Non-canonical quantization of electromagnetic fields and the meaning of $Z_3$

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Non-canonical quantization is based on certain reducible representations of canonical commutation relations. Relativistic formalism for electromagnetic non-canonical quantum fields is introduced. Unitary representations of the Poincaré group at the level of fields and states are explicitly given. Multi-photon and coherent states are introduced. Statistics of photons in a coherent state is Poissonian if an appropriately defined thermodynamic limit is performed. Radiation fields having a correct $S$ matrix are constructed. The $S$ matrix is given by a non-canonical coherent-state displacement operator, a fact automatically eliminating the infrared catastrophe. This, together with earlier results on elimination of vacuum and ultraviolet infinities, suggests that non-canonical quantization leads to finite field theories. Renormalization constant $Z_3$ is identified with a parameter related to wave functions of non-canonical vacua.

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I. INTRODUCTION

The idea of non-canonical quantization of electromagnetic fields was introduced in [1]. A starting point is an observation that even in nonrelativistic quantum mechanics it is natural to treat the frequency $\omega$ characterizing a harmonic oscillator as an eigenvalue and not a parameter. A detailed analysis of a nonrelativistic indefinite-frequency harmonic oscillator given in [2] allowed to better understand physical structures associated with this modification. The main results obtained so far were

(a) Association of an entire spectrum of frequencies with a single indefinite-frequency oscillator.
(b) Non-canonical algebra of creation and annihilation operators.
(c) An ensemble of $N$ indefinite-frequency oscillators having properties of a non-canonical quantum field.
(d) Finite vacuum energy and finite vacuum fluctuations.
(e) Structures of the canonical quantum field theory obtained in the thermodynamic limit $N \to \infty$, but with form-factors regularizing ultraviolet infinities.
(f) The form-factors appearing automatically as a consequence of noncanonical commutation relations and the structure of vacuum, and not introduced in an ad hoc manner.

(g) Space of states in a form of a vector bundle with the set of vacua in a role of a base space and Fock-type fibers. Probabilities have to be associated with a single fiber. In particular, the vacuum is in a given fiber the unique vector obtained by projecting the fiber on the base space.

In [1] it was also concluded that non-canonical quantization may lead to deviations from the Planck black-body law. However, a consistent interpretation of the thermodynamic limit shows that no deviations should be expected (see below).

In the present paper we extend the formalism to the relativistic domain. We begin with quantization in free space. Representation of non-canonical commutation relations and the Poincaré group are explicitly constructed. Multi-photon and coherent states are defined. Their properties make them similar to those from the canonical formalism if one performs a thermodynamic limit. Vacuum states have properties of translation-invariant scalar fields and are Poincaré covariant. Commutation relations for vector potentials are found showing certain deviations from locality due to nontrivial structure of vacua. Radiation fields leading to the correct form of the $S$ matrix are found, the $S$ matrix being proportional to a non-canonical coherent-state displacement operator. The same mechanism that regularized vacuum and ultraviolet infinities is shown to regularize the infrared divergence. Finally, we identify the renormalization constant $Z_3$ with an amplitude related to the electromagnetic vacuum wave function. The latter result is fully consistent with the analysis of perturbation theory given in [1] and [2].

All the results obtained so far lead to the same conclusion: The main difficulty of contemporary quantum field theories is rooted in the fact that they are too classical. The reinterpretation of some parameters in terms of eigenvalues simultaneously removes the ultraviolet, infrared, and vacuum infinities, and explains the origin of renormalization constants. The above conclusions are valid, strictly speaking, for bosons. A work on fermions is in progress.
II. NOTATION

In order to control covariance properties of fields in generalized frameworks it is best to work in a manifestly covariant formalism. The most convenient is the one based on spinors and passive unitary transformations.

A. Spinor convention and fields

We take $c = 1$ and $\hbar = 1$. The index notation we use in the paper is consistent with the Penrose-Rindler spinor and world-tensor convention [4]. The electromagnetic field-tensor and its dual are

$$ F_{ab} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_2 & B_3 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \quad (1) $$

$$ *F_{ab} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix} \quad (2) $$

Self-dual and anti-self-dual parts of $F_{ab}$ are related to the electromagnetic spinor by

$$ +F_{ab} = \frac{1}{2}(F_{ab} - i*F_{ab}) = \varepsilon_{AB}\bar{\varphi}_{A'B'} = \begin{pmatrix} 0 & F_1 & F_3 \\ -F_1 & 0 & iF_2 \\ -F_2 & -iF_3 & 0 \\ -F_3 & iF_2 & -iF_1 & 0 \end{pmatrix} \quad (3) $$

$$ -F_{ab} = \frac{1}{2}(F_{ab} + i*F_{ab}) = \varepsilon_{A'B'}\varphi_{AB} \quad (4) $$

where $F = (E + iB)/2$ is the Riemann-Silberstein vector [4]. Denote $k \cdot x = k_a x^a$. The electromagnetic spinor has the following Fourier representation [4][4]

$$ \varphi_{AB}(x) = \int d\Gamma(k)\pi_A(k)\pi_B(k) \left( f(k, -)e^{-ik \cdot x} + \bar{f(k, +)}e^{ik \cdot x} \right) \quad (5) $$

$$ \bar{\varphi}_{A'B'}(x) = \int d\Gamma(k)\bar{\pi}_{A'}(k)\bar{\pi}_{B'}(k) \left( f(k, +)e^{-ik \cdot x} + \bar{f(k, -)}e^{ik \cdot x} \right) \quad (6) $$

where the spinor field $\pi_A(k)$ is related to the future-pointing 4-momentum by

$$ k^a = \pi^A(k)\bar{\pi}^{A'}(k) = (k_0, k) = (|k|, k) \quad (7) $$

and the invariant measure on the light-cone is $d\Gamma(k) = [(2\pi)^3 2k_0]^{-1}d^3k$. Anti-self-dual and self-dual parts of the field tensor are

$$ -F_{ab}(x) = \int d\Gamma(k)\varepsilon_{A'B'}\pi_{A}(k)\pi_{B}(k) \left( f(k, -)e^{-ik \cdot x} + \bar{f(k, +)}e^{ik \cdot x} \right) \quad (8) $$

$$ = \int d\Gamma(k)\varepsilon_{ab}(k) \left( f(k, -)e^{-ik \cdot x} + \bar{f(k, +)}e^{ik \cdot x} \right) \quad (9) $$

$$ +F_{ab}(x) = \int d\Gamma(k)\varepsilon_{AB}\bar{\pi}_{A'}(k)\bar{\pi}_{B'}(k) \left( f(k, +)e^{ik \cdot x} + \bar{f(k, -)}e^{-ik \cdot x} \right) \quad (10) $$

$$ = \int d\Gamma(k)\varepsilon_{ab}(k) \left( f(k, +)e^{ik \cdot x} + \bar{f(k, -)}e^{-ik \cdot x} \right) \quad (11) $$

The latter form is used by the Bialynicki-Birulas in [4]. The sign in the amplitude $f(k, \pm)$ corresponds to the value of helicity of positive-frequency fields. The four-vector potential $A_a(x)$ is related to the electromagnetic spinor by

$$ \varphi_{XY}(x) = \nabla\times_{(X}^{Y')} A_{Y'}(x) \quad (12) $$
In the Lorenz gauge $\nabla^a A_a = 0$ we do not have to symmetrize the unprimed indices and

$$\varphi_{XY}(x) = \nabla_X^\prime Y^\prime A_{Y^\prime}(x).$$  \hfill (13)

One of the possible Lorenz gauges is

$$A_a(x) = i \int d\Gamma(k) \left( m_a(k) \left( f(k,+) e^{-ik \cdot x} - \bar{f}(k,-) e^{ik \cdot x} \right) - \bar{m}_a(k) \left( \bar{f}(k,+) e^{ik \cdot x} - f(k,-) e^{-ik \cdot x} \right) \right)$$ \hfill (14)

where $\omega_A \pi^A = 1$, i.e. $\omega_A = \omega_A(k)$ is a spin-frame partner of $\pi^A(k)$. In (14) we have introduced the null vectors

$$m_a(k) = \omega_A(k) \bar{\pi}^A(k)$$ \hfill (15)

$$\bar{m}_a(k) = \pi_A(k) \bar{\omega}^A(k)$$ \hfill (16)

which, together with

$$k_a = \pi_A(k) \bar{\pi}^A(k)$$ \hfill (17)

$$\omega_a(k) = \omega_A(k) \bar{\omega}^A(k)$$ \hfill (18)

form a null tetrad [3].

A change of gauge is in the Fourier domain represented by a shift by a multiple of $k^a$. The form (14) shows that gauge freedom is related to the nonuniqueness of $\omega_A(k)$ which can be shifted by a multiple of $\pi_A(k)$.

