Any nonsingular action of the full symmetric group is isomorphic to an action with invariant measure

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Abstract

Let $\mathfrak{S}_\infty$ denote the set of all bijections of natural numbers. Consider the action of $\mathfrak{S}_\infty$ on a measure space $(X, \mathcal{M}, \mu)$, where $\mu$ is $\mathfrak{S}_\infty$-quasi-invariant measure. We prove that there exists $\mathfrak{S}_\infty$-invariant measure equivalent to $\mu$.

1 Introduction

Let $\mathbb{N}$ be the set of all natural numbers and let $\mathfrak{S}_\infty$ be the group of all bijections of $\mathbb{N}$. This group is called infinite full symmetric group. To the given element $s \in \mathfrak{S}_\infty$ we put $\text{supp } s = \{ n \in \mathbb{N} : s(n) \neq n \}$. Element $s \in \mathfrak{S}_\infty$ is called finite if $\#\text{supp } s < \infty$. The set of all finite elements form infinite symmetric group $\mathfrak{S}_\infty$.

Let $\text{Aut } (X, \mathcal{M}, \mu)$ be the set of all nonsingular automorphisms of the measure space $(X, \mathcal{M}, \mu)$. Throughout this paper we suppose that $\mathcal{M}$ is separable $\sigma$-algebra of measurable subsets of $X$. A homomorphism $\alpha$ from a group $G$ into $\text{Aut } (X, \mathcal{M}, \mu)$ is called an action of $G$ on $(X, \mathcal{M}, \mu)$. For convenience we consider $\alpha$ as the right action of the group $G$ on $X$: $X \ni x \mapsto xg \in X$, $g \in G$. We suppose that

$$\mu (\{ x \in X : x(gh) \neq (xg)h \}) = 0 \text{ for each fixed pair } g, h \in G \text{ and }$$

$Ag^{-1} \in \mathcal{M}$ for all $A \in \mathcal{M}$, $g \in G$. Introduce measure $\mu \circ g$ by

$$\mu \circ g(A) = \mu(Ag), A \in \mathcal{M}.$$
Suppose that measures $\mu$ and $\mu \circ g$ are equivalent (i.e. mutually absolutely continuous) for every $g \in G$. In this case measure $\mu$ is called $G$-quasi-invariant. Considering the whole equivalence class of measures $\nu$, equivalent to $\mu$ (the measure class $\mu$), it is also the same to say that the action preserves the class as a whole, mapping any such measure to another such. Let $\frac{d\mu \circ g}{d\mu}$ denote the Radon-Nikodym density of $\mu \circ g$ with respect to $\mu$. For convenience we put $\rho(g, x) = \sqrt{\frac{d\mu \circ g}{d\mu}(x)}$.

**Theorem 1.** Let the action of $\mathcal{G}_\infty$ on $(X, \mathcal{M}, \mu)$ is measurable. If measure $\mu$ is $\mathcal{G}_\infty$-quasi-invariant and $\sigma$-algebra $\mathcal{M}$ is separable then there exists $\mathcal{G}_\infty$-invariant measure $\nu$ (finite or infinite) equivalent to $\mu$.

### 1.1 Outline of the proof of Theorem 1

Since the action $X \ni x \mapsto xg \in X$, $g \in \mathcal{G}_\infty$ preserves the measure class $\mu$, we can to define the Koopman representation of $\mathcal{G}_\infty$ associated to this action. It is given in the space $L^2(X, \mu)$ by the unitary operators

$$(\mathcal{K}(g)\eta)(x) = \rho(g, x)\eta(xg), \text{ where } \eta \in L^2(X, \mu).$$

