A CLASSIFICATION OF THE IRREDUCIBLE ADMISSIBLE GENUINE MOD $p$
REPRESENTATIONS OF $p$-ADIC $\widetilde{SL}_2$

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ABSTRACT. We classify the irreducible, admissible, smooth, genuine mod $p$ representations of the metaplectic double cover of $SL_2(F)$, where $F$ is a $p$-adic field and $p \neq 2$. We show, using a generalized Satake transform, that each such representation is isomorphic to a certain explicit quotient of a compact induction from a maximal compact subgroup by an action of a spherical Hecke operator, and we define a parameter for the representation in terms of this data. We show that our parameters distinguish genuine nonsupercuspidal representations from genuine supercuspidals, and that every irreducible genuine nonsupercuspidal representation is in fact an irreducible principal series representation. In particular, the metaplectic double cover of $p$-adic $SL_2$ has no special mod $p$ representations.

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1. Introduction

1.1. Context. Central extensions of algebraic groups appear in several key roles in number theory, perhaps first in representation-theoretic studies of theta functions and more generally of automorphic forms of half-integral weight. In that context, the relevant covering group is a metaplectic group, a central extension $\widetilde{Sp}_{2n}$ of an even-dimensional symplectic group by the square roots of unity. The work of Weil [27] on the representations of metaplectic groups, in particular a distinguished (Weil) representation, led to the theory of dual reductive pairs and theta correspondence. A central work in this area is the parametrization by Waldspurger of the irreducible genuine representations of $\widetilde{SL}_2(F)$, where $F$ is a local field of characteristic 0 (different from $\mathbb{C}$) and a genuine representation is one which does not factor through a representation of $SL_2(F)$. This parametrization, by the irreducible representations of $PGL_2(F)$ and of the units of the projectivization of a quaternion algebra, has been shown by Waldspurger and others (cf. §2.2 of [11]) to encode the local factors of automorphic representations which appear in the statement of local-global compatibility for the classical local Langlands correspondence (LLC).

The past several years have seen much investigation of the role of covering groups in the Langlands Program. Recently, Waldspurger’s parametrization has been generalized to $\widetilde{Sp}_{2n}(F)$ (for $F$ a local field of...
characteristic 0 and odd residual characteristic) by Gan and Savin [9], who also define a local Langlands correspondence for $\tilde{Sp}_{2n}$ via the existing LLC for the special orthogonal groups whose representations provide the parametrizing set. In the general setting of Brylinski-Deligne covers, Weissman [28] has constructed $L$-groups and $L$-parameters for covers of split groups. The recent preprint [8] of Gan and Gao, incorporating Weissman’s refinement of his $L$-group construction, defines an unramified LLC for this class of covering groups.

Another recent focus of activity has been the mod $p$ local Langlands program, which aims to relate the irreducible mod $p$ representations of a reductive group over a $p$-adic field $F$ to certain mod $p$ representations of the absolute Galois group $\text{Gal}(\overline{F}/F)$. Such a correspondence exists for $GL_2(\mathbb{Q}_p)$ by work of Barthel-Livné [3], [4] and Breuil [5], and has been shown to be induced by Colmez’s functor [7] which realizes the $p$-adic LLC for $GL_2(\mathbb{Q}_p)$. Outside of the specific case of $GL_2(\mathbb{Q}_p)$, it quickly becomes difficult even to formulate a precise conjecture. As a prerequisite, one should have a classification of the irreducible admissible mod $p$ representations of the desired reductive group. Such a classification exists up to supercuspidals for split reductive groups due to Abe [2], building on Herzig’s classification (likewise up to supercuspidals) for $GL_n(F)$ [13].

A parallel question, whose relationship to the mod $p$ Langlands program is so far unclear but intriguing, is the existence of a mod $p$ theta correspondence. Shin [20] notes that several objects appearing in the classical theta correspondence fail to carry over to the mod $p$ setting, and even once reconstructed in geometric terms, appear to behave differently from their classical counterparts [25]. However, a weak version of Howe duality for unramified representations, defined in terms of the respective unramified spherical Hecke algebras, holds in the case of a type II dual reductive pair ([25] Theorem 5.14).

Our main goal in this paper is to provide a detailed study of the mod $p$ representations of $\tilde{SL}_2(F)$, where $F$ is a $p$-adic field of odd residual characteristic. As $SL_2(F)$ is a rank-one group with a unique topological central extension of degree 2, it is possible to work very concretely while keeping track of the effects of all choices involved in the parametrization of representations. In particular, we give a classification of the irreducible admissible genuine mod $p$ representations of $\tilde{SL}_2(F)$ along the lines of Barthel and Livné’s classification of the irreducible mod $p$ representations of $GL_2(F)$. This classification may be viewed as a generalized Satake parameterization, incorporating all admissible irreducible genuine representations (not only the unramified ones). Similar parametrizations already exist for $PGL_2(F)$ (by Barthel-Livné [3] and [4]), $SL_2(F)$ (by Abdellatif [1]), and $PD^\times$ where $D$ is a quaternion algebra (by Cheng, [5]). Thus we hope this work will provide most of the information needed to use $\tilde{SL}_2(F)$ as a test case for conjectures in the above-mentioned mod $p$ programs.

This hope has informed our choice of techniques. For example, Hecke algebras and their modules play a key role both in the mod $p$ Langlands program and in Gan-Savin’s formulation of the classical theta correspondence [10]. On the way to our classification, we give explicit descriptions of the genuine spherical mod $p$ Hecke algebras of $\tilde{SL}_2(F)$ using a generalized Satake transform. In a companion note [23] we also compute the action of spherical Hecke operators on local systems on the Bruhat-Tits tree of $SL_2(F)$.

1.2. Main Results. Let $F$ be a $p$-adic field with $p \neq 2$ and residue field of order $q = p^f$, let $E$ be an algebraic closure of $\mathbb{F}_q$, and from now on write $\tilde{G}$ for the metaplectic group $\tilde{SL}_2(F)$. (See [2,13] for a detailed definition.) The main result is a classification theorem for irreducible admissible genuine mod $p$ representations of $\tilde{G}$. To each such representation, we attach a parameter $(\vec{r}, \lambda)$ consisting of a vector $\vec{r} \in \{0, \ldots, p-1\}^f$ and a value $\lambda \in E$. 
In fact we attach two such parameters, one for each of the two conjugacy classes of maximal compact subgroups in $\tilde{G}$. We briefly explain how this is done. For any maximal compact subgroup $\tilde{K}$ of $\tilde{G}$, the irreducible genuine representations (which we call weights) of $\tilde{K}$ are indexed by vectors $\vec{r} \in \{0, \ldots, p - 1\}^f$ (Proposition 3.11 and Lemma 3.3). Let $\tilde{\sigma}_\vec{r}$ denote the weight of $\tilde{K}$ with index $\vec{r}$. The endomorphism algebra $\text{End}_{\tilde{G}}(\text{ind}\tilde{K}\tilde{\sigma}_\vec{r})$, a genuine spherical Hecke algebra, acts on the compact induction $\text{ind}\tilde{K}\tilde{\sigma}_\vec{r}$. We show (Corollary 4.15) that each genuine spherical Hecke algebra is a polynomial algebra in a single operator $\mathcal{T}$, the precise form of which depends on $\vec{r}$ and $\tilde{K}$. The generator $\mathcal{T}$ of the genuine spherical Hecke algebra then must act by a scalar on the compact induction. An irreducible genuine representation $\pi$ is given the parameter $(\vec{r}, \lambda)$ with respect to $\tilde{K}$ if $\pi$ is isomorphic to a quotient of the $\tilde{G}$-module $\text{ind}\tilde{K}\tilde{\sigma}_\vec{r}$.

The parameter attached to $\pi$ does not depend on the choice of maximal compact subgroup within a given conjugacy class, but does depend on the choice of conjugacy class. (This dependence is described for nonsupersingular parameters by Theorem 5.13 below.) A parameter is called supersingular if $\lambda \neq 0$. We show that if $\pi$ has a supersingular parameter with respect to one maximal compact subgroup, then all of its parameters (with respect to any maximal compact subgroup) are supersingular. Moreover, the property of having a supersingular parameter is equivalent to supercuspidality. This is our first main result:

**Theorem (Theorem 5.10).** The smooth, genuine, irreducible, admissible $E$-representations of $\tilde{G}$ fall into two disjoint, nonempty classes:

1. those which have only nonsupersingular parameters,
2. those which have only supersingular parameters.

The representations in the first class are exactly the genuine principal series representations of $\tilde{G}$. All representations in the second class are supercuspidal.

In particular, we show that all genuine principal series representations of $\tilde{G}$ are irreducible. We are able to give a more refined description of these representations, proving (Theorem 6.10) that a representation having a nonsupersingular parameter $(\vec{r}, \lambda)$ with respect to a maximal compact $\tilde{K}$ is isomorphic to the cokernel module $\text{ind}\tilde{K}\tilde{\sigma}_\vec{r}/\mathcal{T}$, where $\mathcal{T}$ is an appropriate Hecke operator, described in detail in (1).

We fix representatives $\tilde{K}$ and $\tilde{K}'$ for the two conjugacy classes of maximal compact subgroups, where $\tilde{K}$ is the preimage in $\tilde{G}$ of the hyperspecial maximal compact of $SL_2(F)$. We denote the cokernel module $\text{ind}\tilde{K}\tilde{\sigma}_\vec{r}/\mathcal{T}$ by $\pi(\vec{r}, \lambda)$, while the analogous module with respect to $\tilde{K}'$ is denoted by $\pi'(\vec{r}, \lambda)$; a cokernel module is called supersingular if $\lambda = 0$ and nonsupersingular otherwise. As part of the proof of Theorem 5.10 we show that no supersingular cokernel module is isomorphic to a nonsupersingular one. The following theorem gives all equivalences between nonsupersingular cokernel modules.

**Theorem (Theorem 6.13).** Given $\vec{r} \in \{0, \ldots, p - 1\}^f$, let $\vec{r}'$ denote any vector in $\{0, \ldots, p - 1\}^f$ such that

$$\sum_{i=0}^{f-1} r'_i p^i \equiv \sum_{i=0}^{f-1} \left( r_i + \frac{p - 1}{2} \right) p^i \pmod{q}$$

(\text{thus } \vec{r}' \text{ is uniquely determined if } \vec{r} \neq \frac{p - 1}{2}, \text{ and } \vec{r}' \in \{0, \frac{p - 1}{2}\} \text{ if } \vec{r} = \frac{p - 1}{2}.)
(1) For any \( \vec{r} \in \{0, \ldots, p-1\}^f \) and \( \lambda \in E^\times \),
\[
\pi(\vec{r}, \lambda) \cong \pi'(\vec{r}, \lambda).
\]
Let \( \vec{p}-\vec{1} \) (resp., \( \vec{p}+\vec{1} \)) denote the vector in \( \{0, \ldots, p-1\}^f \) with all entries equal to \( p-1 \) (resp., to \( p+1 \)). Then in particular,
\[
\pi(\vec{0}, \lambda) \cong \pi(\vec{p}-\vec{1}, \lambda) \cong \pi'(\frac{p-1}{2}, \lambda)
\]
and
\[
\pi'(\vec{0}, \lambda) \cong \pi'(\vec{p}-\vec{1}, \lambda) \cong \pi'(\frac{p-1}{2}, \lambda).
\]

(2) The isomorphisms of (1) are the only equivalences between nonsupersingular cokernel modules.

Thus a genuine principal series representation \( \pi \) is in one of the following three situations: (i) \( \pi \) has a unique parameter with respect to each of \( \tilde{K} \) and \( \tilde{K}' \), or (ii) \( \pi \) has a unique parameter with respect to \( \tilde{K}' \) and has exactly two parameters with respect to \( \tilde{K} \), or (iii) \( \pi \) has a unique parameter with respect to \( \tilde{K} \) and has exactly two parameters with respect to \( \tilde{K}' \). The metaplectic cover is split over \( K \) (resp., over \( K' \)); we denote the image of this splitting in \( \tilde{G} \) by \( K^* \) (resp., by \( (K')^* \)). Then case (ii) occurs exactly when \( \pi \) has a \( K^* \)-fixed vector, and case (iii) occurs exactly when \( \pi \) has a \( (K')^* \)-fixed vector.

Alternatively, one can parametrize genuine principal series representations by genuine characters of the metaplectic torus \( \tilde{T} \). In fact the genuine principal series representations can be parametrized by characters of \( F^\times \), but there is an ambiguity equivalent to the choice of a class in \( (F^\times)/(F^\times)^2) \). This choice appears also in the complex representation theory of \( \tilde{SL}_2(F) \), where it is tied to the choice of a nontrivial additive character \( \psi \) of \( F \). In the complex theory, this character \( \psi \) determines a genuine character \( \chi_\psi \) of \( \tilde{T} \), which then gives a bijection between genuine characters of \( \tilde{T} \) and characters of \( F^\times \). Any other nontrivial additive character of \( F \) is equal to \( \psi_a := (x \mapsto \psi(ax)) \) for some \( a \in F^\times \), and \( \psi \) and \( \psi_a \) give identical parametrizations of the genuine characters of \( \tilde{T} \) if and only if \( a \in (F^\times)^2 \).

The additive character \( \psi \) plays an important role in Waldspurger’s correspondence, and indeed the correspondence depends on the choice of \( \psi \). But a nontrivial additive continuous character of \( F \) does not even exist in the mod \( p \) setting, much less the theory of Whittaker models which motivates the use of \( \psi \) in the complex case. On the other hand, it is still possible (Lemma A.3) to define a mod \( p \) multiplicative genuine character \( \chi_\psi \) of \( \tilde{T} \) using a complex-valued additive character \( \psi \) of \( F \). One can then proceed to parametrize the mod \( p \) genuine characters of \( \tilde{T} \) by mod \( p \) characters of \( F^\times \) using \( \chi_\psi \). The dependence of this parametrization on the choice of \( \psi \) is the same as in the complex case.

We give a complete cross-indexing between the extant parametrizations of a genuine principal series representation, namely the two parametrizations with respect to each of the two conjugacy classes of maximal compact subgroups, and the parametrizations by characters of \( F^\times \) with respect to nontrivial additive \( C^* \)-valued characters. In the following statement, \( (\cdot, \cdot)_F \) denotes the degree-2 Hilbert symbol on \( F^\times \times F^\times \), \( \varpi \) is a choice of uniformizer of \( F \) (fixed throughout the paper), and \( \overline{B} \) is the full preimage in \( \tilde{SL}_2(F) \) of the lower-triangular Borel subgroup of \( SL_2(F) \).

**Corollary** (Corollary 6.13). Let \( \psi : F \to \mathbb{C}^\times \) be a nontrivial additive character of conductor \( m \), let \( \mu : F^\times \to E^\times \) be a smooth multiplicative character, let \( \chi_\psi \) be the genuine \( E \)-valued character of \( \tilde{T} \) defined in (6.2), and let \( \lambda_{\mu, \psi} \) be the element of \( E^\times \) defined in (6.7).
The main points which have to be developed are the following:

1. Technical prerequisites for applying the methods of [13] in the new context of the covering group $p$-representations of $\tilde{\text{SL}}_2$.

2. Focusing work [4], [3] on the modular representations of $\text{GL}_2$ along the lines of Breuil [5], and we plan to do this in future work. However, the case of supercuspidals exists and must appear as quotients of the cokernel modules for $\text{GL}_2$.

3. We define a local system on the Bruhat-Tits tree of $\text{SL}_2$ and use the fact that the remaining dependence on $\psi$ can be encoded by the second components of the parameters with respect to $K$ and $K'$. Therefore, one can index genuine principal series representations by $E^\times$-valued characters of $F^\times$ and express the dependence of this indexing on the choice of a class in $(F^\times)/(F^\times)^2$ without any reference to a $\mathbb{C}^\times$-valued character.

As for supercuspidals (equivalently, supersingular) genuine representations of $\tilde{\text{SL}}_2(F)$, we show that supercuspidals exist and must appear as quotients of the cokernel modules $\pi(\vec{r}, 0)$ or $\pi'(\vec{r}, 0)$. This paper and [23] lay the ground for classifying the irreducible quotients of these cokernel modules for $\tilde{\text{SL}}_2(F)$ along the lines of Breuil [3], and we plan to do this in future work. However, the case of $\text{GL}_2(F)$, $F \neq \mathbb{Q}_p$, leads us to expect difficulties in classifying supercuspidal representations outside the case of $\tilde{\text{SL}}_2(F)$.

1.3. Plan of the paper. The general outline of the paper is modelled on that of Barthel and Livné’s founding work [1, 3] on the mod $p$ representations of $\text{GL}_2(F)$, and of Herzig’s classification of the mod $p$ representations of $\text{GL}_n(F)$ up to supercuspidal representations [13]. Sections 2, 3, and 4 develop the technical prerequisites for applying the methods of [13] in the new context of the covering group $\tilde{\text{SL}}_2(F)$.

The main points which have to be developed are the following:

1. Weight theory ([3]). A key fact of mod $p$ representation theory is that every representation of a pro-$p$ group on a vector space of characteristic $p$ must have a fixed vector. Therefore every mod $p$ representation of a maximal compact $\mathcal{K} \subset \text{SL}_2(F)$ is tamely ramified. A mod $p$ irreducible representation of $\mathcal{K}$ (or equivalently, of $\tilde{\text{SL}}_2(F)$) is called a weight, and there are exactly $q$ inequivalent such representations. We have defined a weight of a maximal compact subgroup $\tilde{\mathcal{K}} \subset \tilde{\text{SL}}_2(F)$ to be an irreducible genuine mod $p$ representation of $\tilde{\mathcal{K}}$. Then the weights of a maximal compact $\tilde{\mathcal{K}} \subset \tilde{\text{SL}}_2(F)$ have a simple classification in terms of the weights of $\mathcal{K}$, and we recover a useful criterion (Proposition 3.2) for irreducibility of a genuine mod $p$ representation of $\tilde{\text{SL}}_2(F)$.

2. Local systems on the tree of $\text{SL}_2(F)$. Given a weight $\sigma$ of a maximal compact subgroup $\mathcal{K} \subset \text{SL}_2(F)$, one has a local system on the Bruhat-Tits tree of $\text{SL}_2(F)$. In the work of Barthel-Livné, the local system defined by $\sigma$ (a weight of the maximal compact $\text{GL}_2(F)$ in their context) provides a useful framework for understanding the spherical Hecke algebra attached to $\sigma$. For each weight of a maximal compact $\tilde{\mathcal{K}}$ we define a local system on the tree of $\text{SL}_2(F)$, and use
this to deduce several properties of the genuine spherical Hecke algebras. (Some of this work is outsourced to the note [22], with the results used here as needed.) We also use the tree to make a comparison (§2.5) of the Cartan and Iwasawa decompositions in $\tilde{SL}_2(F)$, which is necessary for the following point.

3) **Satake transform** (Proposition 4.8). We define a Satake transform from a genuine spherical Hecke algebra of $\tilde{SL}_2(F)$ to a certain genuine spherical Hecke algebra of the torus $\tilde{T}$. (In fact we define the transform for a more general class of genuine spherical Hecke bimodules, which is needed to establish the “change-of-weight” isomorphism discussed in the next point). The definition of the transform is almost identical to that of the map $\tilde{S}$ defined by Herzig in [13], and we have used ideas from [12] to streamline some arguments. We use the comparison of Cartan and Iwasawa decompositions mentioned above in order to give an explicit formula for the transform in terms of Hecke operators for $\tilde{T}$. This formula is then used to find explicit generators of the spherical Hecke algebras of $\tilde{G}$ and to analyze their action on universal modules (§5). Various properties of the mod p Satake transform for unramified reductive groups are discussed in [14] and apply to our transform as well: in particular, the image is not invariant under the Weyl group.

4) **Change of weight** (Lemma 6.11). A comparison of compact and parabolic inductions (Proposition 6.7) suffices to prove irreducibility of all parabolic inductions which do not contain a one-dimensional weight. A similar situation occurs in the classification of mod p representations of $GL_n(F)$, and in [13] Herzig proceeds by defining a “change-of-weight” map. This map is usually an isomorphism and allows one to embed a higher-dimensional weight into the parabolic induction, thus proving it to be irreducible. The obstruction to injectivity of this change-of-weight map is defined in terms of the explicit Satake transform, and in the case of $GL_2$ injectivity fails only when the representation under consideration is the induction of a character of form $\chi \otimes \chi$, $\chi : F^\times \to \bar{F}_p^\times$. In our situation, however, there is no obstruction to injectivity: due to the behavior of the spherical Hecke operators for $\tilde{SL}_2(F)$, the change-of-weight map is *always* an isomorphism. This allows us to prove irreducibility of all parabolic inductions, and explains the symmetry between the nonsupersingular parameters $(\vec{0}, \lambda)$ and $(\bar{p} - 1, \lambda)$.

Once these pieces are in place, the statement of Theorem 6.10 (and its refinements) follow with only minor adaptations to the strategy of [13]. Consequently, the techniques of this paper should generalize well, modulo the above four points, to tame covering groups of higher rank and/or higher degree. It would be especially interesting to generalize (in either direction) the description of Hecke operators in terms of local systems on Bruhat-Tits buildings.

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2. **Preliminaries**

2.1. **Notation and basic definitions.**

2.1.1. **The base field.** Let $F$ be a $p$-adic field with finite residue field $\mathfrak{f}$ of order $q = p^f$. We assume throughout that $p \neq 2$. For $n \geq 2$ we denote the group of $n^{th}$ roots of unity in $F^\times$ by $\mu_n(F)$. Let $\mathcal{O}_F$ denote the ring of integers of $F$ and fix a uniformizer $\varpi$, hence also an identification of $\mathcal{O}_F/(\varpi \mathcal{O}_F)$ with $\mathfrak{f}$. The image of $a \in \mathcal{O}_F$ under the reduction map $\text{red} : \mathcal{O}_F \to \mathfrak{f}$ will be denoted by $\overline{a}$. The valuation $v_F$ on $F$ is normalized so that $v_F(\varpi) = 1$. 
Let \( \nu : \mathfrak{k} \to \mathcal{O}_F \) denote the Teichmüller map. Then the map \( \mathfrak{k}^n \to \mathcal{O}_F \) defined by
\[
(x_0, \ldots, x_n) \mapsto \sum_{i=0}^{n} \nu(x_i) \omega^i,
\]
is an injection which, abusing notation, we will denote again by \( \nu \).

2.1.2. The Hilbert symbol. The symbol \((\cdot, \cdot)_F\) will denote the degree-2 Hilbert symbol on \( F \). We will frequently use the following formula for \((\cdot, \cdot)_F\) which may be found in, e.g., \[20\] Prop. V.3.4. For \( x \in \mathcal{O}_F^\times \), let \( \omega(x) \) denote the unique element of \( \mu_{p-1}(F) \) such that \( x = \omega(x) \cdot (x) \) with \( (x) \in \mathcal{O}_F^\times \) congruent to 1 (mod \( \varpi \mathcal{O}_F \)). Since \( p \neq 2 \), the degree-2 Hilbert symbol is tame, given by
\[
(a, b)_F = \omega \left( (-1)^{\nu_F(a)\nu_F(b)} \frac{b^{\nu_F(a)}}{a^{\nu_F(b)}} \right)^{\frac{q-1}{2}}
\]
for \( a, b \in F^\times \).

**Remark 2.1.** We will make use of the following properties of \((\cdot, \cdot)_F\):

1. \((\cdot, \cdot)_F\) is symmetric and bilinear.
2. \((-a, a)_F = 1 \) for all \( a \in F^\times \). Together with (1), this implies \((a, a)_F = (-a, a)_F = 1 \) for all \( a \in F^\times \).
3. \((\cdot, \cdot)_F\) is unramified, i.e., \((a, b)_F = 1 \) when \( a, b \in \mathcal{O}_F^\times \).
4. \((d, c)_F = 1 \) for all \( d \in F^\times \) if and only if \( c \in (F^\times)^2 \). In particular, \((-1, \varpi)_F = 1 \) when \( q \equiv 1 \) (mod 4).
5. Let \( (x_0, \ldots, x_{n-1}) \in \mathfrak{k}^n \) with \( x_0 \neq 0 \), so that \( \nu(x_0, \ldots, x_{n-1}) \in \mathcal{O}_F^\times \). Then \( \nu(x_0, \ldots, x_{n-1}), \varpi \) is an injection which, abusing notation, we will denote again by \( \nu \).
6. As a consequence of (5), for any \( n \geq 1 \) the set \( \{ \bar{x} \in \mathfrak{k}^n \times \mathfrak{k}^{n-1} : (\nu(\bar{x}), \varpi)_F = 1 \} \) forms a subgroup of index 2 in \( \mathfrak{k}^n \times \mathfrak{k}^{n-1} \), and the set \( \{ x \in \mathcal{O}_F^\times : (x, \varpi)_F = 1 \} \) forms a subgroup of index 2 in \( \mathcal{O}_F^\times \).

2.1.3. The coefficient field. Let \( E \) be an algebraically closed field of characteristic \( p \) which admits an embedding of \( \mathfrak{k} \). We fix (but suppress from the notation) an embedding \( \mathfrak{k} \hookrightarrow E \). Unless specifically mentioned otherwise, all representations should be assumed to have coefficients in \( E \).

In particular, given a vector \( \mathfrak{r}^\flat = (r_0, \ldots, r_{f-1}) \) with \( 0 \leq r_i \leq p-1 \) for each \( 0 \leq i \leq f-1 \), we define a tamely ramified character \( \delta_\mathfrak{r} : \mathcal{O}_F^\times \to E^\times \) as the composition of the character
\[
\mathcal{O}_F^\times \longrightarrow \mathfrak{t}^\times
\]
with the fixed embedding \( \mathfrak{t} \hookrightarrow E \).

2.1.4. The covering group. Let \( G = SL_2(F) \). In this part we summarize the construction of the metaplectic group, which is a certain extension of \( G \) by the square roots of 1. Everything said here is well-known, and much of the summary is specialized (to the case \( r = n = 2, F \) \( p \)-adic, \( p \neq 2 \)) from \S 0.1 of \[15\].

Since \( H^2_{meas}(G, \mu_2(F)) = \mu_2 \) (cf. \[10\], also \[19\]) there is a unique nontrivial topological central extension of \( G \) by \( \mu_2(F) \), called the metaplectic group and denoted here by \( \tilde{G} \). Kubota \[16\] produced a cocycle \( \Delta \) of nontrivial class in \( H^2_{meas}(G, \mu_2(F)) \). We define \( \tilde{G} \) concretely as the set \( G \times \mu_2 \) with multiplication
\[
(g, \zeta) \cdot (g', \zeta') = (gg', \zeta' \Delta(g, g')).
\]
where $\Delta \in H^2_{\text{meas}}(G, \mu_2(F))$ is Kubota’s cocycle, namely

$$
\Delta(g, g') = \left( \frac{X(gg')}{X(g)} \cdot \frac{X(gg')}{X(g')} \right)_F, \quad X \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0. \end{cases}
$$

(This formula for $\Delta$ is simpler than the one appearing in [15], but is equivalent, cf. [15] or [18].) Let $\Pr: \tilde{G} \to G$ denote the projection $(g, \zeta) \mapsto g$. Given a subgroup $H \subset G$, we denote its full preimage in $\tilde{G}$ by $\tilde{H}$.

The covering $\tilde{G} \to G$ may be extended to a nontrivial double cover of $GL_2(F)$ defined by the following cocycle (which we also denote by $\Delta$):

$$
\Delta(g, g') = \left( \frac{X(gg')}{X(g)} \cdot \frac{X(gg')}{X(g')} \right)_F \cdot \left( \det(g), \frac{X(gg')}{X(g)} \right)_F.
$$

The extension $\tilde{GL}_2(F) \to GL_2(F)$ corresponding to $\Delta$ is the unique nontrivial topological extension of $GL_2(F)$ by $\mu_2(F)$ if and only if $F$ contains a primitive fourth root of unity, but all such extensions are indistinguishable in applications for which $\mu_2(F)$ is identified with a subgroup of a field of coefficients containing a primitive fourth root of unity (cf. [15] §0.1, Remarks, p.41-42). Hence there is no loss of generality, from the point of view of representation theory over a sufficiently large field of coefficients, in fixing $\tilde{GL}_2(F)$ as the preferred double cover of $GL_2(F)$.

2.1.5. Splittings and preferred lifts. The covering $\tilde{G} \to^{\Pr} G$ is canonically split over any unipotent subgroup $\mathcal{U} \subset G$ by the map $u \mapsto (u, 1)$, whose image we denote by $\mathcal{U}^*$. We write $U$ (resp., $\overline{U}$) for the upper-triangular (resp., lower-triangular) unipotent subgroup of $\tilde{G}$, and write $u(x)$ (resp., $\overline{u}(x)$) for the element of $U$ (resp., $\overline{U}$) with off-diagonal entry equal to $x \in F$.

For $x \in F^\times$, elements of the form $w(x) := \begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}$ are generated by unipotent elements of $G$, so we may fix a preferred lift of $w(x)$ using the canonical sections of $U$ and $\overline{U}$. This preferred lift is

$$
\tilde{w}(x) := (u(x), 1) \cdot (\overline{u}(-x^{-1}), 1) \cdot (u(x), 1) = (w(x), 1).
$$

The restriction of the covering $\tilde{G} \to^{\Pr} G$ does not split over the diagonal torus $T$. Letting $h(x)$ denote the element of $T$ with diagonal entries $(x, x^{-1})$, we have $\Delta(h(x), h(y)) = (x, y)_F$. Since $h(x) = w(x)w(-1)$, we again choose a preferred section $T \to \tilde{G}$ using the canonical sections of the unipotent subgroups: for $x \in F^\times$, put

$$
\tilde{h}(x) := \tilde{w}(x)\tilde{w}(-1) = (h(x), (\tilde{x}, 1)).
$$

For future convenience, we define a map $\phi: \mathbb{Z} \to \mu_2$ by

$$
\phi(n) = \begin{cases} (-1, \varpi)^n_F & \text{if } n \equiv 1 \text{ or } 2 \pmod{4}, \\ 1 & \text{if } n \equiv 0 \text{ or } 3 \pmod{4}. \end{cases}
$$

In particular, $\phi(n) = 1$ for all $n$ if and only if $q \equiv 1 \pmod{4}$.

(1) $\phi(n)\phi(-n) = (-1, \varpi)^n_F$ for all $n$.

(2) $\phi(n+1)\phi(-n) = (\overline{\varpi})^n_F$ for all $n$.

Let $B = TU$ and $\overline{B} = T\overline{U}$, and let $\tilde{B}$ and $\overline{\tilde{B}}$ denote the respective preimages in $\tilde{G}$. We have $\tilde{B} = \tilde{T}U^*$ and $\overline{\tilde{B}} = \tilde{T}\overline{U}^*$. The covering $\tilde{G} \to^{\Pr} G$ does not split over $B$ or $\overline{B}$.

Remark 2.2. Some easy properties of $\phi$ are collected here.

(1) $\phi(n) = \begin{cases} (-1, \varpi)^n_F & \text{if } n \equiv 1 \text{ or } 2 \pmod{4}, \\ 1 & \text{if } n \equiv 0 \text{ or } 3 \pmod{4}. \end{cases}$

(2) $\phi(n)\phi(-n) = (\overline{\varpi})^n_F$ for all $n$.

(3) $\phi(n+1)\phi(-n) = (-1, \varpi)^n_F$ for all $n$.
Finally we describe splittings over maximal compact subgroups of $\tilde{G}$. Let $K = SL_2(O_F)$, and let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} \in GL_2(F)$ and $K' = \alpha K\alpha^{-1} \subset G$. Then $K$ and $K'$ represent the two conjugacy classes of maximal compact subgroups in $G$, and likewise $\tilde{K}$ and $\tilde{K}'$ represent the two conjugacy classes of maximal compact subgroups in $\tilde{G}$. There exists a unique splitting $K \to \tilde{K}$, whose image we denote by $K^\ast$, of the restriction of $Pr$ to $\tilde{K}$. Concretely, if $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ then this splitting sends $k$ to $(k, \theta(k))$, where

$$\theta(k) := \begin{cases} 1 & \text{if } c = 0 \text{ or } c \in O_F^\times \\ (c, d)_F & \text{otherwise.} \end{cases}$$

From $\theta$ one can also construct a unique splitting of the extension over $K'$, as follows. Let $\tilde{\alpha}$ be any lift of $\alpha$ to $GL_2(F)$. For any $k \in K$, the product $\tilde{\alpha} \cdot (k, \theta(k)) \cdot (\tilde{\alpha})^{-1}$, taken in $GL_2(F)$ using the cocycle $\alpha$, is independent of the choice of lift of $\alpha$ and we may define $$(\alpha k \alpha^{-1}, \theta'(\alpha k \alpha^{-1})) = \tilde{\alpha} \cdot (k, \theta(k)) \cdot (\tilde{\alpha})^{-1}.$$ Then $g \mapsto (g, \theta'(g))$ defines a splitting of $G \to Pr$ over $K'$. Explicitly, for $k' \in K'$ and $k = \alpha^{-1}k'\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$, we have

$$\theta'(k') = \begin{cases} (c, d)_F & \text{if } c \neq 0 \text{ and } c \notin O_F^\times, \\ (d, \varpi)_F & \text{if } c = 0, \\ 1 & \text{if } c \in O_F^\times. \end{cases}$$

**Remark 2.3.** The canonical splitting of the covering over a unipotent subgroup agrees with $(-, \theta(-))$ (resp., with $(-, \theta'(-))$) on the intersection of $K$ (resp., of $K'$) with that unipotent subgroup. However, $\theta$ and $\theta'$ do not agree on all of $K \cap K'$: for example if $x \in O_F^\times$ is an element whose reduction modulo $\varpi$ is a nonsquare in $\mathfrak{t}$, and if $k = h(x) \in K \cap K'$, then $\theta(k) = 1$ while $\theta'(k) = (x, \varpi)_F = -1$. There is no conflict between these statements, since $T \cap K = T \cap K'$ is generated by unipotent elements which lie in $K$ but not in $K'$.

