Abstract. The invariant integration method for Chern-Simons theory defined on the compact hyperbolic manifold $\Gamma \backslash \mathbb{H}^3$ is verified in the semiclassical approximation. The semiclassical limit for the partition function is presented. We discuss briefly $L^2$--analytic torsion and the eta invariant of Atiyah-Patodi-Singer for compact hyperbolic 3-manifolds.

1. Introduction

It is known that topological invariants associated with 3-manifolds can be constructed within the framework of Chern-Simons gauge theory [1]. These values have been specified in terms of the axioms of topological quantum field theory in [2], whereas the equivalent derivation of invariants has also been presented combinatorially in [3, 4], where modular Hopf algebras related to quantum groups have been used. The Witten’s (topological) invariants have been explicitly calculated for a number of 3-manifolds and gauge groups [5, 6, 7, 8, 9, 10, 11]. The semiclassical approximation for the Chern-Simons partition function may be expressed by the asymptotics for $k \to \infty$ of Witten’s invariant of a 3-manifold $M$ and a gauge group $G$. Typically this expression is a partition function of quadratic functional. This asymptotics leads to a series of $C^\infty$-- invariants associated with triplets $\{M; F; \xi\}$ with $M$ a smooth homology 3--sphere, $F$ a homology class of framings of $M$, and $\xi$ an acyclic conjugacy class of orthogonal representations of the fundamental group $\pi_1(M)$ [12]. In addition the cohomology $H(M; Ad \xi)$ of $M$ with respect to the local system related to $Ad \xi$ vanishes.

Date: May, 1999.

We thank Prof. F.L. Williams for useful discussion. First author partially supported by a CNPq grant (Brazil), RFFI grant (Russia) No 98-02-18380-a, and by GRACENAS grant (Russia) No 6-18-1997.
This note is an extension of the two previous papers [13, 14]. Here our aim is to use again the invariant integration method [15, 16] in its simplest form in order to evaluate the semiclassical approximation in the Chern-Simons theory. We do this analysing the partition function related to compact hyperbolic 3-manifolds $\Gamma \backslash \mathbb{H}^3$, where $\mathbb{H}^3$ is the real hyperbolic space and $\Gamma$ is a co-compact discrete group of isometries (for details see Ref. [17]).

We conclude this section introducing the Witten’s invariant defined by the partition function associated with a Chern-Simons gauge theory

$$\mathfrak{W}(k) = \int \mathcal{D}A e^{ikCS(A)}, \quad k \in \mathbb{Z}. \quad (1.1)$$

The formal integration in (1.1) is one over the gauge fields $A$ in a trivial bundle, i.e. 1-forms on the 3-dimensional manifold $X_\Gamma$ with values in Lie algebra $g$ of a gauge group $\mathfrak{G}$. The Chern-Simons functional $CS(A)$ can be considered as a function on a space of connections on a trivial principal bundle over a compact oriented 3-manifold $X_\Gamma$ given by

$$CS(A) = \frac{1}{4\pi} \int_{X_\Gamma} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (1.2)$$

Let $X$ be a locally symmetric Riemannian manifold with negative sectional curvature. Its universal covering $\tilde{X} \to X$ is a Riemannian symmetric space of rank one. The group of orientation preserving isometries $\tilde{G}$ of $\tilde{X}$ is a connected semisimple Lie group of real rank one and $\tilde{X} = \tilde{G}/\tilde{K}$, where $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$. The fundamental group of $X$ acts by covering transformations on $\tilde{X}$ and gives rise to a discrete, co-compact subgroup $\Gamma \subset \tilde{G}$ such that $X = \Gamma \backslash \tilde{G}/\tilde{K}$.

Let $G$ be a linear connected finite covering of $\tilde{G}$, the embedding $\Gamma \hookrightarrow \tilde{G}$ lifts to an embedding $\Gamma \hookrightarrow G$. Let $K \subset G$ be a maximal compact subgroup of $G$, then $X_\Gamma = \Gamma \backslash G/K$ is a compact manifold. Let $g = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $g$ of $G$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a one-dimensional subspace and $J = K \cap G_{\mathfrak{a}}$ be the centralizer of $\mathfrak{a}$ in $K$.

Fixing a positive root system of $(g, \mathfrak{a})$ we have the Iwasawa decomposition $g = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. For $G = SO(n,1)$ ($n \in \mathbb{Z}_+$), $K = SO(n)$, and $J = SO(n-1)$. The corresponding symmetric space of non-compact type is the real hyperbolic space $\mathbb{H}^n$ of sectional curvature $-1$. Its compact dual space is the unit $n$-sphere.

