We study the ground state of the Pauli Hamiltonian with a magnetic field in $\mathbb{R}^{2d}$. We consider the case where a scalar potential $W$ is present and the magnetic field $B$ is given by $B = 2i \partial \bar{\partial} W$. The main result is that there are no zero modes if the magnetic field decays faster than quadratically at infinity. If the magnetic field decays quadratically then zero modes may appear, and we give a lower bound for the number of them. The results in this paper partly correct a mistake in a paper from 1993.

1. Introduction and main result

The Pauli operator $\mathcal{P}$ in $\mathbb{R}^n$ describes a charged spin-$\frac{1}{2}$ particle in a magnetic field. Along with the Dirac operator, it lies in the base of numerous models in quantum physics. The problem about zero modes, the bound states with zero energy, is one of many questions to be asked about the spectral properties of these operators.

Zero modes were discovered in [AC79] in dimension $n = 2$. Unlike the purely electric interaction, a compactly supported magnetic field can generate zero modes, as soon as the total flux of the field is sufficiently large. Quantitatively, this is expressed by the famous Aharonov-Casher formula. The two-dimensional case is by now quite well studied; the AC formula is extended to rather singular magnetic field, moreover, if the total flux is infinite (and the field has constant sign), there are infinitely many zero modes.

On the other hand, in the three-dimensional case the presence of zero modes is a rather exceptional feature, and the conditions for them to appear are not yet found, see the discussion in [MR03] and references therein.

Even less clear is the situation in the higher dimensions. In [Shi91], for even $n$ some sufficient conditions for the infiniteness of the number of zero modes were found, requiring, in particular, that the field decays rather slowly (more slowly than $r^{-2}$) at infinity. On the other hand, in [Ogu93], again for even $n$, the case where a finite number of zero modes should appear was considered. Under the assumption of a rather regular behavior of the scalar potential of the magnetic field at infinity the number of zero modes was calculated. In particular, for a field with compact support or decaying faster than quadratically at infinity the formula in [Ogu93] implies the absence of zero modes, thus making a difference with the two-dimensional situation.

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Unfortunately, it turned out that the reasoning in [Ogu93] contains an error. A miscalculation in an important integral leads to an erroneous conclusion, thus destroying the final results. This is the reason for us to return to the question on zero modes in the even higher-dimensional case. We try to revive the results in [Ogu93] and succeed partially.

We use the representation of the Pauli and Dirac operators in the terms of multi-variable complex analysis proposed in [Shi91] and used further in [Ogu93]. This approach puts a certain restriction on the class of magnetic field considered, equivalent to the existence of a scalar potential. At the moment it is unclear how to treat the general case.

Under the above condition, the operators are represented as acting on the complex forms, the action expressed via the $\bar{\partial}$ operator. The mistake in [Ogu93] occurs in calculating the $L_2$ norm of the form one gets after applying the $\partial$ operator. We present the detailed analysis of this miscalculation in Section 3.

The strategy of our treatment of zero modes differs from the one in [AC79] and other previous papers including [Ogu93]. Usually, when studying zero modes, one shows first that they, after having been multiplied by some known factor, are holomorphic function in the whole space; then one easily counts the number of such functions. This strategy fails in our case, so we use another one, involving more advanced machinery of complex and real analysis. The main ingredient of the proofs is a combination of the techniques of using the Bochner-Martinelli-Koppelman kernel to solve a $\bar{\partial}$ equation and the use of a weighted Hardy-Littlewood-Sobolev inequality to estimate that solution.

As a result, we establish some of the properties presented in [Ogu93]. We show that there are no zero modes if the magnetic field decays faster than quadratically at infinity (in particular, if it is compactly supported). Another result is that zero modes may exist if the magnetic field decays exactly quadratically, and the formula in [Ogu93] gives a lower bound for their number.

1.1. The Pauli operator. Let $x = (x^1, \ldots, x^{2d})$ denote the usual Euclidean coordinates in $\mathbb{R}^{2d}$. According to the Maxwell equations, a magnetic field $B$ in $\mathbb{R}^{2d}$ is a real closed two-form

$$B(x) = \sum_{j<k} b_{j,k}(x) dx^j \wedge dx^k. \quad (1.1)$$

Throughout this paper we assume that all the coefficient functions $b_{j,k}$ belong to $C^\infty(\mathbb{R}^{2d})$. The condition that the magnetic field $B$ is closed is given by

$$0 = dB = \sum_{j<k<l} \left( \frac{\partial b_{j,k}}{\partial x^l} - \frac{\partial b_{j,l}}{\partial x^k} + \frac{\partial b_{k,l}}{\partial x^j} \right) dx^j \wedge dx^k \wedge dx^l,$$

where $d$ is the usual exterior differential operator. Since $B$ is closed there exists a one-form

$$a(x) = \sum_{j=1}^n a_j(x) dx^j$$

satisfying

$$B = da = \sum_{j<k} \left( \frac{\partial a_k}{\partial x^j} - \frac{\partial a_j}{\partial x^k} \right) dx^j \wedge dx^k.$$
Any such one-form $a$ is called a magnetic one-form or magnetic vector potential. It is not unique. In fact, given one magnetic one-form, another one is obtained by adding $df$ for some regular function $f$. The choice of magnetic one-form $a$ is usually referred to as the choice of gauge.

