Introduction

A vector valued integral is a vector attached to a function $f : X \to V$ from a measure space $X$ to a topological complex vector space $V$, written $\int_X f \, d\mu \in V$ with the property that

$$\alpha \left( \int_X f \, d\mu \right) = \int_X \alpha(f) \, d\mu,$$

for every continuous linear functional $\alpha : V \to \mathbb{C}$. This property determines the vector $\int_X f \, d\mu$ uniquely if the space $V$ is locally convex, as follows from the Hahn-Banach Theorem. For non-locally convex spaces, the notion rarely makes sense, consider for example the case of the space $V = L^p(0,1)$ with $0 < p < 1$, [Rud91]. In this case there are non-zero continuous linear functionals, therefore every vector is an integral for every function.

So from now on we assume the space $V$ to be locally convex. In that case the topology of $V$ is generated by all continuous seminorms. In this note we show the existence and strong continuity of a vector valued integral in many important cases. This includes the case of a continuous function $f : X \to V$ of compact support, where $X$ is a locally compact space equipped with a Radon measure.

For Banach spaces, these integrals have been constructed independently by Bochner [Boc35] and Gelfand-Pettis [Gel36, Pet38]. The latter construction has been generalized to locally convex spaces before, but not the former. In this note we generalize the construction of Bochner, which as additional feature gives strong continuity, more sharply, we get the natural estimate

$$p \left( \int_X f \, d\mu \right) \leq \int_X p(f) \, d\mu,$$

for every continuous seminorm $p$. 
1 Strongly integrable functions

By a topological vector space over \( \mathbb{C} \) we mean a complex vector space with a topology such that addition and scalar multiplication are continuous maps from \( V \times V \) respectively \( \mathbb{C} \times V \) to \( V \). We follow the convention that insists that the set \( \{0\} \) be closed. This implies that \( V \) is a Hausdorff space as can be seen in [Rud91], 1.6 or in greater generality in [DE09], Proposition 1.1.6.

The space is called locally convex, if every point has a neighborhood base consisting of convex open sets. Let \( V \) be a locally convex topological vector space over \( \mathbb{C} \) or locally convex space for short. Let \( (X, \mu) \) be a measure space and \( f : X \to V \) a measurable function. We write \( V' \) for the continuous dual space of \( V \), i.e., the space of all continuous linear functionals \( \alpha : V \to \mathbb{C} \).

**Definition 1.1** We say that \( f \) is strongly integrable, if there exists a vector \( \int_X f \, d\mu \in V \) such that

(a) For every \( \alpha \in V' \) one has

\[
\alpha \left( \int_X f \, d\mu \right) = \int_X \alpha(f) \, d\mu.
\]

(b) For every continuous seminorm \( p \) on \( V \) one has

\[
p \left( \int_X f \, d\mu \right) \leq \int_X p(f) \, d\mu < \infty.
\]

**Lemma 1.2** If \( f : X \to V \) is strongly integrable, then so is \( T(f) = T \circ f \) for every continuous linear map \( T : V \to W \), where \( W \) is another locally convex space.

**Proof:** Let \( g = T(f) \). Define \( \int_X g \, d\mu = T \left( \int_X f \, d\mu \right) \in W \). For \( \alpha \in W' \) one has \( \alpha \circ T \in V' \). Therefore

\[
\alpha \left( \int_X g \, d\mu \right) = \alpha \circ T \left( \int_X f \, d\mu \right) = \int_X \alpha \circ T(f) \, d\mu = \int_X \alpha(g) \, d\mu.
\]

The proof of the estimate (b) is similar. \( \square \)

**Definition 1.3** A measurable function \( f : X \to V \) is called integrally bounded, if one has

\[
\int_X p(f) \, d\mu < \infty
\]

for every continuous seminorm \( p \).
**Definition 1.4** The function \( f \) is called **essentially separable**, if for each continuous seminorm \( p \) there exists a set \( N_p \subset X \) of measure zero and a countable set \( C_p \subset V \) such that \( f(X \setminus N_p) \subset \overline{C_p(p)} \), where the closure is the \( p \)-closure.

