Weak Supersymmetry

A.V. Smilga

SUBATECH, Université de Nantes, 4 rue Alfred Kastler, BP 20722, Nantes 44307, France.

Abstract

We explore “weak” supersymmetric systems whose algebra involves, besides Poincare generators, extra bosonic generators not commuting with supercharges. This allows one to have inequal number of bosonic and fermionic 1–particle states in the spectrum. Coleman–Mandula and Haag–Lopuszanski–Sohnius theorems forbid the presence of such extra bosonic charges in interacting theory for $d \geq 3$. However, these theorems do not apply in one or two dimensions. For $d = 1$, we construct a nontrivial interacting system characterized by weak supersymmetric algebra. It is related to “n–fold” supersymmetric systems and to quasi-exactly solvable systems studied earlier.

1 Introduction.

The basic defining feature of any standard supersymmetric system is double degeneracy of all excited states. This follows from the minimal supersymmetry algebra

$$Q^2 = \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\}_+ = 2H,$$

(1)

where $Q$ is a complex conserved [this is a corollary of Eq.(1)] supercharge. If the superalgebra describing symmetries of the system includes (1) as a subalgebra, double degeneracy of all excited levels (if supersymmetry is spontaneously broken, also the ground state is doubly degenerate) follows.

1 On leave of absence from ITEP, Moscow, Russia.
An interesting question is whether some other “weak” supersymmetric algebras involving Poincare generators and conserved supercharges, but not including (1) as subalgebra, are possible. At the algebraic level, the answer is trivially positive. It is easy also to construct Lagrangians enjoying weak supersymmetry. Indeed, the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + i \bar{\psi} \sigma_\mu \partial_\mu \psi$$

(\(\phi\) is a real scalar and \(\psi\) is a Weyl spinor) is invariant with respect to supersymmetry transformations

$$\begin{align*}
\delta \phi &= \epsilon^\alpha \psi_\alpha - \bar{\psi}^{\dot{\alpha}} \epsilon_{\dot{\alpha}} , \\
\delta \psi_\alpha &= i \left( \sigma^\dagger_\mu \right)_\alpha \bar{\epsilon}^{\dot{\beta}} \partial_\mu \phi , \\
\delta \bar{\psi}^{\dot{\alpha}} &= i \epsilon_\beta \left( \sigma^\dagger_\mu \right)_\beta \partial_\mu \phi .
\end{align*}$$

The corresponding supercharges are

$$\begin{align*}
Q_\alpha &= \int d^3 x \left( \sigma^\dagger_\mu \sigma_0 \psi \right)_\alpha \partial_\mu \phi , \\
\bar{Q}^{\dot{\alpha}} &= \int d^3 x \left( \bar{\psi} \sigma_0 \sigma^\dagger_\mu \right)^{\dot{\alpha}} \partial_\mu \phi .
\end{align*}$$

Now, \(Q_\alpha\) and \(\bar{Q}^{\dot{\alpha}}\) are conserved, but their anticommutators involve besides \(P_\alpha^{\dot{\alpha}}\) also extra terms. In particular, \(\{Q_\alpha, Q_\beta\} \neq 0\). The resulting superalgebra does not include the subalgebra (1) and the number of bosonic and fermionic 1–particle states might be different. And it is: the Lagrangian (2) describes a free real boson (one state \(|B\rangle\) for each 3–momentum \(p\)) and a free Weyl fermion (two states \(|F_\pm\rangle\)). It is interesting (and important !) that, in the sector with given \(p\), the state pairing is restored for two-particle excitations and higher. Thus, at the two–particle level, there are two boson states \(|BB\rangle\) and \(|F_+ F_-\rangle\) and two fermion states \(|BF_+\rangle\) and \(|BF_-\rangle\). Actually, any Lagrangian involving some number of free bosonic and some number of free fermionic fields is supersymmetric. There are a lot of such supersymmetries:

