Dynamics of spatio-temporal defects in the Couette–Taylor flow: comparison of experimental and theoretical results

L Nana 1, A B Ezersky 3, N Abcha 2, I Mutabazi 2

1 Département de Physique, Faculté des Sciences, Université de Douala, B.P. 24157 Douala, Cameroun
2 Laboratoire de Mécanique, Physique et Géosciences (LMPG), Université du Havre, 25, rue Philippe Lebon, F-76058 Le Havre, France
3 Laboratoire de Morphodynamique Continentale et Côtière, UMR 6143 CNRS-Université de Caen, F-14000 Caen, France

Abstract. Recent experiments in the Taylor-Dean system or in the counter-rotating Couette-Taylor system have shown that, spatio–temporal defects were generated periodically in time on the background of traveling vortices. We have found that the dynamics of these defects and many of their features may be described by the one dimensional complex Ginzburg–Landau equation(CGLE) with homogeneous boundary conditions.

1. Introduction

Spiral vortices appearing in the counter-rotating Couette-Taylor system and defects occurring on the background of this vortex pattern have been investigated in a large number of papers [1]. Progress in the understanding of nonlinear dynamics in this system is connected with analysis of amplitude equation for spiral waves [2]. The amplitude equation formalism is well developed in the framework of the complex Ginzburg-Landau equation (CGLE). The CGLE is a prototype equation for a complex field \( A(x,t) \) that exhibits different type of states depending on a reduced number of parameters and on the boundary conditions. It accounts for the slow modulations, in space and time of the oscillatory state in a physical system which undergoes a Hopf bifurcation [3]. One of the landmarks of the CGLE is that it possesses localized defect solutions. Even in a one space dimension, where no topological constraint exists, numerical simulations of the CGLE [4] and analytical [5] work have revealed the existence and importance of various amplitude hole solutions, which are considered as the building blocks of the complex spatiotemporal dynamics observed or of the spatiotemporal chaos. Most of works on the CGLE are based on it analysis in extended systems or in finite geometries with periodic boundary conditions [4]. However, different geometries and different boundary conditions may lead to substantially different dynamical behaviours. Eguiluz et al. [6] carried out extensive analysis of model problems describing different extended dynamical systems in which the dynamics is strongly influenced by the boundaries. They found numerically that boundary effects induce the formation of novel structures, like target and defect solutions.

In hydrodynamic systems, spatially localized perturbations are advected by the flow. Depending on whether the trailing front of a growing perturbation of the basic state propagates in the downstream or in the upstream direction, the system is convectively or absolutely unstable, respectively [7]. It seems
likely that both these effects will depend on lateral boundaries, even if they are largely distant each from the other. The property of absolute or convective instability can be controlled externally by tuning the strength of the imposed flow [8].

The present work deals with the absolutely unstable situation where the nonlinear structures are self-sustained. We consider a system in finite geometry in which we impose the homogeneous Dirichlet boundary conditions: the amplitude of the wave vanishes at the lateral boundaries of the domain. In the absolute unstable regime, for certain values of the parameters, we found a sequence of time periodic defects in a parameter range which was considered as stable to the Benjamin-Feir modes when periodic boundary conditions were used. These defects have been often observed in experiments in the Taylor-Dean system [9].

2. Amplitude equation formalism

The CGLE describes the dynamics of the field amplitude close to onset of the supercritical Hopf bifurcation, and is usually written in one-dimension as:

\[
\frac{\partial A}{\partial t} + s \frac{\partial A}{\partial x} = \mu A + (1 + ic_1) \frac{\partial^2 A}{\partial x^2} - (1 - ic_3) |A|^2 A, \quad 0 \leq x \leq L .
\] (1)

This equation describes the envelope of a traveling wave propagating at the linear group velocity \(s\) towards positive \(x\) (right wave), i.e. in one direction only. The wave is driven by the control parameter \(\mu\), and the coefficients \(c_1\) and \(c_3\) describe linear dispersion and nonlinear frequency detuning, respectively; \(L\) is the length of the system. Formulations such as these are suitable for describing weakly nonlinear waves not only in open flows but also their properties in finite geometries. Here, we are interested in flows evolving in bounded geometries with a finite aspect ratio \(L\). We have chosen the boundary conditions in accordance with experimental data on the Couette-Taylor flows and in the Taylor-Dean system: near lateral boundaries, amplitude of disturbances tends to zero. As a consequence we have imposed the following boundary conditions:

\[
A(x = 0, t) = 0 = A(x = L, t)
\] (2)

on the complex amplitude \(A(x, t)\). It should be noted that, the existence of finite boundaries breaks the space translational invariance, so, it is not possible to eliminate the group velocity \(s\) by a transformation to a moving frame of reference. Although it is possible to eliminate the criticality parameter \(\mu\), we have kept it in the equation for comparison with experimental data. Therefore, the 1D system described by the CGLE is spanned by a three parameter space: \(s, c_1, c_3\). The use of the periodic boundary conditions had the advantage of the simplification of the problem by reducing the number of control parameters to 2 i.e. \((c_1, c_3)\). In the case of the periodic boundary conditions, the length \(L\) takes discrete values while it has a value fixed by experiment in case of finite boundaries.

