COBLE’S QUESTION AND COMPLEX DYNAMICS OF INERTIA
GROUPS ON SURFACES

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Abstract. We study the inertia groups of some smooth rational curves on 2-elementary
K3 surfaces and singular K3 surfaces from the view of topological entropy, with an appli-
cation to a long standing open question of Coble on the inertia group of a generic Coble
surface.

1. Introduction

In this introduction, we assume that the base field is the complex number field \( \mathbb{C} \). For
an algebraic subset \( W \subset V \), we define

\[
\text{Dec}(W) = \text{Dec}(V, W) := \{ f \in \text{Aut}(V) \mid f(W) = W \},
\]

\[
\text{Ine}(W) = \text{Ine}(V, W) := \{ f \in \text{Dec}(V, W) \mid f|_W = \text{id}_W \}.
\]

We call the groups \( \text{Dec}(V, W) \) and \( \text{Ine}(V, W) \) the decomposition group of
\( W \subset V \) and the inertia group of \( W \subset V \). These two groups for curves on surfaces will play essential roles
in this paper.

Let \( C \subset \mathbb{P}^2 \) be a generic nodal sextic plane curve with ten nodes. The classical Coble
surface \( Y = Y_C \) is the blowings up \( \pi : Y = Y_C \rightarrow \mathbb{P}^2 \) at the ten nodes of \( C \). Let \( B \subset Y \) be the proper transform of \( C \). Then \( B \cong \mathbb{P}^1 \) and \( B \) is the unique element of \( | -2K_Y | \) by the adjunction formula and \( (-K_Y)^2 = -4 \). Therefore
\( f(B) = B \) for any \( f \in \text{Aut}(Y) \), i.e., \( \text{Aut}(Y) = \text{Dec}(Y, B) \) and we obtain a natural group homomorphism

\[
\rho_Y : \text{Aut}(Y) \rightarrow \text{Aut}(B) = \text{PGL}(2, \mathbb{C}) ; f \mapsto f|_B.
\]

This homomorphism is first considered by Coble and has attracted many authors since then
(See [AD18] for long history and see also [DZ01], [DK13] for other interesting aspects of
Coble surfaces). In particular, the following natural question ([Co19, Page 245], see also
[AD18, Section 4]) remains unsolved since Coble asked around 1919:

Question 1.1. Is \( \rho_Y \) injective? If otherwise, what can one say about \( \ker \rho_Y \), i.e., \( \text{Ine}(B) \)?

The primary aim of this paper is to give the following answer of alternative type to this
question:

Theorem 1.2. Either \( \rho_Y \) is injective or \( \text{Ine}(B) \) contains a non-commutative free subgroup
isomorphic to \( \mathbb{Z} \ast \mathbb{Z} \) and an element of positive entropy.
Our approach is in some sense indirect. Indeed, we deduce Theorem 1.2 from our study of 2-elementary K3 surfaces (Theorem 1.3). The notion of 2-elementary K3 surface is introduced by Nikulin [Ni81, Section 4].

We call a K3 surface \( X \) 2-elementary if \( \text{NS}(X) \lor \text{NS}(X) \lor (\mathbb{Z}/2\mathbb{Z})^a \) for some non-negative integer \( a = a(X) \). Then \( a(X) \leq \rho(X) := \text{rank} \text{NS}(X) \) and \( X \) has the involution \( \theta \) such that \( \theta^*|_{\text{NS}(X)} = \text{id}_{\text{NS}(X)} \), \( \theta^*\omega_X = -\omega_X \).

Here \( \omega_X \) is a nowhere vanishing holomorphic 2-form on \( X \). Note that \( \theta \) is in the center of \( \text{Aut}(X) \) (See [Ni81, Section 4] and Section 5 for basic properties of 2-elementary K3 surfaces).

Our actual main theorem is the following:

**Theorem 1.3.** Let \( X \) be a 2-elementary K3 surface such that \( \rho(X) + a(X) = 22 \), hence \( \rho(X) \geq 12 \). Then:

1. If \( \rho(X) \geq 12 \), then there exists a smooth rational curve \( C \subset X \) such that \( \text{Ine}(C) \) contains a non-commutative free subgroup isomorphic to \( \mathbb{Z} \lor \mathbb{Z} \) and an element of positive entropy.
2. If \( \rho(X) = 11 \), then \( C := X^\theta \) is a smooth rational curve \( \text{[Ni81, Section 4]} \) and \( \text{Ine}(C) \) contains a non-commutative free subgroup isomorphic to \( \mathbb{Z} \lor \mathbb{Z} \) and an element of positive entropy unless \( \text{Ine}(C) = \{\text{id}_X, \theta\} \).

Let us return back to our classical Coble surface \( Y \). Consider the finite double cover \( p : \tilde{Y} \to Y \) branched along \( B \in |-2K_Y| \). Then \( \tilde{Y} \) is a 2-elementary K3 surface of Picard number \( \rho(\tilde{Y}) = 11 \) and \( a(\tilde{Y}) = 11 \) with the covering involution of \( p \) as \( \theta \). Moreover, we have

\[
\text{Aut}(\tilde{Y}) = \text{Dec}(\tilde{Y}, \tilde{B}) , \text{Dec}(Y, B) = \text{Aut}(Y) = \text{Aut}(\tilde{Y})/\langle \theta \rangle .
\]

Here \( \tilde{B} \simeq B \simeq \mathbb{P}^1 \) is the ramification divisor of \( p \), i.e., \( \tilde{B} = \tilde{Y}^\theta \). Theorem 1.2 is then an obvious consequence of Theorem 1.3 (2).

Our theorem 1.3 (1) is a generalization of [Og18, Theorem 1.2] and also gives a complete affirmative answer to the question in [Og18, Remark 4.4] when \( \rho(X) \geq 12 \).

It happens that \( \text{Aut}(X) \) has no element of positive entropy for some 2-elementary K3 surface \( X \). For instance, any generic K3 surface of degree 2 is a 2-elementary K3 surface with \( \text{Aut}(X) \simeq \mathbb{Z}/2\mathbb{Z} \) (so that no automorphism of positive entropy). The condition \( \rho(X) + a(X) = 22 \) is the condition that guarantees that \( X^\theta \neq \emptyset \) and \( \text{Aut}(X) \) has an element of positive entropy (see Lemma 5.1).

It is also interesting to consider a similar question for inertia groups of singular K3 surfaces, i.e., complex K3 surfaces of maximum Picard number 20 ([SI77]). Recall that the automorphism group of a singular K3 surface always contains an element of positive entropy ([Og07, Theorem 1.6 (1)]).

In this direction, we have the following answer, which is also a generalization of [Og18, Theorem 1.3] (See also [Og18, Remark 5.5]):

**Theorem 1.4.** Every singular K3 surface \( X \) has a smooth rational curve \( C \) such that \( \text{Ine}(C) \) contains a non-commutative free subgroup isomorphic to \( \mathbb{Z} \lor \mathbb{Z} \) and an element of positive entropy.
Theorems 1.3 and 1.4 are proved in a fairly uniform way in Sections 4, 5 as an application of general criteria on the existence of positive entropy element in an inertia group of a K3 surface (Theorem 3.4 and Corollaries 3.5, 3.7) in Section 3. Proof of these criteria are based on the Tits’ alternative type result for K3 surface automorphism groups ([Og06], [Og07], see also Theorem 3.2). We believe that these criteria will be also applicable for dynamical studies of other K3 surfaces. In Section 2, we prove some constraint of the existence of an element of positive entropy in an inertia group Ine \((S, C)\) of a curve \(C\) on a smooth projective surface \(S\) (Theorem 2.1). This will explain one of the reason why we may seek for the inertia group of a smooth rational curve on a K3 surface.

**Acknowledgements.** This work has been done during the authors’ stay at KIAS on March 2019. We would like to thank for KIAS and Professors Jun-Muk Hwang and JongHae Keum for invitation, discussions and warm hospitalities. We would like to thank Professor Igor Dolgachev for sending us a very interesting preprint [AD18] by which our work is much inspired.

2. **Existence of elements of positive entropy in inertia groups of smooth projective surfaces**

We call an irreducible reduced projective curve \(C\) simply a curve. We do not assume that \(C\) is smooth. We denote by \(p_a(C) = h^1(C, \mathcal{O}_C) = h^0(C, \omega_C)\) the arithmetic genus of \(C\) and by \(g(C)\) the geometric genus of the normalization \(\tilde{C}\) of \(C\), i.e., \(g(C) = h^1(C, \mathcal{O}_C) = h^0(\tilde{C}, \omega_{\tilde{C}})\).

