Beads on the torus I: continuous Kasteleyn theory

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Abstract

Consider the semi-discrete torus $T_n := [0, 1) \times \{0, 1, \ldots, n-1\}$ representing $n$ unit length strings running in parallel. A bead configuration is a point process on $T_n$ with the property that between every two consecutive points on same string, there lies a point on each of the neighbouring strings. In this article we develop a continuous version of Kasteleyn theory to show that partition functions for bead configurations on $T_n$ may be expressed in terms of Fredholm determinants of certain operators on $T_n$. We obtain an explicit formula for the volume of bead configurations whose asymptotics confirm a recent prediction of Shlyakhtenko and Tao ([2020] arXiv:2009.01882) in the free probability literature. Thereafter we study random bead configurations on $T_n$, expanding on the works of Boutillier ([2009] Ann. Probab. 37(1), 107-142] and Gordenko ([2020] arXiv:2009.10480]. We make connections with exclusion processes and provide a new probabilistic representation of TASEP on the ring.

1 Introduction and overview

1.1 Bead configurations and occupation processes

Let $Z_n := \mathbb{Z}/n\mathbb{Z}$ be the cyclic group with $n$ elements, and consider the semi-discrete torus $T_n := [0, 1) \times Z_n$. We think of $T_n$ as $n$ unit length strings running in parallel, such that string $h$ lies between $h-1$ and $h+1$ mod $n$. Throughout, for integers $h$, $[h]$ denotes its residue mod $n$. For real $t$, $[t]$ denotes its residue mod 1.

For $k \geq 1$, a bead configuration on $T_n$ is a collection of $nk$ distinct points on $T_n$ such that there are $k$ points on each string, and the $k$ points on neighbouring strings interlace. More specifically, if $t_1 < \ldots < t_k$ are the points on string $h$, and $t'_1 < \ldots < t'_k$ are the points on string $[h+1]$, then we have either

$$t_1 < t'_1 < t_2 < t'_2 < \ldots < t_k < t'_k \quad \text{or} \quad t'_1 < t_1 < t'_2 < t_2 < \ldots < t'_k < t_k.$$ (1.1)

Put simply, between every two beads on one string there is a bead on the neighbouring string. See Figure 1 for a depiction of a bead configuration on $T_n$.

In addition to $n$ (the number of strings) and $k$ (the number of beads per string), an important quantity governing the behaviour of bead configurations is the so-called tilt of a configuration. Before the provision of a precise definition in Section 2, here we will give an informal description and diagram. Let us begin by noting that in a bead configuration with $k$ beads per string, since the strings have unit length, the average distance (across all $nk$ beads in the configuration) to the next bead on the same string is given by $1/k$. Consider now the average distance (again across all beads in the configuration) to the next bead on the string above. By the interlacing property, this quantity is clearly less than $1/k$. We write $\tau$ for this quantity divided by $1/k$. See Figure 2.

2020 Mathematics subject classification. Primary: 82B20, 82B21, 60K35. Secondary: 60J27

Key words and phrases. Bead configuration, interlacing, Kasteleyn theory, Free Probability, free energy, surface tension, Gelfand-Tsetlin pattern, TASEP.
Figure 1: A bead configuration on \( n = 5 \) strings with \( k = 3 \) beads per string. The tilt of this configuration is \( \tau = 2/5 \).

Figure 2: In a bead configuration on \( \mathbb{T}_n \) with \( k \) beads per string, if \((t, h)\) is the location of a uniformly chosen bead (of the \( nk \) possible), the average distance to the next bead on string \( h \) is \( \frac{1}{k} \), whereas the average distance to the next bead to the right of \( t \) on string \( h + 1 \) is \( \tau \frac{1}{k} \) for some \( \tau \in (0, 1) \). We call \( \tau \) the tilt of the configuration.

It is not obvious at this stage (though we hope to make it clear in Section 5), but owing to the toric structure of the configuration, the tilt must take the form

\[
\tau := \frac{\ell}{n}
\]

for some integer \( 1 \leq \ell \leq n - 1 \). We call this integer \( \ell \) the occupation number of the configuration, and now define our central object of study:

**Definition 1.1** \((n, k, \ell)\) configuration. For \( n, k \geq 1 \) and \( 1 \leq \ell \leq n - 1 \), a \((n, k, \ell)\) configuration is a bead configuration on \( \mathbb{T}_n \) with \( k \) beads per string and tilt \( \tau = \ell/n \).

Figure 3: An \((n, k, \ell)\) configuration may be associated with an element \((t_{j,h})_{1 \leq j \leq k, 0 \leq h \leq n-1} \in [0, 1)^{nk}\).

The set of \((n, k, \ell)\) configurations may be regarded as the subset

\[
\mathcal{W}^{(n)}_{k,\ell} := \{(t_{j,h})_{1 \leq j \leq k, 0 \leq h \leq n-1} \in [0, 1)^{nk} : (n, k, \ell) \text{ configuration}\}
\]
of \([0,1)^nk\) by letting \(t_{h,j}\) denote the position of the \(j^{th}\) bead on string \(h\); see Figure 3. Consequently, we can speak of the \(nk\)-dimensional Lebesgue measure of \(W_{k,\ell}^{(n)}\) and write

\[
\text{Vol}_{k,\ell}^{(n)} := \text{Volume of } (n, k, \ell) \text{ configurations} := \text{Leb}_{nk}(W_{k,\ell}^{(n)}).
\] (1.2)

With these definitions at hand, in the remainder of the introduction we overview our results on bead configurations, which fall into three categories: continuous Kasteleyn theory (our method), volumetric properties of bead configurations, and probabilistic properties of bead configurations.

### 1.2 Continuous Kasteleyn Theory

In past work, bead configurations have been studied by numerous authors such as Boutillier [7], Sun [45], Gordenko [17] and Fleming et al. [13] using scaling limits of discrete models in integrable probability such as lozenge tilings and Young diagrams. In this paper, we develop a direct approach to their study using a continuous version of Kasteleyn theory. This direct approach is markedly more efficient, and sidesteps the technical and computational problems of dealing with scaling limits.

Outlining our analogue (Theorem 2.2) of Kasteleyn’s theorem here, if \(\text{Vol}_{k,\ell}^{(n)}\) is the volume of the set of bead processes on \(\mathbb{T}_n\) with \(k\) beads per string and tilt \(\ell/n\), then for complex parameters \(T\) and \(\lambda\) we have

\[
Z_n(\lambda, T) := \sum_{k \geq 0} \sum_{0 \leq \ell \leq n} T^{nk} e^{-\lambda \ell} \text{Vol}_{k,\ell}^{(n)} = \sum_{\theta \in \{0,1\}^2} c_{\theta}^{\lambda,n} \det(I + TC^{\theta_2}_{\lambda+\theta_1\pi i}),
\] (1.3)

where \(c_{\theta}^{\lambda,n}\) are complex numbers and for \(\theta_2 \in \{0,1\}\) and \(\beta \in \mathbb{C}\), \(\det(I + TC^{\theta_2}_{\lambda+\theta_1\pi i})\) is the Fredholm determinant ((2.8)) of the integral operator \(C^{\theta_2}_{\beta} : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{C}\) given by

\[
C^{\theta_2}_{\beta}((t,h),(t',h')) = e^{\theta_2 \pi i/n} 1_{\{h'=[h+1]\}} \frac{e^{-\beta|t'-t|}}{1-e^{-\beta}}.
\]

We then diagonalise the operators \(C^{\theta_2}_{\beta}\), leading to an explicit formula for the partition function. The formulation of \(Z_n(\lambda, T)\) as a sum of Fredholm determinants forms the bedrock of our volumetric and probabilistic work in the remainder of the article: from the partition function we can read off the coefficients to obtain a volume formula \(\text{Vol}_{k,\ell}^{(n)}\) for bead configurations, and thereafter we can utilise the structure of Fredholm determinants to study the correlations. In Section 2 we discuss our continuous Kasteleyn theory in full.

### 1.3 Volumetric properties of bead configurations

In Section 3 we state our volumetric formula for bead configurations, and connect our result with recent progress in free probability. Our chief result here is Theorem 3.1, which states that the volume of the set of \((n, k, \ell)\) configurations is given by

\[
\text{Vol}_{k,\ell}^{(n)} = (-1)^{k(\ell+1)} \sum_{S \subseteq \{w^n=1\}, \#S=\ell} \left( \sum_{w \in S} w \right)^{nk},
\] (1.4)

where the outer sum is taken over all subsets of the \(n^{th}\) roots of unity with cardinality \(\ell\).

We then use this formula to show that the free energy of the bead model with gap \(g\) and tilt \(\tau\) is given by

\[
-\sigma(g, \tau) := \log g + \log \sin(\pi \tau) + 1 - \log \pi.
\] (1.5)
Broadly speaking, this means that in a large bead configuration where the average between consecutive beads on
the same string is \( g \), and the tilt of the configuration is \( \tau \), then the volume of such a configuration grows like
\( e^{-(1+o(1))\sigma(g,\tau)M} \) if the number \( M \) of beads tends to infinity.

The formula (1.5) meets a recent prediction made by Shlyakhtenko and Tao [41] based on formal calculations
in free probability. We discuss in Section 3 how the formula (1.5) for the free energy of the bead model governs a
variational problem connected with various objects of interest in free probability, and make a conjecture on the large
deviation behaviour of Gelfand-Tsetlin patterns.

1.4 Probabilistic aspects of bead processes and exclusion processes on the ring

Finally, in Section 4 we study random bead configurations on \( \mathbb{T}_n \). In short, we define a probability measure \( P^{\lambda,T}_n \)
on bead configurations on \( \mathbb{T}_n \) by choosing a configuration according to its contribution to the partition function
\( Z_n(\lambda,T) \) (1.3).

An integral aspect to our approach to bead configurations is the observation that each bead configuration on \( \mathbb{T}_n \)
gives rise to a time-periodic exclusion process \( (X_t)_{t\in[0,1]} \) taking values in set of subsets of \( \mathbb{Z}_n \), which we call the
occupation process. Giving just a brief verbal definition here (a more careful definition is supplied in Section 5),
for each bead in the configuration at a location \( (t,h) \), draw a thick line travelling to the right from \( (t,h+1) \), and
continue until a bead is hit on string \( h+1 \). We may define a piecewise constant process \( (X_t)_{t\in[0,1]} \) by letting \( X_t \) be
the subset of \( \mathbb{Z}_n \) consisting of \( h \) in \( \mathbb{Z}_n \) such that \( (t,h) \) is occupied by a thick line. The right panel of Figure 4 depicts
the occupation process associated with the bead configuration in the left panel.

![Figure 4](image_url)

Figure 4: On the left we have a bead configuration on \( n = 5 \) strings with \( k = 3 \) beads per string. On the
right we have its associated occupation process, which has occupation number \( \ell = 2 \).

We utilise the structure of Fredholm determinants in the partition function \( Z_n(\lambda,T) \) to show that under \( P^{\lambda,T}_n \),
the locations of the random beads have a determinantal structure. In fact, by developing a continuous analogue of
the complementation principle [6] in determinantal processes, we show that the beads and the associated occupation
process have a joint determinantal structure under \( P^{\lambda,T}_n \), by which we mean the mixed probabilities such as

\[
P^{\lambda,T}_n((dt_1,h_1) \text{ contains a bead}, h_2 \in X_{t_2}, h_3 \notin X_{t_3})
\]

have tractable representations in terms of determinants of certain operators on \( \mathbb{T}_n \).

![Figure 5](image_url)

Figure 5: A depiction of \( X_t \) at a snapshot in time \( t \) in \( \mathbb{R} \).
Thereafter, under a scaling limit of these probability measures for bead configurations on $\mathbb{T}_n$, we obtain a collection of probability measures $P_{n,\ell}$ recently discovered by Gordonko [17] for bead configurations on $\mathbb{R} \times \mathbb{Z}_n$. Alternatively, $P_{n,\ell}$ governs a Markov process $(X_t)_{t \in \mathbb{R}}$ taking values in the set of cardinality $\ell$ subsets of $\mathbb{Z}_n$, with dynamics similar to TASEP.

Our results on the joint determinantal structure of $P_{n,T}^\lambda$ carry through to the limit to $P_{n,\ell}$, and we use this structure to compute the transition rates and stationary distribution of the chain $(X_t)_{t \in \mathbb{R}}$ under $P_{n,\ell}$. Our work in this section culminates in Theorem 4.6, which states that the law of TASEP on the ring may be recovered as an exponential martingale from $P_{n,\ell}$, i.e.

$$\frac{dP_{n,\ell}^{\text{TASEP}}}{dP_{n,\ell}} \bigg| \mathcal{F}_t = \frac{\Delta(X_0)}{\Delta(X_t)} \exp \left\{ \int_0^t \left( \text{Traffic}(X_s) - \mu_{n,\ell} \right) ds \right\},$$

(1.7)

where $\mu_{n,\ell}$ is a constant, $\text{Traffic}(X_s) := \# \{ h \in X_s : [h + 1] \text{ is also in } X_s \}$ counts the number of particles ‘waiting in traffic’ at time $s$, $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of $(X_t)_{t \geq 0}$, and

$$\Delta(E) := \prod_{1 \leq j < k \leq \ell} \left| e^{2\pi i h_k/n} - e^{2\pi i h_j/n} \right| \quad E = \{ h_1, \ldots, h_{\ell} \}$$

is a circular variant of the Vandermonde determinant.

The transition probabilities of TASEP on $\mathbb{Z}_n$ (and on other sets for that matter) are notoriously difficult to describe, with the best formulas to date appearing in the groundbreaking recent article of Baik and Liu [4]. These formulas are highly complicated, involving multiple determinants and contour integrals. Since $P_{n,\ell}$ has fairly simple transition probabilities, the change of measure (1.7) provides an alternative probabilistic representation for the transition probabilities of TASEP on the ring.

### 1.5 Overview

We now overview the paper. As mentioned above, in Sections 2, 3 and 4 we state our main results:

- In Section 2 we state our results on continuous Kasteleyn theory. In Section 3 we state our volume formulas for bead configurations, and make connections with free probability. In Section 4 we state our results on correlations for random bead configurations, making connections with exclusion processes on the ring.

The remaining sections are dedicated to proving the results stated in Sections 2, 3 and 4:

- In Section 5 we develop our continuous analogue of Kasteleyn theory, and proving the results stated in Section 2. In Section 6 we use our work in Section 2 to prove our volume formula for bead configurations. In Section 7 we use standard techniques from determinantal processes to initiate our work on correlations for random bead configurations. In Section 8 we initiate a new differential version of the complementation principle for determinantal processes, allowing us to prove that random bead configurations on the torus and their associated occupation processes have a mixed determinantal structure. Finally, in Section 9 we study scaling limits of our bead configurations on the torus, establishing connections between random bead configurations on $\mathbb{R} \times \mathbb{Z}_n$ and TASEP on the ring.

### 2 Continuous Kasteleyn theory: main results

Our first main contribution is a general framework for a continuous analogue of Kasteleyn theory, which may be defined as the art of writing partition functions in terms of determinants. The benefits of these determinantal expressions are twofold: firstly, these operators may be diagonalised, often leading to explicit formulas for the partition
functions in question. Secondly, these operators may be inverted, providing explicit formulas for the correlation functions of probability measures associated with these partition functions. Kasteleyn theory has been an extremely fruitful tool in the last half century for the study of various discrete processes in integrable probability, including lozenge tilings, domino tilings, and random matchings in general.

2.1 A focal problem

Recalling our definitions from Section 1.1, consider, starting from scratch, the following problem.

**Problem 2.1.** Compute the volume of the set $\mathcal{W}_{k,\ell}^{(n)} \subset [0,1]^{nk}$ of $(n, k, \ell)$ configurations, i.e. bead configurations on $n$ unit length strings with $k$ beads per string and tilt $\ell/n$.

The reader may alternatively wish to alternative (and initially more natural) problem of computing the volume of bead configurations on $\mathbb{T}_n$ with $k$ beads per string and *any* tilt. Either way, aside for some basic cases where one of $n, k$ or $\ell$ is small, Problem 2.1 seems impossible to solve using elementary means, say by an inductive method. We therefore require a more sophisticated technology and to this end, in the next section we take a brief interlude, drawing inspiration from a discrete analogue of our problem of computing the volume of $\mathcal{W}_{k,\ell}^{(n)}$.

2.2 An interlude: Domino tilings

Consider the following fundamental problem in integrable probability. For even $n$, how many ways are there of tiling an $n \times n$ chessboard with $2 \times 1$ dominoes?

![Image of a 6x6 chessboard with 2x1 dominoes]

Figure 6: How many ways are there of tiling a $6 \times 6$ chessboard with $2 \times 1$ dominoes?

Like its continuous cousin Problem 2.1, the problem of counting the number of ways of tiling the $n \times n$ chessboard $S_n := \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ with $2 \times 1$ dominoes seems impossible to approach via elementary means, with elementary and inductive arguments falling short due to complicated interdependencies between the positions of the dominoes.

In 1961, Kasteleyn [25] came up with an ingenious method for attacking this problem: Declare a square $(m, n)$ of the chessboard to be black if $m + n = 0 \mod 2$, and white otherwise. Every domino of the chessboard then covers a black square and a white square. Consider now taking a bijection $\sigma : S_n \to S_n$ of the chessboard mapping squares to neighbouring squares (i.e. $\sigma(x) = x \in \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$). Kasteleyn noted that given such a bijection, we may associate two different tilings of the chessboard: one from following $\sigma$ from black squares to white, the other from following $\sigma$ from white squares to black. With an intelligently chosen collection of weights for the adjacency operator on the chessboard (such a weighting is known as a Kasteleyn weighting) one can counteract the differences in sign for various bijections of $S_n$, and accordingly, show that the square of the number of domino tilings for the chessboard is equal to the determinant of the weighted adjacency operator. Contemporaneously to
Temperly and Fisher [47], Kasteleyn proved the following remarkable formula

\[
\# \{\text{Domino tilings of the } n \times n \text{ square}\} = \prod_{j=1}^{n/2} \prod_{k=1}^{n/2} \left( 4 \cos^2 \left( \frac{\pi j}{n+1} \right) + 4 \cos^2 \left( \frac{\pi k}{n+1} \right) \right).
\] (2.1)

for the number of tilings of the chessboard. We refer the reader to the opening few pages of Kenyon [26] for a more detailed exposition of this problem.

### 2.3 Continuous Kasteleyn theory

Our approach to tackling Problem 2.1 is using a continuous version of the technique used by Kasteleyn to count the number of domino tilings of a square. It turns out to be fruitful to package the volumes \( \text{Vol}^{(n)}_{k,\ell} \) of the set of \((n, k, \ell)\) configurations in terms of a generating function. To this end, for \( \lambda \in \mathbb{C} \), we define

\[
g^\lambda_N(y_1, \ldots, y_N) := \begin{cases} e^{-\lambda \ell} & \text{if for some } k, \ell, N = nk, \text{ and } y_1, \ldots, y_{nk} \text{ is a } (n, k, \ell) \text{ config.} \\ 0 & \text{otherwise.} \end{cases}
\] (2.2)

We clarify that if \((y_1, \ldots, y_{nk})\) is a bead configuration, in our construction so is \((y_{\sigma(1)}, \ldots, y_{\sigma(nk)})\) for any permutation \(\sigma\) of the indices, and \(g^\lambda_N(y_1, \ldots, y_N)\) is always zero when \(N\) is not a multiple of \(k\). For complex parameters \(\lambda\) and \(T\) we then define the partition function

\[
Z_n(\lambda, T) := \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}_n^N} g^\lambda_N(y_1, \ldots, y_N) dy_1 \cdots dy_N = \sum_{k \geq 0} \sum_{0 \leq \ell \leq n} T^{nk} e^{-\lambda \ell} \text{Vol}^{(n)}_{k,\ell},
\] (2.3)

where the latter equality in (2.3) follows from the respective definitions (1.2) and (2.2) of \( \text{Vol}^{(n)}_{k,\ell} \) and \( g^\lambda_N \). It turns out that \( g^\lambda_N \) may be decomposed as a linear combination of functions which have a compliant determinantal structure. For \( \theta \in \{0, 1\}^2 \) we define the variant \( g^\lambda_{\theta} \) of \( g^\lambda_N \) by

\[
g^\lambda_{\theta}(y_1, \ldots, y_N) := \begin{cases} \frac{1}{2}(-1)^{(\theta_1+k+1)(\theta_2+n+\ell+1)} e^{-\lambda \ell} & \text{if } N = nk, \text{ and } y_1, \ldots, y_{nk} \text{ is a } (n, k, \ell) \text{ config.} \\ 0 & \text{otherwise.} \end{cases}
\] (2.4)

In Section 5.1 we show that

\[
g^\lambda_N = \sum_{\theta \in \{0, 1\}^2} g^\lambda_{\theta}.
\] (2.5)

The value in introducing such functions lies in the fact that they may be expressed in terms of determinants of certain operators on the torus. The following is the main result of Section 2:

**Theorem 2.2.** For \( N \geq 0 \) and points \((y_1, \ldots, y_N)\) of \( \mathbb{T}_n \) we have

\[
g^\lambda_{\theta}(y_1, \ldots, y_N) = \frac{1}{2}(-1)^{(\theta_1+1)(\theta_2+n+1)} (1 - e^{-(\lambda+\pi \theta_1)\ell})^n \det_{i,j=1}^{N} C_{\lambda+\theta_1,\pi}(y_i, y_j),
\] (2.6)

where for nonzero complex \( \beta \) and \( \theta_2 \in \{0, 1\} \), the operator \( C_{\beta}^{\theta_2} : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{C} \) is given by

\[
C^{\theta_2}_{\beta}( (t, h), (t', h') ) = e^{\theta_2 \pi i / n} \frac{1}{1-e^{-\beta}} e^{-\beta |t'-t|},
\] (2.7)

where \([t' - t] \) and \([h + 1]\) are \( t' - t \mod 1 \) and \( h + 1 \mod n \).
We remark that when $\lambda$ is a positive real number and $\theta = (\theta_1, \theta_2) = (0, 0)$, we have the operator $C_{1,\lambda}$, which is simply the transition operator of a discrete-time Markov chain on $\mathbb{T}_n := [0, 1) \times \mathbb{Z}_n$ with the following dynamics: if the location of the chain is $(h, t)$ at time $m$, then at time $m + 1$ the location is $([h + 1], [t + W])$ where $W$ is exponentially distributed with parameter $\lambda$. For nonzero $\theta$, the operators $C_{\lambda+\theta_1,\pi}$ are somehow complex analogues of this Markov transition operator.

Now we define the Fredholm determinant of an operator $C : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{C}$ to be

$$\det(I + TC) := \sum_{N=0}^{\infty} \frac{T^N}{N!} \int_{\mathbb{T}_n} \det C(y_i, y_j) dy_1 \ldots dy_N. \quad (2.8)$$

Summing (2.6) against $T^N/N!$, we see that

$$Z_{n,\theta}(\lambda, T) := \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}_n^N} g_N^\lambda(y_1, \ldots, y_N) dy_1 \ldots dy_N
= \frac{1}{2}(-1)^{(\theta_1+1)(\theta_2+n+1)}(1-e^{-\lambda+\pi i})^n \det(I + TC_{\lambda+\theta_1,\pi}). \quad (2.9)$$

Using (2.5) and the definition of $Z_n(\lambda, T)$, we see that $Z_n(\lambda, T)$ may be written as a linear combination of four Fredholm determinants:

$$Z_n(\lambda, T) = \frac{1}{2} \sum_{\theta \in \{0,1\}^2} (-1)^{(\theta_1+1)(\theta_2+n+1)}(1-e^{-\lambda+\pi i})^n \det(I + TC_{\lambda+\theta_1,\pi}). \quad (2.10)$$

This recovers the equation (1.3) stated in the introduction.

Considered as integral operators on $L^2(\mathbb{T}_n)$, the operators $C_{\lambda+\theta_1,\pi}$ (and accordingly, the operators $I + TC_{\lambda+\theta_1,\pi}$) have an orthonormal basis of eigenfunctions, and hence we can diagonalise them to compute the Fredholm determinants explicitly. Ultimately, we prove the following:

**Theorem 2.3.** Let $V_{\alpha, k}\lambda, T\ell$ denote the volume of the set of bead configurations on $n$ unit length strings with $k$ beads per string and occupation number $\ell$. Then

$$Z_n(\lambda, T) := \sum_{k \geq 0}^n T^k e^{-\lambda T} V_{\alpha, k}\lambda, T\ell = \frac{1}{2} \sum_{\theta \in \{0,1\}^2} (-1)^{(\theta_1+1)(\theta_2+n+1)} \prod_{w^n=(-1)^\theta_2} (e^{Tw} - (-1)^{\theta_1} e^{-\lambda}). \quad (2.11)$$

The beady-eyed reader might note that we already stipulated that with every bead configuration the associated occupation number $\ell$ takes values in $\{1, \ldots , n-1\}$, where as the sum in (2.11) is over $0 \leq \ell \leq n$. For reasons that will become apparent in the sequel, it turns out that strictly speaking, (2.11) holds with the convention $W_{0,\ell}^{\lambda, T, \ell} := \binom{n}{\ell}$.

In any case, Theorems 2.2 and 2.3 provide the foundation for our work in the remainder of the article.

