ON THE JONES POLYNOMIAL OF QUASI-ALTERNATING LINKS

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Abstract. We prove that twisting any quasi-alternating link $L$ with no gaps in its Jones polynomial $V_L(t)$ at the crossing where it is quasi-alternating produces a link $L^*$ with no gaps in its Jones polynomial $V_{L^*}(t)$. This leads us to conjecture that the Jones polynomial of any prime quasi-alternating link, other than $(2,n)$-torus links, has no gaps. This would give a new property of quasi-alternating links and a simple obstruction criterion for a link to be quasi-alternating. We prove that the conjecture holds for quasi-alternating Montesinos links as well as quasi-alternating links with braid index 3.

1. Introduction

The class of quasi-alternating links was introduced first by Ozsváth and Szabó in [14] as a natural generalization of the class of alternating links. This class is defined recursively as follows:

Definition 1.1. The set of quasi-alternating links $Q$ is the smallest set satisfying the following properties.

- The unknot belongs to $Q$.
- If $L$ is a link with a diagram $D$ containing a crossing $c$ such that
  - (1) both smoothings of the diagram $D$ at the crossing $c$, $L_0$ and $L_\infty$ as given in Figure 1 belong to $Q$,
  - (2) $\det(L_0), \det(L_\infty) \geq 1$,
  - (3) $\det(L) = \det(L_0) + \det(L_\infty)$; then $L$ is in $Q$ and in this case we say that $L$ is quasi-alternating at the crossing $c$ with quasi-alternating diagram $D$.

![Figure 1](attachment:image.png)

Figure 1. The diagram of the link $L$ with crossing $c$ and its smoothings $L_0$ and $L_\infty$ respectively.

Using this definition, one can easily see that any non-split alternating link is quasi-alternating at any crossing in any reduced alternating diagram. The knot $8_{20}$ is the first example of quasi-alternating non-alternating knot in the knot table. It is worth mentioning here that it is impossible to determine that a given link is not quasi-alternating using the above recursive definition since one has to consider all crossings in all possible diagrams of this given link.

Date: 23/05/2018.

Key words and phrases. quasi-alternating links, Jones polynomial, Montesinos links, 3-braids.

The first author was supported by a research grant from United Arab Emirates University, UPAR grant #G90000260.
A different approach to address this problem is to study the behavior of the invariants of quasi-alternating links in order to find obstruction criteria for a link to be quasi-alternating. Several such obstruction criteria have been introduced over the past fifteen years. For instance, it was shown that these links are homologically thin in both Khovanov and link Floer homology \cite{12,14,15}. In addition, the Heegaard Floer Homology of their branched double covers depends only on the determinant of the link, \cite{14}. On the other hand, a simple obstruction criterion has been introduced in terms of the degree of the $Q$-polynomial and the determinant of the link, \cite{16}. This obstruction has been sharpened by Teragaito in \cite{19}, then extended by the same author to the two-variable Kauffman polynomial \cite{20}.

A simple way to produce new examples of quasi-alternating links from old ones was introduced by Champanerkar and Kofman \cite{5}. Given a link $L$ with quasi-alternating diagram $D$ at a crossing $c$. Then any link diagram obtained from $L$ by replacing the crossing $c$ by an alternating rational tangle of the same type is quasi-alternating at any of the new crossings.

This construction was generalized to links obtained by replacing a crossing by a product of rational tangles and applied to study quasi-alternating Montesinos links \cite{17}. In this paper, we investigate how does the Jones polynomial interact with this twisting property. More precisely, we prove that if the Jones polynomial of a quasi-alternating link $L$ has no gaps, then so is the Jones polynomial of any link $L'$ obtained from it by replacing the quasi-alternating crossing by a product of rational tangles. Based on this fact, in addition to other computational evidences, we conjecture that if $L$ is a prime quasi-alternating link other than the $(2,n)$-torus link, then its Jones polynomial $V_L(t)$ has no gaps. It is well known that this condition is satisfied by alternating links as it was proved in \cite{21}.

This paper is organized as follows. In Section 2, the main theorem is proved and a new obstruction for a link to be quasi-alternating is conjectured. This conjecture is proved for quasi-alternating Montesinos links in Section 3. In Section 4, we prove that the Conjecture holds for quasi-alternating links of braid index 3.

2. MAIN CONJECTURE AND SOME CONSEQUENCES

The Jones polynomial $V_L(t)$ is an invariant of oriented links. It is a Laurent polynomial with integral coefficients that might be defined in several ways. In this section, we shall briefly introduce this polynomial and review some of its properties needed in the sequel. Let us first define the Kauffman bracket polynomial.