2.1.6. **Conventions on representations.** As already mentioned, all representations should be assumed (unless noted otherwise) to have coefficients in the field $E$ of characteristic $p$. A representation of $G$ is called smooth if the subgroup of $G$ fixing each vector is open, and called genuine if it does not factor through a representation of $G$. In terms of our explicit construction of $\tilde{G}$, a representation $\rho$ is genuine if and only if $\rho((g, \zeta)) = \zeta \rho((g, 1))$ for all $g \in G$, $\zeta \in \mu_p$. We say that a function $f$ on $\tilde{G}$ is genuine if the same relation holds of $f$. We will work only with smooth, genuine representations of $\tilde{G}$. We do not assume that representations are admissible unless this hypothesis is specifically mentioned.

The symbol Ind will denote smooth induction (of a smooth representation of a closed subgroup). We do not normalize smooth induction. The symbol ind will denote compact induction (of a smooth representation of an open subgroup). We use the following standard notation for certain elements of a compact induction: given a smooth representation $\sigma$ of a group $H$ which is an open subgroup of a group $H'$, $g \in H$, and $v \in \sigma$, the symbol $[g, v]$ denotes the element of $\text{ind}_H^{H'} \sigma$ which is supported on $Hg^{-1}$ and which takes the value $[g, v](h) = \sigma(hg) \cdot v$ on $h \in Hg^{-1}$.

2.2. **Commutators in $\tilde{G}$.** For $H$ a subgroup of $\tilde{G}$, let $[H, H]$ denote the subgroup of $\tilde{G}$ generated by commutators in $H$. The following fact is well-known.
Lemma 2.4. (1) $[\widetilde{B}, \widetilde{B}] = U^*$. 
(2) $[\widetilde{B}, \widetilde{B}] = \overline{U}^*$. 
(3) $[\widetilde{G}, \widetilde{G}] = \widetilde{G}$.

Proof. (1) and (2) are straightforward calculations using the cocycle $\widetilde{(2.2)}$. For (3), recall that $[G, G] = G$, so it suffices to show that $(1, -1)$ is also generated by commutators in $\widetilde{G}$. By (1) and (2) the subgroups $U^*$ and $\overline{U}^*$ are generated by commutators in $\widetilde{G}$, which implies that for any $a \in F^\times$ we have $\tilde{h}(a) \in [\widetilde{G}, \widetilde{G}]$.

Pick $u \in O_F^\times$ so that $(u, \varpi)_F = -1$ (such a unit exists by Remark 2.1 (6)). We have

$$\tilde{h}(u) \cdot \tilde{h}(\varpi) = \left(\begin{array}{ll} u\varpi & 0 \\ 0 & (u\varpi)^{-1} \end{array}\right), (1, u\varpi)_F(u, \varpi)_F \in [\widetilde{G}, \widetilde{G}],$$

and also

$$\tilde{h}(u\varpi)^{-1} = \left(\begin{array}{ll} (u\varpi)^{-1} & 0 \\ 0 & u\varpi \end{array}\right), (1, u\varpi)_F(u, \varpi)_F \in [\widetilde{G}, \widetilde{G}],$$

so

$$\left(\begin{array}{ll} u\varpi & 0 \\ 0 & (u\varpi)^{-1} \end{array}\right), (1, u\varpi)_F(u, \varpi)_F \left(\begin{array}{ll} u\varpi & 0 \\ 0 & (u\varpi)^{-1} \end{array}\right), (1, u\varpi)_F = (1, (u, \varpi)_F) = (1, -1) \in [\widetilde{G}, \widetilde{G}].$$

Since $\widetilde{B} = \widetilde{T}U^* = U^* \widetilde{T}$ (resp., $\overline{\widetilde{B}} = \overline{T}U^* = U^* \overline{T}$), we obtain:

Corollary 2.5. The abelianization of $\widetilde{B}$ (resp., of $\overline{\widetilde{B}}$) is $U^*$ (resp., $\overline{U}^*$), and the abelianization of $\widetilde{G}$ is trivial.

2.3. Cartan decompositions of $\widetilde{G}$. Recall that $G$ has the following Cartan decompositions:

(2.6) $G = \coprod_{n \geq 0} K h(\varpi)^n K = \coprod_{n \geq 0} K' h(\varpi)^n K'$.

Lifting these decompositions to $\widetilde{G}$, we have

(2.7) $\widetilde{G} = \coprod_{n \geq 0} \widetilde{K} h(\varpi)^n \widetilde{K} = \coprod_{n \geq 0} \widetilde{K}' h(\varpi)^n \widetilde{K}'$.

It will be useful to refine (2.7) to a disjoint union of $K^*$- (resp., $K'^*$-) double cosets.

Lemma 2.6. $\widetilde{G}$ has the Cartan decompositions

$$\widetilde{G} = \coprod_{n \geq 0, \zeta \in \mu_2} K^* h(\varpi)^n (1, \zeta) K^* = \coprod_{n \geq 0, \zeta \in \mu_2} K'^* h(\varpi)^n (1, \zeta) K'^*.$$  

Proof. We first prove the decomposition with respect to $K^*$. By (2.7) we have

$$\widetilde{G} = \coprod_{n \geq 0} \left(\bigcup_{\zeta \in \mu_2} K^* h(\varpi)^n (1, \zeta) K^* \right).$$

We must show that each union over $\mu_2$ is also disjoint. Since $(1, -1) \notin K^*$, the claim is true for $n = 0$. Suppose that $K^* h(\varpi)^n K^* = K^* h(\varpi)^n (1, -1) K^*$ for some $n > 0$. Then $h(\varpi)^n (1, -1) \in K^* h(\varpi)^n K^*$, so there exist $k_1, k_2 \in K$ such that

(2.8) $(k_1, \theta(k_1)) h(\varpi)^n = h(\varpi)^n (1, -1) (k_2, \theta(k_2))$. 


Write \( k_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( k_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \). The equality \((2.12)\) can be rewritten in terms of these matrix entries as

\[
(2.9) \quad \left( \begin{array}{cc} a\varpi^n & b\varpi^{-n} \\ c\varpi^n & d\varpi^{-n} \end{array} \right) \cdot \theta(k_1) \cdot \phi(n) \cdot \Delta(k_1, h(\varpi)^n) = \left( \begin{array}{cc} e\varpi^n & f\varpi^n \\ g\varpi^{-n} & h\varpi^{-n} \end{array} \right) \cdot \theta(k_2) \cdot (-\phi(n)) \cdot \Delta(h(\varpi)^n, k_2) .
\]

Equality of the \( G \)-parts implies that \( a = e, b = f\varpi^n, c = g\varpi^{-2n}, \) and \( d = h \). Applying the formula for \( \theta \), we have

\[
(2.10) \quad \theta(k_1) = \begin{cases} (c, d)_F & \text{if } 0 < v_F(c) < \infty, \\ 1 & \text{otherwise}, \end{cases}
\]

\[
(2.11) \quad \theta(k_2) = \begin{cases} (g, h)_F = (c\varpi^{2n}, d)_F = (c, d)_F & \text{if } 0 < v_F(g) < \infty, \\ 1 & \text{otherwise}. \end{cases}
\]

Since \( n > 0 \) and \( c = g\varpi^{-2n} \in \mathcal{O}_F \), the entry \( g \) is never a unit, so the second case of \((2.11)\) occurs if and only if \( g = 0 \). If \( g = 0 \), then \( c = 0 \), so \( \theta(k_1) = \theta(k_2) = 1 \). If \( g \neq 0 \), then \( c \neq 0 \), and if also \( c \notin \mathcal{O}_F^\times \) then \( \theta(k_1) = \theta(k_2) = (c, d)_F \).

It remains to show that \( \theta(k_1) = \theta(k_2) \) when \( c \in \mathcal{O}_F^\times \) as well. If \( c \in \mathcal{O}_F^\times \), then \( v(g) = v(c\varpi^{2n}) = 2n \). Considering that \( \det(k_2) = 1 \), we have

\[
v(eh - f\varpi^{2n}) = 0,
\]

so \( e \) and \( h \) are both units, hence \( d = h \) is a unit as well. Now, since the Hilbert symbol on \( F \) is unramified,

\[
\theta(k_2) = (c\varpi^{2n}, d)_F = (c, d)_F = 1,
\]

and \( \theta(k_1) = 1 \) by definition. Thus \( \theta(k_1) = \theta(k_2) \) for all values of \( g \in \mathcal{O}_F \).

Next we show that the values of the cocycle \( \Delta \) agree on the two sides of \((2.9)\). On the right-hand side, we have

\[
(2.12) \quad \Delta(h(\varpi)^n, k_2) = \begin{cases} (\varpi^{-n}, g)_F = (\varpi^n, c\varpi^{2n})_F = (\varpi^n, c)_F & \text{if } c \neq 0, \\ (\varpi^{-n}, h)_F = (\varpi^n, d)_F & \text{if } c = 0, \end{cases}
\]

which is exactly the value of the term \( \Delta(k_1, h(\varpi)^n) \) on the left-hand side.

If the two sides of the equation \((2.9)\) have equal projections to \( G \), then their projections to \( \mu_2 \) are, respectively,

\[
(2.13) \quad \theta(k_1) \cdot \Delta(k_1, h(\varpi)^n)
\]

and

\[
(2.14) \quad -\theta(k_2) \cdot \Delta(h(\varpi)^n, k_2).
\]

Since \( \theta(k_1) = \theta(k_2) \) and \( \Delta(k_1, h(\varpi)^n) = \Delta(h(\varpi)^n, k_2) \), we have

\[
-\theta(k_2) \cdot \Delta(h(\varpi)^n, k_2) = -\theta(k_1) \cdot \phi(n) \cdot \Delta(k_1, h(\varpi)^n),
\]

and so \((2.13) \neq (2.14)\). Hence there do not exist \( k_1, k_2 \) which satisfy the equation \((2.9)\), implying that \( \hat{h}(\varpi)^n(1, -1) \notin K^*\hat{h}(\varpi)^nK^* \). We conclude that \( K^*\hat{h}(\varpi)^n(1, -1)K^* \) and \( K^*\hat{h}(\varpi)^nK^* \) are disjoint for all \( n \geq 0 \).
Now suppose that there exist \( k_1, k_2 \in K \) such that
\[
\hat{h}(\varpi)^n(1, -1) = (ak_1\alpha^{-1}, \theta'(ak_1\alpha^{-1})) \cdot \hat{h}(\varpi)^n \cdot (ak_2\alpha^{-1}, \theta'(ak_2\alpha^{-1})).
\]

Then
\[
(\hat{\alpha})^{-1}\hat{h}(\varpi)^n(1, -1)\hat{\alpha} = (k_1, \theta(k_1)) \cdot (\hat{\alpha})^{-1}\hat{h}(\varpi)^n\hat{\alpha} \cdot (k_2, \theta(k_2)),
\]
or equivalently
\[
\hat{h}(\varpi)^n(1, -(\varpi, \varpi^n)F) = (k_1, \theta(k_1)) \cdot \hat{h}(\varpi)^n(1, (\varpi, \varpi^n)F) \cdot (k_2, \theta(k_2)),
\]
which is impossible since \( K^*\hat{h}(\varpi)^n K^* \cap K^* \hat{h}(\varpi)^n(1, -1)K^* = \emptyset \). Hence \( K^*\hat{h}(\varpi)^n K^* \cap K^* \hat{h}(\varpi)^n(1, -1)K^* = \emptyset \) for all \( n \geq 0 \). \( \square \)

2.4. Systems of \( \tilde{K} \)- and \( K^* \)-coset representatives in \( \tilde{G} \). We next set up a system of representatives for the left \( K \)-cosets in \( G \) and for left \( \tilde{K} \)- and \( K^* \)-cosets in \( \tilde{G} \).

Let \( K = GL_2(O_F) \) and \( Z = Z(GL_2(F)) \). In \([21]\), the following set of representatives for \( GL_2(F)/KZ \) is given:
\[
\{1, \left( \begin{array}{cc} 0 & 1 \\ \varpi & 0 \end{array} \right) \} \cup \left\{ \left( \begin{array}{cc} \varpi^n & -\nu(\bar{x}) \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ \varpi^{n+1} & -\varpi\nu(\bar{x}) \end{array} \right) \right\}_{x \in \mathfrak{t}^n, n \geq 1}
\]

This set of representatives is identified with the set of vertices of the tree \( \mathcal{X} \) of \( PGL_2(F) \), and the set of vertices \( \text{Ver}(\mathcal{X}) \) of \( \mathcal{X} \) inherits a transitive left \( GL_2(F) \)-action from the action of \( GL_2(F) \) on \( GL_2(F)/KZ \). Let \( v_0 \) denote the vertex of \( \mathcal{X} \) corresponding to the trivial coset \( KZ \), and let \( d \) denote the distance function on \( \text{Ver}(\mathcal{X})^2 \). The circle \( C_1 \) of vertices of \( \mathcal{X} \) lying at distance 1 from \( v_0 \) is identified with the set of \( q + 1 \) left \( KZ \)-coset representatives
\[
\left\{ \left( \begin{array}{cc} \varpi & -\nu(x_0) \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ \varpi & 0 \end{array} \right) \right\}_{x_0 \in \mathfrak{t}}
\]

while for \( m \geq 2 \) the circle \( C_m \) of vertices of \( \mathcal{X} \) lying at distance \( m \) from \( v_0 \) is identified with the set of \( q^{m-1}(q+1) \) left coset representatives
\[
\left\{ \left( \begin{array}{cc} \varpi^m & -\nu(x_0, \ldots, x_{m-1}) \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ \varpi^m & -\varpi\nu(x_0, \ldots, x_{m-2}) \end{array} \right) \right\}_{x_i \in \mathfrak{t}}.
\]

The action of \( GL_2(F) \) on \( \text{Ver}(\mathcal{X}) \) restricts to an \( G \)-action, which is no longer transitive but instead has two orbits consisting, respectively, of the sets of vertices lying at even and odd distances from \( v_0 \). The left \( KZ \)-coset corresponding to a vertex of \( C_m \) has a representative lying in \( G \) if and only if \( m \) is even, hence the transitive action of \( G \) on the vertices of \( \bigcup_{m \geq 0} C_m \) is identified with that of \( G \) on \( G/K \).

For \( n \geq 0 \) and \( x_i \in \mathfrak{t} \), let
\[
h_{x_{2n}}^0 = \left( \begin{array}{cc} \nu(x_0, \ldots, x_{2n-1}) & 1 \\ -1 & 0 \end{array} \right), h_{x_{2n}} = \left( \begin{array}{cc} \varpi^{-n}\nu(x_0, \ldots, x_{2n-1}) & \varpi^n \\ -\varpi^{-n} & 0 \end{array} \right),
\]
\[
h_{x_{2n-1}}^1 = \left( \begin{array}{cc} 1 & 0 \\ -\varpi\nu(x_0, \ldots, x_{2n-2}) & 1 \end{array} \right), h_{x_{2n-1}} = \left( \begin{array}{cc} \varpi^{-n} & 0 \\ -\varpi^{-n+1}\nu(x_0, \ldots, x_{2n-2}) & \varpi^n \end{array} \right).
\]
Then for each $n \geq 1$,
\[
\begin{align*}
\tilde{h}^0_{x_{2n}} KZ & = \left( \begin{array}{cc}
\varpi^{2n} & -\nu(x_0, \ldots, x_{2n-1}) \\
0 & 1
\end{array} \right) KZ, \\
\tilde{h}^1_{x_{2n-1}} KZ & = \left( \begin{array}{cc}
0 & 1 \\
\varpi^{n+1} & -\varpi \nu(x_0, \ldots, x_{2n-1})
\end{array} \right) KZ,
\end{align*}
\]
so $S := \{ \tilde{h}^0_{x_{2n}}, \tilde{h}^1_{x_{2n-1}} \}_{n \geq 1} \cup \{1\}$ is a complete set of representatives for $G/K$.

Let $S_0 = \{1\}$ and for $n \geq 1$,
\[
S_{2n} = \{ \tilde{h}^0_{x_{2n}}, \tilde{h}^1_{x_{2n-1}} \}_{x_i \in t}.
\]
Then
\[
C_{2n} = \{ h \cdot v_0 : h \in S_{2n} \}.
\]
For $n \geq 1$ we define the standard vertex $v_{2n}$ to be the vertex of $X$ corresponding to the coset $h^0_{x_{2n}} K$ if $n < 0$, and to the coset $h^1_{x_{2n-1}} K$ if $n > 0$. If $n > 0$ then $h^0_{x_{2n}} K = h(\varpi)^{-n} K$ and $h^1_{x_{2n-1}} K = h(\varpi)^n K$, so $w(1) \cdot v_{-2n} = v_{2n}$, and in fact $S_{2n}$ is a complete set of representatives for $Kh(\varpi)^n K/K$ for each $n \geq 0$.

We next fix a preferred lift of $S$ to $\widetilde{G}$. For $n \geq 1$, let
\[
\begin{align*}
\tilde{h}^0_{x_{2n}} & = \left( \begin{array}{cc}
\nu(x_0, \ldots, x_{2n-1}) & 1 \\
-1 & 0
\end{array} \right), \\
\tilde{h}^1_{x_{2n-1}} & = \left( \begin{array}{cc}
1 \\
-\varpi \nu(x_0, \ldots, x_{2n-2}) & 0
\end{array} \right),
\end{align*}
\]
and put
\[
S^* = \{ \tilde{h}^0_{x_{2n}}, \tilde{h}^1_{x_{2n-1}} \}_{n \geq 1} \cup \{ (1, 1) \},
\]
\[
S^*_{2n} = \{ h \in S^* : Pr(h) \in S_{2n} \}.
\]
Let $\widetilde{S}$ (resp., $\widetilde{S}_{2n}$) denote the full preimage of $S$ (resp., $S_{2n}$) in $\widetilde{G}$. For $h, h' \in S^*$, clearly $hK^* \neq h'K^*$ if $Pr(h) \neq Pr(h')$, and we have checked (Lemma 2.6) that $K^* \tilde{h}(\varpi)^{-n} K^* \cap K^* \tilde{h}(\varpi)^{-n} K^* (1, -1) K^* = \emptyset$ for all $n \geq 0$. Hence $\widetilde{S}$ is a complete set of representatives for $\widetilde{G}/K^*$, while $S^*$ is a complete set of representatives for $\widetilde{G}/K^*$.

For $n \geq 0$, $S^*_{2n}$ is a complete set of representatives for $K^* \tilde{h}(\varpi)^{-n} K^* / K^*$, and since $K^* \tilde{h}(\varpi)^n (1, \phi(n) \phi(-n)) K^* = K^* \tilde{h}(\varpi)^{-n} K^*$, the set
\[
\{(1, \phi(n) \phi(-n))h : h \in \widetilde{S}_{2n}\}
\]
is a complete set of representatives for $K^* \tilde{h}(\varpi)^n K^* / K^*$.

2.5. Comparison of Cartan and Iwasawa decompositions in $\widetilde{G}$. The Iwasawa decomposition $G = BK = TUK$ lifts to a decomposition $\widetilde{G} = \widetilde{B}\widetilde{U}\widetilde{K}$. For later applications, in particular the derivation of an explicit formula for a Satake-type transform (Proposition 4.8), we make a comparison of the Iwasawa decomposition of $G$ with the Cartan decomposition with respect to $K^*$.

Lemma 2.7. (1) Let $n \geq 0$ and $m \in \mathbb{Z}$. Then
\[
\#\{ u \in U/(U \cap K) : h(\varpi)^m uK \subset K\varpi^{-n} K \} = \begin{cases} 
0 & \text{if } |m| > n, \\
1 & \text{if } m = -n, \\
q^{n+m-1}(q-1) & \text{if } -n < m < n.
\end{cases}
\]
Let $n \geq 0$, $m \in \mathbb{Z}$, and $\zeta \in \mu_2$. Then
\[
\# \{ \bar{u} \in U^*/(U^* \cap K^*) : \tilde{h}(\varpi)^m \bar{u}K^* \subset K^* \tilde{h}(\varpi)^n (1, \zeta)K^* \}
\]
is equal to
\[
\begin{cases}
1 & \text{if } m = -n \text{ and } \zeta = 1, \\
q^{n+m-1}(q-1) & \text{if } -n < m \leq 0 \text{ and } 2|n \text{ and } m - n \equiv 0 \text{ or } 3 \pmod{4}, \text{ and } \zeta = 1 \\
q^{n+m-1}(q-1) & \text{if } -n < m \leq 0, \text{ and } 2|n \text{ and } m - n \equiv 1 \text{ or } 2 \pmod{4}, \text{ and } \zeta = (-1, \varpi)_F, \\
q^{n+m-1} \left( \frac{q-1}{2} \right) & \text{if } -n < m \leq 0 \text{ and } 2 \nmid n, \\
q^{n+m-1}(q-1) & \text{if } 0 < m < n \text{ and } 2|m \text{ and } m - n \equiv 0 \text{ or } 3 \pmod{4} \text{ and } \zeta = 1, \\
q^{n+m-1}(q-1) & \text{if } 0 < m < n \text{ and } 2|m \text{ and } m - n \equiv 1 \text{ or } 2 \pmod{4} \text{ and } \zeta = (-1, \varpi)_F, \\
q^{n+m-1} \left( \frac{q-1}{2} \right) & \text{if } 0 < m < n \text{ and } 2 \nmid m, \\
q^n & \text{if } m = n \text{ and } \zeta = (-1, \varpi)_F \\
0 & \text{otherwise}.
\end{cases}
\]

**Remark 2.8.** In this section we work with $\mathbb{Z}$-coefficients rather than with $E$-coefficients as elsewhere in this paper. Although in the proof of Proposition 4.8 we only take advantage of the reduction mod $q$ of the calculations in Lemma 2.7, the lemma may also be useful for determining the image of the Satake transform defined in Proposition 4.8 for $\tilde{G}$ when the coefficient field has characteristic 0 or dividing $q-1$.

**Proof of 2.7** (1) The statement of Lemma 2.7 (1) for $m < -n$ follows from Theorem 2.6.11(3) of [17].

For $-n \leq m$, the proof uses the action of $G$ on the vertices of the tree $\mathcal{X}$ as described in [2.4]. We refer to that section for the definitions of the sets $S$ and $S^*$ of representatives for, respectively, $G/K$ and $\tilde{G}/K^*$.

For $r \geq 1$ and $x_i \in \mathfrak{f}$, let
\[
u(x_0, \ldots, x_{r-1}) = \left( \begin{array}{cc} 1 & \varpi^{-r} \nu(x_0, \ldots, x_{r-1}) \\ 0 & 1 \end{array} \right),
\]
and put
\[
U_r = \left\{ u(x_0, \ldots, x_{r-1}) : (x_0, \ldots, x_{r-1}) \in \mathfrak{f}^r, x_0 \neq 0 \right\} \text{ for } r \geq 1,
\]
\[
U_0 = \{1\}.
\]
Then $\bigcup_{r \geq 0} U_r$ is a complete set of representatives for $U/(U \cap K)$.

Let $r \geq 1$ and consider $u(x_0, \ldots, x_{r-1}) \in U_r$. Since $x_0 \neq 0$, the coefficient $\nu(x_0, \ldots, x_{r-1})$ of $\varpi^{-r}$ is a unit of $\mathcal{O}_F$. We determine the element of $S$ which represents the coset $u(x_0, \ldots, x_{r-1})K$ as follows:
\[
u(x_0, \ldots, x_{r-1})K = \left( \begin{array}{cc} 1 & 0 \\ \varpi^r \nu(x_0, \ldots, x_{r-1})^{-1} & 1 \end{array} \right) h(\varpi)^{-r} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) h(\nu(x_0, \ldots, x_{r-1}))^{-1} \left( \begin{array}{cc} 1 & 0 \\ \varpi^r \nu(x_0, \ldots, x_{r-1})^{-1} & 1 \end{array} \right) K
\]
\[
= \left( \begin{array}{cc} 1 & 0 \\ \varpi^r \nu(x_0, \ldots, x_{r-1})^{-1} & 1 \end{array} \right) h(\varpi)^{-r} K.
\]

For any $a \in \mathcal{O}_F^\times$ and $k \in \mathbb{Z}$,
\[
\left( \begin{array}{cc} 1 & 0 \\ a \varpi^r + k & 1 \end{array} \right) h(\varpi)^{-r} K = h(\varpi)^{-r} \left( \begin{array}{cc} 1 & 0 \\ a \varpi^k & 1 \end{array} \right) K,
\]
\[
\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) h(\varpi)^{-r} K = h(\varpi)^{-r} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) K.
\]
which is equal to $h(\varpi)^{-r}K$ if and only if $k \geq 0$. Hence

$$h(\varpi)^{-r}K = \left( \begin{array}{cc} 1 & 0 \\ \varpi^r \nu(x_0, \ldots, x_{r-1})^{-1} & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \varpi^r \nu(x'_0, \ldots, x'_{r-1})^{-1} & 1 \end{array} \right) h(\varpi)^{-r}K$$

where $\nu(x'_0, \ldots, x'_{r-1}) \equiv \nu(x_0, \ldots, x_{r-1})^{-1} \pmod {\varpi^r \mathcal{O}_F}$.

Noting that

$$h(\varpi)^{-r}K = \left( \begin{array}{cc} 1 & 0 \\ \varpi^r \nu(x'_0, \ldots, x'_{r-1})^{-1} & 1 \end{array} \right) h(\varpi)^{-r}K = h^1_{(0, \ldots, 0, -x'_0, \ldots, -x'_{r-1})} K,$$

we have

$$u_{(x_0, \ldots, x_{r-1})} K = h^1_{(0, \ldots, 0, -x'_0, \ldots, -x'_{r-1})} K \subset K h(\varpi)^{-r} K.$$

Hence $u_{(x_0, \ldots, x_{r-1})} v_0 \in C_{2r}$ and $d(u_{(x_0, \ldots, x_{r-1})} v_0, v_{2r}) = 2r$. As the index $(x_0, \ldots, x_{r-1})$ runs over the $q^{-1}(q - 1)$ elements of $\mathbb{F}^* \times \mathbb{F}^{r-1}$, the vertex $u_{(x_0, \ldots, x_{r-1})} v_0$ runs over the set of $q^{-1}(q - 1)$ vertices of $\mathcal{X}$ lying at distance $2r$ from both $v_0$ and $v_{2r}$. Thus for $n \geq 0$,

$$(2.16) \quad \# \{ u \in U/(U \cap K) : u K \subset K \alpha(\varpi)^{-n} K \} = |U_n| = \begin{cases} q^{n-1}(q - 1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Next we consider $h(\varpi)^m u K$ for $m \in \mathbb{Z}$ and $u \in \bigcup_{n \geq 0} U_r$. This is done in two cases, first for $m \leq 0$ and then for $m > 0$. Each of these two cases divides in turn into an “extremal” case (where $|m| = |n|$) and a “generic” case (where $|m| \leq |n|$).

- Suppose $m \leq 0$. If $u \in U_0$, i.e., $u = 1$, then $h(\varpi)^m u K \subset K h(\varpi)^{-m} K$. Otherwise, $u = u_{(x_0, \ldots, x_{r-1})}$ for some $r \geq 1$, and we calculate:

$$h(\varpi)^m u_{(x_0, \ldots, x_{r-1})} K = h(\varpi)^m h^1_{(0, \ldots, 0, -x'_0, \ldots, -x'_{r-1})} K$$

$$= h(\varpi)^m \left( \begin{array}{cc} 1 & 0 \\ \varpi^r \nu(0, \ldots, 0, -x'_0, \ldots, -x'_{r-1}) & 1 \end{array} \right) h(\varpi)^{-r} K$$

$$= \left( \begin{array}{cc} 1 & 0 \\ \varpi^{r-2m} \nu(0, \ldots, 0, -x'_0, \ldots, -x'_{r-1}) & 1 \end{array} \right) h(\varpi)^{m-r} K$$

$$= h^1_{(0, \ldots, 0, -x'_0, \ldots, -x'_{r-1})} K \subset K h(\varpi)^{m-r} K.$$
So as \( u(x_0, \ldots, x_{r-1}) \) runs over the \( q^{r-1}(q-1) \) elements of \( U_r \), \( h(\pi)^m u(x_0, \ldots, x_{r-1}) v_0 \) runs over the set of \( q^{r-1}(q-1) \) vertices of \( C_{2(r-m)} \) which lie at distance 2r from \( v_{r-2m} \).

Combining the calculation just done for \( 0 \neq -n \leq m \leq 0 \) with the result in (2.16) for the case \( n = m = 0 \), we have for \( m \leq 0 \) and \( n \geq 0 \):

\[
\#\{u \in U/(U \cap K) : h(\pi)^m u K \subset Kh(\pi)^{-n}K\} = |U_{n+m}| = \begin{cases} q^{n+m-1}(q-1) & \text{if } -n < m \leq 0, \\ 1 & \text{if } m = -n. \end{cases}
\]

This completes the proof of Part (1) of Lemma 2.7 for all cases in which \( -n \leq m \leq 0 \).

- Now suppose that \( 0 < m \) and that \( h(\pi)^m u K \subset Kh(\pi)^{-n}K \) for some fixed \( u \in U \). Let \( r \) be the unique integer such that \( u \in U_r \). We divide into two cases depending on \( r \).

  - \( 0 \leq r \leq 2m \). We will show that this condition is equivalent to the condition that \( 0 < m = n \). If \( r = 0 \), then \( h(\pi)^m u K = h(\pi)^m K \subset Kh(\pi)^{-n}K = Kh(\pi)^n K \), and \( 0 < m \) so \( m \) must be equal to \( n \). So suppose that \( 0 < r \leq 2m \) and write \( u = u(x_0, \ldots, x_{r-1}) \) with \( x_0, \ldots, x_{r-1} \in \mathfrak{t}^r \times \mathfrak{t}^{r-1} \). Then

\[
h(\pi)^m u K = h(\pi)^m \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \end{array} \right) K
\]

\[
\left( \begin{array}{c} \omega^m \\ \omega^m \cdot r \nu(x_0, \ldots, x_{r-1}) \\ \omega^m \end{array} \right) K
\]

\[
= \left( \begin{array}{c} \omega^m \\ \omega^m \cdot r \nu(x_0, \ldots, x_{r-1}) \\ \omega^m \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) K
\]

\[
= \left( \begin{array}{c} \omega^m \nu(x_0, \ldots, x_{r-1}) \\ \omega^m \nu(x_0, \ldots, x_{r-1}) \\ \omega^m \nu(x_0, \ldots, x_{r-1}) \\ \omega^m \nu(x_0, \ldots, x_{r-1}) \end{array} \right) K
\]

\[
= h(0, \ldots, 0, -x_0, \ldots, -x_{r-1}) K.
\]

Since \( h(0, \ldots, 0, -x_0, \ldots, -x_{r-1}) K \subset Kh(\pi)^{-m}K \) and on the other hand \( h(\pi)^m u K \subset Kh(\pi)^{-n}K \), we must have \( m = n \).

Suppose that \( u' \in U_{r'} \) also satisfies \( h(\pi)^n u' K \subset Kh(\pi)^{-n}K \). It is impossible that \( r' > 2n \), since in that case after writing \( u' = u(y_0, \ldots, y_{r'-1}) \) with \( y_0 \neq 0 \), we would have

\[
h(\pi)^n u' K = \left( \begin{array}{c} \omega^n \\ 0 \\ \omega^n \cdot r' \nu(y_0, \ldots, y_{r'-1}) \\ \omega^n \end{array} \right) K
\]

\[
= \left( \begin{array}{c} \omega^n \\ 0 \\ \omega^{-(n-r')} \nu(y_0, \ldots, y_{r'-1}) \end{array} \right) \left( \begin{array}{cc} 0 & \omega^{-(n-r')} \\ 1 & 0 \end{array} \right) K
\]

\[
\subset Kh(\pi)^{r'-n}K
\]
and \( r' - n > n \); contradiction. So \( 0 \leq r' \leq 2n \), and the calculations done just above for \( r \) imply that

\[
h(\varpi)^nu'K = \begin{cases} 
    h^{0}(0, \ldots, 0, -y_{0}, \ldots, -y_{r'-1})_{2n}K & \text{if } r' > 0, \\
    h^{0}(0, \ldots, 0)_{2n}K & \text{if } r' = 0.
\end{cases}
\]

We conclude that \( h(\varpi)^nuK \subset Kh(\varpi)^nK \) if and only if \( u \in U_r \) for \( 0 \leq r \leq 2n \). Thus for \( n \geq 0 \),

\[
\#\{ u \in U/(U \cap K) : h(\varpi)^nu \in Kh(\varpi)^{-n}K \} = \sum_{0 \leq r \leq 2n} |U_r| 
\]

\[
= 1 + \sum_{1 \leq r \leq 2n} q^{r-1}(q - 1) 
\]

\[
= q^{2n},
\]

proving Part (1) of Lemma 2.7 in the case \( m = n \).

- \( 0 < 2m < r \). We will show that this condition is equivalent to the condition that \( 0 < m < n = r - m \).

Suppose that \( 0 < 2m < r \). In particular \( r \geq 1 \), so \( u = u_{(x_0, \ldots, x_{r-1})} \) with \( x_0 \neq 0 \). Then

\[
h(\varpi)^muK = \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{pmatrix} \begin{pmatrix} 1 & \varpi^{-r} \nu(x_0, \ldots, x_{r-1}) \\ 0 & 1 \end{pmatrix} K
\]

\[
= \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{pmatrix} \begin{pmatrix} -\varpi^{-r+1} \nu(0, \ldots, 0, -x_0', \ldots, -x_{r-1}') & 0 \\ \varpi^{-r} & \varpi^{-m} \end{pmatrix} K
\]

\[
= \begin{pmatrix} -\varpi^{-m-r+1} \nu(0, \ldots, 0, -x_0', \ldots, -x_{r-1}') & 0 \\ \varpi^{-r} & \varpi^{-m} \end{pmatrix} K
\]

\[
= h^{1}_{(0, \ldots, 0, -x_0', \ldots, -x_{r-1}')} K \subset Kh(\varpi)^{m-r}K.
\]

By assumption \( h(\varpi)^muK \subset Kh(\varpi)^{-n}K \), so \( r = n + m \). And since \( 0 < 2m < r \), we must have \( 0 < m < n \).

