On Extension of Functors

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Abstract

A.Chigogidze defined for each normal functor on the category Comp an extension which is a normal functor on the category Tych. We consider this extension for any functor on the category Comp and investigate which properties it preserves from the definition it preserves from the definition of normal functor. We investigate as well some topological properties of such extension.

Introduction. The general theory of functors acting in the category Comp of compact Hausdorff spaces (compacta) and continuous mappings was founded by E.V. Shchepin. He distinguished some elementary properties of such functors and defined the notion of normal functor that has become very fruitful. The class of normal functors includes many classical constructions: the hyperspace exp, the functor of probability measures $P$, the power functor and many other functors (see [13],[9] for more details). But some important functors do not satisfy some of the properties from the Shchepin list. Omitting some properties we obtain wider classes of functors such as weakly normal functors and almost normal functors.

The properties from the definition of normal functor could be easily generalized for the functors on the category Tych of Tychonov spaces and continuous maps. Let us remark that Tych contains Comp as a subcategory. A.Chigogidze defined for each normal functor on the category Comp an extension which is a normal functor on the category Tych. This extension could be considered for any functor on the category Comp. But the situation is more complicated for wider classes of functors. For example, the extension of the projective power functor (which is weakly normal) does not preserve embeddings, which makes such extension useless (see for
However, if we apply the Chigogidze extension to such weakly normal functors as the functor $O$ of order-preserving functionals, the functor $G$ of inclusion hyperspaces, the superextension, we obtain functors on the category $Tych$ which preserve embeddings.

The main aim of this paper is to investigate which properties from the definition of normal functor are preserved by Chigogidze extension, specially we concentrate our attention on the preserving of embeddings. The results devoted to this problem are contained in Section 2. We define in this section the 1-preimages preserving property which is crucial for preserving of embeddings. In Section 3 we consider which functors have the 1-preimages preserving property.

T.Banakh and R.Cauty obtained topological classification of the Chigogidze extension of the functor of probability measures for separable metric spaces. We generalize this result for convex functors in Section 4.

1. All spaces are assumed to be Tychonov, all mappings are continuous. All functors are assumed to be covariant. In the present paper we will consider functors acting in two categories: the category $Tych$ and its subcategory $Comp$.

Let us recall the definition of normal functor. A functor $F : Comp \to Comp$ is called monomorphic (epimorphic) if it preserves embeddings (surjections). For a monomorphic functor $F$ and an embedding $i : A \to X$ we shall identify the space $F(A)$ and the subspace $F(i)(F(A)) \subset F(X)$.

A monomorphic functor $F$ is said to be preimage-preserving if for each map $f : X \to Y$ and each closed subset $A \subset Y$ we have $(F(f))^{-1}(F(A)) = F(f^{-1}(A))$.

For a monomorphic functor $F$ the intersection-preserving property is defined as follows: $F(\cap\{X_\alpha \mid \alpha \in A\}) = \cap\{F(X_\alpha) \mid \alpha \in A\}$ for every family $\{X_\alpha \mid \alpha \in A\}$ of closed subsets of $X$.

A functor $F$ is called continuous if it preserves the limits of inverse systems $S = \{X_\alpha, p_\alpha, A\}$ over a directed set $A$. Let us also note that for any continuous functor $F : Comp \to Comp$ the map $F : C(X,Y) \to C(FX,FY)$ (the space $C(X,Y)$ is considered with the compact-open topology) is continuous.

Finally, a functor $F$ is called weight-preserving if $w(X) = w(F(X))$ for every infinite $X \in Comp$.

A functor $F$ is called normal [15] if it is continuous, monomorphic, epimorphic, preserves weight, intersections, preimages, singletons and the empty space. A functor $F$ is said to be weakly normal (almost normal) if it satisfies all the properties from the definition of a normal functor excepting perhaps the preimage-preserving property (epimorphism)(see [13] for more details).

Similarly, one can define the same properties for a functor $F : Tych \to Tych$ with the only difference that the property of preserving surjections is replaced by the property of sending $k$-covering maps to surjections (recall that $f : X \to Y$ is a $k$-covering map if for any compact set $B \subset Y$ there exists a compact set $A \subset X$ with $f(A) = B$) (see [13], Def.2.7.1).