3. Numerical algorithm

The numerical method used throughout is the finite-difference scheme in space and fourth-order Runge-Kutta algorithm in time. The CGLE (Eq.(1)) takes the simple form

\[
\frac{dA_j}{dt} = \mu A_j - (1 - ic_3) |A_j|^2 A_j - s \frac{A_{j+1} - A_{j-1}}{2\Delta x} + (1 + ic_1) \frac{A_{j+1} - 2A_j + A_{j-1}}{(\Delta x)^2}
\] (3.a)
where $A_j = A(x_j,t)$ with $2 \leq j \leq N - 1$ and the homogeneous boundary conditions (Eq.(2)) were

$$A_1 = A_N = 0.$$  \hfill (3.b)

Thus we have reduced the partial differential equation (PDE) to a system of ordinary differential equations (ODE). The structure of such systems implies that their dynamics is the result of interaction between $N$ individual dynamical entities. We integrate numerically system (3) using standard fourth-order Runge-Kutta method. In our numerical simulations, we consider a system involving $N$ sites with $N = 301$. The accuracy of the numerical experiment is examined by testing different time and space steps. Finally, the space and time increments were chosen as $\Delta x = 0.2$ and $\Delta t = 0.025$ respectively. In fact, the time step must be small enough at a given spacing, for the preceding algorithm to be conditionally stable. We have used as initial condition a pulse-like solution given by

$$A(x,j,0) = A_0 (1 + i) / \cosh(\gamma(x_j - x_a)),$$

where $x_a$ is the position of the pulse maximum.

### 4. Numerical and experimental results

We have restricted the study to the dynamics of the pattern for parameters corresponding to the absolute instability regime. The parameters $c_1$ and $s$ are fixed equal to $c_1 = 0.50, s = 0.50$ and we have varied the parameters $\mu$ and $c_3$. This variation enabled us to identify several areas in the parameters space in which the waves presented different behaviors. There is a rather narrow region where we have found the presence of point defects on the waves. These defects appear at equal time intervals: thus we will call them time periodic defects (PD). In order to characterize these defects and to compare them with those observed in experiments, we have constructed the hydrodynamic field $u(x,t)$ defining the wave packet in the simplest form: $u(x,t) = A(x,t) \exp[i(\omega_0 t - k_0 x)]$ with $\omega_0 = 1.5$ and $k_0 = 2.0$. Figure 1-a represents the space-time diagram of the real part of the field $u(x,t)$ plotted for $c_3 = -4.60$ and $\mu = 0.15$. It shows dislocations of the spirals on defects location: unfolding of spiral in the core of a defect.
Figure 1. (a) Spatio-temporal diagram of the real part of the hydrodynamic field $u(x,t)$ constructed by numerical simulations of the CGLE and (b) the space – time diagram of interpenetrating spiral waves from experimental results [10]: $Ro = -622$, $Ri = 347$. The time duration $\Delta t = 51.2$ s and the length $L = 23$ cm.

The experiments were carried out in a horizontal Couette-Taylor flow between two coaxial counter-rotating cylinders, the description of the flow system is given in [1]. Increasing of frequency of rotation of the inner cylinders leads to the regime corresponding to the excitation of a dominant right-traveling wave (the amplitude of the left propagating wave is very weak). The characteristics of patterns have been investigated by recording the intensity distribution $I(x)$ of the light reflected by Kalliroscope flakes from a line along the axis in the central part of the system. The superimposition of these lines at regular time interval leads to space-time diagram $I(x,t)$ shown in Figure 1-b. Processing of space – time diagram allows to determine the characteristics of the patterns. This flow pattern is characterized by the appearance of point defects for $Ro = -622$ and $Ri = 347$ ($\mu = 0.048$) where $Ro$ and $Ri$ are the Reynolds number defined for the outer and inner cylinders respectively, $\mu = Ri - Ri_c/Ri_c$, $Ri_c$ being the critical value at which the base Couette flow becomes unstable. Thus, sequence of periodic defects found in our numerical simulations was also observed in the experiments. Therefore, results of numerical simulations have allowed explaining regimes observed in most of physical experiments on pattern formation.

