Probe branes on dynamical brane backgrounds

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Abstract

We consider the dynamics of a probe brane on various dynamical brane backgrounds. The consistency condition of the static gauge leads to the supersymmetric intersection rules for the static branes, while the dynamical backgrounds preserve no supersymmetry. This is a specific feature of the dynamical backgrounds since the static gauge condition gives rise to no constraint on the brane configuration in the case of the static background. Furthermore, due to the consistency condition, it follows that there is no velocity-independent force for the probe brane even on the dynamical backgrounds.

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1 Introduction

String theory contains higher-dimensional branes as well as strings. The low-energy dynamics of the branes are described in supergravity theories. An innumerable number of brane solutions have been discovered so far. Although most of them are static, time-dependent brane solutions also exist. For example, colliding brane solutions, which are found by Gibbons, Lu and Pope [1], are time-dependent solutions (For the related progress, see [2–8]). The colliding solutions have some applications in realistic cosmology [9–12]. However, those have not succeeded yet because some unwanted things happen such as the formation of a naked singularity.

An interesting issue is to consider the brane dynamics on the dynamical brane backgrounds. It is well studied on static brane backgrounds [13]. The brane dynamics is an important subject in M-theory and string theory. For example, the matrix model formulations of M-theory and type IIB string theory are intimately concerned with the brane dynamics [14, 15].

Our purpose here is to consider the dynamics of probe branes on various dynamical brane backgrounds. This is formally a generalization of the work by Tseytlin [13], where “static” source brane backgrounds are considered. To study the brane dynamics, first of all, it is necessary to consider the gauge-fixing condition. In particular, the static gauge is consistently taken even for the dynamical backgrounds but it gives rise to an additional constraint as the consistency condition. This constraint leads to the supersymmetric intersection rules for the static branes, while the dynamical backgrounds preserve no supersymmetry. This is a specific feature of the dynamical backgrounds since the static gauge condition gives rise to no constraint on the brane configuration in the case of the static background. Furthermore, due to the constraint, it follows that there is no velocity-independent force for the probe brane even on the dynamical backgrounds.

This paper is organized as follows. In section 2, we introduce dynamical brane solutions. In section 3 we consider the dynamics of a probe $p$-brane moving on the dynamical brane solutions. In particular, we argue the condition under which the velocity-independent force vanishes. Section 4 considers probe M-branes on the dynamical backgrounds in the eleven dimensional supergravity. Section 5 is devoted to probe NS-branes and D-branes on the dynamical brane backgrounds in type IIB and IIA supergravities. In section 6 we argue a generalization of the dynamical background by allowing the metric to depend on the spatial direction along which the branes extend. For a specific choice of parameters, the static gauge leads to no constraint. This fact is explained with an example of M2-brane in detail. Section 7 is devoted to conclusion and discussion.
2 Dynamical brane backgrounds

This section introduces dynamical brane backgrounds used in later discussion.

The gravitational theory considered here includes the metric in the Einstein frame $\tilde{g}_{MN}$, a scalar field (dilaton) $\phi$, and a $(p+1)$-form field $A_p$ where the field strength is $F_{(p+2)} = dA_p$. This theory is realized by imposing some ansatz in type IIB and IIA supergravities. It is enough to describe charged $p$-branes.

The action (in the Einstein frame) is given by [16]

$$ S = \frac{1}{2\kappa^2} \int \left[ \tilde{R} \ast 1_D - \frac{1}{2} \ast d\phi \wedge d\phi - \frac{1}{2} \frac{e^{c\epsilon c\phi}}{(p+2)!} \ast F_{(p+2)} \wedge F_{(p+2)} \right], \quad (2.1) $$

where $\kappa^2$ is the gravitational constant in $D$ dimensions and $\ast$ is the Hodge dual operator, $\tilde{R}$ denotes the Ricci scalar constructed from the $D$-dimensional metric $\tilde{g}_{MN}$, and the metric $\tilde{g}_{MN}$ is related to the $g_{MN}$ denotes the $D$-dimensional metric in the string frame in terms of a Weyl rescaling of the metric:

$$ g_{MN} = e^{\phi/2} \tilde{g}_{MN}. \quad (2.2) $$

Then the constants $c$ and $\epsilon$ are defined as

$$ c^2 \equiv 4 - \frac{2(p+1)(D-p-3)}{D-2}, \quad (2.3a) $$

$$ \epsilon \equiv \begin{cases} + & \text{for electric } p\text{-branes} \\ - & \text{for magnetic } p\text{-branes} \end{cases}. \quad (2.3b) $$

From the classical action (2.1), the field equations are obtained as

$$ \tilde{R}_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + \frac{1}{2} \frac{e^{c\epsilon c\phi}}{(p+2)!} \left[ (p+2) F_{MA_2 \cdots A_{p+2}} F_{N A_2 \cdots A_{p+2}} - \frac{p+1}{D-2} \tilde{g}_{MN} F_{(p+2)}^2 \right], \quad (2.4a) $$

$$ d \ast d\phi - \frac{e\epsilon}{2(p+2)!} e^{c\epsilon c\phi} \ast F_{(p+2)} \wedge F_{(p+2)} = 0, \quad (2.4b) $$

$$ d \left[ e^{c\epsilon c\phi} \ast F_{(p+2)} \right] = 0, \quad (2.4c) $$

where $\tilde{R}_{MN}$ denotes the Ricci tensor with respect to the $D$-dimensional metric $\tilde{g}_{MN}$.

Let us suppose the metric as follows:

$$ ds^2 = [h(x,z)]^a q_{\mu\nu}(X) dx^\mu dx^\nu + [h(x,z)]^b u_{ab}(Z) dz^a dz^b. \quad (2.5) $$

Here $ds^2$ denotes the $D$-dimensional metric in the Einstein frame, $X$ is a $(p+1)$-dimensional spacetime with the metric $q_{\mu\nu}$ and the coordinates $x^\mu$, and $Z$ is a $(D-p-1)$-dimensional space with the metric $u_{ab}$ and the coordinates $z^a$. The parameters $a$ and $b$ are given by

$$ a = -\frac{D-p-3}{D-2}, \quad b = \frac{p+1}{D-2}. \quad (2.6) $$
Then the parameter $c$ in (2.3a) is rewritten as

$$c^2 = 4 \left[ 1 - \frac{1}{2} ab(D - 2) \right].$$

(2.7)

The metric ansatz (2.5) is a generalization of static $p$-branes with a dilaton coupling [16]. The dynamical brane solution can be obtained only in the particular case with (2.6), while the static brane solution does not necessarily require the condition (2.6).

For $\phi$ and $F(p+2)$, suppose the following forms,

$$e^{\phi} = h^{c/2},$$

(2.8a)

$$F(p+2) = d(h^{-1}) \wedge \Omega(X),$$

(2.8b)

where $\Omega(X)$ is the volume $(p+1)$-form,

$$\Omega(X) = \sqrt{-q} \, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p, \quad q \equiv \det q_{\mu \nu}.$$  

(2.9)

Under the supposition, the metric in (2.5) should satisfy

$$R_{\mu \nu}(X) = 0, \quad R_{ab}(Z) = 0,$$

(2.10a)

$$h(x, z) = h_0(x) + h_1(z), \quad D_\mu D_\nu h_0 = 0, \quad \Delta_Z h_1 = 0,$$

(2.10b)

where $D_\mu$ is the covariant derivative with $q_{\mu \nu}$ and the Laplacian $\Delta_Z$ is defined on the $Z$ space. Similarly, $R_{\mu \nu}(X)$ and $R_{ab}(Z)$ are the Ricci tensors associated with $q_{\mu \nu}$ and $u_{ab}$, respectively.

For later argument, we concentrate on a simple case specified with

$$q_{\mu \nu} = \eta_{\mu \nu}, \quad u_{ab} = \delta_{ab},$$

where $\eta_{\mu \nu}$ is the $(p-1)$-dimensional Minkowski metric, and $\delta_{ab}$ are the $(D-p-1)$-dimensional Euclidean metric. The general solution of (2.10) is given by [17, 18]

$$h(x, z) = \begin{cases} 
\beta_\mu x^\mu + \bar{\beta} + \sum_l \frac{M_l}{|z^a - z^a_l|^{D-p-3}} & \text{(for } D - p \neq 3) \\
\beta_\mu x^\mu + \bar{\beta} + \sum_l M_l \ln |z^a - z^a_l| & \text{(for } D - p = 3) 
\end{cases}$$

(2.11)

where $\beta_\mu$, $\bar{\beta}$ and $M_l$ are constant parameters. The distance $|z^a - z^a_l|$ is defined as

$$|z^a - z^a_l| = \sqrt{(z^1 - z^1_l)^2 + (z^2 - z^2_l)^2 + \cdots + (z^{D-p-1} - z^{D-p-1}_l)^2}.$$  

When $\beta_0 \neq 0$, the solution becomes time-dependent. For $\beta_\mu = 0$, the solution describes static BPS $p$-branes with charges $M_l$, which are aligned in parallel.
In general, the dilaton does not vanish. There is no dilaton contribution in the special case with \( c = 0 \), which contains

\[
\begin{align*}
    p &= 2 \quad \text{and} \quad p = 5 \quad \text{for} \quad D = 11, \\
    p &= 3 \quad \text{for} \quad D = 10.
\end{align*}
\]

In the following argument, we will set that \( \beta_0 \neq 0 \) and the other components are zero, for simplicity. In Sec. 6, the contribution of the spatial components of \( \beta_\mu \) will be discussed.

## 3 Probe branes on dynamical brane backgrounds

The dynamics of a probe \( p \)-brane is investigated on the dynamical brane backgrounds without the scalar-field contribution.

### 3.1 General setup

We consider a \( p \)-brane moving on a \( D \)-dimensional dynamical brane background with the \((p+1)\)-form \( A_{(p+1)} \). The analysis includes probe D-branes in ten dimensions with a constant dilaton and M-branes in eleven dimensions. For these cases, there is no distinction between the string frame and the Einstein frame. The world-volume gauge field is not induced by an NS-NS two-form field and it is not taken into account.

