Effect of $U_A(1)$ Breaking on Chiral Phase Structure and Pion Superfluidity at Finite Isospin Chemical Potential

Lianyi He, Meng Jin and Pengfei Zhuang

Physics Department, Tsinghua University
Beijing 100084, China
hely04@mails.tsinghua.edu.cn

We investigate the isospin chemical potential effect in the frame of $SU(2)$ Nambu-Jona-Lasinio model. When the isospin chemical potential is less than the vacuum pion mass, the phase structure with two chiral phase transition lines does not happen due to $U_A(1)$ breaking of QCD. When the isospin chemical potential is larger than the vacuum pion mass, the ground state of the system is a Bose-Einstein condensate of charged pions.

Keywords: Isospin Chemical Potential; Pion Superfluidity; NJL Model .

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1. Introduction

It is generally believed that there exists a rich phase structure of Quantum Chromodynamics (QCD) at finite temperature and baryon density, for instance, the deconfinement process from hadron gas to quark-gluon plasma, the transition from chiral symmetry breaking phase to the symmetry restoration phase, the color superconductivity at low temperature and high baryon density. Recently, the study on the QCD phase structure is extended to finite isospin density. The physical motivation to study isospin spontaneous breaking and the corresponding pion superfluidity is related to the investigation of compact stars, isospin asymmetric nuclear matter and heavy ion collisions at intermediate energies. While there is not yet precise lattice result at finite baryon density due to the Fermion sign problem, it is in principle no problem to do lattice simulation at finite isospin density. It is found that the critical isospin chemical potential for pion condensation is about the pion mass in the vacuum, $\mu_I^c \approx m_\pi$. The QCD phase structure at finite isospin density is also investigated in many low energy effective models, such as chiral perturbation theory, ladder QCD, random matrix method, strong coupling lattice QCD and Nambu–Jona-Lasinio (NJL) model.

One of the models that enables us to see directly how the dynamic mechanisms of chiral symmetry breaking and restoration operate is the NJL model applied to quarks. Within this model, one can obtain the hadronic mass spectrum and the static properties of mesons remarkably well, and the chiral phase transition line in $T - \mu_B$ plane is very close to the one calculated with lattice.
QCD. Recently, this model is also used to investigate the color superconductivity at moderate baryon density. In this letter we report our results on the phase structure of the general SU(2) NJL model with \( U_A(1) \) breaking term at finite isospin chemical potential. At small isospin chemical potential, we argue that there is only one chiral phase transition line in the \( T - \mu_B \) plane. At isospin chemical potential larger than vacuum pion mass, the ground state of the system is a Bose-Einstein condensate of charged pions.

2. Mean Field Approximation at Finite Isospin Chemical Potential

We start with the flavor SU(2) NJL model defined by

\[
\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m_0 + \mu \gamma_0) \psi + \mathcal{L}_{int},
\]

where \( m_0 \) is the current quark mass, \( \mu \) the chemical potential matrix in flavor space,

\[
\mu = \text{diag}(\mu_u, \mu_d) = \text{diag} \left( \mu_B + \frac{1}{2} \mu_I, -\frac{1}{2} \mu_I \right)
\]

with \( \mu_B \) and \( \mu_I \) being the baryon and isospin chemical potential, respectively, and the interaction part includes the normal four Fermion couplings corresponding to scalar mesons \( \sigma, a_0, a_+ \) and \( a_- \) and pseudoscalar mesons \( \eta, \pi_0, \pi_+ \) and \( \pi_- \) excitations, and the 't-Hooft determinant term for \( U_A(1) \) breaking,

\[
\mathcal{L}_{int} = \frac{G}{2} \sum_{a=0}^{3} \left[ (\bar{\psi} \tau_a \psi)^2 + (\bar{\psi} i \gamma_5 \tau_a \psi)^2 \right] + \frac{K}{2} \left[ \text{det} (1 + \gamma_5) \psi + \text{det} (1 - \gamma_5) \psi \right]
\]

\[
= \frac{1}{2} \left( G + K \right) \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \gamma_0 \psi)^2 \right] + \frac{1}{2} \left( G - K \right) \left[ (\bar{\psi} \tau_1 \gamma_5 \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \right].
\]

At zero isospin chemical potential \( \mu_I = 0 \), for \( K = 0 \) and \( m_0 = 0 \) the Lagrangian is invariant under \( U_B(1) \otimes U_A(1) \otimes SU_V(2) \otimes SU_A(2) \) transformations, but for \( K \neq 0 \), the symmetry is reduced to \( U_B(1) \otimes SU_V(2) \otimes SU_A(2) \) and the \( U_A(1) \) breaking leads to \( \sigma \) and \( a \) mass splitting and \( \pi \) and \( \eta \) mass splitting. If \( G = K \), we come back to the standard NJL model with only \( \sigma, \pi_0, \pi_+ \) and \( \pi_- \) mesons.

