Stoner instability revisited near Anderson localization: Emergence of localized magnetic moments in the smearing transition

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Introducing the information of wave-function multifractality into a mean-field theory for Stoner instability near Anderson localization, we find the phenomena of a smearing phase transition above a crossover temperature, characterized by exponential scaling of thermodynamic quantities. The smearing-transition behavior originates from the physics that a power-law tail in the distribution function of the critical temperature does not persist in the regime of low temperatures because the Stoner instability occurs above a critical interaction parameter in the clean case, differentiated from either the Kondo-Anderson or the BCS-Anderson transition, where the power-law tail survives at the lowest temperature in principle, responsible for quantum Griffiths phenomena. An unexpected discovery is that localized magnetic moments emerge above a critical interaction parameter inside the ferromagnetic phase, detected only in the local spin susceptibility which shows a qualitative enhancement at the critical interaction parameter. The underlying mechanism turns out to be that effects of rare regions with a pseudogap become more pronounced when the ferromagnetic interaction exceeds its critical value. Emergence of localized magnetic moments with the smearing transition is an essential feature in our mean-field theory.

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I. INTRODUCTION

It is one of the central issues in modern condensed matter physics how to generalize the Landau-Ginzburg framework for phase transitions when there exist additional ingredients besides correlations between electrons. For example, recent advances on topological states of matter cast an interesting question how to incorporate the topological structure into the Landau-Ginzburg theory consistently, where topological terms play an essential role\cite{1}. In addition, it is highly desirable to construct the Landau-Ginzburg-type theory when the concept of electron quasiparticles is no longer valid\cite{2}. Here, we touch on another long standing problem, that is, how to generalize the Landau-Ginzburg theory when dynamics of electrons lies near Anderson localization.

The essential feature of Anderson localization is that the statistics of eigenfunctions shows multifractality, where moments of eigenstates show power-law scaling with anomalous exponents, instead of an exponential decay, not only for relative distances but also for relative energy differences\cite{3}. Our problem is how to encode the information of the eigenfunction multifractality into the Landau-Ginzburg-type theory for quantum phase transitions, where the wave-function statistics is well known from either the random-matrix theory or the supersymmetric nonlinear σ model approach\cite{3}. Recently, the fractal nature of localized eigenfunctions near the mobility edge has been introduced to generalize the BCS (Bardeen-Cooper-Schrieffer) mean-field theory near the Anderson localization in Fermi liquids\cite{4}, where pseudogap physics plays an important role. Furthermore, the Kondo effect has been revisited near Anderson localization, where the multifractal nature turns out to change how to screen a localized magnetic moment, giving rise to a power-law tail in the distribution function of the Kondo temperature\cite{4}. This power-law tail is shown to result from rare events with a pseudogap, responsible for the quantum Griffiths phenomena detected in divergent behaviors of specific heat and uniform spin susceptibility.

In this study, we extend the previous theoretical framework to the case with spin-rotational symmetry breaking, where the role of an order parameter in thermodynamics is clarified near Anderson localization. Introducing the eigenfunction multifractality into a mean-field theory for Stoner instability near Anderson localization, we observe a smearing type of ferromagnetic transitions instead of quantum Griffiths phenomena in itinerant Ising ferromagnets, where not only the magnetization order parameter but also both specific heat and uniform spin susceptibility show exponential scaling\cite{7} above a crossover temperature characterized by the maximum of the specific heat. The appearance of the smearing transition is rather unexpected since the existence of a power-law tail in the distribution function of the critical temperature is regarded to be the origin of quantum Griffiths physics\cite{8}. However, we uncover that the power-law tail does not persist below a certain temperature because the Stoner instability occurs above a critical interaction parameter in the clean case, differentiated from either the Kondo-Anderson transition\cite{5,6} or the
FIG. 1: A schematic phase diagram in the parameter space of $(W, U, T)$ based on our generalized mean-field theory near Anderson localization, where $W$ is the strength of randomness, $U$, an interaction parameter for Stoner instability, and $T$, temperature. $U_c$ is the critical strength of the interaction parameter for ferromagnetism in the clean case, and $W_c$ is the critical point for the Anderson metal-insulator transition, characterized by the wave-function multifractality. We find two essential features from our generalized mean-field theory. One is that the ferromagnetic transition becomes smeared, the hallmark of which is exponential scaling for thermodynamic quantities such as magnetization, specific heat, and uniform spin susceptibility. The other is that the itinerant character of ferromagnetism disappears, shown in the upturn behavior of the local spin susceptibility above a certain critical value, although the uniform spin susceptibility becomes suppressed. We suspect that there may exist a quantum critical point inside the ferromagnetic phase, judging from drastic enhancement in the local spin susceptibility (Left inset in Fig. 5).

