Kolmogorov-Zakharov Model for Optical Fiber Communication

Mansoor I. Yousefi, Frank R. Kschischang, Fellow, IEEE, and Gerhard Kramer, Fellow, IEEE

Abstract—A power spectral density based on the theory of weak wave turbulence is suggested for calculating the interference power in dense wavelength-division multiplexed optical systems. This power spectrum, termed Kolmogorov-Zakharov (KZ) model, results in a better estimate of the signal spectrum in optical fiber, compared with the so-called Gaussian noise (GN) model.

Index Terms—Power spectral density, perturbation theory, weak turbulence, GN model.

I. INTRODUCTION

This paper studies the power spectral density (PSD) and the probability distribution of a signal propagating according to the one-dimensional cubic nonlinear Schrödinger (NLS) equation.

A PSD known as the Gaussian noise (GN) model has been proposed for optical fiber communications [1], [2]. The GN model results from a first-order perturbation approach to four-wave mixing in the NLS equation [1].

Although the GN model has appeared in the fiber-optic communications literature, there also exists a satisfactory and well-developed theory of PSD for nonlinear dispersive equations in the context of wave turbulence in mathematical physics. Here a statistical approach is taken by averaging over uniform signal phase or random input. This leads to a hierarchy of recursive differential equations for n-point spectral cumulants. In strong turbulence, encountered e.g., in the Navier-Stokes equations of hydrodynamics, nonlinearity is strong and this hierarchy does not truncate. However under the weak nonlinearity assumption, which holds in optical fiber, this hierarchy is truncated and a differential equation for PSD (a 2-point cumulant), known as kinetic equation, is obtained [3]–[5]. The aim of weak wave turbulence (WWT) theory is to study energy distribution in the frequency domain via kinetic equations. The kinetic equations of WWT can often be solved using, e.g., Zakharov conformal transformations to obtain Kolmogorov-Zakharov (KZ) stationary spectra.

Turbulence theory helps us to understand how user data is transported to other users in a multiuser communication system. For instance, kinetic equations provide accurate predictions of interference power. More importantly, the theory gives indispensable insights into inter-channel interactions in wavelength-division multiplexing (WDM). WWT also yields a suitable mathematical framework for statistical signal analysis in optical fiber, such as PSD calculation.

In this paper we introduce the KZ model and explain how it relates to the GN model. The standard kinetic equation of the WWT predicts a stationary spectrum for the integrable NLS equation. However, the analysis can easily be carried out to the next order in the nonlinearity level to account for deviations from the stationary spectrum. We compare the KZ and GN PSDs and show that the KZ PSD is equally simple yet provides better estimates of WDM interference. The KZ model describes energy fluxes correctly; for instance, unlike the GN model, the KZ spectrum of the lossless optical fiber is energy-preserving. The KZ model also predicts a quasi-Gaussian distribution, which is close to a Gaussian one. However, this small deviation from the Gaussian distribution, not captured by the GN model, is responsible for spectrum evolution and interference.

The power spectral density is a central object in statistical studies of nonlinear dispersive waves and is widely studied in mathematical physics [6]. One aim of this paper is to point this out and show that some of the elaborate PSD calculations in communications engineering can be succinctly modified and methodically generalized in the more fundamental framework of WWT. The reader is referred to [3], [5] for an introduction to WWT and to [6] for a survey of optical turbulence (mostly in higher dimensional and non-integrable models). Turbulence in integrable systems, which is the focus of this paper, is discussed in [4], [7].

II. NOTATION AND PRELIMINARIES

The Fourier transform of \( q(t) \) is defined without factor \( 1/2\pi \)

\[
F(q)(\omega) \triangleq \int_{-\infty}^{\infty} q(t) e^{j\omega t} dt.
\]

(1)

We use subscripts to denote the frequency variable, e.g., \( q_\omega = F(q)(\omega) \). Fourier series coefficients of a periodic signal \( q(t) \) are similarly denoted by \( q_k = F_q(q)(k) \). When there are multiple frequencies \( \omega_i \) in an expression, for brevity we often use shorthand notations \( q_i := q(\omega_i) \) and \( d\omega_1 \ldots n := \prod_i d\omega_i \). In such cases, it will be clear from the context whether \( q_i \) corresponds to a discrete or a continuous frequency variable. To avoid confusion, we do not use subscripts to denote the time variable.

The following notation is used throughout the paper

\[
\delta_{k_1 \ldots k_{2n}} \triangleq \delta(s_1 k_1 + \ldots + s_{2n} k_{2n}), \quad k_i \in \mathbb{Z},
\]

In physics, the PSD can be recognized in relation with terms: wave number, wave-action density, particle number, occupation number, pair correlator, energy density, etc.
where $\delta(m)$ is the Kronecker delta and
\[
    s_i = \begin{cases} 
        1, & 1 \leq i \leq n, \\
        -1, & n + 1 \leq i \leq 2n.
    \end{cases}
\]

For continuous frequencies, the corresponding real subscript $\omega_1 \cdots \omega_{2n}$ is shortened to the integer subscript $1 \cdots 2n$
\[
    \delta_{1 \cdots 2n} \triangleq \delta(s_1 \omega_1 + \cdots + s_{2n} \omega_{2n}), \quad \omega_i \in \mathbb{R},
\]
(2)
where, with notation abuse, $\delta(\omega)$ is the Dirac delta function.

Let $q(t)$ be a zero-mean stochastic process. The symmetric 2n-point correlation functions in time (temporal moments) are
\[
    R(t_1 \cdots t_{2n}) \triangleq \mathbb{E}\{q(t_1) \cdots q(t_n)q^*(t_{n+1}) \cdots q^*(t_{2n})\},
\]
(3)
where $\mathbb{E}$ denotes expected value with respect to the corresponding joint probability distribution. Similarly, the 2n-point spectral moments are
\[
    \mu_{1 \cdots 2n} \triangleq \mathbb{E}\{q_1 \cdots q_nq_{n+1}^* \cdots q_{2n}^*\}.
\]
(4)
The asymmetric correlation functions and moments, in which the number of conjugate and non-conjugate variables are not the same, is assumed to be zero. If $n = 1$, $\mu_{12}$ corresponds to the correlation between $q(\omega_1)$ and $q(\omega_2)$.

If a stochastic process $q(t)$ is (strongly) stationary, then
\[
    R(t_1, \cdots, t_{2n}) = R(t_1 - t_0, \cdots, t_{2n} - t_0),
\]
(5)
for any reference point $t_0$ and $n \geq 1$. It is shown in the Appendix [A-A] that if $q(t)$ is stationary, then
\[
    \mu_{1 \cdots 2n} = S_{1 \cdots 2n}\delta_{1 \cdots 2n},
\]
(6)
where
\[
    S_{1 \cdots 2n} = \mathcal{F}(R(0, t_2, \cdots, t_{2n}))(0, s_2 \omega_2, \cdots, s_{2n} \omega_{2n}),
\]
is the moment density function. Thus $\mu_{1 \cdots 2n}$ is non-zero only on the stationary manifold
\[
    s_1 \omega_1 + \cdots + s_{2n} \omega_{2n} = 0.
\]
(7)

For a Gaussian distribution, only the mean and the 2-point cumulants are non-zero. Consequently, $\mu_{1 \cdots 2n}$ is concentrated on normal manifolds
\[
    s_1 \omega_l + s_k \omega_k = 0, \quad 1 \leq l \leq n, \quad n + 1 \leq k \leq 2n,
\]
which are subsets of the stationary manifold. All other distributions have infinitely many non-zero cumulants. As a result, cumulants are used in this paper to measure deviations from the Gaussian distribution. A zero-mean distribution is defined to be quasi-Gaussian if
\[
    \tilde{S}_{1 \cdots n} \approx 0, \quad \forall n \geq 6.
\]
(10)
That is to say, at most 4-point cumulants are significant.

We will often make use of the trilinear integral and sum of the signals $q_\omega(z)$ and $q_k(z)$, defined as
\[
    N_\omega(q, q, q)(z) \triangleq \int_{-\infty}^{\infty} q_1(z)q_2(z)q_3^*(z)\delta_{123\omega}d\omega_{123},
\]
(11)
\[
    N_k(q, q, q)(z) \triangleq \sum_{lmn \in nr_k} q_l(z)q_m(z)q_n^*(z),
\]
where $nr_k$ is the set of the non-resonant frequencies
\[
    nr_k \triangleq \{(l, m, n) \mid l + m = n + k, \quad l \neq k, \quad m \neq k\}.
\]

These expressions help to factor out part of the complexity. The following simple lemma is frequently used in Section [IV] when passing from the zero-order to the first-order in perturbation expansions.

**Lemma 1.** Let $q_\omega(z) = \exp(j \omega^2 z) q_\omega(0)$. Then
\[
    \int_0^{\infty} e^{-j \omega z} N_\omega(q, q, q)(z')dz' =
\]
\[
    j \int H_{123\omega}(z)q_1(0)q_2(0)q_3^*(0)\delta_{123\omega}d\omega_{123},
\]
where the $H$-function is
\[
    H_{123\omega}(z) \triangleq \begin{cases} 
        (1 - e^{j \Omega_{123\omega} z})/\Omega_{123}, & \Omega_{123} \neq 0, \\
        -jz, & \Omega_{123} = 0.
    \end{cases}
\]
(12)
in which
\[
    \Omega_{123} = \omega_1^2 + \omega_2^2 - \omega_3^2 - \omega^2.
\]

**Proof:** The result follows by substitution.

Depending on the context, we may write the $H$-function as $H_{123\omega}$, $H_{123\omega}(z)$ or $H(\Omega_{123\omega})(z)$. A similar lemma can be stated for the trilinear sum.
III. CHANNEL MODEL

A. Continuous-frequency NLS Equation

We consider the one-dimensional cubic dimensionless NLS equation on the real line

$$j\partial_z q = \partial_t q + 2|q|^2 q, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^+,$$  \quad (13)

where \(q(t, z)\) is the signal as a function of space \(z\) and time \(t\). To focus on main ideas, in this section we consider only a single span lossless fiber. Loss and amplification in multi-span systems are introduced later in Section VII.D.

