Zero Energy anomaly in one-dimensional Anderson lattice with exponentially correlated weak diagonal disorder

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Abstract

We calculated numerically the localization length of one-dimensional Anderson model with correlated diagonal disorder. For zero energy point in the weak disorder limit, we showed that the localization length changes continuously as the correlation of the disorder increases. We found that higher order terms of the correlation must be included into the current perturbation result in order to give the correct localization length, and to connect smoothly the anomaly at zero correlation with the perturbation result for large correlation.

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I. INTRODUCTION

Electronic transport properties, the motion of electrons, in a random potential are closely related to the phenomenon of Anderson localization [1]. The phenomena of Anderson localization have been studied in various fields including photonics [2], cold atoms [3], circuits [4], and DNA molecules [5, 6]. Many accurate numerical approaches have been developed, by the quantum transfer matrix renormalization group method for finite temperature systems [7], the density matrix renormalization group method for interacting systems [8], and the integral equation method for systems in the thermodynamic limit [9, 10], respectively. In this work we will study the zero energy behavior for the one-dimensional model with correlated weak diagonal disorder. We first extend the numerical method we developed earlier in Ref. [9] for uncorrelated disorder to correlated system. Our numerical method was an application of the transfer matrix method [11] in localized phase in the thermodynamic limit.

In one-dimensional Anderson model [1] with diagonal disorder is described by,

\[ \psi_{i+1} + \psi_{i-1} = (E - \epsilon_i)\psi_i, \]  

where hopping term is set to unity and \( \psi_i \) is the electron wavefunction at site-\( i \). \( \epsilon_i \) is the on-site energy with a certain type of random distribution which satisfying an exponential correlation: \( \langle \epsilon_i^2 \rangle = \sigma^2 \) and \( \langle \epsilon_i \epsilon_j \rangle = \sigma^2 \exp[-|i-j|/l_{cor}] \) for different sites. \( \sigma^2 \) and \( l_{cor} \) are the strength and correlation length for the disordered on-site energy, respectively. Uncorrelated disorder is given by \( l_{cor} \to 0 \). Recently the anomaly around the band edge \( E = \pm 2 \) has been carefully investigated. [12] In the following we focus on the zero energy anomaly with exponentially correlated diagonal disorder.

All the eigenstates are exponentially localized for one-dimensional uncorrelated disordered systems. [13] The Lyapunov exponent \( \gamma \) is the inverse of the localization length. It is well known that for the zero energy anomaly of the uncorrelated disorder system, the Lyapunov exponent \( \gamma \) is singular at \( E = \sigma = 0 \). [14, 15]. The physical picture behind was also clear [16]. For a box distribution of uncorrelated disorder with width \( W \) and height \( 1/W \), the perturbation result revealed that the Lyapunov exponent depends only on energy \( E \) and disorder strength \( W \) [17]

\[ \gamma = \frac{W^2}{96(1 - E^2/4)}. \] (2)
At the band center, another perturbation yielded \[ \gamma = \frac{W^2}{105.045} \cdots \tag{3} \] The standard variance of the disorder is \( \sigma^2 = \frac{W^2}{12} \). In uncorrelated systems, order by order perturbation expansion in \( \sigma^2 \) and \( E/\sigma^2 \) has been demonstrated \[10\].

For exponentially correlated disorder, the formula for the Lyapunov exponent at finite energy and in the weak disorder strength limit is given by \[ \gamma = \frac{\sigma^2}{8(1 - \frac{E^2}{4})} \cdot \frac{\sinh \frac{1}{l_{\text{cor}}}}{1 + \cosh \frac{1}{l_{\text{cor}}} - \frac{E^2}{2}}. \tag{4} \]

It is straightforward to take the uncorrelated limit \( l_{\text{cor}} \to 0 \) of formula Eq. (4), then obtain \( \gamma/\sigma^2 = 1/8 \) when \( E \) approaches to 0, i.e. \( \lim_{E \to 0} \lim_{\sigma^2 \to 0} \frac{\gamma}{\sigma^2} = 1/8 \). On the other hand, if we stay at \( E = 0 \), we should have \( \gamma/\sigma^2 = 1/8.754 \) in the uncorrelated limit in accordance to Eq. (3), which implies \( \lim_{\sigma^2 \to 0} \lim_{E \to 0} \frac{\gamma}{\sigma^2} = 1/8.754 \). Therefore, we found that the order of the limiting processes for \( E \to 0 \) and \( \sigma^2 \to 0 \) can not be interchanged. It means that the point \( E = \sigma = 0 \) remains singular for perturbation expansions in \( \sigma^2 \) and \( E \) for correlated disorders.

