JETS AND CONNECTIONS IN COMMUTATIVE AND NONCOMMUTATIVE GEOMETRY

LUIGI MANGIAROTTI
Department of Mathematics and Physics, University of Camerino, 62032 Camerino (MC), Italy

GENNADI SARDANASHVILY
Department of Theoretical Physics, Physics Faculty, Moscow State University, 117234 Moscow, Russia

It is emphasized that equivalent definitions of connections on modules over a commutative ring are not so in noncommutative geometry.

1 Introduction

The jet modules $\mathcal{J}^k(P)$ of a module $P$ over a commutative ring $\mathcal{A}$ are well-known to be a representative object of linear differential operator on $P$ [1]. Furthermore, a connection on a module $\mathcal{A}$ is defined to be a splitting of the exact sequence

$$0 \rightarrow \mathcal{D}^1 \otimes P \rightarrow \mathcal{J}^1(P) \xrightarrow{\sigma_1} P \rightarrow 0,$$

where $\mathcal{D}^1$ is the module of differentials of $\mathcal{A}$. In the case of structure modules of smooth vector bundles, these notions of jets and connections coincide with those in differential geometry of fibre bundles where connections on a fibre bundle $Y \rightarrow X$ are sections of the affine jet bundle $J^1Y \rightarrow Y$ [2]. In general, the notion of jets of modules fails to be extended to modules over a noncommutative ring $\mathcal{A}$ since it implies a certain commutativity property of a differential calculus $\mathcal{D}^*$ over $\mathcal{A}$. In relation to this circumstance, we match different definitions of connections which being equivalent for modules over a commutative ring are not so in noncommutative geometry.

1E-mail: mangiaro@camserv.unicam.it
2E-mail: sard@grav.phys.msu.su
2 Modules in noncommutative geometry

Let $\mathcal{A}$ be an associative unital algebra over a commutative ring $\mathcal{K}$, i.e., $\mathcal{A}$ is a $\mathcal{K}$-ring. One considers right [left] $\mathcal{A}$-modules and $\mathcal{A}$-bimodules (or $\mathcal{A}-\mathcal{A}$-bimodules in the terminology of [3]). A bimodule $P$ over an algebra $\mathcal{A}$ is called a central bimodule if

$$pa = ap, \quad \forall p \in P, \quad \forall a \in \mathcal{Z}(\mathcal{A}),$$

where $\mathcal{Z}(\mathcal{A})$ is the centre of the algebra $\mathcal{A}$. By a centre of a $\mathcal{A}$-bimodule $P$ is called a $\mathcal{K}$-submodule $\mathcal{Z}(P)$ of $P$ such that

$$pa \overset{\text{def}}{=} ap, \quad \forall p \in \mathcal{Z}(P), \quad \forall a \in \mathcal{A}.$$ 

If $\mathcal{A}$ is a commutative algebra, every right [left] module $P$ over $\mathcal{A}$ becomes canonically a central bimodule by putting

$$pa = ap, \quad \forall p \in P, \quad \forall a \in \mathcal{A}.$$ 

If $\mathcal{A}$ is a noncommutative algebra, every right [left] $\mathcal{A}$-module $P$ is also a $\mathcal{Z}(\mathcal{A})-\mathcal{A}$-bimodule [a $\mathcal{A}-\mathcal{Z}(\mathcal{A})$-bimodule] such that the equality (2) takes place, i.e., it is a central $\mathcal{Z}(\mathcal{A})$-bimodule. From now on, by a $\mathcal{Z}(\mathcal{A})$-bimodule is meant a central $\mathcal{Z}(\mathcal{A})$-bimodule. For the sake of brevity, we say that, given an associative algebra $\mathcal{A}$, right and left $\mathcal{A}$-modules, central $\mathcal{A}$-bimodules and $\mathcal{Z}(\mathcal{A})$-modules are $\mathcal{A}$-modules of type $(1,0)$, $(0,1)$, $(1,1)$ and $(0,0)$, respectively, where $A_0 = \mathcal{Z}(\mathcal{A})$ and $A_1 = \mathcal{A}$.

Using this notation, let us recall a few basic operations with modules.

- If $P$ and $P'$ are $\mathcal{A}$-modules of the same type $(i,j)$, so is its direct sum $P \oplus P'$.

- Let $P$ and $P'$ be $\mathcal{A}$-modules of types $(i,k)$ and $(k,j)$, respectively. Their tensor product $P \otimes P'$ (see [3]) defines an $\mathcal{A}$-module of type $(i,j)$.

- Given an $\mathcal{A}$-module $P$ of type $(i,j)$, let $P^* = \text{Hom}_{\mathcal{A}_1-\mathcal{A}_j}(P, \mathcal{A})$ be its $\mathcal{A}$-dual. One can show that $P^*$ is the module of type $(i+1,j+1)\mod 2$ [3]. In particular, $P$ and $P^{**}$ are $\mathcal{A}$-modules of the same type. There is the natural homomorphism $P \rightarrow P^{**}$. For instance, if $P$ is a projective module of finite rank, so is its dual $P^*$ and $P \rightarrow P^{**}$ is an isomorphism [3].
There are several equivalent definitions of a projective module. One says that a right [left] module $P$ is projective if $P$ is a direct summand of a right [left] free module, i.e., there exists a module $Q$ such that $P \oplus Q$ is a free module $\mathcal{F}$. Accordingly, a module $P$ is projective if and only if $P = pS$ where $S$ is a free module and $p$ is an idempotent, i.e., an endomorphism of $S$ such that $p^2 = p$. We will refer to projective $\mathbb{C}^\infty(X)$-modules of finite rank in connection with the Serre–Swan theorem below. Recall that a module is said to be of finite rank or simply finite if it is a quotient of a finitely generated free module.

Noncommutative geometry deals with unital complex involutive algebras (i.e., unital $\ast$-algebras) as a rule. Let $\mathcal{A}$ be such an algebra (see [3]). It should be emphasized that one cannot use right or left $\mathcal{A}$-modules, but only modules of type $(1,1)$ and $(0,0)$ since the involution of $\mathcal{A}$ reverses the order of product in $\mathcal{A}$. A central $\mathcal{A}$-bimodule $P$ over $\mathcal{A}$ is said to be a $\ast$-module over a $\ast$-algebra $\mathcal{A}$ if it is equipped with an antilinear involution $p \mapsto p^\ast$ such that

$$(apb)^\ast = b^\ast p^\ast a^\ast, \quad \forall a, b \in \mathcal{A}, \quad p \in P.$$ 

A $\ast$-module is said to be a finite projective module if it is a finite projective right [left] module.

As well-known, noncommutative geometry is developed in main as a generalization of the calculus in commutative rings of smooth functions. Let $X$ be a locally compact topological space and $\mathcal{A}$ a $\ast$-algebra $\mathbb{C}^0_0(X)$ of complex continuous functions on $X$ which vanish at infinity of $X$. Provided with the norm

$$||f|| = \sup_{x \in X} |f|, \quad f \in \mathcal{A},$$

this algebra is a $C^\ast$-algebra [3]. Its spectrum $\hat{\mathcal{A}}$ is homeomorphic to $X$. Conversely, any commutative $C^\ast$-algebra $\mathcal{A}$ has a locally compact spectrum $\hat{\mathcal{A}}$ and, in accordance with the well-known Gelfand–Naïmark theorem, it is isomorphic to the algebra $\mathbb{C}^0_0(\hat{\mathcal{A}})$ of complex continuous functions on $\hat{\mathcal{A}}$ which vanish at infinity of $\hat{\mathcal{A}}$ [3]. If $\mathcal{A}$ is a unital commutative $C^\ast$-algebra, its spectrum $\hat{\mathcal{A}}$ is compact. Let now $X$ be a compact manifold. The $\ast$-algebra $\mathbb{C}^\infty(X)$ of smooth complex functions on $X$ is a dense subalgebra of the unital $C^\ast$-algebra $\mathbb{C}^0(X)$ of continuous functions on $X$. This is not a $C^\ast$-algebra, but it is a Fréchet algebra in its natural locally convex topology of compact convergence for all derivatives. In noncommutative geometry, one does not use the theory of locally convex algebras (see [3]), but considers dense unital subalgebras of $C^\ast$-algebras in a purely algebraic fashion.
Let $E \to X$ be a smooth $m$-dimensional complex vector bundle over a compact manifold $X$. The module $E(X)$ of its global sections is a $*$-module over the ring $\mathbb{C}^\infty(X)$ of smooth complex functions on $X$. It is a projective module of finite rank. Indeed, let $(\phi_1, \ldots, \phi_q)$ be a smooth partition of unity such that $E$ is trivial over the sets $U_\zeta \supset \text{supp} \phi_\zeta$, together with the transition functions $\rho_\zeta\xi$. Then $p_\zeta\xi = \phi_\zeta \rho_\zeta\xi \phi_\xi$ are smooth $(m \times m)$-matrix-valued functions on $X$. They satisfy

$$\sum_\kappa p_{\zeta\kappa} p_{\kappa\xi} = p_{\zeta\xi},$$

and so assemble into a $(mq \times mq)$-matrix $p$ whose entries are smooth complex functions on $X$. Because of (3), we obtain $p^2 = p$. Then any section $s$ of $E \to X$ is represented by a column $(\phi_\zeta s^i)$ of smooth complex functions on $X$ such that $ps = s$. It follows that $s \in p\mathbb{C}(X)^mq$, i.e., $E(X)$ is a projective module. The above mentioned Serre–Swan theorem [7, 8] provides a converse assertion.

