On the Limiting Shape of Markovian Random Young Tableaux

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Abstract

Let \((X_n)_{n \geq 0}\) be an irreducible, aperiodic, homogeneous Markov chain, with state space an ordered finite alphabet of size \(m\). Using combinatorial constructions and weak invariance principles, we obtain the limiting shape of the associated Young tableau as a multidimensional Brownian functional. Since the length of the top row of the Young tableau is also the length of the longest (weakly) increasing subsequence of \((X_k)_{1 \leq k \leq n}\), the corresponding limiting law follows. We relate our results to a conjecture of Kuperberg by showing that, under a cyclic condition, a spectral characterization of the Markov transition matrix delineates precisely when the limiting shape is the spectrum of the traceless GUE. For \(m = 3\), all cyclic Markov chains have such a limiting shape, a fact previously known for \(m = 2\). However, this is no longer true for \(m \geq 4\).

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1 Introduction

The identification of the limiting distribution of \( LI_n \), the length of the longest increasing subsequence of a random word of length \( n \), whose letters are iid and chosen uniformly from an ordered, \( m \)-letter alphabet, was first made by Tracy and Widom [28]. They showed that the limiting distribution of \( LI_n \), properly centered and normalized, is that of the largest eigenvalue of the traceless \( m \times m \) Gaussian unitary ensemble (GUE). In the non-uniform iid case, Its, Tracy, and Widom [20, 21] described the corresponding limiting distribution as that of the largest eigenvalue of one of the diagonal blocks (corresponding to the highest probability) in a direct sum of certain independent GUE matrices. The number and respective dimensions of these matrices are determined by the multiplicities of the probabilities of choosing the letters, and the direct sum is subject again to an overall zero-trace type of condition.

The well-known Robinson-Schensted-Knuth (RSK) correspondence between sequences and pairs of Young tableaux led Tracy and Widom [28] to conjecture that the (necessarily \( m \)-row) Young tableau of a random word generated by an \( m \)-letter, uniform iid sequence has a limiting shape given by the joint distribution of the eigenvalues of a \( m \times m \) traceless element of the GUE. Since the length of the longest row of the Young tableau is precisely \( LI_n \), this appears to be a natural generalization. Johansson [22] proved this conjecture using orthogonal polynomial methods. Further, Okounkov [27], and Borodin, Okounkov, and Olshankii [6], as well as Johansson [22], also answered a conjecture of Baik, Deift, and Johansson [1, 2] regarding the limiting shape of the Young tableau associated to a random permutation of \( \{1, 2, \ldots, n\} \). In particular, as \( n \) grows without bound, the lengths \( \lambda_1, \lambda_2, \ldots, \lambda_k \) of the first \( k \) rows of the Young tableau, appropriately centered and scaled, have the same limiting law as the \( k \) largest eigenvalues of a \( n \times n \) element of the GUE, a result first proved, for \( k = 2 \), in [1, 2].

The extension to the non-uniform iid case was addressed to some degree in Its, Tracy, and Widom [20, 21], who focused primarily on the top row of the Young tableau. Here the obvious conjecture is that the limiting shape has rows whose suitably centered and normalized lengths have a joint distribution which is that of the whole spectrum of the direct sum of GUE matrices described above. Below, we prove this result as a special case of the Markovian framework.

Kuperberg [24] conjectured that if the word is generated by an irreducible, doubly-stochastic, cyclic Markov chain, then the limiting distribution of the
shape is still that of the joint distribution of the eigenvalues of a traceless $m \times m$ element of the GUE. For $m = 2$, this was shown to be true by Chistyakov and Götte [9], who, in view of further simulations, expressed doubts concerning the validity for $m \geq 4$. For $m = 3$, we will show that the conjecture holds as well. However, for $m \geq 4$, this is no longer the case. Indeed, some, but not all, cyclic Markov chains lead to a limiting law as in the iid uniform case already obtained by Johansson [22].

The precise class of homogeneous Markov chains with which Kuperberg’s conjecture is concerned is more specific than the ones we shall study. The irreducibility of the chain is a basic property we certainly must demand: each letter has to occur at some point following the occurrence of any given letter. Moreover, the doubly-stochastic hypothesis ensures that we have a uniform stationary distribution. However, the cyclic criterion, i.e., the Markov transition matrix $P$ has entries satisfying $p_{i,j} = p_{i+1,j+1}$, for $1 \leq i, j \leq m$ (where $m+1 = 1$), is more restrictive: cyclicity implies but is not equivalent to $P$ being doubly stochastic. Kuperberg was led to introduce this latter restriction via simulations [24] inspired by mathematical physics considerations, which appear to show that at least some irreducible, doubly-stochastic, non-cyclic Markov chains do not produce such limiting behavior.

Let us also note that Kuperberg implicitly assumes the Markov chain to also be aperiodic. Indeed, the simple 2-state Markov chain for the letters $\alpha_1$ and $\alpha_2$ described by $\mathbb{P}(X_{n+1} = \alpha_i | X_n = \alpha_j) = 1$ for $i \neq j$, produces a sequence of alternating letters, so that $LI_n$ is always either $n/2$ or $n/2 + 1$, for $n$ even, and $(n+1)/2$, for $n$ odd, and so has a degenerate limiting distribution. Even though this Markov chain is irreducible, doubly-stochastic, and cyclic, it is periodic.

The paper is organized in the following manner. In Section 2, we present the simple combinatorial formulation of the $LI_n$ problem the authors developed in [18]. Next, in Section 3, we use this formulation to rederive the two-letter Markov case first studied by Chistyakov and Götte [9]. Then, in order to extend these results to alphabets of size $m \geq 3$, we introduce, in Section 4, a slight modification of our original combinatorial development, and so obtain a functional of combinatorial quantities which describes the shape of the entire Young tableau with $n$ cells, along with a concise expression for the associated asymptotic covariance structure. In Section 5, we apply Markovian Invariance Principles to express the limiting shape of the Young tableau as a Brownian functional for all irreducible, aperiodic, homogeneous Markov chains (without the cyclic or even the doubly-stochastic constraint.)
Using this functional we are then able to answer Kuperberg’s conjecture. In Section 6, we investigate, in further detail, various symmetries exhibited by the Brownian functional. In particular, we clarify the asymptotic covariance structure in the cyclic case, and obtain, for $m$ arbitrary, a precise description of the class of cyclic Markov chains having the same limiting law as in the uniform iid case. In Section 7, we further explore connections between the various Brownian functionals obtained as limiting laws and eigenvalues of random matrices. Finally, in Section 8, we conclude with a brief discussion of natural extensions and complements to some of the ideas and results presented in the paper.

## 2 Combinatorics

As in [18], one can express $LI_n$ in a fundamentally combinatorial manner. For convenience, this section recapitulates that development.

Let $(X_n)_{n \geq 1}$ consist of a sequence of values taken from an $m$-letter ordered alphabet, $A_m = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\}$. Let $a^r_k$ be the number of occurrences of $\alpha_r$ among $(X_i)_{1 \leq i \leq k}$. Each increasing subsequence of $(X_i)_{1 \leq i \leq k}$ consists simply of consecutive identical values, with these values forming an increasing subsequence of $\alpha_r$. Moreover, the number of occurrences of $\alpha_r \in \{\alpha_1, \ldots, \alpha_m\}$ among $(X_i)_{k+1 \leq i \leq \ell}$, where $1 \leq k < \ell \leq n$, is simply $a^r_\ell - a^r_k$. The length of the longest increasing subsequence of $X_1, X_2, \ldots, X_n$ is thus given by

$$LI_n = \max_{0 \leq k_1 < \cdots < k_m \leq n} [(a^1_{k_1} - a^1_0) + (a^2_{k_2} - a^2_{k_1}) + \cdots + (a^m_{k_m} - a^m_{k_{m-1}})], \quad (2.1)$$

i.e.,

$$LI_n = \max_{0 \leq k_1 < \cdots < k_m \leq n} [(a^1_{k_1} - a^1_{k_1}) + (a^2_{k_2} - a^2_{k_2}) + \cdots + (a^m_{k_m} - a^m_{k_{m-1}}) + a^m_n], \quad (2.2)$$

where $a^r_0 = 0$.

For $i = 1, \ldots, n$ and $r = 1, \ldots, m - 1$, let
\[ Z^r_i = \begin{cases} 1, & \text{if } X_i = \alpha_r, \\ -1, & \text{if } X_i = \alpha_{r+1}, \\ 0, & \text{otherwise,} \end{cases} \quad (2.3) \]

and let \( S^r_k = \sum_{i=1}^{k} Z^r_i, k = 1, \ldots, n \), with also \( S^r_0 = 0 \). Then clearly \( S^r_k = a^r_k - a^{r+1}_k \). Hence,

\[ LI_n = \max_{0 \leq k_1 \leq \ldots \leq k_{m-1} \leq n} \{ S^1_{k_1} + S^2_{k_2} + \cdots + S^{m-1}_{k_{m-1}} + a^m_n \}. \quad (2.4) \]

By the telescoping nature of the sum \( \sum_{k=r}^{m-1} S^k_n = \sum_{k=r}^{m-1} (a^k_n - a^{k+1}_n) \), we find that, for each \( 1 \leq r \leq m - 1 \), \( a^r_n = a^m_n + \sum_{k=r}^{m-1} S^k_n \). Since \( a^1_k, \ldots, a^m_k \) must evidently sum to \( k \), we have

\[
\begin{aligned}
n & = \sum_{r=1}^{m} a^r_n \\
& = \sum_{r=1}^{m-1} \left( a^m_n + \sum_{k=r}^{m-1} S^k_n \right) + a^m_n \\
& = \sum_{r=1}^{m-1} r S^r_n + ma^m_n.
\end{aligned}
\]

Solving for \( a^m_n \) gives us

\[ a^m_n = \frac{n}{m} - \frac{1}{m} \sum_{r=1}^{m-1} r S^r_n. \]

Substituting into (2.4), we finally obtain

\[ LI_n = \frac{n}{m} - \frac{1}{m} \sum_{r=1}^{m-1} r S^r_n + \max_{0 \leq k_1 \leq \ldots \leq k_{m-1} \leq n} \{ S^1_{k_1} + S^2_{k_2} + \cdots + S^{m-1}_{k_{m-1}} \}. \quad (2.5) \]

As was emphasized in [18], (2.5) is of a purely combinatorial nature or, in more probabilistic terms, is of a pathwise nature. We now proceed to analyze (2.5) in the case of a Markovian sequence.
3 Markovian Alphabet: 2-Letter Case

We begin our study of Markovian alphabets by concentrating on the 2-letter case. Here \((X_n)_{n \geq 0}\) is described by the following transition probabilities between the two states (which we identify with the two letters \(\alpha_1\) and \(\alpha_2\):

\[
P(X_{n+1} = \alpha_2 | X_n = \alpha_1) = a \quad \text{and} \quad P(X_{n+1} = \alpha_1 | X_n = \alpha_2) = b,
\]

where 0 < \(a + b < 2\). We later examine the degenerate cases \(a = b = 0\) and \(a = b = 1\).

In keeping with the common usage within the Markov chain literature, we begin our sequence at \(n = 0\), although our focus will be on \(n \geq 1\).Denoting by \((p_1^n, p_2^n)\) the vector describing the probability distribution on \(\{\alpha_1, \alpha_2\}\) at time \(n\), we have

\[
(p_{n+1}^1, p_{n+1}^2) = (p_n^1, p_n^2) \begin{pmatrix} 1 - a & a \\ b & 1 - b \end{pmatrix}.
\]

The eigenvalues of the matrix in (3.1) are \(\lambda_1 = 1\) and \(-1 < \lambda_2 = 1 - a - b < 1\), with respective left eigenvectors \((\pi_1, \pi_2) = (b/(a + b), a/(a + b))\) and \((1, -1)\). Moreover, \((\pi_1, \pi_2)\) is also the stationary distribution. Given any initial distribution \((p_0^1, p_0^2)\), we find that

\[
(p_n^1, p_n^2) = (\pi_1, \pi_2) + \lambda_n \frac{ap_0^1 - bp_0^2}{a + b} (1, -1) \to (\pi_1, \pi_2),
\]

as \(n \to \infty\), since \(\lambda_2 < 1\).

Our goal is now to use these probabilistic expressions to describe the random variables \(Z_k^1\) and \(S_k^1\) defined in the previous section. (We retain the redundant superscript “1” in \(Z_k^1\) and \(S_k^1\) in the interest of uniformity.) Setting \(\beta = ap_0^1 - bp_0^2\), we easily find that

\[
\mathbb{E}Z_k^1 = (+1) \left( \pi_1 + \frac{\beta}{a + b} \lambda_2^k \right) + (-1) \left( \pi_2 - \frac{\beta}{a + b} \lambda_2^k \right)
= \frac{b - a}{a + b} + 2 \frac{\beta}{a + b} \lambda_2^k,
\]

for each \(1 \leq k \leq n\). Thus,

\[
\mathbb{E}S_k^1 = \frac{b - a}{a + b} k + 2 \left( \frac{\beta \lambda_2^k}{a + b} \right) \left( \frac{1 - \lambda_2^k}{1 - \lambda_2} \right),
\]

for each \(1 \leq k \leq n\).
and so $\mathbb{E}S_k^1/k \to (b - a)/(a + b)$, as $k \to \infty$.

Turning to the second moments of $Z_k^1$ and $S_k^1$, first note that $\mathbb{E}(Z_k^1)^2 = 1$, since $(Z_k^1)^2 = 1$ a.s. Next, we consider $\mathbb{E}Z_k^1Z_{\ell}^1$, for $k < \ell$. Using the Markovian structure of $(X_n)_{n \geq 0}$, it quickly follows that

$$
\mathbb{P}((X_k, X_{\ell}) = (x_k, x_{\ell})) = \begin{cases}
\left(\pi_1 + \lambda_2^{\ell-k} \frac{a}{a+b}\right) \pi_1, & \text{if } (x_k, x_{\ell}) = (\alpha_1, \alpha_1), \\
\left(\pi_1 - \lambda_2^{\ell-k} \frac{b}{a+b}\right) \pi_2, & \text{if } (x_k, x_{\ell}) = (\alpha_1, \alpha_2), \\
\left(\pi_2 - \lambda_2^{\ell-k} \frac{a}{a+b}\right) \pi_1, & \text{if } (x_k, x_{\ell}) = (\alpha_2, \alpha_1), \\
\left(\pi_2 + \lambda_2^{\ell-k} \frac{b}{a+b}\right) \pi_2, & \text{if } (x_k, x_{\ell}) = (\alpha_2, \alpha_2).
\end{cases}
$$

(3.5)

For simplicity, we will henceforth assume that our initial distribution is the stationary one, i.e., $(p_0^1, p_0^2) = (\pi_1, \pi_2)$. Later, (see Concluding Remarks) we drop this assumption and deal with initial distributions concentrated on an arbitrary state. Under this assumption, $\beta = 0$, $\mathbb{E}S_k^1 = k\mu$, where $\mu = \mathbb{E}Z_k^1 = (b - a)/(a + b)$, and (3.5) simplifies to

$$
\mathbb{P}((X_k, X_{\ell}) = (x_k, x_{\ell})) = \begin{cases}
\left(\pi_1 + \lambda_2^{\ell-k} \frac{a}{a+b}\right) \pi_1, & \text{if } (x_k, x_{\ell}) = (\alpha_1, \alpha_1), \\
\left(\pi_1 - \lambda_2^{\ell-k} \frac{b}{a+b}\right) \pi_2, & \text{if } (x_k, x_{\ell}) = (\alpha_1, \alpha_2), \\
\left(\pi_2 - \lambda_2^{\ell-k} \frac{a}{a+b}\right) \pi_1, & \text{if } (x_k, x_{\ell}) = (\alpha_2, \alpha_1), \\
\left(\pi_2 + \lambda_2^{\ell-k} \frac{b}{a+b}\right) \pi_2, & \text{if } (x_k, x_{\ell}) = (\alpha_2, \alpha_2).
\end{cases}
$$

(3.6)

We can now compute $\mathbb{E}Z_k^1Z_{\ell}^1$:

$$
\mathbb{E}Z_k^1Z_{\ell}^1 = \mathbb{P}(Z_k^1Z_{\ell}^1 = +1) - \mathbb{P}(Z_k^1Z_{\ell}^1 = -1) = \mathbb{P}((X_k, X_{\ell}) \in \{(\alpha_1, \alpha_1), (\alpha_2, \alpha_2)\}) - \mathbb{P}((X_k, X_{\ell}) \in \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\})
= \left(\pi_1^2 + \lambda_2^{\ell-k} \frac{a}{a+b}\pi_1 + \pi_2^2 + \lambda_2^{\ell-k} \frac{b}{a+b}\pi_2\right)
- \left(\pi_1\pi_2 - \lambda_2^{\ell-k} \frac{b}{a+b}\pi_2 + \pi_1\pi_2 - \lambda_2^{\ell-k} \frac{a}{a+b}\pi_1\right)
= \left(\pi_1^2 + \pi_2^2 + \frac{2ab}{(a+b)^2}\lambda_2^{\ell-k}\right) - \left(2\pi_1\pi_2 - \frac{2ab}{(a+b)^2}\lambda_2^{\ell-k}\right)
$$
\[
= \frac{(b-a)^2}{(a+b)^2} + \frac{4ab}{(a+b)^2} \lambda^{\ell-k}. \tag{3.7}
\]

Hence, recalling that \( \beta = 0, \)
\[
\sigma^2 := \text{Var}Z_k^1 = 1 - \left(\frac{b-a}{a+b}\right)^2
= \frac{4ab}{(a+b)^2}, \tag{3.8}
\]
for all \( k \geq 1, \) and, for \( k < \ell, \) the covariance of \( Z_k^1 \) and \( Z_\ell^1 \) is
\[
\text{Cov}(Z_k^1, Z_\ell^1) = \frac{(b-a)^2}{(a+b)^2} + \sigma^2 \lambda^{\ell-k} - \left(\frac{b-a}{a+b}\right)^2 = \sigma^2 \lambda^{\ell-k}. \tag{3.9}
\]

Proceeding to the covariance structure of \( S_k^1, \) we first find that
\[
\text{Var}S_k^1 = \sum_{j=1}^{k} \text{Var}Z_j^1 + 2 \sum_{j<\ell} \text{Cov}(Z_j^1, Z_\ell^1)
= \sigma^2 k + 2\sigma^2 \sum_{j<\ell} \lambda^{\ell-j}
= \sigma^2 k + 2\sigma^2 \left(\frac{\lambda_{\ell+1}^k - k\lambda_2^2 + (k-1)\lambda_2}{(1-\lambda_2)^2}\right)
= \sigma^2 \left(\frac{1 + \lambda_2}{1 - \lambda_2}\right) k + 2\sigma^2 \left(\frac{\lambda_2(\lambda_2^k - 1)}{(1-\lambda_2)^2}\right). \tag{3.10}
\]

Next, for \( k < \ell, \) and using (3.9) and (3.10), the covariance of \( S_k^1 \) and \( S_\ell^1 \) is given by
\[
\text{Cov}(S_k^1, S_\ell^1) = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \text{Cov}(Z_i^1, Z_j^1)
= \sum_{i=1}^{k} \text{Var}Z_i^1 + 2 \sum_{i<j<k} \text{Cov}(Z_i^1, Z_j^1) + \sum_{i=1}^{k} \sum_{j=k+1}^{\ell} \text{Cov}(Z_i^1, Z_j^1)\]
\begin{align*}
&= \text{Var} S^1_k + \sum_{i=1}^k \sum_{j=k+1}^\ell \text{Cov}(Z^1_i, Z^1_j) \\
&= \text{Var} S^1_k + \sigma^2 \left( \frac{\lambda_2(1 - \lambda_2^k)(1 - \lambda_2^{\ell-k})}{(1 - \lambda_2)^2} \right) \\
&= \sigma^2 \left( \frac{1 + \lambda_2}{1 - \lambda_2} k - \frac{\lambda_2(1 - \lambda_2^k)(1 + \lambda_2^{\ell-k})}{(1 - \lambda_2)^2} \right).
\end{align*}

(3.11)

From (3.10) and (3.11) we see that, as \( k \to \infty \),

\[
\frac{\text{Var} S^1_k}{k} \to \sigma^2 \left( \frac{1 + \lambda_2}{1 - \lambda_2} \right),
\]

(3.12)

and, moreover, as \( k \land \ell \to \infty \),

\[
\frac{\text{Cov}(S^1_k, S^1_\ell)}{(k \land \ell)} \to \sigma^2 \left( \frac{1 + \lambda_2}{1 - \lambda_2} \right).
\]

(3.13)

When \( a = b \), \( \mathbb{E} S^1_k = 0 \), and in (3.12) the asymptotic variance becomes

\[
\frac{\text{Var} S^1_k}{k} \to \frac{4a^2}{(2a)^2} \left( \frac{1 + (1 - 2a)}{1 - (1 - 2a)} \right)
\]

\[
= \frac{1}{a} - 1.
\]

For \( a \) small, we have a "lazy" Markov chain, that is, a Markov chain which tends to remain in a given state for long periods of time. In this regime, the random variable \( S^1_k \) has long periods of increase followed by long periods of decrease. In this way, linear asymptotics of the variance with large constants occur. If, on the other hand, \( a \) is close to 1, the Markov chain rapidly shifts back and forth between \( \alpha_1 \) and \( \alpha_2 \), and so the constant associated with the linearly increasing variance of \( S^1_k \) is small.

As in [13], Brownian functionals play a central rôle in describing the limiting distribution of \( LI_n \). By a Brownian motion on \([0, 1]\) we shall mean an a.s. continuous, centered Gaussian process having stationary, independent increments, and which is zero at the origin. By a standard Brownian motion \( B(t), 0 \leq t \leq 1 \), we shall further require that \( \text{Var} B(t) = t, \ 0 \leq t \leq 1 \), i.e., we endow \( C[0, 1] \) with the Wiener measure. A standard \( m \)-dimensional Brownian motion will be defined to be a multivariate process consisting of \( m \) independent Brownian motions. More generally, an \( m \)-dimensional Brownian
motion shall refer to a linear transformation of a standard \( m \)-dimensional Brownian motion. Throughout the paper, we assume that our underlying probability space is rich enough so that all the Brownian motions and sequences we study can be defined on it.

To move towards a Brownian functional expression for the limiting law of \( L I_n \), define the polygonal function

\[
\hat{B}_n(t) = \frac{S_{[nt]}^1 - [nt]\mu}{\sigma \sqrt{n(1 + \lambda_2)/(1 - \lambda_2)}} + \frac{(nt - [nt])(Z_{[nt]+1}^1 - \mu)}{\sigma \sqrt{n(1 + \lambda_2)/(1 - \lambda_2)}},
\]

for \( 0 \leq t \leq 1 \). In our finite-state, irreducible, aperiodic, stationary Markov chain setting, we may conclude that \( \hat{B}_n \Rightarrow B \), as desired. (See, for example, Gordin’s martingale approach to dependent invariance principles [15], and the stationary ergodic invariance principle found in Theorem 19.1 of Billingsley [5].)

