \to \infty \), such solution becomes the one-soliton solution (7.5)–(7.7) again, but with the replacement

\[
e^{\Gamma_2(z_1)} \to \left( \frac{z_1 - z_2}{z_1 + z_2} \right) e^{\Gamma_2(z_1)}
\]
(8.96)

and with the ratios \( \tau_1/\tau_0, \tau_2/\tau_0, \tau_1^{\alpha_2}/\tau_2, \tau_1^{\alpha_3}/\tau_1, \) and \( \tau_2^{\alpha_3}/\tau_0 \). If we track the soliton \( \alpha_3 \) instead of \( \alpha_2 \), we get the same results (interchanging \( \Gamma_2 \leftrightarrow \Gamma_3 \)), but the ratios of tau-functions changing signs are \( \tau_2/\tau_0, \tau_1^{\alpha_2}/\tau_0, \tau_1^{\alpha_3}/\tau_1, \) and \( \tau_2^{\alpha_3}/\tau_2 \). Therefore, comparing with (6.7), we observe that the lateral displacement and time delay for this two-soliton solution is given by (6.10) and (6.11), respectively, with \( n = 1 \) and \( \omega = 1 \). Consequently, the solitons of species \( \alpha_2 \) and \( \alpha_3 \) experience an attractive force.

### 8.7 Summary of time delay results

The calculations of the time delays of the two-soliton solutions have shown that the solitons of the same species experience an attractive force. The pair of solitons of species \( \alpha_1 \) and \( \alpha_3 \), as well as \( \alpha_2 \) and \( \alpha_3 \) also experience attractive forces. However, the pair of solitons \( \alpha_1 \) and \( \alpha_2 \) suffer a repulsive force. In addition, the force between solitons of the same species is twice as strong than that between solitons of different species. Consequently, the sign \( \omega \) and the integer \( n \) appearing in the time delay (6.11) is determined by the scalar product of the roots by the formula \( (\alpha_i|\alpha_j) = \omega n \), by normalizing the roots as \( \alpha_i^2 = 2 \). Therefore, the time delay suffered by a soliton of species \( \alpha_i \), with rapidity \( \theta_1 \) when scattering with a soliton of species \( \alpha_j \), and rapidity \( \theta_2 \), with \( \theta_1 > \theta_2 \), is given by the formula

\[
\Delta^{(i,j)}_1(t) = \frac{(\alpha_i|\alpha_j)}{m_i \cosh \theta_1} \ln \left[ \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \right].
\]
(8.97)

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A The affine Kac-Moody algebra $\hat{\mathfrak{sl}}_3(\mathbb{C})$

In this appendix we give the necessary information about the affine Kac–Moody algebra $\hat{\mathfrak{sl}}_3(\mathbb{C})$. More details can be found in the book by Kac [11] which we follow in our presentation of basic facts and definitions, or in the review paper by Goddard and Olive [12] intended for physicists.

Recall that the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ is the Lie algebra formed by all $3 \times 3$ complex matrices with zero trace. We use the standard choice for a Cartan subalgebra $\mathfrak{h}$ and for a basis of the corresponding root system $\Delta$. Denote the three positive roots by $\alpha_i$, $i = 1, 2, 3$, with $\alpha_a$, $a = 1, 2$, being the simple roots and $\alpha_3 = \alpha_1 + \alpha_2$. The Cartan generators are

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (A.1)$$

and the basis vectors of the root subspaces corresponding to the positive roots are chosen as

$$E_{+\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{+\alpha_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (A.2)$$

For negative roots we have

$$E_{-\alpha} = (E_{+\alpha})^T. \quad (A.3)$$

We use the invariant bilinear form on $\mathfrak{sl}_3(\mathbb{C})$ defined by

$$(x \ | \ y) = \text{tr}(xy), \quad x, y \in \mathfrak{sl}_3(\mathbb{C}). \quad (A.4)$$