A.Chigogidze defined an extension construction of a functor in $Comp$ onto $Tych$ the following way [6]. For any normal functor $F : Comp \to Comp$ and any $X \in Tych$ the
space $F_\beta(X) = \{a \in F(\beta X) | \text{there exists a compact set } A \subset X \text{ with } a \in F(A)\}$ is considered with the topology induced from $F(\beta X)$, where $\beta X$ is the Stone-Cech compactification of the space $X$. Next, given any continuous mapping $f : X \to Y$ between Tychonov spaces, put $F_\beta(f) = F(\beta f)|_{F_\beta(X)}$. Then $F_\beta$ forms a covariant functor in the category $Tych$. Chigogidze showed that in case $F$ is normal, the functor $F_\beta$ is also normal.

2. Let us modify the Chigogidze construction for any functor $F : Comp \to Comp$. For $X \in Tych$ we put $F_\beta(X) = \{a \in F(\beta X) | \text{there exists a compact set } A \subset X \text{ with } a \in F(i_A)(F(A))\}$ where by $i_A$ we denote the natural embedding $i_A : A \hookrightarrow X$ (we do not assume that the map $F(i_A)$ is an embedding). Evidently $F_\beta$ preserves empty set and one-point space iff $F$ does.

Now we consider the problem when $F_\beta$ preserves embeddings. Extension of any normal functor preserves embeddings, but, if we drop the preimage preserving property, the situation could be different. However, the examples from the introduction show that the preimage-preserving property is not necessary. We define some weaker property which will give us a necessary and sufficient condition.

**Definition 1.** We say that a monomorphic functor $F : Comp \to Comp$ preserves 1-preimages, if for any $f : X \to Y$, where $X,Y \in Comp$, any closed $A \subset Y$ such that $f|_{f^{-1}(A)}$ is a homeomorphism, we have that $(Ff)^{-1}(FA) = F(f^{-1}(A))$. (Let us remark it is equivalent to the condition that the map $Ff | (Ff)^{-1}(FA)$ is a homeomorphism.)

Let us note that this definition was independently introduced by T.Banakh and A.Kucharski [3].

**Proposition 1.** If $F$ is a monomorphic functor that preserves 1-preimages in the class of open mappings, then $F$ preserves 1-preimages.

**Proof.** Take any mapping $f : X \to Y$ such that $f|_{f^{-1}(A)}$ is a homeomorphism for some closed subset $A \subset Y$. Let $i_1 : X \to X \times Y$ be an embedding defined by the formula $i_1(x) = (x, f(x))$. Denote $Z = X \times Y/\varepsilon$, where the relation $\varepsilon$ is given by $\varepsilon = \{\text{pr}_Y^{-1}(a) | a \in A\}$ ($\text{pr}_Y : X \times Y$ is the respective projection). Let $q : X \times Y \to Z$ be the quotient mapping. The map $h : Z \to Y$ given by the conditions $h(z) = y$ for any $z = (x,y) \in Z \setminus q(X \times A)$ and $h(z) = a$ for any $z = q(\text{pr}_Y^{-1}(a))$, $a \in A$, is open and satisfies the following two conditions: $\text{pr}_Y = h \circ q$, $h|_{h^{-1}(A)}$ is a homeomorphism. Apparently, the map $i = q \circ i_1$ is an embedding, moreover, $h \circ i = f$. Since $F$ preserves 1-preimages in the class of open mappings, we have $(Fh)^{-1}(FA) = F(h^{-1}(A))$, which gives us the equality $(Ff)^{-1}(FA) = F(f^{-1}(A))$.

**Proposition 2.** If $F$ is a monomorphic functor that preserves 1-preimages, then $F_\beta$ preserves embeddings.

**Proof.** Take any embedding $f : X \to Y$. Then the map $F_\beta(f)$ is closed as the restriction of a closed map onto a full preimage and, moreover, injective, hence an embedding.
For any \( X \in \text{Tych} \) and any its compactification \( bX \) we can define \( F_b(X) = \{ a \in F(bX) \mid a \in F(A) \} \subset F(bX) \) and consider it with the respective subspace topology.

**Corollary 1.** If \( F \) is a monomorphic, 1-preimage-preserving functor, then \( F_\beta(X) \cong F_b(X) \) for any Tychonov space \( X \) and its compactification \( bX \).

**Proposition 3.** If \( F \) is monomorphic, preserves 1-preimages and weight, then \( F_\beta \) preserves weight.

**Proof.** The statement follows from the previous corollary and the fact that for any \( X \in \text{Tych} \) there exists its compactification \( bX \) which has the same weight as \( X \). \( \square \)

As the following proposition shows, the reverse implication to that of Proposition 2 also holds.