The analysis of the Pauli operator was successful in [Shi91] using complex analysis under a condition that the magnetic field is a complex $(1,1)$-type form. It is not clear what this condition means physically, but to be able to use the theory of complex analysis in several variables, we will throughout use the same assumption. Thus, the coefficient functions in (1.1) of the closed 2-form $B$ must satisfy the $d(d-1)$ equations

$$
\begin{align*}
\left\{\begin{array}{ll}
  b_{j+1,k} = b_{j,k-1}, & \\
  b_{j+1,k} = -b_{j,k-1}
\end{array}\right.
\end{align*}
$$

for $j+1 \leq k \leq d$, $1 \leq j \leq d-1$.

The spinless Schrödinger operator $H$ in $\mathbb{R}^{2d}$ corresponding to the magnetic field $B$ is defined in $L_2(\mathbb{R}^{2d})$ as

$$
H = \sum_{j=1}^{2d} \left( -i \frac{\partial}{\partial x^j} - a_j \right)^2.
$$

We are interested in spin-$\frac{1}{2}$ particles (including the electron). Such systems are described by the Pauli operator $\mathfrak{P}$, acting in $L_2(\mathbb{R}^{2d}) \otimes \mathbb{C}^{2d}$. Let $\{\gamma^i\}_{i=0}^{2d}$ be Hermitian $2^d \times 2^d$ matrices satisfying

$$
\gamma^j \gamma^k + \gamma^k \gamma^j = 2\delta^{jk}I_{2^d},
$$

where $I_{2^d}$ denotes the $2^d \times 2^d$ identity matrix. These matrices $\{\gamma^i\}_{i=0}^{2d}$ generate a Clifford algebra, and are usually called the Dirac matrices. The Pauli operator $\mathfrak{P}$ is defined by

$$
\mathfrak{P} = HI_{2^d} + \sum_{0<j<k} ib_{jk}(x)\gamma^j \gamma^k.
$$

To be more precise, $\mathfrak{P}$ is first defined on $C^\infty_0 \otimes \mathbb{C}^{2^d}$, where it is essential self-adjoint (see [Che73]). We denote the self-adjoint closure by $\mathfrak{P}$. The Pauli operator $\mathfrak{P}$ can also be written as $\mathfrak{P} = \mathfrak{D}^2$, where $\mathfrak{D}$ is the self-adjoint Dirac operator

$$
\mathfrak{D} = \sum_{j=1}^{2d} \left( -i \frac{\partial}{\partial x^j} - a_j \right) \gamma^j.
$$

From this it follows that the Pauli operator is non-negative.

1.2. The main result.

**Theorem 1.1.** Assume that the equations in (1.2) are satisfied, and that there exist constants $C > 0$ and $\rho > 2$ such that

$$
|B(x)| \leq \frac{C}{(1+|x|)^\rho}
$$

for all $x \in \mathbb{R}^{2d}$.

Then

$$
\dim \ker \mathfrak{P} = 0.
$$
We will prove this theorem in Section 2. The case \(|B(x)| \sim 1/|x|^2\) as \(|x| \to \infty\) is more complicated. In section 3 we give an example of a magnetic field satisfying

\[|B(x)| = \frac{\Phi(d-1)}{2|x|^2},\]  

for large values of \(|x|\),

such that \(\dim \ker \Psi = 0\) if \(|\Phi| < d\) and \(\dim \ker \Psi > 0\) otherwise. This result is somehow strange and suggests that the situation for magnetic fields with a quadratic decay is quite complicated and unstable.

1.3. Complex analysis and Differential forms. Let us now switch to the complex analysis viewpoint. We identify \(x = (x^1, \ldots, x^{2d})\) in \(\mathbb{R}^{2d}\) with \(z = (z^1, \ldots, z^d)\) in \(\mathbb{C}^d\), where \(z^j = x^{2j-1} + ix^{2j}\). We define tangent and cotangent vectors by

\[
\frac{\partial}{\partial z^j} = \frac{1}{2} \left( \frac{\partial}{\partial x^{2j-1}} - i \frac{\partial}{\partial x^{2j}} \right),
\]

\[
\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2} \left( \frac{\partial}{\partial x^{2j-1}} + i \frac{\partial}{\partial x^{2j}} \right),
\]

\[dz^j = dx^{2j-1} + idx^{2j},\]  

and

\[d\bar{z}^j = dx^{2j-1} - idx^{2j}.
\]

Written in complex terms, the magnetic field \(B\) can be written as a sum of (1, 1), (2, 0), and (0, 2) type forms as

\[B(z) = \sum_{j,k=1}^{d} b_{j,k}(z) dz^j \wedge d\bar{z}^k + \sum_{j,k=1}^{d} b'_{j,k}(z) dz^j \wedge dz^k + \sum_{j,k=1}^{d} b''_{j,k}(z) d\bar{z}^j \wedge d\bar{z}^k.\]

The equations in (1.2) state that \(B\) is of type (1, 1) which means that all coefficient functions \(b'_{j,k}\) and \(b''_{j,k}\) in the representation above vanish, so the magnetic field \(B\) has the form

(1.4) \[B(z) = \sum_{j,k=1}^{d} b_{j,k}(z) dz^j \wedge d\bar{z}^k.\]

To magnetic fields that are (1, 1)-type forms there exist scalar potentials \(W \in C^\infty(\mathbb{C}^d \to \mathbb{R})\) satisfying

(1.5) \[B = 2i \partial \bar{\partial} W,\]

see [We80]. In [Sh91] it was shown that the Dirac and Pauli operators can be defined in terms of \(W\) and operators acting on differential forms in a very nice way. For the sake of completeness we show how this is done.

