Lower estimates for linear operators with smooth range

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Abstract
We introduce a new method to prove lower estimates for the approximation error of general linear operators with smooth range in terms of classical moduli of smoothness and related $K$-functionals. In addition, we explicitly show how to derive lower estimates for positive linear operators with smooth range and apply this result to classical approximation operators. We finish with some remarks on the eigenvalues of Schoenberg’s spline operator.

Keywords: Converse inequality, positive linear operator, modulus of smoothness, $K$-functional, Bernstein polynomials, splines

1. Introduction
A convenient way to relate the decay rate of a sequence of approximations $T_n$ on a Banach space $X$ with the smoothness of the approximated function $f \in X$ is to establish lower estimates in terms of classical moduli of smoothness and related $K$-functionals: There exists constants $C_1, C_2 > 0$ independent on $n$ such that

$$C_1 \cdot \omega_r(f, \delta_n) \leq \|T_n f - f\| \quad \text{and} \quad C_2 \cdot K_r(f, \delta_n) \leq \|T_n f - f\|$$

holds for all $f \in X$ and $\delta_n \to 0$ for $n \to \infty$. Although there exist already several methods to derive such estimates, see e.g. Knoop and Zhou ([13], [14]), Ditzian and Ivanov [6] and Totik [23], these methods still require many restrictions and therefore are not applicable for a large number of linear operators.

In this article, we introduce a new method to derive such lower estimates for arbitrary compact operators with smooth range satisfying a spectral property. The approximation operator can be defined on arbitrary bounded domains $\Omega \subset \mathbb{R}^d$ with a suitably smooth boundary. As underlying function spaces we consider the space of continuous functions and $L^p$-spaces for $1 \leq p < \infty$. Consequently, we use the space of $r$-times continuously differentiable functions and classical Sobolev spaces as their corresponding smooth subspaces.

We will prove lower estimates for linear operators based on a functional analytic framework depending on the fixed points of the operator and the smoothness of the range. The key idea is to estimate the semi-norm occurring in the $K$-functional by the approximation error using the convergence of the iterates of the operator. The only requirements of this approach are that the seminorms of the $K$-functionals are bounded on the range of the approximation operator and annihilate its fixed points. It will be shown that the degree of the modulus of smoothness or the used $K$-functional depends only on the smoothness of the range and the fixed points of $T$. Note that these results are an extension of the method shown in [17], where lower estimates for Schoenberg’s variation diminishing spline operator have been shown. Here, we establish a very flexible framework to prove lower estimates for very general linear approximation operators.

We finish this article by discussing applications of these results. First, we show how to derive lower estimates for general positive linear operators with smooth range. Then, we provide concrete lower estimates for the Bernstein operator, the Kantorovič operator, the Schoenberg operator and the integral Schoenberg operator. As the eigenvalues of the Schoenberg operator play an important role in the corresponding lower estimate, we give a characterization of them in the end of this article.

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2. Preliminaries

Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with suitable smooth boundary.

2.1. Function spaces

We use the multi-index notation of Schwartz [22] to introduce derivatives. Accordingly, we denote by \( D^\alpha \) the differential operator

\[
D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a multi-index with modulus \(|\alpha| = \sum_{i=1}^n \alpha_i\). We denote by \( C^r(\Omega) \) the space of all complex valued functions \( f \) that have continuous and bounded derivatives \( D^\alpha f \) up to order \( r \), i.e., \(|\alpha| \leq r\).

The norm on \( C^r(\Omega) \) is given by \( \|f\| := \sup_{|\alpha| \leq r} \|D^\alpha f\|_\infty \).

By \( L^p(\Omega), 1 \leq p < \infty \), we denote the space of Lebesgue measurable functions defined on \( \Omega \) whose \( p \)-th power is integrable with respect to the measure \( dz = dx_1 \cdots dx_n = d\mu \). The Sobolev space \( W^{p,r}(\Omega) \) corresponding to \( L^p(\Omega) \) contains all functions \( f \in L^p(\Omega) \) where \( D^\alpha f \in L^p(\Omega) \) for all orders \(|\alpha| \leq r\).

To simplify notation and to combine the previously mentioned spaces, we introduce the spaces \( X^{p,r}(\Omega) \) for \( 1 \leq p \leq \infty \) and \( r = 0, 1, 2, \ldots \) as follows:

\[
X^{p,0}(\Omega) := L^p(\Omega), \quad 1 \leq p < \infty; \quad X^\infty,0(\Omega) := C(\Omega),
\]
\[
X^{p,r}(\Omega) := W^{p,r}(\Omega), \quad 1 \leq p < \infty; \quad X^\infty,r(\Omega) := C^r(\Omega),
\]

Finally, we define the semi-norms

\[
|f|_{r,p} := \sup_{|\alpha| = r} \|D^\alpha f\|_p
\]

for all smooth functions \( f \in X^{p,r}(\Omega) \).