### B. Momentum representation

Consider the momentum-space basis normalized by

$$\langle p | p' \rangle = (2\pi)^3 2 p_0 \delta^{(3)}(p - p') = \delta_T(p, p')$$ \hfill (19)

The identity operator in momentum space is $\int d\Gamma(p) |p\rangle \langle p|$. We can use the following explicit realization of $|p\rangle = |f_p\rangle$ in terms of distributions

$$f_p(k) = (2\pi)^3 2 p_0 \delta^{(3)}(p - k) = \delta_T(p, k)$$ \hfill (20)

Since

$$\int d\Gamma(k) F(k) \delta_T(p, k) = \int d^3 k \delta^{(3)}(p - k) F(k) = F(p)$$ \hfill (21)

the Fourier transform of $f_p(k)$ is

$$\hat{f}_p(x) = \int d\Gamma(k) f_p(k) e^{-ik \cdot x} = e^{-ip \cdot x}$$ \hfill (22)

If 1 is the identity operator occurring at the right-hand-side of CCR $[a_s, a^\dagger_{s'}] = \delta_{s s'} 1$, we denote

$$I_k = |k\rangle \langle k| \otimes 1,$$ \hfill (23)

$$I = \int d\Gamma(k) |k\rangle \langle k| \otimes 1.$$ \hfill (24)

### C. Multi-particle conventions

Let $A$ be an operator $A : \mathcal{H} \rightarrow \mathcal{H}$ where $\mathcal{H}$ is a one-particle Hilbert space. The multi-particle Hilbert space

$$\mathcal{H} = \otimes_{n=1}^{\infty} \otimes_n \mathcal{H}$$ \hfill (25)
is the Hilbert space of states corresponding to an indefinite number of bosonic particles; \( \otimes^n H \) stands for a space of symmetric states in \( H \otimes \ldots \otimes H \). We introduce the following notation for operators defined at the multi-particle level:

\[
\oplus_{\alpha_n} A = \alpha_1 A \oplus \alpha_2 (A \otimes I \otimes I \otimes A) \oplus \alpha_3 (A \otimes I \otimes I \otimes A \otimes I \otimes I \otimes A) \oplus \ldots
\]

(26)

Here \( \oplus_{\alpha_n} A : \mathcal{H} \rightarrow \mathcal{H} \) are real or complex parameters, and \( I \) is the identity operator in \( \mathcal{H} \).

The following properties follow directly from the definition

\[
[\oplus_{\alpha_n} A, \oplus_{\beta_n} B] = \oplus_{\alpha_n \beta_n}[A, B]
\]

(27)

\[
e^{\oplus_{\alpha_n} A} = \oplus_{n=1}^{\infty} e^{\alpha_n A} \otimes \ldots \otimes e^{\alpha_n A}
\]

(28)

\[
e^{\oplus_{\alpha_n} A} \oplus_{\beta_n} B e^{-\oplus_{\alpha_n} A} = \oplus_{\beta_n} e^A B e^{-A}
\]

(29)

Identity operators in \( \mathcal{H} \) and \( \mathcal{H} \) are related by

\[
\mathcal{L} = \oplus_n I.
\]

(30)

We will often use the operator

\[
\mathcal{L}_k = \oplus_n I_k.
\]

(31)

**III. POINCARÉ TRANSFORMATIONS OF CLASSICAL ELECTROMAGNETIC FIELDS**

Denote, respectively, by \( \Lambda \) and \( y \) the \( SL(2, C) \) and 4-translation parts of a Poincaré transformation \( \mathcal{S} (\Lambda, y) \). The spinor representation of the Poincaré group acts in the space of anti-self-dual electromagnetic fields in 4-position representation as follows:

\[
\hat{F}_{ab}(x) \rightarrow \left(T_{\Lambda, y} \hat{F}\right)_{ab}(x) = \Lambda_a^c \Lambda_b^d \hat{F}_{cd}(\Lambda^{-1}(x-y))
\]

(32)

\[
\Lambda_a^c \Lambda_b^d \hat{F}_{cd}(\Lambda^{-1}(x-y)) = \int d\Gamma(k) \varepsilon_{A'B'} \Lambda_A^C \pi_C(k) \Lambda_B^D \pi_D(k) \left(f(k, -) e^{-ik\cdot\Lambda^{-1}(x-y)} + \overline{f(k, +)} e^{ik\cdot\Lambda^{-1}(x-y)}\right)
\]

(33)

\[
= \int d\Gamma(k) \varepsilon_{A'B'} \Lambda_A^C \pi_C(\Lambda^{-1}k) \Lambda_B^D \pi_D(\Lambda^{-1}k) \left(f(\Lambda^{-1}k, -) e^{-ik\cdot(x-y)} + \overline{f(\Lambda^{-1}k, +)} e^{ik\cdot(x-y)}\right)
\]

(34)

where \( \Lambda^{-1}k \) is the spacelike part of \( \Lambda^{-1}_{a'b} \). The transformed field

\[
(\Lambda \pi)_A(k) = \Lambda_A^C \pi_C(\Lambda^{-1}k)
\]

(36)

satisfies

\[
k^a = \pi^A(k) \pi^{A'}(k) = (\Lambda \pi)_A(k) \Lambda_A^A'(k)
\]

(37)

Now, if \( \omega_A(k) \) is a spin-frame partner of \( \pi_A(k) \), i.e. \( \omega_A(k) \pi_A(k) = 1 \), one can write

\[
\pi^A(k) = (\Lambda \pi)_A(k) \omega_A(k) \Lambda_A^A'(k)
\]

(38)

which shows that \( \pi^A(k) \) and \( (\Lambda \pi)_A(k) = \Lambda_A^A \pi_C(\Lambda^{-1}k) \) are proportional to each other, the proportionality factor being

\[
\lambda(\Lambda, k) = \omega_A(k) \Lambda_A^A'(k).
\]

(39)

The form \( [3] \) showed that the gauge freedom is related to shifts

\[
\omega_A(k) \rightarrow \omega_A(k) + \text{scalar} \times \pi_A(k)
\]

(40)

which do not affect \( \lambda(\Lambda, k) \) making it independent of gauge. Using again
one concludes that \( \lambda(\Lambda, \mathbf{k}) \) is a phase factor

\[
\lambda(\Lambda, \mathbf{k}) = e^{i\Theta(\Lambda, \mathbf{k})}
\]

and we find

\[
(T_{\Lambda,y} \tilde{F})_{ab}(x) = \int d\Gamma(k)\pi_A(k)\pi_B(k)e^{-2i\Theta(\Lambda, \mathbf{k})} \left( f(\Lambda^{-1}k, -)e^{-ik\cdot(x-y)} + f(\Lambda^{-1}k, +)e^{ik\cdot(x-y)} \right)
\]

\[
= \int d\Gamma(k)e_{ab}(k) \left( e^{-2i\Theta(\Lambda, \mathbf{k})}e^{ik\cdot y} f(\Lambda^{-1}k, -)e^{-ik\cdot x} + e^{-2i\Theta(\Lambda, \mathbf{k})}e^{-ik\cdot y} f(\Lambda^{-1}k, +)e^{ik\cdot x} \right)
\]

\[
= \int d\Gamma(k)e_{ab}(k) \left( (T_{\Lambda,y}f)(k, -)e^{-ik\cdot x} + (T_{\Lambda,y}f)(k, +)e^{ik\cdot x} \right)
\]

We have obtained therefore the passive transformation of the classical wave function

\[
f(\mathbf{k}, \pm) \mapsto (T_{\Lambda,y}f)(\mathbf{k}, \pm) = e^{-2i\Theta(\Lambda, \mathbf{k})}e^{ik\cdot y} f(\Lambda^{-1}\mathbf{k}, \pm).
\]

which is simply the unitary zero-mass spin-1 representation the Poincaré group. The above derivation clearly shows that the rule (46) is obtained without any particular assumption about the choice of \( f(\mathbf{k}, \pm) \). In particular, the derivation remains valid even if one replaces functions \( f \) by operators, independently of their algebraic properties. The above ‘passive’ viewpoint on the structure of unitary representations is particularly useful if one aims at generalizations of CCR. The passive derivation of all the non-tachyonic unitary representations of the Poincaré group can be found in [7].

**IV. NON-CANONICAL QUANTIZATION**

We follow the strategy described in [1] and [2]. Let \( a_s \) be canonical annihilation operators satisfying CCR \([a_s, a^\dagger_{s'}] = \delta_{ss'}1\). Define the 1-oscillator non-canonical creation and annihilation operators \([\mathbb{I}]\)

\[
a(f)^\dagger = \sum_s \int d\Gamma(k)f(k, s)|k\rangle|k\rangle \otimes a^\dagger_s
\]

\[
= \sum_s \int d\Gamma(k)f(k, s)a(k, s)^\dagger
\]

\[
a(f) = \sum_s \int d\Gamma(k)f(k, s)|k\rangle|k\rangle \otimes a_s
\]

\[
= \sum_s \int d\Gamma(k)f(k, s)a(k, s)
\]

satisfying the non-CCR algebra

\[
[a(k, s), a(k', s')]^\dagger = \delta_{ss'}\delta_{\Gamma}(k, k')|k\rangle \otimes 1
\]

\[
= \delta_{ss'}\delta_{\Gamma}(k, k')I_k.
\]

Taking, in particular, \( f_{p,r}(k, s) = \delta_{ss}\delta_{\Gamma}(p, k) \) one finds

\[
a(f_{p,r}) = |p\rangle\langle p| \otimes a_r = a(p, r).
\]

The one-oscillator quantization is

\[
\tilde{F}_{ab}(x) = \varepsilon_{AB}\varphi_{AB}(x)
\]

\[
= \int d\Gamma(k)\varepsilon_{AB}\pi_A(k)\pi_B(k) \left( a(k, -)e^{-ik\cdot x} + a(k, +)^\dagger e^{ik\cdot x} \right)
\]

\[
= \int d\Gamma(k)e_{ab}(k) \left( a(k, -)e^{-ik\cdot x} + a(k, +)^\dagger e^{ik\cdot x} \right)
\]

\[
= \int d\Gamma(k)e_{ab}(k) \left( a(k, -)e^{-ik\cdot x} + a(k, +)^\dagger e^{ik\cdot x} \right)
\]
Spinor transformations of $\hat{F}_{ab}(x)$ lead to the passive transformation

$$a(k, \pm) \mapsto (T_{\Lambda,y}a)(k, \pm) = e^{\pm 2i\Theta(\Lambda,k)}e^{ik\cdot x}a(\Lambda^{-1}k, \pm).$$

(57)

The quantization procedure is gauge independent since we work at the gauge-independent level of $\hat{F}_{ab}(x)$.