**Proposition 4.** Let \( F \) be a continuous functor such that \( F_\beta \) preserves embeddings. Then \( F \) preserves 1-preimages.

**Proof.** Assume the contrary. Then there exist a map \( f : X \to Y \) and a closed subset \( A \subset Y \) such that \( f|_{f^{-1}(A)} \) is a homeomorphism and \( Ff^{-1}(FA) \neq F(f^{-1}(A)) \). Hence we can choose \( \nu \in FA \) and \( \mu \in FX \setminus F(f^{-1}(A)) \) such that \( Ff(\mu) = \nu \). We will construct a space \( S \in \text{Tych} \) and its compactification \( \gamma S \) such that the map \( F_\beta(id_S) : F_\beta(S) \to F_\beta(\gamma S) \equiv F(\gamma S) \) is not an embedding, where \( id_S : S \to (\gamma S) \) is an identity embedding.

First put \( Z = X \times \alpha \mathbb{N} \), where the space of natural numbers \( \mathbb{N} \) is considered with the discrete topology and \( \alpha \mathbb{N} = \mathbb{N} \cup \{ \xi \} \) is the one-point compactification of \( \mathbb{N} \). Define a continuous function \( g : Z \to Y \) by \( g(x, n) = f(x) \) for any \( x \in X \), \( n \in \alpha \mathbb{N} \). Let \( T = Z/\varepsilon \) be a quotient space, where \( \varepsilon \) is an equivalence relation defined by its classes of equivalence \( \{ \{ x \} | x \in (X \setminus A) \times \mathbb{N} \} \cup \{ g^{-1}(y) \cap X \times \{ \xi \} | y \in Y \setminus A \} \cup \{ \{ a \} \times \alpha \mathbb{N} | a \in A \} \). By \( q : Z \to T \) we denote the respective quotient mapping. Then the map \( h : T \to Y \) defined by the equality \( g = h \circ q \) is continuous. The set \( D = q(X \times \{ \xi \}) \) is compact as a continuous image of a compact set and moreover \( h|_D \) is one-to-one, hence a homeomorphism between \( D \) and \( Y \). We denote by \( j : Y \to T \) the inverse embedding. Also, for any \( n \in \mathbb{N} \) the space \( S_n = q(X \times \{ n \}) \) is homeomorphic to \( X \) and we denote by \( j_n : X \to T \) the inverse embedding. Then we have \( h \circ j_n = f \). Finally note that \( T \) is a compactification of the space \( S = T \setminus q((X \setminus A) \times \{ \xi \}) \).

Put \( \mu_n = F(j_n)(\mu) \) for \( n \in \mathbb{N} \). The sequence \( j_n \) converges to \( j \circ f \) in the space \( C(X, T) \). Since \( F \) is continuous, the sequence \( F(j_n) \) converges to \( F(j \circ f) \) in the space \( C(FX, FT) \). Hence the sequence \( \mu_n \) converges to \( F(j \circ f)(\mu) = F(j)(\nu) \in F(q(A \times \alpha \mathbb{N})) \).

Now consider \( F_\beta(S) \) as a subspace of \( F(\beta S) \). There exists a map \( s_1 : S \to X \) such that \( s_1 \circ j_n = id_X \). Let \( s : \beta S \to X \) be the extension of \( s_1 \). Then \( Fs(\mu_n) = \mu \notin F(f^{-1}(A)) \). Then the sequence \( \mu_n \) does not converge to any element of \( F(q(A \times \alpha \mathbb{N})) \). The proposition is proved. \( \square \)
Propositions 2 and 4 yield the following

**Theorem 1.** For any continuous monomorphic functor $F$ the functor $F_\beta$ preserves embeddings if and only if $F$ preserves $1$-preimages.

The proof of the following proposition is a routine checking and we omit it.

**Proposition 5.** Let $F : \text{Comp} \to \text{Comp}$ be a functor.

1) if $F$ preserves embeddings, $1$-preimages and intersections then $F_\beta$ preserves intersections;
2) if $F$ preserves embeddings and preimages then $F_\beta$ preserves preimages;
3) if $F$ preserves surjections then $F_\beta$ sends $k$-covering maps to surjections;

Now let us consider continuity of the Chigogidze extension. The following example shows that in the absence of the preimage-preserving property of the functor $F$, it is difficult to speak of continuity of $F_\beta$, since even the extension of such known weakly normal functor as $G$ does not possess it.