Conversely, if \( u \in U_{n+m} \) with \( 0 < m < n \), then the calculation just above with \( r = n + m \) shows that \( h(\varpi)^muK \subset Kh(\varpi)^{-n}K \). Thus if \( 0 < m < n \) and \( u \in \bigcup_{r \geq 0} U_r \), then \( h(\varpi)^muK \subset Kh(\varpi)^{-n}K \) if and only if \( u \in U_{n+m} \), and

\[
\#\{ u \in U/(U \cap K) : h(\varpi)^muK \subset Kh(\varpi)^{-n}K \} = |U_{n+m}| = q^{n+m-1}(q - 1),
\]

while if \( 0 \leq n < m \), then

\[
\#\{ u \in U/(U \cap K) : h(\varpi)^muK \subset Kh(\varpi)^{-n}K \} = 0.
\]
In (2.19) we have the statement of Part (1) of Lemma 2.7 for $0 < m < n$, and (2.20) is the statement of Part (1) for all $n < m$. This completes the proof of Part (1).

(2) To prove Part (2) of Lemma 2.7 we carry out the analogous calculations in the covering group. Part (1) of the lemma has already determined the sum over $\zeta \in \mu_2$ of $\{ \tilde{u} \in U^* / (U^* \cap K^*) : \tilde{h}(\varpi)^m \tilde{u}K^* \subset K^* \tilde{h}(\varpi)^{-n}(1, \zeta)K^* \}$ for each pair of $n$ and $m$. Thus the purpose of these calculations is to determine how, for fixed $n$ and $m$, the $(U^* \cap K^*)$-representatives in $\{ \tilde{u} \in \bigcup_{\zeta \in \mu_2} U^* / (U^* \cap K^*) : \tilde{h}(\varpi)^m \tilde{u}K^* \subset K^* \tilde{h}(\varpi)^{-n}(1, \zeta)K^* \}$ are distributed between the two components of the union.

Let $r \geq 1$ and let $u = u(x_0, \ldots, x_{r-1}) \in U_r$. Consider the coset $\tilde{u}(x_0, \ldots, x_{r-1})K^*$ in $\tilde{G}$. The decomposition of $\tilde{u}(x_0, \ldots, x_{r-1})$ in (2.15) above lifts to the following decomposition of $\tilde{u}(x_0, \ldots, x_{r-1})$ in $O^* \tilde{K}$:

$$
\tilde{u}(x_0, \ldots, x_{r-1})^* = (1, \varpi \nu(x_0, \ldots, x_{r-1}), \varpi) \left( \begin{array}{cc}
\varpi \nu(x_0, \ldots, x_{r-1}) & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
\varpi & 0 \\
0 & \varpi
\end{array} \right) \cdot (1, \varpi \nu(x_0, \ldots, x_{r-1})^{-1}, 0) \left( \begin{array}{cc}
\varpi & 0 \\
0 & \varpi
\end{array} \right) \cdot (1, \varpi \nu(x_0, \ldots, x_{r-1})^{-1}, 0) \left( \begin{array}{cc}
\varpi & 0 \\
0 & \varpi
\end{array} \right) \cdot (1, \varpi \nu(x_0, \ldots, x_{r-1})^{-1}, 0).
$$

The three rightmost factors in the decomposition belong to $K^*$, so (using (2) of 2.2 for the second equality):

$$
\tilde{u}(x_0, \ldots, x_{r-1})^* K^* = (1, \varpi \nu(x_0, \ldots, x_{r-1}), \varpi) \left( \begin{array}{cc}
\varpi \nu(x_0, \ldots, x_{r-1}) & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
\varpi & 0 \\
0 & \varpi
\end{array} \right) \cdot \tilde{h}(\varpi)^{-r} K^*
$$

For any $a \in O^*_f$ and $k \in \mathbb{Z}$,

$$
\left( \begin{array}{cc}
1 & 0 \\
\frac{a \varpi}{\varpi} & 1
\end{array} \right) \tilde{h}(\varpi)^{-r} K^* = \tilde{h}(\varpi)^{-r} \left( \begin{array}{cc}
1 & 0 \\
\frac{a \varpi}{\varpi} & 1
\end{array} \right) K^*
$$

which is equal to $\tilde{h}(\varpi)^{-r} K^*$ if and only if $k \geq 0$, hence

$$
(2.21) \quad \tilde{u}(x_0, \ldots, x_{r-1})^* K^* = (1, \phi(r) (\nu(x_0, \ldots, x_{r-1}), \varpi), \varpi) \tilde{h}(\varpi)^{-r} \left( \begin{array}{cc}
1 & 0 \\
\frac{a \varpi}{\varpi} & 1
\end{array} \right) K^*,
$$

where again $(x'_0, \ldots, x'_{r-1}) \in \mathfrak{k}^* \times \mathfrak{k}^* \times 1$ satisfies $\nu(x'_0, \ldots, x'_{r-1}) \equiv \nu(x_0, \ldots, x_{r-1})^{-1} \pmod{\varpi O^*_f}$. We obtain Part (2) of Lemma 2.7 for $m = 0$ and $n > 0$:

$$
(2.22) \quad \# \{ \tilde{u} \in U^* / (U^* \cap K^*) : \tilde{u}K^* \subset K^* \tilde{h}(\varpi)^{-n}(1, \zeta)K^* \}
$$
\begin{align*}
\# \{ (x_0, \ldots, x_{n-1}) \in \mathfrak{t}^n \times \mathfrak{t}^{n-1} : \langle \nu(x_0, \ldots, x_{n-1}), \varpi^n \rangle_F = \zeta \phi(n) \} \\
&= q^{n-1} \cdot \# \{ x_0 \in \mathfrak{t}^n : \langle \nu(x_0), \varpi^n \rangle_F = \zeta \phi(n) \} \\
&= \begin{cases} 
q^{n-1} \cdot \# \{ x_0 \in \mathfrak{t}^n : \langle \nu(x_0), \varpi^n \rangle_F = \zeta \} & \text{if } n \equiv 0 \text{ or } 3 \pmod{4}, \\
q^{n-1} \cdot \# \{ x_0 \in \mathfrak{t}^n : \langle \nu(x_0), \varpi^n \rangle_F = \zeta(-1, \varpi)_F \} & \text{if } n \equiv 1 \text{ or } 2 \pmod{4}, \\
q^{n-1} (q-1) & \text{if } n \equiv 0 \pmod{4} \text{ and } \zeta = 1, \text{ or if } n \equiv 2 \pmod{4} \text{ and } \zeta = (-1, \varpi)_F, \\
0 & \text{otherwise.}
\end{cases}
\end{align*}

Now assume that \( m < 0, n > 0 \), and that \( u \in \bigcup_{r \geq 0} \mathcal{U}_r \) satisfies \( \hat{h}(\varpi)^n \hat{u} \mathcal{K} \subset \mathcal{K} \hat{h}(\varpi)^{-n} \mathcal{K} \). By the calculation (Part 1) done in \( G \) for this case, we must have \( r = n + m \). There are two possibilities.

- \( m = -n \). If \( m = -n \), then \( r = 0 \) and \( u = 1 \). Thus

\begin{equation}
\# \{ \hat{u} \in U^*/(U^* \cap K^*) : \hat{h}(\varpi)^{-n} \hat{u} K^* \subset K^* \hat{h}(\varpi)^{-n} (1, \zeta) K^* \} = \begin{cases} 
1 & \text{if } \zeta = 1, \\
0 & \text{if } \zeta = -1.
\end{cases}
\end{equation}

This proves Part (2) of the lemma in the case \( m = -n \).

- \( -n < m \leq 0 \). If \( -n < m \leq 0 \), then \( u \in \mathcal{U}_r \) with \( r = n + m \geq 1 \). Write \( u = u(x_0, \ldots, x_{n+m-1}) \) with \( x_0 \neq 0 \), and calculate, using (2.21) for the first step:

\begin{align*}
\hat{h}(\varpi)^m \hat{u} K^* &= (1, \phi(n+m)) \langle \nu(x_0, \ldots, x_{n+m-1}), \varpi^{n+m} \rangle_F h(\varpi)^m \left( \begin{pmatrix} 1 & 0 \\ \varpi^{n+m} & 1 \end{pmatrix} \right) h(\varpi)^{-n-m} K^* \\
&= (1, \phi(-n-m)) \langle \nu(x_0, \ldots, x_{n+m-1}), \varpi^{n+m} \rangle_F \left( \begin{pmatrix} 1 & 0 \\ \varpi^{n+m} & 1 \end{pmatrix} \right) h(\varpi)^{-n-m} K^* \\
&= (1, \phi(-m)) \langle \nu(x_0, \ldots, x_{n+m-1}), \varpi^{n+m} \rangle_F \left( \begin{pmatrix} 1 & 0 \\ \varpi^{m} & 1 \end{pmatrix} \right) h(\varpi)^{-n-m} K^* \\
&= (1, \phi(-m)) \langle \nu(x_0, \ldots, x_{n+m-1}), \varpi^{n+m} \rangle_F \left( \begin{pmatrix} 1 & 0 \\ \varpi^{m} & 1 \end{pmatrix} \right) h(\varpi)^{-n-m} K^* \\
&= (1, \phi(-m)) \langle \nu(x_0, \ldots, x_{n+m-1}), \varpi^{n+m} \rangle_F \left( \begin{pmatrix} 1 & 0 \\ \varpi^{m} & 1 \end{pmatrix} \right) h(\varpi)^{-n-m} K^*.
\end{align*}

Thus we have

\[ \hat{h}(\varpi)^m \hat{u}(x_0, \ldots, x_{n+m-1}) K^* \subset K^* \hat{h}(\varpi)^{-n} (1, \phi(m-n)) \langle \nu(x_0, \ldots, x_{n+m-1}), \varpi^n \rangle_F K^* \]

which together with (2.23) implies that for \( -n < m \leq 0, n \geq 0 \), and \( \zeta \in \mu_2 \),

\[ \# \{ \hat{u} \in U^*/(U^* \cap K^*) : \hat{h}(\varpi)^m \hat{u} K^* \subset K^* \hat{h}(\varpi)^{-n} (1, \zeta) K^* \} \]

\[ = \# \{ (x_0, \ldots, x_{n+m-1}) \in \mathfrak{t}^n \times \mathfrak{t}^{n+m-1} : \langle \nu(x_0, \ldots, x_{n+m-1}), \varpi^n \rangle_F = \zeta \phi(m-n) \} \\
= q^{n+m-1} \cdot \# \{ x_0 \in \mathfrak{t}^n : \langle \nu(x_0), \varpi^n \rangle_F = \zeta \phi(m-n) \} \]

We have

\[ \langle \nu(x_0), \varpi^n \rangle_F = \begin{cases} 1 & \text{if } 2 \mid n \\
\langle \nu(x_0), \varpi \rangle_F & \text{if } 2 \nmid n,
\end{cases} \]

while

\[ \phi(m-n) = \begin{cases} (-1, \varpi)_F & \text{if } m-n \equiv 1 \text{ or } 2 \pmod{4}, \\
1 & \text{if } m-n \equiv 0 \text{ or } 3 \pmod{4}.
\end{cases} \]
So
\[
\#\{x_0 \in \mathbb{T}^\times : (\nu(x_0), \varpi^n) = \zeta \phi(m - n)\} = \begin{cases}
q - 1 & \text{if } m - n \equiv 0 \text{ or } \equiv 3 \pmod{4} \text{ and } 2 \mid n \text{ and } \zeta = 1, \\
n-1 & \text{if } 2 \not\mid n, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus for \(\zeta \in \mu_2\) and \(n, m\) such that \(-n < m \leq 0\),
\[
(2.24) \quad \#\{u \in U^*/(U^* \cap K^*) : \tilde{h}(\varpi)^m \tilde{u} K^* \} \subset K^* \tilde{h}(\varpi)^{-n}(1, \zeta) K^*
\]
\[
= \begin{cases}
q^n m^{-1} (q - 1) & \text{if } 2 \mid n \text{ and } m - n \equiv 0 \text{ or } \equiv 3 \pmod{4} \text{ and } \zeta = 1, \\
n^m m^{-1} (q - 1) & \text{if } 2 \mid n \text{ and } m - n \equiv 1 \text{ or } 2 \pmod{4} \text{ and } \zeta = (-1, \varpi)_F \\
n^m m^{-1} (\frac{\varpi^n}{n}) & \text{if } 2 \not\mid n \\
0 & \text{otherwise}
\end{cases}
\]

To finish the proof of Part (2) for all cases in which \(m \leq 0\), we note that \(\tilde{h}(\varpi)^m \tilde{u} K \cap \tilde{K} \tilde{h}(\varpi)^n \tilde{K} = \emptyset\) if \(|m| > n\) since by Part (1) the projection to \(G\) of the intersection is empty. Collecting this fact together with (2.23) and (2.24) gives Part (2) of Lemma 2.7 for all cases in which \(m \leq 0\).

Now let \(0 < m < n\). We again divide into an “extremal” and a “generic” case.

- \(0 < m = n\). Suppose \(0 < m = n\). It was shown in the proof of Part (1) that when \(n > 0\) and \(u \in U/(U \cap K)\), we have \(h(\varpi)^n u K \subset K h(\varpi)^{-n} K\) if and only if \(u \in U_r\) with \(0 \leq r \leq 2n\). Thus \(u \in U/(U \cap K)\) satisfies \(\tilde{h}(\varpi)^n \tilde{u} K \subset \tilde{K} \tilde{h}(\varpi)^{-n} \tilde{K}\) if and only if \(u \in U_r\) for some \(0 \leq r \leq 2n\).

If \(u = 1\), we have
\[
\tilde{h}(\varpi)^n K^* = \tilde{h}(\varpi)^n \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) K^*
\]
\[
= \left( \begin{array}{cc} 0 & \varpi^n \\ -\varpi^{-n} & 0 \end{array} \right), \phi(n)(-1, \varpi)_F \right) \right) K^*,
\]
while
\[
\tilde{h}_0^n K^* = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \tilde{h}(\varpi)^{-n} K^*
\]
\[
= \left( \begin{array}{cc} 0 & \varpi^n \\ -\varpi^{-n} & 0 \end{array} \right), \phi(-n)(-1, \varpi)_F \right) \right) K^*,
\]
so
\[
(2.25) \quad \tilde{h}(\varpi)^n K^* = (1, \phi(n)\phi(-n)) \tilde{h}_0^n K^* \subset K^* \tilde{h}(\varpi)^{-n}(1, (-1, \varpi)_F) K^*.
\]
Next let \( u = u(x_0, \ldots, x_{r-1}) \in U_r \) with \( 0 < r \leq 2n \). Then

\[
\hat{h}(\varpi)r\tilde{u}(x_0, \ldots, x_{r-1})K^* = \left( \begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{pmatrix}, \phi(n) \right) \left( \begin{pmatrix} 1 & \varpi^{-r}v(x_0, \ldots, x_{r-1}) \\ 0 & 1 \end{pmatrix} \right), 1 \right) K^*
\]

\[
= \left( \begin{pmatrix} \varpi^n & -\varpi^{-r}v(x_0, \ldots, x_{r-1}) \\ 0 & \varpi^{-n} \end{pmatrix}, \phi(n) \right) \left( \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \right), 1 \right) K^*
\]

\[
= \left( \begin{pmatrix} -\varpi^{-n}v(x_0, \ldots, x_{r-1}) & \varpi^n \\ 0 & -\varpi^{-n} \end{pmatrix}, \phi(n)(-1, \varpi^{-n})_F \right) K^*
\]

On the other hand,

\[
\hat{h}(\varpi)0_{n-r}K^* = \left( \begin{pmatrix} \nu(0, \ldots, 0, -x_0, \ldots, -x_{r-1}) & 1 \\ 2n-r & -1 \end{pmatrix} \right), 1 \right) \left( \begin{pmatrix} \varpi^{-n} & 0 \\ 0 & \varpi^n \end{pmatrix}, \phi(-n) \right) K^*
\]

\[
= \left( \begin{pmatrix} \varpi^{-n}v(0, \ldots, 0, -x_0, \ldots, -x_{r-1}) & \varpi^n \\ 2n-r & -1 \end{pmatrix} \right), \phi(-n)(-1, \varpi^{-n})_F \right) K^*,
\]

so

\[
(2.26) \quad \hat{h}(\varpi)r\tilde{u}(x_0, \ldots, x_{r-1})K^* = (1, \phi(n)\phi(-n)) \cdot \hat{h}(\varpi)0_{n-r}K^* \subset K^* h(\varpi)^{-n}(1, (-1, \varpi)_F) K^*.
\]

Combining the results \(2.25\) for \( r = 0 \) and \(2.26\) for \( 0 < r \leq 2n \), we have

\[
(2.27) \quad \# \{ u \in U^*/(U^* \cap K^*) : \hat{h}(\varpi)^r\tilde{u}K^* \subset K^* h(\varpi)^{-n}(1, \zeta) K^* \} = \begin{cases} \sum_{r=0}^{2n} |U_r| = q^{2n} & \text{if } \zeta = (-1, \varpi)_F, \\ 0 & \text{if } \zeta = (-1, \varpi)^*_F. \end{cases}
\]

- \( 0 < m < n \). Suppose \( 0 < m < n \) and assume that \( u \in \bigcup_{r \geq 0} U_r \) satisfies \( \hat{h}(\varpi)^m\tilde{u}K \subset \bar{K} h(\varpi)^{-n}\bar{K} \).

By the calculation in the analogous case for \( G \), we know that then \( u \in U_{n+m} \). Write \( u = u(x_0, \ldots, x_{n+m-1}) \) with \( x_0 \neq 0 \) and consider

\[
\hat{h}(\varpi)^m\tilde{u}K^* = \left( \begin{pmatrix} \varpi^m & 0 \\ 0 & \varpi^{-m} \end{pmatrix}, \phi(m) \right) \left( \begin{pmatrix} 1 & \varpi^{-m-n}v(x_0, \ldots, x_{n+m-1}) \\ 0 & 1 \end{pmatrix} \right), 1 \right) K^*
\]

\[
= \left( \begin{pmatrix} \varpi^m & \varpi^{-n}v(x_0, \ldots, x_{n+m-1}) \\ 0 & \varpi^{-m-n} \end{pmatrix}, \phi(m) \right) K^*
\]

\[
= (1, \phi(m)\phi(-n)) \cdot \hat{h}(\varpi)^m\tilde{u}K^* \subset \bar{K} h(\varpi)^{-n}\bar{K}.
\]
where $\nu(x_0', \ldots, x_{n+m-1}) \equiv \nu(x_0, \ldots, x_{n+m-1})^{-1} \pmod{\mathbb{Z}^{n+m}}$.

On the other hand, the coset $\tilde{h}_0(0, \ldots, 0, -x_0', \ldots, -x_{n+m-1})_{n-m-1} K^*$ has the following representative:

$$
\tilde{h}_0(0, \ldots, 0, -x_0', \ldots, -x_{n+m-1})_{n-m-1} K^* = \left(\begin{array}{cc} -\nu(0, \ldots, 0, -x_0', \ldots, -x_{n+m-1})^{-1} & 1 \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} \nu & 0 \\ 0 & \nu \end{array}\right) \phi(-n) K^*
$$

Thus for $0 < m < n$ and $u \in U_{n+m}$,

$$
\tilde{h}(\varpi)^m \tilde{u} K^* = (1, \phi(m)\phi(-n)(\nu(x_0, \ldots, x_{n+m-1}), \varpi^m)_F) \cdot \tilde{h}_0(0, \ldots, 0, -x_0', \ldots, -x_{n+m-1})_{n-m-1} K^*,
$$

so

$$
\tilde{h}(\varpi)^m \tilde{u} K^* \subset K^* \tilde{h}(\varpi)^{-n} (1, \phi(m)\phi(-n)(\nu(x_0, \ldots, x_{n+m-1}), \varpi^m)_F) K^*.
$$

giving the formula

$$
\#\{u \in U^*(U^* \cap K^*) : \tilde{h}(\varpi)^m \tilde{u} K^* \subset K^* \tilde{h}(\varpi)^{-n}(1, \zeta) K^*\}
$$

(2.28)

$$
= \#\{(x_0, \ldots, x_{n+m-1}) \in G^k \times \mathfrak{t}^{n+m-1} : (\nu(x_0, \ldots, x_{n+m-1}), \varpi^m)_F = \zeta \phi(m)\phi(-n)\}
$$

$$
= \#\{(x_0, \ldots, x_{n+m-1}) \in G^k \times \mathfrak{t}^{n+m-1} : (\nu(x_0, \ldots, x_{n+m-1}), \varpi^m)_F = \zeta \phi(m-n)(\varpi^m, \varpi^m)_F\}
$$

$$
= q^{n+m-1} \cdot \#\{(x_0, \ldots, x_{n+m-1}) \in G^k \times \mathfrak{t}^{n+m-1} : (\nu(x_0), \varpi^m)_F = \zeta \phi(m-n)(\varpi^m, \varpi^m)_F\}
$$

$$
= \begin{cases} 
q^{n+m-1}(q-1) & \text{if } 2 \text{ divides } m + n \equiv 0 \text{ or } 3 \pmod{4} \text{ and } \zeta \equiv 1, \\
q^{n+m-1}(q-1) & \text{if } 2 \text{ divides } m - n \equiv 1 \text{ or } 3 \pmod{4} \text{ and } \zeta \equiv (-1, \varpi)_F, \\
q^{n+m-1} & \text{otherwise}.
\end{cases}
$$

This concludes the proof of Lemma 2.28.

In applications (e.g. Proposition 3.12) it will also be convenient to have a comparison of the Cartan decomposition $\tilde{G} = \Pi_{n \geq 0} K^* \tilde{h}(\varpi)^{-n} K^*$ with the “opposite” Iwasawa decomposition $\tilde{G} = \overline{B} K^*$.

The transpose operation on $G$ gives a bijection between the sets $\{u \in U/(U^* \cap K) : h(\varpi)^m u K \subset K \tilde{h}(\varpi)^{-n} K\}$ and $\{\tilde{u} \in (\overline{U} \cap K) \setminus \overline{U} : K \tilde{h}(\varpi)^m \subset K h(\varpi)^{-n} K\}$. The following lemma gives the analogous statement for $\tilde{G}$.

**Lemma 2.9.** Let $n \geq 0$, $m \in \mathbb{Z}$, and $\zeta \in \mu_2$. Then

$$
\#\{\tilde{u} \in (\overline{U} \cap K^*) \setminus \overline{U} : K^* \tilde{u} h(\varpi)^m \subset K^* \tilde{h}(\varpi)^{-n}(1, \zeta) K^*\}
$$
is equal to
\[
\begin{cases}
1 & \text{if } m = -n \text{ and } \zeta = 1, \\
q^{n+m-1}(q-1) & \text{if } -n < m \leq 0 \text{ and } 2 | n \text{ and } m - n \equiv 0 \pmod{3} \quad \text{(mod 4), and } \zeta = (-1, \varpi^n)_F, \\
q^{n+m-1}(q-1) & \text{if } -n < m \leq 0, \text{ and } 2 | n \text{ and } m - n \equiv 1 \text{ or } 2 \quad \text{(mod 4), and } \zeta = (-1, \varpi^{m+1})_F, \\
q^{n+m-1}(q-1) & \text{if } -n < m \leq 0 \text{ and } 2 \nmid n, \\
q^{n+m-1}(q-1) & \text{if } 0 < m < n \text{ and } 2 | m, \text{ and } m - n \equiv 0 \text{ or } 3 \quad \text{(mod 4), and } \zeta = (-1, \varpi^n)_F, \\
q^{n+m-1}(q-1) & \text{if } 0 < m < n \text{ and } 2 | m, \text{ and } m - n \equiv 1 \text{ or } 2 \quad \text{(mod 4), and } \zeta = (-1, \varpi^{n+1})_F, \\
q^{2n} & \text{if } m = n \text{ and } \zeta = (-1, \varpi^n)_F, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof of Lemma 2.25. Since \(\tilde{G}\) is generated by \(U^*\) and \(\overline{U}\) (Lemma 2.4 (3)), we may define a transpose operation on \(\tilde{G}\) by setting \(t(u, 1) := (u^t, 1)\) and \(t(\tilde{u}, 1) := (\tilde{u}^t, 1)\) for \(u \in U\) and \(\tilde{u} \in \overline{U}\), and then \(t(g, \zeta) := t(u_1, 1) \cdots t(u_r, 1)\) where \(u_i \in U \cup \overline{U}\) and \((g, \zeta) = \prod_{i=1}^{\infty} t(u_i, 1)\). Then \(t(g_1 g_2) = t(g_2^t g_1)\) for all \(g_1 \text{ and } g_2 \in \tilde{G}\), \(t k \in K^*\) for all \(k \in K^*\), and \(t h(a) = h(a)(1, -1, a)_F\) for all \(a \in F^\times\).

Putting \(\overline{U}_r = \{ u : u \in U_r \}\) for \(r \geq 0\), the set \(\bigcup_{r \geq 0} \overline{U}_r\) is a complete set of representatives for \((\overline{U} \cap K^*) \setminus \overline{U}\). We have \(\tilde{u} \in U_r\) and \(h(\varpi^n) \tilde{u} K^* \subset K^* h(\varpi)^{-n}(1, \zeta) K^*\) if and only if \(\tilde{u} \in U_r\) and \(K^* (\tilde{u}^t) h(\varpi)^m (1, -1, \varpi^n)_F \subset K^* h(\varpi)^{-n} (1, -1, \varpi^n)_F K^*\). Hence the transpose operation on \(\tilde{G}\) gives a bijection between \(\{ \tilde{u} \in U^*/(U \cap K^*) : h(\varpi)^m \tilde{u} K^* \subset K^* h(\varpi)^{-n} K^* \}\) and \(\{ \tilde{u} \in (\overline{U} \cap K^*) \setminus \overline{U} : K^* \tilde{u} h(\varpi)^m \subset K^* h(\varpi)^{-n} (1, -1, \varpi^{n+m})_F K^*\}\). Lemma 2.9 then follows from Lemma 2.7 (2) by a simple calculation. \(\square\)

3. Weights of \(\tilde{K}\). A weight of \(\tilde{K}\) is a smooth, genuine, irreducible representation of \(\tilde{K}\) on an \(E\)-vector space. The extension which defines the metaplectic cover of \(G\) is split over \(K\), so the classification of weights of \(\tilde{K}\) reduces to the classification of weights of \(K\), i.e., of smooth irreducible \(E\)-representations of \(K\). Recall that \(K^*\) denotes the image of the map \(k \mapsto (k, \theta(k))\) (defined in 2.5) which uniquely splits the extension over \(K\), that \(\epsilon\) denotes the embedding \(\mu_2(F) \to E^\times\), and that \(f\) is the residual degree of \(F/\mathbb{Q}_p\).

Proposition 3.1. For each vector \(\bar{r} = (r_0, \ldots, r_{f-1}) \in \{0, \ldots, p-1\}^f\), let \(\sigma_{\bar{r}}\) denote the inflation to \(K\) of the following \(E\)-representation of \(G(\mathfrak{t})\), likewise denoted by \(\sigma_{\bar{r}}\):
\[
\sigma_{\bar{r}} = \bigotimes_{i=0}^{f-1} (\text{Sym}^{r_i} E^2)^{F_{r_i}}
\]
where \(F_r\) is the Frobenius map \(x \mapsto x^p\).

(1) Any smooth irreducible representation of \(K^*\) is isomorphic to \(\sigma_{\bar{r}}\) for exactly one \(\bar{r} \in \{0, \ldots, p-1\}^f\), and the weights for \(\tilde{K}\) are exactly the representations \(\sigma_{\bar{r}} := \sigma_{\bar{r}} \otimes \epsilon, \bar{r} \in \{0, \ldots, p-1\}^f\).

(2) Let \(\rho \neq 0\) be a smooth, genuine representation of \(\tilde{G}\). Then for any weight \(\tilde{\sigma}_{\bar{r}}\) of \(\tilde{K}\),
\[
\dim E \text{Hom}_{\tilde{K}}(\tilde{\sigma}_{\bar{r}}, \rho|_{\tilde{K}}) = \dim_E \text{Hom}_K(\sigma_{\bar{r}}, \rho|_{K^*})
\]
(where $\sigma_\mathcal{F}$ is viewed as a representation of $K^*$ on the right-hand side). In particular, there is a weight $\tilde{\sigma}_\mathcal{F}$ of $\tilde{K}$ such that
\[
\dim_E \text{Hom}_{\tilde{K}}(\tilde{\sigma}_\mathcal{F}, \rho|_{\tilde{K}}) \geq 1.
\]

Proof of Proposition 3.2. (1) Let $\pi$ be a weight of $\tilde{K} \cong K^* \times \mu_2$. Then $\pi|_{K^*}$ is a smooth irreducible representation of $K^* \cong K$. By the classification of weights of $K$ (a reference is \[1\] Lemme 3.5.1), we have $\pi|_{K^* \times \{1\}} \cong \sigma_\mathcal{F}$ for a unique $\tilde{r} \in \{0, \ldots, p - 1\}$. Since $\pi$ is genuine, the restriction $\pi|_{\{1\} \times \mu_2}$ is nontrivial. Thus $\pi \cong \sigma_\mathcal{F} \otimes \epsilon$ for a unique $r \in \{0, \ldots, p - 1\}$. Conversely, given $\tilde{r} \in \{0, \ldots, p - 1\}$, the inflation of $\sigma_\mathcal{F}$ to $K^*$ is smooth and irreducible. The product $\sigma_\mathcal{F} \otimes \epsilon$ is likewise smooth and irreducible as a representation of $\tilde{K}$, and moreover is genuine, so $\tilde{\sigma}_\mathcal{F} = \sigma_\mathcal{F} \otimes \epsilon$ is a weight of $\tilde{K}$.

(2) Every $E$-representation of $K$ contains a weight of $K$, so, identifying $K$ with $K^*$,
\[
\dim_E \text{Hom}_{K^*}(\sigma_\mathcal{F}, \rho|_{K^*}) \geq 1
\]
for some weight $\sigma_\mathcal{F}$ of $K$. A map of $K^*$-representations has a unique extension to a map of genuine $\tilde{K}$ representations, so
\[
\dim_E \text{Hom}_{\tilde{K}}(\tilde{\sigma}_\mathcal{F}, \rho|_{\tilde{K}}) \geq 1.
\]

From now on we parametrize the weights of $\tilde{K}$ by $\tilde{r} \in \{0, \ldots, p - 1\}$. The underlying vector space of $\tilde{\sigma}_\mathcal{F}$ (equivalently, of $\sigma_\mathcal{F}$) will be denoted by $V_{\tilde{r}}$.

Proposition 3.2(2) provides a convenient irreducibility criterion. The following is a standard argument in mod $p$ representation theory:

**Proposition 3.2.** Suppose that $\rho \neq 0$ is a smooth, genuine representation of $\tilde{G}$, and let $\tilde{\sigma}_\mathcal{F}$ be a weight of $\tilde{K}$ contained in $\rho$. If
\[
\dim_E \text{Hom}_{\tilde{K}}(\tilde{\sigma}_\mathcal{F}, \rho|_{\tilde{K}}) = \begin{cases} 1 & \text{if } \tilde{s} = \tilde{r}, \\ 0 & \text{otherwise}, \end{cases}
\]
then the image of the inclusion $\tilde{\sigma}_\mathcal{F} \hookrightarrow \rho$ generates an irreducible $\tilde{G}$-subrepresentation of $\rho$.

Proof of Proposition 3.2. Let $\tilde{G} \cdot \tilde{\sigma}_\mathcal{F}$ denote the $\tilde{G}$-subrepresentation of $\rho$ generated by the image of $\tilde{\sigma}_\mathcal{F}$. Suppose that $\pi \subset \tilde{G} \cdot \tilde{\sigma}_\mathcal{F}$ is again a $\tilde{G}$-subrepresentation. Then $\pi$ is a smooth genuine $\tilde{G}$-representation, so by Proposition 3.1 $\pi$ contains a $\tilde{K}$-weight. This $\tilde{K}$-weight must be $\tilde{\sigma}_\mathcal{F}$, since it is contained in $\rho$. Since also $\dim_E \text{Hom}_{\tilde{K}}(\tilde{\sigma}_\mathcal{F}, \rho|_{\tilde{K}}) = 1$, the $\tilde{G}$-subrepresentation of $\pi$ generated by $\tilde{\sigma}_\mathcal{F}$ is equal to $\tilde{G} \cdot \tilde{\sigma}_\mathcal{F}$. Thus $\tilde{G} \cdot \tilde{\sigma}_\mathcal{F}$ is irreducible. \qed

3.2. Weights of $\tilde{K}'$. Recall that $\tilde{K}' = \hat{\alpha} \tilde{K} \hat{\alpha}^{-1}$ where $\hat{\alpha}$ is a lift to $\tilde{GL}_2(F)$ of $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$, and that the extension defining $\tilde{G}$ is split over $K'$. We define a weight of $\tilde{K}'$ to be a smooth, irreducible, genuine representation of $\tilde{K}'$. The classification of weights of $\tilde{K}'$ (in Lemma 3.3 below) will reduce to that of the weights of $\tilde{K}$, following the argument used in \[1\] Cor. 3.5.2 to classify weights of $K'$ in terms of weights of $K$.

Given a representation $\pi$ of a subgroup $H$ of $\tilde{GL}_2(F)$ and an element $g \in \tilde{GL}_2(F)$, we define a conjugate representation $\pi^g$ of $gHg^{-1}$ by
\[
\pi^g(h) = \pi(g^{-1}hg)
\]
Lemma 3.3. Let \( \tilde{\alpha} \) be any lift of \( \alpha = \left( \begin{array}{cc} 1 & 0 \\ 0 & \varpi \end{array} \right) \) to \( \widetilde{GL}_2(F) \).

1. Any weight of \( \tilde{K}' \) is isomorphic to \( (\tilde{\sigma})^{\tilde{\alpha}} \) for a unique vector \( \tilde{r} \in \{0, \ldots, p-1\}^f \). (The conjugate representation does not depend on the choice of lift of \( \alpha \).)