2.2. Moduli of smoothness and K-functional

Now, we will introduce the modulus of smoothness and Peetre’s K-functional for the previously defined spaces according to Johnen and Scherer [10]. Let us denote for \( h \in \mathbb{R}^d \) the set

\[
\Omega(h) := \{ x \in \Omega : x + th \in \Omega \quad \text{for} \quad 0 \leq t \leq 1 \}.
\]

Then we define the \( r \)-th modulus of smoothness \( \omega_{r,p} : X^{p,0}(\Omega) \times (0, \infty) \to [0, \infty), 1 \leq p \leq \infty \), as follows:

\[
\omega_{r,p}(f,t) := \begin{cases} \|f\|_p, & r = 0 \\ \sup_{0 < |h| \leq t} \|\chi_{\Omega(h)} \Delta_h^r f(x)\|_p, & r = 1, 2, \ldots \end{cases}
\]

where \( \Delta_h^r \) is the forward difference operator into direction \( h \in \mathbb{R}^d \),

\[
\Delta_h^r f(x) = \sum_{l=0}^r (-1)^{r-l} \binom{r}{l} f(x + lh).
\]

Similarly, the K-functional \( K_{r,p} : X^{p,0}(\Omega) \times (0, \infty) \to [0, \infty), 1 \leq p \leq \infty \) is defined on the spaces \( X^{p,r}(\Omega) \) as follows (19, 10):

\[
K_{r,p}(f,t^r) := \inf \left\{ \|f - g\|_p + t^r |g|_{p,r} : g \in X^{p,r}(\Omega) \right\}.
\]

As shown in Johnen and Scherer [10, Lem. 1], the modulus of smoothness can be bounded from above by the related K-functional in the following way: for all \( 0 < t < \infty \) there holds

\[
\omega_{r,p}(f,t) \leq 2^r \|f - g\|_p + d^{r/2} t^r |g|_{r,p},
\]

for \( f \in X^{p,0}(\Omega), g \in X^{p,r}(\Omega) \) and \( 1 \leq p \leq \infty \). Moreover, the equivalence of the modulus of smoothness to the K-functional have been shown, see Butzer and Berens [1] for the one-dimensional case and Johnen and Scherer [10] for arbitrary Lipschitz domains.
2.3. Projections and Iterates

In order to prove lower estimates in a general setting, we will utilize the convergence of the iterates to a projection operator. We will provide here the necessary results that characterize this behaviour. To this end, let $X$ be a complex Banach space and let us denote by $\mathcal{L}(X)$ the set of all linear operators on $X$. Note that the results shown here are also applicable on real Banach spaces using a standard complexification scheme as outlined, e.g., in Ruston [20, pp. 7–16].

In the following, we consider a bounded linear contraction $T \in \mathcal{L}(X)$, i.e., $\|T\|_{op} \leq 1$. Dunford [8, Thm. 3.16] has shown that the iterates converge to a projection onto the corresponding fixed point space:

**Proposition 1** (Convergence of Iterates). Let $T \in \mathcal{L}(X)$ be a compact operator such that $\|T^{m+1} - T^m\|_{op} \to 0$ for $m \to \infty$. Then there exists $P \in \mathcal{L}(X)$ with $P^2 = P$, and $P(X) = \ker(T - I)$ such that $T^m \to P$.

The necessary criteria, $\|T^{m+1} - T^m\|_{op} \to 0$ for $m \to \infty$, has been further characterized in the work of Katznelson and Tzafriri [12, Thm. 1], where they provided a sufficient and necessary criterion based on the spectral location of $T$.

**Proposition 2** (Spectral Location). Let $T \in \mathcal{L}(X)$ be a contraction. Then

$$\lim_{m \to \infty} \|T^{m+1} - T^m\|_{op} = 0$$

if and only if

$$\sigma(T) \subset B(0, 1) \cup \{1\}.$$  \hfill (4)

The spectrum has to be contained in the unit ball with the only intersection at 1.

Finally, it can be shown, that the convergence rate depends only on the second largest spectral value in the modulus:

**Lemma 1** (Convergence Rate). Let $T \in \mathcal{L}(X)$ be a compact operator with $r(T) = \|T\|_{op} = 1$ satisfying the spectral condition $\sigma(T) \subset B(0, 1) \cup \{1\}$. Define

$$\gamma := \sup \{\|\gamma\| : \gamma \in \sigma(T) \setminus \{1\}\}.$$  

Then there exists a constant $1 \leq C \leq \gamma^{-1}$, such that for all $m \in \mathbb{N}$

$$\|T^m - P\|_{op} \leq C \cdot \gamma^m,$$

where $P \in \mathcal{K}(X)$ is the operator defined in Proposition 1.

