Some results on relative dual Baer property

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Abstract: Let $R$ be a ring. In this article, we introduce and study relative dual Baer property. We characterize $R$-modules $M$ which are $R$-$R$-dual Baer, where $R$ is a commutative principal ideal domain. It is shown that over a right noetherian right hereditary ring $R$, an $R$-module $M$ is $N$-dual Baer for all $R$-modules $N$ if and only if $M$ is a direct summand of $M$. It is also shown that for $R$-modules $M_1, M_2, \ldots, M_n$ such that $M_i$ is $M_j$-projective for all $i > j \in \{1, 2, \ldots, n\}$, an $R$-module $N$ is $\bigoplus_{i=1}^{n} M_i$-dual Baer if and only if $N$ is $M_i$-dual Baer for all $i \in \{1, 2, \ldots, n\}$. We prove that an $R$-module $M$ is dual Baer if and only if $S = \text{End}_R(M)$ is a Baer ring and $IM = r_M(l_S(IM))$ for every right ideal $I$ of $S$.

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1. Introduction

Throughout this paper, $R$ will denote an associative ring with identity, and all modules are unitary right $R$-modules. Let $M$ be an $R$-module. We will use the notation $N \ll M$ to indicate that $N$ is small in $M$ (i.e., $L + N \neq M$ for every proper submodule $L$ of $M$). By $E(M)$ and $\text{End}_R(M)$, we denote the injective hull of $M$ and the endomorphism ring of $M$, respectively. By $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the set of rational numbers, integers and natural numbers, respectively. For a prime number $p$, $\mathbb{Z}(p^\infty)$ denotes the Prüfer $p$-group.

The concept of Baer rings was first introduced in [6] by Kaplansky. Since then, many authors have studied this kind of rings (see, e.g., [2] and [3]). A ring $R$ is called Baer if the right annihilator of any nonempty subset of $R$ is generated by an idempotent. In 2004, Rizvi and Roman extended the notion of Baer rings to a module theoretic version [10]. According to [10], a module $M$ is called a Baer module if for every left ideal $I$ of $\text{End}_R(M)$, $\bigcap_{\phi \in I} \ker \phi$ is a direct summand of $M$. This notion was recently dualized by Keskin Tütüncü-Tribak in [14]. A module $M$ is said to be dual Baer if for every right ideal $I$ of $\text{End}_R(M)$, $\bigcap_{\phi \in I} \ker \phi$ is a direct summand of $M$. This notion was recently dualized by Keskin Tütüncü-Tribak in [14]. A module $M$ is said to be dual Baer if for every right ideal $I$ of $\text{End}_R(M)$, $\bigcap_{\phi \in I} \ker \phi$ is a direct summand of $M$. This notion was recently dualized by Keskin Tütüncü-Tribak in [14]. A module $M$ is said to be dual Baer if for every right ideal $I$ of $\text{End}_R(M)$, $\bigcap_{\phi \in I} \ker \phi$ is a direct summand of $M$. This notion was recently dualized by Keskin Tütüncü-Tribak in [14]. A module $M$ is said to be dual Baer if for every right ideal $I$ of $\text{End}_R(M)$, $\bigcap_{\phi \in I} \ker \phi$ is a direct summand of $M$. This notion was recently dualized by Keskin Tütüncü-Tribak in [14]. A module $M$ is said to be dual Baer if for every right ideal $I$ of $\text{End}_R(M)$, $\bigcap_{\phi \in I} \ker \phi$ is a direct summand of $M$. This notion was recently dualized by Keskin Tütüncü-Tribak in [14]. A module $M$ is said to be dual Baer if for every right ideal $I$ of $\text{End}_R(M)$, $\bigcap_{\phi \in I} \ker \phi$ is a direct summand of $M$. This notion was recently dualized by Keskin Tütüncü-Tribak in [14].
\[ I \text{ of } S = \text{End}_R(M), \sum_{\phi \in I} \text{Im} \phi \text{ is a direct summand of } M. \] Equivalently, for every nonempty subset \( A \) of \( S \), \( \sum_{\phi \in A} \text{Im} \phi \) is a direct summand of \( M \) (see [14, Theorem 2.1]).