The probe \( p \)-brane action (with a constant dilaton or without a dilaton) is given by

\[
S_p = \int d\tau d^p\sigma \mathcal{L} \left( \partial_\tau x^M, \partial_\alpha x^M \right) = -T_p \int d^{p+1}\sigma \sqrt{-\det \bar{g}_{\mu\nu}} + T_p \int \bar{A},
\]

where \( T_p \) is the \( p \)-brane tension. Here \( \bar{g}_{\mu\nu} \) and \( \bar{A} \) are the pull-back of the metric and the \( p \)-form,

\[
\bar{g}_{\mu\nu} = g_{MN} \partial_\mu x^M \partial_\nu x^N, \quad \bar{A}_{\mu_1 \cdots \mu_{p+1}} = A_{M_1 \cdots M_{p+1}} \partial_{\mu_1} x^{M_1} \cdots \partial_{\mu_{p+1}} x^{M_{p+1}}.
\]

The \( p \)-brane world-volume is described by the coordinate \( \sigma^\mu (\mu = 0, \ldots, p) \), and it is embedded into the target spacetime in \( D \) dimensions via the functions \( x^M(\tau, \sigma^\alpha) (M = 0, \ldots, D - 1) \).

It should be noted that the static gauge \( x^\mu = \sigma^\mu \) can be taken even for the dynamical backgrounds, as explained in Sec. 3.2 in detail, though an additional constraint appears in comparison to the static backgrounds. Assuming that the static gauge is available, the induced metric and the \( p \)-form gauge-field are expanded as

\[
\begin{align*}
    \bar{g}_{\mu\nu} &= g_{\mu\nu} + g_{ab} \partial_\mu x^a \partial_\nu x^b \quad (a, b = p + 1, \ldots, D - 1), \\
    \bar{A}_{\mu_1 \cdots \mu_{p+1}} &= A_{\mu_1 \cdots \mu_{p+1}} + A_{\mu_1 \cdots \mu_p} \partial_{\mu_{p+1}} x^a + \cdots.
\end{align*}
\]
Note that \( g_{\mu a} = 0 \) for the dynamical backgrounds concerned with the present analysis.

Then the \( p \)-brane action (3.1a) is expanded as
\[
S_p = -T_p \int d^{p+1}\sigma \left[ V + \frac{1}{2} \sqrt{-\text{det} g_{\mu\nu}} g^{\mu\sigma} g_{ab} \partial_\mu x^a \partial_\sigma x^b + \cdots \right],
\]
(3.4a)
\[
V = \sqrt{-\text{det} g_{\mu\nu} - \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1}} A_{\mu_1 \cdots \mu_{p+1}}.}
\]
(3.4b)

The potential \( V \) gives rise to the velocity-independent force that acts on the probe \( p \)-brane. The condition for no velocity-independent force implies that \( V = \text{constant} \). This condition is equivalent to the constraint coming from the static gauge, as shown later.

### 3.2 Static gauge and constraint condition

The static gauge is argued for the dynamical brane backgrounds. The consistency of the static gauge leads to an additional constraint, which is equivalent to the supersymmetric intersecting rules for the static branes.

From the \( p \)-brane action (3.1a), the classical equations of motion are obtained,
\[
\frac{\partial}{\partial \tau} \frac{\partial L}{\partial x^M} + \frac{\partial}{\partial \sigma^\alpha} \frac{\partial L}{\partial x^M} = 0,
\]
(3.5)
with the coordinates \( \tau \) and \( \sigma^\alpha (\alpha = 1, \cdots, p) \). Here \( \tau \) and \( \sigma^\alpha \) denote time and spatial directions of the \( p \)-brane world-volume and the range of \( \sigma^\alpha \) is taken as \( -\infty < \sigma^\alpha < +\infty \).

It is necessary to impose suitable boundary conditions at the spatial infinity of the \( p \)-brane world-volume. A possible choice is the free endpoints for all of the components of \( M \)
\[
\lim_{\sigma_0^\alpha \to \infty} \frac{\partial L}{\partial x^M}(\tau, \pm \sigma_0^\alpha) = 0,
\]
(3.6)
where \( \sigma_0^\alpha \) denote the endpoints of the world-volume. Any constraints are not imposed on the variation \( \delta x^M(\tau, \sigma_0) \) of the world-volume coordinate at the spatial endpoints of the \( p \)-brane.

### A constraint coming from the static gauge

Let us argue the static gauge for the probe \( p \)-brane on dynamical brane backgrounds. As a concrete example of the dynamical backgrounds, consider the dynamical \( p \)-brane background (2.5). The probe \( p \)-brane is supposed to be parallel to the background \( p \)-brane for simplicity.

As a consistency of the static gauge for the background (2.5), it is necessary to check the following static configuration
\[
t = \tau, \quad x^\alpha = \sigma^\alpha, \quad x^a = \text{const.}
\]
(3.7)
satisfies the classical equations of motion for the probe $p$-brane. By substituting the configuration $\xi = -\frac{(p + 1)(D - p - 3)}{D - 2}$ into the equations of motion (3.5), the following condition is obtained,

$$\partial_t \left( h^{\frac{\xi}{2}} - h^{-1} \right) = 0,$$

where $\xi$ is defined as

$$\xi \equiv -\frac{(p + 1)(D - p - 3)}{D - 2}. \tag{3.9}$$

For the dynamical background, the constraint (3.8) is not satisfied in general, while it is trivially satisfied for the static backgrounds. The special case is $\xi = -2$. The static gauge is available if and only if $\xi = -2$. Note that the first contribution in (3.8) comes from the Nambu-Goto part of the action and the second one from the coupling to the $p$-form gauge field. Thus, if the gauge-field contribution is ignorable (for example, when the probe $p$-brane is not parallel to the background $p$-brane), then $\xi = 0$ is required for the static gauge.

The condition $\xi = -2$ is realized for the following cases:

- $p = 2\text{ or } 5$ for $D = 11$, \tag{3.10a}
- $p = 3$ for $D = 10$. \tag{3.10b}

Thus the potential is trivial when a probe M2-brane (M5-brane) is parallel to the background M2-brane (M5-brane) in eleven dimensions, and when a probe D3-brane is parallel to the background D3-brane.

**Expanding the probe $p$-brane action**

Assuming that $\xi = -2$ and the static gauge, let us expand the full action (3.1a).

The metric in (2.5) is diagonal and the determinant part in (3.1a) is expanded as

$$\sqrt{-\det (g_{\mu\nu})} = \sqrt{-\det g_{\mu\nu}} \left[ 1 + \frac{1}{2} g^{\rho\sigma} g_{ab} \partial_\rho x^a \partial_\sigma x^b + \frac{1}{8} (g^{\rho\sigma} g_{ab} \partial_\rho x^a \partial_\sigma x^b)^2 - \frac{1}{4} g^{\rho\sigma} g^{\alpha\beta} g_{cd} g_{ab} \partial_\rho x^a \partial_\alpha x^b \partial_\sigma x^c \partial_\beta x^d + \cdots \right]$$

$$= h^{\xi/2} \left[ 1 + \frac{1}{2} h \eta^{\rho\sigma} \delta_{ab} \partial_\rho x^a \partial_\sigma x^b + \frac{1}{8} h^2 \eta^{\rho\sigma} \eta^{\alpha\beta} \delta_{cd} \delta_{ab} \partial_\rho x^a \partial_\sigma x^c \partial_\alpha x^b \partial_\beta x^d + \cdots \right], \tag{3.11}$$

where “...” denotes higher-order terms in derivatives. Note that the condition $\xi = -2$ indicates that the second-order terms in derivatives vanish and the higher-order terms start from the fourth order.
Then the coupling term to the gauge-field in (3.1a) is expanded as
\[
T_p \int \bar{A} = T_p \int d^{p+1} \sigma \frac{1}{(p+1)!} \epsilon^{\nu_0 \nu_1 \cdots \nu_p} A_{\mu_0 \mu_1 \cdots \mu_p} \frac{\partial x^\mu_0}{\partial \sigma^{\nu_0}} \frac{\partial x^\mu_1}{\partial \sigma^{\nu_1}} \cdots \frac{\partial x^\nu_p}{\partial \sigma^{\nu_p}}
\]
\[
= T_p \int d^{p+1} x \ h^{-1},
\]
(3.12)
where we have used (2.8b).

In total, the full action (3.4) is expanded as
\[
S_p = T_p \int d^{p+1} x \left[ -h^{\xi/2} \left\{ 1 + \frac{1}{2} h \eta^{\rho \sigma} \delta_{ab} \partial_\rho x^a \partial_t x^b \\
+ \frac{1}{8} h^2 \eta^{\alpha \beta} \delta_{ab} \delta_{cd} \left( \partial_\rho x^a \partial_\sigma x^b \partial_\alpha x^c \partial_\beta x^d - 2 \partial_\rho x^a \partial_\alpha x^b \partial_\sigma x^c \partial_\beta x^d \right) + \cdots \right\} + h^{-1} \right] \]
\[
= T_p \int d^{p+1} x \left[ -h^{\xi/2} + h^{-1} \right] + \text{derivative terms},
\]
(3.13)
and the potential is obtained as
\[
V = h^{\xi/2} - h^{-1}.
\]

For the static gauge condition, \( \xi = -2 \). Thus the non-derivative corrections are canceled out and the force starts at the fourth order in derivatives. This is the same result as in the static case [13]. This indicates that the RR charge is equal to the tension of the probe \( p \)-brane. In other words, the BPS condition is obtained from the cancellation of the non-derivative part of the potential even for the dynamical backgrounds.

When the probe \( p \)-brane is not parallel to the background \( p \)-branes, the potential (3.14) receives the contribution only from the Nambu-Goto part because the coupling to \( A_{(p+1)} \) vanishes. The static gauge implies that \( \xi = 0 \) and then the velocity-independent force vanishes. The condition \( \xi = 0 \) implies \( p = 7 \) (D7-brane) in \( D = 10 \). However, the D7-brane background contains a non-trivial dilaton and hence this case is not included in the analysis here. The analysis including the dilaton is performed in Sec. 5.

