DYNAMICAL SYSTEMS DEFINING JACOBI’S $\vartheta$-CONSTANTS

YURI V. BREZHNEV, SIMON L. LYAKHOVICH, AND ALEXEY A. SHARAPOV

Abstract. We propose a system of equations that defines Weierstrass–Jacobi’s eta- and theta-constant series in a differentially closed way. This system is shown to have a direct relationship to a little-known dynamical system obtained by Jacobi. The classically known differential equations by Darboux–Halphen, Chazy, and Ramanujan are the differential consequences or reductions of these systems. The proposed system is shown to admit the Lagrangian, Hamiltonian, and Nambu formulations. We explicitly construct a pencil of nonlinear Poisson brackets and complete set of involutive conserved quantities. As byproducts of the theory, we exemplify conserved quantities for the Ramanani dynamical system and quadratic system of Halphen–Brioschi.

Contents

1. Introduction 2
1.1. Motivation for the work 3
1.2. The paper content 4
2. ODEs defining $\vartheta$-constants 5
2.1. On symmetrical system (8) 5
2.2. An integrable modification of system (8) 7
3. Explicit solutions and technicalities 8
3.1. Associated linear ODEs 9
3.2. Solution to the Jacobi system 10
3.3. Solution to system (19) 11
4. Derivation of integrals 12
5. Integrals and Lagrangian 13
5.1. Conserved quantities 13
5.2. Action and Lagrangians 15
6. Poisson structures 17
7. A generalization 20
8. Appendix: The Jacobi system 22
References 22

Key words and phrases. Jacobi’s theta-constants, Darboux–Halphen and Chazy differential equations, Lagrangian, Hamiltonian, conserved quantities, Poisson–Nambu structures, hypergeometric functions.

The research was supported by the Federal Targeted Program under state contracts 02.740.11.0238, #P1337 and #P22. The work by SLL and AASh was supported by the RFBR grant 09–02–00723-a and SLL had a partial support from the RFBR grant 08–01–00737-a. SLL and AASh appreciate the hospitality of the Erwin Schroedinger Institute for Mathematical Physics, Vienna.
1. Introduction

In this work we propose a description of the classical Jacobi’s $\vartheta$-constants and Weierstrass’ $\eta$-function by means of closed and Lagrangian/Hamiltonian ordinary differential equations (ODEs). By simple transformations or reductions these equations lead to many well-known differential systems. Among these are the Darboux–Halphen system [11, 16], some its modifications [25, 3], the Chazy equation [10], and also a Jacobi system of ODEs [18] which has not received mention in the modern literature in the context. For both Jacobi’s system and equations defining the $\vartheta, \eta$-series we work out the Hamiltonian formalism and show that they admit a pencil of (compatible) Poisson structures in the sense of Magri [20] and formulation as the generalized Nambu mechanics [22] with a certain 4-bracket.

The three Jacobi’s theta-constants are defined by the classical series

$\vartheta_2(\tau) := \sum_{k=-\infty}^{\infty} e^{(k^2+k)\pi i \tau}$,
$\vartheta_3(\tau) := \sum_{k=-\infty}^{\infty} e^{k^2\pi i \tau}$,
$\vartheta_4(\tau) := \sum_{k=-\infty}^{\infty} (-1)^k e^{k^2\pi i \tau}$

and the Weierstrass $\eta$-function is defined by the series

$\eta(\tau) := 2\pi i \sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{2k\pi i \tau}}{1 - e^{2k\pi i \tau}}$.

Here, the ‘time’ $\tau$ is considered to be a complex variable belonging to the upper half-plane $\mathbb{H}^+: \Im(\tau) > 0$. These series appear in various problems of theoretical physics because of their numerous and deep differential properties [3, 10]. Let us mention some of them.

Three $\vartheta$-constant series satisfy the following differential identities for logarithmic derivatives of their ratios:

$\frac{d}{d\tau} \ln \frac{\vartheta_2}{\vartheta_3} = \frac{\pi}{4} i \vartheta_4$,
$\frac{d}{d\tau} \ln \frac{\vartheta_3}{\vartheta_4} = \frac{\pi}{4} i \vartheta_2$,
$\frac{d}{d\tau} \ln \frac{\vartheta_2}{\vartheta_4} = \frac{\pi}{4} i \vartheta_3$.

Yet another and very well-known identity is the sum of logarithmic derivatives:

$\dot{\vartheta}_3 + \dot{\vartheta}_4 = \frac{3i}{\pi} \eta$.

(dot stands for the $\tau$-derivative). If we introduce a notation for these derivatives, say

$$(X, Y, Z) := 2 \left( \frac{\dot{\vartheta}_2}{\vartheta_2}, \frac{\dot{\vartheta}_3}{\vartheta_3}, \frac{\dot{\vartheta}_4}{\vartheta_4} \right),$$

then the quantities $(X, Y, Z)$ satisfy the 3rd order differential system

$$\dot{X} = (Y + Z)X - YZ, \quad \dot{Y} = (X + Z)Y - XZ, \quad \dot{Z} = (X + Y)Z - XY,$$

which is widely known as the Halphen system [16, p. 330–331]. This system is frequently named as the Darboux–Halphen system though Darboux himself wrote down only differentials [11, p. 149]:

$$C(dA + dB) = B(dA + dC) = A(dB + dC).$$

These can be written in the form

$$\frac{dA}{A(B + C) - BC} = \frac{dB}{B(A + C) - AC} = \frac{dC}{C(A + B) - AB} = dt$$

which is equivalent to the system (2).

Remarkable applications of Eqs. (2) were initiated in the 1990’s by M. Ablowitz et al [7, 1] in connection with reductions of the self-dual Yang–Mills equations. These equations
usually provide the main physical motivation for studying both the \( \eta \), \( \vartheta \)-series and allied modular objects. However, applications go beyond the Yang–Mills theory. In succeeding years the system appeared in the vacuum cosmological Bianchi–IX model [10, p. 143, 147], [3, p. 577], [1], theory of 2-monopole moduli spaces [5], and many other areas of mathematical physics [19]. System (2) has also varieties. One of them is the Weierstrass–Halphen dynamical system for Weierstrass’ invariants \( g_2 \), \( g_3 \), and \( \eta \)-series:

\[
\frac{dg_2}{d\tau} = \frac{i}{\pi} \left( 8g_2\eta - 12g_3 \right), \quad \frac{dg_3}{d\tau} = \frac{i}{\pi} \left( 12g_3\eta - \frac{2}{3}g_2^3 \right), \quad \frac{d\eta}{d\tau} = \frac{i}{\pi} \left( 2\eta^2 - \frac{1}{6}g_2 \right).
\]

It is known that these invariants

\[
g_2(\tau) := 60 \sum_{n,m} \frac{1}{(2mn + 2n)^4}, \quad g_3(\tau) := 140 \sum_{n,m} \frac{1}{(2mn + 2n)^6}, \quad ((n, m) \neq (0, 0))
\]

are related to the \( \vartheta \)-series by the standard polynomial formulae

\[
g_2(\tau) = \frac{\pi^4}{24} \left\{ \vartheta_2^8(\tau) + \vartheta_3^8(\tau) + \vartheta_4^8(\tau) \right\},
\]

\[
g_3(\tau) = \frac{\pi^6}{432} \left\{ \vartheta_2^4(\tau) + \vartheta_3^4(\tau) \right\} \left\{ \vartheta_2^4(\tau) + \vartheta_3^4(\tau) \right\} \left\{ \vartheta_4^4(\tau) - \vartheta_2^4(\tau) \right\}
\]

and the series themselves satisfy the well-known Jacobi identity

\[
\vartheta_4^4(\tau) = \vartheta_2^4(\tau) + \vartheta_3^4(\tau).
\]

In different notation and (number-theoretic) definition for function series, system (3) is known as the Ramanujan system of differential equations [25, 9] for modular forms

\[
E_2(\tau) = \frac{12}{\pi^2} \eta(\tau), \quad E_4(\tau) = \frac{12}{\pi^4} g_2(\tau), \quad E_6(\tau) = \frac{216}{\pi^6} g_3(\tau).
\]

Ramanujan’s system is sometimes referred to as the Eisenstein system of differential equations [9], though Eisenstein himself had not derived it [12]. Further discussions of and bibliography to the systems mentioned above can be found in [1, 3, 4, 7, 10, 13] and references therein.

1.1. Motivation for the work. Dynamical variables for all the systems above are rationally expressed through the \( \vartheta \)-variables. Therefore, the inverse transformations will involve the multi-valued functions, as further examples show. Recently Ablowitz, Chakravarty, and Hahn [4] called attention to yet another instance which is more interesting and comes from the equations for modular forms on group \( \Gamma_0(2) \). This is the Ramamani system [24] (Sect. 5.1.1); it was also considered in works [21, 13]. In this case, the relation between dynamical variables and the \( \vartheta, \eta \)-variables is not obvious because it is given by a duplication of the \( \tau \)-argument in forms (6) [4, 13]. If we make use of the duplication rules

\[
\eta(2\tau) = \frac{1}{2} \eta(\tau) + \frac{\pi^2}{48} \left\{ \vartheta_2^4(\tau) + \vartheta_3^4(\tau) \right\}, \quad g_2(2\tau) = -\frac{1}{4} g_2(\tau) + \frac{5\pi^4}{192} \left\{ \vartheta_2^4(\tau) + \vartheta_3^4(\tau) \right\}^2
\]

we arrive again at substitutions of a rational type (see Sect. 5.1.1 for further details). Though these rules have not appeared explicitly in the literature known to us, they can be established by standard techniques. In a \( q \)-series notation these identities can be found in Ramanujan’s notebooks and (in number-theoretic notation) implicitly have been tabulated in [21, Table 1]. Apart from inversions of that kind substitutions one should mention the fact that 3-dimensional systems, e.g., (3), present generically subsystems or reductions of the 4-dimensional ones because differentiations intertwine equally all the four objects \( \vartheta_k \).
and $\eta$. In particular, we display here a version of relations which, besides their symmetrical form, close differentially these objects.

