Two types of variational integrators and their equivalence

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Abstract

In this paper, we introduce two types of variational integrators, one originating from the discrete Hamilton’s principle while the other from Galerkin variational approach. It turns out that these variational integrators are equivalent to each other when they are used for integrating the classical mechanical system with Lagrangian function $L(q, \dot{q}) = \frac{1}{2}q^T M \dot{q} - U(q)$ ($M$ is an invertible symmetric constant matrix). They are symplectic, symmetric, possess super-convergence order $2s$ (which depends on the degree of the approximation polynomials), and can be related to continuous-stage partitioned Runge-Kutta methods.

Keywords: Euler-Lagrange equations; Hamilton’s equations; Hamilton’s principle; Galerkin methods.

1. Introduction

As is well known, many physical problems can be modeled by using Lagrangian mechanics or Hamiltonian mechanics \cite{1}, which implies that the same physical phenomena can be described in different ways. To be specific, in the context of Lagrangian mechanics the classical problem of computing the dynamics of general mechanical systems attributes to study the solution of the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} ; \quad (1.1)$$

where $q, \dot{q}$ represent generalized coordinates and velocities respectively, and the Lagrangian function $L(q, \dot{q}): \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is often of the form

$L = T(q, \dot{q}) - U(q),$

namely defined as the difference between kinetic energy $T$ and potential energy $U$. A related variational principle called Hamilton’s principle (see, e.g., \cite{13}) states that the true trajectory $q(t)$ of the system \[(1.1)\] between two specified states $q(t_0) = q_0$ and $q(t_N) = q_N$ at two specified time points $t_0$ and $t_N$ is bound to extremize the action functional

$$S(q) = \int_{t_0}^{t_N} L(q(t), \dot{q}(t)) \, dt. \quad (1.2)$$
On the other hand, if the Lagrangian is regular and admits a Legendre transformation via

\[ p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}), \]  

that is, it defines a continuously differentiable bijection \( \dot{q} \leftrightarrow p \) for every \( q \) \[13\], then by introducing

the Hamiltonian function

\[ H(q, p) = p^T \dot{q} - L(q, \dot{q}), \quad \text{with} \quad \dot{q} = \dot{q}(q, p), \]

we have

\[ \dot{q} = \frac{\partial H}{\partial p}(q, p), \] \[1.4\]

And then, it leads to the so-called Hamiltonian system

\[ \dot{q} = \frac{\partial H}{\partial p}(q, p), \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p), \] \[1.5\]

which simplifies the structure of Lagrange’s equations. In fact, \[1.5\] gives an equivalent description of \[1.1\] in Hamiltonian mechanics as stated by the following theorem.

**Theorem 1.1.** \[13\] Lagrange’s equations \[1.1\] are equivalent to Hamilton’s equations \[1.5\].

Generally speaking, it is not easy to obtain the analytical solutions of the system, hence in most situations we resort to numerical methods for deriving approximate solutions. There are two starting points for the numerical discretization of mechanical systems, one from Hamilton’s equations \[1.5\] and the other from Lagrange’s equations \[1.1\]. Different treatments may result in different numerical methods.

According to the relevant theory of Hamiltonian mechanics, the exact flow of the system \[1.5\] is a symplectic transformation in phase space which characterizes the Hamiltonian vector fields \[1.13\]. For the sake of respecting such property, symplectic integration was suggested independently by de Vogelaere (1956) \[34\], Ruth (1983) \[20\] and Feng Kang (1984) \[8\], and in early times the approaches for constructing symplectic methods developed by different authors can be essentially grouped into the following three categories \[23\]: methods based on generating functions \[9, 10\], methods within the framework of standard methods like (partitioned) Runge-Kutta methods \[15, 21, 25, 26, 27\], and methods built around the idea of splitting \[2, 13\]. It is evidenced that symplectic methods are suitable for long-time integration of Hamiltonian systems and we refer the interested readers to \[2, 10, 13, 16, 22\] and references therein. Recently, some new classes of symplectic methods based on Galerkin variational approaches are developed within the framework of Hamiltonian mechanics \[11, 29, 32\].

