ACYLINDRICAL HYPERBOLICITY OF GROUPS ACTING ON QUASI-MEDIAN GRAPHS AND EQUATIONS IN GRAPH PRODUCTS

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Abstract. In this paper we study group actions on quasi-median graphs, or ‘CAT(0) prism complexes’, generalising the notion of CAT(0) cube complexes. We consider hyperplanes in a quasi-median graph $X$ and define the contact graph $C_X$ for these hyperplanes. We show that $C_X$ is always quasi-isometric to a tree, generalising a result of Hagen [Hag14], and that under certain conditions a group action $G \acts X$ induces an acylindrical action $G \acts C_X$, giving a quasi-median analogue of a result of Behrstock, Hagen and Sisto [BHS17].

As an application, we exhibit an acylindrical action of a graph product on a quasi-tree, generalising results of Kim and Koberda for right-angled Artin groups [KK13, KK14]. We show that for many graph products $G$, the action we exhibit is the ‘largest’ acylindrical action of $G$ on a hyperbolic metric space. We use this to show that the graph products of equationally noetherian groups over finite graphs of girth $\geq 5$ are equationally noetherian, generalising a result of Sela [Sel10].

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1. Introduction

Group actions on CAT(0) cube complexes occupy a central role in geometric group theory. Such actions have been used to study many interesting classes of groups, such as right-angled Artin and Coxeter groups, many small cancellation and 3-manifold groups, and even finitely presented infinite simple groups, constructed by Burger and Mozes in [BM97]. Study of CAT(0) cube complexes is aided by their rich combinatorial structure, introduced by Sageev in [Sag95].

In the present paper we study quasi-median graphs, which can be viewed as a generalisation of CAT(0) cube complexes; see Definition 2.1. In particular, one may think of quasi-median graphs as ‘CAT(0) prism complexes’, consisting of prisms – cartesian products of (possibly infinite dimensional) simplices – glued together in a non-positively curved way. In his PhD thesis [Gen17], Genevois introduced cubical-like combinatorial structure and geometry to study a wide class of groups acting on quasi-median graphs, including graph products, certain wreath products, and dihedral products.

In particular, given a quasi-median graph $X$, we study hyperplanes in $X$: that is, the equivalence classes of edges of $X$, under the equivalence relation generated by letting two edges be equivalent if they induce a square or a triangle. Two hyperplanes are said to intersect if two edges defining those hyperplanes are adjacent in a square, and osculate if two edges defining those hyperplanes are adjacent but do not belong to a square; see Definition 2.2. This allows us to define two other graphs related to $X$, which turn out to be useful in the study of groups acting on $X$. 
**Definition 1.1.** Let $X$ be a quasi-median graph. We define the contact graph $CX$ and the crossing graph $\Delta X$ as follows. For the vertices, let $V(CX) = V(\Delta X)$ be the set of hyperplanes of $X$. Two hyperplanes $H, H'$ are then adjacent in $\Delta X$ if and only if $H$ and $H'$ intersect; hyperplanes $H, H'$ are adjacent in $CX$ if and only if $H$ and $H'$ either intersect or osculate.

For a CAT(0) cube complex $X$, Hagen has shown that the contact graph $CX$ is a quasi-tree – that is, it is quasi-isometric to a tree [Hag14, Theorem 4.1]. Here we generalise this result to quasi-median graphs.

**Theorem A.** Let $X$ be a quasi-median graph. Then the contact graph $CX$ is a quasi-tree.

We prove Theorem A in Section 3.2.

In this paper we study acylindrical hyperbolicity of groups acting on quasi-median graphs.

**Definition 1.2.** Suppose a group $G$ acts on a metric space $(X, d)$ by isometries. Such an action is said to be *acylindrical* if for every $\varepsilon > 0$, there exist constants $D_\varepsilon, N_\varepsilon > 0$ such that for all $x, y \in X$ with $d(x, y) \geq D_\varepsilon$, the number of elements $g \in G$ satisfying

$$d(x, x^g) \leq \varepsilon \quad \text{and} \quad d(y, y^g) \leq \varepsilon$$

is bounded above by $N_\varepsilon$. Moreover, an action $G \acts X$ by isometries on a hyperbolic metric space $X$ is said to be *non-elementary* if orbits under this action is unbounded and $G$ is not virtually cyclic.

A group $G$ is then said to be *acylindrically hyperbolic* if it possesses a non-elementary acylindrical action on a hyperbolic metric space.

Acylindrically hyperbolic groups form a large family, including hyperbolic and relatively hyperbolic groups, mapping class groups of most surfaces, and $\Out(F_n)$ for $n \geq 3$ [Osi16]. This family also includes ‘most’ hierarchically hyperbolic groups [BHS17, Corollary 14.4], and in particular ‘most’ groups $G$ that act properly and cocompactly on a $\CAT(0)$ cube complex with a ‘factor system’: see [BHS17]. The following result shows that, more generally, many groups acting on quasi-median graphs are acylindrically hyperbolic.

In the following theorem, we say a group action $G \acts X$ is *special* if there are no two hyperplanes $H, H'$ of $X$ such that $H$ and $H'$ intersect but $H^g$ and $H'^g$ osculate for some $g \in G$, and there is no hyperplane $H$ that intersects or osculates with $H^g \neq H$ for some $g \in G$. We say a collection $S$ of sets is *uniformly finite* if there exists a constant $D \in \mathbb{N}$ such that each $S \subseteq S$ has cardinality $\leq D$.

**Theorem B.** Let $G$ be a group acting specially on a quasi-median graph $X$, and suppose vertices in $\Delta X/G$ have uniformly finitely many neighbours.

(i) If $\Delta X$ is connected and $\Delta X/G$ has finitely many vertices, then the inclusion $\Delta X \hookrightarrow CX$ is a quasi-isometry.

(ii) If stabilisers of vertices under $G \acts X$ are uniformly finite, then the induced action $G \acts CX$ is acylindrical. In particular, if the orbits under $G \acts CX$ are unbounded, then $G$ is either virtually cyclic or acylindrically hyperbolic.

We prove part (i) of Theorem B in Section 3.1 and part (ii) in Section 4.

Note that a large class of examples of group actions on $\CAT(0)$ cube complexes with a factor system comes from special actions [BHS17, Corollaries 8.8 and 14.5]. Theorem B(ii) generalises this result to quasi-median graphs. We also show that several other hierarchically hyperbolic space-like results on $\CAT(0)$ cube complexes generalise to quasi-median graphs: for instance, existence of ‘hierarchy paths’, see [BHS17, Theorem A (2)] and Proposition 3.1.

The main application of Theorems A and B we give is to study graph products of groups. In particular, let $\Gamma$ be a simplicial graph and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of non-trivial groups. The graph product $\Gamma \mathcal{G}$ of the groups $G_v$ over $\Gamma$ is defined as the group

$$\Gamma \mathcal{G} = \left( \ast_{v \in V(\Gamma)} \langle G_v \rangle \right) / \langle g_v^{-1} g_w^{-1} g_v g_w \mid g_v \in G_v, g_w \in G_w, (v, w) \in E(\Gamma) \rangle.$$
For example, for a complete graph \( \Gamma \) we have \( \Gamma G \cong \prod_{v \in V(\Gamma)} G_v \), while for discrete \( \Gamma \) we have \( \Gamma G \cong \ast_{v \in V(\Gamma)} G_v \). The applicability of the results above to graph products follows from the following result of Genevois.

**Theorem 1.3** (Genevois [Gen17 Propositions 8.2 and 8.11]). Let \( \Gamma \) be a simplicial graph, let \( \mathcal{G} = \{ G_v \mid v \in V(\Gamma) \} \) be a collection of non-trivial groups, and let \( S = \bigcup_{v \in V(\Gamma)} G_v \setminus \{ 1 \} \subseteq \Gamma G \). Then the Cayley graph \( X \) of \( \Gamma G \) with respect to \( S \) is quasi-median. Moreover, the action of \( \Gamma G \) on \( X \) is free on vertices, special, and the quotient \( \Delta X/\Gamma G \) is isomorphic to \( \Gamma \).

An important subclass of graph products are right-angled Artin groups (RAAGs): indeed, if \( G_v \cong \mathbb{Z} \) then \( \Gamma G \) is the RAAG associated to \( \Gamma \). In this case, a vertex \( v \in V(\Gamma) \) is usually identified with a generator of \( G_v \). In [KK13] Kim and Koherda constructed the *extension graph* \( \Gamma^e \) of a RAAG \( G = \Gamma G \) as a graph with vertex set \( V(\Gamma^e) = \{ v^g \mid g \in G, v \in V(\Gamma) \} \), where \( g^v \) and \( h^w \) are adjacent in \( \Gamma^e \) if and only if they commute as elements of \( G \). This graph turns out to be the same as the crossing graph \( \Delta X \) of the Cayley graph \( X \) defined in Theorem 1.3.

In fact, Kim and Koherda showed that, given that \( |V(\Gamma)| \geq 2 \) and both \( \Gamma \) and its complement \( \Gamma^C \) are connected, \( \Gamma^e \) is quasi-isometric to a tree [KK13] and the action of \( G \) on \( \Gamma^e \) by conjugation is non-elementary acylindrical [KK14]. In this paper we generalise these results to arbitrary graph products; this follows as a special case of Theorems A and B. As a special case, we recover hyperbolicity of the extension graph \( \Gamma^e \) and acylindricity of the action \( \Gamma G \curvearrowright \Gamma^e \), providing an alternative (shorter and more geometric) argument to the ones presented in [KK13, KK14]. In the following corollary, a graph \( \Gamma \) is said to have bounded degree if there exists a constant \( D \in \mathbb{N} \) such that each vertex of \( \Gamma \) has degree \( \leq D \).

**Corollary C.** Let \( \Gamma \) be a simplicial graph, let \( \mathcal{G} = \{ G_v \mid v \in V(\Gamma) \} \) be a collection of non-trivial groups, and let \( X \) be the quasi-median graph defined in Theorem 1.3. Then \( CX \) is a quasi-tree, and if \( \Gamma \) has bounded degree then the induced action \( \Gamma G \curvearrowright CX \) is acylindrical. Moreover, if \( |V(\Gamma)| \geq 2 \) and the complement \( \Gamma^C \) of \( \Gamma \) is connected, then either \( \Gamma G \cong \mathbb{Z}_2 * \mathbb{Z}_2 \) is the infinite dihedral group, or this action is non-elementary.

The hyperbolicity of \( CX \) and the acylindricity of the action follow immediately from Theorems A, B and 1.3 while non-elementarity is shown in Section 5.1.

It is worth noting that Minasyan and Osin have already shown in [MO15] that if \( |V(\Gamma)| \geq 2 \) and the complement of \( \Gamma \) is connected, then \( \Gamma G \) is either infinite dihedral or acylindrically hyperbolic. However, their proof is not direct and does not provide an explicit acylindrical action on a hyperbolic space. The aim of Corollary C is to describe such an action.

We also show that in many cases the action of \( \Gamma G \) on \( CX \) is, in the sense of Abbott, Balasubramanya and Osin [ABO17], the 'largest' acylindrical action of \( \Gamma G \) on a hyperbolic metric space: see Section 5.2. In particular, we show that many graph products are strongly \( \mathcal{AH} \)-accessible. This generalises the analogous result for right-angled Artin groups [ABO17, Theorem 2.19 (c)].

**Corollary D.** Let \( \Gamma \) be a finite simplicial graph and let \( \mathcal{G} = \{ G_v \mid v \in V(\Gamma) \} \) be a collection of infinite groups. Suppose that for each isolated vertex \( v \in V(\Gamma) \), the group \( G_v \) is strongly \( \mathcal{AH} \)-accessible. Then \( \Gamma G \) is strongly \( \mathcal{AH} \)-accessible. Furthermore, if \( \Gamma \) has no isolated vertices, then the action \( \Gamma G \curvearrowright CX \), where \( X \) is as in Theorem 1.3, is the largest acylindrical action of \( \Gamma G \) on a hyperbolic metric space.

We prove Corollary D in Section 5.2.

**Remark 1.4.** After the first version of this preprint was made available, it has been brought to the author’s attention that most of the results stated in Corollary C follow from the results in [Gen16, Gen18, GM18]. Moreover, a special case of Corollary D (when the vertex groups \( G_v \) are hierarchically hyperbolic) follows from the results in [ABD17, BR18]. See Remarks 5.1 and 5.3 for details.

As an application, we use Corollary C to study the class of equationally noetherian groups, defined as follows.
Definition 1.5. Given $n \in \mathbb{N}$, let $F_n$ denote the free group of rank $n$ with a free basis $X_1, \ldots, X_n$. Given a group $G$, an element $s \in F_n$ and a tuple $(g_1, \ldots, g_n) \in G^n$, we write $s(g_1, \ldots, g_n) \in G$ for the element obtained by replacing every occurrence of $X_i$ in $s$ with $g_i$, and evaluating the resulting word in $G$. Given a subset $S \subseteq F_n$, the solution set of $S$ in $G$ is

$$V_G(S) = \{(g_1, \ldots, g_n) \in G^n \mid s(g_1, \ldots, g_n) = 1 \text{ for all } s \in S\}.$$ 

A group $G$ is said to be *equationally noetherian* if for any $n \in \mathbb{N}$ and any subset $S \subseteq F_n$, there exists a finite subset $S_0 \subseteq S$ such that $V_G(S_0) = V_G(S)$.

Many classes of groups are known to be equationally noetherian. For example, groups that are linear over a field – in particular, right-angled Artin groups – are equationally noetherian \cite{BM99} Theorem B1. It is easy to see that the class of equationally noetherian groups is preserved under taking subgroups and direct products; a deep and non-trivial argument shows that the same is true for free products:

Theorem 1.6 (Sela \cite{Sel10} Theorem 9.1). Let $G$ and $H$ be equationally noetherian groups. Then $G \ast H$ is equationally noetherian.

Using methods of Groves and Hull developed for acylindrically hyperbolic groups \cite{GH17}, we generalise Theorem 1.6 to a wider class of graph products.

Theorem E. Let $\Gamma$ be a finite simplicial triangle-free and square-free graph, and let $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ be a collection of equationally noetherian groups. Then the graph product $\Gamma \mathcal{G}$ is equationally noetherian.

We prove Theorem E in Section 6.

The paper is structured as follows. In Section 2 we define quasi-median graphs and give several results that are used in later sections. In Section 3 we analyse the geometry of the contact graph and its relation to crossing graph, and prove Theorem A and Theorem B (i) \cite{BKM16}. In Section 4 we consider the action of a group $G$ on a quasi-median graph $X$, and prove Theorem B (ii) \cite{BKM16}. In Section 5 we consider the particular case when $G = \Gamma \mathcal{G}$ is a graph product and $X$ is the quasi-median graph associated to it, and deduce Corollaries C and D. In Section 6 we apply these results to prove Theorem E.

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2. Preliminaries

Throughout the paper, we use the following conventions and notation. By a graph $X$, we mean an undirected simple (simplicial) graph, and we write $V(X)$ and $E(X)$ for the vertex and edge sets of $X$, respectively. Moreover, we write $d_X(\cdot, \cdot)$ for the combinatorial metric on $X$ – thus, we view $X$ as a geodesic metric space. We consider the set $\mathbb{N}$ of natural numbers to include 0.

Given a group $G$, all actions of $G$ on a set $X$ are considered to be right actions, $\theta : X \times G \to X$, and are written as $\theta(x, g) = x^g$ or $\theta(x, g) = xg$. Note that this results in perhaps unusual terminology when we consider a Cayley graph $\text{Cay}(G, S)$: in our case it has edges of the form $(g, sg)$ for $g \in G$ and $s \in S$.

2.1. Quasi-median graphs. In this section we introduce quasi-median graphs and basic results that we use throughout the paper. Most of the definitions and results in this section were introduced by Genevois in his thesis \cite{Gen17}. We therefore refer the interested reader to \cite{Gen17} for further discussion and results on applications of quasi-median graphs to geometric group theory.
Definition 2.1. Let $X$ be a graph, let $x_1, x_2, x_3 \in V(X)$ be three vertices, and let $k \in \mathbb{N}$. We say a triple $(y_1, y_2, y_3) \in V(X)^3$ is a \textit{k-quasi-median} of $(x_1, x_2, x_3)$ if (see Figure 1(a)):

(i) $y_i$ and $y_j$ lie on a geodesic between $x_i$ and $x_j$ for any $i \neq j$;
(ii) $k = d_X(y_1, y_2) = d_X(y_1, y_3) = d_X(y_2, y_3)$; and
(iii) $k$ is as small as possible subject to (i) and (ii).

We say $(y_1, y_2, y_3) \in V(X)^3$ is a \textit{quasi-median} of $(x_1, x_2, x_3) \in V(X)^3$ if it is a $k$-quasi-median for some $k$. A 0-quasi-median is called a \textit{median}.

We say a graph $X$ is a \textit{quasi-median graph} if (see Figure 1(b)):

(i) every triple of vertices has a unique quasi-median;
(ii) $K_{1,1,2}$ is not isomorphic to an induced subgraph of $X$; and
(iii) if $Y \cong C_6$ is a subgraph of $X$ such that the embedding $Y \hookrightarrow X$ is isometric, then the convex hull of $Y$ in $X$ is isomorphic to the 3-cube.

![Figure 1. Graphs appearing in Definition 2.1](image)

(3-cube)

There are many equivalent characterisations of quasi-median graphs: see [BMW94, Theorem 1]. In this paper we think of quasi-median graphs as generalisations of median graphs. Recall that a graph $X$ is called a \textit{median graph} if every triple of vertices of $X$ has a unique median. In particular, every median graph is quasi-median; more precisely, it is known that a graph is median if and only if it is quasi-median and triangle-free: see [Gen17, Corollary 2.92], for instance.

In what follows, a \textit{clique} is a maximal complete subgraph, a \textit{triangle} is a complete graph on 3 vertices, and a \textit{square} is a complete bipartite graph on two sets of 2 vertices each.

