ANSATZ OF HANS BETHE FOR A TWO-DIMENSIONAL BOSE GAS

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ABSTRACT. The method of \(q\)-oscillator lattices, proposed recently in [hep-th/0509181], provides the tool for a construction of various integrable models of quantum mechanics in \(2+1\) dimensional space-time. In contrast to any one dimensional quantum chain, its two dimensional generalizations – quantum lattices – admit different geometrical structures. In this paper we consider the \(q\)-oscillator model on a special lattice. The model may be interpreted as a two-dimensional Bose gas. The most remarkable feature of the model is that it allows the coordinate Bethe Ansatz: the \(p\)-particles’ wave function is the sum of plane waves. Consistency conditions is the set of \(2p\) equations for \(p\) one-particle wave vectors. These “Bethe Ansatz” equations are the main result of this paper.

1. Introduction

Integrable models of quantum mechanics in discrete space-time describe a system of interacting quantum observables (spin operators, oscillators, etc.) situated in sites of an one dimensional chain (the models in \(1+1\) dimensional space-time), or in the vertices of a two dimensional lattice (models in \(2+1\) dimensional space-time). The subject of this paper is the second variant – the quantum integrable lattices.

Formulation of a realistic integrable model of quantum mechanics in \(2+1\) dimensional space-time was a long-standing problem in the theory of integrable systems. Examples of higher dimensional integrable systems are known, but the integrability seemed to be a very high price paid to the detriment of a physical interpretation.

Both known classes of integrable quantum lattices may be specified by the definition of the algebra of observables. In contrast to the quantum chains, where the algebra of observables may be e.g. an evaluation representation of any quantum group, in the case of quantum lattices the algebra of observables may be either the local\(^1\) Weyl algebra \([\text{II}]\) or the local \(q\)-oscillator algebra \([\text{2}]\). Other examples of integrable quantum lattices are not known.

When the algebra of observables (and its representation) is fixed, there exists an almost unique way to produce the key notion of the integrability – a complete set of commutative

\(^1\)The term “local” means that the quantum observables in different vertices commute, as well as it is in the quantum chains case
operators. For the spin chains it is the way of Lax operators and transfer matrices with e.g. fundamental representations in the auxiliary space. In both classes of quantum lattices the integrals of motion may be produced by a decomposition of a determinant of a certain big operator-valued matrix with respect to two spectral parameters \[3, 4\]. Therefore the integrals of motion rather have a combinatorial nature. Both classes of quantum lattices allow a definition of another kind of operators: the layer-to-layer transfer matrices, related to intertwining relations, tetrahedron equations etc. Such representation-dependent layer-to-layer transfer matrices are the kind of discrete-time evolution operators for the quantum lattices, their structure is much more complicated, but since their integrals of motion are known, it is not necessary to consider them at the first step.

The concept of quantum lattices implies a certain feature which is not applicable to the quantum chains. One may vary not only a size of the lattice, but its geometry and topology as well. This is the reason why we discuss the classes of models. The methods of Weyl algebra and \(q\)-oscillator algebra are the frameworks for construction of various quantum lattices.

Mainly the simplest geometry was considered before – the square lattices with periodical boundary conditions\(^2\). The simple square lattice with the sizes \(n \times m\) may be regarded as the length-\(m\) chain of its length-\(n\) lines, and therefore such quantum lattice is effectively a quantum chain. In particular, the quantum lattice with the Weyl algebra at a root of unity corresponds to auxiliary transfer matrices \([1]\) of the \(U_q(\hat{gl}_n)\) generalized chiral Potts model \([6, 7]\). The \(q\)-oscillator quantum lattice corresponds to the auxiliary transfer matrices for reducible oscillator representation of \(U_q(\hat{gl}_n)\) \([2]\). From the point of view of quantum mechanics in 2 + 1 dimensional space-time, the square lattices provide a non-realistic models since the translation operators do not belong to the set of integrals of motion and therefore momenta of eigenstates can not be defined. In the quantum chain interpretation, it means that \(U(1)\) charges are the variables, and the chain is homogeneous only on subspaces where all \(U(1)\) charges are the same in all sites. Otherwise, the translation invariance is lost.

