LYAPUNOV EXPONENTS, PATH-INTEGRALS AND FORMS

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ABSTRACT

In this paper we use a path-integral approach to represent the Lyapunov exponents of both deterministic and stochastic dynamical systems. In both cases the relevant correlation functions are obtained from a (one-dimensional) supersymmetric field theory whose Hamiltonian, in the deterministic case, coincides with the Lie-derivative of the associated Hamiltonian flow. The generalized Lyapunov exponents turn out to be related to the partition functions of the respective super-Hamiltonian restricted to the spaces of fixed form-degree.

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1. INTRODUCTION

During the past two decades a lot of efforts have been devoted to the study of the so-called "stochastic" properties of deterministic dynamical systems. Most of the work concentrated on finding appropriate order parameters which could be used to classify dynamical systems according to their degree of stochasticity\textsuperscript{[1]} \textsuperscript{[2]}. Rigorous definitions of integrable, KAM-, ergodic, weakly mixing, mixing, C-systems, etc. were given and it was established that this ordering of the systems amounts to an increasing "chaoticity" or "stochasticity" of the motion. It also became clear that some of the stochasticity properties of the system (like ergodicity) were encoded in the spectrum of its Liouville operator.

One of the order parameters which has been studied most extensively is the Kolmogoroff-Sinai (KS) entropy and similar entropy-like quantities\textsuperscript{[3]}. A priori, the KS entropy is defined in a rather abstract manner \textsuperscript{*}. It has been shown\textsuperscript{[4]}, however, that it is related to the sum of the positive Lyapunov exponents\textsuperscript{[5]}. These exponents, loosly speaking, are a measure of the rate at which nearby trajectories fly away from each other. For deterministic systems the Lyapunov exponents are computed from the properties of a single trajectory, \textit{i.e.}, they are labelled by the initial point of the respective trajectory. More recently the concept of Lyapunov exponents has been generalized also to stochastic systems\textsuperscript{[6]}. In this case the Lyapunov exponents, like all observables, are obtained by averaging over (infinitely) many trajectories. This averaging can be done in different, inequivalent ways [see ref.9]. It is well known\textsuperscript{[7]} that stochastic systems with a Langevin dynamics can be formulated via path-integrals and, once the noise is integrated away, they are equivalent to a one-dimensional supersymmetric field theory. As was pointed out before\textsuperscript{[8]}\textsuperscript{[9]} there exists a special class of generalized Lyapunov exponents which acquires a natural interpretation in terms of this supersymmetric field theory. In fact they are simply given by the lowest eigenvalues of the super-Hamiltonian at fixed fermion number. Hence they can be deduced from the asymptotic behaviour of the corresponding partition function.

The relation between Lyapunov exponents and supersymmetry is much more general than it might appear from the previous work on stochastic systems. In fact, let us consider a trajectory $\phi(t)$ of some dynamical system, and let us visualize $\phi(t)$ as a "bosonic field" in a one-dimensional field theory with $t$ and $\phi$ parametrizing the "spacetime" and the "target-space", respectively. Let us assume that the lagrangian $\mathcal{L}$ of the system is invariant

\textsuperscript{*} See ref.[3] for the definition and the original references.
under a supersymmetry transformation of the form

$$\delta \phi(t) = \epsilon c(t)$$  \hspace{1cm} (1.1)

where $c(t)$ is the anticommuting "super-partner" of $\phi(t)$. Furthermore, assume that $\mathcal{L}$ leads to an equation of motion for $\phi$ which is of the general form

$$\dot{\phi}(t) = V(\phi(t)) + \eta(t)$$  \hspace{1cm} (1.2)

where $V$ is some vector field on the "target-space" and $\eta(t)$ is any function independent of $\phi$, and which does not change under the supersymmetry transformation. Then supersymmetry implies that the "fermionic partner" $c(t)$ evolves according to

$$\dot{c}(t) = V'(\phi(t))c(t)$$  \hspace{1cm} (1.3)

where $V'$ is the Jacobi matrix (see below) of $V$. Obviously the equation of motion of the super-partner $c(t)$ is the linearization of the "bosonic" equation of motion (1.2). Hence the dynamics of $c(t)$ contains information about the stability properties of the $\phi$—trajectories. In particular, if two nearby trajectories fly away from each other exponentially fast, this will manifest itself by exponentially growing eigenmodes of $c(t)$.

The above argument about the connection between supersymmetry and (generalized) Lyapunov exponents is very general. It applies to stochastic and deterministic systems alike. However, only for stochastic systems the field theory approach we mentioned has been widely used in the literature\cite{8,9}, whereas for deterministic systems the relevant field theory was introduced only recently\cite{10}. In the stochastic case the pertinent field theory is the euclidean-time version of supersymmetric quantum mechanics (see ref.\cite{16} for details). In this context the equation of motion (1.2) is the Langevin equation with a white noise $\eta(t)$. From the point of view of supersymmetric field theory, eq. (1.2) is the Nicolai mapping (see the third of ref.\cite{16}) relating $\phi$ to the Gaussian field $\eta$. In the deterministic case $\eta(t) = 0$, so that (1.2) becomes a generic first order evolution equation. We are particularly interested in Hamiltonian systems, in which case the vector field $V$ is the symplectic gradient of some Hamiltonian $H$. In ref.\cite{10} we have set up a field theoretic formalism for such systems, in particular we introduced a path-integral formulation for deterministic systems. The relevant Lagrangian is indeed invariant under a supersymmetry of the form (1.1).
The paper is organized as follows. In section 2 we briefly review some basic facts about Lyapunov exponents. Then, in section 3, we introduce the path-integral for classical hamiltonian systems. In section 4 we present the observables of this theory which measure the (ordinary) Lyapunov exponents referring to a single trajectory. Next we show, in section 5, that the partition function of the classical super-hamiltonian is related to a set of generalized Lyapunov exponents which constitute a classical analog of the one found by Graham\cite{9} in the stochastic case. In appendix A we give the corresponding discussion for stochastic systems. There we generalize previous work for one dimensional configuration space\cite{8,9} to higher dimensions in order to elucidate the geometrical structure underlying the higher dimensional Lyapunov exponents. In a second appendix (B) instead we briefly indicate the relation between our path-integral and the thermodynamic formalism of Ruelle\cite{23} with the hope to come back in the future to a more complete study of the relations between the two.

2. BRIEF REVIEW ABOUT LYAPUNOV EXPONENTS

Let us consider the differential equation

\[
\frac{d}{dt} \phi^a(t) = h^a(\phi(t)) \tag{2.1}
\]

where \( h \) is a vector field on some N-dimensional manifold \( \mathcal{M}_N \) with local coordinates \( \phi^a \). (Later on \( h \) will become the hamiltonian vector field and \( \mathcal{M}_N = \mathcal{M}_{2n} \) a symplectic manifold.) Let \( \Phi^a_{cl}(t; \phi_0) \) (the subscript ”\( cl \)” is for classical) be the solution of (2.1) with initial condition \( \Phi^a_{cl}(t = 0; \phi_0) = \phi_0^a \). We can then define a matrix associated to \( \Phi_{cl} \)

\[
S^a_b(t; \phi_0) \equiv \frac{\partial}{\partial \phi_0^a} \Phi^a_{cl}(t; \phi_0) \tag{2.2}
\]

This matrix is usually known as ”Jacobi matrix”\cite{10} and it describes how small changes of the initial point affect the solution at time \( t \). The Jacobi matrix is a solution of the linear equation

\[
\left[ \partial_t \delta_0^a - \partial_b h^a(\Phi_{cl}(t; \phi_0)) \right] S^b_c(t; \phi_0) = 0 \tag{2.3}
\]

with \( S^a_b(0; \phi_0) = \delta_0^a \). Eq. (2.3) is also called the ”equation of the first variations”, in fact if we displace the initial point \( \phi_0 \) by an infinitesimal amount \( \delta \phi^a(0) \) then the displacement
at later times is given, to first order, by the Jacobi field
\[
\delta \phi^a(t) = S_a^b(t; \phi_0) \delta \phi^b(0)
\] (2.4)

Let us pick some smooth Riemannian metric on \(M\) and let \(e^a\) denote a unit vector in the tangent space \(T_{\phi}M\). Then \(\lambda(t; \phi, e) = \|S(t; \phi)e\|\) is called the coefficient of expansion in the direction of \(e\). If
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \lambda(t; \phi, e) > 0
\] (2.5)
there is then an exponential divergence in the direction of \(e\). In ref.[5] (and summarized in the second of ref.[3]) it was shown, under rather general conditions, that:

\begin{enumerate}
\item[i)] the one dimensional Lyapunov exponent for the direction of \(e\), as defined by
\[
\lambda^{(1)}(e; \phi) = \limsup_{t \to \infty} \frac{1}{t} \ln \|S(t; \phi)e\|
\] (2.6)
exists for every vector \(e \in T_{\phi}M\).
\item[ii)] there exists a basis (called "normal basis"[5]) \(\{e_i, 1 \leq i \leq N\}\) for \(T_{\phi}M_N\) such that
\[
\sum_{i=1}^{N} \lambda^{(1)}(e_i, \phi) = \inf \sum_{i}^{N} \lambda^{(1)}(\tilde{e}_i, \phi)
\] (2.7)
where the "infimum" is taken over all possible bases \(\{\tilde{e}_i\}\) of \(T_{\phi}M_N\). It is then easy to realize that for any \(e \in T_{\phi}M\), we have
\[
\lambda^{(1)}(e; \phi) \in \{\lambda^{(1)}(e_i, \phi); 1 \leq i \leq N\}
\]
The numbers \(\lambda^{(1)}(e_i; \phi), 1 \leq i \leq N\), are called the one-dimensional Lyapunov characteristic numbers or exponents at \(\phi\). We shall write \(\lambda^{(1)}_i(\phi) \equiv \lambda^{(1)}(e_i, \phi)\) and choose the labelling such that
\[
\lambda^{(1)}_1(\phi) \geq \lambda^{(1)}_2(\phi) \geq \cdots \geq \lambda^{(1)}_N(\phi)
\] (2.8)
The numbers \(\{\lambda^{(1)}_i|1 \leq i \leq N\}\) are not necessarily distinct.
The higher dimensional Lyapunov exponents $\lambda^{(p)}$ are defined in a manner similar to the 1-dimensional case. Let \( \{e_1, e_2, \ldots, e_p\} \) be a set of \( p \leq N \) orthonormal vectors in \( T_{\phi}M \). They span a \( p \)-dimensional parallelepiped \( V_p \equiv e_1 \wedge e_2 \wedge \cdots \wedge e_p \). The exponents \( \lambda^{(p)} \) are a measure for the exponential growth of the volume of \( V_p \):

\[
\lambda^{(p)}(V_p; \phi) = \limsup_{t \to \infty} \frac{1}{t} \ln \| S(t; \phi)e_1 \wedge S(t; \phi)e_2 \wedge \cdots \wedge S(t; \phi)e_p \| \quad (2.9)
\]

It was shown\[5\] that, for almost all initial \( V_p \)'s, \( \lambda^{(p)} = \lambda_1^{(p)} \) where \( \lambda_1^{(p)} \) is given by the sum of the \( p \) largest one-dimensional Lyapunov exponents

\[
\lambda_1^{(p)}(\phi) \equiv \sum_{i=1}^{p} \lambda_i^{(1)}(\phi) \quad (2.10)
\]

Besides the numbers \( \{\lambda_i^{(1)}(\phi)\} \), which are uniquely associated to the flow \( h^a \) and the point \( \phi \), there is a quantity that is associated to the flow itself and not to any point in particular. It is the Kolmogorov-Sinai (KS) entropy which we mentioned at the beginning and which has played a leading role in detecting the transition from ordered to stochastic motion\[3\]. According to a theorem by Piesin\[4\], the KS entropy can be related to the positive Lyapunov exponents:

\[
KS = \int_{M^N} d\phi \left[ \sum_{\lambda_i(\phi) > 0} \lambda_i(\phi) \right]
\]

This is the main reason why the central objects to study are the Lyapunov exponents.

