Research Article

Analysis of Stochastic Predator-Prey Model with Disease in the Prey and Holling Type II Functional Response

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A stochastic predator-prey model with disease in the prey and Holling type II functional response is proposed and its dynamics is analyzed. We discuss the boundedness of the dynamical system and find all feasible equilibrium solutions. For the stochastic systems, we obtain the conditions for the existence of the global unique solution, boundedness, and uniform continuity. We derive the conditions for extinction and permanence of species. Moreover, we construct appropriate Lyapunov functions and discuss the asymptotic stability of equilibria. To illustrate our theoretical findings, we have performed numerical simulations and presented the results.

1. Introduction

Mathematical models are used to study the interrelationship among species and their environment. The study of disease transmission has turned out to be a valuable field of research after the fundamental work of Kermac and McKendric [1] on susceptible-infected framework. Hadeler and Freedman [2] first proposed a disease spread model within interacting populations. Initially, epidemics are created if there are some people susceptible to the infection and some infected people in the population. It is especially essential to view the ecosystem with the influence of epidemiological factors to control the disease in the species. From the ecological point of view, the spread of disease can not be disregarded because its effects are serious. So, various authors have paid attention to the study of transmissible disease in ecology, see for example [3–6] and the references therein. Mondal [7] has examined the disease model with two species and analyzed the dynamical properties of the fractional order system. Haque and Venturino [8] investigated the stability behavior of the deterministic Holling-Tanner predator-prey model. In this paper, we propose the predator-prey model and consider the Holling type II response for predation.

In an ecological model, the interactions between two or more species and their dynamics are influenced by each other. So, the growth of one species depends on another and is described by the prey-predator system. Three primary kinds of interaction between the species are: predator-prey, mutualism, and competition. In all predator-prey interactions, Holling functions do not allow the growth of predators to very large extent even if the density of the prey is more. Specifically, Holling type II functional response is defined by a decelerating intake rate which follows from the assumption that the consumer is limited by its capacity to process food. In other words, Holling type II represents the fact that when prey density is small, the predator can take less time for handling prey and if the prey density increases, more prey are attacked so that the handling time also increases. In this article, we have used Holling type II response for both infected and susceptible prey interactions with the predator. This kind of functional response has been widely utilized as a part of biological systems, see few epidemic models [9–11] and chemostat model [12].

During the past decades, a study of dynamical behavior of the population species with stochastic impacts has been growing steadily. The interesting situation occurs at the
global stability of all feasible equilibria. Pitchaimani and Rajaji [13] constructed the stochastic Nowak-May model and investigated the asymptotic stability. In addition to stability, for every population model, the problem of permanence and boundedness property is also important. Solutions of the population model are called ultimately bounded if they satisfy the following condition: if we find the existence of bounded region in the solution space of our system such that each solution enters the bounded region in limited time and remains within the region forever. The permanence gives a guarantee that if initially the density of all species is positive, then after a specific time the density of each species will be present in some sizeable amount. Ghosh et al. [14] illustrated a seasonally perturbed stochastic environment, which is described as noise. These fluctuations can be outlined as natural disasters, human intervention, or animal fluctuations. Accordingly, as time tends to be large, every equilibrium solution does not achieve a steady-state value accurately but it fluctuates continuously around the steady state. Recently, Liu et al. [21] developed and analyzed a population model with Holling II response and random effect. To study the model with fluctuations, several authors have introduced ecological fluctuations into every population model to accentuate the reality [11, 13, 16–18, 22–25]. The predator-prey model with two species and ratio dependence is discussed to examine its stability of equilibrium solutions in [26]. Ji et al. [9] introduced two types of functional response and stochastic perturbation into the system. Zhang et al. [27] found the critical value for the stochastic predator-prey system which can be used to determine the extinction and persistence in the mean of the predator population. Zhang and Meng [6] developed the nonautonomous SIRI epidemic model with random disturbance. The above researchers used various noises and different types of functional response depending on the population model. By the above motivation, we consider the predator-prey model with environmental changes in this article.

The article is arranged as follows. In Section 2, we present few definitions, lemmas, and theorems which are utilized in further analysis. In Section 3, we discuss the detailed explanation about the formulation and the condition that solution of the deterministic model is bounded. For the stochastic system, we derive the existence of positive solution of the system and its uniqueness and also explore the conditions for stochastic boundedness in Section 4. In addition, we prove that the solution is uniformly continuous. In Section 5, stochastic permanence and extinction under certain parametric restriction are established. Using the corresponding Lyapunov function, we have examined the conditions on global asymptotic stability in Section 6. Next, we have obtained some figures to justify the results in Section 7. Finally, the conclusion based on our results is presented in Section 8.

2. Preliminaries

Here, we give certain notations, definitions, theorems, and lemmas which are used in the following analysis. For more details, see [28–31].

Consider the stochastic model (SM) of $d$-dimension of the form

$$dZ(t) = g(Z(t), t)dt + h(Z(t), t)dB(t), t_0 \leq t \leq T < \infty, \quad (1)$$

with $Z(t_0) = Z_0$. The functions $g : R^d \times [0, T] \longrightarrow R^d$ and $h : R^d \times [t_0, T] \longrightarrow R^{d \times m}$ are Borel measurable, $B = \{B(t)\}_{t \geq t_0}$ is an $R^{m \times n}$-valued Wiener process, and $Z_0$ is an $R^d$-valued random variable.

The differential operator $\mathcal{L}$ corresponding to the SM (1) is defined as

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} g_i(z, t) \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{i,j=1}^{m} \left(h(z, t)g^T(z, t)\right)_{ij} \frac{\partial^2}{\partial z_i \partial z_j}. \quad (2)$$

Along with the existence and uniqueness assumptions, we make the assumption that $g$ and $h$ satisfy $g(z^*, t) = 0$ and $h(z^*, t) = 0$ for an equilibrium solution $z^*$, for $t \geq t_0$.

Definition 1. The equilibrium solution $z^*$ of the SM (1) is stochastically stable if it satisfies for every $\varepsilon > 0$ and $s \geq t_0$,

$$\lim_{z_0 \rightarrow z^*} \mathbb{P}\left(\sup_{t \leq s} \|Z_{z_0}(t) - z^*\| \leq \varepsilon\right) = 0, \quad (3)$$

where $Z_{z_0}(t)$ represents the solution of (1) with $Z(s) = z_0$ at time $t \geq s$.

Definition 2. The equilibrium solution $z^*$ of the SM (1) is said to be stochastically asymptotically stable if it satisfies the stochastic stability condition and

$$\lim_{z_0 \rightarrow z^*} \mathbb{P}\left(\lim_{t \rightarrow \infty} Z_{z_0}(t) = z^*\right) = 1. \quad (4)$$

Definition 3. The equilibrium solution $z^*$ of the SM (1) is said to be globally stochastically asymptotically stable if it
satisfies the stochastic stability condition and for every \( z_0 \) and every \( s \),

\[
P\left( \lim_{t \to \infty} Z(t) = z^* \right) = 1. \tag{5}\]

**Theorem 4** (see [28]). Let the functions \( g \) and \( h \) have continuous coefficients with respect to \( t \) and satisfy the existence and uniqueness properties.

