SCALE-FREE AND QUANTITATIVE UNIQUE CONTINUATION
FOR INFINITE DIMENSIONAL SPECTRAL SUBSPACES OF
SCHRÖDINGER OPERATORS

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Abstract. We prove a quantitative unique continuation principle for infinite
dimensional spectral subspaces of Schrödinger operators. Let \( \Lambda_L = (-L/2, L/2)^d \)
and \( H_L = -\Delta_L + V_L \) be a Schrödinger operator on \( L^2(\Lambda_L) \) with a
bounded potential \( V_L : \Lambda_L \to \mathbb{R}^d \) and Dirichlet, Neumann, or periodic bound-
dary conditions. Our main result is of the type
\[
\int_{\Lambda_L} |\phi|^2 \leq C_{\text{sfuc}} \int_{W_\delta(L)} |\phi|^2,
\]
where \( \phi \) is an infinite complex linear combination of eigenfunctions of
\( H_L \) with exponentially decaying coefficients, \( W_\delta(L) \) is some union of equidistributed \( \delta \)-
balls in \( \Lambda_L \) and \( C_{\text{sfuc}} > 0 \) an \( L \)-independent constant. The exponential decay
condition on \( \phi \) can alternatively be formulated as an exponential decay condition
of the map \( \lambda \mapsto \|\chi_{[\lambda, \infty)}(H_L)\phi\|^2 \). The novelty is that at the same time
we allow the function \( \phi \) to be from an infinite dimensional spectral subspace
and keep an explicit control over the constant \( C_{\text{sfuc}} \) in terms of the parameters.
Moreover, we show that a similar result cannot hold under a polynomial decay
condition.

1. Introduction. Starting with the pioneering work [5], there has been plenty of
research concerning unique continuation properties for elliptic operators \( L \) with
non-analytic coefficients. That is, if the solution \( u \) of \( Lu = 0 \) in \( \Omega \subset \mathbb{R}^d \)
vanesishes in a non-empty open set \( \omega \subset \Omega \), then \( u \) will be identically zero, see e.g. [7] and the
references therein. More than this, there are several quantitative formulations of
unique continuation which proved to be useful in a variety of applications, see e.g.
[3, 11, 4, 15, 14]. For instance, Bourgain and Kenig [3] showed that if \( \Delta u = Vu \in \mathbb{R}^d \), \( u(0) = 1 \) and \( u, V \in L^\infty(\mathbb{R}^d) \) then for all \( x \in \mathbb{R}^d \) with \( |x| > 1 \) we have
\[
\max_{|y-x| \leq 1} |u(y)| > c \cdot \exp \left( -c' (\log |x|)^{4/3} \right).
\]

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This quantitative formulation has been crucial for the proof of Anderson localization for the continuum Anderson model with Bernoulli-distributed coupling constants. An $L^2$-variant of Ineq. (1) has been shown in [4] in order to study the density of states of Schrödinger operators. A similar quantitative formulation is an estimate of the type
\[ \|u\|_{L^2(\omega)}^2 \geq C\|u\|_{L^2(\Omega)}^2, \quad (2) \]
where $u$ is in the range of some spectral projector of a Schrödinger operator with potential $V$, and $C$ is some positive constant depending on the geometry of $\omega$ and the potential $V$. Such quantitative unique continuation principles have been applied to control theory for the heat equation and spectral theory of random Schrödinger operators, see e.g. the recent [16] and the references therein. Let us emphasize that the dependence of $C$ on the geometry of $\omega$ turned out to be important for some of these applications. To be more specific, let $\Omega \subset \mathbb{R}^d$ be a finite, open and non-empty connected set, $W \in L^\infty(\Omega)$ and $H_{\Omega} = -\Delta + W$ on $L^2(\Omega)$ with Dirichlet boundary conditions. Then Ineq. (2) has been obtained in [11] in the case $W \equiv 0$, $\omega \subset \Omega$ open and non-empty, $u$ a (finite or infinite) linear combination of eigenfunctions of $H_{\Omega}$. However, the dependence of $C$ on the geometry of $\omega$ is not known. In [15] Ineq. (2) is proven for $\Omega = (-L/2, L/2)^d$, $\omega$ an equidistributed arrangement of $\delta$-balls, $u \in W^{2,2}(\Omega)$ satisfying $|\Delta u| \leq |Wu|$, and with
\[ C = \delta^{N(1+\|W\|_\infty^{2/3})} \]
where $N > 0$ depends only on the dimension. For the application to random Schrödinger operators it is crucial that the result is scale-free, i.e. $C$ is independent of $L$. In [15] the question was raised whether a similar estimate holds for finite linear combinations of eigenfunctions $u \in \text{Ran} \chi\{(-\infty,0)\}(H_{\Omega})$. A partial answer to this question was given in [9]. The full answer has been announced in [13], and full proofs have been given in [14]. There, the constant
\[ C = \delta^{N(1+\|W\|_\infty^{2/3}+\sqrt{\|\omega\|})} \]
is derived. Let us emphasize that this was the missing step to study localization for random Schrödinger operators with non-linear dependence on the random parameters.

The aim of this note is to extend the main result of [14] to the natural setting of infinite dimensional spectral subspaces. For this purpose, we first extend the strategy of [14] to prove Ineq. (2) for infinite linear combinations of eigenfunctions with exponentially decaying coefficients, cf. Theorem 2.3. In a second step we show that Ineq. (2) holds if $|\chi_{(\lambda,\infty)}(H_{\Omega})\phi|^2$ decays exponentially in $\lambda$, cf. Theorem 2.2. In order not to lose the explicit control over the constant $C_{\text{sfuc}}$, in particular its $L$-independence, this step requires a detailed analysis using precise knowledge of the $\Delta$-eigenvalues and eigenfunctions, cf. Lemma 3.1. While the proofs are given in Section 3, we will show in Section 4 that our results are optimal in the sense that they cannot hold under polynomial decay conditions.