### 3.3 Probe \( p_s \)-branes on dynamical \( p_r \)-brane backgrounds

The next case we consider is a probe \( p_s \)-brane moving on dynamical \( p_r \)-brane backgrounds with a constant dilaton or without the scalar field.

The \( p_s \)-brane action is now given by
\[
S_{p_s} = -T_{p_s} \int d^{p_s+1} \sigma \sqrt{- \det \bar{g}_{\mu \nu}} + T_{p_s} \int \bar{A} \equiv \int d\tau d^{p_s} \sigma \mathcal{L} \left( \partial_\tau x^M, \partial_\sigma x^M \right),
\]
(3.14)
where $T_p$ is the $p_s$-brane tension. Here $\bar{g}_{\mu\nu}$ and $\bar{A}$ are the pull-back of the spacetime metric and the $p$-form gauge field,

$$\bar{g}_{\mu\nu} = g_{MN}\partial_\mu x^M \partial_\nu x^N, \quad \bar{A}_{\mu_1\ldots\mu_{p_s+1}} = A_{M_1\ldots M_{p_s+1}} \partial_{\mu_1} x^{M_1} \ldots \partial_{\mu_{p_s+1}} x^{M_{p_s+1}}. \quad (3.15)$$

Suppose that the probe $p_s$-brane overlaps with the background $p_r$-brane in $\bar{p}$ spatial directions. Then it is helpful to express the dynamical $p_r$-brane background as follows:

$$ds^2 = \left[ h_r(t, z)^{\alpha\rho} \left[ \eta_{\mu\nu}(X) dx^\mu dx^\nu + \delta_{ij}(Y) dy^i dy^j \right] \right] + h_r^{-1}(t, z) \Omega(X) \wedge \Omega(Y). \quad (3.16a)$$

$$A_{(p_r+1)} = h_r^{-1}(t, z) \Omega(X) \wedge \Omega(Y). \quad (3.16b)$$

Here $\eta_{\mu\nu}$ is the $(\bar{p} + 1)$-dimensional Minkowski metric and $\delta_{ij}(Y)$ is the $(p_r - \bar{p})$-dimensional flat metric. Then $\delta_{mn}(W)$ and $\delta_{ab}(Z)$ are the $(p_s - p_r)$-dimensional and the $(9 - p_s)$-dimensional flat metrics, respectively. The volume forms on the $X$ and $Y$ spaces are given by $\Omega(X)$ and $\Omega(Y)$, respectively. The constants $a_r$ and $b_r$ are given by

$$a_r = -\frac{D - p_r - 3}{D - 2}, \quad b_r = \frac{p_r + 1}{D - 2}. \quad (3.17)$$

The background $p_r$-branes extend on the $X$ and $Y$ spaces, while the probe $p_s$-brane extends on the $X$ and $W$ spaces.

For the static configuration,

$$\sigma^0 = x^0 = t, \quad \sigma^\alpha = x^\alpha (\alpha = 1, \ldots, \bar{p}),$$

$$\sigma^m = v^m (m = \bar{p} + 1, \ldots, p_s), \quad x^a = \text{const.} \quad (3.18)$$

the equation of motion for the probe $p_s$-brane leads to the following condition,

$$\partial_\rho h^{-\chi'/2} = 0. \quad (3.19)$$

Here $\chi'$ is defined as

$$\chi' \equiv p_r + 1 - \frac{(p_r + 1)(p_s + 1)}{D - 2}. \quad (3.20)$$

For the dynamical background, this condition is not automatically satisfied in comparison to the static backgrounds. Therefore the condition (3.19) is satisfied if and only if $\chi' = 0$.

Then the total action (3.14) is expanded as

$$S_{p_s} = -T_{p_s} \int d^{p_s+1}x h^{\chi'/2} \left[ 1 + \frac{1}{2} h \eta^{\rho\sigma} \delta_{ab} \partial_\rho x^a \partial_\sigma x^b 
+ \frac{1}{8} h^2 \eta^{\rho\sigma} \eta^{\alpha\beta} \delta_{ab} \delta_{cd} (\partial_\rho x^a \partial_\sigma x^b \partial_\alpha x^c \partial_\beta x^d - 2 \partial_\rho x^a \partial_\alpha x^b \partial_\sigma x^c \partial_\beta x^d) + \ldots \right] \quad (3.21)$$

$$= -T_{p_s} \int d^{p_s+1}x h^{\chi'/2} + \text{derivative terms}. \quad (3.21)$$
“· · ·” denotes higher derivative terms. One can read off the potential $V$ as
\[ V(t, z) = h_r^{-\chi'/2}. \] (3.22)

From the consistency to the static gauge, $\chi' = 0$ and the velocity-independent force vanishes. Note that the derivative corrections start from the second-order.

The condition $\chi' = 0$ implies that the overlapping dimension $p$ is described as
\[ \bar{p} = \frac{(p_r + 1)(p_s + 1)}{D - 2} - 1. \] (3.23)

The relation (3.23) is equivalent to the supersymmetric intersecting condition for the static branes with a constant dilaton or without the scalar field.

In the next section, we will discuss the M-brane cases in the eleven-dimensional supergravity, where there is no contribution from the scalar field.

## 4 Probe M-branes

This section considers the dynamics of probe M-branes on dynamical M-brane backgrounds in eleven dimensions. We argue the consistency condition for the static gauge.

### 4.1 M2-brane probes

The first case is a probe M2-brane moving on dynamical M-brane backgrounds.

#### 4.1.1 An M2-brane probe on the dynamical M2-brane background

Let us first consider a probe M2-brane moving on the dynamical M2-brane background,
\[ ds^2 = h_2^{1/3}(t, z) \left[ h_2^{-1}(t, z)\eta_{\mu\nu}(X)dx^\mu dx^\nu + \delta_{ab}(Z)dz^a dz^b \right], \] (4.1a)
\[ F(4) = dA(3) = d \left[ h_2^{-1}(t, z) \right] \wedge \Omega(X), \] (4.1b)

where $\eta_{\mu\nu}$ is a three-dimensional Minkowski spacetime metric, and $\delta_{ab}$ is the eight-dimensional flat metric, and $\Omega(X)$ denotes the volume three-form.

Suppose that the probe M2-brane and the background M2-branes extend on the $X$ space. Then, with the static gauge,
\[ t = \tau, \quad x^\alpha = \sigma^\alpha \quad (\alpha = 1, 2), \quad x^a = \text{const.}, \] (4.2)
the equation of motion for the probe M2-brane leads to the trivial condition,

$$\partial_t \left( h_2^{-1} - h_2^{-1} \right) = 0. \quad (4.3)$$

With the static gauge (4.2), the potential $V$ vanishes:

$$V = \sqrt{-\det \bar{g}} - \frac{1}{3!} \epsilon_{\mu_1 \mu_2 \mu_3} A_{\mu_1 \mu_2 \mu_3}$$
$$= h_2^{-1} - h_2^{-1} = 0. \quad (4.4)$$

Thus there is no velocity-independent force in this case.

Next, let us consider the case that the probe M2-brane is completely orthogonal to the dynamical M2-brane background. It is helpful to rewrite the metric and the gauge field as [18]

$$ds^2 = h_2^{1/3} (t, z) \left[ h_2^{-1} (t, z) \left\{ -dt^2 + \delta_{ij} (Y) dy^i dy^j \right\} + \delta_{mn} (W) dv^m dv^n + \delta_{ab} (Z) dz^a dz^b \right], \quad (4.5a)$$
$$F(4) = dA(3) = d \left[ h_2^{-1} (t, z) \right] \wedge dt \wedge \Omega (Y), \quad (4.5b)$$

where $\delta_{ij} (Y)$ and $\delta_{mn} (W)$ are two-dimensional flat metrics, and $\delta_{ab} (Z)$ is the six-dimensional flat metric. Then $\Omega (Y)$ denotes the volume two-form.

Suppose that the probe M2-brane extends on the $W$ space, while the background M2-branes extend on the $Y$ space. Then for the static configuration,

$$t = \tau, \quad v^m = \sigma^m \quad (m = 1, 2), \quad x^a = \text{const.} \quad (4.6)$$

the equation of motion for the probe M2-brane becomes

$$\partial_t (h_2) = 0. \quad (4.7)$$

Then Eq. (4.7) is automatically satisfied and hence the potential $V$ becomes a constant. Therefore there is no velocity-independent force again. This result agrees with the intersection rule [19] (See also [18] for the dynamical brane background),

$$M2 \cap M2 = 0. \quad (4.8)$$

Finally, let us consider the case that the probe M2-brane probe shares one spatial direction with the background M2-brane. It is convenient to rewrite the background metric and the gauge field as

$$ds^2 = h_2^{1/3} (t, z) \left[ h_2^{-1} (t, z) \left\{ -dt^2 + dx^2 + dy^2 \right\} + dv^2 + \delta_{ab} (Z) dz^a dz^b \right], \quad (4.9a)$$
$$F(4) = dA(3) = d \left[ h_2^{-1} (t, z) \right] \wedge dt \wedge dx \wedge dy, \quad (4.9b)$$
where $\delta_{ab}(Z)$ is the seven-dimensional flat metric.

Suppose that the probe M2-brane extends on $x$ and $v$, while the background M2-branes extend on $x$ and $y$. The static gauge condition
\[
\tau = t, \quad \sigma^1 = x^1 = x, \quad \sigma^2 = x^2 = v, \quad x^a = \text{const.} \quad (4.10)
\]
leads to the following condition,
\[
\partial_t h_2^{-1/2} = 0. \quad (4.11)
\]
This condition is not satisfied in general and the static gauge is not applicable. This fact implies that the velocity-independent force remains.

4.1.2 An M2-brane probe on the dynamical M5-brane background

Let us consider a probe M2-brane moving on the dynamical M5-brane background.