The restriction of this form to the Cartan subalgebra $\mathfrak{h}$ is nondegenerate, therefore it induces a nondegenerate bilinear form on $\mathfrak{h}^*$ which we denote by $(\cdot \ | \cdot)$ as well. Note that with this definition one has

$$(\alpha_1 | \alpha_1) = 2, \quad (\alpha_2 | \alpha_2) = 2, \quad (\alpha_1 | \alpha_2) = -1. \quad (A.5)$$

The affine Kac-Moody algebra $\hat{\mathfrak{sl}}_3(\mathbb{C})$ is constructed in the following way. Consider the loop algebra

$$\mathcal{L}(\mathfrak{sl}_3(\mathbb{C})) = \mathbb{C}[\zeta, \zeta^{-1}] \otimes \mathfrak{sl}_3(\mathbb{C}), \quad (A.6)$$

where $\mathbb{C}[\zeta, \zeta^{-1}]$ is the algebra of Laurent polynomials in $\zeta$. An element of $\mathcal{L}(\mathfrak{sl}_3(\mathbb{C}))$ is a finite linear combination of the elements of the form $\zeta^m \otimes x$, where $m \in \mathbb{Z}$ and $x \in \mathfrak{sl}_3(\mathbb{C})$. The structure of a Lie algebra in $\mathcal{L}(\mathfrak{sl}_3(\mathbb{C}))$ is introduced by the relation

$$[\zeta^m \otimes x, \zeta^n \otimes y] = \zeta^{m+n} \otimes [x, y]. \quad (A.7)$$

We identify the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ with the subalgebra of $\mathcal{L}(\mathfrak{sl}_3(\mathbb{C}))$ formed by the elements of the form $1 \otimes x$, $x \in \mathfrak{sl}_3(\mathbb{C})$. This allows us to write $\zeta^m x$ instead of $\zeta^m \otimes x$. Similar identifications are used below.

Define a $\mathbb{C}$-valued 2-cocycle on $\mathcal{L}(\mathfrak{sl}_3(\mathbb{C}))$ by

$$\psi(\zeta^m x, \zeta^n y) = m (x \ | y) \delta_{m+n,0} \quad (A.8)$$
and denote by $\tilde{\mathcal{L}}(\mathfrak{sl}_3(\mathbb{C}))$ the extension of $\mathcal{L}(\mathfrak{sl}_3(\mathbb{C}))$ by a one-dimensional center associated to the cocycle $\psi$. Denote by $C$ the corresponding center element, then the commutation relations in the Lie algebra $\tilde{\mathcal{L}}(\mathfrak{sl}_3(\mathbb{C})) = \mathcal{L}(\mathfrak{sl}_3(\mathbb{C})) \oplus \mathbb{C} C$ are given by

$$[\zeta^m x, \zeta^n y] = \zeta^{m+n}[x, y] + mC \delta_{m+n,0}.$$  \hspace{1cm} (A.9)

Now denote by $\widehat{\mathfrak{sl}_3(\mathbb{C})}$ the Lie algebra which is obtained by adjoining to $\tilde{\mathcal{L}}(\mathfrak{sl}_3(\mathbb{C}))$ a derivation $D = \zeta (d/d\zeta)$. The commutation relations for the Lie algebra $\widehat{\mathfrak{sl}_3(\mathbb{C})}$ are defined by relations \hspace{1cm} (A.10)

where $m \in \mathbb{Z}$ and $\alpha \in \Delta$, together with the elements $C$ and $D$ form a basis for $\widehat{\mathfrak{sl}_3(\mathbb{C})}$. These elements satisfy the commutation relations

$$[H_a^m, H_b^n] = m(\alpha_a|\alpha_b)C \delta_{m+n,0},$$  \hspace{1cm} (A.12)

$$[H_a^m, E_{a_i}^n] = \pm(\alpha_a|\alpha_{a_i})E_{a_i}^{m+n},$$  \hspace{1cm} (A.13)

$$[E_{a_i}^m, E_{-a_i}^n] = H_a^{m+n} + mC \delta_{m+n,0},$$  \hspace{1cm} (A.14)