**Example 1.5** Suppose that the image \( f(X) \) is relatively compact. Then \( f \) is essentially separable, since for given \( n \in \mathbb{N} \) there are \( x_1(n), \ldots, x_{k_n}(n) \in f(X) \) such that

\[
f(X) \subset \bigcup_{j=1}^{k_n} x_j(n) + \frac{1}{n} U_p,
\]

where

\[
U_p = \{ v \in V : p(v) < 1 \}
\]

is the convex balanced open zero neighborhood attached to the semi-norm \( p \). Let \( C_p \) be the set of all \( x_j(n) \), where \( n \) and \( j \) vary. Then \( f(X) \subset \overline{C_p(p)} \), so \( f \) is essentially separable.

**Definition 1.6** The function \( f \) is called **essentially bounded**, if there exists a set \( N \subset X \) of measure zero, such that \( f|_{X \setminus N} \) is bounded, which means it is bounded in every continuous seminorm.

**Definition 1.7** The space \( V \) is called **complete** if every Cauchy-net converges. It is called **quasi-complete**, if every bounded Cauchy-net converges.

**Theorem 1.8** Let \( V \) be a locally convex space and \( f : X \to V \) a measurable function from a measure space \((X, \mu)\). Suppose that \( f \) is essentially separable and integrally bounded.

(a) If \( V \) is complete, then \( f \) is strongly integrable.

(b) If \( V \) is quasi-complete and \( f \) is essentially bounded, then \( f \) is strongly integrable.

(c) If \( \mu(X) < \infty \) and the closure of the convex hull of \( f(X) \) is complete, then \( f \) is strongly integrable.

**Example 1.9** As an example, consider the case when \( X \) is a locally compact Hausdorff space and \( \mu \) a Radon measure. The space \( V \) is assumed to be quasi-complete, or even weaker, have the property that the closure of the convex hull of a compact set is complete. Then any compactly supported continuous map \( f : X \to V \) is Bochner-integrable. We thus get a map

\[
\int_X : C_c(X, V) \to V.
\]

This map is continuous when \( C_c(X, V) \) is equipped with the usual inductive limit topology.
The proof of the theorem will occupy the rest of the paper.

2 Bochner-approximable functions

Definition 2.1 Let $V$ be a locally convex topological vector space over the complex field. Let $(X, \mu)$ be a measure space. A simple function is a function $s : X \to V$ of the form

$$s = \sum_{j=1}^{n} 1_{A_j} v_j$$

for some measurable sets $A_j \subset X$ of finite measure and some $v_j \in V$. The integral of the simple function $s$ equals

$$\int_X s \, d\mu = \sum_{j=1}^{n} \mu(A_j) v_j \in V.$$ 

A measurable function $f : X \to V$ is called Bochner-approximable, if there exists a net $(s_j)_{j \in J}$ of simple functions such that for every continuous seminorm $p$ on $V$ one has

$$\int_X p(f - s_j) \, d\mu \to 0.$$ 

In that case the net $(s_j)_j$ is called an approximating net.

Lemma 2.2 (net-free formulation) A measurable function $f : X \to V$ is Bochner-approximable if and only if for every continuous seminorm $p$ there exists a simple function $s_p$ such that

$$\int_X p(f - s_p) \, d\mu < 1.$$ 

Proof: If an approximating net exists, the condition in the lemma is obvious. Now suppose that the condition of the lemma is satisfied. The set of all continuous seminorms has a natural partial order. Note that $p \leq q$ is equivalent to $U_p \supset U_q$. As every zero neighborhood contains a convex balanced zero neighborhood, the set of continuous seminorms is directed. So the simple functions $(s_p)_p$ form a net, and this net will do the job. This follows from the fact that for every continuous seminorm $p$ and any $\varepsilon > 0$ the function $\frac{1}{\varepsilon} p$ is again a continuous seminorm and one has

$$\int_X p(f - s_p) \, d\mu < \varepsilon.$$

Theorem 2.3 Let $f : X \to V$ be Bochner-approximable. Then for each approximating net $(s_j)_{j \in J}$, the net of integrals $(\int_X s_j \, d\mu)_{j}$ is a Cauchy net. If this
net converges for one approximating net, then it converges for every approximating net and the limit is uniquely a determined vector \( \int_X f \, d\mu \) in \( V \). In that case we say that \( f \) is Bochner-integrable. If \( f \) is Bochner-integrable, then so is \( T(f) = T \circ f \) for every continuous linear map \( T : V \to W \) into another locally convex space \( W \). One then has

\[
T \left( \int_X f \, d\mu \right) = \int_X T(f) \, d\mu.
\]

For every continuous seminorm \( p \) on \( V \) one has

\[
p \left( \int_X f \, d\mu \right) \leq \int_X p(f) \, d\mu.
\]

In the case \( V = \mathbb{C} \), a function is Bochner integrable if and only if it is Lebesgue integrable, in which case the two integrals coincide.

