Born-Infeld black-body radiation

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Abstract – The problem of black-body radiation is considered in the Born-Infeld theory of electrodynamics. In particular, at 2-loop order the deviation from the Planck expression due to the self-interaction of photons is calculated. It is seen that the system of interacting photons of the theory, opposed to its non-Abelian counterpart, has higher internal energy at this order of perturbation. Possible implications of the result on the evolution of very-high-temperature systems, including various stellar media and the early universe, are briefly discussed.

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The Born-Infeld theory of electrodynamics is given by the Lagrangian density \cite{1}
\begin{equation}
\mathcal{L} = -\sqrt{|\det (\gamma \eta_{\mu\nu} + F_{\mu\nu})|} + \sqrt{-\det (\gamma \eta_{\mu\nu})}
\end{equation}
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1) \) represents the space-time metric. In the above \( \gamma \) is a new constant with dimension (energy) \( ^2 \) in the units \( \hbar = c = 1 \), by which one can recover the Lagrangian of Maxwell electrodynamics in the limit \( \gamma \to \infty \). The original motivation to propose the non-linear theory above was to overcome the problem of infinite field strength and energy associated with the point charge source \cite{1}. In particular, the electrostatic field of a spherically symmetric solution of the theory at the origin is finite and of order of \( \gamma \). It is shown that the Born-Infeld theory, though non-linear, possesses exact plane-wave solutions \cite{2}.

The purpose of this letter is to study the problem of black-body radiation in the electrodynamics theory of Born-Infeld. Due to the non-linear character of the theory we expect interaction among the photons. Including lowest-loop corrections, we compute the deviation from the black-body radiation of the Maxwell theory. Possible implications of the result on the evolution of extremely-high-temperature systems, including various stellar media and the early universe, are briefly discussed.

In 1 + 3 space-time dimensions the Lagrangian density (1) reduces to
\begin{equation}
\mathcal{L} = \gamma^2 \left( 1 - \sqrt{1 + \frac{2Q}{\gamma^2} - \frac{P^2}{\gamma^2}} \right)
\end{equation}
in which \( Q \) and \( P \) are given by
\begin{equation}
Q = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (B^2 - E^2),
\end{equation}
\begin{equation}
P = \frac{1}{4} \ast F_{\mu\nu} F^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}
\end{equation}
with \( * F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} / 2 \). Evidently the Lagrangian is invariant under the Abelian gauge transformation \( A \to A + \partial \alpha \), for any differentiable function \( \alpha(x) \).

The Lagrangian may be expanded in powers of \( \gamma^{-1} \), for which, by keeping the lowest orders, one gets
\begin{equation}
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2 \gamma^2} (Q^2 + P^2) + O(\gamma^{-4}).
\end{equation}
The first term above is simply the Maxwell term, by which the free theory is presented. The other terms introduce the interaction among the photons. By keeping the lowest order, in the language of Feynman diagrams, we have a 4-leg vertex which is given by
\begin{equation}
V_4 = \frac{1}{\gamma^2} \left[ (p \cdot q)(r \cdot s) \eta^{\rho\sigma} \eta^{\mu\nu} + (p \cdot r)(q \cdot s) \eta^{\rho\sigma} \eta^{\mu\nu} + \eta^{\rho\sigma} \eta^{\mu\nu} + \eta^{\rho\sigma} \eta^{\mu\nu} \right]
\end{equation}
where the momenta and space-time indices are introduced in fig. 1. Also, the full quantum theory is given after

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adding a gauge fixing term, which may be taken as usual 
$-\frac{1}{4}(\partial \cdot A)^2$, and the corresponding ghost sector. Since in
the present theory the gauge group is Abelian, the ghosts
do not interact with the photons, hence the ghost sector
simply can be integrated out.

Since the coupling constant $\gamma^{-2}$ has negative mass
dimension, the present theory, as a quantum field theory,
is not renormalizable. This suggests that the proper
interpretation of the Born-Infeld theory would be as an
effective theory of an underlying finite or renormalizable
theory. In lack of the full underlying theory one still
may try to explore the implications of the Born-Infeld
theory at higher orders of perturbation, for which one
has to give a recipe for the problem of infinities. As
mentioned, here we consider the black-body radiation
problem, and the natural framework for dealing with
the interaction contribution is the finite-temperature field
theory [3,4]. As known, and as we will see in more detail
below, the problem of infinities finds a natural solution
in this framework [3,4]. According to the recipe, the
regulated finite-temperature expressions are given simply
after the subtraction of the zero-temperature counterpart
expressions [4]. By this, the study of thermodynamical
properties may give a basis to explore the implications
of the effective theory at higher orders of perturbations.

