Robust stabilization of the multilinked object

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Abstract. We consider the problem of robust control for a multilink object in the presence of the signal and parametrical uncertainty when the measurement of the derivatives of input and output signals of the local subsystems is not available. Proposed and justified the use of auxiliary circuits for each of the subsystems that eliminates the use of the vector of regression during the generate of the control signal, where by the reduced order closed-loop control system. The advantage of the proposed robust control law is that it remain unchanged when exposed to unknown disturbances, and in their absence. In the first case, the system is dissipative, while in the second case, it one is asymptotically stable.

1 Introduction

Problem of control of uncertain objects is one of the classical problems of control theory. Complex multiply linked objects take a special place among object of classical problems of control theory. There are developed method extended for subclass of parametric uncertainty when the measurement of the derivatives of input and output signals of the local subsystems is not available. Proposed and justified the use of auxiliary circuits for each of the subsystems that eliminates the use of the vector of regression during the generate of the control signal, where by the reduced order closed-loop control system. The advantage of the proposed robust control law is that it remain unchanged when exposed to unknown disturbances, and in their absence. In the first case, the system is dissipative, while in the second case, it one is asymptotically stable.

2 Statement of the problem

Let us examine the interconnected system with its dynamic processes which are described by the equations in the local subsystems so:

\[ Q_i(P)y_i(t) = R_i(P)\left(u(t) + f_i(y_i,\dot{y}_i) + \sum_{j=1,j\neq i}^{n} y_j(t)\right), \]

\[ s_i = A_i s_i + B_i y_j, \quad y_i = L_i s_i, \]

where \( Q_i(P), R_i(P) \) - are linear differentiation operators whose elements depend on the vector of unknown parameters \( \xi \in \Xi; \Xi = \{1, 2, \ldots, n\} \) - is a known aggregate of possible values of the vector \( \xi; \deg Q_n = n, \deg R_m = m; f_i(y, \dot{y}) \) - is an unknown bounded function; \( y_i(t) \) - is a scalar control variable of \( i-th \) subsystem which is accessible for measurement; \( u(t) \) - is a scalar control action; \( s_j \in \mathbb{R}^n, y_j \in \mathbb{R} \); \( A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n} \) - are unknown numerical matrices; \( L_i = [1,0,\ldots,0] \) - is a row matrix of the appropriate orders. In this case, the equation of the control object is presented in canonical form.

The equations (1) describe the dynamic processes in the local subsystems, and (2) in the cross coupling. Their transfer functions look like this as in

\[ W_i(\lambda) = L_i(A_{ij} - A_{ii})^{-1}B_{ij} = R_{ij}(\lambda)\cdot Q_{ij}(\lambda) \]

where \( i \neq j \), the orders of polynomials \( R_{ij}(\lambda), Q_{ij}(\lambda) \) are respectively equal to the following \( m_{ij}, n_{ij}; I_{ij} \) is an identity matrix of order.
Hypothesis. 
A1. Set $\Xi$ are known.
A2. Polynomials $R_{i} (\lambda)$ are Hurwitz.
A3. The orders of Polynomials are known, the relative degree a object is known too $n_i - m_i > 1$.
A4. The perturbation actions are the bounded quantity, that is $|f_i (y_i, t)| \le C_i , \ C_i > 0$.
A5. The matrix $A_{si} B_{si}$ is unknown, but they are Polynomials $Q_{si} (\lambda)$ Hurwitz.

### 3 Robust stabilization of the multilinked dynamic object

We first solve the problem of the asymptotic robust stabilization when there are no external disturbances in the object (1), i.e. $f_i (y_i, t) = 0$. We have to do is to design the control system for which one condition will meet

$$\lim_{t \to \infty} |y_i (t)| = 0, \quad (4)$$

but at the same time, use of the measurable values of the other subsystems isn’t allow in the local control subsystems. Decentralized robust control is implemented only if there are used only the current information about the system for construction of local control units for each subsystem. Have the equation of the control object (1), written in canonical form as

$$Q(P)y_i (t) = R(P) \left( u_i (t) + \sum_{j \neq i} y_{ij} (t) \right)$$

where $Q(P), R(P)$ – is a linear differentiation operator with a constant rates. In the generation of control law will take estimate of the output signal with opposite sign. So it form is

$$u_i (t) = - \theta T_i (P) \bar{y}_i (t) \quad (5)$$

where a number $\theta > 0$ and a linear differentiation operator $T_i (P)$ of order $n_i - m_i - 1$ are chosen, according to the Hurwitz of polynomial

$$Q_{si} (\lambda) = Q_i (\lambda) + \theta R_i (\lambda) T_i (\lambda) \quad (6)$$

the function $\bar{y}_i (t)$ is an estimate of a signal $y_i (t)$ which is realized by the observer [12]

$$\bar{x}_i = F_i x + H_i (y_i - \bar{y}_i), \quad \bar{x}_i = L_{si} x_i. \quad (7)$$