Let \(S\) be a smooth projective surface defined over an algebraically closed field \(k\) and let \(C \subset S\) be a curve.

We denote by \(\text{Bir}(S)\) the birational automorphism group of \(S\), i.e., the group of birational selfmaps of \(S\). We define the subgroups \(\text{BirDec}(C)\) and \(\text{BirIne}(C)\) of \(\text{Bir}(S)\) called the (birational) decomposition group and the (birational) inertia group of \(C \subset S\) by

\[
\text{BirDec}(C) := \{ f \in \text{Bir}(S) \mid f_*(C) = C \},
\]

\[
\text{BirIne}(C) := \{ f \in \text{BirDec}(S) \mid f|_C = \text{id}_C \}.
\]

Here \(f_*(C)\) is the proper transform of \(C\), i.e., the Zariski closure of \(f(C \setminus I(f))\) where \(I(f)\) is the indeterminacy locus of \(f\). Note that \(I(f)\) consists of at most finitely many closed points. So, the condition \(f_*(C) = C\) and the condition \(f|_C = \text{id}_C\) are well-defined conditions for \(f \in \text{Bir}(S)\). By definition, we have a natural group homomorphism

\[
\tilde{\rho} : \text{BirDec}(C) \to \text{Aut}(C) \ ; \ f \mapsto f|_C
\]

and

\[
\text{BirIne}(C) = \ker \tilde{\rho}.
\]

In particular, \(\text{BirIne}(C)\) is a normal subgroup of \(\text{BirDec}(C)\). We denote by \(\text{Aut}(S)\) the biregular automorphism group of \(S\) and define

\[
\text{Dec}(C) := \text{BirDec}(C) \cap \text{Aut}(S), \quad \text{Ine}(C) := \text{BirIne}(C) \cap \text{Aut}(S).
\]

By restricting \(\tilde{\rho}\) to \(\text{Dec}(C)\), we obtain the group homomorphism

\[
\rho = \tilde{\rho}|_{\text{Dec}(C)} : \text{Dec}(C) \to \text{Aut}(C).
\]
Then again, by definition,
\[ \text{Ine}(C) = \ker \rho \]
and Ine(C) is also a normal subgroup of Dec(C).

Coble’s question ([Co19, Page 245], see also [AD18, Section 4]) is the one asking the complexity of the actions of
\[ \text{Im} \bar{\rho}, \text{BirIne}(C) = \ker \bar{\rho}, \text{Im} \rho, \text{Ine}(C) = \ker \rho \]
on \( C \subset S \).

Recall that the first dynamical degree \( d_1(f) \) of \( f \in \text{Bir}(S) \) is a fundamental measure of the complexity of the action of the itarations \( f^n (n \in \mathbb{Z}_{\geq 0}) \). It is defined by
\[
d_1(f) := \lim_{n \to \infty} \left| \left( f^n \right)^* \right|_{\text{End}_{\mathbb{R}}(\text{NS}(S)_\mathbb{R})}^{1/n}.
\]
Here \( \text{NS}(S)_\mathbb{R} = \text{NS}(S) \otimes \mathbb{R} \) and \( |*|_{\text{End}_{\mathbb{R}}(\text{NS}(S)_\mathbb{R})} \) is any norm of the vector space \( \text{End}_{\mathbb{R}}(\text{NS}(S)_\mathbb{R}) \) consisting of the linear selfmaps of \( \text{NS}(S)_\mathbb{R} \). By the Gromov-Yomdin’s theorem, the topological entropy of \( f \in \text{Aut}(S) \) (for a smooth complex projective surface \( S \)) is given by
\[
h_{\text{top}}(f) = \log d_1(f).
\]
See [DS05, Pages 1637–1639] for generalities of dynamical degrees and entropy (see also [DF01] for surface case and [ES13] in positive characteristic). Taking this into account, we set
\[
h(f) := \log d_1(f)
\]
also for \( f \in \text{Bir}(S) \) and call \( h(f) \) the algebraic entropy (or just entropy) of \( f \). We are particularly interested in the existence of positive entropy element of BirIne(C) and Ine(C).

The following theorem, which should be known to the experts, shows that the existence of positive entropy element of BirIne(C) already poses a fairly strong constraint on the pair \( C \subset S \):

**Theorem 2.1.** Let \( S \) be a smooth projective surface defined over an algebraically closed field of characteristic \( 0 \) and let \( C \subset S \) be a curve (hence irreducible and reduced by our conventions) on \( S \). Assume that there exists \( f \in \text{BirIne}(S) \) such that \( h(f) > 0 \). Then, one of the following (1) or (2) holds:

1. \( S \) is birational to a K3 surface or an Enriques surface and \( C \) is a smooth rational curve;
2. \( S \) is a rational surface and \( g(C) = 0 \) or \( 1 \). In particular, \( C \) is either a rational curve or an elliptic curve.

**Remark 2.2.**

1. There exist a smooth rational surface and a smooth elliptic curve \( C \subset S \) such that Ine(C) has an element of positive entropy. This is proved by Blanc [BI13, Section 2].
2. There exist a smooth rational surface and a rational curve \( C \subset S \) such that \( p_a(C) \geq 2 \) (hence singular) and BirIne(C) has an element of positive entropy. For instance, Allcock and Dolgachev [AD18, Theorem 6.2] give some explicit examples. This paper is much inspired by their examples.
Proof. The existence of \( f \in \text{Bir}(S) \) with \( h(f) > 0 \) implies that \( S \) is birational to either (i) an abelian surface, (ii) a K3 surface, (iii) an Enriques surface or (iv) a rational surface.

In fact, if the Kodaira dimension \( \kappa(S) \) is greater than or equal to 0, then the minimal model \( S_{\text{min}} \) of \( S \) is unique up to isomorphisms and \( f \) induces a birational automorphism \( f_{\text{min}} \in \text{Aut}(S_{\text{min}}) \). By the birational invariance of the dynamical degrees due to Dinh-Sibony [DS03, Corollaire 7] (See also [DF01] for surface case), we have

\[
h_{\text{top}}(f_{\text{min}}) = h(f_{\text{min}}) = h(f) > 0.
\]

Therefore, if \( \kappa(S) \geq 0 \), then \( S_{\text{min}} \) is a surface in (i)-(iii) by [Ca99, Proposition 1].

If \( \kappa(S) = -\infty \), then \( S \) is either a rational surface or a birationally ruled surface \( \pi : S \to C \) over a smooth curve \( C \) of \( g(C) > 0 \). In the second case, the fibration \( \pi \) is preserved by \( \text{Bir}(S) \), because all rational curves on \( S \) are in fibers of \( \pi \) by \( g(C) > 0 \). Therefore, in the second case, \( h(f) = 0 \) for all \( f \in \text{Bir}(S) \) by the product formula due to Dinh-Nguyen [DNT1, Theorem 1.1].

Thus, \( S \) is birational to a surface in (i)-(iv).

First we consider the cases (i), (ii), (iii).

Let \( \pi : S \to T := S_{\text{min}} \) be the minimal model of \( S \). Then \( \pi \) is a composition of blowings down of (-1)-curves. As remarked above, \( f \in \text{Bir}(S) \) descends to \( f_T \in \text{Aut}(T) \) equivariantly with respect to \( \pi \) and \( h(f_T) = h(f) > 0 \). Set \( C_T := \pi(C) \).

Assume first that \( C_T \) is a point. Then \( C \) is one of the exceptional curves of \( \pi \). Therefore \( C \cong \mathbb{P}^1 \) on \( S \).