**Proposition 1.** The canonical Jacobi’s $\vartheta$-constant series satisfy the closed differential identities upon adjoining the Weierstrass $\eta$-series:

\[
\frac{d\vartheta_2}{d\tau} = \frac{i}{\pi} \left\{ \eta + \frac{\pi^2}{12} (\vartheta_2^4 + \vartheta_4^4) \right\} \vartheta_2, \quad \frac{d\vartheta_3}{d\tau} = \frac{i}{\pi} \left\{ \eta - \frac{\pi^2}{12} (\vartheta_2^4 + \vartheta_4^4) \right\} \vartheta_3, \quad \frac{d\vartheta_4}{d\tau} = \frac{i}{\pi} \left\{ \eta - \frac{\pi^2}{12} (\vartheta_2^4 + \vartheta_4^4) \right\} \vartheta_4.
\]

(8)

Derivation of these formulae uses computation of the theta-derivatives through the derivatives of $g_2$, $g_3$, that is Eqs. (3) and (4), followed by applying the symmetrical identity (5).

To all appearances, these identities, direct consequences of standard relations as they are, do not appear explicitly in so extensive literature on theta-functions though we found first three of them in Appendix A to monograph [19].

On the other hand, considering (8) as a dynamical system, its integration, as we shall see, encounters serious difficulties. Moreover, system (2), under the definition (1), is not a consequence of Eqs. (8) but holds only upon restricting to the constant level surface (5). To put it differently, the modes of embedding functions $\vartheta, \eta$ into differential systems are not unique and Eqs. (8) require modifications. All this will be the subject matter of further consideration.

For the reasons given above it is essential to have a comprehensive description for differential properties of the canonical $\vartheta, \eta$-series as such. In particular, it is of interest to find a Lagrangian and Hamiltonian formulation for these systems. This would provide fresh insight into properties of the theta-constants. This circle of questions, as applied to an equivalent of the system (3), is addressed in the work [9] by D. Chudnovsky & G. Chudnovsky and, to the best of our knowledge, this is the first paper\(^1\) where the problem of Hamiltonian treatment for dynamical systems of modular type was raised. These authors proposed a 4th order differential system [9] and its reduction to equations of the order 3. Although their Hamilton function is rather ingenious and correct, the proposed reduction for system (3) is not preserved; in notation of work [9] on p. 111 the reduction is defined by the constraint $\lambda = 1$. In other words, this reduction is satisfied only by a trivial solution. It should also be noticed that system (3) and its varieties have a nice interpretation as a halfway between integrable systems and the ones with a chaotic behavior. However, we do not touch on this kind of problems here because questions of dynamics and transitions from ‘exactly solvable but not completely integrable’ flows to the ergodic ones are the main subject of the work [9]. In the same place the extended bibliography is given.

1.2. The paper content. The subsequent material is organized as follows. In the next section (Sect. 2) we discuss the correlation between identities like (8) and a little-known Jacobi dynamical system for Legendre’s complete elliptic integrals. We give explanations as to why the change from differential $\vartheta, \eta$-identities to those that should be thought of as defining ODEs is not a trivial question. We write down the simplest version of such (‘integrable’)

\(^1\)See also important comments on p. 5709–5711 of [14] concerning the Darboux–Halphen system (2) and its relation to Euler’s equations and the Lotka–Volterra system; in the same place the detailed comments on Poisson structures for the 3-dimensional system (2) are presented. In work [8] their construction by means of multivalued integrals is discussed and a generalization on Nambu’s brackets is proposed.
ODEs. Section 3 is technical; it is devoted to explicit integration of system (19) and Jacobi’s system (9). Method of solution invokes standard Legendre’s and modular techniques and we present results in both of these forms: the ‘linear’ $k$- and ‘nonlinear’ (modular) $\tau$-representation. In Sect. 4 we explain how these techniques can be exploited in order to derive the transcendental multivalued integrals. In Sect. 5 we exhibit explicitly these objects for all of the systems mentioned above and use them when constructing Lagrangians and the action functional. Complete Hamiltonian formulation to the systems under study is expounded in Sect. 6. The found Poisson structures turn out to be non-obvious (none are simplectic) and may form compatible pencils; we also describe the genesis of a rational degenerate Poisson bracket from a Nambu 4-bracket and possible transitions between various Poisson brackets. Section 7 contains a generalization; we complete the theory for the Halphen–Brioschi quadratic ODEs. The last section 8 (Appendix) contains some historical remarks on Jacobi’s system.

2. ODEs defining $\vartheta$-constants

2.1. On symmetrical system (8). As we mentioned above all the varieties of dynamical systems under consideration are algebraically related to each other. In this respect equations (8) stand out because this system alone represents the $\eta, \vartheta$-constants. However, point transformations between dynamical variables are not unique and resulting ODEs for $\vartheta, \eta$-variables may contain parameters. In connection with this ambiguity it is of interest to consider an elegant dynamical system which was derived by Jacobi. In Jacobi’s record [18] it is as follows:

$$\begin{align*}
\frac{\partial A}{\partial h} &= 2A^2B, \\
\frac{\partial B}{\partial h} &= bA^3, \\
\frac{\partial a}{\partial h} &= -\frac{16bA^2}{b^2}, \\
\frac{\partial b}{\partial h} &= abA^2,
\end{align*}$$

where $h = \frac{1}{4}\pi i \tau$ and the restriction

$$a^2 = 16(1 - 2b)$$

is assumed to be imposed. All the information concerning this system (including solution) has been detailed in the next section and Appendix (Sect. 8) contains additional comments on original motivation of Jacobi. Jacobi deduced Eqs. (9) as a set of differential identities between classical objects of Legendre’s ‘elliptic theory’ [18, 16, 27]:

$$K(k) = \int_0^1 \frac{d\lambda}{\sqrt{(1 - \lambda^2)(1 - k^2\lambda^2)}}, \quad K'(k) = \int_k^1 \frac{d\lambda}{\sqrt{(1 - \lambda^2)(\lambda^2 - k^2)}},$$

$$E(k) = \int_0^1 \frac{1 - k^2\lambda^2}{1 - \lambda^2} d\lambda, \quad E'(k) = \int_0^1 \frac{1 - (1 - k^2)\lambda^2}{1 - \lambda^2} d\lambda.$$  

A simple computation, based on the $\vartheta, \eta$-representations of objects (11)–(12) appearing in Jacobi’s definition of variables $\{A, B, a, b\}$—this is Eqs. (59)—shows that

$$A = \vartheta_3, \quad B = \frac{4}{\pi^2 \vartheta_3} \left\{ \eta + \frac{\pi^2}{12} (\vartheta_2^4 - \vartheta_4^4) \right\}, \quad a = 4 - 8 \frac{\vartheta_4^4}{\vartheta_3^4}, \quad b = 2 \frac{\vartheta_2^4 \vartheta_4^4}{\vartheta_3^4 \vartheta_3^4}.$$
Now, if we drop out the fix constraint (10) and consider (13) just as a point change in
Eqs. (9), we shall not arrive at symmetrical system (8). We may also insert into the change
(13) some parameters, say
\[ a = 4 - \alpha \frac{\vartheta_2^4}{\vartheta_3^4}, \quad b = \beta \frac{\vartheta_2^4}{\vartheta_3^4} \frac{\vartheta_4^4}{\vartheta_3^4}, \] (14)
and yield different forms to resulting ODEs but we never get the system (8) in this way.
(Converse is of course also true: symmetrical system (8) does not entail Jacobi’s equations
(9)). Any of such ODEs will be integrable in terms of \( \vartheta, \eta \)-series since they were obtained
from (9) by coordinate changes of dynamical (phase) variables. The changes are generally
algebraic, i.e., multi-valued in both directions. In this respect Jacobi’s system (9) is not the
best variant because choice of the phase variables, in this case, would lead to replacing the
‘simple’ \( K(k) \) with the ‘cumbrous’ \( K\left( \sqrt{\frac{1}{2} - \frac{\alpha \vartheta_3^2}{\sqrt{\vartheta_3^2 + 12}} \right) \) (see Proposition 6 further below).

For this purpose, however, symmetrical form (8) is apt to be not a good candidate
because it is not amenable to integration and we failed to find out its complete integral.
That such a strong distinction between systems is inherent in the nature of the case (8)
will be apparent from the consideration of their algebraic integrals as algebraic curves in
homogeneous coordinates \( \vartheta_2 : \vartheta_3 : \vartheta_4 \).

**Proposition 2.** The identities (8), being considered as a dynamical system, have an alge-
braic integral \( U \) given by the following rational function of \( \vartheta \)’s:
\[ U \cdot \vartheta_2^4 \vartheta_3^4 \vartheta_4^4 = \left( \vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4 \right)^3. \] (15)
This integral generalizes Jacobi’s identity (5) if \( U \neq 0 \).

Turning now the restriction (10) into algebraic integral (see Eq. (21) in Sect. 3), we
observe that equation (21), under generalization (14), has genus 9, whereas integral (15)
is a curve of genus 19. The best we have succeed in solution of system (8) is its partial
resolution in terms of elliptic functions. In a nutshell, this procedure is as follows.

Let us change notation \( U \mapsto U^2 \) and rewrite integral (15) in form of the elliptic curve\textsuperscript{2}
\[ 2U^2 xy = (y - x - 2)^3, \quad x = 2 \frac{\vartheta_2^4}{\vartheta_4^4}, \quad y = 2 \frac{\vartheta_3^4}{\vartheta_4^4}. \]
Hence it follows that the pair \((x, y)\) is parametrized by Weierstrass’ \((\wp, \wp’)\)-functions and
this curve can be transformed into the canonical Weierstrassian form
\[ \wp’(u)^2 = 4\wp^3(u) - g_2 \wp(u) - g_3. \]
The computation is rather simple and we obtain
\[ x = \frac{1}{U} \wp’(u) - \wp(u) + \frac{U^2}{12} - 1, \quad y = \frac{1}{U} \wp’(u) + \wp(u) - \frac{U^2}{12} + 1, \] (16)
\textsuperscript{2}An analogous transformation to the fourth powers of \( \vartheta \)’s in integral (21) for Eqs. (9) leads to a zero genus curve and no elliptic functions appear in this case (see Sect. 3.2).
where constants $g_2$, $g_3$ are expressed through the integral $U$:

$$
g_2 = \frac{U^4}{12} - 2U^2, \quad g_3 = -\frac{U^6}{216} + \frac{U^4}{6} - U^2.
$$

Therefore formulae (16) substituted into Eqs. (8) must cause this system to become a $\tau$-evolution of the uniformizer $u = u(\tau)$. Indeed, after some algebra we derive that

$$
\frac{36U}{\pi^4} \cdot \frac{1}{\vartheta^3 d\tau} = 12 \varphi(u) - U^2
$$

and therefore

$$
\int \frac{du}{12 \varphi(u) - U^2} = \frac{\pi i}{36U} \int \vartheta^3 d\tau + \text{const}.
$$

The left hand side of this equation is easily integrated because

$$
\frac{12U}{12 \varphi(u) - U^2} = \zeta(u - \kappa) - \zeta(u + \kappa) + 2\zeta(\infty),
$$

where $12 \varphi(\infty) = U^2$, $\varphi'(\infty) = \pm U$, and $\zeta(u)$, $\sigma(u)$ are the standard Weierstrassian functions associated with the basis $\varphi(u)$, $\varphi'(u)$ [16, 27]. We get

$$
\frac{3}{\pi^4} \ln \left\{ \frac{\sigma(u - \kappa)}{\sigma(u + \kappa)} \right\} e^{2\zeta(\infty)u} = \int \vartheta^3 d\tau + \text{const}
$$

but integral in the right hand side requires further integration of the system. This last step is unknown.