2. Let \( \rho \neq 0 \) be a smooth, genuine representation of \( G \). Then for any weight \( \tilde{\sigma}^{\tilde{\alpha}} \) of \( \tilde{K}' \),

\[
\text{Hom}_{\tilde{K}'}(\tilde{\sigma}^{\tilde{\alpha}}, \rho|_{\tilde{K}'}) = \text{Hom}_{\tilde{K}} \left( \tilde{\sigma}, \rho^{(\tilde{\alpha})^{-1}|_{\tilde{K}}} \right).
\]

In particular, there is at least one weight \( \tilde{\sigma}^{\tilde{\alpha}} \) of \( \tilde{K}' \) such that

\[
\dim F \text{Hom}_{\tilde{K}'}(\tilde{\sigma}^{\tilde{\alpha}}, \rho|_{\tilde{K}'}) \geq 1.
\]

Proof of Lemma 3.3

1. Let \( \pi \) be a weight of \( \tilde{K}' \). Then \( \pi^{(\tilde{\alpha})^{-1}} \) is a weight of \( \tilde{K} \), so there is a unique vector \( \tilde{r} \in \{0, \ldots, p-1\}^f \) such that \( \pi^{(\tilde{\alpha})^{-1}} \cong \tilde{\sigma} \). Then \( \pi = (\pi^{(\tilde{\alpha})^{-1}})^{\tilde{\alpha}} \cong (\tilde{\sigma})^{\tilde{\alpha}} \).

2. From the definition of the conjugate representations it follows that each element of \( \text{Hom}_{\tilde{K}'}(\tilde{\sigma}^{\tilde{\alpha}}, \rho|_{\tilde{K}'}) \) belongs to \( \text{Hom}_{\tilde{K}} \left( \tilde{\sigma}, \rho^{(\tilde{\alpha})^{-1}|_{\tilde{K}}} \right) \), and vice versa. By Proposition 3.1 (2) the \( \widetilde{G} \)-representation \( \rho^{(\tilde{\alpha})^{-1}} \) contains some weight \( \tilde{\sigma} \) of \( \tilde{K} \), so \( \rho \) contains the weight \( \tilde{\sigma}^{\tilde{\alpha}} \) of \( \tilde{K}' \).

\( \square \)

3.3. \( \mathcal{U}(\mathfrak{t}) \)-coinvariants of \( \tilde{K} \)- and \( \tilde{K}' \)-weights. Let \( U(\mathfrak{t}) \) denote the upper-triangular unipotent subgroup of \( G(\mathfrak{t}) \), and let \( \tilde{\sigma} \) be a weight of \( \tilde{K} \). The restriction of \( \tilde{\sigma} \) to \( U^* \cap K^* \) factors through the composition

\[
U^* \cap K^* \to U \cap K \to \text{red} \ U(\mathfrak{t})
\]

so we may view \( \tilde{\sigma}|_{U(\cap K)} \) as a representation of \( U(\mathfrak{t}) \). The same is true if \( U \) is replaced with \( \overline{U} \), so we may likewise view \( \tilde{\sigma}|_{\overline{U}(\cap K)} \) as a representation of \( \overline{U}(\mathfrak{t}) \). Let \( (\tilde{\sigma})^{U(\mathfrak{t})} \) denote the subspace of \( U(\mathfrak{t}) \)-invariants in \( V_\tilde{\sigma} \), let \( (\tilde{\sigma})^{\overline{U}(\mathfrak{t})} \) denote the \( \overline{U}(\mathfrak{t}) \)-coinvariants of \( \tilde{\sigma} \), and let \( p^{U(\mathfrak{t})} \) denote the projection \( V_\tilde{\sigma} \to (\tilde{\sigma})^{U(\mathfrak{t})} \).

As an \( E \)-vector space, \( (\tilde{\sigma})^{U(\mathfrak{t})} \) is equal to the invariants of \( V_{\tilde{\sigma}} \) by the action of \( U(\mathfrak{t}) \) under \( \sigma_{\tilde{\sigma}} \) via inflation through \( \text{red} \) alone; this \( U(\mathfrak{t}) \)-invariant space is well-known to be the one-dimensional highest weight space of \( \sigma_{\tilde{\sigma}} \). Furthermore, \( (\sigma_{\tilde{\sigma}})\overline{U}(\mathfrak{t}) \) is stable by \( T \cap K \), and is isomorphic to (the inflation to \( T \cap K \) of) the character \( \delta_{\tilde{\sigma}} \) of \( O_{\tilde{K}}^\times \). The \( \overline{U}(\mathfrak{t}) \)-coinvariants \( (\sigma_{\tilde{\sigma}})\overline{U}(\mathfrak{t}) \) likewise carry a \( T \cap K \)-representation isomorphic to \( \delta_{\tilde{\sigma}} \). Let \( \tilde{\delta}_{\tilde{\sigma}} \) denote the genuine character \( \delta_{\tilde{\sigma}} \otimes \epsilon \) of \( T \cap \tilde{K} \). The following lemma is a consequence of the preceding comment together with well-known facts for weights of \( K \) (cf. [14] Lemma 2.5 for (2) in the general setting of mod \( p \) representations of unramified reductive groups).

Lemma 3.4.

1. \( (\tilde{\sigma})^{U(\mathfrak{t})} \) is generated as an \( E \)-vector space by the highest-weight vector of \( \sigma_{\tilde{\sigma}} \), and

\[
(\tilde{\sigma})^{U(\mathfrak{t})} \cong \delta_{\tilde{\sigma}}
\]

as \( \tilde{T} \cap \tilde{K} \)-representations.

2. The composition

\[
j_{\tilde{\sigma}} : (\tilde{\sigma})^{U(\mathfrak{t})} \hookrightarrow (\tilde{\sigma})^{\overline{U}(\mathfrak{t})} \xrightarrow{j_{\tilde{\sigma}}} (\tilde{\sigma})_{\overline{U}(\mathfrak{t})}
\]

is an isomorphism of \( \tilde{T} \cap \tilde{K} \)-representations.

Since \( \delta_{\tilde{\sigma}} \cong \delta_{\tilde{\sigma}} \) if and only if \( \sum_{i=0}^{f-1} r_i p^i \equiv \sum_{i=0}^{f-1} s_i p^i \) (mod \( q \)), we obtain:
Corollary 3.5. Let \( \hat{\sigma}_F \) and \( \hat{\sigma}_x \) be two weights of \( \tilde{K} \). There exists a \( \tilde{T} \cap \tilde{K} \)-linear isomorphism

\[
(\hat{\sigma}_F)_{\tilde{T}(t)} \rightarrow (\hat{\sigma}_x)_{\tilde{T}(t)}
\]

if and only if \( \vec{r} = \vec{s} \) or \( \{\vec{r}, \vec{s}\} = \{0, p-1\} \).

Now consider the \( \tilde{K}' \)-weight \( \hat{\sigma}_F^{\tilde{F}} \). The subgroup \( (U^* \cap K^*)^{\tilde{F}} \) of \( \tilde{K}' \) consists of the elements \((\tilde{\alpha})(u(x), 1)\tilde{\alpha}^{-1} = (u(\varpi^{-1}x), (-1, \varpi)_F)\) such that \( x \in O_F \), and \( \hat{\sigma}^{\tilde{F}}(u(\varpi^{-1}x), 1) = \hat{\sigma}_F(u(x), 1) \). Hence the restriction \( \hat{\sigma}^{\tilde{F}}|_{(U^* \cap K^*)^{\tilde{F}}} \) is the inflation of a representation of \( U(t) \) through the composition

\[
(U^* \cap K^*)^{\tilde{F}} \rightarrow U \cap K \rightarrow^{red} U(t) \\
(u(\varpi^{-1}x), 1) \mapsto u(x) \mapsto u(\bar{x}),
\]

and we write \((\hat{\sigma}^{\tilde{F}})_U^{(t)}\) for \((\hat{\sigma}^{\tilde{F}})(U^* \cap K^*)^{\tilde{F}}\). Likewise, the restriction \( \hat{\sigma}^{\tilde{F}}|_{(U^* \cap K^*)^{\tilde{F}}} \) is the inflation of a representation of \( U(t) \) through the composition

\[
(U^* \cap K^*)^{\tilde{F}} \rightarrow \tilde{T} \cap K \rightarrow^{red} \tilde{U}(t) \\
(u(\varpi^{-1}x), 1) \mapsto u(\bar{x}) \mapsto u(\bar{x}),
\]

and we denote \((\hat{\sigma}^{\tilde{F}})_\tilde{T}(t)\) by \((\hat{\sigma}^{\tilde{F}})^{\tilde{T}(t)}\).

Lemma 3.6.  
(1) \((\hat{\sigma}^{\tilde{F}})_U^{(t)}\) is generated as an \( E \)-vector space by the highest-weight vector of \( \sigma^{\tilde{F}}_p \).

(2) If \( \vec{r} \in \{0, \ldots, p-1\}^f \) then there is an isomorphism of \( \tilde{T} \cap \tilde{K} \)-representations

\[
(\hat{\sigma}^{\tilde{F}})_U^{(t)} \cong \hat{\sigma}_F
\]

if and only if \( \vec{r} \) satisfies

\[
\sum_{i=0}^{f-1} r_i p^i \equiv \sum_{i=0}^{f-1} \left( r_i + \frac{p-1}{2} \right) \pmod{q}.
\]

(3) The composition

\[
\hat{j}_F^{\tilde{F}} : (\hat{\sigma}^{\tilde{F}})_U^{(t)} \rightarrow \hat{\sigma}_F \rightarrow (\hat{\sigma}^{\tilde{F}})_{\tilde{T}(t)}
\]

is an isomorphism of \( \tilde{T} \cap \tilde{K} \)-representations.

Proof of Lemma 3.6. All three statements follow from Lemma 3.4 under conjugation by \( \tilde{\alpha} \). The calculation for (2) is the following: let \( a \in O_F^\times \) and \( \zeta \in \mu_2(F) \), so that \((t(a), \zeta) \in \tilde{T} \cap \tilde{K} \), and let \( v \in (\hat{\sigma}_F)_{U(t)} \). Then

\[
\hat{\sigma}_F(t(a), \zeta)v = \hat{\sigma}_F((\tilde{\alpha})^{-1}(t(a), \zeta)\tilde{\alpha})v \\
= \hat{\sigma}_F((t(a), \zeta \cdot (a, \varpi)_F))v \\
= \zeta \cdot (a, \varpi)_F \cdot \delta_F(a)v.
\]

By 3.1, the character \( a \mapsto (a, \varpi)_F \) of \( O_F^\times \) is equal to \( a \mapsto (a^{-1})^{\frac{p-1}{2}} \), which in terms of our parametrization of smooth characters of \( O_F^\times \) is equal to \( \delta_F^{-\frac{p+1}{2}} \). Hence

\[
\hat{\sigma}^{\tilde{F}}_F(t(a), \zeta)v = \zeta \cdot \delta_F^{-\frac{p+1}{2}}(a) \cdot \delta_F(a)v.
\]

We have \( \delta_F^{-\frac{p+1}{2}}(a) \cdot \delta_F(a) = \delta_F(a) \) for all \( a \in O_F^\times \) if and only if \( \vec{r} \) satisfies the condition given in (2). \( \square \)
Finally we state a lemma which will be useful in the definition and application of a Satake transform (Proposition 13). It is the appropriate analogue of the fact that, in the context of representations on \(\mathbb{C}\)-vector spaces, the Jacquet functor is left adjoint to parabolic induction. The proof is taken with only minor adaptations from notes of a course by Herzig [12, Lemma 26], where a similar statement is proven for mod \(p\) representations of \(GL_n(F)\).

**Lemma 3.7.**

1. Let \(\hat{\sigma}_T\) be any weight of \(\hat{K}\). There is a natural isomorphism
   
   \[
   \text{Hom}_{\hat{G}}\left(\text{ind}_{\hat{K}}^\hat{G}\hat{\sigma}_T, \text{Ind}_B^G(-)\right) \cong \text{Hom}_{\hat{T} \cap \hat{K}}\left(\hat{\sigma}_T, \text{Ind}_B^\hat{G}(1)\right),
   \]
   
   of functors from the category of smooth genuine \(\hat{T}\)-representations to the category of \(\mathbb{F}_p\)-vector spaces.

2. Given a smooth genuine \(\hat{T}\)-representation \(\pi\) and \(f \in \text{Hom}_{\hat{G}}(\text{ind}_{\hat{K}}^\hat{G}\hat{\sigma}_T, \text{Ind}_B^G(\pi))\), let \(f_T\) denote the image of \(f\) in \(\text{Hom}_{\hat{T} \cap \hat{K}}(\text{ind}_{\hat{T} \cap \hat{K}}(\hat{\sigma}_T), \pi)\). Let \(f'\) denote the element of \(\text{Hom}_{\hat{K}}(\hat{\sigma}_T, \text{Ind}_B^\hat{G}(\pi)|_{\hat{K}})\) which corresponds to \(f\) by Frobenius reciprocity, and let \((f'_T)'\) denote the element of \(\text{Hom}_{\hat{T} \cap \hat{K}}(\hat{\sigma}_T|_{\hat{T} \cap \hat{K}}, \pi|_{\hat{T} \cap \hat{K}})\) which corresponds to \(f_T\) by Frobenius reciprocity. Then for every \(v \in V_{\hat{T}}\),
   
   \[
   f'(v)(1) = (f'_T)'(f(T\pi))(v).
   \]

**Proof of Lemma 3.7**

1. Compact Frobenius reciprocity gives a natural isomorphism of functors
   
   \[
   \text{Hom}_{\hat{G}}\left(\text{ind}_{\hat{K}}^\hat{G}\hat{\sigma}_T, \text{Ind}_B^G(-)\right) \cong \text{Hom}_{\hat{K}}\left(\hat{\sigma}_T, \text{Ind}_B^\hat{G}(1)\right).
   \]

   The Mackey decomposition of the second factor with respect to \(\hat{G} = \hat{B}\hat{K}\) gives another natural isomorphism
   
   \[
   \text{Hom}_{\hat{K}}\left(\hat{\sigma}_T, \text{Ind}_B^\hat{G}(1)|_{\hat{K}}\right) \cong \text{Hom}_{\hat{B} \cap \hat{K}}\left(\hat{\sigma}_T, \text{Ind}_B^\hat{K}(1)|_{\hat{B} \cap \hat{K}}\right).
   \]

   By smooth Frobenius reciprocity,
   
   \[
   \text{Hom}_{\hat{K}}\left(\hat{\sigma}_T, \text{Ind}_B^\hat{K}(1)|_{\hat{K}}\right) \cong \text{Hom}_{\hat{B} \cap \hat{K}}\left(\hat{\sigma}_T, \text{Ind}_B^\hat{K}(1)|_{\hat{B} \cap \hat{K}}\right).
   \]

   The latter is a functor defined on \(\hat{T}\)-representations viewed as \(\hat{B}\)-representations by inflation. Since \(\hat{B} \cap \hat{K} = \hat{T} \cap \hat{K}\), we may replace the restriction to \(\hat{B} \cap \hat{K}\) in the second argument with restriction to \(\hat{T} \cap \hat{K}\). As for the first argument, recall that \(\hat{B} \cap \hat{K} = (\hat{T} \cap \hat{K}) \cdot (\hat{B} \cap \hat{K})^*\) and that \(\hat{\sigma}_T|_{\hat{B} \cap \hat{K}}\) is the inflation of a representation of \(\hat{U}(t)\). Hence by the universal property of the \(\hat{U}(t)\)-coinvariants we have a natural isomorphism
   
   \[
   \text{Hom}_{\hat{B} \cap \hat{K}}\left(\hat{\sigma}_T|_{\hat{B} \cap \hat{K}}, (-)|_{\hat{T} \cap \hat{K}}\right) \cong \text{Hom}_{\hat{T} \cap \hat{K}}\left(\hat{\sigma}_T|_{\hat{T} \cap \hat{K}}, (-)|_{\hat{T} \cap \hat{K}}\right).
   \]

   Finally, compact Frobenius reciprocity gives a natural isomorphism
   
   \[
   \text{Hom}_{\hat{T} \cap \hat{K}}\left(\hat{\sigma}_T|_{\hat{T} \cap \hat{K}}, (-)|_{\hat{T} \cap \hat{K}}\right) \cong \text{Hom}_{\hat{T} \cap \hat{K}}\left(\text{Ind}_T^\hat{T}(1)|_{\hat{T} \cap \hat{K}}, (-)|_{\hat{T} \cap \hat{K}}\right).
   \]

2. We trace the progression of \(f'\) in \(\text{Hom}_{\hat{K}}\left(\hat{\sigma}_T, \text{Ind}_B^\hat{G}(1)|_{\hat{K}}\right)\) through the second, third, and fourth isomorphisms in the proof of Part (1) of the present lemma. The Mackey isomorphism is simply restriction of functions in the second argument. By smooth Frobenius reciprocity, \(f'\) corresponds to the map \(f'' \in \text{Hom}_{\hat{B} \cap \hat{K}}\left(\hat{\sigma}_T|_{\hat{B} \cap \hat{K}}, (-)|_{\hat{T} \cap \hat{K}}\right)\) defined by \(f''(v) = f'(v)(1)\). Finally \((f''_T)'\) is the image of \(f''\) in \(\text{Hom}_{\hat{T} \cap \hat{K}}\left(\hat{\sigma}_T|_{\hat{T} \cap \hat{K}}, (-)|_{\hat{T} \cap \hat{K}}\right)\) via the universal property of the \(\hat{U}(t)\)-coinvariants, i.e.,
   
   \[
   (f''_T)'(f(T\pi))(v) = f''(v).
   \]
Thus $f'(v)(1) = (f_p)'(p_{U(1)}(v))$. \qed

**Lemma 3.8.** The statement of Lemma 3.7 holds when $\tilde{K}$ is replaced by $\tilde{K}'$ everywhere and $\tilde{\sigma}$ is replaced by a weight $\tilde{\sigma}'$ of $\tilde{K}'$.

**Proof of Lemma 3.8.** To prove the $\tilde{K}'$-analogue of Lemma 3.7 (1), one replaces the Mackey decomposition with respect to the decomposition $\tilde{G} = \overline{U} \tilde{K}$ with the Mackey decomposition with respect to the alternative Iwasawa decomposition $\tilde{G} = \overline{B} \tilde{K}'$, and notes that $\overline{B} \cap \tilde{K}' = (\overline{T} \cap \tilde{K}) \cdot \tilde{\alpha}(\overline{U} \cap \tilde{K})^{-1}$ while $\tilde{\sigma}'_{\tilde{\alpha}}|_{\overline{U} \cap \tilde{K}^*}$ is the inflation of a representation of $\overline{U}(t)$. The proof of the $\tilde{K}'$-analogue of Lemma 3.7 (2) goes through with only the obvious adaptations coming from the changes made to the proof of (1). In particular, the formula of Lemma 3.7 (2) is the same for the $\tilde{K}'$-analogue. \qed

### 4. Genuine spherical Hecke algebras and Hecke bimodules

#### 4.1. Intertwining operators for compact inductions of $\tilde{K}$-weights

Let $\tilde{\sigma}$ be a weight of $\tilde{K}$ and let $\text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}$ denote the compact induction. The endomorphism algebra $\text{End}_{\tilde{G}}(\text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma})$ is called the **genuine spherical Hecke algebra of $\tilde{G}$ with respect to $\tilde{\sigma}$** and is denoted by $H(\tilde{G}, \tilde{K}, \tilde{\sigma})$.

More generally, let $\tilde{\sigma}$ and $\tilde{\sigma}'$ be two, possibly distinct, weights of $\tilde{K}$. Then the **genuine spherical Hecke bimodule of $\tilde{G}$ with respect to $\tilde{\sigma}'$ and $\tilde{\sigma}$** is defined to be $\text{Hom}_{\tilde{G}}(\text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}, \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}')$ and is denoted by $H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}')$. Then $H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}')$ is a left $H(\tilde{G}, \tilde{K}, \tilde{\sigma})$-module and a right $H(\tilde{G}, \tilde{K}, \tilde{\sigma}')$-module. Moreover, given three weights $\tilde{\sigma}$, $\tilde{\sigma}'$, and $\tilde{\sigma}'$ of $\tilde{K}$, there is a product

$$H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}') \times H(\tilde{G}, \tilde{K}, \tilde{\sigma}', \tilde{\sigma}) \to H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}')$$

induced by composition.

Let $H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}')$ denote the $E$-vector space of compactly supported functions $f : \tilde{G} \to \text{Hom}_E(\tilde{\sigma}, \tilde{\sigma}')$ such that

$$f(k_1 g k_2) = \tilde{\sigma}(k_1) \circ f(g) \circ \tilde{\sigma}(k_2)$$

for all $k_1, k_2 \in \tilde{K}$ and $g \in \tilde{G}$. In other words, each $f \in H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}')$ is compactly supported and satisfies

$$f(k_1 g(1, \zeta) k_2) = \zeta \cdot \tilde{\sigma}(Pr(k_1)) \circ f(g) \circ \tilde{\sigma}(Pr(k_2))$$

for all $k_1, k_2 \in \tilde{K}^*$, $\zeta \in \mu_2$, and $g \in \tilde{G}$.

Frobenius reciprocity gives a bijection between $H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}')$ and $H(\tilde{G}, \tilde{K}, \tilde{\sigma}', \tilde{\sigma})$, compatible with the bimodule structure on $H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}')$ defined by the following convolution product: for $f_1 \in H(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\sigma}')$ and $f_2 \in H(\tilde{G}, \tilde{K}, \tilde{\sigma}', \tilde{\sigma})$,

$$(f_1 * f_2)(g) = \sum_{x \in \tilde{K} \setminus \tilde{G}} f_1(g x^{-1}) \circ f_2(x).$$

In particular, Frobenius reciprocity gives an $E$-algebra isomorphism between $H(\tilde{G}, \tilde{K}, \tilde{\sigma})$ and the convolution algebra $H(\tilde{G}, \tilde{K}, \tilde{\sigma})$, for each weight $\tilde{\sigma}$ of $\tilde{K}$. 

4.2. Intertwining operators for compact inductions of $\hat{K}'$-weights. We may define genuine spherical Hecke bimodules $\mathcal{H}(\hat{G}, \hat{K}', \hat{\sigma}_F, \hat{\sigma}_S)$ of $\hat{K}'$ in the same way as for $\hat{K}$, replacing $\hat{K}$ with $\hat{K}'$ and $\hat{\sigma}_F$, $\hat{\sigma}_S$ with the conjugate weights $\hat{\sigma}_F^\vee$, $\hat{\sigma}_S^\vee$. The following lemma, a special case of a general fact about conjugate representations (II, Cor. 2.3.6), ensures that every genuine spherical Hecke bimodule of $\hat{K}'$ is isomorphic to a genuine spherical Hecke bimodule of $\hat{K}$.

**Lemma 4.1.** For any weights $\hat{\sigma}_F$ and $\hat{\sigma}_S$ of $\hat{K}$, there is a $\hat{G}$-linear isomorphism

$$\mathcal{H}(\hat{G}, \hat{K}, \hat{\sigma}_F, \hat{\sigma}_S) \to \mathcal{H}(\hat{G}, \hat{K}', \hat{\sigma}_F^\vee, \hat{\sigma}_S^\vee).$$

**Proof of Lemma 4.1** The identity map of $E$-vector spaces

$$\text{Hom}_{\hat{G}} \left( \text{ind}_{\hat{K}}^G (\hat{\sigma}_F), \text{ind}_{\hat{K}}^G (\hat{\sigma}_S) \right) \to \text{Hom}_{\hat{G}} \left( \left( \text{ind}_{\hat{K}}^G (\hat{\sigma}_F) \right)^\alpha, \left( \text{ind}_{\hat{K}}^G (\hat{\sigma}_S) \right)^\alpha \right)$$

is also an isomorphism of $\hat{G}$-modules. For any representation $\pi$ of $\hat{K}$, the map sending $f \in (\text{ind}_{\hat{K}}^G \pi)^\alpha$ to

$$\Phi(f) = (g \mapsto f ((\alpha)^{-1} g\bar{\alpha}))$$

is a $\hat{G}$-linear isomorphism of $(\text{ind}_{\hat{K}}^G \pi)^\alpha$ with $\text{ind}_{\hat{K}}^G (\pi^\alpha)$, inducing a $\hat{G}$-linear isomorphism

$$\text{Hom}_{\hat{G}} \left( \left( \text{ind}_{\hat{K}}^G (\hat{\sigma}_F) \right)^\alpha, \left( \text{ind}_{\hat{K}}^G (\hat{\sigma}_S) \right)^\alpha \right) \to \text{Hom}_{\hat{G}} \left( \text{ind}_{\hat{K}}^G (\hat{\sigma}_F^\vee), \text{ind}_{\hat{K}}^G (\hat{\sigma}_S^\vee) \right).$$

The composition of (4.3) with (4.2) is the desired $\hat{G}$-linear isomorphism. $\square$

4.3. Intertwining operators for compact inductions of $\hat{T} \cap \hat{K}$-representations. Let $\pi_1$ and $\pi_2$ be two irreducible genuine representations of $\hat{T} \cap \hat{K}$. We define the genuine spherical Hecke bimodule of $\hat{T}$ with respect to $\hat{T} \cap \hat{K}$, $\pi_1$, and $\pi_2$ to be $\text{Hom}_{\hat{T}}(\text{ind}_{\hat{T} \cap \hat{K}}^\hat{T} \pi_1, \text{ind}_{\hat{T} \cap \hat{K}}^\hat{T} \pi_2)$, and denote it by $\mathcal{H}(\hat{T}, \hat{T} \cap \hat{K}, \pi_1, \pi_2)$. It is only for formal reasons that we bother to define Hecke bimodules for pairs of nonisomorphic $\hat{T} \cap \hat{K}$-representations $\pi_1$, $\pi_2$: we show immediately (Lemma 4.2) that if $\pi_1 \not\cong \pi_2$, then $\mathcal{H}(\hat{T}, \hat{T} \cap \hat{K}, \pi_1, \pi_2) = 0$.

By Frobenius reciprocity, $\mathcal{H}(\hat{T}, \hat{T} \cap \hat{K}, \pi_1, \pi_2)$ is isomorphic to the bimodule $\mathbb{H}(\hat{T}, \hat{T} \cap \hat{K}, \pi_1, \pi_2)$ of compactly supported functions $f : \hat{T} \to \text{Hom}_E(\pi_1, \pi_2)$ such that

$$f(k_1 t k_2) = \pi_1(k_1) \circ f(t) \circ \pi_2(k_2)$$

for all $k_1, k_2 \in \hat{T} \cap \hat{K}$ and all $t \in \hat{T}$.

**Lemma 4.2.** Let $\pi_1$, $\pi_2$ be two irreducible genuine representations of $\hat{T} \cap \hat{K}$. If $\pi_1 \not\cong \pi_2$ as $\hat{T} \cap \hat{K}$-representations, then $\mathcal{H}(\hat{T}, \hat{T} \cap \hat{K}, \pi_1, \pi_2) = 0$.

**Proof of Lemma 4.2** Since $\pi_1$ and $\pi_2$ are irreducible representations of an abelian group and so are one-dimensional, we have $\text{Hom}_E(\pi_1, \pi_2) \cong E$. Then each $f \in \mathbb{H}(\hat{T}, \hat{T} \cap \hat{K}, \pi_1, \pi_2)$ must satisfy

$$f(k t) = \pi_1(k) f(t) = \pi_2(k) f(t)$$

for all $k \in \hat{T} \cap \hat{K}$ and all $t \in \hat{T}$, which is possible for $f \not\equiv 0$ if and only if $\pi_1 \cong \pi_2$ as representations of $\hat{T} \cap \hat{K}$. Hence $\mathbb{H}(\hat{T}, \hat{T} \cap \hat{K}, \pi_1, \pi_2) = \mathcal{H}(\hat{T}, \hat{T} \cap \hat{K}, \pi_1, \pi_2) = 0$ if $\pi_1 \not\cong \pi_2$. $\square$
Fix once and for all a $\bar{T} \cap \bar{K}$-linear isomorphism $\iota : (\bar{\sigma}_T^{(t)})_{\bar{U}(t)} \to (\bar{\sigma}_{\bar{p}}^{-1})_{\bar{U}(t)}$. For each pair $\vec{r}, \vec{s}$ of vectors in $\{0, \ldots, p-1\}^f$, define a $\bar{T} \cap \bar{K}$-linear map $\iota_{\vec{r}, \vec{s}} : (\bar{\sigma}_T^{(t)})_{\bar{U}(t)} \to (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)}$ as follows:

$$
\iota_{\vec{r}, \vec{s}} = \begin{cases} 
1 & \text{if } \vec{r} = \vec{s} \\
\iota & \text{if } \vec{r} = \vec{0}, \vec{s} = p-1, \\
\iota^{-1} & \text{if } \vec{r} = p-1, \vec{s} = \vec{0}, \\
0 & \text{otherwise.}
\end{cases}
$$

(4.4)

**Lemma 4.3.** Let $\bar{\sigma}_T$, $\bar{\sigma}_{\bar{z}}$ be any weights of $\bar{K}$. Then there is a unique function $\psi^{\vec{r}, \vec{s}}_n \in \mathbb{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}, (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)})$ satisfying

$$
\psi^{\vec{r}, \vec{s}}_n(h(\varpi)^m) = \begin{cases} 
\iota_{\vec{r}, \vec{s}} & \text{if } m = n, \\
0 & \text{if } m \neq n,
\end{cases}
$$

and the set $\{\psi^{\vec{r}, \vec{s}}_n : n \in \mathbb{Z}\}$ is a basis for $\mathbb{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}, (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)})$ as an $E$-vector space.

**Proof of Lemma 4.3.** The statement that $\mathbb{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}, (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)}) = 0$ for $\bar{\sigma}_T$, $\bar{\sigma}_{\bar{z}}$ such that $\vec{r} \neq \vec{s}$ and $\{\vec{r}, \vec{s}\} \neq \{\vec{0}, p-1\}$ follows immediately from Lemma 14.2 and Corollary 3.5. In the remaining cases, the target space of each $f \in \mathbb{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}, (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)})$ is one-dimensional, and $f$ is determined by its values on $h(\varpi)^m$, $m \in \mathbb{Z}$. The function $\psi^{\vec{r}, \vec{s}}_n$ is nonzero $(\bar{T} \cap \bar{K})h(\varpi)^n$ and $\bar{T} = \Pi_{n \in \mathbb{Z}} \text{Supp}(\psi^{\vec{r}, \vec{s}}_n)$, so $\{\psi^{\vec{r}, \vec{s}}_n : n \in \mathbb{Z}\}$ is a basis for $\mathbb{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}, (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)})$ as an $E$-vector space. □

Let $\tau^{\vec{r}, \vec{s}}_n$ denote the element of $\mathcal{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}, (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)})$ which corresponds to $\psi^{\vec{r}, \vec{s}}_n$ by Frobenius reciprocity. It follows from Lemma 4.3 that $\{\tau^{\vec{r}, \vec{s}}_n : n \in \mathbb{Z}\}$ is a basis for $\mathcal{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}, (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)})$ as an $E$-vector space.

Following the convention of previous sections, we avoid duplicate notation when $\vec{r} = \vec{s}$ (so that $\tau^{\vec{r}, \vec{s}}_n \::= \tau^{\vec{r}, \vec{r}}_n$, etc.). In this case $\mathcal{H}_E(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)})$ has an $E$-algebra structure given by composition, and is called a genuine spherical Hecke algebra of $\bar{T}$.

**Lemma 4.4.**

1. For any $\vec{r} \in \{0, \ldots, p-1\}^f$, $(\tau^{\vec{r}}_n)^n = \tau^{\vec{r}}_n$ for all $n \in \mathbb{Z}$, and there is an $E$-algebra isomorphism

$$
\mathcal{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}) \to E[(\tau^{\vec{r}})^{\pm 1}].
$$

2. Suppose that $\{\vec{r}, \vec{s}\} = \{\vec{0}, p-1\}$. Then $\mathcal{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)}, (\bar{\sigma}_{\bar{z}})_{\bar{U}(t)})$ has the following Hecke bimodule structure: for $n, m \in \mathbb{Z}$, $\tau^{\vec{r}, \vec{s}}_n \circ \tau^{\vec{r}, \vec{s}}_m = \tau^{\vec{r}, \vec{r}}_m \circ \tau^{\vec{r}, \vec{s}}_n = \tau^{\vec{r}, \vec{s}}_{n+m}$, $\tau^{\vec{r}, \vec{r}}_n \circ \tau^{\vec{r}, \vec{s}}_m = \tau^{\vec{r}, \vec{r}}_{n+m}$, and $\tau^{\vec{r}, \vec{r}}_n \circ \tau^{\vec{r}, \vec{s}}_m = \tau^{\vec{r}, \vec{s}}_{n+m}$.

**Proof of Lemma 4.4.**

1. We have $(\tau^{\vec{r}})^n = \tau^{\vec{r}}$ if and only if $(\psi^{\vec{r}})^n = \psi^{\vec{r}}$ in $\mathbb{H}(\bar{T}, \bar{T} \cap \bar{K}, (\bar{\sigma}_T^{(t)})_{\bar{U}(t)})$. For any $n, m, k \in \mathbb{Z}$,

$$
(\psi^{\vec{r}}_n \circ \psi^{\vec{s}}_m)(h(\varpi)^k) = \sum_{t \in (\bar{T} \cap \bar{K}) \setminus \bar{T}} \psi^{\vec{r}}_n(h(\varpi)^k t^{-1}) \circ \psi^{\vec{s}}_m(t) = \sum_{j \in \mathbb{Z}} \psi^{\vec{r}}_n(h(\varpi)^{k-j}) \circ \psi^{\vec{s}}_m(h(\varpi)^j).
$$

The summand indexed by $j$ is nonzero only if both $j = m$ and $k - j = n$, so $(\psi^{\vec{r}}_n \circ \psi^{\vec{s}}_m)(h(\varpi)^k) = 0$ unless $k = n + m$. When $k = n + m$, we are left with

$$
(\psi^{\vec{r}}_n \circ \psi^{\vec{s}}_m)(h(\varpi)^{n+m}) = \psi^{\vec{r}}_n(h(\varpi)^n) \circ \psi^{\vec{s}}_m(h(\varpi)^m) = 1.
$$
so \( \psi^n_r \circ \psi^m_s = \psi^{n+m}_r \). Passing back through Frobenius reciprocity, we have \( \tau^n_r \circ \tau^m_s = \tau^{n+m}_{r,s} \) for all \( n, m \in \mathbb{Z} \). Hence \( \tau^n_r \) and \( \tau^n_s \) generate \( \mathcal{H}(T, T \cap \widetilde{K}, (\tilde{\sigma}_r \mid \tilde{\mathcal{U}}(t))) \) over \( E \), and the map \( \mathcal{H}(T, T \cap \widetilde{K}, (\tilde{\sigma}_r \mid \tilde{\mathcal{U}}(t))) \to E[(\tau^n_r)^{-1}] \) sending \( \sum_{n \in \mathbb{Z}} a_n \tau^n_r \) to \( \sum_{n \in \mathbb{Z}} a_n (\tau^n_r)^n \) is an isomorphism of \( E \)-algebras.