2. Notation and main results. Let $d \in \mathbb{N}$. For $L, r > 0$ we denote by $\Lambda_L = (-L/2, L/2)^d \subset \mathbb{R}^d$ the $d$-dimensional cube with side length $L$ and by $B(x, r)$ the ball with center $x$ and radius $r$ with respect to the Euclidean norm. The Laplace operator on $L^2(\Lambda_L)$ with Dirichlet, Neumann or periodic boundary conditions is denoted by $\Delta_L$. For $\Omega \subset \mathbb{R}^d$ open and $\psi \in L^2(\Omega)$ we denote by $\|\psi\|_\Omega = $
we have $\|\psi\|_{L^2(\Omega)}$ the usual $L^2$-norm of $\psi$. If $\Gamma \subset \Omega$ we use the notation $\|\chi_{\Gamma} \psi\|_{\Omega} = \|\psi\|_{\Gamma} = \|\psi\|_{L^2(\Gamma)}$. Moreover, for a measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$ we denote by $V_L : \Lambda_L \to \mathbb{R}$ its restriction to $\Lambda_L$ given by $V_L(x) = V(x)$ for $x \in \Lambda_L$, and by $H_L = -\Delta_L + V_L$ on $L^2(\Lambda_L)$ the corresponding Schrödinger operator. We will also write $V = V_+ - V_-$ for the decomposition into positive and negative part and $\|V\|_{\infty}$ for the $L^{\infty}$-norm of $V$. The operator $H_L$ is lower semibounded and self-adjoint with lower bound $-\|V_+\|_{\infty}$ and purely discrete spectrum.

**Definition 2.1.** Let $G > 0$ and $\delta > 0$. We say that a sequence $Z = (z_j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}^d$ is $(G, \delta)$-equidistributed, if

$$\forall j \in \mathbb{Z}^d : \quad B(z_j, \delta) \subset \Lambda_G + j = \{x + j \in \mathbb{R}^d : x \in \Lambda_L\}.$$  

Corresponding to a $(G, \delta)$-equidistributed sequence $Z$ we define for $L \in \mathbb{N}$ the set

$$W_\delta(L) = \bigcup_{j \in \mathbb{Z}^d} B(z_j, \delta) \cap \Lambda_L,$$

where we suppressed the dependence of $W_\delta(L)$ on $G$ and on the choice of $Z$.

**Theorem 2.2.** There is $N_A = N_A(d) > 0$ such that for all $\kappa > 0$, all $G \in (0, \kappa/(18e\sqrt{d}))$, all $\delta \in (0, G/2)$, all $(G, \delta)$-equidistributed sequences $Z$, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}$, all $D_A \geq 1$ and all $\phi \in L^2(\Lambda_L)$ satisfying

$$\text{for all } \lambda \in [-\|V_+\|_{\infty}, \infty) : \quad \|\chi_{[\lambda, \infty]}(H_L)\phi\|_{\Lambda_L}^2 \leq D_A e^{-\kappa \sqrt{\lambda + \|V_+\|_{\infty}}} \|\phi\|_{\Lambda_L}^2,$$

we have

$$\|\phi\|_{W_\delta(L)}^2 \geq C_{\text{sfuc}}^A \|\phi\|_{\Lambda_L}^2,$$

where

$$C_{\text{sfuc}}^A = C_{\text{sfuc}}^A(d, \delta, D_A, \|V\|_{\infty}) := \left(\frac{\delta}{G}\right)^{N_A \left(1 + G^{4/3} \|V\|_{\infty}^{2/3} + \ln D_A + G/(\kappa - G18e\sqrt{d})\right)}.$$

For every measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$ and every $L \in \mathbb{N}$ we denote the eigenvalues of the corresponding operator $H_L$ by $E_k$, $k \in \mathbb{N}$, enumerated in increasing order and counting multiplicities, and fix a corresponding sequence $\psi_k$, $k \in \mathbb{N}$, of normalized eigenfunctions. Note that we suppress the dependence of $E_k$ and $\psi_k$ on $V$ and $L$. For $\phi \in L^2(\Lambda_L)$ we set $\alpha_k = (\psi_k, \phi)$, whence

$$\phi = \sum_{k \in \mathbb{N}} \alpha_k \psi_k.$$

**Theorem 2.3.** There is $N_B = N_B(d) > 0$ such that for all $\kappa > 0$, all $G \in (0, \kappa/(18e\sqrt{d})$, all $\delta \in (0, G/2)$, all $(G, \delta)$-equidistributed sequences $Z$, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $L \in \mathbb{N}$, all $D_B \geq 1$ and all $\phi \in L^2(\Lambda_L)$ satisfying

$$\sum_{k \in \mathbb{N}} \exp \left(\kappa \sqrt{\max\{0, E_k\}}\right) |\alpha_k|^2 \leq D_B \sum_{k \in \mathbb{N}} |\alpha_k|^2,$$