The issue is evident. One has to consider a lattice which is not a chain of its lines. We define such lattice in the next sections. The framework of \(q\)-oscillator model on our special lattice immediately produces a kind of two dimensional Bose gas with physical dynamics. The Fock vacuum is the natural reference state, the total \(q\)-oscillators occupation number – the number of bosons – is conserved, and the \(p\)-particles wave function is the superposition of one-particle

\(^2\)Examples of Weyl algebra non-square lattices were mentioned in \([5]\). In particular, the Weyl algebra framework on the different shapes of the lattices produces the relativistic Toda chain as well as the quantum discrete Liouville model.
plane waves. Each plane wave is characterized by two components of its momentum, and the consistency conditions give $2p$ equations for $p$ two-components momenta. These consistency conditions look much more complicated than the Bethe Ansatz equations for a quantum chain, we do not investigate them in details here. The aim of this paper is just to present the method, the model, and the “Bethe Ansatz” equations.

2. Framework of $q$-oscillator algebra.

We start with the formulation of a generic $q$-oscillator lattice.

Let $\mathcal{L}$ be a lattice formed by a number of directed lines on a torus. Pairwise intersections of the lines are the vertices of the lattice. Let the vertices are enumerated in some way, for instance by the integer index $j = 1, 2, 3, \ldots, \Delta$, where $\Delta$ is the number of vertices.

The $q$-oscillator generators $x_j, y_j$ and $h_j$ are assigned to vertex $j$. The oscillators for the different vertices commute. Locally, we define the $q$-oscillator algebra by

$$
\begin{align*}
(x_j y_j) &= 1 - q^{2+2h_j}, \\
(y_j x_j) &= 1 - q^{2h_j}, \\
(x_j q^{h_j}) &= q^{h_j+1} x_j, \\
(y_j q^{h_j}) &= q^{h_j-1} y_j.
\end{align*}
$$

Consider next the system of non-self-intersecting paths along the edges of the lattice. The lines of the lattice are directed, and we demand that paths must follow the orientation of the edges. The paths may go through a single vertex in one of six variants shown in Fig. 1. The vertex

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{The “weights” $f_j$ of $j$th vertex, $j = 1, \ldots, \Delta$.}
\end{figure}

“weight” $f_j$ is associated with each variant of bypassing as it is shown in Fig. 1. Coefficients $\lambda_j, \mu_j$ and $\nu_j$ are extra vertex $\mathbb{C}$-valued parameters, they are related by $\nu_j^2 \equiv -q^{-1} \lambda_j \mu_j$. In general, they are free up to a single condition: if two vertices $j$ and $j'$ are formed by the intersections of the same lines, their $\mathbb{C}$-valued parameter must be the same.

For any path $P$ let

$$
\mathbf{t}_P = \prod_{\text{along } P} f_j.
$$

Note, if a path does not touch a vertex, then it vertex’s contribution to $\mathbf{t}_P$ is just the unity according to the leftmost variant of Fig. 1. Recall in addition, the $q$-oscillator algebra is local,
elements of different vertices commute. Therefore the notion of the product along the path in (2) is well defined.

Let \( A \) and \( B \) be the two basic homotopy cycles of the torus. Any path has a certain homotopy class, \( P \sim nA + mB \). Let

\[
\mathbf{t}_{n,m} = \sum_{P: P \sim nA + mB} \mathbf{t}_P .
\]

Here the sum is taken over all possible paths of homotopy class \( nA + mB \). Formula (3) ends the formulation of \( q \)-oscillator lattice’s prescription. The point is that for any lattice \( L \), formed by directed lines on the torus, the operators \( \mathbf{t}_{n,m} \) form the commutative set:

\[
\mathbf{t}_{n,m} \mathbf{t}_{n',m'} = \mathbf{t}_{n',m'} \mathbf{t}_{n,m} \quad \forall \quad n, m, n', m'.
\]

Moreover, if there are no lines of trivial homotopy class in \( L \), the set of integrals of motion is complete.