Since $CS(A)$ does not contain any metric on $X_\Gamma$, the quantity $\mathfrak{W}(k)$ is expected to be metric independent, namely to be a (well-defined) topological invariant of $X_\Gamma$. Indeed, this fact has been proved in Refs. [3, 4]. In the limit $k \to \infty$, the asymptotics of the Witten’s invariant
(semiclassical approximation of Eq. (1.1)), involves only a partition functions of quadratic functionals \[^1\]

\[
\sum_{[A_f]} \exp \left( ikCS(A_f) \right) \int D\omega \exp \left( \frac{ik}{4\pi} \int_{X_\Gamma} \text{Tr}(\omega \wedge dA_f \omega) \right). \tag{1.3}
\]

In above equation the sum is taken over representatives \(A_f\) for each point \([A_f]\) in the moduli-space of flat gauge fields on \(X_\Gamma\). In addition the \(\omega\) are Lie-algebra-valued 1-forms and \(dA_f\) is the covariant derivative determined by \(A_f\),

\[
dA_f \omega = d\omega + [A_f, \omega]. \tag{1.4}
\]

2. Quadratic functional with elliptic resolvent

Let \(M\) be a compact oriented Riemannian manifold without boundary, and \(n = 2m + 1 = \dim M\) is the dimension of the manifold. Let \(\chi : \pi_1(M) \rightarrow O(V, \langle \cdot, \cdot \rangle_V)\) be a representation of \(\pi_1(M)\) on real vector space \(V\). The mapping \(\chi\) determines (on a basis of standard construction in differential geometry) a real flat vectorbundle \(\xi\) over \(M\) and a flat connection map \(\nabla_p\) on the space \(\Omega^p(M, \xi)\) of differential \(p\)-forms on \(M\) with values in \(\xi\). One can say that \(\chi\) determines the space of smooth sections in the vectorbundle \(\Lambda^p(TM)^* \otimes \xi\). One can construct from the metric on \(M\) and Hermitian structure in \(\xi\) a Hermitian structure in \(\Lambda^*(TM)^* \otimes \xi\) and the inner products \(\langle \cdot, \cdot \rangle_m\) in the space \(\Omega^m(M, \xi)\). Thus

\[
S_\Omega = \langle \omega, O\omega \rangle_m, \quad O = \ast \nabla_m, \tag{2.1}
\]

where \((\ast)\) is the Hodge-star map. The map \(O\) is formally self-adjoint with the property \(O^2 = \nabla_m^2 \nabla_m\). Suppose that the quadratic functional (2.1) is defined on the space \(\mathcal{G} = \mathcal{G}(M, \xi)\) of smooth sections in a real Hermitian vectorbundle \(\xi\) over \(M\). There exists a canonical topological elliptic resolvent \(R(S_\Omega)\), related to the functional (2.1), namely

\[
0 \xrightarrow{0} \Omega^0(M, \xi) \xrightarrow{\nabla_0} \ldots \xrightarrow{\nabla_{m-2}} \Omega^{m-1}(M, \xi) \xrightarrow{\nabla_{m-1}} \ker(S_\Omega) \xrightarrow{0} 0. \tag{2.2}
\]

Therefore, for the resolvent \(R(S_\Omega)\), we have \(\mathcal{G}_p = \Omega^{m-p}(M, \xi)\) and \(H^p(R(S_\Omega)) = H^{m-p}(\nabla)\), where \(H^p(\nabla) = \ker(\nabla_p)/\mathbb{H}(\nabla_{p-1})\) are the cohomology space. Note that \(S_\Omega \geq 0\) and therefore \(\ker(S_\Omega) \equiv \ker(O) = \ker(\nabla_m)\).

Let us choose an inner product \(\langle \cdot, \cdot \rangle_{H^p}\) in each space \(H^p(R(S_\Omega))\). The partition function of \(S_\Omega\) with the resolvent (2.2) can be written in the form (see Refs. [18, 16])
\[ \mathcal{W}(k) \equiv \mathcal{W}(k; R(S)\mathcal{O}, \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) = \left( \frac{\pi}{k} \right)^{\zeta(0,|\mathcal{O}|)/2} e^{-\frac{i\pi}{4} \eta(0,\mathcal{O})} \times \tau(M, \chi, \langle \cdot, \cdot \rangle_H)^{1/2}, \] (2.3)

where \( |\mathcal{O}| = \sqrt{O^2} \) is defined via spectral theory. This is the basic formula one has to evaluate. With regard to the quantity \( \tau(M, \chi, \langle \cdot, \cdot \rangle_H) \), it is related to the Ray-Singer torsion. In fact, if \( H^0(\nabla) \neq 0 \) and \( H^p(\nabla) = 0 \) for \( p = 1, ..., m \), then the product

\[ \tau(M, \chi, \langle \cdot, \cdot \rangle_H) = T^{(2)}_{an}(M) \cdot \text{Vol}(M)^{-\dim H^0(\nabla)}, \] (2.4)

is metric independent [19], i.e. the metric dependence of the Ray-Singer torsion \( T^{(2)}_{an}(M) \) factors out as \( V(M)^{-\dim H^0(\nabla)} \).

