Estimation for the change point of the volatility in a stochastic differential equation *

Stefano M. Iacus†
Department of Economics, Business and Statistics, University of Milan
Via Conservatorio 7, 20122 Milan, Italy; stefano.iacus@unimi.it

Nakahiro Yoshida
University of Tokyo, and Japan Science and Technology Agency
Graduate School of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914 Japan; nakahiro@ms.u-tokyo.ac.jp

June 17, 2009

Abstract
We consider a multidimensional Itô process $Y = (Y_t)_{t \in [0,T]}$ with some unknown drift coefficient process $b_t$ and volatility coefficient $\sigma(X_t, \theta)$ with covariate process $X = (X_t)_{t \in [0,T]}$, the function $\sigma(x, \theta)$ being known up to $\theta \in \Theta$. For this model we consider a change point problem for the parameter $\theta$ in the volatility component. The change is supposed to occur at some point $t^* \in (0, T)$. Given discrete time observations from the process $(X, Y)$, we propose quasi-maximum likelihood estimation of the change point. We present the rate of convergence of the change point estimator and the limit theorems of asymptotically mixed type.

keywords Itô processes, discrete time observations, change point estimation, volatility

*This work was in part supported by JST Basic Research Programs PRESTO, Grants-in-Aid for Scientific Research No. 19340021, the Global COE program “The research and training center for new development in mathematics” of Graduate School of Mathematical Sciences, University of Tokyo, and by Cooperative Research Program of the Institute of Statistical Mathematics.
†Corresponding author.
1 Introduction

The problem of change point has been considered initially in the framework of independent and identically distributed data by many authors, see e.g. Hinkley (1971), Csörgő and Horváth (1997), Inclan and Tiao (1994). Recently, it naturally moved to context of time series analysis, see for example, Kim et al. (2000), Lee et al. (2003), Chen et al. (2005) and the papers cited therein.

In fact, change point problems have originally arisen in the context of quality control, but the problem of abrupt changes in general arises in many contexts like epidemiology, rhythm analysis in electrocardiograms, seismic signal processing, study of archeological sites and financial markets. In particular, in the analysis of financial time series, the knowledge of the change in the volatility structure of the process under consideration is of a certain interest.

In this paper we deal with a change-point problem for the volatility of a process solution to a stochastic differential equation, when observations are collected at discrete times. The instant of the change in volatility regime is identified retrospectively by maximum likelihood method on the approximated likelihood. For continuous time observations of diffusion processes Lee et al. (2006) considered the change point estimation problem for the drift. In the present work we only assume regularity conditions on the drift process. De Gregorio and Iacus (2008) considered a least squares approach following the lines of Bai (1994, 1997) of a simplified model also under discrete sampling while Song and Lee (2009) considered a CUSUM approach.

Finally it should be noted that the problems of the change-point of drift for ergodic diffusion processes have been treated by Kutoyants (1994, 2004), but the asymptotics and the sampling schemes are different from this paper.

The paper is organized as follows. Section 2 introduces the model of observation, the regularity conditions and some notation. Section 3 studies consistency and the rate of convergence of estimator of the change while asymptotic distributions are considered in Section 4. A mixture of certain Wiener functionals appears as the limit of the likelihood ratio random field, and it characterizes the limit distribution of the change-point estimator. Those sections assume that consistent estimators of the volatility parameters are available. Section 5 presents some practical considerations and a proposal to obtain first stage estimators of the volatility parameters which allow to obtain all asymptotic properties stated in the previous sections. Finally,
Section 6 presents some numerical analysis to assess the performance of the estimators. Tables are collected at the end of the paper.

2 Estimator for the change-point of the volatility

Consider a d-dimensional Itô process described by the stochastic differential equation
\[
dY_t = b_t dt + \sigma(X_t, \theta) dW_t, \quad t \in [0, T],
\]
where \( W_t \) is an \( r \)-dimensional standard Wiener process, on a stochastic basis, \( b_t \) and \( X_t \) are vector valued progressively measurable processes, and \( \sigma(x, \theta) \) is a matrix valued function.

We assume that there is the time \( t^* \) across which the diffusion coefficient changes from \( \sigma(x, \theta_0) \) to \( \sigma(x, \theta_1) \). The change point \( t^* \in (0, T) \) is unknown and we want to estimate \( t^* \) based on the observations sampled from the path of \( (X, Y) \). The coefficient \( \sigma(x, \theta) \) is assumed to be known up to the parameter \( \theta \), while \( b_t \) is completely unknown and unobservable, therefore possibly depending on \( \theta \) and \( t^* \).

The sample consists of \( (X_i, Y_i) \), \( i = 0, 1, \ldots, n \), where \( t_i = ih \) for \( h = h_n = T/n \). The parameter space \( \Theta \) of \( \theta \) is a bounded domain in \( \mathbb{R}^{d_0} \), \( d_0 \geq 1 \), and the parameter \( \theta \) is a nuisance in estimation of \( t^* \). Denote by \( \theta_i^* \) the true value of \( \theta_i \) for \( i = 0, 1 \).

Let \( \vartheta_n = |\theta_1^* - \theta_0^*| \). We will consider the following two different situations.

(A) \( \theta_0^* \) and \( \theta_1^* \) are fixed and do not depend on \( n \).

(B) \( \theta_0^* \) and \( \theta_1^* \) depend on \( n \), and as \( n \to \infty \), \( \theta_0^* \to \theta^* \in \Theta \), \( \vartheta_n \to 0 \) and \( n\vartheta_n^2 \to \infty \).

In Case (A), \( \vartheta_n \) is a constant \( \vartheta_0 \) independent of \( n \).

We shall formulate the problem more precisely. It will be assumed that the process \( Y \) generating the data is an Itô process realized on a stochastic basis \( \mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, P) \) with filtration \( \mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]} \), and satisfies the stochastic integral equation
\[
Y_t = \left\{ \begin{array}{ll}
Y_0 + \int_0^t b_s ds + \int_0^t \sigma(X_s, \theta_0^*) dW_s & \text{for } t \in [0, t^*) \\
Y_{t^*} + \int_{t^*}^t b_s ds + \int_{t^*}^t \sigma(X_s, \theta_1^*) dW_s & \text{for } t \in [t^*, T].
\end{array} \right.
\]
Here $W_t$ is an $r$-dimensional $\mathcal{F}$-Wiener process on $\mathcal{B}$, and $b_t$, $X_t$ and $\sigma(x, \theta)$ satisfy the conditions below. Let $\mathcal{X}$ be a closed set in $\mathbb{R}^{d_1}$ (possibly $\mathcal{X} = \mathbb{R}^{d_1}$) and denote the modulus of continuity of a function $f : I \to \mathbb{R}^{d_1}$ by

$$w_I(\delta, f) = \sup_{s, t \in I, |s - t| \leq \delta} |f(s) - f(t)|.$$

For matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, we write $A \otimes^2 = A^\top A$, $A[B] = \sum_{ij} a_{ij} b_{ij} = \text{Tr}(A^\top B)$, and the Euclidean norm of $A$ by $|A| = (A[A])^{1/2}$. Set $S(x, \theta) = \sigma(x, \theta) \otimes^2$. $\lambda_1(A)$ denotes the minimum eigenvalue of a symmetric matrix $A$.

\[ [H]_j \]

(i) $\sigma(x, t)$ is a measurable function defined on $\mathcal{X} \times [0, T]$ satisfying

(a) $\inf_{(x, \theta) \in \mathcal{X} \times \Theta} \lambda_1(S(x, \theta)) > 0$,

(b) derivatives $\partial_\theta^\ell \sigma$ ($0 \leq \ell \leq j + \lceil d_0/2 \rceil$) exist and those functions are continuous on $\mathcal{X} \times \Theta$,

(c) there exists a locally bounded function $L : \mathcal{X} \times \mathcal{X} \times \Theta \to \mathbb{R}_+$ such that

$$|\sigma(x, \theta) - \sigma(x', \theta)| \leq L(x, x', \theta)|x - x'|^\alpha \quad (x, x' \in \mathcal{X}, \theta \in \Theta)$$

for some constant $\alpha > 0$.

(ii) $(X_t)_{t \in [0, T]}$ is a progressively measurable process taking values in $\mathcal{X}$ such that

$$w_{[0, T]}\left(\frac{1}{n}, X\right) = o_P(n^{1/\alpha})$$

as $n \to \infty$.

(iii) $(b_t)_{t \in [0, T]}$ is a progressively measurable process taking values in $\mathbb{R}^d$ such that $(b_t - b_0)_{t \in [0, T]}$ is locally bounded.

**Remark 1.** The term “locally bounded” in $[H]_j$ (i) (c) means, as usual, being bounded on every compact set. The case where the drift $b_t$ changes its structure at time $t^*$, or any time in force, is included in our context because $b_t$ admits jumps. The case of time dependent $\sigma$ is included by making $X_t$ have argument $t$. Needless to say, if we set $X$ or a part of $X$ as $Y$, then our
model can express a system with feedback, in particular, a diffusion process. By \([H]_j\) (ii), \(t \mapsto X_t\) is continuous a.s. Also, \([H]_j\) (ii) imposes a restriction on the rate \(\vartheta_n\). For example, when \(\alpha = 1\), for a Brownian motion \(X\), it suffices that \(n\vartheta_n^2/\log n \to \infty\), due to Lévy property. The additional \([d_0/2]\) time differentiability to \(j\) is used only in Step (iii) of the proof of Theorem \(\text{[I]}\). Therefore, it is possible to replace the range of \(\ell\) to “0 \leq \ell \leq j” under a condition that ensures the the Hájek-Renyi type estimate just before going to Inequality \((4)\) below.

