Growth and Bounded Solution of Second-Order of Complex Differential Equations Through of Coefficient Function

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Abstract. The purpose of this research paper, is to present the second- order homogeneous complex differential equation \( f'' + H(z)f = 0 \), which defined on the disk \( D = \{ z \in \mathbb{C} : |z - i| \leq 1 \} \subset \mathbb{C} \), where \( H(z) = e^{p(z)} \), to show it an invariant by applying Liouville and self-adjoint transformation with an examine the convexity property of its coefficient \( H(z) = e^{p(z)} \), in order to study the growing and bounded solution of consider equation.

1. Introduction
Consider \( f'' + H(z)f = 0 \), \( \ldots \ldots (1) \) be a second- order and homogeneous where \( H(z) \) be an analytic function defined on a simply connected domain \( D = \{ z \in \mathbb{C} : |z - i| \leq 1 \} \subset \mathbb{C} \). The growing solution of complex differential equations has obtained great interest for researchers in this field, for instance, CH. Pommerenke et al. [1], [5],[10],[6],[7],[8].

In [1], CH. Pommerenke obtained some interesting results on the growing solution of the equation (1), if assumed that \( H(z) \) belongs to Bergman space, then the solutions of (1) belong to Nevanlinna class and others assumptions as putting suitable conditions on \( H(z) \), such that the solution of (1) be in hardy space.

Subsequently, [5] J. Heittokangas suggested another condition on \( H(z) \) which being \( f \) belongs to Nevanlinna class when gave a good result, it has started to improve it to higher orders differential equation.

Here, there is an intuitive question should arise on the nature of the properties of coefficient function in equation (1), and how it can be effective in the growing and bounded solution to the given equation.

Let \( H(z) = e^{p(z)} \) in equation (1) to be as follow
\[
\quad f'' + e^{p(z)}f = 0, \ldots \ldots \ldots (2)
\]
the choice of such a function is due to its properties, represented by being a non-constant analytic function and primitive of its derivative, hence the starting point has began by depending on two techniques: a certain Canonical form to second-order linear differential equations, and Liouville transformation.

In the following technique which never have been used before, but the nature and environment of this problem required that, so the steps of this technique starts with study the invariance of given equation, by using of Liouville and Self-Adjoint transformation with an examine.
the convexity of $H(z) = e^{p(z)}$ in equation (2) in equation (2) which have a good role to preserve the angles and orientations of given domain especially when we apply the bilinear transformation, in addition to show the compatibility the properties for the coefficient function $H(z) = e^{p(z)}$ with the solution $f$, in order to study and examine the Growth and bounded of the Solution.

2. **Invariance and Convexity of 2nd order-Homogeneous Complex Differential Equation.**
In this section, we have verified of the nature of equation (2) in terms of variation by applying two kinds of transformation, Liouville transform and Self-Adjoint [11] in order to confirm the invariance of equation (2). See the Table 1 below.

**Table 1. Check up the invariance of equation (2) under to type of transformations Liouville and Self- Adjoint**

| Liouville Transformation | Consider the following equation $f'' + e^{p(z)} f = 0$ \( \ldots \ldots \) \((*)_1\) | Consider the following equation $f'' + e^{p(z)} f = 0$ \( \ldots \ldots \) \((*)_1\) |
|-------------------------|-------------------------------------------------|-------------------------------------------------|
| Let \( f = g e^{\int \frac{-1}{2} p(z)} \), where \([B(z)=0]\) is coefficient of \(f'\) | Let \([r(z)g']' + q(z)g = 0, \ldots \ldots \) \((*)_2\) | \(r(z) = e^{\int \frac{a_1}{a_0} dz} \text{ and } q(z) = \frac{a_2}{a_0} r(z)\), where \(a_0, a_1, a_2\) coefficient of \(f'', f', f\) respectively |
| Calculate \(f'\) & \(f''\) as follows \(f' = g'e^c\); \(f'' = g''e^c\) | Then \(r(z) = e^c\) and \(q(z) = e^c e^{p(z)}\) |
| Substituting \(f'\) & \(f''\) in eq\((*)_1\) \(g'' + e^{p(z)} g = 0\) | Substituting \(r(z)\) & \(q(z)\) in eq\((*)_2\) \(g'' + e^{p(z)} g = 0\) |

