Effect of discrete breathers on macroscopic properties of the Fermi-Pasta-Ulam chain

Elena A. Korznikova, Alina Y. Morkina, Mohit Singh, Anton M. Krivtsov, Vitaly A. Kuzkin, Vakhid A. Gani, Yuri V. Bebikhov, and Sergey V. Dmitriev

1 Institute of Molecule and Crystal Physics, Ufa Research Centre of the Russian Academy of Sciences, Ufa 450054, Russia
2 Ufa State Aviation Technical University, Ufa 450008, Russia
3 Indian Institute of Technology Kharagpur, Kharagpur 721302, India
4 Peter the Great Saint Petersburg Polytechnical University, Polytechnicheskaya Street 29, Saint Petersburg, Russia
5 Institute for Problems in Mechanical Engineering RAS, Bolshoy pr. V.O. 61, Saint Petersburg, Russia
6 Department of Mathematics, National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), Moscow 115409, Russia
7 Theory Department, National Research Center Kurchatov Institute, Institute for Theoretical and Experimental Physics, Moscow 117218, Russia
8 North-Eastern Federal University, Polytechnic Institute (branch) in Mirny, 678170 Mirny, Sakha (Yakutia), Russia
9 National Research Tomsk State University, Tomsk 634050, Russia

For a number of crystals the existence of spatially localized nonlinear vibrational modes, called discrete breathers (DBs), has been demonstrated using molecular dynamics and in a few cases the first-principle simulations. High-resolution imaging of DBs is a challenging task due to their relatively short lifetime ranging typically from 10 to $10^3$ atomic oscillation periods (1 to 100 ps). Another way to prove that DBs exist consists in evaluation of their effect on macroscopic properties of crystals. In this paper, we study the effect of DBs on macroscopic properties of the Fermi-Pasta-Ulam chain with symmetric and asymmetric potentials. The specific heat, thermal expansion (stress), and Young’s modulus are monitored during the development of modulational instability of the zone boundary mode. The instability results in the formation of chaotic DBs followed by the transition to thermal equilibrium when DBs disappear due to energy radiation in the form of small-amplitude phonons. Time evolution of the macroscopic properties during this transition is monitored. It is found that DBs reduce the specific heat for all the considered chain parameters. They increase the thermal expansion when the potential is asymmetric and, as expected, thermal expansion is not observed in the case of symmetric potential. The Young’s modulus in the presence of DBs is smaller than in thermal equilibrium for the symmetric potential and for the potential with a small asymmetry, but it is larger than in thermal equilibrium for the potential with greater asymmetry.

PACS numbers: 05.45.Yv, 63.20.-e

Keywords: Crystal lattice, Fermi-Pasta-Ulam chain, modulational instability, discrete breather, intrinsic localized mode, specific heat, thermal expansion, Young’s modulus

I. INTRODUCTION

Discovery of discrete breathers (DBs), or intrinsic localized modes (ILMs) — spatially localized large-amplitude oscillatory modes in nonlinear lattices free of defects — three decades ago $\cite{1, 2}$ has triggered extensive studies devoted to the phenomenon of vibrational energy localization $\cite{3, 4}$. Experimentally DBs have been excited in the physical systems of different nature, including macroscopic periodic systems $\cite{5, 6}$, nonlinear metamaterials, e.g., granular crystals $\cite{7, 8}$ and arrays of micromechanical cantilevers $\cite{9, 10}$, electrical $\cite{11, 12}$ and optical lattices $\cite{13, 14}$, superconducting Josephson junction arrays $\cite{15, 16}$, etc. DBs can be found in crystal lattices $\cite{17}$, as confirmed by measuring vibrational spectra for alpha-uranium $\cite{18, 19}$, helium $\cite{20}$, NaI $\cite{21, 22}$, graphite $\cite{23}$, and PbSe $\cite{24, 25}$. On the other hand, these experimental results in some cases can be interpreted in different ways and they are still debated $\cite{26}$.