(i) Suppose that a positive definite function \( V \in C^{2,1}(U_\varepsilon \times [t_0, \infty)) \) exists, where \( U_\varepsilon = \{ z \in \mathbb{R}^d : \| z - z^* \| < \kappa \} \), for \( k > 0 \), such that for all \( t \geq t_0 \), \( \varepsilon \) is Hölder continuous with \( \varepsilon \) to be stochastically ultimately bounded, if, for any \( S \) and every \( \varepsilon \) satis

(ii) Additionally if \( V \) is decreasing and a positive definite function \( V_1 \) exists such that

\[
V(z, t) \leq V_1(z) \text{ for all } t \geq t_0, \quad z \in U_{\varepsilon} \setminus \mathscr{L}(z, t) \leq 0, \quad (6)
\]

then the equilibrium solution \( z^* \) is stochastically asymptotically stable

(iii) If the assumption (ii) holds for a radially unbounded function \( V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty)) \) defined everywhere, then the equilibrium solution \( z^* \) is globally stochastically asymptotically stable

**Lemma 5** (see [29, 31]. Suppose that a stochastic process \( Z(t) \) on \( t \geq 0 \) of \( n \)-dimension satisfies

\[
E[Z(t) - Z(s)]_{\|t-s\|^\alpha} \leq c[t - s]^{\alpha \nu}, \quad 0 \leq \alpha, \nu, \quad t < \infty, \tag{7}
\]

where \( \alpha, \nu, \) and \( c \) are arbitrarily nonnegative constants and a continuous modification \( \hat{Z}(t) \) of \( Z(t) \) exists satisfying the property that, for every \( \psi \in (0, \alpha/\alpha_j) \), there exists a random variable \( \psi(\omega) > 0 \) such that

\[
P\left\{ \omega : \sup_{0 \leq t-s < \psi(\omega), \| t-s \|^\alpha < \infty} \frac{\|Z(t, \omega) - Z(s, \omega)\|}{\|t - s\|^\alpha} \leq \frac{2}{1 - 2^\nu} \right\} = 1, \tag{8}
\]

that is, each sample path of \( \hat{Z}(t) \) is locally but uniformly Hölder continuous with \( \psi \).

**Definition 6** (see [30]). The solution \( Z(t) \) of model (1) is said to be stochastically ultimately bounded, if, for any \( \varepsilon \in (0, 1) \), there is a constant \( \delta = \delta(\varepsilon) > 0 \), such that for any initial value \( Z_0 \in \mathbb{R}^d \), the solution \( Z(t) \) of (1) satisfies

\[
\limsup_{t \to \infty} P\{ |Z(t)| > \delta \} < \varepsilon. \tag{9}
\]

**Definition 7** (see [30]). The solution \( Z(t) \) of (1) possesses stochastic permanent property, if there exists a pair of constants \( \varphi = \varphi(\nu) > 0 \) and \( \chi = \chi(\nu) > 0 \) for any \( \nu \in (0, 1) \) such that the solution \( Z(t) \) of (1) for any initial value \( Z_0 \in \mathbb{R}^d \) satisfies the property

\[
\liminf_{t \to \infty} P\{ |Z(t)| \geq \varphi \} \geq 1 - \nu, \quad \liminf_{t \to \infty} P\{ |Z(t)| \leq \chi \} \geq 1 - \nu. \tag{10}
\]

**3. Deterministic Model**

In this section, we propose a predator-prey model with disease among the prey population. Chattopadhyay and Bairagi [32] framed the ecoepidemiological model with two species dividing into three compartments in the Salton sea and analyzed the stability of the positive equilibrium. Because of the disease, susceptible prey and infected prey are there as two groups in the prey population. The predator mostly eats infected prey because they are easy to catch. So these infected preys become more attractive to the predator. We have assumed that both the preys are subject to predation by the predator. In our article, we considered the population model as in [32] with the inclusion of the susceptible prey and predator interaction and functional response as Holling type II for interaction in the following form:

\[
\begin{aligned}
\frac{dS}{dt} &= rS \left( 1 - \frac{S + I}{K} \right) - \lambda IS - \frac{aPS}{m + S}, \\
\frac{dI}{dt} &= -\mu I + \lambda IS - \frac{BIP}{a + I}, \\
\frac{dP}{dt} &= -dP + \frac{BIP}{a + I} + \frac{aPS}{m + S}.
\end{aligned} \tag{11}
\]

Here, \( S(t), I(t), \) and \( P(t) \) denote the population densities of susceptible prey, infected prey, and predator at any time \( t \) with \( S(0) = S_0 \geq 0, I(0) = I_0 \geq 0, \) and \( P(0) = P_0 \geq 0. \) \( r, K, \) and \( \lambda \) represent the growth rate of \( S, \) carrying capacity of susceptible prey, and disease transmission coefficient. \( a \) is the search rate of the predator towards susceptible prey and \( \beta \) is the search rate of predator towards infected prey. \( m \) and \( d \) are the natural death rates of infected prey and predator. Parameters \( m \) and \( a \) are half saturation constants. System (11) can have at most five equilibrium solutions:

(i) The trivial equilibrium solution \( E_0 = (0, 0, 0) \)

(ii) The equilibrium solution \( E_1 = (K, 0, 0) \) lying on the boundary

(iii) The planar equilibrium solution \( E_2 = (S_2, I_2, 0) \) on the \( S-I \) plane where \( S_2 = \mu \lambda / \lambda I_2 = (r(KA - c)) / (\lambda (r + KA)) \)

(iv) Another planar equilibrium solution \( E_3 = (S_3, 0, P_3) \) on the \( S-P \) plane where \( S_3 = dm(a - d) \) and \( P_3 = (r(Kma - dm^2 - dKm)) / (K(d - a)^2) \)

(v) The positive equilibrium solution \( E_4 = (S^*, I^*, P^*) \) which is obtained as follows:
Let $S^*$ be a nonnegative root of the following equation

$$g(S^*) = S^*^2 + CS^* + D = 0,$$  \hspace{1cm} (12)

where $C = (1/(r(-d + \alpha + \beta)))(adr + dKr - dmr - ara - Kr\alpha - K\beta + m\beta + adK\lambda)$ and $D = (1/(r(-d + \alpha + \beta)))(admr + dKmr - aK\mu - Kmr\beta + adKm\lambda)$.

The roots of the above quadratic equation are

$$S^* = \frac{-C \pm \sqrt{C^2 - 4D}}{2}.$$  \hspace{1cm} (13)

When any one of the following cases is satisfied, the equilibrium solution $S^*$ can have one or two positive values.

(i) $C < 0$ and $D < 0$

(ii) $C < 0, D > 0$ and $C^2 - 4D > 0$

(iii) $C > 0, D < 0$

where $I^* = (a(d(m + S^*) - aS^*))/((\beta - d)(m + S^*) + aS^*)$, $P^* = ((\lambda S^* - \mu)(a + I^*))/\beta$.

The following relations must hold for the positiveness of $I^*$ and $P^*$:

$$aS^* < d(m + S^*) < \beta(m + S^*) + aS^* \text{ and } \lambda S^* > \mu.$$  \hspace{1cm} (14)

The positive equilibrium solution plays a major role in changing the dynamical behavior. It is the only solution where all the species exist. All other equilibria are the subcases of the coexisting equilibrium solution. Therefore, it is essential to analyze the dynamical properties of positive equilibrium and also it gives the behavior of each species exactly.

Now, we provide certain conditions to bound the solutions of the system through the boundedness of the model equation (11).

**Theorem 8.** All the solutions of system (11) in $R^3_+$ with positive initial conditions are uniformly bounded.

**Proof.** To get the boundedness of solutions of given system (11), we consider the function

$$U = S + I + P.$$  \hspace{1cm} (15)

Differentiate the above equation with respect to time $t$ to obtain

$$\frac{dU}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dP}{dt} = rS \left(1 - \frac{S + I}{K}\right) - \lambda IS - \frac{aSP}{m + S} - \mu I + \frac{\lambda IP}{a + I} - dP + \frac{\lambda IP}{a + I} + \frac{aPS}{m + S},$$

$$= rS \left(1 - \frac{S + I}{K}\right) - \mu I - dP.$$  \hspace{1cm} (16)

For each $0 \leq \eta \leq \min(\mu, d)$, the following inequality holds

$$\frac{dU}{dt} + \eta U \leq \left(r \left(1 - \frac{S}{K}\right) + \eta\right)S + (\eta - \mu)I + (\eta - d)P \leq \frac{K}{4r}(r + \eta)^2.$$  \hspace{1cm} (17)

The maximum value of the quadratic function $ax^2 + bx + c$ is $c - (b^2/4a)$ when $a < 0$. In this way, we get the max $K$-$(r(1 - (S/K)) + \eta)S$ as $(K/4r)(r + \eta)^2$ (refer [33]). Assume that $L = (K/4r)(r + \eta)^2 > 0$, this implies

$$\frac{dU}{dt} + \eta U < L.$$  \hspace{1cm} (18)

By the theory of differential inequalities, we get

$$0 \leq U(S, I, P) < \frac{L}{\eta}(1 - e^{-\eta t}) + U(S(0), I(0), P(0))e^{-\eta t},$$

and letting $t$ tend to infinity, the above solution is of the form

$$0 \leq U(S, I, P) < \frac{L}{\eta}.$$  \hspace{1cm} (19)

From the above discussion, we conclude that the solution space of system (11) lies within

$$\mathcal{D} = \left\{(S, I, P) \in R^3_+ : U = \frac{L}{\eta} + \epsilon, \text{ for every } \epsilon > 0 \right\}.$$  \hspace{1cm} (20)

Hence, the theorem is proved.

