VISUAL PROPERTIES OF GENERALIZED KLOOSTERMAN SUMS

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ABSTRACT. For a positive integer \( m \) and a subgroup \( \Lambda \) of the unit group \((\mathbb{Z}/m\mathbb{Z})^\times\), the corresponding generalized Kloosterman sum is the function

\[
K(a, b, m, \Lambda) = \sum_{u \in \Lambda} e\left(\frac{au + bu^{-1}}{m}\right)
\]

for \( a, b \in \mathbb{Z}/m\mathbb{Z} \), in which \( e(x) = \exp(2\pi ix) \) and \( u^{-1} \) denotes the multiplicative inverse of \( u \) modulo \( m \). Classical Kloosterman sums arise when \( \Lambda = (\mathbb{Z}/m\mathbb{Z})^\times \).

Unlike classical Kloosterman sums, which are real valued, generalized Kloosterman sums display a surprising array of visual features when their values are plotted in the complex plane. In a variety of instances, we identify the precise number-theoretic conditions that give rise to particular phenomena.

1. Introduction

For a positive integer \( m \) and a subgroup \( \Lambda \) of the unit group \((\mathbb{Z}/m\mathbb{Z})^\times\), the corresponding generalized Kloosterman sum is the function

\[
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Unlike their classical counterparts, which are real valued, generalized Kloosterman sums display a surprising array of visual features when their values are plotted in the complex plane; see Figure 1. Our aim here is to initiate the investigation of these sums from a graphical perspective. In a variety of instances, we identify the precise number-theoretic conditions that give rise to particular phenomena.

Like classical Kloosterman sums, generalized Kloosterman sums enjoy a certain multiplicative property. If \( m = m_1 m_2 \), in which \( (m_1, m_2) = 1 \), \( r_1 \equiv m_1^{-1} \pmod{m_2} \), \( r_2 \equiv m_2^{-1} \pmod{m_1} \), \( \omega_1 = \omega \pmod{m_1} \), and \( \omega_2 = \omega \pmod{m_2} \), then

\[
K(a, b, m, \langle \omega \rangle) = K(r_2a, r_2b, m_1, \langle \omega_1 \rangle)K(r_1a, r_1b, m_2, \langle \omega_2 \rangle).
\]

This follows immediately from the Chinese Remainder Theorem. Consequently, we tend to focus on prime or prime power moduli; see Figure 2. Since the group of units modulo an odd prime power is cyclic, most of our attention is restricted to the case where \( \Gamma = \langle \omega \rangle \) is a cyclic group of units.

Additional motivation for our work stems from the fact that generalized Kloosterman sums are examples of supercharacters. The theory of supercharacters, introduced in 2008 by P. Diaconis and I.M. Isaacs, has emerged as a powerful tool in

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(a) \( m = 4820 = 2^2 \cdot 5 \cdot 241, \Lambda = \langle 1209 \rangle, |\Lambda| = 12 \)

(b) \( m = 9015 = 3 \cdot 5 \cdot 601, \Lambda = \langle 596 \rangle, |\Lambda| = 12 \)

(c) \( m = 4820 = 2^2 \cdot 5 \cdot 241, \Lambda = \langle 257 \rangle, |\Lambda| = 12 \)

(d) \( m = 4820 = 2^2 \cdot 5 \cdot 241, \Lambda = \langle 497 \rangle, |\Lambda| = 12 \)

(e) \( m = 3087 = 3^3 \cdot 7^3, \Lambda = \langle 1010 \rangle, |\Lambda| = 6 \)

(f) \( m = 890 = 2 \cdot 5 \cdot 89, \Lambda = \langle 479 \rangle, |\Lambda| = 4 \)

(g) \( m = 9015 = 3 \cdot 5 \cdot 601, \Lambda = \langle 2284 \rangle, |\Lambda| = 12 \)

(h) \( m = 1413 = 3^2 \cdot 157, \Lambda = \langle 13 \rangle, |\Lambda| = 6 \)

(i) \( m = 9015 = 3 \cdot 5 \cdot 601, \Lambda = \langle 577 \rangle, |\Lambda| = 12 \)

Figure 1. Plots in \( \mathbb{C} \) of the values of generalized Kloosterman sums \( K(a, b, m, \Lambda) \) for \( 0 \leq a, b < m \). These plots are symmetric across the real line since \( K(a, b, p, d) = K(-a, -b, p, d) \). The images are colored according to the value of \( a + b \mod k \) for some \( k \) which divides \( m \).

