Relaxing the I.I.D. Assumption: Adaptively Minimax Optimal Regret via Root-Entropic Regularization

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Abstract

We consider prediction with expert advice when data are generated from distributions varying arbitrarily within an unknown constraint set. This semi-adversarial setting includes (at the extremes) the classical i.i.d. setting, when the unknown constraint set is restricted to be a singleton, and the unconstrained adversarial setting, when the constraint set is the set of all distributions. The Hedge algorithm—long known to be minimax (rate) optimal in the adversarial regime—was recently shown to be simultaneously minimax optimal for i.i.d. data. In this work, we propose to relax the i.i.d. assumption by seeking adaptivity at all levels of a natural ordering on constraint sets. We provide matching upper and lower bounds on the minimax regret at all levels, show that Hedge with deterministic learning rates is suboptimal outside of the extremes, and prove that one can adaptively obtain minimax regret at all levels. We achieve this optimal adaptivity using the follow-the-regularized-leader (FTRL) framework, with a novel adaptive regularization scheme that implicitly scales as the square root of the entropy of the current predictive distribution, rather than the entropy of the initial predictive distribution. Finally, we provide novel technical tools to study the statistical performance of FTRL along the semi-adversarial spectrum.

1 Introduction

In this work, we are concerned with obtaining guarantees on the quality of methods used to make decisions in light of data. Often, such guarantees are obtained via assumptions on the distribution of data. One important example of such an assumption is that data are independent and identically distributed (i.i.d.). While this type of assumption on the joint dependence structure of data may be pragmatic, and can motivate methods that seem to perform well in practice, it is impossible to be sure that apparent structure observed in past data will continue. This impossibility highlights the inherent limitations of such assumptions: any guarantees about performance may fail in practice if the assumed dependency does not hold, and statistical methods that are optimal under a specific family of dependence structures may be far from optimal under another. It is of practical interest to determine when the performance of statistical methods is robust to the dependence structures that they are designed for, and to quantify how performance guarantees degrade as assumptions on the dependence structure are relaxed. Thus, contrary to guarantees that hold only under a specific dependence modelling assumption, guarantees should, ideally, hold regardless of the true nature of the data.

One way to formalize such guarantees is through the lens of adaptation theory (e.g., [13]). We do so by first introducing a new notion of regularity that quantifies the degree to which a sequence

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deviates from being i.i.d. The natural question that must be answered when one introduces a new
notion of regularity is whether adaptivity is even possible; it may be the case that no single method
obtains minimax optimal rates in every setting simultaneously. Our main contribution in this
work is answering this question in the affirmative for the specific type of regularity we introduce,
demonstrating it is possible to optimally adapt to a specific relaxation of the i.i.d. assumption. In
particular, we introduce the novel semi-adversarial spectrum, which is an ordering of dependence
structures characterized by their deviation from the i.i.d. assumption, and quantify the performance
of statistical methods at all levels of this spectrum. This new notion of regularity can be applied
to a wide range of decision tasks, and can be combined with existing notions of regularity (such as
smoothness).

Without the i.i.d. assumption, future observations may depend on both past observations and
predictions, and so we study performance in a sequential decision making context. While the
relaxation of the i.i.d. assumption that we introduce is generically applicable to sequential decision
making, in this work we consider specifically its application to the problem of prediction with
expert advice [39, 60]. Prediction with expert advice is a classical problem in statistics dating back
to Cover [21], with close connections to empirical process theory [14] and statistical aggregation
[4, 49, 56, 57]. We show that several state-of-the-art methods for prediction with expert advice
cannot be optimal at all deviations from i.i.d. without oracle knowledge of the deviation, but provide
a novel algorithm that adaptively achieves minimax optimal rates at all deviations from i.i.d. along
the semi-adversarial spectrum.

Finally, we remark on the existing literature that studies benign data-generating mechanisms with-
out relying on the i.i.d. assumption (for a detailed survey, see Section 11.2). Many of these works
obtain performance guarantees in terms of data-dependent (random) quantities; examples include
error bounds that replace the dependence on the number of observations with the $\ell_\infty$ norm or
empirical variance of the incurred losses. In the present work, we take the perspective that per-
formance guarantees should provide guidance on the quality of methods in advance of their use.

Data-dependent guarantees are not immediately satisfactory when viewed through this lens, since
one must still have a prior belief of which data is likely in order to evaluate the quality of the
method in advance. The choice of prior belief is important, since notions of data that make a
data-dependent guarantee “good” (e.g., a small error bound) may not be compatible with which
data is likely under a prior belief that the setting is “easy”. As a concrete example, error bounds in
terms of the empirical variance are large when the observed losses vary significantly, yet this may
occur even when the data is truly i.i.d., a setting for which much smaller error bounds than those
prescribed by the empirical variance bounds are possible.

To address this discrepancy, we examine how the best possible performance degrades as the data-
generating mechanism varies between the i.i.d. and adversarial cases. We explicitly incorporate the
notion that the i.i.d. case should be “easiest”, and performance should degrade smoothly as we relax
the i.i.d. assumption towards the adversarial worst-case. This perspective distinguishes our work
from existing work: (a) we describe a formal spectrum of beliefs characterizing likely observations
with i.i.d. and adversarial data as its extremes, (b) we apply this spectrum to a novel data-dependent
guarantee for a family of methods, identifying precisely which plausible data-generating mechanisms
lead to better performance, and (c) we leverage this spectrum to understand performance when
data is “nearly i.i.d.”, and how performance degrades as the data-generating mechanism varies
between i.i.d. and adversarial.
Contributions  First, we formalize a relaxation of the i.i.d. assumption for prediction with expert advice and a corresponding notion of adaptive minimax optimality, which requires identifying the optimal performance at each element of this semi-adversarial spectrum. Then, our main contribution is to show it is possible to optimally adapt along the entire semi-adversarial spectrum, achieving minimax regret at each level of the spectrum without any advance knowledge of the data-generating mechanism. The Decreasing Hedge algorithm (D.HEDGE), which corresponds to prediction via a tempered Bayesian posterior for an expert-valued parameter, was recently shown to be simultaneously optimal for i.i.d. and adversarial data [44]. However, we show that D.HEDGE (and its variants) requires oracle knowledge of the nature of the data-generating mechanism to optimally tune its learning rate (a.k.a. the tempering parameter), and hence does not adapt along the semi-adversarial spectrum between these endpoints. In light of this negative result, we introduce a novel algorithm Meta-CARE, which implicitly and adaptively adjusts the learning rate of Hedge without the need for oracle knowledge of the nature of data-generating mechanism, and prove that it is adaptively minimax optimal along the entire semi-adversarial spectrum. Meta-CARE consists of boosting our novel follow-the-regularized-leader (FTRL) algorithm FTRL-CARE with D.HEDGE using a second application of D.HEDGE, and hence a major component of our analysis is devoted to a general study of FTRL algorithms along the semi-adversarial spectrum. A pivotal analytic tool that we develop for this analysis is a concentration of measure inequality under our relaxation of the i.i.d. assumption, which we expect to be useful beyond the present setting of prediction with expert advice.

Organization  In Section 2 we formalize the problem setting of interest. In Section 3, we rigorously define the semi-adversarial spectrum and illustrate its relevance via several examples. We present our notion of adaptive minimax optimality and summarize our main results on the minimax rates for the semi-adversarial spectrum in Section 4. In Section 5 we provide our novel concentration of measure inequality for the semi-adversarial spectrum. We precisely state the minimax lower bounds for performance along the semi-adversarial spectrum in Section 6, thus characterizing what an adaptively minimax optimal algorithm must achieve. Section 7 is devoted to quantitative upper and lower bounds for D.HEDGE, including our results on the non-adaptivity of D.HEDGE. Section 8 introduces FTRL-CARE and provides a quantitative upper bound for its regret. An outline of the proofs of the regret upper bounds for D.HEDGE and FTRL-CARE is given in Section 9. We introduce Meta-CARE in Section 10 along with the corresponding upper bound and proof, and then end with a review of the relevant literature in Section 11. Technical details for the proofs of our results, and a brief simulation study, are deferred to the supplementary material.

2 Notation and problem setup

Prediction with expert advice is characterized by the manner in which experts and the player make their predictions and the mechanism by which a response observation is generated. At every time \( t \in \mathbb{N} \), each of the \( N \in \mathbb{N} \) experts (arbitrarily indexed by \( [N] = \{1, \ldots, N\} \)) formulate their predictions for the \( t \)th round, jointly denoted by \( x(t) \in \hat{Y}^N \), the player makes a prediction for the \( t \)th round, \( \hat{y}(t) \in \hat{Y} \), and the environment generates a response observation for the \( t \)th round, \( y(t) \in Y \). The history of the game up to time \( t \) is summarized by \( h(t) = (x(s), \hat{y}(s), y(s))_{s \in [t]} \in \mathcal{H}^t \), where \( \mathcal{H} = \hat{Y}^N \times \hat{Y} \times Y \), with the convention that \( h(0) \) is the empty tuple. For each time \( t \in \mathbb{N} \), the prediction \( \hat{y}(t) \) and response observation \( y(t) \) are conditionally independent given the history \( h(t - 1) \) and the recent expert predictions, \( x(t) \). This conditional independence reflects the fact that the player does not have access to the response until after making their prediction, and that
the player has some private source of stochasticity with which to randomize their predictions.

The conditional distribution of the experts’ predictions and the data observed at round \( t \) given \( h(t-1) \) is uniquely described by a probability kernel \( \pi_t \in \mathcal{K}(\mathcal{H}^{t-1}, \hat{Y}^N \times \mathcal{Y}) \), where \( \mathcal{K}(\mathcal{A}, \mathcal{B}) \) denotes the set of probability kernels (regular conditional distributions) from \( \mathcal{A} \) to \( \mathcal{B} \). Letting \( \mathcal{P}_N = \prod_{t \in \mathbb{N}} \mathcal{K}(\mathcal{H}^{t-1}, \hat{Y}^N \times \mathcal{Y}) \), a data-generating mechanism is any sequence \( \pi = (\pi_t)_{t \in \mathbb{N}} \in \mathcal{P}_N \). Similarly, the conditional distribution of the player’s prediction at time \( t \) given \( h(t-1) \) and \( x(t) \) is uniquely described by a probability kernel \( \hat{\pi}_t \in \mathcal{K}(\mathcal{H}^{t-1} \times \hat{Y}^N, \hat{Y}) \). Letting \( \hat{\mathcal{P}}_N = \prod_{t \in \mathbb{N}} \mathcal{K}(\mathcal{H}^{t-1} \times \hat{Y}^N, \hat{Y}) \), a prediction policy is any sequence \( \hat{\pi} = (\hat{\pi}_t)_{t \in \mathbb{N}} \in \hat{\mathcal{P}}_N \). Finally, a prediction algorithm is any sequence \( a = (\hat{\pi}(N))_{N \in \mathbb{N}} \) with \( \hat{\pi}(N) \in \hat{\mathcal{P}}_N \) for each \( N \).

In a sequential prediction task, prior to any data being generated or predictions being made, the player selects a prediction algorithm and the environment determines a data-generating mechanism. Without loss of generality, the player knows the number of experts \( N \), and so they predict according to the prediction policy \( \hat{\pi} = a(N) \) based on their prediction algorithm. Due to the conditional independence assumption for \( \hat{y}(t) \) and \( y(t) \) given \( h(t-1) \) and \( x(t) \), the joint distribution of \( (x(t), \hat{y}(t), y(t))_{t \in \mathbb{N}} \) is fully determined by the data-generating mechanism and the prediction policy selected by each party. For a data-generating mechanism \( \pi \) and a prediction policy \( \hat{\pi} \), expectation under this joint law is denoted by \( \mathbb{E}_{\pi, \hat{\pi}} \). When the prediction policy is determined by the prediction algorithm \( a \), for any number of experts \( N \) and data-generating mechanism \( \pi \in \mathcal{P}_N \) we use \( \mathbb{E}_{\pi, a} \) to denote \( \mathbb{E}_{\pi, \hat{\pi}} \), where \( \hat{\pi} = a(N) \).

The accuracy of the player and experts is measured on each round using a loss function \( \ell : \hat{Y} \times \mathcal{Y} \to [0, 1] \), and the player’s performance at the end of \( T \in \mathbb{N} \) rounds of the game is measured by regret, defined as the \( \sigma(h(T)) \)-measurable random variable

\[
R(T) = \sum_{t=1}^{T} \ell(\hat{y}(t), y(t)) - \min_{i \in [N]} \sum_{t=1}^{T} \ell(x_i(t), y(t)).
\]

In this work, we focus on bounding the expected regret \( \mathbb{E}_{\pi, a} R(T) \) for three specific prediction algorithms, so we use \( \mathbb{E}_{\pi, D}, \mathbb{E}_{\pi, C}, \) and \( \mathbb{E}_{\pi, M} \) to denote \( \mathbb{E}_{\pi, a} \) under the D.Hedge, FTRL-CARE, and Meta-CARE algorithms respectively (see Sections 7, 8 and 10 for the respective definitions of these prediction algorithms).

Since \( R(T) \) only depends on \( h(T) \) through the loss function, expected regret bounds are often characterized using quantities that push the data-generating distributions forward through the loss function. Specifically, we define the losses \( \ell_i(t) = \ell(x_i(t), y(t)) \) and cumulative losses \( L_i(t) = \sum_{s=1}^{t} \ell_i(s) \) for each expert \( i \in [N] \) and \( t \in \mathbb{N} \). Similarly, we define the loss vector \( \ell(t) = (\ell_i(t))_{i \in [N]} \) and cumulative loss vector \( L(t) = \sum_{s=1}^{t} \ell(s) \).

Let \( \mathcal{M}(\mathcal{A}) \) denote the set of all probability distributions on \( \mathcal{A} \). For a distribution \( \mu \in \mathcal{M}(\mathcal{A}) \) and measurable function \( f : \mathcal{A} \to \mathbb{R} \), we define \( \mu f = \int_{\mathcal{A}} f(a) \mu(da) \). We will frequently use this notation for measures in \( \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \). In particular, for each expert \( i \in [N] \), the expert’s loss \( \ell_i : (x, y) \mapsto \ell(x_i, y) \) is a function on \( \hat{Y}^N \times \mathcal{Y} \), and \( \mu \ell_i \) is the expectation of expert \( i \)’s loss when the expert predictions and response observation are jointly distributed as \( \mu \).

### 3 Semi-adversarial spectrum

Consider a fixed number of experts \( N \). For any time-homogeneous convex constraint \( \mathcal{D} \subseteq \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \), let \( \mathcal{P}(\mathcal{D}) \) denote the collection of data-generating mechanisms \( \pi = (\pi_t)_{t \in \mathbb{N}} \) such that for all \( t \in \mathbb{N} \) and \( h \in \mathcal{H}^{t-1} \), \( \pi_t(h, \cdot) \in \mathcal{D} \). That is, \( \mathcal{P}(\mathcal{D}) \) is the set of data-generating mechanisms under
Figure 1: Visualising the difference between i.i.d. data, adversarial data, and a constraint set in between these two extremes. In each part of the figure, the triangles depict the set of conditional distributions for the tuple of expert predictions and response (an “instance”) given the history at each time. The grey regions depict the space of conditional distributions for the next instance given the history that are possible for a given constraint set.

which the conditional distribution of the expert predictions and response data given the history is constrained to \( D \), but can vary arbitrarily within \( D \) depending on the history.

In Fig. 1, we visualize possible trajectories of data-generating mechanisms for the i.i.d. endpoint, adversarial endpoint, and a constraint set that lies between these. In the i.i.d. case, the conditional distribution of the next instance given the history is fixed, and hence the constraint set corresponds to a single distribution on instances. In the adversarial case, the conditional distribution of the next instance given the history can vary arbitrarily in the space of all probability distributions on instances; in particular, it can be a point-mass at an adversarial instance for the player’s strategy, depicted here as the extreme points of the space of distributions. Since the i.i.d. case corresponds to a singleton set of distributions on instances, and the adversarial case corresponds to the whole space of distributions on instances, a natural concept of “in between” these extremes is a proper subset of the set of distributions on instances. Our relaxation captures this by allowing the conditional distribution of the next instance given the history to vary within some convex constraint set that is not known by the player in advance (visualized here as an ellipse), and measuring performance relative to the properties of that unknown constraint set.

We use two characterizing quantities to describe \( D \). For each expert \( i \in [N] \), let

\[
\Delta_i(D) = \inf_{\mu \in D} \max_{i' \in [N]} \mu[\ell_i - \ell_{i'}],
\]

and define the effective stochastic gap

\[
\Delta_0(D) = \min\{\Delta_i(D) \mid i \in [N], \Delta_i(D) > 0\}.
\]

Second, define the set of effective experts

\[
I_0(D) = \{i \in [N] \mid \Delta_i(D) = 0\}.
\]
\( \mathcal{I}_0(\mathcal{D}) \) contains the experts that could be the best (in conditional expectation given the history) on any particular round. The size of the effective expert set is denoted by \( N_0(\mathcal{D}) = |\mathcal{I}_0(\mathcal{D})| \). \( \Delta_0(\mathcal{D}) \) is the minimal excess expected loss of an ineffective expert over the best effective expert on any round. When \( \mathcal{D} \) is clear, we simplify notation to \( \mathcal{I}_0, N_0, \) and \( \Delta_0 \).

For a fixed \( N, N_0, \) and \( \Delta_0 \), the collection of convex constraint sets that have these characterizing quantities is

\[
\mathcal{V}(N, N_0, \Delta_0) = \left\{ \mathcal{D} \subseteq \mathcal{M}(\mathcal{Y}^N \times \mathcal{Y}) \mid \mathcal{D} \text{ convex}, \ N_0(\mathcal{D}) = N_0, \ \Delta_0(\mathcal{D}) \geq \Delta_0 \right\},
\]

and the corresponding set of data-generating mechanisms is

\[
\mathcal{P}_{N,(N_0,\Delta_0)} = \bigcup_{\mathcal{D} \in \mathcal{V}(N,N_0,\Delta_0)} \mathcal{P}(\mathcal{D}).
\]

Let \( \mathcal{P} = \{ \mathcal{P}_{N,(N_0,\Delta_0)} \mid N_0 \leq N \in \mathbb{N}, \Delta_0 > 0 \} \) denote the collection of all such sets. Together, \( N_0 \) and \( \Delta_0 \) induce a total ordering on constraint sets, and the semi-adversarial spectrum is the collection of equivalence classes this ordering induces.

### 3.1 Motivation for characterizing quantities

The characterizing quantities \( N_0 \) and \( \Delta_0 \) reduce to the standard characterizing quantities for the rate of regret from the i.i.d. setting. To see this, observe that the i.i.d. setting corresponds to \( \mathcal{D} \) defined by a single distribution \( \mu_0 \in \mathcal{M}(\mathcal{Y}^N \times \mathcal{Y}) \); that is, for all \( t \in \mathbb{N} \) and \( h \in \mathcal{H}^{t-1} \), the data-generating mechanism satisfies \( \pi_t(h, \cdot) = \mu_0 \). It is well known that the minimax optimal expected regret under the i.i.d. assumption depends on the stochastic gap. Letting \( \mathcal{I}_0(\mu_0) = \arg \min_{i \in [N]} \mu_0 \ell_i \) be the set of experts that are optimal (w.r.t. \( \ell \)) in expectation under \( \mu_0 \), each expert \( i \in [N] \) has stochastic gap \( \Delta_i(\mu_0) = \mu_0 \ell_i - \min_{\mu \in \mathcal{M}[\mathcal{Y}]} \mu \ell_i \), and the stochastic gap is defined by \( \Delta_0(\mu_0) = \min_{i \in [N]} |\mathcal{I}_0(\mu_0)| \Delta_i(\mu_0) \). The minimax optimal expected regret in the stochastic-with-a-gap setting (i.i.d. with \( |\mathcal{I}_0| = 1 \)) satisfies (cf. [44])

\[
\mathbb{E} R(T) \in \Theta \left( \frac{\log N}{\Delta_0(\mu_0)} \right).
\]

The effective experts and the effective stochastic gap generalize \( \mathcal{I}_0(\mu_0) \) and \( \Delta_0(\mu_0) \) beyond the i.i.d. case, and our expected regret bounds depend on these characterizing quantities in a similar way to the dependence on \( N \) and \( \Delta_0 \) in the stochastic and adversarial settings respectively.

### 3.2 Practical relevance of convex constraints

A standard application of prediction with expert advice is to the setting of statistical aggregation (cf. [4, 45, 64]). We now describe an example of an aggregation task where the time-homogeneous convex constraint setting is the canonical representation of the data-generating mechanism. Suppose the statistician has \( N \) models that map from a covariate space \( \mathcal{X} \) to a response space \( \mathcal{Y} \). Further, suppose that the \( t \)th observation \( (X_t, Y_t) \) is sampled from one of \( K \) unknown distributions on \( \mathcal{X} \times \mathcal{Y} \), where this distribution is selected in a potentially adversarial and non-i.i.d. way using the previous \( t - 1 \) observations. That is, the observed dataset is an adversarial mixture of \( K \) different stochastic sources. The ability of the data-generating mechanism to randomize its selection of the source distribution gives rise to a time-homogeneous convex constraint, where \( \mathcal{D} \) is the convex hull of the \( K \) source distributions. If the source distributions and models are reasonably distinct, this will likely satisfy \( N_0 = K \), which may be much smaller than \( N \).
3.3 Examples of convex constraints

The following examples illustrate the flexibility of time-homogeneous convex constraints and the semi-adversarial spectrum.

**Example 1** (I.I.D.-$\mu_0$, Stochastic-with-a-gap). When the constraint set is the singleton $D_{\mu_0} = \{\mu_0\}$, then there is only one possible data-generating mechanism, and under that data-generating mechanism the data and expert predictions are i.i.d. according to $\mu_0$. Furthermore, if there exists $i_0 \in [N]$ and $\Delta > 0$ such that

$$\inf_{i \in [N] \setminus \{i_0\}} \mu[\ell_i - \ell_{i_0}] = \Delta,$$

(i.e., there is a best expert in expectation under $\mu_0$ and there is a gap of $\Delta$ from the best to the second best expert in expectation) then $I_0(D_{\mu_0}) = \{i_0\}$, $N_0(D_{\mu_0}) = 1$, and $\Delta_0(D_{\mu_0}) = \Delta$. This is called the stochastic-with-a-gap setting. Since any singleton is convex, $D_{\mu_0}$ is convex.

**Example 2** (Adversarial). When the constraint set is the space of all probability measures $D_{\text{adv}} = \mathcal{M}(\hat{Y}^N \times \mathcal{Y})$, then the constrained setting reduces to the fully adversarial setting, since $D$ contains all point-mass distributions. In this case, $I_0(D_{\text{adv}}) = [N]$, $N_0(D_{\text{adv}}) = N$, and $\Delta_0(D_{\text{adv}}) = +\infty$ (by convention, as it is the inf over an empty set). Since the set of all probability measures is convex, $D_{\text{adv}}$ is convex.

**Example 3** (Adversarial-with-an-instantaneous-gap). For any $i_0 \in [N]$ and $\Delta \geq 0$,

$$D_{i_0,\Delta}^{(a.s.)} = \left\{ \mu \in \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \mid \mu(\ell_{i_0} + \Delta \leq \min_{i \in [N] \setminus \{i_0\}} \ell_i) = 1 \right\}$$

is convex (since min is concave), and satisfies $I_0(D_{i_0,\Delta}^{(a.s.)}) = \{i_0\}$, $N_0(D_{i_0,\Delta}^{(a.s.)}) = 1$, and $\Delta_0(D_{i_0,\Delta}^{(a.s.)}) = \Delta$. This contains all mixtures of point-mass distributions with common best expert $i_0$ that satisfy the gap constraint almost surely.

**Example 4** (Adversarial-with-an-$\mathcal{E}$-gap, Mourtada and Gaïffas [44]). For any $i_0 \in [N]$ and $\Delta \geq 0$,

$$D_{i_0,\Delta} = \left\{ \mu \in \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \mid \mu \ell_{i_0} + \Delta \leq \min_{i \in [N] \setminus \{i_0\}} \mu \ell_i \right\}$$

is convex (since min is concave), and satisfies $I_0(D_{i_0,\Delta}) = \{i_0\}$, $N_0(D_{i_0,\Delta}) = 1$ and $\Delta_0(D_{i_0,\Delta}) = \Delta$. This relaxes the adversarial-with-an-instantaneous-gap setting, since $D_{i_0,\Delta}^{(a.s.)} \subseteq D_{i_0,\Delta}$. This constraint set is equivalent to the formulation used in Corollary 6 of Mourtada and Gaïffas [44]; it is also the same setting as Section 4.2 of Wei and Luo [62], although they consider bandit feedback.

**Example 5** (Ball-around-I.I.D.). For any pseudometric $d$, radius $r > 0$, and probability measure $\mu_0$,

$$D_{\mu_0,d,r} = B_d(\mu_0, r) = \left\{ \mu \in \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \mid d(\mu, \mu_0) \leq r \right\}$$

is convex. The exact values of $I_0(D_{\mu_0,d,r})$, $N_0(D_{\mu_0,d,r})$, and $\Delta_0(D_{\mu_0,d,r})$ will depend on $\mu_0$, $r$, and $d$. In general, $I_0$ and $N_0$ are increasing with $r$ (w.r.t. $\subseteq$ and $\leq$ respectively), while $\Delta_0$ will decrease as $r$ increases between the jumps in $N_0$, but increase sharply at the jumps. Thus, the lexicographical ordering on $(N_0, \Delta_0^{-1})$ coincides with increasing the radius, $r$. Since for nested constraint sets it should be more difficult to compete with the larger of the two constraints, it is intuitive that the lexicographical order on $(N_0, \Delta_0^{-1})$ is an assessment of the difficulty of competing with a given constraint set.
Example 6 (Convex hull of basic distributions). As motivated in Section 3.2, a natural setting is where \( D \) is the convex hull of some basic underlying distributions. Suppose \( N = 3 \), and there exist \( \mu, \nu \in \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \) satisfying \( \mu \mathcal{l} = (0, 1, 0.5 + \varepsilon) \) and \( \nu \mathcal{l} = (1, 0, 0.5 + \varepsilon) \), where \( \varepsilon > 0 \) is arbitrary. Set \( D = \{ \alpha \mu + (1 - \alpha) \nu \mid \alpha \in [0, 1] \} \), which gives \( \mathcal{I}_0(D) = \{ 1, 2 \} \) and \( \Delta_0(D) = \varepsilon \).

However, on any given round it is possible for the data to be sampled from either \( \mu \) or \( \nu \), in which case one of the effective experts is as separated (in expectation) as possible from the best expert and separated by an arbitrarily large multiplicative factor of \( \Delta_0 \) from the ineffective expert. That is, this example demonstrates effective experts need not be better or even close to ineffective experts on any given round.

Note that Example 3 is related to the setting of Seldin and Slivkins [53] and Example 5 is related to the setting of Lykouris et al. [41] (both focusing on bandit feedback), with the distinction that the existing literature considers constraints on the cumulative losses. In contrast, our constraints apply to the distributions allowed on any instantaneous round, and are not restricted in how they accumulate. This distinction is subtle, yet crucial to the type of adaptivity we propose in this work. While existing “easy data” results are about adapting to post-hoc summary statistics of the data, we provide adaptivity to the unknown, underlying dependence structure, and propose that statistical methods should be designed to adapt to this as well (beyond adaptivity to model regularity assumptions).

4 Adaptive optimality for the semi-adversarial spectrum

In this section we will state our main results that characterize the minimax regret over time-homogeneous convex constraints. We begin by precisely defining what it means for a prediction algorithm to be adaptively minimax optimal.

4.1 Adaptively minimax optimal prediction algorithms

Informally, an adaptively minimax optimal prediction algorithm achieves the minimax optimal regret (asymptotically in \( T \)) for the characterizing quantities constraining the allowable data-generating mechanism without a priori information on what values these characterizing quantities take. For collections of sequences \( a = \{ (a_N, (N_0, \Delta_0))_{T \in \mathbb{N}} \mid N \in \mathbb{N}, (N_0, \Delta_0) \in [N] \times \mathbb{R}_+ \} \) and \( b = \{ (b_N, (N_0, \Delta_0))_{T \in \mathbb{N}} \mid N \in \mathbb{N}, (N_0, \Delta_0) \in [N] \times \mathbb{R}_+ \} \), we write

\[
a_{N, (N_0, \Delta_0)}(T) \lesssim b_{N, (N_0, \Delta_0)}(T) \quad \text{(abbreviated } a \lesssim b)\]

when

\[
\exists C > 0 \quad \forall N \in \mathbb{N}, \ (N_0, \Delta_0) \in [N] \times \mathbb{R}_+ \quad \exists T_0 \in \mathbb{N} \quad \forall T > T_0 \quad a_{N, (N_0, \Delta_0)}(T) \leq C b_{N, (N_0, \Delta_0)}(T). \tag{1}
\]

If \( a \lesssim b \) and \( b \lesssim a \), we write \( a_{N, (N_0, \Delta_0)}(T) \asymp b_{N, (N_0, \Delta_0)}(T) \) (abbreviated \( a \asymp b \)).