From the separability of $\sigma$-algebra $\mathcal{M}$ follows the separability of the unitary group of the space $L^2(X, \mu)$ in the strong operator topology. Therefore, homomorphism $\mathcal{K}$ induces the separable topology on $\mathcal{G}_\infty$. But, by Theorem 6.26 [1], $\mathcal{G}_\infty$ has exactly two separable group topologies. Namely, trivial and the usual Polish topology, which is defined by fundamental system of neighborhoods $\mathcal{G}(n, \infty) = \{s \in \mathcal{G}_\infty : s(k) = k \text{ for } k = 1, 2, \ldots, n\}$ of unit. Therefore, the representation $\mathcal{K}$ is continuous. It follows that there exist $n \in \mathbb{N} \cup 0$ and non-zero $\xi \in L^2(X, \mu)$ with the property

$$\mathcal{K}(g)\xi = \xi \text{ for all } g \in \mathcal{G}(n, \infty). \quad (1.1)$$

Set $E = \{x \in X : \xi(x) \neq 0\}$. Using (1.1), we obtain

$$\mu(E\Delta(EG)) = 0 \text{ for all } g \in \mathcal{G}(n, \infty). \quad (1.2)$$

For $A \subset E$ we define measure $\nu$ by

$$\nu(A) = \int_X \chi_A(x) \cdot |\xi(x)|^2 \, d\mu.$$

It follows from (1.1) and (1.1) that $\nu$ is $\mathcal{G}(n, \infty)$-invariant measure on $E$. This measure can be extend to the $\mathcal{G}_\infty$-invariant measure on $X$.  

2
2 The properties of the continuous representations of the group $\mathfrak{S}_\infty$.

To the proof of Theorems 1 we will use the general facts about the continuous representations of the group $\mathfrak{S}_\infty$, which have been well studied by A. Lieberman [2] and G. Olshanski [3, 4]. In this section we will give the simple constructions of the important operators and the short direct proofs of their properties.

Let $K$ be the continuous representation of $\mathfrak{S}_\infty$ in Hilbert space $H$. It follows that for each $\eta \in H$

$$\lim_{k \to \infty} \sup_{s \in \mathfrak{S}(k, \infty)} \|K(s)\eta - \eta\| = 0.$$  \hfill (2.3)

Set $^n\sigma_m = (n + 1 \ n + m + 1)(n + 2 \ n + m + 2) \cdots (n + m \ n + 2m)$, where $(k \ j)$ is a permutation that interchanges two numbers $k, j$ and leaves all the others fixed. We will need few auxiliary lemmas.

**Lemma 2.** The sequence of the operators $\{K( ^n\sigma_m)\}_{m \in \mathbb{N}}$ converges in the weak operator topology to a self-adjoint operator $P_n$.

**Proof.** Let us prove that the sequence $\{K( ^n\sigma_m)\}_{m \in \mathbb{N}}$ is fundamental in the weak operator topology. Assuming for the convenience that $M > m$, we write $^n\sigma_M$ in the form $^n\sigma_M = s \cdot ^n\sigma_m \cdot t$, where $s, t \in \mathfrak{S}(n + m, \infty)$. Hence, using (2.3), we have

$$\lim_{m, M \to \infty} \langle (K( ^n\sigma_M) - K( ^n\sigma_m))\eta, \zeta \rangle = 0$$

for all $\eta, \zeta \in H$. \hfill \Box

**Lemma 3.** Operator $P_n$ is a projection.

**Proof.** Using lemma [2] for any fixed $\eta, \zeta \in H$ we find the sequences $\{m_k\}_{k \in \mathbb{N}}$ and $\{M_k\}_{k \in \mathbb{N}}$ such that $m_{k+1} > m_k$, $M_k > 2m_k$ and

$$\lim_{k \to \infty} \left| \langle P_n^2\eta, \zeta \rangle - \langle K( ^n\sigma_{M_k}) \cdot K( ^n\sigma_{m_k}) \eta, \zeta \rangle \right| = 0.$$ \hfill (2.4)

