Finite Difference Method for Inhomogeneous Fractional Dirichlet Problem

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Abstract. We make the split of the integral fractional Laplacian as

\((-\Delta)^s u = (-\Delta)(-\Delta)^{s-1} u,\)

where \(s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1).\) Based on this splitting, we respectively discretize the one- and two-dimensional integral fractional Laplacian with the inhomogeneous Dirichlet boundary condition and give the corresponding truncation errors with the help of the interpolation estimate. Moreover, the suitable corrections are proposed to guarantee the convergence in solving the inhomogeneous fractional Dirichlet problem and an \(O(h^{1+s-2s})\) convergence rate is obtained when the solution \(u \in C^{1,\alpha}(\bar{\Omega}_\delta^n)\), where \(n\) is the dimension of the space, \(\alpha \in \max(0, 2s - 1), 1\), \(\delta\) is a fixed positive constant, and \(h\) denotes mesh size. Finally, the performed numerical experiments confirm the theoretical results.

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Key words: One- and two-dimensional integral fractional Laplacian, Lagrange interpolation, operator splitting, finite difference, the inhomogeneous fractional Dirichlet problem, error estimates.

1. Introduction

Fractional Laplacian is of wide interest to both pure and applied mathematicians, and also has extensive applications in physical and engineering community [7, 19]. Based on the splitting of the integral fractional Laplacian, we provide the finite difference approximations for the one- and two-dimensional cases of the operator. Then the
approximations are used to numerically solve the inhomogeneous fractional Dirichlet problem, i.e.,

\[
\begin{align*}
(-\Delta)^s u(x) &= f(x) \quad \text{in } \Omega_n, \\
u(x) &= g(x) \quad \text{in } \Omega_n^c,
\end{align*}
\]

(1.1)

where \(\Omega_n \subset \mathbb{R}^n (n = 1, 2)\) is a bounded domain and \(\Omega_n^c = \mathbb{R}^n \setminus \Omega_n\) denotes the complement of \(\Omega_n\); \(g(x) = 0\) in \(\Omega_n\), \(g(x) \in L^\infty(\mathbb{R}^n)\), and \(\text{supp } g(x)\) is bounded; \((-\Delta)^s u(x)\) is the integral fractional Laplacian, which can be defined by \([1, 7]\)

\[
(-\Delta)^s u(x) = c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy
\]

(1.2)

with

\[
c_{n,s} = \frac{2^{2s} s \Gamma\left(\frac{n}{2} + s\right)}{\pi^{\frac{n}{2}} \Gamma(1-s)},
\]

and \(s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\). And the Fourier transform of \((-\Delta)^s u(x)\) can be written as \([1, 7]\)

\[
\mathcal{F}((-\Delta)^s u(x))(\xi) = |\xi|^{2s} \mathcal{F}(u),
\]

(1.3)

where \(\mathcal{F}\) stands for the Fourier transform.

Lévy process is one of the most commonly used models for describing anomalous diffusion phenomena \([3, 22]\), especially \(\alpha\)-stable Lévy process. Fractional Laplacian is introduced as the infinitesimal generator of \(\alpha\)-stable Lévy process \([7, 12]\). Since the singularity and non-locality, numerical approximation of fractional Laplacian is still a challenging topic. In the past few decades, finite difference method has been widely used to approximate fractional derivatives \([2, 6, 9, 10, 12–18, 21, 23, 25]\). Among them, \([15–18]\) discretize time fractional Caputo derivative by \(L_1\) method and convolution quadrature method; \([6, 23]\) provide weighted and shifted Grünwald difference method to discretize fractional Riesz derivative; as for fractional Laplacian, \([9, 10, 12, 13]\) propose the finite difference scheme for solving \(d\)-dimensional \((d = 1, 2, 3)\) fractional Laplace equation with homogeneous Dirichlet boundary condition; moreover, the finite difference schemes provided in \([21, 25]\) for tempered fractional Laplacian with \(\lambda = 0\) still apply to fractional Laplacian.

Different from the previous finite difference scheme for fractional Laplacian, we split it into the product of \((-\Delta)\) and \((-\Delta)^{s-1}\) according to its Fourier transform form, where \(-\Delta\) denotes the classical Laplace operator, and \((-\Delta)^{s-1}\) (the exponent \(s-1 < 0\)) is a non-local operator without hyper-singularity (for the detailed definition, see (2.3)). Then we use the Lagrange interpolation to discretize \((-\Delta)^{s-1}\) and the finite difference to \(-\Delta\) for one- and two-dimensional cases, respectively. Moreover, some corrections are made to ensure the convergence when using our discretization to solve Eq. (1.1). Compared with the discretizations in \([9, 10]\), our scheme can deal with the inhomogeneous fractional Dirichlet problem more easily and accurately. Different from the discretizations proposed in \([21, 25]\), the current discretization can produce a Toeplitz matrix in one-dimensional case and a block-Toeplitz-Toeplitz-block for two-dimensional case; so
fast Fourier Transform can be directly used to speed up the evaluation \cite{5}. Besides, we use some examples to verify the effectiveness of the designed scheme, including truncation errors, convergence, and the simulation of the mean exit time of Lévy motion with generator \( A = \nabla P(x) \cdot \nabla + (-\Delta)^s \); the detailed results can refer to Section 5.

The rest of the paper is organized as follows. In Section 2, we discretize one- and two-dimensional fractional Laplacian by using the Lagrange interpolation and the finite difference method. In Section 3, we provide the truncation errors for one- and two-dimensional cases, respectively. In Section 4, we make some corrections to ensure the convergence in solving the inhomogeneous fractional Dirichlet problem. Section 5 provides some numerical experiments to validate the effectiveness of the designed scheme. We conclude the paper with some discussions in the last section. Throughout the paper, \( C \) is a positive constant and may be different at each occurrence.

2. Numerical discretization of the one- and two-dimensional integral fractional Laplacian

In this section, we first introduce a new representation of integral fractional Laplacian according to its Fourier transform form, and then the detailed discretizations of one- and two-dimensional integral fractional Laplacian based on the Lagrange interpolation and finite difference method are provided.

From (1.3), one can split the fractional Laplacian in frequency domain into

\[
\mathcal{F}((-\Delta)^s u)(\xi) = \frac{|\xi|^{2s}}{|\xi|^2} \mathcal{F}(u).
\]  

(2.1)

So for \( s \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \), we get a new representation of fractional Laplacian after recovering (2.1) to the corresponding time domain, i.e.,

\[
(-\Delta)^s u = (-\Delta)^{s-1} u,
\]  

(2.2)

where \((-\Delta)^s\) denotes the classical Laplace operator and \((-\Delta)^{s-1}\) is defined as \cite{24}

\[
(-\Delta)^{s'} u = c_{n,s'} \int_{\mathbb{R}^n} |x - y|^{-2s' - n} u(y) dy, \quad s' \in (-1, 0)
\]  

(2.3)

with

\[
c_{n,s'} = \frac{-2^{2s'} s' \Gamma(\frac{n}{2} + s')} {\pi^{n/2} \Gamma(1 - s')} \quad \text{for} \quad s' \in (-1, 0).
\]

Below, we provide the detailed discretization for one- and two-dimensional fractional Laplacian based on the splitting (2.2), respectively.