**Proof.** Using Dunford [8, Thm. 3.16], we obtain the space decomposition $X = \ker(T - I) \oplus \text{ran}(T - I)$ and $\text{ran}(T - I)$ is closed. Accordingly, we decompose the operator $T$ into

$$T = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} \in \mathcal{L}(\ker(T - I) \oplus \text{ran}(T - I)).$$

Furthermore, we have that $\sigma(S) \subset B(0, 1)$ and therefore we obtain $r(S) = \gamma < 1$. As $r(S) = \lim_{m \to \infty} \|S^m\|^{1/m}$, we obtain that there exists a constant $1 \leq C \leq \gamma^{-1}$ such that

$$\|S^m\| \leq C \cdot \gamma^m$$

for every $m \in \mathbb{N}$. \hfill $\square$

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3. Lower estimates

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a suitably smooth boundary. We consider now a sequence of linear operators $T_n$ defined on $X^{p,0}(\Omega)$ with smooth range $\text{ran}(T_n) \subset X^{p,r}(\Omega)$ whose fixed point space $\ker(T_n - I)$ is annihilated by every differential operator $D^\alpha$ of order $r$ that is bounded on $\text{ran}(T_n)$. In this general setting, we will show that for all $s \geq r$ and $n > 0$ there is $t_n > 0$ and there are constants $M_1, M_2 > 0$ independent of $n$ and $f \in X^{p,0}(\Omega)$ such that

$$M_1 \cdot \omega_{s,p}(f, t_n) \leq \|T_n f - f\|_p \quad \text{and} \quad M_2 \cdot K_{s,p}(f, t_n^*) \leq \|T_n f - f\|_p.$$  

Here, $t_n \to 0$ for $n \to \infty$ provided that $\|f - T_n f\|_p \to 0$.

In order to prove these estimates, we will consider the case where the smooth function $g$ in (2) is replaced by the smooth approximation $T_n f$. Then, we will estimate the semi-norm $\|T_n f\|_{r,p} = \sup \|D^\alpha T_n f\|_p$ with respect to the approximation error $\|T_n f - f\|_p$. The key concept of our approach is to use the limiting operator of the iterates $T^n$ as shown in Proposition 1. Recall that the compactness of the operators $T_n$ combined with a spectral location will guarantee the existence of the limiting operator as seen in Lemma 1 and Proposition 2. With this in mind, we can state the following lemma:

**Lemma 2.** Let $1 \leq p \leq \infty$ and let $T : X^{p,0}(\Omega) \to X^{p,0}(\Omega)$ be a compact contraction, i.e., $\|T\|_{op} \leq 1$. Suppose

1. $\sigma(T) \subset B(0,1) \cup \{1\}$,
2. $\text{ran}(T) \subset X^{p,r}(\Omega)$ for some positive integer $r$,
3. $D^\alpha$ annihilates $\ker(T - I)$ for all $\alpha$ with $|\alpha| = r$.

Then for every $f \in X^{p,0}(\Omega)$,

$$\|T f\|_{r,p} \leq \sup_{|\alpha| = r} \frac{\|D^\alpha \text{ran}(T)\|_{op}}{1 - \gamma} \|T f - f\|_p,$$

where $\|D^\alpha \text{ran}(T)\|_{op}$ is the operator norm of $D^\alpha$ on $\text{ran}(T)$ and

$$\gamma := \sup \{|\lambda| : \lambda \in \sigma(T) \text{ with } |\lambda| < 1\}.$$  

**Proof.** As $T$ is compact and exhibits the spectral property $\sigma(T) \subset B(0,1) \cup \{1\}$, there exists a projection $P$ with $\text{ran}(P) = \ker(T - I)$ and according to Lemma 1 there exists a constant $0 \leq C \leq \gamma^{-1}$ such that

$$\|T^m - P\|_{op} \leq C \gamma^m$$

holds for all integers $m > 0$. As the range of $P$ is exactly the fixed point space of $T$, we have that $D^\alpha P = 0$ whenever $|\alpha| \geq r$.

Using these results we obtain

$$\|T f\|_{r,p} = \sup_{|\alpha| = r} \|D^\alpha T f\|_p = \sup_{|\alpha| = r} \|D^\alpha T f - D^\alpha T^2 f + D^\alpha T^2 f - D^\alpha T^3 f + \ldots\|_p \leq \sup_{|\alpha| = r} \sum_{m=1}^{\infty} \|D^\alpha T^m (f - T f)\|_p \leq \|T f - f\|_p \cdot \sup_{|\alpha| = r} \sum_{m=1}^{\infty} \|D^\alpha T^m\|_{op}$$

