ON TEMPORAL REGULARITY FOR STRONG SOLUTIONS TO
STOCHASTIC p-LAPLACE SYSTEMS

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Abstract. In this article we investigate the temporal regularity of strong solutions to the stochastic p-Laplace system in the degenerate setting, \( p \in [2, \infty) \), driven by a multiplicative nonlinear stochastic forcing. We establish \( 1/2 \)-time differentiability in an exponential Besov-Orlicz space for the solution process \( u \). Furthermore, we prove \( 1/2 \)-time differentiability of the nonlinear gradient \( |\nabla u|^{p - 2} \nabla u \) in a Nikolskii space.

Keywords: SPDEs, Nonlinear Laplace-type systems, Strong solutions, Regularity, Stochastic p-heat equation
MSC: 35K55, 35K65, 35R60, 35D35, 35B65

60H15

1. Introduction

Let \( \mathcal{O} \subset \mathbb{R}^n \) be a bounded domain, \( n, N \in \mathbb{N} \) and \( T > 0 \) be finite. We are interested in the time and spatial regularity of the solution process \( u \) to the stochastic \( p \)-Laplace system. Given an initial datum \( u_0 \) and a stochastic forcing term \( (G, W) \) (for the precise assumptions see Assumption 1), \( u \) is determined by the relations

\[
\begin{align*}
    du - \text{div} S(\nabla u) \, dt &= G(u) \, dW \quad \text{in } (0,T) \times \mathcal{O}, \\
    u &= 0 \quad \text{on } (0,T) \times \partial \mathcal{O}, \\
    u(0) &= u_0 \quad \text{on } \mathcal{O},
\end{align*}
\]

(1.1)

where \( S(\xi) := (\kappa + |\xi|)^{p-2} \xi, \xi \in \mathbb{R}^{n \times N}, p \in [2, \infty) \) and \( \kappa \geq 0 \).

In applications, the time and space regularity of solutions to (1.1) are of great importance. When designing numerical algorithms, the regularity determines the rate of convergence of the scheme. In [BDSW21] we construct an algorithm for the deterministic \( p \)-Laplace system, that is able to approximate rough solutions. There one sees the delicate interplay between regularity of the solution and the rate of convergence of the algorithm.

The existence of analytically weak solutions to (1.1) in the space

\[
L^2(\Omega; C([0,T]; L^2(\mathcal{O}))) \cap L^p(\Omega; L^p((0,T); W^{1,p}_0(\mathcal{O})))
\]

can be established by standard monotonicity arguments [LR10]. In the deterministic setting it is natural to obtain regularity estimates for strong solutions in the spaces

\[
\begin{align*}
    V(\nabla u) &\in L^2(0,T; W^{1,2}(\mathcal{O})) \cap W^{1,2}(0,T; L^2(\mathcal{O})), \\
    u &\in L^\infty(0,T; W^{1,2}(\mathcal{O})) \cap C^{0,1}([0,T], L^2(\mathcal{O})),
\end{align*}
\]

(1.2a, 1.2b)

where \( V(\xi) := (\kappa + |\xi|)^{p - 1} \xi \). For this we test the system formally with \(-\Delta u\) and \( \partial_t^2 u \). It can be made rigorous by a substitution of differentials by difference.

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quotients, cf. [BM19]. In the stochastic case, it is still possible to prove local spatial regularity, i.e.

\begin{align}
(1.3a) \quad & \quad u \in L^2(\Omega; L^\infty(0,T; W^{1,2}_{\text{loc}}(\mathcal{O}))), \\
(1.3b) \quad & \quad V(\nabla u) \in L^2(\Omega \times (0,T); W^{1,2}_{\text{loc}}(\mathcal{O})),
\end{align}

using difference quotients, cf. [Bre15]. However, it is not clear whether the estimate holds up to the boundary under appropriate assumptions on the geometry of \( \mathcal{O} \).

Another method to construct strong solutions is presented in [Ges12]. There the author proves global spatial regularity using a Galerkin ansatz and a special projection operator associated to the energy of the system. He obtains uniform estimates in the space

\begin{align}
(1.4a) \quad & \quad u \in L^\infty(0,T; L^p(\Omega; W^{1,p}(\mathcal{O}))), \\
(1.4b) \quad & \quad \text{div } S(\nabla u) \in L^2((0,T) \times \Omega; L^2(\mathcal{O}))).
\end{align}

The substantial difference between the estimates (1.3) and (1.4) is, that the former corresponds to a formal testing of the equation with \(-\Delta u\) whereas the latter corresponds to testing with \(- \text{div } S(\nabla u)\). The second approach fits naturally to the gradient flow structure of the \(p\)-Laplace system (1.1).

For sufficiently regular or convex domains it is possible to extend the regularity estimate on \(\text{div } S(\nabla u)\) to \(\nabla S(\nabla u)\) as presented in [BCDM12], [CM19] and [CM20]. This allows to establish, at least in the non-degenerate setting \(\kappa > 0\), global gradient regularity for the nonlinear expression \(V(\nabla u)\), cf. Corollary 14.

In contrast to the spatial regularity, where deterministic tools can be applied, we need different techniques in order to investigate the time regularity of strong solutions. In the stochastic setting, the time regularity of the deterministic system (1.2) is unreachable, due to the irregularity of the cylindrical Wiener process \(W\). Even in the linear case, \(p = 2\), the time regularity is substantially lower compared to the deterministic system. The maximal regularity for the stochastic heat equation with an additive forcing has been established in [vNVW12] (see also [DVvN06]). They prove

\begin{equation}
(1.5) \quad \mathbb{E} \left[ \|u\|_{H^{\alpha,q}(0,T; L^2(\mathcal{O}))}^q \right] \lesssim \mathbb{E} \left[ \|G\|_{L^q(0,T; L^2(\mathcal{O})))}^q \right],
\end{equation}

where \(\alpha \in [0,1/2)\), \(q \in (2, \infty)\) and \(H\) denotes a Bessel potential space. The authors use the concept of mild solutions and estimate a stochastic convolution operator.

In the pure nonlinear setting, the approach based on mild solutions does not fit anymore. In [BH16] the authors conjectured, that it is possible to establish

\begin{align}
(1.6a) \quad & \quad u \in L^2(\Omega; C^\alpha(0,T; L^2(\mathcal{O}))), \\
(1.6b) \quad & \quad V(\nabla u) \in L^2(\Omega; W^{\alpha,2}(0,T; L^2(\mathcal{O}))),
\end{align}

\(\alpha \in [0,1/2)\). In this paper, we do not only verify (1.6), but in addition improve the result to a stronger scale of spaces (cf. Theorem 23 and Theorem 26)

\begin{align}
(1.7a) \quad & \quad u \in L^2(\Omega; B^{1,2}_{\alpha,\infty}(0,T; L^2(\mathcal{O}))), \\
(1.7b) \quad & \quad V(\nabla u) \in L^2(\Omega; B^{2,\infty}_{\alpha,\infty}(0,T; L^2(\mathcal{O}))),
\end{align}

where \(B\) denotes a Besov space (for more details see Section 2.1) and \(\Phi_2(t) = e^{t^2} - 1\). The time regularity (1.7a) is optimal, in the sense that it exactly matches the time regularity of the underlying Wiener process \(W\) as presented in [HV08].

A key ingredient is the stability of the stochastic integral generated by a cylindrical Wiener process \(W\) in type 2 Banach spaces as explained in [OV20]. The
gradient regularity (1.7b) heavily relies on the $V$-coercivity,
\begin{equation}
|V(\xi_1) - V(\xi_2)|^2 \approx (S(\xi_1) - S(\xi_2)) : (\xi_1 - \xi_2),
\end{equation}
for all $\xi_1, \xi_2 \in \mathbb{R}^{n \times N}$ and the boundary condition of the nonlinear operator $G$. Furthermore, we show the improved time regularity
\begin{equation}
V(\nabla u) \in L^2(\Omega; B_{q,\infty}^{1/2}(0,T; L^2(\mathcal{O})))
\end{equation}
for $q > 2$, if the diffusion operator has improved time integrability, i.e. $\text{div} S(\nabla u) \in L^2(\Omega; L^q(0,T; L^2(\mathcal{O})))$ (see Theorem 28).

In the past many authors have studied variants of (1.1) under different perspectives. The literature on the numerical analysis of the deterministic system is rich [BL93, Wei92, BL94, EL05, DER07, BDN18].

The corresponding numerical analysis of the stochastic system is not that well developed. First results of fully implementable approximations have been discussed in [GM05, GM09]. Just recently, an algorithm for the stochastic system has been proposed in [BHL21]. They infer convergence of the algorithm under the assumption (1.6) and the global spatial regularity assumption
\begin{align}
(1.10a) & \quad u \in L^2(\Omega; L^\infty(0,T; W^{1,2}(\mathcal{O}))), \\
(1.10b) & \quad V(\nabla u) \in L^2(\Omega; L^2(0,T; W^{1,2}(\mathcal{O}))).
\end{align}

Well-posedness of (1.1) with merely $L^1$-initial data has been addressed in [SZ21]. The more general system, where $p$ is allowed to depend on $(\omega, t, x)$ respectively on $(t, x)$, is considered in [VWZ16], [VZ16] respectively in [BVWZ13]. The singular case $p \in [1, 2]$ has been analyzed in [Liu09], [GT16] and [BRW21].

The paper is organized as follows. Section 2 introduces the mathematical setup and preliminary results on stochastic integrals and strong solutions. Time regularity for strong solutions is addressed in Section 3. It starts with stability results for stochastic integrals in Section 3.1. In Section 3.2 we prove Theorem 23 on Besov regularity of strong solution. Section 3.3 is about regularity of the nonlinear gradient $V(\nabla u)$ as presented in Theorem 26. Lastly, the improved estimates on the nonlinear gradient are discussed in Theorem 28 in Section 3.4.