At 1-loop order (no vertex), the free Maxwell theory
and the contribution from the ghost sector simply give the
standard Planck expression for black-body radiation.
The first contribution from interactions comes at 2-loop
(one vertex) order. As mentioned, to get the relevant
regulated contributions one subtracts an altered form of
the diagrams, the so-called parenthesized diagrams, from
the original ones. At 2-loop order by “parenthesized”
we simply mean that in one of the loops we send the
temperature to zero [4,5]. In the present case we have the
contribution from the diagrams in fig. 2 [4,5].
The internal energy per unit volume is then given by
\[ U = \mathcal{F} + \beta \partial_\beta \mathcal{F} \]
and
\[ U = \int_0^\infty d\omega \ U(\omega) \]
for which we find
\[ \Delta U_{\text{2-loop}}(\omega) = \frac{T^4}{45 \gamma^2} \omega^3 \left[ \frac{3}{\exp(\beta \omega) - 1} + \frac{\beta \omega \exp(\beta \omega)}{(\exp(\beta \omega) - 1)^2} \right] \]
as the energy per unit volume in the frequency interval \((\omega, \omega + d\omega)\). In fact this expression gives the correction to the standard Planck expression,
\[ U_{\text{Planck}}(\omega) = \frac{1}{\pi^2} \frac{\omega^3}{\exp(\beta \omega) - 1}, \]
by which Stefan’s law follows,
\[ U_{\text{Planck}} = \int_0^\infty d\omega \ U_{\text{Planck}}(\omega) = \frac{\pi^2}{15} T^4. \]

Using (12) and
\[ \int_0^\infty s^{2m+2} e^s (e^s - 1)^2 ds = (2m + 2)! \zeta(2m + 2) \]
one finds the following as the correction to Stefan’s law:
\[ \Delta U_{\text{2-loop}} = \frac{7 \pi^4}{675} \frac{T^8}{\gamma^2}. \]

So the total energy is found to be
\[ U_{\text{Born-Infeld}} = \frac{\pi^2}{15} T^4 + \frac{7 \pi^4}{675} \frac{T^8}{\gamma^2}. \]

This expression might be contrasted to the analogous one in non-Abelian gauge theories. For SU(Nc) pure gauge theory, the combination of the free theory and first correction is found to be [5]
\[ U_{\text{pure SU}}(N_c) = \frac{\pi^2}{15} N_c T^4 - \frac{\alpha_c}{3} N_c N_s T^4, \]
in which \( \alpha_c = g^2/(16\pi) \) is the color fine-structure constant, and \( N_c = N_s^2 - 1 \) is the number of the gluons of the theory. Most importantly, due to the minus sign, taking into account the interactions between the gauge bosons would decrease the internal energy and so the pressure in the systems, as befits the name “gluon” for the gauge bosons in the non-Abelian case. In comparison, the photons of the Born-Infeld theory do not exhibit the glue behavior, and in fact act in opposite way.

To get a better idea of the kind of the changes by including the 2-loop correction, in fig. 3 the plots of the Planck expression, \( \Delta U(\omega) \) of (16), and their combination are presented.

The test of the Born-Infeld theory has been the subject of several researches, among them are the tests based on the waveguide physics [6], and explored the consequences of the non-linearity of the theory in the presence of background fields on the induced anisotropy of the electromagnetic waves [7], the change in the measured luminosity of distant objects [8], and the wave dispersion relation of a propagating laser in a ring [9].

One may try to get an idea of the magnitude of the possible deviation by the present results from the classical expressions. There exist lower limits on the parameter \( \gamma \) in the theory. As mentioned \( \gamma \) plays the role of the maximum possible value for the electric field, the so-called absolute field constant [1]. Basing on the electric-field value at the edge of the classical radius of electron, Born and Infeld estimated \( 3 \times 10^{20} \mbox{ V/m} \) in SI units as the lower limit of the absolute constant field, which corresponds to \( \gamma \gtrsim 4 \times 10^{14} \mbox{ eV}^2 \), in the natural units. The order of this lower limit is supported by investigations done on the atomic-energy levels [10]. Having the lower limit for \( \gamma \), there are several ways to have an idea of the upper limit of possible deviations. One is the change in the total radiation energy. By (18) and (20), and after restoring the Boltzmann constant \( k_B = 8.6 \times 10^{-5} \mbox{ eV/K} \), by the above lower limit for \( \gamma \) one finds
\[ \frac{\Delta U_{\text{2-loop}}}{U_{\text{Planck}}} = \frac{7 \pi^2}{45} \frac{(k_B T)^4}{\gamma^2} \lesssim 5.2 \times 10^{-46} T^4. \]

A very close estimation is obtained for the ratio at the maximum frequency, for which one has by Wien’s law \( \beta \omega_{\text{max}} \approx 2.8 \). With \( \gamma \) equal to the lower limit and the temperatures higher than \( 10^{14} \mbox{ K} \) the above ratio finds values that are expected to have remarkable implications on the models, as well as possible observational indications. By the present models for the neutron stars, the cores of young neutron stars have the temperatures in the range \( 10^{10-12} \mbox{ K} \). It should be reminded, due to the perturbative nature of the presented calculation, in the cases where the ratio (23) exceeds unity, the analysis can give only a qualitative description. As the energy distribution of the photons inside a star has direct implications on the
In the ordinary electrodynamics. In particular, at tree level, the effective Lagrangian which depends on the Abelian field strength \( F_{\mu\nu} \) but not on its derivatives, sums up to the Born-Infeld one [15]. In this new rise of the theory the parameter \( \gamma \) is determined by the corresponding string theory, which is supposedly several orders of magnitude higher than the before-mentioned lower limit. Hence, the implications of this interpretation of the Born-Infeld theory are expected to be considerable only in the first minutes of the early universe, when the temperature has been high enough to be comparable with string theory energy scales. By this way of interpretation, various kinds of indirect tests of string-theory–based models might be provided.

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