As the first case, suppose that the probe M2-brane is parallel to the background M5-branes. Then it is convenient to express the metric and the gauge field like
\[
ds^2 = h_2^{2/3}(t, z) \left[ h_5^{-1}(t, z) \left\{ \eta_{\mu\nu}(X) dx^\mu dx^\nu + \delta_{ij}(Y) dy^i dy^j \right\} + \delta_{ab}(Z) dz^a dz^b \right] , \quad (4.12a)
\]
\[
F(4) = dA(3) = * \left[ dh_5^{-1}(t, z) \wedge \Omega(X) \wedge \Omega(Y) \right] . \quad (4.12b)
\]
Here $\eta_{\mu\nu}$ is the three-dimensional Minkowski metric. The flat metrics $\delta_{ij}$ and $\delta_{ab}$ are defined in three and five dimensions, respectively. Then $\Omega(X)$ and $\Omega(Y)$ are the volume three-forms. The background M5-branes extend on the X space and the Y space while the probe M2-brane extends on the X space.

For the static configuration,
\[
\sigma^0 = x^0 = t , \quad \sigma^\alpha = x^\alpha \quad (\alpha = 1, 2) , \quad x^a = \text{const.} , \quad (4.13)
\]
the equation of motion for the probe M2-brane leads to the condition,
\[
\partial_t \left( h_5 \right)^{-1/2} = 0 . \quad (4.14)
\]
This condition is not satisfied in general and the static gauge is not applicable. This fact implies that there is the velocity-independent force.

Next, suppose that the probe M2-brane overlaps with the background M5-branes in one spatial direction. It is helpful to rewrite the background metric and the gauge field like
\[
ds^2 = h_2^{2/3}(t, z) \left[ h_5^{-1}(t, z) \left\{ \eta_{\mu\nu}(X) dx^\mu dx^\nu + \delta_{ij}(Y) dy^i dy^j \right\} + d\tau^2 + \delta_{ab}(Z) dz^a dz^b \right] , \quad (4.15a)
\]
\[
F(4) = dA(3) = * \left[ dh_5^{-1}(t, z) \wedge \Omega(X) \wedge \Omega(Y) \right] . \quad (4.15b)
\]
Here $\eta_{\mu \nu}$ is the two-dimensional Minkowski metric, and $\delta_{ij}$ and $\delta_{ab}$ are the four-dimensional flat metrics. $\Omega(Y)$ is the volume four-form. The background M5-branes extend on the $t$, $x$ coordinates and the Y space while the probe M2-brane extends on the $t$, $x$ and $v$ coordinates.

For the static configuration,

$$\sigma^0 = x^0 = t, \quad \sigma^1 = x^1 = x, \quad \sigma^2 = x^2 = v, \quad x^\alpha = \text{const.}, \quad (4.16)$$

the equation of motion for the probe M2-brane leads to the trivial condition,

$$\partial_t (h_5)^0 = 0. \quad (4.17)$$

Thus the static gauge is available. Then the potential $V$ is evaluated as

$$V = \text{const.}, \quad (4.18)$$

and hence there is no velocity-independent force. This is consistent with the intersection rule for the M2-M5-brane system $[20, 21]$ (See also $[18]$ for the dynamical brane background),

$$M2 \cap M5 = 1. \quad (4.19)$$

### 4.2 M5-brane probes

The next is to consider a probe M5-brane moving on dynamical M-brane backgrounds. Similarly, the consistency condition of the static gauge is equivalent to the supersymmetric intersecting rules for the static M5-branes.

#### 4.2.1 An M5-brane probe on the dynamical M5-brane background

Let us consider a probe M5-brane moving on the dynamical M5-brane background,

$$ds^2 = h_5^{2/3}(t, z) \left[ h_5^{-1}(t, z) \eta_{\mu \nu}(X)dx^\mu dx^\nu + \delta_{ab}(Z)dz^a dz^b \right], \quad (4.20a)$$

$$F(4) = dA(3) = * \left[ dh_5^{-1}(t, z) \wedge \Omega(X) \right], \quad (4.20b)$$

where $\eta_{\mu \nu}$ is the metric of six-dimensional Minkowski spacetime $X$, and $\delta_{ab}$ is the metric of the five-dimensional flat space $Z$. Then $\Omega(X)$ is the volume six-form. The background M5-branes extend on the X space.

First of all, suppose that the probe M5-brane is parallel to the background M5-branes and extends on the X space. Then, for the static configuration,

$$\sigma^0 = x^0 = t, \quad \sigma^\alpha = x^\alpha \quad (\alpha = 1, \cdots, 5), \quad x^\alpha = \text{const.} \quad (4.21)$$
the equation of motion for the probe M5-brane leads to the trivial condition,

\[ \partial_t \left( h_5^{-1} - h_5^{-1} \right) = 0, \]  

(4.22)

and hence the static gauge does not provide any additional constraint. With the static gauge, the potential \( V \) vanishes like

\[ V(t, z) = \left[ h_5^{-1/6}(t, z) \right]^6 - h_5^{-1}(t, z) = 0, \]  

(4.23)

and there is no velocity-independent force.

The next is the case that a probe M5-brane overlaps with the background M5-brane in \( \bar{p} \) spatial directions. It is helpful to rewrite the background metric and the gauge field as [18]

\[
\begin{align*}
    ds^2 &= h_5^{2/3}(t, z) \left[ h_5^{-1}(t, z) \left\{ \eta_{\mu\nu}(X)dx^\mu dx^\nu + \delta_{ij}(Y)dy^i dy^j \right\} ight. \\
    &\quad \left. +\delta_{mn}(W)dv^m dv^n + \delta_{ab}(Z)dz^a dz^b \right], \\
    F_{(4)} &= dA_{(3)} = * \left[ dh_5^{-1}(t, z) \wedge \Omega(X) \wedge \Omega(Y) \right],
\end{align*}
\]  

(4.24a)

(4.24b)

where \( \eta_{\mu\nu} \) is the \((\bar{p} + 1)\)-dimensional Minkowski metric. Then \( \delta_{ij}, \delta_{mn} \) and \( \delta_{ab} \) are flat metrics in \((5 - \bar{p}), (5 - \bar{p})\) and \( \bar{p} \) dimensions, respectively. The volume forms \( \Omega(X) \) and \( \Omega(Y) \) are \((\bar{p} + 1)\)-form and \((5 - \bar{p})\)-form, respectively. The background M5-branes extend on the X space and the Y space.

Suppose that the probe M5-brane extends on the X space and the W space. Then, for the static gauge configuration,

\[
\begin{align*}
    \sigma^0 &= x^0 = t, \quad \sigma^\alpha &= x^\alpha \quad (\alpha = 1, \cdots, \bar{p}), \\
    \sigma^m &= v^m \quad (m = \bar{p} + 1, \cdots, 5), \quad x^a &= \text{const.}
\end{align*}
\]

(4.25)

the equation of motion for the probe M5-brane leads to the non-trivial condition,

\[ \partial_t \left( h_5^{(5 - \bar{p})/2} \right) = 0, \]  

(4.26)

Thus the static gauge is available if and only if \( \bar{p} = 3 \).

With the static gauge with \( \bar{p} = 3 \), the potential \( V \) vanishes like

\[
V(t, z) = \left[ h_5^{-1/6}(t, z) \right]^{\bar{p}+1} \left[ h_5^{1/3}(t, z) \right]^{5-\bar{p}} = [h_5(t, z)]^{(3-\bar{p})/2} = 1,
\]

(4.27)

and there is no velocity-independent force. This result is equivalent to the intersection rule of M5-branes [19] (See also [18] for the dynamical brane background),

\[ M5 \cap M5 = 3. \]  

(4.28)
5 Probe D-branes, F-string and NS5-brane

This section considers the dynamics of a probe brane (such as D-branes, F-string and NS5-brane) on dynamical brane backgrounds in ten dimensions. Note that the dilaton contribution is taken into account, except for the D3-brane background, in comparison to the analysis in Sec. 3.2 and 3.3.

5.1 A D-brane probe

Let us consider a probe \( D_{p_s} \)-brane moving on the dynamical brane backgrounds. First of all, we shall give the outline of our argument in this section.

The \( D_{p_s} \)-brane action, which is concerned with the present analysis, is given by

\[
S_{p_s} = -T_{p_s} \int d^{p_s+1} \sigma e^{-\phi} \sqrt{-\det (\bar{g}_{\mu\nu} + \mathcal{F}_{\mu\nu})} + T_{p_s} \int \bar{C}_{(p_s+1)},
\]

(5.1a)

\[
\bar{g}_{\mu\nu} = g_{M N} \partial_{\mu} x^M \partial_{\nu} x^N, \quad \mathcal{F}_{\mu\nu} = \bar{B}_{\mu\nu} + 2\pi \alpha' F_{\mu\nu}, \quad \bar{B}_{\mu\nu} = B_{M N} \partial_{\mu} x^M \partial_{\nu} x^N, \quad \bar{C}_{\mu_1 \cdots \mu_{p_s+1}} = C_{M_1 \cdots M_{p_s+1}} \partial_{\mu_1} x^{M_1} \cdots \partial_{\mu_{p_s+1}} x^{M_{p_s+1}}.
\]

(5.1b)

(5.1c)

Here \( T_{p_s} \) is the \( D_{p_s} \)-brane tension and \( \bar{g}_{\mu\nu} \) is the induced metric. Then \( \bar{B}_{\mu\nu} \) and \( \bar{C}_{(p_s+1)} \) are the pullback of an NS-NS two-form and a \( (p_s+1) \)-form. The world-volume gauge field is given by \( F_{\mu\nu} \). We work in the string frame hereafter.

We take the static gauge \( x^\mu = \sigma^\mu (\mu = 0, \cdots, p_s) \), though it may give rise to an additional constraint for the configuration of the probe brane as we have seen in the previous sections. Then the metric, the NS-NS two form and the \( p_s \)-form are rewritten as

\[
\bar{g}_{\mu\nu} = g_{\mu\nu} + g_{a b} \partial_{\mu} x^a \partial_{\nu} x^b, \quad \bar{B}_{\mu\nu} = B_{\mu\nu} + B_{a b} \partial_{\mu} x^a \partial_{\nu} x^b, \quad \bar{C}_{\mu_1 \cdots \mu_{p_s+1}} = C_{\mu_1 \cdots \mu_{p_s+1}} + C_{\mu_1 \cdots \mu_{p_s} a} \partial_{\mu_{p_s+1}} x^a + \cdots,
\]

(5.2a)

(5.2b)

where \( g_{a\alpha} = 0 \) is satisfied by the dynamical brane backgrounds concerned with our analysis.

