PERIODIC, ALMOST PERIODIC AND ALMOST
AUTOMORPHIC SOLUTIONS FOR SPDES WITH
MONOTONE COEFFICIENTS

MENGYU CHENG
School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, China

ZHENXIN LIU*
School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, China

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Abstract. In this paper, we use the variational approach to investigate recurrent properties of solutions for stochastic partial differential equations, which is in contrast to the previous semigroup framework. Consider stochastic differential equations with monotone coefficients. Firstly, we establish the continuous dependence on initial values and coefficients for solutions, which is interesting in its own right. Secondly, we prove the existence of recurrent solutions, which include periodic, almost periodic and almost automorphic solutions. Then we show that these recurrent solutions are globally asymptotically stable in square-mean sense. Finally, for illustration of our results we give two applications, i.e. stochastic reaction diffusion equations and stochastic porous media equations.

1. Introduction. Recurrence is an important concept in dynamical systems, which roughly means that a motion returns infinitely often to any small neighborhood of the initial position. The recurrent phenomenon has been found in almost all interesting systems, so it has attracted wide attention. This paper is devoted to studying the recurrence of solutions for stochastic partial differential equations with monotone coefficients. The types of recurrent solutions we investigate in present paper include periodic, almost periodic and almost automorphic solutions.

The analysis of recurrent solutions to ordinary differential equations dates back to Poincaré who studied periodic solutions of the three-body problem. Later, the notion of almost periodic functions was proposed and comprehensively studied by Bohr [11]. Many interesting results were acquired in this subject; see, for example,
Bochner [7, 9], von Neumann [57] and van Kampen [55]. After that, it was found that many differential equations, especially equations arising from physics, possess almost periodic solutions. So extensive investigations concerning almost periodic solutions for differential equations were conducted, following Favard’s pioneering work [22, 23]; see e.g. Amerio and Prouse [2], Fink [24], Yoshizawa [59], Sacker and Sell [53], Levitan and Zhikov [40] for a survey. Subsequently, almost periodicity was further generalized to almost automorphy by Bochner [8]. Veech [56], Johnson [36], Shen and Yi [54] et al studied properties of almost automorphic functions and this kind of solutions for differential equations.

Random factors may have significant impacts on the dynamics, so a natural question is: will recurrent phenomenon still persist when equations are perturbed by noise? Some works have been done to prove the existence of recurrent solutions for stochastic differential equations in both finite and infinite dimensions. For finite dimensional case, among many other works, we mention the following which are closely related to our work. Khasminskii [38] investigated periodic solutions for stochastic ordinary differential equations by Lyapunov’s second method. The existence of periodic and almost periodic solutions to affine stochastic equations were proved by Halanay [34], Arnold and Tudor [3]. Liu and Wang [47] reported the existence of almost periodic solutions for stochastic differential equations by the Favard separation method. For infinite dimensional case, Da Prato and Tudor [19] provided the existence of periodic and almost periodic solutions of semilinear stochastic partial differential equations. Later, studies of periodic, almost periodic and almost automorphic solutions to semilinear stochastic differential equations were performed by Bezandry and Diagana [6], Fu and Liu [25], Wang and Liu [58], Chen and Lin [14], Liu and Sun [46], Gao [26], Cheban and Liu [12], Liu and Liu [45], among others. Note that the almost periodic/automorphic solution in [6, 25] should be in distribution sense instead of square-mean sense, see Kamenskii et al [37] and Liu and Sun [46] for details. It is known that the distribution of solutions for a stochastic differential equation satisfy the corresponding Fokker-Planck equation, so we can also study recurrent solutions through the associated Fokker-Planck equation. See the very recent works of Chen et al [13] and Ji et al [35] on periodic solutions to Fokker-Planck equations.

Despite considerable advances in this direction, as far as we know there is no research so far on recurrent solutions to stochastic partial differential equations with monotone coefficients. Note that if the equation is no longer assumed to be semilinear, it will arise that the semigroup approach does not work any more. So a natural question is: can we still obtain recurrent solutions for stochastic partial differential equations which are not of the semilinear form? One of our main motivations is to partly answer this question. To this end, we adopt in this paper the variational approach which is sometimes called monotone method, to study recurrent solutions for stochastic partial differential equations.

Variational approach is one of basic approaches to analyze nonlinear deterministic/stochastic partial differential equations. For deterministic partial differential equations, the approach originated from the pioneering works of Lions [42] and Agmon [1]. For stochastic partial differential equations, the first work was done by Pardoux [48] who proved the existence of strong solutions for linear stochastic partial differential equations, which was based on Lions [42]. Subsequently, Krylov and Rozovskii [39] further developed this approach to nonlinear equations with continuous martingales as integrators. Fairly rigorous and complete description in a
slightly general form was provided by Prévôt and Röckner [51]. Within this framework, some types of dynamical and probabilistic properties such as existence of invariant measures, ergodicity and existence of random attractors have been studied for nonlinear SPDEs; see, e.g. Bogachev et al [10], Barbu and Da Prato [4], Da Prato et al [18], Gess et al [29], Liu and Tölle [44], Es-Sarhir et al [21], Gess [27, 28].

In the present paper, we investigate recurrent solutions and their global asymptotic stability for SPDEs with monotone coefficients.

Now let us state the framework and our main results more precisely. Let \((H, \langle \cdot, \cdot \rangle_H)\) be a separable Hilbert space and \(H^*\) the dual space of \(H\). As in Zhang [60], we assume that for each \(i = 1, 2\), \((V_i, \|\cdot\|_{V_i})\) is a reflexive Banach space such that \(V_i \subset H\) continuously and densely. Then we get two Gelfand triples \(V_1 \subset H \subset V_1^*\), \(V_2 \subset H \subset V_2^*\).

Consider the following stochastic differential equation on \(H\)

\[
\text{d}X(t) = A(t, X(t))\text{d}t + B(t, X(t))\text{d}W(t),
\]

where \(A := A_1 + A_2, A_i : \mathbb{R} \times V_i \to V_i^*, i = 1, 2\) and \(B : \mathbb{R} \times V \to L_2(U, H)\) satisfy hemicontinuous, monotone, coercive, bounded conditions (see Section 2 for details). Here \(W(t), t \in \mathbb{R}\) is a two-sided cylindrical Winner process on another separable Hilbert space \((U, \langle \cdot, \cdot \rangle_U)\). Under these conditions, the existence and uniqueness of solutions to equation (1) was established in [60]. In the present paper, we first prove that the solutions of (1) depend continuously on the initial value and the coefficients \(A, B\), which is useful to study qualitatively stochastic equations. We next show that (1) admits a unique \(L^2\)-bounded solution when the coefficients \(A\) and \(B\) satisfy some coercive and monotone conditions. Furthermore, with the help of continuous dependence property for solutions, we establish the recurrent properties in distribution sense for this unique \(L^2\)-bounded solution. Indeed, it shares the same recurrent properties with the coefficients \(A\) and \(B\); that is, when \(A\) and \(B\) are stationary (respectively, periodic, almost periodic, almost automorphic), then so is the \(L^2\)-bounded solution in distribution sense. Then we show that this unique recurrent (and bounded) solution is globally asymptotically stable in square-mean sense. This asymptotic stability property is very similar to the ergodicity of homogeneous Markov processes; note that the coefficients \(A\) and \(B\) depend on \(t\), so equation (1) generates an inhomogeneous Markov process. Finally, to illustrate the theoretical results obtained above, we discuss two examples, i.e. stochastic reaction diffusion equations and stochastic porous media equations.

The remainder of this paper is organized as follows. Section 2 gives some definitions and properties of recurrent functions as well as a rough introduction to variational approach. In Section 3, we obtain continuous dependence on initial values and coefficients for solutions of (1), and prove that (1) admits a unique \(L^2\)-bounded solution under suitable conditions. In Section 4, we show that the \(L^2\)-bounded solution has the same recurrent properties as the coefficients. In Section 5, we consider the additive noise case. In this situation, we consider the strongly monotone condition instead of strict one, which can apply to stochastic porous media equations. Section 6 discusses global asymptotic stability of the \(L^2\)-bounded solution. In the last section, we give two applications.

2. Preliminaries. Before turning to our results, we first give some preliminaries. Let \((\mathcal{X}, d)\) be a complete metric space. We write \(C(\mathbb{R}, \mathcal{X})\) to mean the space of all continuous functions \(\varphi : \mathbb{R} \to \mathcal{X}\).
2.1. Recurrent functions. Let us now recall some types of recurrent functions to be studied in this paper.

Definition 2.1. We say \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is \( T \)-periodic, if there exists some nonzero constant \( T \in \mathbb{R} \) such that \( \varphi(t + T) = \varphi(t) \) for all \( t \in \mathbb{R} \). In particular, \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is called stationary provided \( \varphi(t) = \varphi(0) \) for all \( t \in \mathbb{R} \).

Definition 2.2. We say \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is Bohr almost periodic if the set \( \mathcal{T}(\varphi, \varepsilon) \) of \( \varepsilon \)-almost periods of \( \varphi \) is relatively dense for each \( \varepsilon > 0 \), i.e. there exists a constant \( l = l(\varepsilon) > 0 \) such that \( \mathcal{T}(\varphi, \varepsilon) \cap [a, a + l] \neq \emptyset \) for all \( a \in \mathbb{R} \), where

\[
\mathcal{T}(\varphi, \varepsilon) := \{ \tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} d(\varphi(t + \tau), \varphi(t)) < \varepsilon \}.
\]

Let \( \varphi \) be a mapping from \( \mathbb{R} \) to \( \mathcal{X} \). We employ \( \gamma \) to denote a sequence \( \{ \gamma_n \}_{n=1}^{\infty} \) in \( \mathbb{R} \). Denote \( (T_\gamma \varphi)(\cdot) := \lim_{n \rightarrow \infty} \varphi(\cdot + \gamma_n) \), provided the limit exists. The mode of convergence will be pointed out when this symbol is used. Recall the following characterization of almost periodicity that is due to Bochner [7].

Definition 2.3. We say \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is Bochner almost periodic, if for any sequence \( \gamma' = \{ \gamma'_n \} \subset \mathbb{R} \) there exists a subsequence \( \gamma = \{ \gamma_n \} \subset \gamma' \) such that \( T_\gamma \varphi \) exists uniformly on \( \mathbb{R} \).

Theorem 2.4 (Bochner). Assume that \( \varphi : \mathbb{R} \rightarrow \mathcal{X} \) is continuous. Then the following statements are equivalent.

1. \( \varphi \) is Bohr almost periodic.
2. \( \varphi \) is Bochner almost periodic.
3. For any two sequences \( \gamma' = \{ \gamma'_n \} \subset \mathbb{R} \) and \( \beta' = \{ \beta'_n \} \subset \mathbb{R} \) there exist two subsequences \( \gamma = \{ \gamma_n \} \subset \gamma' \) and \( \beta = \{ \beta_n \} \subset \beta' \) with the same indexes such that

\[
T_{\gamma + \beta} \varphi = T_\gamma T_\beta \varphi
\]

uniformly on \( \mathbb{R} \).
4. For any two sequences \( \gamma' = \{ \gamma'_n \} \subset \mathbb{R} \) and \( \beta' = \{ \beta'_n \} \subset \mathbb{R} \) there exist two subsequences \( \gamma = \{ \gamma_n \} \subset \gamma' \) and \( \beta = \{ \beta_n \} \subset \beta' \) with the same indexes such that

\[
T_{\gamma + \beta} \varphi = T_\gamma T_\beta \varphi
\]

pointwise.

Remark 1. It follows from the above theorem that Bohr’s almost periodicity is equivalent to Bochner’s one. Therefore, we just call them almost periodicity below.

Definition 2.5. We say \( \varphi \in C(\mathbb{R}, \mathcal{X}) \) is almost automorphic, if for any sequence \( \gamma' = \{ \gamma'_n \} \subset \mathbb{R} \) there exist a subsequence \( \gamma = \{ \gamma_n \} \subset \gamma' \) and some function \( \psi : \mathbb{R} \rightarrow \mathcal{X} \) such that

\[
\lim_{n \rightarrow \infty} \varphi(t + \gamma_n) = \psi(t) \quad \text{and} \quad \lim_{n \rightarrow \infty} \psi(t - \gamma_n) = \varphi(t)
\]

pointwise for \( t \in \mathbb{R} \).

In order to study recurrent solutions of differential equations, we need to recall the definition of uniformly almost periodic/automorphic functions. Let \( (\mathcal{X}_i, d_i) \), \( i = 1, 2 \) be complete metric spaces. We write \( C(\mathbb{R} \times \mathcal{X}_1, \mathcal{X}_2) \) to mean the set of all continuous functions \( \varphi : \mathbb{R} \times \mathcal{X}_1 \rightarrow \mathcal{X}_2 \).

Definition 2.6. We say \( \varphi \in C(\mathbb{R} \times \mathcal{X}_1, \mathcal{X}_2) \) is \( T \)-periodic in \( t \), if there exists some nonzero constant \( T \in \mathbb{R} \) such that \( \varphi(t + T, x) = \varphi(t, x) \) for all \( t \in \mathbb{R}, x \in \mathcal{X}_1 \).
Definition 2.7 (See Yoshizawa [59]). We say \( \varphi \in C(\mathbb{R} \times X_1, X_2) \) is almost periodic in \( t \) uniformly for \( x \in X_1 \) if for any \( \varepsilon > 0 \) and any compact set \( Q \subset X_1 \), the set \( T(\varphi, \varepsilon, Q) \) is relatively dense, i.e. there exists a constant \( l = l(\varepsilon, Q) > 0 \) such that \( T(\varphi, \varepsilon, Q) \cap [a, a + l] \neq \emptyset \) for all \( a \in \mathbb{R} \), where

\[
T(\varphi, \varepsilon, Q) := \{ \tau \in \mathbb{R} : \sup_{(t,x) \in \mathbb{R} \times Q} d_2(\varphi(t + \tau, x), \varphi(t, x)) < \varepsilon \}.
\]

Similar to Theorem 2.4, we recall the following results.

Theorem 2.8 (See Yoshizawa [59]). Suppose that \( \varphi : \mathbb{R} \times X_1 \to X_2 \) is continuous. Then the following statements are equivalent.

1. \( \varphi \) is almost periodic in \( t \) uniformly for \( x \in X_1 \).
2. For any sequence \( \gamma' = \{ \gamma'_n \} \subset \mathbb{R} \) there exists a subsequence \( \gamma = \{ \gamma_n \} \subset \gamma' \) such that

\[
(T_{\gamma'} \varphi)(t, x) := \lim_{n \to \infty} \varphi(t + \gamma_n, x)
\]

exists uniformly with respect to \( t \in \mathbb{R} \) and \( x \in Q \), where \( Q \) is an arbitrary compact subset of \( X_1 \).
3. For any two sequences \( \gamma' = \{ \gamma'_n \} \subset \mathbb{R} \) and \( \beta' = \{ \beta'_n \} \subset \mathbb{R} \) there exist two subsequences \( \gamma = \{ \gamma_n \} \subset \gamma' \) and \( \beta = \{ \beta_n \} \subset \beta' \) with the same indexes such that

\[
T_{\gamma + \beta} \varphi = T_{\gamma} T_{\beta} \varphi
\]

uniformly on \( \mathbb{R} \times Q \), where \( Q \) is an arbitrary compact subset of \( X_1 \).

Definition 2.9 (See Shen and Yi [54]). We say \( \varphi \in C(\mathbb{R} \times X_1, X_2) \) is almost automorphic in \( t \) uniformly for \( x \in X_1 \), if for any sequence \( \gamma' = \{ \gamma'_n \} \subset \mathbb{R} \) there exist a subsequence \( \gamma = \{ \gamma_n \} \subset \gamma' \) and some function \( \psi : \mathbb{R} \times X_1 \to X_2 \) such that

\[
\lim_{n \to \infty} \varphi(t + \gamma_n, x) = \psi(t, x)
\]

and

\[
\lim_{n \to \infty} \psi(t - \gamma_n, x) = \varphi(t, x)
\]

uniformly on \([a, b] \times Q\), where \([a, b]\) is an arbitrary finite interval and \( Q \) an arbitrary compact subset of \( X_1 \).

Remark 2. When we consider stochastic partial differential equations, \( X_1 \) is a Banach space \( V \). For the sake of simplicity, we call a function \( \varphi \) “uniformly almost periodic (uniformly almost automorphic)”, which means that \( \varphi \) is almost periodic (almost automorphic) in \( t \) uniformly for \( x \in V \).

2.2. Recurrence in distribution. Suppose further that \((X, d)\) is a Polish space, i.e. a separable complete metric space. We write \( Pr(X) \) to mean the set of all Borel probability measures on \( X \). Denote by \( C_b(X) \) the space of all continuous functions \( \varphi : X \to \mathbb{R} \) for which the norm \( \| \varphi \|_\infty := \sup_{x \in X} |\varphi(x)| \) is finite. Let \( \{ \mu_n \} := \{ \mu_n \}_{n=1}^\infty \subset Pr(X) \) and \( \mu \in Pr(X) \). We say \( \mu_n \) converges weakly to \( \mu \) in \( Pr(X) \), provided \( \int \varphi d\mu_n \) converges to \( \int \varphi d\mu \) for all \( \varphi \in C_b(X) \). Let \( \varphi \in C_b(X) \) be Lipschitz continuous, we define

\[
\| \varphi \|_{BL} := Lip(\varphi) + \| \varphi \|_\infty,
\]

where \( Lip(\varphi) := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)} \). We endow \( Pr(X) \) with \( d_{BL} \) metric, where

\[
d_{BL}(\mu, \nu) := \sup \left\{ \left| \int \varphi d\mu - \int \varphi d\nu \right| : \| \varphi \|_{BL} \leq 1 \right\}
\]

for all \( \mu, \nu \in Pr(X) \). It is well known that \( d_{BL} \) generates the weak topology on \( Pr(X) \), i.e. \( \mu_n \to \mu \) weakly in \( Pr(X) \) if and only if \( d_{BL}(\mu_n, \mu) \to 0 \) as \( n \to \infty \). See
Chapter 11 in Dudley [20] for this metric $d_{BL}$ (denoted by $\beta$ there) and its related properties.