We show that \( T \) is not an abelian surface in this case. Assuming otherwise, we choose \( C_T = O \) as the origin of \( T \). We may assume without loss of generality that \( T \) is a complex abelian surface, hence it is a complex 2-torus \( T = \mathbb{C}^2/\Lambda \) and that the exceptional locus of \( \pi : S \to T \) is connected. The automorphism \( f_T \) is then a group automorphism, represented by a linear \( 2 \times 2 \)-matrix, say \( \Lambda \), with respect to the natural holomorphic coordinates \((z_1, z_2)\) of the universal covering space \( \mathbb{C}^2 \). We denote by \( \alpha \) and \( \beta \) the eigenvalues of \( \Lambda \) and arrange so that \(|\alpha| \geq |\beta|\). Let \( \omega_T \) be a nowhere vanishing holomorphic 2-form on \( T \). Then \( f_T^* \omega_T = \alpha \beta \omega_T \). Hence \(|\alpha \beta| = 1\) by the finiteness of the canonical representation of a smooth complex projective variety ([Ue75, Theorem 14.10]). The positivity of the entropy says that the spectral radius of \( f_T^*|_{H^{1,1}(T)} \) is strictly greater than 1. Hence \(|\alpha| > 1 \) and therefore

\[|\alpha| > 1 > |\beta| .\]

Let \( E_1 \) be the exceptional curve of the first blow-up \( T_1 \to T \) at \( O \) in \( \pi \). Then the action of \( f_{T_1} \) on \( T_1 \) preserves \( E_1 \) and has a fixed point \( P_1 \) over which our \( C \) lies. By the property of the blow up, the action of \( f_{T_1}|_{E_1} \) is either one of:

\[
x_1 \mapsto \frac{\alpha}{\beta} \cdot x_1 , \quad x_1 \mapsto \frac{\beta}{\alpha} \cdot x_1
\]

under a suitable affine coordinate \( x_1 \) of \( E_1 \) at \( P_1 \). More precisely, the action \( f_{T_1} \) at \( P_1 \in T_1 \) is biregular and of the form \((x_1, y_1) \mapsto (a_1 x_1, b_1 y_1)\) which is either one of:

\[
(x_1, y_1) \mapsto \left( \frac{\alpha}{\beta} x_1, \beta y_1 \right) , \quad (x_1, y_1) \mapsto \left( \frac{\beta}{\alpha} x_1, \alpha y_1 \right)
\]

under suitable local coordinates \((x, y)\) of \( T_1 \) at \( P_1 \) such that \( E_1 = (y_1 = 0) \) at \( P_1 \). Here we have still either \(|a_1| > 1 > |b_1|\) or \(|b_1| > 1 > |a_1|\). Let \( E_2 \) be the exceptional curve of the
second blow-up $T_2 \to T_1$ at $P_1$ in $\pi$. Then the action of $f_{T_2}$ on $T_2$ of $f$ preserves $E_2$ and has a fixed point $P_2$ over which our $C$ lies. For the same reason as above, the action $f_{T_2}$ at $P_2 \in T_2$ is birational and of the form $(x_2, y_2) \mapsto (a_2 x_2, b_2 y_2)$ which is either one of:

$$(a_1 \frac{x_2}{b_1}, b_2 y_2), \quad (x_2, y_2) \mapsto (b_1 a_1, a_1 y_2)$$

under suitable local coordinates $(x_2, y_2)$ of $T_2$ at $P_2$ such that $E_2 = (y_2 = 0)$ at $P_1$. So, the same condition either $|a_2| > 1 > |b_2|$ or $|b_2| > 1 > |a_2|$ still holds. Now one can repeat this process inductively by reaching the stage that $C$ appears as the exceptional curve $E_n$ of the blow up $T_n \to T_{n-1}$ in $\pi$. The induced action of $f_{T_n}$ on $T_n$ of $f$ preserves $E_n$ and $f_{T_n}|_{E_n}$ is then the multiplication by either

$$\frac{a_{n-1}}{b_{n-1}}, \quad \frac{b_{n-1}}{a_{n-1}}.$$

However, by induction, we see that either $|a_{n-1}| > 1 > |b_{n-1}|$ or $|b_{n-1}| > 1 > |a_{n-1}|$ holds, so that $f_{T_n}|_{E_n} \neq id_{E_n}$. However, this contradicts to the fact that $f|_C = id_C$. Indeed, the actions of $f_{T_n}$ on the generic scheme point of $E_n$ and the action of $f$ on the generic scheme point of $C$ have to be the same. So, $T$ is not an abelian surface when $C_T$ is a point.

Assume that $C_T$ is a curve. Then $f_T^*(C_T) = C_T$ in $NS(T)$. As $h(f_T) > 0$, it follows that $C_T^2 < 0$. Here we used the fact that $f_T^*$ is of finite order if $C_T^2 > 0$ and the eigenvalues of $f_T^*$ are all on the unit circle $S^1$ if $C_T^2 = 0$ and $C_T$ is non-zero effective (See eg. [Og07, Lemma 2.8]).

Since there is no curve with negative self-intersection on an abelian surface, $T$ is not an abelian surface, either. On the other hand, on a K3 surface and an Enriques surface, any curve with negative self-intersection is exactly a $(-2)$-curve and it is isomorphic to $\mathbb{P}^1$. Hence $C_T \simeq \mathbb{P}^1$ on $T$ and hence $C \simeq \mathbb{P}^1$ when $S$ is birational to a K3 surface or an Enriques surface.

We now consider the case where $S$ is a rational surface. Since the statement and the conclusion are birationally invariant ones, we may assume without loss of generality that $S = \mathbb{P}^2$. If $g(C) \geq 2$ and $f \in Bir\text{Ine}(C)$, then, by Castelnuovo’s theorem (See [BPV08 Théorème 1.1] for the statement and a mordern proof), $f$ is either of finite order or a de Jongquiers transformation, which is a birational self map of $\mathbb{P}^2$ preserving a pencil of rational curves. However, then $h(f) = 0$, respectively by the definition of $h(f)$ and by the product formula as above, a contradiction to $h(f) > 0$. Hence $g(C) \leq 1$ as claimed.

This completes the proof of Theorem 2.1. \hfill \square

3. A CRITERION OF THE EXISTENCE OF ELEMENTS OF POSITIVE ENTROPY IN INERTIA GROUPS OF K3 SURFACES

Our main results of this section are Theorem 3.4 and Corollaries 3.5 and 3.7

A group $G$ is called almost abelian, if $G$ is isomorphic to an abelian group up to finite kernel and finite cokernel. More precisely, a group $G$ is called almost abelian if there exist a normal subgroup $G^{(0)}$ of $G$ such that $[G : G^{(0)}] < \infty$ and a finite normal subgroup $K < G^{(0)}$ such that the quotient group $G^{(0)}/K$ is an abelian group. We call $G$ almost abelian of rank $r$ if in addition that we can make $G^{(0)}/K \simeq \mathbb{Z}^\oplus r$. The rank $r$ is well-defined (See eg. [Og08 Section 8]).
Definition 3.1. Let $S$ be a smooth projective surface and let $\phi : S \to B$ a surjective morphism to a smooth projective curve $B$ with connected fibers. We call $\phi$ a genus one fibration if general fibers are of arithmetic genus one. We call $\phi$ an elliptic fibration if general fibers are smooth elliptic curve and $\phi$ admits a global section.

Recall from [Og06, Theorem 1.1] and [Og07, Theorem 1.3] the following:

Theorem 3.2. Let $S$ be a projective K3 surface defined over any algebraically closed field $k$ of characteristic $p \neq 2, 3$. Let $G$ be any subgroup of $\text{Aut}(S)$. Then:

1. Either (i) $G$ is an almost abelian group, necessarily of finite rank, or (ii) $G$ contains a subgroup isomorphic to the non-commutative free group $\mathbb{Z} \ast \mathbb{Z}$, and the two cases (i) and (ii) are exclusive each other.
2. $G$ has an element of positive entropy in the case (ii).
3. In particular, if $G$ has no element of positive entropy, then $G$ is almost abelian, i.e., $G$ belongs to the case (i) (by (1) and (2)). Moreover, if $|G| = \infty$, then $G$ has no element of positive entropy if and only if $G$ preserves a genus one fibration $S \to \mathbb{P}^1$.

Remark 3.3. Statements of [Og06, Theorem 1.1] and [Og07, Theorem 1.3] are formulated over $\mathbb{C}$. However, proofs there are valid without any change over any algebraically closed field $k$ under the assumption that $p \neq 2, 3$. The assumption $p \neq 2, 3$ is used only to guarantee that a general fiber of a genus one fibration is a smooth elliptic curve and that the sum of Euler numbers of singular fibers is exactly 24, the Euler number of a K3 surface, to deduce that a genus one fibration has always at least three singular fibers.

In this section, we prove the following:

Theorem 3.4. Let $S$ be a projective K3 surface defined over any algebraically closed field $k$ of characteristic $p \neq 2, 3$ and $C \subset S$ be a smooth rational curve. Assume that $\text{Dec}(C)$ is not almost abelian and that $|\text{Ine}(C)| = \infty$. Then $\text{Ine}(C)$ contains a non-commutative free subgroup isomorphic to $\mathbb{Z} \ast \mathbb{Z}$ and an element of positive entropy.