If equations (8) are indeed non-integrable then situation is a manifestation of the mere fact that the differential identity for a function and differential equation are not one and the same. The function $u = -\frac{d}{d\varphi} \ln \left( \varphi^2 + \frac{1}{3} \right)$ solves the equation $u'' = 2u^3 + zu - 2$ whose general integral is, however, not representable in terms of any known functions or integrals of them; this is the 2nd Painlevé transcendent [10]. Here is a less trivial example. The function $u = \frac{d}{d\varphi} \ln \{ A\varphi(z) + an(B\varphi(z)) \}$ contains, like our $\vartheta$, $\eta$-solutions, special functions and a free constant; functions $\psi(z)$, $\psi(z)$ satisfy the Airy equation $\psi'' = z \psi$. Here, we again arrive at the $P_2$-transcendent $u'' = 2u^3 - 2zu + 1$.

### 2.2. An integrable modification of system (8).

Returning to the question of canonical representative for ODEs defining $\vartheta$, $\eta$-series, we choose the following modification of equations (8):

$$
\frac{d\vartheta_2}{d\tau} = \frac{i}{\pi} \left\{ \eta + \frac{\pi^2}{12} \left( \vartheta_3^4 + \vartheta_4^4 \right) \right\} \vartheta_2, \quad \frac{d\vartheta_4}{d\tau} = \frac{i}{\pi} \left\{ \eta - \frac{\pi^2}{12} \left( 2\vartheta_3^4 - \vartheta_4^4 \right) \right\} \vartheta_4, \quad \frac{d\vartheta_3}{d\tau} = \frac{i}{\pi} \left\{ \eta + \frac{\pi^2}{12} \left( \vartheta_3^4 - 2\vartheta_4^4 \right) \right\} \vartheta_3, \quad \frac{d\eta}{d\tau} = \frac{i}{\pi} \left\{ 2\eta^2 - \frac{\pi^4}{72} \left( \vartheta_3^4 - \vartheta_4^4 \right) \right\}.
$$

Comprehensive explanation as to why the ‘defining $\vartheta$, $\eta$-equations’ should have such a form has been detailed in work [6]. System (17) has even an (integrable) extension which is described in the same place. It is of interest to observe that all the previous dynamical systems contain in effect only squares of $\vartheta$-constants. For this reason, in the sequel it will be convenient to renormalize variables $\vartheta$, $\eta$ and adopt the following notation:

$$
x = \sqrt{\frac{\pi i}{6}} \vartheta_2, \quad y = \sqrt{\frac{\pi i}{6}} \vartheta_3, \quad z = \sqrt{\frac{\pi i}{6}} \vartheta_4, \quad u = \frac{2i}{\pi} \eta.
$$
Then Eqs. (17) acquire the form

\[
\begin{align*}
\dot{x} &= (u + y^2 + z^2)x, \\
\dot{y} &= (u + y^2 - 2z^2)y, \\
\dot{z} &= (u - 2y^2 + z^2)z, \\
\dot{u} &= u^2 - y^4 + y^2z^2 - z^4,
\end{align*}
\]

which, along with the Jacobi system (9), will be the main subject of further analysis. Apart from simplicity and the symmetry \( y \leftrightarrow \pm z \), there are some additional properties justifying the study of canonical system (19).

First of all, the function \( u \), independently of \((x, y, z)\), satisfies the famous Chazy equation

\[
\dddot{u} = 6(2\dddot{u} - 3 \dot{u}^2),
\]
(proof is a direct calculation) which cannot be said of \( \eta \)-solution to the symmetrical version (8). For the latter, the function \( 2i \pi \eta \) solves this equation only if the \( U \)-integral (15) is equal to zero (consequence of Proposition 2). Similarly, functions \( y \) and \( z \) also satisfy a third (not fourth) order ODE. This is the known Jacobi \( \mathcal{C} \)-equation (60) [18, p. 186]:

\[
\mathcal{C}^4(\ln \mathcal{C}^3 \mathcal{C}_{\tau \tau})_{\tau}^2 = 16 \mathcal{C}^3 \mathcal{C}_{\tau \tau} + 36, \quad \mathcal{C} = \frac{1}{y} \quad \text{or} \quad \frac{1}{z}.
\]

The above mentioned symmetry involves only functions \( y \) and \( z \) other than the function \( x \). Therefore general solution \( 1/x(\tau) \) does not satisfy this Jacobi’s equation. Equations (19) entail that functions \( x(\tau) \) and \( x(\tau) \) times a constant satisfy a common ODE. Hence, making the transformation \( C \mapsto \text{const} \cdot C \) in (20), one infers that the solution \( 1/x(\tau) \) satisfies equation (20) wherein 36 should be replaced by a free constant. One easily derives

\[
\tilde{C}^4(\ln \tilde{C}^3 \tilde{C}_{\tau \tau})_{\tau}^2 - 16 \tilde{C}^3 \tilde{C}_{\tau \tau} = \left( \frac{6}{x^2} \right)^2, \quad \tilde{C} := \frac{1}{x}
\]

but right hand side of this equation is a constant indeed. Explanation to this fact will be apparent from Sect. 3.3 wherein we give a complete integral to the system (19). Thus the function \( \tilde{C} \) satisfies the 4th order ODE

\[
\left( \tilde{C}^4(\ln \tilde{C}^3 \tilde{C}_{\tau \tau})_{\tau}^2 \right)_{\tau} = \left( 16 \tilde{C}^4 \tilde{C}_{\tau \tau} \right)_{\tau}
\]

which is checked by a straightforward substitution.

3. Explicit solutions and technicalities

At first, let us integrate Jacobi’s system. From (9) it follows that \( a \partial a = -16 \partial b \) and this equation yields an algebraic integral that replaces Jacobi’s restriction (10):

\[
I^2 = a^2 + 32b \quad \Rightarrow \quad \dot{I} \equiv 0.
\]

Therefore \( b \) is expressed via the function \( a \) which in turn satisfies a simple differential consequence of (9), namely, the 3rd order equation

\[
\frac{a_{hh}}{a_h} - \frac{3}{2} \frac{a_{hh}^2}{a_h} = -\frac{1}{2} \frac{a^2 + 3I^2}{(a^2 - I^2)^2}.
\]

This is a variety of the standard differential equation for Legendre’s modulus \( \lambda := k^2(\tau) \):

\[
\frac{\lambda_{\tau \tau \tau}}{\lambda_{\tau}} - \frac{3}{2} \frac{\lambda_{\tau}^2}{\lambda_{\tau}} = -\frac{1}{2} \frac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda - 1)^2}.
\]
It immediately follows that there is bound to be a linear fractional change between variables $a$ and $\lambda$ transforming Eq. (22) into equation for $a$ and vice versa. This simple computation gives

$$\lambda = \frac{I - a}{2I}.$$  

Using the well-known $\vartheta$-constant representation for function $\lambda(h)$ [27], we get

$$a = I - 2I \frac{\vartheta_A'(h)}{\vartheta_A^2(h)} \left( \frac{\alpha h + \beta}{\gamma h + \delta} \right),$$  

(23)

where $\{\alpha, \beta, \gamma, \delta\}$ are free constants with $\alpha \delta \neq \beta \gamma$. The further integration for the variables $\{A, B\}$ can be continued in two ways. The first one is to make use of rules for differential computations (17) of the $\vartheta$-series. Applying them to just found expressions for $a(h)$ and $b(h)$, we get expressions for $A(h)$, $B(h)$. The second way is to linearize the system because any Schwarz’s equation is known to be related to a certain linear ODE. We shall give solutions both in $h$- and $k$-representations.

3.1. Associated linear ODEs. Using (23), we have the obvious transformations between pairs $(a, b)$ and $(k, I)$:

$$a = I - 2Ik^2, \quad 8b = I^2k^2(1 - k^2).$$  

(24)

This allows us to bring (9) into the form

$$\dot{A} = 2A^2B, \quad \dot{B} = \frac{1}{8}I^2k^2(1 - k^2)A^3, \quad \dot{k} = \frac{1}{2}Ik(1 - k^2)A^2, \quad \dot{I} = 0,$$

(25)

where we let the dot above a symbol denote an $h$-derivative. We regard this system of equations as an intermediate equivalent of Jacobi’s system (9) because of its relation to linear ODEs. Indeed, as it follows from (25), the quantities $A$ and $B$, as functions of $k$, satisfy the two linear equations

$$\frac{dA}{dk} = \frac{4}{I(1 - k^2)}kB, \quad \frac{dB}{dk} = \frac{I}{4}kA$$

(26)

and their consequences

$$k(k^2 - 1)A_{kk} + (3k^2 - 1)A_k + kA = 0, \quad k(k^2 - 1)B_{kk} - (k^2 - 1)B_k + kB = 0.$$  

(27)

Since $k$ is Legendre’s modulus, it is naturally to expect that these ODEs are integrable in terms of functions (11)–(12).

