A Short Combinatorial Proof of Derangement Identity

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Abstract

The \( n \)-th rencontres number with the parameter \( r \) is the number of permutations having exactly \( r \) fixed points. In particular, a derangement is a permutation without any fixed point. We present a short combinatorial proof for a weighted sum derangement identities.

Keywords: derangements, rencontres numbers, recurrence relation, factorial, binomial coefficient

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1 Introduction

Having a permutation \( \sigma \in S_n \), \( \sigma : [n] \rightarrow [n] \) where \( [n] := \{1, 2, \ldots, n\} \), it is said that \( k \in [n] \) is a fixed point if it is mapped to itself, \( \sigma(k) = k \). Permutations without fixed points are of particular interest and are usually called derangements. We let \( D_n \) denote the number of derangements of the set \( [n] \), \( D_n = |S_n^{(0)}| \),

\[
S_n^{(0)} := \{ \sigma \in S_n : \sigma(k) \neq k, k = 1, \ldots, n \}.
\]

Derangements are usually introduced in the context of inclusion-exclusion principle \([1, 6, 10]\), since this principle is used to provide an interpretation of \( D_n \) as a subfactorial,

\[
D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\]  \( 1 \)

The numbers \( D_0, D_1, D_2, \ldots, D_n, \ldots \) form recursive sequence \( (D_n)_{n \geq 0} \) defined by the recurrence formulae

\[
D_n = (n - 1)(D_{n-1} + D_{n-2})
\]  \( 2 \)

and initial terms \( D_0 = 1, D_1 = 0 \). There is a counting argument to prove this. Let the number \( k \) be mapped by \( \sigma \) to the number \( j, j = 1, \ldots, k - 1, k + 1, \ldots, n \). Note
that there are \((n - 1)\) such permutations \(\sigma\). Now, we separate the set of permutations \(\sigma\) into two disjoint sets \(A\) and \(B\), such that

\[
A := \{ \sigma \in S_n^{(0)} : \sigma(j) \neq k, \sigma(k) = j \}
\]
\[
B := \{ \sigma \in S_n^{(0)} : \sigma(j) = k, \sigma(k) = j \}
\]

This means that

\[
D_n = (n - 1)(|A| + |B|).
\]

The set \(A\) counts \(D_{n-1}\) elements while the set \(B\) counts \(D_{n-2}\) elements. The fact that the number \(k\) in this reasoning is chosen without losing generality, completes the proof of \((2)\).

By a simple algebraic manipulation with \((2)\) we obtain another recurrence for the sequence \((D_n)_{n \geq 0}\),

\[
D_n = nD_{n-1} + (-1)^n. \tag{3}
\]

Namely, it holds true

\[
nD_{n-2} - D_{n-1} - D_{n-2} = -nD_{n-3} + D_{n-2} + 2D_{n-3} = \cdots = (-1)^n
\]

and this immediately gives the above recurrence from \((2)\).

When we iteratively apply recurrence \((3)\) to the derangement number on the r.h.s. of this relation we get

\[
nD_{n-1} + (-1)^n = n[(n - 1)D_{n-2} + (-1)^{n-1}] + (-1)^n
\]

which finally results with

\[
n(n - 1)(n - 2) \cdots 3(-1)^2 + n(n - 1)(n - 2) \cdots 4(-1)^3 + \cdots + (-1)^n. \tag{4}
\]

on the r.h.s. of \((3)\), which completes the proof of \((1)\).

A few identities for the sequence \((D_n)_{n \geq 0}\) are known \([2, 5, 8]\). In \([2]\) Deutsche and Elizalde give a nice identity

\[
D_n = \sum_{k=2}^{n} (k - 1) \binom{n}{k} D_{n-k}. \tag{5}
\]

Recently, Bhatnagar presents families of identities for some sequences including the shifted derangement numbers \([3]\), deriving it using an Euler’s identity \([4]\). In what follows we demonstrate a combinatorial proof for that derangement identity, with weighted sum.
2 A pair of weighted sums for derangements

We define the *rencontres number* $D_n(r)$ as the number of permutations $\sigma \in S_n$ having exactly $r$ fixed points. Thus, $D_n(0) = D_n$. For a given $r \in \mathbb{N}$, we define the sequence $D_0(r), D_1(r), \ldots, D_n(r), \ldots$, denoted $(D_n(r))_{n \geq r}$.

Applying an analogue counting argument that we used when proving relation (2), one can represent rencontres numbers by the derangement numbers,

$$D_n(r) = \binom{n}{r} D_{n-r}. \quad (6)$$

On the other hand, relation (6) follows immediately from the fact that fixed points here are $r$-combinations over the set of $n$ elements.