For a prediction algorithm \( a = (a(N))_{N \in \mathbb{N}} \), we refer to the equivalence class under \( \asymp \) of

\[
N, (N_0, \Delta_0), T \mapsto \sup_{\pi \in \mathcal{P}_{N, (N_0, \Delta_0)}} \mathbb{E}_{\pi, a} R(T)
\]

as the rate of regret or simply the rate of \( a \), and the equivalence class under \( \asymp \) of

\[
N, (N_0, \Delta_0), T \mapsto \inf_{\pi \in \mathcal{P}_{N}} \sup_{\pi \in \mathcal{P}_{N, (N_0, \Delta_0)}} \mathbb{E}_{\pi, \hat{a}} R(T)
\]
as the minimax optimal rate of regret. Then, we say a prediction algorithm \( \mathfrak{a} \) is adaptively minimax optimal if
\[
\sup_{\pi \in \mathcal{P}(N_0, \Delta_0)} \mathbb{E}_{\pi, \mathfrak{a}} R(T) \asymp \inf_{\hat{\pi} \in \mathcal{P}(N)} \sup_{\pi \in \mathcal{P}(N_0, \Delta_0)} \mathbb{E}_{\pi, \hat{\pi}} R(T). \tag{2}
\]
Further, we say that \( \mathfrak{a} \) is adaptive if \( \sup_{\pi \in \mathcal{P}(N)} \mathbb{E}_{\pi, \mathfrak{a}} R(T) \) is always sublinear in \( T \) and, for some \( (N_0, \Delta_0) \), its rate of regret is strictly better than the rate of \( \inf_{\hat{\pi} \in \mathcal{P}(N)} \sup_{\pi \in \mathcal{P}(N_0, \Delta_0)} \mathbb{E}_{\pi, \hat{\pi}} R(T); \) otherwise, we say \( \mathfrak{a} \) is non-adaptive. This definition formalizes the notion that an adaptive prediction algorithm must realize potential benefits from at least some instance of “easier” characterizing quantities and simultaneously have average regret at least converge to zero in all cases.

Importantly, we do not demand that the prediction algorithm perform as well as if they had a priori knowledge of the true data-generating mechanism, since with this information the minimax regret can be quite small (zero or even negative). Instead, the prediction algorithm is only adapting to the problem hardness, as measured by the characterizing quantities, and consequently there is still freedom in the minimax definition for the player to face its worst-case data-generating mechanism subject to these characterizing quantities. Mathematically, this is ensured by placing \( \inf_{\hat{\pi} \in \mathcal{P}(N)} \) after the choice of characterizing quantities, but before the choice of data-generating mechanism (i.e., \( \sup_{\pi \in \mathcal{P}(N_0, \Delta_0)} \)).

More abstractly, our definition of adaptively minimax optimal can be interpreted under a generic adaptive decision problem, with a generic problem size given by \( N \) and a generic problem hardness replacing characterizing quantities. For example, in the case of density estimation, the problem size may correspond to the dimension of the data space, which the statistician knows, and the problem hardness may correspond to the Hölder continuity parameter of the true data-generating density, which the statistician does not know. For a further discussion of our definition of adaptively minimax optimal, see Section 4.3.

### 4.2 Minimax rates

We are now able to state our main result, establishing the minimax optimal rate of regret and that it is achieved by our novel algorithm Meta-CARE, which follows from the conjunction of Theorems 3 and 8 and Proposition 2.

**Theorem 1** (Main result).

\[
\sup_{\pi \in \mathcal{P}(N, (N_0, \Delta_0))} \mathbb{E}_{\pi, \mathcal{M}} R(T) \asymp \inf_{\hat{\pi} \in \mathcal{P}(N)} \sup_{\pi \in \mathcal{P}(N_0, \Delta_0)} \mathbb{E}_{\pi, \hat{\pi}} R(T) \asymp \sqrt{T \log N} + \frac{\log N}{\Delta_0}.
\]

In Theorem 4, we show that D.Hedge using any parametrization that simultaneously achieves the minimax optimal rate of regret in both the stochastic-with-a-gap and adversarial settings is non-adaptive. That is, for \( N_0 \geq 2 \),
\[
\sup_{\pi \in \mathcal{P}(N, (N_0, \Delta_0))} \mathbb{E}_{\pi, \mathcal{H}} R(T) \gtrsim \sqrt{T \log N}.
\]

In fact, from Theorems 4 and 5, we find that without an oracle parametrization of D.Hedge (one where \( N_0 \) is made available to the player in advance), it is only possible to achieve
\[
\log(N_0) \sqrt{T} + \frac{(\log N)}{\Delta_0} \lesssim \sup_{\pi \in \mathcal{P}(N, (N_0, \Delta_0))} \mathbb{E}_{\pi, \mathcal{H}} R(T) \lesssim \log(N_0) \sqrt{T} + \frac{(\log N)^2}{\Delta_0}
\]
or
\[
\sup_{\pi \in \mathcal{P}(N, (N_0, \Delta_0))} \mathbb{E}_{\pi, \mathcal{H}} R(T) \asymp I_{[N_0 \geq 2]} \sqrt{T \log N} + \frac{\log N}{\Delta_0},
\]

where \( I_{[N_0 \geq 2]} \) indicates that this rate is achieved if and only if \( N_0 \geq 2 \).
but not both.

As an intermediary step, we introduce another novel algorithm, FTRL-CARE, and show in Theorem 6 that it adapts with a better rate:

$$
\sup_{\pi \in P_{N,(N_0,\Delta_0)}} \mathbb{E}_{\pi,c} R(T) \lesssim \sqrt{T \log N_0} + \frac{(\log N)^{3/2}}{\Delta_0}.
$$

To also achieve the minimax optimal rate for $N_0 = 1$ (and consequently be adaptively minimax optimal), we introduce Meta-CARE in Theorem 8, which corresponds to another application of D.Hedge to the “meta-experts” corresponding to FTRL-CARE and D.Hedge on all $N$ experts.

Our quantitative upper bounds also explicitly demonstrate how large $T$ must be for algorithms to have adaptive rates (i.e., expected regret that depends on $N_0$ and $\Delta_0$), as opposed to the pessimistic adversarial rate (i.e., $\sqrt{T \log N}$). In particular, for both D.Hedge and FTRL-CARE, roughly $\Delta_0^2$ rounds of adversarial regret are incurred before the level of adaptation is sufficient to reduce the rate of regret accumulation. This demonstrates that as $\Delta_0$ tends to 0, the player does not incur infinite regret from the $\Delta_0^{-1}$ terms, but rather incurs adversarial regret for a longer amount of time. We emphasize that the player does not need to know when they will stop incurring adversarial regret ahead of time to parametrize either algorithm, so knowledge of $N_0$ or $\Delta_0$ is not required.

Our theoretical results are further supported by a simulation study that appears in Appendix G. The simulation study is based on the data-generating mechanisms that achieve the lower bound in the stochastic-with-a-gap setting and the algorithm specific lower bound for D.Hedge with two effective experts. The results of the simulations agree with our theoretical results.

4.3 Discussion on adaptive minimax optimality

One might ask whether it’s possible to strengthen the notion of adaptivity to be *uniform-in-$T$*, where the rate has to be achieved up to a constant at all $T$, rather than only for sufficiently large $T$ depending on $(N_0, \Delta_0)$. This corresponds to replacing the relation $a \precsim b$ with the one defined by

$$
\exists C > 0 \forall T, N \in \mathbb{N}, (N_0, \Delta_0) \in [N] \times \mathbb{R}_+ \quad a_{N,(N_0,\Delta_0)}(T) \leq C b_{N,(N_0,\Delta_0)}(T).
$$

In the context of minimax regret, uniform adaptivity would require understanding the entire path of the regret (over $T$) rather than simply its eventual upper bound. This is not understood even in the stochastic setting; regret bounds of the form $1/\Delta$ in both the bandit and full-information settings [e.g., 8, 25, 44] are all eventual upper bounds that are only known to be tight (i.e., have matching lower bounds) for sufficiently large $T$. Since it remains open to identify the minimax optimal regret uniformly in $T$ even for this basic setting, we do not attempt to also solve this in our more general setting beyond i.i.d. data.

Beyond prediction with expert advice, the lack of uniform adaptivity also persists. For example, the leading constant of the minimax rates for smoothness-adaptation in statistics often depends on the smoothness parameter, which violates uniformity. For general questions of adaptive minimax optimality in sequential prediction, it is not clear how to demonstrate that either form of adaptivity is possible other than by constructing adaptive algorithms, as we have done in the present work.

Finally, one could consider adapting to a different collection of characterizing quantities than $(N_0, \Delta_0)$. For our setting, a natural extension is to consider the individual expectation gaps of each expert, rather than only the smallest gap. While our upper bounds can be extended to handle multiple gaps without much difficulty, tight lower bounds that depend on all the gaps simultaneously are again unknown even in the i.i.d. setting for full-information feedback. Since our work
is about identifying minimax optimality, which would require such lower bounds, we do not consider this refinement. Beyond the extension to multiple gaps, it is an interesting avenue for future work to identify other characterizing quantities that could provide a finer characterization of the data-generating mechanism.

5 Concentration of measure for the semi-adversarial spectrum

In this section, we state and prove a concentration of measure result for data-generating mechanisms permitted by time-homogeneous convex constraints, which we use repeatedly to establish upper bounds on expected regret for D.HEDGE, FTRL-CARE, and Meta-CARE. The result demonstrates that, even though the best expert may vary from round to round, the gap between the best effective expert along the observed data path and any ineffective expert grows like a sum of uniformly sub-Gaussian random variables with mean below $-\Delta_0$.

**Theorem 2.** For all $N \geq 2$, prediction policies $\hat{\pi} \in \hat{\mathcal{P}}_N$, convex sets $D \subseteq \mathcal{M}(Y^N \times Y)$, $\lambda > 0$, $T_0 < T_1$, and $i \in [N] \setminus I_0$,

$$\sup_{\pi \in \mathcal{P}(D)} \min_{i_0 \in I_0} \mathbb{E}_{\pi, \hat{\pi}} \exp \left\{ \lambda \sum_{t=T_0+1}^{T_1} [\ell_{i_0}(t) - \ell_i(t)] \right\} \leq \exp \left\{ (T_1 - T_0) \left[ \lambda^2 / 2 - \lambda \Delta_0 \right] \right\}.$$

Note that we require the constraint set $D$ to be convex. If $D$ is not natively convex, our results clearly apply to its convex hull. There is, however, a natural reason to consider convex constraint sets: given a set $D$ of joint distributions available for the data-generating mechanisms, requiring the set to be convex is equivalent to also allowing mixtures of the original available distributions. That is, the environment and experts together can randomly select a distribution from $D$ to generate data from at each round.

One may wonder whether this result follows from an application of the Azuma–Hoeffding inequality. However, as demonstrated in Example 6, there exist simple constraint sets such that on any round, any effective expert (including the best overall) may have an arbitrarily larger expected loss than any ineffective expert. That is, $L_i(t) - L_{i_0}(t)$ need not be a (sub)martingale, and consequently Azuma–Hoeffding does not directly apply. Instead, the proof of this result first uses a variant of von Neumann's minimax theorem—which is the technical reason why we require the constraint set $D$ to be convex—before applying Hoeffding’s inequality to the instantaneous rounds. We restate the minimax theorem we require for completeness here.

**Proposition 1** (Cesa-Bianchi and Lugosi [15], Theorem 7.1). Let $X$ and $Y$ be convex subsets of linear topological spaces, and suppose that $X$ is compact. Let $f : X \times Y \to \mathbb{R}$ be such that:

(i) for all $y \in Y$, $f(\cdot, y) : X \to \mathbb{R}$ is convex and continuous; and

(ii) for all $x \in X$, $f(x, \cdot) : Y \to \mathbb{R}$ is concave.

Then,

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y).$$

**Proof of Theorem 2.** Let $\mathbb{R}_+ = [0, \infty)$ and $\text{simp}(I_0) = \{v \in \mathbb{R}_{I_0}^\mathbb{Z}_0 : \sum_{i_0 \in I_0} v_{i_0} = 1\}$. First, since at least one optimal solution to a linear program on a compact convex polytope must be at a vertex,

$$\min_{i_0 \in I_0} \sum_{t=T_0+1}^{T_1} \left[ \ell_{i_0}(t) - \ell_i(t) \right] = \inf_{v \in \text{simp}(I_0)} \sum_{t=T_0+1}^{T_1} \left[ \langle v, \ell_{I_0}(t) \rangle - \ell_i(t) \right].$$

11
Further, since \( \exp \) is a monotone function, this identity implies
\[
\min_{i_0 \in I_0} e^{\lambda \sum_{t=T_0+1}^{T_1} [\ell_{i_0}(t) - \ell_i(t)]} = \inf_{v \in \text{simp}(I_0)} e^{\lambda \sum_{t=T_0+1}^{T_1} [\langle v, \ell_{i_0}(t) \rangle - \ell_i(t)]}.
\]

Then, applying Jensen’s and the max–min inequality gives
\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \inf_{v \in \text{simp}(I_0)} e^{\lambda \sum_{t=T_0+1}^{T_1} [\langle v, \ell_{i_0}(t) \rangle - \ell_i(t)]} 
\leq \inf_{v \in \text{simp}(I_0)} \sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} e^{\lambda \sum_{t=T_0+1}^{T_1} [\langle v, \ell_{i_0}(t) \rangle - \ell_i(t)]}.
\]

By the tower rule for conditional expectation and the definition of the kernel \( \pi_{T_1} \),
\[
\mathbb{E}_{\pi, \hat{\pi}} e^{\lambda \sum_{t=T_0+1}^{T_1} [\langle v, \ell_{i_0}(t) \rangle - \ell_i(t)]} 
\leq \left( \mathbb{E}_{\pi, \hat{\pi}} e^{\lambda \sum_{t=T_0+1}^{T_1-1} [\langle v, \ell_{i_0}(t) \rangle - \ell_i(t)]} \right) \left( \sup_{\mu \in \mathcal{D}} \mu \left( e^{\lambda [\langle v, \ell_{i_0} \rangle - \ell_i]} \right) \right).
\]

Iterating this argument \( T_1 - T_0 - 1 \) more times, and using monotonicity of power functions, gives
\[
\inf_{v \in \text{simp}(I_0)} \sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} e^{\lambda \sum_{t=T_0+1}^{T_1} [\langle v, \ell_{i_0}(t) \rangle - \ell_i(t)]} 
\leq \left[ \inf_{v \in \text{simp}(I_0)} \sup_{\mu \in \mathcal{D}} \mu \left( e^{\lambda [\langle v, \ell_{i_0} \rangle - \ell_i]} \right) \right]^{T_1 - T_0}.
\]

Noting that \( \text{simp}(I_0) \) is convex, that \( \mathcal{D} \) is convex, and that the objective function \( f(v, \mu) = \mu(e^{\lambda [\langle v, \ell_{i_0} \rangle - \ell_i]} \right) \) is continuous and convex in \( v \) and linear (and hence concave) in \( \mu \), Proposition 1 gives
\[
\inf_{v \in \text{simp}(I_0)} \sup_{\mu \in \mathcal{D}} f(v, \mu) = \sup_{\mu \in \mathcal{D}} \inf_{v \in \text{simp}(I_0)} f(v, \mu).
\]

Thus,
\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \inf_{v \in \text{simp}(I_0)} e^{\lambda \sum_{t=T_0+1}^{T_1} [\langle v, \ell_{i_0}(t) \rangle - \ell_i(t)]} 
\leq \left[ \sup_{\mu \in \mathcal{D}} \inf_{v \in \text{simp}(I_0)} \mu \left( e^{\lambda [\langle v, \ell_{i_0} \rangle - \ell_i]} \right) \right]^{T_1 - T_0}.
\]

Consider any \( \mu \in \mathcal{D} \), and let \( \ell^*(\mu) \in \arg \min_{i \in [N]} \mu \ell_i \). By the definition of \( \Delta_0 \), \( \mu \left( \ell_{i^*(\mu)} - \ell_i \right) \leq -\Delta_0 \) for every \( i \in [N] \setminus I_0 \). Finally, since \( \ell \in [0, 1]^N \) \( \mu \)-a.s., by Hoeffding’s lemma,
\[
\inf_{v \in \text{simp}(I_0)} \mu \left( e^{\lambda [\langle v, \ell_{i_0} \rangle - \ell_i]} \right) \leq \mu \left( e^{\lambda [\ell_{i^*(\mu)} - \ell_i]} \right) \leq e^{\lambda^2/2 - \lambda \Delta_0}.
\]

Since this holds for all \( \mu \in \mathcal{D} \),
\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \inf_{v \in \text{simp}(I_0)} e^{\lambda \sum_{t=T_0+1}^{T_1} [\langle v, \ell_{i_0}(t) \rangle - \ell_i(t)]} \leq \left[ e^{\lambda^2/2 - \lambda \Delta_0} \right]^{T_1 - T_0} = e^{(T_1 - T_0)(\lambda^2/2 - \lambda \Delta_0)}.
\]

\( \square \)
6 Minimax lower bounds

In this section, we characterize the best possible performance under relaxations of the i.i.d. assumption. In particular, we quantify the best any prediction policy can do with oracle knowledge of the number of effective experts. The proof of this result is found in Appendix E.1. While we do not expect a player to be able to know the nature of the constraint set, we use this oracle lower bound to conclude that since our novel algorithm Meta-CARE achieves the same performance without using oracle knowledge, it is adaptively minimax optimal.

**Theorem 3.** There exist \( \hat{\mathcal{Y}}, \mathcal{Y}, \) and \( \ell \) such that, for all \( N_0 \in \mathbb{N} \), there exists \( t_0 \in \mathbb{N} \) such that for all \( N \in \mathbb{N} \) with \( N \geq N_0 \) and \( T \geq t_0 \),

\[
\sup_{D \in \mathcal{V}(N, N_0, 1/2)} \sup_{\pi \in \mathcal{P}(D)} \inf_{\hat{\pi} \in \mathcal{P}_N} \mathbb{E}_{\pi, \hat{\pi}} R(T) \geq \frac{\sqrt{T \log N_0}}{10}.
\]

Theorem 3 allows us to characterize the minimax optimal dependence on \( T \) and \( N_0 \) of a prediction policy. However, for the case of \( N_0 = 1 \), the leading term instead depends on \( \Delta_0 \). Consequently, to determine the minimax optimal rate of regret at all relaxations of the i.i.d. assumption, we must also use the the following result by Mourtada and Gaïffas [44], which establishes a lower bound for when there is only one effective expert.

**Proposition 2** (Mourtada and Gaïffas [44], Proposition 4). For all \( N \in \mathbb{N} \), there exist \( \hat{\mathcal{Y}}, \mathcal{Y}, \) and \( \ell \) such that for all \( \Delta \in (0, 1/4) \) and \( T \geq \frac{\log N}{16\Delta^2} \),

\[
\inf_{\hat{\pi} \in \mathcal{P}_N} \sup_{D \in \mathcal{V}(N, 1, \Delta)} \sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} R(T) \geq \frac{\log N}{256\Delta}.
\]

These two lower bounds set the bar for what one should hope to achieve. In order to adapt to an unknown number of effective experts \( N_0 \leq N \) and identity of the effective experts, the player can be forced to incur \( \max(\sqrt{T \log N_0}, \Delta_0^{-1} \log N) \) rate of regret. Because \( \max(\sqrt{T \log N_0}, \Delta_0^{-1} \log N) \propto \sqrt{T \log N_0 + \Delta_0^{-1} \log N} \), a prediction algorithm with a rate of regret \( \lesssim \sqrt{T \log N_0 + \Delta_0^{-1} \log N} \) is adaptively minimax optimal.

7 Performance of D.Hedge

In this section, we show that without oracle knowledge of the characterizing quantities, D.Hedge can be parametrized to either (a) be minimax optimal for the special case when \( N_0 \in \{1, N\} \), but incur adversarial regret in between, or (b) adapt suboptimally to every value of the characterizing quantities. Following this section, we introduce FTRL-CARE and prove it adapts minimax optimally when there are multiple effective experts. We then *boost* these two algorithms together in Meta-CARE, and prove this is adaptively minimax optimal.

All of these prediction algorithms produce *proper prediction policies*, which means that rather than picking \( \hat{y} \) from the entirety of \( \hat{\mathcal{Y}} \), at each round the player chooses one of the experts \( i \in [N] \) to emulate and predicts \( \hat{y}(t) = x_i(t) \). To choose the expert to emulate, the history is used to choose a distribution on \( [N] \), and then \( i \) is sampled from this distribution.

Formally, for \( x \in \hat{\mathcal{Y}}^N \) and \( w \in \text{simp}([N]) \), let \( x_i w = \sum_{i \in [N]} w_i \delta_{x_i} \in \mathcal{M}(\hat{\mathcal{Y}}) \) be the pushforward of \( w \in \text{simp}([N]) \) through \( x \), viewing the vector \( x \) as a function \( x : [N] \to \hat{\mathcal{Y}} \) and identifying \( \text{simp}([N]) \) with \( \mathcal{M}([N]) \). A proper prediction policy \( \hat{\pi}^* = (\hat{\pi}_t^*)_{t \in \mathbb{N}} \) is any prediction policy such that, for all \( t \in \mathbb{N} \), there exists a measurable map \( w_t^* : \mathcal{H}^{t-1} \to \text{simp}([N]) \) satisfying, for all \( h \in \mathcal{H}^{t-1} \) and
\[ x \in \hat{Y}^N, \pi_i^*(h, x) = x_i^*[w_i^*(h)]. \] The \( \sigma(h(t-1)) \)-measurable random variable \( w(t) = w_i^*(h(t-1)) \) is called the weight vector, or simply the weights. For each \( i \in [N] \), \( w_i(t) \) corresponds to the probability that the player will emulate the \( i \)th expert’s prediction at time \( t \).

The prediction algorithm HEDGE is parametrized by a sequence of measurable functions \((\hat{\eta}_t)_{t \in \mathbb{N}} \in \prod_{t \in \mathbb{N}} \mathcal{H}^{t-1} \to \mathbb{R}_+\). The \( \sigma(h(t-1)) \)-measurable random variable \( \eta(t) = \hat{\eta}_t(h(t-1)) \) is called the learning rate, and the weights are defined by

\[
w_i^\eta(t) = \frac{\exp\{-\eta(t)L_i(t-1)\}}{\sum_{\nu \in [N]} \exp\{-\eta(t)L_{\nu}(t-1)\}}, \quad i \in [N].
\]

The prediction algorithm DECREASING HEDGE (D.HEDGE) is parametrized by a function \( g : \mathbb{N} \to \mathbb{R}_+ \), and corresponds to HEDGE with the deterministic learning rate \( \eta(t) = g(N)/\sqrt{t} \) for all \( t \in \mathbb{N} \).

It is well-known (see, for example, Theorem 2.3 of Cesa-Bianchi and Lugosi [15]) that D.HEDGE with \( g(N) \propto \sqrt{\log N} \) is minimax optimal in the adversarial setting, which corresponds to \( \mathcal{D} = \mathcal{M}(\hat{Y}^N \times \mathcal{Y}) \). Recently, Mourtada and Gaïffas [44] showed that D.HEDGE with this parametrization is also minimax optimal in the i.i.d. setting, which corresponds to \(|\mathcal{D}| = 1\). One might hope that this stochastic-and-adversarially minimax optimal parametrization would also perform well for all convex \( \mathcal{D} \) in between these two cases. However, part (i) of Theorem 4 shows that, in fact, this parametrization fails to adapt to the number of effective experts when \( N_0 \not\in \{1, N\} \). Further, we show that a different parametrization can adapt in some ways, but does not achieve the minimax optimal dependence on \( T \).

### 7.1 Algorithm-specific lower bounds for D.HEDGE

First, we observe that D.HEDGE with \( g(N) \propto \sqrt{\log N} \), which is minimax optimal for both the stochastic and adversarial cases, does not adapt to an intermediate number of effective experts. Additionally, D.HEDGE with constant \( g \) can do better than the stochastic-and-adversarially minimax optimal parametrization, but still cannot do as well as the oracle knowledge dependence on \( T \) given in Theorem 3. We prove this result in Appendix E.2.

**Theorem 4.** (i) For all \( c > 0 \),

\[
N \geq \exp\left\{ \left( \frac{72 \log 2}{c^2} + 9 \right) e^{c^2/4} \right\}, \quad \text{and} \quad 2 \leq N_0 \leq e^{-c^2/8} N e^{c^2 \exp(c^2/4)/72} - 1,
\]

there exist \( \hat{Y}, \mathcal{Y}, \) and \( \ell \) such that for all \( T \geq 16e^{-2}\log N \), D.HEDGE with \( g(N) = c\sqrt{\log N} \) satisfies

\[
\sup_{\mathcal{D} \in \mathcal{V}(N,N_0,1/2)} \sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi,H} R(T) \geq \frac{c\sqrt{T \log N}}{72 \exp\{c^2/4\}} - \frac{1}{3c^2} - \frac{\log N}{3}.
\]

(ii) Suppose the player is allowed oracle knowledge of \( N_0 \) in addition to \( N \), and consequently can parametrize D.HEDGE by any \( g : \mathbb{N}^2 \to \mathbb{R}_+ \). For all \( 81 < N_0 \leq N \) there exist \( \hat{Y}, \mathcal{Y}, \) and \( \ell \) such that D.HEDGE with \( g(N,N_0) \leq 2\sqrt{\log N_0 - 4\log 3} \) satisfies that for all \( T \geq 32[g(N,N_0)]^{-2} \log N \),

\[
\sup_{\mathcal{D} \in \mathcal{V}(N,N_0,1/2)} \sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi,H} R(T) \geq \frac{\log(N_0)\sqrt{T}}{4g(N,N_0)} - \frac{3 \log N_0}{[g(N,N_0)]^2}.
\]

The proof of Theorem 4 can be used to argue that other “adaptive” variants of HEDGE will also fail to be adaptively minimax optimal along the semi-adversarial spectrum. We highlight this argument
using well-known Hedge-variants from the literature. This is not meant to disparage these works, as they should not be expected to design algorithms for a notion of optimality defined years later, but to exemplify that adapting along the semi-adversarial spectrum is non-trivial and that the objectives of earlier works are insufficient to capture the notion of optimality we introduce.

The algorithm PROD of Cesa-Bianchi et al. [17] is essentially D.Hedge with an adaptive learning rate shared by all experts. This adaptive learning rate is comprised of the reciprocal-square-root of the cumulative squared losses, which will be (essentially) a constant multiple of $t$ under the data-generating mechanism described in the proof of Theorem 4. Thus, the learning rate will behave the same as the data-independent learning rate of D.Hedge, and consequently a similar lower bound on performance applies. A similar argument would also hold for AdaHedge [22].

The refined algorithm ADAPT-ML-PROD of Gaillard et al. [25] is more subtle, since it has a different learning rate for each expert. However, the recommended learning rate (Corollary 4 of their paper) would not achieve this since it uses $\log N$ for all experts, as opposed to an adaptive quantity as in FTRL-CARE. Consequently, for large enough $N_0$ and $t$, the data-generating mechanism of Theorem 4 will make the average loss with respect to the ADAPT-ML-PROD weights roughly $1/2$, and thus ADAPT-ML-PROD inherits the same order of lower bound as D.Hedge.

### 7.2 Upper bounds for D.Hedge

Now, we show that the lower bound of Theorem 4 is tight. For a prediction policy $\hat{\pi}'$ that may be distinct from the actual prediction policy $\hat{\pi}$ the player is using, we define the quasi-regret (with respect to $\hat{\pi}'$) at time $T$ by

$$\hat{R}_{\pi'}(T) = \sum_{t=1}^{T} \int \ell(\hat{g}(t), y(t))\hat{\pi}'\left((h(t-1), x(t)), \hat{g}(t)\right) - \min_{i\in[N]} \sum_{t=1}^{T} \ell(x_i(t), y(t)).$$

Quasi-regret replaces the actual loss at each round $t$ with the conditional expectation of the player’s loss had that player played according to $\hat{\pi}'$ on round $t$; the histories correspond, however, to the actual predictions made by $\hat{\pi}$. This allows us to quantify the performance of $\hat{\pi}'$ even when the entire sequence of predictions is governed by $\hat{\pi}$.

Clearly, $\mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\pi'}(T) = \mathbb{E}_{\pi,\hat{\pi}} R(T)$. However, we can prove almost sure results about $\hat{R}_{\pi'}(T)$ for some prediction policy $\hat{\pi}'$, and then state expectation results of the form $\mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\pi'}(T)$, where the expectation is with respect to a possibly different prediction policy $\hat{\pi}$. Results of this nature are crucial in the proof of Theorem 8, where we use them to control the regret accumulated by D.Hedge and FTRL-CARE when the actual prediction policy is META-CARE.

**Theorem 5.** For all $g : \mathbb{N} \to \mathbb{R}_+$ used to parametrize D.Hedge, all $N \geq 2$, prediction policies $\hat{\pi} \in \hat{\mathcal{D}}_N$, convex $\mathcal{D} \subseteq \mathcal{M}(\mathcal{Y}^N \times \mathcal{Y})$, and $T \in \mathbb{N}$,

$$\sup_{\pi \in \mathcal{D}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\pi}(T) \leq \sqrt{T + 1} \left( \frac{\log N}{g(N)} + g(N) \right).$$

Moreover, when $T > \left[ \frac{8([g(N)]^2/4 + g(N))^2}{[g(N)]^2 \Delta_0} \right]$ the following two cases hold:

* If $N_0 > 1$, then

$$\sup_{\pi \in \mathcal{D}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\pi}(T) \leq \frac{17}{16} \sqrt{T} \left( \frac{\log N_0}{g(N)} + g(N) \right) + \frac{32}{\Delta_0} \left( \frac{\log N}{g(N)} \right) \left( \frac{\log N}{g(N)} + g(N) \right)$$

$$+ \sqrt{2} \left( \frac{\log N}{g(N)} + g(N) \right),$$

* else
and if $N_0 = 1$, then
\[
\sup_{\pi \in \mathcal{P}^D} \mathbb{E}_{\pi, \mathbb{H}} \hat{R}_H(T) \leq \frac{5}{\Delta_0} \left[ \left( \frac{\log N}{g(N)} \right) \left( \frac{\log N}{g(N)} + g(N) \right) + 4 \left( \frac{1}{g(N)^2} + g(N)^2 \right) \right] + \sqrt{2} \left( \frac{\log N}{g(N)} + g(N) \right).
\]

In order to more easily interpret this result, we also state the expected regret of $D$.Hedge for various natural choices of $g$.