we have

$$\|\phi\|_{W_\delta(L)}^2 \geq C_{\text{sfuc}}^B \|\phi\|_{\Lambda_L}^2,$$

where

$$C_{\text{sfuc}}^B = C_{\text{sfuc}}^B(d, G, \delta, D_B, \|V\|_{\infty}) := \left(\frac{\delta}{G}\right)^{N_B \left(1 + G^{4/3} \|V\|_{\infty}^{2/3} + \ln D_B\right)}.$$
A special case of Theorem 2.2 and 2.3 is $\phi \in \text{Ran}(\chi_{(-\infty,b]}(H_L))$ for some $b \geq -\|V_{-}\|_{\infty}$. Let us assume $G = 1$ for convenience. In this case Inequality (5) holds with $\kappa = 18e\sqrt{d}$ and $D_B = \exp(18e\sqrt{d}b + \|V_{-}\|_{\infty})$. Inequality (3) holds, e.g., with $\kappa = 18e\sqrt{d}+1$ and $D_A = \exp((18e\sqrt{d}+1)b + \|V_{-}\|_{\infty})$. Hence, the constants $C_{\text{suc}}^{B,A}$ in Theorem 2.3 and 2.2 can be estimated as

$$C_{\text{suc}}^{B} \geq \delta^{N_B}(1+\|V\|_{\infty}^{2/3}+\sqrt{|b|}), \quad \text{and} \quad C_{\text{suc}}^{A} \geq \delta^{N_A}(1+\|V\|_{\infty}^{2/3}+\sqrt{|b|}),$$

with $N_B$ and $N_A$ depending only on the dimension. This way we recover the original result of [14].

**Remark 1** (Relation between $\kappa$ and $G$). In Theorem 2.2 and 2.3, the parameters $\kappa$ (decay of high energies) and $G$ (grid size) are subject to the relation $G/\kappa \leq 18e\sqrt{d}$ or $G/\kappa < 18e\sqrt{d}$, respectively. This is in accordance with the intuition of uncertainty principles: delocalization in momentum space (large $\kappa$) corresponds to localization in position space, i.e. a fine grid (small $G$) is required in order to obtain an estimate as in Ineq. (4). It also seems that the condition on $G$ and $\kappa$ appears naturally when using Carleman estimates to prove a scale-free quantitative unique continuation result as in Theorem 2.2 and 2.3. Indeed, a similar assumption is possible without much effort to turn this qualitative into a quantitative statement of the form

$$\|\phi\|_{\omega}^2 \geq C\|\phi\|_{\Lambda_L}^2,$$

the method in [11] does not provide any control over the constant $C$ in terms of $\delta$, $L$, and $\kappa$, which we study in this note. One possibility to treat arbitrary $\kappa$ and $G$ might be a so-called chaining argument, as used in [6, 10, 1] in the context of quantitative uniqueness results and nodal sets for solutions of the Schrödinger equation. However, in order to obtain a strong dependence of $C_{\text{suc}}$ on the parameters $\delta$ and $\|V\|_{\infty}$ as in Theorem 2.2 and 2.3, a direct adaptation of these chaining arguments to our setting might not be feasible.

**Remark 2** (Optimality). As observed by Jerison and Lebeau in Proposition 14.9 of [8], the square root in the exponent of Ineq. (3) and (5) is optimal. The exponent $2/3$ of $\|V\|_{\infty}$ in $C_{\text{suc}}$, is known from Meshkov’s example [12] to be optimal in the case of eigenfunctions of Schrödinger operators with complex-valued potentials. It is an open question whether it can be improved for real-valued potentials. Another question is whether one can still expect results as in Theorem 2.2 and 2.3 if the exponential functions in (3) and (5) are replaced by polynomials. We will show in Section 4 that this is not the case. For this purpose, we show that every $\phi \in C_{\text{suc}}^{\infty}(\Lambda_L)$ satisfies such a polynomial condition. Hence polynomial summability of the $|\alpha_k|^2$ does not imply such a quantitative unique continuation principle.
3. Proofs.

3.1. Ghost dimension and interpolation inequalities. In this subsection we restate two interpolation inequalities from [14], on which the proof of Theorem 2.3 relies. For more details we refer to [14].

Given a measurable and bounded \( V : \mathbb{R}^d \to \mathbb{R} \) and \( L \in \mathbb{N} \) we define extensions of \( V_L \) and of the eigenfunctions \( \psi_k \) (defined on \( \Lambda_L \)) to a larger cube \( \Lambda_{RL} \) where \( R \) is the least odd integer larger than \( 18e\sqrt{d} + 2 \). The type of the extension will depend on the boundary conditions, see [14]. In the case of

- periodic boundary conditions we extend both \( V \) and \( \psi_k \) periodically.
- Dirichlet boundary conditions we extend \( V \) iteratively by symmetric reflections with respect to the boundary of \( \Lambda_L \), and \( \psi_k \) by antisymmetric reflections.
- Neumann boundary conditions we extend both \( V \) and \( \psi_k \) iteratively by symmetric reflections with respect to the boundary of \( \Lambda_L \).

We will use the same symbol for the extended \( V_L \) and \( \psi_k \). Note that \( V_L : \Lambda_{RL} \to \mathbb{R} \) takes values in \([-\|V\|_{\infty}, \|V\|_{\infty}]\), the extended \( \psi_k \) are elements of \( W^{2,2}(\Lambda_{RL}) \) with corresponding boundary conditions, they satisfy the eigenvalue equation \( \Delta \psi_k = (V_L - E_k)\psi_k \) on \( \Lambda_{RL} \) and their orthogonality relations remain valid.

