Self-dual formulations of $d = 3$ gravity theories in the path-integral framework

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Abstract

We study the connection, at the quantum level, between $d = 2 + 1$ dimensional self-dual models with actions of growing (from first to fourth) order, governing the dynamics of helicity $\pm 2$ massive excitations. We obtain identities between generating functionals of the different models using the path-integral framework, this allowing to establish dual maps among relevant vacuum expectation values. We check consistency of these v.e.v.’s with the gauge invariance gained in each mapping.

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I. INTRODUCTION

Since its introduction nearly three decades ago \cite{1} 2+1 dimensional topologically massive gravity, with a third order P and T odd action governing the dynamics of a single massive graviton, has attracted a lot of attention and several dual formulations have been found \cite{2}-\cite{16}. Indeed, various 2+1 dimensional models describe, locally, a single massive spin 2 excitation, differing in their gauge symmetries and sharing the common feature of parity and time reversal invariances violation. These properties come from the fact that in 2+1 dimensions the angular momentum tensor $M^{\mu\nu}$ is dual to a pseudo-vector $J^\mu$ so that the Casimir operators of the Poincaré group are $P_\mu P^\mu = -m^2$ and $P_\mu J^\mu = ms$, with $m$ the mass and $s$ the spin of the excitation, identified with its helicity.

Concerning P and T invariant models, they must necessarily have two excitations with the same mass and opposite helicities\cite{17}-\cite{18}. In fact, a fourth order unitary gravity model in which the graviton acquires mass without the introduction of extra fields was recently proposed \cite{19}-\cite{21}. This model is equivalent to the Fierz-Pauli model at the linear level and in this sense it is P and T invariant and describes two excitation of the same mass and opposite helicities.

The equations of motion of the single massive excitation models relate the relevant field with its curl as it happens with self-duality equations arising in vector theories \cite{22}-\cite{23}. Indeed, the minimal description of a spin 2 theory leads to the equation of motion \cite{2}:

\[(h^{Tt})^a_{\mu} = -\frac{1}{m}\varepsilon_{\mu}^{\nu\lambda} \partial_\nu (h^{Tt})^a_{\lambda}\]  

with $(h^{Tt})^a_{\mu}$ a symmetric, transverse and traceless tensor which can be associated to the linear deviation of the dreibein $e^a_{\mu}$. (We are considering a 2+1 metric with signature $(-,+,+)$ and denote $\mu, \nu$ and $a, b$ curved and flat indices respectively.).

Equation (1) can be derived from a first order action with no gauge symmetries, in terms of the tensor field $h^a_{\mu}$. It is resemblant to the self-dual equation for a vector field $A_\mu$:

\[A_\mu = \frac{1}{m}\varepsilon_{\mu}^{\nu\lambda} \partial_\nu A_\lambda\]  

in self-dual (Abelian) gauge theories.

One can extend the model with self-dual equation (1), which we call SD1, endowing the action with a gauge symmetry in such a way that one ends with massive excitations without
breaking the aforementioned symmetry. The first extension that one can envisage is the self-dual *intermediate model* (SD2) introduced in [2] with a second order action having a symmetry that corresponds to the linearization of diffeomorphism invariance. In growing order then comes the third order *linear topologically massive model* [1] (SD3), with a diffeomorphism invariant action also exhibiting Lorentz invariance when written in terms of the dreibeins. Finally there is the so-called *new topological massive model* (SD4) [15], [13] with a fourth order action which, in addition, has a linear conformal invariance.

These four models that describe a single massive excitation of helicity $+2$ or $-2$ can be connected via duality transformations that incorporate the corresponding gauge symmetries in passing from SD1 to SD3 [4], [12], [16] and to SD4 [24]. A link joining the SD4 model was also presented in [13] and in the opposite direction it can be seen that fixing the gauge one can go from SD3 to SD1 models [4], [7], [9]. Such connections have been in general established at the classical level. A quantum analysis has been outlined in [12], [16] for connections SD1 to SD3 and to SD4 in [14], basically analyzing propagators in connection with unitarity.

It is the purpose of the present work to study the dual equivalence between the four massive models SDI (with $I = 1, \ldots, 4$) at the quantum level. The idea is to follow the approach developed in the study of self-dual and the topologically massive vector models, whose equivalence was established at the level of equations of motions in refs. [22]-[23] and discussed at the quantum level within the path-integral approach in [25]-[27].

Basically, we shall introduce successive “interpolating actions” depending on two fields: the one of the departure model and the dual model one. Then, we shall define the generating functional associated to the interpolation actions in the form of a path-integral over the two fields. Integrating over one or the other fields the connection between the generating functionals of the SDI models can be established and, from it, the connections between quantum correlation functions follow.

The plan of the paper is the following: in the next section we introduce the actions governing the dynamics of the different models, analyze their symmetries and exhibit their equivalence at the classical level, showing that the equations of motions for the SD1 to $SD4$ models can be derived from a single equation for a symmetric, transverse and traceless tensor that can be identified in each case with the linear deviation of the dreibein, the linear spin connection, the linear Schouten tensor and the linear Cotton tensor respectively.

We introduce in section 3 the interpolating action for models SD1 and SD2 and define
the associated generating functional in the path-integral framework. Integrating on one or the other field will allow to connect vacuum expectation values for SD1 and SD2 models. An analogous procedure is described in section 4, now for connecting models SD1 and SD2 with SD3. In section 5 we extend the calculation in order to connect the previous models with the new topological massive model SD4. Finally, in section 6 we present our conclusions.

II. DUALITIES AT THE CLASSICAL LEVEL

We take the fields \( h^a_\mu \) as the linear deviation of dreibeins \( e^a_\mu \)

\[
e_\mu^a = \delta^a_\mu + \kappa h^a_\mu
\]  

with \( \kappa \) a parameter related to Newton constant \( G_N, \kappa^2 = 8\pi G_N \). To first order in \( \kappa \) flat indices \( a, b \) in field \( h \) can be replaced by \( \mu, \nu \) curved ones; we shall nevertheless maintain the distinction in order to keep track of this two type of indices. The symmetric part of \( h \) is connected with the linear deviation of the metric

\[
g_{\mu \nu} = \eta_{\mu \nu} + \kappa H_{\mu \nu} \quad \text{with} \quad H_{\mu \nu} = h_{\mu \nu} + h_{\nu \mu}
\]

The dual of the torsion free spin connection is

\[
e^a_\mu \omega^a_\mu = (e^a_\mu e^a_\rho - \frac{1}{2} e^a_\mu e^a_\rho) \varepsilon^{\mu \nu \lambda} \partial_\nu e^b_\lambda
\]

In the linear approximation, the resulting linearized spin connection, becomes

\[
\omega^L_\mu^a (h) = (\eta_{\mu \rho} \delta^a_\rho - \frac{1}{2} \delta^a_\mu \eta_{\rho \delta}) \varepsilon^{\mu \nu \lambda} \partial_\nu h^b_\lambda
\]

\[
= \frac{1}{2} \delta^a_\lambda \varepsilon^{\lambda \nu \rho} (\partial_\nu (h_{\mu \rho} + h_{\rho \mu}) - \partial_\mu h_{\nu \rho})
\]

\[
\equiv [W^a_\mu]_b^\lambda h^b_\lambda = W^a_\mu (h).
\]

From this expression it is easy to verify that

\[
\varepsilon_{abc} \varepsilon^{\mu \nu \lambda} \delta^b_\lambda W^c_\nu (h) = \varepsilon^{\mu \nu \lambda} \partial_\nu h^b_\lambda
\]

The linearized Einstein tensor \( G^{L \mu \nu} (h) \) is then given by

\[
G^{L \mu \nu} (h) = -\varepsilon^{\nu \rho \sigma} \partial_\rho W^c_\sigma (h) \delta^\mu_c = -\varepsilon^{\mu \rho \sigma} \varepsilon^{\nu \beta \sigma} \partial_\alpha \partial_\beta h_{\rho \sigma} = -\frac{1}{2} \varepsilon^{\mu \alpha \rho} \varepsilon^{\nu \beta \sigma} \partial_\alpha \partial_\beta (h_{\rho \sigma} + h_{\sigma \rho}),
\]
while the Cotton tensor can be written in the linear approximation as
\[
C_{\mu\nu}^L = \varepsilon^{\mu\rho\lambda}\partial_{\rho}S_{\lambda\nu}^L = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{\mu\beta\sigma}\partial_{\alpha}\partial_{\beta}(W_{\rho\sigma}(h) + W_{\sigma}(h)) \tag{9}
\]
where \(S_{\mu\nu}^L\) is the linearized Schouten tensor
\[
S_{\mu\nu}^L = G_{\mu\nu}^L - \frac{1}{2}\delta_{\mu\nu}G = -(\delta_{\mu}^{\rho}\eta_{\nu\rho} - \frac{1}{2}\delta_{\nu}^{\rho}\eta_{\mu\rho})\varepsilon^{\rho\sigma\lambda}\partial_{\sigma}W_{\lambda}(h) = -[W_{\mu\nu}]^c_{\lambda}W_{\lambda}(h). \tag{10}
\]