The existence of strong anomalies phenomena in a correlated disorder system was pointed by Titov and Schomerus \[22\]. In this work we study the anomaly at \( E = 0 \).

II. PARAMETRIZATION METHOD

In the transfer matrix method, Eq. (4) can be written as
\[
\Psi_{i+1} = \begin{pmatrix} \psi_{i+1} \\ \psi_i \end{pmatrix} = \begin{pmatrix} E - \epsilon_i & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix} = T_i \Psi_i, \tag{5}
\]
where \( T_i \) is the transfer matrix.

Using a parametrization method of the transfer matrix proposed in our previous work \[9, 10\], we will calculate the Lyapunov exponent in the thermodynamic limit within the localization regime. Let \( M_L = T_L T_{L-1} \cdots T_1 \). Then we parameterize \( MM^t \) as follows
\[
U(\theta_L) M_L M_L^t U(-\theta_L) = \begin{pmatrix} e^{2\lambda_L} & \epsilon^{2\lambda_L} \\ e^{-2\lambda_L} & e^{-2\lambda_L} \end{pmatrix}, \tag{6}
\]
where \( M^t \) is the transpose of \( M \) and
\[
U(\theta_L) = \begin{pmatrix} \cos \theta_L & -\sin \theta_L \\ \sin \theta_L & \cos \theta_L \end{pmatrix}. \tag{7}
\]
The recursion relation of $\theta$ in the large $L$ limit is

$$\tan \theta_{L+1} = \frac{1}{E - \epsilon_{L+1} - \tan \theta_L}.$$  \hspace{1cm} (8)

We introduce the correlations between $\epsilon_i$ through the transformation of a group of independent random variables $\eta_l$ in terms of an identical Gaussian density distribution,

$$p_{\eta}(\eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-\eta^2/2\sigma^2].$$ \hspace{1cm} (9)

Let $q = e^{-1/l_{\text{cor}}}$, the exponentially correlated variable is generated implicitly by $\epsilon_i = \sqrt{1 - q^2} \sum_{l=0}^{\infty} \eta_{l-i} q^l$, or equivalently in the following recursive form,

$$\epsilon_L = \sqrt{1 - q^2} \eta_L + q \epsilon_{L-1}. \hspace{1cm} (10)$$

The three parameters $\lambda$, $\theta$, and $\epsilon$ at a step $L$ are what we need in order to calculate new parameters for the next step $L + 1$.

In the localized region, the equation we obtained for the density distribution function $p(\theta, \epsilon)$ is

$$p(\theta, \epsilon) = \frac{1}{\sin^2 \theta} \int d\eta d\epsilon' d\theta' p_{\eta}(\eta)p(\theta', \epsilon') \delta(\epsilon - \sqrt{1 - q^2} \eta - q \epsilon') \delta(\frac{1}{\tan \theta} + \tan \theta' - E + \epsilon).$$ \hspace{1cm} (11)

After we numerically solve this equation, the Lyapunov exponent $\gamma$ can be calculated through the following formula,

$$\gamma = \frac{1}{2} \int d\eta d\epsilon d\theta p_{\eta}(\eta)p(\theta, \epsilon) \ln[1 - (E - \sqrt{1 - q^2} \eta - q \epsilon) \sin 2\theta + (E - \sqrt{1 - q^2} \eta - q \epsilon)^2 \cos^2 \theta].$$ \hspace{1cm} (12)

By defining the distribution function $p(\theta)$, which is similar to the one in the uncorrelated diagonal disorder case,

$$p(\theta) = \int p(\theta, \epsilon) d\epsilon,$$ \hspace{1cm} (13)

we obtain the same simple relationship between $p(\theta)$ and the Lyapunov exponent,

$$\gamma = -\int p(\theta) \ln |\tan \theta| d\theta.$$ \hspace{1cm} (14)

If we take the limit of $l_{\text{cor}} \to 0$ in the present correlated disorder situation, the equations of $p(\theta)$ in the uncorrelated disorder case will be recovered \cite{9, 10}. However, $p(\theta, \epsilon)$ is not exactly the product of $p(\theta)p_{\eta}(\epsilon)$ in this limit.
FIG. 1: Distribution $p(\theta, \epsilon)$ for $E = l_{\text{cor}} = \sigma = 1$. The forty lines in $\theta$ direction are evenly spaced in the region $[-\pi/2, \pi/2]$. The forty lines in $\epsilon$ direction are scaled to display a better global view.