For a measurable and bounded \( V : \mathbb{R}^d \to \mathbb{R} \), \( L \in \mathbb{N} \) and \( \phi \in L^2(\Lambda_L) \) recall that \( \alpha_k = \langle \psi_k, \phi \rangle \) whence \( \phi = \sum_{k \in \mathbb{N}} \alpha_k \psi_k \). We set \( \omega_k = \sqrt{|E_k|} \) and define for \( n \in \mathbb{N} \) the function \( F_n : \Lambda_{RL} \times \mathbb{R} \to \mathbb{C} \) by

\[
F_n(x, x_{d+1}) = \sum_{k=1}^{n} \alpha_k \psi_k(x) s_k(x_{d+1}),
\]

where \( s_k : \mathbb{R} \to \mathbb{R} \) is given by

\[
s_k(t) = \begin{cases} \sinh(\omega_k t)/\omega_k, & E_k > 0, \\ x, & E_k = 0, \\ \sin(\omega_k t)/\omega_k, & E_k < 0. \end{cases}
\]

Note that we suppress the dependence of \( F_n \) on \( V, L \) and \( \phi \). The function \( F_n \) fulfills the handy relations

\[
\Delta F_n = \sum_{i=1}^{d+1} \partial_i^2 F_n = V_L F_n \quad \text{on} \quad \Lambda_{RL} \times \mathbb{R}
\]

and

\[
\partial_{d+1} F_n(\cdot, 0) = \sum_{k=1}^{n} \alpha_k \psi_k =: \phi_n \quad \text{on} \quad \Lambda_{RL}.
\]

In particular, we have \( \|\partial_{d+1} F_n(\cdot, 0) - \phi\|_{L^2(\Lambda_{RL})} = \|\phi_n - \phi\|_{L^2(\Lambda_{RL})} \to 0 \) for \( n \to \infty \).

In the following, we recall that for \( \Omega \subset \Lambda_{RL} \times \mathbb{R} \),

\[
\|F_n\|_{H^1(\Omega)}^2 = \|F_n\|_{L^2(\Omega)}^2 + \sum_{i=1}^{d+1} \|\partial_i F_n\|_{L^2(\Omega)}^2\]

is the 1-Sobolev norm. Furthermore, in order to avoid confusion we now adapt the notation from [14] and define \( \mathbb{N}_{\text{odd}} = \{1, 3, 5, \ldots\} \), \( X_1 = \Lambda_L \times [-1, 1], \) \( R_3 = 9e\sqrt{d} \).
\[ \hat{X}_{R_3} = \Lambda_{L+2}R_3 \times [-R_3, R_3], \]
\[ S_1 = \left\{ x \in \mathbb{R}^{d+1} : -x_{d+1} + \frac{x_{d+1}^2}{2} - \frac{\sum_{i=1}^{d} x_{i}^2}{4} > -\frac{\delta^2}{16}, x_{d+1} \in [0, 1] \right\} \subset \mathbb{R}^{d+1}, \]
\[ S_3 = \left\{ x \in \mathbb{R}^{d+1} : -x_{d+1} + \frac{x_{d+1}^2}{2} - \frac{\sum_{i=1}^{d} x_{i}^2}{4} > -\frac{\delta^2}{4}, x_{d+1} \in [0, 1] \right\} \subset \mathbb{R}^{d+1}. \]

Moreover, for \( L \in \mathbb{N} \), a \((1, \delta)\)-equidistributed sequence \( Z \) and \( i \in \{1, 3\} \), we define the sets \( U_i(L) = \cup_{j \in \mathbb{N} \setminus \Lambda_i} S_i(z_j) \). The following Propositions are variants of Propositions 3.4, 3.5 and 3.6 from [14], see Remark 3 below.

**Proposition 1.** For all \( \delta \in (0, 1/2) \), all \((1, \delta)\)-equidistributed sequences \( Z \), all measurable and bounded \( V : \mathbb{R}^d \to \mathbb{R} \), all \( L \in \mathbb{N}_{\text{odd}} \), all \( n \in \mathbb{N} \) and all \( \phi \in L^2(\Lambda_L) \) we have
\[ \|F_n\|_{H^1(U_i(L))} \leq D_1 \|\partial_{d+1} F_n\|_{L^2(W_k(L))} \|F_n\|_{H^1(U_5(L))}^{1/2}, \]
where
\[ D_1^{-4} = \delta^{N_1(1+\|V\|_{L^2}^2)}, \]
and \( N_1 = N_1(d) \) is a constant depending on the dimension only.

**Proposition 2.** For all \( \delta \in (0, 1/2) \), all \((1, \delta)\)-equidistributed sequences \( Z \), all measurable and bounded \( V : \mathbb{R}^d \to \mathbb{R} \), all \( L \in \mathbb{N}_{\text{odd}} \), all \( n \in \mathbb{N} \) and all \( \phi \in L^2(\Lambda_L) \) we have
\[ \|F_n\|_{H^1(X_i)} \leq D_2 \|F_n\|_{H^1(U_i(L))} \|F_n\|_{H^1(X_R_3)}^{1-\gamma}, \]
where
\[ \gamma = \left( \log_2 \left( \frac{6e\sqrt{d}}{\frac{1}{2} - \frac{1}{2} \sqrt{16 - \delta^2}} \right) \right)^{-1}, \]
\[ D_2^{-2/\gamma} = \delta^{N_2(1+\|V\|_{L^2}^2)}, \]
and \( N_2 = N_2(d) \) is a constant depending on the dimension only.