An invariant property in the solving of complex or real differential equation is highly useful to ensure the stability of the solution, for instance, if there is a particle moving around the earth, then its orientation be in a plane. One can describe the position of that particle in terms of a distance $z(t)$ from the earth and an angle $\theta$ when it goes around the earth. The existence solution of such as these equations would be is guaranteed and locally. (cf.[2])

Now, should be check the nature of the given coefficient function $H(z) = e^{p(z)}$ in terms of the shape and existence the singularity points on the boundary as follows:

Set, \(p(z) = z\) that is; \(e^{p(z)} = e^z\), which defined on the disk \(D = \{z: |z - i| \leq 1\}\), such that \(z = i + r e^{i\theta}\) thus, \(e^z = e^{i+r e^{i\theta}}\),

\[
e^z = e^{i+r \cos \theta + i r \sin \theta}
\]

\[
= e^{r \cos \theta} [\cos(1 + r \sin \theta) + i \sin(1 + r \sin \theta)],
\]

Let, \(r = 1\), to obtain \(x = e^{\cos \theta} \cos(1 + \sin \theta), \quad y = e^{\cos \theta} \sin(1 + \sin \theta)\).

See figure 1 below.
1. Set, $P(z) = z^2$ that is; $e^{p(z)} = e^{z^2}$ which defined on the disk $D = \{z : |z - i| \leq 1\}$, such that $z = i + re^{i\theta}$ thus
\[ e^{z^2} = e^{(i + re^{i\theta})^2} \]
\[ = e^{-1 + 2i r \cos \theta - 2r \sin \theta + r^2 \cos 2\theta + ir^2 \sin 2\theta} \]
\[ = e^{-1 - 2i r \sin \theta + r^2 \cos 2\theta} [r^2 \cos(2\cos \theta + \sin 2\theta) + ir^2 \sin(2\cos \theta + \sin 2\theta)]. \]
Let, $r = 1$, to obtain $x = e^{-1 - 2i \sin \theta + \cos 2\theta} \cos(2\cos \theta + \sin 2\theta)$, and $y = e^{-1 - 2i \sin \theta + \cos 2\theta} \sin(2\cos \theta + \sin 2\theta)$.
Then, $r = e^{-1 - 2i \sin \theta + \cos 2\theta} = e^{-2 \sin \theta - 2 \sin^3 \theta}$.
See figure 2 below.

2. Set, $P(z) = z^n$ that is; $e^{p(z)} = e^{z^n}$, which defined on the disk $D = \{z : |z - i| \leq 1\}$, where $z = i + re^{i\theta}$; thus

Figure 1. $H(z) = \exp(i + re^{i\theta})$ is a convex function with singular point of outward cusp.

Figure 2. $H(z) = \exp(i + re^{i\theta})^2$ is a convex function with singular point of inward cusp.
\begin{align*}
(i + r e^{i\theta})^n &= \sum_{j=0}^{n} \binom{n}{j} i^{n-j} (r e^{i\theta})^j \\
&= i^n + n i^{n-1} r e^{i\theta} + \frac{(n-1)n}{2} i^{n-2} r^2 e^{i2\theta} + \ldots + r^n e^{in\theta}.
\end{align*}

Let $z = re^{i\theta}$,

\[(i + r e^{i\theta})^n = i^n + z \left[ n i^{n-1} + \frac{(n-1)n}{2} i^{n-2} r z + \ldots + r^{n-1} z^{n-1} \right].
\]