**Example.** Let us define the inclusion hyperspace functor $G$. Recall that a closed subset $A \in \exp^2 X$, where $X \in \text{Comp}$ is called an inclusion hyperspace, if for every $A \in \mathcal{A}$ and every $B \in \exp X$ the inclusion $A \subset B$ implies $B \in \mathcal{A}$. Then $GX$ is the space of all inclusion hyperspaces with the induced from $\exp^2 X$ topology. For any map $f : X \to Y$ define $Gf : GX \to GY$ by $Gf(A) = \{B \in \exp Y | f(A) \subset B \text{ for some } A \in \mathcal{A}\}$. The functor $G$ is weakly normal (see [13] for more details). In the next section we will see that the functor $G$ preserves $1$-preimages.

Let us show that the functor $G_\beta$ is not continuous. Consider the following inverse system. For any $n \in \mathbb{N}$ put $X_n = \mathbb{N} \times \{1, ..., n\}$ (here the spaces $\mathbb{N}$ and $\{1, ..., n\}$ are considered with the discrete topology). Define $p^m_n : X_m \to X_n$, where $m \geq n$, the following way: $p^m_n(x, k) = (x, \min\{k, n\})$. We obtained the inverse system $S = \{X_m, p^m_n, \mathbb{N}\}$. Then the limit space $X = \lim S$ is homeomorphic to the space $\mathbb{N} \times A$ (here $A = \alpha \mathbb{N} = \mathbb{N} \cup \{\xi\}$ is the one-point compactification of $\mathbb{N}$, i.e. a convergent sequence; also we put $\xi$ to be greater than any natural number), and the limit projections $p_n : X \to X_n$ can be given by $p_n(x, k) = (x, \min\{k, n\})$, $k \in A$. The continuity of $G_\beta$ means that $\lim G_\beta(p_n) : G_\beta(\lim S) \to \lim G_\beta(S)$ is a homeomorphism. Here both $G_\beta(\lim S)$ and $\lim G_\beta(S)$ can be thought as subspaces of $G(bX)$, where $b$ is a compactification of $X$ with the property $bX = \lim \beta S$.

Now we will construct $K \in \lim G_\beta(S)$ which does not belong to $\lim G_\beta(p_n)(G_\beta(\lim S))$. Consider the space $X$ imbedded into its compactification $bX$. For any $n \in A \setminus \{\xi\}$ put $K_n = \{1, ..., n\} \times \{n\}$. If we want to obtain a closed family of sets, the set $K_\xi = \overline{\mathbb{N}} \times \{\xi\}$ must be added to the family $\widetilde{K} = \{K_n\}_{n \in \mathbb{N}}$. Now put $K = \{B \subset bX | K_n \subset B \text{ for some } n \in A\}$. Then $K \in \lim G_\beta(S)$. However, there is apparently no element $C \in G_\beta(\lim S)$ with $\lim G_\beta(p_n)(C) = K$. Hence, $\lim G_\beta(p_n)$, being not surjective, is not a homeomorphism.
Let $X$ be compactum. By $C(X)$ we denote the Banach space of all continuous functions $\phi : X \to \mathbb{R}$ with the usual sup-norm. We consider $C(X)$ with natural order. Let $\nu : C(X) \to \mathbb{R}$ be a functional (we do not suppose a priori that $\nu$ is linear or continuous). We say that $\nu$ is 1) non-expanding if $|\nu(\varphi) - \nu(\psi)| \leq d(\varphi, \psi)$ for all $\varphi, \psi \in C(X)$; 2) weakly additive if for any function $\phi \in C(X)$ and any $c \in \mathbb{R}$ we have $\nu(\phi + c_X) = \nu(\phi) + c$ (by $c_X$ we denote the constant function); 3) preserves order if for any $\varphi, \psi \in C(X)$ such that $\varphi \leq \psi$ the inequality $\nu(\varphi) \leq \nu(\psi)$ holds; 4) linear if for any $\alpha, \beta \in \mathbb{R}$ and for any two functions $\psi, \phi \in C(X)$ we have $\nu(\alpha \phi + \beta \psi) = \alpha \nu(\phi) + \beta \nu(\psi)$.