---

2We use the standard Weyl notation where the dotted and undotted indices are raised and lowered with \(\epsilon^{\alpha\beta}(\epsilon^{\dot{\alpha}\dot{\beta}})\) and \(\epsilon_{\alpha\beta}(\epsilon_{\dot{\alpha}\dot{\beta}}) = -\epsilon^{\alpha\beta}(\epsilon^{\dot{\alpha}\dot{\beta}})\); \(\bar{\psi}^{\dot{\alpha}} = (\psi_\alpha)^\dagger\) and \(\psi_\alpha = -(\bar{\psi}^{\dot{\alpha}})^\dagger\); \(\bar{\psi}^2 = \psi_\alpha \psi^{\alpha}\) and \(\bar{\psi}^{\dot{\alpha}} = \bar{\psi}^{\dot{\alpha}} \psi^{\dot{\alpha}}\); \(\sigma_\mu = (1, \sigma)\) and \(\sigma^\dagger_\mu = (1, -\sigma)\).
each bosonic field can be mixed with each fermionic field independently of others. However, this is only true for free theory. As soon as the interaction is switched on, supersymmetry (strong or weak) is broken. Indeed, nonvanishing $\{Q_\alpha, Q_\beta\}$ implies the presence of an extra conserved bosonic charge in the representation $(1, 0)$ of the Lorentz group. It is none other than the self-dual part of the fermion spin operator $S_{\alpha\beta}$. Spin is not conserved, however, in interacting theories. Actually, in any theory involving mass gap and a nontrivial $S$–matrix, the presence of extra nonscalar conserving charges is ruled out by the Coleman–Mandula theorem [2]. Interacting supersymmetric theories can only involve, besides the Poincare generators $P_\mu$, $M_{\mu\nu}$ and supercharges $Q^i$, central charges commuting with everything and some extra global symmetry generators, which can have nontrivial commutators with $Q$ and between themselves, but they cannot appear in this case in the anticommutators of supercharges [3].

This is true if the dimension of space-time is 3 or more. In two dimensions, where scattering can be only forward or backward, the theorems [2, 3] do not apply. In particular, one can have an infinite number of conserved bosonic charges (like, e.g., it is the case in the Sine–Gordon model). Seemingly, nothing prevents one to have an interacting 2d theory enjoying a version of weak supersymmetry.

We tried to construct one, but failed. Probably, one should try harder. What we were able to construct is a weakly supersymmetric quantum mechanical system. It has two complex conserved supercharges with nontrivial anticommutators involving besides $H$ four other bosonic generators which are not central charges — their commutators with supercharges and between themselves do not vanish. The boson-fermion degeneracy is there starting from the second excited level. But not for the first excited level and not for vacuum.

In Sect. 2, we describe a simplest such system — the weak supersymmetric oscillator. In Sect. 3, we present a nontrivial weak supersymmetric Hamiltonian. We find that previously studied quantum systems with so-called “2–fold supersymmetry” [4] are in fact weak supersymmetric systems in disguise. We briefly discuss their relationship to quasi-exactly solvable

---

3 One of the consequences of this is the presence of the so called quasigoldstino branch in the spectrum of collective excitations in quark–gluon plasma [1].

4 For massless theories, Poincare group can be extended to conformal and super-Poincare — to superconformal.
models [5]. Sect. 4 describes our failed attempt to generalize our QM con-
struction to $d = 2$. This experience might be useful in future studies. Sect.
5 is reserved for discussion, conclusions, and acknowledgements.

2 Weak supersymmetric oscillator

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} \dot{x}^2 + i \tilde{\psi}^\alpha \tilde{\psi}_\alpha - \frac{m^2}{2} x^2 - \frac{m}{2} (\tilde{\psi}^\alpha \tilde{\psi}_\alpha + \psi_\alpha \psi^\alpha) , \quad (5)$$

$\alpha = 1, 2$. It can be obtained out of the massive version of the free field
theory Lagrangian (2) by dimensional reduction. It is invariant with respect
to supersymmetry transformations

$$\delta x = \bar{\epsilon} \psi + \tilde{\epsilon} \bar{\psi} ,$$
$$\delta \tilde{\psi}_\alpha = -i \epsilon_\alpha \dot{x} + \bar{\epsilon}_\alpha m x ,$$
$$\delta \psi^\alpha = i \epsilon^\alpha \dot{x} - \epsilon^\alpha m x . \quad (6)$$

The corresponding supercharges are

$$Q_\alpha = p \psi_\alpha + i m x \bar{\psi}_\alpha ,$$
$$\bar{Q}^\alpha = p \bar{\psi}^\alpha + i m x \psi_\alpha , \quad (7)$$

where $p = \dot{x}$ is the bosonic canonical momentum. The canonical Hamiltonian
is

$$H = \frac{1}{2} p^2 + \frac{1}{2} m^2 x^2 + \frac{m}{2} (\bar{\psi}^\alpha \tilde{\psi}_\alpha + \psi_\alpha \psi^\alpha) . \quad (8)$$