Turning now to \( L I_n \), we see that for the present 2-letter situation, (2.5) simply becomes

\[
L I_n = n^2 - 1 \frac{1}{2} S^1_n + \max_{1 \leq k \leq n} S^1_k.
\]

To find the limiting distribution of \( L I_n \) from this expression, recall that 
\( \pi_1 = b/(a + b), \pi_2 = a/(a + b), \mu = \pi_1 - \pi_2 = (b - a)/(a + b), \sigma^2 = 4ab/(a + b)^2 \), and that \( \lambda_2 = 1 - a - b \). Define \( \pi_{\text{max}} = \max\{\pi_1, \pi_2\} \) and \( \tilde{\sigma}^2 = \sigma^2(1 + \lambda_2)/(1 - \lambda_2) \). Rewriting (3.14) as

\[
\hat{B}_n(t) = \frac{S_{[nt]}^1 - [nt]\mu}{\tilde{\sigma} \sqrt{n}} + \frac{(nt - [nt])(Z_{[nt]+1}^1 - \mu)}{\tilde{\sigma} \sqrt{n}},
\]

\( L I_n \) becomes

\[
L I_n = n^2 - 1 \frac{1}{2} \left( \tilde{\sigma} \sqrt{n}\hat{B}_n(1) + \mu n \right) + \max_{0 \leq t \leq 1} \left( \tilde{\sigma} \sqrt{n}\hat{B}_n(t) + \mu t \right)
\]
\[
= n\pi_2 - 1 \frac{1}{2} \left( \tilde{\sigma} \sqrt{n}\hat{B}_n(1) \right) + \max_{0 \leq t \leq 1} \left( \tilde{\sigma} \sqrt{n}\hat{B}_n(t) + (\pi_1 - \pi_2)nt \right)
\]
\[
= n\pi_{\text{max}} - 1 \frac{1}{2} \left( \tilde{\sigma} \sqrt{n}\hat{B}_n(1) \right)
\]
\[
+ \max_{0 \leq t \leq 1} \left( \tilde{\sigma} \sqrt{n}\hat{B}_n(t) + (\pi_1 - \pi_2)nt - (\pi_{\text{max}} - \pi_2)n \right). \tag{3.15}
\]
This immediately gives

\[
\frac{LI_n - \pi_{\text{max}} n}{\tilde{\sigma} \sqrt{n}} = -\frac{1}{2} \hat{B}_n(1) + \max_{0 \leq t \leq 1} \left( \hat{B}_n(t) + \sqrt{\frac{n}{\sigma}} (\pi_1 - \pi_2)t - (\pi_{\text{max}} - \pi_2) \right). \tag{3.16}
\]

Let us examine (3.16) on a case-by-case basis. First, if \( \pi_{\text{max}} = \pi_1 = \pi_2 = 1/2 \), i.e., if \( a = b \), then \( \sigma = 1 \) and \( \tilde{\sigma} = (1 - a)/a \), and so (3.16) becomes

\[
\frac{LI_n - n/2}{\sqrt{(1 - a)n/a}} = -\frac{1}{2} \hat{B}_n(1) + \max_{0 \leq t \leq 1} \hat{B}_n(t). \tag{3.17}
\]

Then, by the Invariance Principle and the Continuous Mapping Theorem,

\[
\frac{LI_n - n/2}{\sqrt{(1 - a)n/a}} \Rightarrow -\frac{1}{2} B(1) + \max_{0 \leq t \leq 1} B(t). \tag{3.18}
\]

Next, if \( \pi_{\text{max}} = \pi_2 > \pi_1 \), (3.16) becomes

\[
\frac{LI_n - \pi_{\text{max}} n}{\tilde{\sigma} \sqrt{n}} = -\frac{1}{2} \hat{B}_n(1) + \max_{0 \leq t \leq 1} \left( \hat{B}_n(t) - \sqrt{\frac{n}{\sigma}} (\pi_{\text{max}} - \pi_1)(1 - t) \right). \tag{3.19}
\]

On the other hand, if \( \pi_{\text{max}} = \pi_1 > \pi_2 \), (3.16) becomes

\[
\frac{LI_n - \pi_{\text{max}} n}{\tilde{\sigma} \sqrt{n}} = -\frac{1}{2} \hat{B}_n(1) + \max_{0 \leq t \leq 1} \left( \hat{B}_n(t) - \hat{B}_n(1) - \sqrt{\frac{n}{\sigma}} (\pi_{\text{max}} - \pi_2)(1 - t) \right) - \frac{1}{2} \hat{B}_n(1) + \max_{0 \leq t \leq 1} \left( \hat{B}_n(t) - \hat{B}_n(1) - \sqrt{\frac{n}{\sigma}} (\pi_{\text{max}} - \pi_2)(1 - t) \right). \tag{3.20}
\]
In both (3.19) and (3.20) we have a term in our maximal functional which is linear in \(t\) or \(1 - t\), with a negative slope. We now show, in an elementary fashion, that in both cases, as \(n \to \infty\), the maximal functional goes to zero in probability.

Consider first (3.19). Let \(c_n = \sqrt{n(\pi_{\text{max}} - \pi_1)}/\tilde{\sigma} > 0\), and for any \(c > 0\), let \(M_c = \max_{0 \leq t \leq 1}(B(t) - ct)\), where \((B(t))\) is a standard Brownian motion. Now for \(n\) large enough,

\[
\hat{B}_n(t) - ct \geq \hat{B}_n(t) - c_nt
\]
a.s., for all \(0 \leq t \leq 1\). Then for any \(z > 0\), and \(n\) large enough,

\[
\mathbb{P}(\max_{0 \leq t \leq 1}(\hat{B}_n(t) - c_nt) > z) \leq \mathbb{P}(\max_{0 \leq t \leq 1}(\hat{B}_n(t) - ct) > z), \tag{3.21}
\]
and so by the Invariance Principle and the Continuous Mapping Theorem,

\[
\limsup_{n \to \infty} \mathbb{P}(\max_{0 \leq t \leq 1}(\hat{B}_n(t) - c_nt) > z) \leq \lim_{n \to \infty} \mathbb{P}(\max_{0 \leq t \leq 1}(\hat{B}_n(t) - ct) > z) = \mathbb{P}(M_c > z). \tag{3.22}
\]

Now, as is well-known, \(\mathbb{P}(M_c > z) \to 0\) as \(c \to \infty\). One can confirm this intuitive fact with the following simple argument. For \(z > 0\), \(c > 0\), and \(0 < \varepsilon < 1\), we have that

\[
\mathbb{P}(M_c > z) \leq \mathbb{P}(\max_{0 \leq t \leq \varepsilon}(B(t) - ct) > z) + \mathbb{P}(\max_{\varepsilon < t \leq 1}(B(t) - ct) > z)
\leq \mathbb{P}(\max_{0 \leq t \leq \varepsilon} B(t) > z) + \mathbb{P}(\max_{\varepsilon < t \leq 1}(B(t) - c\varepsilon) > z)
\leq \mathbb{P}(\max_{0 \leq t \leq \varepsilon} B(t) > z) + \mathbb{P}(\max_{0 < t \leq 1} B(t) > \varepsilon + z)
= 2 \left(1 - \Phi\left(\frac{z}{\sqrt{\varepsilon}}\right)\right) + 2 \left(1 - \Phi(c\varepsilon + z)\right). \tag{3.23}
\]

But, as \(c\) and \(\varepsilon\) are arbitrary, we can first take the limsup of (3.23) as \(c \to \infty\), and then let \(\varepsilon \to 0\), proving the claim.

We have thus shown that

\[
\limsup_{n \to \infty} \mathbb{P}(\max_{0 \leq t \leq 1}(\hat{B}_n(t) - c_nt) > z) \leq 0,
\]

and since the functional clearly is equal to zero when \( t = 0 \), we have
\[
\max_{0 \leq t \leq 1} (\hat{B}_n(t) - c_n t) \xrightarrow{p} 0, \quad (3.24)
\]
as \( n \to \infty \). Thus, by the Continuous Mapping Theorem, and the Converging Together Lemma, we obtain the weak convergence result
\[
\frac{L I_n - \pi_{\text{max}} n}{\bar{\sigma} \sqrt{n}} \xrightarrow{p} -\frac{1}{2} B(1). \quad (3.25)
\]

Lastly, consider (3.20). Here we need simply note the following equality in law, which follows from the stationary and Markovian nature of the underlying sequence \( (X_n)_{n \geq 0} \):
\[
\hat{B}_n(t) - \hat{B}_n(1) - \frac{\sqrt{n}}{\bar{\sigma}} (\pi_{\text{max}} - \pi_2)(1 - t)
\]
\[= -\hat{B}_n(1 - t) - \frac{\sqrt{n}}{\bar{\sigma}} (\pi_{\text{max}} - \pi_2)(1 - t), \quad (3.26)
\]
for \( t = 0, 1/n, \ldots, (n - 1)/n, 1 \). With a change of variables \( u = 1 - t \), and noting that \( B(t) \) and \( -B(t) \) are equal in law, our previous convergence result \(3.24\) implies that
\[
\max_{0 \leq u \leq 1} (-\hat{B}_n(u) - c_n u) \xrightarrow{p} 0, \quad (3.27)
\]
as \( n \to \infty \). Our limiting functional is thus of the form
\[
\frac{L I_n - \pi_{\text{max}} n}{\bar{\sigma} \sqrt{n}} \Rightarrow 1/2 B(1). \quad (3.28)
\]
Since \( B(1) \) is simply a standard normal random variable, the different signs in \(3.25\) and \(3.28\) are inconsequential.

Finally, consider the degenerate cases. If either \( a = 0 \) or \( b = 0 \), then the sequence \( (X_n)_{n \geq 0} \) will be a.s. constant, regardless of the starting state, and so \( L I_n \sim n \). On the other hand, if \( a = b = 1 \), then the sequence oscillates back and forth between \( \alpha_1 \) and \( \alpha_2 \), so that \( L I_n \sim n/2 \). Combining these trivial cases with the previous development, we have proved the following theorem:
Theorem 3.1  Let \((X_n)_{n \geq 0}\) be a 2-state Markov chain, with \(\mathbb{P}(X_{n+1} = \alpha_2 | X_n = \alpha_1) = a\) and \(\mathbb{P}(X_{n+1} = \alpha_1 | X_n = \alpha_2) = b\). Let the law of \(X_0\) be the invariant distribution \((\pi_1, \pi_2) = (b/(a+b), a/(a+b))\), for \(0 < a + b \leq 2\), and \((\pi_1, \pi_2) = (1, 0)\), for \(a = b = 0\). Then, for \(a = b > 0\),

\[
\frac{LI_n - n/2}{\sqrt{n}} \Rightarrow \sqrt{\frac{1-a}{a}} \left( -\frac{1}{2} B(1) + \max_{0 \leq t \leq 1} B(t) \right),
\]

(3.29)

where \((B(t))_{t \in [0,1]}\) is a standard Brownian motion, and for \(a \neq b\) or \(a = b = 0\),

\[
\frac{LI_n - \pi_{\text{max}} n}{\sqrt{n}} \Rightarrow N(0, \tilde{\sigma}^2/4),
\]

(3.30)

where \(N(0, \tilde{\sigma}^2/4)\) is a centered normal random variable with variance \(\tilde{\sigma}^2/4 = ab(2 - a - b)/(a + b)^3\), for \(a \neq b\), and \(\tilde{\sigma}^2 = 0\), for \(a = b = 0\). (If \(a = b = 1\), or \(\tilde{\sigma}^2 = 0\), then the distributions in (3.29) and (3.30), respectively, are understood to be degenerate at the origin.)

To extend this result to the entire Young tableau, let us introduce the following notation. By

\[
(Y_n^{(1)}, Y_n^{(2)}, \ldots, Y_n^{(k)}) \Rightarrow (Y_\infty^{(1)}, Y_\infty^{(2)}, \ldots, Y_\infty^{(k)})
\]

(3.31)

we shall mean the weak convergence of the joint law of the \(k\)-vector \((Y_n^{(1)}, Y_n^{(2)}, \ldots, Y_n^{(k)})\) to that of \((Y_\infty^{(1)}, Y_\infty^{(2)}, \ldots, Y_\infty^{(k)})\), as \(n \to \infty\). Since \(LI_n\) is the length of the top row of the associated Young tableau, the length of the second row is simply \(n - LI_n\). Denoting the length of the \(i^{\text{th}}\) row by \(LY_i\), (3.31), together with an application of the Cramér-Wold Theorem, recovers the result of Chistyakov and Götze [9] as part of the following easy corollary, which is in fact equivalent to Theorem 3.1.

Corollary 3.1  For the sequence in Theorem 3.1, if \(a = b > 0\), then

\[
\left( \frac{LY_1 - n/2}{\sqrt{n}}, \frac{LY_2 - n/2}{\sqrt{n}} \right) \Rightarrow Y_\infty := (Y_\infty^{(1)}, Y_\infty^{(2)}),
\]

(3.32)

where the law of \(Y_\infty\) is supported on the 2nd main diagonal of \(\mathbb{R}^2\), and with
\[ Y^{(1)}_{\infty} \equiv \sqrt{\frac{1-a}{a}} \left( -\frac{1}{2} B(1) + \max_{0 \leq t \leq 1} B(t) \right). \]

If \( a \neq b \) or \( a = b = 0 \), then setting \( \pi_{min} = \min\{\pi_1, \pi_2\} \), we have

\[ \left( \frac{LY^1_n - \pi_{max} n}{\sqrt{n}}, \frac{LY^2_n - \pi_{min} n}{\sqrt{n}} \right) \Rightarrow N((0,0), \tilde{\Sigma}), \quad (3.33) \]

where \( \tilde{\Sigma} \) is the covariance matrix

\[ (\tilde{\sigma}^2/4) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \]

where \( \tilde{\sigma}^2 = 4ab(2-a-b)/(a+b)^3 \), for \( a \neq b \), and \( \tilde{\sigma}^2 = 0 \), for \( a = b = 0 \).

**Remark 3.1** The joint distributions in (3.32) and (3.33) are of course degenerate, in that the sum of the two components is a.s. identically zero in each case. In (3.32), the density of the first component of \( Y_{\infty} \) is easy to find, and is given by (e.g., see [19])

\[ f(y) = \frac{16}{\sqrt{2\pi}} \left( \frac{a}{1-a} \right)^{3/2} y^2 e^{-2ay^2/(1-a)}, \quad y \geq 0. \quad (3.34) \]

As in Chistyakov and Götze [9], (3.32) can then be stated as: For any bounded, continuous function \( g : \mathbb{R}^2 \rightarrow \mathbb{R} \),

\[ \lim_{n \to \infty} \left( g \left( \frac{LY^1_n - n/2}{\sqrt{(1-a)n/a}}, \frac{LY^2_n - n/2}{\sqrt{(1-a)n/a}} \right) \right) = 2\sqrt{2\pi} \int_0^\infty g(x,-x)\phi_{GUE,2}(x,-x)dx, \]

where \( \phi_{GUE,2} \) is the density of the eigenvalues of the \( 2 \times 2 \) GUE, and is given by

\[ \phi_{GUE,2}(x_1, x_2) = \frac{1}{\pi} (x_1 - x_2)^2 e^{-(x_1^2+x_2^2)}. \]

To see the GUE connection more explicitly, consider the \( 2 \times 2 \) traceless GUE matrix
\[ M_0 = \begin{pmatrix} X_1 & Y + iZ \\ Y - iZ & X_2 \end{pmatrix}, \]

where \( X_1, X_2, Y, \) and \( Z \) are centered, normal random variables. Since \( \text{Corr} (X_1, X_2) = -1 \), the largest eigenvalue of \( M_0 \) is

\[ \lambda_{1,0} = \sqrt{X_1^2 + Y^2 + Z^2}, \]

almost surely, so that \( \lambda_{1,0}^2 \sim \chi_3^2 \) if \( \text{Var} X_1 = \text{Var} Y = \text{Var} Z = 1 \). Hence, up to a scaling factor, the density of \( \lambda_{1,0} \) is given by (3.34). Next, let us perturb \( M_0 \) to

\[ M = \alpha GI + \beta M_0, \]

where \( \alpha \) and \( \beta \) are constants, \( G \) is a standard normal random variable independent of \( M_0 \), and \( I \) is the identity matrix. The covariance of the diagonal elements of \( M \) is then computed to be \( \rho := \alpha^2 - \beta^2 \). Hence, to obtain a given value of \( \rho \), we may take \( \alpha = \sqrt{(1 + \rho)/2} \) and \( \beta = \sqrt{(1 - \rho)/2} \). Clearly, the largest eigenvalue of \( M \) can then be expressed as

\[ \lambda_1 = \sqrt{\frac{1 + \rho}{2}} G + \sqrt{\frac{1 - \rho}{2}} \lambda_{1,0}. \quad (3.35) \]

At one extreme, \( \rho = -1 \), we recover \( \lambda_1 = \lambda_{1,0} \). At the other extreme, \( \rho = 1 \), we obtain \( \lambda_1 = Z \). Midway between these two extremes, at \( \rho = 0 \), we have a standard GUE matrix, so that

\[ \lambda_1 = \sqrt{\frac{1}{2}} (G + \lambda_{1,0}). \]

4 Combinatorics Revisited

The original combinatorial development for the \( m \)-letter alphabet resulted in \( m - 1 \) quantities \( S_n^r \), \( 1 \leq r \leq m - 1 \). In the 2-letter case we were then able to proceed with a probabilistic development which involved a single Brownian motion. Using an even more straightforward development which involves \( m \) quantities instead, we can obtain more symmetric expressions for \( LI_n \). This is done next, and will prove useful when studying the shape of the whole
Recall that $a^r_k$ counts the number of occurrences of $\alpha_r$ among $(X_i)_{1 \leq i \leq k}$. Moving beyond the purely combinatorial setting, assume that $(X_k)_{k \geq 0}$ is a doubly-infinite sequence generated by an irreducible homogeneous Markov chain having a stationary distribution $(\pi_1, \pi_2, \ldots, \pi_m)$. (For no $k \geq 0$ is the law of $X_k$ necessarily assumed to be the stationary distribution.) For each $1 \leq r \leq m$, set $T^r_k = a^r_k - \pi_r k$, for $k \geq 1$, and $T^r_0 = 0$. Beginning again with (2.1), we find that

\begin{align*}
LI_n &= \max_{0 \leq k_0 \leq \cdots \leq k_{m-1} \leq n} \left[ (a^1_{k_1} - a^1_0) + (a^2_{k_2} - a^2_{k_1}) + \cdots + (a^m_{k_m} - a^m_{k_{m-1}}) \right] \\
&= \max_{0 \leq k_1 \leq \cdots \leq k_{m-1} \leq n} \left[ (T^1_{k_1} + \pi_1 k_1) - (T^1_{k_0} + \pi_1 k_0) \right] + \left[ (T^2_{k_2} + \pi_2 k_2) - (T^2_{k_1} + \pi_2 k_1) \right] \\
&\quad + \cdots + \left[ (T^m_{k_m} + \pi_m k_m) - (T^m_{k_{m-1}} + \pi_m k_{m-1}) \right] \\
&= \max_{0 \leq k_1 \leq \cdots \leq k_{m-1} \leq n} \left[ (T^1_{k_1} - T^1_{k_0}) + (T^2_{k_2} - T^2_{k_1}) + \cdots + (T^m_{k_m} - T^m_{k_{m-1}}) \right] \\
&\quad + \pi_1(k_1 - k_0) + \pi_2(k_2 - k_1) + \cdots + \pi_m(k_m - k_{m-1}) \tag{4.1}
\end{align*}

Setting $\pi_{\max} = \max\{\pi_1, \pi_2, \ldots, \pi_m\}$, (4.1) becomes

\begin{align*}
LI_n - \pi_{\max} n &= \max_{0 \leq k_0 \leq k_1 \leq \cdots \leq k_{m-1} \leq n} \sum_{r=1}^m \left[ (T^r_{k_r} - T^r_{k_{r-1}}) + (\pi_r - \pi_{\max})(k_r - k_{r-1}) \right]. \tag{4.2}
\end{align*}

For a uniform alphabet, $\pi_{\max} = \pi_r = 1/m$, for all $r$, and (4.2) simplifies to

\begin{align*}
LI_n - \frac{n}{m} &= \max_{0 \leq k_0 \leq k_1 \leq \cdots \leq k_{m-1} \leq n} \sum_{r=1}^m (T^r_{k_r} - T^r_{k_{r-1}}). \tag{4.3}
\end{align*}

To introduce a random walk formalism into the picture, we next set, for $i = 1, \ldots, n$ and $r = 1, 2, \ldots, m$,

\begin{align*}
W^r_i &= \begin{cases} 
1, & \text{if } X_i = \alpha_r, \\
0, & \text{otherwise.} 
\end{cases} \tag{4.4}
\end{align*}
Clearly, $a'_k = \sum_{i=1}^{k} W^r_i$, and so $T^r_k = \sum_{i=1}^{k} (W^r_i - \pi_r)$, for $1 \leq r \leq m$.

To understand the limiting law of (4.2) or (4.3), we must have a more precise description of the underlying Markovian structure. To that end, let $p_{r,s} = \mathbb{P}(X_{k+1} = \alpha_s | X_k = \alpha_r)$ be the transition probability from state $\alpha_r$ to state $\alpha_s$, and let $P = (p_{r,s})$ be the associated Markov transition matrix. In this setting,

$$ (p_{1}^{n+1}, p_{2}^{n+1}, \ldots, p_{m}^{n+1}) = (p_{1}^{n}, p_{2}^{n}, \ldots, p_{m}^{n}) P. $$

Moreover, as usual, let $p_{r,s}^{(k)}$ denote the $k$-step transition probability from $\alpha_r$ to $\alpha_s$; its associated transition matrix is simply $P^k$.

Assume now that the law of $X_0$ is the stationary distribution. Thus, by construction, $\mathbb{E}T^r_k = 0$ for all $1 \leq r \leq m$ and $1 \leq k \leq n$, and our primary task is to describe the covariance structure of these random variables $T^r_k$.

Since $W^r_i$ is, simply, a Bernoulli random variable with parameter $\pi_r$, $\text{Var}W^r_i = \pi_r(1 - \pi_r)$. We then find that, for $k \geq 1$,

$$ \text{Var}T^r_k = \text{Var}\left( \sum_{i=1}^{k} (W^r_i - \pi_r) \right) $$
$$ = \sum_{i=1}^{k} \text{Var}W^r_i + \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} \text{Cov}(W^r_{i_1}, W^r_{i_2}) $$
$$ + \sum_{i_1=2}^{k} \sum_{i_2=1}^{i_1-1} \text{Cov}(W^r_{i_1}, W^r_{i_2}). \quad (4.5) $$

By stationarity, (4.5) becomes

$$ \text{Var}T^r_k = \sum_{i=1}^{k} \text{Var}W^r_i + \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} \text{Cov}(W^r_{0}, W^r_{i_2-i_1}) $$
$$ + \sum_{i_1=2}^{k} \sum_{i_2=1}^{i_1-1} \text{Cov}(W^r_{0}, W^r_{i_1-i_2}) $$
$$ = k\pi_r(1 - \pi_r) + \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} (\pi_r p_{r,r}^{(i_2-i_1)} - \pi_r^2) $$
\[ + \sum_{i_1=2}^{k} \sum_{i_2=1}^{i_1-1} (\pi_r p_{r_1}^{(i_1-i_2)} - \pi_r^2) \]
\[ = k\pi_r - k^2\pi_r^2 + \pi_r \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} e_r P_{i_1-i_2} e_r^T \]
\[ + \pi_r \sum_{i_1=2}^{k} \sum_{i_2=1}^{i_1-1} e_r P_{i_1-i_2} e_r^T, \quad (4.6) \]

where \( e_r = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \) is the \( r \)th standard basis vector of \( \mathbb{R}^m \).

Setting
\[ Q_k = \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} P_{i_1-i_2} = \sum_{i=1}^{k} (k-i)P_i, \quad (4.7) \]
we can rewrite (4.6) in the simple form
\[ \text{Var}T_k^r = k\pi_r - k^2\pi_r^2 + 2\pi_r e_r Q_k e_r^T. \quad (4.8) \]

Our description of the covariance structure can now be completed using the above results. For \( r_1 \neq r_2 \) and \( k \geq 1 \),

\[ \text{Cov}(T_k^{r_1}, T_k^{r_2}) = \sum_{i=1}^{k} \text{Cov}(W_i^{r_1}, W_i^{r_2}) + \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} \text{Cov}(W_i^{r_1}, W_{i_2}^{r_2}) \]
\[ + \sum_{i_1=2}^{k} \sum_{i_2=1}^{i_1-1} \text{Cov}(W_i^{r_1}, W_{i_2}^{r_2}) \]
\[ = \sum_{i=1}^{k} \text{Cov}(W_i^{r_1}, W_i^{r_2}) + \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} \text{Cov}(W_0^{r_1}, W_{i_2}^{r_2}) \]
\[ + \sum_{i_1=2}^{k} \sum_{i_2=1}^{i_1-1} \text{Cov}(W_0^{r_2}, W_{i_1-i_2}^{r_1}) \]
\[ = -k\pi_{r_1}\pi_{r_2} + \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} (\pi_{r_1} p_{i_1}^{(i_2-i_1)} - \pi_{r_1}\pi_{r_2}) \]
\[ k \sum_{i=1}^{k} \sum_{i_2=1}^{i_1-1} (\pi_{r_2} P_{r_2,r_1}^{i_1-i_2} - \pi_{r_1} \pi_{r_2}) \]

\[ = -k^2 \pi_{r_1} \pi_{r_2} + \pi_{r_1} \sum_{i_1=1}^{k-1} \sum_{i_2=i_1+1}^{k} \epsilon_{r_1} P^{i_2-i_1} \epsilon_{r_2}^T \]

\[ + \pi_{r_2} \sum_{i_1=2}^{k} \sum_{i_2=1}^{i_1-1} \epsilon_{r_2} P^{i_1-i_2} \epsilon_{r_1}^T \]

\[ = -k^2 \pi_{r_1} \pi_{r_2} + \pi_{r_1} \epsilon_{r_1} Q_k \epsilon_{r_2}^T + \pi_{r_2} \epsilon_{r_2} Q_k \epsilon_{r_1}^T. \quad (4.9) \]

**Remark 4.1** Both (4.8) and (4.9) appear to be asymptotically quadratic in \( k \). However, since \( Q_k = \sum_{i=1}^{k} (k-i) P^i \), cancellations will show that when the Markov chain is irreducible and aperiodic, the order is, in fact, linear in \( k \).

In order to further analyze the asymptotics of \( Q_k \), we first examine the diagonalization of \( P \) for a very general class of transition matrices.

**Proposition 4.1** Let \( P \) be the \( m \times m \) transition matrix of an irreducible, aperiodic, homogeneous Markov chain with eigenvalues \( 1 > |\lambda_2| \geq \cdots \geq |\lambda_m| \), and let \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \). Let \( P = S^{-1} \Lambda S \) be the diagonalization of \( P \), where the rows of \( S \) consist of the left-eigenvectors of \( P \), with, moreover, the first row of \( S \) being the stationary distribution \((\pi_1, \pi_2, \ldots, \pi_m)\). Then the first column of \( S^{-1} \) is \((1, 1, \ldots, 1)^T\).

**Proof.** Since \( P = S^{-1} \Lambda S \), then \( P S^{-1} = S^{-1} \Lambda \). Denoting the first column of \( S^{-1} \) by \( c_1 \), we have \( P c_1 = c_1 \). But since the rows of \( P \) sum to 1, we see that \( c_1 = (1,1,\ldots,1)^T \) satisfies \( P c_1 = c_1 \). Moreover, \( c_1 \) must be unique, up to normalization, since the irreducibility of \( P \) implies that \( \lambda_1 = 1 \) has multiplicity 1. Finally, since the inner product of the first row of \( S \) and the first column of \( S^{-1} \) is 1, the correct normalization is indeed \((1,1,\ldots,1)^T\).

