The Multifractal Nature of Boltzmann Processes

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Abstract

We consider the spatially homogeneous Boltzmann equation for (true) hard and moderately soft potentials. We study the pathwise properties of the stochastic process \((V_t)_{t \geq 0}\), which describes the time evolution of the velocity of a typical particle. We show that this process is almost surely multifractal and we compute its spectrum of singularities. For hard potentials, we also compute the multifractal spectrum of the position process \((X_t)_{t \geq 0}\).

1 Introduction

The Boltzmann equation is the main model of kinetic theory. It describes the time evolution of the density \(f_t(x,v)\) of particles with position \(x \in \mathbb{R}^3\) and velocity \(v \in \mathbb{R}^3\) at time \(t \geq 0\), in a gas of particles interacting through binary collisions. In the special case where the gas is initially spatially homogeneous, this property propagates with time, and \(f_t(x,v)\) does not depend on \(x\). We refer to the books by Cercignani [6] and Villani [20] for many details on the physical and mathematical theory of this equation, see also the review paper by Alexandre [1].

Tanaka gave in [18] a probabilistic interpretation of the case of Maxwellian molecules: he constructed a Markov process \((V_t)_{t \geq 0}\), solution to a Poisson-driven stochastic differential equation, and such that the law of \(V_t\) is \(f_t\) for all \(t \geq 0\). Such a process \((V_t)_{t \geq 0}\) has a richer structure than the Boltzmann equation, since it contains some information on the history of particles. Physically, \((V_t)_{t \geq 0}\) is interpreted as the time-evolution of the velocity of a typical particle. Fournier and Méleard [9] extended Tanaka’s work to non-Maxwellian molecules, see the last part of paper by Fournier [8] for up-to-date results.

In the case of long-range interactions, that is when particles interact through a repulsive force in \(1/|r|^s\) (for some \(s > 2\)), the Boltzmann equation presents a singular integral (case without cutoff). The reason is that the corresponding process \((V_t)_{t \geq 0}\) jumps infinitely often, i.e. the particle is subjected to infinitely many collisions, on each time interval. In some sense, it behaves, roughly, like a Lévy process.

The Hölder regularity of the sample paths of stochastic processes was first studied by Orey and Taylor [13] and Perkins [16], who showed that the fast and slow points of Brownian motion are located on random sets of times, and they showed that the sets of points with a given pointwise regularity have a fractal nature. Jaffard [13] showed that the sample paths of most Lévy processes are multifractal functions and he obtained their spectrum of singularities. This spectrum is almost surely deterministic: of course, the sets with a given pointwise regularity are extremely complicated, but their Hausdorff dimension is deterministic. Let us also mention the article by Balanço [3], in which he extended the results (and simplified some proofs) of Jaffard [13].

What we expect here is that \((V_t)_{t \geq 0}\) should have the same spectrum as a well-chosen Lévy process. This is of course very natural (having a look at the shape of the jumping SDE satisfied by \((V_t)_{t \geq 0}\)). There are however many complications, compared to the case of Lévy processes, since we loose all the independence and stationarity properties that simplify many computations and arguments. We will also

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compute the multifractal spectrum of the position process \((X_t)_{t\geq 0}\), defined by \(X_t = \int_0^t V_s \, ds\), which appears to have multifractal sample paths as well.

By the way, let us mention that, though there are many papers computing the multifractal spectrum of some quite complicated objects, we are not aware of any work concerning general Markov processes, that is, roughly, solutions to jumping (or even non jumping) SDEs. In this paper, we study the important case of the Boltzmann process, as a physical example of jumping SDE. Of course, a number of difficulties have to be encompassed, since the model is rather complicated. However, we follow, adapting everywhere to our situation, the main ideas of Jaffard \cite{13} and Balançà \cite{3}.

Let us finally mention that Barral, Fournier, Jaffard and Seuret \cite{4} studied a very specific ad-hoc Markov process, showing that quite simple processes may have a random spectrum that depends heavily on the values taken by the process.

1.1 The Boltzmann equation

We consider a 3-dimensional spatially homogeneous Boltzmann equation, which depicts the density \(f_t(v)\) of particles in a gas, moving with velocity \(v \in \mathbb{R}^3\) at time \(t \geq 0\). The density \(f_t(v)\) solves

\[
\partial_t f_t(v) = \int_{\mathbb{R}^3} dv' \int_{S^2} d\sigma B(|v - v'|, \cos \theta)[f_t(v')f_t(v') - f_t(v)f_t(v')], \tag{1.1}
\]

where

\[
v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v_* = \frac{v + v_*}{2} \quad \text{and} \quad \cos \theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle. \tag{1.2}
\]

The cross section \(B(|v - v_*|, \cos \theta) \geq 0\) depends on the type of interaction between particles. It only depends on \(|v - v_*|\) and on the cosine of the deviation angle \(\theta\). Conservations of mass, momentum and kinetic energy hold for reasonable solutions and we may assume without loss of generality that \(\int_{\mathbb{R}^3} f_0(v) dv = 1\). We will assume that there is a measurable function \(\beta : (0, \pi) \to \mathbb{R}_+\) such that

\[
\begin{aligned}
&\exists \, \theta_0, \quad \forall \, \theta \in (0, \pi/2), \quad \cos \theta_0 \leq \beta(\theta) \leq \cos \theta, \\
&\forall \, \theta \in (\pi/2, \pi), \quad \beta(\theta) = 0,
\end{aligned}
\tag{1.3}
\]

for some \(\nu \in (0, 1)\), and \(\gamma \in (-1, 1)\) satisfying \(\gamma + \nu > 0\). The last assumption on the function \(\beta\) is not a restriction and can be obtained by symmetry as in \cite{1}. Note that, when particles collide by pairs due to a repulsive force proportional to \(1/r^s\) for some \(s > 2\), assumption \cite{13} holds with \(\gamma = (s - 5)/(s - 1)\) and \(\nu = 2/(s - 1)\). Here we will be focused on the cases of hard potentials \(s > 5\), Maxwell molecules \(s = 5\) and moderately soft potentials \(3 < s < 5\).

Next, we give the definition of weak solutions of (1.1). We define \(\mathcal{P}_p(\mathbb{R}^3)\) as the set of all probability measures \(f\) on \(\mathbb{R}^3\) such that \(m_p(f) \equiv \int_{\mathbb{R}^3} |v|^p f dv < \infty\).

**Definition 1.1.** Assume \cite{13} is true for some \(\nu \in (0, 1), \gamma \in (-1, 1)\). A measurable family of probability measures \((f_t)_{t \geq 0}\) on \(\mathbb{R}^3\) is called a weak solution to (1.1) if it satisfies the following two conditions.

- For all \(t \geq 0\),

\[
\int_{\mathbb{R}^3} v f_t dv = \int_{\mathbb{R}^3} v f_0 dv \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_t dv = \int_{\mathbb{R}^3} |v|^2 f_0 dv < \infty. \tag{1.4}
\]

- For any bounded globally Lipschitz-continuous function \(\phi : \mathbb{R}^3 \to \mathbb{R}\), any \(t \geq 0\),

\[
\int_{\mathbb{R}^3} \phi(v) f_t dv = \int_{\mathbb{R}^3} \phi(v) f_0 dv + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_B \phi(v, v_*) f_s dv_* f_s dv ds, \tag{1.5}
\]

where \(v'\) and \(\theta\) are defined by (1.2), and

\[
L_B \phi(v, v_*) := \int_{S^2} B(|v - v_*|, \cos \theta)(\phi(v') - \phi(v)) d\sigma.
\]
The existence of a weak solution to (1.1) is now well established (see [19] and [14]). In particular, when $\gamma \in (0, 1)$, it is shown in [14] that for any $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$, there exists a weak solution $(f_t)_{t \geq 0}$ to (1.1) satisfying $\sup_{t \geq t_0} m_p(f_t) < \infty$ for all $p \geq 2$, all $t_0 > 0$. Some uniqueness results can be found in [11].

1.2 The Boltzmann process

We first parameterize (1.2) as in [10]. For each $x \in \mathbb{R}^3 \setminus \{0\}$, we consider the vector $I(x) \in \mathbb{R}^3$ such that $|I(x)| = |x|$ and $I(x) \perp x$. We also set $J(x) = \frac{x}{|x|} \wedge I(x)$. The triplet $(\frac{x}{|x|}, \frac{I(x)}{|I(x)|}, \frac{J(x)}{|J(x)|})$ is an orthonormal basis of $\mathbb{R}^3$. Then for $x, v, \varphi \in \mathbb{R}^3$, $\theta, \varphi \in [0, \pi)$, we set

$$
\Gamma(x, \varphi) := (\cos \varphi I(x) + (\sin \varphi) J(x),
\quad v'(v, v_*, \theta, \varphi) := v - \frac{1}{2} \cos \varphi (v - v_*) + \frac{1}{2} \sin \varphi \Gamma(v - v_*, \varphi),
\quad a(v, v_*, \theta, \varphi) := v'(v, v_*, \theta, \varphi) - v.
$$

(1.6)

Let us observe at once that $\Gamma(x, \varphi)$ is orthogonal to $x$ and has the same norm as $x$, from which it is easy to check that

$$
|a(v, v_*, \theta, \varphi)| = \sqrt{\frac{1 - \cos \theta}{2}} |v - v_*|.
$$

(1.7)

Definition 1.2. Let $(f_t)_{t \geq 0}$ be a weak solution to (1.1). On some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, we consider a $\mathcal{F}_0$-measurable random variable $V_0$ with law $f_0$, a Poisson measure $N(ds, dv, d\theta, d\varphi, du)$ on $[0, \infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $dsf_s(dv)\beta(\theta)d\varphi du$. A càdlàg $(\mathcal{F}_t)_{t \geq 0}$-adapted process $(V_t)_{t \geq 0}$ with values in $\mathbb{R}^3$ is then called a Boltzmann process if it solves

$$
V_t = V_0 + \int_0^t \int_{\mathbb{R}^3} \int_{0}^{\pi/2} \int_{0}^{2\pi} \int_{0}^\infty a(V_s, v, \theta, \varphi) \mathbb{1}_{\{|v| \leq |v_*|\}} N(ds, dv, d\theta, d\varphi, du).
$$

(1.8)

From Proposition 5.1 in [8], we have slightly different results for different potentials: when $\gamma \in (0, 1)$, i.e. hard potentials, we can associate a Boltzmann process to any weak solution to (1.1), but when $\gamma \in (-1, 0)$, i.e. moderately soft potentials, we can only prove existence of a weak solution to (1.1) to which it is possible to associate a Boltzmann process.

Proposition 1.3. Let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. Assume (1.3) for some $\gamma \in (-1, 0)$, $\nu \in (0, 1)$.

- If $\gamma \in (0, 1)$, for any weak solution $(f_t)_{t \geq 0}$ to (1.1) starting from $f_0$ and satisfying

$$
\text{for all } p \geq 2, \text{ all } t_0 > 0, \sup_{t \geq t_0} m_p(f_t) < \infty,
$$

there exist a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, a $(\mathcal{F}_t)_{t \geq 0}$-Poisson measure $N(ds, dv, d\theta, d\varphi, du)$ on $[0, \infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $dsf_s(dv)\beta(\theta)d\varphi du$ and a càdlàg $(\mathcal{F}_t)_{t \geq 0}$-adapted process $(V_t)_{t \geq 0}$ satisfying $\mathcal{L}(V_t) = f_t$ for all $t \geq 0$ and solving (1.8).

- If $\gamma \in (-1, 0)$, assume additionally that $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ for some $p > 2$. There exist a probability space, a Poisson measure $N$ and a càdlàg adapted process $(V_t)_{t \geq 0}$ as in the previous case, satisfying $\mathcal{L}(V_t) = f_t$ for all $t \geq 0$ and solving (1.8).

The Boltzmann equation depicts the velocity distribution of a dilute gas which is made up of a large number of molecules. So, the corresponding Boltzmann process $(V_t)_{t \geq 0}$ represents the time evolution of the velocity of a typical particle. When this particle collides with another, its velocity changes suddenly. It is thus a jump process.
1.3 Recalls on multifractal analysis

In this part, we recall the definition of the main objects in multifractal analysis.

**Definition 1.4.** A locally bounded function $g : [0, 1] \to \mathbb{R}^3$ is said to belong to the pointwise Hölder space $C^\alpha(t_0)$ with $t_0 \in [0, 1]$ and $\alpha \notin \mathbb{N}$, if there exist $C > 0$ and a polynomial $P_\alpha$ of degree less than $[\alpha]$, such that for some neighborhood $I_{t_0}$ of $t_0$,

$$|g(t) - P_\alpha(t)| \leq C|t - t_0|^\alpha, \forall t \in I_{t_0}.$$  

The pointwise Hölder exponent of $g$ at point $t_0$ is given by

$$h_g(t_0) = \sup\{\alpha > 0 : g \in C^\alpha(t_0)\},$$

where by convention $\sup \emptyset = 0$. The level sets of the pointwise Hölder exponent of the function $g$ are called the iso-Hölder sets of $g$, and are denoted, for any $h \geq 0$, by

$$E_g(h) = \{t \geq 0 : h_g(t) = h\}.$$

We now recall the definition of the Hausdorff measures and dimension, see [7] for details.

**Definition 1.5.** Given a subset $A$ of $\mathbb{R}$, given $s > 0$ and $\epsilon > 0$, the $s$-Hausdorff pre-measure $\mathcal{H}_s^\epsilon$ using balls of radius less than $\epsilon$ is given by

$$\mathcal{H}_s^\epsilon(A) = \inf \left\{ \sum_{i \in J} |I_i|^s : (I_i)_{i \in J} \in \mathcal{P}_s(A) \right\},$$

where $\mathcal{P}_s(A)$ is the set of all countable coverings of $A$ by intervals with length at most $\epsilon$. The $s$-Hausdorff measure of $A$ is defined by

$$\mathcal{H}^s(A) = \lim_{\epsilon \to 0} \mathcal{H}_s^\epsilon(A).$$

Finally the Hausdorff dimension of $A$ is defined by

$$\dim_H(A) := \inf \{s \geq 0 : \mathcal{H}^s(A) = 0\} = \sup \{s \geq 0 : \mathcal{H}^s(A) = +\infty\},$$

and by convention $\dim_H \emptyset = -\infty$.