Multi-oscillator fields are defined in terms of

$$a(p, s) = \oplus a(p, s)$$

(58)

and

$$\mathcal{A}(f) = \sum_s \int d\Gamma(k)f(k, s)a(k, s)$$

(59)

$$\mathcal{A}(f) = \sum_s \int d\Gamma(k)f(k, s)a(k, s).$$

(60)

The non-CCR algebra is

$$[\mathcal{A}(f), \mathcal{A}(g)] = \sum_s \int d\Gamma(k)f(k, s)g(k, s)I_k$$

(61)

The right-hand-side of the above formula is in the center of the non-CCR algebra, i.e.

$$[[\mathcal{A}(f), \mathcal{A}(g)], \mathcal{A}(h)] = 0$$

(62)

$$[[\mathcal{A}(f), \mathcal{A}(g)], \mathcal{A}(h)^\dagger] = 0$$

(63)

Useful is also the formula

$$[\mathcal{A}(f_p, r), \mathcal{A}(f_{p', r'})]\dagger = [\mathcal{A}(p, r), \mathcal{A}(p', r')\dagger]$$

(64)

$$= \delta_{r, r'}\delta_{\Gamma(p, p')L_p}.$$  

(65)

The presence of $L_p$ at the right-hand-sides of non-CCR will influence orthogonality properties of multi-photon states, as we shall see later.

At the multi-oscillator level the electromagnetic field tensor operator is

$$F_{ab}(x) = \int d\Gamma(k)e_{ab}(k)\left[\mathcal{A}(k, -)e^{-ik\cdot x} + \mathcal{A}(k, +)^\dagger e^{ik\cdot x}\right]$$

(66)

$$+ \int d\Gamma(k)e_{ab}(k)\left[\mathcal{A}(k, -)^\dagger e^{ik\cdot x} + \mathcal{A}(k, +)e^{-ik\cdot x}\right].$$

(67)

The four-potential operator is (in our gauge)

$$A_a(x) = i \int d\Gamma(k)\left[m_a(k)\left(\mathcal{A}(k, +)e^{-ik\cdot x} - \mathcal{A}(k, -)^\dagger e^{ik\cdot x}\right) + \bar{m}_a(k)\left(\mathcal{A}(k, -)e^{-ik\cdot x} - \mathcal{A}(k, +)^\dagger e^{ik\cdot x}\right)\right].$$

(68)

V. ACTION OF THE POINCARÉ GROUP ON FIELD OPERATORS

We are interested in finding the representation of the group in terms of unitary similarity transformations, i.e.

$$a(k, \pm) \mapsto e^{\pm 2i\Theta(\Lambda,k)}e^{ik\cdot x}a(\Lambda^{-1}k, \pm) = U_{\Lambda, y}a(k, \pm)U_{\Lambda, y}^\dagger$$

(69)

It is sufficient to find an appropriate representation at the one-oscillator level. Indeed, assume we have found $U_{\Lambda, y}$ satisfying

$$e^{\pm 2i\Theta(\Lambda,k)}e^{ik\cdot x}a(\Lambda^{-1}k, \pm) = U_{\Lambda, y}a(k, \pm)U_{\Lambda, y}^\dagger.$$  

(70)

Then

$$U_{\Lambda, y} = \bigoplus_{N=1}^{\infty} U_{\Lambda, y} \otimes \ldots \otimes U_{\Lambda, y}.$$  

(71)
A. Four-translations

The definition of four momentum for a single harmonic oscillator is

\[ P_a = \int d\Gamma(k) a_k |k\rangle \langle k| \otimes h \]  

(72)

where

\[ h = \frac{1}{2} \sum_s (a_s^\dagger a_s + a_s a_s^\dagger) = \sum_s \hat{h}_s. \]  

(73)

One immediately verifies that

\[ e^{iP_x} a(k, s) e^{-iP_x} = a(k, s) e^{ix} \]  

(74)

\[ e^{iP_x} a(k, s) e^{-iP_x} = a(k, s) e^{ix} \]  

(75)

implying

\[ U_{1,y} = e^{iyP}. \]  

(76)

Consequently, the generator of four-translations corresponding to \( U_{1,y} = e^{iyP} \) is \( P_a = \mathbb{1} P_a \) and

\[ e^{iP_x} a(k, s) e^{-iP_x} = a(k, s) e^{ix} \]  

(77)

\[ e^{iP_x} a(k, s) e^{-iP_x} = a(k, s) e^{-ix}. \]  

(78)

The \( x \)-dependence of field operators can be introduced via \( P \):

\[ F_{ab}(x) = e^{iP_x} F_{ab} e^{-iP_x} \]  

(79)

B. Rotations and boosts

To find an analogous representation of \( A(k, \pm) \mapsto e^{\pm 2i\Theta(\Lambda, k)} a(\Lambda^{-1}k, \pm) = U_{\Lambda,0}^\dagger A(k, \pm) U_{\Lambda,0} \) we define

\[ U_{\Lambda,0} = \exp \left( \sum_s 2is \int d\Gamma(k) \Theta(\Lambda, k) |k\rangle \langle k| \otimes h_s \right) \left( \sum_r \int d\Gamma(p) |p, r\rangle \langle \Lambda^{-1}p, r| \otimes 1 \right). \]  

(81)

Finally the transformations of the field tensor are

\[ U_{\Lambda,0}^\dagger F_{ab}(x) U_{\Lambda,0} = \Lambda_a^c \Lambda_b^d F_{cd}^{}(\Lambda^{-1}x) \]  

(82)

\[ U_{\Lambda,0}^\dagger F_{ab}(x) U_{\Lambda,0} = F_{ab}(x - y) \]  

(83)

The zero-energy part of \( P \) can be removed by a unitary transformation leading to a vacuum picture dynamics (cf. [2]). We will describe this in more detail after having discussed the properties of non-canonical states.

VI. STATES AND THEIR POINCARÉ TRANSFORMATIONS

It is clear that in order to control transformation properties of states it is sufficient to discuss single-oscillator representations. We shall start with single-oscillator states and then extend them to many oscillators.
A. Representation in the one-oscillator sector

The one-oscillator Hilbert space consists of functions $f$ satisfying
\[ \sum_{n_+, n_-}^{\infty} \int d\Gamma(k) |f(k, n_+, n_-)|^2 < \infty. \] (84)

We will write them in the Dirac notation as
\[ |f\rangle = \sum_{n_\pm} \int d\Gamma(k) f(k, n_+, n_-) |k, n_+, n_-\rangle. \] (85)

The representation of the Poincaré group is
\[ |f\rangle \rightarrow U_{\Lambda,y} |f\rangle = U_{1,y} U_{\Lambda,0} |f\rangle \]
\[ = \sum_{n_\pm} \int d\Gamma(k) f(\Lambda^{-1} k, n_+, n_-) e^{2i(n_+-n_-)\Theta(\Lambda,k)} e^{ik_y(n_++n_-+1/2)} |k, n_+, n_-\rangle. \] (86)

The latter formula can be written as
\[ f(k, n_+, n_-) \mapsto U_{\Lambda,y} f(k, n_+, n_-) = e^{ik_y(n_++n_-+1/2)} e^{2i(n_+-n_-)\Theta(\Lambda,k)} f(\Lambda^{-1} k, n_+, n_-) \] (87)

or
\[ U_{\Lambda,y} |f\rangle = |U_{\Lambda,y} f\rangle. \] (88)

The form (87) is very similar to the zero-mass spin-1 representation (46), the difference being in the multiplier $n_+ + n_- + 1/2$. One can check by a straightforward calculation that (87) defines a representation of the group.