We assume in the following exposition that $(\Omega, \mathcal{F}, P)$ is a complete probability space and that $(H, \langle \cdot, \cdot \rangle_H)$ is a separable Hilbert space. The space $L^2(\Omega, P; H)$ consists of all $H$-valued random variables $\zeta$ such that $E\|\zeta\|^2_H = \int_{\Omega} \|\zeta\|^2_H dP < \infty$. We say an $H$-valued stochastic process $X = \{X(t) : t \in \mathbb{R}\}$ is $L^2$-bounded provided $\sup_t E\|X(t)\|^2_H < \infty$. Throughout the paper, we denote by $\mathcal{L}(\zeta) \in Pr(H)$ the law or distribution of $H$-valued random variable $\zeta$. A sequence of $H$-valued continuous stochastic processes $\{X_n\}$ is said to converge in distribution to $X$ (on $C(\mathbb{R}, H)$) provided $\mathcal{L}(X_n)$ weakly converges to $\mathcal{L}(X)$ in $Pr(C(\mathbb{R}, H))$, where $\mathcal{L}(X)$ is the law or distribution of $X$ on $C(\mathbb{R}, H)$; if $d_{BL}(\mathcal{L}(X_n(t)), \mathcal{L}(X(t))) \to 0$ as $n \to \infty$ for each $t \in \mathbb{R}$, we simply say that $X_n$ converges in distribution to $X$ on $H$.

Note that $(Pr(C(\mathbb{R}, H)), d_{BL})$ and $(Pr(H), d_{BL})$ are Polish spaces (see, e.g. Theorems 6.2 and 6.5 in Parthasarathy [49, Chapter II]). So, similar to Definitions 2.1, 2.2 and 2.5, we can define recurrence in distribution as follows.

**Definition 2.10.** We say an $H$-valued continuous stochastic process $X$ is $T$-periodic (respectively, almost periodic, almost automorphic) in distribution, if the mapping $t \mapsto \mathcal{L}(X(t))$ is $T$-periodic (respectively, almost periodic, almost automorphic) in $Pr(C(\mathbb{R}, H))$. In particular, $X$ is called stationary provided $X$ is $T$-periodic in distribution for any $T \in \mathbb{R}$.

**Remark 3.** Let $X$ be an $H$-valued continuous stochastic process. Note that $\mu(t) := \mathcal{L}(X(t)), t \in \mathbb{R}$ is $T$-periodic (respectively, almost periodic, almost automorphic) in $Pr(H)$, provided $\mathcal{L}(X)$ is $T$-periodic (respectively, almost periodic, almost automorphic) in $Pr(C(\mathbb{R}, H))$. But the converse is not true in general.

### 2.3. Variational approach.

Recall that $H$ is a separable Hilbert space with norm $\|\cdot\|_H$ and inner product $\langle \cdot, \cdot \rangle_H$, and that $H^*$ is the dual space of $H$. Let $(V, \|\cdot\|_V)$ be a reflexive Banach space such that $V \subset H$ continuously and densely. So we have $H^* \subset V^*$ continuously and densely. Identifying $H$ with its dual $H^*$ via the Riesz isomorphism, then we have

$$V \subset H \subset V^*$$

continuously and densely. We write $\langle \cdot, \cdot \rangle_V$ to denote the pairing between $V^*$ and $V$. It follows that

$$\langle \cdot, \cdot \rangle_V : h, v \mapsto \langle h, v \rangle_H$$

for all $h \in H, v \in V$. $(V, H, V^*)$ is called Gelfand triple. Since $H \subset V^*$ continuously and densely, we deduce that $V^*$ is separable, hence so is $V$. See Prévôt and Röckner [51] for details. Assume that $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ are reflexive Banach spaces and embedded in $H$ continuously and densely. Then we get two triples:

$$V_1 \subset H \simeq H^* \subset V_1^* \quad \text{and} \quad V_2 \subset H \simeq H^* \subset V_2^*.$$ 

We define the norm $\|v\|_V := \|v\|_{V_1} + \|v\|_{V_2}$ on the space $V := V_1 \cap V_2$. Note that $(V, \|\cdot\|_{V})$ is also a Banach space. Since $V_i^*$ and $V_2^*$ can be thought as subspaces of $V^*$, we get a Banach space $W := V_1^* + V_2^* \subset V^*$ with norm

$$\|f\|_W := \inf \{\|f_1\|_{V_1^*} + \|f_2\|_{V_2^*} : f = f_1 + f_2, f_i \in V_i^*, i = 1, 2\}.$$

Similarly, we write $V_i^* \langle \cdot , \cdot \rangle_{V_i}$ to denote the pairing between $V_i^*$ and $V_i$, $i = 1, 2$. Then, for all $v \in V$ and $f = f_1 + f_2 \in W \subset V^*$ we have
\[
V^* (f, v) = V_i^* (f_1, v) + V_2^* (f_2, v).
\]
Note carefully that if $f \in H$ and $v \in V$, then we obtain
\[
V^* (f, v) = V_i^* (f, v) = V_2^* (f, v) = \langle f, v \rangle_H.
\]

Let $W(t), t \in \mathbb{R}$ be a two-sided cylindrical $Q$-Wiener process with $Q = I$ on a separable Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$ with respect to a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Denote by $L_2(U, H)$ the space of all Hilbert-Schmidt operators from $U$ into $H$. Consider the following stochastic partial differential equation on $H$
\[
dX(t) = A(t, X(t))dt + B(t, X(t))dW(t),
\]
where $A = A_1 + A_2, A_i : \mathbb{R} \times V_i \to V_i^*$, $i = 1, 2$ and $B : \mathbb{R} \times V \to L_2(U, H)$.

Let us introduce the following conditions.

(H1) (Hemicontinuity) For all $u, v, w \in V$ and $t \in \mathbb{R}$ the map
\[
\mathbb{R} \ni \theta \mapsto V^* (A(t, u + \theta v), w)
\]
is continuous;

(H2) (Monotonicity) There exists a constant $c \in \mathbb{R}$ such that for all $u, v \in V, t \in \mathbb{R}$
\[
2V^* (A(t, u) - A(t, v), u - v) + \|B(t, u) - B(t, v)\|_{L_2(U, H)}^2 \leq c \|u - v\|_H^2;
\]

(H3) (Coercivity) There exist constants $\alpha_1, \alpha_2 \in (1, \infty), c_1 \in \mathbb{R}, c_2, c'_2 \in (0, \infty)$ and $M_0 \in (0, \infty)$ such that for all $v \in V, t \in \mathbb{R}$
\[
2V^* (A(t, v), v) \leq c_1 \|v\|_H^2 - c_2 \|v\|_{V_1}^{\alpha_1} - c'_2 \|v\|_{V_2}^{\alpha_2} + M_0;
\]

(H4) (Boundedness) There exist constants $c_3, c'_3 \in (0, \infty)$ such that for all $v \in V, t \in \mathbb{R}$
\[
\|A_1(t, v)\|_{V_1^*} \leq c_3 \|v\|_{V_1}^{\alpha_1 - 1} + M_0,
\]
\[
\|A_2(t, v)\|_{V_2^*} \leq c'_3 \|v\|_{V_2}^{\alpha_2 - 1} + M_0,
\]
where $\alpha_i$ is as in (H3).

**Definition 2.11.** We say continuous $H$-valued $(\mathcal{F}_t)$-adapted process $X(t), t \in [0, T]$ is a solution to equation (2), if for its $dt \otimes P$-equivalence class $\hat{X}$ we have $\hat{X} \in \bigcap_{i=1,2} L^\infty ([0, T] \times \Omega, dt \otimes P; V_i) \cap L^2 (0, T \times \Omega, dt \otimes P; H)$ with $\alpha_i$ as in (H3) and $P$-a.s.
\[
X(t) = X(s) + \int_s^t A(\sigma, \hat{X} (\sigma))d\sigma + \int_s^t B(\sigma, \hat{X} (\sigma))dW(\sigma), \quad 0 \leq s \leq t \leq T,
\]
where $\hat{X}$ is any $V$-valued progressively measurable $dt \otimes P$-version of $\hat{X}$.

**Remark 4.**
1. Note that solutions in Definition 2.11 are usually called variational solutions in the literature.

2. By (H3) and (H4), for all $t \in \mathbb{R}$ and $v \in V$ we have
\[
\|B(t, v)\|_{L_2(U, H)}^2 \leq c_1 \|v\|_H^2 + (2c_3 - c_2) \|v\|_{V_1}^{\alpha_1} + (2c'_3 - c'_2) \|v\|_{V_2}^{\alpha_2} + 2M_0 (\|v\|_{V_1} + \|v\|_{V_2}) + M_0.
\]

3. Suppose that (H1)–(H4) hold, then for any $X_0 \in L^2 (\Omega, \mathcal{F}_0, P; H)$ there exists a unique solution to equation (2) in the sense of Definition 2.11 (see Zhang [60] for more general results).
4. For all $0 \leq s \leq t \leq T$ we have the following Itô’s formula (see, e.g. Prévôt and Röckner [51, Theorem 4.2.5]).

$$
\|X(t)\|_{L^2}^2 = \|X(s)\|_{L^2}^2 + \int_s^t \left( 2\langle v, (A(\sigma, \mathbf{X}(\sigma)), \mathbf{X}(\sigma)) \rangle_V + \|B(\sigma, \mathbf{X}(\sigma))\|_{L^2(U,H)}^2 \right) d\sigma + 2 \int_s^t \langle X(\sigma), B(\sigma, \mathbf{X}(\sigma))dW(\sigma) \rangle_H.
$$

3. Continuous dependence and bounded solutions. In this section, we investigate continuous dependence on initial values and coefficients for solutions to equation (2), and show that (2) admits a unique $L^2$-bounded solution under suitable conditions. Despite that the stability of solutions for several types of SPDEs with monotone coefficients was obtained (see Ciotir [15, 16], Ciotir and Tölté [17], Gess and Tölté [33]), the following general stability results are interesting on its own rights but are not known to our knowledge. It turns out to be necessary to consider the following condition.

(HL) There exists a constant $L_B > 0$ such that for all $u, v \in V$, $t \in \mathbb{R}$

$$
\|B(t, u) - B(t, v)\|_{L^2(U,H)} \leq L_B \|u - v\|_H.
$$

**Theorem 3.1.** Suppose that $A_n$, $A$, $B_n$, $B$ satisfy (H1)–(H4) and (HL) with the same constants $c, c_1, c_2, c_3, c'_2, c'_3, M_0, \alpha_i, i = 1, 2$ and $L_B$. Let $X_n$ be a solution of the Cauchy problem

$$
\begin{align*}
\frac{dX(t)}{dt} &= A_n(t, X(t))dt + B_n(t, X(t))dW(t) \\
X(s) &= \zeta^n,
\end{align*}
$$

and $X$ be a solution to the Cauchy problem

$$
\begin{align*}
\frac{dX(t)}{dt} &= A(t, X(t))dt + B(t, X(t))dW(t) \\
X(s) &= \zeta^*.
\end{align*}
$$

Assume further that

1. If $\lim_{n \to \infty} A_n(t, x) = A_i(t, x)$ in $V^*_i$ for all $t \in \mathbb{R}$, $x \in V$, $i = 1, 2$;
2. If $\lim_{n \to \infty} B_n(t, x) = B(t, x)$ in $L^2(U, H)$ for all $t \in \mathbb{R}$, $x \in V$.

Then we have the following conclusions:

1. If $\lim_{n \to \infty} E\|\zeta_n^* - \zeta^*\|_H^2 = 0$, then $\lim_{n \to \infty} E\sup_{s \leq \tau \leq t} \|X_n(\tau) - X(\tau)\|_H^2 = 0$ for any $t > s$;
2. If $\lim_{n \to \infty} \zeta_n^* = \zeta^*$ in probability, then $\lim_{n \to \infty} \sup_{\tau \in [s, t]} \|X_n(\tau) - X(\tau)\|_H = 0$ in probability;
3. If $\lim_{n \to \infty} d_{BL}(\mathcal{L}(\zeta_n^*), \mathcal{L}(\zeta^*)) = 0$ in $Pr(H)$, then $\lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n), \mathcal{L}(X)) = 0$ in $Pr(C([s, \infty), H))$.

**Proof.** (i) Employing Itô’s formula, we obtain

$$
E\sup_{s \leq \tau \leq t} \|X_n(\tau) - X(\tau)\|_H^2
\leq E\sup_{s \leq \tau \leq t} \int_s^\tau \left( 2\langle v, (A_n(\sigma, \mathbf{X}_n(\sigma)) - A_n(\sigma, \mathbf{X}(\sigma)), \mathbf{X}_n(\sigma) - \mathbf{X}(\sigma)) \rangle_V \\
+ 2\|B_n(\sigma, \mathbf{X}_n(\sigma)) - B_n(\sigma, \mathbf{X}(\sigma))\|_{L^2(U,H)}^2 \right) d\sigma + E\|\zeta_n^* - \zeta^*\|_H^2
$$
\[ + E \sup_{s \leq \tau \leq t} \int_s^\tau \left( 2V \cdot \langle A_n(\sigma, \bar{X}(\sigma)) - A(\sigma, \bar{X}(\sigma)), X_n(\sigma) - \bar{X}(\sigma) \rangle_V + 2\|B_n(\sigma, \bar{X}(\sigma)) - B(\sigma, \bar{X}(\sigma))\|^2_{L_2(U, H)} \right) d\sigma \]
\[ + E \sup_{s \leq \tau \leq t} 2 \int_s^\tau \langle X_n(\sigma) - X(\sigma), [B_n(\sigma, X_n(\sigma)) - B_n(\sigma, \bar{X}(\sigma))] \rangle dW(\sigma)_H \]
\[ + E \sup_{s \leq \tau \leq t} 2 \int_s^\tau \langle X_n(\sigma) - X(\sigma), [B_n(\sigma, \bar{X}(\sigma)) - B(\sigma, \bar{X}(\sigma))] \rangle dW(\sigma)_H \]
\[ =: E\|\zeta_n^a - \zeta^a\|^2_H + I_1 + I_2 + I_3 + I_4. \]

For the first term \(I_1\), by (H2) and (HL) we have

\[ I_1 := E \sup_{s \leq \tau \leq t} \int_s^\tau \left( 2V \cdot \langle A_n(\sigma, \bar{X}_n(\sigma)) - A_n(\sigma, \bar{X}(\sigma)), X_n(\sigma) - \bar{X}(\sigma) \rangle_V + 2\|B_n(\sigma, \bar{X}_n(\sigma)) - B_n(\sigma, \bar{X}(\sigma))\|^2_{L_2(U, H)} \right) d\sigma \]
\[ \leq E \sup_{s \leq \tau \leq t} \int_s^\tau (|c| + L^2_B) \|X_n(\sigma) - \bar{X}(\sigma)\|^2_H d\sigma \]
\[ \leq E \int_s^t (|c| + L^2_B) \|X_n(\sigma) - \bar{X}(\sigma)\|^2_H d\sigma. \]

For the second term \(I_2\), by the Hölder inequality we obtain

\[ I_2 := E \sup_{s \leq \tau \leq t} \int_s^\tau \left( 2V \cdot \langle A_n(\sigma, \bar{X}_n(\sigma)) - A(\sigma, \bar{X}(\sigma)), X_n(\sigma) - \bar{X}(\sigma) \rangle_V + 2\|B_n(\sigma, \bar{X}_n(\sigma)) - B(\sigma, \bar{X}(\sigma))\|^2_{L_2(U, H)} \right) d\sigma \]
\[ \leq E \sup_{s \leq \tau \leq t} \int_s^\tau \left( 2\|A_{1,n}(\sigma, \bar{X}(\sigma)) - A(\sigma, \bar{X}(\sigma))\|_{V_1} \|X_n(\sigma) - \bar{X}(\sigma)\|_{V_1} + 2\|A_{2,n}(\sigma, \bar{X}(\sigma)) - A(\sigma, \bar{X}(\sigma))\|_{V_2} \|X_n(\sigma) - \bar{X}(\sigma)\|_{V_2} + 2\|B_n(\sigma, \bar{X}(\sigma)) - B(\sigma, \bar{X}(\sigma))\|^2_{L_2(U, H)} \right) d\sigma \]
\[ \leq 2 \left( E \int_s^t \|A_{1,n}(\sigma, \bar{X}(\sigma)) - A(\sigma, \bar{X}(\sigma))\|_{V_1}^{\alpha_1} \|X_n(\sigma) - \bar{X}(\sigma)\|_{V_1}^{\frac{\alpha_1-1}{\alpha_1}} d\sigma \right)^{\frac{\alpha_1}{\alpha_1-1}} \]
\[ \times \left( E \int_s^t \|X_n(\sigma) - \bar{X}(\sigma)\|_{V_1}^{\frac{\alpha_1}{\alpha_1-1}} d\sigma \right)^{\frac{\alpha_1-1}{\alpha_1}} \]
\[ + 2 \left( E \int_s^t \|A_{2,n}(\sigma, \bar{X}(\sigma)) - A(\sigma, \bar{X}(\sigma))\|_{V_2}^{\alpha_2} \|X_n(\sigma) - \bar{X}(\sigma)\|_{V_2}^{\frac{\alpha_2-1}{\alpha_2}} d\sigma \right)^{\frac{\alpha_2}{\alpha_2-1}} \]
\[ \times \left( E \int_s^t \|X_n(\sigma) - \bar{X}(\sigma)\|_{V_2}^{\frac{\alpha_2-1}{\alpha_2}} d\sigma \right)^{\frac{\alpha_2-1}{\alpha_2}} \]
\[ + E \int_s^t 2\|B_n(\sigma, \bar{X}(\sigma)) - B(\sigma, \bar{X}(\sigma))\|^2_{L_2(U, H)} d\sigma. \]
For the last two terms \(I_3\) and \(I_4\), by Burkholder–Davis inequality (see, e.g. [51, Proposition D.0.1]) and Cauchy’s inequality with \(\epsilon\), we get

\[
I_3 := E \sup_{s \leq \tau \leq t} 2 \int_s^t \langle X_n(\sigma) - X(\sigma), [B_n(\sigma, X_n(\sigma)) - B_n(\sigma, X(\sigma))] \rangle dW(\sigma)_H
\]

\[
\leq 6E \left( \int_s^t \| B_n(\sigma, X_n(\sigma)) - B_n(\sigma, X(\sigma)) \|_{L_2(U,H)}^2 \| X_n(\sigma) - X(\sigma) \|_H^2 d\sigma \right)^{1/2}
\]

\[
\leq C_1 E \int_s^t \| X_n(\sigma) - X(\sigma) \|_H^2 d\sigma + \epsilon E \sup_{s \leq \tau \leq t} \| X_n(\tau) - X(\tau) \|_H^2
\]

and

\[
I_4 := E \sup_{s \leq \tau \leq t} 2 \int_s^t \langle X_n(\sigma) - X(\sigma), [B_n(\sigma, X(\sigma)) - B(\sigma, X(\sigma))] \rangle dW(\sigma)_H
\]

\[
\leq 6E \left( \int_s^t \| B_n(\sigma, X(\sigma)) - B(\sigma, X(\sigma)) \|_{L_2(U,H)}^2 \| X_n(\sigma) - X(\sigma) \|_H^2 d\sigma \right)^{1/2}
\]

\[
\leq C_1 E \int_s^t \| B_n(\sigma, X(\sigma)) - B(\sigma, X(\sigma)) \|_{L_2(U,H)}^2 d\sigma + \epsilon E \sup_{s \leq \tau \leq t} \| X_n(\tau) - X(\tau) \|_H^2.
\]

