Turbulence modeling based on non-Newtonian constitutive laws

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Abstract. This work revisits the analogy between Newtonian turbulence and non-Newtonian laminar flows. Several direct numerical simulations (DNS) data of a plane channel flow, for a large range of Reynolds numbers \(180 \leq Re_\tau \leq 2000\) were explored. The profiles of mean velocity and second moment quantities were used to extract viscometric functions in the non-Newtonian modeling framework. The Reynolds stress tensor is expressed in terms of a set of basis kinematic tensors based on a projection of a nonlinear framework. The coefficients of the model are given as functions of the intensity of the mean strain tensor. The apparent eddy turbulent viscosity, the first and second normal stress differences are presented as function of the shear rate. One of the advantages of the new algebraic nonlinear power law constitutive equation derived in the paper, is that is only dependent on the mean velocity gradient and can be integrated up to the wall.

1. Introduction

The analogy between the turbulent Reynolds stress tensor for Newtonian fluids and the constitutive laws for viscoelastic flows is explored in this paper. Rivlin (1957) was probably the first one to qualitatively investigate the relation between the laminar flows of non-Newtonian fluids and the turbulent flows of Newtonian fluids. Later, such similarities were explored in several other papers Townsend (1966); Crow (1968); Groisman & Steinberg (2000).

The purpose of this work is to propose an approach, alternative to the traditional \(k-\epsilon\) one, to capture the coefficients of the non-linear model expressed by a three-tensor basis that generalizes the Boussinesq hypothesis. Based on the analogy with non-Newtonian fluids, these coefficients are related to the viscosity, first and second normal stress coefficients, which are usual material functions obtained from shear flows of viscoelastic fluids.

We have developed a way to obtain functions from direct numerical simulation (DNS) data to represent the coefficients of non linear eddy viscosity turbulent models. This approach is based on the tensor basis representation of the Reynolds stress. The aim is to go beyond the Boussinesq hypothesis and find, beside the turbulent apparent viscosity, turbulent first and second normal stress functions.
2. Non-linear constitutive equation

Let us denote \( R_{ij} = \frac{2}{3} \rho \delta_{ij} k - \rho u_i u_j \) as the anisotropic traceless stress tensor. \( \rho u_i u_j \) is the Reynolds stress tensor, \( k \) is the turbulent kinetic energy \( (k = \frac{1}{2} \rho u_i u_i) \) and \( \delta_{ij} \) the Kronecker delta function. A well-known expression for a non linear eddy viscosity, using traceless basis tensors, was first proposed by Pope (1975). In the 2-D framework, the anisotropic Reynolds stress tensor \( R \) can be written using three tensor bases, Jongen & Gatski (1998), as follows,

\[
\frac{R}{\rho} = 2 \nu_T S - \beta (SW - WS) - \gamma (S^2 - \frac{1}{3} \{S^2\} I)
\]

where \( S \) and \( W \) are respectively the mean rate of strain tensor and the mean vorticity tensor.

The coefficients \( \nu_T, \beta \) and \( \gamma \) may be written as functions of the basic invariants of the flow Schmitt (2007),

\[
\nu_T = \frac{\{RS\}}{\{S^2\}} = -\frac{\bar{uv}}{a},
\]

\[
\beta = \frac{\{RSW\}}{\{S^2\}\{W^2\}} = \frac{\bar{uu} - \bar{vv}}{a^2},
\]

\[
\gamma = -6 \frac{\{RS^2\}}{\{S^2\}^2} = \frac{6}{a^2} \frac{2}{3} (k - \bar{ww}),
\]

where the symbol \( \{\} \) represents the trace operator, \( \bar{uv}, \bar{uu}, \bar{vv} \) and \( \bar{ww} \), the shear and the three normal Reynolds stress components.

The variable \( a \) is the velocity gradient that is only function of the \( y \) direction, as it is assumed here that the flow is dominate by shear, \( \partial U/\partial y = a(y) \). In this case, the mean rate of strain tensor \( S \) and the mean vorticity tensor \( W \) can be written as:

\[
S = \begin{pmatrix} 0 & \frac{1}{2} a & 0 \\ \frac{1}{2} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & -\frac{1}{2} a & 0 \\ \frac{1}{2} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

3. Viscometric functions

The mechanical properties of the flow are fully determined when three functions are known, \( \tau(a), \ N_1(a), \ N_2(a) \), shear, first and second normal stress difference, respectively.