Definition 2.2. Let $X$ be a quasi-median graph. Let $\sim$ be the equivalence relation on $E(X)$ generated by the equivalences $e \sim f$ when $e$ and $f$ either are two sides of a triangle or opposite sides of a square. A \textit{hyperplane} $H$ is an equivalence class $[e]$ for some $e \in E(X)$; in this case, we say $H$ is the hyperplane dual to $e$ (or, alternatively, $H$ is the hyperplane dual to any clique containing $e$). Given a hyperplane $H$ dual to $e \in E(X)$, the \textit{carrier} of $H$, denoted by $\mathcal{N}(H)$, is the full subgraph of $X$ induced by $[e] \subseteq E(X)$; a \textit{fibre} of $H$ is a connected component of $\mathcal{N}(H) \setminus J$, where $J$ is the union of the interiors of all the edges in $[e]$.

Given two edges $e, e' \in E(X)$ with a common endpoint ($p$, say) that do not belong to the same clique, let $H$ and $H'$ be the hyperplanes dual to $e$ and $e'$, respectively. We then say $H$ and $H'$ \textit{intersect} (or \textit{intersect at p}) if $e$ and $e'$ are adjacent edges in a square, and we say $H$ and $H'$ \textit{osculate} (or \textit{osculate at p}) otherwise.

Finally, given two vertices $p, q \in V(X)$ and a hyperplane $H$, we say $H$ \textit{separates} $p$ from $q$ if every path between $p$ and $q$ contains an edge dual to $H$. More generally, we say $H$ \textit{separates} two subgraphs $P, Q \subseteq X$ if $H$ does not separate any two vertices of $P$ or any two vertices of $Q$, but it separates a vertex of $P$ from a vertex of $Q$. Given a path $\gamma$ in $X$, we also say $H$ \textit{crosses} $\gamma$ if $\gamma$ contains an edge dual to $H$. 


Another important concept in the study of quasi-median graphs are gated subgraphs. Such subgraphs coincide with \textit{convex subgraphs} for median graphs, but in general form a larger class in quasi-median graphs.

\textbf{Definition 2.3.} Let \( X \) be a quasi-median graph, let \( Y \subseteq X \) be a full subgraph, and let \( v \in V(X) \). We say \( p \in V(Y) \) is a \textit{gate} for \( v \) in \( Y \) if, for any \( q \in V(Y) \), there exists a geodesic in \( X \) between \( v \) and \( q \) passing through \( p \). We say a full subgraph \( Y \subseteq X \) is a \textit{gated subgraph} if every vertex of \( X \) has a gate in \( Y \).

The following result says that the subgraphs of interest to us are gated. Here, by convention, given two graphs \( Y \) and \( Z \) we denote by \( Y \times Z \) the 1-skeleton of the square complex obtained as a cartesian product of \( Y \) and \( Z \).

\textbf{Proposition 2.4 (Genevois \cite{Gen17} Proposition 2.15).} Let \( X \) be a quasi-median graph, \( H \) a hyperplane dual to a clique \( C \), and \( F \) a fibre of \( H \). Then \( \mathcal{N}(H) \), \( C \) and \( F \) are gated subgraphs of \( X \). Moreover, there exists a graph isomorphism \( \Psi : \mathcal{N}(H) \to F \times C \), and the cliques dual to \( H \) (respectively the fibres of \( H \)) are precisely the subgraphs \( \Psi^{-1}(\{p\} \times C) \) for vertices \( p \in V(F) \) (respectively \( \Psi^{-1}(F \times \{p\}) \) for vertices \( p \in V(C) \)).

2.2. Special actions. In this section we describe the hypotheses that we impose on group actions on quasi-median graphs. We first define what it means for an action on a quasi-median graph to be special.

\textbf{Definition 2.5.} Let \( X \) be a quasi-median graph, and let \( G \) be a group acting on it by graph isomorphisms. We say the action \( G \acts X \) is \textit{special} if

(i) no two hyperplanes in the same orbit under \( G \acts X \) intersect or osculate; and

(ii) given two hyperplanes \( H \) and \( H' \) that intersect, \( H^g \) and \( H' \) do not osculate for any \( g \in G \).

Special actions on CAT(0) cube complexes were introduced by Haglund and Wise in \cite{HW07}. Notably, there it is shown that, in our terminology, if a group \( G \) acts specially, cocompactly and without ‘orientation-inversions’ of hyperplanes on a CAT(0) cube complex \( X \), then the fundamental group of the quotient \( X/G \) embeds in a right-angled Artin group.

It is clear from Proposition 2.4 that no hyperplane in a quasi-median graph can self-intersect or self-osculate. The next lemma says that, moreover, the action of the trivial group on a quasi-median graph is special. Recall that two hyperplanes are said to \textit{interosculate} if they both intersect and osculate.

\textbf{Lemma 2.6.} In a quasi-median graph \( X \), no two hyperplanes can interosculate.

\textbf{Proof.} Suppose for contradiction that hyperplanes \( H \) and \( H' \) intersect at \( p \) and osculate at \( q \) for some \( p, q \in V(X) \), and assume without loss of generality that \( p \) and \( q \) are chosen in such a way that \( d_X(p, q) \) is as small as possible. It is clear that \( p \neq q \): see, for instance, \cite[Lemma 2.13]{Gen17}. On the other hand, since \( \mathcal{N}(H) \) and \( \mathcal{N}(H') \) are gated (and therefore convex) by Proposition 2.4 and as \( p, q \in \mathcal{N}(H) \cap \mathcal{N}(H') \), it follows that a geodesic between \( p \) and \( q \) lies in \( \mathcal{N}(H) \cap \mathcal{N}(H') \). In particular, if \( r \) is a vertex on this geodesic, then \( H \) and \( H' \) either intersect at \( r \) or osculate at \( r \); by minimality of \( d_X(p, q) \), it then follows that \( d_X(p, q) = 1 \).

Let \( e \) be the edge joining \( p \) and \( q \), and let \( K \) be the hyperplane dual to \( e \). It follows from Proposition 2.4 that \( K \neq H \) and \( K \neq H' \); indeed, if we had \( K = H \) (say), then \( K = H \) and \( H' \) would intersect at \( q \), contradicting the choice of \( q \). Thus \( K \) is distinct from \( H \) and \( H' \), and so \( e \) belongs to a fibre of \( H \) and a fibre of \( H' \). It then follows from Proposition 2.4 that \( K \) intersects both \( H \) and \( H' \) at \( q \), and that the graph \( Y \) shown in Figure 2 is a subgraph of \( X \).

We now claim that the embedding \( Y \hookrightarrow X \) is isometric. Indeed, as \( H \), \( H' \) and \( K \) are distinct hyperplanes, no two vertices \( p', q' \in V(Y) \) with \( d_Y(p', q') = 2 \) can be joined by an edge in \( X \), as that would create a triangle in \( X \) with edges dual to different hyperplanes. It is thus enough to show that if \( p', q' \in V(Y) \) and \( d_Y(p', q') = 3 \), then \( d_X(p', q') = 3 \). Up to relabelling \( H \), \( H' \) and \( K \), we may assume without loss of generality that \( p' = s \) and \( q' = q \). Now it is clear that \( d_X(s, q) \neq 1 \): otherwise, \( p_1s \) and \( q_1q \) are opposite sides in a square in \( X \), contradicting the fact
that $H \neq H'$. Thus, suppose for contradiction that $d_X(s, q) = 2$. But then the triple $(p_1, s, t)$ is a quasi-median of $(p_1, s, q)$ for some vertex $t \in V(X)$, and the edges $p_1 t, q_1 q$ are dual to the same hyperplane, again contradicting the fact that $H \neq H'$. Thus the embedding $Y \hookrightarrow X$ is isometric, as claimed.

But now the embedding of the $C_6 \subseteq Y$ formed by vertices $s, p_1, q_1, q, q_2$ and $p_2$ into $X$ is also isometric, and so the convex hull of this $C_6$ in $X$ is a 3-cube. Thus there exists a vertex $u \in V(X)$ joined by edges to $s, p_2$ and $q_2$. This implies that $H$ and $H'$ intersect at $q$, contradicting the choice of $q$. Thus $H$ and $H'$ cannot interosculate. \hfill \Box

![Figure 2: Proof of Lemma 2.6](image)

**Remark 2.7.** We use Lemma 2.6 in the following setting. Let $\gamma$ be a geodesic in a quasi-median graph $X$, let $e$ and $e'$ be two consecutive edges of $\gamma$, and let $H$ and $H'$ be the hyperplanes dual to $e$ and $e'$, respectively. Suppose that $H$ and $H'$ intersect. It then follows from Lemma 2.6 that $H$ and $H'$ cannot osculate at the common endpoint $p$ of $e$ and $e'$, and therefore $H$ and $H'$ must intersect at $p$. In particular, $X$ contains a square with edges $e$, $e'$, $f$ and $f'$, in which $f$ and $f'$ are the edges opposite to $e$ and $e'$, respectively. We may then obtain another geodesic $\gamma'$ in $X$ (with the same endpoints as $\gamma$) by replacing the subpath $ee'$ of $\gamma$ with $f'f$. We refer to the operation of replacing $\gamma$ by $\gamma'$ as swapping $e$ and $e'$ on $\gamma$.

### 2.3. Geodesics in quasi-median graphs.

Here we record two results on geodesics in a quasi-median graph. The first one of these is due to Genevois.

**Proposition 2.8 (Genevois [Gen17, Proposition 2.30]).** A path in a quasi-median graph $X$ is a geodesic if and only if it intersects any hyperplane at most once. In particular, the distance between two vertices of $X$ is equal to the number of hyperplanes separating them. \hfill \Box

**Lemma 2.9.** Let $p, q, r \in V(X)$ be vertices of a quasi-median graph $X$ such that some hyperplane separates $q$ from $p$ and $r$. Then there exists a hyperplane $C$ separating $q$ from $p$ and $r$ and geodesics $\gamma_p$ (respectively $\gamma_r$) between $q$ and $p$ (respectively $q$ and $r$) such that $q$ is an endpoint of the edges of $\gamma_p$ and $\gamma_r$ dual to $C$.

**Proof.** Let $C$ be a hyperplane separating $q$ from $p$ and $r$, let $\gamma_p$ (respectively $\gamma_r$) be a geodesic between $q$ and $p$ (respectively $q$ and $r$), and let $c_p$ and $c_r$ be the edges of $\gamma_p$ and $\gamma_r$ (respectively) dual to $C$. Let $q_p'$ and $q_r'$, $q_r$ and $q_p'$ be the endpoints of $c_p$, $c_r$ (respectively), labelled so that $C$ does not separate $q_p$ and $q_r$. Suppose, without loss of generality, that $\gamma_p$ and $C$ are chosen in such a way that $d_X(q, q_p)$ is as small as possible, and that $\gamma_r$ is chosen so that $d_X(q, q_r)$ is as small as possible (subject to the choice of $\gamma_p$ and $C$). See Figure 3.

We first claim that $q = q_p$. Indeed, suppose not, and let $c_p' \neq c_p$ be the other edge of $\gamma_p$ with endpoint $q_p$. Let $C_p'$ be the hyperplane dual to $c_p'$. Then $C_p'$ does not separate $q_p$ and $p$ (as $\gamma_p$ is a geodesic), nor $q$ and $r$ (by minimality of $d_X(q, q_p)$), but it separates $q_p$ (and so $p$) from $q$ (and so $r$). On the other hand, $C$ separates $q_p$ from $p$ (as $\gamma_p$ is a geodesic) and $q$ from $r$ (as $\gamma_r$ is a geodesic). Therefore, $C$ and $C_p'$ must intersect. But then we may swap $c_p$ and $c_p'$ on $\gamma_p$ (see Remark 2.7), contradicting minimality of $d_X(q, q_p)$. Thus we must have $q = q_p$. 


We now claim that \( q = q_r \). Indeed, suppose not, and let \( c'_r \neq c_r \) be the other edge of \( \gamma_r \) with endpoint \( q_r \). Let \( C'_r \) be the hyperplane dual to \( c'_r \). Then \( C'_r \) does not separate \( q \) and \( q'_r \) (as \( C \) is the only hyperplane separating \( q = q_p \) and \( q'_p \)), nor \( q_r \) and \( r \) (as \( \gamma_r \) is a geodesic), but it separates \( q \) (and so \( q'_p \)) from \( q_r \) (and so \( r \)). On the other hand, \( C \) separates \( q_r \) from \( r \) (as \( \gamma_r \) is a geodesic) and \( q \) from \( q'_p \). Therefore, \( C \) and \( C'_r \) must intersect. But then we may swap \( c_r \) and \( c'_r \) on \( \gamma_r \), contradicting minimality of \( d_X(q, q_r) \). Thus we must have \( q = q_r \).

\[ \square \]

**Figure 3. Proof of Lemma 2.9**

### 3. Geometry of the contact graph

Here we analyse the geometry of the contact graph \( CX \) of a quasi-median graph \( X \). In Section \[ 3.1 \] we show that, under certain conditions, \( CX \) is quasi-isometric to \( \Delta X \), and prove Theorem \[ B \]. In Section \[ 3.2 \] we prove that \( CX \) is a quasi-tree (Theorem \[ A \]).

#### 3.1. Contact and crossing graphs.

The following proposition allows us to lift geodesics in \( C(X) \) back to \( X \). This generalises the existence of ‘hierarchy paths’ in CAT(0) cube complexes \[ \textbf{BHS17} \] Theorem A(2)] to arbitrary quasi-median graphs. Moreover, the same result applies when \( X \) is replaced by \( \Delta X \), as long as \( \Delta X \) is connected.

**Proposition 3.1.** Let \( \Gamma = CX \) or \( \Gamma = \Delta X \), and let \( A, B \in \mathcal{V}(\Gamma) \) be hyperplanes in the same connected component of \( \Gamma \). Let \( p \in V(X) \) (respectively \( q \in V(X) \)) be a vertex in \( \mathcal{N}(A) \) (respectively \( \mathcal{N}(B) \)). Then there exists a geodesic \( A = A_0, \ldots, A_m = B \in \Gamma \) and vertices \( p = p_0, \ldots, p_{m+1} = q \in V(X) \) such that \( p_i \in \mathcal{N}(A_{i-1}) \cap \mathcal{N}(A_i) \) for \( 1 \leq i \leq m \) and \( d_X(p, q) = \sum_{i=0}^{m} d_X(p_i, p_{i+1}) \).

**Proof.** By assumption, there exists a geodesic \( A = A_0, A_1, \ldots, A_m = B \in \Gamma \). For \( 1 \leq i \leq m \), let \( p_i \in V(X) \) be a vertex in the carriers of both \( A_{i-1} \) and \( A_i \), and let \( p_0 = p, p_{m+1} = q \). Suppose that the \( A_i \) and the \( p_i \) are chosen in such a way that \( d = \sum_{i=0}^{m} d_X(p_i, p_{i+1}) \) is as small as possible. We claim that \( d = d_X(p, q) \).

Let \( \gamma_i \) be a geodesic between \( p_i \) and \( p_{i+1} \) for \( 0 \leq i \leq m \). Suppose for contradiction that \( D > d_X(p, q) \); this means that \( \gamma_0 \gamma_1 \cdots \gamma_m \) is not a geodesic. Therefore, there exists a hyperplane \( C \) separating \( p_i \) and \( p_{i+1} \) as well as \( p_j \) and \( p_{j+1} \) for some \( i < j \). Let \( C_i \) (respectively \( C_j \)) be the edge of \( \gamma_i \) (respectively \( \gamma_j \)) dual to \( C \).

As hyperplane carriers are gated (and therefore convex), any hyperplane separating \( p_i \) and \( p_{i+1} \) either is or intersects \( A_i \) for \( 0 \leq i \leq m \). Now note that \( j - i \leq 2 \): indeed, we have \( d_A(A_i, C) \leq 1 \) and \( d_A(A_j, C) \leq 1 \), so \( j - i = d_A(A_i, A_j) = 1 + 1 = 2 \). In particular, \( j - i \in \{1, 2\} \).

We now claim that \( j = i + 1 \). Indeed, suppose for contradiction that \( j = i + 2 \). Let \( p'_{i+1} \) (respectively \( p'_{i+2} \)) be the endpoint of \( c_i \) (respectively \( c_{i+2} \)) closer to \( p_i \) (respectively \( p_{i+3} \)). Then we have

\[
\begin{align*}
&d_X(p_i, p_{i+1}) + d_X(p_{i+1}, p_{i+2}) + d_X(p_{i+2}, p_{i+3}) \\
&= d_X(p_i, p'_{i+1}) + d_X(p'_{i+1}, p_{i+1}) + d_X(p_{i+1}, p_{i+2}) + d_X(p_{i+2}, p'_{i+2}) + d_X(p'_{i+2}, p_{i+3}) \\
&\geq d_X(p_i, p'_{i+1}) + d_X(p'_{i+1}, p'_{i+2}) + d_X(p'_{i+2}, p_{i+3}),
\end{align*}
\]

with equality if and only if \( \gamma'_i \gamma'_{i+1} \gamma'_{i+2} \) is a geodesic, where \( \gamma'_i \) (respectively \( \gamma'_{i+2} \)) is the portion of \( \gamma_i \) (respectively \( \gamma_{i+2} \)) between \( p'_{i+1} \) and \( p_{i+1} \) (respectively \( p_{i+2} \) and \( p'_{i+2} \)). But \( \gamma'_i \gamma'_{i+1} \gamma'_{i+2} \) cannot be a geodesic as it passes through two edges dual to \( C \), and so strict inequality in \( (1) \) holds. We may then replace \( A_{i+1}, p_{i+1} \) and \( p_{i+2} \) with \( C, p'_{i+1} \) and \( p'_{i+2} \), respectively, contradicting minimality of \( D \). Thus \( j = i + 1 \), as claimed.
Therefore, $C$ separates $p_{i+1}$ from $p_i$ and $p_{i+2}$. By Lemma 2A, we may assume (after modifying $C$, $\gamma_i$ and $\gamma_{i+1}$ if necessary) that $p_{i+1}$ is an endpoint of both $c_i$ and $c_{i+1}$. As $c_i$ and $c_{i+1}$ are dual to the same hyperplane, it follows that they belong to the same clique. In particular (as carriers of hyperplanes are gated and so contain their triangles) this whole clique belongs to $\mathcal{N}(A_i) \cap \mathcal{N}(A_{i+1})$. If $r_{i+1} \neq p_{i+1}$ is the other endpoint of $c_i$, then $d_X(p_i, r_{i+1}) < d_X(p_i, p_{i+1})$ and $d_X(r_{i+1}, p_{i+2}) \leq d_X(p_{i+1}, p_{i+2})$. We may therefore replace $p_{i+1}$ by $r_{i+1}$, contradicting minimality of $D$. Thus $D = d_X(p, q)$, as claimed. 