Now for any space $X$ denote $VX = \prod_{\varphi \in C(X)} [\min \varphi, \max \varphi]$. For any mapping $f : X \to Y$ define the map $Vf$ as follows: $Vf(\nu)(\varphi) = \nu(\varphi \circ f)$ for every $\nu \in VX$, $\varphi \in C(Y)$. Then $V$ is a covariant functor in the category $\text{Comp}$ [11].

Let us remark that the space $VX$ could be considered as the space of all functionals $\nu : C(X) \to \mathbb{R}$ with the only condition $\min \varphi(X) \leq \nu(\varphi) \leq \max \varphi(X)$ for every $\nu \in VX$, $\varphi \in C(Y)$. By $EX$ we denote the subset of $VX$ defined by the condition 1) (non-expanding functionals; see [5] for more details), by $EAX$ the subset defined by the conditions 1) and 2). The conditions 2) and 3) define the subset $OX$ (order-preserving functionals, see [10]); finally, the conditions 3) and 4) define the well-known subset $PX$ (probability measures, see for example [?]). For a map $f : X \to Y$ the mapping $Ff$, where $F$ is one of $P$, $O$, $EA$, $E$, is defined as the restriction of $Vf$ on $FX$. It is easy to check that the constructions $P$, $O$, $EA$ and $E$ define subfunctors of $V$. It is known that the functors $O$ and $E$ are weakly normal (see [10] and [5]). Using the same arguments one can check that $EA$ is weakly normal too.

The question arises naturally which of defined above functors have the property of preserving 1-preimages. It is easy to check that we have the inclusions $PX \subset OX \subset EAX \subset EX \subset VX$. We will show that the functor $EA$ satisfies this property and $E$ does not. Since subfunctors inherit the 1-preimages preserving property, this is the complete answer. Let us also remark that the results of [11] and [12] show that many other known functors could be considered as subfunctors of $EA$, for example the superextension, the hyperspace functor, the inclusion hyperspace functor etc. This shows that the class of functors with the 1-preimages preserving property is wide enough.

We start with a definition of an $AR$-compactum. Recall that a compactum $X$ is called an absolute retract (briefly $X \in AR$) if for any embedding $i : X \to Z$ of $X$ into compactum $Z$ the image $i(X)$ is a retract of $Z$.

The next lemma will be needed in the following discussion.

**Lemma 1.** Let $F$ be a monomorphic subfunctor of $V$ which preserves intersections and $B$ be a closed subset of a compactum $X$. Then $\nu \in FB$ iff $\nu(\varphi_1) = \nu(\varphi_2)$ for each $\varphi_1, \varphi_2 \in C(X)$ such that $\varphi_1|_B = \varphi_2|_B$. 


Proof. Necessity. The inclusion \( \nu \in FB \subset FX \) means that there exists \( \nu_0 \in FB \) with \( F(i_B)(\nu_0) = \nu \), where \( i_B : B \to X \) is a natural embedding. Hence, for any \( \varphi_1, \varphi_2 \in C(X) \) such that \( \varphi_1|_B = \varphi_2|_B \) we have \( \nu(\varphi_1) = \nu_0(\varphi_1 \circ i_B) = \nu_0(\varphi_2 \circ i_B) = \nu(\varphi_2) \).

Sufficiency. We can find an embedding \( j : B \hookrightarrow Y \), where \( Y \in AR \). Define \( Z \) to be the quotient space of the disjoint union \( X \cup Y \) obtained by attaching \( X \) and \( Y \) by \( B \). Denote by \( r : Z \to Y \) the retraction mapping.

Now take any \( \nu \in FX \subset FZ \) with the property \( \nu(\varphi_1) = \nu(\varphi_2) \) for each \( \varphi_1, \varphi_2 \in C(X) \) such that \( \varphi_1|_B = \varphi_2|_B \). We claim that \( F(r)(\nu) = \nu \). Indeed, take any \( \varphi \in C(Z) \). Then \( F(r)(\nu)(\varphi) = \nu(\varphi \circ r) = \nu(\varphi) \) since \( \varphi \circ r|_Y = \varphi|_Y \). Hence, \( \nu \in FX \cap FY = FB \).

Proposition 6. The functor \( EA \) preserves 1-preimages.