A module \( M \) is said to be \textit{Rickart} if for any \( \varphi \in \text{End}_R(M) \), \( \text{Ker} \varphi \) is a direct summand of \( M \) (see [7]). The notion of dual Rickart modules was studied recently in [8] by Lee-Rizvi-Roman. A module \( M \) is said to be \textit{dual Rickart} if for every \( \varphi \in \text{End}_R(M) \), \( \text{Im} \varphi \) is a direct summand of \( M \). In [8], it was introduced the notion of relative dual Rickart property which was used in the study of direct sums of dual Rickart modules. Let \( N \) be an \( R \)-module. An \( R \)-module \( M \) is called \textit{\( N \)-dual Rickart} if for every homomorphism \( \varphi : M \rightarrow N \), \( \text{Im} \varphi \) is a direct summand of \( N \) (see [8]). Similarly, we introduce in this paper the concept of relative dual Baer property. A module \( M \) is called \textit{\( N \)-dual Baer} if for every subset \( A \) of \( \text{Hom}_R(M, N) \), \( \sum_{f \in A} \text{Im} f \) is a direct summand of \( N \). It is clear that if \( M \) is \( N \)-dual Baer, then \( M \) is \( N \)-dual Rickart.

We determine the structure of modules \( M \) which are \( R_R \)-dual Baer for a commutative principal ideal domain \( R \) (Proposition 2.7). Then we show that for an \( R \)-module \( M \), \( R_R \) is \( M \)-dual Baer if and only if \( M \) is a semisimple module (Proposition 2.9). It is shown that over a right noetherian right hereditary ring \( R \), an \( R \)-module \( M \) is \( N \)-dual Baer for all \( R \)-modules \( N \) if and only if \( M \) is an injective \( R \)-module (Corollary 2.17). We prove that if \( \{M_i\}_i \) is a family of \( R \)-modules, then for each \( j \in I \), \( \bigoplus_{i \in I} M_i \) is \( M_j \)-dual Baer if and only if \( M_i \) is \( M_j \)-dual Baer for all \( i \in I \) (Corollary 2.24). It is also shown that for \( R \)-modules \( M_1, M_2, \ldots, M_n \) such that \( M_i \) is \( M_j \)-projective for all \( i > j \in \{1, 2, \ldots, n\} \), an \( R \)-module \( N \) is \( \bigoplus_{i=1}^n M_i \)-dual Baer if and only if \( M_i \) is \( M_j \)-dual Baer for all \( i \in \{1, 2, \ldots, n\} \) (Theorem 2.25). We conclude this paper by showing that an \( R \)-module \( M \) is dual Baer if and only if \( S = \text{End}_R(M) \) is a Baer ring and \( IM = \tau_M(l_S(I)) \) for every right ideal \( I \) of \( S \), where \( l_S(I) = \{ \varphi \in S \mid \varphi I = 0 \} \), \( r_M(l_S(I)) = \{ m \in M \mid l_S(I)m = 0 \} \) and \( IM = \sum_{f \in I} \text{Im} f \) (Theorem 2.31).

2. Main results

**Definition 2.1.** Let \( N \) be an \( R \)-module. An \( R \)-module \( M \) is called \textit{\( N \)-dual Baer} if, for every subset \( A \) of \( \text{Hom}_R(M, N) \), \( \sum_{f \in A} \text{Im} f \) is a direct summand of \( N \).

Obviously, an \( R \)-module \( M \) is dual Baer if and only if \( M \) is \( M \)-dual Baer.

**Example 2.2.** (1) Let \( N \) be a semisimple \( R \)-module. Then for every \( R \)-module \( M \), \( M \) is \( N \)-dual Baer.

(2) If \( M \) and \( N \) are \( R \)-modules such that \( \text{Hom}_R(M, N) = 0 \), then \( M \) is \( N \)-dual Baer. It follows that for any couple of different maximal ideals \( m_1 \) and \( m_2 \) of a commutative noetherian ring \( R \), \( E(R/m_1) \) is \( E(R/m_2) \)-dual Baer (see [12, Proposition 4.21]).

(3) Let \( p \) be a prime number. Note that \( \mathbb{Z}/p\mathbb{Z} \) and \( \mathbb{Z}(p^\infty) \) are dual Baer \( \mathbb{Z} \)-modules. On the other hand, it is clear that \( \mathbb{Z}(p^\infty) \) is \( \mathbb{Z}/p\mathbb{Z} \)-dual Baer but \( \mathbb{Z}/p\mathbb{Z} \) is not \( \mathbb{Z}(p^\infty) \)-dual Baer.

Recall that a module \( M \) is said to have the \textit{strong summand sum property}, denoted briefly by \( \text{SSSP} \), if the sum of any family of direct summands of \( M \) is a direct summand of \( M \).

Following [8, Definition 2.14], a module \( M \) is called \textit{\( N \)-d-Rickart} if, for every homomorphism \( \varphi : M \rightarrow N \), \( \text{Im} \varphi \) is a direct summand of \( N \).