The quantum Hamiltonian can be written by replacing $p$ by $-i \partial / \partial x$ and
$\tilde{\psi}^\alpha$ by $\partial / \partial \psi_\alpha$. Supercharges commute with the Hamiltonian. On the other
hand,

$$\{Q_\alpha, Q_\beta\} = Z_{\alpha\beta} ,$$
$$\{Q_\alpha, \bar{Q}^\beta\} = \delta^\beta_\alpha (2H - Y) , \quad (9)$$

where

$$Z_{\alpha\beta} = m (\psi_\alpha \bar{\psi}_\beta + \psi_\beta \bar{\psi}_\alpha) ,$$
$$Y = \frac{m}{2} (\bar{\psi}^\alpha \tilde{\psi}_\alpha + \psi_\alpha \psi^\alpha) . \quad (10)$$
The operators $Z_{\alpha\beta}$ and $Y$ commute with $H$ and with each other, but the commutators

$$
[Q_\alpha, Z_{\beta\gamma}] = m (\epsilon_{\alpha\beta} Q_{\gamma} + \epsilon_{\alpha\gamma} Q_{\beta}),
$$

$$
[\bar{Q}_\alpha, Z_{\beta\gamma}] = m (\epsilon_{\alpha\beta} \bar{Q}_{\gamma} + \epsilon_{\alpha\gamma} \bar{Q}_{\beta}),
$$

$$
[Q_\alpha, Y] = m Q_\alpha ,
$$

$$
[\bar{Q}_\alpha, Y] = m \bar{Q}_\alpha,
$$

and also

$$
[Z_{\alpha\beta}, Z_{\gamma\delta}] = m (\epsilon_{\beta\gamma} Z_{\alpha\delta} + \epsilon_{\alpha\gamma} Z_{\beta\delta} + \epsilon_{\beta\delta} Z_{\alpha\gamma} + \epsilon_{\alpha\delta} Z_{\beta\gamma})
$$

are nontrivial. The algebra can be presented in a little bit more convenient form if introducing $S_\alpha = Q_\alpha - \bar{Q}_\alpha$, $\bar{S}_\alpha = Q_\alpha + \bar{Q}_\alpha$. Then

$$
\{S_\alpha, \bar{S}_\beta\} = 4H \delta_\alpha^\beta - 2Y \delta_\alpha^\beta + 2Z_\alpha^\beta ,
$$

$$
[S_\alpha, Z_{\beta\gamma}] = m (\epsilon_{\alpha\beta} S_{\gamma} + \epsilon_{\alpha\gamma} S_{\beta}) ,
$$

$$
[\bar{S}_\alpha, Z_{\beta\gamma}] = m (\delta_\alpha^\beta \bar{S}_{\gamma} + \delta_\alpha^\gamma \bar{S}_{\beta}) ,
$$

$$
[S_\alpha, Y] = -m S_\alpha ,
$$

$$
[\bar{S}_\alpha, Y] = m \bar{S}_\alpha,
$$

(13)

to which the commutator (12) should be added. The subalgebra (12) is none other than $sl(2)$, which can be readily seen by identification

$$
Z_{11} \equiv 2i m \sigma_+ , \quad Z_{22} \equiv 2i m \sigma_- , \quad Z_{12} = Z_{21} \equiv -m \sigma_3 .
$$

All other commutators and the anticommutator $\{S_\alpha, S_\beta\}$ vanish.

It is not difficult to find the spectrum of $H$. The eigenstates are

$$
\Phi^\pm_n = \frac{1}{\sqrt{2}} |n\rangle \left(1 \pm \frac{1}{2} \psi_\alpha \bar{\psi}_\alpha^\alpha\right) ; \quad \Phi^\alpha_n = \psi_\alpha |n\rangle ,
$$

(14)

where $|n\rangle$ are the bosonic oscillator eigenstates. Their energies are

$$
E_n^- = \left(-\frac{1}{2} + n\right) m , \quad E_n^\alpha = \left(\frac{1}{2} + n\right) m , \quad E_n^+ = \left(\frac{3}{2} + n\right) m .
$$

(15)

The spectrum is drawn in Fig. 1. We see that there is one vacuum state (its energy can be brought to zero by adding the constant $m/2$ to the Hamiltonian, but for the weak supersymmetric systems with algebra (13), $E_{\text{vac}} = 0$
is an option rather than requirement). There are three first excited states: a bosonic and two fermionic. Starting from the second excited state, there are 2 bosonic and 2 fermionic states at each level.