We use the concept of spectrum of singularities to describe the distribution of the singularities of a function $g$.

**Definition 1.6.** Let $g : [0, 1] \to \mathbb{R}^3$ be a locally bounded function. The spectrum of singularities (or multifractal spectrum) of $g$ is the function $D_g : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{-\infty\}$ defined by

$$D_g(h) = \dim_H(E_g(h)).$$

The iso-Hölder sets $E_g(h)$ are random for most studied stochastic processes, but almost always have an a.s. deterministic Hausdorff dimension, as in the case of Lévy processes [13].

1.4 Main Results

Now, we give the main results of this paper.

**Theorem 1.7.** We assume [13] for some $\gamma \in (-1, 1)$, some $\nu \in (0, 1)$ with $\gamma + \nu > 0$. We consider some initial condition $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ and assume that it is not a Dirac mass. If $\gamma \in (-1, 0]$, we moreover assume that $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ for some $p > 2$. We consider a Boltzmann process $(V_t)_{t \in [0, 1]}$ as introduced in Proposition [13]. Almost surely, for all $h \geq 0$,

$$D_V(h) = \begin{cases} \nu h & \text{if } 0 \leq h \leq 1/\nu, \\ -\infty & \text{if } h > 1/\nu. \end{cases}$$  

(1.9)
The Multifractal Nature of Boltzmann Processes

The condition that $f_0$ is not a Dirac mass is important: if $V_0 = v_0$ a.s. for some deterministic $v_0 \in \mathbb{R}^3$, then $V_t = v_0$ for all $t \geq 0$ a.s. (which is a.s. a $C^\infty$ function on $[0, \infty)$).

It is obvious from the proof that the spectrum of singularities is homogeneous: we could prove similarly that a.s., for any $0 \leq t_0 < t_1 < \infty$, all $h \geq 0$, $\dim_H(E_V(h) \cap [t_0, t_1]) = D_V(h)$.

Finally, it is likely that the same result holds true for very soft potentials. However, there are several technical difficulties, and the proof would be much more intricate.

Now we exhibit the multifractal spectrum of the position process. For simplicity, we only consider the case of hard potentials.

**Theorem 1.8.** We assume (1.9) for some $\gamma \in (0, 1)$ and some $\nu \in (0, 1)$. We consider some initial condition $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ and assume that it is not a Dirac mass. We consider a Boltzmann process $(V_t)_{t \in [0, 1]}$ as introduced in Proposition (1.3) and introduce the associated position process $(X_t)_{t \in [0, 1]}$ defined by $X_t = \int_0^t V_s ds$. Almost surely, for all $h \geq 0$,

$$D_X(h) = \begin{cases} \nu(h - 1) & \text{if } 1 \leq h \leq \frac{1}{\nu} + 1, \\ -\infty & \text{if } h > \frac{1}{\nu} + 1 \text{ or } 0 \leq h < 1. \end{cases} \quad (1.10)$$

This result is very natural once Theorem (1.7) is checked: we expect that at some given time $t$, the pointwise exponent of $X$ is the one of $V$ plus 1. However, this is not always true: for instance, as can be seen on the simple example of the chirp function $g(x) = x \sin(1/x)$: its pointwise exponent at 0 is 1, while its primitive has a pointwise exponent equal to 3 at 0. Balança [3] has shown that such an oscillatory phenomenon may occur for Lévy processes, but on a very small set of points.

**Definition 1.9.** Let $g : [0, 1] \to \mathbb{R}^3$ be a locally bounded function and let $G(t) = \int_0^t g(s) ds$. For all $h \geq 0$, we introduce the sets

$$E_g^{\text{cusp}}(h) = \{ t \in E_g(h) : h_G(t) = 1 + h_g(t) \} \quad \text{and} \quad E_g^{\text{osc}}(h) = \{ t \in E_g(h) : h_G(t) > 1 + h_g(t) \} \quad (1.11)$$

The times $t \in E_g^{\text{cusp}}(h)$ are referred to as cusp singularities, while the times $t \in E_g^{\text{osc}}(h)$ are called oscillating singularities. Observe that $E_g(h) = E_g^{\text{cusp}}(h) \cup E_g^{\text{osc}}(h)$, the union being disjoint: this follows from the fact that obviously, for all $t \in [0, 1]$, $h_G(t) \geq h_g(t) + 1$. We will prove the following.

**Theorem 1.10.** Under the assumptions of Theorem (1.8), we have almost surely:

- for all $h \in [1/(2\nu), 1/\nu)$, $\dim_H\left(E_g^{\text{osc}}(h)\right) \leq 2\nu - 1$,
- for all $h \in [0, 1/(2\nu)) \cup (1/\nu, +\infty]$, $E_g^{\text{osc}}(h) = \emptyset$,
- for all $h \in [0, 1/\nu)$, $\dim_H\left(E_g^{\text{cusp}}(h)\right) = \nu$.

Actually, we will first prove Theorem (1.10) which, together with Theorem (1.7) implies Theorem (1.8)

2 Localization of the problem

In the following sections, we consider a Boltzmann process $(V_t)_{t \in [0, 1]}$ associated to a weak solution $(f_t)_{t \in [0, 1]}$ to (1.1), and driven by a Poisson measure $N(ds, dv, d\theta, d\varphi, du)$ on $[0, 1] \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi] \times [0, \infty)$ with intensity $ds f_s(dv)\beta(d\theta)d\theta d\varphi du$.

For $B \geq 1$, setting $H_B(v) = \frac{|v|v_B}{|v|} v$, we define, for $t \in [0, 1]$,

$$V^B_t := V_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} a(H_B(V_{s-}), v, \theta, \varphi) 1_{\{u \leq |H_B(V_{s-}) - v|\gamma\}} N(ds, dv, d\theta, d\varphi, du), \quad (2.1)$$
where \( a \) is defined in \((1.8)\). We define the corresponding position process, for \( t \in [0,1] \), as

\[
X_t^B = \int_0^t V_s^B \, ds. \tag{2.2}
\]

In the rest of the paper, we will check the following two localized claims.

**Proposition 2.1.** Let \( B \geq 1 \) be fixed. We assume \((1.3)\) for some \( \gamma \in (-1,1) \), some \( \nu \in (0,1) \) with \( \gamma + \nu > 0 \). We consider the localized process introduced in \((2.1)\). Almost surely, for all \( h \geq 0 \),

\[
D_{V_B}(h) = \begin{cases} 
vh & \text{if } 0 \leq h \leq 1/\nu, \\
-\infty & \text{if } h > 1/\nu.
\end{cases}
\]

**Proposition 2.2.** Let \( B \geq 1 \) be fixed. We assume \((1.3)\) for some \( \gamma \in (0,1) \), some \( \nu \in (0,1) \). We consider the localized process \((V_t^B)_{t \geq 0}\) defined in \((2.1)\). Then almost surely,

- for all \( h \in [1/(2\nu), 1/\nu) \), \( \dim_H \left( E_{V^B}^\text{osc}(h) \right) \leq 2h\nu - 1 \),
- for all \( h \in [0, 1/(2\nu)) \cup (1/\nu, +\infty] \), \( E_{V^B}^\text{osc}(h) = \emptyset \),
- for all \( h \in [0, 1/\nu] \), \( \dim_H \left( E_{V^B}^\text{cusp}(h) \right) = vh \).

Once these propositions are verified, Theorems \((1.7)\) and \((1.10)\) are immediately deduced.

**Proof of Theorems \((1.7)\) and \((1.10)\).** Since \( \sup_{t \in [0,1]} |V_t| < +\infty \) a.s. (because \( V \) is a càdlàg process), the event \( \Omega_B = \{ \sup_{t \in [0,1]} |V_t| \leq B \} \) a.s. increases to \( \Omega \) as \( B \) increases to infinity. But on \( \Omega_B \), we obviously have that \((V_t^B)_{t \in [0,1]} = (V_t)_{t \in [0,1]} \). Hence on \( \Omega_B \), it holds that for all \( h \in [0, +\infty] \), \( D_V(h) = D_{V^B}(h) \), \( \dim_H(E_{V^B}^\text{osc}(h)) = \dim_H(E_{V}^\text{osc}(h)) \) and \( \dim_H(E_{V^B}^\text{cusp}(h)) = \dim_H(E_{V}^\text{cusp}(h)) \). The conclusion then follows from the above two propositions.

We thus fix \( B \geq 1 \) for the rest of the paper.

### 3 Study of the velocity process

#### 3.1 Preliminary

First, we need to bound \( f_t \) from below.

**Lemma 3.1.** There exist \( a, b, c > 0 \), such that for any \( w \in \mathbb{R}^3 \), any \( t \in [0,1] \),

\[
f_t(\mathcal{H}_w) \geq b, \tag{3.1}
\]

where \( \mathcal{H}_w = \{ v \in \mathbb{R}^3 : |v - w| \geq a, |v| \leq c \} \).

**Proof.** As \( f_0 \) is not a Dirac mass, there exist \( v_1 \neq v_2 \) such that \( v_1, v_2 \in \text{Supp} f_0 \). We set \( a = \frac{|v_1 - v_2|}{6} \).

**Step 1.** We first show that there exists \( b > 0 \), such that for all \( w \in \mathbb{R}^3 \), \( t \in [0,1] \), \( f_t(\{v : |v - w| \geq a\}) \geq 2b \). First, if \( |w| \geq \sqrt{2m_2(f_0)} + a =: M \), recalling that \( m_2(f_t) = m_2(f_0) \) for all \( t \geq 0 \),

\[
f_t(\{v : |v - w| \geq a\}) \geq f_t(\{v : |v| \leq |w| - a\}) = 1 - f_t(\{v : |v| > |w| - a\}) \geq 1 - \frac{m_2(f_0)}{(|w| - a)^2} \geq 1 - \frac{m_2(f_0)}{2m_2(f_0)} = \frac{1}{2}.
\]

Next, we consider a bounded nonnegative globally Lipschitz-continuous function \( \phi : \mathbb{R}_+ \to [0,1] \), such that for all \( v > 0 \), \( 1_{B(0,a)}(v) = \phi(|v|) \geq 1_{B(0,2a)}(v) \), and define \( F(t, w) = \int_{\mathbb{R}^3} \phi(|w - v|) f_t(\, dv) \). We know
that \( t \to F(t, w) \) is continuous for each \( w \in \mathbb{R}^3 \) by Lemma 3.3 in [8]. Moreover, \( F(t, w) \) is (uniformly in \( t \)) continuous in \( w \) by the Lipschitz-continuity of \( \phi \). So \( F(t, w) \) is continuous on \([0, 1] \times \mathbb{R}^3\). Since for all \( t > 0 \), \( \text{Supp} f_t = \mathbb{R}^3 \) by Theorem 1.2 in [8], we get \( F(t, w) \geq f_t(B(w, 2a)^c) > 0, \forall (t, w) \in [0, 1] \times \overline{B}(0, M) \).

When \( t = 0 \), recalling that \( v_1, v_2 \in \text{Supp} f_0 \) and \( a = \frac{|v_1 - v_2|}{6} \), we easily see that for all \( w \in \mathbb{R}^3 \), either \( B(v_1, a) \subset B(w, 2a)^c \) or \( B(v_2, a) \subset B(w, 2a)^c \), whence \( F(0, w) \geq \min\{f_0(B(v_1, a)), f_0(B(v_2, a))\} > 0 \).

Since \([0, 1] \times \overline{B}(0, M)\) is compact and \( F(t, w) \) is continuous, there exists \( b_1 > 0 \), such that \( f_t(B(w, a)^c) \geq F(t, w) \geq b_1 \) for all \((t, w) \in [0, 1] \times \overline{B}(0, M)\).

So we conclude by choosing \( b = \min(\frac{1}{2}, b_1)/2 \).

**Step 2.** We now conclude. Using Step 1,

\[
\text{Q} \leq 2m(f_0) |a| c.
\]

So, we complete the proof by taking \( c = \sqrt{\frac{m(f_0)}{b}} \).

### 3.2 random fractal sets associated with the Poisson process

First, we introduce some notations. Recall that \( h_{V,a}, E_{V,a}, D_{V,a} \) respectively the Hölder exponent, iso-Hölder set and spectrum of singularities of the Boltzmann process \((V^B_t)_{t \in [0,1]}\). The notation \( \mathcal{L} \) represents the Lebesgue measure. \( \mathcal{J} \) designates the set of the jump times of the process \( V^B \), that is,

\[
\mathcal{J} := \{ s \in [0,1] : |\Delta V_s^B| \neq 0 \}.
\]

For \( m \geq 1 \), we also introduce

\[
\mathcal{J}_m := \{ s \in \mathcal{J} : |\Delta V_s^B| \leq 2^{-m} \}, \; \tilde{\mathcal{J}}_m := \{ s \in \mathcal{J} : 2^{-m-1} < |\Delta V_s^B| \leq 2^{-m} \}.
\]

For \( \delta > 0 \) and \( m \geq 1 \), we define the sets

\[
A_{\delta}^m := \bigcup_{s \in \mathcal{J}_m} [s - |\Delta V_s^B|^{\delta}, s + |\Delta V_s^B|^{\delta}], \; \tilde{A}_{\delta}^m := \bigcup_{s \in \tilde{\mathcal{J}}_m} [s - |\Delta V_s^B|^{\delta}, s + |\Delta V_s^B|^{\delta}].
\]

Finally, for \( \delta > 0 \), we define

\[
A_{\delta} = \limsup_{m \to +\infty} A_{\delta}^m = \limsup_{m \to +\infty} \tilde{A}_{\delta}^m.
\]

The main result of this subsection states that

**Proposition 3.2.** We have a.s. the following properties:

1. for all \( \delta \in (0, \nu) \), \( A_{\delta} \supset [0,1] \),
2. there exists a (random) positive sequence \( (\epsilon_m)_{m \geq 1} \) decreasing to 0, such that

\[
\mathcal{L} \left( A_{\nu} \bigcap [0,1] \right) = 1,
\]

where we use the notation \( A_{\epsilon_{m}} = \limsup_{m \to +\infty} \tilde{A}_{\delta(1-\epsilon_m)}^{m} \), for all \( \delta \in (0, \infty) \).