**Theorem 1.** Let $P$ be a finite projective $*$-module over $\mathbb{C}^\infty(X)$. There exists a complex smooth vector bundle $E$ over $X$ such that $P$ is isomorphic to the module $E(X)$ of global sections of $E$. □

In noncommutative geometry, one therefore thinks of a finite projective $*$-module over a dense unital $*$-subalgebra of a $C^*$-algebra as being a noncommutative vector bundle.

### 3 Commmutative differential calculus

Let us summarize some basic facts on the differential calculus in modules over a commutative $\mathcal{K}$-ring $\mathcal{A}$ [1, 2, 9].

Let $P$ and $Q$ be left $\mathcal{A}$-modules. Right modules are studied in a similar way. The set $\text{Hom}_\mathcal{K}(P, Q)$ of $\mathcal{K}$-module homomorphisms of $P$ into $Q$ is endowed with the $\mathcal{A} - \mathcal{A}$-bimodule structure by the left and right multiplications

$$(a\phi)(p) = a\phi(p), \quad (\phi \star a)(p) = \phi(ap), \quad a \in \mathcal{A}, \quad p \in P.$$  \hfill (4)

However, this is not a central $\mathcal{A}$-bimodule because $a\phi \neq \phi \star a$ in general. Let us denote

$$\delta_a \phi = a\phi - \phi \star a.$$  \hfill (5)
**Definition 2.** An element $\Delta \in \text{Hom}_K(P, Q)$ is called an $s$-order linear differential operator from the $A$-module $P$ to the $A$-module $Q$ if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s} \Delta = 0$$

for arbitrary collections of $s+1$ elements of $A$. It is also called a $Q$-valued differential operator on $P$. □

In particular, a first order linear differential operator $\Delta$ obeys the condition

$$\delta_a \circ \delta_b \Delta(p) = \Delta(abp) - a\Delta(bp) - b\Delta(ap) + ab\Delta(p) = 0$$

(6)

for all $p \in P$, $b, c \in A$.

A first order differential operator $\partial$ from $A$ to an $A$-module $Q$ is called the $Q$-valued derivation of the algebra $A$ if it obeys the Leibniz rule

$$\partial(aa') = a\partial(a') + a'\partial(a), \quad \forall a, a' \in A.$$  

(7)

This is a particular condition (3).

Turn now to the modules of jets. Given an $A$-module $P$, let us consider the tensor product $A \otimes_P K$ of $K$-modules provided with the left $A$-module structure

$$b(a \otimes p) \overset{def}= (ba) \otimes p, \quad \forall b \in A.$$  

(8)

For any $b \in A$, we introduce the left $A$-module morphism

$$\delta^b(a \otimes p) = (ba) \otimes p - a \otimes (bp).$$  

(9)

Let $\mu^{k+1}$ be the submodule of the left $A$-module $A \otimes_P K$ generated by all elements of the type

$$\delta^{b_0} \circ \cdots \circ \delta^{b_k}(1 \otimes p).$$

**Definition 3.** The $k$-order jet module of the $A$-module $P$ is defined to be the quotient $J^k(P)$ of $A \otimes_P K$ by $\mu^{k+1}$. It is a left $A$-module with respect to the multiplication

$$b(a \otimes p \mod \mu^{k+1}) = ba \otimes p \mod \mu^{k+1}.$$  

(10)

□
Besides the left $\mathcal{A}$-module structure induced by (8), the $k$-order jet module $\mathcal{J}^k(P)$ also admits the left $\mathcal{A}$-module structure given by the multiplication

$$b \star (a \otimes p \mod \mu^{k+1}) = a \otimes (bp) \mod \mu^{k+1}.$$  \hfill (11)

It is called the $\star$-left module structure. There is the $\star$-left $\mathcal{A}$-module homomorphism

$$J^k : P \to \mathcal{J}^k(P), \quad J^k p = 1 \otimes p \mod \mu^{k+1},$$

such that $\mathcal{J}^k(P)$ as a left $\mathcal{A}$-module is generated by the elements $J^k p$, $p \in P$. It is readily observed that the homomorphism $\mathcal{J}^k$ (12) is a $k$-order differential operator (compare the relation (8) and the relation (13) below).

**Remark 1.** If $P$ is a $\mathcal{A} - \mathcal{A}$-bimodule, the tensor product $\mathcal{A} \otimes \mathcal{K} P$ is also provided with the right $\mathcal{A}$-module structure

$$(a \otimes p)b \overset{\text{def}}{=} a \otimes pb, \quad \forall b \in \mathcal{A},$$

and so is the jet module $\mathcal{J}^k(P)$:

$$(a \otimes p \mod \mu^{k+1})b = a \otimes (pb) \mod \mu^{k+1}.$$  

If $P$ is a central bimodule, i.e.,

$$ap = pa, \quad \forall a \in \mathcal{A}, \quad p \in P,$$

the $\star$-left $\mathcal{A}$-module structure (11) is equivalent to the right $\mathcal{A}$-module structure (13).

The jet modules possess the properties similar to those of jet manifolds. In particular, since $\mu^r \subset \mu^s$, $r > s$, there is the the inverse system of epimorphisms

$$\mathcal{J}^s(P) \xrightarrow{\pi^s_{s-1}} \mathcal{J}^{s-1}(P) \longrightarrow \cdots \longrightarrow \pi^1_0 \mathcal{J}^1(P).$$

Given the repeated jet module $\mathcal{J}^s(\mathcal{J}^k(P))$, there exists the monomorphism $\mathcal{J}^{s+k}(P) \to \mathcal{J}^s(\mathcal{J}^k(P))$.

In particular, the first order jet module $\mathcal{J}^1(P)$ consists of elements $a \otimes p \mod \mu^2$, i.e., elements $a \otimes p$ modulo the relations

$$\delta^a \circ \delta^b(1 \otimes p) =$$

$$(\delta_a \circ \delta_b \mathcal{J}^1)(p) = 1 \otimes (abp) - a \otimes (bp) - b \otimes (ap) + ab \otimes p = 0.$$  \hfill (13)
The morphism $\pi_0^1 : J^1(P) \to P$ reads

$$\pi_0^1 : a \otimes p \mod \mu^2 \to ap.$$  

(14)

**Theorem 4.** For any differential operator $\Delta \in \text{Diff}_s(P, Q)$ there is a unique homomorphism $f^\Delta : J^s(P) \to Q$ such that the diagram

$$
\begin{align*}
\begin{array}{ccc}
P & \xrightarrow{J^k} & J^s(P) \\
\Delta & \searrow & f^\Delta \\
& Q & \\
\end{array}
\end{align*}
$$

is commutative. $\square$

**Proof.** The proof is based on the following fact [1]. Let $h \in \text{Hom}_A(A \otimes P, Q)$ and

$$\hat{a} : P \ni p \to a \otimes p \in A \otimes P,$$

then

$$\delta_b(h \circ \hat{a})(p) = h(\delta^b(a \otimes p)).$$

QED

The correspondence $\Delta \mapsto f^\Delta$ defines the isomorphism

$$\text{Hom}_A(J^s(P), Q) = \text{Diff}_s(P, Q),$$  

(15)

which shows that the jet module $J^s(P)$ is the representative object of the functor $Q \to \text{Diff}_s(P, Q)$.