**Remark 1.**

Taking $\hat{\pi}$ to be determined by $D$.Hedge,

(i) if $g(N)$ is constant,
\[
\sup_{\pi \in \mathcal{P}_{N, (N_0, \Delta_0)}} \mathbb{E}_{\pi, \mathbb{H}} R(T) \lesssim \log(N_0) \sqrt{T} + \frac{(\log N)^2}{\Delta_0};
\]

(ii) if $g(N) \propto \sqrt{\log N}$,
\[
\sup_{\pi \in \mathcal{P}_{N, (N_0, \Delta_0)}} \mathbb{E}_{\pi, \mathbb{H}} R(T) \lesssim I_{[N_0 \geq 2]} \sqrt{TN} + \frac{\log N}{\Delta_0};
\]

(iii) in the oracle setting for $N_0 \geq 2$, if $g(N, N_0) \propto \sqrt{\log N_0}$,
\[
\sup_{\pi \in \mathcal{P}_{N, (N_0, \Delta_0)}} \mathbb{E}_{\pi, \mathbb{H}} R(T) \lesssim \sqrt{TN_0} + \frac{(\log N)^2}{\Delta_0 \log N_0}.
\]

**Remark 2.** If $g(N) \propto \sqrt{\log N}$, then Theorem 4(i) combined with Remark 1(ii) shows that the dependence on $T$ is tight in Theorem 5. If oracle knowledge of $N_0$ is used to choose $g(N, N_0) \propto \sqrt{\log N_0}$, then Theorem 4(ii) simply matches the oracle lower bound of Theorem 3, confirming the dependence on $T$ is tight in Theorem 5 (see Remark 1(iii)). Finally, if $g$ is constant, then Theorem 4(ii) combined with Remark 1(i) shows that the dependence on $T$ is tight in Theorem 5.

Together with the minimax lower bounds of Section 6, we find that, for the stochastic and adversarial settings, our expected regret bound for $D$.Hedge with $g(N) = \sqrt{\log N}$ is tight up to constants and that the algorithm achieves the minimax optimal rates, as noted by Mourtada and Gaïffas [44]. Furthermore, we have improved upon Corollary 6 of Mourtada and Gaïffas [44] in the “adversarial-with-an-$\mathcal{E}$-gap” setting (see Example 4), having removed the extra $\Delta_0^{-1} \log(\Delta_0^{-1})$ dependence that separated the upper and lower bounds in their work.

### 8 Beating D.Hedge without oracle knowledge

In Section 7, we completed the story of $D$.Hedge by showing that it does not adapt minimax optimally to all possible constraint sets without oracle knowledge of the number of effective experts. It is natural to ask whether we can design an algorithm that adapts to the number of effective experts and has a rate of regret no larger than $\sqrt{T \log N_0}$. 


In this section, we present a modified algorithm that does exactly this. Taking inspiration from the fact that D.HEDGE can be viewed as follow-the-regularized-leader (FTRL) using entropic regularization (see, for example, Section 3.6 of McMahan [42]), we introduce the constraint-adaptive root-entropic (CARE) regularizer. We are able to prove upper bounds for the performance of FTRL for a large class of regularizers, and then use these upper bounds to prove both the upper bound results of Section 7 and the upper bounds for our improved algorithm, by viewing D.HEDGE and FTRL-CARE as FTRL with specifically chosen regularizers. Our bound shows that FTRL-CARE achieves the oracle rate $\sqrt{T \log N_0}$ without requiring knowledge of the characterizing quantities for the constraint set $D$.

8.1 FTRL algorithms

FTRL is a generic method for online optimization. In the setting of sequential prediction with expert advice, FTRL is parametrized by a sequence of regularizers $\{r_t : \text{simp}([N]) \to [0,\infty)\}_t \in \mathbb{Z}_+$. Each such sequence, subject to regularity conditions on the regularizers (see Appendix B), determines a unique proper prediction policy. For each time $t+1$, a player using the FTRL$(\{r_t\}_t \in \mathbb{Z}_+)$ algorithm has a proper prediction policy defined uniquely by the weight vectors given by

$$u(t+1) = \arg\min_{u \in \text{simp}([N])} (\langle L(t), u \rangle + r_{0:t}(u)),$$

where $r_{0:t}(u) = \sum_{s=0}^{t} r_s(u)$, and the existence and uniqueness of the arg min is ensured by the regularity properties of the regularizer. This class of algorithms is well studied in online optimization; for specific results relevant to this work, see Appendix B.

8.2 The constraint-adaptive root-entropic regularizer

First, we note that FTRL directly generalizes D.HEDGE. In particular, letting $H(u) = -\sum_{i \in [N]} u_i \log(u_i)$ denote the entropy function, it is well known that, for $r_{0:t}(u) = -\sqrt{t+1} H(u)/g(N)$, the weights played by FTRL$(\{r_t\}_t \in \mathbb{Z}_+)$ are equal to the weights played by D.HEDGE. We modify the entropic regularizer to achieve improved performance for data-generating mechanisms strictly between stochastic and adversarial.

In order to motivate this new algorithm, we provide the following motivating intuition. First, from Remark 1, playing D.HEDGE with $g(N, N_0) \propto \sqrt{\log N_0}$ achieves the oracle rate. Second, the minimax optimal data-generating mechanism subject to the time-homogeneous convex constraint forces the minimax optimal prediction policy to “concentrate” to Unif($\mathcal{I}_0$). Finally, for $u = \text{Unif}(\mathcal{I}_0)$, $H(u) = \log N_0$. These three observations together suggest that, heuristically, playing Hedge with the “adaptive” learning rate $\eta(t) = \sqrt{H(u(t))/t}$ may lead to an oracle rate of regret. However, $u(t)$ is defined in terms of $\eta(t)$, so this is an implicit system of equations to be solved at each time $t$. In order to define our modification of FTRL, we choose a regularizer such that the solution to the FTRL optimization problem gives rise to a similar system of equations. In particular, for some parameters $c_1, c_2 > 0$, the sequence of regularizers is given by

$$r_{0:t}(u) = -\frac{\sqrt{t+1}}{c_1} \sqrt{H(u) + c_2}.$$  

We call $-r_0$ defined by Eq. (4) a root-entropy function, and regularization with $\{r_t\}_t \in \mathbb{Z}_+$ constraint-adaptive root-entropic (CARE) regularization. We refer to the algorithm FTRL$(\{r_t\}_t \in \mathbb{Z}_+)$ with $r_t$ induced by Eq. (4) as follow-the-regularized-leader with constraint-adaptive root-entropic regularization (or, FTRL-CARE).
Throughout the remainder of the paper, we will use \( u \) for the weights output by the FTRL(\( \{r_t\}_{t \in \mathbb{Z}^+} \)) algorithm with a generic regularizer, \( w^h \) for weights output via entropic regularization (D.Hedge), and \( w^c \) for weights output via root-entropic regularization (FTRL-CARE). Pseudocode for an efficient implementation of FTRL-CARE may be found in Appendix F.

### 8.3 Performance of FTRL-CARE

**Theorem 6.** For all \( c_1, c_2 > 0 \) used to parametrize FTRL-CARE, there exist \( C_1, \ldots, C_4 \) such that for all \( n \geq 2 \), prediction policies \( \pi \in \mathcal{P}_N \), convex \( D \subseteq M(Y \times Y) \), and \( T \in \mathbb{N} \),

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \hat{R}_C(T) \leq C_1 \sqrt{(T + 1)[\log N + c_2]}.
\]

Moreover, when \( T \geq \left\lceil \frac{2[\log N + C_1]^2}{c_1^2 c_2 \Delta_0^2} \right\rceil \), the following two cases hold:

If \( N_0 > 1 \), then

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \hat{R}_C(T) \leq \frac{33C_1}{32} \sqrt{(T + 1)[\log N_0 + c_2]} + C_2 \frac{[\log N + C_4]^{3/2}}{\Delta_0} + C_3 \frac{\Delta_0}{\Delta_0},
\]

and if \( N_0 = 1 \), then

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \hat{R}_C(T) \leq C_2 \frac{[\log N + C_4]^{3/2}}{\Delta_0} + C_3 + 6 \frac{\Delta_0}{\Delta_0}.
\]

The constants \( C_1, \ldots, C_4 \) appearing above are given by:

\[
C_1 = \left( \frac{1}{c_1} + \frac{3c_1}{2} \right), \quad C_2 = \sqrt{2} C_1 \left( \frac{1}{c_1 \sqrt{c_2}} + \frac{1}{c_2} \right), \quad C_3 = \sqrt{2} \frac{8 + 12c_1^2}{3c_1^2 \sqrt{c_2}}, \quad \text{and} \quad C_4 = \max \left\{ c_2, 3c_1 \sqrt{c_2} + \frac{5c_1^2 c_2}{4} \right\}.
\]

With \( c_1 = c_2 = 1 \) this simplifies to: for all \( n \in \mathbb{N} \),

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \hat{R}_C(T) \leq 3 \sqrt{(T + 1)[\log N + 1]},
\]

and when \( T \geq \left\lceil \frac{2[\log N + 5]^2}{\Delta_0^2} \right\rceil \), if \( N_0 > 1 \), then

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \hat{R}_C(T) \leq 3 \sqrt{(T + 1)[\log N_0 + 1]} + 8 \frac{[\log N + 5]^{3/2}}{\Delta_0} + 10 \frac{\Delta_0}{\Delta_0},
\]

and if \( N_0 = 1 \), then

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \hat{R}_C(T) \leq \frac{8 [\log N + 5]^{3/2}}{\Delta_0} + 16 \frac{\Delta_0}{\Delta_0}.
\]

**Remark 3.** Taking \( \hat{\pi} \) to be determined by FTRL-CARE,

\[
\sup_{\pi \in \mathcal{P}(N, (N_0, \Delta_0))} \mathbb{E}_{\pi, C} R(T) \leq \sqrt{T \log N_0} + \frac{[\log N]^{3/2}}{\Delta_0}.
\]

**Remark 4.** Note that in the case \( N_0 = 1 \), this is worse than D.Hedge with learning rate \( g(N) \propto \sqrt{\log N} \), which has \( \Delta_0^{-1} \log N \) rate of regret. We resolve this in Section 10 by introducing a new algorithm, Meta-CARE, that combines the optimality of D.Hedge in the stochastic case and FTRL-CARE elsewhere.
9 Proofs of upper bounds

The proofs of Theorems 5 and 6 rely on several technical results regarding online linear optimization developed in Appendices B and C. In order to simplify notation for FTRL with regularizers that are transformations of the entropy function, we let FTRL\(_H(ψ, β)\) denote FTRL(\(\{r_t\}_{t ∈ ℤ_+}\)) with \(r_{0,t} = -β(t)[ψ ◦ H]\) for any strictly increasing, concave, and twice continuously differentiable function \(ψ : [0, \log N] → ℝ\) and strictly increasing \(β : ℤ_+ → ℝ_+\). The important conclusions from Appendices B and C are summarized in the following result, the proof of which appears in Appendix C.1. The result tells us that the weights played by a player employing the FTRL\(_H(ψ, β)\) strategy are equivalent to the weights played by HEDGE with an implicitly defined, non-deterministic learning rate, and also provides a second-order bound on the quasi-regret incurred.

**Theorem 7.** For every strictly increasing \(β : ℤ_+ → ℝ\), and every strictly increasing, concave, and twice continuously differentiable function \(ψ : [0, \log N] → ℝ\), the solutions to Eq. (3) at time \(t\) for FTRL\(_H(ψ, β)\) given any history and expert predictions satisfy the system of equations

\[
η(t + 1) = \frac{1}{β(t) \cdot [ψ' ◦ H](u(t + 1))}, \quad u(t + 1) = \left(\frac{\exp\{-η(t + 1)L_i(t)\}}{\exp\{-η(t + 1)L_{\hat{\psi}}(t)\}}\right)_{i ∈ [N]}.
\]

Moreover, for any sequence of losses \((ℓ(t))_{t ∈ ℤ} ⊆ [0, 1]^N\), this system has a unique solution satisfying

\[
η(t + 1) ∈ \left[\frac{1}{β(t) \cdot ψ'(0)}, \frac{1}{β(t) \cdot ψ'(\log N)}\right],
\]

and there exists a sequence \(\{α_t\}_{t ∈ ℤ_+} ⊆ [0, 1]\) such that the quasi-regret satisfies

\[
\hat{R}_{FTRL}(T) ≤ -β(T)ψ(0) + β(0)[ψ ◦ H](u(1)) + \sum_{t=1}^{T} [β(t) - β(t - 1)] \cdot [ψ ◦ H](u(t + 1))
\]

\[
+ \sum_{t=1}^{T} \sqrt{\text{Var}_{I ∼ ψ(t+1)} \left(\frac{β(t)}{β(t - 1)} - 1\right) L_I(t - 1) - ℓ_I(t) \times \text{Var}_{I ∼ ψ(t+1)} \left(\ell_I(t) \cdot \frac{1}{β(t) \cdot [ψ' ◦ H](v(t + 1))}\right)}.
\]

where for each \(t ∈ ℤ_+\),

\[
v(t + 1) = v^{(α)}(t + 1),
\]

and for every \(t ∈ ℤ_+\) and \(α ∈ [0, 1]\), we define

\[
v^{(α)}(t + 1) = \arg\min_{v ∈ \text{simp}([N])} \left\{αL(t) + (1 - α)\sqrt{\frac{t + 1}{t} L(t - 1), v} - \sqrt{t + 1} [ψ ◦ H](v)\right\}.
\]

Ultimately, we wish to apply Theorem 7 to both D.HEDGE and FTRL-CARE. Recall that D.HEDGE corresponds to

\[
ψ(s) = \frac{s}{g(N)}, \quad ψ'(s) = \frac{1}{g(N)}, \quad \text{and} \quad β(t) = \sqrt{t + 1},
\]

and therefore, in Eq. (5),

\[
\frac{1}{β(t) \cdot [ψ' ◦ H](v(t + 1))} = \frac{g(N)}{\sqrt{t + 1}}.
\]

FTRL-CARE with parameters \(c_1, c_2 > 0\) corresponds to

\[
ψ(s) = \frac{\sqrt{s + c_2}}{c_1}, \quad ψ'(s) = \frac{1}{2c_1 \sqrt{s + c_2}}, \quad \text{and} \quad β(t) = \sqrt{t + 1},
\]

19
and therefore, in Eq. (5),
\[ \frac{1}{\beta(t) \cdot [\psi' \circ H](v(t + 1))] = 2c_1 \sqrt{\frac{H(v(t + 1)) + c_2}{t + 1}}. \]

Both correspond to the choice $\beta(t) = \sqrt{t + 1}$, so we focus on this rather than continuing to use a generic $\beta(t)$. We leave $\psi$ as generic for the moment, since the following result equally applies to the algorithms’ respective $\psi$ functions. Finally, we wish to move towards proving bounds on the expected regret, which will require taking expectation with respect to a data-generating mechanism $\pi$, so we fix a convex $D \subseteq \mathcal{M}(\mathcal{Y}^N \times \mathcal{Y})$ that characterizes the allowable data-generating mechanisms.

In order to control the quasi-regret using Theorem 7, we need to control the entropy of the FTRL weights $u$ as well as the intermediate weights $v$ (defined in Eq. (6)). The following lemma provides the necessary control, which we prove in Appendix D.1.

**Lemma 1.** For every $u \in \text{simp}(\mathcal{N})$ and $p \in (0, 1)$,
\[ H(u) \leq \frac{2}{e \log 2} \log N_0 + \left(1 + \frac{1}{(1 - p)e}\right) \sum_{i \in [N] \setminus \mathcal{I}_0} [u_i]^p. \] (7)

Our next lemma bounds the expectation of the second term on the RHS of Eq. (7) for the FTRL weights. Combined with the previous result, this allows us to bound the expected entropy of the weights. Crucially, the bound on the expected weights that FTRL would produce holds regardless of whether the actual prediction policy used is FTRL or some other policy, allowing us to control the expected quasi-regret of FTRL when a different policy is used to interact with the environment, as in the statements of Theorems 5 and 6.

**Lemma 2.** Letting $u$ denote the weights output by the FTRL($\psi$, $t \mapsto \sqrt{t + 1}$) algorithm, for every prediction policy $\hat{\pi}$, $t \in \mathbb{N}$, $p > 0$, and $i \in [N] \setminus \mathcal{I}_0$,
\[ \sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \left[ [u_i(t + 1)]^p \right] \leq \exp \left\{ \frac{p^2}{2(\psi'(0))^2} - \frac{\Delta_0 p \sqrt{t}}{\sqrt{2(\psi'(0))}} \right\}, \]
and
\[ \sup_{\pi \in \mathcal{P}(D)} \sup_{\alpha \in [0, 1]} \mathbb{E}_{\pi, \hat{\pi}} \left[ [u_i^{(\alpha)}(t + 1)]^p \right] \leq \exp \left\{ \frac{2p}{\psi'(0)} + \frac{p^2}{2(\psi'(0))^2} - \frac{\Delta_0 p \sqrt{t}}{\sqrt{2(\psi'(0))}} \right\}. \]

The intuition underlying the proof of this result is as follows. First, let $\eta(t + 1) = \frac{1}{\sqrt{t + 1} \psi'(0)}$. Note that for $(u(t))_{t \in \mathbb{N}}$ and $(\eta(t))_{t \in \mathbb{N}}$ given in Theorem 7, $\eta(t + 1) \leq \eta(t + 1)$ for all $t \in \mathbb{N} \cup \{0\}$. Let $L(t) = \arg\min_{i \in [N]} L_i(t)$, so that for any $i \in [N]$, $L_i(t) = L_i(t)$. Thus,
\[ [u_i(t + 1)]^p \leq \left( \frac{u_i(t + 1)}{u_i(t + 1)} \right)^p \leq \min_{i_0 \in \mathcal{I}_0} \exp \left\{ -p [\eta(t + 1)] L_i(t) - L_i(t) \right\}. \]

Applying Theorem 2,
\[ \sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \left[ [u_i(t + 1)]^p \right] \leq \exp \left\{ -t\eta(t + 1)\Delta_0 p + t\eta(t + 1)^2 p^2 / 2 \right\}. \]

The argument for the intermediate weights is similar. For the complete proof, see Appendix D.2.
By combining Lemma 2 with Lemma 1 for $p = 1/2$, and noting that, for all $t \in \mathbb{N}$, $2/(e \log 2) < 17/16$ and $1 + 2/e < 7/4$, it holds that

$$
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\theta}} H(u(t + 1)) \leq \frac{17}{16} \log N_0 + \frac{7}{4} (N - N_0) \exp \left\{ \frac{1}{8(\psi'(0))^2} - \frac{\Delta_0 \sqrt{t}}{2\sqrt{2(\psi'(0))}} \right\},
$$

(8)

and

$$
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\theta}} \sup_{\alpha \in [0,1]} H(\nu(\alpha)(t + 1)) \leq \frac{17}{16} \log N_0 + \frac{7}{4} (N - N_0) \exp \left\{ \frac{1}{\psi'(0)} + \frac{1}{8(\psi'(0))^2} - \frac{\Delta_0 \sqrt{t}}{2\sqrt{2(\psi'(0))}} \right\}.
$$

(9)

These bounds can now be used for the regularizers specific to D.Hedge and FTRL-CARE. Our approach will be to break up the sums of Eq. (5) into the first $t_0$ rounds and then the remaining rounds for some carefully chosen $t_0$. Note that $t_0$ is not a parameter of the algorithm, but rather an artifact of our proof. The rounds after $t_0$ will be handled using our entropy bounds above, but the early rounds we control with the following worst-case bound. The proof of the following result appears in Appendix D.3. Note that it recovers the correct order of standard adversarial bounds for D.Hedge.

**Lemma 3.** For every $t_0 \in \mathbb{N}$ and sequence of losses $\{\ell(t)\}_{t \in \mathbb{N}} \subset [0, 1]^N$, the weights played by FTRL$_H(\psi, t \mapsto \sqrt{t + 1})$ satisfy

$$
\hat{R}_{FTRL_H}(t_0) \leq \left( \psi(\log N) - \psi(0) + \frac{3}{4\psi'((\log N))} \right) \sqrt{t_0 + 1}.
$$

The remainder of the proofs of Theorems 5 and 6 can be found in Appendix A, which consists of substituting in the specific expression for $\psi$ to Theorem 7, Eqs. (8) and (9), and Lemma 3. Then, the variance terms are controlled by a worst case bound for $N_0 > 1$, and by Lemma 9 for $N_0 = 1$, and the summation terms are controlled by an integral comparison (see Lemma 10). Finally, $t_0$ is chosen as specified by the statements of Theorems 5 and 6 respectively.

### 10 CARE if you can, HEDGE if you must, or META-CARE for all

Since we have seen in Theorem 5 that D.Hedge with $g(N) = \sqrt{\log N}$ achieves the minimax optimal order of $\log N$ when $N_0 = 1$, and Theorem 6 shows that FTRL-CARE is minimax optimal in all other cases, it is natural to try to combine these two learners in order to have minimax optimal rate of regret for all values of $N_0$ and $\Delta_0$. To achieve this, we introduce the META-CARE algorithm.

Intuitively, META-CARE plays both D.Hedge and FTRL-CARE, treating them as two meta-experts. META-CARE outputs the weighted average of the predictions made by the two meta-experts, where the weighting output by D.Hedge based on their respective losses. Consequently, META-CARE has four parameters: $c_H, c_{C,1}, c_{C,2}, c_M > 0$. Formally, for each $t \in \mathbb{N}$, let $w^H(t)$ denote the weight vector produced by D.Hedge with $g(N) = c_H \sqrt{\log N}$ at time $t$ and let $w^C(t)$ denote the weight produced by FTRL-CARE with parameters $c_{C,1}, c_{C,2}$ at time $t$. Consider the meta-losses defined by

$$
\ell_H(t) = \langle \ell(t), w^H(t) \rangle, \quad \ell_C(t) = \langle \ell(t), w^C(t) \rangle,
$$

$$
L_H(t) = \sum_{s=1}^{t} \ell_H(t), \quad L_C(t) = \sum_{s=1}^{t} \ell_C(t).
$$

21
Then, for each \( t \in \mathbb{N} \), Meta-CARE produces the weight vector

\[
\hat{w}^M(t + 1) = \frac{\exp \left\{ - \eta_0(t) \ell(t) \right\} w^M(t + 1) + \exp \left\{ - \eta_0(t) L(t) \right\} w^C(t + 1)}{\exp \left\{ - \eta_0(t) \ell(t) \right\} + \exp \left\{ - \eta_0(t) L(t) \right\}},
\]

where \( \eta_0(t) = c_M / \sqrt{t} \). Observe that \( \hat{w}^M(t + 1) \) will be an element of \( \text{simp}([N]) \) since it is a convex combination of \( w^M(t + 1) \) and \( w^C(t + 1) \), both of which are elements of \( \text{simp}([N]) \).

**Theorem 8.** Meta-CARE parametrized by \( c_M = \sqrt{\log N} \) and \( c_{C,1} = c_{C,2} = c_M = 1 \) incurs

\[
\sup_{\pi \in \mathcal{P}_N(N_0, \Delta_0)} \mathbb{E}_{\pi, M} R(T) \lesssim \sqrt{T \log N_0} + \frac{\log N}{\Delta_0}.
\]

We do not state a detailed quantitative form of Theorem 8, since our proof can be easily extended for any arbitrary \( \hat{\pi} \) to a bound on \( \mathbb{E}_{\pi, \hat{\pi}} \hat{R}_M(T) \) with exact constants using the statements and proofs of Theorems 5 and 6.

**Proof of Theorem 8.** For \( N_0 \geq 2 \), we decompose the quasi-regret of Meta-CARE into components coming from the quasi-regret due to meta-learning and the quasi-regret of the better of the two meta-experts. In particular, for any sequence of losses \( \langle \ell(t) \rangle_{t \in \mathbb{N}} \), we can write

\[
\hat{R}_M(T) = \sum_{t=1}^{T} \langle \ell(t), \ w^M(t) \rangle - \min_{i \in [N]} \sum_{t=1}^{T} \ell_i(t)
= \left[ \sum_{t=1}^{T} \langle \ell(t), \ w^M(t) \rangle - \min \left( \sum_{t=1}^{T} \langle \ell(t), \ w^M(t) \rangle, \sum_{t=1}^{T} \langle \ell(t), \ w^C(t) \rangle \right) \right]
+ \min \left( \hat{R}_M(T), \hat{R}_C(T) \right).
\]

Therefore, for any \( N_0 \leq N \) and \( \Delta_0 \),

\[
\sup_{\pi \in \mathcal{P}_N(N_0, \Delta_0)} \mathbb{E}_{\pi, M} R(T)
\leq \sup_{\pi \in \mathcal{P}_N(N_0, \Delta_0)} \mathbb{E}_{\pi, M} \left[ \sum_{t=1}^{T} \langle \ell(t), \ w^M(t) \rangle - \min \left( \sum_{t=1}^{T} \langle \ell(t), \ w^M(t) \rangle, \sum_{t=1}^{T} \langle \ell(t), \ w^C(t) \rangle \right) \right]
+ \sup_{\pi \in \mathcal{P}_N(N_0, \Delta_0)} \mathbb{E}_{\pi, M} \min \left( \hat{R}_M(T), \hat{R}_C(T) \right).
\]

First, we consider the case when \( N_0 \geq 2 \). Since Meta-CARE is D.HEDGE with two experts given by the predictions of D.HEDGE and FTRL-CARE, Theorem 5 implies that

\[
\sup_{\pi \in \mathcal{P}_N(N_0, \Delta_0)} \mathbb{E}_{\pi, M} \left[ \sum_{t=1}^{T} \langle \ell(t), \ w^M(t) \rangle - \min \left( \sum_{t=1}^{T} \langle \ell(t), \ w^M(t) \rangle, \sum_{t=1}^{T} \langle \ell(t), \ w^C(t) \rangle \right) \right]
\leq \sqrt{T + 1} \left( \log(2) / c_M + \frac{3c_M}{4} \right).
\]

Then, since \( (\log N)^{3/2} \Delta_0^{-1} \) is lower order according to our \( \lesssim \) notation when \( N_0 \geq 2 \), from Theorem 6 we obtain

\[
\sup_{\pi \in \mathcal{P}_N(N_0, \Delta_0)} \mathbb{E}_{\pi, M} \min \left( \hat{R}_M(T), \hat{R}_C(T) \right) \leq \sup_{\pi \in \mathcal{P}_N(N_0, \Delta_0)} \mathbb{E}_{\pi, M} \hat{R}_C(T) \lesssim \sqrt{T \log N_0}.
\]
Therefore, let \( t \) be as in Theorem 6 (with \((c_1, c_2) = (c_{c1}, c_{c2})\)), so that \( t_0 \lesssim \frac{\log N}{\Delta_0^2} \). Expanding the quasi-regret of Meta-CARE and using the boundedness of the losses gives

\[
\hat{R}_m(T) = \sum_{t=1}^{t_0} \langle \ell(t), w^M(t) \rangle - \sum_{t=1}^{t_0} \langle \ell(t), w^H(t) \rangle + \hat{R}_H(T) 
\]

\[
\leq \sum_{t=1}^{t_0} \langle \ell(t), w^M(t) \rangle - \min_{\ell=0} \left( \sum_{t=1}^{t_0} \langle \ell(t), w^H(t) \rangle, \sum_{t=1}^{t_0} \langle \ell(t), w^C(t) \rangle \right) + \frac{1}{2} \|w^C(t) - w^H(t)\|_{L^1} + \hat{R}_H(T).
\]

Therefore,

\[
\sup_{\pi \in \mathcal{P}(N, \Delta_0)} \mathbb{E}_{\pi, M} R(T) 
\]

\[
\leq \sup_{\pi \in \mathcal{P}(N, \Delta_0)} \mathbb{E}_{\pi, M} \left[ \sum_{t=1}^{t_0} \langle \ell(t), w^M(t) \rangle - \min_{\ell=0} \left( \sum_{t=1}^{t_0} \langle \ell(t), w^H(t) \rangle, \sum_{t=1}^{t_0} \langle \ell(t), w^C(t) \rangle \right) \right] + \sup_{\pi \in \mathcal{P}(N, \Delta_0)} \mathbb{E}_{\pi, M} \hat{R}_H(T).
\]

Again using the fact that Meta-CARE is D.HEDGE with two experts given by the predictions of D.HEDGE and FTRL-CARE, by Theorem 5 we have

\[
\sup_{\pi \in \mathcal{P}(N, \Delta_0)} \mathbb{E}_{\pi, M} \left[ \sum_{t=1}^{t_0} \langle \ell(t), w^M(t) \rangle - \min_{\ell=0} \left( \sum_{t=1}^{t_0} \langle \ell(t), w^H(t) \rangle, \sum_{t=1}^{t_0} \langle \ell(t), w^C(t) \rangle \right) \right] 
\]

\[
\lesssim \sqrt{t_0} \quad \text{and} \quad \lesssim \frac{\log N}{\Delta_0}.
\]
Next, using the triangle inequality, the fact that $1 - w_{i_0}^H(t) = \sum_{i \in [N] \setminus i_0} w_i^H(t)$ along with the same fact for $w^C$, and Lemmas 2 and 10 (see also the proofs of Theorems 5 and 6 for more details),

$$
\sup_{\pi \in \mathcal{P}_{N,(1,\Delta_0)}} \mathbb{E}_{\pi,M} \sum_{t=t_0}^{\infty} \frac{1}{2} \left\| w^C(t) - w^H(t) \right\|_{L^1}
\leq \sup_{\pi \in \mathcal{P}_{N,(1,\Delta_0)}} \mathbb{E}_{\pi,M} \sum_{t=t_0}^{\infty} \left( \left\| w^C(t) - \delta_{i_0} \right\|_{L^1} + \left\| w^H(t) - \delta_{i_0} \right\|_{L^1} \right)
= \sup_{\pi \in \mathcal{P}_{N,(1,\Delta_0)}} \mathbb{E}_{\pi,M} \sum_{t=t_0}^{T} \sum_{i \in [N] \setminus i_0} \left( w_i^H(t) + w_i^C(t) \right)
\approx \frac{1}{\Delta_0},
$$

where $\delta_{i_0}$ is the point-mass on $i_0$ (equivalently, the weight vector with weight 1 on expert $i_0$ and 0 on the others).