Integrals of motion may be combined into a polynomial of two variables,

\[
\mathbf{T}(u, v) = \sum_{n,m} u^n v^m \mathbf{t}_{n,m} .
\]

Operator \( \mathbf{T} \) has the structure of the layer-to-layer transfer matrix, equations (4) are equivalent to the commutativity of transfer matrices with different spectral parameters \( u, v \). The commutativity may be proven for simple lattices with the help of tetrahedron equation [8, 2, 9].

3. The lattice

Turn now to the definition of our special lattice. It is formed by the intersections of only two lines \( \alpha \) and \( \beta \). With respect to the basic cycles \( A \) and \( B \) of the torus, the lines have the classes

\[
\alpha \sim NA - B , \quad \beta \sim MB - A .
\]

Example of such lattice is given in Fig. 2. There the opposite dashed borders are identified (dashed lines are the cuts of the torus). The line \( \alpha \) is directed up (along the \( A \)-cycle), the line \( \beta \) is directed to the right (along the \( B \)-cycle). The lines are intersecting in

\[
\Delta = NM - 1
\]

points.

We enumerate the vertices by number \( j, j \in \mathbb{Z}\Delta \). The numeration is successive along the reverse direction of \( \alpha \)-line, see Fig. 2. The shift \( j \rightarrow j - 1 \) corresponds to a one-step translation along the \( \alpha \)-line, whereas the sifting \( j \rightarrow j + M \) corresponds to the one-step translation along the \( \beta \)-line.
Now the lattice is formulated, and combinatorial rules of Fig. 1 may be applied. Note, since all the vertices of the lattice are formed by the same lines, all their $\mathbb{C}$-valued parameters are the same, and one may put

\begin{equation}
\lambda = \mu = 1, \quad \nu^2 = -q^{-1}.
\end{equation}

Generating function (5) has the structure

\begin{equation}
T(u, v) = u^N v^{-1} q^N + v^M u^{-1} q^N + \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u^n v^m t_{n,m},
\end{equation}

where $\mathcal{N}$ is the total occupation number

\begin{equation}
\mathcal{N} = \sum_j h_j.
\end{equation}

Elements $q^N$ in (9) are the values of both $t_{N,-1}$ and $t_{-1,M}$, combinatorially they are the paths along all $\alpha$ or $\beta$ lines (the third and the fourth variants of Fig. 1 for all vertices). Elements $t_{0,0}$ and $t_{N-1,M-1}$ correspond to the empty path and the complete path (the first and the second variants of Fig. 1 for all vertices) correspondingly, their values are

\begin{equation}
t_{0,0} = 1, \quad t_{N-1,M-1} = (-q^{-1})^\Delta.
\end{equation}
In what follows, we will consider the elements $t_{0,1}$ and $t_{1,0}$ given by

$$t_{1,0} = -q^{-1} \sum_j x_{j} y_{j+M} q^{h_{j+1} + h_{j+2} + \cdots + h_{j+M-1}},$$

$$t_{0,1} = -q^{-1} \sum_j x_{j} y_{j+1} q^{h_{j+1} + h_{j+2} + M + 2M + \cdots + h_{j+(N-1)M}}.$$

(12)

Combinatorial summand of $t_{0,1}$ is shown in Fig. 3.

![Figure 3. Path of the class B – an element of $t_{0,1}$.](image)

4. Eigenstates.

The total occupation number is the conserving quantity, and therefore we may construct the eigenstates of the model step-by-step just increasing the number of bosons for the Fock space representation of the $q$-oscillator algebra. The Fock vacuum $|0\rangle$ is defined by

$$x_j |0\rangle = 0, \quad h_j |0\rangle = 0, \quad \forall \ j \in \mathbb{Z}_\Delta.$$

(13)

The Fock vacuum is evidently the eigenstate of the model, eigenvalues of all $t_{n,m}$ are zeros (except $t_{0,0}$, $t_{N-1,M-1}$ and $q^N$).

