Multi-dimensional Sum-Up Rounding using Hilbert curve iterates

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Abstract

Mixed-integer optimal control problems can be reformulated by means of partial outer convexification, which introduces binary-valued switching functions for the different realizations of a discrete-valued control variable. They can be relaxed naturally by allowing them to take values in $[0,1]$. Sum-Up Rounding (SUR) algorithms approximate feasible switching functions of the relaxation with binary ones. If the controls are distributed in one dimension, the approximants are known to converge in the weak* topology of $L^\infty$. We show that this still holds true for controls that are distributed in more than one dimension if an appropriate grid refinement strategy is adopted that is coupled with a deliberate ordering of the grid cells is chosen. This condition is satisfied by the iterates of space-filling curves, e.g., the Hilbert curve.

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1 Introduction

We rephrase the SUR algorithm from [7] and the known convergence properties [3–5, 7] for multi-dimensional domains. We state an abstract recursive property for grid refinements and the order of the grid cells as they are approached by SUR in subsequent iterations, which is sufficient to prove the desired weak* convergence from the rephrased convergence properties. We note that the discretization into square cells induced by subsequent Hilbert curve iterates indeed has the desired properties.

2 The multi-dimensional SUR algorithm

We consider a finite partition of a bounded domain $\Omega \subset \mathbb{R}^d$ and compute a binary control $\omega \in L^\infty(\Omega, \mathbb{R}^M)$, i.e., $\sum_{i=1}^{M} \omega_i = 1$ and $\omega_i \in \{0, 1\}^M$ a.e., from a relaxed control $\alpha \in L^\infty(\Omega, \mathbb{R}^M)$, i.e., $\sum_{i=1}^{M} \alpha_i = 1$ and $0 \leq \alpha_i$ a.e., using the SUR algorithm, which follows a given order of the cells that partition $\Omega$.

Definition 2.1 Let $\{S_1, \ldots, S_N\} \subset B(\Omega)$ be a finite partition of $\Omega$. For indices $j \in \{1, \ldots, M\}$, we define recursively

$$\tilde{\omega}_{1,j} := \left\{ \begin{array}{ll} 1: & j = \arg \max_k \int_{S_i} \alpha_k \, d\lambda + \int_{\bigcup_{i=1}^{M} S_i} \alpha_k - \omega(\alpha)_k \, d\lambda \\ 0: & \text{otherwise} \end{array} \right. , \quad \omega(\alpha)|_{S_i} := \tilde{\omega}_i$$

If a tie arises with respect to the maximizing index $k$, the smallest applicable index is chosen.

Proposition 2.2 ([3, 5, 7]) Let $\alpha$ be a relaxed control and $\omega(\alpha)$ be computed by the SUR algorithm of Def. 2.1. Then, $\omega(\alpha)$ is a binary control and there exists $C > 0$ such that for $\phi := \alpha - \omega(\alpha)$ the following estimate holds:

$$\max_{i \in \{1, \ldots, N\}} \left\| \int_{\bigcup_{j=1}^{M} S_j} \phi \, d\lambda \right\|_\infty \leq C \cdot \max_{i \in \{1, \ldots, N\}} \lambda(S_i).$$

3 Weak* approximation for suitable grid refinements

We consider three Hilbert curve iterates on the unit square and its induced partitions in Fig. 1, a facsimile of the figure in [2]. The ordering of the squares along the curve is preserved from an iterate to the next, which allows to conserve a weighted mean of the quantity $\phi$ because the successive averaging always happens on sub-cells of cells we have already under control. We formalize this property below.

![Fig. 1: The first three Hilbert curve iterates on [0, 1]^2. The induced cell orderings are indicated by the small numbers inside the cells.](https://example.com/hilbert_iters.png)

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Definition 3.1 We call \( \left\{ S_1^{(n)}, \ldots, S_N^{(n)} \right\} \subset 2^\mathcal{B}(\Omega) \) an order conserving domain dissection of \( \Omega \) if

1. \( N^{(0)} = 1, S_1^{(0)} = \Omega \),
2. \( \left\{ S_1^{(n)}, \ldots, S_{N(n)}^{(n)} \right\} \) is a finite partition of \( \Omega \) for all \( n \)
3. for all \( n \) and all \( i \in \{1, \ldots, N^{(n-1)}\} \), there exist \( 1 \leq j < k \leq N^{(n)} \) such that \( \bigcup_{j=1}^{k} S_i^{(n)} = S_j^{(n)} \), i.e. the order of the grid cells is preserved from \( n - 1 \) to \( n \).
4. \( \max_{i \in \{1, \ldots, N^{(n)}\}} \lambda(S_i^{(n)}) \to 0 \), i.e. the maximum cell volume tends to zero,
5. the \( \sigma \)-algebra generated by \( \bigcup_{n=1}^{\infty} \left\{ S_1^{(n)}, \ldots, S_{N(n)}^{(n)} \right\} \) is \( \mathcal{B}(\Omega) \).