**Proposition 3.** For all \( T > 0 \), all measurable and bounded \( V : \mathbb{R}^d \to \mathbb{R} \), all \( L \in \mathbb{N}_{\text{odd}} \), all \( n \in \mathbb{N} \) and all \( \phi \in L^2(\Lambda_L) \) we have
\[ \frac{T}{2} \sum_{k=1}^{n} |\alpha_k|^2 \leq \frac{\|F_n\|_{H^1(\Lambda_{R_3} \times [-T, T])}^2}{R^d} \leq 2T(1 + (1 + \|V\|_{L^\infty})T^2) \sum_{k=1}^{n} \beta_k(T)|\alpha_k|^2, \]
where
\[ \beta_k(T) = \begin{cases} 1 & \text{if } E_k \leq 0, \\ e^{2T\sqrt{E_k}} & \text{if } E_k > 0. \end{cases} \]

**Remark 3.** The counterparts of Propositions 1, 2 and 3 in [14] are formulated with \( \phi \in \text{Ran} \chi_{[-\infty, b]}(H_L) \) instead of \( \phi \in L^2(\Lambda_L) \), and
\[ F^b(x, x_{d+1}) := \sum_{k \in \mathbb{N} \cap h} \alpha_k \psi_k(x) s_k(x_{d+1}). \]

Instead of \( F_n \). However, the proofs in [14] do not depend on the particular choice of the index set \( \{k \in \mathbb{N} : E_k \leq b\} \) and apply to arbitrary finite index sets as well.
3.2. Proof of Theorem 2.3. First we consider the case $G = 1$, $\kappa \geq 18e\sqrt{d}$, and $L \in \mathbb{N}_{\text{odd}}$. We note that Proposition 3 remains true if we replace $R$ by 1, i.e. for all $T > 0$, $n \in \mathbb{N}$ and $L \in \mathbb{N}_{\text{odd}}$ we have

\[
\frac{T}{2} \sum_{k=1}^{n} |\alpha_k|^2 \leq \|F_n\|^2_{H^1(\Lambda_L \times [-T,T])} \leq 2T(1 + (1 + \|V\|_{\infty})T^2) \sum_{k=1}^{n} \beta_k(T)|\alpha_k|^2. \tag{7}
\]

We have $\tilde{X}_{R_3} \subset \Lambda_{RL} \times [-R_3, R_3]$. By Ineq. (7) and Proposition 3 we have

\[
\frac{\|F_n\|^2_{H^1(\tilde{X}_{R_3})}}{\|F_n\|^2_{H^1(X_1)}} \leq \frac{\|F_n\|^2_{H^1(\Lambda_{RL} \times [-R_3, R_3])}}{\|F_n\|^2_{H^1(X_1)}} \leq \sum_{k=1}^{n} \theta_k|\alpha_k|^2 \leq D^2_3 \sum_{k=1}^{n} |\alpha_k|^2.
\]

where $D^2_3 = 4R^dR_3(1 + (1 + \|V\|_{\infty})R^2_3)$ and $\theta_k = \beta_k(R_3)$. Now note that $\kappa \geq 18e\sqrt{d} = 2R_3$. Therefore, Assumption (5) yields

\[
\sum_{k \in \mathbb{N}} \theta_k|\alpha_k|^2 = \sum_{k \in \mathbb{N}} \exp \left(2R_3\sqrt{\max\{0, E_k\}}\right)|\alpha_k|^2 \leq D_B \sum_{k \in \mathbb{N}} |\alpha_k|^2.
\]

Since

\[
n \mapsto \frac{\sum_{k=1}^{n} \theta_k|\alpha_k|^2}{\sum_{k=1}^{n} |\alpha_k|^2}
\]

is monotonously increasing, this implies

\[
\text{for all } n \in \mathbb{N}: \sum_{k=1}^{n} \theta_k|\alpha_k|^2 \leq D_B \sum_{k=1}^{n} |\alpha_k|^2.
\]

Hence,

\[
\frac{\|F_n\|^2_{H^1(\tilde{X}_{R_3})}}{\|F_n\|^2_{H^1(X_1)}} \leq D_B D^2_3.
\]

We use Propositions 1 and 2 and obtain

\[
\|F_n\|^2_{H^1(\tilde{X}_{R_3})} \leq D_B^{1/2} D_3 \|F_n\|^2_{H^1(X_1)} \leq D_B^{1/2} D_1 D_2 D_3 \|F_n\|^2_{H^1(\tilde{X}_{R_3})} \left(\|\partial_{d+1} F_n\|_{L^2(W_3(L))}\right)^{\gamma/2} \|F_n\|^2_{H^1(U_3(L))}.
\]

Since $U_3(L) \subset \tilde{X}_{R_3}$ we have

\[
\|F_n\|^2_{H^1(\tilde{X}_{R_3})} \leq D_1^{1/\gamma} D_2^{1/\gamma} D_3^{2/\gamma} \|\partial_{d+1} F_n\|_{L^2(W_3(L))}.
\]

By Ineq. (7), the square of the left hand side is bounded from below by

\[
\|F_n\|^2_{H^1(\tilde{X}_{R_3})} \geq \|F_n\|^2_{H^1(\Lambda_L \times [-R_3, R_3])} \geq \frac{R_3}{2} \sum_{k=1}^{n} |\alpha_k|^2.
\]

