A NEW FORMULATION OF THE TEICHMÜLLER TQFT

JØRGEN ELLEGAARD ANDERSEN AND RINAT KASHAEV

ABSTRACT. By using the Weil–Gel’fand–Zak transform of Faddeev’s quantum dilogarithm, we propose a new state-integral model for the Teichmüller TQFT, where the circle valued state variables live on the edges of oriented leveled shaped triangulations.

1. INTRODUCTION

Topological Quantum Field Theories in dimension $2 + 1$ were discovered and axiomatized in [26, 2, 22], and first constructed in [21, 23] by combinatorial means from the finite dimensional representation category of the quantum group $U^q(sl(2))$ at roots of unity and later in [4] by purely topological means.

Quantum Teichmüller theory [15, 10] produces infinite dimensional representations of surface mapping class groups thus giving an example of a potential TQFT associated with Chern–Simons theory with a non-compact gauge group. The essential ingredients behind quantum Teichmüller theory are Penner’s cell decomposition of decorated Teichmüller space, the associated Ptolemy groupoid [19] (see also the recent book [20] with many other applications), and Faddeev’s quantum dilogarithm [8]. Quantum Teichmüller theory indeed has been extended to a TQFT in [1] along with physical constructions of [13, 14, 7, 5, 6].

In this paper, we propose a new formulation of the Teichmüller TQFT of [1]. We use the same combinatorial setting as in [1], but the new formulation has a number of distinguishing features:

- The state-integral is of the Turaev–Viro type with the state variables living on the edges of a triangulation and taking their values on the real line as in the model suggested in [16].
- Unlike [16], the integrand is periodic in each of the integration variables so that the state integral is performed only over the a compact domain given by a (multi-dimensional) cube. As a consequence, the convergence of the state integral is determined only by the integrability properties of the integrand. Similar integrations over compact sets were used recently in lattice integrable systems [3].
- The tetrahedral weights are quasi-periodic functions of state variables so that geometrically they determine sections of line bundles over tori.
- As in the earlier models of [1, 16], strict positivity of shapes is enough for absolute convergence, but as the tetrahedral weights of the new model admit analytic continuation to meromorphic sections of line bundles over tori.
complex tori, the partition functions can be analytically continued to arbitrary complex shapes so that the theory is well defined without imposing positivity conditions on shapes. As a consequence, the shaped 2-3 and 3-2 Pachner moves are valid without any restriction, and thus the theory becomes truly topologically invariant.

- Similarly to the model of [16], there is no any restriction for the topology of the cobordisms.

The explicit form of our state integral follows. A triangulation \( X \) is a \( CW \)-complex where all cells are simplices. We denote by \( \Delta_i(X) \) the set of \( i \)-dimensional cells of \( X \). We refer to [1] for further notation and the detailed description of the combinatorial setting of leveled shaped triangulated oriented pseudo (LSTOP) 3-manifolds. Let \( T \subset \mathbb{R}^3 \) be a shaped tetrahedron with vertex ordering mapping \( v: \{0, 1, 2, 3\} \to \Delta_0(T) \).

For any

\[
\alpha: \Delta_1(T) \to \mathbb{R},
\]

we define a Boltzmann weight

\[
B(T, x) = g_{\alpha_1, \alpha_3}(x_{02} + x_{13} - x_{03} - x_{12}, x_{02} + x_{13} - x_{01} - x_{23})
\]

if \( T \) is positive and complex conjugate otherwise. Here \( x_{ij} \equiv x(v_i v_j) \).

We have

\[
g_{\alpha, c}(s, t) := \sum_{m \in \mathbb{Z}} \tilde{\psi}_{\alpha, c}(s + m)e^{\pi i t(s + 2m)},
\]

where

\[
(1) \quad \tilde{\psi}_{\alpha, c}(s) := e^{-\pi i s^2} \tilde{\psi}_{\alpha, c}(s),
\]

\[
(2) \quad \tilde{\psi}_{\alpha, c}(s) := \int_{\mathbb{R}} \psi_{\alpha, c}(t)e^{-2\pi i st}dt,
\]

and

\[
(3) \quad \psi_{\alpha, c}(t) := \Phi_b(t - 2c_b(a + c))e^{-4\pi i c_b(t - c_b(a + c))}e^{-\pi i c_b^2(4(a - c) + 1)/6}
\]

\[
= \Phi_b(t + c_b(2b - 1))e^{-4\pi i c_b(t + c_b b)}e^{\pi i c_b^2(4(b - b) + 1)/6},
\]

where \( \Phi_b(x) := 1/\Phi_b(x) \) with Faddeev’s quantum dilogarithm

\[
(5) \quad \Phi_b(x) := \exp \left( \int_C \frac{e^{-2iw}dw}{4\sinh(wb)\sinh(w/b)} \right),
\]

where the contour \( C \) runs along the real axis, deviating into the upper half plane in the vicinity of the origin. It is easily seen that \( \Phi_b(x) \) depends on \( b \) only through the combination \( h \) defined by the formula

\[
(6) \quad h^{-1} := (b + b^{-1})^2 = -4c_b^2
\]

which we suppose to be real positive, but the theory admits an analytic continuation to the complement of the non positive real axis of complex plane.
Theorem 1. Let $X$ be a closed LSTOP 3-manifold. Then, the quantity

$$Z_\hbar(X) := e^{\pi \ell_X/4\hbar} \int_{[0,1]} \Delta_3(X) \left( \prod_{T \in \Delta_3(X)} B(T, x|\Delta_1(T)) \right) dx$$

admits an analytic continuation to a meromorphic function of the complex shapes which is invariant under all shaped $2-3$ and $3-2$ Pachner moves (along balanced edges).