On the other hand, starting from Lagrangian mechanics and using discrete versions of Hamilton’s principle, symplectic integrators can also be constructed and such integrators are conventionally called “variational integrators” by many authors \[13, 17, 18, 28\] — we might as well call them Lagrangian variational integrators in contrast to Galerkin variational integrators. As far as I know, up to now there have not been any discussions concerning the relationship between Galerkin variational integrators and Lagrangian variational integrators. In this paper, we make a first step towards such subject and this is partially done through the consideration of a special mechanical problem.
This paper will be organized as follows. In Section 2, we introduce two types of variational integrators originated from Lagrangian mechanics and Hamiltonian mechanics respectively. Section 3 is devoted to discuss the equivalence of these two integrators. At last, we give some concluding remarks to end our paper in Section 4.

2. Two types of variational integrators

Prior to introducing the so-called variational integrators, we introduce the shifted Legendre polynomial \( \ell_j(x) \) of degree \( j \) by Rodrigues’ formula

\[
\ell_0(x) = 1, \quad \ell_j(x) = \frac{\sqrt{2j + 1}}{j!} \frac{d^j}{dx^j} \left( x^j(x-1)^j \right), \quad j = 1, 2, 3, \cdots.
\]

(2.1)

Note that these polynomials are normalized and orthogonal in \([0, 1]\), i.e.,

\[
\int_0^1 \ell_j(x) \ell_k(x) \, dx = \delta_{jk}, \quad j, k = 0, 1, 2, \cdots,
\]

(2.2)

where \( \delta_{jk} \) is the Kronecker delta. It is known that \( \{ \ell_j(x) : j \geq 0 \} \) constitutes a complete orthogonal basis in the Hilbert space \( L^2([0, 1]) \).

2.1. Lagrangian variational integrators

In what follows, we present a new class of Lagrangian variational integrators based on the discrete Hamilton’s principle by following the idea of \([5, 13]\). It will be seen that our methods are very similar to the averaged Lagrangian (AL) methods presented in \([5]\) but with some slight differences. In fact, all the AL methods of high order in \([5]\) are obtained by using a collocation approach depending on the Lagrangian interpolation, while for our methods, we avoid using Lagrangian interpolatory polynomials and employ the Legendre polynomial expansions instead, which facilitates us to discuss the relationship between Lagrangian and Galerkin variational integrators later on.

2.1.1. Construction of variational integrators

The general approach of constructing Lagrangian variational integrators is replacing the action functional \([1, 2]\) with a discrete version in the first place and then extremizing it with respect to all possible discrete paths. This is known as the discrete version of Hamilton’s principle \([13]\). Specifically, let

\[
L_h(q_n, q_{n+1}) \approx \int_{t_n}^{t_{n+1}} L(q(t), \dot{q}(t)) \, dt,
\]

where \( q(t) \) is the solution of the Euler-Lagrange equations \([1, 1]\) with boundary values \( q(t_n) = q_n \) and \( q(t_{n+1}) = q_{n+1} \), and thus the action functional \([1, 2]\) is approximated by the sum

\[
S_h(\{ q_n \}_{0}^{N}) = \sum_{n=0}^{N-1} L_h(q_n, q_{n+1})
\]

(2.3)

for given \( q_0 \) and \( q_N \). By requiring \( \partial S_h / \partial q_n = 0 \) for an extremum, it yields the discrete Euler-Lagrange equations

\[
\frac{\partial L_h}{\partial y}(q_{n-1}, q_n) + \frac{\partial L_h}{\partial x}(q_n, q_{n+1}) = 0, \quad n = 1, \cdots, N - 1,
\]

(2.4)
where the partial derivatives are with respect to the arguments of the discrete Lagrangian function $L_h = L_h(x, y)$. By defining a discrete Legendre transformation $q_n + 1 \leftrightarrow p_n$ via \[ p_n = -\frac{\partial L_h}{\partial x}(q_n, q_{n+1}), \] it follows from (2.4),
\[ p_n = -\frac{\partial L_h}{\partial x}(q_n, q_{n+1}), \quad p_{n+1} = \frac{\partial L_h}{\partial y}(q_n, q_{n+1}), \]
which then gives a symplectic integrator in the sense of one-step map $\Phi_h : (p_n, q_n) \mapsto (p_{n+1}, q_{n+1})$ (by generating function theory, for more details, see [13]). It is known that Lagrangian variational integrators possess symplectic and momentum conservation properties as well as good energy behavior [13, 18].