\[\text{4} \]
\[ = \|Tf - f\|_p \cdot \sup_{|\alpha| = r} \sum_{m=1}^{\infty} \|D^\alpha (T^m - P)\|_{op}, \]
\[ = \|Tf - f\|_p \cdot \sup_{|\alpha| = r} \sum_{m=1}^{\infty} \|D^\alpha (T^m - P)\|_{op}, \]
as \(D^\alpha\) annihilates \(\ker(T - I)\) and therefore, \(D^\alpha P = 0\). By the boundedness of \(D^\alpha\) on \(\text{ran}(T)\) we get
\[ [Tf]_{r,p} \leq \|Tf - f\|_p \cdot \sup_{|\alpha| = r} \|D^\alpha |\text{ran}(T)|\|_{op} \sum_{m=1}^{\infty} \|T^m - P\|_{op}, \]
\[ \leq \|Tf - f\|_p \cdot \sup_{|\alpha| = r} \|D^\alpha |\text{ran}(T)|\|_{op} \sum_{m=1}^{\infty} C\gamma^m. \]
Using that \(C \leq 1/\gamma\) the series reduces to a convergent geometric series and we conclude the proof with
\[ [Tf]_{r,p} \leq \|Tf - f\|_p \cdot \sup_{|\alpha| = r} \|D^\alpha |\text{ran}(T)|\|_{op} \sum_{m=0}^{\infty} \gamma^m \]
\[ \leq \frac{\sup_{|\alpha| = r} \|D^\alpha |\text{ran}(T)|\|_{op}}{1 - \gamma} \|Tf - f\|_p. \]

Note that the third condition of [Lemma 2] is reflected in the shown estimate as for each \(f \in \ker(T - I)\) we have that \(\|Tf - f\|_p = 0\) and \([Tf]_{r,p} = 0\).

Using this lemma, we can state the main results of this article:

**Theorem 1.** Let \(1 \leq p \leq \infty\) and let \(T : X^{p,0}(\Omega) \to X^{p,0}(\Omega)\) be a compact contraction that satisfies the following conditions:

1. \(\sigma(T) \subset B(0,1) \cup \{1\}\),
2. \(\text{ran}(T) \subset X^{p,r}(\Omega)\) for some positive integer \(r\),
3. \(D^\alpha\) annihilates \(\ker(T - I)\) for all \(\alpha\) with \(|\alpha| = r\).

Then
\[ \omega_{r,p}(f, t) \leq \left(2^r + d^r t^r \frac{\sup_{|\alpha| = r} \|D^\alpha |\text{ran}(T)|\|_{op}}{1 - \gamma}\right) \cdot \|Tf - f\|_p \]
and
\[ K_{r,p}(f, t^r) \leq \left(1 + t^r \frac{\sup_{|\alpha| = r} \|D^\alpha |\text{ran}(T)|\|_{op}}{1 - \gamma}\right) \cdot \|Tf - f\|_p \]
holds for all \(t \in (0,\infty)\), where \(\gamma := \sup \{|\lambda| : \lambda \in \sigma(T)\ \text{with} \ \lambda \neq 1|}\).

**Proof.** We apply \([3]\) and [Lemma 2] to obtain the stated result. \(\square\)

**Corollary 1.** Let \((T_n)\) be a sequence of continuous linear operators on \(X^{p,0}(\Omega)\) that satisfies the conditions of [Theorem 1]. Besides, we assume that \(\|T_n f - f\|_p \to 0\) holds for all \(f \in X^{p,0}(\Omega)\) if \(n\) tends to infinity.

Then, with setting \(\gamma_n := \sup \{|\lambda| : \lambda \in \sigma(T_n) \ \text{with} \ \lambda \neq 1|\}\) the uniform lower estimates
\[ \omega_{r,p}(f, \delta_n) \leq (2^r + d^r t^r) \cdot \|T_n f - f\|_p \quad \text{and} \quad K_{r,p}(f, \delta_n^r) \leq 2 \cdot \|T_n f - f\|_p \]
holds for all \(t \in (0,\infty)\).
holds, where
\[
\delta_n = \left( \frac{1 - \gamma_n}{\sup_{|\alpha|=r} \|D^\alpha|_{\text{ran}(T)}\|_{\text{op}}} \right)^{1/r}
\]
and \(\delta_n \to 0\) if \(n\) tends to infinity.

Remark 1. The property that \(\delta_n \to 0\) if \(n\) tends to infinity follows by \(\|T_n f - f\|_p\) for \(f \in C([0,1])\). To assure that this property holds there are the following two options. Either the second largest eigenvalue tends in the modulus to one, i.e.,
\[
\gamma_n \to 1
\]
which is satisfied as \(T_n\) converges against the identity \(I\) in the strong operator topology, or
\[
\sup_f |\alpha| = r \implies \|D^\alpha|_{\text{ran}(T)}\|_{\text{op}} \to \infty.
\]

Finally, we want to outline a generalization to derive lower estimates for a sequence of linear operators \((T_n)_{n \in \mathbb{N}}\) on arbitrary Banach spaces based on the \(K\)-functional where smoothness of the range is not necessary. The conditions depend on the underlying semi-norms defined on the range of \(T_n\). Accordingly, the semi-norms have to annihilate the fixed points of \(T_n\) and are bounded on the range of \(T_n\).