2. Mathematical setup

In this section we introduce the setup for the system (1.1). Let $\mathcal{O} \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain (further assumptions on $\mathcal{O}$ will be needed for the spatial regularity of solutions). For some given $T > 0$ we denote by $I := [0,T]$ the time interval and write $\mathcal{O}_T := I \times \mathcal{O}$ for the time space cylinder. Moreover let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})$ denote a stochastic basis, i.e. a probability space with a complete and right continuous filtration $(\mathcal{F}_t)_{t \in I}$. We write $f \lesssim g$ for two non-negative quantities $f$ and $g$ if $f$ is bounded by $g$ up to a multiplicative constant. Accordingly we define $\gtrsim$ and $\asymp$. We denote by $c$ a generic constant which can change its value from line to line.

2.1. Function spaces. As usual, $L^q(\mathcal{O})$ denotes the Lebesgue space and $W^{1,q}(\mathcal{O})$ the Sobolev space, where $1 \leq q \leq \infty$. We denote by $W_0^{1,q}(\mathcal{O})$ the Sobolev spaces with zero boundary values. It is the closure of $C_0^\infty(\mathcal{O})$ (smooth functions with compact support) in the $W^{1,q}(\mathcal{O})$-norm. We denote by $W^{-1,q}(\mathcal{O})$ the dual of $W_0^{1,q}(\mathcal{O})$. We do not distinguish in the notation between vector- and matrix-valued functions.

For a Banach space $(X, \|\cdot\|_X)$ let $L^q(I; X)$ be the Bochner space of Bochner-measurable functions $u : I \to X$ satisfying $t \mapsto \|u(t)\|_X \in L^q(I)$. Moreover, $C(T; X)$ is the space of continuous functions with respect to the norm-topology. We
also use $C^0(\mathbb{T}; X)$ for the space of Hölder continuous functions. Given an Orlicz-function $\Phi : [0, \infty] \to [0, \infty]$, i.e. a convex function satisfying $\lim_{t \to 0} \Phi(t)/t = 0$ and $\lim_{t \to \infty} \Phi(t)/t = \infty$ we define the Luxemburg-norm
\[
\|u\|_{L^\Phi(I; X)} := \inf \left\{ \lambda > 0 : \int_I \Phi \left( \frac{\|u\|_X}{\lambda} \right) \,ds \leq 1 \right\}.
\]
The Orlicz space $L^\Phi(I; X)$ is the space of all Bochner-measurable functions with finite Luxemburg-norm. For more details on Orlicz-spaces we refer to [DHHR11]. Given $h \in I$ and $u : I \to X$ we define the difference operator $\tau_h : \{u : I \to X\} \to \{u : I \cap I - \{h\} \to X\}$ via $\tau_h(u)(s) := u(s + h) - u(s)$. The Besov-Orlicz space $B^\Phi_{p,r}(I; X)$ with differentiability $\alpha \in (0, 1)$, integrability $\Phi$ and fine index $r \in (1, \infty]$ is defined as the space of Bochner-measurable functions with finite Besov-Orlicz norm $\| \cdot \|_{B^\Phi_{p,r}(I; X)}$, where
\[
\|u\|_{B^\Phi_{p,r}(I; X)} := \|u\|_{L^\Phi(I; X)} + [u]_{B^\Phi_{p,r}(I; X)},
\]
\[
[u]_{B^\Phi_{p,r}(I; X)} := \left( \int_I h^{-r\alpha} \|\tau_h u\|_{L^\Phi(I \cap I - \{h\}; X)}^r \,dh \right)^{\frac{1}{r}}.
\]
In the case $r = \infty$ the integral in $h$ is replaced by an essential supremum and the space is commonly called Nikolskii-Orlicz space. When $\Phi(t) = t^p$ for some $p \in (1, \infty)$ we call the space $B^\Phi_{p,r}(I; X) = B^p_{r,r}(I; X)$ Besov space. Similarly, given a Banach space $(Y, \|\cdot\|_Y)$, we define $L^\Phi(\Omega; Y)$ as the Bochner space of Bochner-measurable functions $u : \Omega \to Y$ satisfying $\omega \mapsto \|u(\omega)\|_Y \in L^\Phi(\Omega)$. The space $L^\Phi(\Omega \times I; X)$ denotes the subspace of $X$-valued progressively measurable processes. Let $(U, \|\cdot\|_U)$ be a separable Hilbert space. $L^2(U; L^2(\mathcal{O}))$ denotes the space of Hilbert-Schmidt operators from $U$ to $L^2(\mathcal{O})$ with the norm $\|\cdot\|_{L^2(U; L^2)} := \sum_{j \in \mathbb{N}} \|z(u_j)\|_{L^2(\mathcal{O})}$ where $\{u_j\}_{j \in \mathbb{N}}$ is some orthonormal basis of $U$. We abbreviate the notation $L^2_L L^2_U L^2_N := L^2(\mathcal{O}; L^2(I; L^2(\mathcal{O})))$ and $L^r := \bigcap_{r' < q} L^{r'}$.

2.2. Stochastic integrals. In order to construct the stochastic forcing term, we impose the following conditions:

Assumption 1. (a) We assume that $W$ is an $U$-valued cylindrical Wiener process with respect to $(\mathcal{F}_t)_{t \in I}$. Formally $W$ can be represented as
\[
W = \sum_{j \in \mathbb{N}} u_j \beta^j,
\]
where $\{\beta^j\}_{j \in \mathbb{N}}$ are independent $1$-dimensional standard Brownian motions.
(b) Let $v \in L^2_L(\Omega \times I; L^2_U)$. We assume that $G(v)(\cdot) : U \to L^2_L(\Omega \times I; L^2_U)$ is given by
\[
u \mapsto G(v)(\nu) := \sum_{j \in \mathbb{N}} g_j(\cdot, \nu)(u_j, \nu) U,
\]
where $\{g_j\}_{j \in \mathbb{N}} \in C^1(\mathcal{O} \times \mathbb{R}^N; \mathbb{R}^N)$ with
(i) (sublinear growth) for all $x \in \mathcal{O}$ and $\xi \in \mathbb{R}^N$ it holds
\[
\sum_{j \in \mathbb{N}} |g_j(x, \xi)|^2 + |\nabla_x g_j(x, \xi)|^2 \leq c_{\text{growth}}(1 + |\xi|^2),
\]
(ii) (Lipschitz continuity) for all $x \in \mathcal{O}$ and $\xi \in \mathbb{R}^N$ it holds
\[
\sum_{j \in \mathbb{N}} |\nabla_x g_j(x, \xi)|^2 \leq c_{\text{lip}}.
\]
(iii) (boundary data) for all $x \in \partial \mathcal{O}$, $\xi \in \mathbb{R}^N$ and $j \in \mathbb{N}$ it holds $g_j(x, \xi) = 0$.

Commonly the stochastic integral is constructed under weaker assumptions on the coefficient $G$, e.g. $g_j \in C(\mathcal{O} \times \mathbb{R}^N; \mathbb{R}^N)$ with sublinear growth and a direct Lipschitz assumption. Then the operator

$$ G : L^2_x(\Omega \times I; L^2_x) \to L^2_x(\Omega \times I; L^2(U; L^2_x)) $$

is bounded and continuous. However we are interested in strong solutions and thus also need regularity of the gradient. For this we use the more involved stochastic integration theory in type 2 Banach spaces as done by Brzeźniak [Brz95] and Van Neerven and Weis in [vNW05]. Let $(E, \| \cdot \|_E)$ be a Banach space, $\{ \gamma_j \}_{j \in \mathbb{N}} \sim \mathcal{N}(0, 1)$ independent and identically distributed random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $F : U \to E$ a linear operator. We define the norm

$$ \| F \|_{\gamma(U; E)}^2 := \mathbb{E}_\gamma \left( \left\| \sum_{j \in \mathbb{N}} \gamma_j F(u_j) \right\|_E^2 \right) $$

(2.4)

and $\gamma(U; E) := \{ F : U \to E | F \text{ linear, } \| F \|_{\gamma(U; E)} < \infty \}$ as the space of $\gamma$-radonifying operators from $U$ to $E$. For a survey on $\gamma$-radonifying operators see [vN10]. In our application we have $E = W^{1,p}_{0,x}$.

**Lemma 2.** Let Assumption 1 be satisfied. Then

$$ G : L^p_x(\Omega \times I; W^{1,p}_{0,x}) \to L^p_x(\Omega \times I; \gamma(U; W^{1,p}_{0,x})) $$

is bounded.