Finally, from Theorem 5, we have

$$
\sup_{\pi \in \mathcal{P}_{N,(1,\Delta_0)}} \mathbb{E}_{\pi,M} \tilde{R}_H(T) \lesssim \log \frac{N}{\Delta_0}.
$$

Combining Eqs. (12) to (14) shows that, in the case of $N_0 = 1$, we have

$$
\sup_{\pi \in \mathcal{P}_{N,(1,\Delta_0)}} \mathbb{E}_{\pi,M} R(T) \lesssim \frac{\log N}{\Delta_0}.
$$

$\Box$

11 Related work

The existing literature on statistical decision making with sequential data is vast, spanning decades and at least two major fields of study: sequential decision theory began as a sub-field of statistics, and the historical literature is rather exclusive to statistics, while the more recent literature on decision procedures without i.i.d. assumptions has largely been developed within machine learning and computer science. In this section, we highlight the most relevant notions of adaptivity, and how their statistical interpretations differ from each other as well as the present work.

11.1 Distributional assumptions

First, note that while we use the language of prediction to describe our setting, our prediction space $\hat{Y}$ is distinct from the observation space $Y$, so we achieve the same level of generality as allowing for arbitrary decisions. Classically, the statistical literature on sequential hypothesis testing [11, 18, 38, 51, 61] and sequential parameter estimation [3, 26, 50, 63] relies on assumptions on the joint dependence structure of data to obtain performance guarantees. From a minimax perspective, removing the assumptions on the dependence structure reduces the problem to adversarially chosen data. Instead, by characterizing these arbitrary distributions in some way such that performance depends on the characterization, we can design methods for which the performance adapts to the characterization.

Hanneke [29] provides an overview of when classical estimation procedures designed for i.i.d. data will be consistent under various non-stationarity conditions. Additionally, he considers the asymptotic performance of a broader class of algorithms, although there is no notion of adaptivity since
performance is binary: either a dependence structure admits a consistent online learning algorithm or it doesn’t. In contrast, since the present work deals with finite expert classes, there is always a consistent algorithm, and so we focus on the specific performance of algorithms beyond their convergence properties.

Rakhlin et al. [48] consider general constraints on the data-generating mechanism for sequential prediction. We also use constraints on the data-generating mechanism to define relaxations of the i.i.d. assumption, but the specific constraints that we define and study are not ones studied by Rakhlin et al.. Additionally, we focus on developing methods that are minimax optimal under the constraint even when the nature of the constraint is unknown. For each of the constraints analyzed by Rakhlin et al., the authors bound the minimax regret non-constructively, and consequently cannot guarantee the existence of an algorithm that is adaptively minimax optimal. In contrast, we provide an explicit, efficient algorithm that is adaptively minimax optimal for our constraint framework.

11.2 Notions of easy data

Beyond quantifying the minimax performance of decision rules under distributional assumptions, significant progress has been made over the last decade towards regret bounds that depend on key summary statistics of the observed data sequence. While the terminology for these types of bounds varies in the literature, we will follow the nomenclature of Cesa-Bianchi et al. [17], who differentiate between zero-order, first-order, and second-order regret bounds. We use stochastic constraints to link zero- and second-order bounds in a general framework, and hence can compare with results derived in a wide range of settings.

Zero-order bounds refer to those that depend only on the time horizon, the size of the expert class, and an absolute bound on the size of the predictions (alternatively, the losses). Results of this nature have existed for many years, beginning with Littlestone and Warmuth [39] and Vovk [60], and are concisely summarized by Cesa-Bianchi and Lugosi [15]. These bounds are often dubbed worst-case or adversarial, since they hold for any sequence of observations subject to the aforementioned global constraints.

In contrast, first-order bounds control regret in terms of a data-dependent quantity; namely, the sum of the actual observed losses (potentially over all experts, or just the best expert for tighter results). Hence, they may lead to much tighter bounds than zero-order guarantees if the observed losses end up being in a much tighter range than is guaranteed by some absolute bound on the size of the losses. The first bound of this form was by Freund and Schapire [24] for the Hedge algorithm, which was later upgraded to a multiplicative rather than additive dependence on the cumulative best loss [15, Corollary 2.4]. Similar bounds have been developed for the bandit setting [6, 9], algorithms with adaptive parametrization [33, 59], and the combination of adaptive parametrization with partial information [46].

However, a limitation of first-order bounds is that they are not translation-invariant in the losses. In particular, they suggest that every expert incurring loss of 1 on each round is much harder to compete against than every expert incurring loss of zero on each round, which is not the case. One solution is to obtain regret bounds that are similar to first-order, but rather than depending on the sum of the losses, they depend on a single first-order translation-invariant parameter that characterizes the observed loss sequence. In the bandit setting, examples of such a parameter include the effective loss range [16, 55] and the amount of corruption allowed on the mean of the losses [28, 41]. A similar analysis of corruption of experts’ predictions in the full-information setting
has recently appeared by Amir et al. [2].

Beyond these first-order quantities, another line of work has focused on second-order bounds, which depend on some form of variation of the observed losses. The first results of this form were derived by Cesa-Bianchi et al. [17], who obtain a bound in terms of the sum of the squared losses via tuning the learning rate for D.HEDGE. This was extended by both McMahan and Streeter [43] and Hazan and Kale [31] to depend on the sample second moment and variance respectively of the losses (empirically along the trajectory of observations), and again by Hazan and Kale [32] to obtain the same in the bandit setting. Both van Erven et al. [59] and de Rooij et al. [22] obtain similar variation bounds which are smaller for a different notion of “easy” data (defined by the mixability of the loss). Finally, another type of second-order bound was developed by Gaillard et al. [25], where they utilize the squared difference of algorithm losses with expert losses.

A different perspective on easy data is taken by Chaudhuri et al. [19] and Luo and Schapire [40], who develop methods not only to have regret relative to the best expert of size $O(\sqrt{T \log N})$, but to also have regret relative to the $\varepsilon N$-quantile expert of size $O(\sqrt{T \log(1/\varepsilon)})$ for all $\varepsilon \in (1/N, 1)$. The algorithms they propose are more optimistic than D.HEDGE in the sense that they trust the past data more, which leads to suboptimal performance in settings between stochastic and adversarial, exaggerating the shortcomings of the standard parametrization of D.HEDGE in this case.

Several other methods exist that tune the learning rate of Hedge adaptively based on the past interaction with the environment. Generally, these are motivated by improved second order bounds. Examples include Koolen and van Erven [36] and van Erven and Koolen [58], who use a prior on the learning rate and meta-experts for a discrete collection of possible learning rates respectively.

We also derive second-order (in particular, variance) bounds for the observed data sequence (see the intermediary result Theorem 7). However, we are also able to extend this notion due to the stochastic nature of our constraints. In particular, once we take the expectation (with respect to the data-generating mechanism and the player's actions) of our second-order bounds, we obtain bounds directly comparable to (and tighter than) existing zero-order bounds. This provides greater insight than existing second-order bounds, which often leave a direct dependence on the variability of the chosen learning algorithm that is not a priori clear, and do not explicitly characterize what an “easy” data sequence actually looks like.

In the full-information setting, another line of investigation describes “easy” stochastic data by that which satisfies a Bernstein condition; that is, the conditional second moment of the losses are controlled by a concave function of the conditional first moment. This condition was shown to be crucial for achieving fast-rates in the batch setting by Bartlett and Mendelson [10], then in the online convex optimization setting (infinite expert class) by van Erven and Koolen [58], and finally for simultaneously the finite expert and infinite expert online setting by Koolen et al. [37]. Recent work by Grünwald and Mehta [27] provided sufficient conditions to extend these results to unbounded losses.

11.3 Stochastic and adversarially optimal algorithms

In addition to developing bounds for “easy” data, the line of work most relevant to the present paper has focused on developing algorithms that are simultaneously optimal in two key settings: worst-case adversarial observations and i.i.d. (stochastic) observations. These bounds are characterized by matching the adversarial bounds mentioned above and the optimal stochastic bounds for either bandits [8, Theorem 1] or full-information [25, Theorem 11]. Beginning with Audibert and Bubeck [5] and Bubeck and Slivkins [12], the bandit literature is rich in this area; contributions include
removing prior knowledge of the time horizon [53], matching lower bounds [7], and a simultaneously optimal algorithm with respect to a slightly weaker notion of regret [65].

In our discussion of the previous bounds, we have not specifically distinguished between the types of algorithms used to achieve them. However, there is an aesthetic (and computational) desire to find algorithms that achieve regret bounds that are optimal both for worst-case data and some notion of “easy” data, and yet are as simple as the algorithms which perform well in either just the adversarial or just the i.i.d. setting. A recent breakthrough on this front was achieved by Mourtada and Gaïffas [44], who showed the standard parametrization of the D.Hedge algorithm is optimal for both the adversarial and the stochastic settings. For the bandit setting, the $\frac{1}{2}$-Tsallis-INF algorithm of Zimmert and Seldin [65] has a similarly simple aesthetic; namely, it is also an analytic solution to an FTRL problem with an appropriate regularizer. One of the more surprising contributions of our work is that we show every pre-specified parametrization of D.Hedge is not adaptively minimax optimal.

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A Additional details for proofs of upper bounds

In this section, we complete the argument sketched in Section 9.

A.1 Details for Theorem 5

Substituting in that D.Hedge with parameter $g$ corresponds to, for a given $N \in \mathbb{N}$, $\psi(s) = s/g(N)$, Theorem 7 says that the weights $w^i$ lead to quasi-regret bounded by

$$\hat{R}_H(T) \leq \frac{\log N}{g(N)} + \sum_{t=1}^T \frac{\sqrt{t+1} - \sqrt{t}}{g(N)} H(w^i(t+1))$$

$$+ \sum_{t=1}^T g(N) \frac{\operatorname{Var}_{I \sim v(t+1)} \left( \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right)}{\sqrt{t+1}} \frac{\operatorname{Var}_{I \sim v(t+1)} [\ell_I(t)]}{\sqrt{t+1}},$$

where

$$v(t+1) = \arg \min_{v \in \operatorname{simp}(\{N\})} \left( \left( \alpha_t L(t) + (1 - \alpha_t) \frac{\sqrt{t+1}}{\sqrt{t}} L(t-1), v \right) - \frac{\sqrt{t+1}}{g(N)} H(v) \right)$$

for some $\alpha_t \in [0, 1]$. Then, recalling that $\psi'(s) = 1/g(N)$, we can split up Eq. (15) into the rounds before some $t_0 \in \mathbb{N}$ and the rounds after by applying Lemma 3. That is, when $T \leq t_0$, we use the bound of Lemma 3, and if $T > t_0$ we have

$$\hat{R}_H(T) \leq \sqrt{t_0 + 1} \left( \frac{\log N}{g(N)} + \frac{3g(N)}{4} \right) + \sum_{t=t_0+1}^T \frac{\sqrt{t+1} - \sqrt{t}}{g(N)} H(w^i(t+1))$$

$$+ \sum_{t=t_0+1}^T g(N) \frac{\operatorname{Var}_{I \sim v(t+1)} \left( \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right)}{\sqrt{t+1}} \frac{\operatorname{Var}_{I \sim v(t+1)} [\ell_I(t)]}{\sqrt{t+1}}.$$ 

(16)

Next, substituting $\psi$ and $\psi'$ for D.Hedge into Eq. (8), we get

$$\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi,H} H(w^i(t+1))$$

$$\leq \frac{17}{16} \log N_0 + \frac{7}{4} (N - N_0) \exp \left\{ \frac{[g(N)]^2}{8} \right\} \exp \left\{ -\frac{g(N) \Delta_0}{2\sqrt{2}} \sqrt{t} \right\}.$$ 

(17)
Thus,
\[
\sum_{t=t_0+1}^{T} \frac{\sqrt{t+1} - \sqrt{t}}{c} \sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi, \hat{\theta}} H(w^\pi(t+1)) \\
\leq \frac{17 \log N_0 \left[ \sqrt{T+1} - \sqrt{t_0+1} \right]}{16g(N)} \\
+ \frac{7(N-N_0) \exp \left\{ \frac{\lvert g(N) \rvert^2}{8} \right\}}{8g(N)} \sum_{t=t_0+1}^{T} \exp \left\{ -\frac{g(N)\Delta_0}{2\sqrt{2}} \sqrt{t} \right\} \\
\leq \frac{17 \log N_0 \left[ \sqrt{T+1} - \sqrt{t_0+1} \right]}{16g(N)} \\
+ \frac{7(N-N_0) \exp \left\{ \frac{\lvert g(N) \rvert^2}{8} - \frac{g(N)\Delta_0}{2\sqrt{2}} \sqrt{t_0} \right\}}{\sqrt{2} \lvert g(N) \rvert^2 \Delta_0},
\]
where the last step comes from applying Lemma 10 to bound the summation. For the last term of Eq. (16), we consider the cases of \(N_0 > 1\) and \(N_0 = 1\) separately. For both, however, we will use \(t_0 = \left\lceil \frac{8(\log(N)+\lvert g(N) \rvert^2/4+g(N)\rceil^2}{4g(N)}\right\rceil\).

**Hedge upper bound: \(N_0 > 1\).**

If \(N_0 > 1\), using Lemma 8 to bound the variances gives
\[
\sum_{t=t_0+1}^{T} \frac{g(N)}{\sqrt{t+1}} \mathbb{V}_{I \sim \nu(t+1)} \left[ \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right] \mathbb{V}_{I \sim \nu(t+1)} [\ell_I(t)] \\
\leq \frac{3g(N)}{8} \sum_{t=t_0+1}^{T} \frac{1}{\sqrt{t+1}} \\
\leq \frac{3g(N)}{4} \left[ \sqrt{T+1} - \sqrt{t_0+1} \right].
\]

Combining Eqs. (16), (18) and (19) gives that
\[
\sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi, \hat{\theta}} \hat{R}_0(T) \\
\leq \frac{\log N}{g(N)} + \frac{3g(N)}{4} + \frac{17 \log N_0 \left[ \sqrt{T+1} - \sqrt{t_0+1} \right]}{16g(N)} \\
+ \frac{7(N-N_0) \exp \left\{ \frac{\lvert g(N) \rvert^2}{8} - \frac{g(N)\Delta_0}{2\sqrt{2}} \sqrt{t_0} \right\}}{\sqrt{2} \lvert g(N) \rvert^2 \Delta_0} + \frac{3g(N)}{4} \left[ \sqrt{T+1} - \sqrt{t_0+1} \right] \\
= \sqrt{T+1} \left( \frac{3g(N)}{4} + \frac{17 \log N_0}{16g(N)} \right) + \sqrt{t_0+1} \left( \frac{\log N}{g(N)} - \frac{17 \log N_0}{16g(N)} \right) \\
+ \frac{7(N-N_0) \exp \left\{ \frac{\lvert g(N) \rvert^2}{8} - \frac{g(N)\Delta_0}{2\sqrt{2}} \sqrt{t_0} \right\}}{\sqrt{2} \lvert g(N) \rvert^2 \Delta_0}.
\]
Substituting \( t_0 \) into Eq. (20) gives

\[
\sup_{\pi \in \mathcal{D}} \mathbb{E}_{\pi, \hat{R}_i}(T) \leq \sqrt{T + 1} \left( \frac{3g(N)}{4} + \frac{17}{16g(N)} \log N_0 \right) + \sqrt{\frac{8(\log(N) + [g(N)]^2/4 + g(N))^2}{[g(N)]^2 \Delta_0^2}} + 2 \left( \frac{\log(N) - \log(N_0)}{g(N)} \right)
\]

\[
+ 7(N - N_0) \exp \left\{ \frac{[g(N)]^2}{8} - \frac{g(N)\Delta_0}{2\sqrt{2}} \sqrt{\frac{8(\log(N) + [g(N)]^2/4 + g(N))^2}{[g(N)]^2 \Delta_0^2}} \right\}
\]

\[
\leq \sqrt{T} \left( \frac{3g(N)}{4} + \frac{17}{16g(N)} \log(N_0) \right) + \frac{\sqrt{2} \log(N)}{g(N)} + \frac{3g(N)}{4} + \frac{2\sqrt{2} \log(N)}{[g(N)]^2 \Delta_0} + \log(N) + \frac{2\sqrt{2} \log(N)}{g(N)\Delta_0} + \frac{7}{\sqrt{2} [g(N)]^2 \Delta_0}.
\]

**Hedge upper bound:** \( N_0 = 1 \)

If \( I_0 = \{i_0\} \), we control the variance terms using Lemma 9

\[
\mathbb{E}_{\pi, \hat{R}} \left[ \Var_{I \sim \nu(t+1)} \left[ \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t) - \ell_I(t) \right] \right] \leq \frac{9}{4} \mathbb{E}_{\pi, \hat{R}} \left[ \sum_{i \neq i_0} v_i(t+1) \right].
\]

We control this using Lemma 2 with \( p = 1 \), which gives

\[
\mathbb{E}_{\pi, \hat{R}} \left[ \sum_{i \neq i_0} v_i(t+1) \right] \leq (N - 1) \exp \left\{ 2g(N) + \frac{[g(N)]^2}{2} \right\} \exp \left\{ -\frac{g(N)\Delta_0}{\sqrt{2}} \sqrt{t} \right\}.
\]

Thus,

\[
\sup_{\pi \in \mathcal{D}} \mathbb{E}_{\pi, \hat{R}} \sum_{t=t_0+1}^T \frac{g(N) \sqrt{\Var_{I \sim \nu(t+1)} \left[ \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t) - \ell_I(t) \right]}}{\sqrt{t+1}} \leq \frac{9g(N)}{4} (N - 1) \exp \left\{ 2g(N) + \frac{[g(N)]^2}{2} \right\} \sum_{t=t_0+1}^T \frac{1}{\sqrt{t}} \exp \left\{ -\frac{g(N)\Delta_0}{\sqrt{2}} \sqrt{t} \right\}
\]

\[
\leq \frac{9}{\sqrt{2} \Delta_0} (N - 1) \exp \left\{ 2g(N) + \frac{[g(N)]^2}{2} \right\} \exp \left\{ -\frac{g(N)\Delta_0}{\sqrt{2}} \sqrt{t_0} \right\},
\]

where the last step follows from again applying Lemma 10. Combining Eqs. (16), (18) and (22) gives that when \( N_0 = 1 \),

34
\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\mathcal{R}}_n(T)} \leq \sqrt{t_0 + 1} \left( \log N + \frac{3g(N)}{4} \right) + \frac{17(\log 1)}{16g(N)} \left[ \sqrt{T + 1} - \sqrt{t_0 + 1} \right] + 
\frac{7(N - 1) \exp \left\{ \frac{[g(N)]^2}{8} - \frac{g(N)\Delta_0}{2\sqrt{2}} \sqrt{t_0} \right\}}{\sqrt{2} [g(N)]^2 \Delta_0} \left[ 2g(N) + \frac{[g(N)]^2}{2} - \frac{g(N)\Delta_0}{\sqrt{2}} \sqrt{t_0} \right] + 
\frac{9(N - 1) \exp \left\{ 2g(N) + \frac{[g(N)]^2}{2} - \frac{g(N)\Delta_0}{\sqrt{2}} \sqrt{t_0} \right\}}{\sqrt{2} \Delta_0}.
\]

Substituting \(t_0\) into Eq. (23) gives
\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\mathcal{R}}_n(T)} \leq \sqrt{2(\log N)^2 + 2\sqrt{2} \log N + 4\log N + 7/2} \sqrt{\text{Var} \left( \frac{\alpha_t L(t) + (1 - \alpha_t) \frac{\sqrt{T + 1}}{\sqrt{t}} L(t - 1) - \ell_I(t)}{\text{Var} \left[ \ell_I(t) \right]} \right)}.
\]

A.2 Details for Theorem 6

This argument follows the same logical structure as the one for Theorem 5. Using that FTFR-CARE with parameters \(c_1, c_2 > 0\) corresponds to \(\psi(s) = \frac{\sqrt{s + c_2}}{c_1}\), Theorem 7 says that the weights \(w^c\) lead to quasi-regret bounded by
\[
\hat{R}_c(T) \leq -\frac{\sqrt{T + 1} c_2}{c_1} + \sum_{t=0}^{T} \sqrt{\frac{t + 1}{c_1}} \cdot \sqrt{H(w^c(t + 1)) + c_2} + \sum_{t=1}^{T} \frac{2c_1 \sqrt{H(v(t + 1)) + c_2}}{\sqrt{t + 1}}
\]
\[
\times \sqrt{\text{Var} \left[ \frac{\sqrt{T + 1}}{\sqrt{t}} - 1 \right]} \cdot L_I(t - 1) - \ell_I(t) \right] \text{Var} \left[ \ell_I(t) \right],
\]
where
\[
v(t + 1) = \arg \min_{v \in \text{simp}([N])} \left( \left[ \alpha_t L(t) + (1 - \alpha_t) \frac{\sqrt{T + 1}}{\sqrt{t}} L(t - 1), v \right] - \frac{\sqrt{T + 1}}{c_1} \sqrt{H(v) + c_2} \right)
\]

35
for some $\alpha_t \in [0,1]$. Then, recalling that $\psi'(s) = \frac{1}{2c_1\sqrt{s+c_2}}$, we can split up Eq. (25) into the rounds before some $t_0 \in \mathbb{N}$ and the rounds after by applying Lemma 3. That is, when $T \leq t_0$, we use the bound of Lemma 3, and if $T > t_0$ we have

$$\hat{R}_C(T) \leq \sqrt{(t_0 + 1)[\log N + c_2]}\left(\frac{1}{c_1} + \frac{3c_1}{2}\right) - \sqrt{(T+1)c_2}$$

$$+ \sum_{t=t_0}^{T} \frac{\sqrt{t+1} - \sqrt{t}}{c_1} \cdot \sqrt{H(w^c(t+1)) + c_2}$$

$$+ \sum_{t=t_0+1}^{T} \frac{2c_1\sqrt{H(v(t+1)) + c_2}}{\sqrt{t+1}}$$

$$\times \left[ \frac{\text{Var}_{I \sim v(t+1)} \left[ \frac{\sqrt{T+1}}{\sqrt{t}} - 1 \right] L_I(t-1) - \ell_I(t) \right]}{\text{Var}_{I \sim v(t+1)} [\ell_I(t)]} \right].$$

Next, substituting $\psi$ and $\psi'$ for FTRL-CARE into Eq. (8), using Jensen’s inequality with the concavity of square root, and the fact that $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for all $x, y > 0$ gives

$$\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi,\hat{\pi}} \sqrt{H(w^c(t+1)) + c_2}$$

$$\leq \sqrt{\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi,\hat{\pi}} H(w^c(t+1)) + c_2}$$

$$\leq \sqrt{\frac{17 \log N_0}{16} + c_2 + \frac{4\sqrt{(N-N_0)} \exp \left\{ \frac{c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{2\sqrt{2}} \right\}}{3}}. \tag{27}$$

Thus,

$$\sum_{t=t_0+1}^{T} \frac{\sqrt{t+1} - \sqrt{t}}{c_1} \sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi,\hat{\pi}} \sqrt{H(w^c(t+1)) + c_2}$$

$$\leq \sqrt{\frac{17 \log N_0 + c_2}{16} \left[ \sqrt{T+1} - \sqrt{t_0 + 1} \right]}$$

$$+ \sum_{t=t_0+1}^{T} \frac{4\sqrt{(N-N_0)} \exp \left\{ \frac{c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{2\sqrt{2}} \right\}}{3c_1 \sqrt{t}} \tag{28}$$

$$\leq \sqrt{\frac{17 \log N_0 + c_2}{16} \left[ \sqrt{T+1} - \sqrt{t_0 + 1} \right]}$$

$$+ \frac{8\sqrt{2\sqrt{(N-N_0)} \exp \left\{ \frac{c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{2\sqrt{2}} \right\}}}{3c_1^2 \sqrt{c_2} \Delta_0},$$
where the last step used Lemma 10. Similarly, we use these same properties and Eq. (9) to obtain

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \mathbb{H}(v(t + 1)) + c_2 \\
\leq \sqrt{\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \mathbb{H}(v(t + 1)) + c_2} \\
\leq \sqrt{\frac{17 \log N_0}{16} + c_2 + \frac{4}{3} \sqrt{(N - N_0)} \exp \left\{ c_1 \sqrt{c_2} + \frac{c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{2 \sqrt{2}} \right\}.}
\]

For the last term of Eq. (26), we consider the cases of \(N_0 > 1\) and \(N_0 = 1\) separately. For both, however, we will use \(t_0 = \left\lceil \frac{2 \log N + 3c_1 \sqrt{c_2} + \frac{5}{4} c_1^2 c_2}{c_1^2 c_2^{3/2}} \right\rceil\) and the constant \(C = \max \{c_2, 3c_1 \sqrt{c_2} + \frac{5}{4} c_1^2 c_2\}.