Write \(\Delta_i Y = Y_{t_i} - Y_{t_{i-1}}\) and let

\[
\Phi_n(t; \theta_0, \theta_1) = \sum_{i=1}^{[nt/T]} G_i(\theta_0) + \sum_{i=[nt/T]+1}^{n} G_i(\theta_1),
\]

where

\[
G_i(\theta) = \log \det S(X_{t_{i-1}}, \theta) + h^{-1}S(X_{t_{i-1}}, \theta)^{-1}[(\Delta_i Y)^\otimes 2].
\]

Suppose that there exists an estimator \(\hat{\theta}_k\) for each \(\theta_k\), \(k = 0, 1\). Each estimator is based on \((X_t, Y_t)_{i=0,1,\ldots,n}\) and so depends on \(n\). To make our discussion complete, in case \(\theta_k^*\) are known, we define \(\hat{\theta}_k\) just as \(\hat{\theta}_k = \theta_k^*\). This article proposes

\[
\hat{t}_n = \arg\min_{t \in [0,T]} \Phi_n(t; \hat{\theta}_0, \hat{\theta}_1)
\]

for the estimation of \(t^*\). More precisely, \(\hat{t}_n\) is any measurable function of \((X_t)_{i=0,1,\ldots,n}\) satisfying

\[
\Phi_n(\hat{t}_n; \hat{\theta}_0, \hat{\theta}_1) = \min_{t \in [0,T]} \Phi_n(t; \hat{\theta}_0, \hat{\theta}_1).
\]

3 Rate of convergence

We introduce identifiability conditions in order to ensure consistent estimation. In Case (A) we assume

\[
[A] \ P \left[ S(X_{t^*}; \theta_0^*) \neq S(X_{t^*}; \theta_1^*) \right] = 1;
\]

In Case (B) we assume
Remark 2. Since $\Xi(x, \theta^*)$ is the Hessian matrix of the nonnegative function

$$Q(x, \theta^*, \theta) := \text{Tr} \left( S(x, \theta^*)^{-1} S(x, \theta) - I_d \right) - \log \det \left( S(x, \theta^*)^{-1} S(x, \theta) \right)$$

of $\theta$ at $\theta^*$, $\Xi(x, \theta^*)$ is nonnegative-definite.

The following property will be necessary to validate our estimating procedure.

\[ |\hat{\theta}_k - \theta^*_k| = o_p(\vartheta_n) \text{ as } n \to \infty \text{ for } k = 0, 1. \]

In case the parameters are known, $\hat{\theta}_k$ should read $\theta^*_k$, and then Condition [C] requires nothing. Section 5 presents an example of estimator for $\theta_k$ which satisfies Condition [C].

Here we state the result on the rate of convergence of our change-point estimator.

**Theorem 1.** The family \{n\vartheta^2_n(\hat{t}_n - t^*)\}_{n \in \mathbb{N}} is tight under any one of the following conditions.

(a) [H]_1, [A] and [C] hold in Case (A).

(b) [H]_2, [B] and [C] hold in Case (B).

In Case (B), this result gives consistency of $\hat{t}_n$ since $n\vartheta^2_n \to \infty$ by assumption.

The rest of this section will be devoted to the proof of Theorem 1. Define a stopping time $\tau = \tau(K)$ by

$$\tau(K) = \inf \left\{ t; |X_t| + |b_t| > K \right\} \wedge T$$

for $K > 0$. $X^\tau$ denotes the process $X$ stopped at $\tau$. Write $S_i(\theta) = S(X_{t_i}^\tau, \theta)$, and $\Delta_i Y^\tau = Y_{t_i}^\tau - Y_{t_{i-1}}^\tau$. Let

$$\Psi_n(t; \theta_0, \theta_1) = \sum_{i=1}^{[nt/T]} g_i(\theta_0) + \sum_{i=[nt/T]+1}^{n} g_i(\theta_1),$$
where
\[ g_i(\theta) = 1_{\{\tau > 0\}} \left\{ \log \det S_{i-1}(\theta) + h^{-1}S_{i-1}(\theta)^{-1}[(\Delta_iY\tau)^{\otimes 2}] \right\} \]
\[ = 1_{\{\tau > 0\}} \log \det S_{i-1}(\theta) + h^{-1}S_{i-1}(\theta)^{-1}[(\Delta_iY\tau)^{\otimes 2}]. \]

Then \( \sup_{\theta \in \mathcal{K}} |g_i(\theta)| \in L^\infty \) for any compact set \( \mathcal{K} \) in \( \Theta \) under \([H]_1\). Denote by \( E_{i-1}^{\theta_t} \) the conditional expectation with respect to \( \mathcal{F}_{t_{i-1}} \) under the true distribution for \( t_{i-1} \geq t^* \).

**Lemma 1.** For \( t > t^* \),
\[ \Psi_n(t; \theta_0, \theta_1) - \Psi_n(t^*; \theta_0, \theta_1) = M_n(t; \theta_0, \theta_1) + A_n(t; \theta_0, \theta_1) + \rho_n(t; \theta_0, \theta_1), \]
where
\[ M_n(t; \theta_0, \theta_1) = \sum_{i=\lceil nt^*/T \rceil + 1}^{\lceil nt/T \rceil} \left\{ [g_i(\theta_0) - g_i(\theta_1)] - E_{i-1}^{g_i(\theta_0) - g_i(\theta_1)} \right\}, \]
\[ A_n(t; \theta_0, \theta_1) = 1_{\{\tau > 0\}} \sum_{i=\lceil nt^*/T \rceil + 1}^{\lceil nt/T \rceil} \left\{ \text{Tr} \left( S_{i-1}(\theta_0)^{-1}S_{i-1}(\theta_1) - I_d \right) \right. \]
\[ - \log \det \left( S_{i-1}(\theta_0)^{-1}S_{i-1}(\theta_1) \right) \}, \]
\[ \rho_n(t; \theta_0, \theta_1) = 1_{\{\tau > 0\}} \sum_{i=\lceil nt^*/T \rceil + 1}^{\lceil nt/T \rceil} \text{Tr} \left\{ \left( S_{i-1}(\theta_1)^{-1} - S_{i-1}(\theta_0)^{-1} \right) \right. \]
\[ \cdot \left( S_{i-1}(\theta_1) - h^{-1}E_{i-1}^{g_i(\theta_0) - g_i(\theta_1)}[(\Delta_iY\tau)^{\otimes 2}] \right) \}. \]

The proof of Lemma 1 is straightforward and omitted.

**Remark 3.** Later we will consider substitution of estimators \( \hat{\theta}_k \) to \( \theta_k \), \( k = 0, 1 \). Then the expectation \( E_{i-1}^{\theta_t}[g_i(\theta_0) - g_i(\theta_1)] \) is taken before the substitution, and so
\[ M_n(t; \hat{\theta}_0, \hat{\theta}_1) = \sum_{i=\lceil nt^*/T \rceil + 1}^{\lceil nt/T \rceil} \left\{ [g_i(\hat{\theta}_0) - g_i(\hat{\theta}_1)] - E_{i-1}^{\theta_t}[g_i(\theta_0) - g_i(\theta_1)] \right\}_{\theta_0=\hat{\theta}_0, \theta_1=\hat{\theta}_1}. \]

In particular, the second term in the braces is not necessarily \( \mathcal{F}_{t_{i-1}} \)-measurable.
We will need a uniform Hájek-Rényi inequality. Let $D$ be a bounded open set in $\mathbb{R}^d$. The Sobolev norm is denoted by

$$\|f\|_{s,p} = \left\{ \sum_{i=0}^{s} \|\partial_i^s f\|_{L^p(D)}^p \right\}^{1/p}$$

for $f \in W^{s,p}(D)$, the Sobolev space with indices $(s, p)$. Suppose that $p > 1$ and $s > d/p$. The embedding inequality is the following

$$\sup_{\theta \in D} |f(\theta)| \leq C \|f\|_{s,p} \quad (f \in W^{s,p}(D)) \quad (2)$$

where $C$ is a constant depending only on $s, p$ and $D$. We will apply this inequality for $f \in C^s(D)$, and the validity of such an inequality depends on the regularity of the boundary of $D$; see e.g. Yoshida (2005) for the relation to the GRR inequality.

**Lemma 2.** Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_j)_{j \in \mathbb{Z}_+}, P)$ be a stochastic basis. Let $D$ be a bounded domain in $\mathbb{R}^d$ admitting Sobolev’s inequality (2) for some $p \in (1, 2]$ and $s \in \mathbb{N}$ such that $s > d/p$. Let $(c_j)_{j \in \mathbb{Z}_+}$ be a nondecreasing sequence of positive numbers. Let $X = (X_j)_{j \in \mathbb{Z}_+}$ be a sequence of random fields on $D$ for $j \in \mathbb{Z}_+$ satisfying the following conditions:

(i) For each $(w, j) \in \Omega \times \mathbb{Z}_+$, $X_j \in C^s(D)$;

(ii) For each $(\theta, i) \in D \times \{0, 1, ..., s\}$, $(\partial_\theta^i X_j(\theta))_{j \in \mathbb{Z}_+}$ is a zero-mean $L^p$-martingale with respect to $\mathbf{F}$.

Then there exists a constant $C'$ depending only on $s, p$, and $D$, not depending on $X$, such that

$$P\left[ \max_{j \leq n} \frac{1}{c_j} \sup_{D} |X_j(\theta)| \geq a \right] \leq \frac{C'}{a^p} \sum_{j=0}^{n} \frac{1}{c_j^p} E\left[ \|X_j - X_{j-1}\|_{s,p}^p \right]$$

for all $a > 0$ and $n \in \mathbb{Z}_+$.