We would like to point out the limit of our mean-field theory. First of all, our mean-field theory fails to introduce self-consistent renormalizations for both order parameters and disorders below the ordering temperature particularly. The eigenfunction multifractality is expected to disappear in the ordered state due to the self-consistent renormalization. In addition, our mean-field theory fails to describe strong inhomogeneity in spatial fluctuations of order parameters, BCS-Anderson one [4], where the power-law tail survives at the lowest temperature in principle, showing the quantum Griffiths physics. See Fig. 1, where the yellow region is identified with rounding of the ferromagnetic transition. The most unexpected discovery is that localized magnetic moments appear from the itinerant Ising ferromagnet above a critical value of the interaction parameter, detected by the local spin susceptibility only. We suspect the existence of a quantum critical point between itinerant and localized Ising ferromagnets, where the local spin susceptibility shows qualitative enhancement in the critical regime, although our mean-field theory fails to catch possible divergences at the critical point. Emergence of localized magnetic moments within the rounding behavior of a phase transition is a key feature of our generalized mean-field theory for itinerant ferromagnets near the Anderson localization, where effects of rare regions with a pseudogap are incorporated appropriately.
which can be cured if we take into account a full numerical procedure in the Hartree-Fock level for interactions, as will be discussed below. Actually, these two aspects on self-consistent renormalizations for interactions and disorders and strong spatial fluctuations in dynamics of order parameters are deeply related with each other, where the disappearance of eigenfunction multifractality may suppress such strong spatial fluctuations in the distribution of order parameters. Keeping these important problems in our mind, we construct a mean-field theory for the Stoner transition at the Anderson localization.

II. A MEAN-FIELD THEORY NEAR ANDERSON LOCALIZATION

A. Formulation

Reformulating the mean-field theory for ferromagnetism in terms of the eigenfunction basis given by

\[
\left( -\frac{\nabla^2}{2m} - \mu_r + v(r) \right) \Psi_n(r) = \varepsilon_n \Psi_n(r),
\]

where \( \Psi_n(r) \) is an eigenfunction with an eigenvalue \( \varepsilon_n \) and an effective chemical potential \( \mu_r = \mu - \frac{U}{2} \) in a disorder configuration \( v(r) \), we obtain the mean-field free energy

\[
\mathcal{F} \approx -T \int_{-\infty}^{\infty} dv(r) P[v(r)] \left\{ \sum_n \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n - \int d^d r |\Psi_n(r)|^2 \Phi(r)}{T} \right) \right\} + \sum_n \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n + \int d^d r |\Psi_n(r)|^2 \Phi(r)}{T} \right) \right\} - \frac{1}{T} \int d^d r \frac{3}{2U} [\Phi(r)]^2 \right\}.
\]

\( P[v(r)] \) is a distribution function, given by the Gaussian \( P[v(r)] = N_v \exp \left( -\frac{1}{\Gamma_v} \int d^d r \frac{v^2(r)}{2} \right) \) for example, where \( \Gamma_v \) is a variance and \( N_v \) is determined by the normalization condition \( \int_{-\infty}^{\infty} dv(r) P[v(r)] = 1 \). \( U \) is an interaction parameter for ferromagnetism and \( T \) is temperature. \( \Phi(r) \) is the magnetization order parameter, determined by the self-consistent equation

\[
\frac{3}{U} \Phi(r) = \sum_n |\Psi_n(r)|^2 \left( f\left( \varepsilon_n - \int d^d r |\Psi_n(r)|^2 \Phi(r) \right) - f\left( \varepsilon_n + \int d^d r |\Psi_n(r)|^2 \Phi(r) \right) \right).
\]

A general strategy is as follows. For a given disorder configuration \( v(r) \), we obtain an eigenfunction \( \Psi_n(r) \) at each eigenvalue \( \varepsilon_n \), solving Eq. (1). Then, we find a solution \( \Phi(r) \) from Eq. (3) numerically, resorting to the eigenfunction. This procedure should be repeated for various disorder configurations, generated by the Gaussian distribution function \( P[v(r)] \). Performing the summation of free energies for all disorder configurations with \( P[v(r)] \), we can determine thermodynamics for the Stoner transition near Anderson localization, where this strategy incorporates physics of Anderson localization rather accurately.

B. Eigenfunction multifractality

An idea is to replace an integral for the average in disorder configurations \( \int_{-\infty}^{\infty} dv(r) P[v(r)] \) with \( \int_{-\infty}^{\infty} \Pi_n d\alpha_n(r) P[\{\alpha_n(r)\}] \) for the average in the statistics of eigenfunctions. We note that all the information for the statistics of eigenfunctions are encoded into the distribution function of \( P[\{\alpha_n(r)\}] \) with \( \alpha_n(r) = -\frac{\ln |\Psi_n(r)|^2}{\ln L} \), which will be clarified below. An important point is how to perform the integration for the wave-function distribution. Recently, this procedure has been discussed intensively, where an idea is to take into account the so-called joint distribution function which deals with pairs of eigenfunctions \( \{\alpha_n(r)\} \), given by

\[
\int_{-\infty}^{\infty} \Pi_n d\alpha_n(r) P[\{\alpha_n(r)\}] \approx \int_{-\infty}^{\infty} d\alpha(r) P^{(1)}[\alpha(r)] \int_{-\infty}^{\infty} \Pi_n d\alpha_n(r) \frac{P^{(2)}[\alpha_n(r) \neq \alpha(r)]}{P^{(1)}[\alpha(r)]}.
\]