**Proof.** Let $v \in W^{1,p}_{0,x}$ and $J \in \mathbb{N}$. Define the truncated operator

$$ u \mapsto G^J(v)(u) := \sum_{j=1}^J g_j(\cdot, v)(u, u_j)_U. $$
Due to the Kahane-Khintchine inequalities, Fubini’s Theorem and the assumptions (2.2) and (2.3),

$$
\|G^0(v)\|_{\gamma(U;W^{1,p}_x)} = \left( \mathbb{E}_\gamma \left[ \left( \sum_{j=1}^J \gamma_j g_j(\cdot, v) \right)^2 \right] \right)^{\frac{1}{2}}
$$

$$
\approx \left( \mathbb{E}_\gamma \left[ \left( \sum_{j=1}^J \gamma_j g_j(\cdot, v) \right)^p \right] \right)^{\frac{1}{p}}
$$

$$
= \left( \int \mathbb{E}_\gamma \left[ \left( \sum_{j=1}^J \gamma_j g_j(\cdot, v) \right)^p \right] dx \right)^{\frac{1}{p}}
$$

$$
\approx \left( \int \mathbb{E}_\gamma \left[ \left( \sum_{j=1}^J \gamma_j g_j(\cdot, v) \right)^2 \right] + \left( \mathbb{E}_\gamma \left[ \left( \sum_{j=1}^J \gamma_j (\nabla_x g_j(\cdot, v) + \nabla_x g_j(x, v) \nabla v)^2 \right) \right] \right) \frac{1}{p} \right)^{\frac{1}{p}}
$$

$$
\leq \|v\|_{W^{1,p}_x} + 1,
$$

where the constant is independent of $J$. Due to lower semicontinuity of the norm, we can pass to the limit

$$
\|G(v)\|_{\gamma(U;W^{1,p}_x)} \leq \|v\|_{W^{1,p}_x} + 1.
$$

For general $v \in L^p_x(\Omega \times I; W^{1,p}_x)$ we apply the above result pointwise, i.e.

$$
\|G(v)\|_{L^p_x(\Omega \times I; \gamma(U;W^{1,p}_x))} = \mathbb{E} \left[ \int_0^T \|G(v)\|_{\gamma(U;W^{1,p}_x)}^p \right] + 1.
$$

The zero boundary data for $G(v)$ follows by (iii) in Assumption 1. □

In the construction of the stochastic integral, the geometry of the underlying Banach space plays an important role.

**Definition 3.** Let $(E, \|\cdot\|_E)$ be a Banach space and $\{\gamma_j\}_{j \in \mathbb{N}} \sim \mathcal{N}(0, 1)$ independent and identically distributed.

(a) $E$ is of type $q \in [1, 2]$ if there exists a constant $C > 0$ such that for all finite sequences $\{e_j\}_{j \in \mathbb{N}} \subseteq E$

$$
\left( \mathbb{E}_\gamma \left[ \left( \sum_{j \in \mathbb{N}} \gamma_j e_j \right)^2 \right] \right)^{\frac{1}{2}} \leq C \left( \sum_{j \in \mathbb{N}} \|e_j\|^q_E \right)^{\frac{1}{2}}.
$$

(b) $E$ is of cotype $q \in [2, \infty]$ if there exists a constant $C > 0$ such that for all finite sequences $\{e_j\}_{j \in \mathbb{N}} \subseteq E$

$$
\left( \sum_{j \in \mathbb{N}} \|e_j\|^q_E \right)^{\frac{1}{2}} \leq C \left( \mathbb{E}_\gamma \left[ \left( \sum_{j \in \mathbb{N}} \gamma_j e_j \right)^2 \right] \right)^{\frac{1}{2}}.
$$
Hilbert spaces are of type $2$ and of cotype $2$. Even the converse is true, if a Banach space is type $2$ and cotype $2$, then it is isomorphic to a Hilbert space, cf. [Kwa72]. On the classical Lebesgue scale the following result is valid.

**Proposition 4** ([Pis16] Proposition 10.36). If $q \in [1, 2]$, every $L^q$-space is of type $q$ and of cotype $2$. If $q \in [2, \infty)$, every $L^q$-space is of type $2$ and of cotype $q$.

Now we can construct the stochastic integral.

**Proposition 5.** Let Assumption 1 be true. Then the operator $\mathcal{I}$ defined through

$$(2.7) \quad \mathcal{I}(G(v)) := \int_0^1 G(v)(dW_t) := \sum_{j \in \mathbb{N}} \int_0^1 g_j(\cdot, v) d\beta_j$$

defines a bounded linear operator from $L^2_{\mathbb{F}}(\Omega \times \mathbb{R})$ to $L^2(C_0L^2_2)$. Moreover,

- $\mathcal{I}(G(v))$ is an $L^2_{\mathbb{F}}$-valued martingale with respect to $(\mathcal{F}_t)_{t \in T}$.
- (Itô isometry) for all $t \in T$ it holds

$$\mathbb{E} \left[ \|\mathcal{I}(G(v))(t)\|_{L^2_{\mathbb{F}}}^2 \right] = \mathbb{E} \left[ \int_0^t \|G(v)\|_{L^2(U; L^2_1)}^2 \, ds \right].$$

Let $p \geq 2$. Then we have additionally the gradient estimate

$$\left( \mathbb{E} \left[ \sup_{t \in T} \|\mathcal{I}(G(v))(t)\|_{W^{1,p}}^p \right] \right)^{\frac{1}{p}} \lesssim \sqrt{p} \left( \|v\|_{L^p_{\mathbb{F}}L^2_{\mathbb{F}}W^{1,p}} + 1 \right).$$

**Proof.** The first part of Proposition 5 is standard and we skip its proof.

In order to prove (2.8) we employ the stability of the stochastic integral in type 2 Banach spaces as presented in [OV20]. Let $v \in L^p(\Omega \times I; W^{1,p}_0)$. Lemma 2 ensures that $G(v)$ is stochastically integrable on $W^{1,p}_0$. Thus,

$$\left( \mathbb{E} \left[ \sup_{t \in T} \|\mathcal{I}(G(v))(t)\|_{W^{1,p}}^p \right] \right)^{\frac{1}{p}} \lesssim \sqrt{p} \|G(v)\|_{L^p_{\mathbb{F}}L^2_{\mathbb{F}}W^{1,p}} \lesssim \sqrt{p} \left( \|v\|_{L^p_{\mathbb{F}}L^2_{\mathbb{F}}W^{1,p}} + 1 \right).$$

**Remark 6.** In the regime $p \in (1, 2)$ the inequality (2.8) fails. This is directly linked to the fact that $L^r$, $r \in (1, 2)$ is not of type $2$. However, if one uses the stochastic integration theory in UMD (unconditional martingale differences) Banach spaces as done in [vNVW07], it is still possible to estimate

$$\mathbb{E} \left[ \sup_{t \in T} \|\mathcal{I}(G(v))(t)\|_{W^{1,p}}^p \right] \lesssim \mathbb{E} \left[ \int_0^T \left( \int_0^s 1 + |v|^2 + |\nabla v|^2 \, ds \right)^{\frac{p}{2}} \, dx \right],$$

i.e. $\mathcal{I} : L^p_{\mathbb{F}}W^{1,p}_0L^1_t \to L^p_{\mathbb{F}}C_0W^{1,p}_t$ is bounded.

### 2.3. Perturbed gradient flow

Let $\kappa \geq 0$ and $p \in [2, \infty)$. For $\xi \in \mathbb{R}^{n \times N}$ we define

$$(2.9) \quad S(\xi) := \varphi(|\xi|) \cdot \frac{\xi}{|\xi|} = (\kappa + |\xi|)^{p-2} \xi$$

and

$$(2.10) \quad V(\xi) := \sqrt{\varphi(|\xi|)} \cdot \frac{\xi}{|\xi|} = (\kappa + |\xi|)^{\frac{p-2}{2}} \xi,$$
where $\varphi(t) := \int_0^t (\kappa + s)^{p-2} \, ds$. The nonlinear functions $S$ and $V$ are closely related. In particular the following Lemmata are of great importance. The proofs can be found in [DE08]. For more details we refer to [DR07, BDK12, DFTW20].

**Lemma 7** (V-coercivity). Let $\xi_1, \xi_2 \in \mathbb{R}^{n \times N}$. Then it holds

\begin{equation}
(S(\xi_1) - S(\xi_2)) : (\xi_1 - \xi_2) \approx |V(\xi_1) - V(\xi_2)|^2 \\
\approx (\kappa + |\xi_1| + |\xi_1 - \xi_2|)^{p-2} |\xi_1 - \xi_2|^2. 
\end{equation}

**Lemma 8** (generalized Young’s inequality). Let $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^{n \times N}$ and $\delta > 0$. Then there exists $c_\delta \geq 1$ such that

\begin{equation}
(S(\xi_1) - S(\xi_2)) : (\xi_2 - \xi_3) \leq \delta |V(\xi_1) - V(\xi_2)|^2 + c_\delta |V(\xi_2) - V(\xi_3)|^2. 
\end{equation}

**Lemma 9.** Let $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^{n \times N}$ and $\delta > 0$. Then there exists $c_\delta \geq 1$ such that

\begin{equation}
(S(\xi_1) - S(\xi_2)) : (\xi_3) \leq \delta |V(\xi_1) - V(\xi_2)|^2 + c_\delta (\kappa + |\xi_1| + |\xi_1 - \xi_2|)^{p-2} |\xi_3|^2. 
\end{equation}

**Remark 10.** Lemma 7 and 8 are still valid if one replaces $\varphi$ in (2.9) and (2.10) by any uniformly convex $N$-function.

Given some $\mathcal{F}_t$-measurable initial condition $u_0 : \Omega \times \mathcal{O} \rightarrow \mathbb{R}^N$ and a stochastic force $(G, W)$ in the sense of Assumption 1, we are interested in the system

\begin{alignat}{2}
\frac{du}{dt} - \text{div} S(\nabla u) & = G(u) \; dW & \quad & \text{in } \Omega \times \mathcal{O}_T, \\
\frac{dI}{dt} & = 0 & \quad & \text{on } \Omega \times I \times \partial \mathcal{O}, \\
I(0) & = u_0 & \quad & \text{on } \Omega \times \mathcal{O}.
\end{alignat}

The system (2.14a) is a perturbed version of the gradient flow of the energy $\mathcal{J} : W^{1,p}_{0,2} \rightarrow [0, \infty)$ given by

\begin{equation}
\mathcal{J}(u) := \int_\Omega \varphi(|\nabla u|) \, dx.
\end{equation}

2.4. **Weak and strong solutions.** We fix the concept of solutions as follows.