Here $x_i \in \mathbb{R}^{n_i}; L_{si} = [1, 0, ..., 0]; \ \chi_i > 0$ is the little number, $F_i = \begin{bmatrix} 0 & I^{-1} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$; $H_i = \begin{bmatrix} h_{ii} \\ \cdots \\ h_{ii}^{-1} \\ \cdots \\ h_{ii}^{-1} \end{bmatrix}$; $I_{\chi_i}$ – is an identity matrix of order $(n_i - m_i - 1) \times (n_i - m_i - 1)$. It is obvious, that the control law can be technically realizable, as it contains the known or measurable values. At the time, the solution of the problem gives the following statement.

The Statement 1 If the hypotheses A1 – A5 are implemented, then the control law (5) with the observer (7) implements of the target condition (5).

Proof of Proposition 1. Inputting the control (5) to the equation of object (1) with (6), we’ll get the last input-output model in the matrix vectorial form

$$x_i = A_{si} x_i + \bar{B}_i (y_i - \bar{y}_i) + B \sum_{j \neq i} y_{ij}, \quad y_i = L_{si} x_i, \quad (8)$$

where $\bar{B}_i \in \mathbb{R}^{n_i}$ – is a vector whose components are the coefficients $\theta R (P) T_i (P)$; $A_{si}$ – is a Hurwitz matrix in the form of a characteristic polynomial of the Frobenius $Q_{si} (\lambda)$; $B_i = \begin{bmatrix} 0, ..., 0, b_{i1}, ..., b_{in_i} \end{bmatrix}$. Following the [13], we introduce into consideration nullity vector $\eta_i$ as

$$\eta_i = \frac{1}{\chi_i} F_i \eta_i + B_i x_{si} + D_i \bar{B}_i \lambda \eta_i$$

where $D_i = \text{diag} \{ \chi_i^{-1}, ..., \chi_i^{-1} \}$, $x_{si}$ – is a component part of the state vector, namely $x_{si} = [x_{si1}, x_{sii}]$.

We introduce the augmented state vector $z_i = \text{col}(x_i, \eta_i)$ and write the equation (8), (9) in the composite matrix vectorial form

$$z_i = B_i x_{si} + \bar{B}_i \lambda \eta_i +$$

$$+ \frac{1}{\chi_i} A_{si} z_i + B_{si} \sum_{j \neq i} y_{ij} \quad (10)$$

and at the same time matrices $A_{si}$ will be Hurwitz.

Let us take the Lyapunov function in the form of

$$V = \sum_{i=1}^{k} \left[ z_i^T (t) P_i z_i (t) + \sum_{j \neq i} \tilde{z}_i^T (t) P_{ij} \tilde{z}_j (t) \right] \quad (11)$$

in which $P_i, P_{ij}$ – are the positive definite symmetric matrices satisfy the matrix relation:

$$P_i A_{pi} + A_{pi}^T P_i = -Q_{pi}, \quad (12)$$

Because the matrices $A_{pi}$ and $A_{ij}$ are Hurwitz, there are matrices $P_i$ and $P_{ij}$, that satisfy the matrix relation (12). Let us calculate the total derivative of the function (12) on the trajectories of the system (10), (2), with (12). Then the derivative of the Lyapunov function will look like
$V \leq \sum_{i=1}^{k} \left( -\alpha_i V - \frac{p_i}{2\lambda_i} \| k \| \right) \leq -\alpha V,$

calls the synthesized system is stable, because the target condition is satisfied (4).