Remark 1. The state integral (7) extends to arbitrary (non closed) LSTOP 3-manifolds. In that case one has to integrate over only the state variables living on internal edges. The result is a meromorphic section of a line bundle over a complex torus $(\mathbb{C}^*)^{\Delta_1(\partial X)}$. More generally, by fixing the real parts of shape variables, one can treat that section as a distribution over their imaginary parts, the space of test functions being the set of smooth sections of the dual line bundle.

Remark 2. With additional normalization factors associated with regular vertices, the partition function (7) can be shown to be invariant under shaped $2-0$ ($0-2$) moves which remove (add) regular vertices. In particular, this additional property permits to define link invariants in compact oriented 3-manifolds by using arbitrary H-triangulations.

We claim that the proposed model is equivalent to the Teichmüller TQFT of [1], but we postpone the proof for another publication.

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2. Proof of Theorem 1

Along with (4), we also define

$$\tilde{\psi}_{a,c}(t) := \Phi_b(t + 2c_0(a + c))e^{-\pi i c_0(a + c) - \pi i c_1(4(a-c)+1)/6}$$

$$= \Phi_b(t + c_0(1 - 2b))e^{-\pi i c_0(t - c_0b) - \pi i c_1(4(b-a)-1)/6}, \quad b := \frac{1}{2} - a - c.$$

We have two equalities

$$\tilde{\psi}_{a,c}'(x) = e^{\pi i} \tilde{\psi}_{c,a}(x),$$

and

$$\tilde{\psi}_{a,c}(x) = e^{\pi i} \tilde{\psi}_{b,c}(-x) \iff \tilde{\psi}_{a,c}(x) = e^{-\pi i} \tilde{\psi}_{b,c}(-x).$$

2.1. The Pentagon identity.

Proposition 1. For any $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$, the following integral identity is satisfied:

$$\int_{[0,1]} g_{a_2,c_2}(\gamma - \sigma, \alpha + \delta - \sigma) g_{a_3,c_3}(\sigma, \beta - \delta) g_{a_0,c_0}(\alpha - \sigma, \beta + \gamma - \sigma) d\sigma$$

$$= e^{-\pi i P_c} g_{a_0,c_1}(\alpha, \beta) g_{a_2,c_3}(\gamma, \delta),$$

where

$$P_c := 2(c_0 + a_2 + c_4) - \frac{1}{2},$$
and the set of positive reals \( \{ a_i, c_i \mid i = 0, 1, \ldots, 4 \} \subset \mathbb{R}_{>0} \), is such that
\[
b_i := \frac{1}{2} - a_i - c_i > 0, \quad i = 0, 1, \ldots, 4,
\]
and
\[
a_1 = a_0 + a_2, \quad a_3 = a_2 + a_4, \quad c_1 = c_0 + a_4, \quad c_3 = a_0 + c_4, \quad c_2 = c_1 + c_3.
\]

**Proof.** Multiplying (11) by \( e^{-\pi i (\alpha \beta + \gamma \delta)} \), integrating over the unit square in the 1-periodic variables \( \beta \) and \( \delta \), and using definition (1), we obtain an equivalent identity
\[
\int_{[0,1]^3} \sum_{k,l,m \in \mathbb{Z}} \tilde{\psi}'_{a_1,c_4}(\gamma - \sigma + k)\tilde{\psi}'_{a_2,c_2}(\sigma + l)\tilde{\psi}'_{a_0,c_0}(\alpha - \sigma + m)
\times e^{\pi i ((\alpha + \delta - \sigma)(\gamma - \sigma + 2k) + (\beta + \delta)(\sigma + 2l) + (\beta + \gamma - \sigma)(\alpha - \sigma + 2m) - \alpha \beta - \gamma \delta)} d\sigma d\beta d\delta
\]
\[
e^{-\frac{\pi i}{4} P_\beta} \tilde{\psi}'_{a_1,c_1}(\alpha) \tilde{\psi}'_{a_3,c_3}(\gamma)
\]
where the triple sum is absolutely convergent. Exchanging the order of the integrations and summations, the double integral over \( \beta \) and \( \delta \) gives two Kronecker's deltas, \( \delta_{m,-l} \delta_{k,-l} \), which permit to eliminate the summations over \( k \) and \( m \) with the result
\[
\quad \int_{[0,1]} \sum_{l \in \mathbb{Z}} \tilde{\psi}'_{a_1,c_4}(\gamma - \sigma - l)\tilde{\psi}'_{a_2,c_2}(\sigma + l)\tilde{\psi}'_{a_0,c_0}(\alpha - \sigma - l)
\times e^{\pi i ((\alpha - \sigma)(\gamma - \sigma - 2l) + (\gamma - \sigma)(\alpha - \sigma - 2l))} d\sigma.
\]
Now, the identity
\[
e^{\pi i ((\alpha - \sigma)(\gamma - \sigma - 2l) + (\gamma - \sigma)(\alpha - \sigma - 2l))} = e^{2\pi i (\alpha - \sigma - l)(\gamma - \sigma - l)}, \quad \forall l \in \mathbb{Z},
\]
allows us to combine the integral and the sum in (14) into an absolutely convergent integral over the real axis:
\[
\int_{\mathbb{R}} \tilde{\psi}'_{a_1,c_4}(\gamma - \sigma)\tilde{\psi}'_{a_2,c_2}(\sigma)\tilde{\psi}'_{a_0,c_0}(\alpha - \sigma) e^{2\pi i (\alpha - \sigma)(\gamma - \sigma)} d\sigma.
\]
By using definition (2), we eliminate four “primes” out of five and we obtain the following equivalent form of (13)
\[
\int_{\mathbb{R}} \tilde{\psi}_{a_1,c_4}(\gamma - \sigma)\tilde{\psi}_{a_2,c_2}(\sigma)\tilde{\psi}_{a_0,c_0}(\alpha - \sigma) d\sigma
\]
\[
e^{-\frac{\pi i}{4} P_\beta} e^{-2\pi i \gamma} \tilde{\psi}_{a_1,c_1}(\alpha) \tilde{\psi}_{a_3,c_3}(\gamma).
\]
Let us apply now the Fourier transformation over the variable \( \gamma \), namely multiply
(16) by \( e^{2\pi i \beta \gamma} \) and integrate over \( \gamma \) along the real axis. As all integrals are absolutely convergent, we can exchange the order of integrations in the left hand side, so that the integration over \( \gamma \) can be absorbed on both sides of the identity by using definition (3). In this way, we come to the equality:
\[
\int_{\mathbb{R}} \tilde{\psi}'_{a_2,c_2}(\sigma)\tilde{\psi}_{a_0,c_0}(\alpha - \sigma) e^{2\pi i \beta \sigma} d\sigma = e^{-\frac{\pi i}{4} P_\beta} \frac{\tilde{\psi}_{a_1,c_1}(\alpha) \tilde{\psi}_{a_3,c_3}(\beta - \alpha)}{\tilde{\psi}_{a_4,c_4}(\beta)},
\]
which, by using the equalities (9) and (10), can also be written in the form
\[
\int_{\mathbb{R}} \tilde{\psi}_{a_2,b_2}(\sigma)\tilde{\psi}_{b_0,c_0}(\sigma - \alpha) e^{2\pi i \beta \sigma} d\sigma = e^{\frac{\pi i}{4} (1 - \frac{1}{4}) P_\beta} \frac{\tilde{\psi}_{b_1,c_1}(\alpha) \tilde{\psi}_{a_3,c_3}(\beta - \alpha)}{\tilde{\psi}_{a_4,c_4}(\beta)}.
\]
Now, it is a straightforward verification that (18) is a direct consequence of the integral Ramanujan identity. Indeed, for the left hand side of (18), we have that