Under diffeomorphism transformations \(h_{\mu}^a\) and \(\omega_{\mu}^a\) change as
\[
\delta_{\zeta}h_{\mu}^a = \partial_{\mu}\zeta^a, \quad \delta_{\zeta}\omega_{\mu}^a = 0 \tag{11}
\]

Concerning Lorentz transformations, they take the form
\[
\delta_l h_{\mu}^a = \varepsilon^{a}_{bc}l^b\delta_{\mu}^c, \quad \delta_l\omega_{\mu}^a = \partial_{\mu}l^a \tag{12}
\]
and for conformal transformations one has
\[
\delta_{\rho}h_{\mu}^a = \frac{1}{2}\rho\delta_{\mu}^a, \quad \delta_{\rho}\omega_{\mu}^a = \frac{1}{2}\varepsilon^{\nu\sigma}_{\mu}\partial_{\sigma}\rho\delta^a_{\nu}. \tag{13}
\]

In order to see the effect of these transformations on the different objects that we are considering we can make a decomposition of the field \(h\) in its irreducible components
\[
h_{\mu\nu} = (H_{\mu\nu}^{Tt} + \frac{1}{2}P_{\mu\nu}H^{T} + \rho_{\mu}\rho_{\nu}H^{L} + \rho_{\mu}h_{\nu}^{T} + \rho_{\nu}h_{\mu}^{T}) + (\varepsilon_{\mu\nu}^{\lambda}V^{T\lambda} + \varepsilon_{\mu\nu}^{\lambda}\rho^{\lambda}V^{L}) \tag{14}
\]
\[
\equiv h_{\mu\nu}^{S} + h_{\mu\nu}^{A},
\]
with
\[
P_{\nu}^{\mu} = \delta_{\nu}^{\mu} - \rho^{\mu}\rho_{\nu}, \quad \rho_{\mu} = \frac{\partial_{\mu}}{\Box^{1/2}}. \tag{15}
\]
and
\[
h_{\mu\nu}^{S} = h_{\nu\mu}^{S}, \quad h_{\mu\nu}^{A} = -h_{\nu\mu}^{A},
\]
\[
H_{\mu\nu}^{Tt} = 0, \quad \partial_{\mu}H_{\nu}^{Tt} = 0,
\]
\[
\partial_{\mu}h_{\nu}^{T} = 0, \quad \partial_{\mu}V_{\nu}^{T} = 0. \tag{16}
\]

In decomposition (14) \(H_{\mu\nu}^{Tt}\) are the spin 2 components of the field \(h_{\mu\nu}\). The spin 1 components are \(h_{\mu}^{T}\) and \(V_{\mu}^{T}\) and the spin 0 components are \(H^{T}, H^{L}\) and \(V^{L}\). \(H_{\mu\nu}^{Tt}\) is symmetrical, transverse and traceless and it is invariant under diffeomorphism, Lorentz and conformal
transformations. Under diffeomorphism the sensitive components are $H^L$, $h^T_\mu$ and $V^T_\mu$, while under Lorentz transformations the sensitive components are $V^T_\mu$ and $V^L$. Under conformal transformation only $H^T$ and $H^L$ change.

It is important to recall that the absence of the antisymmetrical part of $h^a_\mu$ in a given action is a sign of explicit Lorentz invariance. Similarly, if the antisymmetrical part of its $\omega^a_\mu$ is also absent, the action has an explicit conformal invariance.

In terms of irreducible components, the linearized Einstein tensor takes the form

$$G^L_{\mu\nu} = -\Box(H^T t_{\mu\nu} - \frac{1}{2} P_{\mu\nu} H^T),$$ (17)

showing its explicit invariance under diffeomorphism and Lorentz transformations. Concerning the Cotton tensor, it can be written as

$$C^L_{\mu\nu} = -\Box \varepsilon^{\alpha\beta} \partial_\alpha H^T_{\beta\nu}.$$ (18)

making apparent its invariance under Lorentz and conformal transformations, This tensor is also symmetric, transverse and traceless.

Finally it is straightforward to see that when $h^a_\mu$ is symmetric, transverse and traceless

$$h_{\mu\nu} = h_{\nu\mu}, \quad h^a_\mu \delta^{\nu a} = 0, \quad \partial^{\mu} h^a_\mu = 0$$ (19)

so are the linear spin connection, the linear Einstein tensor and, trivially, the linear Cotton tensor.

Let us introduce a $(2+1)$-dimensional spin 2 excitation $V^T_{\mu\nu}$ which afterwards will be identified either with the spin 2 component of $h_{\mu\nu}$, with the spin connection $W_{\mu\nu}(h^T)$, with the linearized Schouten tensor $S_{\mu\nu}(h^T)$ or with the linearized Cotton tensor $C_{\mu\nu}(h)$. $V^T_{\mu\nu}$ is a symmetric, transverse and traceless tensor satisfying the equation \[17\] - \[18\]

$$\frac{1}{2} \left( [\mathbb{P}_\mu \mathbb{J}^\mu_2] - ms \mathbb{I}_{\alpha\beta}^{\mu\nu} V^T_{\mu\nu} \right) = 0,$$ (20)

with $s = \pm 2$. $\mathbb{P}_\mu$ is the momentum operator and $\mathbb{J}^\mu_2$ is given by \[16\]

$$(\mathbb{J}^\mu_2)_{\alpha\beta}^{\gamma\sigma} = -i \varepsilon^{\mu\lambda} x^\lambda (\mathbb{I}_s)_{\alpha\beta}^{\gamma\sigma} \partial_\mu + \frac{i}{2} (\delta^\gamma_\alpha \varepsilon_\beta^{\mu\sigma} + \delta^\gamma_\beta \varepsilon_\alpha^{\mu\sigma} + \delta^\sigma_\alpha \varepsilon_\beta^{\mu\gamma} + \delta^\sigma_\beta \varepsilon_\alpha^{\mu\gamma})$$ \[21\]

$$\equiv -\varepsilon^{\mu\lambda} x^\lambda (\mathbb{P}_\mu)_{\alpha\beta}^{\gamma\sigma} + (j^\mu)_{\alpha\beta}^{\gamma\sigma},$$ (22)

with $(\mathbb{I}_s)_{\alpha\beta}^{\gamma\sigma} = \frac{1}{2} (\delta^\gamma_\alpha \delta^\sigma_\beta + \delta^\sigma_\alpha \delta^\gamma_\beta)$. One recognizes the first term in \[22\] as the orbital part of
the angular momentum. These operators satisfy the Poincaré algebra

\[ i[\mathbb{P}^\mu, \mathbb{P}^\nu]_{\alpha\beta}^{\rho\sigma} = 0, \]
\[ i[\mathbb{P}^\mu, \mathbb{J}^\nu]_{\alpha\beta}^{\rho\sigma} = -\varepsilon^{\mu\nu\lambda} (\mathbb{P}_\lambda)_{\alpha\beta}^{\rho\sigma}, \] (23)
\[ i[\mathbb{J}^\mu_2, \mathbb{J}^\nu_2]_{\alpha\beta}^{\rho\sigma} = -\varepsilon^{\mu\nu\lambda} (\mathbb{J}^\lambda_2)_{\alpha\beta}^{\rho\sigma}. \]

Equation (20) can be written in the form

\[ -\varepsilon^\lambda_\mu \partial_\lambda V_{Tt}^{\mu\nu} \mp m V_{Tt}^{\mu\nu} = 0 \] (24)

or, using (6),

\[ V_{Tt}^{\mu\nu} = \mp \frac{1}{m} W_{Tt}^{\mu\nu}(V), \] (25)

which we recognize as the self dual equation for \( V_{Tt}^{\mu\nu} \) \([2],[4]\). This equation is sensitive to \( P \) and \( T \) transformations. \( V_{Tt}^{\mu\nu} \) has two independent degrees of freedom but one can see that only one combination propagates \([2],[9]\).