Using Duhamel’s formula, the differential equation (13) can be re-written as an integral equation

$$q_\omega(z) = e^{j2\omega^2/\delta_\omega} q_\omega(0) - 2j \int_0^z e^{j(z-z')\omega^2/\delta_\omega} \mathcal{N}_\omega(q, q, q)(z') dz',$$

(14)

where \(\mathcal{N}_\omega\), defined in (11), represents interaction among all four waves 123\(\omega\) and \(\delta_{123\omega}\) denotes the corresponding frequency matching condition \(\omega_1 + \omega_2 = \omega_3 + \omega_4\). An interacting quartet can be shown schematically as 12 → 3ω. Note that the trivial interactions

$$(\omega_1 = \omega_3, \omega_2 = \omega), \quad (\omega_1 = \omega, \omega_2 = \omega_3),$$  \quad (15)

form a set of zero (Lebesgue) measure and do not contribute to the integral (14).

B. Discrete-frequency NLS Equation

We also consider the NLS equation on a torus \(t \in \mathbb{T} = \mathbb{R}/(T\mathbb{Z})\), corresponding to \(T\)-periodic signals. Partitioning the sums

$$\sum_{lm} = \sum_{(l=k) v (m=k)} + \sum_{(l=k) \wedge (m=k)} + \sum_{(l=k) \vee (m=k)} + \sum_{(l=k) \land (m=k)},$$

(16)

where \(\vee\) and \(\wedge\) are, respectively, or and and operations, we get the identity

$$\mathcal{F}_s(|q|^2)(k) = 2\mathcal{P} q_k - |q_k|^2 q_k + N_k(q, q, q)(z),$$  \quad (17)

where \(\mathcal{P} = ||q||^2/\mathbb{T}\). The NLS equation in the discrete frequency domain is

$$\partial_z q_k = j\omega_0^2 q_k - 4j\mathcal{P} q_k + 2j|q_k|^2 q_k - 2j N_k(q, q, q)(z),$$  \quad (18)

where SPM, XPM and FWM denote self-phase modulation, cross-phase modulation and four-wave mixing. Note that the SPM and XPM indices \((l = k)\) or \((m = k)\) have been removed from \(N_k\). Unlike their continuous version (15), these indices form a set with non-zero measure and have no analogue in (14). Note further that the XPM is a constant phase shift, thanks to conservation of energy.

As in the continuous model, the integral form of (18) is

$$q_k(z) = e^{j(\omega_0^2 k^2 - 4\mathcal{P})z} \left\{ q_k(0) - 2j \int_0^z e^{-j(\omega_0^2 k^2 - 4\mathcal{P})z'} \left\{ -|q_k(z')|^2 q_k(z') + \mathcal{N}_k(q, q, q)(z') \right\} \right\} dz'.$$

(19)

Example 1 (Classification of quartets). Consider the sum

$$S = \left| (x_{-2} + x_{-1} + x_0 + x_1 + x_2) \right|^2 \times \left| (x_{-2} + x_{-1} + x_0 + x_1 + x_2) \right|^2.$$

The interference terms at frequency \(k = 0\) are

$$S_0 = |x_0|^2 x_0 + 2|x_0| \left| x_{-2} \right|^2 + \left| x_{-1} \right|^2 + x_1 \left| x_1 \right|^2 + x_2 \left| x_2 \right|^2$$

degenerate quartet \((l = m) \vee (n = k)\)

$$+ \left\{ 2x^*_0 (x_{-2} x_{-1} + x_{-1} x_0) + (x_{-2} x_{-1} + x_{-1} x_0) \right\}$$

degenerate FWM \(n = k\)

$$+ 2 \left\{ x_{-2} x_{-1} x_1 + x_{-1} x_0 x_0 \right\}.$$  \quad (20)

There are several possibilities for a quartet \(lm \to nk\). If all indices are different, we get non-degenerate FWM. If two indices of the same conjugacy type are equal, \(i.e., l = m\) or \(n = k\), we obtain degenerate FWM. The cases that two indices of the opposite conjugacy type are the same, \(i.e., l = k\) or \(n = k\), are also degenerate quartets. These are the terms with the square brackets in (20). The literature refers to these terms as XPM, not degenerate FWM. The degenerate quartet with multiplicity two where \(l = m = n = k\) is known as the SPM in the literature. However, according to our definition, the SPM in (20) is \(-|x_0|^2 x_0\) and the XPM is the term with the square brackets plus \(2|x_0|^2 x_0\). This simplifies XPM to \(2|x_0|^2 x_0\) and negates the sign of the SPM, as in (17). There are one SPM, ten XPM and ten FWM terms in this example. In general if \(-N \leq k \leq N\), a simple counting shows that there are \(3N^2 + 3N + 2\) (XPM and FWM) interference terms at \(k = 0\). This number decreases as \(k\) approaches the boundaries \(\pm N\). The XPM, degenerate and non-degenerate FWM constitute, respectively, the 1-, 2- and 3-wave interference.

IV. GN MODEL

The GN “model” in the literature refers to a PSD. In this section, we re-derive this PSD for the continuous and discrete...
A. Continuous-frequency NLS Equation

We note that (14) is a fixed-point equation, mapping \( q(z) \) to itself. Iterating the fixed-point map \( q_\omega^{(k)}(z) \rightarrow q_\omega^{(k+1)}(z) \) starting from \( q_\omega^{(0)} = 0 \), we obtain
\[
q_\omega^{(0)}(z) = e^{j\omega z^2} q_\omega(0).
\]
This is just the solution of the linear part of the NLS equation. Iterating one more time and using Lemma 1 the signal to the first-order in nonlinearity level is
\[
q_\omega^{(1)}(z) = e^{j\omega z^2} q_\omega(0) + 2 \int H(\Omega_{123}) q_\omega(0) q_\omega(0) q_\omega'(0) \omega_{123} d\omega_{123} \, dz.
\]
(21)

where \( H(\Omega_{123}) \) is defined in (12). Note that \( \Omega_{123} \neq 0 \) in (12), since \( \Omega_{123} = 0 \) combined with \( \delta_{123w} \) implies trivial interactions which do not have any noticeable impact on PSD. Therefore \( \Omega_{123w} \), and as a result the nonlinear term in (21), are bounded.

It follows that the NLS equation has a simple closed-form solution to the first-order in the perturbation expansion. As a consequence, derivative quantities such as PSD can also be calculated. The PSD is
\[
\mu_\omega(z) = \mu_\omega(0) + 4 \int H(\Omega_{123w}) \delta_{123w} d\omega_{123w} \, dz.
\]
(22)

Equation (22) expresses a 2-point moment as a function of the 4- and 6-point moments. We can close the equation for the 2-point moment if we assume that signal statistics are Gaussian. With this assumption, the 4- and 6-point moments \( \mu_{123w} \) and \( \mu_{123w'} \) break down according to (9), with zero cumulants. Since \( \omega \) is fixed
\[
\delta_{1\omega} = \delta_{2\omega} = \delta_{3\omega} = \delta_{4\omega} = 0.
\]
(23)

Combined with \( \delta_{123w} \) and \( \delta_{123w'} \), we also have
\[
\delta_{13} = \delta_{1'3'} = \delta_{23} = \delta_{2'3'} = 0.
\]
(24)

The 4-point moment concentrates on normal manifolds which, using (23), is zero. Decomposing the 6-point moment and using (23) only two matchings
\[
S_1S_2S_3(\delta_{11'22'} + \delta_{12'21'}) \delta_{33'}
\]
are nonzero. Integrating over primed variables, the resulting first-order PSD is
\[
S_\omega^{GN}(z) = S_\omega^0 + 8 \int |H_{123w}|^2 S_1^0 S_2^0 S_3^0 \delta_{123w} d\omega_{123},
\]
(25)

where \( S_\omega^0 := S_\omega(0) \) is the input PSD. This PSD \( S_\omega^{GN}(z) \) is known as the GN model (PSD) in the literature [1, 2].

Remark 1. Alternatively, the GN PSD can be obtained by simply approximating the nonlinear term \( |q|^2 q \) by \( |q|^2 |q|^2 \) in the NLS equation,
\[
j \hat{z} q = q_{tt} + 2 |q|^2 q \approx q_{tt} + 2 |q|^2 |q|^2,
\]
where \( S_\omega^{GN}(z) = \exp(-j\omega z^2)q_\omega^0(0) \) is the solution of the linear part of the NLS equation.

B. Discrete-frequency NLS Equation

As in the continuous-frequency model, we use the solution of the linear equation
\[
q_k^{(0)}(z) = e^{j(\omega_k^2 k^2 - 4P)z} q_k(0),
\]
(26)
in (19) to obtain the first-order signal
\[
q_k^{(1)}(z) = e^{j(\omega_k^2 k^2 - 4P)z} q_k(0) + 2jz |q_k(0)|^2 q_k(0) + 2 \sum_{lmn \neq n} H(\Omega_{lmnk})(z) q_l(0) q_m(0) q_n^*(0),
\]
(27)

where
\[
\Omega_{lmnk} = \omega_k^2 (\ell^2 + m^2 - n^2 - k^2).
\]
(28)

Note that \( \Omega_{lmnk} \neq 0 \), since singularities \( l = k \) and \( m = k \) have been removed from \( \mathcal{N}_k \).