**Remark 3.3.** We observe at once that for any \( \delta > \delta' > 0 \), \( A_{\delta} \subset A_{\delta'} \subset A_{\nu} \).

The reason why we study \( A_{\delta} \) comes from the following heuristics: if \( t \in A_{\delta} \) with \( \delta \) large, then \( t \) is rather close to many large jump times of \( V^B \), so that \( V^B \) will not be very regular at \( t \). On the contrary, if \( t \) does only belong to those \( A_{\delta} \)'s with \( \delta \) small, this means that \( t \) is rather far away from the jumps of \( V^B \), so that \( V^B \) will be rather regular at \( t \).

We introduce \( A_{\epsilon_{m}} \) (which resembles very much \( A_{\delta} \)) for technical reasons, mainly because at the critical value \( \delta = \nu \), we cannot prove (and it may be false) that \( A_{\nu} \) has a full Lebesgue measure.

The rest of this subsection is devoted to proving this proposition. We first recall the Shepp lemma, first discovered in [17], in the version used in [13].
Lemma 3.4. We consider a Poisson measure \( \pi(ds, dy) = \sum_{s \in \mathcal{D}} \delta_{(s,y_s)} \) on \([0, 1] \times (0, 1)\) with intensity \( ds \mu(dy) \), where \( \mu \) is a measure on \((0, 1)\). We consider the set \( U = \bigcup_{s \in \mathcal{D}} (s - y_s, s + y_s) \). If
\[
\int_0^1 \exp \left( 2 \int_1^1 \mu((y,1)) dy \right) dt = +\infty,
\]
then almost surely, \([0, 1] \subset U\).

We write \( N = \sum_{s \in \mathcal{D}} \delta_{(s,v_s,\varphi_s,u_s)} \), where \( v_s, \varphi_s, u_s \) are the quanta corresponding to the jump time \( s \in \mathcal{D} \). For convenience, we consider this Poisson measure by adding a family of independent variables \((x_s)_{s \in \mathcal{D}}\), which are uniformly distributed in \([0, 1]\) and independent of \( v_s, \varphi_s, u_s \), so that \( O := \sum_{s \in \mathcal{D}} \delta_{(s,v_s,\varphi_s,u_s)} \) is a Poisson measure on \([0, 1] \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi] \times [0, \infty) \times [0, 1]\) with intensity \( ds f_s(\mathcal{H}_m) \beta(\theta) d\theta dy dx \). According to Lemma 3.1, we know that \( f_s(\mathcal{H}_m) \geq b \) for all \( s \in [0, 1] \) and all \( w \in \mathbb{R}^3 \). Then we can get the following lemma.

Lemma 3.5. For \( m \geq 1 \), we introduce
\[
J_m' := \left\{ s \in \mathcal{D} : u_s \leq d^\gamma, \, v_s \in \mathcal{H}_B(V_s-), \, \theta_s \leq K2^{-m}, \, x_s \leq \frac{b}{f_s(\mathcal{H}_B(V_s-))} \right\},
\]
where \( K = 1/(B + c) \) and where \( d = a \) (if \( \gamma \in (0,1) \)) or \( d = B + c \) (if \( \gamma \in (-1,0) \)). Then we have
\[
J_m' \subset J_m \quad \text{and} \quad \bigcup_{s \in J_m'} \left[ s - \left( \frac{a\theta_s}{4} \right)^\delta, s + \left( \frac{a\theta_s}{4} \right)^\delta \right] \subset A_m^\delta.
\]

Proof. We recall that, for all \( s \in [0, 1] \), \( |H_B(V_s-)| = \frac{|V_s-|}{|V_s-|} |V_s-| \leq B \) and that \( v_s \in \mathcal{H}_B(V_s-) \) implies that \( |H_B(V_s-) - v_s| \geq a \) and \( |v_s| \leq c \). Then for all \( m \geq 1 \), for all \( s \in J_m' \), we have (recall (3.3))
\[
|\Delta V_s^B| = \sqrt{\frac{1 - \cos \theta_s}{2}} |H_B(V_s-) - v_s| 1_{(w_s \leq |H_B(V_s-) - v_s|)} \leq \theta_s |H_B(V_s-) - v_s| \leq K2^{-m}(B + c) = 2^{-m}.
\]

In addition, for all \( s \in J_m' \), using that \( |H_B(V_s-) - v_s| \geq a \) and that \( 1 - \cos \theta \geq \theta^2/8 \) on \((0,\pi/2]\),
\[
|\Delta V_s^B| = \sqrt{\frac{1 - \cos \theta_s}{2}} |H_B(V_s-) - v_s| 1_{(w_s \leq |H_B(V_s-) - v_s|)} \geq \frac{a\theta_s}{4}.
\]

Indeed, the indicator equals 1 because we always have \( u_s \leq d^\gamma \leq |H_B(V_s-) - v_s| \) if \( \gamma \in (0,1) \), then \( |H_B(V_s-) - v_s| \geq a \) and \( d = a \), while if \( \gamma \in (-1,0) \), then \( |H_B(V_s-) - v_s| \leq B + c \) and \( d = B + c \). Consequently, for all \( m \geq 1 \), and all \( s \in J_m' \), \( 0 < |\Delta V_s^B| \leq 2^{-m} \): this implies that \( J_m' \subset J_m \). Furthermore, for any \( \delta > 0 \),
\[
A_m^\delta = \bigcup_{s \in J_m'} \left[ s - |\Delta V_s^B|^\delta, s + |\Delta V_s^B|^\delta \right] \supset \bigcup_{s \in J_m'} \left[ s - \left( \frac{a\theta_s}{4} \right)^\delta, s + \left( \frac{a\theta_s}{4} \right)^\delta \right]
\]
as desired.

Lemma 3.6. Let \( m \geq 1 \) and \( \delta > 0 \) be fixed. The random measure
\[
\mu_m^\delta(ds, dy) = \sum_{s \in J_m'} \delta_{(s,a\theta_s/4)^\delta}
\]
is a Poisson measure on \([0, 1] \times (0, \infty)\) with intensity \( ds h_m^\delta(y) dy \), where
\[
h_m^\delta(y) = \frac{8\pi d^\gamma b}{a^\delta} \beta \left( \frac{1}{a} y^{1/\delta} \right) y^{1/2 - 1} 1_{\{y \leq (aK2^{-m+2})^{1/\delta} (\pi/8)^{1/\delta} \}}.
\]
Moreover, we have

\[ c_1 y^{-1+\frac{1}{\nu}} \mathbf{1}_{\{y \leq (aK2^{-(m+2)})^\delta \wedge (a\pi/8)^\delta\}} \leq h_m^\delta(y) \leq C_1 y^{-1+\frac{1}{\nu}} \mathbf{1}_{\{y \leq (aK2^{-(m+2)})^\delta \wedge (a\pi/8)^\delta\}}, \]

for some constants \(0 < c_1 < C_1\) (depending on \(B, \delta\)).

**Proof.** By Jacod-Shiryaev [12] [Chapter 2, Theorem 1.8], it suffices to check that the compensator of the random measure \(\mu_m^\delta(ds, dy)\) is \(dsh_m^\delta(y)dy\), i.e., for any predictable process \(W(s, y)\),

\[
\int_0^t \int_0^\infty W(s, y)(\mu_m^\delta(ds, dy) - dsh_m^\delta(y)dy)
= \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \mathbf{1}_{\{v \in \mathcal{H}_B(v_{\nu-}), \ \theta \leq K2^{-m}, \ \nu \leq d^\nu, \ x \leq b/f_s(\mathcal{H}_B(v_{\nu-}))\}}
\times O(ds, dv, d\theta, d\varphi, du, dx) - \int_0^t \int_0^\infty W(s, y)h_m^\delta(y)dsdy
\]

is a local martingale. Recalling that \(O\) is a Poisson measure with intensity \(dsf_s(dv)\beta(\theta)d\theta d\varphi dudx\), we know that

\[
\int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \mathbf{1}_{\{v \in \mathcal{H}_B(v_{\nu-}), \ \theta \leq K2^{-m}, \ \nu \leq d^\nu, \ x \leq b/f_s(\mathcal{H}_B(v_{\nu-}))\}}
\times O(ds, dv, d\theta, d\varphi, du, dx) = - \int_0^t \int_0^\infty W(s, y)h_m^\delta(y)dsdy
\]

is a local martingale. Thus, we only need to prove that

\[
\int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \mathbf{1}_{\{v \in \mathcal{H}_B(v_{\nu-}), \ \theta \leq K2^{-m}, \ \nu \leq d^\nu, \ x \leq b/f_s(\mathcal{H}_B(v_{\nu-}))\}}
\times dsf_s(dv)\beta(\theta)d\theta d\varphi dudx
\]

\[= \int_0^t \int_0^\infty W(s, y)h_m^\delta(y)dsdy. \]

Actually,

\[
\int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \mathbf{1}_{\{v \in \mathcal{H}_B(v_{\nu-}), \ \theta \leq K2^{-m}, \ \nu \leq d^\nu, \ x \leq b/f_s(\mathcal{H}_B(v_{\nu-}))\}}
\times dsf_s(dv)\beta(\theta)d\theta d\varphi dudx
\]

\[= 2\pi d^\nu b \int_0^t \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty \int_0^1 W(s, (a\theta/4)^\delta) \mathbf{1}_{\{\theta \leq K2^{-m}\}} ds\beta(\theta)d\theta. \]

Using the substitution \(y = (a\theta/4)^\delta\), we conclude that the intensity of \(\mu_m^\delta\) is indeed \(dsh_m^\delta(y)dy\). From [13], we can easily get the bounds for \(h_m^\delta(y)\).

**Proof of Proposition 3.3** We start with (1) and thus fix \(\delta \in (0, \nu)\). By Lemma 3.6, we know that the random measure \(\mu_m^\delta = \sum_{s \in \mathcal{J}_m} \delta_s(a\theta/4)^\delta\) is a Poisson measure on \([0, 1] \times (0, 1)\) with intensity \(ds h_m^\delta(y)dy\), where

\[ h_m^\delta(y) \geq c_1 y^{-1+\frac{1}{\nu}} \mathbf{1}_{\{y \leq (aK2^{-(m+2)})^\delta \wedge (a\pi/8)^\delta\}}. \]

Clearly, for all \(m \geq 1, \delta \in (0, \nu)\),

\[
\int_0^1 \exp \left( 2 \int_0^1 \int_y^1 h_m^\delta(z)dzdy \right) dt = \infty,
\]
since $2 \int_{0}^{1} (\int_{0}^{1} h_{m_{i}}(z) dz) dy \gtrsim 2c_{1} \frac{\delta^{2}}{(\nu-\delta)^{2}} t^{1-\frac{3}{2}}$. Applying Lemma 3.4, we deduce that almost surely, for all $m \geq 1$, 
\[ [0, 1] \subset \bigcup_{s \in J_{m}} \left[ s - \left( \frac{a\theta_{s}}{4} \right)^{\delta}, s + \left( \frac{a\theta_{s}}{4} \right)^{\delta} \right]. \]
Consequently, almost surely, 
\[ [0, 1] \subset \limsup_{m \to +\infty} \bigcup_{s \in J_{m}} \left[ s - \left( \frac{a\theta_{s}}{4} \right)^{\delta}, s + \left( \frac{a\theta_{s}}{4} \right)^{\delta} \right]. \]
Recalling (3.2) and (3.3), we deduce that $[0, 1] \subset A_{\delta}$ almost surely.

We next prove (2). We set $m_{1} = 1$. By (1), we have a.s. $[0, 1] \subset A_{\nu}(1 - \frac{1}{2}) \subset \bigcup_{m \geq m_{1}} \tilde{A}_{m}^{\nu(1-1/2)}$. Hence we can find $m_{2} > m_{1}$ such that 
\[ \mathcal{L} \left( \bigcup_{m_{1} \leq m < m_{2}} \tilde{A}_{\nu(1-1/2)} \cap [0, 1] \right) \geq 1 - \frac{1}{2}. \]
Similarly, we have almost surely, $[0, 1] \subset A_{\nu(1-1/3)} \subset \bigcup_{m \geq m_{2}} \tilde{A}_{\nu(1-1/3)}^{m}$, therefore we can find $m_{3} > m_{2}$ such that 
\[ \mathcal{L} \left( \bigcup_{m_{2} \leq m < m_{3}} \tilde{A}_{\nu(1-1/3)}^{m} \cap [0, 1] \right) \geq 1 - \frac{1}{2^{2}}. \]
By induction, we can find an increasing sequence $(m_{j})_{j \geq 1}$ such that, for all $j \geq 2$,
\[ \mathcal{L} \left( \bigcup_{m_{j-1} \leq m < m_{j}} \tilde{A}_{\nu(1-1/j)}^{m} \cap [0, 1] \right) \geq 1 - \frac{1}{2^{j-1}}. \]
So, from the Fatou lemma, we have 
\[ \mathcal{L} \left( \limsup_{j \to +\infty} \bigcup_{m_{j-1} \leq m < m_{j}} \tilde{A}_{\nu(1-1/j)}^{m} \cap [0, 1] \right) \geq \limsup_{j \to +\infty} \mathcal{L} \left( \bigcup_{m_{j-1} \leq m < m_{j}} \tilde{A}_{\nu(1-1/j)}^{m} \cap [0, 1] \right) \geq 1. \]
We now put $\epsilon_{m} = \frac{1}{j}$ for $m \in [m_{j-1}, m_{j})$ and note that 
\[ \limsup_{j \to +\infty} \bigcup_{m \in [m_{j-1}, m_{j})} \tilde{A}_{\nu(1-\epsilon_{m})}^{m} = \limsup_{m \to +\infty} \tilde{A}_{\nu(1-\epsilon_{m})}^{m}. \]
The conclusion follows.

### 3.3 Study of the Hölder exponent of $V^{B}$

We now study the pointwise Hölder exponent of the localized Boltzmann process $V^{B}$.