**4. Stochastic Model**

In the natural world, each population in an ecosystem is greatly affected by environmental noises which play a major role in population dynamics. By considering the effect of random environment fluctuations, we have included environmental noise in every equation of our deterministic system (11). In our system, the randomness in the environment will directly affect themselves as fluctuations in the growth rate of the susceptible prey, death rate of the infected prey population, and predator population like

$$r \rightarrow r + \sigma_1 B_1(t),$$
$$-\mu \rightarrow -\mu + \sigma_2 B_2(t),$$
$$-d \rightarrow -d + \sigma_3 B_3(t),$$  \hspace{1cm} (21)

where $B_i(t), \ i = 1, 2, 3$ are independent Brownian motions and $\sigma_i^2 (i = 1, 2, 3)$ denote the intensities of the environmental fluctuations and $\sigma_i (i = 1, 2, 3)$ represent the standard
deviation. With this fact, we have framed the stochastic system by using the Itô equations as follows:

\[
\begin{align*}
    dS &= \left( r - \frac{S + I}{K} \right) dt - \alpha P \frac{S}{m + S} dt + \sigma_1 S dB_1(t), \\
    dI &= \left( -\mu - \lambda S - \frac{\beta P}{a + 1} \right) dt + \sigma_2 I dB_2(t), \\
    dP &= \left( -d + \frac{\beta I}{a + 1} + \alpha S \frac{P}{m + S} \right) dt + \sigma_3 P dB_3(t).
\end{align*}
\]

(23)

During the past several years, no work has been reported on the above stochastic model (23). Our aim is to find the dynamics of the stochastic system (23) and show how each population varies with respect to environmental fluctuations.

Now, we discuss some important properties like positivity, boundedness, and continuity of solution of the stochastic model (23).

**Theorem 9.** For \((S_0, I_0, P_0) \in \text{Int}(R^+_1)\), system (23) has a unique positive local solution \((S(t), I(t), P(t))\) for \(t \in [0, \tau_c)\) almost surely, where \(\tau_c\) is the explosion time.

**Proof.** Consider the transformation of variables

\[
\begin{align*}
    x &= \log S, \\
    y &= \log I, \\
    z &= \log P.
\end{align*}
\]

Using the Itô formula,

\[
\begin{align*}
    \mathcal{L}V &= V_x(t, x) + V_y(t, x) f(t, x) + \frac{1}{2} \text{trace} \left( g^T(t, x) V_{xx}(t, x) g(t, x) \right),
\end{align*}
\]

we get

\[
\begin{align*}
    \mathcal{L}x &= \left( r - \frac{e^x + \rho}{K} \right) - \lambda e^x - \frac{\alpha e^x}{m + e^x} - \frac{1}{2} \sigma_1^2, \\
    dx &= \mathcal{L}x dt + \sigma_1 dB_1(t).
\end{align*}
\]

Similarly, we obtain, from system (23),

\[
\begin{align*}
    dy &= \left( \lambda e^x - \frac{\beta e^x}{a + e^x} - \mu - \frac{1}{2} \sigma_2^2 \right) dt + \sigma_2 I dB_2(t), \\
    dz &= \left( \beta e^x \frac{P}{a + e^x} + \alpha S \frac{P}{m + S} - d - \frac{1}{2} \sigma_3^2 \right) dt + \sigma_3 P dB_3(t),
\end{align*}
\]

with \(x(0) = \log S(0), y(0) = \log I(0),\) and \(z(0) = \log P(0)\). Now, the functions corresponding to system (28) have initial growth and they satisfy the local Lipschitz property. Hence, a unique local solution \((x(t), y(t), z(t))\) exists and it is defined in \([0, \tau_c)\). Consequently, there exists a unique positive local solution of (23) as \(S(t) = e^{\text{th}(t)}, I(t) = e^{\text{th}(t)},\) and \(P(t) = e^{\text{th}(t)}\).

**Theorem 10.** System (23) has a unique solution \((S(t), I(t), P(t))\) for \(t \in [0, \tau_c)\) and for any initial condition \((S_0, I_0, P_0) \in \text{Int}(R^+_1)\) and the solution remains in \(\text{Int}(R^+_1)\) with probability one. Therefore, for all \(t \geq 0, (S(t), I(t), P(t)) \in \text{Int}(R^+_1)\) almost surely.

**Proof.** To show that the global solution exists, it is enough to prove that \(\tau = \infty\) almost surely. Assume that \(\kappa_0\) is a large nonnegative integer such that the closed ball \(B(\kappa_0) \in R^+_1\) contains \((S_0, I_0, P_0)\). We choose for any \(\kappa \geq \kappa_0\) and define the stop-time as

\[
\tau_x = \inf \left\{ t \in [0, \tau_c); S(\tau_x) \notin \left( \frac{1}{\kappa}, \kappa \right) \text{ or } I(\tau_x) \notin \left( \frac{1}{\kappa}, \kappa \right) \text{ or } P(\tau_x) \notin \left( \frac{1}{\kappa}, \kappa \right) \right\}.
\]

(30)

Here, \(\inf \emptyset = \infty\) (\(\emptyset\) is the empty set). Therefore, \(\tau_x\) is increasing as \(k \rightarrow \infty\).

Let \(\tau_{x_0} = \lim_{k \rightarrow \infty} \tau_x\), then \(\tau_{x_0} \leq \tau_c\) almost surely. If \(\tau_{x_0} = \infty\) almost surely is true, then \(\tau_x = \infty\) almost surely. If this statement fails, that is, if \(\tau_{x_0} \neq \infty\), then the two constants \(T > 0\) and \(\varepsilon \in (0, 1)\) exist with

\[
P(\tau_{x_0} < T) > \varepsilon.
\]

(31)

Thus, by denoting \(\Omega_x = \{ \tau_x \leq T \}\), then \(\kappa_i \geq \kappa_0\) is an integer such that, for all \(\kappa \geq \kappa_i\),

\[
P(\tau_x \leq T) \geq \varepsilon.
\]

(32)

Define \(V : \text{Int}(R^+_1) \rightarrow \text{Int}(R_+^1)\) by

\[
V(S, I, P) = (S - 1 - \ln S) + (I - 1 - \ln I) + (P - 1 - \ln P),
\]

(33)

where the function \(V(S, I, P) > 0\) for all \((S, I, P) \in \text{Int}(R^+_1)\).

Using Itô’s formula, we get

\[
\begin{align*}
    \mathcal{L}V &= \left( r - \frac{1}{5} \right) \left( rS - \frac{S + I}{K} \right) - \lambda IS - \frac{\alpha PS}{m + S} \\
    &\quad + \frac{1}{2} \sigma_1^2 + \left( r - \frac{1}{5} \right) \left( -\mu I + \lambda IS - \frac{\beta IP}{a + 1} \right) \\
    &\quad + \frac{1}{2} \sigma_2^2 + \left( \mu I - \frac{\beta IP}{a + 1} + \frac{\alpha PS}{m + S} \right) + \frac{1}{2} \sigma_3^2, \\
    &= (S - 1) \left( r - \frac{1}{5} \right) - \lambda IS - \frac{\alpha PS}{m + S} \\
    &\quad + (I - 1) \left( -\mu I + \lambda IS - \frac{\beta IP}{a + 1} \right) + (P - 1) \left( -d + \frac{\beta I}{a + 1} + \frac{\alpha S}{m + S} \right).
\end{align*}
\]
\[
\begin{align*}
+ \frac{1}{2} \sigma_1^2 + \frac{1}{2} \sigma_2^2 + \frac{1}{2} \sigma_3^2, = (S - 1) & \left( r - \frac{r}{K} (S + I) - \lambda I \right) \\
+ (I - 1)(\lambda S - \mu) - d(P - 1) + & \frac{a S (P - 1) - a P (S - 1)}{m + S} \\
+ \frac{\beta I (P - 1) - \beta P (I - 1)}{a + I} & + \sigma_1^2 + \sigma_2^2 + \sigma_3^2.
\end{align*}
\]  