As far as the zeta-function \( \zeta(0,|\mathcal{O}|) \) is concerned, we recall that there exists \( \epsilon, \delta > 0 \) such that for \( 0 < t < \delta \) the heat kernel expansion for self-adjoint Laplace operators \( \mathcal{L}_p \) is given by

\[ \text{Tr} \left( e^{-t\mathcal{L}_p} \right) = \sum_{0 \leq t < \ell_0} a_t(\mathcal{L}_p)t^{-l} + O(t^\epsilon). \] (2.5)

Starting with the formula [18]

\[ \zeta(0, \mathcal{L}_p) = a_0(\mathcal{L}_p) - \dim(\ker(\mathcal{L}_p)) = a_0(\mathcal{L}_p) - \dim H^p(R(S)), \] (2.6)

one can shown that the zeta function \( \zeta(s,|\mathcal{O}|) \) is well-defined and analytic for \( \Re(s) > 0 \) and can be continued to a meromorphic function on \( \mathbb{C} \), regular at \( s = 0 \) and

\[ \zeta(0,|\mathcal{O}|) = \sum_{p=0} (-1)^p (a_0(\mathcal{L}_p) - \dim H^p(R(S))). \] (2.7)

Furthermore, the zeta function \( \zeta(0,|\mathcal{O}|) \) appearing in the partition function (2.3) can be expressed in terms of the dimensions of the cohomology spaces of \( \mathcal{O} \). Indeed, if the dimension of \( M \) is odd \( (n = 2m+1) \) then for all \( p \) \( a_0(\mathcal{L}_p) = 0 \), because we are dealing with manifold without boundary. Since \( H^p(R(S\mathcal{O})) = H^{m-p}(\nabla) \) (the Poincaré duality) for the resolvent (2.2), it follows that

\[ \zeta(0,|\mathcal{O}|) = -\sum_{p=0}^m (-1)^p \dim H^p(R(S)) = (-1)^{m+1} \sum_{p=0}^m (-1)^p \dim H^p(\nabla). \] (2.8)

Finally, the dependence of the eta invariant \( \eta(0|\mathcal{O}) \) of Atiyah-Patodi-Singer on the connection map \( \mathcal{O} \) can be expressed with the help of the formula for the index of the twisted signature operator for a certain
vectorbundle over $M \otimes [0,1]$ (see [20, 21, 22]). Furthermore it can be shown [18] that

$$\eta(s | B) = 2\eta(s | \mathcal{O}),$$  \hspace{1cm} (2.9)

where the $B$ are elliptic self-adjoint maps on $\Omega(M, \xi)$ defined on $p$-forms by

$$B_p = (-i)^{\lambda(p)} \left( * \nabla + (-1)^{p+1} \nabla * \right),$$  \hspace{1cm} (2.10)

In this formula $\lambda(p) = (p+1)(p+2)+m+1$ and for the Hodge star-map we have used $\ast \alpha \wedge \beta = \langle \alpha, \beta \rangle_{\text{vol}}$. From the Hodge theory we have

$$\dim \ker B = \sum_{p=0}^{m} \dim H^p(\nabla).$$  \hspace{1cm} (2.11)

3. THE CASE OF REAL COMPACT HYPERBOLIC MANIFOLDS

In this section, we shall consider the specific case of a compact hyperbolic 3-manifolds of the form $M = X_\Gamma = \Gamma \backslash \mathbb{H}^3$. If the flat bundle, $\xi$ is acyclic, then for $L^2-$ torsion one gets [23]: $[T_{an}^{(2)}(X_\Gamma)]^2 = \mathcal{R}_\chi(0)$, where $\mathcal{R}_\chi(s)\)$ is the Ruelle function. The function $\mathcal{R}_\chi(s)$ is an alternating product of more complicate factors, each of which is a Selberg zeta function $Z_p(s; \chi)$. The relation of Ruelle and Selberg functions is:

$$\mathcal{R}_\chi(s) = \prod_{p=0}^{\dim M-1} Z_p(p+s; \chi)(-1)^j.$$  \hspace{1cm} (3.1)

