On the application of homotopy–perturbation and Adomian decomposition methods to the linear and nonlinear Schrödinger equations

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Abstract

I discuss a recent application of homotopy perturbation and Adomian decomposition methods to the linear and nonlinear Schrödinger equations. I propose a generalization of the procedure for the treatment of a wider class of problems.

1 Introduction

The Homotopy Perturbation Method (HPM) [1–15] (the reader should not mistake this abbreviation with that for the Hypervirial Perturbation Method [16]) is supposed to solve any linear and nonlinear problem in any field of theoretical physics. In a couple of papers I have shown that some of those results are useless and worthless [17,18]. My criticisms have not been welcomed by the referees (most probably homotopy devotees) and therefore they remain unpublished.

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The purpose of this communication is to analyze a recent application of HPM and Adomian decomposition method (ADM) to the linear and nonlinear Schrödinger equations [15]. Since those equations are known to be relevant to many fields of physics, any approach to solve them is always welcome.

2 Linear Schrödinger equation

Sadighi and Ganji [15] consider the linear Schrödinger equation

\[
 u_t(x, t) + iu_{xx}(x, t) = 0 \tag{1}
\]

where \(i^2 = -1\) and the subscripts indicate derivatives with respect to the variables. We immediately recognize this differential equation as a dimensionless version of the Schrödinger equation for the free particle. Its propagator is well known and there is no reason for any special method to solve it. However, Sadighi and Ganji [15] think otherwise and apply the HPM to it.

Most textbooks of quantum mechanics solve equation (1) for a physical square–integrable initial function \(u(x, 0)\)

\[
 \int_{-\infty}^{\infty} |u(x, 0)|^2 dx < \infty \tag{2}
\]

however an homotopy devotee does not care about such a small detail. In fact, Sadighi and Ganji [15] cleverly choose some particular unphysical initial functions, so that they solve the resulting equations much more easily.

Any undergraduate student in a first course of quantum mechanics knows that if one chooses an unphysical initial function of the form

\[
 u(x, 0) = \sum_j c_j e^{\alpha_j x} \tag{3}
\]
then the resulting unphysical solution will be

\[ u(x, t) = \sum_j c_j e^{\alpha_j x - i\alpha_j^2 t} \] (4)

Notice that the initial function (3) does not satisfy (2).

The first example chosen by Sadighi and Ganji [15]

\[ u(x, 0) = 1 + 2 \cosh(2x) \] (5)

is a particular case of (3). If you think that it is a nonsensical quantum–mechanical state then you are not an homotopy devotee. It follows from equation (4) that

\[ u(x, t) = 1 + 2 \cosh(2x) e^{-4it} \] (6)

After a masterful application of the HPM Sadighi and Ganji [15] obtain

\[ u(x, t) = 1 + 2 \cosh(2x) \left[ 1 - 4it + \frac{(4it)^2}{2!} + \ldots \right] \] (7)

and brilliantly conclude that the limit of this series is equation (6).

Example 2 is even simpler than example 1; after some algebraic manipulation Sadighi and Ganji [15] are led to the revelation that when

\[ u(x, 0) = e^{3ix} \] (8)

then

\[ u(x, t) = e^{3ix} \left[ 1 + 9it + \frac{(9it)^2}{2!} + \ldots \right] = e^{3(x+3it)} \] (9)

which is obviously a particular case of equation (4). I insist that if you think that this solution is completely useless from a physical point of view, then you are not an homotopy devotee.
3 Nonlinear Schrödinger equation

The nonlinear Schrödinger equation

\[ iu_t + u_{xx} + \gamma |u|^2 u = 0 \]  \hspace{1cm} (10)

is a much more interesting and challenging problem. However, Sadighi and Ganji [15] cleverly restrict themselves to unphysical solutions with unit modulus \(|u| = 1\) so that the problem reduces to a trivial linear differential equation with constant coefficients:

\[ iu_t + u_{xx} + \gamma u = 0 \]  \hspace{1cm} (11)

If you think that this restriction is ridiculous then I tell you that you are not an homotopy devotee.

