Casting loop quantum cosmology in the spin foam paradigm

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Abstract
The goal of spin foam models is to provide a viable path integral formulation of quantum gravity. Because of background independence, their underlying framework has certain novel features that are not shared by path integral formulations of familiar field theories in Minkowski space. As a simple viability test, these features were recently examined through the lens of loop quantum cosmology (LQC). Results of that analysis, reported in a brief communication (Ashtekar A et al 2009 Phys. Lett. B 681 347–52), turned out to provide concrete arguments in support of the spin foam paradigm. We now present detailed proofs of those results. Since the quantum theory of LQC models is well understood, this analysis also serves to shed new light on some long standing issues in the spin foam and group field theory literature. In particular, it suggests an intriguing possibility for addressing the question of why the cosmological constant is positive and small.

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1. Introduction
Four different avenues to quantum gravity have been used to arrive at spin foam models (SFMs). The fact that ideas from seemingly unrelated directions converge to the same type of structures and models has provided a strong impetus to the spin foam program over the years [2].

The first avenue is the Hamiltonian approach to loop quantum gravity (LQG) [3–5]. By mimicking the procedure that led Feynman [6] to a sum over histories formulation of quantum mechanics, Rovelli and Reisenberger [7] proposed a spacetime formulation of LQG. This work launched the spin foam program. The second route stems from the fact that the starting point in canonical LQG is a rewriting of classical general relativity that emphasizes connections
over metrics [8]. Therefore in the passage to quantum theory it is natural to begin with the path integral formulation of appropriate gauge theories. A particularly natural candidate is the topological BF theory [9] because in three spacetime dimensions it is equivalent to Einstein gravity, and in higher dimensions general relativity can be regarded as a constrained BF theory [10]. The well-controlled path integral formulation of the BF theory provided the second avenue and led to the SFM of Barret and Crane [11]. The third route comes from the Ponzano–Regge model of three-dimensional gravity [12] that inspired Regge calculus in higher dimensions [13–15]. Here one begins with a simplicial decomposition of the spacetime manifold, describes its discrete Riemannian geometry using edge lengths and deficit angles and constructs a path integral in terms of them. If one uses holonomies and discrete areas of loop quantum gravity in place of edge lengths, one is again led to a spin foam. These three routes are inspired by various aspects of general relativity. The fourth avenue starts from approaches to quantum gravity in which gravity is to emerge from a more fundamental theory based on abstract structures that, to begin with, have nothing to do with spacetime geometry. Examples are matrix models for two-dimensional gravity and their extension to three dimensions—the Bouloulatov model [16]—where the basic object is a field on a group manifold rather than a matrix. The Bouloulatov model was further generalized to a group field theory (GFT) tailored to four-dimensional gravity [4, 17, 18]. The perturbative expansion of this GFT turned out be very closely related to the vertex expansions in SFMs. Thus the SFMs lie at a junction where four apparently distinct paths to quantum gravity meet. Through contributions of many researchers it has now become an active research area (see, e.g., [4, 10, 19]).

Let us begin with the first path and examine SFMs from the perspective of LQG. Recall that spin network states are used in LQG to construct a convenient orthonormal basis in the kinematical Hilbert space. A key challenge is to extract physical states from them by imposing constraints. Formally this can be accomplished by the group-averaging procedure which also provides the physical inner product between the resulting states [20, 21]. From the LQG perspective, the primary goal of SFMs is to construct a path integral that leads to this physical Hilbert space.

Heuristically, the main idea behind this construction can be summarized as follows. Consider a 4-manifold $M$ bounded by two 3-surfaces, $S_1$ and $S_2$, and a simplicial decomposition thereof. One can think of $S_1$ as an ‘initial’ surface and $S_2$ as a ‘final’ surface. One can fix a spin network on each of these surfaces to specify an ‘initial’ and a ‘final’ state of the quantum 3-geometry. A quantum 4-geometry interpolating between the two can be constructed by considering the dual triangulation of $M$ and coloring its surfaces with half integers $j$ and edges with suitable intertwiners. The idea is to obtain the physical inner product between the two states by summing first over all the colorings for a given triangulation, and then over triangulations keeping the boundary states fixed. The second sum is often referred to as the vertex expansion because the $M$th term in the series corresponds to a dual triangulation with $M$ vertices. Since each triangulation with a coloring specifies a quantum geometry, the sum is regarded as a path integral over physically appropriate 4-geometries. In ordinary quantum mechanics and Minkowskian field theories where we have a fixed background geometry, such a path integral provides the (dynamically determined) transition amplitude for the first state, specified at initial time, to evolve to the second state at the final time. In the background-independent context of quantum gravity, one does not have access to a time variable and dynamics is encoded in constraints. Therefore the notion of a transition in a pre-specified time interval is not a priori meaningful. Rather, the sum over histories now provides the physical inner product between solutions to the quantum constraints, extracted from the two spin network states.
Over the last 2 years, there have been significant advances in SFMs. While the structure of the path integral is well motivated by the interplay between general relativity and the BF theory, its precise definition requires a key new ingredient—the vertex amplitude. The first proposal for the vertex amplitude was made over 10 years ago [11], but it turned out to have important limitations [22, 23]. New proposals have now been put forward [24–27] and, for the physically interesting regime of the Barbero–Immirzi parameter, they agree. Furthermore, one can regard these SFMs as providing an independent derivation of the kinematics underlying LQG. The detailed agreement between LQG and the new SFMs [28, 29] is a striking development. There are also a number of results indicating that one does recover general relativity in the appropriate limit [32, 33]. Finally, the vertex amplitude is severely constrained by several general requirements which the new proposals meet.

However, so far, the vertex amplitude has not been systematically derived following procedures used in well-understood field theories or, starting from a well-understood Hamiltonian dynamics. Therefore, although the convergence of ideas from several different directions is impressive, a number of issues still remain. In particular, the convergence is not quite as seamless as one would like; some rough edges still remain because of unresolved tensions.

For example, the final vertex expansion is a discrete sum, in which each term is itself a sum over colorings for a fixed triangulation. A priori it is somewhat surprising that the final answer can be written as a discrete sum. Would one not have to take some sort of a continuum limit at the end? One does this in the standard Regge approach [30] which, as we indicated above, is closely related to SFMs. Another route to SFMs emphasizes and exploits the close resemblance to gauge theories. In non-topological gauge theories one also has to take a continuum limit. Why not in SFMs? Is there perhaps a fundamental difference because, while the standard path integral treatment of gauge theories is rooted in the smooth Minkowskian geometry, SFMs must face the Planck scale discreteness squarely?

A second potential tension stems from the fact that the construction of the physical inner product mimics that of the transition amplitude in Minkowskian quantum field theories. As noted above, in a background-independent theory, there is no a priori notion of time evolution and dynamics is encoded in constraints. However, sometimes it is possible to ‘de-parameterize’ the theory and solve the Hamiltonian constraint by introducing an emergent or relational time in the manner of Leibnitz. What would then be the interpretation of the spin foam path integral? Would it yield both the physical inner product and the transition amplitude? Or is there another irreconcilable difference from the framework used in Minkowskian field theories?

There is a also a tension between SFMs and GFTs. Although fields in GFTs live on an abstract manifold constructed from a Lie group, as in familiar field theories the action has a free part and an interaction term. The interaction term has a coupling constant, \( \lambda \), as the coefficient. One can therefore carry out a Feynman expansion and express the partition function, propagators, etc, as a perturbation series in \( \lambda \). If one sets \( \lambda = 1 \), the resulting series can be identified with the vertex expansion of SFMs. But if one adopts the viewpoint that the GFT is fundamental and regards gravity as an emergent phenomenon, one is led to allow \( \lambda \) to run under the renormalization group flow. What then is the meaning of setting \( \lambda = 1 \)? Or do other values of \( \lambda \) have a role in SFMs that has simply remained unnoticed thus far? Alternatively, one can put the burden on GFTs. They appear to be efficient and useful calculational schemes. But if they are to have a direct physical significance on their own, what then would the gravitational meaning of \( \lambda \) be?

Such questions are conceptually and technically difficult. However, they are important precisely because SFMs appear to lie at a junction of several cross-roads and the recent advances bring out their great potential. Loop quantum cosmology (LQC) provides a
physically interesting yet technically simple context to explore such issues. In LQC the principles of LQG are applied to simple cosmological models which have a high degree of symmetry. Thanks to this symmetry, it has been possible to construct and analyze in detail quantum theories in a number of cases [34–47]. Furthermore, LQC shares many of the conceptual problems of LQG and SFMs. Therefore, it provides a fertile ground to test various ideas and conjectures in the full theory. In the Hamiltonian context, LQC has served this role successfully (for a recent review, see [48]). The goal of this paper is to first cast LQC in the spin foam paradigm and then use the results to shed light on the paradigm itself.

In LQC one can arrive at a sum over histories starting from a fully controlled Hamiltonian theory. We will find that this sum bears out the ideas and conjectures that drive the spin foam paradigm. Specifically, we will show that (i) the physical inner product in the timeless framework equals the transition amplitude in the theory that is deparameterized using relational time; (ii) this quantity admits a vertex expansion in the manner of SFMs in which the $M$th term just refers to $M$ volume transitions, without any reference to the time at which the transition takes place; (iii) the exact physical inner product is obtained by summing over just the discrete geometries; no ‘continuum limit’ is involved; and (iv) the vertex expansion can be interpreted as a perturbative expansion in the spirit of GFT, where, moreover, the GFT coupling constant $\lambda$ is closely related to the cosmological constant $\Lambda$. These results were reported in the brief communication [1]. Here we provide the detailed arguments and proofs.

The paper is organized as follows. In section 2 we summarize the salient features of LQC that are needed to arrive at a sum over histories’ formulation. Section 3 establishes the main results in the timeless framework, generally used in SFMs. In particular, we show that the physical inner product can be expressed as a vertex expansion. In section 4 we introduce a deparameterization using the relational time of LQC and obtain an equivalent but distinct vertex expansion, more directly related to the transition amplitude. The existence of distinct vertex expansions which sum to the same result suggests the possibility that there may well be distinct but physically equivalent vertex amplitudes in SFMs, each leading to a perturbative expansion that is tailored to a specific aspect of the physical problem. To avoid repetition, we adopted a strategy that is opposite of that used in [1]: here we provide detailed derivations in the timeless framework (section 3) and leave out the details while discussing analogous results in the deparameterized picture (section 4). Section 5 summarizes the main results and discusses some generalizations and open issues. A number of technical issues are discussed in three appendices.

2. LQC: a brief overview

We will focus on the simplest LQC model that has been analyzed in detail [34–36, 39]: the $k = 0, \Lambda = 0$ Friedmann model with a massless scalar field as a source. However, it should not be difficult to extend this analysis to allow for a non-zero cosmological constant [40, 41] or anisotropies [43, 44] or to the spatially compact $k = 1$ case [37].
In the FRW models, one begins by fixing a (spatial) manifold $S$, topologically $\mathbb{R}^3$, Cartesian coordinates $x^i$ thereon, and a fiducial metric $q_{ab}$ given by $q_{ab} \ dx^a \ dx^b = dx_1^2 + dx_2^2 + dx_3^2$. The physical 3-metric $q_{ab}$ is then determined by a scale factor $a$: $q_{ab} = a^2 q_{ab}$. For the Hamiltonian analysis one fixes a cubical fiducial cell $V$ whose volume with respect to $q_{ab}$ is $V_o$ so that its physical volume is $V = a^3 V_o$. The quantity $v$ defined by $V = 2\pi \gamma \ell_P^2 |v|$ turns out to be a convenient configuration variable, where $\gamma$ is the Barbero–Immirzi parameter of LQG [39].

The kinematical Hilbert space is a tensor product $\mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{kin}}^{\text{grav}} \otimes \mathcal{H}_{\text{kin}}^{\text{mat}}$ of the gravitational and matter Hilbert spaces. Elements $\Psi(v)$ of $\mathcal{H}_{\text{kin}}^{\text{grav}}$ are functions of $v$ with support on a countable number of points and with the finite norm $\|\Psi\|^2 := \sum_v |\Psi(v)|^2$. The matter Hilbert space is the standard one: $\mathcal{H}_{\text{kin}}^{\text{mat}} = L^2(\mathbb{R}, d\phi)$.

Thus, the kinematic quantum states of the model are functions $\Psi(v, \phi)$ with the finite norm $\|\Psi\|^2 := \int d\phi |\Psi(v, \phi)|^2$. A (generalized) orthonormal basis in $\mathcal{H}_{\text{kin}}$ is given by $|v, \phi\rangle$ with

$$\langle v', \phi' | v, \phi \rangle = \delta_{v,v'} \delta(\phi', \phi).$$  \hspace{1cm} (2.1)

To obtain the physical Hilbert space, one first notes that the quantum constraint can be written as

$$-C \Psi(v, \phi) = \phi^2 \Psi(v, \phi) + \Theta \Psi(v, \phi) = 0,$$  \hspace{1cm} (2.2)

where $\Theta$ is a positive and self-adjoint operator on $\mathcal{H}_{\text{kin}}^{\text{grav}}$ [50]. More explicitly, $\Theta$ is a second-order difference operator [43]

$$\Theta \Psi(v) := -\frac{3\pi G}{4\ell_P^2} \left\{ \sqrt{|v(v + 4\ell_o)|} \left( v + 2\ell_o \right) \Psi(v + 4\ell_o) - 2v^2 \Psi(v) + \sqrt{|v(v - 4\ell_o)|} \left( v - 2\ell_o \right) \Psi(v - 4\ell_o) \right\},$$  \hspace{1cm} (2.3)

where $\ell_o$ is related to the ‘area gap’ $\Delta = 4\sqrt{3}\pi \gamma \ell_P^2$ via $\ell_o^2 = \Delta$. The form of $\Theta$ shows that the space of solutions to the quantum constraint can be naturally decomposed into sectors in which the wavefunctions have support on specific ‘$v$-lattices’ [35]. For definiteness, we will restrict ourselves to the lattice $v = 4n\ell_o$ where $n$ is an integer. Details of the expression of $\Theta$ will not be needed in most of our analysis.

The scalar field $\phi$ is monotonic on all classical solutions (also in the cases when $\phi$ is positive and if the orientations are opposite, $v$ is negative. Physics of the model is insensitive to the triad orientation and hence to the sign of $v$). In particular the kinematic and physical quantum states satisfy $\Psi(v, \phi) = \Psi(-v, \phi)$.