**Definition 3.7.** For all $t \in [0, 1]$, the index of approximation of $t$ is defined by 
\[ \delta_{t} := \sup \{ \delta > 0 : t \in A_{\delta} \}. \]

For all $t \in [0, 1]$, the index of approximation of $t$ reflects directly the relation between $t$ and jump times of $V^{B}$. If $\delta_{t}$ is large, then $t$ is close to many large jumps of $V^{B}$.
Proposition 3.9. The inverse of the index of approximation.

Remark 3.8. Recalling Remark 3.3 and Proposition 3.2, we see that almost surely, for all \( t \in [0,1] \),

\[ \delta_t = \sup \{ \delta > 0 : t \in A^*_\delta \} \text{ and } \delta_t \geq \nu. \]

If \( t \in \mathcal{J} \), we know that \( h_{V_B}(t) = 0 \). Then for \( t \in [0,1] \setminus \mathcal{J} \), we claim that the Hölder exponent is the inverse of the index of approximation.

Proposition 3.9. Almost surely, for all \( t \in [0,1] \setminus \mathcal{J} \), \( h_{V_B}(t) = \frac{1}{\delta_t} \).

To prove this claim, we need the following two lemmas. The first lemma, that will give the upper bound for \( h_{V_B}(t) \), can be found in \([13]\) and is as follows.

Lemma 3.10. Let \( f : \mathbb{R} \to \mathbb{R}^3 \) be a function discontinuous on a dense set of points and let \( (t_n)_{n \geq 1} \) be a real sequence converging to some \( t \) and such that \( f \) has left and right limits at each \( t_n \). Then

\[ h_f(t) \leq \liminf_{n \to \infty} \frac{\log |f(t_n^+) - f(t_n^-)|}{\log |t_n - t|}. \]

For the lower bound of \( h_{V_B}(t) \), we will use Lemma 3.11 below, that relies on some ideas of \([3]\). We first introduce, for \( m > 0 \), the following two processes:

\[
\begin{align*}
V^B_{t,m} &:= V_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^\infty \int_0^\infty a(H_B(V_{s-}), v, \theta, \varphi) \, 1_{\{u \leq |H_B(V_{s-})| \}} \\
& \quad \times 1_{\{|a(H_B(V_{s-}), v, \theta, \varphi)| \leq 2^{-m}\}} N(ds, dv, d\theta, d\varphi, du), \\
Z^B_{t,m} &:= \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^\infty \int_0^\infty \theta|H_B(V_{s-}) - v| 1_{\{u \leq |H_B(V_{s-})| \}} \\
& \quad \times 1_{\{|\theta| \leq 2^{-m}\}} N(ds, dv, d\theta, d\varphi, du).
\end{align*}
\]

We can immediately observe that the process \( Z^B_{t,m} \) is almost surely increasing as a function of \( t \). We also notice that a.s., for all \( x, y \in [0,1] \),

\[
|V^B_{x,m} - V^B_{y,m}| \leq |Z^B_{x,m} - Z^B_{y,m}|. \tag{3.4}
\]

This comes from the inequality \( \theta |H_B(V_{s-}) - v| \leq |a(H_B(V_{s-}), v, \theta, \varphi)| \leq \theta |H_B(V_{s-}) - v| \), which follows from \([17]\).

Lemma 3.11. There exists some constant \( C_B > 0 \), such that

1. for all \( \delta > \nu \), all \( m \geq 1 \),

\[
\mathbb{P} \left[ \sup_{x,y \in [0,1], |x - y| \leq 2^{-m}} \left| V^B_{x,m} - V^B_{y,m} \right| \geq m2^{-\frac{m}{2}} \right] \leq C_B e^{-m/4}, \tag{3.5}
\]

2. for all \( m \geq 1 \), all \( \lambda \in [0, 2^m] \),

\[
\mathbb{E} \left[ e^{\lambda Z^B_{1,m}} \right] \leq e^{C_B \lambda 2^{-m(1-\nu)}}. \tag{3.6}
\]

Proof. We first prove (3.5). Setting \( \lambda = 3 \times 2^{m/\delta} \), recalling (3.3), and that \( Z^B_{t,m} \) is almost surely increasing
in \( t \), we get
\[
P \left[ \sup_{x,y \in [0,1],|x-y| \leq 2^{-m}} \left| V_x^{B,\frac{m}{2}} - V_y^{B,\frac{m}{2}} \right| \geq m2^{-\frac{m}{2}} \right] \leq P \left[ \sup_{x,y \in [0,1],|x-y| \leq 2^{-m}} \left| Z_x^{B,\frac{m}{2}} - Z_y^{B,\frac{m}{2}} \right| \geq m2^{-\frac{m}{2}} \right] 
\]
\[
\leq \sum_{k=0}^{2^m-1} \left( Z_{(k+1)2^{-m}}^{B,\frac{m}{2}} - Z_{k2^{-m}}^{B,\frac{m}{2}} \right) \geq \frac{m2^{-\frac{m}{2}}}{3} \right] 
\]
\[
\leq \sum_{k=0}^{2^m-1} e^{-m} \mathbb{E} \left[ \exp \left\{ \lambda \left( Z_{(k+1)2^{-m}}^{B,\frac{m}{2}} - Z_{k2^{-m}}^{B,\frac{m}{2}} \right) \right\} \right] 
\]
\[
= \sum_{k=0}^{2^m-1} e^{-m} I_k. 
\]
We then set
\[
J_k(t) \equiv \mathbb{E} \left[ \exp \left\{ \lambda \left( Z_{t+k2^{-m}}^{B,\frac{m}{2}} - Z_{k2^{-m}}^{B,\frac{m}{2}} \right) \right\} \right]. 
\]
Observe that \( I_k = J_k(2^{-m}). \) For all \( t \geq 0 \), we have
\[
J_k(t) = 1 + 2\pi \mathbb{E} \left[ \int_{k2^{-m}}^{t+k2^{-m}} \int_{\mathbb{R}^3} \left[ \int_0^{\pi/2} \exp \left\{ \lambda \left( Z_{s}^{B,\frac{m}{2}} - Z_{k2^{-m}}^{B,\frac{m}{2}} \right) \right\} (e^{\lambda |H_B(V_s)| - v|} - 1) \right. 
\]
\[
\times |H_B(V_s) - v|^{\gamma+1} \left\{ 1 \leq |H_B(V_s) - v| \leq 2^{2^{-m}} \right\} \beta(\theta) d\theta f_s(dv) \right] ds. 
\]
From \( \lambda \theta |H_B(V_s) - v| \leq 4\lambda 2^{-m/4} = 12 \), we have \( e^{\lambda |H_B(V_s)| - v|} - 1 \leq C\lambda \theta |H_B(V_s) - v| \) for some positive constant \( C \). Using this estimate and recalling \( \|A\| \), we get
\[
J_k(t) \leq 1 + C\mathbb{E} \left[ \int_{k2^{-m}}^{t+k2^{-m}} \int_{\mathbb{R}^3} \left[ \int_0^{\pi/2} \exp \left\{ \lambda \left( Z_{s}^{B,\frac{m}{2}} - Z_{k2^{-m}}^{B,\frac{m}{2}} \right) \right\} \right. 
\]
\[
\times |H_B(V_s) - v|^{\gamma+1} \left\{ 1 \leq |H_B(V_s) - v| \leq 2^{2^{-m}} \right\} \beta(\theta) d\theta f_s(dv) \right] ds. 
\]
Moreover,
\[
|H_B(V_s) - v|^{\gamma+1} \int_0^{\pi/2} \theta^{-\nu} \left\{ 1 \leq |H_B(V_s) - v| \leq 2^{2^{-m}} \right\} d\theta \leq C|H_B(V_s) - v|^{\gamma+1} (|H_B(V_s) - v|2^{2^{-m}})^{-\nu-1} \]
\[
\leq C |H_B(V_s) - v|^{\gamma+\nu} 2^{-m(\gamma+\nu-1)}. 
\]
Since \( \gamma + \nu \in (0,2) \) by assumption, we have \( |H_B(V_s) - v|^{\gamma+\nu} \leq C(1 + |v|^2 + |H_B(V_s)|^2) \), whence
\[
J_k(t) \leq 1 + C\lambda2^{\frac{m(\nu-1)}{4}} \mathbb{E} \left[ \int_{k2^{-m}}^{t+k2^{-m}} \int_{\mathbb{R}^3} \exp \left\{ \lambda \left( Z_{s}^{B,\frac{m}{2}} - Z_{k2^{-m}}^{B,\frac{m}{2}} \right) \right\} \right. 
\]
\[
\left. (1 + |H_B(V_s)|^2 + |v|^2) f_s(dv) \right] ds. 
\]
Since \( |H_B(V_s)| \leq B \), and by conservation of the kinetic energy, we have a.s.
\[
\int_{\mathbb{R}^3} (1 + |H_B(V_s)|^2 + |v|^2) f_s(dv) \leq 1 + B^2 + m_2(f_0). 
\]
Using finally that \( \lambda 2^{\frac{m(\nu-1)}{4}} = 3 \times 2^{\frac{m}{2}} \), we find that
\[
J_k(t) \leq 1 + C_B 2^{\frac{m}{2}} \mathbb{E} \left[ \int_{k2^{-m}}^{t+k2^{-m}} \exp \left\{ \lambda \left( Z_{s}^{B,\frac{m}{2}} - Z_{k2^{-m}}^{B,\frac{m}{2}} \right) \right\} ds \right]. 
\]
Proof of Proposition 3.9. Upper Bound. It follows that \( J_k(t) \leq 1 + C_B2^{-m} \int_0^t J_k(s)ds \). Hence \( J_k(t) \leq \exp(C_B2^{-m}t) \) by the Grönwall inequality, so that \( I_k = J_k(2^{-m}) \leq \exp(C_B2^{-m(1-\frac{1}{2})}) \leq C_B \) because \( \delta \geq \nu \). Finally,

\[
P \left[ \sup_{x,y \in [0,1], \|x-y\| \leq 2^{-m}} \left| V_x^B - V_y^B \right| \geq m2^{-\frac{m}{2}} \right] \leq \sum_{k=0}^{2^m-1} e^{-m} I_k \leq C_B e^{-m/2} \leq C_B \ e^{-m/4}.
\]

This completes the proof of (3.5). We only sketch the proof of (3.6), since it is very similar. First, by Itô Formula,

\[
E \left[ e^{\lambda Z_{t,m}^B} \right] = 1 + 2\pi E \left[ \int_0^t \int_{\mathbb{R}^2} e^{\lambda Z_{t,m}^B} \left( e^{\lambda B[H_B(V_s)-v]} - 1 \right) |H_B(V_s) - v|^r \chi_{\{H_B(V_s) - v \leq 2^{-m}\}} d\theta f_s(dv)ds \right].
\]

Since \( \lambda \theta |H_B(V_s) - v| < 4 \) (because \( \lambda \leq 2^m \)), a similar computation as previously shows that

\[
E \left[ e^{\lambda Z_{t,m}^B} \right] \leq 1 + C_B \lambda 2^m(\nu - 1) E \left[ \int_0^t e^{\lambda Z_{t,m}^B} ds \right] \leq 1 + C_B \lambda 2^m(\nu - 1) \int_0^t E[e^{\lambda Z_{t,m}^B}] ds.
\]

Owing to the Grönwall inequality, we deduce that \( E[e^{\lambda Z_{t,m}^B}] \leq e^{C_B \lambda 2^m(\nu - 1)} \). Taking \( t = 1 \), we obtain the conclusion. \( \square \)

Now, we can proceed to the

Proof of Proposition 3.9. Upper Bound. Here we prove that for all \( t \in [0,1] \), it holds that \( h_{V,V}(t) \leq 1/\delta_t \). To this end, we check that for all \( \delta > 0 \), all \( t \in A_\delta, h_{V,V}(t) \leq 1/\delta \). Let thus \( \delta > 0 \) and \( t \in A_\delta \). By definition of \( A_\delta \), for all \( m \geq 1 \), there exists \( t_m \in \mathcal{J}, \) such that \( |t_m - t| \leq |\Delta V_{t_m}^B|^{\delta} \) and \( |\Delta V_{t_m}^B| \leq 2^{-m} \).

From Lemma 3.10 we directly deduce that

\[
h_{V,V}(t) \leq \liminf_{m \to \infty} \frac{\log |\Delta V_{t_m}^B|}{\log |t_m - t|} \leq \liminf_{m \to \infty} \frac{\log |\Delta V_{t_m}^B|}{\log |\Delta V_{t_m}^B|^{\delta}} = \frac{1}{\delta}.
\]

Lower Bound. In this part we show that almost surely, for all \( t \in [0,1] \setminus \mathcal{J}, h_{V,V}(t) \geq 1/\delta_t \). To get this, we need to check that for all \( \delta > \nu \), if \( t \notin A_\delta \), then \( h_{V,V}(t) \geq 1/\delta \). Let thus \( \delta > \nu \) and \( t \notin A_\delta \).