\[ l.h.s. \text{ of } (18) \]
\[
= \int_{\mathbb{R}} \Phi_b(\sigma - \alpha + 2c_b(b_0 + c_0)) \Phi_b(\sigma - 2c_b(c_2 + b_2)) e^{-4\pi i c_b(\sigma - c_b(c_2 + b_2))} e^{-\pi i c_b^2(4(c_2 - b_2) + 1)^3/6} e^{-4\pi i c_b b_0(\sigma - \alpha + c_b(b_0 + c_0))} e^{\pi i c_b^2(4(b_0 - c_0) + 1)^3/6} e^{2\pi i \beta \sigma} d\sigma
\]
\[
= \int_{\mathbb{R}} \Phi_b(\sigma - \alpha + c_b(1 - 2a_0)) \Phi_b(\sigma - c_b(1 - 2a_2)) e^{2\pi i (\beta - 2a_0(b_0 + c_2))} \sigma d\sigma
\]
\[
= \Phi_b(c_b(1 - 2a_0) - \alpha, c_b(2a_2 - 1), \beta - 2c_b(b_0 + c_2))
\]

while for the right hand side of (18), we have that

\[ r.h.s. \text{ of } (18) \]
\[
= \frac{\Phi_b(c_b(1 - 2(a_0 + a_2)) - \alpha) \Phi_b(c_b(1 - 2(b_0 + c_2)) + \beta)}{\Phi_b(c_b(2(0 + b_2) - 1) - \alpha + \beta)} e^{4\pi i c_b(b_0(\alpha - a_2 \beta)}
\]
\[
\times e^{4\pi i c_b^2(a_0b_0 + a_2c_2 + 2a_0c_2)} e^{2\pi i c_b^2(\alpha - a_2 + c_2) / 3} e^{\pi i (1 + 4c_b^2) / 12},
\]

The final expressions in (19) and (20) coincide provided

\[ a_1 = a_0 + a_2, \ b_0 + c_2 + b_4 = 1, \ b_3 = c_0 + b_2, \ b_0 = b_1 + a_3, \ a_3 = a_2 + a_4 \]

which is equivalent to (12).

\[ \square \]

2.2. Symmetries. We have that

\[ (Gf)(x) := e^{\pi i x^2} f(x), \quad \overline{Gf} := G^{-1} f \]
and

\[ (Ff)(x) := \int_{\mathbb{R}} f(y) e^{2\pi i xy} dy, \quad \overline{Ff} := F^{-1} f \]

If we define the Weil–Gel’fand–Zak (WGZ) transformation \([25, 12, 27, 18]\) by the formula

\[ (Wf)(x, y) = e^{\pi i xy} \sum_{m \in \mathbb{Z}} f(x + m) e^{2\pi imy}, \quad \overline{(Wf)}(x, y) = (W\overline{f})(x, -y) \]

then we have that

\[ g_{a,c} = W\psi_{a,c}', \quad \overline{g}_{a,c}(x, y) = (WGF\overline{\psi}_{a,c})(x, -y) \]
The WGZ-transformation associates to a function in the Schwartz class \( f \in S \equiv S(\mathbb{R}) \) a smooth section \( g = Wf \in \Gamma(L) \) of the complex line bundle \( L \) over the two torus corresponding to the quasi-periodicity properties:

\[
g(x + 1, y) = e^{-\pi iy} g(x, y), \quad g(x, y + 1) = e^{\pi ix} g(x, y),
\]

The inverse of WGZ-transformation is given by the formula:

\[
(W^{-1}g)(x) = \int_0^1 g(x, y)e^{-\pi ixy}dy.
\]

The WGZ-transform is an isometry from \( S \) to \( \Gamma(L) \).