Returning to \( Q_k \), as given in (4.7), and using Proposition 4.1, we then obtain:

**Theorem 4.1** Let \( (X_n)_{n \geq 0} \) be a sequence generated by an \( m \)-letter, aperiodic, irreducible, homogeneous Markov chain with state space \( A_m = \{\alpha_1 < \cdots < \alpha_m\} \), transition matrix \( P \), and stationary distribution \((\pi_1, \pi_2, \ldots, \pi_m)\). Let
also the law of $X_0$ be the stationary distribution. Moreover, for $1 \leq r \leq m$, let $T_r^k = a_r^k - \pi_r k$, for $k \geq 1$, and $T_0^k = 0$, where $a_r^k$ is the number of occurrences of $\alpha_r$ among $(X_i)_{1 \leq i \leq k}$. Then, for $1 \leq r \leq m$,

$$
\lim_{k \to \infty} \frac{\text{Var} T_r^k}{k} = \pi_r \left( 1 + 2 \epsilon_r S^{-1} D \epsilon_r^T \right),
$$

(4.10)

and for $r_1 \neq r_2$,

$$
\lim_{k \to \infty} \frac{\text{Cov} (T_{r_1}^k, T_{r_2}^k)}{k} = \pi_{r_1} \epsilon_{r_1} S^{-1} D \epsilon_{r_2}^T + \pi_{r_2} \epsilon_{r_2} S^{-1} D \epsilon_{r_1}^T,
$$

(4.11)

where $P = S^{-1} \Lambda S$ is the standard diagonalization of $P$ in Proposition 4.1, and $D = \text{diag}(-1/2, \lambda_2/(1 - \lambda_2), \ldots, \lambda_m/(1 - \lambda_m))$. That is, the asymptotic covariance matrix of $(T_1^k, T_2^k, \ldots, T_m^k)$ is given by

$$
\Sigma = \Pi + \Pi (S^{-1} D S) + (S^{-1} D S)^T \Pi,
$$

(4.12)

where $\Pi = \text{diag}(\pi_1, \pi_2, \ldots, \pi_m)$.

**Proof.** Beginning with (4.7), we diagonalize $P$ and find that

$$
Q_k = \sum_{i=1}^{k-1} (k - i)(S^{-1} \Lambda S)^i
= S^{-1} \left( \sum_{i=1}^{k-1} (k - i) \Lambda^i \right) S
= S^{-1} \text{diag}(h(1), h(\lambda_2), \ldots, h(\lambda_m)) S,
$$

(4.13)

where $h(\lambda) := \sum_{k=1}^{n-1} (n - k) \lambda^k$. Now $h(1) = k(k - 1)/2$ is quadratic in $k$, while for $\lambda \neq 1$,

$$
h(\lambda) = k \frac{\lambda}{(1 - \lambda)} + \frac{\lambda(\lambda^k - 1)}{(1 - \lambda)^2},
$$

so that $h(\lambda)$ is linear in $k$. We thus can write $Q_k$ as the sum of terms which are, respectively, quadratic and linear in $k$. Recalling, moreover, that the first row of $S$ contains the stationary distribution, and that the first column of $S^{-1}$ is $(1, 1, \ldots, 1)^T$, we have
\[ Q_k = S^{-1} \text{diag}(h(1), h(\lambda_2), \ldots, h(\lambda_m)) S, \]
\[ = \frac{k^2}{2} S^{-1} \text{diag}(1, 0, \ldots, 0) S \]
\[ + kS^{-1} \text{diag} \left( \frac{1}{2} \frac{\lambda_2}{1 - \lambda_2}, \ldots, \frac{\lambda_m}{1 - \lambda_m} \right) S + o(k) \]
\[ = \frac{k^2}{2} \begin{pmatrix}
\pi_1 & \pi_2 & \cdots & \pi_m \\
\pi_1 & \pi_2 & \cdots & \pi_m \\
\vdots & \vdots & \ddots & \vdots \\
\pi_1 & \pi_2 & \cdots & \pi_m 
\end{pmatrix} + kS^{-1}DS + o(k). \tag{4.14} \]

Starting with the variance in (4.8), we now find that, for each \(1 \leq r \leq m,\)

\[ \text{Var} T_r^k = k\pi_r - \frac{k^2}{2} \pi_r^2 + 2\pi_r e_r Q_k e_r^T \]
\[ = k\pi_r - \frac{k^2}{2} \pi_r^2 + 2\pi_r \left( \frac{k^2}{2} \pi_r + k e_r S^{-1} DSe_r^T \right) + o(k) \]
\[ = k\pi_r \left( 1 + 2e_r S^{-1} DSe_r^T \right) + o(k), \tag{4.15} \]

from which the asymptotic result (4.10) follows immediately.

An identical development shows that, for \(r_1 \neq r_2,\) (4.9) simplifies to

\[ \text{Cov}(T_{r_1}^k, T_{r_2}^k) = -k^2 \pi_{r_1} \pi_{r_2} + \pi_{r_1} e_{r_1} Q_k e_{r_2}^T + \pi_{r_2} e_{r_2} Q_k e_{r_1}^T \]
\[ = -k^2 \pi_{r_1} \pi_{r_2} + \pi_{r_1} \left( \frac{k^2}{2} \pi_{r_2} + k e_{r_2} S^{-1} DSe_{r_2}^T \right) \]
\[ + \pi_{r_2} \left( \frac{k^2}{2} \pi_{r_1} + k e_{r_1} S^{-1} DSe_{r_1}^T \right) + o(k) \]
\[ = k \left( \pi_{r_1} e_{r_1} S^{-1} DSe_{r_2}^T + \pi_{r_2} e_{r_2} S^{-1} DSe_{r_1}^T \right) + o(k), \tag{4.16} \]

from which the asymptotic result (4.11) follows, and so does (4.12). \qed

**Remark 4.2** To see that (4.10) and (4.11) both recover the covariance results for the iid case investigated by the authors in [18], let \(P\) be the transition
matrix whose rows each consist of the stationary distribution \((\pi_1, \pi_2, \ldots, \pi_m)\). In this case \(\lambda_2 = \cdots = \lambda_m = 0\), and so \(D = \text{diag}(-1/2, 0, \ldots, 0)\). Hence,

\[
e_{r_1} S^{-1} D Se_r^T = (1, *, \ldots, *) D (\pi_{r_2}, *, \ldots, *)^T = \frac{-\pi_{r_2}}{2},
\]

for all \(r_1\) and \(r_2\), and so, for each \(r\),

\[
\lim_{k \to \infty} \frac{\text{Var} T_k^r}{k} = \pi_r \left(1 + 2 \left(-\frac{\pi_r}{2}\right)\right) = \pi_r(1 - \pi_r),
\]

while, for \(r_1 \neq r_2\),

\[
\lim_{k \to \infty} \frac{\text{Cov}(T_k^{r_1}, T_k^{r_2})}{k} = \pi_{r_1} \left(-\frac{\pi_{r_2}}{2}\right) + \pi_{r_2} \left(-\frac{\pi_{r_1}}{2}\right) = -\pi_{r_1}\pi_{r_2}.
\]

Note that, in the uniform iid case, we have \(\pi_r = 1/m\), for all \(1 \leq r \leq m\). Hence, for \(r_1 \neq r_2\), the asymptotic correlation between \(T_k^{r_1}\) and \(T_k^{r_2}\) is given by \((-1/(m^2))/(1/m(1-1/m)) = -1/(m-1)\), so that the covariance matrix is indeed the permutation-symmetric one obtained in the iid uniform case in [18]. There is, moreover, another Brownian functional representation for the iid uniform case in [18] in which the Brownian motions have a tridiagonal covariance matrix.

5 The Limiting Shape of the Young Tableau

Thus far, our results have centered on \(LI_n\) alone, essentially ignoring the larger question of the structure of the entire Young tableau. The present section extends the combinatorial development of the previous section to answer the question of the limiting shape of the Young tableau.

Our first result in this direction is a purely combinatorial expression generalizing (2.1). It is standard in the Young tableau literature to have entries chosen from the set \(\{1, 2, \ldots, m\}\). Below, without loss of generality, we allow our entries to be chosen from the \(m\)-letter ordered alphabet \(A_m = \{\alpha_1 < \cdots < \alpha_m\}\).

**Theorem 5.1** Let \(R_{1n}, R_{2n}, \ldots, R_{rn}\) be the lengths of the first \(1 \leq r \leq m\) rows of the Young tableau generated by the sequence \((X_k)_{1 \leq k \leq n}\) whose elements
belong to an ordered alphabet \( A_m = \{\alpha_1 < \cdots < \alpha_m\} \). Then, for each \( 1 \leq r \leq m \), the sum of the lengths of the first \( r \) rows of the Young tableau is given by

\[
\sum_{j=1}^{r} R_n^j = \max_{k_{j,\ell} \in J_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( a_k^j - a_k^j - a_{k_{j+1}}^j \right),
\]  

(5.1)

where \( J_{r,m} = \{(k_{j,\ell}, 1 \leq j \leq r, 0 \leq \ell \leq m) : k_{j,j-1} = 0, k_{j,m-r+j} = n, 1 \leq j \leq r; k_{j,\ell-1} \leq k_{j,\ell}, 1 \leq j \leq r, 1 \leq \ell \leq m; k_{j,\ell} \leq k_{j-1,\ell}, 2 \leq j \leq r, 1 \leq \ell \leq m\} \), and where \( a_k^j \) is the number of occurrences of \( \alpha_k \) among \( \{X_1, X_2, \ldots, X_k\} \).

**Proof.** Recall that the sum of the lengths of the first \( r \) rows of the Young tableau generated by a sequence \((X_k)_{1 \leq k \leq n}\), whose letters arise from an \( m \)-letter alphabet, has an interpretation in terms of the length of certain increasing sequences. Indeed, the sum \( R_n^1 + R_n^2 + \cdots + R_n^r \) is equal to the maximum sum of the lengths of \( r \) disjoint, increasing subsequences of \((X_k)_{1 \leq k \leq n}\), where by *disjoint* it is meant that each element of \((X_k)_{1 \leq k \leq n}\) occurs in at most one of the \( r \) subsequences. (See Lemma 1 of Section 3.2 in [13]). More general results of this sort, involving partial orderings of the alphabet and associated antichains, are known as Greene’s Theorem [17]. However, such results are not enough for our purpose. Below we need a different way of reconstructing disjoint subsequences.

We begin by examining an arbitrary collection of \( r \) disjoint, increasing subsequences of \((X_k)_{1 \leq k \leq n}\), and show that we can always map these \( r \) subsequences onto another collection of \( r \) disjoint, increasing subsequences whose properties will be amenable to our combinatorial analysis.

Specifically, with the number of rows \( r \) fixed, suppose that, for each \( 1 \leq j \leq r \), we have an increasing subsequence \((X_{k_{j,\ell}}^j)_{1 \leq \ell \leq n_j}\) of length \( n_j \leq n \), and that the \( r \) subsequences are disjoint.

We first construct the new subsequence \((\tilde{X}_{k_{j,\ell}}^1)_{1 \leq \ell \leq \tilde{n}_1}\) as follows. First, place all \( \alpha_k \)s occurring among the \( r \) original subsequences into \((\tilde{X}_{k_{j,\ell}}^1)_{1 \leq \ell \leq \tilde{n}_1}\), if there are any. If the last \( \alpha_1 \) occurs at the \( n^{th} \) index, then \((\tilde{X}_{k_{j,\ell}}^1)_{1 \leq \ell \leq \tilde{n}_1}\) is complete. Otherwise, place all \( \alpha_2 \)s which occur after the final \( \alpha_1 \) into \((\tilde{X}_{k_{j,\ell}}^1)_{1 \leq \ell \leq \tilde{n}_1}\), if there are any. If the last \( \alpha_2 \) occurs at the \( n^{th} \) index, then \((\tilde{X}_{k_{j,\ell}}^1)_{1 \leq \ell \leq \tilde{n}_1}\) is complete. Otherwise, continue adding, successively, \( \alpha_3, \ldots, \alpha_{m-r+1} \) in the
same manner. Thus, \((\tilde{X}^1_{k_1})_{1 \leq t \leq \tilde{n}_1}\) consists of a weakly increasing sequence of length \(\tilde{n}_1\) having values in \(\{\alpha_1, \ldots, \alpha_{m-r+1}\}\).

Next, we construct the new subsequence \((\tilde{X}^2_{k_2})_{1 \leq t \leq \tilde{n}_2}\) similarly. By considering only those letters among the \(r\) original subsequences which have not already been moved to the first new subsequence, start with the smallest available letter, \(\alpha_2\), and continue adding, successively, \(\alpha_3, \ldots, \alpha_{m-r-2}\). Note that, crucially, all \(\alpha_2\)s added to \((\tilde{X}^2_{k_2})_{1 \leq t \leq \tilde{n}_2}\) occur before the last index at which \(\alpha_1\) was added to the first subsequence. More generally, each \(\alpha_j\), \(2 \leq j \leq m-r+2\), added to \((\tilde{X}^2_{k_2})_{1 \leq t \leq \tilde{n}_2}\) occurs before the last \(\alpha_{j-1}\) was added to the first subsequence. Thus, \((\tilde{X}^2_{k_2})_{1 \leq t \leq \tilde{n}_2}\) consists of a weakly increasing subsequence of length \(\tilde{n}_2\) having values in \(\{\alpha_2, \ldots, \alpha_{m-r+2}\}\).

The construction of \((\tilde{X}^j_{k_j})_{1 \leq t \leq \tilde{n}_j}\), for \(3 \leq j \leq r\), continues in the same manner, with \((\tilde{X}^j_{k_j})_{1 \leq t \leq \tilde{n}_j}\), constructed from among the entries of the \(r\) original subsequences which were not moved into any of the first \(j-1\) new subsequences, so that \((\tilde{X}^j_{k_j})_{1 \leq t \leq \tilde{n}_j}\), consists of a weakly increasing sequence of length \(\tilde{n}_j\) having values in \(\{\alpha_j, \ldots, \alpha_{m-r+2}\}\). It is possible that beyond some \(j \geq 2\) the new subsequences may be empty.

We claim that, indeed, the construction of the \(r^{th}\) new subsequence exhausts the set of available entries. Indeed, without loss of generality, assume that after we have created the \((r-1)^{th}\) new subsequence, the set of available entries is non-empty, and designate the location of the final \(\alpha_\ell\) to be included in the \(j^{th}\) new subsequence by \(k_{j, \ell}\), for \(1 \leq j \leq r\) and \(1 \leq \ell \leq m\). (If no \(\alpha_\ell\) was available for inclusion, set \(k_{j, \ell} = k_{j, \ell-1}\), where \(k_{j, 0} = 0\), for all \(1 \leq j \leq r\).) Clearly, all \(\alpha_1, \alpha_2, \ldots, \alpha_{r-1}\) have been included in the first \(r-1\) new subsequences. If \(r = m\), we are done: simply put the remaining \(\alpha_\ell\)s into the \(r^{th}\) new subsequence. If \(r < m\), we may still ask whether there was, for some \(r+1 \leq \ell \leq m\), an \(\alpha_\ell\) from among the available entries which occurred before \(k_{r, \ell-1}\). Assume that there is such an \(\alpha_\ell\). Now by construction, \(k_{j+1, \ell-r+j} \leq k_{j, \ell-r-j+1}\), for \(1 \leq j \leq r-1\). Hence, there exist letters \(\alpha_{j_1} < \alpha_{j_2} < \cdots < \alpha_{j_r} \leq \alpha_{\ell-1}\) among the original subsequences which occurred after \(k_{r, \ell-1}\), and, moreover, each letter must come from a different subsequence. But since each original subsequence was increasing, none of them could have contained an \(\alpha_\ell\) before \(k_{r, \ell-1}\), and we have a contradiction.

To better understand this construction, consider the first row of Figure II, which shows an initial sequence of length \(n = 12\), with \(m = 4\) letters, broken
Figure 1: Transformation of $r = 3$ subsequences.

into $r = 3$ disjoint, increasing subsequences of lengths $n_1 = 3, n_2 = 4$, and $n_3 = 3$, and so with total length 10. The final three rows of the diagram show the results of the operations described above, producing 3 new increasing subsequences of length $\tilde{n}_1 = 4, \tilde{n}_2 = 3, \text{and } \tilde{n}_3 = 3$.

Hence, if we wish to find $r$ disjoint, increasing subsequences whose length sum is maximal, it suffices to consider only those disjoint, increasing subsequences for which the final occurrence of the letter $\alpha_\ell$ in the subsequence $i$ happens after the final occurrence in the subsequence $j$, whenever $i < j$. Because such ranges do not overlap, if we wish to count the number of $\alpha_\ell$s in the $j^{th}$ subsequence, it suffices to simply count the number of $\alpha$s in $(X_k)_{1 \leq k \leq n}$ over that range.

Indeed, returning to the fundamental combinatorial objects of our development, the $a_{k}^{j}$, we see that since $a_{k}^{j} - a_{k}^{j-1}$ counts the number of $\alpha_j$s in the range $\ell + 1, \ldots, k$, we can describe the valid index ranges over which to search for the maximal sum as $J_{r,m} = \{(k_{j,t}, 1 \leq j \leq r, 0 \leq \ell \leq m) : k_{j,j-1} = 0, k_{j,m-r+j} = n, 1 \leq j \leq r; k_{j,t-1} \leq k_{j,t}, 1 \leq j \leq r, 1 \leq \ell \leq m; k_{j,t} \leq k_{j-1,t}, 2 \leq j \leq r, 1 \leq \ell \leq m\}$. The constraints on the $k_{j,t}$ follow simply from
the fact that each subsequence is increasing and that, moreover, the intervals associated with a given letter do not overlap. Figure 2 indicates the relative positions of each range, for \( r = 4 \) and \( m = 7 \).

Since the first possible letter of each subsequence grows from \( \alpha_1 \) to \( \alpha_r \), and the last possible letter grows from \( \alpha_{m+r-1} \) to \( \alpha_m \), the result is proved.

We are now ready to apply our asymptotic covariance results (Theorem 4.1), along with a Brownian sample-path approximation, to the combinatorial expression \( (5.1) \), and so obtain a Brownian functional expression for the limiting shape of the Young tableau for all irreducible, aperiodic, homogeneous Markov chains.

Indeed, for each \( 1 \leq r \leq m \), let

\[
V^r_n := \sum_{j=1}^{r} R^j_n = \max_{k,j,\ell \in J_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( a^\ell_{k,j,\ell} - a^\ell_{k,j,\ell-1} \right), \tag{5.2}
\]

where the index set \( J_{r,m} \) is defined as in Theorem 5.1. Define as before \( T^r_k = \sum_{i=1}^{k} (W^r_i - \pi^r) = a^r_k - \pi^r k \), and so rewrite \((5.2)\) as

\[
V^r_n = \max_{k,j,\ell \in J_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( \left( T^\ell_{k,j,\ell} + \pi^\ell k_{j,\ell} \right) - \left( T^\ell_{k,j,\ell-1} + \pi^\ell (k_{j,\ell-1}) \right) \right)
= \max_{k,j,\ell \in J_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( \left( T^\ell_{k,j,\ell} - T^\ell_{k,j,\ell-1} \right) + \pi^\ell (k_{j,\ell} - k_{j,\ell-1}) \right). \tag{5.3}
\]
Next, let $\tau$ be a permutation of the indices $1, 2, \ldots, m$ such that $\pi_{\tau(1)} \geq \pi_{\tau(2)} \geq \cdots \geq \pi_{\tau(m)} > 0$. Moreover, we demand that if $\pi_{\tau(i)} = \pi_{\tau(j)}$ for $i < j$, then $\tau(i) < \tau(j)$. (The permutation so defined is thus unique.) Let $\nu_r = \sum_{j=1}^{r} \pi_{\tau(j)}$ be the sum of the $r$ largest values among $\pi_1, \pi_2, \ldots, \pi_m$. We obtain, below, the limiting distribution of $(V_n^r - \nu_r n)/\sqrt{n}$ as a Brownian functional.

To introduce Brownian sample-path approximations, and for each $1 \leq r \leq m$, we first define the asymptotic variance of $T_n^r$ as in (4.10), by

$$\sigma_r^2 := \lim_{n \to \infty} \frac{\text{Var} T_n^r}{n} = e_r \Sigma e_r, \quad (5.4)$$

and, for $r_1 \neq r_2$, the asymptotic covariance of $T_n^{r_1}$ and $T_n^{r_2}$ by

$$\sigma_{r_1, r_2} := \lim_{n \to \infty} \frac{\text{Cov}(T_n^{r_1}, T_n^{r_2})}{n} = e_{r_1} \Sigma e_{r_2}, \quad (5.5)$$

where $\Sigma$ is the covariance matrix of Theorem 4.1 associated with the transition matrix $P$. For each $1 \leq r \leq m$, we then let

$$\hat{B}_n^r(t) = \frac{T_n^r [nt] + (nt - [nt])(W_n^r [nt] + 1 - \pi_r)}{\sigma_r \sqrt{n}}, \quad (5.6)$$

for $0 \leq t \leq 1$. This rescaling of $[0, n]$ to $[0, 1]$ calls for us to define a new parameter set over which we will maximize a functional arising from the expressions in (5.6). Indeed, for any positive integers $s$ and $d$, with $s \leq d$, define the set

$$I_{s,d} = \left\{(t_{j,\ell}, 1 \leq j \leq s, 0 \leq \ell \leq d) : t_{j,j-1} = 0, t_{j,d-s+j} = 1, 1 \leq j \leq s; \right. $$

$$t_{j,\ell-1} \leq t_{j,\ell}, 1 \leq j \leq s, 1 \leq \ell \leq d; \right. $$

$$t_{j,\ell} \leq t_{j-1,\ell}, 2 \leq j \leq s, 1 \leq \ell \leq d \right\}.$$  

Note that the constraints $t_{j,j-1} = 0$ and $t_{j,d-s+j} = 1$, for $1 \leq j \leq s$, force many of the $t_{j,\ell}$ to be zero or one. We will denote the $s \times (d + 1)$-tuple elements of $I_{s,d}$, by $(t_{\cdot, \cdot})$. Figure 3 shows the structure of $I_{s,d}$, for $s = 4$ and $d = 7$. The locations of $t_{j,\ell}$ are indicated by the horizontal lines within the diagram.

With this notation, (5.3) becomes
Figure 3: Schematic diagram of $I_{s,d}$, for $s = 4, d = 7$.

\[
\frac{V_r^n - \nu_r n}{\sqrt{n}} = \max_{(t,\ldots) \in I_{r,m}} \left\{ \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_\ell \left( \hat{B}_n(\tau_j,\ell) - \hat{B}_n(\tau_j,\ell-1) \right) + \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sqrt{n} (\pi_\ell - \pi_{\tau(j)}) (t_j,\ell - t_{j,\ell-1}) \right\}. 
\] 

(5.7)

Our analysis of (5.7) will yield the following theorem, whose proof we defer to the conclusion of the section. This theorem gives, in particular, a full generalization of the limiting shape of the Young tableau in the non-uniform iid case.

**Theorem 5.2** Let $(X_n)_{n \geq 0}$ be an irreducible, aperiodic, homogeneous Markov chain with finite state space $A_m = \{\alpha_1, \ldots, \alpha_m\}$, transition matrix $P$, and stationary distribution $(\pi_1, \pi_2, \ldots, \pi_m)$. Let $\Sigma = (\sigma_{r,s})_{1 \leq r, s \leq m}$ be the associated asymptotic covariance matrix, as given in (4.12), and let the law of $X_0$ be given by the stationary distribution. Let $\tau$ be the permutation of $\{1, 2, \ldots, m\}$ such that $\pi_{\tau(i)} \geq \pi_{\tau(i+1)}$, and $\tau(i) < \tau(j)$ whenever $\pi_{\tau(i)} = \pi_{\tau(j)}$ and $i < j$. For each $1 \leq r \leq m$, let $V_r^n$ be the sum of the lengths of the first $r$ rows of the associated Young tableau, and let $\nu_r = \sum_{j=1}^{r} \pi_{\tau(j)}$. Finally, let $d_r$ be the multiplicity of $\pi_{\tau(r)}$, and let

\[
m_r = \begin{cases} 
0, & \text{if } \pi_{\tau(r)} = \pi_{\tau(1)}, \\
\max \{i : \pi_{\tau(i)} > \pi_{\tau(r)}\}, & \text{otherwise}.
\end{cases}
\]
Then, for each $1 \leq r \leq m$,

$$
\frac{V^r_n - \nu_r n}{\sqrt{n}} \Rightarrow V^r_\infty := \sum_{i=1}^{m_r} \sigma_\tau(i) \tilde{B}^\tau(i),
$$

(5.8)

$$
+ \max_{I_r-m_r, d_r} \sum_{j=1}^{r-m_r} \sigma_\tau(m_r + j) \left( \tilde{B}^\tau(m_r + j)(t_{j,\ell}) - \tilde{B}^\tau(m_r + j)(t_{j,\ell-1}) \right),
$$

(5.9)

where the first sum on the right-hand side of (5.8) is understood to be 0, if $m_r = 0$. Above, $\sigma^2_r = \sigma_{r,r}$, and $(\tilde{B}^1(t), \tilde{B}^2(t), \ldots, \tilde{B}^m(t))$ is an $m$-dimensional Brownian motion, with covariance matrix $\tilde{\Sigma} = (\tilde{\sigma}_{r,s})_{1 \leq r,s \leq m}$ given by

$$
(\tilde{\sigma}_{r,s}) = \frac{t}{\sigma_r \sigma_s},
$$

(5.9)

for $1 \leq r, s \leq m$. Moreover, for any $1 \leq k \leq m$,

$$
\left( \frac{V^1_n - \nu_1 n}{\sqrt{n}}, \frac{V^2_n - \nu_2 n}{\sqrt{n}}, \ldots, \frac{V^k_n - \nu_k n}{\sqrt{n}} \right) \Rightarrow (V^1_\infty, V^2_\infty, \ldots, V^k_\infty).
$$

(5.10)

**Remark 5.1** The critical indices $d_r$ and $m_r$ in Theorem 5.2 are chosen so that

$$
\pi_\tau(m_r) > \pi_\tau(m_r + 1) = \pi_\tau(r) = \cdots = \pi_\tau(m_r + d_r) > \pi_\tau(m_r + d_r + 1).
$$

Thus, the functional in (5.8) consists of a sum of $m_r$ Gaussian random variables and a maximal functional involving only $d_r$ of the $m$ one-dimensional Brownian motions.