So far we have discussed Lyapunov exponents for deterministic systems only. For stochastic systems various inequivalent versions of Lyapunov-like quantities have been studied in the literature\[6\]. Here we only mention the one employed by Benzi et al.\[8\] and by Graham\[9\]. For the system (1.1) with \( \eta \) a white noise, say, it is not possible to define Lyapunov exponents for individual trajectories, but only for averages. In ref.[8,9] a generalized Lyapunov exponent was defined by

\[
\Lambda_1 = \limsup_{t \to \infty} \frac{1}{t} \ln \langle tr S(t) \rangle \quad (2.11)
\]

where \( \langle \cdot \rangle \) denotes the stochastic average over closed trajectories of length \( t \). Here \( S(t) \) is the Jacobi matrix evaluated at the end-point of the trajectory (actually in ref.[8,9] only systems with one degree of freedom were considered so that \( S(t) \) did not have indices).
3. THE PATH-INTEGRAL FORMULATION OF CLASSICAL HAMILTONIAN DYNAMICS

In classical mechanics (CM) the propagator \( P(\phi_2, t_2|\phi_1, t_1) \), which gives the classical probability for a particle to be at the point \( \phi_2 \) at time \( t_2 \), given that it was at the point \( \phi_1 \) at time \( t_1 \), is just a delta function

\[
P(\phi_2, t_2|\phi_1, t_1) = \delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1))
\]  

(3.1)

where \( \Phi_{cl}(t, \phi_0) \) is a solution of Hamilton’s equation

\[
\dot{\phi}^a(t) = \omega^{ab} \partial_b H(\phi(t)), \quad \text{with} \quad \omega^{ab}\omega_{bc} = \delta^a_c
\]  

(3.2)

subject to the initial conditions \( \phi^a(t_1) = \phi_1^a \) Here \( H \) is the conventional hamiltonian of a dynamical system defined on some phase-space \( \mathcal{M}_{2n} \) with local coordinates \( \phi^a, a = 1 \cdots 2n \) and a constant symplectic structure \( \omega = \frac{1}{2}\omega_{ab}d\phi^a \wedge d\phi^b \).

The delta function in (3.1) can be rewritten as

\[
\delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1)) = \left\{ \prod_{i=1}^{N-1} \int d\phi(i) \delta^{2n}(\phi(i) - \Phi_{cl}(t_i, \phi_0)) \right\} \delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1))
\]  

(3.3)

where we have sliced the interval \([0,t]\) in \( N \) intervals and labelled the various instants as \( t_i \) and the fields at \( t_i \) as \( \phi(i) \). Each delta function contained in the product on the RHS of (3.3) can be written as:

\[
\delta^{2n}(\phi(i) - \Phi_{cl}(t_i, \phi_0)) = \prod_{a=1}^{2n} \delta(\dot{\phi}^a - \omega^{ab} \partial_b H)|_{t_i} \det [\delta^a_b \partial_t - \partial_b (\omega_{ac} \partial_c H(\phi))]|_{t_i}
\]  

(3.4)

where the argument of the determinant is obtained from the functional derivative of the equation of motion (3.2) with respect to \( \phi(i) \). Introducing anticommuting variables \( c^a \) and \( \bar{c}_a \) to exponentiate the determinant, and the commuting auxiliary variables \( \lambda_a \) to exponentiate the delta functions, we can re-write the propagator as a path-integral.

\[
P(\phi_2, t_2|\phi_1, t_1) = \int_{\phi_1}^{\phi_2} D\phi \ D\lambda \ Dc \ D\bar{c} \ \exp i\tilde{S}
\]  

(3.5)

where \( \tilde{S} = \int_{t_1}^{t_2} dt \tilde{\mathcal{L}} \) with

\[
\tilde{\mathcal{L}} \equiv \lambda_a [\dot{\phi}^a - \omega^{ab} \partial_b H(\phi)] + i\bar{c}_a (\delta^a_b \partial_t - \partial_b (\omega^c_{ac} \partial_c H(\phi)))c^b
\]  

(3.6)

In the path-integral (3.5) we have used the using the slicing (3.3) and then taken the limit of \( N \to \infty \). This limit has to be taken with some care and some normalization factors might
appear in eq.[3.5], but they are of no importance for our discussion. Holding $\phi$ and $c$ both fixed at the endpoints of the path-integral, we obtain the kernel\textsuperscript{[10]}, $K(\phi_2, c_2, t_2|\phi_1, c_1, t_1)$, which propagates distributions in the space $(\phi, c)$

$$\tilde{g}(\phi_2, c_2, t_2) = \int d^{2n}\phi_1 \, d^{2n}c_1 \, K(\phi_2, c_2, t_2|\phi_1, c_1, t_1) \tilde{g}(\phi_1, c_1, t_1)$$  \hspace{1cm} (3.7)

The distributions $\tilde{g}(\phi, c)$ are finite sums of monomials of the type

$$\tilde{g}(\phi, c) = \frac{1}{p!} \theta^{(p)}(\phi) \, c^{a_1} \cdots c^{a_p}$$  \hspace{1cm} (3.8)

The kernel $K(\cdot|\cdot)$ is represented by the path-integral

$$K(\phi_2, c_2, t_2|\phi_1, c_1, t_1) = \int \mathcal{D}\phi \mathcal{D}\lambda \mathcal{D}c \mathcal{D}\bar{c} \, \exp i \int_{t_1}^{t_2} dt \tilde{L}$$  \hspace{1cm} (3.9)

with the boundary conditions $\phi^a(t_1, 2) = \phi_{12}^a$ and $c^a(t_1, 2) = c_{12}^a$. The function $\tilde{g}$ of eq. (3.8) is the \textit{classical} analogue of a wave function in the Schrödinger picture. It is also easy from here to build a classical generating functional $Z_{cl}$ from which all correlation-functions can be derived. It is given by

$$Z_{cl} = \int \mathcal{D}\phi^a(t) \mathcal{D}\lambda_a(t) \mathcal{D}c^a(t) \mathcal{D}\bar{c}_a \, \exp i \int dt \{ \tilde{L} + \text{source terms} \}$$  \hspace{1cm} (3.10)

where the lagrangian can be written as

$$\tilde{L} = \lambda_a \dot{\phi}^a + i\bar{c}_a c^b - \tilde{H}$$  \hspace{1cm} (3.11)

with the ”superhamiltonian” given by

$$\tilde{H} = \lambda_a h^a + i\bar{c}_a \partial_b h^a c^b$$  \hspace{1cm} (3.12)

and where $h^a$ is the hamiltonian vector field

$$h^a(\phi) \equiv \omega^{ab} \partial_b H(\phi)$$  \hspace{1cm} (3.13)

From the path-integral (3.10) and (3.11) we can see\textsuperscript{[10]} that the variables $(\phi, \lambda)$ and $(c, \bar{c})$ form conjugate pairs satisfying the ($Z_2$-graded) commutation relations

$$[\phi^a, \lambda_b] = i\delta^a_b$$  \hspace{1cm} [3.14]

$$[c^a, \bar{c}_b] = \delta^a_b$$

The commutators above are defined in precise terms in ref.[10]. Because of these commutators, in the ”Schrödinger-like” picture\textsuperscript{[10]}, the variables $\lambda_a$ and $\bar{c}_a$ are represented by
\( \lambda_a = -i \frac{\partial}{\partial \phi^a} \equiv -i \partial_a \) and by \( \bar{c}_a = \frac{\partial}{\partial c^a} \). In this way the hamiltonian (3.12) becomes

\[
\tilde{H} = -i l_h
\] (3.15)

where

\[
l_h = h^a \partial_a + c^b (\partial_b h^a) \frac{\partial}{\partial c^a}
\] (3.16)

is the Lie derivative operator. Its bosonic part coincides with the Liouvillian \( \tilde{L} = h^a \partial_a \), which gives the evolution of standard distributions \( \varrho^{(0)}(\phi) \) in phase-space:

\[
\partial_t \varrho^{(0)}(\phi, t) = -l_h \varrho^{(0)}(\phi, t) = \tilde{L} \varrho^{(0)}(\phi, t)
\] (3.17)

We see from here that our path-integral is nothing else than the path-integral counterpart of the operator approach to CM pioneered by Koopman and von Neumann\(^{[11]}\). The next step is to understand the meaning of the ghosts \( c^a \). From the lagrangian \( \tilde{L} \) of (3.11) we see that they obey the eq.

\[
c^a(t) = \partial_b h^a(\phi(t)) c^b(t)
\] (3.18)

which is the same equation as the one for the first variations \( \delta \phi^a \) derivable from eq. (3.2). So we can say that there is a one to one correspondence between ghosts \( c^a(t) \) and Jacobi fields \( \delta \phi^a(t) \).

\[
c^a(t) \sim \delta \phi^a(t)
\] (3.19)

Of course we know that the Jacobi field depends on the trajectory \( \phi(t) \) we are varying, but the same is with the ghosts \( c^a \) because in solving (3.18) we had to specify the classical trajectory \( \phi(t) \) to insert in \( h^a \). In a Schroedinger-like picture, in which the ghosts \( c^a(t) \) do not depend on \( t \), but are only arguments of the \( \varrho(\phi, c, t) \), the correspondence of the ghosts is not with the Jacobi fields but with the basis of the cotangent space \( T^*_\phi \mathcal{M}_{2n} \) that is usually indicated as \( d\phi^a \). For the details see ref.\(^{[10]}\). With this interpretation of the ghosts, it is then easy\(^{[10]}\) to re-interpret all the Cartan-Calculus on symplectic manifolds and also to identify \( \tilde{H} \) with the Lie-derivative of the hamiltonian flow \( l_h \) appearing in (3.16). Because the ghosts \( c^a \) form a basis of the cotangent space \( T^*_\phi \mathcal{M}_{2n} \), \( \varrho(\phi, c) \) may be considered a p-form valued field on \( \mathcal{M}_{2n} \). The Lie-derivative acts then on the components of \( \varrho \) (3.8) in the standard
manner
\[ l_h \mathcal{O}_{a_1 a_2 \ldots a_p} = h^b \partial_b \mathcal{O}_{a_1 a_2 \ldots a_p} + \sum_{j=1}^{p} \partial_{a_j} h^b \mathcal{O}_{a_1 \ldots a_{j-1} b a_{j+1} \ldots a_p} \] (3.20)

The kernel \( K \) obeys the Schroedinger-like equation \( i \partial_t K = \tilde{H} K \). Consequently the time evolution of \( \tilde{\rho} \) is governed by the equation
\[ i \partial_t \tilde{\rho}(\phi, c, t) = \tilde{H} \tilde{\rho}(\phi, c, t) \] (3.21)
or
\[ \partial_t \mathcal{O}_{a_1 \ldots a_p}(\phi, t) = -l_h \mathcal{O}_{a_1 \ldots a_p}(\phi, t) \] (3.22)

The explicit evaluation\(^{[10][12]}\) of the path-integral (3.9) yields
\[ K(\phi_2, c_2, t_2|\phi_1, c_1, t_1) = \delta^{(2n)} (\phi_2^a - \Phi_{cl}(t_2; \phi_1)) \delta^{(2n)} (c_2^a - C_{cl}(t_2; c_1, \phi_1)) \] (3.23)

Here \( \Phi_{cl}(t) \) and \( C_{cl}(t) \) are solutions of the classical equations of motion resulting from \( \tilde{\mathcal{L}} \),
\[ \dot{\phi}^a(t) = h^a(\phi(t)) \equiv \omega^{ab} \partial_b H(\phi(t)) \] (3.24)
\[ \dot{c}^a(t) = \partial_b h^a(\phi(t)) c^b(t) \] (3.25)
with the boundary conditions \( \Phi_{cl}(t_1; \phi_1) = \phi_1^a \) and \( C_{cl}(t_1; c_1, \phi_1) = c_1^a \).