By Lemma 3.11(1) and Borel-Cantelli’s lemma, there almost surely exists \( m_0 \geq 1 \) such that for all \( m > m_0 \), for all \( x, y \in [0,1] \) satisfying \( |x - y| \leq 2^{-m} \),

\[
|V_x^B - V_y^B| \leq m2^{-\frac{m}{2}}.
\]

Since \( t \notin A_\delta \), there exists \( m_1 > m_0 \), such that for all \( s \in \mathcal{J} \) satisfying \( |\Delta V_{s}^B| \leq 2^{-m_1} \), we have

\[
|s - t| > |\Delta V_{s}^B|^{\delta}.
\]

For all \( r \in [0,1] \), we define

\[
U_{t,r}^{m_1} := \sum_{s \in [t \wedge r, t \vee r] \cap \mathcal{J}} |\Delta V_{s}^B| 1_{\{|\Delta V_{s}^B| > 2^{-m_1}\}},
\]

and we observe that

\[
|V_t^B - V_r^B| \leq |V_t^{B,m_1} - V_r^{B,m_1}| + U_{t,r}^{m_1}.
\]

Since \( t \notin \mathcal{J} \) and since the process \( V^B \) has almost surely a finite number of jump greater than \( 2^{-m_1} \), we can almost surely find \( \epsilon_1 > 0 \) such that, for all \( r \in (t - \epsilon_1, t + \epsilon_1) \), \( U_{t,r}^{m_1} = 0 \).
Next, we put $\epsilon_2 = 2^{-m_r-1}$. Then for each $r \in (t-\epsilon_2, t + \epsilon_2)$, we set $m_r = \lfloor \log_2 \frac{1}{|t-r|} \rfloor > m_1$, for which $2^{-m_r-1} < |t-r| \leq 2^{-m_r}$. Then for all $r \in (t-\epsilon_2, t + \epsilon_2)$, we write

$$|V_t^{B,m_1} - V_r^{B,m_1}| \leq |V_t^{B,m_r/\delta} - V_r^{B,m_r/\delta}| + \sum_{s \in [t \wedge r, t \vee r] \cap J} |\Delta V_s^B| 1_{\{2^{-\frac{m_r}{\delta}} < |\Delta V_s^B| \leq 2^{-m_1}\}}.$$

According to (3.8), for $s \in [t \wedge r, t \vee r] \cap J$, $|\Delta V_s^B| \leq 2^{-m_1}$ implies that $|\Delta V_s^B| < |s-t|^{1/\delta} \leq |r-t|^{1/\delta} \leq 2^{-\frac{m_r}{2\delta}}$, whence the second term $\sum_{s \in [t \wedge r, t \vee r] \cap J} |\Delta V_s^B| 1_{\{2^{-\frac{m_r}{\delta}} < |\Delta V_s^B| \leq 2^{-m_1}\}}$ vanishes.

To summarize, we have checked that for all $r \in \left(t - (\epsilon_1 \wedge \epsilon_2), t + (\epsilon_1 \wedge \epsilon_2)\right)$,

$$|V_t^B - V_r^B| \leq |V_t^{B,m_r/\delta} - V_r^{B,m_r/\delta}|.$$

Furthermore, since $m_r > m_0$, we conclude from (3.7) that, still for $r \in \left(t - (\epsilon_1 \wedge \epsilon_2), t + (\epsilon_1 \wedge \epsilon_2)\right)$,

$$|V_t^B - V_r^B| \leq m_r 2^{-m_r} \leq \frac{2^{1/\delta}}{\log 2} \log \left(\frac{1}{|t-r|}\right) |t-r|^{1/\delta}.$$

This implies that $h_{\nu}{\sigma}(t) \geq \frac{1}{\delta}$ and ends the proof.

3.4 Hausdorff dimension of the sets $A_0^\alpha$

Now, we compute the Hausdorff dimension of $A_0^\alpha$, which will be used for giving the spectrum of singularities and the proof of Proposition 2.1 in the next subsection.

Proposition 3.12. Almost surely, for all $\delta > \nu$,

$$\dim_H(A_0^\alpha) = \frac{\nu}{\delta} \text{ and } H^{\nu/\delta}(A_0^\alpha) = +\infty.$$

To check this proposition, we need the mass transference principle, proved in [5], Theorem 2 (applied in dimension $k = 1$ and with the function $f(x) = x^\alpha$).

Proposition 3.13. Let $\alpha \in (0,1)$ be fixed. Let $\{F_i = [x_i - r_i, x_i + r_i]\}_{i \in \mathbb{N}}$ be a sequence of intervals in $\mathbb{R}$ with radius $r_i \to 0$ as $i \to +\infty$. Suppose that

$$\mathcal{L}\left(\limsup_{i \to +\infty} F_i^\alpha \cap [0,1]\right) = 1,$$

where $F_i^\alpha := [x_i - r_i^\alpha, x_i + r_i^\alpha]$. Then,

$$H^\alpha\left(\limsup_{i \to +\infty} F_i \cap [0,1]\right) = H^\alpha([0,1]) = +\infty.$$

Proof of Proposition 3.12. Lower Bound. We fix $\delta > \nu$. For all $m \geq 1$, we set

$$N_m := \sharp \mathcal{J}_m = \sharp \{s \in \mathcal{J} : 2^{-m-1} < |\Delta V_s^B| \leq 2^{-m}\}.$$

We can write $\mathcal{J}_m = \{T_1^m, \ldots, T_N^m\}$, ordered chronologically. Then we define a sequence $(F_{\delta,j})_{j \geq 1}$ of intervals as follows. For $j \geq 1$, there is a unique $m \geq 1$ and $i \in \{1, 2, \ldots, N_m\}$ such that $j = \sum_{k=0}^{m-1} N_k + i$ and write

$$F_{\delta,j} := \left[T_i^m - |\Delta V_i^B|^\delta(1-\epsilon_m), T_i^m + |\Delta V_i^B|^\delta(1-\epsilon_m)\right],$$

where $\epsilon_m := \frac{\nu}{\delta} - \epsilon$. Then for all $\delta > \nu$, we have

$$|V_t^B - V_r^B| \leq \frac{2^{1/\delta}}{\log 2} \log \left(\frac{1}{|t-r|}\right) |t-r|^{1/\delta}.$$
where $\epsilon_m$ is defined in Proposition 3.2. By this way, we get a sequence of intervals $(F_{\delta,j})_{j \geq 1}$ of radius tending to 0 and such that, for all $\alpha > 0$, $\limsup_{j \to +\infty} F_{\delta,j}^\alpha = A^*_\alpha \delta$ (this is obvious by definition of $A^*_\delta$, see Remark 3.3). Particularly, taking $\alpha = \frac{\nu}{\delta} \in (0, 1)$, we get

$$\limsup_{j \to +\infty} F_{\nu/\delta,j}^\alpha = A^*_\nu \delta.$$ 

Thus by Proposition 3.2-(2),

$$\mathcal{L} \left( \limsup_{j \to +\infty} F_{\delta,j}^{\nu/\delta} \cap [0, 1] \right) = 1.$$ 

Consequently, by Proposition 3.13, we have

$$H^{\nu/\delta} \left( \limsup_{j \to +\infty} F_{\delta,j}^{\nu/\delta} \cap [0, 1] \right) = +\infty,$$

that is,

$$H^{\nu/\delta} \left( A^*_\nu \cap [0, 1] \right) = +\infty.$$ 

Then $H^{\nu/\delta}(A^*_\nu) = +\infty$ and $\dim_H(A^*_\nu) \geq \frac{\nu}{\delta}$.

Observing that the family of intervals $F_{\nu/\delta,j}^{\nu/\delta}$ does not depend on $\delta$, we can clearly apply Proposition 3.13 simultaneously for all $\delta > \nu$ and we conclude that a.s., for all $\delta > \nu$, $H^{\nu/\delta}(A^*_\nu) = +\infty$ and $\dim_H(A^*_\nu) \geq \frac{\nu}{\delta}$.

**Upper Bound.** Let $\delta > \nu$ be fixed. To get the upper bound for $\dim_H(A^*_\nu)$, we show first that a.s.,

$$H^{\nu/\delta}(A^*_\delta) = +\infty \text{ and } \dim_H(A^*_\delta) \leq \frac{\nu}{\delta}.$$ 

This estimate is obtained by using (3.4). Then

$$P[N_m \geq m^2 m^{\nu}] \leq P[Z^B,m \geq \frac{1}{2} m^2 m^{(\nu-1)}].$$

Setting $\lambda = 2^{m(1-\nu)}$, we get

$$P[Z^B,m \geq \frac{1}{2} m^2 m^{(\nu-1)}] = P[\lambda Z^B,m \geq m/2] \leq e^{-\frac{m}{2}} E[e^{\lambda Z^B,m}].$$

Since $\lambda = 2^{m(1-\nu)} \leq 2^m$, we infer from Lemma 3.11-(2) that

$$E[e^{\lambda Z^B,m}] \leq C_B.$$ 

Hence we obtain

$$P[N_m \geq m^{2 m^\nu}] \leq C_B e^{-m/2}.$$ 

According to the Borel-Cantelli lemma, we know that, almost surely there exists $M > 0$ such that, for all $m > M$, $N_m < m^{2 m^\nu}$.

Next, by definition of $A^*_\delta$,

$$\bigcup_{k \geq m} \tilde{A}^*_k \subset \bigcup_{k \geq m} \bigcup_{s \in \tilde{J}_k} [s - 2^{-k \delta}, s + 2^{-k \delta}],$$

so, recalling Definition 1.5, for all $\alpha > 0$, and all $m > M$, a.s.,

$$H^{\nu/\delta,\alpha} \left( \bigcup_{k \geq m} \tilde{A}^*_k \right) \leq 2^\alpha \sum_{k \geq m} N_k 2^{-k \delta \alpha} \leq 2^\alpha \sum_{k \geq m} k 2^{k(\nu - \delta \alpha)}.$$
But recalling (3.2), \( A_\delta \subset \bigcup_{k \geq m} \tilde{A}_k \), whence, for all \( \alpha > 0 \), and all \( m > M \), a.s.,

\[
\mathcal{H}^{\alpha}_{2^{-m} \delta + 1}(A_\delta) \leq 2^\alpha \sum_{k \geq m} k 2^{k(\nu - \delta \alpha)}.
\]

Consequently,

\[
\mathcal{H}^\alpha(A_\delta) = \lim_{m \to +\infty} \mathcal{H}^{\alpha}_{2^{-m} \delta + 1}(A_\delta) \leq 2^\alpha \lim_{m \to +\infty} \sum_{k \geq m} k 2^{k(\nu - \delta \alpha)}.
\]

It follows that \( \mathcal{H}^\alpha(A_\delta) = 0 \) for all \( \alpha > \nu / \delta \). Thus, \( \dim_H(A_\delta) \leq \nu / \delta \) by Definition 1.5. Since \( A^*_\delta \subset A'_\delta \) for any \( \delta' \in (0, \delta) \), we easily conclude that, a.s.,

\[
\dim_H(A^*_\delta) \leq \nu / \delta.
\]

**3.5 Spectrum of singularity of \( V^B \)**

Using Proposition 3.9, we can easily get the following relationship between \( E_{V^B}(h) \) and \( A^*_\delta \).

**Proposition 3.14.** Almost surely, for all \( h > 0 \),

\[
E_{V^B}(h) = \left( \bigcap_{\delta \in (0, 1/h)} A^*_\delta \right) \setminus \left( \bigcup_{\delta > 1/h} A^*_\delta \right).
\]

and

\[
E_{V^B}(0) = \left( \bigcap_{\delta \in (0, \infty)} A^*_\delta \right).
\]

**Remark 3.15.** Due to Remark 3.3, Proposition 3.14 also holds when replacing everywhere \( A^*_\delta \) by \( A_\delta \).

We now can finally give the

**Proof of Proposition 3.12.** We first deal with the case where \( h \in (0, 1/\nu] \). By Propositions 3.10 and 3.11,

\[
D_{V^B}(h) = \dim_H \left( E_{V^B}(h) \right) \leq \dim_H \left( \bigcap_{\delta \in (0, 1/h)} A^*_\delta \right) \leq \inf_{\delta \in (0, 1/h)} \dim_H(A^*_\delta) = h\nu.
\]

On the other hand, we observe that (recall that \( \delta \mapsto A^*_\delta \) is decreasing)

\[
D_{V^B}(h) = \dim_H \left( E_{V^B}(h) \right) \geq \dim_H \left( A^*_{1/h} \setminus \bigcup_{\delta > 1/h} A^*_\delta \right).
\]

But

\[
\mathcal{H}^{h\nu} \left( A^*_{1/h} \setminus \bigcup_{\delta > 1/h} A^*_\delta \right) = \mathcal{H}^{h\nu}(A^*_{1/h}) - \mathcal{H}^{h\nu} \left( \bigcup_{\delta > 1/h} A^*_\delta \right).
\]

For all \( \delta > 1/h \), \( \dim_H(A^*_\delta) = \nu / \delta < h\nu \), thus \( \mathcal{H}^{h\nu}(A^*_\delta) = 0 \). Moreover, recalling that \( A^*_\delta \) is decreasing when \( \delta > \nu \), hence

\[
\mathcal{H}^{h\nu} \left( \bigcup_{\delta > 1/h} A^*_\delta \right) = 0.
\]

Next, Proposition 3.12 (if \( h\nu < 1 \)) and Proposition 3.2 (if \( h\nu = 1 \)) imply that

\[
\mathcal{H}^{h\nu}(A^*_{1/h}) > 0.
\]
Consequently, \( \dim_H \left( A^*_h \setminus \left( \bigcup_{s > 1/h} A^*_s \right) \right) \geq h \nu \), whence finally, \( D_{V^h}(h) \geq h \nu \). We have checked that for \( h \in (0, 1/e) \), it holds that \( D_{V^h}(h) = h \nu \).

When \( h = 0 \), we immediately get, using Proposition 3.12 that
\[
\dim_H \left( E_{V^h}(0) \right) = \dim_H \left( \bigcap_{\delta \in (0, \infty)} A^*_\delta \right) \leq \inf_{\delta \in (0, \infty)} \frac{\nu}{\delta} = 0.
\]
Since furthermore \( E_{V^h}(0) \supset J \) is a.s. not empty, we conclude that \( \dim_H \left( E_{V^h}(0) \right) = 0 \).

Finally, when \( h > \frac{1}{e} \), we want to show that \( \dim_H \left( E_{V^h}(h) \right) = -\infty \), i.e. that \( E_{V^h}(h) = 0 \). This claim immediately follows from Remark 3.8 and Proposition 3.9, since for all \( t \in [0, 1] \setminus J \), \( h_{V^h}(t) = \frac{1}{e} \leq \frac{1}{e} \).

\[
\square
\]

4 Study of the position process

The goal of this last section is to prove Proposition 2.2. We thus only consider the case of hard potentials \( \gamma \in (0, 1) \). Since \( X^B_t = \int_0^t V_s^B \, ds \), we obviously have a.s., for all \( t \in [0, 1] \),
\[
h_{X^B}(t) = 1 + h_{V^h}(t).
\]
Recall that by Definition, \( t \in E^{\text{cusp}}_{V^h}(h) \) if \( h_{X^B}(t) > 1 + h_{V^h}(t) \) and \( t \in E^{\text{cusp}}_{V^h}(h) \) if \( h_{X^B}(t) = 1 + h_{V^h}(t) \). Inspired by the ideas of Balançá [2], we will prove several technical lemmas to get Proposition 2.2.