If \( V_{Tt}^{\mu\nu} \) is identified with the spin 2 component of the linearization of the dreibein \( h_{\mu\nu} \), then equation (25) coincides, after eliminating the spurious degrees, with the equation of motion of the self-dual (SD) model with action \([2]\)

\[ S_{SD1} = \mp \frac{m}{2} \int d^3 x \left[ h_{\mu\rho} \varepsilon^{\mu\nu\lambda} \partial_\lambda h_{\rho}^\nu \pm m (h_{\mu\nu} h^{\nu\mu} - h_{\mu}^\mu h_{\nu}^\nu) \right] \] (26)
\[ \equiv \frac{1}{2} \int d^3 x h_{\mu}^a K_{-}^{a} \partial_{\nu} h_{\lambda}^b. \] (27)

The ± signs are related to the spin ±2 of the excitations \([10]\); note that changing \( m \) to \(-m\) implies passing from the +2 to the −2 description. Action (27) does not have any gauge invariance and the antisymmetric part of the field \( h_{\mu}^a \) acts as a auxiliary field ensuring the spin 2 content. Taking \( h \) to be symmetric from the start one can see that there will be a spin 1 ghost remaining, associated with \( \partial^{\mu} H_{\mu\nu} \) which propagates with mass \( 2m \) \([16]\).

There are other symmetric, transverse and traceless candidates satisfying equation (20). They are the spin connection \( W_{\mu}^a(H^{Tt}) \), the Schouten tensor \( S_{\mu}^{\nu}(H^{Tt}) \) and the Cotton tensor \( C_{\mu}^{\nu}(H^{Tt}) \) written in terms of a symmetric, transverse and traceless dreibein. When \( V_{Tt}^{\mu\nu} \) is identified with the connection, equation (20) coincides with the equations of motion,
on the physical modes, of the so called intermediate, (second order) action \( S_{SD2} \)

\[
S_{SD2}^\pm = \frac{1}{2} \int d^3x \left( h_\mu^a \varepsilon^{\mu\nu\lambda} \partial_\nu W_{\lambda a}(h) \pm m h_\mu^a \varepsilon^{\mu\nu\lambda} \partial_\nu h_{\lambda a} \right)
\]

\[
= \frac{1}{2} \int d^3x \left[ h_\mu^a K_{\pm a}^{\mu \lambda} \left( \mp \frac{1}{m} W_\lambda^b(h) \right) \right]
\]

\[
\equiv S_E \pm S_{TCS},
\]

where \( K_{\pm a}^{\mu \lambda} \) is the same evolution operator that appears in \( S_{SD1} \), and we have identified the terms \( S_E \) and \( S_{TCS} \) with the Einstein action term and the triadic Chern-Simons term \( S_{CS} \) respectively. The Einstein term can be written in terms of the symmetric part of \( h_\mu^a \) so the contribution of the antisymmetric part in the second term is just to ensure the non-propagation of the spin 1 component of \( H_{\mu\nu} \). This action corresponds to the linearization of the so called Massive vector Chern-Simons gravity \( S_{CS} \).

The intermediate action \( S_{SD2} \) is invariant under diffeomorphism transformations up to a surface term and it can be proved, at the classical level, that it is related with the SD1 action \( S_{SD1} \) after fixing the gauge \( S_{CS} \). It should be noticed, however, that the space of classical solutions of the two models is different (solutions have different topological properties and those associated with action \( SD2 \) include the non trivial solution \( \omega_{\mu\nu}(h) = 0 \) absent in the SD1 model).

If one identifies \( V_{\mu\nu}^{TT} \) with the linearized Schouten tensor, eq. (20) becomes the equations of motion for physical modes of the linear topologically massive action \( S_{SD3} \)

\[
S_{SD3}^\pm = \frac{1}{2} \int d^3x \left[ h_\mu^a \varepsilon^{\mu\nu\lambda} \partial_\nu W_{\lambda a}(h) \pm \frac{1}{2m} W_\mu^a(h) \varepsilon^{\mu\nu\lambda} \partial_\nu W_\lambda^b \right]
\]

\[
= -S_E \pm S_{CS}
\]

\[
= \frac{1}{2} \int d^3x \left[ \mp \frac{1}{m} W_\mu^a(h) K_{\pm a}^{\mu \lambda} \left( \mp \frac{1}{m} W^b_\lambda(h) \right) \right]
\]

\[
= \frac{1}{2} \int d^3x \left[ h_\mu^a K_{\pm a}^{\mu \lambda} \left( - \frac{1}{m^2} s_\lambda^b(h) \right) \right]
\]

We have identified the Einstein term with a minus sign in front so that it leads to the correct propagation of its physical modes \( S_{CS} \). We have also identified the linear gravitational Chern-Simons term, \( S_{CS} \). The expression in the last line, written in terms of the Schouten tensor will be useful later when we make the connection with the other models at the quantum level.

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Action $S_{SD3}^\pm$ is explicitly Lorentz invariant because it depends only on the symmetrical part of $h_{\mu\nu}$ and it also exhibits diffeomorphism invariance up to a surface term. It can be written in terms of the symmetric part of $h$

$$S_{SD3} = \frac{1}{4} \int d^3 x \left[ H_{\mu\nu} G^{L\mu\nu} (H) \pm \frac{1}{m} H_{\mu\nu} C^{L\mu\nu} (H) \right]$$  \hspace{1cm} (32)

$$= \frac{1}{4} \int d^3 x \left[ H_{\mu\alpha} K^{\pm\mu\alpha} \rho^\beta \left( - \frac{1}{m^2} S^{\rho\beta} (H) \right) \right].$$  \hspace{1cm} (33)

It has been shown that SD3 connects with the SD2 model after a gauge fixing [4],[7],[9]. It should be noted that the classical space of solutions of SD2 and SD3 models is different because the later’s one includes the nontrivial solutions of the equation $G^{\mu\nu} (h) = 0$ [3],[7]-[6].

Finally if we identify $V_{T \mu\nu}^T$ in (20) with the linear Cotton tensor, which has the same symmetries, the equation correspond to the linearization of the so called new topological massive model [15],[13]

$$S_{SD4} = \frac{1}{4} \int d^3 x \left[ - \frac{1}{m^2} H_{\mu\alpha} e^{\mu\lambda} \partial_\nu C^{L\lambda} - \frac{1}{m} H_{\mu\alpha} C^{L\mu\alpha} \right]$$  \hspace{1cm} (34)

$$= \frac{1}{4} \int d^3 x \left[ H_{\mu\alpha} K^{\pm\mu\alpha, \rho\beta} \left( \frac{1}{m^3} C^{\rho\beta} (H) \right) \right].$$  \hspace{1cm} (35)

This action is invariant under diffeomorphism and conformal transformations. The sign in front of the first term in (34) is so in order to have the correct description of a massive excitation with helicity $\pm 2$ depending on the sign of the second term [15],[14]. This model can be connected by duality transformation with the other SD models [13].

We have then reviewed the connection, at the classical level, of models with actions $S_{SD1}, S_{SD2}, S_{SD3}$ and $S_{SD4}$ defined in eqs.(27), (30), (31) and (35) respectively. The connections between these models have been established in various ways and in this sense they are considered as dual models [2],[4],[12],[16],[13],[14]. We can summarize these classical dualities as follows

$$S_{SD1} \leftrightarrow S_{SD2} \leftrightarrow S_{SD3} \leftrightarrow S_{SD4}$$  \hspace{1cm} (36)

As stressed in the introduction, the goal of our paper is to analyze dualities at the quantum level using the path-integral approach. Indeed, we shall connect in the following sections the four models at the generating functional level, this allowing to establish identities among correlation functions of the four SDI model’s partition functions. That is, connection (36)
between classical actions will become one between generating functionals. For this purpose we will introduce different *interpolating actions* that allows to pass from one model to the other.

III. DUALITIES AT THE QUANTUM LEVEL: THE PATH-INTEGRAL APPROACH

Following the ideas in [22]-[27], let us introduce the *interpolating action* \( S_I[h, H] \)

\[
S_I[h, H] = \frac{1}{2} \int d^3x (mH_\mu^a \varepsilon^{\mu\nu\lambda} \partial_\nu H_\lambda^a - 2m^2 \delta_\nu^\mu \varepsilon^{\mu\nu\lambda} \partial_\nu H_\lambda^a - m^2 \varepsilon_{abc} \varepsilon^{\mu\nu\lambda} h_\mu^a h_\nu^b \delta_\lambda^c)
\] (37)

where \( h_\mu^a \) is the dreibein deviation field as defined in (3) and \( H_\mu^a \) the corresponding dual field.

Action \( S_I \) is invariant, up to a surface term, under the gauge transformations

\[
\delta H_\mu^a = \partial_\mu \xi^a , \quad \delta h_\mu^a = 0,
\] (38)

which reminds us of the linearized diffeomorphism transformation in \( H_\mu^a \). The signs in the different terms of the action corresponds in our conventions to a +2 helicity description. The description for a \(-2\) helicity is obtained changing \( m \to -m \). We will consider the positive helicity case from here on.