**Definition 11.** Let $u_0 \in L^2_0L^2_T$ be $\mathcal{F}_0$-measurable, $p \geq 2$ and $(G, W)$ be given by Assumption 1. An $(\mathcal{F}_t)$-adapted process $u \in L^2_T$ is called weak solution to (2.14) if

\begin{enumerate}
\item[(a)] $u \in L^2_0C^1_T \cap L^p_\mathcal{O}L^p_\mathcal{O}_T W^{1,p}_{0,2}$,
\item[(b)] for all $t \in I$, $\xi \in C^\infty_{0,\mathcal{O}}$ and $\mathbb{P}$ a.s. it holds
\end{enumerate}

\begin{equation}
\int_0^t (u(t) - u_0) \cdot \xi \, dx + \int_0^t \int_\mathcal{O} S(\nabla u) : \nabla \xi \, dx \, ds = \int_0^t G(u) \, dW_s : \xi \, dx.
\end{equation}

The process $u$ is called strong solution if it is a weak solution and additionally satisfies

\begin{enumerate}
\item[(a)] $\text{div} S(\nabla u) \in L^2_0L^2_T L^2_\mathcal{O}$,
\item[(b)] for all $t \in I$ and $\mathbb{P}$ a.s. it holds
\end{enumerate}

\begin{equation}
\int_0^t (u(t) - u_0) - \int_0^t \text{div} S(\nabla u) \, ds = \int_0^t G(u) \, dW_s
\end{equation}

as an equation in $L^2_\mathcal{O}$.

The existence of strong solutions to gradient flow like equations has been established by Gess in [Ges12]. In particular, it includes the case of the $p$-Laplace system for $p \geq 2$. 
Theorem 12 ([Ges12] Theorem 4.12). Assume $p \geq 2$ and $\Omega$ to be a bounded convex domain. Let $(G,W)$ be given by Assumption 1 and $u_0 \in L^2_{\omega}W^{1,p}_{0,x}$ be $\mathcal{F}_0$-measurable. Then there exists a unique strong solution $u$ to (2.14). Moreover,
\begin{equation}
(2.18) \quad \mathbb{E} \left[ \sup_{t \in I} \|u\|_{W^{1,p}_x}^p + \int_0^T \|\text{div } S(\nabla u)\|_{L^2_x}^2 \, dt \right] \lesssim \mathbb{E} \left[ \|u_0\|_{W^{1,p}_x}^p \right] + 1.
\end{equation}

Proof. We will only sketch the idea of the proof. Similar to the construction of weak solutions one first performs a Galerkin ansatz $V_j \subset W^{1,p}_{0,x}$. Solving the corresponding SDE provides an approximation $u_J$. Apart from the classical a priori estimates $u_J \in L^2_0L^\infty T \cap L^p_0L^\infty T \cap W^{1,p}_0$, one can also get a priori estimates on the gradient level $u_J \in L^p_0L^\infty T \cap W^{1,p}_0$ and $\text{div } S(\nabla u_J) \in L^2_0L^2 T$ by an application of Itô’s formula to $u_J \mapsto J(u_J)$. Finally, one passes to the limit $J \to \infty$ and identifies the nonlinear terms $S$ and $G$ using the monotonicity of $S$ and the Lipschitz assumption on $G$. Uniqueness already holds for weak solutions.

In fact the result can be strengthened to not only give a bound on the divergence of $S(\nabla u)$ but on the full gradient. An optimal result in the vector valued setting has been obtained in [BCDM21] Theorem 2.6. In contrast to the scalar case there exists a threshold $p_c = 2(2 - \sqrt{2})$ such that the regularity transfer to the full gradient is only valid for $p > p_c$. We also want to mention an earlier result of Cianchi and Maz’ya [CM19] where they obtained a similar but suboptimal result.

The geometry of the domain $\Omega$ plays an important role. Let $\Omega$ be a bounded Lipschitz domain such that $\partial \Omega \in W^{2,1}$, i.e. $\Omega$ is locally the subgraph of a Lipschitz continuous function of $n - 1$ variables, which is also twice weakly differentiable. Denote by $B$ the second fundamental form on $\partial \Omega$, by $|B|$ its norm and define
\begin{equation}
(2.19) \quad K_\Omega(r) := \sup_{E \subset \partial \Omega} \frac{\int_E |B| \, d\mathcal{H}^{n-1}}{\text{cap}_{B_1(x)}(E)},
\end{equation}
where $B_1(x)$ denotes the ball of radius $r$ around $x$, $\text{cap}_{B_1(x)}(E)$ is the capacity of the set $E$ relative to the ball $B_1(x)$ and $\mathcal{H}^{n-1}$ is the $n - 1$ dimensional Hausdorff measure.

Lemma 13. Let $p > p_c$. Assume that $\Omega$ is either

(a) bounded and convex,

(b) or bounded, Lipschitz and $\partial \Omega \in W^{2,1}$ with $\lim_{r \to 0} K_\Omega(r) \leq c$.

Let $v \in L^p_0L^p_0W^{1,p}_{0,x}$ with $\text{div } S(\nabla v) \in L^2_0L^2_0$. Then $\nabla S(\nabla v) \in L^2_0L^2_0$ and
\begin{equation}
\mathbb{E} \left[ \|\nabla S(\nabla v)\|_{L^2_0L^2_0}^2 \right] \lesssim \mathbb{E} \left[ \|\text{div } S(\nabla v)\|_{L^2_0L^2_0}^2 \right].
\end{equation}

Proof. First observe that $v$ trivially solves for almost all $(\omega, t)$
\begin{equation}
\text{div } S(\nabla v) = \text{div } S(\nabla v) \quad \text{in } \Omega,
\end{equation}
\begin{equation}
v = 0 \quad \text{on } \partial \Omega.
\end{equation}
The result now follows by Theorem 2.3 respectively Theorem 2.4 in [CM19].

This allows to establish global gradient regularity for $V(\nabla u)$ at least in the non-degenerate setting $\kappa > 0$.

Corollary 14. Let the assumptions of Theorem 12 be satisfied. Denote by $u$ the unique strong solution to (2.14). Then
\begin{equation}
(2.20) \quad \mathbb{E} \left[ \|\nabla V(\nabla u)\|_{L^2_0L^2_0}^2 \right] \lesssim \kappa^{2-p} \left( \mathbb{E} \left[ \|u_0\|_{W^{1,p}_x}^p \right] + 1 \right).
\end{equation}
Hölder’s inequality now implies
\[ C \geq \text{constant} \]

Estimates the geometry of

In particular,
\[ |\nabla V(\nabla u)|^2 \approx (\kappa + |\nabla u|)^{p-2} |\nabla^2 u|^2 \approx \nabla S(\nabla u) : \nabla^2 u, \]
\[ |\nabla S(\nabla u)|^2 \approx (\kappa + |\nabla u|)^{2(p-2)} |\nabla^2 u|^2. \]

Hölder’s inequality now implies
\[ \mathbb{E} \left[ |\nabla V(\nabla u)|^2 \right] \lesssim \left( \mathbb{E} \left[ |\nabla S(\nabla u)|^2 \right] \mathbb{E} \left[ |\nabla^2 u|^2 \right] \right)^{\frac{1}{2}} \]
Due to the non-degeneracy we have
\[ \mathbb{E} \left[ |\nabla^2 u|^2 \right] = \mathbb{E} \left[ \int_0^T \int_{\Omega} \frac{(\kappa + |\nabla u|)^{2(p-2)}}{(\kappa + |\nabla u|)^{2(p-2)}} |\nabla^2 u|^2 \, dx \, dt \right] \lesssim \kappa^{2(p-2)} \mathbb{E} \left[ |\nabla S(\nabla u)|^2 \right]. \]

Since we assume that \( p \geq 2 > p_c \) and \( \mathcal{O} \) is bounded and convex, we can apply Lemma 13. The estimate (2.20) now follows by the a priori estimate of Theorem 12.

\[ \square \]

Remark 15. Naturally one obtains regularity for \( \nabla V(\nabla u) \) when testing (2.14) by the Laplacian, as presented by Breit in [Bre15], rather then the \( p \)-Laplacian. In this way one can establish uniform bounds in at least locally in space. For global estimates the geometry of \( \mathcal{O} \) is the crucial ingredient.

3. Time regularity for strong solutions

The next paragraph is devoted to results on the time regularity of strong solutions to (2.14). Especially important are the mapping properties of the stochastic integral operator in type 2 Banach spaces. The regularity of the stochastic integral also improves when the regularity of the integrand improves. This is a key ingredient when deriving time regularity of solutions to SPDEs as presented by Ondreját and Veraar in [OV20].

3.1. Stability of stochastic integrals. First we need a regularity statement for conditional expectations of stochastic integrals.

Lemma 16 ([OV20] Lemma 3.1). Let \((E, \|\cdot\|_E)\) be a separable type 2 Banach space. Let \( T > 0, q \in [1, \infty) \) and \( r \in (2, \infty] \). Let \( G \in L^{q, \gamma}_{L^r} (U; E) \). Then there exists a constant \( C > 0 \) such that for all \( 0 \leq s \leq t \leq T \) and \( \mathbb{P} \)-a.s.

\[ (1.3) \quad \mathbb{E} \left[ \|\mathcal{I}(G)(t) - \mathcal{I}(G)(s)\|_E^q \left| \mathcal{F}_s \right\|^\frac{q}{2} \right] \leq C \sqrt{q} \|G\|_{L^{q, \gamma}_{L^r}} (t-s)^{\frac{1}{2} - \frac{1}{q}}. \]

Lemma 17. Let \((E, \|\cdot\|_E)\) be a separable Banach space of type 2. Then there exists a constant \( C > 0 \) such that for all \( q \in [1, \infty) \), \( r \in (2, \infty] \), \( \theta \in (1, 2] \) and \( G \in L^{q, \theta}_{\mathbb{F}^0} L^r (U; E) \) we have

\[ (3.2) \quad \mathbb{E} \left[ \|\mathcal{I}(G)\|_{L^{q, \theta}_{\mathbb{F}^0} (0, T; E)}^q \right] \leq C (\theta - 1)^{\frac{q}{p^*}} \sqrt{q} \|G\|_{L^{q, \theta}_{\mathbb{F}^0} L^r (U; E)}, \]

where \( \alpha = \frac{1}{2} - \frac{1}{p} \).