2. Hypocycloids

In what follows, we let \( \phi \) denote the Euler totient function. If \( q = p^\alpha \) is an odd prime power, then \( (\mathbb{Z}/q\mathbb{Z})^\times \) is cyclic. Thus, for each divisor \( d \) of \( \phi(q) = p^{\alpha-1}(p-1) \),

combinatorial representation theory. Certain exponential sums of interest in number theory, such as Ramanujan, Gauss, Heilbronn, and classical Kloosterman sums, arise as supercharacter values on abelian groups [1,5–8]. In the terminology of [1], the functions [1] arise by letting \( n = m, d = 2, \) and \( \Gamma = \{ \text{diag}(u, u^{-1}) : u \in \Lambda \} \).
there is a unique subgroup $\Lambda$ of $(\mathbb{Z}/q\mathbb{Z})^\times$ of order $d$. In this case, we write

$$K(a, b, q, d) = \sum_{u^d=1} e\left(\frac{au + bu^{-1}}{q}\right)$$

instead of $K(a, b, q, \Lambda)$. If $d$ is a fixed odd prime and $q = p^\alpha$ is an odd prime power with $p \equiv 1 \pmod{d}$, then the values $K(a, b, q, d)$ for $0 \leq a, b < q$ are contained in the closure $\mathbb{H}_d$ of the bounded region determined by the $d$-cusped hypocycloid given by

$$\theta \mapsto (d-1)e^{i\theta} + e^{(1-d)i\theta};$$

see Figure 3. As the prime power $q = p^\alpha$ for $p \equiv 1 \pmod{d}$ tends to infinity, the values $K(a, b, q, d)$ “fill out” $\mathbb{H}_d$; see Figure 4 and Theorem 7. Similar asymptotic behavior has been observed in Gaussian periods [5,7] and certain exponential sums related to the symmetric group [2].
To be more precise, we require a few words about uniformly distributed sets. A sequence \( S_1, S_2, \ldots \) of finite subsets of \([0, 1)^k\) is uniformly distributed if

\[
\lim_{n \to \infty} \sup_B \frac{|B \cap S_n|}{|S_n|} - \mu(B) = 0,
\]

where the supremum runs over all boxes \( B = [a_1, b_1] \times \cdots \times [a_k, b_k] \) in \([0, 1)^k\) and \( \mu \) denotes \( k \)-dimensional Lebesgue measure. If \( S_1, S_2, \ldots \) is a sequence of finite subsets of \( \mathbb{R}^k \), then it is uniformly distributed modulo 1 if the sets \( S_n \) are uniformly distributed in \([0, 1)^k\).

Lemma 4. Fix a positive integer \( d \). For each odd prime power \( q = p^\alpha \) with \( p \equiv 1 \) (mod \( d \)), let \( \omega_q \) denote any primitive \( d \)-th root of unity modulo in \( \mathbb{Z}/q\mathbb{Z} \). Let \( \omega_q^{-1} \) denote the inverse of \( \omega_q \) modulo \( q \). For each fixed \( b \in \{0, 1, \ldots, q - 1\} \), the sets

\[
S_q^b = \left\{(a + b, \frac{a\omega_q + b\omega_q^{-1}}{q}, \ldots, \frac{a\omega_q^{d-1} + b\omega_q^{-d+1}}{q}) : 0 \leq a \leq q - 1 \right\}
\]

in \( \mathbb{R}^{\phi(d)} \) are uniformly distributed modulo 1 as \( q \to \infty \).

Proof. Fix a positive integer \( d \) and \( b \in \{0, 1, \ldots, q - 1\} \). Weyl’s criterion asserts that \( S_q^b \) is uniformly distributed modulo 1 if and only if

\[
\lim_{q \to \infty} \frac{1}{|S_q^b|} \sum_{x \in S_q^b} e(x \cdot y) = 0
\]

for all nonzero \( y \in \mathbb{Z}^{\phi(d)} \) [15]. Fix a nonzero \( y \in \mathbb{Z}^{\phi(d)} \). For \( x \in S_q^b \), write \( x = x_1 + x_2 \), in which

\[
x_1 = \left(\frac{a}{q}, \frac{a\omega_q}{q}, \ldots, \frac{a\omega_q^{d-1}}{q}\right), \quad x_2 = \left(\frac{b}{q}, \frac{b\omega_q^{-1}}{q}, \ldots, \frac{b\omega_q^{-d+1}}{q}\right);
\]

Note that \( x_1 \) depends on \( a \) whereas \( x_2 \) is fixed since we regard \( b \) as constant. A result of Myerson [12] Thm. 12 (see also [5] Lem. 6.2) asserts that the sets

\[
\left\{\frac{a}{q} (1, \omega_q, \omega_q^2, \ldots, \omega_q^{d-1}) : 0 \leq a \leq q - 1 \right\} \subseteq [0, 1)^{\phi(d)}
\]

are uniformly distributed modulo 1 as \( q = p^\alpha \) tends to infinity; this requires the assumption that \( p \equiv 1 \) (mod \( d \)). Thus,

\[
\frac{1}{|S_q^b|} \sum_{x \in S_q^b} e(x \cdot y) = \frac{1}{|S_q^b|} \sum_{a=0}^{q-1} e\left((x_1 + x_2) \cdot y\right) = e(x_2 \cdot y) \frac{1}{|S_q^b|} \sum_{a=0}^{q-1} e(x_1 \cdot y)
\]

tends to zero, so Weyl’s criterion ensures that the sets \( S_q^b \) are uniformly distributed modulo 1 as \( q \to \infty \). \( \square \)

Theorem 7. Fix an odd prime \( d \).