(34)

Taking the differential of \( V(S, I, P) \), one gets

\[
dV(S, I, P) = f(S, I, P) dt + g(S, I, P) dB(t),
\]  

(35)

where

\[
\begin{align*}
g(S, I, P) &= \sigma_1 (S - 1) + \sigma_2 (I - 1) + \sigma_3 (P - 1), \\
(S, I, P) &= \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} - (S - 1) \left( r - \frac{r}{K} (S + I) + \lambda I - r \right) \\
&\quad - (I - 1)(\mu - \lambda S) - d(P - 1) \\
&\quad + \frac{a S (P - 1) - a P (S - 1)}{m + S} + \frac{\beta I (P - 1) - \beta P (I - 1)}{a + I}.
\end{align*}
\]  

(36)

From [29], it is easy to show that \( f(S, I, P) \) is bounded above, say by \( N \), in \( R^2_1 \) (that is to say \( |f(S, I, P)| \leq N \), for all \( (S, I, P) \in R^2_1 \)). From equation (35), we have

\[
\int_{\tau_k}^{\tau_k+T} dV(S, I, P) \leq \int_{\tau_k}^{\tau_k+T} N dt + \int_{\tau_k}^{\tau_k+T} g(S, I, P) dB(t),
\]  

(37)

where \( \tau_k + T = \min \{ \tau_k, T \} \). Taking the expectations of the above inequality, one gets

\[
EV(S(\tau_k \land T), I(\tau_k \land T), P(\tau_k \land T)) \leq V(S(0), I(0), P(0)) + NT.
\]  

(38)

Note that no less than one of \( S(\tau_k, \omega) \), \( I(\tau_k, \omega) \), and \( P(\tau_k, \omega) \) belongs to the set \( \{ \kappa, 1/\kappa \} \), for every \( \omega \in \Omega_k \); therefore, we get

\[
V(S(\tau_k \land T), I(\tau_k \land T), P(\tau_k \land T)) \geq (\kappa - 1 - \ln \kappa) \wedge \left( \frac{1}{\kappa} - 1 - \ln \frac{1}{\kappa} \right).
\]  

(39)

Hence, from (32),

\[
EV(S(\tau_k \land T), I(\tau_k \land T), P(\tau_k \land T)) \geq \mathbb{E} \left[ I_{\Omega_k(\omega)} V(S(\tau_k \land T), I(\tau_k \land T), P(\tau_k \land T)) \right],
\]  

(40)

\[
\geq \varepsilon (\kappa - 1 - \ln \kappa) \wedge \left( \frac{1}{\kappa} - 1 - \ln \frac{1}{\kappa} \right),
\]

in which \( I_{\Omega_k(\omega)} \) denotes the indicator function of \( \Omega_k \). It follows from (39) that

\[
V(S(0), I(0), P(0)) + NT \geq \varepsilon (\kappa - 1 - \ln \kappa) \wedge \left( \frac{1}{\kappa} - 1 - \ln \frac{1}{\kappa} \right).
\]  

(41)

\( \kappa \to \infty \) leads to a contradiction: \( \infty > V(S(0), I(0), P(0)) + NT = \infty \). Therefore, \( T_\infty = \infty \) almost surely. Thus, \( \tau = \infty \) then \( (S(t), I(t), P(t)) \in R^2_1 \) almost surely.

With the existence of solution, next, we analyze how the solution changes in \( R^2_1 \).

**Theorem 11.** For any initial value \( W_0 = (S_0, I_0, P_0) \in R^2_1 \), the solutions of system (23) are stochastically ultimately bounded.

**Proof.** By Theorem 10, the solution \( W(t) \) remains in \( R^2_1 \) for all \( t \geq 0 \). Consider the function \( V_1(t, S) = e^{\theta \tau} \) for \( \theta > 0 \). Using the Itô formula, we compute

\[
\mathcal{L} V_1 = e^{\theta \tau} \left[ 1 + \left( 1 - S + \frac{r}{K} \right) - \lambda I - \frac{a S}{m + S} \right] + \theta \left( \frac{\theta - 1}{2} \right) \sigma_1^2,
\]

(42)

By considering the integral and expectation on two sides of the above equation, we get \( e^{\theta \tau} \mathbb{E} (S^\theta(t)) - \mathbb{E} (S^\theta_0) \leq M_1 (\theta) e^\theta \). So, we have \( \lim_{t \to \infty} \mathbb{E} (S^\theta(t)) \leq \lim_{t \to \infty} M_1 (\theta) < +\infty \).

Define the function \( V_2(t, I) = e^{\theta I} \) for \( \theta > 0 \); using the Itô formula, we get

\[
\mathcal{L} V_2 = e^{\theta I} \left[ 1 + \left( -\mu - \lambda S - \frac{\beta I}{a + I} \right) \theta + \frac{\theta (\theta - 1)}{2} \sigma_2^2 \right],
\]

(43)

Then, \( e^{\theta I} (\mathbb{E} (I^\theta(t)) - \mathbb{E} (I^\theta_0)) \leq M_2 (\theta) e^\theta \). So, we have \( \lim_{t \to \infty} \mathbb{E} (I^\theta(t)) \leq \lim_{t \to \infty} M_2 (\theta) < +\infty \). Similarly, defining the function \( V_3(t, P) = e^{\theta P} \) for \( \theta > 0 \) and applying the Itô formula, we get

\[
\mathcal{L} V_3 = e^{\theta P} \left[ 1 + \left( -d - \frac{\beta P}{a + I} + \frac{a S}{m + S} \right) \theta + \frac{\theta (\theta - 1)}{2} \sigma_3^2 \right] \leq M_3 (\theta) e^\theta.
\]  

(44)

Then, \( e^{\theta P} (\mathbb{E} (P^\theta(t)) - \mathbb{E} (P^\theta_0)) \leq M_3 (\theta) e^\theta \). So, we have \( \lim_{t \to \infty} \mathbb{E} (P^\theta(t)) \leq \lim_{t \to \infty} M_3 (\theta) < +\infty \).
For $W(t) = (S(t), I(t), P(t)) \in R^3$, we may get

$$|W(t)|^\theta \leq \left(3 \max \left\{S^\theta(t), I^\theta(t), P^\theta(t)\right\}\right)^{\theta/3} \leq 3^{\theta/3} \left(S^\theta(t) + I^\theta(t) + P^\theta(t)\right).$$  \hspace{1cm} (45)

Consequently,

$$\limsup_{t \to \infty} E|W(t)|^\theta \leq M_4(\theta) < +\infty,$$

where $M_4(\theta) = 3^{\theta/3}(M_1(\theta) + M_2(\theta) + M_3(\theta))$. Applying Chebyshev inequality, we get that all solutions are stochastically bounded.

Using fundamental properties and suitable Lyapunov functions, we continue to show that the positive solution $W(t) = (S(t), I(t), P(t))$ is uniformly Hölder continuous.

**Theorem 12.** Every sample path of $(S(t), I(t), P(t))$ is uniformly continuous, where $(S(t), I(t), P(t))$ is a solution of system (23) on $t \geq 0$ with $(S_0, I_0, P_0) \in R^3_+.$

**Proof.** The modified form of the first equation of system (23) is

$$S(t) = S_0 + \int_0^t S(u) \left(1 - \frac{S + I}{K} - \lambda - \frac{\alpha P}{m + S}\right) du$$

$$+ \int_0^t \sigma_1 S(u) dB_1(u).$$

Assume that $f_1(u) = S(u)(1 - ((S(u) + I(u))/K) - \lambda (u) - (\alpha P(u)/(m + S(u))))$ and $f_2(u) = \sigma_1 S(u)$.

