Prime forms and higher genus deformed Eisenstein series

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Abstract. Using the theory of Szegő kernel on a genus $g$ Riemann surfaces obtained as a result of the multiple $\rho$-parameter formalism of sewing of $g$ handles to the complex sphere, we derive new formulas related prime forms, theta functions, and deformed Eisenstein series. We establish recurrent formulas for genus $g$ prime forms and Szegő kernel as well as further identities. Using the above results, we introduce finally another definition of genus $g$ counterpart of genus one deformed Eisenstein series. The results obtained are then useful in computation of vertex algebra related cohomologies.

1. Introduction: The Szegő Kernel on a Riemann Surface

Consider a compact Riemann surface $\Sigma^{(g)}$ of genus $g$ with canonical homology cycle basis $a_1, \ldots, a_g, b_1, \ldots, b_g$. In general there exists $g$ holomorphic one-forms $\nu^{(g)}_i$, $i = 1, \ldots, g$ which we may normalize by, e.g., [3]

$$\oint_{a_i} \nu^{(g)}_j = 2\pi i \delta_{ij}. \quad (1)$$

The genus $g$ period matrix $\Omega^{(g)}$ is defined by

$$\Omega^{(g)}_{ij} = \frac{1}{2\pi i} \oint_{b_i} \nu^{(g)}_j, \quad (2)$$

for $i, j = 1, \ldots, g$. $\Omega^{(g)}$ is symmetric with positive imaginary part, i.e., $\Omega^{(g)} \in \mathbb{H}_g$, the Siegel upper half plane.

It is useful to introduce the normalized differential of the second kind defined by [5], [1]:

$$\omega^{(g)}(x, y) \sim \frac{dx \, dy}{(x - y)^2}, \quad (3)$$

for local coordinates $x \sim y$, with normalization $\int_{a_i} \omega^{(g)}(x, \cdot) = 0$ for $i = 1, \ldots, g$. Using the Riemann bilinear relations, one finds that $\nu^{(g)}_i(x) = \oint_{b_i} \omega^{(g)}(x, \cdot)$.

We also introduce the normalized differential of the third kind

$$\omega^{(g)}_{p_2-p_1}(x) = \int_{p_1}^{p_2} \omega^{(g)}(x, \cdot), \quad (4)$$
for which \( \oint_{\gamma} \omega^{(g)}_{p_{a-1}p_{a}} = 0 \) and \( \omega^{(g)}_{p_{a-1}p_{a}}(x) \sim \frac{(-1)^{a}}{x-p_{a}} \) \( dx \) for \( x \sim p_{a} \) and \( a = 1, 2 \).

We recall the definition of the theta function with real characteristics, e.g., [5], [1], [3]

\[
\vartheta^{(g)} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (z|\Omega^{(g)}) = \sum_{m \in \mathbb{Z}^{g}} \exp \left( i\pi (m+\alpha).\Omega^{(g)}.(m+\alpha) + (m+\alpha).(z+2\pi i\beta) \right),
\]

for \( \alpha = (\alpha_{i}), \beta = (\beta_{i}) \in \mathbb{R}^{g}, \ z = (z_{i}) \in \mathbb{C}^{g} \) and \( i = 1, \ldots, g \).

There exists a (non-singular and odd) character \[ \pi \] such that [5], [1]

\[
\vartheta^{(g)} \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0|\Omega) = 0, \quad \partial_{z_{i}}\vartheta^{(g)} \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0|\Omega) \neq 0.
\]

Let

\[
\zeta(x) = \sum_{i=1}^{g} \partial_{z_{i}}\vartheta^{(g)} \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (0|\Omega^{(g)})\nu^{(g)}_{i}(x),
\]

a holomorphic one-form, and let \( \zeta(x)^{\frac{1}{2}} \) denote the form of weight \( \frac{1}{2} \) on the double cover \( \Sigma \) of \( \Sigma \). We also refer to \( \zeta(x)^{\frac{1}{2}} \) as a (double-valued) \( \frac{1}{2} \)-form on \( \Sigma \). We define the prime form \( E^{(g)}(x, y) \) by

\[
E^{(g)}(x, y) = \frac{\vartheta^{(g)} \left[ \begin{array}{c} \gamma \\ \delta \end{array} \right] (j_{x}^{-1} \nu^{(g)}_{i}) (x)}{\zeta(x)^{2} \zeta(y)^{\frac{1}{2}}} \sim (x - y)dx^{-\frac{1}{2}}dy^{-\frac{1}{2}} \quad \text{for } x \sim y,
\]

where \( j_{x}^{-1} \nu^{(g)}_{i} = (j_{x}^{-1} \nu_{i}) \in \mathbb{C}^{g} \). \( E^{(g)}(x, y) = -E^{(g)}(y, x) \) is a holomorphic differential form of weight \( (-\frac{1}{2}, -\frac{1}{2}) \) on \( \Sigma \times \Sigma \). Note that our definition differs from that of refs. [5], [1] by a factor of \( -1 \).