### 2.4 A discrete alternative to the continuous Kasteleyn theory approach

We take a moment to note that the results of the current article were born out of more standard (and less efficient) methods, using scaling limits of discrete form of the dimer model. We record this alternative approach in a companion paper: [22]. In this companion paper, we study a dimer model on the discrete torus, which is essentially a form of lozenge tiling ([18]) translated into a favourable coordinate system. In [22], we think of lozenge tilings as a random permutation $\sigma$ on the discrete torus $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ with the property that for all $x = (x_1, x_2) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ we have $\sigma(x) = x + e_1$ or $\sigma(x) + e_2 \text{ mod } (m_1, m_2)$. Here $e_1 = (1,0)$ and $e_2 = (0,1)$. We call such a $\sigma$ a dimer matching. See Figure 7 for a depiction of such a matching.

These random permutations on $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ may be thought of as discrete analogues of bead configurations on the semi-discrete torus $\mathbb{T}_n := [0, 1) \times \mathbb{Z}_n$. Indeed, given a permutation $\sigma : \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \to \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$, we can label the squares of $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ according to the following rule
Figure 7: A dimer matching on $\mathbb{T}_m$ with $m_1 = 12$ and $m_2 = 5$. The squares containing a hollow (resp. solid) circle correspond to the $x \in \mathbb{T}_m$ for which $\sigma(x) = x + e_2$ (resp. $\sigma(x) = x + e_1$). The empty squares are fixed by $\sigma$.

- $x$ is white if $\sigma(x) = x$.
- $x$ has a solid circle if $\sigma(x) = x + e_1$.
- $x$ has a hollow circle if $\sigma(x) = x + e_2$.

The hollow circles in $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ may be thought of as discrete analogues of beads, with the squares containing solid circles forming an occupation process.

Using standard techniques of Kasteleyn theory (see, e.g. [48, Section 2]), in [22] it is shown that the partition function

$$Z := \sum_{\sigma} \alpha^{\#\{x: \sigma(x) = x\}} \beta^{\#\{x: \sigma(x) = e_1\}} \gamma^{\#\{x: \sigma(x) = e_2\}}$$

can be written as a sum of four determinants, and thereafter show that a probability measure associated with these weightings has a determinantal structure. In the remainder of the companion paper, we exploit the scaling limit $m_2 \to \infty$, $\gamma = T/m_2$ to recover the bead process on the torus. This scaling limit highly delicate, and we meet significant both computational and technical challenges in proving Theorem 2.3 and Theorem 4.2 via this route. In fact, it takes 27 pages in [22] (and at a brisk pace at that) to accomplish what is done in the present article in 16. We hope the reader will regard this as a testament to the suitability of the continuous analogue of Kasteleyn theory.

3 Volumetric properties of bead configurations and free probability

3.1 Volume formulas for bead configurations

We recall Definition 1.1 and the following discussion about relating the set of $(n, k, \ell)$ configurations to a subset of $[0, 1)^{nk}$ of volume $\text{Vol}_{k,\ell}^{(n)}$. As a straightforward consequence of Theorem 2.3 we are able to obtain the following explicit formula for these volumes:

**Theorem 3.1.** Let $\text{Vol}_{k,\ell}^{(n)}$ denote the volume of the subset of $[0, 1)^{nk}$ consisting of $(n, k, \ell)$ configurations, i.e. bead configurations on $\mathbb{T}_n$ with $k$ beads per string and tilt $\ell/n$. Then

$$\text{Vol}_{k,\ell}^{(n)} = (-1)^{k(\ell+1)} \sum_{S \subseteq \{w^n = 1\}, \#S = \ell} \left( \sum_{w \in S} w \right)^{nk},$$

where the sum is taken over all subsets of the $n^{th}$ roots of unity with cardinality $\ell$. 


While the formula for $\text{Vol}_{k,\ell}^{(n)}$ may be somewhat opaque, it has amenable asymptotics. We remark that in an $(n, k, \ell)$ configuration, since there are $k$ beads on each string, the average distance between consecutive beads along one string is $1/k$, and consequently it is natural to consider the renormalised quantity $k^{nk}\text{Vol}_{k,\ell}^{(n)}$ measuring the volume of bead configurations on $n$ strings of length $k$ (rather than unit length), with $k$ beads per string and tilt $\ell/n$.

Let

$$\mathcal{P}(x) := \sum_{k \geq 1} \sum_{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k \geq 0} e^{-(\mu_1 + \ldots + \mu_k)x}$$

be the generating function of the integer partitions.

The following result says that the renormalised quantity $k^{nk}\text{Vol}_{k,\ell}^{(n)}$ grows exponentially in the number $(nk)$ of beads.

**Theorem 3.2.** Let $k = [pn]$ and $\ell = [\tau n]$. Then we have

$$\lim_{n \to \infty} k^{nk} \left( \frac{e \sin(\pi \ell/n)}{\pi} \right)^{-nk} \text{Vol}_{k,\ell}^{(n)} = \frac{e^{p\pi^2/6}}{\sqrt{2\pi p}} \mathcal{P}(e^{-pc^+}) \mathcal{P}(e^{-pc^-}),$$

where $c^\pm = 2\pi^2 \pm 2\pi^2 \frac{\cos(\pi \tau)}{\sin(\pi \tau)}$.

The exact form of the constant in (3.2) appears to be a curiosity at this stage, with the more important aspect of the formula being the characterisation of the exponential growth of the volume $\text{Vol}_{k,\ell}^{(n)}$ in the number $nk$ of beads. Indeed, taking very rough logarithmic asymptotics through Theorem 3.2, we obtain the following corollary characterising the free energy of the bead model:

**Corollary 3.3.** With $k$ growing linearly in $n$ (i.e. $k/n$ uniformly bounded away from zero and infinity), and $\ell = [\tau n]$, we have

$$\lim_{n \to \infty} \frac{1}{nk} \log \left\{ k^{nk} \text{Vol}_{k,\ell}^{(n)} \right\} = 1 + \log \sin(\pi \tau) - \log \pi.$$  

(3.3)

That is, the per-bead free energy of the unit-density bead model at tilt $\tau$ is $1 + \log \sin(\pi \tau) - \log \pi$.

Let us remark briefly that $\text{Vol}_{k,\ell}^{(n)}$ is the volume of toric bead configurations on $n$ strings with $k$ beads per string, where as — at least in the setting in which $n = k$ — the surface tension (i.e. minus the free energy) is usually formulated as a limiting measure of the volume of bead configurations with a fixed diamond-shaped boundary [15]. The author has verified in unpublished calculations that the two formulations lead to the same formula for the per-bead free energy.

In any case, as stated in the introduction, the asymptotic formula (3.3) agrees with several predictions made in the free probability and statistical physics literature. Based on formal calculations in free probability, Schlyakhtenko and Tao [41] conjectured the $1 + \log \sin(\pi \tau) - \log \pi$ formula explicitly. A less explicit prediction of the form $\log \sin(\pi \tau) + \text{Const.}$ was made earlier work by the author and Neil O’Connell [23] using a random-matrix-theoretic argument. Sun [45] gave a sketch argument for an analogous result in a different coordinate system using a scaling limit of the dimer model and an appeal to Legendre duality.

### 3.2 Free energy of bead configurations and free probability

We now explain in more detail the connection of the formula (3.3) with free probability. The story here begins with Gelfand-Tsetlin patterns, which are fundamental objects in representation theory and algebraic combinatorics. Gelfand-Tsetlin patterns are essentially bead configurations on $\mathbb{R} \times \{1, \ldots, n\}$ (i.e. $n$ infinite strings) with 1 bead on
the top string, 2 beads on the next string down, and so on, so that there are \( n \) beads on the bottom string. More precisely, a Gelfand-Tsetlin pattern is a collection of real numbers \((t_{k,j})_{1 \leq j \leq k \leq n}\) satisfying the inequalities \( t_{k+1,j} \leq t_{k,j} \leq t_{k+1,j+1} \) for all valid \( k, j \). See Figure 8a for a depiction of a Gelfand-Tsetlin pattern.

Consider the set \( \text{GT}(s_1, \ldots, s_n) := \{(t_{k,j})_{1 \leq j \leq k \leq n} \in \mathbb{R}^{n(n+1)/2} : t_{n,j} = s_j\} \) of Gelfand-Tsetlin patterns with a fixed bottom row. \( \text{GT}(s_1, \ldots, s_n) \) may be associated with a compact subset of \( \mathbb{R}^{n(n-1)/2} \). The compactness of \( \text{GT}(s_1, \ldots, s_n) \) may be seen from the fact that \( \text{GT}(s_1, \ldots, s_n) \subset [s_1, s_n]^{n(n-1)/2} \). It is a consequence of the Weyl dimension formula that the \( \frac{n(n-1)}{2} \)-dimensional Lebesgue measure of this set is given by

\[
\text{Vol} \text{GT}(s_1, \ldots, s_n) = \frac{\prod_{1 \leq j < k \leq n} (s_k - s_j)}{H(n)} \quad H(n) := \prod_{j=1}^{n-1} j!.
\]

One way in which Gelfand-Tsetlin patterns arise is in the eigenvalue processes of Hermitian matrices. Indeed, if \( A \) is an \( n \times n \) Hermitian matrix with eigenvalues \( s_1 \leq \ldots \leq s_n \), then the eigenvalues of its \( (n-1) \times (n-1) \) principle minor interlace those of \( A \) [46, Section 1.3]. Consequently, taking \( n \) strings and plotting the \( k \) eigenvalues of each \( k \times k \) principal minor on the \( k \)-th string up, we obtain a Gelfand-Tsetlin pattern \((t_{k,j})_{1 \leq j \leq k \leq n}\) from the matrix \( A \), which we call the eigenvalue process of \( A \). Here \( t_{k,j} \) is the \( j \)-th largest eigenvalue of the \( k \times k \) minor.

The connection with free probability begins with an observation often attributed to Baryshnikov [5], but most likely part of the folklore dating back to Weyl. Suppose now we take an \( N \times N \) Hermitian random matrix with eigenvalues given by \( s_1 \leq \ldots \leq s_N \). More specifically, let \( U \) be a Haar distributed unitary matrix of dimension \( N \), and consider the random matrix \( A := U \Lambda U^* \), where \( \Lambda \) is the diagonal matrix with entries \((s_1, \ldots, s_N)\). Baryshnikov’s observation states that the Gelfand-Tsetlin pattern associated with the eigenvalue process of \( A \) is uniformly distributed on the set \( \text{GT}(s_1, \ldots, s_N) \).

---

(a) A Gelfand-Tsetlin pattern

(b) A Gelfand-Tsetlin interface

Figure 8: On the left we have a Gelfand-Tsetlin pattern given by the eigenvalue process of a \( n \times n \) Hermitian matrix with \( n = 5 \). On the right we have the associated Gelfand-Tsetlin interface: the heights of the spikes along the \( k \)-th diagonal correspond to the positions of beads on the \( k \)-th string.

From the perspective of a probabilist, free probability is concerned with the asymptotic behaviour of eigenvalues of large random matrices under operations of addition, multiplication and taking minors. With the latter in mind, we are naturally interested in the macroscopic behaviour of uniformly chosen Gelfand-Tsetlin patterns with a fixed bottom row \((s_1, \ldots, s_N)\) approximating a certain measure \( \mu \) in the sense that \( \frac{1}{N} \sum_{i=1}^{N} \delta_{s_i} \approx \mu \). Free probability [51] gives an initial (if slightly obscure) way of describing the asymptotic shape of a Gelfand-Tsetlin pattern under these conditions: when \( N \) is large, for \( \alpha \in (0, 1) \), the empirical distribution of the beads on the \([\alpha N]\)-th row from the bottom of the pattern approximate a measure \( \mu_{\alpha} \) satisfying the free functional equation

\[
(1 - \alpha)\delta_0 + \mu_{\alpha} = ((1 - \alpha)\delta_0 + \alpha\delta_1) \boxtimes \mu,
\]

where \( \boxtimes \) is free multiplicative convolution [36]. See Metcalfe [32, Section 1.3] for further discussion on this front.
A more intuitive perspective may be gained by reformulating random Gelfand-Tsetlin patterns in terms of stochastic interfaces [15, 43], as was done in [23]. A two-dimensional stochastic interface is simply a random function \( \phi : \Gamma \to \mathbb{R} \) defined on a subset \( \Gamma \) of the two dimensional lattice \( \mathbb{Z}^2 \) and whose distribution is proportional to the exp\( \{ - \sum_{x,y} V(\phi(y) - \phi(x)) \} \), where \( V \) is a potential, and the sum is taken over all pairs of directed neighbouring edges in \( \Gamma \). Given a Gelfand-Tsetlin pattern \( (t_{k,j})_{1 \leq j \leq k \leq n} \), we can associate with this pattern a function \( \phi : \{ 0 \leq s \leq t \leq 1 \} \to \mathbb{R} \) by setting \( \phi(\frac{k}{n}, \frac{j}{k}) := t_{k,j} \), and linearly interpolating. A uniform Gelfand-Tsetlin pattern with a fixed bottom row is equivalent to a stochastic interface defined on a triangle in the lattice with a fixed diagonal and hard-core interaction potential \( V(x) = \infty 1_{x<0} \). With this construction we can reformulate our question about the behaviour of a large Gelfand-Tsetlin pattern with a fixed bottom row: given a fixed diagonal \( \Gamma \), was conjectured by Schlyakhtenko and Tao, namely that the cost of the compression interface \( \phi : \Delta \to \mathbb{R} \) is connected with volumetric properties of bead configurations. The authors of [41] studied laws of projections, or compressions, of free random variables, which are essential the measures \( \{ \mu_\alpha : \alpha \in [0,1] \} \) occurring in (3.5). The total mass of each measure \( \mu_\alpha \) is given by \( (1 - \alpha) \), and as such we may associate with \( \mu_\alpha \) the distribution function \( F_\alpha : \mathbb{R} \to [0,1-\alpha] \) given by \( F_\alpha(y) := \int_{-\infty}^y \mu_\alpha(dx) \). We now associate with \( \{ \mu_\alpha : \alpha \in [0,1] \} \) a compression interface \( \phi : \Delta \to \mathbb{R} \) with \( \Delta := \{ 0 \leq s \leq t \leq 1 \} \), where each diagonal \( \phi(\alpha + s, s) : s \in [0,1-\alpha] \) is a function of \( \mu_\alpha \) defined by setting
\[
\phi(\alpha + s, s) := F_\alpha^{-1}(s),
\]
where, in the case of a jump in the support of \( \mu_\alpha \), we take the right-continuous inverse of \( F_\alpha \). In other words, \( \phi(\alpha + s, s) \) is the \( s \)th quantile of the measure \( \mu_\alpha \). In particular, write \( \rho_\mu(s) := \phi(s,s) \) for the inverse of the distribution function of \( \mu \). Based on the understanding that the compression is a free analogue of taking minors, the resulting compression interfaces may be understood as hydrodynamic limits of Gelfand-Tsetlin patterns.

Schlyakhtenko and Tao [41] showed in formal calculations that these compression interfaces satisfy a certain partial differential equation, which in turn characterises these limit interfaces as the minimisers of a certain variational problem. To describe this variational formulation, define
\[
\sigma(g, \tau) := -\log g - \log \sin(\pi \tau) - 1 + \log \pi. \tag{3.6}
\]
Given a function \( \phi : \Delta \to \mathbb{R} \), we define \( \sigma(\nabla \phi) = \sigma(g^\phi, \tau^\phi) \) and \( g^\phi, \tau^\phi : \Delta \to \mathbb{R} \) are functions of the gradient of \( g \) given by
\[
g^\phi := \frac{\partial_s \phi + \partial_t \phi}{\sqrt{2}} \quad \text{and} \quad \tau^\phi := \frac{\partial_s \phi}{\partial_s \phi + \partial_t \phi}
\]
are respectively the gap (which is simply the divergence up to a scalar) and tilt of \( \phi \). We note that \( \sigma(\nabla \phi) \) is symmetric in \( \partial_s \phi \) and \( \partial_t \phi \).

The variational formulation in [41] states that the compression interface \( \phi : \Delta \to \mathbb{R} \) minimise the energy integral
\[
\mathcal{I}[\phi] := \int_\Delta \sigma(\nabla \phi) ds dt
\]
subject to the boundary condition \( \phi(s,s) = \rho_\mu(s) \).

The reader should compare at this stage (3.6), the surface tension minimised by the compression interface \( \phi : \Delta \to \mathbb{R} \), with the free energy of the bead model computed in Corollary 3.3. The agreement of the two formulas was conjectured by Schlyakhtenko and Tao, namely that the cost of the compression interface \( \phi : \Delta \to \mathbb{R} \) to lie at a certain gradient is due to macroscopic volumetric properties of bead configurations.
Translating this variational formulation back into the terminology of Gelfand-Tsetlin patterns, this means that for a non-decreasing function \( \rho : [0, 1] \to \mathbb{R} \) that if one were to take a large random Gelfand-Tsetlin pattern with a fixed bottom row at positions 
\[
    s_1 := \rho(1/N), \ldots, s_N := \rho(N/N),
\]
then one may expect the position \( t_{k,j} \) of the \( j \)th bead on the \( k \)th string to be approximately \( \phi(k/N, j/N) \), where \( \phi \) minimises \( \mathcal{I}[\phi] \) subject to the \( \phi(s, s) = \rho(s) \).

Let us relate our discussion to free entropy, which was introduced and studied by Voiculescu in a series of papers commencing with [50]. As above, let \( \rho_\mu : [0, 1] \to \mathbb{R} \) be the inverse of the distribution function of \( \mu \). Then, by taking logarithmic asymptotics of the Weyl dimension formula (3.4), we define the free energy of a measure \( \mu \) by setting
\[
    \mathcal{F}[\mu] := \lim_{N \to \infty} \frac{1}{N^2} \log GT(N\rho_\mu(1/N), N\rho_\mu(2/N), \ldots, N\rho_\mu(N/N)) = \int_{\mathbb{R} \times \mathbb{R}} \log |t - s| \mu(dt) \mu(ds) + \frac{3}{4},
\]
where we used the fact that \( \lim_{N \to \infty} \frac{1}{N^2} \log H(N) = \frac{1}{2} \log N - \frac{3}{4} + o(1) \) [23]. According to our heuristics here, it seems that if \( \phi^* \) is the minimiser of \( \mathcal{I}[\phi] \) given the fixed boundary condition \( \rho_\mu \) along the diagonal then we should have
\[
    \mathcal{I}[\phi^*] = \mathcal{F}[\mu].
\]

While the works of Voiculescu [51] and Schlyakhtenko and Tao [41] give a variational characterisation of the typical behaviour of a large Gelfand-Tsetlin pattern, we now make a conjecture expanding this idea in the direction of large deviations.

**Conjecture 3.4.** Let \( \mu \) be a probability measure, and let \( \rho_\mu : [0, 1] \to \mathbb{R} \) denote the right-continuous inverse function of \( F_\mu(s) := \int_{-\infty}^s \mu(dx) \). Let \( \phi_N : \Delta \to \mathbb{R} \) be the (linear interpolation of) the stochastic interface associated with a uniformly chosen Gelfand-Tsetlin pattern with bottom row \( \rho_\mu(1/N), \ldots, \rho(N/N) \). Then under a suitable topology the random function \( \phi_N \) satisfies a large deviation principle with speed \( N^2 \) and rate function
\[
    \mathcal{R}_\mu[\phi] = \begin{cases} 
    \int_\Delta \tilde{\sigma}(\nabla \phi) ds dt - \int_{\mathbb{R} \times \mathbb{R}} \log |t - s| \mu(dt) \mu(ds) + \frac{3}{4}, & \phi \in C^1(\hat{\Delta}) \text{ with } \phi(s, s) = \rho_\mu(s), \\
    +\infty, & \text{otherwise},
    \end{cases}
\]
Here \( \hat{\Delta} \) denotes the interior of \( \Delta \) and \( \tilde{\sigma}(\nabla \phi) \) is the surface tension defined below (3.6).

We close this section by touching on some related literature. In the pair of papers [10, 11], Metcalfe and Duse study the asymptotic shapes of discrete Gelfand-Tsetlin patterns, identifying the local correlations in different regions of the pattern.

Perhaps surprisingly, a connected subject of investigation is the asymptotic behaviour of zeroes of polynomials under repeated differentiation. Here we make a simple observation: if one takes a polynomial with \( n \) real roots, then the \( n - 1 \) roots of its first derivative interlace those of the original polynomial [42]. If one studies the asymptotic distribution of the zeroes of the \( \lceil \alpha N \rceil \)th derivative of a polynomial with \( N \) real roots approximating a certain measure \( \mu \), then the roots of this derivative are asymptotically distributed according to the measure \( \mu_{\alpha} \) appearing in the previous section. The flow of polynomial roots under repeated differentiation is an extremely active area of research [20, 24, 37, 44], as is the related emerging field of finite free probability [2, 19, 31], which Marcus, Spielman and Srivastava have exploited to prove spectacular results in other areas of mathematics [29, 30].
4 Probabilistic aspects of bead configurations and interacting particle systems on $\mathbb{Z}_n$

For $\lambda \in \mathbb{R}$, recall the functions $g_N^\lambda$ and the associated partition function $Z_n(\lambda, T)$ introduced in Section 2. We now define a probability measure $P_n^{\lambda,T}$ on bead configurations by setting

$$P_n^{\lambda,T}(\text{There are } nk \text{ beads, they have locations in } dy_1, \ldots, dy_{nk}) := \frac{T^{nk}}{(nk)!} g_n^\lambda(y_1, \ldots, y_{nk}) \frac{Z_n(\lambda, T)}{Z_n(\lambda, T)} dy_1 \ldots dy_{nk}.$$ 

It turns out to be profitable to decompose $P_n^{\lambda,T}$ as an affine combination of signed measures. (We recall that a signed measure is a countably additive real-valued function on a sigma algebra.) Indeed, for $\theta \in \{0, 1\}^2$ we define a signed measure $P_n^{\lambda,\theta,T}$ on bead configurations on $\mathbb{T}_n$ by setting

$$P_n^{\lambda,\theta,T}(\text{There are } nk \text{ beads, they have locations in } dy_1, \ldots, dy_{nk}) := \frac{T^{nk}}{(nk)!} g_n^{\lambda,\theta}(y_1, \ldots, y_{nk}) \frac{Z_n(\theta, \lambda, T)}{Z_n(\theta, \lambda, T)} dy_1 \ldots dy_{nk},$$

where $Z_n(\theta, \lambda, T)$ is a normalising factor so that $P_n^{\lambda,\theta,T}$ has unit total mass. According to (2.5), we can then write

$$P_n^{\lambda,T} = \sum_{\theta \in \{0, 1\}^2} \frac{Z_n(\theta, \lambda, T)}{Z_n(\lambda, T)} P_n^{\lambda,\theta,T}. \quad (4.1)$$

This decomposition may seem somewhat artificial at first, but the value lies in the fact that each $P_n^{\lambda,\theta,T}$ has a highly tractable determinantal structure which we now describe.

As noted in the introduction, given a random bead configuration on $\mathbb{T}_n$ distributed according to $P_n^{\lambda,T}$ or $P_n^{\lambda,\theta,T}$, we have a stochastic occupation process $(X_t)_{t \in [0, 1]}$ taking values in the set of subsets of $\mathbb{Z}_n$ with a fixed (random) cardinality $\ell$.

With a view to expressing probabilities of the form (1.6), we have the following definition.

**Definition 4.1.** Let $(y_i)_{i \in B \cup O \cup U}$ be distinct points of $\mathbb{T}_n$ indexed by disjoints sets $B$, $O$ and $U$. Given a bead configuration and its associated occupation process, let $\Gamma(y_i : i \in B \cup O \cup U)$ be the event that

- Each $dy_i$ with $i \in B$ contains a bead.
- Each $y_i$ with $i \in O$ is occupied.
- Each $y_i$ with $i \in U$ is unoccupied.

In Section 8 we develop a differential version of the complementation principle [6] for discrete determinantal processes, and use this differential version to prove the following result on the mixed correlation structure of the bead configuration and the associated occupation process under $P_n^{\lambda,\theta,T}$:

**Theorem 4.2.** Define the kernels $H_n^{\lambda,\theta,T}$, $K_n^{\lambda,\theta,T} : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{C}$ by

$$H_n^{\lambda,\theta,T}(y, y') = \frac{T}{n} \sum_{z^n = (-1)^q_2} z^{1+h-h'} e^{-(\lambda+\theta_1\pi i+Tz)[t'-t]} \frac{1}{1 - e^{-(\lambda+\theta_1\pi i+Tz)}}. \quad (4.2)$$

$$K_n^{\lambda,\theta,T}(y, y') := -\frac{1}{n} \sum_{z^n = (-1)^q_2} z^{h-h'} e^{-(\lambda+\theta_1\pi i+Tz)([t'-t]+1_{t'=t})} \frac{1}{1 - e^{-(\lambda+\theta_1\pi i+Tz)}}. \quad (4.3)$$
Then with $\Gamma(y_i : i \in B \cup O \cup U)$ as in Definition 4.1 we have

$$P^{\lambda,\theta,T}_n(\Gamma(y_i : i \in B \cup O \cup U)) = \left(\det_{i,j \in O \cup U \cup B} K^*_i,j\right) \prod_{i \in B} dy_i$$

where, setting $K_{i,j} := K^{\lambda,\theta,T}_n(y_i, y_j)$ and $H_{i,j} := H^{\lambda,\theta,T}_n(y_i, y_j)$ we have

$$K^*_i,j = \begin{cases} K_{i,j} & \text{if } i \in O, \\ \delta_{i,j} - K_{i,j} & \text{if } i \in U, \\ H_{i,j} & \text{if } i \in B. \end{cases}$$

The remainder of the article is concerned with studying the signed measures $P^{\lambda,\theta,T}_n$ under the scaling limit where the parameter $T$ controlling the density of beads per string is sent to infinity. We begin by defining a collection of probability measures $P_{n,\ell}$ on bead configurations on $\mathbb{R} \times \mathbb{Z}_n$ recently discovered by Gordenko [17] through studying scaling limits of Young diagrams. To this end, consider the following subsets of the roots of unity:

**Definition 4.3.** Given integers $\ell, n$ with $1 \leq \ell \leq n - 1$, let $\theta_2 = n + \ell + 1 \mod 2$, we define

$$\mathcal{L}_{n,\ell} := \text{The } \ell \text{ elements of } \{z^n = (-1)^{\theta_2}\} \text{ with least real part} \quad \text{and} \quad \mathcal{R}_{n,\ell} := \{z^n = (-1)^{\theta_2}\} - \mathcal{L}_{n,\ell}.$$  

We remark that the parity $\theta_2$ in Definition 4.3 is chosen so that the two sets are well defined and symmetric with respect to the horizontal axis. See Figure 9 for a depiction of the sets $\mathcal{L}_{n,\ell}$ and $\mathcal{R}_{n,\ell}$.