Taking $\Gamma = \Delta X$ and $p = q$ in Proposition 3.1 immediately gives the following.

**Corollary 3.2.** Let $A, B \in V(\Delta X)$ be hyperplanes in the same connected component of $\Delta X$ osculating at a point $p \in V(X)$. Then there exists a geodesic $A = A_0, \ldots, A_m = B$ in $\Delta X$ such that $A_{i-1}$ and $A_i$ intersect at $p$ for $1 \leq i \leq m$.

**Lemma 3.3.** Suppose a group $G$ acts on $X$ specially with $N$ orbits of hyperplanes. Let $A$ and $B$ be hyperplanes that osculate and belong to the same connected component of $\Delta X$. Then $d_{\Delta X}(A, B) \leq \text{max}(2, N - 1)$.

**Proof.** Let $p \in V(X)$ be such that $A$ and $B$ osculate at $p$. By Corollary 3.2, there exists a geodesic $A = A_0, A_1, \ldots, A_m = B$ in $\Delta X$ such that $A_{i-1}$ and $A_i$ intersect at $p$ for each $i$. Let $i_1, \ldots, i_k \in \mathbb{N}$, satisfying $0 = i_1 < i_2 < \cdots < i_k = m + 1$, be such that $A_{i_j}^G = A_{i_j+1}^G$ for $1 \leq j \leq k - 1$ (for instance, we may take $i_j = j - 1$). Suppose this is done so that $k$ is as small as possible. Clearly, this implies $A_{i_j}^{G} \neq A_{i_{j'}}^{G}$ whenever $1 \leq j < j' \leq k$: otherwise, we may replace $i_1, \ldots, i_k$ by $i_1, i_j, i_{j+1}, \ldots, i_k$, contradicting minimality of $k$. In particular, $k \leq N + 1$; as $m \geq 2$, note also that $k \geq 2$. We will consider the cases $k = 2$ and $k \geq 3$ separately.

Suppose first that $k \geq 3$. We claim that $i_{j+1} - i_j \leq 1$ whenever $1 \leq j \leq k - 1$. Indeed, note that whenever $1 \leq j < k - 2$, $p \in \mathcal{N}(A_{i_j}) \cap \mathcal{N}(A_{i_{j+1}})$, and so $A_{i_j}$ and $A_{i_{j+1}}$ must either intersect or osculate. But $A_{i_{j+1}}$ intersects $A_{i_{j+1}+1}$, and $A_{i_j}^G = A_{i_{j+1}+1}^G$: therefore, as the action $G \curvearrowright X$ is special, it follows that $A_{i_j}$ and $A_{i_{j+1}+1}$ must intersect. In particular, $i_{j+1} - i_j = d_{\Delta X}(A_{i_{j+1}}, A_{i_{j+1}+1}) \leq 1$ for $1 \leq j \leq k - 2$. For $j = k - 1$, we may similarly note that $\mathcal{N}(A_{i_{k-1}}) \cap \mathcal{N}(A_{i_{k-1}+1}) \neq \emptyset$ and so $A_{i_{k-1}}$ and $A_m = A_{i_{k+1}+1}$ must intersect: thus $i_{k+1} - i_k = d_{\Delta X}(A_{i_{k+1}}, A_{i_{k+1}+1}) \leq 1$ in this case as well. In particular, we get

$$d_{\Delta X}(A, B) = m = i_k - i_1 = \left(\sum_{j=1}^{k-1} (i_{j+1} - i_j)\right) - 1 \leq k - 2 \leq N - 1,$$

as required.

Suppose now that $k = 2$. Similarly to the case $k \geq 3$, we may note that $p \in \mathcal{N}(A_1) \cap \mathcal{N}(A_m)$, and so, as $A_0$ and $A_1$ intersect and as $A_0^G = A_1^G$, it follows that $A_1$ and $A_m$ intersect. Thus $m - 1 = d_{\Delta X}(A_1, A_m) \leq 1$ and so $d_{\Delta X}(A, B) = m \leq 2$, as required.

**Proof of Theorem 3.1**. It is clear that $d_{CX}(A, B) \leq d_{\Delta X}(A, B)$ for any hyperplanes $A$ and $B$, as $\Delta X$ is a subgraph of $CX$. Conversely, Lemma 3.3 implies that $d_{\Delta X}(A, B) \leq \max(2, N - 1)$. 

**Remark 3.4.** We note that all the assumptions for Theorem 3.1 are necessary. Indeed, it is clear that $\Delta X$ needs to be connected. To show necessity of the other two conditions, consider the following. Let $G_0 = \langle S \mid R \rangle$ be the group with generators $S = \{a_{i,j} \mid (i, j) \in \mathbb{Z}^2\}$ and relators $R = \bigcup_{(i, j) \in \mathbb{Z}^2} \{a_{i,j}^2, [a_{i,j}, a_{i,j+1}], [a_{i,j}, a_{i+1,j}]\}$; this is the (infinitely generated) right-angled Coxeter group associated to a ‘grid’ in $\mathbb{R}^2$: a graph $\Gamma$ with $V(\Gamma) = \mathbb{Z}^2$, where $(i, j)$ and $(i', j')$ are adjacent if and only if $|i - i'| + |j - j'| = 1$. Let $X$ be the Cayley graph of $G_0$ with respect to $S$.

Then $X$ is a quasi-median (and, indeed, median) graph by [Gen17] Proposition 8.2. Furthermore, by the results in [Gen17] Chapter 8, $\Delta X$ is connected, and if $H_{i,j}$ is the hyperplane dual to the edge $(1, a_{i,j})$ of $X$ (for $(i, j) \in \mathbb{Z}^2$) then $d_{CX}(H_{0,0}, H_{i,j}) \leq 1$ but $d_{\Delta X}(H_{0,0}, H_{i,j}) = |i| + |j|$.
for all \((i, j)\). Thus the inclusion \(\Delta X \hookrightarrow CX\) cannot be a quasi-isometry. Moreover, by Theorem A we know that \(CX\) is a quasi-tree, whereas the inclusion into \(\Delta X\) of the subgraph spanned by \(\{H_{i,j} \mid (i, j) \in \mathbb{Z}^2\}\) (which is isomorphic to the ‘grid’ \(\Gamma\)) is isometric, and so \(\Delta X\) cannot be hyperbolic – therefore, \(\Delta X\) and \(CX\) are not quasi-isometric in this case.

It follows from \cite[Proposition 8.11]{Gen17} that the usual action of \(G_0\) on \(X\) is special – however, there are infinitely many orbits of hyperplanes under this action. On the other hand, let \(G = G_0 \times \mathbb{Z}^2\), where the action of \(\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle\) on \(G_0\) is given by \(a_{i,j}^{x^n y^m} = a_{i+n, j+m}\); this can be thought of as an example of a graph-wreath product, see \cite{KM16} for details. Then it is easy to see that the action of \(G\) on \(G_0\) extends to an action of \(G\) on \(X\). This action is transitive on hyperplanes, and therefore not special.

### 3.2. Hyperbolicity

We show here that \(CX\) is a quasi-tree, proving Theorem A.

**Proposition 3.5.** Let \(A, B \in V(CX)\) be two hyperplanes such that \(d_{CX}(A, B) \geq 2\). Then there exists a midpoint \(M\) of a geodesic between \(A\) and \(B\) in \(CX\) and a hyperplane \(C\) separating \(N(A)\) and \(N(B)\).

**Proof.** By Proposition 2.4 we know that \(N(A)\) and \(N(B)\) are gated. It then follows from \cite[Lemma 2.36]{Gen17} that there exist vertices \(p \in V(N(A))\) and \(q \in V(N(B))\) such that any hyperplane separating \(p\) and \(q\) also separates \(N(A)\) from \(N(B)\). Let \(A = A_0, \ldots, A_m = B \in V(\Delta X)\) and \(p = p_0, \ldots, p_{m+1} = q \in V(X)\) be as given by Proposition 3.4 in the case \(\Gamma = CX\), and let \(M\) be the midpoint of the former geodesic. It is clear that \(N(A_i) \cap N(A_j) = \emptyset\) whenever \(|i - j| \geq 2\); in particular, \(p_i \neq p_{i+1}\) whenever \(1 \leq i \leq m - 1\).

Now let \(i = \lfloor m/2 \rfloor \in \{1, \ldots, m - 1\}\), and let \(C\) be any hyperplane separating \(p_i\) and \(p_{i+1}\). By the choice of the \(p_j\), there exists a geodesic between \(p\) and \(q\) passing through \(p_i\) and \(p_{i+1}\); therefore, \(C\) separates \(p\) and \(q\). Therefore, by the choice of \(p\) and \(q\), \(C\) also separates \(N(A)\) from \(N(B)\). Finally, note that as \(C\) separates \(p_i, p_{i+1} \in N(A_i)\), we have \(d_{CX}(A_i, C) \leq 1\). Therefore,

\[
d_{CX}(M, C) \leq d_{CX}(M, A_i) + d_{CX}(A_i, C) = \left\lfloor \frac{m}{2} - i \right\rfloor + d_{CX}(A_i, C) \leq \frac{1}{2} + 1 = \frac{3}{2},
\]

as required. \(\square\)

**Definition 3.6.** For a connected graph \(\Gamma\) and two vertices \(v, w \in V(\Gamma)\) we say a point \(m \in \Gamma\) is a midpoint between \(v\) and \(w\) if \(d_{\Gamma}(m, v) = d_{\Gamma}(m, w) = \frac{1}{2}d_{\Gamma}(v, w)\).

Let \(D \in \mathbb{N}\). A connected graph \(\Gamma\) is said to satisfy the \(D\)-bottleneck criterion if for any vertices \(v, w \in V(\Gamma)\), there exists a midpoint \(m\) between \(v\) and \(w\) such that any path between \(v\) and \(w\) passes through a point \(p\) such that \(d_{\Gamma}(p, m) \leq D\).

**Theorem 3.7** (Manning \cite[Theorem 4.6]{Man05}). A connected graph \(\Gamma\) is a quasi-tree if and only if there exists a constant \(D\) such that \(\Gamma\) satisfies the \(D\)-bottleneck criterion. \(\square\)

**Remark 3.8.** In \cite{Man05}, the statement of this theorem is given for a general geodesic metric space (not necessarily a graph), and the definition of bottleneck criterion given there is stronger: instead of taking \(v, w\) to be vertices of \(\Gamma\) in Definition 3.6 Manning allows \(v, w\) to be any points of \(\Gamma\). However, as any point in a graph is within distance \(\frac{1}{2}\) of a vertex, it is easy to see that in our setting the definition given here is equivalent to the one given in \cite{Man05} (up to possibly modifying the constant \(D\)).

**Proof of Theorem A.** We claim that \(CX\) satisfies the \(5/2\)-bottleneck criterion.

Let \(A, B \in V(CX)\) be two hyperplanes. If \(d_{CX}(A, B) < 2\), then any path between \(A\) and \(B\) passes through \(A\), and \(d_{CX}(A, M) = d_{CX}(A, B)/2 < 1 < 5/2\) for any midpoint \(M\) between \(A\) and \(B\), so the \(5/2\)-bottleneck criterion is satisfied.

On the other hand, if \(d_{CX}(A, B) \geq 2\), then let \(M\) and \(C\) be as given by Proposition 3.5. Let \(A = A_0, B_1, \ldots, B_n = B\) be any path in \(CX\) between \(A\) and \(B\), and choose vertices \(q_1, \ldots, q_n \in V(X)\) such that \(q_i \in N(B_{i-1}) \cap N(B_i)\) for all \(i\). As \(q_1 \in N(A)\), \(q_n \in N(B)\), and as \(C\) separates \(A\) and \(B\), it follows that \(C\) separates \(q_1\) and \(q_n\), and so it separates \(q_i\) and \(q_{i+1}\) for some \(i\). But as \(q_i, q_{i+1} \in N(B_i)\), it follows that \(d_{CX}(C, B_i) \leq 1\). In particular,

\[
d_{CX}(M, B_i) \leq d_{CX}(M, C) + d_{CX}(C, B_i) \leq \frac{3}{2} + 1 = \frac{5}{2},
\]
and so again the 5/2-bottleneck criterion is satisfied.

In particular, Theorem 3.7 implies that $CX$ is a quasi-tree. □

4.ACYLINDRICITY

In this section we prove Theorem 4.2 (ii). To do this, in Section 4.1 we introduce the notion of contact sequences (see Definition 1.2) and show the main technical result we need to prove Theorem 4.2 (ii), namely, Proposition 4.3. In Section 4.2 we use this to deduce Theorem 4.2 (ii).

Throughout this section, let $X$ be a quasi-median graph.

4.1. Contact sequences.

**Lemma 4.1.** Let $Y \subseteq X$ be a gated subgraph and let $\mathcal{H}$ be a collection of hyperplanes in $X$. Let $Y'_\mathcal{H} \subseteq V(X)$ be the set of vertices $v \in V(X)$ for which there exists a vertex $p_v \in V(Y)$ such that all hyperplanes separating $v$ from $p_v$ are in $\mathcal{H}$. Then the full subgraph $Y'_\mathcal{H}$ of $X$ spanned by $Y'_\mathcal{H}$ is gated.

**Proof.** Suppose for contradiction that $Y'_\mathcal{H}$ is not gated, and let $v \in V(X)$ be a vertex that does not have a gate in $Y'_\mathcal{H}$. Let $p$ be the gate for $v$ in $Y$. Let $\tilde{p}$ be a vertex of $Y'_\mathcal{H}$ on a geodesic between $v$ and $p$ with $d_X(v, \tilde{p})$ minimal. By our assumption, $\tilde{p}$ is not a gate for $v$ in $Y'_\mathcal{H}$, and so there exists a vertex $\tilde{q} \in V(Y'_\mathcal{H})$ such that no geodesic between $v$ and $\tilde{q}$ passes through $\tilde{p}$. Let $q$ be the gate of $\tilde{q}$ in $Y$. Let $\gamma_p, \gamma_q, \delta, \delta'$ be geodesics between $\tilde{p}$ and $p$, $\tilde{q}$ and $q$, $p$ and $q$, $\tilde{p}$ and $\tilde{q}$, $v$ and $\tilde{p}$ (respectively), as shown in Figure 4.

Since both $\eta$ and $\delta$ are geodesics, and since $\eta \delta$ is not (by the choice of $\tilde{q}$), it follows from Lemma 2.9 that we may assume, without loss of generality, that there exists a hyperplane $C$ and edges $e_1, e_2$ of $\eta, \delta$ (respectively) both of which are dual to $C$ and have $\tilde{p}$ as an endpoint. But as $p$ is the gate for $v$ in $Y$, as $\eta \gamma_p$ is a geodesic by the choice of $\tilde{p}$, and as $q \in Y$, it follows that $\eta \gamma_p \delta$ is a geodesic. Therefore, by Proposition 2.8 $H$ cannot cross $\gamma_p \delta$, and so $H$ does not separate $\tilde{p}$ and $q$. As $H$ separates $\tilde{p}$ and $\tilde{q}$, it follows that $H$ separates $\tilde{q}$ and $q$ and so crosses $\gamma_q$. In particular, since $\tilde{q} \in V(Y'_\mathcal{H})$ and since $q \in V(Y)$ is a gate for $\tilde{q}$ in $Y$, it follows that all hyperplanes crossing $\gamma_q$ are in $\mathcal{H}$, and therefore $H \in \mathcal{H}$. But then the endpoint $p' \neq \tilde{p}$ of $e_1$ is separated from $p \in V(Y)$ only by hyperplanes in $\mathcal{H}$; this contradicts the choice of $\tilde{p}$. Thus $Y'_\mathcal{H}$ is gated, as claimed. □

**Figure 4.** Proof of Lemma 4.1.

Now let a group $G$ act on a quasi-median graph $X$. This induces an action of $G$ on the crossing graph $\Delta X$. Let $\mathcal{H}$ be the set of orbits of vertices under $G \acts \Delta X$ – alternatively, the set of orbits of hyperplanes under $G \acts X$. We may regard each element of $\mathcal{H}$ as a collection of hyperplanes – thus, for instance, given $\mathcal{H}_0 \subseteq \mathcal{H}$ we may write $\bigcup \mathcal{H}_0$ for the set of all hyperplanes whose orbits are elements of $\mathcal{H}_0$.

Let $n \in \mathbb{N}$, and let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be subsets of $\mathcal{H}$. Pick a vertex (a ‘basepoint’) $o \in V(X)$, and define the subgraphs $Y_0, \ldots, Y_n \subseteq X$ inductively: set $Y_0 = \{o\}$ and $Y_i = (Y_{i-1}) \cup \mathcal{H}_i$ for $1 \leq i \leq n$. By Lemma 4.1 $Y_n$ is a gated subgraph. We denote $Y_n$ as above by $Y(o, \mathcal{H}_1, \ldots, \mathcal{H}_n)$, and we denote the gate for $v \in V(X)$ in $Y_n$ by $g(v; o, \mathcal{H}_1, \ldots, \mathcal{H}_n)$. 
Moreover, let $\hat{H}, H' \in V(CX)$, and let $p, p' \in V(X)$ be such that $p \in N(H)$ and $p' \in N(H')$. Let $n = d_{CX}(H, H')$. Given any geodesic $H = H_0, \ldots, H_n = H' \in CX$ and vertices $p = p_0, p_1, \ldots, p_{n+1} = p' \in V(X)$ such that $p_i, p_{i+1} \in N(H_i)$ for $0 \leq i \leq n$, we call $\mathcal{S} = (H_0, H_1, H_2, \ldots, H_n, p_0, \ldots, p_{n+1})$ a contact sequence for $(H, H', p, p')$. A contact sequence $\mathcal{S} = (H_0, H_1, H_2, \ldots, H_n, p_0, \ldots, p_{n+1})$ for $(H, H', p, p')$ and a vertex $v \in V(X)$, we say $(g_0, \ldots, g_n) \in V(X)^{n+1}$ is the $v$-gate for $\mathcal{S}$ if $g_i$ is the gate for $v$ in $N(H_i)$ for $0 \leq i \leq n$. We furthermore denote the tuples $(d_X(p_0, g_n), \ldots, d_X(p_0, g_0))$ and $(d_X(p_0, g_n), d_X(p_{n+1}, g_n))$ by $\mathcal{C}_{\mathcal{S}}(v)$ and $\mathcal{C}_{\mathcal{S}}(v)$, respectively. We say a contact sequence $\mathcal{S}$ for $(H, H', p, p')$ is $v$-minimal if for any other contact sequence $\mathcal{S}'$ for $(H, H', p, p')$ we have either $\mathcal{C}_{\mathcal{S}}(v) \leq \mathcal{C}_{\mathcal{S}'}(v)$ or $\mathcal{C}_{\mathcal{S}'}(v) \leq \mathcal{C}_{\mathcal{S}}(v)$ in the lexicographical order.