Let us consider the particular jet modules $J^s(A)$ of the algebra $A$, denoted simply by $J^s$. The module $J^s$ can be provided with the structure of a commutative algebra with respect to the multiplication

$$ (aJ^s b) \cdot (a'J^s b) = aa'J^s(bb').$$

For instance, the algebra $J^1$ consists of the elements $a \otimes b$ modulo the relations

$$a \otimes b + b \otimes a = ab \otimes 1 + 1 \otimes ab.$$  

(16)

It has the left $A$-module structure

$$c((a \otimes b) \mod \mu^2) = (ca) \otimes b \mod \mu^2$$  

(17)
and the \(\star\)-left \(\mathcal{A}\)-module structure
\[
c \star ((a \otimes b) \mod \mu^2) = a \otimes (cb) \mod \mu^2
\] (18)
which coincides with the right \(\mathcal{A}\)-module structure (13). We have the canonical monomorphism of left \(\mathcal{A}\)-modules
\[
i_1 : \mathcal{A} \to \mathfrak{J}^1, \quad i_1 : a \mapsto a \otimes 1 \mod \mu^2
\] (19)
and the corresponding projection
\[
\mathfrak{J}^1 \to \mathfrak{J}^1 / \text{Im} i_1 = (\text{Ker } \mu^1) \mod \mu^2 = \mathfrak{D}^1,
\]
\[
a \otimes b \mod \mu^2 \mapsto (a \otimes b - ab \otimes 1) \mod \mu^2.
\] (20)
The quotient \(\mathfrak{D}^1\) consists of the elements
\[
(a \otimes b - ab \otimes 1) \mod \mu^2, \quad \forall a, b \in \mathcal{A}.
\]
It is provided both with the central \(\mathcal{A}\)-bimodule structure
\[
c((a \otimes b - ab \otimes 1) \mod \mu^2) = (ca \otimes b - cab \otimes 1) \mod \mu^2,
\]
\[
((1 \otimes ab - b \otimes a) \mod \mu^2)c = (1 \otimes abc - b \otimes ac) \mod \mu^2
\] (21)
and the \(\star\)-left \(\mathcal{A}\)-module structure
\[
c \star ((a \otimes b - ab \otimes 1) \mod \mu^2) = (a \otimes cb - acb \otimes 1) \mod \mu^2.
\] (23)
It is readily observed that the projection (20) is both the left and \(\star\)-left module morphisms. Then we have the \(\star\)-left module morphism
\[
d^1 : \mathcal{A} \xrightarrow{J^1} \mathfrak{J}^1 \to \mathfrak{D}^1,
\]
\[
d^1 : b \mapsto 1 \otimes b \mod \mu^2 \to (1 \otimes b - b \otimes 1) \mod \mu^2,
\] (24)
such that the central \(\mathcal{A}\)-bimodule \(\mathfrak{D}^1\) is generated by the elements \(d^1(b), b \in \mathcal{A}\), in accordance with the law
\[
ad^1 b = (a \otimes b - ab \otimes 1) \mod \mu^2 = (1 \otimes ab) - b \otimes a) \mod \mu^2 = (d^1 b) a.
\] (25)

**Proposition 5.** The morphism \(d^1\) (24) is a derivation from \(\mathcal{A}\) to \(\mathfrak{D}^1\) seen both as a left \(\mathcal{A}\)-module and \(\mathcal{A}\)-bimodule. \(\square\)

8
Proof. Using the relations (16), one obtains in an explicit form that
\[ d^1(ba) = (1 \otimes ba - ba \otimes 1) \mod \mu^2 = \]
\[ (b \otimes a + a \otimes b - ba \otimes 1 - ab \otimes 1) \mod \mu^2 = bd^1 a + ad^1 b. \] (26)

This is a \( \mathfrak{D}^1 \)-valued first order differential operator. At the same time,
\[ d^1(ba) = (1 \otimes ba - ba \otimes 1 + b \otimes a - b \otimes a) \mod \mu^2 = (d^1 b)a + bd^1 a. \]

QED

With the derivation \( d^1 \) (24), we get the left and \( \star \)-left module splitting
\[ \mathfrak{J}^1 = \mathcal{A} \oplus \mathfrak{D}^1, \] (27)
\[ a\mathfrak{J}^1(cb) = ai_1(cb) + ad^1(cb). \] (28)

Accordingly, there is the exact sequence
\[ 0 \to \mathfrak{D}^1 \to \mathfrak{J}^1 \to \mathcal{A} \to 0 \] (29)
which is split by the monomorphism (19).

**Proposition 6.** There is the isomorphism
\[ \mathfrak{J}^1(P) = \mathfrak{J}^1 \otimes P, \] (30)
where by \( \mathfrak{J}^1 \otimes P \) is meant the tensor product of the right (\( \star \)-left) \( \mathcal{A} \)-module \( \mathfrak{J}^1 \) (18) and the left \( \mathcal{A} \)-module \( P \), i.e.,
\[ [a \otimes b \mod \mu^2] \otimes p = [a \otimes 1 \mod \mu^2] \otimes bp. \]

\[ \square \]

**Proof.** The isomorphism (30) is given by the assignment
\[ (a \otimes bp) \mod \mu^2 \leftrightarrow [a \otimes b \mod \mu^2] \otimes p. \] (31)

QED

The isomorphism (27) leads to the isomorphism
\[ \mathfrak{J}^1(P) = (\mathcal{A} \oplus \mathfrak{D}^1) \otimes P, \]
\[ (a \otimes bp) \mod \mu^2 \leftrightarrow [(ab + ad^1(b)) \mod \mu^2] \otimes p, \]
and to the splitting of left and \( \ast \)-left \( \mathfrak{A} \)-modules

\[
\mathfrak{J}^1(P) = (\mathfrak{A} \otimes P) \oplus (\Omega^1 \otimes P),
\]

(32)

Applying the projection \( \pi^1_0 \) to the splitting (32), we obtain the exact sequence of left and \( \ast \)-left \( \mathfrak{A} \)-modules (1)

\[
0 \to [(a \otimes b - ab \otimes 1) \mod \mu^2] \otimes p \to [(c \otimes 1 + a \otimes b - ab \otimes 1) \mod \mu^2] \otimes p
\]

\[
= (c \otimes p + a \otimes bp - ab \otimes p) \mod \mu^2 \to cp,
\]

similar to the exact sequence (29). This exact sequence has the canonical splitting by the \( \ast \)-left \( \mathfrak{A} \)-module morphism

\[
P \ni ap \mapsto a \otimes p + d^1(a) \otimes p.
\]

However, the exact sequence (1) needs not be split by a left \( \mathfrak{A} \)-module morphism. Its splitting by a left \( \mathfrak{A} \)-module morphism (see (40) below) implies a connection. One can treat the canonical splitting (19) of the exact sequence (29) as being the canonical connection on the algebra \( \mathfrak{A} \).

In the case of \( \mathfrak{J}^s \), the isomorphism (13) takes the form

\[
\text{Hom}_\mathfrak{A}(\mathfrak{J}^s, Q) = \text{Diff}_s(\mathfrak{A}, Q).
\]

(33)

Then Theorem 4 and Proposition 5 lead to the isomorphism

\[
\text{Hom}_\mathfrak{A}(\Omega^1, Q) = \mathfrak{d}(\mathfrak{A}, Q).
\]

(34)

In other words, any \( Q \)-valued derivation of \( \mathfrak{A} \) is represented by the composition \( h \circ d^1 \), \( h \in \text{Hom}_\mathfrak{A}(\Omega^1, Q) \), due to the property \( d^1(1) = 0 \).

For instance, if \( Q = \mathfrak{A} \), the isomorphism (34) reduces to the duality relation

\[
\text{Hom}_\mathfrak{A}(\Omega^1, \mathfrak{A}) = \mathfrak{d}(\mathfrak{A}),
\]

(35)

\[
u(a) = u(d^1 a), \quad a \in \mathfrak{A},
\]

i.e., the module \( \mathfrak{d} \mathfrak{A} \) coincides with the left \( \mathfrak{A} \)-dual \( \Omega^{1*} \mathfrak{A} \) of \( \Omega^1 \).

Let us define the modules \( \Omega^k \) as the skew tensor products of the \( K \)-modules \( \Omega^1 \).

**Proposition 7.** There are the isomorphisms

\[
\text{Hom}_\mathfrak{A}(\Omega^k, Q) = \mathfrak{d}_k(\mathfrak{A}, Q),
\]

(36)

\[
\text{Hom}_\mathfrak{A}(\mathfrak{J}^1(\Omega^k), Q) = \mathfrak{d}_k(\text{Diff}_1(Q)).
\]

(37)
The isomorphism (36) is the higher order extension of the isomorphism (34). It shows that the module $\mathcal{D}^k$ is a representative object of the derivation functor $Q \rightarrow \mathfrak{d}_k(A, Q)$.

The isomorphism (37) implies the homomorphism

$$h^k : \mathcal{J}^1(\mathcal{D}^{k-1}) \rightarrow \mathcal{D}^k$$

and defines the operators of exterior differentiation

$$d^k = h^k \circ J^1 : \mathcal{D}^{k-1} \rightarrow \mathcal{D}^k.$$ (38)

These operators constitute the De Rham complex

$$0 \rightarrow A \xrightarrow{d^1} \mathcal{D}^1 \xrightarrow{d^2} \cdots \mathcal{D}^k \xrightarrow{d^{k+1}} \cdots.$$ (39)

4 Connections on commutative modules

There are several equivalent definition of connections on modules over a commutative ring.