\( P^{(2)}[\alpha_n(r) \neq \alpha(r)] \) is the joint distribution function, where one eigenfunction \( \alpha(r) \) is at the mobility edge \( \varepsilon_m \) and the other wave function \( \alpha_n(r) \) is away from the mobility edge. On the other hand, \( P^{(1)}[\alpha(r)] \) is the distribution function
for the single eigenfunction at the mobility edge. Both distribution functions are given by the Gaussian distribution function (the log-normal distribution function for the intensity of an eigenfunction), constructed to reproduce the wave-function multifractality of the random matrix theory or the supersymmetric nonlinear \( \sigma - \) model approach. Then, this integral means to perform the integral for \( \alpha_n(r) \) with a fixed \( \alpha(r) \) first, based on the mutual distribution function, and to do for \( \alpha(r) \) next, based on the single eigenfunction distribution function. As a result, we reach the following expression

\[
\mathcal{F} \approx -T \int_{-\infty}^{\infty} d\alpha(r) P^{(1)}[\alpha(r)] \sum_{n} \left[ \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n - \int d^d r \left\{ |\Psi_n(r)|^2 \right\}}{T} \right) \right\} \right]
\]

\[
+ \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n + \int d^d r \left\{ |\Psi_n(r)|^2 \right\}}{T} \right) \right\} \] + \int_{-\infty}^{\infty} d\alpha(r) P^{(1)}[\alpha(r)] \int d^d r \frac{3}{2T} |\Phi(r)|^2, \tag{5}
\]

where the average for the mutual distribution function gives rise to

\[
\left\langle |\Psi_n(r)|^2 \right\rangle_{|\Psi_m(r)|^2=L^{-\alpha(r)}} \equiv \int_{-\infty}^{\infty} \Pi_n d\alpha_n(r) \frac{P^{(2)}[\alpha_n(r) \neq \alpha(r)]}{P^{(1)}[\alpha(r)]} |\Psi_n(r)|^2 = L^{-d} |\varepsilon_n - \varepsilon_m|_{\varepsilon_c}^{r_{\alpha(r)}}, \tag{6}
\]

with an exponent

\[
r_{\alpha(r)} = \frac{\alpha(r) - \alpha_0}{d} - \frac{\eta}{2d} g_{nm}, \quad g_{nm} = \frac{\ln |(\varepsilon_n - \varepsilon_m)/\varepsilon_c|}{d \ln L}, \quad \eta = 2(\alpha_0 - d). \tag{7}
\]

\( L \) is the size of a system, \( d \) is a space dimension, \( \varepsilon_c \) is a cutoff, which shows strong correlations of eigenfunctions with different energies up to \( \varepsilon_c \), and \( \alpha_0 \) is a typical value of the logarithm of an eigenfunction.

Taking the integral for discrete energies as follows \( \sum_{n} \approx \rho_m \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \), we obtain

\[
\mathcal{F} \approx -T \rho_m \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \int_{-\infty}^{\infty} d\alpha(r) P^{(1)}[\alpha(r)] \left[ \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n - \Delta_n}{T} \right) \right\} \right]
\]

\[
+ \int_{-\infty}^{\infty} d\alpha(r) P^{(1)}[\alpha(r)] \int d^d r \frac{3}{2T} |\Phi(r)|^2 \tag{8}
\]

with \( \Delta_n \equiv \int d^d r \left| \frac{\varepsilon_n}{\varepsilon_c} \right|^{r_{\alpha(r)}} |\Phi(r)| \approx L^{-d} \int d^d r \left| \frac{\varepsilon_n}{\varepsilon_c} \right|^{r_{\alpha(r)}} |\Phi(r)| \), the magnetization order parameter is determined by the self-consistent equation for a given multifractal function \( \alpha(r) \)

\[
\Delta_l = \int d^d r \left[ \frac{\varepsilon_l}{\varepsilon_c} \right]^{r_{\alpha(r)}} \frac{U}{3} \rho_m \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \left| \varepsilon_n \right|^{r_{\alpha(r)}} \left\{ f(\varepsilon_n - \Delta_n) - f(\varepsilon_n + \Delta_n) \right\}. \tag{9}
\]

In order to solve this self-consistent equation, a general strategy is as follows. We prepare all configurations for \( \alpha_n(r) \), based on the single-particle intensity distribution function \( P^{(1)}[\alpha_n(r)] \) and the mutual intensity distribution function \( P^{(2)}[\alpha_n(r), \alpha_n'(r)]; \alpha_n \neq \alpha_n' \) \( \tilde{G} \). It is not that difficult to solve the self-consistent equation for each configuration of \( \alpha_n(r) \). When all sets of solutions \( \Delta_n[\{\alpha(r)\}] \) are prepared, Eq. (8) allows thermodynamics near the Anderson transition.