In the spin foam literature, by contrast, one does not have access to such a preferred time and therefore one chooses to work with the timeless formalism. Therefore, let us first forgo the emphasis on using $\phi$ as internal time and simply implement the group-averaging procedure which uses the constraint operator as a whole, without having to single out a preferred time variable [20, 21]. This procedure plays an important role in sections 3 and 4. Therefore it is useful to summarize it in some detail. One begins by fixing a dense sub-space $S$ of $\mathcal{H}_{\text{kin}}$.

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1 In LQG the basic geometric variable is an orthonormal triad and the physical metric $q_{ab}$ is constructed from it. If the triad has the same orientation as the fiducial one, given by the coordinates $x^i$, the configuration variable $v$ is positive and if the orientations are opposite, $v$ is negative. Physics of the model is insensitive to the triad orientation and hence to the sign of $v$. In particular the kinematic and physical quantum states satisfy $\Psi(v, \phi) = \Psi(-v, \phi)$.

2 One can also use a ‘polymer quantization’ of the scalar field at the kinematical level but the final physical theory turns out to be the same.
In LQC, this is generally taken to be the Schwartz space of smooth functions $f(\nu, \phi)$ which fall off to zero at infinity faster than any polynomial. The first step in the group-averaging procedure is to extract a solution $\Psi_f(\nu, \phi)$ to the quantum constraint operator (2.2) from each $f \in S$. These solutions are not normalizable in $\mathcal{H}_{\text{kin}}$ because the spectrum of the constraint $C$ on $\mathcal{H}_{\text{kin}}$ is continuous. The second step of the group-averaging procedure provides an appropriate inner product between solutions $\Psi_f(\nu, \phi)$.

Denote by $e_k(\nu)$, with $k \in (-\infty, \infty)$, a complete set of orthonormal eigenfunctions of $\Theta$ on $\mathcal{H}_{\text{grav}}$. We will denote the eigenvalues by $\omega_k^2$ and, without loss of generality, assume that $\omega_k \geq 0$ [35, 36]. (Eigenfunctions and operator functions of $\Theta$ are discussed in appendix C.) Any $f(\nu, \phi) \in S$ can be expanded as

$$f(\nu, \phi) = \int dk \frac{1}{2\pi} \int dp_\phi \tilde{f}(k, p_\phi) e^{i\nu_\phi e_k(\nu)}.$$  

(2.4)

Here and in what follows the range of integrals will be from $-\infty$ to $\infty$ unless otherwise stated.

Using this expansion, we can group average any $f(\nu, \phi)$ to obtain a distributional solution (in $S^*$) $\Psi_1 f(\nu, \phi)$ to the quantum constraint:

$$\Psi_f(\nu, \phi) := \int d\alpha [e^{i\alpha C} |p_\phi| f(\nu, \phi)] = \int dk \int dp_\phi \delta(p^2_\phi - \omega_k^2) 2|p_\phi| \tilde{f}(k, p_\phi) e^{i\nu_\phi e_k(\nu)}.$$

(2.5)

where the operator $2|p_\phi|$ has been introduced just for later technical simplification. Had we dropped it, we would have associated with $f$ the solution $(2|p_\phi|)^{-1} \Psi_f$ and, in the end, obtained a unitarily equivalent representation of the algebra of Dirac observables.

By carrying out the integral over $p_\phi$, the expression of $\Psi_f$ can be brought to the desired form:

$$\Psi_f(\nu, \phi) = \Psi_f^+(\nu, \phi) + \Psi_f^-(\nu, \phi).$$

(2.6)

By their very definition, $\Psi_f^\pm(\nu, \phi)$ satisfy

$$\Psi_f^\pm(\nu, \phi) = e^{\pm i\sqrt{\Theta}(\phi - \phi_0)} \Psi_f^\pm(\nu, \phi_0),$$

(2.7)

whence they can be interpreted as ‘positive and negative frequency solutions’ to (2.2) with respect to the relational time $\phi$. Thus the group average of $f$ is a solution $\Psi_f$ to the quantum constraint (2.2) which, furthermore, is naturally decomposed into positive and negative frequency parts. $\Psi_f$ is to be regarded as a distribution in $S^*$ which acts on elements $g \in S$ via the kinematic inner product [20, 21]:

$$(\Psi_f | g) := \langle \Psi_f | g \rangle$$

$$= \int dk \int dp_\phi \delta(p^2_\phi - \omega_k^2) 2\omega_k \tilde{g}(k, p_\phi) \tilde{f}(k, p_\phi)$$

$$= \int dk [\tilde{f}(k, \omega_k) \tilde{g}(k, \omega_k) + \tilde{f}(k, -\omega_k) \tilde{g}(k, -\omega_k)].$$

(2.8)

Finally, the group-averaged scalar product on solutions $\Psi_f$ is given just by this action [20, 21]. Thus, given any elements $f, g \in S$, the scalar product between the corresponding group-averaged states $\Psi_f, \Psi_g$ is given by

$$(\Psi_f, \Psi_g) := (\Psi_f | g) = (\Psi_g | f).$$

(2.9)

In section 3 we will obtain a vertex expansion for this scalar product.
A conceptually important observation is that, as in the Klein–Gordon case, there is a superselection. A complete set of Dirac observables is given by the scalar field momentum $p_\phi = -i \partial_\phi$ and the volume $V|_{\phi_0}$ (or, equivalently, the energy density operator $\rho|_{\phi_0}$ [35, 36, 39].) The action of these Dirac observables as well as time evolution leaves the space of positive and negative frequency solutions invariant. Therefore, as in the Klein–Gordon theory, we are led to work with either set. In LQC, one generally works with the positive frequency ones. Then the physical Hilbert space $\mathcal{H}_{\text{phy}}$ of LQC consists of positive frequency solutions $\Psi_+(v, \phi)$ to the quantum constraint (2.2), i.e. solutions satisfying

$$-i \partial_\phi \Psi_+(v, \phi) = \sqrt{\Theta} \Psi_+(v, \phi) \equiv H \Psi_+(v, \phi)$$

(2.10)

with the inner product (2.9). This inner product can be re-expressed simply as

$$(\Psi_+, \Phi_+)_{\text{phy}} = \sum_{v = -\infty}^{\infty} \Psi_+(v, \phi_0) \Phi_+(v, \phi_0)$$

(2.11)

and is independent of the value $\phi_0$ of $\phi$ at which the right side is evaluated.

While this construction of $\mathcal{H}_{\text{phy}}$ does not require us to think of $\phi$ as internal time in quantum theory, this interpretation is natural in the light of final equations (2.10) and (2.11), for these equations suggest that we can think of $\nu$ as the sole configuration variable and introduce 'Schrödinger states' $\Psi(\nu)$ through the physical inner product (2.11). These 'evolve' via (2.10). This is the 'deparameterized' description to which we will return in section 4. In this picture, the restriction to positive frequency states has direct interpretation: $p_\phi \equiv \sqrt{\Theta}$ is now a positive operator on $\mathcal{H}_{\text{phy}}$ just as $p_0$ is a positive operator on the traditional Klein–Gordon–Hilbert space.

3. The timeless framework

Recall that in the spin foam literature, one works with the timeless framework because a natural deparameterization is not available in general. To mimic the general spin foam constructions in LQC, in this section we will largely disregard the fact that the scalar field can be used as relational time and that the final constraint has the form of the Schrödinger equation. Instead, we will use the group-averaging procedure for the full constraint

$$C = -\partial^2_{\phi} - \Theta \equiv p^2_{\phi} - \Theta$$

(3.1)

and incorporate the positive frequency condition in a second step. None of the steps in this analysis refer to the evolution in relational time mentioned above. Thus, the primary object of interest will be the physical scalar product, rather than the transition amplitude for a Schrödinger state $\Psi(v, \phi_i)$ at an initial ‘time instant’ $\phi_i$ to evolve to another state $\Phi(v, \phi_f)$ at a final ‘time instant’ $\phi_f$.

In section 2 we considered general kinematic states $f(v, \phi)$. In this section, by contrast, we will focus on the basis vectors $|v, \phi_i\rangle$ in $\mathcal{H}_{\text{kin}}$ which are the LQC analogs of spin networks that are used to specify the boundary states in SFMs. Following the setup introduced in section 1, let us then fix two kinematic states $|v_i, \phi_i\rangle$ and $|v_f, \phi_f\rangle$. For notational simplicity, we will denote the group-averaged solutions to (2.2) that they define by $|[v_i, \phi_i]\rangle$ and $|[v_f, \phi_f]\rangle$. The group-averaged inner product between these states is given by

$$\langle [v_f, \phi_f], [v_i, \phi_i] \rangle = 2 \int dv_f \partial [v_f, \phi_f] |\phi\rangle^a C |p_\phi| |v_i, \phi_i\rangle.$$  

(3.2)

Our goal is to express this scalar product as a vertex expansion in the manner of SFMs and study its properties. In section 3.1 we will begin by rewriting it as a sum over histories in the manner
of Feynman [6] and then rearrange the sum as a vertex expansion. In section 3.2 we will arrive at the same expansion using perturbation theory in a suitably defined interaction picture. This procedure is reminiscent of the perturbation expansion used in GFTs. As an important consistency check, in section 3.3 we verify that this perturbative expansion does satisfy the constraint order by order. Finally, in section 3.4 we observe that, in this simple example, the coupling constant $\Lambda$ used in the expansion is intimately related to the cosmological constant $\Lambda$. Although the precise relation we obtain is tied to LQC, the observation illustrates in a concrete fashion how one may be able to provide a gravitational interpretation to $\lambda$ in GFTs and suggests an avenue for GFT to account for the smallness of $\Lambda$.

### 3.1. Sum over histories

Following Reisenberger and Rovelli [7], let us first focus on the amplitude

$$A(v_f, \phi_f; v_i, \phi_i; \alpha) = 2\langle v_f, \phi_f | e^{\alpha C} | p_\phi | v_i, \phi_i \rangle$$

which constitutes the integrand of (3.2). Mathematically one can choose to regard $\alpha C$ as a Hamiltonian operator. Then $A(v_f, \phi_f, v_i, \phi_i, \alpha)$ can be interpreted as the probability amplitude for an initial kinematic state $|v_i, \phi_i \rangle$ to evolve to a final kinematic state $|v_f, \phi_f \rangle$ in a unit 'time interval' and we can follow Feynman’s procedure [6] to express it as a sum over histories. Technically, a key simplification comes from the fact that the constraint $C$ is a sum of two commuting pieces that act separately on $H_{\text{kin}}$ and $H_{\text{grav}}$. Consequently, the amplitude (3.3) factorizes as

$$A(v_f, \phi_f; v_i, \phi_i; \alpha) = A_\phi(\phi_f, \phi_i; \alpha) A_G(v_f, v_i; \alpha)$$

with

$$A_\phi(\phi_f, \phi_i; \alpha) = 2\langle \phi_f | e^{\alpha p_\phi^2} | p_\phi | \phi_i \rangle$$

and

$$A_G(v_f, v_i; \alpha) = \langle v_f | e^{-i\omega \Theta} | v_i \rangle.$$ (3.5)

It is easy to cast the first amplitude, $A_\phi$, in the desired form using either a standard Feynman expansion or simply evaluating it by inserting a complete eigenbasis of $p_\phi$. The result is

$$A_\phi(\phi_f, \phi_i; \alpha) = 2 \int dp_\phi e^{i p_\phi^2} e^{i p_\phi (\phi_f - \phi_i)} |p_\phi|.$$ (3.6)

The expansion of the gravitational amplitude $A_G$ is not as simple. We will first express it as a sum over histories. In a second step, we will evaluate the total amplitude (3.3) by integrating over $\alpha$ for each history separately. Although it is not a priori obvious, we will find that the amplitude associated with each history is manifestly finite and the total amplitude can be written as a discrete sum that mimics the vertex expansion in SFMs.

#### 3.1.1. The gravitational amplitude $A_G$

As mentioned above, to apply the standard Feynman procedure, we will regard $e^{-i\omega \Theta}$ as an ‘evolution operator’ with ‘Hamiltonian’ $\alpha \Theta$ and a ‘time interval’ $\Delta \tau = 1$. We emphasize that this ‘evolution’ is just a convenient mathematical construct and does not correspond to the physical evolution with respect to the relational time variables $\phi$ normally used in LQC. Rather, since it is generated by the constraint $C$, physically it represents gauge transformations (or time reparameterizations).

Let us divide the interval $\Delta \tau = 1$ into $N$ parts each of length $\epsilon = 1/N$ and write the gravitational amplitude $A_G(v_f, v_i; \alpha)$ as

$$\langle v_f | e^{-i\omega \Theta} | v_i \rangle = \sum_{\psi_{N-1}, \ldots, \psi_1} \langle v_f | e^{-i\omega \Theta} | \psi_{N-1} \rangle \langle \psi_{N-1} | e^{-i\omega \Theta} | \psi_{N-2} \rangle \ldots \langle \psi_1 | e^{-i\omega \Theta} | v_i \rangle,$$ (3.7)

where we have first split the exponential into $N$ identical terms and, then introduced a decomposition of the identity operator at each intermediate ‘time’ $\tau = n\epsilon, n = 1, 2, \ldots, N-1$. 


For notational simplicity, we will denote the matrix element $\langle \vec{v}_n | e^{-i\omega t} \rho | \vec{v}_{n-1} \rangle$ by $U_{\vec{v}_n \vec{v}_{n-1}}$ and set $v_f = \vec{v}_N$ and $v_i = \vec{v}_0$. We then have

$$A_G(v_f, v_i; \alpha) = \sum_{\vec{v}_{M-1}, \ldots, \vec{v}_1} U_{\vec{v}_N \vec{v}_{N-1}} U_{\vec{v}_{N-1} \vec{v}_{N-2}} \cdots U_{\vec{v}_1 \vec{v}_0}. \quad (3.8)$$

The division of $\Delta \tau$ provides a skeletonization of this ‘time interval’. An assignment $\sigma_N = (\vec{v}_N, \ldots, \vec{v}_0)$ of volumes to the $N+1$ time instants $\tau = \epsilon n$ can be regarded as a discrete (gauge) history associated with this skeletonization since one can envision the universe going from $\vec{v}_{n-1}$ to $\vec{v}_n$ under a finite ‘evolution’. The matrix element is given by a sum of amplitudes over these discrete histories with fixed endpoints:

$$A_G(v_f, v_i; \alpha) = \sum_{\sigma_N} A(\sigma_N) \equiv \sum_{\sigma_N} U_{\vec{v}_N \vec{v}_{N-1}} U_{\vec{v}_{N-1} \vec{v}_{N-2}} \cdots U_{\vec{v}_1 \vec{v}_0}. \quad (3.9)$$

The next step in a standard path integral construction is to take the ‘continuum’ limit, $N \to \infty$, of the skeletonization. In particle mechanics at this stage, one uses a continuous basis (say the position basis $|x\rangle$) to carry out this expansion. By contrast, our basis $|v_n\rangle$ is discrete. As a result, one can make rigorous sense of the well-defined sum (3.9) according to the number of volume transitions. The remainder of section 3.1.1 is devoted to carrying out this step.