Let \(\Lambda^{0,q}(\mathbb{C}^d)^*\) denote the space of \((0,q)\)-type differential forms and let \(\Lambda^{0,*}(\mathbb{C}^d)^* = \bigoplus_{q=0}^{\infty} \Lambda^{0,q}(\mathbb{C}^d)^*\). The Dirac operator \(\mathcal{D}\) is realized as an operator in the Hilbert space \(\mathcal{H} := L_2(\mathbb{C}^d; d\lambda) \otimes \Lambda^{0,*}(\mathbb{C}^d)^*\) in the way

(1.6) \[\mathcal{D} = 2(\bar{\partial}_W + \partial^*_W).\]
Here
\[ \bar{\partial}_W = \bar{\partial} + \text{ext}(\bar{\partial} W) = \sum_{j=1}^{d} \text{ext}(d\bar{z}^j) \left( \frac{\partial}{\partial \bar{z}^j} + \frac{\partial W}{\partial \bar{z}^j} \right), \]
\[ \partial_\alpha W = \partial^* + \text{int}(\partial W) = \sum_{j=1}^{d} \text{int}(d\bar{z}^j) \left( -\frac{\partial}{\partial z^j} + \frac{\partial W}{\partial z^j} \right), \]
\( \text{ext}(d\bar{z}^j) \) is the operator on \( \bigwedge^{0,*}(\mathbb{C}^d)^* \) acting as
\[ \text{ext}(d\bar{z}^j)\eta = d\bar{z}^j \wedge \eta, \quad \text{for } \eta \in \bigwedge^{0,*}(\mathbb{C}^d)^*, \]
and \( \text{int}(d\bar{z}^j) \) is the adjoint operator of \( \text{ext}(d\bar{z}^j) \) in \( \mathcal{H} \).
To see that (1.6) is true we use the anti-commutation relations
\[ [\text{ext}(d\bar{z}^j), \text{ext}(d\bar{z}^k)]_+ = 0; \]
\[ [\text{int}(d\bar{z}^j), \text{int}(d\bar{z}^k)]_+ = 0; \]
\[ [\text{ext}(d\bar{z}^j), \text{int}(d\bar{z}^k)]_+ = \delta^{jk}. \]
By defining
\[ \gamma^{2j-1} = i(\text{ext}(d\bar{z}^j) - \text{int}(d\bar{z}^j)); \]
\[ \gamma^{2j} = -(\text{ext}(d\bar{z}^j) + \text{int}(d\bar{z}^j)) \]
on one can easily check that
\[ [\gamma^j, \gamma^k]_+ = 2\delta^{jk}. \]
Hence \( \{\gamma^j\} \) so defined satisfies the relation (1.3) of a Clifford algebra. Now it is easy to see that (1.6) holds:
\[ 2 \left( \bar{\partial}_W + \partial_\alpha W \right) = \sum_{j=1}^{d} \left( -i\gamma^{2j-1} - \gamma^{2j} \right) \left( \frac{\partial}{\partial \bar{z}^j} + \frac{\partial W}{\partial \bar{z}^j} \right) + \left( i\gamma^{2j-1} - \gamma^{2j} \right) \left( -\frac{\partial}{\partial z^j} + \frac{\partial W}{\partial z^j} \right) \]
\[ = \sum_{j=1}^{d} i\gamma^{2j-1} \left( \frac{\partial}{\partial x^{2j-1}} - i\frac{\partial W}{\partial x^{2j-1}} \right) + \gamma^{2j} \left( -i\frac{\partial}{\partial x^{2j}} - \frac{\partial W}{\partial x^{2j}} \right) \]
\[ = \sum_{j=1}^{2d} \gamma^{2j} \left( -i\frac{\partial}{\partial x^j} - a_j(x) \right) = \mathcal{D} \]
where \( a_{2j-1}(x) = \frac{\partial W}{\partial x^{2j-1}} \) and \( a_{2j}(x) = -\frac{\partial W}{\partial x^{2j}} \), so \( a = i(\bar{\partial} - \partial)W \), which fits well with (1.5), since \( B = da = (\bar{\partial} + \partial)a = (\bar{\partial} + \partial)i(\bar{\partial} - \partial)W = 2i\partial W \).
For a form \( \alpha \in \mathcal{H} \) to belong to the kernel of \( \mathfrak{F} \) it is necessary and sufficient that \( \alpha \) belongs to the kernel of the quadratic form
\[ \mathfrak{p}[\alpha] = 4 \int_{\mathbb{C}^d} \left( |\bar{\partial}_W \alpha|^2 + |\partial_\alpha W|^2 \right) d\lambda(z), \quad \alpha \in \mathcal{H}. \]
Let \( U : \mathcal{H} \to \mathcal{H} \text{ } W := L_2(\mathbb{C}^d; e^{-2W}d\lambda) \otimes \bigwedge^{0,*}(\mathbb{C}^d)^* \) be the unitary operator \( U : \alpha \mapsto e^W \alpha \). Then \( \mathfrak{F} \) and \( \mathfrak{F}' = U\mathfrak{F}U^* \) are unitarily equivalent. The quadratic form \( \mathfrak{p} \) on \( \mathcal{H}_W \) corresponding to \( \mathfrak{F} \) is given by
\[ (1.7) \quad \mathfrak{p}[\alpha] = 4 \int_{\mathbb{C}^d} \left( |\bar{\partial} \alpha|^2 + |\partial^* \alpha|^2 \right) e^{-2W}d\lambda(z), \quad \alpha \in \mathcal{H}_W. \]
Here $\bar{\partial}^*$ is the adjoint operator to $\bar{\partial}$ in $\mathcal{H}_W$.