**Lemma 1.** For any function \( f \in S \), the following identities hold true:

\[
(WFf)(x, y) = (Wf)(-y, x), \quad (WF^{-1}f)(x, y) = (Wf)(y, -x),
\]

and

\[
(WGf)(x, y) = (Wf)(x, x + y + 1/2) e^{-\pi ix/2},
\]

\[
(WG^{-1}f)(x, y) = (Wf)(x, y - x - 1/2) e^{\pi ix/2}.
\]

**Proof.** For (27), we have that

\[
Ff(x) = \int_\mathbb{R} f(y)e^{2\pi ixy}dy = \sum_{m \in \mathbb{Z}} \int_0^{1} f(y)e^{2\pi ixy}dy
\]

\[
= \sum_{m \in \mathbb{Z}} \int_0^1 f(y + m)e^{2\pi i(x + m)y}dy = \int_0^1 (Wf)(y, x)e^{\pi ixy}dy = \int_0^1 h(x, y)dy,
\]

where the function \( h(x, y) := (Wf)(y, x)e^{\pi ixy} \) has the following properties:

\[
h(x + 1, y) = e^{2\pi iy} h(x, y), \quad h(x, y + 1) = h(x, y).
\]

In particular, it can be expanded in Fourier series

\[
h(x, y) = \sum_{m \in \mathbb{Z}} h_m(x)e^{2\pi iym},
\]

where

\[
h_m(x) = \int_{[0,1]} h(x, y)e^{-2\pi iym}dy = \int_{[0,1]} h(x - m, y)dy = Ff(x - m).
\]

Thus,

\[
(Wf)(y, x) = h(x, y)e^{-\pi ixy} = e^{-\pi ixy} \sum_{m \in \mathbb{Z}} Ff(x - m)e^{2\pi iym} = (WFf)(x, -y).
\]

For (28), we have that

\[
(WGf)(x, y) = e^{\pi iy} \sum_{m \in \mathbb{Z}} e^{\pi(x^2 + 2xm + m)} f(x + m)e^{2\pi imy}
\]

\[
= e^{-\pi i/2}(Wf)(x, x + y + 1/2) \square
\]
Lemma 2. The following equalities hold true:

\( g_{a,c}(x, y) = \xi_a, b(-x, y - x - 1/2)e^{\pi i x/2}, \)

and

\( g_{a,c}(x, y) = e^{-\pi i} \xi b, c(x - y - 1/2, -y)e^{-\pi y/2}. \)

Proof. From (26) and Lemma 1 we have that

\( g_{a,c}(x, y) = (WG^{-1}F^{-1} \xi a, c)(x, y) = (WF^{-1} \xi a, c)(x, y - x - 1/2)e^{\pi i x/2} = (W^2 \xi a, c)(y - x - 1/2, -x)e^{\pi i x/2}. \)

On the other hand, equality (9) implies that

\( g_{a,c} = W^2 \xi a, c = e^{-\pi i} W \xi b, c \Leftrightarrow W \xi a, c = e^{-\pi i} \xi b, c. \)

Combining (31) and (32), we obtain that

\( g_{a,c}(x, y) = e^{-\pi i} \xi b, c(y - x - 1/2, -x)e^{\pi i x/2} \)

\( \Leftrightarrow g_{a,c}(x, y) = e^{-\pi i} \xi b, c(-y, x - y + 1/2)e^{\pi y/2} \)

From (26), (10), (32), and Lemma 1 it follows that

\( g_{a,c}(x, y) = e^{-\pi i} \xi b, c(WF^2 \xi a, c)(x, y + x + 1/2)e^{-\pi i x/2} = e^{-\pi i} \xi b, c(-x, y - x - 1/2)e^{-\pi i x/2} = g_{a,b}(-x, y - x - 1/2)e^{-\pi i x/2} \)

which is equivalent to (29). Combining (34) and (33) we also have that

\( g_{a,c}(x, y) = g_{a,b}(-y, x - y + 1/2)e^{-\pi y/2} = e^{-\pi i} \xi b, c(-y + x + 1/2, -y)e^{-\pi y/2} \)

which is equivalent to (30). \( \square \)

2.3. Analytical continuation. Analytical properties of \( g_{a,c}(s, t) \) are determined by those of the function \( \phi_b(x, y) = (W \Phi_b)(x, y) \). The relation (33) implies that the following equalities hold true

\( \phi_b(x, y) = \zeta e^{\pi i(x(y - \frac{1}{2}) - y/2)} \phi_b(c_b + y, 1/2 - x - y) \)

\( \quad = \zeta \epsilon \exp(i(\epsilon y + (x - c)/2)) \phi_b(c_b + 1/2 - x - y, x - c_b), \quad \zeta = \exp(\pi i(1 - 4c^2)/12). \)

Besides that, the functional equations on \( \Phi_b(x) \) imply the following functional equations on \( \phi_b(x, y) \):

\( \phi_b(x, y) + e^{\pi \epsilon^2 x - \pi \epsilon^2 y} \phi_b(x, y - i \epsilon^2) = e^{-\pi \epsilon^2 x} \phi_b(x - i \epsilon^2). \)

Define functions

\( \xi(x) := \prod_{(m, n) \in \mathbb{Z}^2} \left( 1 - e^{2\pi i (mx - bn - b^{-1}n)} \right) \)

and

\( \chi(x, y) := \xi(c_b - x)\xi(-y)\xi(x + y + 1/2). \)