The third argument of \( C_{cl}(t; c_1, \phi_1) \) indicates that \( C_{cl}^a \) is the Jacobi field for the classical trajectory \( \Phi_{cl}^a(t) \) emanating from the initial point \( \phi_1 \). Eq. (3.25) is solved by
\[ c^a(t) = S_{b}^{a}(t)c^{b}(t_1) \] (3.26)
if the Jacobi matrix obeys the differential equation
\[ [\partial_t \delta_{b}^{a} - M_{b}^{a}(t)]S_{c}^{b}(t) = 0 \] (3.27)
with the initial condition \( S_{b}^{a}(t_1) = \delta_{b}^{a} \) and where
\[ M_{b}^{a}(t) \equiv \partial_{b} h^{a}(\phi(t)) \equiv \omega^{ac} \partial_{c} h(\phi(t)) \] (3.28)

The formal solution to eq. (3.27) reads
\[ S(t) = \hat{T} \exp \int_{t_1}^{t} dt' M(t') \] (3.29)
where \( \hat{T} \) denotes the time-ordering operator. Note that \( S(t) \) functionally depends on the trajectory \( \phi(t) \) chosen. Since the latter is uniquely characterized by its initial point \( \phi_1 \) we
shall write \( S(t) \equiv S(t; \phi_1) \) for the Jacobi matrix of the trajectory emanating from \( \phi_1 \). The function \( S(t; \phi_1) \) defines what is called\(^{[13]}\) a "multiplicative cocycle":

\[
S_b^a(t + \tau; \phi_0) = S^a_c(t; \Phi_d(\tau; \phi_0))S^d_f(\tau; \phi_0)
\] (3.30)

It is well known\(^{[14]}\) that \( S \) is symplectic, \(*\), \( S \in Sp(2n) \), i.e.,

\[
S^a_c \omega_{ab} S^b_d = \omega_{cd}
\]

As a consequence we have that \( \text{det} S = 1 \). Using (3.26) we write for the Kernel (3.23)

\[
K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1) = \delta^{(2n)}(\phi_2^a - \Phi_d^a(t_2; \phi_1)) \delta^{(2n)}(c_2^a - S_b^a(t_2; \phi_1)c_1^b)
\] (3.31)

In what follows we shall frequently exploit the fact that \( K \) is normalized,

\[
\int d^2n \phi_1 d^2n c_1 K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1) = 1
\] (3.32)

\[
\int d^2n \phi_2 d^2n c_2 K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1) = 1
\] (3.33)

and that it conserves the Grassmannian delta-function \( \delta^{2n}(c) \):

\[
\int d^2n \phi_1 d^2n c_1 K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1) \delta^{(2n)}(c_1) = \delta^{(2n)}(c_2)
\] (3.34)

In fact, eq. (3.34) can be regarded as an expression of Liouville’s theorem\(^{[13]}\). Recall that the Liouville measure on phase space is given by

\[
d\mu = \omega^n = (\frac{1}{2} \omega_{ab} d\phi^a \wedge d\phi^b)^n
\] (3.35)

Consequently, if \( \omega_{ab} \) assumes its canonical, i.e., \( \phi \)-independent form, we have

\[
d\mu = \omega^n = n! \, d\phi^1 \wedge d\phi^2 \wedge \cdots \wedge d\phi^{2n}
\] (3.36)

Invoking the correspondence\(^{[10]}\) \( c^a \leftrightarrow d\phi^a \) between ghosts and differential forms on \( \mathcal{M}_{2n} \),

\* This is most easily seen by noting that \( M \) is of the form \( \omega^{ab} \) times a symmetric matrix\(^{[15]}\) and therefore lies in the Lie-algebra of \( \text{Sp}(2n) \).
we see that $\delta^{2n}(c)$ corresponds to the Liouville measure

$$
\delta^{(2n)}(c) \equiv c^1 c^2 c^3 \cdots c^{2n} \longleftrightarrow d\mu
$$

(3.37)

Eq. (3.34) expresses the fact that $d\mu$ is invariant under the Hamiltonian flow. The infinitesimal version of eq. (3.34) is

$$
l_h \delta^{(2n)}(c) = 0
$$

(3.38)

with the lie-derivative $l_h$ defined in (3.16). Eq. (3.38) follows from

$$
c^b \frac{\partial}{\partial c^a} \delta^{(2n)}(c) = 0
$$

(3.39)

and $\partial_a h^a = 0$. It implies that not only zero-forms evolve according to the Liouville equation (3.17) but also the coefficient functions of the 2n-forms

$$
\tilde{q}(\phi, c) = q_{1\cdots 2n}(\phi) c^1 c^2 \cdots c^{2n} \equiv q(\phi) \delta^{(2n)}(c)
$$

(3.40)

i.e., the equation $\partial_t q = -\hat{L} q$ can be immediately derived from (3.38). In the following we shall use the 2n-form (3.40) in order to represent conventional scalar densities on phase-space. From a mathematical point of view the factor $\delta^{(2n)}(c)$ provides the volume form on $\mathcal{M}_{2n}$, whereas from a physical point of view $\delta(c)$ is the vacuum of the fermionic Fock space. In the Schroedinger-like picture mentioned before the condition $\hat{c}^a |\text{vac}\rangle = 0$ translates into $c^a < c|\text{vac}\rangle = 0$, i.e., the ”position representation” of $|\text{vac}\rangle$ is $< c|\text{vac}\rangle = \delta^{(2n)}(c)$.

To summarize this section we can say that in the classical path-integral the ghosts play a double role: a dynamical one, in the Heisenberg-like picture of classical mechanics, because their equation of motion is the Jacobi equation and a geometrical one, in the Schroedinger-like picture of CM, because the time-independent $c^a$ span the cotangent space $T^*_\phi \mathcal{M}_{2n}$. This double role will be heavily exploited in our discussion of the Lyapunov exponents. The dynamical role of the ghosts had already been partly exploited in the second and the last of ref.[10]. There it was shown that for any hamiltonian $H(\phi)$ the action $\tilde{L}$ is invariant under a set of universal (graded) symmetries which form an ISp(2) algebra. Part of this

† See the appendix and ref.[16] for more details about this.
‡ We called it Heisenberg-like because the variables $\phi, c$ depend on $t$ in this ”picture".
algebra consists of a BRS-like operator $Q = ic^a\lambda_a$ and an anti-BRS operator $\bar{Q} = i\bar{c}\omega^{ab}\lambda_b$, respectively. $\bar{L}$ was also shown to be invariant (up to surface terms) under a supersymmetry generated by the charges

\begin{align}
Q_H &= c^a(\partial_a - \partial_b H) \\
\bar{Q}_H &= \bar{c}_a\omega^{ab}(\partial_b + \partial_b H)
\end{align}

which are nilpotent, $Q_H^2 = \bar{Q}_H^2 = 0$ and close on the superhamiltonian:

$$i\tilde{H} = [Q_H, \bar{Q}_H]$$

(3.42)

In the second of ref.[10] it was shown that the phase of a dynamical hamiltonian system with this supersymmetry unbroken was the same as the ergodic phase of the system. Moreover in the last of ref.[10] the supersymmetry and its relation to ergodicity were applied in a study of the Toda criterion, which was the first criterion put forward in 1974 to detect transition from ordered to stochastic motion. The next step, in the physics-history§ of dynamical systems, was taken in 1974-75¶ and it consisted in using more refined tools to study these transitions. These tools were the KS-entropy and Lyapunov exponents to which we turn now.

4. OBSERVABLES FOR HIGHER DIMENSIONAL LYAPUNOV EXPONENTS

As we have already stressed the path-integral formulation$^{[10]}$ of classical hamiltonian dynamics naturally involves the Jacobi fields $c^a(t)$. Therefore it seems plausible that it should be possible to relate quantities like the generalized Lyapunov exponents to certain observables in this theory.

For any observable $\mathcal{O} = \mathcal{O}(\phi, \lambda, c, \bar{c})$ we define the ”vacuum expectation value” as

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi \mathcal{D}\lambda \mathcal{D}c \mathcal{D}\bar{c} \mathcal{O}(\phi, \lambda, c, \bar{c}) \delta^{(2n)}(c(-\infty)) \exp i \int_{-\infty}^{\infty} dt \tilde{L}$$

(4.1)

where an integration over $\phi^a(\pm\infty)$ and $c^a(\pm\infty)$ is understood¶. Note that here we are dealing with a trace-formalism, rather than a bra-ket formalism. Therefore, contrary

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§ We say ”physics-history” because in the mathematics literature (Kolmogoroff-Sinai) these tools had been developed before.

¶ Here $t=+\infty$ ($t=-\infty$) is a symbolic notation for some finite time which is larger (smaller) than any time argument of the fields in $\mathcal{O}$. 
to quantum mechanics, the "state", a generalized density \( \tilde{\rho} \), appears only once under the path-integral\(^{10}\). Expectation values of the type (4.1) can be reduced to strings of propagation kernels \( K(\phi_j, c_j, t_j|\phi_{j-1}, c_{j-1}, t_{j-1}) \) connecting field monomials with different time arguments. An identity which we will often use is

\[
\int \mathcal{D}\phi \mathcal{D}\lambda \mathcal{D}c \mathcal{D}\bar{c} \ A(\phi(t_2), \lambda(t_2), c(t_2), \bar{c}(t_2)) \cdot B(\phi(t_1), \lambda(t_1), c(t_1), \bar{c}(t_1))
\]

\[
\cdot \exp\{i \int_{t_1}^{t_2} dt \tilde{\mathcal{L}}\} \tilde{\rho}(\phi(t_1), c(t_1)) = \int d\phi_1 \ dc_1 \ A(\phi_2, -i \frac{\partial}{\partial \phi_2}, c_2, \frac{\partial}{\partial c_2}) \cdot K(\phi_2, c_2, t_2|\phi_1, c_1, t_1) B(\phi_1, -i \frac{\partial}{\partial \phi_1}, c_1, \frac{\partial}{\partial c_1}) \tilde{\rho}(\phi_1, c_1)
\]

where \( A \) and \( B \) are arbitrary functions and where the functional integration is subject to the boundary conditions \( \phi(t_2) = \phi_2 \) and \( c(t_2) = c_2 \) fixed, while \( \phi(t_1) \) and \( c(t_1) \) are integrated over. For the observables considered in the following, \( A \) and \( B \) will be free from ordering ambiguities. Eq.(4.2) can be proven by discretizing the path-integral in the usual way.