**Definition 2.1.** The Kauffman bracket polynomial is a function from unoriented link diagrams in the oriented plane to the ring of Laurent polynomials with integer coefficients in an indeterminate $A$. It maps a diagram $D$ to $\langle D \rangle \in \mathbb{Z}[A^{-1}, A]$ and it is defined by the following relations:

1. $\langle \bigcirc \rangle = 1$,
2. $\langle \bigcirc \cup D \rangle = (-A^{-2} - A^2) \langle D \rangle$,
3. $\langle L \rangle = A \langle L_0 \rangle + A^{-1} \langle L_{\infty} \rangle$,

where $\bigcirc$ denotes a trivial circle, and $L, L_0,$ and $L_{\infty}$ represent unoriented link diagrams that are identical except in a small region where they look as in Figure 1.

**Definition 2.2.** The Jones polynomial $V_L(t)$ of an oriented link $L$ is the Laurent polynomial in $t^{1/2}$ with integer coefficients defined by

$$V_L(t) = ((-A)^{-3w(D)} \langle D \rangle)_{t^{1/2} = A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}],$$

where the writh of the oriented diagram $D$, denoted by $w(D)$, is the number of crossings of the first type minus the number of crossings of second type as pictured in Figure 2.
We always can write the Jones polynomial of any link \( L \) as follows:

\[
V_L(t) = t^m \sum_{i=0}^{m} a_i t^i,
\]
where \( m \geq 0, a_0 \neq 0 \) and \( a_m \neq 0 \).

Therefore, the Kauffman bracket of the diagram \( D \) of the link \( L \) can be written as follows:

\[
\langle D \rangle(A) = A^m \sum_{i=0}^{m} a_i A^{4i},
\]
where \( m \geq 0, a_0 \neq 0 \) and \( a_m \neq 0 \).

We aim to study the Jones polynomial of quasi-alternating links. In \([12]\), it was proved that the reduced ordinary Khovanov homology group of any quasi-alternating link is thin. Consequently, the Jones polynomial of any quasi-alternating link is alternating. In other words, its coefficients satisfy

\[
a_i a_{i+1} < 0 \quad \text{for all} \quad 0 \leq i \leq m - 1.
\]

It is known that quasi-alternating links share the same homological properties with alternating links. A natural question is to ask whether we can extend the above result about the Jones polynomial to the class of prime quasi-alternating links, other than \((2, n)\)-torus links. We conjecture the following.

**Conjecture 2.3.** If \( L \) is a prime quasi-alternating link, other than \((2, n)\)-torus link, then the coefficients of the Jones polynomial of \( L \) satisfy

\[
a_i a_{i+1} < 0 \quad \text{for all} \quad 0 \leq i \leq m - 1.
\]

If this conjecture is true, then we will get a positive solution of \([16\text{ Conjecture 3.8}] \) that states that \( \det(L) \geq \br(L) \), where \( \det(L) \) and \( \br(L) \) denote the determinant and the breadth of the Jones polynomial of the quasi-alternating link \( L \), respectively. The last conjecture is a weaker version of \([18\text{ Conjecture 1.1}] \) which states that \( \det(L) \geq c(L) \), where \( c(L) \) denotes the crossing number of the quasi-alternating link \( L \).

The following theorem makes use of the twisting construction introduced in \([3\text{ Page. 2452}] \) which consists of replacing a crossing by a rational tangle that extends it. This construction has been generalized later to product of rational tangles in \([17\text{ Def. 2.5}] \).

**Theorem 2.4.** Let \( L \) be a quasi-alternating link at some crossing \( c \) and let \( L^* \) be the link obtained from \( L \) by replacing the crossing \( c \) by a product of rational tangles that extends it. This construction has no gaps. If \( V_{L}(t) \) has no gaps, then so is \( V_{L^*}(t) \).

Before proving the above theorem, we state the following lemma which is easy to prove.

**Lemma 2.5.** Suppose \( h \) is a product of two alternating polynomials then the monomial \( x^n \) in \( h \) has nonzero coefficient if at least one \( a_i b_{n-i} \neq 0 \) where \( a_i \) is the coefficient of \( x^i \) and \( b_{n-i} \) is the coefficient of \( x^{n-i} \) in the first and second polynomials respectively.

Now, we shall prove Theorem 2.4.

**Proof.** Let \( L \) be a quasi-alternating link at the crossing \( c \). We may assume that \( \sign(c) > 0 \) as shown in Figure 1 by taking the mirror image if it is needed. For a positive integer \( n \), we let
$L^n$ denote the link diagram with $n$ vertical or horizontal positive half-twists crossings replacing the crossing $c$.