![Figure 9: The sets $\mathcal{L}_{n,\ell}$ (solid circles) and $\mathcal{R}_{n,\ell}$ (hollow circles) depicted for $n = 10$ and $\ell = 3$.](image)

With Definition 4.3 at hand, we now introduce the kernels $H_{n,\ell}$ and $K_{n,\ell}$ on $\mathbb{R} \times \mathbb{Z}_n$ by setting for $y, y' = (s, h), (s', h')$ in $\mathbb{R} \times \mathbb{Z}_n$

$$K_{n,\ell}(y, y') = \begin{cases} +\frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}} z^{h-h'} e^{-z(s'-s)} & \text{if } s' \leq s, \\ -\frac{1}{n} \sum_{z \in \mathcal{R}_{n,\ell}} z^{h-h'} e^{-z(s'-s)} & \text{if } s' > s, \end{cases}$$

and

$$H_{n,\ell}(y, y') = \begin{cases} -\frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}} z^{1+h-h'} e^{-z(s'-s)} & \text{if } s' < s, \\ +\frac{1}{n} \sum_{z \in \mathcal{R}_{n,\ell}} z^{1+h-h'} e^{-z(s'-s)} & \text{if } s' \geq s. \end{cases}$$

The following result was proven by Gordenko.
Theorem 4.4 (Gordenko [17]). For each \( \ell, n \) with \( 1 \leq \ell \leq n - 1 \) and a given density of beads per string, there is a unique Gibbs probability measure on bead configurations on \( \mathbb{R} \times \mathbb{Z}_n \) with occupation number \( \ell \). Writing \( P_{n,\ell} \) for the law of the process at density \( \frac{1}{n} \frac{\sin(\pi\ell/n)}{\sin(\pi/n)} \), the beads have a joint determinantal structure:

\[
P_{n,\ell}(dy_1, \ldots, dy_N \text{ contain beads}) = \det_{j,k=1}^{N} H_{n,\ell}(y_j, y_k) dy_1 \ldots dy_N
\]

Alternatively, if \((X_t)_{t \in \mathbb{R}}\) is the associated occupation process, then under \( P_{n,\ell} \) is Markovian, and we have

\[
P_{n,\ell}(y_1, \ldots, y_N \text{ are occupied}) = \det_{j,k=1}^{N} K_{n,\ell}(y_j, y_k),
\]

where, as above, we say \( y = (t, h) \) is occupied if \( h \in X_t \). Moreover, under \( P_{n,\ell} \), \((X_t)_{t \in \mathbb{R}}\) has the law of \( \ell \) independent asymmetric walkers on the ring conditioned never to collide.

The following result, which we prove in Section 9 using a scaling limit of the measures \( P_{n}^{\lambda, \theta, T} \), is an analogue of Theorem 4.2, supplying an extension and consolidation of the previous result by Gordenko.

Theorem 4.5. With \( \Gamma(y_i : i \in B \cup O \cup U) \) as in Definition 4.1 we have

\[
P_{n,\ell}(\Gamma(y_i : i \in B \cup O \cup U)) = \det_{i,j \in \mathcal{O} \cup \mathcal{U} \cup \mathcal{B}} K_{i,j}^* \prod_{i \in \mathcal{B}} dy_i,
\]

where setting \( K_{i,j} := K_{n,\ell}(y_i, y_j) \) and \( H_{i,j} := H_{n,\ell}(y_i, y_j) \) we have

\[
K_{i,j}^* = \begin{cases} K_{i,j} & \text{if } i \in \mathcal{O}, \\ \delta_{i,j} - K_{i,j} & \text{if } i \in \mathcal{U}, \\ H_{i,j} & \text{if } i \in \mathcal{B}. \end{cases}
\]

The formulation in Theorem 4.5 allows us to gain an explicit hold on the transition rates of the Markov chain \((X_t)_{t \in \mathbb{R}}\) under \( P_{n,\ell} \). Indeed, define a function \( \Delta \) on the subsets of \( \mathbb{Z}_n \) of cardinality \( \ell \) by setting

\[
\Delta(E) := \prod_{1 \leq j < k \leq \ell} |e^{2\pi i h_k/n} - e^{2\pi i h_j/n}| \quad E = \{h_1, \ldots, h_\ell\}.
\]

Then we use the mixed correlations under \( P_{n,\ell} \) to show that the transition rates of \((X_t)_{t \in \mathbb{R}}\) under \( P_{n,\ell} \) are given by

\[
\lim_{t \downarrow 0} \frac{1}{t} P_{n,\ell}(X_t = E' | X_0 = E) = \frac{\Delta(E')}{\Delta(E)} \quad E' = E \cup \{h + 1\} - \{h\} \text{ for } [h + 1] \notin E,
\]

and the stationary distribution is given by

\[
P_{n,\ell}(X_0 = E) = \Delta(E)/n^\ell.
\]

The Markov chain \((X_t)_{t \in \mathbb{R}}\) under \( P_{n,\ell} \) is similar to, but — we stress — distinct in law from, a well-studied Markov chain known as TASEP (the totally asymmetric exclusion process) on the ring. Write \( P_{n,\ell}^{\text{TASEP}} \) for the law of TASEP with \( \ell \) particles on the ring. Under \( P_{n,\ell}^{\text{TASEP}} \) the dynamics of \((X_t)_{t \in \mathbb{R}}\) are straightforward to describe: we have \( \ell \) walkers on the ring, and a walker at \( h \) jumps to \([h + 1]\) at rate 1 if \([h + 1]\) is unoccupied, and otherwise stays put. In other words, in parallel to (4.7) we have

\[
\lim_{t \downarrow 0} \frac{1}{t} P_{n,\ell}^{\text{TASEP}}(X_t = E' | X_0 = E) = 1 \quad E' = E \cup \{h + 1\} - \{h\} \text{ for } [h + 1] \notin E.
\]
Despite the somewhat innocuous description of TASEP on the ring, and its fairly simple transition rates, it is notoriously difficult to provide formulas for the transition probabilities $P_{n,\ell}^{\text{TASEP}}(X_t = E' | X_0 = E)$ for fixed times $t$. A recent breakthrough on this front was made by Baik and Liu [4], where they provide a complicated but explicit description of these transition probabilities in terms of contour integrals. Our last result provides an alternative construction of $P_{n,\ell}^{\text{TASEP}}$ through $P_{n,\ell}$ and an exponential martingale. To this end, we define the traffic of a configuration $E$ of $\ell$ points on $\mathbb{Z}_n$ to be the number of neighbouring pairs of points of $E$, i.e.

$$\text{Traffic}(E) := \# \{ h \in E : h + 1 \text{ is also in } E \}.$$ 

In Section 9, we prove the following result:

**Theorem 4.6.** TASEP on the ring may be recovered from $P_{n,\ell}$ via the exponential martingale change of measure

$$\frac{dP_{n,\ell}^{\text{TASEP}}}{dP_{n,\ell}} \bigg|_{\mathcal{F}_t} = \frac{\Delta(X_0)}{\Delta(X_t)} \exp \left\{ \int_0^t (\text{Traffic}(X_s) - \mu_{n,\ell}) \, ds \right\},$$

where $\mu_{n,\ell} = \ell - \frac{1}{n} \frac{\sin(\pi \ell / n)}{\sin(\pi / n)}$ and $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$ is the natural filtration of the chain.

In essence, Theorem 4.6 states that TASEP behaves like a version of $P_{n,\ell}$ under which particles are encouraged to spend more time in traffic. The value of this representation lies in the fact that $P_{n,\ell}$ has relatively simple transition probabilities, as well as an agreeable determinantal structure, and as such TASEP may be understood as a simple transform of this process.

The probability measures $P_{n,\ell}$ govern bead configurations on $\mathbb{R} \times \mathbb{Z}_n$, and are obtained from a scaling limit of $P_{n,T}^{n,T}$ as $n \to \infty$. It is possible to study an alternate scaling limit, where the number $n$ of strings (rather than the lengths of the strings) is sent to infinity, ultimately obtaining the probability measures for bead configurations on $[0,1) \times \mathbb{Z}$ obtained by Metcalfe, Warren and O’Connell [35]. (This model is also discussed in detail in [33].) See Figure 10.

![Figure 10: The cylindrical bead process of Metcalfe, O’Connell and Warren [35] is a determinantal point process on $\mathbb{R} \times \mathbb{Z}$.](image)

We close this section by discussing Boutillier’s doubly infinite bead configurations on $\mathbb{R} \times \mathbb{Z}$. These may be obtained as either an $n \to \infty$ scaling limit of the probability measures $P_{n,\ell}$ of Gordenko’s process [17] on $\mathbb{R} \times \mathbb{Z}_n$ under the scaling limit $\ell = [\tau n]$, or as scaling limit of the cylindrical bead process of Metcalfe, O’Connell and
Warren [35] on $[0, 1) \times \mathbb{Z}$ in which the number of beads-per-string is sent to infinity (and space is rescaled suitably). (The former calculation is supplied by Gordenko in [17].) Boutillier [7] showed that at a given density of beads per string, there is a one-parameter family $\mathcal{P}_\gamma : \gamma \in \mathbb{R}$ of Gibbs measures for random bead configurations on the doubly infinite set $\mathbb{R} \times \mathbb{Z}$ (which we think of as a collection of infinitely long strings indexed by $\mathbb{Z}$). The parameter $\gamma$ controls the tilt $\tau$ of the configuration via the relation $\gamma = \cos \pi \tau$. Boutillier showed under each $\mathcal{P}_\gamma$, the beads form a determinantal process on $\mathbb{R} \times \mathbb{Z}$, with correlation kernel $H_\gamma : (\mathbb{R} \times \mathbb{Z}) \times (\mathbb{R} \times \mathbb{Z}) \rightarrow \mathbb{R}$ given by

$$H_\gamma((t, h), (t', h')) = J_\gamma(t' - t, h' - h)$$

where $J_\gamma(t, h)$ is given by

$$J_\gamma(t, h) = \begin{cases} \int_{[-1,1]} e^{-it\phi} (\gamma + \sqrt{1 - \gamma^2 i\phi})^h \frac{d\phi}{2\pi} & \text{if } h \geq 0 \\ \int_{\mathbb{R} \times [-1,1]} e^{-it\phi} (\gamma + \sqrt{1 - \gamma^2 i\phi})^h \frac{d\phi}{2\pi} & \text{if } h < 0 \end{cases}$$

(4.9)

for $z = (t, h), z' = (t', h')$.

Figure 11: Boutillier’s [7] bead process on $\mathbb{R} \times \mathbb{Z}$ involves infinitely many beads on infinitely many strings.

Let us highlight in particular that Boutillier’s kernels for the correlations for the beads along any given string $\mathbb{R} \times \{h\}$ of $\mathbb{R} \times \mathbb{Z}$ take the celebrated sine kernel form, i.e. for any fixed $h$ we have

$$\mathcal{P}_\gamma((dt_1, h), \ldots, (dt_N, h) \text{ contain beads}) = \det_{j,k=1}^N \left( \frac{\sin(\pi(t_k - t_j))}{\pi(t_k - t_j)} \right) dt_1 \ldots dt_N.$$

In fact, this observation of Boutillier — namely that the process along any given string of the doubly infinite bead process behaves like a sine process — helps explain the ubiquity of the sine process across random matrix theory: usually the underlying source of this sine behaviour is an asymptotic doubly-infinite bead process in the form of an exclusion process or a Gelfand-Tsetlin pattern.

As well as in the setting of Gordenko’s process on $\mathbb{R} \times \mathbb{Z}_n$, it is also possible to associate with a bead process on either $[0, 1) \times \mathbb{Z}$ or $\mathbb{R} \times \mathbb{Z}$ an infinite occupation process $(X_t)_t$ indexed by a suitable set and prove that under that the occupation process has a mixed determinantal structure with the beads, in the spirit of Theorem 4.2. We leave the details to the interested reader.

That concludes the statements of our main results. Before commencing with our proofs in Section 5, we touch here on some further literature related to Boutillier’s bead model. Several authors have studied dynamic versions of the bead model. Adler, Nordenstam and Van Moerbeke [1] show that in the large $n$ limit, in the bulk the Dyson
Brownian minor process forms a determinantal point process in $\mathbb{R} \times \mathbb{Z} \times \text{Time}$ whose correlation kernel at fixed times is given by that of Boutillier’s bead process. It appears that the speed of growth of certain anisotropic stochastic models related to the KPZ equation are governed by related quantities to the Boutillier’s bead model at equilibrium; see Toninelli [49] and Chhita and Ferrari [9]. Boutillier’s bead model has appeared as a limit in countless other models [8, 14, 34, 38].

5 Continuous Kasteleyn theory: proofs

In this section we begin our continuous Kasteleyn theory, culminating in a proof of Theorem 2.2.

For the sake of generality, we will make a slight amendment to our definition of a bead configuration given in the introduction. A (non-empty) bead configuration $(y_1, \ldots, y_{nk})$ is a collection of a $nk$ points on $\mathbb{T}_n$ such that there are $k$ points on each string, and if $t_1 < \ldots < t_k$ are the points on string $h$, and $t_1' < \ldots < t_k'$ are the points on string $h + 1 \pmod{n}$, then we have either

$$t_1 \leq t_1' < t_2 < \ldots < t_k < t_k'$$

or

$$t_1' < t_1 < t_2 < \ldots < t_k' < t_k.$$  

As emphasised in the introduction, our definition does not depend on the ordering of the points: if $(y_1, \ldots, y_{nk})$ is a bead configuration, so is $(y_{\sigma(1)}, \ldots, y_{\sigma(nk)})$ for any permutation $\sigma$ of $\{1, \ldots, nk\}$.

The reader will recall from Section 2.3 that given a non-empty bead configuration on $\mathbb{T}_n$, we may construct an occupation process $(X_t)_{t \in [0, 1)}$. In Section 2.3 we gave a somewhat informal definition, pointing to the picture in Figure 4. We now furnish a more precise definition. We declare a point $z = (t, h)$ to be occupied, and write $h \in X_t$, if the first bead on string $[h + 1]$ after $t$ occurs before the bead on string $h$ after $t$. That is,

$$h \in X_t \iff \inf\{[t_i - t] \neq 0 : h_i = h\} < \inf\{[t_i - t] \neq 0 : h_i = [h - 1]\}. \quad (5.1)$$

If $(t, h)$ is the location of a bead, then $(t, h + 1)$ is occupied but $(t, h)$ is unoccupied. We write $X_t$ for the collection of $h$ in $\mathbb{Z}_n$ such that $(t, h)$ is occupied. As such, each non-empty bead configuration gives rise to an occupation process $(X_t)_{t \in [0, 1)}$ taking values in the collection of subsets of $\mathbb{Z}_n$. Conversely, the occupation process determines the bead configuration: the discontinuities of the process correspond to locations of beads.

It is clear from the diagram that for each $t \in [0, 1)$, $X_t$ contains the same number of elements of $\mathbb{Z}_n$ as $X_0$ does. We note further that $X_0$ is the set of strings $h$ for which the first bead on string $h$ occurs strictly before the first bead on string $h - 1$.

Consequently we recall from Definition 5.1 we have:

**Definition 5.1.** The occupation number $\ell(y_1, \ldots, y_{nk})$ of a bead process $(y_1, \ldots, y_{nk})$ is the number of strings $h$ for which the first bead on string $h$ occurs strictly before the first bead on string $h - 1$. That is,

$$\ell(y_1, \ldots, y_{nk}) = \#\{h \in \mathbb{Z}_n : \inf\{t_i : h_i = h\} < \inf\{t_i : h_i = h - 1\}\}. \quad (5.2)$$

Equivalently, the occupation number is the cardinality of the subset $X_t$ of $\mathbb{Z}_n$, for any choice $t \in [0, 1)$.

The occupation number of the bead configuration in Figure 4 is $\ell = 3$, since at each vertical time slice not crossing through a bead, exactly three strings are occupied.

**Definition 5.2.** For $k \geq 1$, $1 \leq \ell \leq n - 1$, an $(n, k, \ell)$ configuration is a bead configuration on $n$ strings with $k$ beads per string and occupation number $\ell$. 

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The occupation number $\ell$ characterises other important functionals of the bead process. We note first that the torus $\mathbb{T}_n = [0, 1) \times \mathbb{Z}_n$ has a natural Lebesgue measure which gives the set $(t, u) \times \{h\}$ measure $u - t$. We denote integration against this measure by $dz$, and and note that the total mass of $\mathbb{T}_n$ is $n$. The total Lebesgue measure of the occupied region in the torus is given by

$$\int_{\mathbb{T}_n} 1\{y \text{ is occupied}\} \, dy := \sum_{h \in \mathbb{Z}_n} \int_0^1 1\{X_t = j\} \, dt = \int_0^1 \ell \, dt = \ell, \quad (5.3)$$

where $\ell = \ell(y_1, \ldots, y_N)$ is the occupation number of the bead process.

Finally, consider that for a bead configuration on $n$ strings with $k$ beads per string, the average distance between consecutive beads on the same string is $1/k$. That is

$$\frac{1}{nk} \sum_{i=1}^{nk} \inf\{[t_j - t_i] : 1 \leq j \neq i \leq nk \text{ with } h_j = h_i\} = \frac{1}{k}. \quad (5.4)$$

The occupation number may be used to characterise the average distance a bead on one string and the next bead on the string above. Namely,

$$\frac{1}{nk} \sum_{i=1}^{nk} \inf\{[t_j - t_i] : 1 \leq j \neq i \leq nk \text{ with } [h_j = h_i + 1]\} = \frac{\ell}{nk}. \quad (5.5)$$

To see that (5.5) holds, note that along each string, there are exactly $k$ segments of occupied string. The quantity $\inf\{[t_j - t_i] : h_j = [h_i + 1], j \neq i\}$ is the length of the occupied segment of string $[h_i + 1]$ starting at $(t_i, [h_i + 1])$. By (5.3), the sum of all these segment lengths is $\ell$. Dividing through by $nk$, we obtain (5.5) as written.

Figure 12: For an $(n, k, \ell)$ configuration, the average distance between consecutive beads on the same string is $\frac{1}{k}$, whereas the average distance between a given bead and the next bead on the string above is $\frac{\ell}{nk}$.

Figure 12 depicts the quantities in (5.4) and (5.5).

### 5.1 Proof of Theorem 2.2

In this section it will lighten notation to write

$$\zeta := e^{-\lambda}.$$ 

The reader will recall from Section 2 the claimed relation

$$g_N^\lambda = \sum_{\theta \in \{0, 1\}^2} g_N^{\lambda, \theta}. \quad (5.6)$$

We now prove (5.6). Inspecting the definitions (2.2) and (2.4), one can see that (5.6) boils down to the three-vs-one identity, which states that

$$\frac{1}{2} \sum_{\theta \in \{0, 1\}^2} (-1)^{(\theta_1 + k + 1)(\theta_2 + n + \ell + 1)} = 1 \quad k, l \in \mathbb{Z}. \quad (5.7)$$
To see that (5.7) holds, note that \((\theta_1 + k + 1)(\theta_2 + n + \ell + 1)\) is an odd number precisely when \(\theta_1 = k \mod 2\), and \(\theta_2 = n + \ell \mod 2\), and is an even number for the other three elements \(\theta = (\theta_1, \theta_2)\) of \((0,1)^2\). It then follows that three of the summands in (5.7) are equal to \(+1\), and one is equal to \(-1\), proving (5.7) and in turn (5.6).

So far we have defined sequences of functions \(\{g_N^{\lambda,\theta} : N \geq 1\}\) and \(\{g_N^{\lambda,\theta} : N \geq 1\}\), each in \(N \geq 1\) variables. It is useful to extend this definition by defining the constants

\[
g_0^{\lambda,\theta} := \frac{1}{2} (\theta_1 + 1)(\theta_2 + n + 1)(1 - e^{-(\lambda + \theta_1 \pi i)})^n \quad \text{and} \quad g_0^\lambda := (1 + e^{-\lambda})^n. \tag{5.8}
\]

We note that with (5.8), (5.6) continues to hold when \(N = 0\). (The terms with \(\theta_1 = 0\) cancel out.) We will interpret this convention further in the sequel.

Our work towards proving Theorem 2.2 begins with the following adaptation of an idea from Warren [52] (whose work foreshadowed work by Adler et al. [1] cited above).

**Lemma 5.3.** Let \(\zeta \in \mathbb{C}\), and let \(x_1 \leq \ldots \leq x_k\) and \(y_1 \leq \ldots \leq y_k\) be \(2k\) points in \([0,1]\). Then

\[
\det_{i,j=1}^k (z_{ij}^{x_i < x_i}) = \begin{cases} (1 - \zeta)^k - 1 & \text{if } x_1 \leq y_1 < \ldots < x_k \leq y_k \\ (-1)^k - 1 \zeta (1 - \zeta)^{k-1} & \text{if } y_1 < x_1 \leq y_2 < \ldots \leq y_k < x_k \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Consider the \(k \times k\) matrix \((z_{ij}^{x_i < x_i})\) \(i \leq j \leq k\). If either \(x_1 \leq y_1 < \ldots < x_k \leq y_k\) nor \(y_1 < x_1 \leq y_2 < \ldots \leq y_k < x_k\), then two rows or columns of the matrix are identical, in which case the determinant of this matrix is zero.

Suppose now \(x_1 \leq y_1 < \ldots < x_k \leq y_k\), then \(\det_{i,j=1}^k (z_{ij}^{x_i < x_i})\) is given by

\[
det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \zeta & 1 & \ldots & : \\ \vdots & \vdots & \ddots & \vdots \\ \zeta & \zeta & \ldots & 1 \\ \zeta & \zeta & \ldots & \zeta \end{bmatrix} = \det \begin{bmatrix} 1 - \zeta & 1 - \zeta & \ldots & 1 - \zeta & 0 \\ 0 & 1 - \zeta & \ldots & 1 - \zeta & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 - \zeta & 0 \\ \zeta & \zeta & \ldots & \zeta & 1 \end{bmatrix} = (1 - \zeta)^{k-1},
\]

where the former equality above follows from subtracting the bottom row of the matrix from each of the other rows, and the latter equality follows from the fact that the determinant of an upper triangular matrix is the product of the diagonal entries.

We omit the proof of the case \(y_1 < x_1 \leq y_2 < \ldots \leq y_k < x_k\), which is similar. \hfill \square

Given a complex parameter \(\zeta\), define the auxiliary operator \(A_\zeta : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{C}\) by

\[
A_\zeta(z, z') = A_\zeta((t, h), (t', h')) := 1_{\{h' = h + 1\}} \zeta^{1(t' < t)}, \tag{5.9}
\]

for \(t, t' \in [0,1]\), \(h, h' \in \mathbb{Z}_n\).

Our next step in proving Theorem 2.2 is the following lemma.

**Lemma 5.4.** Let \(\zeta = e^{-\lambda}\), and let \((y_1, \ldots, y_N)\) be points of \(\mathbb{T}_n\). Then for \(k := N/n\) we have

\[
\det_{i,j=1}^N A_\zeta(y_i, y_j) = (-1)^{N-k-1}(1 - \zeta)^{N-k-1} g_N^\lambda(y_1, \ldots, y_N), \tag{5.10}
\]

where \(g_N^\lambda\) is defined in (2.2) and \(\ell(y_1, \ldots, y_N)\) is the occupation number defined in (5.2).
Proof. Before delving into the proof, let us recognise immediately that the determinant on the left-hand-side of (5.10) may only be nonzero if \((y_1, \ldots, y_N)\) form a bead configuration. To see this, first we note that the determinant may only be nonzero when there is at least one bead on every string, otherwise there is a row of zeroes. Suppose now there exist two consecutive beads indexed by \(i\) and \(i'\) on the same string \(h\) say with horizontal coordinates \(t_i < t_{i'}\) such that there is no bead on string \(h + 1\) with horizontal coordinate in \([t_i, t_{i'}]\). In this case the \(i^{th}\) and \(i'^{th}\) row of the matrix are identical, and the determinant is zero. It follows that in order for the determinant to be nonzero there must be at least one bead on every string, and between every two consecutive beads on one string there is a bead on the string above.

We therefore assume without loss of generality through the remainder of the proof that \(N = nk\) and \((y_1, \ldots, y_N) = (y_1, \ldots, y_{nk})\) is a bead configuration with \(k \geq 1\) beads per string. We further assume without loss of generality that the points are ordered so that the vertical coordinates are increasing, and points with the same vertical coordinate are ordered so that there horizontal coordinates are increasing.