Any student in a first course of calculus will immediately realize that a particular solution is

\[ u(x, t) = e^{i\alpha x} e^{i(\gamma - \alpha^2)t} \]  \hspace{1cm} (12)

When \(\alpha = 1\) and \(\gamma = 2\) we obtain

\[ u(x, t) = e^{i(x+t)} \]  \hspace{1cm} (13)

which is exactly the example 3 of Sadighi and Ganji [15]. After masterfully solving the tedious HPM equations Sadighi and Ganji [15] obtain

\[ u(x, t) = e^{ix} \left[ 1 + i t + \frac{(it)^2}{2!} + \ldots \right] \]  \hspace{1cm} (14)

which obviously converges to (13).
When $\alpha = 1$ and $\gamma = -2$ equation (12) reduces to the solution of example 4:

$$u(x, t) = e^{i(x-3t)}$$

(15)

As the reader may have guessed, after solving the HPM equations Sadighi and Ganji [15] obtain the power series

$$u(x, t) = e^{ix} \left[ 1 - 3it + \frac{(3it)^2}{2!} + \ldots \right]$$

(16)

that clearly converges towards (15).

It is pointless telling an homotopy devotee that all those calculations are useless and worthless, he is content with showing that his calculation yields the exact result known to everybody.

The reader should not think that Sadighi and Ganji [15] merely restrict to the calculation of the power series of exponential functions $e^{ibt}$ by means of HPM. They also do exactly the same by means of another of the most fashionable approaches: the Adomian decomposition method (ADM) [15].

4 Conclusions

The discussion of the preceding sections show that Sadighi and Ganji [15] study the simplest linear and nonlinear Schrödinger equations. They choose unphysical solutions that do not correspond to actual quantum–mechanical states but are more tractable. One such choice converts the nonlinear Schrödinger equation into a linear one and consequently the authors never solve the nonlinear problem. Then they apply two approximate methods, the HPM and ADM, and obtain a power series approximation to the solutions. As shown above, the calculation reduces to the Taylor expansion of exponential functions of the form $e^{ibt}$ about $t = 0$. If the reader thinks that it is not a great achievement I
will prove him/her wrong. First of all, notice that the authors do not use the obvious formula

$$u(x, t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \frac{\partial^j u}{\partial t^j} \bigg|_{t=0}$$  \hfill (17)

but apply two cumbersome methods in order to obtain this expansion. Second, they managed to publish their remarkably useless and nonsensical contribution in a research journal [15]. In another contribution I will analyze how one can do it.

Finally, I will like to propose a generalization of the great achievement of Sadighi and Ganji [15]. Instead of the simple linear equation (17) one may consider the more general one

$$u_t + i\hat{A}u = 0$$ \hfill (18)

where $\hat{A}$ is a linear operator. According to the textbooks on quantum mechanics, its formal solution is

$$u = e^{-it\hat{A}}u_0$$ \hfill (19)

where $u_0 = u(t = 0)$. A successful application of the HPM or ADM will lead to the series

$$u = \left[ 1 - it\hat{A} + \frac{(-it\hat{A})^2}{2!} + \ldots \right] u_0$$ \hfill (20)

If you do not succeed in publishing your results you may try a simpler problem; for example, when

$$\hat{A}u_0 = au_0$$ \hfill (21)
you have

\[ u = e^{-ita}u_0 \]  

(22)

and

\[ u = \left[ 1 - ita + \frac{(-ita)^2}{2!} + \ldots \right] u_0 \]  

(23)

Hopefully, the referee (most probably an homotopy devotee) will understand your simplified problem and then accept your manuscript.

If your manuscript is accepted then you become a member of the homotopy club and then you can try something bolder, for example

\[ u_0 = \sum_k c_k f_k \]  

(24)

where

\[ \hat{A}f_k = a_k f_k \]  

(25)

I leave the result as an exercise for the future homotopy devotee.

Notice that you can thus write a paper for every solvable or unsolvable problem which you can think of. Most probably, all of them will be accepted in any one of the homotopy journals [1–15] (unless you dare to criticize another homotopy result).

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