**Proposition 2.3.** Let \( M \) and \( N \) be two \( R \)-modules. If \( M \) is \( N \)-dual Baer, then \( M \) is \( N \)-d-Rickart. The converse holds when \( N \) has the \( \text{SSSP} \).

**Proof.** This follows from the definitions of “\( M \) is \( N \)-d-Rickart” and “\( M \) is \( N \)-dual Baer”.

The next example shows that the assumption “\( N \) has the \( \text{SSSP} \)” is not superfluous in Proposition 2.3.

**Example 2.4.** Let \( R \) be a von Neumann regular ring which is not semisimple (e.g., \( R = \prod_{l=1}^\infty \mathbb{Z}/2\mathbb{Z} \)). By [8, Proposition 2.26], the \( R \)-module \( R_R \) does not have the \( \text{SSSP} \). On the other hand, \( R_R \) is \( R_R \)-d-Rickart, but it is not \( R_R \)-dual Baer (see [14, Corollary 2.9] and [8, Remark 2.2]).
Proposition 2.5. Let $N$ be an indecomposable $R$-module. Then the following conditions are equivalent for an $R$-module $M$.

(i) $M$ is $N$-dual Baer;

(ii) $M$ is $N$-d-Rickart;

(iii) Every nonzero $\varphi \in \text{Hom}_R(M, N)$ is an epimorphism.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear.

(ii) $\Rightarrow$ (iii) Let $0 \neq \varphi \in \text{Hom}_R(M, N)$. By assumption, $\text{Im}\varphi$ is a direct summand of $N$. But $N$ is indecomposable. Then $\text{Im}\varphi = N$. This completes the proof.

Proposition 2.6. Let $M$ and $N$ be modules such that $\text{Hom}_R(M, N) \neq 0$ (e.g., $N$ is $M$-generated). Then the following conditions are equivalent:

(i) $M$ is $N$-dual Baer and $N$ is indecomposable;

(ii) Every nonzero homomorphism $\varphi \in \text{Hom}_R(M, N)$ is an epimorphism.

Proof. (i) $\Rightarrow$ (ii) This follows from Proposition 2.5.

(ii) $\Rightarrow$ (i) It is clear that $M$ is $N$-dual Baer. Now let $K$ be a nonzero direct summand of $N$. Let $K'$ be a submodule of $N$ such that $N = K \oplus K'$. Since $\text{Hom}_R(M, N) \neq 0$, there exists a nonzero homomorphism $\varphi \in \text{Hom}_R(M, N)$. Let $\pi': N \to K'$ be the projection map and let $i': K' \to N$ be the inclusion map. Then $i'\pi'\varphi \in \text{Hom}_R(M, N)$. Assume that $i'\pi'\varphi \neq 0$. By hypothesis, $\text{Im}i'\pi'\varphi = N$. So $K' = N$. Thus $K = 0$, a contradiction. Therefore $i'\pi'\varphi = 0$. Hence $K' = 0$ and $K = N$. It follows that $N$ is indecomposable.

The following result describes the structure of $R$-modules which are $R_R$-dual Baer, where $R$ is a commutative principal ideal domain which is not a field.

Proposition 2.7. Let $R$ be a commutative principal ideal domain which is not a field. Then the following conditions are equivalent for an $R$-module $M$:

(i) $M$ is $R_R$-dual Baer;

(ii) $M$ is $R_R$-d-Rickart;

(iii) $M$ has no nonzero cyclic torsion-free direct summands;

(iv) $\text{Hom}_R(M, R_R) = 0$.

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Assume that $M$ has an element $x$ such that $xR$ is a direct summand of $M$ and $R_R \cong xR$. Let $\pi : M \to xR$ be the projection map and let $f : xR \to R_R$ be an isomorphism. Then $f\pi : M \to R_R$ is an epimorphism. Let $\alpha$ be a nonzero element of $R$ which is not invertible. Consider the homomorphism $g : R_R \to R_R$ defined by $g(r) = \alpha r$ for all $r \in R$. Then $gf\pi \in \text{Hom}_R(M, R_R)$ and $\text{Im}gf\pi = \alpha R$. It is clear that $\alpha R \neq 0$ and $\alpha R \neq R$. Thus $\alpha R$ is not a direct summand of $R$. So $M$ is not $R_R$-d-Rickart, a contradiction.

