BOUNDARY AMENABILITY OF RELATIVELY HYPERBOLIC GROUPS

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Abstract. Let $K$ be a fine hyperbolic graph and $\Gamma$ be a group acting on $K$ with finite quotient. We prove that $\Gamma$ is exact provided that all vertex stabilizers are exact. In particular, a relatively hyperbolic group is exact if all its peripheral groups are exact. We prove this by showing that the group $\Gamma$ acts amenably on a compact topological space. We include some applications to the theories of group von Neumann algebras and of measurable orbit equivalence relations.

1. Introduction

A discrete group is said to be exact if it acts amenably on some compact topological space [GH][Oz]. Higson and Roe [HR] proved that this property is equivalent to the property $A$ of Yu [Yu] and is a coarse equivalence invariant. They also proved that a group with finite asymptotic dimension is exact. It was proved by Adams [Ad] that the boundary action of a hyperbolic group on its ideal boundary is amenable and hence hyperbolic groups are exact. (See AR[Ka] for simpler proofs.) In this paper, we study the case of relatively hyperbolic groups. The notion of relative hyperbolicity was proposed by Gromov [Gr1] and has been developed in [Bo][DS][Fa][Os1], just to name a few. In particular, Bowditch [Bo] introduced the notion of a fine hyperbolic graph $K$ and studied its compactification $\Delta K$. Bowditch characterized a relatively hyperbolic group as a group which admits a suitable action on a fine hyperbolic graph $K$. We will prove the following generalization of Adams’s and Kaimanovich’s theorems [Ad][Ka].

Theorem 1. Let $\Gamma$ be a group acting on a countable fine hyperbolic graph $K$ with finite quotient. Let $Y$ be a compact space on which $\Gamma$ acts. We assume that, for every vertex $x$ in $K$, the restricted action of the vertex stabilizer $\Gamma^x$ on $Y$ is amenable. Then, the diagonal action of $\Gamma$ on $\Delta K \times Y$ is amenable.

We include the Hausdorff property in the definition of compactness and assume that group actions on topological spaces are continuous. We note that if $\Lambda$ is an
exact subgroup of $\Gamma$, then the left multiplication action of $\Lambda$ on the Stone-Čech compactification $\beta\Gamma$ of $\Gamma$ is amenable. Since a tree is a fine hyperbolic graph, we obtain Dykema’s theorem \cite{Dy} as a corollary. (See also \cite{Tu}.)

**Corollary 2.** Amalgamated free products of exact groups are exact.

Bowditch’s characterization of relatively hyperbolic groups implies the following.

**Corollary 3.** Let $\Gamma$ be a finitely generated group which is hyperbolic relative to a family of subgroups $\mathcal{G}$. Then, $\Gamma$ is exact provided that each element of $\mathcal{G}$ is exact.

Alternative proofs of this corollary are obtained by Dadarlat and Guentner \cite{DG} and Osin \cite{Os2}. This result should be compared with Osin’s result \cite{Os2} on finiteness of asymptotic dimension. Osin kindly pointed out that the results need not be true for weakly relatively hyperbolic groups; the boundedly generated universal finitely presented group constructed in \cite{Os2} is not exact since there exists a finitely presented group that is not exact \cite{Gr2}.

Amenable action has several applications. Besides those follow from exactness (e.g., the Novikov conjecture \cite{Yu, Hi}), there are applications to the classification of orbit equivalence relations (this was the motivation of Adams \cite{Ad}) as well as the classification of group von Neumann algebras. For instance, we show that a group which is hyperbolic relative to a family of amenable groups satisfies the property (AO) defined in \cite{Os2} (and is in the class $\mathcal{C}$ defined in \cite{Oz3}). Hence, the relevant results in \cite{Os2, OP, Oz3} are applicable to such $G$. Examples of such groups include the fundamental groups of complete non-compact finite-volume Riemannian manifolds with pinched negative sectional curvature (which are hyperbolic relative to nilpotent cusp subgroups) \cite{Fa} and limit groups (which are hyperbolic relative to maximal non-cyclic abelian subgroups) \cite{Da2, Ab}.