Now from the definition (5.9) of \(A_\zeta\), we see that \(A_\zeta(z, z')\) is nonzero only when \(z'\) lies on the string above \(z\), mod \(n\). For each \(h \in \{0, 1, \ldots, n - 1\}\), consider the \(k \times k\) submatrix \(A_h := (A_\zeta(y_i, y_j))_{i,h_i = h,j,h_j = h+1}\). Then we can depict \((A(y_i, y_j))_{1 \leq i,j \leq N}\) as the block matrix:

\[
(A(y_i, y_j))_{1 \leq i,j \leq N} = \begin{bmatrix}
0 & A_0 & 0 & \ldots & 0 \\
0 & 0 & A_1 & \ldots & \vdots \\
\vdots & \ddots & & & \\
0 & 0 & \ldots & 0 & A_{n-2} \\
A_{n-1} & \ldots & 0 & \end{bmatrix}
\] (5.11)

where each 0 denotes a \(k \times k\) matrix of zeroes.

We now note that swapping two columns of a matrix changes to parity of the determinant by \(-1\). It takes \((n - 1)k\) column swaps in order to rearrange the matrix in (5.11) so that \(A_0\) occupies the first \(k\) rows and \(k\) columns of the matrix, \(A_1\) occupies the next \(k\) rows and next \(k\) columns, etc. It then follows from (5.11) that

\[
\det_{i,j=1}^{N} A(y_i, y_j) = (-1)^{(n-1)k} \prod_{h \in \mathbb{Z}_n} \det(A_h).
\] (5.12)

Now each matrix \(A_h\) is a matrix of the form occurring in Lemma 5.3, where \(x_1 \leq \ldots \leq x_k\) are the entries of \((y_1, \ldots, y_N)\) lying on string \(h\), and \(y_1 \leq \ldots \leq y_k\) are the entries lying on string \(h + 1\). Since \((y_1, \ldots, y_N)\) is a bead configuration, the \(x_1 \leq \ldots \leq x_k\) interlace \(y_1 \leq \ldots \leq y_k\). Recall now that \(\ell(y_1, \ldots, y_N)\) is the number of strings for which the first bead on string \(h\) occurs strictly before the first bead on string \(h - 1\). It follows from Lemma 5.3 that

\[
\det(A_h) = \begin{cases} 
(1 - \zeta)^{k-1} & \text{first bead on string } h \text{ before the first bead on string } h + 1 \\
(-1)^{k-1}\zeta(1 - \zeta)^{k-1} & \text{first bead on string } h + 1 \text{ occurs strictly before the first bead on string } h.
\end{cases}
\] (5.13)

Using the definition (5.2) of \(\ell(y_1, \ldots, y_N)\), we immediately see from plugging (5.13) into (5.14) that

\[
\det_{i,j=1}^{N} A(y_i, y_j) = (-1)^{(n-1)k}(1 - \zeta)^{\ell(y_1, \ldots, y_N)}(1 - \zeta)^{n(k-1)} g_N^0(y_1, \ldots, y_N),
\] (5.14)

where we note that \(g_N^0\) is the indicator function of bead configurations.

Using the definition (2.2) of \(g_N^\theta\) seals the result. \(\square\)

For \(\theta = (\theta_1, \theta_2) \in \{0, 1\}^2\), define the operator \(B_\zeta^\theta : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{C}\) by

\[
B_\zeta^\theta(z, z') = \frac{(-1)^{\theta_2} 1_{\zeta' = 0}}{1 - (-1)^{\theta_1} \zeta} A_{(-1)^{\theta_1} \zeta}(z, z'),
\] (5.15)
where \( h' \) is the vertical coordinate of \( z' \), and \( A \) is defined in (5.9). We now state and prove our final lemma before finally giving a proof of Theorem 2.2.

**Lemma 5.5.** We have

\[
(-1)^{(\theta_1+1)(\theta_2+n+1)}(1 - (-1)^{\theta_1}\zeta)^n \det_{i,j=1}^{N} B^\theta_{\zeta}(y_i, y_j) = (-1)^{(\theta_1+\ell+1)(\theta_2+n+\ell+1)} g^\lambda_N(y_1, \ldots, y_N)
\]

**Proof.** For \( 1 \leq i \leq N \) let \( y_i = (t_i, h_i) \). It follows from the definition of \( B^\theta_{\zeta} \) that

\[
\det_{i,j=1}^{N} B^\theta_{\zeta}(y_i, y_j) = \frac{(-1)^{\theta_1 \bar{k}}}{(1 - (-1)^{\theta_1}\zeta)^{nk}} \det_{i,j=1}^{N} A(-1)^{\theta_1}\zeta(y_i, y_j),
\]

where \( \bar{k} = \{1 \leq i \leq N : h_i = 0 \} \). In particular, by Lemma 5.4, \( \det_{i,j=1}^{N} B^\theta_{\zeta}(y_i, y_j) \) is zero unless \((y_1, \ldots, y_N)\) is a bead configuration. For the remainder of the proof we now assume that \((y_1, \ldots, y_N)\) is a bead configuration with \( N = nk \), that is, \( k \) beads per string, and we abbreviate \( \ell := \ell(y_1, \ldots, y_N) \) for the occupation number of the configuration. We note that since there are \( k \) beads per string, \( \bar{k} = k \). Under these assumptions we now compute the determinant explicitly.

Replacing \( \zeta \) with \(-1)^{\theta_1}\zeta \) (or equivalently, \( \lambda \) with \( \lambda + \theta_1\pi i \)) in (5.10) we have

\[
\det_{i,j=1}^{N} A(-1)^{\theta_1}\zeta(y_i, y_j) = (-1)^{(n-1)k+(k-1)\ell}(1 - (-1)^{\theta_1}\zeta)^n g^\lambda_N(y_1, \ldots, y_N)
\]

\[
= (-1)^{(n-1)k+(k-1+\theta_1)\ell}(1 - (-1)^{\theta_1}\zeta)^n g^\lambda_N(y_1, \ldots, y_N),
\]

where the latter equality above follows from relating \( g^\lambda_N \) and \( f(-1)^{\theta_1}\zeta \) using the definition of \( g^\lambda_N \) in (2.2).

Combining (5.16) with (5.17), and using the fact that \( \bar{k} = k \), we obtain

\[
(1 - (-1)^{\theta_1}\zeta)^n \det_{i,j=1}^{N} B^\theta_{\zeta}(y_i, y_j) = (-1)^{(n-1)k+(k-1+\theta_1)\ell+\theta_2k} g^\lambda_N(y_1, \ldots, y_N),
\]

The result follows from noting that

\[
(\theta_1+1)(\theta_2+n+1)+(n-1)k+(k-1+\theta_1)\ell+\theta_2k = (\theta_1+1+k)(\theta_2+n+1+\ell) \mod 2.
\]

The proof of the final lemma is now complete.

We are now equipped to prove Theorem 2.2.

**Proof of Theorem 2.2.** Setting \( \zeta = e^{-\lambda} \) in Lemma 5.5 we have

\[
g^\lambda_{\varphi}(y_1, \ldots, y_N) = \frac{1}{2} (-1)^{(\theta_1-1)(\theta_2+n-1)} (1 - e^{-(\lambda+\theta_1\pi i)})^n \det_{i,j=1}^{N} B_{\varphi \gamma}(y_i, y_j).
\]

Since conjugation by a diagonal matrix leaves a determinant unchanged, we make the simple observation that

\[
\det_{i,j=1}^{N} (B_{\varphi \gamma}(y_i, y_j)) = \det_{i,j=1}^{N} \left( e^{\frac{\theta_2\pi i}{n}(h_j-h_i)-(\lambda+\pi i)(t_j-t_i)} B_{\varphi \gamma}(y_i, y_j) \right).
\]

We now note from (2.7), (5.9) and (5.15) that

\[
e^{\frac{\theta_2\pi i}{n}(h'-h)-(\lambda+\pi i)(t'-t)} B_{\varphi \gamma}(y, y') = \frac{1_{h'=h+1} e^{\frac{\theta_2\pi i}{n}(h'-h+n_1 h'_{=0})-(\lambda+\theta_1\pi i)(t'-t+1, t_{<t})}}{1 - e^{-(\lambda+\theta_1\pi i)}}.
\]
Now whenever \( h' \equiv h + 1 \) we have \( h' - h + n1_{h'=0} = 1 \). Moreover, \( t' - t + 1_{t'<t} = [t' - t] \). Therefore

\[
e^{-\frac{\theta_2 \pi i}{n} (h' - h + t + \theta_1 \pi i)(t' - t)} B_{e^{-\lambda}}^\theta (y, y') = \frac{1_{h'=h+1} e^{-\frac{\theta_2 \pi i}{n} (\lambda + \theta_1 \pi i)(t' - t)}}{1 - e^{-\lambda} e^{\lambda \theta_1 \pi i}}.
\]

we see that

\[
e^{-\frac{\theta_2 \pi i}{n} (h' - h + t + \theta_1 \pi i)(t' - t)} B_{e^{-\lambda}}^\theta (y, y') = C_{\lambda}^\theta ((t, h), (t', h')). \tag{5.21}
\]

Plugging (5.21) and (5.20) into (5.19), we obtain the statement of Theorem 2.2. \( \square \)

### 5.2 Diagonalisation of the Fredholm operators

With Theorem 2.2 now proved, the reader will recall from Section 2 the relation

\[Z_n(\lambda, T) = \frac{1}{2} \sum_{\theta \in \{0, 1\}^2} (-1)^{\theta_1 + 1}(\theta_2 + n + 1)(1 - e^{-(\lambda + \pi \theta_1 i)})^n \det(I + TC_{\lambda + \theta_1 \pi i}^\theta). \tag{5.22}\]

where \( C_{\beta}^\theta : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{C} \) is the operator defined in the statement of Theorem 2.2. In this section we diagonalise this operator, ultimately leading to a proof of Theorem 2.3.

Consider the linear space \( L^2(\mathbb{T}_n) := \{ \phi : \mathbb{T}_n \to \mathbb{C} \text{ measurable} : ||\phi||_2^2 := \int_{\mathbb{T}_n} |\phi(z)|^2 dz < \infty \} \) of measurable and square-integrable complex-valued functions on \( \mathbb{T}_n \), which is a Hilbert space when endowed with the inner product

\[\langle \phi, \psi \rangle := \int_{\mathbb{T}_n} \overline{\phi(z)} \psi(z) dz.\]

Define the functions \( \phi_{w,m} : \mathbb{T}_n \to \mathbb{C} \) by

\[\phi_{w,m}(t, h) := \frac{1}{\sqrt{n}} w^h e^{-2\pi im t} \quad w^n = 1, m \in \mathbb{Z}. \tag{5.23}\]

The countable collection of functions \( \Gamma := \{ \phi_{w,m} : w^n = 1, m \in \mathbb{Z} \} \) form an orthonormal basis of \( L^2(\mathbb{T}) \). (This follows quickly from transferring standard results concerning orthonormal bases of \( L^2([0, 1]) \) to \( L^2(\mathbb{T}_n) \), see e.g. [40].)

**Lemma 5.6.** Each \( \phi_{w,m} \) is an eigenfunction of \( C_{\beta}^\theta \) with eigenvalue \( e^{\theta_2 \pi i/n}w(\beta + 2\pi im)^{-1} \).

**Proof.** Using the definition of \( C_{\beta}^\theta \), we have

\[
C_{\beta}^\theta \phi_{w,m}(t, h) := \sum_{h' \in \mathbb{Z}_n} \int_0^1 C_{\beta}^\theta ((t, h), (t', h')) \phi_{w,m}(t', h') dt'
= e^{\theta_2 \pi i/n} \sum_{h' \in \mathbb{Z}_n} \int_0^1 1_{h'=h+1} \frac{e^{-\beta |t'-t|}}{1 - e^{-\beta}} w^{h'} e^{-2\pi im t'} dt'
= e^{\theta_2 \pi i/n} \left( \sum_{h' \in \mathbb{Z}_n} 1_{h'=h+1} w^{h'} \left( \int_0^1 \frac{e^{-\beta |t'-t|}}{1 - e^{-\beta}} w^{h'} e^{-2\pi im t'} dt' \right) \right). \tag{5.24}\]

Direct computation (in the latter case using \( [t' - t] = t' - t + 1_{t'<t} \)) tells us that

\[
\sum_{h' \in \mathbb{Z}_n} 1_{h'=h+1} w^{h'} = w^{h+1} \quad \text{and} \quad \int_0^1 \frac{e^{-\beta |t'-t|}}{1 - e^{-\beta}} e^{-2\pi im t'} dt' = \frac{e^{-2\pi im t}}{\beta + 2m \pi i}. \tag{5.25}\]
Plugging (5.25) into (5.24), we see that
\[ C_{\beta}^{(2)} \phi_{w,m}(t, h) = e^{\theta_2 \pi i / n} w^{h+1}(\beta + 2\pi im)^{-1} e^{-2\pi im t} = e^{\theta_2 \pi i / n} w(\beta + 2\pi im)^{-1} \phi_{w,m}(t, h), \]
sealing the result.

We now prove Theorem 2.3.

Proof of Theorem 2.3. According to Lemma 5.6, each \( \phi_{w,m} \) is an eigenfunction of \( I + TC_{\beta}^{(2)} \) with eigenvalue
\[ \mu_{w,m} := 1 + \frac{e^{\theta_2 \pi i / n} T w}{\beta + 2\pi im}. \]
Since the operator \( I + TC_{\beta}^{(2)} \) has an orthonormal basis of eigenfunctions, it follows that the Fredholm determinant is alternatively characterised through the infinite product
\[ \det(I + TC_{\beta}^{(2)}) = \prod_{w^n = 1} \prod_{m \in \mathbb{Z}} \left( 1 + \frac{e^{\theta_2 \pi i / n} T w}{\beta + 2\pi im} \right) = \prod_{w^n = (-1)^{\theta_2}} \prod_{m \in \mathbb{Z}} \left( 1 + \frac{T w}{\beta + 2\pi im} \right). \tag{5.26} \]
For each \( w \) with \( w^n = (-1)^{\theta_2} \), consider the product over \( m \in \mathbb{Z} \) in (5.26). By pairing each \( m \geq 0 \) with \(-m\), and accounting for the double counting of zero, we have
\[ \prod_{m \in \mathbb{Z}} \left( 1 + \frac{T w}{\beta + 2\pi im} \right) = (1 + T w / \beta)^{-1} \prod_{m = 0}^{\infty} \left( 1 + \frac{2\beta T w + T^2 w^2}{\beta^2 + 4\pi^2 m^2} \right). \tag{5.27} \]
where the former equality is simply the determinant of \( I + TC_{\beta}^{(2)} \) expressed as a product of eigenvalues, and the latter follows from pairing \( m \) and \(-m\) terms.

According to the infinite product identity
\[ \prod_{m = 0}^{\infty} \left( 1 + \frac{\gamma^2 - b^2}{a^2 k^2 + b^2} \right) = \frac{\gamma \sinh(\pi \gamma / a)}{a \sinh(\pi b / a)}, \tag{5.28} \]
(see e.g. [12]), it then follows from (5.26), (5.27) and (5.28) that
\[ \det(I + TC_{\beta}^{(2)}) = \prod_{w^n = (-1)^{\theta_2}} \frac{\sinh(\beta + T w)}{\sinh(\beta / 2)}. \tag{5.29} \]
Using the definition of \( \sinh \) and exploiting the fact that \( \sum_{w^n = (-1)^{\theta_2}} w = 0 \), it is easily verified that
\[ (1 - e^{-\beta})^n \det(I + TC_{\beta}^{(2)}) = \prod_{w^n = (-1)^{\theta_2}} (e^{T w} - e^{-\beta}). \tag{5.30} \]
Plugging (5.30) into Theorem 2.3, we obtain
\[ Z_n(\lambda, T) := \sum_{k \geq 0} \sum_{0 \leq \ell \leq n} \frac{T^{nk}}{(nk)!} \xi^{\ell} V_{k,\ell}^{(n)} = \frac{1}{2} \sum_{\theta_2 \in (0, 1)^2} (-1)^{(\theta_2 + n + 1)} \prod_{w^n = (-1)^{\theta_2}} (e^{T w} - e^{-\lambda - (\theta_1 + \pi)).} \tag{5.31} \]
Replacing \( e^{-\lambda} \) with \( \zeta \) in (5.31), we have
\[ \sum_{k \geq 0} \sum_{0 \leq \ell \leq n} \frac{T^{nk}}{(nk)!} \xi^{\ell} V_{k,\ell}^{(n)} = \frac{1}{2} \sum_{\theta_2 \in (0, 1)^2} (-1)^{(\theta_2 + n + 1)} \prod_{w^n = (-1)^{\theta_2}} (e^{T w} - (-1)^{\theta_1} \zeta). \tag{5.32} \]
It thus follows that in order to compute \( V_{k,\ell}^{(n)} \), we need to find the coefficient of \( T^{nk} \zeta^\ell \) in the right-hand-side of (5.32). This we do in the next section. \( \square \)
6 Proof of Theorem 3.1

In this section we prove Theorem 3.1, giving an explicit formula for the volume $\text{Vol}^n_{k,\ell}$ of bead configurations on $\mathbb{T}_n$ with $k$ beads per string and occupation number $\ell$.

**Proof.** As remarked below (5.32), $\text{Vol}^n_{k,\ell}$ is the coefficient of $\frac{1}{(nk)!}T^{nk}\zeta^\ell$ in the right-hand-side of (5.32). Expanding the product in the right-hand-side of (5.32), we have

$$\sum_{k \geq 0} \sum_{0 \leq \ell \leq n} \frac{T^{nk}}{(nk)!} \zeta^\ell \text{Vol}^n_{k,\ell} = \frac{1}{2} \sum_{\theta \in \{0,1\}^2} (-1)^{(\theta_1+1)(\theta_2+n+1)} \sum_{\ell=0}^{n} (-1)^{(\theta_1-1)\ell} \zeta^\ell \sum_{S \subseteq \{w^n = (-1)^{\theta_2}\}, \#S = n-\ell} e^T \sum_{w \in S} w, \quad (6.1)$$

where the final sum is over subsets $S$ of the $n$th roots of $(-1)^{\theta_2}$ of cardinality $n - \ell$. We now note that we have the equality of sets $\{w : w^n = (-1)^{\theta_2}\} = \{w e^{\theta_2 \pi i/n} : w^n = 1\}$. Using this fact to obtain the first equality below, then expanding as a power series in $T$ to obtain the second, we have

$$\sum_{S \subseteq \{w^n = (-1)^{\theta_2}\}, \#S = n-\ell} e^T \sum_{w \in S} w = \sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} e^{\theta_2 \pi i/n} T \sum_{w \in S} w = \frac{1}{N!} e^{\theta_2 \pi i/n} \sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( \sum_{w \in S} w \right)^N. \quad (6.2)$$

We now claim that $\sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( \sum_{w \in S} w \right)^N$ is zero unless $N$ is a multiple of $n$. To see this, using the rotational symmetry of the roots to obtain the first equality below, and then pulling the new factor to the front to obtain the second, we have

$$\sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( \sum_{w \in S} w \right)^N = \sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( e^{2\pi i/n} \sum_{w \in S} w \right)^N = e^{2\pi i/n} \sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( \sum_{w \in S} w \right)^N. \quad (6.3)$$

In order for the outermost quantities in (6.3) to be nonzero and identical, it must be the case that $N$ is a multiple of $n$, proving our claim. In particular, setting $N = nk$ in (6.2) we have

$$\sum_{S \subseteq \{w^n = (-1)^{\theta_2}\}, \#S = n-\ell} e^T \sum_{w \in S} w = \sum_{k=0}^\infty \frac{T^{nk}}{(nk)!} (-1)^{\theta_2 k} \sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( \sum_{w \in S} w \right)^{nk}. \quad (6.4)$$

Plugging (6.2) into (6.1) and rearranging, we have

$$\sum_{k \geq 0} \sum_{0 \leq \ell \leq n} \frac{T^{nk}}{(nk)!} \zeta^\ell \text{Vol}^n_{k,\ell} = \sum_{k=0}^\infty \frac{T^{nk}}{(nk)!} \sum_{\ell=0}^{n} \zeta^\ell \frac{1}{2} \sum_{\theta \in \{0,1\}^2} (-1)^{(\theta_1+1)(\theta_2+n+1)+(\theta_1-1)\ell+\theta_2 k} \sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( \sum_{w \in S} w \right)^{nk}, \quad (6.5)$$

or more explicitly,

$$\text{Vol}^n_{k,\ell} = \frac{1}{2} \sum_{\theta \in \{0,1\}^2} (-1)^{(\theta_1+1)(\theta_2+n+1)+(\theta_1-1)\ell+\theta_2 k} \sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( \sum_{w \in S} w \right)^{nk}. \quad (6.6)$$

We now account for the $\theta$ terms in (6.6). A brief calculation tells us that we have

$$(\theta_1+1)(\theta_2+n+1)+(\theta_1-1)\ell+\theta_2 k = (\theta_1+1+k)(\theta_2+n+1+\ell)+k(n+\ell+1) \mod 2.$$
In particular, noting that \((\theta_1 + 1 + k)(\theta_2 + n + 1 + \ell)\) is an odd number for exactly one of the elements \(\theta \in \{0, 1\}^2\), and an even number for the other three elements, we have
\[
\frac{1}{2} \sum_{\theta \in \{0,1\}^2} (-1)^{\theta_1+1}(\theta_2+n+1)+(\theta_1-1)\ell+\theta_2k = (-1)^{k+n+\ell+1}.
\] (6.7)

Plugging (6.7) into (6.6) we have
\[
\text{Vol}^{(n)}_{k,\ell} = (-1)^{k+n+\ell+1} \sum_{S \subseteq \{w^n = 1\}, \#S = n-\ell} \left( \sum_{w \in S} w \right)^{nk}.
\] (6.8)

We now make a final adjustment. Note that since the roots of unity sum to zero, for any subset \(S\) of \(\{w^n = 1\}\) we have
\[
\sum_{w \in S} w = -\sum_{w \in \{w^n = 1\} - S} w.
\] (6.9)

Consequently, we can take the internal sum in (6.8) over subsets \(S\) of \(\{w^n = 1\}\) with cardinality \(\ell\) instead of \(n - \ell\), and account for this change by multiplying by a factor of \((-1)^{nk}\). This adjustment yields (3.1) from (6.8), completing the proof. \(\square\)

We remark by (6.9) and the following remark, we have the following symmetry in the volume:
\[
\text{Vol}^{(n)}_{k,\ell} = \text{Vol}^{(n)}_{k,n-\ell}.
\] (6.10)

In fact, (6.10) follows from the following observation. Let \((y_1, \ldots, y_{nk})\) (here \(y_i = (t_i, h_i)\) be a bead configuration with occupation number \(\ell\), and set \(y'_i := (1 - t_i, h_i)\). Then if the \(t_i\) are all distinct (which is the case for almost all bead configurations), then \((y'_1, \ldots, y'_{nk})\) is a bead configuration with occupation number \(n - \ell\).

Since the proof of Theorem 3.2 is both technical and unconnected with our work in the sequel, we relegate its proof to the appendix.

7 Probability measures on bead configurations

In this section we study random bead configurations and their correlations. While a natural approach to the study of random bead configurations may involve choosing a configuration uniformly from the set of \((n, k, \ell)\) configurations, it turns out that we find it valuable to choose configurations randomly according to their contribution to the generating function \(Z_n(\lambda, T)\) defined in (5.22). In the latter case, the configurations have a pliant correlation structure. (It is often the case in integrable probability that the randomisation of a certain parameter leads to more tractable correlation structure. One notable example is in Baik, Deift and Johansson [3], wherein it is shown the cycle lengths of a uniformly chosen permutation of \(S_N\), where \(N\) is Poisson distributed, have a tractable structure.)

7.1 Complex measures on bead processes

Let \(\mathcal{T}_0\) denote the set of (possibly empty) bead configurations, which may be associated with a subset of \(\bigcup_{k \geq 0} \mathcal{T}^{nk}_n\). In this section we study measures on \(\mathcal{T}_0\). Given a variable \(y \in \mathcal{T}_0\), we write \(|y|\) for the length of \(y\), i.e. if \(y = (y_1, \ldots, y_N)\) then \(|y| = N\).
Recall the functions $g_N^\lambda$ and $g_N^{\lambda, \theta}$ defined in Section 2. For $\beta = \lambda$ or $\beta = (\lambda, \theta)$ we define complex measures $Q_n^{\beta, T}$ on the bead configurations in $T$ by setting, for $k \geq 1$,

$$Q_n^{\beta, T}(|y| = nk, y_1 \in A_1, \ldots, y_{nk} \in A_{nk}) := \frac{T^{nk}}{(nk)!} \int_{A_1 \times \cdots \times A_{nk}} g_n^\beta(y_1, \ldots, y_{nk}) dy_1 \cdots dy_{nk}. \quad (7.1)$$

and for $k = 0$

$$Q_n^{\beta, T}(|y| = 0) = g_0^\beta. \quad (7.2)$$

By virtue of (5.6) we have the decomposition of measures

$$Q_n^{\lambda, T} = \sum_{\theta \in \{0,1\}^2} Q_n^{\lambda, \theta, T}. \quad (7.3)$$

Write $Z_n(\lambda, T)$ and $Z_n(\lambda, T)$ for the total mass of the respective complex measures $Q_n^{\lambda, T}$ and $Q_n^{\lambda, \theta, T}$.