(iii) $\Rightarrow$ (iv) Assume that $\text{Hom}_R(M, R_R) \neq 0$. So there exists a nonzero homomorphism $f : M \to R_R$. Thus $\text{Im}f = aR$ for some nonzero $a \in R$ since $R$ is a principal ideal domain. Then $M/\text{Ker}f \cong aR \cong R_R$ is a projective $R$-module. It follows that $\text{Ker}f$ is a direct summand of $M$. Let $Y$ be a submodule of $M$ such that $M = \text{Ker}f \oplus Y$. Therefore $Y \cong R_R$. This contradicts our assumption. Hence $\text{Hom}_R(M, R_R) = 0$.

(iv) $\Rightarrow$ (i) This is immediate.

Example 2.8. Consider a $\mathbb{Z}$-module $M = \mathbb{Q}(I) \oplus T$, where $T$ is a torsion $\mathbb{Z}$-module and $I$ is an index set. Suppose that $M$ is not $\mathbb{Z}$-dual Baer. By Proposition 2.7, there exists a cyclic submodule $L$ of $M$ such that $L \cong \mathbb{Z}$ and $L$ is a direct summand of $M$. Let $N$ be a submodule of $M$ such that $M = L \oplus N$.
Remark 2.11. The following conditions are equivalent for a ring $R$:

(i) $R$ is $M$-dual Baer;

(ii) $M$ is a semisimple module.

Proof. (i) $\implies$ (ii) Let $x \in M$. Consider the $R$-homomorphism $\varphi : R \to M$ defined by $\varphi(r) = xr$ for all $r \in R$. Then $\text{Im} \varphi = xR$. Since $R_R$ is $M$-dual Baer, it follows that for any submodule $L$ of $M$, $L = \sum_{x \in L} xR$ is a direct summand of $M$. Therefore $M$ is semisimple.

(ii) $\implies$ (i) is obvious. \qed

Corollary 2.10. The following conditions are equivalent for a ring $R$:

(i) The $R$-module $R_R$ is $R$-dual Baer;

(ii) The $R$-module $R_R$ is $E(R)$-dual Baer;

(iii) $R$ is a semisimple ring.

Proof. (i) $\iff$ (iii) By [14, Corollary 2.9].

(ii) $\iff$ (iii) This follows from Proposition 2.9. \qed

Remark 2.11. If $K$ is a submodule of an $R$-module $M$ such that $K$ is $M$-dual Baer, then $K$ is a direct summand of $M$. In particular, if the $R$-module $M$ is $E(M)$-dual Baer, then $M$ is an injective module.

The next example shows that even if a module $M$ is injective, the module $M$ need not be $M$-dual Baer.

Example 2.12. Let $R$ be a self injective ring which is not semisimple (e.g., $R = \prod_{n=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$). Then $E(R_R) = R_R$. By [14, Corollary 2.9], the $R$-module $R_R$ is not $R_R$-dual Baer.

Next, we will be concerned with the modules $M$ which are $N$-dual Baer for all modules $N$. We begin with the following proposition which provides a class of rings $R$ whose semisimple modules are $N$-dual Baer for any $R$-module $N$.

Proposition 2.13. Let $R$ be a right noetherian right $V$-ring and let $M$ be a semisimple $R$-module. Then $M$ is $N$-dual Baer for every $R$-module $N$.

Proof. Let $N$ be an $R$-module. It is clear that for any $\varphi \in \text{Hom}_R(M,N)$, $\text{Im} \varphi$ is semisimple. Let $A$ be a subset of $\text{Hom}_R(M,N)$. Then $\sum_{f \in A} \text{Im} f$ is a semisimple submodule of $N$. Since $R$ is a right noetherian right $V$-ring, $\sum_{f \in A} \text{Im} f$ is injective by [4, Proposition 1]. Therefore $\sum_{f \in A} \text{Im} f$ is a direct summand of $N$. So $M$ is $N$-dual Baer. \qed

The next example shows that the condition “$R$ is a right noetherian ring” in the hypothesis of Proposition 2.13 is not superfluous.

Example 2.14. Let $F$ be a field and let $R = \prod_{n \in \mathbb{N}} F_n$ such that $F_n = F$ for all $n \in \mathbb{N}$. Then $R$ is a commutative $V$-ring which is not noetherian. Note that $\text{Soc}(R) = \bigoplus_{n \in \mathbb{N}} F_n$ is an essential proper ideal of $R$. In particular, $\text{Soc}(R)$ is not a direct summand of $R$. So $\text{Soc}(R)$ is not $R_R$-dual Baer.
Proposition 18.13. Therefore since the ring (i) Proof.

\[ R \]

is a right noetherian ring. Then the following conditions are equivalent for an \( R \)-module \( M \):

(i) \( M \) is \( N \)-dual Baer for all \( R \)-modules \( N \);
(ii) Every factor module of \( M \) is an injective \( R \)-module.