Ignoring the SPM term \( 2jz |q_k(0)|^2 q_k(0) \) in (27), squaring and averaging as before, the GN PSD is
\[
S_k^{GN}(z) = S_k^0 + 8 \sum_n \sum_{l \neq k} \sum_{m \neq k} H(\Omega_{lmnk})(z) q_l(0) q_m(0) q_n^*(0).
\]
(29)

The cross term between linear and nonlinear parts in (27) is non-zero, but it is ignored, since it is supported on trivial interactions (34), which are sparse in the whole space.

C. Secular Behavior in the Signal Perturbation

It can be seen that the second term in (27), corresponding to SPM, grows unbounded with \( z \). Had the XPM not been removed, that too would have produced a similar unbounded term. These degenerate FWM terms that tend to infinity with \( z \) are called secular terms and make the series divergent. As a result, direct perturbation expansion fails in the discrete case.

The secular term of SPM can be removed using a multi-scale analysis. For this purpose, we introduce an additional independent slow variable
\[
\ell = \epsilon z, \quad q(t, z) := q(t, z, l),
\]
where now \( 2\epsilon \ll 1 \) is the nonlinearity coefficient. The NLS equation (18) is transformed to
\[
\epsilon \hat{z} q_k + \epsilon \hat{z} |q_k|^2 q_k = j \omega_k^2 k^2 q_k - 2j \epsilon (2P q_k - |q_k|^2 q_k + \mathcal{N}_k).
\]

We expand \( q_k \) in powers of \( \epsilon \) and equate powers of \( \epsilon \) on both sides. We choose \( \hat{z} |q_k|^2 q_k \) to remove the SPM singularity. Omitting details, the zero- and first-order terms (29) and (27) are, respectively, modified to
\[
q_k^{(0)}(z) = e^{j(\omega_k^2 k^2 - 4P + \epsilon |q_k(0)|^2)z} q_k(0),
\]
(26)

and
\[
q_k^{(1)}(z) = e^{j(\omega_k^2 k^2 - 4P + \epsilon |q_k(0)|^2)z} q_k(0) - 2j \epsilon \sum_{lmn \neq n} H(\Omega_{lmnk})(z) q_l(0) q_m(0) q_n^*(0),
\]
(30)
that the fast variable small distance scales. However, as regardless of its angle, dominates the linear term. As a result, $q$ and $p$ and to use. The PSD is given by (25) with $\bar{\Omega}$ as a function of the nonlinearity parameter $\alpha$. Error $\epsilon = \|g - q^{(1)}\|/\|g\|$ as a function of the nonlinearity parameter $\alpha$. (A) Exact (E) and approximate (A) signals when $\alpha = 0.21$ ($A = 0.5$).

\[ \text{Fig. 2. First-order signal approximation in perturbation expansion method when } q(t,0) = A \exp(-t^2/2) \text{ and } z = 1. \]

(A) Error $\epsilon$; (B) Exact (E) and approximate (A) signals when $\alpha = 0.21$ ($A = 0.5$).

where

\[ \bar{\Omega} = \Omega_{lmnk} + j\omega_0^2\epsilon(|q_l(0)|^2 + |q_m(0)|^2 - |q_n(0)|^2 - |q_k(0)|^2). \]

The PSD is given by (25) with $\Omega \to \bar{\Omega}$. It can be seen that the fast variable $z$ describes the rapid evolution of $q_k$ in small distance scales. However, as $z$ is increased, potentially important dynamics on large scales (where $\epsilon z \approx 1$) can be missed. In our example, the SPM term does indeed grow at scales of order $O(\epsilon^{-1})$. The role of the slow variable $\ell$ is to describe dynamics at this long-haul scale.

The secular terms seem to have been neglected in the literature. This is because missing the sum with minus sign in (16) ignores the SPM term in (18). However, typically energy is distributed over many Fourier modes and $z|q_k(z)|^2q_k(z)$ is quite small. As a result, if $z$ is not too large, the singular perturbation signal (27) is a good approximation and is simpler to use.

Figs. 2(a)–(b) demonstrate the accuracy of the first-order perturbation approximation (27). Here the strength of the nonlinearity is measured as the ratio $\alpha$ of the nonlinear and linear parts of the Hamiltonian (3).

\[ \mathcal{H}(z) = \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial t} q_l(z,t) \right)^2 - |q_l(z,t)|^4 \ dt. \]

It can be seen in Figs. 2(a)–(b) that the perturbation series rapidly diverges as $A$ is increased. Even in the pseudo-linear regime where $\alpha < 0.1$, the error may not be small. Note that in the focusing regime, the linear and nonlinear parts of (21) add up destructively so that $\left\|q_k^{(1)}(z)\right\| < \left\|q_k(z)\right\| = \left\|q_k(0)\right\|$ and $q_k^{(1)}(t,1)$ is below $q(t,1)$ in Fig. 2(b). However, in PSD the sign is lost and the linear and nonlinear PSDs add up constructively, so that $S_k(z)$ stands above $S_k^0$ in Fig. 3. As the amplitude is increased, the nonlinear term grows and, regardless of its angle, dominates the linear term. As a result, $q_k^{(1)}(t,z)$ goes above $q(t,z)$ and $\left\|q^{(1)}\right\|$ rapidly diverges to infinity. However, as we will see, the error in PSD is typically smaller due to the squaring and averaging operations.

\[ \text{Fig. 2. First-order signal approximation in perturbation expansion method when } q(t,0) = A \exp(-t^2/2) \text{ and } z = 1. \]

(a) Error $\epsilon$; (B) Exact (E) and approximate (A) signals when $\alpha = 0.21$ ($A = 0.5$).

V. KZ Model Power Spectral Density

A. Stationary Spectrum

In this section, we motivate the subsequent sections by explaining how energy is transferred among Fourier modes and why one might expect a stationary PSD when the signal propagates according to the NLS equation.

We begin with a two-dimensional Fourier series restricted on the dispersion relation $\zeta = \omega_0^2 k^2$.

\[ q(t,z) = \sum_{k=\infty}^{\infty} a_k(z) e^{i(k\omega_0 t + \omega_0^2 k^2 z)}. \]

(31)

Substituting (31) into the NLS equation, we get

\[ \partial_z a_k(z) = -2j \sum_{l,m,n} e^{i\Omega_{lmnk} z} a_l a_m a_n^* \delta_{lmnk}, \]

(32)

where $\Omega_{lmnk}$ is defined in (28) and the sum is over all possible interactions $lm \to nk$. The integrating factor $\exp(j\omega_0^2 k^2 z)$ removes the additive dispersion term from the NLS equation and reveals it as an operator acting on nonlinearity in (32). If $\Omega_{lmnk} \neq 0$ and $z$ is large, the exponential term oscillates rapidly and the nonlinearity $a_l a_m a_n^*$ is averaged out in integration over $z$, following the Riemann-Lebesgue lemma. Therefore only modes lying on the resonant manifold

\[ \left\{ \ell + m = n + k, \right\} \]

(33a)

\[ \ell^2 + m^2 = n^2 + k^2, \]

(33b)

contribute to the asymptotic changes in the Fourier mode $a_k$. This means that energy is transported in the frequency domain primary via the resonant interactions and the influence of the non-resonant interactions on energy transfer is small. The frequency and phase matching conditions (33a) and (33b) respectively represent conservation of the energy and momentum.

In our example, the resonant manifold (33a)–(33b) permits only trivial interactions

\[ \left\{ l = n, m = k \right\}, \text{ or } \left\{ l = k, m = n \right\}, \]

(34)

describing SPM ($\ell = m$) and XPM ($\ell \neq m$). Separating out the resonant indices from the sum in (32), we get

\[ \partial_z a_k(z) = j(-4P + 2|a_k|^2)a_k - 2jN_k^{ne}(a,a,a), \]

(35)

where

\[ N_k^{ne}(a,a,a) \triangleq \sum_{lmn \in nra} e^{i\Omega_{lmnk} z} a_l a_m a_n^* \delta_{lmnk}. \]

contains only non-resonant quartets, outside of (34). Non-resonant interactions constitute the majority of all interactions, and since $N_k^{ne} \approx 0$, we observe that most of the possible interactions are nearly absent.

Ignoring $N_k^{ne}$ in (35), we obtain

\[ a_k(z) \approx j(-4P + 2|a_k|^2)a_k, \]

(36)

which does not imply any inter-modal interactions. In fact, restoring the dispersion, we have

\[ q_k(z) \approx e^{i(\omega_0^2 k^2 - 4P + 2|q_k(0)|^2)} q_k(0), \]
which means $|q_k(z)| \approx |q_k(0)|$. This is because the resonant quartets for the convex dispersion relation $\zeta = \omega^2$, $\omega = \omega_0^2 k^2$, of the integrable NLS equation consists of only trivial quartets \(^{34}\).

It follows that the signal spectrum is almost stationary. There are small oscillations in the spectrum due to small non-resonant effects, but because most of the possible interactions between Fourier modes, responsible for spectral broadening, do not occur, a localized energy stays localized and does not spread to infinite frequencies. This also intuitively explains the lack of the equi-partition and the periodic exchange of the energy among Fourier modes in the Fermi-Pasta-Ulam (FPU) lattice \(^{8}\), and generally in soliton systems.

Fig. 3 shows the evolution of modes $k = 0$ and $k = N/2$, where $N$ is the integer bandwidth, for input signal $q(t, 0) = 2 \exp(-t^2/2)$ (a(0) = 5.65). Despite local changes in distance, globally the signal spectrum is not broadened monotonically, but rather oscillates. Here evolution is continued for a very long distance $z = 50$ (about $10^5$ km in a standard optical system). In the stochastic case that the input is random, these local oscillations are averaged out so that the PSD is asymptotically stationary.