Let, $r \to 0$, we get

\[(i + r e^{i\theta})^n = i^n + z [n i^{n-1}].
\]

The latest formula tell us that there are several ways to know the behavior of the given case as follows.

a) If $n$ is odd number, then

\[\exp\left[(i + r e^{i\theta})^n\right] = \sum_{n=1}^{\infty} \exp\{(−1)^{n+1}[i(2n − 1)z]\} \]

\[= e^{i+x} + e^{−(i+3z)} + \ldots + \ldots.
\]

Let $z = re^{i\theta}$,

\[\exp\left[(i + r e^{i\theta})^n\right] = e^{i+re^{i\theta}} + e^{−(i+3re^{i\theta})} + \ldots
\]

\[= e^{rcos\theta}[\cos(1 + rsin\theta) + isin(1 + rsin\theta)] + e^{−3rcos\theta}[\cos(−1 − 3rsin\theta) + isin(−1 − 3rsin\theta)] + \ldots.
\]

Let $r = 1$, such that

\[x = e^{cos\theta} \cos(1 + sin\theta) + e^{−3cos\theta} \cos(−1 − 3sin\theta) + \ldots,
\]

\[y = e^{cos\theta} \sin(1 + rsin\theta) + e^{−3cos\theta} \sin(−1 − 3sin\theta) + \ldots,
\]

then

\[x = \sum_{n=1}^{\infty} \exp\{(-1)^{n+1}cos\theta\} \cos((-1)^{n+1}(1 + (2n − 1)sin\theta)
\]

\[y = \sum_{n=1}^{\infty} \exp\{(-1)^{n+1}cos\theta\} \sin((-1)^{n+1}(1 + (2n − 1)sin\theta)
\]

Finally, $r = \sum_{n=1}^{\infty} \exp\{(-1)^{n+1}cos\theta\}$. See figure 3a below.

**Figure 3a.** $H(z) = \exp(i + r e^{i\theta})^n$, where $n$ is an odd number, be a convex function with singular point of outward cusp
b) If $n$ is even number, then

$$\exp\left[ (i + r e^{i\theta})^n \right] = \sum_{n=1}^{\infty} \exp \left\{ (-1)^{n+1} \left[ 1 + 2niz \right] \right\}$$

$$= e^{-1+2iz} + e^{1-4iz} + \ldots$$

Let $z = re^{i\theta}$

$$\exp\left[ (i + re^{i\theta})^n \right] = e^{-1+2ire^{i\theta}} + e^{1-4ire^{i\theta}} + \ldots$$

$$= e^{-1-2rsin\theta} \left[ \cos(2rcos\theta) + isin(2rcos\theta) \right] + e^{1+4rsin\theta} \left[ \cos(-4rcos\theta) + isin(-4rcos\theta) \right] + \ldots$$

Let $r = 1$, we obtain

$$x = e^{-1-2rsin\theta} \cos(2rcos\theta) + e^{1+4rsin\theta} \cos(-4rcos\theta) + \ldots$$

$$y = e^{-1-2rsin\theta} \sin(2rcos\theta) + e^{1+4rsin\theta} \sin(-4rcos\theta) + \ldots$$

then

$$x = \sum_{n=1}^{\infty} \exp \left[ (-1)^{n+1} \left[ 1 - 2n\sin\theta \right] \right] \cos(-1)^{n+1}(2n\cos\theta)$$

$$y = \sum_{n=1}^{\infty} \exp \left[ (-1)^{n+1} \left[ 1 - 2n\sin\theta \right] \right] \sin(-1)^{n+1}(2n\cos\theta)$$

Finally, $r = \sum_{n=1}^{\infty} \exp \left[ (-1)^{n+1} \left[ 1 - 2n\sin\theta \right] \right]$. See figure 3b below.