**Proposition 3.** Canonical Legendre’s elliptic integrals (11)–(12) are differentially closed:

$$\frac{dK}{dk} = \frac{K}{k} - \frac{E}{(k^2 - 1)k}, \quad \frac{dK'}{dk} = \frac{kK'}{1 - k^2} + \frac{E'}{(k^2 - 1)k},$$

$$\frac{dE}{dk} = \frac{K}{k} + \frac{E}{k'}, \quad \frac{dE'}{dk} = \frac{kK'}{1 - k^2} + \frac{kE'}{k^2 - 1}.$$  

(28)

This system, being considered as a dynamical one, has the general solution

$$K = \alpha K(k) - \beta K'(k), \quad K' = \gamma K(k) + \delta K'(k),$$

$$E = \alpha E(k) + \beta [E'(k) - K'(k)], \quad E' = \delta E'(k) + \gamma [K(k) - E(k)],$$

where $\{\alpha, \beta, \gamma, \delta\}$ are free constants.
Of course, one should bear in mind that the canonical functions (11)–(12) themselves are not independent. Rather they satisfy the Legendre identity
\[ K(k)E'(k) + K'(k)E(k) - K(k)K'(k) = \frac{\pi}{2} \forall k, \]
which is a particular case of the constant level surfaces for system (28):
\[ KE' + K'E - K K' = \frac{\pi}{2}(\alpha \delta + \beta \gamma). \]
Curiously, this property and Proposition 3 seems to have not been tabulated in the standard texts. The second order differential consequences of this system are known. Both \( K \) and \( K' \) satisfy the same equation
\[
k(k^2 - 1) \frac{d^2 \Psi}{dk^2} + (3k^2 - 1) \frac{d\Psi}{dk} + k \Psi = 0 \quad \Rightarrow \quad \Psi = \{ K(k), K'(k) \}
\]
and common equation solvable by functions \( E \) and \( E' \) reads as follows
\[
k(k^2 - 1) \frac{d^2 \Psi}{dk^2} + (k^2 - 1) \frac{d\Psi}{dk} - k \Psi = 0 \quad \Rightarrow \quad \Psi = \{ E(k), E'(k) - K'(k) \}.
\]
The two last linear ODEs are not identical to (27) but search for solutions to Eqs. (26)–(27) is not a difficult task. In addition to solution pair (24), we obtain that
\[
A = 4\alpha K(k) + 4\gamma K'(k), \quad IB = \alpha \left[ E(k) + (k^2 - 1)K(k) \right] - \gamma \left[ E'(k) - k^2 K'(k) \right]
\]
with some free constants \( \alpha, \gamma \). We can now combine the ‘\( k \)-formulae’ (29) and \( h \)-time dynamics to obtain the complete integral of system (9).

3.2. Solution to the Jacobi system. Let us denote
\[
T := \frac{\alpha h + \beta}{\gamma h + \delta}.
\]
Then, by virtue of (22),
\[
\frac{\alpha h + \beta}{\gamma h + \delta} = \frac{K'(k)}{K(k)} \quad \Leftrightarrow \quad k = \frac{\partial_3^2(T)}{\partial_3^2(T)}.
\]
Make use of the representation for integrals (11)–(12) through Jacobi’s \( \eta, \vartheta \)-constants. The canonical formulae for \( K \) and \( K' \) are well known:
\[
K(k) = \frac{\pi}{2} \vartheta_3^2(h), \quad K'(k) = \frac{\pi}{21} \vartheta_3^2(h), \quad k = \frac{\partial_3^2(h)}{\partial_3^2(h)}.
\]
One can also show that the second pair \( \{ E, E' \} \) has the following modular \( h \)-representation:
\[
E(k) = \frac{2}{\pi} \frac{1}{\vartheta_3^2(h)} \left\{ \eta(h) + \frac{\pi^2}{12} \left[ \vartheta_3^2(h) + \vartheta_4^2(h) \right] \right\},
\]
\[
E'(k) = \frac{2i}{\pi} \frac{1}{\vartheta_3^2(h)} \left\{ h\eta(h) - \frac{\pi^2}{12} \left[ \vartheta_3^2(h) + \vartheta_4^2(h) \right] h - \frac{\pi^2}{2} \right\}.
\]
Modifying these formulae for the general ratio (30), we obtain
\[
K(k) = \frac{\pi}{2} \vartheta_3^2(T) \quad \Rightarrow \quad \alpha K(k) - i\gamma K'(k) = \frac{K(k)}{\gamma h + \delta} = \frac{\pi}{2} \frac{\partial_3^2(T)}{\gamma h + \delta}.
\]
Adjust the free integration constants in (29) with those of (30) and (31). Then we may write
\[ A = \pm \sqrt{\frac{4I}{\pi}} \left\{ \alpha K(k) - i\gamma K'(k) \right\} \] (33)
and therefore
\[ B = \pm \sqrt{\frac{1}{4\pi I^3}} \left\{ \alpha [E(k) + (k^2 - 1)K(k)] - i\gamma [E'(k) - k^2K'(k)] \right\}. \]
P expansions to the $\vartheta, \eta$-representation, we arrive at the final form of the sought-for solution.

**Theorem 4.** General solution to the dynamical system of Jacobi (9) has the form
\[ a = I - 2I \frac{\vartheta_3'(T)}{\vartheta_3(T)}, \quad b = \frac{I^2}{8} \frac{\vartheta_4'(T) \vartheta_4(T)}{\vartheta_3(T)}, \quad A = \pm \sqrt{\frac{\pi I}{\gamma h + \delta}}, \]
\[ B = \pm \sqrt{\frac{iI}{\pi}} \left( \gamma h + \delta \right) \frac{1}{\vartheta_3'(T)} \left\{ \frac{\pi^2}{12} \left[ \vartheta_4'(T) - \vartheta_4(T) \right] + \eta(T) + \frac{\pi}{2} i\gamma (\gamma h + \delta) \right\}, \]
where \( \{I, \alpha, \beta, \gamma, \delta\} \) are free constants subjected to normalization \( \alpha \delta - \beta \gamma = 1 \).

### 3.3. Solution to system (19)

One integral for Eqs. (19) is easily found because \( x \) is absent in three of these equations. Elimination of \( u \) shows that the function
\[ \pi I^2 = \frac{y^2 - z^2}{x^2} \] (34)
is a constant on solutions of (19), that is integral. This integral is much simpler than those we discussed in Sect. 2.1. As for solutions to system (19), these have the most simple form as against the other equations we consider. We shall give these solutions in the next theorem. The last fact we should mention here is a point transformation from Jacobi’s equations to the system (19). The simplest way of getting such a transformation is realized through the ‘linearizing’ systems (25), (26) which can be thought of as intermediate equivalents for Jacobi’s one (9) or (19). Explanation and details have been given in the previous section.

From now on we change Jacobi’s \( h \)-notation and put
\[ T := \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \]
with normalization \( \alpha \delta - \beta \gamma = 1 \).

**Theorem 5.** The canonical dynamical system (19) defining \( \vartheta, \eta \)-constants and Jacobi’s system (9) are equivalent. They are related through the following point transformation
\[ A = \frac{1 - i}{2I} y, \quad a = \frac{12}{\pi I} \frac{y^2 - z^2}{x^2} \frac{\vartheta_4'(T)}{\vartheta_3(T)}, \]
\[ B = \frac{1 + i}{2I} y (u + y^2 - 2z^2), \quad b = -\frac{18}{\pi I} \frac{z^2}{x^4 y^4} (y^2 - z^2)^3. \] (35)
The system (19) has the following general solution:
\[ x = \varepsilon \frac{\vartheta_3'(T)}{\gamma \tau + \delta}, \quad y = \sqrt{\frac{\pi I}{6}} \frac{\vartheta_3'(T)}{\gamma \tau + \delta}, \quad z = \sqrt{\frac{\pi I}{6}} \frac{\vartheta_3'(T)}{\gamma \tau + \delta}, \quad u = \frac{2i}{\pi} \frac{\eta(T)}{(\gamma \tau + \delta)^2} \frac{\gamma}{\gamma \tau + \delta}, \]
where \( \epsilon \neq 0 \) is the fourth free constant. With \( \epsilon = 0 \), the solution decouples into the two parametric elementary one:

\[
x = 0, \quad y = \frac{\pm 1}{\gamma \tau + \delta}, \quad z = \frac{1}{\gamma \tau + \delta}, \quad u = -\frac{\gamma^2 \tau + \gamma \delta - 1}{(\gamma \tau + \delta)^2}.
\]

**Proof.** The most convenient way to get the point transformation is to exploit the known general solution of Jacobi’s C-equation (20) [18, p. 186]:

\[
C^{-1} = \sqrt{\frac{\pi}{6}} \frac{\partial^2}{\partial \tau^2} \left( \frac{\tau + \delta}{\gamma \tau + \delta} \right)
\]

and to pass to the intermediate set of (‘linear’) variables \((A, B, k, I)\) followed by use of identities (24), formulae (31)–(33), and elimination of integration constants \(\{\alpha, \beta, \gamma, \delta\}\) appearing in the general solution given by Theorem 4. Omitting the computation details, we derive the transformation \(\{A, B, k, I\} \to \{x, y, z, u\}\):

\[
x = 1 + i \sqrt{\pi} k A, \quad y = (1+i) I A, \quad z^2 = 2i I^2 (1-k^2) A^2, \quad u = 2 A \{B-i I^2 (2k^2-1) A\},
\]

where \(12i I^2 = I\). This change turns (19) into the system (25). Inverting this change, we obtain

\[
A = \frac{1 - i}{2I} y, \quad B = \frac{1 + i}{2I} I (u + y^2 - 2z^2), \quad I^2 = \frac{1}{\pi} \frac{y^2 - z^2}{x^2}, \quad k^2 = 1 - \frac{z^2}{y^2}
\]

and, subsequently, the substitution (35). Making use of solution given in Theorem 4, we get the solution for variables \(\{x, y, z, u\}\).

**Remark 1.** From the preceding, incidentally, it follows that equations of the system (2) become now the differential identities for all solutions of Eqs. (17) (proof is a calculation). Thus, the Darboux–Halphen system (2) is a subsystem for Eqs. (17) but is a reduction for system (8). See also the last sentence in Appendix (Sect. 8).