B. Generators and vacuum picture

Denote by $K_a$ and $L_{ab} + S_{ab}$ the generators of 4-translations and $SL(2, C)$ of the standard zero-mass spin-1 unitary representation of the Poincaré group. $L_{ab}$ denotes the orbital part of the generator. The generators of (87) are then
\[ P^a = K_a \otimes h \] (89)
\[ J^{ab} = L^{ab} \otimes 1 + S^{ab}_{ss} \otimes h_s. \] (90)

$S^{ab}_{ss'}$ are matrix elements of $S^{ab}$ (which is a diagonal $p$-dependent matrix). Denote by $\tilde{S}_{ab}$ the generators of the $(1/2, 1/2)$ spinor representation of $SL(2, C)$, i.e. $\Lambda = \exp (i \tilde{S}_{ab})/2$. The generators of the unitary representation are defined by
\[ P_a f(k, n_+, n_-) = -i \frac{\partial}{\partial y^a} U_{\Lambda,y} f(k, n_+, n_-) \big|_{\xi,y=0} \] (91)
\[ J_{ab} f(k, n_+, n_-) = -i \frac{\partial}{\partial \xi_{ab}} U_{\Lambda,y} f(k, n_+, n_-) \big|_{\xi,y=0} \] (92)

In what follows we will work in a “vacuum picture”, i.e with unitary transformations
\[ f(k, n_+, n_-) \mapsto V_{\Lambda,y} f(k, n_+, n_-) = e^{i(n_++n_-)k_y} e^{2i(n_+-n_-)\Theta(\Lambda,k)} f(\Lambda^{-1} k, n_+, n_-). \] (93)

The transition
\[ U_{\Lambda,y} \mapsto V_{\Lambda,y} = W_{y}^{\dagger} U_{\Lambda,y} \] (94)

is performed by means of the unitary transformation which commutes with non-CCR creation and annihilation operators.

Let us stress that the fact that we “remove” the zero-energy parts from generators does not mean that energy of vacuum is zero. The vacuum picture is in a sense a choice of representation co-moving with vacuum.
C. Vacuum states

Vacuum states are all the states which are annihilated by all annihilation operators. At the one-oscillator level these are states of the form

\[ |O\rangle = \int d\Gamma(k)O(k)|k,0,0\rangle. \]  

(95)

Even in the vacuum picture the vacuum states are not Poincaré invariant since

\[ V_{\Lambda,y}O(k) = O(\Lambda^{-1}k) \]  

(96)

which means they transform as a 4-translation-invariant scalar field. We will often meet the expression \( Z(k) = |O(k)|^2 \) describing the probability density of the “zero modes”.

D. Coherent states

An analogue of the standard coherent (or “semiclassical”) state is at the 1-oscillator level

\[ |O_\alpha\rangle = \int d\Gamma(k)O(k)|\alpha(k, +), \alpha(k, -)\rangle \]  

(97)

where

\[ a_s|\alpha(k, +), \alpha(k, -)\rangle = \alpha(k, s)|\alpha(k, +), \alpha(k, -)\rangle \]  

(98)

Its explicit form in the basis of eigenstates of the oscillator is

\[ |O_\alpha\rangle = \int d\Gamma(k)O(k) \sum_{n_+, n_-=0}^\infty \frac{\alpha(k, +)^{n_+} \sqrt{n_+!} e^{-|\alpha(k, +)|^2/2} \alpha(k, -)^{n_-} \sqrt{n_-!} e^{-|\alpha(k, -)|^2/2} |k, n_+, n_-\rangle}{\sqrt{n_+!n_-!}} \]  

(99)

\[ = \sum_{n_+, n_-} \int d\Gamma(k)O_\alpha(k, n_+, n_-)|k, n_+, n_-\rangle \]  

(100)

where

\[ O_\alpha(k, n_+, n_-) = \frac{1}{\sqrt{n_+!n_-!}} O(k)^n_+ \alpha(k, +)^n_+ \alpha(k, -)^n_- e^{-\sum_{\pm}^n_+ |\alpha(k, \pm)|^2/2} \]  

(101)

The average of the 1-oscillator field operator evaluated in such a coherent state is

\[ \langle O_\alpha|\hat{F}_{ab}^\dagger(x)|O_\alpha\rangle = \int d\Gamma(k) e_{ab}(k)Z(k) \left( \alpha(k, -) e^{-ik\cdot x} + \alpha(k, +)^* e^{ik\cdot x} \right). \]  

(102)

The Poincaré transformation of the state implies

\[ \langle O_\alpha|V^{\dagger}_{\Lambda,y} - \hat{F}_{ab}(x) V_{\Lambda,y}|O_\alpha\rangle = \Lambda_a^c \Lambda_b^d \langle O_\alpha|\hat{F}_{cd}^\dagger(\Lambda^{-1}(x - y))|O_\alpha\rangle \]  

(103)

The coherent-state wave function transforms by

\[ V_{\Lambda,y}O_\alpha(k, n_+, n_-) = e^{i(n_+ + n_-)k\cdot y} e^{2i(n_+ - n_-)\Theta(\Lambda,k)} O_\alpha(\Lambda^{-1}k, n_+, n_-) \]  

(104)

\[ = O(\Lambda^{-1}k) \prod_{s=\pm} e^{in_s k\cdot y} e^{2i\Theta(\Lambda,k)} \frac{1}{\sqrt{n_s!}} \alpha(\Lambda^{-1}k, s)^{n_s} e^{-|\alpha(\Lambda^{-1}k, s)|^2/2} \]  

(105)

\[ = O(\Lambda^{-1}k) \prod_{s=\pm} \left( T_{\Lambda,y} A(k, s) \right)^{n_s} e^{-|T_{\Lambda,y} \alpha(k, s)|^2/2} \]  

(106)

where

\[ \alpha(k, s) \rightarrow T_{\Lambda,y} \alpha(k, s) \]  

(107)
is the spin-1 massless unitary representation \( \{4\} \). Using this result we get again

\[
\langle O_\alpha | V_{\Lambda,y}^\dagger - \hat{F}_{ab}(x)V_{\Lambda,y} | O_\alpha \rangle = \int d\Gamma(k) e^{ikx} Z(k) \left( T_{\Lambda,y} \alpha(k,+) e^{ikx} + \overline{T_{\Lambda,y} \alpha(k,-)} e^{-ikx} \right)
\]

\[
= \Lambda_a^c \Lambda_b^d \langle O_\alpha | - \hat{F}_{cd}(\Lambda^{-1}(x-y)) | O_\alpha \rangle
\]

showing that those somewhat counter-intuitive forms of \( U_{\Lambda,y} \) and \( V_{\Lambda,y} \) are consistent with passive \( T_{\Lambda,y} \) transformations of classical wave functions.

With \( \alpha(\beta) \) and \( \alpha(\beta)^\dagger \) given by \( \{15\}-\{17\} \) we define the displacement operator

\[
D(\beta) = e^{\alpha(\beta)^\dagger - \alpha(\beta)}
\]

\[
= \exp \left( \sum_s \int d\Gamma(k) \left( \beta(k,s) a(k,s)^\dagger - \overline{\beta(k,s)} a(k,s) \right) \right)
\]

\[
= \int d\Gamma(k) |k,s\rangle \langle k,s| \otimes \exp \left( \sum_s \left( \beta(k,s) a_s^\dagger - \overline{\beta(k,s)} a_s \right) \right)
\]

which performs a shift of the classical wave function

\[
D(\beta)|O_\alpha\rangle = |O_{\alpha+\beta}\rangle
\]

and commutes with \( I_k \):

\[
D(\beta)^\dagger I_k D(\beta) = I_k.
\]

Vacuum states are also coherent states corresponding to \( \alpha = 0 \).

### E. Multi-oscillator coherent states

Consider a family \( \alpha_N(k,s), N = 1,2, \ldots \) of functions and the state

\[
|\Omega_N\rangle = \bigotimes_{N=1}^\infty \sqrt{p_N} |O_{\alpha_N}\rangle \otimes \cdots \otimes |O_{\alpha_N}\rangle
\]

where

\[
|O_{\alpha_N}\rangle = \int d\Gamma(k) O(k)|k\rangle |\alpha_N(k,+), \alpha_N(k,-)\rangle
\]

and \( \sum_{N=1}^\infty p_N = 1 \). Taking, for example, \( \alpha_N(k,s) = \alpha(k,s)/\sqrt{N} \) we find

\[
\langle O_\alpha | - \hat{F}_{ab}(x) | O_\alpha \rangle = \int d\Gamma(k) e^{ikx} Z(k) \left( \alpha(k,-) e^{-ikx} + \overline{\alpha(k,+)} e^{ikx} \right) = \langle O_\alpha | - \hat{F}_{ab}(x) | O_\alpha \rangle
\]

i.e. the same result as in the 1-oscillator case.

A multi-oscillator displacement operator is

\[
D(\beta) = \bigoplus_{N=1}^\infty D(\beta_N) \otimes \cdots \otimes D(\beta_N) = e^{\alpha(\beta)^\dagger - \alpha(\beta)},
\]

\( \beta_N(k,s) = \beta(k,s)/\sqrt{N} \), implying

\[
D(\beta)|\Omega_N\rangle = |\Omega_{\alpha+\beta}\rangle
\]

\[
D(\beta)^\dagger a(p,s) D(\beta) = a(p,s) + \beta(p,s) L_p
\]

\[
D(\beta)^\dagger L_p D(\beta) = L_p.
\]

The fact that \( \alpha_N(k,s) = \alpha(k,s)/\sqrt{N} \) will be shown to be of crucial importance for the question of statistics of excitations of multi-oscillator coherent states. Let us note that a similar property of coherent states was found in \( \{11\} \) when we employed the definition in terms of eigenstates of annihilation operators.
F. Multi-oscillator vacua

Vacuum consists of states with \( n_\pm = 0 \), i.e. with all the oscillators in their ground states. Of particular interest, due to its simplicity, is the following vacuum state

\[
|O\rangle = \bigoplus_{N=1}^{\infty} \sqrt{p_N} |O\rangle \otimes \ldots \otimes |O\rangle
\]  

where

\[
|O\rangle = \int d\Gamma(k) O(k) |k, 0, 0\rangle
\]

Such a vacuum is simultaneously a particular case of a coherent state with \( \alpha(k, s) = 0 \). Coherent states are related to the vacuum state via the displacement operator

\[
D(\alpha)|O\rangle = \bigoplus_{N=1}^{\infty} \sqrt{p_N} |O_{\alpha N}\rangle \otimes \ldots \otimes |O_{\alpha N}\rangle
\]

\[
= |O_{\alpha}\rangle.
\]

G. Normalized 1-photon states

Consider the vector

\[
a(f)\rangle^\dagger |O\rangle.
\]

Choosing the particular form (122) we find

\[
\langle Q|a(f)\rangle a(g)\rangle^\dagger |O\rangle = \sum_s \int d\Gamma(k) Z(k) f(k, s)g(k, s)
\]

\[
= \langle fO|gO\rangle = \langle f|g\rangle Z.
\]

\( fO \) denotes the pointlike product \( fO(k, s) = O(k)f(k, s) \). Since anyway only the modulus \( |O(k)| = Z(k)^+ \) occurs in the above scalar products one can also work with \( f_B(k, s) = Z(k)^+ f(k, s) \). The relation between \( f_B \) and \( f \) resembles the one between the bare and renormalized fields [10]. We believe this is more than just an analogy.