**Definition 8.** By a connection on a $\mathcal{A}$-module $P$ is called a left $\mathcal{A}$-module morphism

$$\Gamma : P \rightarrow \mathcal{J}^1(P),$$

$$\Gamma(ap) = a\Gamma(p),$$

which splits the exact sequence (1). □

This splitting reads

$$J^1p = \Gamma(p) + \nabla\Gamma(p),$$ (42)

where $\nabla\Gamma$ is the complementary morphism

$$\nabla\Gamma : P \rightarrow \mathcal{D}^1 \otimes P,$$ (43)

$$\nabla\Gamma(p) = 1 \otimes p \mod \mu^2 - \Gamma(p).$$

This complementary morphism makes the sense of a covariant differential on the module $P$, but we will follow the tradition to use the terms ”covariant differential”
and “connection” on modules synonymously. With the relation (41), we find that \( \nabla^\Gamma \) obeys the Leibniz rule
\[
\nabla^\Gamma(ap) = da \otimes p + a\nabla^\Gamma(p).
\] (44)

**Definition 9.** By a connection on a \( \mathcal{A} \)-module \( P \) is meant any morphism \( \nabla \) which obeys the Leibniz rule (44), i.e., \( \nabla \) is a \( (\mathcal{O}^1 \otimes P) \)-valued first order differential operator on \( P \). \( \square \)

In view of Definition (9) and of the isomorphism (32), it is more convenient to rewrite the exact sequence (1) into the form
\[
0 \rightarrow \mathcal{O}^1 \otimes P \rightarrow (\mathcal{A} \oplus \mathcal{O}^1) \otimes P \rightarrow P \rightarrow 0.
\] (45)

Then a connection \( \nabla \) on \( P \) can be defined as a left \( \mathcal{A} \)-module splitting of this exact sequence.

In the case of the ring \( C^\infty(X) \) and a locally free \( C^\infty(X) \)-module \( S \) of finite rank, there exist the isomorphisms
\[
\mathcal{O}^1(X) = \text{Hom}_{C^\infty(X)}(\mathcal{O}(C^\infty(X)), C^\infty(X)),
\]
\[
\text{Hom}_{C^\infty(X)}(\mathcal{O}(C^\infty(X)), S) = \mathcal{O}^1(X) \otimes S.
\] (46)

With these isomorphisms, we come to other equivalent definitions of a connection on modules.

**Definition 10.** Any morphism
\[
\nabla : S \rightarrow \text{Hom}_{C^\infty(X)}(\mathcal{O}(C^\infty(X)), S)
\] (47)
satisfying the Leibniz rule (44) is called a connection on a \( C^\infty(X) \)-module \( S \). \( \square \)

**Definition 11.** By a connection on a \( C^\infty(X) \)-module \( S \) is meant a \( C^\infty(X) \)-module morphism
\[
\mathcal{O}(C^\infty(X)) \ni \tau \mapsto \nabla_\tau \in \text{Diff}_1(S, S)
\] (48)
such that the first order differential operators \( \nabla_\tau \) obey the rule
\[
\nabla_\tau(fs) = (\tau df)s + f\nabla_\tau s.
\] (49)
If a $\mathcal{S}$ is a commutative $C^\infty(X)$-ring, Definition 11 can be modified as follows.

**Definition 12.** By a connection on $C^\infty(X)$-ring $\mathcal{S}$ is meant any $C^\infty(X)$-module morphism

$$\mathfrak{d}(C^\infty(X)) \ni \tau \mapsto \nabla_\tau \in \mathfrak{d}\mathcal{S}$$

which is a connection on $\mathcal{S}$ as a $C^\infty(X)$-module, i.e., obeys the Leibniz rule (49).

Two such connections $\nabla_\tau$ and $\nabla'_\tau$ differ from each other in a derivation of the ring $\mathcal{S}$ which vanishes on $C^\infty(X) \subset \mathcal{S}$.

## 5 Noncommutative differential calculus

One believes that a noncommutative generalization of differential geometry should be given by a $\mathbb{Z}$-graded differential algebra which replaces the exterior algebra of differential forms [10]. This viewpoint is more general than that implicit above where a noncommutative ring replaces a ring of smooth functions.

Recall that a graded algebra $\Omega^*$ over a commutative ring $\mathcal{K}$ is defined as a direct sum

$$\Omega^* = \bigoplus_{k=0} \Omega^k$$

of $\mathcal{K}$-modules $\Omega^k$, provided with the associative multiplication law such that $\alpha \cdot \beta \in \Omega^{[\alpha]+[\beta]}$, where $[\alpha]$ denotes the degree of an element $\alpha \in \Omega^{[\alpha]}$. In particular, $\Omega^0$ is a unital $\mathcal{K}$-algebra $\mathcal{A}$, while $\Omega^{k>0}$ are $\mathcal{A}$-bimodules. A graded algebra $\Omega^*$ is called a graded differential algebra if it is a cochain complex of $\mathcal{K}$-modules

$$0 \longrightarrow \mathcal{A} \stackrel{\delta}{\longrightarrow} \Omega^1 \stackrel{\delta}{\longrightarrow} \cdots$$

with respect to a coboundary operator $\delta$ such that

$$\delta(\alpha \cdot \beta) = \delta\alpha \cdot \beta + (-1)^{[\alpha]} \alpha \cdot \delta\beta.$$
A graded differential algebra \((\Omega^*, \delta)\) with \(\Omega^0 = A\) is called the differential calculus over \(A\). If \(A\) is a \(*\)-algebra, we have additional conditions

\[(\alpha \cdot \beta)^* = (-1)^{[\alpha][\beta]} \beta^* \alpha^*,\]
\[(\delta \alpha)^* = \delta(\alpha^*).\]

**Remark 2.** The De Rham complex \((39)\) exemplifies a differential calculus over a commutative ring. To generalize it to a noncommutative ring \(A\), the coboundary operator \(\delta\) should have the additional properties:

- \(\Omega^{k>0}\) are central \(A\)-bimodules,
- elements \(\delta a_1 \cdots \delta a_k, a_i \in Z(A)\), belong to the centre \(Z(\Omega^k)\) of the module \(\Omega^k\).

Then, if \(A\) is a commutative ring, the commutativity condition \((25)\) holds.

Let \(\Omega^* A\) be the smallest differential subalgebra of the algebra \(\Omega^*\) which contains \(A\). As an \(A\)-algebra, it is generated by the elements \(\delta a, a \in A\), and consists of finite linear combinations of monomials of the form

\[\alpha = a_0 \delta a_1 \cdots \delta a_k, \quad a_i \in A.\]  
(51)

The product of monomials \((51)\) is defined by the rule

\[(a_0 \delta a_1) \cdot (b_0 \delta b_1) = a_0 \delta (a_1 b_0) \cdot \delta b_1 - a_0 a_1 \delta b_0 \cdot \delta b_1.\]

In particular, \(\Omega^1 A\) is a \(A\)-bimodule generated by elements \(\delta a, a \in A\). Because of

\[(\delta a)b = \delta(ab) - a\delta b,\]

the bimodule \(\Omega^1 A\) can also be seen as a left [right] \(A\)-module generated by the elements \(\delta a, a \in A\). Note that \(\delta(1) = 0\). Accordingly,

\[\Omega^k A = \Omega^1 A \cdots \Omega^1 A\]

are \(A\)-bimodules and, simultaneously, left [right] \(A\)-modules generated by monomials \((51)\).

The differential subalgebra \((\Omega^* A, \delta)\) is a differential calculus over \(A\). It is called the universal differential calculus because of the following property \([1, 2, 3]\).
Let \((\Omega^*, \delta')\) be another differential calculus over a unital \(K\)-algebra \(A'\), and let \(\rho : A \rightarrow A'\) be an algebra morphism. There exists a unique extension of this morphism to a morphism of graded differential algebras

\[ \rho^k : \Omega^k A \rightarrow \Omega^k \]

such that \(\rho^{k+1} \circ \delta = \delta' \circ \rho^k\).

Our interest to differential calculi over an algebra \(A\) is caused by the fact that, in commutative geometry, Definition 9 of a connection on an \(A\)-module requires the module \(\mathcal{O}^1\) \((20)\). If \(A = C^\infty(X)\), this module is the module of 1-forms on \(X\). To introduce connections in noncommutative geometry, one therefore should construct the noncommutative version of the module \(\mathcal{O}^1\). We may follow the construction of \(\mathcal{O}^1\) in Section 3, but not take the quotient by \(\text{mod} \mu^2\) that implies the commutativity condition \((25)\).