(2) The product calculations are essentially the same as in the proof of (1).
Proposition 4.8. Let $\tilde{\sigma}_T, \tilde{\sigma}_S$ be two weights of $\tilde{K}$. We refer to Lemma 3.7 for the definition of $(-)_{\tilde{T}}$.

1. There is a unique map $S_{f,T}: \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_T, \tilde{\sigma}_S) \to \mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_T)_{\mathcal{U}(T)}, (\tilde{\sigma}_S)_{\mathcal{U}(T)})$

   such that $(f \circ \mathcal{T})_{\tilde{T}} = f_{\tilde{T}} \circ S_{f,T}(\mathcal{T})$ for all $\mathcal{T} \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_T, \tilde{\sigma}_S)$, for all $f \in \text{Hom}_G(\text{ind}_{K}^{\tilde{K}} \tilde{\sigma}_S, \text{Ind}_{B}^{\tilde{G}}(\pi))$, for every smooth genuine representation $\pi$ of $\tilde{T}$.

2. If $(\tilde{\sigma}_T)_{\mathcal{U}(T)} \neq (\tilde{\sigma}_S)_{\mathcal{U}(T)}$, as $\tilde{T} \cap \tilde{K}$-representations, then $S_{f,T} = 0$. Otherwise,

   $S_{f,T}(\mathcal{T})([1, p_{\mathcal{U}(T)}(v)]) = \sum_{t \in (\tilde{T} \cap \tilde{K}) \setminus T} \left[ t^{-1}, \sum_{\tilde{a} \in (\tilde{T} \cap \tilde{K}) \setminus T} p_{\mathcal{U}(T)}(T'(v)(\tilde{a}t)) \right]$

   for $T \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_T, \tilde{\sigma}_S)$, $t \in \tilde{T}$, and $v \in \mathcal{V}_T$. Here $\mathcal{T}'$ denotes the element of $\text{Hom}_{\tilde{K}}(\tilde{\sigma}_T, \text{ind}_{K}^{\tilde{K}} \tilde{\sigma}_S|_{\tilde{K}})$ which corresponds to $T \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_T, \tilde{\sigma}_S)$ by Frobenius reciprocity. The map $p_{\mathcal{U}(T)}$ is the projection $V_{\tilde{a}} \to (\tilde{\sigma})_{\mathcal{U}(T)}$ for a weight $\tilde{\sigma}$ (with $\tilde{\sigma} = \tilde{\sigma}_T$ on the left-hand side of the formula, and $\tilde{\sigma} = \tilde{\sigma}_S$ on the right-hand side). By $\tilde{T}$-equivariance, the given values determine $S_{f,T}(\mathcal{T})$.

3. $S_{f,T}$ is $E$-linear, and if $\tilde{\sigma}_T$ is a third weight of $\tilde{K}$, then $S_{f,T}(\mathcal{T} \circ \mathcal{T}) = S_{f,T}((\mathcal{T}) \circ S_{f,T}(\mathcal{T})$

   for all $\mathcal{T} \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_T, \tilde{\sigma}_S)$, $\mathcal{T} \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_T, \tilde{\sigma}_T)$.

Proof of Proposition 4.8

1. Let $\mathcal{T} \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_T, \tilde{\sigma}_S) = \text{Hom}_G(\text{ind}_{K}^{\tilde{K}} \tilde{\sigma}_S, \text{ind}_{B}^{\tilde{G}}(\pi))$. Precomposition with $\mathcal{T}$ is a natural transformation

   $\text{Hom}_G \left( \text{ind}_{K}^{\tilde{K}} \tilde{\sigma}_S, \text{Ind}_{B}^{\tilde{G}}(\pi) \right) \to \text{Hom}_G \left( \text{ind}_{K}^{\tilde{K}} \tilde{\sigma}_S, \text{Ind}_{B}^{\tilde{G}}(\pi) \right),$

   hence induces, via the natural isomorphism of Lemma 3.7, a natural transformation

   $\text{Hom}_{\tilde{T}} \left( \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T} \cap K} \left( (\tilde{\sigma}_S)_{\mathcal{U}(T)} \right), - \right) \to \text{Hom}_{\tilde{T}} \left( \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T} \cap K} \left( (\tilde{\sigma}_S)_{\mathcal{U}(T)} \right), - \right).$

   By the Yoneda Lemma, there is a unique map $S_{f,T}(\mathcal{T}) \in \text{Hom}_{\tilde{T}} \left( \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T} \cap K} \left( (\tilde{\sigma}_S)_{\mathcal{U}(T)} \right), \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T} \cap K} \left( (\tilde{\sigma}_S)_{\mathcal{U}(T)} \right) \right)$ such that

   $\text{Hom}_{\tilde{T}} \left( \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T} \cap K} \left( (\tilde{\sigma}_S)_{\mathcal{U}(T)} \right), - \right) \to \text{Hom}_{\tilde{T}} \left( \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T} \cap K} \left( (\tilde{\sigma}_S)_{\mathcal{U}(T)} \right), - \right).$

   (4.5) 

   $(f \circ \mathcal{T})_{\tilde{T}} = f_{\tilde{T}} \circ S_{f,T}(\mathcal{T})$

   for all $f \in \text{Hom}_G \left( \text{ind}_{K}^{\tilde{K}} \tilde{\sigma}_S, \text{Ind}_{B}^{\tilde{G}}(\pi) \right)$, for every smooth genuine representation $\pi$ of $\tilde{T}$.

2. If $(\tilde{\sigma}_T)_{\mathcal{U}(T)}$ and $(\tilde{\sigma}_S)_{\mathcal{U}(T)}$ are not isomorphic as $\tilde{T} \cap \tilde{K}$-representations, then $S_{f,T} = 0$ by Lemma 4.2.

   Otherwise, let $f_0$ denote the unique element of $\text{Hom}_G \left( \text{ind}_{K}^{\tilde{K}} \tilde{\sigma}_S, \text{ind}_{B}^{\tilde{G}}(\tilde{\sigma}_S)_{\mathcal{U}(T)} \right)$ such that $(f_0)_{\tilde{T}} = \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T} \cap \tilde{K}}(\tilde{\sigma}_S)_{\mathcal{U}(T)}$ (recall that we have chosen $\tilde{\sigma}_S$ to be the identity map on $(\tilde{\sigma}_S)_{\mathcal{U}(T)}$), and let $f'_0$ denote the element of $\text{Hom}_{\tilde{T} \cap \tilde{K}} \left( \tilde{\sigma}_S, \text{ind}_{B}^{\tilde{G}}(\tilde{\sigma}_S)_{\mathcal{U}(T)} \right)$ which corresponds to $f_0$ by Frobenius reciprocity.

   The following equalities in $\text{Hom}_{\tilde{T} \cap \tilde{K}} \left( (\tilde{\sigma}_S)_{\mathcal{U}(T)} \right)$ are obtained by applying Frobenius reciprocity to both sides of (4.5):

   (4.6) 

   $((f_0 \circ \mathcal{T})_{\tilde{T}})' = \left( \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T} \cap \tilde{K}}(\tilde{\sigma}_S)_{\mathcal{U}(T)} \circ S_{f,T}(\mathcal{T}) \right)' = S_{f,T}(\mathcal{T})'$. 
Let \( v \in V_T \) and let \((f_0 \circ T)'\) denote the element of \( \text{Hom}_{\mathfrak{K}}(\tilde{\sigma}, \text{Ind}_{T \cap K}^{\mathfrak{K}}(\tilde{\sigma}_\pi)_{(\mathcal{U}(t))}) \) which corresponds to \( f_0 \circ T \) by Frobenius reciprocity. By part (2) of Lemma 3.7,

\[
((f_0 \circ T)'(p_{\mathcal{U}(t)}(v))) = (f_0 \circ T)'(v)(1) = (f_0 \ast T')(v)(1)
\]

as elements of \( \text{ind}_{T \cap K}^{\mathfrak{K}}(\tilde{\sigma}_\pi)_{(\mathcal{U}(t))} \). Calculating the convolution product using Lemma 28 of [12] for the second equality and the fact that \( \mathcal{U} \cap \bar{K} = (\mathcal{U} \cap K)^* \) for the third, we have

\[
(f_0 \ast T')(v)(1) = \sum_{g \in \bar{K} \setminus \bar{G}} f_0'(T'(v))(g)(g^{-1})
= \sum_{t \in (T \cap K) \setminus \bar{T}} \sum_{u \in (\mathcal{U} \cap K)^* \setminus \mathcal{U}} t^{-1} \cdot f_0'(T'(\bar{u}t)v)(1)
= \sum_{t \in (T \cap K) \setminus \bar{T}} \sum_{u \in (\mathcal{U} \cap K)^* \setminus \mathcal{U}} t^{-1} \cdot \bar{u}^{-1} \cdot ((f_0)'(p_{\mathcal{U}(t)}(T'(\bar{u}t)v)).
\]

The corresponding equality in \( \text{Hom}_{\mathfrak{K}}(\text{ind}_{T \cap K}^{\mathfrak{K}}(\tilde{\sigma}_\pi)_{(\mathcal{U}(t))}, \text{ind}_{T \cap K}^{\mathfrak{K}}(\tilde{\sigma}_\pi)_{(\mathcal{U}(t))}) \) is

\[
S_{\tilde{\sigma}_\pi}(T) \left[ 1, p_{\mathcal{U}(t)}(v) \right] = \sum_{t \in (T \cap K) \setminus \bar{T}} \left[ t^{-1} \cdot \sum_{u \in (\mathcal{U} \cap K)^* \setminus \mathcal{U}} \bar{u}^{-1} \cdot \sum_{u \in (\mathcal{U} \cap K)^* \setminus \mathcal{U}} t_{\tilde{\sigma}_\pi} \circ p_{\mathcal{U}(t)} \circ T'(\bar{u}t)v \right],
\]

which is the desired formula.

(3) Both claims of (3) will follow from from (1). For the first claim, let \( e \in E \). For every \( T \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r, \tilde{\sigma}_\pi) \), every smooth genuine representation \( \pi \) of \( \tilde{T} \), and every \( f \in \text{Hom}_{\tilde{G}}(\text{ind}_{\mathfrak{K}}^{\tilde{G}}(\tilde{\sigma}_\pi), \text{Ind}_{\mathfrak{K}}^{\tilde{G}}(\tilde{\sigma}_\pi)) \), we have

\[
f_{\tilde{T}} \circ S_{\tilde{\sigma}_\pi}(e \cdot \cdot) = f \circ e T \tilde{T} = (ef) \tilde{T} \circ S_{\tilde{\sigma}_\pi}(T) = f_{\tilde{T}} \circ e \cdot S_{\tilde{\sigma}_\pi}(T).
\]

The uniqueness statement of (1) now implies that \( S_{\tilde{\sigma}_\pi}(e \cdot \cdot) \) and \( e \cdot S_{\tilde{\sigma}_\pi}(\cdot) \) are identical.

For the second claim, let \( \tilde{\sigma}_r, \tilde{\sigma}_t, \) and \( \tilde{\sigma}_\pi \) be three weights of \( \tilde{K} \), and let \( \pi \) be a smooth genuine representation of \( \tilde{T} \). Let \( T \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r, \tilde{\sigma}_t), \tilde{T} \in \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r, \tilde{\sigma}_t), \) and \( f \in \text{Hom}_{\tilde{G}}(\text{ind}_{\mathfrak{K}}^{\tilde{G}}(\tilde{\sigma}_t), \text{Ind}_{\mathfrak{K}}^{\tilde{G}}(\tilde{\sigma}_t)) \).

Then

\[
f_{\tilde{T}} \circ S_{\tilde{\sigma}_\pi}(T \circ T) = (f \circ T) \tilde{T} = (f \circ \tilde{T}) \circ S_{\tilde{\sigma}_\pi}(T) = f_{\tilde{T}} \circ S_{\tilde{\sigma}_\pi}(T).
\]
So for fixed $\mathcal{T}$ the maps $S_{\mathcal{T},\mathcal{T}}(\mathcal{T},-) \text{ and } S_{\mathcal{T},\mathcal{T}}(\mathcal{T},-)$ agree on $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\tau})$, and therefore are identical by the uniqueness statement of (1). Allowing $\mathcal{T}$ to vary over $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\tau})$, we get the desired compatibility.

Next, in Lemma 4.9 and Corollary 4.11 we determine an explicit basis for $\mathbb{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\tau})$ as an $\mathcal{E}$-vector space, getting by proxy an $\mathcal{E}$-basis for $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\tau})$. The basis is normalized so as to be compatible with the system $\{\ell_{\mathcal{T},\mathcal{T}}: \tilde{\mathcal{T}}, \tilde{\mathcal{S}} \in \{0, \ldots, p-1\}^I\}$ of $\tilde{T} \cap \tilde{K}$-linear maps chosen in [1].

**Lemma 4.9.** Let $\tilde{\sigma}, \tilde{\tau}$ be two weights of $\tilde{K}$, and let $\rho_{\mathcal{T},\mathcal{T}}$ denote the following composition:

\[
\begin{array}{ccc}
\tilde{\sigma} & \xrightarrow{\rho_{\mathcal{T},\mathcal{T}}} & \tilde{\tau} \\
(\tilde{\sigma})_{\mathcal{T}(t)} & \xrightarrow{\rho_{\mathcal{T},\mathcal{T}}(t)} & (\tilde{\tau})_{\mathcal{T}(t)} \\
\end{array}
\]

The space of functions in $\mathbb{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\tau})$ with support in a double coset of the form $\tilde{K}h(\mathbb{X})^{-n}\tilde{K}$, $n \geq 0$, is at most one-dimensional and is spanned by the function $\varphi_{\mathcal{T},\mathcal{T}}(\mathbb{X})$ defined as follows:

\[
\varphi_{\mathcal{T},\mathcal{T}}(\mathbb{X}) = \begin{cases} 
1 \text{ if } m = 0 \text{ and } \tilde{\mathcal{T}} = \tilde{\mathcal{S}}, \\
0 \text{ otherwise,}
\end{cases}
\]

\[
\varphi_{\mathcal{T},\mathcal{T}}(\tilde{h}(\mathbb{X})) = \begin{cases} 
\rho_{\mathcal{T},\mathcal{T}} \text{ if } m = -n \text{ and } (\tilde{\sigma})_{\mathcal{T}(t)} \cong (\tilde{\tau})_{\mathcal{T}(t)} \text{ as } \tilde{T} \cap \tilde{K}\text{-representations,} \\
0 \text{ if } |m| \neq n \text{ or } (\tilde{\sigma})_{\mathcal{T}(t)} \not\cong (\tilde{\tau})_{\mathcal{T}(t)} \text{ as } \tilde{T} \cap \tilde{K}\text{-representations.}
\end{cases}
\]

**Proof of Lemma 4.9.** The proof goes along the same lines as that of the similar statement for $G$ in [1] Lemma 3.5.5. Suppose that a function $\varphi \in \mathbb{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\tau})$ has support contained in $\tilde{K}h(\mathbb{X})^{-n}\tilde{K}$. By definition of $\mathbb{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{\tau})$,

\[
(4.7) \quad \tilde{\sigma}_T(k_1) \circ \varphi(\tilde{h}(\mathbb{X})) = \varphi(\tilde{h}(\mathbb{X})) \circ \tilde{\tau}_T(k_2)
\]

whenever $k_1, k_2 \in \tilde{K}$ satisfy

\[
(4.8) \quad k_1 \tilde{h}(\mathbb{X}) = \tilde{h}(\mathbb{X})k_2.
\]

In the case $n = 0$, we have $\tilde{\sigma}_T(k) \circ \varphi((1,1)) = \varphi((1,1)) \circ \tilde{\tau}_T(k)$ for all $k \in \tilde{K}$. Since $\tilde{\sigma}_T$ is an irreducible $\tilde{K}$-representation, either $\varphi((1,1))$ is an isomorphism or is zero. In the former case, i.e., if $\tilde{\sigma}_T = \tilde{\tau}_T$ and $\varphi((1,1)) \neq 0$, Schur’s Lemma implies that $\varphi((1,1)) \in E_\mathcal{T}^\times$.

In the case $n > 0$, two elements $k_1 \in \tilde{K}$ and $k_2 \in K^\times$ satisfy (4.8) if and only if $Pr(k_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ with $v_F(b) \geq 2n$. A calculation shows that then $Pr(k_1) = \begin{pmatrix} a & \mathbb{X}^{-2nb} \\ \mathbb{X}^{2nb} & d \end{pmatrix}$ and $k_1 \in K^\times$. By (4.7) and the definition of $\tilde{\sigma}_T$, $\tilde{\tau}_T$,

\[
(4.9) \quad \sigma_T \begin{pmatrix} a & \mathbb{X}^{-2nb} \\ 0 & d \end{pmatrix} \circ \varphi(\tilde{h}(\mathbb{X})) = \varphi(\tilde{h}(\mathbb{X})) \circ \sigma_T \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}
\]

for all such $a, b, c, d$. For the same reasons as in the proof of [3] Lemma 7, the equality (4.9) is equivalent to $\varphi(\tilde{h}(\mathbb{X})^{-n})$ having the following three properties: (1) the image of $\varphi(\tilde{h}(\mathbb{X})^{-n})$ is contained in $(\tilde{\sigma}_T)_{\mathcal{T}(t)}$, (2) $\varphi(\tilde{h}(\mathbb{X})^{-n})$ factors through the projection $\rho_{\mathcal{T}(t)}: \tilde{\sigma}_T \rightarrow (\tilde{\sigma}_T)_{\mathcal{T}(t)}$, and (3)
\(\dot{\sigma}_F(t) \circ \varphi(h(\varpi)^{-n}) = \varphi(h(\varpi)^{-n}) \circ \dot{\sigma}_F(t)\) for all \(t \in \widetilde{T} \cap \widetilde{K}\). Due to properties (1) and (2), \(\varphi(h(\varpi)^{-n})\) is a composition of the form given in the statement of the lemma, for some map \(\iota : (\dot{\sigma}_F)_{\mathcal{P}(t)} \to (\dot{\sigma}_F)_{\mathcal{P}(t)}\). By property (3) \(\iota\) must be \(\widetilde{T} \cap \widetilde{K}\)-linear, and since \((\dot{\sigma}_F)_{\mathcal{P}(t)}\) and \((\dot{\sigma}_F)_{\mathcal{P}(t)}\) are one-dimensional, \(\iota_{\mathcal{P},\mathcal{F}}\) is either 0 or a \(\widetilde{T} \cap \widetilde{K}\)-isomorphism. Such an isomorphism, if it exists, is unique up to a scalar and thus the choice does not affect the \(\bar{E}\)-span of \(\varphi_\cdot\). Thus we may take \(\iota = \iota_{\mathcal{P},\mathcal{F}}\), and the resulting function \(\varphi_n^{\mathcal{F},\mathcal{S}} := \varphi\) spans the space of functions in \(\mathbb{E}(\mathbb{G}, \widetilde{K}, \dot{\sigma}_F, \dot{\sigma}_G)\) with support in \(\widetilde{K}h(\varpi)^{-n}\widetilde{K}\). \(\square\)

**Remark 4.10.** Since \(\widetilde{K}h(\varpi)^{-n}\widetilde{K} = \widetilde{K}h(\varpi)^{-n}\widetilde{K}\) and

\[\dot{h}(\varpi)^n = (1, (-1, \varpi^n)_{\mathcal{F}}) \cdot \hat{w}(1)\dot{h}(\varpi)^{-n}\hat{w}(-1)\]

for all \(n \in \mathbb{Z}\), an equivalent definition of \(\varphi_n^{\mathcal{F},\mathcal{S}}\) for \(n > 0\) is

\[
\varphi_n^{\mathcal{F},\mathcal{S}}(\dot{h}(\varpi)^m) = \begin{cases} (1, \varpi^n)_{\mathcal{F}} \cdot \sigma_\mathcal{S}(w(1)) \circ \rho_{\mathcal{F},\mathcal{S}} \circ \sigma_\mathcal{F}(w(-1)) & \text{if } m=n \text{ and } \mathcal{F}, \mathcal{S} \text{ are } \widetilde{T} \cap \widetilde{K}\text{-representations} \\ 0 & \text{if } |m| \neq n \text{ or } \mathcal{F}, \mathcal{S} \text{ are } \widetilde{T} \cap \widetilde{K}\text{-representations,} \end{cases}
\]

As a corollary of Lemma 4.9 we have:

**Corollary 4.11.** \(\mathbb{H}(\mathbb{G}, \widetilde{K}, \dot{\sigma}_F, \dot{\sigma}_G) = 0\) if neither \(\mathcal{F} = \mathcal{S}\) nor \(\{\mathcal{F}, \mathcal{S}\} = \{0, p - 1\}\), and otherwise a basis for \(\mathbb{H}(\mathbb{G}, \widetilde{K}, \dot{\sigma}_F, \dot{\sigma}_G)\) as an \(E\)-vector space is given by

\[
\left\{\left(\varphi_n^{\mathcal{F},\mathcal{S}}\right)_{n \geq 0} \text{ if } \mathcal{F} = \mathcal{S}, \right. \\
\left.\left(\varphi_n^{\mathcal{F},\mathcal{S}}\right)_{n > 0} \text{ if } \{\mathcal{F}, \mathcal{S}\} = \{0, p - 1\}.\right.
\]

**Proof of Corollary 4.11** The definition of \(\iota_{\mathcal{P},\mathcal{F}}\) (cf. (4.4)) implies that the functions \(\tilde{\phi}_n^{\mathcal{F},\mathcal{S}}\) are all identically zero if \(\mathcal{F} \neq \mathcal{S}\) and \(\{\mathcal{F}, \mathcal{S}\} \neq \{0, p - 1\}\). Otherwise, there exists a vector \(v \in V_{\mathcal{F}}\) such that \(\varphi_n^{\mathcal{F},\mathcal{S}}(\dot{h}(\varpi)^n)(v) \neq 0\) (for example, any \(v \in V^U(\mathcal{F})\)). Hence \(0 \neq \text{Supp}(\varphi_n^{\mathcal{F},\mathcal{S}})\), and by Lemma 4.9 \(\varphi_n^{\mathcal{F},\mathcal{S}}\) spans the \(E\)-vector space of functions in \(\mathbb{H}(\mathbb{G}, \widetilde{K}, \dot{\sigma}_F, \dot{\sigma}_G)\) with support in \(\widetilde{K}h(\varpi)^{-n}\widetilde{K}\). The Cartan decomposition \(\mathbb{G} = \prod_{n \leq 0} \widetilde{K}h(\varpi)^{-n}\widetilde{K}\) implies that the set \(\{\varphi_n^{\mathcal{F},\mathcal{S}} : n \in \mathbb{Z}\}\) is linearly independent. \(\square\)

Let \(T_n^{\mathcal{F},\mathcal{S}}\) and \((T_n^{\mathcal{F},\mathcal{S}})^\prime\) denote, respectively, the elements of \(\text{Hom}_{\mathbb{G}}(\text{ind}_{\mathbb{K}}^{\mathbb{G}} \dot{\sigma}_F, \text{ind}_{\mathbb{K}}^{\mathbb{G}} \dot{\sigma}_G)\) and of \(\text{Hom}_{\mathbb{K}}(\dot{\sigma}_F, \text{ind}_{\mathbb{K}}^{\mathbb{G}} \dot{\sigma}_G)\) which correspond to \(\varphi_n^{\mathcal{F},\mathcal{S}}\) by Frobenius reciprocity. Explicitly, for \(v \in V_{\mathcal{F}}, g \in \mathbb{G}, \) and \(f \in \text{ind}_{\mathbb{K}}^{\mathbb{G}} \dot{\sigma}_F\),

(4.10)  

\[
(T_n^{\mathcal{F},\mathcal{S}}(f))(g) = \varphi_n^{\mathcal{F},\mathcal{S}}(g)(v),
\]

If \(\mathcal{F} = \mathcal{S}\), we will write \(T_n^{\mathcal{F},\mathcal{F}}\) instead of \(T_n^{\mathcal{F},\mathcal{S}}\). We next explicitly determine the image of \(T_n^{\mathcal{F},\mathcal{S}}\) under the Satake transform \(S_{\mathcal{F},\mathcal{G}}\).

**Proposition 4.12.** If \(\mathcal{F} = \mathcal{S}\) or if \(\{\mathcal{F}, \mathcal{S}\} = \{0, p - 1\}\), then for \(n > 0\),

\[
S_{\mathcal{F},\mathcal{G}}(T_n^{\mathcal{F},\mathcal{S}}) = T_n^{\mathcal{F},\mathcal{S}}(\mathcal{F}, \mathcal{S}) = \gamma_{\mathcal{F},\mathcal{S}}^{-1}.
\]

If \(\mathcal{F} = \mathcal{S}\), then \(S_{\mathcal{F}}(T_{\mathcal{F},\mathcal{G}}) = 1\).

**Proof of Proposition 4.12** In order to de-clutter the notation, set \(T_n := T_n^{\mathcal{F},\mathcal{S}}\) and \(\varphi_n := \varphi_n^{\mathcal{F},\mathcal{S}}\) for the duration of the proof. We will pass to \(\text{Hom}_{\mathbb{G}}(\text{ind}_{\mathbb{K}}^{\mathbb{G}} \dot{\sigma}_F, \text{ind}_{\mathbb{K}}^{\mathbb{G}} \dot{\sigma}_G)\) and show that \(S_{\mathcal{F},\mathcal{S}}(T_n) = T_n^{-1}\). \(\square\)
Fix \( n \geq 0, m \in \mathbb{Z} \), and \( v \in V_{\bar{r}} \). Using the formula for \( S_{\bar{r}, \bar{s}}(T_n)' \) found in the proof of Proposition 4.8 (2) and the definition \( \{4.10\} \) of \( T_n' \),

\[
S_{\bar{r}, \bar{s}}(T_n)'(\bar{r}, \bar{s}; (\bar{h}(\varpi))^m) = \sum_{\bar{u} \in (\bar{U} \cap K)^* \setminus \bar{U}^*} \sum_{\bar{s} \in (\bar{U} \cap K)^* \setminus \bar{U}^*} p_{\bar{T}(t)}(\bar{T}_n'(v)(\bar{h}(\varpi))^m) = \sum_{\bar{u} \in (\bar{U} \cap K)^* \setminus \bar{U}^*} p_{\bar{T}(t)}(\bar{T}_n'(v)(\bar{h}(\varpi))^m)
\]

\[
= \left\{
\begin{array}{ll}
p_{\bar{T}(t)}(v) & \text{if } m = n = 0 \text{ and } \bar{r} = \bar{s}, \\
p_{\bar{T}(t)}(v) \circ \rho_{\bar{r}, \bar{s}} & \text{if } m = -n < 0, \\
0 & \text{otherwise.}
\end{array}
\right.
\]

Lemma 4.13 specifies the inner summands in the above formula when \( \bar{s} \neq 0 \), and Lemma 4.14 does the same for \( \bar{s} = 0 \).

**Lemma 4.13.** Suppose that \( \bar{s} \neq 0 \) and suppose that \((\bar{r}, \bar{s})_{\bar{T}(t)} \cong (\bar{s})_{\bar{T}(t)}\) as \( \bar{T} \cap \bar{K} \)-representations. If \( v \in V_{\bar{r}} \) and if the triple \((n \geq 0, m \in \mathbb{Z}, \bar{u} \in (\bar{U} \cap K)^* \setminus \bar{U}^*)\) satisfies

\[ K^* \tilde{h}(\varpi) \in K^* \tilde{h}(\varpi)^{-n}(1, \zeta)K^*, \]

then

\[ p_{\bar{T}(t)}(\varphi_{\bar{n}, \bar{s}}(\tilde{h}(\varpi))^m(v)) = \begin{cases} p_{\bar{T}(t)}(v) & \text{if } m = n = 0 \text{ and } \bar{r} = \bar{s}, \\
p_{\bar{T}(t)} \circ \rho_{\bar{r}, \bar{s}}(v) & \text{if } m = -n < 0, \\
0 & \text{otherwise.}
\end{cases} \]

**Proof of Lemma 4.13.** We continue to write \( \varphi_n \) for \( \varphi_{\bar{n}, \bar{s}} \) and refer to Lemma 4.9 for the definitions of \( \varphi_n \) and \( \rho_{\bar{r}, \bar{s}} \). The proof breaks up into the following cases:

(1) \( m = n = 0 \). Then \( \tilde{h}(\varpi) = \tilde{u} \in (\bar{U} \cap K)^* \), so \( \zeta = 1 \) and

\[ p_{\bar{T}(t)}(\varphi_n(\tilde{h}(\varpi)^m(v)) = p_{\bar{T}(t)}(\varphi_0(v)) = p_{\bar{T}(t)}(v) \circ \sigma_\bar{s}(\tilde{u}) \circ \varphi_0((1, 1))(v) \]

If \( \bar{r} = \bar{s} \), then

\[ p_{\bar{T}(t)}(v) \circ \sigma_{\bar{s}}(\tilde{u}) \circ \varphi_0((1, 1)) = p_{\bar{T}(t)}(v). \]

If \( \bar{r} \neq \bar{s} \), then \( \varphi_0((1, 1)) = 0 \), so \( p_{\bar{T}(t)}(\varphi_n(\tilde{h}(\varpi)^m(v)) = 0. \)

(2) \( m = -n < 0 \). We have

\[ \tilde{h}(\varpi)^{-n} \in \tilde{K} \tilde{h}(\varpi)^{-n} \]

if and only if \( \tilde{u} \in (\bar{U} \cap K)^* \), so again \( \zeta = 1 \), and

\[ p_{\bar{T}(t)}(\varphi_n(\tilde{h}(\varpi)^m(v)) = p_{\bar{T}(t)}(\varphi_n(\tilde{h}(\varpi)^{-n}(v)) = p_{\bar{T}(t)}(v) \circ \sigma_{\bar{s}}(\tilde{u}) \circ \varphi_n(\tilde{h}(\varpi)^{-n}) \circ p_{\bar{T}(t)}(v) = p_{\bar{T}(t)}(v) \circ \rho_{\bar{r}, \bar{s}}(v) = p_{\bar{T}(t)}(v). \]
Then $\bar{u} = \bar{u}(x_0, \ldots, x_{n+m-1})$ for some $(x_0, \ldots, x_{n+m-1}) \in \mathbb{F}^n \times \mathfrak{p}^{n+m-1}$, and

\[
p_{\mathbb{T}(t)} \left( \varphi_n(\tilde{h}(\omega^m)(v)) \right) = \zeta \cdot p_{\mathbb{T}(t)}(\varphi_n(h(\nu(x_0', \ldots, x_{n+m-1}'))) \circ \sigma_{\mathbb{T}(t)}(w(-1)) \circ \varphi_n(h(\omega)^{-n}) \circ \sigma_{\mathbb{T}(t)}(w(1)) v
\]

\[
= \zeta \cdot p_{\mathbb{T}(t)}(\varphi_n(h(\nu(x_0', \ldots, x_{n+m-1}'))) \circ \sigma_{\mathbb{T}(t)}(w(-1)) \circ \rho_{\mathbb{T}, \tilde{s}} \circ \sigma_{\mathbb{T}(t)}(w(1)) v
\]

\[
= 0.
\]

since the image of $\sigma_{\mathbb{T}}(w(-1)) \circ \rho_{\mathbb{T}, \tilde{s}}$ lies in the kernel of $p_{\mathbb{T}(t)}$.

(4) $0 \leq m = n$. Then $\bar{u} \in \mathbb{U}_r$ for some $0 \leq r \leq 2n$. If $r = 0$, then $\bar{u} = 1$ and

\[
p_{\mathbb{T}(t)} \left( \varphi_n(\tilde{u}\tilde{h}(\omega^m)(v)) \right) = \zeta \cdot p_{\mathbb{T}(t)}(\varphi_n(h(\omega)^{-n}) \circ \sigma_{\mathbb{T}(t)}(w(1)) v
\]

\[
= \zeta \cdot p_{\mathbb{T}(t)}(\varphi_n(h(\omega)^{-n}) \circ \rho_{\mathbb{T}, \tilde{s}} \circ \sigma_{\mathbb{T}(t)}(w(1)) v
\]

\[
= 0.
\]

If instead $\tilde{s} = \tilde{0}$ is taken in the proof of Lemma 4.13, then $\sigma_{\mathbb{T}}$ is the trivial representation of $K$ and we obtain the following statement:

**Lemma 4.14.** Suppose that $(\tilde{\sigma}_{\mathbb{T}})_{\mathbb{T}(t)} \cong (\tilde{\sigma}_0)_{\mathbb{T}(t)}$ as $\tilde{T} \cap \tilde{K}$-representations, i.e., suppose that $\tilde{r} \in \{0, \tilde{p}^{-1} \}$. If $v \in V_T$ and if the triple $(\nu \geq 0, m \in \mathbb{Z}$, and $\bar{u} \in (\mathbb{U} \cap K)^* \setminus \mathbb{U}^*$) satisfies

\[K^* \tilde{h}(\omega)^m \subset K^\ast \tilde{h}(\omega)^{-n}(1, \zeta)K^*,\]

then

\[
p_{\mathbb{T}(t)} \left( \varphi_n(\tilde{u}\tilde{h}(\omega^m)(v)) \right) = \varphi_n(\tilde{u}\tilde{h}(\omega^m)(v) = \begin{cases} v & \text{if } m = n = 0 \text{ and } \tilde{r} = \tilde{s}, \\ \rho_{\mathbb{T}, \tilde{s}} v & \text{if } m = -n < 0, \\ \zeta \cdot \rho_{\mathbb{T}, \tilde{s}} v & \text{if } -n < m < n, \\ \zeta \cdot \rho_{\mathbb{T}, \tilde{s}} \circ \sigma_{\mathbb{T}(t)}(w(1)) v & \text{if } 0 < m = n. \end{cases}
\]

In particular, if $\tilde{r} = \tilde{s} = \tilde{0}$, then $p_{\mathbb{T}(t)} \left( \varphi_n(\tilde{u}\tilde{h}(\omega^m)(v)) \right) = \zeta \cdot v$ for all $n$, $m$, and $\bar{u}$ satisfying the conditions of the lemma.