Thinking of bases in the Hilbert space one can take functions \( f_i \) satisfying

\[
\langle f_i|f_j\rangle_Z = \delta_{ij} = \langle f_B_i|f_B_j\rangle.
\]

H. Normalization of multi-photon states

Normalization of multi-photon states is more complicated. In this section we will discuss this point in detail since the argument we give is very characteristic for the non-canonical framework. It can be used to show that in the thermodynamic limit of a large number of oscillators the non-CCR perturbation theory tends to the CCR one but in a version which is automatically regularized [3]. We will also use a similar trick to show that the multi-oscillator coherent states have, again in the thermodynamic limit, Poissonian statistics of excitations.

Denote by \( \sum_\sigma \) the sum over all the permutations of the set \( \{1, \ldots, m\} \).

Theorem 1. Consider the vacuum state (122) with \( p_N = 1 \) for some \( N \). Then

\[
\lim_{N \to \infty} \langle Q|a(f_1) \ldots a(f_m)a(g_1) \ldots a(g_m)\rangle |O\rangle = \sum_\sigma \langle f_1|g_{\sigma(1)}\rangle z \ldots \langle f_m|g_{\sigma(m)}\rangle z
\]

\[
= \sum_\sigma \sum_{s_1, \ldots, s_m} \int d\Gamma(k_1) Z(k_1) \ldots d\Gamma(k_m) Z(k_m) f_1(k_1, s_1) \ldots f_m(k_m, s_m) g_{\sigma(1)}(k_1, s_1) \ldots g_{\sigma(m)}(k_m, s_m)
\]
Proof: The scalar product of two general unnormalized multi-photon states is

\[
\langle O | \mathcal{A}(f_1) \ldots \mathcal{A}(f_m) \mathcal{A}(g_1) \ldots \mathcal{A}(g_m) | O \rangle
\]

\[
= \sum_{\sigma} \sum_{s_1 \ldots s_m} \int d\Gamma(k_1) \ldots d\Gamma(k_m) f_1(k_1, s_1) \ldots f_m(k_m, s_m) g_1(k_1, s_1) \ldots g_m(k_m, s_m) \langle O | L_{k_1} \ldots L_{k_m} | O \rangle
\]

\[
= \sum_{\sigma} \sum_{s_1 \ldots s_m} \int d\Gamma(k_1) \ldots d\Gamma(k_m) f_1(k_1, s_1) \ldots f_m(k_m, s_m) g_1(k_1, s_1) \ldots g_m(k_m, s_m)
\]

\[
\times \frac{1}{N^m} \langle O | \ldots \langle O | I_{k_1} \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I_{k_1} \ldots (I_{k_m} \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes I_{k_m}) | O \rangle \ldots | O \rangle
\]

(130)

Further analysis of (130) can be simplified by the following notation:

\[
1_{k_j} = I_{k_j} \otimes \ldots \otimes I
\]

\[
2_{k_j} = I \otimes I_{k_j} \otimes \ldots \otimes I
\]

\[
\vdots
\]

\[
N_{k_j} = I \otimes \ldots \otimes I_{k_j}
\]

with \( j = 1, \ldots, m \); the sums-integrals \( \sum_{s_j} \int d\Gamma(k_j) \) are denoted by \( \sum_{k_j} \). Then (130) can be written as

\[
\sum_{\sigma} \sum_{k_1 \ldots k_m} f_1(k_1) \ldots f_m(k_m) g_{\sigma(1)}(k_1) \ldots g_{\sigma(m)}(k_m) \frac{1}{N^m} \sum_{A \ldots Z=1} \langle O | \ldots \langle O | A_{k_1} \ldots Z_{k_m} | O \rangle \ldots | O \rangle
\]

(131)

Since \( m \) is fixed and we are interested in the limit \( N \to \infty \) we can assume that \( N > m \). Each element of the sum over \( A_{k_1} \ldots Z_{k_m} \) in (131) can be associated with a unique point \((A, \ldots, Z)\) in an \( m \)-dimensional lattice embedded in a cube with edges of length \( N \).

Of particular interest are those points of the cube, the coordinates of which are all different. Let us denote the subset of such points by \( C_0 \). For \((A, \ldots, Z) \in C_0\)

\[
\langle O \rangle \ldots \langle O | A_{k_1} \ldots Z_{k_m} | O \rangle \ldots | O \rangle = Z(k_1) \ldots Z(k_m)
\]

(132)

no matter what \( N \) one considers and what are the numerical components in \((A, \ldots, Z)\). (This makes sense only for \( N > m \); otherwise \( C_0 \) would be empty). Therefore each element of \( C_0 \) produces an identical contribution (132) to (131). Let us denote the number of points in \( C_0 \) by \( N_0 \).

The sum (131) can be now written as

\[
\sum_{\sigma} \sum_{k_1 \ldots k_m} f_1(k_1) \ldots f_m(k_m) g_{\sigma(1)}(k_1) \ldots g_{\sigma(m)}(k_m) \mathcal{P}_0 Z(k_1) \ldots Z(k_m)
\]

\[
+ \sum_{\sigma} \sum_{k_1 \ldots k_m} f_1(k_1) \ldots f_m(k_m) g_{\sigma(1)}(k_1) \ldots g_{\sigma(m)}(k_m) \frac{1}{N^m} \sum_{(A \ldots Z) \not\in C_0} \langle O | \ldots \langle O | A_{k_1} \ldots Z_{k_m} | O \rangle \ldots | O \rangle
\]

(133)

The coefficient \( \mathcal{P}_0 = \frac{N_0}{N^m} \) represents a probability of \( C_0 \) in the cube. The elements of the remaining sum over \((A, \ldots, Z) \not\in C_0\) can be also grouped into classes according to the values of \( \langle O | \ldots \langle O | A_{k_1} \ldots Z_{k_m} | O \rangle \ldots | O \rangle \). There are \( m - 1 \) such different classes, each class has its associated probability \( \mathcal{P}_j \), \( 0 < j \leq m - 1 \), which will appear in the sum in an analogous role as \( \mathcal{P}_0 \).

The proof is completed by the observation that

\[
\lim_{N \to \infty} \mathcal{P}_0 = 1,
\]

\[
\lim_{N \to \infty} \mathcal{P}_j = 0, \quad 0 < j.
\]

Indeed, the probabilities are unchanged if one rescales the cube to \([0, 1]^m\). The probabilities are computed by means of an \( m \)-dimensional uniformly distributed measure. \( N \to \infty \) corresponds to the continuum limit, and in this limit the sets of points of which at least two coordinates are equal are of \( m \)-dimensional measure zero.
Comments: (a) The thermodynamic limit is naturally equipped with the scalar product yielding orthogonality relation of the form (129). However, for small N there will be differences if m is large. On the other hand if N is sufficiently large then the values of m for which the corrections are non-negligible must be also large. But then a classical limit will be justified and the use of non-canonical coherent states should again give the correct description. (b) Concrete values of $P_j$ for some small m were given in (115). For $m = 2$: $P_0 = 1 - 1/N$, $P_1 = 1/N$; for $m = 3$: $P_0 = 1 - 3/N + 2/N^2$, $P_1 = 3/N - 3/N^2$, $P_2 = 1/N^2$. In general we do not have to assume that $p_N = 1$. If $p_N$ are general probabilities then the coefficients involve averages. For $m = 2$: $P_0 = 1 - \langle 1/N \rangle$, $P_1 = \langle 1/N \rangle$; for $m = 3$: $P_0 = 1 - \langle 3/N \rangle + \langle 2/N^2 \rangle$, $P_1 = \langle 3/N \rangle - \langle 3/N^2 \rangle$, $P_2 = \langle 1/N^2 \rangle$, where $\langle 1/N \rangle = \sum_N p_N/N$ etc. The normalization in terms of $\langle |·⟩_Z$ is then obtained under the assumption that all those averages vanish, which can hold only approximately, meaning that the probability $p_N$ is peaked in a region of large Ns.