It can be shown that the coefficients of equation (1) can be written as Qiu et al. (2010):

\[
\nu_T(a) = -\frac{\bar{uv}}{a} = \frac{\tau(a)}{\rho a} = -\frac{\eta(a)}{\rho},
\]

\[
\beta(a) = \frac{\bar{uu} - \bar{vv}}{a^2} = \frac{N_1(a)}{\rho a^2} = \frac{\Psi_1(a)}{\rho},
\]

\[
\gamma(a) = \frac{6}{a^2} \frac{2}{3} (k - \bar{ww}) = \frac{2}{\rho} [\Psi_1(a) - 2 \Psi_2(a)].
\]

For an unknown viscometric flow, these functions are experimentally estimated and some general properties of the flow are inferred. For Newtonian flows \( \tau(a)/a \) is constant, so if this ratio depends on \( a \), the flow possesses non-Newtonian (shear thinning or shear thickening) characteristics.
Re-writting the equation (1) using the total shear stress given by $R_{\text{total}} = R_v + R$:

$$\frac{R_{\text{total}}}{\rho} = 2(\nu + \nu_T)S - \beta(SW - WS) - \gamma(S^2 - \frac{1}{3}(S^2)I),$$

where the viscous contribution for the total stress is $R_v/\rho = 2\nu S$.

The quadratic constitutive equation for the total stress can be written in terms of the viscometric functions as

$$\frac{R_{\text{total}}}{\rho} = 2(\nu + \nu_T)S - \frac{\Psi_1(a)}{\rho}T_2 - \frac{2}{\rho}\left[\Psi_1(a) - 2\Psi_2(a)\right]T_3,$$

where

$$T_2 = SW - WS, \quad T_3 = S^2 - \frac{1}{3}(S^2)I.$$ (11)

4. Databases and turbulent quantities

We consider here several DNS databases of shear flows, characterized by a range of Reynolds numbers $Re_\tau$ from 180 to 2000, Kim et al. (1987); Moser et al. (1999); Iwamoto et al. (2005); Makino et al. (2008); Hoyas & Jimenez (2006, 2008). These databases correspond to plane channel flow, Poiseuille flow and channel flow, see table 1.

| Case | $Re_\tau$ | Reference | Description |
|------|-----------|-----------|-------------|
| 1    | 180       | Kim, Moin and Moser 1987. | Plane channel flow |
| 2    | 395       | Moser, Kim and Mansour 1999. | Plane channel flow |
| 3    | 590       | Moser, Kim and Mansour 1999. | Plane channel flow |
| 4    | 640       | Iwamoto, Suzuki and Kasagi 2005 | Plane channel flow |
| 5    | 950       | Hoyas and Jimenez 2008. | Channel flow |
| 6    | 1020      | Makino, Iwamoto and Kawamura 2006 | Poiseuille flow |
| 7    | 2000      | Hoyas and Jimenez 2006. | Channel flow |

Considering now the power law slopes shown in the previous results, we present below the basis to build the model. Theses results lead us to recall one of the Non-Newtonian constitutive equations, the Carreau model which is expressed by following equation Bird et al. (1987):

$$\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = [1 + (\lambda_\alpha a)^2]^{\frac{n-1}{2}},$$

where $\eta_0$ is the zero shear viscosity, $\eta_\infty$ is the limiting viscosity at high shear rates (supposed to be zero here), $n$ the power law index and $\lambda_\alpha$ a time constant.

Here we use this power law approach to build the model for the viscometric functions.

In order to use the DNS data base to derive the material functions, the following non-dimensionalisation is introduced:

$$y^+ = \frac{y}{y_0}, \quad U^+ = \frac{U}{u_\tau}, \quad \tau^+ = \frac{\tau}{\rho u_\tau^2}, \quad a^+ = \frac{dU^+}{d(y/\delta)} = a\frac{\delta}{u_\tau}.$$ (13)
where \( u_r = \sqrt{\tau_0/\rho} \) is the characteristic velocity scale, \( y_0 = \nu/u_r \) the characteristic length scale, and \( \delta \) is the half width of the channel.