Nowadays, numerical simulations are very important for investigation of DB properties in various types of crystals. Using ab initio simulations, the presence of DBs in strained graphene and graphane has been confirmed $\cite{27, 28}$. With the help of molecular dynamics method DBs have been studied in the ionic crystals $\cite{29, 30}$, lattices with pair-wise potentials $\cite{31, 32}$, crystals with covalent bonding $\cite{33, 34}$, metals $\cite{35, 36}$, intermetallic compounds $\cite{37, 38}$, carbon and hydrocarbon nanomaterials $\cite{39, 40}$, h-BN $\cite{41, 42}$, and DNA $\cite{43, 44}$.

It is of great importance to understand how DBs affect macroscopic properties of crystals $\cite{45}$. In the experimental studies, the connection of anomalies in thermal
expansion \[29\] and heat capacity \[27\] of α-uranium at high temperatures to excitation of DBs has been established. In \[76\] it was shown numerically that transition from ballistic to normal heat conduction is a consequence of presence of DBs in a nonlinear chain. DBs increase (decrease) specific heat of the nonlinear chain with soft (hard) type nonlinearity on-site potential and harmonic nearest-neighbour coupling \[77\].

For a chain with on-site potential, one cannot calculate the effect of DBs on thermal expansion and on elastic properties are calculated together with the microscopic properties. For this we use the same approach as in \[77\], namely, we simulate the modulational instability of the zone-boundary mode \((q = \pi)\) which leads first to energy localization in the form of long-lived DBs and subsequent transition to thermal equilibrium \[20, 53, 87\]. The macroscopic characteristics of the chain in the regime when energy is localized on DBs are compared with that in thermal equilibrium, thus revealing the effect of DBs on those properties.

We note that properties of DBs in the FPU chain have been analyzed by Flach and Gorbach in \[88\].

In Sec. II the model and simulation details are described. The simulation results on modulational instability of the zone-boundary mode and macroscopic properties of the chain are presented in Sec. III. Our conclusions are presented in Sec. IV.

II. THE MODEL AND SIMULATION SETUP

We consider the FPU chain of particles having mass \(m\) (see Fig. 1) described by the Hamiltonian

\[
H = K + P = \sum_{n} \frac{m\dot{u}_{n}^{2}}{2} + \sum_{n} U(u_{n+1} - u_{n}),
\]

where \(K\) and \(P\) are the kinetic and potential energy, \(u_{n}\) and \(\dot{u}_{n}\) are displacement from its equilibrium position and velocity of the \(n\)-th particle, overdot stands for differentiation with respect to time. Each particle is anharmonically coupled to their nearest neighbors:

\[
U(\xi) = \frac{k}{2} \xi^{2} + \frac{\alpha}{3} \xi^{3} + \frac{\beta}{4} \xi^{4},
\]

where \(k\), \(\alpha\) and \(\beta\) are constants.

Without loss of generality, we set \(m=1\) and \(k = 1\). We take \(\beta = 3\) and consider three different values for \(\alpha\), namely, \(\alpha = \{0, -1/4, -1/2\}\). For \(\alpha = 0\) the potential is symmetric, while for the chosen negative values of \(\alpha\) it is an asymmetric single-well potential.

From the Hamiltonian defined above, the following equation of motion can be derived

\[
\dot{\mu}_{n} = \frac{\kappa}{3} ([u_{n+1} + u_{n}]^{2} - [u_{n} + u_{n-1}]^{2}) + \frac{\beta}{4}([u_{n+1} - u_{n}]^{3} - [u_{n} - u_{n-1}]^{3}).
\]

The Störmer method of order six with the time step \(\tau = 10^{-3}\) is used for numerical integration of these equations. With such a time step, the relative change in total energy of the chain in a typical numerical run is not greater than \(10^{-5}\).

Substituting the ansatz \(u_{n} \sim \exp[\alpha(q n - \omega_{q} t)]\) into Eq. (3) with \(\alpha = \beta = 0\), one finds the relation between wave number \(q\) and frequency \(\omega_{q}\) for the small-amplitude normal modes in the form

\[
\omega_{q}^{2} = \frac{2k}{m} (1 - \cos q).
\]

The phonon band of the chain ranges from \(\omega_{\text{min}} = 0\) for \(q = 0\) to \(\omega_{\text{max}} = 2\) for \(q = \pm \pi\).