To prove our result, we first show that $\langle L^n \rangle$ satisfies $a_ia_{i+1} \neq 0$ if $V_L(t)$ satisfies $a_ia_{i+1} \neq 0$ for $i = 0, 1, \ldots, m - 1$. It is a simple exercise to prove that:

$$\langle L^n \rangle = A^n \langle L_0 \rangle + \left( \sum_{i=0}^{n-1} (-1)^i A^{n-4i-2} \right) \langle L_{\infty} \rangle,$$

for the case of $n$ vertical and $n$ horizontal crossings respectively. Now, we shall discuss the first case. The second case can be treated in a similar manner.

Note that no monomial of the first term will cancel out with a monomial of the second term because $\det(L^n) = \det(L) + (n - 1) \det(L_{\infty})$ which can be proved since $L^n$ is quasi-alternating as a result of [3, Theorem 2.1].

Moreover, if $A^t$ is a monomial of nonzero coefficient in $\langle L_{\infty} \rangle$, then $A^{n+t-6}$ is a monomial with nonzero coefficient in the second term and at the same time the monomial $A^{n+t-2}$ has nonzero coefficient in the first term. Note that these two monomials will not cancel out in $\langle L^n \rangle$ because of the above note. This proves that if $a_ia_{i+1} = 0$ then either this happens in first term or in the second term.

The first case is impossible because of the assumption on the link $L$. For the second one, suppose the monomial of nonzero coefficient with the lowest degree in the second term such that $a_ia_{i+1} = 0$ is $A^{4t+s}$ for some $s$ and $t$. In other words, the monomial $A^{4t+s}$ has nonzero coefficient and $A^{4t+s+4}$ has zero coefficient in the second term and $t$ is the smallest such number. We show that the coefficient of the monomial $A^{4t+s+4}$ has nonzero coefficient in the first term. As a result of Lemma 2.5 having a nonzero coefficient of the monomial $A^{4t+s}$ implies that at least one of the following monomials $A^{4t+s-n+6}, A^{4t+s-n+10}, \ldots, A^{4t+s+3n-2}$ has to have nonzero coefficient in $\langle L_{\infty} \rangle$ and at the same time having a zero coefficient of the monomial $A^{4t+s+4}$ implies that all of the other monomials have to have zero coefficient $A^{4t+s-n+10}, A^{4t+s-n+14}, \ldots, A^{4t+s+3n+2}$ in $\langle L_{\infty} \rangle$. It is clear in this case that the monomial $A^{4t+s-n+6}$ has nonzero coefficient and all of the other monomials have to have zero coefficient in $\langle L_{\infty} \rangle$. This will imply that the monomial $A^{4t+s+4}$ has nonzero coefficient in the first term.

Now this proves that $\langle L^n \rangle$ satisfies $a_ia_{i+1} \neq 0$ for $i = 0, 1, \ldots, m - 1$. The fact that the link $L^n$ is quasi-alternating implies that $\langle L^n \rangle$ is alternating. Therefore, $\langle L^n \rangle$ satisfies $a_ia_{i+1} < 0$ for $i = 0, 1, \ldots, m - 1$. Finally the result follows since any product of rational tangles can be obtained by a sequence of integer tangles.

\[\square\]

**Corollary 2.6.** If a link is quasi-alternating with a gap in its Jones polynomial, then it cannot be obtained by twisting a quasi-alternating link with no gaps in its Jones polynomial.

**Remark 2.7.** With the aid of [6] and the tables of quasi-alternating knots of at most 12 crossings in [9], we checked by hand that all knots with 12 crossings or less with gap in the
Jones polynomial are either \((2,n)\)-torus knots or not quasi-alternating. This confirms the above conjecture for all knots of at most 12 crossings. As a consequence of the above theorem, we conclude that a counter example to the above conjecture has to be a knot of at least 13 crossings that cannot be obtained by twisting a quasi-alternating knot of at most 12 crossings.

3. The Jones polynomial of Quasi-alternating Montesinos Links

In this section, we prove Conjecture 2.3 for all quasi-alternating Montesinos links. Let \(\alpha\) and \(\beta\) be coprime integers with \(\alpha > \beta \geq 1\) and \([a_1, a_2, \ldots, a_n]\) be the continued fraction of the rational number \(\frac{\beta}{\alpha}\) of positive integers. In [11], Kauffman and Lopez defined a sequence of positive integers recursively by

\[
T(0) = 1, T(1) = a_1, T(m) = a_n T(m-1) + T(m-2),
\]

and showed that \(T(n)\) is equal to the determinant of the rational link \(L_{\alpha, \beta}\) of slope \(\frac{\beta}{\alpha}\).