**Remark 5.2** Another, more natural, way of describing the covariance structure of the $m$-dimensional Brownian motion in Theorem 5.2 is to note that $(\sigma_1 B^1(t), \sigma_2 B^2(t), \ldots, \sigma_m B^m(t))$ has covariance matrix $t\Sigma$.

Let us now examine the case $r = 1$. Here, as previously noted, $V^1_n = LI_n$. Since $m_1 = 0$, (5.8) becomes

$$
\frac{LI_n - \nu_{max} n}{\sqrt{n}} \Rightarrow \max_{(\ell, \ldots) \in I_{1,d_1}} \sum_{\ell=1}^{d_1} \sigma_\tau(\ell) \left( \tilde{B}^\tau(\ell)(t_{1,\ell}) - \tilde{B}^\tau(\ell)(t_{1,\ell-1}) \right),
$$

(5.11)
where we have written $\pi_{\max}$ for $\pi_{\tau(1)}$. The functional in (5.11) is similar to the one obtained in the iid case in [18], the essential difference being, not in the form of the Brownian functional, but rather in the covariance structure of the Brownian motions.

To see precisely where this difference comes into play, note that if the transition matrix $P$ is cyclic, then the covariance matrix of the Brownian motion is also cyclic. Consider then the 3-letter aperiodic, homogeneous, doubly-stochastic Markov case. Since the Brownian covariance matrix is symmetric, and, moreover, degenerate, an additional cyclicity constraint forces it to have the permutation-symmetric structure seen in the iid uniform case. In particular, $LI_n$ will have, up to a scaling factor, the same limiting distribution as in the iid uniform case:

$$\frac{LI_n - n/3}{\sqrt{n}} \Rightarrow \sigma \max_{(\ell,\ldots) \in \mathbb{I}_{1,3}} \sum_{\ell=1}^{3} \left( \tilde{B}^\ell(t_{1,\ell}) - \tilde{B}^\ell(t_{1,\ell-1}) \right), \quad (5.12)$$

where $\sigma = \sigma_\ell$, for all $1 \leq \ell \leq 3$, and with the Brownian covariance matrix given by

$$\tilde{\Sigma} = t \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix},$$

and where we have used the fact that $\tau(\ell) = \ell$, for all $1 \leq \ell \leq 3$.

However, when $m \geq 4$, the cyclicity constraint does not force the Brownian covariance matrix to have the permutation-symmetric structure, as the following example shows for $m = 4$.

**Example 5.1** Consider the following doubly-stochastic, aperiodic, cyclic transition matrix:

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.3 & 0.2 & 0.1 & 0.4 \end{pmatrix}. \quad (5.13)$$

While the doubly-stochastic nature of $P$ ensures that the stationary distribution is uniform, the covariance matrix of the limiting Brownian motion, at three-decimal accuracy, is computed to be
\[ \Sigma = t \begin{pmatrix} 1.000 & -0.357 & -0.287 & -0.357 \\ -0.357 & 1.000 & -0.357 & -0.287 \\ -0.287 & -0.357 & 1.000 & -0.357 \\ -0.357 & -0.287 & -0.357 & 1.000 \end{pmatrix} , \]  
and \[ \sigma^2 = \sigma^2 := 0.263, \text{ for each } 1 \leq r \leq 4. \]  
Thus, the limiting distribution of \( LI_n \) is given by

\[ \frac{LI_n - n/4}{\sqrt{n}} \Rightarrow \sigma \max_{(\ell,\ldots) \in I_1,4} \sum_{\ell=j}^4 \left( \tilde{B}^\ell(t_{1,\ell}) - \tilde{B}^\ell(t_{1,\ell-1}) \right) , \]

for \( 1 \leq r \leq 4 \). However, while the form of the functional is the same as in the iid uniform case (up to the constant), the covariance structure of the Brownian motion in (5.14) differs from that of the uniform iid case, i.e., from

\[ t \begin{pmatrix} 1 & -1/3 & -1/3 & -1/3 \\ -1/3 & 1 & -1/3 & -1/3 \\ -1/3 & -1/3 & 1 & -1/3 \\ -1/3 & -1/3 & -1/3 & 1 \end{pmatrix} , \]  
and so the limiting distribution in (5.15) is not that of the uniform iid case.

We thus see that Kuperberg’s conjecture regarding the shape of the Young tableau for random sequences generated by aperiodic, homogeneous, and cyclic matrices [24] is not true for general \( m \)-alphabets. By simply extending the first-row analysis above to the second and third rows, we see that it is true for \( m = 3 \). However, as could have been anticipated by (5.12), it fails for \( m \geq 4 \), as the previous example showed. Furthermore, in the next section we shall see that for the cyclic case the structure of \( \Sigma \) can be described in an elegant manner which more clearly delineates when we obtain the uniform iid limiting law.

In the more general doubly stochastic case, we have the following corollary:

**Corollary 5.1** Let the transition matrix \( P \) of Theorem 5.2 be doubly stochastic. Then, for every \( 1 \leq r \leq m, m_r = 0, d_r = m, \) and
\[
\frac{V_r - rn/m}{\sqrt{n}} \Rightarrow \max_{(t_{\cdot}) \in I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{r+j} \sigma_\ell \left( \tilde{B}_\ell^t(t_{j,\ell}) - \tilde{B}_\ell^t(t_{j,\ell-1}) \right).
\] (5.17)

If, moreover, the matrix \(P\) has all entries of \(1/m\) (i.e., in the iid uniform alphabet case), then

\[
\frac{V_r - rn/m}{\sqrt{n}} \Rightarrow \frac{\sqrt{m-1}}{m} \max_{(t_{\cdot}) \in I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{r+j} \left( \tilde{B}_\ell^t(t_{j,\ell}) - \tilde{B}_\ell^t(t_{j,\ell-1}) \right),
\] (5.18)

and the covariance matrix in (5.9) has all its off-diagonals equal to \(-1/(m-1)\).

**Proof.** For each \(1 \leq r \leq m\), \(\pi_r = 1/m\), and so \(\nu_r = r/m, m_r = 0\), and the multiplicity \(d_r = m\). Moreover, the permutation \(\tau\) is simply the identity permutation. This proves (5.17). If, moreover, all the transition probabilities are \(1/m\), then the multinomial nature of the underlying combinatorial quantities \(a_{r,k}\) tells us that \(\sigma_r^2 = (1/m)(1 - 1/m)\), for each \(1 \leq r \leq m\), and that \(\rho_{r_1,r_2} = -1/(m - 1)\), for each \(r_1 \neq r_2\), thus proving (5.18).

To see that the functional in (5.17) is generally different from the uniform iid case, even for \(m = 3\), consider the following non-cyclic example:

**Example 5.2** Let a doubly-stochastic (but non-cyclic), aperiodic Markov chain have transition matrix

\[
P = \begin{pmatrix}
0.4 & 0.6 & 0.0 \\
0.6 & 0.0 & 0.4 \\
0.0 & 0.4 & 0.6
\end{pmatrix}.
\] (5.19)

As in Example [5.1], the doubly-stochastic nature of \(P\) ensures that the stationary distribution is uniform. In the present example, the asymptotic covariance matrix, at three-decimal accuracy, is computed to be

\[
\begin{pmatrix}
0.459 & 0.049 & -0.506 \\
0.049 & 0.086 & -0.136 \\
-0.506 & -0.136 & 0.642
\end{pmatrix}.
\] (5.20)
Note that, even though we have a uniform stationary distribution, the asymptotic variances (i.e., the diagonals of (5.20)) have dramatically different values. Moreover, according to Remark 4.2, in the uniform iid case, the only possibility for the Brownian covariance matrix is that the off-diagonals have value \(-1/2\). However, the Brownian motion covariance matrix obtained from (5.20) is

\[
\begin{pmatrix}
1.000 & 0.246 & -0.935 \\
0.246 & 1.000 & -0.577 \\
-0.935 & -0.577 & 1.000
\end{pmatrix}.
\]

(5.21)

Not only are the off-diagonals different from \(-1/2\), but in some cases are even positive. In short, the functional in (5.17) has a distribution which differs from any iid case (even non-uniform).

**Remark 5.3** Generalizing a result of Baryshnikov [4] and of Gravner, Tracy, and Widom [16] on the representation of the maximal eigenvalue of an \(m \times m\) element of the GUE, Doumerc [12] found a Brownian functional expression for all the eigenvalues of an \(m \times m\) element of the GUE. Our expression in (5.18) is similar, with the exception that our \(m\)-dimensional Brownian motion is constrained by a zero-sum condition, and, moreover, has a different covariance structure. (We note, moreover, that the parameters over which his Brownian functional is maximized in [12] might be intended to range over a slightly larger set which corresponds to our \(I_{r,m}\).) Using a path-transformation technique relating the joint distribution of a certain transformation of \(n\) continuous processes to the joint distribution of the processes conditioned never to leave the Weyl chamber, O’Connell and Yor [26] employed queuing-theoretic arguments to obtain Brownian functional representations for the entire spectrum of the \(m \times m\) element of the GUE. In a study of much more general transformations of this type, Bougerol and Jeulin [7] were able to obtain this result as a special case.

If \(d_r = 1\), i.e., if the \(r^{th}\) most probable state is unique, then the following result can be viewed as lying at the other extreme from Corollary 5.1.

**Corollary 5.2** Let \(1 \leq r \leq m\), and let \(d_r = 1\) in Theorem 5.2. Then

\[
\frac{V_n^r - \nu_r n}{\sqrt{n}} \Rightarrow \sum_{i=1}^{r} \sigma_{r(i)} \tilde{B}^{r(i)}(1).
\]

(5.22)
Proof. If \( d_r = 1 \), then \( m_r = r - 1 \), and so the maximal term of (5.8) contains only one summand, namely \( \sigma_{\tau(m_r+1)} \tilde{B}^{\tau(m_r+1)}(1) = \sigma_{\tau(r)} \tilde{B}^{\tau(r)}(1) \). Including this term in the first summation term of (5.8) proves (5.22). ■

Remark 5.4 The maximal term of the functional in (5.8) is that of the doubly-stochastic, \( d_r \)-letter case. Indeed, the maximal term involves precisely \( d_r \) Brownian motions over the \( r - m_r \) rows. Such a functional would arise in a doubly-stochastic \( d_r \)-letter situation with a covariance matrix consisting of the sub-matrix of the original \( \Sigma \) corresponding to the \( d_r \) Brownian motions, as in Corollary 5.1. The Gaussian term corresponds to the functional of Corollary 5.2. That is, in some sense, the limiting law of (5.8) interpolates between these two extreme cases.

Proof. (Theorem 5.2) Since the \( r = m \) case is trivial (\( V^m_n \) is then identically equal to \( n \)), assume that \( r < m \). Recall the approximating functional (5.7):

\[
\frac{V^r_n - \nu_r n}{\sqrt{n}} = \max_{I_{r,m}} \left\{ \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_{\ell} \left( \tilde{B}^\ell_n(t_{j,\ell}) - \tilde{B}^\ell_n(t_{j,\ell-1}) \right) \right. \\
+ \left. \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sqrt{n} (\pi_{\ell} - \pi_{\tau(j)}) (t_{j,\ell} - t_{j,\ell-1}) \right\}.
\]  

(5.23)

Introducing the notation \( \Delta t_{j,\ell} := [t_{j,\ell-1}, t_{j,\ell}] \) and \( M^\ell_n(\Delta t_{j,\ell}) := M^\ell_n(t_{j,\ell}) - M^\ell_n(t_{j,\ell-1}) \), for any \( m \)-dimensional process \( M(t) = (M^1(t), M^2(t), \ldots, M^m(t)) \), \( t \in [0,1] \), we can rewrite (5.23) more compactly as

\[
\frac{V^r_n - \nu_r n}{\sqrt{n}} = \max_{I_{r,m}} \left\{ \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_{\ell} \tilde{B}^\ell_n(\Delta t_{j,\ell}) - \sqrt{n} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} (\pi_{\tau(j)} - \pi_{\ell}) |\Delta t_{j,\ell}| \right\}.
\]  

(5.24)

The main idea of the proof to follow will be to show that the second summation of (5.24) can, in effect, be eliminated by choosing the \( (\Delta t_{j,\ell}) \) in an appropriate manner. Now some of the coefficients \( (\pi_{\tau(j)} - \pi_{\ell}) \) are zero; such terms do not cause any problems. Intuitively, however, the remaining terms should have \( |\Delta t_{j,\ell}| = 0 \). Defining the restricted set of parameters
Moreover, for each \( 1 \leq s \) that \( \Delta \{ I_1, \ldots \} \) for some integer \( \kappa \), we see that, provided \( I_{r,m}^* \neq \emptyset \),

\[
\max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( \sigma_{\ell} \hat{B}_n^\ell(\Delta t_j,\ell) - \sqrt{n} (\pi_{\tau(j)} - \pi_{\ell}) |\Delta t_j,\ell| \right) \\
\geq \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_{\ell} \hat{B}_n^\ell(\Delta t_j,\ell).
\] (5.25)

Moreover, by the Invariance Principle and the Continuous Mapping Theorem,

\[
\max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_{\ell} \hat{B}_n^\ell(\Delta t_j,\ell) \Rightarrow \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_{\ell} \hat{B}_n^\ell(\Delta t_j,\ell).
\] (5.26)

We claim that, indeed, \( I_{r,m}^* \neq \emptyset \), and that, moreover,

\[
\max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( \sigma_{\ell} \hat{B}_n^\ell(\Delta t_j,\ell) - \sqrt{n} (\pi_{\tau(j)} - \pi_{\ell}) |\Delta t_j,\ell| \right) \\
\Rightarrow \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_{\ell} \hat{B}_n^\ell(\Delta t_j,\ell).
\] (5.27)

We will prove that \( I_{r,m}^* \neq \emptyset \) by creating a bijection between \( I_{r,m}^* \) and \( I_{r-m_r,d_r} \).

To this end, for \( 1 \leq i \leq m_r \), let \( I_{\tau(i),i} = [u_{\tau(i),i-1}, u_{\tau(i),i}] = [0, 1] \). Next, choose any \( (u, \ldots) \in I_{r-m_r,d_r} \), and define further intervals \( I_{\tau(m_r+j),\ell} = \Delta u_j,\ell \), for \( 1 \leq j \leq r - m_r \) and \( 1 \leq \ell \leq d_r \).

We now create a partition of these intervals in a manner which relies on the ideas used in the proof of Theorem 5.1. Consider the set of points \( \{ u_{\tau(i),j} \} \), order them as \( s_0 := 0 < s_1 < \cdots < s_{\kappa-1} < s_\kappa := 1 \) for some integer \( \kappa \), and let \( \Delta s_q = [s_{q-1}, s_q] \), for all \( 1 \leq q \leq \kappa \).

Trivially, for each \( 1 \leq q \leq \kappa \), and for each \( 1 \leq i \leq m_r \), \( \Delta s_q \subset I_{\tau(i),i} \).

Moreover, for each \( 1 \leq j \leq r - m_r \), there exists a unique \( \ell(j, q) \) such that \( \Delta s_q \subset I_{\tau(m_r+j),\ell(j,q)} \). For each \( q \), consider the set of indices \( \Lambda_q := \ldots \)

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\{\tau(1), \ldots, \tau(m_r)\} \cup \{\tau(m_r + \ell(1, q)), \ldots, \tau(m_r + \ell(r - m_r, q))\}, and order these \(r\) elements of \(A_q\) as \(1 \leq \ell(1, q) < \cdots < \ell(r, q) \leq m\).

Using these partitions, we examine, with foresight, the following functional of a general \(m\)-dimensional process \((M(t))_{t \geq 0}\):

\[
\sum_{i=1}^{m_r} M^{\tau(i)}(1) + \sum_{j=1}^{(r-m_r)(r-m_r+d_r-1)} \sum_{\ell=j}^{M^{\tau(m_r+\ell)}(\Delta u_{j,\ell})}
= \sum_{i=1}^{m_r} \left( \sum_{q=1}^{\kappa} M^{\tau(i)}(\Delta s_q) \right)
+ \sum_{j=1}^{(r-m_r)(r-m_r+d_r-1)} \sum_{\ell=j}^{M^{\tau(m_r+\ell+j, q)}(\Delta s_q)}
= \sum_{q=1}^{\kappa} \left( \sum_{i=1}^{m_r} M^{\tau(i)}(\Delta s_q) \right)
+ \sum_{j=1}^{r} \sum_{q=1}^{\kappa} M^{\tau(j, q)}(\Delta s_q)
= \sum_{j=1}^{r} \sum_{\ell=1}^{M^{\tau(j, \ell)}(\Delta t_{j,\ell})},
\]

where, for each \(1 \leq j \leq r\), and for each \(1 \leq \ell \leq m\), \(t_{j,\ell} := \max\{s_q : \ell \geq \ell(j, q)\}\). (That is, for each \(j\), we collapse together intervals \(\Delta s_q\) corresponding to the same component \(M^{\ell}\).) Now, since our functional in (5.29) has non-trivial summands only for \(\ell\) such that \(\pi_{\tau(\ell)} \geq \pi_{\tau(r)}\), we have shown that \((t_{\ldots}) \in I^*_{r,m}\).

The following example illustrates this argument. Suppose we have an alphabet of size \(m = 8\), with

\((\pi_1, \pi_2, \ldots, \pi_8) = (0.07, 0.1, 0.2, 0.06, 0.2, 0.06, 0.1, 0.2)\).

Then,

\[
\pi_{\tau(1)} = \pi_{\tau(2)} = \pi_{\tau(3)} = 0.2, \quad m_1 = m_2 = m_3 = 0, \quad d_1 = d_2 = d_3 = 3, \\
\pi_{\tau(4)} = \pi_{\tau(5)} = 0.1, \quad m_4 = m_5 = 3, \quad d_4 = d_5 = 2,
\]

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In particular, note that the two largest, distinct probability values are 0.2 and 0.1, of multiplicities 3 and 2, respectively. Next, consider the case $r = 4$. We now show how $I_{r-m_r,d_r} = I_{4-3,2} = I_{1,2}$ corresponds to an element of $I_{r,m}^* = I_{4,8}^*$. Figure 4 shows a typical element of the unconstrained index set $I_{4,8}$.

Now $\tau(1) = 3, \tau(2) = 5, \tau(3) = 8, \tau(4) = 2$, and $\tau(5) = 7$. Our construction begins with the amalgamation of $m_r = m_4 = 3$ rows, corresponding to the three indices for which $\pi_i$ is strictly less than $\pi_{\tau(r)} = \pi_{\tau(4)} = 0.1$, with $I_{1,2}$. This is shown in Figure 5.

Finally, we simply reorder each vertical column in the original order of the indices, as shown in Figure 6. We see that, first of all, we have constructed an element of $I_{4,8}$. Moreover, since we have three rows whose indices are associated with the maximum value, and a remaining row whose indices are associated with $\pi_{\tau(4)}$, we indeed have an element of $I_{4,8}^*$. Note that the $4 \times 4 = 16$ free indices in $I_{4,8}$ (corresponding to the locations of the 16 vertical bars in Figure 4) have been reduced to a single one in $I_{4,8}^*$.

In addition, we may essentially reverse this construction, starting with an element of $I_{r,m}^* (\neq \emptyset)$, and so obtain an element of $I_{r-m_r,d_r}$. Indeed, from the definitions of $I_{r,m}^*$ and $\nu_r$ we know that

\[
\pi_{\tau(0)} = 0.07, \quad m_6 = 5, \quad d_6 = 1,
\]
\[
\pi_{\tau(7)} = \pi_{\tau(8)} = 0.06, \quad m_7 = m_8 = 6, \quad d_7 = d_8 = 2.
\]
Figure 5: Amalgamating 3 rows with $I_{1,2}$.

Figure 6: Reordering vertically to obtain an element in $I_{4,8}^*$. 
\[ \nu_r = \sum_{j=1}^{r} \pi_{\tau(j)} = \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \pi_{\ell} |\Delta t_{j,\ell}|, \]

for any\( (t_\cdot,.) \in I_{r,m}^* \). However, we also have

\[ \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \pi_{\ell} |\Delta t_{j,\ell}| = \mathbf{1}_{\{m_r > 0\}} \left( \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \mathbf{1}_{\{\pi_{\tau(\ell)} \geq \pi_{\tau(m_r)}\}} \pi_{\ell} |\Delta t_{j,\ell}| \right) \]

\[ + \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \mathbf{1}_{\{\pi_{\tau(\ell)} < \pi_{\tau(m_r)}\}} \pi_{\ell} |\Delta t_{j,\ell}| \]

\[ + \mathbf{1}_{\{m_r = 0\}} \pi_{\tau(1)} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} |\Delta t_{j,\ell}| \]

\[ \leq \mathbf{1}_{\{m_r > 0\}} \left( (\pi_{\tau(1)} + \cdots + \pi_{\tau(m_r)}) + (r - m_r) \pi_{\tau(r)} \right) \]

\[ + \mathbf{1}_{\{m_r = 0\}} r \pi_{\tau(1)} \]

\[ = \nu_r, \]

with equality holding throughout if and only if\( m_r = 0 \) or\( m_r > 0 \) and

\[ \sum_{j=1}^{r} |\Delta t_{j,\ell}| = 1, \] for all\( \ell \) such that\( \pi_{\tau(\ell)} \geq \pi_{\tau(m_r)} \), and that, moreover,

\[ \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \mathbf{1}_{\{\pi_{\tau(\ell)} = \pi_{\tau(m_r)}\}} |\Delta t_{j,\ell}| = r - m_r. \] If\( m_r > 0 \) then, for any\( (t_\cdot,.) \in I_{r,m}^* \), we may start with (5.29), and use again the permutation of the indices employed there. We thus obtain the first term of (5.28), which corresponds to the condition\( \sum_{j=1}^{r} |\Delta t_{j,\ell}| = 1, \) for all\( \ell \) such that\( \pi_{\tau(\ell)} \geq \pi_{\tau(m_r)} \), and also the second term of (5.28), which corresponds to the other condition

\[ \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \mathbf{1}_{\{\pi_{\tau(\ell)} < \pi_{\tau(m_r)}\}} |\Delta t_{j,\ell}| = r - m_r. \] If\( m_r = 0 \) the same reasoning holds, except that the first term in (5.28) is taken to be zero.