Note that the world-volume gauge-field strength \( F_{\mu\nu} \) is chosen as

\[
\mathcal{F}_{\mu\nu} = 0,
\]

(5.3)

so that the probe brane does not carry the F-string charge, unless otherwise noted. It is because we are interested in the force between the probe brane without resolved F-strings and the background branes. As a result, the NS-NS two-form \( B_{M N} \) effectively couples only to the transverse directions, as noted in [13].

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Then the action in \(5.1\) is also expanded as

\[
S_{ps} = -T_{ps} \int d^{p+1}\sigma \left[ V + \frac{1}{2} \sqrt{-\det g_{\mu\nu}} \partial_{c}x^{a}\partial_{d}x^{b} + \cdots \right], \quad (5.4a)
\]

\[
V = e^{-\phi} \sqrt{-\det g_{\mu\nu}} - \frac{1}{(p_{s} + 1)!} \epsilon^{\mu_{1}\cdots\mu_{p_{s}+1}} C_{\mu_{1}\cdots\mu_{p_{s}+1}}. \quad (5.4b)
\]

Recall that the potential \(V\) determines the motion of the probe \(D_{p_{s}}\)-brane. If \(V\) is represented by a non-trivial function, it induces a velocity-independent force. On the other hand, if \(V = \text{constant}\), then there is no velocity-independent force.

We will investigate the dynamics of a probe brane on each of the possible dynamical brane backgrounds.

### 5.1.1 A \(D_{p_{s}}\)-brane probe on the dynamical \(D_{p_{r}}\)-brane background

The first is a probe \(D_{p_{s}}\)-brane moving on the dynamical \(D_{p_{r}}\)-brane background.

As a simple case, suppose that the probe \(D_{p_{s}}\)-brane is parallel to the background \(D_{p_{r}}\)-branes. The background metric and fields are given by

\[
ds^{2} = \left[h_{r}(t, z)\right]^{-1/2} \eta_{\mu\nu}(X)dx^{\mu}dx^{\nu} + \left[h_{r}(t, z)\right]^{1/2} \delta_{ab}(Z)dz^{a}dz^{b}, \quad (5.5a)
\]

\[
e^{\phi} = h_{r}^{(3-p_{r})/4}, \quad (5.5b)
\]

\[
C_{(p_{r} + 1)} = h_{r}^{-1}(t, z) \Omega(X) \wedge \Omega(Y). \quad (5.5c)
\]

Here \(\eta_{\mu\nu}\) is the \((\bar{p} + 1)\)-dimensional Minkowski metric, and \(\delta_{ab}\) are flat \((9 - \bar{p})\)-dimensional metric. Then \(\Omega(X)\) denotes the volume form on the \(X\) space. The probe \(D_{p_{s}}\)-brane and dynamical \(D_{p_{r}}\)-brane background extend on the \(X\) space.

For the static configuration,

\[
\sigma^{0} = x^{0} = t, \quad \sigma^{a} = x^{a} \quad (\alpha = 1, \cdots, \bar{p}), \quad x^{a} = \text{const.}, \quad (5.6)
\]

the equation of motion for the probe \(D_{p_{s}}\)-brane leads to the trivial condition,

\[
\partial_{t} \left(h_{r}^{-1} - h_{r}^{-1}\right) = 0, \quad (5.7)
\]

and the resulting potential \(V\) becomes constant. Thus there is no velocity-independent force.

The next case is that the probe \(D_{p_{s}}\)-brane overlaps with the background \(D_{p_{r}}\)-brane in \(\bar{p}\)
directions. Then it is convenient to rewrite the background as
\[
ds^2 = [h_r(t, z)]^{-1/2} [\eta_{\mu\nu}(X)dx^\mu dx^\nu + \delta_{ij}(Y)dy^i dy^j] \\
+ [h_r(t, z)]^{1/2} [\delta_{mn}(W)dv^m dv^n + \delta_{ab}(Z)dz^a dz^b] ,
\] (5.8a)
\[
e^\phi = h_r^{(3-p_r)/4} ,
\] (5.8b)
\[
C_{(p_r+1)} = h_r^{-1}(t, z) \Omega(X) \wedge \Omega(Y) .
\] (5.8c)

Here \(\eta_{\mu\nu}\) is the \((\bar{p} + 1)\)-dimensional Minkowski metric. The flat metrics \(\delta_{ij}\), \(\delta_{mn}\) and \(\delta_{ab}\) are defined in \((p_r - \bar{p})\), \((p_s - \bar{p})\) and \((9 + \bar{p} - p_r - p_s)\) dimensions. Then \(\Omega(X)\) and \(\Omega(Y)\) are the volume form on the X space and the Y space, respectively. The background \(D_{p_r}\)-branes extend on the X and Y spaces, while the probe \(D_{p_s}\)-brane extends on the X and W spaces.

For the static configuration,
\[
\sigma^0 = x^0 = t , \quad \sigma^\alpha = x^\alpha \quad (\alpha = 1 , \cdots , \bar{p}) ,
\]
\[
\sigma^m = v^m \quad (m = \bar{p} + 1 , \cdots , p_s) , \quad x^a = \text{const.} ,
\] (5.9)
the equation of motion for the probe \(D_{p_s}\)-brane leads to the condition,
\[
\partial_t (h_r)^{(p_r+p_s-2\bar{p}-4)/4} = 0 .
\] (5.10)
This condition is satisfied if and only if
\[
p_r + p_s - 2\bar{p} - 4 = 0 .
\] (5.11)
Under this condition, the resulting potential \(V\) is evaluated as
\[
V(t, z) = h_r^{(p_r+p_s-2\bar{p}-4)/4} = 1 ,
\] (5.12)
and hence there is no velocity-independent force. The condition in (5.11) is equivalent to a supersymmetric intersection rule for the static D-branes [20,22] (See also [23] for the dynamical brane background),
\[
D_{p_r} \cap D_{p_r} = \frac{1}{2} (p_r + p_s) - 2 .
\] (5.13)

### 5.1.2 A \(D_p\)-brane probe on the dynamical F-string background

The next is a probe \(D_p\)-brane on the dynamical F-string background.

First, suppose that the probe \(D_p\)-brane is parallel to the background F-strings. The background is given by
\[
ds^2 = h_1^{-1}(t, z) \left( -dt^2 + dx^2 \right) + \delta_{mn}(W)dv^m dv^n + \delta_{ab}(Z)dz^a dz^b ,
\] (5.14a)
\[
e^\phi = h_1^{-1/2} ,
\] (5.14b)
\[
H(3) = d \left[ h_1^{-1}(t, z) \right] \wedge dt \wedge dx .
\] (5.14c)
Here $\delta_{mn}$ and $\delta_{ab}$ are the flat metric in $(p - 1)$ and $(9 - p)$ dimensions, respectively. The background F-strings extend along the $t$ and $x$ directions, while the probe D$p$-brane extends along the $t$ and $x$ directions and on the W space.

For the static configuration,
\[
\sigma^0 = x^0 = t, \quad \sigma^1 = x, \quad \sigma^m = v^m \quad (m = 2, \cdots, p), \quad x^a = \text{const.},
\]
the equation of motion for the probe D$p$-brane leads to
\[
\partial_t h_1^{-1/2} = 0, \tag{5.16}
\]
where $x$ denotes the coordinate along the background F-strings. This condition is not satisfied and hence the static gauge is not available.

The next case is that the probe D$p$-brane orthogonally intersects the background F-strings at a point. Then it is convenient to rewrite the background like
\[
ds^2 = h_1^{-1}(t, z) \left( -dt^2 + dy^2 \right) + \delta_{mn}(W)dv^mdv^n + \delta_{ab}(Z)dz^adz^b, \tag{5.17a}
\]
\[
e^\phi = h_1^{-1/2}, \tag{5.17b}
\]
\[
H_{(3)} = d \left[ h_1^{-1}(t, z) \right] \wedge dt \wedge dy. \tag{5.17c}
\]
Here $\delta_{mn}$ and $\delta_{ab}$ are the flat $p$- and $(8 - p)$-dimensional metrics, respectively. The background F-strings extend along the $t$ and $y$ directions, while the probe D$p$-brane extends on $t$ and the W space.

For the static configuration,
\[
\sigma^0 = x^0 = t, \quad \sigma^m = v^m \quad (m = 1, \cdots, p), \quad x^a = \text{const.}, \tag{5.18}
\]
the equation of motion for the probe D$p$-brane is automatically satisfied. Thus the potential $V$ becomes a constant under the static gauge and there is no velocity-independent force. This result agrees with the intersection rule $[24]$ (See also $[23]$ for the dynamical brane background),
\[
F_1 \cap Dp = 0. \tag{5.19}
\]

### 5.1.3 A D$p$-brane probe on the dynamical NS5-brane background

Then we consider a probe D$p$-brane probe moving on the dynamical NS5-brane background. The background metric and fields are given by
\[
ds^2 = \eta_{\mu\nu}(X)dx^\mu dx^\nu + \delta_{ij}(Y)dy^idy^j + h_5(t, z) \left[ \delta_{mn}(W)dv^mdv^n + \delta_{ab}(Z)dz^adz^b \right], \tag{5.20a}
\]
\[
e^\phi = h_5^{1/2}, \tag{5.20b}
\]
\[
H_{(3)} = e^{2\phi} \ast d \left[ h_5^{-1}(t, z) \Omega(X) \wedge \Omega(Y) \right]. \tag{5.20c}
\]
Here $\eta_{\mu\nu}$ is the $(\bar{p} + 1)$-dimensional Minkowski metric. The flat metrics $\delta_{ij}$, $\delta_{mn}$ and $\delta_{ab}$ are defined in $(5 - \bar{p})$, $(p - \bar{p})$ and $(4 + \bar{p} - p)$ dimensions. Then $\Omega(X)$ and $\Omega(Y)$ denote the volume $(\bar{p} + 1)$- and $(5 - \bar{p})$-forms, respectively. The background NS5-branes extend on the X and Y spaces, while the probe Dp-brane extends on the X and W spaces.