The steady-state stationary PSD, without much transient spectral broadening, is a consequence of integrability. Consider a non-integrable equation, e.g., by introducing a third-order dispersion to the NLS equation with dispersion relation $\zeta = \omega^3 + 3\omega^2$, $\omega = \omega_0 k$. The resonant manifold is
\[
\begin{aligned}
\ell + m &= n + k, \\
\ell^3 + m^3 + 3(\ell^2 + m^2) &= n^3 + k^3 + 3(n^2 + k^2).
\end{aligned}
\]

Since the dispersion relation $\zeta = \omega^3 + 3\omega^2$ is non-convex, the resonant manifold contains a larger number of quartets in $\mathbb{R}$ than the trivial ones in \(^{34}\). With factor 3 in $\zeta$, there are also non-trivial integer-valued quartets, e.g., $l = 1$, $m = -3$, $n = 0$ and $k = -2$. As before, ignoring $N_k^m$, equation \(^{6}\) now reads
\[
a_k(z) \approx j(-4\mathcal{P} + 2|a_k|^2)a_k - 2j \sum_{n} a_l a_m a_n \zeta,
\]

where the sum is over non-trivial quartets, i.e., resonant quartets in \(^{37a},^{37b}\) excluding the trivial ones \(^{34}\). The coupling introduced by non-trivial interactions creates a strong energy transfer mechanism, causing substantial spectral broadening (or narrowing, depending on the equation) and dispersing a localized energy to higher (lower) frequencies. Unlike the FPU lattice where energy is exchanged periodically among a few Fourier modes, energy partitioning continues until an equilibrium is reached. This can be a flat (equi-partition) or non-flat stationary steady-state PSD, depending on the equation.

Note that if pulses have short duration, then $\omega_0 \gg 1$ and the dispersion operator inside the sum in $\mathcal{N}_k^m$ averages out nonlinearity more effectively. This explains pseudo-linear transmission in the wideband regime.

To summarize, one can divide four-wave interactions into resonant and non-resonant interactions. Transfer of energy takes place primarily among the resonant modes and via the resonance mechanism. The resonant quartets are themselves divided into trivial and non-trivial quartets. Trivial quartets represent SPM and XPM and, in the energy-preserving integrable NLS equation, do not cause interaction. Non-trivial interactions, which are absent in the integrable equation, cause coupling and transfer of energy among all resonant modes. This occurs when higher order dispersion or nonlinear terms are introduced in the integrable NLS equation. The redistribution of energy among Fourier modes continues until an equilibrium is reached after a transient evolution.

B. Kinetic Equation of PSD

We assume that the signal is strongly stationary so that \(^{5}\) holds. In particular
\[
R(t_1, t_2) = R(\tau, z), \quad \tau = t_2 - t_1.
\]

As shown in Appendix \(^{A-A}\) stationarity implies that the signal is uncorrelated in the frequency domain
\[
\mu_{12}(z) = S_k(z)\delta_{12},
\]

where $S_k(z) = \mathcal{F}_s(R(\tau, z))(k)$.

Often the phase of a signal in a nonlinear dispersive equation varies rapidly compared to the slowly-varying amplitude. Furthermore, in some applications such as ocean waves, it is natural to assume that the initial data is random. This suggests a statistical approach, such as that in the turbulence theory. Here the evolution of the $n$-point spectral cumulants is described.

The NLS equation \(^{13}\) consists of a linear term involving $q$ and a nonlinear term $|q|^2q$. As a result, the evolution of 2-point moment is tied to the 4-point moment, the evolution of the 4-point moment is tied to the 6-point moment, and so on. Multivariate moments and cumulants are interchangeable via \(^{66}\) and \(^{67}\) in Appendix \(^{A-B}\). As a result, one obtains recursive differential equations for $2n$-point cumulants, each depending on cumulants up to $(2n + 2)$-point. In strongly nonlinear systems, higher order cumulants are not negligible and the hierarchy of cumulant equations does not truncate. This makes strong turbulence, traditionally encountered in solid-state physics and fluid dynamics, a difficult problem. However, in weakly nonlinear systems, statistics are close
to Gaussian and consequently higher-order cumulants can be neglected. As a result, a closure of the hierarchy of the cumulant equations is reached. This gives rise to a kinetic equation for the PSD. In WWT, kinetic equations can often be solved using, e.g., Zakharov conformal transformations. The resulting solutions are known as Kolmogorov-Zakharov spectra.

For non-integrable equations, kinetic equations indicate a monotonic transfer of energy to higher or lower frequencies (direct and reverse energy cascade) in the first order in nonlinearity. However, for integrable equations, kinetic equations immediately predict a stationary PSD to the first order. Nevertheless, for the NLS equation the kinetic equation can be solved in the second order in the nonlinearity to account for changes in PSD that is observed in numerical and experimental studies of the integral NLS equation.

A differential equation for 2-point moment $\mu_{kk} := S_k$ can be obtained straightforwardly:

$$\frac{dS_k}{dz} = \langle q_k^* \frac{\partial}{\partial z} q_k + \text{c.c.} \rangle = \langle q_k^* \left( j\omega_0 k^2 q_k - 2j \sum q_l q_m^* \delta_{lmn} \delta_{l'm'n'} \right) + \text{c.c.} \rangle = j\omega_0 k^2 S_k - 2j \sum \mu_{lmnk} \delta_{lmnk} + \text{c.c.} = 4 \sum \Im(\mu_{lmnk}) \delta_{lmnk}, \tag{39}$$

where c.c. stands for complex conjugate.

To the zero order in the nonlinearity, the signal distribution is Gaussian and $S_{lmnk} = 0$. As a result, $\Im(\mu_{lmnk}) = 0$ and $dS_k/dz = 0$.

In the first order in nonlinearity, the evolution of the 4-point moment is

$$\frac{d\mu_{lmnk}}{dz} = \langle q_l q_m^* q_n^* q_k^* \rangle + \cdots = \langle j\omega_0^2 k^2 q_l - 2j \sum q_l q_m^* q_n^* \delta_{l'm'n'} \rangle q_m q_n^* q_k + \cdots = j\Omega_{lmnk} \mu_{lmnk} - 2j \sum \left\{ \mu_{l'm'n'm'n}'' \delta_{l'm'n'}'' + \mu_{l'm'n'm'n}'' \delta_{l'm'n'}'' + \mu_{l'm'n'm'n}'' \delta_{l'm'n'}'' - \mu_{l'm'n'm'n}'' \delta_{l'm'n'}'' \right\}. \tag{40}$$

In the discrete model, dispersion is a multiplication by a unitary matrix. The linear and nonlinear parts of the NLS dynamics are mixing processes in time and frequency. When the input signal is quasi-Gaussian and signal phase is uniformly distributed in evolution, these mixings maintain the quasi-Gaussian distribution, in view of the central limit theorem. As long as the signal phase is uniform and nonlinear interactions are weak, this is an excellent approximation. The accuracy of the quasi-Gaussian approximation improves asymptotically as $z \to \infty$, because dispersion has had enough time to take effect. Nevertheless, for very large amplitudes, dispersion can be suppressed by the nonlinearity before it can convert strongly non-Gaussian statistics to Gaussian.

It follows that, under the assumption that there are a large number of Fourier modes in weak interaction, and that the distribution of $q_k(0)$ is quasi-Gaussian, we can assume that the distribution of $q_k(z)$ remains jointly quasi-Gaussian, as defined in (10). Consequently, the four 6-point moments in (40) break down in terms of the 2-point moments

$$\mu_{l'm'n'm'n''k} = S_m S_n S_k \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'm'n''k} + S_m S_n S_k \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'm'n''k} + S_m S_n S_k \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'm'n''k},$$

$$\mu_{l'm'n'l'm'n''k} = S_l S_m S_k \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'l'm'n''k} + S_l S_m S_k \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'l'm'n''k} + S_l S_m S_k \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'l'm'n''k},$$

$$\mu_{l'm'n'm'n'l'm'n''} = S_l S_m S_n \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'm'n'l'm'n''} + S_l S_m S_n \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'm'n'l'm'n''} + S_l S_m S_n \left( \delta_{l'm'n'} \delta_{m'n''} + \delta_{l'm'n'} \delta_{m'n'} \right) \mu_{l'm'n'm'n'l'm'n''}.$$
The integral form of (41) is
\[ S_{lmnk}(z) = e^{i\Omega_{lmnk}z} S_{lmnk}(0) + 4j \int_0^z e^{i\Omega_{lmnk}(z-z')} T_{lmnk}(S, S', S)(z')dz'. \] (42)

Since resonant interactions do not contribute to \( dS_k/dz \), below we include only non-resonant interactions for which \( \Omega_{lmnk} \neq 0 \) and \( S_{lmnk}(0) = \tilde{S}_{lmnk}(0) \). Substituting (42) into (39), we obtain the kinetic equation for \( S_k \)
\[ \frac{dS_k}{dz} = 4 \sum_{l mn \in nr k} \Im \left( e^{i\Omega_{lmnk}z} \tilde{S}_{lmnk}(0) \right) \]
\[ - 16 \epsilon^2 \sum_{l mn \in nr k} \int_0^z \cos(\Omega_{lmnk}(z-z')) T_{lmnk}(z')dz'. \] (43)

The kinetic equation (43) is a nonlinear cubic equation similar to the NLS equations. However, now the rapidly-varying variables are averaged out and the PSD evolves very slowly so that the perturbation theory is better applicable. We thus solve (43) perturbatively, writing
\[ S_k(z) = S_k^{(0)}(z) + \epsilon S_k^{(1)}(z) + \cdots, \]
\[ S_{lmnk}(z) = S_{lmnk}^{(0)}(z) + \epsilon S_{lmnk}^{(1)}(z) + \cdots. \]

For the zero-order term we obtain
\[ S_k^{(0)}(z) = S_k^{(0)} + 4\Re \left( \sum_{l mn \in nr k} H_{lmnk} \tilde{S}_{lmnk}(0) \delta_{lmnk} \right). \]

If the input signal is quasi-Gaussian, \( \tilde{S}_{lmnk}(0) \approx 0 \) and the contribution of the second term to the PSD can be typically ignored. Consequently, we obtain
\[ S_k^{\text{KZ}}(z) = S_k^{(0)} + 8\epsilon^2 \sum_{l \neq k \atop m \neq k} |H_{lmnk}(z)|^2 T_{lmnk}^{(0)} \delta_{lmnk}. \] (44)

Note that if \( z \to \infty \), \( H_{lmnk} \approx -j\pi \delta(\Omega_{lmnk}) \) and \( S_k^{\text{KZ}} \) is stationary.