**4. Derivation of Integrals**

In order to integrate system (9) we made use of its first integral (21). Having a complete solution, we can find the two remaining conserved quantities and the fourth ‘integral’ corresponds to the time shift \(h \mapsto h + \epsilon\). The simplest way to derive the integrals is to use the linear fractional formula (31). Indeed, the \(h\)-derivative of this formula gives the equalities

\[
(\gamma h + \delta)^2 = \left(\frac{d}{dh} \frac{\alpha h + \beta}{\gamma h + \delta}\right)^{-1} = \left(\frac{d}{dh} \frac{K'(k)}{K(k)}\right)^{-1} = \cdots
\]

and therefore expression

\[
\cdots = \left(\frac{I}{2} k (1-k^2) A^2 \cdot \frac{i}{dk} \frac{K'(k)}{K(k)}\right)^{-1} =: \Phi^2(A, B, k, I)
\]
must be a perfect square. Upon rooting, we get the $h$-linear function $\gamma h + \delta$ with coefficients depending on dynamical variables $\{A, B, k, I\} \Leftrightarrow \{A, B, a, b\}$. Its $h$-derivative
\[
\frac{d\Phi}{dh} = 2A^2 B \frac{d\Phi}{dA} + \frac{1}{8} I^2 k^2 (1 - k^2) A^3 \frac{d\Phi}{dB} + \frac{1}{2} I k (1 - k^2) A^2 \frac{\partial \Phi}{\partial k} = \cdots
\]
yields an $h$-independent constant $\gamma$, that is integral
\[
\cdots = J_1(A, B, a, b).
\]
Doing the same for
\[
(\alpha h + \beta)^2 = \left(\frac{d}{dh} \frac{\gamma h + \delta}{\alpha h + \beta}\right)^{-1} = \left(-i \frac{d}{dh} \frac{K(k)}{\alpha h + \beta}\right)^{-1} = \cdots,
\]
we get one more integral $J_2(A, B, a, b) \sim \alpha$. Both of these integrals are independent of each other since $\{\alpha, \gamma\}$ are independent constants. All the calculus with objects $K, k, \ldots$ has been described in the previous section and computations are somewhat lengthy but routine. We therefore omit them entirely.

**Proposition 6.** The Jacobi system (9) has the only algebraic (rational) integral $I^2 = a^2 + 32b$ and the two functionally independent transcendental integrals
\[
J_1 = 4K(k) \cdot B - \{E(k) + (k^2 - 1)K(k)\} \cdot AI,
\]
\[
J_2 = 4K'(k) \cdot B + \{E'(k) - k^2 K'(k)\} \cdot AI,
\]
where $\{I, k\}$, if required, can be expressed via $\{a, b\}$ by the inversion of formulae (24):
\[
I = \sqrt{a^2 + 32b}, \quad k^2 = \frac{1}{2} - \frac{1}{2} \frac{a}{\sqrt{a^2 + 32b}}.
\]
Integrals $J_1, J_2$ are the multi-valued transcendental functions of dynamical variables $\{a, b\}$ and the linear ones of $\{A, B\}$.

Curiously, the ‘very simple’ monomial dynamical system (9) has rather complicated transcendently algebraic integrals. Another way of derivation of integrals exploits the linear equations (26)–(27) and the well-known Wronskian relation for 2nd order linear ODEs. For example, the $\Lambda$-equation in (27) has $K(k)$ as its particular solution. Therefore
\[
\left\{K(k) \frac{d\Lambda}{dk} - \frac{dK(k)}{dk} \cdot \Lambda\right\} (k^2 - 1) k = \text{const.}
\]
Replacing here $\frac{d\Lambda}{dk} A$ by $B$ through (26) and using rules (28), we arrive again at the integral $J_1$. The choice of $K'(k)$ for a particular solution produces the second integral $J_2$ in (39).

5. INTEGRALS AND LAGRANGIAN

5.1. **Conserved quantities.** Lagrangians, Hamiltonians, and Poisson structures for dynamical systems are known to be closely related to integrals of the corresponding ODEs. The Hamiltonian formalism for the systems (9) and (19), which we are about to give is based on construction of conserved quantities and we first tabulate the complete set of such objects associated with equations (19).

**Proposition 7.** The system (19) has the only algebraic (rational) integral (34) and the two transcendental multi-valued ones
\[
J_1 = \frac{1}{y} (u - 2y^2 + z^2) K\left(\frac{z}{y}\right) + 3y E\left(\frac{z}{y}\right), \quad J_2 = \frac{1}{y} (u + y^2 + z^2) K'\left(\frac{z}{y}\right) - 3y E'\left(\frac{z}{y}\right),
\]
that is \( \dot{J}_1 = \dot{J}_2 \equiv 0 \). The integrals satisfy the identity
\[
J_1 K'(\frac{z}{y}) - J_2 K\left(\frac{z}{y}\right) = \frac{3}{2} \pi y.
\]

Proof. Straightforward computation with use of Propositions 3, 6 and system (19) itself. 

It should be emphasized here that integrability of any modular system is always associated with linear ODEs of Fuchsian class; in particular, with the hypergeometric equations [23, 3]. This makes inevitable the appearance of transcendentally multi-valued functions like \( K, K', E, E' \). Transition between such a ‘linear’ and ‘modularly nonlinear’ \( \tau \)-representation was explained in Sect. 3.1.

Insomuch as the famous system (2) has an extensive literature and applications, it is not without of interest to translate just obtained integrals into conserved quantities for this system. The authors of work [14] note properly on p. 5709: ‘These conserved quantities have not appeared in the literature for over a century even though a great deal of works has been done in related areas’. These integrals (including a generalization of (2)) are discussed in Ref. [8] and implicitly represented there in a form of ‘nonalgebraic, transcendental transformation . . . between’ [8, p. 1755] dynamical variables and integrals through solutions of a hypergeometric equation and its derivatives.

Proposition 8. The following is a complete set of (two) conserved quantities for the Darboux–Halphen system (2):
\[
J_1 = \frac{Z}{\sqrt{X-Z}} K\left(\frac{X-Y}{X-Z}\right) + \sqrt{X-Z} \ E\left(\frac{X-Y}{X-Z}\right),
\]
\[
J_2 = \frac{X}{\sqrt{X-Z}} K'(\frac{X-Y}{X-Z}) - \sqrt{X-Z} \ E'\left(\frac{X-Y}{X-Z}\right).
\]

Proof. Taking Remark 1 into account and using the definitions (1) and (18), we may write system (19) as follows
\[
X = u + y^2 + z^2, \quad Y = u + y^2 - 2z^2, \quad Z = u - 2y^2 + z^2. \tag{41}
\]
Hence
\[
u = \frac{1}{3} (X + Y + Z), \quad y^2 = \frac{1}{3} (X - Z), \quad z^2 = \frac{1}{3} (X - Y).
\]
Substituting this into (40), we get the statement of the proposition.

Not so simple but straightforward computation leads transcendental integrals for dynamical system (3). To do this, one uses the standard notation \( P_\nu^\mu(z) \) for Legendre’s functions with indices \((\nu, \mu) = \left(\frac{3}{2}, \frac{1}{2}\right)\). Recall that both of these functions satisfy the equation [27]
\[
(1 - z^2) \psi'' - 2z \psi' + \left\{ \nu (\nu + 1 - \frac{\mu^2}{1 - z^2} \right\} \psi = 0.
\]
Then one can derive and check the following statement.

Proposition 9. The following expressions
\[
J_1 = \sqrt{g_2 g_3} \left\{ P_\nu^\mu(g_3 w) - (g_3 - \frac{2}{3} \eta g_2) w \ P_\nu^\mu(g_3 w) \right\},
\]
\[
J_2 = \sqrt{g_2 g_3} \left\{ Q_\nu^\mu(g_3 w) - (g_3 - \frac{2}{3} \eta g_2) w Q_\nu^\mu(g_3 w) \right\},
\]
provide the two independent transcendental integrals for the Weierstrass system (3).
It is worth noting here that the Halphen system \( (2) \) and Weierstrass’ equations \((3)\) are equivalent by means of the following transformation \((X, Y, Z) \rightarrow (\eta, g_2, g_3)\) [16, p. 331]:

\[
6i\eta = \pi(X + Y + Z), \\
-3g_2 = \pi^2(X^2 + Y^2 + Z^2 - XY - XZ - YZ), \\
-54ig_3 = \pi^3(2X - Y - Z)(2Y - X - Z)(2Z - X - Y).
\]

As already noted, all these systems are rationally representable through the \( \vartheta, \eta \)-constants, i.e., the phase variables of system \((17)\). Hence, in addition to \((1)\), that is the point change \( (41)\), we obtain a ‘correction’ of definition \((4)\); it determines equations \((3)\) as a subsystem of \((17)\):

\[
g_2 = \frac{\pi^4}{12}\{\vartheta_3^5 - \vartheta_3^4\vartheta_4^4 + \vartheta_3^8\}, \quad g_3 = \frac{\pi^6}{432}\{2\vartheta_3^4 - \vartheta_4^4\}\{2\vartheta_3^4 - \vartheta_4^4\}
\]

(no \( \vartheta_2 \) here). These changes entail an equivalence of integrals: the \( \{P_\nu, Q_\nu, P_\nu^{\prime}, Q_\nu^{\prime}\}\)-objects can be written in terms of \( \{K, E, K', E'\} \) (and vice versa) by the formulae given above. Of course, this property manifests itself in the fact that integrals of such a kind may be rewritten solely in terms of hypergeometric \( _2F_1 \)-functions; we comment this point more fully in Sect. 7.

5.1.1. **On the Ramamani system.** By way of illustration to the theory we can also consider Ramamani’s system mentioned in Introduction. This system is satisfied by certain function series \( P(\tau), \tilde{P}(\tau), \) and \( Q(\tau) \) (see formula (3.3) in [4] and formulae (7)–(9) in [13]). Translating these definitions into our notation \((\eta, g_2, g_3)\), we obtain:

\[
P(\tau) = \frac{4}{\pi^2}\{4\eta(2\tau) - \eta(\tau)\}, \quad \tilde{P}(\tau) = \frac{12}{\pi^2}\{2\eta(2\tau) - \eta(\tau)\}, \quad Q(\tau) = \frac{4}{3\pi^4}\{16g_2(2\tau) - g_2(\tau)\}.
\]

Using the duplication rules \((7)\), notation \((18)\), and definitions \((42)\), we may, as pointed out above, turn these formulæ into the rational point transformation. One obtains

\[
\pi i P = 2(u + y^2 + z^2), \quad \pi i \tilde{P} = 3(y^2 + z^2), \quad \pi^2 Q = 36g^2z^2
\]

and this substitution brings the main system \((19)\) into the system

\[
\frac{dP}{d\tau} = \frac{1}{2}\pi i(P^2 - Q), \quad \frac{d\tilde{P}}{d\tau} = \pi i(P\tilde{P} - Q), \quad \frac{dQ}{d\tau} = 2\pi i(P - \tilde{P})Q.
\]

This is the Ramamani dynamical system [24, p. 116], [4, (1.8)], [13, (10)]. Its theory, including the search for conserved quantities \( J_1, J_2(P, \tilde{P}, Q) \) (exercise), results from Proposition 8 and an equivalent of substitution \((43)\) written in terms of Darboux–Halphen variables reads as follows:

\[
\pi i P = 2X, \quad \pi i \tilde{P} = 2X - Y - Z, \quad \pi^2 Q = 4(X - Z)(X - Y).
\]

See Refs. [4, 21] for further information about system \((44)\).