Taking \(\epsilon = \frac{1}{4}\) and combining (6)–(9), we have

\[
E \sup_{s \leq \tau \leq t} \| X_n(\tau) - X(\tau) \|_H^2 \leq 2E\| \zeta_n^s - \zeta^s \|_H^2 + C_1 E \int_s^t \sup_{s \leq u \leq \sigma} \| X_n(u) - X(u) \|_H^2 d\sigma
\]

\[
+ C_2 \left( E \int_s^t \| A_{1,n}(\sigma, X(\sigma)) - A_1(\sigma, X(\sigma)) \|_{V_1^{\alpha_1/\alpha_1}}^{\alpha_1/\alpha_1} d\sigma \right)^{\frac{\alpha_1-1}{\alpha_1}}
\]

\[
+ C_2 \left( E \int_s^t \| A_{2,n}(\sigma, X(\sigma)) - A_2(\sigma, X(\sigma)) \|_{V_2^{\alpha_2/\alpha_2}}^{\alpha_2/\alpha_2} d\sigma \right)^{\frac{\alpha_2-1}{\alpha_2}}
\]

\[
+ C_3 E \int_s^t \| B_n(\sigma, X(\sigma)) - B(\sigma, X(\sigma)) \|_{L_2(U,H)}^2 d\sigma,
\]

where \(C_1, C_2\) and \(C_3\) are different positive constants, depending only on \(\epsilon, c\) and \(L_B\). Then in view of the Gronwall’s lemma, we have

\[
E \sup_{s \leq \tau \leq t} \| X_n(\tau) - X(\tau) \|_H^2 \leq \xi_n e^{C_1(t-s)},
\]

where

\[
\xi_n := 2E\| \zeta_n^s - \zeta^s \|_H^2 + C_3 E \int_s^t \| B_n(\sigma, X(\sigma)) - B(\sigma, X(\sigma)) \|_{L_2(U,H)}^2 d\sigma
\]

\[
+ C_2 \left( E \int_s^t \| A_{1,n}(\sigma, X(\sigma)) - A_1(\sigma, X(\sigma)) \|_{V_1^{\alpha_1/\alpha_1}}^{\alpha_1/\alpha_1} d\sigma \right)^{\frac{\alpha_1-1}{\alpha_1}}
\]
The proof of (i) is complete.

We obtain for

Thus, by Lebesgue’s dominated convergence theorem, (H4), Remark 4 (ii) and (12),

Therefore,

Letting

It follows from (H3) that

Using Itô’s formula and the product rule, we obtain

\[
E \left( e^{-c_1(t-s)} \|X_n(t \wedge \gamma^n(R)) \|_{H}^2 \right)
\]

\[
= E \|\zeta^n_s \|_{H}^2 + E \int_s^t e^{-c_1(s-r)} M_0 \, d\sigma + E \int_s^t e^{-c_1(s-r)} c_1 \| \mathcal{X}_n(\sigma) \|_{V_1} \, d\sigma
\]

\[
\leq E \|\zeta^n_s \|_{H}^2 + E \int_s^t e^{-c_1(s-r)} M_0 \, d\sigma
\]

Letting \( R \to \infty \) in (11) and using Fatou’s lemma, we have

\[
E \left( e^{-c_1(t-s)} \|X_n(t) \|_{H}^2 \right) + E \int_s^t e^{-c_1(s-r)} (c_2 \| \mathcal{X}_n(\sigma) \|_{V_1} + c_2' \| \mathcal{X}_n(\sigma) \|_{V_2}^2) \, d\sigma
\]

\[
\leq E \|\zeta^n_s \|_{H}^2 + E \int_s^t e^{-c_1(s-r)} M_0 \, d\sigma.
\]

Thus, by Lebesgue’s dominated convergence theorem, (H4), Remark 4 (ii) and (12), we obtain for \( i = 1, 2 \)

\[
\lim_{n \to \infty} \left\{ \left( E \int_s^t \|A_{i,n}(\sigma, \mathcal{X}(\sigma)) - A_i(\sigma, \mathcal{X}(\sigma)) \|_{V_i}^{\alpha_i} \, d\sigma \right)^{\frac{\alpha_i - 1}{\alpha_i}} \right\} = 0,
\]

\[
\lim_{n \to \infty} E \int_s^t \|B_i(\sigma, \mathcal{X}(\sigma)) - B(\sigma, \mathcal{X}(\sigma)) \|_{L_2(U,H)}^2 \, d\sigma = 0.
\]

Therefore,

\[
\lim_{n \to \infty} \xi_n = 0.
\]

The proof of (i) is complete.
(ii) According to the characterization of convergence in probability in terms of $P$-a.s. convergent subsequences (see, e.g., Dudley [20, Theorem 9.2.1]), we may assume without loss of generality that $\lim_{n \to \infty} \zeta_n = \zeta^* \ P$-a.s. Similar to the proof of (12), we have

$$E \left( e^{c_2 (\tau - s) - \sup_n \| \zeta_n \|_H} \| X_n(\tau) \|^2_H \right)$$

$$+ E \int_s^\tau e^{c_1 (\sigma - s) - \sup_n \| \zeta_n \|_H} \left( c_2 \| X_n(\sigma) \|_{L_1}^2 + c_2 \| X_n(\sigma) \|_{L_2}^2 \right) d\sigma$$

$$\leq E \left( e^{\sup_n \| \zeta_n \|_H} \| \zeta^* \|^2_H \right) + E \int_s^\tau e^{c_1 (\sigma - s) - \sup_n \| \zeta_n \|_H} M_0 d\sigma.$$  

Applying Itô’s formula and the product rule, we get

$$\| X_n(\tau) - X(\tau) \|^2_H = \| \zeta_n^* - \zeta^* \|^2_H e^{-c_2 (\tau - s) - \sup_n \| \zeta_n \|_H}$$

$$+ \int_s^\tau \left( e^{-c_1 (\sigma - s) - \sup_n \| \zeta_n \|_H} \left( 2\langle A_n, X_n(\sigma) \rangle - A(\sigma, X(\sigma)), X_n(\sigma) - X(\sigma) \rangle_{U,H} \right) \right) d\sigma$$

$$+ 2 \int_s^\tau e^{-c_1 (\sigma - s) - \sup_n \| \zeta_n \|_H} \left( X_n(\sigma) - X(\sigma), [B_n, \bar{X}_n(\sigma)] \right)_{U,H} d\sigma$$

Note that the last item is a real-valued local martingale. Hence localizing it, by Lebesgue’s dominated convergence theorem and (H2) we obtain that

$$E \left( \| X_n(\tau) - X(\tau) \|^2_H e^{-c_2 (\tau - s) - \sup_n \| \zeta_n \|_H} \right)$$

$$\leq E \left( \| \zeta_n^* - \zeta^* \|^2_H e^{-c_2 (\tau - s) - \sup_n \| \zeta_n \|_H} \right) + E \int_s^\tau \left( e^{-c_1 (\sigma - s) - \sup_n \| \zeta_n \|_H} \left( 2\langle A_n, X_n(\sigma) \rangle - A(\sigma, X(\sigma)), X_n(\sigma) - X(\sigma) \rangle_{U,H} \right) \right) d\sigma$$

$$+ 2 \int_s^\tau e^{-c_1 (\sigma - s) - \sup_n \| \zeta_n \|_H} \left( X_n(\sigma) - X(\sigma), [B_n, \bar{X}_n(\sigma)] \right)_{U,H} d\sigma$$

Therefore, in view of Gronwall’s lemma, we get

$$E \left( \| X_n(\tau) - X(\tau) \|^2_H e^{-c_2 \tau - s) - \sup_n \| \zeta_n \|_H} \right) \leq L_{\tilde{H}}^2(t-s), \quad \text{for all } \tau \in [s,t],$$  

(14)
where
\[
\hat{\xi}_n := E \left( \| \zeta_n^s - \zeta^s \|_H^2 e^{-\sup_n \| \zeta_n^s \|_H} \right) \\
+ 2 \left( E \int_s^t \| A_1, n(\sigma, \mathbf{X}(\sigma)) - A_1(\sigma, \mathbf{X}(\sigma)) \|_{V_1^1}^{a_1} \, d\sigma \right)^{\frac{a_1-1}{a_1}} \\
\times \left( E \int_s^t e^{-(\sigma-s) - \sup_n \| \zeta_n^s \|_H} \| X_n(\sigma) - \mathbf{X}(\sigma) \|_{V_1^1}^{a_1} \, d\sigma \right)^{\frac{a_1}{a_1}} \\
+ 2 \left( E \int_s^t \| A_2, n(\sigma, \mathbf{X}(\sigma)) - A_2(\sigma, \mathbf{X}(\sigma)) \|_{V_2^1}^{a_2} \, d\sigma \right)^{\frac{a_2-1}{a_2}} \\
\times \left( E \int_s^t e^{-(\sigma-s) - \sup_n \| \zeta_n^s \|_H} \| X_n(\sigma) - \mathbf{X}(\sigma) \|_{V_2^1}^{a_2} \, d\sigma \right)^{\frac{a_2}{a_2}} \\
+ 2 E \int_s^t e^{-(\sigma-s) - \sup_n \| \zeta_n^s \|_H} \| B_n(\sigma, \mathbf{X}(\sigma)) - B(\sigma, \mathbf{X}(\sigma)) \|_{L^2(U, H)}^2 \, d\sigma.
\]

By Lebesgue's dominated convergence theorem, (13), (H4) and Remark 4 (i), we have
\[
\lim_{n \to \infty} \hat{\xi}_n = 0. \tag{15}
\]

For any \( \epsilon > 0 \), let
\[
\tau_n^\epsilon := \inf \{ s \geq t : \| X_n(\sigma) - \mathbf{X}(\sigma) \|_H^2 e^{-(\sigma-s) - \sup_n \| \zeta_n^s \|_H} \geq \epsilon \} \wedge t.
\]

It follows from [51, Lemma 3.1.3] and (14) that
\[
P \left( \sup_{s \leq \sigma \leq t} \| X_n(\sigma) - \mathbf{X}(\sigma) \|_H^2 e^{-(\sigma-s) - \sup_n \| \zeta_n^s \|_H} \geq \epsilon \right) \\
\leq \frac{1}{\epsilon} E \left( \| X_n(\tau_n^\epsilon) - \mathbf{X}(\tau_n^\epsilon) \|_H^2 e^{-(\tau_n^\epsilon-s) - \sup_n \| \zeta_n^{\tau_n^\epsilon} \|_H} \right) \\
\leq \frac{1}{\epsilon} \hat{\xi}_n e^{L_\mathbf{H}(t-s)}.
\]

So by (15) we have
\[
\lim_{n \to \infty} P \left( \sup_{s \leq \sigma \leq t} \| X_n(\sigma) - \mathbf{X}(\sigma) \|_H^2 e^{-(\sigma-s) - \sup_n \| \zeta_n^s \|_H} \geq \epsilon \right) = 0. \tag{16}
\]

Since P-a.s. \([0, \infty) \ni t \mapsto e^{-(\sigma-s) - \sup_n \| \zeta_n^s \|_H} \) is continuous and strictly positive, (16) implies
\[
\lim_{n \to \infty} \sup_{s \leq \sigma \leq t} \| X_n(\sigma) - \mathbf{X}(\sigma) \|_H^2 = 0 \quad \text{in probability.}
\]

This completes the proof of (ii).

(iii) According to the Skorohod representation theorem, the uniqueness in law of the solutions for equation (2) and (ii), we complete the proof of (iii).

\[\square\]

Remark 5. Note that \( \lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n), \mathcal{L}(X)) = 0 \) in \( Pr(C(\mathbb{R}, H)) \) implies
\[
\lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n(t)), \mathcal{L}(X(t))) = 0 \quad \text{in } Pr(H) \text{ for each } t \in \mathbb{R},
\]
but not vice versa. However, under the conditions of Theorem 3.1, it follows from (iii) that \( \lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n(t)), \mathcal{L}(X(t))) = 0 \) in \( Pr(H) \) for \( t \in \mathbb{R} \) if and only if
Lemma 3.3. Assume that \( M \) is bounded. Then there exists a constant \( \lambda > 0 \) such that for all \( u, v \in V \), \( t \in \mathbb{R} \)

\[
2v \cdot (A(t, u) - A(t, v), u - v)_V + \|B(t, u) - B(t, v)\|_{L^2(\Omega, H)}^2 \leq -\lambda \|u - v\|_H^2.
\]

**Lemma 3.2.** Suppose that (H2'), (H3) and (H4) hold. Let \( \eta \in (0, \lambda) \). Then there exists a constant \( M_{0, \eta} \in (0, \infty) \), depending only on \( \eta, c_2, c_3, c_2', M_0, \alpha_i, i = 1, 2 \) such that

\[
2v \cdot (A(t, v), v)_V + \|B(t, v)\|_{L^2(\Omega, H)}^2 \leq -\eta \|v\|_H^2 + M_{0, \eta}
\]

for all \( v \in V \), \( t \in \mathbb{R} \).

**Proof.** Similar to the proof of Lemma 4.3.8 in [51], (17) can be obtained by Young’s inequality, (H2'), (H3) and (H4).

**Lemma 3.3.** Assume that (H1), (H2'), (H3) and (H4) hold. Let \( \zeta_s \in L^2(\Omega, \mathcal{F}_s, P; H) \) and \( X(t, s, \zeta_s), t \geq s \) be the solution to the following Cauchy problem

\[
\begin{aligned}
&dX(t) = A(t, X(t))dt + B(t, X(t))dW(t) \\
&X(s) = \zeta_s.
\end{aligned}
\]

Then there exists a constant \( M_1 > 0 \), depending only on \( M_{0, \eta} \) as in (17), such that

\[
E\|X(t, s, \zeta_s)\|_H^2 \leq e^{-\eta(t-s)}E\|\zeta_s\|_H^2 + M_1.
\]

**Furthermore, for some process \( X(t), t \in \mathbb{R} \), we have**

\[
X(t, -n, 0) \to X(t) \quad \text{in} \quad L^2(\Omega, P; H).
\]

**Proof.** By the product rule, Itô’s formula and (17), we have

\[
E \left( e^{\eta(t-s)}\|X(t, s, \zeta_s)\|_H^2 \right)
\]

\[
= E\|\zeta_s\|_H^2 + E \int_s^t e^{\eta(s-\sigma)} \left( 2V \cdot (A(\sigma, X(\sigma, s, \zeta_s)), X(\sigma, s, \zeta_s))_V \\
+ \|B(\sigma, X(\sigma, s, \zeta_s))\|_{L^2(\Omega, H)}^2 \right) d\sigma + \int_s^t \eta e^{\eta(s-\sigma)}E\|X(\sigma, s, \zeta_s)\|_H^2 d\sigma
\]

\[
\leq E\|\zeta_s\|_H^2 + \int_s^t e^{\eta(s-\sigma)}M_{0, \eta} d\sigma
\]

\[
\leq E\|\zeta_s\|_H^2 + \frac{M_{0, \eta}}{\eta} e^{\eta(t-s)}.
\]
Lemma 3.4. For this we need some uniform estimates.

Proof. According to (H3), we have

\[ E \left( e^{\lambda(t+m)} \| X(t-n,0) - X(t-m,0) \|^2_H \right) \]

Now using (18) we deduce

\[ E \left( \| X(t-n,0) - X(t-m,0) \|^2_H \right) \leq E \left( \| X(t-n,0) \|^2_H e^{-\lambda(t+m)} \right) \leq \frac{M_1 e^{-\lambda(t+m)}}{m} \]

Letting \( n > m \), we have

\[ E \left( \| X(t-n,0) - X(t-m,0) \|^2_H \right) \to 0. \]

Therefore, there exists a process \( X(t), t \in \mathbb{R} \) such that

\[ X(t-n,0) \to X(t) \text{ in } L^2(\Omega, P; H). \]

And it follows from (18) that \( \sup_{t \in \mathbb{R}} E \| X(t) \|^2_H \leq M_1. \)

We now show that the limit process \( X(\cdot) \) in (19) is a solution to equation (2). For this we need some uniform estimates.