From Theorem (28), we deduce that

$$E|f_1(t)|^\theta = E \left[S^\theta(1 - \frac{S + I}{K} - \lambda - \frac{\alpha P}{m + S})\right]^\theta,$$

$$= E \left[|S|^\theta \left(1 - \frac{S + I}{K} - \lambda - \frac{\alpha P}{m + S}\right)^\theta\right],$$

$$\leq \frac{1}{2} E|S|^\theta + \frac{1}{2} E[r + rS + (r + \lambda)I + \alpha P]^\theta,$$

$$\leq \frac{1}{2} E|S|^\theta + 4^{\theta-3/2} \left[r^\theta + r^\theta E|S|^\theta\right]$$

$$+ (r + \lambda)^\theta E|I|^\theta + \alpha^\theta E|P|^\theta,$$

$$\leq \frac{1}{2} M_1(2\theta) + 4^{\theta-3/2} \left[r^\theta + r^\theta M_1(2\theta)\right]$$

$$+ (r + \lambda)^\theta M_2(2\theta) + \alpha^\theta M_3(2\theta),$$

$$= F_1(\theta),$$

$$E|f_2(t)|^\theta = E|\sigma_1 S(u)|^\theta = \sigma_1^\theta E|S|^\theta \leq \sigma_1^\theta M_1(\theta) \leq F_2(\theta).$$

(48)

For stochastic integrals, we observe the moment inequality and apply for $0 \leq t_1 \leq t_2$ and $\theta > 2$, to get

$$E \left[\int_{t_1}^{t_2} f_2(u) dB_1(u)\right]^\theta \leq \left(\frac{(\theta - 1)\theta}{2}\right)^{\theta/2} (t_2 - t_1)^{\theta-2/2} \int_{t_1}^{t_2} E|f_2(u)|^\theta du,$$

$$\leq \left(\frac{(\theta - 1)\theta}{2}\right)^{\theta/2} (t_2 - t_1)^{\theta/2} F_2(\theta).$$

(49)

Then, for $0 < t_1 < t_2 < \infty$, $t_2 - t_1 \leq 1/(\theta) + 1/\kappa = 1$, we have

$$E|S(t_2) - S(t_1)|^\theta$$

$$= E \left[\int_{t_1}^{t_2} f_1(u) du + \int_{t_1}^{t_2} f_2(u) dB_1(u)\right]^\theta,$$

$$\leq 2^{\theta-1} E \left[\int_{t_1}^{t_2} f_1(u) du\right]^\theta + 2^{\theta-1} E \left[\int_{t_1}^{t_2} f_2(u) dB_1(u)\right]^\theta,$$

$$\leq 2^{\theta-1} (t_2 - t_1)^{\theta-\kappa} E \left[\int_{t_1}^{t_2} |f_1(u)|^\theta du\right] + 2^{\theta-1} \left(\frac{(\theta - 1)\theta}{2}\right)^{\theta/2}$$

$$\cdot (t_2 - t_1)^{\theta/2} (t_1 - t_2)^{\theta/2} F_2(\theta),$$

$$= 2^{\theta-1} (t_2 - t_1)^{\theta} F_1(\theta) + 2^{\theta-1} \left(\frac{(\theta - 1)\theta}{2}\right)^{\theta/2}$$

$$\cdot (t_2 - t_1)^{\theta/2} F_2(\theta),$$

$$\leq 2^{\theta-1} (t_2 - t_1)^{\theta/2} \left(1 + \left(\frac{(\theta - 1)\theta}{2}\right)^{\theta/2}\right)(F_1(\theta) + F_2(\theta)).$$

(50)

where $F(\theta) = \max \{F_1(\theta), F_2(\theta)\}$. By Lemma 5, for each exponent $\nu \in (0, (\theta - 2)/2\theta)$, we get that each sample path of $S(t)$ is uniformly and locally Hölder continuous and which shows the uniform continuity of each sample path of $S(t)$ on $t \in R^3_+$. Similarly, the uniform continuity of $I(t)$ and $P(t)$ is proved on $R^3_+$. Therefore, we get the uniform continuity of each sample path of $(S(t), I(t), P(t))$ to system (23) on $t \geq 0$.

**5. Long Time Behavior of System**

Here, we look at the solution behaviour of system (23) as time becomes very large. For that, we define the hypotheses which are useful in further analysis.

\begin{align}
(H1) : & \frac{L}{\eta} \max \frac{r}{(K - m)} + \frac{1}{2} \max \{\alpha_1^2, \alpha_2^2, \alpha_3^2\} \\
& < \min \{\mu - \rho, \rho\}, \\
(H2) : & r - \frac{\alpha_1^2}{2} < 0, \\
& \lambda - \mu - \frac{\alpha_2^2}{2} < 0, \\
& \alpha + \beta - d - \frac{\alpha_3^2}{2} < 0.
\end{align}

(51)
First, we will prove stochastic permanence which plays an essential part in population dynamics. We discuss this property as follows:

**Theorem 13.** If the assumption (H1) holds, then system (23) is stochastically permanent.

**Proof.** For any initial value \( W(0) = (S(0), I(0), P(0)) \in \mathbb{R}^3 \), we show that there exists a solution \( W(t) = (S(t), I(t), P(t)) \) such that

\[
\lim_{t \to +\infty} \mathbb{E} \left( \frac{1}{W(t)} \right) \leq M,
\]

where arbitrary nonnegative constant \( \gamma \) satisfies

\[
\frac{L}{\eta} \max \left\{ \frac{r}{K}, \frac{\alpha}{m} \right\} + \frac{(y+1)}{2} \max \{ \sigma_1^2, \sigma_2^2, \sigma_3^2 \} < \min \{ r - \mu, r - d \}. \tag{53}
\]

By (53), there is a positive constant \( \rho > 0 \) such that

\[
\min \{ r - \mu, r - d \} - \rho - \frac{L}{\eta} \max \left\{ \frac{r}{K}, \frac{\alpha}{m} \right\} - \frac{1}{2} \max \{ \sigma_1^2, \sigma_2^2, \sigma_3^2 \} > 0.
\]

Define \( V(S, I, P) = S + I + P \) for \( (S, I, P) \in \mathbb{R}^3 \) and \( Z(S, I, P) = 1/V(S, I, P) \); from the Itô formula, we have

\[
dV(S, I, P) = \left\{ S \left[ r \left( 1 - \frac{S+I}{K} \right) - \lambda - \frac{aP}{m+S} \right] + I \left[ -\mu + \lambda S - \frac{bP}{a+I} \right] + P \left[ -d + \frac{\beta I}{a+I} + \frac{aS}{m+S} \right] \right\} dt
+ (\sigma_1 SdB_1(t) + \sigma_2 IdB_2(t) + \sigma_3 PdB_3(t)),
\]

\[
dZ(W) = \left\{ -Z^2(W) \left\{ S \left[ r \left( 1 - \frac{S+I}{K} \right) - \lambda - \frac{aP}{m+S} \right] + I \left[ -\mu + \lambda S - \frac{bP}{a+I} \right] + P \left[ -d + \frac{\beta I}{a+I} + \frac{aS}{m+S} \right] \right\} dt
+ Z^2(W) [\sigma_1 S]^2 + (\sigma_2 I)^2 + (\sigma_3 P)^2 dt
- Z^2(W) [\sigma_1 SdB_1(t) + \sigma_2 IdB_2(t) + \sigma_3 PdB_3(t)],
\]

\[
= Z^2(W) dt - Z^2(W) [\sigma_1 SdB_1(t) + \sigma_2 IdB_2(t) + \sigma_3 PdB_3(t)].
\tag{55}
\]

Under the hypothesis (H1), we introduce \( \gamma \) as a positive constant such that condition (53) holds. By Itô’s formula, we obtain

\[
\mathcal{L}(1 + Z(W))^\gamma = \gamma (1 + Z(W))^{\gamma-1} \mathcal{L}Z(W)
+ \frac{1}{2} \gamma (\gamma-1) (1 + Z(W))^{\gamma-2} \mathcal{L}^2 Z(W)
\times [\sigma_1 S]^2 + (\sigma_2 I)^2 + (\sigma_3 P)^2].
\tag{56}
\]