Proof. (i) \( \Rightarrow \) (ii) By Proposition 2.15.

(ii) \( \Rightarrow \) (i) Let \( N \) be an \( R \)-module. It is clear that \( \text{Im} \varphi \) is injective for every \( \varphi \in \text{Hom}_R(M,N) \). Since the ring \( R \) is right noetherian, \( \sum_{f \in A} \text{Im} f \) is injective for every subset \( A \) of \( \text{Hom}_R(M,N) \) by [1, Proposition 18.13]. Therefore \( \sum_{f \in A} \text{Im} f \) is a direct summand of \( N \). This proves the proposition.

Recall that a ring \( R \) is called right hereditary if each of its right ideals is projective. It is well known that a ring \( R \) is right hereditary if and only if every factor module of an injective right \( R \)-module is injective (see, for example [16, 39.16]). The next result is a direct consequence of Proposition 2.16. It determines the structure of \( R \)-modules \( M \) which are \( N \)-dual Baer for all \( R \)-modules \( N \), where \( R \) is a right noetherian right hereditary ring.

Corollary 2.17. Let \( R \) be a right noetherian right hereditary ring (e.g., \( R \) is a Dedekind domain). Then the following conditions are equivalent for an \( R \)-module \( M \):

(i) \( M \) is \( N \)-dual Baer for any \( R \)-module \( N \);
(ii) \( M \) is an injective \( R \)-module.

Example 2.18. Let \( M \) be a \( \mathbb{Z} \)-module. It is easily seen from Corollary 2.17 that \( M \) is \( N \)-dual Baer for any \( \mathbb{Z} \)-module \( N \) if and only if \( M \) is a direct sum of \( \mathbb{Z} \)-modules each isomorphic to the additive group of rational numbers \( \mathbb{Q} \) or to \( \mathbb{Z}(p^\infty) \) (for various primes \( p \)).

Combining Corollary 2.17 and [8, Corollary 2.30], we obtain the following result.

Corollary 2.19. The following conditions are equivalent for a ring \( R \):

(i) Every injective \( R \)-module is dual Baer;
(ii) Every injective module is \( N \)-dual Baer for every \( R \)-module \( N \);
(iii) \( R \) is a right noetherian right hereditary ring.

The next characterization extends [14, Corollary 2.5].

Theorem 2.20. Let \( M \) and \( N \) be two \( R \)-modules. Then \( M \) is \( N \)-dual Baer if and only if for any direct summand \( M' \) of \( M \) and any submodule \( N' \) of \( N \), \( M' \) is \( N' \)-dual Baer.

Proof. Let \( M' = eM \) for some \( e^2 = e \in \text{End}_R(M) \) and let \( N' \) be a submodule of \( N \). Let \( \{ \varphi_i \}_{i \in I} \) be a family of homomorphisms in \( \text{Hom}_R(M',N') \). Since \( \varphi_i(M) = \varphi_i(M') \subseteq N' \subseteq N \) for every \( i \in I \) and \( M \) is \( N \)-dual Baer, \( \sum_{i \in I} \varphi_i(M) \) is a direct summand of \( N \). Therefore \( \sum_{i \in I} \varphi_i(M) \) is a direct summand of \( N' \). It follows that \( M' \) is \( N' \)-dual Baer. The converse is obvious.
Corollary 2.21. The following conditions are equivalent for a module $M$:

(i) $M$ is a dual Baer module;

(ii) For any direct summand $K$ of $M$ and any submodule $N$ of $M$, $K$ is $N$-dual Baer.

From [14, Example 3.1 and Theorem 3.4], it follows that a direct sum of dual Baer modules is not dual Baer, in general. Next, we focus on when a direct sum of $N$-dual Baer modules is also $N$-dual Baer for some module $N$.

Proposition 2.22. Let $N$ be a module having the SSSP and let $\{M_i\}_i$ be a family of modules. Then $\bigoplus_{i \in I} M_i$ is $N$-dual Baer if and only if $M_i$ is $N$-dual Baer for all $i \in I$.