Lemma 3.4. Consider equation (2). Assume that (H1), (H2'), (H3) and (H4) hold. For any fixed interval \([a, b] \subset \mathbb{R}\), there exists a constant \( M_2 \), depending only on \( M_1, M_0, c_1 \) and \([a, b]\), such that

\[ \sup_{t \in [a, b]} E \| X(t-n,0) \|^2_H + \sum_{i=1,2} \| X_i(t-n,0) \|_{K_1} + \sum_{i=1,2} \| A_i(t-n,0) \|_{K_i^*} \leq M_2 \]

for all \(-n \leq a\), where \( K_i := L^\alpha([a, b] \times \Omega, dt \otimes P; V_i), K_i^* := L^\alpha([a, b] \times \Omega, dt \otimes P; V_i^*), \) \( i = 1, 2. \)

Proof. According to (H3), we have

\[ E \left( e^{-c_1(t-a)} \| X(t-n,0) \|^2_H \right) \]

\[ = E \| X(a-n,0) \|^2_H + \int_a^t e^{-c_1(-\sigma-a)} E \left( 2 \nu_i (A(\sigma, X(\sigma, -n,0)), X(\sigma, -n,0)) \right) d\sigma \]

\[-c_1 e^{-c_1(-\sigma-a)} E \| X(\sigma, -n,0) \|^2_H d\sigma \]

\[ \leq E \| X(a, -n,0) \|^2_H - c_2 \int_a^t e^{-c_1(-\sigma-a)} E \| X(\sigma, -n,0) \|_{V_1}^2 d\sigma \]

\[ -c_2 \int_a^t e^{-c_1(\sigma-a)} E \| X(\sigma, -n,0) \|_{V_2}^2 d\sigma + \int_a^t e^{-c_1(\sigma-a)} M_0 d\sigma. \]
Consider equation Theorem 3.5. Let

\[ E \left( e^{-c_1(t-a)} \| X(t, -n, 0) \|_{H}^2 \right) + c_2 \int_a^t e^{-c_1(\sigma-a)} E \| X(\sigma, -n, 0) \|_{V_1}^2 \sigma \, d\sigma \]

\[ + c_2' \int_a^t e^{-c_1(\sigma-a)} E \| X(\sigma, -n, 0) \|_{V_2}^2 \sigma \, d\sigma \]

\[ \leq E \| X(a, -n, 0) \|_{H}^2 + \int_a^t e^{-c_1(\sigma-a)} M_0 d\sigma. \]

In view of (H4), we complete the proof.

Employing the classical pullback attraction method in random and non-autonomous dynamics, we have the following theorem on the existence of a unique \( L^2 \)-bounded solution to equation (2) whose distribution has special properties; the similar method has also been used in Da Prato and Tudor [19], Prévôt and Röckner [51] and Gess [28] etc.

**Theorem 3.5.** Consider equation (2). Suppose that (H1), (H2'), (H3) and (H4) hold, then there exists a unique \( L^2 \)-bounded continuous \( H \)-valued solution \( X(t) \), \( t \in \mathbb{R} \) to equation (2). Moreover, the mapping \( \tilde{\mu} : \mathbb{R} \to \text{Pr}(H) \), defined by \( \tilde{\mu}(t) := P \circ [X(t)]^{-1} \), is unique with the following properties:

1. **\( L^2 \)-boundedness:** \( \sup_{t \in \mathbb{R}} |\tilde{\mu}(t)| < +\infty \);
2. **Flow property:** \( \mu(t, s, \tilde{\mu}(s)) = \tilde{\mu}(t) \) for all \( t \geq s \).

Here \( \mu(t, s, \mu_0) \) denotes the distribution of \( X(t, s, \zeta_s) \) on \( H \), with \( \mu_0 = P \circ \zeta_s^{-1} \).

**Proof.** For any fixed interval \([a, b] \subset \mathbb{R}\), we denote

\[ J := L^2([a, b] \times \Omega, dt \otimes P; L_2(U, H)) , \quad K_i := L^{\alpha_i}([a, b] \times \Omega, dt \otimes P; V_i), \]

\[ K_i^* := L^{\alpha_i^*}([a, b] \times \Omega, dt \otimes P; V_i^*), \quad i = 1, 2. \]

According to the reflexivity of \( K_i \), \( i = 1, 2 \), we may assume, going if necessary to a subsequence, that

1. \( X(\cdot, -n, 0) \to X(\cdot) \) in \( L^2([a, b] \times \Omega, dt \otimes P; H) \) and \( \overline{X}(\cdot, -n, 0) \to \overline{X}(\cdot) \) weakly in \( K_1 \) and \( K_2 \);
2. \( A_i(\cdot, \overline{X}(\cdot, -n, 0)) \to Y_i(\cdot) \) weakly in \( K_i^* \), \( i = 1, 2 \);
3. \( B(\cdot, \overline{X}(\cdot, -n, 0)) \to Z(\cdot) \) weakly in \( J \) and hence

\[ \int_a^t B(\sigma, \overline{X}(\sigma, -n, 0)) dW(\sigma) \to \int_a^t Z(\sigma) dW(\sigma) \]

weakly* in \( L^\infty([a, b], dt; L^2(\Omega, P; H)) \).

Thus for all \( v \in V, \varphi \in L^\infty([a, b] \times \Omega) \) by Fubini’s theorem we get

\[ E \int_a^b \nu \cdot \langle X(t), \varphi(t)v \rangle dt \]

\[ = \lim_{n \to \infty} E \int_a^b \nu \cdot \langle X(t, -n, 0), \varphi(t)v \rangle dt \]

\[ = \lim_{n \to \infty} E \int_a^b \nu \cdot \langle X(a, -n, 0), \varphi(t)v \rangle dt \]

\[ + \lim_{n \to \infty} E \int_a^b \nu \cdot \langle A(\sigma, \overline{X}(\sigma, -n, 0)), \varphi(t)v \rangle dt d\sigma \]

\[ = \lim_{n \to \infty} E \int_a^b \nu \cdot \langle \tilde{\mu}(t), \varphi(t)v \rangle dt \]

\[ = E \int_a^b \nu \cdot \langle \tilde{\mu}(t), \varphi(t)v \rangle dt \]

\[ = E \int_a^b \nu \cdot \langle \tilde{\mu}(t), \varphi(t)v \rangle dt \],
To this end, for any $\xi \in K_1 \cap K_2 \cap L^2([a,b] \times \Omega, dt \otimes P; H)$, we have

$$E\|X(t,-n,0)\|^2_H - E\|X(a,-n,0)\|_H^2$$

$$= E \int_a^t \left( 2\langle A_1(\sigma, \mathcal{X}(\sigma, -n, 0)), \mathcal{X}(\sigma, -n, 0) \rangle_{V_1} + 2\langle A_2(\sigma, \mathcal{X}(\sigma, -n, 0)), \mathcal{X}(\sigma, -n, 0) \rangle_{V_2} + \|B(\sigma, \mathcal{X}(\sigma, -n, 0))\|_{L^2(U,H)}^2 \right) d\sigma$$

$$\leq E \int_a^t \left( 2\langle A_1(\sigma, \mathcal{X}(\sigma, -n, 0)), \mathcal{X}(\sigma, -n, 0) \rangle_{V_1} + 2\langle A_2(\sigma, \mathcal{X}(\sigma, -n, 0)), \mathcal{X}(\sigma, -n, 0) \rangle_{V_2} + \|B(\sigma, \mathcal{X}(\sigma, -n, 0))\|_{L^2(U,H)}^2 \right) d\sigma$$

For every nonnegative $\psi \in L^\infty([a,b] \times \Omega, dt \otimes P; \mathbb{R})$, it follows from (H2') that

$$E \int_a^b \psi(t) (\|X(t,-n,0)\|_H^2 - \|X(a,-n,0)\|_H^2) dt$$

(20)
\[
\leq E\left( \int_{a}^{b} \psi(t) \left( 2V_{i}^{*} (A_{1} (\sigma, \overline{X} (\sigma, -n, 0)) - A_{1} (\sigma, \phi(\sigma)), \phi(\sigma)) \right) V_{1} \\
+ 2V_{i}^{*} (A_{1} (\sigma, \phi(\sigma)), \overline{X} (\sigma, -n, 0)) V_{1} + 2V_{2}^{*} (A_{2} (\sigma, \phi(\sigma)), \overline{X} (\sigma, -n, 0)) V_{2} \\
+ 2V_{2}^{*} (A_{2} (\sigma, \overline{X} (\sigma, -n, 0)) - A_{2} (\sigma, \phi(\sigma)), \phi(\sigma)) V_{2} \\
+ 2 \langle B (\sigma, \overline{X} (\sigma, -n, 0)), B (\sigma, \phi(\sigma)) \rangle_{L_{2}(V, H)} - \| B (\sigma, \phi(\sigma)) \|_{L_{2}(V, H)}^{2} \right) \right) dt.
\]

Using (1) we obtain
\[
E \int_{a}^{b} \psi(t) \| X(t) \|_{H}^{2} dt = \lim_{n \to \infty} E \int_{a}^{b} \langle \psi(t) X(t), X(t, -n, 0) \rangle_{H} dt \\
\leq \left( E \int_{a}^{b} \psi(t) \| X(t) \|_{H}^{2} dt \right)^{2} \lim_{n \to \infty} \left( E \int_{a}^{b} \psi(t) \| X(t, -n, 0) \|_{H}^{2} dt \right)^{2}.
\]

Then letting \( n \to \infty \) in (20), we have
\[
E \int_{a}^{b} \psi(t) \left( \| X(t) \|_{H}^{2} - \| X(a) \|_{H}^{2} \right) dt \leq E \left( \int_{a}^{b} \psi(t) \left( 2V_{1}^{*} \langle Y_{1} (\sigma) - A_{1} (\sigma, \phi(\sigma)), \phi(\sigma) \rangle V_{1} + 2V_{2}^{*} \langle A_{2} (\sigma, \phi(\sigma)), \overline{X} (\sigma) \rangle V_{2} \\
+ 2V_{2}^{*} (Y_{2} (\sigma) - A_{2} (\sigma, \phi(\sigma)), \phi(\sigma)) V_{2} + 2V_{2}^{*} (A_{2} (\sigma, \phi(\sigma)), \overline{X} (\sigma)) V_{2} \\
+ 2 \langle Z (\sigma), B (\sigma, \phi(\sigma)) \rangle_{L_{2}(V, H)} - \| B (\sigma, \phi(\sigma)) \|_{L_{2}(V, H)}^{2} \right) \right) dt.
\]

And in view of the product rule, we get
\[
E \int_{a}^{b} \psi(t) \left( \| X(t) \|_{H}^{2} - \| X(a) \|_{H}^{2} \right) dt \leq E \left( \int_{a}^{b} \psi(t) \left( 2V_{1}^{*} \langle Y_{1} (\sigma), \overline{X} (\sigma) \rangle V_{1} + 2V_{2}^{*} \langle Y_{2} (\sigma), \overline{X} (\sigma) \rangle V_{2} \\
+ \| Z (\sigma) \|_{L_{2}(V, H)}^{2} \right) \right) dt.
\]

Therefore, (21) and (22) imply
\[
0 \geq E \left( \int_{a}^{b} \psi(t) \left( 2V_{1}^{*} \langle Y (\sigma) - A(\sigma, \phi(\sigma)), \overline{X} (\sigma) - \phi(\sigma) \rangle V \\
+ \| B (\sigma, \phi(\sigma)) - Z (\sigma) \|_{L_{2}(V, H)}^{2} \right) \right) dt.
\]
Taking \( \phi = \overline{X} \) in (23), we have \( Z = B(\cdot, \overline{X}) \), \( dt \otimes P \)-a.e. Then, applying (23) to \( \phi = \overline{X} - \epsilon \tilde{\phi} v \) for \( \epsilon > 0 \) and \( \tilde{\phi} \in L^\infty([a, b] \times \Omega, dt \otimes P; \mathbb{R}) \), \( v \in V \), we have
\[
E \left( \int_a^b \psi(t) \int_a^t 2 \nu^\cdot (Y(\sigma) - A(\sigma, \overline{X}(\sigma)) - \epsilon \tilde{\phi}(\sigma)v, \epsilon \tilde{\phi}(\sigma)v) v d\sigma dt \right) \leq 0.
\]
Dividing both sides by \( \epsilon \) and letting \( \epsilon \to 0 \), according to Lebesgue’s dominated convergence theorem, (H1) and (H4), we obtain
\[
E \left( \int_a^b \psi(t) \int_a^t \tilde{\phi}(\sigma)v^\cdot (Y(\sigma) - A(\sigma, \overline{X}(\sigma)), v) v d\sigma dt \right) \leq 0.
\]
By the arbitrariness of \( \psi \), \( \tilde{\phi} \) and \( v \), we conclude that \( Y = A(\cdot, \overline{X}) \), \( dt \otimes P \)-a.e. This completes the existence proof, i.e.
\[
X(t) = X(a) + \int_a^t A(\sigma, \overline{X}(\sigma)) d\sigma + \int_a^t B(\sigma, \overline{X}(\sigma)) dW(\sigma), \quad dt \otimes P \text{-a.e.}
\]
In view of the arbitrariness of interval \([a, b] \subset \mathbb{R}\), we conclude that \( X(\cdot) \) is a solution on \( \mathbb{R} \). It follows from (18) and (19) that \( \sup_{t \in \mathbb{R}} E\|X(t)\|_H^2 < \infty \).

Now, we prove the uniqueness of \( L^2 \)-bounded solutions. Suppose that \( X(\cdot) \) and \( Y(\cdot) \) are two \( L^2 \)-bounded continuous solutions to equation (2). By (H2') we have
\[
E\|X(t) - Y(t)\|_H^2
= E\|X(t, -n, X(-n)) - Y(t, -n, Y(-n))\|_H^2
\leq e^{-\lambda(t+n)} E\|X(-n) - Y(-n)\|_H^2 \to 0, \quad \text{as } n \to \infty.
\]

The goal next is to prove that \( \tilde{\mu} \) is unique with the properties (i) and (ii). Note that
\[
\sup_{t \in \mathbb{R}} \int_{\mathcal{H}} \|x\|_H^2 \tilde{\mu}(t)(dx) = \sup_{t \in \mathbb{R}} E\|X(t)\|_H^2 < \infty.
\]
In view of the Chapman-Kolmogorov equation, we have
\[
\mu(t, s, L(X(s, -n, 0))) = L(X(t, -n, 0)).
\]
Then according to the Feller property (see, e.g. [51, Proposition 4.2.10]), we get
\[
\mu(t, s, \tilde{\mu}(s)) = \tilde{\mu}(t).
\]
Suppose that \( \mu_1 \) and \( \mu_2 \) satisfy properties (i) and (ii), let \( \zeta_{n,1} \) and \( \zeta_{n,2} \) be random variables with the distributions \( \mu_1(-n) \) and \( \mu_2(-n) \) respectively. Then consider the solutions \( X(t, -n, \zeta_{n,1}) \) and \( X(t, -n, \zeta_{n,2}) \) on \([-n, \infty)\), we have
\[
d_{BL}(\mu_1(t), \mu_2(t))
= d_{BL}(\mu(t, -n, \mu_1(-n)), \mu(t, -n, \mu_2(-n)))
= \sup_{\|f\|_{BL} \leq 1} \left| \int_\Omega f(x) d \left( \mu(t, -n, \mu_1(-n)) - \mu(t, -n, \mu_2(-n)) \right) \right|
= \sup_{\|f\|_{BL} \leq 1} \left| \int_{\mathcal{H}} \left[ f(X(t, -n, \zeta_{n,1})) - f(X(t, -n, \zeta_{n,2})) \right] dP \right|
\leq (E\|X(t, -n, \zeta_{n,1}) - X(t, -n, \zeta_{n,2})\|_H^2)^{1/2}
\leq e^{-\frac{\lambda}{2}(t+n)} (E\|\zeta_{n,1} - \zeta_{n,2}\|_H^2)^{1/2} \to 0, \quad \text{as } n \to \infty.
\]
Thus, \( \mu_1(t) = \mu_2(t) \) for all \( t \in \mathbb{R} \).
Remark 6. Suppose that (H1), (H2), (H3) and (H4) hold. Assume further that there exist constants \( \eta > 0 \) and \( M_0 > 0 \) such that
\[
2\nu \cdot (A(t, v), v)_V + \|B(t, v)\|_{L_2(U, H)}^2 \leq -\eta \|v\|_H^2 + M_0
\]
for all \( v \in V, t \in \mathbb{R} \). Then there exists an \( L^2 \)-bounded solution \( \tilde{X}(t) \), \( t \in \mathbb{R} \) to equation (2). Indeed it can be verified that
\[
\sup_{t \geq -n} E\|X(t, -n, 0)\|_H^2 \leq \hat{M}_1
\]
for some constant \( \hat{M}_1 > 0 \) depending only on \( M_0 \) and \( \eta \). Therefore, we have, for a subsequence \( n_k \to \infty \), \( X(\cdot, -n_k, 0) \to \tilde{X}(\cdot) \) weakly* in \( L^\infty_{\text{loc}}(\mathbb{R}; L^2(\Omega, P; H)) \) and \( \sup t \in \mathbb{R} E\|\tilde{X}(t)\|_H^2 \leq \hat{M}_1 \). Similar to the proof of the existence of \( L^2 \)-bounded solution in Theorem 3.5, we can show that \( \tilde{X}(\cdot) \) is an \( L^2 \)-bounded solution for equation (2). But note that we cannot obtain the uniqueness of \( L^2 \)-bounded solution to equation (2) under the above conditions.

4. Recurrent solutions. In this section, we show that the \( L^2 \)-bounded solution for equation (2) has the same character of recurrence as coefficients \( A \) and \( B \).

4.1. Periodic solutions. The following theorem shows that the \( L^2 \)-bounded solution for equation (2) is periodic in distribution provided the coefficients \( A \) and \( B \) are periodic.

Theorem 4.1. Consider equation (2). Suppose that (H1), (H2'), (H3), (H4) and (HL) hold. Assume further that the mappings \( A \) and \( B \) are \( T \)-periodic in \( t \). Then the unique \( L^2 \)-bounded solution is \( T \)-periodic in distribution.

In particular, the unique \( L^2 \)-bounded solution is stationary, provided \( A \) and \( B \) are independent of \( t \).

Proof. Define the transition probability function as follows
\[
p_{t,s}(x, dy) := P \circ (X(t, s, x))^{-1}(dy), \quad s \leq t, \ x \in H.
\]
We now check that \( p_{t,s}(x, dy) \) is \( T \)-periodic, provided the coefficients of equation (2) are \( T \)-periodic. For any \( t \geq s \)
\[
X(t + T, s + T, x) = x + \int_{s + T}^{t + T} A(\sigma, \overline{X}(\sigma, s + T, x))d\sigma + \int_{s + T}^{t + T} B(\sigma, \overline{X}(\sigma, s + T, x))dW(\sigma)
\]
\[
= x + \int_{s + T}^{t + T} A(\sigma + T, \overline{X}(\sigma + T, s + T, x))d\sigma + \int_{s + T}^{t + T} B(\sigma + T, \overline{X}(\sigma + T, s + T, x))dW(\sigma)
\]
\[
= x + \int_{s}^{t} A(\sigma, \overline{X}(\sigma + T, s + T, x))d\sigma + \int_{s}^{t} B(\sigma, \overline{X}(\sigma + T, s + T, x))d\overline{W}(\sigma),
\]
where \( \overline{W}(\cdot) = W(\cdot + T) - W(T) \).