Remark 3. This is the crucial point that does not enable us to generalize the notion of jets of modules to modules over a noncommutative ring unless the very particular case when \(dA\) belongs to the centre of the module \(\Omega^1\). ●

Given a unital \(K\)-algebra \(A\), let us consider the tensor product \(A \otimes A\) of \(K\)-modules and the \(K\)-module morphism

\[ \mu^1 : A \otimes A \ni a \otimes b \mapsto ab \in A. \]

Following \((20)\), we define the \(K\)-module

\[ \overline{\mathcal{O}}^1[A] = \text{Ker } \mu^1. \] (52)

There is the \(K\)-module morphism

\[ d : A \ni a \mapsto (1 \otimes a - a \otimes 1) \in \overline{\mathcal{O}}^1[A] \] (53)

(cf. \((24)\)). Moreover, \(\overline{\mathcal{O}}^1[A]\) is a \(A\)-bimodule generated by the elements \(da, a \in A\), with the multiplication law

\[ b(da)c = b \otimes ac - ba \otimes c, \quad a, b, c \in A. \]

The morphism \(d\) \((53)\) possesses the property

\[ d(ab) = (1 \otimes ab - ab \otimes 1 + a \otimes b - a \otimes b) = (da)b + adb \] (54)
(cf. (26)), i.e., \( d \) is a \( \mathfrak{D}^1[\mathcal{A}] \)-valued derivation of \( \mathcal{A} \). Due to this property, \( \mathfrak{D}^1[\mathcal{A}] \) can be seen as a left \( \mathcal{A} \)-module generated by the elements \( da, a \in \mathcal{A} \). At the same time, if \( \mathcal{A} \) is a commutative ring, the \( \mathcal{A} \)-bimodule \( \mathfrak{D}^1[\mathcal{A}] \) does not coincide with the bimodule \( \mathfrak{D}^1 \) (24) because \( \mathfrak{D}^1[\mathcal{A}] \) is not a central bimodule (see Remark 2).

To overcome this difficulty, let us consider the \( \mathcal{Z}(\mathcal{A}) \) of derivations of the algebra \( \mathcal{A} \). They obey the rule

\[
    u(ab) = u(a)b + au(b), \quad \forall a, b \in \mathcal{A}.
\]

(55)

It should be emphasized that the derivation rule (55) differs from that

\[
    u(ab) = u(a)b + u(b)a
\]

for a general algebra [14]. By virtue of (55), derivations of an algebra \( \mathcal{A} \) constitute a \( \mathcal{Z}(\mathcal{A}) \)-bimodule, but not a left \( \mathcal{A} \)-module.

The \( \mathcal{Z}(\mathcal{A}) \)-bimodule \( \mathfrak{dA} \) is also a Lie algebra over the commutative ring \( \mathcal{K} \) with respect to the Lie bracket

\[
    [u, u'] = u \circ u' - u' \circ u.
\]

(56)

The centre \( \mathcal{Z}(\mathcal{A}) \) is stable under \( \mathfrak{dA} \), i.e.,

\[
    u(a)b = bu(a), \quad \forall a \in \mathcal{Z}(\mathcal{A}), \quad b \in \mathcal{A}, \quad u \in \mathfrak{dA},
\]

and one has

\[
    [u, au'] = u(a)u' + a[u, u'], \quad \forall a \in \mathcal{Z}(\mathcal{A}), \quad u, u' \in \mathfrak{dA}.
\]

(57)

If \( \mathcal{A} \) is a unital \( \ast \)-algebra, the module \( \mathfrak{dA} \) of derivations of \( \mathcal{A} \) is provided with the involution \( u \mapsto u^\ast \) defined by

\[
    u^\ast(a) = (u(a^\ast))^\ast.
\]

Then the Lie bracket (56) satisfies the reality condition \([u, u']^\ast = [u^\ast, u'^\ast] \).

Let us consider the Chevalley–Eilenberg cohomology (see [15]) of the Lie algebra \( \mathfrak{dA} \) with respect to its natural representation in \( \mathcal{A} \). The corresponding \( k \)-cochain space \( \mathfrak{O}^k[\mathcal{A}], k = 1, \ldots, \) is the \( \mathcal{A} \)-bimodule of \( \mathcal{Z}(\mathcal{A}) \)-multilinear antisymmetric mappings of \( \mathfrak{dA}^k \) to \( \mathcal{A} \). In particular, \( \mathfrak{O}^1[\mathcal{A}] \) is the \( \mathcal{A} \)-dual

\[
    \mathfrak{O}^1[\mathcal{A}] = \mathfrak{dA}^\ast
\]

(58)
of the derivation module $\mathfrak{d}A$ (cf. (46)). Put $\mathfrak{d}^{0}[A] = A$. The Chevalley–Eilenberg coboundary operator

$$d : \mathfrak{d}^{k}[A] \to \mathfrak{d}^{k+1}[A]$$

is given by

$$\begin{align*}
(d\phi)(u_0, \ldots, u_k) &= \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i u_i (\phi(u_0, \ldots, \widehat{u}_i, \ldots, u_k)) + \sum_{0 \leq r < s \leq k} (-1)^{r+s} \phi([u_r, u_s], u_0, \ldots, \widehat{u}_r, \ldots, \widehat{u}_s, \ldots, u_k),
\end{align*}$$

(59)

where $\widehat{u}_i$ means omission of $u_i$. For instance,

$$\begin{align*}
(da)(u) &= u(a), \quad a \in A, \\
(d\phi)(u_0, u_1) &= \frac{1}{2} (u_0(\phi(u_1)) - u_1(\phi(u_0)) - \phi([u_0, u_1])), \quad \phi \in \mathfrak{d}^{1}[A].
\end{align*}$$

(60)

(61)

It is readily observed that $d^2 = 0$, and we have the Chevalley–Eilenberg cochain complex of $K$-modules

$$0 \longrightarrow A \overset{d}{\longrightarrow} \mathfrak{d}^{k}[A] \overset{d}{\longrightarrow} \cdots.$$  

(62)

Furthermore, the $\mathbb{Z}$-graded space

$$\mathfrak{d}^{*}[A] = \bigoplus_{k=0}^{\infty} \mathfrak{d}^{k}[A]$$

(63)

is provided with the structure of a graded algebra with respect to the multiplication $\wedge$ combining the product of $A$ with antisymmetrization in the arguments. Notice that, if $A$ is not commutative, there is nothing like graded commutativity of forms, i.e.,

$$\phi \wedge \phi' \neq (-1)^{|\phi||\phi'|} \phi' \wedge \phi$$

in general. If $A$ is a $*$-algebra, $\mathfrak{d}^{*}[A]$ is also equipped with the involution

$$\phi^*(u_1, \ldots, u_k) \overset{\text{def}}{=} (\phi(u_1^*, \ldots, u_k^*))^*.$$  

Thus, $(\mathfrak{d}^{*}[A], d)$ is a differential calculus over $A$, called the Chevalley–Eilenberg differential calculus.

It is easy to see that, if $A = \mathbb{C}^\infty(X)$ is the commutative ring of smooth complex functions on a compact manifold $X$, the graded algebra $\mathfrak{d}^{*}[\mathbb{C}^\infty(X)]$ is exactly the
complexified exterior algebra $\mathbb{C} \otimes \mathcal{O}^*(X)$ of exterior forms on $X$. In this case, the coboundary operator (59) coincides with the exterior differential, and (62) is the De Rham complex of complex exterior forms on a manifold $X$. In particular, the operations

$$
(u|\phi)(u_1, \ldots, u_{k-1}) = k\phi(u, u_1, \ldots, u_{k-1}), \quad u \in \mathfrak{d}\mathcal{A},
$$

$$
\mathbf{L}_u(\phi) = d(u|\phi) + u|f(\phi),
$$

are the noncommutative generalizations of the contraction and the Lie derivative of differential forms. These facts motivate one to think of elements of $\mathcal{O}^1[\mathcal{A}]$ as being a noncommutative generalization of differential 1-forms, though this generalization by no means is unique.

Let $\mathcal{O}^*\mathcal{A}$ be the smallest differential subalgebra of the algebra $\mathcal{O}^*[\mathcal{A}]$ which contains $\mathcal{A}$. It is generated by the elements $da, a \in \mathcal{A}$, and consists of finite linear combinations of monomials of the form

$$
\phi = a_0 da_1 \wedge \cdots \wedge da_k, \quad a_i \in \mathcal{A}, (cf. (51)).
$$

In particular, $\mathcal{O}^1\mathcal{A}$ is a $\mathcal{A}$-bimodule (52) generated by $da, a \in \mathcal{A}$. Since the centre $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ is stable under derivations of $\mathcal{A}$, we have

$$
bda = (da)b, \quad adb = (db)a, \quad a \in \mathcal{A}, \quad b \in \mathcal{Z}(\mathcal{A}),
$$

$$
da \wedge db = -db \wedge da, \quad \forall a \in \mathcal{Z}(\mathcal{A}).
$$

Hence, $\mathcal{O}^1\mathcal{A}$ is a central bimodule in contrast with the bimodule $\mathcal{O}^1\mathcal{A}$ (52). By virtue of the relation (51), we have the isomorphism

$$
\mathfrak{d}\mathcal{A} = \mathcal{O}^1\mathcal{A}^*\quad (64)
$$

of the $\mathcal{Z}(\mathcal{A})$-module $\mathfrak{d}\mathcal{A}$ of derivations of $\mathcal{A}$ to the $\mathcal{A}$-dual of the module $\mathcal{O}^1\mathcal{A}$ (cf. (51)). Combining the duality relations (58) and (64) gives the relation

$$
\mathcal{O}^1\mathcal{A} = \mathcal{O}^1\mathcal{A}^{**}.
$$

The differential subalgebra $(\mathcal{O}^*[\mathcal{A}], d)$ is a universal differential calculus over $\mathcal{A}$. If $\mathcal{A}$ is a commutative ring, then $\mathcal{O}^*[\mathcal{A}]$ is the De Rham complex (39).
6 Universal connections

Let \((\Omega^*, \delta)\) be a differential calculus over a unital \(\mathcal{K}\)-algebra \(\mathcal{A}\) and \(P\) a left [right] \(\mathcal{A}\)-module. Similarly to Definition 9, one can construct the tensor product \(\Omega^1 \otimes P [P \otimes \Omega^1]\) and define a connection on \(P\) as follows \([8, 13]\).