Within the path integral framework the partition function \( Z_I \) associated to action \( S_I[h, H] \) reads

\[
Z_I = \int \mathcal{D}H_\mu^a \mathcal{D}h_\mu^a \exp(iS_I[h, H]).
\] (39)

We shall define the associated generating function \( Z_I[j] \) by coupling the \( h \)-field minimally to an external source \( j_\mu^a \)

\[
Z_I[j] = \int \mathcal{D}H_\mu^a \mathcal{D}h_\mu^a \exp(iS_I[h, H; j])
\] (40)

with

\[
S_I[h, H; j] = S_I[h, H] + \int d^3x h_\mu^a j_\mu^a
\] (41)

Now, generating functional \( Z_I[j] \) as defined in (40) can be connected with the generating functional for correlation functions for \( h \)-field with dynamics governed by the self-dual (SD1)
action. To see this, let us perform the path integral on $H_\mu^a$. To this end, consider the $H_\mu^a$-dependent part of the path-integral, which we call $I[h]$, 

$$I[h] = \int D\mathcal{H}_\mu^a \exp \left( i \int d^3x \left( \frac{1}{2} H_\mu^a [S_a^\mu]_b^\rho H_\rho^b + H_\mu^a J_a^\mu \right) \right)$$

(42)

where we have defined 

$$[S_a^\mu]_b^\rho = m\eta_{ab} \epsilon^{\mu\nu\rho} \partial_\nu,$$  

(43)

$$J_a^\mu = -m \epsilon^{\mu\nu\lambda} \partial_\nu h_\lambda^a = -[S_a^\mu]_b^\rho h_\rho^b.$$  

(44)

At this point it should be remarked that due to gauge invariance operator $[S_a^\mu]_b^\rho$ is non-invertible. As usual, this problem can be overcome by adding an appropriate gauge fixing term, this leading to a “regulated” invertible operator (which we call $([S_{reg}^\mu]_b^\rho$) so that $I[h]$ can be written as

$$I[h] = \int D\mathcal{H}_\mu^a \exp \left( i \int d^3x \left( \frac{1}{2} H_\mu^a ([S_{reg}^\mu]_b^\rho H_\rho^b + H_\mu^a J_a^\mu) \right) \right)$$

(45)

As usual, at the end of the calculations the regulator can be turned off and a finite result is attained.

Being quadratic in $H_\mu^a$, path-integral (45) can be accommodated as

$$I[h] = \int D\mathcal{H}_\mu^a \exp \left( i \int d^3x \left( \frac{1}{2} H_\mu^a ([S_{reg}^\mu]_b^\rho H_\rho^b + H_\mu^a J_a^\mu) \right) \right)$$

(46)

After a shift $H_\mu^a - h_\mu^a \to H_\mu^a$ in the path-integral variables the factor in the first line of (46) becomes $h$-independent giving a field independent constant factor $N_1$ that will be irrelevant for the calculation of vacuum expectation values from the generating functional.

We can then write eq. (46) in the form

$$I[h] = N_1 \exp \left( -\frac{i}{2} \int d^3x \left( (S_{reg})_a^\mu \rho h_\rho^b \left( (S_{reg})_{-1}^\mu \partial_\nu h_\nu^d \right) \right) \right)$$

(47)

One can now cancel out the $(S_{reg})_{-1}^{-1}$ $S_{reg}$ factor in the integral of the exponential factor and then turn off the regulator so that one finally ends with

$$I[h_\mu^a] = N_1 \exp \left( i \int d^3x \left( -\frac{m}{2} h_\mu^a \epsilon^{\mu\nu\lambda} \partial_\nu h_\lambda^a \right) \right)$$

(48)

Putting all this together we have

$$Z_I[j] = N_1 \int Dh_\mu^a \exp \left( i(S_{SD1}[h] + i \int d^3x h_\mu^a J_a^\mu) \right)$$

(49)
or
\[ Z_I[j] = N_I Z_{SD1}[j], \] (50)

where \( Z_{SD1}[j] \) is the generating functional for a self-dual spin two model, in the presence of a source, with classical action \( S_{SD1} \) as defined in the previous section, eq.(27).

We shall now start again from \( Z_I[j] \) as given by (40) but now we shall perform the \( h^\mu_a \) integral. As before, we write the path integral as
\[ I[H] = \int \mathcal{D} h^\mu_a \exp \left( i \int d^3 x \left( \frac{1}{2} h^\mu_a [D_a^\mu]_b^\rho h^\rho_b + h^\mu_a \mathcal{J}_a^\mu \right) \right), \] (51)

where we have defined
\[ [D_a^\mu]_b^\rho = -m^2 \varepsilon_{abc} \varepsilon^{\mu \rho \lambda} \delta^c_{\lambda}, \] (52)
\[ \mathcal{J}_a^\mu = -m \varepsilon^{\mu \nu \lambda} \partial_\nu \mathcal{H}_{\lambda a} + j^\mu_a = \frac{1}{m} [D_a^\mu]_b^\rho W^b_\rho (\mathcal{H}) + j^\mu_a \] (53)

and \( W^b_\rho (\mathcal{H}) \) is the linear spin connection as defined in (6) but for the dual field \( \mathcal{H}_\lambda a \). The last equality in (53) follows from (7). Now, we can complete squares in (51) getting
\[ I[H^a] = \int \mathcal{D} h^\mu_a \exp \left( i \int d^3 x \left( \frac{1}{2} \varepsilon^{\mu \nu \lambda} \partial_\nu \mathcal{H}_\lambda a W^a_\mu (\mathcal{H}) - \frac{1}{m} W^a_\mu [\mathcal{H}] j^\mu_a - \frac{1}{2} j^\mu_a [D^{-1}_a]_b^\rho j^\rho_b \right) \right], \] (54)

where \([D^{-1}_a]_b^\rho\) is the inverse of \([D_a^\mu]_b^\rho\),
\[ [D^{-1}_a]_b^\rho = \frac{1}{m^2} \left( \frac{1}{2} \delta^a_\mu \delta^b_\rho - \delta^b_\mu \delta^a_\rho \right) \] (55)

The sign of the first term in (54) is the correct sign for the Einstein action and it does not depend of \( m \), so it will be unaffected if we change the signs of the first two terms in the interpolating action. The gaussian path-integration over \( h^\mu_a \) can be easily performed through a shift in \( h^\mu_a \) leading to a multiplicative irrelevant factor \( N_0 \) so that finally we have for \( Z_I \)
\[ Z_I[j] = N_0 \int \mathcal{D} \mathcal{H}_\mu_a \exp \left( i \left( S_{SD2} - \int d^3 x \left( \frac{1}{m} W^a_\mu [\mathcal{H}] j^\mu_a + \frac{1}{2} j^\mu_a [D^{-1}_a]_b^\rho j^\rho_b \right) \right) \right) \] (56)

Using eq.(55) one can see that when computing correlation functions, the term quadratic in \( j \) in eq.(56) will give contact terms which as usual can be handled introducing an appropriate regularization.
In conclusion, we have obtained now that the generating functional \( Z_f[j] \) is also equivalent to the one for the \( S_{SD2} \) action

\[
Z_f[j] = \mathcal{N}_0 Z_{SD2}[j_w]
\]  

(57)

where the coupling of the external source \( j \) to \( \mathcal{H} \) in \( Z_f[j] \) is now that defined in (56),

\[
Z_{SD2}[j_w] = \int \mathcal{D}\mathcal{H} \exp (i S_{SD2}[\mathcal{H}, j_w])
\]  

(58)

\[
S_{SD2}[\mathcal{H}, j_w] = S_{SD2}[\mathcal{H}] - \int d^3 x \left( \frac{1}{m} W^a_\mu [\mathcal{H}] j^\mu_a + \frac{1}{2} j^\mu_a [D^{-1} \mu \cdot a \cdot b] j^\nu_b \right)
\]  

(59)

The subindex \( W \) in \( j_W \) is included to recall that in generating functional \( Z_{SD2} \) the source couples (non-minimally) to \( \mathcal{H}_\mu \cdot a \) through the connection \( W^a_\mu (\mathcal{H}) \).