The eigenvalues of the operator $Y$ is $-m$ for the leftmost tower, 0 for two central and $+m$ for the rightmost one. In other words, the operator $Y/m$ (or rather $Y/m + 1$) plays the role of the fermionic charge. The operators $Z_{\alpha\beta}$ annihilate the states $\Phi^n_\pm$ while the states $\Phi^n_\alpha$ form doublet representations of the $sl(2)$ algebra (12).

To acquire further insights, it is instructive to write the action of the operators $S_\alpha, \bar{S}^\alpha$ on the states (14):

$$S_\alpha \Phi^n_\pm = 2\sqrt{mn} \Phi^{n-1}_\alpha, \quad S_\alpha \Phi^n_\beta = -2\sqrt{mn} \epsilon_{\alpha\beta} \Phi^{n-1}_+ , \quad S_\alpha \Phi^n_+ = 0 ,$$

$$\bar{S}^\alpha \Phi^n_- = 0, \quad \bar{S}^\alpha \Phi^n_\beta = 2\sqrt{m(n+1)} \delta^\alpha_\beta \Phi^{n+1}_-, \quad \bar{S}^\alpha \Phi^n_+ = 2\sqrt{m(n+1)} \epsilon^{\alpha\beta} \Phi^{n+1}_\beta .$$ (16)

We see that $S_\alpha$ annihilates the states from the rightmost column and brings the states from the left columns to the right. The action of $\bar{S}^\alpha$ is opposite.

Now, one can divide all eigenstates in two sets: (i) the states $\Phi^n_- \text{ and } \Phi^n_1$ and (ii) the states $\Phi^n_2 \text{ and } \Phi^n_+$. The states from the subset (i) form the Hilbert
space of the $\mathcal{N} = 1$ supersymmetric oscillator, with $S_1$ playing the role of the supercharge. The same applies to the subset $(ii)$ with the supercharge $S_2$. For ordinary $\mathcal{N} = 2$ supersymmetric quantum mechanics, the Hilbert space can also be divided into two $\mathcal{N} = 1$ subspaces, but the specifics of a weak supersymmetric system is that two sets of states are shifted with respect to each other, i.e. the Hamiltonian for the right subset differs from the Hamiltonian for the left subset by a constant.

3 A class of interactive weak supersymmetric systems.

Consider the Lagrangian

$$L = \frac{\dot{x}^2}{2} + i\bar{\psi}^{\alpha} \dot{\psi}_\alpha - \frac{V^2}{2} - \frac{V'}{2} (\psi^2 + \bar{\psi}^2) - \frac{B'}{2} \psi^2 \bar{\psi}^2,$$  

$$B = \frac{V' - C}{2V},$$  

where $V(x)$ is an arbitrary function and $C$ is an arbitrary constant. One can observe that the corresponding action is invariant with respect to the supersymmetry transformations

$$\begin{align*}
\delta x &= \bar{\epsilon} \psi + \bar{\psi} \epsilon, \\
\delta \psi_\alpha &= -i \epsilon_\alpha \dot{x} + \bar{\epsilon}_\alpha (V + B \psi^2) - 2B \left[ (\bar{\psi} \psi) \epsilon_\alpha + (\bar{\psi} \epsilon) \psi_\alpha \right], \\
\delta \bar{\psi}^{\alpha} &= i \bar{\epsilon}^{\alpha} \dot{x} - \epsilon^{\alpha} (V + B \bar{\psi}^2) - 2B \left[ \bar{\epsilon}^{\alpha} (\bar{\psi} \psi) + \bar{\psi}^{\alpha} (\bar{\psi} \epsilon) \right].
\end{align*}$$

When $V(x) = mx$ and $C = m$, Eq.(17) is reduced to the oscillator Lagrangian (5) considered above. The first four terms in Eq.(17) represent a rather natural generalization of Eq.(5), like in Witten’s supersymmetric quantum mechanics [6]. In our case, we are obliged to add also a 4-fermion term in the Lagrangian and extra nonlinear terms in the transformation law.