5.2. **Action and Lagrangians.** Let us introduce the collective notation for the phase-space coordinates: \( X := (A, B, a, b)^\top \) for system \((9)\) or \( X := (x, y, z, u)^\top \) for \((19)\). We are looking for the action functional

\[
S = \int \mathcal{L}(X, \dot{X})\, d\tau
\]
in the first-order formalism. Then the most general non-singular Lagrangian has the form
\[ \mathcal{L}(X, \dot{X}) = g_k(X) \dot{X}^k - \mathcal{H}(X), \]  
(46)
where \( \mathcal{H}(X) \) is a Hamiltonian and \( g_k(X) \) define the symplectic potential \( \varrho = g_k(X) dX^k \).

As usual, varying (45), we get the Hamiltonian equations
\[ \dot{X}^a = \Omega^{ak}(X) \frac{\partial \mathcal{H}}{\partial X^k} \iff \Omega_{kn}(X) \dot{X}^n = \frac{\partial \mathcal{H}}{\partial X^k} \iff (9), (19), \]
where \( \Omega = \Omega^{-1} \) is the Poisson bi-vector dual to the symplectic 2-form
\[ \Omega_{kn} = \frac{\partial g_n(X)}{\partial X^k} - \frac{\partial g_k(X)}{\partial X^n}. \]

There is a great deal of ambiguity concerning the choice of Lagrangians for a given system of equations. On the other hand, no general method is known for constructing Lagrangians starting from equations of motion (the so-called inverse problem of calculus of variations). Our situation is, however, somewhat special as the systems under consideration are integrable (in the sense that was explained in sects. 3.2 and 3.3). The latter fact allows us to write down the following ansatz for the Lagrangian:
\[ \mathcal{L}(X, \dot{X}) = (\mathcal{N} - 1) \mathcal{H} + I_1 \dot{I}_2. \]  
(47)

Besides the Hamiltonian \( \mathcal{H}(X) \), it involves two additional independent integrals of motion \( I_j = I_j(X) \) and the quantity \( \mathcal{N} = \mathcal{N}(X) \) obeying condition \( \mathcal{N} \equiv 1 \). Clearly, \( \mathcal{N} \) is a linear function of \( \tau \) modulo integrals of motion, that is \( \mathcal{N}(X) = \tau + \text{const}(\mathcal{H}, I_1, I_2) \). We have constructed such a function in Sect. 4. A computation, based on (39) followed by use of (33), (25), and (28), shows that
\[ \mathcal{N} = -2 \frac{K(k)}{AJ_1} \iff \frac{d\mathcal{N}}{dh} \equiv 1, \]
where \( J_1 \) has been defined in (39). The quantity \( \mathcal{N}(X) \) thus becomes
\[ \mathcal{N}(X) = -2 \frac{K(k)}{AJ_1} = \frac{-K(k)}{(u - 2y^2 + z^2) K(k) + 3y^2 E(k)}, \]
(49)
The Lagrangian density is determined up to a total \( \tau \)-derivative and therefore its choice is always accompanied by some heuristic arguments (simplicity of Lagrangians, brackets, etc.) When deriving the objects above we made use of a ‘linearized’ equivalent to systems (9), (19), and equation (25). Therefore we present result in terms of ‘mixed’ phase variables. The most compact Lagrange function we have found is given by the following statement.

**Theorem 10.** The systems (9), (19), and (25) are the Euler–Lagrange equations for the following Lagrangian \( \mathcal{L} \):
\[ \mathcal{L} = J_1^2 (\mathcal{N} - 1) + J_2 \dot{I} - 8 \frac{d}{dx} \left( \frac{B}{A} K^2 \right) 
= 4 \frac{J_1 K}{A^2} \dot{A} - 2 \left\{ k IK^2 + \frac{J_2^2 - 16B^2 K^2}{k(k^2 - 1) A^2} \right\} \cdot k + \left\{ J_2 + 2 \frac{K}{A} (J_1 - 4B K) \right\} \cdot \dot{I} - J_1^2. \]  
(48)

Here, we omitted indication of argument in Legendre’s integral \( K(k) \) and expressions for \( J_1, J_2(A, B, k, I) \) are taken from Proposition 6. Transitions between variables are described by substitutions (35)–(37).
6. Poisson structures

The following statement characterizes a non-trivial property of the dynamical systems under consideration.

**Theorem 11.** Whatever the Hamilton function $H(X)$ is chosen (single- or multi-valued analytic function), none of systems (8), (9), or (17), (19) does admit a constant non-degenerate Poisson bracket $\Omega$.

**Proof.** Denote by $X$ the phase-space coordinate vector for any of these systems: $\dot{X}^j = V^j(X)$. Assuming the availability of the form $\Omega \dot{X} = \nabla H(X)$ with a constant matrix $\Omega$, we apply integrability condition to equations $\nabla H = \Omega \dot{X}$, considered now as equations for the Hamiltonian $H$:

$$\nabla_k H = \Omega_{kj} V^j \Rightarrow \Omega_{kj} \nabla_n V^j = \Omega_{nj} \nabla_k V^j.$$  \hspace{1cm} (49)

It follows that $\Omega \cdot \partial_n V$ must be a symmetric matrix for all $X$. Straightforward computations show that this property is compatible with vector fields $V$’s defining the systems (8), (9), and (17), (19) if and only if $\Omega \equiv 0$. \hfill $\blacksquare$

This proof gives in fact a criteria for availability of a canonical symplectic form (given coordinates) and absence of such a bracket suggests to look for non-canonical one. Insomuch as (34) is the only single-valued function integral, we have to take it (or function of it) as a Hamiltonian. Furthermore, equations of motion do not depend on choice of the Lagrangian $L$ but bracket $\Omega$ does; even though the Hamilton function $H(X)$ and coordinates $X$ have been fixed. We thus have to choose in (47), except for $H(X)$, the two independent integrals $I_1, I_2(X)$ in order that the bracket $\Omega$ be simplest. We put

$$L = (\dot{N} - 1) H + (\lambda J_1)^{-1} J_2,$$  \hspace{1cm} (50)

where $\lambda$ is an arbitrary constant. Formulae of the previous section contain all what we need for computation of the bracket $\Omega$.

**Lemma 12.** Having integrals of motion $\mathcal{H}, I_k$, and the object $N$, the Poisson bi-vector $\Omega$ is calculated by the following computational rule:

$$\Omega = M - M^T, \quad M_{kn} = \nabla_k H \cdot \nabla_n N + \nabla_k I_1 \cdot \nabla_n I_2.$$  

Now, we insert here

$$I_1 := (\lambda J_1)^{-1}, \quad I_2 := J_2$$  \hspace{1cm} (51)

and use Proposition 3.

**Theorem 13.** Denote $X := (x, y, z, u)^T$. Then

1. **Dynamical system** (19) is Hamiltonian:

$$\dot{X} = \omega \nabla \mathcal{H}, \quad \mathcal{H} = \frac{1}{2} y^2 - z^2,$$

where the degenerate rational (single-valued) Poisson bracket is as follows

$$\omega = \frac{x}{2 \mathcal{H}} \begin{pmatrix} 0 & (u+y^2-2z^2)y & (u-2y^2+z^2)z & u^2-y^4+y^2z^2-z^4 \\ -(u+y^2-2z^2)y & 0 & 0 & 0 \\ -(u-2y^2+z^2)z & 0 & 0 & 0 \\ -u^2+y^4-y^2z^2+z^4 & 0 & 0 & 0 \end{pmatrix}.$$
(2) Non-degenerate but transcendental multi-valued extension of the $\omega$ is given by the bracket
$$\Omega = \omega + \lambda \tilde{\omega} \quad (\det \Omega \neq 0),$$
where
$$\tilde{\omega} = \frac{2}{\pi} K^2 \begin{pmatrix}
0 & \frac{x}{y} z^2 & xz & xM_1 \\
-\frac{x}{y} z^2 & 0 & \frac{z}{y} (y^2 - z^2) & \frac{1}{y} M_2 \\
-xz & \frac{z}{y} (z^2 - y^2) & 0 & zM_3 \\
-xM_1 & -\frac{1}{y} M_2 & -z M_3 & 0
\end{pmatrix}$$
and
$$M_1 := 3y^2(EK^{-1} - 1)^2 - z^2, \quad M_3 := y^2(3E^2 K^{-2} - 1) + z^2,$$
$$M_2 := 3y^4(EK^{-1} - 1)^2 + y^2 z^2 (6EK^{-1} - 5) + 2z^4.$$

(3) The matrix $\tilde{\omega}$ is a bracket as well with $\det \tilde{\omega} = 0$. The brackets $\omega, \tilde{\omega}$ are compatible to each other and have the following Casimir’s functions:
$$\omega \nabla J_1 = \omega \nabla J_2 \equiv 0, \quad \tilde{\omega} \nabla H = \tilde{\omega} \nabla N \equiv 0.$$

The system may thus be treated as bi-Hamiltonian in the sense of Magri [20].