Now, comparing the two results obtained by integrating \( h \) and \( H \) in interchanged orders, eqs.(50) and (57), we conclude that, up to a constant multiplicative factor, the following identity holds for spin-2 generating functionals

\[
\mathcal{N}_1 Z_{SD1}[j] = \mathcal{N}_0 Z_{SD2}[j_w]
\]  

(60)

Differentiating both sides of this equation with respect to the source and then making \( j = 0 \) we have the following identity for vacuum expectation values

\[
\langle h^a_\mu(x) \rangle_{SD1} = -\frac{1}{m} \langle W^a_\mu(\mathcal{H}(x)) \rangle_{SD2}
\]  

(61)

We see that we have established a duality relation which holds at the quantum level and which can be written, in terms of fields as

\[
h^a_\mu(x) \rightarrow -\frac{1}{m} W^a_\mu(\mathcal{H}(x))
\]  

(62)

One can see that duality relation (62) also holds when calculating correlation functions of an arbitrary number of \( h \) fields in the self-dual theory SD1 in the sense that the answer can be analogously calculated from products of spin connections \( W^a_\mu(\mathcal{H}) \) in the SD2 theory

\[
\langle h^{a_1}_\mu_1(x_1) \cdots h^{a_n}_\mu_n(x_n) \rangle_{SD1} = \left\langle \left( -\frac{1}{m} \right)^n W^{a_1}_\mu_1 (\mathcal{H}(x_1)) \cdots W^{a_n}_\mu_n (\mathcal{H}(x_n)) \right\rangle_{SD2} + \text{contact terms}
\]  

(63)

We also note that the identification is consistent with the form of the \( SD2 \) action given in (30). This fact will appear again when we consider the connection with the topologically massive model in the next section. Let us end this section by noting that since the coupling term \( W^a_\mu (\mathcal{H}) j^\mu_a \) is invariant under diffeomorphism it does not constraint the the external source.
IV. CONNECTION WITH TOPOLOGICALLY MASSIVE GRAVITY (TMG)

We can now make the connection between the intermediate model (SD2) \(^2\) and the linear topologically massive (SD3) model \(^1\). For this purpose we start with a new interpolating action

\[
S_I'[h_\mu^a, \mathcal{H}_\mu^a] = \frac{1}{2} \int d^3x \left( mh_\mu^a \varepsilon^{\mu\nu\lambda} \partial_\nu h_\lambda^a - H_\mu^\nu G_\nu^\lambda(\mathcal{H}) + \frac{1}{2} \mathcal{H}_\mu^\nu G_\nu^\lambda(\mathcal{H}) \right)
\]  

(64)

where

\[
H_\mu^\nu = h_\mu^\nu + h_\nu^\mu
\]  

(65)

and \(\mathcal{H}_\mu^\nu\) is its (symmetric) dual field (only the symmetric part of the dreibein deviation contributes in the mixing term). Action (64) is invariant, up to a surface term, under the following diffeomorphism transformations

\[
\delta h_\mu^a = \partial_\mu \xi^a, \quad \delta \mathcal{H}_\mu^\nu = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu
\]  

(66)

which corresponds to diffeomorphism transformations. The terms containing the field \(\mathcal{H}\) are explicitly Lorentz invariant.

As in the previous cases, we introduce a partition function associated to \(I'_I\)

\[
Z_I' = \int \mathcal{D}h_\mu^a \mathcal{D}\mathcal{H}_\mu^\nu \exp(i S_I'[h, \mathcal{H}])
\]  

(67)

and the corresponding generating functional

\[
Z_I'[j] = \int \mathcal{D}h_\mu^a \mathcal{D}\mathcal{H}_\mu^\nu \exp(i S_I'[h, \mathcal{H}, j])
\]  

(68)

with

\[
S_I'[h, \mathcal{H}, j] = S_I'[h, \mathcal{H}, ] + \int d^3x h_\mu^a j_a^\mu
\]  

(69)

and \(j_a^\mu\) the external source. This source should satisfy \(\partial_\mu j_a^\mu = 0\) in order to preserve invariance under diffeomorphisms.

To connect \(Z_I'[j]\) with \(Z_{SD}[j]\), the generating functional of the SD2 theory defined in eq.(30), we shall perform the \(\mathcal{H}_\mu^\nu\)-integration in (68)

\[
I[h_\mu^a] = \int \mathcal{D}\mathcal{H}_\mu^\nu \exp \left( i \int d^3x \left( \frac{1}{4} \mathcal{H}_\mu^\nu [\mathcal{G}^\lambda_\lambda^\rho, \mathcal{H}^\lambda^\rho + \mathcal{H}_\mu^\nu J^\mu^\nu] \right) \right)
\]  

(70)

where we have defined now

\[
[\mathcal{G}^\mu^\nu]_\lambda^\rho = G^\mu^\nu(\mathcal{H})
\]  

(71)

\[
J^\mu^\nu = -\frac{1}{2} [\mathcal{G}^\mu^\nu]_\lambda^\rho H^\lambda^\rho
\]  

(72)
The operator $-(1/2)[G^{\mu\nu}]_{\lambda\rho}$, the evolution operator of the linear Einstein action, is non-invertible due to the gauge invariance and should be regularized. After integration is performed the regularization can be turned off as in the previous case (see eqs. (40)-(49)). The final answer is

$$I[h^a] = \exp \left( -\frac{i}{4} \int d^3 x H_{\mu\nu} G^{\mu\nu}(H) \right) \int D\mathcal{H}_{\mu\nu} \exp \left( \frac{i}{4} \int d^3 x (\mathcal{H}_{\mu\nu} - H_{\mu\nu})[G^{\mu\nu}]_{\lambda\rho}(H_{\lambda\rho} - H_{\lambda\rho}) \right)$$

(73)

Making the shift $\mathcal{H}_{\mu\nu} - H_{\mu\nu} \to \mathcal{H}_{\mu\nu}$ any dependence on $H_{\mu\nu}$ in the path-integral factor disappears. Calling $N_2$ this factor, irrelevant when computing vacuum expectation values, we end with

$$I[h^a] = N_2 \exp \left( -\frac{i}{4} \int d^3 x H_{\mu\nu} G^{\mu\nu}(H) \right)$$

(74)

Inserting this result in expression (68) for $Z_I[j]$, we see that the partition function for the SD2 theory (with minimal coupling $h^a_j a^\mu$) takes the form

$$Z_I'[j] = N_2 \int Dh^a \exp [iS_{SD2}[h, j]] = N_2 Z_{SD2}[j]$$

(75)

As in the previous section, we now invert the integration order in (68) starting from the $h^a$ path-integral,

$$I'[\mathcal{H}] = \int Dh^a \exp \left( i \int d^3 x \left( \frac{1}{2} h^a [S^a_{\mu}]_b \rho h^b_{\rho} + h^a J_a^a \right) \right)$$

(76)

where $[S^a_{\mu}]_b \rho$ is defined in eq. (43) and

$$J_a^a = j_a^a + \frac{1}{m} [S^a_{\mu}]_b \rho W^b_{\rho}(\mathcal{H})$$

(77)

Following the same procedure as for the previous calculation, the integration can be made straightforwardly after completing squares and introducing an appropriate regularization of $[S^a_{\mu}]_b \rho$. The answer is

$$I'[\mathcal{H}] = N_1 \exp \left( i \int d^3 x \left( -\frac{1}{m} W^a_{\mu}(\mathcal{H}) \varepsilon^{\mu\nu\lambda} \partial_\nu W_{\lambda a}(\mathcal{H}) - \frac{1}{m} W^a_{\mu}(\mathcal{H}) j_a^\mu - \frac{1}{2} j_a^\mu [S^{-1}_a]_{\nu}^b j_b^\nu \right) \right)$$

(78)

where as before we have factorized a field-independent irrelevant constant $N_1$ arising from the quadratic path-integral. We then have for $Z_I'[j]$

$$Z_I'[j] = N_1 \int D\mathcal{H} \exp (iS_{SD3}[\mathcal{H}, j_W]) = N_1 Z_{SD3}[j_W]$$

(79)
where we have defined

\[ Z_{SD3}[j_w] = \int \mathcal{D}H^a_\mu \exp (iS_{SD3}[\mathcal{H}, j_w]) \]  \hspace{1cm} (80)\]

with

\[ S_{SD3}[\mathcal{H}, j_w] = S_{SD3}[\mathcal{H}] - \int d^3x \left( \frac{1}{m} W^a_\mu (\mathcal{H}) j_a^\mu + \frac{1}{2} j_a^\mu [S^{-1}_a]_{b \nu} j_b^\nu \right) \] \hspace{1cm} (81)\]

and \( S_{SD3}[\mathcal{H}] \) defined in eq. (31). It is worth noting at this point that in general if we couple the connection with a conserved source the only contribution will come from the part in \( W \) that depends on the symmetric part of the dreibein deviation. In fact from (6)

\[ \partial_\mu j_a^\mu = 0 \rightarrow W^a_\mu (h) j_a^\mu = W^a_\mu (H) j_a^\mu + \text{surface term} \] \hspace{1cm} (82)\]

so this coupling term will be explicitly Lorentz invariant if we work with a general dual field \( \mathcal{H} \).