**Definition 13.** A noncommutative connection on a left \(\mathcal{A}\)-bimodule \(P\) with respect to the differential calculus \((\Omega^*, \delta)\) is a \(\mathcal{K}\)-module morphism

\[
\nabla : P \to \Omega^1 \otimes P
\]

which obeys the Leibniz rule

\[
\nabla(ap) = \delta a \otimes p + a \nabla(p).
\]

\[\square\]

If \(\Omega^* = \Omega^* \mathcal{A}\) is a universal differential calculus, the connection (65) is called a universal connection \([8, 13]\).

The curvature of the noncommutative connection (65) is defined as the \(\mathcal{A}\)-module morphism

\[
\nabla^2 : P \to \Omega^2[\mathcal{A}] \otimes P
\]

[13]. Note also that the morphism (65) has a natural extension

\[
\nabla : \Omega^k \otimes P \to \Omega^{k+1} \otimes P,
\]

\[
\nabla(\alpha \otimes p) = \delta \alpha \otimes p + (-1)^{|\alpha|} \alpha \otimes \nabla(p), \quad \alpha \in \Omega^*,
\]

[13, 16].

Similarly, a noncommutative connection on a right \(\mathcal{A}\)-module is defined. However, a connection on a left [right] module does not necessarily exist as it is illustrated by the following theorem.

**Theorem 14.** A left [right] universal connection on a left [right] module \(P\) of finite rank exists if and only if \(P\) is projective \([13, 17]\). \(\square\)

The problem arises when \(P\) is a \(\mathcal{A}\)-bimodule. If \(\mathcal{A}\) is a commutative ring, left and right module structures of an \(\mathcal{A}\)-bimodule are equivalent, and one deals with either a left or right noncommutative connection on \(P\) (see Definition 9). If \(P\) is a
A-bimodule over a noncommutative ring, left and right connections \(\nabla^L\) and \(\nabla^R\) on \(P\) should be considered simultaneously. However, the pair \((\nabla^L, \nabla^R)\) by no means is a bimodule connection since \(\nabla^L(P) \in \Omega^1 \otimes P\), whereas \(\nabla^R(P) \in P \otimes \Omega^1\). As a palliative, one assumes that there exists a bimodule isomorphism
\[
\varrho : \Omega^1 \otimes P \to P \otimes \Omega^1. \tag{66}
\]
Then a pair \((\nabla^L, \nabla^R)\) of right and left noncommutative connections on \(P\) is called a \(\varrho\)-compatible if
\[
\varrho \circ \nabla^L = \nabla^R \tag{13, 16, 18}
\]
(see also \[19\] for a weaker condition). Nevertheless, this is not a true bimodule connection (see the condition \(70\) below).

**Remark 4.** If \(A\) is a commutative ring, the isomorphism \(\varrho\) \([2]\) is naturally the permutation
\[
\varrho : \alpha \otimes p \mapsto p \otimes \alpha, \quad \forall \alpha \in \Omega^1, \ p \in P.
\]

The above mentioned problem of a bimodule connection is not simplified radically even if \(P = \Omega^1\), together with the natural permutations
\[
\phi \otimes \phi' \mapsto \phi' \otimes \phi, \quad \phi, \phi' \in \Omega^1, \tag{4, 18}
\]
\[13, 16, 18\].

Let now \((\mathcal{D}^*[A], d)\) be the universal differential calculus over a noncommutative \(K\)-ring \(A\). Let
\[
\nabla^L : P \to \Omega^1[A] \otimes P, \tag{67}
\]
\[
\nabla^L(ap) = da \otimes p + a\nabla^L(p).
\]
be a left universal connection on a left \(A\)-module \(P\) (cf. Definition \[3\]). Due to the duality relation \(64\), there is the \(K\)-module endomorphism
\[
\nabla_u^L : P \ni p \to u|\nabla^L(p) \in P \tag{68}
\]
of \(P\) for any derivation \(u \in \mathfrak{d}A\). If \(\nabla^R\) is a right universal connection on a right \(A\)-module \(P\), the similar endomorphism
\[
\nabla_u^R : P \ni p \to \nabla^L(p)|u \in P \tag{69}
\]
takes place for any derivation \( u \in \mathfrak{d} \mathcal{A} \). Let \((\nabla^L, \nabla^R)\) be a \( \varphi \)-compatible pair of left and right universal connections on an \( \mathcal{A} \)-bimodule \( P \). It seems natural to say that this pair is a bimodule universal connection on \( P \) if

\[
\left[ u \right] \nabla^L(p) = \nabla^R(p) \left[ u \right]
\]

(70)

for all \( p \in P \) and \( u \in \mathfrak{d} \mathcal{A} \). Nevertheless, motivated by the endomorphisms (68) – (69), one can suggest another definition of connections on a bimodule, similar to Definition 11.

7 The Dubois-Violette connection

Let \( \mathcal{A} \) be \( \mathcal{K} \)-ring and \( P \) an \( A \)-module of type \((i, j)\) in accordance with the notation in Section 2.

**Definition 15.** By analogy with Definition 11, a Dubois-Violette connection on an \( A \)-module \( P \) of type \((i, j)\) is a \( \mathcal{Z}(\mathcal{A}) \)-bimodule morphism

\[
\nabla : \mathfrak{d} \mathcal{A} \ni u \mapsto \nabla u \in \text{Hom}_\mathcal{K}(P, P)
\]

(71)

of \( \mathfrak{d} \mathcal{A} \) to the \( \mathcal{Z}(\mathcal{A}) \)-bimodule of endomorphisms of the \( \mathcal{K} \)-module \( P \) which obey the Leibniz rule

\[
\nabla_u(a_i pa_j) = u(a_i) p a_j + a_i \nabla_u(p) a_j + a_i p u(a_j), \quad \forall p \in P, \quad \forall a_k \in A_k,
\]

(72)

[4, 18]. \( \Box \)

By virtue of the duality relation (34) and the expressions (68) – (69), every left [right] universal connection yields a connection (71) on a left [right] \( \mathcal{A} \)-module \( P \). From now on, by a connection in noncommutative geometry is meant a Dubois-Violette connection in accordance with Definition 15.

A glance at the expression (72) shows that, if connections on an \( A \)-module \( P \) of type \((i, j)\) exist, they constitute an affine space modelled over the linear space of \( \mathcal{Z}(\mathcal{A}) \)-bimodule morphisms

\[
\sigma : \mathfrak{d} \mathcal{A} \ni u \mapsto \sigma_u \in \text{Hom}_{A_i-A_j}(P, P)
\]

of \( \mathfrak{d} \mathcal{A} \) to the \( \mathcal{Z}(\mathcal{A}) \)-bimodule of endomorphisms

\[
\sigma_u(a_i pa_j) = a_i \sigma(p) a_j, \quad \forall p \in P, \quad \forall a_k \in A_k,
\]
of the $A$-module $P$.

**Example 5.** If $P = A$, the morphisms

$$\nabla_u(a) = u(a), \quad \forall u \in \mathfrak{d}A, \quad \forall a \in A, \quad (73)$$

define a canonical connection on $A$ in accordance with Definition 13. Then the Leibniz rule (72) shows that any connection on a central $A$-bimodule $P$ is also a connection on $P$ seen as a $\mathcal{Z}(A)$-bimodule. $ullet$

**Example 6.** If $P$ is a $A$-bimodule and $A$ has only inner derivations

$$\text{ad} \ b(a) = ba - ab,$$

the morphisms

$$\nabla_{\text{ad} \ b}(p) = bp - pb, \quad \forall b \in A, \quad \forall p \in P, \quad (74)$$

define a canonical connection on $P$. $ullet$

By the curvature $R$ of a connection $\nabla$ (71) on an $A$-module $P$ is meant the $\mathcal{Z}(A)$-module morphism

$$R : \mathfrak{d}A \times \mathfrak{d}A \ni (u, u') \rightarrow R_{u,u'} \in \text{Hom}_{A_i - A_j}(P, P),$$

$$R_{u,u'}(p) = \nabla_u(\nabla_{u'}(p)) - \nabla_{u'}(\nabla_u(p)) - \nabla_{[u,u']}(p), \quad p \in P, \quad (75)$$

[4]. We have

$$R_{a\alpha u',a' \alpha'} = a\alpha' R_{u,u'}, \quad a, a' \in \mathcal{Z}(A),$$

$$R_{u,u'}(a_i pb_j) = a_i R_{u,u'}(p)b_j, \quad a_i \in A_j, \quad b_j \in A_j.$$ 

For instance, the curvature of the connections (73) and (74) vanishes.