Finally, suppose a group $G$ acts on $X$. Given a vertex $v \in V(X)$ and a contact sequence $\mathcal{S} = (H_0, \ldots, H_n, p_0, \ldots, p_{n+1})$ for $(H, H', p, p')$ with a $v$-gate $(g_0, \ldots, g_n)$, we say $(\mathcal{H}_0, \ldots, \mathcal{H}_n, \mathcal{H}_0', \ldots, \mathcal{H}_n')$, where $\mathcal{H}_i, \mathcal{H}'_i \subseteq V(CX/G)$, is the $(v, G)$-orbit sequence for $\mathcal{S}$ if

\[ \mathcal{H}_i = \{ H^G \mid H \in V(CX) \text{ separates } p_i \text{ from } g_i \} \]

and

\[ \mathcal{H}'_i = \{ H^G \mid H \in V(CX) \text{ separates } p_{i+1} \text{ from } g_i \} \]

for $0 \leq i \leq n$.

It is clear that given any $H$, $H'$, $p$ and $p'$ as in Definition 4.2 there exists a contact sequence for $(H, H', p, p')$. As the lexicographical order is a well-ordering of $\mathbb{N}^n$, it follows that a $v$-minimal contact sequence exists as well.

**Proposition 4.3.** Suppose a group $G$ acts specially on a quasi-median graph $X$. Let $H, H' \in V(CX)$, let $p, p' \in V(X)$ be such that $p \in N(H)$ and $p' \in N(H')$, and let $v \in V(X)$. Let $\mathcal{S} = (H_0, \ldots, H_n, p_0, \ldots, p_{n+1})$ be a $v$-minimal contact sequence for $(H, H', p, p')$ with $v$-gate $(g_0, \ldots, g_n)$ and $(v, G)$-orbit sequence $(\mathcal{H}_0, \ldots, \mathcal{H}_n, \mathcal{H}_0', \ldots, \mathcal{H}_n')$. Write $g_i := g(v; p, H_0, \ldots, H_i)$ and $g'_i := g(v; p', H'_0, \ldots, H'_i)$ for $0 \leq i \leq n$. Then,

(i) $g_n = g_0$;

(ii) no two hyperplanes in $\bigcup \mathcal{H}_i$ and $\bigcup \mathcal{H}'_i$ (respectively) osculate whenever $i > j$;

(iii) for $1 \leq i \leq n$, the hyperplanes separating $g_{i-1}$ from $g_i$ (respectively $g'_i$ from $g'_{i-1}$) are precisely the hyperplanes separating $p_i$ from $g_i$ (respectively $p_i$ from $g_{i-1}$).

**Proof.** Induction on $n$.

For $n = 0$, we claim that $g_0 = g_0$. Indeed, by definition of $\mathcal{H}_0$ we have $g_0 \in Y(p_0, H_0)$, and so there exists a geodesic $\eta$ between $p$ and $v$ passing through $g_0$ and $g_0$. Suppose for contradiction $g_0 \neq g_0$, let $a \subseteq \eta$ be the edge with endpoint $g_0$ such that the other endpoint $g_0 \neq g_0$ of $a$ satisfies $d_X(v, g_0) > d_X(v, q_0)$, and let $A$ be the hyperplane dual to $a$; see Figure 5(a). Then $g_0 \in N(H_0) \cap N(A)$, and so $H_0$ and $A$ either coincide, or intersect, or osculate. As $A$ separates $p$ and $g_0$, we know that $A^0$ separates $p$ and $g_0$ and so $A^0$ and $H_0$ either coincide or intersect for some $g \in G$. Thus, as the action $G \curvearrowright X$ is special, it follows that $A$ and $H_0$ cannot osculate, and therefore they either coincide or intersect. But then we also have $q_a \in N(H_0)$, contradicting the choice of $g_0$. Therefore, $g_0 = g_0$, as claimed. A symmetric argument shows that $g'_0 = g_0$, and so the conclusion of the proposition is clear.

Suppose now that $n \geq 1$, and let $g'_i = g(v; p, H'_0, \ldots, H'_i)$ for $0 \leq i \leq n$ (so that $g'_n = p_n$). Notice that $(H_0, \ldots, H_{n-1}, p_0, \ldots, p_n)$ is a $v$-minimal contact sequence for $(H, H_{n-1}, p, p_n)$. Thus, by the inductive hypothesis we have

(i') $g_{n-1} = g'_0$;

(ii) no two hyperplanes in $\bigcup \mathcal{H}_i$ and $\bigcup \mathcal{H}'_i$ (respectively) osculate whenever $n - 1 \geq i > j$;

(iii') for $1 \leq i \leq n - 1$, the hyperplanes separating $g_{i-1}$ from $g_i$ (respectively $g'_i$ from $g'_{i-1}$) are precisely the hyperplanes separating $p_i$ from $g_i$ (respectively $p_i$ from $g_{i-1}$).

Moreover, let $g_i = g(v; p, H_1, \ldots, H_i)$ for $0 \leq i \leq n$ (so that $g_0 = p_1$), and notice that $(H_1, \ldots, H_n, p_1, \ldots, p_{n+1})$ is a $v$-minimal contact sequence for $(H_1, H', p_1, p')$. Thus, by the inductive hypothesis we have

(i') $g_{n} = g'_1$:
(ii’’) no two hyperplanes in \( \bigcup \mathcal{H}_i \) and \( \bigcup \mathcal{H}_i' \) (respectively) osculate whenever \( i > j \geq 1 \);

(iii’’) for \( 2 \leq i \leq n \), the hyperplanes separating \( \hat{g}_{i-1} \) from \( \hat{g}_i \) (respectively \( g_i' \) from \( g_{i-1}' \)) are precisely the hyperplanes separating \( p_i \) from \( g_i \) (respectively \( p_i \) from \( g_{i-1} \)).

Finally, the proof of the \( n = 0 \) case above shows that \( g(v; p_i, \mathcal{H}_i) = g_i = g(v; p_{i+1}, \mathcal{H}_i') \) for \( 0 \leq i \leq n \).

Now let \( q = \hat{g}_{n-1} \) and note that we also have \( q = \hat{g}_1' \): this is clear if \( n = 1 \) and follows from the inductive hypothesis if \( n \geq 2 \). Let \( A, B, A', B' \subseteq V(\Delta X) \) be the sets of hyperplanes separating \( q \) from \( g_{n-1} \), \( q \) from \( g_1 \), \( g_{n-1} \) from \( g_n \), \( g_1' \) from \( g_n \), respectively; see Figure 5(b). We claim that \( A = A' \) and \( B = B' \). We will show this in steps, proving part (iii) of the Proposition along the way.

\( A \cap B = \emptyset \): Suppose for contradiction that there exists some hyperplane \( A \in A \cap B \). As \( A \in A \), we know that \( A \) separates \( \hat{g}_1 \) from \( \hat{g}_0' \), and so by (iii’’) above it also separates \( p_1 \) from \( g_0 \); thus \( d_C(X(H_0, A) \leq 1 \). Similarly, as \( A \in B \), by (iii’’) above we know that \( A \) separates \( p_n \) from \( g_n \) and therefore \( d_C(X(H_n, A) \leq 1 \). Hence, \( n = d_{\Delta X}(H_0, H_n) \leq 2 \), and so either \( n = 1 \) or \( n = 2 \).

Let \( a, \beta \) be geodesics between \( p_1 \) and \( g_0 \), \( p_n \) and \( g_n \), respectively, and let \( a \subseteq \alpha \) and \( b \subseteq \beta \) be the edges dual to \( A \). As \( a \) and \( b \) lie on geodesics with endpoint \( v \), we may pick endpoints \( q_a \) and \( q_b \) of \( a \) and \( b \), respectively, such that \( A \) does not separate \( q_a \) and \( q_b \).

Suppose first that \( n = 2 \): see Figure 5(c) Note that in this case \( H_0, A, H_2 \) is a geodesic in \( C \mathcal{X} \) and that \( d_C(X(p_2, g_2)) = d_C(g_0, g_2) \) and \( d_C(X(p_1, g_1)) = d_C(q_a, g_0) \). Moreover, since \( q_a \) lies on a geodesic between \( p_1 \) and \( g_0 \), we have \( q_a \in N(H_0) \); similarly, \( q_b \in N(H_2) \). Furthermore, by the construction we know that \( q_a, q_b \in N(A) \). We may therefore replace \( p_1, p_2, H_1 \) by \( q_a, q_b \) and \( A \), respectively, contradicting \( v \)-minimality of \( \mathcal{S} \). Thus \( n \neq 2 \).

Suppose now that \( n = 1 \). Then \( A \) separates \( p_1 \) from both \( g_0 \) and \( g_1 \). By Lemma 2.9 we may then without loss of generality assume that \( p_1 \) is an endpoint (distinct from \( q_a \) and \( q_b \)) of both \( a \) and \( b \). Now note that both \( a \) and \( b \) are edges on a geodesic between \( p_1 \) and \( v \), so we must have \( a = b \), and in particular \( q_a = q_b \); see Figure 5(d). Since \( A \) separates \( p_1 \) from both \( g_0 \) and \( g_1 \), it intersects or coincides with both \( H_0 \) and \( H_1 \), and so \( q_a \in N(H_0) \cap N(H_1) \). We may therefore replace \( p_1 \) by \( q_a \); but we have \( d_C(X(q_1, g_1)) = d_C(q_a, g_1) \) and \( d_C(X(p_1, g_0)) = d_C(q_a, g_0) \), contradicting \( v \)-minimality of \( \mathcal{S} \). Thus no such hyperplane \( A \in A \cap B \) can exist and so \( A \cap B = \emptyset \), as claimed.

\( A \cap B' = \emptyset \): This is clear, as \( \hat{g}_{n-1} = \hat{g}_0 \) lies on a geodesic between \( q = \hat{g}_1 \) and \( \hat{g}_n \), and so no hyperplane can separate \( g_{n-1} \) from both \( q \) and \( g_n \).

\( A' \cap B = \emptyset \): Let \( C \) be the set of hyperplanes separating \( g_n \) and \( v \). We first claim that \( A' \cap B = A' \cap C \). Indeed, let \( B \in A' \cap B \). Since \( B \in B \), it separates \( q \) and \( g_1 ' \); as \( g_i' = \hat{g}_i \) lies on a geodesic between \( q \) and \( v \), \( B \) cannot separate \( \hat{g}_1 \) and \( v \). But as \( B \in A' \), it separates \( g_i ' \) and \( g_n \), and so \( B \) must separate \( g_n \). Therefore, \( B \in A' \cap C \). Conversely, let \( C \in A' \cap C \). Since \( C \in C \), it separates \( g_n \) and \( v \); as \( g_n \) lies on a geodesic between \( q \) and \( v \), \( C \) cannot separate \( q \) and \( g_n \). But as \( C \in A' \), it separates \( g_1 ' \) and \( g_n \), and so \( C \) must separate \( q \) and \( g_1 ' \). Therefore, \( C \in A' \cap B \), and so \( A' \cap B = A' \cap C \), as claimed.

Now suppose for contradiction that there exists a hyperplane \( A \in A' \cap B = A' \cap B \cap C \). Let \( \gamma \) be a geodesic between \( g_n \) and \( v \), and let \( c \subseteq \gamma \) be the edge dual to \( A \). By Lemma 2.9 we may without loss of generality assume that \( g_n \) is an endpoint of \( c \): see Figure 5(e).

Now let \( q_c \neq g_n \) be the other endpoint of \( c \). Note that since \( A \in B \), we have \( A^G \in H_n \). Therefore, it follows that \( q_c \) is separated from \( g_{n-1} \) only by hyperplanes in \( \bigcup \mathcal{H}_n \); as \( d_C(v, g_n) > d_C(v, q_c) \), this contradicts the definition of \( g_n \). Thus \( A \cap B' = \emptyset \), as claimed.

\( A \subseteq A' \) and \( B \subseteq B' \): As \( A \cap B = \emptyset = A \cap B' \), every hyperplane separating \( q \) and \( g_{n-1} \) does not separate \( q \) and \( g_1 ' \), nor \( g_{n-1} \) and \( g_n \), thus it separates \( g_1 ' \) and \( g_n \). It follows that \( A \subseteq A' \).

Similarly, as \( A' \cap B = \emptyset = A' \cap B' \), we get \( B \subseteq B' \).

Part (ii) By (ii”) and (iii”) above, it is enough to show that no two hyperplanes in \( \bigcup \mathcal{H}_n \) and \( \bigcup \mathcal{H}_0 \) (respectively) osculate. Thus, let \( A \) (respectively \( B \)) be a hyperplane separating
p_1 and g_0 (respectively p_n and g_n), so that A G ∈ H'_0 and B G ∈ H_n. It is now enough to show that A^0 and B^h do not osculate for any g, h ∈ G.

But as A separates p_1 from g_0, we know from [iii] that it also separates g'_1 = q from ̂g_0, that is, A ∈ A. Similarly, as B separates p_n and g_n, we know from [iii] that B ∈ B. But as A ∩ B = ∅ = A ∩ B' and as B ⊆ B', it follows that A separates q and g'_1 from g_n-1 and g_n, while B separates q from g'_1 and g_n-1 from g_n. Therefore, A and B must intersect. But as the action G ∩ X is special, it follows that A^0 and B^h do not osculate for any g, h ∈ G. Thus no two hyperplanes in ∪ H_n and ∪ H'_0 (respectively) osculate, and so part (ii) holds, as required.

\[ A' ∩ B' = ∅: \] Suppose for contradiction that A ∈ A' ∩ B' is a hyperplane. Let α' be a geodesic between ̂g'_1 and g_n, let a ⊆ α' be the edge dual to A, and let q_a, q_a' be the endpoints of a so that A does not separate ̂g'_1 and q_a. Suppose, without loss of generality, that α' and A are chosen in such a way that d_X( ̂g'_1, q_a) is as small as possible.

We now claim that ̂g'_1 = q_a. Indeed, suppose not, and let a' ≠ a be the other edge on α' with endpoint q_a. Let A' ∈ A' be the hyperplane dual to a'; see Figure 5(f). Then A' does not separate q and ̂g'_1 (as A' ∩ B = ∅), nor g_n-1 and g_n (by minimality of d_X( ̂g'_1, q_a)), but it separates ̂g'_1 (and so q) from g_n (and so g_n-1). In particular, A' ∈ A, and so A' ∈ ∪ H'_0. On the other hand, A ∈ B' ⊆ ∪ H_n, and so A and A' cannot osculate by part (ii). It follows that A and A' must intersect, and therefore we may swap a and a' on α', contradicting minimality of d_X( ̂g'_1, q_a). Thus ̂g'_1 = q_a, as claimed.

But now q'_0 is separated from q just by hyperplanes in ∪ H_n. Furthermore, A cannot separate g_n and v (as ̂g_n lies on a geodesic between g_n-1 and v, and as A separates g_n-1 and g_n), nor g_n and q'_0 (as α' is a geodesic), but A separates q'_0 (and so ̂g_n and v) from ̂g'_1. In particular, d_X(v, ̂g'_1) > d_X(v, q'_0), contradicting the fact that ̂g'_1 = ̂g_n. Thus A ∩ B = ∅, as required.

\[ A = A' \text{ and } B = B': \] We have already shown A ⊆ A' and B ⊆ B'. Conversely, as A' ∩ B = ∅ = A' ∩ B', every hyperplane separating ̂g'_1 and g_n does not separate q and ̂g'_1, nor g_n-1 and g_n, thus it separates q and g_n-1. It follows that A' ⊆ A and so A = A'. Similarly, as A ∩ B = ∅ = A' ∩ B', we get B = B'.

Now part (iii) of the Proposition follows immediately. Indeed, given \[ (iii') \] and \[ (iii'') \] it is enough to show that the hyperplanes separating g_n-1 from g_n (respectively ̂g'_1 from ̂g'_0) are precisely the hyperplanes separating g_n-1 from g_n (respectively ̂g'_1 from ̂g'_0). But this, and so \[ (iii) \] follows from the fact that A = A' and B = B'.

Finally, we are left to show part (i). We know that A' = A ⊆ ∪ H'_0, and so g_n ∈ Y( ̂g'_1, H'_0) ⊆ Y(p', H'_0, . . . , H'_n). In particular, there exists a geodesic between v and g_n passing through g(v, ̂g'_1, H'_0, . . . , H'_n) = ̂g_0. But a symmetric argument can show that there exists a geodesic between v and ̂g_0 passing through ̂g_n. Thus g_n = ̂g_0, proving (i).

\[ \square \]

4.2. Consequences of Proposition 4.3

Corollary 4.4. Let a group G act specially on a quasi-median graph X. Let H, H', K, K' ∈ V(CX), and let p, p', v, v' ∈ V(X) be such that p ∈ N(H), p' ∈ N(H'), v ∈ N(K), and v' ∈ N(K'). Suppose that d_X(H, K) ≥ d_X(H, H') + d_X(H, K') + 3.

If S is a v-minimal contact sequence for (H, H', p, p'), then S is also v'-minimal. Furthermore, if (H_0, . . . , H_n, H'_0, . . . , H'_n) is the (v, G)-orbit sequence for S, then g(v, p, H_0, . . . , H_n) = g(v', p, H_0, . . . , H_n).