Theorem 2. Let \((X_1, \|\cdot\|_{X_1})\) be a Banach space and \((X_2, |\cdot|_{X_2})\) be a quasi Banach space with \(X_2 \subset X_1\). Consider a sequence \(T_n : X_1 \to X_2\) of compact contractions, such that the following conditions hold:
\[
\begin{align*}
1. & \quad \sigma(T_n) \subset B(0,1) \cup \{1\}, \\
2. & \quad \text{the semi-norm } |\cdot|_{X_2} \text{ annihilates } \ker(T_n - I).
\end{align*}
\]
Then
\[
\frac{1}{2} \cdot \inf_{g \in X_2} \left( \|f - g\|_p + \delta_n |g|_{X_2} \right) \leq \|T_n f - f\|_p,
\]
where
\[
\delta_n = \left( \frac{1 - \gamma_n}{\sup_{f \in X_2, \|f\|_{X_2} = 1} |T_n f|_{X_2}} \right)^{1/r}.
\]
Proof. Follows directly along the lines of the proof of Theorem 1.

4. Applications to Positive Linear Operators

We conclude this chapter with concrete examples. First we prove lower estimates for general positive linear operators. Afterwards, we prove give concrete estimates for the Bernstein operator, the Kantorović operator, the the Schoenberg operator and the integral Schoenberg operator.

4.1. Lower estimates for general positive finite-rank operators

In the following, let \(\Omega = [0,1]^d\), thus \(X^{p,r}(\Omega)\) contains the constant function 1 with \(\|1\|_p = 1\). We consider a sequence of positive finite-rank operator \(T_n : X^{p,0}(\Omega) \to X^{p,0}(\Omega)\),
\[
T_n f = \sum_{k=1}^{n} \alpha^*_k(f) e_k, \quad f \in X^{p,0}(\Omega),
\]
where \(e_1, \ldots, e_n \in X^{p,r}(\Omega)\) are linearly independent, smooth positive functions that form a partition of unity; \(\alpha^*_k\) are positive linear functionals satisfying \(\|\alpha^*_k\| = \alpha^*_k(1) = 1\) and \(\alpha^*_k(e_k) > 0\) for \(k \in \{1, \ldots, n\}\). It has been shown in [18], that the spectrum of \(T_n\) is characterized by
\[
\sigma(T_n) \subset B(0,1) \cup \{1\}
\]
and 1 is an eigenvalue of \(T_n\) due to the partition of unity property. Thus, to prove lower estimates with the technique shown in this chapter, only last condition have to be checked. Thus, we can restate Corollary 1 as follows:
Corollary 2. Let \((T_n)\) be a sequence of continuous linear operators on \(X^{p,0}(\Omega)\) of the form \((5)\) such that \(\|T_n f - f\|_p \to 0\) holds for all \(f \in X^{p,0}(\Omega)\) if \(n\) tends to infinity. Let us denote

\[
\gamma_n := \sup \{ |\lambda| : \lambda \in \sigma(T_n) \text{ with } \lambda \neq 1 \}.
\]

If every differential operator of order \(r\) annihilates \(\ker(T_n - I)\) then the approximation error can be bounded from below by

\[
\omega_{r,p}(f, \delta_n) \leq (2^r + 1) \cdot \|T_n f - f\|_p \quad \text{and} \quad K_{r,p}(f, \delta_n) \leq 2 \cdot \|T_n f - f\|_p,
\]

where

\[
\delta_n = \left( \frac{1 - \gamma_n}{\sup_{|\alpha| = r} \|D^{|\alpha|}(T_n)\|_{op}} \right)^{1/r}.
\]

and \(\delta_n \to 0\) if \(n\) tends to infinity.

4.2. Lower estimate for the Bernstein operator
Let \(B_n : C([0,1]) \to C([0,1])\) be the Bernstein operator of order \(n > 0\) defined by

\[
B_n f(x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}.
\]

It is well known, see e.g. Lorentz [15], that this operator can reproduce constant and linear functions and interpolates at the endpoints of the unit interval. Therefore

\[
\ker(B_n - I) = \text{span}(1, x)
\]

and \(D \ker(B_n - I) = 0\). As shown by Călugăreanu [4], the eigenvalues \((\lambda_{k,n})\) of \(B_n\) are explicitly known for \(k \in \{0, \ldots, n\}\) by

\[
\lambda_{k,n} = \frac{n!}{(n-k)! n^k}.
\]

A comprehensive discussion on the corresponding eigenfunctions can be found in the work of Cooper and Waldron [3]. Clearly, we have \(\sigma(B_n) \subset B(0,1) \cup \{1\}\), as

\[
1 = \lambda_{0,n} = \lambda_{1,n} > \lambda_{2,n} > \ldots > \lambda_{n,n} = \frac{n!}{n^n},
\]

while this property also follows by [18]. The second largest eigenvalue \(\gamma_n\) of \(B_n\) is \(\gamma_n := \lambda_{2,n} = \frac{n!}{n^n}\).