2. Proof of Theorem 1.1

Lemma 2.1. Under the same conditions as in Theorem 1.1 there exists a scalar potential $W \in L^\infty(\mathbb{C}^d \to \mathbb{R})$ such that $2i \bar{\partial} \partial W = B$.

We know from [Wel80] that solutions $W$ exist, the essential part of this Lemma is that there exist a bounded solution to (1.5).

Proof. First assume that $\rho$ is not an integer. Two applications of Theorem 9’ in [BA82] gives the existence a scalar potential $W$ and a constant $C > 0$ such that

$$|W(z)| \leq C \frac{1}{(1 + |z|)^{\rho - 2}}$$

which means that $W$ is bounded since $\rho > 2$. If $\rho$ is an integer then we can replace $\rho$ by $\rho - 1/2$ and use the same argument again. □

To prove Theorem 1.1 it is clearly enough to prove the following theorem.

Theorem 2.2. Assume that $W \in L^\infty(\mathbb{C}^d \to \mathbb{R})$. Then

$$\dim \ker P = 0.$$

Since $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are unitarily equivalent we will show instead that $\dim \ker \tilde{\mathcal{P}} = 0$. We need some Lemmata.

Lemma 2.3. Let $\Omega : \mathbb{C}^d \to \mathbb{C}$ be a homogeneous function of degree zero, and let $\Omega$ be bounded on the unit sphere $|z| = 1$. Define the operator $T$ as

$$(Tf)(z) = \int_{\mathbb{C}^d} \frac{\Omega(z - \zeta)}{|z - \zeta|^{2d-1}} f(\zeta) d\lambda(\zeta).$$

Then $T$ is bounded as an operator from $L^2(\mathbb{C}^d)$ to $L^2_{d/(d-1)}(\mathbb{C}^d)$.

Proof. This is a special case of the Hardy-Littlewood-Sobolev theorem, see Theorem V.1 in [Ste70]. □

Let $K_q(\zeta, z)$ be the Bochner-Martinelli-Koppelman kernel

$$K_q(\zeta, z) = \frac{(d - 1)!}{2^{q+1} \pi^d} \frac{1}{|\zeta - z|^{2d}} \sum_{|J| = q+1} \varepsilon^J_{\bar{J}} (\zeta^J - \bar{z}^J) (\ast d\zeta^\bar{J}) \wedge d\zeta^J.$$

Here $J$ is a multiindex of length $q$ and if $A$ and $B$ are ordered subsets of $\{1, 2, \ldots, d\}$ then $\varepsilon^A_B$ denotes the sign of the permutation which takes $A$ into $B$ if $|A| = |B|$ and zero if $|A| \neq |B|$. If $A \subset \{1, 2, \ldots, d\}$ and $|A| = q$ then

$$\ast d\zeta_A = \frac{(-1)^{q(q-1)/2}}{2^{q-d} q!} d\zeta^A \wedge \left( \bigwedge_{\nu \in A'} d\zeta^\nu \wedge d\zeta^\nu \right),$$

where $A'$ is the complementary multiindex of $A$. We see that $K_q(\zeta, z)$ is of type $(d, d - q - 1)$ in $\zeta$ and $(0, q)$ in $z$.

Lemma 2.4. Let $\alpha \in \mathcal{H}_W$ be a $(0, q)$-type form, $1 \leq q \leq d - 1$, satisfying $\bar{\partial}\alpha = \bar{\partial}^*\alpha = 0$. Then the $(0, q - 1)$-type form

$$(2.1) \quad \beta(z) = -\int_{\mathbb{C}^d} \alpha(\zeta) \wedge K_{q-1}(\zeta, z)$$
satisfies \( \bar{\partial} \beta = \alpha \) in the sense of distributions. Moreover, there exists a constant \( C > 0 \) such that

\[
(2.2) \quad \int_{|z|<2R} \frac{|eta(z)|^2}{|z|^2} d\lambda(z) \leq C\|\alpha\|^2_{H^w}
\]

for all \( R > 0 \), where the constant \( C \) does not depend on \( \alpha \) or \( R \).

**Proof.** Let \( \eta_k \) be a family of cut-off functions, such that \( \eta_k(\zeta) = 1 \) if \( |\zeta| < k \), \( \eta_k(\zeta) = 0 \) if \( |\zeta| > k+1 \) and \( |\bar{\partial} \eta_k| \leq 2 \) for all \( k = 1, 2, \ldots \). Since the form \( \alpha \) belongs to the kernel of the elliptic Pauli operator (with smooth coefficient functions), it must itself be smooth. The forms \( \eta_k \alpha \) are smooth and have compact support and thus, according to the Bochner-Martinelli-Koppelman formula (see [Ran86]), they satisfy

\[
\eta_k \alpha(z) = -\int \bar{\partial}(\eta_k \alpha) \wedge K_q(\zeta, z) - \bar{\partial} z \int (\eta_k \alpha) \wedge K_{q-1}(\zeta, z), \quad \text{for } k = 1, 2, \ldots.
\]