2.1. One-dimensional discretization

Here we focus on the discretization of \((-\Delta)^s u\) with the inhomogeneous Dirichlet boundary condition in one-dimensional case. Suppose the bounded domain \( \Omega_1 = \)
\([-L, L]\) and \(u = g(x)\) in \(\Omega^s_i\); set \(h = 2L/N\) with \(N \in \mathbb{N}^+\) and \(x_i = -L + ih, i \in \mathbb{Z}\).
Introduce \(I_i = [x_{i-1}, x_{i+1}] \cap \Omega_i, i = 0, 1, \ldots, N\). Denote \(\phi_i(x)\) as the Lagrange basis polynomial on \(I_i\), i.e.,
\[
\phi_i(x) = \bar{\phi}_1(x - x_i) \chi_{I_i}(x),
\]
where \(\chi_{I_i}(x)\) is the characteristic function on \(I_i\) and \(\bar{\phi}_1(x)\) is defined by
\[
\bar{\phi}_1(y) = \begin{cases} 1 - \frac{|y|}{h}, & y \in (-h, h), \\ 0, & y \notin (-h, h). \end{cases}
\]
Thus \(u(x)\) can be approximated by
\[
u(x) \approx \mathbb{I}_1 u(x) = \sum_{i=0}^{N} u_i \phi_i(x) + g(x),
\]
where \(u_i = u(x_i)\) and \(\mathbb{I}_1\) means the interpolation operator here. So we can approximate \((-\Delta)^{s-1} u\) by using
\[
(-\Delta)^{s-1} u(x) = c_{1,s-1} \int_{\Omega_i} |x_i - y|^{1-2s} \sum_{j=0}^{N} u_j \phi_j(y) dy 
+ c_{1,s-1} \int_{\mathbb{R}} |x_i - y|^{1-2s} g(y) dy = \sum_{j=1}^{N-1} \bar{\omega}_{j-i} u_j + R_i,
\]
where for \(0 < i, j < N,\)
\[
\bar{\omega}_{j-i} = c_{1,s-1} \int_{I_j} |x_i - y|^{1-2s} \phi_j(y) dy 
= c_{1,s-1} \int_{-h}^{h} |(j - i)h - y|^{1-2s} \bar{\phi}_1(y) dy
\]
and
\[
R_i = c_{1,s-1} \int_{\mathbb{R}} |x_i - y|^{1-2s} (u_0 \phi_0(y) + u_N \phi_N(y) + g(y)) dy.
\]
As for \((-\Delta)\), we can approximate it by
\[
(-\Delta) u_i \approx (-\Delta)_h u_i = -\frac{1}{h^2} (u_{i-1} - 2u_i + u_{i+1}).
\]
According to (2.2), we obtain the approximation of fractional Laplacian \((-\Delta)^s u\) for \(s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\), i.e.,
\[
(-\Delta)^s u_i \approx (-\Delta)_h^s u_i = -\frac{1}{h^2} \left((-\Delta)_h^{s-1} u_{i-1} - 2(-\Delta)_h^{s-1} u_i + (-\Delta)_h^{s-1} u_{i+1}\right) 
= \sum_{j=1}^{N-1} w_{j-i} u_j + (-\Delta)_h R_i,
\]
where
\[
w_i = (-\Delta)_h \bar{\omega}_i.\]
2.2. Two-dimensional discretization

Here we discretize \((-\Delta)^{\alpha}u\) with the inhomogeneous Dirichlet boundary condition in two-dimensional case. Suppose the bounded domain \(\Omega_2 = [-L, L] \times [-L, L] \cup \Omega_3\), the mesh size \(h = 2L/N, N \in \mathbb{N}^+\), and \((x, y) = (-L + ih, -L + jh), i, j \in \mathbb{Z}\). Denote \(\phi_{i,j}\) as the Lagrange basis polynomial on \(I_{i,j} = [x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}] \cap \Omega_2, i, j = 0, 1, \ldots, N\), i.e.,

\[
\phi_{i,j}(x) = \bar{\phi}_2(x - x_i, y - y_j)\chi_{I_{i,j}}(x, y),
\]

where \(\chi_{I_{i,j}}(x, y)\) is the characteristic function on \(I_{i,j}\) and \(\bar{\phi}_2(x, y)\) is defined by

\[
\bar{\phi}_2(x, y) = \begin{cases} 
1 - \frac{|x|}{h}, & (x, y) \in (-h, h) \times (-h, h), \\
0, & (x, y) \notin (-h, h) \times (-h, h).
\end{cases}
\]

Introducing \(\mathbb{I}_2\) as the interpolation operator in two space dimensions, one has

\[
u \approx \mathbb{I}_2 u = \sum_{i=0}^{N} \sum_{j=0}^{N} u_{i,j} \phi_{i,j} + g(x, y),
\]

where \(u_{i,j} = u(x_i, y_j)\). Similarly, \((-\Delta)^{\alpha-1}u(x, y)\) can be approximated by

\[
(-\Delta)^{\alpha-1} u(x, y) = \sum_{p=1}^{N-1} \sum_{q=1}^{N-1} \bar{\omega}_{p-i, q-j} u_{p,q} + R_{i,j},
\]

where for \(0 < i, j, p, q < N\),

\[
\bar{\omega}_{p-i, q-j} = c_{2, s-1} \int_{I_{p,q}} |(x, y) - (\xi, \eta)|^{-2s} \phi_{p,q}(\xi, \eta) d\xi d\eta = c_{2, s-1} \int_{-h}^{h} \int_{-h}^{h} |(p - i)h, (q - j)h + (\xi, \eta)|^{-2s} \bar{\phi}(\xi, \eta) d\xi d\eta,
\]

and

\[
R_{i,j} = c_{2, s-1} \int_{\mathbb{R}^2} |(x, y) - (\xi, \eta)|^{-2s} g(\xi, \eta) d\xi d\eta + c_{2, s-1} \sum_{pq(p-N)(q-N)=0, 0 \leq p, q \leq N} \int_{\mathbb{R}^2} |(x, y) - (\xi, \eta)|^{-2s} u_{p,q} \phi_{p,q} d\xi d\eta.
\]

Here

\[
|(x, y) - (\xi, \eta)| = \sqrt{(x_i - \xi)^2 + (y_j - \eta)^2}.
\]

Next, using the formula

\[
(-\Delta) u_{i,j} \approx (-\Delta)_{h, 1} u_{i,j} = -\frac{1}{h^2}(u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j})
\]
to approximate \((-\Delta)\), one gets the approximation of \((-\Delta)^s u:\)

\[
(-\Delta)^s_{h,1} u_{i,j} = (-\Delta)_{h,1} (-\Delta)^{s-1}_{h} u_{i,j} \\
= \sum_{p=1}^{N-1} \sum_{q=1}^{N-1} w_{p-i,q-j}^{(1)} u_{p,q} + (-\Delta)_{h,1} R_{i,j},
\]

where

\[
w_{i,j}^{(1)} = (-\Delta)_{h,1} \bar{\omega}_{i,j}.
\]

An alternative approximation for \((-\Delta)u\) can be derived by using the formula

\[
(-\Delta)u_{i,j} \approx (-\Delta)_{h,2} u_{i,j} \\
= -\frac{1}{2h^2} (u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1} - 4u_{i,j}).
\]

Also, \((-\Delta)^s u\) can be discretized as

\[
(-\Delta)^s_{h,2} u_{i,j} = (-\Delta)_{h,2} (-\Delta)^{s-1}_{h} u_{i,j} \\
= \sum_{p=1}^{N-1} \sum_{q=1}^{N-1} w_{p-i,q-j}^{(2)} u_{p,q} + (-\Delta)_{h,2} R_{i,j},
\]

where

\[
w_{i,j}^{(2)} = (-\Delta)_{h,2} \bar{\omega}_{i,j}.
\]

Thus \((-\Delta)^s\) with \(s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\) can be approximated by the convex combination of (2.10) and (2.12), i.e.,

\[
(-\Delta)^s u \approx (-\Delta)^s_{h,1} u = \theta (-\Delta)^s_{h,1} u + (1 - \theta)(-\Delta)^s_{h,2} u, \quad \theta \in [0, 1],
\]

which means

\[
(-\Delta)^s_{h} u_{i,j} = \sum_{p=1}^{N-1} \sum_{q=1}^{N-1} w_{p-i,q-j} u_{p,q} + \theta (-\Delta)^s_{h,1} R_{i,j} + (1 - \theta)(-\Delta)^s_{h,2} R_{i,j},
\]

where

\[
w_{i,j} = \theta w_{i,j}^{(1)} + (1 - \theta) w_{i,j}^{(2)}.
\]

Here \(w_{i,j}^{(1)}\) and \(w_{i,j}^{(2)}\) are defined in (2.11) and (2.13), respectively.