Therefore, a suitable choice of the discrete Lagrangian function $L_h$ is crucial in the construction of Lagrangian variational integrators. Consider the following choice [5]
\[ L_h(q_n, q_{n+1}) = h \int_0^1 L(Q(\tau), \dot{Q}(\tau)) d\tau, \]
where $Q(\tau)$ is assumed to be a $d$-dimensional polynomial of degree $s$ with boundary values $Q(0) = q_n, Q(1) = q_{n+1}$ and by convention here the upper dot stands for the derivative with respect to the time variable $t = t_n + \tau h$. Unlike the interpolatory treatment in [5], we consider the following Legendre expansion for $\dot{Q}(\tau)$:
\[ \dot{Q}(\tau) = \frac{1}{h} Q'(\tau) = \sum_{i=0}^{s-1} \dot{Q}_i \ell_i(\tau), \]
where $\dot{Q}_i$ are formal notations representing for the expansion coefficients. By multiplying $\ell_j(\tau)$ from two sides and taking integral of (2.7), it follows
\[ \dot{Q}_j = \int_0^1 \dot{Q}(\tau) \ell_j(\tau) d\tau, \quad j = 0, \ldots, s - 1. \]
By taking integral of (2.7) with respect to $\tau$, we get
\[ Q(\tau) = q_n + h \sum_{i=0}^{s-1} a_{\tau,i} \dot{Q}_i, \]
where
\[ a_{\tau,i} = \int_0^\tau \ell_i(x) dx, \quad i = 0, \ldots, s - 1. \]
By setting $\tau = 1$ in (2.9), then it gives
\[ q_{n+1} = Q(1) = q_n + h \sum_{i=0}^{s-1} b_i \dot{Q}_i = q_n + h \dot{Q}_0 \quad \text{(2.8)} \]
\[ q_{n+1} = q_n + h \int_0^1 \dot{Q}(\tau) d\tau, \]

\[ ^1 \text{They are also abuse of notations, noting that here the upper dot does not stand for the derivative with respect to time. Later on, } \dot{P}(\tau) \text{ in (2.12) is also an abuse of notation with the upper dot generally not representing the derivative.} \]
where
\[ b_i = a_{1,i} = \int_0^1 \ell_i(x) \, dx = \delta_{i0}, \quad i = 0, \ldots, s - 1. \]

Under the constraint (2.10) (note that the constraint \( Q(0) = q_n \) has been automatically satisfied), we use the method of Lagrange multipliers to obtain extremality conditions. By introducing the Lagrangian multipliers \( \lambda = (\lambda_1, \cdots, \lambda_d) \) and differentiating (2.6) with respect to \( \dot{Q}_i \), it gives
\[ \int_0^1 \frac{\partial L}{\partial q}(Q, \dot{Q}) h a_{\tau,i} + \frac{\partial L}{\partial \dot{q}}(Q, \dot{Q}) \ell_i \, d\tau = b_i \lambda = \delta_{i0} \lambda. \]  
(2.11)

With the notation
\[ \dot{P}(\tau) = \frac{\partial L}{\partial q}(Q(\tau), \dot{Q}(\tau)), \quad P(\tau) = \frac{\partial L}{\partial \dot{q}}(Q(\tau), \dot{Q}(\tau)), \] we can simplify (2.11) to
\[ \int_0^1 \dot{P} a_{\tau,i} + P \ell_i \, d\tau = \delta_{i0} \lambda. \]  
(2.13)

Substituting (2.6) into (2.5) yields
\[ p_n = -\frac{\partial L_h}{\partial x}(q_n, q_{n+1}) \]
\[ = -h \int_0^1 \dot{P} \left[ I + h \sum_{i=0}^{s-1} a_{\tau,i} \frac{\partial \dot{Q}_i}{\partial q_n} \right] + P \left[ \sum_{i=0}^{s-1} \ell_i \frac{\partial \dot{Q}_i}{\partial q_n} \right] \, d\tau \]  
(2.14)
\[ = -h \int_0^1 \dot{P} d\tau + \lambda. \]