(a) For each odd prime power \( q = p^\alpha \) with \( p \equiv 1 \) (mod \( d \)), the values of \( K(a, b, q, d) \) are contained in \( \mathbb{H}_d \), the closure of the region bounded by the \( d \)-cusped hypocycloid centered at 0 and with a cusp at \( d \).
Figure 4. For the primes $d = 3, 5, 7$, the values $K(a, b, p, d)$ with $0 \leq a, b \leq p - 1$ “fill out” the closure $\mathbb{H}_d$ of the bounded region determined by the hypocycloid \( (3) \); see Theorem 7.

(b) Fix $\epsilon > 0$, $b \in \mathbb{Z}$, and let $B_\epsilon(w)$ be an open ball of radius $\epsilon$ centered at $w \in \mathbb{H}_d$. For every sufficiently large odd prime power $q = p^\alpha$ with $p \equiv 1 \pmod{d}$, there exists $a \in \mathbb{Z}/q\mathbb{Z}$ so that $K(a, b, q, d) \in B_\epsilon(w)$.

Proof. (a) Suppose that $q = p^\alpha$ is an odd prime power and $p \equiv 1 \pmod{d}$. Let $g$ be a primitive root modulo $q$ and define $u = g^{\phi(q)/d}$, so that $u$ has multiplicative order $d$ modulo $q$. Since $p$ and $p - 1$ are relatively prime, $p - 1 \mid p^{\alpha - 1}(\frac{p - 1}{d}) = \phi(q)/d$ and hence $u \not\equiv 1 \pmod{p}$. Thus, $u - 1$ is a unit modulo $q$, from which it follows that

$$1 + u + \cdots + u^{d-1} \equiv 0 \pmod{q}. \quad (8)$$

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ and define $f : \mathbb{T}^{d-1} \rightarrow \mathbb{C}$ by

$$f(z_1, z_2, \ldots, z_{d-1}) = z_1 + z_2 + \cdots + z_{d-1} + \frac{1}{z_1 z_2 \cdots z_{d-1}}; \quad (9)$$
it is well-known that the image of this function is the filled hypocycloid defined by (3); see [3, 7, 10]. For \( k = 1, 2, \ldots, d - 1 \), let

\[
\zeta_k = e\left(\frac{a u^{k-1} + b u^{-(k-1)}}{q}\right)
\]

and use [8] to conclude that

\[
K(a, b, q, d) = \sum_{k=0}^{d-2} e\left(\frac{a u^k + b u^{-k}}{q}\right) + e\left(\frac{a u^{d-1} + b u^{-(d-1)}}{q}\right)
\]

\[
= \sum_{k=0}^{d-2} e\left(\frac{a u^k + b u^{-k}}{q}\right) + e\left(-a \sum_{k=0}^{d-2} u^k - b \sum_{k=0}^{d-2} u^{-k}\right)
\]

\[
= \sum_{k=1}^{d-1} \zeta_k + \frac{1}{\zeta_1 \zeta_2 \cdots \zeta_{d-1}}.
\]

Thus \( K(a, b, q, d) \) is contained in \( \mathbb{H}_d \).

(b) Fix \( \epsilon > 0 \), \( b \in \mathbb{Z} \), and let \( B_{\epsilon}(w) \) be an open ball of radius \( \epsilon \) centered at \( w \in \mathbb{H}_d \). Let \( f : \mathbb{T}^d \rightarrow \mathbb{C} \) denote the function defined by (9) and let \( z \in \mathbb{T}^d \) satisfy \( f(z) = w \). The compactness of \( \mathbb{T}^d \) ensures that \( f \) is uniformly continuous, so there exists \( \delta > 0 \) so that \( |f(z) - f(x)| < \epsilon \) whenever \( |x - z| < \delta \) (here we use the norm induced by the standard embedding of the torus \( \mathbb{T}^d \) into \( d \)-dimensional Euclidean space). Since \( d \) is prime, \( \phi(d) = d - 1 \) and hence Lemma 4 ensures that for each fixed \( b \), the sets \( S_q \) in \( \mathbb{R}^d \) defined by (5) are uniformly distributed mod 1. So for \( q \) sufficiently large, there exists

\[
x = \left(\frac{a + b}{q}, \frac{a \omega_q + b \omega_q^{-1}}{q}, \ldots, \frac{a \omega_q^{\phi(d)-1} + b \omega_q^{-\phi(d)+1}}{q}\right) \in S_q^b,
\]

so that \( |x - z| < \delta \) holds. Then \( K(a, b, q, d) = f(x) \) belongs to \( B_{\epsilon}(w) \).