Next, we provide the error analyses under the assumption \(u \in C^{1,\alpha}(\bar{\Omega}_n)\) \((n = 1, 2)\) to verify the effectiveness of our discrete schemes. For the solution \(u\) with lower regularity, the schemes still converge, but the convergence rates decrease correspondingly. Some numerical results can refer to Example 5.3 in Section 5.
3. Truncation errors

In this section, we provide the estimate of \( \|(−Δ)^s u − (−Δ)^h u\|_∞ \) in one- and two-dimensional cases, respectively. In the following, we denote \( \| \cdot \|_∞ \) and \( \| \cdot \|_2 \) as the discrete \( l^∞ \) and \( l^2 \) norms, and \( \| \cdot \|_L^∞ \) as continuous \( L^∞ \) norm.

**Theorem 3.1.** Let \( s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \). Suppose \( (−Δ)^s \) and \( (−Δ)^h \) are defined in (1.2) and (2.6) or (2.14), respectively. If \( u \in C^{1,α}(Ω^n_h) \) with some fixed constant \( δ > 4h > 0 \) and \( α \in (\max(0, 2s − 1), 1] \), then we have

\[
\|((−Δ)^s − (−Δ)^h)u\|_∞ ≤ Ch^{1+α−2s},
\]
\[
\|((−Δ)^s − (−Δ)^h)u\|_2 ≤ Ch^{1+α−2s},
\]

where
\[
Ω^n_h = (−L − δ, L + δ))^n, \quad n = 1, 2.
\]

Here, we only provide the proof in two-dimensional case in detail; and the proof in one-dimensional case can be got similarly.

**Proof.** For fixed \( i, j \), according to (2.14), we have

\[
\begin{align*}
|((−Δ)^s − (−Δ)^h)_{i,j}| & ≤ θ|((−Δ)^s − (−Δ)^h)_{i,1}| \noalign{\vskip5pt}
+ (1 − θ)|((−Δ)^s − (−Δ)^h)_{2,1}|, \quad θ ∈ [0, 1]. \tag{3.1}
\end{align*}
\]

Using the definitions of \( (−Δ)^s \) and \( (−Δ)^h \) results in

\[
\begin{align*}
|((−Δ)^s − (−Δ)^h)_{i,j}| & ≤ |((−Δ)(−Δ)^s−1 − (−Δ)_{h,1}(−Δ)^s−1)_{i,j}| \noalign{\vskip5pt}
+ |((−Δ)_{h,1}(−Δ)^s−1 − (−Δ)_{h,1}(−Δ)^s−1)_{i,j}| \noalign{\vskip5pt}
& ≤ I + II.
\end{align*}
\]

Let
\[
Φ(x_i − ξ, y_j − η) ∈ C_0^2(Ω^d_2),
\]
which satisfies
\[
Φ(x_i − ξ, y_j − η) = (x_i − ξ)^2 + (y_j − η)^2 \]
if
\[
(ξ, η) ∈ Ω^d_2 \setminus (x_i − h, x_i + h) × (y_j − h, y_j + h),
\]
and
\[
\|Φ(x, y)\|_{L^∞(R^2)} ≤ Ch^{−2s},
\]
\[
\left\|\frac{∂Φ(x, y)}{∂x}\right\|_{L^∞(R^2)}, \left\|\frac{∂Φ(x, y)}{∂y}\right\|_{L^∞(R^2)} ≤ Ch^{−1−2s},
\]
Using integration by parts and change of variables, we have

\[ \left\| \frac{\partial^2 \Phi(x, y)}{\partial x^2} \right\|_{L^\infty(\mathbb{R}^2)} \leq C h^{-2s}, \]

\[ \left\| \frac{\partial^4 \Phi(x, y)}{\partial x^4} \right\|_{L^\infty(\mathbb{R}^2 \setminus \Omega_2)} \leq C. \]

Introduce \( \mu^x \) and \( \mu^y \) satisfying

\[ \frac{\partial^2 \mu^x}{\partial x^2} = \frac{\partial^2 \mu^y}{\partial y^2} = u. \]

Here, divide \( I \) into two parts, i.e.,

\[ I \leq C \left( \int_{\mathbb{R}^2} \Phi(x_i, y_j) - (\xi, \eta) \right)^{2s} \Phi(x_i - \xi, y_j - \eta) \]

\[ + C \left( \int_{\mathbb{R}^2} \Phi(x_i, y_j) - (\xi, \eta) \right)^{2s} \Phi(x_i - \xi, y_j - \eta) \]

\[ \leq I^x + I^y, \]

where

\[ (-\Delta)_x = -\frac{\partial^2}{\partial x^2}, \quad (-\Delta)_y = -\frac{\partial^2}{\partial y^2}, \]

\[ (-\Delta)_{x,h,1} v_{i,j} = -\frac{v_{i-1,j} - 2v_{i,j} + v_{i+1,j}}{h^2}, \]

\[ (-\Delta)_{y,h,1} v_{i,j} = -\frac{v_{i,j-1} - 2v_{i,j} + v_{i,j+1}}{h^2}. \]

Introduce

\[ \Psi(x_i - \xi, y_j - \eta) = |(x_i, y_j) - (\xi, \eta)|^{2s} - \Phi(x_i - \xi, y_j - \eta). \]

For \( I^x \), we find

\[ I^x \leq C \left( \int_{\mathbb{R}^2} \Phi(x_i - \xi, y_j - \eta) u(\xi, \eta) d\xi d\eta \right) \]

\[ - C \left( \int_{\mathbb{R}^2} \Phi(x_i - \xi, y_j - \eta) u(\xi, \eta) d\xi d\eta \right) \]

\[ \leq I^x + I^y. \]

Introduce

\[ D^\delta_{i,j} = \left\{ (\xi, \eta) | (x_i - \xi, y_j - \eta) \in \Omega^\delta_2 \right\}, \quad D^\delta_{i,j} = D^\delta_{i,j} \setminus (-h, h) \times (-h, h). \]