In what follows, we concentrate on the diagonalization of $t_{0,1}$. Its eigenstates are defined uniquely. They will be evidently the eigenstates of $t_{1,0}$ and, due to the uniqueness, they must
be the eigenstates of all the other integrals of motion. We will use below the normalized form of (12):

\[ \tau_\alpha = -\frac{q}{1-q^2} t_{0,1}, \quad \tau_\beta = -\frac{q}{1-q^2} t_{1,0}. \]

4.1. **One boson state.** Consider the states with the single boson, \( N = 1 \). Let

\[ |\phi_j\rangle = y_j |0\rangle \]

be the basis of one-boson states. On this subspace both operators (14) are the translation operators:

\[ \tau_\alpha |\phi_j\rangle = |\phi_{j+1}\rangle, \quad \tau_\beta |\phi_j\rangle = |\phi_{j+M}\rangle. \]

Therefore

\[ |\Psi\rangle = \sum_j \omega^j |\phi_j\rangle, \quad \omega^\Delta = 1, \]

is the eigenstate of the model, the eigenvalues of (14) are

\[ \tau_\alpha |\Psi\rangle = |\Psi\rangle \omega^{-1}, \quad \tau_\beta |\Psi\rangle = |\Psi\rangle \omega^{-M}. \]

Since \( \omega^\Delta = 1 \) may take \( \Delta \) different values, the set of (17) is complete on the subspace \( N = 1 \).

Formally, the spectrum of both translation operators (18) is defined by one parameter \( \omega \). Note, eigenvalues \( \omega^{-1} \) and \( \omega^{-M} \) are dual in the sense \( (\omega^{-M})^N = \omega^{-1} \). When \( N, M \to \infty \), one may talk about two independent components of momentum. Namely, let

\[ k = N k_\alpha + k_\beta, \]

where \( k_\alpha \) and \( k_\beta \) are relatively small. Then, when \( N, M \to \infty \),

\[ \omega^{-1} = e^{2\pi i k/\Delta} \to e^{2\pi i k_\alpha/M}, \quad \omega^{-M} = e^{2\pi i M k/\Delta} \to e^{2\pi i k_\beta/N}, \]

i.e. in the thermodynamical limit \( 2\pi k_\alpha/M \) and \( 2\pi k_\beta/N \) play the rôles of independent components of the momentum.

4.2. **Two bosons state.** Turn next to \( N = 2 \). Define

\[ c_{j,k} = \delta_{j,k+M} + \delta_{j,k+2M} + \cdots + \delta_{j,k+(N-1)M}. \]

Then

\[ \tau_\alpha \cdot y_j y_k |0\rangle = (q^{c_{j-k}} y_j y_{j+1} + q^{c_{j-k}} y_j y_{k+1}) |0\rangle, \quad j \neq k, \]

\[ \tau_\alpha \cdot y_k^2 |0\rangle = (1 + q^2) y_{k+1} y_k |0\rangle. \]
Consider a two-particles state with zero momentum of the mass center:

\[ |\Psi\rangle = \sum_j \left( \Psi_0 (1 + q^2)^{-1} y_j^2 |0\rangle + \sum_{k>0} \Psi_k y_j y_{j+k} |0\rangle \right) \]

Eigenvalue equation \( \tau_\alpha |\Psi\rangle = |\Psi\rangle \tau_\alpha \) reads in components

\[
\begin{align*}
\tau_\alpha \Psi_0 &= (1 + q^2) \Psi_1, \\
\tau_\alpha \Psi_{nM-1} &= q \Psi_{nM} + \Psi_{nM-2}, \\
\tau_\alpha \Psi_{nM} &= \Psi_{nM+1} + \Psi_{nM-1},
\end{align*}
\]

in all other cases: \( \tau_\alpha \Psi_k = \Psi_{k-1} + \Psi_{k+1} \).

Periodicity condition is simply

\[ \Psi_k = \Psi_{\Delta-k} \cdot \]

Define the double index: \( j = M n + m \rightarrow j = (n, m), \)

\[ \Psi_{Mn+m} = \Psi_{(n,m)}, \quad 0 \leq m < M, \quad 0 \leq n < N. \]

Equations \(24\) have in the general position the form \( \tau_\alpha \Psi_{(n,m)} = \Psi_{(n,m-1)} + \Psi_{(n,m+1)}. \) Therefore,

\[ \Psi_{(n,m)} = P_n \omega^m + Q_n \omega^{-m}, \quad \tau_\alpha = \omega + \omega^{-1}. \]