4 Robust stabilization of the multilinked dynamic object under bounded disturbances

However, in practice it is difficult to create the situation for control of technical objects in ideal conditions when it is not exposed to external disturbances. The case where external disturbances are bounded is the simplest. Next, let us solve the problem of a decentralized stabilization when the external disturbances are present in the object (1), i.e. $f(y, t) \neq 0$. We had to design a control system for which next condition will be satisfied

$$\lim_{t \to \infty} y(t) < \delta,$$  (13)

In accordance with the approach presented in [13, 14] let us ask one local control law in the form of (5) where a number $\theta_i > 0$ and the linear differentiation operator $T_i(P)$ of order $n - m - 1$ are chosen from the reasons of Hurwitz polynomial (6), and the function $\gamma_i(t)$ is an estimate of output $y_i(t)$. Then the equation (1) will look like

$$Q_i(P)y_i(t) = R(P) \left( \theta_i T_i(P)y_i(t) - \gamma_i(t) \right) + f_j(y, t) + \sum_{j \neq i} y_j(t),$$  (14)

Implementing the control law (5) requires getting the estimate $\gamma_i(t)$ and its $n - m - 1$ derivatives, for which we will use the observer (7). It is obvious, that the control law can be technically realizable, as it contains the known or measurable values.

The Statement 2. If the hypotheses A.1 – A.5 are implemented, then the control law (5) with the observer (7) implements the limited nature of system trajectory (14).

Observation. But it should be noted, that choosing the number $\theta_i$ of greater value, and the value $\chi$ of smaller, we can achieve the target condition (14).

Proof of Proposition 2. Let us convert the equation (14) to the vectorial matrix form

$$x_i(t) = A_{0i}x_i(t) +$$

$$+ B_i \left( \tau_i^j(x_i(t) - \gamma_i(t)) + f_j(y, t) + \sum_{j \neq i} y_j(t) \right) \right) \) \) \( y_i(t) = L_i x_i(t),$$  (15)

where $x_i \in \mathbb{R}^{n_i}$; $y_i \in \mathbb{R}$; $A_0$ – is a Hurwitz matrix in Frobenius form with characteristic polynomial $Q_i(\lambda)$; $\tau_i \in \mathbb{R}^{n_i}$ – is a vector whose components are the coefficients $\theta_i T_i(P)$; $B_i^j = [0, \ldots, b_{i,j}, \ldots, b_{i,1}]$; $L_i = [1, 0, \ldots, 0]$; $x_{i,0}$ – is a component part of the state vector, namely $x_i^T = [x_{i,0}^T, \ldots, x_{i,n_i-1}^T]$.

$$x_i(t) = F_{i,0}x_i(t) + B_{i,0} \left( \frac{1}{b_{i,0}} x_{i,0}^T +$$

$$+ \tau_i^j(x_i(t) - \gamma_i(t)) + f_j(y, t) + \sum_{j \neq i} y_j(t) \right),$$

$$x_i(t) = F_{i,0}x_i(t) + B_{i,0} \left( \tau_i^j(x_i(t) - \gamma_i(t)) + f_j(y, t) + \sum_{j \neq i} y_j(t) \right),$$

$$y_i(t) = L_i x_i(t),$$  (16)

where the matrix $F_i$ has the same structure as $F_{i,0}$; $B_i^j = [0, \ldots, 0, b_{i,j}, \ldots, b_{i,1}]$; $x_{i,0}$ – is a component part of the state vector; $L_i = [1, 0, \ldots, 0]$. Following the hypotheses A.1 – A.5 the known or measurable values.

$$f_j(y, t) + \sum_{j \neq i} y_j(t),$$

$$\eta(t) = \frac{1}{\chi} F_i \eta(t) + B_{i,0} \left( \tau_i^j D_j \eta(t) + \frac{1}{b_{i,0}} x_{i,0}^T +$$

$$= f_j(y, t) + \sum_{j \neq i} y_j(t),$$  (17)

Structure of matrix $F_{i,0}, B_{i,0}, L_i, D_j, H_j$ is such that the equalities are satisfied

$$D_j^{-1}(F_{i,0} - H_j L_j)D_j = \frac{1}{\chi} F_j, \quad D_j^{-1} B_{i,0} = B_{i,0},$$

Let us introduce block diagonal matrix

$$F = \text{diag} \{ F_1, \ldots, F_n \}, \quad B = \text{diag} \{ B_1, \ldots, B_n \},$$

$$D = \text{diag} \{ D_1, \ldots, D_n \}, \quad B = \text{diag} \{ B_1, \ldots, B_n \},$$

and the vectors $\eta = \text{col} \{ \eta_1, \ldots, \eta_n \}; \quad \tau = \text{col} \{ \tau_1, \ldots, \tau_n \}$

$$x = \text{col} \{ x_1, \ldots, x_n \}, \quad x_{21} = \text{col} \{ x_{21}, \ldots, x_{2n} \},$$

$$f = \text{col} \{ f_1, \ldots, f_n \}, \quad y = \text{col} \{ y_1, \ldots, y_n \},$$