This task involves two key ideas. Let us first note that along a path $\sigma_N$, the volume $\vec{v}$ is allowed to remain constant along a number of time steps, and then jump to another value, where it could again remain constant for a certain number of time steps, and so on. The first key idea is to group paths according to the number of volume transitions rather than time steps. Let us then consider a path $\sigma_N^M$ which involves $M$ volume transitions (clearly, $M \leq N$):

$$\sigma_N^M = \left( v_{M}, \ldots, v_{M}; v_{M-1}, \ldots, v_{M-1}; \ldots; v_{1}, \ldots, v_{1}; v_{0}, \ldots, v_{0} \right) \mod N. \quad (3.10)$$

Thus, the volume changes from $v_{m-1}$ to $v_m$ at ‘time’ $\tau = N_m \epsilon$ and remains $v_m$ till time $\tau = N_m+1 \epsilon$. Note that $v_n$ is distinct from $\check{v}_n$ used in (3.9). While $v_m$ is the volume after the $m$th volume transition along the given discrete path, $\check{v}_m$ is the volume at the end of the $m$th time interval, i.e., at $\tau = m \epsilon$.

These discrete histories can be labeled more transparently by two ordered sequences

$$\sigma_N^M = (v_M, v_{M-1}, \ldots, v_1, v_0); (N_M, N_{M-1}, \ldots, N_2, N_1), \quad v_m \neq v_{m-1}, \quad N_m > N_{m-1}, \quad (3.11)$$

where $v_M, \ldots, v_0$ denote the volumes that feature in the history $\sigma_N^M$ and $N_k$ denotes the number of time steps after which the volume changes from $v_{k-1}$ to $v_k$. Note that while no two consecutive volume values can be equal, a given volume value can repeat in the sequence; $v_m$ can equal some $v_n$ if $n \neq m \pm 1$. The probability amplitude for such a history $\sigma_N^M$ is given by

$$A(\sigma_N^M) = [U_{v_M v_{M-1}}]^{N-M-1} U_{v_M v_{M-2}} \cdots [U_{v_1 v_0}]^{N_2 - N_1 - 1} U_{v_1 v_0} [U_{v_0 v_0}]^{N_1 - 1}. \quad (3.12)$$

The second key idea is to perform the sum over all these amplitudes in three steps. First we keep the ordered set of volumes $(v_M, \ldots, v_0)$ fixed but allow the volume transitions to occur at any value $\tau = n \epsilon$ in the interval $\Delta \tau$, subject only to the constraint that the $m$th transition occurs before the $(m+1)$st for all $m$. The sum of amplitudes over this group of histories is given by

$$A_N(v_M, \ldots, v_0; \alpha) = \sum_{N_M=M} A_N^M(v_M, \ldots, v_0; \alpha) = \sum_{N_M=M} \sum_{N_{M-1}=M-1} \cdots \sum_{N_1=1} A(\sigma_N^M). \quad (3.13)$$
Next we sum over all possible intermediate values of \(v_m\) such that \(v_m \neq v_{m-1}\), keeping \(v_0 = v_1, v_M = v_f\), to obtain the amplitude \(A_N(M)\) associated with the set of all paths in which there are precisely \(M\) volume transitions:

\[
A_N(M; \alpha) = \sum_{\nu_{M-1},...,\nu_1} A_N(v_{M}, \ldots, v_0; \alpha).
\]  

(3.14)

Finally the total amplitude \(A_G(v_f, v_i, \alpha)\) is obtained by summing over all volume transitions that are permissible within our initially fixed skeletonization with \(N\) time steps:

\[
A_G(v_f, v_i; \alpha) = \sum_{M=0}^{N} A_N(M; \alpha).
\]  

(3.15)

This concludes the desired re-arrangement of the sum (3.9). The sum on the right side is manifestly finite. Furthermore, since \(A_G(v_f, v_i; \alpha) = \langle v_f | e^{-i\alpha\Theta} | v_i \rangle\), the value of the amplitude (3.15) does not depend on \(N\) at all; the skeletonization was introduced just to express this well-defined amplitude as a sum over histories. Thus, while the range of \(M\) in the sum and the amplitude \(A_N(M; \alpha)\) in (3.15) both depend on \(N\), the sum does not.

Therefore we are well positioned to get rid of the skeletonization altogether by taking the limit \(N \to \infty\). Note first that with our fixed skeletonization, the gravitational amplitude is a finite sum of terms

\[
A_G(v_f, v_i; \alpha) = A_N(0; \alpha) + A_N(1; \alpha) + \cdots + A_N(M; \alpha) + \cdots + A_N(N; \alpha),
\]  

(3.16)

each providing the contribution of all discrete paths that contain a fixed number of volume transitions. Let us focus on the \(M\)th term in the sum

\[
A_N(M; \alpha) = \sum_{\nu_{M-1},...,\nu_1} A_N(v_{M}, \ldots, v_0; \alpha).
\]  

(3.17)

Now, in appendix A, we show that the limit \(\lim_{N \to \infty} A_N(v_{M}, \ldots, v_0; \alpha)\) exists and is given by

\[
A(v_{M}, \ldots, v_0; \alpha) := \lim_{N \to \infty} A_N(v_{M}, \ldots, v_0; \alpha)
\]

\[
= \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \cdots \int_0^{\tau_1} d\tau_1 A(v_{M}, \ldots, v_0; \tau_M, \ldots, \tau_1; \alpha),
\]  

(3.18)

where

\[
A(v_{M}, \ldots, v_0; \tau_M, \ldots, \tau_1; \alpha) := e^{-i(1-\tau_M) \alpha H^{grav}_{\text{lim}}} \left(-i\alpha \Theta\right)_{v_M v_{M-1}} \cdots \left(-i\alpha \Theta\right)_{v_1 v_0} e^{-i\alpha \Theta}_{v_0 v_0}.
\]  

(3.19)

Note that the matrix elements \(\Theta_{v_M v_0} = \langle v_M | \Theta | v_0 \rangle\) of \(\Theta\) in \(H^{grav}_{\text{lim}}\) can be easily calculated from (2.3) and vanish if \((v_m - v_i) \notin [0, \pm 4\ell_0]\). Therefore, explicit evaluation of the limit is rather straightforward. We will assume that the limit \(N \to \infty\) can be interchanged with the sum over \(v_{M-1}, \ldots, v_1\). (This assumption is motivated by the fact that in the expression of \(A(v_{M}, \ldots, v_0; \alpha)\) most matrix elements of \(\Theta\) vanish, and since the initial and final volumes are fixed, the sums over intermediate volumes \(v_{M-1}, \ldots, v_1\) extend over only a finite number of non-zero terms.) Then it follows that

\[
A_G(M; \alpha) := \lim_{N \to \infty} A_N(M; \alpha)
\]

exists for each finite \(M\). Note that the reference to the skeletonization disappears in this limit. Thus, \(A_G(M; \alpha)\) is the amplitude obtained by summing over all paths that contain precisely \(M\) volume transitions within the ‘time interval’ \(\Delta \tau = 1\), irrespective of precisely when and
at what values of volume they occurred. Finally, (3.16) implies that the total gravitational amplitude can be written as an infinite sum:

$$A_G(v_f, v_i; \alpha) = \sum_{M=0}^{\infty} A_G(M; \alpha).$$  \hspace{1cm} (3.20)

While each partial amplitude $A_G(M; \alpha)$ is well defined and finite, it does not ensure that the infinite sum converges. \textit{A priori} the infinite sum on the right-hand side of (3.20) could be, for example, only an asymptotic series to the well-defined left side. Also our derivation assumed that the limit $N \to \infty$ commutes with the partial sums. Both these limitations will be overcome in section 3.2. We will see that $A_G(v_f, v_i; \alpha)$ is indeed given by a convergent sum (3.20).

Expression (3.18) still contains some integrals. These can be performed exactly. The case when all of $(v_M, \ldots, v_0)$ are distinct is straightforward and the result is as given in [1]. The general case is a little more complicated and is analyzed in appendix B. The final result is

$$A(v_M, \ldots, v_0; \alpha) = \Theta_{v_Mv_{M-1}}\Theta_{v_{M-1}v_{M-2}} \ldots \Theta_{v_2v_1}\Theta_{v_1v_0}$$

$$\times \prod_{k=1}^{p} \frac{1}{(n_k-1)!} \left( \frac{\partial}{\partial \Theta_{v_kw_k}} \right)^{n_k-1} \prod_{m=1}^{p} \Delta_{v_mw_m} \sum_{\mu \neq \nu} e^{-i\alpha \Theta_{\mu\nu} \Delta \tau},$$ \hspace{1cm} (3.21)

where, since the volumes can repeat along the discrete path, $w_m$ label the $p$ distinct values taken by the volume and $n_m$ the number of times that each value occurs in the sequence. The $n_m$ satisfy $n_1 + \cdots + n_p = M + 1$.

To summarize, we have written the gravitational part $A_G(v_f, v_i; \alpha)$ of the amplitude as a ‘sum over histories’:

$$A_G(v_f, v_i; \alpha) = \sum_{M=0}^{\infty} \sum_{\substack{v_M, \ldots, v_0; \alpha \text{ satisfy the boundary conditions} \\text{and } v_m \neq v_{m+1}}} A(v_M, \ldots, v_0; \alpha)$$ \hspace{1cm} (3.22)

with $A(v_M, \ldots, v_0; \alpha)$ given by (3.21). This expression consists of a sum over $M$, the number of volume transitions and a sum over the (finite number of) sequences of $M - 1$ intermediate volumes that are consistent with the boundary conditions and the condition that $v_m \neq v_{m+1}$.

In section 3.1.2, we will use this sum to generate the ‘vertex expansion’ of the physical inner product.

3.1.2. Vertex expansion of the physical inner product. Recall that the group-averaged scalar product can be expressed as

$$([v_f, \phi_f], [v_i, \phi_i]) = 2 \int d\alpha A_G(v_f, v_i; \alpha) A_G(v_f, v_i; \alpha).$$ \hspace{1cm} (3.23)

The main assumption in our derivation—the only one that will also be required in section 3.2—is that one can interchange the integration over $\alpha$ and the (convergent but infinite) sum over $M$ in the expression of $A_G(v_f, v_i; \alpha)$. Let us then use expressions (3.6) and (3.22) of $A_G$ and make the interchange and carry out the integral over $\alpha$. The scalar product (3.23) is then re-expressed as a sum of amplitudes associated with discrete paths $(v_M, \ldots, v_0)$:

$$([v_f, \phi_f], [v_i, \phi_i]) = \sum_{M=0}^{\infty} \left[ \sum_{\substack{v_M, \ldots, v_0; \phi_f, \phi_i \text{ satisfy the boundary conditions} \\text{and } v_m \neq v_{m+1}}} A(v_M, \ldots, v_0; \phi_f, \phi_i) \right].$$ \hspace{1cm} (3.24)
where
\[
A(v_M, \ldots, v_0; \phi_f, \phi_i) = 2\theta_{v_M v_{M-1}} \theta_{v_{M-1} v_{M-2}} \cdots \theta_{v_2 v_1} \theta_{v_1 v_0} \prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \phi_{v_k w_k}} \right)^{n_k-1} \int dp \phi \ e^{i[\phi_f - \phi_i]} \rho_p \left( \frac{\partial^2}{\partial \phi_{w_k w_k}} - \frac{\theta_{w_k w_{k+1}}}{\theta_{w_k w_k}} \Delta r \right) \]
\[
\times \sum_{m=1}^{p} \int dp_m e^{i[\phi_f - \phi_i]} \rho_p \left( \frac{\partial^2}{\partial \phi_{w_m w_m}} - \frac{\theta_{w_m w_{m+1}}}{\theta_{w_m w_m}} \Delta r \right). \tag{3.25}
\]

The right side is a sum of distributions, integrated over \( p \). It is straightforward to perform the integral and express \( A(v_M, \ldots, v_0; \phi_f, \phi_i) \) in terms of the matrix elements of \( \Theta \):
\[
A(v_M, \ldots, v_0; \phi_f, \phi_i) = \Theta_{v_M v_{M-1}} \Theta_{v_{M-1} v_{M-2}} \cdots \Theta_{v_2 v_1} \theta_{v_1 v_0} \]
\[
\times \prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \phi_{v_k w_k}} \right)^{n_k-1} \sum_{m=1}^{p} \ e^{i\sqrt{\theta_{w_m w_m}} \Delta \phi} \sum_{m=1}^{p} \ e^{-i\sqrt{\theta_{w_m w_m}} \Delta \phi}, \tag{3.26}
\]

where \( \Delta \phi = \phi_f - \phi_i \). Since by inspection each amplitude \( A(v_M, \ldots, v_0; \phi_f, \phi_i) \) is real, the group-averaged scalar product (3.24) is also real.

Finally, as explained in section 2, the group-averaging procedure yields a solution which has both positive and negative frequency components while the physical Hilbert space consists only of positive frequency solutions. Let us denote the positive frequency parts of the group-averaged ket \([|v, \phi]\) by \([|v, \phi_+\]\). Then, the physical scalar product between these states in \( \mathcal{H}_\text{phy} \) is given by a sum over amplitudes \( A(M) \), each associated with a fixed number of volume transitions:
\[
([v_f, \phi_f]_+, [v_i, \phi_i])_\text{phy} = \sum_{M=0}^{\infty} A(M)
\]
\[
= \sum_{M=0}^{\infty} \left[ \sum_{v_M \ldots v_0; \phi_{v_0} \phi_{v_{M+1}}} \theta_{v_M v_{M-1}} \theta_{v_{M-1} v_{M-2}} \cdots \theta_{v_2 v_1} \theta_{v_1 v_0} \right.
\]
\[
\times \left. \prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \phi_{v_k w_k}} \right)^{n_k-1} \sum_{m=1}^{p} \ e^{i\sqrt{\theta_{w_m w_m}} \Delta \phi} \right]. \tag{3.27}
\]

(Note that the right side is in general complex, a point to which we will return in section 5.) This is the vertex expansion of the physical inner product we were seeking. It has two key features. First, the integral over the parameter \( \alpha \) was carried out and is not divergent. This is a non-trivial and important result if we are interested in computing the physical inner product perturbatively, i.e. order by order in the number of vertices. Second, the summand involves only the matrix elements of \( \Theta \) which are easy to compute. As remarked earlier, significant simplification arises because equation (2.3) implies that \( \Theta_{v_0 v_{n+1}} \) is zero if \( v_0 - v_{n+1} \not\in \{0, \pm 4\ell_0\} \).