Initial conditions (the first of \(24\)) give

\[ \frac{Q_0}{P_0} = \frac{1 - q^2 \omega^2}{q^2 - \omega^2}. \]

The second and third relations of \(24\) give

\[
\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \mathcal{M} \cdot \begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \omega_M q^2 - \omega^2 & \omega^{-M} \omega_n^2 (q^2 - 1) \\ \omega^{-1} q^2 \omega^2 (1 - \omega^2) & \omega_M^{-1} \omega^2 (1 - q^2) \end{pmatrix}.
\]

Let \( \Omega \) and \( \Omega^{-1} \) be the eigenvalues of \( \mathcal{M} \). Then one may choose the basis of eigenvectors of \( \mathcal{M} \) such that

\[ P_n = A_{+-} \Omega^n + A_{-+} \Omega^{-n}, \quad Q_n = A_{+-} \Omega^n + A_{-+} \Omega^{-n}. \]

The boundary condition \( \Psi_k = \Psi_{\Delta-k} \) reads in double-indices \( \Psi_{(N-1,m)} = \Psi_{(0,M-1-m)} \), i.e.

\[
\begin{pmatrix} P_{N-1} \\ Q_{N-1} \end{pmatrix} = \mathcal{M}^{N-1} \cdot \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} = \begin{pmatrix} \omega^{1-M} Q_0 \\ \omega^{M-1} P_0 \end{pmatrix}.
\]
The following pair of equations summarizes all the calculations:

\[
\Omega + \Omega^{-1} = \omega^M \frac{q^2 - \omega^2}{q(1 - \omega^2)} + \omega^{-M} \frac{1 - q^2\omega^2}{q(1 - \omega^2)},
\]
(32)

\[
\omega + \omega^{-1} = \Omega^N \frac{q^2 - \Omega^2}{q(1 - \Omega^2)} + \Omega^{-N} \frac{1 - q^2\Omega^2}{q(1 - \Omega^2)}.
\]

Here the first equation is the definition of \(\Omega\) (it is the characteristic polynomial of \(M\)). The second equation comes from (31) after excluding of \(P_0, Q_0\) via (28). The final answer is the Bethe Ansatz (two-particles wave function is the superposition of one-particle plane waves)

\[
\Psi_{(n,m)} = A_{++} \Omega^n \omega^m + A_{+-} \Omega^{-n} \omega^m + A_{-+} \Omega^n \omega^{-m} + A_{--} \Omega^{-n} \omega^{-m}.
\]

Parameters \(A_{\pm,\pm}\), related to the eigenvectors of \(M\), are defined by

\[
\Omega^{N-1} A_{+-} = \omega^{M-1} A_{-+}, \quad A_{-+} = \Omega^{N-1} \omega^{M-1} A_{++}
\]
and

\[
\frac{A_{+-} + A_{-+}}{A_{++} + A_{--}} = \frac{1 - q^2\omega^2}{q^2 - \omega^2}, \quad \frac{A_{+-} + A_{++}}{A_{++} + A_{--}} = \frac{1 - q^2\Omega^2}{q^2 - \Omega^2}.
\]

The state has the evident \(N \leftrightarrow M\) symmetry:

\[
\tau_\alpha \ket{\Psi} = (\omega + \omega^{-1}) \ket{\Psi}, \quad \tau_\beta \ket{\Psi} = (\Omega + \Omega^{-1}) \ket{\Psi}.
\]

From the point of view of \(\tau_\beta\), the second relation of (32) is the characteristic polynomial, while the first one comes from the \(\beta\)-boundary conditions.

4.3. \(p\)-bosons state. Turn finally to the state with \(N = p\) bosons. Actually, all the results of this section were obtained explicitly for \(p = 3, 4\), and then conjectured for arbitrary \(p\).

