Black holes with complex multi-string configurations

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November 5, 2001

Abstract

New exact solutions of Einstein equations which describe black hole with radial cosmic strings are constructed in the paper. The case of infinitely thin strings and the case of delocalized strings are considered. The case of delocalized strings allows generalization to dimensions greater than 4.

1 Introduction

4D static charged black hole $M_{\text{BH}}$ has a form of warped product with warp factor $r^2$ of 2D charged black hole $B$ with metric

$$
\begin{align*}
    ds_2^B &= - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}, \quad -\infty < t < +\infty, \quad 0 < r < +\infty,

\end{align*}
$$

and 2D unit sphere $S^2$ with metric

$$
    ds_2^{S^2} = \sin^2 \theta \, d\phi^2 + d\theta^2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi,
$$

coordinate $\phi$ is periodic with period $2\pi$, i.e.

$$
    ds^2 = ds_2^B + r^2 \, ds_2^{S^2}.
$$

Exact solution for a static black hole pierced by a single straight infinitely thin cosmic string $M_{\text{cone}}$ has a similar form, but instead of sphere $S^2$ one has $S^2_{\text{cone}}$ with metric

$$
    ds_2^{S^2_{\text{cone}}} = \sin^2 \theta \, d\phi^2 + d\theta^2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi(1 - \alpha),
$$

coordinate $\phi$ is periodic with period $2\pi(1 - \alpha)$ (Riemannian geometry in space-time with conical defects was considered in $[1]$). String tension is proportional to angle deficit $\alpha$. Similarly, if one glues two equal spherical triangles along the coinciding edges to build manifold $S^2_{\text{triangle}}$, then warped product with warp factor $r^2$ of $B$ and $S^2_{\text{triangle}}$ corresponds to configuration $M_{\text{triangle}}$ of black hole with three radial

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cosmic strings [2]. Exact solution $M_{\text{polyhedron}}$, which describe black hole with $N_s$ radial strings directed along symmetry axes of a regular polyhedron [3], is similarly a warped product with warp factor $r^2$ of $B(1)$ and $S^2_{\text{cone}}[N_s]$. Manifold $S^2_{\text{cone}}[N_s]$ can be constructed by gluing of $2(N_s - 2)$ isometric spherical triangles.

Metric of all the solutions above in the vicinity of string has the form $ds^2 = ds^2_B + r^2 ds^2_{S^2_{\text{cone}}}[N_s]$, $0 \leq \theta \leq \theta_0$. Locally the metric outside the strings is black hole metric. So, all solutions of this sort are build from standard blocks.

The motivation of this work comes from the paper of Frolov and Fursaev [3]. They ask the question whether one can obtain a closed 2D surface with conical singularities by gluing a certain set of different spherical triangles to find a general multi-string configuration attached to a black hole. We construct such solutions and their generalization in the limit of infinite number of infinitely light strings (continuous limit).

2 Notation

D-dimensional space-time, i.e. (pseudo)Riemannian manifold $M$ with metric $ds^2 = g_{MN}dX^M dX^N$, $M, N, \ldots = 0, \ldots, D - 1$ is considered.

For differential forms $A$ and $B$ with components $A_{M_1...M_q}$ and $B_{N_1...N_p}$ the following tensor is introduced

$$(A, B)^{(k)}_{M_{k+1}...M_q N_{k+1}...N_p} = \frac{1}{k!} g^{M_1 N_1} \cdots g^{M_k N_k} A_{M_1...M_q} B_{N_1...N_p}.$$ 

Index $(k)$ indicates the number of indices to contract. If it can not lead to ambiguity, $(k)$ is skipped.

For differential form of power $q$ it is convenient to introduce norm $\|A\|^2 = (A, A)^{(q)}$, Hodge duality operation $*A = (\Omega, A)^{(q)}$, where $\Omega = \sqrt{|g|} d^D X$ is form of volume, and operation $\delta = *^{-1}d*$. Here and below $g = \det(g_{MN})$.