In order to indicate the dependence of the solution \( X(t, s, x), t \in [s, \infty) \) of equation (2) on the Wiener process, we write \( X^W(t, s, x) \) instead of \( X(t, s, x) \). Similarly, we write \( p^W_{t,s}(x, dy) \) instead of \( p_{t,s}(x, dy) \). So by the uniqueness of the solutions
to equation (2), for any \( t \in [s, \infty) \), \( X^W(t+T, s+T, x) = X^W(t, s, x) \), P-a.e. In particular, we have
\[
\hat{p}^W_{t+T, s+T}(x, dy) = P \circ (X^W(t+T, s+T, x))^{-1}(dy)
= P \circ (X^W(t, s, x))^{-1}(dy)
= P \circ (X^W(t, s, x))^{-1}(dy) = \hat{p}^W_{t, s}(x, dy),
\]
where the third equality follows from Yamada-Watanabe theorem (see Röckner et al [52]).

Now we prove that \( \hat{\mu}(t) \), \( t \in \mathbb{R} \) is \( T \)-periodic in \( Pr(H) \), recalling that \( \hat{\mu}(t) = \mathcal{L}(X(t)) \), \( t \in \mathbb{R} \) is the distribution of the unique \( L^2 \)-bounded solution \( X(\cdot) \) on \( H \).

For any \( \phi \in C_b(H) \), we have
\[
\int_H \phi(x) \hat{\mu}(t+T)(dx) = \int_H \phi(x)(P \circ (X(t+T)))^{-1})(dx)
= \lim_{n \to \infty} \int_H \phi(x)(P \circ (X(t+T, -n+T, 0)))^{-1})(dx)
= \lim_{n \to \infty} \int_H \phi(x)(P \circ (X(t, -n, 0)))^{-1})(dx)
= \int_H \phi(x) \hat{\mu}(t)(dx).
\]
The \( T \)-periodicity of the distribution of \( X(\cdot) \) on \( C(\mathbb{R}, H) \) now follows from the uniqueness in law of the solutions for equation (2). The proof is complete. \( \square \)

4.2. Almost periodic solutions. In the sequel, we show that the \( L^2 \)-bounded solution of equation (2) is almost periodic in distribution, if the coefficients \( A \) and \( B \) are uniformly almost periodic. To this end, we need the tightness of the family of distributions \( \{P \circ [X(t)]\} \) \( t \in \mathbb{R} \). Note that \( \{P \circ [X(t)]\} \) \( t \in \mathbb{R} \) is tight provided \( \dim H < \infty \). But when \( \dim H = \infty \), we need the following condition (H5) to get the tightness of \( \{P \circ [X(t)]\} \) \( t \in \mathbb{R} \). This condition was used in Liu [43] to study the invariance of subspaces and in Gess et al [29] to study random attractors; the concept of \( S \)-invariance has also been used in Barbu and Röckner [5], Gess and Röckner [30, 31], Gess and Tölle [32] etc to investigate the regularity and related problems.

(H5) Assume that there exists a closed subset \( S \subset H \) equipped with the norm \( \| \cdot \|_S \) such that \( V \subset S \) is continuous and \( S \subset H \) is compact. Let \( T_n \) be a sequence of positive definite self-adjoint operators on \( H \) such that for each \( n \geq 1 \),
\[
\langle x, y \rangle_n := \langle x, T_n y \rangle_H, \quad x, y \in H,
\]
defines a new inner product on \( H \). Assume further that the norms \( \| \cdot \|_n \) generated by \( \langle \cdot, \cdot \rangle_n \) are all equivalent to \( \| \cdot \|_H \) and for all \( x \in S \) we have
\[
\|x\|_n \uparrow \|x\|_S \quad \text{as } n \to \infty.
\]
Furthermore, we suppose that \( T_n : V \to V \) is continuous and there exist constants \( C > 0 \), \( M_0 > 0 \) such that
\[
2v \cdot \langle A(t, v), T_n v \rangle + \|B(t, v)\|_{L^2(U, H_n)}^2 \leq -C\|v\|_n^2 + M_0
\]
for all \( u, v \in V \), \( t \in \mathbb{R} \) and \( n \geq 1 \).

Let \( H_n := \langle H, \langle \cdot, \cdot \rangle_n \rangle \). We denote by \( i_n \) the Riesz isomorphism from \( H_n \) into \( H_n^* \). Similarly, \( i : H \to H^* \).
Lemma 4.2 (See Liu [43]). If $T_n : V \to V$ is continuous, then $i_n \circ i^{-1} : H^* \to H_n^*$ is continuous with respect to $\| \cdot \|_V$. Therefore, there exists a unique extension $I_n$ of $i_n \circ i^{-1}$ on $V^*$ such that for all $f \in V^*, v \in V$

$$v \cdot \langle I_n f, v \rangle_V = v \cdot \langle f, T_n v \rangle_V.$$  

(24)

Proposition 1. Consider equation (2). Suppose that (H1), (H2'), (H3), (H4) and (H5) hold, and that $\zeta_s \in L^2(\Omega, F_s, P; H)$. Let $X(t, s, \zeta_s), t \geq s$ be the solution to equation (2) with initial condition $X(s) = \zeta_s$. Then the $L^2$-bounded solution $X(\cdot)$ satisfies

$$\sup_{t \in \mathbb{R}} E\|X(t)\|_S^2 < \infty.$$  

In particular, the family of distributions $\{P \circ [X(t)]^{-1}\}_{t \in \mathbb{R}}$ is tight.

Proof. Note that $X(t, s, \zeta_s), t \geq s$ satisfies

$$iX(t, s, \zeta_s) = i\zeta_s + \int_s^t A(\sigma, \overline{X}(\sigma, s, \zeta_s))d\sigma + i \left( \int_s^t B(\sigma, \overline{X}(\sigma, s, \zeta_s))dW(\sigma) \right).$$  

(25)

According to Lemma 4.2, applying $I_n$ to (25) we obtain

$$i_nX(t, s, \zeta_s) = i_n\zeta_s + \int_s^t I_nA(\sigma, \overline{X}(\sigma, s, \zeta_s))d\sigma + i_n \left( \int_s^t B(\sigma, \overline{X}(\sigma, s, \zeta_s))dW(\sigma) \right).$$  

Then using Itô’s formula on the new Gelfand triple

$$V \subset H_n \simeq H_n^* \subset V^*,$$

we get

$$\|X(t, s, \zeta_s)\|_n^2 = \|\zeta_s\|_n^2 + \int_s^t \left( 2V \cdot \langle I_n A(\sigma, \overline{X}(\sigma, s, \zeta_s)), \overline{X}(\sigma, s, \zeta_s) \rangle_V + \|B(\sigma, \overline{X}(\sigma, s, \zeta_s))\|_{L_2(U, H_n)}^2 \right) d\sigma + 2 \int_s^t \langle X(\sigma, s, \zeta_s), B(\sigma, \overline{X}(\sigma, s, \zeta_s))dW(\sigma) \rangle_n.$$  

See [43] for details. By the product rule, (24) and (H5) we have

$$E \left( e^{C(t-s)} \|X(t, s, \zeta_s)\|_n^2 \right)$$

$$= E\|\zeta_s\|_n^2 + E\int_s^t e^{C(\sigma-s)} \left( 2V \cdot \langle I_n A(\sigma, \overline{X}(\sigma, s, \zeta_s)), \overline{X}(\sigma, s, \zeta_s) \rangle_V + \|B(\sigma, \overline{X}(\sigma, s, \zeta_s))\|_{L_2(U, H_n)}^2 \right) d\sigma + E\int_s^t Ce^{C(\sigma-s)} \|X(\sigma, s, \zeta_s)\|_n^2 d\sigma$$

$$\leq E\|\zeta_s\|_n^2 + \int_s^t e^{C(\sigma-s)} M_0 d\sigma$$

$$\leq E\|\zeta_s\|_n^2 + \frac{M_0}{C} e^{C(t-s)}.$$  

Therefore, we obtain

$$E\|X(t, s, \zeta_s)\|_n^2 \leq e^{-C(t-s)} E\|\zeta_s\|_n^2 + \frac{M_0}{C}.$$
In particular, we have
\[ E\|X(t, -m, 0)\|^2_n \leq \frac{M_0}{C}. \]
Thus we may assume, going if necessary to a subsequence, that
\[ X(t, -m, 0) \to X(t) \quad \text{weakly in } L^2(\Omega, P; H_n). \]

Then Fatou’s lemma yields that
\[ E\|X(t)\|^2_n = E \liminf_{m \to \infty} E\|X(t, -m, 0)\|^2_n \leq \frac{M_0}{C}. \]
Moreover, we have
\[ E\|X(t)\|^2_n = E \lim_{n \to \infty} \|X(t)\|^2_n \leq \liminf_{n \to \infty} E\|X(t)\|^2_n \leq \frac{M_0}{C}. \]

The tightness of \( \{P \circ [X(t)]^{-1}\}_{t \in \mathbb{R}} \) is an easy consequence of the compactness of the embedding \( S \subset H \).

**Remark 7.** Note that Gess [27] gave a different technique to obtain the compactness of random dynamical systems generated by stochastic singular evolution equations. That is, he got the compactness of the solution mapping for any fixed sample point. Since the recurrence we are concerned with in this paper is in distribution sense instead of in pathwise sense, it seems that the technique in [27] is not applicable to our problem.

**Proposition 2.** Consider equation (2). Suppose that \( A, B, A_n, B_n \) satisfy (H1), (H2’), (H3), (H4) and (HL) with the same constants \( \lambda, c_1, c_2, c_3, c_1', c_3', M_0, \alpha_i, i = 1, 2 \) and \( L_B \). Let \( X(\cdot), X_n(\cdot) \) be the \( L^2 \)-bounded solutions of equation (2) corresponding to \( A, B \) and \( A_n, B_n \) respectively. Assume in addition that
1. \( \lim_{n \to \infty} \|A_{i,n}(t,x) - A_i(t,x)\|_{V^*_i} = 0 \) for all \( x \in V, t \in \mathbb{R}, i = 1, 2; \)
2. \( \lim_{n \to \infty} \|B_n(t,x) - B(t,x)\|_{L_2(U,H)} = 0 \) for all \( x \in V, t \in \mathbb{R}; \)
3. for each \( t \in \mathbb{R} \) the family of distributions \( \{P \circ [X_n(t)]^{-1}\}_{n \in \mathbb{N}} \) is tight.

Then
\[ \lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n), \mathcal{L}(X)) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)). \]

In particular,
\[ \lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n(t+\cdot)), \mathcal{L}(X(t+\cdot))) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)) \text{ for all } t \in \mathbb{R}. \]

**Proof.** According to Remark 5, we only need to prove that
\[ \lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n(t)), \mathcal{L}(X(t))) = 0 \quad \text{in } Pr(H) \]
for every \( t \in \mathbb{R} \). To this end, it suffices to show that for every sequence \( \gamma' = \{\gamma'_k\} := \{\gamma_k\}_{k=1}^\infty \subset \mathbb{N} \), there exists a subsequence \( \gamma = \{\gamma_k\} \) of \( \gamma' \) such that for every \( t \in \mathbb{R} \)
\[ \lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}(t)), \mathcal{L}(X(t))) = 0 \quad \text{in } Pr(H). \]

For every \( r \geq 1 \), according to the tightness of \( \{\mathcal{L}(X_{\gamma_k}(-r))\} \), there exists a subsequence \( \{\gamma_k\} \subset \gamma' \) such that \( \mathcal{L}(X_{\gamma_k}(-r)) \) converges weakly to some probability measure \( \mu_r \) in \( Pr(H) \). Let \( \xi_r \) be a random variable with distribution \( \mu_r \). Define \( Y_r(t) := X(t, -r, \xi_r) \), recalling that \( X(t, -r, \xi_r), t \in [-r, +\infty) \) is the solution to the following Cauchy problem
\[
\begin{align*}
&dX(t) = A(t, X(t))dt + B(t, X(t))dW(t) \\
&X(-r) = \xi_r.
\end{align*}
\]
In view of Theorem 3.1, we have
\[
\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}), \mathcal{L}(Y_r)) = 0 \quad \text{in } \Pr(C([-r, +\infty), H)).
\]
Since \(\{\mathcal{L}(X_{\gamma_k}(r-1))\}\) is tight, going if necessary to a subsequence, we can assume that \(\mathcal{L}(X_{\gamma_k}(r-1))\) converges weakly to some probability measure \(\mu_{r+1}\) in \(\Pr(H)\).

Let \(\xi_{r+1}\) be a random variable with distribution \(\mu_{r+1}\). In light of Theorem 3.1, we have
\[
\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}), \mathcal{L}(Y_{r+1})) = 0 \quad \text{in } \Pr(C([-r-1, +\infty), H)),
\]
where \(Y_{r+1}(t) := X(t, r-1, \xi_{r+1}), t \in [-r-1, +\infty)\). Therefore, we have
\[
d_{BL}(\mathcal{L}(Y_r), \mathcal{L}(Y_{r+1})) = 0 \quad \text{in } \Pr(C([-r, +\infty), H)).
\]
In particular, \(\mathcal{L}(Y_r(t)) = \mathcal{L}(Y_{r+1}(t)),\) for all \(t \geq -r\).

Define \(\nu(t) := \mathcal{L}(Y_r(t)), t \geq -r\). We use a standard diagonal argument to extract a subsequence which we still denote by \(\{X_{\gamma_k}\}\) satisfying
\[
\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}(t)), \nu(t)) = 0 \quad \text{in } \Pr(H)
\]
for every \(t \in \mathbb{R}\). Note that \(\sup_{t \in \mathbb{R}} \int_H ||x||_2^2 \nu(t)(dx) < +\infty\). And we have \(P\text{-a.e.}\)
\[
Y_r(t) = Y_r(s) + \int_s^t A(\sigma, \gamma(\sigma))d\sigma + \int_s^t B(\sigma, \gamma(\sigma))dW(\sigma), \quad \text{where } t \geq s \geq -r.
\]
By the uniqueness in law of the solutions for equation (2), we have \(\mathcal{L}(Y_r(t)) = \mu(t, s, \mathcal{L}(Y_r(s)), t \geq s \geq -r,\) i.e. \(\nu(t) = \mu(t, s, \nu(s)), t \geq s\). In view of Theorem 3.5, we obtain \(\nu = \tilde{\mu}\). Therefore, for every \(t \in \mathbb{R}\), we have
\[
\lim_{k \to \infty} d_{BL}(\mathcal{L}(X_{\gamma_k}(t)), \mathcal{L}(X(t))) = 0 \quad \text{in } \Pr(H).
\]
\[\square\]

The following result shows that the \(L^2\)-bounded solution is almost periodic in distribution provided \(A\) and \(B\) are uniformly almost periodic.

**Theorem 4.3.** Consider equation (2). Suppose that (H1), (H2'), (H3), (H4), (H5) and (HL) hold. Assume further that the mappings \(A_1, A_2\) and \(B\) are uniformly almost periodic. Then the unique \(L^2\)-bounded solution is almost periodic in distribution.

**Proof.** Let \(\gamma' = \{\gamma'_n\}\) and \(\beta' = \{\beta'_n\}\) be two sequences in \(\mathbb{R}\). By Definition 2.10 and Theorem 2.4, it suffices to show that there exist two subsequences \(\gamma = \{\gamma_n\} \subset \gamma' = \{\gamma'_n\}\) and \(\beta = \{\beta_n\} \subset \beta' = \{\beta'_n\}\) with the same indexes such that for every \(t \in \mathbb{R}\)
\[
\lim_{n \to \infty} \lim_{m \to \infty} \tilde{\mu}(t + \gamma_n + \beta_m), \quad \lim_{n \to \infty} \tilde{\mu}(t + \gamma_n + \beta_n)
\]
exist and equal.

In fact, since \(A_1, A_2\) and \(B\) are uniformly almost periodic, there exist \(\gamma = \{\gamma_n\} \subset \gamma'\) and \(\beta = \{\beta_n\} \subset \beta'\) with the same indexes such that \(T_\beta A_i, T_\beta B, T_\gamma T_\beta A_i, T_\beta B, T_{\gamma + \beta} A_i, i = 1, 2\) and \(T_{\gamma + \beta} B\) exist uniformly with respect to \(t \in \mathbb{R}\) and \(x \in Q\), where \(Q\) is an arbitrary compact subset of \(V\). Furthermore, we have
\[
T_\gamma T_\beta A_i = T_{\gamma + \beta} A_i, \quad i = 1, 2, \quad T_\gamma T_\beta B = T_{\gamma + \beta} B.
\]
It can be verified that \(T_\beta A_i, T_\beta B, T_\gamma T_\beta A_i, i = 1, 2\) and \(T_\gamma T_\beta B\) satisfy (H1), (H2'), (H3), (H4), (H5) and (HL) with the same constants \(\lambda, c_1, c_2, c_3, c'_2, c'_3, M_0, \alpha_i, i = 1, 2\) and \(L_B\).
Let $Y(\cdot)$, $Z_1(\cdot)$ be the unique $L^2$-bounded solutions of equation (2) with coefficients $T_\beta A_i$, $T_\beta B$ and $T_\gamma T_\beta A_i$, $T_\gamma T_\beta B$, $i = 1, 2$, respectively. In view of Proposition 2 and the uniqueness in law of the solutions for equation (2), we obtain
\[
\lim_{m \to \infty} d_{BL}(\mathcal{L}(X(\cdot + \beta_m)), \mathcal{L}(Y)) = 0 \quad \text{in } Pr(C(\mathbb{R}, H))
\] (26)
and
\[
\lim_{n \to \infty} d_{BL}(\mathcal{L}(Y(\cdot + \gamma_n)), \mathcal{L}(Z_1)) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)).
\] (27)
Similarly, we have
\[
\lim_{n \to \infty} d_{BL}(\mathcal{L}(X(\cdot + \gamma_n + \beta_n)), \mathcal{L}(Z_2)) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)),
\] (28)
where $Z_2(\cdot)$ is the unique $L^2$-bounded solution to the following equation
\[
dX(t) = T_{\gamma+\beta}A(t, X(t))dt + T_{\gamma+\beta}B(t, X(t))dW(t).
\]
Since the $L^2$-bounded solution to equation (2) is unique, (26)–(28) imply
\[
\lim_{n \to \infty} \lim_{m \to \infty} d_{BL}(\mathcal{L}(X(\cdot + \gamma_n + \beta_m)), \mathcal{L}(Z)) = d_{BL}(\mathcal{L}(\tilde{X}(\cdot + \gamma_n + \beta_n)), \mathcal{L}(Z)) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)),
\]
where $Z := Z_1 = Z_2$. Thus
\[
T_{\gamma} T_{\beta} \tilde{\mu} = T_{\gamma+\beta} \tilde{\mu}.
\]
The proof is complete. \qed

4.3. **Almost automorphic solutions.** In this subsection, suppose that mappings $A$ and $B$ of equation (2) are uniformly almost automorphic. Then we prove that the $L^2$-bounded solution $X(\cdot)$ is almost automorphic in distribution.