## 2. Fine Hyperbolic Graphs

In this section, we collect several facts on a fine hyperbolic graph $K$ and its compactification $\Delta K$. Most of them are taken from \cite{Bo}.

Let $K$ be a graph with vertex set $V(K)$ and edge set $E(K)$. (We allow no loops nor multiple edges). A path of length $n$ connecting $x, y \in V$ is a sequence $x_0x_1 \cdots x_n$ of vertices, with $x_0 = x$ and $x_n = y$, and with each $x_i$ equal to or adjacent to $x_{i+1}$. A circuit is a closed path ($x_0 = x_n$) such that all $x_0, \ldots, x_{n-1}$ are distinct. For a finite path $\alpha = x_0x_1 \cdots x_n$, we set $\alpha_- = x_0$, $\alpha_+ = x_n$ and $\alpha(k) = x_k$.

We put a path metric $d$ on $V(K)$, where $d(x, y)$ is the length of shortest path in $K$ connecting $x$ to $y$. We assume the graph $K$ is connected so that $d(x, y) < \infty$ for every pair of vertices. A finite or infinite path $\alpha$ is geodesic if $d(\alpha(m), \alpha(n)) = |m-n|$ for all $m, n$. For any three point $x, y, z \in V(K)$, we define the Gromov product by

$$\langle x, y \rangle_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y)).$$
Definition 4. We say the graph $K$ is hyperbolic if there exists $\delta > 0$ such that every geodesic triangle is $\delta$-thin; for any geodesic paths $\alpha$ and $\beta$ with $\alpha_- = \beta_- = z$ and any $k \leq \langle \alpha_+, \beta_+ \rangle_z$, we have $d(\alpha(k), \beta(k)) < \delta$.

We say the graph $K$ is fine if for any $n$ and any edge $e \in E(K)$, the set $C(e, n)$ of circuits of length at most $n$ containing the edge $e$ is finite.

From now on, we assume that $K$ is a countable hyperbolic graph which is uniformly fine, i.e., for any $n$, we have $\sup_{e \in E(K)} |C(e, n)| < \infty$. We note that any fine graph which admits a group action with finite quotient is uniformly fine. Two infinite geodesic paths $\alpha$ and $\beta$ are equivalent if their Hausdorff distance is finite. The Gromov boundary $\partial K$ of $K$ is defined as the set of all equivalence classes of infinite geodesic paths in $K$. We write $\Delta K = V(K) \cup \partial K$. For an infinite geodesic path $\alpha = x_0x_1 \cdots$, we denote by $\alpha_+$ the boundary point that is represented by $\alpha$. For a biinfinite geodesic path $\alpha = \cdots x_{-1}x_0x_1 \cdots$, we likewise denote by $\alpha_-$ the boundary point that is represented by the geodesic path $x_0x_{-1}x_{-2} \cdots$. In any case, we say $\alpha$ connects $\alpha_-$ to $\alpha_+$. For every $x, y \in \Delta K$, we denote by $\mathcal{F}(x, y)$ the set of all geodesic paths which connects $x$ to $y$.

Lemma 5. For any $x, y \in \Delta K$, we have $\mathcal{F}(x, y) \neq \emptyset$.

For every $x \in \Delta K$ and a finite subset $A \subset V(K)$, we define

$$M(x, A) = \{z \in \Delta K : \text{any } \alpha \in \mathcal{F}(x, z) \text{ does not intersects with } A \setminus \{x\}\}$$

The following is Bowditch’s theorem (Section 8 in [Bo]).

Theorem 6. The family $\{M(x, A)\}_A$ defines a neighborhood base for $x \in \Delta K$. With this topology, $\Delta K$ is a compact topological space, in which $V(K)$ is dense. Every graph automorphism on $K$ uniquely extends to a homeomorphism on $\Delta K$.