The images in Figure 4 suggest that the values of \( K(a, b, q, d) \) for \( 0 \leq a, b \leq q - 1 \) are not uniformly distributed in \( \mathbb{H}_d \); indeed, they are not. However, they are uniformly distributed with respect to the push-forward measure \( \mu_{\mathbb{H}_d}(\cdot) = \lambda(f^{-1}(\cdot)) \), in which \( \lambda \) denotes Lebesgue measure on \( \mathbb{T}^d \) and \( f \) is defined by (9); the support of the measure \( \mu_{\mathbb{H}_d} \) is \( \mathbb{H}_d \). We briefly sketch the explanation. Let \( T_q = \bigcup_{b=0}^{q-1} S_q^b \) and observe that \( |T_q| = q^2 = |S_q|^2 \) if \( q \geq 2 \). Indeed, if \( (a + b, a \omega_q + b \omega_q^{-1}) = (c + d, c \omega_q + d \omega_q^{-1}) \), then \( (a - c, b - d) \) is in the nullspace of the matrix

\[
\begin{bmatrix}
1 & 1 \\
\omega_q & \omega_q^{-1}
\end{bmatrix}
\]

which is singular if and only if \( \omega_q^{-1} = \omega_q \). Thus, the elements of \( T_q \) are distinct whenever \( \omega_q \neq 1 \). So for \( q > 2 \), we have \( |T_q| = q^2 = |S_q|^2 \) for each \( b \). Using the notation (9) one can rewrite \( \frac{1}{|T_q|} \sum_{x \in T_q} e(x \cdot y) \) as a product of two terms, each of which tend to zero as \( q \rightarrow \infty \) by Myerson’s lemma. Since the sets \( T_q \) are uniformly distributed modulo 1 as \( q \rightarrow \infty \), the uniform distribution of the values \( K(a, b, q, d) \) with respect to the push-forward measure on \( \mathbb{H}_d \) is immediate.
Figure 5. The values \( \{K(a, b, p, 9) : 0 \leq a, b \leq p - 1 \} \) for several primes \( p \equiv 1 \pmod{9} \); see Theorem 10.

3. Variants of hypocycloids

A glance at Figure 1 suggests that hypocycloids are but one of many shapes that the values of generalized Kloosterman sums “fill out.” With additional work, a variety of results similar to Theorem 7 can be obtained. The following theorem is illustrated in Figure 5.

**Theorem 10.** Let \( p \equiv 1 \pmod{9} \) be an odd prime. If \( q = p^\alpha \) and \( \alpha \geq 1 \), then the values \( \{K(a, b, q, 9) : 0 \leq a, b \leq p - 1 \} \) are contained in the threefold sum

\[
\mathbb{H}_3 + \mathbb{H}_3 + \mathbb{H}_3 = \{w_1 + w_2 + w_3 : w_1, w_2, w_3 \in \mathbb{H}_3\}
\]

of the filled deltoid \( \mathbb{H}_3 \). Moreover, as \( q \to \infty \), this shape is “filled out” in the sense of Theorem 7.

**Proof.** Let \( g \) be a primitive root modulo \( q \) and define \( u = g^{\phi(q)/9} \), so that \( u \) has multiplicative order 9 modulo \( q \). Since \( p \) and \( p - 1 \) are relatively prime, \( p - 1 \nmid \phi(q) = \frac{p^\alpha - 1}{3} = \frac{p^\alpha - 1}{3} \), so \( u^3 \neq 1 \pmod{q} \). Thus,

\[
u^6 + u^3 + 1 \equiv (u^9 - 1)(u^3 - 1)^{-1} \equiv 0 \pmod{q},
\]

so that \( u^{6+j} \equiv -u^{3+j} - u^j \pmod{q} \) for \( j = 1, 2, 3 \). Along similar lines, we have \( u^{-(6+j)} \equiv -u^{-(3+j)} - u^{-j} \pmod{q} \) for \( j = 1, 2, 3 \). For \( k = 1, 2, \ldots, 6 \), let

\[
\zeta_k = e\left(\frac{au^k + bu^{-k}}{p}\right)
\]

and observe that

\[
K(a, b, q, 9) = \sum_{k=1}^{9} e\left(\frac{au^k + bu^{-k}}{q}\right)
= \sum_{k=1}^{6} e\left(\frac{au^k + bu^{-k}}{q}\right) + \sum_{j=1}^{3} e\left(\frac{a(-u^{3+j} - u^j) + b(-u^{6+j} - u^{3+j})}{q}\right)
= \zeta_1 + \zeta_2 + \cdots + \zeta_6 + \frac{1}{\zeta_1 \zeta_4} + \frac{1}{\zeta_2 \zeta_5} + \frac{1}{\zeta_3 \zeta_6}
= \left(\zeta_1 + \zeta_4 + \frac{1}{\zeta_1 \zeta_4}\right) + \left(\zeta_2 + \zeta_5 + \frac{1}{\zeta_2 \zeta_5}\right) + \left(\zeta_3 + \zeta_6 + \frac{1}{\zeta_3 \zeta_6}\right).
\]
Thus, $K(a, b, q, 9)$ belongs to $\mathbb{H}_3 + \mathbb{H}_3 + \mathbb{H}_3$. Since $\phi(9) = 6$, Lemma 4 ensures that the sets

$$T_q = \left\{ \left( e\left(\frac{au + bu^{-1}}{q}\right), \ldots, e\left(\frac{au^6 + bu^{-6}}{q}\right) \right) : 0 \leq a \leq q - 1 \right\}$$

are uniformly distributed modulo 1 for any fixed $b$. Thus, the values $\{K(a, b, q, d) : 0 \leq a, b \leq p - 1\}$ “fill out” $\mathbb{H}_3 + \mathbb{H}_3 + \mathbb{H}_3$ as $q \to \infty$. □