Putting everything together we obtain by using $(\partial_{d+1} F_n)_0 = \phi_n$ for all $L \in \mathbb{N}_{\text{odd}}$:

\[
\tilde{C}_{\text{sf}}^B \|\phi_n\|^2_{L^2(\Lambda_L)} \leq \|\phi_n\|^2_{L^2(W_3(L))}
\]

where $\tilde{C}_{\text{sf}}^B = \tilde{C}_{\text{sf}}^B(d, \delta, D_B, \|V\|_{\infty}) = (R_3/2)D_B^{-2/\gamma} D_1^{-4}(D_2 D_3)^{-4/\gamma}$. For $D_1$, $D_2$, and $D_3$ we infer from [14]

\[
D_1^{-4} \geq \delta^{K_1} \|V\|_{\text{odd}}^{2/3}, \quad D_2^{-4/\gamma} \geq \delta^{K_2} \|V\|_{\text{odd}}^{2/3}, \quad \text{and} \quad D_3^{-4/\gamma} \geq \delta^{K_3} \|V\|_{\text{odd}}^{2/3}.
\]
and calculate 
\[ D_{\beta}^{-2/\gamma} = \left( \frac{1}{2} - \frac{1}{2\sqrt{16 - \delta^2}} \right) \frac{2\ln D_n}{\ln 2} \geq \left( \frac{\delta^2/64}{6\sqrt{d}} \right)^{2\ln D_n/\ln 2} \geq \delta^{K_4 \ln D_n} \]
where \( K_i, i \in \{1, 2, \ldots, 4\} \), are constants depending only on the dimension. Hence, 
\[ C_{sfuc}^B \geq \delta^N (1 + \|V\|^{2/3} + \ln D_n) \]
with some constant \( \tilde{N} = \tilde{N}(d) \). Letting \( n \) tend to infinity and using \( \|\phi_n - \phi\|_{L^2(\Lambda_L)} \to 0 \) for \( n \to \infty \), we conclude the statement of the theorem in the case \( G = 1, \kappa \geq 18\sqrt{d} \), and \( L \in \mathbb{N}_{odd} \).

Let now \( \kappa > 0 \) be arbitrary, \( G \in (0, \kappa/(18\sqrt{d})] \), and \( L/G \in \mathbb{N}_{odd} \). We define the map \( g : \Lambda_{L/G} \to \Lambda_L, g(y) = G \cdot y \). Then, on \( \Lambda_{L/G} \), we have 
\[ -\Delta_{L/G}(\psi_k \circ g) + (G^2 V_L \circ g)(\psi_k \circ g) = G^2 E_k(\psi_k \circ g). \]
Hence, the functions \( \psi_k \circ g \) are an orthonormal basis of eigenfunctions of the operator \( \tilde{H}_L = -\Delta_{L/G} + G^2 V_L \circ g \) with eigenvalues \( \tilde{E}_k = G^2 E_k \). We apply our theorem with \( G = 1 \) and \( \kappa \to G \) to the function \( \tilde{\phi} = \phi \circ g \) and obtain, using \( \|\phi\|_{\Lambda_L}^2 = G^d \|\phi \circ g\|_{\Lambda_{L/G}}^2 \),
\[ \|\phi\|_{W_2(L)}^2 = G^d \|\phi \circ g\|_{W_2(L/G)}^2 \geq \left( \frac{\delta}{G} \right) \tilde{N}(1 + G^{1/3} \|V\|^{2/3} + \ln D_n) \|\phi\|_{\Lambda_L}^2. \] (8)
The general case \( L/G \in \mathbb{N} \) follows by a similar scaling argument and the explicit dependence of \( C_{sfuc}^B \) on the parameters; see [14] for details.

3.3. Proof of Theorem 2.2. Recall that for a measurable and bounded \( V : \mathbb{R}^d \to \mathbb{R} \) we denote by \( V_L : \Lambda_L \to \mathbb{R} \) its restriction to \( \Lambda_L \), by \( \Delta_L \) the Laplace operator on \( L^2(\Lambda_L) \) subject to either Dirichlet, Neumann or periodic boundary conditions, and by 
\[ H_L = -\Delta_L + V_L \text{ on } L^2(\Lambda_L). \]
the corresponding Schrödinger operator. Moreover, we denote the eigenvalues of \( H_L \) by \( E_k, k \in \mathbb{N} \), enumerated in increasing order and counting multiplicities, and fix a corresponding sequence \( \psi_k, k \in \mathbb{N} \), of normalized eigenfunctions.

**Lemma 3.1.** Let \( C_1, C_2 > 0 \), \( L \in \mathbb{N} \), \( V : \mathbb{R}^d \to \mathbb{R} \) measurable and bounded and \( \phi \in L^2(\Lambda_L) \) satisfying 
\[ \|\chi_{[-\lambda, \lambda]}(H_L)\phi\|_{\Lambda_L}^2 \leq C_1 e^{-(C_2 + \varepsilon)\sqrt{\lambda + \|V\|_{\infty}}\|\phi\|_{\Lambda_L}^2} \text{ for every } \lambda \in [-\|V\|_{\infty}, \infty). \]
Then we have 
\[ \sum_{k=1}^{\infty} e^{C_2 \max\{0, E_k\}} |\alpha_k|^2 \leq C_3 \sum_{k=1}^{\infty} |\alpha_k|^2, \]
where 
\[ C_3 = e^{C_2 (\pi + \|V\|_{\infty}^{1/2})} \left( 1 + \frac{C_1 C_2 \pi}{1 - e^{-\epsilon \pi}} \right). \]