We note that $\partial K$ is a Borel subset in $\Delta K$ since it is $G_\delta$. We claim that every $M(x, A)$ is open. For simplicity, we just prove that for every $x, y \in V(K)$, the set

$$T(x, y) = M(x, \{y\})^c = \{z \in \Delta K : \exists \alpha \in \mathcal{F}(x, z) \text{ such that } y \in \alpha\}$$

is closed. Let $(z_n)_n$ be a sequence in $T(x, y)$ which converges to $z$ in $\Delta K$. Let $\alpha_n$ be a geodesic path connecting $x$ to $z_n$ which passes on $y$. Let $l$ be the largest integer (possibly $\infty$) such that there exists $w \in V(K)$ with $\{n : \alpha_n(l) = w\}$ is infinite. If $l$ is finite, then $w$ is a limit point of the sequence $(z_n)_n$. (To see this, use the equivalent topology defined by $\{M'(w, B)\}_B$ in Section 8 in [Bo].) This implies that $z = w \in T(x, y)$. Now, suppose that $l$ is infinite. Since the set of geodesic paths connecting any two points is locally finite, we may pass to a subsequence and assume that $\alpha_n(1) = \cdots = \alpha_n(n)$ for every $n$. It follows that the geodesic path $\alpha_0$ given by $\alpha(n) = \alpha_0(n)$, connects $x$ to $z$ and hence $z \in T(x, y)$.

For every $x \in V(K)$, $z \in \partial K$ and $l, k \in \mathbb{N}$, we set

$$S(x, z, l, k) = \{\alpha(l) \in V(K) : \alpha \in \mathcal{F}(x', z) \text{ with } d(x', x) \leq k\}.$$
We observe that $S(x, z, l, k)$ is finite. For a finite subset $S \subset V(K)$, we write $\xi_S = |S|^{-1} \chi_S$ for the normalized characteristic function on $S$. We note that for every $x, y \in V(K)$ and $l, k \in \mathbb{N}$, the set
\[
\{ z \in \partial K : y \in S(x, z, l, k) \} = \partial K \cap \bigcup_{x'} \{ T(x', y) : d(x', x) \leq k, \ d(x', y) = l \}
\]
is Borel in $\Delta K$. It follows that, for every $x, y \in V(K)$ and $l, k \in \mathbb{N}$, the functions
\[
\partial K \ni z \mapsto |S(x, z, l, k)| \in \mathbb{N} \quad \text{and} \quad \partial K \ni z \mapsto \xi_{S(x, z, l, k)}(y) \in \mathbb{R}
\]
are Borel. The following lemma is well-known for uniformly locally finite hyperbolic graph. It also follows from Lemma 16 of [Os2] in special cases.

**Lemma 7.** There exists a constant $C = C(K) > 0$ such that $|S(x, z, l, k)| < Ck$ for all $x \in V(K)$, $z \in \partial K$ and $l, k \in \mathbb{N}$ with $l > k + \delta$.

**Proof.** Let $\alpha$ be any fixed geodesic path connecting $x$ to $z$. It suffices to show
\[
S(x, z, l, k) \subset \bigcup \{ C(e, 6\delta) : e \text{ an edge in } \alpha([l - k - \delta, l + k + \delta]) \},
\]
where $C(e, n)$ is the set of circuits of length at most $n$ containing the edge $e$. Choose $y \in S(x, z, l, k)$. There exists a geodesic path $\beta$ connecting $x'$ to $z$ such that $d(x', x) \leq k$ and $y = \beta(l)$. Since any geodesic triangle is $\delta$-thin, there exists a number $m \in [l - k, l + k]$ such that $d(\alpha(m \pm \delta), \beta(l \pm \delta)) < \delta$. Let $\gamma^\pm$ be any geodesic paths connecting $\alpha(m \pm \delta)$ to $\beta(l \pm \delta)$. Since $d(\gamma^+ \cup \gamma^-, \gamma^- \cup \gamma^+ \cup \alpha([m - \delta, m + \delta])$ of length at most $6\delta$ contains a circuit which includes $y$. Thus, the closed path $\gamma^- \cup \beta([l - \delta, l + \delta]) \cup \gamma^+ \cup \alpha([m - \delta, m + \delta])$ contains both $y$ and an edge in $\alpha([m - \delta, m + \delta])$.