We use the Gaussian distribution $p_\eta$ to solve Eq. (11) and to calculate $\gamma$ numerically. This method is very efficient to yield high precision results for various disorder correlation length $l_{\text{cor}}$, disorder strength $\sigma^2$, and energy $E$ in the thermodynamic limit. Fig. 1 and Fig. 2 are shown the calculated distribution functions $p(\theta, \epsilon)$ and $p(\theta)$, respectively. In these calculations we have set a relative precision $10^{-10}$ for $p(\theta, \epsilon)$. Similar distributions were calculated recently [23] in the dichotomous correlated disorder case.

The structure of the joint density distribution of $\theta$ and $\epsilon$ is demonstrated by the $p(\theta, \epsilon)$ of $E = l_{\text{cor}} = \sigma = 1$ in Fig. 1. The distribution is not so complicated to perceive, but it can not be decomposed into a direct product of a density distribution for $\theta$ and a density distribution for $\epsilon$. Two curves for $p(\theta)$ of $E = 1$ are given in Fig. 2. One of the curve with $l_{\text{cor}} = \sigma = 0.01$ has very small disorder strength $\sigma$ and very small disorder correlation $l_{\text{cor}}$. This curve can be approximated very well by the expression for distribution $p(\theta)$ of uncorrelated disorder at a finite $E$ in the weak disorder limit [10, 18, 20],

$$p(\theta) = \frac{\sin \mu}{\pi(1 - \cos \mu \sin 2\theta)},$$

where $\cos \mu = E/2$. Another curve with $l_{\text{cor}} = \sigma = 1$ is not in the case for small disorder strength or small disorder correlation, which is different from the curve in small disorder strength or small disorder correlation.

The Lyapunov exponent $\gamma(l_{\text{cor}}, \sigma)$ is then calculated by using the two curves shown in
FIG. 2: Distributions $p(\theta)$ at $E = 1$. The full line is for $l_{\text{cor}} = \sigma = 1$; and dotted line for $l_{\text{cor}} = \sigma = 0.01$. Each point on the dotted line is differed from the analytical curve $\frac{\sqrt{3}}{\pi(2 - \sin 2\theta)}$ within no more than a relative error $10^{-4}$.

We see the numerically calculated Lyapunov exponent for the finite energy and in the weak disorder strength limit is well predicted by formula Eq. (4).

The situation for zero energy is different compared to that for the finite energy. There is no analytical result obtained so far for the zero energy anomaly in the presence of correlated disorder in the weak disorder limit; nor the formula predicting the Lyapunov exponent for a finite correlation length. Our method is a good choice to perform calculation in these situations.

III. ANOMALY AT $E = 0$

We will investigate how the localization length changes as the correlation of the disorder varies at $E = 0$ in the weak disorder limit. At finite energy, the disorder strength $\sigma$ and the correlation $l_{\text{cor}}$ are decoupled in function $\gamma(l_{\text{cor}}, \sigma)$ in Eq. (4). Since Eq. (4) was derived without any limitation on the magnitude of the correlation length, with the help of the
Lyapunov exponent $\gamma(0, \sigma)$ for uncorrelated disorder, the ratio $\gamma(l_{\text{cor}}, \sigma)/\gamma(0, \sigma) = \tanh \frac{1}{2l_{\text{cor}}}$ might be exactly held for any $l_{\text{cor}}$. At $E = 0$ anomaly, even if we keep only $\sigma^2$ term in the weak disorder limit, it is not known whether higher order terms from correlation exists beyond the perturbation result. To answer this question, we compare the numerically calculated result with the perturbation one given by Eq. (4) at $E = 0$: 

$$
\gamma_p = \frac{\sigma^2}{8} \tanh \frac{1}{2l_{\text{cor}}}.
$$

The small quantity related to correlation in $\gamma_p$ can be considered in two limit cases. In the short correlation length limit $l_{\text{cor}} \to 0$, i.e. $\tanh \frac{1}{2l_{\text{cor}}} \to 1$, the small quantity for expansion is $1 - \tanh \frac{1}{2l_{\text{cor}}} \sim 2e^{-1/l_{\text{cor}}}$. whereas in the large correlation length limit $l_{\text{cor}} \to \infty$, the small quantity for expansion is $\tanh \frac{1}{2l_{\text{cor}}}$ itself, $\tanh \frac{1}{2l_{\text{cor}}} \sim \frac{1}{2l_{\text{cor}}}$). Therefore, we will calculate for a group of different correlations with $\tanh \frac{1}{2l_{\text{cor}}}$ close to zero as well as to one. In order to neglect the contribution from the higher order terms of $\sigma$ in our calculation in the weak disorder limit, we will calculate only for small $\sigma$. It is sufficient to keep three significant digits for the Lyapunov exponent $\gamma(l_{\text{cor}}, \sigma)$.