To start with we consider the family of observables

\[
\mathcal{O}_b^a(T; \phi_0) = \exp(a(T)\bar{c}_b(0)\delta^{(2n)}(\phi(0) - \phi_0))
\]

for fixed \( T > 0 \) and \( \phi_0 \in \mathcal{M}_{2n} \). The delta-function picks the trajectory \( \phi(t) \) which passes through the prescribed point \( \phi_0 \) at time \( t=0 \). The creation operator \( \bar{c}_b(0) \) creates at \( t=0 \) a one-ghost state from the vacuum \( \delta^{(2n)}(c) \) and \( a(T) \) destroys it at some later time \( t=T \). By using eq.(4.2) we obtain for the expectation value of \( \mathcal{O}_b^a \):

\[
\langle \mathcal{O}_b^a(T, \phi_0) \rangle = \int d\phi(\infty) \ dc(\infty) \ K(\phi(\infty), c(\infty), \infty|\phi(T), c(T), T) \cdot \int d\phi(T) \ dc(T) \ a(T) \int d\phi(0) dc(0) \ K(\phi(T), c(T), T|\phi(0), c(0), 0) \cdot \delta(\phi(0) - \phi_0) \cdot \frac{\partial}{\partial \bar{c}_b(0)} \int d\phi(-\infty) dc(-\infty) \ K(\phi(0), c(0), 0|\phi(-\infty), c(-\infty), -\infty) \cdot \delta^{(2n)}(c(-\infty))
\]

The third factor of \( K \) on the RHS of (4.4) propagates the vacuum \( \delta^{(2n)}(c(-\infty)) \) from \( t = -\infty \) to \( t = 0 \). Because of Liouville’s theorem, eq.(3.34), the result at \( t = 0 \) is
\[ \phi^{(2n)}(c(0)) \]. Then, between \( t=0 \) and \( t=T \), the second factor of \( K \) propagates the ghost excitation created by \( \frac{\partial}{\partial c^b(0)} \) acting on the delta-function. For \( t > T \) we are left with the vacuum again. Using (3.32) we see that, again as a consequence of Liouville’s theorem, the first \( K \) in (4.4) is ineffective. In this way (4.4) boils down to

\[
\langle O^a_b(T, \phi_0) \rangle = \int d\phi(T) \ dc(T) \ c^a(T) \ \int dc(0) K(\phi(T), c(T), T|\phi_0, c(0), 0) \cdot \frac{\partial}{\partial c^b(0)} \delta^{(2n)}(c(0))
\]

\[
= \int dc(T) dc(0) \ c^a(T) \delta(c(T) - S(T; \phi_0)c(0)) \frac{\partial}{\partial c^b(0)} \delta^{(2n)}(c(0))
\]

where also (3.31) has been used. Taking advantage of (3.39) the final result reads

\[
\langle O^a_b(T, \phi_0) \rangle = S^a_b(T; \phi_0) \theta(T)
\]

The step-function \( \theta(T) \) has been included because \( \langle \mathcal{O} \rangle = 0 \) for negative values of \( T \), in this case in fact the destruction operator acts on the vacuum before the creation operator and therefore the probability is zero. We conclude that the two-point function of the ghosts is given by the Jacobi matrix \( S^a_b \).

Later on we shall see that the higher dimensional Lyapunov exponents are related to observables of the form \( 0 \leq f \leq 2n \)

\[
O_f(T; \phi_0) = c^{a_1}(T) \cdots c^{a_f}(T) \bar{c}_{a_f}(0) \cdots \bar{c}_{a_1}(0) \delta(\phi(0) - \phi_0)
\]

Their expectation values

\[
\Gamma_f(T; \phi_0) \equiv \langle O_f(T; \phi_0) \rangle , \ T > 0
\]

can be evaluated with the same method as above. The result is (up to an irrelevant factor)

\[
\Gamma_f(T; \phi_0) = S^{[a_1}_{a_1}(T; \phi_0)S^{a_2}_{a_2}(T; \phi_0) \cdots S^{a_f}_{a_f]}(T; \phi_0)
\]

where the square brackets denote the complete antisymmetrization; for \( f=2 \), for example, (4.9) reads as \( \Gamma_2 = (TrS)^2 - Tr(S^2) \). The interpretation of the observables \( O_f \) is as follows. At time \( t=0 \) the operator \( \bar{c}_{a_f}(0) \cdots \bar{c}_{a_1}(0) \) creates a state with \( f \) ghosts (or better with \( f \) Jacobi-fields) from the vacuum. In the geometric interpretation of the theory this state corresponds to a \( f \)-form \( d\phi^{a_1} \wedge d\phi^{a_2} \wedge \cdots d\phi^{a_f} \), i.e., to a \( f \)-dimensional volume. Hence \( \Gamma_f \) contains information about the rate of growth of \( f \)-dimensional volume elements in tangent space.
The expectation values $\Gamma_f$ are related to the Lyapunov exponents $\lambda^{(1)}_i(\phi_0)$ as follows. Consider first

$$\Gamma_1(t; \phi_0) = Tr S(t; \phi_0)$$

(4.10)

and assume that the initial point $\phi_0$ gives rise to a periodic trajectory with period $\tau$, i.e., $\phi^a(t) = \phi^a(t + \tau)$. Consequently the matrix $M$ of eq.(3.28) is periodic too, and Floquet theory\cite{17} tells us that $S$ can be written as

$$S(t) = P(t) \exp(Rt)$$

(4.11)

where $P$ is a periodic matrix, $P(t) = P(t + \tau)$, and $R$ is a constant one. Because $S(0) = P(0) = 1$ we have $S(\tau) = \exp(R\tau)$. Let us then diagonalize $R$; its eigenvalues $\rho_a$, $a = 1 \cdots 2n$, are the characteristic exponents of $M(t)$ and $\exp(\rho_a t)$ the corresponding Floquet multipliers. In the eigenvector basis of $R$ we have (suppressing the argument $\phi_0$)

$$S^a_b(t) = P^a_b(t) \exp(\rho_a t)$$

(4.12)

We assume the ordering

$$\Re \rho_1 \geq \Re \rho_2 \geq \cdots \geq \Re \rho_{2n}$$

(4.13)

If $\Re \rho_1$ is strictly larger than $\Re \rho_2$, the large-$t$ behaviour of $\Gamma_1$ is

$$\Gamma_1(t; \phi_0) \sim P^1_1 \exp(\rho_1 t)$$

(4.14)

Hence

$$g_1(\phi_0) \equiv \limsup_{t \to \infty} \frac{1}{t} \ln \Gamma_1(t; \phi_0) = \rho_1$$

(4.15)

exists and coincides with the Lyapunov exponent $\lambda^{(1)}_1(\phi_0)$.

Because $S(\tau)$ is a real symplectic matrix, its eigenvalues appear as 4-tuples: if $\mu$ is a complex eigenvalue then\cite{14} $\mu^*, \frac{1}{\mu}$ and $\frac{1}{\mu^*}$ are eigenvalues too (not necessarily different from $\mu$). Since $S(\tau) = \exp(R\tau)$ this means that if $\rho$ is a characteristic exponent, then also $\rho^*,-\rho$ and $-\rho^*$ are characteristic exponents. Let us write $\rho_a \equiv l_a + i\omega_a$ with $l_a$ and $\omega_a$ real. As for the relative magnitude of the numbers $\Re \rho_a = l_a$ two cases have to be distinguished. If some $\rho_a$ has a nonvanishing imaginary part, $\omega_a \neq 0$, then $\rho_a^*$ is different from $\rho_a$ and consequently there exist two exponents with the same real part.
(they give rise to equal contributions to the Lyapunov exponents). On the other hand, if
\( \omega_a = 0 \), the exponent \( \rho_a \) is real and generically there will be no other exponent with the
same real part. In this case \( \Gamma_1 \), say, is dominated by a single real multiplier, i.e., \( \rho_1 \). If
however, \( \rho_1 \equiv l_1 + i\omega_1 \) and \( \rho_2 \equiv l_1 - i\omega_1 \) are a complex conjugate pair, eq.(4.14) is
replaced by
\[
\Gamma_1(t; \phi_0) \sim \left[ R_1^1(t) e^{i\omega_1 t} + R_2^2 e^{-i\omega_1 t} \right] e^{lt} \tag{4.16}
\]

From (4.9) with (4.12) we obtain for the higher correlation functions
\[
\Gamma_f(t; \phi_0) = P_{a_1}^{[a_1]}(t) \cdots P_{a_f}^{[a_f]}(t) \exp[(\rho_{a_1} + \rho_{a_2} + \cdots + \rho_{a_f})t] \tag{4.17}
\]
Due to the antisymmetrization, the indices \( a_j \) of the \( \rho \)'s in the exponential must all be
different. Because of the ordering (4.13) this implies that for \( t \to \infty \)
\[
\Gamma_f(t; \phi_0) \sim p(t) \exp[(\rho_1 + \rho_2 + \cdots + \rho_f)t] \tag{4.18}
\]
for some \( \tau \)-periodic function \( p(t) \). As it stands (4.18) is correct only if the real part
of the last eigenvalue, \( l_f = \text{Re}\rho_f \), is strictly larger than the real part of the following
eigenvalue, \( l_{f+1} \). If, for some reason \( \star l_f = l_{f+1} \), the asymptotic formula consists of two
terms,
\[
\Gamma_f(t; \phi_0) \sim p(t) \exp[(\rho_1 + \rho_2 + \cdots + \rho_{f-1} + \rho_f)t] + \rho(t) \exp[(\rho_1 + \rho_2 + \cdots + \rho_{f-1} + \rho_{f+1})t] \tag{4.19}
\]
or even more terms if \( \rho_f \) is degenerate with \( \rho_{f+2}, \rho_{f+3}, \cdots \), etc. In any case
\[
g_f(\phi_0) \equiv \lim \sup_{t \to \infty} \frac{1}{t} \ln \Gamma_f(t; \phi_0) = \sum_{i=1}^{f} l_i(\phi_0) \tag{4.20}
\]
is the sum of the \( f \) largest real parts of the eigenvalues \( \rho_a \).

In the introduction we mentioned already that the leading Lyapunov exponent governing
the evolution of \( f \)-forms, \( \lambda_1^{(f)}(\phi_0) \), is related to the higher Lyapunov exponents for one forms,
\( \lambda_i^{(1)}(\phi_0) \), according to
\[
\lambda_1^{(f)}(\phi_0) = \sum_{i=1}^{f} \lambda_i^{(1)}(\phi_0) \tag{4.21}
\]
This is exactly the relation we found in eq. (4.20). The correlation function \( \Gamma_f \) is the expect-
tation value of the operator \( O_f \) which creates and destroys a \( f \)-dimensional "parallelootope",

* e.g., because \( \rho_f \) and \( \rho_{f+1} \) form a complex conjugate pair.
and hence $g_f$ describes the rate of exponential growth of $f$-dimensional volume elements in tangent space: $\lambda_f^{(f)}(\phi_0) = g_f(\phi_0)$. Once $g_f$ is known for all values of $f$, the system of equations (4.20) can be solved for $l_i(\phi_0) = \lambda_i^{(1)}(\phi_0)$ in order to obtain the higher Lyapunov exponents for one forms, i.e., $\lambda_i^{(1)}(\phi_0)$.