For every $x \in V(K)$, $z \in \partial K$ and $n$, we set
\[
\zeta^n_{x,z} = \frac{1}{n} \sum_{k=n+1}^{2n} \xi_{S(x, z, 4n, k)} \in \ell_1(V(K)).
\]
We note that the map $\partial K \ni z \mapsto \zeta^n_{x,z}(y) \in \mathbb{R}$ is Borel for every $x, y \in V(K)$. The following lemma is borrowed from [Ka], but we put a proof for completeness.

**Lemma 8.** For any $d > 0$, we have
\[
\lim_{n \to \infty} \sup_{z \in \partial K, d(x, x') \leq d} \sup_{y \in \mathbb{R}} \| \zeta^n_{x,z} - \zeta^n_{x',z} \|_1 = 0.
\]

**Proof.** Let $z \in \partial K$ and $x, x' \in V(K)$ be given. Let $d = d(x, x')$ and $n \geq d + \delta$. For simplicity, we write $S_k = S(x, z, 4n, k)$ and $S'_k = S(x', z, 4n, k)$. Then, we have $S_k \cup S'_k \subset S_{k+d}$ and $S_k \cap S'_k \supset S_{k-d}$ for every $n < k \leq 2n$. It follows that
\[
\| \xi_{S_k} - \xi_{S'_k} \|_1 = 2 \left( 1 - \frac{|S_k \cap S'_k|}{\max\{|S_k|, |S'_k|\}} \right) \leq 2 \left( 1 - \frac{|S_{k-d}|}{|S_{k+d}|} \right)
\]
for $n < k \leq 2n$. Since $|S_k| \leq Ck$ by Lemma 7 we have
\[
\frac{1}{n} \sum_{k=n+1}^{2n} \frac{|S_k-d|}{|S_k+d|} \geq \left( \frac{2n}{\prod_{k=n+1}^{2n} |S_k+d|} \right)^{1/n} = \left( \frac{\prod_{k=n+1}^{n+d} |S_k|}{\prod_{k=2n+1}^{n+d} |S_k|} \right)^{1/n} \geq (3Cn)^{-2d/n}.
\]
It follows that
\[
\|\zeta_{x,z}^n - \zeta_{x',z}^n\|_1 \leq \frac{1}{n} \sum_{k=n+1}^{2n} \|\xi_{S_k} - \xi_{S_k'}\|_1 \leq 2(1 - (3Cn)^{-2d/n}).
\]
Since $(3Cn)^{-2d/n} \to 1$ as $n \to \infty$, this proves the lemma.

Lemma 9. Let $(x_n)_n$ and $(y_n)_n$ be sequences in $V(K)$ which converge to respectively $x$ and $y$ in $\Delta K$. Suppose that $\sup_n d(x_n, y_n) < \infty$. Then, we have either that $x = y$, or that for every sufficiently large $n$, all geodesic paths connecting $x_n$ to $y_n$ intersect with both $x$ and $y$.

Proof. We observe first that if $x \in \partial K$, then $y \in \partial K$ and $x = y$. Thus, we assume $x, y \in V(K)$. Let $C = \sup_n d(x_n, y_n)$. We claim that if $\limsup_n d(x_n, x) = \infty$, then $x = y$. Indeed, passing to a subsequence, we may assume that $d(x_n, x) > 10C$ for all $n$. For any finite subset $A \subset V(K) \setminus \{x\}$, there exists $\alpha_n \in F(x, x_n)$ which does not intersect with $A$. Composing $\alpha_n$ with a geodesic path connecting $x_n$ to $y_n$ and then discarding redundant vertices, we obtain a $10C$-quasigeodesic path connecting $x$ to $y_n$ which does not intersects with $A$. Hence, the sequence $(y_n)_n$ converges to $x$.