**Theorem 4.4.** Consider equation (2). Assume that (H1), (H2'), (H3), (H4), (H5) and (HL) hold. Suppose further that mappings $A_i$, $i = 1, 2$ and $B$ are uniformly almost automorphic. Then the unique $L^2$-bounded solution is almost automorphic in distribution.

**Proof.** Since $A_1$, $A_2$ and $B$ are uniformly almost automorphic, for any sequence $\beta' = \{\beta'_n\}$ in $\mathbb{R}$ there exists a subsequence $\beta = \{\beta_n\} \subset \beta'$ such that
\[
\lim_{n \to \infty} A_i(t + \beta_n, x) = \tilde{A}_i(t, x), \quad i = 1, 2, \quad \lim_{n \to \infty} B(t + \beta_n, x) = \tilde{B}(t, x)
\]
and
\[
\lim_{n \to \infty} \tilde{A}_i(t - \beta_n, x) = A_i(t, x), \quad i = 1, 2, \quad \lim_{n \to \infty} \tilde{B}(t - \beta_n, x) = B(t, x).
\]
These limits exist uniformly with respect to $[a, b] \times Q$, where $[a, b]$ is an arbitrary finite interval and $Q$ an arbitrary compact subset of $V$. Note that $\tilde{A}_i$, $i = 1, 2$ and $\tilde{B}$ satisfy (H1), (H2'), (H3), (H4), (H5) and (HL) with the same constants $\lambda$, $c_1$, $c_2$, $c_3$, $c'_2$, $c'_3$, $M_0$, $\alpha_1$, $i = 1, 2$ and $L_B$.

Let $Y$ be the unique $L^2$-bounded solution of equation (2) corresponding to $\tilde{A}_1$, $i = 1, 2$ and $\tilde{B}$. In view of Proposition 2 and the uniqueness in law of the solutions for equation (2), we obtain
\[
\lim_{n \to \infty} d_{BL}(\mathcal{L}(X(\cdot + \beta_n)), \mathcal{L}(Y)) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)).
\]
Similarly, we have
\[
\lim_{n \to \infty} d_{BL}(\mathcal{L}(Y(\cdot - \beta_n)), \mathcal{L}(X)) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)).
\]
The proof is complete. \qed
5. SPDEs with additive noise. In this section we consider the following stochastic partial differential equation driven by additive noise

\[ dX(t) = A(t, X(t))dt + B(t)dW(t). \]  

Assume that \( V = V_1 = V_2 \). Then we have \( \alpha_1 = \alpha_2 =: \alpha, \ c_2 = c_3 = c_4 =: c_3 \). For equation (29), we will consider the following strongly monotone condition instead of strictly monotone condition (H2'). This condition has a much wider application; see e.g. Gess et al [29].

(H2'') (Strong monotonicity) There exist constants \( r \geq 2 \) and \( \lambda > 0 \) such that for all \( u, v \in V, t \in \mathbb{R} \)

\[ 2V^\ast(A(t, u) - A(t, v), u - v)_V \leq -\lambda \| u - v \|^r_H. \]

Note that when \( r = 2 \), (H2'') is the same as (H2'). Therefore, we only consider the case \( r > 2 \) in this section.

Lemma 5.1. Consider equation (29). Assume that (H1), (H2''), (H3) and (H4) hold, and there exists a constant \( M_3 > 0 \) such that

\[ \| B(t) \|_{L^2(U, H)}^2 \leq M_3. \]

Let \( \zeta_s \in L^2(\Omega, \mathcal{F}_s, P; H) \). Suppose that \( X(t, s, \zeta_s), t \geq s \) is a solution to equation (29) with initial condition \( X(s) = \zeta_s \). Then for any \( \eta > 0 \), there exists a constant \( M_4 \geq 0 \) depending only on \( r, \eta, \alpha, c_1, c_2, M_0 \) and \( M_3 \), such that

\[ E\| X(t, s, \zeta_s) \|_H^2 \leq E\| \zeta_s \|_H^2 e^{-\eta(t-s)} + M_4. \]  

Proof. By (H3), (H4), (H2'') and Young’s inequality, we have

\[
2V^\ast(A(t, u), u)_V + \| B(t) \|_{L^2(U, H)}^2 \\
\leq 2V^\ast(A(t, u) - A(t, 0), u)_V + 2V^\ast(A(t, 0), u)_V + M_3 \\
\leq -\lambda \| u \|^r_H + 2\| A(t, 0) \|_{V^\ast} \| u \|_V + M_3 \\
\leq -\lambda \| u \|^r_H + \frac{2(\alpha - 1)}{\alpha} \| A(t, 0) \|_{V^\ast} \| u \|_V + M_3 \\
\leq -\lambda \| u \|^r_H + \frac{2(\alpha - 1)}{\alpha} M_0 \| u \|_V^\alpha + \frac{2c_1}{\alpha c_2} \| u \|^2_H + \frac{2M_0}{\alpha c_2} \\
- \frac{2}{\alpha c_2} \left( 2V^\ast(A(t, u), u)_V + \| B(t) \|_{L^2(U, H)}^2 \right) + M_3.
\]

Therefore, there exists a constant \( \tilde{M}_0 = \tilde{M}_0(\alpha, c_2, M_0, M_3) \) such that

\[ 2V^\ast(A(t, u), u)_V + \| B(t) \|_{L^2(U, H)}^2 \leq -\frac{\alpha_2 c_2}{\alpha c_2 + 2} \lambda \| u \|^r_H + \frac{2c_1}{\alpha c_2 + 2} \| u \|^2_H + \tilde{M}_0. \]  

(31)

According to Young’s inequality and (31), we obtain

\[
E \left( e^{\eta(t-s)} \| X(t, s, \zeta_s) \|_H^2 \right) \\
= E\| \zeta_s \|_H^2 + \int_s^t \eta e^{\eta(\sigma-s)} E\| X(\sigma, s, \zeta_s) \|_H^2 d\sigma \\
+ E \int_s^t e^{\eta(\sigma-s)} \left( 2V^\ast(A(\sigma, X(\sigma, s, \zeta_s)), X(\sigma, s, \zeta_s))_V + \| B(\sigma) \|_{L^2(U, H)}^2 \right) d\sigma \\
\leq E\| \zeta_s \|_H^2 + \int_s^t \eta e^{\eta(\sigma-s)} \tilde{M}_0 d\sigma
\]

\[ \leq E\| \zeta_s \|_H^2 + \int_s^t \eta e^{\eta(\sigma-s)} \tilde{M}_0 d\sigma. \]


\begin{align*}
&+ E \int_s^t e^{\eta(s-t)} \left( \left( \eta + \frac{2c_1}{\alpha c_2 + 2} \right) \|X(\sigma, s, \zeta_s)\|_H^2 - \frac{\alpha c_2}{\alpha c_2 + 2} \lambda \|X(\sigma, s, \zeta_s)\|_H^2 \right) d\sigma \\
&\leq E \|\zeta_s\|_H^2 + \int_s^t e^{\eta(s-t)} \tilde{M}_0 d\sigma + E \int_s^t e^{\eta(s-t)} \left[ - \frac{\alpha c_2}{\alpha c_2 + 2} \lambda \|X(\sigma, s, \zeta_s)\|_H^2 \right. \\
&\left. + \epsilon \|X(\sigma, s, \zeta_s)\|_H^2 + C_\epsilon \left( \eta + \frac{2c_1}{\alpha c_2 + 2} \right) \right] d\sigma.
\end{align*}

Choosing \( \epsilon < \frac{\alpha c_2}{\alpha c_2 + 2} \lambda \), we get

\[ E \left( e^{\eta(t-s)} \|X(t, s, \zeta_s)\|_H^2 \right) \leq E \|\zeta_s\|_H^2 + \int_s^t e^{\eta(s-t)} C_1 d\sigma, \]

where the constant \( C_1 \) depends only on \( \epsilon, r, \eta, \alpha, c_1, c_2 \) and \( \tilde{M}_0 \). It follows that

\[ E\|X(t, s, \zeta_s)\|_H^2 \leq E\|\zeta_s\|_H^2 e^{-\eta(t-s)} + M_4, \]

where \( M_4 = \frac{C_1}{\eta}. \) \hfill \( \square \)

**Lemma 5.2.** Consider equation (29). Assume that (H1), (H2''), (H3) and (H4) hold. Let \( X \) and \( Y \) be solutions of equation (29). Then for any \( u \leq \sigma \leq t \) we have the estimate

\[ E\|X(t, u, X(u)) - Y(t, \sigma, Y(\sigma))\|_H^2 \leq E\|X(t, u, X(u)) - Y(t, \sigma, Y(\sigma))\|_H^2 \wedge \left\{ \frac{\lambda}{2} (r - 2) (t - \sigma) \right\}^{-\frac{1}{r}}. \]

In particular, for any \( t \in \mathbb{R} \) there exists some random variable \( X(t) \) such that

\[ X(t, -n, 0) \to X(t) \quad \text{in} \quad L^2(\Omega, P; H) \quad \text{as} \quad n \to \infty. \]  

**Proof.** The proof is analogous to Lemma 2.5 in [29]. \hfill \( \square \)

Similar to Section 3, we will prove that the limit process \( X(\cdot) \) in (33) is a solution to equation (29). To this end, we also need some uniform estimates.

**Lemma 5.3.** Consider equation (29). Assume that the conditions of Lemma 5.1 hold. For any fixed interval \([a, b] \subset \mathbb{R}\) denote

\[ K := L^a([a, b] \times \Omega, dt \otimes P; V), \quad K^* := L^{2a}([a, b] \times \Omega, dt \otimes P; V^*). \]

Then there exists a constant \( M_2 \), depending only on \( M_0, M_4, c_1, c_2 \) and \([a, b]\), such that

\[ \sup_{t \in \mathbb{R}} E\|X(t, -n, 0)\|_H^2 + \|X(\cdot, -n, 0)\|_K + \|A(\cdot, X(\cdot, -n, 0))\|_{K^*} \leq M_2 \]

for all \(-n \leq a. \)

**Proof.** In view of (30), we have

\[ E\|X(t, -n, 0)\|_H^2 \leq M_4. \]

Similar to the proof of Lemma 3.4, we obtain

\[ \|X(\cdot, -n, 0)\|_K + \|A(\cdot, X(\cdot, -n, 0))\|_{K^*} \leq M_2. \]

The proof is complete. \hfill \( \square \)

Under the conditions of Lemma 5.1, we obtain the following conclusion which is as same as Theorem 3.5.
Theorem 5.4. Suppose that the conditions of Lemma 5.1 hold, then there exists a unique $L^2$-bounded continuous $H$-valued solution $X(t), t \in \mathbb{R}$ to equation (29). Moreover, the mapping $\hat{\mu} : \mathbb{R} \to Pr(H)$ defined by $\hat{\mu}(t) := P \circ [X(t)]^{-1}$, is unique with the following properties:

1. $L^2$-boundedness: $\sup_{t \in \mathbb{R}} \|x\|_{L^2}^2 \hat{\mu}(t)(dx) < +\infty$;
2. Flow property: $\mu(t, s, \hat{\mu}(s)) = \hat{\mu}(t)$ for all $t \geq s$.

Recalling that $\mu(t, s, \mu_0)$ denotes the distribution of $X(t, s, \zeta_s)$, with $\mu_0 = P \circ \zeta_s^{-1}$.

Proof. For any fixed interval $[a, b] \subset \mathbb{R}$ we denote

$$K := L^\alpha([a, b] \times \Omega, dt \otimes P; V), \quad K^* := L^{2, \infty}([a, b] \times \Omega, dt \otimes P; V^*).$$

Since $K$ is reflexive, we may assume, going if necessary to a subsequence, that

1. $X(\cdot, -n, 0) \to X(\cdot)$ in $L^2([a, b] \times \Omega, dt \otimes P; H)$ and $X(\cdot, -n, 0) \to X(\cdot)$ weakly in $K$;
2. $A(\cdot, X(\cdot, -n, 0)) \to Y(\cdot)$ weakly in $K^*$.

Thus for all $v \in V, \varphi \in L^\infty([a, b] \times \Omega)$ by Fubini’s theorem we get

$$E \int_a^b V \cdot \langle X(t), \varphi(t)v \rangle_V dt$$

$$= \lim_{n \to \infty} E \int_a^b V \cdot \langle X(t, -n, 0), \varphi(t)v \rangle_V dt$$

$$= \lim_{n \to \infty} E \left( \int_a^b V \cdot \langle X(a, -n, 0) + \int_a^t A(\sigma, X(\sigma, -n, 0))d\sigma 
\quad + \int_a^t B(\sigma)dW(\sigma), \varphi(t)v \rangle_V dt \right)$$

$$= E \left( \int_a^b V \cdot \langle X(a) + \int_a^t Y(\sigma)d\sigma + \int_a^t B(\sigma)dW(\sigma), \varphi(t)v \rangle_V dt \right).$$

It follows that

$$X(t) = X(a) + \int_a^t Y(\sigma)d\sigma + \int_a^t B(\sigma)dW(\sigma), \quad dt \otimes P\text{-a.e.}$$

Thus, it remains to verify that

$$Y = A(\cdot, X), \quad dt \otimes P\text{-a.e.}$$

To this end, for any $\phi \in K \cap L^2([a, b] \times \Omega, dt \otimes P; H)$ we have

$$E \left( \|X(t, -n, 0)\|_H^2 - \|X(a, -n, 0)\|_H^2 \right)$$

$$= E \left( \int_a^t \left( 2V \cdot \langle A(\sigma, X(\sigma, -n, 0)), X(\sigma, -n, 0) \rangle_V + \|B(\sigma)\|_{L^2(U,H)}^2 \right) d\sigma \right)$$

$$= E \left( \int_a^t \left( 2V \cdot \langle A(\sigma, X(\sigma, -n, 0)) - A(\sigma, \phi(\sigma)), X(\sigma, -n, 0) - \phi(\sigma) \rangle_V 
\quad + 2V \cdot \langle A(\sigma, \phi(\sigma)), X(\sigma, -n, 0) - \phi(\sigma) \rangle_V 
\quad + 2V \cdot \langle A(\sigma, X(\sigma, -n, 0)) - A(\sigma, \phi(\sigma)), \phi(\sigma) \rangle_V \right) d\sigma \right)$$

$$\leq E \left( \int_a^t \left( 2V \cdot \langle A(\sigma, X(\sigma, -n, 0)) - A(\sigma, \phi(\sigma)), \phi(\sigma) \rangle_V \right) \right).$$
Similarly to the proof of (21), for given nonnegative $\psi \in L^\infty([0, T], dt; \mathbb{R})$, first multiplying $\psi(t)$ on both sides of (5.6), then integrating with respect to $t$ from $a$ to $b$ and letting $n \to \infty$, we have

$$E \int_a^b \psi(t) (\|X(t)\|_H^2 - \|X(a)\|_H^2) \, dt$$

$$\leq E \left(\int_a^b \psi(t) \int_a^t \left(2V \cdot \langle Y(\sigma) - A(\sigma, \phi(\sigma)), \phi(\sigma) \rangle_V + 2V \cdot \langle A(\sigma, \phi(\sigma)), \overline{X}(\sigma) \rangle_V + \|B(\sigma)\|_{L_2(U, H)}^2\right) \, d\sigma \, dt \right).$$

In view of the product rule, we obtain

$$E \int_a^b \psi(t) (\|X(t)\|_H^2 - \|X(a)\|_H^2) \, dt$$

$$= E \left(\int_a^b \psi(t) \int_a^t \left(2V \cdot \langle Y(\sigma) - A(\sigma, \phi(\sigma)), \phi(\sigma) \rangle_V + \|B(\sigma)\|_{L_2(U, H)}^2\right) \, d\sigma \, dt \right).$$

Therefore, (35) and (36) imply

$$E \left(\int_a^b \psi(t) \int_a^t \left(2V \cdot \langle Y(\sigma) - A(\sigma, \phi(\sigma)), \phi(\sigma) \rangle_V \, d\sigma \, dt \right) \leq 0. \tag{37}$$

Similar to the proof of Theorem 3.5, we obtain that $Y = A(\cdot, \overline{X})$, $dt \otimes P$-a.e. This completes the existence proof, i.e.

$$X(t) = X(a) + \int_a^t A(\sigma, \overline{X}(\sigma)) \, d\sigma + \int_a^t B(\sigma) \, dW(\sigma), \quad dt \otimes P$$

By the arbitrariness of interval $[a, b] \subset \mathbb{R}$, we conclude that $X(\cdot)$ is a solution on $\mathbb{R}$.

It follows from Lemma 5.3 that $\sup_{t \in \mathbb{R}} E\|X(t)\|_H^2 < \infty$.