Here we have used (2.13) and the equality
\[ h \sum_{i=0}^{s-1} b_i \frac{\partial \dot{Q}_i}{\partial q_n} = -I, \]
which is deduced by differentiating the constraint (2.10) with respect to \( q_n \). Analogously, for \( p_{n+1} \) we compute
\[ p_{n+1} = \frac{\partial L_h}{\partial y}(q_n, q_{n+1}) \]
\[ = h \int_0^1 \dot{P} \left[ h \sum_{i=0}^{s-1} a_{\tau,i} \frac{\partial \dot{Q}_i}{\partial q_{n+1}} \right] + P \left[ \sum_{i=0}^{s-1} \ell_i \frac{\partial \dot{Q}_i}{\partial q_{n+1}} \right] \, d\tau \]  
(2.15)
\[ = \lambda, \]
where we have employed the following identity
\[ h \sum_{i=0}^{s-1} b_i \frac{\partial \dot{Q}_i}{\partial q_{n+1}} = I, \]
derived by differentiating (2.10) with respect to \( q_{n+1} \).
In summary, we have the following method which closely resembles a partitioned Runge-Kutta scheme
\[
\int_0^1 P(\tau)\ell_i(\tau)\,d\tau = p_n\delta_{i0} + h\int_0^1 \left[\delta_{i0} - a_{r,i}\right]\hat{P}(\tau)\,d\tau, \quad i = 0, \cdots, s - 1, \tag{2.16a}
\]
\[
q_{n+1} = q_n + h\int_0^1 \hat{Q}(\tau)\,d\tau, \quad p_{n+1} = p_n + h\int_0^1 \hat{P}(\tau)\,d\tau, \tag{2.16b}
\]
where
\[
\hat{P}(\tau) = \frac{\partial L}{\partial q}(Q(\tau), \hat{Q}(\tau)), \quad P(\tau) = \frac{\partial L}{\partial q}(Q(\tau), \hat{Q}(\tau)),
\]
with
\[
\hat{Q}(\tau) = \sum_{j=0}^{s-1} \ell_j(\tau)\hat{Q}_j, \quad Q(\tau) = q_n + h\sum_{j=0}^{s-1} a_{r,j}\hat{Q}_j. \tag{2.17}
\]

The implementation of the method above is similar to the AL integrator presented in [5]. We regard \(\hat{Q}_j\) \((j = 0, \cdots, s - 1)\) as the unknowns which can be first solved by using (2.16a) and then one can get the numerical solutions \(q_{n+1}\) and \(p_{n+1}\) by (2.16b). Besides, for computing the integrals of (2.16), we often have to resort to numerical quadrature rules.

**Remark 2.1.** Remark that \(Q(\tau)\) is an \(s\)-dimensional polynomial of degree \(s\), but from (2.17) it is clear that \(P(\tau)\) is not necessarily polynomial and the format of it depends on the Lagrangian function \(L(q, \dot{q})\). It is not easy to analyze the order accuracy of the Lagrangian variational methods, although one can numerically verify the order is always \(2s\) as done in [4].

### 2.1.2. Hamiltonian formalism

The presented method (2.16) can be reshaped in the Hamiltonian formalism by following the technique given in the Appendix of [5]. In fact, by using Theorem 1.3 of [13] and (1.4) (assume the Legendre transformation exists), it gives (cf. Appendix of [5])
\[
\dot{Q}(\tau) = \frac{\partial H}{\partial p}(Q(\tau), P(\tau)), \quad \dot{P}(\tau) = -\frac{\partial H}{\partial q}(Q(\tau), P(\tau)).
\]

This allows us to rewrite (2.16) as
\[
\int_0^1 P(\tau)\ell_i(\tau)\,d\tau = p_n\delta_{i0} - h\int_0^1 \left[\delta_{i0} - a_{r,i}\right]\frac{\partial H}{\partial q}(Q(\tau), P(\tau))\,d\tau, \quad i = 0, \cdots, s - 1, \tag{2.18a}
\]
\[
q_{n+1} = q_n + h\int_0^1 \frac{\partial H}{\partial p}(Q(\tau), P(\tau))\,d\tau, \quad p_{n+1} = p_n - h\int_0^1 \frac{\partial H}{\partial q}(Q(\tau), P(\tau))\,d\tau, \tag{2.18b}
\]
where
\[
\dot{Q}(\tau) = \frac{\partial H}{\partial p}(Q(\tau), P(\tau)), \tag{2.19}
\]
with
\[
\dot{Q}(\tau) = \sum_{j=0}^{s-1} \ell_j(\tau)\hat{Q}_j, \quad Q(\tau) = q_n + h\sum_{j=0}^{s-1} a_{r,j}\hat{Q}_j.
\]