The range of the Bernstein operator is given by the space of all polynomials with degree at most \(n\). Thus, we obtain for \(r < n\) the following upper bound for the operator norm of \(D^r\) on \(\text{ran}(B_n)\) using the representation of \(D^r B_n f\) in Lorentz [15, p.24]:

\[
\|D^r\|_{op} \leq \frac{2^r n!}{(n-r)!}.
\]

Finally, we obtain with [Theorem 1] the lower estimate

\[
\omega_r(f, t) \leq \left( 2^r + t^r \frac{2^r n!}{(n-r)!} \right) \cdot \|T f - f\|_\infty \leq 2^r \left( 1 + n^{r+1} t^r \right) \cdot \|T f - f\|_\infty
\]

for all \(t \in (0, \infty)\). For the case \(r = 2\), we derive accordingly the following uniform estimate:

Corollary 3. The approximation error of the Bernstein operator \(B_n\) can be uniformly bounded for all \(f \in C([0,1])\) by

\[
\frac{1}{8} \omega_2(f, n^{-3/2}) \leq \|B_n f - f\|_\infty, \quad n \to \infty.
\]

Remark 2. Compared to the known lower estimate using the Ditzian-Totik modulus of smoothness as shown by Ditzian and Totik [2] and Knoop and Zhou [14] one would expect a decay rate of \(n^{-1/2}\). The question arises, whether sharper estimates used in the proof can lead to this optimal decay rate or if this is already the best possible lower estimate for the classical modulus of smoothness.
4.3. Lower estimate for the Kantorović operator

Let us consider the Kantorović operator $K_n : L^1([0, 1]) \to C([0, 1])$,

$$K_n f(x) = (n + 1) \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) \, dt, \quad x \in [0, 1]$$

see Kantorović [11]. This operator has a direct relation to the Bernstein operator in the following way [15, p.30]:

$$K_n(D f) = D(B_{n+1} f), \quad \text{for all } f \in C^1([0, 1]). \quad (6)$$

Besides, we have that $\ker(K_n - I) = \text{span}\{1\}$. Infact, $D1 = 0$, hence the differential operator $D$ annihilates $\ker(K_n - I)$. Besides, $D$ is bounded on $\text{ran}(K_n)$ in the same way as the Bernstein operator:

$$\|DK_n f(x)\|_p = \|D^2B_{n+1} F(x)\|_p \leq \|D^2|_{\text{ran}(B_{n+1})}\|_{op} \|f\|_1,$$

where we have used (6) and $F(x) = \int_0^x f(t) \, dt$. Therefore,

$$\|D\|_{op} \leq \|D^2|_{\text{ran}(B_{n+1})}\|_{op} = \frac{4(n+1)!}{(n+1-2)!} = 4(n^2 + n).$$

holds. Combining these results with Theorem 1 we can state the lower estimate

$$\omega_{1,p}(f, t) \leq \left( 2 + t \frac{4(n^2 + n)}{n} \right) \cdot \|T f - f\|_{\infty} \leq (2 + 4(n^3 + n^2)t) \cdot \|T f - f\|_{\infty}$$

for all $t \in (0, \infty)$. Consequently, we get the following uniform estimate:

Corollary 4. The approximation error of the Kantorović operator $K_n$ can be uniformly bounded from below by

$$\frac{1}{6} \omega_{1,p}(f, \frac{1}{n^3 + n^2}) \leq \|K_n f - f\|_{\infty}, \quad n \to \infty,$$

for all $f \in L^1([0, 1])$.

As in the case of the Bernstein-operator, we are not able to derive the optimal lower estimate shown in Chen and Ditzian [2] with the Ditzian-Totik modulus of continuity, but we could still provide an estimate with the classical modulus of continuity.

4.4. Lower estimate for the Schoenberg operator

A lower estimate for the Schoenberg operator has already been shown in Nagler et al. [17] using similar techniques. Thus we state here only the results for the sake of completeness. To this end let $n > 0$ be an integer and $\Delta_n = \{x_j\}_{j=-n}^{n}$ be an extended knot sequence such that

$$0 = x_{-k} = x_0 < x_1 < \ldots < x_n = x_n = 1.$$

According to Schoenberg [21], we consider the variation diminishing spline operator $S_{\Delta_n,k} : C([0, 1]) \to C([0, 1])$ of degree $k$ with respect to the knot sequence $\Delta_n$ for continuous functions $f$ by

$$S_{\Delta_n,k} f = \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k},$$

where $\xi_{j,k}$ are the so called Greville nodes, see the supplement in [21], defined for all $j \in \{-k, \ldots, n-1\}$ by

$$\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k}.$$
The normalized B-splines $N_{j,k}$ are defined for all $j \in \{-k, \ldots, n-1\}$ and $x \in [0,1]$ by

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, \ldots, x_{j+k+1}](x-x)^k,$$

where $[x_j, \ldots, x_{j+k+1}]$ denotes the divided difference operator and $x^k_+$ denotes the truncated power function. We define the minimal mesh gauge as

$$|\Delta n|_{\text{min}} := \min \{(x_{j+1,k} - x_{j,k}) : j \in \{0, \ldots, n-1\}\}$$

and $\gamma_{\Delta n,k} := \sup \{\lambda \in \mathbb{C} : \lambda \in \sigma(S_{\Delta n,k}) \setminus \{1\}\}$. Then we can state the following lower estimate, see [17, Cor. 2]:

**Corollary 5.** Let $f \in C([0,1])$ and $k > r \geq 2$. Then

$$\frac{1}{2r+1} \omega_r(f, t(\Delta n, k)) \leq \|f - S_{\Delta n,k} f\|_{\infty},$$

where

$$t(\Delta n, k) = \frac{|\Delta n|_{\text{min}}}{k} \cdot \left(1 - \frac{\gamma_{\Delta n,k}}{d_k}\right)^{1/r}.$$

Moreover, $t(\Delta n, k) \to 0$ if the approximation error converges to zero.