**Proof:** For a continuous seminorm \( p \) we have

\[
p \left( \int_X s_i \, d\mu - \int_X s_j \, d\mu \right) \leq \int_X p(s_i - s_j) \, d\mu \\
\leq \int_X p(s_i - f) \, d\mu + \int_X p(f - s_j) \, d\mu.
\]

This implies that the net of integrals is a Cauchy net. By the usual argument, the limit does not depend on the choice of the net. Let \( T \) and \( (s_j) \) be as in the theorem. We claim that \( T \circ s_j = T(s_j) \) is a net of simple functions in \( W \) which approximates \( T(f) \). For this let \( q \) be a continuous seminorm on \( W \). As \( T \) is continuous, there exists a continuous seminorm \( p \) on \( V \) such that \( q(T(v)) \leq p(v) \) for every \( v \in V \). We conclude

\[
\int_X q(T(f) - T(s_j)) \, d\mu \leq \int_X p(f - s_j) \, d\mu.
\]

So \( T(s_j) \) indeed approximates \( T(f) \) and so

\[
T \left( \int_X f \, d\mu \right) = T \left( \lim_j \int_X s_j \, d\mu \right) \\
= \lim_j \int_X T(s_j) \, d\mu = \int_X T(f) \, d\mu.
\]

The assertion about the case \( V = \mathbb{C} \) is easy. For a continuous seminorm \( p \) we have

\[
p \left( \int_X f \, d\mu \right) = p \left( \lim_j \int_X s_j \, d\mu \right) \\
= \lim_j p \left( \int_X s_j \, d\mu \right) \leq \lim inf_j \int_X p(s_j) \, d\mu.
\]

Let \( \varepsilon > 0 \). There exists \( j_0 \) such that for every \( j \geq j_0 \) one has \( \int_X p(s_j - f) \, d\mu < \varepsilon \). As \( |p(s_j) - p(f)| \leq p(s_j - f) \) we conclude \( p \left( \int_X f \, d\mu \right) < \int_X p(f) \, d\mu + \varepsilon \). For \( \varepsilon \to 0 \) the claim follows. \( \square \)
3 Integrable functions

Lemma 3.1 If \( f \) is essentially separable, then for each continuous seminorm \( p \), the set \( C_p \) can be chosen inside the image \( f(X) \).

**Proof:** Let \( c \in \mathbb{C}_p \) and let \( n \in \mathbb{N} \). If \( (c+\frac{1}{n}U_p) \cap f(X) \neq \emptyset \), we choose an element \( y(c,n) \) in that set. Let \( D_p \) be the union of all these elements \( y(c,n) \). We claim that \( f(X) \subset \overline{D_p}^{(p)} \). To prove this, let \( x \in X \) and \( n \in \mathbb{N} \). There exists \( c \in C_p \) with \( p(f(x)-c)<\frac{1}{n} \). So \( (c+\frac{1}{n}U_p) \cap f(X) \neq \emptyset \), i.e., the element \( y(c,2n) \in f(X) \) with \( p(y(c,2n)-c)<\frac{1}{2n} \) exists. It follows that \( p(y(c,n)-f(x))<\frac{1}{n} \). \( \square \)

Theorem 3.2 A measurable function \( f: X \to V \) is Bochner-approximable if and only if

(a) \( f \) is essentially separable and
(b) \( f \) is integrally bounded.

If \( f(X) \) is relatively compact and \( \mu(X)<\infty \), then \( f \) is Bochner-approximable.