Here, a chain of \(N = 2048\) particles is considered. Test runs with larger number of particles produced nearly the same results.

Initial conditions are set in the form of the zone-boundary mode \((q = \pi)\) with the amplitude \(A\),

\[
u_{n} = A \sin(\pi n - \omega_{\text{max}} t).
\]

If the amplitude \(A\) is too small, the this mode is modulationally unstable. At \(t = 0\), all the particles have the same energy but the instability entails energy localization which can be characterized by the localization parameter

\[
L = \frac{\sum e_{n}^{2}}{\left(\sum e_{n}\right)^{2}}.
\]

where

\[
e_{n} = \frac{m\mu_{n}^{2}}{2} + \frac{1}{2} U(u_{n+1} - u_{n}) + \frac{1}{2} U(u_{n} - u_{n-1}).
\]

is the energy of \(n\)-th particle.

We define temperature as the averaged kinetic energy per atom,

\[
T = \frac{1}{N} \sum_{n} \frac{m\mu_{n}^{2}}{2}.
\]

Heat capacity of the chain is defined as

\[
C = \lim_{\Delta T \to \infty} \frac{\Delta H}{\Delta T},
\]

where

\[
\Delta H = H(T + \Delta T) - H(T).
\]
where $\Delta H$ is the increment in energy of the chain and $\Delta T$ is the corresponding temperature increment. The specific heat capacity (or simply specific heat) is the heat capacity per particle. Periodic boundary conditions are used meaning that the specific heat at constant volume is calculated.

In our simulations, total energy $\tilde{H}$ is conserved, so that Eq. (9) cannot be used. We characterize the specific heat of the chain at constant volume by the ratio

$$c_V = \frac{\tilde{H}}{K}$$ (10)

where $\tilde{H}$ ($K$) is the total (kinetic) energy of the chain per atom.

In linear systems $\tilde{H} = 2K$ and $c_V = 2$. Due to non-linearity, the kinetic energy can differ from the potential energy resulting in deviation of $c_V$ from this value.

In Section III the time evolution of localization parameter, specific heat, stress in the chain and Young’s modulus of the chain have been calculated. These macroscopic characteristics have been compared in the regime of energy localization by DBs with those in thermal equilibrium.

III. MODULATIONAL INSTABILITY

We excite the zone-boundary mode Eq. (3) in the chain using various amplitudes $A$. From the numerical integration of Eq. (3), we find the change in the localization parameter, specific heat, stress in the chain and Young’s modulus.

A. Energy localization

Figure 2 shows the localization parameter as a function of time for different amplitudes $A$ and three values of the potential asymmetry parameter $\alpha$. For all the curves, the localization parameter is minimal at $t = 0$ that is $L = 1/N = 0.49 \times 10^{-3}$. It increases gradually as a consequence of appearance of modulational instability that leads to formation of DBs and, hence, energy localization. Finally, the system comes to thermal equilibrium and the DBs slowly radiate their energy, the parameter $L$ gradually decreases and oscillates near $1.8 \times 10^{-3}$.

In Fig. 3 the energy distribution over the FPU chain is shown for various values of the potential asymmetry parameter $\alpha$ from $\alpha = 0$ in (a) to $\alpha = -0.5$ in (c) at the time when localization parameter is maximal; here we see sets of highly localized DBs.

The time evolution of the number of DBs produced, $N_{\text{DB}}$, and the corresponding average DB energy, $E_{\text{DB}}$, are shown in Fig. 4 and Fig. 5 respectively, for different values of the zone-boundary mode amplitudes $A$ and for three values of the potential asymmetry parameter $\alpha$. In these plots, by triangles we indicate the points when the localization parameter $L$ reaches its maximum. It can be seen from Fig. 4 that the maximal number of DBs for $\alpha = 0$ very weakly depends on $A$ and for the negative values of $\alpha$ maximal values of $N_{\text{DB}}$ are somewhat greater for larger $A$. The same can be said about the maximal DB energy; it weakly depends on $A$ for $\alpha = 0$ and increases with $A$ for negative $\alpha$.