(To see this, use the equivalent topology defined by $\{M'_{\partial\Omega}(x, A)\}_A$ in Section 8 in [Bo]). This proves the claim. Finally, we suppose that $C' = \sup d(x_n, x) < \infty$. Let $\alpha_n \in F(x, x_n)$, $\beta_n \in F(y, y_n)$ and $\gamma \in FF(x, y)$. Since $x_n \to x$, we have $\alpha_n \cap \gamma = \{x\}$ for sufficiently large $n$. Thus any geodesic path connecting $x_n$ to $y_n$, which does not intersects $x$, gives rise to a circuit of length at most $2(C + C' + d(x, y))$ containing $x$ and edges in $\alpha_n$ and $\gamma$. Since $x_n \to x$, this is only possible for finitely many $n$. The same thing for $y$ and we are done.

3. Amenable Actions

For every locally compact space $\Omega$, we denote by $\text{Prob}(\Omega)$ the space of all regular Borel probability measure on $\Omega$, equipped with the weak*-topology. In particular, if $\Gamma$ is a discrete group, then
\[
\text{Prob}(\Gamma) = \{\mu \in \ell_1(\Gamma) : \mu \geq 0 \text{ and } \sum_{s \in \Gamma} \mu(s) = 1\}
\]
and $\text{Prob}(\Gamma)$ is equipped with the pointwise convergence topology. We note that the pointwise convergence topology coincides with the norm topology on $\text{Prob}(\Gamma)$. The group $\Gamma$ acts on $\text{Prob}(\Gamma)$ from the left; $(s \cdot \mu)(t) = \mu(s^{-1}t)$ for every $s, t \in \Gamma$ and $\mu \in \text{Prob}(\Gamma)$. Let $\Omega$ be a locally compact space. By an action of $\Gamma$ on $\Omega$, we mean a group homomorphism from $\Gamma$ into the group of homeomorphisms on $\Omega$.
Definition 10. Let $\Gamma$ be a discrete group acting on a compact topological space $\Omega$. We say the action of $\Gamma$ on $\Omega$ is amenable if for every finite subset $E \subset \Gamma$ and $\varepsilon > 0$, there exists a continuous map $\mu: \Omega \ni \omega \mapsto \mu_\omega \in \text{Prob}(\Gamma)$ such that

$$\max_{s \in E} \sup_{\omega \in \Omega} \| s \cdot \mu_\omega - \mu_{s \omega} \| \leq \varepsilon.$$ 

We say a group $\Gamma$ is exact if it acts amenably on some compact topological space. A group $\Gamma$ is amenable iff the trivial action on a singleton set is amenable. If $\Gamma$ acts on $K$, we denote the stabilizer subgroup of $a \in K$ by $\Gamma^a = \{ s \in \Gamma : sa = a \}$.

Proposition 11. Let $\Gamma$ be a countable group acting on $X$, $Y$ and $K$, where $X$ and $Y$ are compact and $K$ is countable discrete. Assume that for any finite subset $E \subset \Gamma$ and $\varepsilon > 0$ there exists a Borel map $\zeta: X \to \text{Prob}(K)$ (i.e., the function $X \ni x \mapsto \zeta_x(a) \in \mathbb{R}$ is Borel for every $a \in K$) such that

$$\max_{s \in E} \sup_{x \in X} \| s \zeta_x - \zeta_{sx} \| < \varepsilon.$$ 

Assume moreover that the restricted action of the vertex stabilizer $\Gamma^a$ on $Y$ is amenable for every $a \in K$. Then, the diagonal action of $\Gamma$ on $X \times Y$ is amenable.

Proof. We first claim that we may take $\zeta$ in the statement to be continuous rather than Borel. Fix a finite symmetric subset $E \subset \Gamma$. For every continuous map $\zeta: X \to \text{Prob}(K)$, we define $f_\zeta \in C(X)$ by

$$f_\zeta(x) = \sum_{s \in E} \| s \zeta_x - \zeta_{sx} \| = \sum_{s \in E} \sum_{a \in K} |\zeta_x(s^{-1}a) - \zeta_{sx}(a)|.$$ 