4. OF SQUARES AND SALIÉ SUMS

The preceding subsections concerned the asymptotic behavior of generalized Kloosterman sums $K(a, b, p, \alpha, d)$, in which $d | (p - 1)$ and $p^\alpha$ tends to infinity. Here we turn the tables somewhat and consider the sums $K(a, b, p, p - 1/2)$ for $d = 2^n$. In general, we take $d$ to be the largest power of two that divides $p - 1$; otherwise the cyclic subgroup of $\mathbb{Z}/p\mathbb{Z}$ of order $p - 1$ has even order. This forces $K(a, b, p, p - 1/2)$ to be real valued, which is uninteresting from our perspective.

For a fixed odd prime $p$,

$$T(a, b, p) = \sum_{u=1}^{p-1} \left( \frac{u}{p} \right) e\left(\frac{au + bu^{-1}}{p}\right)$$

is called a Salié sum; here $\left( \frac{u}{p} \right)$ denotes the Legendre symbol. Although they bear a close resemblance to classical Kloosterman sums, the values of Salié sums can be explicitly determined [9]. If $p$ is an odd prime and $\left( \frac{a}{p} \right) = \left( \frac{b}{p} \right) = 1$, then

$$T(a, b, p) = \begin{cases} 2\tau_p \cos\left(\frac{2\pi k}{p}\right) & \text{if } \left( \frac{a}{p} \right) = \left( \frac{b}{p} \right) = 1, \\ -2\tau_p \cos\left(\frac{2\pi k}{p}\right) & \text{if } \left( \frac{a}{p} \right) = \left( \frac{b}{p} \right) = -1, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where $k$ is a square root of $4ab$ in $\mathbb{Z}/p\mathbb{Z}$ and

$$\tau_n = \begin{cases} \sqrt{n} & \text{if } n \equiv 1 \pmod{4}, \\ i\sqrt{n} & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

The following result explains the phenomenon observed in Figure 6.

**Theorem 13.** Let $p \equiv 3 \pmod{4}$ be an odd prime. If $p \nmid ab$, then

$$|\Re K(a, b, p, \frac{p-1}{2})| \leq \frac{\sqrt{p}}{2}, \quad |\Im K(a, b, p, \frac{p-1}{2})| \leq \frac{\sqrt{p}}{2}. \quad \tag{14}$$

If $p | ab$, then

$$K(a, b, p, \frac{p-1}{2}) = \begin{cases} \frac{1}{2}\left(\left( \frac{a}{p} \right)\tau_p - 1 \right) & \text{if } p | a \text{ and } p \nmid b, \\ \frac{1}{2}\left(\left( \frac{b}{p} \right)\tau_p - 1 \right) & \text{if } p | a \text{ and } p | b. \end{cases}$$

**Proof.** Since

$$T(a, b, p) = \sum_{u=1}^{p-1} e\left(\frac{au + bu^{-1}}{p}\right) - \sum_{u=1}^{p-1} e\left(\frac{au + bu^{-1}}{p}\right),$$

$$K(a, b, p) = \sum_{u=1}^{p-1} e\left(\frac{au + bu^{-1}}{p}\right) + \sum_{u=1}^{p-1} e\left(\frac{au + bu^{-1}}{p}\right),$$
Figure 6. Plots of $K(a, b, p, \frac{p-1}{2})$ for primes $p \equiv 3 \pmod{4}$ and $p \nmid ab$. The images are contained in the square with vertices $(\pm\sqrt{p}, \pm i\sqrt{p})$; see Theorem \[13\] Here $K(a, b, p, \frac{p-1}{2})$ is blue if $(\frac{ab}{p}) = 1$ and red if $(\frac{ab}{p}) = -1$. The appearance of the horizontal red line segment is explained by (15) and the third condition in (12). The higher density of points near the top and bottom of the blue square is due to the cosine term in [12].