Proof. If $\text{Ine}(C)$ has no element of positive entropy, then, since $|\text{Ine}(C)| = \infty$, the group $\text{Ine}(C)$ preserves a genus one fibration $\Phi_{|F|} : S \to \mathbb{P}^1$ and $\text{Ine}(C)$ is almost abelian of positive finite rank by Theorem 3.2 (3). Here $F$ be a general fiber of $\Phi_{|F|}$.

Let $g \in \text{Dec}(C)$. We have

$$g^{-1}\text{Ine}(C)g = \text{Ine}(C),$$

because $\text{Ine}(C)$ is a normal subgroup of $\text{Dec}(C)$. Thus, $\text{Ine}(C)$ also preserves a genus one fibration $\Phi_{|g^*F|} : S \to \mathbb{P}^1$.

Assume that there is $g \in \text{Dec}(C)$ such that $g^*F \neq F$ in $\text{Pic}(S) \simeq \text{NS}(S)$. Then the action of $\text{Ine}(C)$ on $\text{Pic}(S)$ has to preserve the class $g^*F + F$. However, since

$$(g^*F + F)^2 = 2(g^*F,F) > 0,$$

the action of $\text{Ine}(C)$ on $\text{Pic}(S)$ is finite (See eg. [Og07, Lemma 2.8]), hence the group $\text{Ine}(C)$ is a finite group, a contradiction.

Now we may assume that $g^*F = F$ in $\text{Pic}(S) \simeq \text{NS}(S)$ for all $g \in \text{Dec}(C)$. Then the group $\text{Dec}(C)$ preserves a genus one fibration $\Phi_{|F|} : S \to \mathbb{P}^1$. However, then $\text{Dec}(C)$ is almost abelian of finite rank by Theorem 3.2 (3), a contradiction to the assumption that $\text{Dec}(C)$ is not almost abelian.
Hence there is an element \( f \in \text{Ine}(C) \) such that \( h(f) > 0 \), as claimed. \( \square \)

We denote \( \text{Dec}(C, P) := \{ g \in \text{Dec}(C) \mid g(P) = P \} \) for \( P \in C \subset S \) for a smooth projective surface \( S \), a curve \( C \subset S \) and a closed point \( P \in C \). Then \( \text{Dec}(C, P) \) is a subgroup of \( \text{Dec}(C) \) and \( \text{Ine}(C) \) is a subgroup of \( \text{Dec}(C, P) \).

**Corollary 3.5.** Let \( S \) be a projective K3 surface defined over any algebraically closed field \( k \) of characteristic \( p \neq 2, 3 \) and let \( C \subset S \) be a smooth rational curve. Assume that \( \text{Dec}(C, P) \) is not almost abelian for some point \( P \in C \). Then \( \text{Ine}(C) \) contains a non-commutative free subgroup isomorphic to \( \mathbb{Z} \times \mathbb{Z} \) and an element of positive entropy.

**Proof.** Note that \( \text{Aut}(C, P) := \{ f \in \text{Aut}(C) \mid f(P) = P \} \) is isomorphic to the group of affine linear transformations \( f(z) = az + b \) (\( a \in k^\times \), \( b \in k \)) of the affine line \( k \). In particular, \( \text{Aut}(C, P) \) fits in with the exact sequence

\[
0 \to k \to \text{Aut}(C, P) \to k^\times \to 1.
\]

Therefore \( \text{Aut}(C, P) \) is solvable, as so are \( k \) and \( k^\times \) (indeed, both are abelian groups). From the natural representaion \( \rho : \text{Dec}(C, P) \to \text{Aut}(C, P) \) defined by \( f \mapsto f|_{C} \), we obtain the exact sequence

\[
1 \to \text{Ine}(C) \to \text{Dec}(C, P) \to \text{Im} \rho \to 1.
\]

Since \( \text{Dec}(C, P) \) is not almost abelian, \( \text{Dec}(C, P) \) contains a subgroup \( G \) isomorphic to \( \mathbb{Z} \times \mathbb{Z} \) by Theorem 3.2 (1). The group \( \rho(G) \) is solvable, as it is a subgroup of the solvable group \( \text{Aut}(C, P) \). Since \( G \) is not solvable, it follows then that \( \text{Ker}(\rho|_{G}) \) is not isomorphic to \( \{0\} \) nor \( \mathbb{Z} \). On the other hand, since \( G \) is a free group, the subgroup \( \text{Ker}(\rho|_{G}) \) is also a free group by the Nielsen-Schreier theorem. Thus \( \text{Ker}(\rho|_{G}) \), which is a free group other than \( \{0\} \) and \( \mathbb{Z} \), has a subgroup isomorphic to \( \mathbb{Z} \times \mathbb{Z} \). Since \( \text{Ker}(\rho|_{G}) \subset \text{Ine}(C) \), it follows that \( \text{Ine}(C) \) also contains a free subgroup isomorphic to \( \mathbb{Z} \times \mathbb{Z} \). Hence, by Theorem 3.2 (2), \( \text{Ine}(C) \) has an element of positive entropy. \( \square \)

**Remark 3.6.** Let \( \Phi_{[F]} : S \to \mathbb{P}^1 \) be a genus one fibration on a K3 surface over an algebraically closed field of characteristic \( p \neq 2, 3 \). The associated Jacobian fibration \( J(\Phi_{[F]}) : J(S) \to \mathbb{P}^1 \) is defined by the relatively minimal model of the compactification of the Jacobian \( \text{Pic}^0(S_{\eta}) \) of the scheme generic fiber \( S_{\eta} \) of \( \Phi_{[F]} \). Then \( J(\Phi_{[F]}) \) is an elliptic fibration. Then

\[
\text{MW}(J(\Phi_{[F]}) := S_{\eta}(k(\mathbb{P}^1))
\]

forms a finitely generated abelian group called the Mordell-Weil group of \( J(\Phi_{[F]}) \) (See [Shi90] for the basic properties of Mordell-Weil groups). The Mordell-Weil group \( \text{MW}(J(\Phi_{[F]})) \) faithfully acts on \( \Phi_{[F]} : S \to \mathbb{P}^1 \) over \( \mathbb{P}^1 \) through the translation action of \( \text{Pic}^0(S_{\eta}) \) on \( S_{\eta} \). Note also that \( \text{MW}(J(\Phi_{[F]})) \) is a finite index abelian subgroup of the group \( \text{Aut}(S)|_{[F]} \) in [Yu18] Section 2.2. This is because \( \Phi_{[F]} \) has at least three singular fibers, so that \( |\text{Im}(\text{Aut}(S)|_{[F]} \to \text{Aut}(\mathbb{P}^1))| < \infty \), and also \( |\text{Aut}_{\text{group}}(S_{\eta})| < \infty \) as \( S_{\eta} \) is a smooth elliptic curve (cf. Remark 3.3).

In what follows, we denote \( \text{MW}(J(\Phi_{[F]})) \) simply by \( \text{MW}(\Phi_{[F]} \) and call the Mordell-Weil group of \( \Phi_{[F]} \) whenever we regard \( \text{MW}(J(\Phi_{[F]})) \subset \text{Aut}(S) \) in the way explained here.

The next corollary will be frequently used in Sections 4 and 5.
Corollary 3.7. Let $S$ be a projective K3 surface defined over any algebraically closed field $k$ of characteristic $p \neq 2, 3$. Assume that there exist a smooth rational curve $C$, smooth rational curves $R_1, R_2$ and effective divisors $D_1$ and $D_2$ possibly $0$ such that

1. The complete linear system $|E_i|$, where

$$E_i := D_i + a_i R_i + b_i C$$

with suitable positive integers $a_i > 0$ and $b_i > 0$ is free and defines a genus one fibration

$$\Phi_i := \Phi_{|E_i|} : S \to \mathbb{P}^1$$

of positive Mordell-Weil rank for $i = 1$ and $2$.

2. Moreover, $\Phi_i$ ($i = 1, 2$) are different genus one fibrations, that is, the two classes $E_i$ ($i = 1, 2$) are not proportional in $\text{Pic}(X) \simeq \text{NS}(X)$.

3. $C \neq R_1, C \neq R_2$ (possibly $R_1 = R_2$) and $C \cap R_1 \cap R_2 \neq \emptyset$.

Then $\text{Ine}(C)$ contains a non-commutative free subgroup isomorphic to $\mathbb{Z} \ast \mathbb{Z}$ and an element of positive entropy.