The canonical classical supercharges and Hamiltonian are

$$\begin{align*}
Q_\alpha &= p\psi_\alpha + iV \bar{\psi}_\alpha + iB \psi^2 \bar{\psi}_\alpha, \\
\bar{Q}^{\alpha} &= p\bar{\psi}^{\alpha} + iV \psi^{\alpha} + iB \bar{\psi}^2 \psi^{\alpha},
\end{align*}$$

(19)
\[ H^{\text{cl}} = \frac{p^2}{2} + \frac{V^2}{2} + \frac{V'}{2} \left( \psi^2 + \bar{\psi}^2 \right) + \frac{B'}{2} \psi^2 \bar{\psi}^2. \]  

(20)

The Poisson brackets \( \{Q_\alpha, H\}_{\text{P.B.}} \) and \( \{\bar{Q}^\alpha, H\}_{\text{P.B.}} \) vanish. A certain care is required when quantizing this theory. We want to fix the ordering ambiguities in \( Q \) and \( H \) such that classical supersymmetry were not spoiled at the quantum level. An experience acquired by fiddling with supersymmetric \( \sigma \)-models and gauge theories \[7\] teaches us that proper quantum supercharges should be obtained by Weyl ordering procedure from the classical expressions (19). In other words, (19) should be Weyl symbols of the quantum supercharges. This does not necessarily apply to the Hamiltonian. Indeed, if choosing the quantum Hamiltonian such that (20) represents the Weyl symbol of \( \hat{H}^{\text{qu}} \), the Weyl symbol of the commutator \( [\hat{Q}_\alpha, \hat{H}] \) would be given by the Moyal bracket \[8, 7\] of \( Q^{\text{cl}}_\alpha \) and \( H^{\text{cl}} \),

\[
2\text{sh}\left\{ \frac{1}{2} \left[ \frac{\partial^2}{\partial \Psi_\alpha \partial \bar{\psi}^\alpha} - \frac{\partial^2}{\partial \psi_\alpha \partial \bar{\Psi}^\alpha} + i \left( \frac{\partial^2}{\partial q \partial P} - \frac{\partial^2}{\partial Q \partial p} \right) \right] \right\}
\]

\[
Q^{\text{cl}}_\alpha(\bar{\psi}, \psi; p, q) H^{\text{cl}}(\bar{\Psi}, \Psi; P, Q) \big|_{\Psi = \bar{\psi}, \Psi = \psi; P = p, Q = q} = iBB' \psi_\alpha \neq 0.
\]  

(21)

To compensate that, we should add to \( H^{\text{cl}} \) the term \( B^2/2 \). The quantum supercharges and Hamiltonian thus obtained are

\[
\hat{Q}_\alpha = p\psi_\alpha + iV\bar{\psi}_\alpha + iB \left( \psi^2 \bar{\psi}_\alpha + \psi_\alpha \right),
\]

\[
\hat{\bar{Q}}^\alpha = p\bar{\psi}^\alpha + iV\psi^\alpha + iB \left( \bar{\psi}^\alpha \bar{\psi}^2 - \bar{\psi}^\alpha \right)
\]

and

\[
\hat{H} = \frac{p^2}{2} + \frac{V^2}{2} + \frac{V'}{2} \left( \psi^2 + \bar{\psi}^2 \right) + \frac{B'}{2} \psi^2 \bar{\psi}^2 + \frac{B^2}{2} + \frac{C}{2},
\]

(23)

where we added for convenience the constant \( C/2 \) in the Hamiltonian. Direct calculation of the commutators (or, which is simpler, of the Moyal brackets of the classical expressions) leads to a remarkable conclusion: the algebra (13, 12) derived for the oscillator is valid also in the interactive case, with \( m \to C, H \to H - C/2 \) and \( Z_{\alpha\beta}, Y \) having the same form (10) as before.

As earlier, the quantum states can be divided into three classes: (i) the states \( |\rangle \propto 1 - \psi^2/2 \), (ii) the states \( |\alpha\rangle \propto \psi_\alpha \) (they are present in two
copies as the Hamiltonian (23) does not feel the index $\alpha$ and (iii) the states $|+\rangle \propto 1 + \psi^2/2$. These states are characterized by a definite value of the “fermion charge” $Y: Y_\pm = \pm m$ and $Y_\alpha = 0$. In each such sector, we have an ordinary Schrödinger equation with the potentials

\begin{align*}
U_- &= \frac{1}{2}(W_-^2 - W'_-) , \\
U_\alpha &= \frac{1}{2}(W_\alpha^2 + W'_\alpha) = \frac{1}{2}(W_-^2 - W'_+) + C , \\
U_+ &= \frac{1}{2}(W_+^2 + W'_+) + C ,
\end{align*}

(24)

where

\begin{equation}
W_\pm = V \pm B .
\end{equation}

(25)

It is clear now that we are dealing with two superimposed ordinary Witten’s SQM systems. The states $|-\rangle$ and $|1\rangle$ are described by such system with superpotential $W_-$ and the states $|2\rangle$ and $|+\rangle$ - by the system with superpotential $W_+$, with the constant $C$ added to the Hamiltonian. Excited states are mostly 4-fold degenerate as for usual $\mathcal{N} = 2$ SQM.