3 Multi-string configuration

The action of the considered system is [4, 5, 6]

$$S = \int_M d^4X \sqrt{-g} \left( \frac{R}{2} - \frac{1}{2} \|F\|^2 \right) - \sum_s \mu_s \int_{V_s} d^2\xi_s \sqrt{-\gamma_s},$$

where index $s$ numerates strings, $\mu_s$ is string tension, $\xi^i_s$, $i = 1, 2$ are world-sheet coordinates, $(\gamma_s)_{ij}$ is metric induced at world-sheet $V_s$. 2-form $F = dA$ is electromagnetic field.

Similar to solutions discussed in Introduction, let us consider space $M_{\text{complex}}$, which is warped product with warp factor $r^2$ of 2D charged black hole $B(1)$ and 2D space $S^2_{\text{complex}}$. Space $S^2_{\text{complex}}$ is built by gluing of spherical triangles cut off a unit sphere. One can identify any two equal edges of triangles considered. Space $M_{\text{complex}}$ outside strings is locally isometric with black hole space $M_{BH}$. In the vicinities of strings, which correspond to vertices of triangles, $M_{\text{complex}}$ isometric with $M_{\text{cone}}$ for certain value of angle deficit $\alpha$. String tension $\mu_s$ and angle deficit $\alpha_s$ are related as $\mu_s = 2\pi\alpha_s$. Manifolds $M_{\text{cone}}$ and $M_{BH}$ are solutions of the model (8). Model (8) is local, so $M_{\text{complex}}$ is also a solution of the same model.
Solution \( M_{\text{complex}} \) may bear no global symmetry, but in the vicinity of every string it is invariant under O(2) rotations about the string.

One can describe the process of building \( S^2_{\text{complex}} \) in terms of gluing of conical defects. In this case there are some non-trivial consistency conditions written in terms of angle deficits. In this paper we are not looking for such conditions. We consider the process of smooth gluing of spherical triangles cut off a unit sphere. In this case the only consistency conditions is equal lengths of edges we glue. The procedure described below is just a formalization of the simple idea above.

One can use the following algorithm of building closed space \( S^2_{\text{complex}} \). Let us introduce closed 2D simplicial complex \( c^2 \) \((\partial c^2 = 0)\), such that every point of the complex belongs to finite number of simplexes, for non-equal 2D simplexes \( \sigma^2_1, \sigma^2_2, \sigma^2_3 \in c^2 \), \( \sigma^2_1 \cap \sigma^2_2 \cap \sigma^2_3 \) is point or empty set, and length function \( L \), which maps all 1D simplexes of \( c^2 \) to segment \((0, \pi/2)\). If \( \partial \sigma^2 = \sigma^1_1 + \sigma^1_2 + \sigma^1_3 \), where \( \sigma^2 \in c^2 \) is 2D simplex, then \( L(-\sigma^1_1) = L(\sigma^1_1) < L(\sigma^1_2) + L(\sigma^1_3) \). It is possible to continuously map every 2D simplex \( \sigma^2 \in c^2 \) with boundary \( \sigma^1_1 + \sigma^1_2 + \sigma^1_3 \) to spherical triangle of unit sphere with edges \( L(\sigma^1_1), L(\sigma^1_2), L(\sigma^1_3) \). To make the algorithm unambiguous one have to choose spherical triangles, which area does not exceed \( \pi/2 \).