Let \( \Phi \) be a test form with support in \( |z| < M \). Fix \( \varepsilon > 0 \). Then

\[
|\langle \beta, \bar{\partial}^* \Phi \rangle - \langle \alpha, \Phi \rangle| = |\langle \beta, \bar{\partial}^* \Phi \rangle + \left\langle \int_\zeta (\eta_k \alpha) \wedge K_{q-1}(\zeta, z), \bar{\partial}^* \Phi \right\rangle + \left\langle \int_\zeta \bar{\partial}(\eta_k \alpha) \wedge K_q(\zeta, z), \Phi \right\rangle + \langle \eta_k \alpha, \Phi \rangle - \langle \alpha, \Phi \rangle| \\
\leq |\left\langle \int_\zeta (\eta_k - 1) \alpha \wedge K_{q-1}(\zeta, z), \bar{\partial}^* \Phi \right\rangle| \\
+ |\left\langle \int_\zeta \bar{\partial} \eta_k \wedge \alpha \wedge K_q(\zeta, z), \Phi \right\rangle| \\
+ |\left\langle \eta_k - 1, \alpha, \Phi \right\rangle| \\
= I_1 + I_2 + I_3.
\]

We will let \( k \) tend to infinity. For \( k > 2M \) we have \( |K_{q-1}(\zeta, z)| \leq C|\zeta|^{1-2d} \). We get

\[
I_1 \leq \int_{|z|<M} \int_{|\zeta|>k} |\eta_k(\zeta) - 1| \cdot |\alpha(\zeta)| \cdot |K_{q-1}(\zeta, z)| d\lambda(\zeta) \cdot |\bar{\partial}^* \Phi(z)| d\lambda(z) \\
\leq C \sup |\bar{\partial}^* \Phi| \cdot \|\alpha\|_{\mathcal{E}} \int_{|z|<M} \left( \int_{|\zeta|>k} |K_{q-1}(\zeta, z)|^2 d\lambda(\zeta) \right)^{1/2} d\lambda(z) \\
\leq C \sup |\bar{\partial}^* \Phi| \cdot \|\alpha\|_{\mathcal{E}} \left( \int_k^\infty r^{2-4d+2d-1} dr \right)^{1/2} \\
\leq C \sup |\bar{\partial}^* \Phi| \cdot \|\alpha\|_{\mathcal{E}} \cdot k^{1-d}
\]
Note that the integral written as \( \frac{\partial}{\partial \alpha} \) where \( \partial \), we have

\[
I_2 \leq C \sup \| \Phi \| \| \alpha \| \mathcal{C} \left( \int \Phi z \right) \int |K_q(\zeta, z)| e^{2W(\zeta)} \lambda(z) \right)^{1/2} \lambda(z) \\
\leq C \sup \| \Phi \| \| \alpha \| \mathcal{C} \left( \int \kappa^{k+1} r^{2-4d+2d-1} dr \right)^{1/2} \\
\leq C \sup \| \Phi \| \| \alpha \| \mathcal{C} \| \mathcal{W} \cdot k^{1/2-d},
\]

so \( I_2 < \varepsilon \) if \( k \) is large enough. \( I_3 \) is equal to zero if \( k \) is large enough, since then \((1 - \eta_k) \) and \( \Phi \) has disjoint support. We conclude that \( \partial \beta = \alpha \) in the sense of distributions.

To show the estimate \( (3.6) \), we use Lemma 2.3. Indeed, note that \( \beta \) can be written as

\[
\beta = \sum \alpha_j T_j \alpha_j
\]

where \( \alpha = \sum \alpha_j d\bar{z}^j \), \( |J| = q \), and all operators \( T_j \) are of the kind in Lemma 2.3. Denote by \( E_R \) the set \( \{ z \in \mathbb{C}^d : R < |z| \leq 2R \} \). Using the Hölder inequality and Lemma 2.3, we have

\[
\int_{E_R} \left| \frac{\beta^2}{|z|^2} \right| d\lambda(z) \leq \left( \int_{E_R} \frac{1}{|z|^{2d}} d\lambda(z) \right)^{1/d} \left( \int_{E_R} |\beta|^{2/(d-1)} d\lambda(z) \right)^{(d-1)/d} \\
\leq C \| \alpha \|_{\mathcal{C}}^2 \leq C \| \alpha \|_{\mathcal{C}}^2.
\]

Note that the integral

\[
\int_{R<|z|<2R} \frac{1}{|z|^{2d}} d\lambda(z) = c_d \int_{R}^{2R} r^{-2d+2d-1} dr = c_d \log(2)
\]

is independent of \( R \), so the constant \( C \) above is also independent of \( R \).

**Proof** (of Theorem 2.3). Let \( 1 \leq q \leq d - 1 \). Assume that \( \alpha \in \mathcal{H}_W \) is a \((0, q)\)-type form in the kernel of \( \Phi \). Then \( \partial \alpha = \partial^* \alpha = 0 \), and so we get the form \( \beta \) from Lemma 2.3. We don’t know, apriori, that \( \beta \) belongs to the domain of the \( \partial \) operator. We introduce a family of cut-off functions to be able to integrate by parts.