The ground state is not necessarily degenerate. If $\exp\{-\int W_-(x)dx\}$ is normalizable, this (being multiplied by $1 - \psi^2/2$) determines the wave function of the unique vacuum state. With the chosen normalization of the Hamiltonian [the term $C/2$ in Eq.(23) !] it has zero energy. Further, if $\exp\{-\int W_+(x)dx\}$ is normalizable, there is also a unique zero-energy ground state for Witten’s Hamiltonian with superpotential $W_+$. Thus, we obtain a state in the sector $|2\rangle$ with energy $C$. Due to $|2\rangle \leftrightarrow |1\rangle$ and $|1\rangle \leftrightarrow |-\rangle$ degeneracies, we have altogether three states with energy $C$ at the first excited level, and the picture is the same as for the oscillator (see Fig.1).

We have obtained free of charge a wide class of quasi-exactly solvable [5] potentials $U_-(x)$ for which the energy of the ground state ($E_0 = 0$) and of the first excited state $E_1 = C$ are exactly known. They depend on an arbitrary function $V(x)$ and an arbitrary constant $C$ with a certain restriction: both $\exp\{-\int W_-(x)dx\}$ and $\exp\{-\int W_+(x)dx\}$ should be normalizable. Probably, the simplest nontrivial choice is $V(x) = mx + \alpha x^3$ with $C = m$ ($m, \alpha > 0$). Note that the potential $U_-(x)$ is not polynomial in this case.
The potentials $U_\pm$ in Eqs.(24, 25) were discussed before (see Eq.(4.18) in Ref.[9] and Eq.(50) in Ref.[10]) in association with the so called 2–fold supersymmetry construction developed in [4]. $N$–fold supersymmetry is a supersymmetry where supercharges are not linear in momentum, but present polynomials of power $N$. In our case, one can define the quadratic in $p$ operator

$$Q = S_1 S_2$$

(26)

with the action $Q|\alpha\rangle = 0$ and $Q|\bar{\alpha}\rangle = |\alpha\rangle$. $Q$ acts in the opposite direction: $\bar{Q}|\alpha\rangle = |\bar{\alpha}\rangle$, $\bar{Q}|\bar{\alpha}\rangle = Q|\alpha\rangle = 0$. The operators $Q$, $\bar{Q}$ commute with $H$ (as $S_{1,2}$ and $\bar{S}_{1,2}$ do). If disregarding the states $|\alpha\rangle$ annihilated by both $Q$ and $\bar{Q}$ and considering only the sectors $|\alpha\rangle$ and $|\bar{\alpha}\rangle$, one can deduce

$$\{Q, \bar{Q}\} = 16H(H - C).$$

(27)

The quadratic polynomial of $H$ appearing on the right side is characteristic of 2–fold supersymmetry. The full algebra of the weak supersymmetry (12, 13) displays itself only if the “central” sector $|\alpha\rangle$ is brought into consideration.

4 A generalization that failed.

As was discussed in the Introduction, it would be very interesting to construct a nontrivial $2D$ weak supersymmetric model. In this section, we explain why a naive generalization of the model (17) to the $2D$ case fails. Let us consider only the fermion part in (17) and assume $x$ to be constant. Such a restricted action is still invariant with respect to $\delta \psi_\alpha = \ldots, \delta \bar{\psi}_\alpha = \ldots$ as dictated by Eq.(18) Consider only the part in $\delta L$ involving the time derivatives $\dot{\psi}_\alpha$ and $\dot{\bar{\psi}}_\alpha$ and cubic in $\psi$. It is determined by the variation of the kinetic term $i \bar{\psi}_\alpha \dot{\psi}_\alpha$ coming from the bits in $\delta \psi_\alpha$ and $\delta \bar{\psi}_\alpha$ that are quadratic in $\psi, \bar{\psi}$. Such a variation represents a total time derivative as it should.