One can consider similar gluing procedure with spherical triangles cut off sphere of any radius \( R \in [1, \infty) \) \((R = \infty \text{ corresponds to planar triangles})\). For any vertex \( v \) of the simplicial complex above one can calculate angle deficit \( \alpha(v, R) = \frac{1}{2R} (2\pi - \phi v_1 (R) - \ldots - \phi v_{k_v} (R)) \), where \( \phi v_1 (R), \ldots, \phi v_{k_v} (R) \) are adjacent to \( v \) angles of spherical triangles adjacent to \( v \). \( \frac{d}{dR} \phi v_n (R) > 0 \), so \( \frac{d}{dR} \alpha(v, R) < 0 \). So, for finite simplicial complex, if \( \min \alpha(v, \infty) > 0 \) (finite convex complex), then there exists radius \( R_0 \in [1, \infty) \), such that for any \( v \) and any \( R \in (R_0, \infty) \) angle deficit is positive \( \alpha(v, R) > 0 \). Obviously \( \alpha(v, R) = \hat{\alpha}(v, 1) \), where \( \hat{\alpha}(v, 1) \) is angle deficit calculated with length function \( \hat{L} = \frac{1}{R} L \) instead of \( L \). It allows as to rescale any finite convex complex to guarantee positive \( \alpha(v, 1) \) and positive string tensions. A simple example of finite convex complex is any convex polyhedron (to convert it into simplicial complex one have to add some extra edges to split faces into triangles).

## 4 Continuous limit

Let us consider the following action

\[
S = \int_M d^4X \sqrt{-g} \left( \frac{R}{2} - \frac{1}{2} ||F||^2 - ||J|| \right),
\]

where 2-form \( F = dA \) is electromagnetic field, and 2-form \( J = d\phi^1 \wedge d\phi^2 \) is string (membrane) field (see [8]-[13]). Norm \( ||J|| \) represents density of string matter. By integration of \( J \) over 2D surface \( U \) one can find the total tension \( \mu_U \) of strings (string flux), which intersect the surface

\[
\mu_U = \int_U J.
\]

One can consider action \([4]\) as continuous limit of thin string action \([3]\) discussed in previous section. Vice versa thin string action \([3]\) corresponds to singular limit of action \([4]\). So, the formulae of this section under appropriate interpretation are applicable to thin strings case too.
By variation of fields $g_{MN}$, $A_M$ and $\varphi^\alpha (\alpha = 1, 2)$ one can find equations of motion

$$R^M_N - \frac{1}{2} \delta^M_N R = (F, F)^N_M - \frac{1}{2} ||F||^2 \delta^M_N + \frac{1}{||J||} (J, J)^N_M - ||J|| \delta^M_N,$$  \hfill (6)

$$\delta F = 0, \quad \delta \frac{J}{||J||} = 0.$$  

Let space-time $\mathcal{M}$ is warped product with warp factor $r^2$ of 2D black hole $\mathcal{B}$ and an arbitrary 2D Riemannian manifold $\mathcal{S}$ with metric $ds^2_\mathcal{S} = e^{2f(x,y)} (dx^2 + dy^2)$.

The following fields are solution of equations of motion (6)

$$ds^2 = - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2 e^{2f(x,y)} (dx^2 + dy^2),$$  \hfill (7)

$$F = \sqrt{2} \frac{Q}{r^2} dt \wedge dr, \quad J = (\mathcal{K} - 1) \Omega_\mathcal{S},$$

where $\Omega_\mathcal{S} = e^{2f(x,y)} dx \wedge dy$ is 2-form of volume at $\mathcal{S}$, and

$$\mathcal{K} = - e^{-2f(x,y)} \Delta f(x, y)$$  \hfill (8)

is Riemannian curvature of $\mathcal{S}$ ($\Delta = \partial_x^2 + \partial_y^2$). In definition of norm $||J||$ one have to choose branch of square root, which corresponds to $||J|| = \frac{\mathcal{K} - 1}{r^2}$, so the case $\mathcal{K} < 1$ corresponds to negative string tension.

For compact $\mathcal{S}$ by integrating of $J$ defined by (7) the total string tension is

$$\mu_\mathcal{S} = 2\pi \chi(\mathcal{S}) - A(\mathcal{S}),$$  \hfill (9)

where $\chi(\mathcal{S})$ is Euler characteristic of $\mathcal{S}$, and $A(\mathcal{S}) > 0$ is area of $\mathcal{S}$. If one require string tension to be positive, then $\mu_\mathcal{S} > 0$, and $\chi(\mathcal{S}) > 0$ is necessary condition on topology of $\mathcal{S}$.