Now we notice, that $^n\sigma_{M_k} \cdot ^n\sigma_{m_k} = ^n\sigma_{m_k} \cdot s_k$, where $s_k \in \mathfrak{S}(n + m_k, \infty)$. Hence, using (2.3) and (2.4), we have

$$0 = \lim_{k \to \infty} \left| \langle P_n^2\eta, \zeta \rangle - \langle K( ^n\sigma_{M_k}) \cdot K( s_k) \eta, \zeta \rangle \right| = \lim_{k \to \infty} \left| \langle P_n^2\eta, \zeta \rangle - \langle K( ^n\sigma_{m_k}) \eta, \zeta \rangle \right|$$

for all $s \in \mathfrak{S}(n, \infty)$. \hfill \Box

**Lemma 4.** The equality $K(s) \cdot P_n = P_n$ holds for any $s \in \mathfrak{S}(n, \infty)$. 3
Proof. Suppose that $m > n$ and $M \geq 2m$. Then $(m \, m + 1) \cdot {^n}\sigma_M = {^n}\sigma_M \cdot (m + M \, m + M + 1)$. Hence, applying lemma 2 and (2.3), we have
\[
\langle \mathcal{K}((m \, m + 1))P_n\eta, \zeta \rangle = \lim_{M \to \infty} \langle \mathcal{K}((m \, m + 1)) \cdot \mathcal{K}(^{n}\sigma_M)\eta, \zeta \rangle
\]
\[
= \lim_{M \to \infty} \langle \mathcal{K}(^{n}\sigma_M) \cdot \mathcal{K}((m + M \, m + M + 1))\eta, \zeta \rangle \overset{(2.3)}{=} \lim_{M \to \infty} \langle \mathcal{K}(^{n}\sigma_M)\eta, \zeta \rangle
\]
for any $\eta, \zeta$ in $\mathcal{H}$. By Lemma 2, $\mathcal{K}((m \, m + 1)) \cdot P_n = P_n$. Since the transpositions $(m \, m + 1) \ (m > n)$ generate the subgroup $\mathcal{S}(n, \infty)$, lemma is proved.

It follows from Lemmas 2 and 4 that
\[
P_n\mathcal{H} = \{ \eta \in \mathcal{H} : \mathcal{K}(s)\eta = \eta \text{ for all } s \in \mathcal{S}(n, \infty) \}.
\]

**Lemma 5.** The sequence $\{ \mathcal{K}((k \, N)) \}_{N \in \mathbb{N}}$ converges in the weak operator topology to the self-adjoint projection $O_k$.

**Proof.** Using (2.3) and the equality $(k \, N_2) = (N_1 \, N_2)(k \, N_1)(k \, N_2)$, we obtain that the sequence $\{ \mathcal{K}((k \, N)) \}_{N \in \mathbb{N}}$ is fundamental. Since $(k \, N_1)(k \, N_2) = (k \, N_2)(N_1 \, N_2)$, operator $P_k$ is a self-adjoint projection.

**Lemma 6.** The projections $P_n$ and $O_k$ commute: $P_nO_k = O_kP_n$.

**Proof.** Since, by Lemma 4, $O_kP_n = P_n$ for $k > n$, we suppose that $k \leq n$. By Lemmas 2 and 5, for any $\eta, \zeta$ in $\mathcal{H}$ there exists the sequence $\{M_l\}_{l \in \mathbb{N}} \subset \mathbb{N}$ such that $M_{k+1} > M_k$ and
\[
\lim_{l \to \infty} |\langle P_nO_k\eta, \zeta \rangle - \langle \mathcal{K}(^{n}\sigma_{M_l})O_k\eta, \zeta \rangle| = 0,
\]
\[
\lim_{l \to \infty} |\langle O_kP_n\eta, \zeta \rangle - \langle O_k\mathcal{K}(^{n}\sigma_{M_l})\eta, \zeta \rangle| = 0.
\]

For the same reason we can to find the sequence $\{N_l\}_{l \in \mathbb{N}} \subset \mathbb{N}$ such that $N_{k+1} > N_k + 2M_k$ and
\[
\lim_{l \to \infty} |\langle \mathcal{K}(^{n}\sigma_{M_l})\mathcal{K}(k \, N_l)\eta, \zeta \rangle - \langle \mathcal{K}(^{n}\sigma_{M_l})O_k\eta, \zeta \rangle| = 0,
\]
\[
\lim_{l \to \infty} |\langle \mathcal{K}(k \, N_l)\mathcal{K}(^{n}\sigma_{M_l})\eta, \zeta \rangle - \langle O_k\mathcal{K}(^{n}\sigma_{M_l})\eta, \zeta \rangle| = 0.
\]

Now, using (2.6), (2.7) and the equality $(k \, N_l) \cdot {^n}\sigma_{M_l} = {^n}\sigma_{M_l} \cdot (k \, N_l)$, we obtain that $P_nO_k = O_kP_n$.