Let us now try to generalize this to $d = 2$. Consider the fermion kinetic term $^5$

$$L_f = i \bar{\psi} \gamma_\mu \partial_\mu \psi$$

(28)

$^5$The convention in this section is $\bar{\psi} = \psi \gamma^0$, $\gamma_\mu \gamma_\nu = g_{\mu\nu} - \epsilon_{\mu\nu} \gamma^5$ (e.g. $\gamma_0 = \sigma^1, \gamma_1 = i\sigma^2, \gamma_5 = \sigma^3$).
and let us require it to be invariant up to a total derivative with respect to the transformation

\[
\begin{align*}
\delta \psi_{\alpha} &= (\bar{\psi}\epsilon)_{\alpha} + A(\bar{\psi}\gamma_{5}\psi)_{\alpha} + B(\bar{\psi}\gamma_{\nu}\psi)_{\alpha}, \\
\delta \bar{\psi}^{\alpha} &= D\bar{\psi}^{2}\epsilon^{\alpha}
\end{align*}
\]

(29)

with arbitrary constants \(A, B, D\) (only the terms \(\propto \epsilon\) are displayed, the terms \(\propto \bar{\epsilon}\) to be restored by hermiticity). Consider the terms \(\propto \partial_{\mu}\bar{\psi}\) in \(\delta L_{f}\):

\[
\delta L_{f} \propto (\bar{\psi}\gamma_{\mu}\psi)(\partial_{\mu}\bar{\psi}\epsilon) + A(\bar{\psi}\gamma_{\mu}\gamma_{5}\psi)(\partial_{\mu}\bar{\psi}\gamma_{5}\epsilon) + B(\bar{\psi}\psi)(\partial_{\mu}\bar{\psi}\gamma_{\mu}\epsilon)
\]

\[-B\epsilon_{\mu\nu}(\bar{\psi}\gamma_{5}\psi)(\partial_{\mu}\bar{\psi}\gamma_{\nu}\epsilon). \quad (30)\]

Introduce the charge conjugation matrix \(C = i\sigma^{2}\) satisfying the properties

\[
C^{T} = -C, \quad C^{2} = -1, \quad C\gamma_{\mu} = -\gamma_{\mu}^{T}C
\]

(31)

and rewrite Eq. (30) in terms of the structures \(\bar{\psi}C\partial_{\mu}\bar{\psi}, \bar{\psi}\gamma_{\nu}C\partial_{\mu}\bar{\psi}\) and \(\bar{\psi}\gamma_{5}C\partial_{\mu}\bar{\psi}\) using the Fierz identity

\[
\bar{\psi}_{\alpha}\partial_{\mu}\bar{\psi}_{\gamma} = -\frac{1}{2}[C_{\alpha\gamma}(\bar{\psi}C\partial_{\mu}\bar{\psi}) + (C\gamma_{\nu})_{\alpha\gamma}(\bar{\psi}\gamma_{\nu}C\partial_{\mu}\bar{\psi}) + (C\gamma_{5})_{\alpha\gamma}(\bar{\psi}\gamma_{5}C\partial_{\mu}\bar{\psi})]. \quad (32)
\]

If we wish the variation (30) to be a total derivative, only the structure \(\bar{\psi}C\partial_{\mu}\bar{\psi} = \frac{1}{2}\partial_{\mu}[\bar{\psi}C\bar{\psi}]\) in \(\delta L_{f}\) is allowed while the coefficients of the structures \(\bar{\psi}\gamma_{\nu}C\partial_{\mu}\bar{\psi}\) and \(\bar{\psi}\gamma_{5}C\partial_{\mu}\bar{\psi}\) should vanish — the matrices \(\gamma_{\nu}C\) and \(\gamma_{5}C\) are symmetric and \(\bar{\psi}\gamma_{\nu}C\psi\) and \(\bar{\psi}\gamma_{5}C\psi\) vanish identically.

The statement is that it is not possible to suppress unwanted structures with any choice of the constants \(A, B\).

5 Discussion.

Our original motivation was the quest for nontrivial supersymmetric systems with mismatch between bosonic and fermionic degrees of freedom. Let us note here that, while it is difficult to find such field theory systems, their presence in quantum mechanics was known for a long time. Most popular SQM systems (Witten's quantum mechanics, standard supersymmetric \(\sigma\) models, etc) have an equal number of bosonic and fermionic phase space coordinates. But the SQM system describing planar motion in transverse
magnetic field involves two pairs of bosonic variables and only one pair of fermionic variables. A class of nonstandard “symplectic” $\mathcal{N} = 2$ supersymmetric $\sigma$ models involving $3r$ bosonic variables and $2r$ fermionic variables ($r$ is an integer) was constructed in Ref.[11]. The Diaconescu-Entin $\mathcal{N} = 4$ symplectic $\sigma$ model [12] generalized in [13] involves $5r$ bosonic and $4r$ fermionic variables. An industrial method to construct SQM models where the number of bosonic variables is less than the number of fermionic ones was suggested in [14]. In SQM, an equal number bosonic and fermionic degrees of freedom is not required by supersymmetry. What is required is the equal number of bosonic and fermionic quantum states. But in field theory, any bosonic or fermionic dynamical field correspond to an asymptotic state (a particle), and bosons and fermions should normally be matched.