Then the equation (15), (17) in the composite form will look like

$$\eta(t) = \frac{1}{\chi} F \eta(t) + B \left( \tau^j D \eta + B x_{21} + f(t) + a \right),$$  (18)

$$x(t) = A x(t) + B \left( \tau^j D \eta + f(t) + a \right),$$

where $\eta(t) \in \mathbb{R}^{n_i}$; $x(t) \in \mathbb{R}^n$; $a$ – is a block matrix with the blocks $I_{n_i}, I_{n_i}$ – is a identity matrix of order $k \times k$. 


\[ a_i = \begin{bmatrix} I_i & 0 & \cdots & 0 \\ 0 & I_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_i \end{bmatrix} \]

Let us take the Lyapunov function

\[ V_i = \eta_i^T(t)P\eta(t) + \sum_{j=1}^{k} \sum_{j=1}^{k} s_{ij}^T M_{w_i}^j s_{ij}, \tag{19} \]

where \( P, M_{w_i}^j \) are the positive definite symmetric matrices, at the same time \( P = \text{diag}\{P_1, \ldots, P_m\} \). To take account the fact that the matrices are block diagonal, we will produce that positive definite symmetric matrices \( P_i \) must satisfy the equations for every subsystem

\[ P_i F_i + F_i^T P_i = -Q_i - \rho_i I_{n_i}, \]
\[ M_{si} A_{sij} + A_{sij}^T M_{si} = -Q_{si} - \rho_{si} I_{n_i}, \]

where \( \rho_i > 0; \rho_{si} > 0; Q_i = Q_i^T > 0; Q_{si} = Q_{si}^T > 0 \). Since the matrices \( F_i \) and \( A_{si} \) are Hurwitz we see that there are the matrices \( P_i \) and \( M_{w_i}^j \), with the ratio (19).

Let us take the Lyapunov function

\[ V_2 = x^T(t)Hx(t), \tag{20} \]

where \( H \) are the positive definite symmetric matrices, at the same time \( H = \text{diag}\{H_1, \ldots, H_m\} \). To take account the fact that the matrix \( H \) is a block diagonal, we will produce that positive definite symmetric matrices \( H_i \) must satisfy the equations for every subsystem

\[ H_i A_{w_i} + A_{w_i}^T H_i = -Q_{w_i} - \rho_{w_i} I_{n_i}, \]

where \( \rho_{w_i} > 0; Q_{w_i} = Q_{w_i}^T > 0 \). Since the matrices \( A_{si} \) are Hurwitz we see that there are the matrices \( H_i \) which satisfy these matrix relation. To take account the estimated derivative of the functions (19), used (20) and the last inequality, we will calculate the total derivative of the Lyapunov function \( \dot{V} = V_1 + V_2 \), and we’ll get the inequality

\[ \dot{V}(x(t), \eta(t)) \leq -\beta_V V + \beta_V, \]

where

\[ \beta_V = \frac{1}{C_i} \sum_{i=1}^{k} \beta_i \text{sup} C_i^i, \]
\[ \beta_i = \min \left\{ 0.25 \alpha_i; \frac{1}{\chi} \frac{\lambda_{\text{min}}(Q_i)}{\lambda_{\text{max}}(P_i)}; \frac{\lambda_{\text{min}}(Q_{w_i})}{\lambda_{\text{max}}(M_{w_i})} \right\}, \]

from which the dissipative system follows

\[ V \leq \frac{\beta_i}{\beta_V} \] and the estimate \( \| x(t) \| \leq \frac{\beta_i}{\beta_V} \).

This value can be reduced by choosing a little number \( \chi \) which leads to an increase of the \( 1/\chi \), and \( \theta_i \) in the (5) greater. Unfortunately, we cannot get a more accurate estimate.