From Fig. 5 one can also see that $N_{\text{DB}}$ reaches its maximum well before the localization parameter $L$ becomes maximal for the case of $\alpha = 0$ and $\alpha = -0.25$, but when
Figure 4: Number of DBs as the function of time for various amplitudes of the initially excited zone-boundary mode at different values of the potential asymmetry parameter $\alpha$. Triangles indicate the points of maximal localization parameter.

Figure 5: Average DB energy as the function of time for various amplitudes of the initially excited zone-boundary mode at different values of the potential asymmetry parameter $\alpha$. Triangles indicate the points of maximal localization parameter.

Figure 6: Specific heat normalized with respect to the zone-boundary mode amplitude $A$ as a function of time for various values of $A$ at three different values of the potential asymmetry parameter $\alpha$. The black triangular markers indicate the corresponding values at maximal localization parameter $L$. It can be seen that specific heat is close to minimum when DBs are in the system and it increases as the system approaches thermal equilibrium.

B. Specific heat

The variation of specific heat with respect to time is plotted for the various mode amplitudes $A$ in Fig. 6 for three different values of the potential asymmetry parameter $\alpha$. We actually present the deviation of specific heat from its theoretical value of 2 for the linear system, normalized by $A$. The black triangular markers represent the values at which the localization parameter is maximal. From the comparison of Fig. 6 and Fig. 6, we can observe that the specific heat is minimal when the localization parameter is close to its maximum. The specific heat increases during the transition to thermal equilibrium. From this, we conclude that the specific heat of the chain is reduced by the DBs resulting from the modulational instability of the zone-boundary mode. This can be explained quite simply as in our system with hard type anharmonicity, the DB frequency increases with its amplitude. Increase in the oscillation frequency results in an increase of particle velocities and thus, in their kinetic energies. Kinetic energy is in the denominator of Eq. (10) (which means there is an inverse proportionality between $c_V$ and $K$) and hence its increase results in a decrease of $c_V$. Analogously, for the case of soft type anharmonicity, DB frequency decreases with its amplitude and the effect is just opposite, i.e., DBs increase the specific heat.

$\alpha = -0.5$, first $L$ reaches a maximal value and then $N_{DB}$. As for $E_{DB}$, as shown in Fig. 4, the time when it attains maximal value is close to the time when $L$ is maximal.

It is also worth pointing out that the maximal number of DBs, i.e. $N_{DB}$, weakly depends on the values of $\alpha$ (see Fig. 4), whereas the maximal DB energy, $E_{DB}$, rapidly decreases with increasing asymmetry of the potential (see Fig. 5).
The results in a comparable range.

A σ equilibrium, σ (a) ing the corresponding thermal equilibrium values localization. The σ values have been normalized by subtracting the corresponding thermal equilibrium values σ* to get the results in a comparable range.

C. Stress

The time-dependence of stress in the linear chain is plotted for the various mode amplitudes A in Fig. 7 for different values of the potential asymmetry parameter: (a) α = 0, (b) α = −1/4, and (c) α = −1/2. We present here the deviation of stress σ from its value in thermal equilibrium, σ*. For the symmetric potential (α = 0) there is no thermal expansion and σ* = 0. For negative values of α the dependence of σ* on A is given in Fig. 8. In the limit of very small A, i.e., for the linear regime with small amplitude phonons, the stress is zero. For greater asymmetry, the absolute value of σ* is larger for the same value of A [cf. (a) and (b)].

Coming back to Fig. 7 once again according to the position of the triangular markers or from the comparison of Fig. 2 and Fig. 7 it can be seen in (b) and (c) that the stress is maximal when the localization parameter is close to its maximum. During the transition to thermal equilibrium, the stress in the chain decreases.

Note that negative (compressive) stress appears in the system because its thermal expansion is suppressed by the use of periodic boundary conditions. In the chain with free ends, for negative α, thermal expansion at zero stress will be observed.

From this, we conclude that the DBs increase the stress in the FPU chain with periodic boundary conditions and in the case of free boundary conditions they will produce thermal expansion at zero stress.