We now define new complex measures $P_n^{\lambda, T}$ and $P_n^{\lambda, \theta, T}$ with unit total mass by setting

$$P_n^{\lambda, T} := \frac{1}{Z_n(\lambda, T)} Q_n^{\lambda, T} \quad \text{and} \quad P_n^{\lambda, \theta, T} := \frac{1}{Z_n(\lambda, T)} Q_n^{\lambda, \theta, T}; \quad (7.4)$$

that is, $P_n^{\lambda, T}$ (resp. $P_n^{\lambda, \theta, T}$) is simply $Q_n^{\lambda, T}$ (resp. $Q_n^{\lambda, \theta, T}$) multiplied by a scalar. When $\lambda \in \mathbb{R}$, $g_N^\lambda$ takes non-negative values, and consequently, $P_n^{\lambda, T}$ is a genuine probability measure on $T$ in that it takes values in $[0, 1]$. Each $P_n^{\lambda, \theta, T}$ on the other hand may give certain events “negative probabilities”.

By (7.3), we can write $P_n^{\lambda, T}$ as an affine combination of the measures $P_n^{\lambda, \theta, T}$ by setting

$$P_n^{\lambda, T} = \sum_{\theta \in \{0,1\}^2} \frac{Z_n(\lambda, T)}{Z_n(\lambda, T)} P_n^{\lambda, \theta, T}. \quad (7.5)$$

The value of this decomposition (7.5) lies in the fact that each measure $P_n^{\lambda, \theta, T}$ has a tractable determinantal structure. Namely, define the correlation functions $\{p_n^{\lambda, \theta, T} : \mathbb{T}_n^N \to \mathbb{C} : N \geq 0\}$ defined on $\mathbb{T}_n$ to be the (symmetric) functions satisfying

$$p_n^{\lambda, \theta, T}(y_1, \ldots, y_N) dy_1 \cdots dy_N = P_n^{\lambda, \theta, T}(\text{Each of } dy_1, \ldots, dy_N \text{ contains a bead}), \quad (7.6)$$

or more explicitly, by summing over configurations with at least $N$ beads,

$$p_n^{\lambda, \theta, T}(y_1, \ldots, y_N) = \frac{1}{Z_n(\lambda, T)} \sum_{j \geq 0} \frac{T^{N+j}}{(N+j)!} \int_{\mathbb{T}_n^N} g_n^{\lambda, \theta, T}(y_1, \ldots, y_N, y_{N+1}, \ldots, y_{N+j}) dy_{N+1} \cdots dy_{N+j}. \quad (7.7)$$

The main result of this section is the following.

**Theorem 7.1.** Define the operator $H^{\lambda, \theta, T} : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{C}$ by

$$H^{\lambda, \theta, T}(y, y') = \frac{T}{n} \sum_{z^n = (-1)^{\theta_2}} z^{1+h-h'} e^{-(\lambda+\theta_1 \pi i + Tz)(t'-t)} \frac{1}{1 - e^{-(\lambda+\theta_1 \pi i + Tz)}}, \quad (7.8)$$

Then

$$p_n^{\lambda, \theta, T}(y_1, \ldots, y_N) = \frac{\det_{i,j=1}^{N} H^{\lambda, \theta, T}(y_i, y_j)}{\det_{i,j=1}^{N} H^{\lambda_1+\theta_1 \pi i, T}(y_i, y_j)}. \quad (7.9)$$

As a result of Theorem 7.1 and (7.5), the correlations of the measure $P_n^{\lambda, T}$ are then given by the affine sum

$$p_n^{\lambda, T}(y_1, \ldots, y_N) = \sum_{\theta \in \{0,1\}^2} \frac{Z_n(\lambda, T)}{Z_n(\lambda, T)} \det_{i,j=1}^{N} H^{\lambda_1+\theta_1 \pi i, T}(y_i, y_j), \quad (7.10)$$

of the correlation functions of the constituent signed measures.
7.2 Correlation functions and inversion

A brief calculation tells us that the correlation functions are the unique functions satisfying

\[
\frac{1}{Z_n(\lambda, T)} \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}^n} \prod_{i=1}^{N} (1 + \phi(y_i)) g_N^{\lambda, \theta, T}(y_1, \ldots, y_N) dy_1 \ldots dy_N
\]

\[
= \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}^n} \prod_{i=1}^{N} \phi(y_i) p_N^{\lambda, \theta, T}(y_1, \ldots, y_N) dy_1 \ldots dy_N.
\]

(7.11)

for all bounded and measurable \( \phi : \mathbb{T}_n \to \mathbb{C} \).

The following lemma, which is a variation on a well-known technique (see e.g. [39, 6]), tells us that the correlation functions of \( \mathbf{P}_n^{\lambda, \theta, T} \) are determinantal.

**Lemma 7.2.** Under the above conditions, the correlation functions of \( \mathbf{P}_n^{\lambda, \theta, T} \) are given by

\[
p_N^{\lambda, \theta, T}(w_1, \ldots, w_N) = \frac{1}{\det(I + TC)} \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}^n} \prod_{i=1}^{N} (1 + \phi(w_i)) g_N^{\lambda, \theta, T}(w_1, \ldots, w_N) dw_1 \ldots dw_N
\]

(7.12)

where \( \bar{H}_{\lambda, \theta, T}^{0, \theta, \pi, i} := TC_{\lambda, \theta, T}(I + TC_{\lambda, \theta, T})^{-1} \).

**Proof.** In order to lighten notation, throughout the proof we will suppress dependence on \( \lambda \) and \( \theta \) by writing \( C := C_{\lambda, \theta, \pi, i} \) and \( \bar{H}_T := H_{\lambda, \theta, T}^{0, \theta, \pi, i} \), etc.

In light of (7.11), we need to establish that for all bounded and measurable \( \phi : \mathbb{T}_n \to \mathbb{C} \) we have

\[
S_\phi := \frac{1}{Z_{n, \theta}(\lambda, T)} \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}^n} \prod_{i=1}^{N} (1 + \phi(w_i)) g_N^{\lambda, \theta, T}(w_1, \ldots, w_N) dw_1 \ldots dw_N
\]

\[
= \frac{1}{\det(I + TC)} \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}^n} \prod_{i=1}^{N} \phi(w_i) \det \bar{H}_T(w_i, w_j) dw_1 \ldots dw_N.
\]

(7.13)

To this end, using (2.6) to obtain the first equality below, and (2.9) to obtain the second, we have

\[
S_\phi = \frac{1}{2}(-1)_{(\theta_1 + 1)(\theta_2 + \theta_1 + 1)} (1 - e^{-(\lambda + \theta_1 + \theta_2 + \pi i)})^n \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}^n} \prod_{i=1}^{N} (1 + \phi(w_i)) \sum_{i,j=1}^{N} \det C(w_i, w_j) dw_1 \ldots dw_N
\]

\[
= \frac{1}{\det(I + TC)} \sum_{N \geq 0} \frac{T^N}{N!} \int_{\mathbb{T}^n} \prod_{i=1}^{N} (1 + \phi(w_i)) \sum_{i,j=1}^{N} \det C(w_i, w_j) dw_1 \ldots dw_N
\]

(7.14)

Now if we define a new operator \( D_\phi \) on \( \mathbb{T}_n \) by setting \( D_\phi(w, w') := (1 + \phi(w))C(w, w') \), (7.14) reads

\[
S_\phi = \frac{\det(I + TD_\phi)}{\det(I + TC)}.
\]

(7.15)

Write \( D_\phi = C + C_\phi \), where \( C_\phi(w, w') := \phi(w)C(w, w') \). Using the multiplicativity of Fredholm determinants, we have

\[
S_\phi = \frac{\det(I + T(C + C_\phi))}{\det(I + TC)} = \det(I + H_\phi^T),
\]

(7.16)

where \( H^T = TC(I + TC)^{-1} \) and \( H_\phi^T(v, v') = \phi(v)H^T(v, v') \). On the other hand, by definition \( \det(I + H_\phi^T) \) is precisely the quantity on the right-hand-side of (7.13), so that completes the proof.
7.3 Inversion

Define \( \tilde{H}^{\lambda,\theta,T} := TC^{\lambda,\theta}(I + TC^{\lambda,\theta})^{-1} \), where \( C^{\lambda,\theta} := C^{\theta_2}_{\lambda + \theta_1 \pi i} \), and

\[
C^{\theta_2}_{\beta}(w, w') = \frac{e^{\theta_2 \pi i/n}}{1 - e^{-\beta}} h_{t' = t + 1} e^{-\beta|t'|t|c},
\]  

(7.17)

The main result of this section computes this operator:

**Lemma 7.3.** Let \( \tilde{H}^{\lambda,\theta,T} := TC^{\lambda,\theta}(I + TC^{\lambda,\theta})^{-1} \). Then

\[
\tilde{H}^{\lambda,\theta,T}(w, w') = e^{\theta_2 \pi i/n} \frac{T}{n} \sum_{z^n = (-1)^{\theta_2}} z^{1 + h - h'} e^{-(\lambda + \theta_1 \pi i + Tz)[t' - t]} \frac{e^{-\beta|t'|t|c}}{1 - e^{-(\lambda + \theta_1 \pi i + Tz)}}.
\]

(7.18)

In our proof of Theorem 7.3, we will use the following identity

\[
\int_{\mathbb{R}/\mathbb{Z}} e^{-\lambda (s-t)} e^{-\lambda [t'-s]} \frac{1}{1 - e^{-\lambda}} \left[ e^{-\lambda [t'-t]} \frac{1}{1 - e^{-\lambda}} - e^{-\lambda [t'-t]} \right].
\]

(7.19)

To prove (7.19), by periodicity, one can assume without loss of generality that \( t = 0 \), and thereafter use the definition \([t' - s] = t' - s + 1_{s > t} \).

We are now equipped to prove Theorem 7.3.

**Proof of Theorem 7.3.** By replacing \( T \) with \( T' = e^{\theta_2 \pi i/n} T \), it suffices to prove the case \( \theta_2 = 0 \). We write \( \beta := \lambda + \theta_1 \pi i \). For the duration of the proof we write \( H := \tilde{H}^{\lambda,\theta,T} \) and \( C := C^{\theta_2}_{\lambda + \theta_1 \pi i} \) for short.

In order to verify that \( H = TC(I + TC)^{-1} \), it is sufficient to establish the identity \( HC = C - \frac{1}{T} H \), or more explicitly, for \( y, y' \in \mathbb{T}_n \), we need to show that

\[
\int_{\mathbb{T}_n} H(y, y'') C(y'', y') dy'' = C(y, y') - \frac{1}{T} H(y, y').
\]

(7.20)

Expanding the left-hand-side of (7.20) more explicitly with \( y = (t, h) \) etc., and using the definitions of \( H \) and \( C \) in (7.17) and (7.25) (recall we are setting \( \beta := \lambda + \theta_1 \pi i \)), we have

\[
HC(y, y') := \sum_{h'' \in \mathbb{Z}_n} \int_{\mathbb{R}/\mathbb{Z}} d\tau'' \frac{T}{n} \sum_{z^n = 1} z^{1 + h - h'} e^{-(\beta + Tz)[t'' - t]} \frac{e^{-\beta|t' - t'|c}}{1 - e^{-(\beta + Tz)}} \frac{1}{1 - e^{-\beta}}.
\]

(7.21)

Note that the sum is supported on \( h'' \equiv h' - 1 \mod n \), so that \( z^{1 + h'' - h} = z^{2 + h' - h} \). Consequently, (7.21) reduces to

\[
HC(y, y') := \int_{\mathbb{R}/\mathbb{Z}} d\tau'' \frac{T}{n} \sum_{z^n = 1} z^{1 + h - h'} e^{-(\beta + Tz)[t' - t]} \frac{e^{-\beta|t' - t'|}}{1 - e^{-(\beta + Tz)}} e^{-\beta|t' - t'|c}.
\]

(7.22)

Examining now the dependence of the integrand in \( t'' \) and using (7.19) with \( \lambda = \beta \) and \( \lambda' = \beta + Tz \), we see that this further reduces to

\[
HC(y, y') := \frac{T}{n} \sum_{z^n = 1} z^{2 + h - h'} \frac{1}{Tz} \left[ \frac{e^{-\beta|t' - t'|}}{1 - e^{-\beta}} - \frac{e^{-\beta|t' - t'|c}}{1 - e^{-(\beta + Tz)}} \right].
\]

(7.23)

Using the definition of \( H \), we can separate out the terms on the right-hand-side of (7.24) to obtain

\[
HC(y, y') := \frac{1}{n} \sum_{z^n = 1} z^{1 + h - h'} \frac{e^{-\beta|t'|t|c}}{1 - e^{-\beta}} - \frac{1}{T} H(y, y').
\]

(7.24)

The result now follows from using the fact that for \( h, h' \in \mathbb{Z}_n \) we have \( \frac{1}{n} \sum_{z^n = 1} z^{1 + h - h'} = 1_{\{h' = h + 1\}}. \)

\( \Box \)
We are now equipped to prove Theorem 7.1.

**Proof of Theorem 7.1.** According to Lemma 7.2, the correlation functions are given by

\[ p_N^{\lambda,\theta,T}(w_1, \ldots, w_N) = \det_{i,j=1}^{N} \tilde{H}^{\theta_2}_{\lambda+\theta_1 i,T}(w_i, w_j), \]

where \( \tilde{H}^{\theta_2}_{\lambda+\theta_1 i,T} := TC^{\theta_2}_{\lambda+\theta_1 i}(I + TC^{\theta_2}_{\lambda+\theta_1 i})^{-1}. \)

In Lemma 7.3 we computed \( \tilde{H}^{\theta_2}_{\lambda+\theta_1 i,T} \) explicitly. The final adjustment we make is to note that since determinants are invariant under conjugation by a diagonal matrix, we have

\[ \det_{i,j=1}^{N} H^{\lambda,\theta,T}(y_i, y_j) = \det_{i,j=1}^{N} H^{\lambda,\theta,T}(y_i, y_j) \]

with

\[ H^{\lambda,\theta,T}(w, w') = \frac{T}{n} \sum_{z^n = (-1)^\theta_2} z^{1 + h - h'} e^{-(\lambda + \theta_1 i + T h z)[t' - t]} \frac{1 - e^{-(\lambda + \theta_1 i + T h z)}}{1 - e^{-(\lambda + \theta_1 i + T z)}}, \] (7.25)

completing the proof of Theorem 7.1. \( \square \)

### 8 Probability measures on occupation processes

In this section we develop the apparatus to prove the results stated in Section 4. Recall that in the last section we constructed complex measures \( P_n^{\lambda,\theta,T} \) on the set \( T_0 \) of (possibly empty) bead configurations on \( T_n \), and thereafter showed in Theorem 7.1 that they have a tractable determinantal structure. In this section we show that the random occupation process \( (X_t)_{t \in [0,1]} \) associated with a such a bead configuration has a joint determinantal structure with the beads.

We say a bead configuration \( (y_1, \ldots, y_{nk}) \) with \( y_i = (t_i, h_i) \) is regular if \( t_i \neq t_j \) for each \( i \neq j \). The measures \( P_n^{\lambda,\theta,T} \) have almost all of their mass on regular configurations.

Let \( \mathcal{P}(\mathbb{Z}_n) \) denote the collection of all subsets of \( \mathbb{Z}_n \). We write \( \mathcal{T} \) for the set of occupation processes. An occupation process is a right-continuous functions \( X : [0,1] \rightarrow \mathcal{P}(\mathbb{Z}_n) \) satisfying \( X_0 = X(1) \) with finitely many (possibly zero) discontinuities, with the additional property that at each discontinuity \( t \) of \( X \) there exists \( h \in X_t \) such that \( X_t = X(t-) \cup \{h+1\} - \{h\} \). In this case we say \((t,h)\) is a bead of \( X \). Clearly any \( X \) of this form has constantly cardinality \( \ell \), which we again call the occupation number. Write \( \mathcal{T}_{k,\ell} \) for the set of elements of \( \mathcal{T} \) with \( nk \) discontinuities and occupation number \( \ell \). We say that \( X \) is trivial if \( X \) lies in some \( \mathcal{T}_{0,\ell} \), in which case there is a cardinality-\( \ell \) subset \( A \) of \( \mathbb{Z}_n \) such that \( X_t = A \) for all \( t \in [0,1] \).

We now explain the relationship between bead configurations and occupation processes:

- There is a bijective relationship between non-empty regular bead configurations and non-trivial occupation processes: each discontinuity with \( X_t = X(t-) \cup \{h+1\} - \{h\} \) corresponds to a bead at \((t,h)\).
- There are \( 2^n \) different occupation processes in correspondence with the empty bead configurations. These are the set of constant functions \( X_t = A \) for all \( t \in [0,1] \), where \( A \) is any given subset of \( \mathbb{Z}_n \).

We now extend the measures \( P_n^{\lambda,\theta,T} \) and \( P_n^{\lambda,T} \) from the set \( T_0 \) of bead configurations to the set \( \mathcal{T} \) of occupation processes by setting, for subsets \( B \) of \( \mathbb{Z}_n \) of cardinality \( \ell \),

\[ P_n^{\lambda,\theta,T}(X_t = B \text{ for all } t \in [0,1]) := \frac{1}{2} (-1)^{(\theta_1+1)(\theta_2+n+\ell+1)} e^{-\lambda \ell} \frac{1}{Z_n^{\theta}(\lambda, T)}. \] (8.1)

Our task in the remainder of this section is to prove Theorem 4.2, which gives an explicit determinantal representation for the joint laws of the beads and the occupation process under \( P_n^{\lambda,\theta,T} \). We begin in the next section by working in a more general setting, studying properties of continuous determinantal processes on the torus.
8.1 A differential complementation principle on $T_n$

**Definition 8.1.** We say a Kernel $K : T_n \times T_n \to \mathbb{C}$ is nice if $K : T_n \times T_n \to \mathbb{C}$ takes the form $K((t, h), (t', h')) = K(t' - t, h' - h)$ for some $K : \mathbb{Z}_n \times \mathbb{Z}_n \times (-1, 1)$. Suppose further that

- $K(\cdot, h)$ is differentiable on $(-1, 1)$ whenever $h \neq 0$.
- $K(\cdot, 0)$ is differentiable on $(-1, 1) - \{0\}$, and is left-continuous and left-sided-differentiable at zero.

The main result of this section is the following:

**Theorem 8.2.** Suppose under a probability measure $P$ we have an occupation process $X$ for which

$$P(y_1, \ldots, y_N \text{ are occupied}) = \det(K(y_i, y_j)),$$

where $K$ is a nice kernel. Write $\gamma := \dot{K}(0, 0) - \dot{K}(0, 0^+)$. Then for all distinct $(y_i)_{i \in \mathcal{O} \cup \mathcal{U} \cup \mathcal{B}}$ we have

$$P(\Gamma(y_i : i \in \mathcal{B} \cup \mathcal{U}) = \left(\frac{\det}{\prod_{i \in \mathcal{B}} dt_i} K_{i,j}^*\right) \prod_{i \in \mathcal{B}} dt_i$$

where, setting $K_{i,j} := K(y_i, y_j)$ and $K'_{i,j} := K'(y_i, y_j)$ we have

$$K_{i,j}^* = \begin{cases} K_{i,j} & \text{if } i \in \mathcal{O} \\ \delta_{i,j} - K_{i,j} & \text{if } i \in \mathcal{U} \\ K'_{i,j} + \gamma K_{i,j} & \text{if } i \in \mathcal{B}. \end{cases}$$

Before giving our proof of Theorem 8.5, we recall the following intermediate result, the complementation principle for determinantal processes, which is attributed in [6] to Kerov. See the [6, Section A.3] for a proof.

**Lemma 8.3 (The complementation principle).** Suppose under $P$ we have a solid determinantal process $O$ with kernel $K$. Then

$$P(t_i \in X \forall i \in \mathcal{O}, t_i \notin X \forall i \in \mathcal{U}) = \det_{i,j \in \mathcal{O} \cup \mathcal{U}} K_{i,j}^*,$$

where, setting $K_{i,j} := K(y_i, y_j)$ we have

$$K_{i,j}^* = \begin{cases} K_{i,j} & \text{if } i \in \mathcal{O} \\ \delta_{i,j} - K_{i,j} & \text{if } i \in \mathcal{U}. \end{cases}$$

Our next corollary provides some characterisation of the behaviour of the kernel at zero.

**Lemma 8.4.** Suppose under $P$ we have a solid determinantal process $O$ with nice kernel $K$. Suppose $K(0) = K(0-) = \lim_{t \to 0} K(t) = \mu$. Then $\mu \in [0, 1]$ and $K(0+) = \lim_{t \searrow 0} K(t) = -(1 - \mu)$.

**Proof.** Clearly, $P(0 \in X) = \mu$. It follows that $\mu \in [0, 1]$. Since $X$ is solid, we have $P(0 \in X, \varepsilon \in X) = P(0 \in X) + o(1)$ as $\varepsilon \downarrow 0$. Computing the determinant, it then follows that we must have

$$\mu + o(1) = P(0 \in X, \varepsilon \in X) = K(0)^2 - K(-\varepsilon)K(\varepsilon) = \mu(\mu - K(-\varepsilon)) + o(1).$$

This implies $K(\varepsilon) = -(1 - \mu) + o(1)$, as required.
We now prove Theorem 8.5.

Proof. We proceed by induction on the cardinality of $B$. The result is clearly true when $B = \emptyset$, since here it is equivalent to the standard complementation principle, Lemma 8.3.

Let $(y_i : i \in \mathcal{O} \sqcup \mathcal{U} \sqcup B)$ denote any collection of distinct points, and write

$$\Gamma := \{ y_i \in X \forall i \in \mathcal{O}, y_i \notin X \forall i \in \mathcal{U}, dy_i \text{ contains a bead } \forall i \in B \}$$

Let $s$ be distinct from the $y_i$. For small $\varepsilon$ we now compute

$$P(\Gamma, [s, s + \varepsilon) \text{ contains a bead}) = P(\Gamma, s \in X, s + \varepsilon \notin X) + o(\varepsilon).$$

Define new indices $a, b$ with $y_a = s$ and $y_b = s + \varepsilon$. Setting $\tilde{O} = \mathcal{O} \cup \{a\}$ and $\tilde{U} = \mathcal{U} \cup \{b\}$ we have

$$P(\Gamma, s \in X, s + \varepsilon \notin X) = \left( \det_{i,j \in \tilde{O} \sqcup \tilde{U} \sqcup B} K^*_i,j \right) \prod_{i \in B} dy_i.$$

We now study the small-$\varepsilon$ behaviour of the matrix $A := (K^*_i,j)_{i,j \in \tilde{O} \sqcup \tilde{U} \sqcup \tilde{B}}$. We begin by looking at $(K^*_i,j)_{i,j \in \tilde{U} \sqcup \tilde{B}}$.

We now compare the $a$th and $b$th row of $K$. For $j \notin \{a, b\}$ we have $K_{a,j} := K(y_j - s)$ and $K_{b,j} := K(y_j - s - \varepsilon) = K(y_j - s) - \varepsilon K'(y_j - t_a) + o(\varepsilon) = K_{a,j} - \varepsilon K'_{a,j} + o(\varepsilon)$. In short

$$K_{b,j} = K_{a,j} - \varepsilon K'_{a,j} + o(\varepsilon) \quad \text{for } j \notin \{a, b\}. \quad (8.3)$$

We now study the $\{a, b\} \times \{a, b\}$ corner. Write $\mu = K(0) = K(0^-)$, $A = K'(0^-)$ and $B = K'(0^+)$. Then using Lemma 8.4 we have

$$K_{a,a} = K_{b,b} = \mu, \quad K_{a,b} = -(1 - \mu) + B\varepsilon + o(\varepsilon), \quad K_{b,a} = \mu - A\varepsilon + o(\varepsilon). \quad (8.4)$$

Recall now $a \in \mathcal{O}$ and $b \in \mathcal{U}$. Using the definition of $K^*_i,j$ in conjunction with (8.3), we see that the $a$th and $b$th row of $K^*$ satisfy

$$K^*_{a,j} = K_{a,j} \quad \text{and} \quad K^*_{b,j} = -K_{a,j} + \varepsilon K'_{a,j} + o(\varepsilon) \quad \text{for } j \notin \{a, b\}. \quad (8.5)$$

Likewise, using the definition of $K^*_i,j$ in conjunction with (8.4), we see that the $\{a, b\} \times \{a, b\}$ corner of $K^*$ takes the form

$$K^*_{a,a} = \mu, \quad K^*_{b,b} = (1 - \mu) \quad K^*_{a,b} = -(1 - \mu) + B\varepsilon + o(\varepsilon), \quad \text{and} \quad K^*_{b,a} = -\mu + A\varepsilon + o(\varepsilon). \quad (8.6)$$

Finally, we also have $K^*_{i,j} = K_{i,a} + o(\varepsilon)$. (We won’t need to compute this object explicitly.)