Proof. Let \( f : X \to Y \) be a continuous open map between compacta \( X \) and \( Y \) and \( B \) be a closed subset of \( Y \) such that \( f|_{f^{-1}(B)} \) is a homeomorphism. Choose any \( \nu \in EA(B) \subset EA(Y) \). Using Lemma 1 we can define \( \mu_0 \in EA(f^{-1}(B)) \) by the condition \( \mu_0(\varphi) = \psi(\varphi) \) for each \( \varphi \in C(X) \) and \( \psi \in C(Y) \) such that \( \psi \circ f|f^{-1}(B) = \varphi|_{f^{-1}(B)} \).

It is enough to show that for each \( \mu \in (EA(f))^{-1}(\nu) \) we have \( \mu = \mu_0 \). Suppose the contrary. Then there exist \( \varphi \in C(X) \) and \( \psi \in C(Y) \) such that \( \psi \circ f|f^{-1}(B) = \varphi|_{f^{-1}(B)} \) and \( \mu(\varphi) \neq \nu(\psi) \). We can suppose that \( \mu(\varphi) > \nu(\psi) \). Define a function \( \psi' : Y \to \mathbb{R} \) by \( \psi'(y) = \max\{\varphi f^{-1}(y)\} \) for any \( y \in Y \). The function \( \psi' \) is continuous since \( f \) is open. Put \( \xi = (\psi' - D) \circ f \), where \( D = \sup\{\max\varphi f^{-1}(y) - \min\varphi f^{-1}(y)\} | y \in Y \}. \) Then \( d(\xi, \varphi) \leq D \) but \( \mu(\varphi) - \mu(\xi) = \mu(\varphi) - \mu((\psi' - D) \circ f) = \mu(\varphi) - \nu(\psi') + D = \mu(\varphi) - \nu(\psi) + D > D \) and we obtain a contradiction. The proof is similar for the case \( \mu(\varphi) < \nu(\psi) \).

Hence, \( EA \) preserves 1-preimages in the class of open mappings, and, by Proposition 1, we are done.

Proposition 7. The functor of nonexpanding functionals \( E \) does not preserve 1-preimages.

Proof. Consider the mapping \( f : X \to Y \) between discrete spaces \( X = \{x, y, s, t\} \) and \( Y = \{a, b, c\} \) which is defined as follows: \( f(x) = a, f(y) = b, f(s) = f(t) = c \). Put \( A = \{\varphi \in C(X)|\varphi(s) = \varphi(t)\} \). Define the functional \( \nu : A \to \mathbb{R} \) as follows: \( \nu(\varphi) = \min\{\varphi(x), \varphi(y)\} \) if \( \varphi|_{x,y} \geq 0, \nu(\varphi) = \max\{\varphi(x), \varphi(y)\} \) if \( \varphi|_{x,y} \leq 0 \), and \( \nu(\varphi) = 0 \) otherwise. One can check that \( \nu \) is nonexpanding. Now take the function \( \psi : X \to \mathbb{R} \) defined as follows \( \psi(x) = 1, \psi(y) = -1, \psi(s) = 0, \psi(t) = 4 \). One can check that we can extend \( \nu \) to a nonexpanding functional on \( A \cup \{\psi\} \) by defining its value on \( \psi \) to be \(-1 \). This new functional can be further extended to a nonexpanding functional on the whole \( C(X) \) \( \Box \). Denote this extension by \( \tilde{\nu} \). Evidently, \( Ef(\tilde{\nu}) \in E(\{a, b\}) \). On the other hand, \( \tilde{\nu} \notin E(\{x, y\}) \). \( \Box \)

4. We consider in this section a monomorphic continuous functor \( F \) which preserves intersections, weight, empty set, point and 1-preimages. We investigate topology of the space
$F_{\beta}Y$ where $Y$ is a metrizable separable non-compact space. We consider $Y$ as a dense subset of metrizable compactum $X$. It follows from Corollary 1 that $F_{\beta}Y$ is homeomorphic to $F_{\beta}Y \subset FX$ (where $X$ is considered as a compactification $bY$ of $Y$) and in what follows we identify $F_{\beta}Y$ with $F_{\beta}Y$. Also, the properties we impose on $F$ imply that $F_{\beta}Y$ is a dense proper subspace of $FX$.

T. Banakh proved in [1] that $F_{\beta}Y$ is $F_\sigma$-subset of $FX$ when $Y$ is locally compact; $F_{\beta}Y$ is $F_{\sigma\delta}$-subset when $Y$ is $G_\delta$-subset. If $Y$ is not a $G_\delta$-subset, then $F_{\beta}Y$ is not analytic.