The subsequent proof is motivated by the proof of Theorem 3.2 in [OV20].
ON TEMPORAL REGULARITY FOR STRONG SOLUTIONS TO STOCHASTIC p-LAPLACE SYSTEMS

Proof. Without loss of generality we assume $T = 1$, the general case follows by a scaling argument. Define

$$Y_{n,q} := 2^{na} \|I(G)(t + 2^{-n}) - I(G)(t)\|_{L^q(I_t; \mathbb{L}_a)}.$$ 

Note, we may rewrite

$$Y_{n,q}^q = \int_0^{1-2^{-n}} 2^{naq} \|I(G)(t + 2^{-n}) - I(G)(t)\|^q_E dt$$

$$= \sum_{m=1}^{2^n} \int_{(m-1)2^{-n}}^{m2^{-n}} 2^{naq} \|I(G)(t + 2^{-n}) - I(G)(t)\|^q_E dt$$

$$= \sum_{m=1}^{2^n} \int_0^{2^{-n}} 2^{naq} \|I(G)((s + m)2^{-n}) - I(G)((s + m - 1)2^{-n})\|^q_E ds$$

$$=: \int_0^{2^{-n}} \sum_{m=1}^{2^n-1} \eta_{n,m,s} ds.$$

Furthermore define

$$Z_{n,q}^q := \int_0^{1-2^{-n}} \frac{1}{2^{naq}} \sum_{m=1}^{2^n-1} \zeta_{n,m,s} ds$$

with $\zeta_{n,m,s} := \mathbb{E}[\eta_{n,m,s} | \mathcal{F}_{(s+m-1)2^{-n}}]$. For a fixed $n \in \mathbb{N}$ and $s \in [0,1]$ we define for $M \in \{1, \ldots, 2^n - 1\}$

$$\mathcal{E}_{n,M,s} := \sum_{m=1}^{M} \eta_{n,m,s} - \zeta_{n,m,s}.$$ 

Observe $\mathcal{E}_{n,,s}$ is a discrete martingale with respect to the filtration $\mathcal{F}_{(s+\cdot)2^{-n}}$. Indeed, for $k \in \{1, \ldots, 2^n - 1\}$, we have

$$\mathbb{E} \left[ \mathcal{E}_{n,M,s} | \mathcal{F}_{(s+k)2^{-n}} \right]$$

$$= \mathbb{E} \left[ \sum_{m=1}^{k} \eta_{n,m,s} - \zeta_{n,m,s} | \mathcal{F}_{(s+k)2^{-n}} \right] + \mathbb{E} \left[ \sum_{m=k+1}^{M} \eta_{n,m,s} - \zeta_{n,m,s} | \mathcal{F}_{(s+k)2^{-n}} \right]$$

$$=: I + II.$$ 

Recall $I(G)$ is adapted to $(\mathcal{F}_t)$. Therefore $\eta_{n,m,s}$ is $\mathcal{F}_{(s+k)2^{-n}}$-measurable for all $m \leq k$. Now, due to measurability, we conclude

$$I = \mathcal{E}_{n,k,s}.$$ 

The tower property of conditional expectation implies, for all $m \geq k + 1$,

$$\mathbb{E} \left[ \zeta_{n,m,s} | \mathcal{F}_{(s+k)2^{-n}} \right] = \mathbb{E} \left[ \mathbb{E} \left[ \eta_{n,m,s} | \mathcal{F}_{(s+m-1)2^{-n}} \right] | \mathcal{F}_{(s+k)2^{-n}} \right]$$

$$= \mathbb{E} \left[ \eta_{n,m,s} | \mathcal{F}_{(s+k)2^{-n}} \right].$$ 

Thus $II = 0$. 
Fix \( \theta \in (1, 2] \). We obtain, due to Jensen’s inequality and Fubini’s Theorem,
\[
E \left[ \left( Y_{n,q}^q - \bar{Z}_{n,q}^q \right)^\theta \right] = E \left[ \int_0^1 2^{-\eta_n \sum_{m=1}^{2^n-1} \eta_{n,m,s} - \zeta_{n,m,s}} \, ds \right]^{\theta}
\]
\[
\leq \int_0^1 2^{-\theta \eta_n} \left[ \sum_{m=1}^{2^n-1} \eta_{n,m,s} - \zeta_{n,m,s} \right] \, ds
\]
\[
= \int_0^1 2^{-\theta \eta_n} \left[ \mathcal{E}_{n,2^n-1,s}^\theta \right] \, ds
\]
\[
\leq \int_0^1 2^{-\theta \eta_n} \max_{M \in \{1, \ldots, 2^n - 1\}} \left[ \mathcal{E}_{n,M,s}^\theta \right] \, ds.
\]
Applying the classical BDG-inequality and \( \ell^\theta \mapsto \ell^2 \) we get
\[
E \left[ \max_{M \in \{1, \ldots, 2^n - 1\}} \left[ \mathcal{E}_{n,M,s}^\theta \right] \right] \leq E \left[ \sum_{m=1}^{2^n-1} \left( \eta_{n,m,s} - \zeta_{n,m,s} \right)^2 \right]^{\frac{\theta}{2}}
\]
\[
\leq \sum_{m=1}^{2^n-1} \left| \eta_{n,m,s} - \zeta_{n,m,s} \right|^\theta.
\]
Jensen’s inequality implies
\[
E \left[ \sum_{m=1}^{2^n-1} \left| \eta_{n,m,s} - \zeta_{n,m,s} \right|^\theta \right] \leq 2^\theta \sum_{m=1}^{2^n-1} E \left[ \left| \eta_{n,m,s} \right|^\theta \right].
\]
It remains to invoke the BDG-inequality in type 2 Banach spaces and Hölder’s inequality to conclude
\[
E \left[ \left| \eta_{n,m,s} \right|^\theta \right] \leq (\theta q)^{\frac{\theta}{2}} E \left[ 2^{n \alpha q^\theta} \left( \int_{(s+m)2^{-n}}^{(s+m+1)2^{-n}} \left\| G \right\|_{r(U,E)}^r \, ds \right)^{\frac{q}{r}} \right]
\]
\[
\leq (\theta q)^{\frac{\theta}{2}} E \left[ 2^{n \alpha q^\theta} \left( \int_{(s+m)2^{-n}}^{(s+m+1)2^{-n}} \left\| G \right\|_{r(U,E)}^r \, ds \right)^{\frac{q}{r}} \right] (2^{-n})^{\frac{\alpha \theta (\alpha - 1/2 - 1/r)}{2r}}
\]
\[
= (\theta q)^{\frac{\theta}{2}} 2^{n q^\theta (\alpha - (1/2 - 1/r))} E \left[ \left( \int_{(s+m)2^{-n}}^{(s+m+1)2^{-n}} \left\| G \right\|_{r(U,E)}^r \, ds \right)^{\frac{q}{r}} \right]
\]
Recall that \( \alpha = 1/2 - 1/r \). Collecting the previous estimates, we obtain
\[
\sum_{m=1}^{2^n-1} E \left[ \left| \eta_{n,m,s} \right|^\theta \right] \leq (\theta q)^{\frac{\theta}{2}} 2^{n} E \left[ \left( \int_0^1 \left\| G \right\|_{r(U,E)}^r \, ds \right)^{\frac{q}{r}} \right].
\]
Thus, the semi-norm estimate now follows, since

\[
\sup_{n \in \mathbb{N}} \left| Y_{n,q}^q - Z_{n,q}^q \right|^\theta \leq \mathbb{E} \left[ \sum_{n \in \mathbb{N}} \left| Y_{n,q}^q - Z_{n,q}^q \right|^\theta \right] \\
\leq 2^\theta (\theta) \mathbb{E} \left[ \left( \frac{1}{0} \left\| G \right\|_{\gamma(U;E)}^r \, ds \right) \right] \sum_{n \in \mathbb{N}} 2^{-n(\theta-1)} \\
\leq 2^\theta (\theta) \mathbb{E} \left[ \left( \frac{1}{0} \left\| G \right\|_{\gamma(U;E)}^r \, ds \right) \right] \\
\frac{1}{1 - 2^{-(\theta-1)}} 
\]

Note, for \( \theta \geq 1 \),

\[
1 - 2^{-(\theta-1)} \geq 2^{-(\theta-1)} \ln(2)(\theta - 1). 
\]

Thus,

\[
(3.3) \quad \mathbb{E} \left[ \sup_{n \in \mathbb{N}} \left| Y_{n,q}^q - Z_{n,q}^q \right|^\theta \right] \leq 2^{2\theta-1}(\theta) \mathbb{E} \left[ \left( \frac{1}{0} \left\| G \right\|_{\gamma(U;E)}^r \, ds \right) \right] \frac{1}{1 - 2^{-(\theta-1)}}. 
\]

Lastly we estimate \( Z_{n,q}^q \). Due to Lemma 16 it holds \( \mathbb{P} \)-a.s.

\[
Z_{n,q}^q \leq \int_0^1 \max_{1 \leq m \leq 2^n - 1} \zeta_{n,m,s} \, ds \lesssim q^\frac{\theta}{r} 2^{nq(\alpha - (1/2 - 1/r))} \left\| G \right\|_{L^r U \gamma(U;E)}^q.
\]