The time-dependence of the Young’s modulus of elasticity E for the FPU chain is plotted for the various mode amplitudes A in Fig. 8 for (a) α = 0, (b) α = −1/4, and (c) α = −1/2. Again, the difference between E and its value in thermal equilibrium, E*, is given. The values of E* for different amplitudes are plotted in Fig. 9 for (a) α = 0, (b) α = −1/4, and (c) α = −1/2. From the comparison of Fig. 2 and Fig. 8 it can be seen that the modulus of elasticity is minimal when the localization parameter is maximal for α = 0 and α = −0.25. In the limit of very small A one has E* = 1 and the compressive rigidity of the chain increases with increasing A almost equally for different values of α.

From Fig. 8 it can be seen that the effect of DBs on the Young’s modulus is more pronounced for the symmetric potential (α = 0). In this case, E − E* is minimal when DBs are in the system and it increases while the system approaches thermal equilibrium. From this, we conclude that the resulting DBs decrease the modulus of elasticity of the FPU chain with symmetric potential.

The effect of the Young’s modulus reduction by DBs is much weaker for α = −1/4 and this trend gets even reversed for α = −1/2, i.e., in this case the modulus of elasticity is maximal when the localization parameter is maximal and decreases in thermal equilibrium. Thus, no definite conclusion can be made about the effect of DBs on the Young’s modulus of the FPU chain with the asymmetric potential since it can be qualitatively different for different values of the asymmetry parameter.
by subtracting the corresponding thermal equilibrium values as the function of amplitudes of the initially excited zone-boundary mode for different values of the potential asymmetry parameter ($\alpha$). The triangular markers indicate the corresponding values at maximal localization. The values of $E$ have been normalized to get the results in a comparable range.

Figure 9: Modulus of elasticity as the function of time for various amplitudes of the initially excited zone-boundary mode at different values of the potential asymmetry parameter ($\alpha$).

In this study, the effect of DBs on different macroscopic properties such as specific heat, internal stress and Young's modulus were measured during the transition from modulationally unstable zone-boundary mode through the regime with high energy localization on DBs to thermal equilibrium.

It was found that for the chain with any set of parameters, specific heat is reduced by DBs. This is due to the hard-type anharmonicity of the chain with DBs having greater vibrational frequencies for greater amplitudes. In the regime of energy localization by DBs, this leads to an increase in particles velocities and thus, their kinetic energies. Then, according to Eq. (10), an increase of kinetic energy in the DB regime results in the reduction of the specific heat. In the chains with soft-type anharmonicity DB frequency drops with its amplitude and the effect is opposite, i.e., DBs increase the specific heat.

Internal stress in the chain with symmetric potential ($\alpha = 0$) does not appear since thermal expansion is observed only for asymmetric potentials. For negative $\alpha$ values considered in this study, negative (compressive) stress appears in the chain with periodic boundary conditions that suppress free thermal expansion. If free boundary conditions were used, thermal expansion of the chain at zero stress would be observed. The compressive stress is greater in the regime with DBs and thus, DBs increase thermal expansion of the FPU chain with negative $\alpha$.

In the chain with symmetric potential, DBs reduce the Young’s modulus, but with increasing asymmetry the effect gets weaker and for $\alpha = -1/2$ it becomes even reverse, i.e, an increase in modulus of elasticity is observed.

The effect of DBs on macroscopic properties of two-dimensional nonlinear lattices supporting vibrational modes leading to modulational instability with formation of DBs could become a subject of future study.

IV. CONCLUSIONS

In this study, the effect of DBs on different macroscopic properties of $\alpha$-$\beta$-FPU chain was discussed. The properties such as specific heat, internal stress and Young’s modulus were measured during the transition from modulationally unstable zone-boundary mode through the regime with high energy localization on DBs to thermal equilibrium.

Figure 10: The Young’s Modulus of elasticity at thermal equilibrium as the function of Amplitudes of the initially excited zone-boundary mode for different values of the potential asymmetry parameter ($\alpha$).

E.A.K. acknowledges the support of the Grant of the President of the Russian Federation for State support of young Russian scientists (No. MD-3639.2019.2). V.A.G. acknowledges the supported of the MEPhI Academic Excellence Project (Contract No. 02.a03.21.0005, 27.08.2013). The work of S.V.D. was supported by the Russian Foundation for Basic Research, Grant No. 19-02-00971. This work was also partly supported by the Tomsk State University competitiveness improvement programme and State assignment of IMSP RAS. V.A.G. also thanks Tatiana Gani for her help in editing the manuscript.