Proof. Let us choose a point $P \in C \cap R_1 \cap R_2$, which exists by the assumption (3). By the assumption (1), we can choose an element $f_i \in \text{MW}(\Phi_i)$ ($i = 1, 2$) of infinite order (and by replacing it by a suitable power if necessary) such that $f_i(C) = C$ and $f_i(P) = P$. Then $G := \langle f_1, f_2 \rangle$ is a subgroup of $\text{Dec}(C, P)$.

We now claim that $G$ does not preserves any genus one fibration on $S$. Our proof below is a slight modification of [Og06, Theorem 1.2] in which the existence of rational sections are assumed.

Indeed, otherwise, there is a genus one fibration $\Phi_{|E_i|} : S \to \mathbb{P}^1$ preserving by both $f_1$ and $f_2$. Here $E$ is a genus one curve on $S$. By renumbering $i = 1$ and $2$ if necessary, we may assume without loss of generality that $\Phi_{|E_1|}$ is a different genus one fibration from $\Phi_2$. Here we used the assumption (2). Then the classes $E$ and $E_2$ are nef. Since $E$ and $E_2$ are not proportional, it follows that

$$B := E + E_2$$

is a nef and big class in $\text{Pic}(S) \simeq \text{NS}(S)$. By the definition of $B$, the class $B$ is preserved by $f_2$. However, then, $f_2$ would be of finite order (see eg. [Og07, Lemma 2.8]). This contradicts to the fact that $f_2$ is of infinite order. Hence $G$ does not preserve any genus one fibration on $S$.

Note that $|G| = \infty$. Then $G$ is not almost abelian by Theorem 3.2 (3). Note that any subgroup of an almost abelian group is again almost abelian. Thus $\text{Dec}(C, P)$ is not almost abelian, either. The result now follows from Corollary 3.5. $\square$

4. Singular K3 case

In this section, we work over the complex number field $\mathbb{C}$. Our main result of this section is the following Theorem 4.1 which is the same as Theorem 1.4 in Introduction. In this section and the next section, we use Kodaira’s notation of singular fibers of genus one fibrations (See [Ko63, Page 565] for the notation). In this section and the next section, by our definition (Definition 3.1), an elliptic fibration always means a genus one fibration with a global section.
Theorem 4.1. Let $X$ be a singular K3 surface. Then $X$ contains a smooth rational curve $C$ such that $\text{Ine}(C)$ contains a non-commutative free subgroup isomorphic to $\mathbb{Z} \ast \mathbb{Z}$ and an element of positive entropy.

Proof. Let $d$ be the discriminant of $X$, i.e., the absolute value of the determinant of the Néron-Severi lattice $\text{NS}(X)$, which is also the determinant of the transcendental lattice $T(X)$ of $X$. Then $d \leq 3$ and $X$ with the smallest two cases $d = 3$ and $d = 4$ are explicitly described in [SI77]. In the case $d = 3$, Theorem 4.1 is proved in [Og18, Theorem 1.3]. In the case $d = 4$, the surface $X$ is a 2-elementary K3 surfaces with $\rho(X) = 20$ and $a(X) = 2$ and Theorem 4.1 in this case will be proved in Section 5.

In what follows, we assume $d \neq 3, 4$ and we will closely follow the construction in [Og07, Page184].

First one can find two elliptic fibrations whose Mordell-Weil groups are infinite. In fact, the Néron-Severi lattice $\text{NS}(X)$ of $X$ of the form:

$$\text{NS}(X) = U \oplus E_8^{\oplus 2} \oplus N,$$

where $N$ is a negative definite lattice of rank 2. The lattice $U$ is an even unimodular hyperbolic lattice of rank 2 and $E_8$ is the even unimodular negative definite lattice of rank 8. Note then that $d = \det(N)$. Using this description of $\text{NS}(X)$, one finds an elliptic fibration $\varphi : X \rightarrow \mathbb{P}^1$ whose reducible singular fibers are either

$$II^* + II^* , \quad II^* + II^* + I_2 , \quad II^* + II^* + III.$$  

See e.g. [Kon92, Lemma 2.1]. Then rank of Mordell-Weil group of $\varphi$ is positive by the Shioda-Tate formula [Shi90, Corollary 5.3]. In particular, $\varphi$ admits at least two sections, say $D_1$ and $D_2$. Join two $II^*$ singular fibers of $\varphi_1$ by the section $D_1$ (resp. $D_2$) and throw out the components of multiplicity 2 at the edge of two $II^*$. Then one obtains a nef divisor of Kodaira’s type $I_1^* + II$, say $E_1$ (resp. $E_2$), on $X$. The pencil $|E_1|$ (resp. $|E_2|$ ) gives rise to an elliptic fibration $\varphi_1 : X \rightarrow \mathbb{P}^1$ (resp. $\varphi_2$). The two smooth rational curves $S_1$ and $S_2$ thrown out are actually the sections of $\varphi_i$ $(i = 1, 2)$. We regard $S_1$ as the zero of $\text{MW}(\varphi_i)$ and denote $P_i \in \text{MW}(\varphi_i)$ the element corresponding to $S_i$. We regard $S_1$ as the zero of $\text{MW}(\varphi_i)$ and denote $P_i \in \text{MW}(\varphi_i)$ the element corresponding to $S_i$. Then

$$\text{rk} \text{MW}(\varphi_i) > 0$$

for $i = 1$ and 2 by [Og07, page 184]. In fact, we checked there that

$$\langle P_i, P_i \rangle > 0$$

for the height pairing on the Mordell-Weil group $\text{MW}(\varphi_i)$ (See [Shi90, Theorem 8.4, Definition 8.5] for the definition and [Shi90, Theorem 8.6, Table (8.16)] for the explicit formula we used). Thus $P_i \in \text{MW}(\varphi_i)$ is of infinite order for $i = 1, 2$ by [Shi90, Page 228, (8.10)].

Let us choose two irreducible components $C$ and $C'$ of $\text{Supp}E_1 \cap \text{Supp}E_2$

such that $C \neq C'$ and $C \cap C' \neq \emptyset$. Such $C$ and $C'$ certainly exist, Now one can apply Corollary 5.4 for $C$, $R_1 = R_2 = C'$ and $E_i$ $(i = 1, 2)$ to conclude the desired result. \qed
5. 2-elementary K3 case

In this section, we assume that the base field is the complex number field \( \mathbb{C} \). The main results of this section are Theorem 5.5 and Theorem 5.8 which are Theorem 1.3 (1), (2).

Let \( X \) be a 2-elementary K3 surface, i.e., \( X \) is a complex projective K3 surface such that \( \text{NS}(X)^*/\text{NS}(X) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus a} \), for some non-negative integer \( a = a(X) \), with Picard number \( \rho = \rho(X) \). 2-elementary K3 surfaces are extensively studied by Nikulin. See for [Ni81, Section 4] about basic facts on 2-elementary K3 surfaces we will use. Since \( \text{NS}(X) \) is primitively embedded into the unimodular lattice \( H^2(X, \mathbb{Z}) \), it follows that \( \rho + a \leq 22 \).

The K3 surface \( X \) has an automorphism \( \theta \) of order 2 such that \( \theta^*|_{\text{NS}(X)} = \text{id}_{\text{NS}(X)} \), \( \theta^* \omega_X = -\omega_X \).

Here \( \omega_X \) is a nowhere vanishing holomorphic 2-form on \( X \). The involution \( \theta \) is in the center of \( \text{Aut}(X) \) and the fixed locus of \( \theta \) is preserved under \( \text{Aut}(X) \). Following [Ni81, Definition 4.2.1], the 2-elementary lattice \( \text{NS}(X) \) has an invariant \( \delta \), where \( \delta = 0 \) or \( 1 \).

**Lemma 5.1.** ([Og18, Remark 4.4]) If \( X \) contains a smooth rational curve and \( \text{Aut}(X) \) has an element of positive entropy, then \( \rho + a = 22 \).