Proposition 2. The function \( \phi_b \chi \) is an analytic function on \( \mathbb{C}^2 \).
Proof. The series $\sum_{m \in \mathbb{Z}} \Phi_b(x + m)e^{2\pi i m y}$ converges absolutely in the domain
\begin{equation}
D_1 = \{(x, y) \in \mathbb{C}^2 | \Re x > -\Re y > 0\}
\end{equation}
to a meromorphic function with poles contained in
\begin{equation}
P_1 := (c_b + \mathbb{Z} + ib\mathbb{Z}_{>0} + ib^{-1}\mathbb{Z}_{\geq 0}) \times \mathbb{C}.
\end{equation}
Relations (36) imply that the domain of meromorphicity of $\phi_b(x, y)$ is extended to the union $\bigcup_{i=1}^3 D_i$ with $D_1$ defined in (40),
\begin{equation}
D_2 = \{(x, y) \in \mathbb{C}^2 | \Re c_b > \Re x > -\Re y\}
\end{equation}
with poles contained in
\begin{equation}
P_2 := \mathbb{C} \times (\mathbb{Z} + ib\mathbb{Z}_{>0} + ib^{-1}\mathbb{Z}_{\geq 0}),
\end{equation}
and
\begin{equation}
D_3 = \{(x, y) \in \mathbb{C}^2 | \Re c_b > \Re x, \Re y < 0\}
\end{equation}
with poles contained in
\begin{equation}
P_3 := \{(x, y) \in \mathbb{C}^2 | x + y \in \frac{1}{2} + \mathbb{Z} + ib\mathbb{Z}_{>0} + ib^{-1}\mathbb{Z}_{\leq 0}\},
\end{equation}
Notice that $\bigcup_{i=1}^3 P_i = \chi^{-1}(0)$. It remains to extend the domain of definition of $\phi_b(x, y)$ to the rest of the space $\mathbb{C}^2$. We have
\begin{equation}
\mathbb{C}^2 \setminus \bigcup_{i=1}^3 D_i = \bigcup_{i=1}^3 G_i,
\end{equation}
where
\begin{equation}
G_1 = \{(x, y) \in \mathbb{C}^2 | \Re x \leq -\Re y \leq 0\},
\end{equation}
\begin{equation}
G_2 = \{(x, y) \in \mathbb{C}^2 | \Re c_b \leq \Re x \leq -\Re y\},
\end{equation}
\begin{equation}
G_3 = \{(x, y) \in \mathbb{C}^2 | \Re x \geq \Re c_b, \Re y \geq 0\}.
\end{equation}
It is easy to see that the triples of points
\begin{equation}
\Delta_\pm(x, y) := ((x, y), (x - ib^{\pm 1}, y), (x - ib^{\pm 1}, y - ib^{\pm 1}))
\end{equation}
are such that for each $i \in \{1, 2, 3\}$, there exists an open set $U_i \subset G_i$, such that for any $(x, y) \in U_i$ there exists a triple, either $\Delta_+(u, v)$ or $\Delta_-(u, v)$, such that one of its components is $(x, y)$ and the other two components are in the union $D_j \cup D_k$ with $\{i, j, k\} = \{1, 2, 3\}$. That means that we can use one of the functional equations (37) to determine the value of $\phi_b(x, y)$. Thus, we extend the domain of definition of $\phi_b(x, y)$ to $\bigcup_{i=1}^3 (D_i \cup G_i)$ with the set of poles still contained in the set $\chi^{-1}(0)$. In this way, one can continue enlarging the domain until we cover the whole space $\mathbb{C}^2$. \hfill \square

2.4. Shape gauge invariance. The gauge transformation in the space of dihedral angles and the level induced by an edge $e$, see [1], is equivalent to an imaginary shift of the integration variable $x(e)$ (this is seen from the formula (4)) which, by using the holomorphicity of the Boltzmann weights, can be compensated by an imaginary shift of the integration path in the complex $x(e)$-plane.

This completes the proof of Theorem 1 except for the needed tetrahedral symmetries, which we treat in the following section.
3. TQFT RULES AND TETRAHEDRAL SYMMETRIES

We consider oriented surfaces with a restricted (oriented) cellular structure: all 2-cells are either bigons or triangles with their natural cellular structures. For example, the unit disk $D$ in $\mathbb{C}$ has four different bigon structures with two 0-cells \( \{e_0^0(*) = \pm 1\} \), two 1-cells \( e_1^\pm : [0, 1] \rightarrow D \) given by

\[
e_1^+(t) = e^{i\pi t} \text{ or } e_1^-(t) = -e^{-i\pi t}, \quad e_1^1(t) = e^{-i\pi t} \text{ or } e_1^\circ(t) = -e^{i\pi t}.
\]

An inessential bigon is one where both edges are parallel. For the unit disk the two cell structures with \( \{e_1^+(t) = e^{i\pi t}, e_1^1(t) = e^{-i\pi t}\} \) and \( \{e_1^-(t) = -e^{-i\pi t}, e_1^\circ(t) = -e^{i\pi t}\} \) are inessential. Inessential bigons can be eliminated by naturally contracting them to a 1-cell, which in the case of the unit disk corresponds to projection to the real axis, $z \mapsto \mathbb{R}z$.

For triangles we forbid cyclic orientation of the 1-cells. For example, the unit disk admits eight different triangle structures with 0-cells at the third roots of unity \( \{e_0^0(*) = 1, e_0^0(*) = e^{\pm 2\pi i/3}\} \), but only six of them are admissible.