Having thus established a bijection between\( I_{r,m}^* \) and\( I_{r-m_r,dr} \), we may thus maximize over these two parameter sets, and so, for any process\( (M(t))_{t \geq 0} \), obtain the general result

\[ \sum_{i=1}^{m_r} M^{\tau(i)}(1) + \max_{I_{r-m_r,dr}} \sum_{j=1}^{r} \sum_{\ell=j}^{r-m_r+dr-1} M^{\tau(m_r+\ell)}(\Delta u_{j,\ell}) \]

\[ = \max_{I_{r,m}^*} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^{\tilde{\ell}(j,q)}(\Delta t_{j,\ell}). \] 

(5.30)
We now proceed to show that (5.27) holds. First, fix \( c > 0 \), and, for each \( 1 \leq \ell \leq m \), set
\[
c_\ell = \begin{cases} c, & \text{if } \pi_\ell < \pi_{\tau(r)}, \\ 0, & \text{otherwise}. \end{cases}
\] (5.31)

Next, let \( \hat{M}_n^\ell(t) = \sigma_\ell \hat{B}_n^\ell(t) - c_\ell t \), and let \( M(t) = \sigma_\ell \tilde{B}_n^\ell(t) - c_\ell t \). Then, for \( n \) large enough, namely, for \( n > c/(\pi_{\tau(r)} - \pi_{\tau(r+1)}) \), we have that, almost surely, for any \( t \in I_{r,m} \),
\[
\sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{M}_n^\ell(\Delta t_{j,\ell}) \geq \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( \sigma_\ell \hat{B}_n^\ell(\Delta t_{j,\ell}) - \sqrt{n} \left( \pi_{\tau(j)} - \pi_\ell \right) |\Delta t_{j,\ell}| \right). \] (5.32)

Hence, almost surely, both
\[
\max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{M}_n^\ell(\Delta t_{j,\ell}) \geq \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( \sigma_\ell \hat{B}_n^\ell(\Delta t_{j,\ell}) - \sqrt{n} \left( \pi_{\tau(j)} - \pi_\ell \right) |\Delta t_{j,\ell}| \right), \] (5.33)

and
\[
\max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{M}_n^\ell(\Delta t_{j,\ell}) = \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_\ell \hat{B}_n^\ell(\Delta t_{j,\ell}). \] (5.34)

Now choose any \( z > 0 \). Then
\[
\begin{align*}
P & \left( \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( \sigma_\ell \hat{B}_n^\ell (\Delta t_{j,\ell}) - \sqrt{n} \left( \pi_{\tau(j)} - \pi_\ell \right) |\Delta t_{j,\ell}| \right) \\ & \quad - \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_\ell \hat{B}_n^\ell (\Delta s_{j,\ell}) > z \right) \\ & \leq P \left( \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{M}_n^\ell (\Delta t_{j,\ell}) - \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{M}_n^\ell (\Delta t_{j,\ell}) > z \right), \quad (5.35)
\end{align*}
\]

so that

\[
\begin{align*}
\limsup_{n \to \infty} P & \left( \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \left( \sigma_\ell \hat{B}_n^\ell (\Delta t_{j,\ell}) - \sqrt{n} \left( \pi_{\tau(j)} - \pi_\ell \right) |\Delta t_{j,\ell}| \right) \\ & \quad - \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_\ell \hat{B}_n^\ell (\Delta s_{j,\ell}) > z \right) \\ & \leq \limsup_{n \to \infty} P \left( \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{M}_n^\ell (\Delta t_{j,\ell}) - \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{M}_n^\ell (\Delta t_{j,\ell}) > z \right) \\ & = P \left( \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell (\Delta t_{j,\ell}) - \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell (\Delta t_{j,\ell}) > z \right), \quad (5.36)
\end{align*}
\]

by the Invariance Principle and the Continuous Mapping Theorem. Next, for any \(0 \leq \varepsilon \leq 1\), let

\[
I_{r,m}(\varepsilon) = \{(t_{j,\ell}) \in I_{r,m} : \sum_{j,\ell} |\Delta t_{j,\ell}| 1_{\{\pi_\ell < \pi_{\tau(r)}\}} \leq \varepsilon r\}.
\]

Thus, \(I_{r,m}^* = I_{r,m}(0) \subset I_{r,m}(\varepsilon) \subset I_{r,m}(1) = I_{r,m}\). We bound (5.36) using this family of subsets as follows:

\[
P \left( \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell (\Delta t_{j,\ell}) - \max_{I_{r,m}^*} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell (\Delta t_{j,\ell}) > z \right)
\]

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and thus establish convergence to zero in probability. Moreover, since

$$\mathbb{P} \left( \max_{I_{r,m}(\varepsilon)} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell(\Delta t_{j,\ell}) - \max_{I_{r,m}^c} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell(\Delta t_{j,\ell}) > z \right)$$

$$+ \mathbb{P} \left( \max_{I_{r,m} \setminus I_{r,m}(\varepsilon)} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell(\Delta t_{j,\ell}) - \max_{I_{r,m}^c} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell(\Delta t_{j,\ell}) > z \right)$$

$$\leq \mathbb{P} \left( \max_{I_{r,m}(\varepsilon)} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{B}^\ell(\Delta t_{j,\ell}) - \max_{I_{r,m}^c} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{B}^\ell(\Delta s_{j,\ell}) > z \right)$$

$$+ \mathbb{P} \left( \max_{I_{r,m} \setminus I_{r,m}(\varepsilon)} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{B}^\ell(\Delta t_{j,\ell}) - \max_{I_{r,m}^c} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{B}^\ell(\Delta s_{j,\ell}) > z + \varepsilon rc \right)$$

$$\leq \mathbb{P} \left( \max_{I_{r,m}(\varepsilon)} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{B}^\ell(\Delta t_{j,\ell}) - \max_{I_{r,m}^c} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{B}^\ell(\Delta s_{j,\ell}) > z \right)$$

$$+ \mathbb{P} \left( \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{B}^\ell(\Delta t_{j,\ell}) - \max_{I_{r,m}^c} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \hat{B}^\ell(\Delta s_{j,\ell}) > z + \varepsilon rc \right).$$

(5.37)

We can now take the limsup in (5.37), as $c \to \infty$, and then, as $\varepsilon \to 0$, and so establish convergence to zero in probability. Moreover, since

$$\mathbb{P} \left( \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell(\Delta t_{j,\ell}) - \max_{I_{r,m}^c} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell(\Delta t_{j,\ell}) \geq 0 \right) = 1,$$

we have in fact shown, with the help of (5.36), that with probability one,

$$\max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell(\Delta t_{j,\ell}) = \max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} M^\ell(\Delta t_{j,\ell}),$$

and thus

$$\max_{I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \left( \sigma_{\ell} \hat{B}_n^\ell(\Delta t_{j,\ell}) - \sqrt{n} (\pi_{\tau(j)} - \pi_{\ell}) |\Delta t_{j,\ell}| \right)$$

$$- \max_{I_{r,m}^c} \sum_{j=1}^{r} \sum_{\ell=1}^{m-r+j} \sigma_{\ell} \hat{B}_n^\ell(\Delta s_{j,\ell}) \overset{\mathbb{P}}{\to} 0.$$  

(5.38)
Since
\[
\max_{I_{r,m}^*} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_\ell \hat{B}_n^\ell (\Delta s_{j,\ell}) \Rightarrow \max_{I_{r,m}^*} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma_\ell \tilde{B}_n^\ell (\Delta s_{j,\ell}), \quad (5.39)
\]
by the Converging Together Lemma, we have proved (5.27). Equation (5.8) of the theorem follows from the bijection between \(I_{r,m}^*\) and \(I_{r-m,r,d,r}\) described in the general result (5.30).

Finally, we can obtain the convergence of the joint distribution in (5.10) in the following manner. Given any \((\theta_1, \theta_2, \ldots, \theta_r) \in \mathbb{R}^r\), we have
\[
\sum_{k=1}^{r} \theta_k \left( \frac{V_n^k - \nu_k n}{\sqrt{n}} \right)
= \sum_{k=1}^{r} \theta_k \left( \max_{I_{k,m}^*} \sum_{j=1}^{k} \sum_{\ell=j}^{m-k+j} \left( \sigma_\ell \hat{B}_n^\ell (\Delta t_{j,\ell}) - \sqrt{n} (\pi_{\tau(j)} - \pi_\ell) |\Delta t_{j,\ell}| \right) \right)
= \sum_{k=1}^{r} \theta_k \left( \max_{I_{k,m}^*} \sum_{j=1}^{k} \sum_{\ell=j}^{m-k+j} \left( \sigma_\ell \hat{B}_n^\ell (\Delta t_{j,\ell}) - \sqrt{n} (\pi_{\tau(j)} - \pi_\ell) |\Delta t_{j,\ell}| \right) \right)
- \max_{I_{k,m}^*} \sum_{j=1}^{k} \sum_{\ell=j}^{m-k+j} \sigma_\ell \hat{B}_n^\ell (\Delta s_{j,\ell}) + \sum_{k=1}^{r} \theta_k \left( \max_{I_{k,m}^*} \sum_{j=1}^{k} \sum_{\ell=j}^{m-k+j} \sigma_\ell \hat{B}_n^\ell (\Delta s_{j,\ell}) \right).
\]
\[
(5.40)
\]

Now from (5.38), the first summation on the right-hand side of (5.40) converges to zero in probability, as \(n \to \infty\). Moreover, the second summation is a continuous functional of \((\hat{B}_n^1, \hat{B}_n^2, \ldots, \hat{B}_n^m)\), and so, by the Invariance Principle and Continuous Mapping Theorem, converges. Then the Converging Together Lemma, along with the bijection result (5.30), gives
\[
\sum_{k=1}^{r} \theta_k \left( \frac{V_n^k - \nu_k n}{\sqrt{n}} \right)
\Rightarrow \sum_{k=1}^{r} \theta_k \left( \max_{I_{k,m}^*} \sum_{j=1}^{k} \sum_{\ell=j}^{m-k+j} \sigma_\ell \tilde{B}_n^\ell (\Delta s_{j,\ell}) \right) = \sum_{k=1}^{r} \theta_k V_n^k. \quad (5.41)
\]
Since (5.41) holds for arbitrary \((\theta_1, \theta_2, \ldots, \theta_r) \in \mathbb{R}^r\), by the Cramér-Wold Theorem, we have the joint convergence result (5.10). 

Since the shape of the Young tableau is more naturally expressed in terms of the \(R^k_n\), rather than of the \(V^k_n\), we may restate the results of the previous theorem as follows:

**Theorem 5.3** Let \((X_n)_{n \geq 0}\) be an irreducible, aperiodic, homogeneous Markov chain with finite state space \(A_m = \{\alpha_1 < \cdots < \alpha_m\}\), and with stationary distribution \((\pi_1, \pi_2, \ldots, \pi_m)\). Then, in the notations of Theorem 5.2,

\[
\left( \frac{R_1^1 - \pi_1 \tau_{(1)} n}{\sqrt{n}}, \frac{R_2^1 - \pi_2 \tau_{(2)} n}{\sqrt{n}}, \ldots, \frac{R_m^1 - \pi_m \tau_{(m)} n}{\sqrt{n}} \right) \Rightarrow \left( R_1^\infty, R_2^\infty, \ldots, R_m^\infty \right),
\]

(5.42)

where

\[
R_1^\infty = \max_{t_1, t_1} \sum_{\ell=1}^{d_1} \sigma_{\tau(\ell)} \left( \tilde{B}_{\tau(\ell)}^r(t_1, \ell) - \tilde{B}_{\tau(\ell)}^r(t_{1, \ell-1}) \right),
\]

(5.43)

and, for each \(2 \leq k \leq m\),

\[
R_k^\infty = \sum_{i=m_{k-1}+1}^{m_k} \sigma_{\tau(i)} \tilde{B}_{\tau(i)}(1)
\]

\[
+ \max_{l_k-m_k \to d_k} \sum_{j=1}^{k-m_k} \sum_{\ell=j}  \sigma_{\tau(m_k+\ell)} \tilde{B}_{\tau(m_k+\ell)}^r(\Delta t_{j, \ell})
\]

\[
- \max_{l_k-1 \to m_{k-1} \to d_{k-1}} \sum_{j=1}^{k-1-m_{k-1}} \sum_{\ell=j}  \sigma_{\tau(m_{k-1}+\ell)} \tilde{B}_{\tau(m_{k-1}+\ell)}^r(\Delta t_{j, \ell}),
\]

(5.44)

where we use the notation \(\tilde{B}_s(\Delta t_{j, \ell}) = \tilde{B}_s(t_{j, \ell}) - \tilde{B}_s(t_{j, \ell-1})\), for any \(1 \leq s \leq m, 1 \leq j \leq k, \) and \(1 \leq \ell \leq m\), and where the first sum on the right-hand side of (5.44) is understood to be 0, if \(m_k = m_{k-1}\).
Proof. First, $R_n^1 = V_n^1$, and, for each $2 \leq k \leq m$, $R_n^k = V_n^k - V_n^{k-1}$.
Expressing these equalities at the multivariate level, we have

\[
\left( \frac{R_n^1 - \pi_1 \tau_1}{\sqrt{n}}, \frac{R_n^2 - \pi_2 \tau_2}{\sqrt{n}}, \ldots, \frac{R_n^m - \pi_m \tau_m}{\sqrt{n}} \right)
= \left( \frac{V_n^1 - \pi_1 \tau_1}{\sqrt{n}}, \frac{V_n^2 - V_n^1 - \pi_2 \tau_2}{\sqrt{n}}, \ldots, \frac{V_n^m - V_n^{m-1} - \pi_m \tau_m}{\sqrt{n}} \right)
= \left( \frac{V_n^1 - \nu_1 n}{\sqrt{n}}, \frac{V_n^2 - \nu_2 n}{\sqrt{n}}, \ldots, \frac{V_n^m - \nu_m n}{\sqrt{n}} \right)
\Rightarrow (V_{\infty}^1, V_{\infty}^2, \ldots, V_{\infty}^m) - (0, V_{\infty}^1, \ldots, V_{\infty}^m)
:= (R_{\infty}^1, R_{\infty}^2, \ldots, R_{\infty}^m),
\]

(5.45)

where the weak convergence follows immediately from the Continuous Mapping Theorem, since the transformation is linear.

Equations (5.43) and (5.44) follow simply from the Brownian expressions for $(V_{\infty}^1, V_{\infty}^2, \ldots, V_{\infty}^m)$ in Theorem 5.2. \(\blacksquare\)

If all $m$ letters have unique stationary probabilities, then we have the following corollary to Theorem 5.3:

**Corollary 5.3** If the stationary distribution of Theorem 5.3 is such that each $\pi_r$ is unique, then

\[
\left( \frac{R_n^1 - \pi_1 \tau_1}{\sqrt{n}}, \frac{R_n^2 - \pi_2 \tau_2}{\sqrt{n}}, \ldots, \frac{R_n^m - \pi_m \tau_m}{\sqrt{n}} \right) \Rightarrow N((0, 0, \ldots, 0), \Sigma).
\]

In other words, the limiting distribution is identical in law to the spectrum of the diagonal matrix $D = \text{diag}\{Z_1, Z_2, \ldots, Z_m\}$, where $(Z_1, Z_2, \ldots, Z_m)$ is a centered normal random vector with covariance matrix $\Sigma$.

**Proof.** Now, for all $1 \leq k \leq m$, $d_k = 1$, and $m_k = k - 1$, so that
\[ R^1_\infty = \max_{I_{1,d_1}} \sum_{\ell=1}^{d_1} \sigma_{\tau(\ell)} \left( \tilde{B}^{\tau(\ell)}(t_{1,\ell}) - \tilde{B}^{\tau(\ell)}(t_{1,\ell-1}) \right) = \sigma_{\tau(1)}\tilde{B}^{\tau(1)}(1), \]

and, for each \( 2 \leq k \leq m, \)

\[ R^k_\infty = \sum_{i=m_{k-1}+1}^{m_k} \sigma_{\tau(i)}\tilde{B}^{\tau(i)}(1) \]

\[ + \max_{I_{k-m_k,d_k}} \sum_{j=1}^{k-m_k} \sum_{\ell=j}^{d_k+m_k-k+j} \sigma_{\tau(m_k+\ell)}\tilde{B}^{\tau(m_k+\ell)}(\Delta t_{j,\ell}) \]

\[ - \max_{I_{k-1-m_k-1,d_k-1}} \sum_{j=1}^{k-1-m_k-1} \sum_{\ell=j}^{d_k-1+m_k-1-k+1+j} \sigma_{\tau(m_k-1+\ell)}\tilde{B}^{\tau(m_k-1+\ell)}(\Delta t_{j,\ell}) \]

\[ = \sum_{i=k-1}^{k-1} \sigma_{\tau(i)}\tilde{B}^{\tau(i)}(1) \]

\[ + \max_{I_{1,1}} \sum_{j=1}^{1} \sum_{\ell=j}^{1} \sigma_{\tau(k-1+\ell)}\tilde{B}^{\tau(k-1+\ell)}(\Delta t_{j,\ell}) \]

\[ - \max_{I_{1,1}} \sum_{j=1}^{1} \sum_{\ell=j}^{1} \sigma_{\tau(k-2+\ell)}\tilde{B}^{\tau(k-2+\ell)}(\Delta t_{j,\ell}) \]

\[ = \sigma_{\tau(k-1)}\tilde{B}^{\tau(k-1)}(1) + \sigma_{\tau(k)}\tilde{B}^{\tau(k)}(1) - \sigma_{\tau(k-1)}\tilde{B}^{\tau(k-1)}(1) \]

\[ = \sigma_{\tau(k)}\tilde{B}^{\tau(k)}(1). \]

Moreover, the joint law result for \((R^1, R^2, \ldots, R^m)\) holds as well, and this is clearly a multivariate normal distribution, with mean \((0, 0, \ldots, 0)\) and covariance matrix \(\Sigma\). Since the spectrum of a diagonal matrix consists of its diagonal elements, the final claim of the corollary holds. \(\blacksquare\)

**Remark 5.5** We know that the joint law of \((R^1, R^2, \ldots, R^m)\) in the iid uniform alphabet case is identical to the joint law of the eigenvalues of an
m \times m$ traceless GUE matrix. Corollary 5.3 also gives a spectral characterization for the unique probability case, in particular, for a non-uniform iid alphabet with unique stationary probabilities. This is consistent with the characterization of the limiting law of $LI_n$ in the non-uniform iid case, due to Its, Tracy, and Widom [20, 21], as that of the largest eigenvalue of the block associated with the most probable letters among a direct sum of independent GUE matrices whose dimensions correspond to the multiplicities $d_r$ of Theorems 5.2 and 5.3, subject to the condition that $\sum_{r=1}^{m} \sqrt{\pi_{\tau(r)}} X_r = 0$, where $X_1, X_2, \ldots, X_m$ are the diagonal elements of the random matrix.

**Remark 5.6** The difference between the zero-trace condition $\sum_{r=1}^{m} X_r = 0$ and the generalized traceless condition $\sum_{r=1}^{m} \sqrt{\pi_{\tau(r)}} X_r = 0$ amounts to nothing more than a difference in the choice of scaling for each row $R_n$. We will find it more natural to express our results using the normalization associated with the zero-trace condition $\sum_{r=1}^{m} X_r = 0$.

## 6 Fine Structure of the Brownian Functional

So far, we have seen that the limiting shape of the random Young tableau generated by an aperiodic, irreducible, homogeneous Markov chain can be expressed as a Brownian functional. The form of this functional is similar to the iid case; the only difference is in the covariance structure of the Brownian motion. We begin our study of the consequences of this difference.

In the iid uniform $m$-alphabet case, Johansson [22] proved that the limiting shape of the Young tableau had a joint law which is that of the spectrum of an $m \times m$ traceless GUE matrix. An immediate consequence of this result is that the limiting shape of the Young tableau contains simple symmetries, e.g., for each $1 \leq r \leq m$, $R^r_\infty \overset{\text{d}}{=} -R^{m-r}_\infty$. Now, as was seen in Corollary 5.1 of Theorem 5.2, the form of the Brownian functional in the doubly stochastic case involved only the maximal term. We will see that there is also a pleasing symmetry to the limiting shape of Young tableaux in the doubly stochastic case by examining a natural bijection between the parameter set $I_{r,m}$ and $I_{m-r,m}$, for any $1 \leq r \leq m - 1$. Indeed, this result will follow as a corollary to the following, more general, theorem:

**Theorem 6.1** The limiting functionals of Theorem 5.2 enjoy the following symmetry property: for every $1 \leq r \leq m - 1$,
\[ V^r_\infty := \sum_{i=1}^{m_r} \sigma_{\tau(i)} \tilde{B}^{\tau(i)}(1) \]

\[ + \max_{t(\cdot) \in I_{r-m_r,d_r}} \sum_{j=1}^{r-m_r} \sum_{\ell=j}^{m_r+d_r-r} \sigma_{\tau(\ell)} \tilde{B}^{\tau(\ell)}(\Delta t_{j,\ell}) \]

\[ \leq \sum_{i=m_r+d_r+1}^{m} \sigma_{\tau(i)} \tilde{B}^{\tau(i)}(1) \]

\[ + \max_{u(\cdot) \in I_{m_r+d_r-r,d_r}} \sum_{j=1}^{m_r+d_r-r} \sum_{\ell=j}^{r-m_r+j} \sigma_{\tau(\ell)} \tilde{B}^{\tau(\ell)}(\Delta u_{j,\ell}), \quad (6.1) \]

where \( B^{\ell}(\Delta) := B^{\ell}(t) - B^{\ell}(s) \), for \( \Delta = [s,t] \), and where the non-maximal terms on the left and right-hand sides of (6.1) are identically zero if \( m_r = 0 \), or \( m_r + d_r = m \), respectively.

**Remark 6.1** Recall that, from the definitions of \( m_r \) and \( d_r \), the non-maximal summation terms on the left and right-hand sides of (6.1) reflect the letters which have, respectively, greater and smaller stationary probabilities than \( \pi_\tau(r) \). Recall, moreover, that the maximal terms are associated with the indices having the same stationary probability as \( \pi_\tau(r) \). The maximal term on the left-hand side of (6.1) involves a summation over \( r-m_r \) rows, while the one on the right-hand side involves \( m_r+1-r \) rows. Thus, in a sense, the two maximal terms in (6.1) split \( d_r = m_r+1-m_r \) rows between themselves. In summary, the functional on the right-hand side of (6.1) corresponds to the sum of the \( m-r \) bottom rows of the Young tableau.

**Proof.** Without loss of generality, we may assume that \( \tau(j) = j \), for all \( 1 \leq j \leq m \). Fix \( 1 \leq r \leq m-1 \), and for any point \( t \) in the index set \( I_{r-m_r,d_r} \), define \( \Delta t_{j+m_r,\ell} = [t_{j+1,\ell-1}, t_{j,\ell}] \), for \( 1 \leq j \leq r-m_r \) and \( 1 \leq \ell \leq d_r \). Furthermore, for each \( 1 \leq j \leq m_r \) or \( m_r+1 < j \leq m \), set \( \Delta t_{j,\ell} = [0,1] \), for \( j = \ell \), \( \Delta t_{j,\ell} = \{0\} \), for \( 0 \leq \ell < j \), and \( \Delta t_{j,\ell} = \{1\} \), for \( j < \ell \leq m \). Next, as in the proof of Theorem 5.2, consider the set of points \( \{ t_{j,\ell} \mid 1 \leq j \leq r-m_r, 1 \leq \ell \leq d_r \} \), and order them as \( s_0 := 0 < s_1 < \cdots < s_{k-1} < s_k := 1 \), for some integer \( k \), and let \( \Delta s_q = [s_{q-1}, s_q] \), for each \( 1 \leq q \leq k \).

Now, for each \( 1 \leq q \leq k \), let \( A_q \) consist of the indices \( \ell \) for which \( \Delta s_q \cap \Delta t_{j,\ell} \neq \emptyset \). Then, almost surely,
\[
\sum_{i=1}^{m_r} \sigma_i \tilde{B}^i(1) + \sum_{j=1}^{r-m_r} \sum_{\ell=j}^{r} \sigma_{m_r+\ell} \tilde{B}^{m_r+\ell}(\Delta t_{j,\ell}) \\
= \sum_{j=1}^{r} \sum_{\ell=1}^{m} \sigma_\ell \tilde{B}^\ell(\Delta t_{j,\ell}) \\
= \sum_{j=1}^{r} \sum_{q=1}^{\kappa} \sum_{\ell=1}^{m} \sigma_\ell \tilde{B}^\ell(\Delta t_{j,\ell} \cap \Delta s_q) \\
= \sum_{j=1}^{r} \sum_{q=1}^{\kappa} \sum_{\ell \in A_q} \sigma_\ell \tilde{B}^\ell(\Delta s_q).
\]

(6.2)

Now by the “stairstep” properties of \(I_{r,m}\) there are precisely \(r\) elements in each \(A_q\). Letting \(\tilde{A}_q = \{1, \ldots, m\} \setminus A_q\), for each \(1 \leq q \leq \kappa\), we thus see that each \(\tilde{A}_q\) contains exactly \(m-r\) elements. Let \(\tilde{\ell}_{j,q}\) be the \(j^{th}\) smallest element of \(\tilde{A}_q\). We claim that for each \(1 \leq j \leq m-r\), the sequence \(\tilde{\ell}_{j,1}, \tilde{\ell}_{j,2}, \ldots, \tilde{\ell}_{j,\kappa}\) is weakly decreasing.

Indeed, fix \(1 \leq j \leq m-r\) and \(1 \leq q \leq \kappa - 1\), and suppose that \(\tilde{\ell}_{j,q}\) is less than all the elements of \(A_q\). Then, by the properties of \(I_{r,m}\), the least element of \(A_{q+1}\) is no smaller, so that the \(j^{th}\) smallest element of \(\tilde{A}_q\), \(\tilde{\ell}_{j,q+1}\) is also \(\tilde{\ell}_{j,q}\). Next, suppose that \(\tilde{\ell}_{j,q}\) is greater than \(k \geq 1\) elements of \(A_q\). Thus, \(\tilde{\ell}_{j,q} = j + k\). Then there are at most \(k\) elements of \(A_{q+1}\) which are less than or equal to \(\tilde{\ell}_{j,q}\), by the properties of \(I_{r,m}\). But this implies that there are at least \(j\) elements of \(\tilde{A}_{q+1}\) which are less than or equal to \(\tilde{\ell}_{j,q}\). Thus, \(\tilde{\ell}_{j,q+1} \leq \tilde{\ell}_{j,q}\), and the claim is proved.

Moreover, since each \(A_q\) contains \(\{1, 2, \ldots, m_r\}\), we see that necessarily each \(\tilde{A}_q\) contains \(\{m_r + d_r + 1, m_r + d_r + 2, \ldots, m\}\).