Comparing (75) and (79) we obtain the main result in this section

\[ N_2 Z_{SD2}[j] = N_1 Z_{SD3}[j_w]. \] \hspace{1cm} (83)\]

We again stress that the Einstein term in \( SD3 \) appears with a minus sign in front as it should be in order to have the correct description of a \( +2 \) helicity excitation with mass \( m \). As we said before this cannot be changed modifying \( S'_I \) because we will lose the connection with \( SD2 \).

Differentiation with respect to the source leads to the duality connection between vacuum expectation values

\[ \langle h^a_\mu \rangle_{SD2} = - \left( \frac{1}{m} W^a_\mu [\mathcal{H}] \right)_{SD3} \] \hspace{1cm} (84)\]

We have again established a duality relation which holds at the quantum level, this time between theories \( SD2 \) and \( SD3 \).

This identification at the quantum level is consistent with the classical duality result. Indeed, considering (6) and (10) in (31)

\[ S_{SD3} = \frac{1}{2} \int d^3x \left( h_\mu^a K^{\pm}_{a b \ lambda} \left( - \frac{1}{m^2} S_{a b \lambda} (h) \right) \right) \]

\[ = \frac{1}{2} \int d^3x \left( h_\mu^a K^{a b \ lambda} \left( - \frac{1}{m} [W^b_\lambda c] \right) \left( - \frac{1}{m} W^c_\sigma (h) \right) \right) \]

\[ = \frac{1}{2} \int d^3x \left( h_\mu^a K^{SD2}_{a b \ lambda} \left( - \frac{1}{m} W^b_\lambda (h) \right) \right). \] \hspace{1cm} (85)\]

\[ (86)\]
where we have identified the evolution operator of the SD2 model with $K_{a b}^{SD2 \mu \lambda}$.

Concerning correlation functions, in this case the presence of the source terms in (81) should be appropriately handled. The conserved source can be decomposed as

$$j_a^\mu = (j_a^0, j_a^i) = (J_a, \varepsilon_{ij} \partial_j J_a^T + \partial_i (-\Delta)^{-1} J_a),$$

and it can be seen that for conserved sources the quadratic term in (81) can be written as

$$\frac{1}{2} \int d^3 x j_a^\mu [S^{-1}_\mu]^b j_b^\nu = \frac{1}{2m} \int d^3 x j_a^\mu \eta^{ab} \varepsilon_{\mu\rho\sigma} \frac{\partial}{\Box} j_b^\nu = \frac{1}{m} \int d^3 x \eta^{ab} J_a J_b^T.$$  \hspace{1cm} (88)

In this sense we get a result equivalent to (63), now relating correlation functions for SD2 and SD3,

$$\langle h_{\mu_1}^{a_1}(x_1) \cdots h_{\mu_n}^{a_n}(x_n) \rangle_{SD2} = (-\frac{1}{m})^n \langle W_{\mu_1}^{a_1} [H(x_1)] \cdots W_{\mu_n}^{a_n} [H(x_n)] \rangle_{SD3} + \text{contact terms}$$  \hspace{1cm} (89)

so that a field mapping reproducing the quantum relations (84)-(89) can be also written for the SD2 $\rightarrow$ SD3 duality relation

$$h^a_\mu(x) \rightarrow -\frac{1}{m} W^a_\mu[H](x)$$  \hspace{1cm} (90)

Let us end by noting that the coupling term $(1/m)W^a_\mu j_a^\mu$ in (81) is invariant under diffeomorphism and, as we pointed, under Lorentz transformations due to the transversality of the source. So the map preserves the gauge invariances without imposing new constraints on the source.

We can extend the connection to the self-dual model SD1 taking into account (56) and changing the source coupling in $S'_J$ in (59). To this end, instead of adding a term of the form $h_{\mu}^{a} j_{a}^{\mu}$, we add a source coupling term of the form $-(1/m)W^a_\mu(h) j_a^\mu$, which also preserves the gauge invariance. We start the process with a new generating interpolating function $\tilde{Z}'_I[j_w]$ and after integrating over $H_{\mu\nu}$ gets the SD2 model with the appropriate coupling

$$\tilde{Z}'_I[j_w] = \int \mathcal{D} h_{\mu}^{a} \mathcal{D} H_{\mu\nu} \exp(i \tilde{S}'_I[h, H, j_w]),$$

$$= \int \mathcal{D} h_{\mu}^{a} \mathcal{D} H_{\mu\nu} \exp \left( i(S'_I[h, H] - \int d^3 x \frac{1}{m} W^a_\mu(h) j_a^\mu) \right)$$

$$= \mathcal{N}_2 \int \mathcal{D} h_{\mu}^{a} \exp \left( i(S_{SD2}[h] - \int d^3 x \frac{1}{m} W^a_\mu(h) j_a^\mu) \right)$$

$$= \mathcal{N}_2 \tilde{Z}_{SD2}[j_w].$$  \hspace{1cm} (91)
Performing instead the path-integration over \( h_{\mu} \), we get an expression like (76) but where \( J_{a}^{\mu} \) now reads
\[
J_{a}^{\mu} = -\frac{1}{m^2} [S_{a}^{\mu}]_{b} (\tilde{j}_{\rho} - mW_{b}^{\rho}(H)), \tag{93}
\]
with
\[
\tilde{j}_{\rho} = (\delta_{\mu}^{b} \delta_{\rho}^{a} - \frac{1}{2} \delta_{\mu}^{a} \delta_{\rho}^{b}) j_{a}^{\mu}. \tag{94}
\]

Then, after the integration over \( h_{\mu} \) we obtain
\[
\tilde{Z}_{1}[j_{W}] = N_{1} \int D\mathcal{H}_{a} \exp[i(S_{SD3} - H - \int d^3x (\frac{1}{2 m^3} \tilde{j}_{\mu}^{a} \tilde{\epsilon}^{\mu\nu\lambda} \partial_{\nu} \tilde{j}_{\lambda}^{a} + \frac{1}{m^2} \tilde{j}_{\mu}^{a} G^{a}[\mathcal{H}])),]
\]
\[
= N_{1} \int D\mathcal{H}_{a} \exp[i(S_{SD3} - H - \int d^3x (\frac{1}{2 m^3} \tilde{j}_{\mu}^{a} \tilde{\epsilon}^{\mu\nu\lambda} \partial_{\nu} \tilde{j}_{\lambda}^{a} + \frac{1}{m^2} j_{a}^{\mu} S_{a}[\mathcal{H}])),] \tag{95}
\]
where we have denoted the direct coupling term in the form \( \tilde{j}_{\mu}^{a} G^{a}(H) = S_{\mu}^{a}(H) j_{a}^{\mu} \), with \( S_{\mu}^{a} \) the Schouten tensor. We conclude that
\[
N_{2} \tilde{Z}_{SD2}[j_{W}] = N_{1} \tilde{Z}_{SD3}[j_{S}] \tag{96}
\]
where the subindex \( S \) in the source indicates that the coupling in \( SD3 \) is that written in eq. (95). The coupling term with the external source is explicitly Lorentz invariant due to the fact that the Schouten tensor depends on the symmetrical part of \( H_{\mu\nu} \), so as before the gauge invariances of the action are regained by the mapping.

The symmetry of the Schouten tensor implies that only the symmetric part of the source \( j_{s}^{\mu\nu} = (1/2)(j_{\mu\nu} + j_{\nu\mu}) \) contributes to the coupling so the connection between correlation functions should be obtained by differentiating with respect to \( j_{s}^{\mu\nu} \). Taking this into account the following relation between vacuum expectation values hold
\[
\frac{1}{2} \langle H_{\mu\nu}(x) \rangle_{SD1} = -\frac{1}{m} \langle W_{s_{\mu\nu}}^{s}(H)(x) \rangle_{SD2} = -\frac{1}{m^2} \langle S_{s_{\mu\nu}}^{s}(H)(x) \rangle_{SD3} \tag{97}
\]
where the supper script \( s \) indicates that symmetrization is assumed. Note that solely \( H_{\mu\nu} \), the symmetric part of \( h_{\mu\nu} \), appears in (97) due to the conservation condition imposed to the source. The result should be compared with (31) where the complete \( h_{\mu\nu} \) tensor contributes.