**Lemma 7.** Let $\mathcal{S}(k, n, \infty)$ denotes the group generated by the transposition $(k \, n + 1)$ and the subgroup $\mathcal{S}(n, \infty)$. Then $O_kP_n$ is the self-adjoint projection on the subspace $\{ \eta \in \mathcal{H} : \mathcal{K}(s)\eta = \eta \text{ for all } s \in \mathcal{S}(k, n, \infty) \}$. In particular, $O_nP_n = P_{n-1}$ (see (2.3)).
Proof. The proof follows from the next chain of the equalities
\[ \langle K((k \ n + 1)) \cdot O_k \cdot P_n \eta, \xi \rangle = \lim_{N \to \infty} \langle K((k \ n + 1) \cdot (k \ N)) \cdot P_n \eta, \xi \rangle = \lim_{N \to \infty} \langle K((k \ N)) \cdot K((n + 1 \ N)) \cdot P_n \eta, \xi \rangle = \lim_{N \to \infty} \langle K((k \ N)) \cdot K((k \ n + 1 \ N)) \cdot P_n \eta, \xi \rangle = \langle O_k \cdot P_n \eta, \xi \rangle. \]

Since the representation \( K \) is continuous, then there exists \( n \in \mathbb{N} \) such that \( P_n \neq 0 \). Set \( \text{depth}(K) = \min \{ n : P_n \neq 0 \} \).

Lemma 8. If \( n = \text{depth}(K) \) and \( g \notin \mathcal{S}(n, \infty) \) then \( P_n K(g) P_n = 0 \).

Proof. Let \( k \leq n \) and \( g(k) = m > n \). Then \( g = (k \ m) \cdot s \), where \( s(m) = m \).

Let \( \mathcal{S} = \{ M \in \mathbb{N} : \min \{ M, s^{-1}(M) \} > n \} \). It is clear that \#\( \mathcal{S} \) = \( \infty \).

Under this condition we have for \( M \in \mathcal{S} \)
\[ P_n K(g) P_n = P_n \cdot K((m \ M)) \cdot K((k \ m)) \cdot K(s) \cdot K((m \ s^{-1}(M))) \cdot P_n = P_n \cdot K((m \ M)) \cdot K((k \ m)) \cdot K((m \ M)) \cdot K(s) \cdot P_n = P_n \cdot K((m \ M)) \cdot K(s) \cdot P_n. \]

But, by (2.5) and Lemma 7
\[ K((k \ n)) \cdot P_n \cdot O_k \cdot K((k \ n)) = P_n \cdot O_n = P_{n-1} \stackrel{\text{depth}(K) = n}{=} 0. \]

Therefore, \( P_n K(g) P_n = 0 \). \( \square \)

3 The Proof of Theorem 1

We follow the notations of the subsection 1.1. Without loss of generality we will to assume that \( \mu \) is a probability measure. Let \( n = \text{depth}(K) \) (see page 5). Fix non-zero \( \xi_1 \in \mathcal{H} \) such that \( \mu \)-almost everywhere
\[ (K(s) \xi_1)(x) = \rho(s, x) \xi_1(xs) = \xi_1(x) \] for each \( s \in \mathcal{S}(n, \infty) \). (3.8)

It follows that for the characteristic function \( \chi_{E_1} \) of the set \( E_1 = \{ x \in X : \xi_1(x) \neq 0 \} \) and \( s \in \mathcal{S}(n, \infty) \) \( \mu \)-almost every holds
\[ \chi_{E_1}(xs) = \chi_{E_1}(x). \] (3.9)

For each measurable \( A \subset X \) we define its measure \( \mu_1(A) \) as follows
\[ \mu_1(A) = \mu(A \setminus E_1) + \int_{E_1} \chi_A(x) \cdot |\xi_1(x)|^2 \, d\mu. \] (3.10)
By definition, the measures \( \mu \) and \( \mu_1 \) are equivalent.