**Proof.** Suppose \( X \) contains a smooth rational curve and \( \text{Aut}(X) \) has an element of positive entropy. Then \( (\rho, a, \delta) \neq (10, 10, 0) \) (otherwise, \( \text{NS}(X) \cong U(2) \oplus E_8(2) \) and \( X \) contains no smooth rational curve, a contradiction) and \( (\rho, a, \delta) \neq (10, 8, 0) \) (otherwise, by [Ni81, Theorem 4.2.2], \( X^\theta \) is the disjoint union of two smooth elliptic curves, then, by [Og07, Theorem 1.4], \( \text{Aut}(X) \) has no element of positive entropy, a contradiction). Then, by [Ni81, Theorem 4.2.2] again,

\[
X^\theta = H + \sum_{i=1}^{k} C_i
\]

where \( H \) is a smooth projective curve of genus \( g_H = (22 - \rho - a)/2 \), \( C_i \) are smooth rational curves, and \( k = (\rho - a)/2 \). If \( \rho + a < 22 \), then \( g_H > 0 \) and, by [Og07, Theorem 1.4], \( \text{Aut}(X) \) no element of positive entropy, a contradiction. Thus, \( \rho + a = 22 \). This completes the proof of the lemma. \( \Box \)

We are interested in an inerita group with an element of positive entropy (see Theorem 2.1). Thus, in the rest of this section, we assume that

\[
\rho + a = 22.
\]

Then the locus of fixed points of \( \theta \)

\[
X^\theta = \bigcup_{i=1}^{k} C_i
\]

where, \( k = (\rho - a + 2)/2 \), and \( C_i \) are disjoint smooth rational curves. Let

\[
C = C_1 + \ldots + C_k.
\]

We use \( A_l (l \geq 1), D_m (m \geq 4), E_n (n = 6, 7, 8) \) to denote a negative definite root lattice whose basis is given by the corresponding Dynkin diagram. By classification of 2-elementary lattices ([Ni81, Theorem 4.3.2]), there are exactly 11 cases for the triple \( (\rho, a, \delta) \) with \( \rho + a = 22 \):

1. \( (\rho, a, \delta) = (11, 11, 1) \) (then \( k = 1 \), \( \text{NS}(X) \cong U(2) \oplus A_1^{\oplus 9} \));
(2) \((\rho, a, \delta) = (12, 10, 1)\) (then \(k = 2\), \(\text{NS}(X) \cong U \oplus A_1^{10}\));
(3) \((\rho, a, \delta) = (13, 9, 1)\) (then \(k = 3\), \(\text{NS}(X) \cong U \oplus D_4 \oplus A_1^{17}\));
(4) \((\rho, a, \delta) = (14, 8, 1)\) (then \(k = 4\), \(\text{NS}(X) \cong U \oplus D_4 \oplus D_4 \oplus A_1^{14}\));
(5) \((\rho, a, \delta) = (15, 7, 1)\) (then \(k = 5\), \(\text{NS}(X) \cong U \oplus D_4^{3} \oplus A_1\));
(6) \((\rho, a, \delta) = (16, 6, 1)\) (then \(k = 6\), \(\text{NS}(X) \cong U \oplus D_6^{2} \oplus A_1^{12}\));
(7) \((\rho, a, \delta) = (17, 5, 1)\) (then \(k = 7\), \(\text{NS}(X) \cong U \oplus D_6 \oplus D_8 \oplus A_1\));
(8) \((\rho, a, \delta) = (18, 4, 0)\) (then \(k = 8\), \(\text{NS}(X) \cong U \oplus D_4 \oplus D_{12}\));
(9) \((\rho, a, \delta) = (18, 4, 1)\) (then \(k = 8\), \(\text{NS}(X) \cong U \oplus D_{14} \oplus A_1^{12}\));
(10) \((\rho, a, \delta) = (19, 3, 1)\) (then \(k = 9\), \(\text{NS}(X) \cong U \oplus D_{16} \oplus A_1\));
(11) \((\rho, a, \delta) = (20, 2, 1)\) (then \(k = 10\), \(\text{NS}(X) \cong U \oplus E_8^{2} \oplus A_1^{12}\)).

The following lemma is due to \cite[Section 4]{Ni81}.

**Lemma 5.2.** Let \(H\) be a smooth rational curve on \(X\) different from \(C_i\) for all \(1 \leq i \leq k\). Then \(\theta(H) = H\), and \(C\) and \(H\) meet at exactly two points transversally. In particular, \(C.H = 2\).

**Proof.** Following \cite[Section 4]{Ni81}, we recall a proof. Since \(\theta\) acts trivially on \(\text{NS}(X)\) and \(H\) is the unique element of the complete linear system \(|H|\), it follows that \(\theta(H) = H\). Note that any involution of \(\mathbb{P}^1\) has exactly two fixed points. Since \(X^\theta = C\), it follows that \(C\) and \(H\) intersect at exactly two points. At each point, say \(P\), of the two intersection points, the tangent direction of \(C\) (resp. \(H\)) corresponds to the eigenvector of the induced action \(\theta^*|T_{X,P}\) with respect to eigenvalue \(1\) (resp. \(-1\)). Thus, \(C\) and \(H\) meet transversally \(P\).

**Lemma 5.3.** Let \(\varphi : X \rightarrow \mathbb{P}^1\) be an elliptic fibration with a section \(H\). Suppose there exists \(i \in \{1, 2, \ldots, k\}\) such that \(C_i\) is not contained in any fiber of \(\varphi\). Then \(H \subset C\). Moreover, \(C.F = 4\) for any fiber \(F\) of \(\varphi\).

**Proof.** Since \(C_i\) is not contained in any fiber of \(\varphi\), it follows that \(C_i\) intersects each fiber of \(\varphi\). Since \(C_i \subset X^\theta\), it follows that \(\theta\) preserves each fiber of \(\varphi\), i.e., \(\varphi \circ \theta = \varphi\). Note also that \(\theta(H) = H\) by Lemma 5.2. Then \(\theta|_H = \text{id}_H\) and \(H \subset C\).

Note that for a general fiber, say \(E\), of \(\varphi\), \(\theta|E\) is just the inversion of the elliptic curve \(E\) (view \(E \cap H\) as zero \(O\) of \(E\)). Thus \(\theta|_E\) has exactly 4 fixed points. Then \(E\) meets with \(C\) at four points transversally. Thus \(C.F = C.E = 4\) for any fiber \(F\) of \(\varphi\).

In the next lemma, we need the notion of the Mordell-Weil group \(\text{MW}(\varphi)\) of a genus one fibration \(\varphi : X \rightarrow \mathbb{P}^1\) explained in Remark 3.6.

**Lemma 5.4.** Let \(E\) be an effective reducible divisor such that the complete linear system \(|E|\) defines a genus one fibration \(\varphi : X \rightarrow \mathbb{P}^1\). Let \(\mathcal{I} = \{i|C_i \subset \text{Supp}E\}\). Let \(r\) be the number of irreducible components of \(E\). If either one of the following (1) or (2) holds, then \(\text{rk } \text{MW}(\varphi) > 0\).

1. The cardinality \(|\mathcal{I}|\) is equal to \(k\) and \(r < \rho(X) - 1\), or
2. \(|\mathcal{I}| = k - 1\), \(C_i.E = 0\) for the unique \(i \in \{1, \ldots, k\} \setminus \mathcal{I}\), and \(r + 2 < \rho(X)\).

**Proof.** Case (1): By Lemma 5.2, \(E\) is the only reducible singular fiber of \(\varphi\) (otherwise, suppose \(H\) is a smooth rational curve in a reducible fiber different from \(E\), then \(H.E = 0\) and \(H.C = 0\), a contradiction). Then by \(r < \rho(X) - 1\) and by \cite[Lemma 2.1]{Yu18}, \(\varphi\) has infinite automorphism group, and \(\text{rk } \text{MW}(\varphi) > 0\).
Weil rank. Thus, by applying Corollary 3.7 for $E$ since $F$ and $C$ are fibres of $\varphi$. As in the previous case, by Lemmas 5.2, 5.3, we may assume $C.H.C$ there exists an elliptic fibration $\varphi$ there exists an elliptic fibration $\varphi$. Then by $r + 2 < \rho(X)$ and by [Yu18] Lemma 2.1, $\text{rk MW}(\varphi) > 0$. □

Theorem 5.5 (1) follows from:

**Theorem 5.5.** If $\rho \geq 12$, then there exists a smooth rational curve $H \subset X$ such that $\text{Ine}(H)$ contains a non-commutative free subgroup isomorphic to $\mathbb{Z} \ast \mathbb{Z}$ and an element of positive entropy.