Lemma 3.1. For any positive rational number \(\frac{\beta}{\alpha}\), there is a continued fraction \([a_1, a_2, \ldots, a_n]\) of positive integers.

Lemma 3.2. Let \([a_1, a_2, \ldots, a_n]\) be a continued fraction of \(\frac{\beta}{\alpha}\) of positive integers, then \(T(n-1) = \beta\) and \(T(n) = \alpha\).

Proof. Using the recursive formula for \(T(n)\), we can see that if \(d | T(n)\) and \(d | T(n-1)\), then \(d | T(n-2)\). If this process is repeated, we obtain that \(d | T(0) = 1\). This shows that \(T(n)\) and \(T(n-1)\) are relatively prime.

Now, we have

\[
\frac{T(n)}{T(n-1)} = \frac{a_n T(n-1) + T(n-2)}{T(n-1)}
\]

\[
= a_n + \frac{T(n-2)}{T(n-1)}
\]

\[
= a_n + \frac{1}{\frac{T(n-1)}{T(n-2)}}
\]

\[
= a_n + \frac{1}{a_{n-1} + \frac{T(n-3)}{T(n-2)}}
\]

\[
\vdots
\]

\[
= a_n + \frac{1}{a_{n-1} + \cdots + \frac{1}{a_1}} = \frac{\alpha}{\beta}.
\]

\[
\square
\]

Lemma 3.3. Let \([a_1, a_2, \ldots, a_n]\) be a continued fraction of \(\frac{\beta}{\alpha}\) of positive integers, then \(\alpha \geq a_1 + a_2 + \ldots + a_n\).

Proof. We use induction on the length of the continued fraction. It is clear that this holds for \(n = 1\). Now suppose this holds for any continued fraction of length \(n-1\). Therefore, we have \(T(n-1) \geq a_1 + a_2 + \ldots + a_{n-1}\). Now \(\alpha = a_n T(n-1) + T(n-2) \geq a_n T(n-1) + 1 \geq a_n + T(n-1) = a_n + a_{n-1} + \ldots + a_1\). The last inequality follows if we apply the result for \(n = 2\). \(\square\)
Definition 3.4. A Montesinos link is a link that has a diagram as shown in Figure 3(a). In this diagram, \( e \) is an integer that represents the number of half twists and the box \([\alpha_i, \beta_i]\) stands for a rational tangle of slope \( \frac{\alpha_i}{\beta_i} \), where \( \alpha_i > 1 \) and \( \beta_i \) are coprime integers.

![Figure 3. Montesinos link diagram in (a) with \( e = 4 \). (b) represents the rational tangle of slope \( -\frac{7}{31} \) with associated continued fraction \([-3, -2, -4]\).](image)

It is worth mentioning here that our conventions for the signs in the picture above are the same as in [8] and [17]. Recall that Montesinos links are classified by the following theorem.

Theorem 3.5. [3, 4] The Montesinos link \( M(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)) \) with \( r \geq 3 \) and \( \sum_{j=1}^{r} \frac{1}{\alpha_j} \leq r - 2 \), is classified by the ordered set of fractions \( \left( \frac{\beta_1}{\alpha_1} \pmod{1} \right), \left( \frac{\beta_2}{\alpha_2} \pmod{1} \right), \ldots, \left( \frac{\beta_r}{\alpha_r} \pmod{1} \right) \) up to cyclic permutations and reversal of order, together with the rational number \( e_0 = e - \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j} \).

Definition 3.6. A Montesinos link \( M(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)) \) is in standard form if \( 0 < \frac{\alpha_i}{\beta_i} < 1 \) for \( i = 1, 2, \ldots, r \).

Remark 3.7. According to the above theorem, every Montesinos link has a unique standard form.

Lemma 3.8. The Montesinos link \( L = M(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)) \) is not a \((2, n)\)-torus link for any integer \( n \).