The function $\mathcal{R}_\chi(s)$ extends meromorphically to the entire complex plane $\mathbb{C}$ [24]. The Ruelle function associated with closed oriented hyperbolic 3-manifold $X_\Gamma = \Gamma \backslash \mathbb{H}^3$ has the form $\mathcal{R}_\chi(s) = Z_0(s; \chi)Z_2(2+s; \chi)/Z_1(1+s; \chi)$. The analytic torsion for manifold $X_\Gamma$ has been calculated (in the presence of non-vanishing Betti numbers $b_i \equiv b_i(X_\Gamma) = \text{rank}_\mathbb{Z}H_i(X_\Gamma; \mathbb{Z})$) in Refs. [13, 14].

Now we consider the evaluation of eta invariant contribution. With regard to this point, a remarkable formula relating $\eta(s, \mathcal{O})$ to the closed geodesics on $X_\Gamma$ has been obtained by Millson [25]. More explicitly, Millson has proved the following result for a Selberg type (Shintani) zeta function $\tilde{Z}(s, \mathcal{O})$.

Let us define a zeta function by the following series, which is absolutely convergent for $\Re(s) > 0$,

$$\log \tilde{Z}(s, \mathcal{O}) \overset{\text{def}}{=} \sum_{|\gamma| \neq 1} \frac{\text{Tr} \tau_\gamma^+ - \text{Tr} \tau_\gamma^-}{|\det(I - P_\mathcal{B}(\gamma))|^{1/2}} \frac{e^{-s\ell(\gamma)}}{m(\gamma)},$$  \hspace{1cm} (3.2)
where \([\gamma]\) runs over the nontrivial conjugacy classes in \(\Gamma = \pi_1(X_\Gamma)\), \(\ell(\gamma)\) is the length of the closed geodesic \(c_\gamma\) (with multiplicity \(m(\gamma)\)) in the free homotopy class corresponding to \([\gamma]\). \(P_\gamma(\gamma)\) is the restriction of the linear Poincaré map \(P(\gamma) = d\Phi_t\) at \((c_\gamma, \dot{c}_\gamma) \in TX_\Gamma\) to the directions normal to the geodesic flow \(\Phi_t\) and \(\tau^\pm_\gamma\) is the parallel translation around \(c_\gamma\) on \(\Lambda^\pm_\gamma = \pm i\) eigenspace of \(\sigma_B(\dot{c}_\gamma)\) (\(\sigma_B\) denoting the principal symbol of \(\mathcal{O}\)). Then \(\tilde{Z}(z, \mathcal{O})\) admits a meromorphic continuation to the entire complex plane, which in particular is holomorphic at \(s = 0\) and

\[
\log \tilde{Z}(0, \mathcal{O}) = \pi i \eta(0, \mathcal{O}). \tag{3.3}
\]

Furthermore, it is possible to show that \(\tilde{Z}(s, \mathcal{O})\) satisfies the functional equation

\[
\tilde{Z}(s, \mathcal{O})\tilde{Z}(-s, \mathcal{O}) = e^{2\pi i \eta(0, \mathcal{O})}. \tag{3.4}
\]

Now we have all the ingredients for the evaluation of the partition function (2.3) in terms of \(L^2-\)analytic torsion and a Selberg type function. The final result is

\[
\mathcal{W}(k) = \left(\frac{\pi}{k}\right)^{\zeta(0,\mathcal{O})/2} \tilde{Z}(0, \mathcal{O})^{-1/4} \left[T_{an}^{(2)}(X_\Gamma)\right]^{1/2} [\text{Vol}(\Gamma\backslash G)]^{-\dim H^0(\nabla)/2}, \tag{3.5}
\]

where \(\zeta(0, \mathcal{O})\) and \(\tilde{Z}(0, \mathcal{O})\) are given by Eqs. (2.8) and (3.2) respectively.

4. Concluding remarks

For a real compact hyperbolic 3-manifold, the formula (3.5) gives the value of the asymptotics of the Chern-Simons-Witten invariant. This is the main result of our paper. The invariant (3.5) involves the \(L^2-\)analytic torsion, which can be expressed by means of Selberg zeta functions and a Shintani zeta function \(\tilde{Z}(0, \mathcal{O})\) associated with the eta invariant of Atiyah-Patodi-Singer [20]. Finally we note that the explicit result (3.5) can be very important for investigating the relation between quantum invariants for an oriented 3-manifold, defined with the help of a representation theory of quantum groups [3, 4], and Witten’s invariant [1], which is, instead, related to the path integral approach.

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