**FTRL-CARE upper bound:** \(N_0 > 1\).

If \(N_0 > 1\), we again use Lemma 8 to control the variance terms. Then, using Eq. (29) and another application of Lemma 10,

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \sum_{t=t_0+1}^{T} \frac{2c_1 \sqrt{\mathbb{H}(v(t + 1)) + c_2}}{\sqrt{t + 1}} \\
\times \sqrt{\text{Var}_{I \sim v(t+1)} \left[ \frac{\sqrt{I + 1} - 1}{\sqrt{t}} \right] L_I(t - 1) - \ell_I(t)} \text{Var}_{I \sim v(t+1)} \ell_I(t) \\
\leq \frac{3c_1}{4} \sum_{t=t_0+1}^{T} \sqrt{\frac{17 \log N_0 + c_2}{t + 1}} \\
+ c_1 \sqrt{(N - N_0)} \sum_{t=t_0+1}^{T} \exp \left\{ c_1 \sqrt{c_2} + \frac{c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{2 \sqrt{2}} \right\} \\
\leq \frac{3c_1 \sqrt{\frac{17 \log N_0 + c_2}{t + 1}}}{2} \\
+ \frac{8 \sqrt{(N - N_0)} \exp \left\{ c_1 \sqrt{c_2} + \frac{c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{2 \sqrt{2}} \right\}}{\sqrt{2c_2 \Delta_0}}.
\]
Combining Eqs. (26), (28) and (30) gives that

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\theta}} \hat{R}_c(T) \leq \sqrt{(t_0 + 1)[\log N + c_2]\left(\frac{1}{c_1} + \frac{3c_1}{2}\right) - \sqrt{(T + 1)c_2}} \\
+ \frac{\sqrt{17}}{16} \log N_0 + c_2 \left[\sqrt{T + 1 - t_0 + 1}\right] \\
+ \frac{3c_1 \sqrt{17} \log N_0 + c_2 \left[\sqrt{T + 1 - t_0 + 1}\right]}{2} \\
+ 16 \sqrt{(N - N_0)} \exp\left\{\frac{c_2^2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t_0}}{2\sqrt{2}}\right\} \\
+ \frac{8 \sqrt{(N - N_0)} \exp\left\{\frac{c_1 \sqrt{c_2} + c_2^2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t_0}}{2\sqrt{2}}\right\}}{\sqrt{2c_2} \Delta_0} \leq \frac{33}{32} \sqrt{(T + 1)[\log N_0 + c_2]\left(\frac{1}{c_1} + \frac{3c_1}{2}\right) \\
+ \sqrt{(t_0 + 1)\left(\frac{1}{c_1} + \frac{3c_1}{2}\right)(\log N + c_2 - \log N_0 + c_2) \\
+ \sqrt{2}(N - N_0)} \exp\left\{\frac{c_2^2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t_0}}{2\sqrt{2}}\right\} \left(\frac{8}{3c_1^2} + 4 \exp\{c_1 \sqrt{c_2}\}\right)\right) .
\]

Substituting \(t_0\) into Eq. (31) gives

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\theta}} \hat{R}_c(T) \leq \left(\frac{1}{c_1} + \frac{3c_1}{2}\right) \left[\frac{33}{32} \sqrt{(T + 1)[\log N_0 + c_2]} \\
+ \sqrt{\left(\frac{2[\log N + 3c_1 \sqrt{c_2} + \frac{5}{4} c_1^2 c_2]^2}{c_1^2 c_2 \Delta_0^2} + 2\right)[\log N + c_2]} \right] \\
+ \frac{\sqrt{2}(N - N_0)}{\sqrt{c_2} \Delta_0} \left(\frac{8}{3c_1^2} + 4 \exp\{c_1 \sqrt{c_2}\}\right) \times \exp\left\{\frac{c_2^2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t_0}}{2\sqrt{2}}\right\} \sqrt{\frac{2[\log N + 3c_1 \sqrt{c_2} + \frac{5}{4} c_1^2 c_2]^2}{c_1^2 c_2 \Delta_0^2}}\right) \leq \left(\frac{1}{c_1} + \frac{3c_1}{2}\right) \left[\frac{33}{32} \sqrt{(T + 1)[\log N_0 + c_2]} \\
+ \frac{\sqrt{2}[\log N + C]^3}{c_1 \sqrt{c_2} \Delta_0} + \sqrt{2[\log N + c_2]}ight] \\
+ \frac{\sqrt{2}(8 + 12c_1^2)}{3c_1^2 \sqrt{c_2} \Delta_0} .
\]
**FTRL-CARE upper bound:** \( N_0 = 1 \)

If \( I_0 = \{i_0\} \), we control the variance terms using Lemma 9. In particular,

\[
\mathbb{E}_{\pi, \hat{\pi}} \left[ \operatorname{Var}_{I \sim \nu(t+1)} \left[ \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right] L_I(t-1) - \ell_I(t) \right] \operatorname{Var}_{I \sim \nu(t+1)} [\ell_I(t)] \\
\leq \frac{27}{32} \mathbb{E}_{\pi, \hat{\pi}} \left[ \sum_{i \neq i_0} v_i(t+1) \right].
\]

We control this using Lemma 2 with \( p = 1 \), which gives

\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \left[ \sum_{i \neq i_0} v_i(t+1) \right] \leq (N - 1) \exp \left\{ 4c_1 \sqrt{c_2} \left( 2c_1 c_2^2 - \sqrt{2} c_1 \sqrt{c_2} \Delta_0 \sqrt{t} \right) \right\}.
\]

Thus, using Cauchy-Schwarz and Eq. (29) (recalling \( \log N_0 = 0 \)), for any \( \pi \in \mathcal{P}(D) \),

\[
\mathbb{E}_{\pi, \hat{\pi}} \left( H(\nu(t+1)) + c_2 \right) \operatorname{Var}_{I \sim \nu(t+1)} \left[ \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right] \operatorname{Var}_{I \sim \nu(t+1)} [\ell_I(t)] \\
\leq \sqrt{\mathbb{E}_{\pi, \hat{\pi}} H(\nu(t+1)) + c_2} \\
\times \sqrt{\mathbb{E}_{\pi, \hat{\pi}} \left[ \operatorname{Var}_{I \sim \nu(t+1)} \left[ \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right] \operatorname{Var}_{I \sim \nu(t+1)} [\ell_I(t)] \right]} \\
\leq \sqrt{\frac{7(N - 1) \exp \left\{ 2c_1 \sqrt{c_2} + \frac{c_1^2 c_2}{2} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{\sqrt{2}} \right\} + c_2}{4}} \\
\times \sqrt{(N - 1) \exp \left\{ 2c_1 \sqrt{c_2} + c_1^2 c_2 - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{\sqrt{2}} \right\}} \\
\leq \frac{3(N - 1) \exp \left\{ 3c_1 \sqrt{c_2} + \frac{5c_1^2 c_2}{4} - \frac{3c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{2\sqrt{2}} \right\}}{2} \\
+ \sqrt{c_2 (N - 1) \exp \left\{ 2c_1 \sqrt{c_2} + c_1^2 c_2 - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{\sqrt{2}} \right\}} \\
\leq (3/2 + \sqrt{c_2}) (N - 1) \exp \left\{ 3c_1 \sqrt{c_2} + \frac{5c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t}}{\sqrt{2}} \right\}.\]
Summing this over $t$ and applying Lemma 10 gives

$$\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \sum_{t=t_0+1}^T \frac{2c_1 \sqrt{H(v(t+1)) + c_2}}{\sqrt{t+1}}$$

$$\times \left[ \text{Var}_{I \sim v(t+1)} \left( \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right) \right] \text{Var}_{I \sim v(t+1)} [\ell_I(t)]$$

$$\leq c_1 (3 + 2 \sqrt{c_2}) (N - 1) \exp \left\{ 3c_1 \sqrt{c_2} + \frac{5c_1^2 c_2}{4} \right\}$$

$$\times \sum_{t=t_0+1}^T \frac{1}{\sqrt{t}} \exp \left\{ - \frac{c_1 \sqrt{c_2} \Delta_0}{\sqrt{2}} \sqrt{t} \right\}$$

$$\leq \frac{\sqrt{2} (3 + 2 \sqrt{c_2}) (N - 1)}{\sqrt{c_2} \Delta_0} \exp \left\{ 3c_1 \sqrt{c_2} + \frac{5c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0}{\sqrt{2}} \sqrt{t_0} \right\}. \tag{32}$$

Combining Eqs. (26), (28) and (32) gives that for $N_0 = 1$,

$$\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{\pi}} \hat{R}_c(T)$$

$$\leq \sqrt{(t_0 + 1) \log N + c_2} \left( \frac{1}{c_1} + \frac{3c_1}{2} \right) - \frac{\sqrt{(T + 1) c_2}}{c_1}$$

$$+ 8 \frac{\sqrt{2} (N - 1)}{3c_1^2 \sqrt{c_2} \Delta_0} \exp \left\{ \frac{c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0}{2 \sqrt{2}} \sqrt{t_0} \right\}$$

$$+ \frac{\sqrt{c_2}}{c_1} \left[ \sqrt{T + 1} - \sqrt{t_0 + 1} \right]$$

$$\leq \sqrt{(t_0 + 1) \log N + c_2} \left( \frac{1}{c_1} + \frac{3c_1}{2} \right)$$

$$+ 8 \sqrt{2} (N - 1) \exp \left\{ \frac{c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0 \sqrt{t_0}}{2 \sqrt{2}} \right\}$$

$$+ \frac{\sqrt{2} (3 + 2 \sqrt{c_2}) (N - 1)}{\sqrt{c_2} \Delta_0} \exp \left\{ 3c_1 \sqrt{c_2} + \frac{5c_1^2 c_2}{4} - \frac{c_1 \sqrt{c_2} \Delta_0}{\sqrt{2}} \sqrt{t_0} \right\}. \tag{33}$$
Substituting $t_0$ into Eq. (33) gives

\[
\sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi, \hat{R}_C(T)} \leq \left( \frac{1}{c_1} + \frac{3c_1}{2} \right) \sqrt{\left( \frac{2[\log N + 3c_1\sqrt{c_2} + \frac{5}{4}c_1^2c_2^2]}{c_1^2c_2\Delta_0^2} + 2 \right)[\log N + c_2]}
+ \frac{8\sqrt{2(N-1)}}{3c_1^2\sqrt{c_2}\Delta_0} \exp \left\{ \frac{c_1^2c_2}{4} - \frac{c_1\sqrt{c_2}\Delta_0}{2\sqrt{2}} \sqrt{\frac{2[\log N + 3c_1\sqrt{c_2} + \frac{5}{4}c_1^2c_2^2]}{c_1^2c_2\Delta_0^2}} \right\}
+ \frac{\sqrt{2}(3 + 2\sqrt{c_2})(N-1)}{\sqrt{c_2}\Delta_0}
\times \exp \left\{ \frac{3c_1\sqrt{c_2} + 5c_1^2c_2}{4} - \frac{c_1\sqrt{c_2}\Delta_0}{\sqrt{2}} \sqrt{\frac{2[\log N + 3c_1\sqrt{c_2} + \frac{5}{4}c_1^2c_2^2]}{c_1^2c_2\Delta_0^2}} \right\}
\leq \left( \frac{1}{c_1} + \frac{3c_1}{2} \right) \left[ \frac{\sqrt{2}[\log N + C]^3/2}{c_1\sqrt{c_2}\Delta_0} + \sqrt{2[\log N + c_2]} \right]
+ \frac{1}{\sqrt{c_2}\Delta_0} \left[ \frac{8\sqrt{2}}{3c_1^2} + \sqrt{2(3 + 2\sqrt{c_2})} \right].
\]

\[\square\]

**B Generic FTRL regret bounds with local norms**

**B.1 Online linear optimization with FTRL**

An online linear optimization (OLO) problem in $\mathbb{R}^d$ is defined by a closed prediction domain $F \subseteq \mathbb{R}^d$ and a loss domain $G \subseteq \mathbb{R}^d$. At each time $t$, the player selects $\mu(t) \in F$, then observes some $\lambda(t) \in G$ and incurs the loss $\langle \lambda(t), \mu(t) \rangle$. For any sequence of losses $\lambda(1), \ldots, \lambda(T) \in G$, the player’s regret is defined by

\[
R_{\text{o1o}}(T) = \sum_{t=1}^{T} \langle \lambda(t), \mu(t) \rangle - \inf_{\mu \in F} \sum_{t=1}^{T} \langle \lambda(t), \mu \rangle.
\]

There are many ways one could choose $\mu(t)$, but in this work we focus specifically on FTRL, which is a generic method for online linear optimization. The FTRL algorithm is parametrized by $F$, $G$, and a sequence of regularizers $\{\rho_t : F \rightarrow \mathbb{R}\}_{t \in \mathbb{Z}_+}$. For each time $t + 1$, a player using the FTRL($F$, $G$, $(\rho_t)_{t \in \mathbb{Z}_+}$) algorithm outputs

\[
\mu(t + 1) = \arg \min_{\mu \in F} (\langle \Lambda(t), \mu \rangle + \rho_{0:t}(\mu)),
\]

(34)

where $\rho_{0:t}(\mu) = \sum_{s=0}^{t} \rho_s(\mu)$ and $\Lambda(t) = \sum_{s=1}^{t} \lambda(s)$.

**B.2 OLO FTRL regret bounds**

The classical regret bound for FTRL consists of a term that is the difference of losses incurred by consecutive player vectors and a term that looks like the regularizer evaluated at the optimal player vector in hindsight. The former is usually bounded using strong-convexity to obtain a norm of the consecutive weight differences. For tighter control, such as that obtained by Abernethy
and Rakhlin [1], this norm may be chosen to be a local norm. A local norm with respect to a function \( f \) will be of the form \( \|x\|_y = \sqrt{\langle x, \nabla^2 f(y)x \rangle} \), and has the property that the dual is \( \|x\|_{y,*} = \sqrt{\langle x, (\nabla^2 f(y))^{-1}x \rangle} \). The natural choice of function to define the local norm with respect to is the regularizer; however, this is generally more challenging for non-constant regularizers.

Surprisingly, while both local norms and time-dependent regularizers are standard in the FTRL literature, we were unable to find an explicit statement that combines them exactly as we needed. The closest seems to be Theorem 1 of McMahan [42], which requires that the regularizers are strongly convex with respect to a norm and then defines the local norm using the time-dependent strong convexity parameter. This strong-convexity argument is insufficient for our analysis, as the CARE regularizer can be at worst only \( 1/\sqrt{\log N} \)-strongly convex in all settings, and consequently would not lead to the adaptive rates we obtain. We begin with a modification of [43, Lemma 1] to combine local norm bounds with time-dependent regularizer bounds.

**Lemma 4.** For any \( F, G, (\rho_t)_{t\in\mathbb{Z}_+}, \) and \( (\lambda(t))_{t\in\mathbb{N}} \subseteq G, \) the FTRL\((F, G, (\rho_t)_{t\in\mathbb{Z}_+})\) algorithm has regret bounded for all \( T \in \mathbb{N} \) by

\[
R_{\text{o1o}}(T) \leq \rho_{0:T}(\mu_*(T)) - \sum_{t=0}^{T} \rho_t(\mu(t + 1)) + \sum_{t=1}^{T} \langle \lambda(t), \mu(t) - \mu(t + 1) \rangle,
\]

for all \( \mu_*(T) \in \arg\min_{\mu \in F} \langle \Lambda(T), \mu \rangle \).

**Proof of Lemma 4.** This follows from directly modifying the proof of [43, Lemma 1] by not dropping the \( \rho_t(\mu(t+1)) \) term at the end of [43, Lemma 7]. We reproduce the argument here for completeness.

As shown by Kalai and Vempala [34], and restated in [43, Lemma 6],

\[
\sum_{t=0}^{T} f_t(x_*(t)) \leq \sum_{t=0}^{T} f_t(x_*(T))
\]

for any sequence of functions \((f_t)_{t\in\mathbb{Z}_+}\) and any sequence \( x_*(t) \in \arg\min_x \sum_{s=0}^{t} f_s(x) \). Thus, by definition of \( \mu(t+1) \) minimizing Eq. (34),

\[
\sum_{t=0}^{T} [(\lambda(t), \mu(t + 1)) + \rho_t(\mu(t + 1))] \leq \sum_{t=0}^{T} [(\lambda(t), \mu(T + 1)) + \rho_t(\mu(T + 1))]
\]

\[
\leq \sum_{t=0}^{T} [(\lambda(t), \mu_*(T)) + \rho_t(\mu_*(T))]
\]

\[
= \langle \Lambda(T), \mu_*(T) \rangle + \rho_{0:T}(\mu_*(T)).
\]

Rearranging gives that

\[
R_{\text{o1o}}(T) = \sum_{t=0}^{T} \langle \lambda(t), \mu(t) \rangle - \langle \Lambda(T), \mu_*(T) \rangle
\]

\[
= \sum_{t=0}^{T} \langle \lambda(t), \mu(t) - \mu(t + 1) \rangle + \sum_{t=0}^{T} \langle \lambda(t), \mu(t + 1) \rangle - \langle \Lambda(T), \mu_*(T) \rangle
\]

\[
\leq \sum_{t=0}^{T} \langle \lambda(t), \mu(t) - \mu(t + 1) \rangle + \rho_{0:T}(\mu_*(T)) - \sum_{t=0}^{T} \rho_t(\mu(t + 1)).
\]

Finally, the indexing of \( t \) in the sums of the lemma statement follows since by convention \( \lambda(0) = 0 \). \( \square \)
An alternative to the regret expansion for FTRL from McMahan and Streeter [43] has appeared in more recent literature such as that of Duchi et al. [23], Hazan [30], Orabona [47], Shalev-Shwartz [54]. This alternative analysis can be tighter in certain cases, but requires controlling three terms instead of two. Additionally, it could only lead to improvements in the constants in our case (bounded losses), so we opted for the simpler approach.

B.3 OLO FTRL regret bounds with local norms

Now, we provide a local-norm control on the inner product from Lemma 4 for time-dependent regularizers which can be defined as a function of time and a constant regularizer. The types of regularizers we will consider are convex functions of the Legendre type, as defined by [52, Sec. 26].

**Definition 1** (Essentially smooth, Rockafellar [52], Section 26). An extended-real-valued function \( f : F \to \mathbb{R} \) for \( F \subseteq \mathbb{R}^d \) is essentially smooth on \( F \) if it satisfies

1. \( \text{interior}(F) \neq \emptyset \),
2. \( f \) is differentiable on \( \text{interior}(F) \), and
3. \( x \in \partial(F) \) and \( \{y_i\}_{i \in \mathbb{N}} \subseteq \text{interior}(F) \) with \( y_i \to x \) implies \( \|\nabla f(y_i)\| \to +\infty \).

**Definition 2** (Legendre type, Rockafellar [52], Section 26). A closed convex function \( f : F \to \mathbb{R} \) for \( F \subseteq \mathbb{R}^d \) is of the Legendre type on \( F \) if

1. \( f \) is strictly convex on \( \text{interior}(F) \),
2. \( \text{interior}(F) \) is convex, and
3. \( f \) is essentially smooth on \( F \).

**Definition 3** (Legendre Transform, Rockafellar [52], Section 26). The Legendre transform of a function \( f : F \to \mathbb{R} \) for \( F \subseteq \mathbb{R}^d \) of the Legendre type on \( F \) is the function \( f^* : \nabla f(\text{interior}(F)) \to \mathbb{R} \) defined by

\[
  f^*(y) = \sup_{x \in F} \langle x, y \rangle - f(x) = \langle [\nabla f]^{-1}(y), y \rangle - f([\nabla f]^{-1}(y)).
\]

**Proposition 3** (Rockafellar [52], Theorem 26.5). If \( f \) is a closed convex function of the Legendre type on \( F \) for \( F \subseteq \mathbb{R}^d \) and \( F^* = \nabla f(\text{interior}(F)) \), then \( F^* \) is convex and \( f^* \) is of the Legendre type on \( F^* \),

\[
  \nabla f : \text{interior}(F) \to F^*
\]

is a continuous bijection with continuous inverse, and \( \nabla[f^*] = [\nabla f]^{-1} \).

**Corollary 1.** If \( F \subseteq \mathbb{R}^d \) is convex with non-empty interior, and if \( f \) is a closed, convex function of the Legendre type on \( F \), then for any \( y \) with \( -y \in \nabla f(\text{interior}(F)) \),

\[
  \arg\min_{x \in F} \langle y, x \rangle + f(x) = \left\{ [\nabla f]^{-1}(-y) \right\} = \left\{ [\nabla f^*](-y) \right\} \in \text{interior}(F).
\]

**Proof.** Since the objective is convex then if a single local minimum occurs in the interior \( F \) then it must be the unique optimizer on \( F \). Taking the gradient of the objective, we see that a local minimum occurs when \( \nabla f(x) = -y \). Since \( f \) is assumed to be of the Legendre type on \( F \) then this equation has a unique solution in \( \text{interior}(F) \) whenever \( -y \in \nabla f(\text{interior}(F)) \). \( \square \)
Lemma 5. Suppose that $F \subseteq \mathbb{R}^d$ is convex with non-empty interior, $G \subseteq \mathbb{R}^d$ is arbitrary, and the regularizer $\rho_0$ is closed, convex, of the Legendre type on $F$, and twice continuously differentiable on interior($F$). For each $t \in \mathbb{N}$, let $\rho_{0,t}(\mu) = \beta(t)\rho_0(\mu)$ for some increasing function $\beta : \mathbb{N} \to \mathbb{R}_+$. Also, for any $y \in G$ and $x \in F$, define the time-dependent local norm by $\|y\|^2_{t,x} = \langle y, \nabla^2 \rho_{0,t}(x)y \rangle$, and its dual time-dependent local norm by $\|y\|_{t,x,*} = \langle y, [\nabla^2 \rho_{0,t}(x)]^{-1}y \rangle$. Then, for any sequence of losses $(\lambda(t))_{t \in \mathbb{N}} \subseteq G$ such that $(-\frac{1}{\beta(t)}\lambda(t)) \in [\nabla \rho_0](\text{interior}(F))$ for all $t \in \mathbb{N}$, there exists a sequence $(\alpha_t)_{t \in \mathbb{N}} \subseteq [0,1]$ such that, for all $t \in \mathbb{N}$, the weights $(\mu(t))_{t \in \mathbb{N}}$ output by the FTRL($F$, $G$, $(\rho_t)_{t \in \mathbb{N}_+}$) algorithm satisfy

\[
\langle \lambda(t), \mu(t) - \mu(t+1) \rangle \leq \frac{1}{\beta(t)} \left( \frac{\beta(t)}{\beta(t-1)} - 1 \right) \Lambda(t-1) - \lambda(t) \| \lambda(t) \|_{0,v(t+1),*},
\]

where $v(t+1) = \text{arg min}_{v \in F} \left( \langle \alpha_t \Lambda(t) + (1-\alpha_t)\frac{\beta(t)}{\beta(t-1)}\Lambda(t-1), v \rangle + \rho_{0,t}(v) \right)$.

Remark 5. In our applications, $[\nabla \rho_0](\text{interior}(F)) = \mathbb{R}^d$ is the whole space, so the assumption

\[
(-\frac{1}{\beta(t)}\Lambda(t)) \in [\nabla \rho_0](\text{interior}(F))
\]

is benign.

Proof of Lemma 5. Fix some $t \in \mathbb{N}$ and observe that by Corollary 1, $\mu(t+1)$ is the unique $\mu$ that solves $\nabla \rho_{0,t}(\mu) = -\Lambda(t)$. Thus, applying a first-order Taylor expansion of $[\nabla \rho_{0,t}]^{-1}$ centered at $\nabla \rho_{0,t}(\mu(t))$,

\[
\mu(t+1) - \mu(t) = [\nabla \rho_{0,t}]^{-1}(\nabla \rho_{0,t}(\mu(t+1))) - [\nabla \rho_{0,t}]^{-1}(\nabla \rho_{0,t}(\mu(t)))
\]

\[
= [J[\nabla \rho_{0,t}]]^{-1}(\zeta(t))[ \nabla \rho_{0,t}(\mu(t+1)) - \nabla \rho_{0,t}(\mu(t))],
\]

where $J$ denotes the Jacobian and $-\zeta(t) = \alpha_t \nabla \rho_{0,t}(\mu(t+1)) + (1-\alpha_t)\nabla \rho_{0,t}(\mu(t))$ for some $\alpha_t \in [0,1]$. Using the inverse function theorem on $\nabla \rho_{0,t}$ gives

\[
[J[\nabla \rho_{0,t}]]^{-1}(\zeta(t)) = [\nabla^2 \rho_{0,t}[\nabla \rho_{0,t}]]^{-1}(\zeta(t))].
\]

Next, observe that

\[
\nabla \rho_{0,t}(\mu(t)) = \beta(t)\nabla \rho_0(\mu(t)) = \frac{\beta(t)}{\beta(t-1)} \nabla \rho_{0,t-1}(\mu(t)) = \frac{\beta(t)}{\beta(t-1)}(-\Lambda(t-1)),
\]

so $\zeta(t)$ can be viewed as a combination of losses defined by

\[
\zeta(t) = \alpha_t \Lambda(t) + (1-\alpha_t)\frac{\beta(t)}{\beta(t-1)}\Lambda(t-1).
\]

Therefore, $-\frac{\zeta(t)}{\beta(t)} = \alpha_t \frac{-\Lambda(t)}{\beta(t)} + (1-\alpha_t)\frac{-\Lambda(t-1)}{\beta(t-1)} \in \nabla \rho_0(\text{interior}(F))$ since $\nabla \rho_0(\text{interior}(F))$ is convex (by Proposition 3). This implies

\[
-\zeta(t) \in [\beta(t)\nabla \rho_0](\text{interior}(F)) = \nabla \rho_{0,t}(\text{interior}(F)),
\]

so $v(t+1) = [\nabla \rho_{0,t}]^{-1}(-\zeta(t)) \in \text{interior}(F)$ by Corollary 1. Further,

\[
\nabla \rho_{0,t}(\mu(t+1)) - \nabla \rho_{0,t}(\mu(t)) = -\Lambda(t) + \frac{\beta(t)}{\beta(t-1)}\Lambda(t-1) = \left( \frac{\beta(t)}{\beta(t-1)} - 1 \right) \Lambda(t-1) - \lambda(t).
\]
Combining these results, along with the fact that $\nabla^2 \rho_{0:t} = \beta(t) \nabla^2 \rho_0$, gives

$$
\mu(t + 1) - \mu(t) = \frac{1}{\beta(t)} [\nabla^2 \rho_0(v(t + 1))]^{-1} \left[ \left( \frac{\beta(t)}{\beta(t) - 1} \right) \Lambda(t - 1) - \lambda(t) \right].
$$

Next, by Holder’s inequality,

$$
\langle \lambda(t), \mu(t) - \mu(t + 1) \rangle \leq \| \mu(t) - \mu(t + 1) \|_{t,v(t+1)} \| \lambda(t) \|_{t,v(t+1),*}
= \| \mu(t) - \mu(t + 1) \|_{0,v(t+1)} \| \lambda(t) \|_{0,v(t+1),*},
$$

where the last equality follows from the fact that a $\beta(t)$ will factor out of the first norm and a $1/\beta(t)$ will factor out of the second norm. Then, substituting in Eq. (35),

$$
\| \mu(t) - \mu(t + 1) \|^2_{0,v(t+1)}
= \langle \nabla^2 \rho_0(v(t + 1)) [\mu(t) - \mu(t + 1)], \mu(t) - \mu(t + 1) \rangle
= \frac{1}{\beta(t)^2} \langle [\nabla^2 \rho_0(v(t + 1))]^{-1} \left[ \left( \frac{\beta(t)}{\beta(t) - 1} \right) \Lambda(t - 1) - \lambda(t) \right], \left( \frac{\beta(t)}{\beta(t) - 1} \right) \Lambda(t - 1) - \lambda(t) \rangle
= \frac{1}{\beta(t)^2} \| \left( \frac{\beta(t)}{\beta(t) - 1} \right) \Lambda(t - 1) - \lambda(t) \|^2_{0,v(t+1),*}.
$$

Thus,

$$
\langle \lambda(t), \mu(t) - \mu(t + 1) \rangle \leq \frac{1}{\beta(t)} \| \left( \frac{\beta(t)}{\beta(t) - 1} \right) \Lambda(t - 1) - \lambda(t) \|_{0,v(t+1),*} \| \lambda(t) \|_{0,v(t+1),*}.
$$

Amir et al. [2] recently made the same observation that closely related bounds have been derived before but not in the explicit form they desire, and they prove a regret bound very similar to Lemmas 4 and 5. However, they rely on a Taylor expansion of the regularizer around the weights output by FTRL, while we have used a Taylor expansion of the Legendre dual of the regularizer around the observed losses. This makes it easier for us to ultimately apply Theorem 2 when controlling the bound of Lemma 5 in expectation. Zimmert and Seldin [65] have a similar expansion in their analysis, and obtain a local norm in the dual space as an intermediate step in the proof of their Lemma 11. However, the object they use this local norm to upper bound is not the same as what we upper bound, and they ultimately use a bound in the primal space to obtain their results.

### C FTRL regret bounds on the simplex

When we restrict consideration to proper prediction policies (see Section 6) and focus on controlling the expected regret, then online linear optimization is a generalization of the online prediction problem in Section 2, which is just the case where $F = \text{simp}([N])$, $G = [0,1]^N$, and we are interested in $E R_{0:16}(T)$. To bound the expected regret, we choose an appropriate sequence of regularizers and then apply generic techniques for analyzing FTRL in online linear optimization problems. For clarity and to distinguish between FTRL in the generic online linear optimization setting and in the specific case of online prediction on the simplex, we use $(r_t)_{t \in \mathbb{Z}_+}$ to denote the sequence of regularizers in the latter. Thus, the FTRL($(r_t)_{t \in \mathbb{Z}_+}$) notation is really shorthand in this case for FTRL($\text{simp}([N])$, $[0,1]^N$, $(r_t)_{t \in \mathbb{Z}_+}$).
A significant portion of the heavy-lifting required for Theorem 7 is done in Appendix B, which proves a very similar result for generic FTRL under some technical constraints. However, we cannot directly apply Lemma 5 when $F = \text{simp}([N])$, since this set has empty interior. Thus, we need a version of that result tailored to the simplex, which we achieve by a reparametrization of the simplex.

In particular, let $i_1 \in [N]$ be arbitrary, and let $\hat{N} = [N] \setminus \{i_1\}$. Let

$$\hat{F} = \left\{ \mu \in [\mathbb{R}_+]^{\hat{N}} \text{ s.t. } \sum_{i \in \hat{N}} \mu_i \leq 1 \right\},$$

and observe that $\text{interior}(\hat{F})$ is non-empty and convex. The canonical bijection $\phi : \text{simp}([N]) \to \hat{F}$ is given by

$$\phi(u) = u_{-i_1}, \quad \text{and} \quad \phi^{-1}(\mu) = \left( \mu_i : i \in [\hat{N}] \right) \left( 1 - \langle 1, \mu \rangle \right)_{i \in [N]}$$

where $u_{-i}$ is the vector obtained from $u$ by dropping the coordinate with index $i$.