Let $S$ be an oriented triangulated surface. We define a line bundle $E_S$ over $T^{\Delta(S)}$ as the identification space of $\mathbb{R}^{\Delta(S)} \times \mathbb{C}$ with respect to the equivalence relation generated by \( (f + \delta_e, z) \sim (f, e^{\pi i \sum_{e' \in \Delta(S)} a(e, e') f(e')}, e) \), where $\delta_e(e') = 1$ if $e = e'$ and 0 otherwise, and $a(e, e') = \sum_{t \in \Delta(S)} a_t(e, e')$ with $a_t(e, e')$ the antisymmetric $e$ and $e'$ function taking the value 1 if $e$ and $e'$ belong to $t$ and $e'$ is next to $e$ in the clockwise order (w.r.t. the orientation of $S$) and the value 0 if $e$ and $e'$ do not belong to $t$. The projection map $p_S: E_S \rightarrow T^{\Delta(S)}$ is given by

\[
p_S([f, z]) = e^{2\pi if}.
\]

We start defining our TQFT by associating to an oriented triangulated surface $S$ the vector space $\Gamma(E_S)$ of smooth sections of $E_S$. Let $T$ be a shaped tetrahedron in $\mathbb{R}^3$ with ordered vertices $v_i$, enumerated by the integers $0, 1, 2, 3$. In the case of a positive tetrahedron, we define a vector $\Psi_T \in \Gamma(E_{\partial T})$ by the following formula

\[
\Psi_T(e^{2\pi i x}) = g_{a,c}(x_{02} + x_{13} - x_{03} - x_{12}, x_{02} + x_{13} - x_{01} - x_{23})
\]

where $x \in \mathbb{R}^{\Delta(T)}$ with $x_{ij} := x(v_i v_j)$, $2\pi a = \alpha_T(v_0 v_1)$, and $2\pi c = \alpha_T(v_0 v_3)$. In the case of a negative tetrahedron, we use the same formula with $g_{a,c}$ replaced by $\overline{g}_{a,c}$. The gluing rule is very simple: take the product of tetrahedral weights and integrate over state variables associated with all internal edges.

Next, we consider cones over essential bigons with induced cellular structure. Overall, the condition of admissibility of triangular faces leaves four isotopy classes of such cones. Let us describe them by using the cone in $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ over the unit disk in $\mathbb{C}$ with the apex at $(0, 1) \in \mathbb{C} \times \mathbb{R}$. The four possible cellular structures are identified by three 0-cells $\{e_0^0(*) = (\pm 1, 0), e_0^0(*) = (0, 1)\}$, and four 1-cells

\[
\{e_1^0(t) = (\pm e^{i\pi t}, 0), \quad e_1^1(t) = (\pm (1 - t), t)\}
\]

or

\[
\{e_1^1(t) = (\mp e^{-i\pi t}, 0), \quad e_1^\circ(t) = (\pm (1 - t), t)\}
\]

or

\[
\{e_0^1(t) = (\pm e^{i\pi t}, 0), \quad e_0^1(t) = (\pm t, 1 - t)\}
\]

or else

\[
\{e_0^1(t) = (\mp e^{-i\pi t}, 0), \quad e_1^\circ(t) = (\pm t, 1 - t)\}.
\]
Let us call them cones of type $A_+, A_-, B_+, \text{ and } B_-$, respectively. We add TQFT rules by associating to these cones the following vectors:

$$\Psi_{Y_{\pm}}(x) = e^{\pm i\epsilon Y_{\pm} \pi} e^{\pi i(x_{0+} - x_{0-})(x_{1+} - x_{1-})} \sqrt{\delta_1(x_{0+} - x_{0-} + 1/2)},$$

where $Y \in \{A, B\}$, $\epsilon_A = -1$, $\epsilon_B = 1$, $x_{i\pm} := x(e_i^1)$, $i \in \{0, 1\}$. Here we use a formal square root of the 1-periodic $\delta$-function. In practice, this is not a problem, since all bigons should be identified pairwise so that all square roots of $\delta$-functions will be squared. Notice that our cones are symmetric with respect to rotation by the angle $\pi$ around the vertical coordinate axis, and this symmetry corresponds to the symmetry with respect to exchange of the arguments $x_{i+} \leftrightarrow x_{i-}$.

Now we give a TQFT description of the tetrahedral symmetries of the tetrahedral partition functions generated by the identities of Lemma 2. Let us take a positive tetrahedron with vertices $v_0, \ldots, v_3$. The edge $v_0v_1$ is incident to two faces opposite to $v_2$ and $v_3$. We can glue two cones, one of type $A_+$ and another one of type $A_-$, to these two faces in the way that one of the edges of the base bigons are glued to the initial edge $v_0v_1$, of course, by respecting all the orientations. Namely, we glue a cone of type $A_+$ to the face opposite to vertex $v_3$ so that the apex of the cone is glued to vertex $v_2$, and we glue a cone of type $A_-$ to the face opposite to vertex $v_2$ so that the apex of the cone is glued to vertex $v_3$. Finally, we can glue naturally the two bigons with each other by pushing continuously the initial edge inside the ball and eventually closing the gap like a book. The result of all these operations is that we obtain a negative tetrahedron, where the only difference with respect to the initial tetrahedron is that the orientation of the initial edge $v_0v_1$ has changed and this corresponds to changing the order of these vertices. Notice that as these vertices are neighbors, their exchange does not affect the orientation of all other edges. Identity (29) corresponds exactly to these geometric operations on the level of the tetrahedral partition functions.

Similarly, we can describe the transformations with respect to change of orientations of the edges $v_1v_2$ and $v_2v_3$ and relate them to the identities (30) and (29), respectively. Altogether, these three transformations correspond to canonical generators of the permutation group of four elements which is the complete symmetry group of a tetrahedron.