The normalized differentials of the second and third kind can be expressed in terms of the prime form [5]

\[
\omega^{(g)}(x, y) = \partial_{x}\partial_{y} \log K^{(g)}(x, y) dx dy,
\]

\[
\omega^{(g)}_{p_{a}q}(x) = \partial_{x} \log K^{(g)}(x, p_{a}) dx.
\]

Conversely, we can also express the prime form in terms of \( \omega^{(g)}_{p_{a}q} \) by [2]

\[
E^{(g)}(x, y) = \lim_{p \to x, \ q \to y} \left[ \sqrt{(x - p)(q - y)} \exp \left( -\frac{1}{2} \int_{y}^{x} \omega^{(g)}_{p_{a}q} \right) \right] dx^{-\frac{1}{2}} dy^{-\frac{1}{2}}.
\]

We define the Szegő kernel [10], [1] for \( \vartheta^{(g)} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (0|\Omega^{(g)}) \neq 0 \) as follows

\[
S^{(g)} \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (x, y|\Omega^{(g)}) = \frac{\vartheta^{(g)} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (j_{x}^{-1} \nu^{(g)}_{i})}{\vartheta^{(g)} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (0|\Omega^{(g)})} \frac{1}{E^{(g)}(x, y)},
\]

where \( \theta = (\theta_{i}), \ \phi = (\phi_{i}) \in U(1)^{n} \) for \( \theta_{j} = -e^{-2\pi i\beta_{j}}, \ \phi_{j} = -e^{2\pi i\alpha_{j}}, \ j = 1, \ldots, g \). The Szegő kernel has multipliers along the \( \alpha_{i} \) and \( \beta_{j} \) cycles in \( x \) given by \( -\phi_{i} \) and \( -\theta_{j} \) respectively and is a meromorphic \((\frac{1}{2}, \frac{1}{2})\)-form on \( \Sigma \times \Sigma \) satisfying:

\[
S^{(g)} \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (x, y) \sim \frac{1}{x - y} dx^{\frac{1}{2}} dy^{\frac{1}{2}} \quad \text{for } x \sim y,
\]

\[
S^{(g)} \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (x, y) = -S^{(g)} \left[ \begin{array}{c} \theta^{-1} \\ \phi^{-1} \end{array} \right](y, x),
\]

\[\text{for } x \sim y,\]

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\[\text{respectively and is a meromorphic } \left( \frac{1}{2}, \frac{1}{2} \right) \text{-form on } \Sigma \times \Sigma \text{ satisfying:}.
\]

\[\text{for } x \sim y,\]

\[\text{respectively and is a meromorphic } \left( \frac{1}{2}, \frac{1}{2} \right) \text{-form on } \Sigma \times \Sigma \text{ satisfying:}.
\]
where $\theta^{-1} = (\varphi^{-1}_i)$ and $\phi^{-1} = (\varphi^{-1}_i)$. Note that the skew-symmetry property (14) implies $S^{(g)}_{\varphi}(x, y)$ has multipliers along the $a_i$ and $b_j$ cycles in $y$ given by $-\phi^{-1}_i$ and $-\theta^{-1}_j$ respectively.

For a Riemann surface of genus one described by an oriented torus $\mathbb{C}/\Lambda$ for lattice $\Lambda = 2\pi i (\mathbb{Z} \tau + \mathbb{Z})$ for $\tau \in \mathbb{H}_1$, the genus one prime form is

$$E^{(1)}(x, y) = K^{(1)}(x - y, \tau) dx \frac{1}{2} dy \frac{1}{2}, \quad K^{(1)}(z, \tau) = \frac{\theta_1(z, \tau)}{\partial_z \theta_1(0, \tau)}, \quad (15)$$

for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}_1$ and where $\theta_1(z, \tau) = \frac{\vartheta \left[ \frac{z}{\tau} \right]}{\vartheta \left[ \frac{1}{\tau} \right]}(z, \tau)$.