Since $f_\sum_k \alpha_k \zeta_k \leq \sum_k \alpha_k f_\zeta_k$ for every $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, if 0 is in the weak closure of $\{ f_\zeta : \zeta \}$ in $C(X)$, then 0 is in the norm closure of $\{ f_\zeta : \zeta \}$ by the Hahn-Banach separation theorem. We note that the dual of $C(X)$ is the space of finite regular Borel measures on $X$ by the Riesz representation theorem. Let $\varepsilon > 0$ and $m \in \text{Prob}(X)$ be given. By assumption, there exists a Borel map $\eta: X \to \text{Prob}(K)$ such that $\sup_{x \in X} \| s \eta_x - \eta_{sx} \| < \varepsilon/|E|$. By the countable additivity of the measure, there exists a finite subset $F \subset K$ such that $\int_X \sum_{a \in F} \eta_x(a) \ dm(x) > 1 - \varepsilon/|E|$. We approximate, for each $a \in F$, the Borel function $x \mapsto \eta_x(a)$ by a continuous function and obtain a continuous map $\zeta: X \to \text{Prob}(K)$ such that $\sup \zeta_x \subset F$ for all $x \in X$ and

$$\int_X \| \zeta_x - \eta_x \| \ dm(x) = \int_X \sum_{a \in K} |\zeta_x(a) - \eta_x(a)| \ dm(x) < 2\varepsilon/|E|.$$
It follows that
\[ \int_X f_\zeta(x) \, dm(x) < \int_X \sum_{x \in E} \| s\eta_x - \eta_{sx}\| \, dm(x) + 4\varepsilon < 5\varepsilon. \]

Thus, we proved our claim.

Now, let a finite subset \( E \subset \Gamma \) and \( \varepsilon > 0 \) be given. By the previous result, there exists a continuous \( \zeta \) such that \( \sup_{x \in X} \| s\zeta_x - \zeta_{sx}\| < \varepsilon \) for every \( s \in E \). We may assume that there exists a finite subset \( F \subset \Gamma \) such that \( \text{supp} \zeta_x \subset F \) for all \( x \in X \). We fix a \( \Gamma \)-fundamental domain \( V \subset K \) with a projection \( v: K \to V \) and a cross section \( \sigma: K \to \Gamma \), i.e., \( K \) decomposes into the disjoint union \( \bigcup_{v \in V} \Gamma v \) and \( a = \sigma(a)v(a) \) for every \( a \in K \). We note that \( \sigma(sa)^{-1}s\sigma(a) \in \Gamma^{\nu(a)} \) for every \( s \in \Gamma \) and \( a \in K \). For each \( v \in V \), we set
\[ E^v = \{ \sigma(sa)^{-1}s\sigma(a) : a \in F \cap \Gamma v \text{ and } s \in E \} \subset \Gamma^v. \]

Since the \( \Gamma^v \) action on \( Y \) is amenable and \( E^v \) is finite, there exists a continuous map \( \nu^v: Y \to \text{Prob}(\Gamma) \) such that
\[ \max_{s \in E^v} \sup_{y \in Y} \| s\nu^v_y - s\nu^v_y \| < \varepsilon. \]

Now, we define \( \mu: X \times Y \to \text{Prob}(\Gamma) \) by
\[ \mu_{x,y} = \sum_{a \in K} \zeta_x(a) \sigma(a)\nu^{(a)}_{\sigma(a)^{-1}y}. \]

The map \( \mu \) is clearly continuous. Moreover, we have
\[ s\mu_{x,y} = \sum_{a \in K} \zeta_x(a) s\sigma(a)\nu^{(a)}_{\sigma(a)^{-1}y} = \sum_{a \in K} \zeta_x(a) \sigma(sa)\left( \sigma(sa)^{-1}s\sigma(a)\nu^{(a)}_{\sigma(a)^{-1}y} \right) \]
\[ \approx \varepsilon \sum_{a \in K} \zeta_x(a) \sigma(sa)\nu^{(sa)}_{\sigma(a)^{-1}sy} \]
\[ \approx \varepsilon \sum_{a \in K} \zeta_{sx}(sa) \sigma(sa)\nu^{(sa)}_{\sigma(a)^{-1}sy} = \mu_{sx,sy} \]
for every \( s \in E \) and \((x,y) \in X \times Y\). \( \square \)

We prove Theorem \( \square \)