**C. Approximation**

An essential simplification is to lose the information on strong spatial inhomogeneity as the zeroth order approximation. Replacing \( \alpha(r) \) with \( \alpha \), we obtain the generalized mean-field theory for the Stoner transition near Anderson localization

\[
L^{-d} \mathcal{F} \approx -T \rho_m \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \int_{-\infty}^{\infty} d\alpha P(\alpha) \left[ \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n - \left| \varepsilon_n \right|^{r_{\alpha}} \Phi(\alpha)}{T} \right) \right\} \right]
\]

\[
+ \int_{-\infty}^{\infty} d\alpha P(\alpha) \frac{3}{2U} \Phi^2(\alpha), \tag{10}
\]

\( T \) is the temperature, and \( U \) is the exchange interaction.
where \( L^{-d} \int d^d r \Phi(r) \) is replaced with \( \Phi(\alpha) \), determined by the “gap” equation for the order parameter

\[
\Phi(\alpha) = \frac{U}{3} \rho_m \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \left\{ f\left(\varepsilon_n - \left|\frac{\varepsilon_n}{\varepsilon_c}\right|^\gamma \Phi(\alpha)\right) - f\left(\varepsilon_n + \left|\frac{\varepsilon_n}{\varepsilon_c}\right|^\gamma \Phi(\alpha)\right)\right\}. \tag{11}
\]

The distribution function is given by

\[
P(\alpha) = N L^{-\frac{(\alpha-\alpha_0)^2}{\eta^2}} \tag{12},
\]

where \( N \) is a positive numerical constant determined from \( \int_0^\infty d\alpha P(\alpha) = 1 \).

We would like to point out that the magnetization order parameter is given by a function of \( \alpha \) and both the \( \alpha \)-dependent \( \Phi(\alpha) \) and the integration for \( \alpha \) with \( P(\alpha) \) are expected to keep correlation effects in the energy space although strong spatial fluctuations in the intensity of eigenfunctions are not introduced in our mean-field theory. A physical picture for this mean-field analysis is as follows. Suppose an island at a position \( r \) with a characteristic length scale, determined by both interactions and disorders, where the intensity of an eigenfunction may be regarded to be uniform, responsible for the uniform magnetization within the island. Then, we consider another island at a position \( r' \) near the previous island, introducing some couplings such as electron hopping and magnetic interaction between these nearest-neighbor islands. Based on this granular picture, one may perform a weak-coupling analysis for interactions between these granules. One may suspect three kinds of possibilities, which correspond to relevance, irrelevance, and marginality of granular interactions, respectively. We believe that the present mean-field analysis focuses on the case of relevant granular interactions, giving rise to “almost” uniform magnetization for each intensity of eigenfunctions. We also speculate that the possible disappearance of the eigenfunction multifractality may suppress strong spatial fluctuations in the configuration of order parameters, as pointed out in the introduction. This granular-medium picture deserves to be investigated more sincerely near future.

### III. ROUNDING OF THE STONER TRANSITION

#### A. Magnetization

Figure 2 displays how the magnetization order parameter behaves as a function of temperature with various interaction parameters, disorder averaged, i.e., \( \Phi_{av} \equiv \langle \Phi(\alpha) \rangle = \int_{-\infty}^{\infty} d\alpha P(\alpha) \Phi(\alpha) \). Sharp transitions become smoothen and the slope remains negative, not saturated until the lowest temperature (Left in Fig. 2). On the other hand, the log-log plot (Left inset in Fig. 2) makes the behavior look rather conventional, where the negative slope saturates to vanishing. An interesting feature is that the order parameter follows exponential scaling \( \tilde{\Phi} \) above a crossover temperature, given by \( \Phi_{av} = C(U) e^{-A_\varepsilon(U) \tilde{T}} \), where \( A_\varepsilon(U) \) turns out to be almost independent of the interaction parameter \( U \) (Right in Fig. 2). The crossover temperature \( T_{av} \) is identified with maxima in either the specific heat or the uniform spin susceptibility.

#### B. Uniform spin susceptibility

Figure 3 shows how the uniform spin susceptibility behaves as a function of temperature with various interaction parameters, disorder averaged, i.e.,

\[
\chi_u(T) = \frac{\rho_m}{4T} \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \int_{-\infty}^{\infty} d\alpha P(\alpha) \left\{ \frac{1}{\cosh^2\left(\frac{\varepsilon_n - \varepsilon_c}{2\eta}\Phi(\alpha)\right)} + \frac{1}{\cosh^2\left(\frac{\varepsilon_n + \varepsilon_c}{2\eta}\Phi(\alpha)\right)} \right\}. \tag{13}
\]

It also confirms rounding of the ferromagnetic transition, where divergent peaks associated with phase transitions become smoothen to show the exponential scaling of \( \chi_u(T; U) = C_\chi(U) e^{-A_\chi(U) \tilde{T}} \) above the crossover temperature. Here, both \( C_\chi(U) \) and \( A_\chi(U) \) are almost independent of \( U \). The uniform spin susceptibility becomes suppressed at low temperatures due to ferromagnetic ordering as expected.
given by $U$ the smearing transition is confirmed by the exponential scaling of $\Phi(\varepsilon)$. The log-log plot (Inset) makes the behavior look rather conventional, where the negative slope saturates to vanish. The nature of $\chi_n$ nets, detected from a local probe beyond the above mentioned bulk measurements. The local spin susceptibility $\chi_n$ is given by $\varepsilon_n = \frac{\varepsilon_n}{\varepsilon_c}$ for different interaction parameters.