Let us summarize. We did not begin by postulating that the physical inner product is given by a formal path integral. Rather, we started with the kinematical Hilbert space and the group-averaging procedure and derived a vertex expansion of the physical inner product. Because the Hilbert space framework is fully under control, we could pinpoint the one assumption that is needed to arrive at (3.27): the sum over vertices and the integral over \( \alpha \) can be interchanged. In the full theory, one often performs formal manipulations which result in divergent individual terms in the series under consideration. (For instance sometimes one starts by expanding the very first amplitude (3.3) in powers of \( \alpha \) even though the \( \alpha \) integral of each term is
then divergent \cite{7, 19}. In our case, individual terms in the series are all finite, and, as we will show in section 3.2, even the full series \eqref{3.22} representing the gravitational amplitude is convergent. Nonetheless, at present, the interchange of the $\alpha$-integral and the infinite sum over $M$ has not been justified. If this gap can be filled, we would have a fully rigorous argument that the well-defined physical inner product admits an exact, convergent vertex expansion \eqref{3.27}. (This assumption is needed only in the timeless framework because the integration over $\alpha$ never appears in the deparameterized framework of section 4.) In particular, there is no need to take a ‘continuum limit’.

3.2. Perturbation series

We will now show that expression \eqref{3.27} of the transition amplitude can also be obtained using a specific perturbative expansion. Structurally, this second derivation of the vertex expansion is reminiscent of the perturbative strategy used in the group field theory \cite{17, 18}.

Let us begin by considering the diagonal and off-diagonal parts $D$ and $K$ of the operator $\Theta_1$ in the basis $|\nu = 4n\ell_o\rangle$. Thus, matrix elements of $D$ and $K$ are given by

\begin{equation}
D_{\nu'\nu} = \Theta_{\nu'\nu}, \\
K_{\nu'\nu} = \begin{cases}
\Theta_{\nu'\nu} & \nu' \neq \nu \\
0 & \nu' = \nu.
\end{cases}
\end{equation}

Clearly $C = p^2_\phi - D - K$. The idea is to think of $p^2_\phi - D$ as the ‘main part’ of $C$ and $K$ as a ‘perturbation’. To implement it, introduce a 1-parameter family of operators

\begin{equation}
C_\lambda = p^2_\phi - \Theta_1 \lambda = p^2_\phi - D - \lambda K
\end{equation}
as an intermediate mathematical step. The parameter $\lambda$ will simply serve as a marker to keep track of powers of $K$ in the perturbative expansion and we will have to set $\lambda = 1$ at the end of the calculation.

Our starting point is again the decomposition \eqref{3.4} of the amplitude $A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha)$ into a scalar field and a gravitational part. The $\lambda$-dependence appears in the gravitational part:

\begin{equation}
A_G^{(\lambda)}(\nu_f, \nu_i, \alpha) := \langle \nu_f | e^{-i\alpha/\Theta_1 \lambda} | \nu_i \rangle.
\end{equation}

Let us construct a perturbative expansion of this amplitude. Again we think of $e^{-i\alpha/\Theta_1 \lambda}$ as a mathematical ‘evolution operator’ defined by the ‘Hamiltonian’ $\alpha/\Theta_1 \lambda$ and a ‘time interval’ $\Delta_1 \tau = 1$. The ‘unperturbed Hamiltonian’ is $\alpha D$ and the ‘perturbation’ is $\lambda \alpha K$. Following the textbook procedure, let us define the ‘interaction Hamiltonian’ as

\begin{equation}
H_I (\tau) = e^{i\alpha D \tau} \alpha K e^{-i\alpha D \tau}.
\end{equation}

Then the evolution in the interaction picture is dictated by the 1-parameter family of unitary operators on $H_{grav}$

\begin{equation}
\hat{U}_I (\tau) = e^{i\alpha D \tau} e^{-i\alpha \Theta_1 \tau}, \quad \text{satisfying} \quad \frac{d\hat{U}_I (\tau)}{d\tau} = -i\lambda H_I (\tau) \hat{U}_I (\tau).
\end{equation}

The solution of this equation is given by a time-ordered exponential

\begin{equation}
\hat{U}_I (\tau) = T e^{-i \int_0^\tau d\tau' H_I (\tau')}.
\end{equation}

Next we use the relation $e^{-i\alpha \Theta_1 \tau} = e^{-i\alpha D \hat{U}_I (\tau)}$, with $\hat{U}_I$ given by \eqref{3.33}, take the matrix element of $e^{i\alpha \Theta_1 \nu}$ between initial and final states $|\nu_i \equiv \nu_0\rangle$ and $|\nu_f \equiv \nu_M\rangle$, and write out...
explicitly the product of the $H_I$’s. The result is

$$A_G^{(i)}(v_f, v_i, \alpha) = \sum_{M=0}^{\infty} \lambda^M \int_0^{r_2} d\tau_1 \ldots \int_0^{r_1} d\tau_M \sum_{v_{M,-1}, \ldots, v_1} [e^{-i(1-\tau_M)}] \sum_{v_{M,-1}, \ldots, v_1} [e^{-i\alpha D_{v_{M}v_{M-1}}}].$$

(3.34)

We can now replace $D$ and $K$ by their definition (3.28). Because $K$ has no diagonal matrix elements, only the terms with $v_m \neq v_{m+1}$ contribute and the sum reduces precisely to

$$A_G(v_f, v_i, \alpha) = \sum_{M=0}^{\infty} \lambda^M \left[ \sum_{v_{M,-1}, \ldots, v_1} A(v_{M,-1}, \ldots, v_1; v_0; \alpha) \right],$$

(3.35)

where $A(v_{M,-1}, \ldots, v_1; v_0; \alpha)$ is given by (3.21) as in the sum over histories’ expansion of section 3.1.1.

We can now construct the total amplitude by including the scalar field factor (3.6) and performing the $\alpha$ integral as in section 3.1.2. Then the group-averaged scalar product is given by

$$((v_f, \phi_f), (v_i, \phi_i))_{\text{phy}} = \infty \sum_{M=0}^{\infty} \lambda^M \left[ \sum_{v_{M,-1}, \ldots, v_1} A(v_{M,-1}, \ldots, v_1; v_0, \phi_f, \phi_i) \right],$$

(3.36)

where $A(v_{M,-1}, \ldots, v_1; v_0, \phi_f, \phi_i)$ is given in (3.26). If we now set $\lambda = 1$, (3.36) reduces to (3.24) obtained independently in section 3.1.2.

Finally, let us restrict ourselves to the positive frequency parts $|v, \phi\rangle$ of $|v, \phi\rangle$ which provide elements of $H_{\text{phy}}$. Reasoning of section 3.1.2 tells us that the physical scalar product $((v_f, \phi_f), (v_i, \phi_i))_{\text{phy}}$ is given by (3.27).

Thus, by formally regarding the volume changing off-diagonal piece of the constraint as a perturbation, we have obtained an independent derivation of the vertex expansion for $((v_f, \phi_f), (v_i, \phi_i))_{\text{phy}}$ as a power series expansion in $\lambda$, the power of $\lambda$ serving as a bookmark that keeps track of the number of vertices in each term. In this sense this alternate derivation is analogous to the vertex expansion obtained using the group field theory. This derivation has a technical advantage. Since $H_I$ is self-adjoint on $H_{\text{kin}}$, it follows that expansion (3.33) of $\hat{U}(\tau)$ is convergent everywhere on $H_{\text{kin}}$ [51]. This in turn implies that the right-hand side of (3.35) converges to the well-defined gravitational amplitude $A_G^{(i)} = (v_f|e^{-i\Theta_f}|v_i)$. However, to arrive at the final vertex expansion starting from (3.35), we followed the same procedure as in section 3.1.2. Therefore, this second derivation of the vertex amplitude also assumes that one can interchange the integral over $\alpha$ with the (convergent but) infinite sum over $M$ in (3.35).

### 3.3. Satisfaction of the constraint

The physical inner product between the basis states defines a 2-point function

$$G(v_f, \phi_f; v_i, \phi_i) := ((v_f, \phi_f), (v_i, \phi_i))_{\text{phy}}$$

(3.37)

and it follows from section 2 that it satisfies the constraint equation in each argument. Since $G(v_f, \phi_f; v_i, \phi_i) = \hat{G}(v_f, \phi_f; v_i, \phi_i)$, it suffices to focus just on one argument, say the final one. Then we have

$$\left[ \partial^2_{\phi_i} - \Theta_f \right] G(v_f, \phi_f; v_i, \phi_i) = 0,$$

(3.38)
where $\Theta_f$ acts as in (2.3) but on $v_f$ in place of $v$. If one replaces $\Theta$ by $\Theta_a$, one obtains a 2-point function $G_\lambda(v_f, \phi_f; v_1, \phi_1)$ which, as we saw in section 3.2, admits a perturbative expansion

$$G_\lambda(v_f, \phi_f; v_1, \phi_1) = \sum_{M=0}^{\infty} \lambda^M A_M(v_f, \phi_f; v_1, \phi_1),$$  

(3.39)

where $A_M$ is the amplitude defined in (3.27):

$$A_M(v_f, \phi_f; v_1, \phi_1) = \sum_{v_{M-1}, \ldots, v_1 \in \Theta(v_{M-1}, \ldots, v_1)} A_+(v_{M-1}, \ldots, v_1; \phi_f, \phi_1) = \sum_{v_{M-1}, \ldots, v_1 \in \Theta(v_{M-1}, \ldots, v_1)} \Theta_{v_{M-1} \Theta_{v_{M-2}} \cdots \Theta_{v_1 \Theta_{v_1}}}$$

$$\times \prod_{k=1}^{p} \frac{1}{(nk-1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k-1} \sum_{m=1}^{p} \Theta_{w_k w_k} \sqrt{\Theta_{w_k w_k} \Delta \phi} \prod_{j \neq m}^{p} \left( \Theta_{w_k w_k} - \Theta_{w_j w_j} \right).$$

(3.40)

The suffix $+$ in $A_+$ emphasizes that we have taken the positive frequency part.

As a non-trivial check on this expansion we will now show that $G_\lambda$ satisfies (3.38) order by order. Since $\Theta_a = D + \lambda K$, our task reduces to showing

$$(\partial^2_\phi - D_f)A_M(v_f, \phi_f; v_1, \phi_1) - K_f A_{M-1}(v_f, \phi_f; v_1, \phi_1) = 0.$$  

(4.11)

We will show that the left-hand side is zero path by path in the sense that for every path acted on by the off-diagonal part, there are two paths acted on the diagonal part that cancel it. Without loss of generality we assume that $v_f = w_p$ in (4.40). Then we have

$$(\partial^2_\phi - D_f)A_+(v_f, v_{M-1}, \ldots, v_1; \phi_f, \phi_1) = \Theta_{v_f v_{M-1}} \Theta_{v_{M-2} v_{M-2}} \cdots \Theta_{v_1 v_1}$$

$$\times \prod_{k=1}^{p} \frac{1}{(nk-1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k-1} \sum_{m=1}^{p} \Theta_{w_k w_k} \sqrt{\Theta_{w_k w_k} \Delta \phi} \prod_{j \neq m}^{p} \left( \Theta_{w_k w_k} - \Theta_{w_j w_j} \right).$$

(3.42)

If $w_p$ occurs with multiplicity $n_p = 1$ i.e. if $v_f$ is the only volume to take the value $w_p$, then there are no derivatives in $\Theta_{w_p w_p}$ in the above equation and it simplifies to

$$(\partial^2_\phi - D_f)A_+(v_f, v_{M-1}, \ldots, v_1; \phi_f, \phi_1) = \Theta_{v_f v_{M-1}} \Theta_{v_{M-2} v_{M-2}} \cdots \Theta_{v_1 v_1}$$

$$\times \prod_{k=1}^{p-1} \frac{1}{(nk-1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k-1} \sum_{m=1}^{p} \left( \Theta_{w_k w_k} - \Theta_{w_j w_j} \right) \sqrt{\Theta_{w_k w_k} \Delta \phi} \prod_{j \neq m}^{p} \left( \Theta_{w_k w_k} - \Theta_{w_j w_j} \right).$$

(3.43)

Thus, on simple paths where the final volume occurs only once in the sequence, the action of $[\partial^2_\phi - D]$ is to give the amplitude of the path without $v_f$, times a matrix element of $\Theta$ related to the transition from $v_{M-1}$ to $v_f$. In general, the value of the final volume can be repeated in the discrete path: $n_p \neq 1$. In that case we need to push $\Theta_{w_p w_p}$ under the derivatives but the final result is the same. Thus, in all cases we have

$$(\partial^2_\phi - D_f)A_+(v_f, v_{M-1}, \ldots, v_1; \phi_f, \phi_1) = \Theta_{v_f v_{M-1}} A_+(v_{M-1}, \ldots, v_1, v_1; \phi_f, \phi_1).$$

(3.44)
Finally, it is straightforward to evaluate the action of the off-diagonal part on $A_{M-1}$ (see (3.41)):

$$K_f A_{M-1}(v_f, v_{M-2}, \ldots, v_1, v_i; \phi_f, \phi_i) = \sum_{v_{M-1}} \Theta_{v_f v_{M-1}} A_{M-1}(v_{M-1}, v_{M-2}, \ldots, v_1, v_i; \phi_f, \phi_i).$$  

(3.45)

Combining these results we see that equation (3.41) is satisfied. Thus the vertex expansion we obtained is a solution to the quantum constraint equation. Further it is a good perturbative solution in the sense that if we only take paths in which the number of volume transitions is less than some $M^\star$, then the constraint is satisfied to the order $\lambda M^\star$:

$$\left[ \frac{\partial^2}{\partial \phi_f^2} - (D_f + \lambda K_f) \right] \sum_{M=0}^{M^\star} \lambda^M A_M(v_f, \phi_f; v_i, \phi_i) = O(\lambda^{M^\star+1}).$$  

(3.46)

Also in this calculation the cancelations occur in a simple manner; the off-diagonal part acting on paths with $M-1$ transitions gives a contribution for each path with $M$ transitions that could be obtained by adding a single additional transition in the original path. These contributions cancel with the action of the diagonal part on the paths with $M$ transitions.