Proof. Let m = d_X(K, K'), and let K = K_0, . . . , K_m = K' be a geodesic in Δ_X. For 1 ≤ i ≤ m, choose a vertex v_i ∈ N(K_i-1) ∩ N(K_i); let also v_0 = v and v_{m+1} = v'. Let n = d_X(H, H').

Given a contact sequence S = (H_0, . . . , H_n, p_0, . . . , p_n) for (H, H', p, p') and any v ∈ V(X), the tuples C_S(G, v) and C_S(G, v) only depend on the gates for v in the N(H_i), 0 ≤ i ≤ n. In particular, if for all hyperplanes A ∈ V(CX) with d_X(A, H) ≤ n the gates for v and v' in N(A) coincide, then the set of v-minimal contact sequences for (H, H', p, p') coincides with the set of v'-minimal ones.
Thus, let $A \in VCX$ be a hyperplane with $d_{\Delta X}(H, A) \leq n$, and suppose for contradiction that $g \neq g'$, where $g$ and $g'$ are the gates for $v$ and $v'$ (respectively) in $N(A)$. Let $B$ be a hyperplane separating $g$ and $g'$. Since $B$ separates two points in $N(A)$, we must have $d_{CX}(A, B) \leq 1$, and so $d_{CX}(H, B) \leq n + 1$. On the other hand, as $B$ separates the gates of $v$ and $v'$ in a gated subgraph, $B$ must also separate $v = v_0$ and $v' = v_{n+1}$. Thus $B$ must separate $v_i$ and $v_{i+1}$ for some $i \in \{0, \ldots, m\}$. As $v_i, v_{i+1} \in N(K)$, it follows that $d_{CX}(B, K) \leq 1$. In particular, $d_{CX}(B, K) \leq d_{CX}(B, K) + d_{CX}(K, K) \leq i + 1 \leq m + 1$. But then $d_{CX}(H, K) \leq d_{CX}(H, B) + d_{CX}(B, K) \leq n + m + 2$, contradicting our assumption. Thus we must have $g = g'$, and so the set of $v$-minimal contact sequences for $(H, H', p, p')$ coincides with the set of $v'$-minimal ones. In particular, $S$ is a $v'$-minimal structural sequence for $(H, H', p, p')$, and so the conclusion of Proposition 4.3 holds if $v$ is replaced by $v'$ as well.

Now suppose for contradiction that $g_n(v) = g(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_n)$ is not equal to $g_n(v') = g(v'; p, \mathcal{H}_0, \ldots, \mathcal{H}_n)$. Let $B$ be a hyperplane separating $g_n(v)$ from $g_n(v')$. Then $B$ separates gates for $v$ and $v'$ in a gated subgraph, and so as above we get $d_{CX}(B, K) \leq m + 1$. On the other hand, since $B$ separates $g_n(v)$ from $g_n(v')$, it follows that $B$ separates $p$ from either $g_n(v)$ or $g_n(v')$: without loss of generality, suppose the former. Then $B$ must separate $g(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_{j-1})$ and $g(v; p, \mathcal{H}_0, \ldots, \mathcal{H}_j)$ for some $j \in \{0, \ldots, n\}$. By Proposition
it then follows that $B$ separates $p_j$ from $g_j$, and so $d_{CX}(B, H_j) \leq 1$; in particular, $d_{CX}(H, B) \leq d_{CX}(H, H_j) + d_{CX}(H_j, B) \leq j+1 \leq n+1$. Therefore, $d_{CX}(H, K) \leq d_{CX}(H, B) + d_{CX}(B, K) \leq n+m+2$, again contradicting our assumption. Thus we must have $g_{\alpha}(v') = g_{\alpha}(v)$, as required.

**Lemma 4.5.** Suppose $G$ acts specially on $X$. Let $D \in \mathbb{N}$, and suppose every vertex of $\Delta X/G$ has at most $D$ neighbours. If $v, w \in V(X)$, then there exist at most $(D+1)^2$ hyperplanes $H \in V(CX)$ such that $w \in N(H)$ and $w$ is not the gate for $v$ in $N(H)$.

**Proof.** Let $U \subseteq V(X)$ be the set of vertices $u \in V(X)$ such that $d_X(u, w) = 1$ and $d_X(v, u) = d_X(v, u) + 1$. We claim that $|U| \leq D+1$. Indeed, suppose there exist $k$ distinct vertices $u_1, \ldots, u_k \in U$, and let $H_i$ be the hyperplane separating $w$ and $u_i$ for $1 \leq i \leq k$. It is clear that $H_i \neq H_j$ whenever $i \neq j$: indeed, if $H_i = H_j = H$ then by Proposition 2.8, $H$ cannot separate $v$ from either $u_i$ or $u_j$, and therefore $u_i = u_j$, hence $i = j$. Since $w \in N(H_i) \cap N(H_j)$ for every $i, j$ and since the action $G \acts X$ is special, it also follows that $H_i^G \neq H_j^G$ whenever $i \neq j$.

We now claim that $H_i$ and $H_j$ intersect for every $i \neq j$. Indeed, $H_i$ cannot separate $u_i$ from $v$ (by Proposition 2.8), nor $w$ from $w_j$ (as $H_i \neq H_j$), but it does separate $w$ (and so $u_j$) from $u_i$ (and so $v$). On the other hand, a symmetric argument shows that $H_j$ separates $w$ and $u_i$ from $u_j$ and $v$. Thus $H_i$ and $H_j$ must intersect, as claimed. Therefore, $d_{\Delta X}(H_i, H_j) = 1$ and so, as $H_i^G \neq H_j^G$, we have $d_{\Delta X/G}^*(H_i^G, H_j^G) = 1$. In particular, $\{H_i^G, \ldots, H_j^G\}$ are vertices of a clique in $\Delta X/G$, and so by our assumption it follows that $k \leq D+1$. Thus $|U| \leq k$, as claimed.

Now let $u \in U$, and let $H \in V(CX)$ be the set of hyperplanes $H \in V(CX)$ such that $u, w \in N(H)$. It is then enough to show that $|H| \leq D+1$. Thus, let $H_1, H_2, \ldots, H_k \in H$ be $k$ distinct hyperplanes, where $H_i$ is the hyperplane separating $u$ and $w$. As $w \in N(H_i) \cap N(H_j)$ for every $i, j$ and as $G \acts X$ is special, it is clear that $H_i^G \neq H_j^G$ for any $i \neq j$. Furthermore, it is clear (see, for instance, Proposition 2.1) that $H_1$ and $H_2$ intersect for every $j \neq 1$. In particular, $d_{\Delta X}(H_1, H_2) = 1$, and so $d_{\Delta X/G}^*(H_1^G, H_2^G) = 1$. As by assumption $H^G$ has at most $D$ neighbours in $\Delta X/G$, it follows that $k \leq D+1$, and so $|H| \leq D+1$, as required.

**Theorem 4.6.** Suppose a group $G$ acts specially on a quasi-median graph $X$, and suppose there exists some $D \in \mathbb{N}$ such that $|\text{Stab}_G(v)| \leq D$ for any $v \in V(X)$ and any vertex of $\Delta X/G$ has at most $D$ neighbours. Then the induced action $G \acts CX$ is acyclic, and the acyclindricity constants $D_e$ and $N_e$ can be expressed as functions of $D$ and $D_e$ only.

**Proof.** Let $\varepsilon \in \mathbb{N}$. We claim that the acyclindricity condition in Definition 1.2 is satisfied for $D_e = 2\varepsilon + 6$ and $N_e = N_e^{2(\varepsilon+3)}D/(N_e - 1)^2$, where $N = (D+1)^2D+1$.

Indeed, let $h, k \in \Delta X$ be such that $d_{\Delta X}(h, k) \geq D_e$. Let $H, K \in V(CX)$ be hyperplanes such that $d_{CX}(H, h) \leq 1/2$ and $d_{CX}(K, k) \leq 1/2$, and note that we have $d_{CX}(H, K) \geq D_e - 1 = 2\varepsilon + 5$. Let $G_{\varepsilon}(h, k) = \{g \in G \mid d_{CX}(g, h)^{\varepsilon} \leq \varepsilon, d_{CX}(g, k)^{\varepsilon} \leq \varepsilon\}$, and note that we have $G_{\varepsilon}(h, k) \subseteq G_{\varepsilon+1}(H, K)$. We thus aim to show that $|G_{\varepsilon+1}(H, K)| \leq N_e$.

Pick vertices $v \in N(K)$ and $p \in N(H)$, and an element $g \in G_{\varepsilon+1}(H, K)$. Let $G = (H_0, \ldots, H_n, p_0, \ldots, p_{n+1})$ be a $\varepsilon$-minimal contact sequence for the tuple $(H, H^g, p, p^g)$ with $v$-gate $(g_0, \ldots, g_n)$ and $(v, G)$-orbit sequence $(H_0, \ldots, H_n, H'_0, \ldots, H'_n)$; as $g \in G_{\varepsilon+1}(H, K)$, we have $n \leq \varepsilon + 1$. For $0 \leq i \leq n$, set $g_i = g(v; p, H_0, \ldots, H_i)$ and $g'_i = g(v; p^g, H'_0, \ldots, H'_i)$; let also $g_{-1} = p$ and $g'_{-1} = p^g$.

We first claim that there exist hyperplanes $A_0, \ldots, A_n \in V(CX)$ such that $g_{i-1}, g_i \in N(A_i)$ for each $i$. Indeed, this is clear if $g_i = p_{i+1}$ for each $i$, as in that case we may simply take $A_i = H_i$ for each $i$. Otherwise, let $k \in \{0, \ldots, n\}$ be minimal such that $g_k \neq p_{k+1}$, and let $A$ be a hyperplane separating $g_k$ and $p_{k+1}$ such that $g_k \in N(A)$. For $0 \leq i \leq k - 1$ we may take $A_i = H_i$, while for $k \leq i \leq n$ we can show (by induction on $i$, say) that $g_i \in N(A)$. Indeed, the base case ($i = k$) is clear by construction; and if $g_{i-1} \in N(A)$ for some $i > k$ and $g_{i-1} = q_0, \ldots, q_m = g_i$ is a geodesic in $X$, then $A$ cannot osculate with the hyperplane separating $q_{i-1}$ and $q_i$ by Proposition 1.3(ii) and (iii), and so $q_i \in N(A)$ by induction on $i$. Thus we may take $A_i = A$ for $k \leq i \leq n$, so that $g_{i-1}, g_i \in N(A_i)$ for each $i$, as claimed. A symmetric argument shows that there exist hyperplanes $B_n, \ldots, B_0 \in V(CX)$ such that $g'_{i+1}, g'_{i} \in N(B_i)$ for each $i$.,
Now, we may pass the sequence \((g_{−1}, \ldots, g_a)\) to a subsequence \((g_{k_0}, \ldots, g_{k_b})\) by removing those \(g_i\) for which \(g_{i−1} \neq g_i\). It then follows that \(g_{k_{i−1}} \neq g_{k_i}\) and that \(g_{k_{i−1}} \in N(A_{k_i})\) for \(1 \leq i \leq a\), where \(a \leq n + 1 \leq \varepsilon + 2\). Similarly, we may pass the sequence \((g_{n+1}, \ldots, g_b)\) to a subsequence \((g_{k'_0}, \ldots, g_{k'_b})\) such that \(g_{k'_{i−1}} \neq g_{k'_i}\) and that \(g_{k'_{i−1}} \in N(B_{k'_i})\) for \(1 \leq i \leq b\), where \(b \leq n + 1 \leq \varepsilon + 2\).

Now as \(d_{CX}(H, H^0) + d_{CX}(K, K^0) + 3 \leq 2(\varepsilon + 1) + 3 = 2\varepsilon + 5 \leq d_{CX}(H, K)\), it follows from Corollary 5.4 that \(G\) is also a \(q\)-minimal contact sequence and that \(\gamma(v; p, H_0, \ldots, H_n) = \gamma(v'; p, H_0, \ldots, H_n)\) as required.

As the stabiliser of \(\gamma(v; p, H_{k_1}, \ldots, H_{h_n})\) has cardinality \(\leq D\), it follows that, given any subsets \(H_{k_1}, \ldots, H_{h_n}, H'_{k_1}, \ldots, H'_{h_n} \subseteq V(X/G)\), there are at most \(D\) elements \(g \in G\) satisfying (2).

But as \(g_{k_i−1} \neq g_{k_i}\), as \(g_k\) lies on a geodesic between \(g_{k_{i−1}}\) and \(v\), and as \(g_{k_{i−1}, g_{k_i}} \in N(A_i)\), it follows from Lemma 4.5 that there are at most \((D + 1)^2\) possible choices for \(A_k\) (for \(1 \leq i \leq a\)). Moreover, given a choice of \(A_{k_i}\), as \(H_{k_i} \subseteq \Delta_{X/G}(A_{k_i}G)\) and by assumption \(\vert \Delta_{X/G}(A_{k_i}G)\vert \leq D + 1\), there exist at most \(2^{D+1}\) choices for \(H_{k_i}\). It follows that there exist at most \(N^a\) choices for the subsets \(H_{k_1}, \ldots, H_{h_n} \subseteq V(X/G)\), where \(N = (D + 1)^22^{D+1}\); similarly, there exist at most \(N^b\) choices for the subsets \(H'_{k_1}, \ldots, H'_{h_n} \subseteq V(X/G)\). In particular,

\[
\vert \mathcal{G}_{\varepsilon+1}(H, K)\vert \leq 2D \left(\sum_{a=0}^{\varepsilon+2} N^a\right) \left(\sum_{b=0}^{\varepsilon+2} N^b\right) < 2D \left(\frac{N^{\varepsilon+3}}{N-1}\right)^2 = N^\varepsilon
\]

as required.

\[\square\]

5. Application to graph products

We use this section to deduce results about graph products from Theorems A and B. Namely, we show Corollary C in Section 5.1 and Corollary D in Section 5.2. Throughout this section, let \(\Gamma\) be a simplicial graph, let \(\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}\) be a collection of non-trivial graphs, and let \(X\) be the quasi-median graph associated to \(\Gamma\mathcal{G}\), as given by Theorem E. We will use the following result.

\textbf{Theorem 5.1} (Genevois [Gen17 Section 8.1]; Genevois–Martin [GM18 Theorem 2.13]). For \(v \in V(\Gamma)\), let \(H_v\) be the hyperplane dual to the clique \(G_v \subseteq X\). Then any hyperplane \(H\) in \(X\) is of the form \(H_v^0\) for some \(v \in V(\Gamma)\) and \(g \in \Gamma\mathcal{G}\). Moreover, the vertices in \(\Delta \mathcal{G}^0\) of \(\Gamma\mathcal{G}\) are precisely \(\Gamma_{\Delta \mathcal{G}^0} \mathcal{G}_{\Delta \mathcal{G}^0} g \subseteq V(X)\).

\textbf{Remark 5.2.} Due to our convention to consider only right actions, the Cayley graph \(X = \text{Cay}(\Gamma\mathcal{G}, S)\) defined in Theorem 1.3 is the left Cayley graph: for \(s \in S\) and \(g \in \Gamma\mathcal{G}\), an edge labelled \(s\) joins \(g \in V(X)\) to \(sg \in V(X)\). Therefore, contrary to the convention in [Gen17, GM18], the vertices in the carrier of a hyperplane will form a right coset of \(\Gamma_{\Delta \mathcal{G}^0} \mathcal{G}_{\Delta \mathcal{G}^0} v\) for some \(v \in V(\Gamma)\).

5.1. Acylindrical hyperbolicity. Here we prove Corollary C. It is clear from Theorem E that we may apply Theorems A and B to the quasi-median graph \(X\) associated to a graph product \(\Gamma\mathcal{G}\). In particular, it follows that the contact graph \(CX\) is a quasi-tree and \(\Gamma\mathcal{G}\) acts on it acylindrically. We thus only need to show that, given that \(\vert V(\Gamma)\vert \geq 2\) and the complement \(\Gamma^C\) of \(\Gamma\) is connected, the action \(\Gamma\mathcal{G} \cap CX\) is non-elementary.

\textbf{Lemma 5.3.} Let \(H\) be a hyperplane in \(X\). Then the following are equivalent:

(i) \(CX\) is unbounded;
(ii) \(\Gamma^C\) is connected and \(\vert V(\Gamma)\vert \geq 2\).

\textbf{Proof.} We first show (i) \(\Rightarrow\) (ii). Indeed, if \(\Gamma\) is a single vertex \(v\), then \(X\) is a single clique and so \(CX\) is a single vertex. On the other hand, if \(\Gamma^C\) is disconnected, then we have a partition
V(Γ) = A \sqcup B$ where $A$ and $B$ are adjacent and non-empty. In particular, $ΓG = ΓAΓ_A \times ΓBΓ_B$, and so any vertex $g \in ΓG$ of $X$ can be expressed as $g = gA g_B$ for some $g_A \in ΓA Γ_A$ and $g_B \in ΓB Γ_B$. Thus, if $H \in V(CX)$ then by Theorem 5.1 $N(H) = Γ_{\text{star}(v)}G_{\text{star}(v)}BAG_B$ for some $g_A \in ΓA Γ_A$, $g_B \in ΓB Γ_B$ and $v \in V(Γ)$: without loss of generality, suppose $v \in A$. Then $g_B \in ΓB Γ_B \leq Γ_{\text{star}(v)}G_{\text{star}(v)}$ and $g_A \in ΓA \leq Γ_{\text{star}(v)}G_{\text{star}(v)}$ for any $u \in B$, and so $g_A \in N(H) \cap N(H_u)$; therefore, $d_{CX}(H, H_u) \leq 1$. Since $1 \in N(H_u) \cap N(H_s)$ and so $d_{CX}(H_u, H_s) \leq 1$ for any $u, v \in V(Γ)$, it follows that $d_{CX}(H, H') \leq 3$ for any $H, H' \in CX$ and so $CX$ is bounded, as required.