We notice that this matching can be absent if relaxing the requirement that the anticommutator of supercharges involves only the Hamiltonian, momentum, and central charges. A lot of free weak supersymmetric models can be written, but, for $d > 3$, interactive weak supersymmetric theories do not exist. This follows from the Haag–Lopuszanski–Sohnius theorem. This theorem does not apply to 2 dimensions, however, and the existence of interactive weak supersymmetric theories cannot be ruled out. The fact that our quest was not successful leaves two options:

• Maybe one should just try harder and such systems will eventually be found. Perhaps, it is reasonable to look at supersymmetric generalizations of exactly solvable $2D$ models with an infinite number of conservation laws (such bosonic systems do not exist for $d > 3$ due to Coleman-Mandula theorem, which is relative to the HLS theorem). However, in known such supersymmetric generalizations [15], there is no mismatch.

• Maybe such models do not exist, indeed, in which case one should be able to prove a strong version of the Haag–Lopuszanski–Sohnius theorem.

Both possibilities look very interesting and only future studies will show which of them is realized.

The main positive result of this paper is the system (17) which enjoys a weak supersymmetry algebra (12), (13). It describes quantum systems which were studied before, but from a different perspective. It would be interesting
to construct and study other, more complicated weak supersymmetric models, especially the models involving several bosonic degrees of freedom. This would allow one to construct new examples of multidimensional quasi-exactly solvable models.

Acknowledgements

I am indebted to N. Dorey, M. Henneaux, N. Nekrasov, M. Plyushchay, M. Shifman, and A. Vainshtein for illuminating discussions and correspondence.

References

[1] V.V. Lebedev and A.V. Smilga, Ann. Phys. 202 (1990) 229.
[2] S.R. Coleman and J. Mandula, Phys. Rev. 159 (1967) 1251.
[3] R. Haag, J.T. Lopuszanski, and M. Sohnius, Nucl. Phys. B88 (1975) 257.
[4] A.A. Andrianov, M.V. Ioffe, and V.P. Spiridonov, Phys. Lett. A174 (1993) 273.
[5] A. Turbiner, Commun. Math. Phys. 118 (1988) 467; A. Ushveridze, Sov. J. Part. Nucl 20 (1989) 504; for a review see M. Shifman, ITEP lectures on particle physics and field theory, World Scientific, 1999, p. 775.
[6] E. Witten, Nucl. Phys. B188 (1981) 513.
[7] A.V. Smilga, Nucl. Phys. B292 (1987) 363.
[8] I.E. Moyal, Proc. Cambr. Phil. Soc. 45 (1949) 99.
[9] H. Ayoama, M. Sato, and T. Tanaka, Nucl. Phys. B619 (2001) 105 [arXive: quant-ph/0106037].
[10] A.A. Andrianov and A.V. Sokolov, Nucl. Phys. B660 (2003) 25 [arXive: hep-th/0301062].
[11] A.V. Smilga, Nucl. Phys. B291 (1987) 241; E.A. Ivanov and A.V. Smilga, Phys. Lett. B257 (1991) 79.

[12] D.-E. Diaconescu and R. Entin, Phys. Rev. D56 (1997) 8045 [arXive: hep-th/9706059].

[13] A.V. Smilga, Nucl. Phys. B652 (2003) 93 [arXive: hep-th/0209187].

[14] E.A. Ivanov, S.O. Krivonos, and A.I. Pashnev, Class. Quant. Grav. 8 (1991) 19; A. Losev and M. Shifman, Mod. Phys. Lett. A16 (2001) 2529 [arXive: hep-th/0108151].

[15] see e.g. supersymmetric Sine–Gordon model built in [P. Di Vechhia and S. Ferrara, Nucl. Phys. B130 (1977) 93; J. Hruby, Nucl. Phys. 131 (1977) 275; R. Shankar and E. Witten, Phys. Rev. D17 (1978) 2134].