VI. KZ MODEL ASSUMPTIONS

In this section, we summarize the assumptions of the KZ model and comment on their validity in the context of fiber-optic data communications.

a) Fourier transforms \( q_k \) and \( S_k \) exist: Particularly, \( R(\tau, 0) \) should vanish as \( |\tau| \to \infty \).

This assumption is valid in data communications because signals have finite energy and time duration.

b) The input signal is strongly stationary: This ensures that the 2n-point moments are concentrated on stationary manifolds. In particular, \( q_k \) are uncorrelated, as stated in (38). The delta functions that follow from this assumption simplify the collision term in (40).

This assumption is valid in uncoded OFDM systems, where sub-carrier symbols are independent and the transmitted signal is cyclo-stationary. Input signals with uniform random phase are also stationary, regardless of the amplitude correlations. However, in coarse WDM systems the time-domain pulse shape can make the transmitted signal non-stationary and cause correlations in the frequency domain.

c) Signal has quasi-Gaussian distribution in the sense of (10): In particular the input signal must be quasi-Gaussian. Under random phase approximation [3], the flow of the NLS equation would then ensure that the signal remains quasi-Gaussian in the weak nonlinearity framework. The breakdown of the 6-point moments is also required in the GN model, in closing (22) for the 2-point moment.

The integrable NLS equation in the focusing regime has stable soliton solutions. As pointed out in [5], the solitonic regime, in which the nonlinearity is strong, can act against the dispersive mixing of the weak nonlinearity regime. We assume that for random input the coherence is not developed. This means that the interference spectrum in the focusing and defocusing regimes are the same.

To summarize, Assumptions b) and c) may fail in data communications. However, the WWT approach can be re-worked out without using these assumptions. The price to pay is that the closure is no longer achieved and the expressions are not as simple. In Section VIII, we obtain the KZ spectrum for a WDM input signal with and without Assumptions b) and c).

VII. COMPARING THE KZ AND GN MODELS

In this section we explain how the KZ model differs from the GN model.

A. Differences in Assumptions

The GN model assumes a perfectly Gaussian distribution compared with the less stringent quasi-Gaussian assumption of the KZ model. Note that in the presence of the four-wave interactions \( lm \to nk \), higher order moments are encountered. If a closure is to be reached, any perturbative method requires a reduction of the higher-order moments to lower order ones, i.e., the quasi-Gaussian assumption at some order.

The GN PSD in some scenarios has been modified to account for a fourth-order non-Gaussian noise [9]. The perturbation expansion can also be carried out to higher orders to improve accuracy and account for deviations from the Gaussian distribution. However, given the same assumptions, the GN and KZ PSDs are still different. Furthermore, to work out these methodically, one ends up using WWT framework.

B. Differences in PSD

To connect the KZ and GN models, we wrote the modified kinetic equation and the KZ spectrum (44) in terms of the same kernel \( H_{lmnk} \) that appears in the GN model. As a result, from (44) it can be readily seen that
\[ S_k^{\text{KZ}}(z) = S_k^{\text{GN}}(z) - \Delta S_k(z), \]
where
\[ \Delta S_k = 8s_k^{(0)} \sum_{l mn \in nr k} |H_{lmnk}(z)|^2 (S_{lmnk}^{(0)} S_{lnnk}^{(0)} + S_{lnk}^{(0)} S_{lmnk}^{(0)} - S_{lmnk}^{(0)} S_{lnnk}^{(0)}) + 8s_k^{(0)} \sum_{l mn \in nr k} |H_{lmnk}(z)|^2 S_{lnk}^{(0)} S_{lmnk}^{(0)}, \]
in which we made use of the symmetry \( lmnk \leftrightarrow klnm \) in (45). That is to say, the KZ PSD modifies the GN PSD.
by subtracting a term from it. That makes KZ PSD at any order \( n \) in perturbation expansion as accurate as GN PSD at order \( n + 1 \). The improvement might be small in current systems operating in pseudo-linear regime, however, as the signal amplitude is increased the GN PSD rapidly diverges from the true PSD.

The KZ model is energy-preserving unlike the GN model. Perturbation expansion in signal breaks down the structure of the NLS equation, so that some important features of the exact equation can be lost. For example, the average signal power according to the GN PSD is

\[
P(z) = P(0) + \int_{-\infty}^{\infty} |H_{123\omega}|^2 S^0_{1} S^0_{2} S^0_{3} \delta_{123\omega} d\omega_{123\omega}.
\]

It is seen that the signal power is not preserved (see also Fig. 2(b)). At any order in perturbation, ignoring the energy of the higher-order terms violates energy conservation.

In contrast, in the KZ model, noting the symmetries

\[
H_{lmnk} = H_{mlnk} = H_{lknm}, \quad |H_{lmnk}| = |H_{nklm}|,
\]

(45) and the similar ones for \( \delta_{lmnk} \), we have

\[
\sum_{lmnk} |H_{lmnk}|^2 S^0_{l} S^0_{m} S^0_{n} \delta_{lmnk} = \sum_{lmnk} |H_{lmnk}|^2 S^0_{l} S^0_{m} S^0_{n} \delta_{lmnk},
\]

where we substituted \( lmnk \leftrightarrow nklm \). In fact, using symmetries (45), it can be seen that the sum over \( lmnk \) of all four terms in \( T_{lmnk} \) are the same. It follows that \( \sum_k S^k_{lmnk} = \sum_k S^0_{kmn} \), i.e., the KZ model is energy-preserving. Other conservation laws exist for kinetic equations [3].

Fig. 4 compares the power spectral densities of the GN and KZ models with the simulated (sim.) PSD.

\[
\frac{1}{2} \int_{-\infty}^{\infty} \left| S_{\omega} \right|^2 d\omega \leq \frac{1}{2} \int_{-\infty}^{\infty} \left| S_{\omega} \right|^2 d\omega,
\]

deviations from Gaussianity are necessary for any spectral change.

In the KZ model \( \Re(\mu_{lmnk}) \) is found via its differential equation and is used to obtain \( S^k_{lmnk} \). Examining \( \Im(\mu_{lmnk}) \), one finds it cubic in \( S_k \) and comparable to \( \Re(\mu_{lmnk}') \). As a result, signal distribution should be non-Gaussian and \( \mu_{lmnk} \) cannot be ignored in the GN model. This interference adds destructively to the last term in \( S^{GN} \) and reduces the GN PSD.

Another problem with the signal perturbation, and consequently with the GN model, is that here one works with moments, not the cumulants as in the KZ model. Note that even for a perfectly Gaussian input, where \( \mu_{123\omega} = 0 \) in (22) and its omission is no longer responsible for the errors in the GN PSD, \( S^{GN} \) is still nonstationary. This is because, unlike cumulants, one cannot ignore higher order moments. For instance, for a scalar Gaussian random variable \( X \),

\[
E[X^{2n}] = (E[X^2])^n (2n-1)!! \quad \text{where} \quad n!! = n(n-2)(n-4) \cdots
\]

The even moments are non-zero, and indeed grow with \( n \). For the Gaussian input, if one includes all higher order moments in the GN model, the first-order PSD \( S^{GN} \) is canceled out and a stationary energy-preserving spectrum is obtained. In contrast, higher-order cumulants are zero for Gaussian distribution and as the amplitude is increased, they are gradually generated sequentially in increasing order. This makes them suitable for a perturbation theory approach.

D. Differences in Probability Distributions

From the previous discussion, it follows that signal distribution in the KZ model is a zero-mean non-Gaussian distribution with the following moments: Asymmetric moments are zero; 2-point moment is given by the KZ PSD (44); 4-point moments are given by (32). Other higher order moments are given in terms of the second moments according to a Gaussian distribution, i.e., generalizations of (39)–(42) with zero cumulants.

VIII. APPLICATION TO WDM

One application of the PSD is to estimate the interference power in WDM systems. In models where the XPM amounts
to a constant phase shift, the interference at frequency $k$ is non-degenerate FWM, as well as part of the degenerate FWM; see Fig.1 and Example 1. However, in the WDM literature often the WDM is treated as interference. That is to say, all of the nonlinearity $N_k$ in the NLS equation \[18\] is treated as noise. The corresponding spectra $S_k^{KZ}$ and $S_k^{SN}$ include self- and cross-channel interference.