Take the \( \theta \)-th power, supremum in \( n \) and expectation

\[
\mathbb{E} \left[ \sup_{n \in \mathbb{N}} \left| Z_{n,q}^q \right|^\theta \right] \lesssim q^\frac{\theta}{r} \left\| G \right\|_{L^r U \gamma(U;E)}^q \leq q^\frac{\theta}{r} \left\| G \right\|_{L^r U \gamma(U;E)}^q.
\]

The semi-norm estimate now follows, since

\[
\mathbb{E} \left[ \mathcal{I}(G)_{B_{q,\infty}(0,1;E)}^{q\theta} \right] = \mathbb{E} \left[ \sup_{n \in \mathbb{N}} \left| Y_{n,q}^q \right|^\theta \right] \\
\leq 2^\theta \left( \mathbb{E} \left[ \sup_{n \in \mathbb{N}} \left| Y_{n,q}^q - Z_{n,q}^q \right|^\theta \right] + \mathbb{E} \left[ \sup_{n \in \mathbb{N}} \left| Z_{n,q}^q \right|^\theta \right] \right) \\
\leq 2^\theta \left( \frac{2^{2\theta-1} \theta \mathbb{E} \left[ \left( \frac{1}{0} \left\| G \right\|_{\gamma(U;E)}^r \, ds \right) \right]}{\ln(2)(\theta - 1) + 1} \right) q^\frac{\theta}{r} \left\| G \right\|_{L^r U \gamma(U;E)}^q.
\]

Applying the BDG-inequality in type 2 Banach spaces and Hölder’s inequality

\[
\mathbb{E} \left[ \left\| \mathcal{I}(G) \right\|_{L^q(0,1;E)}^{q\theta} \right] \lesssim (q\theta)^\frac{\theta}{r} \mathbb{E} \left[ \left( \frac{1}{0} \left\| G \right\|_{\gamma(U;E)}^r \, ds \right) \right] \frac{1}{\theta} \\
\leq (q\theta)^\frac{\theta}{r} \left\| G \right\|_{L^r U \gamma(U;E)}^q.
\]
Overall we have proved
\[
\mathbb{E} \left[ \left\| I(G) \right\|_{B_q^\infty(0,1,E)}^{q\theta} \right] \leq 2^\theta \left( \mathbb{E} \left[ \left\| I(G) \right\|_{B_q^\infty(0,1,E)}^{q\theta} \right] + \mathbb{E} \left[ \left\| I(G) \right\|_{B_q^\infty(0,1,E)}^{q\theta} \right] \right) \\
\leq 2^\theta \left( \frac{2^\theta - 1}{\ln(2)} \right) + \frac{\theta}{\ln(2) + 1} + 1 \right) q^\frac{\theta}{2} \left\| G \right\|_{L_q^\theta L_\theta^\gamma(U,E)} \\
\leq \frac{1}{\theta - 1} 2^\theta \left( \frac{2^\theta - 1}{\ln(2)} \right) + \frac{\theta}{\ln(2) + 1} + 1 \right) q^\frac{\theta}{2} \left\| G \right\|_{L_q^\theta L_\theta^\gamma(U,E)} .
\]

The assertion follows by taking the \(q\theta\)-th root.

\[\square\]

**Remark 18.** The statements for the time regularity in the critical regime \(\alpha = 1/2 - 1/r\) are delicate. For \(\alpha < 1/2 - 1/r\) one easily proves
\[
\left( \mathbb{E} \left[ \left\| I(G) \right\|_{B_q^\infty(0,T;E)}^{q\theta} \right] \right)^{\frac{1}{q\theta}} \leq \left\| G \right\|_{L_q^\theta L_\theta^\gamma(U,E)} ,
\]
using \(\ell^1 \rightarrow \ell^\infty\) and interchanging integration and summation. In [OV20] Theorem 3.2(ii) the authors prove a slightly weaker estimate
\[
\left( \mathbb{E} \left[ \left\| I(G) \right\|_{B_q^\infty(0,T;E)}^{q\theta} \right] \right)^{\frac{1}{q\theta}} \leq C T \sqrt{q} \left\| G \right\|_{L_q^\theta L_\theta^\gamma(U,E)} .
\]

The result can be recovered by our Lemma 17 by setting \(\theta = 2\) in the proof. We do not know whether the limit case \(\theta = 1\) can be proved by a direct estimate on
\[
\mathbb{E} \left[ \sup_{n \in \mathbb{N}} \left| Y_{n,p}^p - Z_{n,p}^p \right| \right] .
\]

**Corollary 19.** Let \(p \geq 2\) and Assumption 1 be satisfied. Then for all \(q \in [1, \infty)\) and \(\theta \in (1/2, 1]\)
\[
(3.4) \quad \left( \mathbb{E} \left[ \left\| I(G(v)) \right\|_{B_q^\infty(0,T;W_1^{1,p})}^{q\theta} \right] \right)^{\frac{1}{q\theta}} \leq \left\| v \right\|_{L_q^\theta L_\theta^\gamma(U;W_1^{1,p})} + 1.
\]

**Proof.** We aim at applying Lemma 17 with \(E = W_1^{1,p}\) and \(r = \infty\). Similar to Proposition 4 one checks that \(W_1^{1,p}\) is a type 2 Banach space. Now due to Lemma 17 and Lemma 2
\[
\left( \mathbb{E} \left[ \left\| I(G(v)) \right\|_{B_q^\infty(0,T;W_1^{1,p})}^{q\theta} \right] \right)^{\frac{1}{q\theta}} \leq \left\| G(v) \right\|_{L_q^\theta L_\theta^\gamma(U;W_1^{1,p})} \leq \left\| v \right\|_{L_q^\theta L_\theta^\gamma(W_1^{1,p})} + 1.
\]

\[\square\]

Lemma 17 and Corollary 19 guarantee the stability of the stochastic integral operator \(I\) respectively the compositional operator \(I \circ G\) under a prescribed integrability in probability \(L_q^\theta\). We observe that we lose a tiny bit of integrability in time, when prescribing the same integrability in probability of the input and the output space. However if we allow for different integrability properties in probability, one can even establish stability of the stochastic integral operator on exponentially integrable Besov spaces in time as done by Ondreját and Veraar in [OV20].

**Lemma 20** ([OV20] Theorem 3.2 (vi)). Let \((E, \left\| . \right\|_E)\) be a separable Banach space of type 2 and \(G \in L_q^\theta L_\theta^\gamma(U;E)\). Then there exists a constant \(C > 0\) such that for all \(q \in [1, \infty)\) and \(r \in (2, \infty)\) we have
\[
(3.5) \quad \left( \mathbb{E} \left[ \left\| I(G) \right\|_{B_{q\Phi_2}(0,T;E)}^{q\theta} \right] \right)^{\frac{1}{q\theta}} \leq C \sqrt{q} \left\| G \right\|_{L_q^\theta L_\theta^\gamma(U;L_r^\infty([0,T;\gamma(U;E)]))} ,
\]
where \(\alpha = \frac{1}{2} - \frac{1}{r}\), \(\Phi_2(t) = e^{t^2} - 1\) and \(N_q(t) = t^q \log^{q/2}(t + 1)\).
An immediate consequence is the stability of the compositional operator.

**Corollary 21.** Let $p \geq 2$ and Assumption 1 be satisfied. Then for all $q \in [1, \infty)$
\begin{equation}
\|I(G(v))\|_{L^q_B1/2,\infty(0,T;L^p_x)} \lesssim \|v\|_{L^\infty_tL^\infty_xW^1_p} + 1.
\end{equation}

Before we state the main result, let us shortly explain where the exponentially integrable spaces are located in terms of the usual scale of Lebesgue spaces. For a general overview we refer to the book [DHHR11].

**Lemma 22** ([DHHR11] Corollary 3.3.4). Let $\Phi_2(t) = e^{t^2} - 1$. Then
\begin{equation}
L^\infty(I) \hookrightarrow L^{\Phi_2}(I) \hookrightarrow L^{-\infty}(I)
\end{equation}
with continuous embeddings, i.e. for all $q \in [1, \infty)$ there exist constants $C, C_q > 0$ such that for all $u \in L^\infty(I)$
\begin{equation}
\|u\|_{L^q(I)} \leq C_q \|u\|_{L^\Phi_2(I)} \leq C \|u\|_{L^{-\infty}(I)}.
\end{equation}

### 3.2. Exponential Besov regularity

We are ready to formulate the first main result on time regularity of strong solutions:

**Theorem 23** (Exponential Besov regularity). Let the assumptions of Theorem 12 be satisfied. Let $u$ be the unique strong solution to (2.14). Additionally assume
\begin{equation}
u \in L^q_xL^\infty_tL^2_x
\end{equation}
for some $q > 2$. Then $u \in L^2_xB^{1/2}_{q,\infty}$ with
\begin{equation}
\|u\|_{L^2_xB^{1/2}_{q,\infty}} \lesssim \|u_0\|^{p/2}_{L^q_xW^1_p} + \|u\|_{L^\infty_tL^\infty_xL^2_x} + 1,
\end{equation}
where $\Phi_2(t) = e^{t^2} - 1$.