4.1 Preliminaries

For any \( m > 0 \) and any interval \([r, t] \subset [0, 1] \), we set
\[
H^m_{[r, t]} := \{ s \in [r, t] \cap J : |\Delta V^B_s| \geq 2^{-m} \}.
\]  

Lemma 4.1. For any \( m \geq 1 \) and any interval \([r, t] \subset [0, 1] \),

1. we have
\[
H^m_{[r, t]} \subseteq R^m_{[r, t]},
\]
where
\[
R^m_{[r, t]} = \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} 1_{\{ (B+|v|) \geq 2^{-m} \}} 1_{\{ u \geq (B+|v|)^\gamma \}} N(ds, dv, d\theta, d\varphi, du);
\]
and, with \( a > 0 \) introduced in Lemma 3.1 (this actually holds true for any value of \( a > 0 \)),
\[
H^m_{[r, t]} \supseteq S^m_{[r, t]},
\]
where
\[
S^m_{[r, t]} = \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} 1_{\{ |v - H_B(V^B, v, \theta, \varphi)| \geq a \}} 1_{\{ \theta \geq 2^{-m} a^\gamma \}} N(ds, dv, d\theta, d\varphi, du).
\]

Proof. By definition of \( V^B \), see (2.1), we have
\[
H^m_{[r, t]} = \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} 1_{\{ |\alpha(H_B(V^B, v, \theta, \varphi))| \geq 2^{-m} \}} 1_{\{ u \leq |H_B(V^B, v, \theta, \varphi)| \}} N(ds, dv, d\theta, d\varphi, du).
\]
Then the claims immediately follow from \( \frac{4}{B} |H_B(V) - v| \leq |\alpha(H_B(V, v, \theta, \varphi))| \leq \theta(B + |v|) \), see (1.7), and \( |H_B(V) - v|^\gamma \leq (B + |v|)^\gamma \).

\[
\square
\]

Remark 4.2. Glancing at their definitions, it is clear that \( S^m_{[r, t]} \) and \( R^m_{[r, t]} \) are \( \mathcal{F} \)-measurable, that \( R^m_{[r, t]} \) is independent of \( \mathcal{F} \) and is a Poisson variable with parameter \( \lambda^m_{[r, t]} \), where
\[
\lambda^m_{[r, t]} = \int_r^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{\infty} 1_{\{ (B+|v|) \geq 2^{-m} \}} 1_{\{ u \leq (B+|v|)^\gamma \}} dsf_s(du)\beta(\theta)d\theta d\varphi du.
\]
Using (1.8) and that \( m_2(f_s) = m_2(f_0) \) for all \( s \in [0, 1] \), one easily checks that there exists a constant \( C_B > 0 \) such that \( \lambda^m_{[r, t]} \leq C_B 2^{m\nu} |t - r| \) for all \( m > 0 \) and all \( 0 \leq t \leq 1 \).
4.2 Refined study of the jumps

The goal of this part is to prove the following crucial fact.

**Lemma 4.3.** Fix \( \epsilon > 0 \) and set \( \alpha = \nu(1 - 2\epsilon) \) and \( \beta = \nu(1 + 4\epsilon) \). Almost surely, there exists \( M \geq 1 \), such that for all \( m \geq M \), for all \( t \in [0, 1] \), there exists \( t_m \in B(t, 2^{-m\beta}) \) such that \( |\Delta V_{t_m}^B| \geq 2^{-m} \) and there is no other jump of size greater than \( 2^{-m(1+\epsilon)} \) in \( B(t_m, 2^{-m\beta}/3) \).

We start with an intermediate result.

**Lemma 4.4.** Fix \( \epsilon > 0 \), \( \alpha = \nu(1 - 2\epsilon) \) and \( \beta = \nu(1 + 4\epsilon) \). For any interval \( I = [t_0, t_1] \subset [0, 1] \) with length \( 2^{-m\beta} \), divide \( I \) into three consecutive intervals with length \( 2^{-m\beta}/3 \). Consider the event

\[
A_{m, \epsilon}^I = \{ H_{t_0, t_1}^{m(1+\epsilon)} = 0 \} \cap \{ H_{t_1, t_2}^{m(1+\epsilon)} = H_{t_2, t_3}^{m(1+\epsilon)} = 1 \} \cap \{ H_{t_3}^{m(1+\epsilon)} = 0 \}.
\]

There exist some constants \( c_B > 0 \) and \( m_\epsilon > 0 \) such that, for all \( m \geq m_\epsilon \), all intervals \( I \subset [0, 1] \) with length \( 2^{-m\beta} \),

\[
\mathbb{P}[A_{m, \epsilon}^I | \mathcal{F}_0] \geq c_B 2^{-4m\epsilon c}. \tag{4.4}
\]

**Proof.** We introduce \( A_1 = \{ H_{t_0, t_1}^{m(1+\epsilon)} = 0 \} \), \( A_2 = \{ H_{t_1, t_2}^{m(1+\epsilon)} = H_{t_2, t_3}^{m(1+\epsilon)} = 1 \} \) and \( A_3 = \{ H_{t_3}^{m(1+\epsilon)} = 0 \} \), so that \( A_{m, \epsilon}^I = A_1 \cap A_2 \cap A_3 \).

**Step 1.** First we write, since \( A_1 \cap A_2 \in \mathcal{F}_2 \),

\[
\mathbb{P}[A_{m, \epsilon}^I | \mathcal{F}_0] = \mathbb{E} \left[ 1_{A_1 \cap A_2} \mathbb{P}[A_3 | \mathcal{F}_2] | \mathcal{F}_0 \right].
\]

But using Lemma 4.1 and Remark 4.2,

\[
\mathbb{P}[A_3 | \mathcal{F}_2] = \mathbb{P}[H_{t_2, t_3}^{m(1+\epsilon)} = 0 | \mathcal{F}_2] \geq \mathbb{P}[H_{t_2, t_3}^{m(1+\epsilon)} = 0 | \mathcal{F}_2] = \exp(-\lambda_{t_2, t_3}^{m(1+\epsilon)}) \geq \frac{1}{2}
\]

for all \( m \) large enough (depending only on \( \epsilon \)), since \( \lambda_{t_2, t_3}^{m(1+\epsilon)} \leq C_B 2^{m\nu(1+4\epsilon)} \). Consequently, for all \( m \) large enough (depending only on \( \epsilon \)), we a.s. have

\[
\mathbb{P}[A_{m, \epsilon}^I | \mathcal{F}_0] \geq \frac{1}{2} \mathbb{P}[A_1 \cap A_2 | \mathcal{F}_0]. \tag{4.5}
\]

**Step 2.** We next write

\[
\mathbb{P}[A_1 \cap A_2 | \mathcal{F}_0] = \mathbb{E} \left[ 1_{A_1} \mathbb{P}[A_2 | \mathcal{F}_1] | \mathcal{F}_0 \right].
\]

But using again Lemma 4.2,

\[
A_2 = \{ H_{t_1, t_2}^{m(1+\epsilon)} \geq 1 \} \setminus \{ H_{t_1, t_2}^{m(1+\epsilon)} \geq 2 \} \setminus \{ S_{t_1, t_2}^{m(1+\epsilon)} \geq 1 \} \setminus \{ R_{t_1, t_2}^{m(1+\epsilon)} \geq 2 \}.
\]

Thus,

\[
\mathbb{P}[A_2 | \mathcal{F}_1] \geq \mathbb{P}[S_{t_1, t_2}^{m(1+\epsilon)} \geq 1 | \mathcal{F}_1] - \mathbb{P}[R_{t_1, t_2}^{m(1+\epsilon)} \geq 2 | \mathcal{F}_1].
\]

First, by Remark 4.2,

\[
\mathbb{P}[R_{t_1, t_2}^{m(1+\epsilon)} \geq 2 | \mathcal{F}_1] = 1 - \left( 1 + \lambda_{t_1, t_2}^{m(1+\epsilon)} \right) \exp \left( -\lambda_{t_1, t_2}^{m(1+\epsilon)} \right) \leq \left( \lambda_{t_1, t_2}^{m(1+\epsilon)} \right)^2 \leq C_B 2^{-6m\epsilon c}.
\]

Next, we put \( Y_t := S_{t_1, t}^{m(1+\epsilon)} \) for \( t \geq t_1 \) and observe, according to Itô’s Formula, that

\[
1_{Y_t = 0} = 1 + \int_{t_1}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty 1_{|v-H_{B}(V_{\cdot})| \geq a} 1_{|u| \leq \frac{a}{\lambda_{t_1, t_2}^{m(1+\epsilon)}}} \times \left( 1 - Y_t - \Delta Y_t = 0 \right) N(ds, dv, d\theta, d\varphi, du)
\]

\[
= 1 - \int_{t_1}^t \int_{\mathbb{R}^3} \int_{2^{m+2}\pi/a}^{\pi/2} \int_0^{2\pi} \int_0^\infty 1_{|v-H_{B}(V_{\cdot})| \geq a} 1_{Y_{t_1} = 0} N(ds, dv, d\theta, d\varphi, du).
\]
Hence, for all \( t \geq t_1 \),
\[
\frac{d}{dt} \mathbb{E}[1_{\{Y_t = 0\}} | \mathcal{F}_{t_1}] = -\mathbb{E} \left[ \int_{\mathbb{R}^3} \int_{t_2}^t \int_{0}^{2\pi} \int_{0}^{\pi/2} 1_{\{|v-H_B(V_s)| \geq a\}} f_1(Zv) \mathbb{E}(v \mathbb{B}(\theta) d\theta d\varphi dv | \mathcal{F}_{t_1}) \right].
\]

Using (1.3) and Lemma 5.1 (which implies that \( f_4(\{v \in \mathbb{R}^3 : |v-H_B(V_s)| \geq a\}) \geq b > 0 \) a.s. for all \( s \in [0,1] \), we easily deduce see that
\[
\frac{d}{dt} \mathbb{E}[1_{\{Y_t = 0\}} | \mathcal{F}_{t_1}] \leq -\kappa 2^{m\nu} \mathbb{E}[1_{\{Y_t = 0\}} | \mathcal{F}_{t_1}],
\]
for some positive constant \( \kappa \). Integrating this inequality, we deduce that a.s., for all \( t \geq t_1 \),
\[
\mathbb{E}[1_{\{Y_t = 0\}} | \mathcal{F}_{t_1}] \leq \exp(-\kappa 2^{m\nu}(t-t_1)).
\]

Consequently,
\[
\mathbb{P}[S_{t_1,t_2}^m \geq 1 | \mathcal{F}_{t_1}] = 1 - \mathbb{E}[1_{\{Y_t = 0\}} | \mathcal{F}_{t_1}] \geq 1 - \exp(-\kappa 2^{m\nu}(t_2-t_1)) = 1 - \exp(-\kappa 2^{-4m\nu}/3).
\]

Finally, for all \( m \) large enough (depending only on \( \epsilon \)), we a.s. have
\[
\mathbb{P}[A_2 | \mathcal{F}_{t_1}] \geq 1 - \exp(-\kappa 2^{-4m\nu}/3) - C_B 2^{-6m\nu} \geq c_B 2^{-4m\nu}.
\]

**Step 3.** Finally, exactly as Step 1, we obtain that for all \( m \) large enough,
\[
\mathbb{P}[A_1 | \mathcal{F}_{t_0}] \geq \frac{1}{2}.
\]

**Step 4.** It suffices to gather Steps 1, 2 and 3 to conclude the proof. \( \square \)

**Proof of Lemma 4.3.** We thus fix \( \epsilon > 0 \) and consider \( \alpha \) and \( \beta \) as in the statement. For \( m > 0 \), we introduce the notation \( r_m = 2^{-m\beta}/3 \). We also introduce the number \( q_m^2 := \lfloor 2^m(\beta-\alpha) \rfloor \), the length \( \ell_m := q_m 2^{-m\beta} \) (we have \( \ell_m \leq 2^{-m\alpha} \) and \( \ell_m \approx 2^{-m\alpha} \)) and the number \( q_m^1 := \lfloor 1/\ell_m \rfloor + 1 \) (we have \( q_m^1 \approx 2^{m\alpha} \)). We consider a covering of \([0,1]\) by \( q_m^1 \) consecutive intervals \( I_{i,m}^1, \ldots, I_{q_m^1,m}^1 \) with length \( \ell_m \). Next, we divide each \( I_{i,m}^1 \) into \( q_m^2 \) consecutive intervals \( I_{i,m}^{1,k} \) with length \( 2^{-m\beta} \). Finally, we divide each \( I_{i,m}^{1,k} \) into three consecutive intervals with length \( r_m \), writing \( I_{i,j}^{1,k} = [t_{i,j}^{1,k} + r_m, t_{i,j}^{1,k} + 2r_m] \cup [t_{i,j}^{1,k} + 2r_m, t_{i,j}^{1,k} + 3r_m] \cup [t_{i,j}^{1,k} + 3r_m, t_{i,j}^{1,k} + 4r_m] \). We consider the event
\[
A_{i,j}^m = \{ H_{t_{i,j}^{1,k} + r_m}^{m(1+\epsilon)}(t_{i,j}^{1,k} + r_m) = 0 \} \cap \{ H_{t_{i,j}^{1,k} + 2r_m}^{m(1+\epsilon)}(t_{i,j}^{1,k} + 2r_m) = 0 \} \cap \{ H_{t_{i,j}^{1,k} + 3r_m}^{m(1+\epsilon)}(t_{i,j}^{1,k} + 3r_m) = 0 \}.
\]

According to Lemma 3.3, we know that if \( m \) is large enough (depending only on \( \epsilon \)), a.s., for all \( i,j \)
\[
\mathbb{P}[A_{i,j}^m | \mathcal{F}_{t_{i,j}^m}] \geq c_B 2^{-4m\nu}.
\]

We now consider, for each \( i \), the event
\[
K_{m,i} = \bigcap_{j=1}^{q_m^2} (A_{i,j}^m)^c.
\]