For any function $f : \text{simp}([N]) \to \mathcal{Y}$ for some set $\mathcal{Y}$, define $\hat{f} : \hat{F} \to \mathcal{Y}$ by

$$\hat{f}(\mu) = f(\phi^{-1}(\mu)).$$

For example, if we let $H : \text{simp}([N]) \to \mathbb{R}_+$ be the entropy function defined by

$$H(u) = -\sum_{i \in [N]} u_i \log(u_i),$$

then $\hat{H} : \hat{F} \to \mathbb{R}$ is defined by

$$\hat{H}(\mu) = H(\phi^{-1}(\mu)) = -\left( \sum_{i \in [\hat{N}]} \mu_i \log(\mu_i) \right) - \langle 1, \mu \rangle \log(1 - \langle 1, \mu \rangle).$$

Note that for any sequence of regularizers $(r_t)_{t \in \mathbb{Z}_+}$ on $\text{simp}([N])$ and any sequence of losses $(\lambda(t))_{t \in \mathbb{N}}$ in an arbitrary $G \subseteq \mathbb{R}^{[N]}$, for all $t \in \mathbb{N}$ we have

$$\langle \Lambda(t), u \rangle + r_{0:t}(u) = \langle \Lambda_{-i_1}(t), u_{-i_1} \rangle + \Lambda_{i_1}(t)(1 - \langle 1, u_{-i_1} \rangle) + \hat{r}_{0:t}(u_{-i_1})$$

$$= \Lambda_{i_1}(t) + \langle \Lambda_{-i_1}(t) - \Lambda_{i_1}(t)1, u_{-i_1} \rangle + \hat{r}_{0:t}(u_{-i_1}).$$

Additionally, for any $(b(t))_{t \in \mathbb{N}} \subseteq \mathbb{R}$,

$$\arg\min_{u \in \text{simp}([N])} \langle \Lambda(t), u \rangle + r_{0:t}(u) = \arg\min_{u \in \text{simp}([N])} \langle \Lambda(t) - b(t)1, u \rangle + r_{0:t}(u)$$

by the requirement that $u \in \text{simp}([N])$. Similarly, for any sequence $(u(t))_{t \in \mathbb{N}} \subseteq \text{simp}([N])$, the regret is unchanged by shifting the loss vectors. That is,

$$\sum_{t=1}^{T} \langle \lambda(t), u(t) \rangle - \inf_{u \in \text{simp}([N])} \sum_{t=1}^{T} \langle \lambda(t), u(t) \rangle$$

$$= \sum_{t=1}^{T} \langle \lambda(t) - b(t)1, u(t) \rangle - \inf_{u \in \text{simp}([N])} \sum_{t=1}^{T} \langle \lambda(t) - b(t), u \rangle.$$
Thus, there exist equivalence classes of the outputs from the FTRL($\text{simp}([N])$, $G$, $(r_t)_{t \in \mathbb{Z}_+}$) algorithm modulo parallel additive shifts of the loss vectors. Further, by transforming the losses via

$$
\Phi(\lambda) = \lambda_{-i_1} - \lambda_{i_1} 1 \quad \text{and} \quad \Phi^+(\lambda) = \begin{cases} 
\lambda_i : i \in [\hat{N}] \\
0 : i = i_1
\end{cases}_{i \in [N]}
$$

and defining $\hat{G} = \{\Phi(\lambda) : \lambda \in G\}$, there is a canonical correspondence between the equivalence classes of the outputs from the FTRL($\text{simp}([N])$, $G$, $(r_t)_{t \in \mathbb{Z}_+}$) algorithm and those of the outputs from the FTRL($\hat{F}$, $\hat{G}$, $(\hat{r}_t)_{t \in \mathbb{Z}_+}$) algorithm. Namely,

$$
\arg\min_{u \in \text{simp}([N])} \langle \Lambda(t), u \rangle + r_{0,t}(u) = \phi^{-1} \left( \arg\min_{\mu \in \hat{F}} \langle \Phi(\Lambda(t)), \mu \rangle + \hat{r}_{0,t}(\mu) \right).
$$

Under this correspondence, if $G = [0, 1]^N$, $\hat{R}_c(T) = R_{o1}(T)$.

**Corollary 2.** Consider a regularizer $r_0 : \text{simp}([N]) \to \mathbb{R}$ for which $\hat{r}_0$ is closed, convex, of the Legendre type on $\hat{F}$ (see Definition 2), and twice continuously differentiable on interior($\hat{F}$). For each $t \in \mathbb{N}$, define $r_{0,t}(u) = \beta(t)r_0(u)$ for some increasing function $\beta : \mathbb{N} \to \mathbb{R}_+$. Also, for any $y \in G$ and $x \in \text{simp}([N])$, define the time-dependent local semi-norm by $\|y\|_{t,x}^2 = \langle \Phi(y), \nabla^2 \hat{r}_{0,t}(\phi(x))\Phi(y) \rangle$, and its dual time-dependent local semi-norm by $\|y\|_{t,x,*}^2 = \langle \Phi(y), [\nabla^2 \hat{r}_{0,t}(\phi(x))\Phi(y)]^{-1}\Phi(y) \rangle$. Then, for any sequence of losses $(\langle \Lambda(t), u \rangle + r_{0,t}(u))_{t \in \mathbb{N}} \subseteq G$ such that $\Phi((-\frac{1}{\beta(t)}\Lambda(t)) \in \nabla \hat{r}_0(\text{interior}($$\hat{F}$$))) for all $t \in \mathbb{N}$, there exists a sequence $(\alpha_t)_{t \in \mathbb{N}} \subseteq [0, 1]$ such that, for all $t \in \mathbb{N}$, the weights $(u(t))_{t \in \mathbb{N}}$ output by the FTRL($\text{simp}([N])$, $G$, $(r_t)_{t \in \mathbb{Z}_+}$) algorithm satisfy

$$
\langle \lambda(t), u(t) - u(t + 1) \rangle \leq \frac{1}{\beta(t)} \left\| \left( \frac{\beta(t)}{\beta(t - 1)} - 1 \right) \Lambda(t - 1) - \lambda(t) \right\|_{0,v(t+1),*} \|\lambda(t)\|_{0,v(t+1),*},
$$

where $v(t + 1) = \arg\min_{u \in \text{simp}([N])} \langle \alpha_t\Lambda(t) + (1 - \alpha_t)\frac{\beta(t)}{\beta(t - 1)}\Lambda(t - 1), u \rangle + r_{0,t}(u)$.

**Proof of Corollary 2.** For all $t \in \mathbb{N}$, since $u(t), u(t + 1) \in \text{simp}([N])$, it holds that for any $\lambda(t) \in G$,

$$
\langle \lambda(t), u(t) - u(t + 1) \rangle = \langle \lambda(t) - \lambda_{i_1} 1, u(t) - u(t + 1) \rangle
= \langle \lambda_{-i_1}(t) - \lambda_{-i_1}(t), u_{-i_1}(t) - u_{-i_1}(t + 1) \rangle
= \langle \Phi(\Lambda(t)), \phi(u(t)) - \phi(u(t + 1)) \rangle.
$$

Thus, using that $\phi(u(t))$ are the weights output by the FTRL($\hat{F}$, $\hat{G}$, $(\hat{r}_t)_{t \in \mathbb{Z}_+}$) algorithm, we can apply Lemma 5. The result then follows from observing that $\Phi$ is linear.

**Lemma 6.** Suppose $r_0 = -\psi \circ H$ for some $\psi : [0, \log N] \to \mathbb{R}$ that is strictly increasing, concave, and twice continuously differentiable on $\text{simp}([N])$. Then $\hat{r}_0$ is closed, strictly convex, twice continuously differentiable on interior($\hat{F}$), and of the Legendre type on $\hat{F}$.

Moreover, for all $x \in \text{simp}([N])$ and $y \in G$,

$$
\|y\|_{0,x,*}^2 \leq \frac{1}{\psi(\varphi(H(x)))}\varphi(y_t).
$$

47
To derive the semi-norm formula, first notice that using the Sherman–Morrison–Woodbury formula
\[
\nabla \nu \cdot \hat{H}(\mu) = \log(\mu_i) - \log (1 - \langle 1, \mu \rangle),
\]
\[
\nabla^2 \hat{H}(\mu) = \frac{1}{\mu_i} + \frac{1}{1 - \langle 1, \mu \rangle}, \quad \text{and}
\]
\[
\nabla \mu \cdot \nabla \hat{H}(\mu) = \frac{1}{1 - \langle 1, \mu \rangle}.
\]

Thus,
\[
\nabla^2 \hat{H}(\mu) = \text{diag}(1/\mu) + \frac{1}{1 - \langle 1, \mu \rangle} \mathbf{1}\mathbf{1}^T,
\]
which is strictly positive-definite on \( \text{interior}(\hat{F}) \).

Therefore \( \hat{H} \) is strictly concave. Since a composition of a strictly concave function with a strictly increasing strictly concave function is strictly concave, \( \psi \circ \hat{H} \) is strictly concave, which means \( \hat{r}_0 \) is strictly convex. Since \( \hat{r}_0 \) is continuous and finite on \( \hat{F} \), and \( \hat{F} \) is closed it must also be a closed function, because a proper convex function is closed if it is lower-semi-continuous. The twice continuous differentiability of \( \hat{r}_0 \) on \( \text{interior}(\hat{F}) \) follows from the twice continuous differentiability of \( H \) on \( \hat{F} \) and the twice differentiability of \( \psi \).

Since we have already observed that \( \text{interior}(\hat{F}) \) is convex and non-empty, to see that \( \hat{r}_0 \) is of the Legendre type on \( \hat{F} \) we need only verify that \( \lim_{n \to \infty} \| \nabla \hat{r}_0(\mu^{(n)}) \| \to \infty \) for any \( \{\mu^{(n)}\}_{n \in \mathbb{N}} \subseteq \text{interior}(\hat{F}) \) such that \( \mu^{(n)} \to \nu \in \partial(\hat{F}) \). The gradient of \( \hat{r}_0 \) is given by
\[
\nabla \hat{r}_0(\mu) = -[\psi' \circ \hat{H}(\mu)] \nabla \hat{H}(\mu).
\]

Now, notice that if \( \nu \in \partial(\hat{F}) \), \( \hat{H}(\nu) \leq \log(N - 1) \). Since \( \psi \) is strictly increasing and concave on \([0, \log(N)]\), this implies \( \psi'(\hat{H}(\nu)) > 0 \). At any \( \nu \in \partial \hat{F} \), either there exists an \( i \in [\hat{N}] \) such that \( \nu_i = 0 \) or \( \langle 1, \nu \rangle = 1 \). In both cases, \( \mu^{(n)} \to \nu \) implies \( \| \nabla \hat{H}(\mu^{(n)}) \| \to +\infty \). Therefore, \( \nabla \hat{r}_0(\mu_i) \to \psi'(\hat{H}(\nu)) \cdot (+\infty) = +\infty \), which confirms that \( \hat{r}_0 \) is of the Legendre type on \( \hat{F} \).

To derive the semi-norm formula, first notice that using the Sherman–Morrison–Woodbury formula gives
\[
- [\nabla^2 \hat{H}(\mu)]^{-1} = \text{diag}(\mu) - \text{diag}(\mu) \mathbf{1} \left( (1 - \langle 1, \mu \rangle) + \mathbf{1}^T \text{diag}(\mu) \mathbf{1} \right)^{-1} \mathbf{1}^T \text{diag}(\mu)
\]
\[
= \text{diag}(\mu) - \text{diag}(\mu) \mathbf{1} \mathbf{1}^T \text{diag}(\mu)
\]
\[
= \text{diag}(\mu) - \mu \mu^T.
\]

Then,
\[
\nabla^2 \hat{r}_0(\mu) = -[\psi'' \circ \hat{H}(\mu)](\nabla \hat{H}(\mu)) (\nabla \hat{H}(\mu))^T - [\psi' \circ \hat{H}(\mu)](\nabla^2 \hat{H}(\mu))
\]
\[
\geq -[\psi' \circ \hat{H}(\mu)](\nabla^2 \hat{H}(\mu)),
\]
where \( A \succeq B \) means \( A - B \) is positive semi-definite. Therefore,
\[
[
abla^2 \hat{r}_0(\mu)]^{-1} \succeq \left( -[\psi' \circ \hat{H}(\mu)](\nabla^2 \hat{H}(\mu)) \right)^{-1}
\]
\[
= \frac{1}{\psi' \circ \hat{H}(\mu)} \left( \text{diag}(\mu) - \mu \mu^T \right).
\]
Applying Eqs. (36) and (37) to an arbitrary \( x \in \text{simp}([N]) \) and \( y \in G \) gives

\[
\|y\|_{0,x,*}^2 = \left\langle \Phi(y), \nabla^2 \hat{r}_0(\phi(x))^{-1} \Phi(y) \right\rangle \\
= \left\langle y_{-i_1} - y_{i_1} \mathbf{1}, \nabla^2 \hat{r}_0(x_{-i_1})^{-1} [y_{-i_1} - y_{i_1} \mathbf{1}] \right\rangle \\
\leq \frac{1}{\psi' \circ \hat{H}(x_{-i_1})} \left\langle y_{-i_1} - y_{i_1} \mathbf{1}, \left[ \text{diag}(x_{-i_1}) - x_{-i_1} x_{-i_1}^T \right] [y_{-i_1} - y_{i_1} \mathbf{1}] \right\rangle \\
= \frac{1}{\psi' \circ \hat{H}(x)} \left\langle y - y_{i_1} \mathbf{1}, \left[ \text{diag}(x) - xx^T \right] [y - y_{i_1} \mathbf{1}] \right\rangle \\
= \frac{1}{\psi' \circ \hat{H}(x)} \var(y_{i_1} - y_{i_1}) \\
= \frac{1}{\psi' \circ \hat{H}(x)} \var(y_i).
\]

\[\square\]

**Lemma 7.** Suppose \( r_0 = -\psi \circ \hat{H} \) for some \( \psi : [0, \log N] \to \mathbb{R} \) that is strictly increasing, concave, and twice continuously differentiable on \( \text{simp}([N]) \). Further, suppose that \( r_{0,t} = \beta(t)r_0 \) for some strictly increasing \( \beta : \mathbb{N} \to \mathbb{R}_+ \). Then, \( \nabla r_0(\text{interior}(\hat{F})) = \mathbb{R}^{|N|} \), and the weight vectors produced by the FTRL(\text{simp}([N])), \( G, (r_t)_{t \in \mathbb{Z}_+} \) algorithm are equivalent to the weights produced by HEDGE with an implicitly defined learning rate. In particular, the learning rate and weights are the solution to the system of equations

\[
\eta(t + 1) = \frac{1}{\beta(t) \cdot \psi' \circ \hat{H}(u(t + 1))} \\
u(t + 1) = \left( \frac{\exp(-\eta(t + 1)\Lambda_i(t))}{\sum_{i' \in [N]} \exp(-\eta(t + 1)\Lambda_{i'}(t))} \right)_{i \in [N]}. \tag{38}
\]

Moreover, for any sequence of losses \( (\lambda(t))_{t \in \mathbb{N}} \subseteq G \), this system has a unique solution satisfying

\[
\eta(t + 1) \in \left[ \frac{1}{\beta(t) \cdot \psi'(0)}, \frac{1}{\beta(t) \cdot \psi'(\log N)} \right].
\]

**Proof of Lemma 7.** First, recall that the weights output by the FTRL(\text{simp}([N])), \( G, (r_t)_{t \in \mathbb{Z}_+} \) algorithm will solve

\[
u(t + 1) = \arg\min_{w \in \text{simp}([N])} \left\{ \langle \Lambda(t), w \rangle - \beta(t)\psi(H(w)) \right\} \\
= \phi^{-1} \left( \arg\min_{\mu \in \hat{F}} \left\{ \langle \Lambda_{-i_1}(t) - 1\Lambda_{i_1}(t), \mu \rangle - \beta(t)\psi(\hat{H}(\mu)) \right\} \right).
\]

By Lemma 6 and Corollary 1, we know that this means \( u(t + 1) = \phi^{-1}(\mu) \) for the unique \( \mu \in \text{interior}(\hat{F}) \) such that

\[
\nabla \hat{H}(\mu) = \frac{\Lambda_{-i_1}(t) - 1\Lambda_{i_1}(t)}{\beta(t) \cdot \psi'(\hat{H}(\mu))}.
\]

Thus, by the definition of \( \phi \) and \( \hat{H} \),

\[
\nabla \hat{H}(u_{-i_1}(t + 1)) = \frac{\Lambda_{-i_1}(t) - 1\Lambda_{i_1}(t)}{\beta(t) \cdot \psi'(\hat{H}(u_{-i_1}(t + 1)))} = \frac{\Lambda_{-i_1}(t) - 1\Lambda_{i_1}(t)}{\beta(t) \cdot \psi'(\hat{H}(u(t + 1)))}. \tag{39}
\]
It is well known that the unique solution to
\[ \nabla \hat{H}(\phi(u)) = \Phi(-X) \]
is given by
\[ u_i = \frac{\exp(-X_i)}{\sum_{i' \in [N]} \exp(-X_{i'})}. \]
Therefore, any and all solutions of Eq. (39) must also be solutions of Eq. (38). Next, we want to show that there is a unique solution, \( \eta(t + 1) \), to the implicit equation
\[ \eta(t + 1) = \frac{1}{\beta(t) \cdot \psi' \circ H \left( \sum_{i' \in [N]} \frac{\exp(-\eta(t+1)\Lambda_i(t))}{\exp(-\eta(t+1)\Lambda_{i'}(t))} i \in [N] \right)} \]  \hspace{1cm} (40)
On the left hand side, we have \( f_1(\eta) = \eta \), which is trivially strictly increasing from 0 to \( \frac{1}{\beta(t) \cdot \psi'(\log N)} \) as \( \eta \) increases from 0 to \( \frac{1}{\beta(t) \cdot \psi'(0)} \). On the right hand side, we have
\[ f_2(\eta) = \frac{1}{\beta(t) \cdot \psi' \circ H \left( \sum_{i' \in [N]} \frac{\exp(-\eta(t+1)\Lambda_i(t))}{\exp(-\eta(t+1)\Lambda_{i'}(t))} i \in [N] \right)}, \]
which is non-increasing with \( f_2(0) = \frac{1}{\beta(t) \cdot \psi'(\log N)} \). Further, by non-negativity of entropy and concavity of \( \psi, f_2(\eta) \geq \frac{1}{\beta(t) \cdot \psi'(0)} \). Thus, \( f_1 \) and \( f_2 \) must intersect at some \( \eta \in \left[ \frac{1}{\beta(t) \cdot \psi'(0)}, \frac{1}{\beta(t) \cdot \psi'(\log N)} \right] \), and this intersection is unique by the monotonicity of both functions and the strict monotonicity of \( f_1 \).
This guarantees at least one interior point solution to the implicit equation defined in Eq. (40). Moreover, since the objective function optimized by the weights output by the FTRL(\( \text{simp}([N]), G, (r_t)_{t \in \mathbb{Z}_+} \)) algorithm is strictly convex, this interior point solution must be the unique optimizer of the objective. Finally, since the sequence of losses was arbitrary and the FTRL(\( \hat{F}, \hat{G}, (\hat{r}_t)_{t \in \mathbb{Z}_+} \)) algorithm outputs a unique weight vector at each time \( t + 1 \), we conclude that \( [\nabla \hat{r}_0](\text{interior}(F)) = \mathbb{R}^{[N]} \) as otherwise there would be some loss vector for which the solution to Eq. (40) does not exist.

\[ \square \]

C.1 Proof of Theorem 7

Theorem 7 is an immediate consequence of the combination of Corollary 2 and Lemmas 4, 6 and 7. In the application of Lemma 4, we can select \( u_*(T) \in \arg \min_{u \in \text{simp}([N])} \langle L(T), u \rangle \) such that \( H(u_*(T)) = 0 \) because at least one arg min occurs at a vertex of the simplex.

\[ \square \]

D Proofs of lemmas in Section 9

D.1 Proof of Lemma 1

First, observe that
\[ H(u) = -\sum_{i \in [N]} u_i \log (u_i) = -\sum_{i_0 \in \mathcal{I}_0} u_{i_0} \log (u_{i_0}) - \sum_{i \in [N] \setminus \mathcal{I}_0} u_i \log (u_i). \]
To bound the first term, consider the optimization problem

\[
\begin{align*}
\min_{\langle 1, u \rangle = 1} & \sum_{i_0 \in I_0} u_{i_0} \log(u_{i_0}), \\
\text{subject to } & \langle 1, u_{I_0} \rangle \leq 1
\end{align*}
\]

where \( u_{I_0} = \{u_{i_0}\}_{i_0 \in I_0} \). This is a convex objective with linear constraints, so it can be solved using the Lagrange multiplier method. The Lagrangian is

\[
L(u; \alpha, \beta) = \sum_{i_0 \in I_0} u_{i_0} \log(u_{i_0}) + \alpha (\langle 1, u \rangle - 1) + \beta (\langle 1, u_{I_0} \rangle - 1),
\]

and the dual problem is

\[
\begin{align*}
\max_{\alpha \in \mathbb{R}} & \min_{\beta \geq 0, u \in \mathbb{R}^N} \sum_{i_0 \in I_0} u_{i_0} \log(u_{i_0}) + \alpha (\langle 1, u \rangle - 1) + \beta (\langle 1, u_{I_0} \rangle - 1).
\end{align*}
\]

This gives, for \( i_0 \in [N] \) and \( i \in [N] \setminus I_0 \),

\[
\begin{align*}
\partial_{i_0} L(u; \alpha, \beta) &= \log(u_{i_0}) + 1 + \alpha + \beta, \\
\partial_i L(u; \alpha, \beta) &= \alpha.
\end{align*}
\]

Then, at the saddle point, \( \alpha = 0 \) and \( \log u_{i_0} = -\frac{1}{1 + \beta} \) for all \( i_0 \in I_0 \).

If \( \beta = 0 \) then \( u_{i_0} = \frac{1}{\exp(1)} \) for all \( i_0 \in I_0 \). This is only feasible if \( N_0 \leq 2 \). In this case

\[
\sum_{i_0 \in I_0} u_{i_0} \log(u_{i_0}) \geq -\sum_{i_0 \in I_0} \frac{\log(\exp(1))}{\exp(1)} = -\frac{N_0}{\exp(1)}.
\]

Otherwise \( \beta > 0 \), and by the K.K.T. condition, \( \langle 1, u_{I_0} \rangle = 1 \), which implies that \( u_{i_0} = \frac{1}{N_0} \) for all \( i_0 \in I_0 \). That is,

\[
\sum_{i_0 \in I_0} u_{i_0} \log(u_{i_0}) \geq -\sum_{i_0 \in I_0} \frac{\log(N_0)}{N_0} = -\log(N_0).
\]

Thus for \( N_0 \geq 3 \)

\[
H(u) \leq \log(N_0) - \sum_{i \in [N] \setminus I_0} u_i \log u_i,
\]

and for \( N_0 \leq 2 \)

\[
H(u) \leq \frac{N_0}{\exp(1)} - \sum_{i \in [N] \setminus I_0} u_i \log u_i.
\]

Further, if \( I_0 = \{i_0\} \), since \( \log(x) \geq 1 - 1/x \) for all \( x \geq 0 \),

\[
H(u) = -\sum_{i \in [N]} u_i \log(u_i)
\]

\[
= -u_{i_0} \log(u_{i_0}) - \sum_{i \in [N] \setminus I_0} u_i \log(u_i)
\]

\[
\leq -u_{i_0} \left(1 - \frac{1}{u_{i_0}}\right) - \sum_{i \in [N] \setminus I_0} u_i \log(u_i)
\]

\[
= (1 - u_{i_0}) - \sum_{i \in [N] \setminus I_0} u_i \log(u_i)
\]

\[
= \sum_{i \in [N] \setminus I_0} u_i - \sum_{i \in [N] \setminus I_0} u_i \log(u_i).
\]
In order to control the sum over ineffective experts we use the technical result of Lemma 11, which says that
\[ - \sum_{i \in [N] \setminus \mathcal{I}_0} u_i \log(u_i) \leq \frac{1}{(1 - p) \exp(1)} \sum_{i \in [N] \setminus \mathcal{I}_0} [u_i]^p \] (44)

Combing Eqs. (41) to (44) gives for \( N_0 \geq 1, \)
\[ H(u) \leq \frac{2}{\exp(1) \log(2)} \log(N_0) + \left( 1 + \frac{1}{(1 - p) \exp(1)} \right) \sum_{i \in [N] \setminus \mathcal{I}_0} u_i^p. \]

\[ \square \]

D.2 Proof of Lemma 2

For the first result, observe that from Theorem 7, \( u(t+1) \) is the unique solution to
\[ \eta(t+1) = \frac{1}{\sqrt{t+1} \cdot \psi' \circ H(u(t+1))} \]
\[ u(t+1) = \left( \frac{\exp(-\eta(t+1)L_i(t))}{\sum_{i' \in [N]} \exp(-\eta(t+1)L_{i'}(t))} \right)_{i \in [N]}, \]
and
\[ \eta(t+1) \in \left[ \frac{1}{\sqrt{t+1} \cdot [\psi'(0)]}, \frac{1}{\sqrt{t+1} \cdot [\psi'(\log N)]} \right]. \]

Now, set \( \eta(t+1) = \frac{1}{\sqrt{t+1} \cdot \psi'(0)}. \) For \( i \in [N] \setminus \mathcal{I}_0, \) since \( \eta(t+1) \leq \eta(t+1) \) and \( L_{I^*(t)}(t) \leq L_i(t) \) by definition,
\[ \left[ u_i(t+1) \right]^p \leq \left( \frac{u_i(t+1)}{u_{I^*(t)}(t+1)} \right)^p \]
\[ = \exp \left\{ -p \eta(t+1) \left[ L_i(t) - L_{I^*(t)}(t) \right] \right\} \]
\[ \leq \exp \left\{ -p \eta(t+1) \left[ L_i(t) - L_{I^*(t)}(t) \right] \right\} \]
\[ \leq \min_{i_0 \in \mathcal{I}_0} \exp \left\{ -p \eta(t+1) \left[ L_i(t) - L_{i_0}(t) \right] \right\}. \]

Thus, using Theorem 2,
\[ \sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi, \hat{\pi}} \left[ u_i(t+1)^p \right] \leq \sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi, \hat{\pi}} \left[ \min_{i_0 \in \mathcal{I}_0} \exp \left\{ -p \eta(t+1) \left[ L_i(t) - L_{i_0}(t) \right] \right\} \right] \]
\[ \leq \exp \left\{ -t \eta(t+1) \Delta_0 p + t \eta(t+1)^2 \frac{p^2}{2} \right\} \]
\[ = \exp \left\{ -t \frac{1}{\sqrt{t+1} \cdot [\psi'(0)]} \Delta_0 p + t \left( \frac{1}{\sqrt{t+1} \cdot [\psi'(0)]} \right)^2 \frac{p^2}{2} \right\} \]
\[ \leq \exp \left\{ \frac{p^2}{2[\psi'(0)]^2} \right\} \exp \left\{ -\frac{\Delta_0 p}{\sqrt{2[\psi'(0)]}} \right\}, \]
where in the last inequality we used the fact that \( \frac{1}{t+1} \geq \frac{1}{2} \) for \( t \in \mathbb{N} \).

For the second result, for each \( \alpha \in [0, 1] \), we define the intermediate losses \( \xi^{(\alpha)}(t) = \alpha L(t) + (1 - \alpha) \sqrt{\frac{t+1}{t}} L(t - 1) \). We define a new random expert by \( \hat{I}^{(\alpha)}(t) = \arg \min_{i \in [N]} \xi^{(\alpha)}(t) \), which is analogous to \( I(t) \) but for \( \xi^{(\alpha)}(t) \). Then, applying Lemma 7 to the intermediate losses, observe that \( v^{(\alpha)}(t+1) \) is the unique solution to

\[
\vartheta^{(\alpha)}(t+1) = \frac{1}{\sqrt{t+1} \cdot \psi'(0) \circ H(v^{(\alpha)}(t+1))} \\
v^{(\alpha)}(t+1) = \left( \frac{\exp \left( -\vartheta^{(\alpha)}(t+1) \xi^{(\alpha)}(t) \right)}{\sum_{i' \in [N]} \exp \left( -\vartheta^{(\alpha)}(t+1) \xi^{(\alpha)}(t) \right)} \right)_{i \in [N]},
\]

and

\[
\vartheta^{(\alpha)}(t+1) \in \left[ \frac{1}{\sqrt{t+1} \cdot \psi'(0)}, \frac{1}{\sqrt{t+1} \cdot [\psi'(\log N)]} \right].
\]

Next, using that \( \ell_i(t) \in [0, 1] \) for all \( i \in [N] \),

\[
\xi^{(\alpha)}_i(t) = \alpha L_i(t) + (1 - \alpha) \sqrt{\frac{t+1}{t}} L_i(t - 1) \geq L_i(t) - 1.
\]

Then, observe that since \( L_i(t) \leq t \) for all \( i \in [N] \), \( \sqrt{\frac{t+1}{t}} L_i(t - 1) \leq L_i(t - 1) + 1 \) for all \( t \in \mathbb{N} \). Thus, for any \( i' \in [N] \),

\[
\xi^{(\alpha)}_{i'}(t) = \alpha L_{i'}(t) + (1 - \alpha) \sqrt{\frac{t+1}{t}} L_{i'}(t - 1) \leq L_{i'}(t) + 1.
\]

Combining these two facts gives that for all \( \alpha \in [0, 1] \),

\[
\xi^{(\alpha)}_i(t) - \xi^{(\alpha)}_{i'}(t) \geq L_i(t) - L_{i'}(t) - 2.
\]

Now, for \( i \in [N] \setminus I_0 \), taking \( \overline{\theta}(t+1) = \frac{1}{\sqrt{t+1} \cdot \psi'(0)} \), and since \( \overline{\theta}(t+1) \leq \vartheta(t+1) \) we have

\[
\left[ v^{(\alpha)}_i(t+1) \right]^p \leq \left( \frac{v^{(\alpha)}_i(t+1)}{v^{(\alpha)}_{I_0}(t+1)} \right)^p \\
= \exp \left\{ -p \vartheta^{(\alpha)}(t+1) \left[ \xi^{(\alpha)}_i(t) - \xi^{(\alpha)}_{I_0}(t) \right] \right\} \\
\leq \exp \left\{ -p \overline{\theta}(t+1) \left[ \xi^{(\alpha)}_i(t) - \xi^{(\alpha)}_{I_0}(t) \right] \right\} \\
\leq \min_{i_0 \in I_0} \exp \left\{ -p \overline{\theta}(t+1) \left[ L_i(t) - L_{i_0}(t) - 2 \right] \right\}.\]
Thus, again using Theorem 2,
\[
\sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{x}} \sup_{a \in [0,1]} \left[ v_i(t+1)^p \right] \leq \sup_{\pi \in \mathcal{P}(D)} \mathbb{E}_{\pi, \hat{x}} \left[ \min_{i_0 \in I_0} \exp \{ -p \, \eta(t+1) \, [L_i(t) - L_{i_0}(t) - 2] \} \right]
\leq \exp \left\{ 2p \eta(t+1) - t \eta(t+1) \Delta_0 p + t \eta(t+1) \frac{p^2 \eta^2}{2} \right\}
= \exp \left\{ \frac{2p - \Delta_0 pt}{\sqrt{t+1} \cdot \psi'(0)} + \left( \frac{1}{\sqrt{t+1} \cdot \psi'(0)} \right)^2 \frac{p^2 t}{2} \right\}
\leq \exp \left\{ \frac{2p}{\psi'(0)} + \frac{p^2}{2(\psi'(0))^2} \right\} \exp \left\{ - \frac{\Delta_0 p}{\sqrt{2(\psi'(0))}} \sqrt{t} \right\},
\]
where in the last inequality we again used the fact that \( \frac{1}{\sqrt{t+1}} \geq \frac{1}{2} \) for \( t \in \mathbb{N} \). \( \square \)

### D.3 Proof of Lemma 3

Substituting the variance bounds of Lemma 8 Eq. (5) using \( \beta(t) = \sqrt{t+1}, \psi \) increasing and concave, and the fact that \( H(u) \leq \log(N) \) gives
\[
\hat{R}_{\text{FTRL}}(t_0) \leq -\psi(0) \sqrt{t_0 + 1} + \sum_{t=0}^{t_0} \left[ \sqrt{t+1} - \sqrt{t} \right] \psi(\log(N)) + \sum_{t=1}^{t_0} \frac{3}{8 \sqrt{t+1} \cdot \psi'(\log(N))}.
\]
\[
= \sqrt{t_0 + 1} \left( \psi(\log(N)) - \psi(0) \right) + \sum_{t=1}^{t_0} \frac{3}{8 \sqrt{t+1} \cdot \psi'(\log(N))}.
\]
Then, since
\[
\sum_{t=1}^{t_0} \frac{1}{\sqrt{t+1}} \leq \int_0^{t_0} \frac{1}{\sqrt{t+1}} dt = 2 \sqrt{t_0 + 1},
\]
we have that
\[
\hat{R}_{\text{FTRL}}(t_0) \leq \sqrt{t_0 + 1} \left( \psi(\log(N)) - \psi(0) + \frac{3}{4 \psi'(\log(N))} \right). \quad \square
\]

### D.4 Miscellaneous stochastic and mathematical results

Here we state a few convenient results that will be used repeatedly, but require none of the assumptions of our setting except boundedness. The first two of these lemmas allow us to control the variance of the experts’ losses.