We introduce the quark condensates

\[
\sigma_u = \langle \bar{u} u \rangle, \quad \sigma_d = \langle \bar{d} d \rangle,
\]

and the pion condensate

\[
\pi = \left( \bar{\psi} i \gamma_5 \tau_+ \psi \right) = \left( \bar{\psi} i \gamma_5 \tau_- \psi \right) = \frac{1}{\sqrt{2}} \left( \bar{\psi} i \gamma_1 \gamma_5 \psi \right),
\]

where we have chosen the pion condensate to be real. The quark condensate and pion condensate are, respectively, the order parameters of chiral phase transition and pion superfluidity. In mean field approximation, the thermodynamic potential of the system is

\[
\Omega = G (\sigma_u^2 + \sigma_d^2) + 2K \sigma_u \sigma_d + \frac{G + K}{2} \pi^2 - \frac{T}{V} \ln \det S_{m_1}(k).
\]
where $S_{m}^{-1}(k)$ is the inverse of the mean field quark propagator, in momentum space it reads

$$S_{m}^{-1}(k) = \begin{pmatrix} \gamma^{\mu}k_{\mu} + \mu_{u}\gamma_{0} - m_{u} & i\gamma_{5}(G + K)\pi \\ i\gamma_{5}(G + K)\pi & \gamma^{\mu}k_{\mu} + \mu_{d}\gamma_{0} - m_{d} \end{pmatrix}$$

with the effective quark masses

$$m_{u} = m_{0} - 2G\sigma_{u} - 2K\sigma_{d}, \quad m_{d} = m_{0} - 2G\sigma_{d} - 2K\sigma_{u}.$$  

The condensates $\sigma_{u}, \sigma_{d}$ and $\pi$ as functions of temperature and baryon and isospin chemical potentials are determined by the minimum thermodynamic potential,

$$\frac{\partial \Omega}{\partial \sigma_{u}} = 0, \quad \frac{\partial \Omega}{\partial \sigma_{d}} = 0, \quad \frac{\partial \Omega}{\partial \pi} = 0.$$  

It is easy to see from the chemical potential matrix and the quark propagator matrix that for $\mu_{B} = 0$ or $\mu_{I} = 0$ the gap equations for $\sigma_{u}$ and $\sigma_{d}$ are symmetric, and one has

$$\sigma_{u} = \sigma_{d} = \sigma/2, \quad m_{u} = m_{d} = m = m_{0} - (G + K)\sigma.$$  

3. Chiral Phase Structure at $\mu_{I} < m_{\pi}$

We now consider the QCD phase structure below the minimum isospin chemical potential $\mu_{I} = m_{\pi}$ for pion superfluidity. Since the pion condensate is zero, there is only chiral phase structure in this region.

It is well known that in absence of isospin chemical potential the chiral condensate jumps down suddenly at a critical baryon chemical potential, which indicates a first order phase transition. At both finite baryon and isospin chemical potentials, in principle we have $\sigma_{u} \neq \sigma_{d}$, they may jump down at the same critical baryon chemical potential or at two different critical points. In the case $K = 0$, the two gap equations for $\sigma_{u}$ and $\sigma_{d}$ decouple and the two critical points do exist, and therefore, there are two chiral phase transition lines in $T - \mu_{B}$ plane at fixed isospin chemical potential, the interval between the two points is just $\Delta \mu_{B} = 3\mu_{I}$. What is the effect of the $U_{A}(1)$ breaking term on the QCD phase structure? If $K \neq 0$, the two gap equations couple to each other and tend to a single phase transition. Especially in chiral limit, one can clearly see that when one of the quark condensates becomes zero, the other one is forced to be zero for any coupling constants $G$ and $K \neq 0$. Therefore, there is only one chiral phase transition line at any $K \neq 0$ in chiral limit. In real world with small current quark mass, numerical solution of the gap equations shows that if there exists a structure of two chiral phase transition lines depends the quantity

$$\alpha = \frac{K}{G + K} = \frac{1}{2} \left(1 - \frac{G - K}{G + K}\right).$$

In real world there are four parameters in the NJL model, the current quark mass $m_{0}$, the three-momentum cutoff $\Lambda$, and the two coupling constants $G$ and
Among them \( m_0, \Lambda \) and the combination \( G + K \) can be determined by fitting the chiral condensate \( \sigma \), the pion mass \( m_\pi \) and the pion decay constant \( f_\pi \) in the vacuum. For \( \sigma = 2(-241.5 \text{ MeV})^3, m_\pi = 140.2 \text{ MeV}, \) and \( f_\pi = 92.6 \text{ MeV}, \) one has \( m_0 = 6 \text{ MeV}, \Lambda = 590 \text{ MeV}, \) and \( (G + K)\Lambda^2/2 = 2.435. \) To determine the two coupling constants separately or the ratio \( \alpha \), one needs to know the \( \eta' \)- or \( \eta \)-meson properties in the vacuum. In the self-consistent mean field approximation, the \( \eta' \) mass \( m_{\eta'} \) can be calculated in random phase approximation (RPA)\(^{13} \)

\[
1 - (G - K)\Pi_{PS}(k_0 = m_{\eta'}, k = 0) = 0
\]  