Next, we choose \( \rho > 0 \) to be small such that condition (54) holds. Then,

\[
\mathcal{L}e^{\rho t} (1 + Z(W))^\gamma = \rho e^{\rho t} (1 + Z(W))^\gamma + e^{\rho t} \mathcal{L} (1 + Z(W))^\gamma,
= e^{\rho t} (1 + Z(W))^{\gamma-2} [\rho (1 + Z(W))^2 + B],
\tag{57}
\]

where

\[
B = -\gamma Z^2(W) \left\{ S \left[ r \left( 1 - \frac{S+I}{K} \right) - \lambda I - \frac{aP}{m+S} \right] + I \left[ -\mu + \lambda S - \frac{bP}{a+I} \right] + P \left[ -d + \frac{\beta I}{a+I} + \frac{aS}{m+S} \right] \right\}
- \gamma Z^2(W) \left\{ S \left[ r \left( 1 - \frac{S+I}{K} \right) - \lambda I - \frac{aP}{m+S} \right] + I \left[ -\mu + \lambda S - \frac{bP}{a+I} \right] + P \left[ -d + \frac{\beta I}{a+I} + \frac{aS}{m+S} \right] \right\}
+ \gamma Z^2(W) [\sigma_1 S]^2 + (\sigma_2 I)^2 + (\sigma_3 P)^2
+ \frac{\gamma (\gamma+1)}{2} Z^2(W) [\sigma_1 S]^2 + (\sigma_2 I)^2 + (\sigma_3 P)^2].
\tag{58}
\]

The upper bound of the function \((1 + Z(W))^{\gamma-2} [\rho (1 + Z(W))^2 + B] \) is defined in the following way:

\[
\gamma Z^2(W) [\sigma_1 S]^2 + (\sigma_2 I)^2 + (\sigma_3 P)^2
\leq \gamma Z(W) \max \{ \sigma_1^2, \sigma_2^2, \sigma_3^2 \},
\]

\[
\frac{\gamma (\gamma+1)}{2} Z^2(W) [\sigma_1 S]^2 + (\sigma_2 I)^2 + (\sigma_3 P)^2
\leq \frac{\gamma (\gamma+1)}{2} Z^2(W) \max \{ \sigma_1^2, \sigma_2^2, \sigma_3^2 \}. \tag{59}
\]

Hence,

\[
\mathcal{L}e^{\rho t} (1 + Z(W))^\gamma = e^{\rho t} (1 + Z(W))^\gamma - \rho e^{\rho t} (1 + Z(W))^\gamma + \frac{\gamma (\gamma+1)}{2} Z^2(W) \max \{ \sigma_1^2, \sigma_2^2, \sigma_3^2 \} Z(W)
+ \left[ \mu - \gamma \min \{ r - \mu, r - d \} + \frac{\gamma L}{\eta} \max \left\{ \frac{r}{K}, \frac{\alpha}{m} \right\} \right] Z^2(W)
+ \frac{\gamma (\gamma+1)}{2} \max \{ \sigma_1^2, \sigma_2^2, \sigma_3^2 \} Z^2(W).
\tag{60}
\]

From (53) and (54), we get a nonnegative constant \( \xi \) satisfying \( \mathcal{L}e^{\rho t} (1 + Z(W))^\gamma \leq Q e^{\rho t} \). This implies that

\[
\mathbb{E} [e^{\rho t} (1 + Z(W))^\gamma] \leq (1 + Z(W))^\gamma + \frac{Q (e^{\rho t} - 1)}{\rho}. \tag{61}
\]
Therefore,
\[
\limsup_{t \to \infty} E[Z^*(W(t))] \leq \limsup_{t \to \infty} E[(1 + Z(W(t)))^r] \leq \frac{Q}{\rho}.
\] (62)

Note that,
\[
(S + I + P)^r \leq 3^r (S^3 + I^3 + P^3)^{\frac{1}{3}} = 3^r |W|^r,
\] (63)

where \( W = (S, I, P) \in R_+^3 \). Accordingly,
\[
\limsup_{t \to \infty} E \left[ \frac{1}{|W|^r} \right] \leq 3^r \limsup_{t \to \infty} E[Z^*(W(t))] \leq 3^r \frac{Q}{\rho} = R.
\] (64)

Assume that \( \varphi = (\nu R)^{\frac{1}{r}} \) for any \( \nu > 0 \); then, by Chebyshev’s inequality, we get
\[
P(|W(t)| \leq \varphi) = P\left( |W(t)|^{-r} \leq \varphi^{-r} \right) \leq E[|W(t)|^{-r}] / \varphi^{-r} = \varphi^r E[|W(t)|^{-r}],
\] (65)

that is,
\[
\liminf_{t \to \infty} P(|W(t)| \geq \varphi) \geq 1 - \nu.
\] (66)

Similarly, we can get \( \chi > 0 \) for any \( \epsilon > 0 \) such that \( \liminf_{t \to \infty} P(|W(t)| \leq \chi) \geq 1 - \nu \). Hence, by Definition 7, system (23) is stochastically permanent.

In population dynamics, there is a chance to lose the species population fully. So, the study of disappearance of species is also much important in the ecosystem.

**Theorem 14.** If the assumption (H2) holds, then the solution \( W(t) = (S(t), I(t), P(t)) \) of system (23) will be extinct with probability one for any given initial value \( W(0) = (S(0), I(0), P(0)) \in R_+^3 \).

**Proof.** Define \( V_4 = \ln S \). By Itô’s formula, we obtain
\[
d(\ln S) = \left( r \left( 1 - \frac{S + I}{K} \right) - \lambda I - \frac{\alpha P}{m + S} - \frac{1}{2} \sigma_1^2 \right) dt + \sigma_1 dB_1(t).
\] (67)

Integrating it from 0 to \( t \) gives
\[
\ln S(t) = \ln S(0) + \left( r - \frac{\sigma_1^2}{2} \right) t - \frac{r}{K} \int_0^t S(u) du + \left( \frac{r}{K} + \lambda \right) \int_0^t I(u) du - \alpha \int_0^t \frac{P(u)}{m + S(u)} du + \sigma_1 \int_0^t dB_1(u).
\] (68)

Then,
\[
\ln S(t) \leq \ln S(0) + \left( r - \frac{\sigma_1^2}{2} \right) t - \frac{r}{K} \int_0^t S(u) du + \sigma_1 B_1(t).
\] (69)

Divide the above inequality on two sides by \( t \) and taking \( t \to \infty \), we get
\[
\limsup_{t \to \infty} \frac{\ln S(t)}{t} \leq r - \frac{\sigma_1^2}{2} < 0,
\] (70)

almost surely.

Also, define the Lyapunov function \( V_5 = \ln I \) use Itô’s formula to obtain
\[
d(\ln I) = \left( -\mu + \lambda S - \frac{\beta P}{a + I} - \frac{1}{2} \sigma_2^2 \right) dt + \sigma_2 dB_2(t).
\] (71)

Integrating this from 0 to \( t \), we have
\[
\ln I(t) = \ln I(0) - \int_0^t \left( -\mu + \frac{\sigma_2^2}{2} \right) dt - \int_0^t \frac{\beta P}{a + I(u)} du - \int_0^t \sigma_2 dB_2(u).
\] (72)

Consequently,
\[
\ln I(t) \leq \ln I(0) + \left( \lambda - \mu - \frac{\sigma_2^2}{2} \right) t + \sigma_2 B_2(t).
\] (73)

Dividing above inequality by \( t \) and taking \( t \to \infty \), we obtain
\[
\limsup_{t \to \infty} \frac{\ln I(t)}{t} \leq \lambda - \mu - \frac{\sigma_2^2}{2} < 0,
\] (74)

almost surely.

Similarly, we define the Lyapunov function \( V_6 = \ln P \) and by Itô’s formula, we get
\[
d(\ln P) = \left( -d - \frac{\beta I}{a + I} + \frac{\alpha S}{m + S} - \frac{1}{2} \sigma_3^2 \right) dt + \sigma_3 dB_3(t).
\] (75)

Integrating it, we get
\[
\ln P(t) = \ln P(0) - \int_0^t \left( d + \frac{\sigma_3^2}{2} \right) + \beta \int_0^t \frac{I(u)}{a + I(u)} du + \sigma_3 \int_0^t dB_3(u).
\] (76)
Accordingly,
\[
\ln P(t) \leq \ln P(0) + \left( \alpha + \beta - d - \frac{\sigma_1^2}{2} \right) t + \sigma_2 B_3(t). \tag{77}
\]

Then, dividing this by \( t \) and taking \( t \to \infty \), we get
\[
\limsup_{t \to \infty} \frac{\ln P(t)}{t} \leq \alpha + \beta - d - \frac{\sigma_1^2}{2} < 0, \tag{78}
\]
almost surely.