This calculation provides an explicit check on our perturbative expansion of the physical inner product. This is a concrete realization, in this simple example, of a central hope of SFMs, to show that the physical inner product between spin networks, expressed as a vertex expansion, does solve the Hamiltonian constraint of LQG order by order.

### 3.4. The ‘coupling constant’ $\lambda$ and the cosmological constant $\Lambda$

So far we have regarded the GFT inspired perturbation theory as a calculational tool and the coupling constant $\lambda$ as a book-keeping device which merely keeps track of the number of vertices in the vertex expansion. From this standpoint, values of $\lambda$ other than $\lambda = 1$ have no physical significance. However, if one regards GFT as fundamental and gravity as an emergent phenomenon, one is forced to change the viewpoint. From this new perspective, the coupling constant $\lambda$ is physical and can, for example, run under a renormalization group flow.

The question we raised in section 1 is: What would then be the physical meaning of $\lambda$ from the gravitational perspective? Surprisingly, in the LQC model under consideration, $\lambda$ can be regarded as (a function of) the cosmological constant $\Lambda$.

Let us begin by noting how the quantum constraint changes in the presence of a cosmological constant $\Lambda$:

$$-C(\Lambda) = \partial^2 \phi + \Theta(\Lambda) = \frac{\partial^2 \phi}{2\ell_0^2} + \Theta - \pi G\gamma^2 \Lambda \nu^2.$$  

(3.47)

Thus, only the diagonal part of $\Theta$ is modified and it just acquires an additional term proportional to $\Lambda$. In the GFT-like perturbation expansion, then, we are led to decompose $\Theta_\lambda(\Lambda)$ as

$$\Theta_\lambda(\Lambda) = D(\Lambda) + \lambda K,$$

where

$$D(\Lambda) = \pi G \left( \frac{3}{2\ell_0^2} - \gamma^2 \Lambda \right) \nu^2.$$  

(3.48)

It is now easy to check that $\Psi(v, \phi)$ satisfies the constraint equation

$$\left[ \partial^2 \phi + D(\Lambda) + \lambda K \right] \Psi(v, \phi) = 0$$  

(3.49)

with the cosmological constant $\Lambda$ if and only if $\tilde{\Psi}(v, \tilde{\phi})$ satisfies

$$\left[ \tilde{\partial}^2 \phi + D(\tilde{\Lambda}) + K \right] \tilde{\Psi}(v, \tilde{\phi}) = 0,$$  

(3.50)

where

$$\tilde{\Lambda} = \frac{\Lambda}{\lambda} + \frac{3}{2\gamma^2\ell_0^2\lambda}(\lambda - 1), \quad \tilde{\phi} = \sqrt{\lambda} \phi \quad \text{and} \quad \tilde{\Psi}(v, \tilde{\phi}) = \Psi(v, \phi).$$  

(3.51)
Consequently, the two theories are isomorphic. Because of this isomorphism, the gravitational meaning of the coupling constant $\lambda$ is surprisingly simple: it is related to the cosmological constant $\Lambda$ via\footnote{Note incidentally that, contrary to what is often assumed, running of a constant under a renormalization group flow is not related to the physical time evolution in cosmology \cite{52}.}

$$\Lambda = \frac{3}{2\gamma^2 \ell_o^2} (1 - \lambda).$$

(3.52)

At $\lambda = 1$, we have $\Lambda = 0$, whence the GFT reproduces the amplitudes of the SFM with zero cosmological constant. The question is: What is the spacetime interpretation of GFT for other values of $\lambda$? From the perturbation theory perspective, $\lambda$ will start out being zero in GFT and, under the renormalization group flow, it will hopefully increase to the desired value $\lambda = 1$. In the weak coupling limit $\lambda \approx 0$, the SFM will reproduce the amplitudes of the theory which has a positive but Planck scale cosmological constant $\Lambda \approx 3/2\gamma^2 \ell_o^2$. This is just what one would expect from the 'vacuum energy' considerations in quantum field theories in Minkowski spacetime. As the coupling constant $\lambda$ increases and approaches the SFM value $\lambda = 1$, the cosmological constant $\Lambda$ decreases. Now, suppose that the renormalization group flow leads us close to but not all the way to $\lambda = 1$. If we are just slightly away from the fixed point $\lambda = 1$, the cosmological constant $\Lambda$ would be small and positive. These considerations are only heuristic, but they suggest an avenue by which a fully developed GFT could perhaps account for the smallness of the cosmological constant.

4. Deparameterized framework

In this section, we will use the deparameterized framework which emphasizes the role of $\phi$ as internal time. As explained in section 2, now we can work in the Schrödinger picture, regarding $\nu$ as the configuration variable and $\phi$ as time. The physical states are now represented as functions $\Psi(\nu)$ with a finite norm

$$\|\Psi\|_{\text{phy}}^2 = \sum_{\nu = 4n\ell_o} |\Psi(\nu)|^2,$$

(4.1)

and they evolve via the Schrödinger equation

$$-i\partial_\phi \Psi(\nu, \phi) = \sqrt{\Theta} \Psi(\nu, \phi) \equiv H \Psi(\nu, \phi).$$

(4.2)

In contrast to section 3, in this section we will not be interested in the kinematical Hilbert space or the group-averaging procedure. The primary object of interest will rather be the transition amplitude

$$\langle \nu_f, \phi_f; \nu_i, 0 | = \langle \nu_f | e^{i H \phi} | \nu_i \rangle$$

(4.3)

for the initial physical state $|\nu_i\rangle$ at time $\phi_i = 0$ to evolve to $|\nu_f\rangle$ at time $\phi_f = \phi$. From our discussion in section 2, one would expect this amplitude to equal the physical scalar product $\langle [\nu_f, \phi_f], [\nu_i, 0]_{\text{phy}} \rangle = G(\nu_f, \phi; \nu_i, 0)$ considered in section 3. This is indeed the case, for the positive frequency solution $\Psi_{\nu, \phi} = [\nu, \phi]_+ = \int d\omega \bar{e}_\omega(\nu) e^{-i \omega \phi} e^{i \omega \phi} e_k(\nu)$

(4.4)

(see equation (2.6)) so that the physical scalar product between positive frequency solutions
[\nu_i, \phi_i]_\alpha$ and $[\nu_f, \phi_f]_\alpha$ is given by

$$\langle \{ \nu_f, \phi_f \}, \{ \nu_i, \phi_i \}_\alpha \rangle \text{phy} = \int dk \ e^{i\omega(k)\phi_f - \phi_i} \bar{e}_k(\nu_i) e_k(\nu_f)$$  \hfill (4.5)

(see equation (2.9)). The right-hand side is precisely the expression of the transition amplitude $\langle \nu_f | e^{iHt} | \nu_i \rangle = \int dk \langle \nu_f | e^{iHt} | k \rangle \langle k | \nu_i \rangle$. Since $e_k(\nu) = \langle \nu | k \rangle$, we have the equality $G(\nu_f, \phi; \nu_i, 0) = A(\nu_f, \phi; \nu_i, 0)$. However, the interpretation now emphasizes the physical time evolution in $\phi$ generated by $H$ whence $A(\nu_f, \phi; \nu_i, 0)$ has the interpretation of a physical transition amplitude. Therefore, we can literally follow—not just mimic—the procedure Feynman used in non-relativistic quantum mechanics [6]. This will again lead to a vertex expansion but one which, if terminated at any finite order, is distinct from that obtained in section 3.

In spite of important conceptual differences, the mathematical procedure used in this section is completely analogous to that used in section 3. Furthermore, this deparameterized framework was discussed in greater detail than the timeless framework in [1]. Therefore, in this section we will present only the main steps.

4.1. Sum over histories

Following Feynman, let us divide the time interval $(\phi, 0)$ into $N$ equal parts, each of length $\epsilon = \phi/N$, and express the transition amplitude $A(\nu_f, \phi; \nu_i, 0)$ as a sum over discretized paths $\sigma_N = (\nu_f = \check{\nu}_N, \check{\nu}_{N-1}, \ldots, \nu_1, \nu_0 = \nu_i)$:

$$A(\nu_f, \phi; \nu_i, 0) = \sum_{\sigma_N} A(\sigma_N) \quad \text{with} \quad A(\sigma_N) = U_{\check{\nu}_N \check{\nu}_{N-1}} U_{\check{\nu}_{N-1} \check{\nu}_{N-2}} \ldots U_{\check{\nu}_1 \nu_0}, \quad (4.6)$$

where now $U_{\check{\nu}_m \nu_n} = \langle \nu_{n+1} | e^{i\epsilon H} | \check{\nu}_m \rangle$. The structure of equation (4.6) parallels that of equation (3.9) in section 3.1. However, the mathematical 'time interval' $\Delta \tau = 1$ in section 3.1 is now replaced by the physical time interval $(\phi, 0)$ and the mathematical 'Hamiltonian' $\alpha \Theta$ by the physical Hamiltonian $H = \sqrt{\Theta}$. Furthermore, we no longer split the amplitude into a gravitational part and a scalar field part and the group-averaging parameter $\alpha$ will never appear in this section.

As in section 3.1, the next step is to make a convenient rearrangement of this sum, emphasizing volume transitions, rather than what happens at each point $\phi_0 = n\epsilon$ of the skeletonized time interval. Thus, we first recognize that the volume could remain constant for a number of time steps and consider histories $\sigma_N^M$ with precisely $M$ volume transitions (where $M < N$):

$$\sigma_N^M = \{(\nu_M, \nu_{M-1}, \ldots, \nu_1, \nu_0); (N_M, N_{M-1}, \ldots, N_2, N_1)\}, \quad v_m \neq v_{m-1}, N_m > N_{m-1}, \quad (4.7)$$

where $\nu_M, \ldots, \nu_0$ denote the volumes that feature in the history $\sigma_N^M$ and $N_k$ denotes the number of time steps after which the volume changes from $\nu_{k-1}$ to $\nu_k$. The probability amplitude for such a history $\sigma_N^M$ is given by

$$A(\sigma_N^M) = \left[ U_{\nu_M \nu_M} \right]^{N-N_M-1} U_{\nu_M \nu_{M-1}} \ldots \left[ U_{\nu_1 \nu_0} \right]^{N_2-N_1-1} U_{\nu_1 \nu_0} \left[ U_{\nu_0 \nu_0} \right]^{N_1-1}. \quad (4.8)$$

As in section 3.1, we carry out the sum over all these amplitudes in three steps. First we keep the ordered set of volumes $(\nu_M, \ldots, \nu_0)$ fixed, but allow the volume transitions to occur at any value $\phi = n\epsilon$ in the interval $\mathcal{I}$, subject only to the constraint that the $m$th transition occurs before the $(m+1)$st for all $m$. The sum of amplitudes over this group of histories is given by

$$A_N(\nu_M, \ldots, \nu_0) = \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \ldots \sum_{N_1=1}^{N_2-1} A(\sigma_N^M). \quad (4.9)$$
Next we sum over all possible intermediate values of $\nu_m$ such that $\nu_m \neq \nu_{m-1}$, keeping $\nu_0 = \nu_i, \nu_M = \nu_f$, to obtain the amplitude $A(M)$ associated with the set of all paths in which there are precisely $M$ volume transitions:

$$A_N(M) = \sum_{\nu_M - 1, \ldots, \nu_1}^{\nu_0} A_N(\nu_M, \ldots, \nu_0).$$

(4.10)

Finally the total amplitude $A(\nu_f, \phi; \nu_i, 0)$ is obtained by summing over all volume transitions that are permissible within our initially fixed skeletonization with $N$ time steps:

$$A(\nu_f, \phi; \nu_i, 0) = \sum_{M=0}^{N} A_N(M) \equiv \sum_{M=0}^{N} \left[ \sum_{\nu_M - 1, \ldots, \nu_1}^{\nu_0} A_N(\nu_M, \ldots, \nu_0) \right].$$

(4.11)

As in section 3.1, since $A(\nu_f, \phi; \nu_i, 0) = \langle \nu_f | e^{iH\phi} | \nu_i \rangle$, the value of the amplitude (4.11) does not depend on $N$ at all; the skeletonization was introduced just to express this well-defined amplitude as a sum over histories. Thus, while the range of $M$ in the sum and the amplitude $A_N(M)$ in (4.11) both depend on $N$, the sum does not. We can get rid of the skeletonization altogether by taking the limit as $N$ goes to infinity, to express the total transition amplitude as a vertex expansion in the spirit of the timeless framework of spin foams. Reasoning analogous to that in appendix A shows that the limit does exist. In this limit the reference to the skeletonization of the time interval disappears and volume changes can now occur at any time in the continuous interval $(\phi_i = 0, \phi_f = \phi)$. The contribution $A_M$ from paths with precisely $M$ volume changes has a well-defined ‘continuous time’ limit and the total amplitude is given by a discrete sum over $M$:

$$A(\nu_f, \phi; \nu_i, 0) = \sum_{M=0}^{\infty} A_M(\nu_f, \phi; \nu_i, 0),$$

(4.12)

where the partial amplitudes $A_M$ are given by

$$A_M(\nu_f, \phi; \nu_i, 0) = \sum_{\nu_M - 1, \ldots, \nu_1}^{\nu_0} A_M(\nu_f, \nu_M - 1, \ldots, \nu_1 \nu_i \nu)$$

$$= \sum_{\nu_M - 1, \ldots, \nu_1}^{\nu_0} H_{\nu_M \nu_{M-1}} H_{\nu_{M-1} \nu_{M-2}} \cdots H_{\nu_2 \nu_1} H_{\nu_1 \nu_0}$$

$$\times \prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial H_{\nu_k \nu_k}} \right)^{n_k - 1} \sum_{m=1}^{p} \prod_{j \neq m}^{p} \left( \frac{e^iH_{\nu_m \nu_m}}{H_{\nu_m \nu_m} - H_{\nu_k \nu_k}} \right).$$

(4.13)

As one might expect, the final expression involves just the matrix elements of the Hamiltonian $H = \sqrt{\Theta}$. These are calculated in Appendix C.

Thus, the total transition amplitude has been expressed as a vertex expansion (4.12) in the manner of SFMs. We provided several intermediate steps because, although the left-hand sides are equal, the final vertex expansion is different from that obtained in section 3.1. While (4.12) features matrix elements of $H = \sqrt{\Theta}$, (3.27) features matrix elements of $\Theta$ itself. The existence of distinct but equivalent vertex expansions is quite surprising. In each case we emphasized a distinct aspect of dynamics: the timeless framework and group averaging in (3.27), and relational time and deparameterization in (4.12).