**C. Specific heat**

Figure 4 shows the disorder-averaged specific heat as a function of temperature with various interaction parameters, given by

$$C_v(T) = \frac{\rho_m}{4T^2} \int_{-c}^{c} d\varepsilon_n \int_{-\infty}^{\infty} d\alpha P(\alpha) \left\{ \left\{ \frac{\varepsilon_n}{\varepsilon_c} \right\}^\alpha T \frac{\partial \Phi(\alpha)}{\partial T} + \left( \varepsilon_n - \frac{\varepsilon_n}{\varepsilon_c} \right)^\alpha \Phi(\alpha) \right\}^2 + \left\{ \left( \frac{\varepsilon_n}{\varepsilon_c} \right)^\alpha T \frac{\partial \Phi(\alpha)}{\partial T} + \left( \frac{\varepsilon_n}{\varepsilon_c} \right)^\alpha \Phi(\alpha) \right\}^2 \right\} \cosh^2 \left( \frac{\varepsilon_n - \frac{\varepsilon_n}{\varepsilon_c} \alpha \Phi(\alpha)}{2T} \right)$$

Similarly, we observe the signature of smearing transitions, fitted with $C_v(T; U) / T = C_{C_v}(U) e^{-A_{C_v}(U)T}$ above the crossover scale, where $A_{C_v}(U)$ is fairly the same for different interaction parameters.

**D. Local spin susceptibility**

It was unexpected to find the emergence of localized magnetic moments from these itinerant Ising ferromagnets, detected from a local probe beyond the above mentioned bulk measurements. The local spin susceptibility of $\chi^{+-}(r, r'; T) = -\langle S^+_r S^-_{r'} \rangle$ is given by

$$\chi^{+-}(r, r'; T) = - \sum_n \sum_{n'} \Psi_n(r) \Psi_n^\dagger(r') \sum_{n''} \Psi_{n''}(r') \Psi_{n''}^\dagger(r) \frac{f[\varepsilon_n + \int d^3r'' |\Psi_n(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r'')] - f[\varepsilon_{n'} + \int d^3r'' |\Psi_{n''}(r'')|^2 \Phi(r')]
FIG. 3: Disorder-averaged uniform spin susceptibility as a function of temperature with various interaction parameters. Divergent peaks associated with phase transitions become smoothen to show an exponential scaling of $\chi_u(T; U) = C_\chi(U)e^{-A_\chi(U)T}$ above the crossover temperature, where both $C_\chi(U)$ and $A_\chi(U)$ are almost independent of $U$. It becomes suppressed at low temperatures due to ferromagnetic ordering, where the flattening signature indicates the Pauli susceptibility with almost vanishing averaged magnetization.

Performing the average for disorder, we obtain

$$\chi^\pm_L(T) = \frac{1}{L^d} \int d^d r \left\langle \chi^+(r, r; T) \right\rangle$$

$$= - r \rho \frac{2}{N} \int_{-\epsilon_c}^{\epsilon_c} d\epsilon_n \int_{-\epsilon_c}^{\epsilon_c} d\epsilon_n' \int_{-\infty}^{\infty} d\alpha \rho P(\alpha) C_{nn'} \frac{f(\epsilon_n + \epsilon_n') \Phi(\alpha) - f(\epsilon_n - \epsilon_n') \Phi(\alpha)}{\epsilon_n - \epsilon_n' + \epsilon_n' - \epsilon_n} |\Phi(\alpha)|^2, \quad (16)$$

where $r = r'$ is concerned. The weighting factor $C_{nn'}$ is given by correlations of eigenfunctions with different energies $\epsilon_n$ and $\epsilon_n'$,

$$C_{nn'} = \frac{1}{L^d} \int d^d r \left\langle |\Psi_n(r)|^2 |\Psi_{n'}(r)|^2 \right\rangle, \quad (17)$$

where $\langle \ldots \rangle$ represents an average in disorder as discussed before. When one of these energies lies at the mobility edge, i.e., $\epsilon_{n'} = \epsilon_m$, we obtain $C_{nn'} = \left\{ \frac{\epsilon_n}{\max(|\epsilon_n - \epsilon_{m}|, \Delta)} \right\}^{n/d}$, where $\Delta$ is a mean level spacing. On the other hand, if both energies are away from the mobility edge, we have $C_{nn'} = \left\{ \frac{\epsilon_n}{\max(|\epsilon_n - \epsilon_{m}|, \Delta_{\xi_n})} \right\}^{n/d}$, where $\Delta_{\xi_n}$ is an effective mean level spacing with an effective sample size $\xi_n$ and $\xi_n \propto |\epsilon_n - \epsilon_m|^{-\nu}$ with an exponent $\nu$ is the correlation/localization length in a metal/insulator side away from the Anderson transition.