![Figure 3b. $H(z) = \exp(i + re^{i\theta})^n$, where $n$ is even number, be a convex function with singular of outward cusp](image)

3. Application: Matching for the coefficient function $H(z) = e^{p(z)}$ with the solution $f$.

J.Clunie, Tumura (cf. [3], [14]) and others introduced a proof of the result that $f(z)$ is of the form $e^{az+b}$ provided $f(z)f^{(n)}(z) \neq 0$ for some integer $n \geq 2$.

Let us show how much the form $e^{az+b}$ works with given equation (2) when we define $f(z) = e^{az+b}$ for $a \in \mathbb{C}$, $a \neq 0$, $b \in \mathbb{N}$, such that

$f'(z) = ae^{az+b}$; $f''(z) = a^2 e^{az+b}$.

Substitute $f$ & $f''$ in the equation above to find $e^{p(z)}$ as follows.

$a^2 e^{az+b} + e^{p(z)} e^{az+b} = 0$,

$e^{az+b} [a^2 + e^{p(z)}] = 0$, Thus $e^{p(z)} = -a^2$. 

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Now, see figure (4) below, which explains precisely the behavior of the solution \( f(z) = e^{az+b} \) for given equation (2) through showing that \( e^{p(z)} = -a^2 \) is a convex function with outward cusp that means \( p(z) \) belongs to the case (3a).

**Figure 4.** \( H(z) = e^{p(z)} = -a^2 \) is a convex function

### 4. Growth of the Solution.

Here, our work requires in this stage to consider the following two theorems: one of them covers the existence length of coefficient function in equation (2) which has deep meaning to proof the growth solution of given equation (2), and another one is concerned with function \( u(r) \) as will be shown (cf. [4],[15]).

**Theorem (4.1).** Let \( H \) be analytic function on \( D \), and is an coefficient function of equation \( f^{(n)} + H(z)f = 0 \), where \( H(z) = e^{p(z)} \) and \( \delta(p(z), \theta) < 0 \). Then \( |H(z)| \leq e^{(c_1+r+r^n)\delta(p(z), \theta)} \).

**Proof.** Given \( f^{(n)} + H(z)f = 0 \), \( H(z) = e^{p(z)} \), so that \( |H(z)| = |e^{p(z)}| \).

Put \( p(z) = p_n(z) \), \( n \in \mathbb{N} \).

Choose, \( p_n(z) = cz^n \), \( n \in \mathbb{N} \); \( c \in \mathbb{C} \) to obtain

\[ |H(z)| = |e^{p(x)}| = |e^{p_n(x)}| = |e^{cz^n}| = |e^{(c_1+ic_2)z^n}|. \]

Let \( z = i + re^{i\theta} \)

\[ P(z) = (c_1 + ic_2)z^n \quad c_2, c_1 \in \mathbb{R}, \quad n \in \mathbb{N} \]

\[ = (c_1 + ic_2) \left[ (i + re^{i\theta})^n \right] \]

\[ = (c_1 + ic_2) \left[ i^n + n i^{n-1} r e^{i\theta} + \cdots + r^n e^{in\theta} \right] \]

\[ = i^n c_1 + n i^{n-1} r c_1 \cos \theta + n i^n r c_1 \sin \theta + \cdots + r^n c_1 \cos \theta + r^n c_1 \sin \theta \]

\[ + i^{n+1} c_2 + n i^n r c_2 \cos \theta + n i^{n+1} r c_2 \sin \theta + \cdots + r^{n+1} c_2 \cos \theta + r^{n+1} c_2 \sin \theta \]

\[ = i^n c_1 + n i^{n-1} r [c_1 \cos \theta - c_2 \sin \theta] + n i^n r [c_1 \sin \theta + c_2 \cos \theta] + \cdots + i^{n+1} c_2 + n i^{n+1} r [c_1 \cos \theta - c_2 \sin \theta] + \cdots. \]

Set, \( \delta(p(z), \theta) = c_1 \sin \theta + c_2 \cos \theta \)

\( \mathcal{A}^+ = \{ z : \delta(p(z), \theta) > 0 \} \)