It is well to notice that (2.19) implicitly determines \(P(\tau)\). Compared with the Lagrangian formalism (2.16), the Hamiltonian formalism (2.18) seems not convenient to use due to the implication of (2.19) for getting \(P(\tau)\).
2.2. Galerkin variational integrators

In order to construct the Galerkin variational integrators, in the following we consider the time discretization of the Hamilton’s equations (1.5) with a Galerkin weak form. We denote the whole integration domain by \( I = [t_0, T] \) and specify the corresponding initial values as
\[
q(t_0) = q_0 \in \mathbb{R}^d, \quad p(t_0) = p_0 \in \mathbb{R}^d.
\]
We use the following temporal grid to cover the whole computational domain \( I \):
\[
t_0 < t_1 < \cdots < t_{N-1} < t_N = T
\]
with the following notations for elements and the step size
\[
I_j = [t_j, t_{j+1}], \quad h_j = \Delta t_j = t_{j+1} - t_j, \quad 0 \leq j \leq N - 1.
\]
In the following we assume \( h_j \equiv h \) and introduce the vector-valued space
\[
(V_h^k(I))^d = V_h^k(I) \times V_h^k(I) \times \cdots \times V_h^k(I),
\]
denoting the set of polynomials of degree up to \( k \).

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In the following we assume \( h_j \equiv h \) and introduce the vector-valued space
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(V_h^k(I))^d = V_h^k(I) \times V_h^k(I) \times \cdots \times V_h^k(I),
\]
denoting the set of polynomials of degree up to \( k \).

We define the Galerkin method for solving (1.5) as follows \[30\]: find \((Q, P) \in (V_h^s(I))^d \times (V_h^{s-1}(I))^d\), such that
\[
\int_{t_n}^{t_{n+1}} [Q \cdot \dot{\phi} + f(Q, P) \cdot \phi] \, dt = \tilde{Q}_{n+1} \phi(t_{n+1}) - \tilde{Q}_n \phi(t_n^+), \quad (2.21a)
\]
\[
\int_{t_n}^{t_{n+1}} [P \cdot \dot{\varphi} + g(Q, P) \cdot \varphi] \, dt = \tilde{P}_{n+1} \varphi(t_{n+1}) - \tilde{P}_n \varphi(t_n^+), \quad (2.21b)
\]
for \( \forall (\phi, \varphi) \in V_h^{s-1}(I) \times V_h^s(I) \) and \( n = 0, 1, \cdots, N - 1 \), where the numerical fluxes \( \tilde{Q}_n \) and \( \tilde{Q}_{n+1} \) are defined by
\[
\tilde{Q}_n = Q(t_n^+), \quad \tilde{Q}_{n+1} = Q(t_n^-), \quad (2.22)
\]
but for \( \tilde{P}_n \) and \( \tilde{P}_{n+1} \), no extra constraints are imposed. If we regard the numerical fluxes as the boundary values of the trial functions in a typical element \( I_n \), then it implies that \( Q \) is continuous at the element interfaces, while \( P \) admits two jump discontinuities, one at time \( t_n \) and one at time \( t_{n+1} \). Besides, we assume that the numerical fluxes at time \( t_0 \) are given by the initial values \( (2.20) \), say
\[
\tilde{Q}_0 = q_0, \quad \tilde{P}_0 = p_0.
\]

Remark 2.2. The Galerkin method (2.21) is equivalent to a class of the time finite element methods presented in \[24\]. However, unlike the format shown in \[24\], here we present a new elegant formulation by virtue of the terminology “numerical fluxes” (which is widely used in the field of discontinuous Galerkin methods \[24\]) to represent the boundary values of the trial functions.