Then, we easily deduce from (4.6), together with the fact that \( A_{i,1}^m, \ldots, A_{i,j-1}^m \in \mathcal{F}_{t_{i,j}^m} \) for all \( j = 1, \ldots, q_m^2 - 1 \), that
\[
\mathbb{P}(K_{m,i}) \leq (1 - c_B 2^{-4m\nu} q_m^2) \leq (1 - c_B 2^{-4m\nu} q_m^2)^{2m(\beta-\alpha)-1}.
\]
Thus for $m$ large enough (depending only on $\epsilon$), we conclude that
\[ \mathbb{P}(K_{m,i}) \leq \exp \left( -c_B 2^{-4m\epsilon} 2^{m(\beta-\alpha)} \right) = \exp \left( -c_B 2^{2m\epsilon} \right). \]

Next, we introduce the event $K_m = \bigcup_{i=1}^{q_m} K_{m,i}$. Clearly, for $m$ large enough, (allowing the value of the constant $c_B > 0$ to change)
\[ \mathbb{P}(K_m) \leq q_m^4 \exp(-c_B 2^{2m\epsilon}) \leq \exp(-c_B 2^{2m\epsilon}). \]

Finally, using the Borel-Cantelli lemma, we conclude that there a.s. exists $M > 0$ such that for all $m \geq M$, the event $K_m^c$ is realized (whence for all $i = 1, \ldots, q_m^1$, there is $j \in \{1, \ldots, q_m^2\}$ such that $A_{i,j}^m$ is realized). This implies that a.s., for all $m \geq M$, for all $t \in [0,1]$, considering $i \in \{1, \ldots, q_m^1\}$ such that $t \in I_{i,m}^m$ and $j \in \{1, \ldots, q_m^2\}$ such that $A_{i,j}^m$ is realized, $V^B$ has exactly one jump greater than $2^{-m(1+\epsilon)}$ in the time interval $I_{i,j}^m$, this jump is greater than $2^{-m}$ and happens at some time $t_m$ located in the middle of $I_{i,j}^m$ (more precisely, the distance between $t_m$ and the extremities of $I_{i,j}^m$ is at least $r_m$). We clearly have $|t_m - t| \leq \ell_m \leq 2^{-m\alpha}$, $|\Delta V_{i,j}^B| \geq 2^{-m}$, and $V^B$ has no other jump of size greater than $2^{-m(1+\epsilon)}$ in $B(t_m, r_m) \subset I_{i,j}^m$. The proof is complete. \qed

### 4.3 Uniform bound for the Hölder exponent of $X^B$

We show here that $D_{X^B}(h) = -\infty$ for all $h > 1 + 1/\nu$. We use a general result for primitives of discontinuous functions. It based on Proposition 1 in [2], recalled in the following lemma.

**Lemma 4.5.** Let $\eta > 0$ and let $N > \eta$ be an integer. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function and let $\psi$ be a $C^\infty$ compactly supported function with its $N$ first moments vanishing, i.e. \( \int_\mathbb{R} x^k \psi(x) dx = 0 \) for $k = 0, \ldots, N-1$. The wavelet transform of $g$ is defined by
\[ W_\psi(g,a,b) = \frac{1}{a} \int_\mathbb{R} g(t) \psi \left( \frac{t-b}{a} \right) dt. \] (4.7)

If $g \in C^0(t_0)$, then there exists a constant $C > 0$ such that for all $a > 0$, all $b \in [t_0 - 1, t_0 + 1]$,\[ |W_\psi(g,a,b)| \leq C (a^\eta + |t_0 - b|^\eta) . \] (4.8)

Now, we give the following general result. For any function $g : \mathbb{R} \rightarrow \mathbb{R}$, and any interval $I \subset \mathbb{R}$, we set\[ \text{osc}_I(g) = \sup_{x \in I} g(x) - \inf_{x \in I} g(x). \]

**Lemma 4.6.** Let $g : [0, \infty) \rightarrow \mathbb{R}$ be a càdlàg function, discontinuous on a dense set of points, let $G(t) = \int_0^t g(s) ds$. Let $t > 0$ and let $(t_m)_{m \geq 1}$ be a sequence of discontinuities of the function $g$ converging to $t$. For all $s \in \mathbb{R}$, all $m \geq 1$, we define
\[ g_m(s) = g(s) - J_m 1_{\{s \geq t_m\}}, \] (4.9)

where $J_m = g(t_m^+) - g(t_m^-)$. Assume that for all $m \geq 1$, there exist $r_m > 0$ and $\delta_m > 0$ such that
\[ \text{osc}_{[t_m-r_m, t_m+r_m]}(g_m) \leq \delta_m \text{ and } \lim_{m \rightarrow \infty} \frac{\delta_m}{|J_m|} = 0. \] (4.10)

Then
\[ h_G(t) \leq \liminf_{m \rightarrow \infty} \frac{\log \left( \frac{r_m |J_m|}{|t_m - t| + r_m} \right)}{\log (\frac{r_m |J_m|}{|t_m - t| + r_m})} \] (4.11)
Proof. Let \( \varphi \) be a positive \( C^\infty \) function, supported on \([0,1]\) satisfying \( \int_R \varphi(x)dx = 1 \).

For \( k \geq 1 \), let \( \psi_k(t) = \varphi^{(k)}(t) \), it is clear that \( \psi_k \) is \( C^\infty \), supported on \([0,1]\) and that its \( k \) first moments vanish, so it is a wavelet.

We now pick an integer \( N \) such that \( N-2 \) is larger than the right hand side of (4.11), and we denote by \( c_N(a,b) := W_{\psi_N}(g,a,b) \) and \( C_{N+1}(a,b) := W_{\psi_N+1}(G,a,b) \) the wavelet transforms of \( g \) and \( G \) using the wavelet \( \psi_N \) and \( \psi_{N+1} \), respectively. An integration by parts shows that

\[
c_N(a,b) = -\frac{1}{a} C_{N+1}(a,b). \tag{4.12}
\]

We fix \( \theta \in (0,1) \) such that \( \psi_{N-1}(\theta) > 0 \). It follows from (4.10) that \( c_N(r_m,t_m - \theta r_m) = P_m + Q_m \), where

\[
P_m = \frac{1}{r_m} \int_{-\infty}^{+\infty} J_m 1(s \geq t_m) \psi_N \left( \frac{s - t_m + \theta r_m}{r_m} \right) ds = \frac{J_m}{r_m} \int_{t_m}^{+\infty} \psi_N \left( \frac{s - t_m + \theta r_m}{r_m} \right) ds = -J_m \psi_{N-1}(\theta) \tag{4.13}
\]

and

\[
Q_m = \frac{1}{r_m} \int_{-\infty}^{+\infty} g_m(s) \psi_N \left( \frac{s - t_m + \theta r_m}{r_m} \right) ds = \frac{1}{r_m} \int_{-\infty}^{+\infty} (g_m(s) - g_m(t_m)) \psi_N \left( \frac{s - t_m + \theta r_m}{r_m} \right) ds,
\]

where we used that \( \psi_N \) has a vanishing integral. Observing that

\[
\text{supp} \left( \psi_N \left( \frac{\cdot - t_m + \theta r_m}{r_m} \right) \right) \subset [t_m - r_m, t_m + r_m]
\]

and recalling (4.10), we deduce that \( |Q_m| \leq 2 \| \psi_N \|_\infty \delta_m \). As a conclusion,

\[
|c_N(r_m, t_m - \theta r_m)| \geq |P_m| - |Q_m| \geq \psi_{N-1}(\theta)|J_m| - 2 \| \psi_N \|_\infty \delta_m \geq c|J_m|
\]

for all \( m \) large enough, since \( \lim_{m \to +\infty} \frac{4}{|J_m|} = 0 \) by assumption. Then we obtain according to (4.13),

\[
|C_{N+1}(r_m, t_m - \theta r_m)| \geq cr_m |J_m|. \tag{4.14}
\]

Assume that \( G \in C^\eta(t) \) for some \( \eta > \lim \inf m \to +\infty \| \log(r_m)|J_m|)/[\| \log(|t_m - t| + r_m)|] \). We apply Lemma 4.7 with \( g = G, \psi = \psi_{N+1}, a = r_m, b = t_m - \theta r_m \). Hence, there is a constant \( C \) such that for all \( m \),

\[
|C_{N+1}(r_m, t_m - \theta r_m)| \leq C \left( r_m^n + |t - t_m + \theta r_m|^n \right) \leq C(r_m^n + |t - t_m|^n).
\]

This contradicts (4.13), so necessarily (4.11) hold true. \( \square \)

We next apply this lemma to our position process to get a uniform upper bound for all pointwise Hölder exponents of \( X^B \).

**Proposition 4.7.** Almost surely, for all \( t \in [0,1] \), the Hölder exponent of \( X^B \) satisfies

\[
h_{X^B}(t) \leq 1 + \frac{1}{\nu}. \tag{4.14}
\]

**Proof.** We fix \( \epsilon > 0 \) and set \( \alpha = \nu(1 - 2\epsilon) \) and \( \beta = \nu(1 + 4\epsilon) \). We show that a.s., \( h_{X^B}(t) \leq (1 + \beta)/\alpha \) for all \( t \in [0,1] \). This clearly suffices since \( \epsilon > 0 \) can be chosen arbitrarily small.

Lemma 4.7 shows that there a.s. exists \( M > 0 \), such that for all \( m \geq M \), for all \( t \in [0,1] \), there exists \( t_m \in B(t, 2^{-m}) \) such that \( |\Delta V^B_{t_m}| \geq 2^{-m} \) and such that there is no other jump of size greater than \( 2^{-m(1+\epsilon)} \) in \( B(t_m, r_m) \), with \( r_m := 2^{-m\beta}/3 \).

We now fix \( t \in [0,1] \). Up to extraction, one can assume that the first coordinate \( \hat{V}^B_t \) of the three-dimensional vector \( V^B_t \) satisfies \( |\Delta \hat{V}^B_{t_m}| \geq 2^{-m}/3 \). We now apply Lemma 4.7 with \( g = \hat{V}^B \) and \( r_m = 2^{-m\beta}/3 \).
2^{−mβ}/3. We thus introduce \( g_m(s) = g(s) − \Delta \tilde{V}_m B_1(s \geq t_m) \). Since \( V^B \) (and so \( \tilde{V}_s^B \)) has no jump with size greater than \( 2^{−m(1+\epsilon)} \) within the interval \( B(t_m, r_m) = (t_m, r_m + r_m) \), we observe that

\[
\text{osc}_{B(t_m, r_m)}(g_m) \leq 2 \sup_{x, y \in [0, 1], |x − y| \leq 2^{−m \beta}} |V^B_{g}(m(1+\epsilon)) − V^B_{y}(m(1+\epsilon))|.
\]

Next, using Lemma 4.9.1 (with \( \delta = \beta/(1+\epsilon) > \nu \)) and the Borel-Cantelli Lemma, we deduce that there is a.s. \( M' > 0 \) such that, for all \( m \geq M' \), all \( 0 < x < y < 1 \) with \( |x − y| < 2^{−m\beta} \), \( |V^B_{g}(m(1+\epsilon)) − V^B_{y}(m(1+\epsilon))| \leq m\beta2^{−m(1+\epsilon)} \). That is,

\[
\text{osc}_{B(t_m, r_m)}(g_m) \leq 2m\beta2^{−m(1+\epsilon)}.
\]

Since furthermore \( \lim_{m \to +\infty} 2m\beta2^{−m(1+\epsilon)} \leq \lim_{m \to +\infty} 2m\beta2^{−m(1+\epsilon)} = 0 \), we can apply Lemma 4.0 with \( \delta_m = 2m\beta2^{−m(1+\epsilon)} \):

\[
h_{\nu, \nu}(t) \leq \liminf_{m \to +\infty} \frac{\log (r_m/|\Delta \tilde{V}^B_{t_m}|)}{\log (|t_m - r_m| + r_m)} \leq \liminf_{m \to +\infty} \frac{\log (2^{−m(1+\beta)/9})}{\log (2^{−m\alpha})} = \frac{1 + \beta}{\alpha}.
\]

We used that \( r_m/|\Delta \tilde{V}^B_{t_m}| \geq (2^{−m/3}(2^{−m\beta}/3) \) and that \( |t_m - r_m| + r_m \leq 2^{−m\alpha} + 2^{−m\beta}/3 \leq 2.2^{−m\alpha} \). This ends the proof.

### 4.4 Study of the oscillating singularities of \( X^B \)

To characterize more precisely the set of oscillating times, we first give the following lemma.

**Lemma 4.8.** Let \( \delta > \nu, \epsilon > 0 \) and \( k \in \mathbb{N} \) satisfy \( \delta > \nu(1+\epsilon)(k+1)/k \). For all \( m \in \mathbb{N} \), let \( (I^m_j)_{j=1,...,|2^m\delta|+1} \) be the covering of \([0,1]\) composed of successive intervals of length \( 2^{−m\delta} \). Almost surely, there exists \( M \geq 1 \) such that for all \( m \geq M \), for all \( j = 1, \ldots, |2^m\delta| \), recalling (4.2),

\[
H^m_{I^m_j \cup I^m_{j+1}} \leq k,
\]

(4.15)

**Proof.** Using Lemma 4.11 and Remark 4.2,

\[
\mathbb{P}\left(H^m_{I^m_j \cup I^m_{j+1}} > k\right) \leq \mathbb{P}\left(R^m_{I^m_j \cup I^m_{j+1}} > k\right) \leq \sum_{\ell=k+1}^{+\infty} \frac{\lambda^m_{I^m_j \cup I^m_{j+1}} (\ell)}{\ell!} e^{-\lambda^m_{I^m_j \cup I^m_{j+1}}} \leq \lambda^m_{I^m_j \cup I^m_{j+1}} (k+1),
\]

where the value of \( \lambda^m_{I^m_j \cup I^m_{j+1}} \) is given by equation (4.3). But, since the length of \( I^m_j \cup I^m_{j+1} \) is \( 2^{−m\delta} \), we apply the upper bound found for \( \lambda^m_{I^m_j \cup I^m_{j+1}} \) in Remark 4.2 to get \( \lambda^m_{I^m_j \cup I^m_{j+1}} \leq 2C_2^{2m(1+\epsilon)−m\delta} \), so that

\[
\mathbb{P}\left(H^m_{I^m_j \cup I^m_{j+1}} > k\right) \leq 2C_2^{2m(1+\epsilon)−m\delta} e^{-\nu(1+\epsilon)(k+1)/k}.
\]

Consequently,

\[
\mathbb{P}\left(\bigcup_{j=1}^{[2^m\delta]+1} \left\{ H^m_{I^m_j \cup I^m_{j+1}} > k\right\}\right) \leq 2C_2^{2m\delta} 2^{m(1+\epsilon)−m\delta} = 2C_2^{2m(1+\epsilon)−m\delta}.
\]

By assumption, this is the general term of a convergent series. We conclude thanks to the Borel-Cantelli lemma.