Proof. Suppose that the link \( L \) is a \((2, n)\)-torus link for some integer \( n \). It is clear that the corresponding Montesinos diagram of \( L \) is reduced alternating diagram, so this diagram has minimal number of crossings among all other diagrams which has to be equal to \( n \) from the assumption. Therefore, we have \( n = \sum_{j=1}^{r} \frac{\alpha_j}{\beta_j} \) with \( [a_1^j, a_2^j, \ldots, a_{s_j}^j] \) being the continued fraction of \( \frac{\alpha_j}{\beta_j} \). Using the fact that \( \det(L) = \left( \prod_{j=1}^{r} \alpha_j \right) \left( -e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j} \right) \), we get:

\[
 n = \det(L) = \left( \prod_{j=1}^{r} \alpha_j \right) \left( -e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j} \right) > \sum_{j=1}^{r} \alpha_j > \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j},
\]

which is impossible. Thus, \( L \) cannot be a \((2, n)\)-torus link. \( \square \)

Lemma 3.9. The Montesinos link \( L = M(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2)) \) with \( \frac{\alpha_2}{\alpha_1} > \frac{\beta_1}{\beta_2} \) is not a \((2, n)\)-torus link for any integer \( n \).
Proof. It is easy to see that $\frac{c_{D_1}}{\alpha_{D_1} - \beta_{D_1}} > \frac{c_{D_2}}{\alpha_{D_2} - \beta_{D_2}}$ is equivalent to say $\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} > 1$. We know that $\alpha_1, \alpha_2 \geq 2$. Hence, we conclude that either $\beta_1$ is not equal to 1 or $\beta_2$ is not equal to 1. Now suppose that the link $L$ is a $(2,n)$-torus link for some integer $n$. It is clear that the corresponding Montesinos diagram of $L$ is reduced alternating diagram, so this diagram has minimal number of crossings among all other diagrams which has to be equal to $n$ from the assumption. Therefore, we have $n = a_1 + a_2 + \ldots + a_s + b_1 + b_2 + \ldots + b_t$ with $|a_1, a_2, \ldots, a_s|, |b_1, b_2, \ldots, b_t|$ being the continued fractions of $\frac{\alpha_i}{\beta_i}$ and $\frac{\alpha_j}{\beta_j}$ of positive integers, respectively. Also we have

$$n = \det(L) = \alpha_1 \beta_2 + \alpha_2 \beta_1 \geq \beta_2(a_1 + a_2 + \ldots + a_s) + \beta_1(b_1 + b_2 + \ldots + b_t)$$

$$> (a_1 + a_2 + \ldots + a_s) + (b_1 + b_2 + \ldots + b_t) = n,$$

which is impossible. In conclusion, $L$ cannot be a $(2,n)$-torus link.

For the next result, we need the following theorem that is obtained by combining \cite{17} Theorem 3.5] and \cite{8} Theorem 1].

**Theorem 3.10.** The Montesinos link $L = M(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r))$ in standard form is quasi-alternating if one of the following conditions is satisfied

1. $e \leq 0$;
2. $e \geq r$;
3. $e = 1$ with $\frac{\alpha_i}{\alpha_i - \beta_i} > \min\{\frac{\alpha_j}{\beta_j} \mid j \neq i\}$ for some $1 \leq i \leq r$;
4. $e = r - 1$ with $\frac{\alpha_i}{\alpha_i - \beta_i} > \min\{\frac{\alpha_j}{\beta_j} \mid j \neq i\}$ for some $1 \leq i \leq r$.

**Theorem 3.11.** The coefficients of the Jones polynomial of any quasi-alternating Montesinos link satisfy $a_i a_{i+1} < 0$ for all $0 \leq i \leq m - 1$.

Proof. The result holds for the first two cases since the Montesinos link in standard form in these two cases is an alternating link that is not a $(2,n)$-torus link for any integer $n$ according to Lemma \cite{8}. For the third case, the Montesinos link can be obtained by replacing the crossing in the middle tangle in $M(0; (\alpha_1, \beta_1), (1, 1), (\alpha_1, \beta_1 - \alpha_1)$ with $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$ by a product of rational tangles that extends it according to \cite{17} Theorem 3.5(3)]. Now we check that $L = M(0; (\alpha_1, \beta_1), (1, 1), (\alpha_1, \beta_1 - \alpha_1)$ has no gap in the Jones polynomial. It is enough to show that we know that $L$ is equivalent to $M(0; (1, 1), (\alpha_1, \beta_1), (\alpha_1, \beta_1 - \alpha_1)$ and this one is equivalent to $M(-1; (\alpha_1, \beta_1), (\alpha_1, \beta_1 - \alpha_1))$. Now the last link is equivalent to the link $M(0; (\alpha_1, \beta_1), (\alpha_1, \beta_1))$ which is alternating since the corresponding diagram is alternating. The last link is not a $(2,n)$-torus link according to Lemma \cite{8} so its Jones polynomial has no gap. The last case follows using the same argument after we put the link in standard form. Finally, the result follows as a direct consequence of Theorem \cite{17}

**Remark 3.12.** The above theorem can be specialized to the case of pretzel links. But rather than using \cite{8} Theorem 1] and \cite{17} Theorem 3.5(3)], we use \cite{17} Theorem 1] and \cite{17} Corollary 3.6(4)].