4.2. Perturbation expansion

This vertex expansion can also be obtained as a perturbation series that mimics GFTs. As in section 3, the perturbative approach avoids skeletonization altogether and has the advantage
that it guarantees a convergent series. Furthermore, since this deparameterization approach does not refer to an integral over \( \alpha \), the assumption of interchange of the integral and the sum over \( M \) that was required in section 3.2 is no longer necessary.

Let us now focus on the Hamiltonian operator \( H = \sqrt{\Theta} \) (rather than on \( \Theta \) used in section 3.2) and decompose it into a diagonal part \( D \) and the remainder, non-diagonal part \( K \) which is responsible for a volume change. Finally, let us set \( H_\lambda = D + \lambda K \), where \( \lambda \) will serve as a marker for powers of \( K \), i.e. the number of volume changes in the expansion. Then, by working in the appropriate interaction picture, we obtain

\[
A_\lambda(v_f, \varphi; v_i, 0) = \sum_{M=0}^{\infty} \lambda^M A_M(v_f, \varphi; v_i, 0),
\]

where \( A_M \) is again given by (4.13). This power series in \( \lambda \) is reminiscent of what one finds in GFTs. If we set \( \lambda = 1 \) at the end of this derivation, we recover the vertex expansion (4.12) in the manner of SFMs. For a discussion of the intermediate steps, see [1] and appendix A.

4.3. Satisfaction of the Schrödinger equation

Recall that in the deparameterization scheme, the Schrödinger equation (4.2) incorporates both the quantum constraint and the positive frequency condition. By its very definition, the exact transition amplitude \( A(v_f, \varphi; v_i, 0) \) satisfies this Schrödinger equation. As a check on the perturbative expansion (4.14), we are led to ask whether the Schrödinger equation would be satisfied in a well-controlled approximate sense if we were to truncate the series on the right side of (4.14) at a finite value, say \( M^* \) of \( M \). We will now show that this is indeed the case.

Since \( H_\lambda = D + \lambda K \), the Schrödinger equation would be solved order by order in the perturbation series if for each \( M \) we have

\[
(i\partial_\varphi + D_f)A_M(v_f, \varphi; v_i, 0) + K_f A_{M-1}(v_f, \varphi; v_i, 0) = 0.
\]

Using the expression of the partial amplitudes \( A_M \) we are then led to ask if

\[
\sum_{v_M^- \neq \ldots \neq v_1} \sum_{v_{M-1} \neq \ldots \neq v_1} \left( (-i\partial_\varphi + D_f)A_M(v_f, v_{M-1}, \ldots, v_1, \varphi) \right)
\]

\[
+ K_f A_{M-1}(v_f, v_{M-2}, \ldots, v_1, v_i; \varphi)
\]

vanishes for each \( M \). Using expression (4.13) of \( A(v_f, v_{M-1}, \ldots, v_1, v_i; \varphi) \), one can readily verify that this is indeed the case. As in section 3.3, the equation is satisfied ‘path by path’, i.e. already by the intermediate amplitudes \( A(v_f, v_{M-1}, \ldots, v_1, v_i; \varphi) \) and \( A(v_f, v_{M-2}, \ldots, v_1, v_i; \varphi) \).

Thus we have shown that the vertex expansion resulting from the perturbation series satisfies quantum dynamics in a well-controlled fashion. If we were to terminate the sum at \( M = M^* \), we would have

\[
(i\partial_\varphi + D_f + \lambda K) \sum_{M=0}^{M^*} \lambda^M A_M(v_f, \varphi; v_i, 0) = \mathcal{O}(\lambda^{M^*+1}).
\]

This brings out the precise sense in which a truncation to a finite order of the vertex expansion incorporates the quantum dynamics of the deparameterized theory approximately.
5. Discussion

Because LQC is well developed in the Hamiltonian framework, it provides an interesting avenue to probe various aspects of the spin foam paradigm. For definiteness we focused on the Friedmann model with a massless scalar field as source. We used the group-averaging procedure that is available for general constrained systems as well as the natural deparameterization, with \( \phi \) as the emergent time variable, that is often employed in LQC.

Group averaging provides Green’s function \( G(v_f, \phi_f; v_i, \phi_i) \), representing the inner product between physical states extracted from the kinematic kets \( |v_f, \phi_f\rangle \) and \( |v_i, \phi_i\rangle \). The Schrödinger evolution of the deparameterized theory provides the transition amplitude \( A(v_f, \phi_f; v_i, \phi_i) \) for the physical state \( |v_i\rangle \) at the initial instant \( \phi_i \) to evolve to the state \( |v_f\rangle \) at the final instant of time \( \phi_f \). We saw in section 4 that the two quantities are equal. But they emphasize different physics. Following the general procedure invented by Feynman to pass from a Hamiltonian theory to a sum over histories, we were able to obtain a series expansion for each of these quantities—equation (3.27) for \( G(v_f, \phi_f; v_i, \phi_i) \) and equation (4.12) for \( A(v_f, \phi_f; v_i, \phi_i) \)—that mimic the vertex expansion of SFMs. In section 3, we had to make one assumption in the derivation of the vertex expansion of \( G(v_f, \phi_f; v_i, \phi_i) \): in the passage from (3.35) to (3.36) we assumed that the integration over \( \alpha \) of the group-averaging procedure commutes with an infinite sum in (3.35). Since the integration over \( \alpha \) is bypassed in the deparameterized framework, this assumption is not necessary in our derivation of the vertex expansion of \( A(v_f, \phi_f; v_i, \phi_i) \) in section 4.

Detailed parallels between our construction and SFMs are as follows. The analog of the manifold \( M \) with boundaries \( S_i, S_f \) in SFMs is the manifold \( \mathbb{V} \times \mathcal{I} \), where \( \mathbb{V} \) is the elementary cell in LQC and \( \mathcal{I} \), a closed interval in the real line (corresponding to \( \tau \in [0,1] \) in the timeless framework and \( \phi \in [\phi_f, \phi_i] \) in the deparameterized). The analog of a triangulation in spin foams is just a division of \( \mathbb{V} \times \mathcal{I} \) into \( M \) parts by introducing \( M - 1 \) time slices. Just as the triangulation in SFMs is determined by the number of 4-simplices, what matters in LQC is the number \( M \); the precise location of slices is irrelevant. The analog of the dual-triangulation in SFMs is just a ‘vertical’ line in \( \mathbb{V} \times \mathcal{I} \) with \( M \) marked points or ‘vertices’ (not including the two endpoints of \( \mathcal{I} \)). Again, what matters is the number \( M \); the precise location of vertices is irrelevant. Coloring of the dual-triangulation in SFMs corresponds to an ordered assignment \((v_M, v_{M-1}, \ldots, v_1, v_0)\) of volumes to edges bounded by these marked points (subject only to the constraints \( v_M = v_f, v_0 = v_i \) and \( v_m \neq v_{m-1} \)). Each vertex signals a change in the physical volume along the quantum history\(^4\). The probability amplitude associated with the given coloring is given by \( A(v_f, \ldots, v_0; \phi_f, \phi_i) \) in the group-averaging procedure (see equation (3.26)) and by \( A(v_f, \ldots, v_0; \psi) \) in the deparameterization procedure (see equation (4.13)). A sum over colorings yields the partial transition amplitude \( A(v_f, \psi; v_i, 0) \) are given by a sum over these \( M \)-vertex amplitudes.

Thus, the physical inner product of the timeless framework and the transition amplitude in the deparameterized framework can each be expressed as a \emph{discrete sum} without the need of a ‘continuum limit’. A countable number of vertices suffices; the number of volume transitions does not have to become continuously infinite. This result supports the view that LQG and SFMs are not quite analogous to quantum field theories on classical spacetimes. Discrete quantum geometry at the Planck scale makes a key difference. In sections 3.2 and 4.2, we were able to obtain the same vertex sum using a perturbative expansion, in a coupling constant \( \lambda \), that is reminiscent of GFTs. In sections 3.3 and 4.3, we showed that this is a useful

\(^{4}\) In the Bianchi models there are additional labels corresponding to anisotropies [53]. These are associated with the faces of the dual graph, and are thus analogs of the spin labels \( j \) associated with faces of general spin foams.
expansion in the sense that Green’s function and the transition amplitude satisfy the dynamical equations order by order in $\lambda$. Thus, if we were to truncate the expansion to order $M$, the truncated Green’s function and transition amplitude would satisfy the dynamical equations up to terms of the order $O(\lambda^{M+1})$. Finally in section 3.4, we showed that the coupling constant $\lambda$ inspired by GFTs is closely related to the cosmological constant. This interpretation opens a possibility that a detailed study of the renormalization group flow in GFT may be able to account for the very small, positive value of the cosmological constant.

Taken together, these results provide considerable concrete support for the general paradigms that underlie SFM and GFT. However, we emphasize that this analysis has a key limitation. We did not begin with a SFM in full general relativity and then arrive at the LQC model through a systematic symmetry reduction of the full vertex expansion. Rather, we began with an already symmetry reduced model and recast the results in the spin foam language. Reciprocally, a key strength of these results is that we did not have to start by postulating that the physical inner product or the transition amplitude is given by a formal path integral. Rather, a rigorously developed Hamiltonian theory guaranteed that these quantities are well defined. We simply recast their expressions as vertex expansions. The complementarity of the two methods is brought to forefront in the recent work [54] on spin foams in the cosmological context. There, one begins with general spin foams, introduces homogeneity and isotropy only as a restriction on the boundary state and calculates just the leading-order terms in the vertex expansion. By contrast, in this work we restricted ourselves to homogeneity and isotropy at the outset but calculated the physical inner product (or, in the deparameterized picture, the transition amplitude) to all orders in the vertex expansion.

It is often the case that exactly soluble models not only provide support for or against general paradigms but they can also uncover new issues whose significance had not been realized before. The LQC analysis has brought to forefront three such issues.

First, it has revealed the advantage of adding matter fields. It is widely appreciated that on physical grounds it is important to extend SFMs beyond vacuum general relativity. However what was not realized before is that, rather than complicating the analysis, this generalization can in fact lead to interesting and significant technical simplifications. This point is brought out vividly by a recent analysis of Rovelli and Vidotto [55]. They considered a simple model on a finite-dimensional Hilbert space where there is no analog of the scalar field or the possibility of deparameterization. There, individual terms in the vertex expansion turn out to be well defined only after a (natural) regularization. In our example, the presence of the scalar field simplified the analysis (in the transition from (3.24) to (3.26)) and individual terms in the vertex expansion are finite without the need of any regularization. Furthermore, this simplification is not an artifact of our restriction to the simplest cosmological model. For example, in the Bianchi I model, the Hamiltonian theory is also well developed in the vacuum case [56]. The work in progress by Campiglia, Henderson, Nelson and Wilson-Ewing shows that technical problems illustrated in [55] arise also in this case, making it necessary to introduce a regularization. These problems simply disappear if one also includes a scalar field. A qualitative argument suggests that the situation would be similar beyond cosmological models as well.

Second, it came as a surprise that there are two distinct vertex expansions. Group averaging provides one that mainly uses the matrix elements of $\Theta$ while the deparameterized framework provides one that uses only the matrix elements of $\sqrt{\Theta}$. This is not an artifact of using the simplest cosmological model. The work in progress indicates that the situation is
similar in the anisotropic Bianchi models. Indeed, from a Hamiltonian perspective, it would appear that distinct vertex expansions can arise whenever a well-defined deparameterization is available. This raises an interesting and more general possibility. Can there exist distinct spin foam models—constructed by using, say, distinct vertex amplitudes—for which the complete vertex expansions yield the same answer? Finite truncations of these expansions could be inequivalent, but each could be tailored to provide an excellent approximation to the full answer for a specific physical question. One may then be able to choose which truncated expansion to use to probe a specific physical effect.

The third issue concerns three related questions in the spin foam literature. (i) Should the physical inner products between states associated with spin networks be real rather than complex [31]? (ii) In the classical limit, should one recover $\cos S$ in place of the usual term $e^{iS}$, where $S$ is the Einstein Hilbert action [32, 33]? (iii) Should the choice of orientation play a role in the sum of histories [49]? In the LQC example we studied in this paper, these three questions are intimately related. The inner product between the physical states $\{v, \phi\}$ determined by the kinematic basis vectors—which are the analogs of spin networks in this example—are in general complex (see equation (3.27)). However, if we had dropped the positive frequency requirement, the group-averaged inner products would have been real (see equation (3.24)). The situation with action is analogous. And, as we show in the next paragraph, the positive frequency condition also selects a time orientation.

Since this is an important issue, we will discuss it in some detail. Let us begin with the classical theory. The phase space is four dimensional and there is a single constraint: $C(v, b; \phi, p_\phi) := Gp_\phi^2 - 3\pi (\ell_\hbar^2 v^3)b^2 = 0$. Dynamics has two conceptually interesting features. First, given a solution $(v(t), \phi(t))$ to the constraint and dynamical equations, $(-v(t), \phi(t))$ is also a solution (where $t$ denotes proper time). They define the same spacetime metric and scalar field; only the parity of the spatial triad is reversed. Therefore, $(v(t), \phi(t)) \rightarrow (-v(t), \phi(t))$ is regarded as a gauge transformation. The second feature arises from the fact that the constraint surface has two ‘branches’, $p_\phi > 0$ and $p_\phi < 0$, joined at points $p_\phi = 0$ which represent Minkowski spacetime. As is usual in quantum cosmology, let us ignore the trivial flat solution. Then each of the two portions $\Gamma^\pm$ of the constraint surface defined by the sign of $p_\phi$ is left invariant by dynamics. Furthermore, there is a symmetry. Given a dynamical trajectory $(v(t), \phi(t))$ in $\Gamma^+$, there is a trajectory $(v(t), -\phi(t))$ which lies in $\Gamma^-$. This represents a redundancy in the description in the sense that we recover all physical spacetime geometries $g_{ab}(t)$ even if we restrict only to one of the two branches $\Gamma^\pm$. In particular, the dynamical trajectories on $\Gamma^+$, for example, include solutions which start with a big bang and expand out to infinity as well as those which start out with infinite volume and end their lives in a big crunch. The difference is only in time orientation. If we regard $\phi$ as an internal or relational time variable and reconstruct spacetime geometries from phase space trajectories, spacetimes obtained from a trajectory on $\Gamma^+$ define the same geometry as the one obtained from the corresponding trajectory on $\Gamma^-$ but with opposite time orientation. As in the Klein–Gordon theory of a free relativistic particle, this redundancy is removed by restricting oneself either to the $p_\phi > 0$ sector or to the $p_\phi < 0$ sector. In the quantum theory, then, the physical Hilbert space is given by solutions $\Psi(v, \phi)$ to the quantum constraint (2.2) which in addition have only positive (or negative) frequency so that the operator $p_\phi$ is positive (or negative) definite. (They are also invariant under parity, $\Psi(v, \phi) = \Psi(-v, \phi)$). Thus, the LQC example suggests that in general SFMs, one should fix the time orientation, lending independent support to the new ideas proposed in [49]. Reality of the physical inner products between spin network states [31] and the emergence of $\cos S$ in place of $e^{iS}$ [32, 33] can be traced back to the fact that in most of the SFM literature, one sums over both orientations. However, our analysis provides only a hint rather than an iron-clad argument.
because all our discussion is tied to LQC models where symmetry reduction occurs prior to quantization.