So far we have shown that the correlation functions $\langle \mathcal{O}(t; \phi_0) \rangle$ encapsulate the information about all the Lyapunov exponents related to a fixed trajectory, namely the one starting at $\phi_0$ a time $t=0$. If we restrict $\phi_0$ to a region in phase-space of connected stochasticity (excluding regions of regular motion) the Lyapunov exponents are independent of $\phi_0$ and we may equally well extract them from the observable (4.7) with the delta-function fixing the initial point omitted, but under a path-integral which is over a restricted class of trajectories only. (As was shown by Oseledec it is not really necessary to insist on closed trajectories.) As we shall see in the next section, expectation values of this type are closely related to the partition function of the superhamiltonian $\widetilde{H}$.

5. PARTITION FUNCTIONS OF THE SUPERHAMILTONIAN

Obviously the superhamiltonian $\widetilde{H}$ of eqs. (3.15), (3.16) does not mix forms of different degree (ghost number). Therefore it makes sense to consider $\widetilde{H}_p$, the restriction of $\widetilde{H}$ to the space of homogeneous $p$-forms, which is spanned by the generalized densities of the type (3.8). Let us evaluate the partition function

$$Z_p(T) = Tr \left[ \exp(-i\widetilde{H}t) \right] = \sum_\alpha <\chi^\alpha_p|\exp(-i\widetilde{H}_pt)|\chi^\alpha_p>$$

(5.1)

where \{|$\chi^\alpha_p$\} is a basis of $p$-forms and \{|$\chi^\alpha_p$\} the dual basis of $p$-vectors (antisymmetric contravariant tensors of degree $p$). In component notation the completeness relation reads

$$\sum_\alpha \chi^\alpha_p(\phi)_{a_1\ldots a_p}\chi^\alpha_{p'}(\phi')^{b_1\ldots b_p} = \delta^{(2n)}(\phi - \phi')\delta_{a_1\ldots a_p}^{b_1\ldots b_p}$$

(5.2)

From (3.7) and (3.31) we obtain for the matrix element of $\exp(-i\widetilde{H}_p t)$

$$\langle \phi, c | \exp(-i\widetilde{H}_p t) | \chi^\alpha_p \rangle = <\Phi_{cl}^{-1}(t; \phi), S^{-1}(t; \phi)c|\chi^\alpha_p> =$$

$$= \frac{1}{p!}\chi^\alpha_p(\Phi_{cl}^{-1}(t; \phi))_{a_1\ldots a_p}\chi^\alpha_{p'}(\phi)_{b_1\ldots b_p}c^{a_1\ldots a_p}b^{b_1\ldots b_p}$$

(5.3)

The partition function (5.1) is obtained by stripping off the ghosts from (5.3), contracting with the dual basis and integrating over $\phi$. Exploiting (5.2) one finds (up to an unimportant

† For an analogous construction in supersymmetric quantum mechanics see ref[18].
constant)

\[ Z_p(T) = \int d\phi \delta(\phi - \Phi^{-1}_{cl}(T; \phi)) S^{-1}(T; \phi)^{[a_1]} \cdots S^{-1}(T; \phi)^{[a_p]} \]
\[ \sim \int d\phi \delta(\Phi_{cl}(T; \phi) - \phi) S(T; \phi)^{[a_1]} \cdots S(T; \phi)^{[a_{2n-p}]} \]  

(5.4)

In the second line we used the identity [A.55] (which is derived in the appendix) and the fact that the Jacobi matrix is unimodular. As it was to be expected, \( Z_p(T) \) receives contributions only from closed trajectories of period \( T \).

Next we show that the partition functions in the \( p \)-form and the \((2n-p)\)-form sector coincide:

\[ Z_p(T) = Z_{2n-p}(T) \]  

(5.5)

The reason is that there exists a duality operation \( \star \) which maps \( p \)-forms on \((2n-p)\)-forms and which commutes with \( \tilde{H} \). On an arbitrary \( p \)-form \( \chi \) it acts as

\[ (\star \chi)_{a_{p+1} \cdots a_{2n}} \sim \epsilon_{a_1 \cdots a_{2n}} \omega^{a_1 b_1} \cdots \omega^{a_p b_p} \chi_{b_1 \cdots b_p} \]  

(5.6)

This kind of \( \star \) operator is analogous to the Hodge operator on Riemannian manifolds. The \( \star \)-operation commutes with \( \tilde{H} \equiv -i l_h \) because the Lie-derivative along the hamiltonian vector field of both the \( \varepsilon \)-tensor and of the \( \omega^{ab} \) vanishes. (The equation \( l_h \varepsilon_{a_1 \cdots a_p} = 0 \) is the component form of eq. (3.38).) Hence the spectra of \( \tilde{H}_p \) and \( \tilde{H}_{2n-p} \) coincide, which is analogous to the well-known Poincaré duality for the Laplacian. From (5.4) with (5.5) and (4.9) it follows that

\[ Z_f(T) = \int d\phi \delta(\Phi_{cl}(T; \phi) - \phi) S(T; \phi)^{[a_1]} \cdots S(T; \phi)^{[a_f]} \]
\[ = \int d\phi \delta(\Phi_{cl}(T; \phi) - \phi) \Gamma_f(T; \phi) \]  

(5.7)

Thus we find that the ratio \( \frac{Z_f(T)}{Z_0(T)} \) can be interpreted as the average of \( \Gamma_f(T) \) over all closed trajectories of (not necessarily primitive) period \( T \). Because \( \Gamma_f = \langle \mathcal{O}_f \rangle \) with \( \mathcal{O}_f \) given in (4.7), we easily could write down a path-integral representation for \( Z_f \) by combining (5.7) with (4.1). Note also that

\[ Z_0(T) = \int d\phi \delta(\Phi_{cl}(T; \phi) - \phi) \]  

(5.8)

"counts" the number of initial points which lead to a closed trajectory of period \( T \). Of course these points are infinite in number. What we should do in (5.8), and in all the
other $Z_f(T)$, is to factor out the equivalence relation which relates two points which are on the same trajectory. This work is in progress\cite{19}. After having factored out the equivalence relation, the new $Z_0(T)$ would count the number of closed orbits of period $T$ and that implies that from this new $Z_0(T)$ we should be able to get the topological entropy of the system\cite{20}. Anyhow the reader should not be worried about the "infinity" produced by $Z_p(T)$ because the physical quantities which we will consider are always given by ratios of $Z_p(T)$ and in these ratios the infinity cancels out.

Until now we were purposely vague about the domain of the $\phi$ integration in eq. (5.7). If the trace in (5.1) is over p-forms defined on the full phase-space $M_{2n}$ then clearly the integration in (5.7) is over all of $M_{2n}$. However, in general one would like to discuss the chaoticity properties of a system in different regions of phase-space separately, and, most importantly one would like to work at fixed energy $E$. The $(2n-1)$-dimensional energy hypersurface $M_{2n-1}(E)$ is the subspace of $M_{2n}$ on which $H(\phi) = E$. If the initial condition of the classical path-integral are fixed such that the initial point $\phi(0)$ lies on $M_{2n-1}(E)$, then the dynamics is such that $\phi(t > 0)$ is still on $M_{2n}$. Correspondingly, if a zero-form $\varrho(\phi, t = 0)$ has support on $M_{2n-1}(E)$ only, this property is conserved under the time evolution. This is not sufficient, however. We also have to make sure that p-forms on $M_{2n-1}(E)$ evolve into p-forms on $M_{2n-1}(E)$. A p-form on $M_{2n-1}(E)$ is a tensor which has no components in the direction perpendicular to the energy hypersurface. Loosely speaking, the exterior algebra on $M_{2n-1}(E)$ is obtained by putting to zero the component of $d\phi^a$ normal to the energy hypersurface: $\partial_a H(\phi)d\phi^a = 0$. Therefore, defining $N(t) \equiv \partial_a H(\phi(t))c^a(t)$, we have to constrain the path-integration to the subspace with $N(t) = 0$. In the second of ref.[10] we have shown that the charge $N$ is conserved under the time evolution: $[N, \tilde{H}] = 0$. In fact, $N$ is the difference between the supersymmetry generator and the BRS-generator: $N = Q_H - Q$. This implies that, imposing $N(t = 0) = 0$, guarantees that $N(t) = 0$ at any later time. As a consequence, if $\tilde{\varrho}(\phi, c, t = 0)$ is a tensor on $M_{2n-1}(E)$, (i.e., if it does not contain any factor of $N$), also $\tilde{\varrho}(\phi, c, t)$ at $t > 0$ is a tensor on $M_{2n-1}(E)$. Thus we can consistently truncate the classical path-integral to the energy hypersurface. In particular we may define partition functions $Z_p(T; E)$ as in eq.(5.1) but with $\chi^\alpha_p$ a complete set of p-forms on $M_{2n-1}(E)$. The next step is to develop a full symplectic and coordinate free formalism. This implies that we will have to decrease the dimension of $M_{2n-1}(E)$ by one unit to go to an even-dimensional subspace of $M_{2n}$. This is done via the so-called Batalin-Fradkin-Vilkovisky method\cite{21}(BFV-formalism) which
implies the introduction of further auxiliary fields and further ghosts. The details of this will be presented elsewhere\textsuperscript{[19]}. 

Let us now go back to $Z_f(T)$. From the asymptotic $T \to \infty$ behaviour of $Z_f(T)$ we define the generalized Lyapunov exponents $\Lambda_f$, $1 \leq f \leq 2n$, according to

$$\frac{Z_f(T)}{Z_0(T)} \sim \exp[(\Lambda_1 + \Lambda_2 + \cdots + \Lambda_f)T]$$

This is the deterministic analogue of eq.(A.57) for stochastic systems which we will find in appendix A. Formally the exponent $\Lambda_1$, say, is defined as in eq.(2.11) with the only difference that the ensemble average $\langle \cdot \rangle$ is not taken with respect to the stochastic measure but with the deterministic one. Because we formulated also the deterministic systems in a path-integral language, this correspondence becomes particularly transparent. Note that generically (for a deterministic system) there is no simple relation between the generalized exponents $\Lambda_i$ and the ordinary ones, $\lambda^{(1)}_i$. In order to obtain the former, one averages the monodromy matrix $S(T)$ for many paths of length $T$ and sends $T$ to infinity afterwards, whereas the latter one is obtained from the large-$T$ behaviour of $S(T)$ on a single trajectory.

We now briefly comment on the eigenvalue problem of the superhamiltonian $\tilde{H}$ and the time evolution operator $\exp(-i\tilde{H}t)$. It is at this point that we encounter the most important differences between stochastic systems and classical hamiltonian systems. In the former case (see appendix A) the superhamiltonian is a second order Schroedinger operator, in the latter it is the first order Lie-derivative operator $\tilde{H} = -il_h$. Let $\chi(\phi, c)$ be an eigenfunction of $\tilde{H}$ in the p-form sector so that

$$\exp(-i\tilde{H}t)\chi(\phi, c) = \exp(-i\tilde{E}t)\chi(\phi, c)$$

for some constant $\tilde{E}$. Using eq. (5.3) we immediately see that the components $\chi_{a_1\cdots a_p}(\phi)$ satisfy the eq.

$$\chi_{a_1\cdots a_p}(\Phi_{cl}(t; \phi))S(t; \phi)_{b_1}^{a_1}\cdots S(t; \phi)_{b_p}^{a_p} = \exp(+i\tilde{E}t)\chi_{b_1\cdots b_p}(\phi)$$

On the LHS of this equation we recognize the usual tensorial transformation law under the Hamiltonian flow. It affects the eigenfunctions $\chi$ only via the overall factor $\exp(i\tilde{E}t)$. Let us look at $p=1$ in more detail, where the relation is

$$\chi_{a}(\Phi_{cl}(t; \phi))S(t; \phi)_{b}^{a} = \exp(i\tilde{E}t)\chi_{b}(\phi)$$

and it has to hold for all $\phi$ and all $t$ if $\chi$ is an eigenfunction. Let us assume we pick a point $\phi_0$ on $\mathcal{M}_{2n}$ which is the initial point of a closed trajectory of duration $\tau$: $\Phi_{cl}(\tau; \phi_0) = \phi_0$. 