Proof. Suppose that $\bigoplus_{i \in I} M_i$ is $N$-dual Baer. By Theorem 2.20, $M_i$ is $N$-dual Baer for all $i \in I$. Conversely, assume that $M_i$ is $N$-dual Baer for all $i \in I$. Let $\{\varphi_i\}_\lambda$ be a family of homomorphisms in $\text{Hom}_R(\bigoplus_{i \in I} M_i, N)$. For each $i \in I$, let $\mu_i : M_i \to \bigoplus_{i \in I} M_i$ denote the inclusion map. Then for every $i \in I$ and every $\lambda \in \Lambda$, $\varphi_i M_i \in \text{Hom}_R(M, N)$. Since $M_i$ is $N$-dual Baer for every $i \in I$, it follows that $\text{Im}(\varphi_i M_i)$ is a direct summand of $N$ for every $(i, \lambda) \in I \times \Lambda$. Note that for each $\lambda \in \Lambda$, $\text{Im}(\varphi_i M_i) = \sum_{i \in I} \text{Im}(\varphi_i M_i)$. As $N$ has the SSSP, $\sum_{\lambda \in \Lambda} \text{Im}(\varphi_i M_i) = \sum_{\lambda \in \Lambda} \sum_{i \in I} \text{Im}(\varphi_i M_i)$ is a direct summand of $N$. Therefore $\bigoplus_{i \in I} M_i$ is $N$-dual Baer.

The following result is taken from [14, Theorem 2.1].

Theorem 2.23. The following conditions are equivalent for a module $M$ and $S = \text{End}_R(M)$:

(i) $M$ is a dual Baer module;

(ii) For every nonempty subset $A$ of $S$, $\sum_{f \in A} \text{Im} f = e(M)$ for some idempotent $e \in S$;

(iii) $M$ has the SSSP and for every $\varphi : M \to M$, $\text{Im}\varphi$ is a direct summand of $M$.

Corollary 2.24. Let $\{M_i\}_i$ be a family of modules and let $j \in I$. Then $\bigoplus_{i \in I} M_i$ is $M_j$-dual Baer if and only if $M_i$ is $M_j$-dual Baer for all $i \in I$.

Proof. The necessity follows from Theorem 2.20. Conversely, by assumption, we have $M_i$ is $M_j$-dual Baer. Then $M_i$ is a dual Baer module. By Theorem 2.23, $M_i$ has the SSSP. Applying Proposition 2.22, $\bigoplus_{i \in I} M_i$ is $M_j$-dual Baer.

In the following result, we present conditions under which a module $N$ is $\bigoplus_{i=1}^n M_i$-dual Baer for some modules $M_i$ ($1 \leq i \leq n$).

Theorem 2.25. Let $M_1, \ldots, M_n$ be $R$-modules, where $n \in \mathbb{N}$. Assume that $M_i$ is $M_i$-projective for all $i > j \in \{1, 2, \ldots, n\}$. Then for any $R$-module $N$, $N$ is $\bigoplus_{i=1}^n M_i$-dual Baer if and only if $N$ is $M_i$-dual Baer for all $i \in \{1, 2, \ldots, n\}$.

Proof. The necessity follows from Theorem 2.20. Conversely, suppose that $N$ is $M_i$-dual Baer for all $i \in \{1, 2, \ldots, n\}$. We will show that $N$ is $\bigoplus_{i=1}^n M_i$-dual Baer. By induction on $n$ and taking into account [9, Proposition 4.33], it is sufficient to prove this for the case $n = 2$. Assume that $N$ is $M_i$-dual Baer for $i = 1, 2$ and $M_2$ is $M_1$-projective. Let $\{\varphi_i\}_\lambda$ be a family of homomorphisms in $\text{Hom}_R(N, M_1 \oplus M_2)$. Let $\pi_2 : M_1 \oplus M_2 \to M_2$ be the projection of $M_1 \oplus M_2$ on $M_2$ along $M_1$. We want to prove that $\sum_{\lambda \in \Lambda} \text{Im}\varphi_\lambda$ is a direct summand of $M_1 \oplus M_2$. Since $N$ is $M_2$-dual Baer, $\sum_{\lambda \in \Lambda} \varphi_\lambda(N)$ is a direct summand of $M_2$. So $\sum_{\lambda \in \Lambda} \varphi_\lambda(N) = M_1 \oplus (\sum_{\lambda \in \Lambda} \varphi_\lambda(N))$ is a direct summand of $M_1 \oplus M_2$, there exists a submodule $L \leq \sum_{\lambda \in \Lambda} \text{Im}\varphi_\lambda$ such that $M_1 \oplus (\sum_{\lambda \in \Lambda} \text{Im}\varphi_\lambda) = M_1 \oplus L$ by [9, Lemma 4.47]. Thus $\sum_{\lambda \in \Lambda} \text{Im}\varphi_\lambda = (M_1 \cap (\sum_{\lambda \in \Lambda} \text{Im}\varphi_\lambda)) \oplus L$ by modularity. It is easily seen that $\sum_{\lambda \in \Lambda} \varphi_\lambda(N)$ is a direct summand of $M_2$. Let $K_2$ be a submodule of $M_2$ such that $M_2 = K_2 \oplus (\sum_{\lambda \in \Lambda} \varphi_\lambda(N))$. Therefore $M_1 \oplus M_2 = M_1 \oplus L \oplus K_2$. Let $\pi_1 : M_1 \oplus (L \oplus K) \to M_1$
be the projection of $M_1 \oplus M_2$ on $M_1$ along $L \oplus K$. Then $\pi_1 \phi_\lambda \in \text{Hom}_R(N, M_1)$ for every $\lambda \in \Lambda$. Moreover, we have