Using integration by parts and change of variables, we have

\[ I^x \leq C \left( \int_{\Omega^\delta_2} \frac{\partial^2 \Phi(x_i - \xi, y_j - \eta)}{\partial \xi^2} \mu^x(\xi, \eta) d\xi d\eta \right). \]
Decomposing $I^x$ we obtain

\[ I^x \leq C \left| (-\Delta)_{x} - (-\Delta)_{x,h,1} \right| \int \int_{D_{i,j}} \frac{\partial^2 \Phi(x,\eta)}{\partial \xi^2} \mu^x(x_i - \xi, y_j - \eta) d\xi d\eta \]

\[ I^x \leq C \left| \int_{-h}^{h} \int_{-h}^{h} \frac{\partial^2 \Phi(x,\eta)}{\partial \xi^2} ((-\Delta)_{x} - (-\Delta)_{x,h,1}) \mu^x(x_i - \xi, y_j - \eta) d\xi d\eta \right| \]

\[ + C \left| \int \int_{D_{i,j}} \frac{\partial^2 \Phi(x,\eta)}{\partial \xi^2} ((-\Delta)_{x} - (-\Delta)_{x,h,1}) \mu^x(x_i - \xi, y_j - \eta) d\xi d\eta \right| \]

\[ \leq I^x_{1,1} + I^x_{1,2}. \]

By Taylor’s expansion, we have

\[ \left| (-\Delta)_{x} - (-\Delta)_{x,h} \right| v(x_i) \leq C h^{1+\alpha} \| v \|_{C^{3,\alpha}(\bar{\Omega})} \]

for $v \in C^{3,\alpha}([x_{i-1}, x_{i+1}])$. Thus there holds

\[ I_{1,1}^x \leq C h^{1+\alpha} \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial^2 \Phi(x,\eta)}{\partial \xi^2} \right| d\xi d\eta \| u \|_{C^{3,\alpha}(\bar{\Omega})} \]

\[ \leq C h^{1+\alpha-2\alpha} \| u \|_{C^{1,\alpha}(\bar{\Omega})}. \]

Using the fact

\[ \left| \frac{\partial^2 (x,y) - (\xi,\eta)}{\partial \xi^2} \right|^{-2s} \leq C |(x,y) - (\xi,\eta)|^{-2s-2} \text{ for } (x,y) \neq (\xi,\eta), \]

we obtain

\[ I_{1,2}^x \leq C h^{1+\alpha} \int \int_{D_{i,j}} \left| \frac{\partial^2 (x_i, y_j) - (\xi,\eta)}{\partial \xi^2} \right| d\xi d\eta \| u \|_{C^{1,\alpha}(\bar{\Omega})} \]

\[ \leq C h^{1+\alpha-2s} \| u \|_{C^{1,\alpha}(\bar{\Omega})}. \]

Decomposing $I^x_2$ into three parts leads to

\[ I^x_2 \leq C \left| (-\Delta)_{x} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \Psi(x_i - \xi, y_j - \eta) u(\xi,\eta) d\xi d\eta \right| \]

\[ + C \left| (-\Delta)_{x,h,1} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \Psi(x_i - \xi, y_j - \eta) u(\xi,\eta) d\xi d\eta \right| \]

\[ + C \left| ((-\Delta)_{x} - (-\Delta)_{x,h,1}) \int \int_{\bar{\Omega}_2} \Psi(x_i - \xi, y_j - \eta) u(\xi,\eta) d\xi d\eta \right| \]

\[ \leq I^x_{2,1} + I^x_{2,2} + I^x_{2,3}. \]

For $I^x_{2,1}$, we get, for some function $C_0(y)$ independent of $x$,

\[ I^x_{2,1} \leq C \left| \frac{\partial}{\partial x} \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} \Psi(x_i - \xi, y_j - \eta) \frac{\partial u(\xi,\eta)}{\partial \xi} d\xi d\eta \right| \]
Simple calculations imply
\[
\left| \frac{\partial}{\partial x} \int_{x_i-h}^{x_i+h} \int_{y_j-h}^{y_j+h} \Psi(x_i - \xi, y_j - \eta) \left( \frac{\partial u(\xi, \eta)}{\partial \xi} - C_0(\eta) \right) d\xi d\eta \right| \leq C \int_{x_i-h}^{x_i+h} \int_{y_j-h}^{y_j+h} \left| \frac{\partial \Psi(x_i - \xi, y_j - \eta)}{\partial x} \right| \left( \frac{\partial u(\xi, \eta)}{\partial \xi} - C_0(\eta) \right) d\xi d\eta.
\]
Choosing
\[
C_0(y) = \frac{\partial u(x, y)}{\partial x} |_{x=x_i}
\]
results in
\[
I_{2,1}^x \leq Ch^{1+\alpha-2s} \|u\|_{C^{1,\alpha}(\overline{\Omega}_2^s)}.
\]
By using \(|\Phi(\xi, \eta)| \leq Ch^{-2s}\) and the Taylor expansion, there holds
\[
I_{2,2}^x \leq C \left| \int_{-h}^{h} \int_{-h}^{h} \Psi(\xi, \eta)(-\Delta)_{x, h, 1} u(x_i - \xi, y_j - \eta) d\xi d\eta \right| \leq C h^{1+\alpha-2s} \|u\|_{C^{1,\alpha}(\overline{\Omega}_2^s)}.
\]
Simple calculations imply
\[
I_{2,3}^x \leq C \left| \int_{\overline{\Omega}_2^s} ((-\Delta)_x - (-\Delta)_{x, h, 1}) \Psi(x_i - \xi, y_j - \eta) u(\xi, \eta) d\xi d\eta \right|
\leq C \left| \int_{\overline{\Omega}_2^s} ((-\Delta)_x - (-\Delta)_{x, h, 1}) \Psi(x_i - \xi, y_j - \eta) d\xi d\eta \right| \|u(\xi, \eta)\|_{L^\infty(\mathbb{R}^2)}
\leq C h^2 \left| \int_{\overline{\Omega}_2^s} \frac{\partial^4 \Psi(x_i - \xi, y_j - \eta)}{\partial x^4} \right| d\xi d\eta \|u(\xi, \eta)\|_{L^\infty(\mathbb{R}^2)}
\leq C h^2 \delta^{-2-2s} \|u\|_{L^\infty(\mathbb{R}^2)}.
\]
Here, according to the definition of \(\Psi, \delta\) is independent of \(h\). Combining above estimates, one has
\[
I_{2}^x \leq Ch^{1+\alpha-2s}.
\]
Similarly, there is
\[
I_{2}^y \leq Ch^{1+\alpha-2s}.
\]
As for \(II\), the fact
\[
\|u(\xi, \eta) - \mathbb{I}_2 u(\xi, \eta)\|_{L^\infty(\Omega_2^s)} \leq Ch^{1+\alpha} \|u\|_{C^{1,\alpha}(\overline{\Omega}_2^s)}
\]
implies
\[
II \leq C \left| (-\Delta)_{h, 1} \int_{-L}^{L} \int_{-L}^{L} |(x_i, y_j) - (\xi, \eta)|^{-2s} (u(\xi, \eta) - \mathbb{I}_2 u(\xi, \eta)) d\xi d\eta \right|
\leq C h^{1+\alpha} \int_{-L}^{L} \int_{-L}^{L} \left| ((-\Delta)_{h, 1} |(x_i, y_j) - (\xi, \eta)|^{-2s}) \right| d\xi d\eta \|u\|_{C^{1,\alpha}(\overline{\Omega}_2^s)}.
\]
Introduce
\[ D_{i,j} = \Omega_2^h \setminus (x_i - 2h, x_i + 2h) \times (y_j - 2h, y_j + 2h). \]