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In the following we show that the presented Galerkin method (2.21)-(2.22) can be interpreted as a class of continuous-stage partitioned Runge-Kutta methods [29, 30, 31]. Without loss of generality, we restrict our analysis on the typical element $I_n$. From (2.21a) and (2.22), employing partial integration yields

$$\int_{t_n}^{t_{n+1}} [\dot{Q} \cdot \phi - f(Q, P) \cdot \phi] \, dt = 0 \implies \int_0^1 [Q_\tau' \cdot \phi_\tau - h f(Q_\tau, P_\tau) \cdot \phi_\tau] \, d\tau = 0,$$

(2.23)

where we have introduced some new notations:

$$Q_\tau = Q(t_n + \tau h), \quad P_\tau = P(t_n + \tau h), \quad \phi_\tau = \phi(t_n + \tau h), \quad \tau \in [0, 1].$$

(2.24)

Consider the expansion along the Legendre polynomial basis $\{\ell_i(\tau)\}$

$$Q_\tau' = \sum_{i=0}^{s-1} \gamma_i \ell_i(\tau), \quad \gamma_i \in \mathbb{R}^d.$$

(2.25)

Letting $\phi_\tau = \ell_0(\tau), \ldots, \ell_{s-1}(\tau) \in V_{h}^{-1}(I)$ in (2.23) and plugging (2.25) into (2.23), yields

$$\gamma_j = h \int_0^1 f(Q_\sigma, P_\sigma) \ell_j(\sigma) \, d\sigma, \quad j = 0, \ldots, s-1.$$

(2.26)

By taking integral of (2.25) and inserting (2.26) into it, we get

$$Q_\tau = \hat{Q}_n + h \int_0^1 \left[ \sum_{i=0}^{s-1} \int_{t_n}^{t_\tau} \ell_i(x) \, dx \ell_j(\sigma) \right] f(Q_\sigma, P_\sigma) \, d\sigma.$$

(2.27)

Setting $\tau = 1$ and noticing

$$\int_0^1 \ell_i(x) \, dx = \delta_{i0}, \quad i = 0, 1, \ldots,$$

(2.28)

follows

$$\hat{Q}_{n+1} = \hat{Q}_n + h \int_0^1 f(Q_\sigma, P_\sigma) \, d\sigma.$$

(2.29)

By using the transformation $t = t_n + \tau h$, (2.21b) becomes

$$\int_0^1 [P_\tau : \varphi_\tau' + hg(Q_\tau, P_\tau) \cdot \varphi_\tau] \, d\tau = \hat{P}_{n+1} \varphi_1 - \hat{P}_n \varphi_0,$$

(2.30)

where $\varphi_\tau = \varphi(t_n + \tau h)$ and other notations are given by (2.24). Letting

$$\varphi_\tau = 1, \quad \int_0^\tau \ell_0(x) \, dx, \quad \ldots, \quad \int_0^\tau \ell_{s-1}(x) \, dx \in V_{h}^{s}(I),$$

and substituting the following expansion

$$P_\tau = \sum_{i=0}^{s-1} \lambda_i \ell_i(\tau), \quad \lambda_i \in \mathbb{R}^d$$

(2.31)

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2 For simplicity, we have omitted the label information $n$ for representing the $n$-th element $I_n$. 

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Substituting (2.32) into (2.31) gives

\[ P_{\tau} = \hat{P}_{n} + h \int_{0}^{1} \left[ 1 - \sum_{i=0}^{s-1} \int_{0}^{\sigma} \ell_{i}(x)dx \ell_{i}(\tau) \right] g(Q_\sigma, P_\sigma) d\sigma. \]  

(2.33)

Collecting (2.27), (2.29), (2.32) and (2.33) yields the following continuous-stage partitioned Runge-Kutta method

\[
Q_\tau = \hat{Q}_{n} + h \int_{0}^{1} A_{\tau, \sigma} f(Q_\sigma, P_\sigma) d\sigma, \quad \tau \in [0, 1], \\
P_\tau = \hat{P}_{n} + h \int_{0}^{1} \hat{A}_{\tau, \sigma} g(Q_\sigma, P_\sigma) d\sigma, \quad \tau \in [0, 1], \\
\hat{Q}_{n+1} = \hat{Q}_{n} + h \int_{0}^{1} B_{\tau} f(Q_\tau, P_\tau) d\tau, \\
\hat{P}_{n+1} = \hat{P}_{n} + h \int_{0}^{1} \hat{B}_{\tau} g(Q_\tau, P_\tau) d\tau,
\]

where

\[
A_{\tau, \sigma} = \sum_{i=0}^{s-1} \int_{0}^{\tau} \ell_{i}(x)dx \ell_{j}(\sigma), \quad \hat{A}_{\tau, \sigma} = 1 - \sum_{i=0}^{s-1} \int_{0}^{\sigma} \ell_{i}(x)dx \ell_{i}(\tau), \quad B_{\tau} = 1, \quad \hat{B}_{\tau} = 1.
\]