The wave function \(\Psi_{j_1,j_2,\ldots,j_p}\) of the eigenstate

\[
\ket{\Psi} = \sum_{j_1 \leq j_2 \leq \ldots \leq j_p} \Psi_{j_1,j_2,\ldots,j_p} y_{j_1} y_{j_2} \cdots y_{j_p} \ket{0}
\]

is the superposition of the plane waves. In the generic point (no coincidence in \(j_1, j_2, \ldots, j_p\)) the wave function is

\[
\Psi_{j_1,j_2,\ldots,j_p} = \sum_{\sigma,\sigma'} A_{\sigma',\sigma} \Omega^{n_1}_{\sigma_1} \Omega^{n_2}_{\sigma_2} \cdots \Omega^{n_p}_{\sigma_p} \omega_{\sigma_1}^{m_1} \omega_{\sigma_2}^{m_2} \cdots \omega_{\sigma_p}^{m_p}
\]

where \(\sigma\) and \(\sigma'\) are independent permutations of the set \((1, 2, \ldots, p)\), and \(n_c, m_c\) are related with \(j_c\) by

\[
j_c = Mn_c + m_c, \quad 0 \leq m_c < M, \quad 0 \leq n_c < N,
\]

\(\sigma\) denoting:

\[
\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_p, \quad 0 \leq \sigma_1, \ldots, \sigma_p < p.
\]

\(\Omega\) and \(\Omega^{-1}\) are the characteristic polynomials of the matrices \(M\) and \(M^{-1}\), respectively, and \(\omega\) and \(\Omega\) are the characteristic polynomials of the matrices \(P\) and \(Q\), respectively.
Two sets of exponential momenta

\[ \omega_c = e^{ik_c}, \quad \Omega_c = e^{ik'_c}, \quad c = 1, 2, \ldots, p, \]

are solutions of two sets of equations. To write out these equation, we need some extra notations.

Firstly, let

\[ G_{a,b}(\{\omega\}) = \frac{q^{-1}\omega_b - q\omega_a}{\omega_b - \omega_a}. \]

Let \( I_k \) be a length-\( k \) subsequence of \((1, 2, \ldots, p)\), \( I_{p-k} \) be the compliment subsequence such that

\[ I_k \cup I_{p-k} = (1, 2, \ldots, p). \]

Let then

\[ P_k[\{\omega, \omega^M\}] = \sum_{I_k} \left( \prod_{a \in I_k} \omega_a^M \prod_{b \in I_{p-k}} G_{b,a}(\{\omega\}) \right), \]

where the sum is taken over all possible subsequences of the length \( k \). E.g. for \( p = 3 \)

\[ \begin{align*}
P_1[\{\omega, \omega^M\}] &= \omega_1^M G_{21} G_{31} + \omega_2^M G_{12} G_{32} + \omega_3^M G_{13} G_{23}, \\
P_2[\{\omega, \omega^M\}] &= \omega_1^M \omega_2^M G_{31} G_{21} + \omega_1^M \omega_3^M G_{21} G_{12} + \omega_2^M \omega_3^M G_{12} G_{13}, \\
P_3[\{\omega, \omega^M\}] &= \omega_1^M \omega_2^M \omega_3^M.
\end{align*} \]

The final step: let

\[ \Psi(\Omega|\{\omega, \omega^M\}) = \sum_{k=0}^{p} (-)^k \Omega^{p-k} P_k[\{\omega, \omega^M\}] = \Omega^p - \Omega^{p-1} P_1 + \Omega^{p-2} P_2 + \cdots. \]

Then the consistency conditions for the Ansatz (38) read

\[ \begin{align*}
\Psi(\Omega_a|\{\omega, \omega^M\}) &= 0, \\
\Psi(\omega_a|\{\Omega, \Omega^N\}) &= 0,
\end{align*} \quad \forall \quad a = 1, 2, \ldots, p. \]

These are the Bethe Ansatz equations for our two dimensional Bose gas.

The values of \( \Psi_{j_1, j_2, \ldots, j_p} \) in the case when some of \( j_a \) coincide, as well as the values of \((p!)^2\) amplitudes \( A_{\sigma, \sigma'} \), can be defined uniquely.

The state \( 38 \) provides the eigenvalues of \( 14 \)

\[ \tau_\alpha |\Psi\rangle = |\Psi\rangle \left( \sum_{a=1}^{p} \omega_a^{-1} \right), \quad \tau_\beta |\Psi\rangle = |\Psi\rangle \left( \sum_{a=1}^{p} \Omega_a^{-1} \right). \]

Note,

\[ \Omega_1 \Omega_2 \cdots \Omega_p = \omega_1^M \omega_2^M \cdots \omega_p^M. \]
(it follows from $P_p = \omega_1^M \cdots \omega_p^M$), therefore the mass center of $p$-particles state moves accordingly to $\Omega$.