### 4. Appendix A. Faddeev’s Quantum Dilogarithm

Following notation of [1], Faddeev’s quantum dilogarithm $\Phi_b(z)$ is defined by the integral

$$\Phi_b(z) = \exp \left( \int_{0-\infty}^{0+\infty} \frac{e^{-i2\pi w} dw}{4 \sinh(w/b) \sinh(w/b) b w} \right)$$

in the strip $|\Im z| < |\Im c_b|$, where $c_b \equiv (b + b^{-1})/2$. Define also

$$\zeta_{inv} \equiv e^{i\pi(1+2c_b^2)/6} = e^{i\pi e_b^2} \zeta_0, \quad \zeta_\infty \equiv e^{i\pi(1-4c_b^2)/12}$$

When $\Im b^2 > 0$, the integral can be calculated explicitly

$$\Phi_b(z) = (e^{2\pi(z+c_b)b}; q^2)_{\infty}/(e^{2\pi(z-c_b)b^{-1}}; q^2)_{\infty}$$

where

$$q \equiv e^{i\pi b^2}, \quad \bar{q} \equiv e^{-i\pi b^{-2}}.$$
Using symmetry properties
\[ \Phi_b(z) = \Phi_{-b}(z) = \Phi_{1/b}(z) \]
we choose
\[ \Re b > 0, \quad \Im b \geq 0. \]
\( \Phi_b(z) \) can be continued analytically in variable \( z \) to the entire complex plane as a meromorphic function with essential singularity at infinity and with the following characteristic properties

**zeros and poles:**
\[ (\Phi_b(z))^{\pm 1} = 0 \iff z = \mp(c_b + mib + nib^{-1}), \quad m, n \in \mathbb{Z}_{\geq 0}; \]

**behavior at infinity:** depending on the direction along which the limit is taken
\[
\Phi_b(z) \bigg|_{|z| \to \infty} \approx \begin{cases} 
1 & |\arg z| > \frac{\pi}{2} + \arg b \\
\frac{-1}{\zeta_{\text{inv}} e^{i\pi z^2}} & |\arg z| < \frac{\pi}{2} - \arg b \\
\frac{\Theta(ib - \frac{b}{2} - z)}{(q^2, q^2)_{\infty}} & |\arg z - \frac{\pi}{2}| < \arg b \\
\frac{\Theta(ib + \frac{b}{2} - z)}{(q^2, q^2)_{\infty}} & |\arg z + \frac{\pi}{2}| < \arg b 
\end{cases}
\]

where
\[ \Theta(z; \tau) \equiv \sum_{n \in \mathbb{Z}} e^{i\pi n^2 + i2\pi nz}, \quad \Im \tau > 0; \]

**inversion relation:**
\[ \Phi_b(z)\Phi_b(-z) = \zeta_{\text{inv}} e^{i\pi z^2}; \]

**functional equations:**
\[ \Phi_b(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\Phi_b(z + ib^{\pm 1}/2); \]

**unitarity:** when \( b \) is real or on the unit circle
\[ (1 - |b|)3\Re b = 0 \Rightarrow \Phi_b(\bar{z}) = 1/\Phi_b(z); \]

**quantum pentagon identity:**
\[ \Phi_b(p)\Phi_b(q) = \Phi_b(q)\Phi_b(p + q)\Phi_b(p), \]

where selfadjoint operators \( p \) and \( q \) in \( L^2(\mathbb{R}) \) satisfy the commutation relation \( pq - qp = (2\pi i)^{-1} \).

### 4.1. Integral analog of \( \psi_1 \)-summation formula of Ramanujan.

Following [9], consider the following Fourier integral
\[ \Psi(u, v, w) \equiv \int_{\mathbb{R}} \frac{\Phi_b(x + u)}{\Phi_b(x + v)} e^{2\pi iux} \, dx \]
where
\[ \Im (v + c_b) > 0, \quad \Im (-u + c_b) > 0, \quad \Im (v - u) < \Im w < 0 \]

Conditions (64) can be relaxed by deforming the integration contour in the complex plane of the variable \( x \), keeping asymptotic directions within the sectors \( \pm (|\arg x| - \pi/2) > \arg b \). Enlarged in this way the domain of the variables \( u, v, w \) has the form
\[ |\arg (iz)| < \pi - \arg b, \quad z \in \{ w, v - u - w, u - v - 2c_b \} \]

Considering \( \Phi_b(z) \) as a non-compact analog the function \( (x; q)_{\infty} \), definition (63) can be interpreted as an integral analog of \( \psi_1 \)-summation formula of Ramanujan.
By the method of residues, (63) can be calculated explicitly
\begin{equation}
\Psi(u, v, w) = \zeta_o \frac{\Phi_b(u - v - c_b)\Phi_b(w + c_b)}{\Phi_b(u - v + w - c_b)} e^{-2\pi i u(v + c_b)}
\end{equation}

\begin{equation}
= \zeta_o^{-1} \frac{\Phi_b(v - u - w + c_b)}{\Phi_b(v - u + c_b)\Phi_b(-w - c_b)} e^{-2\pi i w(u - c_b)},
\end{equation}
where the two expressions in the right hand side are related by the inversion relation (59).

4.2. Fourier transformation formulae. Special cases of \(\Psi(u, v, w)\) lead to the following Fourier transformation formulae for Faddeev’s quantum dilogarithm
\begin{equation}
\phi_+(w) \equiv \int_{\mathbb{R}} \Phi_b(x)e^{2\pi i u x} dx = \Psi(0, v, w)|_{v \to -\infty}
= \zeta_o^{-1} e^{2\pi i w c_b} / \Phi_b(-w - c_b) = \zeta_o e^{-\pi i w^2} \Phi_b(w + c_b)
\end{equation}
and
\begin{equation}
\phi_-(w) \equiv \int_{\mathbb{R}} (\Phi_b(x))^{-1} e^{2\pi i u x} dx = \Psi(u, 0, w)|_{u \to -\infty}
= \zeta_o e^{-2\pi i w c_b} \Phi_b(w + c_b) = \zeta_o^{-1} e^{\pi i w^2} / \Phi_b(-w - c_b).
\end{equation}
The corresponding inverse transformations look as follows
\begin{equation}
(\Phi_b(x))^{\pm 1} = \int_{\mathbb{R}} \phi_{\pm}(y)e^{-2\pi i y y} dy,
\end{equation}
where the pole \(y = 0\) should be surrounded from below.