Lemmas 4.13 and 4.14 imply that for any weights $\tilde{\sigma}_{\mathbb{T}}$, $\tilde{\sigma}_{\mathbb{T}}$ of $\tilde{K}$, and for fixed $n \geq 0$, $m \in \mathbb{Z}$ and $\zeta \in \mu_2$, the value of $p_{\mathbb{T}(t)} \circ \varphi_n(\tilde{h}(\omega)^m)v$ is independent of the choice of a representative $\bar{u}$ from the index set $\{\bar{u} \in (\mathbb{U} \cap K)^* \setminus \mathbb{U}^* : (\mathbb{U} \cap K)^* \tilde{h}(\omega)^m \subset K^\ast \tilde{h}(\omega)^{-n}(1, \zeta)K^* \}$. To finish the evaluation of $S_{\mathbb{T}, \tilde{s}}(\mathbb{T}_n)^i(p_{\mathbb{T}(t)}(\tilde{h}(\omega)^m))$, it only remains to count the order (mod $q$) of each such index set. These
orders were determined over \( \mathbb{Z} \) in Lemma 2.9 Reducing modulo \( q \) in the formulae of Lemma 2.9 (4.11)

\[
\# \{ \tilde{u} \in (\bar{U} \cap K)^* \setminus \bar{U}^* : (\overline{U \cap K})^* \tilde{u} \tilde{h}(\varpi)^m \subset K^* \tilde{h}(\varpi)^{-n} (1, \zeta) K^* \} \equiv \\
\begin{cases} 
1 & \text{if } m = -n \text{ and } \zeta = 1, \\
\frac{q-1}{2} & \text{if } n = 1 \text{ and } m = 0, \\
0 & \text{otherwise}
\end{cases} (\mod q).
\]

Hence if \( s \neq 0 \), we deduce from Lemma 4.13 and (4.11) that

\[
S_{\tilde{r}, \tilde{s}}(T_n)'(p_{\overline{U}(t)} v)(\tilde{h}(\varpi)^m) = \\
\begin{cases} 
p_{\overline{U}(t)} v & \text{if } m = n = 0 \text{ and } \tilde{r} = \tilde{s}, \\
p_{\overline{U}(t)} \circ \rho_{\tilde{r}, \tilde{s}} v & \text{if } m = -n < 0, \\
0 & \text{otherwise}.
\end{cases}
\]

From Lemma 4.14 and (4.11) we get the following formula for \( \tilde{s} = 0 \):

\[
S_{\tilde{r}, \tilde{0}}(T_n)'(p_{\overline{U}(t)} v)(\tilde{h}(\varpi)^m) = \\
\begin{cases} 
v & \text{if } m = n = 0 \text{ and } \tilde{r} = \tilde{0}, \\
(\frac{q-1}{2}) \rho_{\tilde{r}, \tilde{0}} v - (\frac{q-1}{2}) \rho_{\tilde{r}, \tilde{0}} v = 0 & \text{if } n = 1 \text{ and } m = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

On the other hand,

\[
\tau_{-n}'(p_{\overline{U}(t)} v)(\tilde{h}(\varpi)^m) = \psi_{-n}'(\tilde{h}(\varpi)^m)(p_{\overline{U}(t)} v) = \\
\begin{cases} 
p_{\overline{U}(t)} v & \text{if } n = m = 0 \text{ and } \tilde{r} = \tilde{s}, \\
\iota_{\tilde{r}, \tilde{s}}(p_{\overline{U}(t)} v) & \text{if } m = -n \text{ and } \tilde{r} = \tilde{s} \text{ or } \{ \tilde{r}, \tilde{s} \} = \{ 0, \frac{p-1}{2} \}, \\
0 & \text{otherwise}.
\end{cases}
\]

It follows from the definition of \( \rho_{\tilde{r}, \tilde{s}} \) that \( \iota_{\tilde{r}, \tilde{s}} \circ p_{\overline{U}(t)} = p_{\overline{U}(t)} \circ \rho_{\tilde{r}, \tilde{s}} \), and \( p_{\overline{U}(t)} \circ \rho_{\tilde{r}, \tilde{0}} = \rho_{\tilde{r}, \tilde{0}} \). Thus the formulae for \( S_{\tilde{r}, \tilde{s}}(T_n)'(p_{\overline{U}(t)} v)(\tilde{h}(\varpi)^m) \) and \( \tau_{-n}'(p_{\overline{U}(t)} v)(\tilde{h}(\varpi)^m) \) agree, so \( S_{\tilde{r}, \tilde{s}}(T_n)' = \tau_{-n}' \) if \( n > 0 \) and \( S_{\tilde{r}}(T_0)' = \tau_0' \) if \( \tilde{r} = \tilde{s} \). Passing back to \( \mathcal{H}(\bar{T}, \bar{T} \cap \bar{K}, (\tilde{r})_{\overline{U}(t)}, (\tilde{s})_{\overline{U}(t)}) \) through the equivalence of Frobenius reciprocity, we get the statement of the proposition. \( \square \)

In particular, \( S_{\tilde{r}, \tilde{s}} \) is injective. Proposition 4.11 together with the description of \( \mathcal{H}^\leq_0(\bar{T}, \bar{T} \cap \bar{K}, (\tilde{r})_{\overline{U}(t)}, (\tilde{s})_{\overline{U}(t)}) \) from Lemma 4.5 gives the following corollary.

**Corollary 4.15.**

1. If \( \tilde{r} = \tilde{s} \), then \( S_{\tilde{r}, \tilde{s}} \) is an \( E \)-algebra isomorphism

\[
\mathcal{H}(\bar{G}, \bar{K}, \tilde{r}) \rightarrow E[\tau_{-1}^\tilde{r}] \cong \mathcal{H}^\leq_0(\bar{T}, \bar{T} \cap \bar{K}, (\tilde{r})_{\overline{U}(t)}).
\]

Thus \( \mathcal{H}(\bar{G}, \bar{K}, \tilde{r}) \) is a polynomial algebra over \( E \) in the single operator \( S_{\tilde{r}}^{-1}(\tau_{-1}^\tilde{r}) = T_1^\tilde{r} \).

2. For each pair \( \tilde{r} \neq \tilde{s} \), the map \( S_{\tilde{r}, \tilde{s}} \) is an \( E \)-linear bijection

\[
\mathcal{H}(\bar{G}, \bar{K}, \tilde{r}, \tilde{s}) \rightarrow \mathcal{H}^{<0}(\bar{T}, \bar{T} \cap \bar{K}, (\tilde{r})_{\overline{U}(t)}, (\tilde{s})_{\overline{U}(t)}),
\]

and the family of maps \( \{ S_{\cdot, \cdot} \} \) respects the Hecke bimodule structure on each side.

From Corollary 4.13 and Lemma 5.3, we get an analogous description of the genuine spherical Hecke bimodules of \( \bar{G} \) with respect to \( \bar{K}^\prime \):

**Corollary 4.16.** Let \( T_1^{\tilde{r}, \tilde{s}} \) (resp., \( T^{\tilde{r}, \tilde{s}} \)) denote the image of \( T_1^\tilde{r} \) (resp., \( T^\tilde{r} \)) under the isomorphism of Lemma 5.3.
The composition of the inverse of the isomorphism of Lemma 4.17 (taking $\bar{r} = \bar{s}$) with $S_{\bar{r}}$ is an $E$-algebra isomorphism

$$\mathcal{H}(\tilde{G}, \tilde{K}', \tilde{\sigma}_{\bar{r}}) \to \mathcal{H}^{\leq 0}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_{\bar{r}})_{\mathcal{U}(\bar{r})}).$$

Thus $\mathcal{H}(\tilde{G}, \tilde{K}', \tilde{\sigma}_{\bar{r}})$ is isomorphic to a polynomial algebra over $E$ in the single operator $\mathcal{T}_{1}^{\tilde{\sigma}_{\bar{r}}}.$

For each pair $\bar{r} \neq \bar{s},$ there is an $E$-linear bijection

$$\mathcal{H}(\tilde{G}, \tilde{K}', \tilde{\sigma}_{\bar{r}}, \tilde{\sigma}_{\bar{s}}) \to \mathcal{H}^{\leq 0}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_{\bar{r}})_{\mathcal{U}(\bar{r})}, (\tilde{\sigma}_{\bar{s}})_{\mathcal{U}(\bar{s})})$$

which is compatible with the Hecke bimodule structure on each side.

We conclude this section by using the Satake transform to calculate compositions of elements of compatible genuine spherical Hecke bimodules.

Lemma 4.17. \hspace{1em} (1) For each $n \geq 0$ and $\bar{r} \in \{0, \ldots, p-1\},$ the following equality holds in $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_{\bar{r}})$:

$$(\mathcal{T}_{1}^{\bar{r}})^{n} = \mathcal{T}_{n}^{\bar{r}}.$$

(2) Suppose that $\{\bar{r}, \bar{s}\} = \{0, p-1\},$ then for each $n \geq 0$ and $m \geq 0,$

$$\mathcal{T}_{n}^{\bar{r}, \bar{s}} \circ \mathcal{T}_{m}^{\bar{r}, \bar{s}} = (\mathcal{T}_{1}^{\bar{r}})^{n+m} \text{ and } \mathcal{T}_{m}^{\bar{r}, \bar{s}} \circ \mathcal{T}_{n}^{\bar{r}, \bar{s}} = (\mathcal{T}_{1}^{\bar{s}})^{n+m}.$$

Proof of Lemma 4.17. \hspace{1em} (1) Since $S_{\bar{r}}$ is a homomorphism of $E$-algebras (Proposition 4.8 (3)),

$$S_{\bar{r}}((\mathcal{T}_{1}^{\bar{r}})^{n}) = (S_{\bar{r}}(\mathcal{T}_{1}^{\bar{r}}))^{n}.$$

By Proposition 4.12, $S_{\bar{r}}(\mathcal{T}_{1}^{\bar{r}})^{n} = (\mathcal{T}_{1}^{\bar{r}})^{n},$ which is equal to $\tau_{\bar{r}, n}^{\bar{r}}$ by Lemma 4.4 (1).

By Proposition 4.12, $S_{\bar{r}}(\mathcal{T}_{1}^{\bar{r}})^{n} = (\mathcal{T}_{1}^{\bar{r}})^{n},$ which is equal to $\tau_{\bar{r}, n}^{\bar{r}}$ by Lemma 4.4 (1).

(2) Suppose that $\{\bar{r}, \bar{s}\} = \{0, p-1\},$ and let $n \geq 0$ and $m \geq 0.$ By Proposition 4.8 (3) again,

$$S_{\bar{r}}(\mathcal{T}_{n}^{\bar{r}, \bar{s}} \circ \mathcal{T}_{m}^{\bar{r}, \bar{s}}) = S_{\bar{r}, \bar{s}}(\mathcal{T}_{n}^{\bar{r}, \bar{s}}) \circ S_{\bar{r}, \bar{s}}(\mathcal{T}_{m}^{\bar{r}, \bar{s}}).$$

By Proposition 4.12,

$$S_{\bar{r}, \bar{s}}(\mathcal{T}_{n}^{\bar{r}, \bar{s}}) \circ S_{\bar{r}, \bar{s}}(\mathcal{T}_{m}^{\bar{r}, \bar{s}}) = \tau_{\bar{n}, \bar{m}}^{\bar{r}} \circ \tau_{\bar{n}, \bar{m}}^{\bar{s}},$$

and by Lemma 4.14 (2), $\tau_{\bar{n}, \bar{m}}^{\bar{r}} \circ \tau_{\bar{n}, \bar{m}}^{\bar{s}} = \tau_{\bar{n}, \bar{m}}^{\bar{r} \bar{s}}.$ Then $\mathcal{T}_{n}^{\bar{r}, \bar{s}} \circ \mathcal{T}_{m}^{\bar{r}, \bar{s}} = S_{\bar{r}}^{-1}(\tau_{\bar{n}, \bar{m}}^{\bar{r} \bar{s}}) = \mathcal{T}_{n}^{\bar{r} \bar{s}} \circ \mathcal{T}_{m}^{\bar{r} \bar{s}},$ which is equal to $(\mathcal{T}_{1}^{\bar{r}})^{n+m}$ by (1) of the present lemma. The second equality of (2) is proved in the same way after exchanging $\bar{s}$ and $\bar{r}.$

\[\Box\]

5. Universal modules for genuine spherical Hecke algebras

5.1. Universal modules for spherical Hecke algebras of $\tilde{T}$. The universal module for the genuine spherical Hecke algebra $\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_{\bar{r}})_{\mathcal{U}(\bar{r})})$ is the compact induction $\text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T}}((\tilde{\sigma}_{\bar{r}})_{\mathcal{U}(\bar{r})}).$ Its structure is very simple and is presented here mainly for easy reference in the proofs of Proposition 6.1 and Theorem 6.10.

Lemma 5.1. Let $\tilde{\sigma}_{\bar{r}}$ be any weight of $\tilde{K}.$ Then $\text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T}}((\tilde{\sigma}_{\bar{r}})_{\mathcal{U}(\bar{r})})$ is a free $\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_{\bar{r}})_{\mathcal{U}(\bar{r})})$-module. A free basis is given by the single element $[1, p_{\mathcal{U}(\bar{r})}^{\bar{r}} v],$ where $v$ is any nonzero vector in $V_{\bar{r}}^{\mathcal{U}(\bar{r})}.$
Proof of Lemma 5.1. Let $0 \neq v \in \mathbb{V}_{\mathcal{T}^L(t)}$; then $p_{\mathcal{T}^L(t)}v \neq 0$. An $E$-vector space basis for $\text{ind}_{\mathcal{T}^L \cap \hat{K}} (\hat{\sigma}_{\mathcal{T}^L})$ is given by $\{ [\hat{h}(\varpi)^n, p_{\mathcal{T}^L(t)}v] : n \in \mathbb{Z} \}$. For $n \in \mathbb{Z}$,

$$(\varpi^{-1})^n([1, p_{\mathcal{T}^L(t)}v]) = [\hat{h}(\varpi)^n, p_{\mathcal{T}^L(t)}v],$$

so $[1, p_{\mathcal{T}^L(t)}v]$ is a free $\mathcal{H} (\mathcal{T}, \mathcal{T} \cap \hat{K}, (\hat{\sigma}_{\mathcal{T}^L})$-basis for $\text{ind}_{\mathcal{T}^L \cap \hat{K}} (\hat{\sigma}_{\mathcal{T}^L})$.

Given a genuine representation $\pi$ of $\mathcal{T}$, the Hecke algebra $\mathcal{H} (\mathcal{T}, \mathcal{T} \cap \hat{K}, (\hat{\sigma}_{\mathcal{T}^L})$ acts on the weight space $\text{Hom}_{\mathcal{T} \cap \hat{K}} (\hat{\sigma}_{\mathcal{T}^L}, \pi | \mathcal{T} \cap \hat{K})$ from the right. Likewise, $\mathcal{H} (\mathcal{T}, \mathcal{T} \cap \hat{K}', (\hat{\sigma}_{\mathcal{T}^L})$ acts on $\text{Hom}_{\mathcal{T} \cap \hat{K}'} (\hat{\sigma}_{\mathcal{T}^L}, \pi | \mathcal{T} \cap \hat{K}')$. These actions are clearly scalar; we next determine the eigenvalue of the respective generators.

Lemma 5.2. Let $\pi$ be any smooth genuine representation of $\mathcal{T}$ and let $\hat{\sigma}_{\mathcal{T}^L}$ be a weight of $\mathcal{T}$. Then

1. For each $f' \in \text{Hom}_{\mathcal{T} \cap \hat{K}} (\hat{\sigma}_{\mathcal{T}^L}, \pi | \mathcal{T} \cap \hat{K})$,

$$(f' \cdot \varpi^{-1})(p_{\mathcal{T}^L(t)}v) = (h' \ast \varpi^{-1})(p_{\mathcal{T}^L(t)}v)$$

$$= \sum_{n \in \mathbb{Z}} \pi (\varpi^{-1} \cdot f' (\varpi^{-1}(p_{\mathcal{T}^L(t)}v)))$$

$$= \sum_{n \in \mathbb{Z}} \pi (\hat{h}(\varpi)^{-n} \cdot f' (\varpi^{-1}(\hat{h}(\varpi)^{-1}(p_{\mathcal{T}^L(t)}v)))$$

$$= \pi (\hat{h}(\varpi)) \cdot f' (\varpi^{-1}(\hat{h}(\varpi)^{-1}(p_{\mathcal{T}^L(t)}v)))$$

$$= \pi (\hat{h}(\varpi)) \cdot f' (p_{\mathcal{T}^L(t)}v).$$

2. The proof is essentially identical to that of Lemma 5.1 (2.)

Proof of Lemma 5.2 (1) Let $f'$ be any element of $\text{Hom}_{\mathcal{T} \cap \hat{K}} (\hat{\sigma}_{\mathcal{T}^L}, \pi | \mathcal{T} \cap \hat{K})$ and let $v \in \mathbb{V}_{\mathcal{T}^L}$. Consider $f' \cdot \varpi^{-1}(p_{\mathcal{T}^L(t)}v)$

Finally we check that each genuine character of $\mathcal{T}$ can be constructed as the tensor product of a universal module for some $\mathcal{H} (\mathcal{T}, \mathcal{T} \cap \hat{K}, (\hat{\sigma}_{\mathcal{T}^L})$ with a certain character of a spherical Hecke algebra. The following lemma is a prototype for the parametrization of genuine representations of $G$ defined in the next section.

Lemma 5.3. If $\pi$ is a smooth genuine character of $\mathcal{T}$, $\hat{\sigma}_{\mathcal{T}^L}$ is a weight of $\mathcal{T}$ such that $\text{Hom}_{\mathcal{T} \cap \hat{K}} (\hat{\sigma}_{\mathcal{T}^L}, \pi | \mathcal{T} \cap \hat{K}) \neq 0$, $\theta$ is the character of $\mathcal{H} (\mathcal{T}, \mathcal{T} \cap \hat{K}, (\hat{\sigma}_{\mathcal{T}^L})$ defined by $\theta (\varpi^{-1}) = \pi (\hat{h}(\varpi))$, and $\theta^\hat{\sigma}$ is the character of $\mathcal{H} (\mathcal{T}, \mathcal{T} \cap \hat{K}', \hat{\sigma}_{\mathcal{T}^L})$ defined by $\theta^\hat{\sigma} (\varpi^{-1}) = \pi (\hat{h}(\varpi))$, then

$$\pi \cong \text{ind}_{\mathcal{T} \cap \hat{K}} (\hat{\sigma}_{\mathcal{T}^L}) \otimes_{\mathcal{H}(\mathcal{T}, \mathcal{T} \cap \hat{K}, (\hat{\sigma}_{\mathcal{T}^L}))} \hat{\sigma}_{\mathcal{T}^L} \otimes_{\mathcal{H}(\mathcal{T}, \mathcal{T} \cap \hat{K}', \hat{\sigma}_{\mathcal{T}^L})} \theta^\hat{\sigma}$$

where $\varpi^{-1} \in \{ 0, \ldots, p - 1 \}$ satisfies $\sum_{i=0}^{p-1} r_i p^i \equiv \sum_{i=0}^{p-1} (r_i + \frac{p-1}{p}) p^i \pmod{q}$. 

□
Proof of Lemma 5.3. The genuine character \( \pi \) is determined by the data \( \pi|_{\widetilde{T}\cap \tilde{K}} \) and \( \pi(\tilde{h}(\varpi)) \). If and only if \( \operatorname{Hom}_{\tilde{T}\cap \tilde{K}}((\tilde{\sigma})_{T(1)}, \pi|_{\tilde{\tilde{T}}}) \neq 0 \), we have \( \pi|_{\tilde{T}\cap \tilde{K}} \cong (\tilde{\sigma})_{T(1)} \) as \( \tilde{T} \cap \tilde{K} \)-representations. In addition, by Lemma 5.6 (2) we have \( \pi|_{\tilde{T}\cap \tilde{K}} \cong (\tilde{\sigma})_{\tilde{T}(1)} \) as \( \tilde{T} \cap \tilde{K} = \tilde{T} \cap \tilde{K}' \)-representations and only if \( \tilde{r}' \) satisfies the condition given in the lemma. By definition of \( \theta \) and \( \theta^{\tilde{\sigma}} \), the value of each tensor product on \( \tilde{h}(\varpi) \) matches that of \( \pi \). The result follows from the universal property of the tensor product. \( \square \)

5.2. Universal modules for spherical Hecke algebras of \( \tilde{G} \). The universal module for the genuine spherical Hecke algebra \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r) \) is the compact induction \( \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}_r \).

Lemma 5.4. Let \( \tilde{\sigma}_r \) be any weight of \( \tilde{K} \). Then \( \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}_r \) is a flat \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r) \)-module.

Proof of Lemma 5.4. Since \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r) \) is isomorphic to a polynomial ring in one variable over \( E \), \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r) \) is a flat module. More precisely, there exist finite sets \( \{a_n\}_{0 \leq n \leq k} \) and \( \{b_m\}_{0 < m \leq \ell} \) of elements of \( E \) such that \( a_k \neq 0, b_\ell \neq 0 \), \( \mathcal{T} = \sum_{n>0} a_n T_n^r \) and \( \mathcal{I} = \sum_{m>0} b_m T_m^{\tilde{\tilde{\sigma}}_r} \). Then, using (1.17) (2) for the third equality,

\[
0 = \mathcal{T} \circ \mathcal{I} = \sum_{0 \leq n \leq k \atop 0 < m \leq \ell} a_n b_m (T_n^r \circ T_m^{\tilde{\tilde{\sigma}}_r}) = \sum_{j=0}^{k+\ell} \left( \sum_{n>0, m>0 \atop n+m=j} a_n b_m \right) T_j^{\tilde{\tilde{\sigma}}_r}.
\]

Since \( \{T_j^{\tilde{\tilde{\sigma}}_r} : j > 0\} \) is linearly independent, the above equality implies that the leading coefficient \( a_k b_\ell \) is zero, but \( a_k \neq 0 \) and \( b_\ell \neq 0 \).

Furthermore, it is shown in [23] that if \( f \in \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}_r \) is supported on a single vertex \( v \) of \( \mathcal{X} \), then the support of \( \mathcal{T}(f) \) contains at least one vertex \( v' \) of \( \mathcal{X} \) such that \( d(v, v') = 2 \) and \( d(v_0, v') = d(v_0, v) + 2 \).

Remark 5.5. In fact one can use the above-mentioned fact to show that \( \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}_r \) is a free \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r) \)-module. A free basis may be explicitly (though noncanonically) constructed in a similar manner to that of Barthel-Liñé for \( GL_2(F) \) (3 Thm. 19 for nontrivial weights, 4 Thm. 10 for the trivial weight). Fix a basis \( \mathcal{B} \) of \( V_\rho \) and put \( A_0 = \{[1, b] : b \in \mathcal{B}\} \); then \( A_0 \) is a basis for the functions in \( \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}_r \) which are supported on the unit vertex of \( \mathcal{X} \). For \( n \geq 0 \), we inductively construct a basis \( A_{n+1} \) for the functions supported on \( C_{2n+2} \); given such a basis \( A_k \) for each \( n \geq k \geq 0 \), the set

\[
\{ f \circ (T_i^\rho)^k \}_{f \in A_k, \atop 0 \leq i \leq n, \atop k+i \leq n+1}
\]

is linearly independent and consists of functions supported on the circle \( C_{2n+2} \) of \( \mathcal{X} \), so may be completed to the desired basis \( A_{n+1} \). Then \( \Pi_{k \geq 0} A_k \) is a free basis for \( \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}_r \) as an \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r) \)-module. More details are available in [23].

Lemma 5.6. Let \( \tilde{\sigma}_r \) be any weight of \( \tilde{K} \), let \( \lambda \in E \), and let \( \Theta_{\lambda} \) be the character of \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r) \) defined by \( \Theta_{\lambda}(T_i^\rho) = \lambda. \) Set \( \pi(\tilde{r}, \lambda) := \text{ind}_{\tilde{K}}^{\tilde{G}} \tilde{\sigma}_r \otimes_{\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r)} \Theta_{\lambda}. \) Then...
(1) $\pi(\vec{r}, \lambda)$ is an infinite-dimensional genuine representation of $\tilde{G}$.

(2) If $\rho$ is a quotient of $\pi(\vec{r}, \lambda)$, then $\rho$ contains $\tilde{\sigma}_f$ and $T^\vec{r}_f$ acts by $\lambda$ on $\text{Hom}_{\tilde{K}}(\tilde{\sigma}_f, \rho|_{\tilde{K}})$.

Proof of Lemma 5.6

(1) The representation $\pi(\vec{r}, \lambda)$ is isomorphic to the quotient $(\text{ind}_{K}^{\tilde{G}}(\tilde{\sigma}_f)/\langle T^\vec{r}_f - \lambda \rangle)$. The image of $(T^\vec{r}_f - \lambda)$ in $\text{ind}_{K}^{\tilde{G}}(\tilde{\sigma}_f)$ consists of genuine functions on $\tilde{G}$, so the quotient $\pi(\vec{r}, \lambda)$ is genuine as well. The infinite-dimensionality of $\pi(\vec{r}, \lambda)$ follows from the above-mentioned fact (proved in [23]) that $T^\vec{r}_f$ strictly increases the radius in $\mathcal{X}$ of the support of any $f \in \text{ind}_{K}^{\tilde{G}}(\tilde{\sigma}_f)$.

(2) The composition of the quotients $\text{ind}_{K}^{\tilde{G}}(\tilde{\sigma}_f) \to \pi(\vec{r}, \lambda)$ and $\pi(\vec{r}, \lambda) \to \rho$ is nonzero, so by Frobenius reciprocity there exists a corresponding injection $\tilde{\sigma}_f \hookrightarrow \rho$. The Hecke algebra $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_f)$ acts on $\text{Hom}_{\tilde{G}}(\text{ind}_{K}^{\tilde{G}}(\tilde{\sigma}_f), \rho)$ by precomposition, and the image of the identity map of $\text{ind}_{K}^{\tilde{G}}(\tilde{\sigma}_f)$ in $\text{Hom}_{\tilde{G}}(\text{ind}_{K}^{\tilde{G}}(\tilde{\sigma}_f), \rho)$ is an eigenvector for $T^\vec{r}_f$ with eigenvalue $\lambda$. Passing to $\text{Hom}_{\tilde{K}}(\tilde{\sigma}_f, \rho|_{\tilde{K}})$ by Frobenius reciprocity, we get the second statement of the lemma.

Let $\rho$ be any smooth genuine representation of $\tilde{G}$ and let $\tilde{\sigma}_f$ be a $\tilde{K}$-weight. Suppose that $\tilde{\sigma}_f$ is contained in $\rho$, i.e., that $\text{Hom}_{\tilde{K}}(\tilde{\sigma}_f, \rho|_{\tilde{K}}) \neq 0$. Then $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_f)$ acts on $\text{Hom}_{\tilde{K}}(\tilde{\sigma}_f, \rho|_{\tilde{K}})$ from the right via Frobenius reciprocity. If the action of $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_f)$ admits an eigenvector with associated eigenvalue $\lambda$, then there exists a $\tilde{G}$-linear map $\pi(\vec{r}, \lambda) \to \rho$; if $\rho$ is also irreducible, then it follows that $\rho$ is a quotient of $\pi(\vec{r}, \lambda)$.

Let $\rho$ be a smooth genuine irreducible representation of $\tilde{G}$. A pair $(\vec{r}, \lambda)$, with $\vec{r} \in \{0, \ldots, p-1\}$ and $\lambda \in E$, is a parameter for $\rho$ with respect to $\tilde{K}$ if $\rho$ is a quotient of $\pi(\vec{r}, \lambda)$.

Lemma 5.6 has an obvious analogue for $\tilde{K}'$:

Lemma 5.7. Let $\tilde{\sigma}_f^\tilde{\alpha}$ be a weight of $\tilde{K}'$, let $\lambda \in E$, and let $\Theta^\tilde{\lambda}_f$ be the character of $\mathcal{H}_E(\tilde{G}, \tilde{K}', \tilde{\sigma}_f^\tilde{\alpha})$ defined by $\Theta^\tilde{\lambda}_f(T^\tilde{\alpha}_f) = \lambda$. Set $\pi'(\vec{r}, \lambda) := \text{ind}_{\tilde{K}}^{\tilde{G}}(\tilde{\sigma}_f^\tilde{\alpha}) \otimes_{\mathcal{H}(\tilde{G}, \tilde{K}', \tilde{\sigma}_f^\tilde{\alpha})} \Theta^\tilde{\lambda}_f$. Then

(1) $\pi'(\vec{r}, \lambda)$ is an infinite-dimensional genuine representation of $\tilde{G}$.

(2) If $\rho$ is a quotient of $\pi'(\vec{r}, \lambda)$, then $\rho$ contains $\tilde{\sigma}_f^\tilde{\alpha}$ and $T^\tilde{\alpha}_f$ acts by $\lambda$ on $\text{Hom}_{\tilde{K}'}(\tilde{\sigma}_f^\tilde{\alpha}, \rho|_{\tilde{K}'})$.

Lemma 5.7 follows from:

Lemma 5.8. Let $\vec{r} \in \{0, \ldots, p-1\}$ and $\lambda \in E$, and define $\pi'(\vec{r}, \lambda)$ as in the statement of Lemma 5.7. Then $(\pi(\vec{r}, \lambda))^\tilde{\alpha} \cong \pi'(\vec{r}, \lambda)$.

Proof of Lemma 5.8

Recall that in the proof of Lemma 4.4 was defined, for a weight $\tilde{\sigma}_f$ of $\tilde{K}$, an isomorphism $\Phi : \left(\text{ind}_{\tilde{K}}^{\tilde{G}}(\tilde{\sigma}_f)^\tilde{\alpha}\right) \cong \left(\text{ind}_{\tilde{K}}^{\tilde{G}}(\tilde{\sigma}_f)^\tilde{\alpha}\right)$ such that $\Phi(h)(g) = h((\tilde{\alpha})^{-1} g \tilde{\alpha})$ for all $h \in \left(\text{ind}_{\tilde{K}}^{\tilde{G}}(\tilde{\sigma}_f)^\tilde{\alpha}\right)$ and $g \in \tilde{G}$. Via the identity map $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_f) \to \text{End}_{\tilde{G}}\left(\left(\text{ind}_{\tilde{K}}^{\tilde{G}}(\tilde{\sigma}_f)^\tilde{\alpha}\right)\right)$, we may view $T^\vec{r}_f$ as an endomorphism of $\left(\text{ind}_{\tilde{K}}^{\tilde{G}}(\tilde{\sigma}_f)^\tilde{\alpha}\right)$. Then the map $\text{End}_{\tilde{G}}\left(\left(\text{ind}_{\tilde{K}}^{\tilde{G}}(\tilde{\sigma}_f)^\tilde{\alpha}\right)\right) \to \mathcal{H}(\tilde{G}, \tilde{K}', \tilde{\sigma}_f^\tilde{\alpha})$ induced by $\Phi$ sends $T^\vec{r}_f$ to $T^\tilde{\alpha}_f$. Thus $\Phi$ induces an isomorphism $(\pi(\vec{r}, \lambda))^\tilde{\alpha} \cong \pi'(\vec{r}, \lambda)$.
Proof of Proposition 5.9. If \( \rho \) is smooth and genuine, then \( \rho \) contains a \( \tilde{K} \)-weight \( \tilde{\sigma}_\tau \) (by Proposition 3.1 (2)) and a \( \tilde{K}' \)-weight \( \tilde{\sigma}_{\tilde{s}}^\circ \) (by Proposition 3.10 (2)), and for these weights we have \( \text{Hom}_{\tilde{K}}(\tilde{\sigma}_\tau, \rho|_{\tilde{K}}) \neq 0 \) and \( \text{Hom}_{\tilde{K}'}(\tilde{\sigma}_{\tilde{s}}^\circ, \rho|_{\tilde{K}'}) \neq 0 \). If \( \rho \) is admissible, then the weight spaces are finite-dimensional, hence admit eigenvectors for the actions of \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_\tau) \) and \( \mathcal{H}(\tilde{G}, \tilde{K}', \tilde{\sigma}_{\tilde{s}}^\circ) \) respectively. Let \( \lambda \) denote an eigenvalue for the action of \( \mathcal{T}_1^{\tilde{\tau}} \) on \( \text{Hom}_{\tilde{K}}(\tilde{\sigma}_\tau, \rho|_{\tilde{K}}) \) and \( \lambda' \) denote an eigenvalue for the action of \( \mathcal{T}_1^{\tilde{\sigma}_{\tilde{s}}^\circ} \) on \( \text{Hom}_{\tilde{K}'}(\tilde{\sigma}_{\tilde{s}}^\circ, \rho|_{\tilde{K}'}) \).

Then there exist nonzero \( \tilde{G} \)-linear maps \( \pi(\tilde{\tau}, \lambda) \rightarrow \rho \) and \( \pi'(\tilde{s}, \lambda') \rightarrow \rho \), which are surjective if \( \rho \) is irreducible.

A parameter \( (\tilde{\tau}, \lambda) \) (with respect to either \( \tilde{K} \) or \( \tilde{K}' \)) will be called supersingular if \( \lambda = 0 \), and nonsupersingular otherwise. We will call \( \pi(\tilde{\tau}, \lambda) \) and \( \pi'(\tilde{s}, \lambda') \) cokernel modules due to their construction as the cokernel of an element of a Hecke algebra.

The following theorem is the main classification result. The proof is given in [7].

Theorem 5.10. The smooth, genuine, irreducible, admissible \( E \)-representations of \( \tilde{G} \) fall into two disjoint, nonempty classes:

1. those which have only nonsupersingular parameters,
2. those which have only supersingular parameters.

The representations in the first class are exactly the genuine principal series representations of \( \tilde{G} \). All representations in the second class are supercuspidal.

6. Nonsupercuspidal representations of \( \tilde{G} \)

In this section we study the genuine representations of \( \tilde{G} \) which arise by parabolic induction; in particular, we consider smooth inductions to \( \tilde{G} \) of smooth genuine characters of \( \tilde{B} \). We show that every such representation is irreducible (Theorem 6.10) and that there are no intertwiners between \( \tilde{G} \)-representations induced from distinct characters of \( \tilde{B} \). We also give dictionaries (Theorem 6.13, Corollary 6.14, and Corollary 6.15) between parametrizations of the nonsupercuspidal representations: by parameters with respect to \( \tilde{K} \), by parameters with respect to \( \tilde{K}' \), and by characters of \( F^\times \).