### 4. The Jones Polynomial of Quasi-alternating Closed 3-braids

In this section, we prove that Conjecture \cite{2} holds for prime quasi-alternating links with braid index equal to 3. Quasi-alternating links of braid index 3 have been classified by Baldwin \cite{1} based on Murasugi’s classification of 3-braids \cite{13}. For $n \geq 2$, let $B_n$ be the braid group on $n$ strings. This group is generated by the elementary braids $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2$$

$$\sigma_i \sigma_{i+1} = \sigma_{i+1} \sigma_i \sigma_{i+1}, \forall 1 \leq i \leq n - 2.$$
The two generators $\sigma_1$ and $\sigma_2$ of the braid group $B_3$ are pictured in Figure 4. Recall that every link $L$ in $S^3$ can be obtained as the closure of a certain braid $b$, denoted by $L = \hat{b}$. It is well known that 3-braids have been classified, up to conjugation, by Murasugi [13].

**Figure 4.** The generators $\sigma_1$ and $\sigma_2$ of $B_3$ respectively

**Theorem 4.1.** Let $b$ be a 3-braid and let $h = (\sigma_1 \sigma_2)^3$ be a full positive twist. Then $b$ is conjugate to exactly one of the following:

1. $h^n \sigma_1^{p_1} \sigma_2^{-q_1} \ldots \sigma_1^{p_i} \sigma_2^{-q_i}$, where $s, p_i$ and $q_i$ are positive integers.
2. $h^n \sigma_2^m$ where $m \in \mathbb{Z}$.
3. $h^n \sigma_1^m \sigma_2^{-1}$, where $m \in \{-1, -2, -3\}$.

Baldwin classified quasi-alternating closed 3-braids as in the following theorem [1]:

**Theorem 4.2.** Let $L$ be a closed 3-braid, then

1. If $L$ is the closure of $h^n \sigma_1^{p_1} \sigma_2^{-q_1} \ldots \sigma_1^{p_i} \sigma_2^{-q_i}$, where $s, p_i$ and $q_i$ are positive integers, then $L$ is quasi-alternating if and only if $n \in \{-1, 0, 1\}$.
2. If $L$ is the closure of $h^n \sigma_2^m$, then $L$ is quasi-alternating if and only if either $n = 1$ and $m \in \{-1, -2, -3\}$ or $n = -1$ and $m \in \{1, 2, 3\}$.
3. If $L$ is the closure of $h^n \sigma_1^m \sigma_2^{-1}$ where $m \in \{-1, -2, -3\}$. Then $L$ is quasi-alternating if and only if $n \in \{0, 1\}$.

Now we state the main theorem in this section.

**Theorem 4.3.** The coefficients of the Jones polynomial of any prime quasi-alternating link of braid index 3 satisfy $a_i a_{i+1} < 0$ for all $0 \leq i \leq m - 1$.

**Proof.** The Jones polynomial of a closed 3-braid can be computed using a relatively simple formula introduced by Birman [2]. Given a 3-braid $\alpha$, let $e_\alpha$ be the exponent sum of $\alpha$ as a word in the elementary braids $\sigma_1$ and $\sigma_2$. Let $\psi_\alpha : B_3 \rightarrow GL(2, \mathbb{Z}[t, t^{-1}])$ be the Burau representation defined on the generators of $B_3$ by $\psi_\alpha(\sigma_1) = \begin{bmatrix} -t & 1 \\ 0 & 1 \end{bmatrix}$ and $\psi_\alpha(\sigma_2) = \begin{bmatrix} 1 & 0 \\ t & -t \end{bmatrix}$.

Then, the Jones polynomial of the link $\hat{\alpha}$ is given by the following formula

$$V_{\hat{\alpha}}(t) = (-\sqrt{t})^{e_\alpha} (t + t^{-1} + tr(\psi_\alpha(\alpha))),$$

where $tr$ denotes the usual matrix-trace function.