**Proof.** Since we assume that $u$ is a strong solution to (2.14) it holds for almost all $(\omega, t, x)$
\begin{equation}
u(t) = u_0 + \int_0^t \text{div} S(\nabla u) \, ds + I(G(u))(t).
\end{equation}

Therefore, due to the triangle inequality and Lemma 22
\begin{align*}
\|u\|_{L^2_xL^{p_2}_tL^2_x} \leq \|u_0\|_{L^2_xL^{p_2}_tL^2_x} + \left\| \int_0^t \text{div} S(\nabla u) \, ds \right\|_{L^2_xL^{p_2}_tL^2_x} + \|I(G(u))\|_{L^2_xL^{p_2}_tL^2_x} \\
\lesssim \|u_0\|_{L^2_xL^2} + \|\text{div} S(\nabla u)\|_{L^2_xL^2_tL^2_x} + \|I(G(u))\|_{L^2_xL^{p_2}_tL^2_x}.
\end{align*}
The first term is bounded by assumption. The second term is bounded by the a priori estimate (2.18). The last term is controlled by Corollary 21. Thus,
\begin{align*}
\|u\|_{L^2_xL^{p_2}_tL^2_x} \lesssim \|u_0\|_{L^2_xL^2} + \|u_0\|^{p/2}_{L^q_xW^1_p} + 1 + \|u\|_{L^\infty_tL^\infty_xL^2_x}.
\end{align*}

It remains to estimate the Besov-Orlicz semi-norm
\begin{equation}
E \left[ \|u\|^2_{B^{1/2}_{q,\infty}(t,L^2_x)} \right] = E \left[ \left( \sup_h h^{-1/2} \|\nabla u\|_{L^p_x((t-h,-h);L^2_x)} \right)^2 \right].
\end{equation}
Using the representation (3.10)
\[
\mathbb{E} \left[ \frac{u}{\|B_{\|x,\infty}^{1/2}(I;L^2_x)} \right]^2 \lesssim \mathbb{E} \left[ \left( \sup_{h} h^{-1/2} \left\| \int_{t}^{t+h} \text{div} S(\nabla u) \, ds \right\|_{L^2_x((t-h),t)} \right)^2 \right] \\
+ \mathbb{E} \left[ \left( \sup_{h} h^{-1/2} \left\| \tau_h I(G(u)) \right\|_{L^2_x((t-h),t)} \right)^2 \right] \\
=: I + II.
\]

The first term can be estimated due to the \(L^2_x L^2_t\) integrability of the expression \(\|\text{div}(\nabla u)\|_{L^2_x}^2\). Invoking Lemma 22 and Hölder’s inequality
\[
I \leq \mathbb{E} \left[ \left( \sup_{h} h^{-1/2} \sup_{t \in (t-h), t \in (t-h)} \int_{t}^{t+h} \left\| \text{div} S(\nabla u) \right\|_{L^2_x}^2 \, ds \right)^2 \right] \\
\leq \mathbb{E} \left[ \sup_{h} \sup_{t \in (t-h), t \in (t-h)} \int_{t}^{t+h} \left\| \text{div} S(\nabla u) \right\|_{L^2_x}^2 \, ds \right] \\
\leq \left\| \text{div} S(\nabla u) \right\|_{L^2_t L^2_x}^2 \\
\lesssim \|u_0\|_{L_x^p W^{1,p}_t}^2 + 1,
\]
where in the last line we applied the a priori estimate (2.18). The second term is bounded via Corollary 21
\[
II \leq \left\| I(G(u)) \right\|_{L^2_t B_{\|x,\infty}^{1/2}(0,T;L^2_x)}^2 \lesssim \|u\|_{L^\infty_t L^p_x}^2 + 1.
\]
Overall, the claim (3.9) follows and the proof is finished. \(\square\)

**Corollary 24.** Let the assumptions of Theorem 23 be satisfied. Then it holds
\[
\|u\|_{L^2_t C^{1/2}_t; L^\infty_x} \leq \|u_0\|_{L_x^p W^{1,p}_t}^{p/2} + \|u\|_{L^\infty_t L^p_x}^2 + 1.
\]

**Remark 25.** Hytönen and Veraar have obtained a precise statement about time regularity in terms of Besov spaces for Banach space valued Brownian motions in [HV08]. Indeed, they show that any \(E\)-valued Brownian motion \(W\) satisfies
\[
W \in B_{\|x,\infty}^{1/2}(I;E) \quad a.s.
\]

Theorem 23 proves that the \(L^2_x\)-valued solution process \(u\) is as regular in time as a \(L^2_t\)-valued Brownian motion.

Additionally, the divergence of the nonlinear operator in the representation (3.10) has only slightly higher time regularity in comparision to the stochastic term, i.e.
\[
(\omega, t, x) \mapsto \int_{0}^{t} \text{div} S(\nabla u) \, ds \in L^2_x W^{1,2}_t L^2_t \hookrightarrow L^2_x C^{1/2}_t L^2_t \hookrightarrow L^2_x B_{\|x,\infty}^{1/2} L^2_t,
\]
with continuous embeddings. However, as Corollary 21 suggests, the stochastic integral even has Besov regularity on the gradient level.

The assumption (3.8) is not restrictive. It can be verified under suitable integrability assumptions on the initial condition, e.g. \(u_0 \in L^8_x L^2_t\) is sufficient to ensure
\[
\mathbb{E} \left[ \sup_t \|u\|_{L^2_x}^2 \right] \leq c_q \left( \mathbb{E} \left[ \|u_0\|_{L^2_x}^2 \right] + 1 \right).
\]
3.3. **Nikolskii regularity of nonlinear gradient.** The key ingredient while deriving estimates for the nonlinear term \( V(\nabla u) \) is the \( V \)-coercivity Lemma 7 that relates the nonlinear operators \( S \) and \( V \). Now we present the second main result:

**Theorem 26** (Nikolskii regularity for nonlinear gradient). *Let the assumptions of Theorem 12 be satisfied. Let \( u \) be the unique strong solution to (2.14). Additionally assume\(^{11}\)

\[
(3.12) \quad u \in L^q_\omega L^\infty_\omega W^{1,p}_2
\]

for some \( q > p \). Then \( V(\nabla u) \in L^2_\omega B^{1/2}_{2,\infty} L^2_\omega \) with

\[
(3.13) \quad \|V(\nabla u)\|^2_{L^2_\omega B^{1/2}_{2,\infty} L^2_\omega} \lesssim h_0 \|u\|^p_{L^p_\omega L^\infty_\omega} + \|u\|^p_{L^p_\omega L^\infty_\omega} + 1.
\]

**Proof.** Let \( u \) be the strong solution to (2.14). Take \( t, h \in I \) such that \( t + h \in I \). Due to Lemma 7 it holds

\[
(3.14) \quad \|\tau_h (V(\nabla u))(t)\|^2_{L^2_\omega} \approx \int_\Omega \tau_h (S(\nabla u))(t) \cdot \tau_h (\nabla u)(t) \, dx.
\]

Integration by parts and the solution formula (3.10) then imply

\[
(3.15) \quad \|\tau_h (V(\nabla u))(t)\|^2_{L^2_\omega} \approx - \int_\Omega \tau_h (\text{div} \, S(\nabla u))(t) \cdot \tau_h (u)(t) \, dx
\]

\[
= - \int_\Omega \tau_h (\text{div} \, S(\nabla u))(t) \cdot \left( \int_t^{t+h} \text{div} \, S(\nabla u) \, ds + \tau_h (\mathcal{I}(G(u)))(t) \right) \, dx
\]

\[
=: 1 + \Pi.
\]

The first term can be estimated by Hölder’s and Young’s inequalities

\[
I \leq \|\tau_h (\text{div} \, S(\nabla u))(t)\|_{L^2_\omega} \left\| \int_t^{t+h} \text{div} \, S(\nabla u) \, ds \right\|_{L^2_\omega}
\]

\[
(3.16) \quad \leq h \left( \|\tau_h (\text{div} \, S(\nabla u))(t)\|^2_{L^2_\omega} + \int_t^{t+h} \|\text{div} \, S(\nabla u)\|^2_{L^2_\omega} \, ds \right).
\]

The second term needs a more sophisticated analysis. Due to the Assumption 1 (iii) the stochastic integral preserves zero boundary values. Therefore, integration by parts and Lemma 9 reveal

\[
\Pi = \int_\Omega \tau_h (S(\nabla u))(t) \cdot \tau_h (\mathcal{I}(G(u)))(t) \, dx
\]

\[
\leq \delta \|\tau_h V(\nabla u)(t)\|^2_{L^2_\omega}
\]

\[
+ c \int_\Omega (k + |\nabla u(t)| + |\tau_h (\nabla u)(t)|)^{p-2} |\tau_h (\nabla u)(t)|^2 \, dx
\]

\[
=: \Pi_1 + \Pi_2.
\]

Since \( \delta > 0 \) is arbitrary, we can absorb the first term of (3.17) to the left hand side.

Let us first discuss the case \( p > 2 \). Here the latter one is estimated by the weighted Young’s inequality with exponents \( \theta = p/(p - 2) \) and \( \theta' = p/2 \) to get

\[
(3.18) \quad \Pi_2 \lesssim h \left( \frac{p-2}{p} \left( \sup_t \|\nabla u\|^p_{L^p_\omega} + 1 \right) + \frac{2}{p} h^{-p/2} \|\tau_h \mathcal{I}(G(u))(t)\|^p_{W^{1,p}_2} \right).
\]
In the case \( p = 2 \) it trivially holds
\[
(3.19) \quad \Pi_2 \lesssim \| \tau_k I(G(u))(t) \|^2_{L^2_t L^2_x},
\]
which coincides with (3.18) for \( p = 2 \).