4.3. Other integral identities. Faddeev’s quantum dilogarithm satisfies other analogs of hypergeometric identities, see for example [11]. Following [24, 17], for any \(n \geq 1\), we define
\begin{equation}
\Psi_n(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; w) \equiv \int_{\mathbb{R}} dx e^{i2\pi x(w - c_b)} \prod_{j=1}^{n} \frac{\Phi_b(x + a_j)}{\Phi_b(x + b_j - c_b)}
\end{equation}
where \(b_n = i0\),
\[\Im(b_j) > 0, \quad \Im(c_b - a_j) > 0, \quad \sum_{j=1}^{n} \Im(b_j - a_j - c_b) < \Im(w - c_b) < 0.\]
The integral analog of \(1\psi_1\)-summation formula of Ramanujan in this notation takes the form
\begin{equation}
\Psi(u, v, w) = e^{-i2\pi w(u + c_b)} \Psi_1(u - v - c_b; w + c_b), \quad \Psi_1(a; w) = \zeta_o \frac{\Phi_b(a)\Phi_b(w)}{\Phi_b(a + w - c_b)}.
\end{equation}
Equivalently, we can write
\begin{equation}
\bar{\Psi}_1(a; w) \equiv \int_{\mathbb{R}} \frac{\Phi_b(x + c_b - i0)}{\Phi_b(x + a)} e^{-i2\pi x(w + c_b)} dx
= e^{i2\pi(a + c_b)(w + c_b)} \Psi_1(-a; -w) = \zeta_o^{-1} \frac{\Phi_b(a + w + c_b)}{\Phi_b(a)\Phi_b(w)}.
\end{equation}
Using the integral \( \psi_1 \)-summation formula of Ramanujan, we can obtain the integral analog of the transformation of Heine for \( \phi_2 \)

\[
\Psi_2(a, b; c; w) = \frac{\Phi_b(a)}{\Phi_b(c-b)} \Psi_2(c-b, w; a+w; b).
\]

Using here the evident symmetry

\[
\Psi_2(a, b; c; w) = \Psi_2(b, a; c; w)
\]

we obtain the integral analog of Euler–Heine transformation

\[
(74) \quad \Psi_2(a, b; c; w)
\]

\[
= \frac{\Phi_b(a)\Phi_b(b)\Phi_b(w)}{\Phi_b(c-b)\Phi_b(c-a)\Phi_b(a+b+w-c)} \Psi_2(c-a, c-b; a+b+c-w, c).
\]

Performing the Fourier transformation on \( w \) and using equation (72), we obtain the integral version of Saalschütz summation formula:

\[
\Psi_3(a, b, c; d, a+b+c-d-c_b; -c_b) = \zeta_0^3 e^{i\pi d(2c_b-d)}
\]

\[
\times \frac{\Phi_b(a)\Phi_b(b)\Phi_b(c)\Phi_b(a-d)\Phi_b(b-d)\Phi_b(c-d)}{\Phi_b(a+b-d-c_b)\Phi_b(b+c-d-c_b)\Phi_b(c+a-d-c_b)}.
\]

One particular case is obtained by taking the limit \( c \to -\infty \):

\[
(76) \quad \Psi_2(a, b; d; -c_b) = \zeta_0^3 e^{i\pi d(2c_b-d)} \frac{\Phi_b(a)\Phi_b(b)\Phi_b(a-d)\Phi_b(b-d)}{\Phi_b(a+b-d-c_b)}.
\]

### 4.4. Quasi-classical limit of Faddeev’s quantum dilogarithm

We consider the asymptotic limit \( b \to 0 \).

**Proposition 3.** For \( b \to 0 \) and fixed \( x \), one has the following asymptotic expansion

\[
\ln \Phi_b \left( \frac{x}{2\pi b} \right) = \sum_{n=0}^{\infty} (2\pi ib)^{2n-1} \frac{B_{2n}(1/2)}{(2n)!} \frac{\partial^{2n} \text{Li}_2(-e^x)}{\partial x^{2n}}
\]

where \( B_{2n}(1/2) \) — Bernoulli polynomials evaluated at 1/2.

**Proof.** From one hand side, we can write formally

\[
\ln \Phi_b \left( \frac{x - i\pi b^2}{2\pi b} \right) - \ln \Phi_b \left( \frac{x + i\pi b^2}{2\pi b} \right) = -2 \sinh(i\pi b^2 \partial/\partial x) \ln \Phi_b \left( \frac{x}{2\pi b} \right).
\]

On the other hand side,

\[
\ln(1 + e^x) = \frac{\partial}{\partial x} \int_{-\infty}^{x} \ln(1 + e^z)dz = -\frac{\partial}{\partial x} \text{Li}_2(-e^x).
\]

Substituting these expressions into the linearized functional equation (60) with positive exponent of \( b \), we obtain

\[
(78) \quad 2\pi ib^2 \ln \Phi_b \left( \frac{x}{2\pi b} \right) = \frac{i\pi b^2 \partial/\partial x}{\sinh(i\pi b^2 \partial/\partial x)} \text{Li}_2(-e^x).
\]

Using the expansions

\[
\frac{z}{\sinh(z)} = \sum_{n=0}^{\infty} B_{2n}(1/2) \frac{(2z)^{2n}}{(2n)!}
\]

we obtain (77). \( \square \)
Corollary 1. For $b \to 0$, one has
\begin{equation}
\Phi_b \left( \frac{x}{2\pi b} \right) = e^{\frac{x}{2\pi b}} \text{Li}_2 \left( -e^{-x} \right) \left( 1 + O(b^2) \right).
\end{equation}

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Center for Quantum Geometry of Moduli Spaces, University of Aarhus, DK-8000, Denmark

E-mail address: andersen@qgm.au.dk

University of Geneva, 2-4 rue du Lièvre, Case postale 64, 1211 Genève 4, Switzerland

E-mail address: rinat.kashaev@unige.ch