**Lemma 8.** For any \( w \in \text{simp}(N) \), \( (\ell(t))_{t \in \mathbb{N}} \subseteq [0,1]^N \), and \( t \in \mathbb{N} \),
\[
\Var_{I \sim w} \left[ \left( \sqrt{\frac{t+1}{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right] \leq \frac{9}{16} \quad \text{and} \quad \Var_{I \sim w} \left[ \ell_I(t) \right] \leq \frac{1}{4}.
\]

**Proof of Lemma 8.** Since \( \sqrt{t+1} - \sqrt{t} \leq \frac{1}{2 \sqrt{t}} \) for \( t \geq 1 \), \( \left( \sqrt{\frac{t+1}{t}} - 1 \right) \in \left[ 0, \frac{1}{2t} \right] \). Combined with \( \ell(t) \in [0,1]^N \) for all \( t \in \mathbb{N} \), this gives that for all \( i \in [N] \),
\[
\left( \sqrt{\frac{t+1}{t}} - 1 \right) L_i(t-1) - \ell_i(t) \in [ -1, \frac{1}{2} ].
\]
Thus, the result follows since if \( a \leq X \leq b \), then \( \text{Var}(X) \leq (b - a)^2/4 \).

**Lemma 9.** For any \( \pi \in \mathcal{P} \), \( \hat{\pi} \in \hat{\mathcal{P}} \), sequence \((w(t))_{t \in \mathbb{N}}\) such that \( w(t) \) is \( \sigma(h(t - 1))\)-measurable for all \( t, i_0 \in [N] \), and \( t \in \mathbb{N} \),

\[
\mathbb{E}_{\pi, \hat{\pi}} \left[ \frac{\text{Var}_{I \sim w(t+1)} \left[ \left( \frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right] \text{Var}_{I \sim w(t+1)} \left[ \ell_I(t) \right]}{9/4 \mathbb{E}_{\pi, \hat{\pi}} \left[ \sum_{i \neq i_0} w_i(t + 1) \right]} \right] \leq \frac{27}{32} \mathbb{E}_{\pi, \hat{\pi}} \left[ \sum_{i \neq i_0} w_i(t + 1) \right].
\]

**Proof of Lemma 9.** First, let \( \nu \) be any distribution such that \( \text{Support}(\nu) \subset [-y, 1 - y] \) and \( x \in [-y, 1 - y] \), and suppose \( X \sim \alpha \delta_x + (1 - \alpha) \nu \) for some \( \alpha \in [0, 1] \).

Since variance is invariant to shifts, we can suppose \( y = 0 \) without loss of generality. Define \( \mu_\nu = \mathbb{E}_{Z \sim \nu}(Z) \) and \( \sigma^2_\nu = \text{Var}_{Z \sim \nu}(Z) \). Then, using the variance for a mixture distribution,

\[
\text{Var}(X) = \alpha x^2 + (1 - \alpha) \mu_\nu^2 - (\alpha x + (1 - \alpha) \mu_\nu)^2 + (1 - \alpha) \sigma^2_\nu
\]

\[
= \alpha(1 - \alpha)x^2 + \alpha(1 - \alpha) \mu_\nu^2 - 2\alpha(1 - \alpha)x \mu_\nu + (1 - \alpha) \sigma^2_\nu
\]

\[
= \alpha(1 - \alpha)(x - \mu_\nu)^2 + (1 - \alpha) \sigma^2_\nu
\]

Now,

\[
\sup_{x, \nu} \text{Var}(X) = \sup_{x, \mu, \nu : \mu = \mu} \text{Var}(X).
\]

The inner sup is achieved by \( \nu(\mu) = \text{Ber}(\mu) \) and has \( \sigma^2_{\nu(\mu)} = \mu(1 - \mu) \), so that

\[
\sup_{x, \nu} \text{Var}(X) = \sup_{\mu} \alpha(1 - \alpha)(x - \mu)^2 + (1 - \alpha) \mu(1 - \mu).
\]

Now, the inner sup is achieved by \( x = 0 \) when \( \mu \geq 1/2 \) and by \( x = 1 \) when \( \mu < 1/2 \). Due to symmetry we need only consider the case that \( \mu \geq 1/2 \).

\[
\sup_{x, \nu} \text{Var}(X) = \sup_{\mu} \alpha(1 - \alpha) \mu^2 + (1 - \alpha) \mu(1 - \mu)
\]

\[
= \sup_{\mu} \left[ -(1 - \alpha)^2 \mu^2 + (1 - \alpha) \mu \right].
\]

Since \( \alpha \in [0, 1] \) this is a constrained quadratic maximum. If the unconstrained maximum occurs in interior of the region then it is equal to the constrained maximum. Otherwise the constrained maximum occurs at the boundary.

The unconstrained maximum occurs at \( \mu = \frac{1}{2(1 - \alpha)} \) with objective value \( 1/4 \). This in the interior of the constraint region when \( (1 - \alpha) > 1/2 \); equivalently \( \alpha < 1/2 \). The boundary values are 0 and \( \alpha(1 - \alpha) \).
That is,
\[
\Var(X) \leq \begin{cases} 
\alpha(1 - \alpha) & : \alpha \geq 1/2, \\
1/4 & : \alpha < 1/2.
\end{cases}
\tag{45}
\]

Let \(w \in \text{simp}([N])\) be arbitrary. We can apply Eq. (45) to obtain
\[
\Var_{I \sim w}[\ell_I(t)] \leq \frac{1}{4} \mathbb{I}_{[w_0 \leq 1/2]} + (1 - w_0).
\]

Similarly, since \((\sqrt{\frac{t+1}{t}} - 1) L_I(t-1) - \ell_I(t) \in [-1, \frac{1}{2}]\),
\[
\Var_{I \sim w}
\left[
\left(\sqrt{\frac{t+1}{t}} - 1\right) L_I(t-1) - \ell_I(t)
\right]
\leq \left(\frac{3}{2}\right)^2 \left(\frac{1}{4} \mathbb{I}_{[w_0 \leq 1/2]} + (1 - w_0)\right).
\]

Thus, using Markov’s inequality,
\[
\mathbb{E}_{\pi, \hat{\pi}} \sqrt{\frac{1}{4} \Var_{I \sim w(t+1)} \left[\left(\sqrt{\frac{t+1}{t}} - 1\right) L_I(t-1) - \ell_I(t)\right] \Var_{I \sim w(t+1)}[\ell_I(t)]}
\leq \frac{3}{2} \left(\frac{1}{4} \mathbb{P}_{\pi, \hat{\pi}} \left[w_i(t+1) \leq 1/2\right] + \mathbb{E}_{\pi, \hat{\pi}} \left[1 - w_i(t+1)\right]\right)
\leq \frac{9}{4} \mathbb{E}_{\pi, \hat{\pi}} \left[\sum_{i \neq i_0} w_i(t+1)\right].
\]

Alternatively, using \(\Var_{I \sim w}[\ell_I(t)] \leq 1/4\),
\[
\mathbb{E}_{\pi, \hat{\pi}} \left[\frac{1}{16} \Var_{I \sim w(t+1)} \left[\left(\sqrt{\frac{t+1}{t}} - 1\right) L_I(t-1) - \ell_I(t)\right] \Var_{I \sim w(t+1)}[\ell_I(t)]\right]
\leq \frac{9}{16} \left(\frac{1}{4} \mathbb{P}_{\pi, \hat{\pi}} \left[w_i(t+1) \leq 1/2\right] + \mathbb{E}_{\pi, \hat{\pi}} \left[1 - w_i(t+1)\right]\right)
\leq \frac{27}{32} \mathbb{E}_{\pi, \hat{\pi}} \left[\sum_{i \neq i_0} w_i(t+1)\right].
\]

Next, we have a result which controls a summation term which appears often in our proofs.

**Lemma 10.** For any \(\alpha > 0\) and \(t_0 \geq 1\)
\[
\sum_{t=t_0+1}^{T} \frac{1}{\sqrt{t}} \exp \left\{ -\alpha \sqrt{t} \right\} \leq \frac{2}{\alpha} \exp(-\alpha \sqrt{t_0}).
\]
Proof of Lemma 10.

\[ \sum_{t=t_0+1}^T \frac{1}{\sqrt{t}} \exp \{-\alpha \sqrt{t} \} \leq \int_{t_0}^T \frac{1}{\sqrt{t}} \exp \{-\alpha \sqrt{t} \} \, dt \]

\[ \leq \int_{t_0}^\infty \frac{1}{\sqrt{t}} \exp \{-\alpha \sqrt{t} \} \, dt \]

\[ = \sqrt{t_0} \exp \{-\alpha \sqrt{t_0} \} \cdot \int_{t_0}^\infty \frac{1}{\sqrt{t}} \exp \{-\alpha \sqrt{t} \} \, dt \]

\[ = \frac{2}{\alpha} \exp(-\alpha \sqrt{t_0}). \]

Finally, we have a simple fact about logarithms that will be useful when controlling the entropy of weight distributions.

Lemma 11. For \( x \in (0, 1] \) and \( p \in (0, 1) \)

\[ -x \log(x) \leq \frac{1}{(1 - p) \exp(1)} x^p. \]

Proof of Lemma 11. Consider \( f(x) = -x^{1-p} \log(x) \). Then, \( f(0^+) = f(0) = 0 \), and \( f(1) = 0 \), and

\[ f'(x) = -(1 - p)x^{-p} \log(x) - x^{-p} = -x^{-p}((1 - p) \log(x) + 1). \]

Thus, the only critical point of \( f \) occurs at \( x_0 = \exp(-1/(1 - p)) \). This is a local max since \( \text{sign}(f'(x)) = -\text{sign}(x-x_0) \) for \( x \in (0, 1) \). Thus, \( f \) is maximized on the interval \( (0, 1) \) at \( x_0 \). Hence \( f(x) \leq f(x_0) = \frac{1}{(1 - p) \exp(1)} x^p \). Multiplying both sides by \( x^p \) proves the result.

\[ \square \]

E Proofs of lower bounds

E.1 Proof of Theorem 3

Our strategy is to define a simple setting with multiple experts (many of them identical), so that we can show the lower bound holds in the asymptotic limit as \( T \) and \( N \) tends to infinity. Let \( \mathcal{Y} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \), \( \hat{\mathcal{Y}} = \text{simp}([3]) \), and \( \ell(\hat{y}, y) = \frac{1}{2} \sum_{i=1}^3 |\hat{y}_i - y_i| \). Observe that \( \ell(\hat{y}, y) \in [0, 1] \) for all \( \hat{y} \in \hat{\mathcal{Y}} \) and \( y \in \mathcal{Y} \). Let \( N_0 \leq N \in \mathbb{N} \).

In this setting, consider the distribution

\[ \mu_0 = \left( \left( \frac{1}{2} \delta_{(1,0,0)} + \frac{1}{2} \delta_{(0,1,0)} \right) \otimes N_{0_1} \otimes (\delta_{(0,0,1)})^{\otimes (N- N_0)} \right) \otimes \left( \frac{1}{2} \delta_{(1,0,0)} + \frac{1}{2} \delta_{(0,1,0)} \right), \]

and let \( \mathcal{D} = \{\mu_0\} \). Then \( \mathcal{P}(\mathcal{D}) \) contains a single policy, \( \pi_* \), given by

\[ \pi_* = (h(t) \in \mathcal{H}^t \mapsto \mu_0)_{t \in \mathbb{N}}. \]

Intuitively, each of the effective experts flips a coin to play the first or second element, but the observation is also either the first or second element from an independent coin toss, and the ineffective experts always output the third element.
Now, define the pushforward of the distribution through the loss function by \( \mu'_0 = \ell_2 \mu_0 \) to obtain the single loss distribution on the experts. Observe this simplifies to
\[
\mu'_0 = \text{Ber}(1/2)^{\otimes N_0} \otimes \text{Ber}(1)^{\otimes (N - N_0)}.
\]
This singleton policy space satisfies the time-homogeneous convex constraint condition with \( I_0 = [N_0] \), and \( \Delta_0 = 1/2 \).

Note that any prediction \( \hat{y} \) has \( \mathbb{E}_{y \sim \mu_0} \ell(\hat{y}, y) \geq \frac{1}{2} \). For each \( i_0 \in I_0 \), let \( M_{i_0} = \sum_{t=1}^{T} \ell(\hat{y}_{i_0}, y) \) be the random variable corresponding to the cumulative loss of the effective expert. Then, \( M_{i_0} \text{idl} \sim \text{Bin}(T, 1/2) \), and
\[
\inf_{\pi \in \mathcal{P}^N} \sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi, \hat{x}} \max_{i \in [N]} \sum_{t=1}^{T} \left[ \ell(\hat{y}(t), y(t)) - \ell(x_i(t), y(t)) \right] \geq \mathbb{E}_{M \sim \text{Bin}(T, 1/2)^{\otimes N_0}} \max_{i_0 \in I_0} \left( \frac{T}{2} - M_{i_0} \right).
\]

Now, since \( \frac{2}{\sqrt{T}} (T/2 - M_{i_0}) \) are i.i.d. and converge in Wasserstein distance to a \( \text{N}(0, 1) \) as \( T \to \infty \) (from, for example, [20, Theorem 3.1]), and since max is Lipschitz,
\[
\lim_{T \to \infty} \mathbb{E}_{M \sim \text{Bin}(T, 1/2)^{\otimes N_0}} \left( \max_{i_0 \in I_0} \frac{1}{\sqrt{T}} (T/2 - M_{i_0}) \right) = \frac{1}{2} \mathbb{E}_{Z \sim \text{N}(0, 1)^{\otimes N_0}} \left( \max_{i_0 \in I_0} Z_{i_0} \right).
\]
We now turn to the non-asymptotic lower bound of Kamath [35], which states that for all \( N_0 \in \mathbb{N} \)
\[
\mathbb{E}_{Z \sim \text{N}(0, 1)^{\otimes N_0}} \left( \max_{i_0 \in I_0} Z_{i_0} \right) \geq \frac{0.23 \sqrt{\log N_0}}{0.23} \geq 1,
\]
Now, by the definition of limit, for each \( N_0 \) there exists a \( t_0(N_0) \) such that for \( T \geq t_0(N_0) \)
\[
\mathbb{E}_{M \sim \text{Bin}(T, 1/2)^{\otimes N_0}} \left( \max_{i_0 \in I_0} \frac{1}{\sqrt{T}} (T/2 - M_{i_0}) \right) \geq \frac{0.23}{0.23} \left( \frac{1}{2} \right) \mathbb{E}_{Z \sim \text{N}(0, 1)^{\otimes N_0}} \left( \max_{i_0 \in I_0} Z_{i_0} \right) \geq \sqrt{\log N_0} / 100.
\]
Combining these facts, we have that for any \( N \in \mathbb{N}, N_0 \leq N \) and \( T \geq t_0(N_0) \),
\[
\inf_{\pi \in \mathcal{P}^N} \sup_{\pi \in \mathcal{P}(\mathcal{D})} \mathbb{E}_{\pi, \hat{x}} \max_{i \in [N]} \sum_{t=1}^{T} \left[ \ell(\hat{y}(t), y(t)) - \ell(x_i(t), y(t)) \right] \geq \sqrt{(T \log N_0) / 100} \geq 1.
\]
\( \square \)

E.2 Proof of Theorem 4

Fix \( N > 0, N_0 \leq N, \) and \( c > 0 \) within the respective constraints of either (i) or (ii) of Theorem 4. Let \( \mathcal{Y} = \{0, 1\}^N, \mathcal{Y} = \{0, 1\}^N, \) and \( \ell(\hat{y}, y) = \langle \hat{y}, y \rangle, \) and suppose \( T \geq \frac{32 \log N}{c^2} \). In order to prove both cases of the D.HEDGE lower bound, our approach is first to define a specific example of a \( \mathcal{D} \in \mathcal{V}(\langle , N, N_0 \rangle) \). Then, for either case we find a specific policy \( \pi \in \mathcal{P}(\mathcal{D}) \) which forces D.HEDGE to incur at least as much regret as the desired lower bound. It turns out that we do not need anything more complicated than a \( \mathcal{D} \) that consists of convex combinations of deterministic experts.
For simplicity, suppose that $N_0$ is even. (The argument is the same, but with some more housekeeping, when $N_0$ is odd.) We wish to split $[N]$ up so that $\mathcal{I}_0 = [N_0]$, and thus $[N]\setminus \mathcal{I}_0 = [N]\setminus [N_0]$. To do so, we define a set of distributions on $\mathcal{Y}$ by

$$U = \left\{ \delta_m \otimes \delta_{1-m}^{\otimes N_0/2} \otimes \delta_{1}^{\otimes (N-N_0)} \right\} \cup \left\{ \delta_0 \otimes \delta_{1}^{\otimes (N-1)} \right\},$$

and suppose that each expert $i \in [N]$ predicts $(x)_i = e_i$, the unit vector in direction $i$. Thus, the set $U$ induces three different expert loss distributions. In each of these, the incurred loss of any expert incurs loss of 0 (with the rest incurring loss of 1). These options are either: a) the first $N_0/2$ incur loss of 0, b) the experts labelled $N_0/2 + 1$ to $N_0$ incur loss of 0, and c) only the first expert incurs loss of 0.

Then, we define $D$ to be the convex hull of $U$. One can check that any convex combination of the three distributions in $U$ can only lead to an expert in $\mathcal{I}_0$ being optimal in expectation, and additionally note that $\Delta_0 = 1/2$. Consequently, $D \in \mathcal{V}(\epsilon, [N], N_0)$, so it remains to find a $\pi \in \mathcal{P}(D)$ that forces $D\text{Hedge}$ with either parametrization to incur the regret of the theorem.

Before we do this, we first recall the adversarial analysis of $D\text{Hedge}$ by [15, Theorem 2.3]. Similar to that analysis, we will analyze the telescoping series

$$\Psi(t) = \frac{1}{\eta(t+1)} \log(w_{I^*(t)}^\eta(t) + 1) - \frac{1}{\eta(t)} \log(w_{I^*(t-1)}^\eta(t-1)), $$

which, for an arbitrary $t_0$, satisfies

$$\sum_{t=t_0+1}^T \Psi(t) = \frac{1}{\eta(T+1)} \log(w_{I^*(T)}^\eta(T+1) - \frac{1}{\eta(t_0+1)} \log(w_{I^*(t_0)}^\eta(t_0+1)).$$

When upper bounding, Cesa-Bianchi and Lugosi used that the first term was negative and kept the second term, but we now wish to use that the second term is positive to obtain

$$\sum_{t=t_0+1}^T \Psi(t) \geq \frac{1}{\eta(T+1)} \log(w_{I^*(T)}^\eta(T+1)).$$

(46)

Then, we can partition $-\Psi(t)$ into

$$-\Psi(t) = \left( \frac{1}{\eta(t+1)} - \frac{1}{\eta(t)} \right) \log \left( \frac{1}{w_{I^*(t)}^\eta(t+1)} \right) + \frac{1}{\eta(t)} \log \left( \frac{\sum_{i \in [N]} \exp \{-\eta(t)L_i(t)\}}{\sum_{i \in [N]} \exp \{-\eta(t+1)L_i(t)\}} \right),$$

$$+ \frac{1}{\eta(t)} \log \left( \frac{\sum_{i \in [N]} \exp \{-\eta(t)L_i(t)\}}{\sum_{i \in [N]} \exp \{-\eta(t)L_i(t-1)\}} \right) + \left[ L_{I^*(t)}(t) - L_{I^*(t-1)}(t-1) \right].$$
Observe that

\[
\frac{1}{\eta(t)} \log \left( \frac{\sum_{i \in [N]} \exp \{-\eta(t)L_i(t)\}}{\sum_{i \in [N]} \exp \{-\eta(t)L_i(t-1)\}} \right) = \frac{1}{\eta(t)} \log \left( \sum_{i \in [N]} \frac{\exp \{-\eta(t)L_i(t)\}}{\sum_{i' \in [N]} \exp \{-\eta(t)L_{i'}(t-1)\}} \exp \{-\eta(t)\ell_i(t)\} \right) = \frac{1}{\eta(t)} \log \left( \sum_{i \in [N]} w_i^h(t) \exp \{-\eta(t)\ell_i(t)\} \right)
\]

\[
= -\sum_{i \in [N]} w_i^h(t)\ell_i(t) + \frac{1}{\eta(t)} \log \left( \sum_{i \in [N]} w_i^h(t) \exp \{-\eta(t)\ell_i(t)\} \right) \sum_{i \in [N]} w_i^h(t) \exp \{-\eta(t)\ell_i(t)\}
\]

Thus, we can write

\[
-\Psi(t) = \left(\frac{1}{\eta(t+1)} - \frac{1}{\eta(t)}\right) \log \left( \frac{1}{w_{I^*(t)}(t+1)} \right) + \frac{1}{\eta(t)} \log \left( \frac{\sum_{i \in [N]} \exp \{-\eta(t)L_{I^*(t)}(t)\}}{\sum_{i \in [N]} \exp \{-\eta(t)L_{I^*(t)}(t)\}} \exp \{-\eta(t)\ell_{I^*(t)}(t)\} \right)
\]

\[
- \sum_{i \in [N]} w_i^h(t)\ell_i(t) + \frac{1}{\eta(t)} \log \left( \mathbb{E}_{i \sim w^h(t)} \exp \left\{ \eta(t) \left( -\ell_{I^*(t)}(t) - \mathbb{E}_{I' \sim w^h(t)} [-\ell_{I'(t)}] \right) \right\} \right)
\]

\[
+ [L_{I^*(t)}(t) - L_{I^*(t-1)}(t-1)]
\]

\[
= A(t) + B(t) + C_1(t) + C_2(t) + D(t).
\]

First, observe that since \(\eta(t)\) is decreasing in both cases, \(B(t) \geq 0\). Also,

\[
\sum_{t=t_0+1}^T A(t) = \sum_{t=t_0+1}^T \frac{1}{\eta(t+1)} - \frac{1}{\eta(t)} \log \left( \frac{1}{w_{I^*(t)}(t+1)} \right),
\]

\[
\sum_{t=t_0+1}^T C_1(t) = -\sum_{t=t_0+1}^T \sum_{i \in [N]} w_i^h(t)\ell_i(t),
\]

\[
\sum_{t=t_0+1}^T C_2(t) = \sum_{t=t_0+1}^T \frac{1}{\eta(t)} \log \left( \mathbb{E}_{i \sim w^h(t)} \exp \left\{ \eta(t) \left( -\ell_{I^*(t)}(t) - \mathbb{E}_{I' \sim w^h(t)} [-\ell_{I'(t)}] \right) \right\} \right),
\]

\[
\sum_{t=t_0+1}^T D(t) = L_{I^*(T)}(T) - L_{I^*(t_0)}(t_0).
\]
Thus, combining these with Eq. (46) gives
\[-\frac{1}{\eta(T+1)} \log(w_{I^*}(T)(T+1))\]
\[\geq -\sum_{t=t_0+1}^{T} \Psi(t)\]
\[\geq \sum_{t=t_0+1}^{T} \left( \frac{1}{\eta(t+1)} - \frac{1}{\eta(t)} \right) \log \left( \frac{w_{I^*}(t)(t+1)}{w_{I^*}(t)(t+1)} \right) - \sum_{t=t_0+1}^{T} \sum_{i \in [N]} w_i^\ell(t) \ell_i(t)\]
\[+ \sum_{t=t_0+1}^{T} \frac{1}{\eta(t)} \log \left( \mathbb{E}_{I \sim w_i^\ell(t)} \exp \left\{ \eta(t) \left( -\ell_I(t) - \mathbb{E}_{I' \sim w_i^\ell(t)} [-\ell_{I'}(t)] \right) \right\} \right)\]
\[+ L_{I^*}(T) - L_{I^*}(t_0).\]
Rearranging, we see that
\[\hat{R}_{i*}(T) - \hat{R}_{i*}(t_0)\]
\[\geq \frac{1}{\eta(T+1)} \log(w_{I^*}(T)(T+1)) + \sum_{t=t_0+1}^{T} \left( \frac{1}{\eta(t+1)} - \frac{1}{\eta(t)} \right) \log \left( \frac{w_{I^*}(t)(t+1)}{w_{I^*}(t)(t+1)} \right)\]
\[+ \sum_{t=t_0+1}^{T} \frac{1}{\eta(t)} \log \left( \mathbb{E}_{I \sim w_i^\ell(t)} \exp \left\{ \eta(t) \left( -\ell_I(t) - \mathbb{E}_{I' \sim w_i^\ell(t)} [-\ell_{I'}(t)] \right) \right\} \right)\].

The way we bound these terms will depend on the specific parametrization and data-generating mechanism chosen for that parametrization.

E.2.1 D.HEDGE with adversarially optimal parametrization

First, we consider the case of playing D.HEDGE with \(g(N) = c \sqrt{\log N}\). We define the data-generating mechanism \(\pi \in \mathcal{P}(D)\) such that at round \(t\), the distribution on \(\mathcal{Y}\) is
\[\mu_t = \begin{cases} 
\delta_0^\otimes(N_0/2) \otimes \delta_1^\otimes(N - N_0/2) : t \text{ odd} \\
\delta_0^\otimes(N_0/2) \otimes \delta_0^\otimes(N_0/2) \otimes \delta_1^\otimes(N - N_0) : t \text{ even.}
\end{cases}\]

That is, on even and odd rounds the data alternates between the first half of \(I_0\) incurring loss of 0 and the second half of \(I_0\) incurring loss of 0, with the remaining \(N - N_0\) experts always incurring loss of 1. Both of these distributions are actually in \(U\), so they are trivially in \(D\).

