PARTITIONING HADAMARD VECTORS INTO HADAMARD MATRICES

PETER G. CASAZZA AND JANET C. TREMAIN

Abstract. We will show that in a space of dimension $m$, any family of $2^{m-1}$ distinct Hadamard vectors (where you can choose $x$ or $-x$ but not both) can be partitioned into Hadamard matrices if and only if $m = 2^n$ for some $n$. We will solve this problem with a simple algorithm for assigning the vectors to the Hadamard matrices.

1. Introduction

We make a few simple observations.

Observation 2.1. If we can prove that just one choice of a maximal set of distinct Hadamard vectors can be partitioned into Hadamard matrices, then this is also true for any choice of distinct Hadamard vectors.

1991 Mathematics Subject Classification. 42C15.

The authors were supported by NSF DMS 1307685; NSF ATD 1321779; and ARO W911NF-16-1-0008.
This is clear since any second choice of distinct Hadamard vectors has the property that for any vector \( x \) in this set, either \( x \) or \(-x\) is in the first set. And so the partition of the first set is a partition of the second set with perhaps sign changes of some rows of the Hadamard matrix - which is still an orthogonal matrix.

**Observation 2.2.** The moreover part of the theorem is essentially obvious.

Given any \( m = 4n \), if we can partition the \( 2^{4n-1} \) distinct Hadamard vectors into \((4n) \times (4n)\) Hadamard matrices, let \( k \) be the number of such matrices. Then \( k \cdot (4n) \) uses up all the distinct Hadamard vectors and so

\[
k \cdot (4n) = 2^{4n-1}, \quad \text{and so } m \text{ divides } 2^{4n-1}.
\]

We will do the proof by induction on \( n \) with the case \( n=2 \) below:

\[
A = \begin{bmatrix}
+ & + & + & + \\
+ & + & - & - \\
+ & - & + & - \\
+ & - & - & + \\
\end{bmatrix} \quad B = \begin{bmatrix}
+ & + & - \\
+ & + & + \\
+ & - & - \\
\end{bmatrix}
\]

**3. Proof of the Theorem**

We assume the theorem holds for \( m = 2^n \) and we have partitioned a distinct set of Hadamard vectors into Hadamard matrices \( \{A_i\}_{i=1}^{2^{2^n-n-1}} \) and let \( \{x_{ij}\}_{j=1}^{2^n} \) be the row vectors of \( A_i \). We will construct \( 2^{2^n-1-(n+1)-1} \) matrices of distinct Hadamard vectors of order \( 2^n \times 2^n+1 \) so that cutting each of these matrices vertically in half, each of the left halves and the right halves are orthogonal matrices. For each of these, say \([A \ B]\), we then take:

(1)

\[
\begin{bmatrix}
A & B \\
A & -B
\end{bmatrix}
\]

and have a partition of distinct Hadamard vectors into Hadamard matrices. Since the total number of vectors here is equal to the total number of distinct Hadamard vectors for \( 2^{2^n+1} \), we are done. It will be obvious from our construction that the vectors we construct are unique.

For any \( n \), we define the row shift of an \( n \times n \) matrix \( A \) with row vectors \( \{x_i\}_{i=1}^{n} \) by:

\[
T_n = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]

so that \( T_n \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} \)

For the construction, for each \( A_i, A_j \) above, we form the \( 2^{2^n} \times 2^{2^n+1} \) matrices:

\[
\begin{bmatrix} A_i \mid T_k \end{bmatrix}^{n}_{n} A_j \text{ for all } k = 1, \ldots, 2^n.
\]
Since the rows of $A_i$ are orthogonal, the above matrices are pairs of orthogonal matrices and so are orthogonal. Note that each of the $2^{2^n-n-1} A'_i s$ is paired with all the other $2^{2^n-n-1} A'_j s$ and each is paired with $2^n$ shifts of the rows. So the total number of matrices above is:

$$2^{2^n-n-1} \cdot 2^{2^n-n-1} \cdot 2^n = 2^{2^{n+1}-(n+1)-1}.$$ 

I.e. We have used up all the distinct Hadamard vectors in a space of dimension $2^{n+1}$.

References

Department of Mathematics, University of Missouri, Columbia, MO 65211-4100

E-mail address: Casazzap@missouri.edu; Tremainjc@missouri.edu