For the proof of Lemma 3.1 and 4.1 we shall need explicit formulas for the eigenvalues and eigenfunctions of the negative Laplacian \( -\Delta_L \) on \( L^2(\Lambda_L) \). Depending on the boundary conditions we choose the index set \( I = \mathbb{N} \) in the case of Dirichlet boundary conditions, \( I = \mathbb{N}_0 \) in the case of Neumann boundary conditions, and
Since \( I = 2\mathbb{Z} \) in the case of periodic boundary conditions. Then, the eigenvalues of \(-\Delta_L\) are given by

\[
\lambda_y = \left( \frac{\pi}{L} \right)^2 |y|^2, \quad y \in \mathcal{I}^d,
\]

with corresponding normalized eigenfunctions

\[
e_y(x) = \begin{cases} 
\|e_y\|^{-1} \prod_{i=1}^d \sin \left( \frac{\pi y_i}{L} (x_i + L/2) \right) & \text{in the case of Dirichlet b.c.,} \\
\|e_y\|^{-1} \prod_{i=1}^d \cos \left( \frac{\pi y_i}{L} (x_i + L/2) \right) & \text{in the case of Neumann b.c.,} \\
\|e_y\|^{-1} \exp \left( \frac{i\pi}{L} y \cdot x \right) & \text{in the case of periodic b.c.}
\end{cases}
\]

The normalization constants \( \|e_y\|^{-1} \) can be easily calculated, though we will not need them. Moreover, there exists a bijection \( p : \mathbb{N} \to \mathcal{I}^d \) such that

\[
\lambda_{p(k)}, \quad k \in \mathbb{N},
\]

is the \( k \)-th eigenvalue of \(-\Delta_L\) enumerated in increasing order counting multiplicities. This bijection is unique up to permutations of sites \( y \in \mathcal{I}^d \) with the same Euclidean norm.

**Proof of Lemma 3.1.** By the variational principle, we have for all \( k \in \mathbb{N} \)

\[
\lambda_{p(k)} - \|V_-\|_{\infty} \leq E_k \leq \lambda_{p(k)} + \|V_+\|_{\infty}.
\]

Using Ineq. (11) and \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for \( a, b \geq 0 \), and Eq. (9) we obtain

\[
\sum_{k=1}^{\infty} e^{C_2 \sqrt{\max\{0, E_k\}}} |\alpha_k|^2 \leq e^{C_2 \|V_+\|_{\infty}^2 \sum_{k=1}^{\infty} e^{C_2 |p(k)| \pi/L} |\alpha_k|^2} \\
\leq e^{C_2 \|V_+\|_{\infty}^2 \sum_{k=1}^{\infty} |\alpha_k|^2} =: e^{C_2 \|V_+\|_{\infty}^2 S}.
\]

By a telescoping argument we have

\[
S = e^{C_2 \pi/L} \sum_{l=1}^{\infty} \sum_{k \in \mathbb{N} : l-1 \leq |p(k)| < l} |\alpha_k|^2 + \sum_{l=2}^{\infty} \left( e^{C_2 \pi/L} - e^{C_2 l \pi/L} \right) \sum_{m=1}^{\infty} \sum_{k \in \mathbb{N} : m-1 \leq |p(k)| < m} |\alpha_k|^2 \\
= e^{C_2 \pi/L} \sum_{k=1}^{\infty} |\alpha_k|^2 + \sum_{l=2}^{\infty} \left( e^{C_2 \pi/L} - e^{C_2 l \pi/L} \right) \sum_{k \in \mathbb{N} : l-1 \leq |p(k)|} |\alpha_k|^2.
\]

Since \( |y| = \sqrt{\lambda_y L/\pi} \) and \( E_k \geq \lambda_{p(k)} - \|V_\| \) by Ineq. (11), we have

\[
\sum_{l-1 \leq |p(k)|} |\alpha_k|^2 \leq \sum_{k \in \mathbb{N} : \lambda_{p(k)} \geq |(l-1)\pi/L|^2} \|\chi_{I_l}(H_L)\phi\|^2 \\
\leq \sum_{k \in \mathbb{N} : |(l-1)\pi/L|^2 \geq \|V_\|_{\infty}^2} |\alpha_k|^2.
\]

where \( I_l = \left[ ((l-1)\pi/L)^2 - \|V_\|_{\infty}^2 \right] \) and our assumption on the spectral projector to
From Lemma 3.1 and (12) we infer that Assumption (5) of Theorem 2.3 is satisfied. Since the map $N \ni L \mapsto (L(1 - e^{-\varepsilon \pi / L}))^{-1}$ is monotonically decreasing we finally obtain
\[
\sum_{k \in \mathbb{N}} e^{C_2 \sqrt{\max\{0, E_k\}}} |\alpha_k|^2 \leq e^{C_2 \|V_+\|^{1/2}} \left( e^{C_2 \pi / L} \|\varphi\|^2 + \frac{C_2 \pi}{L} \sum_{l=2}^{\infty} e^{C_2 l \pi / L} \|\chi_L(H_L)\varphi\|^2 \right)
\leq e^{C_2 \|V_+\|^{1/2}} \left( e^{C_2 \pi / L} + \frac{C_1 C_2 \pi}{L} \sum_{l=1}^{\infty} e^{-\varepsilon \pi l / L} \right) \|\varphi\|^2
\leq e^{C_2 (\pi \|V_+\|^{1/2})} \left( 1 + \frac{C_1 C_2 \pi}{L(1 - e^{-\varepsilon \pi / L})} \right) \|\varphi\|^2.
\]

Since the map $N \ni L \mapsto (L(1 - e^{-\varepsilon \pi / L}))^{-1}$ is monotonously decreasing we finally obtain
\[
\sum_{k \in \mathbb{N}} e^{C_2 \sqrt{\max\{0, E_k\}}} |\alpha_k|^2 \leq e^{C_2 (\pi \|V_+\|^{1/2})} \left( 1 + \frac{C_1 C_2 \pi}{1 - e^{-\varepsilon \pi}} \right) \|\varphi\|^2.
\]

Now we are in position to prove Theorem 2.2.