For $(\theta, \phi) \neq (1, 1)$ with $\theta = -e^{-2\pi i \beta}$ and $\phi = -e^{2\pi i \alpha}$ the genus one Szegő kernel is

$$S^{(1)}_{\varphi} = P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (x, y|\tau) = \frac{1}{z} \sum_{n \geq 1} E_n \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) z^{n-1}, \quad (16)$$

where

$$P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau) = \frac{\partial}{\partial \alpha} \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (z, \tau) \frac{1}{K(z, \tau)} - \sum_{k \in \mathbb{Z}} \frac{q_z^{k+\lambda}}{1 - \theta^{-1} q^{k+\lambda}},$$

is a deformed Weierstrass function [9] for $q_z = e^z$ and with $\phi = \exp(2\pi i \lambda)$ for $0 \leq \lambda < 1$.

We also have a Laurent expansion [9]

$$E_n \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) = -\frac{B_n(\lambda)}{n!} + \frac{1}{(n-1)!} \sum_{r \geq 0} \frac{(r + \lambda)^{n-1} \theta^{-1} q^{r+\lambda}}{1 - \theta^{-1} q^{r+\lambda}} + \frac{(-1)^n}{(n-1)!} \sum_{r \geq 1} \frac{(r - \lambda)^{n-1} \theta q^{-r-\lambda}}{1 - \theta q^{-r-\lambda}}, \quad (18)$$

for $n \geq 1$ and where $B_n(\lambda)$ is the Bernoulli polynomial defined by

$$\frac{q_z^\lambda}{q_z - 1} = \frac{1}{z} + \sum_{n \geq 1} \frac{B_n(\lambda)}{n!} z^{n-1}.$$

In particular, we have

$$\frac{\vartheta^{(1)}(z, \tau)}{\vartheta^{(1)}(0, \tau)} = -\sum_{k \in \mathbb{Z}} \frac{q_z^{k+\lambda}}{1 - \theta^{-1} q^{k+\lambda}} \left( \frac{1}{z} - \sum_{n \geq 1} E_n \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) z^{n-1} \right),$$

For $(\theta, \phi) = (1, 1)$ and $n \geq 2$ the deformed Eisenstein series reduce to the standard elliptic Eisenstein series with $E_n(\tau) = 0$ for odd. Using the Laurent expansion (17) we find

$$P_1 \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (x - y, \tau) = \frac{1}{x - y} + \sum_{k, l \geq 1} C \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (k, l)x^{k-1}y^{l-1}, \quad (19)$$

where for $k, l \geq 1$ we define

$$C \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (k, l, \tau) = (-1)^l \binom{k + l - 2}{k - 1} E_{k+l-1} \left[ \begin{array}{c} \theta \\ \phi \end{array} \right] (\tau), \quad (20)$$

for deformed Eisenstein series (18).
2. Prime form formula

In [8], in $\rho$-formalism [12], $\rho \in \mathbb{C}$ of self-sewing of Riemann surfaces, for $a, b = 1,2$ and $k, l = 1,2, \ldots$ they define weighted moments

$$Y_{ab}^{(g)}(k, l) = \frac{\rho^{k+l}/2}{r^l} \frac{1}{(2\pi i)^2} \oint_{C_a(u)} \oint_{C_b(v)} u^{-k} v^{-l} \omega^{(g+1)}(u, v).$$

(21)

Note that $Y_{ab}^{(g)}(k, l) = Y_{ba}^{(g)}(l, k)$. They also define $Y^{(g)} = \left( Y_{ab}^{(g)}(k, l) \right)$ to be the infinite matrix indexed by the pairs $a, k$ and $b, l$. They define a set of holomorphic one-forms on $\hat{\Sigma}^{(g)}$

$$a_a(k, x) = \frac{\rho^{k/2}}{2\pi i} \oint_{C_a(z_a)} z_a^{-k} \omega^{(g)}(x, z_a),$$

(22)

and define $a(x) = (a_a(k, x))$ and $\bar{a}(x) = (\bar{a}_a(k, x))$ to be the infinite row vectors indexed by $a, k$. In [8] they prove

$$\omega^{(g+1)}(x, y) = \omega^{(g)}(x, y) - a(x) \left( I - Y^{(g)} \right) \bar{a}(y)^T.$$