Consider a WDM system with $2N + 1$ users, each having bandwidth $\Omega_s$. In WDM the following (baseband) signal is sent over the channel

$$q(t, 0) = \sum_{m=-N}^{N} \left( \sum_{l=1}^{M} a_m^l \phi^l(t) \right) e^{j m \Omega_s t}, \quad \text{(46)}$$

where $l$ and $m$ are time and user indices, $\Omega_s$ is the user bandwidth, and $\phi^l(t)$ is an orthonormal basis for the space of finite-energy signals with Fourier transform in $[-\Omega_s/2, \Omega_s/2]$ and almost time-limited to $[0, T]$, $T \to \infty$. Finally, $a_m^l$ is a sequence of complex-valued random variables, independent between users, but potentially correlated within each user, i.e.,

$$E \rho_m^l \rho_m'^* = \mu_{mm}^l(a) \delta_{m'm'},$$

where $\mu_{mm}^l(a) = E \rho_m^l \rho_m'^*$ is the symbols correlation function of the user $m$, due to, e.g., channel coding. The set of frequencies of the user $m$, $-N \leq m \leq N$, is:

$$A_m = \left\{ m \Omega_s + k \Omega_0 \mid -N_0/2 \leq k < N_0/2 \right\},$$

where $N_0 = |\Omega_s / \Omega_0|$ and $\Omega_0 = 2\pi/T$.

A. Stationary Gaussian WDM Signals

In this case, the assumptions of the GN and KZ models are satisfied. The interference “spectrum” is

$$S^{NL} = 8 \sum_{l,m,n \in \mathbb{N}} |H_{lmnk}|^2 T_{lmnk},$$

where for the KZ model $T_{lmnk}$ is the collision term, and for the GN model $T_{lmnk} = S_m S_n$. The intra-channel interference for the central user $m = 0$ is a part of the sum in $S^{NL}$, where $l,m,n \in A_0$. This is somewhat similar to SPM. The rest of terms, where at least one index is in the complement set $A_0^c$, is the inter-channel interference. This is divided into three parts: 1) exactly two indices are in $A_0$ (1-wave interference) 2) exactly one index is in $A_0$ (2-wave interference) 3) no index is in $A_0$ (3-wave interference). The 1-wave interference has fewer terms than the others and can be ignored. The 2-wave interference is akin to XPM but is not similarly averaged out and should be accounted for.

Note that the net interference is zero in the KZ model, i.e., $S^{NL}$ is negative for some $k$. Furthermore, the interference is non-local and can be picked up from users far apart.

B. Non-stationary non-Gaussian WDM Signals

The correlation function of the WDM signal \[46\] is

$$R(t_1, t_2) = \sum_{m} \mu_{mm}^l(a) \phi^l(t_1) \phi^l(t_2) \exp(jm\Omega_s(t_2 - t_1))$$

$$= \mathcal{PC}_0 \sum_{l=1}^{M} \phi^l(t_1) \phi^l(t_2) E(t_2 - t_1),$$

where $E(x) = \sum_{m} \exp(jm\Omega_s x)$ and step (a) follows under the additional assumption that $a_m^l$ is i.i.d., so that $\mu_{mm}^l = \mathcal{P}_0 \delta_{mm}^l$. $\mathcal{P}_0 = \mathbb{E}[\rho_m^l \rho_m'^*]$. Unless in special cases, e.g., $\phi^l(t) = \exp(jm\Omega_0 t)$, the input signal is not a stationary process. This can be seen in the frequency domain too. The Fourier series coefficients are

$$q_k = \sum_{l,m} a_m^l \phi_m^{l,k},$$

\[47\]

where

$$\phi_k^l = \begin{cases} F_s(\phi^l(t))(k), & -N_0/2 \leq k < N_0/2, \\ 0, & \text{otherwise.} \end{cases}$$

The orthonormality condition reads

$$\sum_{k=-N_0}^{N_0} \phi_k^{n,k} \phi_k^{n',k} = \delta_{nn'},$$

The 2-point spectral moment at $z = 0$ is

$$\mu_{12} = \mathcal{P}_0 \sum_{l,m} \phi_m^{l,k} \phi_m^{l,k} \phi_k^{n,k} \phi_k^{n,k},$$

\[48\]

where $\delta_{12}^{(N)} = 0$ if $|k_1/(N_0/2)| = |k_2/(N_0/2)|$. As before, unless in special cases, e.g., if $k_1$ and $k_2$ belong to two different users, or $\phi_k^l = \phi_0 \delta_k$, or $\phi_k^l = \phi_0 \exp(j \pi/2 k)$, in general $\mu_{12} \neq \mu_{11} \delta_2$.

In addition to the stationarity Assumption b), Gaussianity Assumption c) may also not hold in WDM. Both the GN and KZ models express $S_k(z)$ in terms of input statistics $\mu_{123456}(0)$. Although the signal distribution may converge to a joint Gaussian distribution during the evolution, the input distribution at $z = 0$ (in the frequency domain) is arbitrary. For non-Gaussian inputs, the cumulant $\kappa_{123456}(0)$ should be included.

The correlations and non-Gaussian input statistics can be introduced into the GN and KZ models using $\mu_{ij}$ and $\kappa_{123456}$. We have

$$\mu_{123456} = \sum_{l_1, -m_1} \mu_{123456}^{l_1}(a) \phi_{m_1}^{l_1} \phi_{m_2}^{l_2} \phi_{m_3}^{l_3} \phi_{m_4}^{l_4} \phi_{m_5}^{l_5} \phi_{m_6}^{l_6},$$

\[49\]

where

$$\mu_{123456}^{l_1}(a) = E \rho_{m_1}^{l_1} \rho_{m_2}^{l_2} \rho_{m_3}^{l_3} \rho_{m_4}^{l_4} \rho_{m_5}^{l_5} \rho_{m_6}^{l_6},$$

\[50\]
The resulting spectral cumulants are
\[ S_{12} = E|a|^2, \]
\[ S_{124} = E|a|^4 - 2E|a|^2, \]
\[ S_{123456} = E|a|^6 + 9E|a|^2E|a|^4 - 12E^3|a|^2. \]

The resulting spectral cumulants are
\[ \kappa_{12} = \hat{S}_{12}(a)^2, \]
\[ \kappa_{1234} = \hat{S}_{124}(a)^2, \]
\[ \kappa_{123456} = \hat{S}_{123456}(a)^2. \]

The resulting PSDs are
\[ S_{k}^{GN} = S_0 + 8 \sum_{\eta} |H_{1m nk}|^2 T_{1m nk}, \]
\[ S_{k}^{KZ} = S_0 + 8 \sum_{\eta} |H_{1m nk}|^2 \tilde{T}_{1m nk}, \]

where
\[ T_{1m nk} = \frac{1}{2} \sum_{\eta} \left( \mu_{n k m' n' k} + \mu_{\eta k m' n' k} \right) \]
\[ - \mu_{n k m' n' \eta} - \mu_{\eta k m' n' \eta} \right). \]

The simplifications of Section V-B in the case of uncorrelated and Gaussian signals, due to integration over delta functions, do not occur anymore. If \( \phi(t) = p(t - lT/M) \), where \( p(t) \) is a pulse shape in time interval \( [0, T/M] \), then \( \phi_k = p_k \exp(j2\pi kl/M) \) and
\[ \mu_{12} = \mathbb{P} \sum_{l=1}^{M} \exp(j2\pi l(k_1 - k_2)/M) \]
\[ = \mathbb{P} \delta_{12}. \]

In this case there is no correlation and PSDs are modified only via \( \kappa_{123456} \):
\[ \hat{S}_{k}^{GN} = S_0 + 4 \sum_{\eta} H_{1m nk} H_{1l n' m' k} \hat{T}_{1m nk}, \]
\[ \hat{S}_{k}^{KZ} = S_0 + 8 \sum_{\eta} |H_{1m nk}|^2 \tilde{T}_{1m nk}, \]

where \( \hat{T}_{1m nk} \) is given by (50) with \( \mu_{123456} \) replaced with \( \kappa_{123456} \).

C. Phase Interference

The power spectral density of the nonlinear term in the NLS equation does not suggest that one should consider nonlinearity as additive noise. In fact, the PSD obviously does not capture cross-phase interference. In the energy-preserving NLS equation, XPM is a constant phase shift, as shown, e.g., in [18]. From (47), the signal energy is
\[ \sum_{k=1}^{N} |q_k|^2 = \sum_{l m} |a_l m|^2 \phi_{m N + k} \phi_{m N + k}. \]

Typically, the per-user power is a known constant, however, in an optical mesh network, the power of interfering users may not be known. Furthermore, often back-propagation is performed at the receiver, which removes part of the sum associated with the user of interest. Energy is also not preserved in the presence of loss. In such cases where XPM is no longer a constant phase shift, part of the sum (51) where \( m \neq 0 \) acts as cross-phase interference for the center user. This signal-dependent phase distortion is not reflected in PSD. Such nonlinear phase noise is indeed the hallmark of the nonlinearity: the solution of the zero-dispersion NLS equation (13) is \( q(t, z) = \exp(2jz|q(t, 0)|^2)q(t, 0). \) Plugging in the WDM signal (46), cross-phase interference appears as a significant distortion.

The interference resulting from XPM is discussed in [10]. This is done by substituting the WDM input signal (46) into the approximate solution (21), sorting out interference terms, and naming XPM and FWM. Note that the perturbative signal (21) is not energy-preserving. As a result, it predicts XPM even for the ideal single-user channel. This is an artifact of the method and one has to be careful not to include it in the actual XPM.

D. PSD in Multi-Span Systems

The PSDs (25) and (44) hold for one span of lossless fiber with second order dispersion. In this section, we include loss and higher order dispersion, and generalize (25) and (44) to multi-span links with amplification.