Hence, the desired assertion is proved.

### 6. Stochastic Asymptotic Stability

In this section, we prove that both planar and coexistence equilibrium of system (23) are stochastically asymptotically stable under certain assumptions.

**Theorem 15.** The equilibrium solution \( E_1 = (K, 0, 0) \) of system (23) is stochastically asymptotically stable on \( \mathcal{D} \), if

\[
R_0 \leq 1, \quad R_1 \leq 1, \quad \frac{1}{2} \frac{\sigma_1^2 K}{S^2} \leq \eta_1(S, I, P),
\]

where \( \eta_1(S, I, P) = (r/K)(S - K)^2 - (\lambda K - \mu)I - (\alpha K/(m + K) - d)P \).

**Proof.** We use Theorem (1) and construct the suitable Lyapunov function as follows:

\[
V(S, I, P) = \left( S - K - K \ln \frac{S}{K} \right) + I + P. \tag{80}
\]

We define the infinitesimal generator \( \mathcal{L} \) on the Lyapunov function and get

\[
\mathcal{L}V(S, I, P) = (S - K) \left[ r \left( \frac{1}{K} \right) - \lambda I - \frac{aP}{m + S} \right]
+ \left[ -\mu + \lambda S - \frac{\beta IP}{a + I} \right] + \left[ -dP + \frac{\beta IP}{a + I} + \frac{aPS}{m + S} \right]
+ \frac{1}{2} \frac{\sigma_1^2 K}{S^2}, \quad (S - K) \left[ r \left( \frac{1}{K} \right) - \lambda I \right] - \lambda(S - K)
+ \frac{1}{2} \frac{\sigma_1^2 K}{S^2},

\]

\[
\leq (S - K) \left[ r \left( \frac{1}{K} \right) + (\lambda K - \mu)I + \left( \frac{\alpha K}{m + K} - d \right) P \right]
+ \frac{1}{2} \frac{\sigma_1^2 K}{S^2}, = -\eta_1(S, I, P) + \frac{1}{2} \frac{\sigma_1^2 K}{S^2}. \tag{81}
\]

If \( R_0 \leq 1, R_1 \leq 1 \), it follows that \( \eta_1(S, I, P) \geq 0 \). By assumptions, \( \mathcal{L}V(S, I, P) < 0 \) on \( \mathcal{D} \). Therefore, \( E_1 \) is stochastically asymptotically stable.

**Theorem 16.** The equilibrium solution \( E_2 = (S_2, I_2, 0) \) of system (23) is stochastically asymptotically stable on \( \mathcal{D} \), if

\[
d > d_1, \quad \frac{1}{2} \frac{\sigma_1^2 S_2}{S^2} + \frac{1}{2} \frac{\sigma_2^2 I_2}{I^2} \leq \eta_2(S, I, P), \tag{82}
\]

where \( \eta_2(S, I, P) = (3\sqrt{r/K})(S - S_2)^2 + (r/K)(I - I_2)^2 - \left( (\alpha S_2/m) + (\beta I_2/a) - d \right) \) and \( d_1 = (m r\beta (K\lambda - \mu) + am\mu(K\lambda + r)) / (am\lambda(K\lambda + r)) \).

**Proof.** Consider the Lyapunov function for the equilibrium solution \( E_2 \)

\[
V(S, I, P) = \left( S - S_2 - S_2 \ln \frac{S}{S_2} \right) + \left( I - I_2 - I_2 \ln \frac{I}{I_2} \right) + P. \tag{83}
\]

Define the infinitesimal generator \( \mathcal{L} \) on the above function as follows:

\[
\mathcal{L}V(S, I, P) = (S - S_2) \left[ r \left( \frac{1}{K} \right) - I - \frac{aP}{m + S} \right] + (I - I_2)
\]

\[
- \lambda(S - S_2) \left[ \frac{S_2}{K} - \frac{S + I}{K} \right] - \lambda(I - I_2)
\]

\[
+ \frac{1}{2} \frac{\sigma_1^2 S_2}{S^2} + \frac{1}{2} \frac{\sigma_2^2 I_2}{I^2}, = (S - S_2) \left[ r \left( \frac{S_2}{K} - \frac{S + I}{K} \right) \right]
\]

\[
- \lambda(I - I_2) - \frac{aP}{m + S} \right] + (I - I_2) \left[ \lambda(S - S_2) - \frac{\beta P}{a + I} \right]
\]

\[
+ \left[ \frac{\beta IP}{a + I} + \frac{aPS}{m + S} \right] + \frac{1}{2} \frac{\sigma_1^2 S_2}{S^2} + \frac{1}{2} \frac{\sigma_2^2 I_2}{I^2},
\]

\[
= -r \frac{S_S^2}{K} - \frac{r}{K} (S - S_2)^2 - \frac{r}{K} (S - S_2)^2 - \frac{r}{K} (I - I_2)^2
\]

\[
+ \left[ \frac{aS_2}{m + S} + \frac{\beta I_2}{a + I} - d \right] P + \frac{1}{2} \frac{\sigma_2^2 I_2}{S^2} + \frac{1}{2} \frac{\sigma_1^2 S_2}{I^2}, \tag{84}
\]

If \( d > d_1 \) where \( d_1 = (m r\beta (K\lambda - \mu) + am\mu(K\lambda + r)) / (am\lambda(K\lambda + r)) \), we obtain \( \eta_2(S, I, P) \geq 0 \). By assumptions, \( \mathcal{L}V(S, I, P) < 0 \) on \( \mathcal{D} \). Therefore, \( E_2 \) is stochastically asymptotically stable.
Theorem 17. The equilibrium solution $E_3 = (S_3, 0, P_3)$ of system (23) is stochastically asymptotically stable on $\mathcal{D}$, if

$$m_1 < m < m_2 \text{ and } \frac{1}{2} \sigma_1^2 S_3 + \frac{1}{2} \sigma_2^2 P_3 \leq \eta_3(S, I, P),$$

(85)

where $m_1 = (d-a)(dK + 2r - 2Kr)/2 dr$, $m_2 = \mu(a-d)/d\lambda$, and $\eta_3(S, I, P) = ((r/K) - (aP_3/(m+S_3)) + (aS_3/(2(m+S_3))))(S-S_3)^2 + (aS_3/(2(m+S_3))(P-P_3)^2 - (\lambda S_3 - \mu)I)$.

Proof. The Lyapunov function for the equilibrium solution $E_3$ can be written as

$$V(S, I, P) = \left( S - S_3 - S_3 \ln \frac{S}{S_3} \right) + I + \left( P - P_3 - P_3 \ln \frac{P}{P_3} \right).$$

(86)

Define the infinitesimal generator $\mathcal{L}$ on $V(S, I, P)$ as

$$\mathcal{L} V(S, I, P) = \left( S - S_3 \right) \left[ - \frac{r}{K} \left( 1 - \frac{S + I}{K} \right) - \lambda I - \frac{aP}{m+S} \right] + \left[ - \mu I + \lambda IS - \frac{\beta IP}{a+1} \right] + \left( P - P_3 \right) \left[ - d + \frac{\beta I}{a+1} + \frac{aS}{m+S} \right] + \frac{1}{2} \sigma_1^2 S_3 + \frac{1}{2} \sigma_2^2 P_3, \leq \eta_3(S, I, P).$$

(87)

One can see that $\eta_3(S, I, P) \geq 0$, if $m_1 < m < m_2$ where $m_1 = (d-a)(dK + 2r - 2Kr)/2 dr$ and $m_2 = \mu(a-d)/d\lambda$. By assumptions, $\mathcal{L} V(S, I, P) < 0$ on $\mathcal{D}$. Therefore, $E_3$ of system (23) is stochastically asymptotically stable.