(23)

It follows from the Fay’s representation (11) [2] of the prime form in terms of $\omega$ that

$$K^{(g+1)}(x, y, \Omega^{(g+1)}) = K^{(g)}(x, y, \Omega^{(g)}) e^{-\frac{1}{2} b_a(I - Y^{(g)})}a_a(\rho) \bar{a}_a^T.$$

(24)

Here $b_a(x, y; k) = \int_x^y a_a(\cdot, k)$, and

$$a_a(x) \left( I - Y^{(g)}_{\bar{a}a} \right) (\rho) a_{\bar{a}}^T(y) = \sum_{k,l \geq 1} a_a(x, k) \left( I - Y^{(g)}_{\bar{a}a} \right)(k, l, \rho) a_a(y, l),$$

with $a_a(x, k)$ a certain one-form [7] on the initial Riemann surface $\hat{\Sigma}^{(g_0)}$ of genus $g$. Here \( \left( I - Y^{(g)}_{\bar{a}a} \right)(k, l, \rho) \) is an infinite matrix determined from genus $g$ data (see [6] for details). Thus from (24) for the multiple sewing of the sphere $\Sigma^{(0)}$ to form a genus $g$ Riemann surface and $\rho_i \in \mathbb{C}$, we obtain

Proposition 2.1

$$K^{(g)}(x, y, \Omega^{(g)}) = K^{(1)}(x - y, \tau) \prod_{i=1}^{g-1} e^{-\frac{1}{2} b_a(i - Y^{(i)})}a_a(\rho_i) (\bar{a}_a^i)^T = K^{(1)}(x - y, \tau) Y^{(g)},$$

(25)

where $Y^{(i)}$ corresponds to genus $i$ Riemann surface as above.

3. Recurrent formula for genus $g$ prime forms and Szegő kernel

We have proved in [11] that the Szegő kernel $S^{(g+1)}$ of the genus $g + 1$ Riemann surface obtained as a results of self-sewing a genus $g$ Riemann surface is holomorphic in $\rho$ for $|\rho| < r_1 r_2$ with

$$S^{(g+1)}(x, y) = S^{(g)}(x, y) + D^{(g)},$$

(26)

$$D^{(g)} = \xi h(x) D^\theta \left( I + \xi Y^{(g)} D^\theta \right) \bar{h}(y)^T,$$
for infinite diagonal matrix $D^g(k, l) = \begin{bmatrix} \frac{\theta^{-1}}{2} & 0 \\ -\theta^{-1} & 0 \end{bmatrix} \delta(k, l)$. for $x, y \in \Sigma^g$ where $S^{(g)}_{\kappa}(x, y)$ is defined as follows: For $\kappa \neq -\frac{1}{2}$

$$
S^{(g)}_{\kappa}(x, y) = \frac{(U^{(g)}(x,y))^\kappa}{E^{(g)}(x,y)\theta^{(g)}} \left[ \frac{\alpha^{(g)}}{\beta^{(g)}} \right] \left( \begin{array}{c} \int^x_y \nu^{(g)} + \kappa z_{p_1,p_2} |\Omega^{(g)}| \\ \int^y_x \nu^{(g)} - \frac{1}{2} \kappa z_{p_1,p_2} |\Omega^{(g)}| \end{array} \right),
$$

(27)

where

$$
U^{(g)}(x, y) = \frac{E^{(g)}(x,p_2)E^{(g)}(y,p_1)}{E^{(g)}(x,p_1)E^{(g)}(y,p_2)}
$$

(28)

for prime form $E^{(g)}$ and where $z_{p_1,p_2} = \int_{p_1}^{p_2} \nu^{(g)}$, for holomorphic 1-forms $\nu^{(g)}$. For $\kappa = -\frac{1}{2}$ then $S^{(g)}_{-\frac{1}{2}}(x, y)$ is given by

$$
S^{(g)}_{-\frac{1}{2}}(x, y) = \left( \frac{(U^{(g)}(x,y))^{-\frac{1}{2}}}{E^{(g)}(x,y)\theta^{(g)}} \left[ \frac{\alpha^{(g)}}{\beta^{(g)}} \right] \left( \begin{array}{c} \int^x_y \nu^{(g)} + \frac{1}{2} z_{p_1,p_2} |\Omega^{(g)}| \\ \int^y_x \nu^{(g)} - \frac{1}{2} z_{p_1,p_2} |\Omega^{(g)}| \end{array} \right) \right)^{-1}.
$$