Assumption 3.2 Let \( \alpha \) be a relaxed control and an order conserving domain dissection \( \left\{ S_1^{(n)}, \ldots, S_{N(n)}^{(n)} \right\} \) of \( \Omega \) be given.

Let \( (\omega^{(n)})_n \) denote the binary controls, which are computed by the SUR algorithm on the partitions along the subscript orderings. We define \( \phi^{(n)} := \alpha - \omega^{(n)} \) for all \( n \).

Lemma 3.3 Let Ass. 3.2 hold and \( f \in L^1(\Omega) \). Then, \( \int_\Omega \phi^{(n)} f \to 0 \) for \( 1 \leq i \leq M \).

Proof. We abbreviate \( \phi := \phi_i \) and have to show \( \int_\Omega \phi^{(n)} f \to 0 \), where the codomains of \( f \) and \( \phi^{(n)} \) are subsets of \( \mathbb{R} \). As \( \phi^{(n)} \in L^\infty(\Omega) \), the products \( \phi^{(n)} f \) are integrable. Let \( \varepsilon > 0 \). Then, \( f \) can be approximated with a simple function \( f_1 \) such that \( \| f_1 - f \|_{L^1} < \varepsilon \). Let \( \varepsilon > 0 \). Then, \( f_1 \) can be approximated with a simple function \( f_2 \) that is defined on the generator from Ass. 3.2, i.e. there exist \( n_0 \in \mathbb{N} \) such that \( f_1 = \sum_{i=1}^{N^{(n_0)}} f_i \chi_{S_i^{(n_0)}} \) and \( \| f_2 - f_1 \|_{L^1} < \varepsilon \) with coefficients \( f_i \in \mathbb{R} \).

We study the product \( f^{(3)} \phi^{(n)} \). Let \( n \geq n_0 \). Then, property 3. of Ass. 3.2 gives

\[
\int_\Omega f^{(3)} \phi^{(n)} \, d\lambda = \sum_{i=1}^{N^{(n_0)}} f_i \int_{S_i^{(n_0)}} \phi^{(n)} \, d\lambda.
\]

Here, \( j(i, n) \) is the starting and \( k(i, n) \) the end index the sets \( S_i^{(n_0)} \) that make up \( S_i^{(n_0)} \). This implies \( \int_{S_i^{(n_0)}} \phi^{(n)} \, d\lambda = \int_{S_i^{(n_0)}} \phi^{(n)} \, d\lambda = \int_{S_i^{(n_0)}} \phi^{(n)} \, d\lambda - \int_{S_i^{(n_0)}} \phi^{(n)} \, d\lambda \), which in turn yields the estimate

\[
\left| \int_\Omega f^{(3)} \phi^{(n)} \, d\lambda \right| \leq 2C \max \lambda(S_i^{(n_0)}) \| f^{(1)} \|_{L^1} + \lambda(\Omega) \left( \| f^{(2)} \|_{L^1} + \int_\Omega f^{(3)} \phi^{(n)} \, d\lambda \right) < \varepsilon.
\]

for \( n \geq n_0 \) large enough.

We summarize the weak* convergence, which follows easily from the previous theorem, into the following theorem.

Theorem 3.4 Let Ass. 3.2 hold. Then, \( \phi^{(n)} \to 0 \) in \( L^p(\Omega, \mathbb{R}^M) \) for \( p \in [1, \infty) \) and \( \phi^{(n)} \to 0 \) for \( p \in (1, \infty] \).

Using e.g. the monograph by Sagan [6], one can easily verify the following proposition, which we skip for sake of brevity.

Proposition 3.5 The sequence of sets of squares induced by the Hilbert curve iterates satisfies Def. 3.1.

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