Localization in open quantum systems

I. Yusipov\textsuperscript{1}, T. Laptyeva\textsuperscript{2}, S. Denisov\textsuperscript{3,4}, and M. Ivanchenko\textsuperscript{4}

\textsuperscript{1}Institute of Supercomputing Technologies, Lobachevsky University, Gagarina Av. 23, Nizhny Novgorod, 603950, Russia
\textsuperscript{2}Department of Control Theory and Systems Dynamics, Lobachevsky University, Gagarina Av. 23, Nizhny Novgorod, 603950, Russia
\textsuperscript{3}Institute of Physics, University of Augsburg, Universitätsstraße 1, 86159 Augsburg, Germany
\textsuperscript{4}Department of Applied Mathematics, Lobachevsky State University of Nizhny Novgorod, Gagarina Av. 23, Nizhny Novgorod, 603950, Russia

In an isolated single-particle quantum system a spatial disorder can induce Anderson localization. Being a result of interference, this phenomenon is expected to be fragile in the face of dissipation. Here we show that a proper dissipation can drive a disordered system into a steady state with tunable localization properties. This can be achieved with a set of identical dissipative operators, each one acting non-trivially on a pair of sites. Operators are parametrized by a uniform phase, which controls selection of Anderson modes contributing to the state. On the microscopic level, quantum trajectories of a system in the asymptotic regime exhibit intermittent dynamics consisting of long-time sticking events near selected modes interrupted by inter-mode jumps.

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FIG. 1: Absolute values of the elements of the asymptotic density matrix $\rho_\infty$ expressed in (a) the original basis and (b) the basis of the Anderson modes (a relevant part is shown) for a particular disorder realization of the model system, Eqs. (1-3), and in-phase next-neighbor dissipators, $\alpha = 0$ and $l = 1$ in Eq. (4). Other parameters are $W = 1$, $N = 100$, and $\gamma = 0.1$.

In practice, however, this will require a priori knowledge of state $|\phi_i\rangle$ and synthesis of very peculiar dissipative operator(s). This can be unfeasible. A realistic and not very specific choice of operators is more attractive. Local dissipators, i.e., those which act non-trivially on a finite number of neighboring particles, are natural in many contexts. Such operators is a popular choice in the recent works on open quantum systems [11, 12, 31–34]. This is also our choice here.

We consider an open single-particle model described by Eq. (1) with a Hamiltonian [21]

$$H = \sum_k \epsilon_k b_k^\dagger b_k - (b_k^\dagger b_{k+1} + b_{k+1}^\dagger b_k),$$

where $\epsilon_k \in [-W/2, W/2]$ are random uncorrelated on-site energies, $W$ is the disorder strength, $b_k$ and $b_k^\dagger$ are the annihilation and creation operators of a boson on the $k$-th site. We recall that the eigenvalues of the Hamiltonian are restricted to a finite interval, $\lambda_\nu \in [-2 - W/2, 2 + W/2]$, while the respective eigenstates, $A^{(\nu)}_k$, are exponentially localized. The localization length is approximated by $\xi_\lambda \approx 24(4 - \lambda^2)/W^2$ [38] with corrections about the band edges [39].

A single dissipative operator acts on a pair of sites,

$$V_k = (b_k^\dagger + e^{i\alpha} b_{k+l}^\dagger)(b_k - e^{-i\alpha} b_{k+l}).$$

When $\alpha = 0$, this operator tries to synchronize the dynamics on the $k$ and $k + l$ sites, by constantly recycling anti-symmetric out-of-phase mode into the symmetric in-phase one. This type of dissipation, with $l = 1$, was introduced in Refs. [11, 12]. A physical implementation of a Bose-Hubbard chain with neighboring sites coupled by such dissipators was discussed in Ref. [35]. The proposed set-up consists of an array of superconductive resonators coupled by qubits; a pair-wise dissipator with $l = 1$ and arbitrary phase $\alpha$ can be realized with the same set-up by (i) varying the photonic mode frequency $\omega_k$ from cavity to cavity (to simulate the disorder), and (ii) adjusting position of the qubits with respect to the centers of the corresponding cavities [36]. In principle, this is enough for our purpose. However, to address other possible implementations, we will consider also dissipators with $l > 1$.