(29)

We use also half-order differentials [11]

$$
h_a(k, x) = h_a \left[ \begin{array}{c} \theta^{(g)} \\ \phi^{(g)} \end{array} \right] (\kappa; k, x) = \frac{\rho_{a}^{\frac{1}{2} (k_a - \frac{1}{2})}}{2\pi i} \oint_{C_a(y_a)} y^{-k_a} S^{(g)}_{\kappa}(x, y_a) dy_a^{\frac{1}{2}},
$$

(30)

$$
h_a(k, y) = h_a \left[ \begin{array}{c} \theta^{(g)} \\ \phi^{(g)} \end{array} \right] (\kappa; k, y) = \frac{\rho_{a}^{\frac{1}{2} (k_a - \frac{1}{2})}}{2\pi i} \oint_{C_a(x_a)} x^{-k_a} S^{(g)}_{\kappa}(x_a, y) dx_a^{\frac{1}{2}}.
$$

(31)

and let $h(x) = (h_a(k, x))$ and $h(y) = (h_a(k, y))$ denote the infinite row vectors indexed by $a, k$. These are related by skew-symmetry with

$$
h_a \left[ \begin{array}{c} \theta^{(g)} \\ \phi^{(g)} \end{array} \right] (\kappa; k, x) = -\bar{h}_a \left[ \begin{array}{c} (\theta^{(g)})^{-1} \\ (\phi^{(g)})^{-1} \end{array} \right] (-\kappa; k, x).
$$

(32)

These moments can be inverted to obtain

$$
S^{(g)}_{\kappa}(x, y_a) = \sum_{k \geq 1} \rho^{-\frac{1}{2} (k_a - \frac{1}{2})} h_a(k, x) y_a^{k_a - 1} dy_a^{\frac{1}{2}}
$$

(33)

$$
S^{(g)}_{\kappa}(x_a, y) = \sum_{k \geq 1} \rho^{-\frac{1}{2} (k_a - \frac{1}{2})} x_a^{k_a - 1} h_a(k, y) dx_a^{\frac{1}{2}}.
$$

(34)

From the sewing relation [12] we have $dx_a^{\frac{1}{2}} = (-1)\xi \rho^{\frac{1}{2}} dz_a^{\frac{1}{2}}$, for $\xi \in \{\pm \sqrt{-1}\}$.

On the other hand

$$
\Theta^{(g+1)}(x, y) = \frac{\Theta^{(1)}(x, y) W^{(g)}(x, y) + \mathcal{D}^{(g)}}{E^{(g+1)}(x, y)},
$$

(35)
Now let us express it in terms of prime forms

Proposition 3.2

Thus (assuming $E^{(g+1)}(x, y)E^{(g)}(x, y) \neq 0$) we obtain

Proposition 3.1

Then it follows

Depending how we express the prime form $E^{(1)}(x, y)$ in (35) we obtain identities for theta-functions (15), differentials (11) sum of Eisenstein series (17), or $q$-series (17).

Note that

Thus

We then recurrently obtain

Proposition 3.2

Now let us express it in terms of prime forms $E^{(g)}$ and $E^{(1)}$. We obtain

from which we obtain the following identities.

For relation $E^{(g)}(x, y)$ and $E^{(1)}(x, y)$ again

$$\Theta^{(g)}(x, y)E^{(1)}(x, y) = E^{(g)}(x, y) \left( \Theta^{(1)}(x, y) \prod_{i=1}^{g-1} W^{(i)}(x, y) \right)$$
thus
\[ E^{(g)}(x, y) = \frac{\Theta^{(g)}(x, y) E^{(1)}(x, y)}{\Theta^{(1)}(x, y) \prod_{i=1}^{g-1} W^{(i)}(x, y) + E^{(1)}(x, y) \sum_{j=1}^{g-1} D^{(j)} \prod_{k=j+1}^{g-1} W^{(k)}(x, y)}. \]