Theorem 18. The equilibrium solution $E_4 = (S^*, I^*, P^*)$ of system (23) is stochastically asymptotically stable on $\mathcal{D}$, if

$$\frac{1}{2} \sigma_1^2 S^* + \frac{1}{2} \sigma_2^2 P^* \leq \eta_4(S, I, P).$$

(88)

where $r_1 = (1/2K)((\alpha P^*/(m+S^*))) - (\lambda(n_1-1)/2r))$, $r_2 = (1/2K)((\lambda(n_1-1)/2) - (\beta P^*/(a+1)))$, and $\eta_4(S, I, P) = ((r/K) - (\alpha P^*/(m+S^*))) + (\lambda(n_1-1)/2) - (\beta P^*/(a+1)P_3)(S-S^*)^2 + (\lambda(n_1-1)/2 - (r/2K) - n_1 \beta P^*/(a+1))(I-I^*)^2$.

for $n_1 = (a(m+S^*)/m(a+1)^*)$ and $n_2 = (m+S^*)/m$.

Proof. Construct the suitable Lyapunov function for the existence equilibrium as follows

$$V(S, I, P) = \left( S - S^* - S^* \ln \frac{S}{S^*} \right) + n_1 \left( I - I^* - I^* \ln \frac{I}{I^*} \right) + n_2 \left( P - P^* - P^* \ln \frac{P}{P^*} \right).$$

(89)

where $n_i > 0, (i = 1, 2)$ are suitable constants to be determined in further analysis. Define the infinitesimal generator $\mathcal{L}$ on $V(S, I, P)$ and we get

$$\mathcal{L} V(S, I, P) = \left( S - S^* \right) \left[ - \frac{r}{K} \left( 1 - \frac{S + I}{K} \right) - \lambda I - \frac{aP}{m+S} \right] + n_1 \left( I - I^* - I^* \ln \frac{I}{I^*} \right) + n_2 \left( P - P^* - P^* \ln \frac{P}{P^*} \right).$$

(90)

where $n_1 > 0, (i = 1, 2)$ are suitable constants to be determined in further analysis. Define the infinitesimal generator $\mathcal{L}$ on $V(S, I, P)$ and we get

$$\mathcal{L} V(S, I, P) = \left( S - S^* \right) \left[ - \frac{r}{K} \left( 1 - \frac{S + I}{K} \right) - \lambda I - \frac{aP}{m+S} \right] + n_1 \left( I - I^* - I^* \ln \frac{I}{I^*} \right) + n_2 \left( P - P^* - P^* \ln \frac{P}{P^*} \right).$$

(91)
where $\Delta t$ is a positive time increment parameter and $\xi_k$, $\eta_k$, $\zeta_k$, $(k = 1, 2, 3, \ldots, n)$ represent independent random Gaussian variables $N(0,1)$, it is given with the help of pseudorandom number generators. Using MATLAB software and possible parameters, we have generated the solution curves of the deterministic system (11) and corresponding stochastic system (23).

In Figures 1–3, we take the initial conditions $S(0) = 0.1, I(0) = 0.1$, and $P(0) = 0.1$ and parameters $r = 1, K = 3, \beta = 1, \alpha = 1.15, c = 0.7, d = 1, a = 1, m = 1$, and $\lambda = 0.5$. The only difference between Figures 1–3 is the intensities of environmental changes. Specially, we choose $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$ in Figure 1; in Figure 2, we have chosen $\sigma_1 = \sigma_2 = \sigma_3 = 0.04$ and in Figure 3 $\sigma_1 = \sigma_2 = \sigma_3 = 0.01$. From Figures 1–3, we have observed that the coexistence equilibrium $(S^*, I^*, P^*)$ solution curves of stochastic model (23) always oscillates with respect to the curves of the deterministic model (11). From Figures 1–3, we have arrived at the conclusion that with the decrease in the values of $\sigma_1$, $\sigma_2$, $\sigma_3$ the fluctuations of the solution curves of the stochastic system are reduced and coincide with that of the solution curves of the deterministic system.

In Figure 4, the parametric values are $r = 0.09, K = 4, \beta = 0.09, \alpha = 1.1, c = 0.02, d = 0.8, a = 1, m = 1$, and $\lambda = 0.005$ with initial values $S(0) = 1.5, I(0) = 0.8, P(0) = 0.6$. By Theorem 14, we observe that all the species in the stochastic case (23) disappear completely under the hypothesis (H1) and we have seen this graphically via Figure 4. Further, we have simulated the corresponding deterministic case in Figure 4 which satisfies condition (14) and leading to the permanence of each species. Under parametric values $r = 0.001, K = 2.5, \beta = 0.01, \alpha = 0.04, c = 0.02, d = 0.01, a = 1, m = 1$, and $\lambda = 0.003$ and initial parametric condition $S(0) = 2, I(0) = 1.3, P(0) = 0.5$, by Theorem 14, all the species of the stochastic system (23) as well as corresponding deterministic system (11) disappear fully under the hypothesis (H1) and the solution curves are shown in Figure 5. If the deterministic system goes to extinction, the stochastic system also goes to extinction, in which case solution is not dependent on the intensity of noise value (Compare Figure 4 with Figure 5).

Furthermore, in Figure 6, parameters chosen are $r = 3, K = 3, \beta = 1.5, \alpha = 1, c = 1, d = 1, a = 1, m = 1$, and $\lambda = 1.7$ with initial values $S(0) = 1.5, I(0) = 0.8, P(0) = 0.1$ and simulations also confirm the result that all three species of system (11) and (23) are permanent by Theorem 13.

Finally, Figure 7 illustrates the Theorem 18. We choose the parameters $r = 2.5, K = 3, \beta = 1.5, \alpha = 0.9, c = 0.8, d = 1, a = 1, m = 1, a = 1, m = 1$, and $\lambda = 1.2$ with different initial values satisfying condition (88). Theorem 18 says that $(S^*, I^*, P^*)$ of (23) is stochastically asymptotically stable.

8. Concluding Remarks

In reality, predator-prey system reacts to more consequences in the environmental changes in the ecosystem. So, the study of dynamical behavior of the predator-prey model has been receiving more attention, and many efforts have been taken in the field of population dynamics by several authors. To overcome the effect of random fluctuations and control the
disease in an ecosystem, we have proposed the stochastic Holling type II predator-prey model (23) with disease in the prey. We are interested to know the changes of dynamical behavior of model (23) with respect to the intensities of environmental fluctuations. There is also need to reveal the relationship between the intensities of the environmental fluctuations and parameters associated with the system since the predator-prey model is disturbed by environmental noises.

First, we explain the basic assumptions of our model (11) with boundedness of solutions. We have introduced the stochastic term to the deterministic model (11) and proved that

![Figure 1: Simulation showing the solution curves of deterministic system (11) and corresponding stochastic system (23) with large noise $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$.](image1)

![Figure 2: Simulation showing the solution curves of deterministic system (11) and corresponding stochastic system (23) with noise $\sigma_1 = \sigma_2 = \sigma_3 = 0.04$.](image2)
for any positive initial value there exists a positive global solution of the stochastic system (23). For a stochastic predator-prey model, the boundedness of a solution is also verified because it gives a guarantee of system validity. In Theorem 12, we have checked the property of uniform continuity of the positive solution of system (23). Permanence and extinction property are provided for system (23) since as time tends to be large it validates the long time and short time survival in an ecosystem. Under assumption (H1), our stochastic system (23) attains the permanence. The stochastic system
goes to extinction state if the assumption (H2) holds. Furthermore, with the help of the Lyapunov function theory we have examined the stochastic asymptotic stability of all feasible equilibrium solutions of (23). The parametric restriction for stochastic stability is established. (see the inequalities in Theorem 15, 16, 17, and 18).

In a stochastic system, we have observed that if the intensities of the environmental changes increase, the
fluctuations of the solutions increase. The numerical result shows that if the strong environmental changes happen in an ecosystem making each species to disappear it can occur in reality while the stochastic system (23) can maintain permanence under sufficiently small environmental noise. In permanence, if we decrease the level of environmental fluctuations, solutions become stable. By the above fact, the stochastic results give more accurate results compared to the deterministic results.

Data Availability

Our paper contains numerical experimental results, and values for these experiments are included in the paper. The data is freely available.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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