We consider in the Hilbert cube $Q = [-1, 1]^\omega$ the following subsets: $\Sigma = \{(t_i) \in Q | \sup_i |t_i| < 1\}$; $\sigma = \{(t_i) \in Q | t_i \neq 0 \text{ for finitely many of } i\}$ and $\Sigma^\omega \subset Q^\omega \cong Q$.

It is shown in [2] that any analytic $P_{\beta}Y$ is homeomorphic to one of the spaces $\sigma$, $\Sigma$ or $\Sigma^\omega$. We generalize this result for convex functors.

By $Conv$ we denote the category of convex compacta (compact convex subsets of locally convex topological linear spaces) and affine maps. Let $U : Conv \to Comp$ be the forgetful functor. A functor $F$ is called convex if there exists a functor $F' : Comp \to Conv$ such that $F = UF'$. It is easy to see that the functors $V$, $E$, $EA$, $O$ and $P$ are convex. It is shown in [14] that for each convex functor $F$ there exists a unique natural transformation $l : P \to F$ such that the map $lX : PX \to FX$ is an affine embedding.

**Lemma 2.** $P_{\beta}Y = (lX)^{-1}(F_{\beta}Y)$.

**Proof.** Take any measure $\mu \in P(X)$ such that $lX(\mu) = \mu' \in F_{\beta}Y$. By the definition of $F_{\beta}Y$ it means that $\mu' \in FB$ for some compactum $B \subset Y$. We will show that $\mu \in PB \subset P_{\beta}Y$. Choose an absolute retract $T$ which contains $B$ and define $Z$ to be the quotient space of the disjoint union $X \cup T$ obtained by attaching $X$ and $T$ by $B$. By $r : Z \to T$ denote the retraction. Since $l$ is a natural transformation and $r$ is an identity on $T \subset Z$, we have that $F(r) \circ lZ(\mu) = \mu' = IT \circ P(r)(\mu)$. Hence, $\mu = P(r)(\mu) \in P(T)$ due to injectivity of $lZ$. Therefore, $\mu \in PX \cap PT = PB$. The lemma is proved. \hfill $\Box$

We need some notions from infinite-dimensional topology. See [4] for more details. All spaces are assumed to be metrizable and separable. A closed subset $A$ of a compactum $T$ is called $Z$-set if there exists a homotopy $H : T \times [0; 1] \to T$ such that $H |_{T \times \{0\}} = \text{id}_{T \times \{0\}}$ and $H(T \times \{0\}) = T \times \{0\}$ and $H(T \times \{1\}) \cap A = \emptyset$; a subset $B$ of $T$ is called $\sigma Z$-set if it is contained in countable union of $Z$-sets of $T$. In what follows we will use the following facts.

We don’t know if $F_{\beta}Y$ is a $\sigma Z$-set in $FX$ for any convex functor $F$. Thus, we introduce some additional property. We consider the compactum $FX$ as a convex subset of a locally convex linear space.

**Definition 2.** A convex functor $F : Comp \to Comp$ is called strongly convex if for each compactum $X$, each closed subset $A \subset X$ we have $(FX \setminus FA) \cap \text{aff}FA = \emptyset$.

**Proposition 8.** Each convex subfunctor $F$ of the functor $V$ is strongly convex.
Proof. By Lemma 1 any element from aff\(FA\) takes the same value at any two functions from \(C(X)\) which coincide on \(A\), which is not true for functionals from \(FX \setminus FA\).

**Proposition 9.** Let \(F\) be a strongly convex functor. Then \(F_\beta Y\) is a \(\sigma Z\)-set in \(FX\).

Proof. Take any \(y \in X \setminus Y\). Then \(F_\beta Y \subset F_\beta (X \setminus \{y\})\), and \(X \setminus \{y\}\) can be represented as a countable union of its compact subsets \(A_n\) with the property that \(A_n \subset \text{int} A_{n+1}\), hence, \(F_\beta (X \setminus \{y\}) = \bigcup_{n \in \mathbb{N}} F(A_n)\). Let us show that all \(F(A_n)\) are \(\sigma Z\)-sets in \(FX\). Take any \(\nu \in FX \setminus F_\beta (X \setminus \{y\})\) and the set \(Z = \{t\nu + (1 - t)\mu | t \in (0, 1], \mu \in F_\beta (X \setminus \{y\})\}\). Since \(F\) is strongly convex, we have \(Z \cap F_\beta (X \setminus \{y\}) = \emptyset\). Since \(Z\) is convex and dense subset of \(FX\), there exists a homotopy \(H : FX \times [0, 1] \to FX\) such that \(H(FX \times (0, 1]) \subset Z\) (see, for example, Ex. 12, 13 to section 1.2 in [4]).