Suppose that the probe Dp-brane overlaps with the background NS5-branes in $\bar{p}$ spatial directions (on the X space). Then, for the static configuration,

$$\begin{align*}
\sigma^0 &= x^0 = t, \\
\sigma^\alpha &= x^\alpha \quad (\alpha = 1, \cdots, \bar{p}), \\
\sigma^m &= \nu^m \quad (\alpha = \bar{p} + 1, \cdots, p), \quad x^a = \text{const.},
\end{align*}$$

the equation of motion leads to the condition,

$$\partial_t h_5^{-(\bar{p} - p + 1)/2} = 0.$$  \hspace{1cm} (5.22)

This condition is satisfied if and only if $\bar{p} = p - 1$.

With the static gauge, the potential $V$ is evaluated as

$$V(t, z) = h_5^{-(\bar{p} - p + 1)/2} = 1.$$  \hspace{1cm} (5.23)

This result agrees with the intersection rule [24] (See also [23] for the dynamical brane background),

$$\text{NS5} \cap \text{Dp} = p - 1.$$  \hspace{1cm} (5.24)

### 5.2 F-string probes

The next we consider is to study the dynamics of an F-string probe moving on dynamical brane backgrounds and to clarify the relation between the consistency condition for the static gauge and the supersymmetric intersection rules concerning F-strings.

The action of a single F-string is given by

$$S_1 = -T_1 \int d^2\sigma \sqrt{- \det (\bar{g}_{\mu\nu} + B_{\mu\nu})},$$

where $T_1$ is the F-string tension. Then $\bar{g}_{\mu\nu}$ and $B_{\mu\nu}$ are the pull-back of the spacetime metric and the NS-NS two-form,

$$\bar{g}_{\mu\nu} = g_{MN} \partial_\mu x^M \partial_\nu x^N, \quad B_{\mu\nu} = B_{MN} \partial_\mu x^M \partial_\nu x^N.$$  \hspace{1cm} (5.26)

In the following, we will consider a probe F-string moving on each of the possible dynamical brane backgrounds.
5.2.1 An F-string probe on the dynamical $D_p$-brane background

The first is a probe F-string moving on the dynamical $D_p$-brane background.

First, suppose that the probe F-string is parallel to the background $D_p$-branes. Then it is convenient to express the background like

$$ds^2 = [h(t, z)]^{-1/2} \left[ \eta_{\mu\nu}(X)dx^\mu dx^\nu + \delta_{ij}(Y)dy^i dy^j \right] + [h(t, z)]^{1/2} \delta_{ab}(Z)dz^a dz^b, \quad (5.27a)$$
$$e^\phi = h^{(3-p)/4}, \quad (5.27b)$$
$$C_{(p+1)} = h^{-1}(t, z) \Omega(X) \wedge \Omega(Y). \quad (5.27c)$$

Here $\eta_{\mu\nu}$ is the two-dimensional Minkowski metric, and $\delta_{ij}, \delta_{ab}$ denote the $(p-1)$- $(9-p)$-dimensional flat metrics. Then $\Omega(X), \Omega(Y)$ are the volume forms on the $X$ space and the $Y$ space, respectively. The background $D_p$-branes extend on the $X$ and $Y$ spaces, while the probe F-string extends on the $X$ space.

For the static configuration,

$$\sigma^0 = x^0 = t, \quad \sigma^1 = x^1 = x, \quad x^a = \text{const.}, \quad (5.28)$$

the equation of motion for the probe F-string leads to the condition in (5.16). Thus the static gauge is not available and this result indicates that the velocity-independent force remains.

The next is the case that a probe F-string orthogonally intersects the background $D_p$-branes at points. Then it is helpful to express the background as

$$ds^2 = [h(t, z)]^{-1/2} \left[ -dt^2 + \delta_{ij}(Y)dy^i dy^j \right] + [h(t, z)]^{1/2} \left[ dv^2 + \delta_{ab}(Z)dz^a dz^b \right], \quad (5.29a)$$
$$e^\phi = h^{(3-p)/4}, \quad (5.29b)$$
$$C_{(p+1)} = h^{-1}(t, z) dt \wedge \Omega(Y). \quad (5.29c)$$

Here $\delta_{ij}$ and $\delta_{ab}$ are the flat metrics in $p$ and $(8-p)$ dimensions. Then $\Omega(Y)$ is the volume forms on the $Y$ space. The background $D_p$-branes extend along the $t$ direction and on the $Y$ space, while the probe F-string extends along the $t$ and $v$ directions.

For the static configuration,

$$\sigma^0 = x^0 = t, \quad \sigma^1 = v, \quad x^a = \text{const.}, \quad (5.30)$$

the equation of motion for the probe F-string is trivially satisfied and hence the static gauge is possible. Then the potential $V$ becomes constant. This configuration satisfies the intersection rule for the $F1-Dp$-brane system (5.19).
5.2.2 An F-string probe on the dynamical F-string background

The next is a probe F-string moving on the dynamical F-string background.

First, suppose that the probe F-string is parallel to the background F-strings. Then it is convenient to express the background like

\[ ds^2 = h_1^{-1}(t, z) \left( -dt^2 + dx^2 \right) + \delta_{ab}(Z)dz^adz^b, \]  
\[ e^\phi = h_1^{-1/2}, \]  
\[ H(3) = d \left[ h_1^{-1}(t, z) \right] \wedge dt \wedge dx. \]  

(5.31a)
(5.31b)
(5.31c)

Here \( \delta_{ab} \) is the flat metric in \((9 - p)\) dimensions. The background F-strings and the probe F-string extend along the \( t \) and \( x \) directions.

For the static configuration,

\[ \sigma^0 = x^0 = t, \quad \sigma^1 = x^1 = x, \quad x^a = \text{const.}, \]  

(5.32)

the equation of motion for the probe F-string leads to the condition,

\[ \partial_t (h_1^{-1} - h_1^{-1}) = 0, \]  

(5.33)

and hence the static gauge is available and the velocity-independent force is not induced. Thus the potential \( V \) becomes constant.

The next case is that a probe F-string orthogonally intersects the background F-strings at points. Then we use the background described as

\[ ds^2 = h_1^{-1}(t, z) \left( -dt^2 + dy^2 \right) + dv^2 + \delta_{ab}(Z)dz^adz^b, \]  
\[ e^\phi = h_1^{-1/2}, \]  
\[ H(3) = d \left[ h_1^{-1}(t, z) \right] \wedge dt \wedge dy. \]  

(5.34a)
(5.34b)
(5.34c)

Here \( \delta_{ab} \) is the flat metric in seven dimensions. The background F-strings extend along the \( t \) and \( y \) directions, while the probe F-string extends along the \( t \) and \( v \) ones.

For the static configuration \((5.28)\), the equation of motion for the probe F-string leads to the condition,

\[ \partial_t h_1^{-1/2} = 0, \]  

(5.35)

and the static gauge is not available and the velocity-independent force appears again.
5.2.3 An F-string probe on the dynamical NS5-brane background

The last is a probe F-string moving on the dynamical NS5-brane background. Suppose that the probe F-string overlap with the background NS5-branes in the \( x \)-direction.

The background is given by

\[
\begin{align*}
    ds^2 &= -dt^2 + dx^2 + \delta_{ij}(Y)dy^i dy^j + h_5(t, z)\delta_{ab}(Z)dz^a dz^b, \\
    e^\phi &= h_5^{1/2}, \\
    H(3) &= e^{2\phi} * d \left[ h_5^{-1}(t, z) \, dt \wedge dx \wedge \Omega(Y) \right].
\end{align*}
\]

(5.36a)\hspace{1cm}(5.36b)\hspace{1cm}(5.36c)

Here \( \delta_{ij} \) and \( \delta_{ab} \) are the four-dimensional flat metrics, and \( \Omega(Y) \) denotes the volume four-form.

The background NS5-branes extend along the \( t \) and \( x \) directions and on the \( Y \) space, while the probe F-string extends along the \( t \) and \( x \) ones.

For the static configuration,

\[
\begin{align*}
    \sigma^0 &= x^0 = t, & \sigma^1 &= x^1 = x, & x^a &= \text{const.},
\end{align*}
\]

(5.37)

the equation for probe F-string is trivially satisfied because

\[
    \partial_t(h_5)^0 = 0.
\]

(5.38)

Then the potential \( V \) is evaluated as

\[
V(t, z) = \text{const.},
\]

(5.39)

and thus there is no velocity-independent force, when the probe F-string is parallel to the background NS5-branes. This configuration satisfies the intersection rule for the F1–NS5-brane system \[24\] (See also \[23\] for the dynamical brane background),

\[
F1 \cap \text{NS5} = 1.
\]

(5.40)

Next, suppose that the probe F-string intersects orthogonally the background NS5-branes. Then it is helpful to express the background as

\[
\begin{align*}
    ds^2 &= -dt^2 + \delta_{ij}(Y)dy^i dy^j + h_5(t, z) \left[ dv^2 + \delta_{ab}(Z)dz^a dz^b \right], \\
    e^\phi &= h_5^{1/2}, \\
    H(3) &= e^{2\phi} * d \left[ h_5^{-1}(t, z) \, dt \wedge \Omega(Y) \right].
\end{align*}
\]

(5.41a)\hspace{1cm}(5.41b)\hspace{1cm}(5.41c)
Here $\delta_{ij}$ and $\delta_{ab}$ denote the five-, and three-dimensional flat metrics, and $\Omega(Y)$ denotes the volume five-form. The background NS5-branes extend along the $t$ direction and on the $Y$ space, while the probe F-string extends along the $t$ and $v$ directions.

For the static configuration,

$$\sigma^0 = x^0 = t, \quad \sigma^1 = v, \quad x^a = \text{const.},$$

(5.42)

the equation of motion for the probe F-string leads to the condition,

$$\partial_t h_5^{1/2} = 0,$$

(5.43)

and hence the static gauge is not available and there remains the velocity-independent force.

5.3 NS5 brane probes

We finally consider a probe NS5-brane moving on dynamical brane backgrounds and present the consistency condition and potential for the static gauge involving NS5-brane probes.