Incidentally it should be observed that degenerate but well-defined rational bracket $\omega$ is obtained from non-degenerate but multi-valued bracket $\Omega$ by a passage to the limit $\lambda \to 0$ in transcendental part of the $\Omega$. This procedure can be interpreted as a formal separability of canonically conjugated pairs ($H, N$) and $(J_1, J_2)$ in Lagrangian (50). Their commutation relations (algebra of integrals) are standard:
$$\{ H, J_1 \}_\Omega = \{ H, J_2 \}_\Omega = 0, \quad \{ J_2, (\lambda J_1)^{-1} \}_\Omega \equiv 1.$$

Remark 2. An explicit analog of Theorem 13 for Jacobi’s system (9) is obtained with avail of transformation law for tensor $\Omega(x, y, z, u) \mapsto \tilde{\Omega}(A, B, a, b)$ under the coordinate change $X := (x, y, z, u)^\top \mapsto (A, B, a, b)^\top =: Y$ defined by Theorem 5. Coordinate form of the transformations reads
$$\tilde{\Omega}^{pq} (Y) \frac{\partial Y^j}{\partial X^n} \frac{\partial Y^p}{\partial X^m} \Omega^{nm} (X) \Rightarrow \tilde{\Omega} = T \Omega T^\top, \quad T_{kn} := \frac{\partial Y^k}{\partial X^n}$$
and implies equations
$$\tilde{Y}^j = \tilde{\Omega}^{jp} (Y) \frac{\partial H}{\partial Y^p} \quad \Leftrightarrow \quad (9).$$
We do not display here the explicit formulae since we were unable to find the compact form to them.

It is interesting to note that in addition to the algebraic integral (21) and the rational (but degenerate) Poisson bi-vector, systems (9) and (19) admit a symmetry given by the linear vector field
$$\tilde{G} = 2x \partial_x = A \partial_A - B \partial_B - 2a \partial_a - 4b \partial_b.$$
This vector field is of course non-Hamiltonian for otherwise it would be generated by a new integral. The absence of rational integrals of motion other than $I$ implies that the result of action of $\tilde{G}$ on $I$ should be a function of $I$. Indeed, one can check that
$$\tilde{G} I = -4I.$$
Also, once the Hamiltonian form for a dynamical system has been found, we can determine its invariant volume form $V = \sqrt{\det \Omega} \, d^4X$. Calculating determinant of the matrix $\Omega$, we obtain that
\[
\det \Omega = \frac{4}{\pi^2} \lambda^2 J_1^4 \cdot x^6 y^2 z^2,
\]
and, as in the case of Halphen’s system (2) [14, 8], the volume form is a polynomial function:
\[
\frac{1}{x^3 y^2} \, dxdydzdu \cong V \cong \frac{1}{A^2b} \, dAdBdadb.
\]
Clearly, the invariant volume is not unique as one is free to multiply it on any positive function of integrals.

Let us also comment on the relationship of the degenerate Poisson structure $\omega$ to the Nambu structure\(^3\). The general Nambu 4-bracket in a four dimensional space reads [22, 26]
\[
\{f_1, f_2, f_3, f_4\} = \Xi^{-1} (X) \varepsilon^{jkln} \nabla_j f_1 \nabla_k f_2 \nabla_l f_3 \nabla_n f_4,
\]
where multiplier $\Xi(X)$ transforms as a density and $\varepsilon$ is the Levi–Civita symbol with $\varepsilon^{1234} = 1$. Setting $\Xi = \sqrt{\det \Omega}$ times a function of integrals, one can see that the rational Poisson bracket $\omega$ appearing in Theorem 13 is a reduction of the Nambu 4-bracket with respect to the pair of transcendental integrals $I_1, I_2$:
\[
\{f_1, f_2\}_\omega = \{f_1, f_2, I_2, I_1\}. 
\]
Now the dynamical system (19) can be viewed as a generalized Nambu mechanics with the 4-bracket (53) and the triple of Hamiltonians $H, I_1, I_2$ (two of which are transcendental):
\[
\dot{X} = \{X, H, I_2, I_1\} \quad \Leftrightarrow \quad \dot{X} = \{X, H\}_\omega. 
\]
More explicitly,
\[
\omega^{jk} = \sqrt{\det \Omega} \varepsilon^{jkln} \cdot \nabla_l I_2 \cdot \nabla_n I_1 = \cdots
\]
and expression (52) leads to a polynomial character of the Nambu bracket:
\[
\cdots = \frac{2}{\pi} x^3 y z \varepsilon^{jkln} \cdot \nabla_l J_1 \cdot \nabla_n J_2,
\]
where the integrals $I_k$ and $J_k$ have been determined in Eqs. (40) and (51).

We conclude the section with general remarks concerning other non-constant brackets. All of them are obtainable from each other by general transformation of the quantities appearing in Lagrangian (50):
\[
N \mapsto N + F_1 (H, J_1, J_2), \quad H \mapsto F_2 (H, J_1, J_2), \quad J_2 \mapsto F_3 (H, J_1, J_2), \quad J_2 \mapsto F_4 (H, J_1, J_2) \quad (54)
\]
(re-normalization of integration constants). This defines a function freedom of the three variables $(\alpha, \beta, \gamma) \simeq (H, J_1, J_2)$. On the other hand, all the dependencies $\Omega(X)$, including possible change of the Hamilton function $H$, are determined by the following modification of the line (49):
\[
\nabla_n (\Omega_{kj} V^j) = \nabla_k (\Omega_{nj} V^j) \quad \Rightarrow \quad (\nabla_n \Omega_{kj} - \nabla_k \Omega_{nj}) V^j = \Omega_{nj} W^j_k - \Omega_{kj} W^j_n, \quad (55)
\]
where the tensor field
\[
W^j_k := \frac{\partial V^j}{\partial X^k}
\]
\(^3\)The comment has been added following a suggestion of the anonymous referee who we wish to thank for that.
can be thought of as given. Equations (55) are a set of partial differential equations for \( \Omega(X) \)'s but, thanks to function freedom mentioned above, we may pass from old set of variables, say \((x,y,z,u)\), to the new one \((N,\alpha,\beta,\gamma)\) and thereby turn these equations into ordinary differential equations in variable \(N\).

**Theorem 14.** Denote \( \Omega(N;\alpha,\beta,\gamma) := \Omega(x,y,z,u) \) and matrix \( W = W(N;\alpha,\beta,\gamma) \):

\[
W_{jk} := \left. \frac{\partial V_j}{\partial X_k} \right|_{X = X(N,\alpha,\beta,\gamma)},
\]

where \((\alpha,\beta,\gamma)\) are seen as parameters. Then all the brackets \(\Omega(X) = \Omega(N)\) satisfy the linear matrix dynamical system

\[
\frac{d\Omega}{dN} = W\Omega + \Omega W^\top
\]

supplemented with the arbitrary initial condition (bracket) \(\Omega(0) = \Lambda(\alpha,\beta,\gamma)\).

**Proof.** With use of antisymmetry \(\Omega_{kj} = -\Omega_{jk}\) and Jacobi’s identity \(\nabla_n \Omega_{kj} + \nabla_k \Omega_{jn} + \nabla_j \Omega_{nk} = 0\) equations (55) may be rewritten as \(V_j \nabla_j \Omega_{nk} = \Omega_{kj} W_n^k - \Omega_{nj} W_k^j\). Hence \(\dot{\Omega} = -\Omega W - W^\top \Omega\) and, subsequently, (56) since \(\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0\) and \(\dot{\Omega} \dot{\Omega} = -\dot{\Omega} \dot{\Omega}\).

By this means function freedom (54) with the three functions of three variables \(\alpha = H(X), \beta = J_1(X), \gamma = J_2(X)\) is converted into the coefficients of dynamical system (56) (matrix \(W\)) and its initial condition \(\Omega(0)\). As the latter one may take any particular bracket; for example, the bracket \(\Omega\) from Theorem 13.

### 7. A Generalization

Outlined receipt of derivation of integrals and the ‘linear objects’ like \(N\) is directly extended to more general (Halphen, Brioschi (1881)) quadratic homogenous systems [15]

\[
\dot{x} = x^2 + \Xi, \quad \dot{y} = y^2 + \Xi, \quad \dot{z} = z^2 + \Xi,
\]

\[
\Xi := a(y-x)^2 + b(z-x)^2 + c(z-y)^2,
\]

associated with a hypergeometric equation of the general type

\[
s(s-1)\Psi'' + \left\{ (a+b+1)s - c \right\} \Psi + ab\Psi = 0,
\]

where \(s\) stands for the \(s\)-derivative. Integrability of this system (and its generalizations) in terms of associated linear equations was considered and established independently in the 1990s by many authors: Ablowitz et all [3], Ohyama [23], Harnad & MacKay [17]; see also Refs. [8, 2]. Parameters \((a,b,c)\) are computed via the hypergeometric ones \((a,b,c)\) (correcting a typo in formula (3.6) of Ref. [23]):

\[
4a = ac + bc - 2ab - c, \quad 4b = a^2 + b^2 - ac - bc + c - 1, \quad 4c = c^2 + 2ab - ac - bc - c
\]

and base definitions for variables \((x,y,z)\) and relations between them and the quantities \(\tau, s, \) and \(\Psi\) read as follows

\[
\tau = \frac{\Psi(s)}{\Psi(s)}, \quad s = s\left(s - 1\right)^{a+b-c+1} s^2,
\]

\[
x = \frac{1}{2} d\ln \left(s - 1\right)^{a+b-c+1} s^c = \frac{1}{2} d\left\{ s\left(s - 1\right)^{a+b-c+1} s^2 \right\},
\]
By repeating the arguments above we obtain the second integral for Eqs. (57) and can deduce its integrals. Indeed, passing to the general solution \( s = s^{(\alpha + \beta)} \gamma \tau + s \) we have, instead of (38),

\[
(\gamma \tau + \delta)^2 = \frac{d}{ds} \Psi(s) = (x - y)s - \frac{\Psi'}{\Psi^2}.
\]

Hence

\[
(\gamma \tau + \delta) \sim \sqrt{s^{(1/2)}(s - 1)^{(a + b - c + 1)}} \cdot 2F_1(a, b; c|s).
\]

Take the \( \tau \)-derivative of this expression and make use of the fact that derivative of a hypergeometric series is another hypergeometric series [27]:

\[
\frac{d}{ds} \{2F_1(a, b; c|s)\} = \frac{ab}{c} \cdot 2F_1(a + 1, b + 1; c + 1|s).
\]