This disorder-averaged local spin susceptibility shows an astonishing behavior at low temperatures with increasing interaction parameters. See Fig. 5. When $U < U^* \approx 1.75U_c$, where $U_c$ is the critical interaction parameter for Stoner instability in the clean case, the local spin susceptibility decreases at low temperatures, which means that local fluctuations of spins are suppressed, expected due to ferromagnetic ordering (Left and right). However, an
FIG. 4: Disorder-averaged specific heat as a function of temperature with various interaction parameters. Divergent peaks associated with phase transitions (Inset) become smoothen (Left) to show an exponential scaling of \( C_v(T; U) / T = C_{Cv}(U) e^{-A_{Cv}(U)T} \) (Right) above the crossover temperature, where \( A_{Cv}(U) \) is fairly the same for different interaction parameters. It becomes suppressed at low temperatures due to ferromagnetic ordering, where the flattening signature indicates that \( \gamma_v = C_v / T \) coefficient is proportional to the density of states with almost vanishing averaged magnetization, which corresponds to the clean limit.

E. Mechanism of the smearing transition

We point out that the mechanism for the smearing transition is not transparent. It is straightforward to find the critical temperature as a function of \( \alpha \) associated with the intensity of an eigenfunction. Actually, we uncover that the critical temperature is given by an exponential function of \( \alpha \) (Left in Fig. 6). This exponential dependence gives rise to a power-law distribution function for the critical temperature with an anomalous exponent (Right in Fig. 6). The existence of the power-law tail in the distribution function is suggested to be responsible for quantum Griffiths phenomena conventionally \( \mathbb{R} \). However, the power-law tail fails to persist below a certain temperature because the ferromagnetic transition is allowed above a critical interaction parameter in the clean case. This situation differs from that of either the Kondo-Anderson transition \( \mathbb{R}, \mathbb{R} \) or the BCS-Anderson one \( \mathbb{A} \), where the power-law tail survives at the lowest temperature, giving rise to the quantum Griffiths physics.
FIG. 5: Disorder-averaged local spin susceptibility as a function of temperature with various interaction parameters. There appears a drastic change around $U \approx U^* \sim 1.75U_c$, where an upturn behavior emerges above $U^*$ (Right: log-log plot). In addition, the degree of this upturn behavior reaches a maximum at around $U \approx U^*_c \equiv U^*_c$ and becomes suppressed gradually although the upturn tendency is sustained (Left inset). The underlying mechanism turns out to be that effects of rare regions with a pseudogap ($r_\alpha > 0$) become more pronounced to prohibit spins from ferromagnetic ordering, where they remain rather independent (Right inset). As a result, we suspect that localized magnetic moments appear to show a quantum phase transition, given by the qualitative enhancement of the local spin susceptibility as a function of the interaction parameter around $U^*_c$ at the lowest temperature (Left and left inset). We recall the local spin susceptibility of the Fermi gas, for a reference, given by

$$\chi_{L,FG}(T) = -\rho_n^2 \int_{-\varepsilon_e}^{\varepsilon_e} d\varepsilon_n \int_{-\varepsilon_e}^{\varepsilon_e} d\varepsilon_{n'} \frac{f(\varepsilon_n) - f(\varepsilon_{n'})}{\varepsilon_n - \varepsilon_{n'}} \sim T \ln T.$$ 

Following T. Vojta’s strategy [7], we can address this question in the respect of the Hertz-Moriya-Millis theory [10]

$$S_{loc} \approx \rho_m^2 \sum_{i\Omega} \int_{-\varepsilon_e}^{\varepsilon_e} d\varepsilon_n \int_{-\varepsilon_e}^{\varepsilon_e} d\varepsilon_{n'} C_{nn'} \frac{f(\varepsilon_n) - f(\varepsilon_{n'})}{i\Omega - \varepsilon_n + \varepsilon_{n'}} \sigma(i\Omega) \sigma(-i\Omega)$$

$$\propto \rho_m^2 \sum_{i\Omega} |\Omega|^{-\eta/d} \sigma(i\Omega) \sigma(-i\Omega) \sim \int d\tau \int d\tau' \sigma(\tau) \frac{1}{(\tau - \tau')^{2+\eta/d}} \sigma(\tau'),$$

where electronic degrees of freedom are integrated out to result in an effective theory for ferromagnetic fluctuations. Here, we consider only local spin fluctuations at a position $r$ and focus on their dynamics along the time direction, which shows nonlocal correlations, where $\eta/d$ was already defined in $C_{nn'}$ before. $\sigma(\tau)$ has been used to emphasize dynamics of Ising spins. Applying the Vojta’s argument to this situation, ferromagnetic ordering is expected to occur due to the nonlocal interaction in the time domain. Once the static order is developed on several rare regions, weak interactions between them may be sufficient to cause actual ordering. However, this argument should be taken into account with great caution since there must exist complex interactions between spin fluctuations, nonlocal not only in time but also in space. Actually, it was not easy to handle $\sigma(r, \tau) \sigma(r', \tau') \sigma(r'', \tau'') \sigma(r''', \tau''')$ terms due to their nonlocal properties. In this respect this argument should be regarded as a consistency check applicable only when the constant interaction vertex dominates nonlocal correlations.
different energies. Taking quantum corrections from magnetization fluctuations in the random phase approximation, where the intensity correlation of eigenfunctions at different positions and different energies is given by

\[ \alpha \]