Set $Q := \mathcal{O} \sqcup \mathcal{U} \sqcup \tilde{B}$. Then writing $K^*$ as a $(Q \cup \{a\} \cup \{b\}) \times (Q \cup \{a\} \cup \{b\})$ matrix, it takes the form

$$A := (K^*_i,j)_{i,j \in \tilde{O} \sqcup \tilde{U} \sqcup \tilde{B}} = \left[ \begin{array}{ccc} K^*_{i,j} & K^*_{i,a} & K^*_{i,a} + O(\varepsilon) \\ K_{a,j} & \mu & -1 + \mu + B\varepsilon \\ -K_{a,j} + \varepsilon K'_{a,j} & -\mu + A\varepsilon & 1 - \mu \end{array} \right] + o(\varepsilon).$$

Adding $(1 + \varepsilon(B - A))$ copies of the $a$th row from the $b$th row, and then subtracting the $a$th column from the $b$th column, we see that $A$ has the same determinant as

$$B := \left[ \begin{array}{ccc} K^*_{i,j} & K^*_{i,a} & O(\varepsilon) \\ K_{a,j} & \mu & -1 + B\varepsilon \\ \varepsilon((B - A)K_{a,j} + K'_{a,j}) & \varepsilon((B - A)\mu + A) & 0 \end{array} \right] + o(\varepsilon).$$

Noting that $\mu = K_{a,a}$ and $A = K'_{a,a}$, we see that up to $o(\varepsilon)$, the determinant of $B$ equal to the $\varepsilon$ times the determinant of $(K^*_i,j)_{\tilde{O} \sqcup \tilde{U} \sqcup \tilde{B}}$, where $\tilde{B} = B \cup \{a\}$, completing the proof. \qed
8.2 A differential complementation principle on \( \mathbb{R} \)

For the sake of applicability for other research on determinantal processes, in this section we briefly state without proof here an analogue of the main result of the previous section for mixed determinantal processes on \( \mathbb{R} \).

Suppose we have a solid determinantal process \( X \) on \( \mathbb{R} \) with kernel \( K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \), namely, (8.2) holds. We say that a point \( t \) in \( \mathbb{R} \) is a bead of the process if there exists a (random) \( \varepsilon > 0 \) such that \( (t - \varepsilon, t) \subseteq X \), and \( (t, t + \varepsilon) \subseteq \mathbb{R} - X \). Then we have the following

**Theorem 8.5.** Suppose under \( \mathbb{P} \) we have \( X \), a solid determinantal process on \( \mathbb{R} \) with translation invariant correlation kernel \( K : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \) of the form \( K(t, t') = K(t' - t) \). Suppose further that \( K \) is left-continuous at zero, and \( K \) is differentiable on \( \mathbb{R} - \{0\} \). Set \( \gamma := K'(0^+) - K'(0^-) \). Then for all \( y_i \in \mathbb{R} \) we have

\[
\mathbb{P}(t_i \in X \forall i \in \mathcal{O}, t_j \notin X \forall j \in \mathcal{U}, dt_k \text{ contains a bead } \forall k \in \mathcal{B}) = \left( \prod_{i,j \in \mathcal{O} \cup \mathcal{U} \cup \mathcal{B}} \frac{\det K_{i,j}^*}{\prod_i dt_i} \right)
\]

where, setting \( K_{i,j} := K(t_j - t_i) \) and \( K_{i,j}^* := K'(t_j - t_i) \) we have

\[
K_{i,j}^* = \begin{cases}
K_{i,j} & \text{if } i \in \mathcal{O}, \\
\delta_{i,j} - K_{i,j} & \text{if } i \in \mathcal{U}, \\
K_{i,j} + \gamma K_{i,j} & \text{if } i \in \mathcal{B}.
\end{cases}
\]

8.3 Proof of Theorem 4.2

**Proof of Theorem 4.2.** We now verify that the kernels \( K_{n}^{\lambda, \theta, T} \) and \( H_{n}^{\lambda, \theta, T} \) are nice in the sense of Definition 8.1. We first note that the kernel \( K_{n}^{\lambda, \theta, T} \) is nice. Indeed, consider the function \( K_{n}^{\lambda, \theta, T} : \mathbb{R}/\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C} \) given by

\[
K_{n}^{\lambda, \theta, T}(t, h) = \frac{1}{n} \sum_{z^n = (-1)^{\theta_2}} z^{-h} e^{-(\lambda + \theta_1 \pi i + T z)(t+1_z,0)} \frac{1 - e^{-(\lambda + \theta_1 \pi i + T z)}}{1 - e^{-(\lambda + \theta_1 \pi i + T z)}}.
\]

A brief calculation using the fact that \( \sum_{z^n = (-1)^{\theta_2}} z^{-h} = 0 \) whenever \( h \neq 0 \) tells us that for \( h \neq 0 \) we have \( \lim_{t \uparrow 0} K_{n}^{\lambda, \theta, T}(t, h) = \lim_{t \downarrow 0} K_{n}^{\lambda, \theta, T}(t, h) \).

The reader may check that if

\[
\mu := \lim_{t \uparrow 0} K_{n}^{\lambda, \theta, T}(t, h) = K_{n}^{\lambda, \theta, T}(0, h) = \frac{1}{n} \sum_{z^n = (-1)^{\theta_2}} z^{-h} e^{-(\lambda + \theta_1 \pi i + T z)} \frac{1 - e^{-(\lambda + \theta_1 \pi i + T z)}}{1 - e^{-(\lambda + \theta_1 \pi i + T z)}}.
\]

Then

\[
\lim_{t \downarrow 0} K_{n}^{\lambda, \theta, T}(t, h) = -(1 - \mu),
\]

c.f. Lemma 8.4. A fairly lengthy calculation tells us that

\[
\lim_{t \downarrow 0} \frac{d}{dt} K_{n}^{\lambda, \theta, T}(t, 0) - \lim_{t \uparrow 0} \frac{d}{dt} K_{n}^{\lambda, \theta, T}(t, 0) = \beta.
\]

Moreover, for all \( (t, h) \), \( K_{n}^{\lambda, \theta, T}(t, h) \) solves the ODE

\[
\frac{d}{dt} K_{n}^{\lambda, \theta, T}(t, h) + \beta K_{n}^{\lambda, \theta, T}(t, h) = H_{n}^{\beta, T}(t, h),
\]

where \( H \) is as in (7.8). Since the beads determine the occupation process, it follows that we may reverse the order of implication in Theorem 8.5, completing the proof. \( \square \)
9 Scaling limits and exclusion processes on the ring

Recall that \( P_{n,\theta,T}^\lambda \) is a signed measure on the set \( \mathcal{T} \) of right-continuous functions \( X : \mathbb{R}/\mathbb{Z} \to \mathcal{P}(\mathbb{Z}_n) \), where \( \mathcal{P}(\mathbb{Z}_n) \) is the collection of subsets of \( \mathbb{Z}_n \). In this section we study two scaling limits of these measures that recover genuine \textit{probability} measures as limiting objects:

- The \textit{horizontal} scaling limit involves sending the density parameter \( T \) to infinity, and rescaling space by \( 1/T \), so that we recover a bead process on \( n \) infinitely long strings indexed by \( \mathbb{Z}_n \). Here we build on the recent work of Gordenko [17].

- The \textit{vertical} scaling limit involves sending the number \( n \) of strings to infinity. In this setting we recover the cylindrical bead model of Metcalfe, O’Connell and Warren [35].

- In both cases, we thereafter take a second scaling limit to recover the doubly infinite bead process of Boutillier [7], though with the additional information provided by the replicated correlation functions.

In the remainder of this section we will freely recall definitions from Section 4.

9.1 The horizontal scaling limit

In this section we prove Theorem 4.5.

Recall the measures \( P_{n,\ell} \) introduced in Section 9. Shortly, we will show that the probability measures \( P_{n,\ell} \) may be obtained from the complex measures \( P_{n,\theta,T}^\lambda \) through a scaling limit. This relationship involves taking the density parameter \( T \) to infinity in the setting of Theorem 4.2. When \( T \) becomes large, the density of beads on \( \mathbb{T}_n \) under \( P_{n,\theta,T}^\lambda \) takes the order \( T \), and for this reason we consider correlations at a horizontal distance of order \( 1/T \). More specifically, we consider the scaling limit

\[
\lambda + \theta \pi i = pT \\
y_T = (t, h), y_T' = (t', h') \text{ with } t = 1 + s/T, t' = 1 + s'/T,
\]

where \( p \) is a complex parameter controlling the tilt of the configuration. Assume now that the parameter \( p \) is chosen so that \( \text{Re}(p + z) \neq 0 \) for all \( z^n = (-1)^{\theta_2} \).

Consider now a bead configuration under the probability law \( P_{n,\theta,T}^\lambda \) under the scaling \( \lambda + \theta \pi i = pT \) for some fixed \( p \). We rescale space using the affine map \( \phi : [0, 1] \to [-\frac{T}{2}, \frac{T}{2}] \) mapping \( \phi(t) := (t - \frac{1}{2})T \). This creates a bead configuration on \( \mathbb{T}^*_n := [-\frac{T}{2}, \frac{T}{2}] \times \mathbb{Z}_n \) under a signed measure \( \tilde{P}_{n,\theta,T}^p \). We may further associate with this bead configuration a rescaled occupation process \( (X_T^t)_{t \in [-\frac{T}{2}, \frac{T}{2}]} \) taking values in the set of subsets of \( \mathbb{Z}_n \). This bead configuration on \( [-\frac{T}{2}, \frac{T}{2}] \times \mathbb{Z}_n \) can then be canonically associated with a bead configuration on \( \mathbb{R} \times \mathbb{Z}_n \) having no beads outside of \( [-\frac{T}{2}, \frac{T}{2}] \times \mathbb{Z}_n \). Moreover, we can extend the occupation process \( (X_t)_{t \in [-\frac{T}{2}, \frac{T}{2}]} \) to \( (X_t)_{t \in \mathbb{R}} \) by simply setting \( X_t = X_T/t = X_{-T}/t \) for \( t \) in \( \mathbb{R} \setminus [-\frac{T}{2}, \frac{T}{2}] \).

The following lemma relates our measure \( P_{n,\theta,T}^\lambda \) to the limit measures \( P_{n,\ell} \) appearing in Gordenko [17].

**Lemma 9.1.** Suppose in the setting of (9.1), the parameter \( p \) is distinct from the real part of each \( z^n = (-1)^{\theta_2} \), and exactly \( \ell \) of the numbers \( \{(p + z) : z^n = (-1)^{\theta_2}\} \) have negative real part. Then as \( T \to \infty \), under the probability measure \( P_{n,\theta,T}^\lambda \), the occupation process \( X_T \) converges in distribution to a process \( X^\infty \) with the law \( P_{n,\ell} \).

**Proof.** Since \( (X_T^t)_{t \geq 0} \) is a.s. cadlag and takes values in a compact space, it is sufficient to establish the convergence of the finite dimensional distributions \( \mathbb{P}^\lambda X(t_1) = E_1, \ldots, X(t_n) = E_n \). These quantities may be given explicitly in terms of determinants of the kernel \( K_{n,\theta,T}^\lambda \). It is thus sufficient to establish the convergence of the kernels \( K_{n,\theta,T}^\lambda((s, h), (s', h')) := K_{n,\theta,T}^\lambda((\frac{1}{2} + \frac{s}{T}, h), (\frac{1}{2} + \frac{s'}{T}, h')) \) under the scaling limit (9.1).
Plugging the scaling limit (9.1) into (6.8) and using the definition \([t'-t] + 1_{t' = t} = 1_{t < t'} + t' - t + 1_{t' = t} = 1_{s' \leq s} + s'/s\), the kernels take the form

\[
K_{\lambda, \theta, T}^\lambda(y_T, y_T') = -\frac{1}{n} \sum_{z^n = (-1)^{\theta_2}} z^{h' - h} e^{-T(p + z)} \left(1_{s' \leq s} + \frac{z'}{T}\right) \frac{1 - e^{-T(p + z)}}{1 - e^{-T(p + z)}}. \tag{9.2}
\]

For each \(z\), the asymptotics of the ratio inside the sum in (9.2) depend on the indicator \(1_{s' \leq s}\) and whether the real part \(\text{Re}(p + z)\) of \(p + z\) is positive or negative. Indeed, if \(\text{Re}(p + z) > 0\), then

\[
\lim_{T \to \infty} e^{-T(p + z)} \left(1_{s' \leq s} + \frac{z'}{T}\right) \frac{1 - e^{-T(p + z)}}{1 - e^{-T(p + z)}} = e^{-(p+z)(s'-s)} 1_{s' > s}, \tag{9.3}
\]

whereas if \(\text{Re}(p + z) < 0\) we have

\[
\lim_{T \to \infty} e^{-T(p + z)} \left(1_{s' \leq s} + \frac{z'}{T}\right) \frac{1 - e^{-T(p + z)}}{1 - e^{-T(p + z)}} = -e^{-(p+z)(s'-s)} 1_{s' \leq s}. \tag{9.4}
\]

By our assumptions on \(p\) in the statement of the lemma, with \(\mathcal{L}_{n, \ell}\) and \(\mathcal{R}_{n, \ell}\) as in Definition 4.3 we have

\[
\{ z : \text{Re}(p + z) < 0 \} = \mathcal{L}_{n, \ell} \quad \text{and} \quad \{ z : \text{Re}(p + z) > 0 \}. \tag{9.5}
\]

Consequently, by (9.3) and (9.4) we have

\[
\lim_{T \to \infty} K_{\lambda, \theta, T}^\lambda(y_T, y_T') = \widetilde{K}_{n, \ell}^{p}(s, h, (s', h')) := \left\{ \begin{array}{ll}
\frac{1}{n} \sum_{z \in \mathcal{L}_{n, \ell}} z^{h' - h} e^{-(p+z)(s'-s)} & \text{if } s' < s, \\
-\frac{1}{n} \sum_{z \in \mathcal{R}_{n, \ell}} z^{h' - h} e^{-(p+z)(s'-s)} & \text{if } s' \geq s.
\end{array} \right. \tag{9.6}
\]

While this limiting kernel \(\widetilde{K}_{n, \ell}^{p}\) clearly depends on \(p\), it turns out that the relevant determinants involving \(\widetilde{K}_{n, \ell}^{p}\) do not. Indeed, we note from conjugation by the diagonal operator \(\Delta_{i, j} = \delta_{i, j} e^{ps_i}\) that for any \(y_1, \ldots, y_N \in \mathbb{R} \times \mathbb{Z}_n\) with \(y_i = (s_i, h_i)\), we have

\[
\frac{\det}{\det} \left( \begin{array}{c}
\widetilde{K}_{n, \ell}^{p}(y_i, y_j) \\
\end{array} \right) = \frac{\det}{\det} \left( \begin{array}{c}
K_{n, \ell}(y_i, y_j)
\end{array} \right),
\]

where \(K_{n, \ell}\) is given in (4.4).

That completes the proof of the result. \(\square\)

We emphasise that the parameter \(p := \frac{\lambda + \theta\pi i}{T}\) does affect the limiting behaviour of the measures \(P_{\lambda, \theta, T}^n\), but only through the number of roots \(z\) of \((-1)^{\theta_2}\) whose real parts lie to the left of \(-p\).

We now prove the extension Theorem 4.5.

**Proof of Theorem 4.5.** The proof follows quickly from Lemma 9.1 and Theorem 4.2. Indeed, we need only establish the convergence of the bead kernel \(H_{\lambda, \theta, T}^\lambda\) (rather than, as in the proof of Lemma 9.1, the occupation kernel \(K_{\lambda, \theta, T}^\lambda\)) under the scaling limit (9.1). As in the proof of Lemma 9.1, one can show that

\[
\lim_{T \to \infty} H_{\lambda, \theta, T}^\lambda(y_T, y_T') = e^{p(s - s')} H_{n, \ell}(y, y')
\]

where \(H_{n, \ell}(y, y')\) is as in (4.5), and then note that the factor \(e^{p(s - s')}\) may be removed by conjugation. \(\square\)

In the next section, we study the measure \(P_{n, \ell}\) in more detail. We begin by explicitly showing that its correlation functions ensure that \(P_{n, \ell}\) is supported on the event \(\{ \#X_t = \ell \ \forall t \}\), and provide a point of clarification on \(P_{n, \ell}\) as a limit of complex measures. Thereafter we analyse its stationary distributions and transition rates, and introduce an exponential transform of the process yielding connections with the totally asymmetric exclusion process on \(\mathbb{Z}_n\).
9.2 Basic occupation probabilities under $P_{n,\ell}$

We begin this section by giving a representation for the occupation probabilities of $X_t$ under $P_{n,\ell}$. We begin in Lemma 9.2 and Corollary 9.3 by showing that $P_{n,\ell}(\#X_t = \ell \forall t) = 1$ follows directly from the kernels

**Lemma 9.2.** Taking an enumeration of the roots $L_{n,\ell} = \{z_1, \ldots, z_\ell\}$ and $R_{n,\ell} = \{z_{\ell+1}, \ldots, z_n\}$, and suppose $\{h_1, \ldots, h_n\}$ is an enumeration of the set $\mathbb{Z}_n$. Then for any $t \in \mathbb{R}$ we have

$$P(X_t = \{h_1, \ldots, h_p\}) = \frac{\det B_{j,k} \det C_{j,k}}{\det n_{j,k = 1} B_{j,k} \det C_{j,k}}$$

where

$$B_{j,k} := (1_{j \leq p, k \leq \ell} + 1_{j \geq p+1, k \geq \ell+1}) \frac{z_{h_j}}{\sqrt{n}}$$

and

$$C_{j,k} = \frac{1}{\sqrt{n}} z_j^{-h_k}.$$

**Proof.** According to either Theorem 4.4 or Theorem 4.5 we have

$$P(X_t = \{h_1, \ldots, h_p\}) = P(h_1, \ldots, h_p \text{ are occupied}, h_{p+1}, \ldots, h_n \text{ are unoccupied}) = \frac{\det (A_{j,k})}{\det n_{j,k = 1} A_{j,k}}$$

where $A_{j,k} = 1_{j \geq p+1} \delta_{j,k} + (-1)^{j+1} K_{j,k}$ and $K_{j,k} := K_{n,\ell}((t, h_j), (t, h_k)) = -\frac{1}{n} \sum z \in L_{n,\ell} z^{h_j-h_k}$. Using the fact that for $h, h' \in \mathbb{Z}_n$ we have $\frac{1}{n} \sum z = (1)^{n} z^{h-h'} = 1$, it follows that whenever $j \geq p+1$ we have

$$A_{j,k} := \delta_{j,k} - K_{j,k} = 1_{j=h_k} - \frac{1}{n} \sum z \in L_{n,\ell} z^{h_j-h_k} = \frac{1}{n} \sum z \in R_{n,\ell} z^{h_j-h_k}.$$

Taking an enumeration of the roots $L_{n,\ell} = \{z_1, \ldots, z_\ell\}$ and $R_{n,\ell} = \{z_{\ell+1}, \ldots, z_n\}$, we may now write $A_{j,k} = \frac{1}{n} \sum n_{j,m = 1} (1_{j \leq p, m \leq \ell} + 1_{j \geq p+1, m \geq \ell+1}) z_{h_m}^{h_j-h_k}$.

One can now check that $A = BC$, where $B$ and $C$ are as in the statement of the lemma, completing the proof of the result. □

**Corollary 9.3.** Whenever $E = \{h_1, \ldots, h_p\}$ does not have cardinality $\ell$, for any $t \in \mathbb{R}$ we have $P_{n,\ell}(X_t = E) = 0$.

**Proof.** The matrix $B$ in Lemma 9.2 may only have full rank when $p = \ell$. □

We constructed $P_{n,\ell}$ as a limit of unit-mass complex measures $P_{n,\ell}^{\lambda,\theta,T}$, so that in principle, without appealing to the construction in Gordenko [17], the reader may like some assurance that $P_{n,\ell}$ is a genuine probability measure, i.e. takes values in $[0, 1]$. The following lemma assures the reader that this is the case.

**Corollary 9.4.** The complex measure $P_{n,\ell}$ gives events positive values. That is, $P_{n,\ell}$ is a probability measure.

**Proof.** By Corollary (9.3), $P_{n,\ell}$ is supported on $\{\#X_t = \ell \forall t \in \mathbb{R}\}$. Now $P_{n,\ell}$ was obtained as a scaling limit of $P_{n,\ell}^{\lambda,\theta,T}$, where, according to Definition 4.3, $\theta_2 = n + \ell + 1$. On the other hand, by (7.4) $P_{n,\ell}^{\lambda,\theta,T}$ is a scalar multiple of the complex measure $Q_{n,\ell}^{\lambda,\theta}$ whose Radon-Nikodym derivative against $\mathbb{R}_n$ is given by $T_n g_{n}^{\lambda,\theta}$ (7.1). By (2.4), $Q_{n,\ell}^{\lambda,\theta}$ and hence $P_{n,\ell}^{\lambda,\theta,T}$ take the sign $(-1)^{(n+\bar{k}+1)(\theta_2+n+\ell+1)}$ on bead configurations with $\bar{k}$ beads per string and occupation number $\ell$. Note from Definition 4.3, $\theta_2 + n + \ell + 1 = 0 \mod 2$, and hence when $\ell = \ell P_{n,\ell}^{\lambda,\theta,T}$ is positive on the event $\{\#X_t = \ell \forall t \in [0, 1]\}$, and accordingly, so is the associated limit $P_{n,\ell}$. □
9.3 Stationary probabilities under $P^{n,\ell}$

We now use a well known identity for Vandermonde determinants, (see e.g. Macdonald [28, Section I.3]) which states that for variables $x_1, \ldots, x_m$ we have

$$\det_{j,k=1}^{m} x_j^{k-1} = \prod_{1 \leq j < k \leq m} (x_k - x_j). \quad (9.7)$$

The identity (9.7) allows us to compute the stationary probabilities $P^{n,\ell}(X_t = E)$ explicitly.

We begin with the following lemma on determinants of root matrices. Recall from Definition 4.3 that $L_{n,\ell} := \{z_1, \ldots, z_\ell\}$ where $z_k := e^{2\pi i \left(n - \ell + 1 \over 2 + k - 1\right)}$. Given an $\ell$-tuple $h = (h_1, \ldots, h_\ell)$ of elements of $\mathbb{Z}_n$, define the $\ell \times \ell$ matrix $A^h = A^{h,n,\ell}$ by

$$A^h_{j,k} := z_k^{h_j}. \quad (9.8)$$

Our following lemma computes the determinant of $A^h$.

Define the function $\Delta : \mathbb{Z}_n^\ell \to [0, \infty)$ by

$$\Delta(h) := \prod_{1 \leq j < k \leq \ell} \left| e^{2\pi i h_k / n} - e^{2\pi i h_j / n} \right|$$

We write $\mathbb{Z}_n^{(\ell)}$ for the set of distinct tuples in $\mathbb{Z}_n^\ell$. Clearly $\det(A^h)$ and $\Delta(h)$ are nonzero only when $h \in \mathbb{Z}_n^{(\ell)}$.

**Lemma 9.5.** For $h \in \mathbb{Z}_n^{(\ell)}$ we have

$$\det_{j,k=1}^{\ell} A^h_{j,k} = \text{sgn}(\sigma_h)(-1)^{\sum_{j=1}^{\ell} h_j \ell(\ell-1)/2} \Delta(h_1, \ldots, h_\ell).$$

where $\sigma_h : \{1, \ldots, \ell\} \to \{1, \ldots, \ell\}$ is the permutation ordering the entries of $h$, i.e. $h_{\sigma(1)} < \ldots < h_{\sigma(\ell)}$.

**Proof.** Using the definition $z_k := e^{2\pi i \left(n - \ell + 1 \over 2 + k - 1\right)}$ we have

$$\det_{j,k=1}^{\ell} A^h_{j,k} = e^{2\pi i \sum_{j=1}^{\ell} h_j \ell(\ell-1)/2} \det_{j,k=1}^{\ell} x_j^{k-1}$$

where $x_j := e^{2\pi i h_j / n}$. Applying (9.7) we obtain

$$\det_{j,k=1}^{\ell} A^h_{j,k} = e^{2\pi i \sum_{j=1}^{\ell} h_j \ell(\ell-1)/2} \prod_{1 \leq j < k \leq \ell} \left( e^{2\pi i h_k / n} - e^{2\pi i h_j / n} \right).$$

The result now follows from using the fact that

$$e^{2\pi i h_k / n} - e^{2\pi i h_j / n} = (-1)^{h_j > h_k} i e^{\pi i h_{\sigma(1)}} \left| e^{2\pi i h_1 / n} - e^{2\pi i h_1 / n} \right|$$

and simplifying. \qed

**Corollary 9.6.** For any $t \in \mathbb{R}$

$$P_{n,\ell}(X_t = \{h_1, \ldots, h_\ell\}) = n^{-\ell} \Delta(h_1, \ldots, h_\ell)^2. \quad (9.9)$$

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Proof. Since $P^{n,\ell}$ is supported on the event \{$X_t$ has cardinality \(\ell\)$}, with probability one the event \{$h_1, \ldots, h_\ell \in X_t$} implies the event \{$X_t = \{h_1, \ldots, h_\ell\}$}, i.e.

$$P^{n,\ell}(X_t = \{h_1, \ldots, h_\ell\}) = P^{n,\ell}(h_1, \ldots, h_\ell \text{ occupied}, \, h_{\ell+1}, \ldots, h_n \text{ unoccupied}) = P^{n,\ell}(h_1, \ldots, h_\ell \text{ occupied}).$$

We can compute this probability explicitly using Theorem 4.5. Indeed, set $B = \emptyset = \emptyset$, let $O = \{1, \ldots, \ell\}$, and set $y_j := (t, h_j)$. Then according to (4.6) we have

$$P^{n,\ell}(X_t = \{h_1, \ldots, h_\ell\}) = \det_{j,k=1}^\ell K_{n,\ell}(y_j, y_k) = \det_{j,k=1}^\ell \left( \frac{1}{n} \sum_{z \in \mathbb{L}_{n,\ell}} z^{h_j-h_k} \right) = n^{-\ell} \det_{j,k=1}^\ell A_{j,k} \det_{j,k=1}^\ell A_{j,k}^*,$$

where $A_{j,k} := z_{k}^{h_j}$ is as in the previous lemma, and and $A_{j,k}^* = A_{k,j}$ its conjugate transpose. The result now follows from applying Lemma 9.5. \qed

9.4 Basic densities under $P_{n,\ell}$

We take a moment to perform some basic calculations for probabilities under $P_{n,\ell}$ using Theorem 4.5 (or Theorem 4.4 for that matter). Setting $O = \{1\}$ and $B = \emptyset = \emptyset$, we can compute the probability that $y_1$ is occupied as

$$P_{n,\ell}(y_1 = (t_1, h_1) \text{ is occupied}) = K_{1,1} = \frac{1}{n} \sum_{z \in \mathbb{L}_{n,\ell}} z^0 = \frac{\ell}{n},$$

which reflects the fact that \(\ell\) of the $n$ strings are occupied at any given time, and hence given the stationarity of the limit any given location has a probability $\ell/n$. Of course then

$$P_{n,\ell}(y_1 = (t_1, h_1) \text{ is unoccupied}) = 1 - K_{1,1} = \frac{n - \ell}{n}.$$  

Instead setting $B = \{1\}$ and $O = \emptyset = \emptyset$ we can compute the probability $dy_1$ contains a bead. Indeed, since $\mathcal{R}_{n,\ell}$ consists of $n - \ell$ consecutive roots around the origin, we have

$$P_{n,\ell}(dy_1 \text{ contains a bead}) = H_{1,1} dy_1 = \left( \frac{1}{n} \sum_{z \in \mathbb{R}_{n,\ell}} z \right) dy_1 = \frac{1}{n} \left| \sum_{j=0}^{n-\ell-1} e^{2\pi ij/n} \right| dy_1 = \frac{1}{n} \frac{\sin(\pi \ell/n)}{\sin(\pi/n)} dy_1.$$

In other words, the density of beads per string under $P_{n,\ell}$ is $\frac{1}{n} \frac{\sin(\pi \ell/n)}{\sin(\pi/n)}$.