For each \(1 \leq j \leq m-r\), we may now amalgamate the intervals \(\Delta s_q\) to obtain a partition of the unit interval. Specifically, for each \(1 \leq j \leq m-r\), and each \(1 \leq \ell \leq m\), let \(\tilde{u}_{j,\ell}\) be the smallest \(s_q\) such that \(\tilde{\ell}_{j,q+1} \leq \ell\). (We define \(\tilde{u}_{j,0} = 1\), for all \(1 \leq j \leq m-r\).)

Finally, and most crucially, recall that \(\sum_{\ell=1}^{m} \sigma_\ell \tilde{B}^\ell(t) = 0\), for all \(t\). Then since \((\tilde{B}^1, \tilde{B}^2, \ldots, \tilde{B}^m) \overset{\ell}{=} (\tilde{B}^1, -\tilde{B}^2, \ldots, -\tilde{B}^m)\),

\[
\sum_{j=1}^{r} \sum_{q=1}^{\kappa} \sum_{\ell \in A_q} \sigma_\ell \tilde{B}^\ell(\Delta s_q)
\]

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where \( \Delta u_{j,\ell} = [u_{j,\ell-1}, u_{j,\ell}] \). But, by the way we ordered each \( A_q \), we must have \( \Delta u_{j_1,\ell} \cap \Delta u_{j_2,\ell} = \emptyset \), for any \( j_1 \neq j_2 \). Thus, \( u \in I_{m_r+d_r-r,d_r} \), and so we may restrict the summation over \( \ell \) in (6.3) to \( \ell = j, \ldots, r - m_r + j \), since the remaining terms are zero. Equation (6.1) follows immediately by taking the maxima over \( I_{r-m_r,d_r} \) and \( I_{m_r+d_r-r,d_r} \) over the left-hand and right-hand sides, respectively, of (6.3).

For doubly stochastic transition matrices, the symmetry is even more apparent:

**Corollary 6.1** Let the transition matrix \( P \) of Theorem 5.2 be doubly stochastic. Then, for every \( 1 \leq r \leq m-1 \),

\[
V_r^\infty := \max_{u(\cdot) \in I_{r,m}} \sum_{j=1}^{r} \sum_{\ell=j}^{m-r+j} \sigma(\tilde{B}^{\ell}(t_{j,\ell}) - \tilde{B}^{\ell}(t_{j,\ell-1}))
\]

\[\leq \max_{u(\cdot) \in I_{m-r,m}} \sum_{j=1}^{m-r} \sum_{\ell=j}^{r+j} \sigma(\tilde{B}^{\ell}(u_{j,\ell}) - \tilde{B}^{\ell}(u_{j,\ell-1})) := V_m^{m-r}, \quad (6.4)
\]

and so

\[
\lim_{n \to \infty} \frac{\sum_{j=1}^{r} R_j^n - rn/m}{\sqrt{n}} \leq \lim_{n \to \infty} \frac{rn/m - \sum_{j=m-r+1}^{m} R_j^n}{\sqrt{n}}. \quad (6.5)
\]

Moreover,

\[
(V_1^\infty, \ldots, V_r^\infty) \leq (V_m^{m-1}, \ldots, V_m^{m-r}). \quad (6.6)
\]
**Proof.** Since \( m_r = 0 \) and \( d_r = m \) for all \( 1 \leq r \leq m \), the non-maximal terms on both sides of (6.1) disappear, and we have (6.4).

To prove (6.5), recall that \( V_m = \sum_{j=1}^{m} R_n^j = n \). Then, from the result just proved,

\[
\frac{V_n^{m-r} - (m-r)n/m}{\sqrt{n}} = \frac{\sum_{j=1}^{m-r} R_n^j - (m-r)n/m}{\sqrt{n}}
\]

\[
= \frac{(n - \sum_{j=m-r+1}^{m} R_n^j) - (m-r)n/m}{\sqrt{n}}
\]

\[
= \frac{rn/m - \sum_{j=m-r+1}^{m} R_n^j}{\sqrt{n}}
\]

\[
\Rightarrow V_{\infty}^{m-r} \overset{D}{=} V_{\infty}^r,
\]

and we have established the claimed symmetry.

Finally, the extension of (6.4) to (6.6) follows from a standard Cramér-Wold argument.

Turning again to the cyclic case, recall that, for \( m \geq 4 \), the limiting shape of the Young tableau in general differs from that of the iid uniform case. The following theorem characterizes the asymptotic covariance matrices of such Markov chains.

**Theorem 6.2** Let \( P \) be the \( m \times m \) transition matrix of an aperiodic, irreducible, cyclic Markov chain on an \( m \)-letter, ordered alphabet, \( A_m = \{\alpha_1 < \alpha_2 < \cdots < \alpha_m\} \), with

\[
P = \begin{pmatrix}
a_1 & a_m & \cdots & a_3 & a_2 \\
a_2 & a_1 & \cdots & a_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
a_{m-1} & \cdots & a_1 & a_m \\
a_m & a_{m-1} & \cdots & a_2 & a_1
\end{pmatrix},
\]

Then, for \( 1 \leq j \leq m \), \( \lambda_j = \sum_{k=1}^{m} a_k \omega^{(k-1)(j-1)} \) is an eigenvalue of \( P \), where \( \omega = \exp(2\pi i/m) \) is the \( m \)th principal root of unity. Moreover, letting \( \gamma_j = \lambda_j/(1 - \lambda_j) \), for \( 2 \leq j \leq m \), and \( \beta_j = \cos(2\pi j/m) \), for \( 0 \leq j \leq m \),
the asymptotic covariance matrix $\Sigma$ is given by:

For $m = 2m_0 + 1$, 

$$
\Sigma = \frac{m-1}{m^2} M^{(1)} + \frac{4}{m^2} \sum_{j=2}^{m_0+1} \text{Re}(\gamma_j) M^{(j)},
$$

(6.9)

and for $m = 2m_0$, 

$$
\Sigma = \frac{m-1}{m^2} M^{(1)} + \frac{4}{m^2} \sum_{j=2}^{m_0} \text{Re}(\gamma_j) M^{(j)} + \frac{2}{m^2} \gamma_{m_0+1} M^{(m_0+1)},
$$

(6.10)

where $M^{(j)}$ is an $m \times m$ Toeplitz matrix with entries $(M^{(j)})_{k,\ell} = \beta_{(j-1)|k-\ell|}$, for $2 \leq j \leq m$, and $(M^{(1)})_{k,\ell} = \delta_{k,\ell} - (1 - \delta_{k,\ell})/(m - 1)$, for $j = 1$.

**Proof.** It is straightforward, and classical, to verify that, for each $1 \leq j \leq m$, $(1, \omega^{j-1}, \omega^{2(j-1)}, \ldots, \omega^{(m-1)(j-1)})$ is a left eigenvector of $P$, with eigenvalue $\lambda_j = \sum_{k=1}^{m} a_k \omega^{(k-1)(j-1)}$. We can thus write our standard diagonalization of $P$ as $P = S^{-1} \Lambda S$, where $\Lambda = \text{diag}(1, \lambda_2, \ldots, \lambda_m)$,

$$
S = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{m-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(m-1)} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \omega^{m-1} & \omega^{2(m-1)} & \cdots & \omega^{(m-1)^2}
\end{pmatrix},
$$

(6.11)

and

$$
S^{-1} = \frac{1}{m} \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 \\
1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(m-1)} \\
1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-(2(m-1))} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \omega^{-(m-1)} & \omega^{-(2(m-1))} & \cdots & \omega^{-(m-1)^2}
\end{pmatrix},
$$

(6.12)

In the present cyclic, and hence, doubly stochastic case, we know that $\Sigma = (1/m)(I + S^{-1}DS + (S^{-1}DS)^T)$, where, as usual, $D = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m) = \text{diag}(-1/2, \lambda_2/(1-\lambda_2), \ldots, \lambda_m/(1-\lambda_m))$. We can then compute the entries of $S^{-1}DS$ as follows:
\[(S^{-1}DS)_{j_1,j_2} = \sum_{k,\ell} (S^{-1})_{j_1,k}(D)_{k,\ell}(S)_{\ell,j_2} \]
\[= \sum_{k,\ell} \frac{1}{m} (\omega^{-1})^{(k-1)}(\delta_{k,\ell}\gamma_k)(\omega^{j_2-1})^{(\ell-1)} \]
\[= \sum_{k=1}^{m} \frac{\gamma_k}{m} \omega^{j_2-j_1}(k-1) \]
\[= \frac{1}{m} \left( -\frac{1}{2} + \sum_{k=2}^{m} \gamma_k \omega^{j_2-j_1}(k-1) \right), \quad (6.13) \]

for all \(1 \leq j_1, j_2 \leq m\). The entries of the asymptotic covariance matrix can thus be written as

\[\sigma_{j_1,j_2} = \frac{1}{m} \left( \delta_{j_1,j_2} + (S^{-1}DS)_{j_1,j_2} + (S^{-1}DS)_{j_2,j_1} \right) \]
\[= \frac{1}{m} \left( \delta_{j_1,j_2} + \frac{1}{m} \left( -1 + \sum_{k=2}^{m} \gamma_k (\omega^{j_2-j_1}(k-1) + \omega^{j_1-j_2}(k-1)) \right) \right) \]
\[= \frac{m-1}{m^2} M^{(1)}_{j_1,j_2} + \frac{2}{m^2} \sum_{k=2}^{m} \gamma_k \beta_{j_2-j_1}(k-1), \quad (6.14)\]

for all \(1 \leq j_1, j_2 \leq m\).

Next, note that since \(\lambda_{m+2-k} = \bar{\lambda}_k\), we have \(\gamma_{m+2-k} = \bar{\gamma}_k\), for all \(2 \leq k \leq m\). Moreover, since \(\beta_{j_2-j_1}|_{(k-1)} = \beta_{j_2-j_1}|((m+2-k)-1, \ldots, 1)\), we can write \((6.14)\) more symmetrically as \((6.9)\) or \((6.10)\), depending on whether \(m\) is odd or even, respectively, and in the latter case, we also use that \(\gamma_{m_0+1}\) is real, since \(\omega^{m_0} = -1\).

Let us again examine the cases \(m = 3\) and \(m = 4\). In the former case, we have

\[M^{(1)} = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}. \]

But for \(m = 3\), \(\beta_1 = -1/2 = \beta_2\), and so \(M^{(1)} = M^{(2)}\). Hence
\[ \Sigma = \frac{2}{9} M^{(1)} + \frac{4}{9} Re(\gamma_2) M^{(2)} = \frac{2}{9} (1 + 2 Re(\gamma_2)) M^{(1)}. \] (6.15)

Hence, for \( m = 3 \), cyclicity always produces a rescaled version of the uniform iid case, with the rescaling factor given by \( 1 + 2 Re(\gamma_2) \).

For \( m = 4 \), however,

\[
M^{(1)} = \begin{pmatrix}
1 & -1/3 & -1/3 & -1/3 \\
-1/3 & 1 & -1/3 & -1/3 \\
-1/3 & -1/3 & 1 & -1/3 \\
-1/3 & -1/3 & -1/3 & 1
\end{pmatrix},
\]

and \( \beta_1 = 0, \beta_2 = -1, \) and \( \beta_3 = 0 \). Thus,

\[
M^{(2)} = \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix},
\]

and

\[
M^{(3)} = \begin{pmatrix}
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1
\end{pmatrix}.
\]

In this case, we have

\[
\Sigma = \frac{3}{16} M^{(1)} + \frac{4}{16} Re(\gamma_2) M^{(2)} + \frac{2}{16} \gamma_3 M^{(3)}.
\]

Next, note that \( 2M^{(2)} + M^{(3)} = 3M^{(1)} \). Then, if \( Re(\gamma_2) = \gamma_3 \),

\[
\Sigma = \frac{3}{16} M^{(1)} + \frac{4}{16} Re(\gamma_2) M^{(2)} + \frac{2}{16} \gamma_3 M^{(3)}
= \frac{3}{16} M^{(1)} + \frac{2}{16} (2 Re(\gamma_2) M^{(1)})
= \frac{3}{16} (1 + 2 Re(\gamma_2)) M^{(1)},
\] (6.16)

so that there is still a rescaled version of the iid case in a non-iid cyclic setting.

Indeed, since we know that \( \lambda_2 = a_1 + ia_2 - a_3 - ia_4 = (a_1 - a_3) + i(a_2 - a_4) \) and \( \lambda_3 = a_1 - a_2 + a_3 - a_4 \), we find that
\[
Re(\gamma_2) = \frac{1 - a_2 - 2a_3 - a_4}{(a_2 + 2a_3 + a_4)^2 + (a_2 - a_4)^2} - 1,
\]
and \(\gamma_3 = 1/(2(a_2 + a_4)) - 1\). A short calculation then shows that \(Re(\gamma_2) = \gamma_3\) if and only if \(a_3^2 = a_2a_4\). We thus have a complete characterization of all 4-letter, cyclic Markov chains whose Young tableaux have the same limiting shape as the uniform iid case. In particular, choosing \(a_2 = a_4 = a\), for some \(0 < a < 1/3\), leads to \(a_3 = a\) and \(a_1 = 1 - 3a\). If, moreover, \(a = 1/4\), we have again the iid uniform case. For \(a \neq 1/4\), however, we may view the Markov chain as a “lazy” version of the uniform iid case.

Note that the scaling factor in both (6.15) and (6.16) is \(1 + 2Re(\gamma_2)\). The following theorem shows that, in fact, such a scaling factor occurs for general \(m\), and gives a spectral characterization of all transition matrices which lead to an iid limiting shape.

**Theorem 6.3** Let \(P\) be the \(m \times m\) transition matrix of an aperiodic, irreducible, cyclic Markov chain on an \(m\)-letter, ordered alphabet given in Theorem 6.2. Then the asymptotic covariance matrix \(\Sigma\) is a rescaled version of the iid uniform covariance matrix \(\Sigma_{iidu} := ((m - 1)/m^2)M^{(1)}\) if and only if the constants \(\gamma_j = \lambda_j/(1 - \lambda_j)\), for \(2 \leq j \leq m\), satisfy the condition

\[
Re(\gamma_j) = \gamma, \quad \text{for all } 2 \leq j \leq m, \quad (6.17)
\]
for some real constant \(\gamma\). Moreover, the scaling is then given by

\[
\Sigma = (1 + 2\gamma)\Sigma_{iidu}. \quad (6.18)
\]

**Proof.** We first claim that the system of matrix equations

\[
\sum_{j=2}^{m} b_j M^{(j)} = M^{(1)} \quad (6.19)
\]
has a unique solution \(b_j = 1/(m - 1)\), for all \(2 \leq j \leq m\). Indeed, revisiting (6.14), we can express each \(M^{(j)}\) as

\[
M^{(j)} = \tilde{M}^{(j)} + \tilde{M}^{(-j)} = \tilde{M}^{(j)} + \tilde{M}^{(m-j+1)}. \quad (6.20)
\]

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where \((\tilde{M}^{(j)})_{k,\ell} = \omega^{(j-1)(\ell-k)/2}\), for all \(1 \leq k, \ell \leq m\), so that (6.19) becomes

\[
M^{(1)} = \sum_{j=2}^{m} b_j \left( \tilde{M}^{(j)} + \tilde{M}^{(m-j+1)} \right) \\
= \sum_{j=2}^{m} (b_j + b_{m-j+1}) \tilde{M}^{(j)} \\
= \sum_{j=2}^{m} \tilde{b}_j \tilde{M}^{(j)},
\]

(6.21)

where \(\tilde{b}_j := (b_j + b_{m-j+1})/2\), for \(2 \leq j \leq m\).

Now, clearly, each \(\tilde{M}^{(j)}\) is cyclic, so that in solving (6.21) we need only examine the \(m\) entries in the first rows of the matrices. We can thus reduce (6.21) to the \(m \times (m-1)\) system of equations

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\omega & \omega^2 & \omega^3 & \cdots & \omega^{m-1} \\
\omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(m-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega^{m-1} & \omega^{2(m-1)} & \omega^{3(m-1)} & \cdots & \omega^{(m-1)^2}
\end{pmatrix}
\begin{pmatrix}
\tilde{b}_2 \\
\tilde{b}_3 \\
\vdots \\
\tilde{b}_m
\end{pmatrix}
= \begin{pmatrix}
1 \\
-m^{-1} \\
-m^{-1} \\
\vdots \\
-m^{-1}
\end{pmatrix}.
\]

(6.22)

Since each of the last \(m-1\) rows of the matrix in (6.22) sums to \(-1\), it is clear that \(\tilde{b}_j = 1/(m-1)\) is a solution to the system. To see that this solution is, in fact, unique, consider the \((m-1) \times (m-1)\) sub-matrix consisting of the last \(m-1\) rows of the matrix in (6.22), namely,

\[
\begin{pmatrix}
\omega & \omega^2 & \omega^3 & \cdots & \omega^{m-1} \\
\omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(m-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega^{m-1} & \omega^{2(m-1)} & \omega^{3(m-1)} & \cdots & \omega^{(m-1)^2}
\end{pmatrix}.
\]

(6.23)

Now this matrix, which is very closely related to the Fourier matrix which arises in discrete Fourier transform problems, is in fact invertible, and can be shown to have one eigenvalue of \(-1\), and \(m-2\) eigenvalues of the form \(\pm \sqrt{m}\) and \(\pm i \sqrt{m}\), so that the modulus of the determinant is \(m^{(m-2)/2} \neq 0\). Thus, the solution \(\tilde{b}_j = 1/(m-1)\) is unique, and since \(b_j = (b_j + b_{m-j+1})/2 = b_{m-j+1}\),
for all $2 \leq j \leq m$, we conclude that $b_j = 1/(m-1)$ as well, for all $2 \leq j \leq m$, and the claim is proved.

We can now use Theorem 6.2 to simplify the asymptotic covariance matrix decomposition as follows:

$$
\Sigma = \frac{m-1}{m^2} M^{(1)} + \frac{2}{m^2} \sum_{k=2}^{m} \gamma_k M^{(k)}
$$

$$
= \frac{m-1}{m^2} M^{(1)} + 2 \gamma \frac{1}{m^2} \sum_{k=2}^{m} M^{(k)}
$$

$$
= \frac{m-1}{m^2} M^{(1)} + 2 \gamma \frac{m-1}{m^2} M^{(2)}
$$

$$
= (1 + 2\gamma) \frac{m-1}{m^2} M^{(1)}
$$

$$
= (1 + 2\gamma) \Sigma_{iidu},
$$

(6.24)

where $\gamma = \text{Re}(\gamma_j)$, for all $2 \leq j \leq m$. If the real parts of $\gamma_j$ are not all identical, then the uniqueness of the solution of (6.19) implies that no such simplification is possible, and the theorem is proved.

**Remark 6.2** To see that the condition in (6.17) is not vacuous for any $m$, recall that for $m=4$, the “lazy” chain has the iid limiting shape. This is true for general $m$: if $a_2 = a_3 = \cdots = a_m = a$, for some $0 < a < 1/(m-1)$, then $\lambda_j = 1-(m-1)a$, for all $2 \leq j \leq m$. Trivially, then, $\gamma_j = 1/((m-1)a)-1 := \gamma$, for all $2 \leq j \leq m$, so that the conditions of Theorem 6.3 are satisfied, and the scaling factor is given by $1 + 2\gamma = (2 - (m-1)a)/((m-1)a)$. Even in the $m=4$ case, however, we saw that there were other, more general, cyclic transition matrices which gave rise to the iid limiting distribution.

The previous theorem indicates precisely when we may expect the limiting shape of a cyclic Markov chain to be identical to that of the iid uniform case. Now the first-order behavior of all rows of the Young tableau is $n/m + O(\sqrt{n})$ for cyclic Markov chains. Although this differs from the first-order behavior in the non-uniform iid case, one may still ask whether the limiting shape for a cyclic Markov chain might still be that of some non-uniform iid case. In fact, this can never occur: cyclicity ensures that the asymptotic covariance
matrix is also cyclic, and thus cannot be equal to the asymptotic covariance matrix of any non-uniform iid case.

Still, we may ask how to relate the iid non-uniform limiting shape to that of a general Markov chain having the same stationary distribution. The following interpolation result describes the asymptotic covariance matrix for a Markov chain whose transition matrix is a convex combination of an iid (uniform or non-uniform) transition matrix and another arbitrary transition matrix having the same stationary distribution:

**Theorem 6.4** For any \( m \geq 3 \), let \( P_0 \) be the \( m \times m \) transition matrix of an irreducible, aperiodic, homogeneous Markov chain, and let its associated asymptotic covariance matrix be given by

\[
\Sigma_0 = \Pi_0 + \Pi_0 (S_0^{-1} D_0 S_0) + (S_0^{-1} D_0 S_0)^T \Pi_0,
\]

in the standard notations of Theorem 4.1. Then, for \( 0 < \delta \leq 1 \), the transition matrix \( P = (1 - \delta)I_m + \delta P_0 \) has an asymptotic covariance matrix given by

\[
\Sigma = \frac{1}{\delta} (\Sigma_0 + (1 - \delta) \Sigma \Pi_0),
\]

where \( \Sigma \Pi_0 \) is the covariance matrix associated with the iid Markov chain having the same stationary distribution as \( P_0 \).

**Proof.** Using the standard notations of Theorem 4.1 we will write

\[
\Sigma = \Pi + \Pi (S^{-1} D S) + (S^{-1} D S)^T \Pi
\]

in terms of the decomposition \( \Sigma_0 \) in (6.25). Now, clearly, the stationary distribution under \( P \) is that of \( P_0 \), so that \( \Pi = \Pi_0 \). We will thus write the stationary distribution simply as \( (\pi_1, \pi_2, \ldots, \pi_m) \). Moreover, the eigenvectors are also unchanged, so that \( S = S_0 \). However, for each eigenvalue \( \lambda_{k,0} \) of \( P_0 \), we have that \( \lambda_k = (1 - \delta) + \delta \lambda_{k,0} \) is an eigenvalue of \( P \), for \( 1 \leq k \leq m \). Thus, for each \( 2 \leq k \leq m \), the diagonal entries of \( D \) are given by

\[
\gamma_k := \frac{\lambda_k}{1 - \lambda_k} = \frac{(1 - \delta) + \delta \lambda_{k,0}}{\delta (1 - \lambda_{k,0})} = \frac{1 - \delta}{\delta} + \gamma_{k,0},
\]
where \( \gamma_{k,0} \) are the diagonal entries of \( D_0 \). We can thus decompose \( D \) as follows:

\[
D = \text{diag}(-1/2, \gamma_2, \ldots, \gamma_m)
= \text{diag}(-1/2, 0, \ldots, 0) + \left( \frac{1 - \delta}{\delta} \right) \text{diag}(0, 1, \ldots, 1)
+ \left( \frac{1}{\delta} \right) \text{diag}(0, \gamma_2, \ldots, \gamma_m, 0)
= \text{diag} \left( -\left( \frac{1 - \delta}{2\delta} \right), 0, \ldots, 0 \right) + \left( \frac{1 - \delta}{\delta} \right) I_m + \left( \frac{1}{\delta} \right) D_0. \tag{6.27}
\]

Next, recall from Proposition 4.1 that the first column of \( S^{-1} \) is \( (1, 1, \ldots, 1)^T \). Hence,

\[
S^{-1}DS = S_0^{-1}DS_0
= \begin{pmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 1 & \cdots & * \end{pmatrix} \begin{pmatrix} -\frac{1 - \delta}{2\delta} & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_m \\ \pi_1 & \pi_2 & \cdots & \pi_m \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_m \end{pmatrix}
+ \left( \frac{1 - \delta}{\delta} \right) S_0^{-1}I_mS_0 + \left( \frac{1}{\delta} \right) S_0^{-1}D_0S_0
= -\left( \frac{1 - \delta}{2\delta} \right) \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_m \\ \pi_1 & \pi_2 & \cdots & \pi_m \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_m \end{pmatrix}
+ \left( \frac{1 - \delta}{\delta} \right) I_m + \left( \frac{1}{\delta} \right) S_0^{-1}D_0S_0,
\]

which gives us

\[
\Pi S^{-1}DS = \Pi_0 S^{-1}DS
\]
\[
\begin{align*}
= & - \left( \frac{1 - \delta}{2\delta} \right) \left( \begin{array}{cccc}
\pi_1 & 0 & \cdots & 0 \\
0 & \pi_2 & & \\
0 & \vdots & \ddots & \\
0 & \cdots & \cdots & \pi_m
\end{array} \right) \left( \begin{array}{cccc}
\pi_1 & \pi_2 & \cdots & \pi_m \\
\pi_1 & \pi_2 & \cdots & \pi_m \\
\vdots & \vdots & \ddots & \\
\pi_1 & \pi_2 & \cdots & \pi_m
\end{array} \right) \\
+ & \left( \frac{1 - \delta}{2\delta} \right) \Pi_0 + \left( \frac{1}{\delta} \right) \Pi_0 S_0^{-1} D_0 S_0 \\
= & - \left( \frac{1 - \delta}{2\delta} \right) \left( \begin{array}{cccc}
\pi_1^2 & \pi_1 \pi_2 & \cdots & \pi_1 \pi_m \\
\pi_2 \pi_1 & \pi_2^2 & \cdots & \pi_2 \pi_m \\
\vdots & \vdots & \ddots & \\
\pi_m \pi_1 & \pi_m \pi_2 & \cdots & \pi_m^2
\end{array} \right) \\
+ & \left( \frac{1 - \delta}{2\delta} \right) \Pi_0 + \left( \frac{1}{\delta} \right) \Pi_0 S_0^{-1} D_0 S_0. \\
\end{align*}
\]

Finally, we can express \( \Sigma \) as

\[
\Sigma = \Pi + \Pi(S^{-1}DS) + (S^{-1}DS)^T \Pi
\]

\[
= \left( \frac{1}{\delta} \right) \Pi_0 + \left( 1 - \frac{1}{\delta} \right) \Pi_0 + \Pi_0(S^{-1}DS) + (\Pi_0(S^{-1}DS))^T
\]

\[
= \left( \frac{1}{\delta} \right) \Sigma_0 + \left( 1 - \frac{1}{\delta} \right) (\Pi_0 - 2\Pi_0)
\]

\[
+ \left( 1 - \frac{1}{\delta} \right) \left( \begin{array}{cccc}
\pi_1^2 & \pi_1 \pi_2 & \cdots & \pi_1 \pi_m \\
\pi_2 \pi_1 & \pi_2^2 & \cdots & \pi_2 \pi_m \\
\vdots & \vdots & \ddots & \\
\pi_m \pi_1 & \pi_m \pi_2 & \cdots & \pi_m^2
\end{array} \right)
\]

\[
= \left( \frac{1}{\delta} \right) \Sigma_0 + \left( 1 - \frac{1}{\delta} \right) (-\Sigma_{\Pi_0})
\]

\[
= \frac{1}{\delta} (\Sigma_0 + (1 - \delta)\Sigma_{\Pi_0})
\]

and we are done.