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Let $S(\tau; \phi_0)$ be the the Jacobi matrix evaluated at $t = \tau^*$. If we evaluate (5.12) for the special values $\phi = \phi_0$ and $t = \tau$, we find

$$\left[S(\tau; \phi_0) \delta^a_b - \exp(i \tilde{E} \tau) \delta^a_b \right] \chi_a(\phi_0) = 0$$  \hspace{1cm} (5.13)

i.e., that $\exp(i \tilde{E} \tau)$ is an eigenvalue of the monodromy matrix provided that $\chi_a(\phi_0) \neq 0$. This shows that the possible values of $\tilde{E}$ are closely related to the periods and the Floquet multipliers of closed orbits.

As a final remark we mention that in a previous paper (the fourth of ref. [10]) we had shown that the alternating sum

$$Z(T) \equiv STr[\exp(-i \tilde{H} t)] = \sum_{p=0}^{2n} (-1)^p Z_p(T)$$  \hspace{1cm} (5.14)

is a topological invariant (the Euler number of $\mathcal{M}_{2n}$) and has no dynamical significance therefore. It was proven in fact that the alternating sum (5.14) is invariant under deformation of the Hamiltonian vector fields. On the other hand, we saw that the individual $Z_p$’s are not invariant under such deformations and therefore can be used to characterize certain properties of the dynamics.

6. CONCLUSIONS

To summarize we can say that, looking back at eqs. (4.7), (4.8), (4.20) for the ordinary Lyapunov exponents and at (5.9) and (5.1) for the generalized ones, both exponents admit a very simple and natural representation in terms of classical path-integrals. The representation of the Lyapunov exponents as expectation values of some observables allows, for example, then for a perturbative calculations of them in the same manner as it is usually done in field-theory. Viceversa the representation via eq. (5.9) allows the use of spectral methods to calculate then. The supersymmetry of the path-integral is crucial in this context: it relates the “bosonic” dynamics of the trajectories $\phi(t)$ to the evolution of its “fermionic” superpartner, the Jacobi-field $c(t)$ and because of this relation it was natural to expect that information on the dynamics of the Jacobi-fields could be extracted from some objects containing only the dynamics of the standard-phase-space variables as $Z_f$ is. This supersymmetry will also produce\[16\] simplifications in the perturbative calculations in the same way as it does in standard field theory.

\* This is what is called in the literature\[22\] ”monodromy matrix” of this loop.
A lot of work remains to be done in order to extract the dependence on the energy of the ordinary (and generalized) Lyapunov exponents and of the various entropy-like quantities that are associated to them. This work is in progress\textsuperscript{[19]} together with an understanding of the relation of our formalism with the thermodynamic formalism of Ruelle\textsuperscript{[23]} on which we briefly comment in appendix B.

APPENDIX A

In this appendix we use the tools of supersymmetric quantum mechanics in order to discuss the generalized Lyapunov exponents for stochastic systems. The reader should compare the various steps of the derivation with their classical counterparts described in the main body of the paper.

Let us consider a stochastic process on an N-dimensional, metrically flat configuration space with (local) coordinates $x_i, i = 1, \ldots, N$. The dynamics of the random variables $x_i(t)$ is given by the Langevin equation

$$\dot{x}_i(t) = -\partial_i U(x(t)) + \eta_i(t) \quad (A.1)$$

where $U(x)$ is a smooth potential and $\eta_i(t)$ is a white noise:

$$\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t - t') \quad (A.2)$$

It is well known\textsuperscript{[7]} that the stochastic correlations $\langle x_\eta \cdots x_\eta \rangle$ derived from (A.1) can be obtained from a supersymmetric generating functional of the form:

$$Z_{susy} = \int DxD\psi D\bar{\psi} \exp(-\int dt L_{susy} + \text{source terms}) \quad (A.3)$$

where

$$L_{susy} = \frac{1}{2} \dot{x}_i^2 + \frac{1}{2} \left(\partial_t U\right)^2 + \bar{\psi}_i \left[\partial_t \delta_{ij} + \partial_i \partial_j U(x)\right] \psi_j \quad (A.4)$$

The $x_i$ are commuting variables while $\psi_i$ and $\bar{\psi}_i$ are Grassmannian ones. The supersymmetry transformations under which (A.4) is invariant are given by:

$$\delta x_i = -\epsilon \psi_i + \bar{\epsilon} \bar{\psi}_i$$

$$\delta \psi_i = \bar{\epsilon}(-\dot{x}_i + \partial_i U)$$

$$\delta \bar{\psi}_i = \epsilon(\dot{x}_i + \partial_i U) \quad (A.5)$$

In a Schroedinger picture formulation of the supersymmetric quantum
mechanics\textsuperscript{[16,18]} defined by (A.4) the states $|\Phi\rangle$ are described by wave functions

$$\Phi(x_i, \psi_i) \equiv \langle x_i, \psi_i | \Phi \rangle \quad (A.6)$$

depending on the variables $x_i, \psi_i$. The operators $\hat{x}_i$ and $\hat{\psi}_i$ act on $\Phi(x, \psi)$ by multiplication and their conjugate momenta by differentiation:

$$\hat{p}_i = \frac{\partial}{\partial x_i} \equiv \partial_i , \quad \hat{\psi}_i = \frac{\partial}{\partial \psi_i} \quad (A.7)$$

Eq.(A.4) gives rise to the (Weyl ordered) superhamiltonian:

$$H_{susy} = H_B + H_F \quad (A.8)$$

where the "bosonic" part is:

$$H_B \equiv -\frac{1}{2} \partial^2 + \frac{1}{2} \partial_i \partial_j U(x) \partial_i \partial_j \quad (A.9)$$

and the "fermionic" part is:

$$H_F \equiv \frac{1}{2} \partial_i \partial_j U(x) [\hat{\psi}_i \hat{\psi}_j - \hat{\psi}_j \hat{\psi}_i]$$

$$= \frac{1}{2} \partial^2 U(x) - \partial_i \partial_j U(x) \psi_j \frac{\partial}{\partial \psi_i} \quad (A.10)$$

In terms of the supercharges

$$Q = [-\partial_i + \partial_i U(x)] \psi_i$$

$$\bar{Q} = [\partial_i + \partial_i U(x)] \frac{\partial}{\partial \psi_i} \quad (A.11)$$

we have

$$H_{susy} = \frac{1}{2} [Q, \bar{Q}] \quad (A.12)$$

In this operatorial formalism\textsuperscript{[16]} a generic wave function $\Phi(x, \psi)$ possesses an expansion of the form

$$\Phi(x, \psi) = \sum_{q=0}^{N} \frac{1}{q!} \Phi^{(q)}(x) \psi^{k_1} \cdots \psi^{k_q} \quad (A.13)$$

which is reminiscent of the expansion of an inhomogeneous differential form in a basis $dx^{k_1} \wedge \cdots \wedge dx^{k_q}$. We say that $\Phi$ has ghost number "p" if on the RHS of (A.13) only the
term with q=p is different from zero (homogeneous form of degree p):
\[
\Phi(x, \psi) = \frac{1}{p!} \Phi_{k_1 \ldots k_p}(x) \psi^{k_1} \ldots \psi^{k_p} \tag{A.14}
\]
The vacuum of the fermionic Fock space, defined by \( \hat{\psi}_i | vac \rangle = 0 \), is represented by a wave function of ghost number n:
\[
<x, \psi| vac > = \Phi_{vac}(x) \delta(\psi) \tag{A.15}
\]
where the Grassmannian delta-function
\[
\delta(\psi) = \psi^1 \psi^2 \ldots \psi^N \tag{A.16}
\]
can be visualized as describing a completely filled "Dirac sea". Multiparticle states are obtained from \( | vac \rangle \) by acting on \( \delta(\psi) \) with the "creation operator" \( \hat{\psi}_i = \frac{\partial}{\partial \psi_i} \). A state containing f "particles" has a wave function with ghost number N-f:
\[
<x, \psi| \Phi_f > = \Phi_{k_1 \ldots k_f}(x) \frac{\partial}{\partial \psi_{k_f}} \ldots \frac{\partial}{\partial \psi_{k_1}} \delta(\psi) \tag{A.17}
\]
The time evolution of the states \( \Phi(x, \psi, t) \) is governed by the Schroedinger equation
\[
H(x, \frac{1}{i} \frac{\partial}{\partial x}, \psi, \frac{\partial}{\partial \psi}) \Phi(x, \psi, t) = -\partial_t \Phi(x, \psi, t) \tag{A.18}
\]
It has the formal solution \( (t > 0) \)
\[
\Phi(x, \psi, t) = \int d^N x_0 \ d^N \psi_0 \ K(x, \psi, t|x_0, \psi_0, t_0) \Phi(x_0, \psi_0, t_0) \tag{A.19}
\]
The evolution kernel \( \textbf{K} \) is a solution of the Schroedinger equation (A.18) with initial condition
\[
\textbf{K}(x, \psi, t_0|x_0, \psi_0, t_0) = \delta^N(x - x_0) \delta^N(\psi - \psi_0) \tag{A.20}
\]
Its path integral representation involves the lagrangian (A.4):
\[
\textbf{K}(x_2, \psi_2, t_2|x_1, \psi_1, t_1) = \int \mathcal{D}x(t) \ \mathcal{D}\psi(t) \ \mathcal{D}\bar{\psi}(t) \ \exp\{-\int_{t_1}^{t_2} dt L_{\text{susy}}\} \tag{A.21}
\]
The boundary conditions are \( x(t_{1,2}) = x_{1,2}, \psi(t_{1,2}) = \psi_{1,2} \) and \( \bar{\psi}(t_{1,2}) \) is integrated over.
Writing the above kernel as

\[ K(x_2, \psi_2, t_2 | x_1, \psi_1, t_1) = \int \mathcal{D}x(t) \exp \left\{ - \int_{t_1}^{t_2} dt \left[ \frac{1}{2} \dot{x}_i^2 + \frac{1}{2} (\partial_i U)^2 \right] \right\} \cdot K_F(\psi_2, t_2 | \psi_1, t_1; [x]) \]  \hspace{1cm} (A.22)

with

\[ K_F(\psi_2, t_2 | \psi_1, t_1; [x]) \equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ - \int_{t_1}^{t_2} dt \bar{\psi}_i \left[ \partial_t \delta_{ij} + \partial_i \partial_j U(x) \right] \psi_j \right\} \]  \hspace{1cm} (A.23)

we can explicitly evaluate the fermionic kernel \( K_F \) which is a functional of the bosonic path \( x_i(t) \). Following the treatment of ref.[12], we first perform the (unconstrained) \( \bar{\psi} \)-integration in eq.(A.23) which leads to

\[ K_F(\psi_2, t_2 | \psi_1, t_1; [x]) = \int \mathcal{D}\psi \delta \left[ (\partial_t \delta_{ij} + \partial_i \partial_j U(x)) \psi_j \right] \]  \hspace{1cm} (A.24)