Simple calculations give
\[
\int_{-L}^{L} \int_{-L}^{L} \left| \frac{(-\Delta)_{h,1}}{h} \right| (x_i, y_j) - (\xi, \eta) \right|^{-2s} \right| \, d\xi \, d\eta \\
\leq C \int_{y_j - 2h}^{y_j + 2h} \int_{x_i - 2h}^{x_i + 2h} \left| \frac{(-\Delta)_{h,1}}{h} \right| (x_i, y_j) - (\xi, \eta) \right|^{-2s} \right| \, d\xi \, d\eta \\
+ C \int \int_{D_{i,j}} \left| \frac{(-\Delta)_{h,1}}{h} \right| (x_i, y_j) - (\xi, \eta) \right|^{-2s} \right| \, d\xi \, d\eta \\
\leq Ch^{-2s},
\]
which leads to
\[
II \leq Ch^{1+\alpha-2s}\|u\|_{C^{1,\alpha}(\Omega_2^h)}.
\]
Thus according to I and II, we have
\[
\left\| ((-\Delta)^s - (-\Delta)^{s}_{h,1}) u \right\|_{\infty} \leq Ch^{1+\alpha-2s}, \\
\left\| ((-\Delta)^s - (-\Delta)^{s}_{h,1}) u \right\|_{2} \leq Ch^{1+\alpha-2s}.
\]
As for \(\left\| ((-\Delta)^s - (-\Delta)^{s}_{h,2}) u \right\|_{\infty}\), by similar arguments, we can get the estimates
\[
\left\| ((-\Delta)^s - (-\Delta)^{s}_{h,2}) u \right\|_{\infty} \leq Ch^{1+\alpha-2s}, \\
\left\| ((-\Delta)^s - (-\Delta)^{s}_{h,2}) u \right\|_{2} \leq Ch^{1+\alpha-2s}.
\]
Collecting the above estimates, the desired results are reached.

**Remark 3.1.** In practice, \(g(x)\) may not be exactly obtained, but one can approximate \(g(x)\) by numerical methods, such as interpolation by piecewise linear polynomials. When the error in the approximation of \(g(x)\) is small enough, the convergence order of our numerical scheme still behaves as \(O(h^{1+\alpha-2s})\).

4. **Convergence in solving the inhomogeneous fractional Dirichlet problem**

In this section, we first propose the sufficient conditions for getting the convergence when using the provided discretizations to solve Eq. (1.1). Then we try to modify the discretizations provided in Section 3 according to the corresponding conditions. Finally, we present the convergence analyses in solving Eq. (1.1).

Now, we first provide a lemma which is useful for the convergence analyses.
Lemma 4.1 ([4]). Let matrix $A$ be

$$
A = \begin{pmatrix}
  a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\
  a_{2,1} & a_{2,2} & \cdots & a_{2,N} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{N,1} & a_{N,2} & \cdots & a_{N,N}
\end{pmatrix}.
$$

Introduce the discs

$$
C_i = \left\{ z \in \mathbb{C}; |z - a_{i,i}| \leq \sum_{j=1,j \neq i}^{N} |a_{i,j}| \right\}, \quad 1 \leq i \leq N,
$$

$$
C'_i = \left\{ z \in \mathbb{C}; |z - a_{i,i}| \leq \sum_{j=1,j \neq i}^{N} |a_{j,i}| \right\}, \quad 1 \leq i \leq N.
$$

(4.1)

The spectrum $\lambda(A)$ of $A$ is enclosed in the union of $C_i$ and $C'_i$.

Below we give two theorems to state the sufficient conditions of achieving the convergence in solving Eq. (1.1) in one and two dimensions, respectively.

Theorem 4.1. Given two vectors $F, G$ and the matrix

$$
B_1 = \begin{pmatrix}
  b_0 & b_1 & \cdots & b_{N-2} \\
  b_1 & b_0 & \cdots & b_{N-3} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{N-2} & b_{N-3} & \cdots & b_0
\end{pmatrix}.
$$

Let $U_h$ be the solution of the linear system

$$
B_1 U_h + G = F.
$$

(4.2)

Assume $U, B_1, G$ and $G$ satisfy the conditions

1. $\|F - (B_1 U + G)\|_{\infty} \leq C h^k, \|F - (B_1 U + G)\|_2 \leq C h^k$;
2. $b_0 > 0; b_i < 0$ for $i \neq 0$;
3. there exists some constant $C_0 > 0$ such that $\inf_{i=1, \ldots, N-1} \sum_{j=1}^{N-1} b_{|i-j|} > C_0$.

Then we obtain

$$
\|U - U_h\|_{\infty} \leq C h^k, \quad \|U - U_h\|_2 \leq C h^k.
$$

Proof. By Lemma 4.1 and the properties of $b_i$, we have

$$
\lambda_{\text{min}}(B_1) > C_0.
$$
Let  
\[ e^U = U_h - U = \{ e_i^U \}_{i=1}^{N-1}, \quad \tilde{F} = F - (B_1 U + G) = \{ \tilde{f}_i \}_{i=1}^{N-1}. \]

Then
\[ CC_0 \| e^U \|_2^2 \leq (B_1 e^U, e^U) = (\tilde{F}, e^U) \leq \| F \|_2 \| e^U \|_2, \]
which leads to  \( \| e^U \|_2 \leq CC_0^{-1} \| \tilde{F} \|_2 \). Assuming  \( \| e^U \|_\infty = |e_p| \), we have
\[ e_p^U (\tilde{f}_p - CC_0 e_p^U) = e_p^U \left( \sum_{i=1}^{N-1} b_{p-i} e_i^U - CC_0 e_p^U \right) \]
\[ = e_p^U \left( \sum_{i=1, i \neq p}^{N-1} b_{p-i} e_i^U + (b_0 - CC_0) e_p^U \right) \]
\[ \geq e_p^U \left( \sum_{i=1, i \neq p}^{N-1} b_{p-i} (e_i^U - e_p^U) \right) \geq 0, \]
which yields
\[ \| e^U \|_\infty \leq CC_0^{-1} |\tilde{f}_p| \leq CC_0^{-1} \| \tilde{F} \|_\infty. \]
Combining the first condition, we can get the desired results. \( \square \)

Similarly, for the two-dimensional case, we find

**Theorem 4.2.** Suppose  \( U, U_h, G, \) and  \( F \) satisfy
\[ B_2 U_h + G = F, \]  
and  
\[ \| F - (B_2 U + G) \|_\infty \leq Ch^l, \quad \| F - (B_2 U + G) \|_2 \leq Ch^l. \]

Here  
\[ B_2 = \begin{bmatrix} T_0 & T_1 & \cdots & T_{N-1} \\ T_{-1} & T_0 & \cdots & T_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ T_{-N+1} & T_{-N+2} & \cdots & T_0 \end{bmatrix}, \quad T_k = \begin{bmatrix} t_{k,0} & t_{k,1} & \cdots & t_{k,N-2} \\ t_{k,1} & t_{k,0} & \cdots & t_{k,N-3} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k,N-2} & t_{k,N-3} & \cdots & t_{k,0} \end{bmatrix}. \]

Assume the following conditions are satisfied:

1. \( t_{k,i} > 0 \) for  \( k = i = 0 \), otherwise,  \( t_{k,i} < 0 \);

2. \( \inf_{p,q=1, \ldots, N-1} \sum_{i,j=1}^{N-1} t_{|p-i|,|q-j|} > C_0 > 0. \)

Then one has
\[ \| U - U_h \|_\infty \leq Ch^l, \quad \| U - U_h \|_2 \leq Ch^l. \]
4.1. Corrections for the one- and two-dimensional discretizations

From the above two theorems, we need to change some properties of the weights produced by the discretization in Section 3 for one- and two-dimensional cases.

4.1.1. One-dimensional case

Here we provide a lemma to state the properties of weights $w_i$ defined in (2.7).