Thus far we have expressed our limiting laws in terms of Brownian functionals whose Brownian motions have a non-trivial covariance structure arising directly from the specific nature of the transition matrix. It is of interest to instead express the limiting laws in terms of standard Brownian motions.
Since the asymptotic covariance matrix $\Sigma$ is non-negative definite, we can find an $m \times m$ matrix $C$ such that $\Sigma = CC^T$. (The matrix $C$ is not unique, since $(CQ)(CQ)^T = CC^T = \Sigma$ for any orthogonal matrix $Q$.) Clearly, we then have

\[
(\sigma_1 \tilde{B}^1(t), \sigma_2 \tilde{B}^2(t), \ldots, \sigma_m \tilde{B}^m(t))^T = C(B^1(t), B^2(t), \ldots, B^m(t))^T, \quad (6.31)
\]

where $(B^1(t), B^2(t), \ldots, B^m(t))^T$ is a standard, $m$-dimensional Brownian motion, since

\[
\mathbb{E}[(\sigma_1 \tilde{B}^1(t), \sigma_2 \tilde{B}^2(t), \ldots, \sigma_m \tilde{B}^m(t))^T (\sigma_1 \tilde{B}^1(t), \sigma_2 \tilde{B}^2(t), \ldots, \sigma_m \tilde{B}^m(t))] = \mathbb{E}[C(B^1(t), B^2(t), \ldots, B^m(t))^T][(C(B^1(t), B^2(t), \ldots, B^m(t))^T)^T]
\]

\[
= C[\mathbb{E}(B^1(t), B^2(t), \ldots, B^m(t))^T](B^1(t), B^2(t), \ldots, B^m(t))]C^T
\]

\[
= C(tI_m)C^T
\]

\[
= t\Sigma.
\]

Next, we can, without loss of generality, assume that $\tau(\ell) = \ell$, for all $\ell$, and so write our main result (5.8) in Theorem 5.2 as

\[
\frac{V_r - \nu_r n}{\sqrt{n}} \Rightarrow \sum_{k=1}^{m_r} \sigma_k \tilde{B}^k(1) + \max_{I_{r-m_r,d_r}} \sum_{j=1}^{r-m_r} \sum_{\ell=j}^{r-m_r+d_r+m_r-r+j} \sigma_{m_r+\ell} \tilde{B}^{m_r+\ell}(\Delta t_{j,\ell})
\]

\[
:= V_\infty^r. \quad (6.32)
\]

Simply substituting (6.31) into (6.32) immediately yields

\[
V_\infty^r = \sum_{k=1}^{m_r} \left( \sum_{i=1}^{m} C_{k,i} B^i(1) \right)
\]

\[
+ \max_{I_{r-m_r,d_r}} \sum_{j=1}^{r-m_r} \sum_{\ell=j}^{r-m_r+d_r+m_r-r+j} \left( \sum_{i=1}^{m} C_{m_r+\ell,i} B^i(\Delta t_{j,\ell}) \right)
\]

\[
= \sum_{i=1}^{m} \sum_{k=1}^{m_r} C_{k,i} B^i(1)
\]

\[
+ \max_{I_{r-m_r,d_r}} \sum_{i=1}^{m} \sum_{j=1}^{r-m_r} \sum_{\ell=j}^{r-m_r+d_r+m_r-r+j} C_{m_r+\ell,i} B^i(\Delta t_{j,\ell}). \quad (6.33)
\]
Now the first term in (6.33) is simply a Gaussian term whose variance can be computed explicitly. Unfortunately, the maximal term does not in general succumb to any significant simplifications. However, in the iid case, we can further simplify (6.33) in a very satisfying way.

Indeed, since, in the iid case, we have $\sigma_k^2 = \pi_k(1 - \pi_k)$ and, for $k \neq \ell$, $\sigma_{k,\ell} = -\pi_k \pi_\ell$, one can quickly check that $C$ can be chosen so that $C_{k,k} = \sqrt{\pi_k} - \sqrt{\pi_k \pi_k}$, and, for $k \neq \ell$, $C_{k,\ell} = -\sqrt{\pi_k \pi_\ell}$. Moreover, for all $m_r + 1 \leq k \leq m_r + d_r$, $\pi_k = \pi_{m_r+1} = \pi_r$. Then, within the maximal term, $C_{m_r+\ell,i} = \sqrt{\pi_r} - \pi_r \sqrt{\pi_i}$, for $i = m_r + \ell$, and $C_{m_r+\ell,i} = -\pi_r \sqrt{\pi_i}$, for $i \neq m_r + \ell$. With the convention that $\nu_0 = 0$, we can then express (6.33) as

$$V^r_\infty = \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) + \sum_{i=1}^{m_r} \sum_{k=1}^{m_r} (-\sqrt{\pi_i \pi_k}) B^i(1) + \max_{I_{r-m_r,dr}} \left\{ \sum_{j=1}^{r-m_r} \sum_{\ell=j}^{m_r} (d_r + m_r - r + j) \sqrt{\pi_r} B^{m_r+\ell}(\Delta t_{j,\ell}) + \sum_{j=1}^{m_r} \sum_{\ell=j}^{m_r} (-\pi_r \sqrt{\pi_i}) B^i(\Delta t_{j,\ell}) \right\} = \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) - \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) \sum_{k=1}^{m_r} \pi_k + \sqrt{\pi_r} \max_{I_{r-m_r,dr}} \left\{ \sum_{j=1}^{r-m_r} \sum_{\ell=j}^{m_r} (d_r + m_r - r + j) \sqrt{\pi_i} B^{m_r+\ell}(\Delta t_{j,\ell}) - \sqrt{\pi_i} \sum_{j=1}^{m_r} \sum_{\ell=j}^{m_r} B^i(\Delta t_{j,\ell}) \right\} = \left\{ \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) - \nu_{m_r} \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) - \pi_r \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) + \sqrt{\pi_r} \max_{I_{r-m_r,dr}} \sum_{j=1}^{r-m_r} \sum_{\ell=j}^{m_r} B^{m_r+\ell}(\Delta t_{j,\ell}) \right\} = \left\{ \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) - (\nu_{m_r} + \pi_r) \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) \right\} = \left\{ \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) - \nu_{m_r} \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) \right\}.
\[
+ \sqrt{\pi} \max_{d_r, m_r} \sum_{j=1}^{r-m_r} \sum_{\ell=0}^{m_r-r+j} B^{m_r+\ell}(\Delta t_{j,\ell}) \\
= \left\{ (1 - \nu_{m_r} - \pi_r) \sum_{i=1}^{m_r} \sqrt{\pi_i} B^i(1) \\
- (\nu_{m_r} + \pi_r) \sum_{i=m_r+1}^{m_r+d_r} \sqrt{\pi_i} B^i(1) \right\} \\
+ \sqrt{\pi} \left\{ -(\nu_{m_r} + \pi_r) \sum_{i=m_r+1}^{m_r+d_r} B^i(1) \\
+ \max_{d_r, m_r} \sum_{j=1}^{r-m_r} \sum_{\ell=0}^{m_r-r+j} B^{m_r+\ell}(\Delta t_{j,\ell}) \right\}, \quad (6.34)
\]

Note that the first two Gaussian term of (6.34) are independent of the remaining two Gaussian-maximal expression terms.

Following Glynn and Whitt[14] and Barishnykov[4], who studied the Brownian functional

\[
D_m = \max_{\ell} \sum_{i=1}^{m} B^i(\Delta t_{\ell}),
\]

we define the following, more general, Brownian functional:

\[
D_{r,m} := \max_{d_r, m_r} \sum_{j=1}^{r} \sum_{\ell=0}^{(m-r+j)} B^{\ell}(\Delta t_{j,\ell}), \quad (6.35)
\]

where \(1 \leq r \leq m\). Clearly, the maximal term in (6.34) has just such a form. We also remark that \(D_{r,m}\) corresponds to the sum of the \(r\) largest eigenvalues of an \(m \times m\) GUE matrix.

To better understand (6.34), we may, without much loss in generality, focus on the first block, that is, values of \(r\) such that \(m_r = 0\). The first Gaussian term of (6.34) thus vanishes, and, writing \(\pi_{\max}\) for \(\pi_r\), we have

\[
V_{\infty}^r = -r \pi_{\max} \sum_{i=d_1+1}^{m} \sqrt{\pi_i} B^i(1)
\]
\[ + \sqrt{\pi_{\text{max}}} \left( -r \pi_{\text{max}} \sum_{i=1}^{d_1} B^i(1) + D_{r,d_r} \right). \tag{6.36} \]

In the uniform iid case, the first Gaussian term of (6.36) itself vanishes, since \( d_r = d_1 = m \), and we have

\[
V_r^\infty = \frac{1}{\sqrt{m}} \left( -\frac{r}{m} \sum_{i=1}^{m} B^i(1) + D_{r,m} \right)
\]
\[
:= \frac{H_{r,m}}{\sqrt{m}}. \tag{6.37} \]

For \( r = 1 \), this result corresponds to Theorem 4.1 of the authors’ previous paper [18]. Furthermore, specializing (6.36) to \( r = 1 \),

\[
\frac{LI_n - \pi_{\text{max}} n}{\sqrt{n}} \Rightarrow -\pi_{\text{max}} \sum_{i=d_1+1}^{m} \sqrt{\pi_i} B^i(1)
\]
\[
+ \sqrt{\pi_{\text{max}}} \left( -\pi_{\text{max}} \sum_{i=1}^{d_1} B^i(1) + D_{1,d_1} \right)
\]
\[
= -\pi_{\text{max}} \sum_{i=d_1+1}^{m} \sqrt{\pi_i} B^i(1)
\]
\[
+ \sqrt{\pi_{\text{max}}} \left( \frac{1}{d_1} - \pi_{\text{max}} \right) \sum_{i=1}^{d_1} B^i(1)
\]
\[
+ \sqrt{\pi_{\text{max}}} H_{1,d_1}. \tag{6.38} \]

One can easily compute the variance of the Gaussian terms in (6.38) to be \( \pi_{\text{max}} (1 - d_1 \pi_{\text{max}}) / d_1 \), which is consistent with Proposition 4.1 of the authors’ previous paper [18].

The iid development above suggests that we can find additional cases which yield simple functionals of standard Brownian motions. Indeed, the first property of the matrix \( C \) in the iid case that allowed the functionals to be simplified was that \( C_{k,\ell} = c_\ell \), for all \( k \neq \ell, m_r + 1 \leq k \leq m_r + d_r \), and \( 1 \leq \ell \leq m \), where \( c_1, c_2, \ldots, c_m \) were real numbers. Then, writing the
diagonal terms of $C$ as $C_{k,k} = b_k + c_k$, for $m_r + 1 \leq k \leq m_r + d_r$, we may revisit (6.33), and write

$$V^r_\infty = \sum_{i=1}^{m_r} \sum_{k=1}^{m_r} C_{k,i} B_i(1) + \max_{I_{r-m_r,d_r}} \left\{ \sum_{j=1}^{r-m_r} \sum_{\ell=1}^{d_r+m_r-r+j} C_{m_r+\ell,i} B_i^{\ell}(\Delta t_{j,\ell}) \right\}$$

$$= \sum_{i=1}^{m_r} \sum_{k=1}^{m_r} C_{k,i} B_i(1) + \max_{I_{r-m_r,d_r}} \left\{ \sum_{j=1}^{r-m_r} \sum_{\ell=1}^{d_r+m_r-r+j} b_{m_r+\ell} B_{m_r+\ell}(\Delta t_{j,\ell}) \right\}$$

$$= \sum_{i=1}^{m_r} \sum_{k=1}^{m_r} C_{k,i} B_i(1) + \sum_{i=1}^{r-m_r} \sum_{\ell=1}^{d_r+m_r-r+j} c_i B^{\ell}(\Delta t_{j,\ell})$$

$$+ \max_{I_{r-m_r,d_r}} \left\{ \sum_{j=1}^{r-m_r} \sum_{\ell=1}^{d_r+m_r-r+j} b_{m_r+\ell} B_{m_r+\ell}(\Delta t_{j,\ell}) \right\}$$

(6.39)

Except for the fact that we have written the functional in terms of standard Brownian motions, the maximal term in (6.39) is no simpler than that of our original functional. However, the second property of the iid case that yielded further simplifications was that $b_k = b$, for all $m_r + 1 \leq k \leq m_r + d_r$. In this case, (6.39) becomes

$$V^r_\infty = \sum_{i=1}^{m_r} \sum_{k=1}^{m_r} C_{k,i} B_i(1) + \sum_{i=1}^{r-m_r} \sum_{\ell=1}^{d_r+m_r-r+j} c_i B^{\ell}(\Delta t_{j,\ell})$$

$$+ b \max_{I_{r-m_r,d_r}} \sum_{j=1}^{r-m_r} \sum_{\ell=1}^{d_r+m_r-r+j} B_{m_r+\ell}(\Delta t_{j,\ell})$$

(6.40)

Again, by focusing on the first block, we no longer have the initial Gaussian term, and (6.40) becomes

$$V^r_\infty = \sum_{i=d_1+1}^{m} c_i B^{i}(1)$$
\[ + r \sum_{i=1}^{d_1} c_i B^i(1) + b \max_{I_{r,d_1}} \sum_{j=1}^{r \cdot (d_1-r+j)} \sum_{\ell=j} B^\ell(\Delta t_{j,\ell}) \]
\[ = r \sum_{i=d_1+1}^{m} c_i B^i(1) + \left( r \sum_{i=1}^{d_1} c_i B^i(1) + b D_{r,d_1} \right) \]
\[ = r \sum_{i=d_1+1}^{m} c_i B^i(1) + r \sum_{i=1}^{d_1} \left( c_i + \frac{b}{d_1} \right) B^i(1) \]
\[ + b \left( -\frac{r}{d_1} \sum_{i=1}^{d_1} B^i(1) + D_{r,d_1} \right) \]
\[ = r \sum_{i=d_1+1}^{m} c_i B^i(1) + r \sum_{i=1}^{d_1} \left( c_i + \frac{b}{d_1} \right) B^i(1) + b H_{r,d_1}. \quad (6.41) \]

We restate these results in the following theorem:

**Theorem 6.5** Assume, without loss of generality, that \( \tau(\ell) = \ell \), for all \( 1 \leq \ell \leq m \), in the notations of Theorem 5.2. Moreover, let the asymptotic covariance matrix be given by \( \Sigma = CC^T \), where \( C \) is an \( m \times m \) matrix whose first \( d_1 \) rows are given by

\[
\begin{cases}
C_{k,\ell} = c_\ell, & k \neq \ell, 1 \leq k \leq d_1, 1 \leq \ell \leq m \\
C_{k,k} = b + c_k, & 1 \leq k \leq d_1,
\end{cases}
\]

for some real constants \( c_1, c_2, \ldots, c_m \) and \( b \). Then, for \( 1 \leq r \leq d_1 \),

\[ V_\infty^r = r \sum_{i=d_1+1}^{m} c_i B^i(1) + r \sum_{i=1}^{d_1} \left( c_i + \frac{b}{d_1} \right) B^i(1) + b H_{r,d_1}, \quad (6.43) \]

where \( H_{r,d_1} \) is the maximal functional

\[
H_{r,d_1} := \frac{1}{\sqrt{d_1}} \left( -\frac{r}{d_1} \sum_{i=1}^{d_1} B^i(1) + \max_{I_{r,d_1}} \sum_{j=1}^{r \cdot (d_1-r+j)} \sum_{\ell=j} B^\ell(\Delta t_{j,\ell}) \right).
\]

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Remark 6.3 One can generalize Theorem 6.5 to non-initial blocks (i.e., to \( r > d_1 \)) by extending the conditions in (6.42) to non-initial blocks and then applying the theorem to \( V^r_\infty - V^m_\infty \).

To better understand which asymptotic covariance matrices \( \Sigma \) can be decomposed in this manner, the conditions \( C_k,\ell = c_\ell, \) for all \( k \neq \ell, 1 \leq k \leq d_1, 1 \leq \ell \leq m, \) and \( b_k = b, \) for all \( 1 \leq k \leq d_1, \) imply that

\[
\sigma^2_k = b^2 + 2bc_k + \sum_{i=1}^{m} c_i^2, \tag{6.44}
\]

for \( 1 \leq k \leq d_1, \) and

\[
\sigma_{k,\ell} = bc_k + bc_\ell + \sum_{i=1}^{m} c_i^2, \tag{6.45}
\]

for \( 1 \leq k < \ell \leq d_1. \)

If we let \( (Z_1, Z_2, \ldots, Z_m) \) be a centered Gaussian random vector with covariance matrix \( \Sigma \), then (6.44) and (6.45) give us

\[
\mathbb{E}(Z_k - Z_\ell)^2 = \sigma^2_k - 2\sigma_{k,\ell} + \sigma^2_\ell = 2b^2, \tag{6.46}
\]

for all \( 1 \leq k < \ell \leq d_1. \) That is, the \( L^2 \)-distance between any pair \((Z_k, Z_\ell)\) is the same, for \( 1 \leq k < \ell \leq d_1. \)

Notice that if \( \sigma^2_k = \sigma^2 \), for all \( 1 \leq k \leq d_1, \) then in fact (6.46) implies that \( \rho_{k,\ell} = \sigma_{k,\ell}/\sigma_k\sigma_\ell = 1 - b^2/\sigma^2 := \rho, \) for all \( 1 \leq k < \ell \leq d_1. \) That is, the \( d_1 \times d_1 \) submatrix of \( \Sigma \) must be permutation-symmetric.

Next, we note that, for \( 1 \leq k < \ell \leq d_1, \)

\[
\sigma^2_k - \sigma^2_\ell = 2b(c_k - c_\ell), \tag{6.47}
\]

so that \( c_k = \sigma^2_k/(2b) + c_0, \) for some constant \( c_0. \) Substituting this expression into (6.44) and, writing \( \Gamma = \sum_{i=d_1+1}^{m} c_i^2, \) we obtain

\[
\sigma^2_k = b^2 + 2b \left( \frac{\sigma^2_k}{2b} + c_0 \right) + \sum_{i=1}^{d_1} \left( \frac{\sigma^2_i}{2b} + c_0 \right)^2 + \Gamma
\]

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\[ b^2 + \sigma_k^2 + 2bc_0 + \sum_{i=1}^{d_1} \left( \frac{\sigma_i^2}{2b} + c_0 \right)^2 + \Gamma. \tag{6.48} \]

Writing \( \overline{\sigma} = (\sum_{i=1}^{d_1} \sigma_i^2)/d_1 \), for any \( r > 0 \), (6.48) gives us

\[
\begin{align*}
 b^2 + 2bc_0 + \sum_{i=1}^{d_1} \left( \frac{\sigma_i^2}{2b} + c_0 \right)^2 + \Gamma \\
= d_1 c_0^2 + \left( 2b + \frac{d_1\overline{\sigma^2}}{b} \right) c_0 + \left( b^2 + \frac{d_1\overline{\sigma^4}}{4b^2} + \Gamma \right) \\
= 0. \tag{6.49}
\end{align*}
\]

In order for \( c_0 \) to be a real number, the discriminant of the quadratic equation in (6.49) must satisfy

\[
\left( 2b + \frac{d_1\overline{\sigma^2}}{b} \right)^2 - 4d_1 \left( b^2 + \frac{d_1\overline{\sigma^4}}{4b^2} + \Gamma \right) \geq 0, \tag{6.50}
\]

which leads to the inequality

\[
(d_1 - 1)b^4 - d_1(\overline{\sigma^2} - \Gamma) + \frac{d_1^2}{4} \left( \overline{\sigma^4} - (\overline{\sigma^2})^2 \right) \leq 0. \tag{6.51}
\]

This inequality, in turn, gives us constraints on \( b^2 \). Indeed, the necessary and sufficient condition needed for such a \( b^2 \) to exist is given by examining the quadratic in \( b^2 \) in (6.51) at its extremal point, namely, at \( b^2 = d_1(\overline{\sigma^2} - \Gamma))/(2(d_1 - 1)) \). Doing so leads to the condition

\[
- \left( \frac{d_1^2(\overline{\sigma^2} - \Gamma)^2}{4(d_1 - 1)} \right) + \frac{d_1^2}{4} \left( \overline{\sigma^4} - (\overline{\sigma^2})^2 \right) \leq 0, \tag{6.52}
\]

or simply,

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\[
\sigma^4 - (\bar{\sigma}^2)^2 \leq \left( \frac{(\sigma^2 - \Gamma)^2}{d_1 - 1} \right),
\tag{6.53}
\]

since \( \sigma^4 \geq (\bar{\sigma}^2)^2 \). The closer that \( \Gamma \) is to \( \bar{\sigma}^2 \), the more similar that the \( d_1 \) variances must be. Thus, (6.53) functions as a bound on the variability among these \( d_1 \) variances. Provided that the variances satisfy (6.53), the condition on \( b^2 \) is given by

\[
b^2 \in \left( \frac{d_1}{2(d_1 - 1)} \right) \left\{ (\sigma^2) - \Gamma - \sqrt{(\sigma^2 - \Gamma)^2 - (d_1 - 1) \left( \sigma^4 - (\bar{\sigma}^2)^2 \right)} \right\},
\]

\[
b^2 \in \left( \frac{d_1}{2(d_1 - 1)} \right) \left\{ (\sigma^2) - \Gamma + \sqrt{(\sigma^2 - \Gamma)^2 - (d_1 - 1) \left( \sigma^4 - (\bar{\sigma}^2)^2 \right)} \right\}.
\tag{6.54}
\]

Now consider the doubly stochastic case, where \( d_1 = m \). Applying the general fact that each row of \( \Sigma \) must necessarily sum to zero, we use (6.44) and (6.45) to find that, for each \( 1 \leq k \leq m \),

\[
\sum_{\ell=1}^{m} \sigma_{k,\ell} = \sigma_k^2 + \sum_{\ell \neq k} \sigma_{k,\ell}
\]

\[
= \left( b^2 + 2bc_k + \sum_{i=1}^{m} c_i^2 \right) + \sum_{\ell \neq k} \left( bc_k + bc_\ell + \sum_{i=1}^{m} c_i^2 \right)
\]

\[
= b^2 + b \left( mc_k + \sum_{\ell=1}^{m} c_\ell \right) + m \sum_{i=1}^{m} c_i^2
\]

\[
= 0,
\tag{6.55}
\]

so that \( c_k = c \in \mathbb{R} \), for all \( 1 \leq k \leq m \). Substituting \( c \) back into (6.55) gives us

\[
\sum_{\ell=1}^{m} \sigma_{k,\ell} = b^2 + b(mc + mc) + m(mc^2)
\]

\[
= (b + mc)^2 = 0,
\tag{6.56}
\]

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so that $b = -mc$. This then implies that $\sigma_k^2 = m(m - 1)c^2$ and $\sigma_{k, \ell} = -mc^2$, for all $1 \leq k \leq m$, $\ell \neq k$. But this is precisely a permutation-symmetric covariance matrix, which in the iid case corresponds to the class of Markov chains having a uniform stationary distribution.

We summarize these results in the following:

**Theorem 6.6** In order that the asymptotic covariance matrix $\Sigma$ have a decomposition $\Sigma = CC^T$, where

$$
\begin{cases}
C_{k, \ell} = c_\ell, & k \neq \ell, \quad 1 \leq k \leq d_1, \quad 1 \leq \ell \leq m, \\
C_{k, k} = b + c_k, & 1 \leq k \leq d_1,
\end{cases}
$$

for some real constants $c_1, c_2, \ldots, c_m$ and $b$, it is necessary and sufficient that

$$
\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2 \leq \frac{\overline{\sigma^2} - \Gamma}{\sqrt{d_1 - 1}},
$$

where $\Gamma = \sum_{i=d_1+1}^{m} c_i^2$, and $\overline{\sigma^r} = (\sum_{i=1}^{d_1} \sigma_i^r)/d_1$, for any $r > 0$. In this case,

$$
b^2 \in \left(\frac{d_1}{2(d_1 - 1)} \left\{ \overline{\sigma^2} - \Gamma - \sqrt{\overline{\sigma^2} - \Gamma)^2 - (d_1 - 1) \left(\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2\right)} \right\} \right).
$$

$$
\frac{d_1}{2(d_1 - 1)} \left\{ \overline{\sigma^2} - \Gamma + \sqrt{\overline{\sigma^2} - \Gamma)^2 - (d_1 - 1) \left(\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2\right)} \right\}.
$$

In particular, if $d_1 = m$, the asymptotic covariance matrix must be permutation-symmetric, with $c_k = c$, for all $k$, and $b = -mc$, so that the common variance is $m(m - 1)c^2$ and the common covariances are all $-mc^2$.