Let \( \mathcal{K}_1 \) denotes Koopman representation, corresponding to \( \mu_1 \), and let \( \rho_1 \) be its cocycle: 
\[
\rho_1(g, x) = \sqrt{\frac{d\mu_1}{dp_1}}, \quad g \in \mathfrak{S}_\infty.
\]
Since \( \mathcal{K} \) and \( \mathcal{K}_1 \) are unitary equivalent, \( \text{depth}(\mathcal{K}_1) = n \).

If \( A \subset E_1 \) then, applying (3.8) and (3.9), we obtain
\[
\mu_1(A) = \mu_1(As) \text{ for all } s \in \mathcal{S}(n, \infty). \tag{3.11}
\]
It follows that for each \( s \in \mathcal{S}(n, \infty) \)
\[
\rho_1(s, x) = 1 \text{ for } \mu\text{-almost every } x \in E_1. \tag{3.12}
\]
Now we will to prove the equality
\[
\mu_1(E_1 \Delta (E_1g)) = 0 \text{ for all } g \notin \mathcal{S}(n, \infty). \tag{3.13}
\]
Using (3.9) and (3.12), we have \( \mathcal{K}_1(s)\chi_{E_1} = \chi_{E_1} \) for all \( s \in \mathcal{S}(n, \infty) \).
Hence, applying Lemma 7, we obtain
\[
0 = \langle \mathcal{K}_1(g)\chi_{E_1}, \chi_{E_1} \rangle = \int_X \rho_1(g^{-1}, x)\chi_{E_1}(x) \cdot \chi_{E_1}(x) \, d\mu_1.
\]
But \( \mu(\{x \in X : \rho_1(g^{-1}, x) = 0\}) = 0 \). Therefore, \( 0 = \int_X \chi_{E_1}(x) \cdot \chi_{E_1}(x) \, d\mu_1 \)
\[
= \mu_1(E_1 \Delta (E_1g)), \quad \text{and (3.13) is proved.}
\]

For the construction of the \( \mathfrak{S}_\infty \)-invariant measure we consider the right coset \( H \setminus G \), where \( H = \mathcal{S}(n, \infty) \) and \( G = \mathfrak{S}_\infty \). Since every bijection \( s \in G \) can be write as \( s = hf \), where \( h \in H \) and \( f \in \mathfrak{S}_\infty \) is the finite permutation, then there exists a countable full set \( g_1, g_2, \ldots \) of the representatives in \( G \) of the cosets \( H \setminus G \). Define the map \( \tau : H \setminus G \mapsto G \) as follows: \( \tau(x) = g_i \), if \( x = Hg_i \). We will to assume that \( \tau(H) \) is the identity \( e \) of \( G \). Set \( \tilde{E}_1 = \bigcup_{i=1}^\infty E_1 g_i \).

For each measurable \( A \subset X \) we define its measure \( \nu_1(A) \) as follows
\[
\nu_1(A) = \mu \left( A \setminus \tilde{E}_1 \right) + \sum_{y \in H \setminus G} \mu_1 \left( (A \cap (E_1 \tau(y))) (\tau(y))^{-1} \right) \tag{3.14}
\]
Let us prove that
\[
\nu_1(A) = \nu_1(Ag) \text{ for all } g \in G \text{ and } A \subset \tilde{E}_1. \tag{3.15}
\]
For this we notice that
\[
\nu_1(Gg) = \sum_{y \in H \setminus G} \mu_1 \left( (A \cap (E_1r(y))) (r(y))^{-1} \right)
\]
\[= \sum_{y \in H \setminus G} \mu_1 \left( (A \cap (E_1r(yg^{-1}))) g(r(y))^{-1} \right)
\]
\[= \sum_{y \in H \setminus G} \mu_1 \left( (A \cap (E_1r(y))) (r(y))^{-1} \cdot r(yg^{-1})g(r(y))^{-1} \right)
\]
where \(r(y)g(r(yg))^{-1} \in H = \mathfrak{G}(n, \infty)\). Hence, using (3.12), and (3.14), we obtain
\[
\nu_1(Gg) = \sum_{y \in H \setminus G} \mu_1 \left( (A \cap (E_1r(y))) (r(y))^{-1} \right) = \nu_1(A).
\]
The equality (3.15) is proved.