\( \mathcal{A}^- = \{ z : \delta(p(z), \theta) < 0 \} \)

Take \( \mathcal{A}^- \) since \( \delta(p(z), \theta) < 0 \) to reach our required which means there exist \( r(\theta) \) works to make \( \log H(z) \) decreasing on \( [r(\theta), \infty) \).
Now, we are in need to define function depends on $r$ since as mentioned above $r(\theta)$ will force \( \log|H(z)| \) to be decreasing on the half open interval for $r(\theta) \& \infty$.

\[
\log|H(z)| = (c_1 + r + r^n) \Im(\delta(p(z), \theta))
\]

Thus, $H(r) = (1 + nr^{-n-1}) \Im(\delta(p(z), \theta))$. Finally, $|H(z)| \leq e^{(c_1 + r + r^n) \Im(\delta(p(z), \theta))}$.

The proof is complete $\blacksquare$

**Theorem 4.2.**

Let $e^{p(z)}$ be analytic function in $D = \{z \in \mathbb{C} : |z - i| \leq 1\}$. Set, $|e^{p(z)}| \leq H(r)$ for $|z - i| \leq r, 0 \leq r < 1$, where $H(r) = (c_1 + r + r^n) \Im(\delta(r, z))$. If $f$ is a solution of (1) and if

\[
u(x) = 4^2H(x^2u(x) \quad (0 \leq x < 2) \quad \ldots \ldots \quad (3)
\]

then $\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt \leq u(r)$ for $0 \leq r < 1$.

**Proof.** Define $f(z) = \sum_{n=0}^{\infty} a_n(z - i)^n \quad (z \in D) \ldots \ldots (5)$

\[
|f(z)| = \left| \sum_{n=0}^{\infty} |a_n|(|z - i|^n) \right| \leq \sum_{n=0}^{\infty} |a_n| r^n
\]

\[
f'(z) = \sum_{n=1}^{\infty} nan_n(z - i)^{n-1}
\]

\[
f''(z) = \sum_{n=2}^{\infty} n(n-1)a_n(z - i)^{n-2}
\]

\[
|f''(z)|^2 = \sum_{n=2}^{\infty} |n(n-1)a_n|^2(z - i)^{2(n-2)}
\]

In view of Parseval’s Formula,

\[
f(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt \quad \ldots \ldots (6)
\]

Derive equation (7) to obtain,

\[
f'(r) = \sum_{n=1}^{\infty} 2n|a_n|^2 r^{2n-1} \quad \ldots \ldots (7)
\]

\[
f''(r) = \sum_{n=1}^{\infty} 2n(2n-1)|a_n|^2 r^{2n-2} \quad \ldots \ldots (8)
\]

\[
f'''(r) = \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)|a_n|^2 r^{2n-3} \quad \ldots \ldots (9)
\]

Parseval’s Formula and equation (6) provide of the following

\[
f^{(4)}(r) = \sum_{n=2}^{\infty} 2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-4}
\]

\[
\leq k \sum_{n=2}^{\infty} |n(n-1)a_n|^2 r^{2(n-2)} = \frac{k}{2\pi} \int_0^{2\pi} |f''(re^{it})|^2 \, dt \quad \ldots \ldots (9)
\]

One can find the value of $k$, from equation (9)

\[
2n(2n-1)(2n-2)(2n-3)|a_n|^2 r^{2n-2} \leq k n^2(n - 1)^2 |a_n|^2 r^{2(n-2)}
\]

\[
4(4n^2 - 8n + 3) \leq k(n^2 - n)
\]

\[
4n^2 - 8n + 3 - \frac{k}{4} n^2 + \frac{k}{4} n \leq 0
\]
by the constitution law could be get

\[(8 - \frac{k}{4})^2 + 12\left(\frac{k}{4} - 4\right) = 0, \text{ where } k = 4^2.\]