Figure 7. Generalized Kloosterman sums $K(a, b, p, \frac{p-1}{4})$ with $p \equiv 5 \pmod{8}$, $1 \leq a, b \leq p - 1$. Write $a = g^r, b = g^s$, where $g$ is a primitive root modulo $p$. $K(a, b, p, \frac{p-1}{4})$ is colored blue, red, green, or purple if $r - s \equiv 0, 1, 2, \text{ or } 3 \pmod{4}$ respectively.

it follows that

$$K(a, b, p, \frac{p-1}{2}) = \frac{1}{2} (T(a, b, p) + K(a, b, p));$$

we thank Bill Duke for pointing this out to us. Since $|T(a, b, p)| \leq 2\sqrt{p} |\cos(\theta)| \leq 2\sqrt{p}$, we have $\frac{1}{2}|T(a, b, p)| \leq \sqrt{p}$. On the other hand, the Weil bound for classical Kloosterman sums ensures that $\frac{1}{2}|K(a, b, p)| \leq \sqrt{p}$ if $p \nmid ab$ [14]. Since $p \equiv 3 \pmod{4}$, the Salié sums $T(a, b, p)$ are purely imaginary (or zero), which yields (14). The evaluation of $K(a, b, p, \frac{p-1}{2})$ when $p|ab$ is straightforward and omitted. □
5. Boxes in boxes

The images for \( d = \frac{p-1}{2} \) resemble a rectangle inside a larger square; see Figure 7. This differs significantly from the \( d = \frac{p-1}{4} \) case. The following lemma and theorem partially explain the “box-in-a-box” behavior of \( d = \frac{p-1}{4} \) plots.

**Lemma 16.** Let \( p \) be an odd prime of the form \( p = 2^n d + 1 \), with \( d \) odd and \( n \geq 1 \). Then \( K(a, b, p, \frac{p-1}{2^n}) = 2 \Re K(a, b, p, \frac{p-1}{4}) \).

**Proof.** Note that \( p = 2^n d + 1 = 2^{n-1}(2d) + 1 \), so \( K(a, b, p, \frac{p-1}{2^n}) \) is real-valued.

Therefore, it suffices to show that \( 2K(a, b, p, \frac{p-1}{2^n}) - K(a, b, p, \frac{p-1}{4}) \) is purely imaginary.

\[
2K(a, b, p, \frac{p-1}{2^n}) - K(a, b, p, \frac{p-1}{4})
= 2 \sum_{u^d=1} e\left(-\frac{au - bu^{-1}}{p}\right) - \sum_{v^{2d}=1} e\left(-\frac{av - bv^{-1}}{p}\right)
= \sum_{u^d=1} e\left(-\frac{au - bu^{-1}}{p}\right) - \sum_{v^{d}=(-1)^d} e\left(-\frac{av + bv^{-1}}{p}\right)
= \sum_{u^d=(-1)^d} e\left(-\frac{au + bu^{-1}}{p}\right) - 2 \sum_{v^{d}=(-1)^d} e\left(-\frac{av + bv^{-1}}{p}\right).
\]

Since \( d \) is odd, \(-(-1)^d = 1\), so the above term simplifies to \( K(a, b, p, \frac{p-1}{2^n}) - 2K(a, b, p, \frac{p-1}{4^n}) \). Then \( 2K(a, b, p, \frac{p-1}{2^n}) - K(a, b, p, \frac{p-1}{4^n}) \) is purely imaginary. \( \square \)

**Theorem 17.** Let \( p \equiv 1 \pmod{4} \) be prime, \( 1 \leq a, b \leq p - 1 \). Let \( g \) be a primitive root of \((\mathbb{Z}/p\mathbb{Z})^\times\) and write \( a = g^r, b = g^s \). Then

\[
|\Re K(a, b, p, \frac{p-1}{4})| \leq \begin{cases} \sqrt{p} & \text{if } r \equiv s \pmod{2}, \\ \sqrt{p} & \text{if } r \not\equiv s \pmod{2}. \end{cases}
\]

Furthermore, if \( r - s \equiv 2 \pmod{4} \), then \( \Im K(a, b, p, \frac{p-1}{4}) = 0 \).

**Proof.** By Theorem 13 we know that

\[
K(a, b, p, \frac{p-1}{2^n}) = \frac{T(a, b, p) + K(a, b, p)}{2}.
\]

Using Lemma 16 we write

\[
\Re K(a, b, p, \frac{p-1}{4}) = \frac{T(a, b, p) + K(a, b, p)}{4}.
\]

The first half of this fraction is simply a traditional Kloosterman sum, which is real valued and bounded by \( \pm 2\sqrt{p} \). Since \( p \equiv 1 \pmod{4} \), \( T(a, b, p) \) is also real valued and bounded by \( \pm 2\sqrt{p} \). Note that \( 4ab \) is a quadratic residue modulo \( p \) if and only if \( r + s \), and therefore \( r - s \), is even. Thus, when \( r - s \equiv 0 \pmod{4} \) or \( r - s \equiv 2 \pmod{4} \), we can say