A few other notable properties of the rencontres numbers is also known. The difference between numbers in the sequences $(D_n)_{n \geq 0}$ and $(D_n(1))_{n \geq 1}$ alternate for the value $1$, which follows from (3). According to the definition of rencontres numbers, the sum of the $n$-th row in the array of numbers $(D_n(r))_{n \geq r}$ is equal to $n!$,

$$n! = \sum_{k=0}^{n} D_n(k). \quad (7)$$

Moreover, identity (6) shows that $D_n$ can be interpreted as a weighted sum of rencontres numbers in the $n$-th row of the array, by means of relation (5),

$$D_n = \sum_{k=2}^{n} (k-1) D_n(k). \quad (8)$$

The number $D_n/(n-1)$ is also a weighted sum of previous consecutive derangement numbers. For example, $24 + 12D_2 + 4D_3 + D_4 = \frac{D_5}{5}$. In general we have

$$n! + \sum_{k=2}^{n} \frac{n!}{k!} D_k = \frac{D_{n+2}}{n+1}. \quad (9)$$

as follows from Theorem 1.

**Theorem 1.** For $n \in \mathbb{N}$ and the sequence of derangement numbers $(D_n)_{n \geq 0}$ we have

$$1 + \sum_{k=1}^{n} \frac{D_k}{k!} = \frac{D_{n+2}}{(n+1)!}. \quad (10)$$

**Proof.** Within a derangement $\sigma$, the number $k$, $k = 1, \ldots, n$ can be mapped to any $j$, $j = 1, \ldots, k-1, k+1, \ldots, n$. We let $\mathcal{A}_n$ denote the set of derangements with $\sigma(k) = j$, where $j \neq k$,

$$\mathcal{A}_n := \{ \sigma \in S_n^{(0)} : \sigma(k) = j \}. $$
Obviously, cardinality of the set $A_n$ is invariant to the choice of $j$, $j \neq k$. More precisely,  

\[ |A_n| = \frac{D_n}{n-1}. \]

Furthermore, we separate the set $A_n$ into two disjoint sets of derangements, sets $B_n$ and $C_n$,  

\[ B_n := \{ \sigma \in A_n : \sigma(j) = k \} \]
\[ C_n := \{ \sigma \in A_n : \sigma(j) \neq k \}. \]

Obviously, the set $B_n$ counts $D_{n-2}$ elements. For derangements in $C_n$ there are now $(n-2)$ equivalent ways to map $j$ (excluding $j$ and $k$), as Figure 1 illustrates. Thus, we have  

\[ |C_n| = (n-2)|A_{n-1}|, \]

which gives the recurrence relation  

\[ |A_n| = D_{n-2} + (n-2)|A_{n-1}|. \quad (11) \]

After repeating usage of (11) we get identity (9) which completes the proof.  

\[ \]

Figure 1: In case of derangements in the set $C_n$ there are $(n-2)$ equivalent ways to map $j$.

In order to prove Theorem 1 algebraically, we apply recurrence (2) to get  

\[ \frac{D_{n+2}}{(n+1)!} = \frac{(n+1)(D_{n+1} + D_n)}{(n+1)!} = \frac{D_{n+1}}{n!} + \frac{D_n}{n!} \]
\[ = \frac{n(D_n + D_{n-1})}{n!} + \frac{D_n}{n!} = \frac{D_n}{(n-1)!} + \frac{D_{n-1}}{(n-1)!} + \frac{D_n}{n!} \]
\[ = 1 + \frac{D_1}{1!} + \cdots + \frac{D_n}{n!} = 1 + \sum_{k=1}^{n} \frac{D_k}{k!}. \]

**Theorem 2.** For $n \in \mathbb{N}$ and the sequence of derangement numbers $(D_n)_{n \geq 0}$ we have  

\[ 1 + \sum_{k=1}^{n} \frac{(-1)^k D_{k+3}}{k+2} = (-1)^n D_{n+2}. \quad (12) \]

\[ \]
Proof. By applying recurrence (2) we have
\[
\sum_{k=0}^{n} \frac{(-1)^k D_{k+3}}{k+2} = \frac{2(D_2 + D_1)}{2} - \frac{3(D_3 + D_2)}{3} + \cdots (-1)^n \frac{(n+2)(D_{n+2} + D_{n+1})}{n+2}
\]
\[
= (D_2 + D_1) - (D_3 + D_2) + \cdots (-1)^n (D_{n+2} + D_{n+1})
\]
\[
= (-1)^n D_{n+2}
\]
which completes the proof. \qed

Once having Theorem 1, substitution of (6) in identity (10) gives the generalization (13).
\[
1 + \sum_{k=1}^{n} \frac{D_{k+r}(r)}{k!(r_k^r)} = \frac{D_{n+r+2}(r)}{(n+1)!\left(\frac{n+r+2}{r}\right)}.
\]

The identity (14) follows by substitution of (6) in (12),
\[
1 + \sum_{k=1}^{n} \frac{(-1)^k D_{k+r+2}(r)}{(k+2)(r_{k+r+2}^r)} = \frac{(-1)^n D_{n+r+2}(r)}{\left(\frac{n+r+2}{r}\right)}.
\]

Note that the terms in identity (14) are always integers, which can be seen as a consequence of recurrence relation (2).

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