Let \( \varphi_k(r) \), \( k = 1, 2, \ldots \), be a \( C^\infty \) family of cut-off functions, such that \( \varphi_k(r) = 1 \) if \( 0 < r \leq 2^k \), \( \varphi_k(r) = 0 \) if \( r \geq 2^{k+1} \) and such that \( 0 \leq \varphi_k \) and \( |\varphi'_k| \leq 2^{1-k} \). Let \( \chi_k(z) = \varphi_k(|z|) \). We have

\[
0 = \langle \partial^* \alpha, \chi_k \beta \rangle \mathcal{H}_W \\
= \langle \alpha, \partial(\chi_k \beta) \rangle \mathcal{H}_W \\
= \int |\alpha|^2 \chi_k e^{-2W} d\lambda + \int \alpha \cdot \partial \chi_k \cdot \beta e^{-2W} d\lambda \\
= I_k + II_k.
\]

The integration by parts above is permitted thanks to the cut-off function \( \chi_k \). It is clear that \( I_k \rightarrow \| \alpha \|_{\mathcal{C}}^2 \) as \( k \rightarrow \infty \). We shall prove that \( II_k \rightarrow 0 \) as \( k \rightarrow \infty \).
Let \( m_k^2 = \int_{E_k} |\alpha|^2 e^{-2W} d\lambda \). Then it holds that \( \sum_k m_k^2 = ||\alpha||^2_{g_{\chi_k}} < \infty \) so \( m_k \to 0 \) as \( k \to \infty \). Since \( \bar{\partial} \chi_k \) has support in \( E_k \) and \( |\bar{\partial} \chi_k| \leq C 2^{-k} \) we have
\[
|I_k| \leq \int_{E_k} |\alpha| \cdot |\beta| \cdot |\bar{\partial} \chi_k| e^{-2W} d\lambda \\
\leq C 2^{-k} \left( \int_{E_k} |\alpha|^2 d\lambda \right)^{1/2} \left( \int_{E_k} |\beta|^2 d\lambda \right)^{1/2} \\
\leq C m_k \left( \int_{E_k} \frac{|\beta|^2}{|z|^2} d\lambda \right)^{1/2} \\
\leq C m_k ||\alpha||_{g_{\chi_k}} \to 0, \quad \text{as} \quad k \to \infty.
\]
The first inequality is just the triangle inequality. The second one is the inequality for \( \chi_k \) and the Cauchy-Schwarz inequality. In the third inequality we use the fact that \( |z| \approx 2^k \), and in the fourth we use Lemma 2.4.

Next let \( q = 0 \), and assume that \( \alpha \) is a \((0, 0)\)-type form in the kernel of \( \tilde{\mathcal{P}} \). According to (1.7) \( \alpha \) has to be an entire function in \( z^1, \ldots, z^d \). Since the function \( \alpha \) also belongs to \( L^2(C^d, e^{-2W} d\lambda) \) a Liouville-type argument gives that it must be zero.

Finally let \( q = d \). Then (1.7) implies that \( \bar{\partial}^* \alpha = 0 \). If \( \alpha = \hat{\alpha} dz^1 \wedge \cdots \wedge d\bar{z}^d \), then this means that
\[
-\frac{\partial \hat{\alpha}}{\partial z_j} + 2 \frac{\partial W}{\partial z_j} \hat{\alpha} = 0, \quad j = 1, \ldots, d.
\]
If we put \( f(z) = e^{-2W(z)} \hat{\alpha}(z) \) we obtain
\[
\frac{\partial f}{\partial z_j} = 0, \quad j = 1, \ldots, d,
\]
that is the function \( f \) is an entire function in \( \bar{z}^1, \ldots, \bar{z}^d \). Moreover the function \( f \) belongs to \( L^2(C^d, e^{2W} d\lambda) \) so it must be zero. \( \square \)

3. Quadratically decaying magnetic fields

The case of determining the kernel of the Pauli operator for potentials with a logarithmic growth, which includes quadratically decaying magnetic fields, is more complicated. Given a real number \( \Phi \), denote by \( N_d(\Phi) \) the number of all monomials in \( d \) variables with degree less than \( |\Phi| - d \). The following Theorem was proposed in \cite{Ogu93}.

**Theorem 3.1.** Assume that \( W \in C^\infty(C^d \to \mathbb{R}) \) and that there exists a real constant \( \Phi \) such that the limit
\[
\lim_{|z| \to \infty} \frac{e^{-W(z)}}{|z|^\Phi}
\]
exists and is greater than zero. Then
\[
\dim \ker \tilde{\mathcal{P}} = N_d(\Phi).
\]

Let us sketch the idea of the proof in the case \( d = 2 \). First, assume that \( \Phi > 0 \), and that
\[
\alpha = \alpha_{00} + \alpha_{10} dz^1 + \alpha_{01} d\bar{z}^2 + \alpha_{11} d\bar{z}^1 \wedge d\bar{z}^2
\]
is an element of \( \ker \tilde{\mathcal{P}} \). Then
\[ \bar{\partial} \alpha = \frac{\partial \alpha_{00}}{\partial \bar{z}^1} d\bar{z}^1 + \frac{\partial \alpha_{00}}{\partial \bar{z}^2} d\bar{z}^2 + \left( \frac{\partial \alpha_{01}}{\partial \bar{z}^1} - \frac{\partial \alpha_{10}}{\partial \bar{z}^2} \right) d\bar{z}^1 \wedge d\bar{z}^2 \]

and thus

\[ 0 = \int_{\mathbb{C}^2} |\bar{\partial} \alpha|^2 e^{-2W} d\lambda \]

\[ = \int_{\mathbb{C}^2} \left( \left| \frac{\partial \alpha_{00}}{\partial \bar{z}^1} \right|^2 + \left| \frac{\partial \alpha_{00}}{\partial \bar{z}^2} \right|^2 + \left| \frac{\partial \alpha_{01}}{\partial \bar{z}^1} \right|^2 - \left| \frac{\partial \alpha_{10}}{\partial \bar{z}^2} \right|^2 \right) e^{-2W} d\lambda. \]