To demonstrate the anomalous behavior at $E = 0$, we plot in Fig. 3 the distribution $p(\theta, \epsilon)$ for $E = 0$, $\tanh \frac{1}{2l_{\text{cor}}} = 0.475$, and $\sigma = 0.1$; and in Fig. 4 the distributions $p(\theta)$ for...
FIG. 4: Distributions $p(\theta)$ for $E = 0$ and $\sigma = 0.1$. The forty lines in $\theta$ direction are evenly spaced in between $[-\pi/2, \pi/2]$. The twenty lines in $x$ direction are for $x = 1 - \tanh \frac{1}{2l_{\text{cor}}} = 0.025, 0.075, 0.125, \ldots, 0.925, 0.975$, respectively.

$E = 0$ and $\sigma = 0.1$ with $x = 1 - \tanh \frac{1}{2l_{\text{cor}}} = 0.025, 0.075, 0.125, \ldots, 0.925, \text{ and } 0.975$, respectively. It shows clearly in Fig. 3 that the joint distribution for $\theta$ and $\epsilon$ has some inner structure. We have observed the flattening of the distribution $p(\theta, \epsilon)$ when increasing the correlation length $l_{\text{cor}}$ in the weak disorder limit. The flattening will not be presented in Fig. 3.

The flattening of $p(\theta)$ can be seen in Fig. 4. In the figure, when correlation length is small, the distribution $p(\theta)$ turns out to be similar to the distribution for the uncorrelated disorder [10, 20, 24]:

$$p(\theta) = \frac{1}{K(1/2)\sqrt{3 + \cos 4\theta}}, \quad (17)$$

where $K$ is the complete elliptic integral of the first kind. As the correlation increases, we see that the distribution $p(\theta)$ flattened towards $1/\pi$. Let’s take $x = 1 - \tanh \frac{1}{2l_{\text{cor}}}$ as a new parameter of the correlation in disorder. In the limit of $x \to 0$, which corresponds to the uncorrelated limit $l_{\text{cor}} \to 0$, both the anomalous distribution $1/\sqrt{3 + \cos 4\theta}$ and the anomalous Lyapunov exponent $\gamma = \sigma^2/8.754$ of the uncorrelated disorder will be recovered. In the limit of $x \to 1$, which is equivalent to the large correlation limit $l_{\text{cor}} \to \infty$, $p(\theta) = 1/\pi$ will correctly give a zero Lyapunov exponent.
FIG. 5: The Lyapunov exponent $\gamma$ for $E = 0$ and $\sigma = 0.1$. The variable $x$ used for different correlations is $x = 1 - \tanh \frac{1}{2l_{cor}}$. The function is $y = \gamma / (\sigma^2 \tanh \frac{1}{2l_{cor}})$. When $x$ is close to zero, $y$ is close to $1/8.754$; and when $x$ is close to one, $y$ is close to $1/8$.

IV. HIGH ORDER TERMS OF CORRELATION AT $E = 0$

Now we analyze the contribution from higher order terms of the correlation in the weak disorder limit. In Ref. [25] the authors gave analyses, which cover not only the localization length, but also all the higher moments of the distribution of the Lyapunov exponent for uncorrelated finite systems. For correlated systems we expect the deviation from Eq. (16) comes from higher order terms of correlation too.

In the perturbation result $\gamma_p$ in Eq. (16), by using variable $x = 1 - \tanh \frac{1}{2l_{cor}}$ to denote the correlation, we see that $\gamma_p$ included the first order correction of small $x$ when $x \to 0$ and also the first order correction of small $1 - x$ when $x \to 1$. $\gamma_p$ has included only the first order term. From the discussion on $E = 0$ anomaly in the previous section we know that $\gamma / (\sigma^2 \tanh \frac{1}{2l_{cor}}) = 1/8.754$ for $x \to 0$, while $\gamma / (\sigma^2 \tanh \frac{1}{2l_{cor}}) = 1/8$ is predicted by perturbation result for $E \to 0$. The question on how $\gamma$ really behaves at $E = 0$ is still not answered: whether $\gamma / (\sigma^2 \tanh \frac{1}{2l_{cor}}) = 1/8.754$ always holds, or there is a crossing to $\gamma / (\sigma^2 \tanh \frac{1}{2l_{cor}}) = 1/8$ as $l_{cor}$ increases. We plot Fig. 5 to answer this question.