We thus obtain the first integral \( J_1 \sim \gamma \) for Eqs. (57):

\[
J_1 = CA \cdot 2F_1\left( a, b; c \mid \frac{x - y}{z - y} \right) + CB \cdot 2F_1\left( a + 1, b + 1; c + 1 \mid \frac{x - y}{z - y} \right),
\]

where

\[
A = (a + b - 1)x - cy - (a + b - c + 1)z, \quad B = \frac{2ab}{c} \frac{(x - y)(z - x)}{z - y},
\]

\[
C = (y - x)^{1/2(a + b - c)}(z - y)^{-1/2(b(a + b - c)}(z - x)^{1/2(c - 1)}.
\]

Assume now that the second (linearly independent of \( \Psi \)) solution to (58) has no logarithmic behavior in the vicinity of point \( s = \infty \); otherwise we can reorder variables \( (x, y, z) \) with the help of the linear transformation \( s \mapsto 1 - s \) or \( s \mapsto s^{-1} \). If the logarithm presents at each of the points \( s \in \{0, 1, \infty\} \), we fall into Proposition 8. This case corresponds to parameters \((a, b, c) = (\frac{1}{2}, \frac{1}{2}, 1)\) and is equivalent to system (2) up to a simple linear transformation [23] of the triples \((x, y, z) \to (X, Y, Z)\). Then we may take the following form for the second solution to (58) [27]:

\[
\tilde{\Psi} = s^{1-c(s - 1)^{(c-a-b)} \cdot 2F_1(1 - a, 1 - b; 2 - c(z - x))}.
\]

By repeating the arguments above we obtain the second integral for Eqs. (57):

\[
J_2 = \tilde{C} \tilde{A} \cdot 2F_1\left( 1 - a, 1 - b; 2 - c \mid \frac{z - y}{z - x} \right) + \tilde{C} \tilde{B} \cdot 2F_1\left( 2 - a, 2 - b; 3 - c \mid \frac{z - x}{z - y} \right),
\]

where

\[
\tilde{A} = A + 2(z + y), \quad \tilde{B} = \frac{(a - 1)(b - 1)}{c - 2} \frac{(x - y)(z - x)}{z - y}, \quad \tilde{C} = \frac{C}{(z - y)^2}.
\]

This completes an integration procedure considered in Refs. [2, 3, 8, 17, 23].
8. Appendix: The Jacobi system

Since the late 1850’s C. Borchardt, being the Editor-in-Chief of Creille’s Journal, began to edit and publish the manuscript material kept after Jacobi’s death in 1851. In particular, in 1857 he published calculations [18, p. 383–394] where Jacobi constructed power series developments for his \( \theta(z|\tau) \)-functions. The power \( \theta \)-series are of interest in their own rights but not a less remarkable fact is that they produce the nice dynamical systems integrable in terms of \( \theta \)-constants.

Jacobi introduces the four variables (we keep completely to Jacobi’s notation in [18, p. 386])

\[
A = \frac{2K}{\pi}, \quad B = \frac{2E}{\pi} - k'^2 \frac{2K}{\pi}, \quad a = 4(1 - k^2), \quad b = 2k^2k'^2, \quad (59)
\]

and shows that these satisfy the dynamical system (9); in doing so Jacobi imposes the condition (10) which is of course an equivalent of the relation \( k^2 + k'^2 = 1 \) or, which is the same, the \( \theta \)-identity (5). Halphen does not mention system (9) and, to all appearances, it has not received mention in the later literature on theta-functions. Jacobi does not restrict his consideration to variables (59) and exhibits what is called presently the canonical transformations, i.e., transformations of dynamical variables preserving the form of equations. Here are his versions of the transformations [18, p. 387]:

\[
A = \frac{2kK}{\pi}, \quad B = \frac{1}{k} \left( \frac{2E}{\pi} - k'^2 \frac{2K}{\pi} \right), \quad a = \frac{4(1 + k'^2)}{k^2}, \quad b = -\frac{2k'^2k^4}{k^4},
\]

\[
A = \frac{2k'K}{\pi}, \quad B = \frac{1}{k'} \left( \frac{2E}{\pi} - \frac{2K}{\pi} \right), \quad a = -\frac{4(1 + k'^2)}{k'^2}, \quad b = -\frac{2k^2k'^2}{k'^4}.
\]

Complete set of differential relations between these and auxiliary variables \( \{k, k', K, E\} \) was written down by Jacobi earlier [18, p. 176–177]. As in the previous differential systems (2) and (3), dynamical variables \( \{A, B, a, b\} \) are expressed through the \( \eta, \vartheta \)-series rationally; see formulae (13).

We also note that system (9) is notable for its homogenous monomial structure. Jacobi exploits intensively this fact when deriving the power \( \theta \)-series; the pages 388–391 of his Werke [18] contain a lot of useful formulae along these lines. System (9) is not the only dynamical system that was derived by Jacobi in connection with \( \theta \)-functions; see also [18, p. 173–190]. Jacobi did not pose a question about integration of (9) as ODEs, however earlier, in 1847, he obtained a complete integral for the 3rd order differential equation

\[
C^4(\ln C^3C_{\tau \tau})^2 = 16C^3C_{\tau \tau} - \pi^2 \quad (60)
\]

satisfied by each of the \( \theta \)-constants: \( C = \vartheta(\tau)^{-2} \) (Jacobi’s notation [18, p. 179]). On the other hand, this equation must be a certain consequence of equations (8) whose solutions are not only the \( \vartheta, \eta \)-series. Invoking integral (15), we conclude that Jacobi’s equation (60) is indeed the consequence of equations (8) with the proviso that \( U = 0 \). It is also clear that this condition is a necessary one in order that the Darboux–Halphen system (2) be a consequence of symmetrical identities (8) as well.

References

[1] Ablowitz, M. J., Chakravarty, S. & Takhtajan, L. A. A self-dual Yang–Mills hierarchy and its reduction to integrable systems in 1 + 1 and 2 + 1 dimensions. Comm. Math. Phys. (1993), 158, 289–314.
DYNAMICAL SYSTEMS DEFINING $\theta$-CONSTANTS

[2] Ablowitz, M. J., Chakravarty, S. & Halburd, R. The Generalized Chazy Equation from the Self-Duality Equations. Stud. Appl. Math. (1999), 103(1), 75–88.

[3] Ablowitz, M. J., Chakravarty, S. & Halburd, R. On Painlevé and Darboux–Halphen-Type Equations. In: [10], p. 573–589.

[4] Ablowitz, M. J., Chakravarty, S. & Hain, H. Integrable systems and modular forms of level 2. J. Phys. A: Math. Gen. (2006), 39(50), 15341–15353.

[5] Atiyah, M. & Hitchin, N. The Geometry and Dynamics of Magnetic Monopoles. Princeton University Press: Princeton (1988).

[6] Brezhnev, Yu. V. Non-canonical extension of $\theta$-functions and modular integrability of $\theta$-constants. http://arXiv.org/abs/1011.1643.

[7] Chakravarty, S., Ablowitz, M. J & Clarkson, P. A. Reductions of self-dual Yang–Mills fields and classical systems. Phys. Rev. Lett. (1990), 65(9), 1085–1087.

[8] Chakravarty, S. & Halburd, R. First integrals of a generalized Darboux–Halphen system. J. Math. Phys. (2003), 44(4), 1751–1762.

[9] Chudnovsky, D. V. & Chudnovsky, G. V. Note on Eisenstein's system of differential equations: an example of “exactly solvable but not completely integrable system of differential equations”. In: Lecture Notes in Pure and Applied Math. (1984), 92, 99–115. D. V. Chudnovsky & G. V. Chudnovsky (eds.), Dekker.

[10] Conte, R. (Ed.) The Painlevé property. One century later. CRM Series in Mathematical Physics (1999). Springer–Verlag: New York (1999). Conte, R. The Painlevé Approach to Nonlinear Ordinary Differential Equations, 77–180.

[11] Darboux, G. Sur la théorie des coordonnées curvilignes et les systèmes orthogonaux. Annales scientifiques de l’École Normale Supérieure. 2e Série, (1878), VII, 101–150.

[12] Eisenstein, G. Mathematische Abhandlungen. Georg Olms: Hildesheim (1967).

[13] Guha, P. & Mayer, D. Ramanujan Eisenstein Series, Faà di Bruno Polynomials and Integrable Systems. Max Planck Institute: Preprint (2007), 87.

[14] Gümral, H. & Nutku, Y. Poisson structure of dynamical systems with three degrees of freedom. J. Math. Phys. (1993), 34(12), 5691–5723.

[15] Halphen, G.-H. Sur certains systèmes d’équations différentielles. Compt. Rend. Acad. Sci. Paris (1881), 92, 1404–1406.

[16] Halphen, G.-H. Traité des Fonctions Elliptiques et de Leurs Applications. I. Gauthier–Villars: Paris (1886).

[17] Harnad, J. & McKay, J. Modular solutions to equations of generalized Halphen type. Proc. Royal Soc. London A (2000), 456(1994), 261–294.

[18] Jacobi, C. G. J. Gesammelte Werke. H. Verlag von G. Reimer: Berlin (1882).

[19] Kriitis, L. Introduction to superstring theory. Leuven Notes in Mathematical and Theoretical Physics. Cornell University Press (1998).

[20] Magri, F. A simple model of the integrable Hamiltonian equation. J. Math. Phys. (1978), 19(5), 1156–1162.

[21] Maier, R. S. Nonlinear differential equations satisfied by certain classical modular forms. Manuscripta Mathematica (2011), 134(1/2), 1–42.

[22] Namík, Y. Generalized Hamiltonian Dynamics. Phys. Rev. D (1973), 7(8), 2405–2421.

[23] Ohya, Y. Systems of nonlinear differential equations related to second order linear equations. Osaka J. Math. (1996). 33(4), 927–949.

[24] Ramamani, V. On some identities conjectured by Srinivasa Ramanujan in his lithographed notes connected with partition theory and elliptic modular functions— their proofs— interconnection with various other topics in the theory of numbers and some generalizations. PhD-Thesis, University of Mysore: Mysore (1970).

[25] Ramanujan, S. On certain arithmetical functions. Trans. Cambridge Phil. Soc. (1916), 22(9), 159–184; Collected papers. Cambridge University Press (1927).

[26] Takhtajan, L. On foundation of the generalized Nambu mechanics. Comm. Math. Phys. (1994), 160, 265–315.

[27] Whittaker, E. T. & Watson, G. N. A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions, with an Account of the Principal Transcendental Functions. Cambridge University Press: Cambridge (1996).