The classical action of a single NS5-brane is given by

$$S_{\text{NS5}} = -T_5 \int d^6 \sigma \left[ e^{-2\phi} \sqrt{-\det \bar{g}_{\mu\nu}(X) - \frac{1}{6!} \epsilon_{\mu_1 \cdots \mu_6} B_{\mu_1 \cdots \mu_6} + \cdots} \right],$$

(5.44)

where $T_5$ is the NS5-brane tension, and $\bar{g}_{\mu\nu}$ is the induced metric and $B_{\mu_1 \cdots \mu_6}$ is the pull-back of the Hodge dual of an NS-NS two-form.

In the following, we will consider a probe NS5-brane on each of the possible dynamical brane backgrounds.

5.3.1 An NS5-brane probe on the dynamical $D_p$-brane background

The first we consider is an NS5-brane probe moving on the dynamical $D_p$-brane background.

First, suppose that the probe NS5-brane shares the background $D_p$-brane with $\bar{p}$ spatial dimensions. Then it is convenient to express the $D_p$-brane background like

$$ds^2 = [h(t, z)]^{-1/2} \left[ \eta_{\mu\nu}(X) dx^\mu dx^\nu + \delta_{ij}(Y) dy^i dy^j \right]$$

$$+ [h(t, z)]^{1/2} \left[ \delta_{mn}(W) dv^m dv^n + \delta_{ab}(Z) dz^a dz^b \right],$$

(5.45a)

$$e^\phi = h^{(3-p)/4},$$

(5.45b)

$$C_{(p+1)} = h^{-1}(t, z) \Omega(X) \wedge \Omega(Y).$$

(5.45c)
η_{\mu\nu} is the \((\bar{p} + 1)\)-dimensional Minkowski metric, \( \delta_{ij}, \delta_{mn}, \delta_{ab} \) denote the \((p - \bar{p})\)-, \((5 - \bar{p})\)-, \((4 + \bar{p} - p)\)-dimensional flat metrics. Then \( \Omega(X) \) and \( \Omega(Y) \) are the volume forms on the X and Y spaces, respectively. The background Dp-branes extend on the X and Y spaces, while the probe NS5-brane extends on the X and W spaces.

For the static configuration,

\[
\begin{align*}
\sigma^0 &= x^0 = t, \quad \sigma^\alpha &= x^\alpha \quad (\alpha = 1, \ldots, \bar{p}), \\
\sigma^m &= v^m \quad (m = \bar{p} + 1, \ldots, 5), \quad x^a = \text{const.},
\end{align*}
\]

the equation of motion for the probe NS5-brane leads to

\[
\partial_t h^{-(\bar{p} - p + 1)/2} = 0.
\]

Thus the static gauge is available if and only if \( \bar{p} = p - 1 \).

With the static gauge and the condition \( \bar{p} = p - 1 \), the potential \( V \) is evaluated as

\[
V(t, z) = h^{-(\bar{p} - p + 1)/2}(t, z) = 1,
\]

and hence the velocity-independent force vanishes. The condition \( \bar{p} = p - 1 \) also agrees with the intersection rule for the NS5–Dp-brane system given in (5.24).

### 5.3.2 An NS5-brane probe on the dynamical F-string background

The next is a probe NS5-brane moving on the dynamical F-string background.

Suppose that the NS5-brane probe intersects orthogonally the background F-strings. It is helpful to express the background as

\[
\begin{align*}
\text{d}s^2 &= h_1^{-1}(t, z) \left( -\text{d}t^2 + \text{d}y^2 \right) + \delta_{mn}(W)\text{d}v^m\text{d}v^n + \delta_{ab}(Z)dz^a\text{d}z^b, \\
e^\phi &= h_1^{-1/2}, \\
H(3) &= d \left[ h_1^{-1}(t, z) \right] \wedge \text{d}t \wedge \text{d}y.
\end{align*}
\]

Here \( \delta_{mn} \) and \( \delta_{ab} \) are the flat metrics in five and three dimensions, respectively. Now the background F-strings extend along the \( t \) and \( y \) directions, while the probe NS5-brane extends along the \( t \) direction and on the W space.

For the static configuration,

\[
\begin{align*}
\sigma^0 &= x^0 = t, \quad \sigma^m &= v^m \quad (m = 1, \ldots, 5), \quad x^a = \text{const.},
\end{align*}
\]

23
the equation of motion for the probe NS5-brane leads to

\[ \partial_t h_1^{1/2} = 0, \]

and this condition is not satisfied. Hence the static gauge is not available and thus the velocity-independent force remains.

Let us next consider the case that the probe NS5-brane overlaps with the background F-strings in one spatial dimension. Then it is convenient to express the background as

\[ ds^2 = h_1(t, z) \left( -dt^2 + dx^2 \right) + \delta_{mn}(W)dv^mdv^n + \delta_{ab}(Z)dz^adz^b, \]

\[ e^\phi = h_1^{-1/2}, \]

\[ H_{(3)} = d \left[ h_1^{-1}(t, z) \right] \wedge dt \wedge dx. \]

Here \( \delta_{mn} \) and \( \delta_{ab} \) are the flat metrics in four dimensions. Now the background F-strings extend along the \( t \) and \( x \) coordinates, while the probe NS5-brane extends along the \( t \) and \( x \) coordinates and on the W space.

For the static configuration,

\[ \sigma^0 = x^0 = t, \quad \sigma^1 = x^1 = x, \]

\[ \sigma^m = v^m \quad (m = 1, \cdots, 4), \quad x^a = \text{const.}, \]

the equation of motion for the probe NS5-brane is trivially satisfied and the potential becomes a constant. That is, there is no velocity-independent force. This configuration satisfies the intersection rule given in (5.40).

### 5.3.3 An NS5-brane probe on the dynamical NS5-brane background

The last case is that a probe NS5-brane moves on the dynamical NS5-brane background.

Let us first consider the case that the probe NS5-brane is parallel to the background NS5-branes. Then it is helpful to express the background as

\[ ds^2 = \eta_{\mu\nu}(X)dx^\mu dx^\nu + h_5(t, z)\delta_{ab}(Z)dz^adz^b, \]

\[ e^\phi = h_5^{1/2}, \]

\[ H_{(3)} = e^{2\phi} \ast d \left[ h_5^{-1}(t, z) \Omega(X) \right]. \]

Here \( \eta_{\mu\nu} \) is the six-dimensional Minkowski metric, \( \delta_{ab} \) denotes the four-dimensional flat metric, and \( \Omega(X) \) denotes the volume six-form. The background NS5-branes and the probe NS5-brane extend on the X space.
For the static configuration,

\[
\sigma^0 = x^0 = t, \quad \sigma^\alpha = x^\alpha \quad (\alpha = 1, \cdots, 5), \quad x^a = \text{const.}, \quad (5.55)
\]

the equation of motion for the probe NS5-brane is trivially satisfied like

\[
\partial_t (h_5^{-1} - h_5^{-1}) = 0. \quad (5.56)
\]

The contribution to the potential comes not only from the Dirac-Born-Infeld part of the NS5-brane action but also from the Wess-Zumino term. The resulting potential is given by

\[
V = h_5^{-1} - h_5^{-1} = 0. \quad (5.57)
\]

Thus there is no velocity-independent force.

Let us next the case that the probe NS5-brane probe not parallel to the background NS5-branes. The probe NS5-brane shares the background NS5-brane with \( \bar{p} \) spatial dimensions. It is convenient to express the background as

\[
ds^2 = \eta_{\mu\nu}(X)dx^\mu dx^\nu + \delta_{ij}(Y)dy^i dy^j + h(t, z) \left[ \delta_{mn}(W)dv^m dv^n + \delta_{ab}(Z)dz^a dz^b \right], \quad (5.58a)
\]

\[
e^\phi = h^{1/2}, \quad (5.58b)
\]

\[
H(3) = e^{2\phi} * d \left[ h_5^{-1}(t, z) \Omega(X) \wedge \Omega(Y) \right]. \quad (5.58c)
\]

Here \( \eta_{\mu\nu} \) is the \((\bar{p} + 1)\)-dimensional Minkowski metric, and \( \delta_{ij}, \delta_{mn} \) and \( \delta_{ab} \) denote the \((5 - \bar{p})\)-, \((5 - \bar{p})\)- and \((\bar{p} - 1)\)-dimensional flat metrics, respectively. Then \( \Omega(X) \) and \( \Omega(Y) \) are the volume forms of the X and Y spaces, respectively. The background NS5-branes extend on the X and Y spaces, while the probe NS5-brane extends on the X and W spaces.

For the static configuration,

\[
\sigma^0 = x^0 = t, \quad \sigma^\alpha = x^\alpha \quad (\alpha = 1, \cdots, \bar{p}), \quad \sigma^m = v^m \quad (\alpha = \bar{p} + 1, \cdots, 5), \quad x^a = \text{const.}, \quad (5.59)
\]

the equation of motion for the probe NS5-brane leads to the condition,

\[
\partial_t h_5^{-(\bar{p} - 3)/2} = 0, \quad (5.60)
\]

Thus the static gauge is available if and only if \( \bar{p} = 3 \). Then the potential \( V \) is evaluated as

\[
V(t, z) = h_5^{-(\bar{p} - 3)/2}(t, z) = 1, \quad (5.61)
\]

and hence the velocity-independent force vanishes. This also agrees with the intersection rule for two NS5-branes \[24\] (See also \[23\] for the dynamical brane background)

\[
\text{NS5} \cap \text{NS5} = 3. \quad (5.62)
\]
For general dynamical brane backgrounds

So far, we have allowed only the time-dependence for the dynamical brane backgrounds (i.e., $\beta_0 = 0$ and the other components of $\beta_\mu = 0$). However, as we have shown in Sec. 2, one may consider the case with $\beta_\mu \neq 0$. Actually, for the general case with $\beta_\mu \neq 0$, there is an interesting mechanism for the gauge-fixing of the probe brane action.

To make our argument clear, let us revisit an example: a probe M2-brane moving on dynamical M2-brane backgrounds. Then we present that the equation of motion for the probe M2-brane may not lead to any constraint if we choose $\beta_\mu$ appropriately.