**Proof:** Let \( f \) be Bochner-approximable. We show that \( f \) is essentially separable. So let \( p \) be a continuous seminorm. For each given \( n \in \mathbb{N} \), there exists a simple function \( s_n: X \to V \) with \( \int_X p(f-s_n) \, d\mu < \frac{1}{n} \). Let \( E_p \) be the \( p \)-closure of the vector space spanned by the union of the images of all \( s_n \), \( n \in \mathbb{N} \). Then \( E_p \) is the \( p \)-closure of some countable set \( C_p \), for instance, one can take the \( \mathbb{Q}(i) \)-vector space spanned by the images of all \( s_n \). For each \( n \in \mathbb{N} \) the set

\[
N_n = \left\{ x \in X : p(f(x), E_p) > \frac{1}{n} \right\}
\]

is a set of measure zero, where

\[
p(v, E_p) = \inf \{ p(v-e) : e \in E_p \}.
\]

The complement in \( X \) of the set \( f^{-1}(E_p) \) is the union of all \( N_n \), therefore a set of measure zero, so \( f \) is essentially separable. As \( \int_X p(f-s_j) \, d\mu < \infty \) it follows that \( \int_X p(f) \, d\mu \leq \int_X p(f-s_j) + p(s_j) \, d\mu < \infty \), so \( p(f) \) is integrable.

Now for the converse direction. Let \( p \) be a continuous seminorm. We will attach to \( p \) a simple function \( s_p \) with \( \int_X p(f-s_p) \, d\mu < 1 \). Then \( f \) is Bochner-approximable by Lemma 2.2. In order to construct \( s_p \), let \( C_p = \{ c_1, c_2, \ldots \} \) be the countable set and let \( N_p \subset X \) be the nullset attached to \( p \). Write \( X_p = X \setminus N_p \). For \( n \in \mathbb{N} \) and \( \delta > 0 \) let \( A_n^\delta \) be the set of all \( x \in X_p \) such that \( p(f(x)) > \delta \) and \( p(f(x)-c_n) < \delta \). To have a sequence of pairwise disjoint sets, define

\[
D_n^\delta = A_n^\delta \setminus \bigcup_{k<n} A_k^\delta.
\]
The set $\bigcup_n A^n = \bigcup_n D^n$ equals $f^{-1}(f(X_p) \setminus \delta U_p)$. Since $p(f)$ is integrable, the set $\bigcup D^n$ is of finite measure. Let $s_{p,n} = \sum_{j=1}^n 1_{D^n/n}c_j$. This is a simple function. It is easy to see that the sequence $p(s_{p,n} - f)$ converges to 0 pointwise on the set $X_p$. On that set we also have $p(s_{p,n}) \leq 2p(f)$ by construction. So we get $p(f - s_{p,n}) \leq p(f) + p(s_{p,n}) \leq 3p(f)$, and by dominated convergence,$$
int_X p(f - s_{p,n}) \, d\mu \rightarrow 0.$$In particular, there exists $n_0 \in \mathbb{N}$, such that for $s_p = s_{p,n_0}$ one has $\int_X p(f - s_p) \, d\mu < 1$.

Finally, assume that $f(X)$ is relatively compact and $\mu(X) < \infty$. Then for any given continuous seminorm $p$, the set $p(f(X))$ is relatively compact, hence bounded, so $p(f)$ is integrable since $\mu(X) < \infty$. Further, $f$ is essentially separable by Example 2.1.

\begin{theorem}
\begin{enumerate}
\item If $V$ is complete, then every Bochner-approximable function in $V$ is Bochner-integrable.
\item If $V$ is quasi-complete, then every bounded Bochner-approximable function in $V$ is Bochner-integrable.
\item Let $f : X \rightarrow V$ be Bochner-approximable. If $\mu(X) < \infty$ and the closure of the convex hull of $f(X)$ is complete, then $f$ is Bochner-integrable.
\end{enumerate}
\end{theorem}

\textbf{Proof:} (i) and (ii) are clear. For (iii) we may assume $\mu(X) = 1$. By Lemma 3.1 the set $C_p$ can be chosen inside $f(X)$. According to the proof of Theorem 3.2 there exists an approximating net $(s_j)_{j \in J}$ such that each $s_j$ takes values in the sets $C_p$ for varying $p$, hence $s_j(X) \subset f(X)$. Now write
$$s_j = \sum_{k=1}^n 1_{A_k}v_k,$$then each $v_k$ lies in $f(X)$ and $X = \bigcup_k A_k$. Therefore,$$
int_X s_j \, d\mu = \sum_{k=1}^n \mu(A_k)v_k$$is a convex-combination of elements of $f(X)$, so lies in the convex hull of $f(X)$. The closure of this convex hull being complete, the Cauchy-net $s_j$ converges. $\square$

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