\[
|\Re K(a, b, p, \frac{p-1}{4})| = \frac{K(a, b, p) + T(a, b, p)}{4} \leq \frac{2\sqrt{p} + 2\sqrt{p}}{4} = \sqrt{p}.
\]
Alternatively, if \( r - s \equiv 1 \pmod{4} \) or \( r - s \equiv 3 \pmod{4} \), then \( T(a, b, p) = 0 \) and
\[
|\text{Re} K(a, b, p, \frac{p-1}{4})| = \frac{K(a, b, p)}{4} \leq \frac{2\sqrt{p}}{4} = \frac{\sqrt{p}}{2}.
\]
Now suppose that \( r - s \equiv 2 \pmod{4} \). First, we rewrite \( \text{Im} K(a, b, p, \frac{p-1}{4}) \) using Lemma 16.
\[
\text{Im} K(a, b, p, \frac{p-1}{4}) = iK(a, b, p, \frac{p+1}{4}) - i \text{Re} K(a, b, p, \frac{p-1}{4})
\]
\[
= iK(a, b, p, \frac{p+1}{4}) - \frac{i}{2} K(a, b, p, \frac{p-1}{4})
\]
\[
= i \sum_{k=1}^{d} e\left(\frac{ag^{4k} + bg^{-4k}}{p}\right) - \frac{i}{2} \sum_{k=1}^{2d} e\left(\frac{ag^{2k} + bg^{-2k}}{p}\right)
\]
\[
= \frac{i}{2} \left( K(a, b, p, \frac{p+1}{4}) - K(a, b, p, \frac{p-1}{4}) \right).
\]
Since \( r - s \equiv 2 \pmod{4} \), we have \( r \equiv 0 \pmod{4} \) and \( s \equiv 2 \pmod{4} \), or \( r \equiv 1 \pmod{4} \) and \( s \equiv 3 \pmod{4} \), up to permutation. Suppose the first case holds. Then we can write \( a = g^{4j}, b = g^{4k+2} \) for some integers \( j, k \). It is easy to check that \( K(a, b, p, d) = K(\alpha v, bv^{-1}, p, d) \) for all \( v^d \equiv 1 \pmod{p} \). Using this fact, we obtain
\[
K(a, b, p, \frac{p+1}{4}) - K(a, b, p, \frac{p-1}{4})
\]
\[
= K(g^{4j}, g^{4k+2}, p, \frac{p+1}{4}) - K(g^{4j+2}, g^{4k}, p, \frac{p-1}{4})
\]
\[
= K(g^{4k}, g^{4j+2}, p, \frac{p+1}{4}) - K(g^{4j+2}, g^{4k}, p, \frac{p-1}{4}) = 0.
\]
The second case is similar. \( \square \)

It is apparent from Figure 7 that different bounds are obeyed by \( K(g^r, g^s, p, \frac{p-1}{4}) \) depending on the value of \( r - s \pmod{4} \). Theorem 17 confirms this observation for the real part of the plot. As seen in Figure 7, the bound of \( \text{Im} K(a, b, p, \frac{p+1}{4}) \) seems dependent on the value of \( r - s \pmod{p} \). We have established the imaginary bound for one value of \( r - s \pmod{p} \). We have the following conjecture.

**Conjecture:** Let \( p = 4d + 1 \) be a prime, \( g \) a primitive root modulo \( p \). Then
\[
|\text{Im} K(g^r, g^s, p, \frac{p-1}{4})| \leq \begin{cases} \frac{\sqrt{p}}{2} & \text{if } r - s \equiv 1, 3 \pmod{4}, \\ \frac{\sqrt{p}}{4} & \text{if } r - s \equiv 0 \pmod{4}. \end{cases}
\]

6. **Sporadic spiders**

We conclude this note with an investigation of a peculiar and intriguing phenomenon. Numerical evidence suggests that for a fixed odd prime \( d \), the spider-like image depicted in Figure 8 appears abruptly for only one specific modulus. Figure 9 illustrates the swift coming and going of the ephemeral spider.

The moduli that generate the spiders are all of the form \( L_{p(n)} \), where \( p(n) \) is the \( n \)th prime and \( L_k \) is the \( k \)th Lucas number; see Table 1. Recall that the Lucas numbers (sequence \( \text{A000032} \) in the OEIS) are defined by the initial conditions \( L_0 = 2, L_1 = 1 \), and the recurrence relation \( L_n = L_{n-1} + L_{n-2} \) for \( n > 1 \). It is not immediately clear that our pattern can continue indefinitely since for prime \( d \) we
Figure 8. Generalized Kloosterman sums $K(a, b, m, \Lambda)$ for some $\Lambda \leq (\mathbb{Z}/m\mathbb{Z})^\times$ of order $d$. These images resemble fat spiders with a horrifically increasing number of legs.

Figure 9. 3469, 3571, 3673 are three consecutive primes congruent to 1 (mod 17). Theorem 7 tells us that the values of the corresponding generalized Kloosterman sums are on their way to filling out $\mathbb{H}_{17}$. The appearance of the spider at $p = 3571$ is as surprising as it is fleeting.

Lemma 18. If $p \geq 5$ is an odd prime, then there is an odd prime $q$ so that $q|L_p$.