The relation between the vacuum expectation values that we have obtained is consistent with the form of the classical actions in (30) and (31). We note that in each mapping the v.e.v. of \( h \) is mapped in a v.e.v. of gauge invariant objects of each model.
V. CONNECTION WITH THE NEW TOPOLOGICALLY MASSIVE GRAVITY

As before, we shall work in terms of the deviation of the metric (see (32) and (34)). The interpolating action with minimal coupling is now

$$S''_I[H, \mathcal{H}, j] = \frac{1}{4} \int d^3x \left( -\frac{1}{m} \mathcal{H}_{\mu\nu} C^{\mu\nu}(\mathcal{H}) + \frac{2}{m} \mathcal{H}_{\mu\nu} C^{\mu\nu}(H) + H_{\mu\nu} G^{\mu\nu}(H) + 2H_{\mu\nu} j^{\mu\nu} \right),$$

(98)

where $H_{\mu\nu}$ is the deviation of the metric and $\mathcal{H}_{\mu\nu}$ the dual field, which is assumed to be symmetric; $j^{\mu\nu}$ is a symmetric conserved source. Action (98) is invariant, up to surface terms, under diffeomorphism transformations in the two fields and under conformal transformations in the dual field

$$\delta H_{\mu\nu} = \partial_{\mu} \xi_{\nu} + \partial_{\nu} \xi_{\mu}; \quad \delta \mathcal{H}_{\mu\nu} = \partial_{\mu} \tilde{\xi}_{\nu} + \partial_{\nu} \tilde{\xi}_{\mu} + \eta_{\mu\nu} \rho$$

(99)

We now introduce the generating functional

$$Z_I[j] = \int D\mathcal{H}_{\mu\nu} D\mathcal{H}_{\mu\nu} \exp(iS''_I[H, \mathcal{H}, j])$$

(100)

and, as in the previous sections we shall integrate in the two opposite orders. First we shall consider the integral over the dual field $\mathcal{H}$, this allowing to connect the generating functional $Z_I[j]$ with that of the linear topologically massive model $SD3$. We start defining the functional

$$I[H_{\mu\nu}] = \int D\mathcal{H}_{\mu\nu} \exp \left( \frac{i}{4} \int d^3x \left( -\frac{1}{m} \mathcal{H}_{\mu\nu} C^{\mu\nu,\lambda\rho}(\mathcal{H}) \mathcal{H}_{\lambda\rho} + \frac{2}{m} \mathcal{H}_{\mu\nu} J^{\mu\nu} \right) \right)$$

(101)

where operator $C^{\mu\nu,\lambda\rho}$, acting on $\mathcal{H}_{\mu\nu}$ gives the linear Cotton tensor

$$C^{\mu\nu,\lambda\rho} \mathcal{H}_{\lambda\rho} = C^{L,\mu\nu}(\mathcal{H})$$

(102)

and

$$J^{\mu\nu} \equiv C^{\mu\nu,\lambda\rho} \mathcal{H}_{\lambda\rho}.$$  

(103)

As it stands, operator $C^{\mu\nu,\lambda\rho}$ is non-invertible due to gauge invariance, so it must be regularized. As before, we shall adopt some regularization which can be safely turned off at the end of the calculation. The answer is

$$I[H_{\mu\nu}] = \exp \left( \frac{i}{4m} \int d^3x H_{\mu\nu} C^{\mu\nu}(H) \right) \times \int D\mathcal{H}_{\mu\nu} \exp \left( -\frac{i}{4m} \int d^3x (\mathcal{H}_{\mu\nu} - H_{\mu\nu}) C^{\mu\nu,\lambda\rho}(\mathcal{H}_{\lambda\rho} - H_{\lambda\rho}) \right)$$

(104)

$$= N_3 \exp \left( \frac{i}{4m} \int d^3x H_{\mu\nu} C^{\mu\nu}(H) \right),$$

(105)
with \( \mathcal{N}_3 \) an irrelevant constant factor.

From the calculation above, a connection with the generating functional of the linear topologically massive model \( SD3 \) is established

\[
Z''_I[j] = \mathcal{N}_3 Z_{SD3}[j] \tag{106}
\]

We shall now follow the inverse road, first integrating in \( H_{\mu \nu} \), this allowing to connect \( Z''_I[j] \) with the partition function of the new topologically massive model. The integral to be performed can be written as

\[
I[H_{\mu \nu}] = \int \mathcal{D}H_{\mu \nu} \exp \left( \frac{i}{2} \int d^3x \left( \frac{1}{2} H_{\mu \nu} [G^{\mu \nu}]^{\lambda \rho} H_{\lambda \rho} + H_{\mu \nu} \mathcal{J}^{\mu \nu} \right) \right) \tag{107}
\]

with \(-(1/2)[G^{\mu \nu}]^{\lambda \rho}\) the evolution operator of the linear Einstein term and

\[
\mathcal{J}^{\mu \nu} = \frac{1}{2} j^{\mu \nu} + \frac{1}{4m}[G^{\mu \nu}]^{\lambda \rho} (\epsilon^{\lambda \alpha \beta} \partial_\beta \mathcal{H}_{\beta \rho} + \epsilon^{\alpha \beta \rho} \partial_\beta \mathcal{H}_{\beta \lambda}). \tag{108}
\]

To obtain this last equation we have used the identity

\[
\mathcal{H}_{\mu \nu} C^{\mu \nu}(H) = \frac{1}{2} [H_{\mu \nu} [G^{\mu \nu}]^{\lambda \rho} (\epsilon^{\lambda \alpha \beta} \partial_\beta \mathcal{H}_{\beta \rho} + \epsilon^{\alpha \beta \rho} \partial_\beta \mathcal{H}_{\beta \lambda}) + \text{surface term}] \tag{109}
\]

The operator \([G^{\mu \nu}]^{\lambda \rho}\) requires a regularization in order to become invertible. Once this is done one can safely integrate and then turn off the regulator. The answer is

\[
I[H_{\mu \nu}] = \mathcal{N}_2 \times \exp \left( -i \int d^3x \left( \frac{1}{4m^2} \epsilon^{\mu \nu \lambda \rho} \mathcal{H}_{\mu \nu} [G_{\lambda \rho}]^{\gamma \sigma} \partial_\gamma \partial_\sigma \mathcal{H}_{\gamma \sigma} 
+ \frac{1}{2m} \epsilon^{\lambda \alpha \beta} \partial_\alpha \mathcal{H}_{\rho \beta \rho} j^{\lambda \rho} + \frac{1}{4} j^{\mu \nu} [G^{-1}_{\mu \nu}]^{\lambda \rho} j^{\lambda \rho} \right) \right). \tag{110}
\]

The direct coupling term between the source and \( \mathcal{H}_{\mu \nu} \) is just \(-(1/m)W_{\mu \nu}(\mathcal{H})j^{\mu \nu}\). Due to the symmetry of the source it is explicitly conformal invariant since it depends solely on the symmetric part of the linear connection. Hence, conformal invariance is regained in the map without imposing new constraints on the source.

After these calculations we can then write

\[
Z''_I[j] = \mathcal{N}_2 Z_{SD4}[j_w] \tag{111}
\]

with

\[
S_{SD4}[j_w] = S_{SD4} - \int d^3x \left( \frac{1}{m} W_{\lambda \rho}(\mathcal{H}) j^{\lambda \rho} + \frac{1}{4} j^{\mu \nu} [G^{-1}_{\mu \nu}]^{\lambda \rho} j^{\lambda \rho} \right) \tag{112}
\]
Then, from eqs. (106) and (111) we finally have the duality identity

\[ N_3 Z_{SD3}[j] = N_2 Z_{SD4}[j_w] \]  

(113)

which allows to write the following identity between vacuum expectation values in the two models

\[ \frac{1}{2} \langle H_{\mu\nu}(x) \rangle_{SD3} = - \frac{1}{m} \langle W^s_{\mu\nu}[\mathcal{H}(x)] \rangle_{SD4} \]  

(114)

As in the previous sections this relation is consistent with the classical results. Indeed, from eq. (10) we can write the \( SD4 \) action in the form

\[ S_{SD4} = \frac{1}{4} \int d^3 x \left( H_{\mu\alpha} K^{\mu\alpha}_{\rho\beta} \left( - \frac{1}{m^2} \varepsilon^{\rho\sigma\lambda} \partial_\sigma [W^s_\lambda]^c W^c_\gamma (H) \right) \right) \]

\[ = \frac{1}{4} \int d^3 x \left( H_{\mu\alpha} K^{\mu\alpha}_{\rho\beta} \left( \frac{1}{m^2} G^{\rho\beta} W(H) \right) \right) \]

\[ = \frac{1}{4} \int d^3 x \left( H_{\mu\alpha} K^{\mu\alpha}_{\rho\beta} \left( - \frac{1}{m^2} S^{\rho\beta}_{\nu\gamma} \left( \frac{1}{m} \varepsilon^{\nu\sigma\lambda} \partial_\sigma H_\lambda \gamma \right) \right) \right) \]

\[ = \frac{1}{4} \int d^3 x \left( H_{\mu\alpha} K^{SD3\mu\alpha}_{\rho\beta} \left( - \frac{1}{m} \varepsilon^{\rho\sigma\lambda} \partial_\sigma H_\lambda \beta \right) \right) \]  

(115)

where \( S^{\rho\beta}_{\nu\gamma} \) as the differential operator that acting on \( H^{\nu\gamma} \) gives the linear Schouten tensor and \( K^{SD3\mu\alpha}_{\rho\beta} \) the evolution operator associated to action \( SD3 \). Written the classical action in this form, the relation between classical and quantum relations becomes clear.