In Fig. 5 the Lyapunov exponent $\gamma$ for $E = 0$ and $\sigma = 0.1$ are presented. We plot for dif-
ferent correlations by using the parameter \( x = 1 - \tanh \frac{1}{2l_{\text{cor}}} \), and we plot \( y = \gamma/(\sigma^2 \tanh \frac{1}{2l_{\text{cor}}}) \) as the function of \( x \). When \( x \) is close to zero, \( y \) is close to \( 1/8.754 \); and when \( x \) is close to one, \( y \) is close to \( 1/8 \). We see a crossover between the anomalous value \( 1/8.754 \) and the perturbation result \( 1/8 \). In the weak disorder limit, besides the term \( \tanh \frac{1}{2l_{\text{cor}}} \), there are higher order terms in \( x \) or \( 1 - x \) from the correlation. The higher order terms connect smoothly the anomalous \( 1/8.754 \) at zero correlation with the perturbation result \( 1/8 \) for large correlation length.

The physical picture is rich behind a finite magnitude of \( \sigma \) and a large correlation length. The \( \sigma \) in Fig. 5 is not a small enough disorder strength. The higher order terms in \( \sigma^2 \) contributes when \( x \) approaches one in Fig. 5. We have calculated for much smaller \( \sigma \) and confirmed that the contribution of higher order terms in \( \sigma^2 \) goes to zero in the weak disorder limit. Our observation suggests further perturbation investigations.

To numerically provide the next leading term of the correlation closed to the uncorrelated limit, we fit \( \gamma \) for \( x \) close to zero in Fig. 6. In Fig. 6 the Lyapunov exponent \( \gamma \) for \( E = 0 \) and \( \sigma = 0.01 \) is plotted. \( y \) represents the difference between the Lyapunov exponent for a finite correlation and for zero correlation: \( y = \gamma(l_{\text{cor}}, \sigma)/(\sigma^2 \tanh \frac{1}{2l_{\text{cor}}}) - 1/8.754 \). The variable \( x \) used for different correlations is \( x = 1 - \tanh \frac{1}{2l_{\text{cor}}} \). We obtain a fitting line \( y = 0.01533x \). Therefore the perturbation expansion of \( \gamma \) to the sub-leading order of the correlation in power of \( x \) is obtained,

\[
\gamma = (1 + 0.1342x) \frac{\sigma^2}{8.754} \tanh \frac{1}{2l_{\text{cor}}}. \tag{18}
\]

In Fig. 6 it is clear that higher order terms contributes when \( x \) is even bigger. In the weak disorder limit, when \( 1 - x \) close zero, the next order term in the correction factor used to multiply to \( \gamma_p \) in Eq. (16) is \( (1 - x)^2 \).

V. CONCLUSION

In summary, we calculated the inverse localization length in one-dimensional Anderson model with correlated diagonal disorder. We obtained numerically the curve of the inverse localization length for correlations at zero energy in the case of weak disorder. A nonsingular curve was obtained for different correlation lengths in the weak disorder limit at zero energy.

The variable used to plot the unifying curve is \( \tanh \frac{1}{2l_{\text{cor}}} \), which has correspondence to the Poisson process of the phase accumulation. The inverse localization length will be singular
FIG. 6: Fitting of the Lyapunov exponent $\gamma$ for $E = 0$ and $\sigma = 0.01$. $y$ is certain the difference between the Lyapunov exponent for finite correlation and zero correlation: $y = \gamma(l_{cor}, \sigma)/\sigma^2 / \tanh \frac{1}{2l_{cor}} - 1/8.754$. The variable $x$ used for different correlations is $x = 1 - \tanh \frac{1}{2l_{cor}}$. The fitting line is $y = 0.01533x$.

as the function of other variables as $l_{cor}$, $1/l_{cor}$, or $e^{-1/l_{cor}}$. We suggest further studies on the inverse localization length in perturbation expansions or functional expansions with the parameter $\tanh \frac{1}{2l_{cor}}$. We have obtained numerically in this work the next leading term for comparison.

We also saw rich behavior for finite disorder strength and large correlation length. A unifying description of the band center anomaly and the correlated disorder will be very interesting.

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