I. Space of states as a vector bundle

An analysis of perturbation theory given in [13] suggested that non-canonical probabilities have to be computed as if the space of vacua was one dimensional (a “unique vacuum”). In the non-canonical theory it is essential that vacuum is represented by an infinite dimensional subspace of all the states annihilated by all annihilation operators. Otherwise it is impossible to associate the entire spectrum of frequencies with a single oscillator.

The way one removes the zero-energy part from the four-momentum shows that the role of a reference state is a “moving vacuum” whose dynamics is given by the zero-energy part of four-momentum (the vacuum picture).

The construction of multi-photon states shows that, having a vacuum state $|O⟩$ representing an ensemble of ground-state oscillators, one can introduce a Fock-type structure representing excitations of the ensemble. Obviously, the Fock spaces of multiphoton states are different for different vacua.

All of this suggests a vector-bundle structure where the set of vacuum states is a base space and Fock spaces are fibers. The probabilities are calculated in fibers. The Poincaré group acts in the entire bundle. The zero-energy part of four-momenta generates a motion in the base space and plays essentially a role of a bundle connection.

VII. STATISTICS OF EXCITATIONS

It is an experimental fact that laser beams produce Poissonian statistics of photocounts. At the theoretical level of canonical quantum optics the Poisson distribution follows trivially from the form of canonical coherent states. In the non-canonical case the exact Poisson statistics is characteristic of the single-oscillator ($N = 1$) sector. For $1 < N < \infty$ the statistics of excitations is non-Poissonian. At the other extreme is the thermodynamic limit for multi-oscillator states. In what follows we will show that in the limit $N \rightarrow \infty$ one recovers the same Poisson distribution as for $N = 1$. This, at a first glance unexpected, result justifying our definitions in terms of displacement operators is a consequence of certain classical universality properties of the Poisson distribution.

We will also return to the question of thermal states and the Planck formula. In [11] it was argued that non-CCR quantization implies deviations from the black-body law. However, a consistent interpretation of the field in terms of the thermodynamic limit shows that no deviations should be expected.

A. Multi-oscillator coherent states

To study the thermodynamic limit of multi-oscillator coherent states we simplify the discussion by taking an exactly N-oscillator coherent state $|O⟩$ ($N \gg 1$ is fixed and $p_N = 1$), i.e.

$$|O⟩ = |O_{αN}⟩ \otimes \cdots \otimes |O_{αN}⟩$$

(136)

where

$$|O_{αN}⟩ = \int dΓ(k)|O(k)|α(k, +)/\sqrt{N}|α(k, −)/\sqrt{N}⟩.$$

(137)

The average number of excitations in this state is

$$⟨n⟩ = \sum_s \int dΓ(k) Z(k)|α(k, s)|^2.$$

(138)
The simplest case is the one where $\alpha(k, s) = \alpha = \text{const.}$ Then $\langle n \rangle = |\alpha|^2$ and the statistics of excitations of single-oscillator coherent states $|O_{\alpha x}\rangle$ is Poissonian with the distribution $p_n = e^{-|\alpha|^2}[\alpha N]^{2n}/n!$

$m$ excitations distributed in the ensemble of $N$ oscillators can be represented by the ordered $m$-tuple $(j_1, \ldots, j_m)$, $1 \leq j_1 \leq \ldots \leq j_m \leq N$. For example, for $m = 10$, $N = 12$, the point $(2, 2, 2, 5, 5, 7, 7, 7, 11, 11)$ represents 10 excitations distributed in the ensemble of 12 oscillators as follows: 3 excitations in 2nd oscillator, 2 in the 5th one, 3 in the 7th, and 2 in the 11th. Such points form a subset of the cube $[0, N]^m$, the interior of the set corresponding to points whose all the indices are different. The latter means that the interior represents situations where there are $m$ oscillators excited, and each of them is in the first excited state. The boundary of this set consists of points representing at least one oscillator in a higher excited state. Probabilities of events represented by points with the same numbers of repeated indices must be identical due to symmetries. Intuitively, the Poissonian statistics of the lowest energy levels of a single oscillator in a coherent state.

To make the argument more formal we introduce the following notation:

$$X_m^{(N)} = \{x \in N^m; m \geq 1, x = (j_1, \ldots, j_m), 1 \leq j_1 \leq \ldots \leq j_m \leq N\}$$
$$X_{n_1 \ldots n_k}^{(N)} = \{x \in X_m^{(N)}; x = (i_1, \ldots, i_1, \ldots, i_k, \ldots, i_k), i_1 < \ldots < i_k\}$$
$$Y_m^{(N)} = \bigcup_{(n_1 \ldots n_k) \neq (1,1)} X_{n_1 \ldots n_k}^{(N)}$$ (139)

If we add a single-element set $X_0^{(N)}$ containing the event representing $N$ oscillators in their ground states we can represent the set of all the events by the disjoint sum

$$X^{(N)} = \bigcup_{m=0}^{\infty} X_m^{(N)}$$ (140)

The probability of finding the partition $m = n_1 + \ldots + n_k$ is

$$P(X_{n_1 \ldots n_k}^{(N)}) = N_{n_1 \ldots n_k} p_{n_1} \ldots p_{n_k} p_0^{N-k}$$
$$= N_{n_1 \ldots n_k} e^{-N|\alpha|^2}[\alpha N]^{2m}/n_1! \ldots n_k!$$ (141)

where $N_{n_1 \ldots n_k}$ is the number of elements of $X_{n_1 \ldots n_k}^{(N)} \subset X_m^{(N)}$. The probability that $m$ excitations are found is

$$P(X_m^{(N)}) = \sum_{n_1 \ldots n_k} P(X_{n_1 \ldots n_k}^{(N)}),$$ (142)

the sum being over all the partitions of $m$. Denote by $P(Y_m^{(N)}|X_m^{(N)})$ the conditional probability of finding at least one oscillator in the 2nd or higher excited state under the condition that the sum of excitations is $m > 1$. We first prove the following

**Lemma 1.**

$$\lim_{N \to \infty} P(Y_m^{(N)}|X_m^{(N)}) = 0.$$ (143)

**Proof:** Since $Y_m^{(N)} \cap X_m^{(N)} = Y_m^{(N)}$ one finds

$$P(Y_m^{(N)}|X_m^{(N)}) = \frac{\sum_{(n_1 \ldots n_k) \neq (1,1)} P(X_{n_1 \ldots n_k}^{(N)})}{\sum_{n_1 \ldots n_k} P(X_{n_1 \ldots n_k}^{(N)})}$$
$$= \left[1 + N_{n_1 \ldots n_k} \left( \sum_{(n_1 \ldots n_k) \neq (1,1)} \frac{N_{n_1 \ldots n_k}}{n_1! \ldots n_k!} \right)^{-1} \right]^{-1}$$
$$< \left[1 + N_{n_1 \ldots n_k} \left( \sum_{(n_1 \ldots n_k) \neq (1,1)} N_{n_1 \ldots n_k}^{-1} \right)^{-1} \right]^{-1}$$ (144)
However,
\[
\lim_{N \to \infty} \frac{\sum_{(n_1 \ldots n_k) \neq (1 \ldots 1)} N_{n_1 \ldots n_k}}{N_{1 \ldots 1}} = 0
\]  
(145)
on the basis of the geometric argument we gave in the previous section. This ends the proof. ■

The main result of this section is the following version of the well known Poisson theorem:

**Theorem 2.** Assume that \( \alpha(k, s) = \alpha = \text{const.} \) Then
\[
\lim_{N \to \infty} P(X^{(N)}_m) = e^{-|\alpha|^2} |\alpha|^{2m} m!.
\]  
(146)

**Proof:** As an immediate consequence of the lemma we find
\[
\lim_{N \to \infty} P(X^{(N)}_m) = \lim_{N \to \infty} P(X^{(N)}_m - Y^{(N)}_m)
\]  
(147)
which means that in the thermodynamic limit we can treat excitations of the oscillators to 2nd and higher excited levels as events whose probability is zero. The probabilities of ground and first excited states follow from the single-oscillator Poisson distributions but conditioned by the fact that only the lowest two levels are taken into account.

We thus arrive at the standard Poisson process with
\[
P_N = \frac{p_1}{p_0 + p_1} = \frac{|\alpha_N|^2}{1 + |\alpha_N|^2} = \frac{|\alpha|^2/N}{1 + |\alpha|^2/N}
\]  
(148)
and \( \lim_{N \to \infty} N P_N = |\alpha|^2. \) ■

**B. Thermal states**

A single-oscillator free-field Hamiltonian \( H \) has the usual eigenvalues
\[
E(\omega, n) = \hbar \omega \left( n + \frac{1}{2} \right).
\]  
(149)

The eigenvalues of the free-field Hamiltonian \( H \) at the multi-oscillator level are sums of the single-oscillator ones. In it was assumed that the Boltzmann-Gibbs distribution of thermal radiation should be constructed in terms of \( H \). Let us note, however, that such a construction makes use of \( H \) as if it was a Hamiltonian of a single element of a statistical ensemble. The discussion of the thermodynamic limit we have given above, as well as the results of [2], suggest that \( H \) is the Hamiltonian of the entire ensemble of systems described by \( H \), and it is \( H \) and not \( H \) which should be used in the Boltzmann-Gibbs distribution. Then, of course, the result will be the standard one and no deviations from the Planck formula will occur.