However, in [Ogu93] this is written as

\[ 0 = \int_{\mathbb{C}^2} |\bar{\partial} \alpha|^2 e^{-2W} d\lambda \]

(3.1)

\[ = \int_{\mathbb{C}^2} \left( \left| \frac{\partial \alpha_{00}}{\partial \bar{z}^1} \right|^2 + \left| \frac{\partial \alpha_{00}}{\partial \bar{z}^2} \right|^2 + \left| \frac{\partial \alpha_{01}}{\partial \bar{z}^1} \right|^2 - \left| \frac{\partial \alpha_{10}}{\partial \bar{z}^2} \right|^2 \right) e^{-2W} d\lambda, \]

which is not correct. The rest of the proof uses (3.1) and some arguments to show that \( \alpha_{00} \), \( \alpha_{10} \) and \( \alpha_{01} \) must vanish. Then it is shown, correctly, that the term \( \alpha_{11} d\bar{z}^1 \wedge d\bar{z}^2 \) contains elements in the kernel if \( \Phi \) is big enough. It is similar if \( \Phi < 0 \).

So, we know from [Ogu93] that if \( W \sim -\Phi \log |z| \), as \( |z| \to \infty \), for \( |\Phi| > d \), then the kernel is non-empty, and the dimension of the kernel is at least \( N_d(\Phi) \). We are not able to prove the Theorem proposed in [Ogu93], but we can show the following Theorem.

**Theorem 3.2.** Assume that the limit

\[ \lim_{|z| \to \infty} \frac{e^{-W(z)}}{|z|^\Phi} \]

exists and is positive. If \( |\Phi| < d \) then \( \dim \ker \mathcal{P} = 0 \). If \( |\Phi| \geq d \) then \( \dim \ker \mathcal{P} \geq N_d(\Phi) \).

The proof goes in the same way as the proof of Theorem 2.2, so we will just point out the main differences. First of all we can assume that \( \Phi \geq 0 \). If \( \Phi \) is negative we can apply a unitary transform that changes the sign of \( W \).

We need a replacement of Lemma 2.4 where weights of polynomial growth are allowed. To prepare for this we introduce the Muckenhoupt weight class.

**Definition 3.3.** A non-negative function \( \psi \) is said to belong to the Muckenhoupt class \( A(p,q) \), \( 1 < p, q < \infty \), if there exists a constant \( C > 0 \) such that

\[ \sup_{B \subset \mathbb{C}^d} \left( \frac{1}{|B|} \int_B \psi^q d\lambda(z) \right)^{1/q} \left( \frac{1}{|B|} \int_B \psi^{-p/(p-1)} d\lambda(z) \right)^{(p-1)/p} \leq C. \]

Here the supremum is taken over all balls in \( \mathbb{C}^d \) and \( |B| \) denotes the Lebesgue measure of the ball \( B \).

**Lemma 3.4.** If \( W \) satisfies the properties in Theorem 3.2 then the weight function \( e^{-W} \) belongs to the Muckenhoupt class \( A(2, 2d/(d-1)) \).
Proof. Let $\gamma = 2d/(d - 1)$. We should show that

$$I := \left( \frac{1}{|B|} \int_B e^{-\gamma W} d\lambda(z) \right)^{1/\gamma} \left( \frac{1}{|B|} \int_B e^{2W} d\lambda(z) \right)^{1/2} \leq C,$$

where $C$ does not depend on the ball $B$. From the assumptions on $e^{-W}$ we know that there exist positive constants $c_1$, $c_2$, $c_3$ and $c_4$ such that

\begin{align*}
(3.2) \quad c_1 |z|^\Phi &\leq e^{-W(z)} \leq c_2 |z|^\Phi, &\text{if } |z| \geq 1 \\
(3.3) \quad c_3 &\leq e^{-W(z)} \leq c_4, &\text{if } |z| < 5.
\end{align*}

We divide the balls into different classes. Say that a ball $B = B(z_0, R)$ is of Type 1 if $|z_0| > 3/2R$ and otherwise of Type 2.

First, assume that $B$ is of Type 1. Then for $z \in B$ we have $|z| \leq |z_0| + R \leq 5/3 |z_0|$ and $|z| \geq |z_0| - R \geq 1/3 |z_0|$. If $|z_0| \geq 3$ we can use $(3.2)$ and get

$$I \leq C \left( \frac{1}{|B|} \int_B |z|^\gamma \Phi d\lambda(z) \right)^{1/\gamma} \left( \frac{1}{|B|} \int_B \frac{1}{|z|^2 \Phi} d\lambda(z) \right)^{1/2} \leq C \left( \frac{1}{|B|} \int_B |z_0|^\gamma \Phi d\lambda(z) \right)^{1/\gamma} \left( \frac{1}{|B|} \int_B \frac{1}{|z_0|^2 \Phi} d\lambda(z) \right)^{1/2} = C (|z_0|^\gamma \Phi)^{1/\gamma} \left( \frac{1}{|z_0|^2 \Phi} \right)^{1/2} = C.$$

If $|z_0| \leq 3$ then $|z| \leq 5$, so we can easily use $(3.3)$ to get that $I \leq C$ independent of $R$.