Now, due to the deterministic nature of \(\pi\), we can exactly determine what \(w_i^\ell(t)\) will look like. In particular, we have that
\[L_i(t) = \begin{cases} 
\frac{t-1}{2} : t \text{ odd, and } i \in [N_0/2] \\
\frac{t+1}{2} : t \text{ odd, and } i \in [N_0] \setminus [N_0/2] \\
\frac{t}{2} : t \text{ even, and } i \in [N_0] \\
t : i \notin [N_0].
\end{cases}\] (48)

Thus, recognizing that \(w_i^\ell(t)\) uses \(L_i(t-1)\) and letting \(\theta(t) = \exp \left\{ -\eta(t) \frac{(t-1)}{2} \right\}\), we can define \(w_i^\ell(t)\)
by
\[
\begin{cases}
[N_0 + (N - N_0)\theta(t)]^{-1} & : t \text{ odd, } i \in [N_0] \\
\theta(t)[N_0 + (N - N_0)\theta(t)]^{-1} & : t \text{ odd, } i \not\in [N_0] \\
\exp(\eta(t)/2)[N_0 \cosh(\eta(t)/2) + (N - N_0)\theta(t)]^{-1} & : t \text{ even, } i \in [N_0/2] \\
\exp(-\eta(t)/2)[N_0 \cosh(\eta(t)/2) + (N - N_0)\theta(t)]^{-1} & : t \text{ even, } i \in [N_0] \setminus [N_0/2] \\
\theta(t)[N_0 \cosh(\eta(t)/2) + (N - N_0)\theta(t)]^{-1} & : t \text{ even, } i \not\in [N_0].
\end{cases}
\]

The next thing to observe is that for all \(t, I^*(t) \in [N_0] \) a.s., and \( w_{I^*(t)}^n(t + 1) \) equals
\[
\begin{cases}
[N_0 + (N - N_0)\theta(t + 1)]^{-1} & : t + 1 \text{ odd} \\
\exp(\eta(t + 1)/2)[N_0 \cosh(\eta(t + 1)/2) + (N - N_0)\theta(t + 1)]^{-1} & : t + 1 \text{ even.}
\end{cases}
\tag{49}
\]

Now, let \( t_0 = \left\lceil \frac{16 \log N}{c^2} \right\rceil \) and suppose \( t \geq t_0 + 1 \). Then, using \( \frac{x}{\sqrt{\pi + 1}} \geq \frac{1}{2}\sqrt{x} \) for \( x \geq 1 \),
\[
\begin{align*}
\theta(t) & \leq \theta(t + 1) \\
& = \exp \left\{ -\frac{c \sqrt{\log N \ t}}{\sqrt{t + 1}} \right\}
\leq \exp \left\{ -\frac{c t \log N}{4} \right\}
\leq \exp \left\{ -\frac{c \sqrt{16(\log N)^2}}{4c} \right\}
= \frac{1}{N}.
\end{align*}
\]
This gives
\[
\frac{1}{N_0 + (N - N_0)\theta(t + 1)} \geq \frac{1}{N_0 + (N - N_0)/N} \geq \frac{1}{N_0 + 1}. \tag{50}
\]
Also, \( \exp \{\eta(t + 1)/2\} \geq 1 \), so
\[
\cosh \left( \frac{\eta(t + 1)}{2} \right) = \frac{1}{2} \left[ \exp \left\{ \frac{c \sqrt{\log N}}{2 \sqrt{t + 1}} \right\} + \exp \left\{ -\frac{c \sqrt{\log N}}{2 \sqrt{t + 1}} \right\} \right]
\leq \frac{1}{2} \left[ \exp \left\{ \frac{c^2 \sqrt{\log N}}{2 \sqrt{16 \log N}} \right\} + 1 \right]
\leq \exp \left\{ c^2/8 \right\}.
\]
Thus,
\[
\frac{\exp \{\eta(t + 1)/2\}}{N_0 \cosh(\eta(t + 1)/2) + (N - N_0)\theta(t + 1)} \geq \frac{1}{\exp \{c^2/8\} N_0 + 1},
\]
which combined with Eq. (50) gives that for all \( t \geq t_0 + 1 \),
\[
w_{I^*(t)}^n(t + 1) \geq \frac{1}{\exp \{c^2/8\} N_0 + 1}.
\]
This observation shows that if \( T \geq t_0 + 1 \),
\[
\frac{1}{\eta(T + 1)} \log(w_{I^*(T)}^n(T + 1)) \geq -\frac{\sqrt{T + 1}}{c \sqrt{\log N}} \left[ c^2/8 + \log(N_0 + 1) \right]. \tag{51}
\]
In order to control the terms of Eq. (47), we first observe that
\[
\sum_{t=t_0+1}^{T} \left( \frac{1}{\eta(t+1)} - \frac{1}{\eta(t)} \right) \log \left( \frac{1}{w_{I^*}(t+1)} \right) \geq 0.
\]

Then, we will use Eq. (51) to lower bound the first term on the RHS of Eq. (47). We now turn to controlling the third term, again supposing \( t \geq t_0 + 1 \). Notice that if \( t \) is odd, then \( I \sim w(t) \) means \( \ell_I(t) = \text{Ber} \left( \frac{N_0/2 + (N-N_0)\theta(t)}{N_0 + (N-N_0)\theta(t)} \right) \). Therefore,
\[
\frac{1}{\eta(t)} \log \left( \mathbb{E}_{I \sim w(t)} \exp \left\{ \eta(t) \left( -\ell_I(t) - \mathbb{E}_{I' \sim w(t)} [-\ell_{I'}(t)] \right) \right\} \right)
\]
\[
= \frac{1}{\eta(t)} \log \left( \frac{N_0/2}{N_0 + (N-N_0)\theta(t)} \exp \left\{ \eta(t) \frac{N_0/2 + (N-N_0)\theta(t)}{N_0 + (N-N_0)\theta(t)} \right\} \right)
\]
\[
\quad + \frac{N_0/2 + (N-N_0)\theta(t)}{N_0 + (N-N_0)\theta(t)} \exp \left\{ -\eta(t) \frac{N_0/2}{N_0 + (N-N_0)\theta(t)} \right\} \right)
\]
\[
= \frac{1}{\eta(t)} \log \left( \frac{N_0}{N_0 + (N-N_0)\theta(t)} \cosh \left\{ \eta(t) \frac{N_0/2}{N_0 + (N-N_0)\theta(t)} \right\} \right).
\]

Now, we observe that \( \log(\cosh(x)) \) is \( \frac{1}{\cosh(x)} \)-strongly convex on \( x \in [0, x_1] \). Thus,
\[
\log(\cosh(x)) - \log(\cosh(0)) \geq \frac{d}{dy} \log(\cosh(y)) \bigg|_{y=0} + \frac{x^2}{2 \cosh^2(x_1)},
\]
so \( \cosh(x) \geq \exp \left\{ \frac{x^2}{2 \cosh^2(x_1)} \right\} \) on this interval. Then, notice that if \( t \geq t_0 + 1, \eta(t) \leq c^2/4 \). So, \( \theta(t) \geq 0 \) gives
\[
\eta(t) \frac{N_0/2}{N_0 + (N-N_0)\theta(t)} \leq \frac{\eta(t)}{2} \leq \frac{c^2}{8}.
\]

Using this strong-convexity bound on \( \cosh(x) \) along with the two inequalities \( \theta(t) \leq 1/N \) and \([2 \cosh^2(c^2/8)]^{-1} \geq (1/2) \exp \{-c^2/4\} \) results in
\[
\frac{1}{\eta(t)} \log \left( \frac{N_0}{N_0 + (N-N_0)\theta(t)} \cosh \left\{ \eta(t) \frac{N_0/2}{N_0 + (N-N_0)\theta(t)} \right\} \right)
\]
\[
\quad \geq \frac{1}{\eta(t)} \log \left( \frac{N_0}{N_0 + (N-N_0)\theta(t)} \right) + \frac{\eta(t)}{2 \exp \{c^2/4\}} \left( \frac{N_0/2}{N_0 + (N-N_0)\theta(t)} \right)^2
\]
\[
\quad \geq \frac{1}{\eta(t)} \log \left( \frac{N_0}{N_0 + (N-N_0)\theta(t)} \right) + \frac{\eta(t)}{2 \exp \{c^2/4\}} \left( \frac{N_0}{2N_0 + 1} \right)^2
\]
\[
\quad \geq \frac{1}{\eta(t)} \log \left( \frac{N_0}{N_0 + (N-N_0)\theta(t)} \right) + \frac{\eta(t)}{18 \exp \{c^2/4\}}.
\]
Finally, using \( \log(x) \geq 1 - 1/x \),

\[
\frac{1}{\eta(t)} \log \left( \frac{N_0}{N_0 + (N - N_0)\theta(t)} \right) \geq \frac{1}{\eta(t)} \left( 1 - \frac{N_0 + (N - N_0)\theta(t)}{N_0} \right) \geq -\frac{N}{N_0} \theta(t).
\]

Thus, when \( t \geq t_0 + 1 \) and \( t \) is odd,

\[
\frac{1}{\eta(t)} \log \left( \frac{\mathbb{E}_{t \sim u^H(t)} \exp \left\{ \eta(t) \left( -\ell_I(t) - \mathbb{E}_{t' \sim u^H(t)} [-\ell_I(t')] \right) \right\}}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right) 
\geq -\frac{N}{N_0} \theta(t) + \frac{\eta(t)}{18 \exp \{c^2/4\}}.
\]

Otherwise, if \( t \) is even, then \( I \sim u^H(t) \) implies

\[
\ell_I(t) \sim \text{Ber} \left( \frac{(N_0/2) \exp \{\eta(t)/2\} + (N - N_0)\theta(t)}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right).
\]

So, using \( \cosh(x) \geq 1 \),

\[
\frac{1}{\eta(t)} \log \left( \frac{\mathbb{E}_{t \sim u^H(t)} \exp \left\{ \eta(t) \left( -\ell_I(t) - \mathbb{E}_{t' \sim u^H(t)} [-\ell_I(t')] \right) \right\}}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right) 
\geq \frac{1}{\eta(t)} \log \left( \frac{(N_0/2) \exp \{-\eta(t)/2\}}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right)
\times \exp \left\{ \eta(t) \frac{(N_0/2) \exp \{-\eta(t)/2\}}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right\}
\geq \frac{1}{\eta(t)} \log \left( \frac{N_0 \exp \{-\eta(t)/2\}}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right)
\times \cosh \left\{ \eta(t) \frac{(N_0/2) \exp \{-\eta(t)/2\}}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right\}
\geq \frac{1}{\eta(t)} \log \left( \frac{N_0 \exp \{-\eta(t)/2\}}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right).
\]
Then, using $\log(x) \geq 1 - 1/x$, 

\[
\frac{1}{\eta(t)} \log \left( \frac{N_0 \exp \{-\eta(t)/2\}}{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)} \right) \geq \frac{1}{\eta(t)} \left( 1 - \frac{N_0 \cosh \{\eta(t)/2\} + (N - N_0)\theta(t)}{N_0 \exp \{-\eta(t)/2\}} \right) \\
= \frac{1}{\eta(t)} \left( \frac{N_0 \sinh \{-\eta(t)/2\} - (N - N_0)\theta(t)}{N_0 \exp \{-\eta(t)/2\}} \right) \\
\geq - \frac{1}{\eta(t)} \frac{N}{N_0} \theta(t) \exp \{-\eta(t)/2\} \\
= - \frac{N}{N_0} \frac{\exp \{-\eta(t)(t - 2)\}}{\eta(t)}.
\]

(53)

Combing Eqs. (52) and (53) and recognizing $\theta(t) \leq \exp \{-\eta(t)(t - 2)\}$ gives us 

\[
\frac{1}{\eta(t)} \log \left( \mathbb{E}_{t \sim \mu(t)} \exp \left\{ \eta(t) \left( -\ell(t) - \mathbb{E}_{t \sim \mu(t)} [-\ell(t)] \right) \right\} \right) \\
\geq - \frac{N}{N_0} \frac{\exp \{-\eta(t)(t - 2)\}}{\eta(t)} + \frac{\eta(t)}{18 \exp \{c^2/4\}} \mathbb{I}[t \text{ is odd}].
\]

(54)

Now, we wish to sum the two terms in Eq. (54). First, using that $\frac{t - 2}{2} \geq \frac{3\sqrt{t}}{2}$ when $t \geq 6$ and $t_0 \geq 5$ since $\log N \geq 5/2$, as well as crudely lower bounding $t_0$ by dividing by 2, 

\[
\sum_{t = t_0 + 1}^{T} \frac{\exp \{-\eta(t)(t - 2)\}}{\eta(t)} \\
= \sum_{t = t_0 + 1}^{T} \frac{\sqrt{t}}{c\sqrt{\log N}} \exp \left\{ -c \sqrt{\log N} \left( t - \frac{2}{2} \right) \right\} \\
\leq \sum_{t = t_0 + 1}^{T} \frac{\sqrt{t}}{c\sqrt{\log N}} \exp \left\{ -\frac{3c}{4} \sqrt{t \log N} \right\} \\
\leq \frac{1}{c\sqrt{\log N}} \int_{0}^{\infty} \sqrt{t} \exp \left\{ -\frac{3c}{4} \sqrt{t \log N} \right\} dt \\
= \frac{128 \left( t_0 \frac{3c^2 \log N}{16} + 2 \sqrt{t_0 \frac{3c \log N}{4}} + 2 \right) \exp \left\{ -\left( \frac{3c}{4} \right) \sqrt{\log N} \right\}}{27c^4 \left( \log N \right)^2} \\
\leq \frac{128 \left( 16 \log N \frac{9c^2 \log N}{16} + 2 \sqrt{16 \log N \frac{3c \log N}{4}} + 2 \right) \exp \left\{ -\left( \frac{3c}{4} \right) \sqrt{\log N} \right\}}{c^4 \left( \log N \right)^2} \\
\leq \frac{128 \left( 16 + \frac{9}{\log N} + \frac{2}{(\log N)^2} \right)}{c^4 N^2}.
\]

Then, supposing the worst case where both $t_0 + 1$ and $T$ are even, and crudely upper bounding
\[ t_0 + 2 \text{ by multiplying by } \frac{3}{2}, \]

\[
\sum_{t = t_0 + 1}^{T} \eta(t) \mathbb{1}_{[t \text{ is odd}]} = \sum_{t = (t_0 + 2)/2}^{(T - 1)/2} \eta(2t) \\
= \sum_{t = (t_0 + 2)/2}^{(T - 1)/2} \frac{c \sqrt{\log N}}{2t} \\
\geq \int_{(t_0 + 2)/2}^{(T - 1)/2} \frac{c \sqrt{\log N}}{2t} \, dt \\
= c \sqrt{\log N} \left[ \sqrt{T - 1} - \sqrt{t_0 + 2} \right] \\
\geq c \sqrt{(T - 1) \log N} - \frac{c \sqrt{24 \log N}}{c} \\
= c \sqrt{(T - 1) \log N} - 2 \log N \sqrt{6}. \tag{56}
\]

Thus, combining Eqs. (51), (55) and (56), we have shown that for \( T \geq \frac{16 \log N}{c^2} \),

\[
\hat{R}_n(T) \geq \hat{R}_n(T) - \hat{R}_n(t_0) \\
\geq -\frac{\sqrt{T + 1}}{c \sqrt{\log N}} \left[ c^2/8 + \log(N_0 + 1) \right] - \frac{128 \left( 16 + \frac{9}{\log N} + \frac{2}{(\log N)^2} \right)}{c^4 N_0 N} \\
+ \frac{c \sqrt{(T - 1) \log N}}{18 \exp \{c^2/4\}} - \frac{2 \log N \sqrt{6}}{18 \exp \{c^2/4\}}.
\]

Finally, rearranging the restriction on the size \( N_0 \) and using \( \frac{\sqrt{T - 1}}{\sqrt{T + 1}} \geq 1/2 \), since \( \log(N_0 + 1) < \frac{c^2 \log N}{72 \exp \{c^2/4\}} - \frac{c^2}{8} \) it holds that

\[
\frac{1}{c \sqrt{\log N}} \left[ c^2/8 + \log(N_0 + 1) \right] < \frac{1}{c \sqrt{T + 1}} \frac{c \sqrt{\log N}}{18 \exp \{c^2/4\}}.
\]

Thus, using \( N \geq e^9 \) and \( N_0 \geq 1 \),

\[
\hat{R}_n(T) \geq \frac{c \sqrt{(T - 1) \log N}}{36 \exp \{c^2/4\}} - \frac{128 \left( 16 + \frac{9}{\log N} + \frac{2}{(\log N)^2} \right)}{c^4 N_0 N} - \frac{2 \log N \sqrt{6}}{18 \exp \{c^2/4\}} \\
\geq \frac{c \sqrt{T \log N}}{72 \exp \{c^2/4\}} - \frac{1}{3c^2} - \frac{\log N}{3}.
\]

### E.2.2 D.Hedge with stochastically optimal parametrization

Now, we consider the case of playing D.Hedge with the oracle-informed parameter \( g(N, N_0) \). We define the data-generating mechanism \( \pi \in \mathcal{P}(\mathcal{D}) \) such that for some even \( t_1 \), at round \( t \) the distribution on \( \mathcal{Y} \) is

\[
\mu_t = \begin{cases} 
\delta_0^{\otimes (N_0/2)} \otimes \delta_1^{\otimes (N-N_0/2)} : t \text{ odd and } t \leq t_1 \\
\delta_0^{\otimes (N_0/2)} \otimes \delta_0^{\otimes (N_0/2)} \otimes \delta_1^{\otimes (N-N_0)} : t \text{ even and } t \leq t_1 \\
\delta_0 \otimes \delta_1^{\otimes (N-1)} : t \text{ even and } t > t_1.
\end{cases}
\]
That is, the data is the same as for D.Hedge in Appendix E.2.1 up to \( t = t_1 \), and then afterwards all experts incur loss of 1 except the first expert, which incurs zero loss. Once again, all of these distributions are actually in \( U \), so they are trivially in \( D \).

Since \( t_1 \) is even, for \( t > t_1 \) we expand on Eq. (48) to obtain

\[
L_i(t) = \begin{cases} 
\frac{t_1}{2} : i = 1 \\
\frac{2t-t_1}{2} : i \in [N_0] \setminus \{1\} \\
t : i \not\in [N_0]. 
\end{cases}
\]

Thus, when \( t > t_1 \),

\[
w_i^{ll}(t) = \left[ 1 + (N_0 - 1) \exp\{-\eta(t)(t - t_1 - 1)\} + (N - N_0) \exp\{-\eta(t)(t - t_1/2 - 1)\} \right]^{-1},
\]

and for \( i \neq 1 \), \( w_i^{ll}(t) \) equals

\[
\left\{ \begin{array}{l}
\left[ \exp\{-\eta(t)(t_1 - t + 1)\} + N_0 - 1 + (N - N_0) \exp\{-\eta(t)(t_1/2)\} \right]^{-1} : i \in [N_0] \setminus \{1\} \\
\left[ \exp\{-\eta(t)(t_1/2 - t + 1)\} + (N_0 - 1) \exp\{\eta(t)(t_1/2)\} + N - N_0 \right]^{-1} : i \not\in [N_0].
\end{array} \right.
\]

The next thing to observe is that for \( t > t_1 \), \( w_{I^*}^{ll}(t + 1) \) equals

\[
\left[ 1 + (N_0 - 1) \exp\{-\eta(t + 1)(t - t_1)\} + (N - N_0) \exp\{-\eta(t + 1)(t - t_1/2)\} \right]^{-1}. \tag{57}
\]

Now, define

\[
t_2 = \left[ \frac{4}{g(N, N_0) + t_1^2} \right].
\]

If \( t > t_2 \), it holds that

\[
\sqrt{t} > \frac{2 \log N}{g(N, N_0)} + 2t_1 \\
\implies \sqrt{t} > \frac{2 \log N_0}{g(N, N_0)} + 2t_1 \\
\implies g(N, N_0) \sqrt{t} - g(N, N_0)t_1 > \log N_0 \\
\implies \frac{g(N, N_0)[t - t_1]}{\sqrt{t+1}} > \log(N_0 - 1) \\
\implies (N_0 - 1) \exp\{-\eta(t + 1)(t - t_1)\} < 1. \tag{58}
\]

Similarly,

\[
\sqrt{t} > \frac{2 \log N}{g(N, N_0)} + 2t_1 \\
\implies \sqrt{t} > \frac{2 \log N}{g(N, N_0)} + t_1 \\
\implies \sqrt{t} - t_1 > \frac{2 \log N}{g(N, N_0)} \\
\implies \frac{c[t - t_1/2]}{\sqrt{t+1}} > \log(N - N_0) \\
\implies (N - N_0) \exp\{-\eta(t + 1)(t - t_1/2)\} < 1. \tag{59}
\]
Combining Eqs. (57) to (59) shows that when \( t > t_2 \), since \( t_2 > t_1 \) by definition, we have

\[ w^H_{\ell^*(t)}(t + 1) \geq 1/3. \]

This observation controls the first term of Eq. (47). For the second term of Eq. (47), we note that by Jensen’s inequality,

\[
\sum_{t=t_0+1}^{T} \frac{1}{\eta(t)} \log \left( \frac{\mathbb{E}_{T \sim w^H(t)} \left[ \exp \left\{ \eta(t) \left( -\ell_{T}(t) - \mathbb{E}_{\ell' \sim w^H(t)} \left[ -\ell_{\ell'}(t) \right] \right) \right\} \right)}{\eta(t)} \right) \geq 0.
\]

Define \( t_0 = \left\lfloor \frac{16 \log N}{[g(N, N_0)]^2} \right\rfloor \). Now, when \( t_0 + 1 \leq t \leq t_1 \), \( w^H_{\ell^*(t)}(t + 1) \) behaves as in Eq. (49). Thus, when \( t + 1 \) is odd, \( w^H_{\ell^*(t)}(t + 1) \leq 1/N_0 \) since \( \theta(t + 1) \geq 0 \). Otherwise, when \( t + 1 \) is even, we use that since \( \log N > 5/2 \),

\[
\exp \left\{ \frac{\eta(t + 1)}{2} \right\} \leq \exp \left\{ \frac{[g(N, N_0)]^2}{8 \log N} \right\} \leq \exp \left\{ \frac{[g(N, N_0)]^2/8}{N_0} \right\},
\]

as well as \( \theta(t + 1) \geq 0 \) and \( \cosh(x) \geq 1 \) to obtain \( w^H_{\ell^*(t)}(t + 1) \leq \frac{\exp\left\{ \frac{[g(N, N_0)]^2/8}{N_0} \right\}}{N_0} \). Thus,

\[
\sum_{t=t_0+1}^{t_1} \left( \frac{1}{\eta(t + 1)} - \frac{1}{\eta(t)} \right) \log \left( \frac{1}{w^H_{\ell^*(t)}(t + 1)} \right) \geq \sum_{t=t_0+1}^{t_1} \left( \frac{1}{\eta(t + 1)} - \frac{1}{\eta(t)} \right) \left[ \log N_0 - \frac{[g(N, N_0)]^2/8}{g(N, N_0)} \right] = \frac{\log N_0 - [g(N, N_0)]^2/8}{g(N, N_0)} \left[ \sqrt{T + 1} - \sqrt{t_1 + 1} - \sqrt{t_0 + 1} \right].
\]

Set \( t_1 = T/2 \), and suppose \( T > \frac{32 \log N}{[g(N, N_0)]^2} \) to ensure \( t_1 > t_0 + 1 \). Then, substituting Eq. (60) into Eq. (47) gives

\[
\hat{R}_n(T) \geq \hat{R}_n(T) - \hat{R}_n(t_0) \geq -\frac{\sqrt{T + 1}}{g(N, N_0)} \log(3) + \frac{\log N_0 - [g(N, N_0)]^2/8}{g(N, N_0)} \left[ \sqrt{T + 1} - \sqrt{t_1 + 1} - \sqrt{t_0 + 1} \right] \geq -\frac{\sqrt{T + 1}}{g(N, N_0)} \log(3) + \frac{\log N_0 - [g(N, N_0)]^2/8}{2g(N, N_0)} \left[ \sqrt{T + 1} - \sqrt{32 \log N}/g(N, N_0) \right] \geq \frac{\log N_0}{4g(N, N_0)} \sqrt{T + 1} - \frac{3 \log N_0 - [g(N, N_0)]^2/8 \log N}{4g(N, N_0)^2} \geq \frac{\log N_0}{4g(N, N_0)} \sqrt{T} - \frac{3 \log N_0}{4g(N, N_0)^2},
\]

where we also used \( \log N_0 > [g(N, N_0)]^2/4 + 4 \log(3) \).
F  Implementing FTRL-CARE and Meta-CARE

The following algorithm efficiently implements FTRL-CARE; its validity follows from Theorem 7.

Algorithm 1: Implementation of FTRL-CARE

| Inputs: |
| --- |
| • constants $c_1, c_2 > 0$, number of experts $N$; |
| • a function $\text{root} : (f, (a, b)) \in (\mathbb{R} \to \mathbb{R}) \times \mathbb{R}^2 \to \text{maybe}(\mathbb{R})$ which returns a root of the function $f$ on the interval $[a, b]$ when $f(a)f(b) < 0$, and returns nothing otherwise. |

| Result: | Infinite list of weight vectors, $\{w^c\}_{c \in \mathbb{N}}$. |
| $H = \text{Function}(u \mapsto \{\sum_{i \in [N]}[-u_i \log(u_i)]\})$; |
| $w = \text{Function}(\eta, \xi \mapsto \{\sum_{i' \in [N]} \exp(-\eta \xi_i) \exp(-\eta \xi_{i'})\}_{i \in [N]}$); |
| $L(0) = \text{zeroes}(N)$; |
| $w^c(1) = \text{ones}(N)/N$; |
| for $t \in \mathbb{N}$ do |
| Receive Data: | vector of expert losses from round $t$, $\ell(t) \in [0, 1]^N$ |
| $L(t) = L(t-1) + \ell(t)$; |
| $\eta(t+1) = \text{root}\left(\text{Function}(\eta \mapsto \left\{\eta - \frac{2c_1 \sqrt{c_2 + H(w(\eta, L(t)))}}{\sqrt{t+1}}\right\})\right)$, $\left(\frac{2c_1 \sqrt{c_2 + \log(N)}}{\sqrt{t+1}}, \frac{2c_1 \sqrt{c_2 + \log(N)}}{\sqrt{t+1}}\right)$; |
| $w^c(t+1) = w(\eta(t+1), L(t))$ |
| end |

Meta-CARE only requires the above implementation of FTRL-CARE and a standard implementation of D.Hedge. The parameters of Meta-CARE can be tuned to optimize the $N_0 = 1$ bound of Theorem 5 and the leading term of Theorem 6, hence improving the universal constants, but it does not affect the order of the bound.

G  Simulations

In this section, we present a brief simulation analysis of the performance of D.Hedge, FTRL-CARE, and Meta-CARE to provide intuition for how the algorithms differ and to demonstrate the effectiveness of Meta-CARE that we have proved in our analysis. Since the weights of all three algorithms can be completely determined by the expert losses, we specify each scenario using only the loss distributions rather than the distributions on $\mathcal{Y}$ and $\hat{\mathcal{Y}}^N$. In Fig. 2, we plot the expected regret against the number of rounds $T$ for two data-generating mechanisms: the left column ($N_0 = 1$) corresponds to the stochastic setting, where the losses of the first expert are i.i.d. Ber(1/2) and the losses of all the other experts are i.i.d. Ber(1); the right column ($N_0 = 2$) corresponds to an adversarial setting with two effective experts, where on the $t$th round the loss of the first expert is deterministically $t$ mod 2, the loss of the second expert is deterministically $(t + 1)$ mod 2, and the losses of the remaining experts are all deterministically 1. In Fig. 3, we plot the expected regret against the number of experts $N$ for various $T$. The data-generating mechanism has $N_0 = 2$, and is the same as for the right column of Fig. 2. For both settings, the gap between the expected losses of the best effective and ineffective experts under distributions in the convex hull of those produced by the data-generating mechanism is $\Delta_0 = 1/2$. For all of the simulations, the algorithms are parametrized using $c_{it} = c_{c,1} = \sqrt{8}$, $c_{c,2} = 1$, and $c_m = 100$. All of the plots display expected regret; for the $N_0 = 1$ case of Fig. 2 this is approximated by averaging over 10 simulations, and for the remaining plots this is exact since the losses are all deterministic.
Figure 2: Comparing expected regret as a function of time $T$, for number of effective experts $N_0 \in \{1, 2\}$ and varying total number of experts $N$. Plots are on a log-log scale; slopes of lines correspond to polynomial powers, and intercepts of lines correspond to log-(constants of proportionality).

Beginning with Fig. 2, for $N_0 = 1$, expected regret levels-off at a higher constant for FTRL-CARE than for D.HEDGE. As anticipated by the theory, the period for which adversarial regret is accumulated before the regret levels off increases with $N$ for both D.HEDGE and FTRL-CARE, and is longer for FTRL-CARE, leading to higher total expected regret. For $N_0 = 2$, the gap between the expected regret of FTRL-CARE and D.HEDGE widens as $N$ increases, corresponding to the $\sqrt{T \log N}$ rate of regret for D.HEDGE v.s. the $\sqrt{T \log N_0}$ rate of regret for FTRL-CARE. As anticipated by our theoretical results, there is a phase transition in the regret accumulation for both FTRL-CARE and D.HEDGE at roughly the time when the respective expected regrets level off in the $N_0 = 1$ case. In all cases, the expected regret of META-CARE closely tracks the better of D.HEDGE and FTRL-CARE.
Figure 3: Comparing expected regret as a function of the number of experts $N$ for $N_0 = 2$ effective experts at varying times $T$. Plots are on a log-log scale; slopes of lines correspond to polynomial powers, and intercepts of lines correspond to log-(constants of proportionality). Note that since the $x$-axis variable in each case is $\log_2(N)$, the second tick on the $x$-axis corresponds to $N = 2^{10^{1.00}} = 1024$, and the last tick on the $x$-axis corresponds to $N = 2^{10^{2.00}} \approx 1.27 \times 10^{30}$.

For Fig. 3, when $T$ is small relative to $\log N$, both FTRL-CARE and D.HEDGE have expected regret growing with $N$ according to the adversarial rate, corresponding to a slope of $1/2$. When $T$ is large relative to $N$, so that $\sqrt{T \log N_0} \gg (\log N)^{3/2}/\Delta_0$, the expected regret of FTRL-CARE is approximately constant in $N$ while the expected regret of D.HEDGE grows like $\sqrt{\log N}$, as anticipated by our theoretical results. Once again, the expected regret of Meta-CARE closely tracks the better of D.HEDGE and FTRL-CARE.