**Proof.** Case $(\rho, a, \delta) = (12, 10, 1)$: Then $k = 2$ and $C = C_1 + C_2$. By NS($X$) $\cong U \oplus A_1^{\oplus 10}$, there exists an elliptic fibration $\varphi : X \longrightarrow \mathbb{P}^1$ with exactly 10 reducible fibers: $H_i + H'_i$, $1 \leq i \leq 10$, where $H_i$ and $H'_i$ are smooth rational curves. By Lemma 5.2 at least one of $C_1$ and $C_2$ is not contained in any fiber of $\varphi$. Then, by Lemma 5.3, one of the two components of $C$ is a section of $\varphi$. Interchanging $C_1$ and $C_2$ if necessary, we may assume $C_1$ is a section of $\varphi$. Then, for any $i$, interchanging $H_i$ and $H'_i$ if necessary, we may assume $C_1.H_i = 1$. By Lemma 5.2, $C.H_i = 2$. Thus $C_2.H_i = 1$. Let

$$E_1 = C_1 + H_1 + C_2 + H_2$$

and

$$E_2 = C_1 + H_1 + C_2 + H_3,$$

which are of Kodaira type $I_4$. By Lemma 5.3 (1), $|E_1|$ and $|E_2|$ define two genus one fibrations of positive Mordell-Weil rank. Thus, by applying Corollary 3.7 for $E_1, E_2, C = C_1, R_1 = R_2 = H_1$, we deduce that $\text{Ine}(C_1)$ contains a non-commutative free subgroup isomorphic to $\mathbb{Z} \ast \mathbb{Z}$ and an element of positive entropy. Thus, this case is proved.

Note that the choice of $H$ in Theorem 5.5 is not necessarily unique. In fact, for the same reason, each of $\text{Ine}(C_2)$ and $\text{Ine}(H_1)$ contains a non-commutative free subgroup isomorphic to $\mathbb{Z} \ast \mathbb{Z}$ and an element of positive entropy.

Case $(\rho, a, \delta) = (13, 9, 1)$: Then $k = 3$ and $C = C_1 + C_2 + C_3$. By NS($X$) $\cong U \oplus D_4 \oplus A_1^{\oplus 7}$, there exists an elliptic fibration $\varphi : X \longrightarrow \mathbb{P}^1$ with exactly 8 reducible fibers:

$$2F_0 + F_1 + F_2 + F_3 + F_4 \ (\text{Kodaira type } I_0^*), H_1 + H'_1, \ldots, H_7 + H'_7.$$  

As in the previous case, by Lemmas 5.2, 5.3 we may assume $C_1$ is a section of $\varphi$. Then we may assume $C_1.F_1 = 1$ and $C_1.H_i = 1$ for all $i$. Since $\theta$ preserves any smooth rational curve on $X$ (Lemma 5.2), it follows that each of the four points $F_0 \cap F_i$, $1 \leq i \leq 4$, must be fixed by $\theta$. Then $F_0$ must be fixed by $\theta$ pointwisely. Thus $F_0 \subset C$, and we may assume $F_0 = C_2$. Thus $C_2.H_i = 0$. Then by Lemma 5.2, $C_3.H_i = 1$ for all $1 \leq i \leq 7$. Note that since $C_1$ is a section of $\varphi$ and $C_1.F_1 = 1$, it follows that $C_1.F_i = 0$, for all $i = 2, 3, 4$. Then $C_3.F_i = 1$, for all $i = 2, 3, 4$. Note that $C_3$ is a 3-section of $\varphi$. Let

$$E_1 := C_2 + F_1 + C_1 + H_1 + C_3 + F_2 \ (\text{Kodaira type } I_6)$$

and

$$E_2 := C_2 + F_1 + C_1 + H_2 + C_3 + F_2 \ (\text{Kodaira type } I_6).$$

By Lemma 5.3 (1), $|E_1|$ and $|E_2|$ define two different genus one fibrations of positive Mordell-Weil rank. Thus, by applying Corollary 3.7 for $E_1, E_2, C = C_1, R_1 = R_2 = F_1$, we deduce
that \( \text{Ine}(C_1) \) contains a non-commutative free subgroup isomorphic to \( \mathbb{Z} \ast \mathbb{Z} \) and an element of positive entropy. Thus, this case is proved.

All of the remaining cases (i.e., cases \( \rho \geq 14 \)) can be proved similarly as in the case \( (\rho, a, \delta) = (13, 9, 1) \). For each of the remaining cases, we just list the reducible singular fibers of an elliptic fibration \( \varphi \) on \( X \), from which we start as in the previous case. It turns out that, in all of these remaining cases, a component, say \( C_1 \), of \( C \) is a section of \( \varphi \), and another component, say \( C_k \), of \( C \) is a 3-section of \( \varphi \). Then we give the definition of two effective divisors \( E_1 \) and \( E_2 \) for which we apply Corollary 3.7. Then by Corollary 3.7, we see that \( \text{Ine}(C_1) \) contains a non-commutative free subgroup isomorphic to \( \mathbb{Z} \ast \mathbb{Z} \) and an element of positive entropy.

In the rest of the proof, for \( 1 \leq i, j \leq k \), we use \( G_{ij}, F_{ij}, F'_{ij}, \) etc. to denote smooth rational curves which intersect both \( C_i \) and \( C_j \) (\( i = j \) means intersecting with \( C_i \) at two distinct points).

Case \( (\rho, a, \delta) = (14, 8, 1) \): Then \( k = 4 \) and \( C = C_1 + C_2 + C_3 + C_4 \). By \( \text{NS}(X) \cong U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 4} \), there exists an elliptic fibration (with a section \( C_1 \)) \( \varphi : X \to \mathbb{P}^1 \) with exactly 6 reducible fibers: \( 2C_2 + F_{12} + F_{24} + F'_{24} + F''_{24} \) (type \( I_0' \)), \( 2C_3 + F_{13} + F_{34} + F'_{34} + F''_{34} \) (type \( I_0'' \)), \( F_{14} + F_{14} + F'_{14} + F''_{14} + F'''_{14} + F''''_{14} \). Let \( E_1 := C_2 + F_{12} + C_1 + F_{14} + C_4 + F_{24} \) (type \( I_6 \)) and \( E_2 := C_2 + F_{12} + C_1 + F'_{14} + C_4 + F_{24} \) (type \( I_6 \)). Note that in this case, we use Lemma 5.4 (2) to prove the positivity of Mordel-Weil ranks.

Case \( (\rho, a, \delta) = (15, 7, 1) \): Then \( k = 5 \) and \( C = C_1 + \ldots + C_5 \). By \( \text{NS}(X) \cong U \oplus D_4^{\oplus 3} \oplus A_1 \), there exists an elliptic fibration (with a section \( C_1 \) \( \varphi : X \to \mathbb{P}^1 \) with exactly 4 reducible fibers: \( 2C_2 + F_{12} + F_{25} + F'_{25} + F''_{25} \) (type \( I_0' \)), \( 2C_3 + F_{13} + F_{35} + F'_{35} + F''_{35} \) (type \( I_0'' \)), \( 2C_4 + F_{14} + F_{45} + F'_{45} + F''_{45} \) (type \( I_0''' \)), \( F_{15} + F_{15} \). Let \( E_1 := C_4 + F_{45} + C_5 + F_{25} + C_2 + F_{12} + C_1 + F_{14} \) (type \( I_8 \)) and \( E_2 := C_4 + F_{45} + C_5 + F'_{25} + C_2 + F_{12} + C_1 + F_{14} \) (type \( I_8 \)). Note that in this case, we use Lemma 5.4 (2) to prove the positivity of Mordel-Weil ranks.

Case \( (\rho, a, \delta) = (16, 6, 1) \): Then \( k = 6 \) and \( C = C_1 + \ldots + C_6 \). By \( \text{NS}(X) \cong U \oplus D_6^{\oplus 2} \oplus A_1^{\oplus 2} \), there exists an elliptic fibration (with a section \( C_1 \) \( \varphi : X \to \mathbb{P}^1 \) with exactly 4 reducible fibers: \( 2C_2 + 2G_{23} + 2C_3 + F_{13} + F_{36} + F_{26} + F'_{26} \) (type \( I_2' \)), \( 2C_4 + 2G_{45} + 2C_5 + F_{15} + F_{56} + \)
$F_{46} + F'_{46}$ (type $I_2'$), $F_{16} + F'_{66}$, $F'_{16} + F''_{66}$. Let $E_1 := C_4 + F_{13} + C_1 + F_{15} + C_5 + G_{45} + C_4 + F_{46} + C_6 + F_{26} + C_2 + G_{23}$ (type $I_{12}$) and $E_2 := C_3 + F_{13} + C_1 + F_{15} + C_5 + G_{45} + C_4 + F_{46} + C_6 + F''_{26} + C_2 + G_{23}$ (type $I_{12}$).