In order to get concrete values, it would be very interesting to have an exact representation of the eigenvalues of $S_{\Delta n,k}$.

### 4.5. Lower estimate for the integral Schoenberg operator

The integral Schoenberg operator is defined by

$$V_{\Delta n,k} f(x) := DS_{\Delta n,k+1} F(x) = \sum_{j=-k}^{n-1} \int_{\xi_{j-1,k+1}}^{\xi_j,k+1} f(t) dt \frac{N_{j,k}(x)}{\xi_j,k+1 - \xi_{j-1,k+1}},$$

where $F(x) = \int_0^x f(t) dt$. More details are shown in Müller [16]. We have that $\ker(V_{\Delta n,k} - I) = \text{span} \{1\}$ and $D1 = 0$ holds. By [18], we can conclude that

$$\sigma(V_{\Delta n,k}) \subset B(0,1) \cup \{1\},$$

holds. The operator norm of the differential operator $D$, can be obtained similarly to the Kantorovič operator. To this end, we utilize a similar relation as in [18] between the Schoenberg operator and its counterpart for the $L^p$-spaces:

**Lemma 3.** For all $f \in C^1([0,1])$ the relation

$$DS_{\Delta n,k} f = V_{\Delta n,k-1} f$$

holds.

**Proof.** Follows directly by the definition of the integral Schoenberg operator, as

$$V_{\Delta n,k} f(x) = DS_{\Delta n,k+1} F(x) = \int_0^x f(t) dt.$$

Then a simple calculation yields

$$V_{\Delta n,k-1} f(x) = DS_{\Delta n,k} \int_0^x f(t) dt = DS_{\Delta n,k} (f(x) - f(0)) = DS_{\Delta n,k} f(x).$$

In the last step, we used the linearity of $S_{\Delta n,k}$ and that $S_{\Delta n,k}$ can reproduce constants. □
Now we can use this relation between $V_{\Delta n,k}$ and $S_{\Delta n,k}$ to derive
\[
\|DV_{\Delta n,k}f\|_p = \|D^2S_{\Delta n,k+1}F\|_p = \|D^2\|_{op,\text{ran}(S_{\Delta n,k+1})} \|f\|_1,
\]
where $F(x) = \int_0^x f(t)dt$. Using (7) and the shown operator norm of $D^2$ on $\text{ran}(S_{\Delta n,k+1})$, see [17], we obtain the following bound on $\text{ran}(V_{\Delta n,k})$:
\[
\|D\|_{op,\text{ran}(V_{\Delta n,k})} \leq \|D^2\|_{op,\text{ran}(S_{\Delta n,k+1})} = \left(\frac{2(k+1)}{|\Delta n_{\min}|}\right)^2 d_{k+1}.
\]
As all conditions of Corollary 1 are satisfied, we can state the following lower estimates:

**Corollary 6.** Lower estimates for the integral Schoenberg operator $V_{\Delta n,k}$ are given by
\[
\frac{1}{6} \omega_{1,p}(f, t(\Delta n, k)) \leq \|V_{\Delta n,k}f - f\|_p,
\]
where
\[
t(\Delta n, k) = \frac{|\Delta n_{\min}|^2}{(k+1)^2} \left(\frac{1 - \gamma_{\Delta n,k}}{d_{k+1}}\right).
\]

5. Remarks on the eigenvalues of the Schoenberg operator

The eigenvalues of the Bernstein operator have been revealed already in 1966 by the Russian Călugăreanu [4]. Up to our knowledge results on the eigenvalues of the Schoenberg operator are not known explicitly. In the following, we show that 1 is a simple eigenvalue of $V_{\Delta n,k}$ and all the other eigenvalues are distinct non-negative, real numbers. Finally, we will show that the Schoenberg operator has the same eigenvalues as $V_{\Delta n,k}$ with the exception that 1 is not a simple eigenvalue as the Schoenberg operator reproduces constants and linear functions.