We conclude with an observation. We have recast LQC as a sum over histories. However, this is different from a Feynman path integral in which the integrand is expressed as $e^{iS}$ for a suitable action $S$. This step was not necessary for the goals of this paper. However, it is of considerable interest, especially in the cosmological context, for certain physical issues such as the emergence of the classical universe and semi-classical corrections to the classical theory. Such a path integral formulation of LQC does exist \[57\] and will be discussed elsewhere\[6\].

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**Appendix A. Limit in equation (3.18)**

It is convenient to rewrite $A_N(v_M, \ldots, v_0; \alpha)$ defined in (3.13) in the following way:

$$A_N(v_M, \ldots, v_0; \alpha) = U_{v_M v_{M-1}} \cdots U_{v_1 v_0} \left[ U_{v_M v_{M-1}} \right]^N \left[ U_{v_M v_{M-1}} \right]^{-1} \times \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N-M-1} \cdots \sum_{N_1=1}^{N-1} \left[ U_{v_M v_{M-1}} \right] \times \left[ U_{v_M v_{M-1}} \right]^{N}.$$

(A.1)

Our aim is to calculate the limit $N \to \infty$ of (A.1) and show that it is given by $A(v_M, \ldots, v_0; \alpha)$, of equation (3.18) which we rewrite as

$$A(v_M, \ldots, v_0; \alpha) = (-i\alpha)^M \Theta v_{M-1} \cdots \Theta v_0 e^{-i\alpha \Theta v_M}$$

$$\times \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \cdots \int_0^{\tau_2} d\tau_1 e^{\tau_M b_M} \cdots e^{\tau_1 b_1},$$

(A.2)

where

$$b_m := -i\alpha \left( \Theta v_{m-1} v_{m-1} - \Theta v_m v_m \right).$$

(A.3)

We start by calculating the $N \gg 1$ behavior of the terms appearing in (A.1). These are

$$U_{v_M v_{M-1}} = -i\frac{\alpha}{N} \Theta v_{M-1} v_{M-1} + O(N^{-2}),$$

(A.4)

$$\left[ U_{v_M v_{M-1}} \right]^N = e^{N \log U_{v_M v_{M-1}}}$$

$$= e^{N(-i\alpha \Theta v_{M-1} v_{M-1} + O(N^{-2}))}$$

$$= e^{-i\alpha \Theta v_{M-1} v_{M-1} v_{M-1} + O(N^{-1})},$$

(A.5)

$$\left[ U_{v_M v_{M-1}} \cdots U_{v_1 v_0} \right]^{-1} = 1 + O(N^{-1}),$$

(A.6)

$$\left[ U_{v_M v_{M-1}} \cdots U_{v_1 v_0} \right]^{N_m} = e^{N_m \log U_{v_M v_{M-1} v_{M-1} v_{M-1} v_{M-1} v_{M-1}}}$$

$$= e^{N_m(b_m/N + O(N^{-2}))}$$

$$= e^{N_m b_m/N + O(N^{-2})},$$

(A.7)

6 A path integral formulation of polymer quantum mechanics was carried out independently by Husain and Winkler \[58\].
with \( b_m \) given in (A.3). In (A.5) and (A.7) we have used the fact that the multivalued nature of the log function does not affect the final result: \( e^{N \log x + 2\pi i k} = e^N \log x \) where \( k \in \mathbb{Z} \) reflects the multiple values that log can take.

We now substitute expressions (A.4) to (A.7) into (A.1) to obtain

\[
A_N(v_M, \ldots, v_0; \alpha) = \left[ (-i\alpha)^M \Theta_{\nu_M\nu_{M-1}} \cdots \Theta_{\nu_i\nu_0} e^{-i\alpha x_M} \right] \times \prod_{m=1}^{M-1} \left( \sum_{N_m=m}^{N_m-1} \sum_{N_{m-1}=m}^{N_{m-1}-1} \sum_{N_{M-1}=\cdot\cdot\cdot} \sum_{N_0=0}^{N_0-1} N^M O(N^{-M-1}) \right) + O(N^{-1}).
\]

where the product denotes the \( M \) nested sums in (A.1). Each sum in (A.8) has two terms. The first one gives a contribution of \( N_m N_{m-1} \cdots N_{M-1} N_0 \sim O(N) \) while the second one is \( N_m N_{m-1} \cdots N_{M-1} N_0 \sim O(1) \). The \( M \) sums then give a contribution of order \( O(N) + O(1) \sim O(N^M) + O(N^{-M-1}) \). By combining this with the first factor of (A.8), we find that the non-vanishing contribution comes from the first terms of the sums

\[
A_N(v_M, \ldots, v_0; \alpha) = (-i\alpha)^M \Theta_{\nu_M\nu_{M-1}} \cdots \Theta_{\nu_i\nu_0} e^{-i\alpha x_M} \sum_{N_m=m}^{N_{M-1}} \sum_{N_{M-1}=m}^{N_{M-1}-1} \sum_{N_{M-2}=\cdot\cdot\cdot} \sum_{N_0=0}^{N_0-1} N^M O(N^{-M-1}) + O(N^{-1}).
\]

Equation (A.9) has all the pre-factors appearing in (A.2). It then remains to show that \( N^{-M} \) times the sums in (A.9) limits to the integrals in (A.2). But this is rather obvious, as the sums can be seen as Riemann sums for the integrals. Specifically,

\[
\lim_{N \to \infty} \prod_{m=1}^{M-1} \left( \sum_{N_m=m}^{N_{M-1}} \sum_{N_{M-1}=m}^{N_{M-1}-1} \sum_{N_{M-2}=\cdot\cdot\cdot} \sum_{N_0=0}^{N_0-1} e^{\frac{2\pi i}{N} b_m} \right) = \lim_{N \to \infty} \prod_{m=1}^{M-1} \left( \sum_{N_m=m}^{N_{M-1}} \sum_{N_{M-1}=m}^{N_{M-1}-1} \sum_{N_{M-2}=\cdot\cdot\cdot} \sum_{N_0=0}^{N_0-1} e^{\frac{2\pi i}{N} b_m} \right)
\]

where, in the second line, we have slightly changed the limits on the sums, introducing an \( O(N^{-1}) \) term which vanishes in the limit. This concludes the proof of the limit (3.18).

**Appendix B. General integrals in equation (3.18)**

The integrals over \( \tau \) appearing in the amplitude for a single discrete path (3.18) can be evaluated for a general sequence of volumes \((v_M, \ldots, v_0)\) with the result given by (3.21). In this appendix we will perform these integrals first for the case where all \( v_i \) are distinct and then for the general case. The amplitude for a single discrete path given by (3.18) and (3.19) is

\[
A(v_M, \ldots, v_0, \alpha) = \int_0^{\Delta \tau} d\tau_M \int_0^{\Delta \tau} d\tau_M-1 \cdots \int_0^{\Delta \tau} d\tau_1 e^{-i\alpha (\Delta \tau - \tau_M) \Theta_{\nu_M\nu_{M-1}} \cdots \Theta_{\nu_1\nu_0}}.
\]

This expression can be written in terms of the following integral:

\[
I(x_M, \ldots, x_0, \Delta \tau) = \int_0^{\Delta \tau} d\tau_M \int_0^{\Delta \tau} d\tau_M-1 \cdots \int_0^{\Delta \tau} d\tau_1 (i)^M e^{i(\Delta \tau - \tau_M) x_M} \times e^{i(\Delta \tau - \tau_{M-1}) x_{M-1}} \cdots e^{i(\Delta \tau - \tau_1) x_1} e^{i\tau_0 x_0}.
\]

(B.2)
We will first evaluate this integral for the case where all \( x_i \) are distinct. By induction on \( M \)—the number of vertices or the number of times that \( x \) changes value—we will show that when the \( x_i \) are all distinct, the integral is given by

\[
I(x_M, \ldots, x_0, \Delta \tau) = \sum_{i=0}^{M} \frac{e^{i x_i \Delta \tau}}{\prod_{j \neq i} (x_i - x_j)}.
\]  

(3.3)

This is true by inspection for \( M = 0 \). If we assume that (3.3) holds for \( M \) we can evaluate the integral with \( M + 1 \) vertices:

\[
I(x_{M+1}, x_M, \ldots, x_0, \Delta \tau) = \int_0^{\Delta \tau} d\tau_{M+1} i e^{i (\Delta \tau - \tau_{M+1}) x_{M+1}} I(x_M, \ldots, x_0, \tau_{M+1})
\]

\[
= \int_0^{\Delta \tau} d\tau_{M+1} i e^{i (\Delta \tau - \tau_{M+1}) x_{M+1}} \sum_{i=0}^{M} \frac{e^{i x_i \tau_{M+1}}}{\prod_{j \neq i} (x_i - x_j)}
\]

\[
= \sum_{i=0}^{M} \frac{e^{i x_i \Delta \tau}}{\prod_{j \neq i} (x_i - x_j)} - e^{i \Delta \tau x_{M+1}} \sum_{i=0}^{M} \frac{1}{\prod_{j \neq i} (x_i - x_j)}.
\]  

(3.4)

In the first step, we recognized that the \((M + 1)\)st integral contains the \( M \)th and then, in the second step, we inserted the assumed result for the \( M \)th integral. In the second step, the integral over \( \tau_{M+1} \) is carried out. Finally using the identity

\[
\sum_{i=1}^{M+1} \frac{1}{\prod_{j \neq i} (x_i - x_j)} = 0,
\]  

(3.5)

the integral can be written as

\[
I(x_{M+1}, x_M, \ldots, x_0, \Delta \tau) = \sum_{i=0}^{M+1} \frac{e^{i x_i \Delta \tau}}{\prod_{j \neq i} (x_i - x_j)}.
\]  

(3.6)

Therefore if (3.3) holds for \( M \) it also holds for \( M + 1 \), thus by induction it holds for all \( M \geq 0 \).

If the \( x_i \) are not distinct, if there exist \( i, j \) such that \( x_i = x_j \), then the proof follows in a similar way. The key element is that the integral \( I(x_M, \ldots, x_0) \) is independent of the order of the \( x_i \)'s. This can be seen by rewriting the integral in terms of the time intervals \( \Delta \tau_i = \tau_{i+1} - \tau_i \) where \( \tau_0 = 0 \) and \( \tau_{M+1} = \Delta \tau \):

\[
I(x_0, x_1, \ldots, x_M, \Delta \tau) = \int_0^{\Delta \tau} d\tau_M d\Delta \tau_{M-1} \ldots d\Delta \tau_0 \delta(\Delta \tau_M + \cdots + \Delta \tau_0 - \Delta \tau)
\]

\[
\int_M e^{i \Delta \tau x_M} e^{i \Delta \tau_{M-1} x_{M-1}} \ldots e^{i \Delta \tau_1 x_1} e^{i \Delta \tau_0 x_0}.
\]  

(3.7)

It is clear that this is symmetric under the interchange of \( x_i \) with \( x_j \) for all \( i, j \), so the integral is independent of the order of the sequence \( x_i \). Since the integral is independent of the order of the values \( x_i \), it should be characterized by the distinct values, labeled by \( y_i \) and their multiplicity \( n_i \), where \( n_1 + \cdots + n_p = M + 1 \). Given a set of values \( x_i \), we will evaluate the integral for the case where they are organized such that any \( x_i \) sharing the same value are grouped together. Doing so the integral simplifies to

\[
I(y_p, n_p, y_1, n_1, \Delta \tau) = \int_0^{\Delta \tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_1} d\tau_1 (i) M e^{i (\Delta \tau - \tau_{n_p+1} - \cdots - \tau_{n_1+1}) y_p}
\]

\[
\times e^{i (\tau_{n_p+1} - \tau_{n_p+2} - \cdots - \tau_{n_1} - \tau_{n_1+1}) y_1} \ldots e^{i (\tau_{n_1+1} - \tau_{n_1}) y_1} e^{i \tau_1 y_1}.
\]  

(3.8)
By induction on $p$, the number of distinct values, we show that this integral is given by

$$I(y_p, n_p, \ldots, y_1, n_1, \Delta \tau) = \frac{1}{(n_p - 1)!} \left( \frac{\partial}{\partial y_p} \right)^{n_p - 1} \left( \frac{1}{(n_1 - 1)!} \right) \times \left( \frac{\partial}{\partial y_1} \right)^{n_1 - 1} \sum_{i=1}^{p} \prod_{j \neq i}^{p} (y_i - y_j)$$

$$\times \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial y_1} \right)^{n_k - 1} \prod_{i=1}^{p} \sum_{j \neq i}^{p} \prod_{k \neq j}^{p} (y_i - y_j).$$  \hspace{1cm} (B.9)

For $p = 1$, (B.8) can be easily evaluated giving

$$I(y_1, n_1) = \int_0^{\Delta \tau} d\tau_{n_1-1} \ldots \int_0^{\tau_1} d\tau_1 (i)^{n_1-1} e^{iy_1 \Delta \tau} = \frac{(i \Delta \tau)^{n_1-1}}{(n_1 - 1)!} e^{iy_1 \Delta \tau}$$

$$= \left( \frac{\partial}{\partial y_1} \right)^{n_1-1} \frac{1}{(n_1 - 1)!} e^{iy_1 \Delta \tau}. \hspace{1cm} (B.10)$$

