ON THE NUMBER OF MAXIMUM INDEPENDENT SETS
IN DOOB GRAPHS

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Abstract. The Doob graph $D(m, n)$ is a distance-regular graph with the same parameters as the Hamming graph $H(2m+n, 4)$. The maximum independent sets in the Doob graphs are analogs of the distance-2 MDS codes in the Hamming graphs. We prove that the logarithm of the number of the maximum independent sets in $D(m, n)$ grows as $2^{2m+n-1}(1+o(1))$. The main tool for the upper estimation is constructing an injective map from the class of maximum independent sets in $D(m, n)$ to the class of distance-2 MDS codes in $H(2m+n, 4)$.

Keywords: Doob graph, independent set, MDS code, latin hypercube.

1. Introduction

The Cartesian product $D(m, n) \equiv Sh^m \times K_4^n$ of $m$ copies of the Shrikhande graph $Sh$ (see Figure 1) and $n$ copies of the complete graph $K_4$ of order $q = 4$ is called a Doob graph if $m > 0$, while $D(0, n)$ is the Hamming graph $H(n, 4)$ (in general $H(n, q) \equiv K_q^n$). The Doob graph $D(m, n)$ is a distance-regular graph with the same parameters as $H(2m+n, 4)$, see e.g. [1, §9.2.B]. It is easy to see that the independence number of this graph is $4^{2m+n-1}$. The maximum independent sets in the Hamming graphs are known as the distance-2 MDS codes (below, simply MDS codes), or the latin hypercubes (in the last case, one of the coordinates is usually considered as a function of the others). It is naturally to call the maximum independent sets in Doob graphs by the same notion, the MDS codes. Indeed, the
maximum independent sets in $D(m, n)$ and the MDS codes in $H(2m + n, 4)$ have the same parameters being considered as error-correcting codes (see [5] for the background on error-correcting codes) and as completely regular codes (see e.g. [1, §11.3]). (The concept of the latin hypercubes can also be generalized to $D(m, n)$; however, to do this, we need at least one $K_4$ coordinate to treat as dependent, i.e., $n > 0$.) There are 4 trivial MDS codes in $D(0, 1)$; 24 equivalent MDS codes in $D(0, 2)$ (16 of them can be found in Figure 2); 16 MDS codes in $D(1, 0)$ (see Figure 2), which form two equivalence classes (with 4 and 12 representatives, respectively).

The main result of the current correspondence is the following.

**Theorem 1.** The number of the maximum independent sets (distance-2 MDS codes) in the Doob graph $D(m, n)$ grows as $2^{2^m+n-1+o(1)}$ as $(2^m+n) \to \infty$.

The statement of the theorem is straightforward from Corollaries 1 (an upper bound) and 2 (a lower bound) proven in the next two sections.

2. An upper bound

In this section, we describe a rather simple recursive way to map injectively the set $MDS_{m,n}$ of MDS codes in $D(m, n)$ into $MDS_{0,2m+n}$. At first, we define the map $\xi$ from $MDS_{1,0}$ into $MDS_{0,2}$, see Figure 2.

For arbitrary $m, n \geq 0$, the action of $\kappa : MDS_{m+1,n} \to MDS_{m,n+2}$ is defined as follows:

$$\kappa M \overset{\text{def}}{=} \{ (x_1, ..., x_m, z_1, z_2, y_1, ..., y_n) \in D(m, n + 2) \mid \exists M_{x_1,...,x_m,y_1,...,y_n} \}$$

where

$$M_{x_1,...,x_m,y_1,...,y_n} \overset{\text{def}}{=} \{ v \in Sh \mid (x_1, ..., x_m, v, y_1, ..., y_n) \in M \}$$

**Lemma 1.** For every MDS code in $D(m+1,n)$, the set $\kappa M$ is an MDS code in $D(m, n + 2)$.

**Proof.** The map $\xi$ has the following important property, which can be checked directly, see Figure 2: two MDS codes $M'$ and $M''$ in $D(1,0)$ intersect if and only if their images $\xi M'$ and $\xi M''$ intersect. Since $M$ is an independent set, $M_{x_1,...,x_m,y_1,...,y_n}$ and $M_{u_1,...,u_m,w_1,...,w_n}$ (and hence, also $\xi M_{x_1,...,x_m,y_1,...,y_n}$ and $\xi M_{u_1,...,u_m,w_1,...,w_n}$) are disjoint for any two neighbor vertices $(x_1, ..., x_m, y_1, ..., y_n)$ and $(u_1, ..., u_m, w_1, ... , w_n)$.
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ξ : \( \cdots, w_n \) \( \rightarrow \) \( \cdots, w_n \) of \( D(m, n) \). It follows that \( \kappa M \) is also an independent set. Moreover, it has the same cardinality as \( M \), i.e., \( 4^{2m+n+1} \).