The one-dimensional limit of (46) must be mentioned. In the case $N = 1$ our system becomes the chain of the length $M - 1$. Polynomial $\mathcal{P}(\omega; \Omega, \Omega)$ has a simple structure, all $\Omega_a$ may be excluded from (46). The resulting equations for $\omega_a$ are

$$\omega_a^{M-1} = \prod_{b \neq a} G_{ab}(\{\omega\}) \frac{G_{ba}(\{\omega\})}{G_{ba}(\{\omega\})},$$

what are the usual Bethe Ansatz equations for the quantum chain.

The last important note is that equations (46) provide real momenta $k_c$ and $k'_c$ of (40) if $q$ is real. In addition, numerical estimations for not too big lattices and for two particles shows that the Ansatz is complete.

4.4. Sketch derivation of (46). Equations (46) may be obtained in the same way as (32). One may start with the case when $m_1 < m_2 < \cdots < m_p$ and $m_p < m_1 + M$. Then the eigenstate of $\tau_\alpha$, $\tau_\alpha |\Psi\rangle = |\Psi\rangle\tau_\alpha$, $\tau_\alpha = \sum_a \omega_a^{-1}$, is given by

$$|\Psi_{m_1,\ldots,m_p}\rangle = \sum_\sigma A_\sigma^{(0)} \omega_{\sigma_1}^{m_1} \omega_{\sigma_2}^{m_2} \cdots \omega_{\sigma_p}^{m_p}.$$

The eigenvalue equation for $\tau_\alpha$ provide the way to interpolate $|\Psi_{m_1,m_2,\ldots,m_p}\rangle$ for the larger values of $m_\alpha$. For instance, when $m_p \sim m_1 + M$, the eigenvalue equation modifies as

$$\tau_\alpha |\Psi_{m_1,\ldots,m_p}\rangle = q^{\delta_{m_p,m_1+M}} |\Psi_{m_1-1,\ldots,m_p}\rangle + q^{\delta_{m_p,m_1+M}} |\Psi_{m_1,\ldots,m_p-1}\rangle + \text{all the rest}.$$

Extra $q$-factors produce some linear transformation of the amplitudes $A_\sigma$ of $|\Psi_{m_1,\ldots,m_p}\rangle$ in the same way as in eq. (29). Repeating this procedure for $m_p \sim m_2 + M$, $m_p \sim m_3 + M$ etc., one comes to

$$|\Psi_{m_1,m_2,\ldots,m_p+M}\rangle = \sum_\sigma A_\sigma^{(1)} \omega_{\sigma_1}^{m_1} \omega_{\sigma_2}^{m_2} \cdots \omega_{\sigma_p}^{m_p},$$

where $A_\sigma^{(1)}$ is a linear combination of $A_\sigma^{(0)}$:

$$A_\sigma^{(1)} = \sum_{\sigma'} \mathcal{M}_{\sigma,\sigma'} A_{\sigma'}^{(0)},$$

cf. (29). The Ansatz (28) corresponds to (52) in the basis of eigenvectors of $\mathcal{M}$. The miracle of the exact integrability is that $p! \times p!$ matrix $\mathcal{M}$ has only $p$ eigenvalues:

$$\det (\Omega - \mathcal{M}) = \mathcal{P}(\Omega; \omega, \omega^M)^{(p-1)!}.$$

The second equation in (46), providing the $N \leftrightarrow M$ symmetry of the Bethe Ansatz equations, solves the boundary condition analogously to (31).
5. Conclusion

This paper presents a model of two-dimensional Bose gas and the conjecture for the equations describing its eigenstates. The model has the main features of a physical model. The states are described by the system of plane waves – at least the notion of the momenta is well defined. In particular, the sum of cosines of momenta is a candidate for the Hamiltonian. Equations, in the case when the number of bosons $p$ is big, are the subject of further investigations.

We believe, this model may have some interest for the theory of integrable systems and for the condensed matter physics.

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