We first study the case where \( h \in [0,1/(2\nu)) \).

**Lemma 4.9.** Almost surely, for all \( h \in [0,1/(2\nu)) \), \( E_{\tilde{V}_s^B}^\text{osc}(h) = \emptyset \).
We used that
Letting \( \varepsilon \) (that is Lemma 4.10.

times that are very close to each other.

Proof. According to (4.1), it is sufficient to check that for

with

By Remark 3.15, we know that

We conclude from Lemma 4.6 that

By Remark 4.2, we know that

As we did before, up to extraction, we can e.g. assume that the first coordinate

We then apply Lemma 4.6 with

Hence, applying the Borel-Cantelli lemma, we know that almost surely , there exists

The Multifractal Nature of Boltzmann Processes 23

and

Denote by

while

We used that

Letting \( \epsilon \to 0 \) (whence \( \delta \to 1/\nu \)), we conclude that \( h_{X^\nu}(t) \leq 1 + h \) as desired.

Before computing the dimension of \( E_{V^\nu}(h) \) when \( h \in [1/(2\nu), 1/\nu) \), we need to count those jump times that are very close to each other.

Lemma 4.10. For \( \epsilon > 0 \) and \( m > 0 \), denote by \( 0 < T_{1,\epsilon,m}^c < \cdots < T_{K_{\epsilon,m}}^c < 1 \) the successive instants of jumps of \( V^\nu B \) with size greater than \( 2^{-m(1+\epsilon)} \). For \( \delta > 0 \), we introduce

with the convention that \( T_0^c = 0 \). For any fixed \( \epsilon > 0 \) and \( \delta > 0 \), there a.s. exists \( M > 0 \) such that for all \( m > M \),

Proof. Recalling Lemma 4.6, we see that

where \( 0 < S_{1,\epsilon,m}^c < \cdots < S_{K_{\epsilon,m}}^c \) are the successive instants of jump of the counting process \( R^m_{[\alpha \nu]}(1+\epsilon) \). Consequently,

By Remark 4.2, we know that \( R^m_{[\alpha \nu]}(1+\epsilon) \) is an inhomogeneous Poisson process with intensity bounded by \( C_B 2^{m(1+\epsilon)\nu} \). Consequently,

Hence, applying the Borel-Cantelli lemma, we know that almost surely, there exists \( M' \geq 1 \) such that for all \( m \geq M' \),

and thus

\[
N_{m, \epsilon}^{\delta, \epsilon} \leq \sum_{i=1}^{L_{\epsilon,m}} 1\{S_i^{c, m} - S_{i-1}^{c, m} \leq 2^{-m \delta}\}.
\]
Lemma 4.10 allows to bound by above the cardinality of such sets. Hence, choosing $T$ Proposition 4.11. Almost surely, for $\nu$, $\delta > 0$ such that for all $m \geq M''$,

$$\sum_{i=1}^{2^{m\nu(1+2\epsilon)}} 1 \{ S_{i, m}^\epsilon - S_{i-1, m}^\epsilon \leq 2^{-m\delta} \} \leq 2^{-m\delta + 2m\nu(1+2\epsilon)} C_B^2 2^{m(1+\nu)\nu - m\delta}$$

By the Borel-Cantelli lemma again, there exists a.s. a constant $M'' > 0$ such that for all $m \geq M''$,

$$\sum_{i=1}^{2^{m\nu(1+2\epsilon)}} 1 \{ S_{i, m}^\epsilon - S_{i-1, m}^\epsilon \leq 2^{-m\delta} \} \leq 2^{-m\delta + 2m\nu(1+2\epsilon)}.$$

As a conclusion, a.s. we have $N_{m}^{\delta, \epsilon} \leq 2^{-m\delta + 2m\nu(1+2\epsilon)}$ for all $m \geq M' \lor M''$. Choosing $M = M' \lor M''$ completes the proof.

Now we treat the case where $h \in [1/(2\nu), 1/\nu]$.

**Proposition 4.11.** Almost surely, for $h \in [1/(2\nu), 1/\nu]$, $\dim_H \left( E_{\nu, \delta}^\epsilon(h) \right) \leq 2\nu - 1$.

**Proof.** We divide the proof into several steps.

**Step 1.** For any fixed $\epsilon > 0$, $\delta \in (\nu, 2\nu)$ and $m \geq 1$, we consider the sets

$$F_m(\delta, \epsilon) = \bigcup_{i \in \mathbb{Z}} \left( \left[ T_{i-1, m}^\epsilon - 2^{-m\delta}, T_{i-1, m}^\epsilon + 2^{-m\delta} \right] \lor \left[ T_{i, m}^\epsilon - 2^{-m\delta}, T_{i, m}^\epsilon + 2^{-m\delta} \right] \right),$$

where the family $T_{i, m}^\epsilon$ has been introduced in Lemma 4.10 and the associated limsup set

$$G(\delta, \epsilon) = \limsup_{m \to +\infty} F_m(\delta, \epsilon).$$

For every $n \geq 1$, $\bigcup_{m \geq n} F_m(\delta, \epsilon)$ forms a covering of $G(\delta, \epsilon)$ by sets of diameter less than $2^{-m\delta + 2}$, and Lemma 4.10 allows to bound by above the cardinality of such sets. Hence, choosing $s > \frac{2\nu(1+2\epsilon)}{\delta} - 1$, a.s. for every $n$ large enough one has

$$\mathcal{H}^s_{2^{-n\delta + 2}}(G(\delta, \epsilon)) \leq \sum_{m \geq n} 2^{-m\delta + 2s} N_{m}^{\delta, \epsilon} \leq \sum_{m \geq n} 2^{2s} 2^{-m(s+1)\delta + 2m\nu(1+2\epsilon)}.$$

We deduce that $\lim_{n \to +\infty} \mathcal{H}^s_{2^{-n\delta + 2}}(G(\delta, \epsilon)) = 0$, hence $\mathcal{H}^s(G(\delta, \epsilon)) = 0$. Therefore, $\dim_H \left( G(\delta, \epsilon) \right) \leq \frac{2\nu(1+2\epsilon)}{\delta} - 1$.

**Step 2.** Here we fix $h \in [1/(2\nu), 1/\nu]$, we consider $\epsilon > 0$ such that $1/[h(\epsilon)(1+\epsilon)] > \nu$, we set $\delta_{\epsilon} = 1/(h + \epsilon)$ and we prove that $E_{\nu, \delta}^\epsilon(h) \subset G(\delta_{\epsilon}, \epsilon)$.

We consider $t \in E_{\nu, \delta}^\epsilon(h) \lor \nu$, and we show that $h(t) = 1 + h$, which will imply indeed that $t \in E_{\nu, \delta}^\epsilon(h)$.

Since $t \not\in G(\delta_{\epsilon}, \epsilon)$, there exists $N \geq 1$ such that for all $m \geq N$, $t \not\in F_m(\delta_{\epsilon}, \epsilon)$. Moreover, for any $0 < \eta \leq \epsilon$, since $t \in E_{\nu, \delta}^\epsilon(h)$, by Remark 3.13, we know that $t \in B_{\delta_{\epsilon}}(h)$ (because $\delta_{\epsilon} = 1/(h + \eta) < 1/h$), so that for all $n \geq 1$, there exist $m_{n} \geq n$ and $t_{n} \in B(t, 2^{-m_{n}\delta_{\eta}})$ such that $|\Delta V_{t_{n}}^B| \geq 2^{-m_{n}}$. Observing that $F_{m_{n}}(\delta_{\eta}, \eta) \subset F_{m_{n}}(\delta_{\epsilon}, \epsilon)$ since $0 < \eta \leq \epsilon$ and $\delta_{\epsilon} \geq \delta_{\eta}$, hence $t \not\in F_{m_{n}}(\delta_{\eta}, \eta)$ (for all $n$ large enough), whence, there is also no other jump in $B(t, 2^{-m_{n}\delta_{\eta}})$ with size greater than $2^{-m_{n}(1+\eta)}$.

As in the previous proofs, up to extraction, we deduce that $|\Delta V_{t_{n}}^B| \geq 2^{-m_{n}/3}$ for all $n$, where $V_{t}^B$ is one of the three coordinates of $V$. Since $V_{t}^B$ (and so $\tilde{V}_{t}^B$) has no jump with size greater than $2^{-m_{n}(1+\eta)}$
in \( B(t_n, 2^{-m_n} δ_n) \), we may use Lemma 3.11 (1) because \( δ_n/(1 + δ_n) = \frac{1}{(h+η)(1+η)} ≥ \frac{1}{(h+η)(1+η)} > ν \) and the Borel-Cantelli Lemma, we deduce that a.s. for all \( n \) sufficiently large, setting \( r_n = 2^{-m_n} δ_n \),

\[
\text{osc}_{B(t_n, r_n)}(\tilde{V}^B) ≤ 2 \times \sup_{x,y \in [0,1],|x-y| ≤ 2^{-m_n} η} |V^B_{x,m_n(1+η)} - V^B_{y,m_n(1+η)}| ≤ 2m_n δ_n 2^{-m_n(1+η)}.
\]

Moreover,

\[
\lim_{n \to +\infty} \frac{2m_n δ_n 2^{-m_n(1+η)}}{2^{-m_n}/3} \leq 0.
\]

Applying Lemma 4.6 with \( g = \tilde{V}^B \), \( r_n = 2^{-m_n} δ_n \) and \( δ_n = 2m_n δ_n 2^{-m_n(1+η)} \), we obtain

\[
h_{X^B}(t) ≤ \liminf_{n \to +\infty} \frac{\log \left( \frac{r_n |\Delta \tilde{V}^B_{t_n}|}{2^{-m_n(1+η)/3}} \right)}{\log(2^{-m_n})} = 1 + \frac{δ_n}{η} = 1 + h + η \tag{4.16}
\]

because \( r_n |\Delta \tilde{V}^B_{t_n}| ≥ 2^{-m_n(1+η)/3} \) and \( r_n + |t_n - t| ≤ 2.2^{-m_n} δ_n \). Since (4.16) is satisfied for any \( 0 < η ≤ ε \), then a.s. \( h_{X^B}(t) ≤ 1 + h \). That is, \( E_{X^B}^{osc}(h) ≤ G(δ, t) \).

Step 3. From step 2 we deduce that \( E_{X^B}(h) ≤ \bigcap_{c \in [0,1]} G(δ, t) \). Hence,

\[
\dim_H \left( E_{X^B}^{osc}(h) \right) ≤ \dim_H \left( \bigcap_{c \in [0,1]} G(δ, t) \right) = \inf_{c \in [0,1]} \left( 2ν(1 + 2ε)(h + ε) - 1 \right) = 2hν - 1.
\]

This ends the proof.

\[
\square
\]

4.5 Conclusion

Proof of Proposition 2.2. First, we now from Proposition 2.1 that \( E_{X^B}(h) = \emptyset \) for \( h > 1/ν \), so that obviously \( E_{X^B}^{osc}(h) = \emptyset \). If now \( h = 1/ν \), then we deduce from Proposition 4.7 that \( E_{X^B}^{osc}(h) = \emptyset \), simply because a.s., for all \( t \in [0,1] \), \( h_{X^B}(t) ≤ 1 + 1/ν \).

As shown in Lemma 4.9 we also know that \( E_{X^B}^{osc}(h) = \emptyset \) for all \( h \in [0,1/(2ν)] \) and as seen in Proposition 4.11 \( \dim_H(E_{X^B}^{osc}(h)) ≤ 2hν - 1 \) for all \( h \in [1/(2ν), 1/ν] \).

It remains to verify that for all \( h \in [0,1/(2ν)] \), \( \dim_H(E_{X^B}^{osc}(h)) = hν \). If \( h \in [0,1/(2ν)] \) or \( h = 1/ν \), it is obvious because \( E_{X^B}(h) = \emptyset \) and by Proposition 2.2. If \( h \in [1/(2ν), 1/(ν)] \), we use Theorem 4.7 and \( \dim_H(E_{X^B}^{osc}(h)) < 2hν - 1 < hν \).

Finally, we verify that Theorems 1.7 and 1.10 imply Theorem 1.8.

Proof of Theorem 1.8. For any \( h \in [1, 1 + 1/ν] \), we have \( E_{X}(h) ≥ E_{X^B}^{osc}(h - 1) \), whence \( \dim_H(E_{X}(h)) ≥ \dim_H(E_{X^B}^{osc}(h - 1)) = (h - 1)ν \) by Theorem 1.10.

Next we obviously have a.s., for all \( t \in [0,1] \),

\[
h_{X}(t) ≥ h_{V}(t) + 1, \tag{4.17}
\]

whence \( E_{X}(h) ≤ \bigcup_{h ≤ h - 1} E_{V}(h') \). We thus infer from Theorem 1.7 that \( E_{X}(h) = \emptyset \) when \( h < 1 \). But when \( h \in [1, 1 + 1/ν] \), recalling Proposition 3.11 and the fact that \( A^*_4 \) is decreasing with \( δ \), we deduce that \( \bigcup_{h ≤ h - 1} E_{V}(h') \) is contained in \( \bigcap_{h < h - 1} A^*_4 \). Whence we derive \( \dim_H(E_{X}(h)) ≤ (h - 1)ν \) from Proposition 3.12.

It only remains to verify that \( E_{X}(h) = \emptyset \) when \( h > 1 + 1/ν \). But in such a case, we know from Proposition 4.7 that \( E_{X^B}(h) = \emptyset \), whence \( E_{X}(h) = \bigcup^∞_{t=1} E_{X^B}(h) = \emptyset \).
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