Let us consider the Jones polynomial of a braid of type 1 in Baldwin’s Theorem. If $n = 0$, then the link is alternating and the Conjecture holds. We will now consider the case $n = 1$. The case $n = -1$ is treated in a similar way. Assume that $\alpha = h\beta$, where $\beta = \sigma_1^{p_1} \sigma_2^{-q_1} \ldots \sigma_1^{p_i} \sigma_2^{-q_i}$. Since $\psi_\beta(h) = t^3 I_2$, then $tr(\psi_\beta(\alpha)) = t^3 tr(\psi_\beta(\beta))$. For any positive integers $p_i$ and $q_i$, elementary computations show that:

$$\psi_\beta(\sigma_1^{p_i}) = \begin{bmatrix} (-t)^{p_i} & 1 - (-t)^{p_i} \\ 0 & 1 + t \end{bmatrix}, \quad \text{and} \quad \psi_\beta(\sigma_2^{-q_i}) = \begin{bmatrix} 1 & 0 \\ 1 - (-t)^{-q_i} & (-t)^{-q_i} \end{bmatrix}.$$
\[ \psi_t(\sigma_1^{p_1}\sigma_2^{-q_1}) = \begin{pmatrix}
(-t)^{p_1} + \frac{1 - (-t)^{p_1}}{1 + t} & \frac{1 - (-t)^{-q_1}}{1 + t} & \frac{(-t)^{-q_1}}{1 + t} & \frac{1 - (-t)^{-q_1}}{1 + t}
\end{pmatrix} \]

Note that the matrix \( \psi_t(\sigma_1^{p_1}\sigma_2^{-q_1}) \) above is of the form \( \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} \), where the four entries are Laurent polynomials, each of which is alternating and has no gaps. Moreover, in each of these polynomials the coefficient of the monomial \( t^i \) has the same sign as \( (-1)^i \). Since \( \beta \) is a product of braids of the form \( \sigma_1^{p_1}\sigma_2^{-q_1} \), then each entry of the matrix \( \psi_t(\beta) \) is indeed a sum of products of elements of type \( A(t), B(t), C(t) \) and \( D(t) \). Consequently, each of the entries of \( \psi_t(\beta) \) is alternating with no gaps and for each \( i \) the sign of the coefficient of the monomial \( t^i \) is the same as \( (-1)^i \). Moreover, one can see that \( \text{tr}(\psi_t(\beta)) \) is a Laurent polynomial which is alternating. When considering the sum of diagonal entries of \( \psi_t(\beta) \), no cancellation will happen because of the observation above about the sign of each monomial.

Consequently, the polynomial \( \text{tr}(\psi_t(\beta)) \) will be alternating with no gaps. In addition, from the computation above, we can see that the maximum power of \( t \) that appears in \( \text{tr}(\psi_t(\beta)) \) is \( p = p_1 + p_2 + \cdots + p_s \), while the minimum power is \(-q_1 + q_2 + \cdots + q_s = -q\). If we consider the expression \( t + t^{-1} + t^3 \psi_t(\beta) \) then it is clear that no gaps will appear in this expression if \( q > 2 \) because the sign of the monomials \( t \) and \( t^{-1} \) is positive in \( t^3 \psi_t(\beta) \). Moreover, the minimum power in \( t^3 \psi_t(\beta) \) is \(-q + 3 \leq 0 \).

It remains now to check the cases \( q = 1 \) and \( q = 2 \). In these cases, the Jones polynomials will have gaps. Indeed, the constant term in \( t^{-1} + t + t^3 \psi_t(\beta) \) will be zero. We shall prove that in both cases the link \( \alpha \) will be either a \((2,n)\)-torus link or a connected sum of such links. Two cases are to be considered.

If \( q = 1 \), then
\[
(\sigma_1\sigma_2)^3\sigma_1^{p_1}\sigma_2^{-1} = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{p_1}\sigma_2^{-1} = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{p_1}\sigma_2^{-1} = \sigma_1\sigma_2\sigma_1^{p_1}\sigma_2^{-1} = \sigma_1\sigma_2\sigma_1^{p_1+1} = \sigma_1^{p_1+1}.
\]

where the symbol \( = \) is used to indicate that the closure of the braids are the same. Thus, \( \alpha \) is the torus link of type \((2,p_1+4)\). If \( q = 2 \), then there are two subcases to be considered. If \( \alpha = (\sigma_1\sigma_2)^3\sigma_1^{p_1}\sigma_2^{-2} \), we have:
\[
(\sigma_1\sigma_2)^3\sigma_1^{p_1}\sigma_2^{-2} = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{p_1}\sigma_2^{-2} = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^{p_1}\sigma_2^{-2} = \sigma_1\sigma_2\sigma_1^{p_1}\sigma_2^{-1} = \sigma_1\sigma_2\sigma_1\sigma_2^{p_1}\sigma_2^{-1} = \sigma_1\sigma_2\sigma_1^{p_1+1} = \sigma_2^{p_1+2}.
\]