$$[E_{a_1}^m, E_{a_2}^n] = H_{a_1}^{m+n} + H_{a_2}^{m+n} + mC \delta_{m+n,0},$$  \hspace{1cm} (A.15)

$$[E_{a_1}^m, E_{a_2}^n] = E_{a_3}^{m+n},$$  \hspace{1cm} (A.16)

$$[E_{-a_1}^m, E_{-a_2}^n] = -E_{-a_2}^{m+n},$$  \hspace{1cm} (A.17)

$$[E_{-a_1}^m, E_{-a_2}^n] = E_{a_1}^{m+n},$$  \hspace{1cm} (A.18)

where $a, b = 1, 2, i = 1, 2, 3$ and $m, n \in \mathbb{Z}$. The remaining non-vanishing commutation relations are obtained by using relation \hspace{1cm} (A.3) which implies that

$$[E_{\alpha}^m, E_{\beta}^n] = -[E_{-\alpha}^m, E_{-\beta}^n].$$  \hspace{1cm} (A.19)

We use in the paper the principal $\mathbb{Z}$-gradation of $\widehat{\mathfrak{sl}_3(\mathbb{C})}$ which can be defined by the grading operator

$$Q_{\text{ppal}} = H_1 + H_2 + 3D$$ \hspace{1cm} (A.20)

in the following way. Consider the following subspaces of $\widehat{\mathfrak{sl}_3(\mathbb{C})}$:

$$\mathfrak{g}_m = \{x \in \widehat{\mathfrak{sl}_3(\mathbb{C})} \mid [Q_{\text{ppal}}, x] = m x\}, \quad m \in \mathbb{Z}.$$ \hspace{1cm} (A.21)

It is easy to get convinced that

$$\widehat{\mathfrak{sl}_3(\mathbb{C})} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m,$$  \hspace{1cm} (A.22)

and, moreover,

$$[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}.$$  \hspace{1cm} (A.23)

Hence, we really have a $\mathbb{Z}$-gradation of $\widehat{\mathfrak{sl}_3(\mathbb{C})}$. The subspace $\mathfrak{g}_0$ is a actually a subalgebra of $\widehat{\mathfrak{sl}_3(\mathbb{C})}$ given by

$$\mathfrak{g}_0 = \mathbb{C} H_1 \oplus \mathbb{C} H_2 \oplus \mathbb{C} C \oplus \mathbb{C} D = \mathbb{C} H_1 \oplus \mathbb{C} H_2 \oplus \mathbb{C} C \oplus \mathbb{C} Q_{\text{ppal}},$$ \hspace{1cm} (A.24)
and for other subspaces $g_m$ we have

$$g_{3m} = CH_1^m \oplus CH_2^m, \quad m \neq 0,$$

$$g_{3m+1} = CE_{\alpha_1}^m \oplus CE_{\alpha_2}^m \oplus CE_{-\alpha_3}^m,$$  

$$g_{3m+2} = CE_{\alpha_1}^{m+1} \oplus CE_{\alpha_2}^{m+1} \oplus CE_{\alpha_3}^{m+1}.  \tag{A.27}$$

Among the representations of $g$ there are three which play an important role. They are the fundamental highest weight representations. Each one has a highest weight state $|\lambda_j\rangle$, $j = 0, 1, 2$, satisfying

$$H_a |\lambda_0\rangle = 0, \quad H_a |\lambda_b\rangle = \delta_{a,b} |\lambda_b\rangle, \quad C |\lambda_j\rangle = |\lambda_j\rangle$$  

for $a, b = 1, 2$ and $j = 0, 1, 2$. Such states are annihilated by all positive grade subspaces

$$g_m |\lambda_j\rangle = 0, \quad m > 0,$$  

and all the states of the representation spaces are obtained by acting on the corresponding highest weight state $|\lambda_j\rangle$, with negative grade generators. The representation spaces can be supplied with a scalar product with respect to which