Applying the above reasonings to the restriction of the action \(X \ni x \rightarrow xs, s \in \mathfrak{S}_\infty\) to the \(\mathfrak{S}_\infty\)-invariant set \(X \setminus \tilde{E}_1\), we find \(\mathfrak{S}_\infty\)-invariant set \(\tilde{E}_2 \subset X \setminus \tilde{E}_1\) and the measure \(\nu_2\), equivalent to \(\mu\), such that
\[
\nu_2(A) = \nu_2(Gg) \quad \text{for all } g \in \mathfrak{S}_\infty, \quad A \subset \tilde{E}_1 \cup \tilde{E}_2,
\]
\[
\nu_1(A) = \nu_2(A) \quad \text{for all } A \in \tilde{E}_1 \text{ and } \nu_2(A) = \mu(A) \quad \text{for all } A \subset X \setminus \left( \tilde{E}_1 \cup \tilde{E}_2 \right).
\]
The continuation of these reasonings gives the family of the sets \(\{\tilde{E}_1, \tilde{E}_2, \ldots\}\) (finite or countable) and the measures \(\nu_1, \nu_2, \ldots\), equivalent to \(\mu\), such that
\[ \widetilde{E}_k \subset X \setminus \left( \bigcup_{i=1}^{k-1} \widetilde{E}_i \right), \quad \mu \left( X \setminus \left( \bigcup_i \widetilde{E}_i \right) \right) = 0, \]

\[ \nu_k(A) = \nu_k(Ag) \text{ for all } g \in \mathfrak{T}_\infty, \ A \subset \bigcup_{i=1}^{k} \widetilde{E}_i, \]  \hspace{1cm} (3.16)

\[ \nu_l(A) = \nu_k(A) \text{ for all } k > l \text{ and } A \subset \bigcup_{i=1}^{k} \widetilde{E}_i, \]

\[ \nu_k(A) = \mu(A) \text{ for all } A \subset X \setminus \left( \bigcup_{i=1}^{k} \widetilde{E}_i \right). \]

Therefore, for each measurable \( A \subset X \) the sequence \( \nu_k(A) - \mu \left( X \setminus \left( \bigcup_{i=1}^{k} \widetilde{E}_i \right) \right) \) is monotone increasing. Since \( \lim_{n \to \infty} \mu \left( X \setminus \left( \bigcup_{i=1}^{k} \widetilde{E}_i \right) \right) = 0 \), we obtain that there exists the limiting measure

\[ \nu(A) = \lim_{k \to \infty} \nu_k(A). \]  \hspace{1cm} (3.17)

Using the equivalence of the measures \( \mu \) and \( \nu_k \) for all \( k \), we conclude that if \( \mu(A) = 0 \) then \( \nu(A) = 0 \). Conversely, assuming that \( \nu(A) = 0 \), we have from (3.16)

\[ \nu \left( A \cap \left( \bigcup_{i=1}^{k} \widetilde{E}_i \right) \right) = \nu_k \left( A \cap \left( \bigcup_{i=1}^{k} \widetilde{E}_i \right) \right) = 0. \]

Therefore, by (3.17), \( \mu(A) = 0 \). Thus the measures \( \nu \) and \( \mu \) are equivalent. By (3.16) and (3.17), the measure \( \nu \) is \( \mathfrak{T}_\infty \)-invariant. Theorem II is proved.

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