Next we assume $\rho_0 = \rho_{N+1} = 0$ at the boundaries and analyze properties of the stationary solution $\rho_\infty$ of Eq. (1), which is unique due to the absence of relevant symmetries [27, 40]. To find it, we use a column-wise vectorization of the density matrix and define the asymptotic solution, a super-vector $\rho_\infty$, as the kernel of a Liouvillian-induced super-operator $\Pi$, $\Pi \rho_\infty = 0$. After folding it into the

FIG. 2: Averaged absolute values at the diagonal of $\rho_\infty$, expressed in the Anderson basis, as functions of averaged eigenvalues for the case of in-phase next-neighbor dissipation, $\alpha = 0$ and $l = 1$ in Eq. (4). Averaging was performed over $N_r = 10^2$ disorder realizations. Theoretical prediction, Eq. (9), is shown by a thick solid line. The other parameters are the same as in Fig. 1.
FIG. 3: (a) Absolute values of the elements of the stationary density matrix $\varrho_\infty$ in the direct basis and (b) the basis of the Anderson modes (a relevant part is shown) for a particular disorder realization of the model, Eqs. (1-3), and out-of-phase next-nearest-neighbor dissipators, $\alpha = \pi$ and $l = 2$ in Eq. (4). Other parameters are $W = 1$, $N = 100$, $\gamma = 0.1$.

matrix form and trace-normalizing, we get the asymptotic state density matrix $\varrho_\infty$.

For the choice $\alpha = 0$ and $l = 1$ in Eq. (4), that is the case of in-phase next-neighbor dissipative coupling [11,12,35], the asymptotic density matrix exhibits a patchy structure with several 'hot' localization spots, Fig. 1a. Remarkably, expressing the density operator in the basis of Anderson modes $A_\nu$, we get a near diagonal matrix with a strong contributions from the eigenstates from the lower part of the spectrum, Fig. 1b. To evaluate this finding analytically, we rewrite Eq. (1) in the Anderson basis and neglect the off-diagonal elements. Under this approximation, the evolution of the diagonal elements is governed by dissipative terms only, 

$$\dot{\varrho}_{p,p} = \gamma \left( \sum_q I_{p,q} \varrho_{q,q} - \varrho_{p,p} \sum_q I_{q,p} \right),$$  

(5) where the overlap coefficients $I_{p,q} = \sum_k |(\hat{V}_k)_{q,p}|^2$ are given by the dissipators in the Anderson basis, $\{\hat{V}_k\}$. Explicitly,

$$I_{p,q} = \sum_k (A_{p,k+1} e^{i\alpha} A_{p,k})^2 \left( A_{q,k+l} e^{-i\alpha} A_{q,k} \right)^2. $$  

(6)

Denoting $I_{p,k}^\pm = (A_{p,k+l} \pm e^{\pm i\alpha} A_{p,k})^2$, we obtain for the stationary solution:

$$\varrho_{p,p} = \frac{\sum_k I_{p,k}^+ \sum_q I_{q,k}^- \varrho_{q,q}}{\sum_k I_{p,k}^- \sum_q I_{q,k}^+}. $$  

(7)

The inner sums in the numerator and denominator do not depend on the index $p$, and subjected to the averaging over all eigenstates spanned by $q$. Because the disorder is spatially homogeneous, the ensemble average makes the result also $k$-independent and so it corresponds to a normalization constant. With that we arrive at the following expression for the asymptotic density matrix:

$$\varrho_{p,p} \propto \sum_k I_{p,k}^+ \sum_q I_{q,k}^- \varrho_{q,q} \sum_k I_{p,k}^- \sum_q I_{q,k}^+, $$  

(8)

which is fully determined by the type of dissipation and a spatial structure of a particular Anderson eigenstate.

For in-phase next-neighbor dissipators, $\alpha = 0$ and $l = 1$ in Eq. (4), it follows $\sum_k (A_{p,k+1} \pm A_{p,k})^2 = 2 \pm 2 \sum_k A_{p,k+1} A_{p,k} = 2 \mp \lambda_p \mp \sum_\epsilon |A_{p,k}^2 |$. Except for the case of strong localization ($\xi_p \sim 1$, for $W > 4$ or about the band edges), the last term averages out due to spatial disorder and can therefore be neglected, and so...
we end up with
\[ \tilde{\varrho}_{p,p} \propto \frac{2 - \lambda_p}{2 + \lambda_p} \]  
(9)

This result explains the quick decay of the contribution from the eigenstates away from the lower band edge. We numerically calculate, for different disorder strengths, the average distribution of the diagonal elements of $\varrho_\infty$, expressed in the Anderson basis, and plot them as functions of the average eigenvalues, see Fig. 2. The obtained results are in a good agreement with the theoretical prediction, Eq. (9). Note that the mismatch increases with the disorder strength $W$ and near the band edges; these effects follow from the nature of the made approximations.

It is straightforward to see that in case of anti-phase next-neighbor dissipators, $\alpha = \pi$ and $l = 1$, the symmetry of the problem leads to the inverse expression, $\tilde{\varrho}_{p,p} \propto \frac{2 + \lambda_p}{2 - \lambda_p}$, and the asymptotic state localized near the upper band edge. A choice $\alpha = \pi/2$ makes all dissipative operators $V_k$, Eq. (4), Hermitian. This leads to the complete delocalization of the asymptotic state, $\varrho_\infty = \frac{1}{N}$. Intermediate phase values, $0 < \alpha < \pi/2$ ($\pi/2 < \alpha < \pi$), produce asymptotic density matrix (expressed in the Anderson basis) with dominating diagonal elements localized near the lower (upper) band edge.