Concerning correlation functions of products of fields one has to take care of contact terms resulting from the term quadratic in the source (see eq. (112)). So also in this case we shall write

\[ \frac{1}{2^n} \langle (H_{\mu_1\nu_1}(x_1) \cdots H_{\mu_n\nu_n}(x_n))_{SD3} = \left( - \frac{1}{m} \right)^n \langle W^s_{\mu_1\nu_1}[\mathcal{H}(x_1)] \cdots W^s_{\mu_n\nu_n}[\mathcal{H}(x_n)] \rangle_{SD4} + \text{contact terms} \]  

(116)

where the superscript \( s \) indicates that the connection \( W \) must be symmetrized.

Connections with generating functionals of the other models can be established by changing couplings to the source in the interpolating action. In particular, taking into account eq. (56) one can change the minimal coupling term in (98) to the following one

\[ - \frac{1}{2m^2} \tilde{S}_{\mu\nu}(H) j^{\mu\nu} = - \frac{1}{2m^2} H_{\mu\nu} G^{\mu\nu,\lambda\rho} \tilde{j}_{\lambda\rho} + \text{surface term} \]  

(117)

with \( \tilde{j}_\rho = (\delta^{b}_\mu \delta^{a}_\rho - \frac{1}{2} \delta^{a}_\mu \delta^{b}_\rho) j_{a\mu} \) as in (94). If we now perform the integration over \( \mathcal{H} \) we get the linear topologically massive generating functional with the appropriate coupling

\[ Z'_{\tilde{j}}[j_s] = N_3 \tilde{Z}_{SD3}[j_s] \]  

(118)
The integration over $H_{\mu \nu}$ is analogous to (107) with
\[ J_{\mu \nu} = \frac{1}{2m} [G_{\mu \nu}]^{\lambda \rho} \left( \varepsilon^{\alpha \beta} \partial_\beta H_{\alpha \rho} - \frac{1}{m} \tilde{\gamma}_{\lambda \rho} \right). \] (119)
and leads to a generating functional in which there is a non-minimal coupling to the Cotton tensor
\[ Z''[j_s] = N_2 \tilde{Z}_{SD4}[j_c] \] (120)
with
\[ S_{SD4}[j_c] = S_{SD4}[H] + \frac{1}{2} \int d^3 x \left( \frac{1}{m^3} C_{\mu \nu}(H) j^{\mu \nu} - \frac{1}{2m^2} \tilde{\gamma}_{\mu \nu} G_{\mu \nu, \lambda \rho} j^{\lambda \rho} \right). \] (121)
This coupling is consistent with the classical expression of action (35) and it is clearly conformal invariant.

We can now make the correspondence of the v.e.v. from SD1 to SD4, completing (97)
\[ \frac{1}{2} \langle H_{\mu \nu}(x) \rangle_{SD1} = -\frac{1}{m} \langle W_{\mu \nu}(H)(x) \rangle_{SD2} = -\frac{1}{m^2} \langle S_{\mu \nu}(H)(x) \rangle_{SD3} = \frac{1}{m^3} \langle C_{\mu \nu}(H)(x) \rangle_{SD4}. \] (122)

The connection with the SD2 could be obtained straightforward if we start the process with a nonminimall coupling in (98) with the symmetrized connection. In this case the integration in $H_{\mu \nu}$ will give the SD3 generating function with the appropriate coupling and the corresponding field independent factor. The integration in $H_{\mu \nu}$ will take us to the generating function of the SD4 model with a coupling with $-\frac{1}{m^2} S_{\mu \nu}(H)$ as we would expect from the classical description. The correspondence of the v.e.v. will be
\[ \frac{1}{2} \langle H_{\mu \nu}(x) \rangle_{SD2} = -\frac{1}{m} \langle W_{\mu \nu}(H)(x) \rangle_{SD3} = -\frac{1}{m^2} \langle S_{\mu \nu}(H)(x) \rangle_{SD4}. \] (123)

VI. CONCLUSIONS

We have discussed the duality between four self-dual linearized formulations for massive gravity in $2 + 1$ dimensions at the quantum level. To this end, we have established generating functional connections in a path-integral framework, this allowing to find identities for vacuum expectations values which are consistent with the gauge invariance of the classical actions.

We have found that the v.e.v. for the field $h^a_{\mu}$, the linear deviation of the dreibein in the self-dual model SD1, is mapped (apart from a constant factor) onto a the v.e.v of the linear connection $-(1/m)W^a_{\mu}(h)$ in the intermediate model SD2, which is invariant under...
diffeomorphisms. In turn, this v.e.v. is mapped onto the vacuum expectation value the Schouten tensor $- (1/m^2) S_{\mu}^a(h)$ in the topologically massive model SD3, which is invariant under diffeomorphism transformations and explicitly under Lorentz transformations as it depends on the symmetric part of the dreibein. Finally, this last v.e.v. is mapped into the v.e.v. of the Cotton tensor, $(1/m^3) C_{\mu}^a(h)$ which is clearly invariant under diffeomorphism and conformal transformations. These correspondences are consistent with the classical equations of motion and ensure the non-propagation of the lower spin parts of $h_{\mu}^a$ so that up to contact terms the mapping connects the spin 2 components of each object.

Following the same route, we have also connected the intermediate model with the linear topologically massive and new topologically massive models. In this case the $h_{\mu}^a$ coupling term is invariant, up to a surface term, under diffeomorphisms if the source is conserved $(\partial_\mu j_a^\mu = 0)$ and so is the non-minimal coupling terms in the SD3 and SD4 models after the dual map. The v.e.v. of $h_{\mu}^a$ is mapped first onto $-(1/m) W_{\mu}^a(H)$ and then onto $-(1/m^2) S_{\mu}^a(h)$.

The two linear topologically massive models are also connected and again, after the dual map, the coupling term is invariant under conformal transformations. The v.e.v. of $H_{\mu\nu}$ is mapped onto $-(1/m) W^a_{\mu\nu}(H)$.

The quantum identities that we have obtained are consistent with those one would expect from classical actions, as can be seen writing them in terms of the evolution operator of the self-dual model SD1, as in (30), (31) and (35). The dual theories differ in their gauge invariances and the couplings obtained when passing from one to the other are consistent.

The results described above can be schematized as follows

\[ \begin{align*}
\text{Self - dual model} & \rightarrow \text{2nd order intermediate model} \rightarrow \text{TMG} \rightarrow \text{New TMG} \\
S_{SD1} &= \frac{1}{2} \int hK \, h \quad S_{SD2} = -\frac{1}{2m} \int hKW[h] \quad S_{SD3} = -\frac{1}{4m^2} \int HKS[H] \quad S_{SD4} = \frac{1}{4m^3} \int HKC[H] \\
\langle h_{\mu}^a \rangle_{SD1} \rightarrow & \quad -\frac{1}{m} \langle W_{\mu}^a[h] \rangle_{SD2} \rightarrow \quad -\frac{1}{m} \langle S_{\mu}^a[H] \rangle_{SD3} \quad \frac{1}{m} \langle C_{\mu}^a[H] \rangle_{SD4} \\
\langle h_{\mu}^a \rangle_{SD2} \rightarrow & \quad -\frac{1}{m} \langle W_{\mu}^a[H] \rangle_{SD3} \quad -\frac{1}{m} \langle S_{\mu}^a[H] \rangle_{SD4} \\
\frac{1}{2} \langle H_{\mu\nu} \rangle_{SD3} \rightarrow & \quad -\frac{1}{m} \langle W^a_{\mu\nu}[H] \rangle_{SD4}
\end{align*} \]

with $K, W, S$ and $C$ defined in eqs.(27), (6), (10) and (9) respectively. The form of the actions
appear in \((27), (28), (31)\) and \((34)\).

There are several extensions of this work that could be considered. It would be interesting to determine how these models behave when interactions are included. Also, one could analyze whether the dualities we established can be also found in the gravity models that are connected via quadratic linearization with the SD2, SD3 and SD4 models \[5, 1, 15-13\]. Since there are quadratic linearizations of topologically massive supergravity \[29\] and new topologically massive supergravity \[15\], it could be of interest to study the possible equivalence between these models. Finally the extension of the quantum duality connections to the case of higher spin theories \[30-32\] which have self-dual formulations for massive excitations. We hope to discuss these issues in a future work.

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