(12)

with polarization function \( \Pi_{PS} \) defined as

\[
\Pi_{PS}(k_0, k) = -i \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ \gamma_5 S_{mf}(p + k)\gamma_5 S_{mf}(p) \right].
\]

(13)

The quantity \( \alpha \) can be written as

\[
\alpha = \frac{1}{2} \left( 1 - \frac{1}{G + K} \frac{1}{12} \frac{\int \frac{d^3k}{(2\pi)^3} \frac{E_k}{E_k^2 - m_{\eta'}^2}}{d^3k/(2\pi)^3} \right)
\]

(14)

where \( E_k = \sqrt{k^2 + m^2}. \) Combining with the above known parameters \( m_0, \Lambda \) and \( G + K \) obtained and choosing \( m_{\eta'} = 958 \text{ MeV}, \) we have \( \alpha = 0.29 \) which is much larger than the critical value 0.11 for the two-line structure at \( \mu_f = 60 \text{ MeV}. \) In fact, for a wide mass region \( 540 \text{ MeV} < m_{\eta'} < 1190 \text{ MeV}, \) we have \( \alpha > 0.11, \) and there is no two-line structure.

To fully answer the question if the two-line structure exists before the pion condensation happens, we consider the limit \( \mu_f = m_\pi. \) In real world with \( m_0 \neq 0 \) and \( K \neq 0, \) with increasing coupling constant \( K \) or the ratio \( \alpha \) the two lines approach to each other and finally coincide at about \( \alpha = 0.21 \) which is still less than the value 0.29 calculated by fitting \( m_{\eta'} = 958 \text{ MeV}. \) Therefore, the two-line structure disappears if we choose \( m_{\eta'} = 958 \text{ MeV}. \) In fact, for \( 720 \text{ MeV} < m_{\eta'} < 1140 \text{ MeV}, \) we have \( \alpha > 0.21 \) and the two phase transition lines are cancelled in this wide mass region. When the \( \eta' \)-meson mass is outside this region, the \( U_A(1) \) breaking term is not strong enough to cancel the two-line structure, but the two lines are already very close to each other. Considering the relation between the \( \eta' \) mass and the number of flavors, \( m_{\eta'}^2 \propto N_f, \) one can estimate the \( \eta' \) mass in the case of two flavors \( m_{\eta'} \approx 780 \text{ MeV} \) which is in the region where \( \alpha > 0.21. \)

4. Pion Superfluidity at \( \mu_f > m_\pi \)

In this section we concentrate on the case \( \mu_B = 0. \) The gap equations for chiral condensate \( \sigma = \sigma_u + \sigma_d \) and pion condensate \( \pi \) are derived as following

\[
\sigma = 6m \int \frac{d^3k}{(2\pi)^3} \frac{1}{E_k} \left( \frac{E_k^-}{E^-} (2f(E^-) - 1) + \frac{E_k^+}{E^+} (2f(E^+) - 1) \right),
\]

\[
\pi \left[ 1 + 12H\pi \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{E_k^-} (2f(E^-) - 1) + \frac{1}{E_k^+} (2f(E^+) - 1) \right) \right] = 0. \]

(15)
where \( H = (G + K)/2, E_k^\pm = E_k \pm \mu_I/2, E^\pm_\pi = \sqrt{(E_k^\pm)^2 + 4H^2\pi^2}. \) The isospin density \( n_I \) can be derived from thermodynamic potential \( \Omega \),

\[
n_I = 3 \int \frac{d^3k}{(2\pi)^3} \left( \frac{E_k^+}{E_\pi^+} (2f(E_\pi^+)^2 - 1) - \frac{E_k^-}{E_\pi^-} (2f(E_\pi^-)^2 - 1) \right).
\]

Numerical solution of the gap equations at zero temperature is shown in Fig. 1. We found that when \( \mu_I < m_\pi \), the ground state is the same as the vacuum and the isospin density is zero, while pion condensate becomes nonzero when \( \mu_I > m_\pi \) and isospin density becomes nonzero. The phase transition is of second order. The critical isospin chemical potential \( \mu^c_I \) is exactly the vacuum pion mass \( m_\pi \), which can be proved analytically. The critical isospin chemical potential is determined by

\[
1 - 12H \int \frac{d^3k}{(2\pi)^3} \frac{1}{E_k^2 - (\mu_I^c)^2/4} = 0.
\]

which is just the same as the mass equation for pion in vacuum.