Then, \( \kappa^m \), the \( m \)th iteration of \( \kappa \), maps MDS\(_{m,n} \) into MDS\(_{0,2m+n} \).

**Corollary 1.** The number of the MDS codes in \( D(m, n) \) does not exceed \( 2^{2^{2m+n}+1 (1+o(1))} \).

**Proof.** Since \( \kappa \) obviously maps different MDS codes to different MDS codes, the statement of the corollary in the general case can be inductively reduced to the partial case \( m = 0 \), which was proven in [8], see also [2].

\( \square \)

3. A lower bound

In this section, we consider a simple way to construct doubly exponential (with respect to the graph diameter \( 2m+n \)) number of MDS codes in the Doob graph \( D(m, n) \).

The vertices of \( \text{Sh} \) will be identified with the pairs \( ab \) (considered as a short notation for \( (a, b) \)), where \( a, b \in \{0, 1, 2, 3\} \), see Figure 1. The vertices of \( K_4 \) will be identified with the pairs \( ab \), where \( a, b \in \{0, 1\} \). For every function \( \lambda \) from \( \{0, 1, 2, 3\}^m \times \{0, 1\}^n \) to \( \{0, 1\} \), we define the set

\[
M_\lambda \overset{\text{def}}{=} \{(x'_1 x''_1, \ldots, x'_{m+n} x''_{m+n}) \in D(m, n) \mid \sum_{i=1}^{m+n} x'_i \equiv 0 \mod 2, \\
\sum_{i=1}^{m+n} x''_i \equiv \lambda(x'_1, \ldots, x'_{m+n}) \mod 2 \}. 
\]

**Lemma 2.** For any function \( \lambda : \{0, 1, 2, 3\}^m \times \{0, 1\}^n \rightarrow \{0, 1\} \), the set \( M_\lambda \) is an MDS code in \( D(m, n) \).

**Proof.** It is easy to see that if \( m < i \leq m+n \), then for any values of \( x'_1 x''_1, \\
\ldots, x'_{i-1} x''_{i-1}, x'_{i+1} x''_{i+1}, \ldots, x'_{m+n} x''_{m+n} \), there is a unique pair \( x'_i x''_i \) such that \( (x'_1 x''_1, \ldots, x'_{m+n} x''_{m+n}) \in M_\lambda \). If \( 1 \leq i \leq m \), then there are four such pairs (two possibilities for \( x'_i \), of the same parity, and for each choice of \( x'_i \), two possibilities for
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$x''_i$), but they correspond to pairwise independent vertices of the Shrikhande graph. Consequently, at first, $|M_\lambda| = 4^{2m+n-1}$, and at second, $M_\lambda$ is an independent set.

Corollary 2. There are at least $2^{2^{2m+n-1}}$ different MDS codes in $D(m,n)$.

Proof. We will say that two functions from $\{0, 1, 2, 3\}^m \times \{0, 1\}^n$ to $\{0, 1\}$ are essentially different if their values are different in at least one point $(x'_1, \ldots, x'_{m+n})$ satisfying $x'_1 + \ldots + x'_{m+n} \equiv 0 \mod 2$. The number of essentially different functions is $2^{2^{2m+n-1}}$. Obviously, essentially different functions $\lambda$ lead to different MDS codes $M_\lambda$. □

4. Conclusion

We have established the asymptotics of $\log |\text{MDS}_{m,n}|$, generalizing the similar result for the MDS codes in the Hamming graph $H(n, 4)$ [2], [8]. Note that the case $q = 4$ is the only nontrivial case when the asymptotics of the double logarithm of the number of MDS codes is known ($n \to \infty$, $q$ is fixed). Known bounds for the other cases can be found in [4], [9]; the exact values for small $q$ and $n$, in [6], [7], [9].

A constructive characterization of the class $\text{MDS}_{0,n}$ can be found in [3]. A possibility to relate the MDS codes (maximum independent sets) in $D(m,n)$ with MDS codes in $D(2m+n, 4)$ using the map $\kappa^m$ suggests that a similar characterization might be possible for $\text{MDS}_{m,n}$ with arbitrary $m$. However, it is not completely clear if the map $\kappa^m$ itself can be helpful for a reasonable proof of such characterization. Since the map $\kappa$ is not point-to-point, the result of the $m$th iteration of $\kappa$ can depend on the order of coordinates. As a result, it is not easy to track which subclass of $\text{MDS}_{0,2m+n}$ we obtain as the image of $\text{MDS}_{m,n}$ under $\kappa^m$ and to describe this subclass in terms of the known characterization of $\text{MDS}_{0,2m+n}$. In any case, finding a characterization of the class $\text{MDS}_{m,n}$, using $\kappa$ or not, will be a natural continuation of the current research.

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*Further reading:

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