$$(H_a^m)^\dagger = H_{-a}^{-m}, \quad (E_{\alpha}^m)^\dagger = E_{-\alpha}^{-m},$$  

$$C^\dagger = C, \quad D^\dagger = D.  \tag{A.31}$$

It follows from (A.24)–(A.27) that in this case

$$(g_m)^\dagger = g_{-m},  \tag{A.32}$$

and one has

$$|\lambda_j | g_{-m} = 0, \quad m > 0.  \tag{A.33}$$

Recall that $g_0$ is a subalgebra of $\widehat{sl}_3(\mathbb{C})$. It is convenient also to consider two additional subalgebras

$$g_{<0} = \bigoplus_{m>0} g_m, \quad g_{>0} = \bigoplus_{m>0} g_{-m}.  \tag{A.34}$$

These subalgebras and the corresponding Lie groups play important role in the dressing transformation method.
B Mathematica program implementing Hirota method

In this appendix we give the source of a Mathematica program which allow to obtain and test soliton solutions of equations (2.1)–(2.5) using the Hirota method.

First of all note that to use it one has fix the number of solitons and the soliton species. This information is saved in the variable n and in the list s respectively. The Hirota ansatz for tau-functions is given by the function \( t \). The corresponding coefficients are collected into the arrays \( b \). The function \( \text{eqs} \) extracts from the equations for the tau-functions the terms with the fixed total power of \( e^{\Gamma_i} \). The function \( \text{calcbs} \) calculates the coefficients of the Hirota ansatz iteratively. The function \( \text{testsol} \) test that the equations for the tau-functions are satisfied by the solution found. The concrete code follows.

```mathematica
g[i_, z_, xp_, xm_] := m[i] (z xp - xm/z)/2;
m[3] = m[1] + m[2];
c[1_][i_] := 1 /; i <= 3 && Plus[1] == 0;
c[1_][i_] := 0 /; i > 3 && Plus[1] == 0;
Ind[p_] := Select[AllInd, Plus @@ #1 == p &];
b[1_] := Array[c[1], 21];
t[k_, xp_, xm_] :=
  Sum[
    Product[
      (e[i] Exp[g[s[[i]], z[i], xp, xm]])^AllInd[[j]][[i]],
      {i, n}]
    (c @@ AllInd[[j]][k], {j, Length[AllInd]}];
eqs[p_, l_] :=
  Block[{temp = Ind[p][[1]]},
    Table[
      D @@ Prepend[Table[{e[i], temp[[i]]}, {i, n}], taueq[k, xp, xm]] /. {e[i_] -> 0}, {k, 21}];
  ];
calcbs :=
  Block[{temp = Ind[p][[1]]},
    Table[
      Evaluate[b @@ Ind[p][[k]]] =
      Factor[temp /. Solve[eqs[p, k] == 0, temp][[1]], {p, 2 n},
        {k, Length[Ind[p]]}];
    ];
    outf = ToString[n] <> "sol" <> ToString[SequenceForm @@ s] <> ".m";
    If[Length[FileNames[outf]] != 0, DeleteFile[outf], Null];
    Save[outf, c];
    Print["The results of calculations were saved in file " <> outf];
    On[Solve::svars];
  ];
testsol :=
  If[Table[Simplify[taueq[k, xp, xm] /. {xm -> 0, xp -> 0}], {k, 21}]
    == Table[0, {21}],
    Print["Equations are satisfied"],
    Print["Equations are not satisfied"]];
```

Below we give the script of a sample Mathematica session. It is supposed that the above code is in the file gensol.m and the equations for the tau-functions are in the file taueqs.m.
In[1]:= << taueqs.m
In[2]:= << gensol.m
In[3]:= n = 2;
In[4]:= s = {1, 3};
In[5]:= calcbs
The results of calculations were saved in file 2sol13.m
In[6]:= testsol
Equations are satisfied

In conclusion note that the program above works in principle for any number of solitons, but already for three-soliton solutions the procedure of testing the solution is very time consuming.
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