**Proof.** We may assume that \( \Gamma \) is countable and \( K \) is uniformly fine. Let a finite subset \( E \subset \Gamma \) and \( \varepsilon > 0 \) be given. Fix an origin \( o \in V(K) \). By Lemma \( \square \) there exists \( n \) such that \( \zeta: \partial K \ni z \mapsto \zeta_z = \zeta^n_{0,z} \in \text{Prob}(K) \) satisfies
\[ \max_{s \in E} \sup_{z \in \partial K} \| s\zeta_z - \zeta_{sz}\| = \max_{s \in E} \sup_{z \in \partial K} \| \zeta^n_{so,sz} - \zeta^n_{0,so,\partial K} \| < \varepsilon. \]

For \( x \in V(K) \), we set \( \zeta_x = \delta_x \in \text{Prob}(K) \). Then, the map \( \zeta: \Delta K \to \text{Prob}(K) \) is Borel (see the remark preceding Lemma \( \square \)) and satisfies the assumptions given in Proposition \( \square \) Thus the action on \( \Delta K \times Y \) is amenable. \( \square \)
4. Applications

Recall that a compactification of a discrete group $\Gamma$ is a compact topological space $\bar{\Gamma}$ which contains $\Gamma$ as a discrete open dense subset. We assume that the left translation action of $\Gamma$ on $\Gamma$ extends to a continuous action of $\Gamma$ on $\bar{\Gamma}$. We have well-known one-to-one correspondence between such compactification $\bar{\Gamma}$ and a $C^*$-subalgebra $c_0(\Gamma) \subset C(\bar{\Gamma}) \subset \ell_\infty(\Gamma)$ which is invariant under left translation. We say the compactification $\bar{\Gamma}$ is small at infinity if the right translation action of $\Gamma$ on $\Gamma$ extends to a continuous action on $\bar{\Gamma}$ and if its restriction to $\partial \Gamma = \bar{\Gamma} \setminus \Gamma$ is trivial. In other words, $f^t - f \in c_0(\Gamma)$ for all $f \in C(\bar{\Gamma}) \subset \ell_\infty(\Gamma)$ and $t \in \Gamma$, where $f^t(s) = f(st^{-1})$ for $s \in \Gamma$.

**Proposition 12.** Let $\Gamma$ be a group which is hyperbolic relative to a family $\mathcal{G}$ of amenable subgroups. Then, $\Gamma$ acts amenably on some compactification $\bar{\Gamma}$ which is small at infinity.

**Proof.** By [Bo], the group $\Gamma$ admits a finite quotient action on a fine hyperbolic graph $K$ in a way that every infinite vertex stabilizer is in $\mathcal{G}$ and that $\Gamma^x \cap \Gamma^y$ is finite for any $x, y \in V(K)$ with $x \neq y$. By Theorem [1], the action of $\Gamma$ on $\Delta K$ is amenable. We fix an origin $o$ and consider the map $\Gamma \ni s \mapsto so \in \Delta K$. This map induces a compactification $\bar{\Gamma}$ such that $\bar{\Gamma} \to \Delta K$ is continuous. We note that $\Gamma$ acts on $\bar{\Gamma}$ amenably. To prove $\bar{\Gamma}$ is small at infinity, it suffices to show that for any $t \in \Gamma$ and $f \in C(\Delta K)$, we have $f_o - f^t_o \in c_0(\Gamma)$, where $f_o \in \ell_\infty(\Gamma)$ is defined by $f_o(s) = f(so)$ for $s \in \Gamma$. We fix $\varepsilon > 0$ and show that $A = \{s \in \Gamma : |f(so) - f(st^{-1}o)| \geq \varepsilon\}$ is finite. Let $(s_n)_n \subset A$ be any sequence such that all $s_n$ are distinct. Since $\Delta K$ is compact (and first countable), we may assume that $s_n o \to x$ and $s_n t^{-1}o \to y$. We note that $x \neq y$ since $|f(x) - f(y)| \geq \varepsilon$. Let $\alpha$ be a geodesic path connecting $o$ to $t^{-1}o$. Then, by Lemma [9] for all sufficiently large $n$, the geodesic paths $s_n \alpha$ intersect with both $x$ and $y$. Since $\Gamma^x \cap \Gamma^y$ is finite, $A$ is finite. 