**Proof.** Let $B = L^p(D)$, then $B$ is $p$-uniformly smooth; see Example 2.2 of Woyczyński (1975), p. 247. We apply Theorem in Shixin (1997) to conclude

$$P\left[ \max_{j \leq n} \frac{1}{c_j} \|\partial_\theta^i X_j\|_B \geq a \right] \leq \frac{C_1}{a^p} \sum_{j=0}^{n} \frac{1}{c_j^p} E\left[ \|\partial_\theta^i X_j - \partial_\theta^i X_{j-1}\|_B^p \right]$$

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for \( i \in \{0, 1, \ldots, s\} \) for some constant \( C_1 \). Therefore (2) yields the result.

**Proof of Theorem**

For the proof, we may assume \( T = 1 \) for notational simplicity without loss of generality.

(i) Let \( \epsilon \) be an arbitrary positive number. Set
\[
H(x) = 4Q(x, \theta_0^*, \theta_1^*) \theta_0^{-2}
\]
in Case (A), and set \( H(x) = \lambda_1(\Xi(x, \theta^*)) \) in Case (B). We denote \( \sigma(t; \theta) = \sigma(X_t^*, \theta) \) and \( h(t) = H(X_t^*) \) in what follows. Those processes depend on \( K \) by definition while it is suppressed from the symbols. Set \( B_K = \{ \tau = 1 \} \) and fix a sufficiently large \( K \) so that \( P[B_K^c] < \epsilon/4 \).

We notice that \( h(s) \geq 0 \) and that \( h(t^*) > 0 \) a.s. on \( B_K \) from the identifiability condition [A]/[B] since \( X_{t^*}^* = X_{t^*} \) on \( B_K \). We will show that there exists a positive constant \( c_\epsilon \) such that
\[
P\left[ \inf_{t \in [t^*, 1]} \frac{1}{t - t^*} \int_{t^*}^t h(s) \, ds \leq 5c_\epsilon \right] < \epsilon.
\]
Define the event \( \mathcal{A}_\delta \) by
\[
\mathcal{A}_\delta = \left\{ \inf_{t \in [t^*, t^*+\delta]} h(s) \geq \frac{1}{2} h(t^*) \right\}
\]
for \( \delta \in (0, 1 - t^*) \). On \( \mathcal{A}_\delta \), it holds that
\[
\inf_{t \in [t^*, t^*+\delta]} \frac{1}{t - t^*} \int_{t^*}^t h(s) \, ds \geq \frac{1}{2} h(t^*) \geq \frac{\delta}{2(1 - t^*)} h(t^*)
\]
and also that, for \( t \in [t^*+\delta, 1] \),
\[
\frac{1}{t - t^*} \int_{t^*}^t h(s) \, ds \geq \frac{1}{1 - t^*} \int_{t^*}^t h(s) \, ds \geq \frac{1}{1 - t^*} \int_{t^*+\delta}^{t^*+\delta} h(s) \, ds \geq \frac{\delta}{2(1 - t^*)} h(t^*).\]
Choose a $\delta$ so that $P[A_3] > 1 - \epsilon/2$ by the continuity of $h$, and next choose a positive number $c_\epsilon = c(\epsilon, \delta)$ such that

$$P\left[ \frac{\delta}{2(1-t^*)} h(t^*) > 5c_\epsilon \right] \geq P\left[ \frac{\delta}{2(1-t^*)} h(t^*) > 5c_\epsilon \right] \cap B_K \geq 1 - \frac{\epsilon}{2}.$$  

Then

$$P\left[ \inf_{t \in [t^*, 1]} \frac{1}{t - t^*} \int_{t^*}^t h(s) \, ds \leq 5c_\epsilon \right] \leq P[A_3^c] + P\left[ A_4, \inf_{t \in [t^*, 1]} \frac{1}{t - t^*} \int_{t^*}^t h(s) \, ds \leq 5c_\epsilon \right] < \epsilon.$$

(ii) With Lemma 11 we decompose $\Psi_n(t; \hat{\theta}_0, \hat{\theta}_1) - \Psi_n(t^*; \hat{\theta}_0, \hat{\theta}_1)$ as follows:

$$\Psi_n(t; \hat{\theta}_0, \hat{\theta}_1) - \Psi_n(t^*; \hat{\theta}_0, \hat{\theta}_1) = M_n(t; \hat{\theta}_0, \hat{\theta}_1) + A_n(t; \hat{\theta}_0, \hat{\theta}_1) + \rho_n(t; \hat{\theta}_0, \hat{\theta}_1).$$

Let $M \geq 1$. We have

$$P[n \theta_n^2 (\hat{\theta}_n - t^*) > M] \leq P\left[ \inf_{t: n \theta_n^2 (t - t^*) > M} \Phi_n(t; \hat{\theta}_0, \hat{\theta}_1) \leq \Phi_n(t^*; \hat{\theta}_0, \hat{\theta}_1) \right] \leq P\left[ \inf_{t: n \theta_n^2 (t - t^*) > M} \Psi_n(t; \hat{\theta}_0, \hat{\theta}_1) \leq \Psi_n(t^*; \hat{\theta}_0, \hat{\theta}_1) \right] + P[B_K^c] < P_{1,n} + P_{2,n} + P_{3,n} + \epsilon,$$

where

$$P_{1,n} = P\left[ \sup_{t: n \theta_n^2 (t - t^*) > M} \frac{1}{n t - [n t]} \left| M_n(t; \hat{\theta}_0, \hat{\theta}_1) \right| \geq c_\epsilon \theta_n^2/3 \right],$$

$$P_{2,n} = P\left[ \inf_{t: n \theta_n^2 (t - t^*) > M} \frac{1}{n t - [n t]} \left| A_n(t; \hat{\theta}_0, \hat{\theta}_1) \right| \leq c_\epsilon \theta_n^2 \right],$$

$$P_{3,n} = P\left[ \sup_{t: n \theta_n^2 (t - t^*) > M} \frac{1}{n t - [n t]} \left| \rho_n(t; \hat{\theta}_0, \hat{\theta}_1) \right| \geq c_\epsilon \theta_n^2/3 \right].$$

Here we read $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. We will estimate these terms.

(iii) Estimate of $P_{1,n}$. In Case (B), let

$$\mathcal{M}_n(t; \theta) = \sum_{i=\lceil nt^* \rceil + 1}^{\lceil nt \rceil} \left\{ \partial_\theta g_i(\theta) - E_{i-1}^{\theta_n^2} [\partial_\theta g_i(\theta)] \right\}.$$
Let \( \hat{\Theta} \) be an open ball such that \( \theta^* \in \hat{\Theta} \) and \( \overline{\Theta} \subset \Theta \). Since
\[
\sup_{\theta_0, \theta_1 \in \hat{\Theta}} |M_n(t; \theta_0, \theta_1)| |\theta_0 - \theta_1|^{-1} \leq \sup_{\theta \in \hat{\Theta}} |M_n(t; \theta)|,
\]
one has
\[
P_{1,n} \leq P \left[ \sup_{t : n\theta_0^*(t - t^*) > M \left[ nt \right] - \left[ nt^* \right]} \frac{1}{n\theta_0^*(t - t^*) \left[ nt \right] - \left[ nt^* \right]} |M_n(t; \hat{\theta}_0, \hat{\theta}_1)| |\hat{\theta}_0 - \hat{\theta}_1|^{-1} \geq \frac{c_\epsilon \theta_0^*}{6}, \; \hat{\theta}_0, \hat{\theta}_1 \in \hat{\Theta} \right]
+ P[|\hat{\theta}_0 - \hat{\theta}_1| \geq 2\theta_0^*] + P[\hat{\theta}_0 \notin \hat{\Theta}] + P[\hat{\theta}_1 \notin \hat{\Theta}]
\leq P \left[ \sup_{t : n\theta_0^*(t - t^*) > M \left[ nt \right] - \left[ nt^* \right]} \frac{1}{n\theta_0^*(t - t^*) \left[ nt \right] - \left[ nt^* \right]} \sup_{\theta \in \hat{\Theta}} |M_n(t; \theta)| \geq \frac{c_\epsilon \theta_0^*}{6} \right]
+ P[|\hat{\theta}_0 - \hat{\theta}_1| \geq 2\theta_0^*] + P[\hat{\theta}_0 \notin \hat{\Theta}] + P[\hat{\theta}_1 \notin \hat{\Theta}].
\]

By the uniform version of the Hájek-Rényi inequality in Lemma 2 applied to the case \( p = 2 \), \( s = 2 + [d_0/2] \) and \( D = \hat{\Theta} \), we see under \([H]_2\) that
\[
P \left[ \sup_{t : n\theta_0^*(t - t^*) > M \left[ nt \right] - \left[ nt^* \right]} \frac{1}{n\theta_0^*(t - t^*) \left[ nt \right] - \left[ nt^* \right]} \sup_{\theta \in \hat{\Theta}} |M_n(t; \theta)| \geq \frac{c_\epsilon \theta_0^*}{6} \right] \leq \frac{C}{c_\epsilon^2 M} =: \rho_\epsilon(M),
\]
therefore
\[
\lim_{n \to \infty} P_{1,n} \leq \rho_\epsilon(M)
\]
thanks to
\[
P[|\hat{\theta}_0 - \hat{\theta}_1| \geq 2\theta_0^*] \leq P[|\hat{\theta}_0 - \theta_0^*| \geq \frac{1}{3}\theta_0^*] + P[|\hat{\theta}_1 - \theta_1^*| \geq \frac{1}{3}\theta_0^*]
\]
for large \( n \).