Overall, we integrate (3.14) over \( I \cap I - \{ h \} \) and multiply by \( h^{-1} \). Applying the estimates (3.16) and (3.18), we arrive at
\[
h^{-1} \int_{I \cap I - \{ h \}} \| \tau_k V(\nabla u)(t) \|^2_{L^2_t} \, dt \\
\lesssim \int_{I \cap I - \{ h \}} \left( \| \tau_k (\text{div } S(\nabla u))(t) \|^2_{L^2_t} + \int_{t}^{t+h} \| \text{div } S(\nabla u) \|^2_{L^2_t} \, ds \right) \, dt \\
+ \int_{I \cap I - \{ h \}} \left( \frac{p-2}{p} \left( \sup_{t} \| \nabla u \|_{L^p_t L^p_x}^p + 1 \right) + \frac{2}{p} h^{-p/2} \| \tau_k I(G(u))(t) \|_{W^{1,p}_x}^p \right) \, dt.
\]

Finally, take the supremum in \( h \) and expectation. Due to Fubini’s theorem
\[
E \left[ \| V(\nabla u) \|^2_{L^{1/2}_{t} L^2_x} \right] = E \left[ \left( \sup_{h} h^{-1/2} \left( \int_{I \cap I - \{ h \}} \| \tau_k V(\nabla u)(t) \|^2_{L^2_t} \, dt \right) \right)^{\frac{p}{2}} \right]^{2}
\]
\[
\lesssim \| \text{div } S(\nabla u) \|^2_{L^{1/2}_{t} L^2_x} + \frac{p-2}{p} \left( \| \nabla u \|_{L^p_t L^p_x}^p + 1 \right)
\]
\[
+ \frac{2}{p} E \left[ \left( \sup_{h} h^{-1/2} \left( \int_{I \cap I - \{ h \}} \| \tau_k I(G(u)) \|_{W^{1,p}_x}^p \, ds \right) \right)^{\frac{p}{2}} \right]^{2}
\]
\[
\leq \| \text{div } S(\nabla u) \|^2_{L^{1/2}_{t} L^2_x} + \frac{p-2}{p} \left( \| \nabla u \|_{L^p_t L^p_x}^p + 1 \right) + \frac{2}{p} \| I(G(u)) \|_{L^p_t B^{1/2}_{1,2} L^2_x}^p.
\]

Lastly, it remains to use Lemma 22 and Corollary 21 with \( q = p + \varepsilon \) to bound the third term in the estimate
\[
(3.20) \quad \| I(G(u)) \|_{L^p_t B^{1/2}_{1,2} W^{1,p}_x}^p \lesssim \| I(G(u)) \|_{L^p_t B^{1/2}_{1,2} W^{1,p}_x}^p \lesssim \| u \|_{L^p_t L^p_x}^p + 1.
\]

Alltogether, we have proven
\[
E \left[ \| V(\nabla u) \|^2_{L^{1/2}_{t} L^2_x} \right] \lesssim \| \text{div } S(\nabla u) \|^2_{L^{1/2}_{t} L^2_x} + \| u \|_{L^p_t L^p_x}^p W^{1,p}_x + 1.
\]

The remaining part of the norm is controlled by Hölder’s inequality
\[
\| V(\nabla u) \|^2_{L^2_t L^2_x} = \| \nabla u \|^2_{L^p_t L^p_x} \lesssim \| \nabla u \|^2_{L^p_t L^p_x}.
\]

Overall, using the a priori estimate (2.18), we arrive at
\[
\| V(\nabla u) \|^2_{L^{1/2}_{t} L^2_x} \lesssim \| \text{div } S(\nabla u) \|^2_{L^{1/2}_{t} L^2_x} + \| u \|_{L^p_t L^p_x}^p W^{1,p}_x + 1
\]
\[
\lesssim \| u \|_{L^p_t L^p_x}^p W^{1,p}_x + \| u \|_{L^p_t L^p_x}^p W^{1,p}_x + 1.
\]

The assertion is proved. \( \square \)

**Remark 27.** The only reason, why we assume condition (iii) in Assumption 1, is, so that we can revert the partial integration after using the strong formulation of the equation (2.14).
The additional assumption (3.12) is also not very restrictive. It can be verified with appropriate assumptions on the initial condition, e.g. $u_0 \in L^q_x W^{1,p}_x$ for some $q > p$ and follows in the same line as the proof of Theorem 12.

If we do not want to allow for an increased integrability assumption on $u$ as (3.12), we still obtain regularity estimates by using Corollary 19 instead of Corollary 21 in the proof of Theorem 26

$$\|V(\nabla u)\|_{L^2_t B^{1/2}_{q_2,\infty} L^2_x} \lesssim \|u_0\|_{L^p_x W^{1,p}_x}^p + \|u\|_{L^p_t L^\infty_x W^{1,p}_x}^p + 1.$$

We want to point out, that one can use the regularity estimate (3.9) together with (3.13) in the analysis of numerical algorithms. In [BHL21] the authors use $\alpha$-Hölder regularity of the solution process $u$ and $\alpha$-fractional Sobolev regularity of $V(\nabla u)$ to obtain convergence of order $\tau^\alpha$, $\alpha < 1/2$. If one instead uses the exponential Besov regularity of $u$ and the Nikolskii regularity of $V(\nabla u)$ this can be improved to $\sqrt{\tau \ln(1 + \tau^{-1})}$.

3.4. Higher order Nikolskii regularity of nonlinear gradient. Surprisingly the time regularity of the nonlinear gradient $V(\nabla u)$ is restricted due to lack of time integrability of the nonlinear diffusion $\|\operatorname{div} S(\nabla u)\|_{L^q_x} \in L^q(0, T)$ for $q > 2$ rather than the reduced time regularity of the stochastic integral.

**Theorem 28** (Higher gradient regularity). Let the Assumption 1 be satisfied. Additionally, assume

\begin{equation}
\begin{aligned}
(3.21a) & \quad u \in L^{q_1}_t L^\infty_x W^{1,p}_x, \\
(3.21b) & \quad \operatorname{div} S(\nabla u) \in L^2_t L^p_x L^2_x,
\end{aligned}
\end{equation}

for some $q_1 > p$ and $q_2 \geq 2$. Let $u$ be a strong solution to (2.14). Then $V(\nabla u) \in L^2_t B^{1/2}_{q_2,\infty} L^2_x$ with

\begin{equation}
\|V(\nabla u)\|_{L^2_t B^{1/2}_{q_2,\infty} L^2_x} \lesssim \|\operatorname{div} S(\nabla u)\|_{L^2_t L^p_x L^2_x}^2 + \|\nabla u\|_{L^{q_2}_t L^\infty_x L^{p_2}_x}^p + 1.
\end{equation}

**Proof.** The proof proceeds similar to the proof of Theorem 26. Let $t, h \in I$ such that $t + h \in I$. Recalling (3.15), we arrive at

$$\|\tau_h V(\nabla u)\|_{L^2_x} \approx (1 + \Pi)^{q_2/2} \lesssim \Pi^{q_2/2} + \Pi^{q_2/2}.$$  

Due to (3.16),

$$\Pi^{q_2/2} \lesssim h^{q_2/2} \left( \|\tau_h (\operatorname{div} S(\nabla u))(t)\|_{L^2_x}^2 + \int_t^{t+h} \|\operatorname{div} S(\nabla u)\|_{L^2_x}^2 \, ds \right).$$

The second term is estimated as in (3.17) and (3.18)

$$\Pi^{q_2/2} \lesssim \|\tau_h V(\nabla u)\|_{L^2_x}^{q_2}$$

$$+ h^{q_2/2} \left( \frac{p-2}{p} \left( \sup_t \|\nabla u\|_{L^p_x}^{p_2/2} + 1 \right) + \frac{2}{p} \left( h^{-1/2} \|\tau_h \mathcal{I}(G(u))(t)\|_{W^{1,p}_x}^{pq_2/2} \right) \right).$$

Choosing $\delta > 0$ sufficiently small, we may absorb the first term to the left hand side. Multiplication by $h^{-q_2/2}$ yields

$$h^{-q_2/2} \|\tau_h V(\nabla u)\|_{L^2_x}^{q_2} \lesssim \|\tau_h (\operatorname{div} S(\nabla u))(t)\|_{L^2_x}^{q_2} + \int_t^{t+h} \|\operatorname{div} S(\nabla u)\|_{L^2_x}^{q_2} \, ds$$

$$+ \sup_t \|\nabla u\|_{L^p_x}^{pq_2/2} + 1 + \left( h^{-1/2} \|\tau_h \mathcal{I}(G(u))(t)\|_{W^{1,p}_x}^{pq_2/2} \right).$$
Finally, integrate the parameter $t$ in time, take the $2/q_2$-th power, supremum over $h$ and expectation
\[
\mathbb{E} \left[ |V(\nabla u)|^2_{L^2_B H^1 L^2_{\omega}} \right] \lesssim \|\text{div} \, S(\nabla u)\|^2_{L^2_B L^2_{\omega} L^2_{\omega}} + \|\nabla u\|^p_{L^p_B L^p_{\omega} L^p_{\omega}} + \mathbb{E} \left[ |\mathcal{I}(G(u))|^p_{B^{1/2}_{q_2/2, \infty} L^1_{\omega}} \right] + 1.
\]

The seminorm estimate now follows by an application of (3.20). Hölder’s inequality establishes the remaining estimate
\[
\|V(\nabla u)|^2_{L^2_B L^2_{\omega} L^2_{\omega}} = \|\nabla u\|^p_{L^p_B L^{p+2/2}_{\omega} L^p_{\omega}} \lesssim \|\nabla u\|^p_{L^p_B L^\infty_{\omega} L^p_{\omega}}.
\]

The assertion is proved. □

**Remark 29.** It is not know whether the assumption (3.21b) can be verified under appropriate assumptions on the domain and the noise coefficient $G$ as done in Theorem 12 for the case $q_2 = 2$.

An immediate consequence is the Hölder regularity of the map $t \mapsto V(\nabla u)(t)$ as an $L^2_{\omega}$-valued process, since
\[
\|V(\nabla u)|^2_{L^2 B^{1/2}_{q_2/2, \infty} L^1_{\omega}} \lesssim \|V(\nabla u)|^2_{L^2_B H^1 L^2_{\omega}}.
\]

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