This exponential dependence is responsible for the existence of a power-law tail in the distribution function of the critical temperature (Right). Although there exists a power-law tail in the distribution function, regarded to be responsible for quantum Griffiths physics, it fails to persist below a certain temperature scale, which results from the fact that the Stoner transition is allowed above a critical interaction parameter in the clean case. This situation differs from that of either the Kondo-Anderson or the BCS-Anderson transition, where the power-law tail survives at the lowest temperature, giving rise to the quantum Griffiths physics. Compare \( P(T_c) \) with \( P(T_K) \) in the inset figure, where an analytic form for \( P(T_K) \) comes from Ref. 10.

**FIG. 6:** A distribution function for the critical temperature. The critical temperature is given by an exponential function of \( \alpha \) (Left). This exponential dependence is responsible for the existence of a power-law tail in the distribution function of the critical temperature (Right). Although there exists a power-law tail in the distribution function, regarded to be responsible for quantum Griffiths physics, it fails to persist below a certain temperature scale, which results from the fact that the Stoner transition is allowed above a critical interaction parameter in the clean case. This situation differs from that of either the Kondo-Anderson or the BCS-Anderson transition, where the power-law tail survives at the lowest temperature, giving rise to the quantum Griffiths physics. Compare \( P(T_c) \) with \( P(T_K) \) in the inset figure, where an analytic form for \( P(T_K) \) comes from Ref. 10.

**IV. DISCUSSION AND SUMMARY**

**A. Beyond the mean-field theory**

Our mean-field theory can be generalized systematically to incorporate spatial correlations of eigenfunctions with different energies. Taking quantum corrections from magnetization fluctuations in the random phase approximation, we obtain

\[
L^{-d} F \approx -T \rho \int_{-\infty}^{\infty} d\varepsilon \int_{-\varepsilon_n}^{\varepsilon_n} d\alpha P(\alpha) \left[ \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n - \left| \frac{r_n}{\varepsilon_c} \Phi(\alpha) \right|}{T} \right) \right\} + \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n + \left| \frac{r_n}{\varepsilon_c} \Phi(\alpha) \right|}{T} \right) \right\} \right] + \int_{-\infty}^{\infty} d\alpha P(\alpha) \frac{3}{2U} \Phi^2(\alpha) + T \int_{-\infty}^{\infty} d\alpha P(\alpha) \frac{3}{2} \sum_{n,m} \sum_{r,r'} t_{r,r'} \ln \left\{ \frac{3}{U} + \rho_m \right\} \int_{-\varepsilon_n}^{\varepsilon_n} d\varepsilon_n \int_{-\varepsilon_n}^{\varepsilon_n} d\varepsilon_n' C_{nn'}(r,r') \right]
\]

\[
\left\{ f \left( \varepsilon_n - \left| \frac{r_n}{\varepsilon_c} \Phi(\alpha) \right| \right) - f \left( \varepsilon_n' - \left| \frac{r_n'}{\varepsilon_c} \Phi(\alpha) \right| \right) \Phi(\alpha) + \Phi(\alpha) \right\} \right]
\]

where the intensity correlation of eigenfunctions at different positions and different energies is given by

\[
C_{nn'}(r,r') = L^{2d} \langle |\Psi_n(r)|^2 |\Psi_n'(r')|^2 \rangle
\]

\[
= \begin{cases} 
\left( \frac{L_{\text{int}}}{|r-r'|} \right)^d, & 0 < |r-r'| < L_{\text{int}} \\
\left( \frac{L_{\text{int}}}{|r-r'|} \right)^{2d}, & L_{\text{int}} < |r-r'| 
\end{cases}
\]

where \( L_{\text{int}} \) is the integral length scale.
with \( L_{\omega_{nm}} = 1/(\rho_m|\varepsilon_n - \varepsilon_m'|)^{1/d} \). Although we believe that dissipation effects from such quantum corrections do not modify the physical picture of the smearing transition, we suspect the emergence of a spin glass phase, where magnetization may be frozen randomly.