(2.34)

(2.35)

Obviously, (2.33) naturally gives a one-step method \(z_{n+1} = \Phi_h(z_n)\) with \(z_n = (\hat{Q}_n, \hat{P}_n)^T\) associated with the boundary values (numerical fluxes) of trial functions. Consequently, the Galerkin method (2.21) can be implemented step by step via solving a standard one-step partitioned method in each element \(I_n\).

**Theorem 2.1.** If the coefficients of a continuous-stage partitioned Runge-Kutta method with the format (2.34) satisfy

\[ B_{\tau} \hat{A}_{\tau, \sigma} + \hat{B}_{\sigma} A_{\sigma, \tau} = B_{\tau} \hat{B}_{\sigma} \quad \text{and} \quad B_{\tau} = \hat{B}_{\tau}, \quad \text{for} \ \forall \ \tau, \ \sigma \in [0, 1], \]

then the method is symplectic for solving Hamiltonian systems (1.5).

**Proof.** We omit the proof since it is very similar to the classical case (cf. Theorem 6.2 of [22] and Theorem 3.1 of [33]). \(\square\)

**Corollary 2.1.** [29] The continuous-stage partitioned Runge-Kutta method (2.34) with coefficients (2.35) is symplectic, symmetric and of order \(2s\). Besides, by using a quadrature formula to approximate the integrals of (2.34) it gives a symplectic partitioned Runge-Kutta method.
Proof. The symplecticity of the method is straightforward from Theorem 2.1. The symmetry property of the method can be easily verified by using the definition of symmetric methods \[13\]. The order is straightforward by using Theorem 3.2 of \[31\] and noticing the fact that symmetric methods possess an even order \[13\]. For the last statement, please refer to \[31\], where the more general cases were studied in detail.

Remark 2.3. If we interchange the role of \(p\)-variable and \(q\)-variable in the format of the Galerkin method \(2.21\), then the resulting method can also be recast as \(2.34\) but with the coefficients \((A_{\tau, \sigma}, B_{\tau})\) and \((\hat{A}_{\tau, \sigma}, \hat{B}_{\tau})\) exchanged in \(2.35\). The corresponding result as given in Corollary 2.1 still holds.

3. Discussion of the equivalence

It is known that for the Galerkin variational integrator \(2.21)-(2.22)\), both \(P(\tau)\) and \(Q(\tau)\) are polynomials approximating to the exact solutions \(p\) and \(q\), while for the Lagrangian variational integrator \(2.16)-(2.17)\), this fact may not always holds (see Remark (2.1)). Therefore, in general the two variational integrators are not the same. However, this does not precludes the possibility of the other side being realized in some special cases. For instance, let us consider the classical mechanical system with an invertible symmetric constant matrix \(M\), say,

\[
L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q} - U(q).
\]

In such a case, the Legendre transformation is legal, i.e.,

\[
p = \frac{\partial L}{\partial \dot{q}} = M \dot{q} \iff \dot{q} = \frac{\partial H}{\partial p}(q, p) = M^{-1} p,
\]

where

\[
H(q, p) = \frac{1}{2} p^T M^{-1} p + U(q),
\]

and hence the corresponding Lagrangian variational integrator possesses the Hamiltonian formalism \(2.18)-(2.19)\). By using \(2.17\), it follows

\[
P(\tau) = \frac{\partial L}{\partial \dot{q}}(Q(\tau), \dot{Q}(\tau)) = M \dot{Q}(\tau),
\]

which means \(P(\tau)\) is also a \(d\)-dimensional polynomial with the same degree of \(\dot{Q}(\tau)\), say, \(s - 1\). Hence it can be expanded along the Legendre polynomial basis \(\{\ell_i(\tau)\}\) in the form

\[
P(\tau) = \sum_{i=0}^{s-1} \gamma_i \ell_i(\tau), \quad \gamma_i \in \mathbb{R}^d,
\]

where

\[
\gamma_i = \int_0^1 P(\tau) \ell_i(\tau) \, d\tau, \quad i = 0, \cdots, s - 1.
\]