**Theorem 4.3.** Let $w_i$ be defined in (2.7). Then $w_i$ satisfies

$$w_i < 0, \quad |i| \geq 2, \quad w_i = w_{-i}, \quad i \geq 0,$$

$$\sum_{i=-N+1}^{N-1} w_i \geq CL^{-2s},$$

where $2L$ means the length of $\Omega_1$.

**Proof.** The definition of $c_{1,s-1}$ and simple calculations give, for $\zeta > h$,

$$c_{1,s-1} < 0, \quad (\zeta - h)^{1-2s} - 2s^{1-2s} + (\zeta + h)^{1-2s} < 0 \quad \text{for} \quad s < \frac{1}{2},$$

$$c_{1,s-1} > 0, \quad (\zeta - h)^{1-2s} - 2s^{1-2s} + (\zeta + h)^{1-2s} > 0 \quad \text{for} \quad s > \frac{1}{2},$$

which leads to $w_i < 0, |i| \geq 2$. As for $w_1$, simple calculations give

$$w_1 = -c_{1,s-1}h^{2s} \frac{7 - 25 - 2s + 3^{2s}}{(2s - 3)(2s - 2)}. \tag{4.4}$$

Summing $w_i$ from $-N + 1$ to $N - 1$ gives

$$\sum_{i=-N+1}^{N-1} w_i = \frac{1}{h^2} \sum_{i=-N+1}^{N-1} (2\bar{\omega}_i - \bar{\omega}_{i+1} - \bar{\omega}_{i-1}) = 2 \frac{\bar{\omega}_{N-1} - \bar{\omega}_N}{h^2}.$$

According to the definitions of $\bar{\omega}_N$, we have

$$\frac{1}{h^2} (\bar{\omega}_{N-1} - \bar{\omega}_N) = \frac{1}{h^2} \left( c_{1,s-1} \int_{-h}^{h} \left( ((N-1)h + \zeta)^{1-2s} - (Nh + \zeta)^{1-2s} \right) \bar{\phi}_1(\zeta) d\zeta \right) \geq C \frac{1}{h} \int_{-h}^{h} L^{-2s} \bar{\phi}_1(\zeta) d\zeta \geq CL^{-2s},$$

which leads to desired results. \qed

From Theorem 4.1 and the fact $w_1 > 0$ for some $s \in (0, \frac{1}{2})$ (see (4.4)), we find that the numerical scheme constructed by (2.6) may not be effective. To make the $w_i$
satisfy the condition of Theorem 4.1 and get an effective numerical scheme, we do the modifications for \( \bar{\omega}_0 \), i.e.,

\[
\bar{\omega}_0^M = \begin{cases} 
0, & \text{if } w_1 \geq 0, \\
\bar{\omega}_0, & \text{if } w_1 < 0.
\end{cases}
\] (4.5)

Then we obtain a modified scheme

\[
(-\Delta)_h^s u(x) \approx (-\Delta)_h^s u(x) = \sum_{j=1}^{N-1} w_j^M u_j + R_i,
\] (4.6)

where

\[
w_0^M = -\frac{\bar{\omega}_0 - 2\bar{\omega}_0^M + \bar{\omega}_1}{h^2}, \quad w_i^M = w_i, \quad |i| \geq 2,
\]

\[
w_i^M = -\frac{\bar{\omega}_0^M - 2\bar{\omega}_1 + \bar{\omega}_2}{h^2}, \quad |i| = 1.
\]

By the definitions of \( w_i^M \) and \( \bar{\omega}_i \), it is easily checked that \( w_1^M < 0 \).

Next, we present the truncation error of the modified discretization (4.6).

**Theorem 4.4.** Let \( s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \), \( (-\Delta)^s \) and \( (-\Delta)^s_{h,M} \) are defined in (1.2) and (4.6), respectively. If \( u \in C^{1,\alpha}(\bar{\Omega}_1^\delta) \) with a fixed constant \( \delta > 4h > 0 \) and \( \alpha \in (\max(0, 2s - 1), 1) \), then

\[
\|(\Delta)^s_{h,M} - (-\Delta)^s_{h,M})u\|_{\infty} \leq Ch^{1+\alpha-2s},
\]

\[
\|(\Delta)^s_{h,M} - (-\Delta)^s_{h,M})u\|_2 \leq Ch^{1+\alpha-2s},
\]

where \( \Omega_1^\delta = (-L - \delta, L + \delta) \).

**Proof.** For fixed \( i \), by triangle inequality and Theorem 3.1, we obtain

\[
\left| (\Delta)^s - (-\Delta)^s_{h,M})u_i \right| \leq \left| (\Delta)^s - (-\Delta)^s_{h,M})u_i \right| + \left| (-\Delta)^s_{h,M} - (-\Delta)^s_{h,M})u_i \right| \leq Ch^{1+\alpha-2s} + \vartheta.
\]

As for \( \vartheta \), if \( \bar{\omega}_0^M = \bar{\omega}_0 \), there is \( \vartheta = 0 \). Otherwise, we have

\[
\vartheta \leq \left| (-\Delta)^s_{h}u_i \int_{-h}^{h} |y|^{1-2s} \tilde{\phi}_1(y)dy \right| \leq Ch^{1+\alpha-2s} \| u \|_{C^{1,\alpha}(\bar{\Omega}_1^\delta)},
\]

which leads to the desired results. \( \square \)

Thus we can get the following convergence results for one-dimensional case by Theorem 4.1.
Theorem 4.5. Assume \( s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \). Let \( u \) and \( U_h \) be solutions of Eqs. (1.1) and (4.2) with \( b_i = w_i^M \) and

\[
U = \{ u(x_i) \}_{i=1}^{N-1}, \quad G = \{ (-\Delta)R_i \}_{i=1}^{N-1}, \quad F = \{ f(x_i) \}_{i=1}^{N-1}.
\]

If \( u \in C^{1,\alpha}(\Omega_\delta^2) \) with some fixed constant \( \delta > 4h > 0 \) and \( \alpha \in (\max(0, 2s - 1), 1] \), then we have

\[
\| U - U_h \|_\infty \leq Ch^{1+\alpha - 2s}, \quad \| U - U_h \|_2 \leq Ch^{1+\alpha - 2s}.
\]

4.1.2. Two-dimensional case

According to Theorem 4.2, to obtain an effective numerical scheme, we need to make \( w_{i,j} \) satisfy the following requirements:

\[
w_{0,0}^M > 0, \quad w_{i,j}^M < 0,
\]

where \( (i, j) \in \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\} \). To be specific, we modify \( \tilde{\omega}_{i,j} \) as

\[
\tilde{\omega}_{0,0}^M = \tilde{\omega}_{0,0} + c_0, \quad c_0 \geq 0, \quad \tilde{\omega}_{i,j}^M = \tilde{\omega}_{i,j},
\]

and take \( w_{i,j}^M \) as

\[
w_{i,j}^M = (\theta(-\Delta)_{h,1} + (1 - \theta)(-\Delta)_{h,2}) \tilde{\omega}_{i,j}^M.
\]

(4.7)

Thus the two-dimensional discretization scheme can be modified as

\[
(-\Delta)^s_{h,M} u_{i,j} = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} w_{i,j}^M \sum_{p=1}^{N-1} \sum_{q=1}^{N-1} w_{p,q-j}^M u_{p,q-j} + (\theta(-\Delta)_{h,1} + (1 - \theta)(-\Delta)_{h,2}) R_{i,j}.
\]

(4.8)

Similar to the proofs of Theorems 4.3 and 4.4, there hold

Theorem 4.6. Let \( w_{i,j}^M \) be defined in (4.7). Then

\[
\sum_{i=-N+1}^{N-1} \sum_{j=-N+1}^{N-1} w_{i,j}^M \geq CL^{-2s}.
\]

Theorem 4.7. Let

\[
\Omega_\delta^2 = (-L - \delta, L + \delta) \times (-L - \delta, L + \delta)
\]

and \( s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \). Suppose \((-\Delta)^s \) and \((-\Delta)^s_{h,M} \) are defined in (1.2) and (4.8), respectively. If \( u \in C^{1,\alpha}(\Omega_\delta^2) \) with some fixed constant \( \delta > 4h > 0 \) and \( \alpha \in (\max(0, 2s - 1), 1] \), then we have

\[
\| ((-\Delta)^s - (-\Delta)^s_{h,M}) u \|_\infty \leq Ch^{1+\alpha - 2s}, \quad \| ((-\Delta)^s - (-\Delta)^s_{h,M}) u \|_2 \leq Ch^{1+\alpha - 2s}.
\]
Thus the corresponding convergence results can be obtained by Theorem 4.2.