6.1. Hecke action on weight spaces of nonsupercuspidal representations. By Lemma 2.4 (2), the abelianization of \( \tilde{B} \) is the quotient \( \tilde{B}/U' = \tilde{T} \). Therefore any genuine character of \( \tilde{B} \) arises by inflation of a genuine character of \( \tilde{T} \).

Proposition 6.1. Let \( \pi \) be a smooth genuine character of \( \tilde{T} \), let \( \tilde{\sigma}_\tau \) be a weight of \( \tilde{K} \), let \( \tau_{\tilde{\tau}}^{\tilde{\tau}_{\tilde{\tau}}} \) denote the generator of \( \mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_\tau)_{U(4)}) \) defined in (4.3), and let \( \mathcal{T}_1^{\tilde{\tau}} \) denote the generator of \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_\tau) \) defined in (4.10).

1. For each \( f' \in \text{Hom}_{\tilde{K}}(\tilde{\sigma}_\tau, \text{Ind}_{\tilde{B}}^{\tilde{G}}(\pi)|_{\tilde{K}}) \),
   \[ f' \cdot \mathcal{T}_1^{\tilde{\tau}} = \pi(\tilde{h}(\varpi)) \cdot f' . \]

2. For each \( f' \in \text{Hom}_{\tilde{K}'}(\tilde{\sigma}_{\tilde{s}}^\circ, \text{Ind}_{\tilde{B}}^{\tilde{G}}(\pi)|_{\tilde{K}'}) \),
   \[ f' \cdot \mathcal{T}_1^{\tilde{\sigma}_{\tilde{s}}^\circ} = \pi(\tilde{h}(\varpi)) \cdot f' . \]

Proof of Proposition 6.1. To ease notation, put \( T_1 := \mathcal{T}_1^{\tilde{\tau}} \), \( T_1^{\tilde{\sigma}_{\tilde{s}}^\circ} := \mathcal{T}_1^{\tilde{\sigma}_{\tilde{s}}^\circ} \), \( \tau_{-1} := \tau_{\tilde{\tau}}^{-1} \), and \( \psi_{-1} := \psi_{\tilde{\tau}}^{-1} \) for the duration of the proof.
(1) Let \( f' \) be any element of \( \text{Hom}_K(\tilde{\sigma}, \text{Ind}_B^G \pi|_K) \), and let \( f \) denote the element of \( \text{Hom}_G(\text{ind}_K^G \tilde{\sigma}, \text{Ind}_B^G \pi) \) which corresponds to \( f' \) by Frobenius reciprocity. Let \( \mathcal{F}_0 \) denote the unique element of
\[
\text{Hom}_G\left(\text{ind}_K^G \tilde{\sigma}, \text{Ind}_B^G \left( \text{ind}_K^G \tilde{\sigma}(\pi) \right) |_{\tilde{T} \cap K}\right)
\]
satisfying
\[
(\mathcal{F}_0)_{\tilde{T}} = 1 \in \text{End}_{\tilde{T}} \left( \text{ind}_K^G \left( \text{ind}_K^G \tilde{\sigma}(\pi) \right) |_{\tilde{T} \cap K} \right).
\]
Then \( f_{\tilde{T}} = f_{\tilde{T}} \circ (\mathcal{F}_0)_{\tilde{T}} \), so \( f = \text{Ind}_B^G (f_{\tilde{T}}) \circ \mathcal{F}_0 \).

By Proposition 4.8 (1),
\[
(\mathcal{F}_0 \circ \mathcal{T}_1)_{\tilde{T}} = S_f(\mathcal{T}_1) = S_f(\mathcal{T}_1) \circ (\mathcal{F}_0)_{\tilde{T}},
\]
so
\[
\mathcal{F}_0 \circ \mathcal{T}_1 = \text{Ind}_B^G (S_f(\mathcal{T}_1)) \circ \mathcal{F}_0.
\]

Thus \( f \circ \mathcal{T}_1 = \text{Ind}_B^G (f_{\tilde{T}}) \circ \mathcal{F}_0 \circ \mathcal{T}_1 = \text{Ind}_B^G (f_{\tilde{T}}) \circ \text{Ind}_B^G (S_f(\mathcal{T}_1)) \circ \mathcal{F}_0 = \text{Ind}_B^G (f_{\tilde{T}} \circ \mathcal{T}_1) \circ \mathcal{F}_0 \), using Proposition 4.12 for the last equality.

By Frobenius reciprocity \( f_{\tilde{T}} \circ \mathcal{T}_1 \) corresponds to \( (f_{\tilde{T}})' \cdot \mathcal{T}_1 \in \text{Hom}_{\tilde{T} \cap K}(\tilde{\sigma} |_{\tilde{T} \cap K}, \pi |_{\tilde{T} \cap K}) \).

By (1) of this proposition, \( (f_{\tilde{T}})' \cdot \mathcal{T}_1 = \pi(\tilde{\sigma}(\pi)) \cdot (f_{\tilde{T}})' \), so \( f_{\tilde{T}} \circ \mathcal{T}_1 = \pi(\tilde{\sigma}(\pi)) \cdot f_{\tilde{T}} \).

Then \( \text{Ind}_B^G (f_{\tilde{T}} \circ \mathcal{T}_1) = \pi(\tilde{\sigma}(\pi)) \cdot \text{Ind}_B^G (f_{\tilde{T}}) \), so
\[
f \circ \mathcal{T}_1 = \pi(\tilde{\sigma}(\pi)) \cdot \text{Ind}_B^G (f_{\tilde{T}}) \circ \mathcal{F}_0 = \pi(\tilde{\sigma}(\pi)) \cdot f.
\]
Passing back to \( \text{Hom}_K(\tilde{\sigma}, \text{Ind}_B^G \pi|_K) \), we have \( f' \cdot \mathcal{T} = \pi(\tilde{\sigma}(\pi)) \cdot f' \).

(2) We will show that \( \mathcal{T}_1^\alpha \) acts on \( \text{Hom}_G\left(\text{ind}_K^G \tilde{\sigma}, \text{Ind}_B^G (\pi)\right) \) by the scalar \( \pi(\tilde{\sigma}(\pi)) \); then (3) follows by Frobenius reciprocity. We again use the map \( \Phi \) defined in the proof of Lemma 4.11 precomposing with the identity map of vector spaces to obtain an isomorphism \( \mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}) \rightarrow \mathcal{H}(\tilde{G}, \tilde{K}', \tilde{\sigma}) \) which we again denote by \( \Phi \). This isomorphism satisfies \( \mathcal{T}_1^\alpha(\Phi(h)) = \Phi(\mathcal{T}_1(h)) \) for all \( h \in \text{ind}_K^G \tilde{\sigma} \).

Given \( f \in \text{Hom}_G\left(\text{ind}_K^G \tilde{\sigma}, \text{Ind}_B^G (\pi)\right) \), let \( \tilde{f} = \Phi^{-1}(f) \in \text{Hom}_G\left(\text{ind}_K^G \tilde{\sigma}, \text{Ind}_B^G (\pi)\right) \); i.e.,
\[
f(\Phi(h)) = \tilde{f}(h)
\]
for all \( h \in \text{ind}_K^G \tilde{\sigma} \). We have to show that
\[
(f \circ \mathcal{T}_1^\alpha)(\Phi(h)) = \pi(\tilde{\sigma}(\pi)) \cdot f(\Phi(h))
\]
for all \( h \in \text{ind}_K^G \tilde{\sigma} \). But the left-hand side of (6.1) is equal to
\[
f \circ \Phi(\mathcal{T}_1)(\Phi(h)) = f(\Phi(\mathcal{T}_1)(h)) = (f \circ \mathcal{T}_1)(h),
\]
while the right-hand side of (6.1) is equal to \( \pi(\tilde{\sigma}(\pi)) \cdot \tilde{f}(\Phi(h)) \). By (2) we have \( f \circ \mathcal{T}_1(h) = \pi(\tilde{\sigma}(\pi)) \cdot \tilde{f}(\Phi(h)) \), so (6.1) holds.

\( \square \)

6.2. Genuine characters of \( \tilde{T} \). We begin by recalling the parametrization of genuine complex characters of \( \tilde{T} \).
6.2.1. **Genuine \( \mathbb{C} \)-valued characters of \( \tilde{T} \).** The genuine \( \mathbb{C} \)-valued characters of \( \tilde{T} \) have been classified in terms of a certain map \( \gamma : F^\times \times \hat{F} \to \mu_4(\mathbb{C}) \), the *Weil index* defined in [27]. Here \( \hat{F} \) denotes the group of \( \mathbb{C} \)-valued continuous additive characters of \( F \). Fixing a nontrivial character \( \psi \in \hat{F} \), one obtains a function \( \gamma_F(-, \psi) : F^\times \to \mu_4(\mathbb{C}) \). Define

\[
\chi_\psi : \tilde{T} \to \mu_4(\mathbb{C}) \\
(h(x), \zeta) \mapsto \zeta \cdot \gamma_F(x, \psi)^{-1}.
\]

Then \( \chi_\psi \) is a smooth genuine \( \mathbb{C} \)-character of \( \tilde{T} \). The following fact is mentioned in [9]:

**Lemma 6.2.** Let \( \psi \) be a nontrivial \( \mathbb{C} \)-valued additive character of \( F \). There is a bijection, depending on \( \psi \), of

\[
\{ \text{smooth } E \text{-valued characters of } F^\times \} \to \{ \text{smooth genuine } \mathbb{C} \text{-valued characters of } \tilde{T} \},
\]

given by \( \mu \mapsto \mu \cdot \chi_\psi \).

6.2.2. **Genuine \( E \)-valued characters of \( \tilde{T} \).** We would like to have a similar parametrization of the genuine \( E \)-valued characters of \( \tilde{T} \). It is well-known that there is no nontrivial \( E \)-valued continuous additive character of \( F \). However, given a nontrivial \( \mathbb{C} \)-valued additive character \( \psi \) of \( F \), we may postcompose the resulting \( \mu_4(\mathbb{C}) \)-valued function \( \gamma_F(-, \psi)^{-1} \) with an isomorphism of \( \mu_4(\mathbb{C}) \) with \( \mu_4(E) \) to obtain a \( \mu_4(E) \)-valued function on \( F^\times \). We denote this function again by \( \gamma_F(-, \psi)^{-1} \), and define an \( E \)-valued character of \( \tilde{T} \) by

\[
(6.2) \quad \chi_\psi(h(a)), \zeta) := \zeta \gamma_F(a, \psi)^{-1}.
\]

Thus we directly transport Lemma 6.2 to obtain a parametrization of genuine \( E \)-valued characters of \( \tilde{T} \):

**Lemma 6.3.** Let \( \psi \) be a \( \mathbb{C} \)-valued additive character of \( F \), and let \( \chi_\psi \) be the genuine \( E \)-valued character of \( \tilde{T} \) defined in (6.2). Then there is a bijection

\[
\{ \text{smooth } E \text{-valued characters of } F^\times \} \to \{ \text{smooth genuine } E \text{-valued characters of } \tilde{T} \}
\]

given by \( \mu \mapsto \mu \cdot \chi_\psi \), where \( \mu \) is viewed as a (non-genuine) character of \( \tilde{T} \) by inflation.

6.2.3. **The Weil index.** Here we recall some properties of the Weil index and develop explicit expressions for certain values, which will be needed in later sections.

Let \( L \) be either a local field (e.g., \( F \)) or a finite field (e.g., \( \mathfrak{f} \)), and let \( \psi \) be a nontrivial additive \( \mathbb{C} \)-valued character of \( L \). Let \( \gamma_{L, \psi} \) denote the Weil index defined in [27] Theorem 2. This is a \( \mu_4(\mathbb{C}) \)-valued function, depending on \( \psi \), defined on the Witt group of quadratic forms over \( L \). For \( a \in L^\times \), let \( \gamma_{L, \psi}(a) \) denote the value of \( \gamma_{L, \psi} \) on the class of the quadratic form \( q(x) = ax^2 \). The following theorem, first proven by Weil [27], is stated here as in Rao [24].

**Theorem 6.4 (24 Theorem A.4).** Let \( \gamma_{L}(a, \psi) = \gamma_{L, \psi}(a)/\gamma_{L, \psi}(1) \). Then

1. \( \gamma_{L}(ac^2, \psi) = \gamma_{L}(a, \psi) \) for any \( a, c \in L^\times \),
2. \( \gamma_{L}(a, \psi)\gamma_{L}(b, \psi) = \gamma_{L}(ab, \psi)(a,b)F \).

Another theorem stated by Rao [24] gives a more detailed description of \( \gamma_F(a, \psi) \) when \( a \in \mathcal{O}_F \):
Theorem 6.5 ([24] Theorem A.11 with correction). Suppose $F$ is a nonarchimedean local field with finite residue field $\mathfrak{p}$ of odd characteristic, and let $m$ denote the conductor of $\psi$. Given any integer $n$, let $\delta(n) \in \{0, 1\}$ denote its parity. For $y \in \mathcal{O}_F$, define

$$\tilde{\psi}(x + \varpi y) = \psi(\varpi^{-m-1}x).$$

Then

1. $\tilde{\psi}$ is a nontrivial character of $\mathfrak{p}$ and

$$\gamma_F, \tilde{\psi}(1) = (\gamma_{\mathfrak{t}, \tilde{\psi}}(1))^{\delta(m)},$$

2. If $a \in \mathcal{O}_F$ and $a = u \cdot \varpi^{v_F(a)}$ with $u \in \mathcal{O}_F^\times$,

$$\gamma_F(a, \tilde{\psi}) = \begin{cases} 
1 & \text{if } \delta(m) = \delta(v_F(a)) = 0, \\
(u, \varpi)_F \cdot \gamma_{\mathfrak{t}, \tilde{\psi}} & \text{if } \delta(m) = 0 \text{ and } \delta(v_F(a)) = 1, \\
(u, \varpi)_F & \text{if } \delta(m) = 1 \text{ and } \delta(v_F(a)) = 0, \\
\gamma_{\mathfrak{t}, \psi}^{-1} & \text{if } \delta(m) = \delta(v_F(a)) = 1.
\end{cases}$$

In particular, if $a \in \mathcal{O}_F^\times$,

$$\gamma_F(a, \psi) = (a, \varpi)^m_{\mathfrak{p}}.$$ 

Proof of Theorem 6.5. The proof of (1) is given in [24]. We give a proof of (2) since our formula differs from the one given in [24] Thm. A.11 (and appears to disagree with it; for example, we claim that $\gamma_F(\mathfrak{t}, \psi)$ is nontrivial on units when the conductor of $\psi$ is odd; our formula is consistent with Gan-Savin’s remark in §2.6 of [9], while the apparent statement in Rao is not). If $a = u \cdot \varpi^{v_F(a)} \in \mathcal{O}_F^\times$ and $\psi$ is a nontrivial additive $\mathbb{C}$-character of $F$ with conductor $m$, then the conductor of $\psi_a = (x \mapsto \psi(ax))$ is $m + v_F(a)$. By part (1) of the theorem, we have

$$\gamma_F(a, \tilde{\psi}) = \frac{\gamma_F(\tilde{\psi}_a)}{\gamma_F(\tilde{\psi})} = \left(\gamma_{\mathfrak{t}, \tilde{\psi}}(1)^{-1}\gamma_{\mathfrak{t}, \tilde{\psi}}(a)\right)^{\delta(m + v_F(a))}.$$ 

The character $\tilde{\psi}_a$ sends $x + \varpi\mathcal{O}_F$ to $\psi(\varpi^{-m-v_F(a)-1}ax) = \psi(\varpi^{-m-1}ax)$, so

$$\tilde{\psi}_a = \tilde{\psi}_u = \tilde{\psi}_u.$$ 

We have

$$\gamma_{\mathfrak{t}, \tilde{\psi}}(\tilde{u}, \tilde{\psi}) = \gamma_{\mathfrak{t}, \tilde{\psi}}(\tilde{u}) \gamma_{\mathfrak{t}, \tilde{\psi}}(\tilde{\psi}) = \left(\frac{\tilde{u}}{\mathfrak{p}}\right),$$

where the first equality is the definition of $\gamma_{\mathfrak{t}, \tilde{\psi}}(\tilde{u}, \tilde{\psi})$ and the second is [24] Thm. A.9 (i). Under our assumptions on $F$, the Legendre symbol $\left(\frac{\tilde{u}}{\mathfrak{p}}\right)$ is equal to the Hilbert symbol $(u, \varpi)_F$, so

$$\gamma_{\mathfrak{t}, \tilde{\psi}}(\tilde{u}) = \gamma_{\mathfrak{t}, \tilde{\psi}}(\tilde{u}) \cdot \gamma_{\mathfrak{t}, \tilde{\psi}} = (u, \varpi)_F \cdot \gamma_{\mathfrak{t}, \tilde{\psi}}.$$ 

Substituting $(u, \varpi)_F \cdot \gamma_{\mathfrak{t}, \tilde{\psi}}$ in (6.3) gives

$$\gamma_F(a, \psi) = \frac{(u, \varpi)_F \cdot \gamma_{\mathfrak{t}, \psi}(a)}{(\gamma_{\mathfrak{t}, \psi})^{\delta(m + v_F(a))}},$$

which depends on both the parity of $m$ and also of $v_F(a)$. Considering (6.4) in each of the four cases, we get (2). □
6.3. Weight spaces of principal series representations. By Lemma 2.4 (2), any genuine character of $\overline{B}$ is the inflation of a genuine character of $\overline{T}$. Fix a nontrivial additive $C$-valued character $\psi$ and for any smooth character $\mu$ of $F^\times$, let $\mu \cdot \chi_\psi$ also denote the inflation to $\overline{B}$ of the $E$-valued character of $\overline{T}$ considered in 3.2. By Lemma 6.3, every genuine $E$-valued character of $\overline{T}$ appears as $\mu \cdot \chi_\psi$ for some character $\mu$ of $F^\times$. Hence any genuine principal series representation of $\overline{G}$ is isomorphic to $\text{Ind}_{\overline{B}}(\mu \cdot \chi_\psi)$ for some smooth character $\mu$ of $F^\times$.

**Proposition 6.6.** Fix a nontrivial additive character $\psi : F \to C$ and denote the conductor of $\psi$ by $m$. Let $\mu$ be a smooth $E$-valued character of $F^\times$ and view $\mu \cdot \chi_\psi$ (defined in 6.2) as a genuine character of $\overline{B}$ by inflation from $\overline{T}$.

1. Suppose $\mu |_{\mathcal{O}_F^\times} \neq (-, \omega)_F^m$. Then there is a unique weight $\bar{\sigma}_\tau$ of $\bar{K}$ such that $\bar{\sigma}_\tau$ is any weight of $\bar{K}$.

$$\dim_E \text{Hom}_{\bar{K}}(\bar{\sigma}_\tau, \text{Ind}_{\bar{B}}(\mu \cdot \chi_\psi)|_{\bar{K}}) = \begin{cases} 1 & \text{if } \bar{s} = \bar{r}, \\ 0 & \text{otherwise.} \end{cases}$$

The weight $\bar{\sigma}_\tau$ satisfies $1 < \dim_E \bar{\sigma}_\tau < q$, i.e., $\bar{r} \notin \{0, p - 1\}$.

2. Suppose $\mu |_{\mathcal{O}_F^\times} = (-, \omega)_F^m$. Then

$$\dim_E \text{Hom}_{\bar{K}}(\bar{\sigma}_\tau, \text{Ind}_{\bar{B}}(\mu \cdot \chi_\psi)|_{\bar{K}}) = \begin{cases} 1 & \text{if } \bar{s} = 0 \text{ or } \bar{s} = p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

3. Suppose $\mu |_{\mathcal{O}_F^\times} \neq (-, \omega)_F^{m+1}$. Then there is a unique weight $\bar{\sigma}_\tau$ of $\bar{K}'$ such that $\bar{\sigma}_\tau$ is any weight of $\bar{K}'$.

$$\dim_E \text{Hom}_{\bar{K}'}(\bar{\sigma}_\tau, \text{Ind}_{\bar{B}}(\mu \cdot \chi_\psi)|_{\bar{K} glimpse)} = \begin{cases} 1 & \text{if } \bar{s} = \bar{r}, \\ 0 & \text{otherwise.} \end{cases}$$

The weight $\bar{\sigma}_\tau$ satisfies $1 < \dim_E \bar{\sigma}_\tau < q$, i.e., $\bar{r} \notin \{0, p - 1\}$.

4. Suppose $\mu |_{\mathcal{O}_F^\times} = (-, \omega)_F^{m+1}$. Then

$$\dim_E \text{Hom}_{\bar{K}'}(\bar{\sigma}_\tau, \text{Ind}_{\bar{B}}(\mu \cdot \chi_\psi)|_{\bar{K} glimpse}) = \begin{cases} 1 & \text{if } \bar{s} = 0 \text{ or } \bar{s} = p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof of Proposition 6.6.** Let $\mu$ be any smooth character of $F^\times$ and let $\bar{\sigma}_\tau$ be any weight of $\bar{K}$. Each of $\mu \cdot \chi_\psi$ and $(\bar{\sigma}_\tau|_{T \cap \bar{K}})$ is a one-dimensional representation of $\overline{T} \cap \bar{K}$, so the space $\text{Hom}_{\overline{T} \cap \bar{K}}((\bar{\sigma}_\tau|_{T \cap \bar{K}}), \mu \cdot \chi_\psi|_{\overline{T} \cap \bar{K}})$ is one-dimensional if $((\bar{\sigma}_\tau|_{T \cap \bar{K}}) \cong \mu \cdot \chi_\psi|_{\overline{T} \cap \bar{K}})$ as $\overline{T} \cap \bar{K}$-representations, and zero otherwise.

By Frobenius reciprocity,

$$\dim_E \text{Hom}_{\overline{T} \cap \bar{K}}((\bar{\sigma}_\tau|_{T \cap \bar{K}}), \mu \cdot \chi_\psi|_{\overline{T} \cap \bar{K}}) = \dim_E \text{Hom}_{\overline{T}}(\text{ind}_{\overline{T} \cap \bar{K}}(\bar{\sigma}_\tau|_{T \cap \bar{K}}), \mu \cdot \chi_\psi).$$

By Lemma 3.7 (1) followed by Frobenius reciprocity again,

$$\dim_E \text{Hom}_{\overline{T}}(\text{ind}_{\overline{T} \cap \bar{K}}(\bar{\sigma}_\tau|_{T \cap \bar{K}}), \mu \cdot \chi_\psi) = \dim_E \text{Hom}_{\overline{G}}(\text{ind}_{\bar{K}}(\bar{\sigma}_\tau), \text{Ind}_{\bar{B}}(\mu \cdot \chi_\psi)) = \dim_E \text{Hom}_{\bar{K}}(\bar{\sigma}_\tau, \text{Ind}_{\bar{B}}(\mu \cdot \chi_\psi)|_{\bar{K} glimpse}).$$
Thus we have
\[
\dim_E \text{Hom}_K(\tilde{\sigma}, \text{Ind}_{\bar{\mathfrak{g}}}^G(\mu \cdot \chi_\psi)|_{\bar{K}}) = \begin{cases} 
1 & \text{if } (\tilde{\sigma})_{\mathcal{U}(t)} \cong \mu \cdot \chi_\psi |_{\tilde{T} \cap \bar{K}} \text{ as } \tilde{T} \cap \bar{K}-\text{representations,} \\
0 & \text{otherwise.} 
\end{cases}
\]

Recall that every smooth character of $\mathcal{O}_F^\times$ is isomorphic $\delta_{\bar{\tau}}$ for exactly one $\bar{s} \in \{0, \ldots, p-1\} \setminus \{p-1\}$ (see [2.1.3] for the definition of $\delta_{\bar{s}}$), and that $\delta_{\bar{0}} = \delta_{p-1}$. Furthermore, each smooth genuine representation of $\tilde{T} \cap \bar{K}$ is isomorphic to $\delta_{\bar{s}} := \delta_{\bar{s}} \otimes \epsilon$, where $\delta_{\bar{s}}$ is viewed as a character of $\tilde{T} \cap K^*$ by inflation. In particular, the $\tilde{T} \cap \bar{K}$-representation on $(\tilde{\sigma})_{\mathcal{U}(t)}$ is $\delta_{\bar{r}}$.

By Theorem [5.5] $\gamma_F(a, \psi)^{-1} = (a, \varpi)^{\nu}_{\mathcal{O}_F}$ for all $a \in \mathcal{O}_F^\times$. Under the condition of case (1) of the proposition, the restriction of $\mu \cdot \chi_\psi$ to $\tilde{T} \cap K^*$ factors through a nontrivial smooth character of $\mathcal{O}_F^\times$, i.e., through $\delta_{\bar{s}}$ for some $\bar{s} \notin \{\bar{0}, \bar{p}-1\}$, so we have a $\tilde{T} \cap \bar{K}$-isomorphism $(\tilde{\sigma})_{\mathcal{U}(t)} \cong \mu \cdot \chi_\psi$ if and only if $\bar{s} = \bar{r}$. Since $\bar{r}$ is neither $\bar{0}$ nor $\bar{p}-1$, the dimension of $\tilde{\sigma}$ is strictly between 1 and $q$.

In case (2) of the proposition, the restriction of $\mu \cdot \chi_\psi$ to $\tilde{T} \cap K^*$ factors through the trivial character of $\mathcal{O}_F^\times$. Thus we have a $\tilde{T} \cap \bar{K}$-isomorphism $(\tilde{\sigma})_{\mathcal{U}(t)} \cong \mu \cdot \chi_\psi$ exactly when $\bar{r} = \bar{0}$ or $\bar{r} = \bar{p}-1$, i.e., exactly when $\dim_E \tilde{\sigma} = 1$ or $q$.

The proofs of (3) and (4) are similar. The difference between the statements (1) and (3) (resp., between (2) and (4)) is a consequence of the fact (Lemma [3.3]) that for a weight $\tilde{\sigma}^\circ_{\mathfrak{g}}$ of $\bar{K}$, the $\tilde{T} \cap \bar{K}$-representation on $(\tilde{\sigma}_{\mathfrak{g}})_{\mathcal{U}(t)}$ is $\delta_{\bar{r}} \cdot (-, \varpi)_{\mathcal{O}_F}$ rather than $\delta_{\bar{r}}$.

We maintain the notations $\mathcal{T}_n := \mathcal{T}_n^\epsilon$, $\tau_{-n} := \tau_{-n}^\epsilon$, $S := S_{\mathfrak{g}}$ for the remainder of this section, and continue to refer to the map $\mathcal{F}_0$ defined in the proof of Proposition [6.7]. Define another map

\[(5.6) \quad \mathcal{F} : \text{ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma}) \otimes_{\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}), \mathcal{S}_\mathfrak{g}} \mathcal{H}(\tilde{T}, \tilde{T} \cap \bar{K}, (\tilde{\sigma})_{\mathcal{U}(t)}) \to \text{Ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma})_{\mathcal{U}(t)}|_{\tilde{T} \cap \bar{K}}.
\]

by setting $\mathcal{F}(f \cdot \tau_{-n}) := \left(\text{Ind}_{\bar{\mathfrak{g}}}^G(\tau_{-n}) \circ \mathcal{F}_0\right)(f)$ for $f \in \text{ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma}), n \in \mathbb{Z}$. Then $\mathcal{F}$ is clearly $\tilde{G}$-linear, and has the following $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_{\mathfrak{g}})$-linearity:

\[(\mathcal{F} \circ \mathcal{T}_1)(f \cdot \tau_{-n}) = \left(\text{Ind}_{\bar{\mathfrak{g}}}^G(\tau_{-n}) \circ \mathcal{F}_0 \circ \mathcal{T}_1\right)(f) = \left(\text{Ind}_{\bar{\mathfrak{g}}}^G(\tau_{-n} \circ \mathcal{S}(\mathcal{T}_1) \circ \mathcal{F}_0\right)(f) = \left(\text{Ind}_{\bar{\mathfrak{g}}}^G(\tau_{-(n+1)} \circ \mathcal{F}_0\right)(f).
\]

Proposition 6.7. The map $\mathcal{F} : \text{ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma}) \otimes_{\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}), \mathcal{S}_\mathfrak{g}} \mathcal{H}(\tilde{T}, \tilde{T} \cap \bar{K}, (\tilde{\sigma})_{\mathcal{U}(t)}) \to \text{Ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma})_{\mathcal{U}(t)}|_{\tilde{T} \cap \bar{K}}$ defined in (6.5) is an injective map of $(\tilde{G}, \mathcal{H}(\tilde{T}, \tilde{T} \cap \bar{K}, (\tilde{\sigma})_{\mathcal{U}(t)}))$-modules. If $\bar{r} \neq \bar{0}$, then $\mathcal{F}$ is an isomorphism.

Proof of Proposition 6.7. The argument of [12] Theorem 32 goes through with only the obvious adaptations. We repeat the argument here for completeness. For injectivity of $\mathcal{F}$, it suffices to show that $\mathcal{F}_0 : \text{ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma}) \to \text{Ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma})_{\mathcal{U}(t)}$ is injective. Suppose that $\mathcal{F}_0$ has a nonzero kernel. Then $\text{ker}(\mathcal{F}_0)$ is a genuine subrepresentation of $\text{ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma})$, so $\text{ker}(\mathcal{F}_0)$ contains some weight $\tilde{\sigma}_{\mathfrak{g}}$ of $\bar{K}$. The $\bar{K}$-linear injection $\tilde{\sigma}_{\mathfrak{g}} \hookrightarrow \text{ker}(\mathcal{F}_0)|_{\bar{K}}$ corresponds by Frobenius reciprocity to a nonzero $\tilde{G}$-linear homomorphism $\text{ind}_{\bar{\mathfrak{g}}}^G(\tilde{\sigma}) \to \text{ker}(\mathcal{F}_0)$, which we may compose
with the $\tilde{G}$-linear inclusion $\ker(F_0) \hookrightarrow \text{Ind}_{K}^{\tilde{G}} \sigma_{\hat{r}}$ to get a nonzero element $\Phi \in \mathcal{H}_c(\tilde{G}, \tilde{K}, \tilde{\sigma}_{\hat{r}}, \tilde{\sigma}_{r})$ such that $F_0 \circ \Phi = 0$. Then

$$0 = (F_0 \circ \Phi)_{\tilde{T}} = 1_{\text{Ind}_{K}^{\tilde{G}}((\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)})} \circ S_{\tilde{G}, \tilde{F}}(\Phi),$$

so $S_{\tilde{G}, \tilde{F}}(\Phi) = 0$. But $S_{\tilde{G}, \tilde{F}}$ is injective, so $\Phi = 0$, contradicting the assumption that $\ker(F_0) \neq 0$.

Next we show surjectivity under the assumption that $\hat{r} \neq 0$. Let $0 \neq v \in V_{\tilde{r}}^{U(t)}$ and write

$$f_0 = F([1, v]).$$

Then, unwinding the chain of isomorphisms in Lemma [3.7] we have the following formula for $f_0$:

$$f_0(g) = [t, p_{\mathcal{U}(t)} \circ \tilde{\sigma}(k)v],$$

where $g = tuk$ for $t \in \tilde{T}$, $u \in \mathcal{U}$, and $k \in \tilde{K}$, and where $p_{\mathcal{U}(t)}$ denotes the projection $\tilde{\sigma}_{\hat{r}} \rightarrow (\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)}$.

Let $e_{\hat{g}}$ (resp., $e_{r}$) denote a basis for the one-dimensional highest-weight (resp., lowest-weight) space of $\tilde{\sigma}_{r}$. Let $k \in \tilde{K}$ and consider $Pr(k) \in K$. By the Bruhat decomposition of $G(t)$, the image of $Pr(k)$ in $G(t)$ lies in either $w(-1)B(t)$ or in $B(t)U(t)$. In the former case, then $\tilde{\sigma}(r)(k)e_{\hat{g}} = \zeta \tilde{\sigma}(r)(w(-1)) \sigma_{r}(b)e_{\hat{g}}$ for some $\zeta \in \mu_2$ and $b \in B \cap K$. Since $\sigma_{r}(b)e_{\hat{g}} \subset E \cdot e_{\hat{g}}$, we then have $\tilde{\sigma}_{\hat{r}}(k)e_{\hat{g}} \subset E \cdot e_{\hat{g}}$, hence (since $\hat{r} \neq 0$), $p_{\mathcal{U}(t)} \circ \tilde{\sigma}_{\hat{r}}(k)v = 0$.

Suppose $k \in \tilde{K}$ is in the latter case, i.e., that the image of $Pr(k)$ in $G(t)$ lies in $B(t)U(t)$. Then $Pr(k) \in (B \cap K) \cdot (U \cap K)$ and $\tilde{\sigma}_{\hat{r}}(k)e_{\hat{g}} \subset E \cdot e_{\hat{g}}$, so $p_{\mathcal{U}(t)} \circ \tilde{\sigma}_{\hat{r}}(k)v \neq 0$. Hence the support of $f_0$ is equal to $B \cdot (U \cap K)^*$. In particular, if $a \in \mathcal{O}_F^\times$, then

$$f_0 \left( \begin{array}{cc} 1 & 0 \\ \varpi^r a & 1 \end{array} \right), 1 = \begin{cases} [1, p_{\mathcal{U}(t)}(v)] \\ \left[ \left( \begin{array}{cc} \varpi^r a & 0 \\ 0 & \varpi^{-r} a^{-1} \end{array} \right), (-1, \varpi)^{\tilde{r}}_F, p_{\mathcal{U}(t)} \circ \sigma_{\hat{r}}(w(1)) v \right] \end{cases} = 0 \quad \text{if } r \geq 0 \quad \text{if } r < 0.
$$

(6.6)

We proceed in two steps: first, we will show that $f_0$ generates the subspace of $\text{Ind}_{\mathcal{B}}^{\tilde{G}} \left( \text{Ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{G}} \left( (\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)} \right) \right)$ consisting of functions with support in $\mathcal{B} \cdot (U \cap K)^*$. Then we will show that functions in this subspace generate all of $\text{Ind}_{\mathcal{B}}^{\tilde{G}} \left( \text{Ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{G}} \left( (\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)} \right) \right)$ under the $\tilde{G}$- and $\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)})$-actions.