Obviously only the solutions of the equation

\[ \dot{\psi}_i = -\partial_i \partial_j U(x(t)) \psi_j \]  \hspace{1cm} (A.25)

obeying the boundary conditions \( \psi(t_{1,2}) = \psi_{1,2} \) contribute to \( K_F \). This is a rather remarkable fact, because eq.(A.25) is precisely the Jacobi eq. pertaining to the Langevin equation (A.1) \( i.e., \) if \( x_i(t) \) is a solution of (A.1), then the variation \( \delta x_i(t) \equiv \psi_i(t) \) is a solution of (A.25). The explicit solution of eq.(A.25) reads

\[ \psi_i(t) = S_{ij}(t; [x]) \psi_j(t_1) \]  \hspace{1cm} (A.26)

with the Jacobi matrix

\[ S(t; [x]) = \hat{T} \exp \left\{ - \int_{t_1}^{t} dt' M(x(t')) \right\} \]  \hspace{1cm} (A.27)

(\( \hat{T} \) denotes the time ordering operator) and where

\[ M_{ij}(x(t)) \equiv \partial_i \partial_j U(x(t)) \]  \hspace{1cm} (A.28)

\( S \) is a solution of the matrix equation \( \dot{S} = -MS \) with \( S(t_1) = 1 \). We obtain for the kernel (A.24)

\[ K_F(\psi_2, t_2 | \psi_1, t_1; [x]) = \int \mathcal{D}\psi \delta \left[ \psi(t) - S(t; [x]) \psi_{t_1} \right] \det \left[ \partial_i \delta_{ij} + \partial_i \partial_j U(x) \right] \]  \hspace{1cm} (A.29)

In order to give a well-defined meaning to the determinant, we have to specify a discretization scheme for the functional integral. Because the Hamiltonian in (A.8) was chosen to be Weyl
ordered, we must use the mid point rule for the discretization.\textsuperscript{[24]} For this discretization, it is known that\textsuperscript{[25]}

\[
det \left[ \partial_t \delta_{ij} + \partial_i \partial_j U \right] = e^{\frac{1}{2} \int_{t_1}^{t_2} dt \partial^2 U(x(t))} \tag{A.30}
\]

Hence the final result for the fermionic kernel is

\[
K_F(\psi_2, t_2 \mid \psi_1, t_1; [x]) = \delta(\psi_2 - S(t_2; [x])\psi_1) e^{\frac{1}{2} \int_{t_1}^{t_2} dt \partial^2 U(x(t))} \tag{A.31}
\]

It is possible to check eq.\textsuperscript{(A.31)} also without referring to path-integral manipulations. The path-integral on the RHS of eq. (A.23) is the formal solution of the Schroedinger equation:

\[
\left[ \frac{1}{2} \partial_i^2 U(x) - \partial_i \partial_j U(x) \partial_j \partial \psi_i + \partial_t \right] K_F(\psi, t|\psi_1, t_1; [x]) = 0 \tag{A.32}
\]

with \(K_F(\psi, t_1|\psi_1, t_1; [x]) = \delta(\psi - \psi_1)\). It can be checked that the RHS of \textsuperscript{(A.31)} does indeed solve this initial value problem (for \(t > 0\)). For future reference we note that \(K_F\) can also be written as

\[
K_F(\psi_2, t_2 \mid \psi_1, t_1; [x]) = \delta(S(t_2; [x])^{-1}\psi_2 - \psi_1) e^{\frac{1}{2} \int_{t_1}^{t_2} dt \partial^2 U(x(t))} \tag{A.33}
\]

because \(\dot{S} = -MS\) implies that

\[
det [S(t)] = e^{\int_{t_1}^{t} dt' \partial^2 U(x(t'))} \tag{A.34}
\]

Obviously the Hilbert space \(\mathcal{H}\) of our theory is a direct sum of subspaces with a fixed ghost number \(p, 0 \leq p \leq N\). Instead of labelling the various sectors by their ghost number (or, equivalently, their degree as a differential form) we shall use \(f \equiv N - p, \ 0 \leq f \leq N\), so that the subspace \(\mathcal{H}_f\) of \(\mathcal{H}\) consists of wave functions of the form (A.17). Consequently

\[
\mathcal{H} = \bigoplus_{f=0}^{N} \mathcal{H}_f \tag{A.35}
\]
The partition function decomposes correspondingly

\[ Z_{\text{susy}}(T) \equiv Tr \left[ e^{\exp(-TH_{\text{susy}})} \right] \]

\[ = \sum_{f=0}^{N} Z_f(T) \]  \hspace{1cm} (A.36)

with

\[ Z_f(T) = \sum_{\alpha} \langle \Phi_\alpha^f | e^{\exp(-TH_f)} | \Phi_\alpha^f \rangle \]  \hspace{1cm} (A.37)

where \( H_f \) denotes the restriction of \( H_{\text{susy}} \) to \( \mathcal{H}_f \), and \( \{ \Phi_\alpha^f \} \) is a complete set of states with ghost number \( p=N-f \). The Hamiltonians are particularly simple for \( f=0 \) and \( f=N \). In these cases they are diagonal in the tensor indices:

\[ H_0 = -\frac{1}{2} \partial^2 + \frac{1}{2} \partial_i U \partial_i U - \frac{1}{2} \partial^2 U \]

\[ H_N = -\frac{1}{2} \partial^2 + \frac{1}{2} \partial_i U \partial_i U + \frac{1}{2} \partial^2 U \]  \hspace{1cm} (A.38)

The respective partition functions can be represented by purely bosonic path-integrals:

\[ Z_{0,N}(T) = \int_{pbc} Dx(t) \exp \left\{ -\int_0^T dt \ L_{0,N} \right\} \]  \hspace{1cm} (A.39)

where

\[ L_0 = \frac{1}{2} x_i^2 + \frac{1}{2} \partial_i U \partial_i U - \frac{1}{2} \partial^2 U \]

\[ L_N = \frac{1}{2} x_i^2 + \frac{1}{2} \partial_i U \partial_i U + \frac{1}{2} \partial^2 U \]  \hspace{1cm} (A.40)

are the restriction of \( L_{\text{susy}} \) to \( \mathcal{H}_0 \) and \( \mathcal{H}_N \), respectively. In (A.39) we used periodic boundary conditions (pbc): \( x(0) = x(T) \). As a preparation for the discussion of the Lyapunov exponents, we shall now give an alternative representation of \( Z_0 \) and \( Z_f \) in terms of the complete supersymmetric lagrangian \( L_{\text{susy}} \) of (A.4). For \( f=0 \) we may write

\[ Z_0(T) = \int_{pbc} Dx(t) \int D\psi(t) D\bar{\psi}(t) \exp \left\{ -\int_0^T dt \ L_{\text{susy}} \right\} \delta(\psi(0)) \]  \hspace{1cm} (A.41)

Again, periodic boundary conditions are used for the \( x_i \)-integration, but \( \psi(0) \) and \( \psi(T) \) are treated as independent integration variables. Note that these boundary conditions are different from the ones usually used in supersymmetric quantum mechanics. The equivalence
of (A.41) and (A.39) is established by noting that:

\[
\int \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp \left\{ - \int_0^T \bar{\psi} \left[ \partial_t \delta_{ij} + \partial_i \partial_j U \right] \psi \right\} \delta(\psi(0)) = \\
= \int d\psi(T) \ d\psi(0) \ K_F(\psi(T), T|\psi(0), 0; [x]) \delta(\psi(0)) \quad \text{(A.42)} \\
= \exp \left\{ \frac{1}{2} \int_0^T dt \partial^2 U(x(t)) \right\}
\]

where (A.31) has been used in the last line. The term \( \sim \partial^2 U \) found in (A.42), when combined with the first two terms on the RHS of eq.(A.4), yields exactly the lagrangian \( L_0 \) of (A.40). In the same way one can show that

\[
Z_N(T) = \int_{\text{pbc}} \mathcal{D} x(t) \int \mathcal{D} \psi(t) \mathcal{D} \bar{\psi}(t) \ \delta(\psi(T)) \ \exp \left\{ - \int_0^T dt L_{\text{susy}} \right\} \quad \text{(A.43)}
\]

(Here and in the following we ignore multiplicative constants.)

In deriving eq. (A.31) we realized already that the dynamics of the Grassmannian variables \( \psi_i(t) \) is essentially deterministic. In fact, given a fixed trajectory \( x_i(t) \), the kernel \( K_F \) is non-zero only if the initial and final value of \( \psi \) are related by the ” classical” Jacobi matrix \( S(t_2; [x]) \). Because all the information about Lyapunov exponents is contained in \( S \), this suggests that it should be possible to extract them from certain correlation functions involving ghosts and antighosts as we did in the deterministic case. In fact, let us consider

\[
\Gamma_f(T) = \langle \psi_{k_1}(T) \psi_{k_2}(T) \cdots \psi_{k_f}(T) \bar{\psi}_{k_f}(0) \cdots \bar{\psi}_{k_2}(0) \bar{\psi}_{k_1}(0) \rangle \quad \text{(A.44)}
\]

with the expectation value \( \langle O \rangle \) of observables \( O \) defined by

\[
\langle O \rangle \equiv Z_0(T)^{-1} \int_{\text{pbc}} \mathcal{D} x \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \ \delta(\psi(0)) \ \exp \left\{ - \int_0^T dt L_{\text{susy}} \right\} \quad \text{(A.45)}
\]

where \( \psi(T) \) and \( \psi(0) \) are independent integration variables. As we discussed in section 4, the operator \( \bar{\psi}_{k_f} \cdots \bar{\psi}_{k_1} \) creates at time \( t=0 \) a f-volume from the vacuum which is propagated by the supersymmetric dynamics until it is destroyed by \( \psi_{k_1} \cdots \psi_{k_f} \) at time
t = T. This becomes obvious if we use (A.4) in (A.45), in order to write

$$\Gamma_f(T) = Z_0^{-1} \int_{pbc} Dx \exp\left\{ -\int_0^T \left[ \frac{1}{2} \dot{x}_i^2 + \frac{1}{2} (\partial_i U)^2 \right] \right\} G_f(T; [x]) \quad (A.46)$$

with

$$G_f(T; [x]) = \int D\psi D\bar{\psi} \exp\left\{ -\int_0^T dt \bar{\psi}_i \left[ \partial_t \delta_{ij} + \partial_i \partial_j U \right] \psi_j \right\} \cdot \psi_k(T) \cdots \psi_{k_f}(T) \bar{\psi}_{k_f}(0) \delta(\psi(0))$$

$$(A.47)$$

Using (A.31) it is easy to obtain:

$$G_f(T; [x]) = \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left\{ -\int_0^T dt \bar{\psi}_i \partial_i^2 U(x) \right\} \cdot S_{k_1}^{k_f} \cdots S_{k_f}^{k_f} \quad (A.48)$$

with $S_i^j \equiv S_{ij}(T; [x])$ (we do not distinguish between upper and lower indices here). Inserting (A.48) into (A.46), and making use of (A.40), we obtain the final result

$$\Gamma_f(T) = \left\langle \left\langle S_{k_1}^{k_f} \cdots S_{k_f}^{k_f} \right\rangle \right\rangle$$

$$(A.49)$$

with the average $\langle\langle \cdots \rangle\rangle$ performed by means of the lagrangian $L_0$, i.e.,

$$\langle\langle O[x]\rangle\rangle \equiv Z_0(T)^{-1} \int_{pbc} Dx \, O[x] \exp\left\{ -\int_0^T dt L_0 \right\} \quad (A.50)$$