Let us provide some standard operations with the connections (71).

(i) Given two modules $P$ and $P'$ of the same type $(i, j)$ and connections $\nabla$ and $\nabla'$ on them, there is an obvious connection $\nabla \oplus \nabla'$ on $P \oplus P'$.

(ii) Let $P$ be a module of type $(i, j)$ and $P^*$ its $A$-dual. For any connection $\nabla$ on $P$, there is a unique dual connection $\nabla'$ on $P^*$ such that

$$u(\langle p, p' \rangle) = \langle \nabla_u(p), p' \rangle + \langle p, \nabla'(p') \rangle, \quad p \in P, \quad p' \in P^*, \quad u \in \mathfrak{d}A.$$
Let $P_1$ and $P_2$ be $A$-modules of types $(i, k)$ and $(k, j)$, respectively, and let $\nabla^1$ and $\nabla^2$ be connections on these modules. For any $u \in \mathfrak{d}A$, let us consider the endomorphism

$$ (\nabla^1 \otimes \nabla^2)_u = \nabla^1_u \otimes \text{Id} P_1 + \text{Id} P_2 \otimes \nabla^2_u $$

of the tensor product $P_1 \otimes P_2$ of $\mathcal{K}$-modules $P_1$ and $P_2$. This endomorphism preserves the subset of $P_1 \otimes P_2$ generated by elements $p_1 a \otimes p_2 - p_1 \otimes ap_2$, with $p_1 \in P_1$, $p_2 \in P_2$ and $a \in A_k$. Due to this fact, the endomorphisms (76) define a connection on the tensor product $P_1 \otimes P_2$ of modules $P_1$ and $P_2$.

(iv) If $\mathcal{A}$ is a unital $\ast$-algebra, we have only modules of type $(1, 1)$ and $(0, 0)$, i.e., $\ast$-modules and $\mathcal{Z}(\mathcal{A})$-bimodules. Let $P$ be a module of one of these types. If $\nabla$ is a connection on $P$, there exists a conjugate connection $\nabla^\ast$ on $P$ given by the relation

$$ \nabla^\ast_u(p) = (\nabla^{u\ast}(p^\ast))^\ast. \tag{77} $$

A connection $\nabla$ on $P$ is said to be real if $\nabla = \nabla^\ast$.

Let now $P = \mathfrak{D}^1[\mathcal{A}]$. A connection on $\mathcal{A}$-bimodule $\mathfrak{D}^1[\mathcal{A}]$ is called a linear connection $\mathfrak{D}$. Note that this is not the term for an arbitrary left [right] connection on $\mathfrak{D}^1[\mathcal{A}]$. If $\mathfrak{D}^1[\mathcal{A}]$ is a $\ast$-module, a linear connection on it is assumed to be real. Given a linear connection $\nabla$ on $\mathfrak{D}^1[\mathcal{A}]$, there is a $\mathcal{A}$-bimodule homomorphism, called the torsion of the connection $\nabla$,

$$ T : \mathfrak{D}^1[\mathcal{A}] \rightarrow \mathfrak{D}^2[\mathcal{A}], $$

$$ (T\phi)(u, u') = (d\phi)(u, u') - \nabla_u(\phi)(u') + \nabla_{u'}(\phi)(u), \tag{78} $$

for all $u, u' \in \mathfrak{d}A, \phi \in \mathfrak{D}^1[\mathcal{A}]$.

8 Matrix geometry

This Section gives a standard example of linear connections in matrix geometry when $\mathcal{A} = M_n$ is the algebra of complex $(n \times n)$-matrices [20, 21, 22].

Let $\{\varepsilon_r\}, 1 \leq r \leq n^2 - 1$, be an anti-Hermitian basis of the Lie algebra $su(n)$. Elements $\varepsilon_r$ generate $M_n$ as an algebra, while $u_r = \text{ad} \varepsilon_r$ constitute a basis of the right
Lie algebra \( \mathfrak{d} M_n \) of derivations of the algebra \( M_n \), together with the commutation relations

\[
[u_r, u_q] = c_{rq}^s u_s,
\]

where \( c_{rq}^s \) are structure constants of the Lie algebra \( su(n) \). Since the centre \( \mathcal{Z}(M_n) \) of \( M_n \) consists of matrices \( \lambda 1 \), \( \mathfrak{d} M_n \) is a complex free module of rank \( n^2 - 1 \).

Let us consider the universal differential calculus \( (\mathcal{O}^*[M_n], d) \) over the algebra \( M_n \), where \( d \) is the Chevalley–Eilenberg coboundary operator (59). There is a convenient system \( \{\theta^r\} \) of generators of \( \mathcal{O}^1[M_n] \) seen as a left \( M_n \)-module. They are given by the relations

\[
\theta^r(u_q) = \delta^r_q 1.
\]

Hence, \( \mathcal{O}^1[M_n] \) is a free left \( M_n \)-module of rank \( n^2 - 1 \). It is readily observed that elements \( \theta^r \) belong to the centre of the \( M_n \)-bimodule \( \mathcal{O}^1[M_n] \), i.e.,

\[
a\theta^r = \theta^r a, \quad \forall a \in M_n.
\] (79)

It also follows that

\[
\theta^r \wedge \theta^q = -\theta^q \wedge \theta^r.
\] (80)

The morphism \( d : M_n \rightarrow \mathcal{O}^1[M_n] \) is given by the formula (60). It reads

\[
d\varepsilon_r(u_q) = \text{ad} \varepsilon_q(\varepsilon_r) = c_{qr}^s \varepsilon_s,
\]

that is,

\[
d\varepsilon_r = c_{qr}^s \varepsilon_s \theta^q.
\] (81)

The formula (61) leads to the Maurer–Cartan equations

\[
d\theta^r = -\frac{1}{2} c_{qs}^r \theta^q \wedge \theta^s.
\] (82)

If we define \( \theta = \varepsilon_r \theta^r \), the equality (81) can be rewritten as

\[
da = a\theta - \theta a, \quad \forall a \in M_n.
\]

It follows that the \( M_n \)-bimodule \( \mathcal{O}^1[M_n] \) is generated by the element \( \theta \). Since \( \mathfrak{d} M_n \) is a finite free module, one can show that the \( M_n \)-bimodule \( \mathcal{O}^1[M_n] \) is isomorphic to the \( M_n \)-dual \( \mathcal{O}^1[M_n] \) of \( \mathfrak{d} M_n \).
Turn now to connections on the \( M_n \)-bimodule \( \mathcal{O}^1[M_n] \). Such a connection \( \nabla \) is given by the relations
\[
\begin{align*}
\nabla_u & = c^u_r, \\
\nabla_r(\theta^p) & = \omega^p_{rq} \theta^q, \quad \omega^p_{rq} \in M_n.
\end{align*}
\] (83)

Bearing in mind the equalities (79) – (80), we obtain from the Leibniz rule (72) that
\[
a \nabla_r(\theta^p) = \nabla_r(\theta^p) a, \quad \forall a \in M_n.
\]

It follows that elements \( \omega^p_{rq} \) in the expression (83) are proportional \( 1 \in M_n \), i.e., complex numbers. Then the relations
\[
\nabla_r(\theta^p) = \omega^p_{rq} \theta^q, \quad \omega^p_{rq} \in \mathbb{C},
\] (84)
define a linear connection on the \( M_n \)-bimodule \( \mathcal{O}^1[M_n] \).

Let us consider two examples of linear connections.

(i) Since all derivations of the algebra \( M_n \) are inner, we have the curvature-free connection (74) given by the relations
\[
\nabla_r(\theta^p) = 0.
\]

However, this connection is not torsion-free. The expressions (78) and (82) result in
\[
(T \theta^p)(u_r, u_q) = -c^p_{rq}.
\]

(ii) One can show that, in matrix geometry, there is a unique torsion-free linear connection
\[
\nabla_r(\theta^p) = -c^p_{rq} \theta^q.
\]

9 Connes’ differential calculus

Connes’ differential calculus is based on the notion of a spectral triple [8, 13, 23, 24].