## 5 Multidimensional continuous case

In multidimensional case straightforward generalization of multi-string solutions is not possible, but one can generalize spherically symmetric version of continuous solution (7). Let us consider the action

$$S = \int_\mathcal{M} d^D X \sqrt{-g} \left( \frac{R}{2} - \frac{1}{2} ||F||^2 - ||J|| \right),$$  \hfill (10)

where 2-form $F = dA$ is electromagnetic field, and $(D - 2)$-form $J = d\varphi^1 \wedge \ldots \wedge d\varphi^{D-2}$ is string field. By variation of fields $g_{MN}$, $A_M$ and $\varphi^\alpha (\alpha = 1, \ldots, D - 2)$ one can find equations of motion, which have the form identical to (6).

The following fields are solution of equations of motion (6)

$$ds^2 = - \left( 1 - \frac{2(M + \mu r)}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}} \right) dt^2 + \frac{dr^2}{1 - \frac{2(M + \mu r)}{r^{D-3}} + \frac{Q^2}{r^{2(D-3)}}} + r^2 g^\mathcal{S}_{ij} dx^i dx^j,$$  \hfill (11)

$$F = \sqrt{(D - 3)(D - 2)} \frac{Q}{r^{D-2}} dt \wedge dr, \quad J = (D - 2) \mu \Omega_\mathcal{S},$$
where \( x^i (i, j = 1 \ldots D - 2) \) are coordinates at \((D - 2)\)-dimensional Riemannian manifold \( S \) with metric \( ds_S^2 = g_{ij}^S dx^i dx^j \), which satisfies the condition (Einstein equations at \( S \))

\[
R_{ij}^S = (D - 3) g_{ij}^S \tag{12}
\]

(e.g. unit sphere \( S^{D-2} \)). \( \Omega_S \) is volume form at \( S \). In definition of norm \( \| J \| \) one have to choose branch of square root, which corresponds to \( \| J \| = (D - 2) r^{D-2} \).

6 Uniqueness of 4D case

The effective mass of solution (11) at large \( r \) is linear in \( r \). One can expect this feature of solution a priori, because string energy is proportional to string length and radial strings pierce every surface \( r = const > r_0 \). Newtonian gravitational potential is \( M + \mu r \), so only in 4D case one can express radial uniform string distribution in terms of deformation of surface \( S \) (this surface up to scale factor is isometric to horizon).

In 4D case surface \( S \) is 2D. 2D Riemannian tensor has only one independent component \( R_{1212} \), it forces manifold \( S \) in (11) to be a unit sphere (locally), but in solution (7) one is able to choose any manifold to be the horizon. So, using solutions (7), (11) it is possible to choose any horizon geometry in 2D case. If \( D > 4 \) horizon geometry is restricted by equation (12). In 4D case 2D horizon geometry defines the density of string matter in the bulk. If \( D > 4 \) string matter density is controlled by a single parameter \( \mu \).

7 Discussion

The purpose of this work was to find exact static solutions which describe black hole with radial strings. In 4D case the solutions were built using thin strings or string field (7). 4D case appears to be very special (see discussion in the previous section).

In multidimensional cases \((D > 4)\) solutions (11) were found only in terms of string field. String matter density \( \| J \| = (D - 2) r^{D-2} \) is constant along \( r = const \) surfaces, so solutions of form (11) \((D > 4)\) do not describe thin strings.

It is an interesting problem to find a static black hole solution with radial thin strings in multidimensional case \((D > 4)\) or to prove absence of solutions of this sort.

Another problem is to study static black p-brane solutions with radial membranes of different dimensions.

The author is grateful to I.V. Volovich, M.O. Katanaev, V.P. Frolov and D.V. Fursaev. The work was partially supported by grant RFFI 99-01-00866.

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