It follows from (the proof of) Lemma 5.2 in [HG] that the group $\Gamma$ satisfying the assumptions of Proposition 12 is in the class $\mathcal{C}$ defined in [Oz3] (i.e., the left and right translation action of $\Gamma \times \Gamma$ on the Stone-Cech remainder $\partial^3 \Gamma$ is amenable). Hence, the corresponding results in [Oz2], [OP], [Oz3] are applicable to such $\Gamma$.

**References**

[Ab] E. Alibegović, *A Combination Theorem for Relatively Hyperbolic Groups*. Preprint 2003.

[Ad] S. Adams, *Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups*. Topology 33 (1994), 765–783.

[AD] C. Anantharaman-Delaroche, *Systèmes dynamiques non commutatifs et moyennabilité*. Math. Ann. 279 (1987), 297–315.

[AR] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*. With a foreword by Georges Skandalis and Appendix B by E. Germain. Monographies de L’Enseignement Mathématique 36. Geneva, 2000.
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[Bo] B.H. Bowditch, Relatively hyperbolic groups. Preprint. 1999.

[DG] M. Dadarlat and E. Guentner, Uniform embeddability of relatively hyperbolic groups. Preprint 2005.

[Da1] F. Dahmani, Classifying spaces and boundaries for relatively hyperbolic groups. Proc. London Math. Soc. (3) 86 (2003), 666–684.

[Da2] F. Dahmani, Combination of convergence groups. Geom. Topol. 7 (2003), 933–963.

[DS] C. Drutu, M.V. Sapir, Tree graded spaces and asymptotic cones. Preprint 2004.

[Dy] K. J. Dykema, Exactness of reduced amalgamated free product C*-algebras. Forum Math. 16 (2004), 161–180.

[GH] E. Ghys and P. de la Harpe, Sur les groupes hyperboliques d’après Mikhael Gromov. Progress in Math., 83, Birkhäuser, 1990.

[Gr1] M. Gromov, Hyperbolic groups. Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987.

[Gr2] M. Gromov, Random walk in random groups. Geom. Funct. Anal. 13 (2003), 73–146.

[GK] E. Guentner and J. Kaminker, Exactness and the Novikov conjecture. Topology 41 (2002), 411–418.

[Fa] B. Farb, Relatively hyperbolic groups. Geom. Funct. Anal. 8 (1998), 810–840.

[Hi] N. Higson, Bivariant K-theory and the Novikov conjecture. Geom. Funct. Anal. 10 (2000), 563–581.

[HG] N. Higson and E. Guentner, Group C*-algebras and K-theory. Noncommutative geometry, 137–251, Lecture Notes in Math., 1831, Springer, Berlin, 2004.

[HR] N. Higson and J. Roe, Amenable group actions and the Novikov conjecture. J. Reine Angew. Math. 519 (2000), 143–153.

[Ka] V. Kaimanovich, Boundary amenability of hyperbolic spaces. Discrete geometric analysis, 83–111, Contemp. Math., 347, Amer. Math. Soc., Providence, RI, 2004.

[Os1] D.V. Osin, Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems. Memoirs of Amer. Math. Soc., to appear.

[Os2] D.V. Osin, Asymptotic dimension of relatively hyperbolic groups. Preprint 2004.

[Oz1] N. Ozawa, Amenable actions and exactness for discrete groups. C. R. Acad. Sci. Paris Sr. I Math. 330 (2000), 691–695.

[Oz2] N. Ozawa, Solid von Neumann algebras. Acta Math. 192 (2004), 111–117.

[Oz3] N. Ozawa, A Kurosh type theorem for type II1 factors. Preprint 2004.

[OP] N. Ozawa and S. Popa, Some prime factorization results for type II1 factors. Invent. Math. 156 (2004), 223–234.

[Tu] J.-L. Tu, Remarks on Yu’s “property A” for discrete metric spaces and groups. Bull. Soc. Math. France 129 (2001), 115–139.

[Yu] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. Invent. Math. 139 (2000), 201–240.

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