In Case (A), Let \( \hat{\Theta}_k \) be an open ball such that \( \overline{\Theta}_k \subset \Theta \) and \( \theta_k^* \in \hat{\Theta}_k \) for each \( k = 0, 1 \).

\[
P_{1,n} \leq P \left[ \sup_{t : n\theta_0^*(t - t^*) > M \left[ nt \right] - \left[ nt^* \right]} \frac{1}{n\theta_0^*(t - t^*) \left[ nt \right] - \left[ nt^* \right]} \sup_{\theta_0 \in \Theta_0} \sup_{\theta_1 \in \Theta_1} |M_n(t; \theta_0, \theta_1)| \geq \frac{c_\epsilon \theta_0^2}{3} \right]
+ P[\hat{\theta}_0 \notin \hat{\Theta}_0] + P[\hat{\theta}_1 \notin \hat{\Theta}_1].
\]
We apply the Hájek-Renyi inequality for $M_n(t; \theta_0, \theta_1)$, which is a difference of two random fields on $\hat{\Theta}_k$ to be done with one by one, in order to obtain (i) under $[H]_1$.

(iv) Estimation of $P_{2,n}$. First we consider Case (B). There is a positive constant $c_2$ independent of $n$ such that

$$\begin{align*}
\text{Tr} \left( S_{i-1}(\hat{\theta}_0)^{-1} S_{i-1}(\hat{\theta}_1) - I_d \right) - \log \det \left( S_{i-1}(\hat{\theta}_0)^{-1} S_{i-1}(\hat{\theta}_1) \right) \\
\geq \Xi(X^T_{t_{i-1}}, \theta^*)[(\hat{\theta}_1 - \hat{\theta}_0)^{\otimes 2}] + r_{n,i-1} |\hat{\theta}_1 - \hat{\theta}_0|^2 \\
\geq \{\lambda_1(\Xi(X^T_{t_{i-1}}, \theta^*)) + r_{n,i-1}\} |\hat{\theta}_1 - \hat{\theta}_0|^2
\end{align*}$$

for all $i$, where $\max_i |r_{n,i-1}| \leq c_2 \vartheta_n$, on the event

$$B_{K,n} = B_K \cap \{\hat{\theta}_0, \hat{\theta}_1 \in \hat{\Theta}, |\hat{\theta}_k - \theta^*| \leq \vartheta_n (k = 0, 1)\}.$$ 

Thus

$$P_{2,n} \leq P \left[ \inf_{t : n \vartheta_n^2 (t - t^*) > M} \frac{1}{[nt] - [nt^*]} A_n(t; \hat{\theta}_0, \hat{\theta}_1) |\hat{\theta}_1 - \hat{\theta}_0|^2 \leq 4c_\epsilon, B_{K,n} \right] + P \left[ |\hat{\theta}_1 - \hat{\theta}_0| \leq \frac{1}{2} \vartheta_n \right] + P[B^c_{K,n}]$$

$$\leq P \left[ \inf_{t : n \vartheta_n^2 (t - t^*) > M} \frac{1}{[nt] - [nt^*]} \sum_{i = [nt^*] + 1}^{[nt]} \{\lambda_1(\Xi(X^T_{t_{i-1}}, \theta^*)) + r_{n,i-1}\} \leq 4c_\epsilon \right] + \epsilon$$

for large $n$. The scaled summation converges to the corresponding scaled integral uniformly in $t$ a.s., hence from Step (i) we have

$$\lim_{n \to \infty} P_{2,n} \leq P \left[ \inf_{t \in [t^*, t]} \frac{1}{t - t^*} \int_{t^*}^t h(s) \, ds \leq 5c_\epsilon \right] + \epsilon < 2\epsilon$$

for large $n$. 

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We will consider Case (A). There is a positive constant $c_2$ independent of $n$ such that

$$\text{Tr} \left( S_{i-1}(\hat{\theta}_0)^{-1} S_{i-1}(\hat{\theta}_1) - I_d \right) - \log \det \left( S_{i-1}(\hat{\theta}_0)^{-1} S_{i-1}(\hat{\theta}_1) \right)$$

$$\geq \text{Tr} \left( S_{i-1}(\theta_0^*)^{-1} S_{i-1}(\theta_1^*) - I_d \right) - \log \det \left( S_{i-1}(\theta_0^*)^{-1} S_{i-1}(\theta_1^*) \right)$$

$$- c_2(|\hat{\theta}_1 - \theta_1^*| + |\hat{\theta}_0 - \theta_0|)$$

for all $i$ on the event $B_{K,n}' = B_K \cap \{\hat{\theta}_0 \in \hat{\Theta}_0, \hat{\theta}_1 \in \hat{\Theta}_1\}$ because there exists a continuous derivative $\partial_0 \sigma$ by $[H]_1$. In this way,

$$P_{2,n} \leq P \left[ \inf_{t_n(t-t^*) > M \overline{nt} - \overline{nt^*}} A_n(t; \hat{\theta}_0, \hat{\theta}_1) \leq c_2 \vartheta_0^2, B_{K,n}' \right] + P[B_{K,n}^c]$$

Therefore,

$$\lim_{n \to \infty} P_{2,n} \leq P \left[ \inf_{t \in [t^*, 1]} \frac{1}{t - t^*} \int_{t^*}^t h(s) \, ds \leq 5c \epsilon \right] + \epsilon$$

by Step (i).

(v) Estimation of $P_{3,n}$. We have

$$\sup_{t \in [t^*, 1]} \left| S(X_t, \hat{\theta}_k) - S(X_t, \theta_k^*) \right| 1_{\{\hat{\theta}_k - \theta_k^* < 2\vartheta_n\} \cap B_K} \leq C \vartheta_n \quad (k = 0, 1),$$

$$\sup_{t \in [t^*, 1]} \left| S(X_t, \hat{\theta}_k) - S(X_t, \theta_k^*)^{-1} \right| 1_{\{\hat{\theta}_k - \theta_k^* < 2\vartheta_n\} \cap B_K} \leq C \vartheta_n \quad (k = 0, 1)$$

and

$$\sup_{i \geq \overline{\|nt^*\|} 2} \left| S_{i-1}(\theta_k^*) - h^{-1} E_{t-i}^{\theta_k^*}[(\Delta_i Y)^{\otimes 2}] \right|_{1_{B_K}} \leq C w_{[0,T]}(X, \frac{1}{n})^\alpha.$$
because of $[H]_j$ (ii). Consequently, we see $\lim_{n \to \infty} P_{\epsilon,n} \leq \epsilon$ due to $[C]$ and the localization by $B_K$.

(vi) From the estimates in Steps (ii)-(iv) and making $K$ sufficiently large, we have

$$\lim_{n \to \infty} P[n \psi_n^2(\hat{t}_n - t^*) > M] \leq \rho_\epsilon(M) + 5\epsilon$$

for any $M \geq 1$ and $\epsilon > 0$. Therefore, we have

$$\lim_{M \to \infty} \lim_{n \to \infty} P[n \psi_n^2(\hat{t}_n - t^*) > M] \leq 5\epsilon,$$

which shows the tightness of $\{n \psi_n^2(\hat{t}_n - t^*)\}_n$. In a quite similar way, we can show that $\{n \psi_n^2(\hat{t}_n - t^*)\}_n$ is tight, and hence the family $\{n \psi_n^2(\hat{t}_n - t^*)\}_n$ is tight.

\[ \square \]

4 Asymptotic distribution of the change point estimator

This section discusses limit theorems for the distribution of the estimators. First we consider Case (B).

Let

$$\mathbb{H}(v) = -2 \left( \frac{1}{2} \Gamma_\eta \mathcal{W}(v) - \frac{1}{2} \Gamma_\eta |v| \right)$$

for $\Gamma_\eta = (2T)^{-1} \Xi(X_{t^*}, \theta^*| \eta \otimes 2]$. Here $\mathcal{W}$ is a two-sided standard Wiener process independent of $X_{t^*}$.

**Theorem 2.** Suppose that the limit $\eta = \lim_{n \to \infty} \psi_n^{-1}(\hat{\theta}_1 - \theta_0^*)$ exists. Suppose that $[H]_2$, $[C]$ and $[B]$ are fulfilled in Case (B). Then $n \psi_n^2(\hat{t}_n - t^*) \to^{d} \arg\min_{v \in \mathbb{R}} \mathbb{H}(v)$ as $n \to \infty$.

We will prove Theorem 2 and assume for a while that $T = 1$ to simplify the notation. Introduce a new parameter $v$ as $t = t_v^\dagger := t^* + v(n \psi_n^2)^{-1}$. Let

$$D_n(v) = \left\{ \Psi_n(t_v^\dagger; \hat{\theta}_0, \hat{\theta}_1) - \Psi_n(t^*; \hat{\theta}_0, \hat{\theta}_1) \right\} - \left\{ \Psi_n(t_v^\dagger; \hat{\theta}_0^*, \hat{\theta}_1^*) - \Psi_n(t^*; \hat{\theta}_0^*, \hat{\theta}_1^*) \right\}$$

$$= \left\{ M_n(t_v^\dagger; \hat{\theta}_0, \hat{\theta}_1) - M_n(t_v^\dagger; \hat{\theta}_0^*, \hat{\theta}_1^*) \right\} + \left\{ A_n(t_v^\dagger; \hat{\theta}_0, \hat{\theta}_1) - A_n(t_v^\dagger; \hat{\theta}_0^*, \hat{\theta}_1^*) \right\}$$

$$+ \{ \rho_n(t_v^\dagger; \hat{\theta}_0, \hat{\theta}_1) - \rho_n(t_v^\dagger; \hat{\theta}_0^*, \hat{\theta}_1^*) \}.$$
Lemma 3. For every $L > 0$,

$$\sup_{v \in [-L, L]} |D_n(v)| \rightarrow^p 0$$

as $n \rightarrow \infty$.