First, suppose that the probe M2-brane is parallel to the background M2-branes. Recall that the background is given by

$$ds^2 = h_2^{1/3}(x, z) \left[ h_2^{-1}(x, z) \eta_{\mu\nu}(X) dx^\mu dx^\nu + \delta_{ab}(Z) dz^a dz^b \right],$$

(6.1a)

$$F(4) = dA(3) = d \left[ h_2^{-1}(x, z) \right] \wedge \Omega(X),$$

(6.1b)

where $\eta_{\mu\nu}$ is the three-dimensional Minkowski spacetime metric, $\delta_{ab}$ is the eight-dimensional flat metric, and $\Omega(X)$ denotes the volume three-form. The background M2-branes and the probe M2-brane extend on the X space.

For the static configuration given in (4.2), the equation of motion for the probe M2-brane gives the trivial condition,

$$\left( \partial_t + \sum_{\alpha=1}^2 \partial_\alpha \right) (h_2^{-1} - h_2^{-1}) = 0,$$

(6.2)

where $\partial_\alpha$ is the partial derivatives with respect to $x^\alpha$. Thus the result is not changed by the general value of $\beta_\mu$.

Let us next consider the case that the probe M2-brane is completely orthogonal to the dynamical M2-brane background. Then it is convenient to express the background as

$$ds^2 = h_2^{1/3}(t, y, z) \left[ h_2^{-1}(t, y, z) \left\{ -dt^2 + \delta_{ij}(Y) dy^i dy^j \right\} + \delta_{mn}(W) dv^m dv^n + \delta_{ab}(Z) dz^a dz^b \right],$$

(6.3a)

$$F(4) = dA(3) = d \left[ h_2^{-1}(t, y, z) \right] \wedge dt \wedge \Omega(Y),$$

(6.3b)

where $\delta_{ij}(Y)$ and $\delta_{mn}(W)$ are the two-dimensional flat metrics, and $\delta_{ab}(Z)$ is the six-dimensional flat metric. Then $\Omega(Y)$ denotes the volume two-form. Now the background M2-branes extend on the Y space, while the probe M2-brane extends on the W space.
For the static configuration (4.6), the equation of motion for the probe M2-brane leads to
\[ \left( \partial_t + \sum_{i=1}^{2} \partial_i \right) (h_2)^0 = 0 , \] (6.4)
where \( \partial_i \) is the partial derivative with respect to \( y^i \). Then Eq. (6.4) is automatically satisfied and thus the previous argument is not changed again.

The last case is that the probe M2-brane shares one spatial direction with the background M2-branes. Then it is helpful to express the background as
\[ ds^2 = h_2^{1/3}(t, x, y, z) \left[ h_2^{-1}(t, x, y, z) \left( -dt^2 + dx^2 + dy^2 \right) + dv^2 + \delta_{ab}(Z)dz^a dz^b \right] , \] (6.5a)
\[ F(4) = dA(3) = d \left[ h_2^{-1}(t, x, y, z) \right] \wedge dt \wedge dx \wedge dy , \] (6.5b)
where \( \delta_{ab}(Z) \) is the seven-dimensional flat metric. The background M2-branes extend along the \( x \) and \( y \) directions, while the probe M2-brane extends along the \( x \) and \( v \) directions.

For the static gauge configuration given in (4.10), the equation of motion for the probe M2-brane leads to the condition,
\[ (\partial_t + \partial_x + \partial_y) h_2^{-1/2} = 0 . \] (6.6)
Recall that the function \( h_2 \) in the dynamical M2-brane background is given by
\[ h_2(t, x, y, z) = \beta_0 t + \beta_1 x + \beta_2 y + \beta_3 + \sum_l \frac{M_l}{|z^a - z^a_l|^3} . \] (6.7)
Then the condition (6.6) is reduced to
\[ h_2^{-3/2} \sum_{\mu=0}^{2} \beta_\mu = 0 . \] (6.8)
Thus the condition (6.6) is satisfied by requiring an additional condition for \( \beta_\mu \),
\[ \sum_{\mu=0}^{2} \beta_\mu = 0 . \] (6.9)
This condition may be interpreted as a dispersion relation for the dynamical M2-brane background. Under the additional condition (6.9), one may take the static gauge condition properly.

However, the potential \( V \) is given by a non-trivial function,
\[ V(t, x, y, z) = \left[ h_2^{-1/3}(t, x, y, z) \right]^2 h_2^{1/6}(t, x, y, z) = h_2^{-1/2}(t, x, y, z) . \] (6.10)
Thus the velocity-independent force remains. This is also consistent with the rule (4.8).

So far, we have discussed a specific example, but the argument is the same for the other dynamical backgrounds and the result is also not changed. In summary, by imposing an appropriate condition for $\beta_\mu$, the constraint for the static gauge can be removed. But the velocity-independent force still remains.

7 Conclusion and Discussion

We have considered the dynamics of a probe brane on various dynamical brane backgrounds. This is a generalization of the work on static brane backgrounds [13]. The static gauge on the dynamical backgrounds leads to an additional constraint. It leads to the supersymmetric intersection rules for the static branes, while the dynamical backgrounds preserve no supersymmetry. This is a specific feature of the dynamical backgrounds since the static gauge condition gives rise to no constraint on the brane configuration in the case of the static background. Furthermore, due to the constraint, it follows that there is no velocity-independent force for the probe brane even on the dynamical backgrounds.

The dynamics of branes has continued to give a new insight in gravitational theories. Nowadays, it is of great interest to apply the brane dynamics to realistic cosmology. Time-dependent brane solutions would provide a bridge between string theory and cosmology. Those would be available to construct realistic cosmological models. The brane dynamics would shed light on the origin of the Universe.

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References

[1] G. W. Gibbons, H. Lu and C. N. Pope, “Brane worlds in collision,” Phys. Rev. Lett. 94 (2005) 131602 [arXiv:hep-th/0501117].

[2] W. Chen, Z. -W. Chong, G. W. Gibbons, H. Lu and C. N. Pope, “Horava-Witten stability: Eppur si muove,” Nucl. Phys. B 732 (2006) 118 [hep-th/0502077].
[3] H. Kodama and K. Uzawa, “Moduli instability in warped compactifications of the type IIB supergravity,” JHEP 0507 (2005) 061 [arXiv:hep-th/0504193].

[4] H. Kodama and K. Uzawa, “Comments on the four-dimensional effective theory for warped compactification,” JHEP 0603 (2006) 053 [arXiv:hep-th/0512104].

[5] M. Minamitsuji and K. Uzawa, “Cosmology in p-brane systems,” Phys. Rev. D 83 (2011) 086002 [arXiv:1011.2376 [hep-th]].

[6] M. Minamitsuji and K. Uzawa, “Dynamics of partially localized brane systems,” Phys. Rev. D 84 (2011) 126006 [arXiv:1109.1415 [hep-th]].

[7] M. Minamitsuji and K. Uzawa, “Cosmological brane systems in warped spacetime,” Phys. Rev. D 87 (2013) 046010 [arXiv:1207.4334 [hep-th]].

[8] K. Uzawa and K. Yoshida, “Dynamical Lifshitz-type solutions and aging phenomena,” Phys. Rev. D 87 (2013) 106003 [arXiv:1302.5224 [hep-th]].

[9] D. R. Brill, G. T. Horowitz, D. Kastor and J. H. Traschen, “Testing cosmic censorship with black hole collisions,” Phys. Rev. D 49 (1994) 840 [gr-qc/9307014].

[10] K. i. Maeda, M. Minamitsuji, N. Ohta and K. Uzawa, “Dynamical p-branes with a cosmological constant,” Phys. Rev. D 82 (2010) 046007 [arXiv:1006.2306 [hep-th]].

[11] K. i. Maeda and K. Uzawa, “Dynamical brane with angles: Collision of the universes,” Phys. Rev. D 85 (2012) 086004 [arXiv:1201.3213 [hep-th]].

[12] K. Uzawa and K. Yoshida, “Dynamical F-strings intersecting D2-branes in type IIA supergravity,” Phys. Rev. D 88 (2013) 066005 [arXiv:1307.3093 [hep-th]].

[13] A. A. Tseytlin, “‘No force’ condition and BPS combinations of p-branes in eleven-dimensions and ten-dimensions,” Nucl. Phys. B 487 (1997) 141 [hep-th/9609212].

[14] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A Conjecture,” Phys. Rev. D 55 (1997) 5112 [hep-th/9610043].

[15] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A Large N reduced model as superstring,” Nucl. Phys. B 498 (1997) 467 [hep-th/9612115].

[16] H. Lu, C. N. Pope, E. Sezgin and K. S. Stelle, “Stainless super p-branes,” Nucl. Phys. B 456 (1995) 669 [hep-th/9508042].
[17] P. Binetruy, M. Sasaki and K. Uzawa, “Dynamical D4-D8 and D3-D7 branes in supergravity,” Phys. Rev. D 80 (2009) 026001 [arXiv:0712.3615 [hep-th]].

[18] K. i. Maeda, N. Ohta and K. Uzawa, “Dynamics of intersecting brane systems – Classification and their applications –,” JHEP 0906 (2009) 051 [arXiv:0903.5483 [hep-th]].

[19] G. Papadopoulos and P. K. Townsend, “Intersecting M-branes,” Phys. Lett. B 380 (1996) 273 [hep-th/9603087].

[20] A. Strominger, “Open p-branes,” Phys. Lett. B 383 (1996) 44 [hep-th/9512059].

[21] P. K. Townsend, “D-branes from M-branes,” Phys. Lett. B 373 (1996) 68 [hep-th/9512062].

[22] M. R. Douglas, “Branes within branes,” In *Cargese 1997, Strings, branes and dualities* 267-275 [hep-th/9512077].

[23] M. Minamitsuji, N. Ohta and K. Uzawa, “Cosmological intersecting brane solutions,” Phys. Rev. D 82 (2010) 086002 [arXiv:1007.1762 [hep-th]].

[24] R. Argurio, “Brane physics in M-theory,” arXiv:hep-th/9807171.