**Theorem 4.8.** Let \( u \) and \( U_h \) be solutions of (1.1) and (4.3) with \( t_{i,j} = w_{i,j}^M \) and

\[
\mathbf{U} = \{u(x_i, y_j)\}_{i,j=1}^{N-1}, \quad \mathbf{F} = \{f(x_i, y_j)\}_{i,j=1}^{N-1},
\]

\[
\mathbf{G} = \{((\theta(-\Delta))h, 1 + (1 - \theta)(-\Delta)h, 2)R_{i,j}\}_{i,j=1}^{N-1}.
\]

After choosing suitable \( \theta \) and \( c_{0,0} \), we have if \( u \in C^{1,\alpha}(\bar{\Omega}_\delta^2) \) with some fixed constant \( \delta > 4h > 0 \) and \( \alpha \in (\max(0, 2s - 1), 1] \),

\[
\|U - U_h\|_\infty \leq Ch^{1+\alpha-2s}, \quad \|U - U_h\|_2 \leq Ch^{1+\alpha-2s},
\]

where \( s \in \left[\frac{1}{250}, \frac{1}{2}\right] \cup (\frac{1}{2}, 1) \).

**Remark 4.1.** By numerical experiments, we give the range of \( \theta \) with different \( s \in \left[\frac{1}{250}, \frac{1}{2}\right] \cup (\frac{1}{2}, 1) \) and \( c_{0,0} \) in Fig. 1 (shown in the shaded area), which makes above estimates hold. But for a smaller \( s \in (0, \frac{1}{250}) \), we do not find a suitable \( \theta \) to make \( u_{i,j}^M \) satisfy Theorem 4.2.

**Remark 4.2.** It is easy to check that the coefficient \( c_{n,s-1} \) in (2.3) can tend to \( \infty \) when \( s = \frac{1}{7} \) in one-dimensional case, but it does not for the two-dimensional case.

![Figure 1: Range of \( \theta \) for different \( s \) and \( c_{0,0} \).](image-url)
5. Numerical experiments

In this section, we first verify the convergence of the numerical method in discretizing \((-\Delta)^s\) and solving Eq. (1.1). Then we simulate the mean exit time of Lévy motion with generator \(\mathcal{A} = \nabla P(x) \cdot \nabla + (-\Delta)^s\). From [11], we have

\[
 u = \begin{cases} 
 (1 - x^2)^{P+s}, & x \in (-1, 1), \\
 0, & \text{otherwise}
\end{cases} \tag{5.1}
\]

with \(P \in \mathbb{R}\) and

\[
 (-\Delta)^s u = \frac{2^{2s} \Gamma(1/2 + s) \Gamma(P + 1 + s)}{\sqrt{\pi} \Gamma(P + 1)} \, {}_2F_1 \left( \frac{1}{2} + s, -P; \frac{1}{2}; x^2 \right), \quad x \in (-1, 1)
\]

with \(\, _2F_1\) being the Gauss hypergeometric function. Using this result, we test the truncation errors and the convergence rates (the right hand side and boundary terms of Eq. (1.1) are taken as the corresponding expressions).

**Example 5.1.** In this example, we consider the truncation error in one-dimensional case. Here we choose \(\Omega_1 = (-1, 1), g(x) = 0\), and \(P = 2 - s\) in (5.1). All the results presented in Table 1 agree with Theorem 3.1.

### Table 1: \(l^\infty(\Omega)\) truncation errors and convergence rates with \(P = 2 - s\).

| \(s\) \( \backslash \) \(2/h\) | 128     | 256      | 512       | 1024      |
|----------------|--------|----------|-----------|-----------|
| 0.2            | 2.313E-03 | 8.321E-04 | 2.930E-04 | 1.016E-04 |
| Rates          | 1.4749  | 1.5061   | 1.5274    |           |
| 0.4            | 4.168E-03 | 1.723E-03 | 7.215E-04 | 3.057E-04 |
| Rates          | 1.2742  | 1.2559   | 1.2388    |           |
| 0.6            | 1.776E-02 | 9.859E-03 | 5.541E-03 | 3.142E-03 |
| Rates          | 0.8495  | 0.8314   | 0.8185    |           |
| 0.8            | 6.368E-02 | 4.724E-02 | 3.536E-02 | 2.662E-02 |
| Rates          | 0.4309  | 0.4179   | 0.4098    |           |

**Example 5.2.** In this example, we use numerical scheme (4.6) to solve (1.1) with \(g(x) = 0\) and \(\Omega_1 = (-1, 1)\). Here we choose \(P = 1\) in (5.1) which leads to \(u \in C^{1,s}(\Omega_1)\). The results presented in Table 2 show that the numerical scheme (4.6) has an \(O(h^{1+s})\) convergence rate which is higher than the one \((O(h^{1-s}))\) predicted in Theorem 4.5.

**Example 5.3.** We choose \(P = 0\) in (5.1). We first take \(\Omega_1 = (-0.5, 0.5)\) and

\[
 g(x) = \begin{cases} 
 (1 - x^2)^{P+s}, & x \in (-1, 1) \setminus \Omega_1, \\
 0, & \text{otherwise}
\end{cases}
\]
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Table 2: \( l^\infty(\Omega) \) errors and convergence rates with \( P = 1 \).

| \( s \) \( \frac{2}{h} \) | 128 | 256 | 512 | 1024 |
|-----------------|-----|-----|-----|-----|
| 0.1             | 3.663E-04 | 1.911E-04 | 9.470E-05 | 4.570E-05 |
| Rates           | 0.9389  | 1.0128 | 1.0512 |       |
| 0.2             | 1.005E-03 | 4.965E-04 | 2.329E-04 | 1.061E-04 |
| Rates           | 1.0175  | 1.0922 | 1.1337 |       |
| 0.3             | 4.274E-04 | 1.859E-04 | 7.800E-05 | 3.219E-05 |
| Rates           | 1.2011  | 1.2529 | 1.2770 |       |
| 0.6             | 2.415E-04 | 7.109E-05 | 2.433E-05 | 8.171E-06 |
| Rates           | 1.7644  | 1.5470 | 1.5741 |       |

to verify the convergence when we use (4.6) to solve the inhomogeneous Dirichlet problem. According to (4.5), we have \( \bar{\omega}_0^M = 0 \) when \( s = 0.2 \) and \( \bar{\omega}_0^M = \bar{\omega}_0 \) when \( s = 0.3, 0.6, 0.7 \). From the results presented in Table 3, we find when \( \bar{\omega}_0^M = 0 \), the convergence rates are \( O(h^{2-2s}) \) which are the same as the ones predicted by Theorem 4.5 and when \( \bar{\omega}_0^M \neq 0 \), the convergence rates are \( O(h^2) \) which are higher than the predicted ones.