Proof. We assume that $v > 0$. We have

$$
M_n(t^\dagger_v; \hat{\theta}_0, \hat{\theta}_1) - M_n(t^\dagger_v; \theta^*_0, \theta^*_1) \\
= \int_0^1 \partial_n \partial_\theta M_n(t^\dagger_v; \theta^*_0 + u(\hat{\theta}_0 - \theta^*_0), \theta^*_1 + u(\hat{\theta}_1 - \theta^*_1)) \, du \, [\partial_n^{-1}(\hat{\theta}_0 - \theta^*_0, \hat{\theta}_1 - \theta^*_1)].
$$

For $k = 0, 1$ and $j = 1, 2$,

$$E \left[ \sup_{t \in [t^*, t^* + L(nv^2_n)^{-1}]} |\partial^j_{\theta_k} \partial_{\theta_k} M_n(t; \theta_0, \theta_1)|^2 \right] \leq 8E \left[ |\partial^j_{\theta_k} M_n(t^* + L(nv^2_n)^{-1}; \theta_0, \theta_1)|^2 \right] + O(1)
$$

$$\leq 8L \partial_n^{-2} \sup_{i \geq 1} E \left[ |\partial^j_{\theta_k} g_i(\theta_k)|^2 \right] + O(1)
$$

$$\leq C \partial_n^{-2}.$$

Then Sobolev’s inequality implies

$$\partial_n \sup_{t \in [t^*, t^* + L(nv^2_n)^{-1}], \theta_0, \theta_1 \in \Theta} |\partial_{\theta} M_n(t; \theta_0, \theta_1)| = O_p(1).$$

As a result,

$$\sup_{v \in [0, L]} |M_n(t^\dagger_v; \hat{\theta}_0, \hat{\theta}_1) - M_n(t^\dagger_v; \theta^*_0, \theta^*_1)| \rightarrow^p 0$$

as $n \rightarrow \infty$.

Set $r_n = |\hat{\theta}_0 - \theta^*_0| + |\hat{\theta}_1 - \theta^*_1|$. Simple calculus yields

$$|\{\text{Tr } y - \log \det(I_d + y)\} - \{\text{Tr } x - \log \det(I_d + x)\}| \leq c_3 |y - x| (|x| + |y - x|)$$

for $d \times d$-symmetric matrices $x$ and $y$ whenever $|x|, |y| \leq c'_3$, where $c'_3$ and $c_3$ are some positive constants independent of $x, y$. Indeed, the formula

$$\int \exp(-2^{-1}(I_d + \epsilon x)^\odot 2) dz = (2\pi)^{d/2} \det(I_d + \epsilon x)^{-1/2}$$

is convenient for explicit computation.
Applying this inequality to \( y = S_{i-1}(\hat{\theta}_0)^{-1/2}S_{i-1}(\hat{\theta}_1)S_{i-1}(\hat{\theta}_0)^{-1/2} - I_d \) and 
\( x = S_{i-1}(\theta^*_0)^{-1/2}S_{i-1}(\theta^*_1)S_{i-1}(\theta^*_0)^{-1/2} - I_d \), we see that there exists a constant 
\( c_4 \) such that for large \( n \), on \( B_K \cap \{ |\hat{\theta}_k - \theta^*| < \vartheta_n (k = 0, 1) \} \),
\[
|A_n(t; \hat{\theta}_0, \hat{\theta}_1) - A_n(t; \theta^*_0, \theta^*_1)| \leq c_4 \sum_{i=[nt^*]+1}^{[nt]} r_n(\vartheta_n + r_n).
\]
Therefore, for any \( \epsilon > 0 \), if we take sufficiently large \( K \), then
\[
\lim_{n \to \infty} P \left[ \sup_{t \in [t^*, t^* + L(n\vartheta_n)^{-1}]} |A_n(t; \hat{\theta}_0, \hat{\theta}_1) - A_n(t; \theta^*_0, \theta^*_1)| \geq \epsilon \right] \leq \epsilon.
\]
This implies
\[
\sup_{v \in [0, L]} |A_n(t^1_v; \hat{\theta}_0, \hat{\theta}_1) - A_n(t^1_v; \theta^*_0, \theta^*_1)| \to^p 0
\]
as \( n \to \infty \). The convergence
\[
\sup_{v \in [0, L]} |\rho_n(t^1_v; \hat{\theta}_0, \hat{\theta}_1) - \rho_n(t^1_v; \theta^*_0, \theta^*_1)| \to^p 0
\]
can be shown in the same way as (5).

A similar proof of the uniform convergence on \([-L, 0]\) is possible. After all, we obtained the desired result. \( \square \)

**Remark 4.** When \( \theta^*_k (k = 0, 1) \) are known, we do not need Lemma 3. Thus we can focus only on \( \Psi_n(t^1_v; \theta^*_0, \theta^*_1) - \Psi_n(t^*; \theta^*_0, \theta^*_1) \). For simplicity, we write \( \Psi_n^*(t) \) for \( \Psi_n(t; \theta^*_0, \theta^*_1) \). By assumption, there exists a limit \( \eta = \lim_{n \to \infty} \varphi_n^{-1}(\theta^*_1 - \theta^*_0) \). \( \mathcal{D} \) denotes the \( D \)-space on an interval of \( t \). Let
\[
\mathbb{H}_n(v) = \Psi_n^* (t^* + v(m \vartheta_n^2)^{-1}) - \Psi_n^*(t^*).
\]
and
\[
\mathbb{H}^*(v) = -2 \left( \Gamma_{\eta, \tau}^1 \mathcal{W}(v) - \frac{1}{2} \Gamma_{\eta, \tau} |v| \right)
\]
for \( \Gamma_{\eta, \tau} = 1_{\{\tau > 0\}} (2T)^{-1} \Xi(X^*_\tau, \theta^*) |\eta|^2 \).
Lemma 4. Let \( \eta = \lim_{n \to \infty} \vartheta_n^{-1}(\theta^*_0 - \theta^*_1) \). Suppose that \([H]_2, [C]\) and \([B]\) are fulfilled in Case (B). Then \( \mathbb{H}_n \to_d (\mathcal{F}_T) \mathbb{H}^r \) in \( \mathcal{D}([-L, L]) \) as \( n \to \infty \) for every \( L > 0 \).

Proof. We will only consider positive \( v \) since the argument is essentially the same for negative \( v \). Let \( T = 1 \) as before. It follows from Lemma \( \Pi \) that

\[
\mathbb{H}_n(v) = M_n^\Delta(v) + A_n^\Delta(v) + \rho_n^\Delta(v),
\]

where

\[
\begin{align*}
M_n^\Delta(v) &= M_n(t^* + v(n \vartheta_n^2)^{-1}; \theta^*_0, \theta^*_1), \\
A_n^\Delta(v) &= A_n(t^* + v(n \vartheta_n^2)^{-1}; \theta^*_0, \theta^*_1), \\
\rho_n^\Delta(v) &= \rho_n(t^* + v(n \vartheta_n^2)^{-1}; \theta^*_0, \theta^*_1).
\end{align*}
\]

The evaluation of these terms will be done in the following. As repeated previously, we may proceed discussion on the event \( B_K \) hereafter. First

\[
M_n^\Delta(v) = 1_{\{\tau > 0\}} \sum_{i = [nt^*] + 1}^{[nt^* + \vartheta_n^{-2}v]} \mathbb{E} \left[ (S_{i-1}(\theta^*_0)^{-1} - S_{i-1}(\theta^*_1)^{-1}) \right]
\]

\[
\cdot \left( h^{-1} \left( \int_{t_{i-1}}^{t_i} \sigma(X_t^\tau, \theta^*_1) dW_t + E_{i-1}^\theta \left[ \int_{t_{i-1}}^{t_i} S(X_t^\tau, \theta^*_1) dt \right] \right) \right] + o_p(1)
\]

(6)

where \( U_n(v) = \tilde{\varrho}_p(1) \) means that \( \sup_{v \in [0, L]} |U_n(v)| \to_p 0 \), and we used the hypothesis \( n \vartheta_n^2 \to \infty \) and the fact that \( |S_{i-1}(\theta^*_0)^{-1} - S_{i-1}(\theta^*_1)^{-1}| \leq C \vartheta_n \) with the localization. To obtain \( \tilde{\varrho}_p(1) \), \( L^1 \)-estimate helps. It follows from \([H]_i\)

(i)(c) and (ii) that

\[
\begin{align*}
&\left| h^{-1} \left( \int_{t_{i-1}}^{t_i} S(X_t^\tau, \theta^*_0) dt - E_{i-1}^\theta \left[ \int_{t_{i-1}}^{t_i} S(X_t^\tau, \theta^*_1) dt \right] \right) \right| \\
&= \left| h^{-1} \left( \int_{t_{i-1}}^{t_i} [S(X_t^\tau, \theta^*_0) - S(X_t^\tau, \theta^*_1)] dt \\
- E_{i-1}^\theta \left[ \int_{t_{i-1}}^{t_i} [S(X_t^\tau, \theta^*_0) - S(X_t^\tau, \theta^*_1)] dt \right] \right) \right| \\
&\leq C \|w_{[0,T]}(n^{-1}, X)\| \alpha \\
&= o_p(\vartheta_n).
\end{align*}
\]
Moreover, with the Burkholder-Davis-Gundy inequality, the first terms on the right-hand side of (6) equals $\tilde{M}_n^\Delta(v) + \bar{\sigma}_p(1)$ with $\tilde{M}_n^\Delta(v) = \sum_{i=|nt^*|+1}^{nt^*} \xi_n, i$, where

$$\xi_{n, i} = 1_{\{r > 0\}} \frac{1}{(h^{-1}(\Delta_i W)^\otimes 2 - I_r)}$$

(7)

and $\sigma_{i-1}(\theta) = \sigma(X^\tau_{i-1}, \theta)$.