Definition 16. A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is given by a \( \ast \)-algebra \( \mathcal{A} \subset B(\mathcal{H}) \) of bounded operators on a Hilbert space \( \mathcal{H} \), together with an (unbounded) self-adjoint operator \( D = D^* \) on \( \mathcal{H} \) with the following properties:

- the resolvent \((D - \lambda)^{-1}, \lambda \neq \mathbb{R}\), is a compact operator on \( \mathcal{H} \),
• $[D, \mathcal{A}] \in \mathcal{B}(\mathcal{H})$.

The couple $(\mathcal{A}, D)$ is also called a $K$-cycle over $\mathcal{A}$. In many cases, $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space equipped with a projector $\Gamma$ such that

$$\Gamma D + D \Gamma = 0, \quad [a, \Gamma] = 0, \quad \forall a \in \mathcal{A},$$

i.e., $\mathcal{A}$ acts on $\mathcal{H}$ by even operators, while $D$ is an odd operator.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, let $(\Omega^* \mathcal{A}, \delta)$ be a universal differential calculus over the algebra $\mathcal{A}$. Let us construct a representation of the graded differential algebra $\Omega^* \mathcal{A}$ by bounded operators on $\mathcal{H}$ when the Chevalley–Eilenberg derivation $\delta$ (59) of $\mathcal{A}$ is replaced with the bracket $[D, a], a \in \mathcal{A}$:

$$\pi : \Omega^* \mathcal{A} \to \mathcal{B}(\mathcal{H}),$$

$$\pi(a_0 \delta a_1 \cdots \delta a_k) \overset{\text{def}}{=} a_0[D, a_1] \cdots [D, a_k]. \quad (85)$$

Since

$$[D, a] = -[D, a^*],$$

we have $\pi(\phi) = \pi(\phi^*)$, $\phi \in \Omega^* \mathcal{A}$. At the same time, $\pi$ (59) fails to be a representation of the graded differential algebra $\Omega^* \mathcal{A}$ because $\pi(\phi) = 0$ does not imply that $\pi(\delta \phi) = 0$. Therefore, one should construct the corresponding quotient in order to obtain a graded differential algebra of operators on $\mathcal{H}$.

Let $J_0$ be the graded two-sided ideal of $\Omega^* \mathcal{A}$ where

$$J_0^k = \{ \phi \in \Omega^k \mathcal{A} : \pi(\phi) = 0 \}.$$ 

Then it is readily observed that $J = J_0 + \delta J_0$ is a graded differential two-sided ideal of $\Omega^* \mathcal{A}$. By Connes’ differential calculus is meant the pair $(\Omega^*_D \mathcal{A}, d)$ such that

$$\Omega^*_D \mathcal{A} = \Omega^* \mathcal{A} / J,$$

$$d[\phi] = [\delta \phi],$$

where $[\phi]$ denotes the class of $\phi \in \Omega^* \mathcal{A}$ in $\Omega^*_D \mathcal{A}$. It is a differential calculus over $\Omega^*_D \mathcal{A} = \mathcal{A}$. Its $k$-cochain submodule $\Omega^*_D \mathcal{A}$ consists of the classes of operators

$$\sum_j a_0^j [D, a_1^j] \cdots [D, a_k^j], \quad a_i^j \in \mathcal{A},$$

26
modulo the submodule of operators
\[ \left\{ \sum_j [D, b_0^j][D, b_1^j] \cdots [D, b_{k-1}^j] : \sum_j b_0^j [D, b_1^j] \cdots [D, b_{k-1}^j] = 0 \right\}. \]

Let now \( P \) be a right finite projective module over the \(*\)-algebra \( \mathcal{A} \). We aim to study a right connection on \( P \) with respect to Connes’ differential calculus \((\Omega_D^1, \mathcal{A}, d)\). As was mentioned above in Theorem 14, a right finite projective module has a connection. Let us construct this connection in an explicit form.

Given a generic right finite projective module \( P \) over a complex ring \( \mathcal{A} \), let
\[
p : \mathbb{C}^N \otimes \mathcal{A} \to P,
\]
\[
i_P : P \to \mathbb{C}^N \otimes \mathcal{A},
\]
be the corresponding projection and injection, where \( \otimes \) denotes the tensor product over \( \mathbb{C} \). There is the chain of morphisms
\[ P \xrightarrow{i_P} \mathbb{C}^N \otimes \mathcal{A} \xrightarrow{\text{Id} \otimes \delta} \mathbb{C}^N \otimes \Omega^1 \mathcal{A} \xrightarrow{p} P \otimes \Omega^1 \mathcal{A}, \tag{86} \]
where the canonical module isomorphism
\[ \mathbb{C}^N \otimes \Omega^1 \mathcal{A} = (\mathbb{C}^N \otimes \mathcal{A}) \otimes \Omega^1 \mathcal{A} \]
is used. It is readily observed that the composition (86) denoted briefly as \( p \circ \delta \) is a right universal connection on the module \( P \).

Given the universal connection \( p \circ \delta \) on a right finite projective module \( P \) over a \(*\)-algebra \( \mathcal{A} \), let us consider the morphism
\[ P \xrightarrow{p \circ \delta} P \otimes \Omega^1 \mathcal{A} \xrightarrow{\text{Id} \otimes \pi} P \otimes \Omega^1_D \mathcal{A}. \]

It is readily observed that this is a right connection \( \nabla_0 \) on the module \( P \) with respect to Connes’ differential calculus. Any other right connection \( \nabla \) on on \( P \) with respect to Connes’ differential calculus takes the form
\[ \nabla = \nabla_0 + \sigma = (\text{Id} \otimes \pi) \circ p \circ \delta + \sigma \tag{87} \]
where \( \sigma \) is an \( \mathcal{A} \) module morphism
\[ \sigma : P \to P \otimes \Omega^1_D \mathcal{A}. \]
A components \( \sigma \) of the connection \( \nabla \) (87) is called a noncommutative gauge field.
References

[1] I.Krasil’shchik, V.Lychagin and A.Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations* (Gordon and Breach, Glasgow, 1985).

[2] G.Giachetta, L.Mangiarotti and G.Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory* (World Scientific, Singapore, 1997).

[3] S.Mac Lane, *Homology* (Springer-Verlag, Berlin, 1967).

[4] M.Dubois-Violette and P.Michor, Connections on central bimodules in noncommutative differential geometry, *J. Geom. Phys.* 20 (1996) 218.

[5] J.Dixmier, *C*-Algebras (North-Holland, Amsterdam, 1977).

[6] E.Michael, *Locally Multiplicatively Convex Topological Algebras* (Am. Math. Soc., Providence, 1974).

[7] R.Swan, Vector bundles and projective modules, *Trans. Am. Math. Soc.* 105 (1962) 264.

[8] J.Várilly and J.Grasia-Bondia, Connes’ noncommutative differential geometry and the Standard Model, *J. Geom. Phys.* 12 (1993) 223.

[9] J.Koszul, *Lectures on Fibre Bundles and Differential Geometry* (Tata University, Bombay, 1960).

[10] G. Maltsiniotis, Le langage des espaces et des groupes quantiques, *Commun. Math. Phys.* 151 (1993) 275.

[11] A.Connes, Non-commutative differential geometry, *Publ. I.H.E.S* 62 (1986) 257.

[12] M.Karoubi, Connexion, courbures et classes caracteristique en K-theorie algebrique, *Can. Nath. Soc. Conf. Proc.* 2 (1982) 19.

[13] G.Landi, *An Introduction to Noncommutative Spaces and their Geometries*, Lect. Notes in Physics, New series m: Monographs, 51 (Springer-Verlag, Berlin, 1997).
[14] S.Lang, *Algebra* (Addison–Wisley, N.Y., 1993).

[15] I.Vaisman, *Lectures on the Geometry of Poisson Manifolds* (Birkhäuser Verlag, Basel, 1994).

[16] M.Dubois-Violette, J.Madore, T.Masson and J.Morad, On curvature in noncommutative geometry, *J. Math. Phys.* 37 (1996) 4089.

[17] J.Cuntz and D.Quillen, Algebra extension and nonsingularity, *J. Amer. Math. Soc.* 8 (1995) 251.

[18] J.Mourad, Linear connections in noncommutative geometry, *Class. Quant. Grav.* 12 (1995) 965.

[19] L.Dabrowski, P.Hajac, G.Lanfi and P.Siniscalco, Metrics and pairs of left and right connections on bimodules, *J. Math. Phys.* 37 (1996) 4635.

[20] M.Dubois-Violette, R.Kerner and J.Madore, Noncommutative differential geometry of matrix algebras, *J. Math. Phys.* 31 (1990) 316.

[21] J.Madore, T.Masson and J.Mourad, Linear connections on matrix geometries, *Class. Quant. Grav.* 12 (1995) 1429.

[22] J.Madore, Linear connections on fuzzy manifolds, *Class. Quant. Grav.* 13 (1996) 2109.

[23] A.Connes, *Noncommutative Geometry* (Academic Press, N.Y., 1994).

[24] J.Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications* (Cambridge Univ. Press, Cambridge, 1995).