To simplify notation, we define the B-splines $M_{j,k}$ for $j \in \{-k, \ldots, n-1\}$ as in [5] by:
\[
M_{j,k}(x) := \frac{N_{j,k}(x)}{\xi_{j,k+1} - \xi_{j-1,k+1}}.
\]
Note that these functions are normalized to have integral one, i.e. $\int_0^1 M_{j,k}(x)dx = 1$, and have finite support:
\[
\text{supp } M_{j,k}(x) = [x_j, x_{j+k+1}].
\]
Using this notation, we state the following theorem:

**Theorem 3.** The collocation matrix of the integral Schoenberg operator with the normalized B-splines as defined in (8)
\[
\left(\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} M_{j,k}(t)dt\right)_{ij}
\]
is an oscillatory matrix. Thus, all eigenvalues are distinct positive real numbers.

**Proof.** Recall, that the Greville nodes $\xi_{j,k}$ are defined as the knot averages as in (4.4) by
\[
\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k}.
\]
First, note that the relations
\[
x_j < \xi_{j-1,k+1} < \xi_{j,k} < \xi_{j,k+1} < x_{j+k+1},
\]
\[ x_{j+1} < \xi_{j,k+1} < \xi_{j+1,k} < \xi_{j+1,k+1} < x_{j+k+2} \]

and

\[ \text{supp} M_{j,k} = [x_{j}, x_{j+k+1}], \quad \text{supp} M_{j+1,k} = [x_{j+1}, x_{j+k+2}] \]

hold. From the continuity of \( M_{j,k} \) and \( M_{j+1,k} \) and the relations

\[ M_{j,k}(\xi_{j,k}) > 0, \quad M_{j+1,k}(\xi_{j,k+1}) > 0, \quad M_{j,k}(\xi_{j,k+1}) > 0, \]

we can follow that

\[ \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} M_{j,k}(t)dt > 0, \quad \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} M_{j+1,k}(t)dt > 0, \quad \text{and} \quad \int_{\xi_{j,k+1}}^{\xi_{j+1,k+1}} M_{j,k}(t)dt > 0 \]

holds. Moreover, the matrix

\[ \left( \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} M_{j,k}(t)dt \right)_{ij} \]

is non-singular as the B-splines \( M_{j,k}(x) \) are linearly independent and so are the functionals \( \int_{\xi_{j-1,k+1}}^{\xi_{j,k+1}} \cdot dt \) due to their distinct support. Using the well known result of [9, Thm. 10, p.100], which states that a totally positive matrix is oscillatory. By [9, Thm. 6, p.87] it follows that the eigenvalues of the collocation matrix \( A \) are distinct positive real numbers, i.e., \( \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 \).

Thus, besides \( 0 \) and \( 1 \) the Schoenberg operator has \( n + k - 1 \) distinct positive real eigenvalues.

\[ 1 = \lambda_0 = \lambda_1 > \lambda_2 > \cdots > \lambda_{n+k-1} > \lambda_{n+k} = 0. \]

**Proof.** We use that \( V_{\Delta_n,k-1} \) has \( n + k - 1 \) distinct positive eigenvalues combined with the eigenvalue 0 coming from the finite-dimensional range of \( V_{\Delta_n,k-1} \) and Lemma 3 saying that

\[ DS_{\Delta_n,k} f = V_{\Delta_n,k-1} Df \]

holds for all \( f \in C([0,1]) \).

We show first that \( 0 \in \sigma_p(S_{\Delta_n,k}) \). To this end, let \( f \in C([0,1]) \) be a function, such that

\[ f(\xi_j) = 0 \quad \text{for all} \quad j \in \{-k, \ldots, n-1\} \]

and such that there exists \( x \in [0,1] \) \( \{ \xi_j : j \in \{-k, \ldots, n-1\} \} \) with \( f(x) \neq 0 \). For example, consider the polynomial \( f(x) = \prod_{i=-k}^{n-1} (x - \xi_i) \). Clearly, \( f \in C([0,1]) \) and we obtain \( S_{\Delta_n,k} f = 0 \cdot f = 0 \), because for all \( x \in [0,1] \)

\[ S_{\Delta_n,k} f(x) = \sum_{j=-k}^{n-1} \prod_{i=-k}^{n-1} (\xi_j - \xi_i) N_{j,k}(x) = 0. \]

We now construct the set of eigenvalues and eigenfunctions of \( S_{\Delta_n,k} \) by their relation to the integral Schoenberg operator \( V_{\Delta_n,k-1} \). To this end, let us consider now an eigenfunction \( s \in S(\Delta_n,k) \) of \( S_{\Delta_n,k} \) corresponding to some eigenvalue \( \lambda \in \sigma_p(S_{\Delta_n,k}) \setminus \{0,1\} \). Then we calculate

\[ V_{\Delta_n,k-1} Ds = DS_{\Delta_n,k} s = \lambda Ds. \]
Now we use that all eigenfunctions \( s \) of the Schoenberg operator with corresponding eigenvalue \( \lambda \neq 0 \) are linearly independent, to conclude that the same holds true for the functions \( Ds_1, \ldots, Ds_{n+k-1} \). Consequently, the positive numbers \( \lambda_1, \ldots, \lambda_{n+k-1} \) are exactly the \( n+k-1 \) distinct eigenvalues of \( S_{\Delta_n,k} \).

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