Hence,

\[J^{(t)}(r) = \sum_{n=2}^{\infty} 2n(2n - 1)(2n - 2)(2n - 3)|a_n|^2 r^{2n-4} \leq 4^{2} \sum_{n=2}^{\infty} |n(n-1)a_n|^2 r^{2(n-2)} = \frac{2^2}{\pi} \int_{0}^{2\pi} |f'(re^{it})|^2 \, dt,\]

From equations (1),(2) and (10) to obtain

\[J^{(t)}(r) \leq \frac{2^2}{\pi} \int_{0}^{2\pi} |e^{p(re^{it})}|^2 |f(re^{it})|^2 \, dt \leq \frac{2^2}{\pi} |e^{(c_1 + r^n)} \delta(p(z),\theta)|^2 \int_{0}^{2\pi} |f'(re^{it})|^2 \, dt \leq 4^2 H(r)^2 J(r),\]

since \(|H(z)| = |e^{(c_1 + r^n)} \delta(p(z),\theta)|, to obtain

\[J^{(t)}(r) \leq \frac{2^2}{\pi} \int_{0}^{2\pi} |e^{p(re^{it})}|^2 |f'(re^{it})|^2 \, dt \leq 4^2 H(r)^2 J(r) \ldots \ldots \ldots (10)\]

It follows that equations (6), (7) and (8) for \(r \to 0\). Since,

\[|f(z)| = \sum_{n=0}^{\infty} |a_n| |(z - i)^n| \leq \sum_{n=0}^{\infty} |a_n| r^n \]

\[|f(z)|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \]

\[\therefore |f(0)|^2 = f(0), \text{ and from (4) that } f^{(yr)}(0) \leq u^{(yr)}(0) \text{ for } y = 0,1,2,3.\]

The solution of given equation is found on the disk \(D = \{ z \in \mathbb{C} : |z - i| \leq 1 \}\), and clearly is an uniform convergence, absolute convergence with fixed point theorem. Hence, limit points on the boundary of given domain are \{(0,0), (1, i), (-1, i), (0, 2i)\}. That’s why we can conclude that \(r \to 0\); their purpose, find a relation between the characteristics of the solution mentioned above and the comparison theorem. See figure 5 below.

\[\text{Figure 5. Unit Disk}\]
\[ u^{(4)}(r) = 4^2 H(r)^2 u(r), \quad (0 \leq r < 1), \text{then} \quad I^{(4)}(r) \leq u^{(4)}(r) \]

Therefore conclude that \( I(r) \leq u(r) \) for \( 0 \leq r < 1 \), which means that
\[
\frac{1}{2\pi} \int_0^{2\pi} |f( re^{it})|^2 \, d \theta \leq u(r) \quad \text{for} \quad 0 \leq r < 1.
\]

The proof is complete ■

Now about to present a next theorem that covers one of the families of transformation namely, bilinear transformation, whose inverse property. In this case, it will be the guarantee to preserve all inverse elements on the convex domain in order to prove that there is related between the solution and the coefficient function in the given equation (2).

5. **Bounded of the Solution**

Let \( f^\dagger(z) + e^{p(z)} f(z) = 0 \quad (z \in D) \), be analytic function in \( D = \{ z \in \mathbb{C} : \ |z - i| \leq 1 \} \subseteq \mathbb{C} \). Furthermore, see [10], [13], [15].