### 7 Connections to Random Matrix Theory

For iid uniform $m$-letter alphabets, the limiting law of the Young tableau corresponds to the joint distribution of the eigenvalues of an $m \times m$ matrix from the traceless GUE [22]. In the non-uniform iid case, we further noted that Its, Tracy, and Widom [20, 21] have essentially described the limiting
shape as that of the joint distribution of the eigenvalues of a random matrix consisting of independent diagonal blocks, each of which is a matrix from the GUE. The size of each block depends upon the multiplicity of the corresponding stationary probability. In addition, there is a zero-trace condition involving the stationary probabilities on the composite matrix.

As a first step in extending these connections between Brownian functionals and spectra of random matrices, recall the general case when the stationary probabilities are all distinct (see Remark 5.5). Our Brownian functionals then have no true maximal terms, so that the limiting shape, $(R_{1\infty}, R_{2\infty}, \ldots, R_{m\infty})$ is simply multivariate normal, with covariance matrix $\Sigma$ (or, more precisely, the matrix obtained by permuting the rows and columns of $\Sigma$ using $\tau$, the permutation of $\{1, 2, \ldots, m\}$ previously defined). Trivially, this limiting law corresponds to the spectrum of a diagonal matrix whose elements are multivariate normal with the same covariance matrix $\Sigma$.

We can see that this general result is consistent with the non-uniform iid case having distinct probabilities. Indeed, each block is of size 1, and is rescaled so that the variance is $\pi_{\tau(i)}(1 - \pi_{\tau(i)})$, for $1 \leq i \leq m$. Because of this rescaling, instead of having a generalized zero-trace condition, as in the non-rescaled matrices used in [20, 21], our condition is rather a true zero-trace condition. This zero-trace condition is clear, since the covariance matrix for any iid case (uniform and non-uniform alike) is that of a multinomial distribution with parameters $(n = 1; \pi_{\tau(1)}, \pi_{\tau(2)}, \ldots, \pi_{\tau(m)})$, and any $(Y_1, Y_2, \ldots, Y_m)$ having such a distribution of course satisfies $\sum_{i=1}^{m} Y_i = 1$, so that $Var(\sum_{i=1}^{m} Y_i) = 0$, which implies the zero-trace condition for $(R_{1\infty}, R_{2\infty}, \ldots, R_{m\infty})$.

Next, consider the case when each stationary probability has multiplicity no greater than 2. We conjecture that the limiting shape $(R_{1\infty}, R_{2\infty}, \ldots, R_{m\infty})$ is that of the spectrum of a direct sum of certain $1 \times 1$ and/or $2 \times 2$ random matrices. Specifically, let $\kappa \leq m$ be the number of distinct probabilities among the stationary distributions. Then the composite matrix consists of a direct sum of $\kappa$ GUE matrices which are as follows. First, the overall diagonal $(X_1, X_2, \ldots, X_m)$ of the matrix has a $N(0, \Sigma)$ distribution. Next, if $d_r = 1$, then the GUE matrix is simply the $1 \times 1$ matrix $(X_r)$. Finally, if $d_r = 2$, then the GUE matrix is the $2 \times 2$ matrix

$$
\begin{pmatrix}
X_{m_r+1} & Y_{m_r+1} + iZ_{m_r+1} \\
Y_{m_r+1} - iZ_{m_r+1} & X_{m_r+2}
\end{pmatrix},
$$

whose off-diagonal random variables $Y_{m_r+1}$ and $Z_{m_r+1}$ are iid, centered, nor-
mal random variables, independent of all other random variables in the overall matrix, with variance

\[
(\sigma^2_{m_{r+1}} - 2\rho_m \sigma_{m+1}\sigma_{m+2} + \sigma^2_{m_{r+2}})/4.
\]

If such a conjecture were true, it would imply the following, more modest marginal result regarding a single block of such a matrix, which without loss of generality we take to be the first block. Specifically, if \(d_1 = 2\) and \(\tau(r) = r\), for all \(1 \leq r \leq m\), we claim that \((R^1_\infty, R^2_\infty) = (V^1_\infty, V^2_\infty - V^1_\infty)\) is distributed as the spectrum \((\lambda_1, \lambda_2)\) of the \(2 \times 2\) GUE matrix

\[
A_1 := \begin{pmatrix}
X_1 & Y_1 + iZ_1 \\
Y_1 - iZ_1 & X_2
\end{pmatrix},
\]

where \(\lambda_1 \geq \lambda_2\). Equivalently, we will show that \((V^1_\infty, V^2_\infty)\) is distributed as \((\lambda_1, \lambda_1 + \lambda_2)\).

Let the \(2 \times 2\) submatrix \(\Sigma_2\) of \(\Sigma\) be written as

\[
\Sigma_2 = \begin{pmatrix}
\tilde{\sigma}_1^2 & \tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2 \\
\tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2 & \tilde{\sigma}_2^2
\end{pmatrix}.
\]

(7.2)

Then

\[
(V^1_\infty, V^2_\infty) = \left(\max_{0 \leq t \leq 1} \left(\tilde{\sigma}_1 \tilde{B}_1^1(t) + \tilde{\sigma}_2 \tilde{B}_2^2(t) - \tilde{\sigma}_2 \tilde{B}_2^2(1)\right)\right),
\]

\[
\tilde{\sigma}_1 \tilde{B}_1^1(1) + \tilde{\sigma}_1 \tilde{B}_2^2(1)
\]

\[
= \left(\tilde{\sigma}_2 \tilde{B}_2^2(1) + \max_{0 \leq t \leq 1} \left(\tilde{\sigma}_1 \tilde{B}_1^1(t) - \tilde{\sigma}_2 \tilde{B}_2^2(t)\right)\right),
\]

\[
\tilde{\sigma}_1 \tilde{B}_1^1(1) + \tilde{\sigma}_1 \tilde{B}_2^2(1).
\]

(7.3)

We simplify (7.3), by introducing new Brownian motions and then decomposing the resulting expression into two independent parts. To do so, begin by defining the new variances and correlation coefficients \(\sigma_1^2 := \tilde{\sigma}_1^2\), \(\sigma_2^2 := \tilde{\sigma}_1^2 - 2\tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2 + \tilde{\sigma}_2^2\), and \(\rho := (\tilde{\rho}\tilde{\sigma}_1 - \tilde{\sigma}_2)/\sqrt{\tilde{\sigma}_1^2 - 2\tilde{\rho}\tilde{\sigma}_1\tilde{\sigma}_2 + \tilde{\sigma}_2^2}\). Then it is easily verified that \(B_1^1(t) := \tilde{B}_2^2(t)\), and \(B_2^2(t) := (\tilde{\sigma}_1 \tilde{B}_1^1(t) - \tilde{\sigma}_2 \tilde{B}_2^2(t))/\sigma_2\) are (dependent) standard Brownian motions, and (7.3) becomes
\[(V_1^1, V_2^2) = (\sigma_1 B^1(1) + \sigma_2 \max_{0 \leq t \leq 1} B^2(t), 2\sigma_1 B^1(1) + \sigma_2 B^2(1))
= \left( (\sigma_1 B^1(1) - \rho \sigma_1 B^2(1)) + \sigma_2 \left( \frac{\sigma_1}{\sigma_2} + \max_{0 \leq t \leq 1} B^2(t) \right),
2(\sigma_1 B^1(1) - \rho \sigma_1 B^2(1)) + (\sigma_2 + 2\rho \sigma_1) B^2(1) \right). \tag{7.4} \]

Note that \(B^1(t) - \rho B^2(t)\) is independent of \(B^2(t)\) and has variance \(\sigma_2^2(1 - \rho^2)\).

Introducing the Brownian functional
\[U(\beta) = \left( \beta - \frac{1}{2} \right) B^2(1) + \max_{0 \leq t \leq 1} B^2(t), \tag{7.5} \]
\(\beta \in \mathbb{R}\), and using \(\sigma_1^2, \sigma_2^2\), and \(\rho\) above, (7.4) becomes
\[(V_1^1, V_2^2) \overset{\mathcal{L}}{=} \sigma_1 \sqrt{1 - \rho^2} Z(1, 2) + \left( \sigma_2 U \left( \frac{1}{2} - \rho \frac{\sigma_1}{\sigma_2} \right), (\sigma_2 + 2\rho \sigma_1) B^2(1) \right)
= \frac{\tilde{\sigma}_1 \tilde{\sigma}_2 \sqrt{1 - \tilde{\rho}^2}}{\sqrt{\tilde{\sigma}_1^2 - 2\tilde{\rho} \tilde{\sigma}_1 \tilde{\sigma}_2 + \tilde{\sigma}_2^2}} Z(1, 2)
+ \left( \sqrt{\tilde{\sigma}_1^2 - 2\tilde{\rho} \tilde{\sigma}_1 \tilde{\sigma}_2 + \tilde{\sigma}_2^2} \ U \left( \frac{\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2}{2\sqrt{\tilde{\sigma}_1^2 - 2\tilde{\rho} \tilde{\sigma}_1 \tilde{\sigma}_2 + \tilde{\sigma}_2^2}} \right),
2(\tilde{\sigma}_1^2 - \tilde{\sigma}_2^2) B^2(1) \right), \tag{7.6} \]

where \(Z\) is a standard normal random variable independent of the sigma-field generated by \(B^2\).

Turning now to the eigenvalues’ distributions, we first consider the centered, multivariate normal random variables \((W_1, W_2)\), having covariance matrix
\[\begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}, \]
and let \(W_3\) and \(W_4\) be two iid, centered, normal random variables, independent of \((W_1, W_2)\), with variance \(\sigma_2^2\). Then it is classical that
\[\left( W_2, \sqrt{W_2^2 + W_3^2 + W_4^2} \right) \overset{\mathcal{L}}{=} \sigma_2 \left( B(1), 2\max_{0 \leq t \leq 1} B(t) - B(1) \right), \]
or, equivalently,

\[
(W_2, \beta W_2 + \frac{1}{2} \sqrt{W_2^2 + W_3^2 + W_4^2}) \overset{d}{=} \sigma_2(B(1), U(\beta)),
\]

(7.7)

where \( B \) is a standard Brownian motion, and \( U(\beta), \beta \in \mathbb{R}, \) is defined in terms of \( B, \) rather than in terms of \( B^2, \) as in (7.5). Then consider the random variable

\[
\tilde{\lambda} := W_1 + \sqrt{W_2^2 + W_3^2 + W_4^2}
\]

\[
= \left( W_1 - \rho \frac{\sigma_1}{\sigma_2} \right) + \left( \rho \frac{\sigma_1}{\sigma_2} + \sqrt{W_2^2 + W_3^2 + W_4^2} \right).
\]

(7.8)

Using (7.7), and noting that the variance of the first term in (7.8) is \( \sigma_1^2(1-\rho^2), \) it is easy to see that

\[
\tilde{\lambda} \overset{d}{=} \sigma_1 \sqrt{1-\rho^2} Z + 2 \sigma_2 U \left( \frac{\rho \sigma_1}{2 \sigma_2} \right),
\]

(7.9)

where \( Z \) is a standard normal random variable independent of \( B. \)

We now apply this result to the eigenvalues of the matrix \( A_1 \) in (7.1), namely, to

\[
\lambda_1 = \left( \frac{X_1 + X_2}{2} \right) + \sqrt{\left( \frac{X_1 - X_2}{2} \right)} + Y_1^2 + Z_1^2;
\]

(7.10)

and

\[
\lambda_2 = \left( \frac{X_1 + X_2}{2} \right) - \sqrt{\left( \frac{X_1 - X_2}{2} \right)} + Y_1^2 + Z_1^2.
\]

(7.11)

Letting \( W_1 = (X_1 + X_2)/2, W_2 = (X_1 - X_2)/2, W_3 = Y_1, \) and \( W_4 = Z_1, \) we have
\[(\lambda_1, \lambda_1 + \lambda_2) = \left( W_1 + \sqrt{W_2^2 + W_3^2 + W_4^2}, 2W_1 \right) \]
\[
= \left( W_1 - \hat{\rho}\hat{\sigma}_1 W_2 + 2\sqrt{\frac{W_2^2 + W_3^2 + W_4^2}{2}}, 2\hat{\rho}\hat{\sigma}_1 W_2 \right) \]
\[
= \left( W_1 - \hat{\rho}\hat{\sigma}_1 W_2 \right) (1, 2) \]
\[
+ \left( \hat{\rho}\hat{\sigma}_1 W_2 + \frac{1}{2} \sqrt{W_2^2 + W_3^2 + W_4^2}, 2\hat{\rho}\hat{\sigma}_1 W_2 \right), \]
\]
\[
(7.12)
\]

where \(\hat{\sigma}_1^2 = (\hat{\sigma}_1^2 + 2\hat{\rho}\hat{\sigma}_1 + \hat{\sigma}_2^2)/4, \hat{\sigma}_2^2 = (\hat{\sigma}_1^2 - 2\hat{\rho}\hat{\sigma}_1 + \hat{\sigma}_2^2)/4\), and \(\hat{\rho}\hat{\sigma}_1^2\hat{\sigma}_2^2 = (\hat{\sigma}_1^2 - \hat{\sigma}_2^2)/4\). Noting that the variance of \(W_1 - (\hat{\rho}\hat{\sigma}_1)/\hat{\sigma}_2) W_2\) is \(\hat{\sigma}_1^2(1 - \hat{\rho}^2) = \sigma_1^2(1 - \rho^2)\), and that, moreover, \(\beta := \hat{\rho}\hat{\sigma}_1/2\hat{\sigma}_2 = (\hat{\sigma}_1^2 - \hat{\sigma}_2^2)/(2\sqrt{\hat{\sigma}_1^2 - 2\hat{\rho}\hat{\sigma}_1 + \hat{\sigma}_2^2})\), we find that

\[(\lambda_1, \lambda_1 + \lambda_2) = \hat{\sigma}_1 \sqrt{1 - \hat{\rho}^2} Z(1, 2) + \left( 2\hat{\sigma}_2 U\left( \frac{\hat{\rho}\hat{\sigma}_1}{2\hat{\sigma}_2} \right), 2\hat{\rho}\hat{\sigma}_1 B^2(1) \right) \]
\[
= \sigma_1 \sqrt{1 - \rho^2} Z(1, 2) + \sigma_2 \left( U(\beta), 4\beta B^2(1) \right) \]
\[
\equiv (V_1^1, V_2^1), \]
\]
\[
(7.13)
\]

and we have our identity in law.

To illustrate the ways in which random matrix interpretations might potentially illuminate other, apparently unrelated, Brownian functionals, consider the following example. Let \((\varepsilon_k)_{k \geq 1}\) be a sequence of positive numbers decreasing to zero. Then it is possible to find an increasing sequence of integers \((m_k)_{k \geq 1}\) so that, for each \(k\), there is a Markov chain on \(m_k\) letters such that:

- the maximal stationary probability \(\pi_{\text{max}}(k)\) is of multiplicity 3, and
- the \(3 \times 3\) covariance submatrix \(\Sigma_3(k)\) governing the associated Brownian functional \(V_1^1(k)\) is of the form

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That is, the variance of $B^{\tau(2)}$ becomes arbitrarily large in comparison to that of $B^{\tau(1)}$ and $B^{\tau(3)}$.

Then, since $L_{I_n}(k) = V_{n}^{1}(k)$, we have, as $n \to \infty$,

$$\frac{L_{I_n}(k) - \pi_{\text{max}}(k)}{\sqrt{n}} \Rightarrow \max_{i, \ell} \sum_{\ell=1}^{3} \sigma_{\tau(\ell)} B^{\tau(\ell)}(\Delta t_{\ell})$$

$$= \sigma(k) \max_{i, \ell} (\epsilon_{k}(B^{\tau(1)}(t_{1}) - B^{\tau(1)}(0)) + (B^{\tau(2)}(t_{2}) - B^{\tau(2)}(t_{1}))$$

$$+ (\epsilon_{k}(B^{\tau(3)}(1) - B^{\tau(3)}(t_{2})))$$

$$:= V_{\infty}^{1}(k),$$

so that, as $k \to \infty$,

$$\frac{V_{\infty}^{1}(k)}{\sigma(k)} \Rightarrow \max_{0 \leq t_{1} \leq t_{2} \leq 1} (B(t_{2}) - B(t_{1})), \ (7.16)$$

where $B(t)$ is a standard Brownian motion. The right-hand side of (7.16) is known as the local score, and describes the largest positive increase that $B$ makes within the unit interval. Such functionals are of great importance in sequence comparison, particularly in bioinformatics (e.g., see Daudin, Ettienne, and Vallois [10].) Moreover,

$$\max_{0 \leq t_{1} \leq t_{2} \leq 1} (B(t_{2}) - B(t_{1})) \overset{\mathbb{P}}{=} \max_{0 \leq t \leq 1} |B(t)|, \ (7.17)$$

which follows immediately from the classical equality in law, due to Lévy, $\{ |B(t)| \}_{t \geq 0} \overset{\mathbb{P}}{=} \{ \max_{0 \leq s \leq t} B(s) - B(t) \}_{t \geq 0}$. Thus, if we have a random matrix connection to $V_{\infty}^{1}(k)$, we can extend it to $\max_{0 \leq t \leq 1} |B(t)|$, at least in some limiting sense. This is also interesting from the following point of view. Classically, the Brownian functional $\max_{0 \leq t \leq 1} B(t) \overset{\mathbb{P}}{=} |B(1)|$, and a trivial random matrix connection can be seen by examining the eigenvalues of the random matrix.
\[
\begin{pmatrix}
Z & 0 \\
0 & -Z
\end{pmatrix},
\] (7.18)

where \( Z \) is a standard normal random variable. Then, clearly, \( \lambda_{\text{max}} \) has
the half-normal law, since \( \lambda_{\text{max}} = \max(Z, -Z) = |Z| \). Thus, the functional \( \max_{0 \leq t \leq 1} B(t) \) has a random matrix interpretation, one which is considerably simpler than any potential random matrix interpretation for \( \max_{0 \leq t \leq 1} |B(t)| \).

8 Concluding Remarks

In this paper, we have obtained the limiting shape of Young tableaux generated by an aperiodic, irreducible, homogeneous Markov chain on a finite state alphabet. The following remarks indicate natural directions in which our results in some cases can, and in other cases, may hope to, be extended.

- Our limiting theorems have all been proved assuming that the initial distribution is the stationary one. However, such results as Theorem 2 of Derriennic and Lin [11] allow to extend our framework to initial distributions started at a specified state. Indeed, in this case, \( i.e., \) if for some \( k = 1, \ldots, m \), \( P(X_0 = \alpha_k) = 1 \), the asymptotic covariance matrix is still given by (4.12), and, for example, Theorem 5.2 remains valid. For an arbitrary initial distribution, what is needed in this non-stationary context is an invariance principle. More generally, our results continue to hold for \( k^{th} \)-order Markov chains, and in fact, they extend to any sequence for which both an asymptotic covariance matrix and an invariance principle exist.

- Our limiting theorems have only been proved for finite alphabets. However, from the authors’ previous work [18], it is known that for countably infinite iid alphabets, \( LI_n \) has a limiting law corresponding to that of a non-uniform, finite-alphabet. Hence, for a countably infinite-alphabet Markov chain (subject to additional constraints such as Harris recurrence?), we might still be able to obtain limiting laws of the form developed in this paper.

- By using appropriate existing concentration inequalities, one can expect to establish the convergence of the moments of the rows of the tableaux.
One field in which the connection between Brownian functionals and random matrix theory has been exploited is in Queuing Theory. The development below, following O’Connell and Yor [25], shows how Brownian functionals of the sort we have studied arise as generalizations of standard queuing models.

Let \( A(s, t] \) and \( S(s, t] \), \( -\infty < s < t < \infty \), be two independent Poisson point processes on \( \mathbb{R} \), with intensity measures \( \lambda \) and \( \mu \), respectively, with \( 0 < \lambda < \mu \). Here \( A \) represents the arrivals process, and \( S \) the service time process, at a queue consisting of a single server. The condition \( \lambda < \mu \) ensures that the queue length

\[
Q(t) = \sup_{-\infty < s \leq t} \{ A(s, t] - S(s, t] \}, \tag{8.1}
\]

is a.s. finite, for any \( t \in \mathbb{R} \). Then, defining the departure process

\[
D(s, t] = A(s, t] - (Q(t) - Q(s)), \tag{8.2}
\]

which is simply the number of arrivals during \( (s, t] \) less the change in the queue length during \( (s, t] \), the classical problem is to determine the distribution of \( D(s, t] \). The answer to this problem is given by Burke’s Theorem [8] (see Theorem 1 of [25]):

**Theorem 8.1** \( D \) is a Poisson process with intensity \( \lambda \), and \( \{D(s, t], s \leq t\} \) is independent of \( \{Q(s), s \geq t\} \).

That is, \( D \) has the same law as the arrivals process \( A \). Moreover, since the queue length after time \( t \) is independent of the process \( D \) up to time \( t \), one may take the departures from the first queue and use them as inputs to a second queue, and observe that the departure process from the second queue also has the law of \( A \). Proceeding in this way, one generalizes to a tandem queue of \( n \) servers, each taking the departures from the previous queue as its arrivals process.

One can further generalize this model to a Brownian queue in tandem in the following manner. Let \( B, B^1, B^2, \ldots, B^n \) be independent, standard Brownian motions on \( \mathbb{R} \), and write \( B^k(s, t] = B^k(t) - B^k(s) \), for each \( k \) and \( s < t \), and similarly for \( B \). Let \( m > 0 \) be a constant, and define, in complete analogy to (8.1) and (8.2),

\[
q_1(t) = \sup_{-\infty < s \leq t} \{ B(s, t] + B^1(s, t] - m(t - s) \}, \tag{8.3}
\]
and, for \( s < t \),
\[
d_1(s, t) = B(s, t) - (q_1(t) - q_1(s)).
\]  
(8.4)

For \( k = 2, 3, \ldots, n \), let
\[
q_k(t) = \sup_{-\infty < s \leq t} \left\{ d_{k-1}(s, t) + B^k(s, t) - m(t - s) \right\},
\]  
(8.5)

and, for \( s < t \),
\[
d_k(s, t) = d_{k-1}(s, t) - (q_k(t) - q_k(s)).
\]  
(8.6)

Here \( B \) is the arrivals process for the first queue, \( d_{k-1} \) is the arrivals process for the \( k \)th queue \( (k \geq 2) \), and \( mt - B^k(t) \) is the service process for the \( k \)th queue, for all \( k \). Using the ideas employed in Burke’s Theorem, it can be shown that the generalized queue lengths \( q_1(0), q_2(0), \ldots, q_n(0) \) are iid random variables. Moreover, they are exponentially distributed with mean \( 1/m \).

Using the definitions in (8.3)-(8.6), and a simple inductive argument, one finds that
\[
\sum_{k=0}^{n} q_k(0) = \sup_{t > 0} \left\{ B(-t, 0) - mt + L_n(t) \right\},
\]  
(8.7)

where
\[
L_n(t) = \sup_{0 \leq s_1 \leq \cdots \leq s_{m-1} \leq t} \{ B^1(-t, -s_{n-1}) + \cdots + B^n(-s_1, 0) \}.
\]  
(8.8)

By Brownian rescaling, we observe that
\[
L_n(t) \overset{D}{=} \sqrt{t} V^1_{\infty},
\]  
(8.9)

where the functional \( V^1_{\infty} \) is as in Theorem 5.2 with associated \( n \times n \) covariance matrix \( \Sigma = tI_n \) and parameter set \( I_{1,n} \). Thus, \( L_n(t) \) may be thought of as a process version of this \( V^1_{\infty} \).
The generalized Brownian queues in (8.3)-(8.6) involved independent Brownian motions. These can be extended with Brownian motions \( B^1, \ldots, B^n \) for which \( (\sigma_1 B^1(t), \ldots, \sigma_n B^n(t)) \) has (nontrivial) covariance matrix \( t\Sigma \). Whether or not we keep the initial arrival process \( B(t) \) independent of \( (B^1, \ldots, B^n) \), we now no longer have that \( q_1(0), q_2(0), \ldots, q_n(0) \) are iid random variables, due to the dependence among the service times \( mt - B^k(t) \), but we do still have the identity (8.7) and (8.9) relating the total occupancy of the queue at time zero to \( V_1^\infty \). More importantly, our generalizations of the Brownian functionals \( L_n(t) \) above can be used to describe the joint law of the input/output of each queue.

• An important topic connecting much of random matrix theory to other problems, such as the shape of random Young tableaux, is the field of orthogonal polynomials. (See, e.g., [22].) It would be of great interest to see what, if any, classes of orthogonal polynomials are associated with the present paper.

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