Table 3: \( l^\infty(\Omega) \) errors and convergence rates with \( P = 0 \).

| \( s \) \( \frac{2}{h} \) | 128 | 256 | 512 | 1024 |
|-----------------|-----|-----|-----|-----|
| 0.2             | 5.20E-05 | 1.74E-05 | 5.82E-06 | 1.95E-06 |
| Rates           | 1.5763  | 1.5834 | 1.5799 |       |
| 0.3             | 6.432E-06 | 1.622E-06 | 4.064E-07 | 1.007E-07 |
| Rates           | 1.9870  | 1.9973 | 2.0123 |       |
| 0.6             | 6.647E-06 | 1.749E-06 | 4.547E-07 | 1.167E-07 |
| Rates           | 1.9265  | 1.9434 | 1.9617 |       |
| 0.7             | 6.474E-06 | 1.718E-06 | 4.505E-07 | 1.166E-07 |
| Rates           | 1.9143  | 1.9307 | 1.9502 |       |

Afterwards, we show the numerical results that use (4.6) to solve (1.1) with \( \Omega_1 = (-1, 1) \) and \( g(x) = 0 \) in Table 4. Due to \( P = 0 \), the exact solution has a low regularity. The results presented in Table 4 show the numerical scheme (4.6) is still effective.

Example 5.4. Here we present some examples in two dimensions. We choose

\[
    u = \begin{cases} 
    ((1 - x^2)(1 - y^2))^2, & (x, y) \in \Omega_2, \\
    0, & (x, y) \in \Omega_2^c, 
    \end{cases}
\]

where \( \Omega_2 = (-1, 1) \times (-1, 1) \) and \( g(x, y) = 0 \). Table 5 shows the truncation errors when using (2.14) with \( \theta = 0 \) and 1 to approximate \( (-\Delta)^s u \). Since \( (-\Delta)^s u \) is unknown, the
Table 4: $l^\infty(\Omega)$ errors and convergence rates with $P = 0$.

| $s \| 2/h | 256    | 512    | 1024   | 2048   |
|-------|--------|--------|--------|--------|
| 0.2   | 7.681E-02 | 6.680E-02 | 5.812E-02 | 5.058E-02 |
| Rates | 0.2016 | 0.2008 | 0.2004 |        |
| 0.4   | 8.422E-03 | 6.387E-03 | 4.842E-03 | 3.670E-03 |
| Rates | 0.3990 | 0.3995 | 0.3997 |        |
| 0.6   | 2.459E-03 | 1.621E-03 | 1.069E-03 | 7.053E-04 |
| Rates | 0.6010 | 0.6005 | 0.6002 |        |
| 0.8   | 4.966E-04 | 2.859E-04 | 1.644E-04 | 9.450E-05 |
| Rates | 0.7964 | 0.7982 | 0.7991 |        |

Truncation errors are calculated by

$$e_h = \| (-\Delta)^s_h u - (-\Delta)^s_t u \|_\infty.$$  

All the results validate Theorem 3.1.

In Table 6, we show the convergence of the numerical scheme (4.8). Since $(-\Delta)^s u$ is unknown, we use $(-\Delta)^s_t u$ with $h = \frac{1}{2048}$ and $\theta = 1$ to approximately represent it. For $s = 0.2, 0.3$, we take $c_{0,0} = 1$ and $\theta = 0.5$; the convergence rates presented in Table 6 are the same as the ones predicted by Theorem 4.8. For $s = 0.4, 0.8$, we choose $c_{0,0} = 0$ and $\theta = 1$; the convergence rates are higher than the predicted ones.

Table 5: $l^\infty(\Omega)$ truncation errors and convergence rates in two dimensions.

| $(s, \theta) \| 2/h | 64    | 128    | 512    | 1024   |
|--------|--------|--------|--------|--------|
| (0.3,0) | 1.238E-03 | 4.994E-04 | 1.968E-04 | 7.645E-05 |
| Rates  | 1.3099 | 1.3437 | 1.3641 |        |
| (0.3,1) | 1.324E-03 | 5.208E-04 | 2.021E-04 | 7.778E-05 |
| Rates  | 1.3461 | 1.3656 | 1.3777 |        |
| (0.8,0) | 1.358E-01 | 9.929E-02 | 7.399E-02 | 5.562E-02 |
| Rates  | 0.4516 | 0.4244 | 0.4116 |        |
| (0.8,1) | 1.332E-01 | 9.868E-02 | 7.384E-02 | 5.559E-02 |
| Rates  | 0.4330 | 0.4183 | 0.4097 |        |

**Example 5.5.** Finally, we use the discretization (4.8) to simulate the mean exit time $u(x)$ of an orbit starting at $x$, from a two-dimensional bounded interval $\Omega_2$. According to Dynkin formula [3, 22] of Markov processes, $u(x)$ satisfies [8, 20],

$$\begin{cases} Au(x) = 1, & \text{in } \Omega_2, \\
 u(x) = 0, & \text{in } \Omega_2^c. \end{cases}$$
Table 6: $l^\infty(\Omega)$ errors and convergence rates in two dimensions.

| $s/2/h$ | 64      | 128     | 256     | 512     |
|---------|---------|---------|---------|---------|
| 0.2     | 6.837E-03 | 2.371E-03 | 8.040E-04 | 2.654E-04 |
| 0       | 1.5281  | 1.5600  | 1.5993  |         |
| 0.3     | 7.525E-03 | 3.030E-03 | 1.179E-03 | 4.419E-04 |
| 0       | 1.3125  | 1.3618  | 1.4157  |         |
| 0.4     | 1.122E-03 | 2.826E-04 | 7.286E-05 | 1.834E-05 |
| 0       | 1.9886  | 1.9557  | 1.9901  |         |
| 0.8     | 1.222E-03 | 3.049E-04 | 7.550E-05 | 1.837E-05 |
| 0       | 2.0030  | 2.0138  | 2.0393  |         |

where

$$\mathcal{A} = \nabla P(x) \cdot \nabla + (-\Delta)^s,$$

$\nabla$ denotes gradient operator, and $P(x)$ is a given potential. Here, we take $h = \frac{1}{64}$, $c_{0,0} = 100$, $\theta = \frac{1}{2}$, $\Omega_2 = ((-1, 1))^2$, and $P(x) = \kappa(x_1^2 + x_2^2)$ with $x = (x_1, x_2)$. In Fig. 2, we show the mean exit time when taking $s = 0.2, 0.4, 0.6, 0.8$, and $\kappa = 0.5$. Comparing Fig. 2(a) with Figs. 2(b)-2(d), we find the mean exit time becomes longer and boundary
layer phenomena become weaker as \( s \) increases. In Fig. 3, we show the mean exit time with \( s = 0.6 \) and different \( \kappa \). We find that the boundary layer phenomena become stronger and the mean exit time becomes longer as \( \kappa \) increases.

6. Conclusions

A fundamentally new idea of discretizing the fractional Laplacian is introduced and used to solve the inhomogeneous fractional Dirichlet problem. The effectiveness of the designed scheme is ensured by the completely theoretical analyses and verified by numerical experiments. Specific applications for simulating the mean exit time of Lévy processes under harmonic potential are provided, the effects of the strengths of the potential and the Lévy exponents are uncovered.

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