To show (ii) ⇒ (i), suppose that $Γ$ is a graph with at least 2 vertices and connected complement. Then, there exists a closed walk $(v_0, v_1, \ldots, v_{\ell})$ on the complement of $Γ$ that visits every vertex — in particular, we have $v_i \in V(Γ')$ with $v_0 = v_0$ and $v_{i-1} \neq v_i$; $(v_{i-1}, v_i) \not\in E(Γ)$ for $1 \leq i \leq \ell$. Pick arbitrary non-identity elements $g_i \in G_{v_i}$ for $i = 1, \ldots, \ell$, and consider the element $g = g_1 \cdots g_{\ell} \in ΓG$.

Now let $n \in \mathbb{N}$, and let $A, B \in V(CX)$ be such that $1 \in N(A)$ and $g^n \in N(B)$. Let $A = A_0, \ldots, A_m = B$ be the geodesic in $CX$ and let $1 = p_0, \ldots, p_m+1 = g^n$ be the vertices in $X$ given by Proposition 5.1. It follows from the normal form theorem for graph products $ΓG$.

Theorem 3.9 that $(g_1 \cdots g_\ell) \cdots (g_1 \cdots g_\ell)$ is the unique normal form for the element $g^n$. In particular, as geodesics in $X$ are precisely the words spelling out normal forms of elements of $ΓG$, we have $p_i = g_{\ell n-c_i+1} g_{\ell n-c_i+2} \cdots g_{\ell n}$, where $0 = c_0 \leq c_1 \leq \cdots \leq c_{m+1} = \ell n$ and indices are taken modulo $\ell$.

We now claim that $c_{i+1} - c_i < \ell$ for each $i$. Indeed, suppose $c_{i+1} - c_i \geq \ell$ for some $i$. Note that, as $p_i, p_{i+1} \in N(A_i)$, it follows from Theorem 5.1 that $Γ_{\text{star}(v_i)}G_{\text{star}(v_i)}p_i = V(N(A_i)) = Γ_{\text{star}(v_i)}G_{\text{star}(v_i)}p_{i+1}$ for some $v \in V(Γ)$, and therefore we have $p_{i+1}p_i^{-1} = Γ_{\text{star}(v_i)}G_{\text{star}(v_i)}$. But as $g_{\ell n-c_i+1} g_{\ell n-c_i+2} \cdots g_{\ell n-c_{i+1}}$ is a normal form for $p_{i+1}p_i^{-1}$ (where indices are taken modulo $\ell$), it follows that $v_j \in \text{star}(v)$ for $\ell n - c_{i+1} \leq j < \ell n - c_i$ (with indices again modulo $\ell$). But as by assumption $c_{i+1} - c_i \geq \ell$ and as $(v_1, \ldots, v_{\ell}) = V(Γ)$, this implies that $\text{star}(v) = V(Γ)$, and so $v$ is an isolated vertex of $Γ^n$. This contradicts the fact that $Γ^n$ is connected; thus $c_{i+1} - c_i < \ell$ for each $i$, as claimed.

In particular, we get $\ell n = \sum_{i=0}^{m}(c_{i+1} - c_i) < (m+1)\ell$, and so $m+1 > n$. Thus $d_{CX}(A, B) = m \geq n$ and so $CX$ is unbounded, as required.

It is now easy to deduce when the action of $ΓG$ on $CX$ is non-elementary acylindrical.

Proof of Corollary. By the argument above, we only need to show the last part. Thus, suppose that $Γ$ is a graph with at least 3 vertices and connected complement. Then, by Lemma 5.3 the graph $CX$ is unbounded. In particular, given any $H \in V(CX)$ and $n \in \mathbb{N}$, we may pick $H' \in V(CX)$ such that $d_{CX}(H, H') \geq n+1$. Since the action $ΓG \acts X$ is transitive on vertices, it follows that given any vertex $p \in N(H)$ there exists $g \in ΓG$ such that $g^n \in N(H')$, and in particular $d_{CX}(H^n, H') \leq 1$. Thus $d_{CX}(H, H^n) \geq n$, and so the action $ΓG \acts CX$ has unbounded orbits.

We now claim that $ΓG$ is not virtually cyclic. Indeed, since $|V(Γ)| \geq 3$ and $Γ^n$ is connected, $Γ^n$ contains a path of length $2$, and so there exist vertices $v_1, v_2, w \in Γ$ such that $v_1 \sim w \sim v_2$. Let $A = \{v_1, v_2, w\}$ and $H = Γ\{v_1, v_2\}G_{\{v_1, v_2\}}$ (so either $H \cong G_{v_1} \times G_{v_2}$ or $H \cong G_{v_1} \ast G_{v_2}$). Since the groups $G_v$ are non-trivial for each $v \in V(Γ)$, we have $|H| \geq 4 > 2$ and so $ΓG \not\cong G_w \ast H$ has infinitely many ends. In particular, since the subgroup $ΓAΓ_A \leq ΓG$ is not virtually cyclic, neither is $ΓG$, as required.

Remark 5.4. After appearance of the first version of this preprint, it has been brought to the author’s attention that most of the results stated in Corollary C have already been proved by Genevois. In [Gen18 Theorem 2.39], Genevois shows that $ΔX$ is quasi-isometric to a tree whenever it is connected and $Γ$ is finite, so in particular, by Theorem B(Γ) CX is a quasi-tree as well. Moreover, methods used by Genevois to prove [Gen16 Theorem 22] can be adapted to show that the action of $ΓG$ on $CX$ is non-uniformly acylindrical; here, the non-uniform acylindricity of an action $Γ \acts X$ is a weaker version of acylindricity, defined by replacing the
phrase ‘is bounded above by $N_ε$’ by ‘is finite’ in Definition \ref{def:bounded}. Corollary \ref{cor:bounded} strengthens this statement.

5.2. $AH$-accessibility. Here we study $AH$-accessibility, introduced in \cite{ABO17} by Abbott, Balasubramanya and Osin, of graph products. In particular, we show that if $Γ$ is connected, non-trivial, and the groups in $G$ are infinite, then the action of $ΓG$ on $CX$ is the ‘largest’ acylindrical action of $ΓG$ on a hyperbolic metric space. Hence we prove Corollary \ref{cor:acylindrical}.

We briefly recall the terminology of \cite{ABO17}. Given two isometric actions $G ∼ X$ and $G ∼ Y$ of a group $G$, we say $G ∼ X$ dominates $G ∼ Y$, denoted $G ∼ Y ≤ G ∼ X$, if there exist $x ∈ X$, $y ∈ Y$ and a constant $C$ such that

$$d_Y(y, y^g) ≤ C d_X(x, x^g) + C$$

for all $g ∈ G$. The actions $G ∼ X$ and $G ∼ Y$ are said to be weakly equivalent if $G ∼ X ≤ G ∼ Y$ and $G ∼ Y ≤ G ∼ X$. This partitions all such actions into equivalence classes.

It is easy to see that $≤$ defines a preorder on the set of all isometric actions of $G$ on metric spaces. Therefore, $≤$ defines a partial order on the set of equivalence classes of all such actions.

We may restrict this to a partial order on the set $AH(G)$ of equivalence classes of acylindrical actions of $G$ on a hyperbolic space. We then say the group $G$ is $AH$-accessible if the partial order $AH(G)$ has a largest element (which, if exists, must necessarily be unique), and we say $G$ is strongly $AH$-accessible if a representative of this largest element is a Cayley graph of $G$.

Recall that for an action $G ∼ X$ by isometries with $X$ hyperbolic, an element $g ∈ G$ is said to be loxodromic if, for some (or any) $x ∈ X$, the map $Z → X$ given by $n → x^n$ is a quasi-isometric embedding. It is clear from the definitions that the ‘largest’ action $G ∼ X$ will also be universal, in the sense that every element of $G$, that is loxodromic with respect to some acylindrical action of $G$ on a hyperbolic space, will be loxodromic with respect to $G ∼ X$.

In \cite{ABO17} Theorem 2.19 (c)], it is shown that the all right-angled Artin groups are $AH$-accessible (and more generally, so are all hierarchically hyperbolic groups – in particular, groups acting properly and cocompactly on a CAT(0) cube complex possessing a factor system \cite[Theorem A]{ABD17}). Here we generalise this result to ‘most’ graph products of infinite groups. The proof is very similar to that of \cite{ABO17} Lemma 7.16.

**Proof of Corollary \ref{cor:acylindrical}** It is easy to show – for instance, by Theorem \ref{thm:acylindrical} – that $CX$ is ($G$-equivariantly) quasi-isometric to the Cayley graph of $ΓG$ with respect to $\bigcup_{v ∈ V(Γ)} G_{\Gamma \star (v)} G_{\Gamma \star (v)}$.

We prove the statement by induction on $|V(Γ)|$. If $|V(Γ)| = 1$, then $V(Γ) = \{v\}$, say, then $v$ is an isolated vertex of $Γ$ and so, by the assumption, $ΓG_v$ is strongly $AH$-accessible.

Suppose now that $|V(Γ)| ≥ 2$. If $Γ$ has an isolated vertex $Γ = Γ_A ∪ \{v\}$, then $ΓG ≅ ΓG_A * G_v$ is hyperbolic relative to $\{ΓG_A, G_v\}$. By the induction hypothesis, both $ΓG_A$ and $G_v$ are strongly $AH$-accessible, and hence, by \cite[Theorem 7.9]{ABO17}, so is $ΓG$. If, on the other hand, the complement $ΓC$ of $Γ$ is disconnected ($ΓC = ΓA_C ∪ ΓB_C$ for some partition $V(Γ) = A ∪ B$, say), then $ΓG ≅ ΓG_A × ΓG_B$ is not acylindrically hyperbolic by \cite{Osi16} Corollary 7.2, as both $ΓG_A$ and $ΓG_B$ are infinite. It then follows from \cite[Example 7.8]{ABO17} that $ΓG$ is strongly $AH$-accessible; it also follows that any acylindrical action of $ΓG$ on a hyperbolic metric space $(ΓG ∼ CX)$, say) represents the largest element of $AH(ΓG)$.

Hence, we may without loss of generality assume that $Γ$ is a graph with no isolated vertices and connected complement. It then follows that $|V(Γ)| ≥ 4$, and so by Corollary \ref{cor:acylindrical} $CX$ is a hyperbolic metric space and $ΓG$ acts on it non-elementarily acylindrically. It is easy to see from Theorem \ref{thm:acylindrical} that, given two hyperplanes $H, H' ∈ V(CX)$, they are adjacent in $CX$ if and only if there exist distinct $u, v ∈ V(Γ)$ and $g ∈ ΓG$ such that $H = H^u_g$ and $H' = H^v_g$. It follows that the quotient space $CX/ΓG$ is the complete graph on $|V(Γ)|$ vertices, and in particular, the action $ΓG ∼ CX$ is cocompact.

Moreover, it follows from Theorem \ref{thm:acylindrical} that the stabiliser of an arbitrary vertex $H^u_g$ of $CX$ is precisely $G ≅ (Γ_{\Gamma \star (v)} G_{\Gamma \star (v)})^g ≅ G^g_v × (Γ_{\Gamma \star (v)} G_{\Gamma \star (v)})^g$. Since $Γ$ has no isolated vertices, $\Gamma \star (v) ≠ ∅$, and so, as all groups in $G$ are infinite, both $G^g_v$ and $(Γ_{\Gamma \star (v)} G_{\Gamma \star (v)})^g$ are infinite groups. Thus, $G$ is a direct product of two infinite groups, and so – by \cite[Corollary 7.2]{Osi16},
say – $G$ does not possess a non-elementary acylindrical action on a hyperbolic space. Since $G$ is not virtually cyclic, for every acylindrical action of $\Gamma G$ on a hyperbolic space $Y$, the induced action of $G$ on $Y$ has bounded orbits. It then follows from [ABO17, Proposition 4.13] that $\Gamma G$ is strongly $\mathcal{AH}$-accessible and – in particular, $\Gamma G \simeq CX$ represents the largest element of $\mathcal{AH}(\Gamma G)$. □

Remark 5.5. Corollary [D] gives some explicit descriptions for the class of hierarchically hyperbolic groups, introduced by Behrstock, Hagen and Sisto in [BHS17]. In particular, a result by Berlai and Robbio [BR18, Theorem C] says that if all vertex groups $G_v$ are hierarchically hyperbolic with the intersection property and clean containers, then the same can be said about $\Gamma G$. Moreover, Abbott, Behrstock and Durham show in [ABD17, Theorem A] that all hierarchically hyperbolic groups are $\mathcal{AH}$-accessible, which implies Corollary [D] in the case when the vertex groups $G_v$ are hierarchically hyperbolic with the intersection property and clean containers.

More precisely, every hierarchically hyperbolic group $G$ comes with an action on a space $X$, such that there exist projections $\pi_Y : X \to 2^Y$ to some collection of $\delta$-hyperbolic spaces $\{\hat{C}Y \mid Y \in \mathcal{S}\}$, where $\mathcal{S}$ is a partial order that contains a (unique) largest element, $S \in \mathcal{S}$, say. Moreover, the action of $G$ on $X$ induces an action of $G$ on (a space quasi-isometric to) $U = \bigcup_{v \in \mathcal{V}} \pi_S(x) \subseteq CS$, and in [BHS17] Theorem 14.3) it is shown that this action is acylindrical. In [ABD17], this construction is modified so that the action $G \acts U$ represents the largest element of $\mathcal{AH}(G)$. If $\Gamma$ is connected, non-trivial, and the groups $G_v$ are infinite and hierarchically hyperbolic (with the intersection property and clean containers), then the proof of Corollary [D] gives this action $\Gamma G \acts U$ explicitly. This is potentially useful for studying hierarchical hyperbolicity of graph products.

Remark 5.6. Note that the condition on the $G_v$ being infinite is necessary for the proof to work. Indeed, suppose $\Gamma = \langle a, b, c, d \rangle$ is a path of length 3, and $G_v = \langle g_v \rangle \cong C_2$ for each $v \in V(\Gamma)$, so that $\Gamma G$ is the right-angled Coxeter group over $\Gamma$. Notice that $\Gamma G \cong A \ast_\mathcal{C} B$, where $A = G_a \times (G_a \ast G_c), B = G_c \times G_d$ and $C = G_c$. In particular, since $C$ is finite, $\Gamma G$ is hyperbolic relative to $\{A, B\}$. Hence the Cayley graph Cay($\Gamma G, A \cup B$) is hyperbolic and the usual action of $\Gamma G$ on it is acylindrical.

It is easy to verify from the normal form theorem for amalgamated free products that the element $g_b g_d$ will be loxodromic with respect to $\Gamma G \acts \text{Cay}(\Gamma G, A \cup B)$. However, as $g_b g_d \in \Gamma_{\text{star}(c)} G_{\text{star}(c)}$, we know that $g_b g_d$ stabilises the hyperplane dual to $G_c \subseteq V(X)$ under the action of $\Gamma G$ on $CX$, and so $g_b g_d$ is not loxodromic with respect to $\Gamma G \acts CX$. In particular, the equivalence class of $\Gamma G \acts CX$ cannot be the largest element of $\mathcal{AH}(\Gamma G)$. It is straightforward to generalise this argument to show that if $c \in V(\Gamma)$ is a separating vertex of a connected finite simplicial graph $\Gamma$, then for any graph product $\Gamma G$ with $G_c$ finite, the action $\Gamma G \acts CX$ will not be the ‘largest’ one.

On the other hand, note that this particular group $\Gamma G$ (and indeed any right-angled Coxeter group) will be $\mathcal{AH}$-accessible: see [ABD17] Theorem A (4)].

6. EQUATIONAL NOETHERIANITY OF GRAPH PRODUCTS

In this section we prove Theorem [E]. To do this, we use the methods that Groves and Hull exhibited in [GH17]. Here we briefly recall their terminology.

The approach to equationally noetherian groups used in [GH17] is through sequences of homomorphisms. In particular, let $G$ be any group, let $F$ be a finitely generated group and let $\varphi_i : F \to G$ be a sequence of homomorphisms ($i \in \mathbb{N}$). Let $\omega : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ be a non-principal ultrafilter. We say a sequence of properties $(P_i)_{i \in \mathbb{N}}$ holds $\omega$-almost surely if $\omega\{i \in \mathbb{N} \mid P_i \text{ holds}\} = 1$. We define the $\omega$-kernel of $F$ with respect to $(\varphi_i)$ to be

$$F_{\omega,(\varphi_i)} = \{ f \in F \mid \varphi_i(f) = 1 \text{ $\omega$-almost surely}\};$$

we write $F_\omega$ for $F_{\omega,(\varphi_i)}$ if the sequence $(\varphi_i)$ is clear. It is easy to check that $F_\omega$ is a normal subgroup of $F$. We say $\varphi_i$ factors through $F_\omega$ $\omega$-almost surely if $F_\omega \subseteq \ker(\varphi_i)$ $\omega$-almost surely.

The idea behind all these definitions is the following result.
Theorem 6.1 (Groves and Hull [GH17, Theorem 3.5]). Let \( \omega \) be a non-principal ultrafilter. Then the following are equivalent for any group \( G \):

(i) \( G \) is equationally noetherian;
(ii) for any finitely generated group \( F \) and any sequence of homomorphisms \( (\varphi_i : F \to G) \), \( \varphi_i \) factors through \( F_\omega \) \( \omega \)-almost surely.

Remark 6.2. Note that Definition 6.3 differs from the usual definition of equationally noetherian groups, as we do not allow ‘coefficients’ in our equations: that is, we restrict to subsets \( S \subseteq F_n \) instead of \( S \subseteq G * F_n \). However, the two concepts coincide when \( G \) is finitely generated – see [BMR99, §2.2, Proposition 3]. We use this (weaker) definition of equationally noetherian groups as it is more suitable for our methods. In particular, we use an equivalent characterisation of equationally noetherian groups given by Theorem 6.1.

In this section we prove Theorem E. In Section 6.1, we introduce ‘admissible’ graphs and show that being equationally noetherian is preserved under taking graph products over admissible graphs. In Section 6.3, we show that indeed all graphs of girth \( \geq 5 \) are admissible.