B. Role of density fluctuations in the Stoner instability near Anderson transition

An important issue is on the role of repulsive interactions between density fluctuations in the smearing ferromagnetic transition. Incorporating the Hartree channel into the mean-field theory, we obtain

\[
L^{-d}F \approx -T\rho_m \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \int_{-\infty}^{\infty} d\alpha P(\alpha) \left[ \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n + \frac{\varepsilon_n}{\varepsilon_c} \varphi(\alpha) - \frac{\varepsilon_n}{\varepsilon_c} \Phi(\alpha)}{T} \right) \right\} \right]
+ \ln \left\{ 1 + \exp \left( -\frac{\varepsilon_n + \frac{\varepsilon_n}{\varepsilon_c} \varphi(\alpha) + \frac{\varepsilon_n}{\varepsilon_c} \Phi(\alpha)}{T} \right) \right\}
+ \int_{-\infty}^{\infty} d\alpha P(\alpha) \left( \frac{3}{2U} \varphi^2(\alpha) - \frac{1}{2U} \varphi^2(\alpha) \right),
\]

where self-consistent equations for both the magnetization \( \Phi(\alpha) \) and effective chemical potential \( \varphi(\alpha) \) are given by

\[
\Phi = \frac{U}{3} \rho_m \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \left[ \frac{\varepsilon_n}{\varepsilon_c} \right] r_\alpha \left\{ f \left( \varepsilon_n + \frac{\varepsilon_n}{\varepsilon_c} \varphi - \frac{\varepsilon_n}{\varepsilon_c} \Phi \right) - f \left( \varepsilon_n + \frac{\varepsilon_n}{\varepsilon_c} \varphi + \frac{\varepsilon_n}{\varepsilon_c} \Phi \right) \right\},
\]

\[
\varphi = U \rho_m \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon_n \left[ \frac{\varepsilon_n}{\varepsilon_c} \right] r_\alpha \left\{ f \left( \varepsilon_n + \frac{\varepsilon_n}{\varepsilon_c} \varphi - \frac{\varepsilon_n}{\varepsilon_c} \Phi \right) + f \left( \varepsilon_n + \frac{\varepsilon_n}{\varepsilon_c} \varphi + \frac{\varepsilon_n}{\varepsilon_c} \Phi \right) \right\}.
\]

Although one may introduce the Fock channel into this mean-field theory in principle, we believe that the Altshuler-Aronov suppression of the density of states [11] can be described qualitatively within this mean-field theory. Then, an essential question is how the suppression of the density of states affects the smearing transition. Recalling the emergence of localized magnetic moments driven by the pseudogap physics, we suspect that localized magnetic moments dominate physics in most of the parameter space, particularly, at low temperatures. It is not clear whether quantum Griffiths phenomena emerge or not, overcoming the smearing transition.

C. Summary

In conclusion, we generalized the Landau-Ginzburg-type mean-field theory near Anderson localization, incorporating the wave-function multifractality. As a result, we found that the nature of the ferromagnetic transition becomes rounding instead of showing quantum Griffiths physics, where the power-law tail of the distribution function of the critical temperature is cut at a certain temperature scale. It is quite surprising to observe that the nature of ferromagnetism becomes localized above a critical value of the interaction parameter in spite of itinerant ferromagnets. We revealed that this hidden transition originates from the pseudogap physics. It is important to perform the full numerical analysis for the Hartree-Fock theory, where both self-consistent renormalizations for interactions and disorders and strong spatial fluctuations in dynamics of order parameters can be taken into account.

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Appendix: Analytic solution for magnetization

It is straightforward to find an analytic solution for magnetization. In the clean limit of \( r_\alpha = 0 \) we obtain

\[
m = 2 \sqrt{\frac{3 f_c}{T_c}} \left( \frac{\partial f(\epsilon_n)}{\partial \epsilon_n} \right)_{\epsilon_n = \frac{\epsilon_c}{T_c}} \left( 1 - \frac{T}{T_c} \right)^{r_\alpha + \frac{1}{2}},
\]

(A.1)

reproducing the mean-field result of the clean case. In the case with a pseudogap given by \( r_\alpha > 0 \), we obtain

\[
m = 3 \sqrt{\frac{3 f_c}{T_c}} \left( \frac{\partial f(\epsilon_n)}{\partial \epsilon_n} \right)_{\epsilon_n = \frac{\epsilon_c}{T_c}} \left( 1 - \frac{T}{T_c} \right)^{r_\alpha + \frac{1}{2}},
\]

(A.2)

where the temperature dependence of the magnetization order parameter remains the same as that of the clean case. In the region where the density of states is enhanced moderately, given by \(-\frac{1}{4} < r_\alpha < 0\), we obtain

\[
m = 2 \frac{3 f_c}{T_c} \left( \frac{\partial f(\epsilon_n)}{\partial \epsilon_n} \right)_{\epsilon_n = \frac{\epsilon_c}{T_c}} \left( 1 - \frac{T}{T_c} \right)^{r_\alpha + \frac{1}{2}}.
\]

(A.3)

When the density of states is much enhanced, given by \( r_\alpha < -\frac{1}{4} \), we obtain

\[
m = \frac{1}{T_c^{r_\alpha}} \sqrt{\frac{6}{f_c}} \left( \frac{\partial f(\epsilon_n)}{\partial \epsilon_n} \right)_{\epsilon_n = \frac{\epsilon_c}{T_c}} \left( 1 - \frac{T}{T_c} \right)^{-r_\alpha}.
\]

(A.4)

Based on these analytic solutions, one can find the magnetization order parameter, disorder averaged. Our numerical solutions turn out to coincide with these analytic ones.

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