Proof. If we observe $L_0, L_1, \ldots, L_{11}$ modulo 8, we get 2, 1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2, 1. Because the first two digits of this sequence are the same as the last and $L_n = \frac{\prod_{i=0}^{n-1} L_i}{h}$, we see that $L_n$ is divisible by 7 for all $n \geq 2$. Therefore, $q = 2$ or 3 is a divisor of $L_n$.
Further, how does the structure of Lucas numbers for prime indices influence the phenomena be formalized? One can immediately recognize a spider when one sees it, but it is more difficult to express the irregularity in a mathematical manner. However, many questions remain. How can the spider phenomena be formalized? One can immediately recognize a spider when one sees it, but it is more difficult to express the irregularity in a mathematical manner. Further, how does the structure of Lucas numbers for prime indices influence the spider-like images? These are questions we hope to return to at a later time.

\[
L_{n-1} + L_{n-2}, \text{ the sequence repeats. Thus, } L_n \text{ is never divisible by } 8, \text{ and furthermore } L_n > 8 \text{ for all } n \geq 5. \text{ Any integer greater than } 8 \text{ and not divisible by } 8 \text{ cannot be a power of two. Thus, there exists an odd prime } q \text{ such that } q | L_p. \]

**Theorem 19.** If \( p \geq 5 \) is an odd prime, then \( p | \phi(L_p) \).

**Proof.** Let \( F_n \) denote the \( n \)-th Fibonacci number and let \( z(n) \) denote the order of appearance of \( n \) \([13, \text{ p. } 89]\). By Lemma \([18]\) there is an odd prime \( q \) such that \( q | L_p \). Using the fact that \( F_{2p} = L_p F_p \) \([13, \text{ p. } 25]\), we know that \( q | F_{2p} \). Furthermore, \( q | F_{z(q)} \) by \([13, \text{ p. } 89]\). Consequently, \( q | \gcd(F_{2p}, F_{z(q)}) = F_{\gcd(2p, z(q))} \),

where we have used that \( \gcd(F_a, F_b) = F_{\gcd(a,b)} \) for all \( a, b \in \mathbb{Z}^+ \).\(^{11}\) Theorem 16.3. Now, set \( d = \gcd(2p, z(q)) \), and observe that since \( p \) is prime, \( d = 1, 2, p \) or \( 2p \). However, \( q \) is an odd prime and \( q | F_{2p} \). If \( d = 1 \) or \( 2 \), this implies \( q | 1 \) because \( F_1 = F_2 = 1 \), which is impossible because \( q \) is an odd prime. Thus, \( d = p \) or \( 2p \).

Now, consider the case \( d = p \), implying that \( q | F_n \). However, by \([13, \text{ p. } 29]\) we know

\[
L_p^2 - 5F_p^2 = 4(-1)^p,
\]

thus implying that \( q | 4 \) which is impossible because \( q \) is an odd prime. Thus, \( d = \gcd(2p, z(q)) = 2p \) and therefore \( 2p | z(q) \). Furthermore, we know \( z(q) | q - (\frac{q}{p}) \). Now, \( q \) cannot be 5, because the Lucas numbers are always coprime to 5 \([13, \text{ p. } 89]\). We would like to show that \( (\frac{q}{5}) = 1 \), because then

\[
p | 2p | z(q) | (q - 1) = \phi(q) | \phi(L_p).
\]

Thus we must show that \( q \) is a quadratic residue modulo 5. For this, we must again use the fact that \( L_p^2 - 5F_p^2 = 4(-1)^p \). Reducing this modulo \( q \), we get that

\[
-5F_p^2 \equiv -4 \pmod{q}.
\]

Thus \( (\frac{2}{5}) = 1 \), and furthermore \( (\frac{q}{5}) = 1 \) by quadratic reciprocity. \( \square \)

For \( n \geq 3 \), Theorem \([19]\) guarantees the existence of an order \( p(n) \) subgroup \( \Lambda \) of \((\mathbb{Z}/L(p(n)) \mathbb{Z})^\times\). In principle, this permits the patterns hinted at in Figure 9 to continue indefinitely. However, many questions remain. How can the spider phenomena be formalized? One can immediately recognize a spider when one sees it, but it is more difficult to express the irregularity in a mathematical manner. Further, how does the structure of Lucas numbers for prime indices influence the spider-like images? These are questions we hope to return to at a later time.

| \( n \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| \( p(n) \) | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 |
| \( L_{p(n)} \) | 11 | 29 | 199 | 521 | 3571 | 9349 | 64079 | 1149851 | 3010348 |
| \( \phi(L_{p(n)}) \) | 10 | 28 | 198 | 520 | 3570 | 9348 | 63480 | 1130304 | 3010348 |

**Table 1.** This sequence \( L_{p(n)} \), in which \( p(n) \) is the \( n \)-th prime and \( L_k \) is the \( k \)-th Lucas number; see [A180363] in the OEIS. Although the initial terms in this sequence are prime, they are not all so. Theorem 19 ensures that \( p(n) \) divides \( L_{p(n)} \) for \( n \geq 3 \).
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