Hence, \( \alpha \) is the connected sum of the torus link of type \((2,p_1+2)\) and the Hopf link. The second subcase concerns the closure of \( \alpha = (\sigma_1\sigma_2)^3\sigma_1^{p_1}\sigma_2^{-1}\sigma_2^{p_2}\sigma_2^{-1} \). Similar elementary calculations in the braid group show that:
Finally, it is not difficult to see that the closure of \( \sigma_2 \sigma_1^{p_1+2} \sigma_2^{-1} \sigma_1^{p_2+2} \) is the connected sum of the torus links of type \((2, p_1 + 2)\) and \((2, p_2 + 2)\), see Figure 5.

The conjecture then holds for closed 3-braids as in the first case of Baldwin’s Theorem, with \( n = 1 \). For the case \( n = -1 \), the proof can be done in a similar way. It is easy to check that cases 2 and 3 in Baldwin’s Theorem yield either a torus link of type \((2, n)\) for some \( n \) or a connected sum of two links of this type.  

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References

[1] J. Baldwin, Heegaard Floer homology and genus one, one boundary component open books, J. Topol. 1 (4): 963-992, 2008.
[2] J. Birman, On the Jones polynomial of closed 3-braids, Inventiones Mathematicae 81: 287–294, 1985.
[3] F. Bonahon, Involution et fibrés de Sefert dans les variétés de dimension 3, Thèse de 3e cycle, Orsay 1979.
[4] M. Boileau and L. Siebenmann, A planar classification of pretzel knots and Montesinos knots, Prepublication Orsay 1980. Gruter, New York, 2003.
[5] A. Champaerkar and I. Kofman, Twisting Quasi-alternating Links, Proc. Amer. Math. Soc., 137(7):2451–2458, 2009.
[6] J. C. Cha and C. Livingston, KnotInfo: Table of Knot Invariants, [http://www.indiana.edu/~knotinfo](http://www.indiana.edu/~knotinfo), April 15, 2018.
[7] J. Greene, Homologically thin, non-quasi-alternating links, Math. Res. Lett. 17 (1): 39–49, 2010.
[8] A. Issa, The classification of quasi-alternating Montesinos links, Preprint: arXiv:1701.08425.
[9] S. Jablan, Tables of Quasi-alternating knots with at most 12 crossings, arXiv:1404.4965v2, 2014.
[10] L. Kaufman, New invariants in the theory of knots, Amer. Math. Monthly 3:195–242, 1988.
[11] L. Kaufman, Formal Knot Theory, Dover Publications, 2006.
[12] L. Kaufman and P. Lopes, Determinants of rational knots, Discrete Math. Theoret Comput. Sci., 11(2):111-122, 2009.
[13] C. Manolescu and P. Ozsváth, On the Khovanov and knot Floer homologies of quasi-alternating links. In Proceedings of Gökova Geometry-Topology Conference (2007), 60-81. Gökova Geometry/Topology Conference (GGT), Gökova, 2008.
[14] K. Murasugi, On closed 3-braids, Memoirs of the American Mathematical Society 151 (AMS, Providence, RI 1974).
[15] P. Ozsváth and Z. Szabó, On the Heegaard Floer homology of branched double-covers, Adv. Math. 194 (1):1-33, 2005.
[15] P. Ozsváth, J. Rasmussen and Z. Szabó, *Odd Khovanov homology*. Algebr. Geom. Topol. **13** (3): 1465-1488, 2013.

[16] K. Qazaqzeh and N. Chbili, *A new obstruction of quasi-alternating links*, Algebr. Geom. Topol. **15** (2015) 1847-1862.

[17] K. Qazaqzeh, N. Chbili, and B. Qublan, *Characterization of quasi-alternating Montesinos links*, J. Knot Theory Ramifications **24** (1):155000 (13 pages), 2015.

[18] K. Qazaqzeh, B. Qublan, and A. Jaradat, *A remark on the determinant of quasi-alternating links*, J. Knot Theory and its Ram. **22** (6): 1350031 (13 pages), 2013.

[19] M. Teragaito, *Quasi-alternating Links and $Q$-polynomials*, J. Knot Theory Ram. **23**, No.12, 1450068 (2014).

[20] M. Teragaito, *Quasi-alternating Links and Kauffman polynomials*, J. Knot Theory Ram. Vol. **24**, No. 07, 1550038 (2015).

[21] M. Thistlewaite, *A spanning tree expansion of the Jones polynomial*, Topology 26: 297-309, 1988.