Using the DNS data presented in Fig. 1 to obtain \( \nu_{\text{apparent}} \), we can write that:

\[
\nu_{\text{apparent}}(a^+) = \nu_0(1 + (\lambda a^+)^2)^{n_0-1},
\]

where

\[
\nu_0 = 0.0655, \quad n_0 = 0.0064, \quad \lambda = 0.0678.
\]

The comparison between the DNS result for the large Reynolds number and the model fit for the apparent viscosity is shown in Fig. 3.

For the first viscometric function \( \Psi_1^+ \), since there are two slopes (see Fig. 2-a), one for the small \( a^+ \) region and one for the large \( a^+ \) region, the following expression to fit \( \Psi_1^+ \) is used:

\[
\Psi_1^+(a^+) = \psi_1 \exp(-\lambda_1 a^+) + \psi_2 \exp(-\lambda_2 a^+),
\]

\[
(14)
\]

For small \( a^+ (a^+ < 4) \), \( \Psi_1^+ \) tends to be \( \psi_1 a^{(m-1)} \), and for larger \( a^+ \), \( \Psi_1^+ \) tends to be \( \psi_2 a^{(n-1)} \). Based on the two slopes (see Fig. 2-a) the coefficients of the model can be obtained as below:

\[
\begin{align*}
\psi_1 & = 0.3501, \quad \psi_2 = 1.0957, \quad \lambda_1 = 0.2, \quad \lambda_2 = 0.11, \\
m_1 = -1.9836, \quad n_1 = -1.6854.
\end{align*}
\]

(17)

The comparison between the DNS results and the model fit for the first normal stress difference is shown in Fig. 4 (a).

For the viscometric function \( \Psi_2^+ \), also using the same previous approach for \( \Psi_1 \), we get:

\[
\Psi_2^+(a^+) = \phi_1 \exp(-\lambda d_1 a^+) + \phi_2 \exp(-\lambda d_2 a^+),
\]

\[
(18)
\]

For small \( a^+ (a^+ < 4) \), \( \Psi_2^+ \) tends to be \( \phi_1 a^{(m-1)} \), and for larger \( a^+ \), \( \Psi_2^+ \) tends to be \( \phi_2 a^{(n-1)} \). Based on the two slopes, we have the following coefficients:

\[
\begin{align*}
\phi_1 & = 0.0117, \quad \phi_2 = 0.2605, \quad \lambda d_1 = 1, \quad \lambda d_2 = 0.05, \\
m_2 = -1.9434, \quad n_2 = -1.6760.
\end{align*}
\]

(19)

The comparison between the DNS results and the model fit for the second normal stress difference is shown in Fig. 4 (b). Figures 3 and 4(a-b) shows that the proposed model is quite close to DNS data for the largest Reynolds number case.

The model in dimensionless form is given by the following explicit algebraic equation:

\[
R_{\text{total}}^+ = 2\nu_{\text{apparent}}^+(a^+)S^+(a^+) - \Psi_1^+(a^+)T_2^+(a^+) - 2\Psi_1^+(a^+)T_2^+(a^+) - 2\Psi_1^+(a^+)T_3^+(a^+),
\]

\[
(20)
\]

where

\[
\begin{align*}
T_2^+ & = S^+ \textbf{W}^+ - \textbf{W}^+ S^+, \quad T_3^+ = S^{2+} - \frac{1}{3} \{S^{2+}\} \textbf{I}.
\end{align*}
\]

(21)

are the basis.

For this model, non-Newtonian material functions were introduced to derive the parameters \( \nu_{\text{apparent}} \), \( \beta \) and \( \gamma \), instead of using the turbulent kinetic energy and dissipation rate scales in classical turbulent models, as \( k - \epsilon \) model.

An important advantage of the resulted formulation is that the coefficients can be used to reproduce nearly viscometric flows, and do not need the solution of evolution equations for \( k \) and for \( \epsilon \) as in the classical approach.
Figure 1. Profiles apparent viscosity $\nu_{\text{apparent}}$ vs. shear rate $a^+$. 

Figure 2. Viscometric functions (a) $\Psi_1^+(a^+)$ and (b) $\Psi_2^+(a^+)$ vs. shear rate $a^+$. 

Figure 3. Apparent viscosity vs. shear rate ($Re_\tau = 2000$).

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