6.1. Reduction to sequences of linking homomorphisms. Suppose now that the group \( G \) acts by isometries on a metric space \( (X, d) \). As before, let \( \Gamma \) be a finite simplicial graph and let any non-divergent sequence of homomorphisms \( (\varphi_i : F \to G) \), \( \varphi_i \) factors through \( F_\omega \) \( \omega \)-almost surely. It turns out that in this case we may reduce any non-divergent sequence of linking homomorphisms to one of the following form: see the proof of Theorem 6.6.

\[
\liminf_{\omega} \max_{x \in X} d(x, x^{\varphi_i(s)}) < \infty.
\]

We say that \( \varphi_i \) is divergent otherwise. It is easy to see that this does not depend on the choice of a generating set for \( F \).

The main technical result of [GH17] states that in case \( X \) is hyperbolic and the action of \( G \) on \( X \) is non-elementary acylindrical, it is enough to consider non-divergent sequences of homomorphisms (cf Theorem 6.1).

Theorem 6.3 (Groves and Hull [GH17, Theorem B]). Let \( X \) be a hyperbolic metric space and \( G \) a group acting non-elementarily acylindrically on \( X \). Suppose that for any finitely generated group \( F \) and any non-divergent sequence of homomorphisms \( (\varphi_i : F \to G) \), \( \varphi_i \) factors through \( F_\omega \) \( \omega \)-almost surely. Then \( G \) is equationally noetherian.

We now consider the particular case when \( G \) is a graph product and \( X \) is the extension graph. Thus, as before, let \( \Gamma \) be a finite simplicial graph and let \( \mathcal{G} = \{ G_v \mid v \in V(\Gamma) \} \) be a collection of non-trivial groups. It turns out that in this case we may reduce any non-divergent sequence of homomorphisms to one of the following form: see the proof of Theorem 6.6.

Definition 6.4. Let \( \mathcal{F} = \{ F_v \mid v \in V(\Gamma) \} \) be a collection of finitely generated groups, and let \( \varphi : \Gamma \mathcal{F} \to \Gamma \mathcal{G} \) be a homomorphism. We say \( \varphi \) is linking if \( \varphi(F_v) \subseteq \Gamma \mathcal{G}_{\text{link}(v)} \) for each \( v \in V(\Gamma) \). We say the graph \( \Gamma \) is admissible if for every collection of non-trivial equationally noetherian groups \( \mathcal{G} = \{ G_v \mid v \in V(\Gamma) \} \) and every sequence of linking homomorphisms \( (\varphi_i : \Gamma \mathcal{F} \to \Gamma \mathcal{G})_{i=1}^\infty \), \( \varphi_i \) factors through \( (\Gamma \mathcal{F})_\omega \) \( \omega \)-almost surely.

The proof of Theorem 6.6 uses the following result.

Lemma 6.5. Full subgraphs of admissible graphs are admissible.

Proof. Let \( \Gamma \) be a admissible graph, let \( \mathcal{G} = \{ G_v \mid v \in V(\Gamma) \} \) be a collection of non-trivial equationally noetherian groups, and let \( \mathcal{F} = \{ F_v \mid v \in V(\Gamma) \} \) be a collection of finitely generated groups. Let \( A \subseteq V(\Gamma) \), so that \( \Gamma_A \) is a full subgraph of \( \Gamma \), and let \( (\varphi^A_i : \Gamma_A \mathcal{F}_A \to \Gamma_A \mathcal{G}_A)_{i=1}^\infty \) be a sequence of linking homomorphisms. Let \( \omega \) be a non-principal ultrafilter. We aim to show that \( \varphi^A_i \) factors through \( (\Gamma_A \mathcal{F}_A)_\omega \) \( \omega \)-almost surely.

\( A \) factors through \( (\Gamma_A \mathcal{F}_A)_\omega \) \( \omega \)-almost surely.

By a canonical retraction \( \rho_A : \Gamma \mathcal{F} \to \Gamma_A \mathcal{F}_A \), defined on vertex groups by \( \rho_A(f) = f \) if \( f \in F_v \) for \( v \in A \), and \( \rho_A(f) = 1 \) if \( f \in F_v \) for \( v \not\in A \). We also have a canonical inclusion of subgroup \( \iota_A : \Gamma_A \mathcal{G}_A \to \Gamma \mathcal{G} \) for each \( A \). For each \( i \), let \( \varphi_i = \iota_A \circ \varphi^A_i \circ \rho_A : \Gamma \mathcal{F} \to \Gamma \mathcal{G} \). It is easy
to see that the $\varphi_i$ are linking homomorphisms. In particular, since $\Gamma$ is admissible, we have $(\Gamma F) \omega \subseteq \ker \varphi_i \omega$-almost surely. Moreover, since $\iota_A$ is injective, we obtain
\[ \ker \varphi_i = \rho_A^{-1}(\ker \varphi_i^A) \text{ for each } i \quad \text{and} \quad (\Gamma F) \omega = \rho_A^{-1}((\Gamma A F_A) \omega). \]
As $\rho_A$ is surjective, it follows that $((\Gamma A F_A) \omega \subseteq \ker \varphi_i^A \omega$-almost surely, and so $\Gamma_A$ is admissible, as required.

\[ \Box \]

Theorem 6.6. For any admissible graph $\Gamma$ and any collection $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ of equationally noetherian groups, the graph product $\Gamma \mathcal{G}$ is equationally noetherian.

Proof. We proceed by induction on $|V(\Gamma)|$. If $|V(\Gamma)| = 1 = V(\Gamma) = \{v\}$, then $\Gamma \mathcal{G} \cong G_v$, and so the result is clear. Thus, assume that $|V(\Gamma)| \geq 2$.

If $\Gamma$ is disconnected, then we have a partition $V(\Gamma) = A \sqcup B$ into non-empty subsets such that $\Gamma = \Gamma_A \sqcup \Gamma_B$, and so $\Gamma \mathcal{G} \cong \Gamma_A \mathcal{G}_A \times \Gamma_B \mathcal{G}_B$. By Lemma 6.5, both $\Gamma_A$ and $\Gamma_B$ are admissible, and so by the induction hypothesis, both $\Gamma_A \mathcal{G}_A$ and $\Gamma_B \mathcal{G}_B$ are equationally noetherian. By Theorem 6.5, $\Gamma \mathcal{G}$ is equationally noetherian as well, as required. Thus, without loss of generality, we may assume that $\Gamma$ is connected.

Similarly, if the complement of $\Gamma$ is disconnected, then we have a partition $V(\Gamma) = A \sqcup B$ such that $\Gamma \mathcal{G} \cong \Gamma_A \mathcal{G}_A \times \Gamma_B \mathcal{G}_B$. As before, $\Gamma_A \mathcal{G}_A$ and $\Gamma_B \mathcal{G}_B$ are equationally noetherian by the induction hypothesis. It follows from Lemma 6.3 that a direct product $G \times H$ of equationally noetherian groups $G$ and $H$ is equationally noetherian: indeed, this follows from the cartesian product decomposition $V_G \times H(S) = V_G(S) \times V_H(S)$, for any $S \subseteq F_n$. Thus $\Gamma \mathcal{G}$ is equationally noetherian in this case as well.

Therefore, we may without of loss of generality assume that $\Gamma$ is a connected graph with a connected complement and $|V(\Gamma)| \geq 2$ (and, therefore, $|V(\Gamma)| \geq 4$). In this case, Corollary C shows that $C_X$ is a hyperbolic metric space and the action of $\Gamma \mathcal{G}$ on it is non-elementary acylindrical. We thus may use Theorem 6.3 to show that $\Gamma \mathcal{G}$ is equationally noetherian.

In particular, let $F$ be a finitely generated group and let $(\varphi_i : F \to \Gamma \mathcal{G})_{i \in \mathbb{N}}$ be a non-divergent sequence of homomorphisms. By Theorem 6.3, it is enough to show that $\varphi_i$ factors through $F_\omega \omega$-almost surely.

We proceed as in the proof of [GHT7 Theorem D]. Let $S$ be a finite generating set for $F$. Note that, by Theorem 5.1, we may conjugate each $\varphi_i$ (if necessary) to assume that the minimum (over all hyperplanes $H$ in $X$) of $\max_{x \in S} d_{C_X}(H, H^{\varphi_i}(s))$ is attained for $H = H_{\tilde{n}_s}$ for some $s \in V(\Gamma)$. Moreover, it is easy to see from Theorem 5.1 that $\|g\|_{i, \varphi} = d_{C_X}(H_u, H_{\tilde{n}}^\varphi) \leq 1$ for any $g \in \mathcal{G}$ and $u \in V(\Gamma)$, where we write $\|g\|_i$ for the minimal integer $\ell \in \mathbb{N}$ such that $g = g_1 \cdots g_{\ell}$ and $g_i \in \Gamma_{\varphi(v)} \mathcal{G} \mathcal{G}_{\varphi(v)}$ for some $v \in V(\Gamma)$. In particular, since the sequence $(\varphi_i)$ is non-divergent, it follows that
\[ \lim \max_{s \in S} \|\varphi_i(s)\|_i < \infty. \]

It follows that for each $s \in S$, there exists $n_s \in \mathbb{N}$ such that $\|\varphi_i(s)\|_i \leq n_s$ $\omega$-almost surely. Moreover, for each $s \in S$, there exist $v_{s,1}, \ldots, v_{s,n_s} \in V(\Gamma)$ such that we have
\[ \varphi_i(s) = g_{i,s,1} \cdots g_{i,s,n_s} \]
with $g_{i,s,j} \in \Gamma_{\varphi(v)} \mathcal{G} \mathcal{G}_{\varphi(v)}$ $\omega$-almost surely. But since we have $\Gamma_{\varphi(v)} \mathcal{G} \mathcal{G}_{\varphi(v)} = G_v \times \Gamma_{\link(v)} \mathcal{G} \mathcal{G}_{\link(v)}$ for each $v \in V(\Gamma)$, we can write $g_{i,s,j} = g_{i,s,2j-1}^{-1}g_{i,s,2j}$, where $g_{i,s,2j} \in \Gamma_{\link(v)} \mathcal{G} \mathcal{G}_{\link(v)}$ with any choice of vertex $v_{s,2j-1} \in \link(\tilde{v}_{s,j})$ (which exists since $\Gamma$ is connected and $|V(\Gamma)| \geq 2$), and $g_{i,s,2j} \in \Gamma_{\link(v_{s,2j})} \mathcal{G} \mathcal{G}_{\link(v_{s,2j})}$ with $v_{s,2j} = \tilde{v}_{s,j}$. It follows that, after setting $n_s = 2n_s$, we may write
\[ \varphi_i(s) = g_{i,s,1} \cdots g_{i,s,n_s} \]
with $g_{i,s,j} \in \Gamma_{\link(v_{s,j})} \mathcal{G} \mathcal{G}_{\link(v_{s,j})}$ $\omega$-almost surely.

Now for each $s \in S$, define abstract letters $h_{s,1}, \ldots, h_{s,n_s}$. For each $v \in V(\Gamma)$, let
\[ H_v = \{h_{s,j} \mid v_{s,j} = v\}, \]
and let $F_v = F(H_v)$, the free group on $H_v$. Let $\mathcal{F} = \{F_v \mid v \in V(\Gamma)\}$, and consider the graph product $\Gamma \mathcal{F}$. We can define a map from $S$ to $\Gamma \mathcal{F}$ by sending $s \in S$ to $h_{s,1} \cdots h_{s,n_s}$. Let $N$ be
the normal subgroup of $\Gamma F$ generated by images of all the relators of $F$ under this map. This gives a group homomorphism $\rho : F \to \Gamma F / N$.

The map $\hat{\varphi}_i : \Gamma F / N \to \Gamma G$, obtained by sending $h_{s,j}N$ to $g_{i,s,j}$, is $\omega$-almost surely a well-defined homomorphism. Indeed, all the relators in $\Gamma F / N$ are either of the form $[h_{s_1,j_1}, h_{s_2,j_2}] = 1$ if $v_{s_1,j_1} \sim v_{s_2,j_2}$ in $\Gamma$, or of the form $\phi(\{h_{s_1,1} \cdots h_{s,n_s} \mid s \in S\})$, where $\phi(S)$ is a relator in $F$. Both of these $\omega$-almost surely map to the identity under $\hat{\varphi}_i$: the former because $[g_{i,s_1,j_1}, g_{i,s_2,j_2}] = 1$ in $G$ if $v_{s_1,j_1} \sim v_{s_2,j_2}$ in $\Gamma$, and the latter because $\varphi_i$ is a well-defined homomorphism. It is also clear that $\varphi_i = \hat{\varphi}_i \circ \rho$ $\omega$-almost surely.

Now let $\pi : \Gamma F \to \Gamma F / N$ be the quotient map. Then, by construction, the homomorphisms $\varphi_i = \hat{\varphi}_i \circ \pi : \Gamma F \to \Gamma G$ are linking (when they are well-defined). Since $\Gamma$ is admissible and the groups $G_v$ are equationally noetherian, it follows that $\varphi_i$ factors through $(\Gamma F / N)_\omega$ $\omega$-almost surely.

Since $\pi$ is surjective, this implies that $(\Gamma F / N)_\omega \subseteq \ker \varphi_i$ $\omega$-almost surely. Thus $\varphi_i = \hat{\varphi}_i \circ \rho$ $\omega$-almost surely, as required.

We expect that the class of equationally noetherian groups is closed under taking arbitrary graph products. Although we are not able to show this in full generality, in the next subsection we show that any triangle-free and square-free graph $\Gamma$ is admissible, and therefore, by Theorem 6.2, the class of equationally noetherian groups is closed under taking graph products over such graphs $\Gamma$.

6.2. Digression: dual van Kampen diagrams. Before embarking on a proof of Theorem 6, let us define the following notion. Following methods of [CW04] and [KK14], we consider dual van Kampen diagrams for words representing the identity in $\Gamma G$; recently, dual van Kampen diagrams for graph products have been independently introduced by Genevois in [Gen19]. Here we explain their construction and properties.

We consider van Kampen diagrams in the quasi-median graph $X$ given by Theorem 1.3 viewed as a Cayley graph. In particular, note that we have a presentation

\begin{equation}
\Gamma G = \langle S \mid R_\Delta \cup R_\square \rangle
\end{equation}

with generators

\[ S = \bigcup_{v \in V \Gamma} (G_v \setminus \{1\}) \]

and relators of two types: the ‘triangular’ relators

\[ R_\Delta = \bigcup_{v \in V \Gamma} \{ghk^{-1} \mid g, h, k \in G_v \setminus \{1\}, gh = k \text{ in } G_v\} \]

and the ‘rectangular’ relators

\[ R_\square = \bigcup_{(v, w) \in E \Gamma} \{[g_v, g_w] \mid g_v \in G_v \setminus \{1\}, g_w \in G_w \setminus \{1\}\}. \]

We now dualise the notion of van Kampen diagrams with respect to the presentation 3. Let $D \subseteq \mathbb{R}^2$ be a van Kampen diagram with boundary label $w$, for some word $w \in S^*$ representing the identity in $\Gamma G$, with respect to the presentation 3. It is convenient to pick a colouring $V(\Gamma) \to \mathbb{N}$ and to colour edges of $D$ according to their labels. Suppose that $w = g_1 \cdots g_n$ for some syllables $g_i$, and let $e_1, \ldots, e_n$ be the corresponding edges on the boundary of $D$. We add a ‘vertex at infinity’ $\infty$ somewhere on $\mathbb{R}^2 \setminus D$, and for each $i = 1, \ldots, n$, we attach to $D$ a triangular ‘boundary’ face whose vertices are the endpoints of $e_i$ and $\infty$. We get the dual van Kampen diagram $\Delta$ corresponding to $D$ by taking the dual of $D$ as a polyhedral complex and removing the face corresponding to $\infty$: thus, $\Delta$ is a tessellation of a disk. See Figure 6.

We lift the colouring of edges in $D$ to a colouring of edges of $\Delta$: this gives a corresponding vertex $v \in V \Gamma$ for each internal edge of $\Delta$. We say a 1-subcomplex (a subgraph) of $\Delta$ is a $v$-component (or just a component) for some $v \in V \Gamma$ if it is a maximal connected subgraph each of whose edges correspond to the vertex $v$. We call a vertex of $\Delta$ an intersection point (respectively branch point, boundary point) if it comes from a triangular (respectively rectangular,
Figure 6. Van Kampen diagram ($D$, left) and its dual ($\Delta$, right) with the word $a_1b_1c_1a_2b_2c_2a_3^{-1}c_2^{-1}a_3b_2^{-1}c_3^{-1}a_4^{-1}c_4^{-1}b_1^{-1}$ as its boundary label, where $a_i \in G_a$ with $a_1a_2 = a_4$, $b_i \in G_b$, $c_i \in G_c$ with $c_4c_3 = c_1$, and $b \sim a \sim c$ in $\Gamma$. The black edges on $D$ represent the boundary faces attached: the non-visible endpoint of each black edge is the point $\infty$. The dual van Kampen diagram $\Delta$ contains 6 components in total: 2 components corresponding to each of the vertices $a$, $b$ and $c$.

boundary) face in $D$. It is easy to see that boundary, intersection and branch points lying on a component $C$ will be precisely the vertices of $C$ of degree 1, 2 and 3, respectively.

The following Lemma says that, without loss of generality, we may always assume that components of dual van Kampen diagrams do not contain cycles. It is a special case of [Gen19, Proposition 1.1].

**Lemma 6.7.** Let $w \in S^*$ be a word representing the identity element in $\Gamma G$. Then there exists a dual van Kampen diagram $\Delta$ for $w$ such that each component of $\Delta$ is a tree.

**Proof.** Let $D$ be a van Kampen diagram for $w$ with the corresponding dual van Kampen diagram $\Delta$. Suppose a $v$-component $C$ of $\Delta$ (for some $v \in V(\Gamma)$) contains a cycle $C_0 \subseteq C$. Then $C_0$ corresponds to a ‘corridor’ $K_0 \subseteq D$: that is, a subcomplex $K_0$ homeomorphic to an annulus or, in ‘degenerate’ cases, a disk. The interior $\text{int}(K_0)$ of $K_0$ will consist of faces and edges that correspond to vertices and edges of $C_0$. Note that his will not have the usual meaning if $K_0$ is homeomorphic to a disk, as vertices contained in the ‘usual’ interior of $K_0$ and edges joining them will not belong to $\text{int}(K_0)$. Thus $\text{int}(K_0)$ separates $D$ into two connected components: the inside and the outside of $K_0$.