**Theorem (5.1).** Let \( T : D \to D \), be holomorphic function in \( D \), where \( |H(z)| \geq 1 \), and let \( f^\dagger(z) + H(z) f(z) = 0 \) for \( z \in D \). Then there exist a self - conformal mapping which preserves the boundary of the area of \( \Omega \) such that

\[
\int_{\Omega} \frac{(1 - |z|^2)(1 - |\zeta|^2)}{|1 - \zeta z|^2} \, dx dy \\
\leq \int_{\Omega} (1 - |t|^2)^3 |H(t)|^2 \frac{|1 + \zeta|}{|1 - |t|^2|^2 |1 - |\zeta|^2|} ds dr \quad \ldots \ldots \quad (11)
\]

**Proof.** Define \( T(z) = \frac{z + \zeta}{1 + \zeta z} : D \to D \), where \( z, t \in D \exists |z| < 1 \) and \( |t| < 1 \). Derive \( T(z) \) which respect to \( z \) on \( D \) such that

\[
d(T(z)) = \frac{1 - \zeta^2}{(1 + \zeta z)^2} \, dz, \quad \text{which implies to}
\]

\[
T'(z) = \frac{dT}{dz} = \frac{(1 - |\zeta|^2)}{(1 + \zeta z)^2} \quad \ldots \ldots \quad (12)
\]

Given \( f^\dagger + H(z) f = 0 \), where \( H(z) = e^{p(z)} \) with condition \( |H(z)| \geq 1 \).

![Diagram](D) \rightarrow T \rightarrow D \rightarrow H(H(T(z))) = S

Let us define \( (H \circ T)(z) = H(T(z)) \) in \( \Omega \).

So, should be calculate the codomain of \( H(T(z)) \) in \( \Omega \) through finding its area.

Hence, Take double integral for the inequality (2) of Schwarz – Pick’s theorem, and apply co – area formula to get
\[\int_D \frac{(1-|z|^2)(1-|t|^2)}{|1-tz|^2} \, dx \, dy = \int_D \frac{(1-|z|^2)(1-|t|^2)}{|1-\bar{t}z|^2} \, ds \, dr \]
\[\leq \int_D (1-|z|^2) \, |H(T(z))|^2 \frac{(1-|T(z)|^2)(1-|T(t)|^2)}{|1-T(t)T(z)|^2} \, ds \, dr \]
where \(|H(T(z))| \geq 1, T(z) = t\) and clearly \(|z| = |t| < 1\).
\[= \int_\Omega (1-|t|^2) \, |H(t)|^2 \frac{(1-|t|^2)(1-|T(t)|^2)}{|1-T(t)t|^2} \, ds \, dr \]
\[= \int_\Omega (1-|t|^2)^2 \, |H(t)|^2 \frac{(1-|T(t)|^2)}{|1-T(t)t|^2} \, ds \, dr \]
\[= \int_\Omega (1-|t|^2)^3 \, |H(t)|^2 \frac{|1+\bar{\xi}z|}{|1-|t|^2|^2 |1-|\xi|^2|} \, ds \, dr.
\]
Set \(T(\zeta) = t\) since \(\zeta \in D\).
\[= \int_\Omega (1-|t|^2)^3 \, |H(t)|^2 \frac{(1+\bar{\xi}z)}{|1-|t|^2|^2 |1-|\xi|^2|} \, ds \, dr\]

The proof is complete. Theorem 5.2. If \(H(z)\) be a holomorphic function in the given equation (1), and there is \((H \circ T)(z)\): \(D \to \Omega\) then the solution of given equation can be growth by satisfies the condition below
\[\frac{1}{2\pi} \int_\Omega \frac{1}{2} \, (1-|z|^2)^3 \, |H(t)|^2 \, d\Omega \leq O(1 + r^2).
\]
Proof. The previous theorem (1.3), satisfies the inequality (11) hence, the right side of given inequality will be a base of our result with Gutzmer identity to obtain
\[\frac{1}{2\pi} \int_\Omega \frac{1}{2} \, (1-|z|^2)^3 \, |H(t)|^2 \, d\Omega \leq O(1 + r^2).
\]

\[\frac{1}{2\pi} \int_\Omega \frac{1}{2} \, (1-|z|^2)^3 \, |H(t)|^2 \, d\Omega \leq \frac{1}{2\pi} \int_\Omega (1-|r|^2)^3 \, |f''|^2 \, d\Omega,
\]
where \(ds \, dr = d\Omega\).