Now we introduce the backward approximation

$$\tilde{\xi}_{n, i} = 1_{\{r > 0\}} \frac{1}{(h^{-1}(\Delta_i W)^\otimes 2 - I_r)}$$

to $\xi_{n, i}$ for $\epsilon_n = 2Ln^{-1}\vartheta_n^{-2}$. After all, by $\tilde{M}_n^\Delta(v) = \sum_{i=|nt^*|+1}^{nt^*} \tilde{\xi}_{n, i}$, we have

$$M_n^\Delta(v) = \tilde{M}_n^\Delta(v) + \bar{\sigma}_p(1), \quad (8)$$

Since

$$1_{\{r > 0\}} 2\vartheta_n^{-2}v^1 \sigma_{i-1}(\theta^*_1) \sigma_{i-1}(\theta^*_1) \sigma_{i-1}(\theta^*_1) \lvert S(X^\tau_{i-1}, \theta^*_1) - S(X^\tau_{i-1}, \theta^*_1) - (\bar{\sigma}_p(1)\eta^2) v,$$

the central limit theorem ensures the convergence $\tilde{M}_n^\Delta \rightarrow^d -2\Gamma_{\eta, \tau} W$ in $D([0, L])$. Indeed, the joint convergence of $X^\tau_{i-1}, \tau_{i-1} \bar{\sigma}_{i-1}(X^\tau_{i-1}, \theta^*_1) -(S(X^\tau_{i-1}, \theta^*_1) - S(X^\tau_{i-1}, \theta^*_1) - (\bar{\sigma}_p(1)\eta^2) v,$

In the same fashion, we can show $\tilde{M}_n^\Delta \rightarrow^d -2\Gamma_{\eta, \tau} W \cdot (+L)$ in $D([-L, 0])$ if $\tilde{M}_n^\Delta$ is defined in a natural way over negative $v$, where $(\mathcal{W}'(v))_{v \in [0, L]}$ is a standard Wiener process independent of $(\mathcal{W}(v))_{v \in [0, L]}$ and $\mathcal{F}$. Since $\Xi(X^\tau_{i}, \theta^*_1) \lvert \eta^2 \gamma$ is independent of $\mathcal{W}'$, we can replace the stochastic integral with respect to $\mathcal{W}'$ in the representation of the limit distribution of $\tilde{M}_n^\Delta$ by the one with respect to the negative-time part of the two sided Wiener process $\mathcal{W}$ reversible in time. Easy calculations yield $sup_{v \in [-L, L]} |A^\Delta_n(v) - \Gamma_{\eta, \tau} v| \rightarrow^p 0$ and $sup_{v \in [-L, L]} |p^\Delta_n(v)| \rightarrow^p 0$ for
extended $A^\Delta_n$ and $\rho^\Delta_n$ to $[-L, L]$, which completes the proof.

Proof of Theorem \[2\]. We have supposed that $T = 1$ to state the lemmas, and we start with this case. Write $\hat{v} = \arg\min_{v \in \mathbb{R}} \mathbb{H}(v)$. For $\epsilon > 0$, take large $K$ so that $P[\tau = T] > 1 - \epsilon$. It follows from Lemma \[1\] that for every $x \in \mathbb{R}$,

$$\lim_{n \to \infty} P[n\theta^2_n(\hat{t}_n - t^*) \leq x] = \epsilon$$

$$\leq \lim_{n \to \infty} P[\inf_{v \in [-L,x]} \mathbb{H}^T_1(v) \leq \inf_{v \in [x,L]} \mathbb{H}^T_1(v) + \sup_n P[n\theta^2_n(\hat{t}_n - t^*) \notin [-L,L]]$$

$$= P[\inf_{v \in [-L,x]} \mathbb{H}^T(v) \leq \inf_{v \in [x,L]} \mathbb{H}^T(v) + \sup P[n\theta^2_n(\hat{t}_n - t^*) \notin [-L,L]]$$

$$\leq \epsilon + P[\hat{v} \leq x] + P[\hat{v} \notin [-L,L]] + \sup P[n\theta^2_n(\hat{t}_n - t^*) \notin [-L,L]]$$

As $L \to \infty$, the last two terms of the right-hand side of the above inequality tend to 0 thanks to Theorem \[1\](b). So we have obtained

$$\lim_{n \to \infty} P[n\theta^2_n(\hat{t}_n - t^*) \leq x] \leq P[\hat{v} \leq x].$$

The estimate of $P[n\theta^2_n(\hat{t}_n - t^*) \leq x]$ from below can be done in a similar manner, which concludes the proof in case $T = 1$.

For general $T$, we introduce a stochastic basis $\tilde{\mathcal{B}} = (\Omega, \mathcal{F}, \tilde{\mathcal{F}}, P)$ with $\tilde{\mathcal{F}} = (\mathcal{F}_T)_{u \in [0,1]}$, and the processes $\hat{b}_u = b_{Tu}$, $\tilde{X}_u = X_{Tu}$ and $\hat{Y}_u = Y_{Tu}$, $u \in [0,1]$, to scale the time as $t = Tu$. Those stochastic processes satisfy the stochastic integral equation

$$\tilde{Y}_u = \tilde{Y}_0 + \int_0^u \tilde{b}_r dr + \int_0^u \tilde{\sigma}(\tilde{X}_r, \theta) d\tilde{W}_r,$$

where $\tilde{\sigma}(x, \theta) = \sqrt{T}\sigma(x, \theta)$ and $\tilde{W}$ is an $r$-dimensional $\tilde{\mathcal{F}}$-Wiener process. The sampling times $(iT/n)_{i=0}^{n}$ now change to $(i/n)_{i=0}^{n}$ in the new setting after scaling time. For the change point estimator $\hat{u}_n$ for $u^* = T^{-1}t^*$, we know

$$n\theta^2_n(\hat{u}_n - u^*) \rightarrow_{d^*} \arg\min_{v \in \mathbb{R}} \mathbb{H}(\hat{v}),$$

where $\mathbb{H}(\hat{v}) = -2(\tilde{\Gamma}_\eta \hat{W}(\hat{v}) - 2^{-1} \tilde{\Gamma}_\eta |\hat{v}|)$, $\tilde{\Gamma}_\eta = 2^{-1} \Xi(\hat{X}_{u^*}, \theta^*)[\eta^{\otimes 2}]$ and $\hat{W}$ is a two-sided Wiener process independent of $\tilde{\sigma}(\hat{X}_{u^*}, \theta^*) = \sqrt{T}\sigma(X_{t^*}, \theta^*)$. Since

$$T \arg\min_{v \in \mathbb{R}} \mathbb{H}(\hat{v}) = \arg\min_{v \in \mathbb{R}} \mathbb{H}(\frac{v}{T})$$

$$= \arg\min_{v \in \mathbb{R}} \tilde{\mathbb{H}}(v)$$

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thanks to $\mathcal{W}(\cdot) \overset{\text{d}}{=} T^{1/2} \tilde{\mathcal{W}}(\cdot/T)$. Thus (9) gives the desired convergence of $\hat{t}_n$ since $\hat{t}_n = T \hat{u}_n$.

Let us investigate the limit distribution of the estimator in Case (A). By nature of the sampling scheme, only the set $\mathcal{G}_n = \{kT/n; k \in \mathbb{Z}\}$ has essential meaning for the optimization with respect to the parameter $t$. Without loss of generality, we modify $\hat{t}_n$ so that it takes values in $\mathcal{G}_n$, and set $\hat{k}_n = n \hat{t}_n / T$.

Let $K(v) = \sum_{i=1}^v \left\{ \text{Tr} \left[ \sigma(X_{t^*}, \theta^*_0) \left( S(X_{t^*}, \theta^*_0)^{-1} - S(X_{t^*}, \theta^*_1)^{-1} \right) \sigma(X_{t^*}, \theta^*_1) \right] \right\} - \log \det \left( S(X_{t^*}, \theta^*_0)^{-1} S(X_{t^*}, \theta^*_1) \right)$, where $\zeta_i$ are independent $r$-dimensional standard normal variables independent of $X_{t^*}$.

**Theorem 3.** Suppose that [H]1, [C] and [A] are fulfilled in Case (A). Then $\hat{k}_n - [n \hat{t}_n/T] \rightarrow^d \arg\min_{v \in \mathbb{Z}} K(v)$ as $n \rightarrow \infty$.

**Proof.** We change the definition of $t^*$ and newly set $t^*_v = [n \hat{t}_n/T] + T v$. Lemma 3 is still valid by essentially the same proof and hence we may only consider $\Psi_n(t^*_v; \theta^*_0, \theta^*_1) - \Psi_n(t^*_0; \theta^*_0, \theta^*_1)$. Writing $\Psi_n^*(t)$ for $\Psi_n(t; \theta^*_0, \theta^*_1)$, we will investigate the behavior of the random field

$$K_n(v) = \Psi_n^*(t^*_v) - \Psi_n^*(t^*_0)$$

on $v \in \mathbb{Z}$. For a while, we consider nonnegative $v$. The argument is similar for negative $v$. According to Lemma 1 we have the decomposition

$$K_n(v) = M_n(v) + A_n(v) + \varrho_n(v),$$

where $M_n(v) = M_n(t^*_v; \theta^*_0, \theta^*_1)$, $A_n(v) = A_n(t^*_v; \theta^*_0, \theta^*_1)$ and $\varrho_n(v) = \rho_n(t^*_v; \theta^*_0, \theta^*_1)$.