Now, we are going to obtain the complete topological classification of the pair \((FX, F_\beta Y)\) where \(X\) is a metrizable compactum and \(Y\) its proper dense \(G_\delta\)-subset. We need some characterization theorems.

**Theorem A.** [8] Let \(C\) be an infinite-dimensional dense \(\sigma Z\) convex subspace of a convex metrizable compactum \(K\), and additionally let \(C\) be a countable union of its finite-dimensional compact subspaces. Then the pair \((K, C)\) is homeomorphic to \((Q, \sigma)\).

**Theorem B.** [7] Let \(K\) be a convex metrizable compactum, and let \(C \subset K\) be its proper dense \(\sigma Z\) convex \(\sigma\)-compact subspace that contains an infinite-dimensional convex compactum. Then the pair \((K, C)\) is homeomorphic to the pair \((Q, \Sigma)\).

The following theorem follows from 5.3.6, 5.2.6, 3.1.10 in [4].

**Theorem C.** Let \(K\) be a convex compact subset locally convex linear metric space, and let \(C \subset K\) be its proper dense \(\sigma Z\) convex \(F_{\sigma\delta}\) subspace such that \(K \setminus C) \cap \text{aff} C = \emptyset\), and additionally there exists a continuous embedding \(h : Q \to K\) such that \(h^{-1}(C) = \Sigma^\omega\). Then the pair \((K, C)\) is homeomorphic to the pair \((Q, \Sigma^\omega)\).

**Theorem 2.** Let \(F\) be a strongly convex functor, \(X\) is a metrizable compactum and \(Y\) is its proper dense \(G_\delta\)-subset. The pair \((FX, F_\beta Y)\) is homeomorphic to

1. \((Q, \sigma)\), if \(Y\) is discrete subspace of \(X\) and \(F(n)\) is finite-dimensional for each \(n \in \mathbb{N}\);
2. \((Q, \Sigma)\), if \(Y\) is discrete subspace of \(X\) and \(F(n)\) is infinite-dimensional for some \(n \in \mathbb{N}\) or \(Y\) is locally compact non-discrete subspace of \(X\);
3. \((Q, \Sigma^\omega)\), if \(Y\) is not locally compact.
Proof. It is easy to see that $F_\beta Y$ is a convex subset of $FX$.

We prove the first assertion. Since $X$ is metrizable, $Y$ is countable. We can represent $Y = \bigcup_{n=1}^{\infty} Y_n$ where $|Y_n| = n$. Then $F_\beta Y = \bigcup_{n=1}^{\infty} FY_n$. Since $PY_n$ could be considered as an $n - 1$-dimensional subspace of $FY_n$, the space $F_\beta Y$ is infinite-dimensional. Moreover, $F_\beta Y$ is a $\sigma Z$-set by Proposition 9. Since each $FY_n$ is a finite-dimensional compactum, we can apply Theorem A.

We prove the second assertion. In the case when $Y$ is discrete, $FY_n$ is infinite-dimensional convex compactum for some $n$. When $Y$ is not discrete, it contains an infinite compactum $Y'$ and $FY'$ is infinite-dimensional convex compactum. We apply Proposition 9 and Theorem B.

For the third assertion, note that the pair $(PX, P_\beta Y)$ is homeomorphic to $(Q, \Sigma^\omega)$ [2]. Since $F$ is strongly convex, we have $(FX \setminus F_\beta Y) \cap \text{aff} F_\beta Y = \emptyset$. We apply Lemma 2, Proposition 9 and Theorem C.

Corollary 2. Suppose that $F$ is a strongly convex functor. Then for any separable metrizable space $X$

1) $X \cong \mathbb{N}$ implies $F_\beta(X) \cong Q_f$ in case $F(n)$ is finite-dimensional for any $n \in \mathbb{N}$ or $F_\beta(X) \cong \Sigma$ otherwise;

2) if $X$ is locally compact non-discrete and non-compact then $F_\beta(X) \cong \Sigma$;

3) if $X$ is topologically complete not locally compact then $F_\beta(X) \cong \Sigma^\omega$.

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