Fix $e$ a directed edge $e$ in $\text{int}(K_0)$ with initial vertex in the inside of $K_0$, and let $g \in G_v$ be the label of $e$. We then construct a new van Kampen diagram $D'$ from $D$ as follows. Given any directed edge $e'$ in $\text{int}(K_0)$ with initial vertex in the inside of $K_0$ and label $g' \in G_v$, we replace the label of $e'$ with $g^{-1}g'$. By construction, the resulting diagram will have one or more edges labelled by the trivial element. Each face containing such an edge (we call it a bad face) will either be a triangular face with other two edges having the same (non-identity) labels, or a rectangular one with two opposite edges labelled by the trivial element. In either case we can remove such a face by gluing the two edges labelled by non-identity elements. We remove all the bad faces in such a way, and call the resulting diagram $D'$. The corresponding dual van Kampen diagram $\Delta'$ will be identical to $\Delta$ apart from some of the edges of $C_0$ removed (along with vertices that would otherwise have degree 2 in $\Delta'$). Thus $\Delta'$ has strictly fewer cycles
contained in a single component than \( \Delta \), and so we may repeat this procedure to obtain a dual van Kampen diagram in which each component is a tree.

6.3. Graphs of large girth. Here we aim to show that all (finite simplicial) graphs of girth \( \geq 5 \) — that is, triangle-free and square-free graphs — are admissible. Thus, let \( \Gamma \) be a finite simplicial graph, and let \( \mathcal{F} = \{ F_v \mid v \in V(\Gamma) \} \) and \( \mathcal{G} = \{ G_v \mid v \in V(\Gamma) \} \) be two collections of groups, with all \( F_v \) finitely generated and all \( G_v \) equationaly noetherian. Let \( \omega \) be a non-principal ultrafilter. For each \( i \in \mathbb{N} \), let \( \varphi_i : \Gamma \mathcal{F} \to \Gamma \mathcal{G} \) be a linking homomorphism (in the sense of Definition 6.4).

Notice that, given a homomorphism \( \varphi : \Gamma \mathcal{F} \to \Gamma \mathcal{G} \), there are only finitely many choices for the subsets \( \text{supp}(\varphi(F_v)) \subseteq \text{link}(v) \) for \( v \in V(\Gamma) \). Therefore, there exist subsets \( A_v \subseteq \text{link}(v) \) such that \( A_v = \text{supp}(\varphi_i(F_v)) \) for all \( v \in V(\Gamma) \) \( \omega \)-almost surely. We will fix these subsets \( A_v \) throughout this subsection. The next result characterizes combinatorial restrictions that must be imposed on the \( A_v \).

**Lemma 6.8.** If \( \Gamma \) has girth \( \geq 4 \), then for any \( v \sim w \) we have \( a_v \sim a_w \) for all \( a_v \in A_v \) and \( a_w \in A_w \). In particular, if \( \Gamma \) has girth \( \geq 5 \), then either \( A_v \subseteq \{ w \} \) or \( A_w \subseteq \{ v \} \) whenever \( v \sim w \).

**Proof.** First, we prove the first statement. Let \( i \in \mathbb{N} \) be such that \( A_u = \text{supp}(\varphi_i(F_u)) \) for \( u \in \{ v, w \} \), and let \( g_u \in \varphi_i(F_u) \) be an element such that \( a_u \in \text{supp}(g_u) \) for \( u \in \{ v, w \} \). Since \( \varphi_i \) is a homomorphism, \( [g_v, g_w] = 1 \). Let \( \Delta \) be a dual van Kampen diagram corresponding to the word \( p_v^{-1} p_w^{-1} p_v p_w \) for some reduced words \( p_v, p_w \) representing \( g_v, g_w \), respectively, and let \( \partial_v \) and \( \partial'_v \) (respectively \( \partial_w \) and \( \partial'_w \)) be the intervals on the boundary of \( \Delta \) that spell out \( p_v \) (respectively \( p_w \)).

Let \( P_v \) (respectively \( P_w \)) be a \( a_v \)-component (respectively \( a_w \)-component) of \( \Delta \) that has a boundary point on \( \partial_v \) (respectively \( \partial_w \)). Notice that no other boundary point of \( P_v \) lies on \( \partial_v \); since \( p_v \) is reduced. Notice also that as \( A_v \subseteq \text{link}(v) \) and \( A_w \subseteq \text{link}(w) \), and as by assumption \( \Gamma \) is triangle-free, we have \( A_v \cap A_w = \emptyset \) — in particular, \( a_v \notin A_w \). Thus \( P_v \) cannot have boundary points on either \( \partial_w \) or \( \partial'_w \).

As \( P_v \) must have at least two boundary points, this implies that \( P_v \) must have a boundary point on \( \partial'_v \). Similarly, \( P_w \) must have a boundary point on \( \partial'_w \). But then \( P_v \) and \( P_w \) intersect, implying that \( a_v \sim a_w \), as required. This proves the first statement.

The second statement of the Lemma now follows from the first one under the additional assumption that \( \Gamma \) is square-free.

As an immediate consequence, we obtain the following result.

**Corollary 6.9.** If \( \Gamma \) has girth \( \geq 5 \) and \( v \in V(\Gamma) \) has \( |A_v| \geq 2 \), then \( |A_w| \leq 1 \) for all \( w \sim v \).

This implies the existence of ‘non-rigid’ vertices if \( \Gamma \) has girth \( \geq 5 \), in the following sense. The idea behind this is that there are transformations that allow us to move boundary points of components corresponding to non-rigid vertices in certain dual van Kampen diagrams: see Lemma 6.11.

**Definition 6.10.** We call a vertex \( v \in V(\Gamma) \) \( (\varphi_i) \)-rigid (or simply rigid) if there exists \( w \in V(\Gamma) \) such that \( v \in A_w \) and \( |A_w| \geq 2 \). Otherwise, \( v \) is called non-rigid.

Given a subset \( A \subseteq V(\Gamma) \), we write \( \iota_A : \Gamma A \mathcal{F}_A \to \Gamma \mathcal{F} \) for the canonical inclusion, and \( \rho_A : \Gamma \mathcal{G} \to \Gamma A \mathcal{G}_A \) for the canonical retraction. We then may define further homomorphisms

\[
\varphi_i^{(v,1)} = \rho_{V(\Gamma) \setminus \text{link}(v)} \circ \varphi_i : \Gamma \mathcal{F} \to \Gamma_{V(\Gamma) \setminus \text{link}(v)} \mathcal{G}_{V(\Gamma) \setminus \text{link}(v)}
\]

and

\[
\varphi_i^{(v,2)} = \rho_{V(\Gamma) \setminus \{v\}} \circ \varphi_i : \Gamma \mathcal{F} \to \Gamma_{V(\Gamma) \setminus \{v\}} \mathcal{G}_{V(\Gamma) \setminus \{v\}}.
\]

In addition, given any \( v \in V(\Gamma) \), we define

\[
B(v) = \{ w \in V(\Gamma) \mid A_w = \{ v \} \}.
\]
If \( v \) is non-rigid, then we may ‘decompose’ the homomorphisms \( \varphi_i \) into ones with a ‘smaller’ domain. In particular, \( \varphi_i, \omega \)-almost surely restricts to homomorphisms

\[
\varphi_i^{(v,3)} = \varphi_i \circ t_{B(v)} : \Gamma_{B(v)}F_{B(v)} \to G_v
\]

and

\[
\varphi_i^{(v,4)} = \varphi_i \circ t_{V(\Gamma)\setminus B(v)} : \Gamma_{V(\Gamma)\setminus B(v)}F_{V(\Gamma)\setminus B(v)} \to \Gamma_{V(\Gamma)\setminus \{v\}}G_{V(\Gamma)\setminus \{v\}}.
\]

For \( j \in \{1, 2, 3, 4\} \), let \( (\mathcal{F}_i)^{(v,j)} \) be the \( \omega \)-kernel for the sequence of homomorphisms \( \left( \varphi_i^{(v,j)} \right)_{i=1}^\infty \).

**Lemma 6.11.** Suppose \( v \in V(\Gamma) \) is non-rigid. Then \( \omega \)-almost surely we have

\[
(4) \quad \ker(\varphi_i) = \left\langle t_{B(v)}\left( \ker \varphi_i^{(v,3)} \right) \cup t_{V(\Gamma)\setminus B(v)}\left( \ker \varphi_i^{(v,4)} \right) \cup \left[ \ker \varphi_i^{(v,1)}, \ker \varphi_i^{(v,2)} \right] \right\rangle.
\]

Moreover, the \( \omega \)-kernel for the sequence \( (\varphi_i^{(v,j)})_{i=1}^\infty \) is

\[
(\mathcal{F}_i)^\omega = \left\langle t_{B(v)}\left( (\mathcal{F}_i^{(v,3)}) \right) \cup t_{V(\Gamma)\setminus B(v)}\left( (\mathcal{F}_i^{(v,4)}) \right) \cup \left[ (\mathcal{F}_i^{(v,1)}), (\mathcal{F}_i^{(v,2)}) \right] \right\rangle.
\]

**Proof.** We first prove that (4) holds \( \omega \)-almost surely. The inclusion \( (\supseteq) \) is clear, and so we only need to prove the inclusion \( (\subseteq) \).

Let \( i \in \mathbb{N} \) be such that \( \text{supp}(\varphi_i(F_w)) = A_w \) for all \( w \in V(\Gamma) \): this happens \( \omega \)-almost surely. Let \( g \in \ker(\varphi_i) \) be a cyclically reduced element. Consider an expression \( g = g_1 \cdots g_n \), with \( g_j \in F_{v_j} \) for some \( v_1, \ldots, v_n \in V(\Gamma) \). We will look at \( g_1 \cdots g_n \) as a cyclic word throughout, that is, we will not distinguish between \( g_1 \cdots g_n \) and its cyclic permutations.

We will perform two types of transformations of the cyclic word \( g_1 \cdots g_n \), which will not change whether or not the resulting element is contained in either side of (4).

(A) **Transpositions:** if, for some \( k \leq \ell \leq m \), we have \( \varphi_i(g_k \cdots g_\ell) \in \Gamma_{\text{link}(v)}G_{\text{link}(v)} \) and \( \varphi_i(g_{\ell+1} \cdots g_m) \in G_v \), then we may transpose the corresponding subwords of \( g_1 \cdots g_n \): replace the (cyclic) word \( g_k \cdots g_\ell \) with the word \( g_{\ell+1} \cdots g_m g_k \cdots g_\ell g_{m+1} \cdots g_n \). By construction, we have

\[
g_{k+1} \cdots g_\ell \in \ker \varphi_i^{(v,1)}
\]

and

\[
g_{\ell+1} \cdots g_m \in \ker \varphi_i^{(v,2)},
\]

so this transformation multiplies \( g \) by a conjugate of the element

\[
[g_{\ell+1} \cdots g_m, g_{k+1} \cdots g_\ell] \in \left[ \ker \varphi_i^{(v,1)}, \ker \varphi_i^{(v,2)} \right].
\]

(B) **Removals:** if, for some \( k, \ell \), we have \( \varphi_i(g_k \cdots g_\ell) = 1 \) and \( v_j \in B(v) \) for \( j = k + 1, \ldots, \ell \), then we may remove the corresponding subword of \( g_1 \cdots g_n \): that is, replace the (cyclic) word \( g_k \cdots g_\ell \) with the word \( g_{k+1} \cdots g_\ell \). By construction, this transformation multiplies \( g \) by a conjugate of the element

\[
(g_{k+1} \cdots g_\ell)^{-1} \in t_{B(v)}\left( \ker \varphi_i^{(v,3)} \right).
\]

Let \( \Delta \) be a dual van Kampen diagram for the word \( \varphi_i(g_1) \cdots \varphi_i(g_n) \), where the elements \( \varphi_i(g_j) \) are represented by reduced words. We will prove that \( g \) is contained in the right-hand side of (4) by induction on \( n \). The base case, \( n = 0 \), is clear. Without loss of generality, we may assume that \( \varphi_i(g_j) \neq 1 \) for each \( j \). Indeed, if \( \varphi_i(g_j) = 1 \) for some \( j \) then we may replace the cyclic word \( g_{j+1} \cdots g_j \) with \( g_{j+1} \cdots g_{j-1} \) by multiplying \( g \) by a conjugate of \( g_j^{-1} \in t_{B(v)}\left( \ker \varphi_i^{(v,3)} \right) \) or \( g_j^{-1} \in t_{V(\Gamma)\setminus B(v)}\left( \ker \varphi_i^{(v,4)} \right) \), depending on whether or not \( v_j \in B(v) \). This reduces the length of the representing word \( g \), and so we are done by the induction hypothesis.

If \( \Delta \) does not contain any \( v \)-components, then we are done: indeed, this means that \( v_j \notin B(v) \) for all \( j \) and so \( g \in t_{V(\Gamma)\setminus B(v)}\left( \ker \varphi_i^{(v,4)} \right) \). Otherwise, let \( P \) be a \( v \)-component of \( \Delta \).

Since \( v \) is non-rigid, it follows that we may write \( g_1 \cdots g_n \) (or some its cyclic permutation) as \( h_1 k_1 \cdots h_m k_m \), where any boundary point on the interval on the boundary of \( \Delta \) corresponding to \( \varphi_i(h_j) \) (respectively \( \varphi_i(k_j) \)) is (respectively is not) a boundary point of \( P \). Notice that the
Suppose now that \( m \geq 2 \). If \( Q \) is any component of \( \Delta \) having a boundary point on the interval \( \partial_j \) corresponding to \( \varphi_i(k_j) \), then either \( Q \) intersects \( P \), or all other boundary points of \( Q \) are on \( \partial_i \). It follows that \( \varphi_i(k_j) \in \Gamma_{\text{link}(v)} \Gamma_{\text{link}(v)} \); as \( P \) is a \( v \)-component, it is also clear that \( \varphi_i(h_j) \in G_\circ \). Thus we may transpose subwords \( h_j \) and \( k_j \) of \( g_1 \cdots g_n \) for any \( j \), as explained in \([A]\) above. This also can be done with minimal changes to \( \Delta \); see Figure 7. In particular, this rearranges boundary points in \( \Delta \) without changing whether or not a specific boundary point belongs to \( P \). This reduces the value of \( m \) for the corresponding word, and so after \( m - 1 \) such transpositions we return to the case \( m = 1 \). We are then done by the previous paragraph. This proves \((4)\).

Finally, for the second statement, notice that in the proof above, the only operations we do to the cyclic word \( g_1 \cdots g_n \) are transpositions \([A]\) or removals \([B]\) of its subwords, and there are finitely many operations of this form. The number of these operations is also bounded as a function of \( n \); for instance, we may assume that no permutation of syllables of \( g_1 \cdots g_n \) is obtained more than once while performing the procedure, and so there are at most \( n! \) transpositions of subwords performed until we remove a subword. Thus some particular sequence of transpositions and removals of subwords happens \( \omega \)-almost surely, which implies the second statement. \( \square \)

By combining Corollary 6.9 with Lemma 6.11 we obtain the following.

**Theorem 6.12.** Any finite graph \( \Gamma \) of girth \( \geq 5 \) is admissible.

**Proof.** We will induct on \( |V(\Gamma)| \); the base case, \( |V(\Gamma)| = 1 \), is clear. Now assume that \( \Gamma \) is a graph of girth \( \geq 5 \) with \( |V(\Gamma)| \geq 2 \) and that every graph \( \hat{\Gamma} \) of girth \( \geq 5 \) with \( |V(\hat{\Gamma})| < |V(\Gamma)| \) is admissible.

Note that \( \Gamma \) has at least one non-rigid vertex. Indeed, it is clear that any vertex \( v \) such that \( |A_w| \leq 1 \) for all \( w \sim v \) is non-rigid. Thus, if \( \Gamma \) contains a vertex \( v \) with \( |A_v| \geq 2 \) then, by Corollary 6.9, \( v \) is non-rigid. On the other hand, if \( \Gamma \) contains no vertices \( v \) with \( |A_v| \geq 2 \), then no vertices of \( \Gamma \) are rigid.

Without loss of generality, we can assume that \( \Gamma \) is connected – indeed, if it is not then \( \Gamma G \cong \Gamma A \Gamma A \Gamma B \Gamma B \Gamma B \) for some partition \( V(\Gamma) = A \cup B \). By the inductive hypothesis, \( \Gamma A \) and \( \Gamma B \) are admissible, and therefore, by Theorem 6.6, \( \Gamma A \Gamma A \) and \( \Gamma B \Gamma B \Gamma B \) are equationally noetherian. It then follows from Theorem 1.6 that \( \Gamma G \) is equationally noetherian as well, and so (by Theorem 6.11) \( \Gamma G \) is admissible, as required. We will therefore assume here that \( \Gamma \) is connected.

Now let \( v \) be a non-rigid vertex of \( \Gamma \). As \( \Gamma \) is connected, \( \text{link}(v) \neq \emptyset \). Therefore, by inductive hypothesis, the graphs \( \Gamma \Lambda(\Gamma) \Gamma_{\text{link}(v)} \) and \( \Gamma \Lambda(\Gamma) \Gamma_{\{v\}} \) are admissible, and consequently, by Theorem 6.6 the groups \( \Gamma \Lambda(\Gamma) \Gamma_{\text{link}(v)} \Gamma \Lambda(\Gamma) \Gamma_{\text{link}(v)} \) and \( \Gamma \Lambda(\Gamma) \Gamma_{\{v\}} \Gamma \Lambda(\Gamma) \Gamma_{\{v\}} \) are equationally noetherian. As

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Proof of Lemma 6.11: transposing \( k_j \) and the last syllable \( g_\ell \) of \( h_j \).
We transpose \( h_j \) and \( k_j \) by performing finitely many operations like these. \( P \) is shown in red, other components in other colours.}
\end{figure}
$G_v$ is also equationally noetherian, it follows that for every $j \in \{1, 2, 3, 4\}$ we have 

$$(\Gamma F)(v,j) \subseteq \ker \varphi^{(v,j)}$$

$\omega$-almost surely. The result now follows from Lemma 6.11. □

Proof of Theorem 7. This is immediate from Theorems 6.6 and 6.12. □

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