Now we prove the uniqueness of $L^2$-bounded solution. Let $X(\cdot)$, $Y(\cdot)$ be two $L^2$-bounded solutions, then by (32) we have

$$E\|X(t) - Y(t)\|_H^2$$

$$= E\|X(t, -n, X(-n)) - Y(t, -n, Y(-n))\|_H^2$$

$$\leq \left\{\frac{\lambda}{2} (r - 2) (t + n)\right\}^{-\frac{1}{r-2}} \to 0 \quad \text{as} \quad n \to \infty.$$

Finally, we show that $\mu$ is unique with properties (i) and (ii). Note that

$$\sup_{t \in \mathbb{R}} \int_H \|x\|_H^2 \mu(t)(dx) = \sup_{t \in \mathbb{R}} E\|X(t)\|_H^2 < \infty.$$ 

According to the Chapman-Kolmogorov equation, we have

$$\mu(t, s, L(X(s, -n, 0))) = L(X(t, -n, 0)). \tag{38}$$

In view of (32), we get

$$E\|X(t, s, X(s)) - X(t, s, X(s, -n, 0))\|_H^2 \leq E\|X(s) - X(s, -n, 0)\|_H^2.$$
This inequality and (38) yield
\[ \mu(t, s, \hat{\mu}(s)) = \hat{\mu}(t). \]

It remains to prove the uniqueness of \( \hat{\mu} \). Let \( \mu_1, \mu_2 \) be two mappings which satisfy (i) and (ii), and let \( \zeta_{n,1}, \zeta_{n,2} \) be random variables with distributions \( \mu_1(-n), \mu_2(-n) \) respectively. Consider the solutions \( X(t, -n, \zeta_{n,1}) \), \( X(t, -n, \zeta_{n,2}) \) on \([-n, \infty)\), then we have
\[
d_{BL}(\mu_1(t), \mu_2(t)) = d_{BL}(\mu(t, -n, \mu_1(-n)), \mu(t, -n, \mu_2(-n)))
\]
\[
= \sup_{\|f\|_{BL} \leq 1} \left| \int_H f(x) d(\mu(t, -n, \mu_1(-n)) - \mu(t, -n, \mu_2(-n))) \right|
\]
\[
= \sup_{\|f\|_{BL} \leq 1} \left| \int_H [f(X(t, -n, \zeta_{n,1})) - f(X(t, -n, \zeta_{n,2}))] dP \right|
\]
\[
\leq (E\|X(t, -n, \zeta_{n,1}) - X(t, -n, \zeta_{n,2})\|_{H}^2)^{1/2}
\]
\[
\leq \left\{ \frac{\lambda}{2} (r - 2)(t + n) \right\}^{-1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
Thus, \( \mu_1(t) = \mu_2(t) \) for all \( t \in \mathbb{R} \).

Completely similar to Proposition 2, Theorems 4.1, 4.3 and 4.4, we can get the following proposition and theorems.

**Proposition 3.** Consider equation (29). Suppose that \( A, B, A_n, B_n \) satisfy (H1), (H2′′), (H3) and (H4) with the same constants \( \lambda, r, c_1, c_2, c_3, M_0, \alpha \). Let \( X(\cdot) \), \( X_n(\cdot) \) be the \( L^2 \)-bounded solutions of equation (29) corresponding to \( A, B \) and \( A_n, B_n \) respectively. Assume in addition that

1. \( \lim_{n \to \infty} \|A_n(t, x) - A(t, x)\|_{V^*} = 0 \) for all \( x \in V, t \in \mathbb{R} \);
2. \( \lim_{n \to \infty} \|B_n(t) - B(t)\|_{L_2(U, H)} = 0 \) for all \( x \in V, t \in \mathbb{R} \);
3. for each \( t \in \mathbb{R} \) the family of distributions \( \{P \circ [X_n(t)]^{-1}\}_{n \in \mathbb{N}} \) is tight.

Then
\[
\lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n), \mathcal{L}(X)) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)).
\]

In particular,
\[
\lim_{n \to \infty} d_{BL}(\mathcal{L}(X_n(t + \cdot)), \mathcal{L}(X(t + \cdot))) = 0 \quad \text{in } Pr(C(\mathbb{R}, H)) \quad \text{for all } t \in \mathbb{R}.
\]

**Theorem 5.5.** Consider equation (29). Suppose that (H1), (H2′′), (H3) and (H4) hold. Assume further that the mappings \( A \) and \( B \) are \( T \)-periodic in \( t \). Then the unique \( L^2 \)-bounded solution is \( T \)-periodic in distribution.

In particular, this unique \( L^2 \)-bounded solution is stationary provided the mappings \( A \) and \( B \) are independent of \( t \).

**Theorem 5.6.** Consider equation (29). Suppose that (H1), (H2′′), (H3), (H4) and (H5) hold. Assume further that the coefficients \( A \) and \( B \) are uniformly almost periodic (uniformly almost automorphic). Then the unique \( L^2 \)-bounded solution is almost periodic in distribution (almost automorphic in distribution).
6. Stability of the bounded solution. In this section, we prove that the $L^2$-bounded solutions of equations (2) and (29) are globally asymptotically stable.

**Definition 6.1** (See Fu and Liu [25]). We say that a solution $X(\cdot)$ of equation (2) or (29) is stable in square-mean sense, if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \geq 0$

$$E\|X(t,0) - X(t)\|_H^2 < \epsilon,$$

whenever $E\|\zeta_0 - X(0)\|_H^2 < \delta$. The solution $X(\cdot)$ is said to be asymptotically stable in square-mean sense if it is stable in square-mean sense and

$$\lim_{t \to \infty} E\|X(t,0) - X(t)\|_H^2 = 0. \tag{39}$$

We say $X(\cdot)$ is globally asymptotically stable in square-mean sense provided (39) holds for any $\zeta_0 \in L^2(\Omega,F_0,P;H)$.

**Theorem 6.2.** Consider equation (2). Suppose that (H1), (H2'), (H3) and (H4) hold, then the unique $L^2$-bounded solution of equation (2) is globally asymptotically stable in square-mean sense. Moreover, for any $t \geq s$ and $\zeta_s \in L^2(\Omega,F_s,P;H)$, we have

$$E\|X(t,s) - X(t)\|_H^2 \leq e^{-\lambda(t-s)}E\|\zeta_s - X(s)\|_H^2.$$

**Proof.** In view of Itô's formula, the product rule and (H2'), we have

\[
E\left(e^{\lambda(t-s)}\|X(t,s) - X(t)\|_H^2\right) \\
= E\|\zeta_s - X(s)\|_H^2 + \int_s^t \lambda e^{\lambda(s-\sigma)}E\|X(\sigma,s) - X(\sigma)\|_H^2 d\sigma \\
+ E \int_s^t e^{\lambda(s-\sigma)} \left(2\langle A(\sigma,\bar{X}(\sigma,s,\zeta_s)) - A(\sigma,\bar{X}(\sigma)), \bar{X}(\sigma,s,\zeta_s) - \bar{X}(\sigma) \rangle_v \\
+ \|B(\sigma,\bar{X}(\sigma,s,\zeta_s)) - B(\sigma,\bar{X}(\sigma))\|_{L_2(U,H)}^2\right) d\sigma \\
\leq E\|\zeta_s - X(s)\|_H^2.
\]

It follows that

$$E\|X(t,s) - X(t)\|_H^2 \leq e^{-\lambda(t-s)}E\|\zeta_s - X(s)\|_H^2, \quad \text{for all } t \geq s.$$

The proof is complete. 

Applying Lemma 5.2 we obtain the following result:

**Theorem 6.3.** Consider equation (29). Suppose that (H1), (H2''), (H3) and (H4) hold, then the unique $L^2$-bounded solution of equation (29) is globally asymptotically stable in square-mean sense. Moreover, for any $t \geq s$ and $\zeta_s \in L^2(\Omega,F_s,P;H)$, we have

$$E\|X(t,s) - X(t)\|_H^2 \leq E\|\zeta_s - X(s)\|_H^2 \wedge \left(\frac{\lambda}{2(r-2)(t-s)}\right)^{-\frac{1}{2}}.$$

7. Applications. In this section, we illustrate our theoretical results by two examples. For simplicity, we mainly consider the additive type noise in these examples.
7.1. Stochastic reaction diffusion equations. Let $\Lambda$ be an open bounded subset of $\mathbb{R}^n$, $n \in \mathbb{N}$. Consider the equation

$$
\frac{du}{dt} = (\Delta u - au|u|^{p-2} + \phi(t)u) \ dt + B(t) dW(t),
$$

where $W(\cdot)$ is a two-sided cylindrical Q-Wiener process with $Q = I$ on a separable Hilbert space $(U, \langle \cdot, \cdot \rangle_U)$ and $p \in [2, \infty)$. Here $a > 0$ is a constant and $\phi(\cdot)$ is bounded, i.e. there exists a constant $C_1 > 0$ such that $|\phi(t)| \leq C_1$ for all $t \in \mathbb{R}$. We define $V_1 := H^{1,2}_0(\Lambda)$, $V_2 := L^p(\Lambda)$, $H := L^2(\Lambda)$, $V_1^* := (H^{1,2}_0(\Lambda))^*$, $V_2^* := (L^p(\Lambda))^*$, $V := V_1 \cap V_2$ and

$$
A_1(t, u) := \Delta u + \phi(t)u, \quad A_2(u) := -au|u|^{p-2}.
$$

**Theorem 7.1.** Let $\lambda_*$ be the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition and assume that $\lambda_* - C_1 > 0$.

1. If $\|B(t)\|_{L^2(U, H)}^2 \leq M$ for some constant $M > 0$, then there exists a unique $L^2$-bounded solution $X(\cdot)$ to equation (40), which is globally asymptotically stable in square-mean sense. Furthermore, $X(\cdot)$ is $T$-periodic in distribution (stationary) if $B$ and $\phi$ are $T$-periodic (independent of $t$).

2. Let $S = H^{1,2}_0(\Lambda)$. Suppose that there exists a constant $\hat{M} > 0$ such that $\|B(t)\|_{L^2(U, S)}^2 \leq \hat{M}$. Then the $L^2$-bounded solution $X(\cdot)$ is almost periodic (almost automorphic) in distribution if $B$ and $\phi$ are almost periodic (almost automorphic).

**Proof.** (1) In order to prove (1), by Theorems 3.5, 6.2 and 4.1, it suffices to show that $A$ and $B$ satisfy (H1), (H2'), (H3) and (H4).

(H1) $A_1$ is obviously hemicontinuous. We now prove that $A_2$ is hemicontinuous. Let $u, v, w \in V$. For $\theta \in \mathbb{R}$, without loss of generality, we assume $|\theta| \leq 1$, then we have

$$
\begin{aligned}
\langle v, A_2(u + \theta v) - A_2(u), w \rangle_V &= 
= \int_{\Lambda} (-\langle u(\xi) + \theta v(\xi) \rangle \ |u(\xi) + \theta v(\xi)|^{p-2} w(\xi) + u(\xi) |u(\xi)|^{p-2} w(\xi)) \ d\xi \\
&\leq \int_{\Lambda} \left( 2^{p-2} \left( |u(\xi)|^{p-1} + |v(\xi)|^{p-1} \right) |w(\xi)| + |u(\xi)|^{p-1} |w(\xi)| \right) \ d\xi < \infty.
\end{aligned}
$$

The last inequality holds since $u, v, w \in L^p(\Lambda)$. Then $\langle v, A_2(u + \theta v) - A_2(u), w \rangle_V$ converges to zero as $\theta \to 0$ by Lebesgue’s dominated convergence theorem. So, (H1) holds.

(H2') If $u, v \in V$ then there exist $u_n, v_n \in C^\infty_c(\Lambda)$, $n \in \mathbb{N}$ such that $u_n \to u$, $v_n \to v$ as $n \to \infty$ in $V_1$. Hence we get

$$
\begin{aligned}
v_1^* \langle A_1(t, u) - A_1(t, v), u - v \rangle_{V_1} &= 
= \lim_{n \to \infty} v_1^* \langle \Delta u_n - \Delta v_n, u_n - v_n \rangle_{V_1} + \phi(t) \langle u - v, u - v \rangle_H \\
&\leq -\lambda_* \lim_{n \to \infty} \langle u_n - v_n, u_n - v_n \rangle_H + \phi(t) \|u - v\|^2_H \\
&\leq -\lambda_* (u - v, u - v)_{V_1}
\end{aligned}
$$

and

$$
\begin{aligned}
v_2^* \langle A_2(u) - A_2(v), u - v \rangle_{V_2} &= 
= -a \int_{\Lambda} (u(\xi) |u(\xi)|^{p-2} - v(\xi) |v(\xi)|^{p-2}) (u(\xi) - v(\xi)) \ d\xi \leq 0.
\end{aligned}
$$
Then
\[2v \cdot \langle A(t, u) - A(t, v), u - v \rangle_v + \|B(t) - B(t)\|_{L^2}(u, v)^2 \leq -2(\lambda_\ast - C_1)\|u - v\|_H^2.\]
So (H2') holds with \(\lambda = 2(\lambda_\ast - C_1) > 0\).

(H3) For all \(v \in V, t \in \mathbb{R}\) we have
\[
v_1^* \langle A_1(t, v), v \rangle_v^1 = \lim_{n \to \infty} v_1^* \langle \Delta v_n, v_n \rangle_{V^1} + \phi(t) \langle v, v \rangle_H
= -\int_\Lambda |\nabla v(\xi)|^2 d\xi + \phi(t)\|v\|_H^2
= \|v\|_H^2 - \|v\|_{V_1^2}^2 + \phi(t)\|v\|_H^2
\leq (C_1 + 1)\|v\|_H^2 - \|v\|_{V_1^2}^2,
\]
and
\[
v_2^* \langle A_2(v), v \rangle_v^2 = -a \int_\Lambda |v(\xi)|^p d\xi = -a\|v\|_{V_2}^p.
\]
Then
\[2v \cdot \langle A(t, v), v \rangle_v + \|B(t)\|_{L^2}(u, v) \leq 2(C_1 + 1)\|v\|_H^2 - 2\|v\|_{V_1^2}^2 - 2a\|v\|_{V_2}^p + M.
\]
So (H3) holds with \(\alpha_1 = 2, \alpha_2 = p\).

(H4) For all \(u, v \in V, t \in \mathbb{R}\) we have
\[
|v_1^* \langle A_1(t, u), v \rangle_v^1| = |\lim_{n \to \infty} v_1^* \langle \Delta u_n, v_n \rangle_{V^1} + \phi(t)\langle u, v \rangle_H|
\leq \|\nabla u\|_H\|\nabla v\|_H + C\|u\|_H\|v\|_H
\leq (C_1 + 1)\|u\|_{V^1}\|v\|_{V^1},
\]
and
\[
|v_2^* \langle A_2(u), v \rangle_v^2| = \left|a \int_\Lambda -u(\xi)|u(\xi)|^{p-2}v(\xi) d\xi\right| \leq a\|u\|_{V_2}^{p-1}\|v\|_{V_2}.
\]
Therefore, we get
\[
\|A_1(t, u)\|_{V^1} \leq (C_1 + 1)\|u\|_{V^1}, \quad \|A_2(u)\|_{V_2} \leq a\|u\|_{V_2}^{p-1}.
\]
So (H4) holds with \(\alpha_1 = 2, \alpha_2 = p\).

(2) Note that \(\|B(t)\|_{L^2(U, V)} \leq \|B(t)\|_{L^2(U, U)}\) and \(\|B(t)\|_{L^2(U, U)} \leq \|B(t)\|_{L^2(U, V)}\) for all \(t \in \mathbb{R}\) (see, e.g. [51, Remark B.0.6]). So, in order to prove the almost automorphic property of the \(L^2\)-bounded solution, by Theorems 4.3 and 4.4, it suffices to show that (H5) holds. To this end, we define \(T_n = -\Delta (I - \frac{\Delta}{n})^{-1} = n(I - (I - \frac{\Delta}{n})^{-1})\). Note that \(T_n\) are continuous on \(W_0^{1,2}(\Lambda)\). Since the heat semigroup \(\{P_t\}_{t \geq 0}\) (generated by \(\Delta\)) is contractive on \(L^p(\Lambda)\), \(p > 1\) (see Theorem 3.6 on page 215 of Pazy [50]) and \((I - \frac{\Delta}{n})^{-1}u = \int_0^\infty e^{-tP_{\frac{\Delta}{n}}}udt\), \(T_n\) are continuous on \(L^p(\Lambda)\).

For all \(u \in V, t \in \mathbb{R}\) we have
\[
v_1^* \langle \Delta u, T_n u \rangle_v^1 = \lim_{m \to \infty} v_1^* \langle \Delta u_m, T_n u_m \rangle_{V^1}
\]
\[
= \lim_{m \to \infty} \sum_{j=1}^\infty \langle \Delta u_m, e_j \rangle_H e_j, T_n u_m \rangle_H
\]
\[
= \lim_{m \to \infty} \sum_{j=1}^\infty -\lambda_j \langle u_m, e_j \rangle_H e_j, T_n u_m \rangle_H
\]
\[
\leq -\lambda_\ast \lim_{m \to \infty} \langle u_m, T_n u_m \rangle_H
\]
In view of the contractivity of 
\[ - \lambda_* \|u\|_{L^2}^2 \]
and
\[ \phi(t)(u, T_n u)_H = \phi(t)\|u\|_{L^2}^2 \leq C_1 \|u\|_{L^2}^2. \]
In view of the contractivity of \{P_t\}_{t \geq 0} on \(L^p(\Omega)\), we have
\[ v_2(A_2(u), T_n u)_{V_2} = n \int_0^\infty e^{-t} \left( \int_{\Lambda} -au(\xi)|u(\xi)|^{p-2} \left( u(\xi) - P_\xi u(\xi) \right) d\xi \right) dt \leq 0. \]
Then we obtain
\[ 2v_2(A(t, u), T_n u)_{V_2} + \|B(t)\|_{L^2(U, H_n)}^2 \leq -2(\lambda_* - C_1) \|u\|_{L^2}^2 + M. \]
That is, (H5) holds.

The proof is complete. \( \square \)

**Remark 8.**

1. In the above stochastic reaction diffusion equation, we can also consider multiplicative noise case. Here we give a simple example; see e.g. Liu [43]. Consider
\[ B(t, v)u := B_0(t)u + \sum_{i=1}^N \phi_i(t)(u, u_i)_{U, v}, \ u \in U, \ v \in V \]
where \( B_0 : \mathbb{R} \rightarrow L_2(U, S) \) is progressively measurable, \( u_i \in U, \ \phi_i : \mathbb{R} \rightarrow \mathbb{R} \) and there exist constants \( C_{1,i} > 0 \) such that \( |\phi_i(t)| \leq C_{1,i} \) for all \( t \in \mathbb{R}, \ i = 1, 2, ..., N \). Suppose that there exists a constant \( M > 0 \) such that
\[ \|B_0(t)\|_{L^2(U, S)}^2 \leq M \text{ for all } t \in \mathbb{R} \]
and
\[ \lambda_* - C_1 - \frac{N+1}{2} \sum_{i=1}^N C_{1,i} \|u_i\|_U^2 > 0. \]
Then there exists a unique \( L^2 \)-bounded solution \( X(\cdot) \), which is globally asymptotically stable in square-mean sense. Furthermore, if \( \phi, \phi_i, i = 1, 2, ..., N \) and \( B_0 \) are \( T \)-periodic (respectively, almost periodic, almost automorphic), then the \( L^2 \)-bounded solution \( X(\cdot) \) is \( T \)-periodic (respectively, almost periodic, almost automorphic) in distribution. In particular, \( X(\cdot) \) is stationary provided \( \phi, \phi_i, i = 1, 2, ..., N \) and \( B_0 \) are independent of \( t \).