Since, \(|H(t)|^2 \left[ \frac{1+\bar{\xi}}{|1-|t|^2|^2 |1-|\xi|^2|} \right] \) explains the behavior of the holomorphic function \(H(z)\) with function \(f\) in given equation (13) which is generated from bilinear transformation.

Now, define
\[f(z) = \sum_{n \geq 0} c_n (z - i)^n, \quad f'(z) = \sum_{n \geq 1} nc_n (z - i)^{n-1},\]
and
\[f''(z) = \sum_{n \geq 2} n(n-1)c_n (z - i)^{n-2}\]
such that
\[\frac{1}{2\pi} \int_\Omega \frac{1}{2} \, (1-|z|^2)^3 \, |H(t)|^2 \, d\Omega \]
\[= \frac{1}{2\pi} \int_0^1 (1-r^2)^3 \, dr \sum_{n \geq 2} n^2(n-1)^2 |c_n|^2 |z - i|^{2n-4} \, d\Omega,
\]
Integration by parts to calculate the integral \( (1) \) with respect to \( \rho \) in order to obtain,

\[
\frac{1}{2\pi} \int_0^1 (1 - \rho^2)^3 d\rho \int_0^{2\pi} \sum_{n=0}^{2\pi} n^2(n-1)^2 |c_n|^2 \rho^{2n-4} d\Omega_	heta.
\]

\[
= \frac{1}{2\pi} \int_1^1 (1 - \rho^2)^3 \rho^{2n-4} d\rho \int_0^{2\pi} \sum_{n=2}^{\infty} n^2(n-1)^2 |c_n|^2 d\Omega_	heta.
\]

Short and simple comparison for the power series of the equation’s coefficients (14) with its equivalent of the given equation.

\[
\sum_{n=0}^{\infty} |c_n|^2 = |c_0|^2 + |c_1|^2 + |c_2|^2 + \cdots \cdots \cdots (15)
\]

In our case, the calculation gives

\[
\frac{1}{2\pi} \int_{\Omega} (1 - |z|^2)^3 |H(t)|^2 \frac{1 + \xi z}{|1 - |t|^2|^2 |1 - |\xi|^2|} d\Omega_	heta
\]

\[
= \sum_{n=2}^{\infty} \frac{n^2[6(2n-1) - (2n-3)(2n+1)](n-1)^2}{(2n-3)(4n^2 - 1)} |c_n|^2 \cdots \cdots (16)
\]

Consequently, from (15) and (16) we obtain that

\[
|c_0|^2 + |c_1|^2 + |c_2|^2 \leq |c_0|^2 + |c_1|^2 + \frac{1}{2\pi} \int_{\Omega} (1 - |z|^2)^3 |H(t)|^2 \frac{1 + \xi z}{|1 - |t|^2|^2 |1 - |\xi|^2|} d\Omega_	heta \cdots \cdots (17)
\]

Since the bilinear transformation starts to vanish when \( \xi \to \Omega \), and set \( |t|^2 = r \), \( r > 0 \) to get

\[
\frac{1}{2\pi} \int_{\Omega} (1 - |z|^2)^3 |H(t)|^2 \frac{1 + \xi z}{|1 + |t|^2|^2 |1 + |\xi|^2|} d\Omega_	heta \leq \frac{1}{2\pi} \int_{\Omega} (1 - |z|^2)^3 |H(t)|^2 d\Omega_	heta \leq O(1 + r^2).
\]

Finally, rewrite the inequality above as follows.

\[
\frac{1}{2\pi} \int_{\Omega} (1 - |z|^2)^3 |H(t)|^2 d\Omega_	heta \leq O(1 + r^2).
\]

The proof is complete \( \blacksquare \)

**Conclusion.**

In the end of this work, we have to show that invariant property of coefficient function in the linear complex differential equation plays good rule to ensure the stability of solution toward the convexity which succeeded to show the growing solution of given equation.
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