**Proof of Theorem 2.2.** First we consider the case $\kappa > 18e \sqrt{d} = 2R_3$ and $G = 1$. Hence we have $\varepsilon := \kappa - 2R_3 > 0$. Note that (3) implies
\[
\|\chi_{[\lambda, \infty)}(H_L)\varphi\|_{\Lambda_L}^2 \leq D_A e^{-(2R_3 + \varepsilon) \sqrt{\lambda + \|V_+\|_{\infty}}} \|\varphi\|_{\Lambda_L}^2.
\]  

From Lemma 3.1 and (12) we infer that Assumption (5) of Theorem 2.3 is satisfied with
\[
D_B = e^{2R_3 (\pi \|V_+\|^{1/2})} \left( 1 + \frac{2D_A R_3 \pi}{1 - e^{-\varepsilon \pi}} \right).
\]

Hence, we can apply Theorem 2.3 and obtain
\[
\|\varphi\|^2_{W_4(L)} \geq \delta^{N_h} (1 + \|V\|_{2/3} + \ln D_B) \|\varphi\|^2_{L^2(\Lambda_L)}.
\]

As an upper bound for $\ln D_B$ we use $x^{1/2} \leq 1 + x^{2/3}$, $R_3, D_A \geq 1$ and
\[
-\ln(1 - e^{-\varepsilon \pi}) = \ln \left( 1 + \frac{e^{-\varepsilon \pi}}{1 - e^{-\varepsilon \pi}} \right) \leq \frac{e^{-\varepsilon \pi}}{1 - e^{-\varepsilon \pi}} = \frac{1}{e^{\varepsilon \pi} - 1} \leq \frac{1}{\varepsilon \pi},
\]
and find
\[
\ln D_B = 2R_3 \pi + 2R_3 \|V\|_{\infty}^{1/2} + \ln(1 - e^{-\varepsilon \pi} + 2D_A R_3 \pi) - \ln(1 - e^{-\varepsilon \pi})
\leq 2R_3 (\pi + 1) + 2R_3 \|V\|_{\infty}^{2/3} + \ln(D_A) + \ln(3R_3 \pi) + (\varepsilon \pi)^{-1}.
\]

Hence there is a constant $\tilde{N}$ depending only on the dimension such that
\[
\ln D_B \leq \tilde{N} \left( 1 + \|V\|_{\infty}^{2/3} + \varepsilon^{-1} + \ln D_A \right).
\]

This shows the statement of the theorem in the case $\kappa > 18e \sqrt{d}$ and $G = 1$. The general case follows by scaling, analogously to the end of the proof of Theorem 2.3.\]
4. Discussion on optimality. In Remark 2 we discussed whether the class of functions $\phi$ satisfying
\[ \sum_{k \in \mathbb{N}} \max\{0, E_k\}^\kappa |\alpha_k|^2 \leq D_B \sum_{k \in \mathbb{N}} |\alpha_k|^2 \tag{13} \]
for some $D_B, \kappa > 0$ can still exhibit a unique continuation principle as in Theorem 2.3. The following lemma leads to a counterexample in the case $V \equiv 0$.

**Lemma 4.1.** Let $L > 0$, $V \equiv 0$, $\kappa > 0$, $\phi \in C_0^\infty(\Lambda_L)$. Then there is $C = C(\phi, L, \kappa) > 0$ such that
\[ \sum_{k \in \mathbb{N}} |E_k|^{\kappa} |\alpha_k|^2 < C. \]

Now let $\phi \in C_0^\infty(\Lambda_L)$ be non-zero and vanishing on $W_0(L)$. By Lemma 4.1, $\phi$ satisfies (13) with $D_B := C/\|\phi\|_{\Lambda_L}^2$, but not $\|\phi\|^2_{W_0(L)} \geq C_{suc} \|\phi\|^2_{\Lambda_L}$. This shows the following corollary.

**Corollary 1.** The statement of Theorem 2.3 with $\exp(\kappa \sqrt{\max\{0, E_k\}})$ replaced by $\max\{0, E_k\}^\kappa$ cannot hold.

**Proof of Lemma 4.1.** Since the eigenfunctions and eigenvalues of $-\Delta$ on $\Lambda_L$ are explicitly known, cf. Section 3.3, we can replace the sum on the left hand side by
\[ \sum_{k \in \mathbb{N}} |E_k|^\kappa |\alpha_k|^2 = \sum_{y \in \mathbb{Z}^d} \left( \frac{\pi}{L} \right)^{2\kappa} |y/2^N| \langle e_y, \phi \rangle|^2 \leq \left( \frac{\pi}{L} \right)^{2\kappa} \sum_{y \in \mathbb{Z}^d} |y/2^N| \langle e_y, \phi \rangle|^2 \]
where $N \in 2\mathbb{N}$ is the least even integer larger than $\kappa$. For the eigenfunctions, see Eq. (10), we have $\partial_i^N e_y = -(\pi/L)^N y_i |y| y_i$ for $i \in \{1, \cdots, d\}$. We calculate using integration by parts
\[ \sum_{y \in \mathbb{Z}^d} |y/2^N| \langle e_y, \phi \rangle|^2 \leq N \sum_{i=1}^d \sum_{y \in \mathbb{Z}^d} |y_i| |y| \langle e_y, \phi \rangle|^2 = N \left( \frac{L}{\pi} \right)^{2N} \sum_{i=1}^d \sum_{y \in \mathbb{Z}^d} |\partial_i^N \phi(y)|^2 = N \left( \frac{L}{\pi} \right)^{2N} \sum_{i=1}^d \|\partial_i^N \phi\|_{\Lambda_L}^2. \]
\[ \square \]

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