### 5 Example

Consider the multilinked control plant, dynamic processes which are described by the equations

\[
(P^i + a_i P^i + a_{2i} P + a_{3i}) y_i(t) = \\
= (b_i P + b_{3i}) u_i(t) + s_i y_i(t) \\
(P^i + a_i P^i + a_{2i} P + a_{3i}) y_i(t) = \\
= (b_i P + b_{3i}) u_i(t) + s_i y_i(t)
\]

Aggregate of possible values of the vector

\[ \Xi = \{a, s, b, l = \|\Xi\|: a_i \in [-8.5, b_i \in [1.8]) \} . \]

The equation of the auxiliary circuit has the form

\[ (P^i + 2P + 1) y_i = u_i, i = 1, 2. \]

Let’s used observer

\[ \frac{v_i}{\theta_i} \left[ \begin{array}{c} v_i(t) + 3/2 \xi_i(t) \\
3/2 \xi_i(t) - \vartheta_i(t) \end{array} \right] \]

The control law is in the form of

\[ u_i(t) = -\frac{\alpha}{b_{2i}} (v_i(t) + 2 \xi_i(t) + \xi_{\vartheta}(t)), i = 1, 2. \]

General block diagram of the control object shown in fig. 1. Here we have the multilinked control object consisting of \( K \) local subsystems. Control signal and the output signals from all the other \( K-1 \) local subsystems are inputs of each local subsystem. The local control unit generates a control signal, using an estimate of the state vector of the object that is obtained with the observer. Thus is implemented a decentralized control.

![Fig.1. Structural scheme of stabilisation of the i-th subsystem](image)

This object is synthesized with a decentralized robust controller for stabilization of each of the local subsystems, the structure of which is shown in Fig.2. The control signal is generated with a opposite sign to
the assessment of variables of the internal state. Compensation of external and structural disturbances is possible because introducing the auxiliary circuit. The disadvantage of the proposed control law is that the absolute magnitude of the coefficients $K$ and $\alpha$ is impossible to find analytically. Their value is chosen at the modeling stage.

![Fig.2. Structural scheme of robust control of the $i$-th subsystem](image)

Fig. 2 shows the simulation results of the stabilization system of the unstable object in the presence of external disturbances

$$
\begin{align*}
f_1(t) &\neq 0, \quad f_2 \neq 0, \\
y_1(0) &= y_1(0) = y_2(0) = 2, \\
y_2(0) &= y_2(0) = y_2(0) = -1
\end{align*}
$$

There are follows external disturbances in the object

$$
\begin{align*}
f_1(t) &= 1 + \sin 0.1t + \sin 10t, \\
f_2(t) &= 1 + 2\sin 0.3t + \sin 10t.
\end{align*}
$$

Consider the control object, dynamic processes in which case are described by the equation

$$
\begin{align*}
(P^3 + a_1P^2 + a_2P + a_3)y_1(t) &= b_1(y_1(t) + f_1(y_1(t) + y_2(t)), \\
y_1 &= 3(p + 2)y_2, \\
(P^3 + a_1P^2 + a_2P + a_3)y_2(t) &= b_1(y_1(t) + f_2(y_1(t) + y_2(t)), \\
y_2 &= 5p^2 + p + 1y_1.
\end{align*}
$$

![Fig.3. The trajectories of the system’s outputs in robust stabilization](image)

Aggregate of possible values of the vector

$$
\Xi = \{a_i, b_i, i = 1, 2 : a \in [-5;5], b \in [1;5]\}.
$$

External disturbances are present as $f_1(t) = \sin 2t, f_2(t) = 2\sin 3t$.

We define the control law in the form (5), where $T_i(P) = \tau_i P^2 + \tau_2 P + \tau_3$.

Get $\theta_1 = 20, \tau_{u_1} = 16, \tau_{u_2} = 8, \tau_{u_3} = 2$.

In this case, the polynomials $Q_i(\lambda), (i = 1, 2)$ are Hurwitz. To obtain the observer (7) will take the value $\chi = 0.01$, and vector $\Pi_i = [-3; -3; -1], \ i = 1, 2$.

Fig. 4 shows a model diagram of the first subsystem of the object (1) dissipative stabilization.

![Fig.4. Structural scheme of robust control of the $i$-th subsystem.](image)

Fig. 5 shows the simulation results of dissipative system of stabilization with perturbations $f_1(t) = \sin 2t, f_2(t) = 2\sin 3t$. 

![Fig.5.](image)
Fig. 5. The trajectories of the system’s outputs in robust stabilization.

5 Discussion

The advantage of the proposed robust control law (5) is that it remains the same in the presence of unknown perturbation actions and in their absence. Only in the first case the system will be dissipative, while the second case it will be asymptotically robust.

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