9.5 Transition probabilities under $P_{n,\ell}$

Let $E = \{h_1, \ldots, h_\ell\}$ and $E' = \{h'_1, \ldots, h'_\ell\}$ be subsets of $\mathbb{Z}_n$. Then with the case $O = \{1, \ldots, 2\ell\}$, $B = \emptyset = \emptyset$ of Theorem 4.5 we have

$$P(X_0 = E, X_t = E') = \det_{j,k=1}^{2\ell} K_{n,\ell}(y_j, y_k),$$

where for $1 \leq j \leq \ell$, $y_j := (0, h_j)$, and for $\ell + 1 \leq j \leq 2\ell$ we have $y_j := (t, h'_{j-\ell})$. In particular, using Corollary 9.6 we have the transition probabilities

$$P(X_t = E'|X_0 = E) = \frac{n^{\ell}}{\Delta(E)^2} \det_{j,k=1}^{2\ell} K_{n,\ell}(y_j, y_k).$$

For small $t$ however, we utilise the complementary form in the formulation Theorem 4.5 to study the transition rates.
Lemma 9.7. Let $h = (h_1, \ldots, h_\ell)$ and $h' = (h'_1, \ldots, h'_\ell)$ be $\ell$-tuples of distinct elements of $\mathbb{Z}_n$, with
\[ h'_j = h_j \text{ for } j \leq \ell - 1 \quad \text{and} \quad h'_\ell := [h_\ell + 1]. \]

Then
\[ \lim_{t \to 0} \frac{1}{t} \mathbb{P}^{n,\ell}(X_t = \{h'_1, \ldots, h'_\ell\}|X_0 = \{h_1, \ldots, h_\ell\}) = \frac{\Delta(h')}{{\Delta(h)}}. \]

Proof. Without loss of generality, let $E = \{h_1, \ldots, h_\ell\}$ and $E' = \{h_1, \ldots, h_{\ell-1}, h_{\ell+1}\}$ where $h_{\ell+1} = h_\ell + 1 \mod n$. We then have
\[ \mathbb{P}^{n,\ell}(X_0 = E, X_t = E') = \mathbb{P}^{n,\ell}(X_0 = E, h_\ell \notin X_t) + o(t). \]

According to Theorem 4.5 with $y_j = (0, h_j)$ for $1 \leq j \leq \ell$ and $y_{\ell+1} := (t, h_\ell)$, $\mathcal{O} = \{1, \ldots, \ell + 1\}$, $\mathcal{U} = \{\ell + 1\}$ and $\mathcal{B} = \emptyset$, we may write $\mathbb{P}^{n,\ell}(X_0 = E, h_\ell \notin X_t)$ as a determinant of an $(\ell + 1) \times (\ell + 1)$ matrix
\[ \mathbb{P}^{n,\ell}(X_0 = E, h_\ell \notin X_t) = \det \begin{pmatrix} K_{j,k} & K_{j,\ell+1} \\ -K_{\ell+1,k} & 1 - K_{\ell+1,\ell+1} \end{pmatrix} \] (9.10)

where for $1 \leq j, k \leq \ell + 1$ we have $K_{j,k} := K_{n,\ell}(y_j, y_k)$. We now compare the $\ell$th and $(\ell + 1)$th rows and columns. First we note that whenever $1 \leq j, k \leq \ell$, \[ K_{j,k} = \frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}} z^{h_j - h_k}. \]

Now for any $k \leq \ell$ we have
\[ K_{\ell+1,k} := K_{n,\ell}(y_{\ell+1}, y_k) = K_{n,\ell}((t, h_\ell), (0, h_k)) := \frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}} z^{h_\ell - h_k} e^{zt} = K_{\ell,k} + tT_{\ell,k} + O(t^2). \] (9.11)

Whereas for any $j \leq \ell$ and $K_{j,\ell+1}$, we require a less sharp estimate. Indeed, we have
\[ K_{j,\ell+1} = K_{n,\ell}(y_j, y_{\ell+1}) = K_{n,\ell}((0, h_j), (t, h_\ell)) := -\frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}} z^{h_j - h_\ell} e^{-zt} \]
\[ = -\frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}} z^{h_j - h_\ell} + O(t) \]
\[ = -\delta_{j,\ell} + K_{j,\ell} + O(t), \] (9.12)

where in the penultimate equality above, we used the fact that for $h, h' \in \mathbb{Z}_n$, \[ \frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}} z^{h - h'} = \delta_{h,h'} - \frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}}. \]

Finally,
\[ K_{\ell+1,\ell+1} = K_{\ell,\ell} = \ell/n. \] (9.13)

Plugging (9.11), (9.12) and (9.13) into (9.10) can write $\mathbb{P}^{n,\ell}(X_0 = E, h_\ell \notin X_t)$ as a determinant of the $((\ell - 1) + 1+1) \times ((\ell - 1) + 1+1)$ matrix
\[ \mathbb{P}^{n,\ell}(X_0 = E, h_\ell \notin X_t) = \det \begin{pmatrix} K_{j,k} & K_{j,\ell} & K_{j,\ell+1} + O(t) \\ K_{j,\ell} & K_{\ell,\ell} & -1 + K_{\ell,\ell} - tT_{\ell,\ell} + O(t^2) \\ -K_{\ell,k} - tT_{\ell,k} + O(t^2) & -K_{\ell,\ell} - tT_{\ell,\ell} + O(t^2) & 1 - K_{\ell,\ell} \end{pmatrix} \] (9.14)

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Adding the $\ell$th row to the $(\ell + 1)$th row, we obtain from (9.14) the representation (again depicting a $((\ell - 1) + 1 + 1) \times ((\ell - 1) + 1 + 1)$ matrix)

$$
P_{n,\ell}(X_0 = E, h_\ell \not\in X_\ell) = \det \begin{pmatrix}
K_{j,k} & K_{j,\ell} & K_{j,\ell} + O(t) \\
K_{j,\ell} & K_{\ell,\ell} & -1 + K_{\ell,\ell} + O(t) \\
-tT_{\ell,k} + O(t^2) & -tT_{\ell,\ell} + O(t^2) & -tT_{\ell,\ell} + O(t^2)
\end{pmatrix}.
$$

(9.15)

Subtracting now the $\ell$th column from the $(\ell + 1)$th column, we have

$$
P_{n,\ell}(X_0 = E, h_\ell \not\in X_\ell) = \det \begin{pmatrix}
K_{j,k} & K_{j,\ell} & O(t) \\
K_{j,\ell} & K_{\ell,\ell} & -1 + O(t) \\
-tT_{\ell,k} + O(t^2) & -tT_{\ell,\ell} + O(t^2) & O(t^2)
\end{pmatrix}.
$$

(9.16)

Upon examination, we see that to first order in $t$, the determinant in (9.16) concentrates on cycles going through the $(\ell, \ell + 1)$ slot. Consequently, we can write $P_{n,\ell}(X_0 = E, h_\ell \not\in X_\ell)$ in terms of the $((\ell - 1) + 1) \times ((\ell - 1) + 1)$ determinant

$$
P_{n,\ell}(X_0 = E, h_\ell \not\in X_\ell) = -t \det \begin{pmatrix} K_{j,k} & K_{j,\ell} \end{pmatrix} + O(t^2).
$$

(9.17)

We note that the $(j, k)^{th}$ entry of the matrix in (9.17) may be written

$$
1_{j \leq \ell - 1} K_{j,k} + 1_{j = \ell} T_{j,k} = \frac{1}{n} \sum_{z \in \mathcal{L}_{n,\ell}} z^{1_{j = \ell} + h_j - h_k} = \frac{1}{n} (A^h A^{h',*})_{j,k},
$$

(9.18)

where $A^h$ is defined in (9.8), $A^{h',*}$ is the conjugate transpose of $A^{h'}$, and $h' := (h_1, \ldots, h_{\ell - 1}, h_\ell + 1)$. Plugging (9.18) into (9.17) we obtain

$$
P_{n,\ell}(X_0 = E, h_\ell \not\in X_\ell) = -t n^{-\ell} \prod_{j,k=1}^{\ell} (A^h)_{j,k} (A^{h'})_{j,k} + O(t^2).
$$

(9.19)

Appealing now to Lemma 9.5 we have

$$
\prod_{j,k=1}^{\ell} A^h_{j,k} = \text{sgn}(\sigma_h)(-1)^{\sum_{j=1}^{\ell} h_j} \ell^{(\ell - 1)/2} \Delta(h_1, \ldots, h_\ell)
$$

and

$$
\prod_{j,k=1}^{\ell} (A^{h'})_{j,k} = \text{sgn}(\sigma_{h'})(-1)^{\sum_{j=1}^{\ell} h'_j} \ell^{(\ell - 1)/2} \Delta(h'_1, \ldots, h'_\ell).
$$

By rotational symmetry, we may assume at this point without any loss of generality that $h_\ell \leq n - 2$, so that $h_{\ell + 1} \leq n - 1$. This convention ensures that the permutations $\sigma_h$ and $\sigma_{h'}$ ordering $h$ and $h'$ are identical, and hence noting that $\sum_{j=1}^{\ell} h_j + \sum_{j=1}^{\ell} h'_j$ is odd we have

$$
\prod_{j,k=1}^{\ell} A^h_{j,k} \prod_{j,k=1}^{\ell} (A^{h'})_{j,k} = -\Delta(h_1, \ldots, h_\ell) \Delta(h'_1, \ldots, h'_\ell).
$$

(9.20)

Plugging (9.20) into (9.21) we obtain

$$
P_{n,\ell}(X_0 = E, h_\ell \not\in X_\ell) = t n^{-\ell} \Delta(h) \Delta(h') + O(t^2).
$$

(9.21)

Dividing through by $P_{n,\ell}(X_0 = E) = n^{-\ell} \Delta(h)^2$, we seal the result.
9.6 Eigenfunctions of Markov chains

Write $\mathbb{Z}_n^\ell$ for the $\ell$ dimensional product of $\mathbb{Z}_n$. $\mathbb{Z}_n^\ell$ is itself a commutative group. Let $e_j = (0, \ldots, 0, 1, 0, \ldots)$ be the element of $\mathbb{Z}_n^\ell$ with a 1 in the $j$th slot and zeroes in the other slots. We write $\mathbb{Z}_n^{(\ell)}$ for the set of distinct tuples of elements of $\mathbb{Z}_n^\ell$. For $h$ in $\mathbb{Z}_n^{(\ell)}$ we write $P_h^{n,\ell}$ for the law of $(X_t)_{t \in \mathbb{R}}$ conditional on the event $\{X_0 = \{h_1, \ldots, h_\ell\}\}$.

According to Lemma 9.7, the generator of $(X_t)_{t \in \mathbb{R}}$ is the linear map $G_{\text{Gordenko}}$ acting on the set of functions $\{f : \mathbb{Z}_n^\ell \to \mathbb{R}\}$ and defined by

$$G_{\text{Gordenko}, n, \ell} f(h) := \lim_{t \to 0} \frac{1}{t} \left( P_h^{n,\ell}[f(X_t)] - f(h) \right) = \sum_{j=1}^\ell \frac{\Delta(h + e_j)}{\Delta(h)} f(h + e_j).$$

(9.22)

We note that if $h_k = h_j + 1$ for some $k \neq j$, the $\Delta(h + e_j)$ is zero. In other words, the sum in (9.22) is supported on those $j$ for which $h_j + 1$ is unoccupied by the configuration $\{h_1, \ldots, h_\ell\}$, so we may alternatively write

$$G_{\text{Gordenko}, n, \ell} f(h) := \sum_{j : h_k \neq h_j + 1 \forall k} \frac{\Delta(h + e_j)}{\Delta(h)} f(h + e_j).$$

(9.23)

We now introduce TASEP (the totally asymmetric exclusion process) on $\mathbb{Z}_n$ to be the Markov chain $(X_t)_{t \in \mathbb{R}}$ with the following dynamics. Suppose at some moment we have $X_t = \{h_1, \ldots, h_\ell\}$. Then we imagine $h_1, \ldots, h_\ell$ to be the positions of $\ell$ independent walkers, each of which jumps from a position $h$ to $[h + 1]$ at rate 1, provided that $[h + 1]$ is unoccupied.

$$G_{\text{TASEP}, n, \ell} f(h) := \sum_{j : h_k + 1 \neq h_j \forall k} f(h + e_j).$$

(9.24)

Namely, the generator of TASEP is similar to that of the noncolliding walkers, but without the factor of $\frac{\Delta(h + e_j)}{\Delta(h)}$.

It is notoriously difficult to describe the transition probabilities of TASEP on the ring $\mathbb{Z}_n$ (as well as on other domains, for that matter). Recent work by Baik and Liu [4] provided a complicated but explicitly formula involving various contour integrals and determinants. The main result of this section is an alternative viewpoint, showing that the probability law of TASEP can be recovered from the simple probability laws $P_{h, \ell}$ via a simple exponential transform. To this end, for $h \in \mathbb{Z}_n^{(\ell)}$ let us write

$$P_h^{n,\ell} \equiv P_h^{n,\ell,\text{Gordenko}} \quad \text{and} \quad Q_h^{n,\ell} \equiv Q_h^{n,\ell,\text{TASEP}}$$

for the respective probability laws of the noncolliding walkers and of TASEP associated with $\ell$ walkers on $\mathbb{Z}_n$ starting from a configuration $X_0 = \{h_1, \ldots, h_\ell\}$.

Recall from the introduction we defined the Traffic of a configuration $h \in \mathbb{Z}_n^{(\ell)}$ to be the number of elements of $\{h_1, \ldots, h_\ell\}$ waiting in traffic:

$$\text{Traffic}(h_1, \ldots, h_\ell) := \#\{(j, k) : 1 \leq j \neq k \leq \ell : h_k = [h_j + 1]\}.$$ 

Alternatively, writing $E$ for the set $\{h_1, \ldots, h_\ell\}$, we may also write

$$\text{Traffic}(E) := \#\{h \in E : [h + 1] \in E\}.$$ 

The main result of this section is the following.

**Theorem 9.8.** Writing $Q_h = Q_h^{n,\ell,\text{TASEP}}$ and $P_h = P_h^{n,\ell,\text{Gordenko}}$ for short, we have

$$\frac{dQ_h}{dP_h} \bigg|_{X_t} = \frac{\Delta(h)}{\Delta(X_t)} \exp \left\{ \int_0^t (\text{Traffic}(X_s) - \mu) \, ds \right\}.$$
In other words, given a measurable event \( A \), we have

\[
Q_h(A) = P_h \left[ \frac{\Delta(h)}{\Delta(X_t)} \exp \left\{ \int_0^t \left( \text{Trafic}(X_s) - \mu \right) \, ds \right\} 1_A \right].
\]

Before proving Theorem 9.8, we recall some general facts about Markov chain generators and their associated exponential transforms. Our work in this section is related to that of König, O’Connell and Roch [27], who show under moment conditions that the Vandermonde determinant in \( k \) variables is harmonic for any random walk in \( \mathbb{R}^k \). Their observation is somehow dual to what we prove in the next few pages, where we work in continuous-time but discrete space.

Suppose \( \{P_x : x \in V\} \) is a collection of probability laws for a \((X_t)_{t \geq 0}\) be a Markov chain taking values in a finite set \( V \). Let \( \mathbb{R}^V := \{\phi : V \to \mathbb{R}\} \). The generator \( \mathcal{G} \) of \((X_t)_{t \geq 0}\) is the linear operator on \( \mathbb{R}^V \) given by

\[
\mathcal{G}\phi(x) := \lim_{t \downarrow 0} \frac{1}{t} P_x [\phi(X_t) - \phi(x)].
\]

Suppose we have a positive function \( \Delta : V \to (0, \infty) \) of \( \mathcal{G} \) satisfying

\[
\mathcal{G} \Delta(x) = p(x) \Delta(x).
\]

Then the stochastic process \( M_t := \frac{\Delta(X_t) e^{-\int_0^t p(X_s) \, ds}}{\Delta(X_0)} \) is a non-negative unit-mean \( P_x \)-martingale for each \( x \in V \), and accordingly we can define a change of measure

\[
\frac{dP^\Delta}{dP_x} \bigg|_{\mathcal{F}_t} := M_t.
\]

It is easily verified that if, for \( x \neq y \) in \( V \), we set \( q(x, y) := \lim_{t \downarrow 0} \frac{1}{t} P_x (X_t = y) \), then

\[
\lim_{t \downarrow 0} \frac{1}{t} P^\Delta_x (X_t = y) = \frac{\Delta(y)}{\Delta(x)} q(x, y).
\]

Moreover, the measures \( P_x \) and \( P^\Delta_x \) are absolutely continuous with respect to one another.

The statement of the Theorem 9.8 then follows from a statement about applying the generator of TASEP to the function \( \Delta : \mathbb{Z}^{(\ell)}_n \to (0, \infty) \). Our main computation is the following lemma.

**Lemma 9.9.** For \( h = (h_1, \ldots, h_\ell) \) in \( \mathbb{Z}^{(\ell)}_n \), define the function

\[
\phi(h) := \det_{j,k=1}^\ell A^h_{j,k},
\]

where \( A^h_{j,k} \) is the matrix introduced in (9.8). Then

\[
\sum_{j=1}^\ell \phi(h + e_j) = -\mu_n,\ell \phi(h) \quad \text{where} \quad \mu_n,\ell := \frac{\sin(\pi\ell/n)}{\sin(\pi/n)}.
\]
Proof. Expanding the determinant as a sum over permutations to obtain the second equality below, and then interchanging the order of summation to obtain the fourth, we have

\[
\sum_{i=1}^{\ell} \phi(h_1, \ldots, h_{i-1}, h_i + 1, h_{i+1}, \ldots, h_{\ell}) = \sum_{i=1}^{\ell} \det_{j,k=1}^{\ell} \left( z_{h_j}^{h_{j+1}} \right) \\
= \sum_{i=1}^{\ell} \sum_{\sigma \in S_\ell} \text{sgn}(\sigma) \prod_{j=1}^{\ell} z_{\sigma(j)}^{h_{j+1}} \\
= \sum_{i=1}^{\ell} \sum_{\sigma \in S_\ell} \text{sgn}(\sigma) z_{\sigma(i)}^{h_j} \prod_{j=1}^{\ell} z_{\sigma(j)}^{h_j} \\
= \left( \sum_{z \in \mathcal{L}_{n,\ell}} z \right) \phi(h_1, \ldots, h_{\ell}).
\]

The result now follows from noting that

\[
\sum_{z \in \mathcal{L}_{n,\ell}} z = -\frac{\sin(\pi \ell/n)}{\sin(\pi/n)} = -\mu_{n,\ell}
\]

which is a result of that fact that \(\sum_{z \in \mathcal{L}_{n,\ell}} z\) is a negative real number with the same modulus as the geometric sum \(\sum_{j=0}^{\ell-1} e^{2\pi i j/n}\).

We are now ready to prove Theorem 9.8.

Proof of Theorem 9.8. We actually prove the reciprocal of the statement in Theorem 9.8, namely that

\[
\frac{\text{d}P_h}{\text{d}Q_h} \bigg|_{\mathcal{F}_t} = \frac{\Delta(X_t)}{\Delta(h)} \exp \left\{ \int_0^t (\mu_{n,\ell} - \text{Free}(X_s)) \text{d}s \right\}.
\]

In light of the discussion preceding Lemma 9.9, and the Lemma 9.7 relating the transition rates of \(Q_h^{n,\ell,\text{TASEP}}\) and \(P_h^{n,\ell,\text{TASEP}}\), in order to prove Theorem 9.8 it is sufficient to establish that if \(G_{n,\ell,\text{TASEP}}\) is the generator of TASEP with \(\ell\) particles on \(\mathbb{Z}_n\), defined in (9.24), we have

\[
G_{n,\ell,\text{TASEP}} \Delta(E) = (\mu_{n,\ell} - \text{Free}(E)) \Delta(E),
\]

where \(\text{Free}(E) := \#\{h \in E : h + 1 \notin E\}\) is the number of particles not in traffic.

To this end, let \(h = (h_1, \ldots, h_{\ell})\) be an enumeration of \(E\), i.e. \(E := \{h_1, \ldots, h_{\ell}\}\).

We calculate

\[
G_{n,\ell,\text{TASEP}} \Delta(E) = -\text{Free}(E) \Delta(E) + \sum_{j : h_j + 1 \notin E} \Delta(h + e_j) = -\text{Free}(E) \Delta(E) + \sum_{j=1}^{\ell} \Delta(h + e_j)
\]

where the final equality above follows from simply using the fact that \(\Delta(h + e_j)\) is zero if \(h + e_j\) is nondistinct.

It remains to show \(\sum_{j=1}^{\ell} \Delta(h + e_j) = -\mu_{n,\ell} \Delta(h)\). Using Lemma 9.5 and the definition of \(\phi\) is Lemma 9.9 to obtain the first equality below, we have

\[
\sum_{j=1}^{\ell} \Delta(h + e_j) = \sum_{j=1}^{\ell} \frac{\phi(h + e_j)}{\text{sgn}(\sigma_{h+e_j})(-1)^{\sum_{k=1}^{\ell}(h_k + \delta_{j,k})} \ell(\ell-1)/2}.
\]

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For reasons of convenience we assume without loss of generality (thanks to rotationally symmetry) that in our enumeration of $E$ we have $h_1 < \ldots < h_\ell \leq n - 2$. Then $\sigma_h$ in the setting of Lemma 9.5 is the identity matrix, and moreover if for some $1 \leq j \leq \ell$, $h + e_j$ is a tuple of distinct elements $\sigma_{h+e_j}$ is also the identity matrix. It follows that $\text{sgn}(\sigma_{h+e_j})$ does not depend on $j$ in the sum in (9.26). Using this fact to obtain the first equality below, then Lemma 9.9 to obtain the second, we have

$$\sum_{j=1}^{\ell} \Delta(h + e_j) = -\frac{1}{\text{sgn}(\sigma_h)(-1)^{j+\ell}} \sum_{j=1}^{\ell} \delta(h + e_j) = \mu_{n,\ell} \Delta(h),$$

(9.27)

where in the final equality above we simply repackaged everything using Lemma 9.5. That completes the proof of (9.25), and thus the proof of Theorem 9.8.

\[\square\]

### 9.7 The vertical scaling limit

While in the majority of this section we studied the horizontal scaling limit of the measures $P_n^{\lambda,\theta,T}$ under a scaling limit $T \to \infty$, and derived a collection of probability measures $P_{n,\ell}$ for bead configurations and the associated occupation processes on $\mathbb{R} \times \mathbb{Z}_n$, in this section we outline an orthogonal scaling limit, taking the number $n$ of strings to infinity, to create the bead process on $[0, 1) \times \mathbb{Z}$ obtained by Metcalfe, O’Connell and Warren in [35].

In short, we are simply sending $n \to \infty$ in the setting of Theorem 4.2. (In this case, no scaling limit is needed.) Respectively taking $n \to \infty$ in (4.4) and (4.5) we obtain

$$H_{\infty}^{\lambda,\theta,T}(y, y') = \int \frac{dz}{2\pi i z} z^{1+h-h'} e^{-(\lambda + \theta_1 \pi i + Tz)[t'-t]} \frac{1}{1 - e^{-(\lambda + \theta_1 \pi i + Tz)}}.$$  

(9.28)

and

$$K_{\infty}^{\lambda,\theta,T}(y, y') := -\int \frac{dz}{2\pi i z} z^{h-h'} e^{-(\lambda + \theta_1 \pi i + Tz)[t'-t] + h_{t,t'}} \frac{1}{1 - e^{-(\lambda + \theta_1 \pi i + Tz)}}.$$  

(9.29)

The following lemma gives an explicit description for these integrals.