[1] A. S. Dolgov, Sov. Phys. Solid State 28, 907 (1986).
[2] A. J. Sievers and S. Takeno, Phys. Rev. Lett. 61, 970 (1988).
[3] J. B. Page, Phys. Rev. B 41, 7835 (1990).
[4] S. Flach and C. R. Willis, Phys. Rep. 295, 181 (1998).

[arXiv:patt-sol/9704004]
[36] G. M. Chechin, S. V. Dmitriev, I. P. Lobzenko, and D. S. Ryabov, Phys. Rev. B 90, 045432 (2014).

[37] I. P. Lobzenko et al., Phys. Solid State 58, 633 (2016).

[38] S. A. Kiselev and A. J. Sievers, Phys. Rev. B 55, 5755 (1997).

[39] L. Z. Khadeeva and S. V. Dmitriev, Phys. Rev. B 81, 214306 (2010).

[40] A. Riviere, S. Lepri, D. Colognesi, and F. Piazza, Phys. Rev. B 99, 024307 (2019).

[41] A. Kistanov et al., JETP Lett. 99, 353 (2014).

[42] E. A. Korznikova, S. Yu. Fomin, E. G. Soboleva, and S. V. Dmitriev, JETP Lett. 103, 277 (2016).

[43] N. K. Voulgarakis, G. Hadjisavvas, P. C. Kileres, and G. P. Tsironis, Phys. Rev. B 69, 113201 (2004).

[44] R. T. Murzaev, D. V. Bachurin, E. A. Korznikova, and S. V. Dmitriev, Phys. Lett. A 381, 1003 (2017).

[45] M. Haas et al., Phys. Rev. B 84, 144303 (2011).

[46] O. V. Bachurina, Comp. Mater. Sci. 160, 217 (2019).

[47] O. V. Bachurina, Model. Simul. Mater. Sci. Eng. 27, 055001 (2015).

[48] K. T. Murzaev et al., Comp. Mater. Sci. 98, 88 (2015).

[49] D. A. Terentyev et al., Model. Simul. Mater. Sc. 23, 085007 (2015).

[50] R. T. Murzaev et al., Eur. Phys. J. B 89, 168 (2016).

[51] O. V. Bachurina et al., Phys. Solid State 60, 989 (2018).

[52] N. N. Medvedev, M. D. Starostenkov, and M. E. Manley, J. Appl. Phys. 114, 213906 (2013).

[53] M. D. Starostenkov et al., Russ. Phys. J. 58, 1353 (2016).

[54] V. Dubinko et al., Assessment of discrete breathers in the metallic hydrides, Comp. Mater. Sci. 158, 389 (2019).

[55] P. V. Zakharov et al., Surf. Sci. 679, 1 (2019).

[56] J. A. Baimova, E. A. Korznikova, I. P. Lobzenko, and S. V. Dmitriev, Rev. Adv. Mater. Sci. 42, 68 (2015).

[57] E. A. Korznikova, J. A. Baimova, and S. V. Dmitriev, Europhys. Lett. 102, 60004 (2013).

[58] B. Liu et al., J. Phys. D: Appl. Phys. 46, 305302 (2013).

[59] J. A. Baimova, S. V. Dmitriev, and K. Zhou, Europhys. Lett. 100, 36005 (2012).

[60] E. A. Korznikova et al., JETP Lett. 96, 222 (2012).

[61] A. V. Savin and Yu. S. Kivshar, Phys. Rev. B 85, 125427 (2012).

[62] T. Shimada, D. Shirasaki, and T. Kitamura, Phys. Rev. B 81, 035401 (2010).

[63] Y. Yanase et al., Europhys. Lett. 80, 40008 (2007).

[64] Y. Kinoshita et al., J. Phys. B 44, 032206 (2019).

[65] Y. Doi and A. Nakatani, J. Solid Mech. Mater. Eng. 6, 71 (2012).

