Uchiyama’s conjecture on sums of squares

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To the Australian Test Cricket selectors, to whom I am indebted a slice of humble pie, after they ignored the strident remonstrances of TS Trudgian and included MR Marsh in the team.

Abstract

Uchiyama showed that every interval \((n, n + cn^{1/4})\) contains an integer that is the sum of two squares, where \(c = 2^{3/2}\). He also conjectured a minimal value of \(c\) such that the above statement still holds. We investigate this claim.

1 Introduction

We are in the market for short intervals \((x, x + h(x)]\) where \(h(x) = o(x)\) such that each interval (or at least each interval with \(x\) sufficiently large) contains a number that is the sum of two squares. The analogous question for primes in short intervals is well-known. The best result is \(h(x) = x^{0.525}\) by Baker, Harman and Pintz \([1]\), which is true for sufficiently large \(x\).

Anything that can be done for primes can presumably be done for the sum of squares. After all, every prime \(p \equiv 1 \pmod{4}\) is the sum of two squares. Indeed, we have that

\[
\pi(x) \sim \frac{x}{\log x}, \quad R(x) \sim C \frac{x}{\sqrt{\log x}}, \tag{1}
\]

where \(\pi(x)\) denotes the number of primes not exceeding \(x\), and \(R(x)\) the number of \(n \leq x\) that are the sum of two squares. Here \(C\) is a constant \(\approx 0.74\) the exact form of which is known but does not concern us here. Therefore, by \([1]\) there are many more sums of squares than there are primes.

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†Indeed, this result is effective in the sense that one could calculate the \(x_0\) — presumably astronomically large — such that the result holds for all \(x \geq x_0\). Setting this problem to a graduate student could well violate the 8th Amendment to the US Constitution.

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Perhaps, then, it is not surprising that one can do better in the sum-of-squares case. Indeed, one can show easily that \( h = cx^{1/4} \) is admissible, for some constant \( c \) — see, e.g., Heath-Brown [6, p. 2]. Bambah and Chowla [2] proved that one may take \( c = 2^{3/2} + \varepsilon \) where \( \varepsilon \to 0 \) as \( x \to \infty \). Uchiyama [8] neatened this up and showed (in less than two pages, using nothing more than introductory calculus) that one could take \( c = 2^{3/2} \) for all \( x \). Uchiyama’s proof is generalised to other quadratic forms in [7]. Remarkably, the elementary methods used in these papers have not been improved upon. Therefore, one may pose the following.

**Problem 1.** Find a value of \( c < 2^{3/2} \) such that for all \( n \geq 1 \) (or even all \( n \) sufficiently large) there are integers \( x, y \) such that

\[
 n < x^2 + y^2 < n + cn^{1/4}.
\]  

(2)

At the 61st Annual Meeting of the Australian Mathematical Society, held recently at Macquarie University, I offered a bottle of wine (or cricket merchandise vouchers for non-drinkers) to the tune of $50 AUD for a resolution of Problem 1. Freed from the pressure of prizes, one may also wish to consider the following.

**Problem 2.** Assume the Riemann hypothesis. Find a function \( h(x) = o(x^{1/4}) \) such that for all \( x \) sufficiently large there is a sum of squares in \((x, x + h(x))\].

Problem 2 is presumably much harder than Problem 1 and hence deserves a larger prize for its resolution. Unfortunately the margins in my chequebook are too small to write down a full amount.

Again, on analogy with primes in short intervals, one may examine Cramér’s conjecture, viz. that one may take \( h = c \log^2 x \) for some constant \( c \). It is known that \( h(x) \) cannot be smaller than a constant times \( \log^2 x / (\log \log x)^{1/2} \).

**Problem 3.** Assume the Riemann hypothesis (and any other high-powered conjecture you like). Give a conjecture on the smallest \( h(x) \) such that for all \( x \) sufficiently large there is a sum of squares in \((x, x + h(x))\].

Combining Erdös’ result with Cramér’s conjecture, it seems reasonable that one should look for a function \( h(x) \) in Problem 3 such that

\[
 \log x / (\log \log x)^{1/2} \ll h(x) \ll \log^2 x.
\]
2 Uchiyama’s conjecture

We now return to Problem 1. Uchiyama shows that were one looking to improve the constant $c$ in (2) for all $n \geq 1$, one must have $c > 2^{-1/2}5^{3/4} = 2.364 \ldots$. This comes about from inserting $n = 20$ into (2) and noting that none of 21, 22, 23, 24 is a sum of two squares. Uchiyama conjectures that this is the worst value of $c$, that is, (2) should be true for all $c > 2^{-1/2}5^{3/4}$. It is remarked on [8, p. 126] that this has been verified for all $n \leq 1000$.

One can easily extend this and, indeed, note that at $n = 1493$ we run into a problem. Since none of 1494, 1495, . . . , 1507 is a sum of two squares, we find that if (2) is true then we must have $c > 15/(1493)^{1/4} = 2.413 \ldots$. We find that this repaired conjecture of Uchiyama’s is valid for all $n \leq 10^8$. To summarise, we have

**Theorem 1.** There is a constant $c \in (15/(1493)^{1/4}, 2^{3/2}] = (2.413 \ldots, 2.828 \ldots)$ such that for all $n \geq 1$ there are integers $x, y$ such that

$$n < x^2 + y^2 < n + cn^{1/4}. \quad (3)$$

Moreover, for any constant $c > 15/(1493)^{1/4}$ the inequality (3) is true for all $n \leq 10^8$.

It should not be difficult to extend the range $10^8$ in Theorem 1. Indeed, the code used was slovenly in the extreme. We merely enumerated all intervals in (3): once we found a value inside an interval that was the sum of two squares, we ticked off this $n$, and went to the next one. This took 6 hours using *Mathematica* on a standard desktop machine. This overlooks entirely the fact that checking one interval will suffice for many others. For example, with $n = 400$ we have the interval $(401, 410.79 \ldots)$. Since 410 is a sum of two squares this would serve to verify (2) for $n = 400, 401, \ldots, 409$ — ten for the price of one!

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