In our numerical calculations, we have observed a sequence of three defects on an interval of time $t \approx 83$. It is possible to observe more or less than three point defects in the same time interval if other values of control parameters $\mu$ and $c_3$ were chosen. The slope of the defects observed in figure 1-a, indicates the right traveling wave and it is related the linear group velocity $s$. 
Figure 2. Space-time diagram of the field (a) the amplitude $|A|$ and (b) phase $\text{arg}(u(x,t))$: the phase ($-\pi \rightarrow +\pi$) is mapped to grey scale (black $\rightarrow$ white), a jump of $\pi$ is observed around a defect.

Figure 3. (a) The profile of the amplitude of the wave of a single defect and (b) cross section of the amplitude along defects.

We found that multi-stability occurred in the system: for different initial conditions it was possible to obtain different coordinate $x$ at which periodically organized point defects were situated. Appearance of periodic sequence of point defects seems to be typical for the systems with rotation. In the light to complete our numerical study of the CGLE, we built the spatiotemporal diagrams of the amplitude and phase of the hydrodynamic field. The space-time of the Figure 2-a shows that point defects are the points where the amplitude of the wave vanishes ($|A| = 0$) and the phase ($\phi = \text{arg}(A)$) is no longer defined. The space-time diagram of the Figure 2-b shows the phase of the hydrodynamic field $u(x,t)$. The phase varies from $-\pi$ (black colour) to $+\pi$ (white colour) and undergoes a jump of $\pi$ in the position of the core of a defect. The amplitude profile of the wave of a single defect is shown in Figure 3-a. The core of defect is located at $x \approx 40$. In fact, these point defects appear near the boundary of the system and evolve to the interior when one varies the physical parameters of the flow. The cross section of the amplitude along defects is plotted in Figure 3-b; one can determine the period of appearance of the defects.
5. Discussion
It should be emphasized that in most of papers devoted to the numerical simulations of the CGLE, periodic boundary conditions are used and described completely different results from ours. In this work, we have pointed out the importance of boundary conditions on the solutions of the CGLE. We have shown that homogeneous boundary conditions change sufficiently the pattern generated in the system [13]. A particular attention was focused on the investigation of the regime with time-periodic defects generation in the CGLE in a finite geometry. Defect generation and defect chaos in the CGLE with periodic boundary conditions have been reported in many papers [4, 13] in the region of Benjamin-Feir instability when the condition \(1 - c_1 c_3 < 0\) is satisfied. Corresponding experiments that mimic periodic boundary conditions have been especially designed to verify the numerical simulations of the CGLE [14]. In finite geometries, our numerical simulations have shown that periodic sequence of defects occurred in the parameter range corresponding to the Benjamin-Feir stable zone \((1 - c_1 c_3 > 0)\). We should emphasis that periodic defects can be obtained only for homogeneous boundary conditions, and cannot be observed in case of periodic boundary conditions. Wave incident to a boundary cannot be reflected and the lateral boundaries naturally behave as sources of defects. In fact, for solution with defect, the phase of the complex amplitude \(A\) at the time \(t\) is not periodic and one has \(\phi(L) - \phi(0) \neq 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \cdots\). It is then impossible to glue this solution at the ends to satisfy periodic boundary conditions. The variation of the length of the system does not modify the results presented, but rather modifies the coordinates \(x\) at which the sequence of periodically organized point defects appears. The topological or point defects have been reported in many experiments for example in the Taylor-Dean system [9] or in binary mixture convection [11]. Numerical simulations of the anisotropic CGLE have presented a class of solutions where the defects align spontaneously along chains [12]. A further work is needed to investigate the coupled 1-D complex Ginzburg-Landau equations with homogeneous boundary conditions, these equations describe counter-propagating waves observed in many finite-length systems.

6. Conclusion
We have solved numerically the 1-D complex Ginzburg-Landau equation (CGLE) with homogenous boundary conditions (Dirichlet problem). In this case, the dynamics of the system is controlled by at least 4 parameters while it was described by 2 parameters in the case of the periodic boundary conditions. We have found time-periodic defects in the parameter range for which the Benjamin-Feir condition predicts the stability. The occurrence of time periodic defects has been reported in the many experiments with a finite aspet ratio, in particular in the spiral pattern in the Couette-Taylor flow with counter-rotating cylinders and in the Taylor-Dean system.

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