Now assume that $B$ is of Type 2. Then $B \subset B' := B(0, 3R)$. Since $B$ and $B'$ are of comparable size, we have

\begin{equation}
(3.4) \quad I \leq C \left( \frac{1}{|B'|} \int_{B'} e^{-\gamma W} d\lambda(z) \right)^{1/\gamma} \left( \frac{1}{|B'|} \int_{B'} e^{2W} d\lambda(z) \right)^{1/2} =: J.
\end{equation}

If $R \leq 5/3$ we can use $(3.3)$ to get that $J \leq C$ independent of $R$. If $R > 5/3$ we have

$$J \leq C \left( \frac{1}{R^{2d}} \left( \int_{|z| < 5} \frac{(1/c_4) \gamma \Phi}{c_2^2} d\lambda(z) + \int_{5 < |z| < 3R} \frac{|z|^\gamma \Phi}{c_2^2} d\lambda(z) \right) \right)^{1/\gamma} \times \left( \frac{1}{R^{2d}} \left( \int_{|z| < 5} c_3^2 d\lambda(z) + \int_{5 < |z| < 3R} \frac{c_1^2}{|z|^2 \Phi} d\lambda(z) \right) \right)^{1/2}.$$

In this product the first factor is $O(R^\Phi)$ while the second factor is $O(R^{-\min(d, \Phi)})$ as $R \to \infty$. Since the expression clearly is bounded for bounded values of $R$ there exists a constant $C$ such that $J \leq C$ independent of $R$.

We conclude that $e^{-W} \in A(2, 2d/(d - 1))$. \hfill \Box

The following Lemma replaces Lemma 2.3.
Lemma 3.5. Let \( \Omega : \mathbb{C}^d \to \mathbb{C} \) be a homogeneous function of degree zero, and let \( \Omega \) be bounded on the unit sphere \( |z| = 1 \). Define the operator \( T \) as

\[
(Tf)(z) = \int_{\mathbb{C}^d} \frac{\Omega(z - \zeta)}{|z - \zeta|^{2d-1}} f(\zeta) d\lambda(\zeta).
\]

If the weight function \( \psi \) belongs to the Muckenhoupt class \( A(2, 2d/(d-1)) \) then there exists a constant \( C > 0 \), independent of \( f \), such that

\[
\left( \int_{\mathbb{C}^d} (|Tf(z)| \cdot |\psi(z)|)^{2d/(d-1)} \right)^{(d-1)/(2d)} \leq C \left( \int_{\mathbb{C}^d} |f(z)\psi(z)|^2 \right)^{1/2}.
\]

**Proof.** This is a special case of Theorem 1 in [DL98]. \( \square \)

Finally we get the result that replaces Lemma 2.4.

Lemma 3.6. Let \( \alpha \in \mathcal{K}_W \) be a \((0,q)\)-type form, \( 1 \leq q \leq d-1 \), satisfying \( \bar{\partial}\alpha = \bar{\partial}^*\alpha = 0 \). Then the \((0,q-1)\)-type form

\[
\beta(z) = -\int_{\mathbb{C}^d} \alpha(\zeta) \wedge K_{q-1}(\zeta,z)
\]

satisfies \( \bar{\partial}\beta = \alpha \) in the sense of distributions. Moreover, there exists a constant \( C > 0 \) such that

\[
\int_{R<|z|<2R} \frac{|eta(z)|^2}{|z|^2} e^{-2W(z)} d\lambda(z) \leq C\|\alpha\|^2_{\mathcal{K}_W}
\]

for all \( R > 0 \), where the constant \( C \) does not depend on \( \alpha \) or \( R \).

**Proof.** The part that \( \beta \) solves \( \bar{\partial}\beta = \alpha \) is just the same as in the proof of Lemma 2.4. For the estimate we can use Lemma 3.5 to get

\[
\int_{E_R} |eta|^2 e^{-2W} d\lambda(z) \leq \left( \int_{E_R} |z|^{-2d} \right)^{1/d} \left( \int_{E_R} (|eta| e^{-W})^{2d/(d-1)} d\lambda(z) \right)^{(d-1)/d} \leq C\|\alpha\|^2_{\mathcal{K}_W} \square
\]

**Proof (of Theorem 3.2).** First, let \( 1 \leq q \leq d-1 \). The proof runs in the same way as in the proof of Theorem 2.2 but with the use of Lemmata with weights.

Next, let \( q = 0 \), and assume that \( \alpha \) is a \((0,0)\)-type form in the kernel of \( \tilde{\Phi} \). According to (1.7) \( \alpha \) has to be an entire function in \( \bar{z}_1, \ldots, \bar{z}_d \). Also belonging to \( L_2(\mathbb{C}^d, e^{-2W} d\lambda) \), it must tend to zero at infinity. Hence it must be constant equal to zero by a Liouville type argument.

Finally, let \( q = d \). Then (1.7) implies that \( \bar{\partial}^*\alpha = 0 \). If \( \alpha = \tilde{\alpha} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_d \), then this means that the function \( f(z) = e^{-2W(z)}\tilde{\alpha}(z) \) is an entire function in \( \bar{z}_1, \ldots, \bar{z}_d \). Moreover there exist constants \( c_1 \) and \( c_2 \) such that

\[
\frac{c_1}{|z|^{\Phi}} \leq e^{W(z)} \leq \frac{c_2}{|z|^{\Phi}}
\]

if \( |z| \) is large enough. Now the condition that \( \alpha \in \mathcal{K}_W \) means that \( e^W f \in L_2(\mathbb{C}^d) \). This is the case if and only if \( f \) is a polynomial in \( \bar{z}_1, \ldots, \bar{z}_d \) of degree strictly less than \( \Phi - d \). The dimension of the space of such polynomials is exactly \( N_d(\Phi) \). \( \square \)
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