Now, $M_n(v)$ admits a similar expansion as before:

$$M_n(v) = \sum_{i=[n \hat{t}_n/T]+1}^{[n t^*/T]+v} \xi_{n,i} + \tilde{o}_p(1)$$

with $\xi_{n,i}$ given by (7). Moreover, for $\epsilon_n = n^{-1/2}$ this time, we consider the backward approximation of $\xi_{n,i}$, that is,

$$\xi_{n,i} = \tilde{\xi}_{n,i} + o_p(1).$$
Here $v \in [0, L] \cap \mathbb{Z}$, however this approximation is available when we consider $v \in [-L, 0]$. Let $L_0$ be the maximum integer in $[0, L]$. By continuity of $\sigma$ and because $W$ is an $F$-Wiener process, we have

$$
(X_{t^* - n}, \langle h^{-1}(\Delta_i W) \otimes 2 \rangle_{i=[nt^*/T]-L_0}^{[nt^*/T]+L_0} ) \to d (X_{t^*}, (\zeta_i \otimes 2)_{i=-L_0}^{L_0}),
$$

where $\zeta_i$ are independent $r$-dimensional standard normal variables independent of $X_{t^*}$; we use the same symbol $\zeta_i$ as in the statement. Consequently,

$$
(X_{t^*}, M_n(v))_{v=-L_0}^{L_0} \to_d (X_{t^*}, M_\infty(v))_{v=-L_0}^{L_0},
$$

where

$$
M_\infty(v) = \sum_{i=[nt^*/T]+1}^{[nt^*/T]+v} \xi_{\infty,i}
$$

and $\xi_{\infty,i}$ is given by

$$
\xi_{\infty,i} = 1_{\{\tau > 0\}} \text{Tr} \left[ \sigma(X_{t^*}, \theta_1^*) \left( S(X_{t^*}, \theta_0^*)^{-1} - S(X_{t^*}, \theta_1^*)^{-1} \right) \sigma(X_{t^*}, \theta_1^*) \cdot (\zeta_i \otimes 2 - I_r) \right].
$$

For $A_n$, we have $A_n(v) \to A_\infty(v)$ with

$$
A_\infty(v) = 1_{\{\tau > 0\}} \sum_{i=[nt^*/T]+1}^{[nt^*/T]+v} \left\{ \text{Tr} \left( S(X_{t^*}, \theta_0^*)^{-1} S(X_{t^*}, \theta_1^*) - I_d \right) - \log \det \left( S(X_{t^*}, \theta_0^*)^{-1} S(X_{t^*}, \theta_1^*) \right) \right\}.
$$

On the other hand, $\vartheta_n(v)$ tends to 0 uniformly in $v$. Therefore,

$$
(\mathbb{K}_{n}(v))_{v=-L_0}^{L_0} \to_d (\mathbb{K}^\tau(v))_{v=-L_0}^{L_0},
$$

where $\mathbb{K}^\tau(v) = M_\infty(v) + A_\infty(v)$. Removing $\tau$ by letting $K \to \infty$, and using Theorem 1 we obtain the limit distribution of $\hat{t}_n$. \hfill \Box
5 Initial estimator for $\theta_k$

In this section, we will briefly discuss the construction of the initial estimators. There are two situations according to the prior knowledge of the parameter space $\mathbb{T}$ of the change point. The first one is the case where $\mathbb{T} = [t_0, t_1] \subset (0,1)$ for given numbers $t_0$ and $t_1$. In the second case, we do not assume a prior information of $t_0$ and $t_1$, instead the precision of the initial estimator will be lost. Let

$$\Phi^0_n(t; \theta_0) = \sum_{i=1}^{[nt/T]} G_i(\theta_0) \quad \text{and} \quad \Phi^1_n(t; \theta_1) = \sum_{i=[nt/T]+1}^{n} G_i(\theta_1).$$

Suppose that $t_0$ and $t_1$ are known. Let $\hat{\theta}_0$ and $\hat{\theta}_1$ satisfy

$$\Phi_k^n(\hat{t}_n; \hat{\theta}_k) = \min_{\theta_k} \Phi_k^n(\hat{t}_n; \theta_k)$$

for $k = 0, 1$. To validate asymptotic properties of the estimators, it is sufficient that these relations are satisfied asymptotically. Under suitable regularity conditions as well as the identifiability conditions that

$$\int^0_{t_0} Q(X_t, \theta^*, \theta) \, dt > 0 \quad \text{a.s. and} \quad \int^{t_1}_{t} Q(X_t, \theta^*, \theta) \, dt > 0 \quad \text{a.s. (10)}$$

for every $\theta \neq \theta^*$, it is possible to show that $\hat{\theta}_k - \theta^*_k = O_p(n^{-1/2})$, therefore Condition [C] is satisfied in both cases (A) and (B). Based on $\hat{\theta}_k$, the estimator $\hat{t}_n$ are defined. According to the previous sections, $\hat{t}_n$ possesses $n\theta^2$-consistency and the asymptotic distribution in each case is already known.

We can also construct the second stage estimators. Let $b_n$ be a sequence of positive numbers such that $b_n = (n\theta^2)^{-1}$, where $\delta \in (2, \infty)$ is a constant satisfying $n\theta^\delta \to \infty$ as $n \to \infty$. Construct $\tilde{\theta}_k$ so that

$$\Phi^k_n(\hat{t}_n + (-1)^{k+1}b_n; \tilde{\theta}_k) = \min_{\hat{\theta}_k} \Phi^k_n(\hat{t}_n + (-1)^{k+1}b_n; \theta_k)$$

for $k = 0, 1$. The new estimators $\tilde{\theta}_k$ are expected to improve $\hat{\theta}_k$ since they utilize up to the data near $t^*$. Further, it is possible to construct a new change-point estimator with those estimators. Based on $\tilde{\theta}_k$, we define $\tilde{t}_n$ for $t^*$ as

$$\tilde{t}_n = \arg\min_{t \in [0,T]} \Phi_n(t; \tilde{\theta}_0, \tilde{\theta}_1).$$
Since it is usually easy to verify Condition [C] for \( \tilde{\theta}_k \), we will then obtain the same asymptotic results for \( \tilde{t}_n \) as \( \hat{t}_n \).

Next, let us consider the second situation. The knowledge of \( t_k \) is not available and it means that any data set sampled over a fixed time interval \([0, a]\) since \( t^* \) may be less than \( a \) and then the data over \((t^*, a]\) causes bias in general. A similar notice is also for the estimation of \( \theta_1 \). This consideration suggests the use of estimators \( \hat{\theta}_k \) based on the data over time interval \([0, a_n]\) for \( k = 0 \) and the one over \([T-a_n, T]\) for \( k = 1 \), respectively, for some sequence \( a_n \) tending to zero. We assume that there exist a constant \( \beta \in (0, 1/2) \) such that \( a_n \geq \frac{1}{n\vartheta_1^{1/\beta}} \) and that \( |\hat{\theta}_k - \theta_k^*| = o_p((a_n)^{-\beta}) \) for \( k = 0, 1 \). When \( \lim_{n \to \infty} \vartheta_n > 0 \), we also assume \( na_n \to \infty \). In particular, the first condition implies \( n\vartheta^2_n \to \infty \). The second condition is natural because the number of data is proportional to \( na_n \). To obtain \( \hat{\theta}_k \), we may need the identifiability condition that \( \sigma(\theta, x) = \sigma(\theta', x) \) implies \( \theta = \theta' \); it is a strong condition like monotonicity of \( \sigma(\theta, x) \) in \( \theta \). Under the assumptions, [C] holds and after that it is possible to construct \( \hat{t}_n, \tilde{\theta}_k \) and \( \hat{t}_n \) in turn as mentioned above. The asymptotic properties of \( \tilde{t}_n \) are the same as \( \hat{t}_n \) because \( \hat{\theta}_k \)'s satisfy Condition [C]. It is expected that the new estimator \( \tilde{t}_n \) possesses equal or better precision than \( \hat{t}_n \) as numerical studies in Section 6 suggest.

6 Numerical studies

In this section we run some simulation experiments to assess the quality of the estimator of the change point and of the volatilities, under two different models. We first consider the following diffusion model without drift

\[
X_t = \begin{cases} 
X_0 + \int_0^t (1 + X_s^2)\theta_0^* dW_s & \text{for } t \in [0, t^*) \\
X_{t^*} + \int_{t^*}^t (1 + X_s^2)\theta_1^* dW_s & \text{for } t \in [t^*, T].
\end{cases}
\]

(11)

where \( t^* \) is the true change point assumed to be \( t^* = 0.6 \). The true value of the parameters are \( \theta_0^* = 0.2 \) and \( \theta_1^* = \theta_0^* + n^{-\gamma} \), with \( \gamma = \frac{1}{4} \). \( n \) is the sample size and \( T = nh = 1 \). The initial value is \( X_0 \) assumed to be constant, in particular we take \( X_0 = 5 \). The sequences \( a_n = b_n = \frac{1}{n\vartheta_1} \) with \( \delta = 3 \) so that they satisfy the properties required in Section 3. The first stage estimator of \( \theta_0^* \) (resp. \( \theta_1^* \)) is obtained using the first \( na_n \) observations from the left (resp. \( na_n \) from the right). We denote the first stage estimators with \( \hat{\theta}_i \).
Once the first stage estimators of $\theta_0^*$ and $\theta_1^*$ are available, the first stage estimator of $t^*$, i.e. $\hat{t}_n$ is obtained via

$$
\Phi_n(\hat{t}_n; \hat{\theta}_0, \hat{\theta}_1) = \min_{t \in [0, T]} \Phi_n(t; \hat{\theta}_0, \hat{\theta}_1).
$$