[66] L. Z. Khadeeva, S. V. Dmitriev, and Yu. S. Kivshar, JETP Lett. 94, 539 (2011).

[67] I. Evazzade et al., Phys. Rev. B 95, 035423 (2017).

[68] E. Barani et al., Eur. Phys. J. B, 90, 38 (2017).

[69] E. Barani et al., Phys. Lett. A 381, 3553 (2017).

[70] B. Juanico, Y.-H. Sanejouand, F. Piazza, and A. Riviere, Phys. Rev. B 99, 024307 (2019).

[71] A. J. Sievers, Sci. Rep., 044102 (2003).

[72] A. Riviere, S. Lepri, D. Colognesi, and F. Piazza, Phys. Rev. Lett. 96, 125427 (2012).

[73] A. J. Sievers et al., Phys. Rev. B 80, 024307 (2009).

[74] A. J. Sievers et al., Phys. Rev. B 79, 134304 (2009).

[75] A. J. Sievers et al., Phys. Rev. B 84, 035423 (2017).

[76] A. J. Sievers et al., Phys. Rev. Lett. 102, 224101 (2009).

[77] A. J. Sievers et al., Europhys. Lett. 95, 035423 (2017).

[78] A. J. Sievers et al., JETP Lett. 99, 238104 (2007).

[79] A. J. Sievers et al., Phys. Rev. B 90, 045432 (2014).

[80] A. J. Sievers et al., Phys. Rev. B 90, 035423 (2017).

[81] A. J. Sievers et al., Phys. Rev. B 89, 024307 (2019).

[82] A. J. Sievers et al., Phys. Rev. B 90, 035423 (2017).

[83] A. J. Sievers et al., Phys. Rev. B 89, 024307 (2019).

[84] A. J. Sievers et al., Phys. Rev. B 90, 035423 (2017).

[85] A. J. Sievers et al., Phys. Rev. B 89, 024307 (2019).

[86] A. J. Sievers et al., Phys. Rev. B 90, 024307 (2019).

[87] A. J. Sievers et al., Phys. Rev. B 90, 035423 (2017).

[88] A. J. Sievers et al., Phys. Rev. B 90, 035423 (2017).
[74] M. E. Manley, Acta Mater. 58, 2926 (2010) 
   [arXiv:1001.4683].
[75] Y. Ming, D.-B. Ling, H.-M. Li, and Z.-J. Ding, 
   Chaos 27, 063106 (2017) [arXiv:1706.03437].
[76] D. Xiong, D. Saudatmand, and S. V. Dmitriev, 
   Phys. Rev. E 96, 042109 (2017) [arXiv:1706.04750].
[77] M. Singh et al., [arXiv:1907.03280]
[78] V. M. Burlakov and S. Kiselev, 
   Sov. Phys. JETP 72, 854 (1991)
[79] V. V. Mirnov, A. J. Lichtenberg, and H. Guclu, 
   Physica D 157, 251 (2001)
[80] K. Ullmann, A. J. Lichtenberg, and G. Corso, 
   Phys. Rev. E 61, 2471 (2000)
[81] Yu. A. Kosevich and S. Lepri, 
   Phys. Rev. B 61, 299 (2000) 
   [arXiv:cond-mat/9910192].
[82] T. Cretegny, T. Dauxois, S. Ruffo, and A. Torcini, 
   Physica D 121, 109 (1998) [arXiv:cond-mat/9709204].
[83] K. Ikeda, Y. Doi, B. F. Feng, and T. Kawahara, 
   Physica D 225, 184 (2007)
[84] L. Kavitha et al., J. Magn. Magn. Mat. 404, 91 (2016)
[85] L. Kavitha et al., J. Magn. Magn. Mat. 401, 394 (2016)
[86] B. Tang and K. Deng, Nonlinear Dyn. 88, 2417 (2017)
[87] E. A. Korznikova et al., Eur. Phys. J. B 90, 23 (2017)
[88] S. Flach and A. Gorbach, Chaos 15, 015112 (2005) 
   [arXiv:cond-mat/0410026].
[89] D. Saadatmand et al., Phys. Rev. E 97, 022217 (2018) 
   [arXiv:1711.03485].