Define a map

$$\left\{ f \in \text{Ind}_{\mathcal{B}}^{\tilde{G}} \left( \text{Ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{G}} \left( (\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)} \right) \right) : \text{supp}(f) \subset \mathcal{B} \cdot (U \cap K)^* \rightarrow C^\infty_c (F, \text{Ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{G}} \left( (\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)} \right) ) \right\}$$

by

$$f \mapsto \left( f : c \mapsto f(u(c), 1) \right),$$

and note that extension by zero is an inverse. Hence $f \mapsto f$ defines an isomorphism of $E$-vector spaces, and is also a map of $\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_{\hat{r}})$-modules. The subspace of $\text{Ind}_{\mathcal{B}}^{\tilde{G}} \left( \text{Ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{G}} \left( (\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)} \right) \right)$ with support in $\mathcal{B} \cdot (U \cap K)^*$ is $\mathcal{B}$-stable, and the induced $\mathcal{B}$-module structure on $C^\infty_c (F, \text{Ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{G}} \left( (\tilde{\sigma}_{\hat{r}})_{\mathcal{U}(t)} \right) )$ is given by

$$(u(c), 1) f'(d) = f(c + d),$$

$$(h(x), \zeta) f'(d) = f(x^{-2}d).$$
By (6.6), for \( c \in F \) we have

\[
\begin{cases} 
1, P_T(t)(c) & \text{if } c \in O_F, \\
0 & \text{otherwise}.
\end{cases}
\]

Then for \( n \in \mathbb{Z} \),

\[
(f_0 \cdot T^n)(c) = (\hat{h}(\varpi)^n \cdot f_0)(c) = \begin{cases} 
\hat{h}(\varpi)^n, P_T(t)(c) & \text{if } c \in O_F, \\
0 & \text{otherwise},
\end{cases}
\]

so the \( \mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)}) \)-module generated by \( f_0 \) contains all constant functions in \( \mathcal{C}_c^\infty(F, \text{ind}_{\tilde{T} \cap \tilde{K}}^\tilde{T}((\tilde{\sigma}_r)_{\U(t)}) \) which are supported on \( O_F \). Acting by \( \tilde{T} \) and by \( U^* \) gives all constant functions on scalings and translations of \( O_F \), and the set of these functions spans \( \mathcal{C}_c^\infty(F, \text{ind}_{\tilde{T} \cap \tilde{K}}^\tilde{T}((\tilde{\sigma}_r)_{\U(t)}) \). Hence \( f_0 \) generates all functions in \( \text{Ind}_B^\tilde{T}((\text{ind}_{\tilde{T} \cap \tilde{K}}^\tilde{T}((\tilde{\sigma}_r)_{\U(t)})) \) which have support in \( \mathcal{B} : (U \cap K)^* \). The set of such functions generates all of \( \text{Ind}_B^\tilde{T}((\tilde{\sigma}_r)_{\U(t)}) \) under the action of \( \tilde{G} \), so \( \mathcal{F} \) is surjective.

By Proposition 6.1 the action of the Hecke operator \( T_i^\tau \) on any nontrivial weight space of a principal series representation \( \text{Ind}_B^\tilde{T}(\mu \cdot \chi_\psi) \) has a unique eigenvalue, namely \( \mu \cdot \chi_\psi(\hat{h}(\varpi)) \). From now on, write

(6.7)

\[
\lambda_{\mu, \psi} := \mu \cdot \chi_\psi(\hat{h}(\varpi)).
\]

**Corollary 6.8.** Let \( \tilde{\tau}_r \) be a weight of \( \tilde{K} \) with \( \tilde{r} \neq \tilde{0} \), and let \( \mu \) be a smooth character of \( F^* \) such that \( \dim_{E} \text{Hom}_{\tilde{K}}(\tilde{\sigma}_r, \text{Ind}_B^\tilde{T}(\mu \cdot \chi_\psi)) = 1 \). Define a character \( \theta_{\mu, \psi} \) of \( \mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)} \) by setting

\[
\theta_{\mu, \psi}(\tau_{-1}) = \lambda_{\mu, \psi}.
\]

Then there is a \( \tilde{G} \)-linear isomorphism

\[
\pi(\tilde{r}, \lambda_{\mu, \psi}) \longrightarrow \text{Ind}_{B}^{\tilde{G}}((\text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T}}(\tilde{\sigma}_r)_{\U(t)}) \otimes_{\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r)} S_{\tilde{r}} \theta_{\mu, \psi}.
\]

**Proof of Corollary 6.8.** Since \( \tilde{r} \neq 0 \), Proposition 6.7 demonstrates that the map \( \mathcal{F} : \text{ind}_{\tilde{K}}^{\tilde{G}}((\tilde{\sigma}_r) \otimes_{\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r)} S_{\tilde{r}}) \mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)}) \rightarrow \text{Ind}_{B}^{\tilde{G}}((\text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T}}(\tilde{\sigma}_r)_{\U(t)}) \) defined in (6.5) is an isomorphism of \( \mathcal{G} \)-modules. Specializing the action of \( \mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)} \) to \( \theta_{\mu, \psi} \) on each side, we obtain an isomorphism of \( \mathcal{G} \)-modules

\[
\text{ind}_{\tilde{K}}^{\tilde{G}}((\tilde{\sigma}_r) \otimes_{\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r)} S_{\tilde{r}} \theta_{\mu, \psi} \rightarrow \text{Ind}_{B}^{\tilde{G}}((\mu \cdot \chi_\psi) \otimes_{\mathcal{H}(\tilde{G}, \tilde{K}, \tilde{\sigma}_r)} S_{\tilde{r}} \theta_{\mu, \psi},
\]

and domain of this isomorphism is equal to \( \pi(\tilde{r}, \lambda_{\mu, \psi}) \) since \( \theta_{\mu, \psi} \circ S_{\tilde{r}}^{-1}(T_{i}^{\tau}) = \lambda_{\mu, \psi}. \)

**Lemma 6.9.** Under the hypotheses of Corollary 6.8, there is a \( \tilde{G} \)-linear isomorphism

\[
\text{Ind}_{B}^{\tilde{G}}((\mu \cdot \chi_\psi) \rightarrow \text{Ind}_{B}^{\tilde{G}}((\text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T}}(\tilde{\sigma}_r)_{\U(t)}) \otimes_{\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)})} \theta_{\mu, \psi}.
\]

**Proof of Lemma 6.9.** Since \( \mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)}) \) is Noetherian and \( \text{Ind}_{B}^{\tilde{G}}(-) \) is an exact functor, and also (by Lemma 5.11) the universal module \( \text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T}}(\tilde{\sigma}_r)_{\U(t)} \) is a flat \( \mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)}) \)-module, the proof of the Corollary to Theorem 32 of [12] goes through to give a \( \tilde{G} \)-linear isomorphism

\[
\text{Ind}_{B}^{\tilde{G}}((\text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T}}((\tilde{\sigma}_r)_{\U(t)}) \otimes_{\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)})} \theta_{\mu, \psi}) \rightarrow \text{Ind}_{B}^{\tilde{G}}((\text{ind}_{\tilde{T} \cap \tilde{K}}^{\tilde{T}}((\tilde{\sigma}_r)_{\U(t)}) \otimes_{\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, (\tilde{\sigma}_r)_{\U(t)})} \theta_{\mu, \psi}.
\]
Under the hypotheses of Corollary \[\text{6.8}\] Lemma \[\text{6.3}\] implies that we have also have an $\tilde{G}$-linear isomorphism $\text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi) \rightarrow \text{Ind}_{\tilde{B}}^\tilde{G}\left(\text{ind}_{\tilde{T} \cap \tilde{K}}^\tilde{T}\left(\tilde{\sigma}_\tilde{r}\|_{\tilde{T}(\tilde{t})}\right) \otimes \text{H}(_{}\tilde{T} \cap \tilde{K},(\tilde{\sigma}_\tilde{r})|_{\tilde{T}(\tilde{t})}) \theta_{\mu,\psi}\right)$.

\[\textbf{Theorem 6.10.}\] Let $\psi : F \rightarrow \mathbb{C}$ be a nontrivial additive character, let $\mu : F^* \rightarrow E^*$ be a smooth character, and let $\chi_\psi$ be as defined in \[\text{(6.2)}\].

1. $\text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)$ is an irreducible $\tilde{G}$-representation.

2. Let $\tilde{\sigma}_\tilde{r}$ be any weight of $\tilde{K}$ such that $\text{Hom}_{\tilde{K}}(\tilde{\sigma}_\tilde{r}, \text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)) \neq 0$, and let $\lambda_{\mu,\psi} \in E$ be as defined in \[\text{(6.7)}\]. Then there is a $\tilde{G}$-linear isomorphism

$$\pi(\tilde{r}, \lambda_{\mu,\psi}) \rightarrow \text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi).$$

\[\text{Proof of Theorem 6.10.}\] First suppose that $\text{Hom}_{\tilde{K}}(\tilde{\sigma}_\tilde{r}, \text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)|_{\tilde{K}}) \neq 0$ for some weight $\tilde{\sigma}_\tilde{r}$ of $\tilde{K}$ with $\tilde{r} \notin \{0, \overline{p-1}\}$. Then Proposition \[\text{6.6}\] (1) and (2) imply that $\tilde{\sigma}_\tilde{r}$ is the unique weight of $\tilde{K}$ contained in $\text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)$ and that $\dim_E \text{Hom}_{\tilde{K}}(\tilde{\sigma}_\tilde{r}, \text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)|_{\tilde{K}}) = 1$. Composing the map of Corollary \[\text{6.8}\] with the inverse of the map of Lemma \[\text{6.9}\] we have a $\tilde{G}$-linear isomorphism

$$\pi(\tilde{r}, \lambda_{\mu,\psi}) \rightarrow \text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi).$$

Hence $\pi(\tilde{r}, \lambda_{\mu,\psi})$ likewise contains a unique weight of $\tilde{K}$, namely $\tilde{\sigma}_\tilde{r}$, and has a 1-dimensional $\tilde{\sigma}_\tilde{r}$-weight space. Thus by Proposition \[\text{3.2}\] the weight $\tilde{\sigma}_\tilde{r}$ generates an irreducible $\tilde{G}$-submodule of $\pi(\tilde{r}, \lambda_{\mu,\psi})$. The image of $\tilde{\sigma}_\tilde{r}$ under the composition $\tilde{\sigma}_\tilde{r} \rightarrow \text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_\tilde{r} \rightarrow \pi(\tilde{r}, \lambda_{\mu,\psi})$ generates all of $\pi(\tilde{r}, \lambda_{\mu,\psi})$ as a $\tilde{G}$-module, so both $\pi(\tilde{r}, \lambda_{\mu,\psi})$ and $\text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)$ are irreducible. We have proved both parts of Theorem \[\text{6.10}\] in the case where $\text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)$ contains a $\tilde{K}$-weight different from $\tilde{\sigma}_0$ and $\tilde{\sigma}_{\overline{p-1}}$.

Suppose that $\text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)$ does not contain any $\tilde{K}$-weight $\tilde{\sigma}_\tilde{r}$ such that $\tilde{r} \notin \{0, \overline{p-1}\}$. Then by Proposition \[\text{6.6}\] (1) and (2), the $\tilde{K}$-weights contained in $\text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)$ are exactly $\tilde{\sigma}_0$ and $\tilde{\sigma}_{\overline{p-1}}$, and both weight spaces are 1-dimensional. The map of Corollary \[\text{6.8}\] composes with the inverse of the map of Lemma \[\text{6.9}\] to give an isomorphism $\pi_0(\overline{p-1}, \lambda_{\mu,\psi}) \rightarrow \text{Ind}_{\tilde{B}}^\tilde{G}(\mu \cdot \chi_\psi)$. The following lemma gives the extra information needed to prove irreducibility of the principal series representations which contain these “extremal” $\tilde{K}$-weights.

\[\textbf{Lemma 6.11.}\] If $\theta$ is a character of $\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, \tilde{\sigma}_{\overline{p-1}})$, and if $\theta'$ is the character of $\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, \tilde{\sigma}_0)$ defined by $\theta'(\tilde{r}) = \theta(\tau_{\overline{p-1}}^{-1})$, then there is a $\tilde{G}$-linear isomorphism

$$\text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_{\overline{p-1}} \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_{\overline{p-1}}), S_{\overline{p-1}}^{-1}\theta \rightarrow \text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_0 \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_0), S_0^{-1} \theta'.$$

\[\text{Proof of Lemma 6.11.}\] We have $\theta' \circ S_0 = \theta \circ S_{\overline{p-1}}^{-1} \circ S_0$ and $\theta \circ S_{\overline{p-1}} = \theta' \circ S_{\overline{p-1}}^{-1} \circ S_{\overline{p-1}}$, so the Hecke operators $T_{\overline{p-1}}\delta$ and $T_{\overline{p-1}}\delta_{\overline{p-1}}$ induce $\tilde{G}$-linear homomorphisms

$$\text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_{\overline{p-1}} \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_{\overline{p-1}}), S_{\overline{p-1}}^{-1}\theta \rightarrow \text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_0 \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_0), S_0^{-1} \theta',$$

$$\text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_0 \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_0), S_0^{-1} \theta' \rightarrow \text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_{\overline{p-1}} \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_{\overline{p-1}}), S_{\overline{p-1}}^{-1}\theta$$

respectively. By Lemma \[\text{6.12}\] $T_{\overline{p-1}}\delta_{\overline{p-1}} \circ T_{\overline{p-1}}^{-1}\delta = (T_{\overline{p-1}}^{-1})^2$, so the composition of induced maps

$$\text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_{\overline{p-1}} \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_{\overline{p-1}}), S_{\overline{p-1}}^{-1}\theta \rightarrow \text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_0 \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_0), S_0^{-1} \theta' \rightarrow \text{ind}_{\tilde{K}}^\tilde{G}\tilde{\sigma}_{\overline{p-1}} \otimes \text{H}(_{}\tilde{G}, \tilde{K}, \tilde{\sigma}_{\overline{p-1}}), S_{\overline{p-1}}^{-1}\theta$$
is given by $\theta \circ S_{p^{-1}}(T_{1}^{p-1})^2$, i.e., by $(\theta(r_{p^{-1}}^{-1}))^2$. Likewise, the map induced by $T_{1}^{p-1,\delta} \circ T_{1}^{0,p^{-1}} = (T_{1}^0)^2$ is multiplication by $(\theta'(r_{p^{-1}}^{p-1}))^2 = (\theta(r_{p^{-1}}^{p-1}))^2$. Since $\theta$ is a character of $\mathcal{H}(\tilde{T}, \tilde{T} \cap \tilde{K}, \delta_{p^{-1}})$, the value $\theta(r_{p^{-1}}^{-1})$ lies in $E^\times$. Hence the maps induced by $T_{1}^{p-1,\delta}$ and $(\theta(r_{p^{-1}}^{-1}))^{-2} \cdot T_{1}^{0,p^{-1}}$ are inverse to each other, and the map induced by $T_{1}^{p-1,\delta}$ is the desired isomorphism.

Let $\rho$ be a nonzero $\tilde{G}$-subrepresentation of $\text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$. Then $\rho$ contains at least one $\tilde{K}$-weight. Since $\rho \subset \text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$, the $\tilde{K}$-weights of $\rho$ lie in the set $\{\tilde{\sigma}_{\tilde{K}}, \tilde{\sigma}_{p^{-1}}\}$, and the corresponding weight spaces are at most 1-dimensional. Suppose that $\rho$ contains $\tilde{\sigma}_{p^{-1}}^{-1}$. Then the image of the inclusion $\rho \hookrightarrow \text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ contains the image of $\tilde{\sigma}_{p^{-1}}$ in $\text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ under the composition

$$
\tilde{\sigma}_{p^{-1}}^{-1} \mapsto \text{ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi}) \mapsto \pi(p^{-1}, \lambda_{\mu}, \psi) \rightarrow \text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi}).
$$

Since the latter image generates $\text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ as a $\tilde{G}$-module, we have $\rho = \text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$.

Suppose towards a contradiction that $\rho$ does not contain $\tilde{\sigma}_{p^{-1}}$. Then $\rho$ contains $\tilde{\sigma}_{\tilde{K}}$, so Frobenius reciprocity provides a nonzero $\tilde{G}$-linear map $\text{ind}_{\tilde{K}}^{\tilde{G}}(\tilde{\sigma}_{\tilde{K}}) \rightarrow \rho$. Pulling back through the quotient $\text{ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ we have already seen that $\pi(\tilde{\sigma}_{\tilde{K}}, \lambda_{\mu}, \psi)$ fits into an exact sequence of principal series representations containing $\tilde{\sigma}_{\tilde{K}}$ and $\tilde{\sigma}_{p^{-1}}^{-1}$. Thus $\rho = \text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$. This completes the proof of (1) for principal series representations containing $\tilde{\sigma}_{\tilde{K}}$ and $\tilde{\sigma}_{p^{-1}}^{-1}$.

In the case that the weights of $\text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ are $\{\tilde{\sigma}_{\tilde{K}}, \tilde{\sigma}_{p^{-1}}^{-1}\}$, we have already seen that $\pi(p^{-1}, \lambda_{\mu}, \psi) \cong \text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$. By Lemma 4.11, also $\pi(\tilde{\sigma}_{\tilde{K}}, \lambda_{\mu}, \psi) \cong \text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$. This proves (2) for principal series representations which contain the extremal $\tilde{K}$-weights. \hfill $\square$

**Remark 6.12.** Alternatively, one can prove irreducibility of genuine principal series representations without any comparison to compact inductions. This was done by the author in [22] following the strategy of Abdellatif [1] for $SL_2(F)$. The idea is to show that $\text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ fits into an exact sequence of $\tilde{B}$-modules

$$
0 \rightarrow W_{\mu, \psi} \rightarrow \text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi}) \rightarrow \mu \cdot \chi_{\psi} \rightarrow 0
$$

in which $W_{\mu, \psi}$ is a certain irreducible infinite-dimensional $\tilde{B}$-module, so that each $\text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ is of length 2 as a $\tilde{B}$-module and admits $\mu \cdot \chi_{\psi}$ as its unique 1-dimensional subquotient. If $\text{Ind}_{\tilde{B}}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ were reducible as a $\tilde{G}$-module, then by smooth Frobenius reciprocity would admit $\mu \cdot \chi_{\psi}$ as a 1-dimensional subquotient. But this is impossible, since the abelianization of $\tilde{G}$ is trivial (cf. Lemma 2.4) and on the other hand $\mu \cdot \chi_{\psi}$ is genuine and therefore nontrivial.
Next we give a dictionary between parameters with respect to \( \widetilde{K} \) and to \( \widetilde{K}' \).

**Theorem 6.13.** Given \( \vec{r} \in \{0, \ldots, p-1\}^f \), let \( \vec{r}' \) denote any vector in \( \{0, \ldots, p-1\}^f \) such that

\[
\sum_{i=0}^{f-1} r'_i p^i \equiv \sum_{i=0}^{f-1} \left( r_i + \frac{p-1}{2} \right) p^i \pmod{q}
\]

(thus \( \vec{r}' \) is uniquely determined by \( \vec{r} \) if \( \vec{r} \neq \overrightarrow{\frac{p-1}{2}} \), and \( \vec{r}' \in \{0, p-1\} \) if \( \vec{r} = \overrightarrow{\frac{p-1}{2}} \)).

1. For any \( \vec{r} \in \{0, \ldots, p-1\}^f \) and \( \lambda \in E^\times \),

\[
\pi(\vec{r}, \lambda) \cong \pi'(\vec{r}', \lambda);
\]

in particular,

\[
\pi(\vec{0}, \lambda) \cong \pi'(\overrightarrow{p-1}, \lambda) \cong \pi'(\overrightarrow{\frac{p-1}{2}}, \lambda)
\]

and

\[
\pi'(\vec{0}, \lambda) \cong \pi'(\overrightarrow{p-1}, \lambda) \cong \pi'(\overrightarrow{\frac{p-1}{2}}, \lambda);
\]

2. the isomorphisms of (1) are the only equivalences between nonsupersingular cokernel modules.

**Proof of Theorem 6.13**

1. Let \( \vec{r} \in \{0, \ldots, p-1\}^f \) and \( \lambda \in E^\times \), and fix an additive \( \mathbb{C} \)-valued character \( \psi \) of conductor \( m \). Let \( \psi \) be a nontrivial additive \( \mathbb{C} \)-valued character of \( F \) of conductor \( m \). Denote the parity of \( m \) by \( s(m) \in \{0,1\} \). There exists a character \( \mu \) of \( F^\times \) such that \( \mu|_{\mathcal{O}_F^\times} \cong \delta_F \cdot (-, \varpi)_F^m \) and \( \mu(\varpi) = (-1, \varpi)_F \cdot \lambda \cdot \gamma_T(\psi)^{1-2s(m)} \). Then by Theorem 6.5 (2), the genuine character \( \mu \cdot \chi_\psi \) of \( \mathcal{T} \) satisfies

\[
\mu \cdot \chi_\psi|_{\mathcal{T} \cap K} \equiv \epsilon \cdot \delta_F \cdot (-, \varpi)_F^{2m} = \overrightarrow{\delta_F} = (\overrightarrow{\delta_F})_T(t)
\]

and

\[
\mu \cdot \chi_\psi(\overrightarrow{h}(\varpi)) = \lambda.
\]

The isomorphism (6.9) implies, due to Proposition 6.6 (1) and (2), that the \( \widetilde{K} \)-weight \( \delta_F \) is contained in \( \text{Ind}_{\mathcal{B}}^\mathcal{G} L(\mu \cdot \chi_\psi \cdot ) \). Hence (6.9) and (6.10) together with Theorem 6.10 (2) imply that \( \pi(\vec{r}, \lambda) \cong \text{Ind}_{\mathcal{B}}^\mathcal{G} \mu \cdot \chi_\psi \).

Now let \( \vec{r}' \in \{0, \ldots, p-1\}^f \) satisfy the condition of the theorem with respect to \( \vec{r} \). Then \( \mu' := \delta_F^{-1} \cdot \mu = (-, \varpi)_F \cdot \mu \) is a character of \( F^\times \) such that \( \mu'|\mathcal{O}_F^\times \cong \delta_F \cdot (-, \varpi)_F^m \) and \( \mu'(\overrightarrow{\varpi}) = \mu(\overrightarrow{\varpi}) \).

By Theorem 6.10 \( \pi(\vec{r}', \lambda) \cong \text{Ind}_{\mathcal{B}}^\mathcal{G} (\mu' \cdot \chi_\psi) \). And by Lemma 5.5 we have \( \pi'(\vec{r}', \lambda) \cong \left( \pi(\vec{r}, \lambda) \right)^{\overrightarrow{\alpha}} \), so

\[
\pi'(\vec{r}', \lambda) \cong \left( \text{Ind}_{\mathcal{B}}^\mathcal{G} (\mu' \cdot \chi_\psi) \right)^{\overrightarrow{\alpha}} \cong \text{Ind}_{\mathcal{B}}^\mathcal{G} (\mu' \cdot \chi_\psi)^{\overrightarrow{\alpha}}.
\]
Corollary 6.14. Let $\psi : F \to \mathbb{C}$ be a nontrivial additive character of conductor $m$, let $\mu : F^\times \to E^\times$ be a smooth multiplicative character, and let $\chi$ be the genuine $E$-valued character of $T$ defined in $(6.2)$.

(1) Suppose either that $2 | m$ and $\mu \mid_{O_F^\times} = 1$, or that $2 \nmid m$ and $\mu \mid_{O_F^\times} = (-, \varpi)_F$. Then the parameters of $\text{Ind}^G_B(\mu \cdot \chi \psi)$ with respect to $\tilde{K}$ are $(\tilde{0}, \lambda_{\mu, \psi})$ and $(\tilde{p}, \lambda_{\mu, \psi})$, and $\text{Ind}^G_B(\mu \cdot \chi \psi)$ has the unique parameter $(\tilde{p}, \lambda_{\mu, \psi})$ with respect to $\tilde{K}'$.

(2) Suppose either that $2 | m$ and $\mu \mid_{O_F^\times} = (-, \varpi)_F$, or that $2 \nmid m$ and $\mu \mid_{O_F^\times} = 1$. Then $\text{Ind}^G_B(\mu \cdot \chi \psi)$ has the unique parameter $(\tilde{p}, \lambda_{\mu, \psi})$ with respect to $\tilde{K}$, and the parameters of $\text{Ind}^G_B(\mu \cdot \chi \psi)$ with respect to $\tilde{K}'$ are $(\tilde{0}, \lambda_{\mu, \psi})$ and $(\tilde{p}, \lambda_{\mu, \psi})$. 

\[ (\rho' \cdot \chi \psi) \equiv \text{Ind}^G_B(\mu \cdot \chi \psi) \equiv \rho(T, \lambda). \]

(2) Let $\rho$, $\tilde{s} \in \{0, \ldots, p - 1\}$ and let $\lambda, \nu \in E^\times$. Suppose that $\rho(T, \lambda) \equiv \rho(s, \nu)$. Then $\rho(T, \lambda)$ is a nonsupercuspidal $G$-representation containing the $\tilde{K}$-weights $\tilde{\sigma}_T$ and $\tilde{\sigma}_F$, so by Proposition 6.1, we must have $\rho(T, \lambda) \equiv \rho(s, \nu)$. Finally, let $\rho, \tilde{s} \in \{0, \ldots, p - 1\}$ and let $\lambda, \nu \in E^\times$, and suppose that $\rho(T, \lambda) \equiv \rho(s, \nu)$. By Theorem 6.10 there is a genuine character $\mu \cdot \chi \psi$ of $\mathcal{T}$ sucht that $\rho(T, \lambda) \equiv \text{Ind}^G_B(\mu \cdot \chi \psi)$. Then by Proposition 6.1 we must have $\lambda = \nu = $ $\mu \cdot \chi \psi(\tilde{h}(\varpi))$. And by Lemma 5.11 $\rho(T, \lambda) \equiv (\tilde{\rho}(s, \lambda))^{\tilde{\delta}}$, so

$$
\rho(T, \lambda) \equiv (\rho(s, \lambda))^{\tilde{\delta}}^{-1} \equiv (\text{Ind}^G_B(\mu \cdot \chi \psi))^{\tilde{\delta}}^{-1} \equiv (\text{Ind}^G_B(\mu \cdot \chi \psi))^{\tilde{\delta}}^{-1}.
$$

By a similar calculation to the one in the proof of (1), $(\mu \cdot \chi \psi)^{\tilde{\delta}}^{-1} \equiv \delta_{\tilde{p}} \cdot \mu \cdot \chi \psi$. Since $\tilde{\sigma}_T$ is a $\tilde{K}$-weight of $\text{Ind}^G_B(\mu \cdot \chi \psi)$, by Lemma 3.8 we have an isomorphism of $\tilde{T} \cap K$-representations $(\tilde{\sigma}_T)_{\tilde{T}(t)} \cong \delta_{\tilde{p}} \cdot \mu \cdot \chi \psi |_{\tilde{T} \cap \tilde{K}}$. And since $\tilde{\sigma}_T$ is a $\tilde{K}$-weight of $\text{Ind}^G_B(\mu \cdot \chi \psi)$, we have $(\tilde{\sigma}_T)_{\tilde{T}(t)} \cong \mu \cdot \chi \psi |_{\tilde{T} \cap \tilde{K}}$. Hence $\delta_{\tilde{p}} = \delta_{\tilde{p}} \cdot \tilde{\delta}_T$, which is only true if $s$ satisfies the condition

$$
\sum_{i=0}^{f-1} s_i q^i \equiv \sum_{i=0}^{f-1} \left( r_i + \frac{p-1}{2} \right) q^i \pmod{q}.
$$

\[ \square \]
(3) Otherwise, $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ has a unique parameter with respect to $\tilde{K}$, and this parameter is of the form $(\tilde{r}, \lambda_{\mu, \psi})$ for some $\tilde{r} \notin \{0, \frac{p^{i}}{2}, p - 1\}$. $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ also has a unique parameter with respect to $\tilde{K}'$, equal to $(\tilde{r}', \lambda_{\mu, \psi})$ where $\tilde{r}'$ is the unique vector in $\{0, \ldots, p - 1\}$ such that $\sum_{i=0}^{r'-1} r'_i p^i \equiv \sum_{i=0}^{r-1} (r_i + \frac{p-1}{2}) p^i \pmod{q}$.

Proof of Corollary 6.14 All points follow from Theorem 6.13 together with Proposition 6.6.

Finally, we hold the character $\mu : F^\times \to E$ fixed and put the statement of Corollary 6.14 in a form which highlights the dependence on $\psi$ of the parameters for principal series representations. For $a \in F^\times$, let $\psi_a$ denote the character $x \mapsto \psi(ax)$. A consequence of Theorem 6.4 is the formula $\gamma_F(c, \psi_a) = (a, c). \gamma_F(c, \psi)$. Thus $\chi_{\psi}$ and $\chi_{\psi_a}$ define identical bijections

$$\left\{ \text{smooth characters } F^\times \to E^\times \right\} \leftrightarrow \left\{ \text{genuine characters } \tilde{T} \to E^\times \right\}$$

if and only if $a \in (F^\times)^2$. Likewise, the principal series representations $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ and $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi_a})$ have identical parameters with respect to $\tilde{K}$ and $\tilde{K}'$ if and only if $a \in (F^\times)^2$. The full dependence is as follows:

Corollary 6.15. Keep the notation of Corollary 6.14 and in addition let $a = u \cdot \varpi^{v_F(a)} \in F^\times$ with $u \in O_F^\times$. Suppose that $(\tilde{r}, \lambda)$ is a parameter of $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ with respect to $\tilde{K}$, and that $(\tilde{r}', \lambda)$ is a parameter of $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi_a})$ with respect to $\tilde{K}'$. Then

1. If $2 \mid v_F(a)$, then $(\tilde{r}, (u, \varpi)^{-1} \cdot \lambda)$ is a parameter of $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi_a})$ with respect to $\tilde{K}$ and $(\tilde{r}', (u, \varpi)^{-1} \cdot \lambda)$ is a parameter of $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi_a})$ with respect to $\tilde{K}'$.

2. If $2 \nmid v_F(a)$, then $(\tilde{r}', (u, \varpi)^{-1} \cdot \lambda)$ is a parameter of $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi_a})$ with respect to $\tilde{K}$ and $(\tilde{r}, (u, \varpi)^{-1} \cdot \lambda)$ is a parameter of $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi_a})$ with respect to $\tilde{K}'$.

Proof of Corollary 6.15 The conductors of $\psi$ and $\psi_a$ have equal parity if $2 \mid v_F(a)$ and opposite parity otherwise. The statement about the first components of the given parameters then follow from 6.14

The second component of each parameter of $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi})$ is equal to $\mu \cdot \chi_{a}(\tilde{h}(\varpi))$. Since $(\tilde{r}, \lambda)$ and $(\tilde{r}', \lambda)$ are parameters for $\text{Ind}_{F}^{\tilde{G}}(\mu \cdot \chi_{\psi})$, we have $\mu \cdot \chi_{\psi}(\tilde{h}(\varpi)) = \lambda$. We also have $\mu \cdot \chi_{\psi}(\tilde{h}(\varpi)) = (\tilde{r}, \lambda) F \cdot \mu(\varpi) \cdot \gamma_F(\varpi, \psi)^{-1}$, while $\mu \cdot \chi_{\psi}(\tilde{h}(\varpi)) = (\tilde{r}', \lambda) F \cdot \mu(\varpi) \cdot \gamma_F(\varpi, \psi_a)^{-1}$, so

$$\mu \cdot \chi_{\psi}(\tilde{h}(\varpi)) = \frac{\gamma_F(\varpi, \psi_a)}{\gamma_F(\varpi, \psi)} \cdot \lambda = (a, \varpi) F \cdot \lambda.$$  

If $2 \mid v_F(a)$ then $(a, \varpi) F = (u, \varpi) F$, and if $2 \nmid v_F(a)$ then $(a, \varpi) F = (u \varpi, \varpi) F = (-u, \varpi) F$.

7. Proof of the classification theorem

Proof of Theorem 6.10 Let $\pi$ be a smooth, genuine, irreducible, admissible representation of $\tilde{G}$. Then by Proposition 6.9 $\pi$ has a parameter with respect to each of $\tilde{K}$ and $\tilde{K}'$. Suppose that $\pi$ is nonsupercuspidal. Then by Corollary 6.14 $\rho$ has a nonsupersingular parameter with respect to each of $\tilde{K}$ and $\tilde{K}'$.

Conversely, suppose that $\pi$ is a smooth, genuine, irreducible $\tilde{G}$-representation which has a nonsupersingular parameter. If $\pi$ has the nonsupersingular parameter $(\tilde{r}, \lambda)$ with respect to $\tilde{K}$, then by definition $\pi$ is a quotient of $\pi(\tilde{r}, \lambda)$. Since $\lambda \in E^\times$, there exists a character $\mu : F^\times \to E^\times$ such that $\mu(\varpi) = \lambda$ and such that $|\mu|_{O_F^\times} \cong \delta_F$. If $\psi : F \to \mathbb{C}$ is a nontrivial additive character with even conductor $m$, then by
Proposition 6.10 (1) the genuine $E$-valued character $\mu \cdot \chi_\psi$ of $\tilde{T}$ satisfies $\mu \cdot \chi_\psi(\tilde{h}(\pi)) = \lambda$, so by Theorem 6.10 (2) we have $\pi(\vec{r}, \lambda) \cong \text{Ind}_{\mathcal{P}}^G(\mu \cdot \chi_\psi)$. Thus $\pi$ is a quotient of a principal series representation, and in fact $\pi \cong \text{Ind}_{\mathcal{P}}^G(\mu \cdot \chi_\psi)$ since by Theorem 6.10 (1) the latter is also irreducible. If $\pi$ has a nonsupersingular parameter $(\vec{r}, \lambda)$ with respect to $\tilde{K}'$, then the same argument (only changed to take $\psi$ of odd conductor) shows again that $\pi$ is nonsupercuspidal. Thus an irreducible admissible genuine representation $\pi$ of $\tilde{G}$ has a nonsupersingular parameter if and only if all of its parameters are nonsupersingular, if and only if $\pi$ is nonsupercuspidal. Since there exist irreducible genuine representations with supersingular parameters, namely the irreducible quotients of the supersingular cokernel modules $\pi(\vec{r}, 0)$ for $\vec{r} \in \{0, \ldots, p - 1\}^I$, the class of supersingular representations is nonempty. \qed

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