We consider a multi-span optical system with \( N \) spans, each of length \( \epsilon \), in a fiber of total length \( z = N\epsilon \). Pulse propagation in the overall link is governed by
\[ \partial_z q_\omega(z) = j \left( \frac{j\alpha(z)}{2} - \beta(\omega) \right) q_\omega(z) - j\gamma N_\omega q_\omega(q, q, q)(z) \]
\[ + \left( \sum_{n=1}^{N} G_n(z) \delta(z - n\epsilon) \right) q_\omega(z), \]

where \( \alpha(z) \) is (power) loss exponent, \( G_n(z) = \int_{(n-1)\epsilon}^{n\epsilon} \alpha(l) dl \) is the lumped gain exponent at the end of span \( n \), \( \gamma \) is the nonlinearity coefficient and
\[ \beta(\omega) = \beta_0 + \beta_1 (\omega - \omega_0) + \frac{\beta_2}{2} (\omega - \omega_0)^2 + \cdots, \]
is the dispersion function (also known as the wavenumber or propagation constant).

Lumped power amplification at the end of each span restores the linear part of PSD, however, since loss is distributed, it does not normalize the nonlinear part. As a result, signal amplification leads to a growth of FWM interference, which we calculate in this section.

1) GN Model: Consider the NLS equation (52) with loss, dispersion \( \beta(\omega) \), and amplification. Let
\[ F(z) = \mathbb{P} \int_{0}^{z} \left( \alpha(l) - \sum_{n=1}^{N} G_n(l) \delta(l - n\epsilon) \right) dl. \]
Comparing (59) with the dimensionless NLS equation, we identify $\omega^2 z \rightarrow jF(z)/2 - \beta(\omega) z$. Therefore

$$\Omega_{123}\omega z = (\omega_1^2 + \omega_2^2 - \omega_3^2 - \omega^2) z$$

in signal (21) and PSD (44) is replaced with

$$\Omega_{123}\omega z = jF(z) - \Omega_{123}\omega z,$$

where

$$\Omega_{123}\omega = \beta(\omega_1) + \beta(\omega_2) - \beta(\omega_3) - \beta(\omega).$$

The GN PSD (25) is modified to

$$S_\omega(z) = e^{-F(z)} \left( S_\omega(0) + 2\gamma^2 \int |\tilde{H}_{123}\omega|^2 S_1 S_2 S_3 \delta(z - \omega_{123}) d\omega_{123} \right)$$

where

$$\tilde{H}_{123}\omega = -j \int \frac{z}{0} e^{j\tilde{H}_{123}\omega} d\ell = -j \int \frac{z}{0} e^{-(F(l) + j\Omega_{123}\omega) l} d\ell. \quad (53)$$

Several cases can be derived from (53).

a) Single-span lossy fiber: In a single-span fiber with constant loss $\alpha$ and no amplification, $G = 0$ and $F(z) = \alpha z$. Thus $H_{123}\omega = H_{123}(\alpha - \Omega_{123}\omega)(z)$. This shows the effect of loss and higher order dispersion.

b) Multi-span links: In a multi-span link with constant loss $\alpha$,

$$F(l) = \alpha l - \alpha N \sum_{i=1}^{N} U(l - ie) = \alpha(l - ne), \quad n = \lfloor l/e \rfloor, \quad (54)$$

where $U(x)$ is the Heaviside step function. Thus

$$\tilde{H}_{123}\omega = -j \int \frac{z}{0} e^{-(\alpha l + \alpha N \sum_{i=1}^{N} U(l - ie) - j\Omega_{123}\omega) l} d\ell$$

$$= -j \sum_{n=0}^{N-1} e^{-(\alpha(l - ne) - j\Omega_{123}\omega) l} \int \frac{z}{0} e^{-j\Omega_{123}\omega l} d\ell$$

$$= -j \sum_{n=0}^{N-1} e^{j\alpha(l - ne) - j\Omega_{123}\omega l} \int \frac{z}{0} e^{j\alpha - j\Omega_{123}\omega} l^2 d\ell$$

$$= H(j\alpha - \Omega_{123}\omega)(e) G_{123}\omega \quad (55)$$

where

$$G_{123}\omega = \sum_{n=0}^{N-1} e^{-jn\Omega_{123}\omega} = \frac{1 - e^{-j\Omega_{123}\omega}}{1 - e^{-j\Omega_{123}\omega}}$$

$$= e^{-j\Omega_{123}\omega/2} \sin(\Omega_{123}\omega/2) / \sin(\Omega_{123}\omega/2). \quad (56)$$

Therefore, the GN PSD of multi-span link is given by the same equation (25), with $H(\Omega_{123}\omega)(z)$ replaced with $(\gamma/2)H(j\alpha - \Omega_{123}\omega)(e) G_{123}\omega$. Note that $\tilde{H}$ describes the FWM growth in both the signal and PSD. For further clarification, see Appendix B.

2) KZ Model: Considering the analysis of Section 4-B factors $-(dF(z)/dz)S_k$ and $-2(dF(z)/dz)S_{lmnk}$ appear, respectively, in the right hand sides of moment equations (39) and (41). The KZ PSD is

$$S_k(z) = e^{-F(z)} \left( S_k^0 + 2\gamma \int \frac{z}{0} e^{F(z')} \Im(S_{lmnk}(z')) d\ell \right)^2, \quad (57)$$

where

$$\Im(S_{lmnk}(z)) = 2\gamma \int \frac{z}{0} e^{-2(F(z') - F(z))} \cos \left( \Omega_{lmnk}(z - z') \right) \times T_{lmnk}(z' \ell) d\ell'. \quad (58)$$

Here, as in (42) and (43), we assumed that $S_{lmnk}(0)$ is real-valued for quasi-Gaussian input.

We substitute (58) into (57) and solve the resulting fixed-point equation iteratively starting from $S_k(z) = 0$. The first iterate gives $S_k(0)(z) = \exp(-F(z))S_k^0$. In the next iterate, the collision term is found to be $T_{lmnk}(z) = \exp(-3F(z))T_{lmnk}(0)$, which is no longer constant. Using this collision term in (58), and subsequently in (57), we obtain

$$S_k^{KZ} = e^{-F(z)} \left( S_k^0 + 2\gamma \sum_{lmnk \in \Omega_{lmnk}} |T_{lmnk}|^2 T_{lmnk} \right), \quad (59)$$

where

$$|\tilde{H}_{lmnk}|^2 = 2 \int \frac{z}{0} e^{-(F(z') + F(l))} \cos \left( \Omega(z' - l) \right) d\ell d\ell'. \quad (60)$$

The integration in (60) is over a triangle. However the function under integration is symmetric in $l$ and $z'$, i.e., around the line $l = z'$. Thus integration can be extended to the rectangle:

$$|\tilde{H}_{lmnk}|^2 = \int \frac{z}{0} e^{-(F(z') + F(l))} \cos \left( \Omega(z' - l) \right) d\ell d\ell'$$

$$= \int \frac{z}{0} e^{-(F(z') + F(l))} \cos \left( \Omega(z' - l) \right) d\ell d\ell'. \quad (61)$$

This is the same as $|\tilde{H}_{123}\omega|^2$ in (53) for the GN model, with $123\omega \rightarrow \omega_{lmnk}$. It follows that in all cases the PSD kernels $|H_{123}\omega|^2$ and $|\tilde{H}_{lmnk}|^2$ in GN and KZ models are the same.

a) Single-span lossy fiber: For constant $\alpha$, the PSD is given by (59) with $F(z) = \alpha z$ and

$$|\tilde{H}_{lmnk}|^2 = |H(j\alpha - \Omega_{lmnk})(z)|^2 = 2e^{-\alpha z} \left( \cos\left( \frac{\alpha z}{2} \right) - \cos \left( \frac{\Omega}{2} \right) \right).$$

This shows that in the presence of loss and physical parameters, just as in the GN model, the KZ PSD, after amplification $\exp(\alpha z)$ at the end of the link, is the same as (44), with $H(\Omega_{lmnk})(z)$ replaced with $(\gamma/2)H(j\alpha - \Omega_{lmnk})$. z).

b) Multi-span links: In the multi-span link, at the end of the link $F(z) = 0$. The PSD is given by (59) with $F(z) = 0$ and

$$\tilde{H}_{123}\omega = H(j\alpha - \Omega_{123}\omega)(e) G_{123}\omega.$$
IX. Conclusions

The GN model suggests a spectrum evolution, in agreement with numerical and experimental fiber-optic transmissions. That seemingly conflicts with the WWT which predicts a stationary spectrum for integrable models. It is shown that if the kinetic equation of WWT is solved to the next order in nonlinearity, a PSD is obtained which is similar to \( S_k^\text{GN} \). The two models are explained mathematically and connected with each other. The analysis shows that the kinetic equation of the NLS equation better describes the PSD. This is not surprising, because the GN model applies perturbation theory to the signal equation, while the WWT applies it directly to the PSD differential equation. The assumptions of the WWT are verified in data communications and the KZ spectra are extended to various cases encountered in communications. The GN model is also simplified for clarity and comparison.

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APPENDIX A

Moments and Cumulants

A. Property of the Stationary Processes

We make use of the following simple lemma throughout the paper, which says that the spectral moments of a stationary process are supported on the stationary manifold \( \mathbb{M} \).

Lemma 2. If \( q(t) \) is a strongly stationary stochastic process with finite power and an existing Fourier transform, then \( \mu_{1...2n} = S_{1...2n} \delta_{1...2n} \).