Note that this is equivalent to
\[ \Theta^{(g)}(x, y) = \mathcal{Y} \left[ \Theta^{(1)}(x, y) \prod_{i=1}^{g-1} W^{(i)}(x, y) + E^{(1)}(x, y) \sum_{j=1}^{g-1} D^{(j)} \prod_{k=j+1}^{g-1} W^{(k)}(x, y) \right], \]
or
\[ E^{(1)}(x, y) = \frac{\Theta^{(g)}(x, y) \mathcal{Y}^{-1} - \Theta^{(1)}(x, y) \prod_{i=1}^{g-1} W^{(i)}(x, y)}{\sum_{j=1}^{g-1} D^{(j)} \prod_{k=j+1}^{g-1} W^{(k)}(x, y)}. \]

Using (35) we obtain the identity
\[ \left[ \Theta^{(g+1)}(x, y) \mathcal{Y}^{-1} - \Theta^{(1)}(x, y) W^{(g)}(x, y) \right] \mathcal{Y}^{(g)} = \frac{\Theta^{(g)}(x, y) \left( \mathcal{Y}^{(g)} \right)^{-1} - \Theta^{(1)}(x, y) \prod_{i=1}^{g-1} W^{(i)}(x, y)}{\sum_{j=1}^{g-1} D^{(j)} \prod_{k=j+1}^{g-1} W^{(k)}(x, y)} D. \]

4. Higher genus deformed Eisenstein series
In [4] they introduce a version of genus two counterpart of Eisenstein series. Here we give another possible definition of the deformed Eisenstein series in terms of the multiple-sewing construction. Note that
\[ S^{(1)} \left[ \frac{\theta}{\phi} \right] (x, y; \tau) = \frac{\varphi^{(1)}}{\varphi^{(1)}} \left[ \frac{\alpha}{\beta} \right] \left( x \tau, \nu \right) \left[ \frac{1}{z} - \sum_{n \geq 1} E_n \left[ \frac{\theta}{\phi} \right] (\tau) z^{n-1} \right] dx^{-\frac{1}{2}} dy^{-\frac{1}{2}}, \quad (38) \]
i.e., the torus Szegő kernel can be expressed via deformed Eisenstein series. Using (37) we obtain by expanding the right hand side
\[ S^{(g)}(x, y) = \left[ \frac{1}{x-y} + \sum_{k,l \geq 1} C \left[ \frac{\theta}{\phi} \right] (k, l) x^{k-1} y^{l-1} \right] \prod_{i=1}^{g-1} W^{(i)}(x, y) dx^{-\frac{1}{2}} dy^{-\frac{1}{2}} + \sum_{j=1}^{g-1} D^{(j)} \prod_{k=j+1}^{g-1} W^{(k)}(x, y) \]
\[ = \frac{\varphi^{(g)}}{\varphi^{(g)}} \left[ \frac{\alpha}{\beta} \right] \left( x \tau, \nu \right) \left[ A(x, y) - \sum_{n, m \geq 1} E_{n,m}^{(g)} \left[ \frac{\theta}{\phi} \right] (\Omega^{(g)}) x^{n-1} y^{m-1} \right], \quad (39) \]

where
\[ A^{(g)}(x, y) = \frac{\varphi^{(g)}}{\varphi^{(g)}} \left[ \frac{\alpha}{\beta} \right] \left( x \tau, \nu \right) \prod_{i=1}^{g-1} W^{(i)}(x, y) dx^{-\frac{1}{2}} dy^{-\frac{1}{2}}, \]
and by $E_{n,m}^{(g)}\left[\frac{\theta}{\phi}\right](\Omega^{(g)})$ we call the elements of the higher genus Eisenstein series

$$
\sum_{n,m \geq 1} E_{n,m}^{(g)}\left[\frac{\theta}{\phi}\right](\Omega^{(g)}) \cdot x^{n-1}y^{m-1} = -\frac{\phi^{(g)}}{\psi^{(g)}}\left[\begin{array}{c} \alpha \\ \beta \end{array}\right] (z,\Omega^{(g)}) \left[\begin{array}{c} g-1 \\ g-1 \end{array}\right] \sum_{j=1}^{g-1} D^{(j)} \prod_{k=j+1}^{g} W^{(k)}(x,y) \\
+ \sum_{k,l \geq 1} C\left[\begin{array}{c} \theta \\ \phi \end{array}\right] (k,l)x^{k-1}y^{l-1} \prod_{i=1}^{g-1} W^{(i)}(x,y)dx^{-1}dy^{-1} \right].
$$

(40)

5. Acknowledgments
Research of the author was supported by the GACR project 18-00496S and RVO: 67985840.

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