The collective meson excitations can be calculated in the framework of random phase approximation. Different from normal RPA approach in vacuum and at finite temperature, the meson modes in the pion superfluid will mix with each other and the dispersion relations of each eigen modes are determined by

\[
\det(1 - 2H\Pi(k_0, k)) = 0
\]

where the polarization function matrix \( \Pi(k_0, k) \) is defined as

\[
1 - 2H\Pi(k_0, k) = \begin{pmatrix}
1 - 2H\Pi_{\sigma\sigma} & -2H\Pi_{\sigma\pi_+} & -2H\Pi_{\sigma\pi_-} & -2H\Pi_{\sigma\pi_0} \\
-2H\Pi_{\pi_+\sigma} & 1 - 2H\Pi_{\pi_+\pi_+} & -2H\Pi_{\pi_+\pi_-} & -2H\Pi_{\pi_+\pi_0} \\
-2H\Pi_{\pi_-\sigma} & -2H\Pi_{\pi_-\pi_+} & 1 - 2H\Pi_{\pi_-\pi_-} & -2H\Pi_{\pi_-\pi_0} \\
-2H\Pi_{\pi_0\sigma} & -2H\Pi_{\pi_0\pi_+} & -2H\Pi_{\pi_0\pi_-} & 1 - 2H\Pi_{\pi_0\pi_0}
\end{pmatrix}.
\]

The polarization functions are defined as

\[
\Pi_{MM'}(k_0, k) = i \int \frac{d^4p}{(2\pi)^4} \text{Tr} [\Gamma^*_M S_{mf}(p + k) \Gamma_{M'} S_{mf}(p)],
\]

with the vertexes \( \Gamma_M \) and \( \Gamma^*_M \)

\[
\Gamma_M = \begin{pmatrix}
1 & M = \sigma \\
i\tau_+\gamma_5 & M = \pi_+ \\
i\tau_-\gamma_5 & M = \pi_- \\
i\tau_3\gamma_5 & M = \pi_0
\end{pmatrix}, \quad \Gamma^*_M = \begin{pmatrix}
1 & M = \sigma \\
i\tau_+\gamma_5 & M = \pi_+ \\
i\tau_-\gamma_5 & M = \pi_- \\
i\tau_3\gamma_5 & M = \pi_0
\end{pmatrix}
\]

where \( \tau_\pm = (\tau_1 \pm i\tau_2)/\sqrt{2} \). From detailed calculation one has

\[
\Pi_{\sigma\pi_\pm} = \Pi_{\pi_\pm\sigma} \propto \sigma \pi, \quad \Pi_{\pi_+\pi_-} = \Pi_{\pi_-\pi_+} \propto \pi^2, \quad \Pi_{\pi_0\pi_\pm} = \Pi_{\pi_\pm\pi_0} = 0 (I = \sigma, \pi_\pm).
\]

Thus \( \pi_0 \) does not mix with other modes, while \( \sigma, \pi_+, \pi_- \) mix with each other. However, mixing between \( \sigma \) and \( \pi_\pm \) is only strong near the phase transition point and can be neglect at large \( \mu_I \). The meson mass spectrum as a function of \( \mu_I \) is shown in Fig. 2. We found that the full calculation (solid line) is consistent with
chiral perturbation theory, and the mixing between \( \sigma \) and \( \pi \pm \) plays a very important role. On the other hand, in the superfluid phase with \( \pi \neq 0 \), \( \det(1 - 2H\Pi(0,0)) = 0 \) is always satisfied, which indicates that Goldstone’s theorem is satisfied in the whole superfluid phase at RPA level.

5. Summary
We have investigated the two flavor NJL model with \( U_A(1) \) breaking term at finite isospin chemical potential. The main conclusions are:

1) The two chiral phase transition lines in the \( T - \mu_B \) plane predicated by the NJL model without \( U_A(1) \) breaking term are cancelled by the strong \( U_A(1) \) breaking term at low isospin chemical potential. Therefore, in relativistic heavy ion collisions where the typical \( \mu_I \) value is much less than \( m_\pi \), it looks impossible to realize the two-line structure.

2) The critical isospin chemical potential for pion condensation in NJL model is exactly the pion mass in the vacuum, \( \mu_I^c = m_\pi \), independent of the model parameters, the regularization scheme, the \( U_A(1) \) breaking term. When \( \mu_I \) exceeds \( m_\pi \),
The ground state of the system is in a Bose-Einstein condensate of charged pions. In this pion superfluid, the meson modes $\sigma, \pi_+, \pi_-$ mixed with each other and there is a Goldstone mode due to the spontaneous breaking of $U_I(1)$ symmetry.

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