Then, with the first stage estimator of $t^*$ in hands, we calculate the second stage estimator of $\theta_i^*$ using observations in the interval $[0, \hat{t}_n - b_n]$ for $\theta_0^*$ and observations in the interval $[\hat{t}_n + b_n, T]$ for $\theta_1^*$. We denote the second stage estimators of $\theta_i^*$ by $\hat{\theta}_i$. Finally, the second stage estimator of $t^*$, i.e. $\hat{t}_n$, is obtained as

$$
\Phi_n(\hat{t}_n; \hat{\theta}_0, \hat{\theta}_1) = \min_{t \in [0, T]} \Phi_n(t; \hat{\theta}_0, \hat{\theta}_1).
$$

For comparison, we also report the value of the estimator $\tilde{t}_n$ obtained plugging the true parameter values in the contrast function, i.e. when the volatilities are supposed to be known

$$
\Phi_n(\tilde{t}_n; \theta_0^*, \theta_1^*) = \min_{t \in [0, T]} \Phi_n(t; \theta_0^*, \theta_1^*),
$$

and this can be considered as a benchmark. For the Monte Carlo setup, we consider different sample sizes $n = 1000, 2000, 5000$ and for each sample size $n$, we run $M = 10000$ Monte Carlo replications. Under this choice of $n$ the value of $\theta_1^* = 0.3778$, $0.3495$, and $0.3189$ respectively. The values of the sequences $a_n$ and $b_n$ are reported in Table 1. Observations are supposed to be sampled at sample rate $h = 1/n$. Table 1 also reports Monte Carlo estimates (i.e. average over the $M$ replications) of the volatility parameters $\theta_0$ and $\theta_1$ and the change point $t^*$. In parenthesis are the standard deviations of the Monte Carlo estimates. In the second experiment we consider a Cox-Ingersoll-Ross (1985) model

$$
X_t = \begin{cases} 
X_0 + \int_0^t \sqrt{\theta_0} X_s dW_s & \text{for } t \in [0, t^*) \\
X_{t^*} + \int_{t^*}^t \sqrt{\theta_1} X_s dW_s & \text{for } t \in [t^*, T].
\end{cases}
$$

(12)

with change point $t^* = 0.7$ and all remaining experimental conditions are the same as in previous experiment. The results are reported in Table 2. The difference in the two experiments is only in the regularity of the diffusion coefficient term. Comparing the two simulation results, it is possible to see
that the second stage estimators in the second experiment performs slightly better in term of the standard deviation.

We also consider the behaviour of the asymptotic distribution of the change point estimator for second stage estimator in the first model, for sample size $n = 5000$. In particular, due to mixed-normal limit, we studied the distribution of the studentized limiting distribution of $n\theta_n^2(\hat{t}_n - t^*)$ under the true model, i.e.

$$Z = n\theta_n^2(\hat{t}_n - t^*)\hat{\Gamma}(X_{t^*}, \theta_0),$$

with $\hat{\Gamma}(X_{t^*}, \theta_0) = (\log(1 + X_{t^*}^2))^2$. Then $Z$ converges to $W(v) - \frac{1}{2}|v|$ with density

$$f(x) = \frac{3}{2}e^{x^2}\left(1 - \Phi\left(\frac{3}{2}\sqrt{|x|}\right)\right) - \frac{1}{2}\left(1 - \Phi\left(\frac{1}{2}\sqrt{|x|}\right)\right)$$

and distribution function

$$F(x) = \begin{cases} g(x), & x > 0 \\ 1 - g(-x), & x \leq 0 \end{cases}$$

with $\Phi(x)$ the distribution function of the gaussian random variable, and

$$g(x) = 1 + \sqrt{\frac{x}{2\pi}}e^{-\frac{x^2}{8}} - \frac{1}{2}(x + 5)\Phi\left(-\frac{\sqrt{x}}{2}\right) + \frac{3}{2}e^{x}\Phi\left(-\frac{3}{2}\sqrt{x}\right)$$

(see e.g. Csörgő and Horváth, 1997). In Figure 1 we report the graphical representation of the histogram and empirical distribution function of $Z$ (over 10000 Monte Carlo replications) against their theoretical counterparts which looks quite reasonable.

References

[1] Bai, J. (1994) Least squares estimation of a shift in linear processes, *Journal of Times Series Analysis*, 15, 453-472.

[2] Bai, J. (1997) Estimation of a change point in multiple regression models, *The Review of Economics and Statistics*, 79, 551-563.

[3] Chen, G., Choi, Y.K., Zhou, Y. (2005) Nonparametric estimation of structural change points in volatility models for time series, *Journal of Econometrics*, 126, 79-144.
Figure 1: Histogram versus theoretical density function (up) and empirical distribution function versus theoretical distribution function (bottom) for the second stage change point estimator. Results of 10000 Monte Carlo replications and sample size $n = 5000$ for the first model.
[4] Cox, J.C., Ingersoll, J.E., Ross, S.A. (1985) A theory of the term structure of interest rates, *Econometrica*, **53**, 385–408.

[5] Csörgő, M., Horváth, L. (1997) *Limit Theorems in Change-point Analysis*. New York: Wiley.

[6] De Gregorio, A., Iacus, S.M. (2008) Least squares volatility change point estimation for partially observed diffusion processes, *Communications in Statistics, Theory and Methods*, **37**(15), 2342-2357.

[7] Hinkley, D.V. (1971) Inference about the change-point from cumulative sum tests, *Biometrika*, **58**, 509-523.

[8] Inclan, C., Tiao, G.C. (1994) Use of cumulative sums of squares for retrospective detection of change of variance, *Journal of the American Statistical Association*, **89**, 913-923.

[9] Kim, S., Cho, S., Lee, S. (2000) On the cusum test for parameter changes in GARCH(1,1) models. *Commun. Statist. Theory Methods*, **29**, 445-462.

[10] Kutoyants, Y. (1994) *Identification of Dynamical Systems with Small Noise*, Kluwer, Dordrecht.

[11] Kutoyants, Y. (2004) *Statistical Inference for Ergodic Diffusion Processes*, Springer-Verlag, London.

[12] Lee, S., Ha, J., Na, O., Na, S. (2003) The Cusum test for parameter change in time series models, *Scandinavian Journal of Statistics*, **30**, 781-796.

[13] Lee, S., Nishiyama, Y., Yoshida, N. (2006) Test for parameter change in diffusion processes by cusum statistics based on one-step estimators, *Ann. Inst. Statist. Mat.*, **58**, 211-222.

[14] Shixin, G. (1997) The Hájek-Rényi inequality for Banach space valued martingales and the $p$ smoothness of Banach spaces, *Statistics and Probability Letters*, **32**, 245-248.

[15] Song, J., Lee, S. (2009) Test for parameter change in discretely observed diffusion processes, forthcoming in *Statistical Inference for Stochastic Processes*. 

27
[16] Woyczyński, W.A. (1975) Geometry and Martingales in Banach Spaces, in Winter School on Probability, Kapracz, Springer Lecture Notes in Mathematics, Vol. 472, 235-275

[17] Yoshida, N.: Polynomial type large deviation inequality and its applications. reprint (2005), to appear in Annals of the Institute of Statistical Mathematics
\begin{table}[
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\textit{n} & \textit{a}_n & \bar{\textit{t}}_n & \hat{\theta}_0 & \hat{\theta}_1 & \hat{\textit{t}}_n & \bar{\theta}_0 & \bar{\theta}_1 & \bar{\textit{t}}_n \\
\hline
5000 & 0.1189 & 0.601 & 0.200 & 0.319 & 0.601 & 0.200 & 0.319 & 0.601 \\
& & (0.005) & (0.009) & (0.014) & (0.011) & (0.005) & (0.013) & (0.012) \\
2000 & 0.1495 & 0.601 & 0.200 & 0.349 & 0.601 & 0.200 & 0.349 & 0.601 \\
& & (0.008) & (0.013) & (0.020) & (0.014) & (0.008) & (0.017) & (0.015) \\
1000 & 0.1778 & 0.601 & 0.199 & 0.377 & 0.601 & 0.200 & 0.377 & 0.602 \\
& & (0.011) & (0.017) & (0.025) & (0.019) & (0.011) & (0.026) & (0.018) \\
\hline
\end{tabular}
\caption{Monte Carlo estimates for model (11) over 10000 replications. True values: \( \theta_0^* = 0.2, \theta_1^* = 0.378, 0.350, \) and 0.319 for different sample sizes \textit{n} = 1000, 2000 and 5000. True change point \textit{t}^* = 0.6.}
\end{table}

\begin{table}[
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\textit{n} & \textit{a}_n & \bar{\textit{t}}_n & \hat{\theta}_0 & \hat{\theta}_1 & \hat{\textit{t}}_n & \bar{\theta}_0 & \bar{\theta}_1 & \bar{\textit{t}}_n \\
\hline
5000 & 0.1189 & 0.701 & 0.200 & 0.319 & 0.701 & 0.200 & 0.319 & 0.701 \\
& & (0.010) & (0.012) & (0.018) & (0.011) & (0.018) & (0.012) & (0.010) \\
2000 & 0.1495 & 0.702 & 0.200 & 0.350 & 0.701 & 0.200 & 0.350 & 0.701 \\
& & (0.016) & (0.016) & (0.029) & (0.024) & (0.009) & (0.030) & (0.021) \\
1000 & 0.1778 & 0.703 & 0.200 & 0.378 & 0.701 & 0.200 & 0.377 & 0.701 \\
& & (0.025) & (0.021) & (0.040) & (0.038) & (0.012) & (0.056) & (0.040) \\
\hline
\end{tabular}
\caption{Monte Carlo estimates for model (12) over 10000 replications. True values: \( \theta_0^* = 0.2, \theta_1^* = 0.378, 0.350, \) and 0.319 for different sample sizes \textit{n} = 1000, 2000 and 5000. True change point \textit{t}^* = 0.7.}
\end{table}