**VIII. COMMUTATORS OF 4-POTENTIALS AND LOCALITY**

A straightforward calculation shows that the multi-oscillator vector potential operator satisfies, for any space-time points \( x, y, z \), the commutators
\[
[A_a(x), A_b(y)] = \int d\Gamma(k) L_k \left( m_a(k) \bar{m}_b(k) e^{ik \cdot (y-x)} - \bar{m}_a(k) m_b(k) e^{ik \cdot (x-y)} \right)
\]
\[
+ \int d\Gamma(k) L_k \left( \bar{m}_a(k) m_b(k) e^{ik \cdot (y-x)} - m_a(k) \bar{m}_b(k) e^{ik \cdot (x-y)} \right)
\]  
(150)
\[
[A_a(x), A_b(y), A_c(z)] = 0.
\]  
(151)
To obtain more insight as to the meaning of the commutator (150) consider its coherent-state average evaluated in a state of the form (114):
\[ (Q_a| [A_b(x), A_b(y)]|Q_a) = \int d\Gamma(k) Z(k) \left( m_a(k)\bar{m}_b(k)e^{ik\cdot(y-x)} - \bar{m}_a(k)m_b(k)e^{ik\cdot(x-y)} \right) \]
\[ + \int d\Gamma(k) Z(k) \left( \bar{m}_a(k)m_b(k)e^{ik\cdot(y-x)} - m_a(k)\bar{m}_b(k)e^{ik\cdot(x-y)} \right). \]

The Minkowski-space metric tensor can be decomposed in terms of null tetrads as follows
\[ g_{ab} = k_a\omega_b + \omega_a k_b - m_a\bar{m}_b - m_b\bar{m}_a. \]

With the help of this identity we can write
\[ (Q_a| [A_b(x), A_b(y)]|Q_a) = \int d\Gamma(k) Z(k) \left( k_a\omega_b(k) + k_b\omega_a(k) \right) \left( e^{ik\cdot(y-x)} - e^{ik\cdot(x-y)} \right) \]
\[ + g_{ab} \int d\Gamma(k) Z(k) \left( e^{ik\cdot(x-y)} - e^{ik\cdot(y-x)} \right). \]

It is known that terms such as the first integral vanish if the potential couples to a conserved current. The same property guarantees gauge independence of the formalism. Therefore we can concentrate only on the explicitly gauge independent term proportional to \( g_{ab} \). Denote
\[ D_Z(x) = i \int d\Gamma(k) Z(k) \left( e^{-ik\cdot x} - e^{ik\cdot x} \right) \]

For \( Z(k) = \text{const} = Z \) we get \( D_Z(x) \) proportional to the Jordan-Pauli function,
\[ D_Z(x) = Z \, D(x) \]

which vanishes for spacelike \( x \). However, the choice of constant \( Z(k) \) is excluded by the requirement of square-integrability of \( O \). Therefore the requirement that the vacuum state be square-integrable seems to introduce some kind of nonlocality into the formalism.

There are two possibilities one can contemplate. First of all, one can perform the calculations with arbitrary \( O \) and then perform a renormalization (we have seen that such a step is necessary even in the nonrelativistic case). After the renormalization we can go to the “flat” pointwise limit \( Z(k) \to 0, \| O \| = 1 \), corresponding to the uniform distribution of all the frequencies. Second, performing the calculations in a preferred frame we can consider equal-time commutation relations
\[ (Q_a| [A_b(t,x), A_b(t,y)]|Q_a) = \int d\Gamma(k) Z(k) \left( k_a\omega_b(k) + k_b\omega_a(k) \right) \left( e^{ik\cdot(y-x)} - e^{ik\cdot(x-y)} \right) \]
\[ + g_{ab} \int d\Gamma(k) Z(k) \left( e^{ik\cdot(x-y)} - e^{ik\cdot(y-x)} \right). \]

The last term will vanish if
\[ Z(k) = Z(-k) \]
i.e. if the vacuum is 3-inversion invariant. This can hold, however only in one reference frame unless \( O \) is constant, which we exclude.

One can conclude that non-canonical quantized electrodynamics is not a local quantum field theory, at least in the strict standard sense. This is not very surprising if one recalls that the thermodynamic limit of nonrelativistic theories discussed in [1] and [2] was equivalent to their canonical versions but with cut-offs. The presence of \( O \) in the integrals introduces some kinds of effective extended structures, a consequence of nontrivial structures of non-canonical vacua. The issue requires further studies. In particular, it is important to understand an influence of the thermodynamic limit \( N \to \infty \) on locality problems in the context of relativistic perturbation theory.

There is an intriguing analogy between the kind of non-locality we have obtained and the one encountered in field theory in non-commutative space-time [11].
IX. RADIATION FIELDS ASSOCIATED WITH CLASSICAL CURRENTS

The problem of radiation fields is interesting for several reasons. First of all, the radiation fields satisfy homogeneous Maxwell equations so that the theory we have developed can be directly applied. Second, this is one of the simplest ways of addressing the question of infrared divergences within the non-canonical framework.

It is widely known that in the canonical theory the scattering matrix corresponding to radiation fields produced by a classical transversal current is given, up to a phase, by a coherent-state displacement operator $e^{-i \int d^4y J(y) \cdot A_{in}(y)}$. One of the consequences of such an approach is the Poissonian statistics of photons emitted by classical currents. An unwanted by-product of the construction is the infrared catastrophe.

Starting with the same $S$ matrix but expressed in terms of non-canonical “in” fields we obtain the non-canonical displacement operator. Photon statistics is Poissonian in the thermodynamic limit but the infrared catastrophe is automatically eliminated.

A. Classical radiation field

Let us assume that we deal with a classical transversal current $J_a(x)$ whose Fourier transform is $\tilde{J}_a(k) = \int d^4x e^{ik \cdot x} J_a(x)$. Transversality means here that

$$\tilde{J}_a(|k|, k) = \bar{m}_a(k) \tilde{J}_{10'}(|k|, k) + m_a(k) \tilde{J}_{01'}(|k|, k).$$

Formally, a solution of Maxwell equations

$$\Box A_a(x) = J_a(x) I$$

can be written as

$$A_a(x) = A_{a in}(x) + \int d^4y D_{ret}(x - y) J_a(y) L$$

$$= A_{a out}(x) + \int d^4y D_{adv}(x - y) J_a(y) L$$

Here $A_{a in}$ and $A_{a out}$ are solutions of homogeneous equations. $D_{ret}$ and $D_{adv}$ are the retarded and advanced Green functions whose difference is the Jordan-Pauli function

$$D(x) = i \int d\Gamma(k) (e^{-ik \cdot x} - e^{ik \cdot x}).$$

The 4-potential of the radiation field is

$$A_{a rad}(x) = A_{a out}(x) - A_{a in}(x) = \int d^4y D(x - y) J_a(y) L$$

and leads to the field spinors

$$\varphi_{X Y \cdot \text{rad}}(x) = \int d\Gamma(k) \left( \pi(X \pi^Y \bar{J}_Y Y') (k) e^{-ik \cdot x} + \pi(X \pi^Y \bar{J}_Y Y') (k) e^{ik \cdot x} \right) L$$

$$\varphi_{Y' \cdot \text{rad}}(x) = \int d\Gamma(k) \left( \pi(X \pi^Y \bar{J}_Y Y') (k) e^{ik \cdot x} + \pi(X \pi^Y \bar{J}_Y Y') (k) e^{-ik \cdot x} \right) L$$

Comparing these formulas with expressions (6) and (11) valid for all solutions of free Maxwell equations one finds

$$f(k, +) = -\bar{m}_a(k) \tilde{J}_a(|k|, k) = \tilde{J}_{01'}(|k|, k) = j(k, +)$$

$$f(k, -) = -m_a(k) \tilde{J}_a(|k|, k) = \tilde{J}_{10'}(|k|, k) = j(k, -)$$

$$\varphi(k, s)_{out} = \varphi(k, s)_{in} + j(k, s) L$$
B. Non-canonical radiation field

Formula (170) is analogous to the one from the canonical theory. It is clear that although \(a(k,s)_{\text{in}}\) and \(a(k,s)_{\text{out}}\) cannot be simultaneously of the form given by (153) and (158), they do satisfy the non-CCR algebra (159). In spite of this the result (170) is not very satisfactory. Indeed, one expects that the scattering matrix describing a non-canonical quantum field interacting with a classical current is

\[
S = e^{i\phi} e^{-i \int d^4 y J(y) A_{\text{in}}(y)}
\]

with some phase \(\phi\). Then

\[
A_{\text{out}}(x) = S^\dagger A_{\text{in}}(x) S
\]

Employing (150), (168), (169) one can write

\[
A_{\text{rad}}(x) = i \int d^4 y J^b(y) \int d\Gamma(k) L_k \left( e^{ik(x-y)} \tilde{m}_a m_b - e^{-ik(x-y)} \tilde{m}_a \tilde{m}_b + e^{ik(x-y)} m_a \tilde{m}_b - e^{-ik(x-y)} m_a m_b \right)
\]

where \(m_a = m_a(k)\), and

\[
a(k,s)_{\text{out}} = a(k,s)_{\text{in}} + j(k,s) L_k
\]

Consequently, the \(S\) matrix is in the non-canonical theory proportional to the non-canonical displacement operator

\[
S = e^{i\phi} \mathcal{D}(j).
\]

This fact will be shown to eliminate the infrared catastrophe.