Proof: Define

\[
E(t_1, \ldots, t_{2n}) = \exp \left( j (s_1 \omega_1 t_1 + \cdots + s_{2n} \omega_{2n} t_{2n}) \right).
\]

Shifting the \( t_i \) variables by \( t_1 \), it can be verified that

\[
E(t_1, \ldots, t_{2n}) = \exp(j \Delta \omega t_1) E(0, t_2 - t_1, \ldots, t_{2n} - t_1),
\]

where \( \Delta \omega = \sum_{i=1}^{2n} s_i \omega_i \). Expressing \( q(t) \) in the inverse Fourier transform, the (strong) stationarity property implies that

\[
\mu_{1...2n} = \mathbb{E} \left( q_1 \cdots q_{2n}^* \right) = \int R(t_1, \ldots, t_{2n}) E(t_1, \ldots, t_{2n}) dt_{1...2n}
\]

\[
= \int R(0, t_2 - t_1, \ldots, t_{2n} - t_1) \times E(0, t_2 - t_1, \cdots, t_{2n} - t_1) \exp(j \Delta \omega t_1) dt_{1...2n}
\]

\[
= \delta(\Delta \omega) \int R(0, \tau_2, \cdots, \tau_{2n}) E(0, \tau_2, \cdots, \tau_{2n}) d\tau_{2...2n}
\]

\[
= S_{1...2n} \delta(\Delta \omega),
\]

where

\[
S_{1...2n} = \mathcal{F}(R(0, \tau_2, \cdots, \tau_{2n}))(0, s_2 \omega_2, \cdots, s_{2n} \omega_{2n}).
\]

The lemma essentially follows from the shift property of the exponential function (62). A more general statement is the Wiener-Khinchin theorem.

B. Cumulants

Let \( q = (q_1, \cdots, q_n) \) be a complex-valued random vector. We define the joint moment generating function of \( q, \Phi : \mathbb{C}^n \to \mathbb{R}, \)

\[
\Phi(\zeta) \triangleq \mathbb{E} \exp \left( \mathbb{R} (\zeta^H q) \right)
\]

\[
= \sum_{r,s=0}^{\infty} \frac{1}{r!s!} \mu_r^* \zeta^r \zeta^s^*,
\]

where \( r, s \) are \( n \)-dimensional multi-indices, \( r! = \prod_{k=1}^{n} r_k! \), \( \zeta^r = \prod_{k=1}^{n} \zeta_k^{r_k} \), and

\[
\mu_r = \mathbb{E} q_r^*,
\]

is the \( |r|+|s| \)-point moment. It can be verified that

\[
\mu_s^r = \frac{\partial^{r+s} \Phi(\zeta)}{\partial \zeta^r \partial \zeta^s} \bigg|_{\zeta=0}.
\]

The joint cumulant generating function is \( \Psi(\tau) = \log \Phi(\tau) \). The cumulants \( \kappa_r \) are defined from multi-variate Taylor expansion similar to (63) and (65).

Note that with the notation of Section II, \( \mu_{1...2n} := \mu_r^s \) with \( |r| = |s| = n \). Cumulants \( \kappa_r \) relate to cumulant densities \( \tilde{S} \) via

\[
\kappa_{1...2n} = \tilde{S}_{1...2n} \delta_{1...2n}.
\]

Joint moments can be obtained from joint cumulants by applying chain rule of differentiation to \( \Phi = \exp \Psi \), obtaining

\[
\mu_{1...2n} = \sum_{p \in P} \prod_{a \in p} \kappa_a,
\]

where \( P \) is the set of all partitions of \( (1, 2, \cdots, 2n) \). The expression is considerably simplified for a zero-mean process, which is assumed throughout this paper. Further simplifications occur by noting that the asymmetric moments and cumulants are zero. With these assumption, the first four relations are:

\[
\begin{align*}
\mu_1 &= \kappa_1 = 0, \\
\mu_{12} &= \kappa_{12}, \\
\mu_{1234} &= \kappa_{11} \kappa_{24} + \kappa_{14} \kappa_{23} + \kappa_{1234}, \\
\mu_{123456} &= \left( \kappa_{14} \kappa_{25} \kappa_{36} + \kappa_{14} \kappa_{26} \kappa_{35} + \kappa_{15} \kappa_{24} \kappa_{36} \\
&\quad + \kappa_{15} \kappa_{26} \kappa_{34} + \kappa_{16} \kappa_{24} \kappa_{35} + \kappa_{16} \kappa_{25} \kappa_{34} \right) \\
&\quad + \left( \kappa_{14} \kappa_{23} \kappa_{56} + \kappa_{15} \kappa_{24} \kappa_{36} + \kappa_{16} \kappa_{25} \kappa_{34} + \kappa_{24} \kappa_{12} \kappa_{35} + \kappa_{25} \kappa_{14} \kappa_{36} + \kappa_{26} \kappa_{15} \kappa_{34} \right) \\
&\quad + \kappa_{34} \kappa_{12} \kappa_{56} + \kappa_{35} \kappa_{14} \kappa_{26} + \kappa_{36} \kappa_{15} \kappa_{24} \\
&\quad + \kappa_{123456}.
\end{align*}
\]
For a stationary process $\kappa_{ij} = \mu_{ij} = S_{ii} \delta_{ij}$ and we obtain (8) and (9). Since higher-order cumulants are smaller than the second-order cumulant for quasi-Gaussian distributions, we can assume $\kappa_{lmnk} = \kappa_{lmn'}\nu_{n'} = 0$, thereby obtaining (9).

Cumulants can be obtained from moments by applying the chain rule of differentiation to $\Psi = \log \Phi$, obtaining

$$\kappa_{1\ldots 2n} = \sum_{p \in P} \prod_{a \in p} (-1)^{|p|} |p| - 1 |\mu_a|, \quad (67)$$

where $|p|$ is the number of the sets in the partition $p$, i.e., the number of products in $\mu_a$. Setting asymmetric moments to zero, we have

$$\kappa_{12} = \mu_{12}$$

$$\kappa_{1234} = \mu_{1342} + \mu_{1423} - \mu_{1234} \quad (68)$$

$$\kappa_{123456} = -2 \left( \mu_{142536} + \mu_{145263} + \mu_{154263} + \mu_{154263} + \mu_{156243} + \mu_{156243} + \mu_{156243} + \mu_{156243} \right)$$

$$+ \mu_{142536} + \mu_{145263} + \mu_{154263} + \mu_{154263} + \mu_{156243} + \mu_{156243} + \mu_{156243} + \mu_{156243}$$

$$+ \mu_{134256} + \mu_{135426} + \mu_{136426} + \mu_{136426} \quad (69)$$

For i.i.d. zero-mean random variables, any variables matching in $\mu_{1\ldots 2n}$ is canceled by terms prior to $\mu_{1\ldots 2n}$ in (68), except when all variables are equal. Thus $\kappa_{1234} = \left( \sum_{q} q \kappa_q^2 \right)$ and so on.

APPENDIX B

GN PSD IN MULTI-SPAN LINKS

In Section VIII-D the multi-span PSDs were obtained in a unified manner by introducing function $F(\omega)$ and modifying kernels $H_{123\omega}$. One consequence is that multi-span PSDs can (expectedly) be obtained from single-span PSDs, regardless of whether the interference is added coherently or not in the signal picture. The multi-span GN PSD (55) is known in the literature [2]. It is presumably obtained in the manner described below; however, it is often intuitively explained rather than fully derived.

At the end of the first span, after amplification, we have

$$q_\omega(e) = e^{-j\beta_1(e) \omega} \left\{ q_\omega(0) - j \gamma N_\omega(q, q, q | H)(0, e) \right\},$$

where

$$N_\omega(q, q, q | H)(z, z') = \int H_{123\omega}(j \alpha - \Omega_{123\omega})(z' - z) \times q_1(z)q_2(z)q_3^*(z) d\omega_{123}.$$ 

At the end of the second span

$$q_\omega(2e) = e^{-j\beta_2(e) \omega} \left\{ q_\omega(e) - j \gamma N_\omega(q, q, q | H)(e, 2e) \right\}$$

$$= e^{-j2\beta_2(e) \omega} q_\omega(0) - j \gamma e^{-j2\beta_2(e) \omega} N_\omega(q, q, q | H)(0, e)$$

$$- j \gamma e^{-j\beta_2(e) \omega} N_\omega(q, q, q | H)(e, 2e) \quad (70)$$

The last term contains $q_\omega(e)$, which itself is the sum of a linear and a nonlinear term. In agreement with the first-order approach of the GN model, all FWM terms are evaluated at the linear solution; the contribution of the nonlinear term to $q_\omega(2\epsilon)$ is of second order $\gamma^2$. We thus evaluate the last term at $exp(-j\epsilon \beta_2(\omega)) q_\omega(0)$:

$$N_\omega(q, q, q | H)(e, 2e) = \int H_{123\omega}(e) q_1(e)q_2(e)q_3^*(e) d\omega_{123}$$

$$= e^{-j\beta_2(\omega)} \int e^{-j\Omega H_{123\omega}(e)} q_1(0)q_2(0)q_3^*(0) d\omega_{123}$$

$$= e^{-j\beta_2(\omega)} N_\omega(q, q, q | e^{-j\Omega H}(0, e)).$$

Thus

$$q_\omega(2\epsilon) = e^{-2j\epsilon \beta_2(\omega)} \left\{ q_\omega(e) - j \gamma N_\omega(q, q, q | G_{1H})(0, e) \right\},$$

where $G_{1F} = 1 + e^{-j\Omega F}$. By induction, we have

$$q_\omega(z) = e^{-jz \beta_2(e) \omega} \left\{ q_\omega(0) - j \gamma N_\omega(q, q, q | G_{1H})(0, e) \right\}, \quad (70)$$

where $G_{1F}$ is given by (55). Squaring and averaging (70), we obtain the multi-span GN PSD.

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