2. Note that when the noise is additive or of linear form as in (42), the main result of Gao [26] is a special case of the above example for \( p = 4 \).

### 7.2. Stochastic porous media equations.

Let \( \Lambda \) be an open bounded subset of \( \mathbb{R}^n, \ n \in \mathbb{N} \). Consider the equation
\[ du = (\Delta(|u|^{p-2}u) + \phi(t)u) dt + B(t)dW(t), \]
where \( W(\cdot) \) is a two-sided cylindrical \( Q \)-Wiener process with \( Q = I \) on a separable Hilbert space \( (U, (\cdot, \cdot)_U), p > 2 \). And there exist constants \( C_1 > C_2 > 0 \) such that \( -C_1 < \phi(t) < -C_2 \) for all \( t \in \mathbb{R} \). We define
\[ V := L^p(\Lambda) \subset H := W_0^{-1,2}(\Lambda) \subset V^*. \]

**Theorem 7.2.**

1. If \( \|B(t)\|_{L^2(U, H)}^2 \leq M \) for some constant \( M > 0 \), then there exists a unique \( L^2 \)-bounded solution \( X(\cdot) \) to equation (43), which is globally asymptotically stable in square-mean sense. Furthermore, \( X(\cdot) \) is \( T \)-periodic in distribution (stationary) if \( \phi \) and \( B \) are \( T \)-periodic (independent of \( t \)).
(2) Let \( S = L^2(\Lambda) \). Suppose that there exists a constant \( \hat{M} > 0 \) such that \( \|B(t)\|_{L^2(U,S)}^2 \leq \hat{M} \). Then the \( L^2 \)-bounded solution \( X(\cdot) \) is almost periodic (almost automorphic) in distribution provided \( \phi \) and \( B \) are almost periodic (almost automorphic).

**Proof.** Fix \( u \in V, t \in \mathbb{R} \), for all \( v \in V \) we denote

\[
V \cdot \langle A(t,u), v \rangle_V := -\int_{\Lambda} u(\xi)|u(\xi)|^{p-2}v(\xi)d\xi + \int_{\Lambda} \phi(t)u(\xi)v(\xi)d\xi.
\]

We first show that \( A : \mathbb{R} \times V \to V^* \) is well-defined. Indeed, for all \( u, v \in V, t \in \mathbb{R} \)

\[
|V \cdot \langle A(t,u), v \rangle_V| \\
\leq \int_{\Lambda} |u(\xi)|^{p-1}|v(\xi)|d\xi + C_1 \int_{\Lambda} |u(\xi)||v(\xi)|d\xi \\
\leq \left( \int_{\Lambda} |u(\xi)|^{p}d\xi \right)^{\frac{p-1}{p}} \left( \int_{\Lambda} |v(\xi)|^{p}d\xi \right)^{\frac{1}{p}} \\
+ C_1 \left( \int_{\Lambda} |u(\xi)|d\xi \right)^{\frac{p-1}{p}} \left( \int_{\Lambda} |v(\xi)|^{p}d\xi \right)^{\frac{1}{p}} \\
\leq \|u\|_{L^p}^p \|v\|_{L^p} + C_1 \|u\|_{L^p} \|v\|_{L^p} \\
\leq \|u\|_{L^p}^p \|v\|_{L^p} + C_1 \left( \frac{1}{p-1} \|u\|_{L^p}^{p-1} + \frac{p-2}{p-1} (|\Lambda|)^{\frac{p-1}{p}} \right) \|v\|_{L^p}.
\]

Therefore, \( A : \mathbb{R} \times V \to V^* \) is well-defined and we have

\[
\|A(t,u)\|_{V^*} \leq \left( 1 + C_1 \frac{1}{p-1} \right) \|u\|_{L^p}^{p-1} + \frac{p-2}{p-1} (|\Lambda|)^{\frac{p-1}{p}} C_1. \tag{44}
\]

Next we verify assertions (1) and (2).

(1) It suffices to show that (H1), (H2''), (H3) and (H4) hold.

(H1) follows immediately from (41).

(H2'') For all \( u, v \in V, t \in \mathbb{R} \) we have

\[
V \cdot \langle A(t,u) - A(t,v), u - v \rangle_V \\
= -\langle u|u|^{p-2} - v|v|^{p-2}, u - v \rangle_{L^2} + \phi(t)||u - v||_{L^2}^2 \\
\leq -2^{2-p} ||u - v||_{L^p}^2 + \phi(t)||u - v||_{L^2}^2 \\
\leq -2^{2-p} ||u - v||_{H^s}^2.
\]

Therefore, (H2'') holds with \( r = p, \lambda = 2^{3-p} \).

(H3) Note that for all \( u \in V, t \in \mathbb{R} \)

\[
V \cdot \langle A(t,u), u \rangle_V = -\int_{\Lambda} u(\xi)|u(\xi)|^{p-2}u(\xi)d\xi + \phi(t)\int_{\Lambda} u(\xi)u(\xi)d\xi \leq -\|u\|_{L^p}^p,
\]

so we have

\[
2V \cdot \langle A(t,u), u \rangle_V + \|B(t)\|_{L^2(U,H)}^2 \leq -2\|u\|_{L^p}^p + M.
\]

That is, (H3) holds with \( \alpha = p \).

(H4) holds by (44) with \( \alpha = p \).

(2) Like the proof of Theorem 7.1, it suffices to verify (H5). Let \( S = L^2(\Lambda) \) and \( \Delta \) be the Laplace operator on \( L^2(\Lambda) \) with the Dirichlet boundary condition. We define \( T_n = -\Delta (I - \frac{\Lambda}{\Lambda})^{-1} = n (I - (I - \frac{\Lambda}{\Lambda})^{-1}) \). The continuity of \( T_n \) on \( L^p(\Lambda) \), \( p > 1 \)
was already shown in the proof of Theorem 7.1. Then recalling the contractivity of \( \{P_t\}_{t \geq 0} \) on \( L^p(\Lambda) \), \( p > 1 \), we obtain
\[
\begin{align*}
V^* \langle \Delta (u|u|^{p-2}) + \phi(t)u, -\Delta (I - \frac{\Delta}{n})^{-1}u \rangle_V \\
&= -\langle u|u|^{p-2}, nu - n(I - \frac{\Delta}{n})^{-1}u \rangle_{L^2} + \phi(t)\|u\|_n^2 \\
&= -\langle u|u|^{p-2}, nu - n \int_0^\infty e^{-t}P_{\frac{\Delta}{n}}u(\xi)dt \rangle_{L^2} + \phi(t)\|u\|_n^2 \\
&= -n \int_0^\infty e^{-t} \left( \int_\Lambda |u(\xi)|^p d\xi - \int_\Lambda |u(\xi)|^{p-2}u(\xi) \cdot P_{\frac{\Delta}{n}}u(\xi)d\xi \right) dt + \phi(t)\|u\|_n^2 \\
&\leq -C_2\|u\|_n^2.
\end{align*}
\]
Then we have
\[
2V^* \langle A(t, u), T_n u \rangle_V + \|B(t)\|_{L^2(\Lambda, H_n)}^2 \leq -2C_2\|u\|_n^2 + \dot{M}.
\]
That is, (H5) holds.

The proof is complete. \( \square \)

REFERENCES

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2 D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965. vi+291 pp.
[2] L. Amerio and G. Prouse, *Almost-Periodic Functions and Functional Equations*, Van Nostrand Reinhold Co., New York-Toronto-Ont.-Melbourne, 1971. viii+184 pp.
[3] L. Arnold and C. Tudor, *Stationary and almost periodic solutions of almost periodic affine stochastic differential equations*, Stochastics Stochastics Rep., 64 (1998), 177–193.
[4] V. Barbu and G. Da Prato, *Ergodicity for nonlinear stochastic equations in variational formulation*, Appl. Math. Optim., 53 (2006), 121–139.
[5] V. Barbu and M. R"ockner, *Stochastic variational inequalities and applications to the total variation flow perturbed by linear multiplicative noise*, Arch. Ration. Mech. Anal., 209 (2013), 797–834.
[6] P. H. Bezandry and T. Diagana, *Existence of almost periodic solutions to some stochastic differential equations*, Appl. Anal., 86 (2007), 819–827.
[7] S. Bochner, *Beiträge zur theorie der fastperiodischen funktionen*, I. Funktionen einer Veränderlichen, Math. Ann., 96 (1921), 119–147 (in German).
[8] S. Bochner, *Curvature and Betti numbers in real and complex vector bundles*, Rend. Semin. Mat. Univ. Politec. Torino, 15 (1955/1956), 225–253.
[9] S. Bochner, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S.A., 48 (1962), 2039–2043.
[10] V. I. Bogachev, G. Da Prato and M. Röckner, *Invariant measures of generalized stochastic equations of porous media*, Dokl. Akad. Nauk, 396 (2004), 7–11 (in Russian).
[11] H. Bohr, *Zur theorie der fastperiodischen funktionen*, I. Acta Math., 45 (1924), 29–127; II, Acta Math., 46 (1925), 101–214; III, Acta Math., 47 (1926), 237–281. (All in German)
[12] D. Cheban and Z. Liu, *Periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent and Poisson stable solutions for stochastic differential equations*, J. Differential Equations, 269 (2020), 3652–3685.
[13] F. Chen, Y. Han, Y. Li and X. Yang, *Periodic solutions of Fokker-Planck equations*, J. Differential Equations, 263 (2017), 285–298.
[14] Z. Chen and W. Lin, *Square-mean weighted pseudo almost automorphic solutions for non-autonomous stochastic evolution equations*, J. Math. Pures Appl., 100 (2013), 476–504.
[15] I. Ciotir, *A Trotter type result for the stochastic porous media equations*, Nonlinear Anal., 71 (2009), 5606–5615.
[16] I. Ciotir, *A Trotter-type theorem for nonlinear stochastic equations in variational formulation and homogenization*, Differential Integral Equations, 24 (2011), 371–388.
[17] I. Ciotir and J. M. Tölle, Convergence of invariant measures for singular stochastic diffusion equations, *Stochastic Process. Appl.*, **122** (2012), 1998–2017. Corrigendum: *Stochastic Process. Appl.*, **123** (2013), 1178–1181.

[18] G. Da Prato, M. Röckner, B. L. Rozovskii and F. Wang, Strong solutions of stochastic generalized porous media equations: Existence, uniqueness, and ergodicity, *Comm. Partial Differential Equations*, **31** (2006), 277–291.

[19] G. Da Prato and C. Tudor, Periodic and almost periodic solutions for semilinear stochastic equations, *Stochastic Anal. Appl.*, **13** (1995), 13–33.

[20] R. M. Dudley, *Real Analysis and Probability*, Revised reprint of the 1989 original. Cambridge Studies in Advanced Mathematics, 74. Cambridge University Press, Cambridge, 2002. x+555 pp.

[21] A. Es-Sarhir, M. K. von Renesse and W. Stannat, Estimates for the ergodic measure and polynomial stability of plane stochastic curve shortening flow, *NoDEA Nonlinear Differential Equations Appl.*, **19** (2012), 663–675.

[22] J. Favard, Sur les équations différentielles linéaires à coefficients presque-périodiques, *Acta Math.*, **51** (1928), 31–81 (in French).

[23] J. Favard, *Lecons sur les Fonctions Presque-Périodiques*. Gauthier-Villars, Paris, 1933.

[24] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Math., vol. 377, Springer-Verlag, Berlin-New York, 1974. viii+336 pp.

[25] M. Fu and Z. Liu, Square-mean almost automorphic solutions for some stochastic differential equations, *Proc. Amer. Math. Soc.*, **138** (2010), 3689–3701.

[26] P. Gao, Some periodic type solutions for stochastic reaction-diffusion equation with cubic nonlinearities, *Comput. Math. Appl.*, **74** (2017), 2281–2297.

[27] B. Gess, Random attractors for singular stochastic evolution equations, *J. Differential Equations*, **255** (2013), 524–559.

[28] B. Gess, Random attractors for degenerate stochastic partial differential equations, *J. Dynam. Differential Equations*, **25** (2013), 121–157.

[29] B. Gess, W. Liu and M. Röckner, Random attractors for a class of stochastic partial differential equations driven by general additive noise, *J. Differential Equations*, **251** (2011), 1225–1253.

[30] B. Gess and M. Röckner, Singular-degenerate multivalued stochastic fast diffusion equations, *SIAM J. Math. Anal.*, **47** (2015), 4058–4090.

[31] B. Gess and M. Röckner, Stochastic variational inequalities and regularity for degenerate stochastic partial differential equations, *Trans. Amer. Math. Soc.*, **369** (2017), 3017–3045.

[32] B. Gess and J. M. Tölle, Multi-valued, singular stochastic evolution inclusions, *J. Math. Pures Appl. (9)*, **101** (2014), 789–827.

[33] B. Gess and J. M. Tölle, Stability of solutions to stochastic partial differential equations, *J. Differential Equations*, **260** (2016), 4973–5025.

[34] A. Halanay, Periodic and almost periodic solutions to affine stochastic systems. *Proceedings of the Eleventh International Conference on Nonlinear Oscillations (Budapest, 1987)*, 94–101, János Bolyai Math. Soc., Budapest, 1987.

[35] M. Ji, W. Qi, Z. Shen and Y. Yi, Existence of periodic probability solutions to Fokker-Planck equations with applications, *J. Funct. Anal.*, **277** (2019), Art. 108281, 41 pp.

[36] R. A. Johnson, A linear, almost periodic equation with an almost automorphic solution, *Proc. Amer. Math. Soc.*, **82** (1981), 199–205.

[37] M. Kamenskiı̈, O. Mellah and P. Raynaud de Fitte, Weak averaging of semilinear stochastic differential equations with almost periodic coefficients, *J. Math. Anal. Appl.*, **427** (2015), 336–364.

[38] R. Has’minskiı̈, *Stochastic Stability of Differential Equations*, Translated from the Russian by D. Louvish, Sijthoff & Noordhoff, Alphen aan den Rijn–Germantown, Md., 1980. xvi+344 pp. (see also 2nd ed., Springer, New York, 2012. xvii+339)

[39] N. V. Krylov and B. L. Rozovskiı̈, Stochastic evolution equations, *Current Problems in Mathematics, Vol. 14 (Russian)*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, **256** (1979), 71–147.

[40] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Translated from the Russian by L. W. Longdon. Cambridge University Press, Cambridge-New York, 1982. xi+211 pp.

[41] Y. Li, Z. Liu and W. Wang, Almost periodic solutions and stable solutions for stochastic differential equations, *Discrete Contin. Dyn. Syst. Ser. B*, **24** (2019), 5927–5944.
[42] J. L. Lions, *Équations Différentielles Opérationelles et Problèmes aux Limites*, Die Grundlehren der mathematischen Wissenschaften, Bd. 111 Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961. ix+292 pp (in French).

[43] W. Liu, Invariance of subspaces under the solution flow of SPDE, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 13 (2010), 87–98.

[44] W. Liu and J. M. Tölle, Existence and uniqueness of invariant measures for stochastic evolution equations with weakly dissipative drifts, *Electron. Commun. Probab.*, 16 (2011), 447–457.

[45] X. Liu and Z. Liu, Poisson stable solutions for stochastic differential equations with Lévy noise, *Acta Math. Sin. (Engl. Ser.)*, to appear.

[46] Z. Liu and K. Sun, Almost automorphic solutions for stochastic differential equations driven by Lévy noise, *J. Funct. Anal.*, 266 (2014), 1115–1149.

[47] Z. Liu and W. Wang, Favard separation method for almost periodic stochastic differential equations, *J. Differential Equations*, 260 (2016), 8109–8136.

[48] E. Pardoux, Équations aux dérivées partielles stochastiques de type monotone, Séminaire sur les Équations aux Dérivées Partielles (1974–1975), III, Exp. No. 2, 10 pp. Collège de France, Paris, 1975 (in French).

[49] K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Probability and Mathematical Statistics, No. 3 Academic Press, Inc., New York-London, 1967. xi+276 pp.

[50] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983. viii+279 pp.

[51] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Lecture Notes in Mathematics, 1905. Springer, Berlin, 2007. vi+144 pp.

[52] M. Röckner, B. Schmuland and X. Zhang, Yamada-Watanabe Theorem for stochastic evolution equations in infinite dimensions, *Cond. Matt. Phys.*, 11 (2008), 247–259.

[53] R. J. Sacker and G. R. Sell, Lifting properties in skew-product flows with applications to differential equations, *Mem. Amer. Math. Soc.*, 11 (1977), iv+67 pp.

[54] W. Shen and Y. Yi, Almost automorphic and almost periodic dynamics in skew-product semiflows, *Mem. Amer. Math. Soc.*, 136 (1998), x+93 pp.

[55] E. R. van Kampen, Almost periodic functions and compact groups, *Ann. of Math.*, 37 (1936), 78–91.

[56] W. A. Veech, Almost automorphic functions on groups, *Amer. J. Math.*, 87 (1965), 719–751.

[57] J. von Neumann, Almost periodic functions in a group. I, *Trans. Amer. Math. Soc.*, 36 (1934), 445–492.

[58] Y. Wang and Z. Liu, Almost periodic solutions for stochastic differential equations with Lévy noise, *Nonlinearity*, 25 (2012), 2803–2821.

[59] T. Yoshizawa, *Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions*, Applied Mathematical Sciences, Vol. 14. Springer-Verlag, New York-Heidelberg, 1975. viii+233 pp.

[60] X. Zhang, On stochastic evolution equations with non-Lipschitz coefficients, *Stoch. Dyn.*, 9 (2009), 549–595.

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E-mail address: mengyucheng@mail.dlut.edu.cn
E-mail address: zxliu@dlut.edu.cn