$$\sum_{\lambda \in \Lambda} \pi_1 \phi_\lambda(N) = \pi_1 \left( \sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda \right) = \left( \left( \sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda \right) + (L \oplus K) \right) \cap M_1.$$ 

But $\sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda = (M_1 \cap (\sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda)) \oplus L$. Then,

$$\sum_{\lambda \in \Lambda} \pi_1 \phi_\lambda(N) = \left( M_1 \cap \left( \sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda \right) \right) \oplus L \oplus K \cap M_1 = M_1 \cap \left( \sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda \right).$$

Since $N$ is $M_1$-dual Baer, $\sum_{\lambda \in \Lambda} \pi_1 \phi_\lambda(N) = M_1 \cap \left( \sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda \right)$ is a direct summand of $M_1$. It follows that $(M_1 \cap (\sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda)) \oplus L$ is a direct summand of $M_1 \oplus L \oplus K_2$. So $\sum_{\lambda \in \Lambda} \text{Im} \phi_\lambda$ is a direct summand of $M_1 \oplus M_2$. Consequently, $N$ is $M_1 \oplus M_2$-dual Baer. This completes the proof.

**Corollary 2.26.** Let $M_1, \ldots, M_n$ be $R$-modules, where $n \in \mathbb{N}$. Assume that $M_i$ is $M_j$-projective for all $i > j \in \{1, 2, \ldots, n\}$. Then $M = \bigoplus_{i=1}^n M_i$ is a dual Baer module if and only if $M_i$ is $M_j$-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$.

**Proof.** The necessity follows from Theorem 2.20. Conversely, suppose that $M_i$ is $M_j$-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$. By Corollary 2.24, $M$ is $M_j$-dual Baer for all $j \in \{1, 2, \ldots, n\}$. Since $M_i$ is $M_j$-projective for all $i > j \in \{1, 2, \ldots, n\}$, $M$ is $\bigoplus_{i=1}^n M_i$-dual Baer by Theorem 2.25. Thus $M$ is a dual Baer module.

Note that the sufficiency in Corollary 2.26 can be proved by using [14, Theorem 3.10].

**Proposition 2.27.** Let $M_1, \ldots, M_n$ be $R$-modules, where $n \in \mathbb{N}$. Assume that $M_i$ is $M_j$-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$. Then $\bigoplus_{i=1}^n M_i$ is a dual Baer module if and only if $M_i$ is $M_j$-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$ and $M$ has the SSSP.

**Proof.** ($\Rightarrow$) By [8, Theorem 5.11], $M_i$ is $M_j$-Rickart for all $i, j \in \{1, 2, \ldots, n\}$. Note that $M_i$ has the SSSP for every $i \in \{1, 2, \ldots, n\}$ (see Theorem 2.23). Applying Proposition 2.3, it follows that $M_i$ is $M_j$-dual Baer for all $i, j \in \{1, 2, \ldots, n\}$.

($\Leftarrow$) This follows easily from [8, Theorem 5.11], Proposition 2.3 and Theorem 2.23.

**Theorem 2.28.** Let $M = \bigoplus_{i \in I} M_i$ be the direct sum of fully invariant submodules $M_i$. Then $M$ is a dual Baer module if and only if $M_i$ is a dual Baer module for all $i \in I$.