(3.1)

Multiplying \(\ell_i(\sigma)\) from both sides of \(2.18\) for \(i = 0, \cdots, s - 1\) and adding all the resulting formulas gives

\[
\sum_{i=0}^{s-1} \int_0^1 P(\tau) \ell_i(\tau) \, d\tau \ell_i(\sigma) = \frac{p_n}{\hbar} \sum_{i=0}^{s-1} \delta_{i0} \ell_i(\sigma) - \frac{1}{\hbar} \int_0^1 \sum_{i=0}^{s-1} \left[ \delta_{i0} \ell_i(\sigma) - a_{\tau, i} \ell_i(\sigma) \right] \frac{\partial H}{\partial q}(Q(\tau), P(\tau)) \, d\tau,
\]

\[
= \frac{1}{\hbar} \int_0^1 \sum_{i=0}^{s-1} \left[ \delta_{i0} \ell_i(\sigma) - a_{\tau, i} \ell_i(\sigma) \right] \frac{\partial H}{\partial q}(Q(\tau), P(\tau)) \, d\tau.
\]
which can be further simplified to
\[
P(\sigma) = p_n - h \int_0^1 \left[ 1 - \sum_{i=0}^{s-1} \int_0^\tau \ell_i(x) \, dx \ell_i(\sigma) \right] \frac{\partial H}{\partial q}(Q(\tau), P(\tau)) \, d\tau.
\]

By interchanging the notations \( \tau \leftrightarrow \sigma \), we then get
\[
P(\tau) = p_n - h \int_0^1 \left[ 1 - \sum_{i=0}^{s-1} \int_0^\sigma \ell_i(x) \, dx \ell_i(\tau) \right] \frac{\partial H}{\partial q}(Q(\sigma), P(\sigma)) \, d\sigma.
\] (3.2)

Besides, by combining (2.8), (2.9) and (2.19), we have
\[
Q(\tau) = q_n + h \sum_{i=0}^{s-1} a_{\tau,i} \int_0^1 \frac{\partial H}{\partial p}(Q(\sigma), P(\sigma)) \ell_i(\sigma) \, d\sigma
\]
\[
= q_n + h \int_0^1 \left[ \sum_{i=0}^{s-1} \int_0^\tau \ell_i(x) \, dx \ell_i(\sigma) \right] \frac{\partial H}{\partial p}(Q(\sigma), P(\sigma)) \, d\sigma.
\] (3.3)

Gathering the important formulas (3.2), (3.3) and (2.18b) gives a continuous-stage partitioned Runge-Kutta method which coincides with the scheme (2.34)–(2.35) except for \( \tilde{P}_n, \tilde{Q}_n, \tilde{P}_{n+1}, \tilde{Q}_{n+1} \) being replaced by \( p_n, q_n, p_{n+1}, q_{n+1} \) respectively. This implies that the two types of variational integrators are equivalent to each other. Clearly, the points \( q_n \) (see (2.3)) along the discrete paths and the discrete momenta \( p_n \) can be understood as the numerical fluxes of the Galerkin methods.

Besides, by using the Lagrangian variational technique given in Section 2.1 one should not be able to acquire the Galerkin method mentioned in Remark 2.3 since in order to employ the discrete Hamilton’s principle one usually has to fix the boundary values.

4. Concluding remarks

This paper is concerned with two types of variational integrators and their equivalence. These variational integrators are symplectic methods which are suitable for long-time integration of those systems with the exact flow being a symplectic transformation. For a special class of mechanical systems, they are shown to be equivalent to each other but the same result may not holds in a more general case. Therefore, the close relationship between Lagrangian and Galerkin variational integrators is partially observed. The study in this paper helps us to further understand a profound viewpoint proposed by Kang Feng (see, e.g., [10]) saying that different representations for the same physical law can lead to different computational techniques in solving the same problem, which may produce different numerical results.

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