A qualitatively different picture is observed with next nearest neighbor dissipators, $\alpha = \pi$ and $l = 2$. In this case the asymptotic state becomes delocalized in the original basis, Fig. 3a. At the same time, it remains localized in the Anderson basis, though shifted to the center of the spectrum, see Fig. 3b. Delocalization in the direct space occurs due to the substantial spatial overlap of the contributing Anderson states, which are much weaker localized at the band center than at the edges. Analytically, that corresponds to $I_p = \sum_k \epsilon_k A_p k^2 (A_{p,k + 2} + A_{p,k})^2 = \frac{\lambda_p^2}{8} - 2\lambda_p \sum \epsilon_k A_p k^2 A_{p,k + 1} + \sum \epsilon_k A_{p,k}^2 \approx \frac{\lambda_p^2}{8} + \frac{W^2}{12}$ and $I_p^+ = 4 - I_p^-$, and it follows
\[ \tilde{\varrho}_{p,p} \propto \frac{4}{\lambda_p^2 + W^2/12} - 1. \]  
(10)

This expression indicates that the dominant contribution comes from the central Anderson modes. We also calculate the average profiles of asymptotic states for different disorder strengths to compare them with the analytical solution, which depends now explicitly on $W$, Fig. 4. Again, we find a good agreement with the numerical results, while the mismatch increases with $W$ and distance from the band center.

It is noteworthy that in the limit of strong localization, $W \gg 1$, when all eigenstates are essentially single-site localized, dissipation induces strong delocalization. As it follows from Eq. (6), in this limit all overlap coefficients become $I_{p,q} \sim \delta_{p,q}$, and the distribution of the values of the diagonal elements of the density matrix expressed in the Anderson basis should become near uniform. This means also delocalization in the direct space. As disorder strength increases, this trend can be seen on both Figs. 2 and 4.

We gain further insight in the dissipative effects by unraveling deterministic equation (1) into an ensemble of quantum trajectories $|\psi(t)\rangle$. This allows us to recast the evolution of the model system in terms of pure states, governed by an effective non-Hermitian Hamiltonian, $\hat{H} = H - \frac{i}{2} \sum_k V_k^\dagger V_k$, and random jumps induced by dissipators $V_k$. This is not a formal step only; for some quantum optics realizations it may properly model the reality of the experiment [24]. For $W = 1$, $\gamma = 0.1$ and $N = 100$, we choose transient time $t_{tr} = 10^4$ and performed an averaging over $M = 10^6$ realizations to calculate the probability density function (pdf) on the position-energy plane, $\{n(t) = \sum_k |\langle \psi(t) | k \rangle|^2 n_k \langle \psi(t) | k \rangle \}$ and $E(t) = \langle \psi(t) | H | \psi(t) \rangle$.

Figure 5(a) presents the obtained pdf for the case of in-phase next-neighbor dissipators, $\alpha = 0$ and $l = 1$. On the trajectory level, the asymptotic dynamics of the system is remarkable. Several localized states are selected from the part of the spectrum specified by the phase properties of dissipators (here it is the lower band edge). The intermittent dynamics is a mixture of sticky-like beating
near one of the localized eigenstates (non-Hermitian evolution with $\hat{H}$ [47]), which are interrupted by quantum jumps (induced by a randomly selected operator $V_k$). Every jump throws the system into the high-energy region from where it quickly relaxes, through a fine-structured network, to one of the eigenstates. The structure of the network is specific to the disorder realization; however, it does not change with further increase of $M$. Marginal distribution over $n(t)$ recovers the diagonal elements of the asymptotic density matrix $\varrho_\infty$, expressed in the direct basis, see Fig. [3].

We have shown that dissipation can be used to create steady states dominated by a few localized modes of a spatially disordered Hamiltonian. Anderson modes are selected according to their spatial-phase properties inherited from the seeding plane waves, the eigenstates of the Hamiltonian in the zero-disorder limit [48], by using phase-parametrized dissipative operators. It is possible to steer the system into a desired asymptotic state, with footprints of localization or completely delocalized, by changing phase parameter of the dissipative operators.

Our findings pose several interesting issues for future investigations. First, it is a synergy between dissipation and modulation effects, such as dynamical localization [39]. A quantum chaos, induced by strong periodic modulations [50–54] or quasi-periodic modulations [55], can also play a role of an effective disorder and lead to AL-like effects. A controllable dissipation, added to such systems, can lead to nontrivial results. Next direction is many-body localization [28–30] when the dissipative operators are phase-controlled? It is possible to create a MBL steady state with local dissipators? To answer these questions, not only spectra of the MBL Hamiltonians and such integral characteristics as, e.g., the inverse participation ratio, have to be analyzed, but also spatial phase structure of MBL eigenstates.

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