**Lemma 9.10.** For $t \in [0, 1]$ and $h \in \mathbb{Z}$ we have

$$M_{\beta}^{T}(t, h) := \int \frac{dz}{2\pi i z} z^{-h} e^{-(\beta + Tz)t} \frac{1}{1 - e^{-(\beta + Tz)}} = T^h \left\{ \frac{1}{h!} \left( \frac{\partial}{\partial z} \right)^h \frac{e^{-(\beta + Tz)t}}{1 - e^{-(\beta + Tz)}} \bigg|_{z=0} + \sum_{|k| \leq k_0} \left( -\beta + 2\pi ik \right)^{-(h+1)} e^{-2\pi i k t} \right\}.$$  

**Proof.** We begin by noting that if $h \geq 0$, the integrand has a pole at $z = 0$ of the order $h + 1$. The integrand also has poles of order 1 at all $z$ of the form

$$z = \frac{-\beta + 2\pi ik}{T}, \quad k \in \mathbb{Z};$$  

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only those poles for which $|z| < 1$ contribute however. As such, by Cauchy’s integral formula we have
\[
\oint_{|z|=1} \frac{dz}{2\pi i z} z^{-h} \frac{e^{-(\beta + T z)t}}{1 - e^{-(\beta + T z)}} = \left. \frac{1}{h!} \left( \frac{\partial}{\partial z} \right)^h \frac{e^{-(\beta + T z)t}}{1 - e^{-(\beta + T z)}} \right|_z=0 + \frac{1}{T} \sum_{|k| \leq k_0} \left( \frac{-\beta + 2\pi i k}{T} \right)^{-(h+1)} e^{-2\pi i k t},
\]
where the sum over $k$ is taken over all $k$ such that $|\beta + 2\pi i k| < T$. After a small rewriting we obtain the result. \hfill $\square$

We remark that from conjugate symmetry, $M^T_{\beta}(t, h)$ takes real values whenever $\beta$ is equal to $\lambda + \theta_1 \pi i$ with $\lambda \in \mathbb{R}$ and $\theta_1 \in \{0, 1\}$. In particular, we have the real kernels
\[
H^L_{\infty,\beta,T}((t, h), (t', h')) = T M^T_{\lambda + \theta_1 \pi i}([t' - t], h' - h - 1)
\]
and
\[
K^L_{\infty,\beta,T}(y, y') = M^T_{\lambda + \theta_1 \pi i}([t' - t] + 1_{t'=t}, h' - h).
\]

Considering just the determinantal structure of the beads for a moment (as opposed to the joint structure of the beads and the occupation process), we see that the beads form a determinantal process on $[0, 1] \times \mathbb{Z}$ with, for $y_i = (t_i, h_i)$, $t_i \in [0, 1)$, $h_i \in \mathbb{Z}$,
\[
\lim_{n \to \infty} P^\beta_n(dy_1, \ldots, dy_N \text{ occupied}) = \frac{\det}{\prod_{i,j=1}^N \left( T M^T_{\lambda + \theta_1 \pi i}([t_j - t_i], h_j - h_i - 1) \right)}.
\]

Now
\[
T M^T_{\lambda + \theta_1 \pi i}([t_j - t_i], h_j - h_i - 1) = T \cdot T^{h_j, -h_i - 1} \left\{ \frac{1}{(h_j - h_i - 1)!} \left( \frac{\partial}{\partial z} \right)^{h_j - h_i - 1} \frac{e^{-(\beta + z)t}}{1 - e^{-(\beta + z)}} \right|_{z=0} + \sum_{|k| \leq k_0} \frac{(-\lambda - \theta_1 \pi i + 2\pi i k)^{-(h_j - h_i + 1)} e^{-2\pi i k [t_j - t_i]}}{\lambda - \theta_1 \pi i + 2\pi i k} \right\},
\]
where we recall from above that the sum over $k$ is taken over all $k$ such that $|\lambda - \theta_1 \pi i + 2\pi i k| < T$. Now when $\theta_1 = 0$, the sum over $k$ has an odd number of terms; when $\theta_1 = 0$, an even number. Moreover, we can conjugate by a diagonal matrix to remove the factor $T \cdot T^{h_j, -h_i - 1}$ at the front. As such, $T$ has no asymptotic effect other than to determine $k_0$ (in conjunction with $\theta_1$ and $\lambda$). In short, writing $p := \# \{ k \in \mathbb{Z} : |\lambda - \theta_1 \pi i + 2\pi i k| < T \}$ for the cardinality of the sum over $k$, for each $p \geq 1$ and each $\lambda \in \mathbb{R}$ we have a probability measure $P_{p,\lambda}$ governing a determinantal bead process on $[0, 1] \times \mathbb{Z}$ with correlation kernel
\[
K((t, h), (t', h')) = \frac{1}{(h' - h - 1)!} \left( \frac{\partial}{\partial z} \right)^{h' - h - 1} \frac{e^{-(\beta + z)t}}{1 - e^{-(\beta + z)}} \bigg|_{z=0} + \sum_{k=1}^p \frac{(-\lambda - 2\pi i j_k)^{-(h' - h - 1)} e^{-2\pi i j_k [t_j - t_i]}}{\lambda - 2\pi i j_k},
\]
where $j_k := k - \frac{p-1}{2}$. This coincides with the determinantal representation in Metcalfe, O’Connell and Warren [35] (who give the higher derivative in terms of a Fourier series).

\section{Appendix}

\subsection{Proof of Theorem 3.2}

In light of (6.10) we will assume without loss of generality that $\ell \leq n/2$. More specifically, we will fix a constant $\tau_0$ such that
\[
\tau_0 \leq \frac{\ell - 1}{n} \leq \frac{\ell}{n} \leq \frac{1}{2} \quad \text{and} \quad \tau_0 \leq \frac{k}{n} \leq \tau_1.
\]
To understand the asymptotics of $\text{Vol}^{(n)}_{k,\ell}$, we will require a fine understanding of the moduli of the quantities $\sum_{w \in S} w$ as $S$ ranges across the subsets of $\{w \in \mathbb{C} : w^n = 1\}$ with cardinality $\ell$. There is a canonical association between subsets $S$ of $\{w^n = 1\}$ and subsets of $\mathbb{Z}_n$ which we use freely in this section, namely, every $S \subseteq \{w^n = 1\}$ may be written $S = \{e^{2\pi ij/n} : j \in S\}$ for a subset $S$ of $\mathbb{Z}_n$. We write $\mathbb{Z}_n^{(\ell)}$ for the collection of subsets of $\mathbb{Z}_n$ with size $\ell$.

For $S$ in $\mathbb{Z}_n^{(\ell)}$ we write $||S|| := |\sum_{j \in S} e^{2\pi ij/n}|$.

We define an equivalence relation on $\mathbb{Z}_n^{(\ell)}$ by declaring $S \sim S'$ if there exists $j \in \mathbb{Z}_n$ such that

$$S' = \{k + j : k \in S\}. \quad (A.2)$$

We note that in the setting of $(A.2)$ we have $\sum_{w \in S'} w = e^{2\pi ij/n} \sum_{w \in S} w$, and consequently

$$\left(\sum_{w \in S'} w\right)^{nk} = \left(\sum_{w \in S} w\right)^{nk} \quad \text{whenever } S' \sim S. \quad (A.3)$$

Write $\mathbb{Z}_n^{(\ell)} := \mathbb{Z}_n^{(\ell)} / \sim$ for the set of equivalence classes. In light of $(A.3)$ and $(1.4)$ we have

$$\text{Vol}^{(n)}_{k,\ell} = \frac{(-1)^k(e^{(\ell+1)n} - 1)}{(nk)!} \sum_{S \subseteq \mathbb{Z}_n^{(\ell)}} \left(\sum_{w \in S} w\right)^{nk}. \quad (A.4)$$

The reader will note that intuitively speaking, the subsets $S$ in $\mathbb{Z}_n^{(\ell)}$ maximising $||S||$ should consist of consecutive elements around the circle. We say a subset $C$ in $\mathbb{Z}_n$ is consecutive if $C = C_{j_0} := \{j_0, j_0 + 1, \ldots, j_0 + \ell - 1\}$ for some $j_0 \in \mathbb{Z}_n$. The consecutive sets are precisely the subsets $S$ for which the modulus the sum $\sum_{w \in S} w$ is maximised; we may compute this quantity explicitly in this case:

$$\sum_{w \in C_{j_0}} w = \sum_{p=0}^{\ell-1} e^{2\pi ij_0 + p/n} = e^{2\pi ij_0/n} e^{2\pi i/n} - 1 = e^{2\pi i/n} - 1 = e^{2\pi i/n} - 1 = \frac{e^{2\pi i(\ell-1)+j_0} \sin(\pi \ell/n)}{\sin(\pi/n)}.$$ 

For sets $S$ which are not consecutive, we might expect that their modulus $||S||$ is related to how far they are from being consecutive in some suitable sense.

In this direction, for any element $S$ of $\mathbb{Z}_n^{(\ell)}$ let $s_0 := \lfloor n \arg S/2\pi \rfloor$. $s_0$ may be regarded as approximating the centre of mass of $S$, though of course we need not have $s_0 \in S$. In any case, set

$$\tilde{\rho}(S) := \sum_{j \in S} |j - s_0|_n,$$

where $|a|_n := \inf_{r \in \mathbb{Z}} |a + rn|$. The quantity $\tilde{\rho}(S)$ is minimised when $C$ is consecutive. In this case, we have

$$\tilde{\rho}(C) = \sum_{-\lfloor \frac{\ell}{2}\rfloor \leq j \leq \lfloor \frac{\ell}{2}\rfloor} |j|.$$ 

We now define

$$\rho(S) := \tilde{\rho}(S) - \tilde{\rho}(C),$$

which measures how far $S$ is from being consecutive.

Let $S$ be an element of $\mathbb{Z}_n^{(\ell)}$ with $\arg S \in [0, 2\pi/n)$, and let $C$ be the unique consecutive set with argument in $[0, 2\pi/n)$. Let $\phi$ be a bijection from $C - S$ to $S - C$. Then

$$\rho(S) = \sum_{j \in C - S} (|\phi(j)|_n - |j|_n).$$

The following tells us that $||S||$ decreases as $\rho(S)$ increases.
Lemma A.1. There is a constant \( C(\tau_0) \) depending on \( \tau_0 \) but independent of \( \ell \) and \( n \) such that whenever \( S \) and \( C \) have sum arguments in \([0, 2\pi/n]\) we have

\[
||S|| \leq ||C|| - C(\tau_0) \frac{\rho(S)}{n} + \frac{4\pi^2}{n^2}.
\]

for all subsets \( S \) of \( \mathbb{Z}_n^{(\ell)} \), with \( n \geq 7 \) and \( \ell \) satisfying (A.1).

Proof. We begin by studying the real part of \( \sum_{w \in S} w \). We have

\[
\sum_{w \in S} \text{Re} \ w = \sum_{w \in C} \text{Re} \ w - \left( \sum_{w \in S} \text{Re} \ w - \sum_{w \in C} \text{Re} \ w \right)
\]

\[
\leq ||C|| - \left( \sum_{w \in S-C} \text{Re} \ w - \sum_{w \in C-S} \text{Re} \ w \right).
\]

Let \( \phi \) be a bijection between \( C - S \) and \( S - C \). Then for each \( j \in C - S \) we have

\[
\cos(2\pi j/n) \geq \cos(\pi \tau_0) \geq \cos(2\pi \phi(j)/n).
\]

Equivalently,

\[
|j|_{\mathbb{Z}_n} \leq \tau_0 n \leq |\phi(j)|_{\mathbb{Z}_n}.
\]

In particular, using the simple bound \( \sum_{w \in C} \text{Re}(w) \leq ||C|| \) we have

\[
\sum_{w \in S} \text{Re} \ w \leq ||C|| - \sum_{j \in C-S} \left( \cos(2\pi j/n) - \cos(2\pi \phi(j)/n) \right).
\]

It is easily verified that

\[
\frac{\cos \theta - \cos \phi}{\phi - \theta} = \frac{1 - \cos(\pi \tau_0)}{\pi \tau_0} \quad \text{for } 0 \leq \theta \leq \pi \tau_0 \leq \phi \leq \pi.
\]

In particular, we have

\[
\sum_{w \in S} \text{Re} \ w \leq ||C|| - \frac{1 - \cos(\pi \tau_0)}{\pi \tau_0} \sum_{j \in C-S} \left( \frac{2\pi |\phi(j)|_{\mathbb{Z}_n}}{n} - \frac{2\pi |j|_{\mathbb{Z}_n}}{n} \right)
\]

\[
= ||C|| - \frac{C(\tau_0)}{n} \rho(S),
\]

(A.5)

where \( C(\tau_0) = \frac{2(1 - \cos(\pi \tau_0))}{\pi \tau_0} \).

Now since \( \arg(S) \in [0, 2\pi/n] \), for all \( n \geq 7 \) we have

\[
||S|| \leq \frac{1}{\cos(2\pi/n)} \text{Re}(S) \leq \frac{1}{1 - \frac{4\pi^2}{n^2}} \text{Re}(S) \leq (1 + \frac{4\pi^2}{n^2}) \text{Re}(S).
\]

(A.6)

Combining (A.5) and (A.6) we have

\[
||S|| \leq ||C|| - \frac{C(\tau_0)}{n} \rho(S) + \frac{4\pi^2}{n^2}.
\]

\( \square \)
For positive integers $N$ let $L_N : = \lfloor \frac{N-1}{2} \rfloor$ and $R_N = \lfloor N/2 \rfloor$. We note then that $L_N + R_N = N - 1$. We can then write

$$Z_n = \{-L_n, -L_n + 1, \ldots, -1, 0, 1, \ldots, R_n - 1, R_n\}.$$  

Let $C$ be the unique consecutive element of $Z_n^{(\ell)}$ whose argument lies in $[0, 2\pi/n)$. We can write

$$C = \{-L_\ell, -L_\ell + 1, \ldots, -1, 0, 1, \ldots, R_\ell\} \subseteq Z_n.$$  

For integers $j$, we now obtain an upper bound on the number of equivalence classes $[S]$ for which $\rho(S) = j$. Let $S$ and $C$ have arguments in $[0, 2\pi/n)$, and embed them as subsets of $\{-L_n, \ldots, 0, \ldots, R_n\}$. Then there are collections of non-negative integers $(\lambda_1, \ldots, \lambda_a), (\beta_1, \ldots, \beta_b), (\gamma_1, \ldots, \gamma_c), (\delta_1, \ldots, \delta_d)$ such that

$$S - C = \{-L_n - \lambda_1, \ldots, L_n - \lambda_a\} \cup \{R_n + \beta_1, \ldots, R_n + \beta_b\}$$

and

$$C - S = \{-L_n + \gamma_1, \ldots, L_n + \gamma_n\} \cup \{R_n - \delta_1, \ldots, R_n + \delta_d\}.$$  

Here $\lambda_1 > \ldots > \lambda_a > 0, \beta_1 > \ldots > \beta_b > 0$ and $\gamma_1 > \ldots > \gamma_c \geq 0$ and $\delta_1 > \ldots > \delta_d \geq 0$. Moreover, $\rho(S) = \lambda_1 + \ldots + \lambda_a + \beta_1 + \ldots + \beta_b + \gamma_1 + \ldots + \gamma_c + \delta_1 + \ldots + \delta_d$.

In particular, there is an injection from the collection of $[S]$ in $Z_n^{(\ell)}$ with $\rho(S) = m$ to the set of quadruplets of integer partitions $\lambda, \beta, \gamma, \delta$ whose masses sum to $m$. In particular,

$$\# \{[S] \in Z_n^{(\ell)} : \rho(S) = m\} \leq \sum_{e+f+g+h=m} P(e)P(f)P(g)P(h).$$

where $P(m)$ is the number of integer partitions with total mass $m$. According to Ramanujan’s bound, there are constants $C, c$ such that

$$P(m) \leq C e^{\sqrt{m}}.$$  

In particular, taking a very rough bound we have

$$\# \{[S] \in Z_n^{(\ell)} : \rho(S) = m\} \leq C^4 m^4 e^{4\sqrt{m}} \leq C' e^{c' \sqrt{m}}, \tag{A.7}$$

for possibly different constants $c', C'$. The constants here are independent of $n$ and $\ell$.

We may now control the contribution to $Vol^{(n)}_{k,\ell}$ from subsets $S$ for which $\rho(S) \geq m$. Decomposing (A.4) we have

$$Vol^{(n)}_{k,\ell} = \frac{(-1)^{k(\ell+1)n}}{(nk)!} \left\{ \sum_{S \in Z_n^{(\ell)}, \rho(S) \leq m} \left( \sum_{w \in S} w \right)^{nk} + \sum_{S \in Z_n^{(\ell)}, \rho(S) > m} \left( \sum_{w \in S} w \right)^{nk} \right\} \tag{A.8}$$

We look at the latter term inside the braces in (A.8), renormalised by the maximal sumsize $||C||$. Indeed, using (A.7) and Lemma A.1 to obtain the second line below we have

$$\frac{1}{||C||^{nk}} \sum_{S \in Z_n^{(\ell)}, \rho(S) > m} \left( \sum_{w \in S} w \right)^{nk} = \frac{1}{||C||^{nk}} \sum_{p > m} \frac{1}{\sum_{S \in Z_n^{(\ell)}, \rho(S) = m} \left( \sum_{w \in S} w \right)^{nk}}$$

$$= O \left( \sum_{p > m} e^{c' \sqrt{p}} \left( 1 - C(\tau_0) \frac{\rho(S)}{||C||^{kn}} + O\left(1/n^2\right) \right)^{nk} \right)$$

$$= O \left( \sum_{p > m} \exp \left\{ c' \sqrt{p} - C(\tau_0) \frac{kp}{||C||} + O(k/n) \right\} \right),$$
where the final line above is simply using $1 - s \leq e^{-s}$. Now $||C|| = \sin(\pi \ell/n) / \sin(\pi/n) = O_{\tau_0}(n)$. In particular, we have

\[
\frac{1}{||C||^{nk}} \sum_{S \in \tilde{\mathcal{Z}}_n^{(\ell)} : \rho(S) > m} \left( \sum_{w \in S} w \right)^{nk} = \sum_{p > m} \exp \{ O(\sqrt{p}) - O(pk/n) + O(k/n) \}, \tag{A.9}
\]

where the $O$ terms depend on $\tau_0$. By (A.1) we have

\[
\frac{1}{||C||^{nk}} \sum_{S \in \tilde{\mathcal{Z}}_n^{(\ell)} : \rho(S) > m} \left( \sum_{w \in S} w \right)^{nk} = O_{\tau_0} \left( e^{-cn} \right),
\]

for some constant $c$ depending only on $\tau_0$.

Suppose $m$ is small compared to $k, \ell, n$. Then if the argument of $S$ is in $[0, 2\pi/n)$, $\rho(S) \leq m$ guarantees that $S$ has all of the same elements as $C$ except for a discrepancy of at most $m$ elements. In fact, $\rho(S) \leq m$ contains all of $\{-L_\ell + m, \ldots, R_\ell - m\}$. In particular, we may associate with $S$ a pair of integer partitions $\mu$ and $\mu$ of length at most $m$ such that

\[
S = \{ \mu_j + R_\ell - j + 1 : 0 \leq j \leq R_\ell \} \cup \{-L_\ell - \mu_j + j - 1 : 1 \leq j \leq L_\ell \}. \tag{A.10}
\]

For such $m$, we have $\rho(S) = |\mu| + |\mu|$, where $|\mu| := \mu_1 + \mu_2 + \ldots$ etc.

For $S$ of the form in (A.10), we have

\[
\sum_{j \in S} e^{2\pi i j/n} = \sum_{j \in C} e^{2\pi i j/n} + \sum_{j=0}^{R_\ell} e^{2\pi i (R_\ell - j + 1)/n} (e^{2\pi i \mu_j/n} - 1) + \sum_{j=1}^{L_\ell} e^{2\pi i (-L_\ell - j + 1)/n} (e^{-2\pi i \mu_j/n} - 1)
\]

\[
= \sum_{j \in C} e^{2\pi i j/n} + \frac{2\pi i}{n} \cos(\pi \tau)(|\mu| - |\mu|) - \frac{2\pi \sin(\pi \tau)}{n}(|\mu| + |\mu|) + O(n^{-3/2}).
\]

Note that $\sum_{j \in C} e^{2\pi i j/n} = \frac{e^{2\pi i j even/n} \sin(\pi \ell/n)}{\sin(\pi/n)} = (1 + O(1/n)) \frac{n \sin(\pi \tau)}{\pi}$.

When $m \leq n^{1/4}$, we have $|\mu|$ and $|\mu|$ $\leq n^{1/4}$, in which case for $S$ of the form (A.10) we have

\[
\frac{\sum_{j \in S} e^{2\pi i j/n}}{\sum_{j \in C} e^{2\pi i j/n}} = \exp \left\{ \frac{\pi}{n \sin(\pi \tau)} \left( -\frac{2\pi \sin(\pi \tau)}{n} (|\mu| + |\mu|) + \frac{2\pi i}{n} \cos(\pi \tau)(|\mu| - |\mu|) \right) + O(n^{-5/2}) \right\}
\]

\[
= \exp \left\{ \frac{1}{n^2} \left( \left( -2\pi^2 + \frac{\cos(\pi \tau)}{\sin(\pi \tau)} \right) |\mu| + \left( -2\pi^2 - \frac{\cos(\pi \tau)}{\sin(\pi \tau)} \right) |\mu| \right) + O(n^{-5/2}) \right\}. \tag{A.11}
\]

We now wrap everything together. Using (A.9) in (A.8), and setting $m = n^{1/4}$ we have

\[
\text{Vol}^{(n)}_{k,\ell} = \frac{n}{(nk)!} \left( \frac{\sin(\pi \ell/n)}{\sin(\pi/n)} \right)^{nk} \left\{ \sum_{S \in \tilde{\mathcal{Z}}_n^{(\ell)} : \rho(S) \leq m} \left( \sum_{w \in S} w \right)^{nk} + O(e^{-cn^{1/4}}) \right\}. \tag{A.13}
\]

Now noting that for $m = n^{1/4}$, the set of $S \in \tilde{\mathcal{Z}}_n^{(\ell)}$ for which $\rho(S) = m$ is in bijection with the pairs of integer partitions $(\mu, \mu)$ with $m = |\mu| + |\mu|$. In particular, by (A.11), when $k = \lambda n$ we have

\[
\text{Vol}^{(n)}_{k,\ell} = \frac{n}{(nk)!} \left( \frac{\sin(\pi \ell/n)}{\sin(\pi/n)} \right)^{nk} \left\{ \sum_{|\mu| + |\mu| \leq n^{1/4}} e^{-\lambda \mu_1^2 |\mu| - \lambda \mu_2^2 |\mu|} + O(\lambda n^{-1/2}) + O(e^{-cn^{1/4}}) \right\}, \tag{A.14}
\]

50
where \( c_1^\tau := 2\pi^2 - \frac{\cos(\pi\tau)}{\sin(\pi\tau)} \) and \( c_2^\tau = 2\pi^2 + \frac{\cos(\pi\tau)}{\sin(\pi\tau)} \). Let
\[
\mathcal{P}(x) := \sum_{\mu} e^{-|\mu|x}
\]
be the generating functions of the integer partitions. Then appealing to the fact that there are \( e^{O(\sqrt{m})} \) integer partitions of mass \( m \), by (A.14) we have
\[
\text{Vol}_{k,\ell}^{(n)} = \frac{n}{(nk)!} \left( \frac{\sin(\pi\ell/n)}{\sin(\pi/n)} \right)^{nk} \left\{ \mathcal{P}(e^{-\lambda c_1^\tau})\mathcal{P}(e^{-\lambda c_2^\tau}) + O(n^{-1/2}) \right\}, \tag{A.15}
\]
where we’ve gathered all of the lower order terms under the \( O(n^{-1/2}) \) umbrella. Finally, multiplying by a factor of \( k^{nk} \) and using Stirling’s formula in (A.15) we have
\[
k^{nk}\text{Vol}_{k,\ell}^{(n)} = (1 + O(n^{-1/2})) \frac{n}{\sqrt{2\pi nk}(nk/e)^{nk}} \left( k \frac{\sin(\pi\ell/n)}{\sin(\pi/n)} \right)^{nk} \mathcal{P}(e^{-\lambda c_1^\tau})\mathcal{P}(e^{-\lambda c_2^\tau}) \tag{A.16}
\]
Now noting \( \sin(\pi/n) = \frac{\pi}{n} - \frac{\pi^3}{6n^3} + O(1/n^5) = \frac{\pi}{n} e^{-\pi^2/6n^2} + O(n^4) \), we have
\[
k^{nk}\text{Vol}_{k,\ell}^{(n)} = (1 + O(n^{-1/2})) \frac{e^{\pi^2/6}}{\sqrt{2\pi \lambda}} \left( \frac{e\sin(\pi\ell/n)}{\pi} \right)^{nk} \mathcal{P}(e^{-\lambda c_1^\tau})\mathcal{P}(e^{-\lambda c_2^\tau}), \tag{A.17}
\]
completing the proof.

**Acknowledgements**

The author is immensely grateful to Elia Bisi, Dominik Schmid and Piotr Dyszewski for their valuable comments, and to Neil O’Connell for pointing out several references.

This research is supported by the EPSRC funded Project EP/S036202/1 Random fragmentation-coalescence processes out of equilibrium.

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