C. Propagators

Evaluating the average of (174) in a coherent state \(|Q_a\rangle\) one finds

\[
\langle Q_a | A_{\text{rad}}(x) | Q_a \rangle = \int d^4 y D_Z(x-y) J_a(y) + \text{gauge term.}
\]

The irrelevant gauge term is a remainder of the first part of (153). As expected the radiation field does not depend on what \(\alpha\) one takes in the coherent state, but does depend on the vacuum structure. The presence of the regularized function \(D_Z(x-y)\) instead of \(D(x-y)\) implies that the radiation signal propagates in a neighborhood of the light cone. Any deviations from \(c\) in velocity of signal propagation can be regarded as indications of a non-constant vacuum wave function \(O\). However, even in the orthodox canonical quantum electrodynamics a detailed analysis of signal propagation leads to small deviations from velocity of light, especially at small distances [15]. It may be difficult to experimentally distinguish between the two effects. A similar effect was predicted for Maxwell fields in non-commutative space-time [14-13].

Using (154) one can rewrite (155) as

\[
[A_a(x), A_b(y)] = g_{ab} \int d\Gamma(k) L_k \left( e^{ik(x-y)} - e^{-ik(x-y)} \right) + \ldots
\]

where the dots stand for all the terms which are gauge dependent and do not contribute to physically meaningful quantities. We can therefore identify

\[
\mathcal{D}(x) = i \int d\Gamma(k) L_k \left( e^{-ik\cdot x} - e^{ik\cdot x} \right)
\]
as the operator responsible for the appearence of the smeared out Jordan-Pauli function $D_Z$ in the coherent-state average \( \langle 179 \rangle \). $D(x)$ is a translation-invariant scalar-field operator solution of the d’Alambert equation, i.e.

\[
\square D(x) = 0, \\
V_{\Lambda,y}^\dagger D(x) V_{\Lambda,y} = D(\Lambda^{-1}x).
\]

The operator analogues of retarded and advanced Green functions are

\[
\begin{align*}
&D_{\text{ret}}(x) = \Theta(x_0) D(x), \\
&D_{\text{adv}}(x) = -\Theta(-x_0) D(x), \\
&D(x) = D_{\text{ret}}(x) - D_{\text{adv}}(x).
\end{align*}
\]

Eq. \((182)\) implies that the operators

\[
\int d^4y D_{\text{ret}}(x - y) J_a(y)
\]

and

\[
\int d^4y D_{\text{adv}}(x - y) J_a(y)
\]

differ by at most a solution of the homogeneous equation

\[
\square A_a(x) = 0.
\]

Eq. \((182)\) implies also that one can define

\[
\delta(x) = \square D_{\text{adv}}(x) = \square D_{\text{ret}}(x).
\]

It follows that having a solution $A_{\text{in}}(x)$ of \((180)\) one can define another solution $A_{\text{out}}(x)$ of \((180)\) by means of

\[
A_a(x) = A_{\text{in}}(x) + \int d^4y D_{\text{ret}}(x - y) J_a(y)
\]

\[
= A_{\text{out}}(x) + \int d^4y D_{\text{adv}}(x - y) J_a(y),
\]

simultaneously guaranteeing that the correct $S$-matrix conditions \((171), (172)\) are fulfilled up to, perhaps, a gauge transformation. $A_a(x)$ is a solution of

\[
\square A_a(x) = J_a(x)
\]

where

\[
J_a(x) = \int d^4y \delta(x - y) J_a(y).
\]

D. The problem of infrared catastrophe

To close the discussion let us consider the issue of infrared catastrophe. We have to compute an average number of photons in the state $D(j)|O\rangle$. The number-of-photons operator is

\[
n = \oplus_1 (1 \otimes \sum_s a_s^\dagger a_s).
\]

The 1 in the above formula is the identity in the $k$ space and $a_s$ satisfy CCR. The average reads

\[
\langle n \rangle = \langle O_j|n|O_j\rangle = \sum_s \int d\Gamma(k)Z(k)|j(k,s)|^2.
\]
The four-momentum of the radiation field is
\[ \langle P_a \rangle = \langle Q_a | P_a | Q_a \rangle = \sum_s \int d\Gamma(k)k_a Z(k)|j(k,s)|^2. \] (197)

\( O(k) \) belongs to a carrier space of an appropriate unitary representation of the Poincaré group. As such this is a differentiable function vanishing at the origin \( k = 0 \) of the light cone. This is a consequence of the fact that the cases \( k = 0 \) and \( k \neq 0, k^2 = 0 \), correspond to representations of the Poincaré group induced from \( SL(2, C) \) and \( E(2) \), respectively (for another justification see \([3]\)).

Hence, the regularization of the infrared divergence is implied by relativistic properties of the field. It is quite remarkable that all the divergences are regularized automatically by the same property of the formalism: The nontrivial structure of the vacuum state. In the case of ultraviolet and vacuum divergences the regularization is a consequence of square integrability of \( O \).

X. ON THE MEANING OF \( Z_3 \)

Consider the class of vacua whose probability densities \( Z(k) \) are constant over some region in the \( k \)-space, say \( Z(k) = Z \) for \( k_{\text{min}} < |k| < k_{\text{max}} \), and decay to zero outside of this plateau region. For \( Z(k) \) in the class we find
\[ \langle n \rangle \approx Z \sum_s \int_{\text{cut-off}} d\Gamma(k)|j(k,s)|^2 \] (198)
\[ \langle P_a \rangle \approx Z \sum_s \int_{\text{cut-off}} d\Gamma(k)k_a |j(k,s)|^2. \] (199)

The index “cut-off” means restricting the integral to the plateau region of \( Z \). Although the pointwise limit \( Z(k) \to 0 \) does not belong to the class, one can nevertheless consider the limits (in general of the \( 0 \cdot \infty \) type)
\[ \langle n \rangle_{Z=0} = \lim_{Z \to 0, \text{pointwise}} \sum_s \int_{\text{cut-off}} d\Gamma(k)Z|j(k,s)|^2, \] (200)
\[ \langle P_a \rangle_{Z_3=0} = \lim_{Z \to 0, \text{pointwise}} \sum_s \int_{\text{cut-off}} d\Gamma(k)k_a Z|j(k,s)|^2, \] (201)
which are essentially the formulas for renormalized quantities from the standard formalism if one identifies \( Z \) with the renormalization constant \( Z_3 \). Actually, all the above formulas involving \( Z(k) \) contain \( Z \) in exactly those places where one expects \( Z_3 \) to appear in renormalized quantities \([3]\). The (smooth) cut-offs appear automatically through the asymptotic properties of \( Z(k) \) at zero and infinity. One can thus speak of (finite!) self-renormalization of the non-canonical theory. Let us note that both perturbative and nonperturbative calculations presented in \([3]\) and \([4]\) were pointing into a reinterpretation of \( Z(k)^2 e_0 \) in terms of an effective physical charge.

The formulas (200), (201) have to be supplemented by
\[ \lim_{Z \to 0, \text{pointwise}} \int d\Gamma(k)Z = \int d\Gamma(k)Z(k) = 1. \] (202)

The fact that the pointwise limit gives predictions of the renormalized theory suggests that the physical vacuum is indeed “flat” and \( Z = Z_3 \approx 0 \). The interpretation suggests also that the value and even a functional form of \( Z \) may depend on the state of the entire Universe and, hence, change during its evolution. Now, since
\[ \alpha = \frac{e^2}{\hbar c} = \frac{e_0^2 Z_3}{\hbar c} \] (203)
one may may treat the observed astrophysical changes of \( \alpha \) \([5]\) as an indication of evolving \( Z(k) \).

XI. CONCLUSIONS AND FURTHER PERSPECTIVES

We have formulated a manifestly covariant version of non-canonical quantized electromagnetic fields. A thermodynamic limit of non-canonical theories looks like a finite canonical theory, explaining the success of the latter. In
the limit of an “infinitely flat” vacuum (meaning the cut-off at infinity and $Z \to 0$) the theory becomes local. The elements such as ultraviolet and infrared formfactors do not have to be introduced in ad hoc manners but follow from the more fundamental non-canonical level.

An issue which has not been addressed so far is how to quantize fermions. Some results on the Dirac equation are known already and will be presented in a separate paper. Another problem is to embed the concrete non-canonical quantization procedure we have proposed into a more abstract scheme of quantizations in a $C^*$-algebraic setting. The fact that the right-hand-side of commutation relations is not an identity but rather an operator belonging to the center of the algebra suggests directions for generalizations. It seems there is a link with the work of Streater on non-abelian cocycles [16]. An appropriate version of a coherent-state quantization based on the formalism of Naudts and Kuna [17] is in preparation.

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[8] In this paper when we speak of the Poincaré group we mean the semidirect product of 4-translations and $SL(2,C)$, i.e. the universal covering space of the Poincaré group.
[9] This representation is in-between those introduced in [1] and [2]. The difference is that here two CCR operators are used to describe the polarization degree of freedom, as opposed to the one operator used in [1]. The reason for such a modification becomes clear when one introduces non-canonical (anti)commutation relations for massive particles (M. Czachor, in preparation). In the massless case there is no essential difference between the two approaches. That two CCR operators should be employed in the context of polarizations was stressed by J. Naudts (private communication).
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