If we assume that (B.9) holds for $p$ distinct values, then we can evaluate it for $p + 1$ distinct values as follows:

$$I(y_{p+1}, n_{p+1}, y_p, n_p, \ldots, y_1, n_1, \Delta \tau) = \int_0^{\Delta \tau} d\tau_{M-\nu_{p+1}+2} \ldots \int_0^{\tau_M} d\tau_{M-n_{p+1}+1}$$

$$(i)^{p+1} e^{(i \Delta \tau - \tau_{M-\nu_{p+1}+1}) y_{p+1}} I(y_p, n_p, \ldots, y_1, n_1, \tau_{M-n_{p+1}+1}). \hspace{1cm} (B.11)$$

Plugging in the assumed result for distinct values and performing the integrals over $\tau$, we obtain

$$I(y_{p+1}, n_{p+1}, y_p, n_p, \ldots, y_1, n_1, \Delta \tau) = \prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial y_k} \right)^{n_k - 1} \sum_{i=1}^{p} \prod_{j \neq i}^{p} (y_i - y_j)$$

$$\times \left[ \frac{e^{iy_{p+1} \Delta \tau}}{(y_1 - y_{p+1})^{p+1}} - \sum_{m=0}^{p-1} (y_1 - y_{p+1})^m (n_{p+1} - m)! \right]. \hspace{1cm} (B.12)$$

We recognize the term in brackets can be written as derivatives with respect to $y_{p+1}$ of a simple function

$$I(y_{p+1}, n_{p+1}, y_p, n_p, \ldots, y_1, n_1, \Delta \tau) = \prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial y_k} \right)^{n_k - 1} \sum_{i=1}^{p} \prod_{j \neq i}^{p} (y_i - y_j)$$

$$\times \left[ \frac{1}{(n_{p+1} - 1)!} \left( \frac{\partial}{\partial y_{p+1}} \right)^{n_{p+1} - 1} \left( \frac{e^{iy_{p+1} \Delta \tau}}{(y_1 - y_{p+1})} \right) \right]. \hspace{1cm} (B.13)$$

Finally simplifying the expression and using equation (B.5), we obtain

$$I(y_{p+1}, n_{p+1}, \ldots, y_1, n_1, \Delta \tau) = \prod_{k=1}^{p+1} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial y_k} \right)^{n_k - 1} \sum_{i=1}^{p+1} \prod_{j \neq i}^{p+1} (y_i - y_j). \hspace{1cm} (B.14)$$

Thus if (B.9) holds for $p$ then it also holds for $p + 1$, so it is true for all $p \geq 0$. Using this result we find that the contribution due to each discrete path is

$$A(\nu_M, \ldots, \nu_0, \alpha) = (\Theta_{\nu_M \nu_{M-1}}) \ldots (\Theta_{\nu_0 \nu_1})$$

$$\times \prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{\nu_k \nu_k}} \right)^{n_k - 1} \sum_{i=1}^{p} \prod_{j \neq i}^{p} e^{-x_0 \Theta_{\nu_k \nu_k} \Delta \tau} \hspace{1cm} (B.15)$$

where $w_j$ label the distinct values taken by $\nu$ along the path and $n_i$ the multiplicity of each value.
Appendix C. Eigenstates and operator functions of $\Theta$

In the timeless framework of section 3, the vertex expansion mostly featured matrix elements $\Theta_{\nu_m\nu_k} = \langle \nu_m | \Theta | \nu_k \rangle$. These are easy to evaluate directly from definition (2.3) of $\Theta$. In the deparameterized framework of section 4, on the other hand, the vertex expansion involves matrix elements of $\sqrt{\Theta}$. To evaluate these one needs the spectral decomposition of $\Theta$. In the first part of this appendix we construct eigenstates of $\Theta$ and discuss their relevant properties. In the second part we use these eigenstates to evaluate the matrix elements functions of $\Theta$, including $\sqrt{\Theta}$.

C.1. Eigenstates of $\Theta$

Recall that $\Theta$ is a positive self-adjoint operator on $\mathcal{H}_{\text{kin}}^{\text{grav}}$. By its definition (2.3), it follows that $\Theta$ preserves each of the three sub-spaces in the decomposition $\mathcal{H}_{\text{kin}}^{\text{grav}} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$, spanned by functions with support on $\nu < 0$, $\nu = 0$ and $\nu > 0$, respectively. In particular, $|\nu = 0\rangle$ is the unique eigenvector of $\Theta$, with eigenvalue 0, and $\mathcal{H}_0$ is one dimensional. Our first task is to solve the eigenvalue equation for a general eigenvalue $\omega_k^2$:

$$\Theta \chi_k(b) = \omega_k^2 \chi_k(b).$$  \hfill (C.1)

This task becomes simpler in the representation in which states are functions $\chi(b)$ of the variable $b$ conjugate to $\nu$:\footnote{Our normalization is different from that in [39]. The wavefunction $\tilde{\Psi}(v)$ in [39] is related to the one here by $\tilde{\Psi}(v) = \sqrt{\frac{\ell_o}{2\pi \nu}} \Psi(v)$.}

$$\chi(b) := \sqrt{\frac{\ell_o}{\pi}} \sum_{v = -\infty}^{\infty} e^{i\nu b} \frac{\Psi(v)}{\sqrt{|v|}}.$$

(C.2)

In this representation, the eigenvalue equation (C.1) takes the form of a simple differential equation

$$(\Theta \chi_k)(b) = -12\pi G \left( \sin \frac{\ell_o b}{\ell_o} \right)^2 \chi_k(b) = \omega_k^2 \chi_k(b),$$

where $A(k)$ is a normalization factor and $k \in (-\infty, \infty)$. $k = 0$ yields a discrete eigenvalue $\omega_k = 0$ and in the $\nu$ representation the eigenvector can be expressed simply as $\chi_k(\nu) = \delta_{0,\nu}$. Eigenvectors with non-zero eigenvalues can also be expressed in the $\nu$ representation by applying the inverse transformation of (C.2) to (C.4):

$$\chi_k(b) = A(k) \sqrt{\frac{\ell_o}{\pi}} \int_0^{\frac{\pi}{\ell_o}} db \ e^{-\frac{i}{2} \nu b} e^{i k \log(\tan \frac{\omega_b}{2})} \quad \text{where} \quad k \neq 0.$$  \hfill (C.5)

Let us note two properties of these eigenvectors. First, $\chi_k$ and $\chi_{-k}$ have the same eigenvalue and so the $\omega_k^2$-eigenvalue is two dimensional. Second, the vectors $\chi_0(\nu)$ we have obtained have support on both $\nu > 0$ and $\nu < 0$. However, since $\Theta$ preserves the sub-spaces $\mathcal{H}_\pm$, it is natural to seek linear combinations $e_{\nu k}(\nu)$ of $\chi_k(\nu)$ and $\chi_{-k}(\nu)$ which lie in these sub-spaces. In particular, this will simplify the problem of normalization of eigenfunctions.

Let us begin by rewriting the integral in (C.5) as a contour integral in the complex plane. Recalling that $\nu = 4\pi z$ and setting $z = e^{\rho t}$, we obtain

$$\int_0^{\frac{\pi}{\ell_o}} db \ e^{-2i\rho n} e^{i k \log(\tan \frac{\omega_b}{2})} = \frac{e^{-\pi k/2}}{\pi i} \int_0^{\pi} e^{-2\pi n - 1} \frac{1 - z}{1 + z} \ dz =: J(k, n),$$

where $J(k, n)$ is a normalization factor and $n$ is an integer. The solutions of the eigenvalue equation (C.1) are expressed in the $\nu$ representation by

$$\chi_k(\nu) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{\pi}{\ell_o}} db \ e^{-2i\rho n} e^{i k \log(\tan \frac{\omega_b}{2})} \Psi(v)$$

where $\Psi(v)$ is a wavefunction. The explicit form of $\Psi(v)$ in [39] is related to the one here by $\Psi(v) = \sqrt{\frac{\ell_o}{2\pi \nu}} \tilde{\Psi}(v)$. The eigenstates are expressed as linear combinations of $\chi_k(\nu)$:

$$\chi(\nu) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{\pi}{\ell_o}} db \ e^{-2i\rho n} e^{i k \log(\tan \frac{\omega_b}{2})} \chi_k(\nu).$$

(C.6)
where $\mathcal{C}$ is the unit semicircle in the counterclockwise direction in the upper half, $\Im z > 0$, of the complex plane. As remarked earlier, $e_k(v) = A(k)\sqrt{\ell_o}|v|/\pi J(k, v/4\ell_o)$ has support on both positive and negative values of $v = 4\ell_o n$. Now, the second independent eigenfunction $e_{-k}(v)$ with the same eigenvalue $\omega_k^2$ can be represented in a similar fashion by setting $z = -e^{ib\ell_o}$. The result is a contour integral along the unit semicircle in the counterclockwise direction in the lower half, $\Im z < 0$, of the complex plane. By combining the two integrals, we obtain a closed integral along the unit circle:

$$\frac{1}{2\pi i} \oint z^{-2n-1} \frac{1 - z}{1 + z} dz = \frac{1}{2} (e^{\pi k/2} J(k, n) + e^{-\pi k/2} J(-k, n)) =: I(k, n).$$

(C.7)

Being a linear combination of $e_k(v)$ and $e_{-k}(v)$, this $I(k, n)$ also gives an eigenfunction of $\Theta$ with the eigenvalue $\omega_k^2$. Moreover, using an elementary complex analysis, one finds that it has support only on positive $n$:

$$I(k, n) = \begin{cases} \frac{1}{12\pi^2} \frac{d^2}{db^2} |_{b=0} \left( \frac{1 - e}{1 + e} \right)^k & n \geq 0 \\ 0 & n < 0. \end{cases}$$

(C.8)

Repeating the argument but taking $z = e^{-ib\ell_o}$ and $z = -e^{ib\ell_o}$, one obtains

$$\frac{1}{2}(e^{-\pi k/2} J(k, n) + e^{\pi k/2} J(-k, n)) = \frac{1}{2\pi i} \oint z^{2n-1} \frac{1}{1 + z} dz = I(k, -n)$$

(C.9)

which has support only on negative $n$. Thus, the basis we are looking for is given by

$$e^\pm_k(v) := \frac{1}{2}(e^{\pm\pi k/2} e_k(v) + e^{\mp\pi k/2} e_{-k}(v)) = A(k) \sqrt{\pi/\ell_o} I(k, \pm v/4\ell_o).$$

(C.10)

By construction, $e^\pm_k \in \mathcal{H}_\pm$.

Next, let us calculate the normalization of these vectors. It is convenient to introduce kets $|k\pm\rangle$ such that $\langle v|k\pm\rangle = e^\pm_k(v)$. Then, it is clear that $\langle k'^{\mp}|k\mp\rangle = 0$. To calculate the nontrivial inner product, $\langle k' \pm | k\pm \rangle$, let us return to the $b$ representation. There, the functions describing the states $|k\pm\rangle$ are

$$\chi^\pm_k(b) = \frac{A(k)}{2} \left( e^{\pm\pi k/2} e^{i k \log(\tan \frac{\pi b}{\ell_o})} + e^{\mp\pi k/2} e^{-i k \log(\tan \frac{\pi b}{\ell_o})} \right)$$

(C.11)

and their inner product is given by [39]

$$\langle k' \pm | k\pm \rangle = \int_0^{\pi/\ell_o} db \ |A(k)|^2 \chi^\pm_k(b) \chi^{\mp*}_k(b),$$

(C.12)

where $|2i\partial_b|$ is the absolute value of the volume operator $\hat{v} = 2i\partial_b$. Simplification occurs because $e^\pm_k(v)$ have support only on positive/negative $v$ values. Because of this property, one can replace $|\partial_b|$ in (C.12) by $\pm \partial_b$. The calculation now reduces to a straightforward integration. The result is

$$\langle k' \pm | k\pm \rangle = |A(k)|^2 2\pi k \sinh(\pi k) \delta(k', k).$$

(C.13)

### C.2. Matrix elements for $f(\Theta)$

We will now use the eigenbasis $|k\pm\rangle$ of $\Theta$ to calculate the matrix elements $\langle 4n\ell_o | f(\Theta)| 4m\ell_o \rangle$, of the operators of the form $f(\Theta)$, for a measurable function $f$. Throughout this section, the normalization factor $A(k)$ is chosen to be unity. From the normalization condition (C.13) with $A(k) = 1$, we have the following decomposition of the identity:

$$I = \int_0^\infty \frac{dk}{2\pi k \sinh(\pi k)} \langle k+| + \langle k-| (k-) \rangle,$$

(C.14)
which can be inserted in $\langle 4n\ell_n f(\hat{\Theta}) | 4m\ell_m \rangle$. If $m$ and $n$ have different signs, the result is zero. It suffices to consider the case where both are positive. By writing $\langle 4n\ell_n | k \rangle$ in terms of derivatives (see equations (C.10) and (C.8)), one obtains

$$\langle 4n\ell_n | f(\hat{\Theta}/\Theta_1) | 4m\ell_m \rangle = 2\sqrt{mn} \frac{\Gamma(2n)}{\Gamma(2m)} \frac{d^2}{dx^2} \frac{d^2}{dt^2} F_{f(\hat{\Theta}/\Theta_1)}(1 + s - s^2 + t - t^2),$$

(C.15)

with $F_{f(\hat{\Theta})}$ being the ‘generating function’ given by

$$F_{f(\hat{\Theta})}(x) = \int_0^\infty dk \frac{f(12\pi Gk^2)}{k \sinh(\pi k)}.$$  

(C.16)

We now give the generating function for $\sqrt{\Theta}$. It is also useful (at least to check normalization factors) to write down the generating functions for operators whose matrix elements are known, namely $\Theta$ and the identity $I$. These generating functions are given by

$$F_{\sqrt{\Theta}}(x) = -i\sqrt{12\pi G} \frac{d}{dx} F_{\Theta}(x),$$  

(C.20)

$$F_{\Theta}(x) = -i\sqrt{12\pi G} \frac{d}{dx} F_{\sqrt{\Theta}(x)},$$  

(C.21)

which can be derived from (C.16).

We will conclude by noting that the matrix elements for the evolution operator $U(\varphi) = e^{i\varphi\sqrt{\Theta}}$ are easy to find. From (C.16) one sees that $F_{U(\varphi)}(x) = F_{f(e^{\sqrt{12\pi G}\varphi}x)}$.

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8 For a general $f$, integral as defined may diverge. However the divergent terms (e.g. those which are $x$-independent) do not contribute to the expression of the matrix element and can therefore be discarded. This ‘finite part extraction’ is implicit in going from (C.16) to (C.17), (C.18) and (C.19).
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