Case $(p, a, \delta) = (17, 5, 1)$: Then $k = 7$ and $C = C_1 + C_2 + \ldots + C_7$. By $\text{NS}(X) \cong U \oplus D_6 \oplus D_8 \oplus A_1$, then there exists an elliptic fibration (with a section $C_1$) $\varphi : X \to \mathbb{P}^1$ with exactly 3 reducible fibers: $F_{27} + F'_{27} + 2C_2 + 2G_{23} + 2C_3 + F_{37} + F_{31}$ (type $I_2'$), $F_{47} + F'_{47} + 2C_4 + 2G_{45} + 2C_5 + 2G_{56} + 2C_6 + F_{61} + F_{67}$ (type $I_1$), $F_{17} + F_{77}$. Let $E_1 := C_4 + G_{45} + C_5 + G_{56} + C_6 + F_{61} + C_1 + F_{31} + C_3 + G_{23} + C_2 + F_{27} + C_7 + F_{47}$ (type $I_{14}$) and $E_2 := C_4 + G_{45} + C_5 + G_{56} + C_6 + F_{61} + C_1 + F_{31} + C_3 + G_{23} + C_2 + F_{27} + C_7 + F'_{47}$ (type $I_{14}$).

Case $(p, a, \delta) = (18, 4, 0)$: This is the case studied in [Og18]. Then $k = 8$ and $C = C_1 + C_2 + \ldots + C_8$. By $\text{NS}(X) \cong U \oplus D_4 \oplus D_{12}$, then there exists an elliptic fibration (with a section $C_1$) $\varphi : X \to \mathbb{P}^1$ with exactly 2 reducible fibers: $F_{28} + F'_{28} + 2C_2 + F''_{28} + F_{12}$ (type $I_6$), $F_{38} + F'_{38} + 2C_3 + 2G_{34} + 2C_4 + 2G_{45} + 2C_5 + 2G_{56} + 2C_6 + 2G_{67} + 2C_7 + F_{78}$ (type $I_8$). Let $E_1 := C_3 + G_3 + C_{14} + G_{45} + C_5 + G_{56} + C_6 + F_{61} + C_1 + F_{31} + C_3 + G_{23} + C_2 + F_{27} + C_7 + F_{47}$ (type $I_{16}$) and $E_2 := C_3 + G_{34} + C_4 + G_{45} + C_5 + G_{56} + C_6 + F_{61} + C_1 + F_{31} + C_3 + G_{23} + C_2 + F_{27} + C_7 + F_{47}$ (type $I_{16}$).

Case $(p, a, \delta) = (18, 4, 1)$: Then $k = 8$ and $C = C_1 + C_2 + \ldots + C_8$. By $\text{NS}(X) \cong U \oplus D_{14} \oplus A_2^2$, then there exists an elliptic fibration (with a section $C_1$) $\varphi : X \to \mathbb{P}^1$ with exactly 3 reducible fibers: $F_{28} + F'_{28} + 2C_2 + 2G_{23} + 2C_3 + 2G_{34} + 2C_4 + 2G_{45} + 2C_5 + 2G_{56} + 2C_6 + 2G_{67} + 2C_7 + F_{78}$ (type $I_{16}$), $F_{18} + F_{88}, F'_{18} + F'_{88}$. Let $E_1 := C_2 + G_{23} + C_3 + G_{34} + C_4 + G_{45} + C_5 + G_{56} + C_6 + G_{67} + C_7 + F_{17} + C_1 + F_{18} + C_8 + F_{28}$ (type $I_{16}$) and $E_2 := C_2 + G_{23} + C_3 + G_{34} + C_4 + G_{45} + C_5 + G_{56} + C_6 + G_{67} + C_7 + F_{17} + C_1 + F_{18} + C_8 + F_{28}$ (type $I_{16}$).

Case $(p, a, \delta) = (19, 3, 1)$: Then $k = 9$ and $C = C_1 + C_2 + \ldots + C_9$. By $\text{NS}(X) \cong U \oplus D_{18} \oplus A_1$, then there exists an elliptic fibration (with a section $C_1$) $\varphi : X \to \mathbb{P}^1$ with exactly 2 reducible fibers: $F_{29} + F'_{29} + 2C_2 + 2G_{23} + 2C_3 + 2G_{34} + 2C_4 + 2G_{45} + 2C_5 + 2G_{56} + 2C_6 + 2G_{67} + 2C_7 + 2G_{78} + 2C_8 + F_{89} + F_{18}$ (type $I_{12}'$), $F_{19} + F_{99}$. Let $E_1 := C_2 + G_{23} + C_3 + G_{34} + C_4 + G_{45} + C_5 + G_{56} + C_6 + G_{67} + C_7 + G_{78} + C_8 + F_{18} + C_1 + F_{19} + C_9 + F_{29}$ (type $I_{18}$) and $E_2 := C_2 + G_{23} + C_3 + G_{34} + C_4 + G_{45} + C_5 + G_{56} + C_6 + G_{67} + C_7 + G_{78} + C_8 + G_{89} + C_9 + F_{90} + C_9 + F_{29}$ (type $I_{18}$). Note that in this case, we use Lemma 4.3.4 (2) to prove the positivity of Mordell-Weil ranks.

This completes the proof of the theorem.

For the rest of this section, we consider the case $(p, a, \delta) = (11, 11, 1)$. Then $k = 1$ and $C = X^\theta$ is an irreducible smooth rational curve. Note that $\text{Aut}(X) = \text{Dec}(C)$, because $\theta$ is in the center of $\text{Aut}(X)$.
Lemma 5.6. Let $\phi : \text{Aut}(X) \to O(\NS(X))$ be the map given by $\phi(f) = f^\ast|_{\NS(X)}$. Then $\ker \phi = \{id_X, \theta\}$.

Proof. Since $\text{rk} \ T(X) = 11$ an odd number, it follows that, for any automorphism $g$ of $X$, the induced action $g^\ast$ on $T(X)$ must be $\pm id_{T(X)}$. Thus, $\ker \phi = \{id_X, \theta\}$. \hfill $\Box$

Lemma 5.7. There exist eleven smooth rational curves such that their classes in $\NS(X)$ form a $\mathbb{Q}$-basis of $\NS(X)_Q$.

Proof. By $\NS(X) \cong U(2) \oplus \mathbb{A}_1^{\oplus 9}$, there exists genus one fibration, say $\varphi : X \to \mathbb{P}^1$, such that this fibration has exactly 9 reducible singular fibres and each of them consists of exactly two irreducible components, say $H_i$ and $H_i'$, $1 \leq i \leq 9$, where $H_i$ and $H_i'$ are smooth rational curves. By Lemma 5.2, $C$ cannot be contained in any fiber of $\varphi$. Then

$$[H_1] + [H_1'], [C], [H_2], ..., [H_9]$$

form $\mathbb{Q}$-basis of $\NS(X)_Q$ since the corresponding Gram matrix has nonzero determinant. Thus

$$[H_1'], [C], [H_2], ..., [H_9]$$

also form a $\mathbb{Q}$-basis of $\NS(X)_Q$. This completes the proof of the lemma. \hfill $\Box$

Theorem 1.3 (2), hence Theorem 1.2, follows from:

Theorem 5.8. Let $\rho : \text{Aut}(X) = \text{Dec}(C) \to \text{Aut}(C)$ be the restriction map. If $\ker \rho \neq \{id_X, \theta\}$, i.e., $\text{Ine}(C) \neq \{id_X, \theta\}$, then $\text{Ine}(C)$ contains a non-commutative free subgroup isomorphic to $\mathbb{Z} \ast \mathbb{Z}$ and an element of positive entropy.

Proof. Suppose $\text{Ine}(C) \neq \{id_X, \theta\}$. Let $g \in \text{Ine}(C) \setminus \{id_X, \theta\}$. By Lemmas 5.6 and 5.7, there exists a smooth rational curve $H$ on $X$ such that $g(H) \neq H$. By $g|_C = id_C$ and Lemma 5.2, $H \cap g(H) \cap C$ consists of exactly two points, say $P_1$ and $P_2$. Let $E_1 := H + C$ and $E_2 := g(H) + C$. Then complete linear systems $|E_i|$, $i = 1, 2$, define two genus one fibrations, say $\varphi_i : X \to \mathbb{P}^1$, of positive Mordell-Weil rank (Lemma 5.4). Then by Corollary 3.7, there exists $f \in \text{Ine}(C)$ such that $h(f) > 0$. This completes the proof of the theorem. \hfill $\Box$

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