Before relating (A.49) to the Lyapunov exponents, let us briefly discuss another representation of $\Gamma_f(T)$. We shall show that $\Gamma_f$ is the ratio of the partition functions of $H$ and $H_0$, respectively:

$$\Gamma_f(T) = \frac{Z_f(T)}{Z_0(0)} = \frac{Tr[exp(-TH_f)]}{Tr[exp(-TH_0)]}$$

$$(A.51)$$

In order to evaluate $Z_f$ as written down in eq. (A.37), we first note that the components
of the vector \( \exp(-TH_f)\Phi_f^\alpha \) can be represented by the bosonic functional integral

\[
(\exp(-TH_f)\Phi_f^\alpha)(x,\psi) = \int Dx \exp\{ - \int_0^T dt L_N \} \Phi_f^\alpha(x(0), S^{-1}(T, [x])\psi) \tag{A.52}
\]

with \( x(T) = x \) and where \( Dx \) includes an integration over \( x(0) \). To arrive at eq. (A.52), we used (A.20) with (A.22) and (A.33), as well as the definition of \( L_N \) in (A.40). Furthermore, the inner product of two \( p \)-forms of the type (A.14) is given by

\[
< \Phi^{(1)}|\Phi^{(2)} > = \frac{1}{p!} \int dN x \Phi^{(1)*}_{k_1\cdots k_p}(x)\Phi^{(2)}_{k_1\cdots k_p}(x) \tag{A.53}
\]

If we use (A.52) with (A.53) in (A.37), and exploit the completeness relation of the \( \Phi_f^\alpha \), we arrive at:

\[
Z_f(T) = \int_{pbc} Dx \exp\{ - \int_0^T L_N \} (S^{-1})^{[k_1}_{k_1} \cdots (S^{-1})^{k_N-f}_{k_N-f}] \tag{A.54}
\]

The occurrence of \( N-f \) factors of the Jacobi matrix is due to the fact that \( H_f \) consists of monomials with \( N-f \) factors of \( \psi \), i.e., differential forms of degree \( N-f \). However, if we use the identity

\[
S^{[k_1} \cdots S^{k_f]}_{k_1} = C^f_N \det(S)(S^{-1})^{[k_1}_{k_1} \cdots (S^{-1})^{k_N-f}_{k_N-f}] \tag{A.55}
\]

with \( \det(S) \) given by eq. (A.34), and \( C^f_N \) some constants, we see that \( Z_f(T) \) coincides with \( Z_0(T) \cdot \Gamma_f(T) \) as given in eq. (A.49). This completes the proof of (A.51).

Benzi et al.\cite{8} and Graham\cite{9} have studied eq. (A.49) for the special case of a one-dimensional configuration space, i.e., \( N=1 \). They determined a generalized Lyapunov exponent \( \Lambda_1 \) from the behaviour of \( \Gamma_1(T) \) for \( T \to \infty : \Gamma_1(T) \sim \exp(\Lambda_1 T) \). If we represent \( \Gamma_1 \) as the ratio of two partition functions, like in eq. (A.51), we see that \( \Lambda_1 \) can be expressed in terms of the lowest eigenvalues \( E^\text{min}_0 \) and \( E^\text{min}_1 \) of \( H_0 \) and \( H_1 \) , respectively:

\[
\Lambda_1 = E^\text{min}_0 - E^\text{min}_1 \tag{A.56}
\]

If supersymmetry is unbroken, the vacuum is non-degenerate so that either \( E^\text{min}_0 < E^\text{min}_1 \) or \( E^\text{min}_0 > E^\text{min}_1 \). Only in the second case, when the vacuum is in the one-form sector, a positive Lyapunov exponent is obtained.
In the work of Benzi et al.\cite{8} and Graham\cite{9} only one dimensional systems were considered, so that only the exponent $\Lambda_1$ related to the evolution of one-dimensional volume elements could be defined. In the present paper we suggest the following interpretation of the correlation function $\Gamma_f(T)$ of (A.49). The antisymmetrized product of $S$-matrices describes the evolution of f-dimensional volume elements. Therefore the large-$T$ behaviour of $\Gamma_f(T)$ can be used to define higher dimensional Lyapunov-like exponents: $\Lambda_f$, $1 \leq f \leq N$.

In view of eq.(2.10) and the discussion in section 4, it is natural to define

$$\Gamma_f(T) \sim \exp\left[(\Lambda_1 + \Lambda_2 + \cdots + \Lambda_f)T\right]$$

(A.57)

for $T \to \infty$. This is the higher dimensional generalization of eq. (2.11) discussed in the introduction. By virtue of eq. (A.51) the sum of the first $f$ $\Lambda$’s is given by the lowest eigenvalue $E_{\text{min}}^f$ in the $f$-form sector:

$$\Lambda_1 + \Lambda_2 + \cdots + \Lambda_f = E_{\text{0}}^\text{min} - E_{\text{f}}^\text{min}$$

(A.58)

The above relation was conjectured in ref.[9] but no proof was provided, while here, using the machinery developed above, we have given a proof of it. For the deterministic case we cannot have a relation like (A.58) because the spectrum of the relative Hamiltonian $\tilde{H}_f$ is not bounded below.
In this appendix we briefly comment on the relation between our path-integral formalism and the thermodynamic one of Ruelle\(^{[23]}\). The basic object of ref.\([23]\) is a Fredholm determinant that is by now called the Ruelle Zeta-function \(\zeta(z)\) is defined as

\[
\zeta(z) = \det[1 - zK_{(0)}] = \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} \text{Tr}(K_{(0)}^n)\right]
\]  

(B.1)

where \(z\) is a complex variable and \(K_0\) (for the classical Hamiltonian systems) is the kernel represented in eq. (3.9) with the ghosts \(c^a\) put to zero and the interval of time taken to be a finite one which we will call \(\Delta\). So, roughly speaking, \(K(\Delta)\) are the matrix elements of \(\exp[-i\hat{H}_0\Delta]\) which is the operator of evolution for zero-forms. The above series can be defined for a wide class of maps and it has a finite radius of convergence. The operator \(\exp[-i\hat{H}_0\Delta]\) is also called the Koopman operator and its inverse the Perron-Frobenius operator\(^{[26]}\). Using the formalism of Ruelle, it is easy to prove that the series (B.1) is also equal to the following one

\[
\zeta(z) = \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n} a_n\right]
\]  

(B.2)

where, (for the Hamiltonian evolution of zero-forms),

\[
a_n = \text{number of periodic trajectories of period } n\Delta
\]  

(B.3)

Note that, using eqn. (5.8), we can say that

\[
a_n = Z_o(n\Delta)
\]  

(B.4)

Here \(Z_0\) has to be evaluated with all the cares we indicated under equations (5.8), basically following the lines of the last of refs.\([23]\). Ruelle proved the relation (B.2) for very general maps and to do that he had to use a highly sophisticated mathematical machinery. At a formal level, for classical hamiltonian systems, the proof is straightforward if we use our path-integral representation\(^{[10]}\). In fact, following the steps (3.4) through (3.9) we see that the trace of the Koopman operator boils down to an integral over a Dirac delta function

\[
\text{Tr} \left[\exp(-i\hat{H}_0(n\Delta))\right] = Z_o(n\Delta)
\]

and this proves the theorem above. We do not pretend any mathematical rigorosity in the proof we have given above, but we have provided it to give to the reader some intuition.
about the Ruelle Zeta-function. Another point to be cautious in the above proof is the problem of the continuum limit. The Ruelle zeta function was always built for discrete maps and not for the continuous time evolution given by our path-integral. So in the proof above, $Z_0(n\Delta)$ should be considered the finite-lattice approximation to our path-integral, and actually that is how path-integrals are defined. Taking the limit of $\Delta$ going to zero has to be done with the same care as explained in the book of Feynman-Hibbs[27] for standard path-integral. If we want to avoid that, then we could choose $\Delta$ as long as the whole interval of time we are interested. Doing that we can then use the Ruelle Zeta-function to derive our $Z_0(\Delta)$, in fact it is easy to see that

$$Z_0(\Delta) = Tr \left[ e^{i(\tilde{H}_0\Delta)} \right] = -\oint \frac{dz}{2\pi i} \frac{\ln \zeta(z)}{z^2} \tag{B.5}$$

So one see that, using the appropriate zeta-function, we can extract one of the partition functions we have used in the paper.

The same construction can also be applied to the kernel of evolution of higher forms and this was also already envisioned by Ruelle in 1976[23]. We can define a generalized Ruelle zeta-function as

$$\zeta_{(p)}(z) = det \left[ 1 - zK_{(p)} \right] = e^{\exp \left[ -\sum_{n=1}^{\infty} \frac{z^n}{n} Tr (K_{(p)}^n) \right]} \tag{B.6}$$

where we called $K_{(p)}$ the operator $e^{i(\tilde{H}_p\Delta)}$ appearing in (5.1) which makes the evolution of p-forms. Like for the zero-form case, for these generalized zeta functions $\zeta_{(p)}(z)$ there exists an alternative representation, namely

$$\zeta_{(p)}(z) = e^{\exp \left[ -\sum_{n=1}^{\infty} \frac{z^n}{n} a_n^{(p)} \right]} \tag{B.7}$$

where the coefficients $a_n^{(p)}$ are given by

$$a_n^{(p)} = \sum_{\phi_{per}} \left[ (S^{-1}(n\Delta; \phi_{per}) a_1 \cdots S^{-1}(n\Delta; \phi_{per}) a_p) \right] \tag{B.8}$$

Here $S^a_b$ is the Jacobi matrix we introduced in eq. (2.3) and $\phi_{per}$ denotes all points from which a periodic orbit of period $n\Delta$ originates. From eq. (B.8) and (5.4) we get immediately that

$$a_n^{(p)} = Z_p(n\Delta) \tag{B.9}$$

The proof of the above theorem is basically what is contained in formulas (5.1) through (5.4). One could have also used directly the $K$ of eq. (3.9) and have all the higher forms included...
in a single formula:

\[ \tilde{\zeta}(z) = \det[1 - zK] = \exp[- \sum_{n=1}^{\infty} Tr(K^n)] \]  

The trace \( TrK^n \) can be taken either by choosing periodic boundary conditions or antiperiodic ones for the ghosts. In the second case, which corresponds to Ruelle’s choice\(^{[23]} \), we will get an expression which is an alternating sum (according to the order of the forms) and which will not depend at all on the Hamiltonian. This has been proved in the third of refs.[10] along the lines of modern topological field theory\(^{[28]} \). It is remarkable that the same trick of getting something which does not depend on the Hamiltonian, by working with an alternating sum, was already employed by Ruelle in 1976. In our paper on the contrary, we have exploited also the individual terms of this alternating sum, and seen that they are related to higher order Lyapunov exponents. Finally we mention also the work of Christiansen et al.\(^{[29]} \) where forms were used in order to get information on the dynamics and not just on topological features of the manifold. In this work\(^{[29]} \) forms have a different meaning than in our formalism, however.

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