**Proof.** The necessity follows from [14, Corollary 2.5]. Conversely, let $S = \text{End}_R(M)$ and let $\{\varphi_\lambda\}_\Lambda$ be a family of homomorphisms in $S$. For each $i \in I$, let $\pi_i : M \to M_i$ be the projection map and let $\mu_i : M_i \to M$ be the inclusion map. Note that for each $\lambda \in \Lambda$, $\varphi_\lambda(M) = \sum_{i \in I} \varphi_\lambda \mu_i(M_i)$. Since each $M_i$ ($i \in I$) is fully invariant in $M$, it follows that $\varphi_\lambda(M) = \sum_{i \in I} \pi_i \varphi_\lambda \mu_i(M_i)$ for all $\lambda \in \Lambda$. For every $i \in I$ and every $\lambda \in \Lambda$, let $N_{i,\lambda} = \pi_i \varphi_\lambda \mu_i(M_i)$. Therefore,

$$\sum_{\lambda \in \Lambda} \varphi_\lambda(M) = \sum_{\lambda \in \Lambda} \sum_{i \in I} \pi_i \varphi_\lambda \mu_i(M_i) = \sum_{\lambda \in \Lambda} \left( \sum_{i \in I} N_{i,\lambda} \right) = \bigoplus_{i \in I} \left( \sum_{\lambda \in \Lambda} N_{i,\lambda} \right).$$

Since each $M_i$ ($i \in I$) is dual Baer, each $M_i$ ($i \in I$) has the SSSP by Theorem 2.23. Thus $\sum_{\lambda \in \Lambda} N_{i,\lambda}$ is a direct summand of $M_i$ for every $i \in I$. So $\sum_{\lambda \in \Lambda} \varphi_\lambda(M)$ is a direct summand of $M$. Consequently, $M$ is a dual Baer module.
We conclude this paper by showing a new characterization of dual Baer modules.

Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). Then for every nonempty subset \( A \) of \( S \), we denote \( l_S(A) = \{ \phi \in S \mid \phi A = 0 \} \) and \( r_M(A) = \{ m \in M \mid Am = 0 \} \). We also denote \( l_S(N) = \{ \phi \in S \mid \phi(N) = 0 \} \) for any submodule \( N \) of \( M \).

Recall that a ring \( R \) is called a Baer ring if for every nonempty subset \( I \subseteq R \), there exists an idempotent \( e \in R \) such that \( l_S(I) = Re \).

**Proposition 2.29.** ([5, Proposition 2.3]) For an \( R \)-module \( M, S = \text{End}_R(M) \) is a Baer ring if and only if \( r_M(l_S(\sum_{\varphi \in A} \text{Im} \varphi)) \) is a direct summand of \( M \) for all nonempty subsets \( A \) of \( S \).

The next example shows that if \( M \) is a module such that \( S = \text{End}_R(M) \) is a Baer ring, then \( M \) is not a dual Baer module, in general.

**Example 2.30.** Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z} \). Then \( S = \text{End}_\mathbb{Z}(M) \cong \mathbb{Z} \). Clearly, \( \mathbb{Z} \) is a Baer ring. On the other hand, it is easily seen that \( M \) is not a dual Baer module.

Note that if \( M \) is an \( R \)-module with \( S = \text{End}_R(M) \), then for any nonempty subset \( A \) of \( S \), \( l_S(A) = l_S(AM) \), where \( AM = \sum_{f \in A} \text{Im} f \). The next result can be considered as an analogue of [8, Theorem 3.5].

**Theorem 2.31.** The following are equivalent for an \( R \)-module \( M \) and \( S = \text{End}_R(M) \):

(i) \( M \) is a dual Baer module;

(ii) \( S \) is a Baer ring and \( AM = r_M(l_S(AM)) \) for every nonempty subset \( A \) of \( S \);

(iii) \( S \) is a Baer ring and \( IM = r_M(l_S(IM)) \) for every right ideal \( I \) of \( S \).

**Proof.**

(i) \( \Rightarrow \) (ii) From [15, Theorem 3.6], it follows that \( S \) is a Baer ring. Moreover, we have \( r_M(l_S(AM)) = r_M(l_S(A)) = r_M(S(1 - e)) = e(M) = AM \) for all nonempty subsets \( A \) of \( S \).

(ii) \( \Rightarrow \) (iii) This is obvious.

(iii) \( \Rightarrow \) (i) Let \( I \) be a right ideal of \( S \). Since \( S \) is a Baer ring, \( r_M(l_S(IM)) \) is a direct summand of \( M \) by Proposition 2.29. But \( IM = r_M(l_S(IM)) \). Then \( IM \) is a direct summand of \( M \). By Theorem 2.23, it follows that \( M \) is a dual Baer module.

Combining Theorem 2.31 and [10, Theorem 4.1], we get the following result.

**Corollary 2.32.** Let \( M \) be an \( R \)-module such that \( IM = r_M(l_S(IM)) \) for every right ideal \( I \) of \( S = \text{End}_R(M) \). If \( M \) is a Baer module, then \( M \) is a dual Baer module.

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